(a) Q. $I = \int_0^\infty \frac{x^{n-1}}{1+x} dx$

Answer:

Let, $x = \tan^2\theta$

$\Rightarrow dx = 2\tan\theta \sec^2\theta \, d\theta$

x	0	∞
θ	0	$\pi/2$

Now,

$$I = \int_0^{\pi/2} \frac{(\tan^2\theta)^{n-1}}{1+\tan^2\theta} \, 2\tan\theta \sec^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\tan^{2n-2}\theta}{\sec^2\theta} \cdot \tan\theta \cdot \sec^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \tan^{2n-2}\theta \tan\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \tan^{2n-1}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{\sin\theta}{\cos\theta}\right)^{2n-1} d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{1-2n}\theta \, d\theta$$

$$= 2 \cdot \frac{1}{2} \frac{\Gamma\frac{2n-1+1}{2} \, \Gamma\frac{1-2n+1}{2}}{\Gamma\frac{2n-1+1-2n+2}{2}}$$

$$= \frac{\Gamma\frac{2n}{2} \quad \Gamma\frac{2-2n}{2}}{\Gamma\frac{2}{2}}$$

$$= \frac{\Gamma n \, \Gamma 1-n}{\Gamma 1}$$

$$= \Gamma n \, \Gamma 1-n$$ Answer.

Q. $I = \int_0^{\pi/2} (\tan\theta)^{1/2} \, d\theta$

Answer: $\int_0^{\pi/2} (\tan\theta)^{1/2} \, d\theta$

$$= \int_0^{\pi/2} \sin\theta^{1/2} \cos\theta^{-1/2} \, d\theta$$

$$= \frac{\Gamma\frac{1/2+1}{2} \quad \Gamma\frac{-1/2+1}{2}}{2\,\Gamma\frac{1/2 - 1/2 + 2}{2}}$$

$$= \frac{\Gamma 3/4 \quad \Gamma 1/4}{2}$$

$$= \frac{\Gamma 1/4 \, \Gamma 1-1/4}{2}$$

$$= \frac{1}{2} \frac{\pi}{\sin \pi/4}$$

$$= \frac{\sqrt{2}}{2} \pi$$

$$= \frac{1}{\sqrt{2}} \pi$$ Answer.

(2) $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$

$= \int_0^{\pi/2} \sqrt{\frac{\cos\theta}{\sin\theta}}\, d\theta$

$= \int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta\, d\theta$.

$= - - -$ same

Q. $0! = \Gamma(0+1) = \Gamma 1 = 1$

Answer:

We know, $n! = \Gamma n+1$

so, $0! = \Gamma 0+1$

$= \Gamma 1$

$= 1 \quad \because \left[\Gamma 1 = 1\right]$

Answer.

Q. $\Gamma m\, \Gamma(1-m) = \dfrac{\pi}{\sin m\pi}$

Answer: $\cancel{\Gamma m\, \Gamma(1-m)}$ $=$

Taking by Region $0 < Re(m) < 1$ [By definition of Analytic Continuation Property]

Now, Taking

$$\frac{\Gamma m\, \Gamma 1-m}{\Gamma m+(1-m)} = \beta(m, 1-m) \quad \left[\because \frac{\Gamma m \Gamma n}{\Gamma m+n} = \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx\right.$$

$$= \int_0^1 t^{m-1} (1-t)^{-m} dt$$

$$= \int_0^1 \frac{t^{m-1}}{(1-t)^m} dt$$

$$= \int_0^1 \left\{\frac{t}{1-t}\right\}^m \frac{dt}{t}$$

Let $y = \frac{t}{1-t} \Rightarrow t = \frac{y}{1+y}$

$$\Rightarrow dy = \left\{\frac{1}{(1-t)} + \frac{t}{(1-t)^2}\right\} dt$$

$$dy = \frac{1}{(1-t)^2} dt$$

when,

$t = 0,\ y = 0$

$t = 1 \Rightarrow y = \infty$

$$\Gamma m\, \Gamma 1-m = \int_0^\infty \frac{y^m (1-t)^2 \, dy}{t} = \int_0^\infty y^m \left\{1 - \frac{y}{y+1}\right\}^2 \frac{dy}{\frac{y}{(1+y)}}$$

$$\Gamma m\, \Gamma 1-m = \int_0^\infty \frac{y^{m-1}}{1+y} dy \quad \text{------ } (1)$$

Now, evalute integral

$\int_0^\infty \frac{y^{m-1}}{1+y} dy$ by Complex variable method.

