Lecture 2

* basic Bayes
* two-level model for rates
* two-level normal-normal model

Gaussian - Gaussian

Lecture 2 Outline

Today we will focus on the case where we have a outcome Y_{ij} measured for i = 1, …, m clusters and n_i observations within i.

2-level: level 1 j, level 2 i

The analysis goals are to:

1. Understand the relative size of the within and between-cluster variation in Y
2. Estimate the cluster-specific mean of Y_{ij}

j i μ_i ij $Y_{ij} = \mu_i + \varepsilon_{ij}$

To accomplish these goals we will

- Review Bayes theorem within context of diagnostic testing
- Look at the simplest binary two-level model
- Introduce the two-level normal-normal model
- Review the calculations for random effects within an example: on testing in schools

Bayes Rule in Diagnostic Testing

you learned it from World's Smartest person

born in 1946

IQ max = 228

Diagnostic Testing

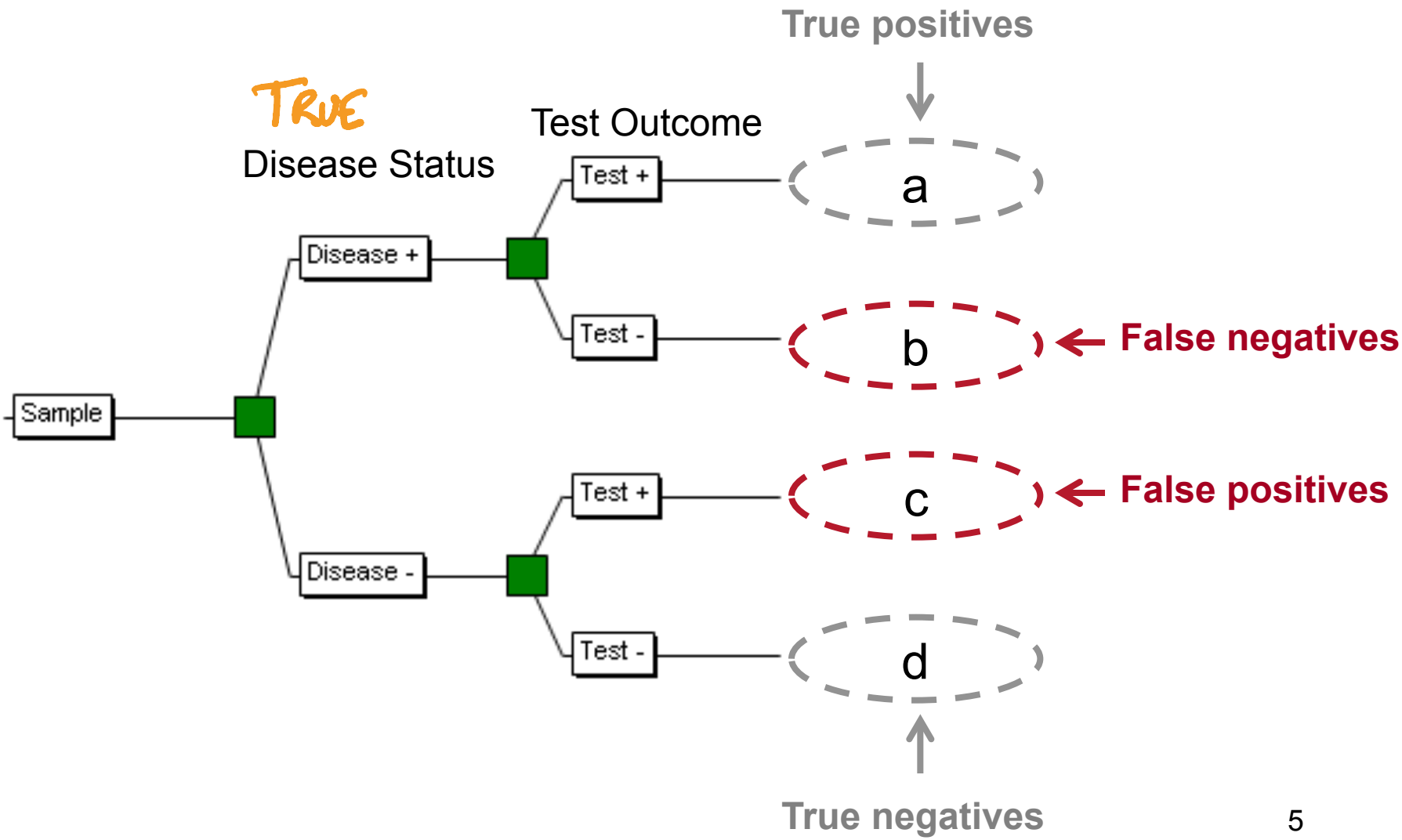
Ask

Marilyn®

BY MARILYN VOS SAVANT

A particularly interesting and important question today is that of testing for drugs. Suppose it is assumed that about 5% of the general population uses drugs. You employ a test that is 95% accurate, which we'll say means that if the individual is a user, the test will be positive 95% of the time, and if the individual is a nonuser, the test will be negative 95% of the time. A person is selected at random and is given the test. It's positive. What does such a result suggest? Would you conclude that the individual is a drug user? What is the probability that the person is a drug user?

Diagnostic Testing



Diagnostic Testing

- "The workhorse of Epi": The 2 × 2 table

	Disease +	**Disease -**	**Total**
Test +	***a***	***b***	***a + b***
Test -	***c***	***d***	***c + d***
Total	***a + c***	***b + d***	***a + b + c + d***

Diagnostic Testing

- "The workhorse of Epi": The 2 × 2 table

	Disease +	Disease -	Total
Test +	*a*	*b*	*a + b*
Test -	*c*	*d*	*c + d*
Total	*a + c*	*b + d*	*a + b + c + d*

$$Sens = P(+ \mid D) = \frac{a}{a+c} \qquad Spec = P(- \mid \overline{D}) = \frac{d}{b+d}$$

Diagnostic Testing

- "The workhorse of Epi": The 2 × 2 table

	Disease +	Disease -	Total
Test +	***a***	***b***	***a + b***
Test -	***c***	***d***	***c + d***
Total	***a + c***	***b + d***	***a + b + c + d***

Positive Predictive Value

$$PPV = P(D \mid +) = \frac{a}{a+b}$$

$$NPV = P(\overline{D} \mid -) = \frac{d}{c+d}$$

Negative Predictive Value

$$Sens = P(+ \mid D) = \frac{a}{a+c}$$

$$Spec = P(- \mid \overline{D}) = \frac{d}{b+d}$$

Diagnostic Testing

- Marilyn's Example { Sens = 0.95; Spec = 0.95 }

	Disease + (D)	Disease - ($\bar{D}$)	Total
Test + (+)	*48*	*47*	*95*
Test - (-)	*2*	*903*	*905*
Total	*50*	*950*	*1000*

PPV = 51%

NPV = 99%

P(D) = 0.05

Diagnostic Testing

- Marilyn's Example { Sens = 0.95; Spec = 0.95 }

	Disease +	Disease -	Total
Test +	***190***	***40***	***230***
Test -	***10***	***760***	***770***
Total	***200***	***800***	***1000***

