Lecture 8

Advanced inference in multiple linear regression

Review of where we left off

1. We have established the multiple linear regression model:

$$\underbrace{Y_{n\times 1}} = \underbrace{X_{n\times(p+1)}\beta_{(p+1)\times 1}}_{\mu_{n\times 1}} + \underbrace{\epsilon_{n\times 1}}, \epsilon_{n\times 1} \sim MVN(0_{n\times 1}, \sigma^2 I_{n\times n})$$

$$V = \begin{bmatrix} \sigma^2 & & 0 \\ & \sigma^2 & \\ 0 & & \ddots \\ & & & \sigma^2 \end{bmatrix}$$

2. We know that:

$$\hat{\beta} \text{ satisfies } \underbrace{X'(Y - X\beta) = 0} \text{ and minimizes } \underbrace{\sum_{i=1}^{n}(y_i - x_i'\beta)^2}_{\text{sums of squared residuals}}$$

3. We have defined:

- $\hat{Y} = X\hat{\beta} = HY$, where $H = \underbrace{X(X'X)^{-1}X'}$
- $\hat{R} = Y - \hat{Y} = Y - X\hat{\beta} = \underbrace{(I - H)Y}$

4. Then we showed that:

- $\hat{\beta} \sim MVN(\beta, \sigma^2(X'X)^{-1})$
- $\hat{Y} \sim MVN(X\beta, \sigma^2 H)$
- $\hat{R} \sim MVN(0, \sigma^2(I - H))$

Possible inference: single regression coefficient

Target	Estimate $\sim$ Sampling Distn	95% CI for target	Test statistic for H0: Target = 0
β_j	$\hat\beta_j \sim N(\beta_j, [\sigma^2(X'X)^{-1})]_{jj})$	$\hat\beta_j \pm t \times \hat{se}(\hat\beta_j)$	$\frac{\hat\beta_j}{\hat{se}(\hat\beta_j)}$

$$\hat{\beta} = \begin{pmatrix} \hat\beta_0 \\ \hat\beta_1 \\ \vdots \\ \hat\beta_p \end{pmatrix} \sim MVN\left(\begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}, \underbrace{\begin{bmatrix} Var(\beta_0) & Cov(\beta_0,\beta_1) & \\ Cov(\beta_0,\beta_1) & Var(\beta_1) & \\ & & \ddots \\ & & & Var(\beta_p) \end{bmatrix}}_{\sigma^2 (X'X)^{-1}} \right)$$

$$\hat\beta_j \sim N\left(\beta_j, V_{jj} = [\sigma^2 (X'X)^{-1}]_{jj}\right)$$

Example: inference for single regression coefficient

Nepali data Y_i = arm circumference

$$\text{age} = \begin{cases} \text{age} \\ (\text{age}-6)^+ \end{cases}$$

linear spline model with knot at 6 months

$$Y_i = \beta_0 + \beta_1 \text{age}_i + \beta_2 (\text{age}_i - 6)^+ + \varepsilon_i$$

β_1 = linear growth rate per month of age among children under 6 months

$H_0: \beta_1 = 0$

$H_A: \beta_1 \neq 0$

95% CI for β_1

test statistic $= .31/.09 = 3.361$

$.31 \pm t \times .09$

$df = n-p-1-1$

$n-p-2$

```
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 11.12089    0.50959  21.823  < 2e-16 ***
## age          0.31141    0.09264   3.361 0.000945 ***
## agesp6      -0.27958    0.09441  -2.961 0.003473 **
## ---
```

Possible inference: linear combination of coefficients

Target	Estimate $\sim$ Sampling Distn	95% CI for target	Test statistic for H0: Target = 0
β_j	$\hat{\beta}_j \sim N(\beta_j, [\sigma^2(X'X)^{-1})]_{jj})$	$\hat{\beta}_j \pm t \times \hat{se}(\hat{\beta}_j)$	$\frac{\hat{\beta}_j}{\hat{se}(\hat{\beta}_j)}$
$\boxed{A\beta}$	$A\hat{\beta} \sim N(A\beta, \sigma^2 A(X'X)^{-1}A')$	$A\hat{\beta} \pm t \times \hat{se}(A\hat{\beta})$	$\frac{A\hat{\beta}_j}{\hat{se}(A\hat{\beta}_j)}$

