Complex differentiation and the Cauchy-Riemann equations.

Complex variables.

Complex number: A number of the form, $a \pm ib$ Where, a, b are real and $i = \sqrt{-1}$ is called complex number.

Polar form: Put,

$a = r\cos\theta$ $\quad$ $r \to$ modulus.

$b = r\sin\theta$ $\quad$ $\theta \to$ argument.

$\therefore a^2 + b^2 = r^2$

$\Rightarrow r = \sqrt{a^2+b^2}$

and, $\theta = \tan^{-1}\frac{b}{a}$.

$e^{ix} = \cos x + i\sin x$

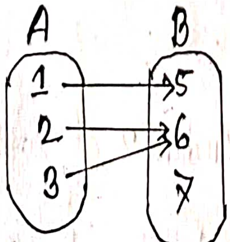
$e^{-ix} = \cos x - i\sin x$

$\therefore a \pm ib = r(\cos\theta \pm i\sin\theta)$

$= re^{\pm i\theta} \to$ Polar form.

- Set of natural numbers, $N = \{1, 2, 3, \dots\}$ $\quad$ $N \in Z$
- Set of integer numbers, $Z = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- Set of rational numbers, $\{p/q;\ p, q \in Z,\ q \neq 0,\ (p,q) = 1\}$ $\to$ coprime.
- Set of irrational numbers, $\{\sqrt{2}, \sqrt{3}, \sqrt{5}, \dots\}$
- Set of real numbers $= \{Q \cup Q'\}$
- $a, b, c, \dots$; Constants: A constant is a quantity which is fixed.
- $x, y, z, \dots$, Variables: A variable is a quantity which can take any value assign to it.
- $z \to$ Complex variable: A complex variable is a quantity which can stand for any one of a set of complex number.

Function: Function is a correspondence between two sets (A and B say). Such that for every element in A there is a unique element in B.



Complex function: Each value that a complex variable z can assume, there corresponds one or more values of a complex variable ω. We can say that ω is a function of z and write $w = f(z)$ or, $\omega = G(z)$.

Analytic function: A complex function $f(z)$ is said to be analytic at a point z_0 if its derivative exists at each point z in some neiborhood of z_0.
or, A complex function $f(z)$ is said to be analytic at a point z_0 if it's Taylor's series exists about z_0.

Cauchy-Riemann equation: A necessary condition that $\omega = f(z) = u(x,y) + iv(x,y)$ be analytic in a region R is that, in R, u and v satisfy the Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Necessary and sufficient condition.

Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Harmonic function: If any function satisfy the Laplace equation then, the function is called Harmonic function.

Singular Point: A point at which $f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$. Example:

$$f(z) = \frac{1}{z-1} \qquad z = 1 \rightarrow \text{singular point.}$$

$$f(z) = \frac{1}{(z-2)^3} \qquad z = 2 \rightarrow \text{singular point.}$$

Poles: If z_0 is not a singular and we can find a positive integer n such that $\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$. then $z = z_0$ is called a pole of order n.
If $n = 1$, z_0 is called a simple pole.

Examples: (a) $f(z) = \frac{1}{(z-2)^3}$ has a pole of order 3 at $z = 2$.

(b) $f(z) = \frac{3z-2}{(z-1)^2(z+1)(z-4)}$ has a pole of order 2 at $z = 1$, and simple poles at $z = -1$ and $z = 4$.

Show that $u = 2x - x^3 + 3xy^2$ is harmonic. Also find v (harmonic conjugate). Such that $u+iv$ is analytic.

Soln: Here,

$$u = 2x - x^3 + 3xy^2$$

$$\therefore \frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 \qquad \therefore \frac{\partial^2 u}{\partial x^2} = -6x$$

$$\therefore \frac{\partial u}{\partial y} = 6xy \qquad \therefore \frac{\partial^2 u}{\partial y^2} = 6x$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2 - 3x^2 + 3y^2$$

$$\therefore \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 6xy.$$

Integrating $\frac{\partial v}{\partial y}$ with respect to y.

$$v = 2y - 3x^2y + y^3 + c$$

Differentiating with respect to ~~$\frac{\partial v}{\partial x} = -6xy$~~ x,

$$\frac{\partial v}{\partial x} = -6xy$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -6y + 6y = 0 \quad \text{hence, } v \text{ is harmonic}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0 \quad \text{hence, } u \text{ is also harmonic.}$$

Again we see,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, $u+iv$ is analytic.

(a) Prove that, $u = e^{-x}(x\sin y - y\cos y)$ is harmonic.

(b) Find v such that $f(z) = u+iv$ is analytic.

(c) Find $f(z)$.

Soln: (a) Here, $u = e^{-x}(x\sin y - y\cos y)$

$$\therefore \frac{\partial u}{\partial x} = e^{-x}(\sin y) + (-e^{-x})(x\sin y - y\cos y)$$

$$= e^{-x}\sin y - e^{-x}x\sin y + ye^{-x}\cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y)$$

$$= -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y \qquad \text{——— (1)}$$

$$\therefore \frac{\partial u}{\partial y} = e^{-x}(x\cos y + y\sin y - \cos y)$$
$$= xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y.$$
$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y\right)$$
$$= -xe^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y \quad \text{———} (2)$$

Adding equ.(1) and (2) we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } u \text{ is harmonic.}$$

(b) From the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y \quad \text{———} (3)$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y \quad \text{———} (4)$$

Integrating equ.(3) with respect to y, keeping x constant. Then,

$$v = -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}(y\sin y + \cos y) + F(x)$$
$$= ye^{-x}\sin y + xe^{-x}\cos y + F(x) \quad \text{———} (5)$$

Where $F(x)$ is an arbitary real function of x.

~~Substitute equ.(5) in equ.(4) and obtain~~,

Differentiating equ(5) with respect to x,

$$\frac{\partial v}{\partial x} = -ye^{-x}\sin y - xe^{-x}\cos y + e^{-x}\cos y + F'(x)$$

Putting the value $\frac{\partial v}{\partial x}$ in equ.(4);

$$+ye^{-x}\sin y + xe^{-x}\cos y - e^{-x}\cos y + F'(x) = xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y$$
$$\Rightarrow F'(x) = 0$$

Putting $F'(x) = 0$ and $F(x) = c$ we get;

$$v = ye^{-x}\sin y + xe^{-x}\cos y + c$$
$$= e^{-x}(y\sin y + x\cos y) + c.$$

Find $f(z)$ in $u+iv = e^{-x}(x\sin y - y\cos y) + i e^{-x}(y\sin y + x\cos y)$.

