

Lecture 12 | 3

Finish case-study of log-linear regression applied to binned survival data

Continuous time survival analysis

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- The data contains information about time to death for inpatients hospitalized for a severe mental disorder. Survival time from hospitalization is in years.
- Patients are censored: i.e. we don't get to follow patients long enough to see when the event occurs for all patients.
- In the data, "censor" is 1 if censored; 0 if the patient died; "age" of hospitalization for mental disorder is in years; "male" is 1 for males and 0 for females.

age male event ## 15



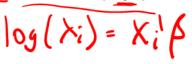
▶ We "binned" the information about survival into 10-year increments of follow-up

##		survive	censor	age	${\tt male}$	event
##	1	1	0	58	0	1
##	2	1	0	51	0	1
##	3	2	0	55	0	1
##	4	11	0	48	0	1
##	5	14	0	47	0	1
##	6	22	0	28	0	1
##	7	24	0	45	0	1
##	8	26	0	43	0	1
##	9	31	1	31	0	0
##	10	32	0	25	0	1
##	11	35	1	35	0	0
##	12	35	1	33	0	0
##	13	36	1	25	0	0
##	14	37	1	30	0	0
##	15	40	0	36	0	1

							•							
##		Cuto	off	male	pyears	n	event	rate	midp					
##	1	1	-10	0	124	15	3	0.024	5					
/##	2	11-	20	0	105	12	2	0.019	15					
##	3	21-	-30	0	82	10	3	0.037	25	- /				
##	4	31-	-40	0	36	7	2	0.056	35					
##	5	12-	10	1	110	11	0	0.000	5					
##	6	11-	-20	1	110	11	0	0.000	15					
##	7	21-	-30	1	95	11	3	0.032	25					
##	8	31-	-40	1	25	6	1	0.040	35					
(_	ן ס)	N	~2 0	21-30			1						
1-10/11-20/21-30/ Janissum yours of hospitali														
	1+1+2+12×10=124													

- ▶ Incidence: risk per unit time of the event occurring among those that enter the interval
- ▶ Hazard: the limit of the incidence rate as the interval width goes to zero
 - ► Crude estimate: number of events divided by the person-time experienced in the interval
- ullet Want a smooth estimate of incidence/hazard using a log linear model: $\lambda_i = exp(X_i^{ extsf{ iny i}}eta)$
- Assume the number of events per interval $Y_i \sim Poisson(PT_i\lambda_i)$

$$\underbrace{E(Y_i)}_{\text{form}} = \lambda_i P T_i = exp(log(PT_i) + X_i^{\text{I}}\beta)$$

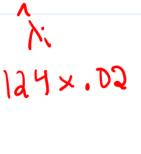




```
Model A:
                         Model A: E(Y_i) = \lambda_i PT_i = exp(log(PT_i) + \beta_0)
fitA = glm(event~1,offset=log(pyears),data=binned,family="poisson")
summary(fitA)
## Coefficients:
               Estimate Std. Error z value Pr(>|z|)
                            0.2673 -14.57 <2e-16 ***
## (Intercept) -3.8933
## ---
                   0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## Signif. codes:
lincom(fitA,"(Intercept)",eform=TRUE)
##
                 Estimate
                              2.5 %
                                                           Pr(>Chisq)
                                                   Chisq
                   estimated incidence of huturd of denth
per year is .07 2 dante
  (Intercept) 0.02037846 0.0120692 0.03440838 212.2069 4.533119e-48
```

▶ Model fitted values, i.e. Expected deaths per interval time

##		Cutoff	${\tt male}$	pyears	\mathbf{n}	event	rate	midp	expected
##	1	0-10	0	124	15	3	0.024	5	2.5269287
##	2	11-20	0	105	12	2	0.019	15	2.1397380
##	3	21-30	0	82	10	3	0.037	25	1.6710335
##	4	31-40	0	36	7	2	0.056	35	0.7336245
##	5	0-10	1	110	11	0	0.000	5	2.2416303
##	6	11-20	1	110	11	0	0.000	15	2.2416303
##	7	21-30	1	95	11	3	0.032	25	1.9359534
##	8	31-40	1	25	6	1	0.040	35	0.5094614



