1st Chapter : Special Function

Beta function :

Sir

The function denoted by $\beta(m,n)$, where m and n are positive values $[m,n>0]$ and it defined by

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \;;$$

is known as Beta function. It is also called the first Eulerian integral.

Some other form of Beta function

(i) $$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

(ii) $$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

Properties of beta function :

Symmetric function of beta function :

Sir

We know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad \text{——— (i)}$$

If we let,

$$1-x=y$$

$\Rightarrow x = 1-y$

$\therefore dx = -dy$

x	0	1
y	1	0

Putting the value in equation (i),

$$\beta(m,n) = \int_1^0 (1-y)^{m-1} \cdot y^{n-1} (-dy)$$

$$= \int_0^1 (1-y)^{m-1} \cdot y^{n-1} dy$$

$$= \beta(n,m)$$

Therefore, $\beta(m,n) = \beta(n,m)$

This is the symmetric function of Beta function.

Another form of Beta function:

We know,

No need

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad \text{——— (i)}$$

If we let,

Wall's formula

$$x = \sin^2\theta$$

$$dx = 2\sin\theta \cos\theta \, d\theta$$

Limit:

x	0	1
θ	0	$\pi/2$

Putting the value in equation (i),

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1}\, 2\sin\theta \cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (\sin^2\theta)^{m-1} \cdot (\cos^2\theta)^{n-1} \sin\theta \cos\theta\, d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta$$

Sin

Prove that, (Wallis formula)

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

Soln:
We know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \quad \text{——(i)}$$

Let,
$x = \sin^2\theta$

$\Rightarrow dx = 2\sin\theta \cos\theta\, d\theta$

x	0	1
θ	0	$\pi/2$

Putting this value in the eqn (i), we get,

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1}\, 2\sin\theta \cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \sin\theta\cos\theta \, d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\Rightarrow \frac{\beta(m,n)}{2} = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$ [Another form of Beta function]

Let, $2m-1 = p$ and, $2n-1 = q$

$\therefore m = \frac{p+1}{2}$ $\therefore n = \frac{q+1}{2}$

and we know,

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

Therefore,

$$\frac{\Gamma m \, \Gamma n}{2\Gamma(m+n)} = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\Rightarrow \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{2\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)} = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$ [Another formula of Beta function]

$$\Rightarrow \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{2\Gamma\frac{p+1+q+1}{2}} = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$$\therefore \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}} = \int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta$$

Another form of Beta function:

We know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put, $x = \frac{y}{1+y}$

$$dx = \frac{(1+y)1 - y.1}{(1+y)^2}dy$$

$$= \frac{1}{(1+y)^2}dy$$

Here $x = \frac{y}{1+y}$

$\Rightarrow y = x + xy$

$\Rightarrow y - xy = x$

$\Rightarrow y(1-x) = x$

$\therefore y = \frac{x}{1-x}$ (i)

x	0	1
y	0	∞

Putting the value in eq^n (i)

$$\beta(m,n) = \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1}\left(1-\frac{y}{1+y}\right)^{n-1}\frac{1}{(1+y)^2}.dy$$

$$= \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1}\left(\frac{1}{1+y}\right)^{n-1}.\frac{1}{(1+y)^2}dy$$

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}.dy$$

we can also write,

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

limit ঠিক থাকলে চলক পরিবর্তন করলে মানের পরিবর্তন হয় না

Gamma function :

The function, ~~is~~ denoted by Γn and it defined by $\Gamma n = \int_0^\infty e^{-x} . x^{n-1} dx$; where, $n > 0$ is known as gamma function.

It is also called second Eulerian integral.

Properties of Beta function and Gamma function.

Sir?

① Prove that $\Gamma(n+1) = n\Gamma n$

Soln :-

We know,

$$\Gamma n = \int_0^\infty e^{-x} . x^{n-1} dx$$

$$\therefore \Gamma n+1 = \int_0^\infty e^{-x} . x^{n+1-1} dx$$

$$= \int_0^\infty e^{-x} . x^{n} dx$$

$$= \cancel{\left[x^n \int e^{-x} dx\right]_0^\infty - \int_0^n n x^{n-1}(-e^{-x})} dx$$

$$= \left[-x^n e^{-x}\right]_0^\infty + \int_0^\infty n x^{n-1} . e^{-x} dx$$

$$= 0 + n\int_0^\infty e^{-x} . x^{n-1} dx$$

$$\therefore \Gamma n+1 = n\Gamma n$$

Gamma function is known as generalized factorial function,

A quick derivation of Euler Integrals equation, we know,

$$n! \text{ or } \lfloor n = \int_0^{\infty} e^{-x} \cdot x^{n}\, dx \quad \text{——— (i)}$$

Again, the equation of gamma function,

$$\Gamma n = \int_0^{\infty} e^{-x}\, x^{n-1}\, dx$$

$$\therefore \Gamma n+1 = \int_0^{\infty} e^{-x} \cdot x^{n+1-1}\, dx$$

$$= \int_0^{\infty} e^{-x} \cdot x^{n}\, dx$$

$$\Gamma n+1 = \lfloor n \quad \text{proved}$$

Factorial function to Gamma function, $\lfloor n = \Gamma n+1$

The equation of gamma function is,

$$\Gamma n = \int_0^{\infty} e^{-x} \cdot x^{n-1}\, dx$$

A quick derivation of Eular equation, we know that

$$\lfloor n = \int_0^{\infty} e^{-x} \cdot x^{n}\, dx$$

$$\therefore \lfloor n-1 = \int_0^{\infty} e^{-x} \cdot x^{n-1}\, dx = \Gamma n$$

If, $n = n+1$, then,

$$\lfloor n+1-1 = \Gamma n+1$$

$$\therefore \lfloor n = \Gamma n+1$$

Sir) Prove that, $\Gamma(n+1) = n! = \lfloor n = n\Gamma n$

Here we get,

$$\Gamma(n+1) = n\Gamma n$$

When, $n=1$; we get,

$$\Gamma 2 = 1\Gamma 1 = 1.1 = 1!$$

When, $n=2$; we get,

$$\Gamma 3 = 2\Gamma 2 = 2.1 = 2!$$

When, $n=3$, we get,

$$\Gamma 4 = 3\Gamma 3 = 3.2.1 = 3!$$

... ...

... ...

... ...

Therefore,

$$\Gamma n+1 = (n+1-1)! = n!$$

Hence,

$$\Gamma n+1 = n! = \lfloor n = n\Gamma n \quad \text{(proved)}$$

Prove that $\Gamma(n) = (n-1)\,\Gamma(n-1)$

Soln:-

We know,

$$\Gamma(n) = \int_0^\infty e^{-x}\cdot x^{n-1}\,dx$$

$$\therefore \Gamma(n+1) = \int_0^\infty e^{-x}\cdot x^{n+1-1}\cdot dx$$

$$= \int_0^\infty e^{-x}\cdot x^{n}\,dx$$

$$= \left[-x^{n}\cdot e^{-x}\right]_0^\infty - n\int_0^\infty e^{-x}\cdot x^{n-1}\,dx$$

$$= 0 - n\,\Gamma(n)$$

$$\therefore \Gamma(n+1) = n\,\Gamma(n)$$

Replacing n by $n-1$, we get,

$$\Gamma(n-1+1) = (n-1)\,\Gamma(n-1)$$

$$\therefore \Gamma(n) = (n-1)\,\Gamma(n-1) \quad \text{Proved}$$

IIly, $n = n-2$;

$$\Gamma(n-1) = (n-2)\,\Gamma(n-2)$$

IIly $n = n-3$

$$\Gamma(n-2) = (n-3)\,\Gamma(n-3)$$

Sir

Prove that $\frac{\Gamma n}{\lambda^n} = \int_0^\infty e^{-\lambda y} \cdot y^{n-1}\, dy$

Soln:-

We know that,

$$\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx \quad ; [n>0] \quad \text{—(i)}$$

Let, $x = \lambda y$

$\therefore dx = \lambda\, dy$

x	0	∞
y	0	∞

Putting these value in eqn (i),

$$\Gamma n = \int_0^\infty e^{-\lambda y} \cdot (\lambda y)^{n-1} \cdot \lambda\, dy$$

$$= \int_0^\infty e^{-\lambda y} \cdot \lambda^{n-1} \cdot y^{n-1} \cdot \lambda \cdot dy$$

$$= \int_0^\infty e^{-\lambda y} \cdot y^{n-1} \cdot \lambda^{n-1+1} \cdot dy$$

$$\Rightarrow \Gamma n = \int_0^\infty e^{-\lambda y} \cdot y^{n-1} \cdot \lambda^n\, dy$$

$$\therefore \frac{\Gamma n}{\lambda^n} = \int_0^\infty e^{-\lambda y} \cdot y^{n-1}\, dy \quad \text{proved}$$

Prove that. (H.W)

(i) $\Gamma n = \int_0^1 \log\left(\frac{1}{y}\right)^{n-1} dy$

(ii) $\Gamma n = (m+1)^n \int_0^1 y^m \left(\log\frac{1}{y}\right)^{n-1} dy$

H.W Soln:- Sir

We know that,

$$\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1} dx \quad \text{——— (i)}$$

Let,

$e^{-x} = y$

Limits:

x	0	∞
y	1	0

$\Rightarrow -e^{-x} dx = dy$

$\Rightarrow -y\, dx = dy$

$\Rightarrow dx = -\frac{1}{y} dy$

$\Rightarrow -\int dx = \int \frac{dy}{y}$

$\Rightarrow -x = \log y$

$\Rightarrow x = -\log y$

$\Rightarrow x = \log y^{-1}$

$\Rightarrow x = \log\left(\frac{1}{y}\right)$

Now From, (i)

$$\Gamma n = \int_1^0 -y\left(\log\frac{1}{y}\right)^{n-1}\frac{dy}{y}$$

$$= \cancel{\int_0^1 \left\{\log\left(\frac{1}{y}\right)\right\}^{n-1} d}$$

$$\therefore \Gamma n = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy \quad \text{proved}$$

H.W

(ii) Solⁿ:-

We know that,

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \quad \text{——— (1)}$$

Let, $e^{-x} = y^{m+1}$

$\Rightarrow -e^{-x} dx = (m+1) y^m dy$

$\Rightarrow -y^{m+1} dx = (m+1) y^m dy$

$\Rightarrow -dx = \frac{(m+1) y^m}{y^{m+1}} dy$

$\Rightarrow -\int dx = \int \frac{m+1}{y} dy$

$\Rightarrow -x = (m+1) \cancel{\log\frac{1}{y}} \log y$

$\Rightarrow x = -(m+1)\log y = (m+1)\log y^{-1}$

$$\Rightarrow x = (m+1)\log\frac{1}{y}$$

Now, from eq^n (i),

$$\Gamma n = -\int_1^0 y^{m+1}\left\{(m+1)\log\frac{1}{y}\right\}^{n-1}\frac{m+1}{y}\,dy$$

$$= \int_0^1 y^{m+1-1}\cdot(m+1)^{n-1}\left(\log\frac{1}{y}\right)^{n-1}\cdot(m+1)\,dy$$

$$\therefore \Gamma n = \int_0^1 y^m\,(m+1)^n\left(\log\frac{1}{y}\right)^{n-1}dy$$

$$\therefore \Gamma n = (m+1)^n\int_0^1 y^m\left(\log\frac{1}{y}\right)^{n-1}dy \quad \text{(proved)}$$

H.W ▣ Prove that $\pi(n) = \Gamma n+1$ [Gauss's pi function]

Sir

Soln:-

From gauss pi function we have,

$$\pi(n) = \int_0^\alpha x^n e^{-x}\,dx$$

$$= \int_0^\alpha e^{-x}\cdot x^{\{(n+1)-1\}}\,dx$$

$$\therefore \pi(n) = \Gamma n+1$$

Sir

The relation between Beta and gamma function or prove that,

$$\beta(m,n) = \frac{\sqrt{m}\,\sqrt{n}}{\sqrt{m+n}} \quad ; \text{ where } m>0;\ n>0$$

Soln:-

We know that,

$$\sqrt{m} = \int_0^{\alpha} e^{-x}.x^{m-1}\,dx$$

$$\text{and, } \sqrt{n} = \int_0^{\alpha} e^{-y}.y^{n-1}\,dy = \int_0^{\alpha} e^{-y}.y^{n-1}dy$$

Now,

$$\sqrt{m}.\sqrt{n} = \int_0^{\alpha} e^{-x}.x^{m-1}dx . \int_0^{\alpha} e^{-y}.y^{n-1}dy$$

$$= \int_0^{\alpha}\int_0^{\alpha} e^{-(x+y)}.x^{(m-1)}.y^{(n-1)}.\,dx\,dy$$

Let,

$$x+y = u \quad \text{and, } v = \frac{x}{x+y}$$

Therefore,

$$uv = x \quad \text{and, } y = u - uv. \quad [\because y = u-x]$$

$$= u.(1-v).$$

$$\text{and, } dy\,dx = |J|\,du\,dv$$

As x and y range 0 to ∞

Therefore,

u ranges 0 to ∞

v ranges 0 to 1

Here,

$$|J| = \left\| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right\|$$

$$= \left\| \begin{matrix} v & u \\ (1-v) & -u \end{matrix} \right\|$$

$$= |\{-uv - u(1-v)\}|$$

$$= |\{-uv - u + uv\}|$$

$$= |-u|$$

$$= u$$

Therefore,

$$\sqrt{m}.\sqrt{n} = \int_{u=0}^{\alpha}\int_{v=0}^{1} e^{-u}.(uv)^{m-1}.\{u(1-v)\}^{n-1}.|J|\,du\,dv$$

$$= \int_{0}^{\alpha}\int_{0}^{1} e^{-u}.u^{m-1}.v^{m-1}.u^{n-1}.(1-v)^{n-1}.u\,du\,dv$$

$$= \int_{0}^{\alpha} e^{-u}.u^{m-1}.u^{n-1}.u\,du \int_{0}^{1} v^{m-1}.(1-v)^{n-1}dv$$

$$= \int_{0}^{\alpha} e^{-u}.u^{m+n-1}du \int_{0}^{1} v^{m-1}.(1-v)^{n-1}dv$$

$\therefore \sqrt{m}\sqrt{n} = \sqrt{(m+n)} \,.\, \beta(m,n).$

$\Rightarrow \beta(m,n).\sqrt{m+n} = \sqrt{m}.\sqrt{n}$

$$\therefore \beta(m,n) = \frac{\sqrt{m}.\sqrt{n}}{\sqrt{(m+n)}}$$

田 The relation between Gamma and Beta function,

$$\beta(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma m+n} \quad [\text{prove}]$$

Soln:-

We know that from the Gamma function,

$$\frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} \cdot x^{n-1}\, dx$$

$$\Rightarrow \Gamma n = \int_0^\infty e^{-zx} \cdot x^{n-1} \cdot z^n\, dx$$

multiplying by e^{-z} and z^{m-1} both the side of the equation,

$$\Gamma n\, e^{-z} z^{m-1} = \int_0^\infty e^{-zx} \cdot e^{-z} z^{m-1} z^n x^{n-1}\, dx$$

Integrating w.r.t z on both side between the limit 0 to ∞

$$\Gamma n \int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty x^{n-1} dx \int_0^\infty e^{-z(x+1)} \cdot z^{m+n-1} dz$$

$$\Rightarrow \Gamma n . \Gamma m = \int_0^\infty x^{n-1} dx \cdot \frac{\Gamma m+n}{(x+1)^{m+n}}$$

$$\Rightarrow \frac{\Gamma n\, \Gamma m}{\Gamma m+n} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$\Rightarrow \frac{\Gamma m\, \Gamma n}{\Gamma m+n} = \beta(m,n) \quad (\text{proved})$$

Sir

Prove that $\Gamma\frac{1}{2} = \sqrt{\pi}$

Soln:-

We know that,

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \quad \text{——— (i)}$$

and $$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta \quad \text{——— (ii)}$$

We can write the equation (i) and (ii),

$$\frac{\Gamma m \, \Gamma n}{\Gamma m+n} = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta \quad \text{——— (3)}$$

Let,
$$m = n = \frac{1}{2}$$

Putting this value in equation (3), we get,

$$\frac{\Gamma 1/2 \cdot \Gamma 1/2}{\Gamma 1/2 + 1/2} = 2\int_0^{\pi/2} \sin^{2\cdot\frac{1}{2}-1}\theta \cdot \cos^{2\cdot\frac{1}{2}-1}\theta \, d\theta$$

$$\Rightarrow \frac{(\Gamma 1/2)^2}{1} = 2\int_0^{\pi/2} \sin^0\theta \cdot \cos^0\theta \, d\theta$$

$$\Rightarrow \frac{(\Gamma 1/2)^2}{1} = 2\int_0^{\pi/2} 1 \cdot 1 \, d\theta$$

$$\Rightarrow (\Gamma 1/2)^2 = 2\,[\theta]_0^{\pi/2}$$

$$\Rightarrow (\Gamma 1/2)^2 = 2 \cdot \frac{\pi}{2}$$

$$\Rightarrow (\Gamma 1/2)^2 = \pi$$

$$\therefore \Gamma 1/2 = \sqrt{\pi}$$ (proved)

Evaluate / Prove

① $\Gamma 3 = 2$ ④ $\Gamma -1/2$

② $\Gamma 4 = 6$ ⑤ $\Gamma 1/2$

③ $\Gamma 5 = 24$ ⑥ $\Gamma -3/2$

Sol^n :-

(i) We know that,

$$\Gamma n = \int_0^\alpha e^{-x} x^{n-1} dx$$

$$\therefore \Gamma 3 = \int_0^\alpha e^{-x} \cdot x^{3-1} dx$$

$$= \int_0^\alpha e^{-x} \cdot x^2 dx$$

$$= \left[-x^2 e^{-x}\right]_0^\alpha + \int_0^\alpha 2x \cdot e^{-x} dx$$

$$= 0 + 2\int_0^\alpha x e^{-x} dx$$

$$= 2\left\{\left[-xe^{-x}\right]_0^\alpha + \int_0^\alpha e^{-x} dx\right\}$$

$$= 2\left\{0 + \int_0^\alpha e^{-x} dx\right\}$$

$$= 2 \cdot \left[-e^{-x}\right]_0^\alpha$$

$$= 2 \cdot \left[-(0-1)\right]$$

$$= 2$$

(iii) We know that,

$$\Gamma n = \int_0^{\alpha} e^{-x} \cdot x^{n-1} dx$$

$$\therefore \Gamma 4 = \int_0^{\alpha} e^{-x} \cdot x^{4-1} dx$$

$$= \int_0^{\alpha} e^{-x} \cdot x^3 dx$$

$$= \left[-x^3 e^{-x}\right]_0^{\alpha} + 3\int_0^{\alpha} x^2 e^{-x}$$

$$= 0 + 3\left\{\left[-x^2 \cdot e^{-x}\right]_0^{\alpha} + 2\int_0^{\alpha} x e^{-x} dx\right\}$$

$$= (3\times 0) + 3\times 2\left\{\left[-x e^{-x}\right]_0^{\alpha} + \int_0^{\alpha} e^{-x} dx\right\}$$

$$= (3\times 2\times 0) + 3\times 2 \int_0^{\alpha} e^{-x} dx$$

$$= 6 \left[-e^{-x}\right]_0^{\alpha}$$

$$= 6\times(-0+1)$$

$$= 6$$

(iii) We know that,

$$\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx$$

$$\Gamma 5 = \int_0^\infty e^{-x} \cdot x^{4}\, dx$$

$$= \left[-x^4 e^{-x}\right]_0^\infty + 4\int_0^\infty x^3 e^{-x}\, dx$$

$$= 4\left\{\left[-x^3 e^{-x}\right]_0^\infty + 3\int_0^\infty x^2 e^{-x}\, dx\right\}$$

$$= (4\times 3)\left\{\left[-x^2 e^{-x}\right]_0^\infty + 2\int_0^\infty x e^{-x}\, dx\right\}$$

$$= (4\times 3\times 2)\left\{\left[-x e^{-x}\right]_0^\infty + \int_0^\infty e^{-x}\, dx\right\}$$

$$= (4\times 3\times 2)\left[-e^{-x}\right]_0^\infty$$

$$= (4\times 3\times 2)\left[-0+1\right]$$

$$= 4\times 3\times 2\times 1$$

$$= 24$$

Sir (iv) We know

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

Here, $n = -\frac{1}{2}$

We know that,

$$\Gamma n+1 = n\Gamma n$$

$$\Rightarrow \Gamma(-\tfrac{1}{2}+1) = -\tfrac{1}{2}\Gamma -\tfrac{1}{2}$$

$$\cancel{\Rightarrow \Gamma \tfrac{1}{2} = 2}$$

$$\Rightarrow \Gamma \tfrac{1}{2} = -\tfrac{1}{2}\Gamma -\tfrac{1}{2}$$

$$\Rightarrow 2\Gamma \tfrac{1}{2} = -\Gamma -\tfrac{1}{2}$$

$$\Rightarrow \Gamma -\tfrac{1}{2} = -2\Gamma \tfrac{1}{2}$$

$$\boxed{\therefore \Gamma -\tfrac{1}{2} = -2\sqrt{\pi}}$$

~~(v) We know~~

$\cancel{\beta}$

~~(v) We know,~~

$$\cancel{\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}}$$ (see previous questions)