$e^{iy} = \cos y + i\sin y$

$e^{-iy} = \cos y - i\sin y$

$e^{iy} - e^{-iy} = \cancel{\cos y} + i\sin y - \cancel{\cos y} + i\sin y$

$\Rightarrow 2i\sin y = e^{iy} - e^{-iy}$

$\Rightarrow \sin y = \dfrac{e^{iy} - e^{-iy}}{2i}$

$e^{iy} + e^{-iy} = \cos y + \cancel{i\sin y} + \cos y - \cancel{i\sin y}$

$\Rightarrow 2\cos y = e^{iy} + e^{-iy}$

$\therefore \cos y = \dfrac{e^{iy} + e^{-iy}}{2}$

Solⁿ:

We know,

$$f(z) = u + iv$$

Here, $u = e^{-x}(x\sin y - y\cos y)$

$v = e^{-x}(y\sin y + x\cos y)$

$$\therefore f(z) = e^{-x}(x\sin y - y\cos y) + i e^{-x}(y\sin y + x\cos y)$$

$$= e^{-x}\left\{x\left(\frac{e^{iy} - e^{-iy}}{2i}\right) - y\left(\frac{e^{iy} + e^{-iy}}{2}\right)\right\} + i e^{-x}\left\{y\left(\frac{e^{iy} - e^{-iy}}{2i}\right) + x\left(\frac{e^{iy} + e^{-iy}}{2}\right)\right\}$$

$$= i(x+iy)e^{-(x+iy)}$$

$$= i z e^{-z}.$$

If $W(z) = \phi(x,y) + i\psi(x,y)$, represent the complex potential for an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$. Determine the function ϕ.

Solⁿ:

$$W(z) = \phi(x,y) + i\psi(x,y)$$

and, $\psi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$, $\therefore \dfrac{\partial \psi}{\partial x} = 2x + \dfrac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$

$$= 2x + \frac{y^2 - x^2}{(x^2+y^2)^2}$$

and, $\dfrac{\partial \psi}{\partial y} = -2y - \dfrac{x(2y)}{(x^2+y^2)^2}$

$$= -2y - \frac{2xy}{(x^2+y^2)^2}$$

Now, $d\phi = \dfrac{\partial \phi}{\partial x} dx + \dfrac{\partial \phi}{\partial y} dy$

$$= \frac{\partial \psi}{\partial y} dx - \frac{\partial \psi}{\partial x} dy$$

$$= \left(-2y - \frac{2xy}{(x^2+y^2)^2}\right) dx - \left(2x + \frac{y^2 - x^2}{(x^2+y^2)^2}\right) dy$$

This is an exact differential equation.

Therefore, $\phi = \displaystyle\int \left[-2y - \frac{2xy}{(x^2+y^2)^2}\right] dx + C$

$$= -2xy + \frac{y}{x^2+y^2} + C$$

Construct an analytic function $f(z)$ whose real part is $e^x \cos y$.

Soln: Let, $f(z) = u + iv$.

Here, $u = e^x \cos y$

$$\therefore \frac{\partial u}{\partial x} = e^x \cos y$$

$$\therefore \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

We know,
$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\cancel{\Rightarrow dv = e^x \cos y + e^x \sin y}$$

$$\Rightarrow dv = e^x \sin y \, dx + e^x \cos y \, dy$$

$$\Rightarrow dv = d(e^x \sin y)$$

$$\therefore v = e^x \sin y.$$

$\therefore$ Analytic function,
$$f(z) = u + iv$$
$$= e^x \cos y + i e^x \sin y$$
$$= e^x (\cos y + i \sin y)$$
$$= e^x . e^{iy}$$
$$= e^{(x+iy)}$$
$$= e^z.$$

Find an analytic function $w(z) = u(x,y) + iv(x,y)$, given that $v = \frac{x}{x^2+y^2} + \cosh x \cos y$.

$w(z) = u(x,y) + iv(x,y)$.

Soln: It is given that, $v = \frac{x}{x^2+y^2} + \cosh x . \cos y$

$$\therefore \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2} + \cosh x . \cos y\right)$$
$$= \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} + \sinh x \cos y$$
$$= \frac{y^2 - x^2}{(x^2+y^2)^2} + \sinh x \cos y$$

$$\therefore \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2} + \cosh x . \cos y\right)$$
$$= \frac{-x.(2y)}{(x^2+y^2)^2} - \cosh x \sin y$$
$$= \frac{-2xy}{(x^2+y^2)^2} - \cosh x \sin y.$$

Using Cauchy-Riemann equations,
$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$
$$= \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$$

$$\therefore du = \left[\frac{-2xy}{(x^2+y^2)^2} - \cosh x \sin y\right] dx - \left[\frac{y^2 - x^2}{(x^2+y^2)^2} + \sinh x \cos y\right] dy$$

This is an exact differential equation.

$$\int du = \int\left(\frac{-2xy}{(x^2+y^2)^2} - \cosh x \sin y\right)dx - \int\left(\frac{y^2-x^2}{(x^2+y^2)^2} + \sinh x \cos y\right)dy.$$

Ignoring the term containing x:

$$v = \frac{y}{x^2+y^2} - \sinh x \sin y.$$

$\therefore W = u + iv$

$$= \frac{y}{x^2+y^2} - \sinh x \sin y + i\left[\frac{x}{x^2+y^2} + \cosh x \cos y\right]$$

$$= \frac{y+ix}{x^2+y^2} - \sinh x \sin y + i \cosh x \cos y.$$

Find the values of constants a, b, c and d such that the function $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$ is analytic.

Soln:

$$f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$$
$$= u + iv \text{ (say)}.$$

Where, $u = x^2 + axy + by^2$, $v = cx^2 + dxy + y^2$

$\therefore \frac{\partial u}{\partial x} = 2x + ay$ $\qquad$ $\therefore \frac{\partial v}{\partial x} = 2cx + dy$

$\therefore \frac{\partial u}{\partial y} = ax + 2by$ $\qquad$ $\therefore \frac{\partial v}{\partial y} = dx + 2y$

Since, $f(z) = u + iv$ is analytic, so Cauchy-Riemann equation must be satisfied.

i.e; $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Now, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 2x + ay = dx + 2y$ ——— (1)

and, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow ax + 2by = -2cx - dy$ ——— (2)

From equ.(1);

$$2x + ay - dx - 2y = 0$$
$$\Rightarrow 2x - dx + ay - 2y = 0$$
$$\Rightarrow x(2-d) + y(a-2) = 0 \text{ —— (3)}$$

From equ(2);

$$ax + 2by + 2cx + dy = 0$$
$$\Rightarrow ax + 2cx + 2by + dy = 0$$
$$\Rightarrow x(a+2c) + y(2b+d) = 0 \text{ —— (4)}$$

Equation (3) and (4) will hold good if;

$$2-d=0,\ a-2=0$$
$$a+2c=0,\ 2b+d=0$$

i.e; $a=2, d=2, c=-1, b=-1$.

Construct an analytic function $f(z) = u(x,y) + iv(x,y)$, where $v(x,y) = 6xy - 5x + 3$. Express the result as a function of z.

