9

DIFFRACTION

9.1 INTRODUCTION

It is a matter of common experience that the path of light entering a dark room through a hole in the window illuminated by sunlight is straight. Similarly, if an opaque obstacle is placed in the path of light, a sharp shadow is cast on the screen, indicating thereby that light travels in straight lines. Rectilinear propagation of light can be easily explained on the basis of Newton's corpuscular theory. But it has been observed that when a beam of light passes through a small opening (a small circular hole or a narrow slit) it spreads to some extant into the region of the geometrical shadow also. If light energy is propagated in the form of waves, then similar to sound waves, one would expect bending of a beam of light round the edges of an opaque obstacle or illumination of the geometrical shadow.

Each progressive wave, according to Huygens wave theory produces secondary waves, the envelope of which forms the secondary wavefront. In Fig. 9.1 (*a*), *S* is a source of monochromatic light and *MN* is a small aperture. *XY* is the screen placed in the path of light. *AB* is the illuminated portion of the screen and above *A* and below *B* is the region of the geometrical shadow. Considering *MN* as the primary wavefront, according to Huygens' construction, if secondary wavefronts are drawn, one would expect encroachment of light in the geometrical shadow. Thus, the shadows formed by small obstacles are not sharp. This bending of light round the edges of an obstacle or the encroachment of light within the geometrical shadow is called diffraction. Similarly, If an opaque obstacle *MN* is placed in the path of light [Fig. 9.1 (b), there should be illumination in the geometrical shadow region *AB* also. But the illumination in the geometrical shadow of an obstacle is not commonly observed because the light sources are not point sources and secondly the obstacles used are of very large size compared to the wavelength of light. If a shadow of an obstacle is cast by an extended source, say a frosted electric bulb, light from every point on the surface of the bulb forms its own diffraction pattern (bright and dark diffraction bands) and these overlap such that no single pattern can be identified. The term diffraction is referred to such problems in which one considers the resultant effect produced by a limited portion of a wavefront.

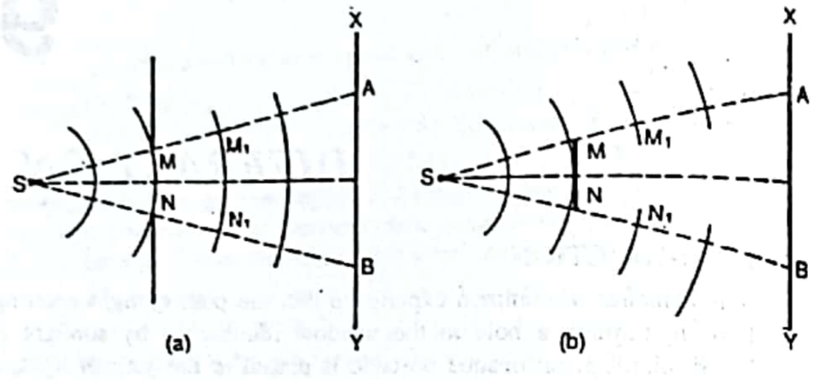


Fig. 9.1

Diffraction phenomena are part of our common experience. The luminous border that surrounds the profile of a mountain just before the sun rises behind it, the light streaks that one sees while looking at a strong source of light with half shut eyes and the coloured spectra (arranged in the form of a cross) that one sees while viewing a distant source of light through a fine piece of cloth are all examples of diffraction effects.

Augustin Jean Fresnel in 1815, combined in a striking manner Huygens wavelets with the principle of interference and could satisfactorily explain the bending of light round obstacles and also the rectilinear propagation of light.

9.2 FRESNEL'S ASSUMPTIONS

According to Fresnel, the resultant effect at an external point due to a wavefront will depend on the factors discussed below :-

In Fig. 9.2, S is a point source of monochromatic light and MN is a small aperture. XY is the screen and SO is perpendicular to XY. MCN is the incident spherical wavefront due to the point source S. To obtain the resultant effect at a point P on the screen, Fresnel assumed that (1) a wavefront can be divided into a large number of strips or zones called Fresnal's zones of small area and the resultant effect at any point will depend on the combined effect of all the secondary waves emanating from the various zones ; (2) the effect at a point due to any particular zone will depend on the distance of the point from the zone ; (3) the effect at P will also depend on the obliquity of the point with reference to the zone under consideration, *e.g.* due to the part of the wavefront at C, the

effect will be maximum at O and decreases with increasing obliquity., It is maximum in a direction radially outwards from C an it decreases in the opposite direction. The effect at a point due to the obliquity factor is proportional to $(1 + \cos\theta)$ where $\angle PCO = \theta$. Considering an elementary wavefront at C, the effect is maximum at O because $\theta = 0$ and $\cos\theta = 1$. Similarly, in a direction tangential to the primary wavefront at C (along CQ) the resultant effect is one half of that along

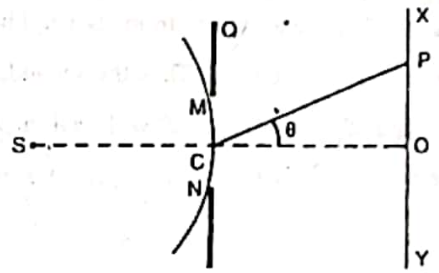


Fig. 9.2

CO because $\theta = 90°$ and $\cos 90 = 0$. In this direction CS, the resultant effect is zero since $\theta = 180°$ and $\cos 180 = -1$ and $1 + \cos 180 = 1 - 1 = 0$. This property of the secondary waves eliminates one of the difficulties experienced with the simpler form of Huygens principle viz., that if the secondary waves spread out **in all directions** from each point on the primary wavefront, they should give a wave travelling forward as well as backward. as the amplitude at the rear of the wave is zero there will evidently be no back wave.

