NUMERICAL METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

FIRST MANDATORY EXERCISE

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Problem 1a

Considering a ordinary differential equation (ODE) problem as following

$$u'' + \omega^2 u = f, \quad u(0) = I, u'(0) = V, t \in (0, T] \tag{1}$$

We start off by discretizing the ODE according to $\left[D_t D_t u + \omega^2 u = f\right]^n$ using centered difference scheme

$$[D_t u]^n = \frac{u^{n+\frac{1}{2}} - u^{n-\frac{1}{2}}}{\Delta t}$$

$[D_t D_t u]^n$ thus becomes

$$\begin{aligned}
[D_t D_t u]^n &= \frac{\frac{u^{n+\frac{1}{2}+\frac{1}{2}} - u^{n-\frac{1}{2}+\frac{1}{2}}}{\Delta t} - \frac{u^{n+\frac{1}{2}-\frac{1}{2}} - u^{n-\frac{1}{2}-\frac{1}{2}}}{\Delta t}}{\Delta t} \\
&= \frac{\frac{u^{n+1} - u^n}{\Delta t} - \frac{u^n - u^{n-1}}{\Delta t}}{\Delta t} \\
&= \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}
\end{aligned}$$

Adding this to the discretization gives following expression

$$\begin{aligned}
\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + \omega^2 u = f \\
u^{n+1} = \Delta t^2 f^n - \Delta t^2 \omega^2 u^n + 2u^n - u^{n-1} \\
u^{n+1} = (2 - \Delta t^2 \omega^2) u^n - u^{n-1} + \Delta t^2 f^n
\end{aligned}$$

To find the equation for the first time step (u^1) we need to know two previous steps, u^0 and u^{-1}. The latter expression is not among the given initial conditions, thus we need to discretize $u'(0)$

$$\begin{aligned}
u'(0) = \frac{u^1 - u^{-1}}{2\Delta t} = V \\
u^{-1} = u^1 - 2\Delta t V
\end{aligned}$$

Now, we can find the equation for the first time step by replacing u^{-1} with the new expression

$$\begin{aligned}
u^1 &= (2 - \Delta t^2 \omega^2) u^0 - u^{-1} + \Delta t^2 f^0 \\
u^1 &= (2 - \Delta t^2 \omega^2) I - (u^1 - 2\Delta t V) + \Delta t^2 f^0 \\
u^1 &= \frac{1}{2}(2 - \Delta t^2 \omega^2) I + \Delta t V + \frac{1}{2}\Delta t^2 f^0 \\
u^1 &= I - \frac{1}{2}\Delta t^2 \omega^2 I + \Delta t V + \frac{1}{2}\Delta t^2 f^0
\end{aligned}$$

Problem 1b

For verification, the method of manufactured solutions (MMS) are used with the choice of $u_e = ct + d$. We are going to find restrictions for c and d from the initial conditions

$$u_e(0) = c \cdot 0 + d = I \Rightarrow d = I$$
$$u_e'(0) = c = V$$

The linear function becomes

$$u_e(x,t) = Vt + I$$

Further, we compute the corresponding source term f

$$\left[D_t D_t u_e + \omega^2 u_e = f\right]^n$$
$$\left[[D_t D_t (It + V)]^n + \omega^2 u_e = f\right]^n$$

Using the fact that $D_t D_t$ operator is linear gives

$$0 + \omega^2 u_e^n = f^n$$
$$f^n = \omega^2 (Vt_n + I)$$

Next, we are going to show that $[D_t D_t t] = 0$

$$[D_t t]^n = \frac{t_{n+\frac{1}{2}} - t_{n-\frac{1}{2}}}{\Delta t} = \frac{\left(n+\frac{1}{2}\right)\Delta t - \left(n-\frac{1}{2}\right)\Delta t}{\Delta t}$$
$$[D_t D_t t]^n = \frac{\frac{t_{n+\frac{1}{2}+\frac{1}{2}} - t_{n-\frac{1}{2}+\frac{1}{2}}}{\Delta t} - \frac{t_{n+\frac{1}{2}-\frac{1}{2}} - t_{n-\frac{1}{2}-\frac{1}{2}}}{\Delta t}}{\Delta t}$$
$$= \frac{\frac{t_{n+1} - t_n}{\Delta t} - \frac{t_n - t_{n-1}}{\Delta t}}{\Delta t}$$
$$= \frac{t_{n+1} - 2t_n + t_{n-1}}{\Delta t^2}$$
$$= \frac{(n+1)\Delta t - (2n)\Delta t + (n-1)\Delta t}{\Delta t^2}$$
$$= \frac{n+1-2n+n-1}{\Delta t^2} = 0$$

Further we want to show that u_e is a perfect solution to the discrete equations. To do so, we need to show that the residual of the discrete equation with u_e inserted becomes zero

$$R = D_t D_t u_e^n + \omega^2 u_e^n - f^n$$

Replacing $D_t D_t u_e^n$ and f^n with the derived expressions gives

$$R = \frac{u_e^{n+1} - 2u_e^n + u_e^{n-1}}{\Delta t^2} + \omega^2 u_e^n - \omega^2 u_e^n$$

Since $u_e = It_n + V$, with the corresponding timestep the equation becomes

$$R = \frac{I(n+1)\Delta t + V - 2(In\Delta t + V) + I(n-1)\Delta t + V}{\Delta t^2}$$

$$R = \frac{In\Delta t + I\Delta t + V - 2In\Delta t - 2V + In\Delta t - I\Delta t + V}{\Delta t^2} = 0$$

We have thus showed that u_e is a perfect solution of the discrete equations.