Solⁿ: It is given that $v(x,y) = 6xy - 5x + 3$

$$\therefore \frac{\partial v}{\partial x} = 6y - 5 \quad \therefore \frac{\partial v}{\partial y} = 6x$$

$$\therefore du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$= \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy \quad \text{[Using C-R equation]}$$

$$= 6x\,dx - (6y - 5)dy$$

$$\therefore u = \int 6x\,dx - \int (6y-5)\,dy$$

$$= 3x^2 - 3y^2 + 5y + c$$

$$\therefore f(z) = u + iv = 3x^2 - 3y^2 + 5y + i(6xy - 5x + 3) + c$$

$$= 3x^2 - 3y^2 + 5y + 6ixy - 5ix + 3i + c$$

$$= 3(x^2 - y^2 + 2ixy) + (-5ix + 5y) + 3i + c$$

$$= 3(x+iy)^2 - 5i(x+iy) + 3i + c$$

$$= 3z^2 - 5iz + 3i + c.$$

Prove that the function $f(z) = u + iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} \ (z \neq 0),\ f(0) = 0.$$ Is continuous and that Cauchy-Riemann equations are satisfied at the origin, yet $f'(z)$ does not exist there.

Solⁿ: Here, $f(z) = u + iv = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$

So, $u = \dfrac{x^3 - y^3}{x^2+y^2}$, $v = \dfrac{x^3+y^3}{x^2+y^2}$ (where $z \neq 0$)

Here, we see that both u and v are rational and finite for all values of $z \neq 0$, so u and v are continuous at all those points for which $z \neq 0$. Hence $f(z)$ is continuous where $z \neq 0$.

At the origin $u = 0, v = 0$. [Since $f(0) = 0$]

Hence, u and v are both continuous at the origin, therefore, $f(z)$ is continuous at the origin.

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0}\left(\frac{x}{x}\right) = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0}\left(\frac{-y}{+y}\right) = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0}\left(\frac{x}{x}\right) = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0}\left(\frac{y}{y}\right) = 1.$$

Thus, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Therefore, Cauchy-Riemann equations are satisfied at $z=0$. Again, $f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0}\left[\frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2}\cdot\frac{1}{x+iy}\right]$

Let $z\to 0$ along $y=x$ then we have

$$f'(0) = \lim_{z\to 0} \frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2}\cdot\frac{1}{x+iy}$$

$$= \lim_{z\to 0} \frac{2i}{2(1+i)} = \tfrac{1}{2}(1-i).$$

Further, let $z\to 0$ along $y=0$, then we have $f'(0) = \lim_{z\to 0} \frac{x^3(1+i)}{x^3} = 1+i$. Hence, $f'(0)$ is not unique. Thus, $f'(z)$ does not exist at the origin.

Show that the function $u = \cos x \cosh y$ is harmonic and find it's harmonic conjugate.

Soln: It is given that, $u = \cos x \cosh y$

then, $\frac{\partial u}{\partial x} = -\sin x \cosh y$, $\frac{\partial u}{\partial y} = \cos x \sinh y$

$\therefore \frac{\partial^2 u}{\partial x^2} = -\cos x \cosh y$ $\therefore \frac{\partial^2 u}{\partial y^2} = \cos x \cosh y$

Now, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\cos x \cosh y + \cos x \cosh y = 0 \Rightarrow u$ satisfied Laplace's equation, so, u is a harmonic function.

Let v be its conjugate harmonic function, then we have

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$= -\cos x \sinh y\, dx + -\sin x \cosh y\, dy$$

$$= -(\cos x \sinh y\, dx + \sin x \cosh y\, dy)$$

Integrating, we obtain

$$v = -\sin x \sinh y + c,$$

where c is a real constant.

Prove that, $u = y^3 - 3x^2y$ is a harmonic function. Determine it's harmonic Conjugate, hence find the corresponding analytic function $f(z)$ in terms of z.

Solⁿ: Given that, $u = y^3 - 3x^2y$

$$\therefore \frac{\partial u}{\partial x} = \cancel{y^3} - 6xy \quad , \quad \frac{\partial u}{\partial y} = 3y^2 - 3x^2$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = -6y \qquad \therefore \frac{\partial^2 u}{\partial y^2} = 6y$$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0 \Rightarrow u$ satisfies Laplace's equation, so u is a harmonic function. Further, let v be the harmonic conjugate to u, then we have,

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$= -(3y^2 - 3x^2)dx - 6xy\,dy$$

$$= -(3y^2dx + 6xy\,dy) + 3x^2dx$$

Integrating,

$$v = -3xy^2 + x^3 + c$$

The analytic function,

$$f(z) = u + iv$$

$$= y^3 - 3x^2y + i(-3xy^2 + x^3 + c)$$

$$= y^3 - 3x^2y - 3xy^2i + x^3i + ic$$

$$= (y - ix)^3 + ic$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

Prove that, if $u = x^2 - y^2$, $v = \frac{-y}{(x^2+y^2)}$, both u and v satisfy Laplace's equation, but $u+iv$ is not an analytic function of z.

Soln: Given that,

$$u = x^2 - y^2, \quad v = \frac{-y}{(x^2+y^2)}$$

$$\therefore \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial y} = \frac{-(x^2+y^2).1 + y.2y}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(2x) = 2$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(-2y) = -2$$

$$\therefore \frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{2xy}{(x^2+y^2)^2}\right)$$

$$= 2y\left[\frac{(x^2+y^2)^2.1 - 2(x^2+y^2).2x.x}{(x^2+y^2)^4}\right]$$

$$= 2y.(x^2+y^2).\frac{x^2+y^2-4x^2}{(x^2+y^2)^4}$$

$$= \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$\therefore \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{y^2-x^2}{(x^2+y^2)^2}\right]$$

$$= \frac{\partial}{\partial y}\left[\frac{-(x^2-y^2)}{(x^2+y^2)^2}\right]$$

$$= -\left[\frac{(x^2+y^2)^2.(-2y) - 2(x^2+y^2).2y(x^2-y^2)}{(x^2+y^2)^4}\right]$$

$$= -(x^2+y^2).\frac{x^2+y^2.(-2y) - 2.2y(x^2-y^2)}{(x^2+y^2)^4}$$

$$= \cancel{\frac{-2y(x^2+y^2+2y(x^2-y^2))}{}}$$

$$= \frac{-[2y(-x^2-y^2-2x^2+2y^2)]}{(x^2+y^2)^3}$$

$$= \frac{-2y(y^2-3x^2)}{(x^2+y^2)^3}$$

Now, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$, and, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{2y(y^2-3x^2) - 2y(y^2-3x^2)}{(x^2+y^2)^3}$

$= 0.$

Hence, both u and v satisfy Laplace's equation.

But, we see that $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$ i.e; C-R equations are not satisfied.

Hence, $u+iv$ is not an analytic function of z.

Complex integration and Cauchy's theorem.

Cauchy's theorem: Let $f(z)$ be analytic in a region R and on its boundary C. Then

$$\oint_C f(z)\,dz = 0.$$

This fundamental theorem, often called Cauchy's integral theorem o briefly Cauchy's theorem.

