

高等数学

积分表

公式推导

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(一) 含有 $ax+b$ 的积分 (1~9)

1. $\int \frac{dx}{ax+b} = \frac{1}{a} \cdot \ln |ax+b| + C$

证明: 被积函数 $f(x) = \frac{1}{ax+b}$ 的定义域为 $\{x | x \neq -\frac{b}{a}\}$

令 $ax+b=t$ ($t \neq 0$), 则 $dt = adx$, $\therefore dx = \frac{1}{a} dt$

$$\begin{aligned}\therefore \int \frac{dx}{ax+b} &= \frac{1}{a} \int \frac{1}{t} dt \\ &= \frac{1}{a} \cdot \ln |t| + C\end{aligned}$$

将 $t=ax+b$ 代入上式得: $\int \frac{dx}{ax+b} = \frac{1}{a} \cdot \ln |ax+b| + C$

2. $\int (ax+b)^\mu dx = \frac{1}{a(\mu+1)} \cdot (ax+b)^{\mu+1} + C \quad (\mu \neq -1)$

证明: 令 $ax+b=t$, 则 $dt = adx$, $\therefore dx = \frac{1}{a} dt$

$$\begin{aligned}\therefore \int (ax+b)^\mu dx &= \frac{1}{a} \int t^\mu dt \\ &= \frac{1}{a(\mu+1)} \cdot t^{\mu+1} + C\end{aligned}$$

将 $t=ax+b$ 代入上式得: $\int (ax+b)^\mu dx = \frac{1}{a(\mu+1)} \cdot (ax+b)^{\mu+1} + C$

3. $\int \frac{x}{ax+b} dx = \frac{1}{a^2} (ax+b - b \cdot \ln |ax+b|) + C$

证明: 被积函数 $f(x) = \frac{x}{ax+b}$ 的定义域为 $\{x | x \neq -\frac{b}{a}\}$

令 $ax+b=t$ ($t \neq 0$), 则 $x = \frac{1}{a}(t-b)$, $dx = \frac{1}{a} dt$

$$\begin{aligned}\therefore \int \frac{x}{ax+b} dx &= \int \frac{\frac{1}{a}(t-b)}{t} \cdot \frac{1}{a} dt = \frac{1}{a^2} \int \left(1 - \frac{b}{t}\right) dt \\ &= \frac{1}{a^2} \int dt - \frac{1}{a^2} \int \frac{b}{t} dt \\ &= \frac{t}{a^2} - \frac{b}{a^2} \cdot \ln |t| + C \\ &= \frac{1}{a^2} (t - b \cdot \ln |t|) + C\end{aligned}$$

将 $t=ax+b$ 代入上式得: $\int \frac{x}{ax+b} dx = \frac{1}{a^2} (ax+b - b \cdot \ln |ax+b|) + C$

$$4. \int \frac{x^2}{ax+b} dx = \frac{1}{a^3} \left[\frac{1}{2} (ax+b)^2 - 2b(ax+b) + b^2 \cdot \ln |ax+b| \right] + C$$

$$\begin{aligned} \text{证明: } \int \frac{x^2}{ax+b} dx &= \frac{1}{a^2} \int \frac{(ax+b)^2 - 2abx - b^2}{ax+b} dx \\ &= \frac{1}{a^2} \int (ax+b) dx - \frac{1}{a^2} \int \frac{2abx}{ax+b} dx - \frac{1}{a^2} \int \frac{b^2}{ax+b} dx \end{aligned}$$

$$\because \frac{1}{a^2} \int (ax+b) dx = \frac{1}{2a^3} (ax+b)^2 + C_1$$

$$\begin{aligned} \frac{1}{a^2} \int \frac{2abx}{ax+b} dx &= \frac{2b}{a^3} \int \frac{ax+b-b}{ax+b} d(ax) \\ &= \frac{2b}{a^3} \int dx - \frac{2b^2}{a^3} \int \frac{1}{ax+b} d(ax+b) \\ &= \frac{2b}{a^3} x - \frac{2b^2}{a^3} \ln |ax+b| + C_2 \end{aligned}$$

$$\frac{1}{a^2} \int \frac{b^2}{ax+b} dx = \frac{b^2}{a^3} \int \frac{1}{ax+b} d(ax+b) = \frac{b^2}{a^3} \ln |ax+b| + C_3$$

$$\text{由以上各式整理得: } \int \frac{x^2}{ax+b} dx = \frac{1}{a^3} \left[\frac{1}{2} (ax+b)^2 - 2b(ax+b) + b^2 \cdot \ln |ax+b| \right] + C$$

$$5. \int \frac{dx}{x(ax+b)} = -\frac{1}{b} \cdot \ln \left| \frac{ax+b}{x} \right| + C$$

证明: 被积函数 $f(x) = \frac{1}{x \cdot (ax+b)}$ 的定义域为 $\{x/x \neq -\frac{b}{a}\}$

$$\text{设 } \frac{1}{x \cdot (ax+b)} = \frac{A}{x} + \frac{B}{ax+b}, \text{ 则 } 1 = A(ax+b) + Bx = (Aa+B)x + Ab$$

$$\therefore \text{ 有 } \begin{cases} Aa+B=0 \\ Ab=1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{b} \\ B = -\frac{a}{b} \end{cases}$$

$$\text{于是 } \int \frac{dx}{x(ax+b)} = \int \left[\frac{1}{bx} - \frac{a}{b \cdot (ax+b)} \right] dx = \frac{1}{b} \int \frac{1}{x} dx - \frac{a}{b} \int \frac{1}{ax+b} dx$$

$$= \frac{1}{b} \int \frac{1}{x} dx - \frac{1}{b} \int \frac{1}{ax+b} d(ax+b)$$

$$= \frac{1}{b} \cdot \ln |x| - \frac{1}{b} \cdot \ln |ax+b| + C$$

$$= \frac{1}{b} \cdot \ln \left| \frac{x}{ax+b} \right| + C$$

提示: $\log_a b^{-1} = -\log_a b$

$$= -\frac{1}{b} \cdot \ln \left| \frac{ax+b}{x} \right| + C$$

$$6. \int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2} \cdot \ln \left| \frac{ax+b}{x} \right| + C$$

证明：被积函数 $f(x) = \frac{1}{x^2 \cdot (ax+b)}$ 的定义域为 $\{x \mid x \neq -\frac{b}{a}\}$

$$\text{设 } \frac{1}{x^2 \cdot (ax+b)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{ax+b}, \text{ 则 } 1 = Ax(ax+b) + B(ax+b) + Cx^2$$

$$\text{即 } x^2(Aa+C) + x(Ab+aB) + Bb = 1$$

$$\therefore \text{ 有 } \begin{cases} Aa+C=0 \\ Ab+aB=0 \\ Bb=1 \end{cases} \Rightarrow \begin{cases} A=-\frac{a}{b^2} \\ B=\frac{1}{b} \\ C=\frac{a^2}{b^2} \end{cases}$$

$$\begin{aligned} \text{于是 } \int \frac{dx}{x^2(ax+b)} &= -\frac{a}{b^2} \int \frac{1}{x} dx + \frac{1}{b} \int \frac{1}{x^2} dx + \frac{a^2}{b^2} \int \frac{1}{ax+b} dx \\ &= -\frac{a}{b^2} \int \frac{1}{x} dx + \frac{1}{b} \int \frac{1}{x^2} dx + \frac{a}{b^2} \int \frac{1}{ax+b} d(ax+b) \\ &= -\frac{a}{b^2} \cdot \ln|x| - \frac{1}{bx} + \frac{a}{b^2} \cdot \ln|ax+b| + C \\ &= -\frac{1}{bx} + \frac{a}{b^2} \cdot \ln \left| \frac{ax+b}{x} \right| + C \end{aligned}$$

$$7. \int \frac{x}{(ax+b)^2} dx = \frac{1}{a^2} \left(\ln|ax+b| + \frac{b}{ax+b} \right) + C$$

证明：被积函数 $f(x) = \frac{x}{(ax+b)^2}$ 的定义域为 $\{x \mid x \neq -\frac{b}{a}\}$

$$\text{设 } \frac{x}{(ax+b)^2} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2}, \text{ 则 } x = A(ax+b) + B$$

$$\text{即 } x \cdot Aa + (Ab+B) = x$$

$$\therefore \text{ 有 } \begin{cases} Aa=1 \\ Ab+B=0 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{a} \\ B=-\frac{b}{a} \end{cases}$$

$$\begin{aligned} \text{于是 } \int \frac{x}{(ax+b)^2} dx &= \frac{1}{a} \int \frac{1}{ax+b} dx - \frac{b}{a} \int \frac{1}{(ax+b)^2} dx \\ &= \frac{1}{a^2} \int \frac{1}{ax+b} d(ax+b) - \frac{b}{a^2} \int \frac{1}{(ax+b)^2} d(ax+b) \\ &= \frac{1}{a^2} \cdot \ln|ax+b| + \frac{b}{a^2(ax+b)} + C \\ &= \frac{1}{a^2} \left(\ln|ax+b| + \frac{b}{ax+b} \right) + C \end{aligned}$$

$$8. \int \frac{x^2}{(ax+b)^2} dx = \frac{1}{a^3} \left(ax+b - 2b \cdot \ln |ax+b| - \frac{b^2}{ax+b} \right) + C$$

证明：被积函数 $f(x) = \frac{x^2}{(ax+b)^2}$ 的定义域为 $\{x/x \neq -\frac{b}{a}\}$

$$\text{令 } ax+b=t \quad (t \neq 0), \text{ 则 } x = \frac{1}{a}(t-b), \quad dx = \frac{1}{a} dt$$

$$\therefore \frac{x^2}{(ax+b)^2} = \frac{(b-t)^2}{a^2 t^2} = \frac{b^2 + t^2 - 2bt}{a^2 t^2}$$

$$\begin{aligned} \therefore \int \frac{x^2}{(ax+b)^2} dx &= \int \frac{b^2 + t^2 - 2bt}{a^3 t^2} dt = \frac{b^2}{a^3} \int \frac{1}{t^2} dt + \frac{1}{a^3} \int dt - \frac{2b}{a^3} \int \frac{1}{t} dt \\ &= -\frac{b^2}{a^3 t} + \frac{1}{a^3} \cdot t - \frac{2b}{a^3} \cdot \ln |t| + C \\ &= \frac{1}{a^3} (t - 2b \cdot \ln |t| - \frac{b^2}{t}) + C \end{aligned}$$

$$\text{将 } t = ax+b \text{ 代入上式得: } \int \frac{x^2}{(ax+b)^2} dx = \frac{1}{a^3} \left(ax+b - 2b \cdot \ln |ax+b| - \frac{b^2}{ax+b} \right) + C$$

$$9. \int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} - \frac{1}{b^2} \cdot \ln \left| \frac{ax+b}{x} \right| + C$$

证明：被积函数 $f(x) = \frac{1}{x(ax+b)^2}$ 的定义域为 $\{x/x \neq -\frac{b}{a}\}$

$$\text{设: } \frac{1}{x(ax+b)^2} = \frac{A}{x} + \frac{B}{ax+b} + \frac{D}{(ax+b)^2}$$

$$\begin{aligned} \text{则 } 1 &= A(ax+b)^2 + Bx(ax+b) + Dx \\ &= Aa^2 x^2 + Ab^2 + 2Aabx + Bax^2 + Bbx + Dx \\ &= x^2(Aa^2 + Ba) + x(2Aab + Bb + D) + Ab^2 \end{aligned}$$

$$\therefore \text{ 有 } \begin{cases} Aa^2 + Ba = 0 \\ 2Aab + Bb + D = 0 \\ Ab^2 = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{b^2} \\ B = -\frac{a}{b^2} \\ D = -\frac{a}{b} \end{cases}$$

$$\begin{aligned} \text{于是 } \int \frac{dx}{x(ax+b)} &= \frac{1}{b^2} \int \frac{1}{x} dx - \frac{a}{b^2} \int \frac{1}{ax+b} dx - \frac{a}{b} \int \frac{1}{(ax+b)^2} dx \\ &= \frac{1}{b^2} \cdot \ln |x| - \frac{1}{b^2} \cdot \ln |ax+b| + \frac{1}{b} \cdot \frac{1}{ax+b} + C \\ &= \frac{1}{b(ax+b)} - \frac{1}{b^2} \cdot \ln \left| \frac{ax+b}{x} \right| + C \end{aligned}$$

(二) 含有 $\sqrt{ax+b}$ 的积分 (10~18)

$$10. \int \sqrt{ax+b} dx = \frac{2}{3a} \cdot \sqrt{(ax+b)^3} + C$$

$$\begin{aligned} \text{证明: } \int \sqrt{ax+b} dx &= \frac{1}{a} \int (ax+b)^{\frac{1}{2}} d(ax+b) = \frac{1}{a} \cdot \frac{1}{1+\frac{1}{2}} \cdot (ax+b)^{\frac{1}{2}+1} + C \\ &= \frac{2}{3a} \cdot \sqrt{(ax+b)^3} + C \end{aligned}$$

$$11. \int x\sqrt{ax+b} dx = \frac{2}{15a^2} \cdot (3ax-2b) \cdot \sqrt{(ax+b)^3} + C$$

$$\text{证明: 令 } \sqrt{ax+b} = t \quad (t \geq 0), \text{ 则 } x = \frac{t^2-b}{a}, \quad dx = \frac{2t}{a} dt, \quad x\sqrt{ax+b} = \frac{t^2-b}{a} \cdot t$$

$$\begin{aligned} \therefore \int x\sqrt{ax+b} dx &= \int \frac{t^2-b}{a} \cdot t \cdot \frac{2t}{a} dt = \frac{2}{a^2} \int (t^4 - bt^2) dt \\ &= \frac{2}{5a^2} \int d(t^5) - \frac{2b}{3a^2} \int d(t^3) = \frac{2}{5a^2} \cdot t^5 - \frac{2b}{3a^2} \cdot t^3 + C \\ &= \frac{2t^3}{15a^2} (3t^2 - 5b) + C \end{aligned}$$

$$\begin{aligned} \text{将 } t = \sqrt{ax+b} \text{ 代入上式得: } \int x\sqrt{ax+b} dx &= \frac{2}{15a^2} [3(ax+b) - 5b] \cdot \sqrt{(ax+b)^3} + C \\ &= \frac{2}{15a^2} \cdot (3ax-2b) \cdot \sqrt{(ax+b)^3} + C \end{aligned}$$

$$12. \int x^2 \sqrt{ax+b} dx = \frac{2}{105a^3} \cdot (15a^2x^2 - 12abx + 8b^2) \cdot \sqrt{(ax+b)^3} + C$$

$$\text{证明: 令 } \sqrt{ax+b} = t \quad (t \geq 0), \text{ 则 } x = \frac{t^2-b}{a}, \quad dx = \frac{2t}{a} dt,$$

$$x^2 \sqrt{ax+b} = \frac{(t^2-b)^2}{a^2} \cdot t = \frac{t^5 + b^2t - 2bt^3}{a^2}$$

$$\begin{aligned} \therefore \int x^2 \sqrt{ax+b} dx &= \frac{2}{a^3} \int t \cdot (t^5 + b^2t - 2bt^3) dt \\ &= \frac{2}{a^3} \int t^6 dt - \frac{2b^2}{a^3} \int t^2 dt - \frac{4b}{a^3} \int t^4 dt \\ &= \frac{2}{a^3} \cdot \frac{1}{1+6} \cdot t^{6+1} + \frac{2b^2}{a^3} \cdot \frac{1}{1+2} \cdot t^{1+2} - \frac{4b}{a^3} \cdot \frac{1}{1+4} \cdot t^{4+1} + C \\ &= \frac{2}{7a^3} \cdot t^7 + \frac{2b^2}{3a^3} \cdot t^3 - \frac{4b}{5a^3} \cdot t^5 + C \\ &= \frac{2t^3}{105a^3} \cdot (15t^4 + 35b^2 - 42bt^2) + C \end{aligned}$$

将 $t = \sqrt{ax+b}$ 代入上式得:

$$\begin{aligned} \int x^2 \sqrt{ax+b} dx &= \frac{2}{105a^3} \cdot \sqrt{(ax+b)^3} [15a^2x^2 + 15b^2 + 30abx + 35b^2 - 42b \cdot (ax+b)] \\ &= \frac{2}{105a^3} \cdot (15a^2x^2 - 12abx + 8b^2) \cdot \sqrt{(ax+b)^3} + C \end{aligned}$$

$$13. \int \frac{x}{\sqrt{ax+b}} dx = \frac{2}{3a^2} \cdot (ax-2b) \cdot \sqrt{(ax+b)} + C$$

证明：令 $\sqrt{ax+b} = t$ ($t > 0$)，则 $x = \frac{t^2-b}{a}$ ， $dx = \frac{2t}{a} dt$ ，

$$\begin{aligned} \therefore \int \frac{x}{\sqrt{ax+b}} dx &= \int \frac{t^2-b}{at} \cdot \frac{2t}{a} dt \\ &= \frac{2}{a^2} \int t^2 dt - \frac{2}{a^2} \int b dt \\ &= \frac{2}{a^2} \cdot \frac{1}{1+2} \cdot t^{2+1} - \frac{2b}{a^2} \cdot t + C \\ &= \frac{2}{3a^2} \cdot t^3 - \frac{2b}{a^2} \cdot t + C \end{aligned}$$

$$\begin{aligned} \text{将 } t = \sqrt{ax+b} \text{ 代入上式得：} \int \frac{x}{\sqrt{ax+b}} dx &= \frac{2}{3a^2} \cdot (ax+b) \cdot \sqrt{(ax+b)} - \frac{2b}{a^2} \cdot \sqrt{(ax+b)} + C \\ &= \frac{2}{3a^2} \cdot (ax-2b) \cdot \sqrt{(ax+b)} + C \end{aligned}$$

$$14. \int \frac{x^2}{\sqrt{ax+b}} dx = \frac{2}{15a^3} \cdot (3a^2x^2 - 4abx + 8b^2) \cdot \sqrt{(ax+b)} + C$$

证明：令 $\sqrt{ax+b} = t$ ($t > 0$)，则 $x = \frac{t^2-b}{a}$ ， $dx = \frac{2t}{a} dt$ ，

$$\begin{aligned} \therefore \int \frac{x^2}{\sqrt{ax+b}} dx &= \int \left(\frac{t^2-b}{a}\right)^2 \cdot \frac{1}{t} \cdot \frac{2t}{a} dt \\ &= \frac{2}{a^3} \int (t^4 + b^2 - 2bt^2) dt \\ &= \frac{2}{a^3} \int t^4 dt + \frac{2}{a^3} \int b^2 dt - \frac{4b}{a^3} \int t^2 dt \\ &= \frac{2}{a^3} \left(\frac{1}{5} t^5 + b^2 t - \frac{2b}{3} t^3 \right) + C \\ &= \frac{2t}{15a^3} \cdot (3t^4 + 15b^2 - 10bt^2) + C \end{aligned}$$

将 $t = \sqrt{ax+b}$ 代入上式得：

$$\begin{aligned} \int \frac{x^2}{\sqrt{ax+b}} dx &= \frac{2}{15a^3} \cdot \sqrt{(ax+b)} \cdot [3(a^2x^2 + b^2 + 2abx) + 15b^2 - 10b \cdot (ax+b)] \cdot \sqrt{(ax+b)} + C \\ &= \frac{2}{15a^3} \cdot (3a^2x^2 - 4abx + 8b^2) \cdot \sqrt{(ax+b)} + C \end{aligned}$$

$$15. \int \frac{dx}{x\sqrt{ax+b}} = \begin{cases} \frac{1}{\sqrt{b}} \cdot \ln \left| \frac{\sqrt{ax+b} - \sqrt{b}}{\sqrt{ax+b} + \sqrt{b}} \right| + C & (b > 0) \\ \frac{2}{\sqrt{-b}} \cdot \arctan \sqrt{\frac{ax+b}{-b}} + C & (b < 0) \end{cases}$$

证明：令 $\sqrt{ax+b} = t$ ($t > 0$)，则 $x = \frac{t^2 - b}{a}$ ， $dx = \frac{2t}{a} dt$ ，

$$\begin{aligned} \therefore \int \frac{dx}{x\sqrt{ax+b}} &= \int \frac{1}{\frac{t^2 - b}{a} \cdot t} \cdot \frac{2t}{a} dt \\ &= \int \frac{2}{t^2 - b} dt \end{aligned}$$

$$\begin{aligned} 1. \text{当 } b > 0 \text{ 时, } \int \frac{2}{t^2 - b} dt &= 2 \int \frac{1}{t^2 - (\sqrt{b})^2} dt \\ &= \frac{1}{\sqrt{b}} \cdot \ln \left| \frac{t - \sqrt{b}}{t + \sqrt{b}} \right| + C \end{aligned}$$

$$\text{公式 21: } \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \cdot \ln \left| \frac{x-a}{x+a} \right| + C$$

$$\text{将 } t = \sqrt{ax+b} \text{ 代入上式得: } \int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}} \cdot \ln \left| \frac{\sqrt{ax+b} - \sqrt{b}}{\sqrt{ax+b} + \sqrt{b}} \right| + C$$

$$\begin{aligned} 2. \text{当 } b < 0 \text{ 时, } \int \frac{2}{t^2 - b} dt &= 2 \int \frac{1}{t^2 + (\sqrt{-b})^2} dt \\ &= \frac{2}{\sqrt{-b}} \cdot \arctan \frac{t}{\sqrt{-b}} + C \end{aligned}$$

$$\text{公式 19: } \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \cdot \arctan \frac{x}{a} + C$$

$$\text{将 } t = \sqrt{ax+b} \text{ 代入上式得: } \int \frac{dx}{x\sqrt{ax+b}} = \frac{2}{\sqrt{-b}} \cdot \arctan \sqrt{\frac{ax+b}{-b}} + C$$

$$\text{综合讨论 1, 2 得: } \int \frac{dx}{x\sqrt{ax+b}} = \begin{cases} \frac{1}{\sqrt{b}} \cdot \ln \left| \frac{\sqrt{ax+b} - \sqrt{b}}{\sqrt{ax+b} + \sqrt{b}} \right| + C & (b > 0) \\ \frac{2}{\sqrt{-b}} \cdot \arctan \sqrt{\frac{ax+b}{-b}} + C & (b < 0) \end{cases}$$

$$16. \int \frac{dx}{x^2 \sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b} \int \frac{dx}{x \sqrt{ax+b}}$$

证明：设 $\frac{1}{x^2 \cdot \sqrt{ax+b}} = \frac{A}{x\sqrt{ax+b}} + \frac{B\sqrt{ax+b}}{x^2}$ ，则 $1 = Ax + B(ax+b)$

$$\therefore \text{有} \begin{cases} A + Ba = 0 \\ Bb = 1 \end{cases} \Rightarrow \begin{cases} A = -\frac{a}{b} \\ B = \frac{1}{b} \end{cases}$$

$$\begin{aligned} \text{于是} \int \frac{dx}{x^2 \sqrt{ax+b}} &= -\frac{a}{b} \int \frac{1}{x\sqrt{ax+b}} dx + \frac{1}{b} \int \frac{\sqrt{ax+b}}{x^2} dx \\ &= -\frac{a}{b} \int \frac{1}{x\sqrt{ax+b}} dx - \frac{1}{b} \int \sqrt{ax+b} d\frac{1}{x} \\ &= -\frac{a}{b} \int \frac{1}{x\sqrt{ax+b}} dx - \frac{\sqrt{ax+b}}{bx} + \frac{1}{b} \int \frac{1}{x} d\sqrt{ax+b} \\ &= -\frac{a}{b} \int \frac{1}{x\sqrt{ax+b}} dx - \frac{\sqrt{ax+b}}{bx} + \frac{1}{b} \int \frac{1}{x} \cdot \frac{a}{2} (ax+b)^{-\frac{1}{2}} dx \\ &= -\frac{a}{b} \int \frac{1}{x\sqrt{ax+b}} dx - \frac{\sqrt{ax+b}}{bx} + \frac{a}{2b} \int \frac{1}{x\sqrt{ax+b}} dx \\ &= -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b} \int \frac{dx}{x\sqrt{ax+b}} \end{aligned}$$

$$17. \int \frac{\sqrt{ax+b}}{x} dx = 2\sqrt{ax+b} + b \int \frac{dx}{x\sqrt{ax+b}}$$

证明：令 $\sqrt{ax+b} = t$ ($t \geq 0$)，则 $x = \frac{t^2-b}{a}$ ， $dx = \frac{2t}{a} dt$

$$\begin{aligned} \therefore \int \frac{\sqrt{ax+b}}{x} dx &= \int \frac{at}{t^2-b} \cdot \frac{2t}{a} dt = 2 \int \frac{t^2}{t^2-b} dt \\ &= 2 \int \frac{t^2-b^2+b^2}{t^2-b} dt = 2 \int dt + 2b \int \frac{1}{t^2-b} dt \\ &= 2t + 2b \int \frac{1}{t^2-b} dt \end{aligned}$$

$\because b$ 取值为 R ，符号可正可负 $\therefore \int \frac{1}{t^2-b} dt$ 不能明确积分

$$\begin{aligned} \therefore \int \frac{\sqrt{ax+b}}{x} dx &= 2t + 2b \int \frac{1}{t^2-b} dt \\ &= 2t + 2b \int \frac{1}{t^2-b} \cdot \frac{a}{2t} dx \end{aligned}$$

$$\begin{aligned} \text{将 } t = \sqrt{ax+b} \text{ 代入上式得：} \int \frac{\sqrt{ax+b}}{x} dx &= 2\sqrt{(ax+b)} + 2b \int \frac{1}{ax+b-b} \cdot \frac{a}{2\sqrt{ax+b}} dx \\ &= 2\sqrt{ax+b} + b \int \frac{dx}{x\sqrt{ax+b}} \end{aligned}$$

$$18. \int \frac{\sqrt{ax+b}}{x^2} dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2} \int \frac{dx}{x\sqrt{ax+b}}$$

$$\begin{aligned} \text{证明: } \int \frac{\sqrt{ax+b}}{x^2} dx &= -\int \sqrt{ax+b} d\frac{1}{x} \\ &= -\frac{\sqrt{ax+b}}{x} + \int \frac{1}{x} d\sqrt{ax+b} \\ &= -\frac{\sqrt{ax+b}}{x} + \int \frac{1}{x} \cdot (ax+b)^{-\frac{1}{2}} \cdot \frac{a}{2} dx \\ &= -\frac{\sqrt{ax+b}}{x} + \frac{a}{2} \int \frac{dx}{x\sqrt{ax+b}} \end{aligned}$$

(三) 含有 $x^2 \pm a^2$ 的积分 (19~21)

$$19. \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \cdot \arctan \frac{x}{a} + C$$

证明: 令 $x = a \cdot \tan t$ ($-\frac{\pi}{2} < t < \frac{\pi}{2}$), 则 $dx = d(a \cdot \tan t) = a \cdot \sec^2 t dt$

$$\frac{1}{x^2 + a^2} = \frac{dx}{a^2 \cdot (1 + \tan^2 t)} = \frac{1}{a^2 \sec^2 t}$$

$$\begin{aligned} \therefore \int \frac{dx}{x^2 + a^2} &= \int \frac{1}{a^2 \sec^2 t} \cdot a \cdot \sec^2 t dt \\ &= \frac{1}{a} \int dt \\ &= \frac{1}{a} \cdot t + C \end{aligned}$$

$$\because x = a \cdot \tan t \quad \therefore t = \arctan \frac{x}{a}$$

$$\text{将 } t = \arctan \frac{x}{a} \text{ 代入上式得: } \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \cdot \arctan \frac{x}{a} + C$$

$$20. \int \frac{dx}{(x^2 + a^2)^n} = \frac{x}{2(n-1) \cdot a^2 \cdot (x^2 + a^2)^{n-1}} + \frac{2n-3}{2(n-1) \cdot a^2} \int \frac{dx}{(x^2 + a^2)^{n-1}}$$

$$\text{证明: } \int \frac{dx}{(x^2 + a^2)^n} = \frac{x}{(x^2 + a^2)^n} - \int x d \frac{1}{(x^2 + a^2)^n}$$

$$= \frac{x}{(x^2 + a^2)^n} - \int x \cdot (-n) \cdot (x^2 + a^2)^{-n-1} \cdot 2x dx$$

$$= \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{x^2}{(x^2 + a^2)^{n+1}} dx$$

$$= \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{x^2 + a^2 - a^2}{(x^2 + a^2)^{n+1}} dx$$

$$= \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{1}{(x^2 + a^2)^n} dx - 2na^2 \int \frac{1}{(x^2 + a^2)^{n+1}} dx$$

$$\text{移项并整理得: } (1-2n) \int \frac{dx}{(x^2 + a^2)^n} = \frac{x}{(x^2 + a^2)^n} - 2na^2 \int \frac{1}{(x^2 + a^2)^{n+1}} dx$$

$$\therefore \int \frac{1}{(x^2 + a^2)^{n+1}} dx = \frac{1}{2na^2} \left[\frac{x}{(x^2 + a^2)^n} + (2n-1) \int \frac{dx}{(x^2 + a^2)^n} \right]$$

$$\begin{aligned} \text{令 } n+1=n, \text{ 则 } \int \frac{dx}{(x^2 + a^2)^n} &= \frac{1}{2(n-1) \cdot a^2} \left[\frac{x}{(x^2 + a^2)^{n-1}} + (2n-3) \int \frac{dx}{(x^2 + a^2)^{n-1}} \right] \\ &= \frac{x}{2(n-1) \cdot a^2 \cdot (x^2 + a^2)^{n-1}} + \frac{2n-3}{2(n-1) \cdot a^2} \int \frac{dx}{(x^2 + a^2)^{n-1}} \end{aligned}$$

$$21. \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \cdot \ln \left| \frac{x-a}{x+a} \right| + C$$

$$\text{证明: } \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \int \left[\frac{1}{x-a} - \frac{1}{x+a} \right] dx$$

$$= \frac{1}{2a} \int \frac{1}{x-a} dx - \frac{1}{2a} \int \frac{1}{x+a} dx$$

$$= \frac{1}{2a} \cdot \ln |x-a| - \frac{1}{2a} \cdot \ln |x+a| + C$$

$$= \frac{1}{2a} \cdot \ln \left| \frac{x-a}{x+a} \right| + C$$

(四) 含有 $ax^2 + b$ ($a > 0$) 的积分 (22~28)

$$22. \int \frac{dx}{ax^2 + b} = \begin{cases} \frac{1}{\sqrt{ab}} \cdot \arctan \sqrt{\frac{a}{b}} \cdot x + C & (b > 0) \\ \frac{1}{2\sqrt{-ab}} \cdot \ln \left| \frac{\sqrt{a} \cdot x - \sqrt{-b}}{\sqrt{a} \cdot x + \sqrt{-b}} \right| + C & (b < 0) \end{cases} \quad (a > 0)$$

证明:

$$1. \text{当 } b > 0 \text{ 时, } \frac{1}{ax^2 + b} = \frac{1}{x^2 + \frac{b}{a}} \cdot \frac{1}{a} = \frac{1}{x^2 + (\sqrt{\frac{b}{a}})^2} \cdot \frac{1}{a}$$

$$\begin{aligned} \therefore \int \frac{dx}{ax^2 + b} &= \frac{1}{a} \int \frac{1}{x^2 + (\sqrt{\frac{b}{a}})^2} dx \\ &= \frac{1}{a} \cdot \sqrt{\frac{a}{b}} \cdot \arctan \sqrt{\frac{a}{b}} \cdot x + C \\ &= \frac{1}{\sqrt{ab}} \cdot \arctan \sqrt{\frac{a}{b}} \cdot x + C \end{aligned}$$

$$2. \text{当 } b < 0 \text{ 时, } \frac{1}{ax^2 + b} = \frac{1}{x^2 - (-\frac{b}{a})} \cdot \frac{1}{a} = \frac{1}{x^2 - (\sqrt{-\frac{b}{a}})^2} \cdot \frac{1}{a}$$

$$\begin{aligned} \therefore \int \frac{dx}{ax^2 + b} &= \frac{1}{a} \int \frac{1}{x^2 - (\sqrt{-\frac{b}{a}})^2} dx \\ &= \frac{1}{2\sqrt{-\frac{b}{a}}} \cdot \frac{1}{a} \cdot \ln \left| \frac{x - \sqrt{-\frac{b}{a}}}{x + \sqrt{-\frac{b}{a}}} \right| + C \\ &= \frac{1}{2\sqrt{-ab}} \cdot \ln \left| \frac{\sqrt{a} \cdot x - \sqrt{-b}}{\sqrt{a} \cdot x + \sqrt{-b}} \right| + C \end{aligned}$$

$$\text{综合讨论 1, 2 得: } \int \frac{dx}{ax^2 + b} = \begin{cases} \frac{1}{\sqrt{ab}} \cdot \arctan \sqrt{\frac{a}{b}} \cdot x + C & (b > 0) \\ \frac{1}{2\sqrt{-ab}} \cdot \ln \left| \frac{\sqrt{a} \cdot x - \sqrt{-b}}{\sqrt{a} \cdot x + \sqrt{-b}} \right| + C & (b < 0) \end{cases}$$

$$23. \int \frac{x}{ax^2 + b} dx = \frac{1}{2a} \cdot \ln |ax^2 + b| + C \quad (a > 0)$$

$$\begin{aligned} \text{证明: } \int \frac{x}{ax^2 + b} dx &= \frac{1}{2} \int \frac{1}{ax^2 + b} d(x^2) \\ &= \frac{1}{2a} \int \frac{1}{ax^2 + b} d(ax^2 + b) \\ &= \frac{1}{2a} \cdot \ln |ax^2 + b| + C \end{aligned}$$

$$24. \int \frac{x^2}{ax^2+b} dx = \frac{x}{a} - \frac{b}{a} \int \frac{dx}{ax^2+b} \quad (a > 0)$$

$$\begin{aligned} \text{证明: } \int \frac{x^2}{ax^2+b} dx &= \frac{b}{a} \int \frac{ax^2}{ax^2+b} \cdot \frac{1}{b} dx \\ &= \frac{b}{a} \int \left(\frac{1}{b} - \frac{1}{ax^2+b} \right) dx \\ &= \frac{b}{a} \int \frac{1}{b} dx - \frac{b}{a} \int \frac{1}{ax^2+b} dx \\ &= \frac{x}{a} - \frac{b}{a} \int \frac{dx}{ax^2+b} \end{aligned}$$

$$25. \int \frac{dx}{x(ax^2+b)} = \frac{1}{2b} \cdot \ln \left| \frac{x^2}{ax^2+b} \right| + C \quad (a > 0)$$

$$\begin{aligned} \text{证明: } \int \frac{dx}{x(ax^2+b)} &= \int \frac{x}{x^2(ax^2+b)} dx \\ &= \frac{1}{2} \int \frac{1}{x^2(ax^2+b)} d(x^2) \end{aligned}$$

$$\text{设: } \frac{1}{x^2(ax^2+b)} = \frac{A}{x^2} + \frac{B}{ax^2+b}$$

$$\text{则 } 1 = A(ax^2+b) + Bx^2 = x^2(Aa+B) + Ab$$

$$\therefore \text{ 有 } \begin{cases} Aa+B=0 \\ Ab=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{b} \\ B=-\frac{a}{b} \end{cases}$$

$$\begin{aligned} \text{于是 } \int \frac{dx}{x(ax^2+b)} &= \frac{1}{2} \int \left[\frac{1}{bx^2} - \frac{a}{b(ax^2+b)} \right] d(x^2) \\ &= \frac{1}{2b} \int \frac{1}{x^2} dx^2 - \frac{a}{2b} \int \frac{1}{ax^2+b} d(x^2) \\ &= \frac{1}{2b} \int \frac{1}{x^2} dx^2 - \frac{1}{2b} \int \frac{1}{ax^2+b} d(ax^2+b) \\ &= \frac{1}{2b} \cdot \ln |x^2| - \frac{1}{2b} \ln |ax^2+b| + C \\ &= \frac{1}{2b} \cdot \ln \frac{x^2}{|ax^2+b|} + C \end{aligned}$$

$$26. \int \frac{dx}{x^2(ax^2+b)} = -\frac{1}{bx} - \frac{a}{b} \int \frac{dx}{ax^2+b} \quad (a > 0)$$