~~Let,~~

$$\cancel{m = n = \frac{1}{2}}$$

sir (vi) we know that,

$$\Gamma(n+1) = n\Gamma n$$

Let, $n = -3/2$

Now, $\Gamma\left(-\frac{3}{2}+1\right) = -\frac{3}{2}\,\Gamma -\frac{3}{2}$

$$\Rightarrow \Gamma\frac{-3+2}{2} = -\frac{3}{2}\,\Gamma -\frac{3}{2}$$

$$\Rightarrow \Gamma -\frac{1}{2} = -\frac{3}{2}\,\Gamma -\frac{3}{2}$$

$$\Rightarrow -2\sqrt{\pi} = -\frac{3}{2}\,\Gamma -\frac{3}{2}$$

$$\Rightarrow \frac{3}{2}\,\Gamma -\frac{3}{2} = 2\sqrt{\pi}$$

$$\Rightarrow \Gamma -\frac{3}{2} = \frac{4}{3}\sqrt{\pi}$$ Ans

8(ix)

Prove that

$$\int_0^1 x^4(1-x)^3\,dx = \frac{1}{280}$$

Soln:-

$$L.H.S = \int_0^1 x^4(1-x)^3\,dx$$

$$= \int_0^1 x^{5-1}\cdot(1-x)^{(4-1)}\,dx$$

$$= \beta(5,4)$$

$$= \frac{\Gamma 5\cdot\Gamma 4}{\Gamma 5+4}$$

$$= \frac{4!\times 3!}{\Gamma 9}$$

$$= \frac{4!\times 3!}{8!}$$

$$= \frac{\cancel{3}\times\cancel{2}\times 1}{8\times\cancel{7}\times\cancel{6}\times 5}$$

$$= \frac{1}{280}$$

$= R.H.S$ (proved)

Evaluate the following functions,

1. $\int_0^{\pi/2} \sin^6 x \, dx$

2. $\int_0^{\pi/2} \cos^5 x \sin^4 x \, dx$

3. $\int_0^{\pi/2} \sin^5 \theta \cos^5 \theta \, d\theta$

4. $\int_0^{\pi/2} \cos^9 \theta \, d\theta$

5. $\int_0^{\pi/2} x^6 \sqrt{1-x^2} \, dx$

6. $\int_0^1 x^3 (1-x)^3 \, dx$

7. $\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta$

8. $\int_0^{\pi/2} \sqrt{\cot\theta} \, d\theta$

9. $\int_0^1 \frac{35 x^3}{32\sqrt{1-x}} \, dx$

10. $\int_0^1 (1-x^n)^{1/n} \, dx$

Prove that,

1. $\int_0^1 \frac{dx}{\sqrt{1-x^n}} \, dx = \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma(1/n)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$

2. $\int_0^1 x^{m-1}(1-x^a)^n \, dx = \frac{1}{a} \frac{n! \, \Gamma(m/a)}{\Gamma(m/a+n+1)}$

Sol^n:

Sir

(i) We know that,

$$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

$$\therefore \int_0^{\pi/2} \sin^6 x \, dx = \int_0^{\pi/2} \sin^6 x \cos^0 x \, dx$$

$$= \frac{\Gamma\frac{6+1}{2}\;\Gamma\frac{0+1}{2}}{2\,\Gamma\frac{6+0+2}{2}}$$

$$= \frac{\Gamma 7/2\;\Gamma 1/2}{2\,\Gamma 4}$$

$$= \frac{\Gamma(5/2+1)\;\Gamma 1/2}{2\,\Gamma(3+1)}$$

$$= \frac{5/2 \cdot 3/2 \cdot 1/2\;\Gamma 1/2 \cdot \Gamma 1/2}{2\cdot 3\cdot 2\cdot 1}$$

$$= \frac{15\;\Gamma 1/2\;\Gamma 1/2}{96}$$

$$= \frac{15\sqrt{\pi}\cdot\sqrt{\pi}}{96} = \frac{5\sqrt{\pi}\cdot\sqrt{\pi}}{32}$$

$$\cancel{= \frac{15}{48}\pi} \quad = \frac{5}{32}\pi$$

sin

(ii) We know that,

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

$$\therefore \int_0^{\pi/2} \cos^5 x \sin^4 x \, dx = \frac{\Gamma\frac{4+1}{2}\,\Gamma\frac{5+1}{2}}{2\,\Gamma\frac{4+5+2}{2}}$$

$$= \frac{\Gamma 5/2 \cdot \Gamma 3}{2\,\Gamma 11/2}$$

$$= \frac{\Gamma(3/2+1) \cdot \Gamma(2+1)}{2\,\Gamma 9/2+1}$$

$$= \frac{3/2 \cdot 1/2 \cdot \Gamma 1/2 \cdot 2 \cdot 1}{2 \cdot 9/2 \cdot 7/2 \cdot 5/2 \cdot 3/2 \cdot 1/2 \cdot \Gamma 1/2}$$

$$= \frac{8}{315}$$ Ans

(iii) We know,

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

$$\therefore \int_0^{\pi/2} \sin^5\theta \cos^5\theta \, d\theta = \frac{\Gamma\frac{5+1}{2}\,\Gamma\frac{5+1}{2}}{2\,\Gamma\frac{5+5+2}{2}}$$

$$= \frac{\Gamma 3 \; \Gamma 3}{2 \Gamma 6}$$

$$= \frac{2! \times 2!}{2 \times 5!}$$

$$= \frac{2! \times 2!}{2 \times 5 \times 4 \times 3 \times 2!}$$

$$= \frac{\cancel{2}\,1}{60}$$

Sir

(4) We know,

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2} \; \Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

$$\therefore \int_0^{\pi/2} \cos^q\theta \, d\theta = \int_0^{\pi/2} \sin^0\theta \cos^q\theta \, d\theta$$

$$= \frac{\Gamma\frac{0+1}{2} \; \Gamma\frac{q+1}{2}}{2\,\Gamma\frac{0+q+2}{2}}$$

$$= \frac{\Gamma\frac{1}{2} \; \Gamma\frac{q+1}{2}}{2\,\Gamma\frac{q+2}{2}}$$

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\frac{q+1}{2}}{\Gamma\frac{q+2}{2}}$$

Sir

(5) Here, $\int_0^1 x^6\sqrt{1-x^2}\,dx$

Let, $x^2 = \sin^2\theta \Rightarrow x = \sin\theta$

$2x\,dx = 2\sin\theta\cos\theta\,d\theta$

$\Rightarrow 2\sin\theta\,dx = 2\sin\theta\cos\theta\,d\theta$

$\Rightarrow dx = \cos\theta\,d\theta$

x	0	1
θ	0	$\pi/2$

Now, $\int_0^1 x^6\sqrt{1-x^2}\,dx$

$= \int_0^{\pi/2} \sin^6\theta\sqrt{1-\sin^2\theta}\;\cancel{d\theta}\cos\theta\,d\theta$

$= \int_0^{\pi/2} \sin^6\theta\cdot\cos\theta\cdot\cos\theta\,d\theta$

$= \int_0^{\pi/2} \sin^6\theta\cdot\cos^2\theta\,d\theta$

$= \dfrac{\Gamma\frac{6+1}{2}\,\Gamma\frac{2+1}{2}}{2\,\Gamma\frac{6+2+2}{2}}$

$= \dfrac{\Gamma\frac{7}{2}\,\Gamma\frac{3}{2}}{2\,\Gamma 5}$

$= \dfrac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{2\cdot 4!}$

$= \dfrac{15\;\sqrt{\pi}\cdot\sqrt{\pi}}{2^4\cdot 2\cdot 4\times3\times2\times1}$

$= \dfrac{5}{256}\pi$ Ans

Sin

(6) Here,

$$\int_0^1 x^3(1-x)^3\,dx = \int_0^1 x^{(4-1)}(1-x)^{(4-1)}\,dx$$

$$= \beta(4,4)$$

$$= \frac{\Gamma 4\,\Gamma 4}{\Gamma 4+4}$$

$$= \frac{\Gamma 4\,\Gamma 4}{\Gamma 8}$$

$$= \frac{3!\times 3!}{7!}$$

$$= \frac{3\times 2\times 1\times 3!}{7\times 6\times 5\times 4\times 3!}$$

$$= \frac{1}{140} \quad \text{Ans}$$

Sin

(7) Here,

$$\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2}\left(\frac{\sin\theta}{\cos\theta}\right)^{1/2} d\theta$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^{-1/2}\theta\,d\theta$$

$$= \frac{\Gamma\frac{1/2+1}{2}\;\Gamma\frac{-1/2+1}{2}}{2\,\Gamma\frac{1/2-1/2+2}{2}}$$

$$= \frac{\Gamma\left(\frac{3/2}{2}\right)\Gamma\left(\frac{1/2}{2}\right)}{2\,\Gamma\left(\frac{2}{2}\right)}$$

$$= \frac{\Gamma(3/4)\,\Gamma(1/4)}{2\,\Gamma(1)} \qquad = \frac{1}{2}\cdot\Gamma(1/4)\cdot\Gamma(3/4) \text{ Ans}$$

$$= \frac{1}{2}\cdot\Gamma(1/4)\cdot\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{2}\cdot\frac{\pi}{\sin \pi/4}$$

$$= \frac{\pi}{2\times\frac{1}{\sqrt{2}}}$$

$$= \frac{\pi}{\sqrt{2}} \text{ Ans}$$

Soln (8) Here,

$$\int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta = \int_0^{\pi/2}\left(\frac{\cos\theta}{\sin\theta}\right)^{1/2} d\theta$$

$$= \int_0^{\pi/2}\sin^{-1/2}\theta\cdot\cos^{1/2}\theta\,d\theta$$

$$= \frac{\Gamma\left(\frac{-1/2+1}{2}\right)\Gamma\left(\frac{1/2+1}{2}\right)}{2\,\Gamma\left(\frac{-1/2+1/2+2}{2}\right)}$$

$$= \frac{\Gamma\frac{1}{4}\ \Gamma\frac{3}{4}}{2\,\Gamma 1}$$

$$= \frac{1}{2}\,\Gamma\tfrac{1}{4}\cdot\Gamma\left(1-\tfrac{1}{4}\right)$$

$$= \frac{1}{2}\,\frac{\pi}{\sin \pi/4}$$

$$= \frac{1}{2}\cdot\frac{\pi}{1/\sqrt{2}}$$

$$= \frac{\pi}{\sqrt{2}} \quad \text{Ans}$$

Sir

9) $\displaystyle\int_0^1 \frac{35\,x^3}{32\sqrt{1-x}}\,dx$

Let,

$x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta.$

Limits:

x	0	1
θ	0	$\pi/2$

$$\therefore \int_0^1 \frac{35}{32}\cdot\frac{x^3}{\sqrt{1-x}}\,dx$$

$$= \int_0^{\pi/2} \frac{35}{32}\cdot\frac{(\sin^2\theta)^3}{\sqrt{1-\sin^2\theta}}\cdot 2\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{35}{32} \cdot \frac{\sin^6\theta}{\sqrt{\cos^2\theta}} \cdot 2\sin\theta\cos\theta \, d\theta$$

$$= \frac{35}{16} \int_0^{\pi/2} \frac{\sin^7\theta}{\cos\theta} \cdot \cos\theta \, d\theta$$

$$= \frac{35}{16} \int_0^{\pi/2} \sin^7\theta \cdot \cos^0\theta \, d\theta$$

$$= \frac{35}{16} \frac{\Gamma\frac{7+1}{2}\,\Gamma\frac{0+1}{2}}{2\Gamma\frac{7+0+2}{2}}$$

$$= \frac{35}{16} \frac{\Gamma 4 \, \Gamma \frac{1}{2}}{2\Gamma \frac{9}{2}}$$

$$= \frac{35}{32} \times \frac{3! \, \Gamma\frac{1}{2}}{\Gamma \frac{7}{2}+1}$$

$$= \frac{35}{32} \times \frac{3! \, \Gamma\frac{1}{2}}{\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\frac{1}{2}}$$

$$= \frac{35}{16} \times \frac{3 \times 2 \times 1}{\frac{7}{2} \times \frac{5}{2} \times \frac{3}{2}}$$

$$= \frac{16}{16}$$

$$= 1$$

sir

(10) Here,

$$\int_0^1 (1-x^n)^{1/n} dx$$

Let,

$x = \sin^2\theta$

$dx = 2\sin\theta \cos\theta \, d\theta$

x	0	1
θ	0	$\pi/2$

Therefore,

$$\int_0^1 (1-x^n)^{1/n} dx$$

$$= \int_0^{\pi/2} \{1-(\sin^n\theta)^2\}^{1/n} \, 2\sin\theta \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} (\cos^{2n}\theta)^{1/n} \, 2\sin\theta \cos\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \cos^2\theta \cdot \sin\theta \cdot \cos\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin\theta \cdot \cos^3\theta \, d\theta$$

$$= 2 \frac{\Gamma\frac{1+1}{2} \, \Gamma\frac{3+1}{2}}{2\,\Gamma\frac{1+3+2}{2}}$$

$$= \frac{\Gamma 1 \, \Gamma 2}{\Gamma 3} = \frac{\Gamma 2}{2\,\Gamma(2+1)} = \frac{\Gamma 2}{2\,\Gamma 2} = \frac{1}{2} \text{ Ans}$$

Sir

1. ~~Here~~ Show that, $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma 1/n}{\Gamma 1/n + 1/2}$

Sol^n:

Let, $x^n = \sin^2\theta \Rightarrow x = \sin^{2/n}\theta$

$\Rightarrow n x^{n-1} dx = 2\sin\theta \cos\theta \, d\theta$

$\Rightarrow n x^n . x^{-1} . dx = 2\sin\theta \cos\theta \, d\theta$

$\Rightarrow n \sin^2\theta \, \sin^{-2/n}\theta \, dx = 2\sin\theta \cos\theta \, d\theta$

$\Rightarrow dx = \frac{2}{n} \sin^{2/n}\theta \, \sin^{-2}\theta \, \sin\theta \cos\theta \, d\theta$

$\Rightarrow dx = \frac{2}{n} \sin^{(2/n - 2 + 1)}\theta \cos\theta \, d\theta$

$\Rightarrow dx = \frac{2}{n} \sin^{(2/n - 1)}\theta \cos\theta \, d\theta$

x	0	1
θ	0	$\pi/2$

Now,

$$L.H.S = \int_0^1 \frac{dx}{\sqrt{1-x^n}}$$

$$= \int_0^{\pi/2} \frac{\frac{2}{n} \sin^{(2/n-1)}\theta \cos\theta \, d\theta}{\sqrt{1-\sin^2\theta}}$$

$$= \int_0^{\pi/2} \frac{2}{n} \cdot \frac{\sin^{(2/n-1)}\theta}{\cos\theta} \cdot \cos\theta \, d\theta$$

$$= \frac{2}{n} \int_0^{\pi/2} \sin^{(2/n-1)}\theta \cdot \cos^0\theta \, d\theta$$

$$= \frac{2}{n} \frac{\Gamma\left(\frac{2/n - 1 + 1}{2}\right) \Gamma\frac{0+1}{2}}{2\,\Gamma\frac{2/n - 1 + 0 + 2}{2}}$$

$$= \frac{1}{n} \frac{\Gamma 1/n \cdot \Gamma 1/2}{\Gamma\frac{2/n + 1}{2}}$$

$$= \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma 1/n}{\Gamma\left(\frac{2/n}{2} + \frac{1}{2}\right)}$$

$$= \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma 1/n}{\Gamma\left(1/n + 1/2\right)}$$

$= R.H.S$ (Showed)

III Prove that, $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Sol^n:-

Here, $\int_0^{\infty} e^{-x^2} dx$

Let, $x^2 = z \Rightarrow x = \mp\sqrt{z}$

$\Rightarrow 2x\,dx = dz$

$\therefore dx = \frac{1}{2x} dz = \frac{1}{2\sqrt{z}} dz$

x	0	∞
z	0	∞

Now,

$$\int_0^\infty e^{-x^2} dx$$

$$= \int_0^\infty e^{-z} \cdot \frac{1}{2\sqrt{z}} dz$$

$$= \frac{1}{2} \int_0^\infty e^{-z} \cdot z^{-1/2} dz$$

$$\cancel{= \frac{1}{2} \int_0^\infty e^{-z} \cdot z^{\left(-\frac{3}{2}-1\right)}} dz$$

$$\cancel{= \frac{1}{2} \cdot \Gamma\left(-\frac{3}{2}\right)}$$

$$= \frac{1}{2} \int_0^\infty e^{-z} \cdot z^{\left(\frac{1}{2}-1\right)} dz$$

$$= \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2} \cdot \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2}$$

III Prove that,

$$\int_0^1 x^{m-1}(1-x^a)^n\,dx = \frac{1}{a}\,\frac{n!\,\Gamma\frac{m}{a}}{\Gamma\left(\frac{m}{a}+n+1\right)}$$

Soln:

Let,

$$x^a = \sin^2\theta \Rightarrow x = \sin^{2/a}\theta$$

$$a\,x^{a-1}dx = 2\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow a.\,x^a.\,x^{-1}dx = 2\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow a\sin^2\theta\cdot\sin^{-2/a}\theta\,dx = 2\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow dx = \frac{2}{a}\sin^{-2}\theta\cdot\sin^{2/a}\theta\,\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow dx = \frac{2}{a}\sin^{2/a-2+1}\theta\,\cos\theta\,d\theta$$

x	0	1
θ	0	$\pi/2$

Now,

$$\int_0^1 x^{m-1}(1-x^a)^n\,dx$$

$$= \int_0^{\pi/2}\left(\sin^{2/a}\theta\right)^{m-1}\cdot(1-\sin^2\theta)^n\cdot\frac{2}{a}\sin^{2/a-2+1}\theta\,\cos\theta\,d\theta$$

$$= \int_0^{\pi/2}\sin^{2m/a-2/a}\theta\cdot\cos^{2n}\theta\cdot\frac{2}{a}\sin^{2/a-1}\theta\,\cos\theta\,d\theta$$

$$= \frac{2}{a}\int_0^{\pi/2}\sin^{2m/a-2/a+2/a-1}\theta\,.\,\cos^{2n+1}\theta\,d\theta$$

$$= \frac{2}{a}\int_0^{\pi/2}\sin^{2m/a-1}\theta\,\cos^{2n+1}\theta\,d\theta$$

$$= \frac{2}{a} \times \frac{\Gamma\frac{\frac{2m}{a}-1+1}{2}\ \Gamma\frac{2n+1+1}{2}}{2\,\Gamma\frac{\frac{2m}{a}-1+2n+1+2}{2}}$$

$$= \frac{1}{a} \times \frac{\Gamma\frac{2m/a}{2}\ \Gamma n+1}{\Gamma\frac{\frac{2m}{a}+2n+2}{2}}$$

$$= \frac{1}{a} \times \frac{\Gamma m/a \times n!}{\Gamma m/a+n+1}$$ Ans

Q) Evaluate, $\int_0^{\infty} \frac{x^{n-1}}{1+x}\,dx$

Soln:

Let,

$x = \tan^2\theta$

$dx = 2\tan\theta\ \sec^2\theta\, d\theta$

x	0	∞
θ	0	$\pi/2$

Now, $\int_0^{\infty} \frac{x^{n-1}}{1+x}\,dx$

$$= \int_0^{\pi/2} \frac{(\tan^2\theta)^{n-1}}{1+\tan^2\theta} \cdot 2\tan\theta \cdot \sec^2\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\tan^{2n-2}\theta}{\sec^2\theta} \cdot \tan\theta\ \cancel{d\theta}\ \sec^2\theta\, d\theta$$

CamScanner

$$= 2\int_0^{\pi/2} \tan^{2n-1}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sin^{2n-1}\theta}{\cos^{2n-1}\theta} \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n-1}\theta \cdot \cos^{-(2n-1)}\theta \, d\theta$$

$$= 2 \, \frac{\Gamma\left(\frac{2n-1+1}{2}\right)\Gamma\left(\frac{-2n+1+1}{2}\right)}{2\,\Gamma\left(\frac{2n-1-2n+1+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{2n}{2}\right)\Gamma\left(\frac{-2n+2}{2}\right)}{\Gamma\left(\frac{2}{2}\right)}$$

$$= \frac{\Gamma n \, \Gamma(1-n)}{\Gamma 1}$$

$$= \Gamma n \, \Gamma(1-n)$$

Application of gamma, beta function in physics:

A particle is attracted toward a fixed point O with a force inversely proportional to its instantaneous distance from O. Applying the knowledge of gamma function, show that if the particle is ~~relad~~ released from rest the time for it to reach O is given by,

$$T = a\sqrt{\frac{\pi m}{2k}}$$

Soln:

At time $t=0$, Let the particle be located on the x-axis at $x = a > 0$ and O be the origin.

Then according to Newton's law,

$$m\frac{d^2x}{dt^2} = -\frac{k}{x} \quad \text{——— (i)}$$

$$\Rightarrow m\frac{d^2x}{dt^2} + \frac{k}{x} = 0$$

where, m is the mass of the particle and k is the proportionality constant.

Multiplying (i) by $2 \cdot \frac{dx}{dt} \cdot dt$,

$$2m\frac{d^2x}{dt^2} \cdot \frac{dx}{dt} \cdot dt = -\frac{2k}{x} \frac{dx}{dt} \cdot dt$$

$$\Rightarrow 2m\frac{d^2x}{dt^2}\cdot\frac{dx}{dt}\cdot dt = -\frac{2k}{x}dx$$

Let,

$v = \frac{dx}{dt}$; be the velocity of the particle.

