Lecture 8

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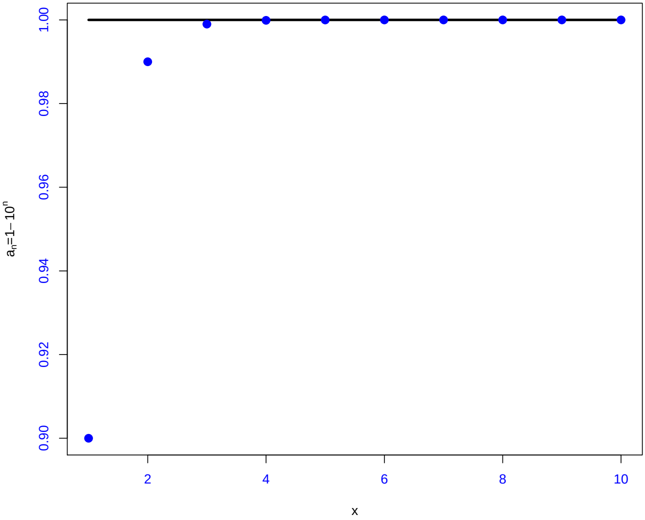
Outline

1. Define convergent series
2. Define the Law of Large Numbers
3. Define the Central Limit Theorem
4. Create Wald confidence intervals using the CLT

Numerical limits

- Imagine a sequence
    - $a_1 = .9$,
    - $a_2 = .99$,
    - $a_3 = .999, \ldots$
- Clearly this sequence converges to 1
- Definition of a limit: For any fixed distance we can find a point in the sequence so that the sequence is closer to the limit than that distance from that point on
- $|a_n - 1| = 10^{-n}$

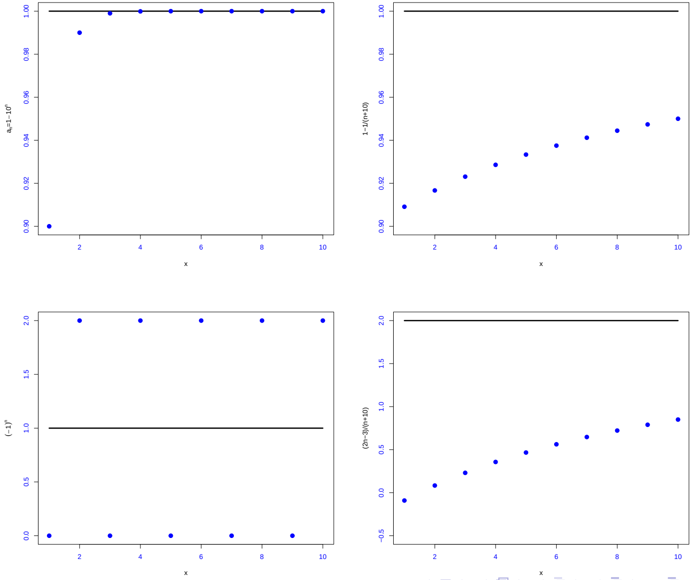


$a_n = 1 - 10^n$
1.00
0.98
0.96
0.94
0.92
0.90
2
4
6
8
10
x

More examples

- Sequence 1: $a_n = 1 - 10^n$
- Sequence 2: $a_n = 1 - \frac{1}{n+10}$
- Sequence 4: $a_n = 1 + (-1)^n$
- Sequence 3: $a_n = \frac{2n-3}{n+10}$

Examples



Limits of random variables

- The problem is harder for random variables
- Consider $\bar{X}_n$ the sample average of the first n of a collection of iid observations
    - Example $\bar{X}_n$ could be the average of the result of n coin flips (i.e. the sample proportion of heads)
- We say that $\bar{X}_n$ **converges in probability** to a limit if for any fixed distance the *probability* of $\bar{X}_n$ being closer (further away) than that distance from the limit converges to one (zero)
- $P(|\bar{X}_n - \text{limit}| < \epsilon) \to 1$

Why RVs are different

- Each experiment is different
- RVs are functions, not numbers
- Only after the experiment outcome is observed do we have a random variable realization
- Two scientists, same experiment
  - obtain different data
  - the summary of their experiments (e.g. the mean) converges to the same limit

The Law of Large Numbers

- Establishing that a random sequence converges to a limit is hard
- Fortunately, we have a theorem that does all the work for us, called the **Law of Large Numbers**
- The law of large numbers states that if $X_1, \ldots X_n$ are iid from a population with mean μ and variance σ^2 then $\bar{X}_n$ converges in probability to μ
- (There are many variations on the LLN; we are using a particularly lazy one)

