140.652
Solutions to 2010 Midterm exam

lstlisting template

```
# ----------------------------------------------------------------------------
# Result (not needed for full credit)
NUMER.restricted.comb
# 17545920
DENOM.all.comb
# 37957920
NUMER.restricted.comb / DENOM.all.comb
# 0.4622466
```

1 Problem 6

6. You collect an iid sample from a population and obtain the data (in ascending order):

```
1, 2, 4, 7
```

(a) List out all of the equally likely bootstrap resamples from this data and calculate the median of each.

(b) Calculate the bootstrapped distribution of the sample median using your answer from question (a).

Note that bootstrap samples **with replications**. There are 256 possible samples in total.

```
dt = c(1,2,4,7)
boot.sample = matrix(NA, nrow = 256, ncol = 4) # All possible bootstrap samples
for (i in 1:4){
    for (j in 1:4){
        for (k in 1:4){
            for (l in 1:4){
                boot.sample[(i-1)*(4^3)+(j-1)*(4^2)+(k-1)*4+l,] = dt[c(i,j,k,l)]
            }
        }
    }
}
# Median of bootstrap samples
boot.med = apply(boot.sample, 1, median)

print(head(boot.sample))
print(summary(boot.med))
```

Run the command above to view results.

2 Problem 7

7. Recall that the Poisson distribution is $P(X = x) = \lambda^x e^{-\lambda}/x!$ for $x = 0, 1, 2, \ldots$ and $\lambda > 0$ is the mean, $E[X] = \lambda$. Consider writing an exponential prior on lambda $f(\lambda) = \beta e^{-\lambda\beta}$ for $\lambda > 0$, where $\beta > 0$ is a specified number. The mean of this distribution is β^{-1}. Suppose that you collect data and obtain $x = 3$.

   (a) Write out the likelihood for λ.

   (b) Write out the posterior for λ; what distribution is it? (Note, I'm not asking you to calculate the distribution function, I'm asking what the name of the distribution is.)

   (c) What is the posterior mean?

2.1 (a)

$$\mathcal{L}(\lambda|x) = P_\lambda(X = x) = \lambda^x e^{-\lambda}/x!$$
$$\mathcal{L}(\lambda|x = 3) = \lambda^3 e^{-\lambda}/3!$$

2.2 (b)

Side note: exponential distribution density $f(x) = \beta e^{-x\beta}$ is a special case of gamma density

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \text{ shape } \alpha > 0, \text{ rate } \beta > 0 \tag{1}$$

for shape $\alpha = 1$ (recall $\Gamma(1) = 1$).

To write out the posterior distribution, we use the rule that:

$$\text{Posterior} \propto \text{Likelihood} \times \text{Prior}$$

We write:

$$\begin{aligned} f(\lambda|x) &\propto \mathcal{L}(\lambda|x) \times f(\lambda) \\ &= \lambda^x e^{-\lambda}/x! \cdot \beta e^{-\lambda\beta} \\ &\propto \lambda^x e^{-(1+\beta)\lambda} \\ &\propto \lambda^{(x+1)-1} e^{-(1+\beta)\lambda} \end{aligned}$$

which is a kernel of **Gamma distribution** density for shape $\alpha^* = (x + 1)$ and rate $\beta^* = (1 + \beta)$ (compare with Gamma density formulation in Eq. 1; note: we use * notation to denote parameters of posterior distribution so as not to confuse with parameters from prior distribution).

Given the data, we have

$$f(\lambda|x = 3) \propto \lambda^{(3+1)-1} e^{-(1+\beta)\lambda}$$

which is a kernel of **Gamma distribution** density for shape $\alpha^* = (3 + 1)$ and rate $\beta^* = (1 + \beta)$.

2.3 (c)

We look up the formula for Gamma distribution mean:

$$\mathbb{E}(\lambda|x) = \frac{\alpha^*}{\beta^*} = \frac{x+1}{1+\beta} \overset{x=3}{=} \frac{4}{1+\beta}$$