$A = {}_{1\times(p+1)}$ vector $= (c_0, c_1, \ldots, c_p)$

$A_{1\times(p+1)}\,\beta_{(p+1)\times 1} = c_0\beta_0 + c_1\beta_1 + \ldots + c_p\beta_p$

Example: linear growth rate per month of age among children greater than 6 months

target: $\beta_1 + \beta_2$ estimate: $\hat{\beta}_1 + \hat{\beta}_2$

$Var(\hat{\beta}_1 + \hat{\beta}_2) = Var(\hat{\beta}_1) + Var(\hat{\beta}_2) + 2\,Cov(\hat{\beta}_1, \hat{\beta}_2)$ *

$Var(aY + bX) = a^2 Var(Y) + b^2 Var(X) + 2ab\,Cov(X, Y)$

Example: inference for linear combination of coefficients

Target	Estimate ∼ Sampling Distn	95% CI for target	Test statistic for H0: Target = 0
β_j	$\hat{\beta}_j \sim N(\beta_j, [\sigma^2(X'X)^{-1})]_{jj})$	$\hat{\beta}_j \pm t \times \hat{se}(\hat{\beta}_j)$	$\frac{\hat{\beta}_j}{\hat{se}(\hat{\beta}_j)}$
$A\beta$	$A\hat{\beta} \sim N(A\beta, \sigma^2 A(X'X)^{-1}A')$	$A\hat{\beta} \pm t \times \hat{se}(A\hat{\beta})$	$\frac{A\hat{\beta}_j}{\hat{se}(A\hat{\beta}_j)}$

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} \qquad A_{1\times 3} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \qquad A\hat{\beta} = 0 \times \hat{\beta}_0 + 1\hat{\beta}_1 + 1\hat{\beta}_2$$

$$C_{ij} = Cov(\hat{\beta}_i, \hat{\beta}_j)$$

$$\hat{\sigma}^2 (X'X)^{-1}_{3\times 3} = \begin{bmatrix} Var(\hat{\beta}_0) & C_{01} & C_{02} \\ C_{01} & Var(\hat{\beta}_1) & C_{12} \\ C_{02} & C_{12} & Var(\hat{\beta}_2) \end{bmatrix}$$

$$Var(A\hat{\beta}) = \underbrace{A_{1\times 3}\, Var(\hat{\beta})_{3\times 3}}\, A'_{3\times 1} =$$

$$\begin{bmatrix} \underbrace{0 \cdot Var(\hat{\beta}_0) + C_{01} + C_{02}} & 0 \times C_{01} + Var(\hat{\beta}_1) + C_{12} & \underbrace{0 \times C_{02} + C_{12} + Var(\hat{\beta}_2)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$Var(\hat{\beta}_1) + C_{12} + C_{12} + Var(\hat{\beta}_2) = Var(\hat{\beta}_1) + Var(\hat{\beta}_2) + 2C_{12}$$

Example: inference for linear combination of coefficients

```
cc=complete.cases(select(d,age,arm))
d.cc=filter(d,cc)
d.cc = arrange(d.cc,age)
reg1<-lm(data=d.cc, arm~age+agesp6)
reg1.coeff = reg1$coeff
reg1.vc = vcov(reg1)
# Define the linear combination of betas
A = matrix(c(0,1,1),nrow=1,ncol=3)
# Estimate the A beta-hat
A %*% reg1.coeff
```

```
##              [,1]
## [1,] 0.03182924
```

```
# What is the statistical variance of the estimate
A %*% reg1.vc %*% t(A)
```

```
##              [,1]
## [1,] 1.985802e-05
```

$Var(\hat{\beta}_1 + \hat{\beta}_2)$

```
# What is the standard error of the estimate
sqrt(A %*% reg1.vc %*% t(A))
```

```
##              [,1]
## [1,] 0.004456234
```

