Contents

CHAPTER 52

INDUCTION

Data Structures, Algorithms, & Applications in Java

This material is essentially Section 2.2 of the book *Concepts in Discrete Mathematics* **by Sartaj Sahni,** *Camelot Publishing*, **1985. It is reproduced here with permission of the publisher.**

52.1 BASIS FOR INDUCTION

Mathematical induction (or simply, induction) is an exceptionally powerful proof method that can be used to establish that a predicate (i.e., a statement that can be either true or false) P is true whenever its parameters are assigned values from given domains. Let us consider a single parameter predicate $P(n)$ and a domain (or universe) D such that D is the set of all values assignable to n. We will assume that the members of D are labeled a, $a+1$, $a+2$, $\cdots$.

Example 52.1 The statement

3 is an even number

is a predicate that has no parameters. It is also a predicate that is false.

The statement

What is your name?

is not a predicate. It makes no sense to ask whether this statement is true or false.

The statement

n is an even number

is a predicate because it is a statement that is either true or false. This predicate has the single parameter n, and the predicate is true whenever n is a multiple of 2 and is false otherwise. Although the statement

n is an even number

is true for all n in the domain $D = \{4, 16, 24\}$, this statement is false for some n in the domain $D = \{3, 4, 9, 16, 24\}$.

The elements of the set

$$A = \{3, 4, 5, 6, \cdots\}$$

may be labeled as described above by assigning the elements 3, 4, 5, $\cdots$, the labels a, $a+1$, $a+2$, $\cdots$, respectively. The labeling could use $a = 0$ or even $a = 3$. The elements of the set

$$B = \{0, 2, 4, 6, \cdots\}$$

may be assigned the labels 0, 1, 2, 3, $\cdots$, respectively. In this labeling element i of B is assigned the label $i/2$.

We may label the elements of the set

$$\{\text{Sunday, Monday, } \cdots, \text{ Saturday}\}$$

1, 2, $\cdots$, 7, respectively.

The elements of the set

$$\{0, \pm 1, \pm 2, \pm 3, \cdots\}$$

may be labeled as follows. Element i is labeled $-2i-1$ if i is negative, and is labeled $2i$ otherwise. The labels for 0, -1, 1, and -2 are 0, 1, 2, and 3, respectively.

The elements of the set of real numbers cannot be labeled in the manner described. ■

Henceforth, we will simply refer to an element of D by its label (i.e., a or $a+1$ or $a+2$ or $\cdots$). Typically, the domain for n is the set of nonnegative integers 0, 1, 2, $\cdots$. So typically, $a = 0$. We will use the notation

$$\forall n \in DP(n)$$

to mean "the predicate $P(n)$ is true for all n in the domain D."

Induction may be used to establish the validity of statements of the form

$$\forall n \in DP(n)$$

Example 52.2 In Chapter 1 we used induction to show that

$$\sum_{i=0}^{n} i = n(n+1)/2, n \geq 0 \tag{52.1}$$

Actually, what we showed was

$$\forall n \in DP(n)$$

where D is the domain $\{0, 1, 2, \cdots\}$ and the predicate $P(n)$ is

$$\sum_{i=0}^{n} i = n(n+1)/2 \tag{52.2}$$

■

The structure of a proof by induction should be familiar to you from Chapter 1. A proof by induction has three components—(1) an induction base in which you prove the predicate $P(n)$ is true for the first (we shall generalize this later) value of n from the domain D, (2) an induction hypothesis in which you assume that $P(n)$ is true for an arbitary (but not last) value $n = m$ of the domain, and (3) an induction step in which you prove $P(m+1)$ is true (this proof will generally use the assumption that $P(m)$ is true). Before proceeding with additional examples of proofs that use this structure, we establish the validity of proofs that use this structure.

An inference rule has the format

$$\{P_1, P_2, \cdots\} \models Q \tag{52.3}$$

where the P_is and Q are predicates. $\{P_1, P_2, \cdots\}$ is the left side of the inference rule and Q is the right side. The inference rule is interpeted as "if P_1, P_2, $\cdots$ are true then you can infer (or conclude) that Q is true." Every valid proof method is based on one or more inference rules.

Some of the following inference rules that induction relies on are given below.

MI1. $\{P(a), \forall m, m+1 \in D(P(m) \Rightarrow P(m+1))\} \models \forall n \in DP(n)$
Inference rule MI1 is to be read as "if P is true for the first value a of the domain D (i.e., $P(a)$ is true) and if for every m and $m+1$ which are in the domain D the truth of $P(m)$ implies the truth of $P(m+1)$, then you can conclude that $P(m)$ is true for all m in the domain D."

This is the inference rule that we used in Chapter 1 to establish Equation 52.1. In the induction hypothesis we showed that the predicate of Equation 52.2 is true for the first value $a = 0$ of the domain $\{0,\ 1,\ 2,\ \cdots\}$. In the induction hypothesis and induction step, we showed

$$\forall m, m+1 \in D(P(m) \Rightarrow P(m+1))$$

by showing that whenever $P(m)$ is true (the induction hypothesis assumes that $P(m)$ is true), $P(m+1)$ is also true.

MI2. $\{P(a) \bigwedge P(a+1), \forall m, m+1, m+2 \in D(P(m) \bigwedge P(m+1) \Rightarrow P(m+2))\} \models \forall n \in DP(n)$
Here $\bigwedge$ denotes the logical and operator. Inference rule MI2 is to be read as "if P is true for the first two values a and $a+1$ of the domain D (i.e., $P(a)$ and $P(a+1)$ are true) and if for every m, $m+1$, $m+2$ which are in the domain D the truth of $P(m)$ and $P(m+1)$ implies the truth of $P(m+2)$, then you can conclude that $P(n)$ is true for all n in the domain D."