■ Prove Cauchy's theorem $\oint_C f(z)\,dz = 0$ if $f(z)$ is analytic with derivative $f'(z)$ which is continuous at all points inside and on a simple closed curve C.

Solⁿ: Since $f(z) = u + iv$ is analytic and has a continuous derivative

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

It follows that the partial derivatives,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

are continuous inside and on C. Thus, Green's theorem can be applied and we have,

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+idy)$$
$$= \oint_C u\,dx - v\,dy + i\oint_C v\,dx + u\,dy$$
$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy = 0$$

■ Evaluate $\int_{(0,3)}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy$ along; (a) the parabola $x = 2t$, $y = t^2+3$; (b) straight lines from $(0,3)$ to $(2,3)$ and then from $(2,3)$ to $(2,4)$; (c) a straight line from $(0,3)$ to $(2,4)$.

Solⁿ: (a) The points $(0,3)$ and $(2,4)$ on the parabola correspond to $t=0$ and $t=1$, respectively. Then the given integral equals,

Here, $x = 2t$
$\therefore dx = 2dt$
$y = t^2+3$
$\therefore dy = 2t\,dt$

$$\int_{(0,3)}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy$$
$$= \int_0^1 (2t^2+6+4t^2)\cdot 2dt + (6t-t^2-3)\cdot 2t\,dt$$
$$= \int_0^1 (4t^2+12+8t^2+12t^2-2t^3-6t)\,dt$$
$$= \int_0^1 (24t^2+12-2t^3-6t)\,dt$$

$$= \left[24 \cdot \tfrac{1}{3} \cdot t^3 + 12t - 2 \cdot \tfrac{1}{4} t^4 - 6 \cdot \tfrac{1}{2} t^2\right]_0^1$$

$$= \left[8t^3 + 12t - \frac{t^4}{2} - 3t^2\right]_0^1$$

$$= 8 + 12 - \tfrac{1}{2} - 3$$

$$= \frac{33}{2}.$$

(b) Along the straight line (0,3) to (2,3);

$y = 3 \quad \therefore dy = 0$

x varies from 0 to 2.

$$\therefore \int_0^2 (6 + x^2)\,dx = \left[6x + \tfrac{1}{3} x^3\right]_0^2$$

$$= \left(12 + \frac{8}{3}\right)$$

$$= \frac{44}{3}$$

Along the straight line (2,3) to (2,4)

$x = 2 \quad \therefore dx = 0$

y varies from 3 to 4

$$\therefore \int_3^4 (6 - y)\,dy = \left[6y - \tfrac{1}{2} y^2\right]_3^4$$

$$= \left\{24 - \tfrac{16}{2}\right\} - \left\{18 - \tfrac{9}{2}\right\}$$

$$= 16 - \frac{27}{2}$$

$$= \tfrac{5}{2}$$

Then, the required value $= \tfrac{44}{3} + \tfrac{5}{2}$

$$= \frac{103}{6}.$$

(c) An equation for the line joining (0,3) and (2,4) is $2y - x = 6$. Solving for x, we have $x = 2y - 6$. Then, the line integral equals,

$x = 2y - 6$
$\therefore dx = 2\,dy$

$$\int_3^4 \left[2y + (2y-6)^2\right] 2\,dy + \left[3(2y-6) - y\right] dy$$

$$= \int_3^4 (2y + 4y^2 - 24y + 36) \cdot 2\,dy + (6y - 18 - y)\,dy$$

$$= \int_3^4 (4y + 8y^2 - 48y + 72 + 6y - 18 - y)\,dy$$

$$= \int_3^4 (8y^2 - 39y + 54)\,dy$$

$$= \left[8\cdot\tfrac{1}{3}y^3 - 39\cdot\tfrac{1}{2}y^2 + 54y\right]_3^4$$

$$= \left(\tfrac{8}{3}\cdot 64 - 39\cdot\tfrac{1}{2}\cdot 16 + 54\cdot 4\right) - \left(\tfrac{8}{3}\cdot 27 - 39\cdot\tfrac{1}{2}\cdot 9 + 54\cdot 3\right)$$

$$= \left(\tfrac{512}{3} - 312 + 216\right) - \left(72 - \tfrac{351}{2} + 162\right)$$

$$= \tfrac{97}{6}.$$

4. Evaluate $\int_C \bar{z}\,dz$ from $z=0$ to $z=4+2i$ along the curve C given by: (a) $z=t^2+it$, (b) the line from $z=0$ to $z=2i$ and then the line from $z=2i$ to $z=4+2i$.

Sol^n:

(a) The Points $z=0$ and $z=4+2i$ on C correspond to $t=0$ and $t=2$, respectively. Then the line integral equals,

$$\int_0^2 \overline{t^2+it}\, d(t^2+it)$$

$$= \int_0^2 (t^2-it)(2t+i)\,dt$$

$$= \int_0^2 (2t^3+it^2-i2t^2-i^2t)\,dt$$

$$= \int_0^2 (2t^3+it^2-2it^2+t)\,dt$$

$$\cancel{= \left[2\cdot\tfrac{1}{4}t^4 + i\tfrac{1}{3}t^3 - 2\right.}$$

$$= \int_0^2 (2t^3-it^2+t)\,dt$$

$$= \left[2\cdot\tfrac{1}{4}t^4 - i\cdot\tfrac{1}{3}t^3 + \tfrac{1}{2}t^2\right]_0^2$$

$$= \tfrac{1}{2}\cdot 2^4 - i\cdot\tfrac{1}{3}2^3 + \tfrac{1}{2}2^2$$

$$= 8+2-i\cdot\tfrac{8}{3}$$

$$= 10 - \tfrac{8i}{3}.$$

(b) The given line integral equals,

$$\int_C (x-iy)(dx+idy) = \int_C x\,dx + y\,dy + i\int_C x\,dy - y\,dx.$$

The line from $z=0$ to $z=2i$ is the same as the line from $(0,0)$ to $(0,2)$ for which $x=0$, $dx=0$ and the line integral equals,

$$\int_{y=0}^{2} (0)\cdot(0) + y\,dy + i\int_{y=0}^{2} (0)\,dy - y\cdot(0)$$

$$= \int_0^2 y\,dy = 2$$

The line from $z=2i$ to $z=4+2i$ is the same as the line from $(0,2)$ to $(4,2)$ for which $y=2$, $dy=0$ and the x varies from 0 to 4.

$$\int_0^4 x\,dx + 2\cdot 0 + i\int_0^4 x\cdot 0 - 2\,dx$$

$$= \int_0^4 x\,dx + i\int_0^4 -2\,dx$$

$$= \left[\tfrac{1}{2}x^2\right]_0^4 + i\left[-2x\right]_0^4$$

$$= \tfrac{1}{2}\cdot 16 + i\cdot -8$$

$$= 8 - 8i$$

Then, the required value $= 2 + (8-8i)$

$$= 10 - 8i.$$

Evaluate $\oint_C \frac{dz}{z-a}$ where C is any simple closed curve C and $z=a$ is (a) Outside C, (b) inside C.