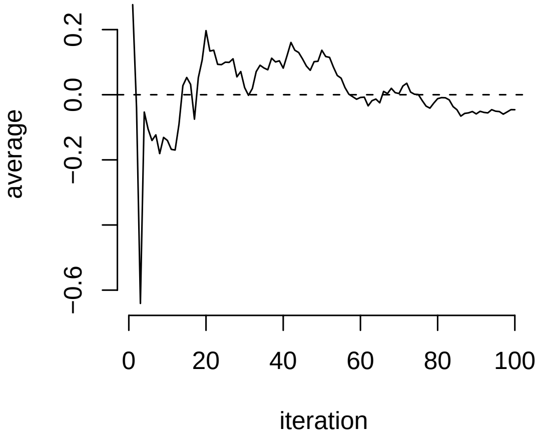
Proof using Chebyshev's inequality

- Recall Chebyshev's inequality states that the probability that a random variable variable is more than k standard deviations from its mean is less than $1/k^2$
- Therefore for the sample mean

$$P\left\{|\bar{X}_n - \mu| \geq k \text{ sd}(\bar{X}_n)\right\} \leq 1/k^2$$

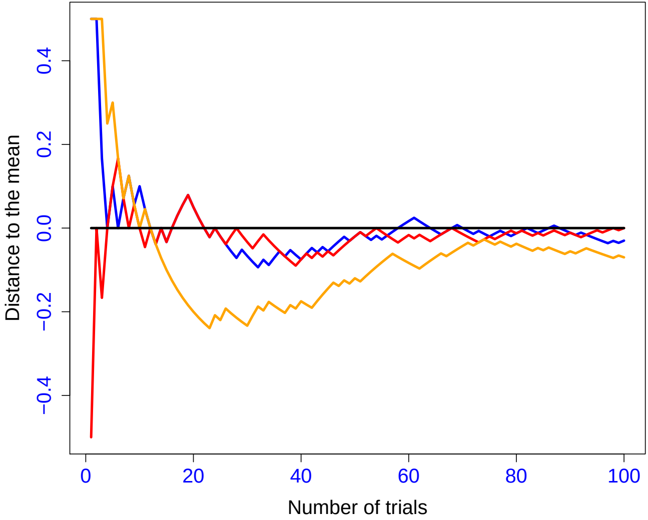
- Pick a distance ϵ and let $k = \epsilon/\text{sd}(\bar{X}_n)$

$$P(|\bar{X}_n - \mu| \geq \epsilon) \leq \frac{\text{sd}(\bar{X}_n)^2}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$$


average
0.2
0.0
−0.2
−0.6
0
20
40
60
80
100
iteration

Generating sequences of means

```
x1=rbinom(100,1,0.5)
x2=rbinom(100,1,0.5)
x3=rbinom(100,1,0.5)
xbar1=rep(0,length(x1))
xbar2=xbar1
xbar3=xbar1

for (i in 1:length(x1))
   {xbar1[i]=mean(x1[1:i])
   xbar2[i]=mean(x2[1:i])
   xbar3[i]=mean(x3[1:i])}
```


Distance to the mean
0.4
0.2
0.0
−0.2
−0.4
0
20
40
60
80
100
Number of trials

The strength of the weak LLN

- Widely used in sampling/polling
- A main reason why Nate Silver (and other Statisticians) was right in the 2012 presidential election when all the "pundits" were wrong
- He might have been a bit "too right"

```
sum(rbinom(50,1,0.8)<1)
```

- A main reason why big data is over-hyped
- Data and scientific complexity $>>>$ Data size

Convergence of transformed data

- If $X_1, \ldots, X_n$ are iid random variables then

$$\frac{1}{n}\sum_{i=1}^{n} f(X_i) \to E[f(X)]$$

- $E[f(X)]$ needs to exists; otherwise, no go
- $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \underset{n}{\longrightarrow} E[X^2]$
- $\frac{1}{n}\sum_{i=1}^{n} X_i^3 \underset{n}{\longrightarrow} E[X^3]$
- $\frac{1}{n}\sum_{i=1}^{n} \exp(X_i) \underset{n}{\longrightarrow} E[\exp(X)]$
- $\frac{1}{n}\sum_{i=1}^{n} \sin(X_i) \underset{n}{\longrightarrow} E[\sin(X)]$