9.3 RECTILINEAR PROPAGATION OF LIGHT

$ABCD$ is a plane wavefront perpendicular to the plane of the paper

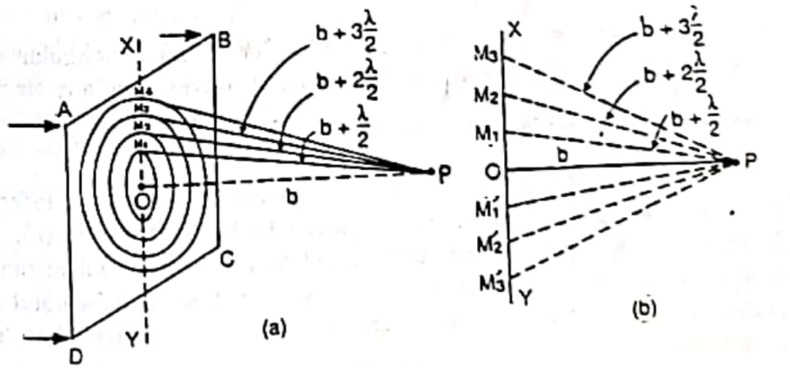


Fig. 9.3

[Fig. 9.3 (*a*)] and P is an external point at a distance b perpendicular to $ABCD$. To find the resultant intensity at P due to the wavefront $ABCD$, Fresnel's method consists in dividing the wavefront into a number of half period elements or zones called Fresnel's zones and to find the effect of all the zones at the point P.

With P as centre and radii equal to $b+\frac{\lambda}{2}$, $b+\frac{2\lambda}{2}$, $b+\frac{3\lambda}{2}$ etc., construct spheres which will cut out circular areas of radii OM_1, OM_2, OM_3 etc., on the wavefront. These circular zones are called half period zones or half period elements. Each zone differs from its neighbour by a phase difference of π or a path difference of $\frac{\lambda}{2}$. Thus the secondary waves starting from the point O and M_1 and reaching P will have a phase difference of π or a path difference of $\frac{\lambda}{2}$. A Fresnel half period zone with respect

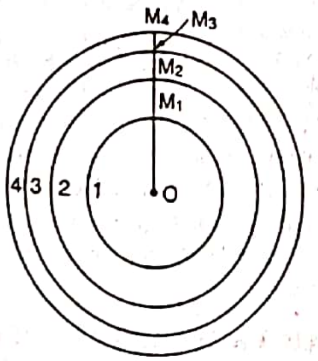


Fig. 9.3 (*c*)

to an actual point P is a thin annular zone (or a thin rectangular strip) of the primary wavefront in which the secondary waves from any two corresponding points of neighbouring zones differ in path by $\frac{\lambda}{2}$.

In Fig. 9.3 (*b*), O is the pole of the wavefront XY with reference to the external point P. OP is perpendicular to XY. In Fig. 9.3 (*c*), 1, 2, 3 etc. are the half period zones constructed on the primary wavefront XY. OM_1 is the radius of the first zone. OM_2 is the radius of the second zone and so on. P is the point at which the resultant intensity has to be calculated.

$$OP = b,\ OM_1 = r_1,\ OM_2 = r_2,\ OM_3 = r_3 \text{ etc.}$$

and
$$M_1P = b+\frac{\lambda}{2},\ M_2P = b+\frac{2\lambda}{2},\ M_3P = \frac{3\lambda}{2} \text{ etc.}$$

The area of the first half period zone is

$$\pi\, OM_1^2 = \pi\,[M_1P^2 - OP^2]$$

$$= \pi\left[\left(b+\frac{\lambda}{2}\right)^2 - b^2\right]$$

$$= \pi\left[b\lambda + \frac{\lambda^2}{2}\right]$$

$$= \pi b\lambda \text{ approximately} \qquad ...(i)$$

(As λ is small, λ^2 term is negligible).

The radius of the first half period zone is

$$r_1 = OM_1 = \sqrt{b\lambda} \qquad ...(ii)$$

The radius of the second half period zone is

$$OM_2 = [M_2P^2 - OP^2]^{1/2}$$

$$= [(b+\lambda)^2 - b^2]^{1/2}$$

$$= \sqrt{2b\lambda} \text{ approximately.}$$

The area of the second half period zone

$$= \pi\,[OM_2^2 - OM_1^2]$$

$$= \pi\,[2b\lambda - b\lambda] = \pi b\lambda$$

Thus, the area of each half period zone is equal to $\pi b\lambda$. Also the radii of the Ist, 2nd, 3rd etc. half period zones are $\sqrt{1b\lambda}$ $\sqrt{2b\lambda}$, $\sqrt{3b\lambda}$ etc. Therefore, the radii are proportional to the square roots of the natural numbers. However, it should be remembered that the area of the zones are not constant but are dependent on (*i*) λ the wavelength of light and (*ii*) b, the distance of the point from the wavefront. The area of the zone increases with increase in the wavelength of light and with increase in the distance of the point P from the wavefront.

As discussed in article 9.2, the effect at a point P will depend on (*i*) the distance of P from the wavefront, (*ii*) the area of the zone, and (*iii*) the obliquity factor.

Here, the area of each zone is the same. The secondary waves reaching the point P are continuously out of phase and in phase with reference to the central or the first half period zone. Let m_1, m_2, m_3 etc

AUGUSTIN FRESNEL (1788-1827)

He established the wave theory of light. He propounded the basic concept that the light waves are transverse. His name is associated with Fresnel's class of diffraction phenomena etc.