证明：设： $\frac{1}{x^2(ax^2+b)} = \frac{A}{x^2} + \frac{B}{ax^2+b}$

则 $1 = A(ax^2+b) + Bx^2 = x^2(Aa+B) + Ab$

$$\therefore \text{有} \begin{cases} Aa+B=0 \\ Ab=1 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{b} \\ B = \frac{a}{b} \end{cases}$$

$$\begin{aligned} \text{于是} \int \frac{dx}{x^2(ax^2+b)} &= \int \left[-\frac{1}{bx^2} + \frac{a}{b(ax^2+b)} \right] dx \\ &= -\frac{1}{b} \int \frac{1}{x^2} dx + \frac{a}{b} \int \frac{1}{ax^2+b} dx \\ &= -\frac{1}{bx} - \frac{a}{b} \int \frac{dx}{ax^2+b} \end{aligned}$$

$$27. \int \frac{dx}{x^3(ax^2+b)} = \frac{a}{2b^2} \ln \frac{|ax^2+b|}{x^2} - \frac{1}{2bx^2} + C \quad (a > 0)$$

证明： $\int \frac{dx}{x^3(ax^2+b)} = \int \frac{x}{x^4(ax^2+b)} dx$

$$= \frac{1}{2} \int \frac{1}{x^4(ax^2+b)} d(x^2)$$

设： $\frac{1}{x^4(ax^2+b)} = \frac{A}{x^2} + \frac{B}{x^4} + \frac{C}{ax^2+b}$

则 $1 = Ax^2(ax^2+b) + B(ax^2+b) + Cx^4$

$$= (Aa+C)x^4 + (Ab+Ba)x^2 + Bb$$

$$\therefore \text{有} \begin{cases} Aa+C=0 \\ Ab+Ba=0 \\ Bb=1 \end{cases} \Rightarrow \begin{cases} B = \frac{1}{b} \\ A = -\frac{a}{b^2} \\ C = \frac{a^2}{b^2} \end{cases}$$

$$\begin{aligned} \text{于是} \int \frac{dx}{x^3(ax^2+b)} &= -\frac{a}{2b^2} \int \frac{1}{x^2} d(x^2) + \frac{1}{2b} \int \frac{1}{x^4} d(x^2) + \frac{a^2}{2b^2} \int \frac{1}{ax^2+b} d(x^2) \\ &= -\frac{a}{2b^2} \ln |x^2| - \frac{1}{2bx^2} + \frac{a}{2b^2} \ln |ax^2+b| + C \\ &= \frac{a}{2b^2} \ln \frac{|ax^2+b|}{x^2} - \frac{1}{2bx^2} + C \end{aligned}$$

$$28. \int \frac{dx}{(ax^2+b)^2} = \frac{x}{2b(ax^2+b)} + \frac{1}{2b} \int \frac{dx}{ax^2+b} \quad (a > 0)$$

$$\begin{aligned} \text{证明: } \int \frac{dx}{(ax^2+b)^2} &= -\int \frac{1}{2ax} d \frac{1}{ax^2+b} = -\frac{1}{2ax} \cdot \frac{1}{ax^2+b} + \int \frac{1}{ax^2+b} d \frac{1}{2ax} \\ &= -\frac{1}{2ax} \cdot \frac{1}{ax^2+b} - \int \frac{1}{ax^2+b} \cdot \frac{1}{2ax^2} dx \end{aligned}$$

$$\text{设: } \frac{1}{2ax^2(ax^2+b)} = \frac{A}{2ax^2} + \frac{B}{ax^2+b}, \text{ 则 } 1 = A(ax^2+b) + 2Bax^2 = (Aa+2Ba)x^2 + Ab$$

$$\therefore \text{ 有 } \begin{cases} Aa+2Ba=0 \\ Ab=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{b} \\ B=-\frac{1}{2b} \end{cases}$$

$$\begin{aligned} \text{于是上式} &= -\frac{1}{2ax(ax^2+b)} - \int \left(\frac{1}{2abx^2} - \frac{1}{2b(ax^2+b)} \right) dx \\ &= -\frac{1}{2ax(ax^2+b)} - \frac{1}{2ab} \int \frac{1}{x^2} dx + \frac{1}{2b} \int \frac{1}{2b(ax^2+b)} dx \\ &= -\frac{1}{2ax(ax^2+b)} + \frac{1}{2abx} + \frac{1}{2b} \int \frac{1}{2b(ax^2+b)} dx = \frac{ax^2+b-b}{2abx(ax^2+b)} + \frac{1}{2b} \int \frac{1}{2b(ax^2+b)} dx \\ &= \frac{x}{2b(ax^2+b)} + \frac{1}{2b} \int \frac{dx}{ax^2+b} \end{aligned}$$

(五) 含有 ax^2+bx+c ($a > 0$) 的积分 (29~30)

$$29. \int \frac{dx}{ax^2+bx+c} = \begin{cases} \frac{2}{\sqrt{4ac-b^2}} \cdot \arctan \frac{2ax+b}{\sqrt{4ac-b^2}} + C & (b^2 < 4ac) \\ \frac{1}{\sqrt{b^2-4ac}} \cdot \ln \left| \frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}} \right| + C & (b^2 > 4ac) \end{cases} \quad (a > 0)$$

$$\text{证明: } \because ax^2+bx+c = \frac{1}{4a} [(2ax+b)^2 + (4ac-b^2)]$$

$$\therefore \int \frac{dx}{ax^2+bx+c} = 4a \int \frac{1}{(2ax+b)^2 + (4ac-b^2)} dx$$

$$1. \text{ 当 } b^2 < 4ac \text{ 时, } \int \frac{dx}{ax^2+bx+c} = 4a \int \frac{1}{(2ax+b)^2 + (\sqrt{4ac-b^2})^2} dx$$

$$\begin{aligned} \boxed{\text{公式 19: } \int \frac{dx}{x^2+a^2} = \frac{1}{a} \cdot \arctan \frac{x}{a} + C} &= \frac{4a}{2a} \int \frac{1}{(2ax+b)^2 + (\sqrt{4ac-b^2})^2} d(2ax+b) \\ &= \frac{2}{\sqrt{4ac-b^2}} \cdot \arctan \frac{2ax+b}{\sqrt{4ac-b^2}} + C \end{aligned}$$

$$2. \text{ 当 } b^2 > 4ac \text{ 时, } \int \frac{dx}{ax^2+bx+c} = 4a \int \frac{1}{(2ax+b)^2 + (4ac-b^2)} dx$$

$$\begin{aligned} &= 4a \int \frac{1}{(2ax+b)^2 - (b^2-4ac)} dx \\ &= \frac{4a}{2a} \int \frac{1}{(2ax+b)^2 - (\sqrt{b^2-4ac})^2} d(2ax+b) \\ \boxed{\text{公式 21: } \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \cdot \ln \left| \frac{x-a}{x+a} \right| + C} &= \frac{1}{\sqrt{b^2-4ac}} \cdot \ln \left| \frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}} \right| + C \end{aligned}$$

$$\text{综合讨论 1, 2 得: } \int \frac{dx}{ax^2+bx+c} = \begin{cases} \frac{2}{\sqrt{4ac-b^2}} \cdot \arctan \frac{2ax+b}{\sqrt{4ac-b^2}} + C & (b^2 < 4ac) \\ \frac{1}{\sqrt{b^2-4ac}} \cdot \ln \left| \frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}} \right| + C & (b^2 > 4ac) \end{cases}$$

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$$30. \int \frac{x}{ax^2+bx+c} dx = \frac{1}{2a} \cdot \ln |ax^2+bx+c| - \frac{b}{2a} \int \frac{dx}{ax^2+bx+c} \quad (a > 0)$$

$$\begin{aligned} \text{证明: } \int \frac{x}{ax^2+bx+c} dx &= \int \frac{1}{2a} \cdot \frac{2ax+b-b}{ax^2+bx+c} dx \\ &= \frac{1}{2a} \int \frac{2ax+b}{ax^2+bx+c} dx + \frac{1}{2a} \int \frac{-b}{ax^2+bx+c} dx \\ &= \frac{1}{2a} \int \frac{1}{ax^2+bx+c} d(ax^2+bx+c) - \frac{b}{2a} \int \frac{1}{ax^2+bx+c} dx \\ &= \frac{1}{2a} \cdot \ln |ax^2+bx+c| - \frac{b}{2a} \int \frac{dx}{ax^2+bx+c} \end{aligned}$$

(六) 含有 $\sqrt{x^2+a^2}$ ($a > 0$) 的积分 (31~44)

$$31. \int \frac{dx}{\sqrt{x^2+a^2}} = \operatorname{arsh} \frac{x}{a} + C_1 = \ln(x + \sqrt{x^2+a^2}) + C \quad (a > 0)$$

证明: 被积函数 $f(x) = \frac{1}{\sqrt{x^2+a^2}}$ 的定义域为 $\{x/x \in R\}$

可令 $x = a \tan t$ ($-\frac{\pi}{2} < t < \frac{\pi}{2}$), 则 $dx = d(a \tan t) = a \sec^2 t dt$, $\sqrt{x^2+a^2} = |a \sec t|$

$$\because -\frac{\pi}{2} < t < \frac{\pi}{2}, \sec t = \frac{1}{\cos t} > 0, \therefore \sqrt{x^2+a^2} = a \sec t$$

$$\begin{aligned} \therefore \int \frac{dx}{\sqrt{x^2+a^2}} &= \int \frac{1}{a \sec t} \cdot a \sec^2 t dt \quad \boxed{\text{公式 87: } \int \sec t dt = \ln |\sec t + \tan t| + C} \\ &= \int \sec t dt \\ &= \ln |\sec t + \tan t| + C_2 \end{aligned}$$

在 $Rt\triangle ABC$ 中, 设 $\angle B = t$, $|BC| = a$, 则 $|AC| = x$, $|AB| = \sqrt{x^2+a^2}$

$$\therefore \sec t = \frac{1}{\cos t} = \frac{\sqrt{x^2+a^2}}{a}, \tan t = \frac{x}{a}$$

$$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \ln |\sec t + \tan t| + C_2$$

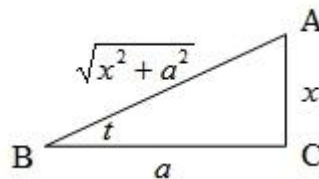
$$= \ln \left| \frac{\sqrt{x^2+a^2} + x}{a} \right| + C_2$$

$$= \ln |\sqrt{x^2+a^2} + x| - \ln a + C_2$$

$$= \ln |\sqrt{x^2+a^2} + x| + C_3$$

$$\because \sqrt{x^2+a^2} + x > 0$$

$$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2}) + C$$



$$32. \int \frac{dx}{\sqrt{(x^2 + a^2)^3}} = \frac{x}{a^2 \sqrt{x^2 + a^2}} + C \quad (a > 0)$$

证明: 被积函数 $f(x) = \frac{1}{\sqrt{(x^2 + a^2)^3}}$ 的定义域为 $\{x | x \in R\}$

可令 $x = a \tan t \quad (-\frac{\pi}{2} < t < \frac{\pi}{2})$, 则 $dx = d(a \tan t) = a \sec^2 t dt$, $\sqrt{(x^2 + a^2)^3} = |a^3 \sec^3 t|$

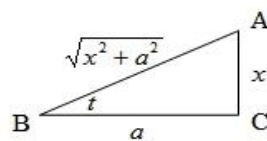
$$\because -\frac{\pi}{2} < t < \frac{\pi}{2}, \sec t = \frac{1}{\cos t} > 0, \therefore \sqrt{(x^2 + a^2)^3} = a^3 \sec^3 t$$

$$\begin{aligned} \therefore \int \frac{dx}{\sqrt{(x^2 + a^2)^3}} &= \int \frac{1}{a^3 \sec^3 t} \cdot a \sec^2 t dt = \frac{1}{a^2} \int \frac{1}{\sec t} dt \\ &= \frac{1}{a^2} \int \cos t dt = \frac{1}{a^2} \sin t + C \end{aligned}$$

在 $Rt\triangle ABC$ 中, 设 $\angle B = t$, $|BC| = a$, 则 $|AC| = x$, $|AB| = \sqrt{x^2 + a^2}$

$$\therefore \sin t = \frac{|AC|}{|AB|} = \frac{x}{\sqrt{x^2 + a^2}}$$

$$\therefore \int \frac{dx}{\sqrt{(x^2 + a^2)^3}} = \frac{1}{a^2} \cdot \sin t + C = \frac{x}{a^2 \sqrt{x^2 + a^2}} + C$$



$$33. \int \frac{x}{\sqrt{x^2 + a^2}} dx = \sqrt{x^2 + a^2} + C \quad (a > 0)$$

证明: 令 $\sqrt{x^2 + a^2} = t \quad (t > 0)$, 则 $x = \sqrt{t^2 - a^2}$

$$\therefore dx = \frac{1}{2}(t^2 - a^2)^{-\frac{1}{2}} \cdot 2t dt = \frac{t}{\sqrt{t^2 - a^2}} dt$$

$$\begin{aligned} \therefore \int \frac{x}{\sqrt{x^2 + a^2}} dx &= \int \frac{\sqrt{t^2 - a^2}}{t} \cdot \frac{t}{\sqrt{t^2 - a^2}} dt \\ &= \int dt = t + C \end{aligned}$$

将 $t = \sqrt{x^2 + a^2}$ 代入上式得: $\int \frac{x}{\sqrt{x^2 + a^2}} dx = \sqrt{x^2 + a^2} + C$

$$34. \int \frac{x}{\sqrt{(x^2 + a^2)^3}} dx = -\frac{1}{\sqrt{x^2 + a^2}} + C \quad (a > 0)$$

$$\text{证明: } \int \frac{x}{\sqrt{(x^2 + a^2)^3}} dx = \int x \cdot (x^2 + a^2)^{-\frac{3}{2}} dx = \frac{1}{2} \int (x^2 + a^2)^{-\frac{3}{2}} dx^2$$

$$= \frac{1}{2} \int (x^2 + a^2)^{-\frac{3}{2}} d(x^2 + a^2)$$

$$= \frac{1}{2} \times \frac{1}{1 - \frac{3}{2}} \cdot (x^2 + a^2)^{1 - \frac{3}{2}} + C$$

$$= -\frac{1}{\sqrt{x^2 + a^2}} + C$$

$$35. \int \frac{x^2}{\sqrt{x^2+a^2}} dx = \frac{x}{2} \cdot \sqrt{x^2+a^2} - \frac{a^2}{2} \ln(x+\sqrt{x^2+a^2}) + C \quad (a>0)$$

$$\text{证明: } \int \frac{x^2}{\sqrt{x^2+a^2}} dx = \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}} dx$$

$$= \int \sqrt{x^2+a^2} dx - a^2 \int \frac{1}{\sqrt{x^2+a^2}} dx$$

$$\therefore \int \sqrt{x^2+a^2} dx = \frac{x}{2} \cdot \sqrt{x^2+a^2} + \frac{a^2}{2} \cdot \ln(x+\sqrt{x^2+a^2}) + C \quad (\text{公式39})$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln(x+\sqrt{x^2+a^2}) + C \quad (\text{公式31})$$

$$\begin{aligned} \therefore \int \frac{x^2}{\sqrt{x^2+a^2}} dx &= \frac{x}{2} \cdot \sqrt{x^2+a^2} + \frac{a^2}{2} \ln(x+\sqrt{x^2+a^2}) - a^2 \cdot \ln(x+\sqrt{x^2+a^2}) + C \\ &= \frac{x}{2} \cdot \sqrt{x^2+a^2} - \frac{a^2}{2} \cdot \ln(x+\sqrt{x^2+a^2}) + C \end{aligned}$$

$$36. \int \frac{x^2}{\sqrt{(x^2+a^2)^3}} dx = -\frac{x}{\sqrt{x^2+a^2}} + \ln(x+\sqrt{x^2+a^2}) + C \quad (a>0)$$

$$\text{证明: 被积函数 } f(x) = \frac{x^2}{\sqrt{(x^2+a^2)^3}} \text{ 的定义域为 } \{x | x \in R\}$$

$$\text{可令 } x = a \tan t \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right), \text{ 则 } dx = d(a \tan t) = a \sec^2 t dt, \frac{x^2}{\sqrt{(x^2+a^2)^3}} = \frac{a^2 \tan^2 t}{|a^3 \sec^3 t|}$$

$$\therefore -\frac{\pi}{2} < t < \frac{\pi}{2}, \sec t = \frac{1}{\cos t} > 0, \therefore \frac{x^2}{\sqrt{(x^2+a^2)^3}} = \frac{\tan^2 t}{a \sec^3 t}$$

$$\therefore \int \frac{x^2}{\sqrt{(x^2+a^2)^3}} dx = \int \frac{\tan^2 t}{a \sec^3 t} \cdot a \sec^2 t dt = \int \frac{\tan^2 t}{\sec t} dt = \int \frac{\sec^2 t - 1}{\sec t} dt$$

$$= \int \sec t dt - \int \frac{1}{\sec t} dt = \int \sec t dt - \int \cos t dt$$

$$= \ln |\sec t + \tan t| - \sin t + C_1 \quad \boxed{\text{公式 87: } \int \sec t dt = \ln |\sec t + \tan t| + C}$$

在Rt△ABC中, 设∠B = t, |BC| = a, 则|AC| = x, |AB| = √(x² + a²)

$$\therefore \sin t = \frac{x}{\sqrt{x^2+a^2}}, \tan t = \frac{x}{a}, \sec t = \frac{1}{\cos t} = \frac{\sqrt{x^2+a^2}}{a}$$

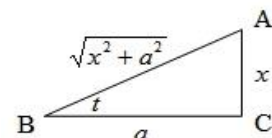
$$\therefore \int \frac{x^2}{\sqrt{(x^2+a^2)^3}} dx = \ln |\sec t + \tan t| - \sin t + C_1$$

$$= \ln \left| \frac{\sqrt{x^2+a^2} + x}{a} \right| - \frac{x}{\sqrt{x^2+a^2}} + C_1$$

$$= \ln |\sqrt{x^2+a^2} + x| - \frac{x}{\sqrt{x^2+a^2}} - \ln a + C_1$$

$$\therefore \sqrt{x^2+a^2} + x > 0$$

$$\therefore \int \frac{x^2}{\sqrt{(x^2+a^2)^3}} dx = -\frac{x}{\sqrt{x^2+a^2}} + \ln(x+\sqrt{x^2+a^2}) + C$$



$$37. \int \frac{dx}{x \cdot \sqrt{x^2 + a^2}} = \frac{1}{a} \cdot \ln \frac{\sqrt{x^2 + a^2} - a}{|x|} + C \quad (a > 0)$$

证明: 令 $\sqrt{x^2 + a^2} = t$ ($t > 0$), 则 $x = \sqrt{t^2 - a^2}$

$$\therefore dx = \frac{1}{2}(t^2 - a^2)^{-\frac{1}{2}} \cdot 2tdt = \frac{t}{\sqrt{t^2 - a^2}} dt$$

$$\therefore \int \frac{dx}{x \cdot \sqrt{x^2 + a^2}} = \int \frac{1}{t \cdot \sqrt{t^2 - a^2}} \cdot \frac{t}{\sqrt{t^2 - a^2}} dt$$

$$= \int \frac{1}{t^2 - a^2} dt$$

$$\text{公式21: } \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \cdot \ln \left| \frac{x-a}{x+a} \right| + C$$

$$= \frac{1}{2a} \cdot \ln \left| \frac{t-a}{t+a} \right| + C$$

$$= \frac{1}{2a} \cdot \ln \left| \frac{(t-a)^2}{t^2 - a^2} \right| + C$$

$$\text{将 } t = \sqrt{x^2 + a^2} \text{ 代入上式得: } \int \frac{dx}{x \cdot \sqrt{x^2 + a^2}} = \frac{1}{2a} \cdot \ln \left| \frac{(\sqrt{x^2 + a^2} - a)^2}{x^2 + a^2 - a^2} \right| + C$$

$$\text{提示: } \log_a b^n = n \log_a b$$

$$= \frac{1}{2a} \cdot \ln \left| \frac{(\sqrt{x^2 + a^2} - a)^2}{x^2} \right| + C$$

$$= \frac{1}{a} \cdot \ln \frac{\sqrt{x^2 + a^2} - a}{|x|} + C$$

$$38. \int \frac{dx}{x^2 \cdot \sqrt{x^2 + a^2}} = -\frac{\sqrt{x^2 + a^2}}{a^2 x} + C \quad (a > 0)$$

$$\text{证明: } \int \frac{dx}{x^2 \cdot \sqrt{x^2 + a^2}} = -\int \frac{1}{\sqrt{x^2 + a^2}} d\frac{1}{x}$$

$$\text{令 } t = \frac{1}{x} \quad (t \neq 0), \text{ 则 } x = \frac{1}{t}$$

$$\therefore -\int \frac{1}{\sqrt{x^2 + a^2}} d\frac{1}{x} = -\int \frac{1}{\sqrt{\frac{1}{t^2} + a^2}} dt = -\int \frac{t}{\sqrt{1 + a^2 t^2}} dt$$

$$= -\frac{1}{2a^2} \int \frac{2a^2 t}{\sqrt{1 + a^2 t^2}} dt$$

$$= -\frac{1}{2a^2} \int \frac{1}{\sqrt{1 + a^2 t^2}} d(1 + a^2 t^2)$$

$$= -\frac{1}{2a^2} \cdot \frac{1}{1 - \frac{1}{2}} (1 + a^2 t^2)^{1 - \frac{1}{2}} + C$$

$$= -\frac{1}{a^2} \cdot \sqrt{1 + a^2 t^2} + C$$

$$\text{将 } t = \frac{1}{x} \text{ 代入上式得: } \int \frac{dx}{x^2 \cdot \sqrt{x^2 + a^2}} = -\frac{\sqrt{x^2 + a^2}}{a^2 x} + C$$

$$39. \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \cdot \ln(x + \sqrt{x^2 + a^2}) + C \quad (a > 0)$$

$$\begin{aligned} \text{证法1: } \because \int \sqrt{x^2 + a^2} dx &= x \sqrt{x^2 + a^2} - \int x d \sqrt{x^2 + a^2} \\ &= x \sqrt{x^2 + a^2} - \int \frac{x^2}{\sqrt{x^2 + a^2}} dx \end{aligned}$$

$$\therefore \int \sqrt{x^2 + a^2} dx + \int \frac{x^2}{\sqrt{x^2 + a^2}} dx = x \sqrt{x^2 + a^2} \quad \text{①}$$

$$\text{又} \int \sqrt{x^2 + a^2} dx - \int \frac{x^2}{\sqrt{x^2 + a^2}} dx = \int \frac{a^2}{\sqrt{x^2 + a^2}} dx \quad \text{②}$$

$$\boxed{\text{公式31:} \int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2}) + C \quad (a > 0)} \quad = a^2 \cdot \ln(x + \sqrt{x^2 + a^2}) + C_1$$

$$\text{由①+②得, } 2 \int \sqrt{x^2 + a^2} dx = x \sqrt{x^2 + a^2} + a^2 \cdot \ln(x + \sqrt{x^2 + a^2})$$

$$\text{即} \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \cdot \ln(x + \sqrt{x^2 + a^2}) + C$$

$$39. \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \cdot \sqrt{x^2 + a^2} + \frac{a^2}{2} \cdot \ln(x + \sqrt{x^2 + a^2}) + C \quad (a > 0)$$

$$\text{证法2: 令 } x = a \cdot \tan t \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right), \text{ 则 } \sqrt{x^2 + a^2} = a \sqrt{1 + \tan^2 t} = |a \cdot \sec t|,$$

$$\because -\frac{\pi}{2} < t < \frac{\pi}{2}, \sec t = \frac{1}{\cos t} > 0, \therefore \sqrt{x^2 + a^2} = a \cdot \sec t$$

$$\begin{aligned} \therefore \int \sqrt{x^2 + a^2} dx &= \int a \cdot \sec t d(a \cdot \tan t) = a^2 \int \sec t dt \quad \boxed{\text{提示: } 1 + \tan^2 t = \sec^2 t} \\ &= a^2 \sec t \cdot \tan t - a^2 \int \tan t d \sec t \end{aligned} \quad \text{①}$$

$$\begin{aligned} \text{又} \int \tan t d \sec t &= \int \tan t \cdot \sec t \cdot \tan t dt = \int \frac{\sin^2 t}{\cos^3 t} dt \\ &= \int \frac{1 - \cos^2 t}{\cos^3 t} dt = \int \frac{1}{\cos t} \cdot \frac{1}{\cos^2 t} dt - \int \frac{1}{\cos t} dt \\ &= \int \sec t dt \tan t - \int \sec t dt \end{aligned} \quad \text{②}$$

$$\text{联立①②有 } a^2 \int \sec t dt \tan t = \frac{1}{2} (a^2 \sec t \cdot \tan t + a^2 \int \sec t dt) \quad \text{③}$$

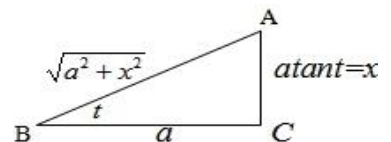
$$\text{又} \int \sec t dt = \ln | \sec t + \tan t | + C_1 \quad (\text{公式 87}) \quad \text{④}$$

$$\text{联立③④有 } a^2 \int \sec t dt \tan t = \frac{1}{2} a^2 \sec t \cdot \tan t + \frac{1}{2} a^2 \ln | \sec t + \tan t | + C_2 \quad \text{⑤}$$

$$\because x = a \cdot \tan t, \therefore \text{在 Rt} \triangle ABC \text{ 中, 可设 } \angle B = t, |BC| = a,$$

$$\text{则 } |AC| = a \cdot \tan t = x, |AB| = \sqrt{a^2 + x^2}$$

$$\therefore \sec t = \frac{1}{\cos t} = \frac{\sqrt{a^2 + x^2}}{a}, \tan t = \frac{x}{a}$$



$$\begin{aligned} \therefore \frac{1}{2} a^2 \sec t \cdot \tan t + \frac{1}{2} a^2 \ln | \sec t + \tan t | &= \frac{x}{2} \cdot \sqrt{x^2 + a^2} + \frac{a^2}{2} \cdot \ln \left| \frac{x + \sqrt{x^2 + a^2}}{a} \right| \\ &= \frac{x}{2} \cdot \sqrt{x^2 + a^2} + \frac{a^2}{2} \cdot \ln(x + \sqrt{x^2 + a^2}) - \frac{a^2}{2} \cdot \ln a \end{aligned}$$

$$\text{综合①②③④⑤得} \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \cdot \sqrt{x^2 + a^2} + \frac{a^2}{2} \cdot \ln(x + \sqrt{x^2 + a^2}) + C$$

$$40. \int \sqrt{(x^2 + a^2)^3} dx = \frac{x}{8} \cdot (2x^2 + 5a^2) \sqrt{x^2 + a^2} + \frac{3}{8} \cdot a^4 \cdot \ln(x + \sqrt{x^2 + a^2}) + C \quad (a > 0)$$

证明: 被积函数 $f(x) = \sqrt{(x^2 + a^2)^3}$ 的定义域为 $\{x | x \in R\}$

$$\text{可令 } x = a \tan t \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right), \quad \text{则 } \sqrt{(x^2 + a^2)^3} = |a^3 \sec^3 t|$$

$$\because -\frac{\pi}{2} < t < \frac{\pi}{2}, \sec t = \frac{1}{\cos t} > 0, \therefore \sqrt{(x^2 + a^2)^3} = a^3 \cdot \sec^3 t$$

$$\begin{aligned} \therefore \int \sqrt{(x^2 + a^2)^3} dx &= \int a^3 \cdot \sec^3 t d(a \tan t) = a^4 \int \sec^3 t d \tan t \\ &= a^4 \sec^3 t \cdot \tan t - a^4 \int \tan t d \sec^3 t \\ &= a^4 \sec^3 t \cdot \tan t - a^4 \int \tan t \cdot 3 \cdot \sec^2 t \cdot \sec t \tan t dt \\ &= a^4 \sec^3 t \cdot \tan t - 3a^4 \int \tan^2 t \cdot \sec^3 t dt \\ &= a^4 \sec^3 t \cdot \tan t - 3a^4 \int \tan^2 t \cdot \sec t d \tan t \\ &= a^4 \sec^3 t \cdot \tan t - 3a^4 \int (\sec^2 t - 1) \cdot \sec t d \tan t \\ &= a^4 \sec^3 t \cdot \tan t - 3a^4 \int \sec^3 t d \tan t + 3a^4 \int \sec t d \tan t \end{aligned}$$

$$\text{移项并整理的: } a^4 \int \sec^3 t d \tan t = \frac{1}{4} (a^4 \sec^3 t \tan t + 3a^4 \int \sec t d \tan t) \quad \text{①}$$

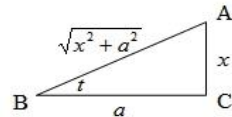
$$\begin{aligned} \therefore \int \sec t d \tan t &= \sec t \cdot \tan t - \int \tan t d \sec t \\ &= \sec t \cdot \tan t - \int \tan^2 t \cdot \sec t dt \\ &= \sec t \cdot \tan t - \int (\sec^2 t - 1) \cdot \sec t dt \\ &= \sec t \cdot \tan t - \int \sec^3 t dt + \int \sec t dt \end{aligned} \quad \text{②}$$

$$\text{又 } \therefore \int \sec t d \tan t = \int \sec^3 t dt \quad \text{③}$$

$$\begin{aligned} \text{联立②③得: } a^4 \int \sec^3 t d \tan t &= \frac{1}{2} \cdot \sec t \cdot \tan t + \frac{1}{2} \int \sec t dt \\ &= \frac{1}{2} \cdot \sec t \cdot \tan t + \frac{1}{2} \ln | \sec t + \tan t | + C_1 \end{aligned} \quad \text{④}$$

$$\text{联立①④得 } a^4 \int \sec^3 t d \tan t = \frac{1}{4} a^4 \sec^3 t \cdot \tan t + \frac{3}{8} a^4 \sec t \cdot \tan t + \frac{3}{8} a^4 \cdot \ln | \sec t + \tan t | + C_1$$

在 $Rt\triangle ABC$ 中, 设 $\angle B = t$, $|BC| = a$, 则 $|AC| = x$, $|AB| = \sqrt{x^2 + a^2}$



$$\therefore \tan t = \frac{x}{a}, \sec t = \frac{1}{\cos t} = \frac{\sqrt{x^2 + a^2}}{a}$$

$$\begin{aligned} \therefore a^4 \int \sec^3 t d \tan t &= \frac{a^4}{4} \cdot \frac{x}{a} \cdot \frac{x^2 + a^2}{a^3} \cdot \sqrt{x^2 + a^2} + \frac{3a^4}{8} \cdot \frac{\sqrt{x^2 + a^2}}{a} \cdot \frac{x}{a} + \frac{3}{8} a^4 \cdot \ln \left| \frac{\sqrt{x^2 + a^2} + x}{a} \right| + C_1 \\ &= \frac{x}{4} (x^2 + a^2) \sqrt{x^2 + a^2} + \frac{3a^2 \cdot x}{8} \cdot \sqrt{x^2 + a^2} + \frac{3a^4}{8} \cdot \ln \left| \sqrt{x^2 + a^2} + x \right| + C \end{aligned}$$

$$\therefore \int \sqrt{(x^2 + a^2)^3} dx = \frac{x}{8} \cdot (2x^2 + 5a^2) \sqrt{x^2 + a^2} + \frac{3}{8} \cdot a^4 \cdot \ln(x + \sqrt{x^2 + a^2}) + C$$

$$41. \int x \cdot \sqrt{x^2 + a^2} dx = \frac{1}{3} \sqrt{(x^2 + a^2)^3} + C \quad (a > 0)$$

$$\begin{aligned} \text{证明: } \int x \cdot \sqrt{x^2 + a^2} dx &= \frac{1}{2} \int (x^2 + a^2)^{\frac{1}{2}} dx^2 \\ &= \frac{1}{2} \int (x^2 + a^2)^{\frac{1}{2}} d(x^2 + a^2) \\ &= \frac{1}{2} \times \frac{1}{1 + \frac{1}{2}} \cdot (x^2 + a^2)^{1 + \frac{1}{2}} + C \\ &= \frac{1}{3} \sqrt{(x^2 + a^2)^3} + C \end{aligned}$$

$$42. \int x^2 \cdot \sqrt{x^2 + a^2} dx = \frac{x}{8} \cdot (2x^2 + a^2) \sqrt{x^2 + a^2} - \frac{a^4}{8} \cdot \ln(x + \sqrt{x^2 + a^2}) + C \quad (a > 0)$$

证明: 被积函数 $f(x) = x^2 \cdot \sqrt{x^2 + a^2}$ 的定义域为 $\{x | x \in R\}$

$$\text{可令 } x = a \tan t \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right), \quad \text{则 } x^2 \cdot \sqrt{(x^2 + a^2)} = a^2 \tan^2 t \cdot a \sec t$$

$$\because -\frac{\pi}{2} < t < \frac{\pi}{2}, \sec t = \frac{1}{\cos t} > 0, \therefore x^2 \cdot \sqrt{(x^2 + a^2)} = a^3 \tan^2 t \cdot \sec t$$

$$\begin{aligned} \therefore \int x^2 \cdot \sqrt{(x^2 + a^2)} dx &= \int a^3 \tan^2 t \cdot \sec t d(a \tan t) = a^4 \int \tan^2 t \cdot \sec t d \tan t = a^4 \int \tan^2 t \cdot \sec^3 t dt \\ &= a^4 \int \tan t \cdot \sec^2 t d \sec t \\ &= a^4 \int \tan t \cdot (1 + \tan^2 t) d \sec t = a^4 \int \tan t d \sec t + a^4 \int \tan^3 t d \sec t \\ &= a^4 \int \tan t d \sec t + a^4 \cdot \tan^3 t \cdot \sec t - a^4 \int \sec t d \tan^3 t \\ &= a^4 \int \tan t d \sec t + a^4 \cdot \tan^3 t \cdot \sec t - 3a^4 \int \sec^3 t \tan^2 t dt \\ &= a^4 \int \tan t d \sec t + a^4 \cdot \tan^3 t \cdot \sec t - 3a^4 \int \sec^2 t \tan t d \sec t \end{aligned}$$

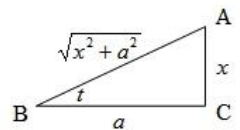
$$\begin{aligned} \text{移项并整理的: } a^4 \int \tan t \cdot \sec^2 t d \sec t &= \frac{1}{4} (a^4 \int \tan t d \sec t + a^4 \cdot \tan^3 t \cdot \sec t) \\ &= \frac{a^4}{4} \int \tan t d \sec t + \frac{a^4}{4} \cdot \tan^3 t \cdot \sec t \quad \text{①} \end{aligned}$$

$$\begin{aligned} \because \int \tan t d \sec t &= \sec t \cdot \tan t - \int \sec t d \tan t = \sec t \cdot \tan t - \int \sec^3 t dt \\ &= \sec t \cdot \tan t - \int (1 + \tan^2 t) \cdot \sec t dt \\ &= \sec t \cdot \tan t - \int \sec t dt - \int \tan^2 t \sec t dt \\ &= \sec t \cdot \tan t - \int \sec t dt - \int \tan t d \sec t \end{aligned}$$

$$\begin{aligned} \text{移项并整理得: } \int \tan t d \sec t &= \frac{1}{2} \cdot \sec t \cdot \tan t - \frac{1}{2} \int \sec t dt \quad \boxed{\text{公式 87: } \int \sec t dt = \ln |\sec t + \tan t| + C} \\ &= \frac{1}{2} \cdot \sec t \cdot \tan t - \frac{1}{2} \ln |\sec t + \tan t| + C_1 \quad \text{②} \end{aligned}$$

$$\text{联立①②得: } a^4 \int \tan t \cdot \sec^2 t d \sec t = \frac{a^4}{8} \sec t \cdot \tan t - \frac{a^4}{8} \ln |\sec t + \tan t| + \frac{a^4}{4} \tan^3 t \cdot \sec t + C$$