MI3. $\{P(a) \bigwedge P(a+1) \bigwedge P(a+2), \forall m, m+1, m+2, m+3 \in D(P(m) \bigwedge P(m+1) \bigwedge P(m+2) \Rightarrow P(m+3))\} \models \forall n \in DP(n)$

Inference rules MI4, MI5, $\cdots$ and so on are defined in an analogous way. The statement of the above rules can be simplified somewhat by realizing that m, $m+1$, $\cdots m+b$ are all in D iff m and $m+b$ are. So for instance, $\forall m, m+1, m+2, m+3 \in D$ is equivalent to $\forall m, m+3 \in D$.

Before attempting to use the above inference rules to prove statements of the form $\forall n \in DP(n)$, we establish the correctness of these inference rules.

Theorem 52.1 *Inference rule MI1 is correct.*

Proof In inference rule MI1, $P_1 = P(a)$, $P_2 = \forall m, m+1 \in D(P(m) \Rightarrow P(m+1))$, and $Q = \forall n \in DP(n)$ (see Equation 52.3). Assume that both P_1 and P_2 are true. Call this assumption, assumption 1 or A1.

To establish the correctness of MI1 we must show that Q follows from A1. The truth of Q will be established by contradiction. As in all proofs by contradiction, we begin with the assumption (A2) that Q is false. Assumption A2 is equivalent to $\exists m \in D\overline{P(m)}$ (read as "there exists an m in D for which $P(m)$ is false). Because of the nature of D, there must exist a least m in D for which $P(m)$ is false. Let this least m be c.

There are two possibilities for c. Either $c = a$ or $c > a$. If $c = a$, then from A1 we obtain $P(a) = P(c)$ is true. But, by choice of c, $P(c)$ is false. So we have a contradiction.

If $c > a$, then since c is the least m for which $P(m)$ is false, $P(c-1)$ is true. From P_2 we obtain

$$P(c-1) \Rightarrow P(c)$$

Therefore, $P(c)$ is true. But, c was chosen such that $P(c)$ is false. Once again, we have a contradiction.

Hence when P_1 and P_2 are true there is no n, $n \in D$, for which $P(n)$ is false. In other words, when P_1 and P_2 are true, Q is also true. Therefore, MI1 is a valid inference rule. ■

As we will see in subsequent examples, MI1 is not adequate to prove the correctness of all predicates of the form $\forall n \in DP(n)$. MI2, MI3, $\cdots$ will enable us to prove additional predicates of this form. Inference rules MI2, MI3, $\cdots$ may be proved correct in essentially the same way as we proved MI1.

EXERCISES

1. Prove inference rule MI2.

2. Prove inference rule MI3.

52.2 PROOF STRUCTURE

A proof by induction can be divided into two distinct parts. If inference rule MIj is being used, then the first part of the proof establishes the truth of

$$P_1 = P(a) \bigwedge P(a+1) \bigwedge \cdots \bigwedge P(a+j-1)$$

This part of the proof is called the **induction base (IB)**. In the second part of the proof we establish the truth of

$$P_2 = \forall m, m+1, \cdots, m+j \in D(P(m) \bigwedge P(m+1) \bigwedge \cdots \bigwedge P(m+j-1) \Rightarrow P(m+j))$$

From parts one and two of the proof and the inference rule MIj, the truth of

$$Q = \forall n \in DP(n)$$

is inferred.

The second part of the proof is a direct proof in which we begin by assuming that

$$P(m) \bigwedge P(m+1) \bigwedge \cdots \bigwedge P(m+j-1)$$

is true for an arbitrary m such that m, $m+1$, $\cdots$, $m+j$ are in D. This is called the **induction hypothesis (IH)**. Next it is shown that the truth of $P(m+j)$ follows from the induction hypothesis. This part of the proof is referred to as the **induction step (IS)**.

52.3 EXAMPLE PROOFS

Equation 52.1 was proved using MI1 in Chapter 1 of the text. Additional examples of proofs by induction are given below.

Example 52.3 Inference rule MI1 may be used to show

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1} \tag{52.4}$$

for all nonnegative integers (i.e., $D = \{0, 1, 2, \cdots\}$) n and $a \neq 1$. The proof is given below.

Induction Base: When $n = 0$ the left and right sides of Equation 52.4 are both equal to 1.

Induction Hypothesis: Let m be any arbitrary nonnegative integer. Assume that Equation 52.4 is correct for $n = m$.

Induction Step: When $n = m + 1$,

$$\begin{aligned}
\text{left side} &= \sum_{i=0}^{m+1} a^i \\
&= a^{m+1} + \sum_{i=0}^{m} a^i \\
&= a^{m+1} + \frac{a^{m+1} - 1}{a - 1} \text{ (from the IH)} \\
&= \frac{a^{m+2} - a^{m+1} + a^{m+1} - 1}{a - 1} \\
&= \frac{a^{m+2} - 1}{a - 1} \\
&= \text{right side}
\end{aligned}$$

■

Example 52.4 [Fibonacci Numbers] The Fibonacci numbers $F_0, F_1, \cdots$ are defined as below.

$$F_0 = 0, F_1 = 1, and F_{n+2} = F_{n+1} + F_n, n \geq 0 \tag{52.5}$$

The first few Fibonacci numbers are therefore $F_0 = 0$, $F_1 = 1$, F_2=1, F_3=2. We wish to show that

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{52.6}$$

for all integer n, $n \geq 0$. This time we will use the inference rule MI2. The proof is given below.

Induction Base: When $n = 0$, the right side of Equation 52.6 evaluates to 0. Also $F_0 = 0$. When $n = 1$, the right side evaluates to 1 which is equal to F_1. So Equation 52.6 is valid when $n = 0$ and $n = 1$.