Soln:

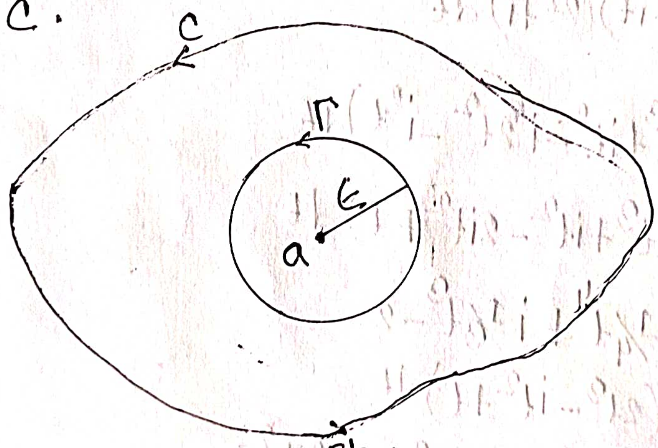


Fig.1

(a) If a is outside C, then $f(z) = \frac{1}{z-a}$ is analytic everywhere inside and on C. Hence, Cauchy's theorem,

$$\oint_C \frac{dz}{z-a} = 0.$$

(b) Suppose a is inside C and let Γ be a circle of radius ϵ with center at $z=a$ so that Γ is inside C. (this can be done since $z=a$ is interior point).

$$\oint_C \frac{dz}{z-a} = \oint_\Gamma \frac{dz}{z-a} \qquad \text{——— (1)}$$

Now, on Γ, $|z-a| = \epsilon$ or, $z-a = \epsilon e^{i\theta}$

i.e., $z = a + \epsilon e^{i\theta}$, $0 \le \theta \le 2\pi$. Thus, since $dz = i\epsilon e^{i\theta}d\theta$, the right side of equ (1) becomes,

$$\int_0^{2\pi} \frac{i\epsilon e^{i\theta}d\theta}{\epsilon e^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i$$

which is required value.

Evaluate $\oint_C \frac{dz}{(z-a)^n}$, $n = 2, 3, 4, \ldots$ where $z=a$ is inside the simple closed curve C.

Soln: From fig. 1 on Γ,

$$|z-a| = \epsilon$$
$$\Rightarrow z-a = \epsilon e^{i\theta}$$
$$\Rightarrow z = a+\epsilon e^{i\theta}, \quad 0 \le \theta \le 2\pi$$
$$\therefore dz = i\epsilon e^{i\theta} d\theta.$$

Hence,

$$\int_0^{2\pi} \frac{i\epsilon e^{i\theta} d\theta}{\epsilon^n e^{i\theta n}}$$
$$= \frac{i}{\epsilon^{n-1}} \int_0^{2\pi} e^{(1-n)i\theta} d\theta$$
$$= \frac{i}{\epsilon^{n-1}} \left[\frac{e^{(1-n)i\theta}}{(1-n)i}\right]_0^{2\pi}$$
$$= \frac{1}{(1-n)\epsilon^{n-1}} \left[e^{(1-n)2\pi i} - 1\right]$$
$$= \frac{1}{(1-n)\epsilon^{n-1}} \left[e^{2(1-n)\pi i} - 1\right]$$
$$= 0.$$

Where, $n \neq 1$.

If $f(\xi) = \int_C \frac{4z^2+z+5}{z-\xi} dz$, where C is the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$. Find the value of $F(3.5)$.

Soln:

$$F(3.5) = \int_C \frac{4z^2+z+5}{z-3.5} dz.$$

Since $\xi = 3.5$ is the only singular point of $\frac{4z^2+z+5}{z-3.5}$ and it lies outside the ellipse C, therefore, $\frac{4z^2+z+5}{z-3.5}$ is analytic everywhere within C.

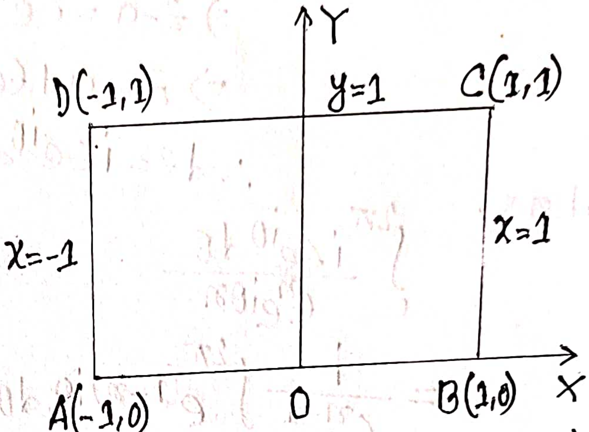
Hence, by Cauchy's theory,

$$\int_C \frac{4z^2+z+5}{z-3.5} dz = 0.$$

Verify Cauchy's theorem for the integral of z^3 taken over the boundary of the rectangle with vertices $-1, 1, 1+i, -1+i$.

Soln: Let $f(z)=z^3$, since $f(z)$ is analytic within and on the boundary of the rectangle (say, C) and also $f'(z)$ is continuous at each point within and on C. Hence, applying Cauchy's theorem, we get

$$\oint_C z^3 dz = 0$$



Now consider,

$$\oint_C z^3 dz$$

$$= \oint_C (x+iy)^3 (dx+idy)$$

$$= \oint_C [(x+iy)^3 dx + i(x+iy)^3 dy]$$

$$= \int_{AB} (x+iy)^3 dx + i(x+iy)^3 dy + \int_{BC} (x+iy)^3 dx + i(x+iy)^3 dy$$

$$+ \int_{CD} (x+iy)^3 dx + i(x+iy)^3 dy + \int_{DA} (x+iy)^3 dx + i(x+iy)^3 dy$$

Along AB, $y=0 \Rightarrow dy=0$ $(-1 \le x \le 1)$
Along BC, $x=1 \Rightarrow dx=0$ $(0 \le y \le 1)$
Along CD, $y=1 \Rightarrow dy=0$ $(-1 \le x \le 1)$
Along DA, $x=-1 \Rightarrow dx=0$ $(0 \le y \le 1)$

$$= \int_{-1}^{1} x^3 dx + i\int_0^1 (1+iy)^3 dy + \int_{+1}^{-1} (x+i)^3 dx + i\int_1^0 (-1+iy)^3 dy$$

$$= 0 + i\int_0^1 (1+iy)^3 dy + \int_1^{-1} (x+i)^3 dx + i\int_1^0 (iy-1)^3 dy$$

$$= i\left[\frac{(1+iy)^4}{4\times i}\right]_0^1 + \left[\frac{(x+i)^4}{4}\right]_1^{-1} + i\left[\frac{(iy-1)^4}{4\times i}\right]_1^0$$

$$= \frac{1}{4}\left[(1+iy)^4\right]_0^1 + \frac{1}{4}\left[(x+i)^4\right]_1^{-1} + \frac{1}{4}\left[(iy-1)^4\right]_1^0$$

$$= \frac{1}{4}\left[\{(1+i)^4 - 1\} + \{(i-1)^4 - (1+i)^4\} + \{1 - (i-1)^4\}\right] = 0$$

$$= 0$$

This verifies the Cauchy's theorem.