Useful facts

- Functions of convergent random sequences converge to the function evaluated at the limit
- This includes sums, products, differences, ...
- Example: $\bar{X}_n^2$ converges to μ^2
- Notice that this is different than $(\sum X_i^2)/n$ which converges to $E[X_i^2] = \sigma^2 + \mu^2$
- We can use this to prove that the sample variance converges to σ^2

Continued

$$
\begin{aligned}
\sum (X_i - \bar{X}_n)^2/(n-1) &= \frac{\sum X_i^2}{n-1} - \frac{n(\bar{X}_n)^2}{n-1} \\
&= \frac{n}{n-1} \times \frac{\sum X_i^2}{n} - \frac{n}{n-1} \times (\bar{X}_n)^2 \\
&\xrightarrow{p} 1 \times (\sigma^2 + \mu^2) - 1 \times \mu^2 \\
&= \sigma^2
\end{aligned}
$$

Hence we also know that the sample standard deviation converges to σ

Quizz!!!

- Example of a sequence of unbiased estimators that is not convergent?
- Example of a convergent sequence of estimators that are not unbiased?

Discussion

- An estimator is **consistent** if it converges to what you want to estimate
- The LLN basically states that the sample mean is consistent
- We just showed that the sample variance and the sample standard deviation are consistent as well
- Recall also that the sample mean and the sample variance are unbiased as well
- (The sample standard deviation is not unbiased, by the way)

The Central Limit Theorem

- The **Central Limit Theorem** (CLT) is one of the most important theorems in statistics
- For our purposes, the CLT states that the distribution of averages of iid variables, properly normalized, becomes that of a standard normal as the sample size increases
- The CLT applies in an endless variety of settings

Convergence in distribution

- Consider a sequence of rvs $X_n, n \geq 1$. We say that X_n converges in distribution to X if

$$P(X_n \leq x) = F_n(x) \underset{n}{\longrightarrow} F(x) = P(X \leq x)$$

  for every x
- This is sometimes referred to as the *weak convergence of random variables*

The CLT

- Let $X_1, \ldots, X_n$ be a collection of iid random variables with mean μ and variance σ^2
- Let $\bar{X}_n$ be their sample average
- Then
$$P\left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq z\right) \to \Phi(z)$$
- Notice the form of the normalized quantity
$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{\text{Estimate} - \text{Mean of estimate}}{\text{Std. Err. of estimate}}.$$
- We say that $Z_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$ converges in distribution to $Z \sim N(0,1)$

Example

- Simulate a standard normal random variable by rolling n six sided dice
- Let X_i be the outcome for die i
- Then note that $\mu = E[X_i] = 3.5$
- $\mathrm{Var}(X_i) = 2.92$
- SE $\sqrt{2.92/n} = 1.71/\sqrt{n}$
- Standardized mean

$$\frac{\bar{X}_n - 3.5}{1.71/\sqrt{n}}$$


1 die rolls
2 die rolls
6 die rolls
0.0
0.2
0.4
−3
−1
1
2
3

R simulations: exponential

Assume that $X_1, \ldots, X_n$ are iid with an exp(1) distribution

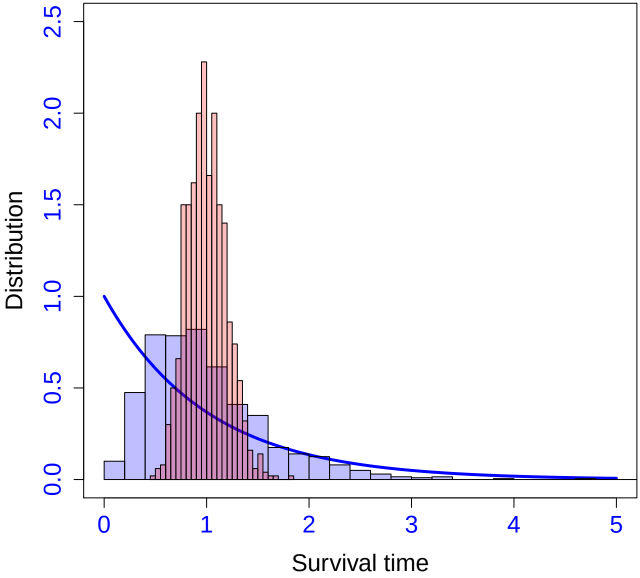
$$f(x) = \exp(-x) \text{ for } x > 0$$

- $E[X_i] = 1$, $\mathrm{Var}(X) = 1$
- Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$
- Simulate $\bar{X}_n$ for $n = 3$, $n = 30$ and plot
- Show histograms of $\bar{X}_n$ and

