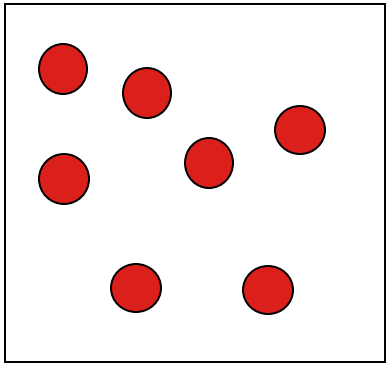
# 11th i-CoMSE Workshop: Mesoscale Particle-Based Modeling

**Mississippi State University**
**July 21–25, 2025**

## Session 5: Molecular Dynamics

# Classical Mechanics

- Classical mechanics: behavior of a system of *N* point particles system is governed by Newton's (2nd) Law



Equation of motion for particle *i*

$$\mathbf{F}_i = m_i \mathbf{a}_i = m_i \frac{d^2 \boldsymbol{r}_i}{dt^2} = \frac{d\boldsymbol{p}_i}{dt} \quad (1)$$

$\mathbf{F}$ = net force
$\mathbf{a}$ = acceleration
$\mathbf{r}$ = position
t = time
$\mathbf{v}$ = velocity
$\mathbf{p} = m\mathbf{v}$ = momentum

In 3D, bold quantities are 3-component Cartesian vectors

$$\boldsymbol{r}_\mathrm{i} = (x_i, \mathrm{y_i}, z_i) \qquad \mathbf{F}_\mathrm{i} = (F_{i,x}, F_{i,y}, F_{i,z}) \qquad \boldsymbol{p}_\mathrm{i} = (p_{i,x}, p_{i,y}, p_{i,z})$$

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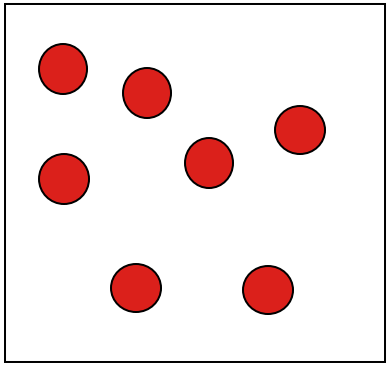

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The net force arises from interactions between particles is described by the *potential energy function* $\mathcal{U}(\mathbf{r_1}, \mathbf{r_2}, \ldots \mathbf{r_N})$ for the system and is, in general, a function of the Cartesian position vectors of all *N* particles

$$\mathbf{F}_i = \mathbf{F}_i(\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_N}) = -\frac{\partial \mathcal{U}(\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_N})}{\partial \boldsymbol{r}_i} \quad (2)$$

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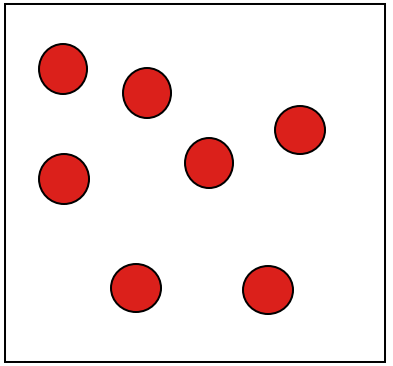

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Newton's Law (Eq. 1) yields:

- 3 differential equations for each particle
- 3*N* differential equations describing evolution of the *N* particle system

Integrating the equations for the *N* particle system:

- *3N* particle positions and *3N* particle momenta (functions of *t*)
- Requires specifying *6N* initial conditions at time *t*=0

We adopted a short-hand notation for these 6*N* functions $\{\mathbf{r}^\mathbf{N}, \mathbf{p}^\mathbf{N}\}$, where $\mathbf{r}^\mathbf{N} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_\mathbf{N})$ and $\boldsymbol{p}^\mathbf{N} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_\mathbf{N})$

# Classical Mechanics

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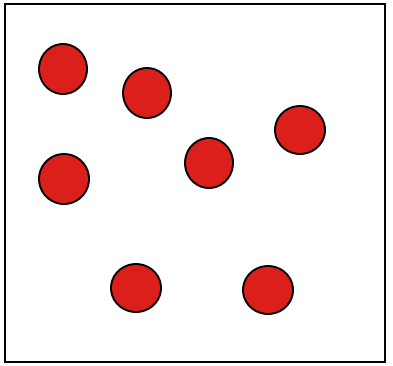


- The *classical state* of the system at a point in time is specified by the instantaneous values of $\{\mathbf{r}^{\mathbf{N}}, \mathbf{p}^{\mathbf{N}}\}$

