NUMERICAL METHODS
FOR
PARTIAL DIFFERENTIAL EQUATIONS

SECOND MANDATORY EXERCISE

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Introduction

In following exercise we are looking at a 1D wave equation with variable wave velocity. The compact version of the problem we are looking at is given as following

$$\left[D_t D_t u = D_x q^{-x} D_x u + f\right]_i^n$$

We start off by discretizing the partial differential equation (PDE) using centered difference scheme

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} = \frac{1}{\Delta x^2}\left[\frac{1}{2}\left(q_i + q_{i+1}\right)\left(u_{i+1}^n - u_i^n\right) - \frac{1}{2}\left(q_i + q_{i-1}\right)\left(u_i^n - u_{i-1}^n\right)\right] + f_i^n$$

$$u_i^{n+1} = -u_i^{n-1} + 2u_i^n + \left(\frac{\Delta t}{\Delta x}\right)^2\left[\frac{1}{2}\left(q_i + q_{i+1}\right)\left(u_{i+1}^n - u_i^n\right) - \frac{1}{2}\left(q_i + q_{i-1}\right)\left(u_i^n - u_{i-1}^n\right)\right] + \Delta t^2 f_i^n \quad (1)$$

Eq. 1 is the general discretization for all the inner points. For the first step we need to modify the general discretization, that is done by adding following condition

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = V \Rightarrow u_i^{n-1} = u_i^{n+1} - 2\Delta t$$

Inserting this condition to the eq. 1 gives the discretization for the first step (n=0)

$$u_i^{n+1} = u_i^n + \Delta t V + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2\left[\frac{1}{2}\left(q_i + q_{i+1}\right)\left(u_{i+1}^n - u_i^n\right) - \frac{1}{2}\left(q_i + q_{i-1}\right)\left(u_i^n - u_{i-1}^n\right)\right] + \frac{1}{2}\Delta t^2 f_i^n \quad (2)$$

Problem 13c

For the one-sided difference method we have been given these conditions: $u_i^n = u_{i-1}^n$ for i = N_x and $u_{i-1}^n = u_i^n$ for i = 0 By adjusting the indices we can rewrite the relations as

$$i = N_x : u_{i+1}^n = u_i^n \quad (3)$$

$$i = 0 : u_i^n = u_{i-1}^n \quad (4)$$

Boundary conditions for the first step

For i = 0 we add eq. 4 into only a specific term in eq. 2 to get rid off q_{i-1} and u_{i-1}. This procedure gives following discretization

$$u_i^{n+1} = u_i^n + \Delta t V + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2\left[\frac{1}{2}\left(q_i + q_{i+1}\right)\left(u_{i+1}^n - u_i^n\right) - \frac{1}{2}\left(q_i + q_{i-1}\right)\left(u_{i-1}^n - u_{i-1}^n\right)\right] + \frac{1}{2}\Delta t^2 f_i^n$$

$$u_i^{n+1} = u_i^n + \Delta t V + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2\left[\frac{1}{2}\left(q_i + q_{i+1}\right)\left(u_{i+1}^n - u_i^n\right)\right] + \frac{1}{2}\Delta t^2 f_i^n$$

The same follows for i = N_x but now we want to get rid off q_{i+1} and u_{i+1}. Using the relation given in eq. 3 the boundary condition becomes

$$u_i^{n+1} = u_i^n + \Delta t V + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2\left[\frac{1}{2}\left(q_i + q_{i+1}\right)\left(u_i^n - u_i^n\right) - \frac{1}{2}\left(q_i + q_{i-1}\right)\left(u_i^n - u_{i-1}^n\right)\right] + \frac{1}{2}\Delta t^2 f_i^n$$

$$u_i^{n+1} = u_i^n + \Delta t V + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2\left[-\frac{1}{2}\left(q_i + q_{i-1}\right)\left(u_i^n - u_{i-1}^n\right)\right] + \frac{1}{2}\Delta t^2 f_i^n$$

Boundary conditions for the general steps

Now the same adjustments are done to the general formula (eq. 1) for i = 0 and i = N_x.