Log-linear model, Model B

Model B

Log-linear model; Model B

```
lincom(fitB,c("(Intercept)","(Intercept)+male", "male"),eform=TRUE)
                                                                                                                                            97.5 %
                                                                                                                                                                                Chisq
                                                                                                                                                                                                             Pr(>Chisq)
                                                                  Estimate 2.5 %
           (Intercept)
                                                                  0.02881844 0.01351409
                                                                                                                                             0.06145458
                                                                                                                                                                                84.26334 4.330714e-20
 ## (<u>Intercept)+ma</u>le 0.01176471 0.003552796 0.03895757 52.88403 3.538347e-13
 ## male
                                                                  0.4082353 0.09899778
                                                                                                                                             1.683432
                                                                                                                                                                                 1.536181 0.2151872
exp(B1) = 1/1 male = 1 = .41 The relative

Nilmale = 0 = .41 The relative
                                                                                                                                                                                                ina giver yer
                                                                                                                  among mules is . 41 times the
                                                                                                                                    risk of death for females.
                              In any guan yer, the rick of death formules is 5900 smaller than the risk for females.
```

Log-linear model; Models C and D

We would expect the hazard of death to depend on how long one has been in the hospital since we do not live forever. Models C and D estimate the relative risk of death for men as compared to women, controlling for a time-varying baseline hazard

women, controlling for a time-varying baseline hazard

In survival analysis, we refer to the "baseline hazard" as the hazard function when setting exposure

Model D:
$$E(Y_i) = \lambda_i PT_i = exp(log(PT_i) + \beta_0 + \beta_1 I(midp_i = 15) + \beta_2 I(midp_i = 25) + \beta_3 I(midp_i = 35) + \beta_4 male_i)$$

Log-linear model; Models C and D

```
fitC = glm(event~1+male+midp,offset=log(pyears),data=binned,family="quasipoisson")
summary(fitC)
##
               Estimate Std. Error t value Pr(>|t|)
                                                            exp (.05) = 1.05
   (Intercept)
                                     -7.548 0.000647 ***
               -4.55525
                            0.60351
   male
               -0.88461
                            0.54687
                                     -1.618 0.166674
                                                         The incidence
## midp
                0.05391
                            0.02444
                                      2.206 0.078504
                                                           rate of druth
lincom(fitC, "male", eform=TRUE)
                                                             increases by 5%
                                          Chisq Pr(>Chisq)
                       2.5 %
                               97.5 %
##
          Estimate
                                                              per additual
## male 0.4128736 0.141358 1.205907 2.616609
                                                  0.1057502
                                                               year of hosp.
 At any the following prospitalization,
the huzard of death for males
is 5906 smaller than that of
                                                                rejardless of
```

Log-linear model; Models C and D

```
fitD = glm(event~1+male+as.factor(midp),offset=log(pyears),data=binned,family="quasipoisson")
summary(fitD)
## Coefficients:
##
                     Estimate Std. Error t value Pr(>|t|)
   (Intercept)
                      -4.0321
                                  0.6024 - 6.694
                                                   0.0068 **
## male
                      -0.8909
                                  0.5975 - 1.491
                                                   0.2327
## as.factor(midp)15
                      -0.2863
                                  0.9187 -0.312
                                                   0.7757
## as.factor(midp)25
                       1.0282
                                  0.7122
                                           1.444
                                                   0.2445
## as.factor(midp)35
                      1.2965
                                  0.8219
                                           1.577
                                                   0.2128
lincom(fitD, "male", eform=TRUE)
##
          Estimate
                       2.5 %
                               97.5 %
                                          Chisq Pr(>Chisq)
   male 0.4102721 0.1271973 1.323323 2.223383
                                                0.1359349
```

