CHAPTER 53

RECURRENCE EQUATIONS

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53.1 INTRODUCTION

The computing time of an algorithm (particularly a recursive algorithm) is often easily expressed recursively (i.e., in terms of itself). This was the case, for instance, for the function `rSum` (Program 1.26). We had determined that $t_{rSum}(n) = c + t_{rSum}(n-1)$ where c is some constant. The worst-case computing time, $t_M^w(n)$, of the merge sort method is easily seen to satisfy the inequality:

$$t_M^w(n) \le \begin{cases} c_1 & n = 1 \\ t_M^w(\lceil n/2 \rceil) + t_M^w(\lfloor n/2 \rfloor) + c_4 n & n > 1 \end{cases} \qquad (53.1)$$

We expect the recurrence (53.1) to be difficult to solve because of the presence of the ceiling and floor functions. If we attempt to solve (53.1) only for values of n that are a power of 2 ($n=2^k$), then (53.1) becomes:

$$t_M^w(n) \le \begin{cases} c_1 & n=1 \\ 2t_M^w(n/2)+c_4 n & n>1 \text{ and } a \text{ } power \text{ } of \text{ } 2 \end{cases} \tag{53.2}$$

If the inequality of (53.2) is converted to the equality:

$$t_M'(n) = \begin{cases} c_1 & n=1 \\ 2t_M'(n/2)+c_4 n & n>1 \textit{ and a power of 2} \end{cases} \tag{53.3}$$

then $t_M'(n)$ is an upper bound on $t_M^W(n)$. So, if $t_M'(n) = f(n)$ then $t_M^w(n) = O(f(n))$. Since it is also the case that there exist constants c_5 and c_6 such that:

$$t_M^w(n) \ge \begin{cases} c_5 & n=1 \\ 2t_M^w(n/2)+c_6 n & n>1 \textit{ and n a power of 2} \end{cases}$$

it follows that $t_M^w(n) = \Omega(f(n))$. Hence, $t_M^w(n) = \Theta(f(n))$.

The entire discussion concerning the worst case complexity t_M^w can be repeated with respect to the best case complexity (i.e. the minimum time spent on any input of n numbers). The conclusion is that $t_M^b(n) = \Theta(f(n))$. Since both the best and worst case complexities are $\Theta(f(n))$, it follows that $t_M^a(n) = \Theta(f(n))$ and $t_M(n) = \Theta(f(n))$.

when analyzing quick sort, we see that the partitioning into left (L) and right (R) segments can be done in Θ(n) time. So,

$$t_Q(n) = \begin{cases} c_1 & n \le 1 \\ c_2 + t_Q(|L|) + t_Q(|R|) & n>1 \end{cases} \tag{53.4}$$

In (53.4), Q has been used as an abbreviation for `QuickSort`. $|L|$ can be any number in the range 0 to $n-1$. For random instances, $|L|$ equals each of 0, 1, ..., $n-1$ with equal probability. So, for the average complexity of `QuickSort`, we obtain:

$$t_Q^a(n) = \begin{cases} c_1 & n \le 1 \\ c_2 n + \frac{1}{n}\left(\sum_{i=1}^{n-1} [t_Q^a(i) + t_Q^a(n-i)]\right] & n>1 \end{cases} \tag{53.5}$$

$$= \begin{cases} c_1 & n \le 1 \\ c_2 n + \frac{2}{n}\sum_{i=1}^{n-1} t_Q^a(i) & n>1 \end{cases}$$

The worst case for `QuickSort` is when one of L and R is empty at all levels of the recursion. In this case, we obtain the recurrence:

$$t_Q^w(n) = \begin{cases} c_1 & n \le 1 \\ c_2 n + t_Q^w(n-1) & n > 1 \end{cases} \tag{53.6}$$

The best case for `QuickSor` is when $|L| \approx |R|$ at all levels of the recursion. The recurrence for this case is:

$$t_Q^b(n) = \begin{cases} c_1 & n \le 1 \\ c_2 n + t_Q^b(\lceil \frac{n-1}{2} \rceil) + t_Q^b(n-1-\lceil \frac{n-1}{2} \rceil) & n > 1 \end{cases} \tag{53.7}$$

A function $g(n)$ such that $t_Q^b(n) = \Theta(g(n))$ for n a power of 2 can be obtained by solving the recurrence:

$$g(n) = t_Q'(n) = \begin{cases} c_1 & n \le 1 \\ c_2 n + 2t_Q'(n/2) & n>1 \textit{ and a power of 2} \end{cases} \tag{53.8}$$

For `select` (Program 19.8), the worst case is when $k = 1$ and $|R| = 0$ at all levels of the recursion. So, the worst-case computing time of `select` is given by the recurrence:

$$t_{select}^w(n) = \begin{cases} c_1 & n = 1 \\ c_2 n + t_{select}^w(n-1) & n > 1 \end{cases} \tag{53.9}$$

To obtain the recurrence for the average computing time of `select`, we need to introduce some new functions. First, we shall assume that all the elements are distinct. Let $t^k(n)$ be the average time to find the kth smallest element. This average is taken over all $n!$ permutations of the elements. The average computing time of `select` is given by:

$$t_{select}^a(n) = \frac{1}{n} \sum_{k=1}^{n} t^k(n)$$

Define $R(n)$ to be the largest $t^k(n)$. That is,

$$R(n) = \max_{1 \le k \le n} \{t^k(n)\}$$

It is easy to see that $t^a_{select}(n) \le R(n)$.

With these definitions in mind, let us proceed to analyze `select` for the case when all elements are distinct. For random input, there is an equal probability that $|L| = 0, 1, 2, ..., n-1$. This leads to the following inequality for $t^k(n)$:

$$t^k(n) \le \begin{cases} c & n=1 \\ cn + \frac{1}{n}\left[\sum_{k<j\le n} t^k(j-1) + \sum_{1\le j<k} t^{k-j}(n-j)\right] & n\ge 2 \end{cases}$$

From this, we conclude that:

$$R(n) \le cn + \frac{1}{n}\max_k\{\sum_{k<j\le n} R(j-1) + \sum_{1\le j<k} R(n-j)\},\ \mathrm{n}\ge 2$$

$$= \mathrm{cn} + \frac{1}{n}\max_k\{\sum_{k}^{n-1} R(i) + \sum_{n-k+1}^{n-1} R(i)\},\ \mathrm{n}\ge 2$$

Since R is an increasing function of n,

$$R(n) \le \begin{cases} c & n=1 \\ 2.5c & n=2 \\ cn + \frac{2}{n}\sum_{i=n/2}^{n-1} R(i) & n \text{ even and } n>2 \\ cn + \frac{2}{n}\sum_{i=(n+1)/2}^{n-1} R(i) & \text{else} \end{cases} \tag{53.10}$$

If $R(n) = \Theta(f(n))$, then it follows from our earlier observation ($t^a_{select}(n) \le R(n)$) that $t^a_{select}(n) = O(f(n))$. We shall later see that $R(n) = \Theta(n)$. This together with the observation $t^a_{select}(n) = \Omega(n)$ leads to the conclusion that $t^a_{select}(n) = \Theta(n)$.

Even though `binarySearch` (Program 3.1) is not a recursive algorithm, its worst-case time complexity is best described by a recurrence relation. It is not too difficult to see that the following recurrence is correct:

$$t^w_B(n) = \begin{cases} c_1 & n=1 \\ c_2 + t^w_B(\lceil \frac{n-1}{2} \rceil) & n>1 \end{cases} \tag{53.11}$$

When n is a power of 2, (53.11) simplifies to:

$$t_B^w(n) = \begin{cases} c_1 & n=1 \\ c_2 + t_B^w(n/2) & n>1 \textit{ and a power of } 2 \end{cases} \tag{53.12}$$

Hopefully, these examples have convinced you that recurrence relations are indeed useful in describing the time complexity of both iterative and recursive algorithms. In each of the above examples, the recurrence relations themselves were easily obtained. Having obtained the recurrence, we must now solve it to determine the asymptotic growth rate of the time complexity. We shall consider four methods of solving recurrence relations:

(a) substitution
(b) induction
(c) characteristic roots
(d) generating functions.

53.2 SUBSTITUTION

In the substitution method of solving a recurrence relation for f(n), the recurrence for $f(n)$ is repeatedly used to eliminate all occurrences of $f()$ from the right hand side of the recurrence. Once this has been done, the terms in the right hand side are collected together to obtain a compact expression for $f(n)$. The mechanics of this method are best described by means of examples.

