→ PS1 grades / feedba

→ PS2 ⇒ Friday 5pm

Lecture 9 /10

Model Checking and Key Extensions

Review of where we left off

1. We have established the multiple linear regression model:

$$\underbrace{Y_{n\times1}} = \underbrace{X_{n\times(p+1)}\beta_{(p+1)\times1}}_{\text{mean model}} + \underbrace{\epsilon_{n\times1}}_{\text{error}}, \underbrace{\epsilon_{n\times1} \sim MVN(0_{n\times1}, \sigma^2 I_{n\times n})}_{Cov(\varepsilon_i, \varepsilon_j)=0}$$

2. We know that:

$$\hat{\beta} \text{ satisfies } X'(Y - X\beta) = 0 \text{ and minimizes } \underbrace{\sum_{i=1}^{n}(y_i - x_i'\beta)^2}$$

3. We have defined:
   - $\hat{Y} = X\hat{\beta} = HY$, where $H = X(X'X)^{-1}X'$
   - $\hat{R} = Y - \hat{Y} = Y - X\hat{\beta} = (I - H)Y$

4. Then we showed that:
   - $\hat{\beta} \sim MVN(\beta, \sigma^2(X'X)^{-1})$
   - $\hat{Y} \sim MVN(X\beta, \sigma^2 H)$
   - $\hat{R} \sim MVN(0, \sigma^2(I - H))$

Review of where we left off

Target	Estimate ~ Sampling Distn	95% CI for target	Test statistic for H0: Target = 0
β_j	$\hat{\beta}_j \sim N(\beta_j, [\sigma^2(X'X)^{-1})]_{jj})$	$\hat{\beta}_j \pm t \times \hat{se}(\hat{\beta}_j)$	$\frac{\hat{\beta}_j}{\hat{se}(\hat{\beta}_j)}$
$A\beta$	$A\hat{\beta} \sim N(A\beta, \sigma^2 A(X'X)^{-1}A')$	$A\hat{\beta} \pm t \times \hat{se}(A\hat{\beta})$	$\frac{A\hat{\beta}_j}{\hat{se}(A\hat{\beta}_j)}$
$g(\beta_j)$	$g(\hat{\beta}_j) \sim N(g(\beta_j), [g'(\beta_j)]^2[\sigma^2(X'X)^{-1}]_{jj})$	$g(\hat{\beta}_j) \pm t \times \hat{se}(g(\hat{\beta}_j))$	$\frac{g(\hat{\beta}_j)}{\hat{se}(g(\hat{\beta}_j))}$
$g(\beta)$	$g(\hat{\beta}) \sim N(g(\beta), g'(\beta)'[\sigma^2(X'X)^{-1}]g'(\beta))$	$g(\hat{\beta}) \pm t \times \hat{se}(g(\hat{\beta}))$	$\frac{g(\hat{\beta})}{\hat{se}(g(\hat{\beta}))}$
$\mu_i = E(Y_i \mid X_i)$	$\hat{Y}_i \sim N(\mu_i, \sigma^2[H]_{ii})$	$\hat{Y}_i \pm t \times \hat{se}(\hat{Y}_i)$	$\frac{\hat{Y}_i}{\hat{se}(\hat{Y}_i)}$
$\mu(x_0) = E(Y \mid x_0)$	$x_0'\hat{\beta} \sim N(x_0'\beta, \hat{\sigma}^2 x_0'(X'X)^{-1}x_0)$	$x_0'\hat{\beta} \pm t \times \hat{se}(x_0'\hat{\beta})$	$\frac{x_0'\hat{\beta}}{\hat{se}(x_0'\hat{\beta}}$

$\to x_0$

Key Assumptions by Order of Importance

$\Rightarrow$ Estimation of and interpretation of β

1. $E(Y|X) = X\beta$ $\Rightarrow$ we have "correctly" specified the mean model
   - omitted a key confounder / covariate
   - incorrectly specified functional form for a continuous X
   - missed key interactions
   - error measurement in X

2. Residuals are independent

   ↳ Design of the study: how is the data generated

   Longitudinal study / clustered design

3. Variance of residuals is constant $Var(\varepsilon_i) = \sigma^2$

   ↳ $f(X_i)$

   Estimation of $\hat{Var}(\hat{\beta})$ ↓ Inference

4. Residuals are normally distributed

   $\rightarrow \hat{Var}(\hat{\beta}) \Rightarrow$ bootstrap procedure

5. There are not a small number of highly influencial observations

   Estimation / interpretation of β

   $\hat{Var}(\hat{\beta})$

Omitted Variable Bias

Exposure of interest : X_1 , Confounder X_2

We will fit : $Y_i = \alpha_0 + \alpha_1 X_{1i} + U_i$

The truth population : $Y_i = \underline{\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i}$

$$\hat{\alpha}_1 = \frac{Cov(X_1, \underline{Y})}{Var(X_1)} = \frac{Cov(X_1, \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \varepsilon)}{Var(X_1)}$$

$$= \frac{\cancel{Cov(X_1, \beta_0)} + Cov(X_1, \beta_1 X_1) + Cov(X_1, \beta_2 X_2) + \cancel{Cov(X_1, \varepsilon)}}{Var(X_1)}$$