- The instantaneous internal energy of the system is given by

$$E(\mathbf{p}^{\mathbf{N}}, \mathbf{r}^{\mathbf{N}}) = K(\mathbf{p}^{\mathbf{N}}) + \mathcal{U}(\mathbf{r}^{\mathbf{N}}) \qquad (3)$$

Kinetic Potential

where

$$K(\mathbf{p}^{\mathbf{N}}) = \sum_i \frac{\mathbf{p}_i \cdot \mathbf{p}_i}{2m_i}$$

- $E(\mathbf{p}^{\mathbf{N}}, \mathbf{r}^{\mathbf{N}})$ is often called the Hamiltonian, denoted $H(\mathbf{p}^{\mathbf{N}}, \mathbf{r}^{\mathbf{N}})$, and viewed as an operator acting on $\{\mathbf{r}^{\mathbf{N}}, \mathbf{p}^{\mathbf{N}}\}$

# Classical Mechanics

- Integration of Newton's Law results in a trajectory $\{\mathbf{r}^{\mathbf{N}}(t), \mathbf{p}^{\mathbf{N}}(t)\}$

- Conserved quantities along the trajectory of an isolated system

  – Total energy: $E(\mathbf{p}^{\mathbf{N}}(t), \mathbf{r}^{\mathbf{N}}(t))$ is conserved (constant)

  – Total linear momentum: $\sum_i p_{i,x}$ is conserved (so are $\sum_i p_{i,y}$ and $\sum_i p_{i,z}$)

$$\sum_i p_{i,x} = \sum_i m_i v_{i,x} = const.$$

  → Center-of-mass velocity: $v_{COM,x} = \sum_i m_i v_{i,x} / \sum_i m_i = const.$

- *Integration must be done numerically for many-body systems!*

# Molecular Dynamics

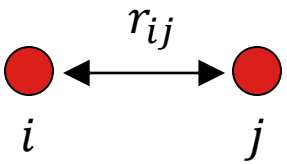
- Molecular dynamics (MD): solve Newton's equation of motion numerically via a finite difference scheme to obtain $\{\mathbf{r}^{\mathbf{N}}(t), \mathbf{p}^{\mathbf{N}}(t)\}$

1. **Input initial conditions** $\{\boldsymbol{r}^{\boldsymbol{N}}(t=0), \boldsymbol{p}^{\boldsymbol{N}}(t=0)\}$
   - Initial configuration $\mathbf{r}^{\mathbf{N}}(0)$
   - Initial momenta / velocities $\mathbf{p}^{\mathbf{N}}(0)$
2. **Compute the initial forces** $\boldsymbol{F}_i(t=0) = -\frac{\partial \mathcal{U}(\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_N})}{\partial \boldsymbol{r}_i}$
3. **Advance** $\{\mathbf{r}^{\mathbf{N}}, \mathbf{p}^{\mathbf{N}}\}$ **by time step** $\delta t$ **using a numerical integrator**
   - $\{\boldsymbol{r}^{\boldsymbol{N}}(t+\delta t), \boldsymbol{p}^{\boldsymbol{N}}(t+\delta t))\} \leftarrow \mathbf{integrator}\big(\{\boldsymbol{r}^{\boldsymbol{N}}(t), \boldsymbol{p}^{\boldsymbol{N}}(t)\}\big)$
   - $\boldsymbol{F}(t+\delta t)$ during each step using $\boldsymbol{r}^{\boldsymbol{N}}(t+\delta t)$
   - Repeated $n$ times to simulate elapsed time $\Delta t = n\delta t$
4. **Output properties periodically: positions, velocities, energies, etc.**

# MD: Initial Conditions

- **Initial configuration:** $\boldsymbol{r}^{\boldsymbol{N}}(t = 0)$:
  - Generating an initial configuration can be non-trivial as overlap of the atoms must be avoid to prevent unphysically high forces that create numerically instability when integrating with MD
  - Overlaps can be removed by performing energy minimization prior to initiating MD integration