$$Z_n = \frac{\bar{X}_n - 1}{1/\sqrt{n}} = \sqrt{n}(\bar{X}_n - 1)$$

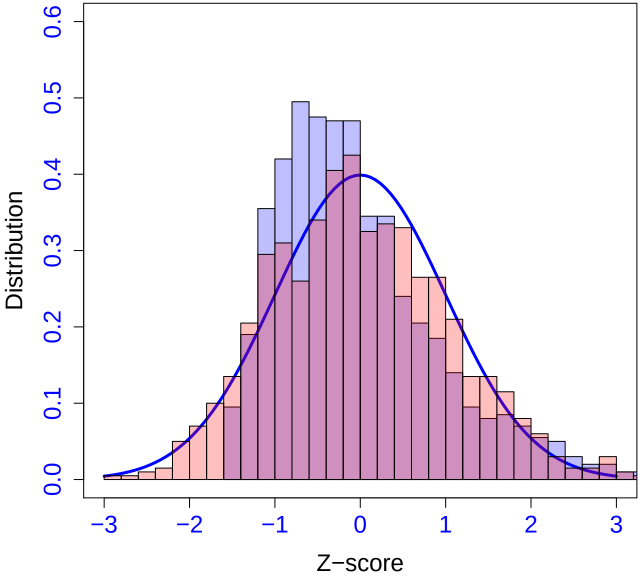
R simulations: exponential

```
xh=seq(0,5,length=101)
he=dexp(xh,rate=1)
n=c(3,30)
mx=matrix(rep(0,2000),ncol=2)

for (i in 1:1000)
   {mx[i,1]=mean(rexp(n[1], rate = 1))
   mx[i,2]=mean(rexp(n[2], rate = 1))}

plot(xh,he,type="l",col="blue",lwd=3,
   ylim=c(0,2.5))
hist(mx[,1],prob=T,add=T,col=rgb(0,0,1,1/4),
   breaks=25)
hist(mx[,2],prob=T,add=T,col=rgb(1,0,0,1/4),
   breaks=25)
```


Distribution
0.0
0.5
1.0
1.5
2.0
2.5
0
1
2
3
4
5
Survival time

R simulations: exponential Z-score

```
zx=mx for (j in 1:2)
   {zx[,j]<-sqrt(n[j])*(mx[,j]-1)}
xx=seq(-3,3,length=101)
yx=dnorm(xx)

plot(xx,yx,type="l",col="blue",lwd=3)
hist(zx[,1],prob=T,add=T,col=rgb(0,0,1,1/4),
   breaks=50)
hist(zx[,2],prob=T,add=T,col=rgb(1,0,0,1/4),
   breaks=50)
```


Distribution
0.0
0.1
0.2
0.3
0.4
0.5
0.6
−3
−2
−1
0
1
2
3
Z−score

Coin CLT

- Let X_i be the 0 or 1 result of the i^{th} flip of a possibly unfair coin
- The sample proportion, say $\hat{p}_n$, is the average of the coin flips
- $E[X_i] = p$ and $\mathrm{Var}(X_i) = p(1-p)$
- Standard error of the mean is $\sqrt{p(1-p)/n}$
- Then

$$z_n = \frac{\hat{p}_n - p}{\sqrt{p(1-p)/n}}$$

will be approximately normally distributed

Coin CLT: z-score

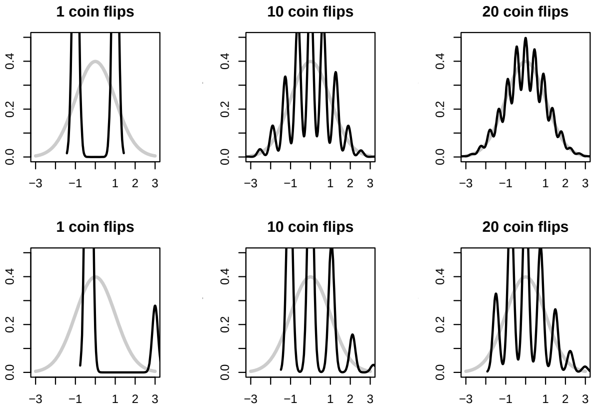
Recall that with n Bernoulli trials $\hat{p}_n$ can take the values $0/n, 1/n, \ldots, n/n$ and

$$P(\hat{p}_n = k/n) = P(\sum_{i=1}^{n} X_i = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