(Fig. 9.4) represent the amplitudes of vibration of the ether particles at P due to secondary waves from the Ist, 2nd, 3rd, etc. half period zones. As we consider the zones outwards from O, the obliquity increases and hence the quantities m_1, m_2, m_3 etc., are of continuously decreasing order. Thus, m_1 is slightly greater than m_2 ; m_2 is slightly greater than m_3 and so on. Due to the phase difference of π between any two consecutive zones, if the displacements of the ether particles due to odd numbered

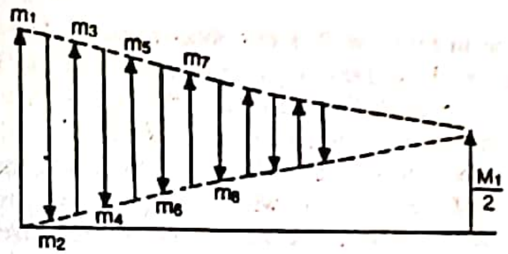


Fig. 9.4

zones is in the positive direction, then due to the even numbered zones the displacement will be in the negative direction at the same instant. As the amplitudes are of gradually decreasing magnitude, the amplitude of vibration at P due to any zone can be approximately taken as the mean of the amplitudes due to the zones preceding and succeeding it.

e.g. $$m_2 = \frac{m_1 + m_3}{2}$$

The resultant amplitude at P at any instant is given by

$$A = m_1 - m_2 + m_3 - m_4 \ldots + m_n \text{ if } n \text{ is odd.}$$

(If n is even, the last quantity is $-m_n$).

$$\therefore \quad A = \frac{m_1}{2} + \left[\frac{m_1}{2} - m_2 + \frac{m_3}{2}\right] + \left[\frac{m_3}{2} - m_4 + \frac{m_5}{2}\right] + \ldots$$

But $$m_2 = \frac{m_1}{2} + \frac{m_3}{2} \text{ and } m_4 = \frac{m_3}{2} + \frac{m_5}{2}$$

$$\therefore \quad A = \frac{m_1}{2} + \frac{m_n}{2} \ldots \text{if } n \text{ is odd}$$

and $$A = \frac{m_1}{2} + \frac{m_{n-1}}{2} - m_n \ldots \text{ if } n \text{ is even.}$$

If the whole wavefront $ABCD$ is unobstructed, the number of half period zones that can be constructed with reference to the point P is infinite *i.e.*, $n \to \infty$. As the amplitudes are of gradually diminishing order, m_n and m_{n-1} tend to be zero.

Therefore, the resultant amplitude at P due to the whole wavefront $= A = \frac{m_1}{2}$. The intensity at a point is proportional to the square of the amplitude.

$$\therefore \qquad I \propto \frac{m_1^2}{4}$$

Thus, the intensity at P is only one-fourth of that due to the first half period zone alone. Here, only half the area of the first half period zone is effective in producing the illumination at the point P. A small obstacle of the size of half the area of the first half period zone placed at O will screen the effect of the whole wavefront and the intensity at P due to the rest of the wavefront will be zero. While considering the rectilinear propagation of light, the size of the obstacle used is far greater than the area of the first half period zone and hence the bending effect of light round corners (diffraction effects) cannot be noticed. In the case of sound waves, the wavelengths are far greater than the wavelength of light and hence the area of the first half period zone for a plane wavefront of sound is very large. If the effect of sound at a point beyond an obstacle is to be shadowed, an obstacle of very large size has to be used to get no sound effect. If the size of the obstacles placed in the path of light is comparable to the wavelength of light, then it is possible to observe illumination in the region of the geometrical shadow also. Thus, rectilinear propagation of light is only approximately true.

9.4 ZONE PLATE

A zone plate is a specially constructed screen such that light is obstructed from every alternate zone. It can be designed so as to cut off light due to the even numbered zones or that due to the odd numbered zones. The correctness of Fresnel's method in dividing a wavefront into half period zones can be verified with its help.

(*a*) (*b*)

Zone Plate

Fig. 9.5

To construct a zone plate, concentric circles are drawn on white paper such that the radii are proportional to the square roots of the natural numbers (as shown in article 9.3, the radii are propartional to the square roots of the natural numbers). The odd numbered zones (*i.e.*, 1st, 3rd, 5th, etc.) are covered with black ink and a reduced photograph is taken. The drawing appears as shown in Fig. 9.5 (*b*). The negative of the photograph will be as shown in Fig. 9.5 (*a*). In the developed negative, the odd zones are transparent to incident light and the even zones will cut off light.

If such a plate is held perpendicular to an incident beam of light and a screen is moved on the other side to get the image, it will be observed that maximum brightness is possible at some position of the screen say b cm from the zone plate (Fig. 9.6). XO is the upper half of the incident plane wavefront. P is the point at which the light intensity is to be considered. The distance of the point P from the wavefront is b. OM_1 ($= r_1$), OM_2 ($= r_2$) etc. are the radii of the zones,

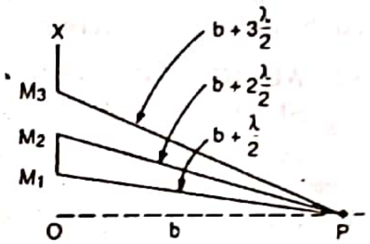


Fig. 9.6

$$r_1 = \sqrt{b\lambda} \quad \text{and} \quad r_2 = \sqrt{2b\lambda}$$

where λ is the wavelength of light.

$$r_n = \sqrt{nb\lambda} \quad \text{or} \quad b = \frac{r_n^2}{n\lambda}$$

If the source is at a large distance from the zone plate, a bright spot will be obtained at P. As the distance of the source is large, the incident wavefront can be taken as a plane one with respect to the small area of the zone plate. The even numbered zones cut off the light and hence the resultant amplitude at $P = A = m_1 + m_3 + \ldots$ etc. In this case the focal length of the zone plate f_n is given by

$$f_n = b = \frac{r_n^2}{n\lambda} \qquad [\because r_n^2 = bn\lambda]$$

Thus, a zone plate has different foci for different wavelengths. The radius of the n th zone increases with increasing value of λ. *It is very interesting to note that as the even numbered zones are opaque, the intensity at P is much greater than that when the whole wavefront is exposed to the point P.*

