STAT3040 Assignment 2

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March 2021

1 Question 1

Firstly let $\{X_{t_1}, X_{t_2}, ..., X_{t_n}\}$ be a stationary time series. To verify that the sample autocovariance function $\hat{\gamma}_X(h)$ is a nonnegative definite function.
Using the moving average proof:
Firstly they state that the sample autocovariance function can be given by:

$$\hat{\gamma}_X(h) = \frac{1}{n} \sum_{t=h+1}^{n} (z_t - \overline{z})(z_{t-h} - \overline{z})$$

where $\overline{z} = \sum \frac{z_t}{n}$ for k ¡ 0 $\hat{\gamma}_h = \hat{\gamma}_{-h}$ and $\hat{\gamma}_h = 0$ for $h \geq n$

Then, it is important to note that the theoretical autocovariance function of a stationary time series can be estimated from n consecutive observations from the sample autocovariance function. A sequence c_k can be stated to be nonnegative definite if the resulting matrices C_m are nonnegative definite.It is clear that the theoretical autocovariance function is clearly nonnegative definite.
To verify the sample autocovariance function is a nonnegative definite function the moving average process is used. Showing that the theoretical autocovariance is the same as the sample autocovariance and proves that the sample autocovariance is nonnegative definite.
Firstly letting $d_t(t = 1, 2, 3, ...)$ be a moving average process of order n defined by the following:

$$d_t = \sum_{i=1}^{n} z_i e_{t-i}$$

where e_t is a white noise sequence with a variance 1/n. So it is a stationary time series which is a linear combination of White Noise from $i = \{1, .., n\}$.
Note that this is also a linear process with $\mu = 0$ and is a linear combination of White Noise variates. So this means the theoretical autocovariance, $\gamma_X(h)$ can be given by:

$$\gamma_x(h) = \sigma_e^2 \sum_{t=h+1}^{n} z_t z_{t-h}$$

But we know that $\sigma^2 = 1/n$ so:

$$\gamma_x(h) = \frac{1}{n} \sum_{t=h+1}^{n} z_t z_{t-h}$$

This theoretical autocovariance function is nonnegative definite as stated earlier. Additionally, now note that $\bar{z} = 0$ and this is clearly equivalent to the sample autocovariance function stated at the beginning.
So since the theoretical autocovarnace function is just the sample autocovariance function given earlier means that the sample autocovariance function is also nonnegative definite.

2 Bonus Question

Question:
Show that if the time series $\{X_{t_i}; t = 0, \pm 1, \pm 2, ...\}$ is a stationary Gaussian Process, then it is strictly stationary as well

Answer:
Note that a time series $\{X_{t_i}; t = 0, \pm 1, \pm 2, ...\}$ is said to be a Gaussian process if the n dimensional random vectors $X = (X_{t_1}, X_{t_2}, ..., X_{t_n})'$, for every collection of time points $t_1, t_2, ..., t_n$, and every positive integer n have a multivariate normal distribution.

Firstly assuming the time series $\{X_{t_i}; t = 0, \pm 1, \pm 2, ...\}$ meets this property and is a stationary Gaussian process.
Note that this multivariate Gaussian distribution will be fully characterised by the first two moments and has the following properties:

- The mean value function, μ_t is constant and does not depend on time t
- $Var(X_t) = \gamma_0; \forall t$
- $Cov(X_{t_1+h}, X_{t_2+h}) = Cov(X_{t_1}, X_{t_2})$ since it is a Gaussian process and depends on only the first two moments

This all means that:

$$f_{t_1+h, t_2+h, ..., t_n+h} = f_{t_1, t_2, ..., t_n}$$

So this means the behaviour of every collection of values $X_{t_1}, X_{t_2}, ..., X_{t_n}$ is identical to that of time shifted set $X_{t_1+h}, X_{t_2+h}, ..., X_{t_n+h}$ so the stationary Gaussian process is also strongly stationary.