Lennard-Jones

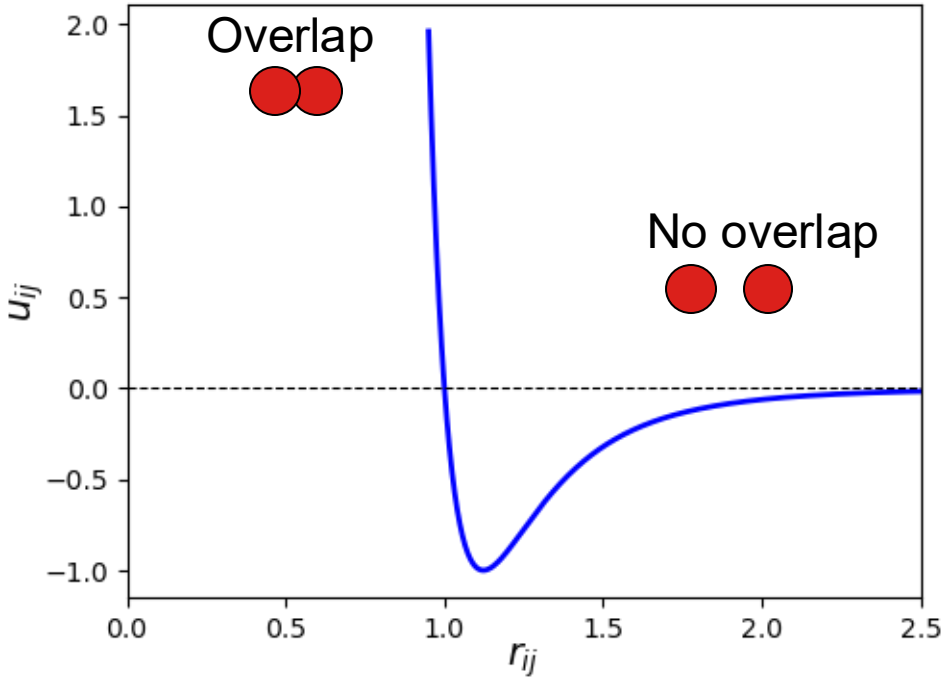



$$u(r_{ij}) = 4\varepsilon\left[\left(\frac{\sigma}{r_{ij}}\right)^{12} - \left(\frac{\sigma}{r_{ij}}\right)^{6}\right]$$

repulsion attraction

$\varepsilon =$ energy scale

$\sigma =$ length scale



# MD: Initial Conditions

- **Initial momenta:** $\boldsymbol{p}^N(t=0)$:
  - Generated by drawing them randomly from the Maxwell-Boltzmann (MB) distribution (the equilibrium distribution)

$$P(p_{i,x}) = \left(\frac{1}{2\pi m_i kT}\right)^{\frac{1}{2}} \exp\left(-\frac{p_{i,x}^2}{2m_i kT}\right)$$

$$P(\boldsymbol{p}) = P(p_x)P(p_y)P(p_z)$$

  - Other distributions (e.g., uniform) can be used but the MB distribution will always be recovered as the system equilibrates

  - Convenient to remove center-of-mass velocity to avoid translational motion of the entire system

$$\boldsymbol{v}_{COM} = \sum_i m_i \boldsymbol{v}_i / \sum_i m_i = 0 \rightarrow \boldsymbol{x}_{COM} = const.$$

# MD: Computing the forces

- **Compute the forces $\mathbf{F}_i = -\frac{\partial \mathcal{U}(\mathbf{r_1},\mathbf{r_2},\ldots,\mathbf{r_N})}{\partial \boldsymbol{r}_i}$ :**
  - Typically, > 90% of the computational work

$$\mathcal{U}(\mathbf{r}_1, \mathbf{r_2}, \ldots \mathbf{r_N}) = \sum_{i,j}^{N} u(r_{ij}) + \sum_{i,j,k}^{N} u_3(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) + \cdots$$

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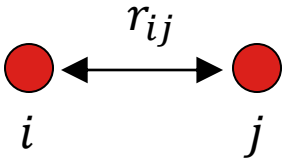
e.g., 2-body interactions involve loops over particle pairs




Non-bonded (Lennard-Jones)

$$u(r_{ij}) = 4\varepsilon\left[\left(\frac{\sigma}{r_{ij}}\right)^{12} - \left(\frac{\sigma}{r_{ij}}\right)^{6}\right]$$

Bonded (harmonic bond)

$$u(r_{ij}) = \frac{1}{2}k(r_{ij} - l)^2$$

$k$ = bond stiffness

$l$ = equilibrium bond length

**F**= 0; $\mathcal{U}$=0;
for particle $i = 1, N-1$
  for particle $j = i+1, N$
    $\boldsymbol{r}_{ij} = \boldsymbol{r}_j - \boldsymbol{r}_i$;  $r_{ij} = |\boldsymbol{r}_{ij}|$
    if $r_{ij} < r_{ij}^{\text{cutoff}}$:
      $\mathcal{U}$ += $u(r_{ij})$
      $\mathbf{f}_{ij} = (-\partial u(r_{ij})/\partial r_{ij})\, \boldsymbol{r}_{ij} / r_{ij}$
      $\mathbf{F}_i$ -= $\mathbf{f}_{ij}$
      $\mathbf{F}_j$ += $\mathbf{f}_{ij}$ (Newton's 3rd law)
    end; end