In the first case the resultant amplitude is given by

$$A = m_1 + m_3 + m_5 + \ldots m_n \ldots (n \text{ is odd})$$

When the whole wavefront is unobstructed the amplitude is given by

$$A = m_1 - m_2 + m_3 - m_4 \ldots + m_n$$

$$= \frac{m_1}{2} \text{ (if } n \text{ is very large and } n \text{ is odd).}$$

If a parallel beam of white light is incident on the zone plate, different colours come to focus at different points along the line OP. Thus, the function of a *zone plate is similar to that of a convex (converging) lens* and a formula connecting the distance of the object and image points can be obtained for a zone plate also.

9.5 ACTION OF A ZONE PLATE FOR AN INCIDENT SPHERICAL WAVEFRONT

Let XY represent the section of the zone plate perpendicular to the plane of the paper. S is a point source of light, P is the position of the screen for a bright image, a is the distance of the source from the zone

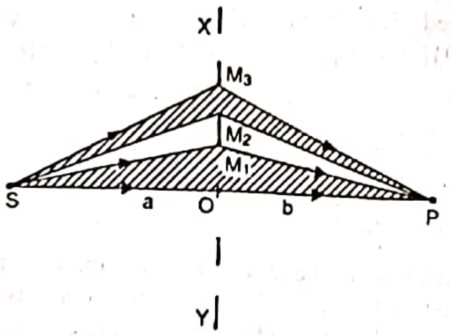


Fig. 9.7

plate and b is the distance of the screen from the plate. OM_1, OM_2, OM_3 (r_1, r_2, r_3) etc. are the radii of the 1 st, 2 nd, 3 rd etc. half period zones. The position of the screen is such, that from one zone to the next there is an increasing path difference of $\frac{\lambda}{2}$.

Thus,
$$SO + OP = a + b$$

$$SM_1 + M_1P = a + b + \frac{\lambda}{2} \qquad \ldots(i)$$

$$SM_2 + M_2P = a + b + \frac{2\lambda}{2} \text{ and so on.}$$

From the ΔSM_1O

$$SM_1 = (SO^2 + OM_1^2)^{1/2}$$
$$= (a^2 + r_1^2)^{1/2}$$

Similarly from the ΔOM_1P

$$M_1P = (OP^2 + OM_1^2)^{1/2}$$
$$= (b^2 + r_1^2)^{1/2}$$

Substituting the values of SM_1 and M_1P in equation (i)

$$(a^2 + r_1^2)^{1/2} + (b^2 + r_1^2)^{1/2} = a + b + \frac{\lambda}{2}$$

$$a\left(1 + \frac{r_1^2}{a^2}\right)^{1/2} + b\left(1 + \frac{r_1^2}{b_2}\right)^{1/2} = a + b + \frac{\lambda}{2}$$

$$a + \frac{r_1^2}{2a} + b + \frac{r_1^2}{2b} = a + b + \frac{\lambda}{2}$$

$$\frac{r_1^2}{2}\left[\frac{1}{a} + \frac{1}{b}\right] = \frac{\lambda}{2}$$

$$r_1^2\left(\frac{1}{a} + \frac{1}{b}\right) = \lambda$$

Similarly for r_n, *i.e.*, the radius of the n th zone, the relation can be written as

$$r_n^2\left[\frac{1}{a} + \frac{1}{b}\right] = n\lambda$$

Applying the sign convention,

or
$$\frac{1}{b} - \frac{1}{a} = \frac{n\lambda}{r_n^2} = \frac{1}{f_n} \qquad \ldots(ii)$$

$$f_n = \frac{r_n^2}{n\lambda}$$

Equation (ii) is similar to the equation $\left(\frac{1}{v} - \frac{1}{u} = \frac{1}{f}\right)$ in the case of lenses with a and b as the object and image distances and f_n the focal length. Thus, a zone plate acts as a converging lens. A zone plate has a number of foci which depend on the number of zones used as well as the wavelength of light employed.

9.6 DIFFERENCE BETWEEN A ZONE PLATE AND A CONVEX LENS

For a given wavelength of light, a convex lens has only one focal length given by

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

where f is the focal length of the lens, μ is the refractive index of the material of the lens and R_1 and R_2 are the radii of curvature. In a convex lens, the violet rays of light come to focus nearer the lens than the red rays of light because for a given material the refractive index for violet rays of light is more than for red rays of light.

In the case of a zone plate, there are a number of foci between the point O and P (Fig. 9.7). Each focus corresponds to the position where, with reference to P an odd number of half period elements can be constructed on each zone. As the screen is moved nearer the zone plate, the area of the half period elements decreases and more half period elements can be present on each zone. If P_m is the position on the image when $(2_m - 1)$ half period elements can be present on each zone, f_m the focal length of the zone plate is given by

$$f_m = \frac{r_n^2}{(2m-1)n\lambda} \qquad ...(i)$$

Putting $m = 1, 2, 3$., etc., the different positions of the screen for a bright image can be obtained. In equation (*i*), r_n is the radius of the n th zone of the wavefront,. λ is the wavelength of light and $2m - 1$) is the number of odd half period elements present on each zone. For example, if the position of the screen is such that with reference to the point P, three half period elements can be constructed on each zone, then the focal length of the zone plate f_3 is given by

$$f_3 = \frac{r_n^2}{5n\lambda}$$

With the decrease in the focal length of the zone plate, the brightness of the image decreases. Let the first zone contain only one half period element. Then, the amplitude at P due to this zone $= m_1$. If the first zone contains three half period elements for a particular position of the screen, then the amplitude at P due to the first zone

$$= m_1 - m_2 + m_3 = \frac{m_1}{2} + \left[\frac{m_1}{2} - m_2 + \frac{m_3}{2}\right] + \frac{m_3}{2} = \frac{m_1}{2} + \frac{m_3}{2}.$$

But, $\frac{m_1 + m_3}{2}$ is less than m_1, because $m_1 > m_3$. Further, in a zone plate (for the same number of odd half period elements contained in each zone) the focal length for violet light is more than for red light, which is reverse in the case of a convex lens.

