AUV Dynamics

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The standard way of modeling an autonomous underwater vehicle is given by Elmokadem et al. As usual to represent the dynamics of a dynamic system, an inertial frame is considered along with a body-fixed frame. The dynamics of the AUV is given by

$$\dot{x} = u\cos\psi - v\sin\psi \tag{1a}$$
$$\dot{y} = u\sin\psi + v\cos\psi \tag{1b}$$
$$\dot{\psi} = r \tag{1c}$$
$$\dot{u} = M_1(X_u u + a_{23}vr + \tau_1) \tag{1d}$$
$$\dot{v} = M_2(Y_v v + a_{13}ur) \tag{1e}$$
$$\dot{r} = M_3(N_r r + a_{12}uv + \tau_2) \tag{1f}$$

Here x, y, ψ represent the position and orientation of the AUV wrt the earth-fixed frame, while u, v are the body-fixed surge and sway linear velocities, and r is the rotational velocity. After substituting typical values for the constants in the dynamics, these equations can be normalized with control coefficients as 1, by considering

$$\mathbf{x} = \left[\frac{x}{0.0318}, \frac{y}{0.0318}, \psi, \frac{u}{0.0318}, \frac{v}{0.0318}, \frac{r}{0.02567}\right]^T \tag{2}$$

Consequently the AUV dynamics can be written in a compact form which will be considered throughout this text.

$$\dot{\mathbf{x}}_1 = \mathbf{x}_4\cos\mathbf{x}_3 - \mathbf{x}_5\sin\mathbf{x}_3 \tag{3a}$$
$$\dot{\mathbf{x}}_2 = \mathbf{x}_4\sin\mathbf{x}_3 + \mathbf{x}_5\cos\mathbf{x}_3 \tag{3b}$$
$$\dot{\mathbf{x}}_3 = c\mathbf{x}_6 \tag{3c}$$
$$\dot{\mathbf{x}}_4 = -a_1\mathbf{x}_4 + b_1\mathbf{x}_5\mathbf{x}_6 + \tau_1 \tag{3d}$$
$$\dot{\mathbf{x}}_5 = -a_2\mathbf{x}_5 - b_2\mathbf{x}_4\mathbf{x}_6 \tag{3e}$$
$$\dot{\mathbf{x}}_6 = -a_3\mathbf{x}_6 - b_3\mathbf{x}_4\mathbf{x}_5 + \tau_2 \tag{3f}$$

All the constants are positive values given by $a_1 = 0.28$, $a_2 = 0.993$, $a_3 = 0.1729$, $b_1 = 0.0.0538$, $b_2 = 0.0122$, $b_3 = 0.0349$, $c = 0.02567$. The aim is to find an appropriate coordinate transformation for the state space system that will allow the resulting dynamics into the Brunovsky canonical form. An important requirement for carrying out the procedure of feedback linearization is to have the same number of outputs as the number of inputs. Considering $\mathbf{y}$ as the output and the input as $\tau = [\tau_1, \tau_2]^T$ the system can thus by represented as

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} \mathbf{x}_4 \cos \mathbf{x}_3 - \mathbf{x}_5 \sin \mathbf{x}_3 \\ \mathbf{x}_4 \sin \mathbf{x}_3 + \mathbf{x}_5 \cos \mathbf{x}_3 \\ c\mathbf{x}_6 \\ -a_1\mathbf{x}_4 + b_1\mathbf{x}_5\mathbf{x}_6 \\ -a_2\mathbf{x}_5 - b_2\mathbf{x}_4\mathbf{x}_6 \\ -a_3\mathbf{x}_6 - b_3\mathbf{x}_4\mathbf{x}_5 \end{bmatrix}}_{f(\mathbf{x})} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{g(\mathbf{x})} \tau \tag{4a}$$

$$\mathbf{y} = \underbrace{\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_3 \end{bmatrix}}_{h(x)} \tag{4b}$$

Such a system has a well-defined vector relative degree $\{r_1, ..., r_m\}$ at a point $\mathbf{x}^\circ$ (where $m =$ number of inputs) if

1.
$$\mathcal{L}_{g_j}\mathcal{L}_f^k h_i(\mathbf{x}) = 0$$
$\forall\, 1 \leq j \leq m, \forall\, 1 \leq i \leq m, \forall\, < r_i - 1$, and $\forall\, \mathbf{x}$ in a neighborhood of $\mathbf{x}^\circ$,