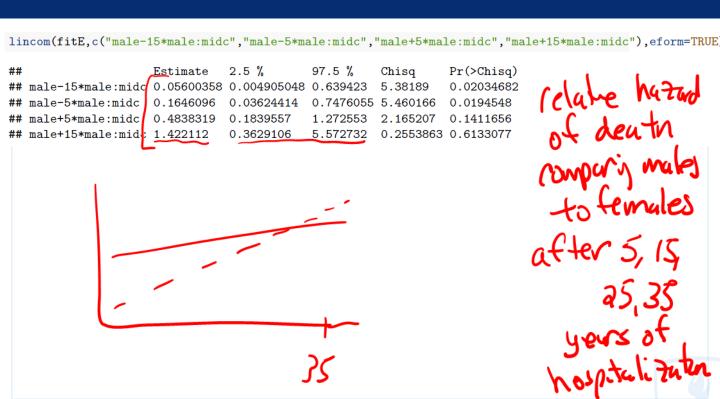
Log-linear models: Model E

- Finally, we look for evidence that the relative rate for men as compared to women changes over the duration of follow-up
 - ► I.e. the proportional hazards assumption is inadequate for our data.
- Model E: we center the midpoint variable at 20 years duration so that the male coefficient has a more reasonable interpretation and include interaction between male and years of hospitalization

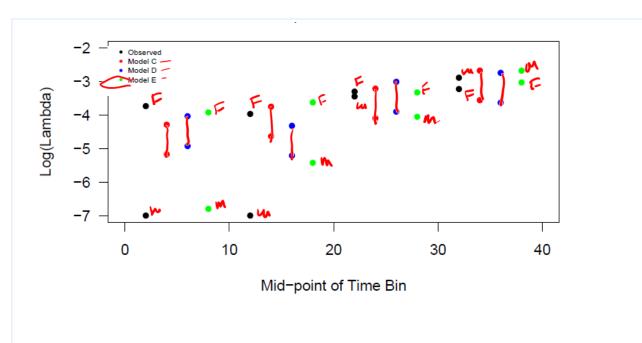
```
Model E: E(Y_i) = \lambda_i PT_i = exp(log(PT_i) + \beta_0 + \beta_1(midp_i - 20) + \beta_2 male_i + \beta_3(midp_i - 20) male_i)
```

```
binned$midc = binned$midp - 20
fitE = glm(event~1+male*midc,offset=log(pyears),data=binned,family="quasipoisson")
summary(fitE)
## Coefficients:
               Estimate Std. Error t value Pr(>|t|)
##
                            0.24520 -14.150 0.000145 ***
## (Intercept) -3.46957
## male
                -1.26510
                            0.58774
                                      -2.152 0.097711
## midc
                0.02981
                            0.02344
                                       1.272 0.272360
   male:midc
                 0.10782
                            0.05454
                                       1.977 0.119211
```

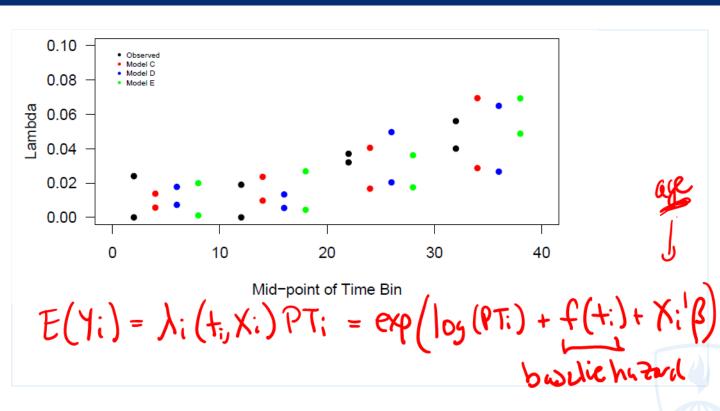
Log-linear models: Model E



Model comparison

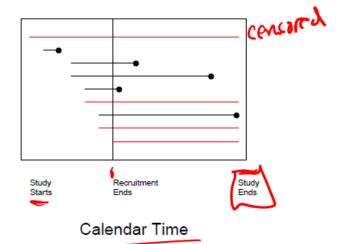


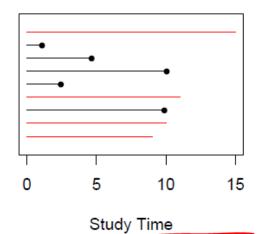
Model comparison



Continuous time survival analysis

- Binning survival times is convenient when working from administrative data or data where you do not have access to individual level data
- Most natural to treat time as continuous
- Review definition of censoring





