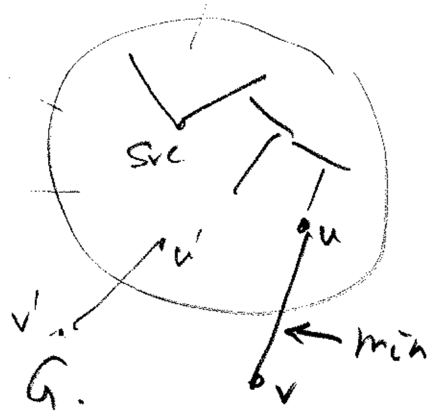
# Correctness of Prim's MST algorithm

<u>Greedy choice theorem:</u>

Let $T$ be a tree that contains the source node such that $T$ is a subset of some MST of $G$.

Let $e = (u,v)$ be a min-weight edge with

$u \in T$ and $v \notin T$.




Then there is an MST of $G$ that contains $T' = T \cup \{(u,v)\}$.

Proof: ~~Let~~ Consider an MST, $T_{opt}$, of $G$.

such that $T \subseteq T_{opt}$.

Case 1: $(u,v) \in T_{opt}$ — done $T_{opt}$ contains $T \cup \{(u,v)\}$

Case 2: $(u,v) \notin T_{opt}$.

Since $T_{opt}$ is a spanning tree, it contains at least one edge $e' = (u', v')$ such that $u' \in T$, $v' \notin T$.

Suppose we add $(u,v)$ to $T_{opt}$. This induces a cycle containing $(u,v)$. The cycle has at least one other edge that connects a node in $T$ to a node outside $T$, such as $e'$. $e = (u,v)$ was chosen to be smallest edge that connects a node in $T$ to a node outside $T$.

$\Rightarrow w(e) \leq w(e')$

Consider $T_{opt} - \{e'\} \cup \{e\} = T_{new}$

$$w(T_{new}) = w(T_{opt}) - w(e') + w(e)$$
$$\leq w(T_{opt}) \quad \text{because } w(e) \leq w(e').$$

$\Rightarrow T_{new}$ is also an MST of $G$.

$T_{new}$ contains $T \cup \{(u,v)\}$. ∎

Correctness of Prim's algorithm:

By induction on the number iterations of the while loop.

At the beginning, $T$ has no edges (just src vertex).

Therefore $T$ is a subset of all MST's of $G$.

Loop invariant: $T \subseteq$ some MST of $G$.

Algorithm grows $T$ by adding a min-weight edge $e = (u,v)$ with $u \in T$, $v \notin T$.

By greedy choice theorem, $T \cup \{e\}$ is a subset of some MST of $G \Leftarrow$ LI of next iteration.

At the end, $T$ is a spanning tree that is a subset of some MST of $G \Rightarrow T$ is an MST.

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RT of Kruskal's algorithm $= O$(Time to sort edge array + Time for remaining steps)

$$= O(|E| \log |E| + |E| \cdot \alpha(|V|))$$

$\alpha$ = inverse Ackermann's function. – $\alpha$ grows slower than almost anything.