Then,

$$\frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt}$$

$$\Rightarrow \frac{d^2x}{dt^2} = v\cdot\frac{dv}{dx}$$

Then eq^n (ii) becomes,

$$2mv\frac{dv}{dx}\cdot dx = -\frac{2k}{x}dx$$

$$\Rightarrow mv\,dv = -\frac{k}{x}dx$$

integrating both sides, we get,

$$\frac{mv^2}{2} = -k\ln x + C$$

where,

$x = a$; $v = \frac{dx}{dt} = 0$

Therefore,

$$0 = -k\ln a + C$$

$$\Rightarrow C = k\ln a$$

Now,

$$\frac{mv^2}{2} = -k\ln x + k\ln a$$

$$\Rightarrow \frac{mv^2}{2} = k\ln\frac{a}{x}$$

$$\Rightarrow \frac{mv^2}{2} = k\ln\frac{a}{x}$$

$$\Rightarrow v^2 = \frac{2k\cdot\ln\frac{a}{x}}{m}$$

$$\Rightarrow v = \sqrt{2k/m}\cdot\sqrt{\ln a/x}$$

$$\Rightarrow \frac{dx}{dt} = \sqrt{2k/m}\cdot\sqrt{\ln a/x}$$

$$\Rightarrow dt = \sqrt{m/2k}\cdot\left(\ln a/x\right)^{-1/2}\cdot dx$$

$$\Rightarrow \int_0^T dt = \sqrt{m/2k}\cdot\int_0^a \left(\ln a/x\right)^{-1/2} dx$$

Let, $\ln a/x = p$

$\Rightarrow \frac{a}{x} = e^p$

$\Rightarrow \frac{x}{a} = e^{-p}$

$\Rightarrow x = ae^{-p}$

$\therefore dx = -ae^{-p}dp$

x	0	a
p	α	0

$$\Rightarrow T = \sqrt{m/2k}\cdot\int_\alpha^0 p^{-1/2}\{-ae^{-p}dp\}$$

$$= a\sqrt{m/2k}\cdot\int_0^\alpha p^{-1/2}\cdot e^{-p}dp$$

$$= a\sqrt{m/2k}\cdot\int_0^\alpha p^{1/2-1}\cdot e^{-p}dp$$

$$= a\sqrt{\frac{m}{2k}}\cdot\Gamma\tfrac{1}{2}$$ [By the definition of gamma function]

$$\Rightarrow T = a\sqrt{\frac{m}{2k}}\cdot\sqrt{\pi}$$

$$\therefore T = a\sqrt{\frac{m\pi}{2k}}$$ Ans

Even function:

A function $f(x)$ is said to be an even function if $f(-x) = f(x)$

For example, $f(x) = x^2$; $f(x) = \cancel{\sin x} \cos x$

Odd function:

A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$.

For example, $f(x) = x$; $f(x) = \sin x$

Fourier series:

It is a mathematical way to represent Non- ~~Trigon~~ Trignometric periodic function as an infinite sum of trignometric function.

A series whose each and every term is either of sine ~~and~~ or cosine,

$$= a_0 \cos 0x + a_1 \cos 1x + a_2 \cos 2x + a_3 \cos 3x + \dots \dots$$
$$+ b_0 \sin 0x + b_1 \sin 1x + b_2 \sin 2x + b_3 \sin 3x + \dots \dots$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Therefore,

The Fourier series for the function $f(x)$ in the interval $[-\pi, \pi]$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (i)}$$

where, a_0, a_n and b_n are called the Fourier constants or co-efficients.

Now,

Integrating both sides of eqn (i) w.r.t. x over $[-\pi, \pi]$, we get,

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx\right]$$

$$= \frac{a_0}{2}[x]_{-\pi}^{\pi} + \sum_{n=1}^{\infty}\left\{\frac{a_n}{n}[\sin nx]_{-\pi}^{\pi} + \frac{b_n}{n}[-\cos nx]_{-\pi}^{\pi}\right\}$$

$$= \frac{a_0}{2}\cdot 2\pi + 0 + 0$$

$$= a_0 \pi$$

$$\therefore \int_{-\pi}^{\pi} f(x)\,dx = a_0 \pi$$

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Now, multiplying both sides of equation (i) by $\cos nx$ and integrating w.r.t x over $[-\pi, \pi]$, we get,

$$\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{a_0}{2}\int_{-\pi}^{\pi} \cos nx\, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos^2 nx\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos nx\, dx$$

$$= \frac{a_0}{2}\cdot 0 + \sum_{n=1}^{\infty} \frac{a_n}{2}\int_{-\pi}^{\pi}(1+\cos 2nx)\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos nx\, dx$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{2}\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \frac{a_n}{2}\int_{-\pi}^{\pi} \cos 2nx\, dx + 0$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{2}[x]_{-\pi}^{\pi} + 0 + 0$$

$$= \frac{a_n}{2}\cdot 2\pi$$

$$= a_n \pi$$

Therefore,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

Similarly, multiplying both sides of (i) by $\sin nx$ and integrating w.r.t x over $[-\pi, \pi]$, we get,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

Hence,

Fourier co-efficients are,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

Fourier cosine and sine series:

We know that, the furier series of the function $f(x)$ in the interval $[-\pi, \pi]$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (i)}$$

where,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

Sir

II Cosine Series : [An even function can have no sine term in its fourier series]

If the function $f(x)$ is even, i.e., $f(-x) = f(x)$

then, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$

$\therefore a_0 = I_1 + I_2$ ——— (ii)

Now, $I_1 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx$

Putting, $x = -y$;

$dx = -dy$

x	0	$-\pi$
y	0	π

$\therefore I_1 = \frac{1}{\pi}\int_{\pi}^{0} f(-y)\cdot -dy$

$= \frac{-1}{\pi}\int_{\pi}^{0} f(y)\cdot dy$

$= \frac{1}{\pi}\int_{0}^{\pi} f(x)\cdot dx$

From eqn (ii),

$a_0 = 2 \times \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx.$ [This is half range cosine series]

Again,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx \, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx \, dx$$

$$= I_3 + I_4$$

Putting, $x = -y$

$\Rightarrow dx = -dy$

x	0	$-\pi$
y	0	π

Now,

$$I_3 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(-y)\cos n(-y) \cdot (-dy)$$

$$= -\frac{1}{\pi}\int_{\pi}^{0} f(y) \cdot \cos ny \cdot dy$$

$$= \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx \, dx$$

Therefore,

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx \, dx$$

And,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= I_5 + I_6$$

Now,

$$I_5 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(-y)\sin n(-y)\,(-dy)$$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(y)\sin ny\,dy$$

$$= -\frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= -I_6$$

Therefore,

$$b_n = I_5 + I_6$$

$$= -I_6 + I_6$$

$$\therefore b_n = 0$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{1}{2}\times\frac{2}{\pi}\int_0^{\pi} f(x)\,dx + \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx\right]\cos nx + 0$$

$$= \frac{1}{\pi}\int_0^{\pi} f(x)\,dx + \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx\right]\cos nx$$

[This is the cosine series of Fourier series of interval $0, \pi$]

Sir

III Sine Series: [An odd function has no cosine term and constant term in its Fourier series.]

If the function $f(x)$ is odd i.e. $f(-x) = -f(x)$

Then,
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

and, $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= I_1 + I_2$$

Now, $$I_1 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(-y)\sin n(-y)\,(-dy)$$

$$= -\frac{1}{\pi}\int_{\pi}^{0} f(y)\sin ny\,dy$$

$$= \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= I_2$$

Therefore,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Therefore Fourier series becomes,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= 0 + 0 + \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx\right]\cdot \sin nx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\sin nx \int_0^{\pi} f(x)\sin nx\,dx$$

This is the Fourier sine series in the Interval $(0, \pi)$.

Example:

Develop $f(x) = x$ in Fourier series.

Sol^n:

Since the function $f(x) = x$ is an odd function, then,

$$a_0 = 0 \quad \text{and} \quad a_n = 0$$

And also,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left\{\left[-\frac{x\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}dx\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{\pi\cos n\pi}{n} + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right\}$$

$$= -\frac{2\cos n\pi}{n}$$

$$= -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\Rightarrow x = \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n \sin nx$$

54

$$= -2\left[\frac{-\sin x}{1} + \frac{\sin 2x}{2} + \frac{-\sin 3x}{3} + \dots \dots\right]$$

$$= 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \dots\right]$$

Example :-

Q.1:- Develop $f(x)$ in the Fourier series where,

$f(x) = x^2 + x$

Q.2:- Find the Fourier series of the function,

i) $f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ \pi/2 & 0 \le x \le \pi \end{cases}$

ii) $f(x) = \begin{cases} -\pi/2 & -\pi \le x < 0 \\ \pi/2 & 0 \le x \le \pi \end{cases}$

Q.3:- Find the Fourier series which represents the function $f(x)$, defined by,

$$f(x) = \begin{cases} -\cos x & \text{for } -\pi \le x \le 0 \\ \cos x & \text{for } 0 \le x \le \pi \end{cases}$$

Q.4:- Find the Fourier series representing the function $f(x) = x \sin x$ in the interval $-\pi < x < \pi$. Hence show that,

$$\pi/4 = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{1.5} + \frac{1}{1.7} - \dots \dots$$

Soln-1:

~~Since the function $f(x) = x^2+x$ is an even function~~, then,

$$\cancel{a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx}$$

$$\cancel{= \frac{2}{\pi}\int_0^{\pi} x^2+x\,dx}$$

$$\cancel{= \frac{2}{\pi}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^{\pi}}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2\pi^3}{\pi 3}$$

$$= \frac{2}{3}\pi^2$$

and,
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2+x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - \int_0^{\pi} 2x\frac{\sin nx}{n}\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n} - \frac{2}{n}\left\{\frac{-x\cos nx}{n} + \int\frac{\cos nx}{n}\,dx\right\}\right]^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{x^2\sin nx}{n} - \frac{2}{n}\left\{\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right\}\right]_0$$

$$= \frac{2}{\pi}\left[0 + \frac{2}{n^2}\cancel{x\cos nx} - 0\right]_0^{\pi}$$

$$= \frac{\cancel{2}}{\pi}$$

$$= \frac{4}{n^2\pi}\left[\pi\cos n\pi - 0\right]$$

$$= \frac{4}{n^2}\cos n\pi$$

$$= \frac{4}{n^2}(-1)^n$$

and also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2+x)\, \sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left\{\left[-\frac{x\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n}\, dx\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{\pi \cos n\pi}{n} + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{\pi \cdot \cos n\pi}{n} + 0\right\}$$

$$= -\frac{2\cos n\pi}{n}$$

$$= -\frac{2}{n}(-1)^n$$

Therefore, the Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{2\pi^2}{2\times 3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

$$+ \sum_{n=1}^{\infty}\left\{-\frac{2}{n}(-1)^n \cdot \sin nx\right\}$$

$$\Rightarrow x^2+x = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx - \sum_{n=1}^{\infty} \frac{2}{n}(-1)^n \sin nx$$

Now, Putting the value of $x=\pi$, we get,

$$\pi+\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n(-1)^n - 0 \quad \text{———} (i)$$

$\boxed{(-1)^n \cdot (-1)^n = \text{even} = 1}$

And, for $x=-\pi$, we get,

$$-\pi+\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cdot (-1)^n \quad \text{———} (ii)$$

$\boxed{(-1)^n \cdot (-1)^n = \text{even} = 1}$

Adding eq^n (i) and (ii) we have,

$$2\pi^2 = \frac{2\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\Rightarrow \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\Rightarrow \cancel{\pi^2 - \frac{\pi^2}{3}}$$

$$\Rightarrow \pi^2 - \frac{\pi^2}{3} = 4\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \dots + \right\}$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right\}$$

$$\Rightarrow \frac{\pi^2}{3} = 2\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right\}$$

$$\Rightarrow \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Soln : 2 (i)

The given function,

$$f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ \pi/2 & 0 \le x \le \pi \end{cases}$$

The fouriers series for the function $f(x)$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{—— (i)}$$

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\, dx$$

$$= 0 + \frac{1}{\pi}\int_{0}^{\pi} \pi/2 \cdot dx$$

$$= \frac{1}{2}\int_{0}^{\pi} 1.dx$$

$$= \left[\frac{x}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

Now,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x) \cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \cos nx\, dx$$

$$= 0 + \frac{1}{\pi}\int_0^{\pi} \frac{\pi}{2} \cdot \cos nx \, dx$$

$$= \frac{1}{2}\int_0^{\pi} \cos nx \, dx$$

$$= \left[\frac{\sin nx}{2n}\right]_0^{\pi}$$

$$= [0 - 0]$$

$$= 0$$

also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x) \sin nx \, dx + \frac{1}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= 0 + \frac{1}{\pi}\int_0^{\pi} \frac{\pi}{2} \cdot \sin nx \, dx$$

$$= \frac{1}{2}\int_0^{\pi} \sin nx \, dx$$

$$= -\frac{1}{2n}\left[\cos nx\right]_0^{\pi}$$

$$\cancel{= -\frac{1}{2n}[-1-1]}$$

$$\cancel{= \frac{1}{n}}$$

$$= -\frac{1}{2n}\left[(-1)^n - 1\right]$$

$$= \frac{(-1)^{n+1}}{2n} + \frac{1}{2n}$$

61

$= 0 + \frac{1}{\pi} \int_0^{\pi} \frac{\pi}{2} \cdot \cos nx \, dx$

$= \frac{1}{2} \int_0^{\pi} \cos nx \, dx$

$= \left[\frac{\sin nx}{2n} \right]_0^{\pi}$

$= [0 - 0]$

$= 0$

also,

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$

$= \frac{1}{\pi} \int_{-\pi}^{0} f(x) \sin nx \, dx + \frac{1}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

$= 0 + \frac{1}{\cancel{\pi}} \int_0^{\pi} \frac{\pi}{2} \cdot \sin nx \, dx$

$= \frac{1}{2} \int_0^{\pi} \sin nx \, dx$

$= -\frac{1}{2n} [\cos nx]_0^{\pi}$

$\boxed{\cancel{= -\frac{1}{2n} [-1 - 1] \quad = \frac{1}{n}}}$

$= -\frac{1}{2n} [(-1)^n - 1]$

$= \frac{(-1)^{n+1}}{2n} + \frac{1}{2n}$

Substituting this values in eqn (1),

$$f(x) = \frac{\pi/2}{2} + \sum_{n=1}^{\infty} 0 \cdot \cos nx + \sum_{n=1}^{\infty} \frac{\pi}{n} \cdot \sin nx$$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty} \cancel{\frac{\sin nx}{n}}$$

$$\frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{2n} + \frac{1}{2n}\right\} \cdot \sin nx$$

~~Ans~~

Soln: 2 (ii)

The given function,

$$f(x) = \begin{cases} -\pi/2 & ; -\pi \le x < 0 \\ \pi/2 & ; 0 \le x \le \pi \end{cases}$$

The Fourier series for the function $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\, dx$$

$$\cancel{= \frac{1}{\pi}\int_{-\pi}^{0} -\frac{1}{\pi}\, dx + \frac{1}{\pi}\int_{0}^{\pi} \frac{1}{\pi}}$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\frac{\pi}{2}\,dx + \frac{1}{\pi}\int_{0}^{\pi}\frac{\pi}{2}\,dx$$

$$= -\frac{1}{2}\int_{-\pi}^{0} dx + \frac{1}{2}\int_{0}^{\pi} dx$$

$$= -\frac{1}{2}[x]_{-\pi}^{0} + \frac{1}{2}[x]_{0}^{\pi}$$

$$= -\frac{1}{2}[0+\pi] + \frac{1}{2}[\pi - 0]$$

$$= \cancel{-\frac{\pi}{2}} + \cancel{\frac{\pi}{2}}$$

$$= 0$$

Now, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\frac{\pi}{2}\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\frac{\pi}{2}\cos nx\,dx$$

$$= -\frac{1}{2}\int_{-\pi}^{0}\cos nx\,dx + \frac{1}{2}\int_{0}^{\pi}\cos nx\,dx$$

$$= -\frac{1}{2}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{1}{2}\left[\frac{\sin nx}{n}\right]_{0}^{\pi}$$

$$= -\frac{1}{2}\times 0 + \frac{1}{2}\times 0$$

$$= 0$$

Also,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\frac{\pi}{2}\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\frac{\pi}{2}\sin nx\,dx$$

$$= -\frac{1}{2}\int_{-\pi}^{0}\sin nx\,dx + \frac{1}{2}\int_{0}^{\pi}\sin nx\,dx$$

$$\cancel{= -\frac{1}{2}\left[-\sin nx\right]}$$

$$= -\frac{1}{2}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{2}\left[\frac{-\cos nx}{n}\right]_{0}^{\pi}$$

$$= \frac{1}{2}\left[\frac{\cos nx}{n}\right]_{-\pi}^{0} - \frac{1}{2}\left[\frac{\cos nx}{n}\right]_{0}^{\pi}$$

$$= \frac{1}{2}\left[\frac{1}{n} + \frac{1}{n}\right] - \frac{1}{2}\left[-\frac{1}{n} - \frac{1}{n}\right]$$

$$= \frac{1}{2}\left[\frac{1}{n} - \frac{\cos n\pi}{n}\right] - \frac{1}{2}\left[\frac{\cos n\pi}{n} - \frac{1}{n}\right]$$

$$= \frac{1}{2n} - \frac{\cos n\pi}{2n} - \frac{\cos n\pi}{2n} + \frac{1}{2n}$$

$$= \frac{1}{n} - \frac{2\cos n\pi}{2n} = \frac{1}{n} - \frac{\cos n\pi}{n} = \frac{1-(-1)^n}{n} = \frac{1+(-1)^{n+1}}{n}$$

$$= \frac{1}{n} + \frac{1}{n}$$

$$= \frac{2}{n}$$

Substituting these values in eqn ①,

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

$$f(x) = \sum_{n=1}^{\infty} \frac{1+(-1)^{n+1}}{n} \sin nx$$

Soln: 3

The given function,

$$f(x) = \begin{cases} -\cos x & ; -\pi \le x \le 0 \\ \cos x & ; 0 \le x \le \pi \end{cases}$$

The Fourier series for the function $f(x)$ is.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\cos x\, dx + \frac{1}{\pi}\int_{0}^{\pi} \cos x\, dx$$

$$= \frac{1}{\pi}\left[-\sin x\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\sin x\right]_{0}^{\pi}$$

$$= 0 + 0$$

$$= 0$$

Now,
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\cos x\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\cos x\cos nx\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0} 2\cos nx\cos x\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 2\cos nx\cos x\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0}\{\cos(nx+x)+\cos(nx-x)\}dx$$
$$+ \frac{1}{2\pi}\int_{0}^{\pi}\{\cos(nx+x)+\cos(nx-x)\}dx$$

$$= -\frac{1}{2\pi}\left[\frac{\sin nx+x}{(n+1)} + \frac{\sin(nx-x)}{(n-1)}\right]_{-\pi}^{0}$$
$$+ \frac{1}{2\pi}\left[\frac{\sin(nx+x)}{(n+1)} + \frac{\sin(nx-x)}{(n-1)}\right]_{0}^{\pi}$$

$$= -\frac{1}{2\pi}\times 0 + \frac{1}{2\pi}\times 0$$

$$= 0$$

$\sin(n\pi+\theta) = (-1)^n\sin\theta$

$\sin(n\pi+\pi) = (-1)^n\sin\pi$
$= 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-\cos x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \cos x\,\sin nx\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0} 2\sin nx\cos x\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 2\sin nx\cos x\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0}\left\{\sin(nx+x) + \overset{\sin}{\cancel{\cos}}(nx-x)\right\}dx$$

$$+ \frac{1}{2\pi}\int_{0}^{\pi}\left\{\sin(nx+x) + \overset{\sin}{\cancel{\cos}}(nx-x)\right\}dx$$

$$= -\frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{(n+1)} + \frac{\overset{-\cos}{-\cancel{\sin}}(n-1)x}{(n-1)}\right]_{-\pi}^{0}$$

$$+ \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{(n+1)} + \frac{\overset{-\cos}{\cancel{\sin}}(n\cancel{+}-1)x}{(n-1)}\right]_{0}^{\pi}$$

$$= \frac{-1}{2\pi}\left\{\left(\frac{-\cos 0}{(n+1)} + 0\right) - \left(-\frac{\cos[(n+1)(-\pi)]}{(n+1)} + 0\right)\right\}$$

$$+ \frac{1}{2\pi}\left\{\left(-\frac{\cos(n+1)\pi}{n+1} + 0\right) - \left(\frac{-\cos 0}{(n+1)} + 0\right)\right\}$$