2. the $m \times m$ matrix
$$A(\mathbf{x}) = \begin{bmatrix} \mathcal{L}_{g_1}\mathcal{L}_f^{r_1-1}h_1(\mathbf{x}) & ... & \mathcal{L}_{g_m}\mathcal{L}_f^{r_1-1}h_1(\mathbf{x}) \\ ... & ... & ... \\ \mathcal{L}_{g_1}\mathcal{L}_f^{r_m-1}h_m(\mathbf{x}) & ... & \mathcal{L}_{g_m}\mathcal{L}_f^{r_m-1}h_m(\mathbf{x}) \end{bmatrix}$$
is nonsingular at $\mathbf{x} = \mathbf{x}^\circ$.w

For the system under consideration, for $\mathbf{x}^\circ = 0$ i.e. the equilibrium of the system, it can be verified that $\mathcal{L}_{g_j}h_i(\mathbf{x}) \equiv 0$ for $i, j = 1, 2$, and the matrix

$$A(\mathbf{x}) = \begin{bmatrix} \mathcal{L}_{g_1}\mathcal{L}_f h_1(\mathbf{x}) & \mathcal{L}_{g_2}\mathcal{L}_f h_1(\mathbf{x}) \\ \mathcal{L}_{g_1}\mathcal{L}_f h_2(\mathbf{x}) & \mathcal{L}_{g_2}\mathcal{L}_f h_2(\mathbf{x}) \end{bmatrix}$$

yields a determinant $-c \cos \mathbf{x}_3$ which is non-zero at the origin. Thus, having verified that the system has a well-defined relative degree, we can now find a coordinate transformation that shall have as the new states, 2 decoupled linear systems of dimension 2, and another 2 states without any control input forming the zero/internal dynamics. The resultant transformation $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is of the form

$$\Phi(\mathbf{x}) = \begin{bmatrix} h_1(\mathbf{x}) \\ \mathcal{L}_f h_1(\mathbf{x}) \\ h_2(\mathbf{x}) \\ \mathcal{L}_f h_2(\mathbf{x}) \\ \phi_5(\mathbf{x}) \\ \phi_6(\mathbf{x}) \end{bmatrix} \tag{5}$$

The transformation Φ must be a diffeomorphism and the functions ϕ_5, ϕ_6 must satisfy the condition $\mathcal{L}_{g_j}\phi_i(\mathbf{x}) = 0 \;\forall\; j = 1, 2$ and $\forall\; i = 5, 6$. Consider

$$\Phi(\mathbf{x}) = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_4 \cos \mathbf{x}_3 - \mathbf{x}_5 \sin \mathbf{x}_3 \\ \mathbf{x}_3 \\ c\mathbf{x}_6 \\ \mathbf{x}_2 \\ \mathbf{x}_5 \end{bmatrix} \quad \text{and} \quad \Phi^{-1}(\mathbf{x}) = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_5 \\ \mathbf{x}_3 \\ \frac{\mathbf{x}_2 + \mathbf{x}_6 \sin \mathbf{x}_3}{\cos \mathbf{x}_3} \\ \mathbf{x}_6 \\ \frac{\mathbf{x}_4}{c} \end{bmatrix} \tag{6}$$

It can be verified that this transformation is a diffeomorphism and ϕ_5, ϕ_6 satisfy the desired conditions. Let the new states be $\mathbf{z} = \Phi(\mathbf{x})$. The dynamics of the transformed system is given by

$$\dot{\mathbf{z}} = \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_4 \cos \mathbf{x}_3 - \mathbf{x}_4 \dot{\mathbf{x}}_3 \sin \mathbf{x}_3 - \dot{\mathbf{x}}_5 \sin \mathbf{x}_3 - \mathbf{x}_5 \dot{\mathbf{x}}_3 \cos \mathbf{x}_3 \\ \dot{\mathbf{x}}_3 \\ c\dot{\mathbf{x}}_6 \\ \dot{\mathbf{x}}_2 \\ \dot{\mathbf{x}}_5 \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} \mathbf{z}_2 \\ p_2(z) + \tau_1 \cos \mathbf{z}_3 \\ \mathbf{z}_4 \\ p_4(z) + c\tau_2 \\ p_5(z) \\ p_6(z) \end{bmatrix} \tag{8}$$