Induction Hypothesis: Let m be any arbitrary nonnegative integer. Assume that Equation 52.6 is valid when $n = m$ and when $n = m + 1$.

Induction Step: (We will show the validity of Equation 52.6 when $n = m + 2$). We will use the following symbols:

$$a = \frac{1+\sqrt{5}}{2} \text{ and } b = \frac{1-\sqrt{5}}{2}$$

When $n = m + 2$, we obtain:

$$\begin{aligned} F_{m+2} &= F_{m+1} + F_m \text{ (from Equation 52.5)} && (52.7) \\ &= \frac{1}{\sqrt{5}}(a^{m+1} - b^{m+1} + a^m - b^m) \text{ (from the IH)} && (52.8) \\ &= \frac{1}{\sqrt{5}}(a^m(a+1) - b^m(b+1)) \\ &= \frac{1}{\sqrt{5}}(a^{m+2} - b^{m+2}) \ (a^2 = a + 1 \text{ and } b^2 = b + 1) \\ &= \text{ right side} \end{aligned}$$

From the proof of the induction step it becomes apparent why MI2 has to be used. In going from Equation 52.7 to Equation 52.8, we used a value for F_{m+1} as well as a value for F_m. If MI1 was used, then the induction hypothesis would contain an assumption for only one F_n. In this case the transition from Equation 52.7 to Equation 52.8 would be invalid. Of course, MI3, MI4, etc. could also be used. Using an MIj for higher j than necessary is of no advantage as more things have to be proved in the induction base. ■

Example 52.5 Consider the sequence N_0, N_1, N_2, $\cdots$ defined as below.

$$N_0 = 0, N_1 = 1, \text{ and } N_{h+2} = N_{h+1} + N_h + 1 \tag{52.9}$$

for all integers h, $h \geq 0$. N_h gives the minimum number of nodes in an AVL tree whose height is h (see Section 16.1).

We wish to show that

$$N_h = F_{h+2} - 1, h \geq 0 \tag{52.10}$$

where the F_is are the Fibonacci numbers of Example 52.4. The validity of Equation 52.10 may be established using inference rule MI2 as is done below.

Induction Base: When $h = 0$, $N_h = 0$, and $F_{h+2} - 1 = F_1 + F_0 - 1 = 0$. When $h = 1$, $N_1 = 1$, and $F_3 - 1 = 2 - 1 = 1$.

Induction Hypothesis: Let m be any arbitrary nonnegative inetger. Assume that $N_h = F_{h+2} - 1$ when $h = m$ as well as when $h = m + 1$.

Induction Step:

$$\begin{aligned} N_{m+2} &= N_{m+1} + N_m + 1 \text{ (Equation 52.9)} \\ &= F_{m+3} - 1 + F_{m+2} - 1 + 1 \text{ (from the IH)} \\ &= F_{m+3} + F_{m+2} - 1 \\ &= F_{m+4} - 1 \text{ (from Equation 52.5)} \end{aligned}$$

Once again, one should note that the induction step cannot be carried through using inference rule MI1. ■

Example 52.6 Suppose that

$$f(0) = 0, f(1) = 1, \text{ and } f(n) = 5f(n-1) - 6f(n-2) + 4 * 3^n, n > 2 \qquad (52.11)$$

We may show that

$$f(n) = 35 * 2^n - 35 * 3^n + 12n3^n \qquad (52.12)$$

for $n \geq 0$ by induction on n. Since $f(n)$ is defined in terms of $f(n-1)$ and $f(n-2)$, the inference rule MI2 will be used. The induction proof is given below.

Induction Base: When $n = 0$, $35 * 2^n - 35 * 3^n + 12n3^n = 0 = f(0)$. When $n = 1$, $35 * 2^n - 35 * 3^n + 12n3^n = 1 = f(1)$. So Equation 52.12 is correct for $n = 0$ and $n = 1$.

Induction Hypothesis: Let m be an arbitrary nonnegative integer. Assume that Equation 52.12 is correct for $n = m$ and $n = m + 1$.

Induction Step: When $n = m + 2$, we get:

$$\begin{aligned} f(m+2) &= 5f(m+1) - 6f(m) + 4 * 3^{m+2} \text{ (Equation 52.11)} \\ &= 175 * 2^{m+1} - 175 * 3^{m+1} + 60(m+1)3^{m+1} \\ &\quad - 210 * 2^m + 210 * 3^m - 72m3^m \end{aligned}$$

$$
\begin{aligned}
& \quad + 4 * 3^{m+2} \text{ (from the IH)} \\
= & \quad 350 * 2^m - 525 * 3^m + 180m3^m + 180 * 3^m \\
& \quad - 210 * 2^m + 210 * 3^m - 72m3^m + 36 * 3^m \\
= & \quad 140 * 2^m - 315 * 3^m + 108m3^m + 216 * 3^m \\
= & \quad 35 * 2^{m+2} - 35 * 3^{m+2} + 12(m+2)3^{m+2}
\end{aligned}
$$

which equals the right side of Equation 52.12 when $n = m + 2$. ■

EXERCISES

3. Prove the following using inference rule MI1. In each case, n is a nonnegative integer.

   (a) $\sum_{i=0}^{n} i^2 = n(n+1)(2n+1)/6$
   (b) $\sum_{i=0}^{n} i^3 = n^2(n+1)^2/4$
   (c) $\sum_{i=0}^{n} (2i-1)^2 = n(2n-1)(2n+1)/3 + 1$
   (d) $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$
   (e) $\sum_{i=0}^{n} (x + i * a) = (n+1)x + an(n+1)/2$
   (f) $\sum_{i=1}^{n} \frac{1}{i(i+1)} = n/(n+1)$
   (g) $\sum_{i=0}^{n} \frac{1}{2^i} = 2 - \frac{1}{2^n}$
   (h) $\sum_{i=0}^{n} \frac{i}{2^i} = 2 - \frac{n}{2^n} - \frac{2}{2^n}$

4. [Harmonic Numbers] Let $H_n = \sum_{i=1}^{n} \frac{1}{i}$, $n \geq 1$. H_n is the nth **Harmonic** number. Use induction to prove the following.

