NON-COOPERATIVE GAMES

A DISSERTATION

Presented to the Faculty of Princeton University in Candidacy for the Degree of Doctor of Philosophy

Recommended for Acceptance by the Department of Mathematics

May, 1950

Abstract

This paper introduces the concept of a non-cooperative game and develops methods for the mathematical analysis of such games. The games considered are n-person games represented by means of pure strategies and pay-off functions defined for the combinations of pure strategies.

The distinction between cooperative and non-cooperative games is unrelated to the mathematical description by means of pure strategies and pay-off functions of a game. Rather, it depends on the possibility or impossibility of coalitions, communication, and side-payments.

The concepts of an equilibrium point, a solution, a strong solution, a sub-solution, and values are introduced by mathematical definitions. And in later sections the interpretation of these concepts in non-cooperative games is discussed.

The main mathematical result is the proof of the existence in any game of at least one equilibrium point. Other results concern the geometrical structure of the set of equilibrium points of a game with a solution, the geometry of sub-solutions, and the existence of a symmetrical equilibrium point in a symmetrical game.

As an illustration of the possibilities for application a treatment of a simple three-man poker model is included.

5-26-50 - Doctoral dissertation

Table of Contents

Introduction

Von Neumann and Morgenstern have developed a very fruitful theory of two-person zero-sum games in their book *Theory of Games and Economic Behavior*. This book also contains a theory of n-person games of a type which we would call cooperative. This theory is based on an analysis of the interrelationships of the various coalitions which can be formed by the players of the game.

Our theory, in contradistinction, is based on the *absence* of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others.

The notion of an *equilibrium* point is the basic ingredient in our theory. This notion yields a generalization of the concept of the solution of a two-person zero-sum game. It turns out that the set of equilibrium points of a two-person zero-sum game is simply the set of all pairs of opposing "good strategies."

In the immediately following sections we shall define equilibrium points and prove that a finite non-cooperative game always has at least one equilibrium point. We shall also introduce the notions of solvability and strong solvability of a non-cooperative game and prove a theorem on the geometrical structure of the set of equilibrium points of a solvable game.

As an example of the application of our theory we include a solution of a simplified three person poker game.

The motivation and interpretation of the mathematical concepts employed in the theory are reserved for discussion in a special section of this paper.

Formal Definitions and Terminology

In this section we define the basic concepts of this paper and set up standard terminology and notation. Important definitions will be preceeded by a sub-title indicating the concept defined. The non-cooperative idea will be implicit, rather than explicit, below.

Finite Game:

For us an n-person game will be a set of n players, or positions, each with an associated finite set of pure strategies; and corresponding to each player, i, a pay-off function, p_i, which maps the set of all n-tuples of pure strategies into the real numbers. When we use the term n-tuple we shall always mean a set of n items, with each item associated with a different player.

Mixed Strategy, s_i :

A mixed strategy of player i will be a collection of non-negative numbers which have unit sum and are in one to one correspondence with his pure strategies.

We write $s_i = \sum_\alpha c_{i\alpha} \pi_{i\alpha}$ with $\sum_\alpha c_{i\alpha} = 1$ and $c_{i\alpha} \geq 0$ to represent such a mixed strategy, where the $\pi_{i\alpha}$'s are the pure strategies of player i. We regard the s_i's as points in a simplex whose vertices are the $\pi_{i\alpha}$'s. This simplex may be regarded as a convex subset of a real vector space, giving us a natural process of linear combination for the mixed strategies.

We shall use the suffixes i, j, k for players and α, β, γ to indicate various pure strategies of a player. The symbols s_i, t_i and r_i, etc. will indicate mixed strategies; $\pi_{i\alpha}$ will indicate the i th player's αth pure strategy, etc.

Pay-off function, p_i :

The pay-off function, p_i, used in the definition of a finite game above, has a unique extension to the n-tuples of mixed strategies which is linear in the mixed strategy of each player [n-linear]. This extension we shall also denote by p_i, writing $p_i(s_1, s_2, \cdots s_n)$.

We shall write $\mathcal{s}$ or $\mathcal{t}$ to denote an n-tuple of mixed strategies and if $\mathcal{s} = (s_1, \cdots s_n)$ then $p_i(\mathcal{s})$ shall mean $p_i(s_1, s_2, \cdots s_n)$ Such an n-tuple, $\mathcal{s}$, will also be regarded as a point in a vector space, which space could be obtained by multiplying together the vector spaces containing the mixed strategies. And the set of all such n-tuples forms, of course, a convex polytope, the product of the simplices representing the mixed strategies.

For convenience we introduce the substitution notation

$(\mathcal{s}; t_i)$ to stand for $(s_1, s_2, \cdots s_{i-1}, t_i, s_{i+1}, \cdots s_n)$

where $\mathcal{s} = (s_1, s_2, \cdots s_n)$. The effect of successive substitutions $((\mathcal{s}; t_i); t_j)$ we indicate by $(\mathcal{s}; t_i; t_j)$, etc.

