# 1st Chapter : Special Function

## Beta function:

The function denoted by $\beta(m,n)$, where m and n are positive values $[m,n>0]$ and it defined by

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \;;$$

is known as Beta function. It is also called the first Eulerian integral.

Some other form of Beta function

(i) $$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

(ii) $$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Properties of Beta function:

## Symmetric function of beta function:

We know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \text{——— (i)}$$

If we let,

$$1-x = y$$

$\Rightarrow x = 1 - y$

$\therefore dx = -dy$

| $x$ | 0 | 1 |
|---|---|---|
| $y$ | 1 | 0 |

Putting the value in equation (i),

$$\beta(m,n) = \int_1^0 (1-y)^{m-1} \cdot y^{n-1} (-dy)$$

$$= \int_0^1 (1-y)^{m-1} \cdot y^{n-1} \, dy$$

$$= \beta(n,m)$$

Therefore,

$$\beta(m,n) = \beta(n,m)$$

This is the symmetric function of Beta function.

## Another form of Beta function:

We know,

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{m-1} \, dx \quad \text{——— (i)}$$

No need

If we let,

$x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta \, d\theta$

Limit:

| $x$ | 0 | 1 |
|---|---|---|
| $\theta$ | 0 | $\pi/2$ |

Wall's formula তে আছে

Putting the value in equation (i),

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (\sin^2\theta)^{m-1} \cdot (\cos^2\theta)^{n-1} \sin\theta\cos\theta\, d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$$

Sir

Prove that, (Walis formula)

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$$

Soln:

We know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \quad \text{——— (i)}$$

Let,

$x = \sin^2\theta$

$\Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

| $x$ | 0 | 1 |
|---|---|---|
| $\theta$ | 0 | $\pi/2$ |

Putting this value in the eqn (i), we get,

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \sin\theta\cos\theta \, d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\Rightarrow \frac{\beta(m,n)}{2} = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$ [Another form of Beta function]

Let, $2m-1 = p$ and, $2n-1 = q$

$\therefore m = \frac{p+1}{2}$ $\quad\quad$ $\therefore n = \frac{q+1}{2}$

and we know,

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

Therefore,

$$\frac{\Gamma m \, \Gamma n}{2\,\Gamma(m+n)} = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\Rightarrow \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)} = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$ [Another formula of Beta function]

$$\Rightarrow \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\left(\frac{p+1+q+1}{2}\right)} = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$$\therefore \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\left(\frac{p+q+2}{2}\right)} = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

## Another form of Beta function:

We know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put, $x = \frac{y}{1+y}$

$$dx = \frac{(1+y)1 - y\cdot 1}{(1+y)^2}\,dy$$

$$= \frac{1}{(1+y)^2}\,dy$$

Here, $x = \frac{y}{1+y}$

$\Rightarrow y = x + xy$

$\Rightarrow y - xy = x$

$\Rightarrow y(1-x) = x$

$\Rightarrow y = \frac{x}{1-x}$ (1)

| $x$ | 0 | 1 |
|---|---|---|
| $y$ | 0 | $\infty$ |

Putting the value in eq$^n$ (i)

$$\beta(m,n) = \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1}\left(1-\frac{y}{1+y}\right)^{n-1}\frac{1}{(1+y)^2}\cdot dy$$

$$= \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1}\cdot\left(\frac{1}{1+y}\right)^{n-1}\cdot\frac{1}{(1+y)^2}\,dy$$

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\cdot dy$$

We can also write,

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

limit ঠিক থাকলে চলক পরিবর্তন করলে আলোক পরিবর্তন হয় না

## Gamma function :

The function, ~~is~~ denoted by $\Gamma n$ and it defined by $\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1} dx$ ; where, $n > 0$ is known as gamma function.

It is also called second Eulerian integral.

## Properties of Beta function and Gamma function.

(1) Prove that $\Gamma(n+1) = n\Gamma n$

**Soln:-**

We know,

$$\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1} dx$$

$$\therefore \Gamma n+1 = \int_0^\infty e^{-x} \cdot x^{n+1-1} dx$$

$$= \int_0^\infty e^{-x} \cdot x^n dx$$

$$\cancel{= \left[x^n \int e^{-x} dx\right]_0^\infty - \int_0^n n x^{n-1} (-e^{-x}) dx}$$

$$= \left[-x^n e^{-x}\right]_0^\infty + \int_0^\infty n x^{n-1} \cdot e^{-x} dx$$

$$= 0 + n \int_0^\infty e^{-x} \cdot x^{n-1} dx$$

$$\therefore \Gamma n+1 = n \Gamma n$$

Gamma function is known as generalized factorial function,

A quick derivation of Euler's Integrals equation, we know,

$$n! \text{ or } \lfloor n = \int_0^\infty e^{-x} \cdot x^n \, dx \qquad \text{——— (i)}$$

Again, the equation of gamma function,

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx$$

$$\therefore \Gamma n+1 = \int_0^\infty e^{-x} \cdot x^{n+1-1} \, dx$$

$$= \int_0^\infty e^{-x} \cdot x^n \, dx$$

$$\Gamma n+1 = \lfloor n \quad \text{proved}$$

Factorial function to Gamma function, $\lfloor n = \Gamma n+1$

The equation of gamma function is,

$$\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1} \, dx$$

A quick derivation of Eular's equation, we know that

$$\lfloor n = \int_0^\infty e^{-x} \cdot x^n \, dx$$

$$\therefore \lfloor n-1 = \int_0^\infty e^{-x} \cdot x^{n-1} \, dx = \Gamma n$$

If, $n = n+1$, then,

$$\lfloor n+1-1 = \Gamma n+1$$

$$\therefore \lfloor n = \Gamma n+1$$

Sir

Prove that, $\Gamma(n+1) = n! = \lfloor n = n\,\Gamma n$

Here we get,

$$\Gamma(n+1) = n\,\Gamma n$$

When, $n = 1$ ; we get,

$$\Gamma 2 = 1\,\Gamma 1 = 1 \cdot 1 = 1!$$

When, $n = 2$ ; we get,

$$\Gamma 3 = 2\,\Gamma 2 = 2 \cdot 1 = 2!$$

When, $n = 3$ , we get,

$$\Gamma 4 = 3\,\Gamma 3 = 3 \cdot 2 \cdot 1 = 3!$$

... ... ...
... ... ...
... ... ...

Therefore,

$$\Gamma n+1 = (n+1-1)! = n!$$

Hence,

$$\Gamma n+1 = n! = \lfloor n = n\,\Gamma n$$ (proved)

Prove that $\Gamma n = (n-1)\Gamma n-1$

Soln:-

We know,

$$\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx$$

$$\therefore \Gamma n+1 = \int_0^\infty e^{-x} \cdot x^{n+1-1} \cdot dx$$

$$= \int_0^\infty e^{-x} \cdot x^n\, dx$$

$$= \left[-x^n \cdot e^{-x}\right]_0^\infty - n\int_0^\infty e^{-x} \cdot x^{n-1}\, dx$$

$$= 0 - n\Gamma n$$

$$\therefore \Gamma n+1 = n\Gamma n$$

Replacing $n$ by $n-1$, we get,

$$\Gamma n-1+1 = (n-1)\Gamma n-1$$

$$\therefore \Gamma n = (n-1)\Gamma n-1 \quad \text{proved}$$

If, $n = n-2$;

$$\Gamma n-1 = (n-2)\Gamma n-2$$

If $n = n-3$

$$\Gamma n-2 = (n-3)\Gamma n-3$$

Lit

Prove that $\frac{\Gamma n}{\lambda^n} = \int_0^\infty e^{-\lambda y} \cdot y^{n-1}\, dy$

Soln:-

We know that,

$$\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx \quad ; [n > 0] \quad \text{—(i)}$$

Let,

$x = \lambda y$

$\therefore dx = \lambda\, dy$

| $x$ | $0$ | $\infty$ |
|---|---|---|
| $y$ | $0$ | $\infty$ |

Putting these value in eqn (i),

$$\Gamma n = \int_0^\infty e^{-\lambda y} \cdot (\lambda y)^{n-1} \cdot \lambda\, dy$$

$$= \int_0^\infty e^{-\lambda y} \cdot \lambda^{n-1} \cdot y^{n-1} \cdot \lambda \cdot dy$$

$$= \int_0^\infty e^{-\lambda y} \cdot y^{n-1} \cdot \lambda^{n-1+1} \cdot dy$$

$$\Rightarrow \Gamma n = \int_0^\infty e^{-\lambda y} \cdot y^{n-1} \cdot \lambda^n\, dy$$

$$\therefore \frac{\Gamma n}{\lambda^n} = \int_0^\infty e^{-\lambda y} \cdot y^{n-1}\, dy \quad \text{proved}$$

Prove that, (H.W)

(i) $\Gamma n = \int_0^1 \log\left(\frac{1}{y}\right)^{n-1} dy$

(ii) $\Gamma n = (m+1)^n \int_0^1 y^m \left(\log \frac{1}{y}\right)^{n-1} dy$

H.W Soln:-

Sir

We know that,

$$\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1} dx \quad \text{——— (i)}$$

Let,

$$e^{-x} = y$$

Limits:

| $x$ | $0$ | $\infty$ |
|---|---|---|
| $y$ | $1$ | $0$ |

$$\Rightarrow -e^{-x} dx = dy$$

$$\Rightarrow -y\, dx = dy$$

$$\Rightarrow dx = -\frac{1}{y} dy$$

$$\Rightarrow -\int dx = \int \frac{dy}{y}$$

$$\Rightarrow -x = \log y$$

$$\Rightarrow x = -\log y$$

$$\Rightarrow x = \log y^{-1}$$

$$\Rightarrow x = \log\left(\frac{1}{y}\right)$$

Now From, (i)

$$\Gamma n = \int_1^0 -y\left(\log\frac{1}{y}\right)^{n-1}\frac{dy}{y}$$

$$= \cancel{\int_0^1 \left\{\log\left(\frac{1}{y}\right)\right\}^{n-1} d}$$

$$\therefore \Gamma n = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy \quad \text{proved}$$

H.W

(ii) Solⁿ:-

We know that,

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \quad \text{——— (1)}$$

Let, $e^{-x} = y^{m+1}$

$$\Rightarrow -e^{-x} dx = (m+1) y^m dy$$

$$\Rightarrow -y^{m+1} dx = (m+1) y^m dy$$

$$\Rightarrow -dx = \frac{(m+1) y^m}{y^{m+1}} dy$$

$$\Rightarrow -\int dx = \int \frac{m+1}{y} dy$$

$$\Rightarrow -x = (m+1) \cancel{\log\frac{1}{y}} \log y$$

$$\Rightarrow x = -(m+1)\log y = (m+1)\log y^{-1}$$

$\Rightarrow x = (m+1) \log \frac{1}{y}$

Now, from eq$^n$ (i),

$$\Gamma n = -\int_1^0 y^{m+1} \left\{(m+1) \log \frac{1}{y}\right\}^{n-1} \frac{m+1}{y} dy$$

$$= \int_0^1 y^{m+1-1} \cdot (m+1)^{n-1} \left(\log \frac{1}{y}\right)^{n-1} \cdot (m+1) dy$$

$$\therefore \Gamma n = \int_0^1 y^m (m+1)^n \left(\log \frac{1}{y}\right)^{n-1} dy$$

$$\therefore \Gamma n = (m+1)^n \int_0^1 y^m \left(\log \frac{1}{y}\right)^{n-1} dy \quad \text{(proved)}$$

H.W Prove that $\pi(n) = \Gamma n+1$ [Gauss's pi function]

Sir

Soln:-

From gauss pi function we have,

$$\pi(n) = \int_0^\infty x^n e^{-x} dx$$

$$= \int_0^\infty e^{-x} \cdot x^{\{(n+1)-1\}} dx$$

$$\therefore \pi(n) = \Gamma n+1$$

Sir

The relation between Beta and gamma function

Or prove that,

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \quad ; \text{ where } m>0;\ n>0$$

Soln:-

We know that,

$$\Gamma m = \int_0^\infty e^{-x} \cdot x^{m-1} \, dx$$

and, $$\Gamma n = \int_0^\infty e^{-y} \cdot y^{n-1} \, dy = \int_0^\infty e^{-y} \cdot y^{n-1} \, dy$$

Now,

$$\Gamma m \cdot \Gamma n = \int_0^\infty e^{-x} \cdot x^{m-1} \, dx \cdot \int_0^\infty e^{-y} \cdot y^{n-1} \, dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x+y)} \cdot x^{(m-1)} \cdot y^{(n-1)} \cdot dx \, dy$$

Let,

$x+y = u$ and, $v = \frac{x}{x+y}$

Therefore,

$uv = x$ and, $y = u - uv$. $[\because y = u - x]$

$= u(1-v)$.

and, $dy\,dx = |J| \, du\,dv$

As $x$ and $y$ range $0$ to $\infty$

Therefore,

$u$ ranges $0$ to $\infty$

$v$ ranges $0$ to $1$

Here,

$$|J| = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right|$$

$$= \left| \begin{vmatrix} v & u \\ (1-v) & -u \end{vmatrix} \right|$$

$$= \left| \{-uv - u(1-v)\} \right|$$

$$= \left| \{-uv - u + uv\} \right|$$

$$= |-u|$$

$$= u$$

Therefore,

$$\Gamma m \cdot \Gamma n = \int_{u=0}^{\infty}\int_{v=0}^{1} e^{-u} \cdot (uv)^{m-1} \cdot \{u(1-v)\}^{n-1} \cdot |J| \, du\, dv$$

$$= \int_0^{\infty}\int_0^{1} e^{-u} \cdot u^{m-1} \cdot v^{m-1} \cdot u^{n-1} \cdot (1-v)^{n-1} \cdot u \, du\, dv$$

$$= \int_0^{\infty} e^{-u} \cdot u^{m-1} \cdot u^{n-1} \cdot u \, du \int_0^{1} v^{m-1} \cdot (1-v)^{n-1} dv$$

$$= \int_0^{\infty} e^{-u} \cdot u^{m+n-1} du \int_0^{1} v^{m-1} \cdot (1-v)^{n-1} dv$$

$$\therefore \Gamma m \Gamma n = \Gamma(m+n) \cdot \beta(m,n)$$

$$\Rightarrow \beta(m,n) \cdot \Gamma(m+n) = \Gamma m \cdot \Gamma n$$

$$\therefore \beta(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma(m+n)}$$

The relation between Gamma and Beta function,

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \quad [\text{prove}]$$

Soln:-

We know that from the Gamma function,

$$\frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} \cdot x^{n-1} dx$$

$$\Rightarrow \Gamma n = \int_0^\infty e^{-zx} \cdot x^{n-1} \cdot z^n dx$$

multiplying by $e^{-z}$ and $z^{m-1}$ both the side of the equation,

$$\Gamma n \, e^{-z} z^{m-1} = \int_0^\infty e^{-zx} \cdot e^{-z} z^{m-1} z^n x^{n-1} dx$$

Integrating w.r.t. $z$ on both side between the limit 0 to $\infty$

$$\Gamma n \int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty x^{n-1} dx \int_0^\infty e^{-z(x+1)} \cdot z^{m+n-1} dz$$

$$\Rightarrow \Gamma n . \Gamma m = \int_0^\infty x^{n-1} dx \cdot \frac{\Gamma m+n}{(x+1)^{m+n}}$$

$$\Rightarrow \frac{\Gamma n \, \Gamma m}{\Gamma m+n} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\Rightarrow \frac{\Gamma m \, \Gamma n}{\Gamma m+n} = \beta(m,n) \quad (\text{proved})$$

Sir

Prove that $\Gamma\frac{1}{2} = \sqrt{\pi}$

Soln:-

We know that,

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \quad \text{——— (i)}$$

and $$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta \quad \text{——— (ii)}$$

We can write the equation (i) and (ii),

$$\frac{\Gamma m \, \Gamma n}{\Gamma m+n} = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta \quad \text{——— (3)}$$

Let,
$$m = n = \frac{1}{2}$$

Putting this value in equation (3), we get,

$$\frac{\Gamma\frac{1}{2} \cdot \Gamma\frac{1}{2}}{\Gamma\frac{1}{2}+\frac{1}{2}} = 2\int_0^{\pi/2} \sin^{2\cdot\frac{1}{2}-1}\theta \cdot \cos^{2\cdot\frac{1}{2}-1}\theta \, d\theta$$

$$\Rightarrow \frac{\left(\Gamma\frac{1}{2}\right)^2}{1} = 2\int_0^{\pi/2} \sin^0\theta \cdot \cos^0\theta \, d\theta$$

$$\Rightarrow \frac{\left(\Gamma\frac{1}{2}\right)^2}{1} = 2\int_0^{\pi/2} 1 \cdot 1 \, d\theta$$

$$\Rightarrow \left(\Gamma\frac{1}{2}\right)^2 = 2\,[\theta]_0^{\pi/2}$$

$$\Rightarrow \left(\Gamma\frac{1}{2}\right)^2 = 2 \cdot \frac{\pi}{2}$$

$$\Rightarrow \left(\Gamma\frac{1}{2}\right)^2 = \pi$$

$$\therefore \Gamma\frac{1}{2} = \sqrt{\pi}$$ (proved)

Evaluate / Prove

① $\Gamma 3 = 2$ ④ $\Gamma -1/2$

② $\Gamma 4 = 6$ ⑤ $\Gamma 1/2$

③ $\Gamma 5 = 24$ ⑥ $\Gamma -3/2$

Soln:-

(i) We know that,

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\therefore \Gamma 3 = \int_0^\infty e^{-x} \cdot x^{3-1} dx$$

$$= \int_0^\infty e^{-x} \cdot x^2 dx$$

$$= \left[-x^2 e^{-x}\right]_0^\infty + \int_0^\infty 2x \cdot e^{-x} dx$$

$$= 0 + 2\int_0^\infty x e^{-x} dx$$

$$= 2\left\{\left[-x e^{-x}\right]_0^\infty + \int_0^\infty e^{-x} dx\right\}$$

$$= 2\left\{0 + \int_0^\infty e^{-x} dx\right\}$$

$$= 2 \cdot \left[-e^{-x}\right]_0^\infty$$

$$= 2 \cdot \left[-(0-1)\right]$$

$$= 2$$

(ii) We know that,

$$\Gamma n = \int_0^\alpha e^{-x} \cdot x^{n-1} dx$$

$$\therefore \Gamma 4 = \int_0^\alpha e^{-x} \cdot x^{4-1} dx$$

$$= \int_0^\alpha e^{-x} \cdot x^3 dx$$

$$= \left[-x^3 e^{-x}\right]_0^\alpha + 3\int_0^\alpha x^2 e^{-x}$$

$$= 0 + 3\left\{\left[-x^2 \cdot e^{-x}\right]_0^\alpha + 2\int_0^\alpha x e^{-x} dx\right\}$$

$$= (3 \times 0) + 3 \times 2\left\{\left[-x e^{-x}\right]_0^\alpha + \int_0^\alpha e^{-x} dx\right\}$$

$$= (3 \times 2 \times 0) + 3 \times 2 \int_0^\alpha e^{-x} dx$$

$$= 6\left[-e^{-x}\right]_0^\alpha$$

$$= 6 \times (-0 + 1)$$

$$= 6$$

(iii) We know that,

$$\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx$$

$$\Gamma 5 = \int_0^\infty e^{-x} \cdot x^4\, dx$$

$$= \left[-x^4 e^{-x}\right]_0^\infty + 4\int_0^\infty x^3 e^{-x}\, dx$$

$$= 4\left\{\left[-x^3 e^{-x}\right]_0^\infty + 3\int_0^\infty x^2 e^{-x}\, dx\right\}$$

$$= (4\times 3)\left\{\left[-x^2 e^{-x}\right]_0^\infty + 2\int_0^\infty x e^{-x}\, dx\right\}$$

$$= (4\times 3\times 2)\left\{\left[-x e^{-x}\right]_0^\infty + \int_0^\infty e^{-x}\, dx\right\}$$

$$= (4\times 3\times 2)\left[-e^{-x}\right]_0^\infty$$

$$= (4\times 3\times 2)\left[-0+1\right]$$

$$= 4\times 3\times 2\times 1$$

$$= 24$$

Sol (iv) We know

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

Hence, $n = -\frac{1}{2}$

We know that,

$$\Gamma n+1 = n\Gamma n$$

$$\Rightarrow \Gamma\left(-\tfrac{1}{2}+1\right) = -\tfrac{1}{2}\Gamma -\tfrac{1}{2}$$

$$\cancel{\Rightarrow \Gamma \tfrac{1}{2} = 2}$$

$$\Rightarrow \Gamma \tfrac{1}{2} = -\tfrac{1}{2}\Gamma -\tfrac{1}{2}$$

$$\Rightarrow 2\Gamma \tfrac{1}{2} = -\Gamma -\tfrac{1}{2}$$

$$\Rightarrow \Gamma -\tfrac{1}{2} = -2\Gamma \tfrac{1}{2}$$

$$\boxed{\therefore \Gamma -\tfrac{1}{2} = -2\sqrt{\pi}}$$

~~(v) We know~~

(v) We know,

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$ (see previous questions)

