Minimum distance to the origin from the intersection of a Sphere and a Plane

Consider the sphere $\mathtt{S} : (x-x_c)^2+(y-y_c)^2+(z-z_c)^2 = r_c^2$ and a plane $\mathtt{P} : ax+by+cz+d=0$. We are interested in finding a point in $\mathtt{S}\cap\mathtt{P}$ that is closest to the origin. To this end, consider the optimization problem

$$\begin{aligned} &\min_{(x,y,z)\in\mathbb{R}^3} x^2+y^2+z^2 \\ \text{subject to } &(x-x_c)^2+(y-y_c)^2+(z-z_c)^2 = r_c^2 \\ &ax+by+cz+d=0 \end{aligned}$$

Formulating the Lagrangian as $L := x^2+y^2+z^2+\lambda_1((x-x_c)^2+(y-y_c)^2+(z-z_c)^2-r_c^2)+\lambda_2(ax+by+cz+d)$, and writing the first order conditions we get

$$\begin{aligned} 2x^*+2\lambda_1(x^*-x_c)+a\lambda_2 = 0 &\Rightarrow x^* = \frac{-a\lambda_2+2\lambda_1 x_c}{2\lambda_1+2} \\ 2y^*+2\lambda_1(y^*-y_c)+b\lambda_2 = 0 &\Rightarrow y^* = \frac{-b\lambda_2+2\lambda_1 y_c}{2\lambda_1+2} \\ 2z^*+2\lambda_1(z^*-z_c)+c\lambda_2 = 0 &\Rightarrow z^* = \frac{-c\lambda_2+2\lambda_1 z_c}{2\lambda_1+2} \end{aligned}$$

Substituting these in $\mathtt{P}$ we get

$$\begin{aligned} &a\left(\frac{-a\lambda_2+2\lambda_1 x_c}{2\lambda_1+2}\right)+b\left(\frac{-b\lambda_2+2\lambda_1 y_c}{2\lambda_1+2}\right)+c\left(\frac{-c\lambda_2+2\lambda_1 z_c}{2\lambda_1+2}\right)+d=0 \\ &\Rightarrow 2d-(a^2+b^2+c^2)\lambda_2+2(ax_c+by_c+cz_c+d)\lambda_1 = 0 \\ &\Rightarrow \lambda_1+1 = \frac{(a^2+b^2+c^2)\lambda_2+2(ax_c+by_c+cz_c)}{2(ax_c+by_c+cz_c+d)} \end{aligned}$$

Substituting the above in $\mathtt{S}$ we get

$$\begin{aligned} (x^*-x_c)^2+(y^*-y_c)^2+(z-z_c)^2-r_c^2 &= \left(\frac{2x_c+a\lambda_2}{2\lambda_1+2}\right)^2+\left(\frac{2y_c+b\lambda_2}{2\lambda_1+2}\right)^2+\left(\frac{2z_c+c\lambda_2}{2\lambda_1+2}\right)^2-r_c^2 = 0 \\ \Rightarrow 4(x_c^2+y_c^2+z_c^2)&+4(ax_c+by_c+dz_c)\lambda_2+(a^2+b^2+c^2)\lambda_2^2-4r_c^2(\lambda_1+1)^2 = 0 \\ 4r_c^2(\lambda_1+1)^2&-\left(a^2+b^2+c^2\right)\lambda_2^2-4(ax_c+by_c+cz_c)\lambda_2 = 4(x_c^2+y_c^2+z_c^2) \end{aligned}$$

Let $\alpha := ax_c+by_c+cz_c+d$, $\beta^2 = a^2+b^2+c^2$, $\gamma^2 = x_c^2+y_c^2+z_c^2$.

Substituting $\lambda_1 + 1 = \frac{\beta^2 \lambda_2 + 2(\alpha - d)}{2\alpha}$ in S we get

$$4r_c^2 \left(\frac{\beta^2 \lambda_2 + 2(\alpha - d)}{2\alpha} \right)^2 - \beta^2 \lambda_2^2 - 4(\alpha - d)\lambda_2 = 4\gamma^2$$
$$\left(\frac{\beta^2 r_c^2}{\alpha^2} - 1 \right) \beta^2 \lambda_2^2 - 4 \frac{\left(\alpha^2 - \beta^2 r_c^2\right) (\alpha - d)}{\alpha^2} \lambda_2 + 4 \frac{r_c^2 (\alpha - d)^2}{\alpha^2} - 4\gamma^2 = 0$$

$$\lambda_2 = \pm \frac{2\alpha \sqrt{(\alpha^2 - \beta^2 r_c^2)\,((\alpha - d)^2 - \beta^2 \gamma^2)}}{\beta^2 \,(\alpha^2 - \beta^2 r_c^2)} - \frac{2\,(\alpha - d)}{\beta^2}$$
$$\lambda_1 + 1 = \pm \frac{\sqrt{(\alpha^2 - \beta^2 r_c^2)\,((\alpha - d)^2 - \beta^2 \gamma^2)}}{\alpha^2 - \beta^2 r_c^2}$$

$$\begin{aligned} x^{*2} + y^{*2} + z^{*2} &= \left(\frac{-a\lambda_2 + 2\lambda_1 x_c}{2\lambda_1 + 2} \right)^2 + \left(\frac{-b\lambda_2 + 2\lambda_1 y_c}{2\lambda_1 + 2} \right)^2 + \left(\frac{-c\lambda_2 + 2\lambda_1 z_c}{2\lambda_1 + 2} \right)^2 \\ &= \frac{-2\,\alpha(\alpha - d) + \beta^2 \left(\gamma^2 + r_c^2\right) \mp 2\sqrt{(\alpha^2 - \beta^2 r_c^2)\,((\alpha - d)^2 - \beta^2 \gamma^2)}}{\beta^2} \end{aligned}$$