PPV = 83%

NPV = 99%

P(D) = 0.20

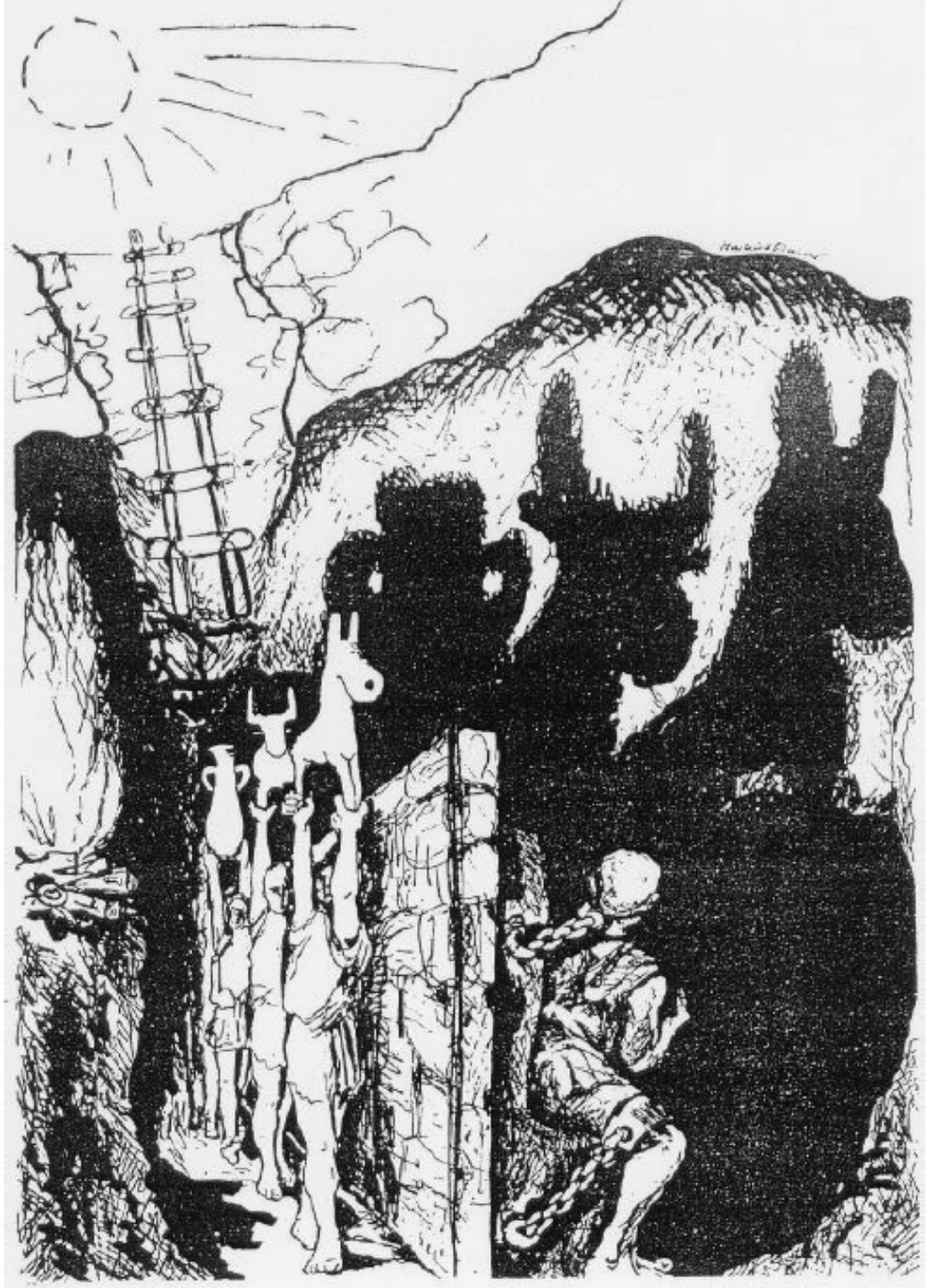
Point: PPV depends on prior probability of disease in the population

Diagnostic Testing Linked to Bayes Theorem

- P(D): prior distribution, that is prevalence of disease in the population
- P(+|D): likelihood function, that is the probability of observing a positive test given that the person has the disease (sensitivity) or doesn't (1-specificity)
- P(D|+), P(D|-) : posterior distribution of the unknown (latent) disease state given the test result

$$P(D \mid +) = \frac{P(+ \mid D)P(D)}{P(+)}$$

$$P(+) = P(+ \mid D)P(D) + P(+ \mid \overline{D})P(\overline{D})$$



Bayes Rule

Unknown state of nature: θ (D or $\bar{D}$)

Data: Y (Test+ or test−)

Distribution of data given state of nature:

$P_r(Y|\theta)$ ($P_r(+|D)$, $P_r(-|\bar{D})$).

sens spec.

Bayes Rule:

$$\frac{P_r(\theta_1|Y)}{P_r(\theta_0|Y)} = \frac{P_r(Y|\theta_1)}{P_r(Y|\theta_0)} \cdot \frac{P_r(\theta_1)}{P_r(\theta_0)}$$

posterior odds | likelihood ratio | prior odds

if $y = Test+$ | $\frac{P_r(D|Test+)}{P_r(\bar{D}|Test+)} = \frac{P_r(Test+|D)}{P_r(Test+|\bar{D})} \cdot \frac{P_r(D)}{P_r(\bar{D})}$

if $y = Test-$ | _____ = _____

Bayes & MLMs…

Terminology

- **Two stage normal normal model**
- **Variance component model**
- **Two-way random effects ANOVA**
- **Hierarchical model with a random intercept and no covariates**

Are all the same thing!

Testing in Schools

- Goldstein and Spiegelhalter JRSS (1996)
- Outcome: Standardized Test Scores
- Sample: 1978 students from 38 schools
  - Hierarchy: students (level 1) clustered within schools (level 2)
- Goal: Estimate overall “quality” of the schools using the results of the standardized test scores
- Possible Analyses:
  1. Calculate each school’s observed average score (approach A)
  2. Calculate an overall average for all schools (approach B)
  3. Borrow strength across schools to improve individual school estimates (Approach C)

Two-stage normal normal model

$$y_{ij} = \theta_i + \varepsilon_{ij}$$

$$i = 1,..., I,\ j = 1,.., n_i$$

$$\varepsilon_{ij} \sim N(0, \sigma^2)$$

$$\theta_i \sim N(\theta, \tau^2)$$

i represents schools

j represents students

So that there are j =1,…, n_i students within school i

y_{ij} is the standardized test score for student j within school i

Two-stage normal normal model

$$y_{ij} = \theta_i + \varepsilon_{ij}$$

$$i = 1,\ldots,I, j = 1,..,n_i$$

$$\varepsilon_{ij} \sim N(0,\sigma^2)$$

$$\theta_i \sim N(\theta,\tau^2)$$

In Bayesian approach, we would specify prior distributions for θ, and τ^2.

