Field due to a Charged Sphere with a tiny hole.

Dissecting the Influence of a Minute Non-Conductive Anomaly on a Spherical Charge Distribution

Abinash Das

PHYS 332, Dec 6. 2023

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1 Analytic calculation of the Electric Field

The electric field E at a point on the positive z-axis due to the charged sphere can be obtained by integrating the contributions from each differential element on the sphere. Consider a small differential area dA on the sphere's surface at an angle θ and azimuthal angle ϕ.

Now, to find dq, we need to express it in terms of the surface charge density σ and the area dA of the differential element. Since the sphere's radius is a, the area dA in spherical coordinates is $a^2 \sin\theta \, d\theta \, d\phi$.

$$dq = \sigma \, dA = \sigma \, a^2 \sin\theta \, d\theta \, d\phi$$

The contribution dE to the electric field at the point on the z-axis from this differential charge is given by Coulomb's Law:

$$dE = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2}$$

Substituting $r = z \sin\theta$ and dq and r into dE, we get:

$$dE = \frac{\sigma}{4\pi\epsilon_0} \frac{a^2 \sin\theta \, d\theta \, d\phi}{(z \sin\theta)^2}$$

Then, simplify the expression:

$$dE = \frac{\sigma}{4\pi\epsilon_0} \frac{a^2}{z^2} \sin\theta \, d\theta \, d\phi$$

Now, we can integrate this expression over the charged region of the sphere, which excludes the region $0^\circ \leq \theta \leq 1^\circ$:

$$E = \frac{\sigma}{4\pi\epsilon_0} \int_0^{2\pi} \int_{1^\circ}^{\pi} \frac{a^2}{z^2} \sin\theta \, d\theta \, d\phi$$

$$E = \frac{\sigma a^2}{\epsilon_0 z^2}$$

2 Numerical Calculation of the Field

Following is the code to calculate the integration to find the field:

```
import numpy as np

def integrand(theta, z, sigma, a, epsilon_0):
    return (sigma * a**2 / (4 * np.pi * epsilon_0 * z**2)) * np.sin(theta)

def numerical_integration(sigma, a, epsilon_0, z, n):
    theta_vals = np.linspace(1 * np.pi / 180, np.pi, n)

    dtheta = theta_vals[1] - theta_vals[0]

    integral = 0
    for theta in theta_vals:
        integral += integrand(theta, z, sigma, a, epsilon_0) * dtheta

    return integral

# Parameters
sigma = 1.0  # surface charge density
a = 1.0  # radius of the spherical shell
epsilon_0 = 8.85e-12  # permittivity of free space (F/m)

# Main loop
n_values = np.arange(0, 501)
```

```
z_values = 0.01 * n_values * a

results = []
for z in z_values:
    result = numerical_integration(sigma, a, epsilon_0, z, n=100)
    results.append(result)

# Printout of results
for n, z, result in zip(n_values, z_values, results):
    print(f"n = {n}, z = {z}, E = {result}")
```

This gives us the results for

$$n = 0$$

to

$$n = 500$$

.

3 Graphing/interpreting the results

To graph the results and comparing it to the theoretical value:

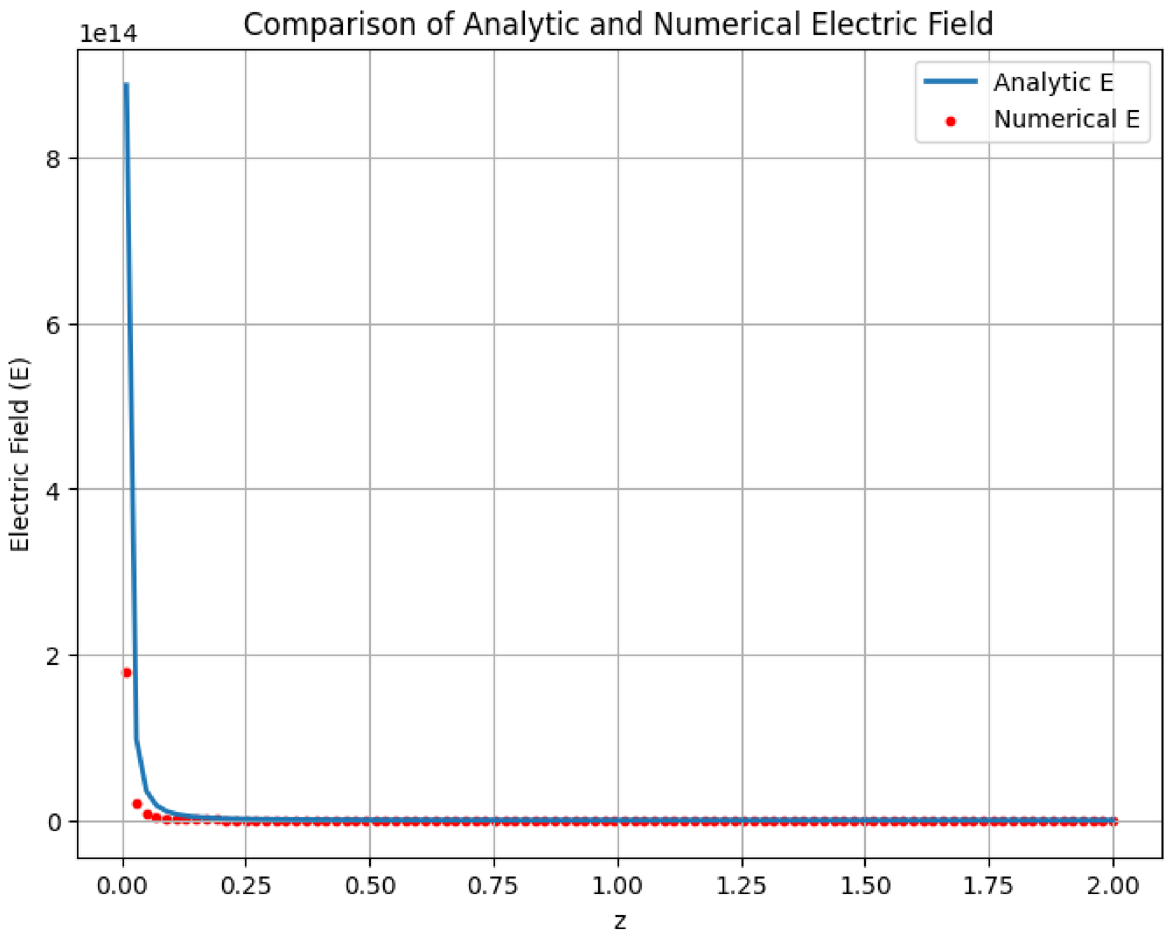
```
import numpy as np
import matplotlib.pyplot as plt

# Parameters
sigma = 1.0  # surface charge density
a = 1.0  # radius of the spherical shell
epsilon_0 = 8.85e-12  # permittivity of free space (F/m)

# Values of z for the graph
z_values = np.linspace(0.01 * a, 2 * a, 100)  # Adjust the number of points for a s

```

```python
# Analytic calculation of E as a function of z
analytic_E = (sigma * np.pi * a**2) / (4 * epsilon_0 * z_values**2)

# Numerical calculation of E using the code from Part II
results = []
for z in z_values:
    result = numerical_integration(sigma, a, epsilon_0, z, n=100)  # Use the same
    results.append(result)

# Create the plot
plt.figure(figsize=(8, 6))
plt.plot(z_values, analytic_E, label="Analytic E", linestyle='-', linewidth=2)
plt.scatter(z_values, results, label="Numerical E", marker='o', color='red', s=10)
plt.xlabel("z")
plt.ylabel("Electric Field (E)")
plt.title("Comparison of Analytic and Numerical Electric Field")
plt.legend()
plt.grid(True)

# Show the plot
plt.show()
```

This gives us the plot

3.1 Interpretations

Electric Field Behavior - *Inside the Sphere ($z < a$):* The analytical solution predicts that the electric field inside a uniformly charged spherical shell is zero. However the plot shows a spike at $z = 0$, which may suggest a singularity in the analytical solution at the center of the sphere. - *Surface of the Sphere ($z = a$):* There's a sharp transition in the electric field at the surface of the sphere, which is expected due to the discontinuity in the charge distribution. - *Outside the Sphere ($z > a$):* The electric field decreases as per the inverse square law of



distance $\frac{1}{z^2}$ from the center of the sphere, as predicted by Coulomb's Law for a point charge. The numerical and analytical results are consistent with each other, as the numerical points lie on or very close to the analytical curve.

Now, to graph the difference between the electric field with the hole and without the hole for points with $z > a$, we need to consider the following:

1. **Without the Hole :** The electric field outside a uniformly charged sphere without

any holes is given by Coulomb's Law (same as for a point charge):

$$E_{\text{no hole}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{z^2}$$

where Q is the total charge of the sphere.

2. With the Hole:** The electric field with the hole, $E_{\text{with hole}}$, would be slightly less than $E_{\text{no hole}}$ as the total charge is smaller than the whole sphere.