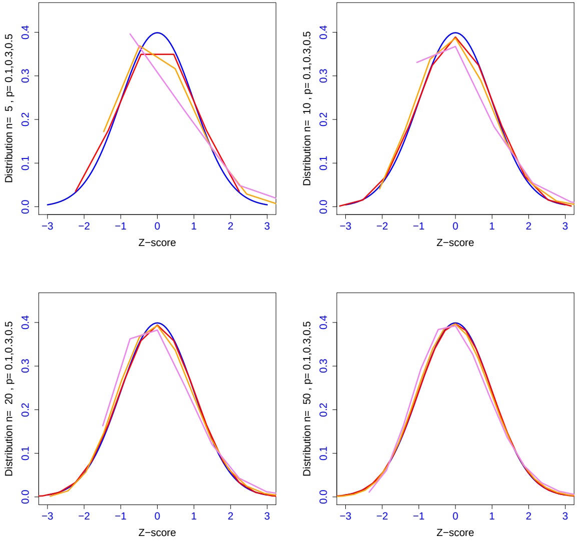
- for $n = 1$: z_n takes the values $\sqrt{(1-p)/p}$ with probability p and $-\sqrt{p/(1-p)}$ with probability $1-p$
- for a general n: z_n takes the values $\sqrt{n}\frac{k/n-p}{\sqrt{p(1-p)}}$ with probability $P(\hat{p}_n = k/n)$.

R simulations: coin flip Z-score

```
n=50
k=0:n
p=c(0.5,0.3,0.1)
values=matrix(rep(0,(n+1)*3),ncol=3)
pr=values
for (i in 1:length(p))
   {values[,i]=sqrt(n)*(k/n-p[i])/sqrt(p[i]*(1-p[i]))
   pr[,i]=dbinom(k,n,prob=p[i])/(values[2,i]-values[1,i])}

xx=seq(-3,3,length=101)
yx=dnorm(xx)
plot(xx,yx,type="l",col="blue",lwd=3)
lines(values[,1],pr[,1],lwd=3,col="red")
lines(values[,2],pr[,2],lwd=3,col="orange")
lines(values[,3],pr[,3],lwd=3,col="violet")
```


1 coin flips
10 coin flips
20 coin flips
1 coin flips
10 coin flips
20 coin flips
0.0
0.2
0.4
−3
−1
1
2
3

CLT for coin flips



CLT in practice

- In practice the CLT is mostly useful as an approximation

$$P\left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq z\right) \approx \Phi(z).$$

- Recall 1.96 is a good approximation to the $.975^{th}$ quantile of the standard normal
- Consider

$$\begin{aligned} .95 &\approx P\left(-1.96 \leq \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq 1.96\right) \\ &= P\left(\bar{X}_n + 1.96\sigma/\sqrt{n} \geq \mu \geq \bar{X}_n - 1.96\sigma/\sqrt{n}\right), \end{aligned}$$

Confidence intervals

- Therefore, according to the CLT, the probability that the random interval

$$\bar{X}_n \pm z_{1-\alpha/2}\sigma/\sqrt{n}$$

  contains μ is approximately 95%, where $z_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution
- This is called a 95% **confidence interval** for μ
- **Slutsky's theorem**, allows us to replace the unknown σ with s

Slutsky's theorem

If X_n and Y_n are random sequences, such that X_n converges in distribution to X and Y_n converges in probability to a constant c then

- $X_n + Y_n \underset{d}{\longrightarrow} X + c$
- $X_n Y_n \underset{d}{\longrightarrow} Xc$
- $X_n Y_n^{-1} \underset{d}{\longrightarrow} Xc^{-1}$

Sample proportions

- In the event that each X_i is 0 or 1 with common success probability p then $\sigma^2 = p(1-p)$
- The interval takes the form

$$\hat{p} \pm z_{1-\alpha/2}\sqrt{\frac{p(1-p)}{n}}$$

- Replacing p by $\hat{p}$ in the standard error results in what is called a Wald confidence interval for p
- Also note that $p(1-p) \leq 1/4$ for $0 \leq p \leq 1$
- Let $\alpha = .05$ so that $z_{1-\alpha/2} = 1.96 \approx 2$ then

$$2\sqrt{\frac{p(1-p)}{n}} \leq 2\sqrt{\frac{1}{4n}} = \frac{1}{\sqrt{n}}$$

- Therefore $\hat{p} \pm \frac{1}{\sqrt{n}}$ is a quick CI estimate for p