Correction

$$= -\frac{1}{2\pi}\left[\left(\frac{-\cos 0}{n+1}+\frac{-\cos(n-1)\cdot 0}{n-1}\right)-\left(\frac{-\cos(n+1)(-\pi)}{n+1}+\frac{-\cos(n-1)(-\pi)}{(n-1)}\right)\right]$$

$$+\frac{1}{2\pi}\left[\left(\frac{-\cos(n+1)\pi}{n+1}+\frac{-\cos(n-1)\pi}{n-1}\right)-\left(\frac{-\cos(n+1)\cdot 0}{(n+1)}+\frac{-\cos(n+1)\cdot 0}{(n-1)}\right)\right]$$

$$= -\frac{1}{2\pi}\left[\left(\frac{-\cos 0}{n+1}+\frac{-\cos 0}{n-1}\right)+\left(\frac{\cos(n+1)\pi}{n+1}+\frac{\cos(n-1)\pi}{(n-1)}\right)\right]$$

$$+\frac{1}{2\pi}\left[\left(\frac{-\cos(n+1)\pi}{(n+1)}+\frac{-\cos(n-1)\pi}{(n-1)}\right)-\left(\frac{-\cos 0}{(n+1)}+\frac{-\cos 0}{(n-1)}\right)\right]$$

$$= -\frac{1}{2\pi}\left[\left(\frac{-1}{(n+1)}+\frac{-1}{(n-1)}\right)+\left(\frac{(-1)^{n+1}}{(n+1)}+\frac{(-1)^{n-1}}{(n-1)}\right)\right]$$

$$+\frac{1}{2\pi}\left[\left(\frac{-(-1)^{n+1}}{(n+1)}+\frac{-(-1)^{n-1}}{(n-1)}\right)-\left(\frac{-1}{n+1}+\frac{-1}{n-1}\right)\right]$$

$$= -\frac{2}{2\pi}\left(\frac{-1}{(n+1)}+\frac{-1}{n-1}\right)-\frac{2}{2\pi}\left(\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{n+1}+\frac{1}{n-1}\right)-\frac{1}{\pi}\left(\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{n+1}-\frac{(-1)^{n+1}}{n+1}\right)+\frac{1}{\pi}\left(\frac{1}{n-1}-\frac{(-1)^{n-1}}{n-1}\right)$$

$$= \frac{1}{\pi}\left(\frac{1-(-1)^{n+1}}{(n+1)}\right)+\frac{1}{\pi}\left(\frac{1-(-1)^{n-1}}{(n-1)}\right)$$

Substituding these values in eq^n (i),

$$f(x) = 0 + 0 + \sum^{\infty} \left[\frac{1}{\pi} \left(\frac{1-(-1)^{n+1}}{n+1} \right) + \frac{1}{\pi} \left(\frac{1-(-1)^{n-1}}{n-1} \right) \right] \sin nx$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \left[\frac{1}{\pi} \left(\frac{1-(-1)^{n+1}}{n+1} \right) + \frac{1}{\pi} \left(\frac{1-(-1)^{n-1}}{n-1} \right) \right] \cdot \sin nx$$

Ans

Soln-4: Given function,

$$f(x) = x \sin x \quad ; \quad -\pi < x < \pi$$

The Fourier series for the function $f(x)$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin x\,dx$$

$$= \frac{1}{\pi}\left\{\left[-x\cos x\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos x\,dx\right\}$$

$$= \frac{1}{\pi}\left(\left\{-\pi\cos\pi - \pi\cos(-\pi)\right\} + \left[\sin x\right]_{-\pi}^{\pi}\right)$$

$$= \frac{1}{\pi}\left(-2\pi\cos\pi\right) + \frac{1}{\pi}\times 0$$

$= \cancel{}$

$$= -2\cos\pi$$

$$= -2\times -1$$

$$= 2$$

And,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x \cdot 2\cos nx \sin x \, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x \cdot \{\sin(nx + x) - \sin(nx - x)\}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x \sin(n+1)x \, dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} x \sin(n-1)x \, dx$$

$$= \frac{1}{2\pi}\left\{\left[-\frac{x\cos(n+1)x}{(n+1)}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(n+1)x}{n+1}\, dx\right\}$$

$$- \frac{1}{2\pi}\left\{\left[\frac{-x\cos(n-1)x}{(n-1)}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(n-1)x}{(n-1)}\, dx\right\}$$

$$= \frac{1}{2\pi}\left\{\left(\frac{-\pi\cos(n+1)\pi}{(n+1)} - \frac{\pi\cos(n+1)(-\pi)}{(n+1)}\right) + \left[\frac{\sin(n+1)x}{(n+1)^2}\right]_{-\pi}^{\pi}\right\}$$

$$- \frac{1}{2\pi}\left\{\left(\frac{-\pi\cos(n-1)\pi}{(n-1)} - \frac{\pi\cos(n-1)(-\pi)}{(n-1)}\right) + \left[\frac{\sin(n-1)x}{(n-1)^2}\right]_{-\pi}^{\pi}\right.$$

$$= \frac{-1}{2\pi}\left(\frac{\pi\cdot(-1)^{n+1}}{(n+1)} + \frac{\pi(-1)^{n+1}}{n+1}\right) + \frac{1}{2\pi}\times 0$$

$$+ \frac{1}{2\pi}\left(\frac{\pi(-1)^{n-1}}{n-1} + \frac{\pi(-1)^{n-1}}{n-1}\right) + \frac{1}{2\pi}\times 0$$

$$= \frac{-1}{2\pi} \times \frac{2\pi(-1)^{n+1}}{n+1} + \frac{1}{2\pi} \times \frac{2\pi(-1)^{n-1}}{(n-1)}$$

$$= \frac{-1\cdot(-1)^{n+1}}{(n+1)} + \frac{(-1)^{n-1}}{(n-1)}$$

$$= \boxed{\frac{(-1)^n}{(n+1)} + \frac{(-1)^{n-1}}{(n-1)}} = \frac{(-1)^{n-1}}{(n-1)} - \frac{(-1)^{n+1}}{n+1}$$

$$= \cancel{\frac{(n-1)(-1)^n + (n+1)(-1)^{n-1}}{(n-1)^2}}$$

$$= (-1)^n\left\{\frac{(-1)^{-1}}{(n-1)} - \frac{(-1)^1}{n+1}\right\}$$

$$= (-1)^n\left\{\frac{1}{n+1} - \frac{1}{n-1}\right\}$$

$$= (-1)^n \frac{n-1-n-1}{n^2-1}$$

$$= (-1)^n \cdot \frac{(-2)}{n^2-1}$$

$$= \frac{(-1)^n \times 2 \times (-1)^1}{n^2-1}$$

$$= \frac{2(-1)^{n+1}}{n^2-1}$$

Now,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x \sin nx\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\cdot 2\sin nx\cdot\sin x\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x \cdot \{\cos(n-1)x - \cos(n+1)x\}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\cos(n-1)x - \frac{1}{2\pi}\int_{-\pi}^{\pi} x\cos(n+1)x\,dx$$

$$= \frac{1}{2\pi}\left\{\left[\frac{x\sin(n-1)x}{n-1}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin(n-1)x}{n-1}\,dx\right\}$$

$$- \frac{1}{2\pi}\left\{\left[\frac{x\sin(n+1)x}{n+1}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin(n+1)x}{n+1}\,dx\right\}$$

$$= -\frac{1}{2\pi}\left[\frac{-\cos(n-1)x}{(n-1)^2}\right]_{-\pi}^{\pi} + \frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{(n+1)^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(n-1)^2}\{\cos(n-1)\pi - \cos(n-1)(-\pi)\}$$

$$- \frac{1}{2\pi(n+1)^2}\{\cos(n+1)\pi - \cos(n+1)(-\pi)\}$$

$$= \frac{1}{2\pi(n-1)^2}\left(\cos(n-1)\pi - \cos(n-1)\pi\right)$$

$$- \frac{1}{2\pi(n+1)^2}\left(\cos(n+1)\pi - \cos(n+1)\pi\right)$$

$$= 0 - 0$$

$$= 0$$

$\therefore b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\Rightarrow x \sin x = \frac{2}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n^2-1} \cdot \cos nx$$

where,

$$n = 1, 2, 3, 4, \ldots \ldots$$

Now,

$$x \sin x = 1 + \left[\frac{2\cdot(-1)^2}{1^2-1} \cos x + \frac{2(-1)^{2+1}}{2^2-1} \cos 2x + \frac{2(-1)^{3+1}}{3^2-1} \cos 3x \right.$$

$$+ \frac{2\cdot(-1)^{4+1}}{4^2-1} \cos 4x + \frac{2(-1)^{5+1}}{5^2-1} \cos 5x + \frac{2(-1)^{6+1}}{6^2-1} \cos 6x$$

$$\left. + \frac{2(-1)^{7+1}}{7^2-1} \cos 7x + \frac{2(-1)^{8+1}}{8^2-1} \cos 8x + \ldots \ldots \right]$$

$$= 1 \cancel{+} - \frac{2}{3} \cos 2x + \frac{2}{8} \cos 3x - \frac{2}{15} \cos 4x$$

$$+ \frac{2}{24} \cos 5x \cancel{+} - \frac{2}{35} \cos 6x + \frac{2}{48} \cos 7x$$

$$- \frac{1}{63} \cos 8x + \ldots$$

Let, $x = \frac{\pi}{2}$; then,

$$\frac{\pi}{2} \sin \frac{\pi}{2} = 1 - \frac{2}{3} \cos \frac{2\pi}{2} + \frac{2}{8} \cos \frac{3\pi}{2} - \frac{2}{15} \cos \frac{4\pi}{2}$$

$$+ \frac{2}{24} \cos \frac{5\pi}{2} - \frac{2}{35} \cos \frac{6\pi}{2} + \frac{2}{48} \cos \frac{7\pi}{2}$$

$$- \frac{1}{63} \cos \frac{8\pi}{2} + \ldots$$

$$\frac{\pi}{2}\times 1 = 1 - \frac{2}{3}(-1) + \frac{2}{8}\times 0 - \frac{2}{15}\times 1 + \frac{2}{24}\times 0 - \frac{2}{35}(-1) + \frac{2}{48}\times 0 - \frac{2}{63}\times 1$$

$$\Rightarrow \frac{\pi}{2} = 1 + \frac{2}{3} - \frac{2}{15} + \frac{2}{35} - \frac{2}{63} + \dots$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{2} + \frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \frac{1}{63} + \dots$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \dots \quad \text{Ans}$$

Complex form of Fourier series:

(Derive Fourier series from complex Fourier series)

The Fourier series may also be written in the complex form as,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{——— (i)}$$

where,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx \quad \text{——— (ii)}$$

And,

$$c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{inx}\, dx \quad \text{——— (iii)}$$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{a_0}{2} \quad \text{——— (iv)}$$

From eq^n (ii),

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x).(\cos nx - i\sin nx)\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx - \frac{1}{2\pi}\, i\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{a_n}{2} - i\frac{b_n}{2}$$

Similarly,

$$c_{-n} = \frac{a_n}{2} + i\frac{b_n}{2}$$

71

$$\therefore f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + \sum_{n=-1} c_n e^{inx}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} - \frac{i\,b_n}{2}\right) e^{inx}$$

$$+ \sum_{n=1}^{\infty} c_{-n} e^{-inx}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} - i\frac{b_n}{2}\right) e^{inx} + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} + i\frac{b_n}{2}\right) e^{-inx}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\left(e^{inx} + e^{-inx}\right) - i\frac{b_n}{2}\left(e^{inx} - e^{-inx}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\cdot 2\cos nx - i\frac{b_n}{2}\, 2\cdot i \sin nx)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

Parseval's identity:

Statement:

If the Fourier series for the function $f(x)$ converges uniformly in $(-l, l)$, then

$$\frac{1}{l}\int_{-l}^{l}\{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2);$$

where, a_0, a_n ~~ab~~ and b_n called Fourier co-efficients or constants.

Proof:

The Fourier series for the function $f(x)$ in the interval $(-l, l)$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \quad \text{---} \quad (i)$$

where,

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cdot\cos\frac{n\pi x}{l}\,dx$$

and, $$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\cdot\sin\frac{n\pi x}{l}\,dx$$

Multiplying both sides of eq^n (i) by $f(x)$ and integrating between limits $-l$ to l we get,

$$\int_{-l}^{l} \{f(x)\}^2 dx = \frac{a_0}{2}\int_{-l}^{l} f(x)\, dx + \sum_{n=1}^{\infty}\left(a_n \int_{-l}^{l} f(x) \cos\frac{n\pi x}{l} dx\right)$$

$$+ \sum_{n=1}^{\infty}\left(b_n \int_{-l}^{l} f(x) \sin\frac{n\pi x}{l} dx\right)$$

$$= \frac{a_0}{2} \cdot a_0 l + \sum_{n=1}^{\infty} a_n \cdot a_n l + \sum_{n=1}^{\infty} b_n \cdot b_n l$$

$$= \frac{a_0^2}{2} \cdot l + \sum_{n=1}^{\infty} a_n^2 \cdot l + \sum_{n=1}^{\infty} b_n^2 \cdot l$$

$$\Rightarrow \int_{-l}^{l} \{f(x)\}^2 dx = \frac{a_0^2}{2} \cdot l + l \cdot \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\Rightarrow \frac{1}{l}\int_{-l}^{l} \{f(x)\}^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

(proved)

Fourier Transformation : (also inverse fourier transform)

The general Fourier transform $f(s)$ or $f(n)$ of a function $f(x)$ is defined as,

$$f(n) = \int_{-\infty}^{\infty} f(x)\, e^{inx}\, dx \;;\; \text{~~where the inv~~}$$

where the inverse Fourier transform, $f(x)$ is given by,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(n)\, e^{-inx}\, dn$$

Example-1: Find the Fourier complex transform of $f(x)$, where,

$$f(x) = \begin{cases} 1-x^2 & ; |x| < 1 \\ 0 & ; |x| > 1 \end{cases}$$

confusion, $|x| \leq 1$ (is'n it better)

Soln:

We have, fourier transform of a function $f(x)$ is

$$f(n) = \int_{-\infty}^{\infty} f(x)\, e^{inx}\, dx$$

$$= \int_{-\infty}^{-1} f(x)\, e^{inx} + \int_{-1}^{1} f(x)\, e^{inx}\, dx + \int_{1}^{\infty} f(x)\, e^{inx}\, dx$$

$$= \int_{-1}^{1} (1-x^2)\, e^{inx}\, dx$$

$$= \left[\frac{(1-x^2)\cdot e^{inx}}{in}\right]_{-1}^{1} - \int_{-1}^{1} \frac{-2x\cdot e^{inx}}{in}\, dx$$

$$= 0 + \frac{2}{in}\int_{-1}^{1} x\, e^{inx}\, dx$$

$$= \frac{2}{in}\left\{\left[\frac{x\, e^{inx}}{in}\right]_{-1}^{1} - \int \frac{e^{inx}}{in}\, dx\right\}$$

$$= \frac{2}{in}\left\{\left(\frac{e^{in}}{in} + \frac{e^{-in}}{in}\right) - \left[\frac{e^{inx}}{i^2 n^2}\right]_{-1}^{1}\right\}$$

$$= \frac{2}{in}\left\{\left(\frac{e^{in}+e^{-in}}{in}\right) + \frac{1}{n^2}\left(e^{in}-e^{-in}\right)\right\}$$

$$= -\frac{2}{n^2}\left(e^{in}+e^{-in}\right) + \frac{2}{n^3}\left(\frac{e^{in}-e^{-in}}{i}\right)$$

$$= -\frac{2}{n^2}\cdot 2\cos n + \frac{2}{n^3}\cdot \frac{2i\sin n}{i}$$

$$= -\frac{4}{n^2}\cos n + \frac{4}{n^3}\sin n$$

Example-2: Find the Fourier transform of

$$f(x) = \begin{cases} 1 & ; \ |x| < a \\ 0 & ; \ |x| > a \end{cases}$$

Soln We know Fourier transform of a function,

$$f(n) = \int_{-\infty}^{\infty} f(x)\, e^{inx}\, dx$$

$$= \int_{-\infty}^{-a} f(x)\, e^{inx}\, dx + \int_{-a}^{a} f(x)\, e^{inx}\, dx + \int_{a}^{\infty} f(x)\, e^{inx}\, dx$$

$$= \int_{-a}^{a} f(x)\, e^{inx}\, dx$$

$$= \int_{-a}^{a} e^{inx}\, dx$$

$$= \left[\frac{e^{inx}}{in}\right]_{-a}^{a}$$

$$= \left(\frac{e^{ina}}{in} - \frac{e^{-ina}}{in}\right)$$

$$= \frac{e^{ina} - e^{-ina}}{in}$$

$$= \frac{(\cos na + i\sin na) - (\cos na - i\sin na)}{in}$$

$$= \frac{2i\sin na}{in}$$

$$= \frac{2\sin na}{n}$$

$$= \frac{2}{n}\sin na$$

Fourier sine transform:

The infinite Fourier sine transform $f_s(n)$ of function $f(x)$ is defined as,

$$f_s(n) = \int_0^{\infty} f(x)\,\sin nx\,dx$$

and its inverse formula is given by,

$$f(x) = \frac{2}{\pi}\int_0^{\pi} f_s(n)\,\sin nx\,dn$$

Example-1: Find the Fourier sine trans-form of e^{-x}.

Soln:

we have,

$$f_s(n) = \int_0^{\infty} f(x)\,\sin nx\,dx$$

$$= \int_0^{\infty} e^{-x}.\sin nx\,dx$$

$$= \lim_{R\to\infty}\int_0^{R} e^{-x}.\sin nx\,dx \longrightarrow (i)$$

~~Let,~~ $\cancel{I = \int_0^{R} e^{-x}\,\sin nx\,dx}$

$\therefore I =$

$$= \lim_{R\to\infty}\left[\frac{e^{-x}}{(-1)^2+n^2}\{(-1)\sin nx - n\cos nx\}\right]_0^R$$

$$= \lim_{R\to\infty}\left\{\left[\frac{e^{-R}}{n^2+1}(-\sin nR - n\cos nR)\right] - \left[\frac{1}{n^2+1}(-0-n\cdot 1)\right]\right\}$$

$$= \lim_{R\to\infty}\left[\frac{1}{e^{R}(n^2+1)}(-\sin nR - n\cos nR)\right] - \lim_{R\to\infty}\left[\frac{1}{n^2+1}(-n)\right]$$

$$= 0 - \frac{(-n)}{n^2+1}$$

$$= \frac{n}{n^2+1}$$ Ans

Finite Fourier Transform :

The Finite Fourier sine transform of a function

সাইন ট্রান্সফর্ম

$f(x)$ is defined as,

$$f_s(n) = \int_0^\pi f(x) \sin nx \, dx ;$$

and its inverse formula $f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} f_s(n) \sin nx$.

Example :-

Using finite Fourier sine transform, solve,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} ;$$

Given, $u(0,t) = 0$; $u(\pi, t) = 0$; $u(x,0) = 2x$

when, $0 < x < \pi$, $t > 0$

Soln:- Given that,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

taking finite Fourier sine transform we get,

$$\int_0^\pi \frac{\partial u}{\partial t} \sin nx \, dx = \int_0^\pi \frac{\partial^2 u}{\partial x^2} \sin nx \, dx \quad \text{——— (i)}$$

Let, $\int_0^\pi u(x,t) \cdot \sin nx \, dx = U(n,t)$ ——— (ii)

$$\therefore \frac{du}{dt} = \int_0^\pi \frac{\partial u}{\partial t} \sin nx \, dx$$

$$= \int_0^\pi \frac{\partial^2 u}{\partial x^2} \sin nx \, dx$$

$$= \left[\frac{\partial u}{\partial x} \cdot \sin nx\right]_0^\pi - n\int_0^\pi \cos nx \frac{\partial u}{\partial x} dx$$

$$= -n\int_0^\pi \cos nx \frac{\partial u}{\partial x} dx$$

$$= -n\left[u \cos nx\right]_0^\pi - n^2\int_0^\pi u \sin nx \, dx$$

$$= 0 - n^2 u \qquad \left[\because \int_0^\pi u(x,t) \sin nx \, dx = u(n,t)\right]$$

$$\therefore \frac{du}{dt} = -n^2 u$$

$$\Rightarrow \frac{du}{u} = -n^2 dt$$

$$\Rightarrow \int \frac{1}{u} du = -n^2 \int dt$$

$$\Rightarrow \log u = -n^2 t + \log c$$

$$\Rightarrow \log \frac{u}{c} = -n^2 t$$

$$\Rightarrow \frac{u}{c} = e^{-n^2 t}$$

$$\Rightarrow u = c \cdot e^{-n^2 t}$$

$$\Rightarrow u(n,t) = c \cdot e^{-n^2 t} \quad \text{——— (ii)}$$

or, $U(n,0) = c$

also, $U(n,0) = \int_0^{\pi} u(x,0) \sin nx \, dx$

$$\Rightarrow c = 2\int_0^{\pi} x \sin nx \, dx$$

$$= 2\left\{\left[-\frac{x\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n}\right\}$$

$$= 2\left\{-\frac{\pi\cos n\pi}{n} + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right\}$$

$$= -\frac{2\pi\cos n\pi}{n}$$

$$= -\frac{2\pi}{n}(-1)^n$$

$$\therefore c = \frac{2\pi}{n}(-1)^{n+1}$$

From eq^n (iii);

$$U(n,t) = \frac{2\pi}{n}(-1)^{n+1} \cdot e^{-n^2 t}$$

This equation is also called [heat equation]

Laplace Transformation :

Laplace transform of a function $f(t)$ is defined by,

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt = f(s)$$

s = frequency domain

t = time domain

$\cancel{= f(s)}$

Example-1: Find Laplace transform of $\sin t$,

Soln:-

Given,

$$f(t) = \sin t$$

$$\therefore L\{f(t)\} = \int_0^{\infty} e^{-st} . \sin t \, dt$$

$$= \int_0^{\infty} e^{-st} . \frac{1}{2i} . 2i \sin t \, dt$$

$$= \int_0^{\infty} e^{-st} . \frac{1}{2i} (e^{it} - e^{-it}) dt$$

$$= \frac{1}{2i} \int_0^{\infty} e^{-st+it} dt - \frac{1}{2i} \int_0^{\infty} e^{-st-it} dt$$

$$= \frac{1}{2i} \int_0^{\infty} e^{-(s-i)t} dt - \frac{1}{2i} \int_0^{\infty} e^{-(s+i)t} dt$$

$$= \frac{1}{2i}\left[\frac{e^{-(s-i)t}}{-(s-i)} - \frac{e^{-(s+i)t}}{-(s+i)}\right]_0^{\infty}$$

$$= \frac{1}{2i}\left[\frac{1}{(s-i)} - \frac{1}{(s+i)}\right]$$

$$= \frac{1}{2i}\left[\frac{(s+i)-(s-i)}{s^2-i^2}\right]$$

$$= \frac{1}{2i}\,\frac{2i}{s^2+1}$$

$$= \frac{1}{s^2+1} \quad \text{Ans}$$

$$\boxed{L\{\sin at\} = \frac{a}{s^2+a^2}}$$

$$\therefore L\{f(t)\} = f(s) = \frac{1}{s^2+1}$$

Example-2: Find the laplace transform of cos at.

