Assignment - 1

Q1: RT analysis of insertion sort : n = arr.length

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for ( i = 1; i < arr.length; i++ ) {          — 1
    key = arr[i]                               — 1
    j = i - 1                                  — 2
    while ( j >= 0 and arr[j] > key ) {        — 2
        arr[j+1] = arr[j]                      — 2
        j = j - 1                              — 2
    }                                          i(2+2+2)
    arr[j+1] = key                             — 2
```

In each iteration of while loop j is decremented.
While loop exits when $j < 0$ (first condition of while)
In the worst-case while loop runs i times
(for $j = i-1, i-2, \ldots, 1, 0$). Total RT of while loop
is $i(2+2+2) = 6i$.
for loop runs for $i = 1, 2, \ldots, n-1$.

Total RT of for loop $= \sum_{i=1}^{n-1} (1+1+2+6i+2)$

$= \sum_{i=1}^{n-1} (6i+6)$ — sum of arithmetic sequence

$= O(\#\text{ of terms} * \text{max element in sequence})$

$= O((n-1) * (6(n-1)+6)) = 6n^2 - 6n$

$= O(n^2)$.

Q2 (cont) By induction, $T(n) \leq n\log n + n = O(n \log n)$

Assignment - 2

Q1: Correctness of power: power(x, n) returns x^n for $n \geq 0$

Proof: By induction on n.
Base: $n = 0$. Program returns $1 = x^0$.
Step: consider $n > 0$.
Program assigns $s = \text{power}(x * x, n/2)$.
For $n > 0$, $n/2 < n$. Therefore by the IH,
power(x * x, n/2) returns $(x^2)^{\lfloor n/2 \rfloor}$
[$\lfloor n/2 \rfloor = \text{floor}(n/2)$ - result of integer division]
Program now checks if n is even.
Case 1: n is even $\Rightarrow \lfloor n/2 \rfloor = n/2$.
Program returns $s = (x^2)^{n/2} = x^n$.
Case 2: n is odd $\Rightarrow \lfloor n/2 \rfloor = \frac{n}{2} - \frac{1}{2} = \frac{n-1}{2}$
Program returns $s * x = (x^2)^{\frac{n-1}{2}} \cdot x = x^{n-1} \cdot x = x^n$.
In all cases program returns x^n - by induction it is correct.

Q2. $T(n) \leq 2T(n/2) + n$ for $n > 1$, $T(1) = 1$
Theorem: $T(n) \leq an\log n + bn$ for $n \geq 1$.
Proof: By induction on n.
Base: $n = 1$ $T(1) = 1 \leq a \cdot 1 \cdot \log 1 + b \cdot 1 = b$
True if $\boxed{1 \leq b}$
Step: consider $n > 1$. By recurrence, $T(n) \leq 2T(n/2) + n$
For $n > 1$, $n/2 < n \Rightarrow$ by IH, $T(n/2) \leq a\left(\frac{n}{2}\right)\log\left(\frac{n}{2}\right) + b\left(\frac{n}{2}\right)$
Substituting in recurrence and simplifying, we get
$T(n) \leq 2\left[a\frac{n}{2}\log(n/2) + b\left(\frac{n}{2}\right)\right] + n = an\log n - an + bn + n$
For proof to work, we need $an\log n - an + bn + n \leq an\log n + bn$
$\equiv -an + n \leq 0 \equiv \boxed{1 \leq a}$ choose $\boxed{b = 1, a = 1}$