Let,

$$m = n = \frac{1}{2}$$

Sir

(vi) we know that,

$$\Gamma(n+1) = n\Gamma n$$

Let, $n = -3/2$

Now,

$$\Gamma\left(-\tfrac{3}{2}+1\right) = -\tfrac{3}{2}\,\Gamma\left(-\tfrac{3}{2}\right)$$

$$\Rightarrow \Gamma\left(\frac{-3+2}{2}\right) = -\tfrac{3}{2}\,\Gamma\left(-\tfrac{3}{2}\right)$$

$$\Rightarrow \Gamma\left(-\tfrac{1}{2}\right) = -\tfrac{3}{2}\,\Gamma\left(-\tfrac{3}{2}\right)$$

$$\Rightarrow -2\sqrt{\pi} = -\tfrac{3}{2}\,\Gamma\left(-\tfrac{3}{2}\right)$$

$$\Rightarrow \tfrac{3}{2}\,\Gamma\left(-\tfrac{3}{2}\right) = 2\sqrt{\pi}$$

$$\Rightarrow \Gamma\left(-\tfrac{3}{2}\right) = \frac{4}{3}\sqrt{\pi}$$

Gire

Prove that,

$$\int_0^1 x^4 (1-x)^3 \, dx = \frac{1}{280}$$

Soln:-

$$L.H.S = \int_0^1 x^4 (1-x)^3 \, dx$$

$$= \int_0^1 x^{5-1} \cdot (1-x)^{(4-1)} \, dx$$

$$= \beta(5, 4)$$

$$= \frac{\Gamma 5 \cdot \Gamma 4}{\Gamma 5+4}$$

$$= \frac{4! \times 3!}{\Gamma 9}$$

$$= \frac{4! \times 3!}{8!}$$

$$= \frac{\cancel{3} \times \cancel{2} \times 1}{8 \times \cancel{7} \times \cancel{6}^{2} \times 5}$$

$$= \frac{1}{280}$$

$= R.H.S$ (proved)

## Evaluate the following functions.

1. $\int_0^{\pi/2} \sin^6 x \, dx$

2. $\int_0^{\pi/2} \cos^5 x \sin^4 x \, dx$

3. $\int_0^{\pi/2} \sin^5 \theta \cos^5 \theta \, d\theta$

4. $\int_0^{\pi/2} \cos^9 \theta \, d\theta$

5. $\int_0^{\pi/2} x^6 \sqrt{1-x^2} \, dx$

6. $\int_0^{1} x^3 (1-x)^3 dx$

7. $\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta$

8. $\int_0^{\pi/2} \sqrt{\cot\theta} \, d\theta$

9. $\int_0^{1} \frac{35 x^3}{82\sqrt{1-x}} dx$

10. $\int_0^{1} (1-x^n)^{1/n} dx$

## Prove that,

1. $\int_0^1 \frac{dx}{\sqrt{1-x^n}} dx = \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma(1/n)}{\Gamma(1/n + 1/2)}$

2. $\int_0^1 x^{m-1} (1-x^a)^n dx = \frac{1}{a} \frac{n! \, \Gamma(m/a)}{\Gamma(m/a + n + 1)}$

Sol$^n$:

Sir

(i) We know that,

$$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

$$\therefore \int_0^{\pi/2} \sin^6 x \, dx = \int_0^{\pi/2} \sin^6 x \cos^0 x \, dx$$

$$= \frac{\Gamma\frac{6+1}{2}\,\Gamma\frac{0+1}{2}}{2\,\Gamma\frac{6+0+2}{2}}$$

$$= \frac{\Gamma\frac{7}{2}\,\Gamma\frac{1}{2}}{2\,\Gamma 4}$$

$$= \frac{\Gamma\frac{5}{2}+1\,\Gamma\frac{1}{2}}{2\,\Gamma 3+1}$$

$$= \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}\cdot\Gamma\frac{1}{2}}{2\cdot 3\cdot 2\cdot 1}$$

$$= \frac{15\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{96}$$

$$= \frac{15\sqrt{\pi}\cdot\sqrt{\pi}}{96} = \frac{5\sqrt{\pi}\cdot\sqrt{\pi}}{32}$$

$$\cancel{= \frac{15}{48}\pi} \quad = \frac{5}{32}\pi$$

Sir

(ii) We know that,

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$\therefore \int_0^{\pi/2} \cos^5 x \sin^4 x \, dx = \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{2\,\Gamma\left(\frac{4+5+2}{2}\right)}$$

$$= \frac{\Gamma(5/2)\cdot\Gamma(3)}{2\,\Gamma(11/2)}$$

$$= \frac{\Gamma(3/2+1)\cdot\Gamma(2+1)}{2\,\Gamma(9/2+1)}$$

$$= \frac{3/2\cdot 1/2\cdot\Gamma(1/2)\cdot 2\cdot 1}{2\cdot 9/2\cdot 7/2\cdot 5/2\cdot 3/2\cdot 1/2\cdot\Gamma(1/2)}$$

$$= \frac{8}{315}$$ Ans

(iii) We know,

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$\therefore \int_0^{\pi/2} \sin^5\theta \cos^5\theta \, d\theta = \frac{\Gamma\left(\frac{5+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{2\,\Gamma\left(\frac{5+5+2}{2}\right)}$$

$$= \frac{\Gamma 3 \, \Gamma 3}{2 \Gamma 6}$$

$$= \frac{2! \times 2!}{2 \times 5!}$$

$$= \frac{2! \times 2!}{2 \times 5 \times 4 \times 3 \times 2!}$$

$$= \frac{1}{60}$$

Ans

(4) We know,

Sir

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}}$$

$$\therefore \int_0^{\pi/2} \cos^q\theta \, d\theta = \int_0^{\pi/2} \sin^0\theta \cos^q\theta \, d\theta$$

$$= \frac{\Gamma\frac{0+1}{2} \, \Gamma\frac{q+1}{2}}{2\Gamma\frac{0+q+2}{2}}$$

$$= \frac{\Gamma\frac{1}{2} \, \Gamma\frac{q+1}{2}}{2\Gamma\frac{q+2}{2}}$$

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\frac{q+1}{2}}{\Gamma\frac{q+2}{2}}$$

Ans

Sin

(5) Here, $\int_0^1 x^6 \sqrt{1-x^2}\, dx$

Let, $x^2 = \sin^2\theta \Rightarrow x = \sin\theta$

$2x\, dx = 2\sin\theta \cos\theta\, d\theta$

$\Rightarrow 2\sin\theta\, dx = 2\sin\theta \cos\theta\, d\theta$

$\Rightarrow dx = \cos\theta\, d\theta$

| $x$ | 0 | 1 |
|---|---|---|
| $\theta$ | 0 | $\pi/2$ |

Now, $\int_0^1 x^6 \sqrt{1-x^2}\, dx$

$= \int_0^{\pi/2} \sin^6\theta \sqrt{1-\sin^2\theta}\, \cancel{d\theta} \cos\theta\, d\theta$

$= \int_0^{\pi/2} \sin^6\theta \cdot \cos\theta \cdot \cos\theta\, d\theta$

$= \int_0^{\pi/2} \sin^6\theta \cdot \cos^2\theta\, d\theta$

$= \dfrac{\Gamma\left(\frac{6+1}{2}\right) \Gamma\left(\frac{2+1}{2}\right)}{2\,\Gamma\left(\frac{6+2+2}{2}\right)}$

$= \dfrac{\Gamma\left(\frac{7}{2}\right) \Gamma\left(\frac{3}{2}\right)}{2\,\Gamma(5)}$

$= \dfrac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)}{2 \cdot 4!}$

$= \dfrac{15 \sqrt{\pi} \cdot \sqrt{\pi}}{2^4 \cdot 2 \cdot 4 \times 3 \times 2 \times 1}$

$= \dfrac{5}{256} \pi$ Ans

Sir

(6) Here,

$$\int_0^1 x^3(1-x)^3\,dx = \int_0^1 x^{(4-1)}(1-x)^{(4-1)}\,dx$$

$$= \beta(4,4)$$

$$= \frac{\Gamma 4\,\Gamma 4}{\Gamma 4+4}$$

$$= \frac{\Gamma 4\,\Gamma 4}{\Gamma 8}$$

$$= \frac{3!\times 3!}{7!}$$

$$= \frac{3\times 2\times 1\times 3!}{7\times 6\times 5\times 4\times 3!}$$

$$= \frac{1}{140}$$

Sir

(7) Here,

$$\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2} \left(\frac{\sin\theta}{\cos\theta}\right)^{1/2} d\theta$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^{-1/2}\theta\,d\theta$$

$$= \frac{\Gamma\left(\frac{1/2+1}{2}\right)\Gamma\left(\frac{-1/2+1}{2}\right)}{2\,\Gamma\left(\frac{1/2-1/2+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{3/2}{2}\right)\Gamma\left(\frac{1/2}{2}\right)}{2\,\Gamma\left(\frac{2}{2}\right)}$$

$$= \frac{\Gamma(3/4)\,\Gamma(1/4)}{2\,\Gamma(1)} = \frac{1}{2}\cdot\Gamma(1/4)\cdot\Gamma(3/4)$$ Ans

$$= \frac{1}{2}\cdot\Gamma(1/4)\cdot\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{2}\cdot\frac{\pi}{\sin\pi/4}$$

$$= \frac{\pi}{2\times\frac{1}{\sqrt{2}}}$$

$$= \frac{\pi}{\sqrt{2}}$$ Ans

Sin

(8) Here,

$$\int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta = \int_0^{\pi/2}\left(\frac{\cos\theta}{\sin\theta}\right)^{1/2}d\theta$$

$$= \int_0^{\pi/2}\sin^{-1/2}\theta\cdot\cos^{1/2}\theta\,d\theta$$

$$= \frac{\Gamma\left(\frac{-1/2+1}{2}\right)\cdot\Gamma\left(\frac{1/2+1}{2}\right)}{2\,\Gamma\left(\frac{-1/2+1/2+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\,\Gamma(1)}$$

$$= \frac{1}{2}\,\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{2}\,\frac{\pi}{\sin \pi/4}$$

$$= \frac{1}{2}\cdot\frac{\pi}{1/\sqrt{2}}$$

$$= \frac{\pi}{\sqrt{2}}$$ Ans

Sir

(9) $$\int_0^1 \frac{35\,x^3}{32\sqrt{1-x}}\,dx$$

Let,

$x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$

Limits:

| $x$ | 0 | 1 |
|---|---|---|
| $\theta$ | 0 | $\pi/2$ |

$$\therefore \int_0^1 \frac{35}{32}\cdot\frac{x^3}{\sqrt{1-x}}\,dx$$

$$= \int_0^{\pi/2} \frac{35}{32}\cdot\frac{(\sin^2\theta)^3}{\sqrt{1-\sin^2\theta}}\cdot 2\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \frac{35}{32} \cdot \frac{\sin^6\theta}{\sqrt{\cos^2\theta}} \cdot 2\sin\theta\cos\theta \, d\theta$$

$$= \frac{35}{16} \int_0^{\pi/2} \frac{\sin^7\theta}{\cos\theta} \cdot \cos\theta \, d\theta$$

$$= \frac{35}{16} \int_0^{\pi/2} \sin^7\theta \cdot \cos^0\theta \, d\theta$$

$$= \frac{35}{16} \frac{\Gamma\left(\frac{7+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{7+0+2}{2}\right)}$$

$$= \frac{35}{16} \frac{\Gamma(4)\,\Gamma(1/2)}{2\,\Gamma(9/2)}$$

$$= \frac{35}{32} \times \frac{3!\,\Gamma(1/2)}{\Gamma(7/2+1)}$$

$$= \frac{35}{32} \times \frac{3!\,\Gamma(1/2)}{7/2 \cdot 5/2 \cdot 3/2 \cdot 1/2 \cdot \Gamma(1/2)}$$

$$= \frac{35}{16} \times \frac{3 \times 2 \times 1}{7/2 \times 5/2 \times 3/2}$$

$$= \frac{16}{16}$$

$$= 1$$

sir

(10) Here,

$$\int_0^1 (1-x^n)^{1/n}\, dx$$

Let,

$x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

| $x$ | 0 | 1 |
|---|---|---|
| $\theta$ | 0 | $\pi/2$ |

Therefore,

$$\int_0^1 (1-x^n)^{1/n}\, dx$$

$$= \int_0^{\pi/2} \left\{1-(\sin^n\theta)^2\right\}^{1/n} 2\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} (\cos^{2n}\theta)^{1/n}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \cos^2\theta \cdot \sin\theta \cdot \cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin\theta \cdot \cos^3\theta\, d\theta$$

$$= 2\,\frac{\Gamma\left(\frac{1+1}{2}\right)\Gamma\left(\frac{3+1}{2}\right)}{2\,\Gamma\left(\frac{1+3+2}{2}\right)}$$

$$= \frac{\Gamma 1\,\Gamma 2}{\Gamma 3} = \frac{\Gamma 2}{\Gamma(2+1)} = \frac{\Gamma 2}{2\Gamma 2} = \frac{1}{2}$$

Sir

1. ~~Here~~ Show that, $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma(1/n)}{\Gamma(1/n+1/2)}$

Soln:

Let,

$$x^n = \sin^2\theta \quad \Rightarrow \quad x = \sin^{2/n}\theta$$

$$\Rightarrow n x^{n-1} dx = 2\sin\theta\cos\theta \, d\theta$$

$$\Rightarrow n x^n . x^{-1} . dx = 2\sin\theta\cos\theta \, d\theta$$

$$\Rightarrow n\sin^2\theta \, \sin^{-2/n}\theta \, dx = 2\sin\theta\cos\theta \, d\theta$$

$$\Rightarrow dx = \frac{2}{n}\sin^{2/n}\theta \, \sin^{-2}\theta \, \sin\theta\cos\theta \, d\theta$$

$$\Rightarrow dx = \frac{2}{n}\sin^{(2/n-2+1)}\theta \cos\theta \, d\theta$$

$$\Rightarrow dx = \frac{2}{n}\sin^{(2/n-1)}\theta \cos\theta \, d\theta$$

| $x$ | 0 | 1 |
|---|---|---|
| $\theta$ | 0 | $\pi/2$ |

Now,

$$L.H.S = \int_0^1 \frac{dx}{\sqrt{1-x^n}}$$

$$= \int_0^{\pi/2} \frac{\frac{2}{n}\sin^{(2/n-1)}\theta \cos\theta \, d\theta}{\sqrt{1-\sin^2\theta}}$$

$$= \int_0^{\pi/2} \frac{2}{n} \cdot \frac{\sin^{(2/n-1)}\theta}{\cos\theta} \cdot \cos\theta \, d\theta$$

$$= \frac{2}{n}\int_0^{\pi/2} \sin^{(2/n-1)}\theta \cdot \cos^0\theta \, d\theta$$

$$= \frac{2}{n} \frac{\Gamma\left(\frac{2/n - 1 + 1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{2/n - 1 + 0 + 2}{2}\right)}$$

$$= \frac{1}{n} \frac{\Gamma(1/n) \cdot \Gamma(1/2)}{\Gamma\left(\frac{2/n + 1}{2}\right)}$$

$$= \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma(1/n)}{\Gamma\left(\frac{2/n}{2} + \frac{1}{2}\right)}$$

$$= \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma(1/n)}{\Gamma(1/n + 1/2)}$$

= R.H.S (Showed)

Prove that, $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Solⁿ:-

Here, $\int_0^\infty e^{-x^2} dx$

Let, $x^2 = z \Rightarrow x = \sqrt{z}$

$\Rightarrow 2x\,dx = dz$

$\therefore dx = \frac{1}{2x} dz = \frac{1}{2\sqrt{z}} dz$

| $x$ | $0$ | $\infty$ |
|---|---|---|
| $z$ | $0$ | $\infty$ |

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Now,

$$\int_0^\infty e^{-x^2}\,dx$$

$$= \int_0^\infty e^{-z} \cdot \frac{1}{2\sqrt{z}}\,dz$$

$$= \frac{1}{2}\int_0^\infty e^{-z} \cdot z^{-1/2}\,dz$$

$$\cancel{= \frac{1}{2}\int_0^\infty e^{-z} \cdot z^{\left(-\frac{3}{2}-1\right)}\,dz}$$

$$\cancel{= \frac{1}{2} \cdot \Gamma\left(-\frac{3}{2}\right)}$$

$$= \frac{1}{2}\int_0^\infty e^{-z} \cdot z^{\left(\frac{1}{2}-1\right)}\,dz$$

$$= \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2} \cdot \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2}$$

III Prove that,

$$\int_0^1 x^{m-1}(1-x^a)^n\,dx = \frac{1}{a}\,\frac{n!\,\Gamma(m/a)}{\Gamma(m/a+n+1)}$$

Soln:

Let,

$$x^a = \sin^2\theta \Rightarrow x = \sin^{2/a}\theta$$

$$a\,x^{a-1}\,dx = 2\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow a\cdot x^a\cdot x^{-1}\,dx = 2\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow a\sin^2\theta\cdot\sin^{-2/a}\theta\,dx = 2\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow dx = \frac{2}{a}\sin^{-2}\theta\cdot\sin^{2/a}\theta\,\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow dx = \frac{2}{a}\sin^{2/a-2+1}\theta\,\cos\theta\,d\theta$$

| $x$ | $0$ | $1$ |
|---|---|---|
| $\theta$ | $0$ | $\pi/2$ |

Now,

$$\int_0^1 x^{m-1}(1-x^a)^n\,dx$$

$$= \int_0^{\pi/2} (\sin^{2/a}\theta)^{m-1}\cdot(1-\sin^2\theta)^n\cdot\frac{2}{a}\sin^{2/a-2+1}\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \sin^{2m/a-2/a}\theta\cdot\cos^{2n}\theta\cdot\frac{2}{a}\sin^{2/a-1}\theta\cos\theta\,d\theta$$

$$= \frac{2}{a}\int_0^{\pi/2} \sin^{2m/a-2/a+2/a-1}\theta\cdot\cos^{2n+1}\theta\,d\theta$$

$$= \frac{2}{a}\int_0^{\pi/2} \sin^{2m/a-1}\theta\,\cos^{2n+1}\theta\,d\theta$$

$$= \frac{2}{a} \times \frac{\Gamma\left(\frac{\frac{2m}{a}-1+1}{2}\right) \Gamma\left(\frac{2n+1+1}{2}\right)}{2\,\Gamma\left(\frac{\frac{2m}{a}-1+2n+1+2}{2}\right)}$$

$$= \frac{1}{a} \times \frac{\Gamma\left(\frac{2m/a}{2}\right) \Gamma(n+1)}{\Gamma\left(\frac{\frac{2m}{a}+2n+2}{2}\right)}$$

$$= \frac{1}{a} \times \frac{\Gamma(m/a) \times n!}{\Gamma(m/a+n+1)}$$

Ans

Evaluate, $\int_0^{\alpha} \frac{x^{n-1}}{1+x}\,dx$

| $x$ | $0$ | $\alpha$ |
|---|---|---|
| $\theta$ | $0$ | $\pi/2$ |

Soln:

Let,

$x = \tan^2\theta$

$dx = 2\tan\theta \sec^2\theta\, d\theta$

Now, $\int_0^{\alpha} \frac{x^{n-1}}{1+x}\,dx$

$$= \int_0^{\pi/2} \frac{(\tan^2\theta)^{n-1}}{1+\tan^2\theta} \cdot 2\tan\theta \cdot \sec^2\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\tan^{2n-2}\theta}{\sec^2\theta} \cdot \tan\theta\, \sec^2\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \tan^{2n-1}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sin^{2n-1}\theta}{\cos^{2n-1}\theta} \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n-1}\theta \cdot \cos^{-(2n-1)}\theta \, d\theta$$

$$= 2 \frac{\Gamma\left(\frac{2n-1+1}{2}\right) \Gamma\left(\frac{-2n+1+1}{2}\right)}{2\,\Gamma\left(\frac{2n-1-2n+1+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{2n}{2}\right) \Gamma\left(\frac{-2n+2}{2}\right)}{\Gamma\left(\frac{2}{2}\right)}$$

$$= \frac{\Gamma n \, \Gamma(1-n)}{\Gamma 1}$$

$$= \Gamma n \, \Gamma(1-n)$$

## Application of gamma, beta function in physics:

A particle is attracted toward a fixed point O with a force inversely proportional to its instantaneous distance from O. Applying the knowledge of gamma function, show that if the particle is ~~relat~~ released from rest the time for it to reach O is given by,

$$T = a\sqrt{\frac{\pi m}{2k}}$$

**Soln:**

At time $t=0$, Let the particle be located on the $x$-axis at $x = a > 0$ and O be the origin.

Then according to Newton's law,

$$m\frac{d^2x}{dt^2} = -\frac{k}{x} \quad \text{——— (i)}$$

$$\Rightarrow m\frac{d^2x}{dt^2} + \frac{k}{x} = 0$$

where, $m$ is the mass of the particle and $k$ is the proportionality constant.

Multiplying (i) by $2\cdot\frac{dx}{dt}\cdot dt$,

$$2m\frac{d^2x}{dt^2}\cdot\frac{dx}{dt}\cdot dt = -\frac{2k}{x}\frac{dx}{dt}\cdot dt$$

$$\Rightarrow 2m \frac{d^2x}{dt^2} \cdot \frac{dx}{dt} \cdot dt = -\frac{2k}{x} dx$$

Let,

$v = \frac{dx}{dt}$ ; be the velocity of the particle.