# MD: Computing the forces

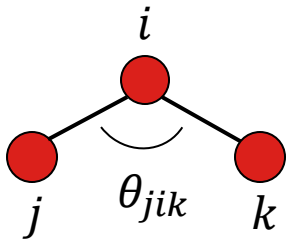
- **Compute the forces** $\mathbf{F}_i = -\frac{\partial \mathcal{U}(\mathbf{r_1},\mathbf{r_2},\ldots,\mathbf{r_N})}{\partial \boldsymbol{r}_i}$ **:**
  - Typically, > 90% of the computational work

$$\mathcal{U}(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_N) = \sum_{i,j}^{N} u(r_{ij}) + \sum_{i,j,k}^{N} u_3(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) + \cdots$$

e.g., 2-body interactions involve loops over particle pairs

e.g., 3-body interactions involve loops over particle triplets

Harmonic bond angle (3-body)




$$u_3(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) = \frac{1}{2} k(\theta_{jik} - \theta_0)^2 \quad \theta_{jik} = \cos^{-1}\left(\frac{\boldsymbol{r}_{ij} \cdot \boldsymbol{r}_{ik}}{\boldsymbol{r}_{ij}\boldsymbol{r}_{ik}}\right)$$

$k$ = bending stiffness

$\theta_0$ = equilibrium bond angle

# MD: Numerical integration

- Finite-difference numerical integrators used to advance trajectory in time. e.g., velocity Verlet method (very common)

$$\boldsymbol{v}_i\left(t+\frac{1}{2}\delta t\right)=\boldsymbol{v}_i(t)+\frac{\delta t}{2}\frac{\boldsymbol{F}_i(t)}{m_i}$$

$$\boldsymbol{r}_i(t+\delta t)=\boldsymbol{r}_i(t)+\delta t\boldsymbol{v}_i\left(t+\frac{1}{2}\delta t\right)$$

Compute $\boldsymbol{F}_i(t+\delta t)$ using $\boldsymbol{r}_i(t+\delta t)$

$$\boldsymbol{v}_i(t)=\boldsymbol{v}_i\left(t+\frac{1}{2}\delta t\right)+\frac{\delta t}{2}\frac{\boldsymbol{F}_i(t+\delta t)}{m_i}$$

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$$\boldsymbol{r}_i(t+\delta t)=\boldsymbol{r}_i(t)+\delta t\boldsymbol{v}_i\left(t+\frac{1}{2}\delta t\right)$$

Compute $\boldsymbol{F}_i(t+\delta t)$ using $\boldsymbol{r}_i(t+\delta t)$

$$\boldsymbol{v}_i(t)=\boldsymbol{v}_i\left(t+\tfrac{1}{2}\delta t\right)+\frac{\delta t}{2}\frac{\boldsymbol{F}_i(t+\delta t)}{m_i}$$

- Derived via Taylor expansions to relate $\boldsymbol{r}_i(t)$ to $\boldsymbol{r}_i(t+\delta t)$ and $\boldsymbol{r}_i(t-\delta t)$

e.g., $\boldsymbol{r}_i(t+\delta t)=\boldsymbol{r}_i(t)+\delta t\dot{\boldsymbol{r}}_i(t)+\frac{1}{2!}(\delta t)^2\ddot{\boldsymbol{r}}_i(t)+O(\delta t^3)$

where $\dot{\boldsymbol{r}}_i(t)=\frac{d\boldsymbol{r}_i}{dt}=\mathbf{v}_i$ and $\ddot{\boldsymbol{r}}_i(t)=\frac{d^2\boldsymbol{r}_i}{dt^2}=\boldsymbol{a}_i=\frac{\boldsymbol{F}_i(t)}{m_i}$