$Cov(X_1, \beta_0) = 0$

$Cov(\underline{X_1}, \varepsilon) = 0$

Omitted Variable Bias

$$\hat{\alpha}_1 = \frac{\mathrm{Cor}(X_1, \beta_1 X_1) + \mathrm{Cov}(X_1, \beta_2 X_2)}{\mathrm{var}(X_1)}$$

$$= \frac{\beta_1 \mathrm{Var}(X_1) + \beta_2 \mathrm{Cov}(X_1, X_2)}{\mathrm{var}(X_1)}$$

$$= \beta_1 + \beta_2 \frac{\mathrm{Cov}(X_1, X_2)}{\mathrm{Var}(X_1)}$$

SLR slope from regression X_2 on X_1

$$X_2 = \delta_0 + \delta_1 X_1 + v$$

$$\hat{\alpha}_1 = \beta_1 + \beta_2 \delta_1$$

the liner relationship between X_2 and X_1] δ_1

← the relationship between Y and X_2

it depends on sign of β_2, δ_1

Simulation exercise

- Within your breakout group, design a simulation study that would numerically demonstrate the result we just derived.
  - You go work for 15 minutes and then we will review together

Correct Functional Form for Continuous X

- To explore the assumption that E(Y|X) = Xβ, you can make the following plots:
  1. Plot $\hat{R}$ vs. $X_j, j = 1, \ldots, p$.
     - Recall that the residuals are independent of X if the model is correctly specified
  2. Plot $\hat{R}$ vs. $\hat{Y}$.
     - The residuals and predicted values are independent if the model is correctly specified

- * Never plot $\hat{R}$ vs. Y because these are correlated!

- Based on the figures from 1. and 2., you could modify the model to increase/decrease the complexity of the functional form of the variables.

* Hypothesis
exploratory data analysis
↓
specify a model
↓
evaluate the fit

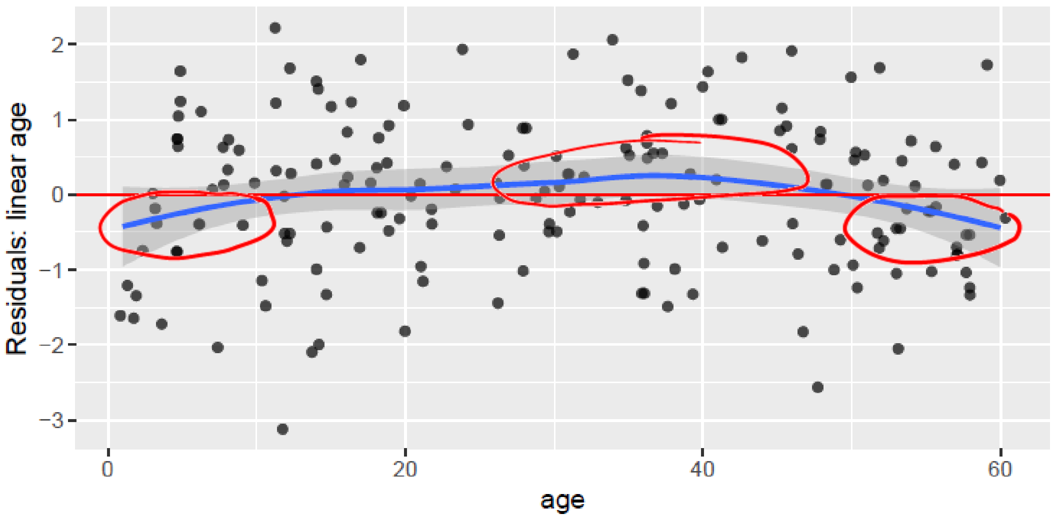
Example: Nepali Anthropometry Data

```r
reg0<-lm(data=d.cc, arm~age)
d.cc$residuals = residuals(reg0)
```

$\hat{R}$ vs X

```r
ggplot(d.cc,aes(x=age, y=residuals)) +
    geom_jitter(alpha = 0.7) +
    geom_smooth() +          geom_hline(yintercept=0,color="red") +
    labs(y="Residuals: linear age")
```

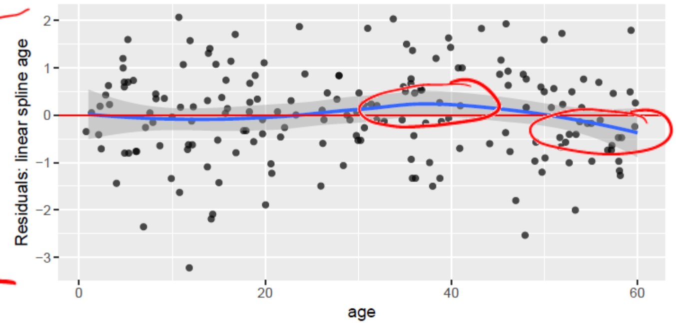
```
## `geom_smooth()` using method = 'loess' and formula 'y ~ x'
```