Then,

$$\frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{d^2x}{dt^2} = v \cdot \frac{dv}{dx}$$

Then eq$^n$ (ii) becomes,

$$2m\, v \frac{dv}{dx} \cdot dx = -\frac{2k}{x} dx$$

$$\Rightarrow m v dv = -\frac{k}{x} dx$$

integrating both sides, we get,

$$\frac{mv^2}{2} = -k \ln x + c$$

where,

$x = a$ ; $v = \frac{dx}{dt} = 0$

Therefore,

$$0 = -k \ln a + c$$

$$\Rightarrow c = k \ln a$$

Now,

$$\frac{mv^2}{2} = -k\ln x + k\ln a$$

$$\Rightarrow \frac{mv^2}{2} = k\ln\frac{a}{x}$$

$$\Rightarrow \frac{mv^2}{2} = k\ln\frac{a}{x}$$

$$\Rightarrow v^2 = \frac{2k\ln\frac{a}{x}}{m}$$

$$\Rightarrow v = \sqrt{2k/m}\cdot\sqrt{\ln a/x}$$

$$\Rightarrow \frac{dx}{dt} = \sqrt{2k/m}\cdot\sqrt{\ln a/x}$$

$$\Rightarrow dt = \sqrt{m/2k}\cdot(\ln a/x)^{-1/2}\cdot dx$$

$$\Rightarrow \int_0^T dt = \sqrt{m/2k}\cdot\int_0^a (\ln a/x)^{-1/2}dx$$

$$\Rightarrow T = \sqrt{m/2k}\cdot\int_\alpha^0 p^{-1/2}\{-ae^{-p}dp\}$$

Let,
$\ln a/x = p$
$\Rightarrow \frac{a}{x} = e^p$
$\Rightarrow \frac{x}{a} = e^{-p}$
$\Rightarrow x = ae^{-p}$
$\therefore dx = -ae^{-p}dp$

$$= a\sqrt{m/2k}\cdot\int_0^\alpha p^{-1/2}\cdot e^{-p}dp$$

| $x$ | $0$ | $a$ |
|---|---|---|
| $p$ | $\alpha$ | $0$ |

$$= a\sqrt{m/2k}\cdot\int_0^\alpha p^{1/2-1}\cdot e^{-p}dp$$

$$= a\sqrt{\frac{m}{2k}}\cdot\Gamma\tfrac{1}{2}$$ [By the definition of gamma function]

$$\Rightarrow T = a\sqrt{\frac{m}{2k}}\cdot\sqrt{\pi}$$

$$\therefore T = a\sqrt{\frac{m\pi}{2k}}$$

Even function:

A function $f(x)$ is said to be an even function if $f(-x) = f(x)$

For example, $f(x) = x^2$; $f(x) = \cancel{\sin x}\ \cos x$

Odd function:

A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$.

For example, $f(x) = x$; $f(x) = \sin x$

Fourier series:

It is a mathematical way to represent Non-~~Trigon~~ Trignometric periodic function as an infinite sum of trignometric function.

A series whose each and every term is either of sine ~~and~~ or cosine.

$$= a_0 \cos 0x + a_1 \cos 1x + a_2 \cos 2x + a_3 \cos 3x + \dots\dots$$
$$+ b_0 \sin 0x + b_1 \sin 1x + b_2 \sin 2x + b_3 \sin 3x + \dots\dots$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Therefore,

The Fourier series for the function $f(x)$ in the interval $[-\pi, \pi]$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (i)}$$

where, $a_0$, $a_n$ and $b_n$ are called the Fourier constants or co-efficients.

Now,

Integrating both sides of eqn (i) w.r.t. $x$ over $[-\pi, \pi]$, we get,

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx\right]$$

$$= \frac{a_0}{2}\left[x\right]_{-\pi}^{\pi} + \sum_{n=1}^{\infty}\left\{\frac{a_n}{n}\left[\sin nx\right]_{-\pi}^{\pi} + \frac{b_n}{n}\left[-\cos nx\right]_{-\pi}^{\pi}\right\}$$

$$= \frac{a_0}{2} \cdot 2\pi + 0 + 0$$

$$= a_0 \pi$$

$$\therefore \int_{-\pi}^{\pi} f(x)\,dx = a_0 \pi$$

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Now,
multiplying both sides of equation (1) by $\cos nx$ and integrating w.r.t $x$ over $[-\pi, \pi]$, we get,

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos nx \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos^2 nx \, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos nx \, dx$$

$$= \frac{a_0}{2} \cdot 0 + \sum_{n=1}^{\infty} \frac{a_n}{2} \int_{-\pi}^{\pi} (1 + \cos 2nx) \, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos nx \, dx$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{2} \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \frac{a_n}{2} \int_{-\pi}^{\pi} \cos 2nx \, dx + 0$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{2} [x]_{-\pi}^{\pi} + 0 + 0$$

$$= \frac{a_n}{2} \cdot 2\pi$$

$$= a_n \pi$$

Therefore,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

Similarly, multiplying both sides of (1) by $\sin nx$ and integrating w.r.t $x$ over $[-\pi, \pi]$, we get,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Hence,

Fourier co-efficients are,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

## Fourier cosine and sine series:

We know that, the fourier series of the function $f(x)$ in the interval $[-\pi, \pi]$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (i)}$$

where,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

Sir

## Cosine Series : [An even function can have no sine term in its fourier series]

If the function $f(x)$ is even, i.e., $f(-x) = f(x)$

then, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$\therefore a_0 = I_1 + I_2 \quad \text{——— (ii)}$$

Now, $I_1 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx$

Putting, $x = -y$;

$dx = -dy$

| $x$ | $0$ | $-\pi$ |
|---|---|---|
| $y$ | $0$ | $\pi$ |

$$\therefore I_1 = \frac{1}{\pi}\int_{\pi}^{0} f(-y) \cdot -dy$$

$$= \frac{-1}{\pi}\int_{\pi}^{0} f(y) \cdot dy$$

$$= \frac{1}{\pi}\int_{0}^{\pi} f(x) \cdot dx$$

From eqn (ii),

$$a_0 = 2 \times \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx.$$ [This is half range cosine series]

Again,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx$$

$$= I_3 + I_4$$

Putting,

$x = -y$

$\Rightarrow dx = -dy$

| $x$ | $0$ | $-\pi$ |
|---|---|---|
| $y$ | $0$ | $\pi$ |

Now,

$$I_3 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(-y)\cos n(-y).(-dy)$$

$$= -\frac{1}{\pi}\int_{\pi}^{0} f(y).\cos ny.\, dy$$

$$= \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx$$

Therefore,

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx$$

And,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= I_5 + I_6$$

Now,

$$I_5 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(-y)\sin n(-y)\,(-dy)$$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(y)\sin ny\,dy$$

$$= -\frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= -I_6$$

Therefore,

$$b_n = I_5 + I_6$$

$$= -I_6 + I_6$$

$$\therefore b_n = 0$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{1}{2}\times\frac{2}{\pi}\int_0^{\pi} f(x)\,dx + \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx\right]\cos nx + 0$$

$$= \frac{1}{\pi}\int_0^{\pi} f(x)\,dx + \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx\right]\cos nx$$

[This is the cosine series of Fourier series of interval $0, \pi$]

Sir

## Sine Series: [An odd function has no cosine term and constant term in its fourier series.]

If the function $f(x)$ is odd i.e. $f(-x) = -f(x)$

Then,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

and, $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= I_1 + I_2$$

Now,

$$I_1 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(-y)\sin n(-y)\,(-dy)$$

$$= -\frac{1}{\pi}\int_{\pi}^{0} f(y)\sin ny\,dy$$

$$= \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= I_2$$

Therefore,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Therefore Fourier series becomes,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= 0 + 0 + \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx\right]\cdot\sin nx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\sin nx\int_0^{\pi} f(x)\sin nx\,dx$$

This is the Fourier sine series in the Interval $(0,\pi)$.

Example:

Develop $f(x) = x$ in Fourier series.

Soln:

Since the function $f(x) = x$ is an odd function, then,

$$a_0 = 0 \quad \text{and} \quad a_n = 0$$

And also,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left\{\left[-\frac{x\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}dx\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{\pi\cos n\pi}{n} + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right\}$$

$$= -\frac{2\cos n\pi}{n}$$

$$= -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\Rightarrow x = \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n \sin nx$$

$$= -2\left[\frac{-\sin x}{1} + \frac{\sin 2x}{2} + \frac{-\sin 3x}{3} + \dots \dots\right]$$

$$= 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \dots\right]$$

Example :-

Q.1 :- Develop $f(x)$ in the Fourier series where,

$f(x) = x^2 + x$

Q.2 :- Find the Fourier series of the function,

i) $f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ \pi/2 & 0 \le x \le \pi \end{cases}$

ii) $f(x) = \begin{cases} -\pi/2 & -\pi \le x < 0 \\ \pi/2 & 0 \le x \le \pi \end{cases}$

Q.3 :- Find the Fourier series which represents the function $f(x)$, defined by,

$$f(x) = \begin{cases} -\cos x & \text{for } -\pi \le x \le 0 \\ \cos x & \text{for } 0 \le x \le \pi \end{cases}$$

Q.4 :- Find the Fourier series representing the function $f(x) = x \sin x$ in the interval $-\pi < x < \pi$. Hence show that,

$$\pi/4 = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{1.5} + \frac{1}{1.7} - \dots \dots$$

**Soln-1:**

~~Since the function $f(x) = x^2+x$ is an even function, then,~~

$$\cancel{a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx}$$

$$\cancel{= \frac{2}{\pi}\int_0^{\pi} x^2+x\,dx}$$

$$\cancel{= \frac{2}{\pi}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^{\pi}}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2\pi^3}{\pi\,3}$$

$$= \frac{2}{3}\pi^2$$

and,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2+x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\, dx$$

$$= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - \int_0^{\pi} 2x\frac{\sin nx}{n}\, dx\right]$$

$$= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n} - \frac{2}{n}\left\{\frac{-x\cos nx}{n} + \int\frac{\cos nx}{n}\, dx\right\}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n} - \frac{2}{n}\left\{\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right\}\right]_0$$

$$= \frac{2}{\pi}\left[0 + \frac{2}{n^2}x\cos nx - 0\right]_0^{\pi}$$

$$= \frac{4}{\pi}\left[\pi\cos n\pi - 0\right]$$

$$= \frac{4}{n^2\pi}\left[\pi\cos n\pi - 0\right]$$

$$= \frac{4}{n^2}\cos n\pi$$

$$= \frac{4}{n^2}(-1)^n$$

and also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left\{\left[-\frac{x\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}dx\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{\pi\cos n\pi}{n} + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{\pi\cos n\pi}{n} + 0\right\}$$

$$= -\frac{2\cos n\pi}{n}$$

$$= -\frac{2}{n}(-1)^n$$

Therefore, the fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$= \frac{2\pi^2}{2\times 3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx$$

$$+ \sum_{n=1}^{\infty}\left\{-\frac{2}{n}(-1)^n.\sin nx\right\}$$

$$\Rightarrow x^2+x = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx - \sum_{n=1}^{\infty} \frac{2}{n}(-1)^n \sin nx$$

Now, Putting the value of $x=\pi$, we get,

$$\pi+\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n(-1)^n - 0 \quad \text{——— (i)}$$

$\boxed{(-1)^n.(-1)^n = \text{even} = 1}$

And, for $x=-\pi$, we get,

$$-\pi+\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n.(-1)^n \quad \text{——— (ii)}$$

$\boxed{(-1)^n.(-1)^n = \text{even} = 1}$

Adding eqn (i) and (ii) we have,

$$2\pi^2 = \frac{2\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\Rightarrow \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\Rightarrow \pi^2 - \frac{\pi^2}{3}$$

$$\Rightarrow \pi^2 - \frac{\pi^2}{3} = 4\left\{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots \dots +\right\}$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\left\{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right\}$$

$$\Rightarrow \frac{\pi^2}{3} = 2\left\{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right\}$$

$$\Rightarrow \frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\dots$$

Solⁿ : 2 (i)

The given function,

$$f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ \pi/2 & 0 \le x \le \pi \end{cases}$$

The fourier series for the function $f(x)$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{—— (i)}$$

Now,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} f(x)\, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x)\, dx$$

$$= 0 + \frac{1}{\pi} \int_{0}^{\pi} \pi/2 \cdot dx$$

$$= \frac{1}{2} \int_{0}^{\pi} 1 . dx$$

$$= \left[\frac{x}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

Now,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} f(x) \cos nx\, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) \cos nx\, dx$$

$$= 0 + \frac{1}{\pi}\int_0^{\pi} \pi/2 \cdot \cos nx \, dx$$

$$= \frac{1}{2}\int_0^{\pi} \cos nx \, dx$$

$$= \left[\frac{\sin nx}{2n}\right]_0^{\pi}$$

$$= [0 - 0]$$

$$= 0$$

also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x) \sin nx \, dx + \frac{1}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= 0 + \frac{1}{\pi}\int_0^{\pi} \pi/2 \cdot \sin nx \, dx$$

$$= \frac{1}{2}\int_0^{\pi} \sin nx \, dx$$

$$= -\frac{1}{2n}\left[\cos nx\right]_0^{\pi}$$

$$\boxed{\cancel{= -\frac{1}{2n}[-1-1] \quad = \frac{1}{n}}}$$

$$= -\frac{1}{2n}\left[(-1)^n - 1\right]$$

$$= \frac{(-1)^{n+1}}{2n} + \frac{1}{2n}$$

$$= 0 + \frac{1}{\pi}\int_0^{\pi} \pi/2 \cdot \cos nx \, dx$$

$$= \frac{1}{2}\int_0^{\pi} \cos nx \, dx$$

$$= \left[\frac{\sin nx}{2n}\right]_0^{\pi}$$

$$= [0 - 0]$$

$$= 0$$

also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x) \sin nx \, dx + \frac{1}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= 0 + \frac{1}{\pi}\int_0^{\pi} \pi/2 \cdot \sin nx \, dx$$

$$= \frac{1}{2}\int_0^{\pi} \sin nx \, dx$$

$$= -\frac{1}{2n}\left[\cos nx\right]_0^{\pi}$$

$$\boxed{\cancel{= -\frac{1}{2n}\left[-1-1\right] \quad = \frac{1}{n}}}$$

$$= -\frac{1}{2n}\left[(-1)^n - 1\right]$$

$$= \frac{(-1)^{n+1}}{2n} + \frac{1}{2n}$$

Substituting this values in eq$^n$ (1),

$$f(x) = \frac{\pi/2}{2} + \sum_{n=1}^{\infty} 0 \cdot \cos nx + \sum_{n=1}^{\infty} \frac{1}{n} \cdot \sin nx$$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty} \cancel{\frac{\sin nx}{n}}$$

$$\frac{\pi}{4} + \sum_{n=1}^{\infty} \left\{ \frac{(-1)^{n+1}}{2n} + \frac{1}{2n} \right\} \cdot \sin nx$$

Ans

**Sol$^n$: 2 (ii)**

The given function,

$$f(x) = \begin{cases} -\pi/2 \; ; & -\pi \le x < 0 \\ \pi/2 \; ; & 0 \le x \le \pi \end{cases}$$

The Fourier series for the function, $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—(1)}$$

Now,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} f(x)\, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x)\, dx$$

$$\cancel{= \frac{1}{\pi} \int_{-\pi}^{0} -\frac{1}{\pi}\, dx + \frac{1}{\pi} \int_{0}^{\pi} \frac{1}{\pi}}$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\frac{\pi}{2}\,dx + \frac{1}{\pi}\int_{0}^{\pi} \frac{\pi}{2}\,dx$$

$$= -\frac{1}{2}\int_{-\pi}^{0} dx + \frac{1}{2}\int_{0}^{\pi} dx$$

$$= -\frac{1}{2}[x]_{-\pi}^{0} + \frac{1}{2}[x]_{0}^{\pi}$$

$$= -\frac{1}{2}[0+\pi] + \frac{1}{2}[\pi - 0]$$

$$= -\cancel{\frac{\pi}{2}} + \cancel{\frac{\pi}{2}}$$

$$= 0$$

Now, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\frac{\pi}{2}\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \frac{\pi}{2}\cos nx\,dx$$

$$= -\frac{1}{2}\int_{-\pi}^{0} \cos nx\,dx + \frac{1}{2}\int_{0}^{\pi} \cos nx\,dx$$

$$= -\frac{1}{2}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{1}{2}\left[\frac{\sin nx}{n}\right]_{0}^{\pi}$$

$$= -\frac{1}{2}\times 0 + \frac{1}{2}\times 0$$

$$= 0$$

Also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\frac{\pi}{2}\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\frac{\pi}{2}\sin nx\,dx$$

$$= -\frac{1}{2}\int_{-\pi}^{0}\sin nx\,dx + \frac{1}{2}\int_{0}^{\pi}\sin nx\,dx$$

$$\cancel{= -\frac{1}{2}\left[-\sin nx\right]}$$

$$= -\frac{1}{2}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{2}\left[\frac{-\cos nx}{n}\right]_{0}^{\pi}$$

$$= \frac{1}{2}\left[\frac{\cos nx}{n}\right]_{-\pi}^{0} - \frac{1}{2}\left[\frac{\cos nx}{n}\right]_{0}^{\pi}$$

$$= \frac{1}{2}\left[\frac{1}{n} + \frac{1}{n}\right] - \frac{1}{2}\left[-\frac{1}{n} - \frac{1}{n}\right]$$

$$= \frac{1}{n} + \frac{1}{n}$$

$$= \frac{2}{n}$$

$$= \frac{1}{2}\left[\frac{1}{n} - \frac{\cos n\pi}{n}\right] - \frac{1}{2}\left[\frac{\cos n\pi}{n} - \frac{1}{n}\right]$$

$$= \frac{1}{2n} - \frac{\cos n\pi}{2n} - \frac{\cos n\pi}{2n} + \frac{1}{2n}$$

$$= \frac{1}{n} - \frac{\cos n\pi}{n} = \frac{1}{n} - \frac{\cos n\pi}{n} = \frac{1-(-1)^n}{n} = \frac{1+(-1)^{n+1}}{n}$$

Substituting these values in eqn ①,

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

$$f(x) = \sum_{n=1}^{\infty} \frac{1+(-1)^{n+1}}{n} \sin nx$$

Soln: 3

The given function,

$$f(x) = \begin{cases} -\cos x & ; -\pi \le x \le 0 \\ \cos x & ; 0 \le x \le \pi \end{cases}$$

The Fourier series for the function $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

Now,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} f(x)\, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} -\cos x\, dx + \frac{1}{\pi} \int_{0}^{\pi} \cos x\, dx$$

$$= \frac{1}{\pi} \left[-\sin x\right]_{-\pi}^{0} + \frac{1}{\pi} \left[\sin x\right]_{0}^{\pi}$$

$$= 0 + 0$$

$$= 0$$

Now,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\cos x\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\cos x\cos nx\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0} 2\cos nx\cos x\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 2\cos nx\cos x\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0}\{\cos(nx+x)+\cos(nx-x)\}dx$$

$$+\frac{1}{2\pi}\int_{0}^{\pi}\{\cos(nx+x)+\cos(nx-x)\}dx$$

$$= -\frac{1}{2\pi}\left[\frac{\sin nx+x}{(n+1)}+\frac{\sin(nx-x)}{(n-1)}\right]_{-\pi}^{0}$$

$$+\frac{1}{2\pi}\left[\frac{\sin(nx+x)}{(n+1)}+\frac{\sin(nx-x)}{(n-1)}\right]_{0}^{\pi}$$

$$= -\frac{1}{2\pi}\times 0 + \frac{1}{2\pi}\times 0$$

$$= 0$$

$\sin(n\pi+\theta) = (-1)^n\sin\theta$

$\sin(n\pi+\pi) = (-1)^n\sin\pi$

$= 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-\cos x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \cos x\sin nx\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0} 2\sin nx\cos x\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 2\sin nx\cos x\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0}\{\sin(nx+x) + \overset{\sin}{\cancel{\cos}}(nx-x)\}\,dx$$

$$+ \frac{1}{2\pi}\int_{0}^{\pi}\{\sin(nx+x) + \overset{\sin}{\cancel{\cos}}(nx-x)\}\,dx$$

$$= -\frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{(n+1)} + \frac{\overset{-\cos}{\cancel{-\sin}}(n-1)x}{(n-1)}\right]_{-\pi}^{0}$$

$$+ \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{(n+1)} + \frac{\overset{-\cos}{\cancel{\sin}}(n-1)x}{(n-1)}\right]_{0}^{\pi}$$

$$= -\frac{1}{2\pi}\left\{\left(\frac{-\cos 0}{(n+1)} + 0\right) - \left(\frac{-\cos[(n+1)(-\pi)]}{(n+1)} + 0\right)\right\}$$

$$+ \frac{1}{2\pi}\left\{\left(-\frac{\cos(n+1)\pi}{n+1} + 0\right) - \left(\frac{-\cos 0}{(n+1)} + 0\right)\right\}$$

correction

$$= -\frac{1}{2\pi}\left[\left(\frac{-\cos 0}{n+1} + \frac{-\cos(n-1)\cdot 0}{n-1}\right) - \left(\frac{-\cos(n+1)(-\pi)}{n+1} + \frac{-\cos(n-1)(-\pi)}{(n-1)}\right)\right]$$

$$+\frac{1}{2\pi}\left[\left(\frac{-\cos(n+1)\pi}{n+1} + \frac{-\cos(n-1)\pi}{n-1}\right) - \left(\frac{-\cos(n+1)\cdot 0}{(n+1)} + \frac{-\cos(n+1)\cdot 0}{(n-1)}\right)\right]$$

$$= -\frac{1}{2\pi}\left[\left(\frac{-\cos 0}{n+1} + \frac{-\cos 0}{n-1}\right) + \left(\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{(n-1)}\right)\right]$$

$$+\frac{1}{2\pi}\left[\left(\frac{-\cos(n+1)\pi}{(n+1)} + \frac{-\cos(n-1)\pi}{(n-1)}\right) - \left(\frac{-\cos 0}{(n+1)} + \frac{-\cos 0}{(n-1)}\right)\right]$$

$$= -\frac{1}{2\pi}\left[\left(\frac{-1}{(n+1)} + \frac{-1}{(n-1)}\right) + \left(\frac{(-1)^{n+1}}{(n+1)} + \frac{(-1)^{n-1}}{(n-1)}\right)\right]$$

$$+\frac{1}{2\pi}\left[\left(\frac{-(-1)^{n+1}}{(n+1)} + \frac{-(-1)^{n-1}}{(n-1)}\right) - \left(\frac{-1}{n+1} + \frac{-1}{n-1}\right)\right]$$

$$= -\frac{2}{2\pi}\left(\frac{-1}{(n+1)} + \frac{-1}{n-1}\right) - \frac{2}{2\pi}\left(\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{n+1} + \frac{1}{n-1}\right) - \frac{1}{\pi}\left(\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{n+1} - \frac{(-1)^{n+1}}{n+1}\right) + \frac{1}{\pi}\left(\frac{1}{n-1} - \frac{(-1)^{n-1}}{n-1}\right)$$

$$= \frac{1}{\pi}\left(\frac{1-(-1)^{n+1}}{(n+1)}\right) + \frac{1}{\pi}\left(\frac{1-(-1)^{n-1}}{(n-1)}\right)$$

