JOHNS HOPKINS
BLOOMBERG SCHOOL
of PUBLIC HEALTH

HW1 → email today
PS2 → Thursday

Lecture 9

Model Checking and Key Extensions

estimate $\hat{\sigma}^2 = \dfrac{\sum_{i=1}^{n} (y_i - \hat{y}_i)^2}{\underbrace{n-p-1}}$

Review of where we left off

1. We have established the multiple linear regression model:

$$\underbrace{Y_{n\times 1}} = \underbrace{X_{n\times(p+1)}\beta_{(p+1)\times 1}}_{\text{mean model}} + \epsilon_{n\times 1},\ \epsilon_{n\times 1} \sim MVN(0_{n\times 1}, \sigma^2 I_{n\times n})$$

error

$Cov(\varepsilon_i, \varepsilon_j) = 0$

2. We know that:

$$\hat{\beta} \text{ satisfies } X'(Y - X\beta) = 0 \text{ and minimizes } \underbrace{\sum_{i=1}^{n}(y_i - x_i'\beta)^2}_{SSR}$$

3. We have defined:

- $\hat{Y} = X\hat{\beta} = HY,$ where $H = X(X'X)^{-1}X'$
- $\hat{R} = Y - \hat{Y} = Y - X\hat{\beta} = (I - H)Y$

4. Then we showed that:

- $\hat{\beta} \sim MVN(\beta, \sigma^2(X'X)^{-1})$
- $\hat{Y} \sim MVN(X\beta, \sigma^2 H)$
- $\hat{R} \sim MVN(0, \sigma^2(I - H))$

Review of where we left off

Target	Estimate ~ Sampling Distn	95% CI for target	Test statistic for H0: Target = 0
β_j	$\hat\beta_j \sim N(\beta_j, [\sigma^2(X'X)^{-1})]_{jj})$	$\hat\beta_j \pm t \times \hat{se}(\hat\beta_j)$	$\frac{\hat\beta_j}{\hat{se}(\hat\beta_j)}$
$A\beta$	$A\hat\beta \sim N(A\beta, \sigma^2 A(X'X)^{-1}A')$	$A\hat\beta \pm t \times \hat{se}(A\hat\beta)$	$\frac{A\hat\beta_j}{\hat{se}(A\hat\beta_j)}$
$g(\beta_j)$	$g(\hat\beta_j) \sim N(g(\beta_j), [g'(\beta_j)]^2[\sigma^2(X'X)^{-1}]_{jj})$	$g(\hat\beta_j) \pm t \times \hat{se}(g(\hat\beta_j))$	$\frac{g(\hat\beta_j)}{\hat{se}(g(\hat\beta_j))}$
$g(\beta)$	$g(\hat\beta) \sim N(g(\beta), g'(\beta)'[\sigma^2(X'X)^{-1}]g'(\beta))$	$g(\hat\beta) \pm t \times \hat{se}(g(\hat\beta))$	$\frac{g(\hat\beta)}{\hat{se}(g(\hat\beta))}$
$\mu_i = E(Y_i \mid X_i)$	$\hat Y_i \sim N(\mu_i, \sigma^2[H]_{ii})$	$\hat Y_i \pm t \times \hat{se}(\hat Y_i)$	$\frac{\hat Y_i}{\hat{se}(\hat Y_i)}$
$\mu(x_0) = E(Y \mid x_0)$	$x_0'\hat\beta \sim N(x_0'\beta, \hat\sigma^2 x_0'(X'X)^{-1}x_0)$	$x_0'\hat\beta \pm t \times \hat{se}(x_0'\hat\beta)$	$\frac{x_0'\hat\beta}{\hat{se}(x_0'\hat\beta)}$