Continuous time survival analysis

Absent censoring, the survival outcome Y_i, is the time from start of an at risk period to when the event of interest occurs.

In the presence of censoring, we get to see $\delta_i = 1$ if the event occurs and 0 if the event is censored

- $T_i = \min(D_i, C_i)$ where D_i is the time when the event occurs and C_i is the time of censoring
- Data for patient i is (T_i, δ_i)

Goals:

- Determine if the survival experience differs across exposure groups
- Predict survival experience

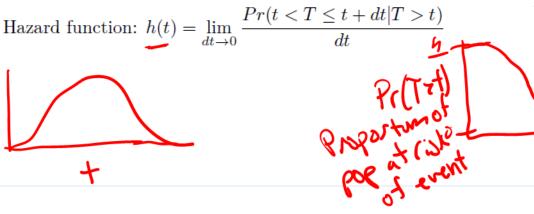
SCOXPH model La muchae learing

Key survival analysis definitions

Let T be a time to event random variable, $T \geq 0$.

Then we will define a series of quantities that can be used to describe the distribution of T.

- Cumulative Distribution Function: $F(t) = Pr(T \le t)$
- Survival Function: S(t) = Pr(T > t) = 1 F(t)
- Density function: $f(t) = \frac{d}{dt}F(T)$



Key survival analysis definitions

$$h(t) = \lim_{dt \to 0} \frac{Pr(t < T \le t + dt | T > t)}{dt}$$

$$= \lim_{dt \to 0} \frac{Pr(t < T \le t + dt \text{ and } T > t)}{Pr(T > t)dt}$$

$$= \lim_{dt \to 0} \frac{Pr(t < T \le t + dt \text{ and } T > t)}{dtS(t)}$$

$$= \frac{f(t)}{S(t)}$$

$$= \frac{f(t)}{1 - F(t)}$$

$$= \frac{dF(t)}{dt} / [1 - F(t)]$$

$$= -\frac{d}{dt} [1 - F(t)] / [1 - F(t)]$$

$$= -\frac{d}{dt} S(t) / S(t)$$

$$= -\frac{d}{dt} log_e S(t)$$

Cumulative hazard function: $H(t) = \int_0^t h(u)du = \log_e S(t)$. This implies: $S(t) = e^{-\int_0^t h(u)du} = e^{-H(t)}$

Common Parametric Models; Exponential

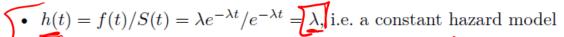
Assume $T \sim Exponential(\lambda)$ then

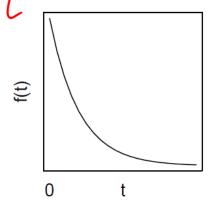
•
$$F(t) = 1 - e^{-\lambda t}$$
, $S(t) = e^{-\lambda t}$

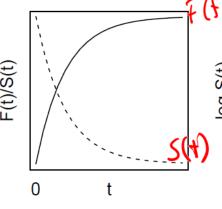
•
$$F(t) = 1 - e^{-\lambda t}, S(t) = e^{-\lambda t}$$

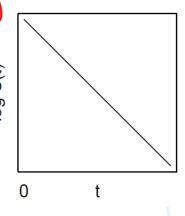
• $f(t) = \frac{d}{dt}(1 - e^{-\lambda t}) = \lambda e^{-\lambda t}$
• $E(T) = 1/\lambda, Var(T) = 1/\lambda^2$