在 $Rt\triangle ABC$ 中, 设 $\angle B = t$, $|BC| = a$, 则 $|AC| = x$, $|AB| = \sqrt{x^2 + a^2}$



$$\therefore \tan t = \frac{x}{a}, \sec t = \frac{1}{\cos t} = \frac{\sqrt{x^2 + a^2}}{a}$$

$$\begin{aligned} \therefore a^4 \int \tan t \cdot \sec^2 t d \sec t &= \frac{a^4}{8} \cdot \frac{x}{a} \cdot \frac{\sqrt{x^2 + a^2}}{a} - \frac{a^4}{8} \cdot \ln \left| \frac{\sqrt{x^2 + a^2} + x}{a} \right| + \frac{a^4}{4} \cdot \frac{x^3}{a^3} \cdot \frac{\sqrt{x^2 + a^2}}{a} + C_1 \\ &= \frac{a^4 x}{8} \cdot \frac{\sqrt{x^2 + a^2}}{a} - \frac{a^4}{8} \cdot \ln \left| \sqrt{x^2 + a^2} + x \right| + \frac{x^3}{4} \cdot \sqrt{x^2 + a^2} + C_2 \\ &= \frac{x}{8} \cdot (2x^2 + a^2) \sqrt{x^2 + a^2} - \frac{a^4}{8} \cdot \ln \left| \sqrt{x^2 + a^2} + x \right| + C \end{aligned}$$

$$\because \sqrt{x^2 + a^2} + x > 0, \therefore \frac{a^4}{8} \cdot \ln \left| \sqrt{x^2 + a^2} + x \right| = \frac{a^4}{8} \cdot \ln (x + \sqrt{x^2 + a^2})$$

$$\therefore \int x^2 \cdot \sqrt{x^2 + a^2} dx = \frac{x}{8} \cdot (2x^2 + a^2) \sqrt{x^2 + a^2} - \frac{a^4}{8} \cdot \ln (x + \sqrt{x^2 + a^2}) + C$$

$$43. \int \frac{\sqrt{x^2+a^2}}{x} dx = \sqrt{x^2+a^2} + a \cdot \ln \left| \frac{\sqrt{x^2+a^2}-a}{x} \right| + C \quad (a > 0)$$

证明: 被积函数 $f(x) = \frac{\sqrt{x^2+a^2}}{x}$ 的定义域为 $\{x | x \neq 0\}$

$$\text{令 } \sqrt{x^2+a^2} = t \quad (t \geq 0 \text{ 且 } t \neq a), \text{ 则 } x = \sqrt{t^2-a^2}$$

$$\therefore dx = \frac{1}{2}(t^2-a^2)^{-\frac{1}{2}} \cdot 2tdt = \frac{t}{\sqrt{t^2-a^2}} dt$$

$$\therefore \int \frac{\sqrt{x^2+a^2}}{x} dx = \int \frac{t}{\sqrt{t^2-a^2}} \cdot \frac{t}{\sqrt{t^2-a^2}} dt = \int \frac{t^2}{t^2-a^2} dt$$

$$= \int \frac{t^2-a^2+a^2}{t^2-a^2} dt = \int dt + a^2 \int \frac{1}{t^2-a^2} dt$$

$$\text{公式21: } \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \cdot \ln \left| \frac{x-a}{x+a} \right| + C$$

$$= t + a^2 \cdot \frac{1}{2a} \cdot \ln \left| \frac{t-a}{t+a} \right| + C = t + \frac{a}{2} \cdot \ln \left| \frac{(t-a)^2}{t^2-a^2} \right| + C$$

$$\text{将 } t = \sqrt{x^2+a^2} \text{ 代入上式得: } \int \frac{\sqrt{x^2+a^2}}{x} dx = \sqrt{x^2+a^2} + \frac{a}{2} \cdot \ln \left| \frac{(\sqrt{x^2+a^2}-a)^2}{x^2+a^2-a^2} \right| + C$$

$$= \sqrt{x^2+a^2} + a \cdot \ln \left| \frac{(\sqrt{x^2+a^2}-a)}{x} \right| + C$$

$$= \sqrt{x^2+a^2} + a \cdot \ln \left| \frac{\sqrt{x^2+a^2}-a}{x} \right| + C$$

$$44. \int \frac{\sqrt{x^2+a^2}}{x^2} dx = -\frac{\sqrt{x^2+a^2}}{x} + \ln(x + \sqrt{x^2+a^2}) + C \quad (a > 0)$$

证明: 被积函数 $f(x) = \frac{\sqrt{x^2+a^2}}{x^2}$ 的定义域为 $\{x | x \neq 0\}$

1. 当 $x > 0$ 时, 可令 $x = a \tan t \quad (0 < t < \frac{\pi}{2})$, 则 $dx = d(a \tan t) = a \sec^2 t dt$,

$$\frac{\sqrt{x^2+a^2}}{x^2} = \frac{|a \sec t|}{a^2 \tan^2 t}, \quad \because 0 < t < \frac{\pi}{2}, \sec t = \frac{1}{\cos t} > 0, \quad \therefore \frac{\sqrt{x^2+a^2}}{x^2} = \frac{\sec t}{a \tan^2 t}$$

$$\therefore \int \frac{\sqrt{x^2+a^2}}{x^2} dx = \int \frac{\sec t}{a \tan^2 t} \cdot a \sec^2 t dt = \int \frac{\sec t}{\tan^2 t} \cdot (1 + \tan^2 t) dt$$

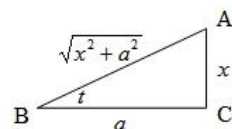
$$= \int \sec t dt + \int \frac{\sec t}{\tan^2 t} dt = \int \sec t dt + \int \frac{1}{\cos t} \cdot \frac{\cos^2 t}{\sin^2 t} dt$$

$$= \int \sec t dt + \int \frac{\cos t}{\sin^2 t} dt = \int \sec t dt + \int \frac{1}{\sin^2 t} d \sin t$$

$$= \ln | \sec t + \tan t | - \frac{1}{\sin t} + C_1 \quad \text{公式 87: } \int \sec t dt = \ln | \sec t + \tan t | + C$$

在 $\text{Rt} \triangle ABC$ 中, 设 $\angle B = t$, $|BC| = a$, 则 $|AC| = x$, $|AB| = \sqrt{x^2+a^2}$

$$\therefore \sin t = \frac{x}{\sqrt{x^2+a^2}}, \tan t = \frac{x}{a}, \sec t = \frac{1}{\cos t} = \frac{\sqrt{x^2+a^2}}{a}$$



$$\therefore \int \frac{\sqrt{x^2+a^2}}{x^2} dx = \ln \left| \frac{\sqrt{x^2+a^2}+x}{a} \right| - \frac{\sqrt{x^2+a^2}}{x} + C_1$$

$$= -\frac{\sqrt{x^2+a^2}}{x} + \ln \left| \sqrt{x^2+a^2}+x \right| - \ln a + C_1$$

$$\because \sqrt{x^2+a^2}+x > 0 \quad \therefore \int \frac{\sqrt{x^2+a^2}}{x^2} dx = -\frac{\sqrt{x^2+a^2}}{x} + \ln(\sqrt{x^2+a^2}+x) + C$$

$$2. \text{ 当 } x < 0 \text{ 时, 同理可证得: } \int \frac{\sqrt{x^2+a^2}}{x^2} dx = -\frac{\sqrt{x^2+a^2}}{x} + \ln(\sqrt{x^2+a^2}+x) + C$$

综合讨论1,2得: $\int \frac{\sqrt{x^2+a^2}}{x^2} dx = -\frac{\sqrt{x^2+a^2}}{x} + \ln(\sqrt{x^2+a^2}+x) + C$

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(七) 含有 $\sqrt{x^2 - a^2}$ ($a > 0$) 的积分 (45~58)

$$45. \int \frac{dx}{\sqrt{x^2 - a^2}} = \frac{x}{|x|} \cdot \operatorname{arsh} \frac{|x|}{a} + C_1 = \ln |x + \sqrt{x^2 - a^2}| + C \quad (a > 0)$$

证法1: 被积函数 $f(x) = \frac{1}{\sqrt{x^2 - a^2}}$ 的定义域为 $\{x/x > a \text{ 或 } x < -a\}$

1. 当 $x > a$ 时, 可设 $x = a \cdot \sec t$ ($0 < t < \frac{\pi}{2}$), 则 $dx = a \cdot \sec t \cdot \tan t dt$

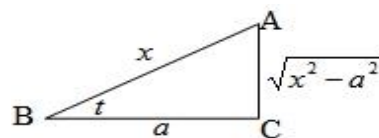
$$\sqrt{x^2 - a^2} = a \sqrt{\sec^2 t - 1} = a \cdot |\tan t| \quad \because 0 < t < \frac{\pi}{2}, \quad \sqrt{x^2 - a^2} = a \cdot \tan t$$

$$\begin{aligned} \therefore \int \frac{dx}{\sqrt{x^2 - a^2}} &= \int \frac{a \cdot \sec t \cdot \tan t}{a \cdot \tan t} dt = \int \sec t dt \quad \boxed{\text{公式 87: } \int \sec t dt = \ln |\sec t + \tan t| + C} \\ &= \ln |\sec t + \tan t| + C_2 \end{aligned}$$

在 $\text{Rt} \triangle ABC$ 中, 可设 $\angle B = t$, $|BC| = a$, 则 $|AB| = x$, $|AC| = \sqrt{x^2 - a^2}$

$$\therefore \sec t = \frac{1}{\cos t} = \frac{x}{a}, \quad \tan t = \frac{|AC|}{|BC|} = \frac{\sqrt{x^2 - a^2}}{a}$$

$$\begin{aligned} \therefore \int \frac{dx}{\sqrt{x^2 - a^2}} &= \ln |\sec t + \tan t| = \ln \left| \frac{x + \sqrt{x^2 - a^2}}{a} \right| \\ &= \ln |x + \sqrt{x^2 - a^2}| + C_3 \end{aligned}$$



2. 当 $x < -a$, 即 $-x > a$ 时, 令 $\mu = -x$, 即 $x = -\mu$

$$\begin{aligned} \text{由讨论 1 可知 } \int \frac{dx}{\sqrt{x^2 - a^2}} &= -\int \frac{d\mu}{\sqrt{\mu^2 - a^2}} = -\ln |\mu + \sqrt{\mu^2 - a^2}| + C_4 \\ &= -\ln |-x + \sqrt{x^2 - a^2}| + C_4 = \ln \frac{1}{|-x + \sqrt{x^2 - a^2}|} + C_4 \\ &= \ln \frac{|-x + \sqrt{x^2 - a^2}|}{a^2} + C_4 \\ &= \ln |-x - \sqrt{x^2 - a^2}| + C_5 \end{aligned}$$

$$\text{综合讨论 1, 2, 可写成 } \int \frac{dx}{\sqrt{x^2 - a^2}} = \frac{x}{|x|} \cdot \operatorname{arsh} \frac{|x|}{a} + C_1 = \ln |x + \sqrt{x^2 - a^2}| + C$$

$$45. \int \frac{dx}{\sqrt{x^2 - a^2}} = \frac{x}{|x|} \cdot \operatorname{arsh} \frac{|x|}{a} + C_1 = \ln |x + \sqrt{x^2 - a^2}| + C \quad (a > 0)$$

证法2: 被积函数 $f(x) = \frac{1}{\sqrt{x^2 - a^2}}$ 的定义域为 $\{x/x > a \text{ 或 } x < -a\}$

1. 当 $x > a$ 时, 可设 $x = a \cdot \operatorname{ch} t$ ($t > 0$), 则 $t = \operatorname{arch} \frac{x}{a}$

$$\sqrt{x^2 - a^2} = \sqrt{a^2 \operatorname{ch}^2 t - a^2} = a \cdot \operatorname{sh} t, \quad dx = a \cdot \operatorname{sh} t dt$$

$$\therefore \int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \cdot \operatorname{sh} t}{a \cdot \operatorname{sh} t} dt = \int dt = t + C_1$$

$$= \operatorname{arch} \frac{x}{a} + C = \ln \left[\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 - 1} \right] + C_2$$

$$= \ln |x + \sqrt{x^2 - a^2}| + C_3$$

2. 当 $x < -a$, 即 $-x > a$ 时, 令 $\mu = -x$, 即 $x = -\mu$

$$\text{由讨论 1 可知 } \int \frac{dx}{\sqrt{x^2 - a^2}} = - \int \frac{d\mu}{\sqrt{\mu^2 - a^2}} = - \ln |\mu + \sqrt{\mu^2 - a^2}| + C_4$$

$$= - \ln(-x + \sqrt{x^2 - a^2}) + C_4 = \ln \frac{1}{|-x + \sqrt{x^2 - a^2}|} + C_4$$

$$= \ln \frac{|-x + \sqrt{x^2 - a^2}|}{a^2} + C_4$$

$$= \ln |-x - \sqrt{x^2 - a^2}| + C_5$$

$$\text{综合讨论 1, 2, 可写成 } \int \frac{dx}{\sqrt{x^2 - a^2}} = \frac{x}{|x|} \cdot \operatorname{arsh} \frac{|x|}{a} + C_1 = \ln |x + \sqrt{x^2 - a^2}| + C$$

$$46. \int \frac{dx}{\sqrt{(x^2 - a^2)^3}} = -\frac{x}{a^2 \cdot \sqrt{x^2 - a^2}} + C \quad (a > 0)$$

证明: 被积函数 $f(x) = \frac{1}{\sqrt{(x^2 - a^2)^3}}$ 的定义域为 $\{x/x > a \text{ 或 } x < -a\}$

1. 当 $x > a$ 时, 可设 $x = a \cdot \sec t$ ($0 < t < \frac{\pi}{2}$), 则 $dx = a \cdot \sec t \cdot \tan t dt$

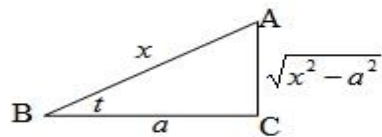
$$\sqrt{(x^2 - a^2)^3} = |a^3 \cdot \tan^3 t| \quad \because 0 < t < \frac{\pi}{2}, \tan t > 0, \sqrt{(x^2 - a^2)^3} = a^3 \cdot \tan^3 t$$

$$\begin{aligned} \therefore \int \frac{dx}{\sqrt{(x^2 - a^2)^3}} &= \int \frac{a \cdot \sec t \cdot \tan t}{a^3 \cdot \tan^3 t} dt = \frac{1}{a^2} \int \frac{\sec t}{\tan^3 t} dt \\ &= \frac{1}{a^2} \int \frac{1}{\cos t} \cdot \frac{\cos^2 t}{\sin^2 t} dt = \frac{1}{a^2} \int \frac{\cos t}{\sin^2 t} dt \\ &= \frac{1}{a^2} \int \frac{1}{\sin^2 t} d \sin t \\ &= -\frac{1}{a^2 \sin t} + C \end{aligned}$$

在 $\text{Rt} \triangle ABC$ 中, 可设 $\angle B = t$, $|BC| = a$, 则 $|AB| = x$, $|AC| = \sqrt{x^2 - a^2}$

$$\therefore \sin t = \frac{\sqrt{x^2 - a^2}}{x}$$

$$\therefore \int \frac{dx}{\sqrt{(x^2 - a^2)^3}} = -\frac{x}{a^2 \cdot \sqrt{x^2 - a^2}} + C$$



2. 当 $x < -a$, 即 $-x > a$ 时, 令 $\mu = -x$, 即 $x = -\mu$

$$\therefore \int \frac{dx}{\sqrt{(x^2 - a^2)^3}} = -\int \frac{d\mu}{\sqrt{(\mu^2 - a^2)^3}}$$

$$\text{由讨论 1 可知 } -\int \frac{d\mu}{\sqrt{(\mu^2 - a^2)^3}} = \frac{\mu}{a^2 \cdot \sqrt{(\mu^2 - a^2)}} + C$$

$$\text{将 } \mu = -x \text{ 代入得: } \int \frac{dx}{\sqrt{(x^2 - a^2)^3}} = -\frac{x}{a^2 \cdot \sqrt{x^2 - a^2}} + C$$

$$\text{综合讨论 1, 2 得: } \int \frac{dx}{\sqrt{(x^2 - a^2)^3}} = -\frac{x}{a^2 \cdot \sqrt{x^2 - a^2}} + C$$

$$47. \int \frac{x}{\sqrt{x^2 - a^2}} dx = \sqrt{x^2 - a^2} + C \quad (a > 0)$$

$$\text{证明: } \int \frac{x}{\sqrt{x^2 - a^2}} dx = \frac{1}{2} \int (x^2 - a^2)^{-\frac{1}{2}} d(x^2)$$

$$= \frac{1}{2} \int (x^2 - a^2)^{-\frac{1}{2}} d(x^2 - a^2)$$

$$= \frac{1}{2} \times \frac{1}{1 - \frac{1}{2}} (x^2 - a^2)^{1 - \frac{1}{2}} + C$$

$$= \sqrt{x^2 - a^2} + C$$

$$48. \int \frac{x}{\sqrt{(x^2-a^2)^3}} dx = -\frac{1}{\sqrt{x^2-a^2}} + C \quad (a > 0)$$

证明: 被积函数 $f(x) = \frac{x}{\sqrt{(x^2-a^2)^3}}$ 的定义域为 $\{x/x > a \text{ 或 } x < -a\}$

1. 当 $x > a$ 时, 可设 $x = a \cdot \sec t$ ($0 < t < \frac{\pi}{2}$), 则 $dx = a \cdot \sec t \cdot \tan t dt$

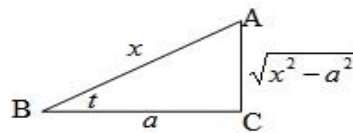
$$\frac{x}{\sqrt{(x^2-a^2)^3}} = \frac{a \cdot \sec t}{|a^3 \cdot \tan^3 t|} \because 0 < t < \frac{\pi}{2}, \frac{x}{\sqrt{(x^2-a^2)^3}} = \frac{\sec t}{a^2 \cdot \tan^3 t}$$

$$\begin{aligned} \therefore \int \frac{x}{\sqrt{(x^2-a^2)^3}} dx &= \int \frac{\sec t}{a^2 \cdot \tan^3 t} \cdot a \cdot \sec t \cdot \tan t dt \\ &= \frac{1}{a} \int \frac{\sec^2 t}{\tan^2 t} dt = \frac{1}{a} \int \frac{1}{\sin^2 t} dt \\ &= -\frac{1}{a} \int -\csc^2 t dt = -\frac{1}{a} \cdot \cot t + C \end{aligned}$$

在 $\text{Rt} \triangle ABC$ 中, 可设 $\angle B = t$, $|BC| = a$, 则 $|AB| = x$, $|AC| = \sqrt{x^2 - a^2}$

$$\therefore \cot t = \frac{a}{\sqrt{x^2 - a^2}}$$

$$\therefore \int \frac{x}{\sqrt{(x^2-a^2)^3}} dx = -\frac{1}{a} \cdot \frac{a}{\sqrt{x^2-a^2}} + C = -\frac{1}{\sqrt{x^2-a^2}} + C$$



2. 当 $x < -a$, 即 $-x > a$ 时, 令 $\mu = -x$, 即 $x = -\mu$

$$\therefore \int \frac{x}{\sqrt{(x^2-a^2)^3}} dx = \int \frac{\mu}{\sqrt{(\mu^2-a^2)^3}} d\mu$$

$$\text{由讨论 1 可知 } \int \frac{\mu}{\sqrt{(\mu^2-a^2)^3}} d\mu = -\frac{1}{\sqrt{\mu^2-a^2}} + C$$

$$\text{将 } \mu = -x \text{ 代入得: } \int \frac{x}{\sqrt{(x^2-a^2)^3}} dx = -\frac{1}{\sqrt{x^2-a^2}} + C$$

综合讨论 1, 2 得: $\int \frac{x}{\sqrt{(x^2-a^2)^3}} dx = -\frac{1}{\sqrt{x^2-a^2}} + C$

$$49. \int \frac{x^2}{\sqrt{x^2-a^2}} dx = \frac{x}{2} \sqrt{x^2-a^2} + \frac{a^2}{2} \cdot \ln \left| x + \sqrt{x^2-a^2} \right| + C \quad (a > 0)$$

$$\text{证明: } \int \frac{x^2}{\sqrt{x^2-a^2}} dx = \int \frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}} dx$$

$$= \int (\sqrt{x^2-a^2} + \frac{a^2}{\sqrt{x^2-a^2}}) dx$$

$$= \int \sqrt{x^2-a^2} dx + a^2 \int \frac{1}{\sqrt{x^2-a^2}} dx$$

$$\therefore \int \sqrt{x^2-a^2} dx = \frac{x}{2} \cdot \sqrt{x^2-a^2} - \frac{a^2}{2} \cdot \ln \left| x + \sqrt{x^2-a^2} \right| + C \quad \text{① (公式53)}$$

$$a^2 \int \frac{dx}{\sqrt{x^2-a^2}} = a^2 \cdot \ln \left| x + \sqrt{x^2-a^2} \right| + C \quad \text{② (公式45)}$$

$$\therefore \text{由①+②得: } \int \frac{x^2}{\sqrt{x^2-a^2}} dx = \frac{x}{2} \sqrt{x^2-a^2} + \frac{a^2}{2} \cdot \ln \left| x + \sqrt{x^2-a^2} \right| + C$$

$$50. \int \frac{x^2}{\sqrt{(x^2-a^2)^3}} dx = -\frac{x}{\sqrt{x^2-a^2}} + \ln \left| x + \sqrt{x^2-a^2} \right| + C \quad (a > 0)$$

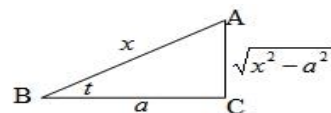
证明: 被积函数 $f(x) = \frac{x^2}{\sqrt{(x^2-a^2)^3}}$ 的定义域为 $\{x/x > a \text{ 或 } x < -a\}$

1. 当 $x > a$ 时, 可设 $x = a \cdot \sec t \quad (0 < t < \frac{\pi}{2})$, 则 $dx = a \cdot \sec t \cdot \tan t dt$

$$\begin{aligned} \frac{x^2}{\sqrt{(x^2-a^2)^3}} &= \frac{a^2 \cdot \sec^2 t}{|a^3 \cdot \tan^3 t|} \because 0 < t < \frac{\pi}{2}, \therefore \frac{x^2}{\sqrt{(x^2-a^2)^3}} = \frac{\sec^2 t}{a \cdot \tan^3 t} \\ \therefore \int \frac{x^2}{\sqrt{(x^2-a^2)^3}} dx &= \int \frac{\sec^2 t}{a \cdot \tan^3 t} \cdot a \cdot \sec t \cdot \tan t dt = \int \frac{\sec^3 t}{\tan^2 t} dt = \int \frac{1}{\cos^3 t} \cdot \frac{\cos^2 t}{\sin^2 t} dt = \int \frac{1}{\sin^2 t \cdot \cos t} dt \\ &= \int \frac{\cos t}{\sin^2 t \cdot \cos^2 t} dt = \int \frac{1}{\sin^2 t (1 - \sin^2 t)} d \sin t = \int \left(\frac{1}{\sin^2 t} + \frac{1}{1 - \sin^2 t} \right) d \sin t \\ &= \int \frac{1}{\sin^2 t} d \sin t + \int \frac{1}{1 - \sin^2 t} d \sin t = \int \frac{1}{\sin^2 t} d \sin t - \int \frac{1}{\sin^2 t - 1} d \sin t \\ &= \int \frac{1}{\sin^2 t} d \sin t + \frac{1}{2} \int \left(\frac{1}{\sin t + 1} - \frac{1}{\sin t - 1} \right) d \sin t \\ &= \int \frac{1}{\sin^2 t} d \sin t + \frac{1}{2} \int \frac{1}{\sin t + 1} d(\sin t + 1) - \frac{1}{2} \int \frac{1}{\sin t - 1} d(\sin t - 1) \\ &= -\frac{1}{\sin t} + \frac{1}{2} \ln |\sin t + 1| - \frac{1}{2} \ln |\sin t - 1| + C_1 \\ &= -\frac{1}{\sin t} + \frac{1}{2} \ln \left| \frac{\sin t + 1}{\sin t - 1} \right| + C_1 = -\frac{1}{\sin t} + \frac{1}{2} \ln \left| \frac{(\sin t + 1)^2}{\sin^2 t - 1} \right| + C_1 \\ &= -\frac{1}{\sin t} + \frac{1}{2} \ln \left| \frac{(\sin t + 1)^2}{\cos^2 t} \cdot (-1) \right| + C_1 = -\frac{1}{\sin t} + \ln |\tan t + \sec t| + C_2 \end{aligned}$$

在 Rt $\triangle ABC$ 中, 可设 $\angle B = t$, $|BC| = a$, 则 $|AB| = x$, $|AC| = \sqrt{x^2 - a^2}$

$$\therefore \sin t = \frac{\sqrt{x^2 - a^2}}{x}, \tan t = \frac{\sqrt{x^2 - a^2}}{a}, \sec t = \frac{x}{a}$$



$$\therefore \int \frac{x^2}{\sqrt{(x^2-a^2)^3}} dx = -\frac{x}{\sqrt{x^2-a^2}} + \ln \left| \frac{x + \sqrt{x^2-a^2}}{a} \right| + C_2 = -\frac{x}{\sqrt{x^2-a^2}} + \ln \left| x + \sqrt{x^2-a^2} \right| + C$$

2. 当 $x < -a$, 即 $-x > a$ 时, 令 $\mu = -x$, 即 $x = -\mu$

$$\therefore \int \frac{x^2}{\sqrt{(x^2-a^2)^3}} dx = -\int \frac{\mu^2}{\sqrt{(\mu^2-a^2)^3}} d\mu$$

$$\text{由讨论 1 可知 } -\int \frac{\mu^2}{\sqrt{(\mu^2-a^2)^3}} d\mu = \frac{\mu}{\sqrt{\mu^2-a^2}} - \ln \left| \mu + \sqrt{\mu^2-a^2} \right| + C$$

$$\text{将 } \mu = -x \text{ 代入得: } \int \frac{x^2}{\sqrt{(x^2-a^2)^3}} dx = -\frac{x}{\sqrt{x^2-a^2}} - \ln \left| -x + \sqrt{x^2-a^2} \right| + C$$

$$= \int \frac{x^2}{\sqrt{(x^2-a^2)^3}} dx = -\frac{x}{\sqrt{x^2-a^2}} - \ln \left| \frac{(\sqrt{x^2-a^2}-x)(\sqrt{x^2-a^2}+x)}{\sqrt{x^2-a^2}+x} \right| + C$$

$$= -\frac{x}{\sqrt{x^2-a^2}} - \ln \left| \frac{x^2-a^2-x^2}{(\sqrt{x^2-a^2}+x)} \right| + C = -\frac{x}{\sqrt{x^2-a^2}} + 2 \ln a - \ln \left| \frac{1}{\sqrt{x^2-a^2}+x} \right| + C$$

提示: $\log_a b^n = n \log_a b$

$$= -\frac{x}{\sqrt{x^2-a^2}} + \ln \left| x + \sqrt{x^2-a^2} \right| + C$$

综合讨论 1, 2 得: $\int \frac{x^2}{\sqrt{(x^2-a^2)^3}} dx = -\frac{x}{\sqrt{x^2-a^2}} + \ln \left| x + \sqrt{x^2-a^2} \right| + C$

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$$51. \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \cdot \arccos \frac{a}{|x|} + C \quad (a > 0)$$

证法 1: 被积函数 $f(x) = \frac{1}{x\sqrt{x^2-a^2}}$ 的定义域为 $\{x/x > a \text{ 或 } x < -a\}$

1. 当 $x > a$ 时, 可设 $x = a \cdot \sec t$ ($0 < t < \frac{\pi}{2}$), 则

$$x\sqrt{x^2-a^2} = a^2 \cdot \sec t \sqrt{\sec^2 t - 1} = a^2 \sec t \cdot \tan t, \quad dx = a \cdot \sec t \cdot \tan t \, dt$$

$$\begin{aligned} \therefore \int \frac{dx}{x\sqrt{x^2-a^2}} &= \int \frac{a \cdot \sec t \cdot \tan t}{a^2 \sec t \cdot \tan t} dt = \int \frac{1}{a} dt \\ &= \frac{1}{a} t + C_1 \end{aligned}$$

$$\because x = a \cdot \sec t, \therefore \cos t = \frac{a}{x}, \therefore t = \arccos \frac{a}{x}$$

$$\therefore \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \cdot \arccos \frac{a}{x} + C$$

2. 当 $x < -a$, 即 $-x > a$ 时, 令 $\mu = -x$, 即 $x = -\mu$

$$\begin{aligned} \text{由讨论 1 可知 } \int \frac{dx}{x\sqrt{x^2-a^2}} &= \int \frac{d\mu}{\mu\sqrt{\mu^2-a^2}} = \frac{1}{a} \cdot \arccos \frac{a}{\mu} + C_2 \\ &= \frac{1}{a} \cdot \arccos \frac{a}{-x} + C \end{aligned}$$

$$\text{综合讨论 1, 2, 可写成 } \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \cdot \arccos \frac{a}{|x|} + C$$

$$51. \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \cdot \arccos \frac{a}{|x|} + C \quad (a > 0)$$

证法2: 被积函数 $f(x) = \frac{1}{x\sqrt{x^2-a^2}}$ 的定义域为 $\{x/x > a \text{ 或 } x < -a\}$

1. 当 $x > a$ 时, 可设 $x = a \cdot \operatorname{cht} \quad (0 < t)$, 则

$$x\sqrt{x^2-a^2} = a \cdot \operatorname{cht} \cdot a \cdot \operatorname{sht} = a^2 \operatorname{cht} \cdot \operatorname{sht}, dx = a \cdot \operatorname{sht} dt$$

$$\begin{aligned} \therefore \int \frac{dx}{x\sqrt{x^2-a^2}} &= \int \frac{a \cdot \operatorname{sht}}{a \cdot \operatorname{cht} \cdot \operatorname{sht}} dt = \int \frac{1}{a} \cdot \frac{1}{\operatorname{cht}} dt \\ &= \frac{1}{a} \int \frac{\operatorname{cht}}{\operatorname{ch}^2 t} dt = \frac{1}{a} \int \frac{1}{1+\operatorname{sh}^2 t} d\operatorname{sht} \\ &= \frac{1}{a} \cdot \arctan(\operatorname{sht}) + C \end{aligned}$$

公式 19: $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + C$

$$\because x = a \cdot \operatorname{cht}, \therefore \operatorname{cht} = \frac{x}{a}, \therefore \operatorname{sht} = \sqrt{1-\operatorname{ch}^2 t} = \frac{\sqrt{x^2-a^2}}{a}$$

在 $\operatorname{Rt}\triangle ABC$ 中, 设 $\tan y = \operatorname{sht} = \frac{\sqrt{x^2-a^2}}{a}, \angle B = y, |BC| = a$

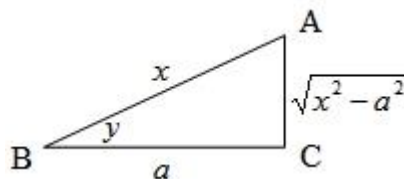
$$\therefore y = \arctan(\operatorname{sht}), |AC| = \sqrt{x^2-a^2}, |AB| = \sqrt{|AC|^2 + |BC|^2} = x$$

$$\therefore \cos y = \frac{|BC|}{|AB|} = \frac{a}{x}$$

$$\text{即 } \cos y = \cos \arctan(\operatorname{sht}) = \frac{a}{x}$$

$$\therefore \arctan(\operatorname{sht}) = \arccos \frac{a}{x} + C$$

$$\therefore \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \cdot \arctan(\operatorname{sht}) + C = \frac{1}{a} \cdot \arccos \frac{a}{x} + C$$



2. 当 $x < -a$, 即 $-x > a$ 时, 令 $\mu = -x$, 即 $x = -\mu$

$$\begin{aligned} \text{由讨论 1 可知 } \int \frac{dx}{x\sqrt{x^2-a^2}} &= \int \frac{d\mu}{\mu\sqrt{\mu^2-a^2}} = \frac{1}{a} \cdot \arccos \frac{a}{\mu} + C_2 \\ &= \frac{1}{a} \cdot \arccos \frac{a}{-x} + C \end{aligned}$$

$$\text{综合讨论 1, 2, 可写成 } \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \cdot \arccos \frac{a}{|x|} + C$$

$$52. \int \frac{dx}{x^2 \sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a^2 x} + C \quad (a > 0)$$

证明：被积函数 $f(x) = \frac{1}{x^2 \sqrt{x^2 - a^2}}$ 的定义域为 $\{x | x > a \text{ 或 } x < -a\}$

$$1. \text{ 当 } x > a \text{ 时, 可设 } x = \frac{1}{t} \quad (0 < t < \frac{1}{a}), \text{ 则 } dx = -\frac{1}{t^2} dt, \quad \frac{1}{x^2 \sqrt{x^2 - a^2}} = \frac{t^3}{\sqrt{1 - a^2 t^2}}$$

$$\begin{aligned} \therefore \int \frac{dx}{x^2 \sqrt{x^2 - a^2}} &= \int \frac{t^3}{\sqrt{1 - a^2 t^2}} \cdot \left(-\frac{1}{t^2}\right) dt \\ &= -\int \frac{t}{\sqrt{1 - a^2 t^2}} dt = -\frac{1}{2} \int (1 - a^2 t^2)^{-\frac{1}{2}} d(t^2) \\ &= \frac{1}{2a^2} \int (1 - a^2 t^2)^{-\frac{1}{2}} d(1 - a^2 t^2) = \frac{1}{2a^2} \cdot \frac{1}{1 - \frac{1}{2}} \cdot (1 - a^2 t^2)^{1 - \frac{1}{2}} + C \\ &= \frac{\sqrt{1 - a^2 t^2}}{a^2} + C \end{aligned}$$

$$\begin{aligned} \text{将 } x = \frac{1}{t}, \text{ 即 } t = \frac{1}{x} \text{ 代入上式得: } \int \frac{dx}{x^2 \sqrt{x^2 - a^2}} &= \frac{1}{a^2} \cdot \sqrt{1 - a^2 \left(\frac{1}{x}\right)^2} + C = \frac{1}{a^2} \cdot \sqrt{\frac{x^2 - a^2}{x^2}} + C \\ &= \frac{1}{a^2} \cdot \frac{\sqrt{x^2 - a^2}}{|x|} + C \end{aligned}$$

$$\because x > a > 0 \quad \therefore \int \frac{dx}{x^2 \sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a^2 x} + C$$

2. 当 $x < -a$, 即 $-x > a$ 时, 令 $\mu = -x$, 即 $x = -\mu$

$$\text{由讨论 1 可知 } \int \frac{dx}{x^2 \sqrt{x^2 - a^2}} = -\int \frac{d\mu}{\mu^2 \sqrt{\mu^2 - a^2}} = -\frac{\sqrt{\mu^2 - a^2}}{a^2 \mu} + C$$

$$\text{将 } \mu = -x \text{ 代入上式得: } \int \frac{dx}{x^2 \sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a^2 x} + C$$

$$\text{综合讨论 1, 2 得: } \int \frac{dx}{x^2 \sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a^2 x} + C$$

$$53. \int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \cdot \ln \left| x + \sqrt{x^2 - a^2} \right| + C \quad (a > 0)$$

证明: 被积函数 $f(x) = \sqrt{x^2 - a^2}$ 的定义域为 $\{x/x > a \text{ 或 } x < -a\}$

1. 当 $x > a$ 时, 可设 $x = a \cdot \sec t$ ($0 < t < \frac{\pi}{2}$), 则 $\sqrt{x^2 - a^2} = |a \cdot \tan t|$