Example 53.1 Consider the recurrence:

$$t(n) = \begin{cases} c_1 & n=0 \\ c_2 + t(n-1) & n \geq 1 \end{cases} \tag{53.13}$$

When $c_1 = c_2 = 2$, $t(n)$ is the recurrence for the step count of `rSum` (Program 1.9). If $n > 2$ then $t(n-1) = c_2 + t(n-2)$. If $n > 3$ then $t(n-2) = c_2 + t(n-3)$ etc. These equalities are immediate consequences of (53.13) and are used in the following derivation of a nonrecursive expression for $t(n)$:

$$\begin{aligned} t(n) &= c_2 + t(n-1) \\ &= c_2 + c_2 + t(n-2) \\ &= c_2 + c_2 + c_2 + t(n-3) \\ &\;\;. \\ &\;\;. \\ &\;\;. \end{aligned}$$

$$= c_2 n + t(0)$$

$$= c_2 n + c_1,\ n \geq 0$$

So, we see that $t(n) = c_2 n + c_1$, $n \geq 0$. From this, we obtain $t_{rSum}(n) = 2n+2$. □

Example 53.2 Recurrence (53.12) may be solved by substitution. Observe that (53.12) defines $t_B^w(n)$ only for values of n that are a power of 2. If n is not a power of 2, then the value of $t(n)$ is not defined by (53.12). For example, if n = 5 then $t_B^w(2.5)$ appears on the right hand side of (53.12). But t_B^w is a function whose domain is the natural numbers. If n = 6, then from (53.12) we obtain (t is used as an abbreviation for t_B^w):

$$t(6) = c_2 + t(3)$$

$$= c_2 + c_2 + t(1.5)$$

But, t(1.5) is undefined. When n is a power of 2, t(n) is always defined (i.e., using the recurrence 7.12). $t(n)$ is of course defined for all $n \in N - \{1\}$ when (53.11) is used. Assuming n is a power of 2 (say, $n = 2^k$), the substitution method leads to the following series of equalities:

$$\begin{aligned}
t(n) &= c_2 + t(n/2) \\
&= c_2 + c_2 + t(n/4) \\
&= c_2 + c_2 + c_2 + t(n/8) \\
&\ \ \vdots \\
&= kc_2 + t(n/2^k) \\
&= kc_2 + t(1) \\
&= c_1 + kc_2 \\
&= c_1 + c_2 \log n,\ n \text{ a power of } 2
\end{aligned}$$

Unless otherwise specified, all logarithms in this chapter are base 2. At this point, we only have an expression for $t_B^w(n)$ for values of n that are a power of 2. If n is between 2^k and 2^{k+1}, then $t_B^w(n)$ will be between $t(2^k)$ and $t(2^{k+1})$. So, $c_1 + c_2 \lfloor \log n \rfloor \leq t_B^w \leq c_1 + c_2 \lceil \log n \rceil$ for all n. This implies that $t_B^w(n) = \Theta(\log n)$. □

Example 53.3 Consider the recurrence:

$$t(n) = \begin{cases} c & n = 1 \\ a * t(n/b) + cn & n \geq 2 \end{cases} \tag{53.14}$$

This recurrence defines $t(n)$ only for values of n that are a power of b. The recurrence (53.3) is the same as (53.14) when $c_1 = c_4$ = c, and a = b = 2. Even though (53.3) is not an instance of (53.14) (as $c_1 \neq c_4$ in general), the solution to (53.14) with a = 2 and b = 2 does give us a function g(n) such that t'_M(n) = O(g(n)). And, from the discussion following (53.3) it should be clear that t^w_M(n) = Θ(g(n)). When c = c_1, and a = b = 2, (53.14) becomes the same as (53.8).

Assume that n = b^k for some natural number k. Solving (53.14) by the substitution method yields:

$$
\begin{aligned}
t(n) &= a*t(n/b)+cn \\
&= a[a*t(n/b^2)+c(n/b)]+cn \\
&= a^2 t(n/b^2)+cn(a/b)+cn \\
&= a^2[a*t(n/b^3)+c(n/b^2)]+cn[a/b+1] \\
&= a^3 t(n/b^3)+cn(a^2/b^2)+cn[a/b+1] \\
&= a^3[a*t(n/b^4)+c(n/b^3)]+cn[a^2/b^2+a/b+1] \\
&= a^4 t(n/b^4)+cn[a^3/b^3+a^2/b^2+a/b+1] \\
&\;\;\vdots \\
&= a^k t(n/b^k)+cn\sum_{i=0}^{k-1}(a/b)^i \\
&= a^k t(1)+cn\sum_{i=0}^{k-1}(a/b)^i \\
&= a^k c+cn\sum_{i=0}^{k-1}(a/b)^i \\
&= (a/b)^k cn+cn\sum_{i=0}^{k-1}(a/b)^i, \; (b^k = b^{\log_b n} = n) \\
&= cn\sum_{i=0}^{k}(a/b)^i.
\end{aligned}
$$

When a = b, $\sum_{0}^{k}(a/b)^i$ = k+1. When a ≠ b, $\sum_{i=0}^{k}(a/b)^i$ = $((a/b)^{k+1}$-1)/(a/b-1). If a<b then a/b<1 and $((a/b)^{k+1}$-1)/(a/b-1) = (1-$(a/b)^{k+1}$)/ (1-a/b)<1/(1-a/b). So,

$\sum_0^k (a/b)^i = \Theta(1)$. When a>b, $((a/b)^{k+1}$-1)/(a/b-1) = $\Theta((a/b)^k) = \Theta(a^k/b^{\log_b n})$ = $\Theta(a^{\log_b n}$/n) = $\Theta(n^{\log_b a}$/n). So, we obtain:

$$t(n) = \begin{cases} \Theta(n) & a<b \\ \Theta(n\log n) & a=b \\ \Theta(n^{\log_b a}) & a>b \end{cases}$$

From this and our earlier discussion, we conclude that t_M^w(n) = Θ(nlog n) and $t_Q^b(n)$ = Θ(nlog n). □

Recurrence (53.14) is a very frequently occurring recurrence form in the analysis of algorithms. It often occurs with the cn term replaced by such terms as c, or cn^2, or cn^3 etc. So, we would like to extend the result of Example 53.3 and obtain a general form for the solution of the recurrence:

$$t(n) = a*t(n/b)+g(n), n \geq b \text{ and } n \text{ a power of } b, \tag{53.15}$$

where a and b are known constants. We shall assume that t(1) is also known. Clearly, (53.15) reduces to (53.14) when t(1)= c and g(n) = cn. Using the substitution method, we obtain:

$$\begin{aligned} t(n) &= a*t(n/b)+g(n) \\ &= a[a*t(n/b^2)+g(n/b)]+g(n) \\ &= a^2 t(n/b^2)+ag(n/b)+g(n) \\ &\;\;\vdots \\ &= a^k t(1)+\sum_{i=0}^{k-1} a^i g(n/b^i) \end{aligned}$$

where $k = \log_b n$. This equation may be further simplified as below:

$$\begin{aligned} t(n) &= a^k t(1)+\sum_{i=0}^{k-1} a^i g(n/b^i) \\ &= a^k t(1)+\sum_{i=0}^{k-1} a^i g(b^{k-i}) \\ &= a^k[t(1)+\sum_{j=1}^{k} a^{-j} g(b^j)] \end{aligned}$$

Since $a^k = a^{\log_b n} = n^{\log_b a}$, the expression for $t(n)$ becomes:

$$\begin{aligned} t(n) &= n^{\log_b a}[t(1)+\sum_{j=1}^{k} a^{-j}\, g(b^j)] \\ &= n^{\log_b a}[t(1)+\sum_{j=1}^{k} \{g(b^j)/(b^j)^{\log_b a}\}] \\ &= n^{\log_b a}[t(1)+\sum_{j=1}^{k} h(b^j)] \end{aligned}$$

where h(n) = g(n)/$n^{\log_b a}$. Our final form for t(n) is:

$$t(n) = n^{\log_b a}[t(1)+f(n)] \tag{53.16}$$

where $f(n) = \sum_{j=1}^{k} h(b^j)$ and h(n) = g(n)/$n^{\log_b a}$. Figure 53.1 tabulates the asymptotic value of $f(n)$ for various h(n)s. This table together with (53.16) allows one to easily obtain the asymptotic value of $t(n)$ for many of the recurrences one encounters when analyzing algorithms.

h(n)	f(n)
$O(n^r)$, r<0	O(1)
$\Theta((logn)^i)$, i≥0	$\Theta(((logn)^{i+1})/(i+1))$
$\Omega(n^r)$, r>0	Θ(h(n))

Figure 53.1 $f(n)$ values for various h(n) values.

Let us consider some examples using this table. The recurrence for t_B^w when n is a power of 2 is:

$$t(n) = t(n/2)+c_2$$

and t(1) = c_1. Comparing with (53.15), we see that a=1, b=2, and g(n) = c_2. So, $\log_b(a) = 0$ and h(n) = g(n)/$n^{\log_b a} = c_2 = c_2(logn)^0 = \Theta((logn)^0)$. From ?F7.1}, we obtain $f(n)$ = Θ(log n). So,

$$t(n) = n^{\log_b a}(c_1+\Theta(\log n))$$
$$= \Theta(\log n)$$

For the recurrence

$$t(n) = 7t(n/2)+18n^2,\ n\geq 2 \text{ and n a power of 2,}$$

we obtain: a=7, b=2, and g(n) = $18n^2$. So, $\log_b a = \log_2 7 \approx 2.81$ and h(n) = $18n^2/n^{\log_2 7} = 18n^{2-\log_2 7} = O(n^r)$ where r = 2-$\log_2 7 < 0$. So, $f(n) = O(1)$. The expression for $t(n)$ is:

$$t(n) = n^{\log_2^7}(t(1)+O(1))$$
$$= \Theta(n^{\log_2 7})$$

as t(1) may be assumed to be constant.

As a final example, consider the recurrence:

$$t(n) = 9t(n/3)+4n^6,\ n \geq 3 \text{ and a power of 3.}$$

Comparing with (53.15), we obtain a=9, b=3, and g(n) = $4n^6$. So, $\log_b a = 2$ and h(n) = $4n^6/n^2 = 4n^4 = \Omega(n^4)$. From Figure 53.1, we see that $f(n) = \Theta(h(n)) = \Theta(n^4)$. So,

$$t(n) = n^2(t(1)+\Theta(n^4))$$
$$= \Theta(n^6)$$

as t(1) may be assumed constant.

53.3 INDUCTION

Induction is more of a verification method than a solution method. If we have an idea as to what the solution to a particular recurrence is then we can verify it by providing a proof by induction.

