Tensor

⇒ **Tensor :-** Tensor is a quantity which needs direction, magnitude and plane to define.

Use of Tensor :-

1. relativity theory
2. differential geometry
3. mechanics
4. elasticity
5. hydrodynamics
6. electromegnetic theory.

Difference between vector and Tensor

Tensor	vector
1. Tensor is a quantity which needs direction magnitude and plane to define.	1. Any quantity that has both magnitude and direction is called vector.
2. zero order tensor is scaler	2. First order tensor is vector.
3. Doesent obey law of vector addition	3. obeys the law of vector addition.
4. Electric current are example	4. velocity are example

Space of N dimension :-

A point in N dimensional space is a set of N number denoted by $(x^1, x^2, \cdots x^N)$ where $1, 2, \cdots N$ are taken not as exponent but as superscript.

$(x, y, z), (\rho, \phi, z), (r, \theta, \phi)$ are coordinates of a point in rectangular, cylindrical, spherical coordinates system respectively.

Coordinates transformation :-

Let $(x^1, x^2, \cdots x^N)$ and $(\bar{x}^1, \bar{x}^2, \cdots \bar{x}^N)$ be coordinates of a point in two different frames of reference. Suppose there exists N independent relations between the coordinate of two system. Let every one of $\bar{x}^1, \bar{x}^2, \cdots \bar{x}^N$ be different single valued function of $x^1, x^2, \cdots x^N$.

i.e.

$$\begin{aligned} \bar{x}^1 &= \bar{f}^1 (x^1, x^2, \cdots x^n) \\ \bar{x}^2 &= \bar{f}^2 (x^1, x^2, \cdots x^n) \qquad (1) \\ \cdots & \quad \cdots \quad \cdots \quad \cdots \\ \bar{x}^n &= \bar{f}^n (x^1, x^2, \cdots x^n) \end{aligned}$$

which can be indicate briefly

$$\bar{x}^K = \bar{f}^K (x^1, x^2, \cdots x^n) \qquad \text{(ii)}$$

Then conversely,

$$x^k = f^k(\bar{x}^1, \bar{x}^2, \dots \bar{x}^n) \quad k = 1, 2, \dots n \quad \text{--- (iii)}$$

equation (ii) and (iii) define a transformation of coordinates from one frame of reference to another.

The Summation convention: In writing an consider the expression such as $a_1x^1 + a_2x^2 + \dots a_Nx^N$ we can use the short notation $\sum_{j=1}^{N} a_j x^j$. we write simply $a_j x^j$ instead of $\sum_{j=1}^{N} a_j x^j$. This is called the summation convention.

dummy index / umbral index: Any index which is repeated in a given term so that the summation convention applies is called dummy index.

free index: An index occurring only one in a given term is called free index.

[4] Contravariant and covariant vector:-

suppose
If n quantities $A^1, A^2 \cdots A^n$ in a coordinates system $(x^1, x^2, x^3 \cdots x^n)$ are related to n another quantities $\bar{A}^1, \bar{A}^2, \cdots \bar{A}^n$ in other coordinates system $(\bar{x}^1, \bar{x}^2, \cdots \bar{x}^n)$ by the transformation equation

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j$$

then A^j are called components of a contravariant vector or contravariant tensor of rank one.

If n quantities $A_1, A_2 \cdots A_n$ in a coordinates system $(x^1, x^2, \cdots x^n)$ are related to n another quantities $\bar{A}_1, \bar{A}_2 \cdots \bar{A}_n$ in other coordinates system $(\bar{x}^1, \bar{x}^2, \cdots \bar{x}^n)$ by the transformation equations,

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j.$$

then A_j are called components covariant vector or covariant tensor of rank one.

Contravariant, covariant and Mixed tensor :-

If N^2 quantities A^{qs} in a coordinates system $(x^1, x^2, \dots x^N)$ are related to N^2 other quantities $\bar{A}^{pr}$ in another coordinates system $(\bar{x}^1, \bar{x}^2 \dots \bar{x}^N)$ by the transformation equations.

$$\bar{A}^{pr} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^{qs}$$

they are called contravariant components of a tensor of the second rank

the N^2 quantities A_{qs} are called covariant components of a tensor of the second rank if

$$\bar{A}_{pr} = \frac{\partial x^q}{\partial \bar{x}^p} \frac{\partial x^s}{\partial \bar{x}^r} A_{qs}$$

Similarly the N^2 quantities A^q_s are called components of a mixed tensor of the second rank if,

$$\bar{A}^p_r = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial x^s}{\partial \bar{x}^r} A^q_s$$

[4] The Kronecker delta :-

written δ^j_k is defined by,

$$\delta^j_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

it is a Mixed tensor of second rank.