Equilibrium Point:

An n-tuple $\mathcal{s}$ is an <u>equilibrium point</u> if and only if for every i

$$p_i(\mathcal{s}) = \max_{\text{all } t_i\text{'s}} \left[p_i(\mathcal{s}; t_i) \right] \tag{1}$$

Thus an equilibrium point is an n-tuple $\mathcal{s}$ such that each player's mixed strategy maximizes his pay-off if the strategies of the others are held fixed. Thus each player's strategy is optimal against those of the others. We shall occasionally abbreviate equilibrium point by eq. pt.

We say that a mixed strategy s_i <u>uses</u> a pure strategy $\pi_{i\beta}$ if $s_i = \sum_\alpha c_{i\alpha} \pi_{i\alpha}$ and $c_{i\beta} > 0$. If $\mathscr{s} = (s_1, s_2, \cdots s_n)$ and s_i uses $\pi_{i\alpha}$ we also say that $\mathscr{s}$ uses $\pi_{i\alpha}$.

From the linearity of $p_i(s_1, \cdots s_n)$ in s_i,

$$(2) \qquad \max_{\text{all } r_i\text{'s}} \left[p_i(\mathscr{s}; r_i) \right] = \max_\alpha \left[p_i(\mathscr{s}; \pi_{i\alpha}) \right] .$$

We define $p_{i\alpha}(\mathscr{s}) = p_i(\mathscr{s}; \pi_{i\alpha})$. Then we obtain the following trivial necessary and sufficient condition for $\mathscr{s}$ to be an equilibrium point:

$$(3) \qquad p_i(\mathscr{s}) = \max_\alpha p_{i\alpha}(\mathscr{s}) .$$

If $\mathscr{s} = (s_1, s_2, \cdots s_n)$ and $s_i = \sum_\alpha c_{i\alpha} \pi_{i\alpha}$ then $p_i(\mathscr{s}) = \sum_\alpha c_{i\alpha} p_{i\alpha}(\mathscr{s})$, consequently for (3) to hold we must have $c_{i\alpha} = 0$ whenever $p_{i\alpha}(\mathscr{s}) < \max_\beta p_{i\beta}(\mathscr{s})$, which is to say that $\mathscr{s}$ does not use $\pi_{i\alpha}$ unless it is an optimal pure strategy for player i. So we write

$$(4) \qquad \text{if } \pi_{i\alpha} \text{ is used in } \mathscr{s} \text{ then } p_{i\alpha}(\mathscr{s}) = \max_\beta p_{i\beta}(\mathscr{s})$$

as another necessary and sufficient condition for an equilibrium point.

Since a criterion (3) for an eq. pt. can be expressed as the equating of two continuous functions on the space of n-tuples $\mathscr{s}$ the eq. pts. obviously form a closed subset of this space. Actually, this subset is formed from a ~~lot~~ number of pieces of albegraic varieties, cut out by other algebraic varieties.

Existence of Equilibrium Points

I have previously published [Proc. N. A. S. 36 (1950) 48–49] a proof of the result below based on Kakutani's generalized fixed point theorem. The proof given here uses the Brouwer theorem.

The method is to set up a sequence of continuous mappings:

$\mathscr{s} \to \mathscr{s}'(\mathscr{s},1)\,;\ \mathscr{s} \to \mathscr{s}'(\mathscr{s},2)\,;\ \cdots$ whose fixed points have an equilibrium point as limit point. A limit mapping exists, but is discontinuous, and need not have any fixed points.

THEO. 1: Every finite game has an equilibrium point.

Proof: Using our standard notation, let $\mathscr{s}$ be an n-tuple of mixed strategies, and $p_{i\alpha}(\mathscr{s})$ the pay-off to player i if he uses his pure strategy $\pi_{i\alpha}$ and the others use their respective mixed strategies in $\mathscr{s}$. For each integer λ we define the following continuous functions of $\mathscr{s}$:

$$q_i(\mathscr{s}) = \max_{\alpha} p_{i\alpha}(\mathscr{s}),$$

$$\phi_{i\alpha}(\mathscr{s},\lambda) = p_{i\alpha}(\mathscr{s}) - q_i(\mathscr{s}) + 1/\lambda, \text{ and}$$

$$\phi^{+}_{i\alpha}(\mathscr{s},\lambda) = \max\left[0, \phi_{i\alpha}(\mathscr{s},\lambda)\right].$$

Now $\sum_{\alpha} \phi^{+}_{i\alpha}(\mathscr{s},\lambda) \geq \max_{\alpha} \phi^{+}_{i\alpha}(\mathscr{s},\lambda) = 1/\lambda > 0$ so that

$$C'_{i\alpha}(\mathscr{s},\lambda) \doteq \frac{\phi^{+}_{i\alpha}(\mathscr{s},\lambda)}{\sum_{\beta} \phi^{+}_{i\beta}(\mathscr{s},\lambda)}$$ is continuous.