Example 53.4 Induction can be used to show that $t(n)$ = 3n+2 is the solution to the recurrence:

$$t(n) = \begin{cases} 2 & n=0 \\ 3+t(n-1) & n>0 \end{cases}$$

For the induction base, we see that when n=0, $t(n)$ = 2 and 3n+2 = 2. Assume that $t(n)$ = 3n+2 for some n, n = m. For the induction step, we shall show that

$t(n)$ = 3n+2 when n = m+1. From the recurrence for t(n), we obtain t(m+1) = 3+t(m). But from the induction hypothesis t(m) = 3m+2. So, t(m+1) = 3+3m+2 = 3(m+1)+2. □

Example 53.5 Consider recurrence (53.10). We shall show that R(n)<4cn, n≥1.

Induction Base: For n=1, and 2, (53.10) yields: R(1)≤c<4cn and R(2)≤2.5c<4cn.

Induction Hypothesis: Let m be an arbitrary natural number, m≥3. Assume that R(n)<4cn for all n, 1≤n<m.

Induction Step: For n = m and m even, (53.10) gives:

$$\mathrm{R(m)} \le \mathrm{cm} + \frac{2}{m}\sum_{m/2}^{m-1} R(i)$$

$$< \mathrm{cm} + \frac{8c}{m}\sum_{m/2}^{m-1} i \quad \text{(from the IH)}$$

$$< 4\mathrm{cm}$$

When m is odd, (53.10) yields:

$$\mathrm{R(m)} \le \mathrm{cm} + \frac{2}{m}\sum_{(m+1)/2}^{m-1} \mathrm{R(i)}$$

$$< \mathrm{cm} + \frac{8c}{m}\sum_{(m+1)/2}^{m-1} i$$

$$< 4\mathrm{cm}$$

Since R(n)<4cn, R(n) = O(n). Hence, the average computing time of procedure `select` is O(n). Since procedure `select` spends at least n units of time on each input of size n, t^{a}_{select}(n) = Ω(n). Combining these two, we get t^{a}_{select}(n) = Θ(n). □

Example 53.6 Consider recurrence (53.9). Let $t(n)$ denote $t^{W}_{select}(n)$. We shall show that $t(n) = c_2 n(n+1)/2 + c_1 - c_2$ = $\Theta(n^2)$.

Induction Base: When n = 1, (53.9) yields $t(n) = c_1$. Also, $c_2 n(n+1)/2 + c_1 - c_2$ = 0.

Induction Hypothesis: Let m be an arbitrary natural number. Assume that $t(n)$

$= c_2 n(n+1)/2 + c_1 - c_2$ when $n = m$.

Induction Step: When $n = m+1$, (53.9) yields:

$$\begin{aligned} t(m+1) &= c_2(m+1) + t(m) \\ &= c_2(m+1) + c_2 m(m+1)/2 + c_1 - c_2 \text{ (from the IH)} \\ &= [2c_2(m+1) + c_2 m(m+1)]/2 + c_1 - c_2 \\ &= c_2(m+1)(m+2)/2 + c_1 - c_2. \ \square \end{aligned}$$

As mentioned earlier, the induction method cannot be used to find the solution to a recurrence equation; it can be used only to verify that a candidate solution is correct.

53.4 CHARACTERISTIC ROOTS

The recurrence equation of $f(n)$ is a **linear recurrence** iff it is of the form:

$$f(n) = \sum_{i=1}^{k} g_i(n) f(n-i) + g(n)$$

where the $g_i(n)$, $1 \le i \le k$ and $g(n)$ are functions of n but not of f. A linear recurrence is of **order** k iff it is of the form given above, k is a constant, and $g_k(n)$ is not identically equal to zero. If $g_k(n)$ is zero for all n, then the order of the recurrence is less than k. A linear recurrence of order k is of **constant coefficients** iff there exists constants $a_1, a_2, \cdots, a_k$ such that $g_i(n) = a_i$, $1 \le i \le k$. In this section, we are concerned only with the solution of linear recurrences of order k that have constant coefficients. These recurrences are of the form:

$$f(n) = a_1 f(n-1) + a_2 f(n-2) + \cdots + a_k f(n-k) + g(n),\ n \ge k \tag{53.17}$$

where $a_k \ne 0$ and $g(n)$ is a function of n but not of f. (53.17) is a **homogeneous recurrence** iff $g(n) = 0$. One may readily verify that for any set $f(0), f(1), \cdots, f(k-1)$ of initial values, the recurrence (53.17) uniquely determines $f(k)$, $f(k+1), \cdots$.

Many of the recurrence equations we have considered in this book are linear recurrences with constant coefficients. Using $t(n)$ to denote t'_M, (53.3) takes the form:

$$t(n) = \begin{cases} c_1 & n = 1 \\ 2t(n/2) + c_4 n & n >= 2 \text{ and a power of } 2 \end{cases} \tag{53.18}$$

This isn't a linear recurrence of order k for any fixed k because of the occurrence of $t(n/2)$ on the right side. However, since n is a power of 2, (53.18) may be rewritten as:

$$t(2^k) = \begin{cases} c_1 & k=0 \\ 2t(2^{k-1})+c_4 2^k & k \geq 1 \end{cases} \tag{53.19}$$

Using $h(k)$ to denote $t(2^k)$, (53.19) becomes:

$$h(k) = \begin{cases} c_1 & k=0 \\ 2h(k-1)+c_4 2^k & k \geq 1 \end{cases} \tag{53.20}$$

Recurrence (53.20) is readily identified as a linear recurrence with constatnt coefficients. It is of order 1 and it is not homogeneous. Since $h(k) = t(2^k) = t(n)$ for n a power of 2, solving (53.20) is equivalent to solving (53.3).

Recurrence (53.5) is a linear recurrence. However, it is not of order k for any fixed k. By performing some algebra, we can transform this recurrence into an order 1 linear recurrence. We use $t(n)$ as an abbreviation for $t_Q^a(n)$. With this, (53.5) becomes (for $n > 1$):

$$t(n) = c_2 n + \frac{2}{n} \sum_{i=1}^{n-1} t(i) \tag{53.21}$$

Multiplying (53.21) by n, we obtain:

$$nt(n) = c_2 n^2 + 2 \sum_{i=1}^{n-1} t(i) \tag{53.22}$$

Substituting $n-1$ for n in (53.22), we get:

$$(n-1)t(n-1) = c_2 (n-1)^2 + 2 \sum_{i=1}^{n-2} t(i) \tag{53.23}$$

Subtracting (53.23) from (53.22) yields:

$$nt(n) - (n-1)t(n-1) = (2n-1)c_2 + 2t(n-1)$$

or

$$nt(n) = (2n-1)c_2 + (n+1)t(n-1)$$

or

$$t(n) = \frac{n+1}{n} t(n-1) + (2 - \frac{1}{n}) c_2 \tag{53.24}$$

Even though (53.24) is not a linear recurrence with constant coefficients, it can be solved fairly easily. Recurrence (53.8) can be transformed into an equivalent constant coefficient linear recurrence of order 1 in much the same way as we transformed (53.3) into such a recurrence. (53.9) is already in the form of (53.17). The recurrence:

$$F(n) = F(n-1) + F(n-2), \ \ n \geq 2$$

defines the Fibonacci numbers when the initial values $F(0) = 0$ and $F(1) = 1$ are used. This is an order 2 homogeneous constant coefficient linear recurrence.

Linear recurrences of the form (53.17) occur frequently in the analysis of computer algorithms, particularly in the analysis of divide-and-conquer algorithms. These recurrences can be solved by first obtaining a general solution for $f(n)$. This general solution caontains some unspecified constants and has the property that for any given set $f(0), f(1), \cdots, f(k-1)$ of initial values, we can assign values to the unspecified constants such that the general solution defines the unique sequence $f(0), f(1), \cdots$.

Consider the recurrence $f(n) = 5f(n-1) - 6f(n-2)$, $n \geq 2$. Its general solution is $f(n) = c_1 2^n + c_2 3^n$ (we shall soon see how to obtain this). The unspecified constants are c_1 and c_2. If we are given that $f(0) = 0$ and $f(1) = 1$, then we substitute into $f(n) = c_1 2^n + c_2 3^n$ to determine c_1 and c_2. Doing this, we get:

$$f(0) = c_1 + c_2 = 0 \text{ and } f(1) = 2c_1 + 3c_2 = 1$$

Solving for c_1 and c_2, we get $c_1 = -c_2 = -1$. Therefore, $f(n) = 3^n - 2^n$, $n \geq 0$, is the solution to the recurrence $f(n) = 5f(n-1) - 6f(n-2)$, $n \geq 2$ when $f(0) = 0$ and $f(1) = 1$. If we change the initial values to $f(0) = 0$ and $f(1) = 10$, then we get:

$$f(0) = c_1 + c_2 = 0 \text{ and } f(1) = 2c_1 + 3c_2 = 10$$

Solving for c_1 and c_2, we get $c_1 = -c_2 = -10$. Therefore, $f(n) = 10(3^n - 2^n)$, $n \geq 0$.