Evaluate $\int_C \frac{z+4}{z^2+2z+5}dz$, where C is the circle $|z+1|=1$.

Sol^n: If $f(z) = \frac{z+4}{z^2+2z+5}$, then poles of $f(z)$ are given by,

$$z^2+2z+5=0 \quad \therefore z = \frac{-2\pm\sqrt{4-20}}{2}$$
$$= \frac{-2\pm\sqrt{16}}{2}$$
$$= \frac{2(-1\pm\sqrt{4})}{2}$$
$$= -1\pm 2i \quad \therefore$$

$$\therefore z = -1+2i, -1-2i$$

When $z=-1+2i$, then $|z+1| = |-1+2i+1| = 2>1$

$\therefore$ The pole $z=-1+2i$ lies outside the circle $|z+1|=1$

Since both the poles lie outside the circle C, hence $f(z)$ is analytic everywhere within C. Also $f'(z)$ is continuous within and on C.

By applying Cauchy's theorem, we get,

$$\int_C f(z)\,dz = 0 \quad \text{i.e.}, \int_C \frac{z+4}{z^2+2z+5}dz = 0$$

Evaluate $\int_C \frac{z^2-z+1}{z-1}dz$, where C is the circle $|z| = \frac{1}{2}$.

Sol^n: Let $f(z) = \frac{z^2-z+1}{z-1}$

Poles of $f(z)$ are given by,

$$z-1=0$$
$$\text{i.e.}, z=1$$

$$|x+iy| = \frac{1}{2}$$
$$\Rightarrow \sqrt{x^2+y^2} = \frac{1}{4}$$
$$\therefore x^2+y^2 = \frac{1}{2}$$

Since the pole $z=1$ lie outside the circle C, hence $f(z)$ is analytic within and on C. Also $f'(z)$ is continuous at each point within and on C.

So, by applying Cauchy's theorem, we get.

$$\int_C f(z)dz = 0 \quad \text{i.e.}, \int_C \frac{z^2-z+1}{z-1} = 0.$$

Cauchy's integral formulae and related theorems.

Cauchy's integral formulae: If $f(z)$ be analytic inside and on a simple closed curve C and a is any point inside C. Then,

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

Also, the nth derivative of $f(z)$ at $z=a$ is given by,

$$f^{n}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \quad , n = 1, 2, 3, \dots$$

Examples,

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} dz$$

$$f''(a) = \frac{2}{2\pi i} \oint_C \frac{f(z)}{(z-a)^3} dz$$

(a) Evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$,

(b) $\oint_C \frac{e^{2z}}{(z+1)^4} dz$ where C is the circle $|z|=3$

Soln:

(a) Since $\frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$, we have

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz$$

$|z| = 3$ is a circle of radius 3.

By Cauchy's integral formulae; $a=2$

$$f(2) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-2} dz$$

$$\Rightarrow 2\pi i . f(2) = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz$$

By Cauchy's integral formulae, $a=1$

$$f(1) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-1} dz$$

$$\Rightarrow 2\pi i . f(1) = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz$$

Since $z=1$ and $z=2$ are inside C and $\sin \pi z^2 + \cos \pi z^2$ is analytic inside C.

Then the required integral;

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i . f(2) - 2\pi i . f(1).$$

$$= 2\pi i(\underset{0}{\sin\pi 2^2} + \underset{1}{\cos\pi 2^2}) - 2\pi i(\underset{0}{\sin\pi 1^2} + \underset{-1}{\cos\pi 1^2})$$

$$= 2\pi i - (-2\pi i)$$

$$= 4\pi i.$$

(b) Let $f(z) = e^{2z}$ and $a = -1$ in the Cauchy integral formula,

$$f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \quad \text{———— (1)}$$

If $n = 3$; then

$$f'(z) = 2e^{2z}$$

$$f''(z) = 4e^{2z}$$

$$f'''(z) = 8e^{2z}$$

$$\therefore f^3(-1) = 8e^{-2}$$

Hence, equ. (1) becomes

$$8e^{-2} = \frac{3!}{2\pi i}\oint_C \frac{e^{2z}}{(z+1)^4}\,dz$$

$$\Rightarrow \oint_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{8e^{-2}\cdot \cancel{2}\pi i}{3\times \cancel{2}}$$

$$= \frac{8\pi i e^{-2}}{3}$$

$\therefore$ The required integral has the value $\frac{8\pi i e^{-2}}{3}$

(4) Evaluate $\int_C \frac{e^{-z}}{z+1}\,dz$, where C is the circle $|z| = \frac{1}{2}$

Solⁿ: Here, $f(z) = e^{-z}$ is an analytic function.

The point $z = -1$ lies outside the circle $|z| = \frac{1}{2}$.

$\therefore$ The function $\frac{e^{-z}}{z+1}$ is analytic within and on C.

By Cauchy's theorem, we have

$$\int \frac{e^{-z}}{z+1}\,dz = 0.$$

Evaluate $\int_C \frac{3z^2+z}{z^2-1}\,dz$, where C is the circle $|z-1|=1$.

Solⁿ: The integral has singularities, where $z^2-1=0$ i.e; at $z=1$ and $z=-1$.
The circle $|z-1|=1$ has center at $z=1$, $f(z)=3z^2+z$, is an analytic function.

Also, $$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$$

$$\therefore \int_C \frac{3z^2+z}{z^2-1}\,dz = \frac{1}{2}\int_C \frac{3z^2+z}{z-1}\,dz - \frac{1}{2}\int_C \frac{3z^2+z}{z+1}\,dz \quad \text{——— (1)}$$

$$= \frac{1}{2}\cdot 2\pi i\cdot f(1) - 0$$

$$= \frac{1}{2}\cdot 2\pi i\,(3\cdot 1^2+1) - 0$$

$$= \frac{1}{\cancel{2}}\cdot \cancel{2}\pi i\cdot 4$$

$$= 4\pi i$$

Evaluate $\oint_C \frac{\sin 3z}{z+\pi/2}\,dz$ if C is the circle $|z|=5$

Solⁿ: $|z|=5$ is a circle of radius 5.

From Cauchy's theorem; $a=-\pi/2$.

$$\therefore f(-\pi/2) = \frac{1}{2\pi i}\oint_C \frac{\sin 3z}{z+\pi/2}\,dz$$

$$\Rightarrow \oint_C \frac{\sin 3z}{z+\pi/2}\,dz = 2\pi i\cdot f(-\pi/2)$$

$$= 2\pi i\cdot \sin(3\cdot -\pi/2)$$

$$= 2\pi i\cdot 1$$

$$= 2\pi i.$$

The residue theorem and applications.