[5] Tensor of rank greater than two

$$\bar{A}^{prm}_{ij} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} \frac{\partial \bar{x}^m}{\partial x^t} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} A^{qst}_{kl}$$

Mixed tensor rank 5.
contravariant of order 3.
covariant of order 2!!

4.1. Fundamental operation with tensor.

Tensor of the same rank and same Type :- If two tensor have same Number of covariant indices and same number of contravariant indices then they said to be same rank and same type.

1) **Addition :** The sum of two or more tensor of the same rank and type is also a tensor of the same rank and type.

$$A^{ij}_{k} + B^{ij}_{k} = C^{ij}_{k}$$

2) **Subtraction :** The difference of two tensor of the same rank and same type is also a tensor of the same rank and type.

$$A^{ij}_{k} - B^{ij}_{k} = D^{ij}_{k}$$

3) **Outer Product / Multiplication :** The Product of two tensors is a tensor whose rank is the sum of the rank of given tensors. ordinary multiplication of the components of tensors is called outer Product. Not every tensor can written product of two tensor of lower rank. For this reason division of tensor is not always possible.

$A^{ij}_{k} B^{l}_{m} = C^{ijl}_{km}$ is the outer Product of A^{ij}_{k} and B^{l}_{m}.

4) <u>Contraction:-</u> If one covariant index and one contravariant index of a tensor are set equal. Summation over according to the summation convention. This resulting sum is a Tensor of rank two less than that of the original Tensor. This Process is called contraction.

A^{ijk}_{lm} is a Tensor of rank 5. set $m=k$ then we get $A^{ijk}_{lk} = B^{ij}_{l}$ is a tensor of rank 3.

again set $l=j$ $B^{ij}_{j} = C^{i}$ is a tensor of rank 1.

5) <u>Inner Multiplication:</u> By the Process of outer Multiplication of two tensor followed by a contraction we obtain a new Tensor called inner Product of the given tensor.

Let A^{ij}_{k} and B^{r}_{st} are two Tensor.

The Product $A^{ij}_{k} B^{r}_{st} = C^{ijr}_{kst}$ rank 6.

Now $t=r$ $C^{ijr}_{ksr} = D^{ij}_{ks}$ rank 4.

6. Quotient law:- suppose it is not known whether a quantity x is a tensor or not. If an inner Product of x with an arbitary tensor is a tensor, Then x is also a Tensor. This is called the quotient law.

Christoffels symbols:-

The symbols,

$$[pq, r] = \frac{1}{2}\left(\frac{\partial g_{qr}}{\partial x^p} + \frac{\partial g_{pr}}{\partial x^q} - \frac{\partial g_{pq}}{\partial x^r}\right)$$

$$\left\{ {s \atop pq} \right\} = g^{rs}[pq, r]$$

are called christoffels symbol first and second kind respectively.

other symbol are used instead of $\left\{ {s \atop pq} \right\}$ are $\{pq, s\}$ and Γ^s_{pq}

Symmetric Tensor:- A Tensor is said to be symmetric with respect to two contravariant indices or two covariant indices if the value of its component remain unchanged due to the interchanging of the indices.

Example: If $A^{ij}_{pqr} = A^{ji}_{pqr}$ Then A^{ij}_{pqr} is called symmetric mixed tensor.

Skew - symmetric Tensor:- A Tensor is said to be skew - symmetric with respect to two contravarient or two covarient indices if the value of its components change the sign due to interchanging of the indices.

Example: If $A_{ijk} = -A_{ikj}$ Then A_{ijk} is a skew symmetric Tensor with respect to indices j and k.

Problem-01

write each of the following using the summation convention.