$se(\hat{\beta}_1 + \hat{\beta}_2)$

Example: inference for linear combination of coefficients

```
# Confirm these values!
summary(glht(reg1, linfct = A))
```

```
##
##   Simultaneous Tests for General Linear Hypotheses
##
## Fit: lm(formula = arm ~ age + agesp6, data = d.cc)
##
## Linear Hypotheses:
##            Estimate Std. Error t value Pr(>|t|)
## 1 == 0 0.031829   0.004456   7.143 2.12e-11 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## (Adjusted p values reported -- single-step method)
```

$\frac{.0318}{.0044} = 7.143 \Rightarrow$ pvalue

```
# 95% CI for beta1 + beta2
A %*% reg1.coeff - qt(0.975,df=summary(reg1)$df[2]) * sqrt(A %*% reg1.vc %*% t(A))
```

```
##              [,1]
## [1,] 0.02303672
```

```
A %*% reg1.coeff + qt(0.975,df=summary(reg1)$df[2]) * sqrt(A %*% reg1.vc %*% t(A))
```

```
##              [,1]
## [1,] 0.04062177
```

95% CI for $\beta_1 + \beta_2$

.023 to .041

Example: inference for linear combination of coefficients

```
# Confirm these values!
summary(glht(reg1, linfct = A))

##
##   Simultaneous Tests for General Linear Hypotheses
##
## Fit: lm(formula = arm ~ age + agesp6, data = d.cc)
##
## Linear Hypotheses:
##          Estimate Std. Error t value Pr(>|t|)
## 1 == 0 0.031829   0.004456   7.143 2.12e-11 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## (Adjusted p values reported -- single-step method)
```

```
# Hypothesis test of H0: beta1 + beta2 = 0
test.stat = (A %*% reg1.coeff) / sqrt(A %*% reg1.vc %*% t(A))
test.stat

##             [,1]
## [1,] 7.142632

2 * pt(abs(test.stat),df=summary(reg1)$df[2],lower.tail=FALSE)

##                 [,1]
## [1,] 2.124636e-11
```

Possible inference: Non-linear function of a coefficient

Target	Estimate $\sim$ Sampling Distn	95% CI for target	Test statistic for H0: Target = 0
β_j	$\hat{\beta}_j \sim N(\beta_j, [\sigma^2(X'X)^{-1})]_{jj})$	$\hat{\beta}_j \pm t \times \hat{se}(\hat{\beta}_j)$	$\frac{\hat{\beta}_j}{\hat{se}(\hat{\beta}_j)}$
$A\beta$	$A\hat{\beta} \sim N(A\beta, \sigma^2 A(X'X)^{-1}A')$	$A\hat{\beta} \pm t \times \hat{se}(A\hat{\beta})$	$\frac{A\hat{\beta}_j}{\hat{se}(A\hat{\beta}_j)}$
$g(\beta_j)$	$g(\hat{\beta}_j) \sim N(g(\beta_j), \underline{[g'(\beta_j)]}^2[\sigma^2(X'X)^{-1}]_{jj})$	$g(\hat{\beta}_j) \pm t \times \hat{se}(g(\hat{\beta}_j))$	$\frac{g(\hat{\beta}_j)}{\hat{se}(g(\hat{\beta}_j))}$

g is a function that is continuous at its 1st derivate

Example: We say that Y follows a log-normal distribution
if $\log(Y) \sim N(\mu, \sigma^2) \Rightarrow E(Y) = \exp(\mu + \sigma^2/2)$
$\text{median}(Y) = \exp(\mu)$

PS2 NMES = Y_i = expenditures$_i$ + 1

$\log(Y_i) = \beta_0 + \beta_1(age-65) + \beta_2(age-75)^+ + \beta_3(age-85)^+ + \varepsilon_i$

Estimate the median expenditure for 65 year olds

Example: inference for non-linear function of a coefficient

median expenditure for 65 year olds

$$E[\log(Y_i)] \text{ at } 65 = \beta_0 \quad = \exp(\beta_0) - 1$$

$$g(\beta_0) = \exp(\beta_0) - 1 \qquad \text{estimate: plugin } \underbrace{\exp(\hat{\beta}_0) - 1}$$

$$\frac{d}{d\beta_0} g(\beta_0) = \frac{d}{d\beta_0} \exp(\beta_0) - 1 = \underbrace{\exp(\beta_0)}$$

$$Var(g(\hat{\beta}_0)) = \exp(\hat{\beta}_0)^2 \left[\hat{\sigma}^2 (X'X)^{-1}\right]_{11}$$

Univariate delta method

Assuming the function g is continuous at its first derivative. The delta method is derived from the first order approximation to Taylor series using Taylor's theorem.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

In statistical applications, we are interested in finding the distribution of $g(\hat{\theta})$ where $\hat{\theta}$ follows a normal distribution.