Define $s'_i(\mathscr{s},\lambda) = \sum_{\alpha} \pi_{i\alpha} C'_{i\alpha}(\mathscr{s},\lambda)$ and $\mathscr{s}'(\mathscr{s},\lambda) = (s'_1, s'_2, \cdots s'_n)$. Since all the operations have preserved continuity, the mapping $\mathscr{s} \to \mathscr{s}'(\mathscr{s},\lambda)$ is continuous; and since the space of n-tuples, s , is a cell, there must be a fixed point for each λ . Hence there will be a subsequence s_μ , converging to s^* , where s_μ is fixed under the mapping $s \to s'(s, \lambda(\mu))$.

Now suppose s^* were not an equilibrium point. Then if $s^* = (s_1^*, \ldots s_n^*)$ some component s_i^* must be non-optimal against the others, which means s_i^* uses some pure strategy $\pi_{i\alpha}$ which is non-optimal. [see (4), pg. 4] This means that

$$p_{i\alpha}(s^*) < q_i(s^*) \qquad \text{which justifies}$$

writing
$$p_{i\alpha}(s^*) - q_i(s^*) < -\epsilon \quad .$$

From continuity, if μ is large enough,

$$\Big| [p_{i\alpha}(s_\mu) - q_i(s_\mu)] - [p_{i\alpha}(s^*) - q_i(s^*)] \Big| < \epsilon/2 \text{ and } 1/\lambda(\mu) < \epsilon/2 .$$

Adding,
$$p_{i\alpha}(s_\mu) - q_i(s_\mu) + 1/\lambda(\mu) < 0 \qquad \text{which is}$$

simply
$$\phi_{i\alpha}(s_\mu, \lambda(\mu)) < 0 \text{, whence } \phi^+_{i\alpha}(s_\mu, \lambda(\mu)) = 0 \text{, whence}$$

$C'_{i\alpha}(s_\mu, \lambda(\mu)) = 0$. From this last equation we know that $\pi_{i\alpha}$ is not used in s_μ since

$$s_\mu = \sum_\alpha \pi_{i\alpha} \, C'_{i\alpha}(s_\mu, \lambda(\mu)) \qquad \text{, because } s_\mu \text{ is a}$$

fixed point.

And since $s_\mu \to s^*$, $\pi_{i\alpha}$ is not used in s^* , which contradicts our assumption.

Hence s^* is indeed an equilibrium point.

<u>Symmetries of Games</u>

An <u>automorphism</u>, or <u>symmetry</u>, of a game will be a permutation of its pure strategies which satisfies certain conditions, given below.

If two strategies belong to a single player they must go into two strategies belonging to a single player. Thus if ϕ is the permutation of the pure strategies it induces a permutation ψ of the players.

Each n-tuple of pure strategies is therefore permuted into another n-tuple of pure strategies. We may call χ the induced permutation of these n-tuples. Let ξ denote an n-tuple of pure strategies and $p_i(\xi)$ the pay-off to player i when the n-tuple ξ is employed. We require that if

$$j = i^{\psi} \quad \text{then} \quad p_j(\xi^{\chi}) = p_i(\xi)$$

which completes the definition of a symmetry.

The permutation ϕ has a unique linear extension to the mixed strategies. If

$$s_i = \sum_{\alpha} c_{i\alpha} \pi_{i\alpha} \quad \text{we define} \quad (s_i)^{\phi} = \sum_{\alpha} c_{i\alpha} (\pi_{i\alpha})^{\phi}.$$

The extension of ϕ to the mixed strategies clearly generates an extension of χ to the n-tuples of mixed strategies. We shall also denote this by χ.

We define a <u>symmetric n-tuple</u> $\mathcal{L}$ of a game by

$$\mathcal{L}^{\chi} = \mathcal{L} \quad \text{for all } \chi\text{'s}$$

it being understood that χ means a permutation derived from a symmetry ϕ.

THEO. 4: Any finite game has a symmetric equilibrium point.

Proof: First we note that

$$s_{i0} = \frac{\sum_\alpha \pi_{i\alpha}}{\sum_\alpha 1}$$ has the property $(s_{i0})^\phi = s_{j0}$ where $j = i^\psi$, so that the n-tuple $\mathscr{s}_0 = (s_{10}, s_{20}, \cdots s_{n0})$ is fixed under any χ; hence any game has at least one symmetric n-tuple.