Soln:

Given that,

$$f(t) = \cos at$$

$$\therefore L\{f(t)\} = f(s) = \int_0^{\infty} e^{-st} \cdot \cos at \; dt$$

$$= \int_0^{\alpha} e^{-st} \cdot \frac{1}{2} \cdot 2\cos at \, dt$$

$$= \frac{1}{2}\int_0^{\alpha} e^{-st} \cdot (e^{iat} + e^{-iat})\, dt$$

$$= \frac{1}{2}\int_0^{\alpha} e^{-st+iat}\, dt + \frac{1}{2}\int_0^{\alpha} e^{-st-iat}\, dt$$

$$= \frac{1}{2}\int_0^{\alpha} e^{-(s-ia)t}\, dt + \frac{1}{2}\int_0^{\alpha} e^{-(s+ia)t}\, dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-ia)t}}{-(s-ia)} + \frac{e^{-(s+ia)t}}{-(s+ia)}\right]_0^{\alpha}$$

$$= \frac{1}{2}\left[\frac{1}{(s-ia)} + \frac{1}{(s+ia)}\right]$$

$$= \frac{1}{2}\left[\frac{(s+ia)+(s-ia)}{s^2 - i^2a^2}\right]$$

$$= \frac{1}{2}\,\frac{2s}{s^2+a^2}$$

$$= \frac{s}{s^2+a^2}$$

Find the laplace ~~t~~transform of $f(t)=1$

we have,

$$f(t)=1$$

$$\therefore L\{f(t)\} = \int_0^{\infty} e^{-st} \cdot f(t) \cdot dt$$

$$= \int_0^{\infty} e^{-st} \cdot 1 \, dt$$

$$= \int_0^{\infty} e^{-st} \, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= 0 - \left(\frac{1}{-s}\right)$$

$$= \frac{1}{s} \quad \text{Ans}$$

[4] Laplace transform of cosh t.

Soln:

$$f(t) = \cosh t$$

$$L\{f(t)\} = \int_0^\infty e^{-st} \cdot \cosh t \cdot dt$$

$$= \int_0^\infty e^{-st} \cdot \left(\frac{e^{t}+e^{-t}}{2}\right) dt \qquad \left[\because \cosh t = \frac{e^{t}+e^{-t}}{2}\right]$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-1)t}\, dt + \frac{1}{2}\int_0^\infty e^{-(s+1)t}\, dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^\infty + \frac{1}{2}\left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{-1}{-(s-1)}\right] + \frac{1}{2}\left[\frac{-1}{-(s+1)}\right]$$

$$= \frac{1}{2(s-1)} + \frac{1}{2(s+1)}$$

$$= \frac{(s+1) + (s-1)}{2(s-1)(s+1)}$$

$$= \frac{s+2+s-2}{2(s-1)(s+1)}$$

$$= \frac{2s}{2(s^2-1)} = \frac{s}{s^2-1}$$

Laplace transform of ~~sin ht~~ sinhat

Soln:
we have,

$$f(t) = \sinh \cancel{t}at$$

$$L\{f(t)\} = \int_0^\infty e^{-st} \cdot \cancel{\sinh t} \cdot \sinh at \, dt$$

$$= \int_0^\infty e^{-st} \cdot \frac{e^{at} - e^{-at}}{2} \, dt$$

$$= \frac{1}{2} \int_0^\infty e^{-(s-a)t} \, dt - \frac{1}{2} \int_0^\infty e^{-(s+a)t} \, dt$$

$$= \frac{1}{2} \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty - \frac{1}{2} \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty$$

$$= \frac{1}{2} \left[\frac{-1}{-(s-a)} - \frac{-1}{-(s+a)} \right]$$

$$= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[\frac{(s+a) - (s-a)}{(s-a)(s+a)} \right]$$

$$= \frac{1}{2} \frac{2a}{s^2 - a^2}$$

$$= \frac{a}{s^2 - a^2}$$

Properties of Laplace transformation :

① Linear property.
② Shifting property.
③ Change of scale property.

Linear property :

The linear property of Laplace transformation is given by,

$$L\{k_1 f_1(t) + k_2 f_2(t)\} = k_1 f_1(s) + k_2 f_2(s)$$
$$= k_1 L\{f_1(t)\} + k_2 L\{f_2(t)\}$$

where, k_1 and k_2 are constants.

Proof:

Let $f_1(s)$ and $f_2(s)$ be the Laplace transform of $f_1(t)$ and $f_2(t)$ respectively.

Then,
$$L\{f_1(t)\} = \int_0^\infty e^{-st} f_1(t)\, dt = f_1(s)$$

and,
$$L\{f_2(t)\} = \int_0^\infty e^{-st} f_2(t)\, dt = f_2(s)$$

$$\therefore L\{k_1 f_1(t) + k_2 f_2(t)\}$$

$$= \int_0^{\infty} e^{-st} (k_1 f_1(t) + k_2 f_2(t))\, dt$$

$$= \int_0^{\infty} e^{-st} \cdot k_1 f_1(t)\, dt + \int_0^{\infty} e^{-st} \cdot k_2 f_2(t)\, dt$$

$$= k_1 f_1(s) + k_2 f_2(s)$$

$$= k_1 L\{f_1(t)\} + k_2 L\{f_2(t)\}$$

Example-1: Evaluate $L\{\sin 2t + 2\cos 3t\}$

Soln:

Using linear property of Laplace transform, we can write,

$$L\{\sin 2t + 2\cos 3t\} = L\{\sin 2t\} + 2L\{\cos 3t\}$$

Here,

$$L\{\sin 2t\} = \int_0^{\infty} e^{-st} \sin 2t\, dt$$

$$= \frac{2}{s^2+2^2} = \frac{2}{s^2+2^2} \quad \text{[direct]}$$

87

$$= \int_0^{\infty} e^{-st} \left(\frac{e^{2it} - e^{-2it}}{2i} \right) dt$$

$$= \frac{1}{2i} \int_0^{\infty} e^{-st+2it} dt - \frac{1}{2i} \int_0^{\infty} e^{-st-2it} dt$$

$$= \frac{1}{2i} \int_0^{\infty} e^{-(s-2i)t} dt - \frac{1}{2i} \int_0^{\infty} e^{-(s+2i)t} dt$$

$$= \frac{1}{2i} \left[\frac{e^{-(s-2i)t}}{-(s-2i)} - \frac{e^{-(s+2i)t}}{-(s+2i)} \right]_0^{\infty}$$

$$= \frac{1}{2i} \left[\frac{1}{(s-2i)} - \frac{1}{s+2i} \right]$$

$$= \frac{1}{2i} \left[\frac{(s+2i) - (s-2i)}{s^2 - 2^2 i^2} \right]$$

$$= \frac{1}{2i} \frac{4i}{s^2 + 2^2}$$

$$= \frac{2}{s^2 + 2^2}$$

$$= \frac{2}{s^2 + 4}$$

Now,

$$L\{\cos 3t\} = \int_0^\infty e^{-st} \cdot \cos 3t \, dt = \frac{s}{s^2+9}$$

$$= \int_0^\infty e^{-st} \cdot \left(\frac{e^{3it} + e^{-3it}}{2} \right) dt$$

$$= \frac{1}{2} \int_0^\infty e^{-st} \left(e^{3it} + e^{-3it} \right) dt$$

$$= \frac{1}{2} \int_0^\infty e^{-(s-3i)t} \, dt + \frac{1}{2} \int_0^\infty e^{-(s+3i)t} \, dt$$

$$= \frac{1}{2} \left[\frac{e^{-(s-3i)t}}{-(s-3i)} + \frac{e^{-(s+3i)t}}{-(s+3i)} \right]_0^\infty$$

$$= \frac{1}{2} \left[\frac{1}{(s-3i)} + \frac{1}{(s+3i)} \right]$$

$$= \frac{1}{2} \left[\frac{(s+3i) + (s-3i)}{(s-3i)(s+3i)} \right]$$

$$= \frac{1}{2} \frac{2s}{s^2+3^2}$$

$$= \frac{s}{s^2+9}$$

Putting these values in equation (i),

$$L\{\sin 2t + 2\cos 3t\} = \frac{2}{s^2+4} + 2\cdot\frac{s}{s^2+9}$$

$$= \frac{2}{s^2+9} + \frac{2s}{s^2+9}$$

(ii) Shifting property :

First shifting property : If $L\{f(t)\} = f(s)$ be the Laplace transform of $f(t)$. Then,

$$L\{e^{kt}\cdot f(t)\} = f(s-k)$$

Proof: We ~~know~~, have,

$$L\{e^{kt}\cdot f(t)\} = \int_0^\infty e^{-st}\cdot e^{kt} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-k)t}\cdot f(t)\,dt$$

$$= f(s-k)$$

30

Example: Evaluate, $L\{e^{5t}\sin 3t\}$

Soln:

We know that,

$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

and,

$$L\{e^{kt}.\sin at\} = \frac{a}{(s-k)^2+a^2}$$

$$\therefore L\{\sin 3t\} = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}$$

Now,

$$L\{e^{5t}.\sin 3t\} = \frac{3}{(s-5)^2+9} \quad \text{Ans}$$

Second shifting property:

If $G(t) = \begin{cases} f(t-k) & ; t > k \\ 0 & ; 0 < t < k \end{cases}$

then, $L\{G(t)\} = e^{-ks} f(s)$

Proof: We have,

$$L\{G(t)\} = \int_0^{\infty} e^{-st}.G(t)\,dt$$

$$= \int_0^{k} e^{-st}.G(t).dt + \int_k^{\infty} e^{-st} G(t)\,dt$$

$$= 0 + \int_{k}^{\infty} e^{-st} \cdot \cancel{g(t)\,dt}\ f(t-k)\,dt$$

$$= \int_{k}^{\infty} e^{-st} \cdot f(t-k)\,dt$$

Let,
$t-k = u \Rightarrow t = u+k$

$\Rightarrow du = dt$

t	k	∞
u	0	∞

$$= \int_{0}^{\infty} e^{-s(u+k)} \cdot f(u) \cdot du$$

$$= \int_{0}^{\infty} e^{-su} \cdot e^{-sk} \cdot f(u)\,du$$

$$= e^{-sk} \int_{0}^{\infty} e^{-su} \cdot f(u)\,du$$

$$= e^{-sk}\ L\{f(u)\}$$

$$= e^{-ks} \cdot f(s) \quad \text{(proved)}$$

Example :-

If $f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & ; \ t > \frac{2\pi}{3} \\ 0 & ; \ t < \frac{2\pi}{3} \end{cases}$

find $L\{f(t)\}$.

02

Sol^n: Given that,

$$f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & ; t > \frac{2\pi}{3} \\ 0 & ; t < \frac{2\pi}{3} \end{cases}$$

~~Now,~~
$\cancel{L\{f(t)\}} =$

We know that,

$$L\{G(t)\} = e^{-ks} \cdot f(s)$$

$$= e^{-\frac{2\pi}{3} \cdot s} \cdot f(s)$$

$$= e^{-\frac{2\pi}{3} s} \cdot L\{\cos t\}$$

$$= e^{-\frac{2\pi}{3} \cdot s} \cdot \frac{s}{s^2+1}$$

$$= \frac{s}{s^2+1} \cdot e^{-\frac{2\pi s}{3}}$$

Ans

(iii) Change of scale property :

If $L\{f(t)\} = f(s)$ then $L\{f(kt)\} = \frac{1}{k} f\left(\frac{s}{k}\right)$

Example: Evaluate $L\{\sin 3t\}$

Soln:

Here,

$$L\{\sin 3t\} = \frac{1}{3} \cdot f\left(\frac{s}{3}\right)$$

$$= \frac{1}{3} \cdot \frac{1}{\left(\frac{s}{3}\right)^2 + 1} \qquad \left[\because f(s) = \frac{1}{s^2+1}\right.$$

$$= \frac{1}{3 \cdot \frac{s^2+3^2}{3^2}}$$

$$= \frac{3^2}{3(s^2+3^2)}$$

$$= \frac{3}{s^2+3^2}$$

$$= \frac{3}{s^2+9}$$

Laplace transformation of derivative :

1st derivative :

$$L\left\{\frac{df(t)}{dt}\right\} = \int_0^{\infty} e^{-st} \cdot \frac{df(t)}{dt} \cdot dt$$

$$= \left[e^{-st} \cdot f(t)\right]_0^{\infty} + s\int e^{-st} f(t)\, dt$$

$$= -f(0) + s f(s)$$

$$= s f(s) - f(0)$$

2nd derivative :

$$L\left\{\frac{d^2 f(t)}{dt^2}\right\} = \int_0^{\infty} e^{-st} \cdot \frac{d^2 f(t)}{dt^2} \cdot dt$$

$$= \left[e^{-st} \cdot \frac{df(t)}{dt}\right]_0^{\infty} + s\int e^{-st} \cdot \frac{df(t)}{dt} \cdot dt$$

$$= -\frac{df(0)}{dt} + s\left[-f(0) + s f(s)\right]$$

$$= -f'(0) - s f(0) + s^2 f(s)$$

▣ Solve the initial value problem $\frac{d^2x}{dt^2}+4\frac{dx}{dt}+4x=4e^{-2t}$

Given $x(0)=-1$ and $x'(0)=4$.

Soln: Given differential equation,

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+4x=4e^{-2t}$$

Taking Laplace transform on both sides, we get,

$$L\left\{\frac{d^2x}{dt^2}\right\}+4L\left\{\frac{dx}{dt}\right\}+4L\{x\}=4L\{e^{-2t}\}$$

$\Rightarrow \cancel{-f'(0)}$

$$\Rightarrow -x'(0)-s\,x(0)+s^2x(s)+4\{-x(0)\}+4s\,x(s)$$

$$+4L\{x(t)\}=\frac{4}{s+2}$$

$$\Rightarrow -4+s+s^2x(s)+4+4s\,x(s)+4\,x(s)=\frac{4}{s+2}$$

$$\Rightarrow x(s)\{s^2+4s+4\}=\frac{4}{s+2}-s$$

$$\Rightarrow x(s)=\frac{4-s^2-2s}{(s+2)(s^2+4s+4)}$$

$$=\frac{4-s(s+2)}{(s+2)(s+2)^2}$$

$$=\frac{4}{(s+2)(s+2)^2}-\frac{s}{(s+2)^2}$$

96.

$$\Rightarrow L\{x(t)\} = \frac{4}{(s+2)^3} - \frac{s+2-2}{(s+2)^2}$$

$$= \frac{4}{(s+2)^3} - \frac{1}{(s+2)} + \frac{2}{(s+2)^2}$$

$$= \frac{4}{(s+2)^3} + \frac{2}{(s+2)^2} - \frac{1}{s+2}$$

$$\therefore x(t) = L^{-1}\left[\frac{4}{(s+2)^3} + \frac{2}{(s+2)^2} - \frac{1}{s+2}\right]$$

$$= 4L^{-1}\left\{\frac{1}{(s+2)^3}\right\} + 2L^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$- L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= 4L^{-1}\left\{\frac{\Gamma(2+1)}{\Gamma(2+1)(s+2)^{2+1}}\right\} + 2L^{-1}\left\{\frac{\Gamma(1+1)}{\Gamma(1+1)(s+2)^{1+1}}\right\}$$

$$- L^{-1}\left\{\frac{\Gamma(0+1)}{\Gamma(0+1)(s+2)^{0+1}}\right\}$$

$$\cancel{=4L^{-1}}$$

$$= \frac{4}{\lceil 2+1}\, L^{-1}\left\{\frac{\lceil 2+1}{(s+2)^{2+1}}\right\} + \frac{2}{\lceil 1+1}\, L^{-1}\left\{\frac{\lceil 1+1}{(s+2)^{1+1}}\right\}$$

$$- L^{-1}\left\{\frac{\lceil 0+1}{(s+2)^{0+1}}\right\}$$

$$= \frac{4}{2!}\cdot e^{-2t}\cdot t^{2} + \frac{2}{1!}\cdot e^{-2t}\cdot t^{1} - e^{-2t}\cdot t^{0}$$

$$= \frac{4}{2}\times e^{-2t} t^{2} + 2e^{-2t}\cdot t - e^{-2t}$$

$$= 2e^{-2t} t^{2} + 2e^{-2t}\cdot t - e^{-2t}$$

$$= e^{-2t}\left(2t^{2}+2t-1\right) \quad \text{Ans}$$

$\because L(t^{n}) = \dfrac{\lceil n+1}{s^{n+1}}$

And,

$L\left(e^{-mt}\cdot t^{n}\right) = \dfrac{\lceil n+1}{(s+m)^{n+1}}$

$\therefore e^{-mt}\cdot t^{n} = L^{-1}\left\{\dfrac{\lceil n+1}{(s+m)^{n+1}}\right\}$

Another easy formula,

$$x(t) = L^{-1}\left[\frac{4}{(s+2)^{3}} + \frac{2}{(s+2)^{2}} - \frac{1}{(s+2)}\right]$$

$$= 4L^{-1}\left(\frac{1}{(s+2)^{3}}\right) + 2L^{-1}\left(\frac{1}{(s+2)^{2}}\right) - L^{-1}\left(\frac{1}{(s+2)}\right)$$

$$= 4e^{-2t}\cdot L^{-1}\left(\frac{1}{s^{3}}\right) + 2\cdot e^{-2t}\cdot L^{-1}\left(\frac{1}{s^{2}}\right) - e^{-2t}\cdot L^{-1}\left(\frac{1}{s}\right)$$

$$= 4\cdot e^{-2t}\cdot\frac{t^2}{\Gamma 3} + 2\cdot e^{-2t}\cdot\frac{t^1}{\Gamma 2} - e^{-2t}\cdot\frac{t^0}{\Gamma 1}$$

$$= \frac{4}{2!}e^{-2t}t^2 + \frac{2}{1!}e^{-2t}\cdot t - e^{-2t}$$

$$= 2e^{-2t}t^2 + 2e^{-2t}\cdot t - e^{-2t}$$

$$= e^{-2t}(2t^2+2t-1)$$

Bessel's Function :

The equation $x^2 \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + (x^2 - n^2)y = 0$ is called Bessel's differential equation of order n.

By solving this equation, we get,

$$\boxed{J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r+n}}{r!\,(n+r)!}}$$

and this function is called Bessel's function.