(a) $d\phi = \frac{\partial\phi}{\partial x^1} dx^1 + \frac{\partial\phi}{\partial x^2} dx^2 + \cdots + \frac{\partial\phi}{\partial x^N} dx^N$

$d\phi = \frac{\partial\phi}{\partial x^j} dx^j$

(b) $\frac{d\bar{x}^K}{dt} = \frac{\partial\bar{x}^K}{\partial x^1}\frac{dx^1}{dt} + \frac{\partial\bar{x}^K}{\partial x^2}\frac{dx^2}{dt} + \cdots + \frac{\partial\bar{x}^K}{\partial x^N}\frac{dx^N}{dt}$

$\frac{d\bar{x}^K}{dt} = \frac{\partial\bar{x}^K}{\partial x^m}\frac{dx^m}{dt}$

(c) $(x^1)^2 + (x^2)^2 + (x^3)^2 + \cdots + (x^N)^2$

$x^K x^K$

(d) $ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$

$ds^2 = g_{KK}\, dx^K dx^K \quad N=3$

(e) $\sum_{p=1}^{3}\sum_{q=1}^{3} g_{pq}\, dx^p dx^q$

$g_{pq}\, dx^p dx^q \quad N=3.$

Problem-02

Write the terms in each of the following indicated sums.

(a) $a_{jk} x^k$

$$\sum_{k=1}^{N} a_{jk} x^k = a_{j1} x^1 + a_{j2} x^2 + \cdots + a_{jN} x^N.$$

(b) $A_{pq} A^{qr}$

$$\sum_{q=1}^{N} A_{pq} A^{qr} = A_{p1} A^{1r} + A_{p2} A^{2r} + \cdots + A_{pN} A^{Nr}$$

(c) $\bar{g}_{rs} = g_{jk} \frac{\partial x^j}{\partial \bar{x}^r} \frac{\partial x^k}{\partial \bar{x}^s}$ $\quad N = 3.$

$$\bar{g}_{rs} = \sum_{j=1}^{3} \sum_{k=1}^{3} g_{jk} \frac{\partial x^j}{\partial \bar{x}^r} \frac{\partial x^k}{\partial \bar{x}^s}$$

$$= \sum_{j=1}^{3} \left(g_{j1} \frac{\partial x^j}{\partial \bar{x}^r} \cdot \frac{\partial x^1}{\partial \bar{x}^s} + g_{j2} \frac{\partial x^j}{\partial \bar{x}^r} \frac{\partial x^2}{\partial \bar{x}^s} + g_{j3} \frac{\partial x^j}{\partial \bar{x}^r} \cdot \frac{\partial x^3}{\partial \bar{x}^s} \right)$$

$$= g_{11} \frac{\partial x^1}{\partial \bar{x}^r} \cdot \frac{\partial x^1}{\partial \bar{x}^s} + g_{21} \frac{\partial x^2}{\partial \bar{x}^r} \cdot \frac{\partial x^1}{\partial \bar{x}^s} + g_{31} \frac{\partial x^3}{\partial \bar{x}^r} \cdot \frac{\partial x^1}{\partial \bar{x}^s} + g_{12} \frac{\partial x^1}{\partial \bar{x}^r} \cdot \frac{\partial x^2}{\partial \bar{x}^s} + g_{22} \frac{\partial x^2}{\partial \bar{x}^r} \cdot \frac{\partial x^2}{\partial \bar{x}^s} + g_{32} \frac{\partial x^3}{\partial \bar{x}^r} \cdot \frac{\partial x^2}{\partial \bar{x}^s} + g_{13} \frac{\partial x^1}{\partial \bar{x}^r} \cdot \frac{\partial x^3}{\partial \bar{x}^s} + g_{23} \frac{\partial x^2}{\partial \bar{x}^r} \cdot \frac{\partial x^3}{\partial \bar{x}^s} + g_{33} \frac{\partial x^3}{\partial \bar{x}^r} \cdot \frac{\partial x^3}{\partial \bar{x}^s}.$$

Problem-04

Write the law of transformation for the Tensors (a) A^{i}_{jk} (b) B^{mn}_{ijk} (c) C^{m}

(a) $$\bar{A}^{p}_{qr} = \frac{\partial \bar{x}^{p}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \bar{x}^{q}} \frac{\partial x^{k}}{\partial \bar{x}^{r}} A^{i}_{jk}$$

p	q	r	s	t
↓	↓	↓	↓	↓
i	j	k	l	m

(b) $$\bar{B}^{pq}_{rst} = \frac{\partial \bar{x}^{p}}{\partial x^{m}} \frac{\partial \bar{x}^{q}}{\partial x^{n}} \frac{\partial x^{i}}{\partial \bar{x}^{r}} \frac{\partial x^{j}}{\partial \bar{x}^{s}} \frac{\partial x^{k}}{\partial \bar{x}^{t}} B^{mn}_{ijk}$$

(c) $$\bar{C}^{p} = \frac{\partial \bar{x}^{p}}{\partial x^{m}} C^{m}$$

Problem-07.