σ^2,

Two-stage normal normal model

$$y_{ij} = \theta_i + \varepsilon_{ij}$$

$$i = 1,\ldots,I,\ j = 1,\ldots,n_i$$

$$\varepsilon_{ij} \sim N(0,\sigma^2)$$

$$\theta_i \sim N(\theta,\tau^2)$$

In **EMPIRICAL** Bayesian approach, we estimate θ and τ^2 from the observed data and make no additional assumptions on these parameters

Shrinkage estimation

- Goal: estimate the school-specific average score θ_i
- Two simple approaches:
  - A) No shrinkage $\bar{y}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij}$

    Maximum likelihood estimates of the cluster means
  - B) Total shrinkage $\hat{\theta} = \frac{\sum_{i=1}^{I} \frac{n_i}{\sigma^2}\bar{y}_i}{\sum_{i=1}^{I} \frac{n_i}{\sigma^2}}$

    Inverse variance weighted average

    NOTE: (a modified version of) this our estimate of θ in a random intercept model

ANOVA and the F test

- To decide which estimate to use, a traditional approach is to perform a classic F test for differences among means
- If the group-means appear significantly different from each other then use A
- If the variance between groups is not significantly greater than what could be explained by individual variability within groups, then use B

Shrinkage Estimation: Approach C

We are not forced to choose between A and B

- An alternative is to use a weighted combination between A and B

$$\hat{\theta}_i = \lambda_i \bar{y}_i + (1 - \lambda_i)\hat{\theta}$$

$$\hat{\theta}_i = \hat{\theta} + \lambda_i(\bar{y}_i - \hat{\theta})$$

$$\lambda_i = \frac{\tau^2}{\tau^2 + \sigma_i^2}; \sigma_i^2 = \sigma^2 / n_i$$

Empirical Bayes estimate

NOTE: assumed variance within groups is constant.

Shrinkage Estimation: Approach C

Empirical Bayes Estimates; THINKING 1st

- First way to think about it
- Take the "some" of the MLE + (1 – "some") of the overall mean
- Result is to get a biased estimate of the cluster specific mean BUT less variable since we are "shrinking" the cluster means towards the overall mean

$$\hat{\theta}_i = \lambda_i \bar{y}_i + (1 - \lambda_i)\hat{\theta}$$

$$\lambda_i = \frac{\tau^2}{\tau^2 + \sigma_i^2}; \sigma_i^2 = \sigma^2 / n_i$$

Shrinkage Estimation: Approach C

Empirical Bayes Estimates

- Second way to think about it
- Application of Bayes rule!
- Take the estimate of the overall population mean
- Add an update based on the separation of variation into between clusters and within clusters

$$\hat{\theta}_i = \hat{\theta} + \lambda_i(\bar{y}_i - \hat{\theta})$$

$$\lambda_i = \frac{\tau^2}{\tau^2 + \sigma_i^2}; \sigma_i^2 = \sigma^2 / n_i$$

Shrinkage estimation

- Approach C reduces to approach A (no pooling) when the shrinkage factor is equal to 1, that is, when the variance between groups is very large
- Approach C reduces to approach B, (complete pooling) when the shrinkage factor is equal to 0, that is, when the variance between group is close to be zero

$$\hat{\theta}_i = \lambda_i \bar{y}_i + (1 - \lambda_i)\hat{\theta}$$

$$\hat{\theta}_i = \hat{\theta} + \lambda_i (\bar{y}_i - \hat{\theta})$$

$$\lambda_i = \frac{\tau^2}{\tau^2 + \sigma_i^2}$$

$$\sigma_i^2 = \sigma^2 / n_i$$

Back to the school example:

- Why borrow across schools?
  - Median # of students per school: 48, Range: 1-198
- Suppose small school (N=3)
  - Scores: 90, 90,10, average = 63
  - Difficult to say, small N $\Rightarrow$ highly variable estimates
- For larger schools we have good estimates, for smaller schools we may be able to borrow information from other schools to obtain more accurate estimates

Two-stage normal normal model

$$y_{ij} = \theta_i + \varepsilon_{ij}$$

$$y_{ij} = \theta + b_i + \varepsilon_{ij}$$

$$i = 1,..., I,\ j = 1,.., n_i$$

$$\varepsilon_{ij} \sim N(0, \sigma^2)$$

$$\theta_i \sim N(\theta, \tau^2), b_i \sim N(0, \tau^2)$$

Two-stage Normal-Normal Model

$$y_{ij} = \theta_i + \varepsilon_{ij}$$

$$y_{ij} = \theta + b_i + \varepsilon_{ij}$$

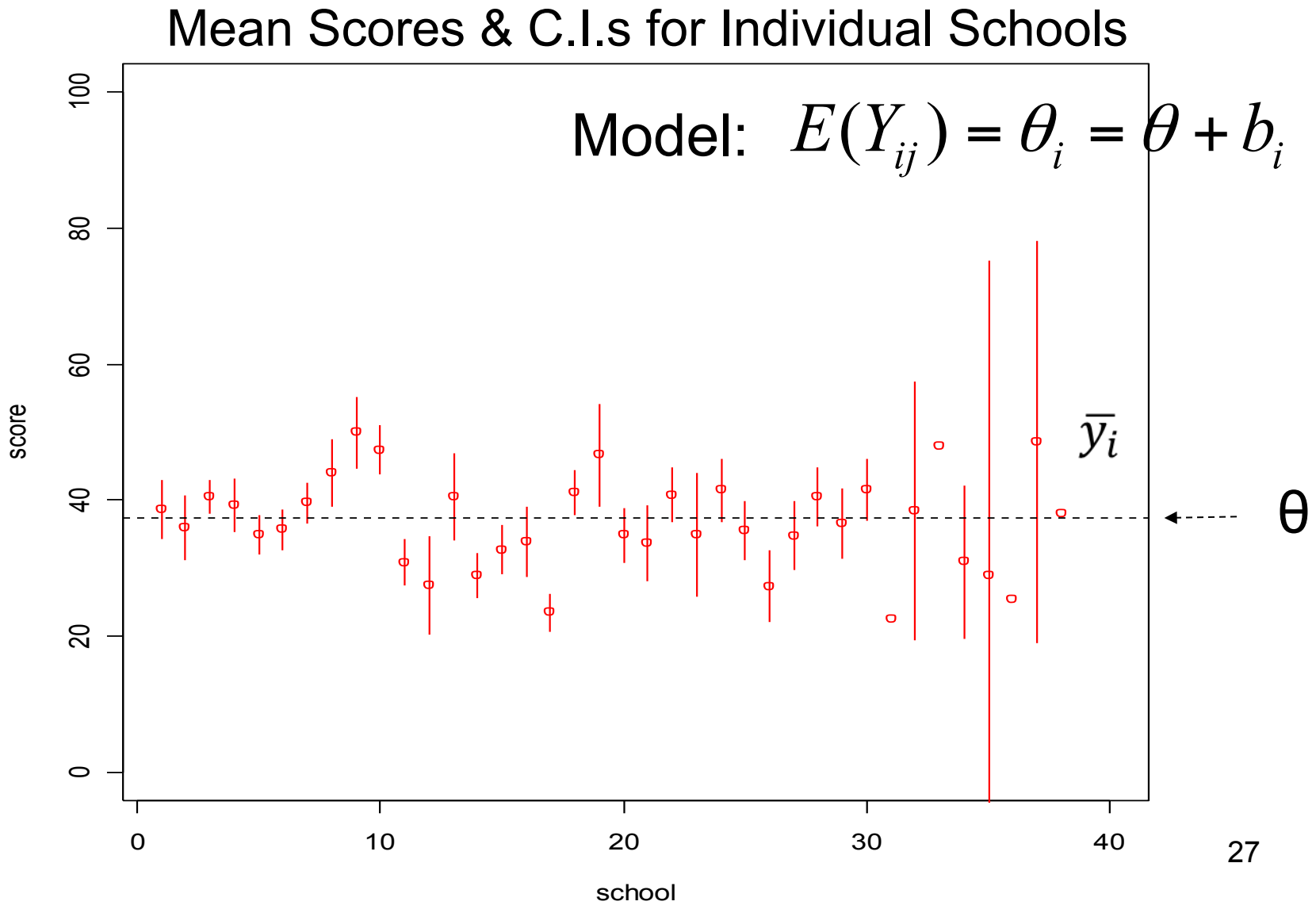
$\hat{\theta}$ = inverse-variance weighted mean