For i = 0:

$$u_i^{n+1} = -u_i^{n-1} + 2u_i^n + \left(\frac{\Delta t}{\Delta x}\right)^2 \left[\frac{1}{2}\left(q_i + q_{i+1}\right)\left(u_{i+1}^n - u_i^n\right) - \frac{1}{2}\left(q_i + q_{i-1}\right)\left(u_{i-1}^n - u_{i-1}^n\right)\right] + \Delta t^2 f_i^n$$

$$u_i^{n+1} = -u_i^{n-1} + 2u_i^n + \left(\frac{\Delta t}{\Delta x}\right)^2 \left[\frac{1}{2}\left(q_i + q_{i+1}\right)\left(u_{i+1}^n - u_i^n\right)\right] + \Delta t^2 f_i^n$$

For i = N_x:

$$u_i^{n+1} = -u_i^{n-1} + 2u_i^n + \left(\frac{\Delta t}{\Delta x}\right)^2 \left[\frac{1}{2}\left(q_i + q_{i+1}\right)\left(u_i^n - u_i^n\right) - \frac{1}{2}\left(q_i + q_{i-1}\right)\left(u_i^n - u_{i-1}^n\right)\right] + \Delta t^2 f_i^n$$

$$u_i^{n+1} = -u_i^{n-1} + 2u_i^n + \left(\frac{\Delta t}{\Delta x}\right)^2 \left[-\frac{1}{2}\left(q_i + q_{i-1}\right)\left(u_i^n - u_{i-1}^n\right)\right] + \Delta t^2 f_i^n$$

Problem 13d

The compact version of the technique described in this task is given below

$$[D_t D_t u]_i^n = \frac{1}{\Delta x}\left([qD_x u]_{i+\frac{1}{2}}^n - [qD_x u]_{i-\frac{1}{2}}^n\right) + [f]_i^n$$

Writing this out gives the following

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} = \frac{1}{\Delta x}\left[q_{i+\frac{1}{2}}\left(\frac{u_{i+1}^n - u_i^n}{\Delta x}\right) - q_{i-\frac{1}{2}}\left(\frac{u_i^n - u_{i-1}^n}{\Delta x}\right)\right] + f_i^n$$

$$u_i^{n+1} = -u_i^{n-1} + 2u_i^n + \left(\frac{\Delta t}{\Delta x}\right)^2 \left[\frac{1}{2}\left(q_i + q_{i+1}\right)\left(u_{i+1}^n - u_i^n\right) - \frac{1}{2}\left(q_i + q_{i-1}\right)\left(u_i^n - u_{i-1}^n\right)\right] + \Delta t^2 f_i^n$$

The discretization ends up being the same as the general formula (eq. 1). Naturally the discretization for the first step becomes as eq. 2

$$u_i^{n+1} = u_i^n + \Delta t V + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 \left[\frac{1}{2}\left(q_i + q_{i+1}\right)\left(u_{i+1}^n - u_i^n\right) - \frac{1}{2}\left(q_i + q_{i-1}\right)\left(u_i^n - u_{i-1}^n\right)\right] + \frac{1}{2}\Delta t^2 f_i^n$$

Now to the boundary conditions for this particular technique. The right boundary is now placed at $x_{N_x - \frac{1}{2}}$, which lets us set $[qD_x u]_{i+\frac{1}{2}}^n = 0$ and for the left boundary $x_{\frac{1}{2}}$ we can set $[qD_x u]_{i-\frac{1}{2}}^n = 0$

Boundary conditions for the first step

For i = 0:

$$u_i^{n+1} = u_i^n + \Delta t V + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 \left[\frac{1}{2}\left(q_i + q_{i+1}\right)\left(u_{i+1}^n - u_i^n\right) - 0\right] + \frac{1}{2}\Delta t^2 f_i^n$$

For i = N_x

$$u_i^{n+1} = u_i^n + \Delta t V + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 \left[0 - \frac{1}{2}\left(q_i + q_{i-1}\right)\left(u_i^n - u_{i-1}^n\right)\right] + \frac{1}{2}\Delta t^2 f_i^n$$

Boundary conditions for the general steps

For i = 0:

$$u_i^{n+1} = -u_i^{n-1} + 2u_i^n + \left(\frac{\Delta t}{\Delta x}\right)^2 \left[\frac{1}{2}\left(q_i + q_{i+1}\right)\left(u_{i+1}^n - u_i^n\right) - 0\right] + \Delta t^2 f_i^n$$

For i = N_x

$$u_i^{n+1} = -u_i^{n-1} + 2u_i^n + \left(\frac{\Delta t}{\Delta x}\right)^2 \left[0 - \frac{1}{2}\left(q_i + q_{i-1}\right)\left(u_i^n - u_{i-1}^n\right)\right] + \Delta t^2 f_i^n$$