•
$$E(T) = 1/\lambda$$
, $Var(T) = 1/\lambda^2$







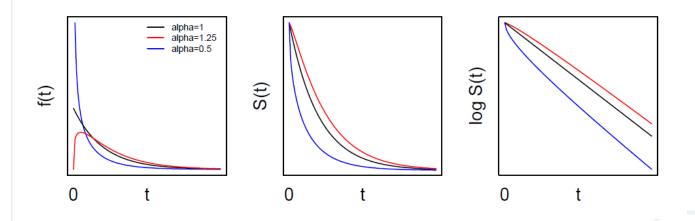


Common Parametric Models: Gamma

Assume $T \sim Gamma(\alpha, \lambda)$, then

•
$$f(t) = \frac{\lambda^{\alpha} t^{\alpha - 1} e^{-\lambda t}}{\Gamma(\alpha)}, t > 0, \Gamma(\alpha) = \int_{0}^{\infty} t^{\alpha - 1} e^{-t} dt$$

• F(t), S(t) and h(t) have to be solved by numerical integration; there are no closed form solutions.



Common Parametric Models: Weibull

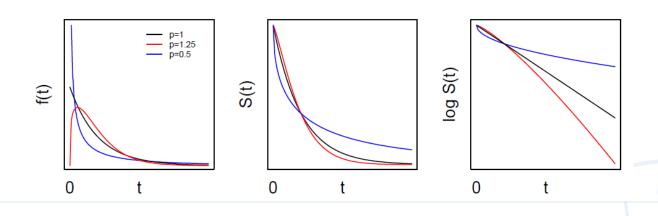
Assume $T \sim Weibull(\lambda, p)$, then

•
$$f(t) = p\lambda t^{p-1}e^{-(\lambda t)^p}$$

•
$$F(t) = 1 - e^{-(\lambda t)^p}, S(t) = e^{-(\lambda t)^p}$$

•
$$h(t) = p\lambda^p t^{p-1}$$

• When p = 1, $Weibull(\lambda, 1) = Exponential(\lambda)$.



Analysis of survival analysis outcomes in continuous time

Estimating S(t) via Kaplan-Meier survival function estimate (now)

Testing whether S1(t) = S2(t), via the log-rank test (Lab 7)

Regression of survival outcomes on exposures via Cox Proportional Hazards regression models (Lecture 14)

Kaplan-Meier estimate of the survival function

The Kaplan-Meier estimate of the survival function S(t) is also known as the **Product-limit** estimator.

This estimator for the survival function assumes that:

- the survival probabilities are the same for subjects recruited early and late in the study
- * the events happened at the times specified

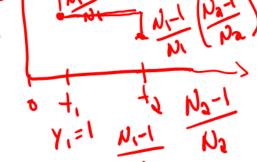
To construct the Kaplan-Meier estimator, you need to order the unique event times and compute:

Event times:
$$t_1 < t_2 < \dots < t_J$$

No. at risk:
$$N_1 > N_2 > \dots > N_J$$

No. of events:
$$y_1$$
 y_2 ... y_J

The estimate of S(t) is 1 if $t < t_1$ and

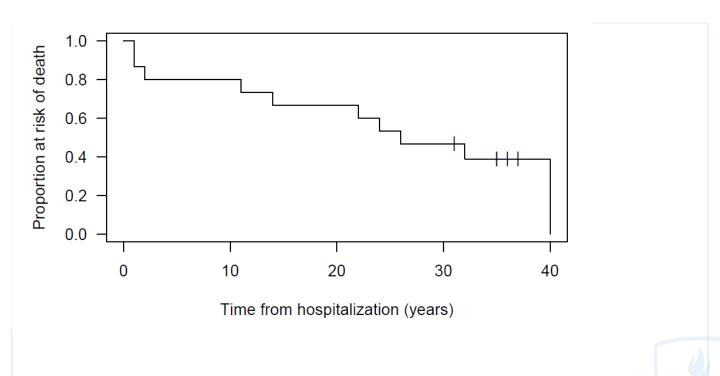


Kaplan-Meier estimate of the survival function

Using the data for inpatients hospitalized for a severe mental disorder, we will be computing the Kaplan-Meier estimate of the survival function for the female patients. Survival time from hospitalization is in years.