$$\hat{\theta}_i = \lambda_i \bar{y}_i + (1 - \lambda_i)\hat{\theta}$$

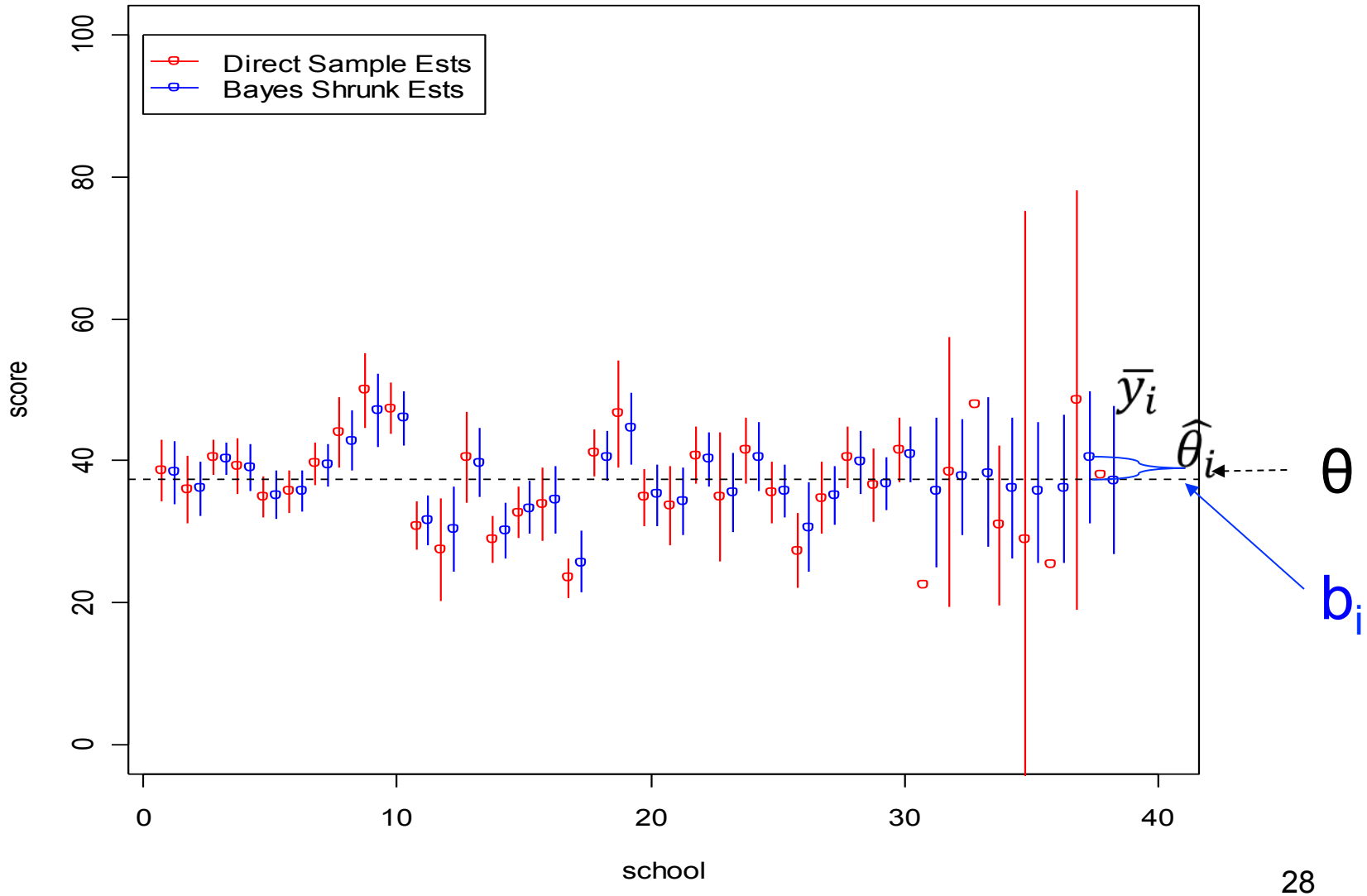
$$\hat{\theta}_i = \hat{\theta} + \lambda_i (\bar{y}_i - \hat{\theta})$$

$$\hat{\theta}_i = \hat{\theta} + \hat{b}_i$$

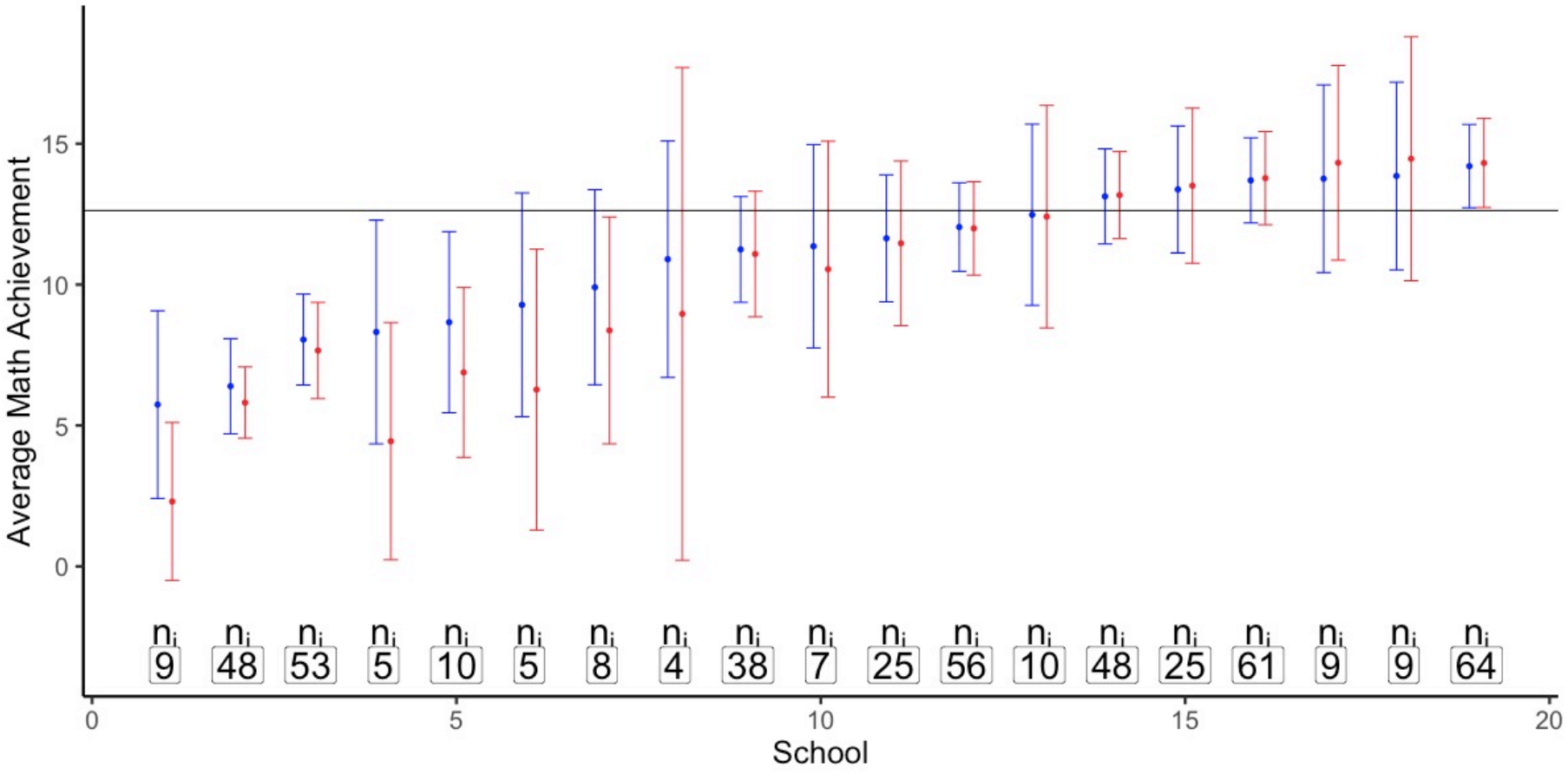
Testing in Schools: MLE School Specific Mean