A covariant tensor has components $xy, 2y-z^2, xz$ in rectangular coordinates. Find its covariant components in spherical coordinates.

Solution:-

Let A_j denote the covariant components in rectangular coordinates.

$$x^1 = x, \quad x^2 = y, \quad x^3 = z \qquad \text{Then}$$

$$A_1 = xy = x^1 x^2 \qquad A_2 = 2y - z^2 = 2x^2 - (x^3)^2 \qquad A_3 = xz = x^1 x^3$$

Let $\bar{A}_K$ denoted the covariant components in spherical coordinates

$$\bar{x}^1 = r,\ \bar{x}^2 = \theta,\ \bar{x}^3 = \phi \quad \text{Then.}$$

$$\bar{A}_K = \frac{\partial x^J}{\partial \bar{x}^K} A_J \quad \text{———} (1)$$

we know,

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$x^1 = \bar{x}^1 \sin\bar{x}^2 \cos\bar{x}^3$, $x^2 = \bar{x}^1 \sin\bar{x}^2 \sin\bar{x}^3$, $x^3 = \bar{x}^1 \cos\bar{x}^2$

we Put $K=1$ and $J=1,2,3$ in (1) then we get,

$$\bar{A}_1 = \frac{\partial x^1}{\partial \bar{x}^1} A_1 + \frac{\partial x^2}{\partial \bar{x}^1} A_2 + \frac{\partial x^3}{\partial \bar{x}^1} A_3$$

$$= (\sin\bar{x}^2 \cos\bar{x}^3)(x^1 x^2) + (\sin\bar{x}^2 \sin\bar{x}^3)(2x^2 - (x^3)^2)$$
$$+ (\cos\bar{x}^2)(x^1 x^3)$$
$$= (\sin\theta\cos\phi)(r^2\sin^2\theta\sin\phi\cos\phi) + (\sin\theta\sin\phi)$$
$$(2r\sin\theta\sin\phi - r^2\cos^2\theta) + (\cos\theta)(r^2\sin\theta\cos\theta\cos\phi)$$

$$\bar{A}_2 = \frac{\partial x^1}{\partial \bar{x}^2} A_1 + \frac{\partial x^2}{\partial \bar{x}^2} A_2 + \frac{\partial x^3}{\partial \bar{x}^2} A_3$$

$$= (r\cos\theta\cos\phi)(r^2\sin^2\theta\sin\phi\cos\phi) + (r\cos\theta\sin\phi)$$
$$(2r\sin\theta\sin\phi - r^2\cos^2\theta) + (-r\sin\theta)(r^2\sin\theta\cos\theta\cos\phi).$$

$$\bar{A}_3 = \frac{\partial x^1}{\partial \bar{x}^3} A_1 + \frac{\partial x^2}{\partial \bar{x}^3} A_2 + \frac{\partial x^3}{\partial \bar{x}^3} A_3$$

$$= (-r\sin\theta\sin\phi)(r^2\sin^2\theta\sin\phi\cos\phi)$$

$$+ (r\sin\theta\cos\phi)(2r\sin\theta\sin\phi - r^2\cos^2\theta)$$

$$+ (0)(r^2\sin\theta\cos\theta\cos\phi)$$

Problem-08

Show that $\frac{\partial A_p}{\partial x^q}$ is not a tensor even though A_p is a covariant tensor of rank one.

By hypothesis, $\bar{A}_j = \frac{\partial x^p}{\partial \bar{x}^j} A_p$. Differentiating with respect to $\bar{x}^k$.

$$\frac{\partial \bar{A}_j}{\partial \bar{x}^k} = \frac{\partial x^p}{\partial \bar{x}^j} \cdot \frac{\partial A_p}{\partial \bar{x}^k} + \frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^j} \cdot A_p$$

$$= \frac{\partial x^p}{\partial \bar{x}^j} \cdot \frac{\partial A_p}{\partial x^q} \cdot \frac{\partial x^q}{\partial \bar{x}^k} + \frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^j} A_p$$

$$= \frac{\partial x^p}{\partial \bar{x}^j} \cdot \frac{\partial x^q}{\partial \bar{x}^k} \cdot \frac{\partial A_p}{\partial x^q} + \frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^j} A_p$$

Due to Preseence of second term on the right hand side. It is evident that $\frac{\partial A_p}{\partial x^q}$ does't transfrom As Tensor should.