If $\mathscr{s} = (s_1, \cdots s_n)$ and $\mathscr{t} = (t_1, \cdots t_n)$ are symmetric then $\frac{\mathscr{s}+\mathscr{t}}{2} = \left(\frac{s_1+t_1}{2}, \cdots \frac{s_n+t_n}{2}\right)$ is so too because $\mathscr{s}^\chi = \mathscr{s} \Leftrightarrow s_j = (s_i)^\phi$ where $j = i^\psi$, hence

$$\frac{s_j+t_j}{2} = \frac{(s_i)^\phi + (t_i)^\phi}{2} = \left(\frac{s_i+t_i}{2}\right)^\phi, \text{ hence } \left(\frac{\mathscr{s}+\mathscr{t}}{2}\right)^\chi = \frac{\mathscr{s}+\mathscr{t}}{2}.$$

This shows that the set of symmetric n-tuples is a convex subset of the space of n-tuples since it is obviously closed.

Now observe that for each λ the mapping $\mathscr{s} \to \mathscr{s}'(\mathscr{s}, \lambda)$ used in the proof of existence theorem was intrinsically defined. Therefore, if $\mathscr{s}_2 = \mathscr{s}'(\mathscr{s}_1, \lambda)$ and χ a permutation [illegible] is an auto morphism of the game we will have $\mathscr{s}_2^\chi = \mathscr{s}'(\mathscr{s}_1^\chi, \lambda)$. If $\mathscr{s}_1$ is symmetric $\mathscr{s}_1^\chi = \mathscr{s}_1$ and therefore $\mathscr{s}_2^\chi = \mathscr{s}'(\mathscr{s}_1, \lambda) = \mathscr{s}_2$. Consequently this mapping maps the set of symmetric n-tuples into itself.

Since this set is a cell there must be a symmetric fixed point $\mathscr{s}_\lambda$. And, as in the proof of the existence theorem we could obtain a limit point $\mathscr{s}^*$ which would have to be symmetric.

Solutions

We define here solutions, strong solutions, and sub-solutions. A non-cooperative game does not always have a solution, but when it does the solution is unique. Strong solutions are solutions with special properties. Sub-solutions always exist and have many of the properties of solutions, but lack uniqueness.

S_i will denote a set of mixed strategies of player i and $\mathcal{S}$ a set of n-tuples of mixed strategies.

Solvability:

A game is solvable if its set, $\mathcal{S}$, of equilibrium points satisfies the condition

(1) $\quad (t;r_i)\in\mathcal{S}$ and $s\in\mathcal{S} \Rightarrow (s;r_i)\in\mathcal{S}$ for all i's.

This is called the <u>interchangeability</u> condition. The <u>solution</u> of a solvable game is its set, $\mathcal{S}$, of equilibrium points.

Strong Solvability:

A game is <u>strongly solvable</u> if it has a solution, $\mathcal{S}$, such that for all i's

$$s\in\mathcal{S} \quad \text{and} \quad p_i(s;r_i)=p_i(s) \Rightarrow (s;r_i)\in\mathcal{S}$$

and then $\mathcal{S}$ is called a <u>strong solution</u>.

Equilibrium Strategies:

In a solvable game let S_i be the set of all mixed strategies s_i

such that for some t the n-tuple $(t; s_i)$ is an equilibrium point. [s_i is the i th component of some equilibrium point.] We call S_i the set of equilibrium strategies of player i.

Sub-solutions:

If $\mathcal{L}$ is a subset of the set of equilibrium points of a game and satisfies condition (1); and if $\mathcal{L}$ is maximal relative to this property then we call $\mathcal{L}$ a sub-solution.

For any sub-solution $\mathcal{L}$ we define the i th factor set, S_i, as the set of all s_i's such that $\mathcal{L}$ contains $(t; s_i)$ for some t.

Note that a sub-solution, when unique, is a solution; and its factor sets are the sets of equilibrium strategies.

THEO. 2: A sub-solution, $\mathcal{L}$, is the set of all n-tuples $(s_1, s_2, \cdots s_n)$ such that each $s_i \in S_i$ where S_i is the i th factor set of $\mathcal{L}$. Geometrically, $\mathcal{L}$ is the product of its factor sets.

Proof: Consider such an n-tuple $(s_1, \cdots s_n)$. By definition $\exists\ t_1, t_2, \cdots, t_n$ such that for each i $(t_i; s_i) \in \mathcal{L}$. Using the condition (1) n-1 times we obtain successively $(t_1; s_1; s_2) \in \mathcal{L}$, $\cdots$, $(t_1; s_1; s_2; s_3; \cdots; s_n) \in \mathcal{L}$ and the last is simply $(s_1, s_2, \cdots s_n) \in \mathcal{L}$, which we needed to show.

THEO. 3: The factor sets $S_1, S_2, \cdots S_n$ of a sub-solution are closed and convex as subsets of the mixed strategy spaces.

Proof: It suffices to show two things: (a) if s_i and $s_i' \in S_i$

then $s_i^* = (s_i + s_i')/2 \in S_i$; (b) if $s_i^{\#}$ is a limit point of S_i then $s_i^{\#} \in S_i$.