Applying the first order Taylor expansion to $\boxed{g(\hat{\theta})}$ about the mean θ, we get:

$$g(\hat{\theta}) \approx g(\theta) + g'(\theta)(\hat{\theta} - \theta)$$

Then, $E(g(\hat{\theta})) = g(\theta) + g'(\theta)(E(\hat{\theta}) - \theta) = g(\theta) + g'(\theta - \theta) = g(\theta)$ and $Var(g(\hat{\theta})) = g'(\theta)^2 Var(\hat{\theta})$.

$E\left[g(\theta) + g'(\theta)(\hat{\theta} - \theta)\right]$

$g'(\theta)(\theta - \theta)$

0

Possible inference: non-linear function of coefficients

Target	Estimate $\sim$ Sampling Distn	95% CI for target	Test statistic for H0: Target = 0
β_j	$\hat{\beta}_j \sim N(\beta_j, [\sigma^2(X'X)^{-1})]_{jj})$	$\hat{\beta}_j \pm t \times \hat{se}(\hat{\beta}_j)$	$\frac{\hat{\beta}_j}{\hat{se}(\hat{\beta}_j)}$
$A\beta$	$A\hat{\beta} \sim N(A\beta, \sigma^2 A(X'X)^{-1}A')$	$A\hat{\beta} \pm t \times \hat{se}(A\hat{\beta})$	$\frac{A\hat{\beta}_j}{\hat{se}(A\hat{\beta}_j)}$
Univariate delta $g(\beta_j)$	$g(\hat{\beta}_j) \sim N(g(\beta_j), [g'(\beta_j)]^2[\sigma^2(X'X)^{-1}]_{jj})$	$g(\hat{\beta}_j) \pm t \times \hat{se}(g(\hat{\beta}_j))$	$\frac{g(\hat{\beta}_j)}{\hat{se}(g(\hat{\beta}_j))}$
$g(\beta)$ multivariate delta	$g(\hat{\beta}) \sim N(g(\beta), g'(\beta)'[\sigma^2(X'X)^{-1}]g'(\beta))$	$g(\hat{\beta}) \pm t \times \hat{se}(g(\hat{\beta}))$	$\frac{g(\hat{\beta})}{\hat{se}(g(\hat{\beta}))}$

Example: Y_i = arm circumference

$$Y_i = \beta_0 + \beta_1 age_i + \beta_2 (age_i - 6)^+ + \varepsilon_i$$

monthly growth rate

< 6 months β_1

≥ 6 months $\beta_1 + \beta_2$

relative growth rate $= \dfrac{\beta_1+\beta_2}{\beta_1}$

$$= 1 + \frac{\beta_2}{\beta_1}$$

Example: inference for non-linear function of coefficients

$$g(\beta) = 1 + \beta_2/\beta_1$$

estimate $\Rightarrow$ plug in estimate

$$1 + \hat{\beta}_2/\hat{\beta}_1$$

$$Var(g(\hat{\beta}))\,??$$

$$g'(\beta) = \begin{bmatrix} \frac{d}{d\beta_0} g \\ \frac{d}{d\beta_1} g \\ \frac{d}{d\beta_2} g \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ -\beta_2/\beta_1^2 \\ 1/\beta_1 \end{bmatrix}}_{3\times 1}$$

$$Var(g(\hat{\beta})) = \underbrace{[g'(\hat{\beta})]'}_{1\times 3}\ \underbrace{V_{\hat{\beta}}}_{3\times 3}\ \underbrace{[g'(\hat{\beta})]}_{3\times 1}$$