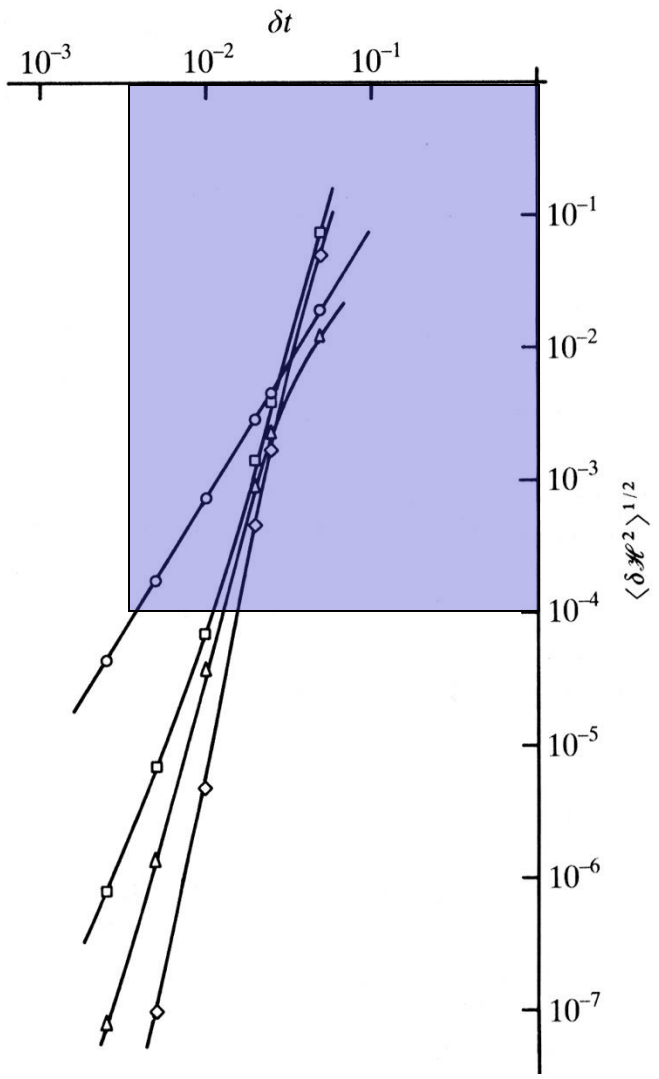
- Different integrators use different numbers of terms in the Taylor expansion; affects the cost, accuracy, and stability of the integrator

# MD: Numerical integration

- The time step $\delta t$ is an important parameter that must be chosen carefully based on the system and integrator
- Tradeoff in choosing the time step $\delta t$ between accuracy and performance → Larger time step generally less accurate, but allows one to simulate longer durations
- We desire:
  - Small enough so that truncation errors from the numerical method are negligible. This is usually tested by monitoring the total energy of the system as a function of time. If energy is not conserved, it means that $\delta t$ is too big.
  - Big enough so that the desired real time is covered with a minimum number of time steps.
- A rule of thumb: total energy $E$ should be conserved within 0.01% (1 part in $10^4$). Typically, this requires us to pick $\delta t$ ~ one tenth of the characteristic time of the fastest vibration in the system. For atoms or rigid molecules this is usually of the order of 10 fs = $10^{-14}$ s. Thus, for a typical system it takes ~ 100000 time steps to model 1 ns of real time. For flexible molecules we may need a time step of about 1 fs = $10^{-15}$s.

# MD: Numerical integration

- How well is is the total energy (or Hamiltonian) conserved?
- Mean square deviation of the total energy/mol. vs. time step (reduced units) for different methods
  - Circles: velocity Verlet (2nd order)
  - Squares: 4th-order Gear
  - Triangles: 5th-order Gear
  - Diamonds: 6th-order Gear
- As time step becomes larger, lower order methods become better



# MD: Numerical integration

- It is possible to look at longer time scales by removing the fastest degrees of freedom from the system. Some ways to do this are:
  - Treat the parts of the system involved in the fast mode (e.g. C-H vibrations) as a single particle, *i.e.* coarse grain out the fast degrees of freedom. This is done, for example, in the "united atoms" force fields.
  - Constrain the fast modes (e.g., make rigid C-H bonds effectively rigid).
- The contribution of the fast modes to the system's properties, if needed, must be added later. This assumes that the dynamics of the faster modes are uncoupled from the rest of the system.

# Exercise

- Molecular dynamics simulations of a 30-mer FENE polymer chain

$$u_{FENE}(r_{ij}) = -\frac{1}{2}kr_0^2\left[1-\left(\frac{r_{ij}}{r_0}\right)^2\right]$$

$$u_{WCA}(r_{ij}) = 4\varepsilon\left[\left(\frac{\sigma_{ij}}{r_{ij}}\right)^{12}-\left(\frac{\sigma_{ij}}{r_{ij}}\right)^{6}\right]+\varepsilon \text{ for } r < 2^{1/6}\sigma_{ij};\ 0 \text{ otherwise}$$

- Determine an appropriate time integration step by examining energy conservation

$$RMSF = \frac{\sqrt{(E(t)-\langle E\rangle)^2}}{\langle E\rangle} \leq 10^{-4}$$

- Compute average kinetic temperature and potential energy

$$\langle KE\rangle = \frac{1}{2}kTN_{dof};\quad N_{dof} = 3N-3$$

- Examine conservation of linear momentum $\sum_i m_i \boldsymbol{v}_i$