Substituting these values in eq$^n$ (i),

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty}\left[\frac{1}{\pi}\left(\frac{1-(-1)^{n+1}}{n+1}\right) + \frac{1}{\pi}\left(\frac{1-(-1)^{n-1}}{n-1}\right)\right]\sin nx$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\left[\frac{1}{\pi}\left(\frac{1-(-1)^{n+1}}{n+1}\right) + \frac{1}{\pi}\left(\frac{1-(-1)^{n-1}}{n-1}\right)\right]\cdot\sin nx$$

Ans

**Soln-4:** Given function,

$$f(x) = x \sin x \quad ; \quad -\pi < x < \pi$$

The Fourier series for the function $f(x)$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—— (1)}$$

Now,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x \, dx$$

$$= \frac{1}{\pi} \left\{ \left[ -x \cos x \right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos x \, dx \right\}$$

$$= \frac{1}{\pi} \left( \left\{ -\pi \cos \pi - \pi \cos(-\pi) \right\} + \left[ \sin x \right]_{-\pi}^{\pi} \right)$$

$$= \frac{1}{\pi} \left( -2\pi \cos \pi \right) + \frac{1}{\pi} \times 0$$

$$= -2 \cos \pi$$

$$= -2 \times -1$$

$$= 2$$

And,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos nx\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\cdot 2\cos nx\sin x\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\cdot\{\sin(nx+x) - \sin(nx-x)\}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sin(n+1)x\, dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sin(n-1)x\, dx$$

$$= \frac{1}{2\pi}\left\{\left[-\frac{x\cos(n+1)x}{(n+1)}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(n+1)x}{n+1}\, dx\right\}$$

$$- \frac{1}{2\pi}\left\{\left[\frac{-x\cos(n-1)x}{(n-1)}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(n-1)x}{(n-1)}\, dx\right\}$$

$$= \frac{1}{2\pi}\left\{\left(\frac{-\pi\cos(n+1)\pi}{(n+1)} - \frac{\pi\cos(n+1)(-\pi)}{(n+1)}\right) + \left[\frac{\sin(n+1)x}{(n+1)^2}\right]_{-\pi}^{\pi}\right\}$$

$$- \frac{1}{2\pi}\left\{\left(\frac{-\pi\cos(n-1)\pi}{(n-1)} - \frac{\pi\cos(n-1)(-\pi)}{(n-1)}\right) + \left[\frac{\sin(n-1)x}{(n-1)^2}\right]_{-\pi}^{\pi}\right.$$

$$= \frac{-1}{2\pi}\left(\frac{\pi\cdot(-1)^{n+1}}{(n+1)} + \frac{\pi(-1)^{n+1}}{n+1}\right) + \frac{1}{2\pi}\times 0$$

$$+ \frac{1}{2\pi}\left(\frac{\pi(-1)^{n-1}}{n-1} + \frac{\pi(-1)^{n-1}}{n-1}\right) - \frac{1}{2\pi}\times 0$$

$$= \frac{-1}{2\pi}\times\frac{2\pi(-1)^{n+1}}{n+1} + \frac{1}{2\pi}\times\frac{2\pi(-1)^{n-1}}{(n-1)}$$

$$= \frac{-1\cdot(-1)^{n+1}}{(n+1)} + \frac{(-1)^{n-1}}{(n-1)}$$

$$= \boxed{\frac{(-1)^{n}}{(n+1)} + \frac{(-1)^{n-1}}{(n-1)}} = \frac{(-1)^{n-1}}{(n-1)} - \frac{(-1)^{n+1}}{n+1}$$

$$= \cancel{\frac{(n-1)(-1)^{n}+(n+1)(-1)^{n-1}}{(n-1)^2}}$$

$$= (-1)^n\left\{\frac{(-1)^{-1}}{(n-1)} - \frac{(-1)^{1}}{n+1}\right\}$$

$$\cancel{= (-1)^n\left\{\frac{\quad}{(n^2-1)}\right.}$$

$$= (-1)^n\left\{\frac{1}{n+1} - \frac{1}{n-1}\right\}$$

$$= (-1)^n\,\frac{n-1-n-1}{n^2-1}$$

$$= (-1)^n\cdot\frac{(-2)}{n^2-1}$$

$$= \frac{(-1)^n\times 2\times(-1)^1}{n^2-1}$$

$$= \frac{2(-1)^{n+1}}{n^2-1}$$

Now,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\cdot 2\sin nx\cdot\sin x\,dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} x \cdot \{\cos(n-1)x - \cos(n+1)x\} \, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} x \cos(n-1)x - \frac{1}{2\pi} \int_{-\pi}^{\pi} x \cos(n+1)x \, dx$$

$$= \frac{1}{2\pi} \left\{ \left[ \frac{x \sin(n-1)x}{n-1} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\sin(n-1)x}{n-1} dx \right\}$$

$$- \frac{1}{2\pi} \left\{ \left[ \frac{x \sin(n+1)x}{n+1} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\sin(n+1)x}{n+1} dx \right\}$$

$$= -\frac{1}{2\pi} \left[ \frac{-\cos(n-1)x}{(n-1)^2} \right]_{-\pi}^{\pi} + \frac{1}{2\pi} \left[ \frac{-\cos(n+1)x}{(n+1)^2} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(n-1)^2} \{\cos(n-1)\pi - \cos(n-1)(-\pi)\}$$

$$- \frac{1}{2\pi(n+1)^2} \{\cos(n+1)\pi - \cos(n+1)(-\pi)\}$$

$$= \frac{1}{2\pi(n-1)^2} (\cos(n-1)\pi - \cos(n-1)\pi)$$

$$- \frac{1}{2\pi(n+1)^2} (\cos(n+1)\pi - \cos(n+1)\pi)$$

$$= 0 - 0$$

$$= 0$$

$$\therefore b_n = 0$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\Rightarrow x \sin x = \frac{2}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n^2-1} \cdot \cos nx$$

where,

$n = 1, 2, 3, 4, \ldots \ldots$

Now,

$$x \sin x = 1 + \left[ \frac{2\cdot(-1)^2}{1^2-1} \cos x + \frac{2(-1)^{2+1}}{2^2-1} \cos 2x + \frac{2(-1)^{3+1}}{3^2-1} \cos 3x \right.$$

$$+ \frac{2\cdot(-1)^{4+1}}{4^2-1} \cos 4x + \frac{2(-1)^{5+1}}{5^2-1} \cos 5x + \frac{2(-1)^{6+1}}{6^2-1} \cos 6x$$

$$\left. + \frac{2(-1)^{7+1}}{7^2-1} \cos 7x + \frac{2(-1)^{8+1}}{8^2-1} \cos 8x + \ldots \ldots \right]$$

$$= 1 - \frac{2}{3} \cos 2x + \frac{2}{8} \cos 3x - \frac{2}{15} \cos 4x$$

$$+ \frac{2}{24} \cos 5x - \frac{2}{35} \cos 6x + \frac{2}{48} \cos 7x$$

$$- \frac{1}{63} \cos 8x + \ldots$$

Let, $x = \frac{\pi}{2}$ ; then,

$$\frac{\pi}{2} \sin \frac{\pi}{2} = 1 - \frac{2}{3} \cos \frac{2\pi}{2} + \frac{2}{8} \cos \frac{3\pi}{2} - \frac{2}{15} \cos \frac{4\pi}{2}$$

$$+ \frac{2}{24} \cos \frac{5\pi}{2} - \frac{2}{35} \cos \frac{6\pi}{2} + \frac{2}{48} \cos \frac{7\pi}{2}$$

$$- \frac{1}{63} \cos \frac{8\pi}{2} + \ldots$$

$$\frac{\pi}{2}\times 1 = 1 - \frac{2}{3}(-1) + \frac{2}{8}\times 0 - \frac{2}{15}\times 1 + \frac{2}{24}\times 0 - \frac{2}{35}(-1)$$

$$+ \frac{2}{48}\times 0 - \frac{2}{63}\times 1$$

$$\Rightarrow \frac{\pi}{2} = 1 + \frac{2}{3} - \frac{2}{15} + \frac{2}{35} - \frac{2}{63} + \dots$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{2} + \frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \frac{1}{63} + \dots$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \dots$$

## Complex form of Fourier series:

(Derive Fourier series from complex Fourier series)

The Fourier series may also be written in the complex form as,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{——— (i)}$$

where,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \quad \text{——— (ii)}$$

And,

$$c_{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} dx \quad \text{——— (iii)}$$

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{a_0}{2} \quad \text{——— (iv)}$$

From eq$^n$ (ii),

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot (\cos nx - j \sin nx) dx =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx - \frac{1}{2\pi} j \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{a_n}{2} - j \frac{b_n}{2}$$

Similarly,

$$c_{-n} = \frac{a_n}{2} + j \frac{b_n}{2}$$

$$\therefore f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + \sum_{n=-1} c_n e^{inx}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} - \frac{i b_n}{2}\right) e^{inx}$$

$$+ \sum_{n=1}^{\infty} c_{-n} e^{-inx}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} - i\frac{b_n}{2}\right) e^{inx}$$

$$+ \sum_{n=1}^{\infty} \left(\frac{a_n}{2} + i\frac{b_n}{2}\right) e^{-inx}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{2}\left(e^{inx} + e^{-inx}\right) - i\frac{b_n}{2}\left(e^{inx} - e^{-inx}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2} \cdot 2\cos nx - i\frac{b_n}{2}\, 2 \cdot i \sin nx)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

## Parseval's identity :

Statement :

If the Fourier series for the function $f(x)$ converges uniformly in $(-l, l)$, then

$$\frac{1}{l}\int_{-l}^{l}\{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2);$$

where, $a_0$, $a_n$ and $b_n$ called Fourier co-efficients or constants.

Proof :

The Fourier series for the function $f(x)$ in the interval $(-l, l)$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \quad \text{——— (i)}$$

where,

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cdot \cos\frac{n\pi x}{l}\,dx$$

and, $$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \cdot \sin\frac{n\pi x}{l}\,dx$$

Multiplying both sides of eq$^n$ (i) by $f(x)$ and integrating between limits $-l$ to $l$ we get,

$$\int_{-l}^{l} \{f(x)\}^2 dx = \frac{a_0}{2}\int_{-l}^{l} f(x)\, dx + \sum_{n=1}^{\infty}\left(a_n \int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\, dx\right)$$

$$+ \sum_{n=1}^{\infty}\left(b_n \int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\, dx\right)$$

$$= \frac{a_0}{2} \cdot a_0 l + \sum_{n=1}^{\infty} a_n \cdot a_n l + \sum_{n=1}^{\infty} b_n \cdot b_n l$$

$$= \frac{a_0^2}{2} \cdot l + \sum_{n=1}^{\infty} a_n^2 \cdot l + \sum_{n=1}^{\infty} b_n^2 \cdot l$$

$$\Rightarrow \int_{-l}^{l} \{f(x)\}^2 dx = \frac{a_0^2}{2} \cdot l + l \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\Rightarrow \frac{1}{l}\int_{-l}^{l} \{f(x)\}^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

(proved)

## Fourier Transformation : (also inverse fourier transform)

The general Fourier transform $f(s)$ or $f(n)$ of a function $f(x)$ is defined as,

$$f(n) = \int_{-\infty}^{\infty} f(x)\, e^{inx}\, dx \; ; \; \cancel{\text{where the inv}}$$

where the inverse Fourier transform, $f(x)$ is given by,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(n)\, e^{-inx}\, dn$$

**Example-1:** Find the Fourier complex transform of $f(x)$, where,

$$f(x) = \begin{cases} 1-x^2 & ; |x| < 1 \\ 0 & ; |x| > 1 \end{cases}$$

confusion, $|x| \leq 1$ (isn't it better)

**Soln:**

We have, fourier transform of a function $f(x)$ is

$$f(n) = \int_{-\infty}^{\infty} f(x)\, e^{inx}\, dx$$

$$= \int_{-\infty}^{-1} f(x)\, e^{inx} + \int_{-1}^{1} f(x)\, e^{inx}\, dx + \int_{1}^{\infty} f(x)\, e^{inx}\, dx$$

$$= \int_{-1}^{1} (1-x^2)\, e^{inx}\, dx$$

$$= \left[\frac{(1-x^2).e^{inx}}{in}\right]_{-1}^{1} - \int_{-1}^{1} \frac{-2x \cdot e^{inx}}{in}\, dx$$

$$= 0 + \frac{2}{in}\int_{-1}^{1} x e^{inx}\, dx$$

$$= \frac{2}{in}\left\{\left[\frac{x e^{inx}}{in}\right]_{-1}^{1} - \int \frac{e^{inx}}{in}\, dx\right\}$$

$$= \frac{2}{in}\left\{\left(\frac{e^{in}}{in} + \frac{e^{-in}}{in}\right) - \left[\frac{e^{inx}}{i^2 n^2}\right]_{-1}^{1}\right\}$$

$$= \frac{2}{in}\left\{\left(\frac{e^{in} + e^{-in}}{in}\right) + \frac{1}{n^2}\left(e^{in} - e^{-in}\right)\right\}$$

$$= -\frac{2}{n^2}\left(e^{in} + e^{-in}\right) + \frac{2}{n^3}\left(\frac{e^{in} - e^{-in}}{i}\right)$$

$$= -\frac{2}{n^2} \cdot 2\cos n + \frac{2}{n^3} \cdot \frac{2i \sin n}{i}$$

$$= -\frac{4}{n^2}\cos n + \frac{4}{n^3}\sin n$$

Example-2: Find the Fourier transform of

$$f(x) = \begin{cases} 1 & ; \ |x| < a \\ 0 & ; \ |x| > a \end{cases}$$

Soln We know Fourier transform of a function,

$$f(n) = \int_{-\alpha}^{\alpha} f(x)\, e^{inx}\, dx$$

$$= \int_{-\alpha}^{-a} f(x)\, e^{inx}\, dx + \int_{-a}^{a} f(x)\, e^{inx}\, dx + \int_{a}^{\alpha} f(x)\, e^{inx}\, dx$$

$$= \int_{-a}^{a} f(x)\, e^{inx}\, dx$$

$$= \int_{-a}^{a} e^{inx}\, dx$$

$$= \left[\frac{e^{inx}}{in}\right]_{-a}^{a}$$

$$= \left(\frac{e^{ina}}{in} - \frac{e^{-ina}}{in}\right)$$

$$= \frac{e^{ina} - e^{-ina}}{in}$$

$$= \frac{(\cos na + i\sin na) - (\cos na - i\sin na)}{in}$$

$$= \frac{2i\sin na}{in}$$

$$= \frac{2\sin na}{n}$$

$$= \frac{2}{n}\sin na$$

## Fourier sine transform:

The infinite Fourier sine transform $f_s(n)$ of function $f(x)$ is defined as,

$$f_s(n) = \int_0^\infty f(x) \sin nx \, dx$$

and its inverse formula is given by,

$$f(x) = \frac{2}{\pi} \int_0^\infty f_s(n) \sin nx \, dn$$

Example-1: Find the Fourier sine trans-form of $e^{-x}$.

Soln:

we have,

$$f_s(n) = \int_0^\infty f(x) \sin nx \, dx$$

$$= \int_0^\infty e^{-x} . \sin nx \, dx$$

$$= \lim_{R \to \infty} \int_0^R e^{-x} . \sin nx \, dx \quad \text{—— (i)}$$

~~Let, $I = \int_0^R e^{-x} \sin nx \, dx$~~

$$= \lim_{R\to\infty}\left[\frac{e^{-x}}{(-1)^2+n^2}\{(-1)\sin nx - n\cos nx\}\right]_0^R$$

$$= \lim_{R\to\infty}\left\{\left[\frac{e^{-R}}{n^2+1}(-\sin nR - n\cos nR)\right] - \left[\frac{1}{n^2+1}(-0-n\cdot 1)\right]\right\}$$

$$= \lim_{R\to\infty}\left[\frac{1}{e^R(n^2+1)}(-\sin nR - n\cos nR)\right]$$

$$- \lim_{R\to\infty}\left[\frac{1}{n^2+1}(-n)\right]$$

$$= 0 - \frac{(-n)}{n^2+1}$$

$$= \frac{n}{n^2+1}$$ Ans

## Finite Fourier Transform:

The Finite Fourier sine transform of a function $f(x)$ is defined as,

$$f_s(n) = \int_0^{\pi} f(x) \sin nx \, dx \, ;$$

and its inverse formula $f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} f_s(n) \sin nx$.

**Example:-**

Using finite Fourier sine transform, solve,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \, ;$$

Given, $u(0,t) = 0$ ; $u(\pi,t) = 0$ ; $u(x,0) = 2x$

when, $0 < x < \pi$ , $t > 0$

**Soln:-** Given that,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

taking finite Fourier sine transform we get,

$$\int_0^{\pi} \frac{\partial u}{\partial t} \sin nx \, dx = \int_0^{\pi} \frac{\partial^2 u}{\partial x^2} \sin nx \, dx \quad \text{——— (i)}$$

Let, $\int_0^{\pi} u(x,t) \cdot \sin nx \, dx = u(n,t)$

$$\therefore \frac{du}{dt} = \int_0^{\pi} \frac{\partial u}{\partial t} \sin nx \, dx$$

$$= \int_0^{\pi} \frac{\partial^2 u}{\partial x^2} \sin nx \, dx$$

$$= \left[\frac{\partial u}{\partial x} \cdot \sin nx\right]_0^{\pi} - n\int_0^{\pi} \cos nx \frac{\partial u}{\partial x} dx$$

$$= -n\int_0^{\pi} \cos nx \frac{\partial u}{\partial x} dx$$

$$= -n\left[u \cos nx\right]_0^{\pi} - n^2 \int_0^{\pi} u \sin nx \, dx$$

$$= 0 - n^2 u \qquad \left[\because \int_0^{\pi} u(x,t) \sin nx \, dx = u(n,t)\right]$$

$$\therefore \frac{du}{dt} = -n^2 u$$

$$\Rightarrow \frac{du}{u} = -n^2 dt$$

$$\Rightarrow \int \frac{1}{u} du = -n^2 \int dt$$

$$\Rightarrow \log u = -n^2 t + \log c$$

$$\Rightarrow \log \frac{u}{c} = -n^2 t$$

$$\Rightarrow \frac{u}{c} = e^{-n^2 t}$$

$$\Rightarrow u = c \cdot e^{-n^2 t}$$

$$\Rightarrow u(n,t) = c \cdot e^{-n^2 t} \qquad \text{——— (ii)}$$

or, $u(n,0) = c$

also, $u(n,0) = \int_0^{\pi} u(x,0) \sin nx \, dx$

$$\Rightarrow c = 2\int_0^{\pi} x \sin nx \, dx$$

$$= 2\left\{\left[\frac{x\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n}\right\}$$

$$= 2\left\{-\frac{\pi\cos n\pi}{n} + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right\}$$

$$= -\frac{2\pi\cos n\pi}{n}$$

$$= -\frac{2\pi}{n}(-1)^n$$

$$\therefore c = \frac{2\pi}{n}(-1)^{n+1}$$

From eq$^n$ (ii);

$$u(n,t) = \frac{2\pi}{n}(-1)^{n+1} \cdot e^{-n^2 t}$$

This equation is also called [heat equation]

## Laplace Transformation :

Laplace transform of a function $f(t)$ is defined by,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = f(s)$$

$s$ = frequency domain

$t$ = time domain

$\cancel{= f(s)}$

**Example-1:** Find Laplace transform of $\sin t$,

**Solⁿ:-** Given,

$$f(t) = \sin t$$

$$\therefore L\{f(t)\} = \int_0^\infty e^{-st} \cdot \sin t \, dt$$

$$= \int_0^\infty e^{-st} \cdot \frac{1}{2i} \cdot 2i \sin t \, dt$$

$$= \int_0^\infty e^{-st} \cdot \frac{1}{2i} \left(e^{it} - e^{-it}\right) dt$$

$$= \frac{1}{2i} \int_0^\infty e^{-st+it} \, dt - \frac{1}{2i} \int_0^\infty e^{-st-it} \, dt$$

$$= \frac{1}{2i} \int_0^\infty e^{-(s-i)t} \, dt - \frac{1}{2i} \int_0^\infty e^{-(s+i)t} \, dt$$

$$= \frac{1}{2i}\left[\frac{e^{-(s-i)t}}{-(s-i)} - \frac{e^{-(s+i)t}}{-(s+i)}\right]_0^\infty$$

$$= \frac{1}{2i}\left[\frac{1}{(s-i)} - \frac{1}{(s+i)}\right]$$

$$= \frac{1}{2i}\left[\frac{(s+i)-(s-i)}{s^2-i^2}\right]$$

$$= \frac{1}{2i}\,\frac{2i}{s^2+1}$$

$$= \frac{1}{s^2+1}$$

$$\boxed{L\{\sin at\} = \frac{a}{s^2+a^2}}$$

$$\therefore L\{f(t)\} = f(s) = \frac{1}{s^2+1}$$

<u>Example-2:</u> Find the laplace transform of $\cos at$.