Example 9.1. *With a zone plate, for a point source of light on the axis, the strongest and the next strongest images are formed at 30 cm and 6 cm respectively from the zone plate. Both the images are on the same side and on the other side of the source. Calculate*

(*i*) *the distance of the source from the zone plate*
(*ii*) *the radius of the first zone and*
(*iii*) *principal focal length*

Here $\lambda = 6000$ Å

For a zone plate

$$f_m = \frac{r_n^2}{(2_m - 1)n\lambda}$$

$$\lambda = 6000\text{ Å} = 6 \times 10^{-5}\text{ cm}$$

For the first zone, $n = 1$
For the strongest image $m = 1$
For the next strongest image $m = 2$

$$\therefore \quad f_1 = \frac{r_1^2}{\lambda} \qquad ...(i)$$

$$f_2 = \frac{r_1^2}{3\lambda} \qquad ...(ii)$$

Also $\quad \frac{1}{v} - \frac{1}{u} = \frac{1}{f}; \quad \frac{1}{v_1} - \frac{1}{u} = \frac{1}{f_1}$

$$\frac{1}{v_2} - \frac{1}{u} = \frac{1}{f_2}; \quad v_1 = +30\text{ cm}, \ v_2 = +6\text{ cm}$$

$$\therefore \quad \frac{1}{30} - \frac{1}{u} = \frac{\lambda}{r_1^2} \qquad ...(iii)$$

$$\frac{1}{6} - \frac{1}{u} = \frac{3\lambda}{r_1^2} \qquad ...(iv)$$

Multiplying equation (*iii*) by 3 and equating we get

$$\frac{1}{10}-\frac{3}{u}=\frac{1}{6}-\frac{1}{u}$$

or $$u = -30 \text{ cm} \quad \text{...(v)}$$

Negative sign shows that the point source is to the left of the zone plate and its distance is 30 cm.

Substituting the value of u in equation (*iii*)

$$\frac{1}{30}+\frac{1}{30}=\frac{6\times10^{-5}}{r_1^2}$$

$$r_1^2 = 9\times10^{-4}$$

$$r_1 = 3\times10^{-2}\text{ cm} \quad \text{...(vi)}$$

Substituting the value of r_1 and λ in equation (*i*)

$$f_1 = \frac{9\times10^{-4}}{6\times10^{-5}}$$

$$f_1 = \mathbf{15\ cm.}$$

Example 9.2. *What is the radius of the first zone in a zone plate of focal length 20 cm for light of wavelength 5000 Å ?* (*Agra 1966*)

Here $f = 20\text{ cm}, \quad \lambda = 5000\times10^{-8}\text{ cm}$

$$n = 1 \qquad f = \frac{r_n^2}{n\lambda}$$

or $$r_1^2 = fn\lambda = 20\times5000\times10^{-8}; \quad r_1 = \mathbf{0.0316\ cm}$$

Example 9.3. *The diameter of the central zone of a zone plate is 2.3 mm. If a point source of light (λ = 5893 Å) is placed at a distance of 6 m from it, calculate the position of the first image* [*I.A.S, 1984*]

Here $d = 2.3\text{ mm} = 2.3\times10^{-3}\text{ m}$

$$r = 1.55\times10^{-3}\text{ m}$$

$$\lambda = 5893\text{ Å} = 5893\times10^{-10}\text{ m}$$

$$f = \frac{r^2}{\lambda}$$

$$f = \frac{(1.55\times10^{-3})^2}{5893\times10^{-10}}$$

$$f = 4\text{ m}$$

Here $$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

$$u = -6\text{ m}, \quad f = +4\text{ m}$$

$$\frac{1}{v}+\frac{1}{6}=\frac{1}{4}$$

$$v = \mathbf{12\ m}$$

The first image is formed at a distance of 12 m

Example 9.4. *Find the radius of the first half period zone on a zone plate, behaving like a convex lens of focal length 60 cm.* (λ = 6000 Å). [*Delhi, 1991*]

Here $f = 60\text{ cm} = 0.6\text{ m}$

$$\lambda = 6000\text{ Å} = 6\times10^{-7}\text{ m}$$

$$n = 1$$

$$f = \frac{r_n^2}{n\lambda}$$

$$r_n^2 = fn\lambda$$

$$r_1^2 = 0.6\times1\times6\times10^{-7}$$

$$r_1^2 = 36\times10^{-8}$$

$$r_1 = 6\times10^{-4}\text{ m}$$

$$r_1 = \mathbf{0.6\ mm}$$

Example 9.5 *A zone plate is found to give series of images of a point source on its axis. If the strongest and the second strongest images are at distances of 0.30 m and 0.06 m respectively from the zone plate (both on the same side remote from the source), calculate the distance of the source from the zone plate, principal focal length and the radius of the first zone. Assume* $\lambda = 5\times10^{-7}$ m (*Delhi, (Hons) 1990*)