→ deriving CI for $\hat{Y}_i$ where i = is in the sample

→ X_0 ⇒ a value potentially outside the sample

Key Assumptions by Order of Importance

- $E(Y|X) = X\beta$ = we have "correctly" specified $X\beta$ / mean model
  - * no omitted variables / confounders
  - * correctly specified the functional form between X and Y
  - * appropriate interactions
  - - no error in ~~measurement~~ in X
  - $\boxed{\text{bias in } \hat{\beta}}$
- Residuals are independent
  - 2. → determined by how the sample / data is derived
    - - Longitudinal design
    - - clustered design
- Variance of residuals is constant
  - 3. $Var(\varepsilon_i) = \sigma^2$ ⇒ $Var(\varepsilon_i) = f(X_i)$
- Residuals are normally distributed
  - 4 $\varepsilon_i \sim N$ ⇒ bootstrap procedure
    - ⇒ CLT
  - 2, 3, 4 ⇒ impact $\hat{Var}(\hat{\beta})$
- There are not a small number of highly influencial observations → impact on $\hat{\beta}$, $\hat{Var}(\hat{\beta})$

Omitted Variable Bias

Exposure of interest is X_1, important confounder X_2

You fit: $Y_i = \alpha_0 + \underbrace{\alpha_1}_{} X_1 + \upsilon_i$

True model: $Y_i = \underbrace{\beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon_i}_{}$

$$\hat{\alpha}_1 = \frac{\mathrm{Cov}(Y, X_1)}{\mathrm{Var}(X_1)} = \frac{\mathrm{Cov}(\beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon, X_1)}{\mathrm{Var}(X_1)}$$

$$= \frac{\cancel{\mathrm{Cov}(\beta_0, X_1)} + \mathrm{Cov}(\beta_1 X_1, X_1) + \mathrm{Cov}(\beta_2 X_2, X_1) + \cancel{\mathrm{Cov}(\varepsilon, X_1)}}{\mathrm{Var}(X_1)}$$

note: $\mathrm{Cov}(\beta_0, X_1) = 0$

note: $\mathrm{Cov}(\varepsilon, X_1) = 0$

Omitted Variable Bias

$$\hat{\alpha}_1 = \frac{Cov(\beta_1 X_1, X_1) + Cov(\beta_2 X_2, X_1)}{Var(X_1)}$$

$$= \frac{\beta_1 Var(X_1) + \beta_2 Cov(X_2, X_1)}{Var(X_1)}$$

$$= \beta_1 + \beta_2 \underbrace{\frac{Cov(X_2, X_1)}{Var(X_1)}}$$

slope of the SLR of X_2 on X_1

$$X_2 = \delta_0 + \delta_1 X_1 + v$$

$$\hat{\alpha}_1 = \beta_1 + \underbrace{\beta_2 \delta_1}$$

$\hat{\alpha}_1$ is unbiased if:

1) X_1 and X_2 are independent
2) $\beta_2 = 0 \Rightarrow Y$ and X_2 are independent

Simulation exercise

- Within your breakout group, design a simulation study that would numerically demonstrate the result we just derived.
  - You go work for 15 minutes and then we will review together

Correct Functional Form for Continuous X

- To explore the assumption that E(Y|X) = Xβ, you can make the following plots:

1. Plot $\hat{R}$ vs. $X_j, j = 1, \ldots, p$.
   - Recall that the residuals are independent of X if the model is correctly specified
2. Plot $\hat{R}$ vs. $\hat{Y}$.
   - The residuals and predicted values are independent if the model is correctly specified

- Never plot $\hat{R}$ vs. Y because these are correlated!

- Based on the figures from 1. and 2., you could modify the model to increase/decrease the complexity of the functional form of the variables.