	1	1	2	11	14	22	24	26	31 +	32	35 +	35 +	36+	37 +	40	Ni	yi	(Ni-yi)/Ni	S(t)
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	15	2	0.867	0.867
2			1	0	0	0	0	0	0	0	0	0	0	0	0	13	1	0.923	0.800
3				0	0	0	0	0	0	0	0	0	0	0	0	12	0	1.000	0.800
4				0	0	0	0	0	0	0	0	0	0	0	0	12	0	1.000	0.800
5				0	0	0	0	0	0	0	0	0	0	0	0	12	0	1.000	0.800
6				0	0	0	0	0	0	0	0	0	0	0	0	12	0	1.000	0.800
7				0	0	0	0	0	0	0	0	0	0	0	0	12	0	1.000	0.800
8				0	0	0	0	0	0	0	0	0	0	0	0	12	0	1.000	0.800
9				0	0	0	0	0	0	0	0	0	0	0	0	12	0	1.000	0.800
10				0	0	0	0	0	0	0	0	0	0	0	0	12	0	1.000	0.800
11				1	0	0	0	0	0	0	0	0	0	0	0	12	1	0.917	0.733
12					0	0	0	0	0	0	0	0	0	0	0	11	0	1.000	0.733
13					0	0	0	0	0	0	0	0	0	0	0	11	0	1.000	0.733
14					1	0	0	0	0	0	0	0	0	0	0	11	1	0.909	0.667

Kaplan-Meier estimate of survival function



Greenwood's formula for variance of S(t)

An estimate of the variance of $\hat{S}(t)$ based on Greenwood's formula (application of Delta method) is:

$$\hat{V}ar(\hat{S}(t)) = \hat{S}(t)^2 \sum_{j:t_i \le t} \frac{y_j}{N_j(N_j - y_j)}$$

A 95% confidence interval for S(t) can be derived as:

$$\hat{S}(t) \pm 1.96 \sqrt{\hat{V}ar(\hat{S}(t))}$$

with imposing the constraint that the confidence interval lies in [0,1], i.e. if the bounds of the confidence interval go outside [0,1], set the values to 0 or 1, respectively. This is unappealing in many respects!

Variance of S(t) estimate based on complementary log-log

An alternative to Greenwoods formula for the variance, a variance estimate can be derived based on the complementary Log-Log transformation.

Let v(t) = log[-logS(t)]. Note that $S(t) \in [0,1]$ and $v(t) \in [-\infty,\infty]$.

$$\hat{V}ar(\hat{v}(t)) = \sum_{j:t_j \le t} \frac{y_j}{N_j(N_j - y_j)} \left[\sum_{j:t_j \le t} log\left(\frac{N_j - y_j}{N_j}\right) \right]^{-2}$$

The 95% confidence interval for v(t) is given by:

$$\hat{v}(t) \pm 1.96 \sqrt{\hat{V}ar(\hat{v}(t))}$$

where we can define the upper and lower bound as $\hat{v}_L(t)$ and $\hat{v}_U(t)$.

NOTE: S(t) = exp(-exp(v(t))), so the 95% confidence interval for S(t) is:

$$[exp(-exp(\hat{v}_U(t))), exp(-exp(\hat{v}_L(t)))]$$

Example calculations: Greenwood's formula

Compute the 95% confidence interval for S(2):