<u>Soln:</u>

Given that,

$$f(t) = \cos at$$

$$\therefore L\{f(t)\} = f(s) = \int_0^\infty e^{-st} \cdot \cos at \, dt$$

$$= \int_0^\infty e^{-st} \cdot \frac{1}{2} \cdot 2\cos at \, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st} \cdot \left(e^{jat} + e^{-jat}\right) dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st+jat} \, dt + \frac{1}{2}\int_0^\infty e^{-st-jat} \, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-ja)t} \, dt + \frac{1}{2}\int_0^\infty e^{-(s+ja)t} \, dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-ja)t}}{-(s-ja)} + \frac{e^{-(s+ja)t}}{-(s+ja)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{(s-ja)} + \frac{1}{(s+ja)}\right]$$

$$= \frac{1}{2}\left[\frac{(s+ja)+(s-ja)}{s^2 - j^2a^2}\right]$$

$$= \frac{1}{2} \frac{2s}{s^2+a^2}$$

$$= \frac{s}{s^2+a^2}$$

Find the laplace transform of $f(t) = 1$

We have,

$$f(t) = 1$$

$$\therefore L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t) \cdot dt$$

$$= \int_0^\infty e^{-st} \cdot 1 \, dt$$

$$= \int_0^\infty e^{-st} \, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= 0 - \left(\frac{1}{-s}\right)$$

$$= \frac{1}{s} \quad \text{Ans}$$

Laplace transform of $\cosh t$.

Soln:

$$f(t) = \cosh t$$

$$L\{f(t)\} = \int_0^\infty e^{-st} \cdot \cosh t \cdot dt$$

$$= \int_0^\infty e^{-st} \cdot \left(\frac{e^t + e^{-t}}{2}\right) dt \quad \left[\because \cosh t = \frac{e^t + e^{-t}}{2}\right]$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-1)t} dt + \frac{1}{2}\int_0^\infty e^{-(s+1)t} dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^\infty + \frac{1}{2}\left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{-1}{-(s-1)}\right] + \frac{1}{2}\left[\frac{-1}{-(s+1)}\right]$$

$$= \frac{1}{2(s-1)} + \frac{1}{2(s+1)}$$

$$= \frac{(s+1) + (s-1)}{2(s-1)(s+1)}$$

$$= \frac{s+2+s-2}{2(s-1)(s+1)}$$

$$= \frac{2s}{2(s^2-1)} = \frac{s}{s^2-1}$$

Laplace transform of ~~sinh t~~ sinh at

Soln:

we have,

$$f(t) = \sinh at$$

$$L\{f(t)\} = \int_0^\infty e^{-st} \cdot \cancel{\sinh t} \cdot \sinh at \, dt$$

$$= \int_0^\infty e^{-st} \cdot \frac{e^{at} - e^{-at}}{2} \, dt$$

$$= \frac{1}{2} \int_0^\infty e^{-(s-a)t} \, dt - \frac{1}{2} \int_0^\infty e^{-(s+a)t} \, dt$$

$$= \frac{1}{2} \left[ \frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty - \frac{1}{2} \left[ \frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty$$

$$= \frac{1}{2} \left[ \frac{-1}{-(s-a)} - \frac{-1}{-(s+a)} \right]$$

$$= \frac{1}{2} \left[ \frac{1}{s-a} - \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[ \frac{(s+a) - (s-a)}{(s-a)(s+a)} \right]$$

$$= \frac{1}{2} \frac{2a}{s^2 - a^2}$$

$$= \frac{a}{s^2 - a^2}$$

## Properties of Laplace transformation:

1. Linear property.
2. Shifting property.
3. Change of scale property.

### Linear property:

The linear property of Laplace transformation is given by,

$$L\{k_1 f_1(t) + k_2 f_2(t)\} = k_1 f_1(s) + k_2 f_2(s)$$

$$= k_1 L\{f_1(t)\} + k_2 L\{f_2(t)\}$$

where, $k_1$ and $k_2$ are constants.

**Proof:**

Let $f_1(s)$ and $f_2(s)$ be the Laplace transform of $f_1(t)$ and $f_2(t)$ respectively.

Then,

$$L\{f_1(t)\} = \int_0^\infty e^{-st} f_1(t)\, dt = f_1(s)$$

and,

$$L\{f_2(t)\} = \int_0^\infty e^{-st} f_2(t)\, dt = f_2(s)$$

$\therefore L\{k_1 f_1(t) + k_2 f_2(t)\}$

$$= \int_0^\infty e^{-st}\left(k_1 f_1(t) + k_2 f_2(t)\right) dt$$

$$= \int_0^\infty e^{-st} \cdot k_1 f_1(t)\, dt + \int_0^\infty e^{-st} \cdot k_2 f_2(t)\, dt$$

$$= k_1 f_1(s) + k_2 f_2(s)$$

$$= k_1 L\{f_1(t)\} + k_2 L\{f_2(t)\}$$

**Example-1:** Evaluate $L\{\sin 2t + 2\cos 3t\}$

**Soln:**

Using linear property of Laplace transform, we can write

$$L\{\sin 2t + 2\cos 3t\} = L\{\sin 2t\} + 2L\{\cos 3t\}$$

Here,

$$L\{\sin 2t\} = \int_0^\infty e^{-st} \sin 2t\, dt$$

$$= \cancel{\frac{2}{s^2+2^2}} = \frac{2}{s^2+2^2}$$ [direct]

$$= \int_0^\infty e^{-st} \cdot \left( \frac{e^{2jt} - e^{-2jt}}{2j} \right) dt$$

$$= \frac{1}{2j} \int_0^\infty e^{-st + 2jt} \, dt - \frac{1}{2j} \int_0^\infty e^{-st - 2jt} \, dt$$

$$= \frac{1}{2j} \int_0^\infty e^{-(s-2j)t} \, dt - \frac{1}{2j} \int_0^\infty e^{-(s+2j)t} \, dt$$

$$= \frac{1}{2j} \left[ \frac{e^{-(s-2j)t}}{-(s-2j)} - \frac{e^{-(s+2j)t}}{-(s+2j)} \right]_0^\infty$$

$$= \frac{1}{2j} \left[ \frac{1}{(s-2j)} - \frac{1}{s+2j} \right]$$

$$= \frac{1}{2j} \left[ \frac{(s+2j) - (s-2j)}{s^2 - 2^2 j^2} \right]$$

$$= \frac{1}{2j} \frac{4j}{s^2 + 2^2}$$

$$= \frac{2}{s^2 + 2^2}$$

$$= \frac{2}{s^2 + 4}$$

Now,

$$L\{\cos 3t\} = \int_0^\infty e^{-st} \cdot \cos 3t \, dt = \frac{s}{s^2+9}$$

$$= \int_0^\infty e^{-st} \cdot \left(\frac{e^{3it} + e^{-3it}}{2}\right) dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st} \left(e^{3it} + e^{-3it}\right) dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-3i)t} \, dt + \frac{1}{2}\int_0^\infty e^{-(s+3i)t} \, dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-3i)t}}{-(s-3i)} + \frac{e^{-(s+3i)t}}{-(s+3i)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{(s-3i)} + \frac{1}{(s+3i)}\right]$$

$$= \frac{1}{2}\left[\frac{(s+3i)+(s-3i)}{(s-3i)(s+3i)}\right]$$

$$= \frac{1}{2} \frac{2s}{s^2+3^2}$$

$$= \frac{s}{s^2+9}$$

Putting these values in equation (i),

$$L\{\sin 2t + 2\cos 3t\} = \frac{2}{s^2+4} + 2\cdot\frac{s}{s^2+9}$$

$$= \frac{2}{s^2+4} + \frac{2s}{s^2+9}$$

(ii) Shifting property :

First shifting property : If $L\{f(t)\} = f(s)$ be the Laplace transform of $f(t)$. Then,

$$L\{e^{kt}\cdot f(t)\} = f(s-k)$$

Proof: We ~~know,~~ have,

$$L\{e^{kt}\cdot f(t)\} = \int_0^\infty e^{-st}\cdot e^{kt} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-k)t}\cdot f(t)\,dt$$

$$= f(s-k)$$

**Example:** Evaluate, $L\{e^{5t}\sin 3t\}$

**Soln:**

We know that, and,

$$L\{\sin at\} = \frac{a}{s^2+a^2} \qquad L\{e^{kt}.\sin at\} = \frac{a}{(s-k)^2+a^2}$$

$$\therefore L\{\sin 3t\} = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}$$

Now,

$$L\{e^{5t}.\sin 3t\} = \frac{3}{(s-5)^2+9}$$

**Second shifting property:**

If $G(t) = \begin{cases} f(t-k) & ; t>k \\ 0 & ; 0<t<k \end{cases}$

then, $L\{G(t)\} = e^{-ks} f(s)$

**Proof:** We have,

$$L\{G(t)\} = \int_0^\infty e^{-st} \cdot G(t)\, dt$$

$$= \int_0^k e^{-st} \cdot G(t) \cdot dt + \int_k^\infty e^{-st} G(t)\, dt$$

$$= 0 + \int_k^\infty e^{-st} \cdot \cancel{g(t)\,dt}\; f(t-k)\,dt$$

$$= \int_k^\infty e^{-st} \cdot f(t-k)\,dt$$

let,
$t-k = u \Rightarrow t = u+k$

$\Rightarrow du = dt$

| $t$ | $k$ | $\infty$ |
|---|---|---|
| $u$ | $0$ | $\infty$ |

$$= \int_{0\,\cancel{k}}^\infty e^{-s(u+k)} \cdot f(u) \cdot du$$

$$= \int_0^\infty e^{-su} \cdot e^{-sk} \cdot f(u)\,du$$

$$= e^{-sk} \int_0^\infty e^{-su} \cdot f(u)\,du$$

$$= e^{-sk}\; L\{f(u)\}$$

$$= e^{-ks} \cdot f(s) \quad \text{(proved)}$$

<u>Example :-</u>

If $f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & ; \; t > \frac{2\pi}{3} \\ 0 & ; \; t < \frac{2\pi}{3} \end{cases}$

find $L\{f(t)\}$.

**Sol$^n$:** Given that,

$$f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & ; t > \frac{2\pi}{3} \\ 0 & ; t < \frac{2\pi}{3} \end{cases}$$

~~Now, $L\{f(t)\} =$~~

We know that,

$$L\{G(t)\} = e^{-ks} \cdot f(s)$$

$$= e^{-\frac{2\pi}{3}s} \cdot f(s)$$

$$= e^{-\frac{2\pi}{3}s} \cdot L\{\cos t\}$$

$$= e^{-\frac{2\pi}{3}s} \cdot \frac{s}{s^2+1}$$

$$= \frac{s}{s^2+1} \cdot e^{-\frac{2\pi s}{3}}$$

Ans

(iii) Change of scale property :

If $L\{f(t)\} = f(s)$ then $L\{f(kt)\} = \frac{1}{k} f\left(\frac{s}{k}\right)$

Example: Evaluate $L\{\sin 3t\}$

Solⁿ:

Here,

$$L\{\sin 3t\} = \frac{1}{3} \cdot f\left(\frac{s}{3}\right)$$

$$= \frac{1}{3} \cdot \frac{1}{\left(\frac{s}{3}\right)^2 + 1} \quad \left[\because f(s) = \frac{1}{s^2+1}\right.$$

$$= \frac{1}{3 \cdot \frac{s^2+3^2}{3^2}}$$

$$= \frac{3^2}{3(s^2+3^2)}$$

$$= \frac{3}{s^2+3^2}$$

$$= \frac{3}{s^2+9}$$

## Laplace transformation of derivative:

1st derivative:

$$L\left\{\frac{df(t)}{dt}\right\} = \int_0^\infty e^{-st} \cdot \frac{df(t)}{dt} \cdot dt$$

$$= \left[e^{-st} \cdot f(t)\right]_0^\infty + s\int e^{-st} f(t)\, dt$$

$$= -f(0) + s f(s)$$

$$= s f(s) - f(0)$$

2nd derivative:

$$L\left\{\frac{d^2f(t)}{dt^2}\right\} = \int_0^\infty e^{-st} \cdot \frac{d^2f(t)}{dt^2} \cdot dt$$

$$= \left[e^{-st} \cdot \frac{df(t)}{dt}\right]_0^\infty + s\int e^{-st} \cdot \frac{df(t)}{dt} \cdot dt$$

$$= -\frac{df(0)}{dt} + s\left[-f(0) + s f(s)\right]$$

$$= -f'(0) - s f(0) + s^2 f(s)$$

Solve the initial value problem $\frac{d^2x}{dt^2}+4\frac{dx}{dt}+4x=4e^{-2t}$ Given $x(0)=-1$ and $x'(0)=4$.

<u>Soln:</u> Given differential equation,

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+4x=4e^{-2t}$$

Taking Laplace transform on both sides, we get,

$$L\left\{\frac{d^2x}{dt^2}\right\}+4L\left\{\frac{dx}{dt}\right\}+4L\{x\}=4L\{e^{-2t}\}$$

$$\Rightarrow -x'(0)-s\,x(0)+s^2x(s)+4\{-x(0)\}+4s\,x(s)+4L\{x(t)\}=\frac{4}{s+2}$$

$$\Rightarrow -4+s+s^2x(s)+4+4s\,x(s)+4\,x(s)=\frac{4}{s+2}$$

$$\Rightarrow x(s)\{s^2+4s+4\}=\frac{4}{s+2}-s$$

$$\Rightarrow x(s)=\frac{4-s^2-2s}{(s+2)(s^2+4s+4)}$$

$$=\frac{4-s(s+2)}{(s+2)(s+2)^2}$$

$$=\frac{4}{(s+2)(s+2)^2}-\frac{s}{(s+2)^2}$$

$$\Rightarrow L\{x(t)\} = \frac{4}{(s+2)^3} - \frac{s+2-2}{(s+2)^2}$$

$$= \frac{4}{(s+2)^3} - \frac{1}{(s+2)} + \frac{2}{(s+2)^2}$$

$$= \frac{4}{(s+2)^3} + \frac{2}{(s+2)^2} - \frac{1}{s+2}$$

$$\therefore x(t) = L^{-1}\left[\frac{4}{(s+2)^3} + \frac{2}{(s+2)^2} - \frac{1}{s+2}\right]$$

$$\cancel{L^{-1}\ 4}$$

$$= 4L^{-1}\left\{\frac{1}{(s+2)^3}\right\} + 2L^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$- L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\cancel{= 4 \cdot e^{-2t}}$$

$$= 4L^{-1}\left\{\frac{\sqrt{2+1}}{\sqrt{2+1}\,(s+2)^{2+1}}\right\} + 2L^{-1}\left\{\frac{\sqrt{1+1}}{\sqrt{1+1}\,(s+2)^{1+1}}\right\}$$

$$- L^{-1}\left\{\frac{\sqrt{0+1}}{\sqrt{0+1}\,(s+2)^{0+1}}\right\}$$

$$= \frac{4}{\Gamma(2+1)} L^{-1}\left\{\frac{\Gamma(2+1)}{(s+2)^{2+1}}\right\} + \frac{2}{\Gamma(1+1)} L^{-1}\left\{\frac{\Gamma(1+1)}{(s+2)^{1+1}}\right\}$$

$$- L^{-1}\left\{\frac{\Gamma(0+1)}{(s+2)^{0+1}}\right\}$$

$$= \frac{4}{2!} \cdot e^{-2t} \cdot t^{2} + \frac{2}{1!} \cdot e^{-2t} \cdot t^{1} - e^{-2t} \cdot t^{0}$$

$$= \frac{4}{2} \times e^{-2t} t^{2} + 2e^{-2t} \cdot t - e^{-2t}$$

$$= 2e^{-2t} t^{2} + 2e^{-2t} \cdot t - e^{-2t}$$

$$= e^{-2t}(2t^{2} + 2t - 1)$$ Ans

$\because L(t^{n}) = \frac{\Gamma(n+1)}{s^{n+1}}$

And,

$L(e^{-mt} \cdot t^{n}) = \frac{\Gamma(n+1)}{(s+m)^{n+1}}$

$\therefore e^{-mt} \cdot t^{n} = L^{-1}\left\{\frac{\Gamma(n+1)}{(s+m)^{n+1}}\right\}$

Another easy formula,

$$x(t) = L^{-1}\left[\frac{4}{(s+2)^{3}} + \frac{2}{(s+2)^{2}} - \frac{1}{(s+2)}\right]$$

$$= 4L^{-1}\left(\frac{1}{(s+2)^{3}}\right) + 2L^{-1}\left(\frac{1}{(s+2)^{2}}\right) - L^{-1}\left(\frac{1}{(s+2)}\right)$$

$$= 4e^{-2t} \cdot L^{-1}\left(\frac{1}{s^{3}}\right) + 2 \cdot e^{-2t} \cdot L^{-1}\left(\frac{1}{s^{2}}\right) - e^{-2t} \cdot L^{-1}\left(\frac{1}{s}\right)$$

$$= 4 \cdot e^{-2t} \cdot \frac{t^2}{\lfloor 3} + 2 \cdot e^{-2t} \cdot \frac{t^1}{\lfloor 2} - e^{-2t} \cdot \frac{t^0}{\lfloor 1}$$

$$= \frac{4}{2!} e^{-2t} t^2 + \frac{2}{1!} e^{-2t} \cdot t - e^{-2t}$$

$$= 2e^{-2t} t^2 + 2e^{-2t} \cdot t - e^{-2t}$$

$$= e^{-2t} (2t^2 + 2t - 1)$$

$\therefore L(t^n) = \frac{\lfloor n+1}{s^{n+1}}$

And $L(e^{-mt} \cdot t^n) = \frac{\lfloor n+1}{(s+m)^{n+1}}$

$\therefore e^{-mt} \cdot t^n = L^{-1}\left\{\frac{\lfloor n+1}{(s+m)^{n+1}}\right\}$

Another way Formula

$$x(t) = L^{-1}\left[\frac{4}{(s+2)^3} + \frac{2}{(s+2)^2} - \frac{1}{(s+2)}\right]$$

$$= 4 L^{-1}\left(\frac{1}{(s+2)^3}\right) + 2 L^{-1}\left(\frac{1}{(s+2)^2}\right) - L^{-1}\left(\frac{1}{(s+2)}\right)$$

$$= 4 e^{-2t} \cdot L^{-1}\left(\frac{1}{s^3}\right) + 2 \cdot e^{-2t} \cdot L^{-1}\left(\frac{1}{s^2}\right) - e^{-2t} \cdot L^{-1}\left(\frac{1}{s}\right)$$

## Bessel's Function :

The equation $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$ is called Bessel's differential equation of order $n$.

By solving this equation, we get,

$$\boxed{J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r+n}}{r!\,(n+r)!}}$$

and this function is called Bessel's function.