Let $t \in \mathcal{L}$. Then we have

$$p_j(t; s_i) \geq p_j(t; s_i; r_j) \quad \text{and} \quad p_j(t; s_i) \geq p_j(t; s_i'; r_j)$$

for any r_j , by using the criterion of (1), pg. 3 for an eq. pt. Adding these inequalities, using the linearity of $p_j(s_1, \cdots s_n)$ in s_i, and dividing by 2, we get $p_j(t; s_i^*) \geq p_j(t; s_i^*; r_j)$ since $s_i^* = (s_i + s_i')/2$. From this we know that $(t; s_i^*)$ is an eq. pt. for any $t \in \mathcal{L}$. If the set of all such eq. pts. $(t; s_i^*)$ is added to $\mathcal{L}$ the augmented set clearly satisfies condition (1), and since $\mathcal{L}$ was to be maximal it follows that $s_i^* \in S_i$.

To attack (b) note that the n-tuple $(t; s_i^{\#})$, where $t \in \mathcal{L}$ will be a limit point of the set of n-tuples of the form $(t; s_i)$ where $s_i \in S_i$, since $s_i^{\#}$ is a limit point of S_i . But this set is a set of eq. pts. and hence any point in its closure is an eq. pt., since the set of all eq. pts. is closed [see pg. 4] . Therefore $(t; s_i^{\#})$ is an eq. pt. and hence $s_i^{\#} \in S_i$ from the same argument as for s_i^* .

Values:

Let $\mathcal{L}$ be the set of equilibrium points of a game. We define

$$v_i^+ = \max_{s \in \mathcal{L}} [p_i(s)] \quad , \quad v_i^- = \min_{s \in \mathcal{L}} [p_i(s)] \quad .$$

If $v_i^+ = v_i^-$ we write $v_i = v_i^+ = v_i^-$. v_i^+ is the upper value to player i of the game; v_i^- the lower value; and

V_i the <u>value</u>, if it exists.

Values will obviously have to exist if there is but one equilibrium point.

One can define <u>associated values</u> for a sub-solution by restricting $\mathcal{L}$ to the eq. pts. in the sub-solution and then using the same defining equations as above.

A two-person zero-sum game is always solvable in the sense defined above. The sets of equilibrium strategies S_1 and S_2 are simply the sets of "good" strategies. Such a game is not generally strongly solvable; strong solutions exist only when there is a "saddle point" in <u>pure</u> strategies.

Geometrical Form of Solutions

In the two-person zero-sum case it has been shown that the set of "good" strategies of a player is a convex polyhedral subset of his strategy space. We shall obtain the same result for a player's set of equilibrium strategies in any solvable game.

THEO. 5: The sets $S_1, S_2, \dots S_n$ of equilibrium strategies in a solvable game are polyhedral convex subsets of the respective mixed strategy spaces.

Proof: An n-tuple $\mathscr{s}$ will be an equilibrium point if and only if for every i

$$(1) \qquad p_i(\mathscr{s}) = \max_{\alpha} p_{i\alpha}(\mathscr{s})$$

which is condition (3) on page 4. An equivalent condition is for every i and α

$$(2) \qquad p_i(\mathscr{s}) - p_{i\alpha}(\mathscr{s}) \geq 0 \ .$$

Let us now consider the form of the set S_j of equilibrium strategies, s_j, of player j. Let $\mathscr{t}$ be any equilibrium point, then $(\mathscr{t}; s_j)$ will be an equilibrium point if and only if $s_j \in S_j$, from Theo. 2. We now apply conditions (2) to $(\mathscr{t}; s_j)$, obtaining

$$(3) \quad s_j \in S_j \iff \text{for all } i, \alpha \quad p_i(\mathscr{t}; s_j) - p_{i\alpha}(\mathscr{t}; s_j) \geq 0 \ .$$

Since p_i is n-linear and $\mathscr{t}$ is constant these are a set of linear inequalities of the form $F_{i\alpha}(s_j) \geq 0$. Each such inequality is either satisfied for all s_j or for those lying on and to one side of some hyperplane passing through the strategy simplex. Therefore, the

Simple Examples

These are intended to illustrate the concepts defined in the paper and display special phenomena which occur in these games.

The first player has the roman letter strategies and the pay-off to the left, etc.

