Lecture 6

Ciprian Crainiceanu

Department of Biostatistics
Johns Hopkins Bloomberg School of Public Health
Johns Hopkins University

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Likelihood

- A common approach to statistics is to assume that data arise from a family of distributions indexed by a parameter that represents a useful summary of the distribution
- The **likelihood** of the data is the joint density evaluated as a function of the parameters with the data fixed
- Likelihood analysis of data uses the likelihood to perform inference regarding the unknown parameter

Examples: Normal

- $X_1, X_2, X_3 \sim N(\mu, 1)$ are independent identically distributed rvs (conditional on μ)
  $f(x,\mu) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2}\right\}$
- Suppose that we observe $X_1 = 5$, $X_2 = 2$, $X_3 = 3$
- $\mathcal{L}(\mu|X_1 = 5, X_2 = 2, X_3 = 3) = f(5,\mu)f(2,\mu)f(3,\mu)$
- $\mathcal{L}(\mu|\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \exp\left\{-\frac{(5-\mu)^2+(2-\mu)^2+(3-\mu)^2}{2}\right\}$

Examples: Normal

- In general if $X_1 = x_1, \ldots, X_n = x_n$

$$\mathcal{L}(\mu|\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \exp\left\{-\frac{\sum_i^n (x_i - \mu)^2}{2}\right\}$$

- Note that $f(\mu|\mathbf{x})$ is not a normalized pdf, that is,

$$\int \mathcal{L}(\mu|\mathbf{x}) d\mu \neq 1$$

- Taking logs typically makes log likelihoods better behaved

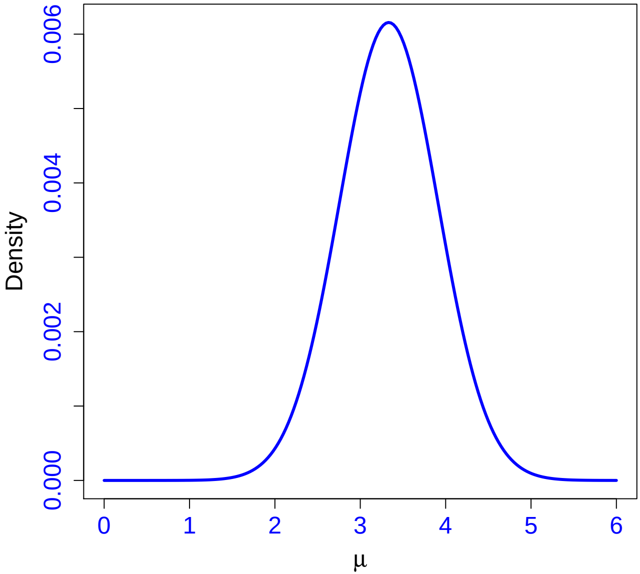
$$-2\log\{\mathcal{L}(\mu|\mathbf{x})\} = \sum_i^n (x_i - \mu)^2 + \text{const.}$$

Examples: Normal

```
mu=4
bx=c(5,2,3)
ebx2=-sum((bx-mu)^2)/2
like=exp(ebx2)/((2*pi)^(length(bx)/2))

mu=seq(0,6,length=201)
likep=rep(0,201)
for (i in 1:201)
   {ebx2=-sum((bx-mu[i])^2)/2
   likep[i]=exp(ebx2)/((2*pi)^(length(bx)/2))}

plot(mu,likep,type="l",col="blue",lwd=3)
mle<-mu[which.max(likep)]
```


Density
0.000
0.002
0.004
0.006
0
1
2
3
4
5
6
μ

Likelihood

Given a statistical probability mass function or density, say $f(x,\theta)$, where θ is an unknown parameter, the **likelihood** is f viewed as a function of θ for a fixed, observed value of x

$$\mathcal{L}(\theta|x) = f(x,\theta)$$

Interpretations of likelihoods

The law of likelihood requires:

1. Ratios of likelihood values measure the relative **evidence** of one value of the unknown parameter to another
2. **Likelihood principle:** Given a statistical model and observed data, all of the relevant information contained in the data regarding the unknown parameter is contained in the likelihood
3. If $\{X_i\}$ are independent random variables, then their likelihoods multiply. That is, the likelihood of the parameters given all of the X_i is simply the product of the individual likelihoods

Log likelihood

- Assume $X_1, \ldots, X_n$ are iid with pdf $f(x, \theta)$
- Likelihood

$$\mathcal{L}(\theta|\mathbf{x}) = \prod_{i=1}^{n} f(x_i, \theta)$$

- Log likelihood

$$\log\{\mathcal{L}(\theta|\mathbf{x})\} = \sum_{i=1}^{n} \log\{f(x_i, \theta)\}$$

Example

- Suppose that we flip a coin with success probability θ
- Recall that the mass function for x

$$f(x,\theta) = \theta^x (1-\theta)^{1-x} \quad \text{for} \quad \theta \in [0,1].$$

where x is either 0 (Tails) or 1 (Heads)

- Suppose that the result is a head
- The likelihood is

$$\mathcal{L}(\theta|1) = \theta^1 (1-\theta)^{1-1} = \theta \quad \text{for} \quad \theta \in [0,1].$$

- Therefore, $\mathcal{L}(.5|1)/\mathcal{L}(.25|1) = 2$,
- There is twice as much evidence supporting the hypothesis that $\theta = .5$ than the hypothesis that $\theta = .25$

Example continued

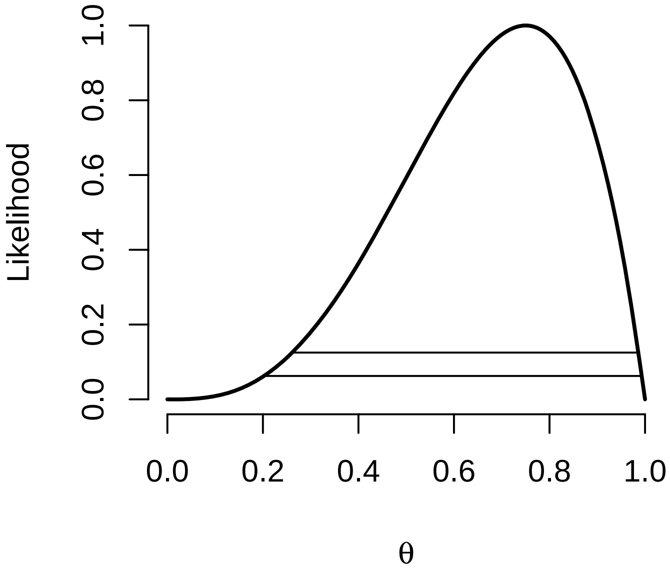
- Suppose now that we flip our coin from the previous example 4 times and get the sequence 1, 0, 1, 1
- The likelihood is:

$$\begin{aligned}\mathcal{L}(\theta|1,0,1,1) &= \theta^1(1-\theta)^{1-1}\theta^0(1-\theta)^{1-0} \\ &\times \theta^1(1-\theta)^{1-1}\theta^1(1-\theta)^{1-1} \\ &= \theta^3(1-\theta)^1\end{aligned}$$

- This likelihood only depends on the total number of heads and the total number of tails; we might write $\mathcal{L}(\theta|1,3)$ for shorthand
- Now consider $\mathcal{L}(.5|1,3)/\mathcal{L}(.25|1,3) = 5.33$
- There is over five times as much evidence supporting the hypothesis that $\theta = .5$ over the hypothesis that $\theta = .25$

Plotting likelihoods

- Generally, we want to consider all the values of θ between 0 and 1
- A **likelihood plot** displays θ by $\mathcal{L}(\theta|x)$
- Usually, it is divided by its maximum value so that its height is 1
- Because the likelihood measures *relative evidence*, dividing the curve by its maximum value (or any other value for that matter) does not change its interpretation