The general solution to any recurrence of the form (53.17) can be represented as the sum of two functions $f_h(n)$ and $f_p(n)$; $f_h(n)$ is the general solution to the homogeneous part of (53.17):

$$f_h(n) = a_1 f_h(n-1) + a_2 f_h(n-2) + \cdots + a_k f_h(n-k)$$

and $f_p(n)$ is a particular solution for:

$$f_p(n) = a_1 f_p(n-1) + a_2 f_p(n-2) + \cdots + a_k f_p(n-k) + g(n)$$

While at first glance it might seem sufficient to determine $f_p(n)$, it should be noted that $f_p(n) + f_h(n)$ is also a solution to (53.17). Since the methods used to determine $f_p(n)$ will give us an $f_p(n)$ form that does not explicitly contain all zeroes of $f(n)$ (i.e., all solutions to $f(n) - \sum_{i=1}^{k} a_i f(n-i) = 0$), it is necessary to determine $f_h(n)$ and add it to $f_p(n)$ to get the general solution to $f(n)$.

53.4.1 Solving For $f_h(n)$

To determine $f_h(n)$ we need to solve a recurrence of the form:

$$f_h(n) = a_1 f_h(n-1) + a_2 f_h(n-2) + \cdots + a_k f_h(n-k)$$

or

$$f_h(n) - a_1 f_h(n-1) - a_2 f_h(n-2) - \cdots - a_k f_h(n-k) = 0 \tag{53.25}$$

We might suspect that (53.25) has a solution of the form $f_h(n) = Ax^n$. Substituting this into (53.25), we obtain:

$$A(x^n - a_1 x^{n-1} - a_2 x^{n-2} - \cdots - a_k x^{n-k}) = 0$$

We may assume that $A \neq 0$. So we obtain:

$$x^{n-k}(x^k - \sum_{i=1}^{k} a_i x^{k-i}) = 0$$

The above equation has n roots. Because of the term x^{n-k}, $n-k$ of these roots are 0. The remaining k roots are roots of the equation:

$$x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_k = 0 \tag{53.26}$$

(53.26) is called the **characteristic equation** of (53.25). From elementary polynomial root theory, we know that (53.26) has exactly k roots $r_1, r_2, \cdots, r_k$.

The roots of the characteristic equation

$$x^2 - 5x + 6 = 0 \tag{53.27}$$

are $r_1 = 2$ and $r_2 = 3$. The characteristic equation

$$x^3 - 8x^2 + 21x - 18 = 0 \tag{53.27}$$

has the roots $r_1 = 2$, $r_2 = 3$, and $r_3 = 3$. As is evident, the roots of a characteristic equation need not be distinct. A root r_i is of **multiplicity** j iff r_i occurs j times in the collection of k roots. Since the roots of (53.27) are distinct, all have multiplicity 1. For (53.28), 3 is a root of multiplicity 2, and the multiplicity of 2 is 1. The distinct roots of (53.28) are 2 and 3. Theorem 53.1 tells us how to determine the general solution to a linear homogeneous recurrence of the form (53.25) from the roots of its characteristic equation.

Theorem 53.1 Let the distinct roots of the characteristic equation:

$$x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_k = 0$$

of the linear homogeneous recurrence

$$f_h(n) = a_1 f_h(n-1) + a_2 f_h(n-2) + \cdots + a_k f_h(n-k)$$

be $t_1, t_2, \cdots, t_s$, where $s \le k$. There is a general solution $f_h(n)$ which is of the form:

$$f_h(n) = u_1(n) + u_2(n) + \cdots + u_s(n)$$

where

$$u_i(n) = (c_{i_0} + c_{i_1} n + c_{i_2} n^2 + \cdots + c_{i_{w-1}} n^{w-1}) t_i^n$$

Here, w is the multiplicity of the root t_i.

Proof See the references for a proof of this theorem. □

The characteristic equation for the recurrence

$$f(n) = 5f(n-1) - 6f(n-2),\ n \ge 2$$

is

$$x^2 - 5x + 6 = 0$$

The roots of this characteristic equation are 2 and 3. The distinct roots are $t_1 = 2$ and $t_2 = 3$. From Theorem 53.1 it follows that $f(n) = u_1(n) + u_2(n)$, where $u_1(n) = c_1 2^n$ and $u_2(n) = c_2 3^n$. Therefore, $f(n) = c_1 2^n + c_2 3^n$.

(53.28) is the characteristic equation for the homogeneous recurrence:

$$f(n) = 8f(n-1) - 21f(n-2) + 18f(n-3)$$

Its distinct roots are $t_1 = 2$ and $t_2 = 3$. t_2 is a root of multiplicity 2. So, $u_1(n) = c_1 2^n$, and $u_2(n) = (c_2 + c_3 n)3^n$. The general solution to the recurrence is $f(n) = c_1 2^n + (c_2 + c_3 n)3^n$.

The recurrence for the Fibonacci numbers is homogeneous and has the characteristic equation $x^2 - x - 1 = 0$. Its roots are $r_1 = (1 + \sqrt{5})/2$ and $r_2 = (1 - \sqrt{5})/2$. Since the roots are distinct, $u_1(n) = c_1((1 + \sqrt{5})/2)^n$ and $u_2(n) = c_2((1 - \sqrt{5})/2)^n$. Therefore

$$F(n) = c_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

is a general solution to the Fibonacci recurrence. Using the initial values F(0) = 0 and F(1) = 1, we get $c_1 + c_2 = 0$ and $c_1(1+5)/2 + c_2(1-5)/2 = 1$. Solving for c_1 and c_2, we get $c_1 = -c_2 = 1/5$. So the Fibonacci numbers satisfy the equality:

$$F(n) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n .$$

Theorem 53.1 gives us a straightforward way to determine a general solution for an order k linear homogeneous recurrence with constant coefficients. We need only determine the roots of its characteristic equation.

53.4.2 Solving For $f_{p(n)}$

There is no known general method to obtain the particular solution f_p(n). The form of f_p(n) depends very much on the form of g(n). We shall consider only two cases. One where g(n) is a polynomial in n and the other where g(n) is an exponential function of n.

When g(n) = 0, the particular solution is f_p(n) = 0.

When g(n) = $\sum_{i=0}^{d} e_i n^i$, and $e_d \neq 0$, the particular solution is of the form:

$$f_p(n) = p_0 + p_1 n + p_2 n^2 + \cdots + p_{d+m} n^{d+m} \tag{53.29}$$

where m = 0 if 1 is not a root of the characteristic equation corresponding to the homogeneous part of (53.17). If 1 is a root of this equation, then m equals the multiplicity of the root 1.

To determine $p_0, p_1, ..., p_{d+m}$, we merely substitute the right hand side of (53.29) into the recurrence for $f_p(\,)$; compare terms with like powers of n on the left and right hand side of the resulting equation and solve for $p_0, p_1, p_2, ..., p_{d+m}$.

As an example, consider the recurrence:

$$f(n) = 3f(n-1) + 6f(n-2) + 3n + 2 \tag{53.30}$$

g(n) = 3n+2. The characteristic equation is x^2-3x-6 = 0. 1 is not one of its roots. So, the particular solution is of the form:

$$f_p(n) = p_0 + p_1 n$$

Substituting into (53.30), we obtain:

$$p_0 + p_1 n = 3(p_0 + p_1(n-1)) + 6(p_0 + p_1 * (n-2)) + 3n + 2$$

$$= 3p_0 + 3p_1 n - 3p_1 + 6p_0 + 6p_1 n - 12p_1 + 3n + 2$$

$$= (9p_0 - 15p_1 + 2) + (9p_1 + 3)n$$

Comparing terms on the left and right hand sides, we see that:

$$p_0 = 9p_0 - 15p_1 + 2$$

and

$$p_1 = 9p_1 + 3.$$

So, p_1 = -3/8 and p_0 = -61/64. The particular solution for (53.30) is therefore:

$$f_p(n) = -\frac{61}{64} - \frac{3}{8}n$$

Consider the recurrence:

$$f(n) = 2f(n-1) - f(n-2) - 6 \tag{53.31}$$

The corresponding characteristic equation is $x^2 - 2x + 1 = 0$. Its roots are $r_1 = r_2 = 1$. So, $f_p(n)$ is of the form:

$$f_p(n) = p_0 + p_1 n + p_2 n^2$$

Substituting into (53.31), we obtain:

$p_0+p_1\mathrm{n}+p_2n^2 \quad = \quad 2(p_0+p_1\mathrm{n}-p_1+p_2(n^2\text{-2n+1}))-p_0-p_1\mathrm{n}+2p_1-p_2(n^2\text{-4n+4})\text{-6}$

$= (2p_0-2p_1+2p_2-p_0+2p_1-4p_2-6)+(2p_1-4p_2-p_1+4p_2)\mathrm{n}+p_2n^2$

$= (p_0-2p_2-6)+p_1\mathrm{n}+p_2n^2$

Comparing terms, we get:

$$p_0 = p_0-2p_2-6$$

or

$$p_2 = -3$$

So, f_p(n) = p_0+p_1n-3n^2. f_h(n) = $(c_0+c_1\mathrm{n})(1)^n$. So, $f(n) = c_0+c_1\mathrm{n}+p_0+p_1\mathrm{n}-3n^2$ = $c_2+c_3\mathrm{n}-3n^2$. c_2 and c_3 can be determined once the intial values f(0) and f(1) have been specifed.

When g(n) is of the form ca^n where c and a are constants, then the particular solution f_p(n) is of the form:

$$f_p(n) = (p_0 + p_1 n + p_2 n^2 + \cdots + p_w n^w) a^n$$

where w is 0 if a is not a root of the characteristic equation corresponding to the homogeneous part of (53.17) and equals the multiplicity of a otherwise.

Consider the recurrence:

$$f(n) = 3f(n-1) + 2f(n-4) - 6 * 2^n \tag{53.32}$$

The corresponding homogeneous recurrence is:

f_h(n) = 3f_h(n-1)+2f_h(n-4)

Its characteristic equation is:

x^4-3x^3-2 = 0

We may verify that 2 is not a root of this equation. So, the particular solution to (53.32) is of the form:

f_p(n) = $p_0 2^n$.