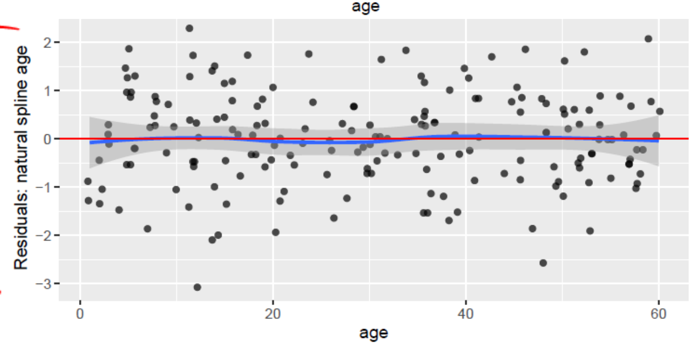


Example: Nepali Anthropometry Data

Update the model to include a smooth function of age via linear splines or natural splines.

```
reg1 = lm(arm~age + agesp6 + agesp12,data=d.cc)
reg2 = lm(arm~ns(age,3),data=d.cc)
d.cc$residuals1 = residuals(reg1)
d.cc$residuals2 = residuals(reg2)
```





Independence Assumption

MLR $\Rightarrow$ ε_i are independent

- Driven by the design of the study
- Longitudinal design : enroll units/people/cell/etc

measure the outcome and covariates on the unit over time at various assessments

Y_{ij} i = unit, $1, \ldots, m$; j = follow-up, $1, \ldots, n_i$ - outcome for individual i at assessment j.

- Clustered design

units/clusters are sampled → individuals within the cluster are assessed

sample of villages → interview each household in village

sample of clinics → interview patients

Y_{ij} = outcome for individual j from cluster i.

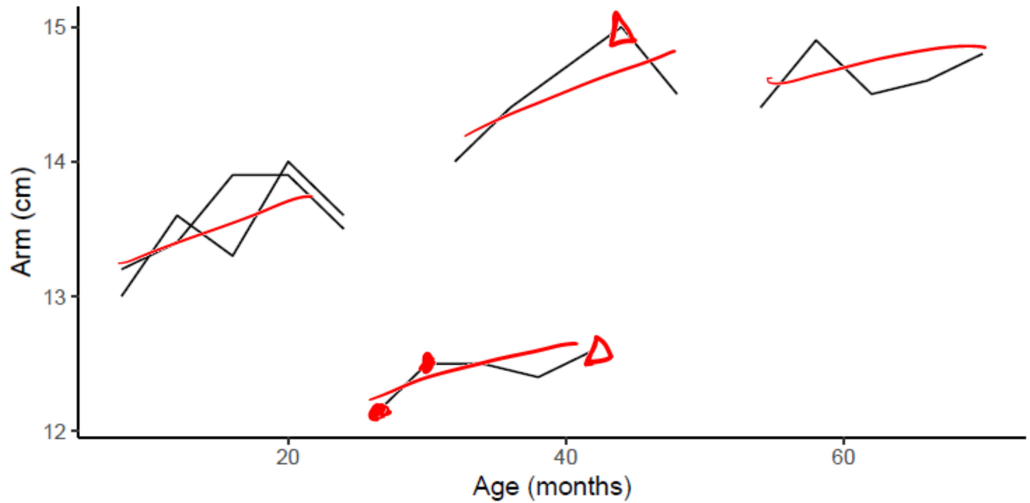
- Why do we care?

$\hat{Var}(\hat{\beta})$ = are wrong

Example: Nepali Anthropometry Data

- Design: i = 1, …, m = 200 children each measured at baseline (j = 1) and then every 4 months for 4 follow-up visits (j = 2, 3, 4, 5).

```
ggplot(d5,aes(x=age,y=arm,group = factor(id))) +
    geom_line() +
    labs(x='Age (months)', y ='Arm (cm)') +
    theme_classic()
```



Checking the Independence Assumption

- Don't need to, we know the independence assumption is violated based on knowledge of the design

- We can explore covariance/correlation in the observed data

- Example: Consider the Nepali Anthropometry data where we have data for i = 1, …, m = 200 children each measured at baseline (j = 1) and then every 4 months for 4 follow-up visits (j = 2, 3, 4, 5).
  - Step 1: Regress *Y* on *X* assuming independence and estimate β and *R*
  - Step 2: Plot $\hat{R}_{ij}$ vs. $\hat{R}_{ik}$ for all j,k
  - Compute $Cov(\hat{R}_{ij}, \hat{R}_{ik}) = \sqrt{Var(\hat{R}_{ij})} \times \sqrt{Var(\hat{R}_{ik})} \times Corr(\hat{R}_{ij}, \hat{R}_{ik})$
  - Or standardize the residuals and plot $Corr(\hat{R}_{sij}, \hat{R}_{sik})$

Lab 5

How do we rethink the model?

What if we apply least squares to correlated data?

What if we apply least squares to correlated data?

Solution: Weighted least squares

Solution: Weighted least squares

Next time....

- More model checking....
  - Robust variance estimation
  - Non-constant variance
  - Non-normal residuals
  - Influence and leverage statistics