Likelihood
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
1.0
θ

Uniform distribution

- Suppose now that we observe three independent realizations from a uniform distribution $U[0,\theta]$
- $X_1 = 5,\ X_2 = 2,\ X_3 = 3$
- The likelihood of one observation

$$f(x,\theta) = \begin{cases} \frac{1}{\theta} & \text{if } 0 \leq x \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

- The likelihood of all three observations

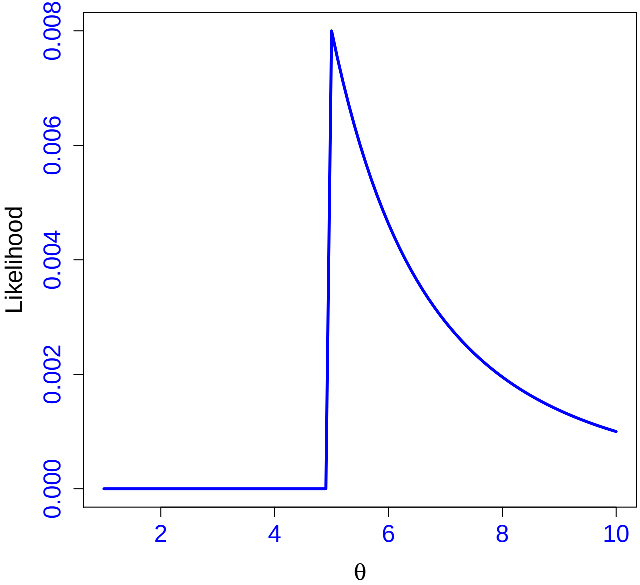
$$\begin{aligned} \mathcal{L}(\theta|\mathbf{x}) &= \frac{1}{\theta^3} I[0 \leq 5 \leq \theta] I[0 \leq 2 \leq \theta] I[0 \leq 3 \leq \theta] \\ &= \frac{1}{\theta^3} I[\theta \geq 5] \end{aligned}$$

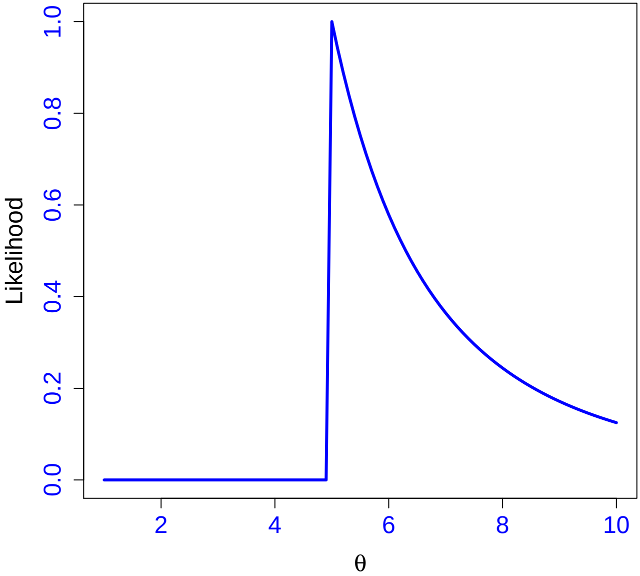
Uniform distribution: R

```
theta=seq(1,10,by=0.1)
like=1/theta^3*(theta>=5)
plot(theta,like,type="l",col="blue",lwd=3)

like[theta==6]/like[theta==5]
like[theta==6]/like[theta==4]
theta[which.max(like)] # maximum likelihood

liken=like/max(like)
plot(theta,liken,type="l",col="blue",lwd=3)
```


0.000
0.002
0.004
0.006
0.008
Likelihood
2
4
6
8
10
θ


Likelihood
0.0
0.2
0.4
0.6
0.8
1.0
2
4
6
8
10
θ

Uniform distribution

- Suppose now we observe n independent realizations from a uniform distribution $U[0,\theta]$
- $X_1 = x_1, \ldots, X_n = x_n$
- The likelihood for all n observations

$$\begin{aligned} \mathcal{L}(\theta|\mathbf{x}) &= \frac{1}{\theta^n}\prod_{i=1}^n I[0 \leq x_i \leq \theta] \\ &= \frac{1}{\theta^n} I[\theta \geq \max_i x_i] \end{aligned}$$

