Lecture 23

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Simpson's (perceived) paradox

		Death penalty		
Victim	Defendant	yes	no	% yes
White	White	53	414	11.3
	Black	11	37	22.9
Black	White	0	16	0.0
	Black	4	139	2.8
	White	53	430	11.0
	Black	15	176	7.9
White		64	451	12.4
Black		4	155	2.5

[1]

[1] From Agresti, Categorical Data Analysis, second edition

Discussion

- Marginally, white defendants received the death penalty a greater percentage of time than black defendants
- Across white and black victims, black defendant's received the death penalty a greater percentage of time than white defendants
- Simpson's paradox refers to the fact that marginal and conditional associations can be opposing
- The death penalty was enacted more often for the murder of a white victim than a black victim. Whites tend to kill whites, hence the larger marginal association.

Example

- Wikipedia's entry on Simpson's paradox gives an example comparing two player's batting averages

	First Half		Second Half		Whole Season	
Player 1	4/10	(.40)	25/100	(.25)	29/110	(.26)
Plater 2	35/100	(.35)	2/10	(.20)	37/110	(.34)

- Player 1 has a better batting average than Player 2 in both the first and second half of the season, yet has a worse batting average overall
- Consider the number of at-bats

Berkeley admissions data

- The Berkeley admissions data is a well known data set regarding Simpsons paradox

```
?UCBAdmissions
data(UCBAdmissions)
     apply(UCBAdmissions, c(1, 2), sum)
          Gender
Admit      Male Female
  Admitted 1198    557
  Rejected 1493   1278
           .445   .304 <- Acceptance rate
```

Acceptance rate by department

```
> apply(UCBAdmissions, 3,
        function(x) c(x[1] / sum(x[1 : 2]),
                      x[3] / sum(x[3 : 4])
                      )
        )
Dept   M    F
   A 0.62 0.82
   B 0.63 0.68
   C 0.37 0.34
   D 0.33 0.35
   E 0.28 0.24
   F 0.06 0.07
```

Why? The application rates by department

```
> apply(UCBAdmissions, c(2, 3), sum)
        Dept
Gender     A   B   C   D   E   F
  Male   825 560 325 417 191 373
  Female 108  25 593 375 393 341
```

Discussion

- Mathematically, Simpson's pardox is not paradoxical

$$\begin{aligned} a/b &< c/d \\ e/f &< g/h \\ (a+e)/(b+f) &> (c+g)/(d+h) \end{aligned}$$

- More statistically, it says that the apparent relationship between two variables can change in the light or absence of a third

Confounding

- Variables that are correlated with both the explanatory and response variables can distort the estimated effect
  - Victim's race was correlated with defendant's race and death penalty
- One strategy to adjust for confounding variables is to **stratify** by the confounder and then combine the strata-specific estimates
  - Requires appropriately weighting the strata-specific estimates
- Unnecessary stratification reduces precision

Aside: weighting

- Suppose that you have two unbiased scales, one with variance 1 lb and and one with variance 9 lbs
- Confronted with weights from both scales, would you give both measurements equal creedance?
- Suppose that $X_1 \sim N(\mu, \sigma_1^2)$ and $X_2 \sim N(\mu, \sigma_2^2)$ where σ_1 and σ_2 are both known
- log-likelihood for μ

$$-(x_1 - \mu)^2/2\sigma_1^2 - (x_2 - \mu)^2/2\sigma_2^2$$

Continued

- Derivative wrt μ set equal to 0

$$(x_1 - \mu)/\sigma_1^2 + (x_2 - \mu)/\sigma_2^2 = 0$$

- Answer

$$\frac{x_1 r_1 + x_2 r_2}{r_1 + r_2} = x_1 p + x_2(1 - p)$$

where $r_i = 1/\sigma_i^2$ and $p = r_1/(r_1 + r_2)$

- Note, if X_1 has very low variance, its term dominates the estimate of μ
- General principle: instead of averaging over several unbiased estimates, take an average weighted according to inverse variances
- For our example $\sigma_1^2 = 1$, $\sigma_2^2 = 9$ so $p = .9$

Mantel/Haenszel estimator

- Let n_{ijk} be entry i, j of table k
- The k^{th} sample odds ratio is $\hat{\theta}_k = \frac{n_{11k}n_{22k}}{n_{12k}n_{21k}}$
- The Mantel Haenszel estimator is of the form $\hat{\theta} = \frac{\sum_k r_k \hat{\theta}_k}{\sum_k r_k}$
- The weights are $r_k = \frac{n_{12k}n_{21k}}{n_{++k}}$
- The estimator simplifies to $\hat{\theta}_{MH} = \frac{\sum_k n_{11k}n_{22k}/n_{++k}}{\sum_k n_{12k}n_{21k}/n_{++k}}$
- SE of the log is given in Agresti (page 235) or Rosner (page 656)

	Center															
	1		2		3		4		5		6		7		8	
	S	F	S	F	S	F	S	F	S	F	S	F	S	F	S	F
T	11	25	16	4	14	5	2	14	6	11	1	10	1	4	4	2
C	10	27	22	10	7	12	1	16	0	12	0	10	1	8	6	1
n	73		52		38		33		29		21		14		13	

S - Success, F - failure
T - Active Drug, C - placebo[2]

$$\hat{\theta}_{MH} = \frac{(11 \times 27)/73 + (16 \times 10)/25 + \ldots + (4 \times 1)/13}{(10 \times 25)/73 + (4 \times 22)/25 + \ldots + (6 \times 2)/13)} = 2.13$$

Also $\log \hat{\theta}_{MH} = .758$ and $\hat{SE}_{\log \hat{\theta}_{MH}} = .303$

[2]Data from Agresti, Categorical Data Analysis, second edition

CMH test

- $H_0 : \theta_1 = \ldots = \theta_k = 1$ versus $H_a : \theta_1 = \ldots = \theta_k \neq 1$
- The CHM test applies to other alternatives, but is most powerful for the H_a given above
- Same as testing conditional independence of the response and exposure given the stratifying variable
- CMH conditioned on the rows and columns for each of the k contingency tables resulting in k hypergeometric distributions and leaving only the n_{11k} cells free

CMH test cont'd

- Under the conditioning and under the null hypothesis
  - $E(n_{11k}) = n_{1+k}n_{+1k}/n_{++k}$
  - $\mathrm{Var}(n_{11k}) = n_{1+k}n_{2+k}n_{+1k}n_{+2k}/n_{++k}^2(n_{++k} - 1)$
- The CMH test statistic is

$$\frac{[\sum_k \{n_{11k} - E(n_{11k})\}]^2}{\sum_k \mathrm{Var}(n_{11k})}$$

- For large sample sizes and under H_0, this test statistic is $\chi^2(1)$ (regardless of how many tables you are summing up)

In R

```
dat <- array(c(11, 10, 25, 27,  16, 22,  4, 10,
               14,  7,  5, 12,   2,  1, 14, 16,
                6,  0, 11, 12,   1,  0, 10, 10,
                1,  1,  4,  8,   4,  6,  2,  1),
             c(2, 2, 8))
mantelhaen.test(dat, correct = FALSE)
```

Results: $CMH_{TS} = 6.38$
P-value: .012
Test presents evidence to suggest that the treatment and response are not conditionally independent given center

Some final notes on CMH

- It's possible to perform an analogous test in a random effects logit model that benefits from a complete model specification
- It's also possible to test heterogeneity of the strata-specific odds ratios
- Exact tests (guarantee the type I error rate) are also possible `exact = TRUE` in R