Testing in Schools: Shrinkage Plot



Using (a subset of) the data from Lab 0



Some Bayes Concepts

- Frequentist: Parameters are “the truth”
- Bayesian: Parameters have a distribution
- In Bayes,
  - we “Borrow Strength” from other clusters
  - we “Shrink Estimates” towards overall averages
  - we compromise between model & data
  - Incorporate prior/other information in estimates
  - Account for other sources of uncertainty that are not measured

Lecture 2 Summary

- Reviewed idea of updating information within Bayes theorem using classic 2x2 table
- Considered idea of "borrowing strength across similar problems" - risk estimation
- Walked slowly through the 2-level normal-normal model
  - The simplest case of multi-level model
  - Model allows for two sources of variation: within and between clusters
  - Overall mean is inverse-variance weighted average
  - Cluster specific estimates are weighted averages of observe cluster specific estimate and the overallmean

    - Weighting depends on the relative size of within and between cluster variance.

Improved Assessment by *Borrowing Strength*; Estimating Local Coverage Rates (and penalty kick scoring) *Using*

Empirical Bayes Estimation

(aka Multi-level models)

Many Related Questions

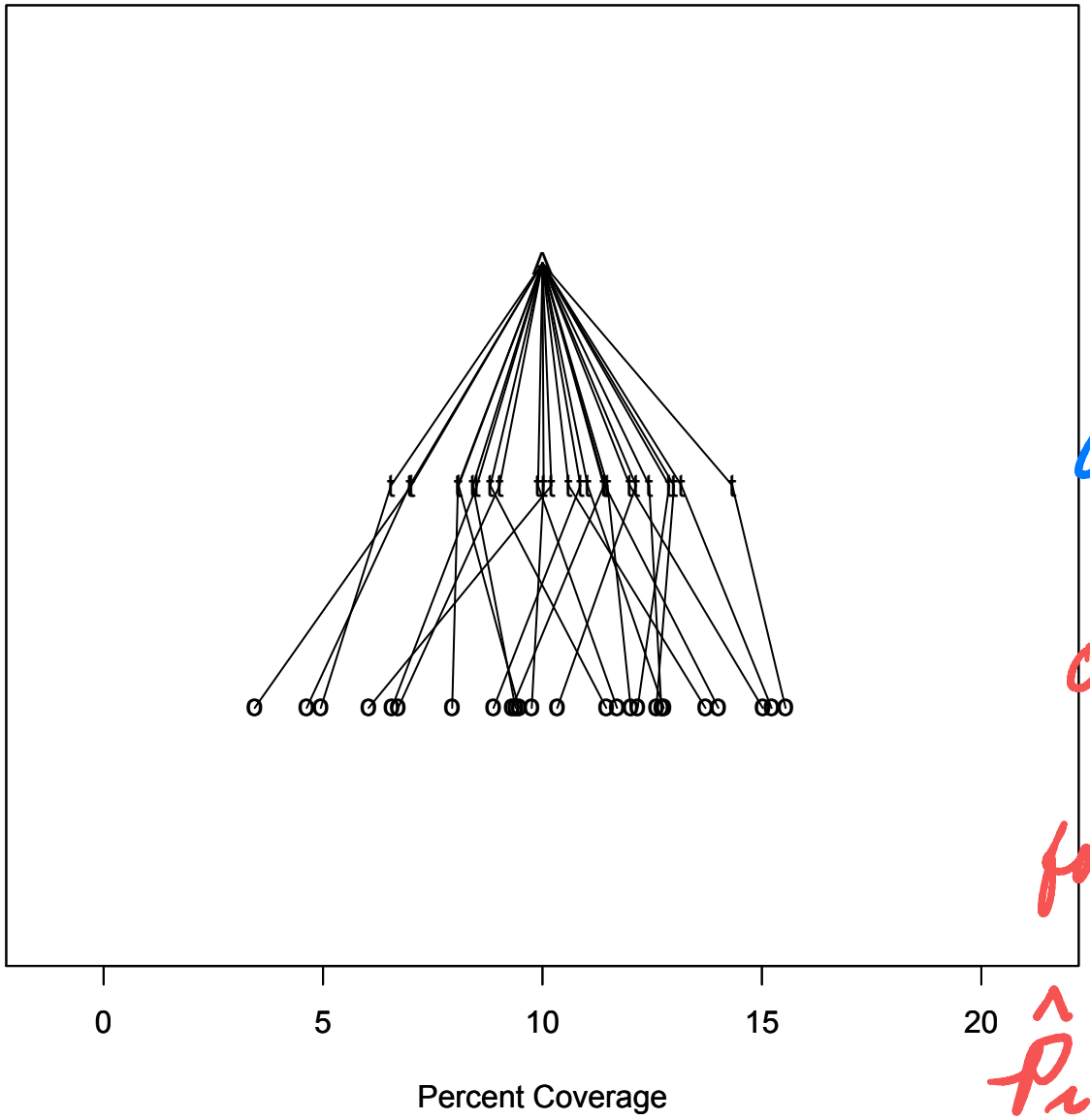
One question: What is the rate of vaccine coverage for children <24 mo in Mali?

Many related questions: What are the rates of measles vaccine coverage for children <24 mo for the 65 districts of Mali?

One question: What is the rate of penalty kick scoring in the world cup?

Many related questions: What are the rates of penalty kick scoring for the 30 top players in the world cup?