Ex. 1

$$\begin{array}{rcr} 5 & a\alpha & -3 \\ -4 & a\beta & 4 \\ -5 & b\alpha & 5 \\ 3 & b\beta & -4 \end{array}$$

Weak Solution: $\left(\frac{9}{16}a+\frac{7}{16}b\,,\ \frac{7}{17}\alpha+\frac{10}{17}\beta\right)$

$v_1=\frac{-5}{17}\ ,\ v_2=+1/2$

Ex. 2

$$\begin{array}{rcr} 1 & a\alpha & 1 \\ -10 & a\beta & 10 \\ 10 & b\alpha & -10 \\ -1 & b\beta & -1 \end{array}$$

Strong Solution: (b,β)

$v_1=v_2=-1$

Ex. 3

$$\begin{array}{rcr} 1 & a\alpha & 1 \\ -10 & a\beta & -10 \\ -10 & b\alpha & -10 \\ 1 & b\beta & 1 \end{array}$$

Unsolvable; equilibrium points (a,α), (b,β), **and** $(a/2+b/2\,,\ \alpha/2+\beta/2)$ **. The strategies in the last case have maxi-min and mini-max properties.**

Ex. 4

$$\begin{array}{rcr} 1 & a\alpha & 1 \\ 0 & a\beta & 1 \\ 1 & b\alpha & 0 \\ 0 & b\beta & 0 \end{array}$$

Strong Solution: all pairs of mixed strategies.

$v_1^+=v_2^+=1\ ,\ v_1^-=v_2^-=0$

Ex. 5

$$\begin{array}{rcr} 1 & a\alpha & 2 \\ -1 & a\beta & -4 \\ -4 & b\alpha & -1 \\ 2 & b\beta & 1 \end{array}$$

Unsolvable; eq. pts. (a,α), (b,β) **and** $(1/4\,a+3/4\,b\,,\ 3/8\,\alpha+5/8\,\beta)$ **. However, empirical tests show a tendency toward** (a,α).

Ex. 6

$$\begin{array}{rcr} 1 & a\alpha & 1 \\ 0 & a\beta & 0 \\ 0 & b\alpha & 0 \\ 0 & b\beta & 0 \end{array}$$

Eq. pts.: (a,α) **and** (b,β) **, with** (b,β) **an example of instability.**

complete set [which is finite] of conditions will all be satisfied simultaneously on some convex polyhedral subset of player j's strategy simplex. [Intersection of half-spaces.]

As a corollary we may conclude that S_k is the convex closure of a finite set of mixed strategies [vertices].

Dominance and Contradiction Methods

We say that s_i' dominates s_i if $p_i(\mathcal{t}; s_i') > p_i(\mathcal{t}; s_i)$ for every $\mathcal{t}$.

This amounts to saying that s_i' gives player i a higher pay-off than s_i no matter what the strategies of the other players are. To see whether a strategy s_i' dominates s_i it suffices to consider only pure strategies for the other players because of the n-linearity of p_i.

It is obvious from the definitions that no equilibrium point can involve a dominated strategy s_i.

The domination of one mixed strategy by another will always entail other dominations. For suppose s_i' dominates s_i and t_i uses all of the pure strategies which have a higher coefficient in s_i than in s_i'. Then for a small enough $\rho > 0$

$$t_i' = t_i + \rho(s_i' - s_i)$$

is a mixed strategy; and t_i' dominates t_i by linearity.

One can prove a few properties of the set of undominated strategies. It is simply connected and is formed by the union of some collection of faces of the strategy simplex.

The information obtained by discovering dominances for one player may be of relevance to the others, insofar as the elimination of classes of mixed strategies as possible components of an equilibrium point is concerned. For the $\mathcal{t}$'s whose components are all undominated are all that need be considered and this eliminating some of the strategies of one player may make possible the elimination of a new class of strategies for another player.

Another procedure which may be used in locating equilibrium points is the contradiction-type analysis. Here one assumes that an equilibrium point exists having component strategies lying within certain regions of the strategy spaces and proceeds to deduce further conditions which must be satisfied if the hypothesis is true. This sort of reasoning may be carried through several stages to eventually obtain a contradiction indicating that there is no equilibrium point satisfying the initial hypothesis.

A Three-Man Poker Game

As an example of the application of our theory to a more or less realistic case we include the simplified poker game given below. The rules are as follows:

(1) The deck is large, with equally many high and low cards, and a hand consists of one card.

(2) Two chips are used to ante, open, or call.

(3) The players play in rotation and the game ends after all have passed or after one player has opened and the others have had a chance to call.

(4) If no one bets the antes are retrieved.

(5) Otherwise the pot is divided equally among the highest hands which have bet.

We find it more satisfactory to treat the game in terms of quantities we call "behavior parameters" than in the normal form of "Theory of Games and Economic Behavior." In the normal form representation two mixed strategies of a player may be equivalent in the sense that each makes the individual choose each available course of action in each particular situation requiring action on his part with the same frequency. That is, they represent the same behavior pattern on the part of the individual.

Behavior parameters give the probabilities of taking each of the various possible actions in each of the various possible situations which may arise. Thus they describe behavior patterns.

In terms of behavior parameters the strategies of the players may be represented as follows, assuming that since there is no point in passing with a high card at one's last opportunity to bet that this will not be done. The greek letters are the probabilities of the various acts.