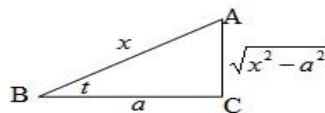
$$\because 0 < t < \frac{\pi}{2}, \therefore \sqrt{x^2 - a^2} = a \cdot \tan t$$

$$\begin{aligned} \therefore \int \sqrt{x^2 - a^2} dx &= \int a \cdot \tan t d(a \cdot \sec t) = a^2 \int \tan t d \sec t \\ &= a^2 \cdot \tan t \cdot \sec t - a^2 \int \sec t d \tan t \\ &= a^2 \cdot \tan t \cdot \sec t - a^2 \int \sec^3 t dt \\ &= a^2 \cdot \tan t \cdot \sec t - a^2 \int \sec t (1 + \tan^2 t) dt \\ &= a^2 \cdot \tan t \cdot \sec t - a^2 \int \sec t dt - a^2 \int \sec t \tan^2 t dt \\ &= a^2 \cdot \tan t \cdot \sec t - a^2 \int \sec t dt - a^2 \int \tan t d \sec t \\ &= a^2 \cdot \tan t \cdot \sec t - a^2 \cdot \ln | \sec t + \tan t | - a^2 \int \tan t d \sec t \end{aligned}$$

$$\text{移项并整理得: } a^2 \int \tan t d \sec t = \frac{a^2}{2} \cdot \tan t \cdot \sec t - \frac{a^2}{2} \cdot \ln | \sec t + \tan t | + C_1$$

在 Rt $\triangle ABC$ 中, 可设 $\angle B = t$, $|BC| = a$, 则 $|AB| = x$, $|AC| = \sqrt{x^2 - a^2}$

$$\therefore \tan t = \frac{\sqrt{x^2 - a^2}}{a}, \sec t = \frac{x}{a}$$



$$\begin{aligned} \therefore \int \sqrt{x^2 - a^2} dx &= a^2 \int \tan t d \sec t \\ &= \frac{a^2}{2} \cdot \frac{\sqrt{x^2 - a^2}}{a} \cdot \frac{x}{a} - \frac{a^2}{2} \cdot \ln \left| \frac{\sqrt{x^2 - a^2} + x}{a} \right| + C_1 \\ &= \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \cdot \ln \left| x + \sqrt{x^2 - a^2} \right| + C \end{aligned}$$

2. 当 $x < -a$ 时, 可设 $x = a \cdot \sec t$ ($-\frac{\pi}{2} < t < 0$) 同理可证

$$\text{综合讨论 1, 2 得: } \int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \cdot \ln \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$54. \int \sqrt{(x^2 - a^2)^3} dx = \frac{x}{8} \cdot (2x^2 - 5a^2) \sqrt{x^2 - a^2} + \frac{3}{8} \cdot a^4 \cdot \ln \left| x + \sqrt{x^2 - a^2} \right| + C \quad (a > 0)$$

$$\text{证明: } \int \sqrt{(x^2 - a^2)^3} dx = x \cdot (x^2 - a^2)^{\frac{3}{2}} - \int x d(x^2 - a^2)^{\frac{3}{2}}$$

$$= x \cdot (x^2 - a^2)^{\frac{3}{2}} - \int x \cdot \frac{3}{2} \cdot (2x) \cdot (x^2 - a^2)^{\frac{1}{2}} dx$$

$$= x \cdot (x^2 - a^2)^{\frac{3}{2}} - 3 \int x^2 (x^2 - a^2)^{\frac{1}{2}} dx$$

$$= x \cdot (x^2 - a^2)^{\frac{3}{2}} - 3 \int (x^2 - a^2 + a^2) (x^2 - a^2)^{\frac{1}{2}} dx$$

$$= x \cdot (x^2 - a^2)^{\frac{3}{2}} - 3 \int (x^2 - a^2)^{\frac{3}{2}} dx - 3a^2 \int (x^2 - a^2)^{\frac{1}{2}} dx$$

$$\text{移项并整理得: } \int \sqrt{(x^2 - a^2)^3} dx = \frac{x}{4} \cdot (x^2 - a^2)^{\frac{3}{2}} - \frac{3a^2}{4} \int (x^2 - a^2)^{\frac{1}{2}} dx \quad \text{①}$$

$$\text{又 } \int (x^2 - a^2)^{\frac{1}{2}} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \cdot \ln \left| x + \sqrt{x^2 - a^2} \right| + C \quad (\text{公式 53}) \quad \text{②}$$

联立①②得:

$$\begin{aligned} \int \sqrt{(x^2 - a^2)^3} dx &= \frac{x}{4} (x^2 - a^2)^{\frac{3}{2}} - \frac{3x}{8} \cdot a^2 \cdot \sqrt{x^2 - a^2} + \frac{3}{8} \cdot a^4 \cdot \ln \left| x + \sqrt{x^2 - a^2} \right| + C \\ &= \left(\frac{x^3}{4} - \frac{a^2 x}{4} \right) \sqrt{x^2 - a^2} - \frac{3x}{8} \cdot a^2 \cdot \sqrt{x^2 - a^2} + \frac{3}{8} \cdot a^4 \cdot \ln \left| x + \sqrt{x^2 - a^2} \right| + C \\ &= \frac{x}{8} \cdot (2x^2 - 5a^2) \sqrt{x^2 - a^2} + \frac{3}{8} \cdot a^4 \cdot \ln \left| x + \sqrt{x^2 - a^2} \right| + C \end{aligned}$$

$$55. \int x \sqrt{x^2 - a^2} dx = \frac{1}{3} \sqrt{(x^2 - a^2)^3} + C \quad (a > 0)$$

$$\text{证明: } \int x \sqrt{x^2 - a^2} dx = \frac{1}{2} \int \sqrt{x^2 - a^2} d(x^2)$$

$$= \frac{1}{2} \int (x^2 - a^2)^{\frac{1}{2}} d(x^2 - a^2)$$

$$= \frac{1}{2} \times \frac{1}{1 + \frac{1}{2}} \cdot (x^2 - a^2)^{1 + \frac{1}{2}} + C$$

$$= \frac{1}{3} \sqrt{(x^2 - a^2)^3} + C$$

$$56. \int x^2 \sqrt{x^2 - a^2} dx = \frac{x}{8} \cdot (2x^2 - a^2) \sqrt{x^2 - a^2} - \frac{a^4}{8} \cdot \ln \left| x + \sqrt{x^2 - a^2} \right| + C \quad (a > 0)$$

证明:被积函数 $f(x) = x^2 \sqrt{x^2 - a^2}$ 的定义域为 $\{x/x > a \text{ 或 } x < -a\}$

1. 当 $x > a$ 时, 可令 $x = a \cdot \sec t$ ($0 < t < \frac{\pi}{2}$), 则 $x^2 \sqrt{x^2 - a^2} = a^2 \sec^2 t | a \tan t |$

$$\because 0 < t < \frac{\pi}{2}, \tan t > 0, \therefore x^2 \sqrt{x^2 - a^2} = a^3 \sec^2 t \cdot \tan t$$

$$\therefore \int x^2 \sqrt{x^2 - a^2} dx = \int a^3 \sec^2 t \cdot \tan t d(a \sec t) = a^4 \int \sec^3 t d \tan^2 t dt \quad ①$$

$$\begin{aligned} &= \frac{a^4}{3} \int \sec t \cdot 3 \cdot \sec^2 t \cdot \tan^2 t dt = \frac{a^4}{3} \int \sec t d \tan^3 t \\ &= \frac{a^4}{3} \cdot \sec t \cdot \tan^3 t - \frac{a^4}{3} \int \tan^3 t d \sec t \\ &= \frac{a^4}{3} \cdot \sec t \cdot \tan^3 t - \frac{a^4}{3} \int \tan t (\sec^2 t - 1) d \sec t \\ &= \frac{a^4}{3} \cdot \sec t \cdot \tan^3 t - \frac{a^4}{3} \int \tan t \cdot \sec^2 t d \sec t + \frac{a^4}{3} \int \tan t d \sec t \\ &= \frac{a^4}{3} \cdot \sec t \cdot \tan^3 t - \frac{a^4}{3} \int \tan^2 t \cdot \sec^3 t dt + \frac{a^4}{3} \int \tan t d \sec t \end{aligned}$$

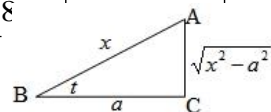
$$\text{移项并整理得: } a^4 \int \sec^3 t d \tan^2 t dt = \frac{a^4}{4} \cdot \sec t \cdot \tan^3 t + \frac{a^4}{4} \int \tan t d \sec t \quad ②$$

$$\begin{aligned} \text{又 } \int \tan t d \sec t &= \sec t \cdot \tan t - \int \sec t dt \tan t = \sec t \cdot \tan t - \int \sec^3 t dt \\ &= \sec t \cdot \tan t - \int (1 + \tan^2 t) \cdot \sec t dt \\ &= \sec t \cdot \tan t - \int \sec t dt - \int \tan^2 t \sec t dt \\ &= \sec t \cdot \tan t - \int \sec t dt - \int \tan t d \sec t \end{aligned}$$

$$\begin{aligned} \text{移项并整理得: } \int \tan t d \sec t &= \frac{1}{2} \cdot \sec t \cdot \tan t - \frac{1}{2} \int \sec t dt \\ &= \frac{1}{2} \cdot \sec t \cdot \tan t - \frac{1}{2} \ln | \sec t + \tan t | + C_1 \quad ③ \end{aligned}$$

$$\text{将③式代入②式得: } a^4 \int \sec^3 t d \tan^2 t dt = \frac{a^4}{4} \tan^3 t \cdot \sec t + \frac{a^4}{8} \sec t \cdot \tan t - \frac{a^4}{8} \ln | \sec t + \tan t | + C_1$$

在 $\text{Rt} \triangle ABC$ 中, 设 $\angle B = t$, $|BC| = a$, 则 $|AB| = x$, $|AC| = \sqrt{x^2 - a^2}$



$$\therefore \tan t = \frac{\sqrt{x^2 - a^2}}{a}, \sec t = \frac{1}{\cos t} = \frac{x}{a}$$

$$\begin{aligned} \therefore a^4 \int \sec^3 t d \tan^2 t dt &= \frac{a^4}{4} \cdot \frac{x}{a} \cdot \frac{x^2 - a^2}{a^3} \cdot \sqrt{x^2 - a^2} + \frac{a^4}{8} \cdot \frac{\sqrt{x^2 - a^2}}{a} \cdot \frac{x}{a} - \frac{a^4}{8} \cdot \ln \left| \frac{x + \sqrt{x^2 - a^2}}{a} \right| + C_1 \\ &= \frac{x}{4} \cdot (x^2 - a^2) \cdot \sqrt{x^2 - a^2} + \frac{a^2 x}{8} \cdot \sqrt{x^2 - a^2} - \frac{a^4}{8} \cdot \ln \left| x + \sqrt{x^2 - a^2} \right| + C_2 \\ &= \frac{x}{8} \cdot (2x^2 - a^2) \sqrt{x^2 - a^2} - \frac{a^4}{8} \cdot \ln \left| x + \sqrt{x^2 - a^2} \right| + C \end{aligned}$$

2. 当 $x < -a$, 即 $-x > a$ 时, 令 $\mu = -x$, 则 $x = -\mu$

$$\text{由讨论1得: } \int x^2 \sqrt{x^2 - a^2} dx = - \int \mu^2 \sqrt{\mu^2 - a^2} d\mu$$

$$= \frac{-\mu}{8} \cdot (2\mu^2 - a^2) \sqrt{\mu^2 - a^2} + \frac{a^4}{8} \cdot \ln \left| -\mu + \sqrt{\mu^2 - a^2} \right| + C$$

$$\text{将 } \mu = -x \text{ 代入上式得: } \int x^2 \sqrt{x^2 - a^2} dx = \frac{x}{8} \cdot (2x^2 - a^2) \sqrt{x^2 - a^2} - \frac{a^4}{8} \cdot \ln \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\text{综合讨论 1 与 2 得: } \int x^2 \sqrt{x^2 - a^2} dx = \frac{x}{8} \cdot (2x^2 - a^2) \sqrt{x^2 - a^2} - \frac{a^4}{8} \cdot \ln \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$57. \int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \cdot \arccos \frac{a}{|x|} + C \quad (a > 0)$$

证法1: 被积函数 $f(x) = \frac{\sqrt{x^2 - a^2}}{x}$ 的定义域为 $\{x/x > a \text{ 或 } x < -a\}$

1. 当 $x > a$ 时, 可设 $x = a \cdot \sec t$ ($0 < t < \frac{\pi}{2}$),

$$\text{则 } \frac{\sqrt{x^2 - a^2}}{x} = \frac{a \cdot \tan t}{a \cdot \sec t}, \quad dx = a \cdot \sec t \cdot \tan t dt$$

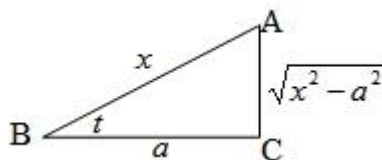
$$\begin{aligned} \therefore \int \frac{\sqrt{x^2 - a^2}}{x} dx &= \int \frac{a \cdot \tan t \cdot a \cdot \sec t \cdot \tan t}{a \cdot \sec t} dt = \int a \cdot \tan^2 t dt \\ &= a \int \frac{\sin^2 t}{\cos^2 t} dt = a \int \frac{1 - \cos^2 t}{\cos^2 t} dt = a \int \frac{1}{\cos^2 t} dt - \int dt \\ &= a \cdot \tan t - a \cdot t + C \end{aligned}$$

$$\therefore x = a \cdot \sec t, \therefore \cos t = \frac{a}{x}, \therefore t = \arccos \frac{a}{x}$$

在 $\text{Rt}\triangle ABC$ 中, 设 $\angle B = t$, $|BC| = a$, 则 $|AB| = x$, $|AC| = \sqrt{x^2 - a^2}$

$$\therefore \tan t = \frac{|AC|}{|BC|} = \frac{\sqrt{x^2 - a^2}}{a}$$

$$\therefore \int \frac{\sqrt{x^2 - a^2}}{x} dx = a \cdot \tan t - a \cdot t + C$$



$$= \sqrt{x^2 - a^2} - a \cdot \arccos \frac{a}{x} + C$$

2. 当 $x < -a$, 即 $-x > a$ 时, 令 $\mu = -x$, 即 $x = -\mu$

$$\begin{aligned} \text{由讨论 1 可知 } \int \frac{\sqrt{x^2 - a^2}}{x} dx &= \int \frac{\sqrt{\mu^2 - a^2}}{\mu} d\mu = \sqrt{\mu^2 - a^2} - a \cdot \arccos \frac{a}{\mu} + C \\ &= \sqrt{x^2 - a^2} - a \cdot \arccos \frac{a}{-x} + C \end{aligned}$$

$$\text{综合讨论 1, 2, 可写成: } \int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \cdot \arccos \frac{a}{|x|} + C$$

$$57. \int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \cdot \arccos \frac{a}{|x|} + C \quad (a > 0)$$

证法 2: 被积函数 $f(x) = \frac{\sqrt{x^2 - a^2}}{x}$ 的定义域为 $\{x/x > a \text{ 或 } x < -a\}$

1. 当 $x > a$ 时, 可设 $x = a \cdot \operatorname{ch} t$ ($0 < t$),

$$\text{则 } \frac{\sqrt{x^2 - a^2}}{x} = \frac{a \cdot \operatorname{sh} t}{a \cdot \operatorname{ch} t} = \frac{\operatorname{sh} t}{\operatorname{ch} t}, dx = a \cdot \operatorname{sh} t dt$$

$$\therefore \int \frac{\sqrt{x^2 - a^2}}{x} dx = \int \frac{\operatorname{sh} t}{\operatorname{ch} t} \cdot a \cdot \operatorname{sh} t dt = a \int \frac{\operatorname{sh}^2 t}{\operatorname{ch} t} dt$$

$$= a \int \frac{\operatorname{ch}^2 t - 1}{\operatorname{ch} t} dt = a \int \operatorname{ch} t dt - a \int \frac{\operatorname{ch} t}{\operatorname{ch}^2 t} dt$$

$$= a \int \operatorname{ch} t dt - a \int \frac{1}{1 + \operatorname{sh}^2 t} d\operatorname{sh} t$$

$$= a \cdot \operatorname{sh} t - a \cdot \arctan(\operatorname{sh} t) + C$$

提示: $\operatorname{ch}^2 t - \operatorname{sh}^2 t = 1$
 $(\operatorname{ch} t)' = \operatorname{sh} t$
 $(\operatorname{sh} t)' = \operatorname{ch} t$

公式 19: $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \cdot \arctan \frac{x}{a} + C$

$$\because x = a \cdot \operatorname{ch} t, \therefore \operatorname{ch} t = \frac{x}{a}, \therefore \operatorname{sh} t = \sqrt{1 - \operatorname{ch}^2 t} = \frac{\sqrt{x^2 - a^2}}{a}$$

在 $\operatorname{Rt} \triangle ABC$ 中, 设 $\tan y = \operatorname{sh} t = \frac{\sqrt{x^2 - a^2}}{a}$, $\angle B = y$, $|BC| = a$

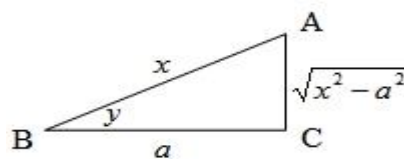
$$\therefore y = \arctan(\operatorname{sh} t), |AC| = \sqrt{x^2 - a^2}, |AB| = \sqrt{|AC|^2 + |BC|^2} = x$$

$$\therefore \cos y = \frac{|BC|}{|AB|} = \frac{a}{x}$$

$$\text{即 } \cos y = \cos \arctan(\operatorname{sh} t) = \frac{a}{x}$$

$$\therefore \arctan(\operatorname{sh} t) = \arccos \frac{a}{x}$$

$$\therefore \int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \cdot \arccos \frac{a}{x} + C$$



2. 当 $x < -a$, 即 $-x > a$ 时, 令 $\mu = -x$, 即 $x = -\mu$

$$\text{由讨论 1 可知 } \int \frac{\sqrt{x^2 - a^2}}{x} dx = \int \frac{\sqrt{\mu^2 - a^2}}{\mu} d\mu = \sqrt{\mu^2 - a^2} - a \cdot \arccos \frac{a}{\mu} + C$$

$$= \sqrt{x^2 - a^2} - a \cdot \arccos \frac{a}{-x} + C$$

$$\text{综合讨论 1, 2, 可写成: } \int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \cdot \arccos \frac{a}{|x|} + C$$

$$58. \int \frac{\sqrt{x^2 - a^2}}{x^2} dx = -\frac{\sqrt{x^2 - a^2}}{x} + \ln \left| x + \sqrt{x^2 - a^2} \right| + C \quad (a > 0)$$

$$\begin{aligned} \text{证明: } \int \frac{\sqrt{x^2 - a^2}}{x^2} dx &= -\int \sqrt{x^2 - a^2} d\frac{1}{x} \\ &= -\frac{\sqrt{x^2 - a^2}}{x} + \int \frac{1}{x} d\sqrt{x^2 - a^2} \\ &= -\frac{\sqrt{x^2 - a^2}}{x} + \int \frac{1}{x} \cdot \frac{1}{2} \cdot 2x \cdot (x^2 - a^2)^{-\frac{1}{2}} dx \\ &= -\frac{\sqrt{x^2 - a^2}}{x} + \int \frac{1}{\sqrt{x^2 - a^2}} dx \quad \boxed{\text{公式45: } \int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| + C} \\ &= -\frac{\sqrt{x^2 - a^2}}{x} + \ln \left| x + \sqrt{x^2 - a^2} \right| + C \end{aligned}$$

(八) 含有 $\sqrt{a^2 - x^2}$ ($a > 0$) 的积分 (59~72)

$$59. \int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C \quad (a > 0)$$

证明: 被积函数 $f(x) = \frac{1}{\sqrt{a^2 - x^2}}$ 的定义域为 $\{x | -a < x < a\}$

$$\therefore \text{可设 } x = a \cdot \sin t \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right), \text{ 则 } dx = a \cdot \cos t dt, \frac{1}{\sqrt{a^2 - x^2}} = \frac{1}{|a \cdot \cos t|}$$

$$\because -\frac{\pi}{2} < t < \frac{\pi}{2}, \cos t > 0 \therefore \frac{1}{\sqrt{a^2 - x^2}} = \frac{1}{a \cdot \cos t}$$

$$\begin{aligned} \therefore \int \frac{dx}{\sqrt{a^2 - x^2}} &= \int \frac{1}{a \cdot \cos t} \cdot a \cdot \cos t dt \\ &= \int dt \\ &= t + C \end{aligned}$$

$$\because x = a \cdot \sin t \quad \therefore t = \arcsin \frac{x}{a}$$

$$\therefore \int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$$

$$60. \int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2 \cdot \sqrt{a^2 - x^2}} + C \quad (a > 0)$$

证明: 被积函数 $f(x) = \frac{1}{\sqrt{(a^2 - x^2)^3}}$ 的定义域为 $\{x | -a < x < a\}$

$$\therefore \text{可设 } x = a \cdot \sin t \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right), \text{ 则 } dx = a \cdot \cos t \, dt, \frac{1}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{|a^3 \cdot \cos^3 t|}$$

$$\because -\frac{\pi}{2} < t < \frac{\pi}{2}, \cos t > 0 \therefore \frac{1}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{a^3 \cdot \cos^3 t}$$

$$\therefore \int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \int \frac{1}{a^3 \cdot \cos^3 t} \cdot a \cdot \cos t \, dt$$

$$= \int \frac{1}{a^2 \cdot \cos^2 t} \, dt$$

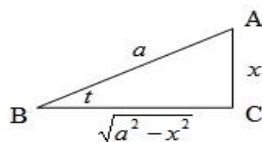
$$= \int \frac{1}{a^2} \cdot \sec^2 t \, dt$$

$$= \frac{1}{a^2} \cdot \tan t + C$$

在 $\text{Rt} \triangle ABC$ 中, 设 $\angle B = t$, $|AB| = a$, 则 $|AC| = x$, $|BC| = \sqrt{a^2 - x^2}$

$$\therefore \tan t = \frac{x}{\sqrt{a^2 - x^2}}$$

$$\therefore \int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2 \cdot \sqrt{a^2 - x^2}} + C$$



$$61. \int \frac{x}{\sqrt{a^2 - x^2}} \, dx = -\sqrt{a^2 - x^2} + C \quad (a > 0)$$

$$\text{证明: } \int \frac{x}{\sqrt{a^2 - x^2}} \, dx = \frac{1}{2} \int (a^2 - x^2)^{-\frac{1}{2}} \, dx^2$$

$$= -\frac{1}{2} \int (a^2 - x^2)^{-\frac{1}{2}} \, d(a^2 - x^2)$$

$$= -\frac{1}{2} \times \frac{1}{1 - \frac{1}{2}} \cdot (a^2 - x)^{1 - \frac{1}{2}} + C$$

$$= -\sqrt{a^2 - x^2} + C$$

$$62. \int \frac{x}{\sqrt{(a^2-x^2)^3}} dx = \frac{1}{\sqrt{a^2-x^2}} + C \quad (a > 0)$$

$$\begin{aligned} \text{证明: } \int \frac{x}{\sqrt{(a^2-x^2)^3}} dx &= \frac{1}{2} \int (a^2-x^2)^{-\frac{3}{2}} d(x^2) \\ &= -\frac{1}{2} \int (a^2-x^2)^{-\frac{3}{2}} d(a^2-x^2) \\ &= -\frac{1}{2} \times \frac{1}{1-\frac{3}{2}} \cdot (a^2-x^2)^{1-\frac{3}{2}} + C \\ &= \frac{1}{\sqrt{a^2-x^2}} + C \end{aligned}$$

$$63. \int \frac{x^2}{\sqrt{a^2-x^2}} dx = -\frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \cdot \arcsin \frac{x}{a} + C \quad (a > 0)$$

证明: 被积函数 $f(x) = \frac{x^2}{\sqrt{a^2-x^2}}$ 的定义域为 $\{x | -a < x < a\}$

$$\therefore \text{可设 } x = a \cdot \sin t \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right), \text{ 则 } dx = a \cdot \cos t dt, \frac{x^2}{\sqrt{a^2-x^2}} = \frac{a^2 \cdot \sin^2 t}{|a \cdot \cos t|}$$

$$\because -\frac{\pi}{2} < t < \frac{\pi}{2}, \cos t > 0 \therefore \frac{x^2}{\sqrt{a^2-x^2}} = \frac{a \cdot \sin^2 t}{\cos t}$$

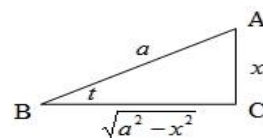
$$\begin{aligned} \therefore \int \frac{x^2}{\sqrt{a^2-x^2}} dx &= \int \frac{a \cdot \sin^2 t}{\cos t} \cdot a \cdot \cos t dt \\ &= a^2 \int \sin^2 t dt \\ &= a^2 \int \frac{1 - \cos 2t}{2} dt \\ &= \frac{a^2}{2} \int dt - \frac{a^2}{4} \int \cos 2t d(2t) \\ &= \frac{a^2}{2} \cdot t - \frac{a^2}{4} \cdot \sin 2t + C \\ &= \frac{a^2}{2} \cdot t - \frac{a^2}{2} \cdot \sin t \cdot \cos t + C \end{aligned}$$

提示: $\cos 2t = \cos^2 t - \sin^2 t$
 $= 1 - 2\sin^2 t$
 $\sin 2t = 2 \cdot \sin t \cdot \cos t$

在 Rt $\triangle ABC$ 中, 设 $\angle B = t$, $|AB| = a$, 则 $|AC| = x$, $|BC| = \sqrt{a^2-x^2}$

$$\therefore \sin t = \frac{x}{a}, \cos t = \frac{\sqrt{a^2-x^2}}{a}$$

$$\therefore \int \frac{x^2}{\sqrt{a^2-x^2}} dx = -\frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \cdot \arcsin \frac{x}{a} + C$$



$$64. \int \frac{x^2}{\sqrt{(a^2 - x^2)^3}} dx = \frac{x}{\sqrt{a^2 - x^2}} - \arcsin \frac{x}{a} + C \quad (a > 0)$$

证明: 被积函数 $f(x) = \frac{x^2}{\sqrt{(a^2 - x^2)^3}}$ 的定义域为 $\{x | -a < x < a\}$

$$\therefore \text{可设 } x = a \cdot \sin t \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right), \text{ 则 } dx = a \cdot \cos t dt, \frac{x^2}{\sqrt{(a^2 - x^2)^3}} = \frac{a^2 \cdot \sin^2 t}{|a^3 \cdot \cos^3 t|}$$

$$\because -\frac{\pi}{2} < t < \frac{\pi}{2}, \cos t > 0 \therefore \frac{x^2}{\sqrt{(a^2 - x^2)^3}} = \frac{\sin^2 t}{a \cdot \cos^3 t}$$

$$\therefore \int \frac{x^2}{\sqrt{a^2 - x^2}} dx = \int \frac{\sin^2 t}{a \cdot \cos^3 t} \cdot a \cdot \cos t dt$$

$$= \int \frac{\sin^2 t}{\cos^2 t} dt$$

$$= \int \frac{1 - \cos^2 t}{\cos^2 t} dt$$

$$= \int \frac{1}{\cos^2 t} dt - \int dt$$

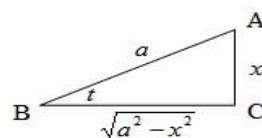
$$= \int d \tan t - \int dt$$

$$= \tan t - t + C$$

在 Rt $\triangle ABC$ 中, 设 $\angle B = t$, $|AB| = a$, 则 $|AC| = x$, $|BC| = \sqrt{a^2 - x^2}$

$$\therefore \tan t = \frac{x}{\sqrt{a^2 - x^2}}$$

$$\therefore \int \frac{x^2}{\sqrt{(a^2 - x^2)^3}} dx = \frac{x}{\sqrt{a^2 - x^2}} - \arcsin \frac{x}{a} + C$$



$$65. \int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a} \cdot \ln \frac{a-\sqrt{a^2-x^2}}{|x|} + C \quad (a > 0)$$

证明: 被积函数 $f(x) = \frac{1}{x\sqrt{a^2-x^2}}$ 的定义域为 $\{x/-a < x < a \text{ 且 } x \neq 0\}$

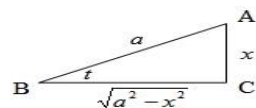
1. 当 $-a < x < 0$ 时, 可设 $x = a \cdot \sin t$ ($-\frac{\pi}{2} < t < 0$), 则 $dx = a \cdot \cos t dt$

$$x\sqrt{a^2-x^2} = a \cdot \sin t \cdot |a \cdot \cos t| \quad \because -\frac{\pi}{2} < t < 0, \cos t > 0 \quad \therefore x\sqrt{a^2-x^2} = a^2 \cdot \sin t \cdot \cos t$$

$$\begin{aligned} \therefore \int \frac{dx}{x\sqrt{a^2-x^2}} &= \int \frac{1}{a^2 \cdot \sin t \cdot \cos t} \cdot a \cdot \cos t dt \\ &= \frac{1}{a} \int \frac{1}{\sin t} dt \\ &= \frac{1}{a} \int \frac{\sin t}{\sin^2 t} dt \\ &= -\frac{1}{a} \int \frac{1}{1-\cos^2 t} d \cos t \\ &= -\frac{1}{2a} \int \left(\frac{1}{1+\cos t} + \frac{1}{1-\cos t} \right) d \cos t \\ &= -\frac{1}{2a} \int \frac{1}{1+\cos t} d(\cos t + 1) + \frac{1}{2a} \int \frac{1}{1-\cos t} d(1-\cos t) \\ &= -\frac{1}{2a} \cdot \ln |1+\cos t| + \frac{1}{2a} \cdot \ln |\cos t - 1| + C_1 \\ &= \frac{1}{2a} \cdot \ln \left| \frac{\cos t - 1}{1+\cos t} \right| + C_1 \\ &= \frac{1}{2a} \cdot \ln \left| \frac{(\cos t - 1)^2}{1-\cos^2 t} \cdot (-1) \right| + C_1 \\ &= \frac{1}{2a} \cdot \ln \left| \frac{(\cos t - 1)^2}{\sin^2 t} \right| + C_2 \\ &= \frac{1}{a} \cdot \ln \left| \frac{\cos t - 1}{\sin t} \right| + C_2 \\ &= \frac{1}{a} \cdot \ln | \cot t - \csc t | + C_2 \end{aligned}$$

在 Rt $\triangle ABC$ 中, 设 $\angle B = t$, $|AB| = a$, 则 $|AC| = x$, $|BC| = \sqrt{a^2-x^2}$

$$\therefore \cot t = \frac{\sqrt{a^2-x^2}}{x}, \csc t = \frac{1}{\sin t} = \frac{a}{x}$$



$$\begin{aligned} \therefore \int \frac{dx}{x\sqrt{a^2-x^2}} &= \frac{1}{a} \cdot \ln \left| \frac{\sqrt{a^2-x^2}-a}{x} \right| + C_2 = \frac{1}{a} \cdot \ln \left| \frac{a-\sqrt{a^2-x^2}}{x} \cdot (-1) \right| + C_2 \\ &= \frac{1}{a} \cdot \ln \left| \frac{a-\sqrt{a^2-x^2}}{x} \right| + C_3 \end{aligned}$$

$$\because a - \sqrt{a^2-x^2} > 0$$

$$\therefore \int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a} \cdot \ln \frac{a-\sqrt{a^2-x^2}}{|x|} + C$$

2. 当 $0 < x < a$ 时, 可设 $x = a \cdot \sin t$ ($0 < t < \frac{\pi}{2}$), 同理可证

$$\text{综合讨论 1, 2 得: } \int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a} \cdot \ln \frac{a-\sqrt{a^2-x^2}}{|x|} + C$$

$$66. \int \frac{dx}{x^2 \sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x} + C \quad (a > 0)$$

证明: 被积函数 $f(x) = \frac{1}{x^2 \sqrt{a^2 - x^2}}$ 的定义域为 $\{x | -a < x < a \text{ 且 } x \neq 0\}$

1. 当 $-a < x < 0$ 时, 可设 $x = a \cdot \sin t$ ($-\frac{\pi}{2} < t < 0$), 则 $dx = a \cdot \cos t dt$,

$$\frac{1}{x^2 \sqrt{a^2 - x^2}} = \frac{1}{a^2 \cdot \sin^2 t} \cdot \frac{1}{|a \cdot \cos t|}$$

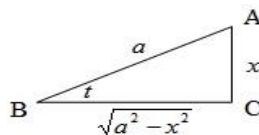
$$\because -\frac{\pi}{2} < t < \frac{\pi}{2}, \cos t > 0 \therefore \frac{1}{x^2 \sqrt{a^2 - x^2}} = \frac{1}{a^3 \cdot \sin^2 t \cdot \cos t}$$

$$\begin{aligned} \therefore \int \frac{dx}{x^2 \sqrt{a^2 - x^2}} &= \int \frac{1}{a^3 \cdot \sin^2 t \cdot \cos t} \cdot a \cdot \cos t dt \\ &= \frac{1}{a^2} \int \frac{1}{\sin^2 t} dt \\ &= -\frac{1}{a^2} \int -\csc^2 t dt \\ &= -\frac{1}{a^2} \cdot \cot t + C \end{aligned}$$

在 $\text{Rt} \triangle ABC$ 中, 设 $\angle B = t$, $|AB| = a$, 则 $|AC| = x$, $|BC| = \sqrt{a^2 - x^2}$

$$\therefore \cot t = \frac{\sqrt{a^2 - x^2}}{x}$$

$$\therefore \int \frac{dx}{x^2 \sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x} + C$$



2. 当 $0 < x < a$ 时, 可设 $x = a \cdot \sin t$ ($0 < t < \frac{\pi}{2}$), 同理可证

$$\text{综合讨论 1, 2 得: } \int \frac{dx}{x^2 \sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x} + C$$

$$67. \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \cdot \arcsin \frac{x}{a} + C \quad (a > 0)$$

证明: 被积函数 $f(x) = \sqrt{a^2 - x^2}$ 的定义域为 $\{x | -a < x < a\}$

$$\therefore \text{可设 } x = a \cdot \sin t \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right), \text{ 则 } dx = a \cdot \cos t dt, \sqrt{a^2 - x^2} = |a \cdot \cos t|$$

$$\because -\frac{\pi}{2} < t < \frac{\pi}{2}, \cos t > 0 \therefore \sqrt{a^2 - x^2} = a \cdot \cos t$$

$$\begin{aligned} \therefore \int \sqrt{a^2 - x^2} dx &= \int a \cdot \cos t \cdot a \cdot \cos t dt \\ &= a^2 \int \cos^2 t dt \\ &= a^2 \int (1 - \sin^2 t) dt \\ &= a^2 \int dt - a^2 \int \sin^2 t dt \end{aligned} \quad ①$$

$$\begin{aligned} \text{又 } \int \sqrt{a^2 - x^2} dx &= a^2 \int \cos^2 t dt \\ &= a^2 \int \cos t d \sin t \\ &= a^2 \cdot \sin t \cdot \cos t - a^2 \int \sin t d \cos t \\ &= a^2 \cdot \sin t \cdot \cos t + a^2 \int \sin^2 t dt \end{aligned} \quad ②$$

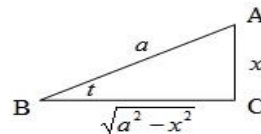
$$\text{由 } ① + ② \text{ 得: } 2 \int \sqrt{a^2 - x^2} dx = a^2 \int dt + a^2 \cdot \sin t \cdot \cos t = a^2 t + a^2 \cdot \sin t \cdot \cos t$$

$$\therefore \int \sqrt{a^2 - x^2} dx = \frac{a^2}{2} t + \frac{a^2}{2} \cdot \sin t \cdot \cos t + C$$