Example: non-linear function of coefficients

```
reg.coeff = reg$coeff
reg.vc = vcov(reg)

# Compute the estimate of g(beta)
g.est = 1 + reg.coeff[3]/reg.coeff[2]
# Define the vector of the derivative of g(beta) wrt beta
g.prime = matrix(c(0,-reg.coeff[3]/reg.coeff[2]^2,1/reg.coeff[2]),nrow=3,ncol=1)
g.prime
```

```
##          [,1]
## [1,] 0.000000
## [2,] 2.883012
## [3,] 3.211236
```

```
# Compute the variance of g(beta.hat)
g.var = t(g.prime) %*% reg.vc %*% g.prime
g.est
```

```
##    agesp6
## 0.1022112
```

```
g.est - qt(0.975,df=summary(reg)$df[2]) * sqrt(g.var)
```

```
##            [,1]
## [1,] 0.02689796
```

```
g.est + qt(0.975,df=summary(reg)$df[2]) * sqrt(g.var)
```

```
##           [,1]
## [1,] 0.1775244
```

$1 + \hat{\beta}_2 / \hat{\beta}_1 = .10$

95% CI for $1 + \beta_2/\beta_1$

.027 to .18

Comparing nested MLR models

model of interest

extended model

$$Y_i = \underbrace{\beta_0 + \beta_1 X_{1i} + \dots + \beta_p X_{pi}}_{p+1} + \underbrace{\beta_{p+1} X_{p+1 i} + \dots + \beta_{p+s} X_{p+s i}}_{s} + \varepsilon_i$$

Interested in comparing the extended model to simpler model nested within the extended model

null model

$$Y_i = \beta_0 + \beta_1 X_{1i} + \dots + \beta_p X_{pi}$$

$H_0: \beta_{p+j} = 0$, for $j = 1, \dots, s$

H_A: at least one $\beta_{p+j} \neq 0$

F-test for nested models; ANOVA method

Define:

- RN $= R_N = (I - H_N)Y$ → residual from fitting the null model
- RE $= R_E = (I - H_E)Y$ → residuals from fitting the extended model
- Δ $= R_N - R_E$

You can show the following results (which we will not do in class):

- $H_E - H_N$ is idempotent with rank s
- $H_E - H_N$ is orthogonal to $\underline{(I - H_E)Y}$
- $\left[\dfrac{\Delta'\Delta/s}{R'_E R_E/(n-p-s-1)}\right. \sim \mathscr{F}_{df1=s, df2=n-p-s-1}$

$$\frac{[SSresidual_{null} - SSresidual_{ext}]/s}{SSresidual\ extended/n-p-s-1}$$

Examples: nested MLR model comparisons

Consider the medical expenditure data you are analyzing for Problem Set 2. Define Y = log(medical expenditures + 1) and let $X_1 = age - 65$ and $X_2 = male$ (indicator 1 = male, 0 = female). Define three models:

Model	Xs	residual df	SS(residual)
A	X_1, X_2	5691	31332.38
B	$X_1, (X_1 - 10)^+, (X_1 - 20)^+, X_2$	5689	31314.59
C	$[X_1, (X_1 - 10)^+, (X_1 - 20)^+] \times X_2$	5686	31299.23

model A: $E(Y|X_1, X_2) = \beta_0 + \beta_1(age-65) + \beta_2\, male$

model B $E(Y|X_1, X_2) = \beta_0 + \beta_1(age-65) + \beta_2\, male$
$+ \beta_3 (age-75)^+ + \beta_4 (age-85)^+$

model C $E(Y|X_1, X_2) =$ model B $+$
$\beta_5(age-65)\,male + \beta_6 (age-75)^+\, male$
$+ \beta_7 (age-85)^+\, male$

Example 1: nested MLR model comparisons

null model A
ext model B

After adjusting for gender, is the average log expenditure a linear function of age?