Prove that $J_{-n}(x) = (-1)^n J_n(x)$

Soln:

We Have,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r+n}}{r!\,(n+r)!}$$

$$\therefore J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{x}{2}\right)^{2r-n}}{r!\,(-n+r)!}$$

Putting $-n+r = s$

$\Rightarrow r = s+n$

Therefore,

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{s+n} \left(\frac{x}{2}\right)^{2s+n}}{(s+n)!\,s!} = (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s \cdot \left(\frac{x}{2}\right)^{2s+n}}{s!\,(s+n)!}$$

$$= (-1)^n J_n(x) \quad \text{(proved)}$$

100

(8) Prove that, $x J_n'(x) = x J_{n-1}(x) - n J_n(x)$

Sol^n:

We know,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r+n}}{r!\,(n+r)!}$$

$$\Rightarrow J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (2r+n)(x/2)^{2r+n-1}}{r!\,(n+r)!} \cdot \frac{1}{2}$$

$$\Rightarrow x J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (2r+2n-n)(x/2)^{2r+n-1} \cdot \left(\frac{x}{2}\right)}{r!\,(n+r)!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2r+2n-n)(x/2)^{2r+n}}{r!\,(n+r)!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \cdot 2(r+n)(x/2)^{2r+n}}{r!\,(n+r)!} - n\sum_{r=0}^{\infty} \frac{(-1)^r \cdot (x/2)^{2r+n}}{r!\,(n+r)!}$$

$$= x\sum_{r=0}^{\infty} \frac{(-1)^r \cdot \cancel{\left(\frac{x}{2}\right)} \cdot \frac{2}{x} \cdot (n+r)(x/2)^{2r+n}}{r!\,(n+r)!} - n J_n(x)$$

$$= x\sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-1} \cdot (n+r)(x/2)^{2r+n}}{r!\,(n+r)!} - n J_n(x)$$

$$= x\sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r+(n-1)} \cdot (n+r)}{r!\,(n+r)\,(n+r-1)!} - n J_n(x)$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r+(n-1)}}{r!\,\{(n-1)+r\}!} - n J_n(x)$$

$$= x J_{n-1}(x) - n J_n(x) \quad \text{(proved)}$$

(3) Prove that, $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

Soln:- We know,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r+n}}{r!\,(n+r)!}$$

$$\Rightarrow J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (2r+n) \left(\frac{x}{2}\right)^{2r+n-1} \cdot \frac{1}{2}}{r!\,(n+r)!}$$

$$\Rightarrow x J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (2r+n) \left(\frac{x}{2}\right)^{2r+n-1} \cdot \frac{x}{2}}{r!\,(n+r)!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2r+n) \left(\frac{x}{2}\right)^{2r+n}}{r!\,(n+r)!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, 2r \cdot \left(\frac{x}{2}\right)^{2r+n}}{r!\,(n+r)!} + n \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{x}{2}\right)^{2r+n}}{r!\,(n+r)!}$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \frac{2}{x} \cdot \left(\frac{x}{2}\right)^{2r+n} \cdot r}{r!\,(n+r)!} + n J_n(x)$$

102

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-1} \left(\frac{x}{2}\right)^{2r+n} \cdot r}{r!\,(n+r)!} + n J_n(x)$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)\cdot(-1)^{r-1} \left(\frac{x}{2}\right)^{2r+n-1} \cdot r}{r\,(r-1)!\,(n+r)!} + n J_n(x)$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)(-1)^{r-1} \cdot \left(\frac{x}{2}\right)^{\{2(r-1)+(n+1)\}}}{(r-1)!\,\{(n+1)+(r-1)\}!} + n J_n(x)$$

$$= -x \sum_{r=0}^{\infty} \frac{(-1)^{r-1} \cdot \left(\frac{x}{2}\right)^{\{2(r-1)+(n+1)\}}}{(r-1)!\,\{(n+1)+(r-1)\}!} + n J_n(x)$$

$$= -x J_{n+1}(x) + n J_n(x)$$

$$= + n J_n(x) - x J_{n+1}(x)$$ (proved)

Prove that $2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

Soln :-

We know that,

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x) \quad \text{--- (i)}$$

and,

$$x J_n'(x) = -n J_n(x) + x J_{n-1}(x) \quad \text{--- (ii)}$$

(i)+(ii),

$$2x J_n'(x) = x\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

$$\Rightarrow 2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$ Ans

III Prove that,

$$2n J_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

we know that,

$$x J_n'(x) = n J_n(x) - x J_{n+1}(x) \quad \text{—— (i)}$$

$$x J_n'(x) = -n J_n(x) + x J_{n-1}(x) \quad \text{—— (ii)}$$

(i)-(ii),

$$0 = 2n J_n(x) - x\left[J_{n+1}(x) + J_{n-1}(x)\right]$$

$$\Rightarrow 2n J_n(x) = x\left[J_{n+1}(x) + J_{n-1}(x)\right]$$

Prove that,

(i) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\, \sin x$

(ii) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\, \cos x$

$$\Rightarrow \boxed{\left(J_{1/2}(x)\right)^2 + \left(J_{-1/2}(x)\right)^2 = \frac{2}{\pi x}}$$

Soln:

(i) We know that,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{x}{2}\right)}{r!\,(n+r)!}$$

Putting the values of r as 0, 1, 2, 3... respectively,

$$J_n(x) = \frac{1}{n!}\cdot\frac{x^n}{2^n} - \frac{1}{1!\,(n+1)!}\,\frac{x^{n+2}}{2^{n+2}} + \frac{1}{2!\,(n+2)!}\,\frac{x^{n+4}}{2^{n+4}} -$$

$$J_n(x) = \frac{1}{\Gamma(n+1)}\,\frac{x^n}{2^n} - \frac{1}{\Gamma(n+2)}\cdot\frac{x^n}{2^n}\cdot\frac{x^2}{4} + \frac{1}{2!\,\Gamma(n+3)}\cdot\frac{x^n}{2^n}\cdot\frac{x^4}{16} -$$

$$\Rightarrow J_n(x) = \frac{1}{\Gamma(n+1)} \frac{x^n}{2^n}\left\{1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4^2(n+2)(n+1)\cdot 2!} - \cdots\cdots\right\}$$

Putting $n = \frac{1}{2}$,

$$J_{1/2}(x) = \frac{1}{\Gamma 3/2} \cdot \frac{x^{1/2}}{2^{1/2}} \cdot \left\{1 - \frac{x^2}{4(3/2)} + \frac{x^4}{4^2 \cdot 2! \,(5/2)\cdot(3/2)} - \cdots\right\}$$

$$J_{1/2}(x) = \frac{1}{1/2\,\Gamma 1/2}\left(\frac{x}{2}\right)^{1/2} \cdot \frac{x}{x}\left\{1 - \frac{x^2}{3\cdot 2} + \frac{x^4}{5\cdot 4\cdot 3\cdot 2} - \cdots\cdots\right\}$$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2^{1/2}} \cdot x^{1/2} \cdot \frac{1}{x}\left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots\right\}$$

$$= \frac{2^{1/2}}{\sqrt{\pi}} \cdot \frac{1}{x^{1/2}}\left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots\right\}$$

$$= \sqrt{\frac{2}{\pi x}} \cdot \sin x \quad \text{(proved)}$$

Solⁿ:-

(ii) We know that,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \cdot \left(\frac{x}{2}\right)^{2r+n}}{r!\,\Gamma(n+r+1)}$$

Putting the values of r as, $0, 1, 2, 3 \ldots$ respectively,

$$J_n(x) = \frac{1}{\Gamma n+1} \cdot \frac{x^n}{2^n} - \frac{1}{1!\,\Gamma n+2} \cdot \frac{x^n}{2^n} \cdot \frac{x^2}{2^2} + \frac{1}{2!\,\Gamma n+3} \cdot \frac{x^n}{2^n} \cdot \frac{x^4}{2^4} - \cdots$$

$$\Rightarrow J_n(x) = \frac{1}{\Gamma n+1} \cdot \frac{x^n}{2^n}\left\{1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4^2(n+2)(n+1)\cdot 2!} - \cdots\right\}$$

Putting $n = -\frac{1}{2}$

$$J_{-1/2}(x) = \frac{1}{\sqrt{-\frac{1}{2}+1}} \cdot \frac{x^{-1/2}}{2^{-1/2}} \left\{1 - \frac{x^2}{4 \cdot \frac{1}{2}} + \frac{x^4}{2! \, 4^2 \, \frac{3}{2} \cdot \frac{1}{2}} - \cdots \right\}$$

$$J_{-1/2}(x) = \frac{1}{\sqrt{1/2}} \, \frac{2^{1/2}}{x^{1/2}} \left\{1 - \frac{x^2}{2!} + \frac{x^4}{4 \cdot 3 \cdot 2!} - \cdots \right\}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{x}} \cdot \cos x$$

$$= \sqrt{\frac{2}{\pi x}} \cos x$$

Extra:-

$$\because J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sin x$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \cos x$$

$$\therefore \{J_{1/2}(x)\}^2 + \{J_{-1/2}(x)\}^2 = \frac{2}{\pi x}(\sin^2 x + \cos^2 x)$$

$$= \frac{2}{\pi x} \cdot 1$$

$$= \frac{2}{\pi x}$$

(iii) We know that,

$$\boxed{J_n(x) = \frac{x}{2n}\left[J_{n+1}(x) + J_{n-1}(x)\right]}$$

Putting, $n = \frac{1}{2}$, we get,

$$J_{1/2}(x) = \frac{x}{2\cdot\frac{1}{2}}\left[J_{1/2+1}(x) + J_{1/2-1}(x)\right]$$

$$\Rightarrow J_{1/2}(x) = x\left[J_{3/2}(x) + J_{-1/2}(x)\right]$$

$$\Rightarrow x J_{3/2}(x) = J_{1/2}(x) - x J_{-1/2}(x)$$

$$\Rightarrow J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$$

$$= \frac{1}{x}\cdot\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cdot\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left\{\frac{\sin x}{x} - \cos x\right\}$$

(iv) We know that,

$$J_n(x) = \frac{1}{2n}\left[J_{n+1}(x) + J_{n-1}(x)\right]$$

now putting the value of $n = -\frac{1}{2}$, we get,

$$J_{-1/2}(x) = \frac{x}{2\cdot\left(\frac{-1}{2}\right)}\left[J_{-1/2+1}(x) + J_{-1/2-1}(x)\right]$$

$$\Rightarrow J_{-1/2}(x) = -x\left[J_{1/2}(x) + J_{-3/2}(x)\right]$$

$$\Rightarrow J_{-1/2}(x) = -xJ_{1/2}(x) - xJ_{-3/2}(x)$$

$$\Rightarrow xJ_{-3/2}(x) = -xJ_{1/2}(x) - J_{-1/2}(x)$$

$$= -x\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= -\sqrt{\frac{2}{\pi x}}\left[x\sin x + \cos x\right]$$

$$\therefore J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left[\frac{x\sin x + \cos x}{x}\right]$$

$$= -\sqrt{\frac{2}{\pi x}}\cdot\left[\sin x + \frac{\cos x}{x}\right]$$

Hypergeometric Equation and Hypergeometric function :

The series, $1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\dots\dots$ —— (i);

is called Hypergeometric series.

The generalisation of this series is,

$$1+x+x^2+\dots\dots$$

And the Hypergeometric equation of this Hypergeometric series is,

$$\boxed{x(1-x)y''+[\gamma-(\alpha+\beta+1)x]y'-\alpha\beta y=0} \quad \text{—— (ii)}$$

we may write the series (i) in the form of,

$$_2F_1(\alpha,\beta;\gamma;x)=\sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{r!(\gamma)_r}\times x^r \quad \text{—— (iii)}$$

~~This equation no (iii) is called~~

This is called the Hyper geometric function.

Properties of Hypergeometric function

1. Symmetry property :

$$ {}_2F_1(\alpha, \beta; \gamma; 0) = 1 \text{ and } {}_2F_1(\alpha, \beta; \gamma; x) = {}_2F_1(\beta, \alpha; \gamma; x) $$

The hypergeometric function does not change if the parameters α and β are interchanged.

Proof:

We know that,

$$ {}_2F_1(\alpha, \beta; \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k \, k!} \cdot x^k $$

$$ = \sum_{k=0}^{\infty} \frac{(\beta)_k (\alpha)_k}{(\gamma)_k \, k!} \cdot x^k $$

$\therefore {}_2F_1(\alpha, \beta; \gamma; x) = {}_2F_1(\beta, \alpha; \gamma; x)$ (proved)

For, $x = 0$ we can write,

$$ {}_2F_1(\alpha, \beta; \gamma; x) = {}_2F_1(\beta, \alpha; \gamma; 0) = \lim_{x \to 0} \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{k! (\gamma)_k} x^k $$

$$ = \lim_{x \to 0} \left[1 + \frac{\alpha . \beta}{1 . \gamma} x + \frac{\alpha(\alpha+1) \cdot \beta(\beta+1)}{1 . 2 . \gamma(\gamma+1)} x^2 + \dots \dots \right] $$

$$ = 1 $$

$$ \boxed{\text{Therefore } {}_2F_1(\alpha, \beta; \gamma; x) = 1 \text{ for } x = 0} $$

Differentiation of H.F.

We know that,

$${}_2F_1(\alpha,\beta;\gamma;x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k \cdot (\beta)_k}{(\gamma)_k \cdot k!} x^k$$

$$\therefore \frac{d}{dx}\left\{{}_2F_1(\alpha,\beta;\gamma;x)\right\} = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k \cdot k!} \cdot k \cdot x^{k-1}$$

$$= \sum_{k=1}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k \cdot \cancel{k}(k-1)!} \cdot \cancel{k} \cdot x^{k-1}$$

$$= \sum_{k=1}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k \cdot (k-1)!} \cdot x^{k-1}$$

When,

$$k-1 = n$$
$$\Rightarrow k = n+1$$

then,

$$\frac{d}{dx}\left\{{}_2F_1(\alpha,\beta;\gamma;x)\right\} = \sum_{n=0}^{\infty} \frac{(\alpha)_{n+1} (\beta)_{n+1}}{(\gamma)_{n+1} \; n!} x^n$$

Since,

$$(\alpha)_{n+1} = \alpha(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+n-1)(\alpha+n)$$

$$= \alpha\left[(\alpha+1)(\alpha+2)\ldots(\alpha+1+n-1)\right]$$

$$= \alpha\,(\alpha+1)_n$$

Similarly,

$$(\beta)_{n+1} = \beta(\beta+1)_n$$

and,

$$(\gamma)_{n+1} = \gamma(\gamma+1)_n$$

Thus,

$$\frac{d}{dx}\left[{}_2F_1(\alpha,\beta;\gamma;x)\right] = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)_n\ \beta(\beta+1)_n}{n!\ \gamma(\gamma+1)_n} \cdot x^n$$

$$= \frac{\alpha\beta}{\gamma}\sum_{n=0}^{\infty} \frac{(\alpha+1)_n\ (\beta+1)_n}{(\gamma+1)_n \cdot n!} \cdot x^n$$

$$= \frac{\alpha\beta}{\gamma}\left[{}_2F_1\{(\alpha+1),(\beta+1);\gamma+1;x\}\right]$$

Similarly,

$$\frac{d^2}{dx^2}\left[{}_2F_1(\alpha,\beta;\gamma;x)\right] = \frac{\alpha(\alpha+1)\ \beta(\beta+1)}{\gamma\ (\gamma+1)} \cdot {}_2F_1(\alpha+2,\beta+2;\gamma+2;x)$$

... ...

... ...

$$\frac{d^m}{dx^m}\left[{}_2F_1(\alpha,\beta;\gamma;x)\right]$$

$$= \frac{\alpha(\alpha+1)\dots(\alpha+m-1)\cdot\beta(\beta+1)\dots(\beta+m-1)}{\gamma(\gamma+1)\dots\dots(\gamma+m-1)} \cdot {}_2F_1(\alpha+m,\beta+m;\gamma+m;x)$$

Now, we have,

$$\frac{d}{dx}\left[{}_2F_1(\alpha,\beta;\gamma;0)\right] = \cancel{\sum_{n=0}} \frac{\alpha\beta}{\gamma}\left[{}_2F_1(\alpha+1,\beta+1;\gamma+1;0)\right]$$

$$= \frac{\alpha\beta}{\gamma} \times \lim_{x\to 0}\sum_{k=0}^{\infty}\frac{(\alpha+1)_k(\beta+1)_k}{(\gamma+1)_k\cdot k!}x^k$$

$$= \frac{\alpha\beta}{\gamma}\times\lim_{x\to 0}\left[1+\frac{(\alpha+1)\cdot(\beta+1)}{1\cdot(\gamma+1)}\cdot x+\frac{(\alpha+1)(\alpha+2)\cdot(\beta+1)(\beta+2)}{(\gamma+1)(\gamma+2)\cdot 1.2}x^2+\dots\right]$$

$$= \frac{\alpha\beta}{\gamma}\times 1$$

$$= \frac{\alpha\beta}{\gamma}$$

Therefore,

$$\frac{d}{dx}\left[{}_2F_1(\alpha,\beta;\gamma;0)\right] = \frac{\alpha\beta}{\gamma}$$

The confluent Hypergeometric equation function:-

The Hypergeometric equation,

$$x(1-x)\frac{d^2y}{dx^2}+\left[\gamma-(\alpha+\beta+1)x\right]\frac{dy}{dx}-\alpha\beta y=0$$

Putting $x=\frac{x}{\beta}$,

$$x\left(1-\frac{x}{\beta}\right)\frac{d^2y}{dx^2}+\left[\gamma-\left(1+\frac{\alpha+1}{\beta}\right)x\right]\frac{dy}{dx}-\alpha y=0 \quad \text{---(i)}$$

the soln of which represented by the function ${}_2F_1(\alpha,\beta;\gamma;x/\beta)$

Letting, $\beta\to\infty$; the function becomes,

$$\lim_{\beta\to\infty} {}_2F_1\left(\alpha,\beta;\gamma;\frac{x}{\beta}\right)$$ and (i) reduces to,

$$x\frac{d^2y}{dx^2}+(\gamma-x)\frac{dy}{dx}-\alpha y=0 \quad \text{---(ii)}$$

This 2nd order D.E. (ii) in which α and γ are constant is called confluent hypergeometric function.

The soln of (ii) is represented by,

$$\lim_{\beta\to\infty} {}_2F_1(\alpha,\beta;\gamma;x/\beta)=\lim_{\beta\to\infty}\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k}\cdot\frac{x^k}{\beta^k}$$

But,
$$\lim_{\beta \to \infty} \frac{(\beta)_k}{\beta^k} = 1$$

Hence,
$$\lim_{\beta \to \infty} {}_2F_1\left(\alpha, \beta; \gamma; x/\beta\right) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!\,(\gamma)_k} \cdot x^k$$

$$= {}_1F_1\left(\alpha; \gamma; x\right)$$

The series,

$(\alpha;\gamma;x)$

$$\sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!\,(\gamma)_k} x^k$$ is denoted by ${}_1F_1(\alpha; \cancel{\beta}; \cancel{x})$

Since it is the solution of confluent hypergeometric equation.

Hermite's Differential equation and Hermite's polynomials :

The differential equation of the form

$$\boxed{y'' - 2xy' + 2\alpha y = 0} \quad \text{---(i)} \quad (\text{where } \alpha \text{ is a constant})$$

is known as Hermite differential equation.

By solving this equation we get,

$$y = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} + \dots \dots + (-1)^r \frac{(2x)^{n-2r}}{r!} n(n-1)(n-2)\dots(n-2r+1) + \dots$$

This value of y is known as Hermite polynomial of degree n and is denoted by $H_n(x)$.

⊞ Legendre's equation and Legendre's polynomials:

The second order differential equation,

$$\boxed{(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\}y = 0};$$

in which m and n are constants is known as Legendre's associated equation and its solutions are called associated Legendre functions.

It reduces to Legendre's equation when $m = 0$.

The Legendre's differential equation is,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \text{--- (i)}$$

And the sol^n of this eq^n is called Legendre's function or Legendre's polynomials.

The sol^n of Legendre's differential equation is,

$$y = C_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n-2)(n-3)}{4!}x^4 \dots\right]$$

$$+ C_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n+2)(n+4)}{5!}x^5 - \dots\right]$$

$$= C_0 u + C_1 v \quad \text{(say)}$$

where, $u = \uparrow$ (ইভ টার্ম)
$v = \uparrow$ (অড় টের্ম)

[iii] Legendre polynomial : (After the first part, then this)

$$\boxed{\therefore P_n(x) = \sum_{r=0}^{N} (-1)^r \cdot \frac{(2n-2r)!}{2^n \cdot r! \cdot (n-r)!\,(n-2r)!}\, x^{n-2r}}$$

Where,

$N = \frac{n}{2}$ for n even

$N = \frac{n-1}{2}$ for n odd

Another form of $P_n(x)$ is given below,

$$P_n(x) = \frac{(2n)!}{2^n\, n!\, n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}\, x^{n-2} + \dots\dots\right]$$

119

Rodrigues formula for Legendre polynomial

$$P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n} (x^2-1)^n$$

Proof:

Let $v = (x^2-1)^n$

Differentiating w.r.t. x, we get,

$$\frac{dv}{dx} = v_1 = n(x^2-1)^{n-1} \cdot 2x$$

$$\Rightarrow v_1 = \frac{2nx(x^2-1)^n}{x^2-1}$$

$$\Rightarrow (x^2-1) v_1 = 2nxv$$

$$\Rightarrow (1-x^2) v_1 = -2nxv$$

$$\Rightarrow (1-x^2) v_1 + 2nxv = 0$$

Again differentiating w.r.t. x,

$$(1-x^2) v_2 - 2xv_1 + 2nxv_1 + 2nv = 0$$

$$\Rightarrow (1-x^2) v_2 + (2nx-2x) v_1 + 2nv = 0$$

$$\Rightarrow (1-x^2) v_2 + 2(n-1)xv_1 + 2nv = 0$$

Differentiating n times using Leibnitz's theorem,

$$(1-x^2)v_{n+2} + {}^nC_1(-2x)v_{n+1} + {}^nC_2(-2)v_n + 2(n-1)xv_{n+1} + {}^nC_1\,2(n-1)v_n + 2nv_n = 0$$

$$\Rightarrow (1-x^2)v_{n+2} + v_{n+1}(-2nx + 2nx - 2x) + v_n\left\{\frac{n(n-1)}{2}\cdot(-2) + 2n(n-1) + 2n\right\} = 0$$

$$\Rightarrow (1-x^2)v_{n+2} - 2xv_{n+1} + \{n(n-1)+2n\}v_n = 0$$

$$\Rightarrow (1-x^2)v_{n+2} - 2xv_{n+1} + n(n+1)v_n = 0$$

$$\Rightarrow (1-x^2)\frac{d^2}{dx^2}(v_n) - 2x\frac{d}{dx}(v_n) + n(n+1)v_n = 0$$

This is a legendre equation of v_n.

So, v_n is a solution of legendre equation.

But,
$$v_n = \frac{d^n}{dx^n}v = \frac{d}{dx^n}(x^2-1)^n$$

Here we see that, v_n is a polynomial of ~~deg~~ degree n.

$P_n(x)$ is also a solution of Legendre equation.

Hence, we get,

$$P_n(x) = k\, V_n(x) = k \frac{d^n}{dx^n}(x^2-1)^n \quad \text{--- (i)}$$

where, k is a constant.