在 $\text{Rt} \triangle ABC$ 中, 设 $\angle B = t$, $|AB| = a$, 则 $|AC| = x$, $|BC| = \sqrt{a^2 - x^2}$

$$\therefore \sin t = \frac{x}{a}, \cos t = \frac{\sqrt{a^2 - x^2}}{a}$$

$$\begin{aligned} \therefore \int \sqrt{a^2 - x^2} dx &= \frac{a^2}{2} \cdot \arcsin \frac{x}{a} + \frac{a^2}{2} \cdot \frac{\sqrt{a^2 - x^2}}{a} \cdot \frac{x}{a} + C \\ &= \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \cdot \arcsin \frac{x}{a} + C \end{aligned}$$



$$68. \int \sqrt{(a^2 - x^2)^3} dx = \frac{x}{8} \cdot (5a^2 - 2x^2) \sqrt{a^2 - x^2} + \frac{3}{8} \cdot a^4 \cdot \arcsin \frac{x}{a} + C \quad (a > 0)$$

$$\begin{aligned} \text{证明: } \int \sqrt{(a^2 - x^2)^3} dx &= x \cdot (a^2 - x^2)^{\frac{3}{2}} - \int x d(a^2 - x^2)^{\frac{3}{2}} \\ &= x \cdot (a^2 - x^2)^{\frac{3}{2}} - \int x \cdot \frac{3}{2} \cdot (-2x) \cdot (a^2 - x^2)^{\frac{1}{2}} dx \\ &= x \cdot (a^2 - x^2)^{\frac{3}{2}} + 3 \int x^2 (a^2 - x^2)^{\frac{1}{2}} dx \\ &= x \cdot (a^2 - x^2)^{\frac{3}{2}} + 3 \int (x^2 - a^2 + a^2) (a^2 - x^2)^{\frac{1}{2}} dx \\ &= x \cdot (a^2 - x^2)^{\frac{3}{2}} - 3 \int (a^2 - x^2)^{\frac{1}{2}} dx + 3a^2 \int (a^2 - x^2)^{\frac{1}{2}} dx \end{aligned}$$

$$\text{移项并整理得: } \int \sqrt{(a^2 - x^2)^3} dx = \frac{x}{4} \cdot (a^2 - x^2)^{\frac{3}{2}} + \frac{3a^2}{4} \int (a^2 - x^2)^{\frac{1}{2}} dx \quad \textcircled{1}$$

$$\text{又 } \int (a^2 - x^2)^{\frac{1}{2}} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \cdot \arcsin \frac{x}{a} + C \quad (\text{公式 67}) \quad \textcircled{2}$$

联立①②得:

$$\begin{aligned} \int \sqrt{(a^2 - x^2)^3} dx &= \frac{x}{4} (a^2 - x^2)^{\frac{3}{2}} + \frac{3x}{8} \cdot a^2 \cdot \sqrt{a^2 - x^2} + \frac{3}{8} \cdot a^4 \cdot \arcsin \frac{x}{a} + C \\ &= \left(\frac{a^2 x}{4} - \frac{x^3}{4} \right) \sqrt{a^2 - x^2} + \frac{3x}{8} \cdot a^2 \cdot \sqrt{a^2 - x^2} + \frac{3}{8} \cdot a^4 \cdot \arcsin \frac{x}{a} + C \\ &= \frac{x}{8} \cdot (5a^2 - 2x^2) \sqrt{a^2 - x^2} + \frac{3}{8} \cdot a^4 \cdot \arcsin \frac{x}{a} + C \end{aligned}$$

$$69. \int x \sqrt{a^2 - x^2} dx = -\frac{1}{3} \sqrt{(a^2 - x^2)^3} + C \quad (a > 0)$$

证明: 被积函数 $f(x) = x \sqrt{a^2 - x^2}$ 的定义域为 $\{x | -a < x < a\}$

$$\therefore \text{可设 } x = a \cdot \sin t \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right), \text{ 则 } dx = a \cdot \cos t dt, x \sqrt{a^2 - x^2} = a \cdot \sin t \cdot |a \cdot \cos t|$$

$$\therefore -\frac{\pi}{2} < t < \frac{\pi}{2}, \cos t > 0 \therefore x \sqrt{a^2 - x^2} = a^2 \cdot \sin t \cdot \cos t$$

$$\therefore \int x \sqrt{a^2 - x^2} dx = \int a^2 \cdot \sin t \cdot \cos t \cdot a \cdot \cos t dt = a^3 \int \cos^2 t \cdot \sin t dt$$

$$= -a^3 \int \cos^2 t d \cos t = -\frac{a^3}{3} \cos^3 t + C$$

$$= -\frac{a^3}{3} (1 - \sin^2 t)^{\frac{3}{2}} + C$$

$$\therefore x = a \cdot \sin t \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right), \therefore \sin t = \frac{x}{a}$$

$$\therefore (1 - \sin^2 t)^{\frac{3}{2}} = \left(\frac{a^2 - x^2}{a^2}\right)^{\frac{3}{2}} = \frac{\sqrt{(a^2 - x^2)^3}}{a^3}$$

$$\begin{aligned} \therefore \int x \sqrt{a^2 - x^2} dx &= -\frac{a^3}{3} (1 - \sin^2 t)^{\frac{3}{2}} + C \\ &= -\frac{1}{3} \sqrt{(a^2 - x^2)^3} + C \end{aligned}$$

$$70. \int x^2 \sqrt{a^2 - x^2} dx = \frac{x}{8} \cdot (2x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \cdot \arcsin \frac{x}{a} + C \quad (a > 0)$$

证明: 被积函数 $f(x) = x^2 \sqrt{a^2 - x^2}$ 的定义域为 $\{x | -a < x < a\}$

$$\therefore \text{可令 } x = a \cdot \sin t \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right), \quad \text{则 } x^2 \sqrt{a^2 - x^2} = a^2 \cdot \sin^2 t \cdot |a \cdot \cos t|$$

$$\because -\frac{\pi}{2} < t < \frac{\pi}{2}, \cos t > 0, \therefore x^2 \sqrt{a^2 - x^2} = a^3 \cdot \sin^2 t \cdot \cos t$$

$$\begin{aligned} \therefore \int x^2 \sqrt{a^2 - x^2} dx &= \int a^3 \sin^2 t \cdot \cos t d(a \cdot \sin t) = a^4 \int \sin^2 t \cdot \cos^2 t dt \\ &= \frac{a^4}{3} \int 3 \cdot \sin^2 t \cdot \cos t \cdot \cos t dt \\ &= \frac{a^4}{3} \int \cos t d \sin^3 t \\ &= \frac{a^4}{3} \cdot \cos t \cdot \sin^3 t - \frac{a^4}{3} \int \sin^3 t d \cos t \\ &= \frac{a^4}{3} \cdot \cos t \cdot \sin^3 t - \frac{a^4}{3} \int \sin t \cdot (1 - \cos^2 t) d \cos t \\ &= \frac{a^4}{3} \cdot \cos t \cdot \sin^3 t - \frac{a^4}{3} \int \sin t d \cos t + \frac{a^4}{3} \int \sin t \cdot \cos^2 t d \cos t \\ &= \frac{a^4}{3} \cdot \cos t \cdot \sin^3 t - \frac{a^4}{3} \int \sin t d \cos t + \frac{a^4}{3} \int \sin^2 t \cdot \cos^2 t dt \end{aligned}$$

$$\begin{aligned} \text{移项并整理得: } \int x^2 \sqrt{a^2 - x^2} dx &= a^4 \int \sin^2 t \cdot \cos^2 t dt \\ &= \frac{a^4}{4} \cdot \cos t \cdot \sin^3 t - \frac{a^4}{4} \int \sin t d \cos t \end{aligned} \quad ①$$

$$\begin{aligned} \because \int \sin t d \cos t &= \sin t \cdot \cos t - \int \cos t d \sin t \\ &= \sin t \cdot \cos t - \int \cos^2 t dt \\ &= \sin t \cdot \cos t - \int (1 - \sin^2 t) dt \\ &= \sin t \cdot \cos t - \int dt + \int \sin^2 t dt \end{aligned} \quad ②$$

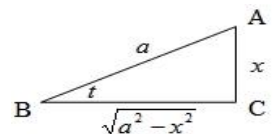
$$\text{又} \because \int \sin t d \cos t = -\int \sin^2 t dt \quad ③$$

$$\begin{aligned} \text{联立②③得: } \int \sin t d \cos t &= \frac{1}{2} \cdot \sin t \cdot \cos t - \frac{1}{2} \int dt \\ &= \frac{1}{2} \cdot \sin t \cdot \cos t - \frac{1}{2} \cdot t + C_1 \end{aligned} \quad ④$$

$$\text{联立①④得: } \int x^2 \sqrt{a^2 - x^2} dx = \frac{a^4}{4} \cdot \cos t \cdot \sin^3 t - \frac{a^4}{8} \cdot \sin t \cdot \cos t - \frac{a^4}{8} \cdot t + C$$

在 $\text{Rt}\triangle ABC$ 中, 设 $\angle B = t$, $|AB| = a$, 则 $|AC| = x$, $|BC| = \sqrt{a^2 - x^2}$

$$\therefore \cos t = \frac{\sqrt{a^2 - x^2}}{a}, \sin t = \frac{x}{a}$$



$$\begin{aligned} \therefore \int x^2 \sqrt{a^2 - x^2} dx &= \frac{a^4}{4} \cdot \frac{\sqrt{a^2 - x^2}}{a} \cdot \frac{x^3}{a^3} - \frac{a^4}{8} \cdot \frac{\sqrt{a^2 - x^2}}{a} \cdot \frac{x}{a} + \frac{a^4}{8} \cdot \arcsin \frac{x}{a} + C \\ &= \frac{x}{8} \cdot (2x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \cdot \arcsin \frac{x}{a} + C \end{aligned}$$

$$71. \int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} + a \cdot \ln \frac{a - \sqrt{a^2 - x^2}}{|x|} + C \quad (a > 0)$$

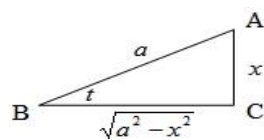
证明: 被积函数 $f(x) = \frac{1}{x\sqrt{a^2 - x^2}}$ 的定义域为 $\{x | -a < x < a \text{ 且 } x \neq 0\}$

1. 当 $-a < x < 0$ 时, 可设 $x = a \cdot \sin t$ ($-\frac{\pi}{2} < t < 0$), 则 $dx = a \cdot \cos t dt$

$$\begin{aligned} \frac{\sqrt{a^2 - x^2}}{x} &= \frac{|a \cdot \cos t|}{a \cdot \sin t} \quad \because -\frac{\pi}{2} < t < 0, \cos t > 0 \quad \therefore \frac{\sqrt{a^2 - x^2}}{x} = \frac{\cos t}{\sin t} \\ \therefore \int \frac{\sqrt{a^2 - x^2}}{x} dx &= \int \frac{\cos t}{\sin t} \cdot a \cdot \cos t dt = a \int \frac{\cos^2 t}{\sin t} dt \\ &= a \int \frac{1 - \sin^2 t}{\sin t} dt = a \int \frac{1}{\sin t} dt - a \int \sin t dt \\ &= a \int \frac{\sin t}{\sin^2 t} dt - a \int \sin t dt = -a \int \frac{1}{1 - \cos^2 t} d\cos t - a \int \sin t dt \\ &= -\frac{a}{2} \int \left(\frac{1}{1 + \cos t} + \frac{1}{1 - \cos t} \right) d\cos t - a \int \sin t dt \\ &= -\frac{a}{2} \int \frac{1}{1 + \cos t} d(\cos t + 1) + \frac{a}{2} \int \frac{1}{\cos t - 1} d(\cos t - 1) - a \int \sin t dt \\ &= -\frac{a}{2} \cdot \ln |1 + \cos t| + \frac{a}{2} \cdot \ln |\cos t - 1| + a \cdot \cos t + C_1 \\ &= \frac{a}{2} \cdot \ln \left| \frac{\cos t - 1}{1 + \cos t} \right| + a \cdot \cos t + C_1 \\ &= \frac{a}{2} \cdot \ln \left| \frac{(\cos t - 1)^2}{1 - \cos^2 t} \cdot (-1) \right| + a \cdot \cos t + C_1 \\ &= \frac{a}{2} \cdot \ln \left| \frac{(\cos t - 1)^2}{\sin^2 t} \right| + a \cdot \cos t + C_2 \\ &= a \cdot \ln \left| \frac{\cos t - 1}{\sin t} \right| + a \cdot \cos t + C_2 \\ &= a \cdot \ln | \cot t - \csc t | + a \cdot \cos t + C_2 \end{aligned}$$

在 Rt $\triangle ABC$ 中, 设 $\angle B = t$, $|AB| = a$, 则 $|AC| = x$, $|BC| = \sqrt{a^2 - x^2}$

$$\begin{aligned} \therefore \cot t &= \frac{\sqrt{a^2 - x^2}}{x}, \csc t = \frac{1}{\sin t} = \frac{a}{x}, \cos t = \frac{\sqrt{a^2 - x^2}}{a} \\ \therefore \int \frac{\sqrt{a^2 - x^2}}{x} dx &= a \cdot \ln \left| \frac{\sqrt{a^2 - x^2} - a}{x} \right| + a \cdot \frac{\sqrt{a^2 - x^2}}{a} + C_2 \\ &= a \cdot \ln \left| \frac{a - \sqrt{a^2 - x^2}}{x} \cdot (-1) \right| + a \cdot \frac{\sqrt{a^2 - x^2}}{a} + C_2 \\ &= \frac{1}{a} \cdot \ln \left| \frac{a - \sqrt{a^2 - x^2}}{x} \right| + a \cdot \frac{\sqrt{a^2 - x^2}}{a} + C_3 \\ \therefore a - \sqrt{a^2 - x^2} &> 0 \\ \therefore \int \frac{\sqrt{a^2 - x^2}}{x} dx &= \sqrt{a^2 - x^2} + a \cdot \ln \frac{a - \sqrt{a^2 - x^2}}{|x|} + C \end{aligned}$$



2. 当 $0 < x < a$ 时, 可设 $x = a \cdot \sin t$ ($0 < t < \frac{\pi}{2}$), 同理可证

$$\text{综合讨论 1, 2 得: } \int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} + a \cdot \ln \frac{a - \sqrt{a^2 - x^2}}{|x|} + C$$

$$72. \int \frac{\sqrt{a^2 - x^2}}{x^2} dx = -\frac{\sqrt{a^2 - x^2}}{x} - \arcsin \frac{x}{a} + C \quad (a > 0)$$

证明: 被积函数 $f(x) = \frac{\sqrt{a^2 - x^2}}{x^2}$ 的定义域为 $\{x | -a < x < a \text{ 且 } x \neq 0\}$

1. 当 $-a < x < 0$ 时, 可设 $x = a \cdot \sin t$ ($-\frac{\pi}{2} < t < 0$), 则 $dx = a \cdot \cos t dt$, $\frac{\sqrt{a^2 - x^2}}{x^2} = \frac{|a \cdot \cos t|}{a^2 \cdot \sin^2 t}$

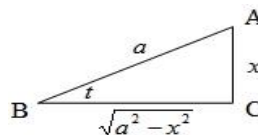
$$\because -\frac{\pi}{2} < t < 0, \cos t > 0 \therefore \frac{\sqrt{a^2 - x^2}}{x^2} = \frac{\cos t}{a \cdot \sin^2 t}$$

$$\begin{aligned} \therefore \int \frac{\sqrt{a^2 - x^2}}{x^2} dx &= \int \frac{\cos t}{a \cdot \sin^2 t} \cdot a \cdot \cos t dt \\ &= \int \frac{\cos^2 t}{\sin^2 t} dt \\ &= \int \frac{1 - \sin^2 t}{\sin^2 t} dt \\ &= \int \csc^2 t dt - \int dt \\ &= -\cot t - t + C \end{aligned}$$

在 $\text{Rt} \triangle ABC$ 中, 设 $\angle B = t$, $|AB| = a$, 则 $|AC| = x$, $|BC| = \sqrt{a^2 - x^2}$

$$\therefore \cot t = \frac{\sqrt{a^2 - x^2}}{x}$$

$$\therefore \int \frac{\sqrt{a^2 - x^2}}{x^2} dx = -\frac{\sqrt{a^2 - x^2}}{x} - \arcsin \frac{x}{a} + C$$



2. 当 $0 < x < a$ 时, 可设 $x = a \cdot \sin t$ ($0 < t < \frac{\pi}{2}$), 同理可证

$$\text{综合讨论 1, 2 得: } \int \frac{\sqrt{a^2 - x^2}}{x^2} dx = -\frac{\sqrt{a^2 - x^2}}{x} - \arcsin \frac{x}{a} + C$$

(九) 含有 $\sqrt{\pm ax^2 + bx + c}$ ($a > 0$) 的积分 (73~78)

$$73. \int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \cdot \ln \left| 2ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c} \right| + C \quad (a > 0)$$

证明: 若被积函数 $f(x) = \frac{1}{\sqrt{ax^2 + bx + c}}$ 成立, 则 $ax^2 + bx + c > 0$ 恒成立

$$\because a > 0 \quad \therefore \Delta = b^2 - 4ac > 0$$

$$\begin{aligned} \because ax^2 + bx + c &= \frac{1}{4a} [(2ax + b)^2 + 4ac - b^2] \\ &= \frac{1}{4a} [(2ax + b)^2 - (\sqrt{b^2 - 4ac})^2] \end{aligned}$$

$$\begin{aligned} \therefore \int \frac{dx}{\sqrt{ax^2 + bx + c}} &= 2\sqrt{a} \int \frac{1}{\sqrt{(2ax + b)^2 - (\sqrt{b^2 - 4ac})^2}} dx \\ &= \frac{2\sqrt{a}}{2a} \int \frac{1}{\sqrt{(2ax + b)^2 - (\sqrt{b^2 - 4ac})^2}} d(2ax + b) \\ &= \frac{1}{\sqrt{a}} \int \frac{1}{\sqrt{(2ax + b)^2 - (\sqrt{b^2 - 4ac})^2}} d(2ax + b) \quad \boxed{\text{公式45: } \int \frac{dx}{\sqrt{x^2 - a^2}} = \ln |x + \sqrt{x^2 - a^2}| + C} \\ &= \frac{1}{\sqrt{a}} \cdot \ln \left| 2ax + b + \sqrt{(2ax + b)^2 - (\sqrt{b^2 - 4ac})^2} \right| + C \\ &= \frac{1}{\sqrt{a}} \cdot \ln \left| 2ax + b + \sqrt{4a \cdot (ax^2 + bx + c)} \right| + C \\ &= \frac{1}{\sqrt{a}} \cdot \ln \left| 2ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c} \right| + C \end{aligned}$$

$$74. \int \sqrt{ax^2 + bx + c} dx = \frac{2ax + b}{4a} \sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8\sqrt{a^3}} \cdot \ln \left| 2ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c} \right| + C \quad (a > 0)$$

证明: 若被积函数 $f(x) = \sqrt{ax^2 + bx + c}$ 成立, 则 $ax^2 + bx + c > 0$ 恒成立

$$\because a > 0 \quad \therefore \Delta = b^2 - 4ac > 0$$

$$\because ax^2 + bx + c = \frac{1}{4a} [(2ax + b)^2 + 4ac - b^2]$$

$$= \frac{1}{4a} [(2ax + b)^2 - (\sqrt{b^2 - 4ac})^2] \quad \boxed{\text{公式53: } \int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \cdot \ln |x + \sqrt{x^2 - a^2}| + C}$$

$$\begin{aligned} \therefore \int \sqrt{ax^2 + bx + c} dx &= \frac{1}{2\sqrt{a}} \int \sqrt{(2ax + b)^2 - (\sqrt{b^2 - 4ac})^2} dx \\ &= \frac{1}{2a \cdot 2\sqrt{a}} \int \sqrt{(2ax + b)^2 - (\sqrt{b^2 - 4ac})^2} d(2ax + b) \\ &= \frac{1}{4a \cdot \sqrt{a}} \cdot \left[\frac{2ax + b}{2} \sqrt{(2ax + b)^2 - (\sqrt{b^2 - 4ac})^2} - \frac{b^2 - 4ac}{2} \cdot \ln \left| 2ax + b + \sqrt{(2ax + b)^2 - (\sqrt{b^2 - 4ac})^2} \right| \right] \\ &= \frac{1}{4\sqrt{a^3}} \cdot \frac{2ax + b}{2} \cdot 2\sqrt{a}\sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8\sqrt{a^3}} \cdot \ln \left| 2ax + b + \sqrt{(2ax + b)^2 - (\sqrt{b^2 - 4ac})^2} \right| + C \\ &= \frac{1}{4\sqrt{a^3}} \cdot \frac{2ax + b}{2} \cdot 2\sqrt{a}\sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8\sqrt{a^3}} \cdot \ln \left| 2ax + b + \sqrt{4a \cdot (ax^2 + bx + c)} \right| + C \\ &= \frac{2ax + b}{4a} \cdot \sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8\sqrt{a^3}} \cdot \ln \left| 2ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c} \right| + C \end{aligned}$$

$$75. \int \frac{x}{\sqrt{ax^2+bx+c}} dx = \frac{1}{a} \sqrt{ax^2+bx+c} - \frac{b}{2\sqrt{a^3}} \cdot \ln \left| 2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c} \right| + C \quad (a>0)$$

证明: $\because d(ax^2+bx+c) = (2ax+b)dx$

$$\therefore \text{ 可将 } \int \frac{x}{\sqrt{ax^2+bx+c}} dx \text{ 变换成 } \int \left[\frac{1}{\sqrt{ax^2+bx+c}} \cdot \left(\frac{2ax+b}{2a} - \frac{b}{2a} \right) \right] dx$$

$$\begin{aligned} \therefore \text{ 上式} &= \frac{1}{2a} \int \frac{1}{\sqrt{ax^2+bx+c}} \cdot (2ax+b) dx - \frac{b}{2a} \int \frac{1}{\sqrt{ax^2+bx+c}} dx \\ &= \frac{1}{2a} \int (ax^2+bx+c)^{-\frac{1}{2}} d(ax^2+bx+c) - \frac{b}{2a} \int \frac{1}{\sqrt{ax^2+bx+c}} dx \\ &= \frac{1}{a} \sqrt{ax^2+bx+c} - \frac{b}{2a} \int \frac{1}{\sqrt{ax^2+bx+c}} dx \end{aligned}$$

$$\begin{aligned} \text{又 } \frac{b}{2a} \int \frac{1}{\sqrt{ax^2+bx+c}} dx &= \frac{b}{2a} \cdot \frac{1}{\sqrt{a}} \cdot \ln \left| 2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c} \right| + C_1 \quad (\text{公式 73}) \\ &= \frac{b}{2\sqrt{a^3}} \cdot \ln \left| 2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c} \right| + C_1 \end{aligned}$$

$$\therefore \int \frac{x}{\sqrt{ax^2+bx+c}} dx = \frac{1}{a} \sqrt{ax^2+bx+c} - \frac{b}{2\sqrt{a^3}} \cdot \ln \left| 2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c} \right| + C$$

$$76. \int \frac{dx}{\sqrt{c+bx-ax^2}} = \frac{1}{\sqrt{a}} \cdot \arcsin \frac{2ax-b}{\sqrt{b^2+4ac}} + C \quad (a>0)$$

证明: 若被积函数 $f(x) = \frac{1}{\sqrt{c+bx-ax^2}}$ 成立, 则 $c+bx-ax^2 > 0$ 有解

$$\because a>0 \quad \therefore \Delta = b^2+4ac > 0$$

$$\begin{aligned} \because c+bx-ax^2 &= \frac{1}{4a} [b^2 - (2ax-b)^2] + c \\ &= \frac{b^2+4ac}{4a} - \frac{(2ax-b)^2}{4a} \end{aligned}$$

$$\begin{aligned} \therefore \int \frac{dx}{\sqrt{c+bx-ax^2}} &= 2\sqrt{a} \int \frac{1}{\sqrt{(b^2+4ac)^2 - (2ax-b)^2}} dx \\ &= \frac{1}{\sqrt{a}} \cdot \arcsin \frac{2ax-b}{\sqrt{b^2+4ac}} + C \end{aligned}$$

原题: $\int \frac{dx}{\sqrt{c+bx-ax^2}} = -\frac{1}{\sqrt{a}} \cdot \arcsin \frac{2ax-b}{\sqrt{b^2+4ac}} + C$ 有误

$$77. \int \sqrt{c+bx-ax^2} dx = \frac{2ax-b}{8a} \sqrt{c+bx-ax^2} + \frac{b^2+4ac}{8\sqrt{a^3}} \cdot \arcsin \frac{2ax-b}{\sqrt{b^2+4ac}} + C \quad (a > 0)$$

证明: 若被积函数 $f(x) = \sqrt{c+bx-ax^2}$ 成立, 则 $c+bx-ax^2 \geq 0$ 有解

$$\because a > 0 \quad \therefore \Delta = b^2 + 4ac \geq 0$$

$$\because c+bx-ax^2 = \frac{1}{4a}[b^2 - (2ax-b)^2] + c$$

$$= \frac{b^2+4ac}{4a} - \frac{(2ax-b)^2}{4a}$$

$$\boxed{\text{公式67: } \int \sqrt{a^2-x^2} dx = \frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \cdot \arcsin \frac{x}{a} + C}$$

$$\begin{aligned} \therefore \int \sqrt{c+bx-ax^2} dx &= \frac{1}{2\sqrt{a}} \int \sqrt{(b^2+4ac)^2 - (2ax-b)^2} dx \\ &= \frac{1}{2\sqrt{a} \cdot 2a} \int \sqrt{(\sqrt{b^2+4ac})^2 - (2ax-b)^2} d(2ax-b) \\ &= \frac{1}{4\sqrt{a^3}} \left[\frac{2ax-b}{2} \sqrt{(\sqrt{b^2+4ac})^2 - (2ax-b)^2} + \frac{b^2+4ac}{2} \cdot \arcsin \frac{2ax-b}{\sqrt{b^2+4ac}} \right] + C \\ &= \frac{2ax-b}{8\sqrt{a^3}} \sqrt{4a \cdot (c+bx-ax^2)} + \frac{b^2+4ac}{8\sqrt{a^3}} \cdot \arcsin \frac{2ax-b}{\sqrt{b^2+4ac}} + C \\ &= \frac{2ax-b}{8a} \sqrt{c+bx-ax^2} + \frac{b^2+4ac}{8\sqrt{a^3}} \cdot \arcsin \frac{2ax-b}{\sqrt{b^2+4ac}} + C \end{aligned}$$

$$78. \int \frac{x}{\sqrt{c+bx-ax^2}} dx = -\frac{1}{a} \sqrt{c+bx-ax^2} + \frac{b}{2\sqrt{a^3}} \cdot \arcsin \frac{2ax-b}{\sqrt{b^2+4ac}} + C \quad (a > 0)$$

证明: 若被积函数 $f(x) = \frac{x}{\sqrt{c+bx-ax^2}}$ 成立, 则 $c+bx-ax^2 > 0$ 有解

$$\because a > 0 \quad \therefore \Delta = b^2 + 4ac > 0$$

$$\because c+bx-ax^2 = \frac{1}{4a}[b^2 - (2ax-b)^2] + c$$

$$= \frac{1}{4a}[b^2 + 4ac - (2ax-b)^2]$$

$$\begin{aligned} \therefore \int \frac{x}{\sqrt{c+bx-ax^2}} dx &= 2\sqrt{a} \int \frac{x}{\sqrt{(\sqrt{b^2+4ac})^2 - (2ax-b)^2}} dx \quad \boxed{\text{公式61: } \int \frac{x}{\sqrt{a^2-x^2}} dx = -\sqrt{a^2-x^2} + C} \\ &= 2\sqrt{a} \cdot \frac{1}{2a} \cdot \frac{1}{2a} \int \frac{2ax-b+b}{\sqrt{(\sqrt{b^2+4ac})^2 - (2ax-b)^2}} d(2ax-b) \\ &= \frac{1}{2\sqrt{a^3}} \int \frac{2ax-b}{\sqrt{(\sqrt{b^2+4ac})^2 - (2ax-b)^2}} d(2ax-b) + \frac{b}{2\sqrt{a^3}} \int \frac{1}{\sqrt{(\sqrt{b^2+4ac})^2 - (2ax-b)^2}} d(2ax-b) \\ &= -\frac{1}{2\sqrt{a^3}} \sqrt{(\sqrt{b^2+4ac})^2 - (2ax-b)^2} + \frac{b}{2\sqrt{a^3}} \cdot \arcsin \frac{2ax-b}{\sqrt{b^2+4ac}} + C \quad \boxed{\text{公式59: } \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C} \\ &= -\frac{1}{2\sqrt{a^3}} \sqrt{4a \cdot (c+bx-ax^2)} + \frac{b}{2\sqrt{a^3}} \cdot \arcsin \frac{2ax-b}{\sqrt{b^2+4ac}} + C \\ &= -\frac{1}{a} \sqrt{c+bx-ax^2} + \frac{b}{2\sqrt{a^3}} \cdot \arcsin \frac{2ax-b}{\sqrt{b^2+4ac}} + C \end{aligned}$$

(十) 含有 $\sqrt{\pm \frac{x-a}{x-b}}$ 或 $\sqrt{(x-a)(b-x)}$ 的积分 (79~82)

$$79. \int \sqrt{\frac{x-a}{x-b}} dx = (x-b) \sqrt{\frac{x-a}{x-b}} + (b-a) \cdot \ln(\sqrt{|x-a|} + \sqrt{|x-b|}) + C$$

证明: $\because \sqrt{\frac{x-a}{x-b}} > 0$ 可令 $t = \sqrt{\frac{x-a}{x-b}} \quad (t > 0)$, 则 $x = \frac{a-bt^2}{1-t^2}$, $dx = \frac{2t \cdot (a-b)}{(1-t^2)^2} dt$

$$\begin{aligned} \therefore \int \sqrt{\frac{x-a}{x-b}} dx &= \int t \cdot \frac{2t \cdot (a-b)}{(1-t^2)^2} dt = 2(a-b) \int \frac{t^2}{(1-t^2)^2} dt \\ &= 2(b-a) \int \frac{1-t^2+1}{(1-t^2)^2} dt = 2(b-a) \int \left[\frac{1}{1-t^2} - \frac{1}{(1-t^2)^2} \right] dt \\ &= 2(b-a) \int \frac{1}{1-t^2} dt - 2(b-a) \int \frac{1}{(1-t^2)^2} dt = 2(a-b) \int \frac{1}{t^2-1} dt + 2(a-b) \int \frac{1}{(1-t^2)^2} dt \\ &= 2(a-b) \cdot \frac{1}{2} \cdot \ln \left| \frac{t-1}{t+1} \right| + 2(a-b) \int \frac{1}{(1-t^2)^2} dt = (a-b) \cdot \ln \left| \frac{t-1}{t+1} \right| + 2(a-b) \int \frac{1}{(1-t^2)^2} dt \end{aligned}$$

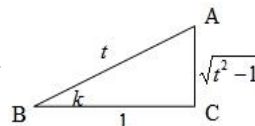
对于 $\int \frac{1}{(1-t^2)^2} dt = \int \frac{1}{(t^2-1)^2} dt \quad (t > 0)$

\therefore 可令 $t = \sec k \quad (0 < k < \frac{\pi}{2})$, 则 $(t^2-1)^2 = \tan^4 k$, $d \sec k = \sec k \cdot \tan k dk$

$$\begin{aligned} \therefore \int \frac{1}{(t^2-1)^2} dt &= \int \frac{1}{\tan^4 k} \cdot \sec k \cdot \tan k dk = \int \frac{\sec k}{\tan^4 k} dk = \int \frac{\cos^2 k}{\sin^3 k} dk \\ &= \int \frac{1-\sin^2 k}{\sin^3 k} dk = \int \frac{1}{\sin^3 k} dk - \int \frac{1}{\sin k} dk = -\frac{1}{2} \cdot \frac{\cos k}{\sin^2 k} + \frac{1}{2} \int \frac{1}{\sin k} dk - \int \frac{1}{\sin k} dk \\ &= -\frac{1}{2} \cdot \frac{\cos k}{\sin^2 k} - \frac{1}{2} \int \frac{1}{\sin k} dk = -\frac{1}{2} \cdot \ln | \csc k - \cot k | - \frac{1}{2} \cdot \frac{\cos k}{\sin^2 k} \end{aligned}$$

在 $\text{Rt} \triangle ABC$ 中, $\angle B = k$, $|BC| = 1$ 则 $|AC| = \sqrt{t^2-1}$, $|AB| = t$

$\therefore \csc k = \frac{1}{\sin k} = \frac{t}{\sqrt{t^2-1}}$, $\cot k = \frac{1}{\tan k} = \frac{1}{\sqrt{t^2-1}}$, $\cos k = \frac{1}{t}$, $\sin k = \frac{\sqrt{t^2-1}}{t}$



$$\begin{aligned} \therefore \int \sqrt{\frac{x-a}{x-b}} dx &= (a-b) \cdot \ln \left| \frac{t-1}{t+1} \right| + 2(a-b) \left[-\frac{1}{2} \cdot \ln \left| \frac{t-1}{\sqrt{t^2-1}} \right| - \frac{t}{2(t^2-1)} \right] + C_1 \\ &= (a-b) \cdot \ln \left| \frac{t-1}{t+1} \right| - (a-b) \cdot \ln \left| \frac{t-1}{\sqrt{t^2-1}} \right| - \frac{(a-b) \cdot t}{t^2-1} + C_1 \\ &= (a-b) \cdot \ln \left| \frac{\sqrt{t^2-1}}{t+1} \right| - \frac{(a-b) \cdot t}{(t^2-1)} + C_1 \end{aligned}$$

将 $t = \sqrt{\frac{x-a}{x-b}}$ 代入上式得: $\therefore \int \sqrt{\frac{x-a}{x-b}} dx = (a-b) \cdot \ln \left| \frac{\sqrt{\frac{b-a}{x-b}}}{\frac{\sqrt{|x-a|} + \sqrt{|x-b|}}{\sqrt{|x-b|}}} \right| - (a-b) \sqrt{\frac{x-a}{x-b}} \cdot \frac{x-b}{b-a} + C_1$

$$\begin{aligned} &= (x-b) \sqrt{\frac{x-a}{x-b}} + (a-b) \ln \left| \frac{\sqrt{b-a}}{\sqrt{|x-a|} + \sqrt{|x-b|}} \right| + C_1 \\ &= (x-b) \sqrt{\frac{x-a}{x-b}} + (a-b) \ln |\sqrt{b-a}| + (b-a) \ln |\sqrt{|x-a|} + \sqrt{|x-b|}| + C_1 \end{aligned}$$

$$80. \int \sqrt{\frac{x-a}{b-x}} dx = (x-b) \sqrt{\frac{x-a}{b-x}} + (b-a) \cdot \arcsin \sqrt{\frac{x-a}{b-a}} + C$$

证明: $\because \sqrt{\frac{x-a}{b-x}} > 0$ 可令 $t = \sqrt{\frac{x-a}{b-x}} \quad (t > 0)$, 则 $x = \frac{a+bt^2}{1+t^2}$, $dx = \frac{2t \cdot (b-a)}{(1+t^2)^2} dt$

$$\begin{aligned} \therefore \int \sqrt{\frac{x-a}{b-x}} dx &= \int t \cdot \frac{2t \cdot (b-a)}{(1+t^2)^2} dt = 2(b-a) \int \frac{t^2}{(1+t^2)^2} dt \\ &= 2(b-a) \int \frac{1+t^2-1}{(1+t^2)^2} dt = 2(b-a) \int \left[\frac{1}{1+t^2} - \frac{1}{(1+t^2)^2} \right] dt \\ &= 2(b-a) \int \frac{1}{1+t^2} dt - 2(b-a) \int \frac{1}{(1+t^2)^2} dt = 2(b-a) \arcsin t - 2(a-b) \int \frac{1}{(1+t^2)^2} dt \\ &= 2(a-b) \cdot \frac{1}{2} \cdot \ln \left| \frac{t-1}{t+1} \right| + 2(a-b) \int \frac{1}{(1-t^2)^2} dt = (a-b) \cdot \ln \left| \frac{t-1}{t+1} \right| + 2(a-b) \int \frac{1}{(1-t^2)^2} dt \end{aligned}$$