## Prove that $J_{-n}(x) = (-1)^n J_n(x)$

**Soln:**

We have,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r+n}}{r!\,(n+r)!}$$

$$\therefore J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{x}{2}\right)^{2r-n}}{r!\,(-n+r)!}$$

Putting $-n+r = s$,

$\Rightarrow r = s+n$

Therefore,

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{s+n} \left(\frac{x}{2}\right)^{2s+n}}{(s+n)!\,s!} = (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{2s+n}}{s!\,(s+n)!}$$

$$= (-1)^n J_n(x)$$ (Proved)

(8) Prove that, $x J_n'(x) = x J_{n-1}(x) - n J_n(x)$

Sol$^n$:

We know,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r+n}}{r!\,(n+r)!}$$

$$\Rightarrow J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (2r+n)(x/2)^{2r+n-1} \cdot \frac{1}{2}}{r!\,(n+r)!}$$

$$\Rightarrow x J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (2r+2n-n)(x/2)^{2r+n-1} \cdot \left(\frac{x}{2}\right)}{r!\,(n+r)!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2r+2n-n)(x/2)^{2r+n}}{r!\,(n+r)!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cdot 2(r+n)(x/2)^{2r+n}}{r!\,(n+r)!} - n\sum_{r=0}^{\infty} \frac{(-1)^r \cdot (x/2)^{2r+n}}{r!\,(n+r)!}$$

$$= x\sum_{r=0}^{\infty} \frac{(-1)^r \cdot \cancel{\left(\frac{x}{2}\right)^{-1}} \cdot \frac{2}{x}(n+r)(x/2)^{2r+n}}{r!\,(n+r)!} - n J_n(x)$$

$$= x\sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-1} \cdot (n+r)(x/2)^{2r+n}}{r!\,(n+r)!} - n J_n(x)$$

$$= x\sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r+n-1} \cdot (n+r)}{r!\,(n+r)(n+r-1)!} - n J_n(x)$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r+(n-1)}}{r!\,\{(n-1)+r\}!} - n J_n(x)$$

$$= x J_{n-1}(x) - n J_n(x)$$ (proved)

(3) Prove that, $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

Soln:- We know,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(x/2\right)^{2r+n}}{r!\,(n+r)!}$$

$$\Rightarrow J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (2r+n)\left(x/2\right)^{2r+n-1} \cdot \frac{1}{2}}{r!\,(n+r)!}$$

$$\Rightarrow x J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (2r+n)\left(x/2\right)^{2r+n-1} \cdot \frac{x}{2}}{r!\,(n+r)!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2r+n)\left(x/2\right)^{2r+n}}{r!\,(n+r)!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, 2r \cdot \left(x/2\right)^{2r+n}}{r!\,(n+r)!} + n \sum_{r=0}^{\infty} \frac{(-1)^r \left(x/2\right)^{2r+n}}{r!\,(n+r)!}$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \frac{2}{x} \cdot \left(x/2\right)^{2r+n} \cdot r}{r!\,(n+r)!} + n J_n(x)$$

$$= x\sum_{r=0}^{\infty}\frac{(-1)^{r}\left(\frac{x}{2}\right)^{-1}\left(\frac{x}{2}\right)^{2r+n}\cdot r}{r!\,(n+r)!} + n J_n(x)$$

$$= x\sum_{r=0}^{\infty}\frac{(-1)\cdot(-1)^{r-1}\left(\frac{x}{2}\right)^{2r+n-1}\cdot r}{r(r-1)!\,(n+r)!} + n J_n(x)$$

$$= x\sum_{r=0}^{\infty}\frac{(-1)(-1)^{r-1}\cdot\left(\frac{x}{2}\right)^{\{2(r-1)+(n+1)\}}}{(r-1)!\,\{(n+1)+(r-1)\}!} + n J_n(x)$$

$$= -x\sum_{r=0}^{\infty}\frac{(-1)^{r-1}\cdot\left(\frac{x}{2}\right)^{\{2(r-1)+(n+1)\}}}{(r-1)!\,\{(n+1)+(r-1)\}!} + n J_n(x)$$

$$= -x J_{n+1}(x) + n J_n(x)$$

$$= +n J_n(x) - x J_{n+1}(x) \quad \text{(proved)}$$

Prove that $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

Soln:-

We know that,

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x) \quad \text{--- (i)}$$

and,

$$x J_n'(x) = -n J_n(x) + x J_{n-1}(x) \quad \text{--- (ii)}$$

(i)+(ii),

$$2x J_n'(x) = x\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

$$\Rightarrow 2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

Prove that,

$$2nJ_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

we know that,

$$x J_n'(x) = nJ_n(x) - xJ_{n+1}(x) \quad \text{—— (i)}$$

$$x J_n'(x) = -nJ_n(x) + xJ_{n-1}(x) \quad \text{—— (ii)}$$

(i)-(ii),

$$0 = 2nJ_n(x) - x\left[J_{n+1}(x) + J_{n-1}(x)\right]$$

$$\Rightarrow 2nJ_n(x) = x\left[J_{n+1}(x) + J_{n-1}(x)\right]$$

Prove that,

(i) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

(ii) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

$$\Rightarrow \left[ \left(J_{1/2}(x)\right)^2 + \left(J_{-1/2}(x)\right)^2 = \frac{2}{\pi x} \right]$$

Soln:

(i) We know that,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r}}{r!\,(n+r)!}$$

Putting the values of $r$ as 0, 1, 2, 3... respectively,

$$J_n(x) = \frac{1}{n!} \cdot \frac{x^n}{2^n} - \frac{1}{1!\,(n+1)!} \frac{x^{n+2}}{2^{n+2}} + \frac{1}{2!\,(n+2)!} \frac{x^{n+4}}{2^{n+4}} -$$

$$J_n(x) = \frac{1}{\lfloor n+1} \frac{x^n}{2^n} - \frac{1}{\lfloor n+2} \cdot \frac{x^n}{2^n} \cdot \frac{x^2}{4} + \frac{1}{2!\,\lfloor n+3} \cdot \frac{x^n}{2^n} \cdot \frac{x^4}{16} -$$

$$\Rightarrow J_n(x) = \frac{1}{\Gamma(n+1)} \frac{x^n}{2^n} \left\{ 1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4^2(n+2)(n+1)\cdot 2!} - \cdots \right\}$$

Putting $n = \frac{1}{2}$,

$$J_{1/2}(x) = \frac{1}{\Gamma(3/2)} \cdot \frac{x^{1/2}}{2^{1/2}} \cdot \left\{ 1 - \frac{x^2}{4(3/2)} + \frac{x^4}{4^2 \cdot 2!\,(5/2)\cdot(3/2)} - \cdots \right\}$$

$$J_{1/2}(x) = \frac{1}{\frac{1}{2}\Gamma(1/2)} \left(\frac{x}{2}\right)^{1/2} \cdot \frac{x}{x} \left\{ 1 - \frac{x^2}{3.2} + \frac{x^4}{5.4.3.2} - \cdots \right\}$$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2^{1/2}} \cdot x^{1/2} \cdot \frac{1}{x} \left\{ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right\}$$

$$= \frac{2^{1/2}}{\sqrt{\pi}} \cdot \frac{1}{x^{1/2}} \left\{ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right\}$$

$$= \sqrt{\frac{2}{\pi x}} \cdot \sin x \quad \text{(proved)}$$

Soln:-

(ii) We know that,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{x}{2}\right)^{2r+n}}{r!\,\Gamma(n+r+1)}$$

Putting the values of $r$ as, 0, 1, 2, 3 ... respectively,

$$J_n(x) = \frac{1}{\Gamma(n+1)} \cdot \frac{x^n}{2^n} - \frac{1}{1!\,\Gamma(n+2)} \cdot \frac{x^n}{2^n} \cdot \frac{x^2}{2^2} + \frac{1}{2!\,\Gamma(n+3)} \cdot \frac{x^n}{2^n} \cdot \frac{x^4}{2^4} - \cdots$$

$$\Rightarrow J_n(x) = \frac{1}{\Gamma(n+1)} \cdot \frac{x^n}{2^n} \left\{ 1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4^2(n+2)(n+1)\cdot 2!} - \cdots \right\}$$

Putting $n = -\frac{1}{2}$

$$J_{-1/2}(x) = \frac{1}{\Gamma\left(-\frac{1}{2}+1\right)} \cdot \frac{x^{-1/2}}{2^{-1/2}} \left\{1 - \frac{x^2}{4 \cdot \frac{1}{2}} + \frac{x^4}{2! \, 4^2 \, \frac{3}{2} \cdot \frac{1}{2}} - \cdots\right\}$$

$$J_{-1/2}(x) = \frac{1}{\Gamma\frac{1}{2}} \frac{2^{1/2}}{x^{1/2}} \left\{1 - \frac{x^2}{2!} + \frac{x^4}{4 \cdot 3 \cdot 2!} - \cdots\right\}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{x}} \cdot \cos x$$

$$= \sqrt{\frac{2}{\pi x}} \cos x$$

Extra:-

$$\because J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sin x$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \cos x$$

$$\therefore \{J_{1/2}(x)\}^2 + \{J_{-1/2}(x)\}^2 = \frac{2}{\pi x}(\sin^2 x + \cos^2 x)$$

$$= \frac{2}{\pi x} \cdot 1$$

$$= \frac{2}{\pi x}$$

(iii) We know that,

$$\boxed{J_n(x) = \frac{x}{2n}\left[J_{n+1}(x) + J_{n-1}(x)\right]}$$

Putting, $n = \frac{1}{2}$, we get,

$$J_{1/2}(x) = \frac{x}{2\cdot\frac{1}{2}}\left[J_{1/2+1}(x) + J_{1/2-1}(x)\right]$$

$$\Rightarrow J_{1/2}(x) = x\left[J_{3/2}(x) + J_{-1/2}(x)\right]$$

$$\Rightarrow x J_{3/2}(x) = J_{1/2}(x) - x J_{-1/2}(x)$$

$$\Rightarrow J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$$

$$= \frac{1}{x}\cdot\sqrt{\frac{2}{\pi x}}\cdot\sin x - \sqrt{\frac{2}{\pi x}}\cdot\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left\{\frac{\sin x}{x} - \cos x\right\}$$

(iv) We know that,

$$J_n(x) = \frac{1}{2n}\left[J_{n+1}(x) + J_{n-1}(x)\right]$$

now putting the value of $n = -\frac{1}{2}$, we get,

$$J_{-1/2}(x) = \frac{x}{2\cdot\left(\frac{-1}{2}\right)}\left[J_{-1/2+1}(x) + J_{-1/2-1}(x)\right]$$

$$\Rightarrow J_{-1/2}(x) = -x\left[J_{1/2}(x) + J_{-3/2}(x)\right]$$

$$\Rightarrow J_{-1/2}(x) = -xJ_{1/2}(x) - xJ_{-3/2}(x)$$

$$\Rightarrow xJ_{-3/2}(x) = -xJ_{1/2}(x) - J_{-1/2}(x)$$

$$= -x\sqrt{\frac{2}{\pi x}}$$

$$= -x\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= -\sqrt{\frac{2}{\pi x}}\left[x\sin x + \cos x\right]$$

$$\therefore J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left[\frac{x\sin x + \cos x}{x}\right]$$

$$= -\sqrt{\frac{2}{\pi x}}\cdot\left[\sin x + \frac{\cos x}{x}\right]$$

Ans

## Hypergeometric Equation and Hypergeometric function :

The series, $1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\dots\dots$ —(i);

is called Hypergeometric series.

The generalisation of this series is,

$$1+x+x^2+\dots\dots$$

And the Hypergeometric equation of this Hypergeometric series is,

$$\boxed{x(1-x)y''+[\gamma-(\alpha+\beta+1)x]y'-\alpha\beta y=0} \quad \text{—(ii)}$$

we may write the series (i) in the form as,

$${}_2F_1(\alpha,\beta;\gamma;x)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n}\times x^n \quad \text{—(iii)}$$

~~This equation no (ii) is called~~

This is called the Hyper geometric function.

## Properties of Hypergeometric function

### 1. Symmetry property :

$${}_2F_1(\alpha,\beta;\gamma;0)=1 \text{ and } {}_2F_1(\alpha,\beta;\gamma;x)={}_2F_1(\beta,\alpha;\gamma;x)$$

The hypergeometric function does not change if the parameters $\alpha$ and $\beta$ are interchanged.

**Proof:**

We know that,

$$\boxed{{}_2F_1(\alpha,\beta;\gamma;x)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k\,k!}\cdot x^k}$$

$$=\sum_{k=0}^{\infty}\frac{(\beta)_k(\alpha)_k}{(\gamma)_k\,k!}\cdot x^k$$

$$\therefore {}_2F_1(\alpha,\beta;\gamma;x)={}_2F_1(\beta,\alpha;\gamma;x) \quad \text{(proved)}$$

For, $x=0$ we can write,

$${}_2F_1(\alpha,\beta;\gamma;x)={}_2F_1(\beta,\alpha;\gamma;0)=\lim_{x\to 0}\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k}x^k$$

$$=\lim_{x\to 0}\left[1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\dots\dots\right]$$

$$=1$$

$$\boxed{\text{Therefore } {}_2F_1(\alpha,\beta;\gamma;x)=1 \text{ for } x=0}$$

## Differentiation of H.F.

We know that,

$$ {}_2F_1(\alpha,\beta;\gamma;x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k \cdot (\beta)_k}{(\gamma)_k \cdot k!} x^k $$

$$ \therefore \frac{d}{dx}\left\{{}_2F_1(\alpha,\beta;\gamma;x)\right\} = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k \cdot k!} \cdot k \cdot x^{k-1} $$

$$ = \sum_{k=1}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k \cdot k(k-1)!} \cdot k \cdot x^{k-1} $$

$$ = \sum_{k=1}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k \cdot (k-1)!} \cdot x^{k-1} $$

When,

$$ k-1 = n $$

$$ \Rightarrow k = n+1 $$

then,

$$ \frac{d}{dx}\left\{{}_2F_1(\alpha,\beta;\gamma;x)\right\} = \sum_{n=0}^{\infty} \frac{(\alpha)_{n+1} (\beta)_{n+1}}{(\gamma)_{n+1}\, n!} \cdot x^n $$

Since,

$$ (\alpha)_{n+1} = \alpha(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+n-1)(\alpha+n) $$

$$ = \alpha\left[(\alpha+1)(\alpha+2)\ldots(\alpha+1+n-1)\right] $$

$$ = \alpha\,(\alpha+1)_n $$

Similarly,

$$(\beta)_{n+1} = \beta(\beta+1)_n$$

and,

$$(\gamma)_{n+1} = \gamma(\gamma+1)_n$$

Thus,

$$\frac{d}{dx}\left[{}_2F_1(\alpha, \beta; \gamma; x)\right] = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)_n \, \beta(\beta+1)_n}{n! \, \gamma(\gamma+1)_n} \cdot x^n$$

$$= \frac{\alpha\beta}{\gamma} \sum_{n=0}^{\infty} \frac{(\alpha+1)_n \, (\beta+1)_n}{(\gamma+1)_n \cdot n!} \cdot x^n$$

$$= \frac{\alpha\beta}{\gamma}\left[{}_2F_1\{(\alpha+1), (\beta+1); \gamma+1; x\}\right]$$

Similarly,

$$\frac{d^2}{dx^2}\left[{}_2F_1(\alpha, \beta; \gamma; x)\right] = \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{\gamma(\gamma+1)} \cdot {}_2F_1(\alpha+2, \beta+2; \gamma+2; x)$$

$$\cdots \quad \cdots$$

$$\frac{d^m}{dx^m}\left[{}_2F_1(\alpha, \beta; \gamma; x)\right]$$

$$= \frac{\alpha(\alpha+1)\ldots(\alpha+m-1) \cdot \beta(\beta+1)\ldots(\beta+m-1)}{\gamma(\gamma+1)\ldots\ldots(\gamma+m-1)} \cdot {}_2F_1(\alpha+m, \beta+m; \gamma+m; x)$$

Now, we have,

$$\frac{d}{dx}\left[{}_2F_1(\alpha,\beta;\gamma;x)\right] = \frac{\alpha\beta}{\gamma}\left[{}_2F_1(\alpha+1,\beta+1;\gamma+1;x)\right]$$

$$= \frac{\alpha\beta}{\gamma} \times \lim_{x\to 0} \sum_{k=0}^{\infty} \frac{(\alpha+1)_k (\beta+1)_k}{(\gamma+1)_k \cdot k!} x^k$$

$$= \frac{\alpha\beta}{\gamma} \times \lim_{x\to 0}\left[1 + \frac{(\alpha+1)\cdot(\beta+1)}{1\cdot(\gamma+1)}\cdot x + \frac{(\alpha+1)(\alpha+2)\cdot(\beta+1)(\beta+2)}{(\gamma+1)(\gamma+2)\cdot 1.2} x^2 + \dots\right]$$

$$= \frac{\alpha\beta}{\gamma} \times 1$$

$$= \frac{\alpha\beta}{\gamma}$$

Therefore,

$$\frac{d}{dx}\left[{}_2F_1(\alpha,\beta;\gamma;x)\right] = \frac{\alpha\beta}{\gamma}$$

## The confluent Hypergeometric equation function:-

The Hypergeometric equation,

$$x(1-x)\frac{d^2y}{dx^2}+\left[\gamma-(\alpha+\beta+1)x\right]\frac{dy}{dx}-\alpha\beta y=0$$

Putting $x=\frac{\alpha}{\beta}$,

$$x\left(1-\frac{\alpha}{\beta}\right)\frac{d^2y}{dx^2}+\left[\gamma-\left(1+\frac{\alpha+1}{\beta}\right)x\right]\frac{dy}{dx}-\alpha y=0 \quad \text{——(i)}$$

the soln of which represented by the function ${}_2F_1(\alpha,\beta;\gamma;x/\beta)$

Letting, $\beta\to\infty$; the function becomes,

$$\lim_{\beta\to\infty} {}_2F_1\left(\alpha,\beta;\gamma;\frac{x}{\beta}\right)$$ and (i) reduces to,

$$x\frac{d^2y}{dx^2}+(\gamma-\alpha)\frac{dy}{dx}-\alpha y=0 \quad \text{——(ii)}$$

This 2nd order D.E. (ii) in which $\alpha$ and $\gamma$ are constant is called confluent hypergeometric function.

The soln of (ii) is represented by,

$$\lim_{\beta\to\infty} {}_2F_1\left(\alpha,\beta;\gamma;x/\beta\right)=\lim_{\beta\to\infty}\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k}\cdot\frac{x^k}{\beta^k}$$

But,

$$\lim_{\beta \to \infty} \frac{(\beta)_k}{\beta^k} = 1$$

Hence,

$$\lim_{\beta \to \infty} {}_2F_1\left(\alpha, \beta; \gamma; x/\beta\right) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!\,(\gamma)_k} \cdot x^k$$

$$= {}_1F_1\left(\alpha; \gamma; x\right)$$

The series,

$$\sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!\,(\gamma)_k} x^k \text{ is denoted by } {}_1F_1\left(\alpha; \cancel{\beta}\,\gamma; \cancel{x}\,x\right)$$

Since it is the solution of confluent hypergeometric equation.

## Hermite's Differential equation and Hermite's polynomials :

The differential equation of the form

$$\boxed{y'' - 2xy' + 2\alpha y = 0 \; ;} \quad \text{(i)} \quad \text{(where } \alpha \text{ is a constant)}$$

is known as Hermite differential equation.

By solving this equation we get,

$$\boxed{y = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} + \dots \dots + (-1)^r \frac{(2x)^{n-2r}}{r!} n(n-1)(n-2)\dots(n-2r+1) + \dots}$$

This value of $y$ is known as Hermite polynomial of degree $n$ and is denoted by $H_n(x)$.

## Legendre's equation and Legendre's polynomials:

The second order differential equation,

$$\boxed{(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\}y = 0}$$

in which m and n are constants is known as Legendre's associated equation and its solutions are called associated Legendre functions.

It reduces to Legendre's equation when $m=0$.

Then Legendre's differential equation is,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \text{——— (i)}$$

And the solⁿ of this eqⁿ is called Legendre's function or Legendre's polynomials.

The solⁿ of Legendre's differential equation is,

$$y = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n-2)(n-3)}{4!}x^4 - \dots\right]$$
$$+ c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n+2)(n+4)}{5!}x^5 - \dots\right]$$

$= c_0 u + c_1 v$ (say) where, $u = \uparrow$ (বড় পৌর্ণ)
$v = \uparrow$ (বড় উর্ধ্ব)

## Legendre polynomial : (After the first part, then this)

$$\boxed{\therefore P_n(x) = \sum_{r=0}^{N} (-1)^r \cdot \frac{(2n-2r)!}{2^n \cdot r! \cdot (n-r)!(n-2r)!} x^{n-2r}}$$

Where,

$N = \frac{n}{2}$ for $n$ even

$N = \frac{n-1}{2}$ for $n$ odd

Another form of $P_n(x)$ is given below,

$$P_n(x) = \frac{(2n)!}{2^n \cdot n! \cdot n!} \left[ x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \dots \dots \right]$$

Rodrigues formula for Legendre polynomial,

$$P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n} (x^2-1)^n$$

**Proof:**

Let,

$$v = (x^2-1)^n$$

Differentiating w.r.t. $x$, we get,

$$\frac{dv}{dx} = v_1 = n(x^2-1)^{n-1} \cdot 2x$$

$$\Rightarrow v_1 = \frac{2nx(x^2-1)^n}{x^2-1}$$

$$\Rightarrow (x^2-1)v_1 = 2nxv$$

$$\Rightarrow (1-x^2)v_1 = -2nxv$$

$$\Rightarrow (1-x^2)v_1 + 2nxv = 0$$

Again differentiating w.r.t. $x$,

$$(1-x^2)v_2 - 2xv_1 + 2nxv_1 + 2nv = 0$$

$$\Rightarrow (1-x^2)v_2 + (2nx-2x)v_1 + 2nv = 0$$

$$\Rightarrow (1-x^2)v_2 + 2(n-1)xv_1 + 2nv = 0$$

Differentiating $n$ times using Leibnitz's theorem,

$$(1-x^2)v_{n+2} + {}^nC_1(-2x)v_{n+1} + {}^nC_2(-2)v_n$$

$$+ 2(n-1)x v_{n+1} + {}^nC_1 2(n-1)v_n + 2n v_n = 0$$

$$\Rightarrow (1-x^2)v_{n+2} + v_{n+1}(-2nx + 2nx - 2x)$$

$$+ v_n\left\{\frac{n(n-1)}{2}(-2) + 2n(n-1) + 2n\right\} = 0$$

$$\Rightarrow (1-x^2)v_{n+2} - 2x v_{n+1} + \{n(n-1) + 2n\}v_n = 0$$

$$\Rightarrow (1-x^2)v_{n+2} - 2x v_{n+1} + n(n+1)v_n = 0$$

$$\Rightarrow (1-x^2)\frac{d^2}{dx^2}(v_n) - 2x\frac{d}{dx}(v_n) + n(n+1)v_n = 0$$

This is a legendre equation of $v_n$.