   (a) $H_{2^m} \geq 1 + m/2, m \geq 0$
   (b) $(m+1)H_m - m = \sum_{i=1}^{m} H_i, m \geq 1$
   (c) $\sum_{i=2}^{m} \frac{1}{i(i-1)} H_i = 2 - \frac{H_{m+1}}{m} - \frac{1}{m+1}, m \geq 2$
   (d) $\sum_{i=0}^{m} \frac{1}{2i+1} = H_{2m+1} - \frac{1}{2} H_m, m \geq 0$
   (e) $\sum_{i=1}^{m} H_i^2 = (m+1)H_m^2 - (2m+1)H_m + 2m, m \geq 1$

5. Prove the following using induction. In each case, n is a nonnegative integer.

   (a) If $t(0) = 2$ and $t(n) = 3 + t(n-1)$, $n > 0$, then $t(n) = 3n + 2$, $n \geq 0$.
   (b) If $t(1) = c_1$ and $t(n) = c_2 n + t(n-1)$, $n > 1$, then $t(n) = c_2 n(n+1)/2 + c_1 - c_2$, $n \geq 1$.
   (c) If $t(0) = 0$, $t(1) = 4\sqrt{5}$, and $t(n) = 6t(n-1) - 4t(n-2)$, $n \geq 2$, then $t(n) = 2(3+\sqrt{5})^n - 2(3-\sqrt{5})^n, n \geq 0$.

(d) If $f(0) = 2.5$, $f(1) = 4.5$, $f(n) = 5f(n-1) - 6f(n-2) + 3n^2$, $n \geq 2$, then $f(n) = 22.5 + 10.5n + 1.5n^2 - 30 * 2^n + 10 * 3^n$, $n \geq 0$.

(e) If $f(0) = f(1) = f(2) = f(3) = 1$ and $f(n) = 10f(n-1) - 37f(n-2) + 60f(n-3) - 36f(n-4) + 4$, $n \geq 4$, then $f(n) = 1$, $n \geq 0$.

(f) If $f(0) = f(1) = f(2) = 1$, $f(3) = 4$, and $f(n) = 10f(n-1) - 37f(n-2) + 60f(n-3) - 36f(n-4) + 4$, $n \geq 4$, then $f(n) = (6+1.5n)2^n + (n-6)3^n + 1$, $n \geq 0$.

(g) If $f(0) = 1$, $f(1) = 2$, $f(2) = 20$, and $f(n) = 6f(n-1) - 12f(n-2) + 8f(n-3)$, $n \geq 3$, then $f(n) = (2n^2 - 2n + 1)2^n$, $n \geq 0$.

(h) If $f(0) = 0$ and $f(n) = 2f(n-1) + 7n$, $n \geq 1$, then $f(n) = 14 * 2^n - 7n - 14$, $n \geq 0$.

(i) If $f(0) = 0$ and $f^2(n) - 2f^2(n-1) = 1$, then $f(n) = \sqrt{2^n - 1}$, $n \geq 0$.

6. Let m_i, $1 \leq i \leq n$ be n positive integers. Let $p = \sum_{i=1}^n m_i$. Show that:

$$\sum_{i=1}^{n} \frac{m_i}{\sum_{j=i}^{n} m_j} \leq \sum_{i=1}^{p} \frac{1}{i}$$

Use induction on n.

7. What is wrong with the following proof (reproduced from Opus 1961):

> *Theorem: All horses are the same color.*
> *Proof:* It is obvious that one horse is the same color. Let us assume the predicate $P(k)$ that k horses are the same color and use this to imply that $k+1$ horses are the same color. Given the set of $k+1$ horses, we remove one horse; then the remaining k horses are the same color by hypothesis. We remove another horse and replace the first; the k horses, by hypothesis, are again the same color. We repeat this until by exhaustion the $k+1$ sets of k horses have each been shown to be the same color. It follows then that since every horse is the same color as every other horse, $P(k)$ entails $P(k+1)$. But since we have shown $P(1)$ to be true, P is true for all succeeding values of k, that is, all horses are the same color. ■

52.4 A GENERALIZATION

The inference rules for mathematical induction are readily generalized to permit different types of proofs by induction. Suppose that the domain D of the parameter n in the predicate $P(n)$ is $\{k, k-1, k-2, \cdots\}$. Then we may obtain the following inference rules which are analogous to the rules MI1, MI2, etc.

MD1. $\{P(k), \forall m, m-1 \in D[P(m) \Rightarrow P(m-1)]\} \models \forall n \in DP(n)$

MD2. $\{P(k) \bigwedge P(k-1), \forall m, m-1, m-2 \in D[P(m) \bigwedge P(m-1) \Rightarrow P(m-2)]\} \models \forall n \in DP(n)$

MD3. $\{P(k) \bigwedge P(k-1) \bigwedge P(k-2), \forall m, m-1, m-2, m-3 \in D[P(m) \bigwedge P(m-1) \bigwedge P(m-2) \Rightarrow P(m-3)]\} \models \forall n \in DP(n)$

and so on.