DHS Observed Coverage Rates for 25 Imaginary Districts



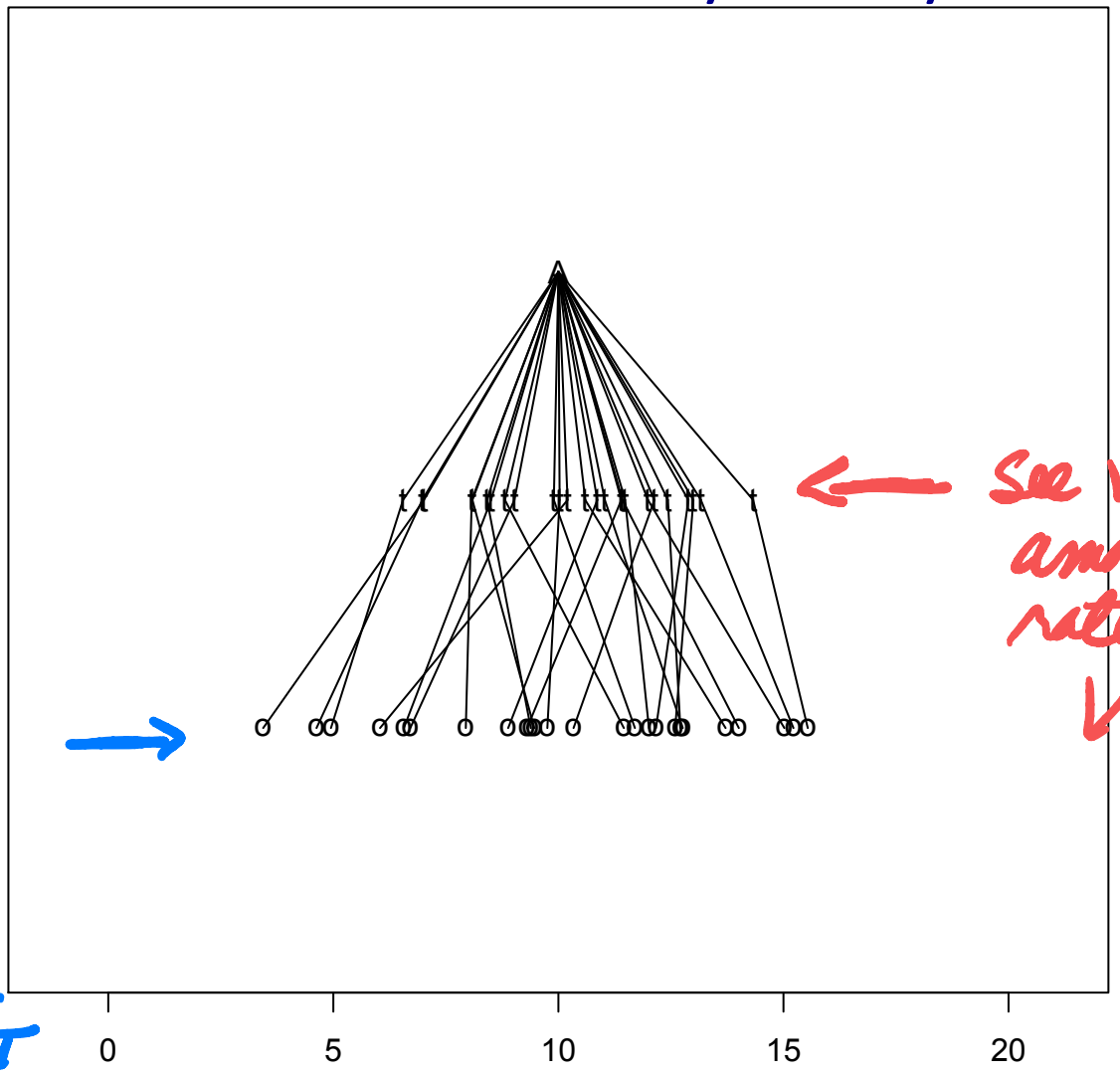
nation

district (i)

observed rates $\hat{p}_i$ for n_i children

$$\hat{p}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}$$

Q1. Are the observed rates more or less variable across districts than the true rates? Why and by how much?



See variance among true rates p_i

$Var(p_i) = V_p$

See Total variance among $\hat{p}_i$

$Var(\hat{p}_i) \equiv V_T$

$= V_p + Ave(Var(\hat{p}_i | p_i))$

Hierarchical (or Multi (2)-level) Statistical Model

Level I: Observed rate (o) = true rate (t) for given district + deviation attributable to the sampling of only a subset of the population

$$\hat{p}_i = \bar{Y}_{i\cdot} = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}, \quad Y_{ij} = \begin{cases} 1 & \text{caused} \\ 0 & \text{not} \end{cases}$$

child j in district i

Level II: True value for given district = average true value for the country + real deviation specific to the district

Model: $Y_{ij} \sim B(p_i, 1)$; $p_i = p_\cdot + \delta_i$

$p_\cdot$: national rate; δ_i: district i deviation

Understanding "Borrowing Strength" Idea
You do it every day; it is how our minds work

Q2. Identify a prediction you have made recently in answering a question and explain what "data" you used. How did you "borrow strength" from related questions and their data?

Answer

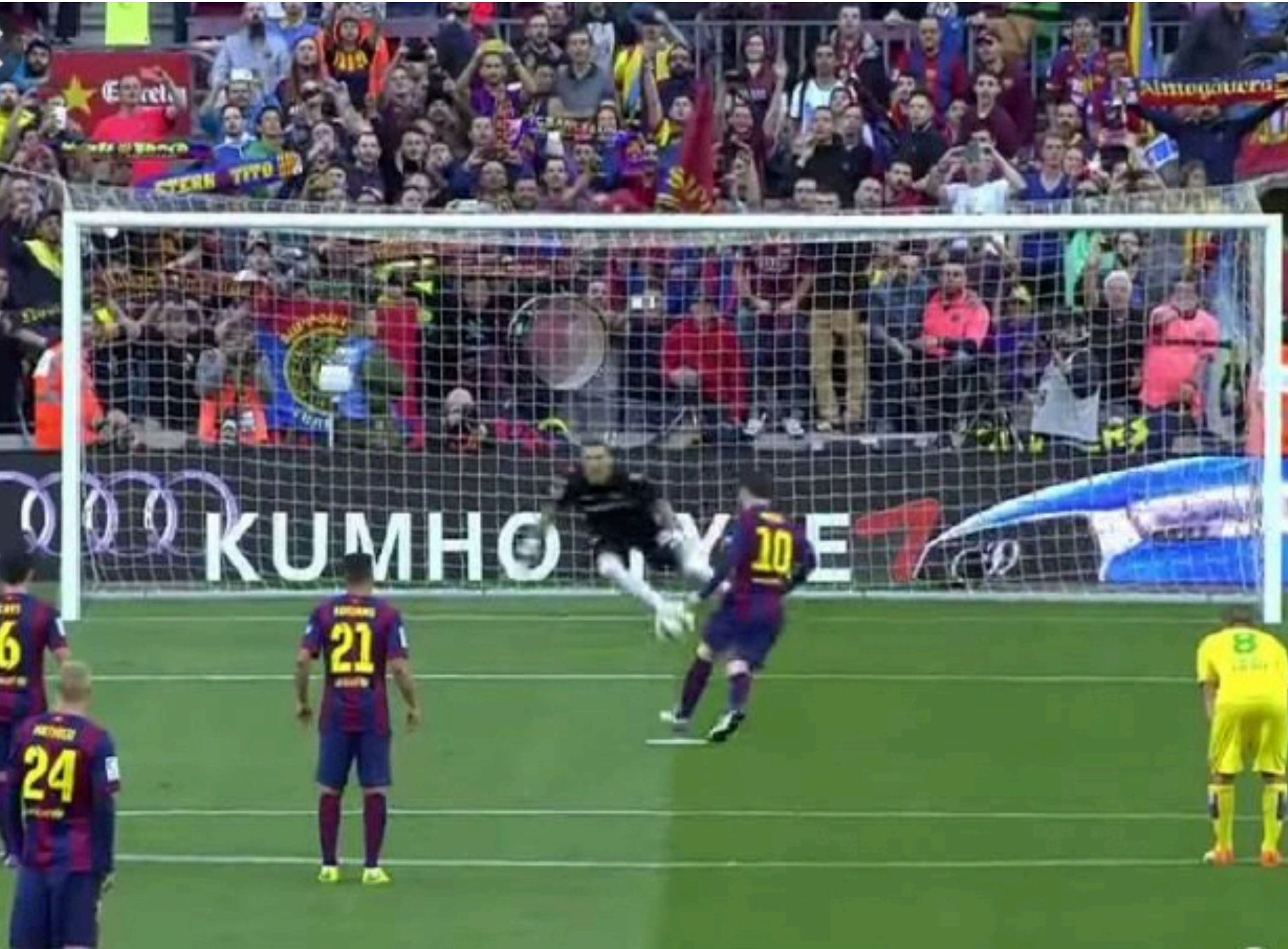
- Question: Will my boss give me a >3% raise this year?
- Related questions: Will my boss give each of my co-workers >3% raises this year
- Data
  - My last 3 raises have been <1%
  - The 4 other people in my group have all gotten <3% in each of the last three years; the boss favors Francine who told me she got a 2.5% raise
- Prediction: Probably, I will not get a raise >3%!