Substituting this into (53.32), we obtain:

$p_0 2^n = 3p_0 2^{n-1}+2p_0 2^{n-4}-6*2^n$

Dividing out by 2^{n-4}, we obtain:

$$16p_0 = 24p_0+2p_0-96 = 26p_0-96$$

So, p_0 = 96/10 = 9.6. The particular solution to (53.32) is $f_p(n) = 9.6*2^n$

The characteristic equation corresponding to the homogeneous part of the recurrence:

$$f(n) = 5f(n-1) - 6f(n-2) + 4 * 3^n \tag{53.33}$$

is $x^2 - 5x + 6 = 0$. Its roots are $r_1 = 2$ and $r_2 = 3$. Since 3 is a root of multiplicity 1 of the characteristic equation, the particular solution is of the form:

$$f_p(\mathrm{n}) = (p_0+p_1\mathrm{n})3^n.$$

Substituting into (53.33), we obtain:

$$p_0 3^n+p_1 n\, 3^n$$
$$= 5(p_0+p_1(n-1))3^{n-1}-6(p_0+p_1(\mathrm{n}-2))3^{n-2}+4*3^n$$

Dividing by 3^{n-2}, we get:

$$9p_0+9p_1\mathrm{n} = 15p_0+15p_1\mathrm{n}-15p_1-6p_0-6p_1\mathrm{n}+12p_1+36$$

$$= (9p_0-3p_1+36)+9p_1\mathrm{n}$$

Comparing terms, we obtain:

$$9p_1 = 9p_1$$

and

$$9p_0 = 9p_0-3p_1+36$$

These equations enable us to determine that p_1 = 12. The particular solution to (53.33) is:

$$f_p(n) = (p_0+12\mathrm{n})3^n$$

The homogeneous solution is:

$$f_h(\mathrm{n}) = c_1 2^n+c_2 3^n$$

The general solution for $f(n)$ is therefore:

$$f(n) = f_h(n)+f_p(n)$$

$$= c_1 2^n+(c_2+p_0)3^n+12n3^n$$

$$= c_1 2^n+c_3 3^n+12n3^n$$

Given two initial values, f(0) and f(1), we can determine c_1 and c_3.

53.4.3 Obtaining The Complete Solution

We know that $f_h(n)+f_p(n)$ is a general solution to the recurrence:

$$f(n) = a_1 f(n-1)+a_2 f(n-2)+ \cdots +a_k f(n-k)+g(n),\ n \ge k \tag{53.34}$$

By using the initial values f(0), f(1), ..., f(k-1), we can solve for the k undetermined coefficients in $f_h(n) + f_p(n)$ to obtain the unique solution of (53.34) for which f(0),...,f(k-1) have the given values.

53.4.4 Summary

The characteristic roots method to solve the linear recurrence (53.34) consists of the following steps:

1. Write down the characteristic equation:

$$x^k - \sum_{i=1}^{k} a_i x^{k-i} = 0$$

2. Determine the distinct roots $t_1, t_2, ..., t_s$ of the characteristic equation. Determine the multiplicity m_i of the root t_i, 1≤i≤s.
3. Write down the form of $f_h(n)$. I.e.,

$$f_h(n) = u_1(n) + u_2(n) + \cdots + u_s(n)$$

where

$$u_i(n) = (c_{i_0} + c_{i_1} n + c_{i_2} n^2 + \cdots + c_{i_{w-1}} n^{w-1}) t_i^n$$

and w = m_i = multiplicity of the root t_i.

4. Obtain the form of the particular solution f_p(n).
   (a) If g(n) = 0 then f_p(n) = 0.
   (b) If g(n) = $\sum_{i=0}^{d} e_i n^i$ and $e_d \neq 0$, then f_p(n) has the form:

$$f_p(n) = p_0 + p_1 n + p_2 n^2 + \cdots + p_{d+m} n^{d+m}$$

   where m = 0 if 1 is not a root of the characteristic equation. m is the multiplicity of 1 as a root of the characteristic equation otherwise.
   (c) If g(n) = c*a^n then

$$f_p(n) = (p_0 + p_1 n + p_2 n^2 + \cdots + p_w n^w) a^n$$

   where w is zero if a is not a root of the characteristic equation. If a is a root of the characteristic equation, then w is the multiplicity of a.
5. If g(n) ≠ 0, then use the f_p(n) obtained in (4) above to eliminate all occurrences of f(n-i), 0≤i≤k from (53.34). This is done by substituting the value of f_p(n-i) for f(n-i), 0≤i≤k in (53.34). Following this substitution, a system of equations equating the coefficients of like powers of n is obtained. This system is solved to obtain the values of as many of the p_is as possible.
6. Write down the form of the answer. I.e., $f(n) = f_h(n)+f_p(n)$. Solve for the remaining unknowns using the initial values f(0), f(1), ..., f(k-1).

Theorem 53.2 The six step procedure outlined above always finds the unique solution to (53.34) with the given initial values.

Proof See the text by Brualdi that is cited in the reference section. □

53.4.5 EXAMPLES

Example 53.7 The characteristic equation for the homogeneous recurrence:

$$t(n) = 6t(n-1) - 4t(n-2), n \geq 2$$

is

$$x^2 - 6x + 4 = 0$$

Its roots are $r_1 = 3+\sqrt{5}$ and $r_2 = 3-\sqrt{5}$. The roots are distinct and so $t(n) = c_1(3+\sqrt{5})^n + c_2(3-\sqrt{5})^n$. Suppose we are given that t(0) = 0 and t(1) = $4\sqrt{5}$.

Substituting n = 0 and 1 into t(n), we get:

$$0 = c_1 + c_2$$

and

$$45 = c_1(3+5) + c_2(3-5)$$

The first equality yields $c_1 = -c_2$. The second then gives us 45 = c_1(3+5-3+5) = 25c_1 or c_1 = 2. The expression for t(n) is therefore: $t(n) = 2(3+5)^n - 2(3-5)^n$, n≥0. □

Example 53.8 In this example we shall obtain a closed form formula for the sum s(n) = $\sum_{i=0}^{n} i$. The recurrence for s(n) is easily seen to be:

$$s(n) = s(n-1) + n, n \geq 1$$

Its characteristic equation is x-1 = 0. So, $s_h(n) = c_1(1)^n = c_1$. Since g(n) = n and 1 is a root of multiplicity 1, the particular solution is of the form:

$$s_p(n) = p_0 + p_1 n + p_2 n^2$$

Substituting into the recurrence for s(n), we obtain:

$$\begin{aligned} p_0+p_1n+p_2n^2 &= p_0+p_1(n-1)+p_2(n-1)^2+n \\ &= p_0+p_1n-p_1+p_2n^2-2p_2n+p_2+n \\ &= (p_0-p_1+p_2) + (p_1-2p_2+1)n + p_2n^2 \end{aligned}$$

Equating the coefficients of like powers of n and solving the resulting equations, we get $p_1 = p_2$, and $2p_2 = 1$ or $p_2 = 1/2$. The particular solution is $s_p(n) = p_0+n/2+n^2/2$ The general solution becomes s(n) = $(c_1+p_0) + n/2 + n^2/2$. Since s(0) = 0, $c_1+p_0 = 0$. Hence, s(n) = n(n+1)/2, n≥0. □

Example 53.9 Consider the recurrence:

$$f(n) = 5f(n-1) - 6f(n-2) + 3n^2, n \geq 2$$

and

$$f(0) = 2.5; f(1) = 4.5.$$

The characteristic equation for the homogeneous part is:

$$x^2 - 5x + 6 = 0$$

Its roots are $r_1 = 2$ and $r_2 = 3$. The general solution to the homogeneous part is therefore:

$$f_h(n) = c_1 2^n + c_2 3^n .$$

Since g(n) = $3n^2$ and 1 is not a root of the characteristic equation, the particular solution has the form:

$$f_p(n) = p_0 + p_1 n + p_2 n^2$$

Substituting into the recurrence for f( ), we obtain:

$$p_0 + p_1 n + p_2 n^2$$

$$= 5(p_0 + p_1(n-1) + p_2(n-1)^2) - 6(p_0 + p_1(n-2) + p_2(n-2)^2) + 3n^2$$

$$= (7p_1 - p_0 - 19p_2) + (14p_2 - p_1)n + (3 - p_2)n^2$$

Comparing terms, we get:

$$p_2 = 3 - p_2$$

$$p_1 = 14p_2 - p_1$$

$$p_0 = 7p_1 - p_0 - 19p_2$$

Hence, $p_2 = 1.5$, $p_1 = 7p_2 = 10.5$, and $p_0 = 22.5$. So, the general solution for $f(n)$ is:

$$f(n) = c_1 2^n + c_2 3^n + 22.5 + 10.5n + 1.5n^2$$

Since f(0) and f(1) are known to be 2.5 and 4.5, respectively, we obtain:

$$2.5 = c_1 + c_2 + 22.5$$

and

$$4.5 = 2c_1 + 3c_2 + 34.5$$

Solving for c_1 and c_2, we get: $c_1 = -30$ and $c_2 = 10$. The solution to our recurrence is therefore:

$$f(n) = 22.5 + 10.5n + 1.5n^2 - 30*2^n + 10*3^n. \ \square$$

Example 53.10 Let us solve the recurrence:

$$f(n) = 10f(n-1)-37f(n-2)+60f(n-3)-36f(n-4)+4, n\geq 4$$

and

$$f(0) = f(1) = f(2) = f(3) = 1 .$$

The characteristic equation is:

$$x^4-10x^3+37x^2-60x+36 = 0$$

or

$$(x-2)^2(x-3)^2 = 0$$

The four roots are $r_1 = r_2 = 2$, and $r_3 = r_4 = 3$. Since each is a root of multiplicity 2, $u_1(n) = (c_1+c_2 n)2^n$ and $u_2(n) = (c_3+c_4 n)3^n$. The solution to the homogeneous part is:

$$f_h(n) = (c_1+c_2 n)2^n + (c_3+c_4 n)3^n$$

Since $g(n) = 4 = 4*n^0$ and 1 is not a root of the characteristic equation, $f_p(n)$ is of the form:

$$f_p(n) = p_0$$

Substituting into the recurrence for f(n), we get:

$$p_0 = 10p_0-37p_0+60p_0-36p_0+4$$

or

$$p_0 = 1$$

The general solution for $f(n)$ is:

$$f(n) = (c_1+c_2 n)2^n + (c_3+c_4 n)3^n + 1$$

Substituting for n = 0, 1, 2, and 3, and using f(0) = f(1) = f(2) = f(3) = 1, we get:

$$0 = c_1 + c_3 \tag{53.35a}$$

$$0 = 2c_1 + 2c_2 + 3c_3 + 3c_4 \tag{53.35b}$$

$$0 = 4c_1 + 8c_2 + 9c_3 + 18c_4 \tag{53.35c}$$

$$0 = 8c_1 + 24c_2 + 27c_3 + 81c_4 \tag{53.35d}$$

Solving for c_1, c_2, c_3, and c_4, we obtain $c_1 = c_2 = c_3 = c_4 = 0$. So, $f(n) = 1$, n≥1.

We may verify that $f(n) = 1$, n≥0 does indeed satisfy the given recurrence. We proceed by induction. For the induction base, we need to show that $f(n) = 1$, 0≤n≤3. This is true by definition of f(). So, let m be an arbitrary natural number such that m≥3. Assume $f(n) = 1$, for n≤m. When n = m+1, f(m+1) = 10f(m) - 37f(m-1) + 60f(m-2) - 36f(m-3) + 4 = 10-37+60-36+4 = 1.

Let us change the initial values to f(0) = f(1) = f(2) = 1, and f(3) = 4. Now, only equation (53.35d) changes. It becomes:

$$3 = 8c_1 + 24c_2 + 27c_3 + 81c_4 \tag{53.35e}$$

Solving (53.35 a to c) and (53.35e) for c_1, c_2, c_3, and c_4, we obtain $c_1 = 6$, $c_2 = 1.5$, $c_3 = -6$, and $c_4 = 1$. So,

$$f(n) = (6+1.5n)2^n + (n-6)3^n \;\; 1, \;\; n \geq 0$$

Once again, one may verify the correctness of this formula using induction on n. □

53.4.6 Solving Other Recurrences

Certain non-linear recurrences as well as linear ones with non constant coefficients may also be solved using the method of this section. In all cases, we need to first perform a suitable transformation on the given recurrence so as to obtain a linear recurrence with constant coefficients. For example, recurrences of the form:

$$f^c(n) = \sum_{i=1}^{k} a_i f^c(n-i) + g(n), \text{ n≥k}$$

may be solved by first substituting $f^c(n)$ = q(n) to obtain the recurrence:

$$q(n) = \sum_{i=1}^{k} a_i q(n-i) + g(n), \text{ n≥k}$$

This can be solved for q(n) as described earlier. From q(n) we may obtain $f(n)$ by noting that $f(n) = (q(n))^{1/c}$.

Recurrences of the form:

$$nf(n) = \sum_{i=1}^{k} (n-i)a_i f(n-i) + g(n), \text{ n≥k}$$

may be solved by substituting q(n) = nf(n) to obtain:

$$q(n) = \sum_{i=1}^{k} a_i q(n-i) + g(n), \text{ n≥k}$$

Since $f(n)$ = q(n)/n, $f(n)$ is determined once q(n) is.

53.5 GENERATING FUNCTIONS

A generating function G(z) is an infinite power series

$$G(z) = \sum_{i=0}^{\infty} c_i z^i \tag{53.36}$$

We shall say that the generating function $G(z)$ corresponds to the function f: N -> R iff c_i = f(i), i≥0.

Example 53.11 G(z) = $\sum_{i\geq 0} 2z^i$ generates the function $f(n) = 2$, n≥0; G(z) = $\sum_{i\geq 0} iz^i$ generates the function $f(n)$ = n, n≥0; G(z) = $\sum_{i\geq 8} 2z^i$ generates the function:

$$f(n) = \begin{cases} 0 & 0\leq n\leq 7 \\ 2 & n\geq 8 \end{cases} \quad \square$$

A generating function may be specified in two forms. One of these is called the *power series* form. This is the form given in equation (53.36). The other form is called the *closed form*. In this form there are no occurrences of the symbol $\sum$.

Example 53.12 The power series form for the generating function for $f(n) = 1$, n≥0 is G(z) = $\sum_{i\geq 0} z^i$. So, zG(z) = $\sum_{i\geq 1} z^i$. Subtracting, we obtain: G(z) - zG(z) = $\sum_{i\geq 0} z^i - \sum_{i\geq 1} z^i = 1$. So, G(z) = $\frac{1}{1-z}$. The closed form for the power series $\sum_{i\geq 0} z^i$ is therefore $\frac{1}{1-z}$.

Note that $\frac{1}{1-z} = \sum_{i \geq 0} z^i$ only for those values of z for which the series $\sum_{i \geq 0} z^i$ converges. The values of z for which this series converges are not relevant to our discussion here. □

Example 53.13 Let n be an integer and i a natural number. The *binomial coefficient* $\begin{bmatrix} n \\ i \end{bmatrix}$ is defined to be:

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{n(n-1)(n-2)...(n-i+1)}{i(i-1)(i-2)...(1)}$$

So, $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{3*2}{2*1} = 3$; $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{4*3}{2*1} = 6$; $\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \frac{(-3)(-4)}{2*1} = 6$.

The *binomial theorem* states that:

$$(1+z)^n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} z^i, \ n \geq 0$$

A more general form of the binomial theorem is:

$$(1+z)^n = \sum_{i=0}^{m} \begin{bmatrix} n \\ i \end{bmatrix} z^i \qquad (53.37)$$

where $m = n$ if $n \geq 0$ and $m = \infty$ otherwise.

(53.37) leads us to some important closed forms. When n = -2, we obtain:

$$\frac{1}{(1+z)^2} = \sum_{i \geq 0} \begin{bmatrix} -2 \\ i \end{bmatrix} z^i$$

But,

$$\begin{bmatrix} -2 \\ i \end{bmatrix} = \frac{(-2)(-3)...(-i-1)}{i(i-1)...1} = (-1)^i(i+1)$$

So,

$$\frac{1}{(1+z)^2} = \sum_{i \geq 0} (-1)^i (i+1) z^i \qquad (53.38)$$

Substituting $-z$ for z in (53.38), we obtain:

$$\frac{1}{(i-z)^2} = \sum_{i\geq 0}(i+1)z^i$$

Hence, $\frac{1}{(1-z)^2}$ is the closed form for $\sum_{i\geq 0}(i+1)z^i$. (53.37) may be used to obtain the power series form for $\frac{1}{(1-z)^n}$, n≥1. □

As we shall soon see, generating functions can be used to solve recurrence relations. First, let us look at the calculus of generating functions.

53.5.1 Generating Function Operations

Addition and Subtraction: If $G_1(z) = \sum_{i\geq 0} c_i z^i$ and $G_2(z) = \sum_{i\geq 0} d_i z^i$ are the generating functions for f_1 and f_2, then the generating function for $f_1 + f_2$ is:

$$G_3(z) = \sum_{i\geq 0}(c_i+d_i)z^i$$

and that for f_1-f_2 is:

$$G_4(z) = \sum_{i\geq 0}(c_i-d_i)z^i\ .$$

These two equalities follow directly from the definition of a generating function.

Multiplication: If $G_1(z) = \sum_{i\geq 0} c_i z^i$ is the generating function for f, then $G_2(z)$ = a$G_1(z) = \sum_{i\geq 0}(ac_i)z^i$ is the generating function for a*f (a is a constant).

Since $z^k G_1(z) = \sum_{i\geq 0} c_i z^{k+i}$, it is the generating function for a function g such that g(j) = 0, 0≤j<k and g(j) = f(j - k), j≥k. So, multiplying a generating function by z^k corresponds to shifting the function it generates by k.

Example 53.14 In Example 53.13, we showed that $\frac{1}{(1-z)^2} = \sum_{i\geq 0}(i+1)z^i$. Multiplying both sides by z, we obtain $\frac{z}{(1-z)^2} = \sum_{i\geq 0}(i+1)z^{i+1} = \sum_{i\geq 0} iz^i$. So, $\frac{z}{(1-z)^2}$ is the closed form for the generating function for f(i) = i, i≥0. □

The product $G_1(z)*G_2(z)$ of the two generating functions $G_1(z) = \sum_{i \ge 0} c_i z^i$ and $G_2(z) = \sum_{i \ge 0} d_i z^i$ is a third generating function $G_3(z) = \sum_{i \ge 0} e_i z^i$. One may verify that e_i is given by:

$$e_i = \sum_{j=0}^{i} c_j d_{i-j}, \ \ i \ge 0 \tag{53.39}$$

Note that * is commutative (i.e., $G_1(z) * G_2(z) = G_2(z) * G_1(z)$).