对于 $\int \frac{1}{(1+t^2)^2} dt \quad (t > 0)$

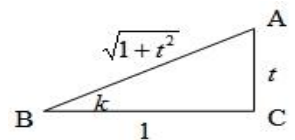
\therefore 可令 $t = \tan k \quad (0 < k < \frac{\pi}{2})$, 则 $(t^2+1)^2 = \sec^4 k$, $dt = \sec^2 k dk$

$$\begin{aligned} \therefore \int \frac{1}{(1+t^2)^2} dt &= \int \frac{1}{\sec^4 k} \cdot \sec^2 k dk = \int \frac{1}{\sec^2 k} dk = \int \cos^2 k dk \\ &= \frac{1}{2} \int (1 + \cos 2k) dk = \frac{1}{2} \int dk + \frac{1}{2} \int \cos 2k dk \\ &= \frac{k}{2} + \frac{1}{4} \cdot \sin 2k + C_1 \end{aligned}$$

$$\begin{aligned} \therefore \int \sqrt{\frac{x-a}{b-x}} dx &= 2(b-a)k - 2(b-a) \left[\frac{k}{2} + \frac{1}{4} \cdot \sin 2k \right] + C_1 \\ &= (b-a)k - (b-a) \sin k \cdot \cos k + C_1 \end{aligned}$$

在 $\text{Rt}\triangle ABC$ 中, $\angle B = k$, $|BC| = 1$ 则 $|AC| = t$, $|AB| = \sqrt{t^2+1}$

$$\therefore \cos k = \frac{1}{\sqrt{t^2+1}}, \sin k = \frac{t}{\sqrt{t^2+1}}$$



$$\begin{aligned} \therefore \int \sqrt{\frac{x-a}{b-x}} dx &= (b-a) \arcsin \frac{t}{\sqrt{t^2+1}} - (b-a) \cdot \frac{1}{\sqrt{t^2+1}} \cdot \frac{t}{\sqrt{t^2+1}} + C_1 \\ &= (b-a) \arcsin \frac{t}{\sqrt{t^2+1}} - (b-a) \cdot \frac{t}{t^2+1} + C_1 \end{aligned}$$

$$\begin{aligned} \text{将 } t = \sqrt{\frac{x-a}{b-x}} \text{ 代入上式得: } \therefore \int \sqrt{\frac{x-a}{b-x}} dx &= (b-a) \arcsin \left(\sqrt{\frac{b-x}{b-a}} \sqrt{\frac{x-a}{b-x}} \right) - (b-a) \cdot \sqrt{\frac{x-a}{b-x}} \sqrt{\frac{b-x}{b-a}} + C_1 \\ &= (b-a) \arcsin \sqrt{\frac{x-a}{b-a}} - (b-x) \cdot \sqrt{\frac{x-a}{b-x}} + C_1 \\ &= (b-a) \arcsin \sqrt{\frac{x-a}{b-a}} + (x-b) \cdot \sqrt{\frac{x-a}{b-x}} + C \end{aligned}$$

$$81. \int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2 \arcsin \sqrt{\frac{x-a}{b-a}} + C \quad (a < b)$$

$$\text{证明: } \int \frac{dx}{\sqrt{(x-a)(b-x)}} = \int \frac{1}{|x-a|} \cdot \sqrt{\frac{x-a}{b-x}} dx$$

$$\text{令 } t = \sqrt{\frac{x-a}{b-x}}, \text{ 则 } x = \frac{a+bt^2}{1+t^2}, |x-a| = \left| \frac{(b-a)t^2}{1+t^2} \right|, dx = \frac{2t(b-a)}{(1+t^2)^2} dt$$

$$\because b > a, \therefore |x-a| = (b-a) \cdot \frac{t^2}{1+t^2}$$

$$\begin{aligned} \text{于是 } \int \frac{1}{|x-a|} \cdot \sqrt{\frac{x-a}{b-x}} dx &= \int \frac{1}{b-a} \cdot \frac{1+t^2}{t^2} \cdot t \cdot \frac{2t \cdot (b-a)}{(1+t^2)^2} dt \\ &= 2 \int \frac{1}{1+t^2} dt = 2 \arctan t + C \quad (\text{公式19}) \\ &= 2 \arctan \sqrt{\frac{x-a}{b-x}} + C \end{aligned}$$

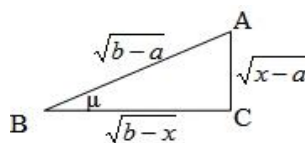
$$\text{令 } \tan \mu = \sqrt{\frac{x-a}{b-x}}, \text{ 则 } \mu = \arctan \sqrt{\frac{x-a}{b-x}}$$

在 Rt $\triangle ABC$ 中, $\angle B = \mu$, $|AC| = \sqrt{x-a}$

$$\therefore |BC| = \sqrt{b-x}, |AB| = \sqrt{|AC|^2 + |BC|^2} = \sqrt{b-a}$$

$$\therefore \sin \mu = \frac{\sqrt{x-a}}{\sqrt{b-a}}, \therefore \mu = \arcsin \sqrt{\frac{x-a}{b-a}}$$

$$\therefore \int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2 \arcsin \sqrt{\frac{x-a}{b-a}} + C$$



$$82. \int \sqrt{(x-a)(b-x)} dx = \frac{2x-a-b}{4} \sqrt{(x-a)(b-x)} + \frac{(b-a)^2}{4} \cdot \arcsin \sqrt{\frac{x-a}{b-x}} + C \quad (a < b)$$

$$\text{证明: } \int \sqrt{(x-a)(b-x)} dx = \int |x-a| \sqrt{\frac{b-x}{x-a}} dx$$

$$\because \sqrt{\frac{b-x}{x-a}} > 0 \text{ 可令 } t = \sqrt{\frac{b-x}{x-a}} \quad (t > 0), \text{ 则 } x = \frac{b+at^2}{1+t^2}, \quad dx = \frac{2at \cdot (1+t^2) - 2t(at^2+b)}{(1+t^2)^2} dt = \frac{2t(a-b)}{(1+t^2)^2} dt$$

$$|x-a| = \left| \frac{at^2+b-a-at^2}{1+t^2} \right| = \left| \frac{b-a}{1+t^2} \right|$$

$$\because a < b \quad \therefore |x-a| = \frac{b-a}{1+t^2}$$

$$\begin{aligned} \therefore \int \sqrt{(x-a)(b-x)} dx &= \int \frac{b-a}{1+t^2} \cdot t \cdot \frac{2t(a-b)}{(1+t^2)^2} dt \\ &= -2(a-b)^2 \int \frac{t^2}{(1+t^2)^3} dt \end{aligned}$$

$$\text{对于 } \int \frac{t^2}{(1+t^2)^3} dt \quad (t > 0) \quad \therefore \text{可令 } t = \tan k \quad (0 < k < \frac{\pi}{2}), \text{ 则 } (t^2+1)^3 = \sec^6 k, \quad dt = \sec^2 k dk$$

$$\begin{aligned} \therefore \int \frac{t^2}{(1+t^2)^3} dt &= \int \frac{\tan^2 k}{\sec^6 k} \cdot \sec^2 k dk = \int \frac{\tan^2 k}{\sec^4 k} dk = \int \sin^2 k \cdot \cos^2 k dk \\ &= \frac{1}{4} \int (2 \sin k \cdot \cos k)^2 dk = \frac{1}{4} \int \sin^2 2k \cdot dk \\ &= \frac{1}{8} \left[\frac{2k}{2} - \frac{1}{4} \cdot \sin 4k \right] + C \\ &= \frac{k}{8} - \frac{1}{32} \cdot \sin 4k + C \\ &= \frac{k}{8} - \frac{1}{32} \cdot (4 \sin k \cdot \cos^3 k - 4 \sin^3 k \cdot \cos k) + C \\ &= \frac{k}{8} - \frac{1}{8} \cdot \sin k \cdot \cos^3 k + \frac{1}{8} \sin^3 k \cdot \cos k + C \end{aligned}$$

$$\begin{aligned} \text{联立以上两式得: } \int \sqrt{(x-a)(b-x)} dx &= -2(b-a)^2 \cdot \frac{1}{8} \cdot (k - \sin k \cdot \cos^3 k + \sin^3 k \cdot \cos k) + C \\ &= -\frac{(b-a)^2}{4} \cdot (k - \sin k \cdot \cos^3 k + \sin^3 k \cdot \cos k) + C \end{aligned}$$

$$\text{在 Rt} \triangle ABC \text{ 中, } \angle B = k, |BC|=1 \text{ 则 } |AC|=t, |AB|=\sqrt{t^2+1}$$

$$\therefore \cos k = \frac{t}{\sqrt{t^2+1}}, \sin k = \frac{1}{\sqrt{t^2+1}}$$

$$\begin{aligned} \therefore \int \sqrt{(x-a)(b-x)} dx &= -\frac{(b-a)^2}{4} \cdot \left(\arcsin \frac{t}{\sqrt{t^2+1}} - \frac{1}{\sqrt{t^2+1}} \cdot \frac{t^2}{t^2+1} \cdot \frac{t}{\sqrt{t^2+1}} \right. \\ &\quad \left. + \frac{1}{t^2+1} \frac{1}{\sqrt{t^2+1}} \cdot \frac{t}{\sqrt{t^2+1}} \right) \\ &= -\frac{(b-a)^2}{4} \cdot \left(\arcsin \frac{t}{\sqrt{t^2+1}} - \frac{t^3}{(t^2+1)^2} + \frac{t}{(t^2+1)^2} \right) + C \\ &= -\frac{(b-a)^2}{4} \cdot \left(\arcsin \frac{t}{\sqrt{t^2+1}} - \frac{t(t^2-1)}{(t^2+1)^2} \right) + C \end{aligned}$$

$$\text{将 } t = \sqrt{\frac{b-x}{x-a}} \text{ 代入上式得: } \int \sqrt{(x-a)(b-x)} dx = \frac{2x-a-b}{4} \sqrt{(x-a)(b-x)} + \frac{(b-a)^2}{4} \cdot \arcsin \sqrt{\frac{x-a}{b-x}} + C$$

(十一) 含有三角函数的积分 (83~112)

$$83. \int \sin x \, dx = -\cos x + C$$

$$\text{证明: } \int \sin x \, dx = -\int (-\sin x) \, dx$$

$\because (\cos x)' = -\sin x$ 即 $\cos x$ 为 $-\sin x$ 的原函数

$$\begin{aligned}\therefore \int \sin x \, dx &= -\int d\cos x \\ &= -\cos x + C\end{aligned}$$

$$84. \int \cos x \, dx = \sin x + C$$

证明: $\because (\sin x)' = \cos x$ 即 $\sin x$ 为 $\cos x$ 的原函数

$$\begin{aligned}\therefore \int \cos x \, dx &= \int d\sin x \\ &= \sin x + C\end{aligned}$$

$$85. \int \tan x \, dx = -\ln |\cos x| + C$$

$$\begin{aligned}\text{证明: } \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\ &= -\int \frac{1}{\cos x} d\cos x \\ &= -\ln |\cos x| + C\end{aligned}$$

$$86. \int \cot x \, dx = \ln |\sin x| + C$$

$$\begin{aligned}\text{证明: } \int \cot x \, dx &= \int \frac{\cos x}{\sin x} \, dx \\ &= \int \frac{1}{\sin x} d\sin x \\ &= \ln |\sin x| + C\end{aligned}$$

$$87. \int \sec x \, dx = \ln \left| \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right| + C = \ln |\sec x + \tan x| + C$$

$$\begin{aligned}\text{证明: } \int \sec x \, dx &= \int \frac{1}{\cos x} \, dx = \int \frac{\cos x}{\cos^2 x} \, dx \\ &= \int \frac{1}{1 - \sin^2 x} d\sin x = \frac{1}{2} \int \frac{1}{1 + \sin x} d\sin x + \frac{1}{2} \int \frac{1}{1 - \sin x} d\sin x \\ &= \frac{1}{2} \cdot \ln |1 + \sin x| - \frac{1}{2} \cdot \ln |1 - \sin x| + C \\ &= \frac{1}{2} \cdot \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C = \frac{1}{2} \cdot \ln \left| \frac{(1 + \sin x)^2}{1 - \sin^2 x} \right| + C \\ &= \frac{1}{2} \cdot \ln \left| \frac{(1 + \sin x)^2}{\cos^2 x} \right| + C = \ln \left| \frac{1 + \sin x}{\cos x} \right| + C \\ &= \ln \left| \frac{1}{\cos x} - \frac{\sin x}{\cos x} \right| + C \\ &= \ln |\sec x + \tan x| + C\end{aligned}$$

$$88. \int \csc x \, dx = \ln \left| \tan \frac{x}{2} \right| + C = \ln | \csc x - \cot x | + C$$

$$\text{证法1: } \because \csc x = \frac{1}{\sin x} = \frac{1}{2 \cdot \sin \frac{x}{2} \cdot \cos \frac{x}{2}} = \frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}{2 \cdot \sin \frac{x}{2} \cdot \cos \frac{x}{2}} = \frac{1 + \tan^2 \frac{x}{2}}{2 \tan \frac{x}{2}}$$

$$\text{又 } \because d \tan \frac{x}{2} = \frac{1}{2} \cdot \frac{1}{\cos^2 \frac{x}{2}} dx$$

$$\therefore dx = 2 \cdot \cos^2 \frac{x}{2} d \tan \frac{x}{2}$$

$$\begin{aligned} \therefore \int \csc x \, dx &= \int \frac{1}{2 \cdot \sin \frac{x}{2} \cdot \cos \frac{x}{2}} \cdot 2 \cdot \cos^2 \frac{x}{2} d \tan \frac{x}{2} \\ &= \int \frac{1}{\tan \frac{x}{2}} d \tan \frac{x}{2} \\ &= \ln \left| \tan \frac{x}{2} \right| + C \end{aligned}$$

$$\because \tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \frac{\sin^2 \frac{x}{2}}{\sin \frac{x}{2} \cdot \cos \frac{x}{2}} = \frac{2 \sin^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cdot \cos \frac{x}{2}} = \frac{1 - \cos x}{\sin x} = \csc x - \cot x$$

$$\therefore \int \csc x \, dx = \ln \left| \tan \frac{x}{2} \right| + C = \ln | \csc x - \cot x | + C$$

$$\begin{aligned} \text{证法2: } \int \csc x \, dx &= \int \frac{1}{\sin t} dt \\ &= \int \frac{\sin t}{\sin^2 t} dt \\ &= - \int \frac{1}{1 - \cos^2 t} d \cos t \\ &= - \frac{1}{2} \int \left(\frac{1}{1 + \cos t} + \frac{1}{1 - \cos t} \right) d \cos t \\ &= - \frac{1}{2} \int \frac{1}{1 + \cos t} d(\cos t + 1) + \frac{1}{2} \int \frac{1}{1 - \cos t} d(1 - \cos t) \\ &= - \frac{1}{2} \cdot \ln |1 + \cos t| + \frac{1}{2} \cdot \ln |\cos t - 1| + C_1 \\ &= \frac{1}{2} \cdot \ln \left| \frac{\cos t - 1}{1 + \cos t} \right| + C_1 \\ &= \frac{1}{2} \cdot \ln \left| \frac{(1 - \cos t)^2}{1 - \cos^2 t} \cdot (-1) \right| + C_1 \\ &= \frac{1}{2} \cdot \ln \left| \frac{(1 - \cos t)^2}{\sin^2 t} \right| + C_2 \\ &= \ln \left| \frac{1 - \cos t}{\sin t} \right| + C_2 \\ &= \ln | \csc x - \cot x | + C \end{aligned}$$

$$89. \int \sec^2 x \, dx = \tan x + C$$

证明: $\because (\tan x)' = \sec^2 x$ 即 $\tan x$ 为 $\sec^2 x$ 的原函数

$$\begin{aligned} \therefore \int \sec^2 x \, dx &= \int d \tan x \\ &= \tan x + C \end{aligned}$$

$$90. \int \csc^2 x \, dx = -\cot x + C$$

证明: $\int \csc^2 x \, dx = -\int (-\csc^2 x) \, dx$

$\because (\cot x)' = -\csc^2 x$ 即 $\cot x$ 为 $-\csc^2 x$ 的原函数

$$\begin{aligned} \therefore \int \csc^2 x \, dx &= -\int d \cot x \\ &= -\cot x + C \end{aligned}$$

$$91. \int \sec x \cdot \tan x \, dx = \sec x + C$$

证明: $\because (\sec x)' = \sec x \cdot \tan x$ 即 $\sec x$ 为 $\sec x \cdot \tan x$ 的原函数

$$\begin{aligned} \therefore \int \sec x \cdot \tan x \, dx &= \int d \sec x \\ &= \sec x + C \end{aligned}$$

$$92. \int \csc x \cdot \cot x \, dx = -\csc x + C$$

证明: $\int \csc x \cdot \cot x \, dx = -\int (-\csc x \cdot \cot x) \, dx$

$\because (\csc x)' = -\csc x \cdot \cot x$ 即 $\csc x$ 为 $-\csc x \cdot \cot x$ 的原函数

$$\begin{aligned} \therefore \int \csc x \cdot \cot x \, dx &= -\int d \csc x \\ &= -\csc x + C \end{aligned}$$

$$93. \int \sin^2 x \, dx = \frac{x}{2} - \frac{1}{4} \cdot \sin 2x + C$$

证明: $\int \sin^2 x \, dx = \int \left(\frac{1}{2} - \frac{1}{2} \cdot \cos 2x \right) dx$

$$= \frac{1}{2} \int dx - \frac{1}{4} \int \cos 2x \, d2x$$

$$= \frac{x}{2} - \frac{1}{4} \sin 2x + C$$

提示: $\sin^2 x = \frac{1 - \cos 2x}{2}$

$$94. \int \cos^2 x \, dx = \frac{x}{2} + \frac{1}{4} \cdot \sin 2x + C$$

证明: $\int \cos^2 x \, dx = \int \left(\frac{1}{2} + \frac{1}{2} \cdot \cos 2x \right) dx$

$$= \frac{1}{2} \int dx + \frac{1}{4} \int \cos 2x \, d2x$$

$$= \frac{x}{2} + \frac{1}{4} \sin 2x + C$$

提示: $\cos^2 x = \frac{1 + \cos 2x}{2}$

$$95. \int \sin^n x dx = -\frac{1}{n} \cdot \sin^{n-1} x \cdot \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx$$

$$\begin{aligned} \text{证明: } \int \sin^n x dx &= \int \sin^{n-1} x \cdot \sin x dx \\ &= -\int \sin^{n-1} x d \cos x \\ &= -\cos x \cdot \sin^{n-1} x + \int \cos x d \sin^{n-1} x \\ &= -\cos x \cdot \sin^{n-1} x + \int \cos x \cdot (n-1) \cdot \sin^{n-2} x \cdot \cos x dx \\ &= -\cos x \cdot \sin^{n-1} x + (n-1) \int \cos^2 x \cdot \sin^{n-2} x dx \\ &= -\cos x \cdot \sin^{n-1} x + (n-1) \int (1 - \sin^2 x) \cdot \sin^{n-2} x dx \\ &= -\cos x \cdot \sin^{n-1} x + (n-1) \int \sin^{n-2} x dx - (n-1) \int \sin^n x dx \end{aligned}$$

$$\text{移项并整理得: } n \int \sin^n x dx = -\cos x \cdot \sin^{n-1} x + (n-1) \int \sin^{n-2} x dx$$

$$\therefore \int \sin^n x dx = -\frac{1}{n} \cdot \sin^{n-1} x \cdot \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx$$

$$96. \int \cos^n x dx = \frac{1}{n} \cdot \cos^{n-1} x \cdot \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$$

$$\begin{aligned} \text{证明: } \int \cos^n x dx &= \int \cos^{n-1} x \cdot \cos x dx \\ &= \int \cos^{n-1} x d \sin x \\ &= \sin x \cdot \cos^{n-1} x - \int \sin x d \cos^{n-1} x \\ &= \sin x \cdot \cos^{n-1} x + \int \sin x \cdot (n-1) \cdot \cos^{n-2} x \cdot \sin x dx \\ &= \sin x \cdot \cos^{n-1} x + (n-1) \int \sin^2 x \cdot \cos^{n-2} x dx \\ &= \sin x \cdot \cos^{n-1} x + (n-1) \int (1 - \cos^2 x) \cdot \cos^{n-2} x dx \\ &= \sin x \cdot \cos^{n-1} x + (n-1) \int \cos^{n-2} x dx - (n-1) \int \cos^n x dx \end{aligned}$$

$$\text{移项并整理得: } n \int \cos^n x dx = \sin x \cdot \cos^{n-1} x + (n-1) \int \cos^{n-2} x dx$$

$$\therefore \int \cos^n x dx = \frac{1}{n} \cdot \sin x \cdot \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x dx$$

$$97. \int \frac{dx}{\sin^n x} = -\frac{1}{n-1} \cdot \frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\sin^{n-2} x}$$

$$\text{证明: } \int \frac{dx}{\sin^n x} = -\int \frac{1}{\sin^{n-2} x} \cdot \frac{1}{-\sin^2 x} dx$$

$$= -\int \frac{1}{\sin^{n-2} x} d \cot x$$

$$= -\frac{\cot x}{\sin^{n-2} x} + \int \cot x d \frac{1}{\sin^{n-2} x}$$

$$= -\frac{\cot x}{\sin^{n-2} x} + \int \cot x \cdot (2-n) \cdot \sin^{1-n} x \cdot \cos x dx$$

$$= -\frac{\cot x}{\sin^{n-2} x} + (2-n) \int \frac{\cos^2 x}{\sin^n x} dx$$

$$= -\frac{\cot x}{\sin^{n-2} x} + (2-n) \int \frac{1-\sin^2 x}{\sin^n x} dx$$

$$= -\frac{\cot x}{\sin^{n-2} x} + (2-n) \int \frac{dx}{\sin^n x} - (2-n) \int \frac{1}{\sin^{n-2} x} dx$$

$$\text{移项并整理得: } (n-1) \int \frac{dx}{\sin^n x} = -\frac{\cot x}{\sin^{n-2} x} - (2-n) \int \frac{1}{\sin^{n-2} x} dx$$

$$= -\frac{\cos x}{\sin^{n-1} x} + (n-2) \int \frac{1}{\sin^{n-2} x} dx$$

$$\therefore \int \frac{dx}{\sin^n x} = -\frac{1}{n-1} \cdot \frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\sin^{n-2} x}$$

$$98. \int \frac{dx}{\cos^n x} = -\frac{1}{n-1} \cdot \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}$$

$$\text{证明: } \int \frac{dx}{\cos^n x} = \int \frac{1}{\cos^{n-2} x} \cdot \frac{1}{\cos^2 x} dx$$

$$= \int \frac{1}{\cos^{n-2} x} d \tan x$$

$$= \frac{\tan x}{\cos^{n-2} x} + \int \tan x d \frac{1}{\cos^{n-2} x}$$

$$= \frac{\tan x}{\cos^{n-2} x} + \int \tan x \cdot (2-n) \cdot \cos^{1-n} x \cdot \sin x dx$$

$$= \frac{\tan x}{\cos^{n-2} x} - (n-2) \int \frac{\sin^2 x}{\cos^n x} dx$$

$$= \frac{\tan x}{\cos^{n-2} x} - (n-2) \int \frac{1-\cos^2 x}{\cos^n x} dx$$

$$= \frac{\sin x}{\cos^{n-1} x} - (n-2) \int \frac{dx}{\cos^n x} + (n-2) \int \frac{1}{\cos^{n-2} x} dx$$

$$\text{移项并整理得: } (n-1) \int \frac{dx}{\cos^n x} = \frac{\sin x}{\cos^{n-1} x} + (n-2) \int \frac{1}{\cos^{n-2} x} dx$$

$$= \frac{\sin x}{\cos^{n-1} x} + (n-2) \int \frac{1}{\cos^{n-2} x} dx$$

$$\therefore \int \frac{dx}{\cos^n x} = -\frac{1}{n-1} \cdot \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}$$

$$99. \int \cos^m x \cdot \sin^n x dx = \frac{1}{m+n} \cdot \cos^{m-1} x \cdot \sin^{n+1} x + \frac{m-1}{m+n} \int \cos^{m-2} x \cdot \sin^n x dx \quad \textcircled{1}$$

$$= -\frac{1}{m+n} \cdot \cos^{m+1} x \cdot \sin^{n-1} x + \frac{n-1}{m+n} \int \cos^m x \cdot \sin^{n-2} x dx \quad \textcircled{2}$$

证明①: $\because d \sin^{m+n} x = (m+n) \cdot \sin^{m+n-1} x \cdot \cos x dx$

$$\begin{aligned} \therefore \int \cos^m x \cdot \sin^n x dx &= \frac{1}{m+n} \int \cos^{m-1} x \cdot \sin^{1-m} x d \sin^{m+n} x \\ &= \frac{1}{m+n} \cdot \cos^{m-1} x \cdot \sin^{1-m} x - \frac{1}{m+n} \int \sin^{m+n} x d(\cos^{m-1} x \cdot \sin^{1-m} x) \end{aligned}$$

$$\begin{aligned} \because d(\cos^{m-1} x \cdot \sin^{1-m} x) &= [-(m-1) \cdot \cos^{m-2} x \cdot \sin x \cdot \sin^{1-m} x + (1-m) \cdot \sin^{1-m-1} x \cdot \cos x \cdot \cos^{m-1} x] dx \\ &= [(1-m) \cdot \sin^{-m} x \cdot \cos^m x \cdot (\sin^2 x \cdot \cos^{-2} x + 1)] dx \\ &= [(1-m) \cdot \sin^{-m} x \cdot \cos^m x \cdot (\frac{\sin^2 x + \cos^2 x}{\cos^2 x})] dx \\ &= [(1-m) \cdot \sin^{-m} x \cdot \cos^{m-2} x] dx \end{aligned}$$

$$\therefore -\frac{1}{m+n} \int \sin^{m+n} x d(\cos^{m-1} x \cdot \sin^{1-m} x) = \frac{m-1}{m+n} \int \cos^{m-2} x \cdot \sin^n x dx$$

$$\therefore \int \cos^m x \cdot \sin^n x dx = \frac{1}{m+n} \cdot \cos^{m-1} x \cdot \sin^{n+1} x + \frac{m-1}{m+n} \int \cos^{m-2} x \cdot \sin^n x dx$$

证明②: $\because d \cos^{m+n} x = -(m+n) \cdot \cos^{m+n-1} x \cdot \sin x dx$

$$\begin{aligned} \therefore \int \cos^m x \cdot \sin^n x dx &= \frac{-1}{m+n} \int \cos^{1-n} x \cdot \sin^{n-1} x d \cos^{m+n} x \\ &= \frac{-1}{m+n} \cdot \sin^{n-1} x \cdot \cos^{m+1} x + \frac{1}{m+n} \int \cos^{m+n} x d(\sin^{n-1} x \cdot \cos^{1-n} x) \end{aligned}$$

$$\begin{aligned} \because d(\sin^{n-1} x \cdot \cos^{1-n} x) &= [(n-1) \cdot \sin^{n-2} x \cdot \cos x \cdot \cos^{1-n} x - (1-n) \cdot \cos^{1-n-1} x \cdot \sin x \cdot \sin^{n-1} x] dx \\ &= [(n-1) \cdot \cos^{-n} x \cdot \sin^n x \cdot (\sin^{-2} x \cdot \cos^2 x + 1)] dx \\ &= [(n-1) \cdot \cos^{-n} x \cdot \sin^n x \cdot (\frac{\sin^2 x + \cos^2 x}{\sin^2 x})] dx \\ &= [(n-1) \cdot \cos^{-n} x \cdot \sin^{n-2} x] dx \end{aligned}$$

$$\therefore \frac{1}{m+n} \int \cos^{m+n} x d(\sin^{n-1} x \cdot \cos^{1-n} x) = \frac{n-1}{m+n} \int \cos^m x \cdot \sin^{n-2} x dx$$

$$\therefore \int \cos^m x \cdot \sin^n x dx = -\frac{1}{m+n} \cdot \cos^{m+1} x \cdot \sin^{n-1} x + \frac{n-1}{m+n} \int \cos^m x \cdot \sin^{n-2} x dx$$

$$100. \int \sin ax \cdot \cos bx \, dx = -\frac{1}{2(a+b)} \cdot \cos(a+b)x - \frac{1}{2(a-b)} \cdot \cos(a-b)x + C$$

证明: $\int \sin ax \cdot \cos bx \, dx = \int \frac{1}{2} [\sin(a+b)x + \sin(a-b)x] dx$ 提示: $\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$

$$= \frac{1}{2} \int \sin(a+b)x \, dx + \frac{1}{2} \int \sin(a-b)x \, dx$$

$$= \frac{1}{2(a+b)} \int \sin(a+b)x \, d(a+b)x + \frac{1}{2(a-b)} \int \sin(a-b)x \, d(a-b)x$$

$$= -\frac{1}{2(a+b)} \cdot \cos(a+b)x - \frac{1}{2(a-b)} \cdot \cos(a-b)x$$

$$101. \int \sin ax \cdot \sin bx \, dx = -\frac{1}{2(a+b)} \cdot \sin(a+b)x + \frac{1}{2(a-b)} \cdot \sin(a-b)x + C$$

证明: $\int \sin ax \cdot \sin bx \, dx = \int \frac{1}{2} [\cos(a-b)x - \cos(a+b)x] dx$ 提示: $\sin \alpha \sin \beta = -\frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$

$$= \frac{1}{2} \int \cos(a-b)x \, dx - \frac{1}{2} \int \cos(a+b)x \, dx$$

$$= \frac{1}{2(a-b)} \int \cos(a-b)x \, d(a-b)x - \frac{1}{2(a+b)} \int \cos(a+b)x \, d(a+b)x$$

$$= \frac{1}{2(a-b)} \cdot \sin(a-b)x - \frac{1}{2(a+b)} \cdot \sin(a+b)x + C$$

$$102. \int \cos ax \cdot \cos bx \, dx = \frac{1}{2(a+b)} \cdot \sin(a+b)x + \frac{1}{2(a-b)} \cdot \sin(a-b)x + C$$

证明: $\int \cos ax \cdot \cos bx \, dx = \int \frac{1}{2} [\cos(a+b)x + \cos(a-b)x] dx$ 提示: $\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$

$$= \frac{1}{2} \int \cos(a+b)x \, dx + \frac{1}{2} \int \cos(a-b)x \, dx$$

$$= \frac{1}{2(a+b)} \int \cos(a+b)x \, d(a+b)x + \frac{1}{2(a-b)} \int \cos(a-b)x \, d(a-b)x$$

$$= \frac{1}{2(a+b)} \cdot \sin(a+b)x + \frac{1}{2(a-b)} \cdot \sin(a-b)x + C$$

$$103. \int \frac{dx}{a+b \cdot \sin x} = \frac{2}{\sqrt{a^2-b^2}} \cdot \arctan \frac{a \cdot \tan \frac{x}{2} + b}{\sqrt{a^2-b^2}} + C \quad (a^2 > b^2)$$

证明: 令 $t = \tan \frac{x}{2}$, 则 $\sin x = 2 \cdot \sin \frac{x}{2} \cdot \cos \frac{x}{2} = \frac{2 \cdot \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{2t}{1+t^2}$

$$dt = (\tan \frac{x}{2}) dx = \frac{1}{2} \cdot \sec^2 \frac{x}{2} dx = \frac{1}{2} (1 + \tan^2 \frac{x}{2}) dx = \frac{1}{2} (1 + t^2) dx$$

$$\therefore dx = \frac{2}{1+t^2} dt, \quad a + b \cdot \sin x = a + \frac{2bt}{1+t^2} = \frac{a(1+t^2) + 2bt}{1+t^2}$$

$$\begin{aligned} \therefore \int \frac{dx}{a+b \cdot \sin x} &= \int \frac{1+t^2}{a(1+t^2) + 2bt} \cdot \frac{2}{1+t^2} dt \\ &= 2 \int \frac{1}{at^2 + 2bt + a} dt \\ &= 2 \int \frac{1}{a(t + \frac{b}{a})^2 - \frac{b^2}{a} + a} dt \\ &= 2a \int \frac{1}{(at+b)^2 + (a^2-b^2)} dt \\ &= 2 \int \frac{1}{(at+b)^2 + (a^2-b^2)} d(at+b) \end{aligned}$$

当 $a^2 > b^2$, 即 $a^2 - b^2 > 0$ 时

$$2 \int \frac{1}{(at+b)^2 + (a^2-b^2)} d(at+b) = 2 \int \frac{1}{(at+b)^2 + (\sqrt{a^2-b^2})^2} d(at+b)$$

$$\boxed{\text{公式19: } \int \frac{dx}{x^2+a^2} = \frac{1}{a} \cdot \arctan \frac{x}{a} + C} = \frac{2}{\sqrt{a^2-b^2}} \cdot \arctan \frac{at+b}{\sqrt{a^2-b^2}} + C$$

$$\text{将 } t = \tan \frac{x}{2} \text{ 代入上式得: } \int \frac{dx}{a+b \sin x} = \frac{2}{\sqrt{a^2-b^2}} \cdot \arctan \frac{a \cdot \tan \frac{x}{2} + b}{\sqrt{a^2-b^2}} + C$$

$$104. \int \frac{dx}{a+b \sin x} = \frac{1}{\sqrt{b^2-a^2}} \cdot \ln \left| \frac{a \cdot \tan \frac{x}{2} + b - \sqrt{b^2-a^2}}{a \cdot \tan \frac{x}{2} + b + \sqrt{b^2-a^2}} \right| + C \quad (a^2 < b^2)$$

证明: 令 $t = \tan \frac{x}{2}$, 则 $\sin x = 2 \cdot \sin \frac{x}{2} \cdot \cos \frac{x}{2} = \frac{2 \cdot \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{2t}{1+t^2}$

$$dt = \left(\tan \frac{x}{2} \right) dx = \frac{1}{2} \cdot \sec^2 \frac{x}{2} dx = \frac{1}{2} (1 + \tan^2 \frac{x}{2}) dx = \frac{1}{2} (1+t^2) dx$$

$$\therefore dx = \frac{2}{1+t^2} dt, \quad a + b \sin x = a + \frac{2bt}{1+t^2} = \frac{a(1+t^2) + 2bt}{1+t^2}$$

$$\begin{aligned} \therefore \int \frac{dx}{a+b \sin x} &= \int \frac{1+t^2}{a(1+t^2) + 2bt} \cdot \frac{2}{1+t^2} dt \\ &= 2 \int \frac{1}{at^2 + 2bt + a} dt \\ &= 2 \int \frac{1}{a(t + \frac{b}{a})^2 - \frac{b^2}{a} + a} dt \\ &= 2a \int \frac{1}{(at+b)^2 + (a^2-b^2)} dt \\ &= 2 \int \frac{1}{(at+b)^2 + (a^2-b^2)} d(at+b) \end{aligned}$$

当 $a^2 < b^2$, 即 $a^2 - b^2 < 0$ 时

$$\begin{aligned} 2 \int \frac{1}{(at+b)^2 + (a^2-b^2)} d(at+b) &= 2 \int \frac{1}{(at+b)^2 - (b^2-a^2)} d(at+b) \\ &= 2 \int \frac{1}{(at+b)^2 - (\sqrt{b^2-a^2})^2} d(at+b) \end{aligned}$$

$$\boxed{\text{公式21: } \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \cdot \ln \left| \frac{x-a}{x+a} \right| + C} = 2 \times \frac{1}{2\sqrt{b^2-a^2}} \cdot \ln \left| \frac{at+b-\sqrt{b^2-a^2}}{at+b+\sqrt{b^2-a^2}} \right| + C$$

$$\text{将 } t = \tan \frac{x}{2} \text{ 代入上式得: } \int \frac{dx}{a+b \sin x} = \frac{1}{\sqrt{b^2-a^2}} \cdot \ln \left| \frac{a \cdot \tan \frac{x}{2} + b - \sqrt{b^2-a^2}}{a \cdot \tan \frac{x}{2} + b + \sqrt{b^2-a^2}} \right| + C$$

$$105. \int \frac{dx}{a+b \cdot \cos x} = \frac{2}{a+b} \cdot \sqrt{\frac{a+b}{a-b}} \arctan \left(\sqrt{\frac{a-b}{a+b}} \cdot \tan \frac{x}{2} \right) + C \quad (a^2 > b^2)$$