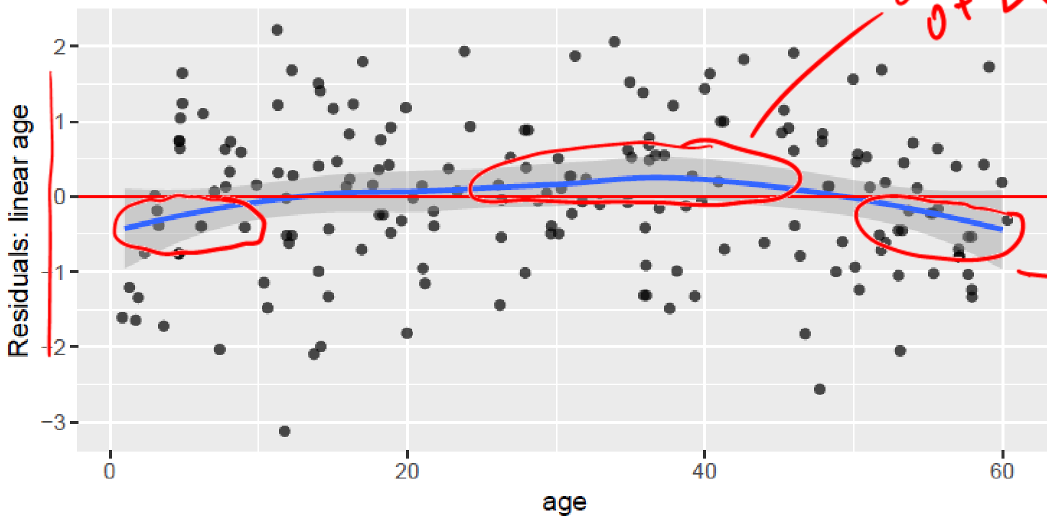
Example: Nepali Anthropometry Data

```r
reg0<-lm(data=d.cc, arm~age)
d.cc$residuals = residuals(reg0)
```


```r
ggplot(d.cc,aes(x=age, y=residuals)) +
    geom_jitter(alpha = 0.7) +
    geom_smooth() +          geom_hline(yintercept=0,color="red") +
    labs(y="Residuals: linear age")
```

```
## `geom_smooth()` using method = 'loess' and formula 'y ~ x'
```

overestimatn of E(arm)



Cm

under estimatn of E(arm)

Example: Nepali Anthropometry Data

Update the model to include a smooth function of age via linear splines or natural splines.

```
reg1 = lm(arm~age + agesp6 + agesp12,data=d.cc)
reg2 = lm(arm~ns(age,3),data=d.cc)
d.cc$residuals1 = residuals(reg1)
d.cc$residuals2 = residuals(reg2)
```

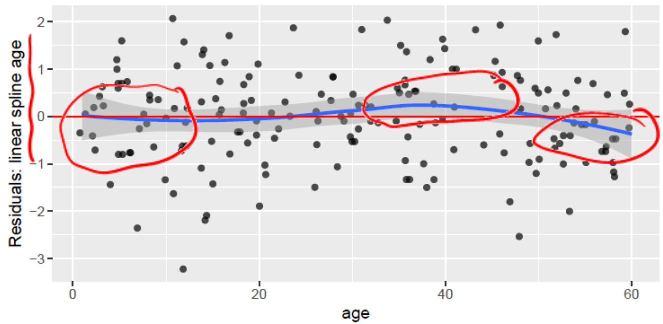
exploratory analysis

↳ working mean model

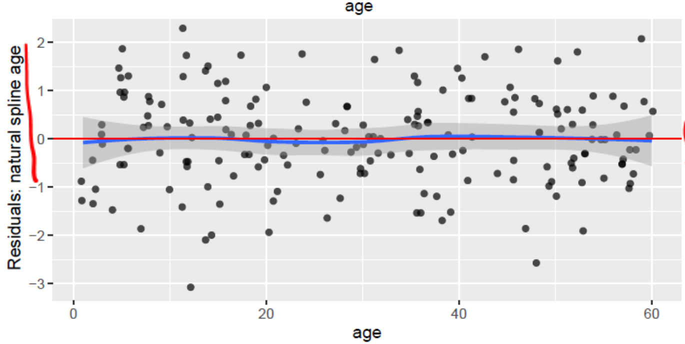
→ Fit model

→ evaluation

update



mean 0



Independence Assumption

- Driven by the design of the study
- Longitudinal design = enroll / recruit units / people