Laurent series:

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz \quad n = 0, \pm 1, \pm 2, \ldots \qquad \text{--------} (1)$$

With an appropriate change of notation, we can write the above as

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \ldots \qquad \text{--------} (2)$$

This is called a Laurent series or expansion.

Residues:

Let $f(z)$ be single valued and analytic inside and on a circle C except at the point $z=a$ chosen at the center of circle C. Then $f(z)$ has a Laurent's series about $z=a$ given by,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \ldots \qquad \text{--------} (1)$$

Where,

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz \; ; n = 0, \pm 1, \pm 2, \ldots \qquad \text{--------} (2)$$

In the special case, when $n=-1$

$$a_{-1} = \frac{1}{2\pi i}\oint_C f(z)\,dz$$

$$\Rightarrow \oint_C f(z)dz = 2\pi i\, a_{-1} \qquad \text{--------} (3)$$

This a_{-1} is known as the residue of $f(z)$ at $z=a$.

Calculation of residues:

If $z=a$ is a pole of order k, there is a simple formula for a_{-1} given by,

$$a_{-1} = \lim_{z\to a} \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left\{(z-a)^k f(z)\right\}$$

If $k=1$ (simple pole), then the result is especially simple and is given by,

$$a_{-1} = \lim_{z\to a}(z-a)\,f(z).$$

Residue theorem:

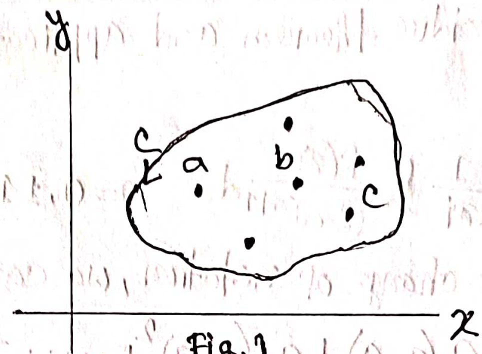


Fig. 1

Let $f(z)$ be single-valued and analytic inside and on a simple closed curve C except at the singularities $a, b, c, \ldots$ inside C, which have residues given by $a_{-1}, b_{-1}, c_{-1}, \ldots$ [in Fig. 1]

Then the residue theorem states that,

$$\oint_C f(z)\,dz = 2\pi i\,(a_{-1} + b_{-1} + c_{-1} + \cdots)$$

i.e; the integral of $f(z)$ around C is $2\pi i$ times the sum of the residues of $f(z)$ at the singularities enclosed by C.

Find the residues of (a) $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ and (b) $f(z) = e^z \operatorname{cosec}^2 z$ at all its poles in the finite plane.

Soln: (a) Here, $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+4)}$

Now, $f(z)$ has three poles; $z=-1$ is a pole of order 2.

$z^2+4=0 \Rightarrow z = \pm 2i$ are simple pole.

Residue at $z=-1$:

$$\lim_{z\to -1} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \left\{ (z+1)^2 \cdot \frac{z^2-2z}{(z+1)^2(z^2+4)} \right\}$$

$$= \lim_{z\to -1} \frac{d}{dz}\left(\frac{z^2-2z}{z^2+4}\right)$$

$$= \lim_{z\to -1} \frac{(z^2+4)\frac{d}{dz}(z^2-2z) - (z^2-2z)\frac{d}{dz}(z^2+4)}{(z^2+4)^2}$$

$$= \lim_{z\to -1} \frac{(z^2+4)(2z-2) - (z^2-2z)\cdot 2z}{(z^2+4)^2}$$

$$= \frac{-20+6}{25}$$

$$= -14/25$$

Residue at $z=2i$:

$$\lim_{z\to 2i}\left\{(z-2i)\cdot\frac{(z^2-2z)}{(z+1)^2(z-2i)(z+2i)}\right\}$$

$$=\lim_{z\to 2i}\left\{\frac{z^2-2z}{(z+1)^2(z+2i)}\right\}$$

$$=\frac{-4-4i}{(2i+1)^2\cdot 4i}=\frac{7+i}{25}$$

Residue at $z=-2i$:

$$\lim_{z\to -2i}\left\{(z+2i)\cdot\frac{(z^2-2z)}{(z+1)^2(z-2i)(z+2i)}\right\}$$

$$=\lim_{z\to -2i}\left\{\frac{(z^2-2z)}{(z+1)^2(z-2i)}\right\}$$

$$=\frac{-4+4i}{(-2i+1)^2\cdot(-4i)}=\frac{7-i}{25}$$

(b) Here, $f(z)=e^z\,\mathrm{cosec}^2 z=\frac{e^z}{\sin^2 z}$ has double Poles at $z=0,\pm\pi,\pm 2\pi,\ldots$ i.e; $z=\pm m\pi$

Where, $m=0,\pm 1,\pm 2,\ldots$

Residue at $z=m\pi$ is,

$$\lim_{z\to m\pi}\frac{1}{1!}\frac{d}{dz}\left\{(z-m\pi)^2\cdot\frac{e^z}{\sin^2 z}\right\}$$

$$=\lim_{z\to m\pi}\frac{e^z\left[(z-m\pi)^2\sin z+2(z-m\pi)\sin z-2(z-m\pi)^2\cos z\right]}{\sin^3 z}$$

Letting $z-m\pi=u$ or; $z=u+m\pi$, this limit can be written

$$=\lim_{u\to 0}e^{u+m\pi}\left\{\frac{u^2\sin u+2u\sin u-2u^2\cos u}{\sin^3 u}\right\}$$

$$=e^{m\pi}\left\{\lim_{u\to 0}\frac{u^2\sin u+2u\sin u-2u^2\cos u}{\sin^3 u}\right\}$$

This limit in brackets can be obtained using L'Hospital's rule. However, it is clear to first note that,

$$\lim_{u\to 0}\frac{u^3}{\sin^3 u}=\lim_{u\to 0}\left(\frac{u}{\sin u}\right)^3=1$$

and thus write the limit as

$$e^{m\pi} \lim_{u\to 0}\left(\frac{u^2\sin u + 2u\sin u - 2u^2\cos u}{u^3} \cdot \frac{u^3}{\sin^3 u}\right)$$

$$= e^{m\pi} \lim_{u\to 0} \frac{u^2\sin u + 2u\sin u - 2u^2\cos u}{u^3}$$

$$= e^{m\pi}.$$

Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz$ around the circle C with equation $|z|=3$.

Soln: Here, the integral, $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz$ has three poles,

$z=0$ is a pole of order 2. and, $z^2+2z+2=0$

$\Rightarrow z=\frac{-2\pm\sqrt{-4}}{2} = -1\pm i$ are simple poles.