But,

$$P_n(x) = \frac{(2n)!}{2^n (n!)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)} \cdot x^{n-2} + \dots\right] \quad \text{--- (ii)}$$

From eq^n (i) and (ii) we get

$$k \frac{d^n}{dx^n}(x^2-1)^n = \frac{(2n)!}{2^n (n!)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \dots\right]$$

$$\Rightarrow \frac{(2n)!}{2^n (n!)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \dots\right] = k \frac{d^n}{dx^n}(x^2-1)^n$$

equating the co-efficient of x^n from both sides,

$$\frac{(2n)!}{2^n (n!)^2} = k \cdot 2n(2n-1)(2n-2)\dots(n+1)$$

$$\Rightarrow \frac{(2n)!}{2^n (n!)^2} = k \frac{(2n)!}{n!}$$

$$\Rightarrow \frac{1}{2^n \cdot n!} = k$$

$\therefore k = \frac{1}{2^n n!}$

putting the value $k = \frac{1}{2^n n!}$ in equation (i),

$$\boxed{P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n} (x^2-1)^n}$$

⊞ ~~$P_0(x)$~~ [All of these are with Legendre's polynomials]

⊞ For $P_0(x)$; $n=0$ is even and $N = \frac{n}{2} = 0$

$$\therefore P_0(x) = \sum_{r=0}^{0} (-1)^0 \frac{(2\times 0 - 2\times 0)!}{2^0 \times 0! \times (0-0)! \times (0-2\times 0)!} x^{0-2\times 0}$$

$$= 1 \cdot x^0$$

$$= 1$$

⊞ ~~P_1 For, $P_1(x) = \sum_{r=0}^{0} (-1)^0$~~

⊞ For, $P_1(x)$; $n=1$ is odd number and $N = \frac{n-1}{2} = 0$

$$\therefore P_1(x) = \sum_{r=0}^{0} (-1)^0 \cdot \frac{(2\times 1 - 2\times 0)!}{2^1 \times 0! \times (1-0)! (1-2\times 0)!} x^{1-2\times 0}$$

$$= 1 \times \frac{2!}{2} \cdot x^1$$

$$= x$$

⊞ $P_2(x) =$ $n=2$ is even number and $N = \frac{n}{2} = \frac{2}{2} = 1$

$$\therefore P_2(x) = \sum_{r=0}^{1} (-1)^r \frac{(2\times2-2\times0)!}{2^2 \times 0! \times (2-0)! \times (2-2\times0)!} x^{2-2\times0}$$

$$= \sum_{r=0}^{1} 1 \cdot \frac{24}{4\times2!\times2!} \times x^2$$

$$= \sum_{r=0}^{1} \frac{3}{2} \cdot x^2$$

⊞ For $P_3(x)$ $n=3$ is odd numbers and $N = \frac{n-1}{2} = \frac{3-1}{2} = 1$

$$P_3(x) = \sum_{r=0}^{1} (-1)^0 \frac{(2\times3-2\times0)!}{2^3 \times 0! \times (3-0)! \times (3-2\times0)!} \cdot x^{3-2\times0}$$

$$= \sum_{r=0}^{1} 1 \times \frac{6!}{8\times3!\times3!} x^3$$

$$= \sum_{r=0}^{1} \frac{720}{288} x^3$$

By Rodrigue's formula:

$$\boxed{P_n(x) = \frac{1}{2^n n!} \times \frac{d^n}{dx^n}(x^2-1)^n}$$

(1) $P_0(x) = \frac{1}{2^0 \cdot 0!} \times \frac{d^0}{dx^0}(x^2-1)^0 = 1$

(2) $P_1(x) = \frac{1}{2^1 \cdot 1!} \times \frac{d^1}{dx^1}(x^2-1)^1 = \frac{1}{2} \times 2x = x$

(3) $P_2(x) = \frac{1}{2^2 \cdot 2!} \times \frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8} \times \frac{d^2}{dx^2}(x^4-2x^2+1)$

$= \frac{1}{8} \times \frac{d}{dx}(4x^3-4x)$

$= \frac{1}{8} \times (12x^2-4)$

$= \frac{1}{2}(3x^2-1)$

(4) $P_3(x) = \frac{1}{2^3 \cdot 3!} \times \frac{d^3}{dx^3}(x^2-1)^3$

$= \frac{1}{48} \times \frac{d^3}{dx^3}(x^6-3x^4+3x^2-1)$

$= \frac{1}{48} \times \frac{d^2}{dx^2}(6x^5-12x^3+6x)$

$= \frac{6}{48} \times \frac{d}{dx}(5x^4-6x^2+1)$

$= \frac{1}{8} \times (20x^3-12x) = \frac{1}{2}(5x^3-3x)$ Ans

Express the following function in terms of Legendre's polynomials,

$$f(x) = x^4 + 2x^3 + 2x^2 - x - 3$$

Solⁿ:-

We know from Legendre polynomials,

$$P_4(x) = \frac{1}{2^4 . 4!} \times \frac{d^4}{dx^4} (x^2-1)^4$$

$$= \cancel{\frac{1}{384}}$$ I don't know

$$\Rightarrow P_4(x) = \frac{1}{8} \times (35x^4 - 30x^2 + 3)$$

$$\Rightarrow \cancel{\frac{4}{x}}$$

$$\Rightarrow 8P_4(x) = 35x^4 - 30x^2 + 3$$

$$\Rightarrow 35x^4 = 8P_4(x) + 30x^2 - 3$$

$$\Rightarrow x^4 = \frac{8}{35} P_4(x) + \frac{6}{7} x^2 - \frac{3}{35}$$

And, $P_3(x) = \frac{1}{2} \times (5x^3 - 3x)$

$$\Rightarrow 2P_3(x) = 5x^3 - 3x$$

$$\Rightarrow 5x^3 = 2P_3(x) + 3x$$

$$\Rightarrow x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x$$

also,

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$\Rightarrow 2P_2(x) = 3x^2 - 1$$

$$\Rightarrow 3x^2 = 2P_2(x) + 1$$

$$\Rightarrow x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}$$

and also,

$$P_1(x) = x$$

$$\therefore x = P_1(x)$$

also,

$$P_0(x) = 1$$

$$\therefore 1 = P_0(x)$$

Putting the values of x^4, x^3, x^2 and x we get,

$$f(x) = \frac{8}{35}P_4(x) + \frac{4}{5}P_3(x) + \frac{40}{21}P_2(x) + \frac{1}{5}P_1(x) - \frac{224}{105}P_0(x)$$

Problem: Show that, $P_n(-x) = (-1)^n P_n(x)$

We know that,

$$P_n(x) = \frac{(2n)!}{2^n (n!)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \dots \dots\right]$$

Therefore,

$$P_n(-x) = \frac{(2n)!}{2^n (n!)^2}\left[(-x)^n - \frac{n(n-1)}{2(2n-1)} (-x)^{n-2} + \dots \dots\right]$$

$$= \frac{(2n)!}{2^n (n!)^2}\left[(-1)^n \cdot x^n - \frac{n(n-1)}{2(2n-1)} (-1)^{n-2} \cdot (x)^{n-2} + \dots\right]$$

$$= \frac{(2n)!}{2^n (n!)^2}\left[(-1)^n \cdot x^n - \frac{n(n-1)}{2(2n-1)} \cdot \frac{(-1)^n}{(-1)^2} \cdot x^{n-2} + \dots\right]$$

$$= \frac{(2n)!}{2^n (n!)^2} \times (-1)^n \left[x^n - \frac{n(n-1)}{2(2n-1)} \cdot x^{n-2} + \dots\right]$$

$$= (-1)^n \frac{(2n)!}{2^n (n!)^2} \times \left[x^n - \frac{n(n-1)}{2(2n-1)} \cdot x^{n-2} + \dots\right]$$

$\therefore P_n(-x) = (-1)^n \cdot P_n(x)$ (Showed)

4. Find the values of $P_n(1)$, $P_n(-1)$ and $P_n(0)$.

Sol^n:

We know from generating function,

$$(1-2hx+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \quad \text{—— (i)}$$

Putting $x=1$ we get,

$$(1-2h+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(1)$$

$$\Rightarrow \{(1-h)^2\}^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(1)$$

$$\Rightarrow (1-h)^{-1} = \sum_{n=0}^{\infty} h^n P_n(1)$$

$$\Rightarrow 1+h+h^2+h^3+\dots \dots + h^n + \dots = \sum_{n=0}^{\infty} h^n P_n(1)$$

Equating the co-efficient of h^n, we get,

$$1 = P_n(1)$$

$$\therefore P_n(1) = 1 \quad \text{Ans.}$$

229

For, $x = -1$, from eq^n (i),

$$\{1 - 2h(-1) + h^2\}^{-1/2} = \sum_{n=0}^{\alpha} h^n P_n(-1)$$

$$\Rightarrow (1 + 2h + h^2)^{-1/2} = \sum_{n=0}^{\alpha} h^n P_n(-1)$$

$$\Rightarrow \{(1+h)^2\}^{-1/2} = \sum_{n=0}^{\alpha} h^n P_n(-1)$$

$$\Rightarrow (1+h)^{-1} = \sum_{n=0}^{\alpha} h^n P_n(-1)$$

$$\Rightarrow 1 - h + h^2 - h^3 + \dots + (-1)^n h^n + \dots = \sum_{n=0}^{\alpha} h^n P_n(-1)$$

Equating the co-efficients of h^n,

$$(-1)^n = P_n(-1)$$

$$\therefore P_n(-1) = (-1)^n$$

For, $x = 0$,

$$(1 + h^2)^{-1/2} = \sum_{n=0}^{\alpha} h^n P_n(0)$$

???

$$\therefore P_n(0) = 0$$

III Prove that $P_5'(x) = 9P_4(x) + 5P_2(x) + P_0(x)$

Soln:-

From recurrence relation we have,

$$\boxed{P'_{n+1}(x) - P'_{(n-1)}(x) = (2n+1)\,P_n(x)}$$

Putting $n=4$ and $n=2$ we have,

$$P_5'(x) - P_3'(x) = 9P_4(x) \quad \text{——— (i)}$$

$$P_3'(x) - P_1'(x) = 5P_2(x) \quad \text{——— (ii)}$$

~~Therefore~~,

But, $P_1(x) = x$

$\therefore P_1'(x) = 1$

$\therefore P_1'(x) = P_0(x) \qquad [\because P_0(x) = 1]$ ——— (iii)

Now from (i), (ii), and (iii) we have,

$$P_5'(x) = 9P_4(x) + 5P_2(x) + P_0(x)$$

(Showed)

Laguerre Differential equation and Legendre polynomials :

The equation of the form $xy'' + (1-x)y' + \alpha y = 0$; —(i)

where α = constant, is known as Laguerre's differential equation.

The sol^n of this eq^n is,

$$L_n(x) = (-1)^n \left[x^n - \frac{n^2}{1!} x^{n-1} + \frac{n^2(n-1)^2}{2!} x^{n-2} + \dots\dots + (-1)^n n! \right] \quad \text{—(ii)}$$

This is the expression for Legendre polynomials.

From the equation (ii),

$$L_n(0) = n!$$

$$L_0(x) = 1$$

$$L_1(x) = 1 - x$$

$$L_2(x) = x^2 - 4x + 2$$

$$L_3(x) = -x^3 + 9x^2 - 36x + 6$$

$$L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 48$$

Chapter-02 : Complex Variables

The real number system :

1. Natural number :

1, 2, 3, 4, ..., also called positive integers, were first used in counting. If a and b are natural numbers the sum $(a+b)$ and product $(a \cdot b)$ are also natural number. For this reason the set of natural number is said to be closed under the operations of addition and multiplication.

2. Negative integers and zero :

$-1, -2, -3, \ldots$ and 0 are arose to permit solutions of equations such as $x+b=a$ where a and b are any natural numbers. This leads to the operation of subtraction, or inverse of addition, and we write $x = a-b$

> The set of positive and negative integers and zero is called the set of integers and closed under the operations of addition, multiplication and subtraction

3. Radional numbers / fractions :

$3/4$, $-8/3$, ... permit solutions of equations such as $bx = a$ for all integers a and b where $b \neq 0$. This leads to the operation of division or inverse of multiplication, and we write $x = a/b$ or $a \div b$ where a is the ~~number~~ numerator and b is the denominator.

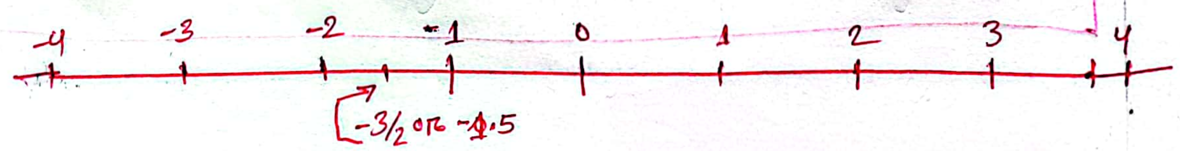
The set of integers is a part or subset of the rational numbers, since integers correspond to rational numbers a/b where b = 1.

4. Irrational numbers :

$\sqrt{2}$ and π are such numbers that cannot be expressed as a/b where a and b are integers and $b \neq 0$

The set of rational and irrational ~~an~~ numbers is called the set of real numbers. ~~It is assumed that the stude~~

Graphical representation of real numbers :



The complex numbers system:

Complex number is a combination of a real number and and imaginary number.

We can consider a complex number as having the form $a+bi$ where a and b are real numbers and i, which is called the imaginary unit, has the property that $i^2=-1$.

If $z=a+bi$, then a is called the real part of z and b is called the imaginary part of the z and are denoted by $Re\{z\}$ and $Im\{z\}$, respectively.

The symbol z, which can stand for any complex number, is called a complex variable.

The complex conjugate, or briefly conjugate, of a complex number $a+bi$ is $a-bi$. The complex conjugate of a complex number z is often indicated by $\bar{z}$ or z^*

⑮ Perform the indicated operations both analytically and graphically:

(a) $(3+4i)+(5+2i)$

(b) $(6-2i)-(2-5i)$

Soln:

(a) Analytically :-

$$(3+4i)+(5+2i)$$

$$= 3+5+4i+2i$$

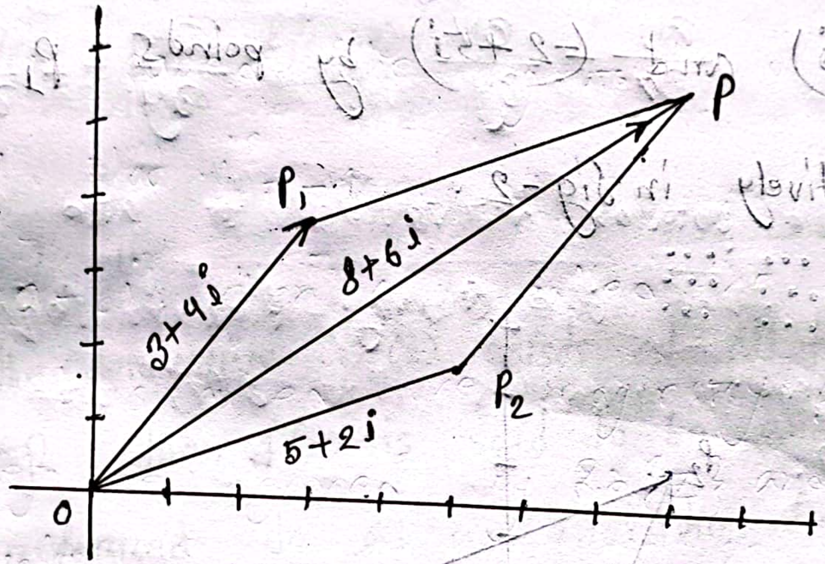
$$= (3+5)+(4+2)i$$

$$= 8+6i$$

Graphically:
Let us represent the two complex number $(3+4i)$ and $(5+2i)$ by points P_1 and P_2 respectively as in fig-1.

~~Now~~ Then complete the parallelogram with OP_1 and OP_2 as adjacent sides. Point P represent the sum, $8+6i$, of the two given complex numbers. Note the similarity with the parallelogram law for addition of vectors OP_1 and OP_2 to obtain vector OP

For this reason it is often convenient to consider a complex number a+bj as a vector components a and b in the directions of the position x and y axes respectively.



ⓑ Analytically :

$$(6-2j)-(2-5j)$$
$$= 6-2-2j+5j$$
$$= (6-2)-(2-5)j$$
$$= 4-(-3)j$$
$$= 4+3j$$

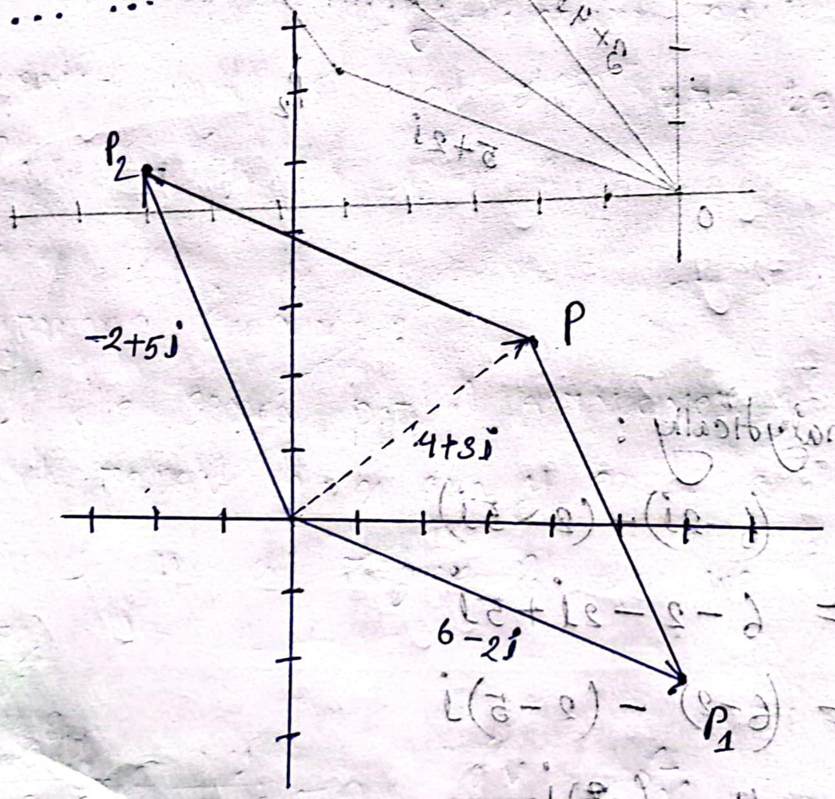
137

Graphically :

$$(6-2j)-(2-5j)$$

$$=(6-2j)+(-2+5j)$$

Now ~~the the~~ let us represent the two complex num ~~vector~~ $(6-2j)$ and $(-2+5j)$ by points P_1 and P_2 respectively in fig-2.



Complex - Differentiation and Cauchy-Rieman Equation :

Derivative :

If $f(z)$ is single-valued in some region R of the Z plane, the derivative of $f(z)$ is defined as,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Analytic function :

If the derivative $f'(z)$ exists at all points z of a region R, then $f(z)$ is said to be analytic in R and is referred to as an analytic function in R or a function analytic in R.

A function $f(z)$ is said to be analytic at a point z_0 if there exists a neighbourhood $|z - z_0| < \delta$ at all the points of which $f'(z)$ exists.

Analytic function also called the Regular function or Holomorphic function.

Cauchy – Riemann Equation

A necessary condition that $w = f(z) = u(x,y) + iv(x,y)$ be analytic in a region R is that, in R, u and v satisfy the Cauchy-Riemann equation.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{—(1)}$$

If the partial derivatives are continuous in R, then the Cauchy-Riemann equations are sufficient conditions that $f(z)$ ~~are~~ be analytic in R.

Harmonic functions:

If the second partial derivatives of u and v with respect to x and y exists and are continuous in a region R, then we find from eq^n (i),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \;,\; \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

It follows that under these conditions the real and imaginary part of an analytic function satisfy Laplace's equation denoted by,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{or} \quad \vec{\nabla}^2 \psi = 0$$

where, $\vec{\nabla}^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

The operator ∇^2 is often called Laplacian

Geometric Interpretation of the derivative,

Let z_0 (in fig-1) be a point P in the z plane and let w_0 (Fig-2) be its image P' in the w plane under the transformation $w = f(z)$. Since we suppose that $f(z)$ is single-valued, the point z_0 maps into only one point w_0.