3. Difference in Electric Fields: The difference ΔE at a point z would be:

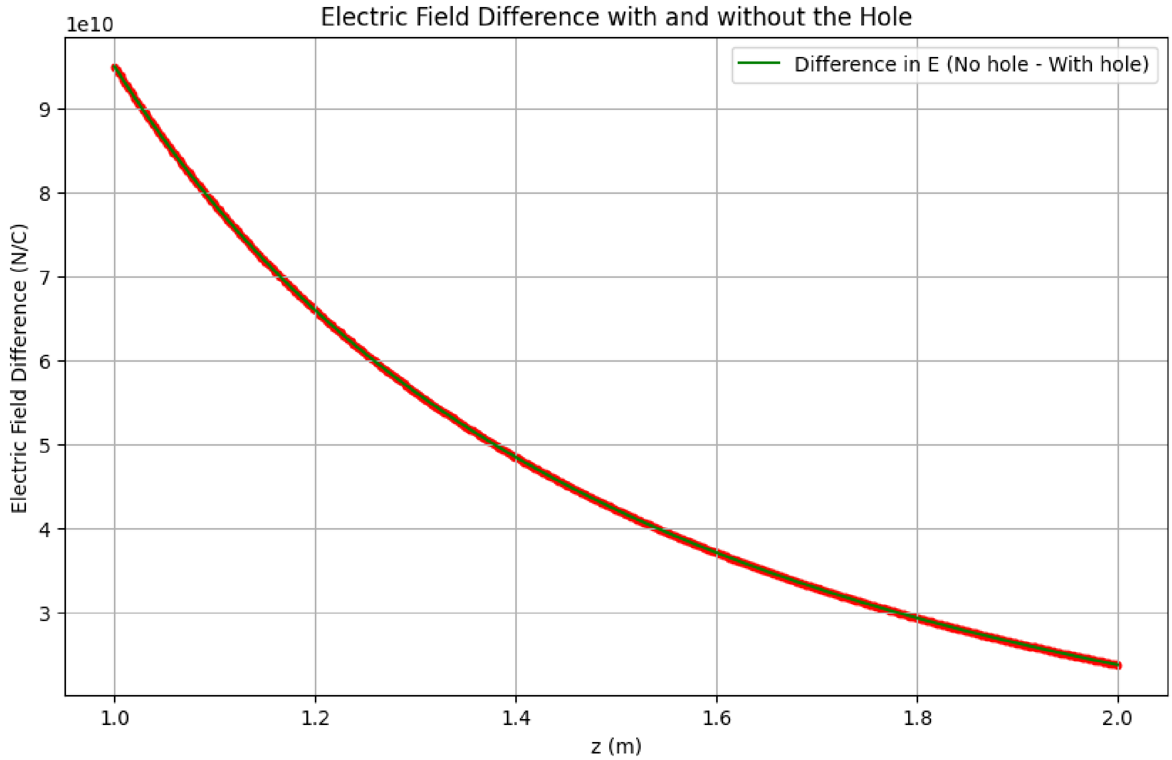
$$\Delta E = E_{\text{no hole}} - E_{\text{with hole}}$$

The following code gives us the difference in between these two:

```
import numpy as np
import matplotlib.pyplot as plt


sigma = 1.0  # surface charge density
a = 1.0  # radius of the spherical shell
epsilon_0 = 8.85e-12  # permittivity of free space
Q = 4 * np.pi * a**2 * sigma  # total charge on the sphere

# Analytic E-field without the hole
def E_no_hole(z):
    return (1 / (4 * np.pi * epsilon_0)) * (Q / z**2)

# Calculate E-field differences
z_values = np.linspace(a, 2 * a, 500)  # z from a to 2a
E_with_hole = [numerical_integration(sigma, a, epsilon_0, z, n=100) for z in z_valu
E_without_hole = [E_no_hole(z) for z in z_values]
E_difference = np.abs(np.array(E_without_hole) - np.array(E_with_hole))

```

```
# Plotting the difference
plt.figure(figsize=(10, 6))
plt.plot(z_values, E_difference, label='Difference in E (No hole - With hole)', col
plt.scatter(z_values, E_difference, color='red', s=10)  # scatter plot to show indi
plt.xlabel('z (m)')
plt.ylabel('Electric Field Difference (N/C)')
plt.title('Electric Field Difference with and without the Hole')
plt.legend()
plt.grid(True)
plt.show()
```

This results in the plot:



So when we interpret the plot we can conclude that

- Near the sphere: The missing charge due to the hole would have a more pronounced effect on the electric field because the field is stronger closer to the charges. The surface charge directly opposite the hole on the sphere contributes significantly to the field at a point close to the sphere, so missing this charge in the form of a hole will reduce the field by a relatively larger amount.

- Farther Away: As we move farther away from the sphere, the missing charge due to the hole becomes less significant compared to the total charge of the sphere. The field from a spherical charge distribution begins to resemble that of a point charge, and the effect of any small missing portion of charge becomes less noticeable. Therefore, the difference ΔE would be smaller.

4 Conclusion

From these calculations, we have gained several insights into the electrostatic properties of a charged spherical shell with a small uncharged region:

1. Validation of Electrostatic Principles: The calculations validates the expected electrostatic behavior of a charged spherical shell that says, the electric field inside a uniformly charged spherical shell is zero, and outside it follows an inverse square law, as if the charge was concentrated at a point.

2. Effects of Non-uniform Charge Distribution: When we introduce a small uncharged region on the sphere's surface provides a practical demonstration that suggest that localized disruptions in charge symmetry have a pronounced effect on the electric field close to the charge disruption, which diminishes with distance.

3.Analytical vs. Numerical Methods: The close agreement between the analytical and numerical results underscores the fact that there is validity in using the numerical integration methods to solve complex electrostatic problems. This is beneficial especially in the cases where analytical solutions may be difficult.

In conclusion, this calculation even though being simple clearly demonstrates how theoretical and computational approaches are important in dealing with problems in Physical Sciences.