$$\int_C \frac{w^{m-1}}{1+w} dw$$

Pole of the given integral

$$1+w=0$$

$$\Rightarrow w=-1 \quad \text{simple pole}$$

Then by Cauchy Residue theorem

$$\int_C \frac{w^{m-1}}{(1+w)}\,dw = 2\pi i \times \text{Residue at } w=-1$$

$$= 2\pi i \times \left\{\lim_{w\to -1} (w+1) f(w)\right\}$$

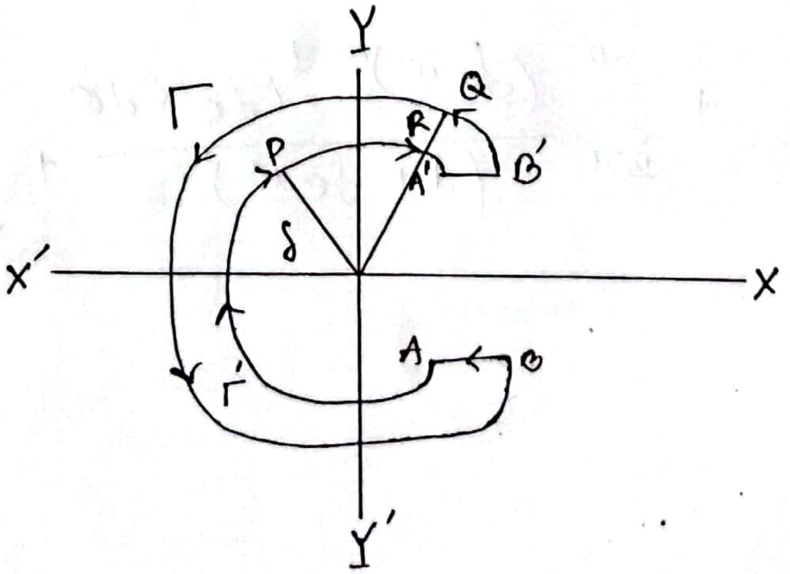
$$= 2\pi i \times \left\{\lim_{w\to -1} (w+1) \frac{w^{m-1}}{(1+w)}\right\}$$

$$= 2\pi i \times (-1)^{m-1}$$

$$\int_C \frac{w^{m-1}}{(1+w)}\,dw = 2\pi i \left\{e^{\pi i(m-1)}\right\} \quad [\because e^{\pi i}=(-1)]$$

$$\text{———————— (2)}$$

We draw the figure then



Let radius of circle Γ and radius of circle of Γ' are R and δ respectively.

we show that,

$$|w|=\delta \Rightarrow w=\delta e^{i\theta}$$

$$dw=\delta i e^{i\theta}\, d\theta$$

$$|w|=R \Rightarrow w=Re^{i\theta}$$

$$dw=Rie^{i\theta}\, d\theta$$

from equation (2)

$$\int_0^{2\pi} \frac{(Re^{i\theta})^{m-1}}{1+Re^{i\theta}} Rie^{i\theta}\, d\theta + \int_R^{\delta} \frac{y^{m-1} e^{2\pi i(m-1)} e^{2\pi i} dy}{(1+y)}$$

$$+ \int_{2\pi}^{0} \frac{(\delta e^{i\theta})^{m-1} i\delta e^{i\theta} d\theta}{(1+\delta e^{i\theta})} + \int_{\delta}^{R} \frac{y^{m-1}}{1+y} dy =$$

$$2\pi i \{ e^{\pi i(m-1)}$$

Let $\delta \to 0$ and $R \to \infty$ and also $0 < Re(m) < 1$

then,

$$0 + \int_{\infty}^{0} \frac{y^{m-1} e^{2\pi i m}}{(1+y)} \cdot dy + 0 + \int_{0}^{\infty} \frac{y^{m-1}}{1+y} dy = -2\pi i \{e^{\pi i m}\}$$

$$\left[\because 2\pi i \{e^{\pi i m}\} \cdot e^{-\pi i}\right]$$

$$-\int_{0}^{\infty} \frac{e^{2\pi i m} \cdot y^{m-1}}{(1+y)} dy + \int_{0}^{\infty} \frac{y^{m-1} dy}{1+y} = -2\pi i \, e^{\pi i m}$$

$$\Rightarrow \int_{0}^{\infty} \frac{y^{m-1}}{(1+y)} dy = \frac{-2\pi i \, e^{\pi i m}}{1 - e^{2\pi i m}}$$

$$= \frac{-2\pi i \, e^{\pi i m}}{-e^{\pi i m}\{e^{\pi i m} - e^{-\pi i m}\}}$$

$$= \frac{\pi \{2i\}}{(e^{\pi i m} - e^{-\pi i m})}$$

$$\int_{0}^{\infty} \frac{y^{m-1}}{(1+y)} dy = \frac{\pi}{\sin(\pi m)} \quad \text{——— (3)}$$

From equ. (1) and (3)

$$\Gamma m \, \Gamma 1-m = \frac{\pi}{\sin m\pi}$$

Answer.