Here $\lambda = 5\times10^{-7}\text{ m}$

For a zone plate,

$$f_m = \frac{r_n^2}{(2m-1)\,n\lambda}$$

For the first zone, $n = 1$

For the strongest image, $m = 1$

For the next strongest image, $m = 2$

$$\therefore \qquad f_1 = \frac{r_1^2}{\lambda} \qquad ...(i)$$

$$f_2 = \frac{r_1^2}{3\lambda} \qquad ...(ii)$$

Also
$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

$$\therefore \qquad \frac{1}{v_1} - \frac{1}{u} = \frac{1}{f_1}$$

$$\frac{1}{v_2} - \frac{1}{u} = \frac{1}{f_2}$$

Here $v_1 = 0.3\text{ m}$ and $v_2 = 0.6\text{ m}$

$$\frac{1}{0.3} - \frac{1}{u} = \frac{\lambda}{r_1^2} \qquad ...(iii)$$

$$\frac{1}{0.06} - \frac{1}{u} = \frac{3\lambda}{r_1^2} \qquad ...(iv)$$

Multiplying equation (*iii*) by 3 and equating with (*iv*),

$$\frac{1}{0.1} - \frac{3}{u} = \frac{1}{0.06} - \frac{1}{u}$$

$$u = -0.3\text{ m} \qquad ...(v)$$

Negative sign shows that the point source is to the left of the zone plate and its distance is 0.3 m.

Substituting the value of u and λ in equation (*iii*)

$$\frac{1}{0.3} + \frac{1}{0.3} = \frac{5 \times 10^{-7}}{r_1^2}$$

$$r_1 = 2.74 \times 10^{-4}\text{ m} \qquad ...(vi)$$

From equation (*i*)

$$f_1 = \frac{(2.74 \times 10^{-4})^2}{5 \times 10^{-7}} = 0.15\text{ m}$$

Example 9.6. *A zone plate is made by arranging the radii of the circles which define the zones such that they are the same as the radii of newton's rings formed between a plane surface and the surface having radius of curvature 200 cm. Find the principal focal length of the zone plate.* [*Delhi (Hons) 1992*]

For Newton' rings,

radius of the n th ring,

$$r_n = \sqrt{n \lambda R}$$

$$r_1 = \sqrt{\lambda R} \qquad ...(i)$$

For a zone plate, the principal focal length

$$f_1 = \frac{r_1^2}{\lambda} \qquad ...(ii)$$

From (*i*) and (*ii*)

$$f_1 = \frac{\lambda R}{\lambda} = R$$

But $R = 200\text{ cm} = 2\text{ m}$

$$\therefore \qquad f_1 = 2\text{ m}$$

9.7 FRESNEL AND FRAUNHOFER DIFFRACTION

Diffraction phenomena can conveniently be divided into two groups viz, (*i*) Fresnel diffraction phenomena and (*ii*) Fraunhofer diffraction phenomena. In the Fresnel class of diffraction, the source or the screen or both are at finite distances from the aperture or obstacle causing diffraction. In this case, the effect at a specific point on the screen due to the exposed incident wavefront is considered and no modification is made by lenses and mirrors. In such a case, the phenomenon observed on the screen is called Fresnel diffraction pattern. In the Fraunhofer class of diffraction phenomena, the source and the screen on which the pattern is observed are at infinite distances from the aperture or the obstacle causing diffraction. Fraunhofer diffraction pattem can be easily observed in practice. The incoming light is rendered parallel with a lens and the diffracted beam is focussed on the screen with another lens. Observation of Fresnel diffraction phenomena do not require any lenses. Theoretical treatment of Fraunhofer diffraction phenomena is simpler. Fresnel class of diffraction phenomena are treated first in this chapter.

9.8 DIFFRACTION AT A CIRCULAR APERTURE

Let AB be a small aperture (say a pin hole) and S is a point source of monochromatic light. XY is a screen perpendicular to the plane of the paper and P is a point on the screen. SP is perpendicular to the screen. O is the centre of the aperture and r is the radius of the aperture. Let the distance of the source from the aperture be a ($SO = a$) and the distance of the screen from the aperture be b ($OP = b$). P_1OQ_1 is the incident spherical wavefront and with reference to the point P, O is the pole of the wavefront (Fig. 9.8). To consider the intensity at P, half period zones can be constructed with P as centre and radii $b+\frac{\lambda}{2}, b+\frac{2\lambda}{2}$ etc., on the exposed wavefront AOB. Depending on the distance of P from the aperture (*i.e.*, the distance b) the number of half period zones that can be constructed may be odd or even. If the distance a is such that only one half period zone can be constructed, then the intensity at P will be proportional to m_1^2 (where m_1 is the amplitude due to the first zone at P). On the other hand, if the whole of the wavefront is exposed to the point P, the resultant amplitude is $\frac{m_1}{2}$ or the intensity at P will be proportional to $\frac{m_1^2}{4}$. The position of the screen can be altered so as to construct 2, 3 or more half period zones for the same area of the aperture. If only 2 zones are exposed, the resultant amplitude at $P = m_1 - m_2$ (minimum) and if 3 zones are exposed, the amplitude $= m_1 - m_2 + m_3$ (maximum) and so on. Thus, by continuously altering the value of b, the point P becomes alternately bright and dark depending on whether odd or even number of zones are exposed by the aperture.