where

$$\begin{aligned} p_2(z) &= -a_1\mathbf{z}_2 + (a_2 - a_1)\mathbf{z}_6 \sin \mathbf{z}_3 - \frac{1 - b_2}{c}\mathbf{z}_2\mathbf{z}_4 \tan \mathbf{z}_3 + \\ &\quad \frac{\mathbf{z}_4\mathbf{z}_6}{c}[(b_2 - 1)\sin \mathbf{z}_3 \tan \mathbf{z}_3 + (b_1 - 1)\cos \mathbf{z}_3] \\ p_4(z) &= -a_3\mathbf{z}_4 - \frac{b_3 c}{\cos \mathbf{z}_3}\mathbf{z}_2\mathbf{z}_6 - b_3 c\mathbf{z}_6^2 \tan \mathbf{z}_3 \\ p_5(z) &= \mathbf{z}_2 \tan \mathbf{z}_3 + \mathbf{z}_6(\cos \mathbf{z}_3 + \sin \mathbf{z}_3 \tan \mathbf{z}_3) \\ p_6(z) &= a_2\mathbf{z}_6 - \frac{b_2\mathbf{z}_4}{c \cos \mathbf{z}_3}(\mathbf{z}_2 + \mathbf{z}_6 \sin \mathbf{z}_3) \end{aligned}$$

Consider the control inputs

$$\tau_1 = \frac{1}{\cos \mathbf{z}_3}(-p_2(z) + v_1) \tag{9}$$

$$\tau_2 = \frac{1}{c}(-p_4(z) + v_2) \tag{10}$$

Clearly, these inputs will cancel the nonlinearities in $\mathbf{z}_2$ and $\mathbf{z}_4$ dynamics. Thus, the states $\mathbf{z}_1, \mathbf{z}_2$ form a double integrator system with control v_1, while $\mathbf{z}_3, \mathbf{z}_4$ form the same with control v_2. Now, if we speak about the question of controlling the system, i.e. bringing the states to zero, then $v_1 = -k_1\mathbf{z}_1 - k_2\mathbf{z}_2$ and $v_2 = -k_3\mathbf{z}_3 - k_4\mathbf{z}_4$ can asymptotically bring the states $\mathbf{z}_1, ..., \mathbf{z}_4$ to zero as per any desired decay rate. Note that $\mathbf{x} \to 0 \leftrightarrow \mathbf{z} \to 0$. But, the two remaining states $\mathbf{z}_5, \mathbf{z}_6$ cannot be controlled. Hence, we want to check, as the other states go to zero, whether the internal dynamics is stable? To do this, we substitue $\mathbf{z}_1 = \mathbf{z}_2 = \mathbf{z}_3 = \mathbf{z}_4 = 0$ into the internal dynamics. The two equations then boil down to

$$\dot{\mathbf{z}}_5 = \mathbf{z}_6 \tag{11}$$

$$\dot{\mathbf{z}}_6 = -a_2\mathbf{z}_6 \tag{12}$$

Even though the internal dynamics shall not go to zero, but atleast it doesn't blow up. Next, we move on to the problem of tracking a reference trajectory. Typical games of pursuit evasion considers agents such that no state appears in the Hamiltonian. This results in constant costate dynamics and consequently in constant optimal controls. This leads to a constant heading direction, i.e. optimal trajectories for either agents involves travelling in a straight line. Thus, the aim here is to track a desired straight line trajectory using feedback linearization based control.

Let the current position of the AUV pursuing an opponent at (x_f, y_f) be (x_0, y_0). The optimal trajectory for a holonomic pursuer at the same position would be to head straight along the line joining the two points. This trajectory can be parametrized with time, and is given by

$$x(t) = x_0 + v(x_f - x_0)t \tag{13}$$

$$y(t) = y_0 + v(y_f - y_0)t \tag{14}$$

The term v can be associated with the maximum allowable speed of the AUV. To be able to track this trajectory, the linearized system must be fed reference output trajectories that generate this straight line. Since the two outputs for the AUV system are $\mathbf{x}_1, \mathbf{x}_3$ i.e. x, ψ the reference output trajectories are given by

$$x_d(t) = x_0 + (x_f - x_0)t \tag{15}$$

$$\psi_d(t) = \tan^{-1}\left(\frac{y_f - y_0}{x_f - x_0}\right) \tag{16}$$

In place of the controls v_1, v_2 that asymptotically brought the states to zero, the following inputs are given so as to bring the error between desired and actual output trajectories to zero.