So, $v_n$ is a solution of legendre equation.

But,

$$v_n = \frac{d^n}{dx^n}v = \frac{d^n}{dx^n}(x^2-1)^n$$

Here we see that, $v_n$ is a polynomial of ~~deg~~ degree $n$.

$P_n(x)$ is also a solution of Legendre equation.

Hence we get,

$$P_n(x) = k\, V_n(x) = k \frac{d^n}{dx^n}(x^2-1)^n \quad \text{——— (i)}$$

where, $k$ is a constant.

But,

$$P_n(x) = \frac{(2n)!}{2^n (n!)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)} \cdot x^{n-2} + \dots\right] \quad \text{——— (ii)}$$

From eqn (i) and (ii) we get,

$$k \frac{d^n}{dx^n}(x^2-1)^n = \frac{(2n)!}{2^n (n!)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \dots\right]$$

$$\Rightarrow \frac{(2n)!}{2^n (n!)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \dots\right] = k \frac{d^n}{dx^n}(x^2-1)^n$$

equating the co-efficient of $x^n$ from both sides,

$$\frac{(2n)!}{2^n \cdot (n!)^2} = k \cdot 2n\,(2n-1)(2n-2)\dots(n+1)$$

$$\Rightarrow \frac{(2n)!}{2^n (n!)^2} = k \frac{(2n)!}{n!}$$

$$\Rightarrow \frac{1}{2^n \cdot n!} = k$$

$$\therefore k = \frac{1}{2^n n!}$$

putting the value $k = \frac{1}{2^n n!}$ in equation (i),

$$\boxed{P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n} (x^2-1)^n}$$

[All of these are with Legendre's polynomials]

~~$P_0(x) =$~~

For $P_0(x)$ ; $n=0$ is even and $N = \frac{n}{2} = 0$

$$\therefore P_0(x) = \sum_{r=0}^{0} (-1)^0 \frac{(2\times0-2\times0)!}{2^0 \times 0! \times (0-0)! \times (0-2\times0)!} x^{0-2\times0}$$

$$= 1 \cdot x^0$$

$$= 1$$

~~For, $P_1(x) = \sum_{r=0}^{0} (-1)^0$~~

For, $P_1(x)$ ; $n=1$ is odd number and $N = \frac{n-1}{2} = 0$

$$\therefore P_1(x) = \sum_{r=0}^{0} (-1)^0 \cdot \frac{(2\times1-2\times0)!}{2^1 \times 0! \times (1-0)! (1-2\times0)!} x^{1-2\times0}$$

$$= 1 \times \frac{2!}{2} \cdot x^1$$

$$= x$$

For $P_2(x) =$ $n=2$ is even number and $N = \frac{n}{2} = \frac{2}{2} = 1$

$$\therefore P_2(x) = \sum_{r=0}^{1} (-1)^r \frac{(2\times2-2\times0)!}{2^2 \times 0! \times (2-0)! \times (2-2\times0)!} x^{2-2\times0}$$

$$= \sum_{r=0}^{1} 1 \cdot \frac{24}{4\times2!\times2!} \times x^2$$

$$= \sum_{r=0}^{1} \frac{3}{2} \cdot x^2$$

For $P_3(x)$ $n=3$ is odd number and $N = \frac{n-1}{2} = \frac{3-1}{2} = 1$

$$P_3(x) = \sum_{r=0}^{1} (-1)^0 \frac{(2\times3-2\times0)!}{2^3 \times 0! \times (3-0)! \times (3-2\times0)!} \cdot x^{3-2\times0}$$

$$= \sum_{r=0}^{1} 1 \times \frac{6!}{8\times3!\times3!} x^3$$

$$= \sum_{r=0}^{1} \frac{720}{288} x^3$$

By Rodrigue's formula:

$$\boxed{P_n(x) = \frac{1}{2^n n!} \times \frac{d^n}{dx^n}(x^2-1)^n}$$

① $P_0(x) = \frac{1}{2^0 \cdot 0!} \times \frac{d^0}{dx^0}(x^2-1)^0 = 1$

② $P_1(x) = \frac{1}{2^1 \cdot 1!} \times \frac{d^1}{dx^1}(x^2-1)^1 = \frac{1}{2} \times 2x = x$

③ $P_2(x) = \frac{1}{2^2 \cdot 2!} \times \frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8} \times \frac{d^2}{dx^2}(x^4-2x^2+1)$

$= \frac{1}{8} \times \frac{d}{dx}(4x^3-4x)$

$\Rightarrow = \frac{1}{8} \times (12x^2-4)$

$= \frac{1}{2}(3x^2-1)$

④ $P_3(x) = \frac{1}{2^3 \cdot 3!} \times \frac{d^3}{dx^3}(x^2-1)^3$

$= \frac{1}{48} \times \frac{d^3}{dx^3}(x^6-3x^4+3x^2-1)$

$= \frac{1}{48} \times \frac{d^2}{dx^2}(6x^5-12x^3+6x)$

$= \frac{6}{48} \times \frac{d}{dx}(5x^4-6x^2+1)$

$= \frac{1}{8} \times (20x^3-12x) = \frac{1}{2}(5x^3-3x)$

Express the following function in terms of Legendre's polynomials,

$$f(x) = x^4 + 2x^3 + 2x^2 - x - 3$$

Solⁿ:-

We know from Legendre polynomials,

$$P_4(x) = \frac{1}{2^4 \cdot 4!} \times \frac{d^4}{dx^4}(x^2-1)^4$$

$$= \cancel{\frac{1}{384}}$$ I don't know

$$\Rightarrow P_4(x) = \frac{1}{8} \times (35x^4 - 30x^2 + 3)$$

$$\Rightarrow \cancel{x^4 =}$$

$$\Rightarrow 8P_4(x) = 35x^4 - 30x^2 + 3$$

$$\Rightarrow 35x^4 = 8P_4(x) + 30x^2 - 3$$

$$\Rightarrow x^4 = \frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}$$

And, $P_3(x) = \frac{1}{12} \times (5x^3 - 3x)$

$$\Rightarrow 2P_3(x) = 5x^3 - 3x$$

$$\Rightarrow 5x^3 = 2P_3(x) + 3x$$

$$\Rightarrow x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x$$

also,

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$\Rightarrow 2P_2(x) = 3x^2 - 1$$

$$\Rightarrow 3x^2 = 2P_2(x) + 1$$

$$\Rightarrow x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}$$

and also,

$$P_1(x) = x$$

$$\therefore x = P_1(x)$$

also,

$$P_0(x) = 1$$

$$\therefore 1 = P_0(x)$$

Putting the values of $x^4, x^3, x^2$ and $x$ we get,

$$f(x) = \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \frac{40}{21}P_2(x) + \frac{1}{5}P_1(x) - \frac{224}{105}P_0(x)$$

**Problem:** Show that, $P_n(-x) = (-1)^n P_n(x)$

We know that,

$$P_n(x) = \frac{(2n)!}{2^n (n!)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \dots\dots\right]$$

Therefore,

$$P_n(-x) = \frac{(2n)!}{2^n (n!)^2}\left[(-x)^n - \frac{n(n-1)}{2(2n-1)}(-x)^{n-2} + \dots\dots\right]$$

$$= \frac{(2n)!}{2^n (n!)^2}\left[(-1)^n \cdot x^n - \frac{n(n-1)}{2(2n-1)}(-1)^{n-2}\cdot(x)^{n-2} + \dots\right]$$

$$= \frac{(2n)!}{2^n (n!)^2}\left[(-1)^n \cdot x^n - \frac{n(n-1)}{2(2n-1)}\cdot\frac{(-1)^n}{(-1)^2}\cdot x^{n-2} + \dots\right]$$

$$= \frac{(2n)!}{2^n (n!)^2} \times (-1)^n \times \left[x^n - \frac{n(n-1)}{2(2n-1)}\cdot x^{n-2} + \dots\right]$$

$$= (-1)^n \frac{(2n)!}{2^n (n!)^2} \times \left[x^n - \frac{n(n-1)}{2(2n-1)}\cdot x^{n-2} + \dots\right]$$

$\therefore P_n(-x) = (-1)^n \cdot P_n(x)$ (Showed)

Find the values of $P_n(1)$, $P_n(-1)$ and $P_n(0)$.

Solⁿ:

We know from generating function,

$$(1-2hx+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \quad \text{——(i)}$$

Putting $x=1$ we get,

$$(1-2h+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(1)$$

$$\Rightarrow \{(1-h)^2\}^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(1)$$

$$\Rightarrow (1-h)^{-1} = \sum_{n=0}^{\infty} h^n P_n(1)$$

$$\Rightarrow 1+h+h^2+h^3+\dots\dots+h^n+\dots = \sum_{n=0}^{\infty} h^n P_n(1)$$

Equating the co-efficient of $h^n$, we get,

$$1 = P_n(1)$$

$$\therefore P_n(1) = 1$$

For, $x=-1$, from eqn (i),

$$\{1-2h(-1)+h^2\}^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(-1)$$

$$\Rightarrow (1+2h+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(-1)$$

$$\Rightarrow \{(1+h)^2\}^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(-1)$$

$$\Rightarrow (1+h)^{-1} = \sum_{n=0}^{\infty} h^n P_n(-1)$$

$$\Rightarrow 1-h+h^2-h^3+\dots+(-1)^n h^n+\dots = \sum_{n=0}^{\infty} h^n P_n(-1)$$

Equating the co-efficients of $h^n$,

$$(-1)^n = P_n(-1)$$

$$\therefore P_n(-1) = (-1)^n$$

For, $x=0$,

$$(1+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(0)$$

$$\therefore P_n(0) = 0$$

Prove that $P_5'(x) = 9P_4(x) + 5P_2(x) + P_0(x)$

Soln:-

From recurrence relation we have,

$$\boxed{P'_{n+1}(x) - P'_{(n-1)}(x) = (2n+1)P_n(x)}$$

Putting $n=4$ and $n=2$ we have,

$$P_5'(x) - P_3'(x) = 9P_4(x) \quad \text{——— (i)}$$

$$P_3'(x) - P_1'(x) = 5P_2(x) \quad \text{——— (ii)}$$

~~Therefore~~, But,

$$P_1(x) = x$$

$$\therefore P_1'(x) = 1$$

$$\therefore P_1'(x) = P_0(x) \quad [\because P_0(x) = 1] \quad \text{——— (iii)}$$

Now from (i), (ii), and (iii) we have

$$P_5'(x) = 9P_4(x) + 5P_2(x) + P_0(x)$$

Showed

## Laguerre Differential equation and Legendre polynomials :

The equation of the form $xy'' + (1-x)y' + \alpha y = 0$ ; —(i)
where $\alpha$ = constant, is known as Laguree's differential equation.

The sol$^n$ of this eq$^n$ is,

$$L_n(x) = (-1)^n \left[ x^n - \frac{n^2}{n!} x^{n-1} + \frac{n^2(n-1)^2}{2!} x^{n-2} + \dots + (-1)^n \cdot n! \right] \quad \text{—(ii)}$$

This is the expression for Legendre polynomials.

From the equation (ii),

$L_n(0) = n!$

$L_0(x) = 1$

$L_1(x) = 1 - x$

$L_2(x) = x^2 - 4x + 2$

$L_3(x) = -x^3 + 9x^2 - 36x + 6$

$L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 48$

# Chapter-02 : Complex Variables

## The real numbers system:

### 1. Natural numbers:

1, 2, 3, 4, ... ... ; also called positive integers, were first used in counting. If a and b are natural numbers the sum $(a+b)$ and product $(a \cdot b)$ are also natural number. For this reason the set of natural number is said to be closed under the operations of addition and multiplication.

### 2. Negative integers and zero:

$-1, -2, -3, \ldots$ and $0$ are arose to permit solutions of equations such as $x+b=a$ where a and b are any natural numbers. This leads to the operation of subtraction or inverse of addition, and we write $x = a-b$

The set of positive and negative integers and zero is called the set of integers and closed under the operations of addition, multiplication and subtraction

3. Rational numbers / fractions:

$3/4$, $-8/3$, ... permit solutions of equations such as $bx = a$ for all integers $a$ and $b$ where $b \neq 0$.

This leads to the operation of division or inverse of multiplication, and we write $x = a/b$ or $a \div b$ where $a$ is the ~~number~~ numerator and $b$ is the denominator.

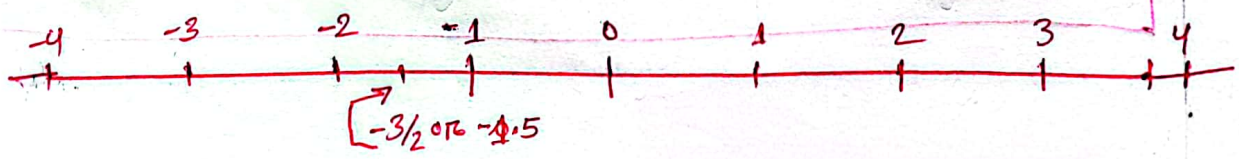
The set of integers is a part or subset of the rational numbers, since integers correspond to rational numbers $a/b$ where $b = 1$

4. Irrational numbers:

$\sqrt{2}$ and $\pi$ are such numbers that cannot be expressed as $a/b$ where $a$ and $b$ are integers and $b \neq 0$

The set of rational and irrational ~~an~~ numbers is called the set of real numbers. ~~It is assumed that the state~~

Graphical representation of real numbers:



## The complex numbers system:

Complex number is a combination of a real number and an imaginary number.

We can consider a complex numbers as having the form $a+bi$ where $a$ and $b$ are real numbers and $i$, which is called the imaginary unit, has the property that $i^2=-1$.

If $z=a+bi$, then $a$ is called the real part of $z$ and $b$ is called the imaginary part of the $z$ and are denoted by $Re\{z\}$ and $Im\{z\}$, respectively.

The symbol $z$, which can stand for any complex number, is called a complex variable.

The complex conjugate, or briefly conjugate, of a complex numbers $a+bi$ is $a-bi$. The complex conjugate of a complex numbers $z$ is often indicated by $\bar{z}$ or $z^*$

Perform the indicated operations both analytically and graphically:

(a) $(3+4j)+(5+2j)$

(b) $(6-2j)+(2-5j)$

Soln:

(a) Analytically :-

$$(3+4j)+(5+2j)$$
$$= 3+5+4j+2j$$
$$= (3+5)+(4+2)j$$
$$= 8+6j$$

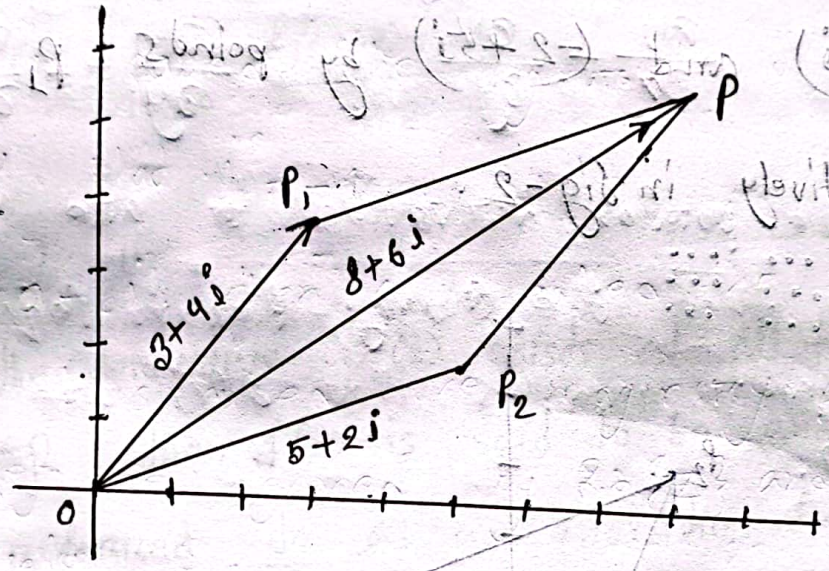
Graphically:

Let us represent the two complex numbers $(3+4j)$ and $(5+2j)$ by points $P_1$ and $P_2$ respectively as in fig-1.

~~Now~~ Then complete the parallelogram with $OP_1$ and $OP_2$ as adjacent sides. Point $P$ represent the sum, $8+6j$, of the two given complex numbers.

Note the similarity with the parallelogram law for addition of vectors $OP_1$ and $OP_2$ to obtain vector $OP$

For this reason it is often convenient to consider a complex number $a+bj$ as a vector components $a$ and $b$ in the directions of the position $x$ and $y$ axes respectively.




ⓑ Analytically :

$$(6-2j)-(2-5j)$$

$$= 6-2-2j+5j$$

$$= (6-2)-(2-5)j$$

$$= 4-(-3)j$$

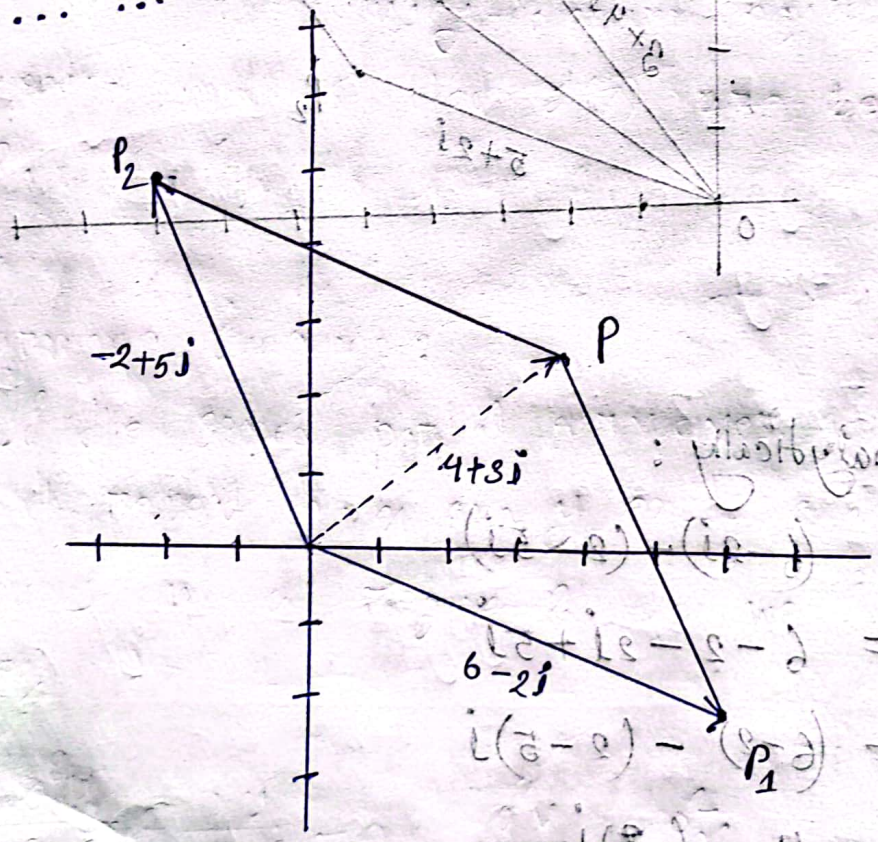
$$= 4+3j$$

Graphically:

$$(6-2j)-(2-5j)$$

$$=(6-2j)+(-2+5j)$$

Now ~~the the~~ let us represent the two ~~vector~~ complex numbers $(6-2j)$ and $(-2+5j)$ by points $P_1$ and $P_2$ respectively in fig-2.



## III Complex - Differentiation and Cauchy - Riemann Equation :

**Derivative :**

If $f(z)$ is single-valued in some region $R$ of the $Z$ plane, the derivative of $f(z)$ is defined as,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

**Analytic function :**

If the derivative $f'(z)$ exists at all points $z$ of a region $R$, then $f(z)$ is said to be analytic in $R$ and is referred to as an analytic function in $R$ or a function analytic in $R$.

A function $f(z)$ is said to be analytic at a point $z_0$ if there exists a neighbourhood $|z - z_0| < \delta$ at all the points of which $f'(z)$ exists.

Analytic function also called the Regular function or Holomorphic function.