Example 52.7 Consider the sequence G_{100}, G_{99}, $\cdots$, G_0 defined as below.

$$G_{100} = 9900, G_i = G_{i+1} - 2i, 0 \leq i < 100 \tag{52.13}$$

We wish to show that

$$G_i = i(i-1), 0 \leq i \leq 100 \tag{52.14}$$

We can prove this using MD1 as is done below.

Induction Base: When $i = 100$, $i(i-1) = 9900 = G_{100}$.

Induction Hypothesis: Let m be an arbitrary nonnegative integer such that $0 < m \leq 100$. Assume that $G_m = m(m-1)$.

Induction Step: When $i = m-1$ the left side of Equation 52.14 is

$$\begin{aligned} G_{m-1} &= G_m - 2(m-1) \text{ (from Equation 52.13)} \\ &= m(m-1) - 2(m-1) \text{ (IH)} \\ &= (m-1)(m-2) \end{aligned}$$

which equals the right side of Equation 52.14. ■

EXERCISES

8. Prove that inference rule MD1 is correct.
9. Prove that inference rule MD2 is correct.
10. Prove the following using either MD1 or MD2.

(a) $\sum_{i=-n}^{-1} i = -\frac{n(n+1)}{2}$. Here n is a nonnegative integer.

(b) $[F_{1000} = 1000] \bigwedge [F_i = F_{i+1} - 1, i < 1000] \Rightarrow F_i = i$. Here i is an integer that is less than or equal to 1000.

(c) $[F_{1000} = 1000 \bigwedge F_{999} = 1998 \bigwedge F_i = F_{i+2} - 2, i < 1000] \Rightarrow [F_i = i,$ if $i \leq 1000$ and even and $F_i = 2i$ if $i \leq 1000$ and odd$]$

11. Prove that the following inference rules are correct ($D = \{a, a+1, \cdots\}$ and $E = \{b, b+1, \cdots\}$).

(a) $\{P(a,b) \bigwedge \forall x, x+1 \in D(P(x,b) \Rightarrow P(x+1,b) \bigwedge \forall m \in D \forall y, y+1 \in E(P(x,y) \Rightarrow P(x,y+1))\} \models \forall x \in D \forall y \in EP(x,y)$

(b) $\{P(a,b) \bigwedge \forall y, y+1 \in E(P(a,y) \Rightarrow P(a,y+1)) \bigwedge \forall x \in D, \forall y \in E(P(x,y) \Rightarrow P(x+1,y))\} \models \forall x \in D, \forall y \in EP(x,y)$

52.5 MULTIPLE PARAMETER PREDICATES

The principle of mathematical induction can be extended to predicates with more than one parameter. Suppose that $P(n_1, n_2, \cdots, n_k)$ is a predicate and that D_1, D_2, $\cdots$, D_k are the domains of n_1, n_2, $\cdots$, n_k, respectively. Assume that $D_i = \{a_i, a_i + 1, \cdots\}$. We can prove that $P(n_1, n_2, \cdots, n_k)$ is true for all $n_i \in D_i$, $1 \leq i \leq k$ by using MI1 (say) and regarding P as if it were only a one parameter predicate (say $Q(n_1)$). In this case the proof would take the form given below.

Induction Base: Show that $P(a_1, n_2, n_3, \cdots, n_k)$ is true for all n_i in D_i, $2 \leq i \leq k$.

Induction Hypothesis: Let m be any arbitrary element of D_1 such that $m+1$ is also in D_1. Assume that $P(m, n_2, \cdots, n_k)$ is true for all n_i in D_i, $2 \leq i \leq k$.

Induction Step: Show that $P(m+1, n_2, n_3, \cdots, n_k)$ is true for all $n_i \in D_i$, $2 \leq i \leq k$.

The general form of a proof by induction for a k parameter predicate (when the proof is based on MI1) takes the form given below.

Induction Base(1): Show that $P(a_1, n_2, n_3, \cdots, n_k)$ is true for all n_i in D_i, $2 \leq i \leq k$.

Induction Hypothesis(1): Let m_1 be any arbitrary element in D_1 such that m_1+1 is also in D_1. Assume that $P(m_1, n_2, \cdots, n_k)$ is true for all n_i in D_i, $2 \leq i \leq k$.

Induction Step(1): Show that $P(m_1+1, n_2, \cdots, n_k)$ is true for all n_i in D_i, $2 \leq i \leq k$.

Induction Base(2): Show that $P(m_1+1, a_2, n_3, \cdots, n_k)$ is true for all n_i in D_i, $3 \leq i \leq k$.

Induction Hypothesis(2): Let m_2 be any arbitrary element of D_2 such that m_2+1 is also in D_2. Assume that $P(m_1+1, m_2, \cdots, n_k)$ is true for all $n_i \in D_i$, $3 \leq i \leq k$.

Induction Step(2): Show that $P(m_1+1, m_2+1, n_3, \cdots, n_k)$ is true for all $n_i \in D_i$, $3 \leq i \leq k$.

Induction Base(3): Show that $P(m_1+1, m_2+1, a_3, n_4, \cdots, n_k)$ is true for all n_i in D_i, $4 \leq i \leq k$.

Induction Hypothesis(3): Let m_3 be any arbitrary element of D_3 such that m_3+1 is also in D_3. Assume that $P(m_1+1, m_2+1, m_3, n_4, \cdots, n_k)$ is true for all n_i in D_i, $4 \leq i \leq k$.

Induction Step(3): Show that $P(m_1+1, m_2+1, m_3+1, n_4, \cdots, n_k)$ is true for all n_i in D_i, $4 \leq i \leq k$.

$\vdots$

Induction Base(k): Show that $P(m_1+1, m_2+1, \cdots, m_k-1+1, a_k)$ is true.