An examination of (53.39) indicates that the product of generating functions might be useful in computing sums. In particular, if $G_2(z) = \sum_{i \ge 0} z^i = \frac{1}{1-z}$ (Example 53.11) (i.e., $d_i = 1$, $i \ge 0$), then (53.39) becomes

$$e_i = \sum_{j=0}^{i} c_j \tag{53.40}$$

Example 53.15 Let us try to find the closed form for the sum $s(n) = \sum_{i=0}^{n} i$. From Example 53.14, we know that $\frac{z}{(1-z)^2}$ is the closed form for the generating function for $f(i) = i$, $i \ge 0$. Also, from Example 53.12, we know that $\frac{1}{1-z}$ generates $f(i) = 1$, $i \ge 0$. So, $\frac{z}{(1-z)^2} * \frac{1}{1-z} = \frac{z}{(1-z)^3}$ is the closed form for $(\sum_{i \ge 0} iz^i)(\sum_{i \ge 0} z^i)$. Let the power series form of $\frac{z}{(1-z)^3}$ be $\sum_{i \ge 0} e_i z^i$. From (53.40), it follows that $e_n = \sum_{i=0}^{n} i = s(n)$, $n \ge 0$. Let us proceed to determine e_n. Using the binomial theorem (53.37), we obtain:

$$(1-z)^{-3} = \sum_{i \ge 0} \begin{bmatrix} -3 \\ i \end{bmatrix} (-1)^i z^i$$

The coefficient of z^{n-1} in the expansion of $(1-z)^{-3}$ is therefore:

$$\begin{aligned} \begin{bmatrix} -3 \\ n-1 \end{bmatrix} (-1)^{n-1} &= \frac{(-3)(-4)...(-3-n+2)}{(n-1)(n-2)...(1)} (-1)^{n-1} \\ &= \frac{(n+1)n(n-1)...(3)}{(n-1)(n-2)...(1)} \end{aligned}$$

$$= \frac{n(n+1)}{2}$$

So, the coefficient e_n of z^n in the power series form of $\frac{z}{(1-z)^3}$ is n(n+1)/2 = s(n), n≥0. □

Differentiation: Differentiating (53.36) with respect to z gives:

$$\frac{d}{dz}G(z) = \sum_{i \ge 0} i c_i z^{i-1}$$

or

$$z\frac{d}{dz}G(z) = \sum_{i \ge 0} (i c_i) z^i \tag{53.41}$$

Example 53.16 In Example 53.13, the binomial theorem was used to obtain the closed form for $\sum_{i \ge 0}(i+1)z^i$. This closed form can also be obtained using differentiation. From Example 53.12, we know that $\frac{1}{1-z} = \sum_{i \ge 0} z^i$. From (53.41), it follows that:

$$\frac{d}{dz}\frac{1}{1-z} = \sum_{i \ge 0} i z^{i-1}$$

or

$$\frac{1}{(1-z)^2} = \sum_{i \ge 0}(i+1)z^i \quad \square$$

Integration: Integrating (53.36), we get

$$\int_0^z G(u)du = \sum_{j \ge 1} c_{j-1} z^j / j \tag{53.42}$$

Example 53.17 The closed form of the generating function for $f(n)$ = 1/n, n≥1 can be obtained by integrating the generating function for $f(n)$ = 1. From Example 53.12, we obtain:

$$\frac{1}{1-u} = \sum_{i\geq 0} u^i$$

Therefore

$$\int_0^z \frac{1}{1-u}du = \sum_{i\geq 0}\int_0^z u^i du$$

$$= \sum_{i\geq 0}\frac{1}{i+1}z^{i+1}$$

$$= \sum_{i>0}\frac{1}{i}z^i$$

But,

$$\int_0^z \frac{1}{1-u}du = -ln(1-z)$$

So, the generating function for $f(n) = 1/n$, n≥1 and f(0) = 0, is -ln(1-z). □

The five operations: addition, subtraction, multiplication, differentiation, and integration prove useful in obtaining generating functions for f:N → R.

Figure 53.2 lists some of the more important generating functions in both power series and closed forms.

53.5.2 Solving Recurrence Equations

The generating function method of solving recurrences is best illustrated by an example. Consider the recurrence:

$$F(n) = 2F(n-1) + 7, n\geq 1; F(0) = 0$$

The steps to follow in solving any recurrence using the generating function method are:

1. Let G(z) = $\sum_{i\geq 0} a_i z^i$ be the generating function for F(). So, $a_i = F(i)$, i≥0.
2. Replace all occurrences of F() in the given recurrence by the corresponding a_i. Doing this on the example recurrence yields:

$$a_n = 2a_{n-1} + 7, n\geq 1$$

	Closed Form	Power Series
1	$(1-az)^{-1}$	$\sum_{i\geq 0} a^i z^i$
2	$(1-az)^{-2}$	$\sum_{i\geq 0} (i+1) a^i z^i$
3	$(1+az)^n$	$\sum_{i=0}^{m} \binom{n}{i} a^i z^i$ m = n if n≥0 m = ∞ otherwise
4	ln(1+az)	$\sum_{i\geq 1} \frac{(-1)^{i+1}}{i} a^i z^i$
5	-ln(1-az)	$\sum_{i\geq 1} \frac{1}{i} a^i z^i$
6	e^{az}	$\sum_{i\geq 0} \frac{1}{i!} a^i z^i$

Figure 53.2 Some power series.

3. Multiply both sides of the resulting equation by z^n and sum up both sides for all n for which the equation is valid. For the example, we obtain:

$$\sum_{n\geq 1} a_n z^n = 2\sum_{n\geq 1} a_{n-1} z^n + \sum_{n\geq 1} 7z^n$$

4. Replace all infinite sums involving the a_is by equivalent expressions involving only G(z), z, and a finite number of the a_is. For a degree k recurrence only a_0, a_1, ..., a_{k-1} will remain. The example yields:

$$G(z) - a_0 = 2zG(z) + \sum_{n\geq 1} 7z^n$$

5. Substitute the known values of a_0, a_1, ...,a_{k-1} (recall that F(i) = a_i, 0≤i<k. Our example reduces to:

$$G(z) = 2zG(z) + \sum_{n\geq 1} 7z^n$$

6. Solve the resulting equation for G(z). The example equation is easily solved for G(z) by collecting the G(z) terms on the left and then dividing by the coefficient of G(z). We get:

$$G(z) = \sum_{n\geq 1} 7z^n * \frac{1}{1-2z}$$

7. Determine the coefficient of z^n in the power series expansion of the expression obtained for G(z) in step 6. This coefficient is a_n = F(n). For our example, we get:

$$\text{G(z)} = \sum_{n\geq 1} 7z^n * \frac{1}{1-2z}$$

$$= \sum_{n\geq 1} 7z^n * \sum_{i\geq 0} 2^i z^i$$

The coefficient of z^n in the above series product is:

$$\sum_{i=1}^{n} 7*2^{n-i} = 7(2^n-1)$$

So, F(n) = $7(2^n-1)$, n≥0.

The next several examples illustrate the technique further.

Example 53.18 Let us reconsider the recurrence for the Fibonacci numbers:

F(n) = F(n-1) + F(n-2), n≥2

and

F(0) = 0, F(1) = 1.

Let G(z) = $\sum_{i\geq 0} c_i z^i$ be the generating function for F. From the definition of a generating function, it follows that F(j) = c_j, j≥0. So, F(n) = c_n, F(n-1) = c_{n-1}, and F(n-2) = c_{n-2}. From the recurrence relation for F, we see that:

$c_n = c_{n-1} + c_{n-2}$, n≥2

Multiplying both sides by z^n and summing from n=2 to ∞, we get:

$$\sum_{n\geq 2} c_n z^n @=@ \sum_{n\geq 2} c_{n-1} z^n + \sum_{n\geq 2} c_{n-2} z^n \qquad (53.43)$$

Observe that the sum cannot be performed from n = 0 to ∞ as the recurrence F(n) = F(n-1) + F(n-2) is valid only for n≥2. (53.43) may be rewritten as:

$$\begin{aligned} G(z)-c_1 z-c_0 &= z\sum_{n\geq 2} c_{n-1}z^{n-1} + z^2\sum_{n\geq 2} c_{n-2}z^{n-2} \\ &= z\sum_{i\geq 1} c_i z^i + z^2\sum_{i\geq 0} c_i z^i \\ &= zG(z)-c_0 z+z^2 G(z) \end{aligned}$$

Collecting terms and substituting c_0 = F(0) = 0 and c_1 = F(1) = 1, we get:

$$\begin{aligned} G(z) &= \frac{z}{1-z-z^2} \\ &= \frac{z}{(1-az)(1-bz)}, \ a = \frac{1+\sqrt{5}}{2} \text{ and } b = \frac{1-\sqrt{5}}{2} \\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{1-az} - \frac{1}{1-bz}\right) \end{aligned}$$

From Figure 53.2, we see that the power series expansion of $(1\text{-}az)^{-1}$ is $\sum_{i\geq 0}(az)^i$. Using this, we obtain:

$$\begin{aligned} G(z) &= \frac{1}{\sqrt{5}}\left(\sum_{i\geq 0} a^i z^i - \sum_{i\geq 0} b^i z^i\right) \\ &= \sum_{i\geq 0}\frac{1}{\sqrt{5}}\left(a^i - b^i\right) z^i \end{aligned}$$