	First Moves	Second Moves
I	α Open on <u>high</u> β Open on <u>low</u>	κ Call III on <u>low</u> λ Call II on <u>low</u> μ Call II and III on <u>low</u>
II	γ Call I on <u>low</u> δ Open on <u>high</u> ϵ Open on <u>low</u>	ν Call III on <u>low</u> ξ Call III and I on <u>low</u>
III	ζ Call I and II on <u>low</u> η Open on <u>low</u> θ Call I on <u>low</u> ι Call II on <u>low</u>	Player III never gets a second move.

We locate all possible equilibrium points by first showing that most of the greek paramenters must vanish. By dominance mainly with a little contradiction-type analysis β is eliminated and with it go γ, ζ, and θ by dominance. Then contradictions eliminate $\mu, \xi, \iota, \lambda, \kappa$, and ν in that order. This leaves us with α, δ, ϵ and η. Contradiction analysis shows that none of these can be zero or one and thus we obtain a system of simultaneous algebraic equations. The equations happen to have but one solution with the variables in the range $(0,1)$. We get

$$\alpha = \frac{21-\sqrt{321}}{10}, \quad \eta = \frac{5\alpha+1}{4}, \quad \delta = \frac{5-2\alpha}{5+\alpha}, \text{ and}$$

$\epsilon = \frac{4\alpha-1}{\alpha+5}$. These yield $\alpha = .308$, $\eta = .635$, $\delta = .826$, and $\epsilon = .044$.

Since there is only one equilibrium point the game has values; these are

$$V_1 = -.147 = \frac{-(1+17\alpha)}{8(5+\alpha)}, \qquad V_2 = -.096 = -\frac{1-2\alpha}{4}, \text{ and}$$

$$V_3 = .243 = \frac{79}{40}\left(\frac{1-\alpha}{5+\alpha}\right).$$

Investigation of the coalition powers yields the following "good strategies" and values for the various coalitions. Paramenters not mentioned are zero.

I and II	versus III
$\alpha = 3/4$	$\iota = 1/4, \; 0 \le \eta \le 2/3$
$\delta = \epsilon = 1$	value to III: $.03125 = 1/32$

II and III	versus I
$\delta = 1, \epsilon = 0$	$\alpha = 2/3$
$\eta = 2/3$	value to I: $-.1667 = -1/6$

I and III versus II

high	low	III	
bet	pass	$\eta = 0$	3/11
pass	pass	$\eta = 13/16$	8/11

II: $\delta = 7/11, \; \epsilon = 3/11$

value to II: $-.1136 = -5/44$

The coalition members have the power to agree upon a pattern of play before the game is played. This advantage becomes significant only in the case of coalition I III where III may open after two passes when I had planned to pass on both <u>high</u> and <u>low</u> but will not open if

I had planned to bet if he got <u>high</u>. The values given are, of course, what the single player assures himself with his "safe" strategy.

A more detailed treatment of this game is being prepared for publication elsewhere. This will consider different relative sizes of ante and bet.

Motivation and Interpretation

In this section we shall try to explain the significance of the concepts introduced in this paper. That is, we shall try to show how equilibrium points and solutions can be connected with observable phenomena.

The basic requirements for a non-cooperative game is that there should be no pre-play communication among the players [unless it has no bearing on the game]. Thus, by implication, there are no coalitions and no side-payments. Because there is no extra-game utility [pay-off] transfer, the pay-offs of different players are effectively incomparable; if we transform the pay-off functions linearly: $p_i' = a_i p_i + b_i$, where $a_i > 0$ the game will be essentially the same. Note that equilibrium points are preserved under such transformations.

We shall now take up the "mass-action" interpretation of equilibrium points. In this interpretation solutions have no great significance. It is unnecessary to assume that the participants have full knowledge of the total structure of the game, or the ability and inclination to go through any complex reasoning processes. But the participants are supposed to accumulate empirical information on the relative advantages of the various pure strategies at their disposal.

To be more detailed, we assume that there is a population [in the sense of statistics] of participants for each position of the game. Let us also assume that the "average playing" of the game involves n participants selected at random from the n populations, and that there is a stable average frequency with which each pure strategy is employed by the "average member" of the appropriate population.

Since there is to be no collaboration between individuals playing in different positions of the game, the probability that a particular n-tuple of pure strategies will be employed in a playing of the game should be the product of the probabilities indicating the chance of each of the n pure strategies to be employed in a random playing.

Let the probability that $\pi_{i\alpha}$ will be employed in a random playing of the game be $c_{i\alpha}$, and let $s_i = \sum_{\alpha} c_{i\alpha}\pi_{i\alpha}$, $\mathcal{S}=(s_1, s_2, \ldots s_n)$. Then the expected pay-off to an individual playing in the i th position of the game and employing the pure strategy $\pi_{i\alpha}$ is $p_i(\mathcal{S};\pi_{i\alpha}) = p_{i\alpha}(\mathcal{S})$.