Q3. What is the best way to estimate the true population rate for a particular district?

- Two extremes to consider
  - What if the true rates of coverage all are the same for all districts?
    - Best estimator of that common rate is:
  - What if there is large heterogeneity among districts so that one teaches us little about the other
    - Best estimator of a districts true rate is:

Optimal Solution - Big Scientific Idea

- Compromise between the two extremes
  - Estimate:
    1. average rate (more precise; more biased); and
    2. district specific rate (less precise; less biased)
  - Estimate the degree of heterogeneity among districts in their **true** rates – (tricky)
  - Combine the average and district specific estimates, upweight the average when the heterogeneity is estimated to be small; upweight the district rate when heterogeneity is large


KUMHO
10
21
24

	goals	shots	$\hat{p}_i$
Valdivia	0	1	0.00
Forlan	2	6	0.33
Hernandez	4	9	0.44
Cavani	11	17	0.65
Lukaku	3	5	0.60
Samara	0	3	0.00
Benzema	2	3	0.67
Ozil	1	6	0.17
Giroud	5	10	0.50
Dempsey	4	7	0.57
Odemwinge	2	4	0.50
Johannsson	2	8	0.25
Martinez	6	9	0.67
Hulk	12	23	0.52
Robben	7	10	0.70
Ruaz	3	5	0.60
van Persie	5	17	0.29
Messi	22	41	0.54
Rodriguez	4	7	0.57
Muller	8	14	0.57
Aguero	5	10	0.50
Hazard	13	19	0.68
Vidal	11	21	0.52
Neymar	5	6	0.83
Guardado	3	6	0.50
Borges	1	1	1.00
Salpingidis	0	1	0.00
Moses	3	3	1.00
Drmie	2	2	1.00
Shaquiri	0	1	0.00

p_i - true rate

$\hat{p}_i$ - observed rate ($\bar{Y}_{i\cdot}$).

Y_{ij} - player i makes (1) or misses (0) shot j

Optimal Solution- Implemented for World Cup Futbol

- Estimate the average goal scoring and the player specific rates $\hat{p}_\cdot$
- Estimate the degree of heterogeneity among players in their true rates $\widehat{Var}(p_i)$
- Compromise between the average and player-specific estimates, closer to the average when the player-to-player heterogeneity is estimated to be small.

$$E(p_i \mid \hat{p}_i) = \tilde{p}_i = (1-\alpha_i)\hat{p}_\cdot + \alpha_i \hat{p}_i$$

Bayes estimate; $\hat{p}_\cdot$ biased precise; $\hat{p}_i$ unbiased imprecise

Do the Empirical Bayes Calculation

Task	Calculations and Results
1. Calculate total variance, V_T, of observed rates, p_i	0.0792
2. Calculate variance among true rates = V_T – mean of person-specific sampling variances (se_i^2) $= V_P$	0.0792 - .0673
3. Calculate $\alpha_i = V_P/(V_P + se_i^2)$ – weight to give p_i	
4. Calculate $p.eb_i = (1-\alpha_i)\ p.bar + \alpha_i\ p_i$	p.bar = 146 total goals out of 275 shots = 0.53