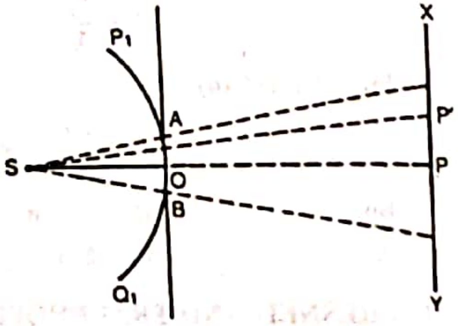


Fig 9.8

Now consider a point P' on the screen XY (Fig. 9.9) Join S to P'. The line SP' meets the wavefront at O'. O' is the pole of the wavefront

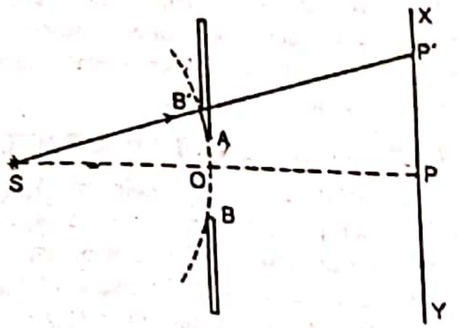


Fig. 9.9

with reference to the point P'. Construct half period zones with the point O' as the pole of the wavefront. The upper half of the wavefront is cut off by the obstacle. If the first two zones are cut off by the obstacle between the points O' and A and if only the 3 rd, 4 th and 5 th zones are exposed by the aperture AOB, then the intensity at P' will be maximum. Thus, if odd number of half period zones are exposed, point P' will be of maximum intensity and if even number of zones are exposed, the point P' will be of minimum intensity. As the distance of P' from P increases, the intensity of maxima and minima gradually decreases, because, with the point P' far removed from P, the most effective central half period zones are cut off by the obstacle between the points O' and A. With the outer zones, the obliquity increases with reference to the point P' and hence the intensity of maxima and minima also will be less. If the point P' happens to be of maximum intensity, then all the points lying on a circle of radius PP' on the screen will also be of maximum intensity. Thus, with a circular aperture, the diffraction pattern will be concentric bright and dark rings with the centre P bright or dark depending on the distance b. The width of the rings continuously decreases.

9.9 MATHEMATICAL TREATMENT OF DIFFRACTION AT A CIRCULAR APERTURE

In Fig. 9.10, S is a point source of monochromatic light, AB is the circular aperture and P is a point on the screen. O is the centre of the circular aperture. The line SOP is perpendicular to the circular aperture AB and the screen at P. The screen is perpendicular to the plane of the paper.

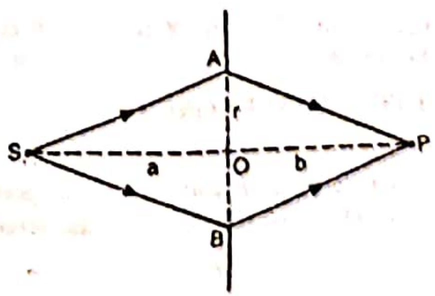


Fig. 9.10

Let δ be the path difference for the waves reaching P along the paths SAP and SOP.

$SO = a;\quad OP = b;\quad OA = r$

$$\delta = SA + AP - SOP$$

$$= (a^2 + r^2)^{1/2} + (b + r^2)^{1/2} - (a+b)$$

$$= a\left(1+\frac{r^2}{a^2}\right)^{1/2} + b\left(1+\frac{r^2}{b^2}\right)^{1/2} - (a+b)$$

$$= a\left(1+\frac{r^2}{2a^2}\right)+b\left(1+\frac{r^2}{2b^2}\right) \quad a+b)$$

$$\delta = \frac{r^2}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$$

$$\therefore \quad \frac{1}{a}+\frac{1}{b} = \frac{2\delta}{r^2} \quad \text{...(i)}$$

If the position of the screen is such that n full numbe alf period zones can be constructed on the aperture, then the path di nce

$$\delta = \frac{n\lambda}{2} \quad \text{or} \quad 2\delta = n\lambda$$

Substituting this value of 2δ in (*i*)

$$\frac{1}{a}+\frac{1}{b} = \frac{n\lambda}{r^2} \quad \text{...(ii)}$$

The point P will be of maximum or minimum ir depending on whether n is odd or even. If the source is at infini nce (for an incident plane wavefront), then $a = \infty$ and

$$\frac{1}{b} = \frac{1}{f} = \frac{n\lambda}{r^2} \quad \text{...(iii)}$$

If n is odd, P will be a bright point. The idea of fo P does not mean that if is always a bright point.

9.10 INTENSITY AT A POINT AWAY FROM T CENTRE

In Fig. 9.11, AB is a circular aperture and P and wo points on the screen. $PP' = x$ and $OP = b$. OP is perpendic the screen.

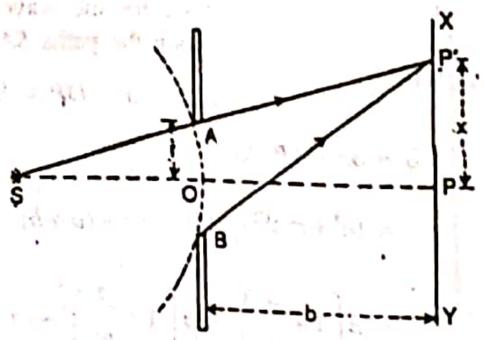


Fig. 9.11

Let r be the radius of the aperture. The path difference between the secondary waves from A and B and reaching P' can be given by

$$\delta = BP' - AP'$$

$$= \sqrt{b^2+(x+r)^2} - \sqrt{b^2+(x-r)^2}$$

$$= b\left(1+\frac{(x+r)^2}{2b^2}\right) - b\left(1+\frac{(x-r)^2}{2b^2}\right)$$

$$= b + \frac{(x+r)^2}{2b} - b - \frac{(x-r)^2}{2b^2}$$

$$= \frac{1}{2b}\left[(x+r)^2 - (x-r)^2\right]$$

$$\delta = \frac{1}{2b}(4xr) = \frac{2rx}{b} \quad \text{...(i)}$$

The point P' will be dark if the path difference $\delta = 2n\frac{\lambda}{2}$

(2 n means even number of zones).

$$2n\frac{\lambda}{2} = \frac{2rx_n}{2}$$

or
$$x_n = \frac{nb\lambda}{2r} \quad \text{...(ii)}$$

where x_n gives the radius of the n th dark ring.