Q. Evaluate $\int_0^{\pi/2} \cos^q \theta \, d\theta$

$$= \int_0^{\pi/2} \sin^0\theta \cos^q\theta \, d\theta \quad [\because \sin^0\theta = 1]$$

$$= \frac{\Gamma\frac{0+1}{2} \, \Gamma\frac{q+1}{2}}{\Gamma\frac{0+q+2}{2}}$$

$$= \frac{\Gamma\frac{1}{2} \, \Gamma\frac{q+1}{2}}{2\Gamma\frac{q+2}{2}}$$

$$= \frac{\Gamma\frac{q+1}{2} \, \sqrt{\pi}}{2\Gamma\frac{q+2}{2}}$$

Answer.

Q. Evaluate $\int_0^{\pi/2} \sin^p\theta \, d\theta$

$$= \int_0^{\pi/2} \sin^p\theta \cos^0\theta \, d\theta$$

$$= \frac{\Gamma\frac{p+1}{2} \Gamma\frac{0+1}{2}}{2\Gamma\frac{p+0+2}{2}}$$

$$= \frac{\Gamma\frac{p+1}{2} \sqrt{\pi}}{2\Gamma\frac{p+2}{2}}$$ Answer.

Q. Evaluate $\int_0^{\pi/2} \sin^6 x \, dx$

Answer:

$$I = \int_0^{\pi/2} \sin^6 x \, dx$$

$$= \frac{\Gamma\frac{6+1}{2}}{\Gamma\frac{6+2}{2}} \frac{\sqrt{\pi}}{2}$$

$$= \frac{\Gamma\frac{7}{2} \cdot \sqrt{\pi}}{\Gamma 4 \cdot 2}$$

$$= \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \cdot \sqrt{\pi}}{3 \cdot 2 \cdot 1 \cdot 2}$$

$$= \frac{5 \cdot 3 \cdot 1 \cdot \pi}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 8}$$

$$= \frac{5\pi}{32}$$ Answer.

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Q. Evaluate $\int_0^{\pi/2} \cos^5 x \sin^4 x \, dx$

Answer:

$$I = \int_0^{\pi/2} \cos^5 x \sin^4 x \, dx$$

$$= \frac{\Gamma\frac{5+1}{2} \, \Gamma\frac{4+1}{2}}{2 \, \Gamma\frac{5+4+2}{2}}$$

$$= \frac{\Gamma 3 \, \Gamma\frac{5}{2}}{2 \, \Gamma\frac{11}{2}}$$

$$= \frac{2 \cdot 1 \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{2 \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}$$

$$= \frac{3 \cdot 32}{9 \cdot 7 \cdot 5 \cdot 3 \cdot 4}$$

$$= \frac{8}{315}$$ Answer.

Q. Evaluate $\int_0^1 x^3 (1-x)^3 dx$

Answer:

$$I = \int_0^1 x^{4-1} (1-x)^{4-1} dx$$

$$= \beta(4,4)$$

$$= \frac{\Gamma 4 \, \Gamma 4}{\Gamma 4+4}$$

$$= \frac{3 \cdot 2 \cdot 1 \, . \, 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \frac{1}{140}$$ Answer.

Q. Evaluate $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

Answer:

$$I = \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta \quad \text{——— (1)}$$

$$I = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta \quad \text{——— (2)}$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} (\sqrt{\tan\theta} + \sqrt{\cot\theta})\, d\theta$$

$$= \sqrt{2} \int_0^{\pi/2} \frac{\sin\theta + \cos\theta}{\sqrt{\sin 2\theta}}\, d\theta$$

$$= \sqrt{2} \int_0^{\pi/2} \frac{\sin\theta + \cos\theta}{\sqrt{1 - (\sin\theta - \cos\theta)^2}}\, d\theta$$

$$= \sqrt{2} \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} \quad (\text{Where } \sin\theta - \cos\theta = t)$$

$$= 2\sqrt{2} \int_0^{1} \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\pi$$

or

$$I = \frac{\pi}{\sqrt{2}}$$

Answer.

Evalute, $\int_0^1 x^6 \sqrt{1-x^2}\, dx$.

Q. Evaluate $\int_0^1 x^6 \sqrt{1-x^2}\, dx$

Soln: We have,

Lt.		
x	0	1
θ	0	$\pi/2$

$$\int_0^1 x^6 \sqrt{1-x^2}\, dx$$

$$= \int_0^{\pi/2} \sin^6\theta \cos^2\theta \, d\theta$$

Let, $\sin\theta = x$

$$= \frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{6+2+2}{2}\right)}$$

$$= \frac{\Gamma(7/2)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(5)}$$

$$= \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} \times \frac{1}{2}\sqrt{\pi}}{2\times 24}$$

$$= \frac{5\pi}{256}$$