## Cauchy – Riemann Equation

A necessary condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic in a region $R$ is that, in $R$, $u$ and $v$ satisfy the Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{—— (1)}$$

If the partial derivatives are continuous in $R$, then the Cauchy-Riemann equations are sufficient conditions that $f(z)$ be analytic in $R$.

## Harmonic functions:

If the second partial derivatives of $u$ and $v$ with respect to $x$ and $y$ exists and are continuous in a region $R$, then we find from eqn (1),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

It follows that under these conditions the real and imaginary part of an analytic

function satisfy Laplace's equation denoted by,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{or} \quad \vec{\nabla}^2 \psi = 0$$

where, $\vec{\nabla}^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

The operator $\vec{\nabla}^2$ is often called Laplacian

## Geometric Interpretation of the derivative,

Let $z_0$ (in fig-1) be a point $P$ in the $z$ plane and let $w_0$ (Fig-2) be its image $P'$ in the $w$ plane under the transformation $w = f(z)$. Since we suppose that $f(z)$ is single-valued, the point $z_0$ maps into only one point $w_0$.

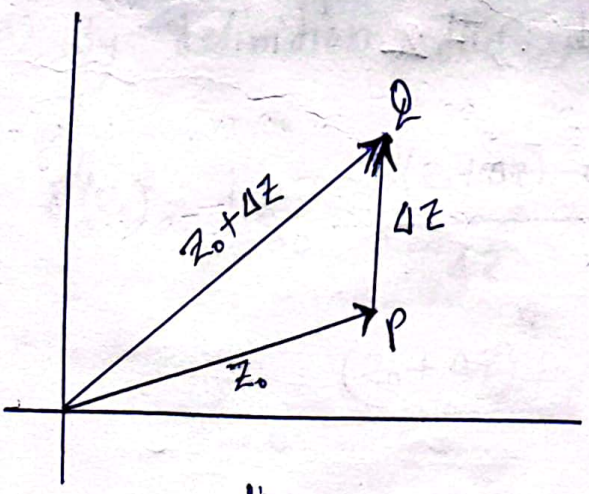



fig-1

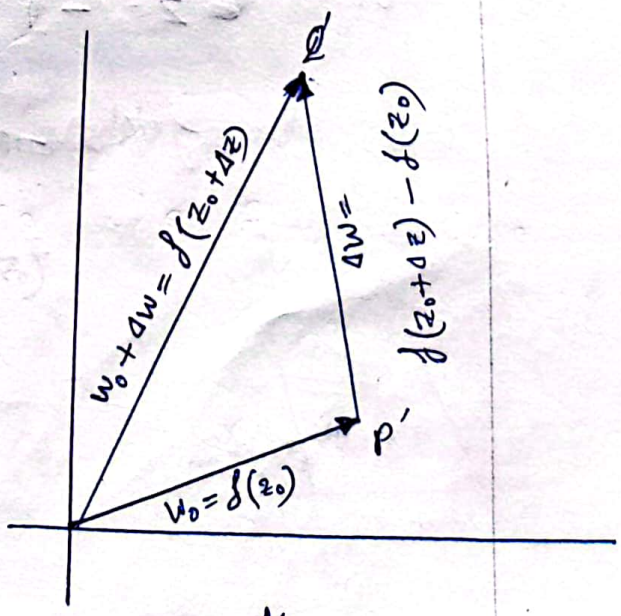



fig-2

If we give $z_0$ an increment $\Delta z$ we obtain the point $Q$ of fig-1. This point has image $Q'$ in the $w$ plane.

Thus from fig-2 we see that $P'Q'$ represents the complex number $\Delta w = f(z_0+\Delta z) - f(z_0)$. It follows that the derivative at $z_0$ is given by

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \lim_{Q \to P} \frac{Q'P'}{QP}$$

i.e. the limit of the ratio $Q'P'$ to $QP$ as point $Q$ approaches point $P$. The above interpretation clearly holds when $z_0$ is replaced by any point $z$.

## L'Hospital's Rule:

Let $f(z)$ and $g(z)$ be analytic in a region containing the point $z_0$ and suppose that $f(z_0)=g(z_0)=0$ but $g'(z_0)\neq 0$. Then L'Hospital's rule states that,

$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\frac{f'(z_0)}{g'(z_0)}$$

In case $f'(z_0)=g'(z_0)=0$, the rule may be extended.

## Solved Problems:

**3.1** Using the definition find the derivative of $w=f(z)=z^3-2z$ at the point where,

(a) $z=z_0$ and

(b) $z=-1$.

Solⁿ

(a) By definition, the derivative at $z=z_0$ is,

$$f'(z_0)=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

$$=\lim_{\Delta z\to 0}\frac{(z_0+\Delta z)^3-2(z_0+\Delta z)-\{(z_0^3-2z_0)\}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z_0^3} + 3z_0^2\Delta z + 3z_0(\Delta z)^2 + (\Delta z)^3 - \cancel{2z_0} - 2\Delta z - \cancel{z_0^3} + \cancel{2z_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3z_0^2\Delta z + 3z_0(\Delta z)^2 + (\Delta z)^3 - 2\Delta z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} 3z_0^2 + 3z_0\Delta z + (\Delta z)^2 - 2$$

$$= 3z_0^2 - 2$$

In general $f'(z) = 3z^2 - 2$ for all $z$

(b) We get from (a),

$$f'(z) = 3z^2 - 2$$

for, $z = -1$,

(a) By definition $f'(-1) = 3(-1)^2 - 2$

$$= 1$$

**3.2** – Show that $\frac{d\bar{z}}{dz}$ does not exist anywhere i.e. $f(z) = \bar{z}$ is non-analytic anywhere.

**Solⁿ:**

By definition,

$$\frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

If this limit exists independent of the manner in where $\Delta z = \Delta x + j\Delta y$ approaches zero.

Then,

$$\frac{d\bar{z}}{dz} = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{x+jy+\Delta x+j\Delta y} - \overline{x+jy}}{\Delta x + j\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x-jy+\Delta x-j\Delta y - (x-jy)}{\Delta x + j\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - j\Delta y}{\Delta x + j\Delta y}$$

If $\Delta y = 0$, the required limit is,

$$\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

If $\Delta x = 0$, the required limit is

$$\lim_{\Delta y \to 0} \frac{-j\Delta y}{j\Delta y} = -1$$

Then, since the limit depends on the manner in which $\Delta z \to 0$, the derivative does not exist; i.e. $f(z) = \bar{z}$ is non-analytic anywhere.

3.3 – Given $w = f(z) = \frac{(1+z)}{(1-z)}$, find,

(a) $\frac{dw}{dz}$ and,

(b) determine where $f(z)$ is non-analytic

Soln:-

(a) using definition,

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\frac{1+(z+\Delta z)}{1-(z+\Delta z)} - \frac{1+z}{1-z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\{1+(z+\Delta z)\}\cdot(z-1) - (1+z)\{1-(z+\Delta z)\}}{\{1-(z+\Delta z)\}\{z-1\}}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{1} - \cancel{z} + z - z^2 + \Delta z - z\Delta z - \cancel{1} - \cancel{z} + \cancel{z} + z^2 + \Delta z + z\Delta z}{\{1-(z+\Delta z)\}\{z-1\}\,\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2\Delta z}{(1-z-\Delta z)(z-1)\,\Delta z}$$

$$= \frac{2}{(1-z)(z-1)}$$

$$= \frac{2}{(z-1)^2}$$

Using differentiation rules,

$$\frac{d}{dz}\left\{\frac{(1+z)}{(1-z)}\right\} = \frac{(1-z)\frac{d}{dz}(1+z) - (1+z)\frac{d}{dz}(1-z)}{(1-z)^2}$$

$$= \frac{(1-z)\times 1 - (1+z)\times(-1)}{(1-z)^2}$$

$$= \frac{1-z+1+z}{(1-z)^2}$$

$$= \frac{2}{(1-z)^2}$$ Ans

Q.7 – (a) Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic

(b) Find $v$ such that $f(z) = u + iv$ is analytic

Soln:

(a) Here,

$$\frac{\partial u}{\partial x} = e^{-x}(\sin y) + (-e^{-x})\cdot(x\sin y - y\cos y)$$

$$= e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = -e^{-x}\sin y - (e^{-x} - xe^{-x})\sin y - ye^{-x}\cos y$$

$$= -e^{-x}\sin y - e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y \quad \text{—— (i)}$$

Now,

$$\frac{\partial u}{\partial y} = e^{-x}\{x\cos y - (y\sin y + \cos y)\}$$

$$= e^{-x}(x\cos y + y\sin y - \cos y)$$

$$= xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = -xe^{-x}\sin y + ye^{-x}\cos y + e^{-x}\sin y + e^{-x}\sin y$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = -xe^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y \quad \text{—— (ii)}$$

adding eqn (i) and (ii) we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Therefore, $u = e^{-x}(x\sin y - y\cos y)$ is harmonic.

(b) From (a),

$$\frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y \quad \text{—— (i)}$$

and $$\frac{\partial u}{\partial y} = xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y$$

From the Cauchy - Riemann equation,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y \quad \text{—— (iii)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{-x}\cos y - xe^{-x}\cos y - ye^{-x}\sin y \quad \text{—— (iv)}$$

Integrate (iii) with respect to $y$ keeping $x$ constant, then

$$v = -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}(y\sin y + \cos y) + F(x)$$

$$= -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}y\sin y + e^{-x}\cos y + F(x)$$

$$= xe^{-x}\cos y + ye^{-x}\sin y + F(x) \quad \text{—— (v)}$$

where $F(x)$ is an arbitrary real function of $x$.
Substitute (v) into (iv) and obtain,

$$\frac{\partial}{\partial x}\left\{ye^{-x}\sin y + xe^{-x}\cos y + F(x)\right\} = e^{-x}\cos y - xe^{-x}\cos y - ye^{-x}\sin y$$

$$\Rightarrow -e^{-x} y \sin y + e^{-x}\cos y - x e^{-x}\cos y + F'(x) = e^{-x}\cos y - x e^{-x}\cos y - y e^{-x}\sin y$$

$$\Rightarrow F'(x) = 0$$

$$\Rightarrow F(x) = \text{constant}$$

$$\Rightarrow F(x) = c$$

Then from eq$^n$ (v),

$$v = y e^{-x}\sin y + x e^{-x}\cos y + c$$ Ans

**3.12** – Construct an analytic function $f(z)$ whose real part is $e^x \cos y$

**Soln:**

Let,

$$f(z) = u(x,y) + i\,v(x,y)$$

It is given that,

$$u(x,y) = e^x \cos y$$

$$\therefore \frac{\partial u}{\partial x} = e^x \cos y \quad \text{and} \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

~~Accordingly Cauchy - Rieman~~

According by Cauchy - Rimann,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x \cos y$$

and,

$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} = -(-$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(-e^x \sin y)$$

$$= e^x \sin y$$

We know that,

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$\cancel{dv = e^x \cos y \, dx +}$$

$$dv = e^x \sin y \, dx + e^x \cos y \, dy$$

This is an exact differential equation,

$$\therefore v = \int e^x \sin y \, dx + \int e^x \cos y \, dy$$

Ignoring the term containing $x$,

$$v = e^x \sin y$$

$$\therefore f(z) = u + iv = e^x \cos y + i e^x \sin y$$

$$= e^x (\cos y + i \sin y)$$

$$= e^x \cdot e^{iy} = e^{x+iy} = e^z$$

**3.13** Find an analytic function, $w(z) = u(x,y) + iv(x,y)$ given that, $v = \frac{x}{x^2+y^2} + \cosh x \cos y$

Soln:-

Now,

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} + \sinh x \cos y$$

and,

$$\frac{\partial v}{\partial y} = \frac{-x\cdot 2y}{(x^2+y^2)^2} - \cosh x \sin y$$

Now, we know that,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\Rightarrow du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy \quad \text{[Using C-R equation]}$$

$$\Rightarrow du = \left[\frac{-2xy}{(x^2+y^2)^2} - \cosh x \sin y\right] dx - \left[\frac{y^2-x^2}{(x^2+y^2)^2} + \sinh x \cos y\right] dy$$

This is an exact differential equation,

$$\int du = \int \left[\frac{-2xy}{(x^2+y^2)^2} - \cosh x \sin y\right] dx - \int \left[\frac{y^2-x^2}{(x^2+y^2)^2} + \sinh x \cos y\right] dy$$

$$\therefore u = \frac{y}{x^2+y^2} - \sinh x \sin y \quad \text{[Ignoring the term containing } x\text{]}$$

$\therefore W = u + iv$

$$= \frac{y}{x^2+y^2} - \sinh x \sin y + i\left[\frac{x}{x^2+y^2} + \cosh x \cos y\right]$$

$$= \frac{y+ix}{x^2+y^2} - \sinh x \sin y + i\cosh x \cos y$$

**3.14** - Construct an analytic function $f(z) = u(x,y) + iv(x,y)$, where $v(x,y) = 6xy - 5x + 3$. Express the result as a function of $z$.

**Soln:**

Given that,

$$v(x,y) = 6xy - 5x + 3$$

$$\therefore \frac{\partial v}{\partial x} = 6y - 5 \quad \text{and} \quad \frac{\partial v}{\partial y} = 6x.$$

we know,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\Rightarrow du = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy \quad \text{[using C-R equation]}$$

$$\Rightarrow du = 6x\,dx - (6y-5)\,dy$$

$$\therefore u = \int 6x\,dx - \int (6y-5)\,dy + c$$

$$= 3x^2 - 3y^2 - 5y + c$$

$\therefore f(z) = u + iv$

$= 3x^2 - 3y^2 + 5y + c + i(6xy - 5x + 3)$

$= 3x^2 - 3y^2 + 5y + 6ixy - 5ix + 3i + c$

$= 3(x^2 - y^2 + 2ixy) + (-5ix + 5y) + 3i + c$

$= 3(x+iy)^2 - 5i(x+iy) + 3i + c$

$= 3z^2 - 5iz + 3i + c$ Ans

3.16- Find the value of constants a, b, c and d such that the function $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$ is analytic.

Solⁿ:

Given that,

$$f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$$

where,

$$u = x^2 + axy + by^2 \quad \text{and} \quad v = cx^2 + dxy + y^2$$

$$\therefore \frac{\partial u}{\partial x} = 2x + ay \quad ; \quad \frac{\partial v}{\partial x} = 2cx + dy$$

$$\frac{\partial u}{\partial y} = ax + 2by \quad ; \quad \frac{\partial v}{\partial y} = dx + 2y$$

Since $f(z) = u + iv$ is analytic, so Cauchy-Riemann equation must be satisfied.

i.e, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Therefore,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow 2x + ay = dx + 2y \quad \text{——— (i)}$$

and,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow ax + 2by = -2cx - dy \quad \text{——— (ii)}$$

Now from (i) and (ii),

$$(2-d)x + (a-2)y = 0 \quad \text{——— (iii)}$$

and, $(a+2c)x + (2b+d)y = 0$ ——— (iv)

(iii) and (iv) will be true when,

$$(2-d) = 0 \; ; \; (a-2) = 0$$

and, $(a+2c) = 0$ ; $(2b+d) = 0$

Therefore,

$$\boxed{a = 2, \; d = 2, \; b = -1 \text{ and } c = -1}$$

3.20 – Show that the function $u = \cos x \cosh y$ is harmonic and find its harmonic conjugate.

Solⁿ: Given that,

$$u = \cos x \cosh y$$

Now, $\frac{\partial u}{\partial x} = -\sin x \cosh y$ and, $\frac{\partial u}{\partial y} = \cos x \sinh y$

$\frac{\partial^2 u}{\partial x^2} = -\cos x \cosh y$ $\quad \frac{\partial u}{\partial y} = \cos x \cosh y$

Now, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\cos x \cosh y + \cos x \cosh y = 0$

Therefore $u = \cos x \cosh y$ is a harmonic function.

Let $v$ be its conjugate harmonic function, then,

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$= -\cos x \sinh y \, dx - \sin x \cosh y \, dy$$

$$= -(\cos x \sinh y \, dx + \sin x \cosh y \, dy)$$

Integrating,

$v = -\sin x \sinh y + c$, where $c$ is a real constant.

**3.21** – Prove that $u = y^3 - 3x^2y$ is a harmonic function. Determine its harmonic conjugate. hence find the corresponding analytic function $f(z)$ in terms of $z$.

**Soln:** Given that,

$$u = y^3 - 3x^2y$$

$$\therefore \frac{\partial u}{\partial x} = -6xy \quad ; \quad \frac{\partial u}{\partial y} = 3y^2 - 3x^2$$

$$\frac{\partial^2 u}{\partial x^2} = -6y \quad ; \quad \frac{\partial^2 u}{\partial y^2} = 6y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0$$

Therefore $u = y^3 - 3x^2y$ is a harmonic function.

Let,

$v$ be the harmonic conjugate to $u$, then,

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$= -(3y^2 - 3x^2)\, dx - 6xy\, dy$$

$$= -(3y^2 dx + 6xy\, dy) + 3x^2 dx$$

Integrating,

$$v = -3xy^2 + x^3 + c$$

**3.38** Evaluate,

(a) $\lim_{z \to i} \dfrac{z^{10}+1}{z^6+1}$

(b) $\lim_{z \to 0} \dfrac{1-\cos z}{z^2}$

(c) $\lim_{z \to 0} \dfrac{1-\cos z}{\sin z^2}$

**Soln**

(a) Let $f(z) = z^{10}+1$ and $g(z) = z^6+1$

then, $f(i) = g(i) = 0$.

Also $f(z)$ and $g(z)$ are analytic at $z = i$

Hence, by L'Hospital rule,

$$\lim_{z \to i} \frac{z^{10}+1}{z^6+1} = \lim_{z \to i} \frac{10z^9}{6z^5} = \lim_{z \to i} \frac{5}{3} z^4$$

$$= \frac{5}{3} \quad \text{Ans}$$

(b) Let, $f(z) = 1-\cos z$ and $g(z) = z^2$

then, $f(0) = g(0) = 0$. Also $f(z)$ and $g(z)$ are analytic at $z=0$.

Hence by L'Hospital rule,

$$\lim_{z\to 0} \frac{1-\cos z}{z^2}$$

$$= \lim_{z\to 0} \frac{\sin z}{2z}$$

Since $f_1(z) = \sin z$ and $g_1(z) = 2z$ are analytic and equal to zero when $z=0$, therefore we can apply L'Hospital rule again,

$$\lim_{z\to 0} \frac{\sin z}{2z} = \lim_{z\to 0} \frac{\cos z}{2} = \frac{1}{2}$$ Ans

(c) Let, $f(z) = 1-\cos z$ and $g(z) = \sin z^2$, then,

$f(0) = g(0) = 0$ also $f(z)$ and $g(z)$ are analytic at $z=0$.

Hence by repeated application of L'Hospital rule,

$$\lim_{z\to 0} \frac{1-\cos z}{\sin z^2} = \lim_{z\to 0} \frac{\sin z}{2z\cos z^2}$$

$$= \lim_{z \to 0} \frac{\cos z}{2\cos z^2 - 4z^2 \sin z^2}$$

$$= \frac{1}{2}$$

Another method,

$$\lim_{z \to 0} \frac{1-\cos z}{\sin z^2} = \lim_{z \to 0} \frac{\sin z}{2z\cos z^2}$$

$$= \lim_{z \to 0} \frac{\sin z}{z} \lim_{z \to 0} \frac{1}{2\cos z^2}$$

$$= 1 \times \frac{1}{2}$$

$$= \frac{1}{2}$$

**3.52**: Prove that in polar form the Cauchy-Riemann equation can be written,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

**Soln:**

We have,

$$x = r\cos\theta \quad ; y = r\sin\theta$$

$$r = \sqrt{x^2+y^2} \text{ and } \theta = \tan^{-1}\frac{y}{x}$$

then,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cdot\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial u}{\partial \theta}\cdot\frac{-y}{x^2+y^2}$$

$$= \frac{\partial u}{\partial r}\cdot\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta$$

And,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$$= \frac{\partial u}{\partial r}\cdot\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial u}{\partial \theta}\cdot\frac{x}{x^2+y^2}$$

$$= \frac{\partial u}{\partial r}\cdot\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cdot\cos\theta$$

Similarly,

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= \cancel{\frac{\partial v}{\partial r}\frac{x}{\sqrt{x^2+y^2}} + \frac{1}{r}\frac{\partial v}{\partial \theta}}$$

$$= \frac{\partial v}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta$$

and,

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta$$

From Cauchy-Rieman equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta = \frac{\partial v}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial v}{\partial \theta}\cos\theta$$

$$\Rightarrow \left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\cos\theta - \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\sin\theta = 0 \quad \text{——— (5)}$$

And also,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta = -\frac{\partial v}{\partial r}\cos\theta + \frac{1}{r}\frac{\partial v}{\partial \theta}\sin\theta$$

$$\Rightarrow \left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\sin\theta + \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta = 0 \quad \text{——— 6}$$

Multiplying (5) by $\cos\theta$ and (6) by $\sin\theta$ and adding,

$$\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta} = 0$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Multiplying (5) by $-\sin\theta$ and (6) by $\cos\theta$ and adding

$$\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} = 0$$

$$\Rightarrow \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$