证明：令 $t = \tan \frac{x}{2}$, 则 $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1 - t^2}{1 + t^2}$

$$\therefore a + b \cdot \cos x = a + b \cdot \frac{1 - t^2}{1 + t^2} = \frac{(a+b) + t^2(a-b)}{1 + t^2}$$

$$\therefore dt = d \tan \frac{x}{2} = \frac{1}{2} \cdot \sec^2 \frac{x}{2} dx = \frac{1}{2 \cos^2 \frac{x}{2}} dx = \frac{1}{1 + \cos x} dx = \frac{1 + t^2}{2} dx$$

$$\therefore dx = \frac{2}{1 + t^2} dt$$

提示: $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$

$$\therefore \int \frac{dx}{a+b \cdot \cos x} = \int \frac{2}{(a+b) + t^2(a-b)} dt$$

当 $|a| > |b|$, 即 $a^2 > b^2$ 时

$$\int \frac{2}{(a+b) + t^2(a-b)} dt = \frac{2}{a-b} \int \frac{1}{\left(\sqrt{\frac{a+b}{a-b}} \right)^2 + t^2} dt$$

公式19: $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \cdot \arctan \frac{x}{a} + C$

$$= \frac{2}{a-b} \cdot \sqrt{\frac{a-b}{a+b}} \cdot \arctan \left(\sqrt{\frac{a-b}{a+b}} \cdot t \right) + C$$

$$= \frac{2}{a-b} \cdot \sqrt{\frac{a-b}{a+b}} \cdot \arctan \left(\sqrt{\frac{a-b}{a+b}} \cdot t \right) + C$$

$$= 2 \sqrt{\frac{1}{(a+b) \cdot (a-b)}} \cdot \arctan \left(\sqrt{\frac{a-b}{a+b}} \cdot t \right) + C$$

$$= \frac{2}{a+b} \cdot \sqrt{\frac{a+b}{a-b}} \cdot \arctan \left(\sqrt{\frac{a-b}{a+b}} \cdot t \right) + C$$

将 $t = \tan \frac{x}{2}$ 代入上式得: $\int \frac{dx}{a+b \cdot \cos x} = \frac{2}{a+b} \cdot \sqrt{\frac{a+b}{a-b}} \arctan \left(\sqrt{\frac{a-b}{a+b}} \cdot \tan \frac{x}{2} \right) + C$

$$106. \int \frac{dx}{a+b \cdot \cos x} = \frac{1}{a+b} \cdot \sqrt{\frac{a+b}{b-a}} \cdot \ln \left| \frac{\tan \frac{x}{2} + \sqrt{\frac{a+b}{b-a}}}{\tan \frac{x}{2} - \sqrt{\frac{a+b}{b-a}}} \right| + C \quad (a^2 < b^2)$$

证明: 令 $t = \tan \frac{x}{2}$, 则 $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1 - t^2}{1 + t^2}$

$$\therefore a + b \cdot \cos x = a + b \cdot \frac{1 - t^2}{1 + t^2} = \frac{(a+b) + t^2(a-b)}{1 + t^2}$$

$$\therefore dt = d \tan \frac{x}{2} = \frac{1}{2} \cdot \sec^2 \frac{x}{2} dx = \frac{1}{2 \cos^2 \frac{x}{2}} dx = \frac{1}{1 + \cos x} dx = \frac{1 + t^2}{2} dx$$

$$\therefore dx = \frac{2}{1 + t^2} dt$$

提示: $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$

$$\therefore \int \frac{dx}{a+b \cdot \cos x} = \int \frac{2}{(a+b) + t^2(a-b)} dt$$

当 $a^2 < b^2$, 即 $|a| < |b|$, $\therefore b-a > 0$

$$\int \frac{2}{(a+b) + t^2(a-b)} dt = \int \frac{2}{(a+b) - t^2(b-a)} dt$$

$$= \frac{2}{b-a} \int \frac{1}{\left(\sqrt{\frac{a+b}{b-a}}\right)^2 - t^2} dt = \frac{2}{a-b} \int \frac{1}{t^2 - \left(\sqrt{\frac{a+b}{b-a}}\right)^2} dt$$

$$= \frac{2}{a-b} \cdot \frac{1}{2} \cdot \sqrt{\frac{b-a}{a+b}} \cdot \ln \left| \frac{t - \sqrt{\frac{a+b}{b-a}}}{t + \sqrt{\frac{a+b}{b-a}}} \right| + C = \frac{1}{a-b} \cdot \sqrt{\frac{b-a}{a+b}} \cdot \ln \left| \frac{t - \sqrt{\frac{a+b}{b-a}}}{t + \sqrt{\frac{a+b}{b-a}}} \right| + C$$

公式21: $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \cdot \ln \left| \frac{x-a}{x+a} \right| + C$

$$= (-1) \sqrt{\frac{1}{(a+b) \cdot (b-a)}} \cdot \ln \left| \frac{t - \sqrt{\frac{a+b}{b-a}}}{t + \sqrt{\frac{a+b}{b-a}}} \right| + C = -\frac{1}{a+b} \cdot \sqrt{\frac{a+b}{b-a}} \cdot \ln \left| \frac{t - \sqrt{\frac{a+b}{b-a}}}{t + \sqrt{\frac{a+b}{b-a}}} \right| + C$$

$$= \frac{1}{a+b} \cdot \sqrt{\frac{a+b}{b-a}} \cdot \ln \left| \frac{t + \sqrt{\frac{a+b}{b-a}}}{t - \sqrt{\frac{a+b}{b-a}}} \right| + C$$

将 $t = \tan \frac{x}{2}$ 代入上式得: $\int \frac{dx}{a+b \cdot \cos x} = \frac{1}{a+b} \cdot \sqrt{\frac{a+b}{b-a}} \cdot \ln \left| \frac{\tan \frac{x}{2} + \sqrt{\frac{a+b}{b-a}}}{\tan \frac{x}{2} - \sqrt{\frac{a+b}{b-a}}} \right| + C$

$$107. \int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{1}{ab} \cdot \arctan \left(\frac{b}{a} \cdot \tan x \right) + C$$

$$\begin{aligned} \text{证明: } \int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} &= \int \frac{1}{\cos^2 x} \cdot \frac{1}{a^2 + b^2 \tan^2 x} dx \\ &= \int \frac{1}{a^2 + b^2 \tan^2 x} d \tan x \\ &= \frac{1}{b^2} \int \frac{1}{\left(\frac{a^2}{b^2} + \tan^2 x\right)} d \tan x \\ &= \frac{1}{b^2} \int \frac{1}{\left(\frac{a}{b}\right)^2 + \tan^2 x} d \tan x \\ &= \frac{1}{b^2} \cdot \frac{b}{a} \cdot \arctan \left(\frac{b}{a} \cdot \tan x \right) + C \\ &= \frac{1}{ab} \cdot \arctan \left(\frac{b}{a} \cdot \tan x \right) + C \end{aligned}$$

$$\text{公式19: } \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \cdot \arctan \frac{x}{a} + C$$

$$108. \int \frac{dx}{a^2 \cos^2 x - b^2 \sin^2 x} = \frac{1}{2ab} \cdot \ln \left| \frac{b \cdot \tan x + a}{b \cdot \tan x - a} \right| + C$$

$$\begin{aligned} \text{证明: } \int \frac{dx}{a^2 \cos^2 x - b^2 \sin^2 x} &= \int \frac{1}{\cos^2 x} \cdot \frac{1}{a^2 - b^2 \tan^2 x} dx \\ &= \int \frac{1}{a^2 - b^2 \tan^2 x} d \tan x \\ &= \frac{1}{b} \int \frac{1}{a^2 - (b \cdot \tan x)^2} d (b \cdot \tan x) \\ &= -\frac{1}{b} \int \frac{1}{(b \cdot \tan x)^2 - a^2} d (b \cdot \tan x) \\ &= -\frac{1}{b} \cdot \frac{1}{2a} \cdot \ln \left| \frac{b \cdot \tan x - a}{b \cdot \tan x + a} \right| + C \\ &= -\frac{1}{2ab} \cdot \ln \left| \frac{b \cdot \tan x - a}{b \cdot \tan x + a} \right| + C \\ &= \frac{1}{2ab} \cdot \ln \left| \frac{b \cdot \tan x + a}{b \cdot \tan x - a} \right| + C \end{aligned}$$

$$\text{公式21: } \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \cdot \ln \left| \frac{x-a}{x+a} \right| + C$$

$$\text{提示: } \log_a b^{-1} = -\log_a b$$

$$109. \int x \cdot \sin ax \, dx = \frac{1}{a^2} \cdot \sin ax - \frac{1}{a} \cdot x \cdot \cos ax + C$$

$$\begin{aligned} \text{证明: } \int x \cdot \sin ax \, dx &= -\frac{1}{a} \int x \, d \cos ax \\ &= -\frac{1}{a} \cdot x \cdot \cos ax + \frac{1}{a} \int \cos ax \, dx \\ &= -\frac{1}{a} \cdot x \cdot \cos ax + \frac{1}{a^2} \int \cos ax \, dax \\ &= -\frac{1}{a} \cdot x \cdot \cos ax + \frac{1}{a^2} \cdot \sin ax + C \end{aligned}$$

$$110. \int x^2 \cdot \sin ax \, dx = -\frac{1}{a} \cdot x^2 \cdot \cos ax + \frac{2}{a^2} \cdot x \cdot \sin ax + \frac{2}{a^3} \cdot \cos ax + C$$

$$\begin{aligned} \text{证明: } \int x^2 \cdot \sin ax \, dx &= -\frac{1}{a} \int x^2 \, d \cos ax \\ &= -\frac{1}{a} \cdot x^2 \cdot \cos ax + \frac{1}{a} \int \cos ax \, dx^2 \\ &= -\frac{1}{a} \cdot x^2 \cdot \cos ax + \frac{2}{a} \int x \cdot \cos ax \, dx \\ &= -\frac{1}{a} \cdot x^2 \cdot \cos ax + \frac{2}{a^2} \cdot \int x \, d \sin ax \\ &= -\frac{1}{a} \cdot x^2 \cdot \cos ax + \frac{2}{a^2} \cdot x \cdot \sin ax - \frac{2}{a^3} \cdot \int \sin ax \, dax \\ &= -\frac{1}{a} \cdot x^2 \cdot \cos ax + \frac{2}{a^2} \cdot x \cdot \sin ax + \frac{2}{a^3} \cdot \cos ax \end{aligned}$$

$$111. \int x \cdot \cos ax \, dx = \frac{1}{a^2} \cdot \cos ax - \frac{1}{a} \cdot x \cdot \sin ax + C$$

$$\begin{aligned} \text{证明: } \int x \cdot \cos ax \, dx &= \frac{1}{a} \int x \, d \sin ax \\ &= \frac{1}{a} \cdot x \cdot \sin ax - \frac{1}{a} \int \sin ax \, dx \\ &= \frac{1}{a} \cdot x \cdot \sin ax - \frac{1}{a^2} \int \sin ax \, dax \\ &= \frac{1}{a} \cdot x \cdot \sin ax + \frac{1}{a^2} \cdot \cos ax + C \end{aligned}$$

$$112. \int x^2 \cdot \cos ax \, dx = \frac{1}{a} \cdot x^2 \cdot \sin ax + \frac{2}{a^2} \cdot x \cdot \cos ax - \frac{2}{a^3} \cdot \sin ax + C$$

$$\begin{aligned} \text{证明: } \int x^2 \cdot \cos ax \, dx &= \frac{1}{a} \int x^2 \, d \sin ax \\ &= \frac{1}{a} \cdot x^2 \cdot \sin ax - \frac{1}{a} \int \sin ax \, d(x^2) \\ &= \frac{1}{a} \cdot x^2 \cdot \sin ax + \frac{2}{a} \int x \cdot \sin ax \, dx \\ &= \frac{1}{a} \cdot x^2 \cdot \sin ax - \frac{2}{a^2} \cdot \int x \, d \cos ax \\ &= \frac{1}{a} \cdot x^2 \cdot \sin ax + \frac{2}{a^2} \cdot x \cdot \cos ax - \frac{2}{a^3} \cdot \int \cos ax \, dax \\ &= \frac{1}{a} \cdot x^2 \cdot \sin ax + \frac{2}{a^2} \cdot x \cdot \cos ax - \frac{2}{a^3} \cdot \sin ax + C \end{aligned}$$

(十二) 含有反三角函数的积分 (其中 $a > 0$) (113~121)

113. $\int \arcsin \frac{x}{a} dx = x \cdot \arcsin \frac{x}{a} + \sqrt{a^2 - x^2} + C \quad (a > 0)$

证明: $\int \arcsin \frac{x}{a} dx = x \cdot \arcsin \frac{x}{a} - \int x d \arcsin \frac{x}{a}$
 $= x \cdot \arcsin \frac{x}{a} - \int x \cdot \frac{1}{\sqrt{1 - (\frac{x}{a})^2}} \cdot \frac{1}{a} dx$
 $= x \cdot \arcsin \frac{x}{a} - \int \frac{x}{\sqrt{a^2 - x^2}} dx$
 $= x \cdot \arcsin \frac{x}{a} - \frac{1}{2} \int \frac{1}{\sqrt{a^2 - x^2}} d(x^2)$
 $= x \cdot \arcsin \frac{x}{a} + \frac{1}{2} \int (a^2 - x^2)^{-\frac{1}{2}} d(a^2 - x^2)$
 $= x \cdot \arcsin \frac{x}{a} + \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} \cdot (a^2 - x^2)^{1 - \frac{1}{2}} + C$
 $= x \cdot \arcsin \frac{x}{a} + \sqrt{a^2 - x^2} + C$

114. $\int x \cdot \arcsin \frac{x}{a} dx = (\frac{x^2}{2} - \frac{a^2}{4}) \cdot \arcsin \frac{x}{a} + \frac{x}{4} \sqrt{a^2 - x^2} + C \quad (a > 0)$

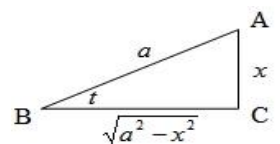
证明: 令 $t = \arcsin \frac{x}{a}$, 则 $x = a \cdot \sin t$

$\therefore \int x \cdot \arcsin \frac{x}{a} dx = \int a \cdot \sin t \cdot t d(a \cdot \sin t) = a^2 \int t \cdot \sin t \cdot \cos t dt$
 $= \frac{a^2}{2} \int t \cdot \sin 2t dt = -\frac{a^2}{4} \int t d \cos 2t$
 $= -\frac{a^2}{4} \cdot t \cdot \cos 2t + \frac{a^2}{4} \int \cos 2t dt$
 $= -\frac{a^2}{4} \cdot t \cdot \cos 2t + \frac{a^2}{8} \int \cos 2t d2t$
 $= -\frac{a^2}{4} \cdot t \cdot \cos 2t + \frac{a^2}{8} \cdot \sin 2t + C$
 $= -\frac{a^2}{4} \cdot t \cdot (2 \cos^2 t - 1) + \frac{a^2}{4} \cdot \sin t \cdot \cos t + C$
 $= -\frac{a^2}{2} \cdot t \cdot \cos^2 t + \frac{a^2}{4} \cdot t + \frac{a^2}{4} \cdot \sin t \cdot \cos t + C$

提示: $\sin 2x = 2 \cdot \sin x \cdot \cos x$
 $\cos 2x = \cos^2 x - \sin^2 x$
 $= 2 \cos^2 x - 1$

在 $\text{Rt} \triangle ABC$ 中, 可设 $\angle B = t$, $|AB| = a$, 则 $|AC| = x$, $|BC| = \sqrt{a^2 - x^2}$

$\therefore \cos t = \frac{\sqrt{a^2 - x^2}}{a}, \sin t = \frac{x}{a}$



$\therefore \int x \cdot \arcsin \frac{x}{a} dx = -\frac{a^2}{2} \cdot \arcsin \frac{x}{a} \cdot \frac{a^2 - x^2}{a^2} + \frac{a^2}{4} \cdot \arcsin \frac{x}{a} + \frac{a^2}{4} \cdot \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} + C$
 $= \frac{x^2 - a^2}{2} \cdot \arcsin \frac{x}{a} + \frac{a^2}{4} \cdot \arcsin \frac{x}{a} + \frac{x}{4} \cdot \sqrt{a^2 - x^2} + C$

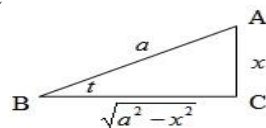
$$115. \int x^2 \cdot \arcsin \frac{x}{a} dx = \frac{x^3}{3} \cdot \arcsin \frac{x}{a} + \frac{1}{9}(x^2 + 2a^2)\sqrt{a^2 - x^2} + C \quad (a > 0)$$

证明: 令 $t = \arcsin \frac{x}{a}$, 则 $x = a \cdot \sin t$

$$\begin{aligned} \therefore \int x^2 \cdot \arcsin \frac{x}{a} dx &= \int a^2 \cdot \sin^2 t \cdot t \, d(a \cdot \sin t) = a^3 \int t \cdot \sin^2 t \cdot \cos t \, dt \\ &= \frac{a^3}{3} \int t \, d \sin^3 t \\ &= \frac{a^3}{3} \cdot t \cdot \sin^3 t - \frac{a^3}{3} \int \sin^3 t \, dt \\ &= \frac{a^3}{3} \cdot t \cdot \sin^3 t - \frac{a^3}{3} \int \sin t (1 - \cos^2 t) \, dt \\ &= \frac{a^3}{3} \cdot t \cdot \sin^3 t - \frac{a^3}{3} \int \sin t \, dt + \frac{a^3}{3} \int \sin t \cdot \cos^2 t \, dt \\ &= \frac{a^3}{3} \cdot t \cdot \sin^3 t + \frac{a^3}{3} \cdot \cos t - \frac{a^3}{3} \int \cos^2 t \, d \cos t \\ &= \frac{a^3}{3} \cdot t \cdot \sin^3 t + \frac{a^3}{3} \cdot \cos t - \frac{a^3}{3} \cdot \frac{1}{1+2} \cdot \cos^3 t + C \\ &= \frac{a^3}{3} \cdot t \cdot \sin^3 t + \frac{a^3}{3} \cdot \cos t - \frac{a^3}{9} \cdot \cos^3 t + C \end{aligned}$$

在 Rt $\triangle ABC$ 中, 可设 $\angle B = t$, $|AB| = a$, 则 $|AC| = x$, $|BC| = \sqrt{a^2 - x^2}$

$$\therefore \cos t = \frac{\sqrt{a^2 - x^2}}{a}, \quad \sin t = \frac{x}{a}$$



$$\begin{aligned} \therefore \int x^2 \cdot \arcsin \frac{x}{a} dx &= \frac{a^3}{3} \cdot \arcsin \frac{x}{a} \cdot \frac{x^3}{a^3} + \frac{a^3}{3} \cdot \frac{\sqrt{a^2 - x^2}}{a} - \frac{a^3}{9} \cdot \frac{a^2 - x^2}{a^3} \cdot \sqrt{a^2 - x^2} + C \\ &= \frac{x^3}{3} \cdot \arcsin \frac{x}{a} + \frac{a^2}{3} \cdot \sqrt{a^2 - x^2} - \frac{a^2 - x^2}{9} \cdot \sqrt{a^2 - x^2} + C \\ &= \frac{x^3}{3} \cdot \arcsin \frac{x}{a} + \frac{1}{9}(x^2 + 2a^2)\sqrt{a^2 - x^2} + C \end{aligned}$$

$$116. \int \arccos \frac{x}{a} dx = x \cdot \arccos \frac{x}{a} - \sqrt{a^2 - x^2} + C \quad (a > 0)$$

证明: $\int \arccos \frac{x}{a} dx = x \cdot \arccos \frac{x}{a} - \int x d \arccos \frac{x}{a}$

$$= x \cdot \arccos \frac{x}{a} + \int x \cdot \frac{1}{\sqrt{1 - (\frac{x}{a})^2}} \cdot \frac{1}{a} dx$$

$$= x \cdot \arccos \frac{x}{a} + \int \frac{x}{\sqrt{a^2 - x^2}} dx$$

$$= x \cdot \arccos \frac{x}{a} + \frac{1}{2} \int \frac{1}{\sqrt{a^2 - x^2}} d(x^2)$$

$$= x \cdot \arccos \frac{x}{a} - \frac{1}{2} \int (a^2 - x^2)^{-\frac{1}{2}} d(a^2 - x^2)$$

$$= x \cdot \arccos \frac{x}{a} - \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} \cdot (a^2 - x^2)^{1 - \frac{1}{2}} + C$$

$$= x \cdot \arccos \frac{x}{a} - \sqrt{a^2 - x^2} + C$$

$$117. \int x \cdot \arccos \frac{x}{a} dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right) \cdot \arccos \frac{x}{a} - \frac{x}{4} \sqrt{a^2 - x^2} + C \quad (a > 0)$$

证明: 令 $t = \arccos \frac{x}{a}$, 则 $x = a \cdot \cos t$

$$\therefore \int x \cdot \arccos \frac{x}{a} dx = \int a \cdot \cos t \cdot t d(a \cdot \cos t) = -a^2 \int t \cdot \cos t \cdot \sin t dt$$

$$= -\frac{a^2}{2} \int t \cdot \sin 2t dt = \frac{a^2}{4} \int t d \cos 2t$$

$$= \frac{a^2}{4} \cdot t \cdot \cos 2t - \frac{a^2}{4} \int \cos 2t dt$$

$$= \frac{a^2}{4} \cdot t \cdot \cos 2t - \frac{a^2}{8} \int \cos 2t d2t$$

$$= \frac{a^2}{4} \cdot t \cdot \cos 2t - \frac{a^2}{8} \cdot \sin 2t + C$$

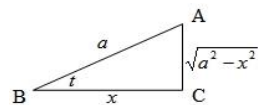
$$= \frac{a^2}{4} \cdot t \cdot (2 \cos^2 t - 1) - \frac{a^2}{4} \cdot \sin t \cdot \cos t + C$$

$$= \frac{a^2}{2} \cdot t \cdot \cos^2 t - \frac{a^2}{4} \cdot t - \frac{a^2}{4} \cdot \sin t \cdot \cos t + C$$

提示: $\sin 2x = 2 \cdot \sin x \cdot \cos x$
 $\cos 2x = \cos^2 x - \sin^2 x$
 $= 2 \cos^2 x - 1$

在 Rt $\triangle ABC$ 中, 可设 $\angle B = t$, $|AB| = a$, 则 $|BC| = x$, $|AC| = \sqrt{a^2 - x^2}$

$$\therefore \sin t = \frac{\sqrt{a^2 - x^2}}{a}, \quad \cos t = \frac{x}{a}$$



$$\therefore \int x \cdot \arccos \frac{x}{a} dx = \frac{a^2}{2} \cdot \arcsin \frac{x}{a} \cdot \frac{x^2}{a^2} - \frac{a^2}{4} \cdot \arcsin \frac{x}{a} - \frac{a^2}{4} \cdot \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} + C$$

$$= \frac{x^2}{2} \cdot \arcsin \frac{x}{a} - \frac{a^2}{4} \cdot \arcsin \frac{x}{a} - \frac{x}{4} \cdot \sqrt{a^2 - x^2} + C$$

$$= \left(\frac{x^2}{2} - \frac{a^2}{4}\right) \cdot \arcsin \frac{x}{a} + \frac{x}{4} \sqrt{a^2 - x^2} + C$$

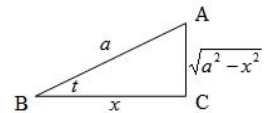
$$118. \int x^2 \cdot \arccos \frac{x}{a} dx = \frac{x^3}{3} \cdot \arccos \frac{x}{a} - \frac{1}{9}(x^2 + 2a^2)\sqrt{a^2 - x^2} + C \quad (a > 0)$$

证明: 令 $t = \arccos \frac{x}{a}$, 则 $x = a \cdot \cos t$

$$\begin{aligned} \therefore \int x^2 \cdot \arccos \frac{x}{a} dx &= \int a^2 \cdot \cos^2 t \cdot t \cdot d(a \cdot \cos t) = -a^3 \int t \cdot \cos^2 t \cdot \sin t dt \\ &= \frac{a^3}{3} \int t d \cos^3 t \\ &= \frac{a^3}{3} \cdot t \cdot \cos^3 t - \frac{a^3}{3} \int \cos^3 t dt \\ &= \frac{a^3}{3} \cdot t \cdot \cos^3 t - \frac{a^3}{3} \int \cos t (1 - \sin^2 t) dt \\ &= \frac{a^3}{3} \cdot t \cdot \cos^3 t - \frac{a^3}{3} \int \cos t dt + \frac{a^3}{3} \int \cos t \cdot \sin^2 t dt \\ &= \frac{a^3}{3} \cdot t \cdot \cos^3 t - \frac{a^3}{3} \cdot \sin t + \frac{a^3}{3} \int \sin^2 t d \sin t \\ &= \frac{a^3}{3} \cdot t \cdot \cos^3 t - \frac{a^3}{3} \cdot \sin t + \frac{a^3}{3} \cdot \frac{1}{1+2} \cdot \sin^3 t + C \\ &= \frac{a^3}{3} \cdot t \cdot \cos^3 t - \frac{a^3}{3} \cdot \sin t + \frac{a^3}{9} \cdot \sin^3 t + C \end{aligned}$$

在 Rt $\triangle ABC$ 中, 可设 $\angle B = t$, $|AB| = a$, 则 $|BC| = x$, $|AC| = \sqrt{a^2 - x^2}$

$$\therefore \sin t = \frac{\sqrt{a^2 - x^2}}{a}, \quad \cos t = \frac{x}{a}$$



$$\begin{aligned} \therefore \int x^2 \cdot \arccos \frac{x}{a} dx &= \frac{a^3}{3} \cdot \arcsin \frac{x}{a} \cdot \frac{x^3}{a^3} - \frac{a^3}{3} \cdot \frac{\sqrt{a^2 - x^2}}{a} + \frac{a^3}{9} \cdot \frac{a^2 - x^2}{a^3} \cdot \sqrt{a^2 - x^2} + C \\ &= \frac{x^3}{3} \cdot \arcsin \frac{x}{a} - \frac{a^2}{3} \cdot \sqrt{a^2 - x^2} + \frac{a^2 - x^2}{9} \cdot \sqrt{a^2 - x^2} + C \\ &= \frac{x^3}{3} \cdot \arcsin \frac{x}{a} - \frac{1}{9}(x^2 + 2a^2)\sqrt{a^2 - x^2} + C \end{aligned}$$

$$119. \int \arctan \frac{x}{a} dx = x \cdot \arctan \frac{x}{a} - \frac{a}{2} \cdot \ln(a^2 + x^2) + C \quad (a > 0)$$

$$\begin{aligned} \text{证明: } \int \arctan \frac{x}{a} dx &= x \cdot \arctan \frac{x}{a} - \int x d \arctan \frac{x}{a} \\ &= x \cdot \arctan \frac{x}{a} - \int x \cdot \frac{1}{1 + (\frac{x}{a})^2} \cdot \frac{1}{a} dx \\ &= x \cdot \arctan \frac{x}{a} - a \int \frac{x}{a^2 + x^2} dx \\ &= x \cdot \arctan \frac{x}{a} - \frac{a}{2} \int \frac{1}{a^2 + x^2} d(x^2) \\ &= x \cdot \arctan \frac{x}{a} - \frac{a}{2} \int \frac{1}{a^2 + x^2} d(a^2 + x^2) \\ &= x \cdot \arctan \frac{x}{a} - \frac{a}{2} \cdot \ln |a^2 + x^2| + C \end{aligned}$$

$$\therefore a^2 + x^2 > 0$$

$$\therefore \int \arctan \frac{x}{a} dx = x \cdot \arctan \frac{x}{a} - \frac{a}{2} \cdot \ln(a^2 + x^2) + C$$

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$$120. \int x \cdot \arctan \frac{x}{a} dx = \frac{1}{2} (a^2 + x^2) \cdot \arctan \frac{x}{a} - \frac{a}{2} \cdot x + C \quad (a > 0)$$

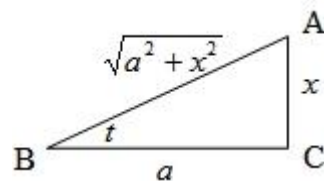
证明: 令 $t = \arctan \frac{x}{a}$, 则 $x = a \cdot \tan t$

$$\begin{aligned} \therefore \int x \cdot \arctan \frac{x}{a} dx &= \int a \cdot \tan t \cdot t d(a \cdot \tan t) = a^2 \int t \cdot \sec^2 t \cdot \tan t dt \\ &= \frac{a^2}{2} \int t d \sec^2 t \\ &= \frac{a^2}{2} \cdot t \cdot \sec^2 t - \frac{a^2}{2} \int \sec^2 t dt \\ &= \frac{a^2}{2} \cdot t \cdot \sec^2 t - \frac{a^2}{2} \cdot \tan t + C \end{aligned}$$

在 $\text{Rt} \triangle ABC$ 中, 可设 $\angle B = t$, $|BC| = a$, 则 $|AC| = x$, $|AB| = \sqrt{a^2 + x^2}$

$$\therefore \sec t = \frac{1}{\cos t} = \frac{\sqrt{a^2 + x^2}}{a}, \quad \tan t = \frac{x}{a}$$

$$\begin{aligned} \therefore \int x \cdot \arctan \frac{x}{a} dx &= \frac{a^2}{2} \cdot \arctan \frac{x}{a} \cdot \frac{a^2 + x^2}{a^2} - \frac{a^2}{2} \cdot \frac{x}{a} + C \\ &= \frac{1}{2} (a^2 + x^2) \cdot \arctan \frac{x}{a} - \frac{a}{2} \cdot x + C \end{aligned}$$



$$121. \int x^2 \cdot \arctan \frac{x}{a} dx = \frac{x^3}{3} \cdot \arctan \frac{x}{a} - \frac{a}{6} \cdot x^2 + \frac{a^3}{6} \ln(a^2 + x^2) + C \quad (a > 0)$$

$$\begin{aligned} \text{证明: } \therefore \int x^2 \cdot \arctan \frac{x}{a} dx &= \frac{1}{3} \int \arctan \frac{x}{a} dx^3 \\ &= \frac{x^3}{3} \cdot \arctan \frac{x}{a} - \frac{1}{3} \int x^3 \cdot \frac{1}{1 + (\frac{x}{a})^2} \cdot \frac{1}{a} dx \\ &= \frac{x^3}{3} \cdot \arctan \frac{x}{a} - \frac{a}{3} \int \frac{x^3}{a^2 + x^2} dx \\ &= \frac{x^3}{3} \cdot \arctan \frac{x}{a} - \frac{a}{6} \int \frac{x^2}{a^2 + x^2} d(x^2) \\ &= \frac{x^3}{3} \cdot \arctan \frac{x}{a} - \frac{a}{6} \int \frac{x^2 + a^2 - a^2}{a^2 + x^2} d(x^2) \\ &= \frac{x^3}{3} \cdot \arctan \frac{x}{a} - \frac{a}{6} \int dx^2 + \frac{a}{6} \int \frac{a^2}{a^2 + x^2} d(x^2) \\ &= \frac{x^3}{3} \cdot \arctan \frac{x}{a} - \frac{a}{6} \int dx^2 + \frac{a^3}{6} \int \frac{1}{a^2 + x^2} d(x^2 + a^2) \\ &= \frac{x^3}{3} \cdot \arctan \frac{x}{a} - \frac{a}{6} \cdot x^2 + \frac{a^3}{6} \ln |a^2 + x^2| + C \end{aligned}$$

$$\because a^2 + x^2 > 0$$

$$\therefore \int x^2 \cdot \arctan \frac{x}{a} dx = \frac{x^3}{3} \cdot \arctan \frac{x}{a} - \frac{a}{6} \cdot x^2 + \frac{a^3}{6} \ln(a^2 + x^2) + C$$

(十三) 含有指数函数的积分 (122~131)

$$122. \int a^x dx = \frac{1}{\ln a} \cdot a^x + C$$

$$\text{证明: } \int a^x dx = \frac{1}{\ln a} \int \ln a \cdot a^x dx$$

$\because (a^x)' = a^x \ln a$, 即 $a^x \ln a$ 的原函数为 a^x

$$\begin{aligned} \therefore \int a^x dx &= \frac{1}{\ln a} \int d(a^x) \\ &= \frac{1}{\ln a} \cdot a^x + C \end{aligned}$$

$$123. \int e^{ax} dx = \frac{1}{a} \cdot e^{ax} + C$$

$$\text{证明: 令 } ax = \mu, \text{ 则 } x = \frac{\mu}{a}, dx = \frac{1}{a} d\mu$$

$$\begin{aligned} \therefore \int e^{ax} dx &= \frac{1}{a} \int e^{\mu} d\mu = \frac{1}{a} \cdot e^{\mu} + C \\ &= \frac{1}{a} \cdot e^{ax} + C \end{aligned}$$

$$124. \int x \cdot e^{ax} dx = \frac{1}{a^2} (ax - 1) e^{ax} + C$$

$$\begin{aligned} \text{证明: } \int x \cdot e^{ax} dx &= \frac{1}{a} \int x d(e^{ax}) \\ &= \frac{1}{a} \cdot x \cdot e^{ax} - \frac{1}{a} \int e^{ax} dx \\ &= \frac{1}{a} \cdot x \cdot e^{ax} - \frac{1}{a^2} \int e^{ax} d(ax) \\ &= \frac{1}{a} \cdot x \cdot e^{ax} - \frac{1}{a^2} e^{ax} + C \\ &= \frac{1}{a^2} (ax - 1) e^{ax} + C \end{aligned}$$

$$125. \int x^n \cdot e^{ax} dx = \frac{1}{a} \cdot x^n \cdot e^{ax} - \frac{n}{a} \int x^{n-1} \cdot e^{ax} dx$$

$$\begin{aligned} \text{证明: } \int x^n \cdot e^{ax} dx &= \frac{1}{a} \int x^n d(e^{ax}) \\ &= \frac{1}{a} \cdot x^n \cdot e^{ax} - \frac{1}{a} \int e^{ax} d(x^n) \\ &= \frac{1}{a} \cdot x^n \cdot e^{ax} - \frac{n}{a} \int x^{n-1} \cdot e^{ax} dx \end{aligned}$$

$$126. \int x \cdot a^x dx = \frac{x}{\ln a} \cdot a^x - \frac{1}{(\ln a)^2} \cdot a^x + C$$

证明: $\int x \cdot a^x dx = \frac{1}{\ln a} \int x d(a^x)$

$$= \frac{1}{\ln a} \cdot x \cdot a^x - \frac{1}{\ln a} \int a^x dx$$

公式122: $\int a^x dx = \frac{1}{\ln a} \cdot a^x + C$

$$= \frac{1}{\ln a} \cdot x \cdot a^x - \frac{1}{(\ln a)^2} \cdot a^x + C$$

$$127. \int x^n \cdot a^x dx = \frac{1}{\ln a} \cdot x^n \cdot a^x - \frac{n}{\ln a} \int x^{n-1} \cdot a^x dx$$

证明: $\int x^n \cdot a^x dx = \frac{1}{\ln a} \int x^n d(a^x)$

$$= \frac{1}{\ln a} \cdot x^n \cdot a^x - \frac{1}{\ln a} \int a^x d(x^n)$$

$$= \frac{1}{\ln a} \cdot x^n \cdot a^x - \frac{n}{\ln a} \int x^{n-1} \cdot a^x dx$$

$$128. \int e^{ax} \cdot \sin bx dx = \frac{1}{a^2 + b^2} \cdot e^{ax} (a \cdot \sin bx - b \cdot \cos bx) + C$$