H0: $\beta_3 = 0$ and $\beta_4 = 0$

HA: at least β_3 or $\beta_4 \neq 0$

Model	Xs	residual df	SS(residual)	MS	F
A	X_1, X_2	5691	31332.38		
B	$X_1, (X_1 - 10)^+, (X_1 - 20)^+, X_2$	5689	31314.59	$\boxed{5.50}$ +	
Change		2	17.79 *	$\boxed{8.90}$	$\frac{8.90}{5.50} = 1.62$

Compute the P-value as: $Pr(\mathscr{F}_{2,5689} > 1.62) = 0.199$.

$$F = \frac{[31332.38 - 31314.58]/2 \; *}{31314.59/5689 \; +} = 1.62 \qquad F(2, 5689)$$

Example 1: nested MLR model comparisons

```
load("C:\\Users\\Elizabeth\\Dropbox\\Biostat6532020\\Problem Set 2\\nmes.rdata")
d = nmes %>% select(names(.)[c(1,2,3,15)]) %>% filter(.,lastage>=65)
d = mutate(d,
logy = log(totalexp+1),
agec=lastage-65,
agesp1 = ifelse(lastage-75>0, lastage-75,0),
agesp2 = ifelse(lastage-85>0, lastage-85,0)
)
reg0 = lm(logy~agec+male,data=d) A
reg1 = lm(logy~agec+agesp1+agesp2+male,data=d) B
reg2 = lm(logy~(agec+agesp1+agesp2)*male,data=d) C
```

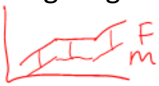
```
# Questoin 1: using anova function
anova(reg0,reg1)
```

null extended

```
## Analysis of Variance Table
##
## Model 1: logy ~ agec + male          - null
## Model 2: logy ~ agec + agesp1 + agesp2 + male   - ext
##   Res.Df   RSS Df Sum of Sq      F Pr(>F)
## 1   5691 31332
## 2   5689 31315  2     17.79 1.6159 0.1988
```

Example 2: nested MLR model comparisons

Is the non-linear relationship of average log expenditures on age the same for males and females? i.e. are the curves parallel?

model B 

model C 

- Equivalently: Is the difference between the average log expenditure for males and females the same at all ages?

HO: $\beta_5 = 0, \beta_6 = 0, \beta_7 = 0$

HA: at least one is non-zero

Model	Xs	residual df	SS(residual)
A	X_1, X_2	5691	31332.38
B	$X_1, (X_1 - 10)^+, (X_1 - 20)^+, X_2$	5689	31314.59
C	$[X_1, (X_1 - 10)^+, (X_1 - 20)^+] \times X_2$	5686	31299.23

$$F = \frac{[31314.59 - 31299.23]/3}{31299.23/5686}$$

Example 2: nested MLR model comparisons

Model	Xs	residual df	SS(residual)
A	X_1, X_2	5691	31332.38
B	$X_1, (X_1 - 10)^+, (X_1 - 20)^+, X_2$	5689	31314.59
C	$[X_1, (X_1 - 10)^+, (X_1 - 20)^+] \times X_2$	5686	31299.23

```
# Question 2:
anova(reg1,reg2)
```

B C

```
## Analysis of Variance Table
##
## Model 1: logy ~ agec + agesp1 + agesp2 + male
## Model 2: logy ~ (agec + agesp1 + agesp2) * male
##   Res.Df   RSS Df Sum of Sq      F Pr(>F)
## 1   5689 31315
## 2   5686 31299  3     15.36 0.9301 0.4252
```

Likelihood ratio tests for nested MLR models

Let $loglike_{ext}$ and $loglike_{null}$ be the values of the log likelihoods evaluated at the parameter estimates from the extended and null models, respectively.

Then to test H0:

Compute $2 \times loglike_{ext} - 2 \times loglike_{null} \sim \chi$, df = s

likelihood ratio tests
⇒ perform better
type I, power

Compared to model F test when n is small

Examples: nested MLR model comparisons using LRT

```r
# Question 1: by hand
lr.test.stat = as.numeric(2 * logLik(reg1) - 2 * logLik(reg0))
pchisq(lr.test.stat,df=2,lower.tail=FALSE)

## [1] 0.1985122
# Question 1: Using lrtest function
#install.packages(lmtest)
library(lmtest)

lrtest(reg0,reg1)

## Likelihood ratio test
##
## Model 1: logy ~ agec + male
## Model 2: logy ~ agec + agesp1 + agesp2 + male
##   #Df LogLik Df  Chisq Pr(>Chisq)
## 1   4 -12934
## 2   6 -12933  2 3.2338     0.1985
```

Next time....

- Model checking for MLR models
- Key extensions for MLR models