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Lecture 7

Vector representation of MLR continued, assessing the impact of Gaussian residuals assumption

MLR model expressed in vector notation

We have for each subject i, $i = 1, \dots, n$

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \dots + \beta_p X_{pi} + \varepsilon_i$$

$$\begin{array}{c|l}
i & \\
\hline
1 & Y_1 = 1 \cdot \beta_0 + \beta_1 X_{11} + \beta_2 X_{21} + \dots \beta_p X_{p1} + \varepsilon_1 \\
2 & Y_2 = 1 \cdot \beta_0 + \beta_1 X_{12} + \beta_2 X_{22} + \dots \beta_p X_{p2} + \varepsilon_2 \\
\vdots & \vdots \\
n & Y_n = 1 \cdot \beta_0 + \beta_1 X_{1n} + \beta_2 X_{2n} + \dots \beta_p X_{pn} + \varepsilon_p
\end{array}$$

$$\underset{\sim}{Y}_{n\times 1} = \underset{\sim}{1}\,\beta_0 + \beta_1 \underset{\sim}{X_1}_{n\times 1} + \beta_2 \underset{\sim}{X_2}_{n\times 1} + \dots \beta_p \underset{\sim}{X_n}_{n\times 1} + \underset{\sim}{\varepsilon}_{n\times 1}$$

$$\underset{\sim}{Y}_{n\times 1} = \underbrace{(\underset{\sim}{1}, \underset{\sim}{X_1}, X_2, \dots, X_p)}_{X_{n\times(p+1)} \text{ design matrix}} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}_{p+1\times 1} + \underset{\sim}{\varepsilon}_{n\times 1}$$

MLR model expressed in vector notation

$$\Rightarrow \underset{\sim}{Y}_{n\times 1} = X_{n\times(p+1)} \underset{\sim}{\beta}_{(p+1)\times 1} + \underset{\sim}{\varepsilon}_{n\times 1}$$

Say $n=3$, $p=2$ $\Rightarrow Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i$

$$\underset{\sim}{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \quad X = \begin{bmatrix} 1 & X_{11} & X_{21} \\ 1 & X_{12} & X_{22} \\ 1 & X_{13} & X_{23} \end{bmatrix} \quad \underset{\sim}{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} \quad \underset{\sim}{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}$$

$$X_{3\times 3} \underset{\sim}{\beta}_{3\times 1} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}_{3\times 1} = \underset{\sim}{\mu}_{3\times 1}$$

Defines the mean of Y_i for each i

MLR model expressed in vector notation

What about distribution of $\underset{\sim}{\varepsilon}$? and $\underset{\sim}{Y}$?

In general, we can define the multivariate normal distribution as: $\underset{\sim}{Y}_{n\times 1} \sim MVN(\underset{\sim}{\mu}_{n\times 1}, V_{n\times n})$

where $\underset{\sim}{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}$ $\underset{\sim}{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}$

$$V = \begin{bmatrix} V_{11} & & V_{ij} \\ & V_{22} & \\ V_{ji} & & \ddots \quad V_{nn} \end{bmatrix}$$

$V_{ii} = Var(Y_i)$

$V_{ij} = V_{ji} = Cov(Y_i, Y_j)$

If $\varepsilon_i \sim N(0, \sigma^2)$, independent

$\underset{\sim}{\varepsilon} \sim MVN(\underset{\sim}{0}, \sigma^2 I)$

$I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix}$ identity matrix

$\underset{\sim}{Y} \sim MVN(X\underset{\sim}{\beta}, \sigma^2 I)$

MLE or LS solution expressed in vector notation

MLR model: $\underset{\sim}{Y} = X\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$, $\underset{\sim}{\varepsilon} \sim MVN(\underset{\sim}{0}, \sigma^2 I)$

MLE or least squares: Going to drop the "$\sim$"

Choose $\hat{\beta}$ and $\hat{\sigma}^2$ to minimize $\sum_{i=1}^{n} (y_i - X_i\beta)^2$

X_i = ith row of X

$$\sum_{i=1}^{n} (y_i - X_i\beta)^2 = \underbrace{(Y - X\beta)'}_{1\times n}\underbrace{(Y - X\beta)}_{n\times 1}$$

$$U_\beta(\beta) = \frac{d}{d\beta}(Y - X\beta)'(Y - X\beta)$$

$$= X'(Y - X\beta)$$

set to 0, solve for $\hat{\beta} = (X'X)^{-1}X'Y$

Predicted values and residuals in vector notation

$$\hat{\beta} = (X'X)^{-1}X'Y$$

$$\hat{Y} = X\hat{\beta} =$$

$$\hat{R} = Y - \hat{Y} =$$

Distribution of $\hat{\beta}$

Note that if $\underset{\sim}{Y} \sim MVN(\underset{\sim}{\mu}, V)$, then

$$A\underset{\sim}{Y} \sim MVN(A\underset{\sim}{\mu}, AVA')$$

$$\hat{\beta} = \underbrace{(X'X)^{-1}X'}_{A}Y$$

$$E(\hat{\beta}) = E(AY) =$$

$$Var(\hat{\beta}) = Var(AY) =$$

Distribution of $\hat{Y}$

$$\hat{Y} = X\hat{\beta} = X(X'X)^{-1}X'Y = HY$$

Properties of the Hat matrix

1) hat matrix is symmetric

$H' =$

2) hat matrix is idempotent: $H \cdot H = H$

Distribution of $\hat{R}$

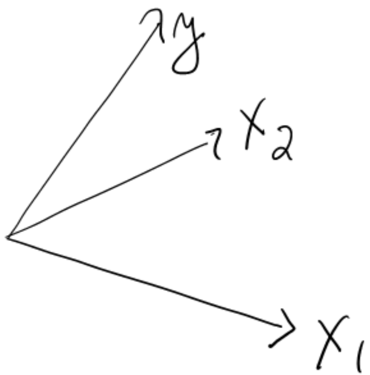
$$\hat{R} = Y - \hat{Y} = Y - HY =$$

Relationship between $\hat{Y}$ and $\hat{R}$

$$\mathrm{Cor}(\underbrace{\hat{Y}}_{HY}, \underbrace{\hat{R}}_{(I-H)Y}) = E\left[HY\{(I-H)Y\}'\right]$$

Geometry of least squares

Consider $\underset{\sim}{Y}_{n\times 1}$, $\underset{\sim}{X}_{1\,n\times 1}$, $\underset{\sim}{X}_{2\,n\times 1}$



H projects y onto the plane spanned by X_1, X_2

$$\Rightarrow \hat{y} = Hy = X\hat{\beta}$$

1) minimize the distance between y and $\hat{y} = X\hat{\beta}$

2) Shortest distance is the one that has a right angle between the predicted value and residual

3) residual is orthogonal to the plane spanned by X

4) Score equations:

$$X'(Y - X\beta) = 0$$

Simulation study

- We derived the distribution of the estimated regression coefficients assuming the residuals were Gaussian.

- Does approximate normality of the estimated regression coefficients hold even when the residuals are non-Gaussian?

Next time....

- Deriving the distribution of linear combinations of regression coefficients
- Deriving the distribution of non-linear combinations of regression coefficients using the Delta method
- LAB: You will generate the distribution of combinations of regression coefficients using bootstrap!