Induction Hypothesis(k): Let m_k be any arbitrary element of D_k such that m_k+1 is also in D_k. Assume that $P(m_1+1, m_2+1, \cdots, m_k)$ is true.

Induction Step(k): Show that $P(m_1+1, m_2+1, \cdots, m_k+1)$ is true.

It is not essential for an induction proof of $P(n_1, n_2, \cdots, n_k)$ to contain all k levels of induction as given above. If it is possible to prove the induction step i for some i, $1 \leq i < k$ using some other proof method (say by contradiction), then levels $i+1$, $i+2$, $\cdots$, k will not be present in the proof. If i levels of induction are present in a proof, the proof is an i level induction proof. When $i=2$, the proof is a double induction; when $i=3$ it is a triple induction; and so on.

Example 52.8 Let $S(n,m)$ be defined as below.

$$S(n,m) = \sum_{j=0}^{n} \sum_{i=0}^{m} i * j \tag{52.15}$$

for every pair of nonnegative integers m and n. We wish to show that

$$S(n,m) = n(n+1)m(m+1)/4 \tag{52.16}$$

A simple noninductive proof is given below.

$$\begin{aligned}
S(n,m) &= \sum_{j=0}^{n}\sum_{i=0}^{m} i * j \\
&= \sum_{j=0}^{n} j * \sum_{i=0}^{m} i \\
&= \sum_{j=0}^{n} jm(m+1)/2 \text{ (Equation 52.1)} \\
&= n(n+1)m(m+1)/4 \text{ (Equation 52.1)}
\end{aligned}$$

We will use this simple example to illustrate the mechanics of a two-level proof by induction. This two-level proof, which is given below, is a proof by induction on n and m.

Induction Base (1): When $n = 0$, $S(n,m) = \sum_{j=0}^{0}\sum_{i=0}^{m} i * j = 0$ and $n(n+1)m(m+1)/4 = 0$.

Induction Hypothesis(1): Let p be an arbitrary nonnegative integer. Assume that $S(p,m) = p(p+1)m(m+1)/4$ for every nonegative integer m.

Induction Step(1): We need to show that $S(p+1,m) = (p+1)(p+2)m(m+1)/4$ for every nonegative integer m. We will do this by induction on m.

Induction Base(2): When $m = 0$, $S(p+1,m) = 0$ and $(p+1)(p+2)m(m+1)/4 = 0$.

Induction Hypothesis(2): Let q be any arbitrary nonnegative integer. Assume that $S(p+1,q) = (p+1)(p+2)q(q+1)/4$.

Induction Step(2):

$$\begin{aligned}
S(p+1,q+1) &= \sum_{j=0}^{p+1}\sum_{i=0}^{q+1} i*j \\
&= S(p+1,q) + \sum_{j=0}^{p+1}(q+1)j \\
&= (p+1)(p+2)q(q+1)/4 + (p+1)(p+2)(q+1)/2 \\
&\quad \text{(From IH(2) and Equation 52.1)} \\
&= (p+1)(p+2)(q(q+1)+2(q+1))/4 \\
&= (p+1)(p+2)(q+1)(q+2)/4
\end{aligned}$$

Since the proof of IS(2) does not use IH(1), it is possible to carry out the proof by using induction only on m as below:

Induction Base: When $m = 0$, $S(n,m) = 0$ and $n(n+1)m(m+1)/4 = 0$.

Induction Hypothesis: Let q be an arbitrary nonnegative integer. Assume that $S(n,q) = n(n+1)q(q+1)/4$ for every nonnegative integer n.

Induction Step: When $m = q+1$ the left side of Equation 52.16 is

$$\begin{aligned}
S(n,q+1) &= \sum_{j=0}^{n}\sum_{i=0}^{q+1} i*j \\
&= S(n,q) + \sum_{j=0}^{n}(q+1)*j \\
&= n(n+1)q(q+1)/4 + n(n+1)(q+1)/2 \\
&\quad \text{(From IH and Equation 52.1)} \\
&= n(n+1)(q+1)(q+2)/4
\end{aligned}$$

which equals the right side of Equation 52.16. ∎

Example 52.9 One definition of Ackermann's function, $A(i, j)$, is

$$A(i,j) = \begin{cases} 2j & i = 0 \\ 0 & i \geq 1 \text{ and } j = 0 \\ 2 & i \geq 1 \text{ and } j = 1 \\ A(i-1, A(i, j-1)) & i \geq 1 \text{ and } j \geq 2 \end{cases} \tag{52.17}$$

Note that this definition of Ackermann's function is different from that given in Chapter 1.

We wish to show that $A(i, j+1) > A(i, j)$ for all integers $i \geq 0$ and $j \geq 0$. We shall establish this by induction of i and j.

Induction Base(1): When $i = 0$, $A(i, j) = 2j$. So $A(0, j+1) > A(0, j)$ for all integers $j \geq 0$.

Induction Hypothesis(1): Let m be any arbitrary nonnegative integer. Assume that $A(m, j+1) > A(m, j)$ for all integers $j \geq 0$.

Induction Step(1): We need to show that $A(m+1, j+1) > A(m+1, j)$ for all $j \geq 0$. We will do this by induction on j.

Induction Base(2): When $j = 0$, $A(m+1, j+1) = 2$ and $A(m+1, 0) = 0$. So $A(m+1, j+1) > A(m+1, j)$.

Induction Hypothesis(2): Let r be any arbitrary nonnegative integer. Assume that $A(m+1, r+1) > A(m+1, r)$.

Induction Step(2): We will show that $A(m+1, r+2) > A(m+1, r+1)$ by considering the two cases: (1) $r = 0$ and (2) $r > 0$. When $r = 0$, we must show that $A(m+1, 2) > A(m+1, 1)$. From Equation 52.17 we see that $A(m+1, 2) = A(m, A(m+1, 1)) = A(m, 2) = 4$ (see Exercise 12). But, $A(m+1, 1) = 2$.