  → measure the outcome of interest for each unit i at several occassions j

  $i = 1, \ldots, m$

  Y_{ij} = outcome for unit i at occassion j

  $j = 1, \ldots, n_i$
- Clustered design Sample individuals nested within a cluster

  heirarchical, multilevel design

  - sample all person living in a given village, villages randomly selected
  - household survey: households randomly selected

    → interview adults in household
- Why do we care? Y_{ij} = outcome for the jth member of cluster i

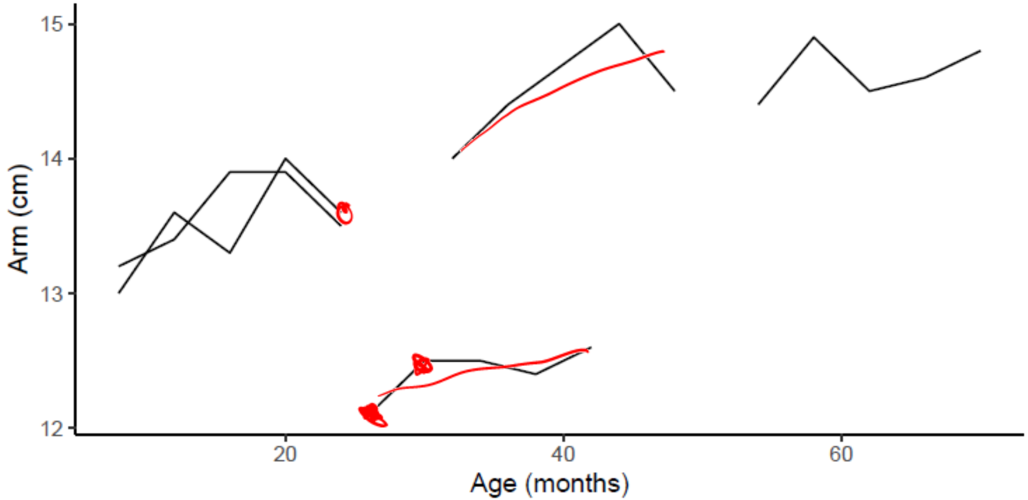
  $i = 1, \ldots, m$

  $j = 1, \ldots, n_i$

Example: Nepali Anthropometry Data

- Design: i = 1, …, m = 200 children each measured at baseline (j = 1) and then every 4 months for 4 follow-up visits (j = 2, 3, 4, 5).

```
ggplot(d5,aes(x=age,y=arm,group = factor(id))) +
    geom_line() +
    labs(x='Age (months)', y ='Arm (cm)') +
    theme_classic()
```



Checking the Independence Assumption

- Don't need to, we know the independence assumption is violated based on knowledge of the design

- We can explore covariance/correlation in the observed data

- Example: Consider the Nepali Anthropometry data where we have data for i = 1, …, m = 200 children each measured at baseline (j = 1) and then every 4 months for 4 follow-up visits (j = 2, 3, 4, 5).
  - Step 1: Regress *Y* on *X* assuming independence and estimate β and *R*
  - Step 2: Plot $\hat{R}_{ij}$ vs. $\hat{R}_{ik}$ for all j,k
  - Compute $Cov(\hat{R}_{ij}, \hat{R}_{ik}) = \sqrt{Var(\hat{R}_{ij})} \times \sqrt{Var(\hat{R}_{ik})} \times Corr(\hat{R}_{ij}, \hat{R}_{ik})$
  - Or standardize the residuals and plot $Corr(\hat{R}_{sij}, \hat{R}_{sik})$

Lab 5

How do we rethink the model?

What if we apply least squares to correlated data?

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Solution: Weighted least squares

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Next time....

- More model checking....
  - Robust variance estimation
  - Non-constant variance
  - Non-normal residuals
  - Influence and leverage statistics