$$v_1(t) = \ddot{x}_d(t) - c_1(\mathbf{z}_2(t) - \dot{x}_d(t)) - c_0(\mathbf{z}_1(t) - x_d(t)) \tag{17}$$

$$v_2(t) = \ddot{\psi}_d(t) - c_3(\mathbf{z}_4(t) - \dot{\psi}_d(t)) - c_2(\mathbf{z}_3(t) - \psi_d(t)) \tag{18}$$

The resulting τ_1, τ_2 cancel the nonlinearities and the dynamics of $\mathbf{z}_2$ and $\mathbf{z}_4$ boils down to

$$\dot{\mathbf{z}}_2 = \ddot{x}_d - c_1(\mathbf{z}_2 - \dot{x}_d) - c_0(\mathbf{z}_1 - x_d) \tag{19}$$

$$\dot{\mathbf{z}}_4 = \ddot{\psi}_d - c_3(\mathbf{z}_4 - \dot{\psi}_d) - c_2(\mathbf{z}_3 - \psi_d) \tag{20}$$

But, we know that $\mathbf{z}_1 = x$ and $\mathbf{z}_3 = \psi$, so the above equations become

$$\ddot{e}_x + c_1\dot{e}_x + c_0 e_x = 0 \tag{21}$$

$$\ddot{e}_\psi + c_3\dot{e}_\psi + c_2 e_\psi = 0 \tag{22}$$

where $e_x(t) = x(t) - x_d(t)$ and $e_\psi(t) = \psi(t) - \psi_d(t)$. Thus, by an appropriate choice for the constants c_0, c_1, c_2, c_3 the error dynamics can be made to decay arbitrarily fast. The only remaining issue is of the internal dynamics. Like earlier, we can plug in the steady state values of the states $\mathbf{z}_1, ..., \mathbf{z}_4$ into those of $\mathbf{z}_5, \mathbf{z}_6$, and then study their behaviour. For $\mathbf{z}_6$, this results in an asymptotically stable system as shown below.

$$\dot{\mathbf{z}}_6 = -a_2\mathbf{z}_6 - \frac{b_2\dot{\psi}_d}{c\cos\psi_d}(\dot{x}_d + \mathbf{z}_6\sin\psi_d) \tag{23}$$

$$\implies \dot{\mathbf{z}}_6 = -a_2\mathbf{z}_6 \quad (\text{as } \psi_d = constant) \tag{24}$$

$$\implies \mathbf{z}_6(t) = \mathbf{z}_6(0)e^{-a_2 t} \tag{25}$$

This leads to the dynamics of $\mathbf{z}_5$ as

$$\dot{\mathbf{z}}_5 = \mathbf{z}_2\tan\mathbf{z}_3 + \mathbf{z}_6(\cos\mathbf{z}_3 + \sin\mathbf{z}_3\tan\mathbf{z}_3) \tag{26}$$

$$\implies \dot{\mathbf{z}}_5 = \dot{x}_d\tan\psi_d + \mathbf{z}_6(0)e^{-a_2 t}(\cos\psi_d + \sin\psi_d\tan\psi_d) \tag{27}$$

$$\implies \dot{\mathbf{z}}_5 = (y_f - y_0) + \mathbf{z}_6(0)\frac{\sqrt{(x_f - x_0)^2 + (y_f - y_0)^2}}{x_f - x_0}e^{-a_2 t} \tag{28}$$

$$\implies \mathbf{z}_5(t) = y_0 + (y_f - y_0)t + \frac{z_6(0)}{a_2\cos\psi_d}(1 - e^{-a_2 t}) \tag{29}$$

As we know that $\mathbf{z}_5(t) = y(t)$, the above equation shows that with the given inputs, not only will the system track $x_d(t)$ but also $y_d(t)$ under the condition that $\mathbf{z}_6(0) = v(0) = 0$. Physically this implies a zero initial sway velocity, which is not unreasonable to assume. Thus, at least theoretically the AUV system with the right control inputs should track a desired straight line trajectory asymptotically.

One issue with this approach for pursuit evasion games is that the optimal controls are not in a feedback form, hence the open-loop optimal trajectory has to be recomputed if the opponent deviates from their optimal trajectory. Such a computation can be done periodically in some specified time interval and the open-loop trajectories be modified as the scenario evolves. But, this approach has a problem, as after each modification the new $\mathbf{z}_6(0)$ may be nonzero.