- Note that often the likelihood depends only on a function of the data (e.g. $\max_i x_i$)
- The evidence is often compressed in a much simpler, easier to understand form

Maximum likelihood

- The value of θ where the curve reaches its maximum has a special meaning
- It is the value of θ that is most well supported by the data
- This point is called the **maximum likelihood estimate** (or MLE) of θ

$$\widehat{\theta}_{\mathsf{ML}} = \mathit{MLE} = \mathrm{argmax}_{\theta}\mathcal{L}(\theta|\mathbf{x})$$

- Another interpretation of the MLE is that it is the value of θ that would make the data that we observed most probable
- Every estimator is a function of the observed data, $\mathbf{x}$

Maximum likelihood, coin example

- The maximum likelihood estimate for θ is always the proportion of heads
- Proof: Let x be the number of heads and n be the number of trials
- Recall

$$\mathcal{L}(\theta|x) = \theta^x(1-\theta)^{n-x}$$

- It's easier to maximize the **log-likelihood**

$$l(\theta, x) = x\log(\theta) + (n-x)\log(1-\theta)$$

Continued

- Taking the derivative we get

$$\frac{d}{d\theta} l(\theta, x) = \frac{x}{\theta} - \frac{n - x}{1 - \theta}$$

- Setting equal to zero implies

$$(1 - \frac{x}{n})\theta = (1 - \theta)\frac{x}{n}$$

- Which is clearly solved at $\theta = \frac{x}{n}$
- Notice that the second derivative

$$\frac{d^2}{d\theta^2} l(\theta, x) = -\frac{x}{\theta^2} - \frac{n - x}{(1 - \theta)^2} < 0$$

provided that x is not 0 or n (do these cases on your own)

What constitutes strong evidence?

- Again imagine an experiment where a person repeatedly flips a coin
- Consider the possibility that we are entertaining three hypotheses: $H_1 : \theta = 0$, $H_2 : \theta = .5$, and $H_3 : \theta = 1$

Outcome X	$P(X \mid H_1)$	$P(X \mid H_2)$	$P(X \mid H_3)$	$\mathcal{L}(H_1)/\mathcal{L}(H_2)$	$\mathcal{L}(H_3)/\mathcal{L}(H_2)$
H	0	.5	1	0	2
T	1	.5	0	2	0
HH	0	.25	1	0	4
HT	0	.25	0	0	0
TH	0	.25	0	0	0
TT	1	.25	0	4	0
HHH	0	.125	1	8	8
HHT	0	.125	0	0	0
HTH	0	.125	0	0	0
THH	0	.125	0	0	0
HTT	0	.125	0	0	0
THT	0	.125	0	0	0
TTH	0	.125	0	0	0
TTT	1	.125	0	0	8

Benchmarks

- Using this example as a guide, researchers tend to think of a likelihood ratio
  - of 8 as being moderate evidence
  - of 16 as being moderately strong evidence
  - of 32 as being strong evidence

  of one hypothesis over another
- Because of this, it is common to draw reference lines at these values on likelihood plots
- Parameter values above the $1/8$ reference line, for example, are such that no other point is more than 8 times better supported given the data

Likelihood for multiple parameters

- So far, we have focused on the case when θ is a scalar
- Many distributions depend on multiple parameters (normal, gamma, beta, t)
- Definitions remain the same
- Likelihood

$$\mathcal{L}(\theta|\mathbf{x}) = \prod_{i=1}^{n} f(x_i, \theta)$$

- MLE

$$\widehat{\theta}_{\mathrm{ML}} = MLE = \mathrm{argmax}_{\theta}\mathcal{L}(\theta|\mathbf{x})$$