Now let's consider the case $r > 0$. From Equation 52.17 we obtain the equality:

$$A(m+1, r+2) = A(m, A(m+1, r+1))$$

From induction hypothesis (1), it follows that $A(m, x) > A(m, y)$ whenever $x > y$. From this and induction hypothesis (2), we conclude that $A(m, A(m+1, r+1)) > A(m, A(m+1, r))$. Since $r > 0$, Equation 52.17 yields the equality:

$$A(m+1, r+1) = A(m, A(m+1, r))$$

Combining these equalities and inequalities, we obtain (for $r > 0$)

$$A(m+1, r+2) > A(m+1, r+1)$$

The proofs of cases (1) and (2) complete the proof of induction step (2). This in turn completes the proof of induction step (1). ■

EXERCISE

12. Let $A(i, j)$ be the Ackermann's function defined in Equation 52.17. Use induction to show the following.

    (a) $A(i, 2) = 4, i \geq 1$

    (b) $A(1, j) = 2^j, j \geq 1$

    (c) $A(2, j) = 2^{2^{2^{-\,-\,-\,2}}} \Big\} \, j \text{ twos}, j \geq 1$

    (d) $A(i, j) \geq j, i \geq 0, j \geq 0$

    (e) $A(i+1, j) \geq A(i, j), i \geq 0, j \geq 0$

52.6 WEAK INDUCTION

Weak mathematical induction differs from the mathematical induction discussed so far (called strong mathematical induction) only in the nature of the induction hypothesis. In the induction hypotheses used so far we assumed that the predicate $P(n)$ is true for some fixed number of ns. For example, when using MI1, we assume that $P(n)$ is true for one arbitrary value $n = m$. We do not assume that $P(n)$ is also true for $n = a, a+1, \cdots, m-1$. Similarly, when MI2 is used, $P(n)$ is assumed true (in the induction hypothesis) for only two values of n (i.e., m and $m+1$).

When weak mathematical induction is used, $P(n)$ is assumed true for all values of n, $a \leq n \leq m$. In the induction step we prove that $P(m+1)$ is true. In the induction base, at least $P(a)$ is shown true. Depending upon the nature of the proof provided in the induction step, it may also be necessary to prove $P(a+1)$, $P(a+2)$, and so on in the induction base.

Theorem 52.2 *Strong mathematical induction is a valid proof method.*

Proof In weak mathematical induction, the truth of $P(n)$ for all n in $D = \{a, a+1, a+2, \cdots\}$ is established by showing $P1$ and $P2$ are true.

$$P1 = P(a) \tag{52.18}$$
$$P2 = \forall m, m+1 \in D(\forall x, a \le x \le mP(x) \Rightarrow P(m+1)) \tag{52.19}$$

We will proceed to show that if $P1$ and $P2$ are true, then there is no $j \in D$ for which $P(j)$ is false. The proof is by contradiction. Suppose there exists a $j \in D$ for which $P(j)$ is false. Since the elements of D are a, $a+1$, $\cdots$, the existence of a j for which $P(j)$ is false, implies the existence of a least j (say r) for which $P(r)$ is false. If $r = a$, then we have a contradiction with the truth of $P1$. So r must be greater than a, and $P(a), \cdots, P(r-1)$ are all true. Hence, $\forall x, a \le x \le r-1P(x)$ is true. Therefore, from $P2$ we can conclude that $P(r)$ is also true. This contradicts the assumption that $P(r)$ is false. Hence, if $P1$ and $P2$ are true, there can be no $j \in D$ for which $P(j)$ is false. ■

The proof of Example 52.5 becomes a proof by weak mathematical induction if we change the induction hypothesis to:

> Assume that $N_h = F_{h+2} - 1$, $0 \le h \le m$ for an arbitrary m.

Note that it is still necessary to prove $N_0 = F_2 - 1$ and $N_1 = F_3 - 1$ in the induction base. If we prove only $N_0 = F_2 - 1$ in the induction base, then we cannot conclude $N_h = F_{h+2} - 1$ for all $h \ge 0$ as the case $h = 1$ does not follow from

$$N_0 = F_2 - 1 \text{ and } \forall m(\forall h, 0 \le h \le m(N_h = F_{h+2} - 1) \Rightarrow N_{m+2} = F_{m+4} - 1$$

To see this, observe that when $m = 0$ we can infer $N_2 = F_4 - 1$ from $N_0 = F_2 - 1$ and $N_1 = F_3 - 1$. But there is no m that allows us to infer $N_1 = F_3 - 1$.

As pointed out in the proof of Theorem 52.2, the inference rule on which weak mathematical induction is based is:

WM1: $\{P(a), \forall m, m+1 \in D[\forall x, a \le x \le mP(x) \Rightarrow P(m+1)]\} \models \forall n \in DP(n)$

Using a proof similar to that used in Theorem 52.2, we may obtain the following additional inference rules for weak mathematical induction on single parameter predicates:

WM2: $\{P(a) \bigwedge P(a+1), \forall m, m+1, m+2 \in D[\forall x, a \le x \le m+1P(x) \Rightarrow P(m+2)]\} \models \forall n \in DP(n)$

WM3: $\{P(a) \bigwedge P(a+1) \bigwedge P(a+2), \forall m, m+1, m+2, m+3 \in D[\forall x, a \le x \le m+2P(x) \Rightarrow P(m+3)]\} \models \forall n \in DP(n)$

and so on.

While every proof by mathematical induction is easily changed to a proof by weak mathematical induction, the reverse is not true.