Now let us consider what effects the experience of the participants will produce. To assume, as we did, that they accumulated empirical evidence on the pure strategies at their disposal is to assume that those playing in position i learn the numbers $p_{i\alpha}(\mathcal{S})$. But if they know these they will employ only optimal pure strategies, i.e., those pure strategies $\pi_{i\alpha}$ such that

$$p_{i\alpha}(\mathcal{S}) = \max_{\beta} p_{i\beta}(\mathcal{S}) .$$

Consequently since s_i expresses their behavior s_i attaches positive coefficients only to optimal pure strategies, so that

$$\pi_{i\alpha} \text{ is used in } s_i \Rightarrow p_{i\alpha}(\mathcal{S}) = \max_{\beta} p_{i\beta}(\mathcal{S}) .$$

But this is simply a condition for $\mathcal{S}$ to be an equilibrium point. [see (4), pg. 7]

Thus the assumptions we made in this "mass-action" interpretation lead to the conclusion that the mixed strategies representing the average behavior in each of the populations form an equilibrium point.

The populations need not be large if the assumptions still make it:

. . . . ill hold. There are situations in economics or international politics in which, effectively, a group of interests are involved in a non-cooperative game without being aware of it; the non-awareness helping to make the situation truly non-cooperative.

Actually, of course, we can only expect some sort of approximate euqilibrium, since the information, its utilization, and the stability of the average frequencies will be imperfect.

We now sketch another interpretation, one in which solutions play a major role, and which is applicable to a game played but once.

We proceed by investigating the question: what would be a "rational" prediction of the behavior to be expected of rational playing the game in question? By using the principles that a rational prediction should be unique, that the players should be able to deduce and make use of it, and that such knowledge on the part of each player of what to expect the others to do should not lead him to act out of conformity with the prediction, one is led to the concept of a solution defined before.

If $S_1, S_2, \dots S_n$ were the sets of equilibrium strategies of a solvable game, the "rational" prediction should be: "The average behavior of rational men playing in position i would define a mixed strategy s_i in S_i if an experiment were carried out."

In this interpretation we need to assume the players know the full structure of the game in order to be able to deduce the prediction for themselves. It is quite strongly a rationalistic and idealising interpretation.

In an unsolvable game it sometimes happens that good heuristic reasons can be found for narrowing down the set of equilibrium points to those in a single sub-solution, which then plays the role of a solution.

In general a sub-solution may be looked at as a set of mutually compatible equilibrium points, forming a coherent whole. The sub-solutions appear to give a natural subdivision of the set of equilibrium points of a game.

Applications

The study of n-person games for which the accepted ethics of fair play imply non-cooperative playing is, of course, an obvious direction in which to apply this theory. And poker is the most obvious target. The analysis of a more realistic poker game than our very simple model should be quite an interesting affair.

The complexity of the mathematical work needed for a complete investigation increases rather rapidly, however, with increasing complexity of the game; so that it seems that analysis of a game much more complex than the example given here would only be feasible using approximate computational methods.

A less obvious type of application is to the study of cooperative games. By a cooperative game we mean a situation involving a set of players, pure strategies, and pay-offs as usual; but with the assumption that the players can and will collaborate as they do in the von Neumann and Morgenstern theory. This means the players may communicate and form coalitions which will be enforced by an umpire. It is unnecessarily restrictive, however, to assume any transferability, or even comparability of the pay-offs [which should be in utility units] to different players. Any desired transferability can be put into the game itself instead of assuming it possible in the extra-game collaboration.

The writer has developed a "dynamical" approach to the study of cooperative games based upon reduction to non-cooperative form. One proceeds by constructing a model of the pre-play negotiation so that the steps of negotiation become moves in a larger non-cooperative game [which will have an infinity of pure strategies] describing the total situation.

This larger game is then treated in terms of the theory of this paper [extended to infinite games] and if values are obtained they are taken as the values of the cooperative game. Thus the problem analyzing a cooperative game becomes the problem of obtaining a suitable, and convincing, non-cooperative model for the negotiation.

The writer has, by such a treatment, obtained values for all finite two person cooperative games, and some special *n*-person games.

Bibliography

(1) von Neumann, Morgenstern, "Theory of Games and Economic Behavior", Princeton University Press, 1944.

(2) J. F. Nash, Jr., "Equilibrium Points in N-Person Games", <u>Proc. N. A. S.</u> 36 (1950) 48-49.

Acknowledgements

Drs. Tucker, Gale, and Kuhn gave valuable criticism and suggestions for improving the exposition of the material in this paper. David Gale suggested the investigation of symmetric games. The solution of the Poker model was a joint project undertaken by Lloyd S. Shapley and the author. Finally, the author was sustained financially by the Atomic Energy Commission in the period 1949-50 during which this work was done.