- It simply requires working with multivariate parameters

Profile likelihood

- Sometimes one is interested in one of the parameters, whereas the others are not the primary focus of the problem
- Example: Effect of air pollution as measured by $PM_{2.5}$ on cardiovascular outcomes in the presence of potential confounders (temperature, secular trends, etc)
- Evidence is hard to visualize with respect to all parameters at once
- Profile likelihood: a way to visualize the evidence with respect to the parameter of interest

Profile likelihood

- $X_1, \ldots, X_n$ iid with pdf $f(x, \theta)$
- The multivariate parameter can be partitioned in $\theta = (\mu, \eta)$
  - μ is a scalar parameter of interest
  - η are the nuisance parameters
- For each value of μ maximize the likelihood with respect to the rest of the parameters η
- Obtain $\widehat{\eta}(\mu, \mathbf{x}) = \max_\eta \mathcal{L}(\mu, \eta | \mathbf{x})$
- The profile likelihood is

$$\mathcal{PL}(\mu | \mathbf{x}) = \mathcal{L}\{\mu, \widehat{\eta}(\mu, \mathbf{x}) | \mathbf{x}\}$$

Profile likelihood: normal

- In general if $X_1 = x_1, \ldots, X_n = x_n$ with mean μ and variance σ^2

$$\mathcal{L}(\mu, \sigma^2|\mathbf{x}) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left\{-\frac{\sum_i^n (x_i - \mu)^2}{2\sigma^2}\right\}$$

- Fix μ and maximize the log likelihood with respect to σ^2
- Log likelihood

$$2\log\{\mathcal{L}(\mu, \sigma^2|\mathbf{x})\} = -\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{\sigma^2} - n\log(\sigma^2) + \text{const.}$$

- Profile estimator of the variance

$$\widehat{\sigma}^2(\mu, \mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$$

Profile likelihood: normal

- The profile log likelihood is

$$2 \log \mathcal{PL}(\mu|\mathbf{x}) = 2 \log \mathcal{L}\{\mu, \widehat{\sigma}^2(\mu, \mathbf{x})|\mathbf{x}\}$$

- Plug-in estimator

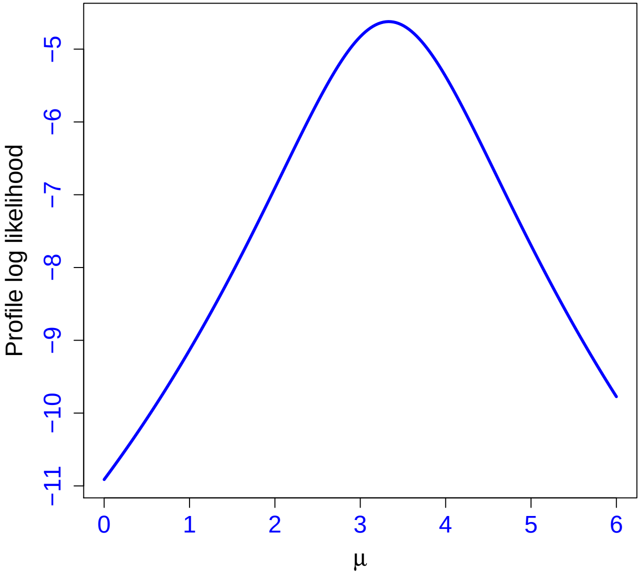
$$2 \log \mathcal{PL}(\mu|\mathbf{x}) = -n \log \left\{ \sum_{i=1}^{n} (x_i - \mu)^2 \right\} + const$$

- Thus, the profile likelihood is, essentially, the sum of squares for the normal distribution

Profile likelihood: R

```
bx=c(5,2,3) # Data
mu=seq(0,6,length=201) # Grid for μ
likep=rep(0,201)
for (i in 1:201)
   {likep[i]=-3*log(sum((bx-mu[i])^2))}

plot(mu,likep,type="l",col="blue",lwd=3)
mlep<-mu[which.max(likep)]
```


Profile log likelihood
−5
−6
−7
−8
−9
−10
−11
0
1
2
3
4
5
6
μ