证明: $\int e^{ax} \cdot \sin bx dx = -\frac{1}{b} \int e^{ax} d \cos bx$

$$= -\frac{1}{b} \cdot e^{ax} \cdot \cos bx + \frac{1}{b} \int \cos bx d(e^{ax})$$

$$= -\frac{1}{b} \cdot e^{ax} \cdot \cos bx + \frac{a}{b^2} \cdot e^{ax} \cdot \sin bx - \frac{a}{b^2} \int \sin bx d(e^{ax})$$

$$= -\frac{1}{b} \cdot e^{ax} \cdot \cos bx + \frac{a}{b^2} \cdot e^{ax} \cdot \sin bx - \frac{a}{b^2} \int \sin bx d(e^{ax})$$

移项并整理得: $\frac{a^2 + b^2}{b^2} \int e^{ax} \cdot \sin bx dx = -\frac{1}{b} \cdot e^{ax} \cdot \cos bx + \frac{a}{b^2} \cdot e^{ax} \cdot \sin bx + C$

$$\therefore \int e^{ax} \cdot \sin bx dx = -\frac{b}{a^2 + b^2} \cdot e^{ax} \cdot \cos bx + \frac{a}{a^2 + b^2} \cdot e^{ax} \cdot \sin bx + C$$

$$= \frac{1}{a^2 + b^2} \cdot e^{ax} (a \cdot \sin bx - b \cdot \cos bx) + C$$

$$129. \int e^{ax} \cdot \cos bxdx = \frac{1}{a^2 + b^2} \cdot e^{ax} (b \cdot \sin bx + a \cdot \cos bx) + C$$

$$\text{证明: } \int e^{ax} \cdot \cos bxdx = \frac{1}{b} \int e^{ax} d \sin bx$$

$$= \frac{1}{b} \cdot e^{ax} \cdot \sin bx - \frac{1}{b} \int \sin bxd(e^{ax})$$

$$= \frac{1}{b} \cdot e^{ax} \cdot \sin bx - \frac{a}{b} \int \sin bx \cdot e^{ax} dx$$

$$= \frac{1}{b} \cdot e^{ax} \cdot \sin bx + \frac{a}{b^2} \int e^{ax} d \cos bx$$

$$= \frac{1}{b} \cdot e^{ax} \cdot \sin bx + \frac{a}{b^2} \cdot e^{ax} \cdot \cos bx - \frac{a}{b^2} \int \cos bxd(e^{ax})$$

$$= \frac{1}{b} \cdot e^{ax} \cdot \sin bx + \frac{a}{b^2} \cdot e^{ax} \cdot \cos bx - \frac{a^2}{b^2} \int e^{ax} \cdot \cos bxdx$$

$$\therefore (1 + \frac{a^2}{b^2}) \int e^{ax} \cdot \cos bxdx = \frac{a^2 + b^2}{b^2} \int e^{ax} \cdot \cos bxdx = \frac{1}{b} \cdot e^{ax} \cdot \sin bx + \frac{a}{b^2} \cdot e^{ax} \cdot \cos bx$$

$$\therefore \int e^{ax} \cdot \cos bxdx = \frac{1}{a^2 + b^2} \cdot e^{ax} (b \cdot \sin bx + a \cdot \cos bx) + C$$

$$130. \int e^{ax} \cdot \sin^n bx \, dx = \frac{1}{a^2 + b^2 n^2} e^{ax} \cdot \sin^{n-1} bx (a \cdot \sin bx - nb \cdot \cos bx)$$

$$+ \frac{n \cdot (n-1) b^2}{a^2 + b^2 n^2} \int e^{ax} \cdot \sin^{n-2} bx \, dx$$

$$\begin{aligned} \text{证明: } \int e^{ax} \cdot \sin^n bx \, dx &= \int e^{ax} \cdot \sin^{n-2} bx \cdot \sin^2 bx \, dx = \int e^{ax} \cdot \sin^{n-2} bx \cdot (1 - \cos^2 bx) \, dx \\ &= \int e^{ax} \cdot \sin^{n-2} bx \, dx - \int e^{ax} \cdot \sin^{n-2} bx \cdot \cos^2 bx \, dx \end{aligned} \quad (1)$$

$$\begin{aligned} \text{又 } \int e^{ax} \cdot \sin^{n-2} bx \cdot \cos^2 bx \, dx &= \frac{1}{b \cdot (n-1)} \int e^{ax} \cdot \cos bx \, d \sin^{n-1} bx \\ &= \frac{1}{b \cdot (n-1)} \cdot e^{ax} \cdot \cos bx \cdot \sin^{n-1} bx - \frac{1}{b \cdot (n-1)} \int \sin^{n-1} bx \, d(e^{ax} \cdot \cos bx) \end{aligned} \quad (2)$$

$$\begin{aligned} \text{又 } \int \sin^{n-1} bx \, d(e^{ax} \cdot \cos bx) &= \int \sin^{n-1} bx (a \cdot e^{ax} \cdot \cos bx - b \cdot \sin bx \cdot e^{ax}) \, dx \\ &= a \int e^{ax} \cdot \sin^{n-1} bx \cdot \cos bx \, dx - b \int \sin^n bx \cdot e^{ax} \, dx \end{aligned} \quad (3)$$

$$\begin{aligned} \text{又 } \int e^{ax} \cdot \sin^{n-1} bx \cdot \cos bx \, dx &= \frac{1}{b} \int e^{ax} \cdot \sin^{n-1} bx \, d \sin bx \\ &= \frac{1}{b} \cdot e^{ax} \cdot \sin^n bx - \frac{1}{b} \int \sin bx \, d(e^{ax} \cdot \sin^{n-1} bx) \\ &= \frac{1}{b} \cdot e^{ax} \cdot \sin^n bx - \frac{1}{b} \int \sin bx [a \cdot e^{ax} \cdot \sin^{n-1} bx + b \cdot (n-1) \sin^{n-2} bx \cdot \cos bx \cdot e^{ax}] \, dx \\ &= \frac{1}{b} \cdot e^{ax} \cdot \sin^n bx - \frac{a}{b} \int \sin^n bx \cdot e^{ax} \, dx - (n-1) \int e^{ax} \cdot \sin^{n-1} bx \cdot \cos bx \, dx \end{aligned}$$

$$\text{移项并整理得: } \int e^{ax} \cdot \sin^{n-1} bx \cdot \cos bx \, dx = \frac{1}{bn} \cdot e^{ax} \cdot \sin^n bx - \frac{a}{bn} \int \sin^n bx \cdot e^{ax} \, dx \quad (4)$$

$$\begin{aligned} \text{将④式代入③式的得: } \int \sin^{n-1} bx \, d(e^{ax} \cdot \cos bx) &= \frac{a}{bn} \cdot e^{ax} \cdot \sin^n bx - \frac{a^2}{bn} \int \sin^n bx \cdot e^{ax} \, dx - b \int \sin^n bx \cdot e^{ax} \, dx \\ &= \frac{a}{bn} \cdot e^{ax} \cdot \sin^n bx - \frac{a^2 + b^2 n}{bn} \int \sin^n bx \cdot e^{ax} \, dx \end{aligned} \quad (5)$$

$$\begin{aligned} \text{将⑤式代入②式得: } \int e^{ax} \cdot \sin^{n-2} bx \cdot \cos^2 bx \, dx &= \frac{1}{b \cdot (n-1)} \cdot e^{ax} \cdot \cos bx \cdot \sin^{n-1} bx \\ &\quad - \frac{a}{b^2 \cdot n \cdot (n-1)} \cdot e^{ax} \cdot \sin^n bx + \frac{a^2 + b^2 n}{b^2 \cdot n \cdot (n-1)} \int \sin^n bx \cdot e^{ax} \, dx \end{aligned} \quad (6)$$

$$\begin{aligned} \text{将⑥式代入①式得: } \int e^{ax} \cdot \sin^n bx \, dx &= \int e^{ax} \cdot \sin^{n-2} bx \, dx - \frac{1}{b \cdot (n-1)} \cdot e^{ax} \cdot \cos bx \cdot \sin^{n-1} bx \\ &\quad + \frac{a}{b^2 \cdot n \cdot (n-1)} \cdot e^{ax} \cdot \sin^n bx - \frac{a^2 + b^2 n}{b^2 \cdot n \cdot (n-1)} \int \sin^n bx \cdot e^{ax} \, dx \end{aligned}$$

$$\begin{aligned} \text{移项并整理得: } \int e^{ax} \cdot \sin^n bx \, dx &= \frac{n \cdot (n-1) b^2}{a^2 + b^2 n^2} \left[\int e^{ax} \cdot \sin^{n-2} bx \, dx - \frac{1}{b \cdot (n-1)} \cdot e^{ax} \cdot \cos bx \cdot \sin^{n-1} bx + \frac{1}{n \cdot (n-1) b^2} \cdot e^{ax} \cdot \sin^n bx \right] \\ &= \frac{n \cdot (n-1) b^2}{a^2 + b^2 n^2} \cdot \int e^{ax} \cdot \sin^{n-2} bx \, dx - \frac{bn}{a^2 + b^2 n^2} \cdot e^{ax} \cdot \cos bx \cdot \sin^{n-1} bx + \frac{a}{a^2 + b^2 n^2} \cdot e^{ax} \cdot \sin^n bx \\ &= \frac{1}{a^2 + b^2 n^2} \cdot e^{ax} \cdot \sin^{n-1} bx (a \cdot \sin bx - nb \cdot \cos bx) \\ &\quad + \frac{n \cdot (n-1) b^2}{a^2 + b^2 n^2} \int e^{ax} \cdot \sin^{n-2} bx \, dx \end{aligned}$$

$$131. \int e^{ax} \cdot \cos^n bx \, dx = \frac{1}{a^2 + b^2 n^2} \cdot e^{ax} \cdot \cos^{n-1} bx (a \cdot \cos bx + nb \cdot \sin bx)$$

$$+ \frac{n \cdot (n-1)b^2}{a^2 + b^2 n^2} \int e^{ax} \cdot \cos^{n-2} bx \, dx$$

$$\begin{aligned} \text{证明: } \int e^{ax} \cdot \cos^n bx \, dx &= \int e^{ax} \cdot \cos^{n-2} bx \cdot \cos^2 bx \, dx = \int e^{ax} \cdot \cos^{n-2} bx \cdot (1 - \sin^2 bx) \, dx \\ &= \int e^{ax} \cdot \cos^{n-2} bx \, dx - \int e^{ax} \cdot \cos^{n-2} bx \cdot \sin^2 bx \, dx \end{aligned} \quad ①$$

$$\begin{aligned} \text{又 } \int e^{ax} \cdot \cos^{n-2} bx \cdot \sin^2 bx \, dx &= \frac{1}{b \cdot (1-n)} \int e^{ax} \cdot \sin bx \, d \cos^{n-1} bx \\ &= \frac{1}{b \cdot (1-n)} \cdot e^{ax} \cdot \sin bx \cdot \cos^{n-1} bx - \frac{1}{b \cdot (1-n)} \int \cos^{n-1} bx \, d(e^{ax} \cdot \sin bx) \end{aligned} \quad ②$$

$$\begin{aligned} \text{又 } \int \cos^{n-1} bx \, d(e^{ax} \cdot \sin bx) &= \int \cos^{n-1} bx (a \cdot e^{ax} \cdot \sin bx + b \cdot \cos bx \cdot e^{ax}) \, dx \\ &= a \int e^{ax} \cdot \cos^{n-1} bx \cdot \sin bx \, dx + b \int \cos^n bx \cdot e^{ax} \, dx \end{aligned} \quad ③$$

$$\begin{aligned} \text{又 } \int e^{ax} \cdot \cos^{n-1} bx \cdot \sin bx \, dx &= -\frac{1}{b} \int e^{ax} \cdot \cos^{n-1} bx \, d \cos bx \\ &= -\frac{1}{b} \cdot e^{ax} \cdot \cos^n bx + \frac{1}{b} \int \cos bx \, d(e^{ax} \cdot \cos^{n-1} bx) \\ &= -\frac{1}{b} \cdot e^{ax} \cdot \cos^n bx + \frac{1}{b} \int \cos bx [a \cdot e^{ax} \cdot \cos^{n-1} bx - b \cdot (n-1) \cos^{n-2} bx \cdot \sin bx \cdot e^{ax}] \, dx \\ &= -\frac{1}{b} \cdot e^{ax} \cdot \cos^n bx + \frac{a}{b} \int \cos^n bx \cdot e^{ax} \, dx - (n-1) \int e^{ax} \cdot \cos^{n-1} bx \cdot \sin bx \, dx \end{aligned}$$

$$\text{移项并整理得: } \int e^{ax} \cdot \cos^{n-1} bx \cdot \sin bx \, dx = -\frac{1}{bn} \cdot e^{ax} \cdot \cos^n bx + \frac{a}{bn} \int \cos^n bx \cdot e^{ax} \, dx \quad ④$$

$$\begin{aligned} \text{将④式代入③式的得: } \int \cos^{n-1} bx \, d(e^{ax} \cdot \sin bx) &= -\frac{a}{bn} \cdot e^{ax} \cdot \cos^n bx + \frac{a^2}{bn} \int \cos^n bx \cdot e^{ax} \, dx + b \int \cos^n bx \cdot e^{ax} \, dx \\ &= -\frac{a}{bn} \cdot e^{ax} \cdot \cos^n bx + \frac{a^2 + b^2 n}{bn} \int \cos^n bx \cdot e^{ax} \, dx \end{aligned} \quad ⑤$$

$$\begin{aligned} \text{将⑤式代入②式得: } \int e^{ax} \cdot \cos^{n-2} bx \cdot \sin^2 bx \, dx &= \frac{1}{b \cdot (1-n)} \cdot e^{ax} \cdot \sin bx \cdot \cos^{n-1} bx \\ &+ \frac{a}{b^2 \cdot n \cdot (1-n)} \cdot e^{ax} \cdot \cos^n bx - \frac{a^2 + b^2 n}{b^2 \cdot n \cdot (1-n)} \int \cos^n bx \cdot e^{ax} \, dx \end{aligned} \quad ⑥$$

$$\begin{aligned} \text{将⑥式代入①式得: } \int e^{ax} \cdot \cos^n bx \, dx &= \int e^{ax} \cdot \cos^{n-2} bx \, dx - \frac{1}{b \cdot (1-n)} \cdot e^{ax} \cdot \sin bx \cdot \cos^{n-1} bx \\ &+ \frac{a}{b^2 \cdot n \cdot (n-1)} \cdot e^{ax} \cdot \cos^n bx - \frac{a^2 + b^2 n}{b^2 \cdot n \cdot (n-1)} \int \cos^n bx \cdot e^{ax} \, dx \end{aligned}$$

$$\begin{aligned} \text{移项并整理得: } \int e^{ax} \cdot \cos^n bx \, dx &= \frac{n \cdot (1-n)b^2}{-a^2 - b^2 n^2} \left[\int e^{ax} \cdot \cos^{n-2} bx \, dx - \frac{1}{b \cdot (1-n)} \cdot e^{ax} \cdot \sin bx \cdot \cos^{n-1} bx - \frac{a}{n \cdot (1-n)b^2} \cdot e^{ax} \cdot \cos^n bx \right] \\ &= \frac{n \cdot (n-1)b^2}{a^2 + b^2 n^2} \cdot \int e^{ax} \cdot \cos^{n-2} bx \, dx + \frac{bn}{a^2 + b^2 n^2} \cdot e^{ax} \cdot \sin bx \cdot \cos^{n-1} bx + \frac{a}{a^2 + b^2 n^2} \cdot e^{ax} \cdot \cos^n bx \\ &= \frac{1}{a^2 + b^2 n^2} \cdot e^{ax} \cdot \cos^{n-1} bx (a \cdot \cos bx + nb \cdot \sin bx) + \frac{n \cdot (n-1)b^2}{a^2 + b^2 n^2} \int e^{ax} \cdot \cos^{n-2} bx \, dx \end{aligned}$$

(十四) 含有对数函数的积分 (132~136)

$$132. \int \ln x dx = x \cdot \ln x - x + C$$

$$\begin{aligned} \text{证明: } \int \ln x dx &= x \cdot \ln x - \int x d \ln x \\ &= x \cdot \ln x - \int x \cdot \frac{1}{x} dx \\ &= x \cdot \ln x - \int dx \\ &= x \cdot \ln x - x + C \end{aligned}$$

$$133. \int \frac{dx}{x \cdot \ln x} = \ln |\ln x| + C$$

$$\begin{aligned} \text{证明: } \int \frac{dx}{x \cdot \ln x} &= \int \frac{1}{\ln x} d \ln x \\ &= \ln |\ln x| + C \end{aligned}$$

提示: $(\ln x)' = \frac{1}{x}$

$$134. \int x^n \cdot \ln x dx = \frac{1}{n+1} \cdot x^{n+1} \left(\ln x - \frac{1}{n+1} \right) + C$$

$$\begin{aligned} \text{证明: } \int x^n \cdot \ln x dx &= \int \frac{\ln x}{n+1} \cdot (n+1) \cdot x^n dx \\ &= \int \frac{\ln x}{n+1} dx^{n+1} \\ &= \frac{\ln x}{n+1} \cdot x^{n+1} - \frac{1}{n+1} \int x^{n+1} d(\ln x) \\ &= \frac{\ln x}{n+1} \cdot x^{n+1} - \frac{1}{n+1} \int x^n dx \\ &= \frac{\ln x}{n+1} \cdot x^{n+1} - \left(\frac{1}{n+1} \right)^2 \cdot x^{n+1} + C \\ &= \frac{1}{n+1} \cdot x^{n+1} \left(\ln x - \frac{1}{n+1} \right) + C \end{aligned}$$

$$135. \int (\ln x)^n dx = x \cdot (\ln x)^n - n \int (\ln x)^{n-1} dx$$

$$= x \sum_{k=0}^n (-1)^{n-k} \cdot \frac{n!}{k!} \cdot (\ln x)^k$$

$$\text{证明: } \int (\ln x)^n dx = x \cdot (\ln x)^n - \int x d(\ln x)^n$$

$$= x \cdot (\ln x)^n - \int x \cdot n \cdot (\ln x)^{n-1} \cdot \frac{1}{x} dx$$

$$= x \cdot (\ln x)^n - n \int (\ln x)^{n-1} dx$$

$$= x \cdot (\ln x)^n - n \cdot x \cdot (\ln x)^{n-1} + n \int x d(\ln x)^{n-1}$$

$$= x \cdot (\ln x)^n - n \cdot x \cdot (\ln x)^{n-1} + n \cdot (n-1) \int (\ln x)^{n-2} dx$$

$$= x \cdot (\ln x)^n - n \cdot x \cdot (\ln x)^{n-1} + n \cdot (n-1) \cdot x \cdot (\ln x)^{n-2} - n \cdot (n-1) \cdot (n-2) \int (\ln x)^{n-3} dx$$

.....

$$= x \cdot (\ln x)^n - n \cdot x \cdot (\ln x)^{n-1} + n \cdot (n-1) \cdot x \cdot (\ln x)^{n-2} - n \cdot (n-1) \cdot (n-2) (\ln x)^{n-3}$$

$$+ \dots + (-1)^{n-k} \cdot n \cdot (n-1) \cdot (n-2) \dots (n-k+1) \cdot (\ln x)^{n-k} + \dots$$

$$+ (-1)^2 \cdot n \cdot (n-1) \cdot (n-2) \dots 5 \times 4 \times 3 \cdot (\ln x)^{3-1} \cdot x$$

$$+ (-1)^1 \cdot n \cdot (n-1) \cdot (n-2) \dots 4 \times 3 \times 2 \cdot (\ln x)^{2-1} \cdot x$$

$$+ (-1)^0 \cdot n \cdot (n-1) \cdot (n-2) \dots 3 \times 2 \times 1 \cdot (\ln x)^{1-1} \cdot x$$

$$= x \sum_{k=0}^n (-1)^{n-k} \cdot \frac{n!}{k!} \cdot (\ln x)^k$$

$$136. \int x^m \cdot (\ln x)^n dx = \frac{1}{m+1} \cdot x^{m+1} \cdot (\ln x)^n - \frac{n}{m+1} \int x^m \cdot (\ln x)^{n-1} dx$$

$$\text{证明: } \int x^m \cdot (\ln x)^n dx = \frac{1}{m+1} \int (\ln x)^n d(x^{m+1})$$

$$= \frac{1}{m+1} \cdot x^{m+1} \cdot (\ln x)^n - \frac{1}{m+1} \int x^{m+1} d(\ln x)^n$$

$$= \frac{1}{m+1} \cdot x^{m+1} \cdot (\ln x)^n - \frac{n}{m+1} \int x^{m+1} \cdot (\ln x)^{n-1} \cdot \frac{1}{x} dx$$

$$= \frac{1}{m+1} \cdot x^{m+1} \cdot (\ln x)^n - \frac{n}{m+1} \int x^m \cdot (\ln x)^{n-1} dx$$

(十五) 含有双曲函数的积分 (137~141)

137. $\int shx dx = chx + C$

证明: $\because (chx)' = shx$, 即 chx 为 shx 的原函数

$$\begin{aligned}\therefore \int shx dx &= \int d chx \\ &= chx + C\end{aligned}$$

138. $\int ch x dx = shx + C$

证明: $\because (shx)' = chx$, 即 shx 为 chx 的原函数

$$\begin{aligned}\therefore \int ch x dx &= \int d shx \\ &= shx + C\end{aligned}$$

139. $\int th x dx = \ln chx + C$

$$\begin{aligned}\text{证明: } \int th x dx &= \int \frac{shx}{chx} dx \\ &= \int \frac{1}{chx} d chx \\ &= \ln chx + C\end{aligned}$$

140. $\int sh^2 x dx = -\frac{x}{2} + \frac{1}{4} sh 2x + C$

$$\begin{aligned}\text{证明: } \int sh^2 x dx &= \int \left(\frac{e^x - e^{-x}}{2} \right)^2 dx \\ &= \frac{1}{4} \int (e^{2x} + e^{-2x} - 2) dx \\ &= \frac{e^{2x}}{8} - \frac{e^{-2x}}{8} - \frac{x}{2} + C \\ &= -\frac{x}{2} + \frac{1}{4} \cdot \frac{e^{2x} - e^{-2x}}{2} + C \\ &= -\frac{x}{2} + \frac{1}{4} \cdot sh 2x + C\end{aligned}$$

$提示: chx = \frac{e^x + e^{-x}}{2}$ (双曲余弦)
$shx = \frac{e^x - e^{-x}}{2}$ (双曲余弦)

141. $\int ch^2 x dx = \frac{x}{2} + \frac{1}{4} \cdot sh 2x + C$

$$\begin{aligned}\text{证明: } \int ch^2 x dx &= \int \left(\frac{e^x + e^{-x}}{2} \right)^2 dx \\ &= \frac{1}{4} \int (e^{2x} + e^{-2x} + 2) dx \\ &= \frac{e^{2x}}{8} - \frac{e^{-2x}}{8} + \frac{x}{2} + C \\ &= \frac{x}{2} + \frac{1}{4} \cdot \frac{e^{2x} - e^{-2x}}{2} + C \\ &= \frac{x}{2} + \frac{1}{4} \cdot sh 2x + C\end{aligned}$$

$提示: chx = \frac{e^x + e^{-x}}{2}$ (双曲余弦)
$shx = \frac{e^x - e^{-x}}{2}$ (双曲余弦)

(十六) 定积分 (142~147)

$$142. \int_{-\pi}^{\pi} \cos nx \, dx = \int_{-\pi}^{\pi} \sin nx \, dx = 0$$

$$\begin{aligned} \text{证明①: } \int_{-\pi}^{\pi} \cos nx \, dx &= \frac{1}{n} \int_{-\pi}^{\pi} \cos nx \, d(nx) \\ &= \frac{1}{n} \cdot (\sin nx) \Big|_{-\pi}^{\pi} \\ &= \frac{1}{n} \cdot \sin(n\pi) - \frac{1}{n} \cdot \sin(-n\pi) \\ &= \frac{2}{n} \cdot \sin(n\pi) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{证明②: } \int_{-\pi}^{\pi} \sin nx \, dx &= \frac{1}{n} \int_{-\pi}^{\pi} \sin nx \, d(nx) \\ &= -\frac{1}{n} \cdot (\cos nx) \Big|_{-\pi}^{\pi} \\ &= -\frac{1}{n} \cdot \cos(n\pi) + \frac{1}{n} \cdot \cos(-n\pi) \\ &= 0 \end{aligned}$$

$$\text{综合证明①②得: } \int_{-\pi}^{\pi} \cos nx \, dx = \int_{-\pi}^{\pi} \sin nx \, dx = 0$$

$$143. \int_{-\pi}^{\pi} \cos mx \cdot \sin nx \, dx = 0 \quad \boxed{\text{公式100: } \int \sin ax \cdot \cos bx \, dx = -\frac{1}{2(a+b)} \cdot \cos(a+b)x - \frac{1}{2(a-b)} \cdot \cos(a-b)x + C}$$

证明: 1. 当 $m \neq n$ 时

$$\begin{aligned} \int_{-\pi}^{\pi} \cos mx \cdot \sin nx \, dx &= -\frac{1}{2(m+n)} \cdot \cos(m+n)x \Big|_{-\pi}^{\pi} - \frac{1}{2(n-m)} \cos(n-m)x \Big|_{-\pi}^{\pi} \\ &= -\frac{1}{2(m+n)} [\cos(m+n)\pi - \cos(m+n)(-\pi)] - \frac{1}{2(n-m)} [\cos(n-m)\pi - \cos(n-m)(-\pi)] \\ &= 0 + 0 = 0 \end{aligned}$$

2. 当 $m = n$ 时

$$\begin{aligned} \int_{-\pi}^{\pi} \cos mx \cdot \sin nx \, dx &= \int_{-\pi}^{\pi} \cos mx \cdot \sin mx \, dx \quad \boxed{\text{提示: } \sin 2x = 2 \cdot \sin x \cdot \cos x} \\ &= \frac{1}{2m} \int_{-\pi}^{\pi} \sin 2mx \, d(mx) \\ &= \frac{1}{4m} \int_{-\pi}^{\pi} \sin 2mx \, d(2mx) \\ &= -\frac{1}{4m} \cdot \cos 2mx \Big|_{-\pi}^{\pi} \\ &= -\frac{1}{4m} \cdot [\cos 2m\pi - \cos(-2m\pi)] \\ &= 0 \end{aligned}$$

$$\text{综合讨论 1, 2 得: } \int_{-\pi}^{\pi} \cos nx \, dx = \int_{-\pi}^{\pi} \cos mx \cdot \sin nx \, dx = 0$$

$$144. \int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

证明：1. 当 $m \neq n$ 时

$$\begin{aligned} \int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx &= \frac{1}{2(m+n)} \cdot \sin(m+n)x \Big|_{-\pi}^{\pi} - \frac{1}{2(m-n)} \sin(m-n)x \Big|_{-\pi}^{\pi} \\ &= \frac{1}{2(m+n)} [\sin(m+n)\pi - \sin(m+n)(-\pi)] - \frac{1}{2(m-n)} [\sin(m-n)\pi + \sin(m-n)(-\pi)] \\ &= 0 - 0 = 0 \end{aligned}$$

$$\text{公式102: } \int \cos ax \cdot \cos bx \, dx = \frac{1}{2(a+b)} \cdot \sin(a+b)x + \frac{1}{2(a-b)} \cdot \sin(a-b)x + C$$

2. 当 $m = n$ 时

$$\begin{aligned} \int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx &= \int_{-\pi}^{\pi} \cos mx \cdot \cos mx \, dx \\ &= \frac{1}{m} \int_{-\pi}^{\pi} \cos^2 mx \, d(mx) \quad \text{公式94: } \int \cos^2 x \, dx = \frac{x}{2} + \frac{1}{4} \cdot \sin 2x + C \\ &= \frac{1}{4m} \cdot \sin 2mx \Big|_{-\pi}^{\pi} + \frac{1}{2m} \cdot mx \Big|_{-\pi}^{\pi} \\ &= \frac{1}{4m} \cdot [\sin 2m\pi - \sin(-2m\pi)] + \frac{\pi}{2} + \frac{\pi}{2} \\ &= \pi \end{aligned}$$

$$\text{综合讨论 1, 2 得: } \int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

$$145. \int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

证明：1. 当 $m \neq n$ 时

$$\begin{aligned} \int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx &= -\frac{1}{2(m+n)} \cdot \sin(m+n)x \Big|_{-\pi}^{\pi} + \frac{1}{2(m-n)} \sin(m-n)x \Big|_{-\pi}^{\pi} \\ &= -\frac{1}{2(m+n)} [\sin(m+n)\pi - \sin(m+n)(-\pi)] + \frac{1}{2(m-n)} [\sin(m-n)\pi - \sin(m-n)(-\pi)] \\ &= 0 + 0 = 0 \end{aligned}$$

$$\text{公式101: } \int \sin ax \cdot \sin bx \, dx = -\frac{1}{2(a+b)} \cdot \sin(a+b)x + \frac{1}{2(a-b)} \cdot \sin(a-b)x + C$$

2. 当 $m = n$ 时

$$\begin{aligned} \int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx &= \int_{-\pi}^{\pi} \sin^2 mx \, dx \quad \text{公式93: } \int \sin^2 x \, dx = \frac{x}{2} - \frac{1}{4} \cdot \sin 2x + C \\ &= \frac{1}{m} \int_{-\pi}^{\pi} \sin^2 mx \, d(mx) \\ &= \frac{1}{2m} \cdot mx \Big|_{-\pi}^{\pi} - \frac{1}{4m} \cdot \sin 2mx \Big|_{-\pi}^{\pi} \\ &= -\frac{1}{4m} \cdot [\sin 2m\pi - \sin(-2m\pi)] + \frac{\pi}{2} + \frac{\pi}{2} \\ &= \pi \end{aligned}$$

$$\text{综合讨论 1, 2 得: } \int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

$$146. \int_0^{\pi} \sin mx \cdot \sin nx \, dx = \int_0^{\pi} \cos mx \cdot \cos nx \, dx \begin{cases} 0, & m \neq n \\ \frac{\pi}{2}, & m = n \end{cases}$$

证明：1. 当 $m \neq n$ 时

$$\begin{aligned} \int_0^{\pi} \sin mx \cdot \sin nx \, dx &= -\frac{1}{2(m+n)} \cdot \sin(m+n)x \Big|_0^{\pi} + \frac{1}{2(m-n)} \sin(m-n)x \Big|_0^{\pi} \\ &= -\frac{1}{2(m+n)} [\sin(m+n)\pi - \sin 0] + \frac{1}{2(m-n)} [\sin(m-n)\pi - \sin 0] \\ &= 0 + 0 = 0 \end{aligned}$$

$$\begin{aligned} \int_0^{\pi} \cos mx \cdot \cos nx \, dx &= \frac{1}{2(m+n)} \cdot \sin(m+n)x \Big|_0^{\pi} + \frac{1}{2(m-n)} \sin(m-n)x \Big|_0^{\pi} \\ &= \frac{1}{2(m+n)} [\sin(m+n)\pi - \sin 0] + \frac{1}{2(m-n)} [\sin(m-n)\pi + \sin 0] \\ &= 0 + 0 = 0 \end{aligned}$$

2. 当 $m = n$ 时

$$\begin{aligned} \int_0^{\pi} \sin mx \cdot \sin nx \, dx &= \int_0^{\pi} \sin^2 mx \, dx \\ &= \frac{1}{m} \int_0^{\pi} \sin^2 mx \, d(mx) \\ &= \frac{1}{2m} \cdot mx \Big|_0^{\pi} - \frac{1}{4m} \cdot \sin 2mx \Big|_0^{\pi} \\ &= -\frac{1}{4m} \cdot [\sin 2m\pi - \sin 0] + \frac{\pi}{2} + 0 \\ &= \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} \int_0^{\pi} \cos mx \cdot \cos nx \, dx &= \int_0^{\pi} \cos^2 mx \, dx \\ &= \frac{1}{m} \int_0^{\pi} \cos^2 mx \, d(mx) \\ &= \frac{1}{4m} \cdot \sin 2mx \Big|_0^{\pi} + \frac{1}{2m} \cdot mx \Big|_0^{\pi} \\ &= \frac{1}{4m} \cdot [\sin 2m\pi - \sin 0] + \frac{\pi}{2} + 0 \\ &= \frac{\pi}{2} \end{aligned}$$

$$\text{综合讨论 1, 2 得: } \int_0^{\pi} \sin mx \cdot \sin nx \, dx = \int_0^{\pi} \cos mx \cdot \cos nx \, dx \begin{cases} 0, & m \neq n \\ \frac{\pi}{2}, & m = n \end{cases}$$

以上所用公式：

$$\text{公式101: } \int \sin ax \cdot \sin bx \, dx = -\frac{1}{2(a+b)} \cdot \sin(a+b)x + \frac{1}{2(a-b)} \cdot \sin(a-b)x + C$$

$$\text{公式102: } \int \cos ax \cdot \cos bx \, dx = \frac{1}{2(a+b)} \cdot \sin(a+b)x + \frac{1}{2(a-b)} \cdot \sin(a-b)x + C$$

$$\text{公式93: } \int \sin^2 x \, dx = \frac{x}{2} - \frac{1}{4} \cdot \sin 2x + C$$

$$\text{公式94: } \int \cos^2 x \, dx = \frac{x}{2} + \frac{1}{4} \cdot \sin 2x + C$$

$$147. I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx$$

$$I_n = \frac{n-1}{n} I_{n-2}$$

$$= \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3} & (n \text{ 为大于1的正奇数}), I_1 = 1 \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} & (n \text{ 为正偶数}), I_0 = \frac{\pi}{2} \end{cases}$$

$$\begin{aligned} \text{证明①: } I_n &= \int_0^{\frac{\pi}{2}} \sin^n x dx = -\frac{1}{n} \cdot \sin^{n-1} x \cdot \cos x \Big|_0^{\frac{\pi}{2}} + \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x dx \\ &= -\frac{1}{n} \left(\sin^{n-1} \frac{\pi}{2} \cdot \cos \frac{\pi}{2} - \sin^{n-1} 0 \cdot \cos 0 \right) + \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x dx \\ &= \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x dx = \frac{n-1}{n} I_{n-2} \end{aligned}$$

当 n 为正奇数时

$$\begin{aligned} I_n &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot \int_0^{\frac{\pi}{2}} \sin x dx \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot (-\cos x) \Big|_0^{\frac{\pi}{2}} \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1 \end{aligned}$$

$$\text{特别的, 当 } n=1 \text{ 时, } I_n = \int_0^{\frac{\pi}{2}} \sin x dx = (-\cos x) \Big|_0^{\frac{\pi}{2}} = 1$$

当 n 为正偶数时

$$\begin{aligned} I_n &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \int_0^{\frac{\pi}{2}} \sin^0 x dx \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot (x) \Big|_0^{\frac{\pi}{2}} \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \end{aligned}$$

$$\text{特别的, 当 } n=0 \text{ 时, } I_n = \int_0^{\frac{\pi}{2}} \sin^0 x dx = (x) \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$\text{证明②: } I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx \cdots \text{亦同理可证}$$

附录：常数和基本初等函数导数公式

$$1. (C)' = 0 \quad (C \text{ 为常数})$$

$$2. (x^\mu)' = \mu \cdot x^{\mu-1} \quad (x \neq 0)$$

$$3. (\sin x)' = \cos x$$

$$4. (\cos x)' = -\sin x$$

$$5. (\tan x)' = \sec^2 x$$

$$6. (\cot x)' = -\csc^2 x$$

$$7. (\sec x)' = \sec x \cdot \tan x$$

$$8. (\csc x)' = -\csc x \cdot \cot x$$

$$9. (a^x)' = a^x \cdot \ln a \quad (a \text{ 为常数})$$

$$10. (e^x)' = e^x$$

$$11. (\log_a x)' = \frac{1}{x \cdot \ln a} \quad (a > 0)$$

$$12. (\ln x)' = \frac{1}{x}$$

$$13. (\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$14. (\arccos x)' = \frac{1}{-\sqrt{1-x^2}}$$

$$15. (\arctan x)' = \frac{1}{1+x^2}$$

$$16. (\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