Hence $\frac{\partial A^p}{\partial x^p}$ is not a tensor.

Problem-09

Show that velocity of a fluid at any Point is a contravariant Tensor of rank one.

The velocity of a fluid at any point has components $\frac{dx^k}{dt}$ in the coordinates system x^k. In the coordinates system $\bar{x}^J$ the velocity is $\frac{d\bar{x}^J}{dt}$. But

$$\frac{d\bar{x}^J}{dt} = \frac{\partial \bar{x}^J}{\partial x^k} \cdot \frac{dx^k}{dt}$$

by the chain rule, and it follows that the velocity is a contravariant tensor of rank one. or a contravariant vector.

Problem - 10 The kronecker delta.

Evalute (a) $\delta^p_q A^{qr}_s$ (b) $\delta^p_q \delta^q_r$

Since $\delta^p_q = 1$ if $p = q$ and 0 if $p \neq q$

(a) $\delta^p_q A^{qr}_s = A^{pr}_s$

(b) $\delta^p_q \delta^q_r = \delta^p_r$

Problem - 11

Show that, $\frac{\delta x^p}{\partial x^q} = \delta^p_q$

If $p = q$ $\quad \frac{\partial x^p}{\partial x^q} = 1$ since $x^p = x^q$

If $p \neq q$ $\quad \frac{\partial x^p}{\partial x^q} = 0$ since x^p and x^q are independent.

Then $\frac{\partial x^p}{\partial x^q} = \delta^p_q$.

Problem - 12

Prove that, $\frac{\partial x^p}{\partial \bar{x}^q} \frac{\partial \bar{x}^q}{\partial x^r} = \delta^p_r$

By the chain rule.

$$\frac{\partial x^p}{\partial x^r} = \frac{\partial x^p}{\partial \bar{x}^q} \cdot \frac{\partial \bar{x}^q}{\partial x^r} = \delta^p_r$$

Problem-14 :- Prove that δ^p_q is a Mixed tensor of second rank.

we know,

$$\bar{\delta}^i_j = \frac{\partial \bar{x}^i}{\partial \bar{x}^j}$$

$$= \frac{\partial \bar{x}^i}{\partial x^p} \cdot \frac{\partial x^p}{\partial \bar{x}^j}$$

$$= \frac{\partial \bar{x}^i}{\partial x^p} \cdot \frac{\partial x^p}{\partial x^q} \cdot \frac{\partial x^q}{\partial \bar{x}^j}$$

$$= \frac{\partial \bar{x}^i}{\partial x^p} \cdot \frac{\partial x^q}{\partial \bar{x}^j} \cdot \frac{\partial x^p}{\partial x^q}$$

$$= \frac{\partial \bar{x}^i}{\partial x^p} \cdot \frac{\partial x^q}{\partial \bar{x}^j} \cdot \delta^p_q .$$

which is transformation law of Mixed Tensor of rank 2. So δ^p_q is a Mixed Tensor of second rank.

Problem-15

If A^{pq}_{r} and B^{pq}_{r} are tensor, Prove that their sum and difference are Tensor.

Solution:

Since A^{pq}_{r} and B^{pq}_{r} are tensor so from the law of transformation we get,

$$\bar{A}^{ij}_{k} = \frac{\partial \bar{x}^{i}}{\partial x^{p}} \frac{\partial \bar{x}^{j}}{\partial x^{q}} \frac{\partial x^{r}}{\partial \bar{x}^{k}} A^{pq}_{r} \quad \text{—— (i)}$$

$$\bar{B}^{ij}_{k} = \frac{\partial \bar{x}^{i}}{\partial x^{p}} \frac{\partial \bar{x}^{j}}{\partial x^{q}} \frac{\partial x^{r}}{\partial \bar{x}^{k}} B^{pq}_{r} \quad \text{—— (ii)}$$