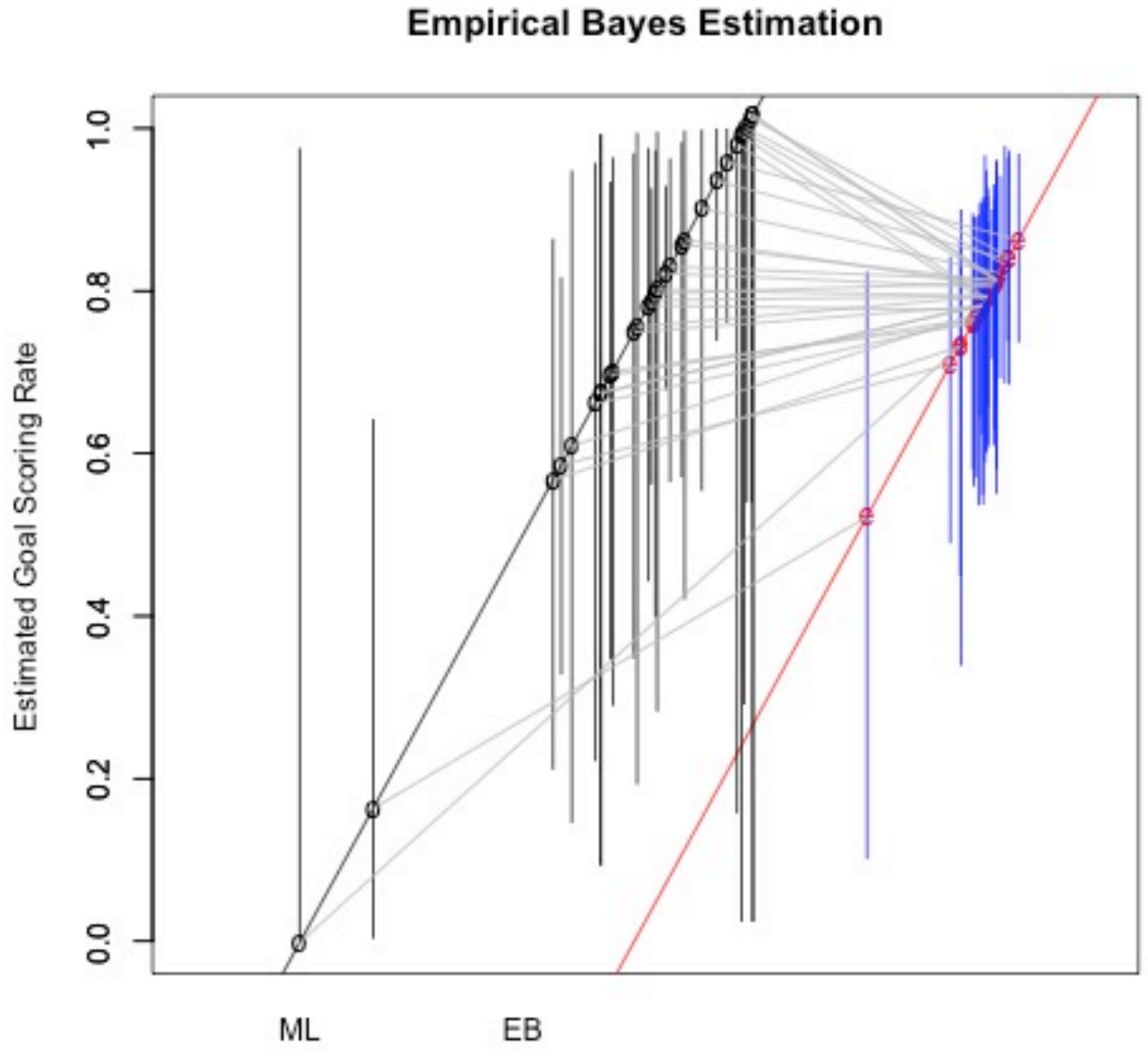
average statistical variance

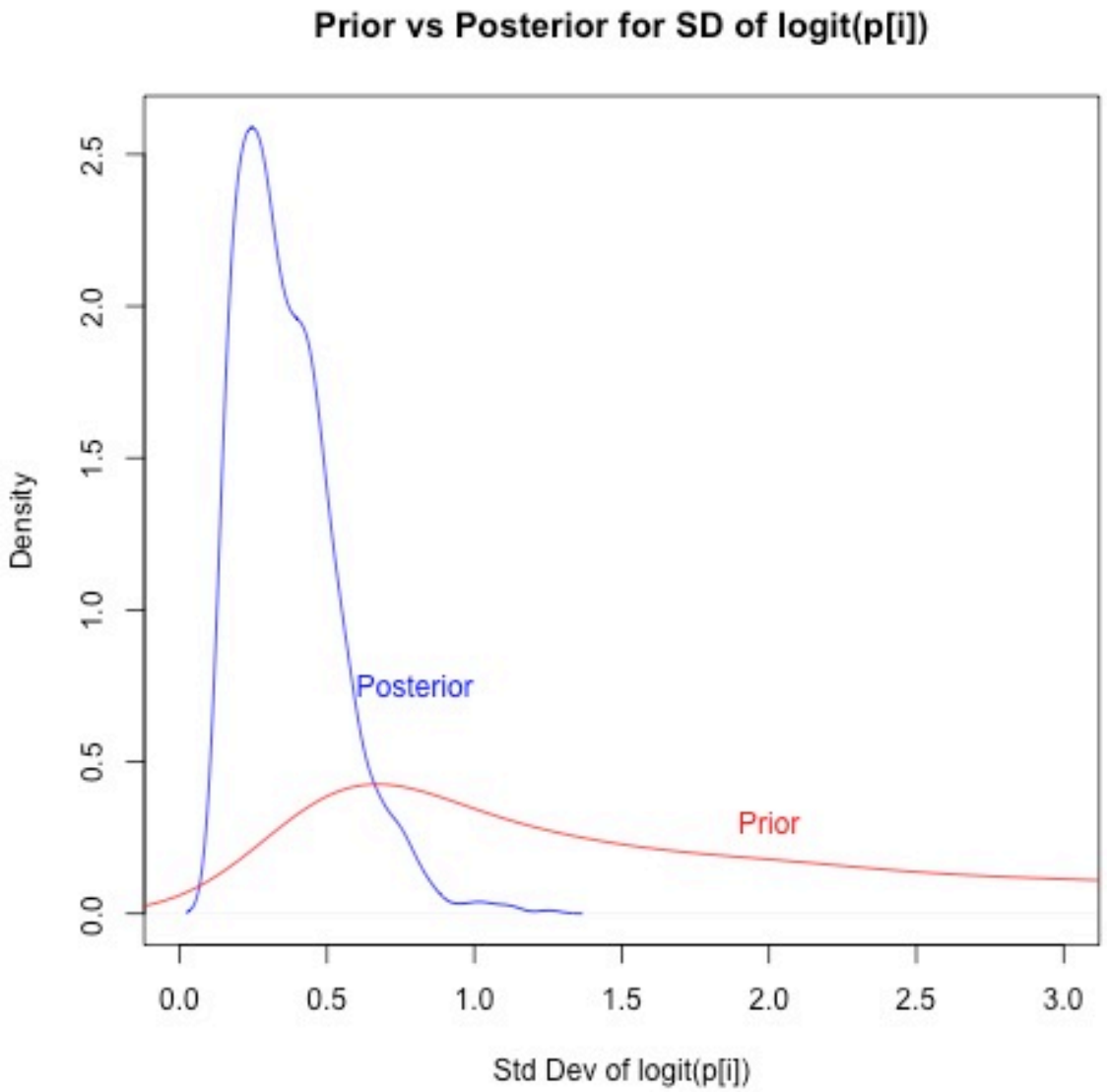
Total Var among $\hat{p}_i$

$$\hat{\alpha}_i = \frac{\hat{V}_P}{\hat{V}_P + se^2_{\hat{p}_i}} \quad \text{where} \quad \hat{V}_P = \frac{\sum_{i=1}^{m}(\hat{p}_i - \bar{\hat{p}})^2}{(m-1)} - \frac{\sum_{i=1}^{m} se^2_{\hat{p}_i}}{m}$$

$= 0$ if negative

	shots	goals	p	se	alpha	p.eb
Valdivia	1	0	0.00	0.499	0.045	0.51
Forlan	6	2	0.33	0.204	0.222	0.49
Hernandez	9	4	0.44	0.166	0.300	0.50
Cavani	17	11	0.65	0.121	0.447	0.58
Lukaku	5	3	0.60	0.223	0.192	0.54
Samara	3	0	0.00	0.288	0.125	0.46
Benzema	3	2	0.67	0.288	0.125	0.55
Ozil	6	1	0.17	0.204	0.222	0.45
Giroud	10	5	0.50	0.158	0.323	0.52
Dempsey	7	4	0.57	0.189	0.250	0.54
Odemwinge	4	2	0.50	0.250	0.160	0.53
Johannsson	8	2	0.25	0.176	0.276	0.45
Martinez	9	6	0.67	0.166	0.300	0.57
Hulk	23	12	0.52	0.104	0.523	0.53
Robben	10	7	0.70	0.158	0.323	0.59
Ruaz	5	3	0.60	0.223	0.192	0.54
van Persie	17	5	0.29	0.121	0.447	0.42
Messi	41	22	0.54	0.078	0.661	0.53
Rodriguez	7	4	0.57	0.189	0.250	0.54
Muller	14	8	0.57	0.133	0.400	0.55
Aguero	10	5	0.50	0.158	0.323	0.52
Hazard	19	13	0.68	0.114	0.475	0.60
Vidal	21	11	0.52	0.109	0.500	0.53
Neymar	6	5	0.83	0.204	0.222	0.60
Guardado	6	3	0.50	0.204	0.222	0.52
Borges	1	1	1.00	0.499	0.045	0.55
Salpingidis	1	0	0.00	0.499	0.045	0.51
Moses	3	3	1.00	0.288	0.125	0.59
Drmie	2	2	1.00	0.353	0.087	0.57
Shaquiri	1	0	0.00	0.499	0.045	0.51


Empirical Bayes Estimation
Estimated Goal Scoring Rate
0.0
0.2
0.4
0.6
0.8
1.0
ML
EB


Prior vs Posterior for SD of logit(p[i])
Density
0.0
0.5
1.0
1.5
2.0
2.5
Posterior
Prior
0.0
0.5
1.0
1.5
2.0
2.5
3.0
Std Dev of logit(p[i])

Multi-level Model – hint of Bayes

- Level I. $[y_i \mid t_i, n_i]$: likelihood of the data given unknowns
  - Given true rate t_i for player $i = 1,\ldots,m,$ the observed number of scored penalty shots, y_i out of n_i attempts is *binomial*(t_i, n_i)
- Level II. $[t_i \mid n_i, m, v]$: prior distribution of unknowns
  - The true rates t_i are independent and identically distributed random variables with distribution *Beta*(m, v)
- Multi-level model calculates $[t_i, m, v \mid n_i, y_i]$: posterior of the unknowns given the data – using Bayes theorem