Residue at $z=0$ is,

$$\lim_{z\to 0} \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left\{z^2 \cdot \frac{e^{zt}}{z^2(z^2+2z+2)}\right\}$$

$$= \lim_{z\to 0} \frac{d}{dz}\left\{\frac{e^{zt}}{z^2+2z+2}\right\}$$

$$= \lim_{z\to 0} \frac{(z^2+2z+2)(te^{zt}) - (e^{zt})(2z+2)}{(z^2+2z+2)^2}$$

$$= \frac{2t-2}{4} = \frac{t-1}{2}$$

Residue at $z=-1+i$ is,

$$\lim_{z\to -1+i}\left\{[z-(-1+i)] \cdot \frac{e^{zt}}{z^2(z^2+2z+2)}\right\}$$

$$\Rightarrow \lim_{z\to -1+i}\left\{\frac{e^{zt}}{z^2}\right\} \cdot \lim_{z\to -1+i}\left\{\frac{z+1-i}{z^2+2z+2}\right\}$$

$$= \frac{e^{(-1+i)t}}{(-1+i)^2} \cdot \frac{1}{2i} = \frac{e^{(-1+i)t}}{4}$$

Residue at $z=-1-i$ is,

$$\lim_{z\to -1-i}\left\{[z-(-1-i)] \frac{e^{zt}}{z^2(z^2+2z+2)}\right\}$$

$$= \frac{e^{(-1-i)t}}{4}$$

Then, by the residue theorem;

$$\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz = 2\pi i(\text{sum of residues})$$
$$= 2\pi i \left\{ \frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4} \right\}$$
$$= 2\pi i \left\{ \frac{t-1}{2} + \tfrac{1}{2} e^{-t} \cos t \right\}$$

that is, $$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz = \frac{t-1}{2} + \tfrac{1}{2} e^{-t}\cos t.$$

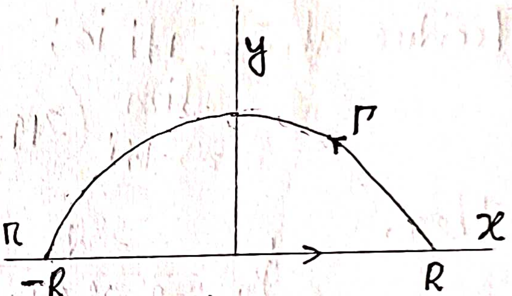


Fig.1

Evaluate $\int_0^\infty \frac{dx}{x^6+1}$.

Solⁿ:

Consider $\oint_C \frac{dz}{z^6+1}$, where C is the close contour of Fig.1 consisting of the line from $-R$ to R and the semicircle Γ, traversed in the positive (counter clockwise) sense.

Since $z^6+1=0$ when $z = e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}$, these are simple poles at $\frac{1}{(z^6+1)}$. Only the poles $e^{\pi i/6}, e^{3\pi i/6}$, and $e^{5\pi i/6}$ lie within C. Then, using L'Hospital's rule,

Residue at $e^{\pi i/6} = \lim_{z\to e^{\pi i/6}} \left\{ (z - e^{\pi i/6}) \frac{1}{(z^6+1)} \right\} = \lim_{z \to e^{\pi i/6}} \frac{1}{6z^5} = \tfrac{1}{6} e^{-5\pi i/6}$.

Residue at $e^{3\pi i/6} = \lim_{z\to e^{3\pi i/6}} \left\{ (z - e^{3\pi i/6}) \frac{1}{z^6+1} \right\} = \lim_{z \to e^{\frac{3\pi i}{6}}} \frac{1}{6z^5} = \tfrac{1}{6} e^{-\frac{5\pi i}{2}}$.

Residue at $e^{5\pi i/6} = \lim_{z\to e^{\frac{5\pi i}{6}}} \left\{ (z - e^{\frac{5\pi i}{6}}) \frac{1}{z^6+1} \right\} = \lim_{z \to e^{\frac{5\pi i}{6}}} \frac{1}{6z^5} = \tfrac{1}{6} e^{-\frac{25\pi i}{6}}$

Thus, $$\oint_C \frac{dz}{z^6+1} = 2\pi i \left\{ \frac{1}{6} e^{-\frac{5\pi i}{6}} + \tfrac{1}{6} e^{-\frac{5\pi i}{2}} + \tfrac{1}{6} e^{-\frac{25\pi i}{6}} \right\}$$
$$= \frac{2\pi}{3}.$$

That is, $$\int_{-R}^{R} \frac{dx}{x^6+1} + \int_\Gamma \frac{dz}{z^6+1} = \frac{2\pi}{3} \quad \text{——— (1)}$$

Taking the limit of both sides of equ.(1) as $R\to\infty$ and we have,

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{x^6+1} = \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3} \quad \text{——— (2)}$$

Since, $$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = 2\int_0^\infty \frac{dx}{x^6+1}$$

the required integral has the value $\pi/3$

■ Show that $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}$

Soln: The poles of $z^2/(z^2+1)^2(z^2+2z+2)$ enclosed by the contour C of Fig. 1 are $z=i$ of order 2 and $z=-1+i$ of order 1.

Residue at $z=i$ is;

$$\lim_{z\to i} \frac{d}{dz}\left\{(z-i)^2 \frac{z^2}{(z+i)^2(z-i)^2(z^2+2z+2)}\right\} = \frac{9i-12}{100}$$

Residue at $z=-1+i$ is;

$$\lim_{z\to -1+i} (z+1-i) \frac{z^2}{(z^2+1)^2(z+1-i)(z+1+i)} = \frac{3-4i}{25}$$

Then,

$$\oint_C \frac{z^2 dz}{(z^2+1)^2(z^2+2z+2)} = 2\pi i\left(\frac{9i-12}{100} + \frac{3-4i}{25}\right) = \frac{7\pi}{50}$$

or, $$\int_{-R}^{R} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} + \int_{\Gamma} \frac{z^2 dz}{(z^2+1)^2(z^2+2z+2)} = \frac{7\pi}{50}.$$

■ Show that $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta = \frac{\pi}{12}$.

Soln: Let $z=i\theta$, then $\cos\theta = (z+z^{-1})/2$, $\cos 3\theta = (z^3+z^{-3})/2$, $dz = iz\,d\theta$ so that

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta = \oint_C \frac{(z^3+z^{-3})/2}{5-4(z+z^{-1})/2} \frac{dz}{iz} = -\frac{1}{2i}\oint_C \frac{z^6+1}{z^3(2z-1)(z-2)} dz$$

The integrand has a pole of order 3 at $z=0$ and a simple pole $z=\frac{1}{2}$.

Residue at $z=0$ is,

$$\lim_{z\to 0} \frac{1}{2!} \frac{d^2}{dz^2}\left\{z^3 \cdot \frac{z^6+1}{z^3(2z-1)(z-2)}\right\} = \frac{21}{8}$$

Residue at $z=\frac{1}{2}$ is,

$$\lim_{z\to 1/2} \left\{(z-\tfrac{1}{2}) \cdot \frac{z^6+1}{z^3(2z-1)(z-2)}\right\} = -\frac{65}{24}$$

Then, $$-\frac{1}{2i}\oint_C \frac{z^6+1}{z^3(2z-1)(z-2)} dz = -\frac{1}{2i}(2\pi i)\left\{\frac{21}{8} - \frac{65}{24}\right\} = \frac{\pi}{12}$$ as required.