Adding and substracting (1) and (2) we get

$$\bar{A}^{ij}_{k} + \bar{B}^{ij}_{k} = \frac{\partial \bar{x}^{i}}{\partial x^{p}} \frac{\partial \bar{x}^{j}}{\partial x^{q}} \frac{\partial x^{r}}{\partial \bar{x}^{k}} (A^{pq}_{r} + B^{pq}_{r})$$

$$\bar{A}^{ij}_{k} - \bar{B}^{ij}_{k} = \frac{\partial \bar{x}^{i}}{\partial x^{p}} \frac{\partial \bar{x}^{j}}{\partial x^{q}} \frac{\partial x^{r}}{\partial \bar{x}^{k}} (A^{pq}_{r} - B^{pq}_{r})$$

~~Then~~ They satisfy the tensor law of Transformation

Then $A^{pq}_{r} + B^{pq}_{r}$ and $A^{pq}_{r} - B^{pq}_{r}$ are tensor of the same rank and type as A^{pq}_{r} and B^{pq}_{r}.

Problem-18. If A^{pq}_{r} and B^{s}_{t} are tensor Prove that $C^{pqs}_{rt} = A^{pq}_{r} \cdot B^{s}_{t}$ is also a Tensor.

Solutions:

$$\bar{A}^{ij}_{k} = \frac{\partial \bar{x}^{i}}{\partial x^{p}} \frac{\partial \bar{x}^{j}}{\partial x^{q}} \frac{\partial x^{r}}{\partial \bar{x}^{k}} A^{pq}_{r}$$

$$\bar{B}^{m}_{n} = \frac{\partial \bar{x}^{m}}{\partial x^{s}} \frac{\partial x^{t}}{\partial \bar{x}^{n}} B^{s}_{t}$$

Multiplying, $\bar{A}^{ij}_{k} \bar{B}^{m}_{n}$

$$= \frac{\partial \bar{x}^{i}}{\partial x^{p}} \frac{\partial \bar{x}^{j}}{\partial x^{q}} \frac{\partial x^{r}}{\partial \bar{x}^{k}} \frac{\partial \bar{x}^{m}}{\partial x^{s}} \frac{\partial x^{t}}{\partial \bar{x}^{n}} A^{pq}_{r} B^{s}_{t}$$

which shows that $A^{pq}_{r} B^{s}_{t}$ is a tensor of rank 5. with contravariant indices p, q, s and covariant indices r, t thus warranting the notations C^{pqs}_{rt}. We call $C^{pqs}_{rt} = A^{pq}_{r} B^{s}_{t}$ the outer Product of A^{pq}_{r} and B^{s}_{t}.

Christoffel's symbols.

Problem - 44

Prove (a) $[pq, r] = [qp, r]$ (b) $\left\{ {s \atop pq} \right\} = \left\{ {s \atop qp} \right\}$

(c) $[pq, r] = g_{rs} \left\{ {s \atop pq} \right\}$

(a) $$[pq, r] = \frac{1}{2}\left(\frac{\partial g_{qr}}{\partial x^p} + \frac{\partial g_{pr}}{\partial x^q} - \frac{\partial g_{pq}}{\partial x^r}\right) \quad —(1)$$

interchanging p and q in (1) then we get.

$$[qp, r] = \frac{1}{2}\left(\frac{\partial g_{pr}}{\partial x^q} + \frac{\partial g_{qr}}{\partial x^p} - \frac{\partial g_{qp}}{\partial x^r}\right)$$

$$= \frac{1}{2}\left(\frac{\partial g_{pr}}{\partial x^q} + \frac{\partial g_{qr}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^r}\right) \quad [\partial g_{pq} = \partial g_{qp}]$$

$$= [pq, r]$$

$\therefore [qp, r] = [pq, r]$ Proved.

(b) $\left\{ {s \atop pq} \right\} = g^{sr}[pq, r] = g^{sr}[qp, r] = \left\{ {s \atop qp} \right\}$

(c) we know,

$$\left\{ {s \atop pq} \right\} = g^{sr}[pq, r]$$

multiplying both side by g_{ks}

$$g_{ks}\left\{ {s \atop pq} \right\} = g_{ks}g^{sr}[pq, r]$$

$$g_{ks}\left\{ {s \atop pq} \right\} = \delta^r_k [pq, r] = [pq, k]$$

$\Rightarrow [pq, k] = g_{ks} \left\{ {s \atop pq} \right\}$

i.e., $[pq, r] = g_{rs} \left\{ {s \atop pq} \right\}$