1. Using Greenwood's formula:

$$\hat{V}ar(\hat{S}(2)) = \hat{S}(2)^2 \sum_{j:t_j \le 2} \frac{y_j}{N_j(N_j - y_j)}$$

$$= \hat{S}(2)^2 \left[\frac{y_1}{N_1(N_1 - y_1)} + \frac{y_2}{N_2(N_2 - y_2)} \right]$$

$$= 0.8^2 \left[\frac{2}{15 \times (15 - 2)} + \frac{1}{13 \times (13 - 1)} \right]$$

$$= 0.0107$$

95% CI for S(2): $0.8 \pm 1.96 * \sqrt{0.0107} \rightarrow (0.598, 1.003)$

Example calculations: Complementary log-log

2. Using the Complementary Log-Log transformation

$$\hat{v}(2) = log(-log(\hat{S}(2)))$$

$$= log(-log(0.8))$$

$$= -1.50$$

$$\hat{V}ar(\hat{v}(2)) = \sum_{j:t_j \le 2} \frac{y_j}{N_j(N_j - y_j)} \left[\sum_{j:t_j \le 2} log\left(\frac{N_j - y_j}{N_j}\right) \right]^{-2} \\
= \left[\frac{y_1}{N_1(N_1 - y_1)} + \frac{y_2}{N_2(N_2 - y_2)} \right] \left[log\left(\frac{N_1 - y_1}{N_1}\right) + log\left(\frac{N_2 - y_2}{N_2}\right) \right]^{-2} \\
= \left[\frac{2}{15 \times 13} + \frac{1}{13 \times 12} \right] \left[log(13/15) + log(12/13) \right]^{-2} \\
= 0.335$$

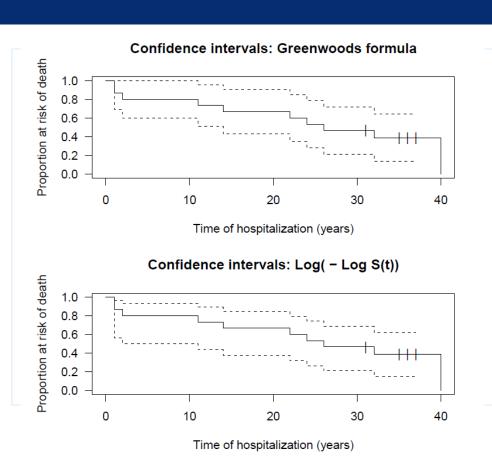
95% CI for
$$v(2)$$
 is: $\hat{v}(2) \pm 1.96\sqrt{\hat{V}ar(\hat{v}(2))}$ is $-1.50 \pm 1.96\sqrt{0.335}$ is $(-2.63, -0.36)$.

95% CI for
$$S(2)$$
 is: $(exp(-exp(-0.36)), exp(-exp(-2.63)))$ is $(0.50, 0.93)$.

R implementation

```
library(survival)
St.green = survfit(Surv(survive, event) ~ 1, data = d.female,
              type = "kaplan-meier",
              conf.type = "plain")
St.cll = survfit(Surv(survive, event) ~ 1, data = d.female,
              type = "kaplan-meier",
              conf.type = "log-log")
summary(St.green)
## Call: survfit(formula = Surv(survive, event) ~ 1, data = d.female,
##
      type = "kaplan-meier", conf.type = "plain")
##
##
   time n.risk n.event survival std.err lower 95% CI upper 95% CI
##
           15
                        0.867 0.0878
                                           0.695
                                                      1.000
##
        13
                   1 0.800 0.1033
                                          0.598
                                                      1.000
##
     11
        12
                   1 0.733 0.1142
                                          0.510
                                                      0.957
##
     14
        11
                   1 0.667 0.1217
                                          0.428
                                                      0.905
##
     22
        10
                   1 0.600 0.1265
                                          0.352
                                                      0.848
     24
                   1 0.533 0.1288
                                          0.281
                                                      0.786
##
##
     26
                   1 0.467 0.1288
                                          0.214
                                                      0.719
                   1 0.389 0.1287
##
     32
                                           0.137
                                                      0.641
                        0.000
##
     40
                                 NaN
                                            NaN
                                                        NaN
```

R implementation



Where to next....

- ► Lab: log-rank test comparing two survival functions
- ► Thursday: Cox proportional hazards model