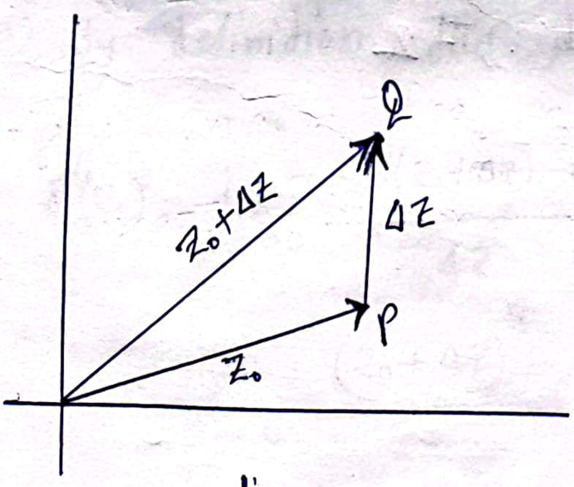


fig - 1

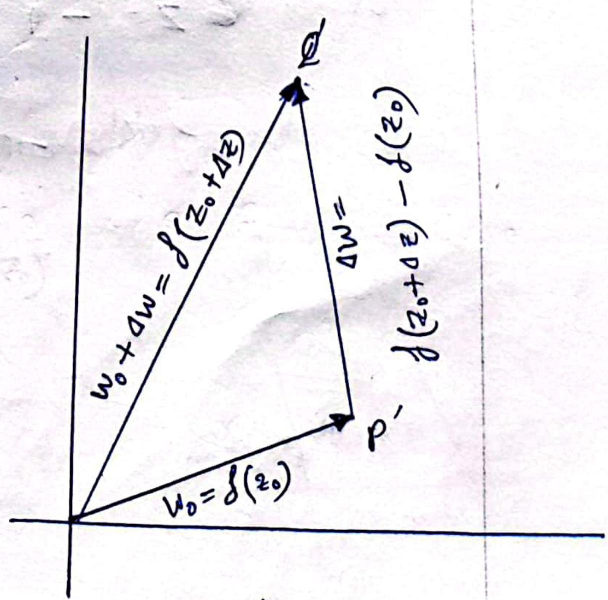


fig - 2

If we give z_0 an increment Δz we obtain the point Q of fig-1. This point has image Q' in the w plane.

Thus from fig-2 we see that $P'Q'$ represents the complex number $\Delta w = f(z_0+\Delta z) - f(z_0)$. It follows that the derivative at z_0 is given by

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \lim_{Q \to P} \frac{Q'P'}{QP}$$

i.e. the limit of the ratio $Q'P'$ to QP as point Q approaches point P. The above interpretation clearly holds when z_0 is replaced by any point z.

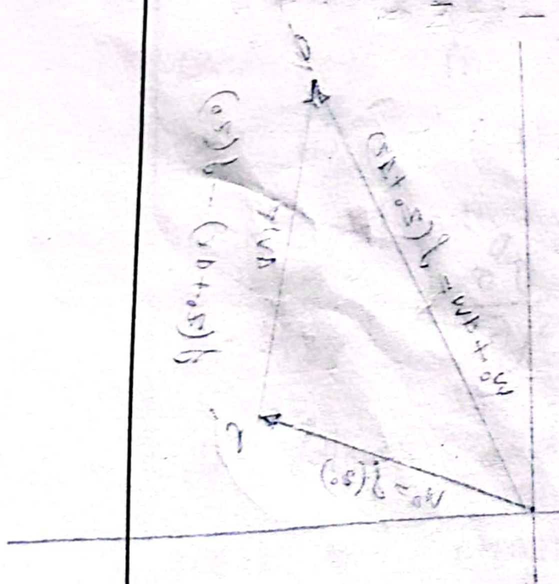


fig-2

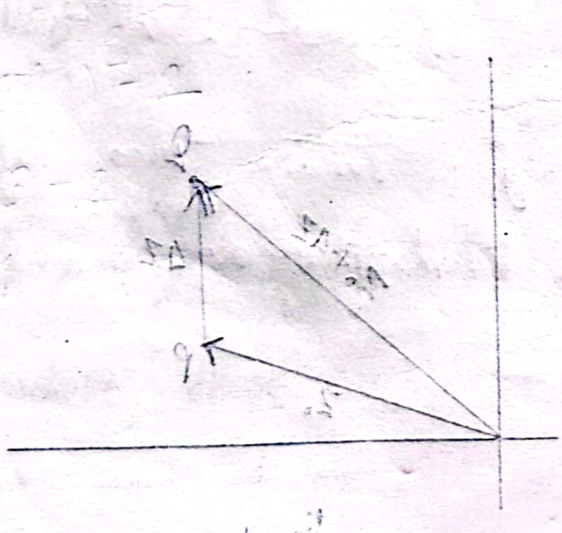


fig-1

L'Hospital's Rule:

Let $f(z)$ and $g(z)$ be analytic in a region containing the point z_0 and suppose that $f(z_0) = g(z_0) = 0$. but $g'(z_0) \neq 0$. Then L'Hospital's rule states that,

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

In case $f'(z_0) = g'(z_0) = 0$, the rule may be extended.

Solved Problems:

3.1 – Using the definition find the derivative of $w = f(z) = z^3 - 2z$ at the point where,

(a) $z = z_0$ and

(b) $z = -1$.

Soln

(a) By definition, the derivative at $z = z_0$ is,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^3 - 2(z_0 + \Delta z) - \{(z_0^3 - 2z_0)\}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z_0^3} + 3z_0^2 \Delta z + 3z_0(\Delta z)^2 + (\Delta z)^3 - \cancel{2z_0} - 2\Delta z - \cancel{z_0^3} + \cancel{2z_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3z_0^2 \Delta z + 3z_0(\Delta z)^2 + (\Delta z)^3 - 2\Delta z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} 3z_0^2 + 3z_0 \Delta z + (\Delta z)^2 - 2$$

$$= 3z_0^2 - 2$$

In general $f'(z) = 3z^2 - 2$ for all z.

(b) We get from (a),

$$f'(z) = 3z^2 - 2$$

for, $z = -1$,

$$f'(-1) = 3(-1)^2 - 2$$

$$= 1$$

3.2 - Show that $\frac{d\bar{z}}{dz}$ does not exist anywhere i.e. $f(z) = \bar{z}$ is non-analytic anywhere.

Sol^n:

By definition,

$$\frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

If this limit exists independent of the manner in where $\Delta z = \Delta x + i\Delta y$ approaches zero.

Then,

$$\frac{d\bar{z}}{dz} = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{x+iy+\Delta x+i\Delta y} - \overline{x+iy}}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x-iy+\Delta x - i\Delta y - (x-iy)}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

If $\Delta y = 0$, the required limit is,

$$\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

If $\Delta x = 0$, the required limit is

$$\lim_{\Delta y \to 0} \frac{-j\Delta y}{j\Delta y} = -1$$

Then, since the limit depends on the manner in which $\Delta z \to 0$, the derivative does not exist; i.e. $f(z) = \bar{z}$ is non-analytic anywhere.

3.3 – Given $w = f(z) = \frac{(1+z)}{(1-z)}$, find,

(a) $\frac{dw}{dz}$ and,

(b) determine where $f(z)$ is non-analytic

Soln:

(a) using definition,

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\frac{1+(z+\Delta z)}{1-(z+\Delta z)} - \frac{1+z}{1-z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\{1+(z+\Delta z)\}\cdot(1-z) - }{1-(z+\Delta z)\ (1-z)}$$

$$= \lim_{\Delta z \to 0} \frac{\dfrac{\{1+(z+\Delta z)\}\cdot(1-z) - (1+z)\,\{1-(z+\Delta z)\}}{\{1-(z+\Delta z)\}\,\{1-z\}}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\dfrac{\cancel{1}-\cancel{z}+\cancel{z}-\cancel{z^2}+\Delta z-\cancel{z\Delta z}-\cancel{1}-\cancel{z}+\cancel{z}+\cancel{z^2}+\Delta z+\cancel{z\Delta z}}{\{1-(z+\Delta z)\}\{1-z\}}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\dfrac{2\Delta z}{(1-z-\Delta z)\,(1-z)}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2}{(1-z-\Delta z)\,(1-z)}$$

$$= \frac{2}{(1-z)\,(1-z)}$$

$$= \frac{2}{(1-z)^2}$$

Using differentiation rules,

$$\frac{d}{dz}\left\{\frac{(1+z)}{(1-z)}\right\} = \frac{(1-z)\frac{d}{dz}(1+z) - (1+z)\frac{d}{dz}(1-z)}{(1-z)^2}$$

$$= \frac{(1-z)\times 1 - (1+z)\times(-1)}{(1-z)^2}$$

$$= \frac{1-z+1+z}{(1-z)^2}$$

$$= \frac{2}{(1-z)^2}$$ Ans

Q.7 – (a) Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic

(b) Find v such that $f(z) = u + iv$ is analytic

Soln:

(a) Here,

$$\frac{\partial u}{\partial x} = e^{-x}(\sin y) + (-e^{-x})\cdot(x\sin y - y\cos y)$$

$$= e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$$

148

$$\therefore \frac{\partial^2 u}{\partial x^2} = -e^{-x}\sin y - (e^{-x} - xe^{-x})\sin y - ye^{-x}\cos y$$

$$= -e^{-x}\sin y - e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y \quad \text{--- (i)}$$

Now,

$$\frac{\partial u}{\partial y} = e^{-x}\{x\cos y - (y\sin y + \cos y)\}$$

$$= e^{-x}(x\cos y + y\sin y - \cos y)$$

$$= xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = -xe^{-x}\sin y + ye^{-x}\cos y + e^{-x}\sin y + e^{-x}\sin y$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = -xe^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y \quad \text{--- (ii)}$$

adding eq^n (i) and (ii), we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Therefore, $u = e^{-x}(x\sin y - y\cos y)$ is harmonic.

(b) From (a),

$$\frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y \quad \text{—— (i)}$$

and $$\frac{\partial u}{\partial y} = xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y$$

From the Cauchy - Riemann equation,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y \quad \text{—— (iii)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{-x}\cos y - xe^{-x}\cos y - ye^{-x}\sin y \quad \text{—— (iv)}$$

Integrate (iii) with respect to y keeping x constant, then

$$V = -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}(y\sin y + \cos y) + F(x)$$

$$= -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}y\sin y + e^{-x}\cos y + F(x)$$

$$= xe^{-x}\cos y + ye^{-x}\sin y + F(x) \quad \text{—— (v)}$$

where $F(x)$ is an arbitrary real function of x.
Substitute (v) into (iv) and obtain,

$$\frac{\partial}{\partial x}\left\{ye^{-x}\sin y + xe^{-x}\cos y + F(x)\right\} = e^{-x}\cos y - xe^{-x}\cos y - ye^{-x}\sin y$$

$$\Rightarrow -e^{-x}y\sin y + e^{-x}\cos y - xe^{-x}\cos y + F'(x) = e^{-x}\cos y - xe^{-x}\cos y - ye^{-x}\sin y$$

$$\Rightarrow F'(x) = 0$$

$$\Rightarrow F(x) = \text{constant}$$

$$\Rightarrow F(x) = C$$

Then from eq^n (vi),

$$V = ye^{-x}\sin y + xe^{-x}\cos y + C$$ Ans

3.12 – Construct an analytic function $f(z)$ whose real part is $e^{x}\cos y$.

Soln:

Let, $f(z) = u(x,y) + iV(x,y)$

It is given that,

$$u(x,y) = e^{x}\cos y$$

$$\therefore \frac{\partial u}{\partial x} = e^{x}\cos y \quad \text{and} \quad \frac{\partial u}{\partial y} = -e^{x}\sin y$$

~~Accordingly Cauchy-Rieman~~

According to Cauchy - Rimenn,

~~$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \cos y$~~

~~and, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -($~~

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x \cos y$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(-e^x \sin y)$$
$$= e^x \sin y$$

we know that,

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

~~$dv = e^x \cos y \, dx +$~~

$$dv = e^x \sin y \, dx + e^x \cos y \, dy$$

This is an exact differential equation,

$$\therefore v = \int e^x \sin y \, dx + \int e^x \cos y \, dy$$

Ignoring the term containing x,

$$v = e^x \sin y$$

$$\therefore f(z) = u + iv = e^x \cos y + i e^x \sin y$$
$$= e^x (\cos y + i \sin y)$$
$$= e^x \cdot e^{iy} = e^{x+iy} = e^z$$

3.13 Find an analytic function, $w(z) = u(x,y) + iv(x,y)$ given that, $v = \frac{x}{x^2+y^2} + \cosh x \cos y$

Soln:-

Now,

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} + \sinh x \cos y$$

and,

$$\frac{\partial v}{\partial y} = \frac{-x\cdot 2y}{(x^2+y^2)^2} - \cosh x \sin y$$

Now, we know that,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\Rightarrow du = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy \quad [\text{Using C-R equation}]$$

$$\Rightarrow du = \left[\frac{-2xy}{(x^2+y^2)^2} - \cosh x \sin y\right]dx - \left[\frac{y^2-x^2}{(x^2+y^2)^2} + \sinh x \cos y\right]dy$$

This is an exact differential equation,

$$\int du = \int\left[\frac{-2xy}{(x^2+y^2)^2} - \cosh x \sin y\right]dx - \int\left[\frac{y^2-x^2}{(x^2+y^2)^2} + \sinh x \cos y\right]dy$$

$$\therefore u = \frac{y}{x^2+y^2} - \sinh x \sin y \quad [\text{Ignoring the term containing } x].$$

$\therefore W = u + iv$

$$= \frac{y}{x^2+y^2} - \sinh x \sin y + i\left[\frac{x}{x^2+y^2} + \cosh x \cos y\right]$$

$$= \frac{y+ix}{x^2+y^2} - \sinh x \sin y + i\cosh x \cos y$$

3.14 - Construct an analytic function $f(z) = u(x,y) + iv(x,y)$, where $v(x,y) = 6xy - 5x + 3$. Express the result as a function of z.

Solⁿ:

Given that,

$$v(x,y) = 6xy - 5x + 3$$

$$\therefore \frac{\partial v}{\partial x} = 6y - 5 \quad \text{and} \quad \frac{\partial v}{\partial y} = 6x.$$

We know,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\Rightarrow du = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy \quad [\text{using C-R equation}]$$

$$\Rightarrow du = 6x\,dx - (6y-5)\,dy$$

$$\therefore u = \int 6x\,dx - \int (6y-5)\,dy + c$$

$$= 3x^2 - 3y^2 - 5y + C$$

$\therefore f(z) = u + iv$

$$= 3x^2 - 3y^2 + 5y + c + i(6xy - 5x + 3)$$

$$= 3x^2 - 3y^2 + 5y + 6ixy - 5ix + 3i + c$$

$$= 3(x^2 - y^2 + 2ixy) + (-5ix + 5y) + 3i + c$$

$$= 3(x+iy)^2 - 5i(x+iy) + 3i + c$$

$$= 3z^2 - 5iz + 3i + c \quad \text{Ans}$$

3.16- Find the value of constants a, b, c and d such that the function $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$ is analytic.

Soln:

Given that,

$$f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$$

where,

$$u = x^2 + axy + by^2 \quad \text{and} \quad v = cx^2 + dxy + y^2$$

$$\therefore \frac{\partial u}{\partial x} = 2x + ay \quad ; \quad \frac{\partial v}{\partial x} = 2cx + dy$$

$$\frac{\partial u}{\partial y} = ax + 2by \quad ; \quad \frac{\partial v}{\partial y} = dx + 2y$$

Since $f(z) = u + iv$ is analytic, so Cauchy-Riemann equation must be satisfied.

i.e., $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Therefore,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow 2x + ay = dx + 2y \quad \text{—— (i)}$$

and,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow ax + 2by = -2cx - dy \quad \text{—— (ii)}$$

Now from (i) and (ii),

$$(2-d)x + (a-2)y = 0 \quad \text{—— (iii)}$$

and, $(a+2c)x + (2b+d)y = 0$ —— (iv)

(iii) and (iv) will be true when,

$$(2-d) = 0 \; ; \; (a-2) = 0$$

and, $(a+2c) = 0$; $(2b+d) = 0$

Therefore,

$$\boxed{a = 2,\ d = 2,\ b = -1 \text{ and } c = -1}$$

3.20 – Show that the function $u = \cos x \cosh y$ is harmonic and find its harmonic conjugate.

Soln: Given that,

$$u = \cos x \cosh y$$

Now, $\frac{\partial u}{\partial x} = -\sin x \cosh y$ and, $\frac{\partial u}{\partial y} = \cos x \sinh y$

$\frac{\partial^2 u}{\partial x^2} = -\cos x \cosh y$ $\frac{\partial u}{\partial y} = \cos x \cosh y$

Now, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\cos x \cosh y + \cos x \cosh y = 0$

Therefore $u = \cos x \cosh y$ is a harmonic function.

Let v be its conjugate harmonic function, then,

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= \frac{-\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$= -\cos x \sinh y \, dx - \sin x \cosh y \, dy$$

$$= -(\cos x \sinh y \, dx + \sin x \cosh y \, dy)$$

Integrating,

$v = -\sin x \sinh y + c$, where c is a real constant.

3.21 – Prove that $u = y^3 - 3x^2y$ is a harmonic function. Determine its harmonic conjugate. hence find the corresponding analytic function $f(z)$ in terms of z.

Soln: Given that,

$$u = y^3 - 3x^2y$$

$$\therefore \frac{\partial u}{\partial x} = -6xy, \quad \frac{\partial u}{\partial y} = 3y^2 - 3x^2$$

$$\frac{\partial^2 u}{\partial x^2} = -6y, \quad \frac{\partial^2 u}{\partial y^2} = 6y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0$$

Therefore $u = y^3 - 3x^2y$ is a harmonic function.

Let,

v be the harmonic conjugate to u, then,

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$= -(3y^2 - 3x^2) dx - 6xy\, dy$$

$$= -(3y^2dx + 6xy\,dy) + 3x^2dx$$

Integrating,

$$v = -3xy^2 + x^3 + c$$

3.38 Evaluate,

(a) $\lim\limits_{z\to i} \dfrac{z^{10}+1}{z^6+1}$

(b) $\lim\limits_{z\to 0} \dfrac{1-\cos z}{z^2}$

(c) $\lim\limits_{z\to 0} \dfrac{1-\cos z}{\sin z^2}$

Soln

(a) Let $f(z) = z^{10}+1$ and $g(z) = z^6+1$

then, $f(i) = g(i) = 0$.

Also $f(z)$ and $g(z)$ are analytic at $z = i$

Hence, by L'Hospital rule,

$$\lim_{z\to i} \frac{z^{10}+1}{z^6+1} = \lim_{z\to i} \frac{10z^9}{6z^5} = \lim_{z\to i} \frac{5}{3} z^4$$

$$= \frac{5}{3} \quad \text{Ans}$$

159

(b) Let, $f(z) = 1 - \cos z$ and $g(z) = z^2$

then, $f(0) = g(0) = 0$. Also $f(z)$ and $g(z)$ are analytic at $z = 0$.

Hence by L'Hospital rule,

$$\lim_{z \to 0} \frac{1 - \cos z}{z^2}$$

$$= \lim_{z \to 0} \frac{\sin z}{2z}$$

Since $f_1(z) = \sin z$ and $g_1(z) = 2z$ are analytic and equal to zero when $z = 0$, therefor we can apply L'Hospital rule again,

$$\lim_{z \to 0} \frac{\sin z}{2z} = \lim_{z \to 0} \frac{\cos z}{2} = \frac{1}{2} \text{ Ans}$$

(c) Let, $f(z) = 1 - \cos z$ and $g(z) = \sin z^2$, then, $f(0) = g(0) = 0$ also $f(z)$ and $g(z)$ are analytic at $z = 0$.

Hence by repeated application of L'Hospital rule,

$$\lim_{z \to 0} \frac{1 - \cos z}{\sin z^2} = \lim_{z \to 0} \frac{\sin z}{2z \cos z^2}$$

$$= \lim_{z\to 0} \frac{\cos z}{2\cos z^2 - 4z^2 \sin z^2}$$

$$= \frac{1}{2}$$

Another method,

$$\lim_{z\to 0} \frac{1-\cos z}{\sin z^2} = \lim_{z\to 0} \frac{\sin z}{2z\cos z^2}$$

$$= \lim_{z\to 0} \frac{\sin z}{z} \lim_{z\to 0} \frac{1}{2\cos z^2}$$

$$= 1 \times \frac{1}{2}$$

$$= \frac{1}{2}$$

3.52: Prove that in polar form the Cauchy-Riemann equation can be written,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Soln:

We have,

$$x = r\cos\theta \quad ; y = r\sin\theta$$

$$r = \sqrt{x^2+y^2} \text{ and } \theta = \tan^{-1}\frac{y}{x}$$

then,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cdot\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial u}{\partial \theta}\cdot\frac{-y}{x^2+y^2}$$

$$= \frac{\partial u}{\partial r}\cdot\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta$$

And,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$$= \frac{\partial u}{\partial r}\cdot\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial u}{\partial \theta}\cdot\frac{x}{x^2+y^2}$$

$$= \frac{\partial u}{\partial r}\cdot\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cdot\cos\theta$$

Similarly,

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= \cancel{\frac{\partial v}{\partial r}\frac{x}{\sqrt{x^2+y^2}} - \frac{1}{r}\frac{\partial v}{\partial \theta}}$$

$$= \frac{\partial v}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta$$

and,

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial v}{\partial \theta}\cos\theta$$

From Cauchy-Rieman equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta = \frac{\partial v}{\partial r}\overset{\sin\theta}{\cancel{\cos\theta}} + \frac{1}{r}\frac{\partial v}{\partial \theta}\overset{\cos\theta}{\cancel{\sin\theta}}$$

$$\Rightarrow \left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\cos\theta - \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\sin\theta = 0 \qquad (5)$$

And also,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta = -\frac{\partial v}{\partial r}\cos\theta + \frac{1}{r}\frac{\partial v}{\partial \theta}\sin\theta$$

$$\Rightarrow \left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\sin\theta + \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta = 0 \qquad 6$$

Multiplying (5) by $\cos\theta$ and (6) by $\sin\theta$ and adding,

$$\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta} = 0$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Multiplying (5) by $-\sin\theta$ and (6) by $\cos\theta$ and adding

$$\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} = 0$$

$$\Rightarrow \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$