Hence, $F(n) = c_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$, n≥0. □

Example 53.19 Consider the recurrence:

$$t(n) = \begin{cases} 0 & n=0 \\ at(n-1)+bn & n\geq 1 \end{cases}$$

Let $G(Z) = \sum_{i\geq 0} c_i z^i$ be the generating function for t(n). So, $t(n) = c_n$, n≥0. From the recurrence, it follows that:

$$c_n = ac_{n-1} + bn,\ n\geq 1$$

Multiplying both sides by z^n and summing from n=1 to ∞ yields:

$$\sum_{n\geq 1} c_n z^n = \sum_{n\geq 1} ac_{n-1} z^n + \sum_{n\geq 1} bnz^n$$

or

$$G(z)-c_0 = az\sum_{n\geq 1} c_{n-1} z^{n-1} + \sum_{n\geq 1} bnz^n$$

$$= azG(z)+\sum_{n\geq 1} bnz^n$$

Substituting $c_0 = 0$ and collecting terms, we get:

$$G(z) = (\sum_{n\geq 1} bnz^n)/(1-az)$$

$$= (\sum_{n\geq 1} bnz^n)(\sum_{i\geq 0} a^i z^i)$$

Using the formula for the product of two generating functions, we obtain:

$$c_n = b\sum_{i=1}^{n} ia^{n-i}$$

$$= ba^n \sum_{i=1}^{n} \frac{i}{a^i}$$

Hence, $t(n) = ba^n \sum_{i=1}^{n} \frac{i}{a^i}$, n≥0. □

Example 53.20 In the previous example, we determined that $c_n = ba^n \sum_{i=0}^{n} \frac{i}{a^i}$. A closed form for c_n can be obtained from a closed form for $d_n = \sum_{i=0}^{n} \frac{i}{a^i}$, n≥0. First, let us find the generating function for $f(i) = i/a^i$. We know that $(1-z)^{-1} = \sum_{i\geq 0} z^i$. So, $(1-z/a)^{-1} = \sum_{i\geq 0} (z/a)^i$. Differentiating with respect to z, we obtain:

$$\frac{d}{dz}\frac{1}{1-z/a} = \sum_{i\geq 0}\frac{d}{dz}(z/a)^i = \sum_{i\geq 0}\frac{i}{a^i}z^{i-1}$$

or

$$\frac{1}{a}\frac{1}{(1-z/a)^2} = \sum_{i\geq 0}\frac{i}{a^i}z^{i-1}$$

Multiplying both sides by z, we get:

$$\frac{z}{a(1-z/a)^2} = \sum_{i\geq 0}\frac{i}{a^i}z^i$$

The generating function for $\sum_{i=0}^{n} i/a^i$ can now be obtained by multiplying by 1/(1-z) (see Equation (53.40)). So,

$$\frac{z}{a(1-z/a)^2(1-z)} = \sum_{n\geq 0}\left(\sum_{i=0}^{n}\frac{i}{a^i}\right)z^n$$

$$= \sum_{n\geq 0} d_n z^n$$

We now need to find the form of the coefficient of z^n in the expansion of

$$\frac{z}{a(1-z/a)^2(1-z)}$$

Expanding this, we get:

$$\frac{z}{a(1-z/a)^2(1-z)} = \frac{z}{a}\sum_{i\geq 0}(z/a)^i\sum_{i\geq 0}(z/a)^i\sum_{i\geq 0}z^i$$

So,

$$d_n = \frac{1}{a}\sum_{i=0}^{n-1}\left[\frac{1}{a^i}\sum_{j=0}^{n-1-i}\frac{1}{a^j}\right]$$

$$= \frac{1}{a}\sum_{i=0}^{n-1}\left[\frac{1}{a^i}\frac{(1/a)^{n-i}-1}{1/a-1}\right], \text{ a} \neq 1$$

$$= \frac{1}{a(1/a-1)}\sum_{i=0}^{n-1}\left(\frac{1}{a^n}-\frac{1}{a^i}\right), \text{ a} \neq 1$$

$$= \frac{1}{1-a}\left[\frac{n}{a^n} - \frac{(1/a)^n - 1}{(1/a) - 1}\right], a \neq 1$$

$$= \frac{n}{(1-a)a^n} - \frac{a(1/a^n - 1)}{(1-a)^2}, a \neq 1$$

When a = 1, $d_n = \sum_{i=0}^{n} i$ = n(n+1)/2. Observe that the recurrence for d_n is:

$$d_n = d_{n-1} + \frac{n}{a^n}, n \geq 1$$

Since the general form of the particular solution is not known when g(n) = n/a^n, it would be difficult to obtain the solution for d_n using the characteristic roots method. □

Example 53.21 An alternate approach to obtain the power series form of G(z) = $(\sum_{n\geq 1} bnz^n)/(1-az)$ (see Example 53.19) is:

$$G(z) = (\sum_{n\geq 1} bnz^n)/(1-az)$$

$$= (b\sum_{n\geq 0} nz^n)/(1-az).$$

$$= \frac{bz}{(1-z)^2(1-az)}$$

When a ≠ 1, we obtain:

$$G(z) = \frac{Az+B}{(1-z)^2} + \frac{C}{1-az}$$

Solving for A, B, and C, we obtain:

$$A = \frac{b}{(1-a)^2}$$

$$B = -\frac{ab}{(1-a)^2}$$

$$C = \frac{ab}{(1-a)^2}$$

So,

$$G(z) = \frac{Az}{(1-z)^2} + \frac{B}{(1-z)^2} + \frac{C}{1-az}$$

$$= A\sum_{i\ge0} iz^i + B\sum_{i\ge0}(i+1)z^i + C\sum_{i\ge0} a^i z^i$$

The coefficient of z^n is therefore:

$$t(n) = An + B(n+1) + Ca^n$$

$$= \frac{b}{(1-a)^2}n - \frac{ab}{(1-a)^2}(n+1) + \frac{ab}{(1-a)^2}a^n$$

$$= \frac{bn}{1-a} - \frac{ab(1-a^n)}{(1-a)^2}, \text{ a} \neq 1, \text{ n} \ge 0$$

When a = 1,

$$G(z) = \frac{bz}{(1-z)^3}$$

$$= \frac{b}{2}\sum_{i\ge0} i(i+1)z^i \quad \text{(Example 53.15)}$$

So, $f(n)$ = bn(n+1)/2, n≥0, a=1. □

Example 53.22 Consider the recurrence:

$$f(n) = 5f(n-1) - 6f(n-2) + 2n, \ n\ge2$$

and

$$f(0) = f(1) = 0$$

Let G(z) = $\sum_{i\ge0} c_i z^i$ be the generating function for f. So, $f(n) = c_n$, f(n-1) = c_{n-1} and f(n-2) = c_{n-2}. Therefore:

$$c_n = 5c_{n-1} - 6c_{n-2} + 2n, \ n\ge2$$

or

$$c_n z^n = 5c_{n-1} z^n - 6c_{n-2} z^n + 2nz^n, \text{ n≥2}$$

Summing up for n from 2 to ∞ yields:

$$\sum_{n\geq 2} c_n z^n = 5\text{z} \sum_{n\geq 2} c_{n-1} z^{n-1} - 6z^2 \sum_{n\geq 2} c_{n-2} z^{n-2} + \sum_{n\geq 2} 2nz^n$$

or

$$\text{G(z)-}c_1 z - c_0 = 5z\,(G\,(\text{z}) - c_0) - 6\text{z}^2 G\,(z) + \sum_{n\geq 2} 2nz^n$$

Substituting $c_1 = c_0 = 0$, we get:

$$\text{G(z)(1-5z+6}z^2) = \sum_{n\geq 2} 2nz^n$$

or

$$\text{G(z)} = \frac{\sum_{n\geq 2} 2nz^n}{(1-3z)(1-2z)}$$

$$= \sum_{j\geq 2} 2jz^j [\frac{3}{1-3z} - \frac{2}{1-2z}]$$

$$= \sum_{j\geq 2} 2jz^j [3 \sum_{i\geq 0} 3^i z^i - 2 \sum_{i\geq 0} 2^i z^i]$$

The coefficient c_n of z^n is now seen to be:

$$c_n = \sum_{j=2}^{n} 6j3^{n-j} - \sum_{j=2}^{n} 4j2^{n-j}$$

$$= 6 * 3^n \sum_{j=2}^{n} (j/3^j) - 4 * 2^n \sum_{j=2}^{n} j/2^j$$

From Example 53.20, we know that:

$$\sum_{2}^{n} \frac{j}{3^j} = -\frac{1}{2}\frac{n}{3^n} - \frac{3(3^{-n} - 1)}{4} - \frac{1}{3}$$

and

$$\sum_{j=2}^{n} \frac{j}{2^j} = -\frac{n}{2^n} - 2(2^{-n} - 1) - \frac{1}{2}$$

So,

$$c_n = -3n - 4.5 + 4.5 * 3^n - 2 * 3^n + 4n + 8 - 8 * 2^n + 2 * 2^n$$

$$= n + 3.5 + 2.5 * 3^n - 6 * 2^n$$

So, $f(n) = n + 3.5 + 2.5 * 3^n - 6 * 2^n$, n≥0. □

53.6 REFERENCES AND SELECTED READINGS

Figure 53.1 is due to J. Bentley, D. Haken, and J. Saxe. They presented it in their solution of recurrence (53.15). Their work appears in the report *A general method for solving divide-and-conquer recurrences*, by J. Bentley, D. Haken, and J. Saxe, *SIGACT News*, 12(3), 1980, pp. 36-44.

A proof of Theorem 53.3 may be found in *Introductory combinatorics*, by R. Brualdi, Elsevier North-Holland Inc., New York, 1977.