Example 52.10 Let $t(n)$ be defined as below.

$$t(n) = \begin{cases} 0 & n = 0 \\ t(\lfloor n/3 \rfloor) + t(\lfloor n/5 \rfloor) + 2t(\lfloor n/7 \rfloor) & n > 0 \end{cases} \tag{52.20}$$

We will use WM1 to show that $t(n) \leq 6n$, $n \geq 0$.

Induction Base: When $n = 0$, $t(n) = 0 \leq 6n$.

Induction Hypothesis: Let m be an arbitrary nonnegative integer. Assume that $t(n) \leq 6n$ for $0 \leq n \leq m$.

Induction Step: When $n = m + 1$, Equation 52.20 yields:

$$t(m+1) = t(\lfloor (m+1)/3 \rfloor) + t(\lfloor (m+1)/5 \rfloor) + 2t(\lfloor (m+1)/7 \rfloor) + m + 1$$

Since $\lfloor (m+1)/3 \rfloor$, $\lfloor (m+1)/5 \rfloor$, and $\lfloor (m+1)/7 \rfloor$ are all less than or equal to m, the induction hypothesis may be used to obtain:

$$\begin{aligned} t(m+1) &\leq 6\lfloor (m+1)/3 \rfloor + 6\lfloor (m+1)/5 \rfloor + 12\lfloor (m+1)/7 \rfloor + m + 1 \\ &\leq 2(m+1) + 6(m+1)/5 + 12(m+1)/7 + m + 1 \\ &= (2 + 6/5 + 12/7 + 1)(m+1) \\ &= 207(m+1)/35 \\ &< 6(m+1) \end{aligned}$$

■

Example 52.11 Suppose that:

$$t(n) = \begin{cases} b & 0 \leq n \leq 1 \\ cn + \sum_{j=0}^{n-1} t(j) & n > 1 \end{cases} \tag{52.21}$$

Equation 52.21 describes the average run time for the quick sort method of Section 19.2.3.

We wish to show that $t(n) \leq 2(b+c)n\log_e n$, $n \geq 2$. This can be done using WM1 as below.

Induction Base: When $n = 2$, we see that $t(2) \leq 2c + t(0) + t(1) \leq 2c + 2b \leq 2(b+c) * 2 * \log_e 2$.

Induction Hypothesis: Let m be an arbitrary integer greater than 1. Assume that $t(n) \leq 2(b+c)n\log_e n$ for $2 \leq n \leq m$.

Induction Step: When $n = m+1$, we get:

$$\begin{aligned}
t(m+1) &\leq c(m+1) + \frac{2}{m+1}\sum_{j=0}^{m} t(j) \\
&\leq c(m+1) + \frac{4b}{m+1} + \frac{2}{m+1}\sum_{j=2}^{m} t(j) \\
&\leq c(m+1) + \frac{4b}{m+1} + \frac{4(b+c)}{m+1}\sum_{j=2}^{m} j\log_e j \text{ (from IH)} \\
&\leq c(m+1) + \frac{4b}{m+1} + \frac{4(b+c)}{m+1}\int_{2}^{m+1} x\log_e x\, dx \\
&\leq c(m+1) + \frac{4b}{m+1} + \frac{4(b+c)}{m+1}\left[\frac{(m+1)^2\log_e(m+1)}{2} - \frac{(m+1)^2}{4}\right] \\
&= c(m+1) + \frac{4b}{m+1} + 2(b+c)(m+1)\log_e(m+1) - (b+c)(m+1) \\
&\leq 2(b+c)(m+1)\log_e(m+1)
\end{aligned}$$

■

Like ordinary (or strong) mathematical induction, weak mathematical induction can also be used to prove predicates with several parameters. The extension to multiple parameter predicates is analogous to the corresponding extension of ordinary mathematical induction. Whether strong or weak mathematical induction is used, the proof is usually just referred to as a proof by induction.

EXERCISES

13. Prove the validity of inference rule WM2.

14. Prove the following (n is a nonnegative integer and c is a positive constant):

    (a) If

    $$t(n) = \begin{cases} 0 & n = 0 \\ t(\lfloor n/5 \rfloor) + t(\lfloor 3n/4 \rfloor) + cn & n > 0 \end{cases}$$

    then $t(n) \leq 20cn$, $n \geq 0$.

    (b) If

    $$t(n) = \begin{cases} 0 & n = 0 \\ t(\lfloor n/9 \rfloor) + t(\lfloor 63n/72 \rfloor) + cn & n > 0 \end{cases}$$

    then $t(n) \leq 72cn$, $n \geq 0$.

    (c) If

    $$t(n) = \begin{cases} 0 & n = 0 \\ t(\lfloor n/7 \rfloor) + t(\lfloor 27n/35 \rfloor) + cn & n > 0 \end{cases}$$

    then $t(n) \leq 35cn$, $n \geq 0$.

    (d) If

    $$t(n) = \begin{cases} 0 & n = 0 \\ t(\lfloor 3n/14 \rfloor) + t(\lfloor 2n/3 \rfloor) + cn & n > 0 \end{cases}$$

    then $t(n) \leq 42cn$, $n \geq 0$.

    (e) If

    $$t(n) = \begin{cases} 2 & n = 2 \\ 2t(n/2) + 3 & n > 2 \end{cases}$$

    then $t(n) = 5n/2 - 3$ for $n = 2^k$, $k \geq 1$.

15. Prove or disprove the following inference rule:

    $$\{P(a), \forall x \in D, x > a[\forall y, a \leq y < xP(y) \Rightarrow P(x)]\} \models \forall x \in DP(x)$$

    where D consists of all real numbers greater than or equal to a.

16. Use induction to show that every nonnegative integer can be written as the product of prime numbers.