Similarly, if
$$\delta = \frac{(2n+1)\lambda}{2},$$

$$\frac{(2n+1)\lambda}{2} = \frac{2rx_n}{b}$$

or
$$x_n = \frac{(2n+1)b\lambda}{4r} \quad \text{...(iii)}$$

where x_n gives the radius of the n th bright ring.

The objective of a telescope consists of an achromatic convex lens and a circular aperture is fixed in front of the lens. Let the diameter of the aperture be D ($= 2r$). While viewing distant objects, the incident wavefront is plane and the diffraction pattern consists of a bright centre surrounded by dark and bright rings of gradually decreasing intensity. The radii of the dark rings is given by

$$x_n = \frac{nb\lambda}{2r} = \frac{nb\lambda}{D} \quad \text{...(iv)}$$

The radius of the first dark ring is

$$x_1 = \frac{b\lambda}{D}$$

For an incident plane wavefront, $b = f$ the focal length of the objective.

$$\therefore \quad x_1 = \frac{f\lambda}{D}$$

The value of x_1 measures the distance of the first secondary minimum from the central bright maximum. However, according to Airy' s theory, the radius of the first dark ring is given by

$$x_1 = \frac{1.22 f\lambda}{D} \quad ...(v)$$

It is interesting to note that the size of the central image depends on λ, the wavelength of light, f the focal length of the lens and D the diameter of the lens aperture.

Example 9.7. *A circular aperture of 1.2* mm *diameter is illuminated by plane waves of monochromatic light. The diffracted light is received on a distant screen which is gradually moved towards the aperture. The centre of the circular patch of light first becomes dark when the screen is 30* cm *from the aperture. Calculate the wavelength of light.*

(Rajasthan)

Diameter $= 1.2\text{ mm} = 0.12\text{ cm}$

Radius $= r = 0.06\text{ cm}$

$$b = 30\text{ cm}$$

Here $(b^2 + r^2) = (b + \lambda)^2$

$$30^2 + (0.06)^2 = (30 + \lambda)^2$$

$$\lambda = \frac{(0.06)^2}{2 \times 30} \text{ approximately, neglecting } \lambda^2$$

$$= 0.00006\text{cm} = \mathbf{6000\ \text{Å}}$$

Example 9.8. *A monochromatic beam of light on passing through a slit 1.6* mm *falls on a screen held close to the slit. The screen is then gradually moved away and the middle of the patch of light on it becomes dark, when the screen is 50* cm *from the slit. Calculate the wavelength of light.*

(Punjab)

Here the width of the slit $= 1.6\text{ mm} = 0.16\text{ cm}$

Half width $= r = 0.08\text{ cm}$

Distance between the slit and the screen $= b = 50\text{ cm}$

Here $(b^2 + r^2) = (b + \lambda)^2$

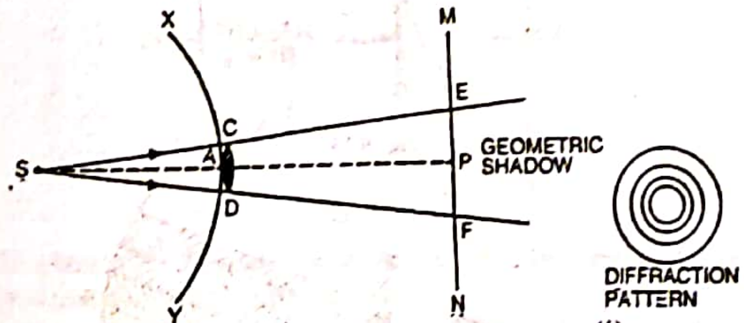
$$\therefore \quad 50^2 + (0.08)^2 = (50 + \lambda)^2$$

or $$\lambda = \frac{(0.08)^2}{2 \times 50} \text{ approximately(neglecting} \lambda^2)$$

$$= 0.000064\text{cm} = 6400\ \text{Å}$$

9.11 DIFFRACTION AT AN OPAQUE CIRCULAR DISC

S is a point source of monochromatic light. CD is an opaque disc and MN is the screen. P is a point on the screen such that SAP is perpendicular to the screen. The screen is perpendicular to the plane of the paper. XY is the incident spherical wavefront. EF is the geometrical shadow and P is the centre of the shadow. With reference to the point P, the wavefront can be divided into half period zones taking the centre of the disc (A) as the pole (Fig. 9.12). If one half period zone can be constructed on the surface of the disc, the rest of the zones are exposed to the point P and the resultant amplitude at $P = \frac{m_2}{2}$ approximately, where m_2 is the amplitude due to the second zone



(a) (b)

Fig. 9.12

$\left[m_2 - m_3 + m_4 - \ldots = \frac{m_2}{2} \text{ approximately}\right]$. Similarly, if two half period zones can be constructed on the surface of the disc, the resultant amplitude at P due to the exposed zones will be $\frac{m_3}{2}$ and so an. Thus, the point P will always be bright but the intensity at P decreases with increase in the diameter of the disc. That is, with a large diameter of the disc, the most effective central zones will be cut off by the disc and the exposed outer zones are more oblique with reference to the point P. Thus (at P) the centre of the geometrical shadow will be bright as if the disc were absent. The diffraction pattern consists of a central bright spot surrounded by alternate bright and dark rings as shown in Fig. 9.12 (*b*).

9.12 DIFFRACTION PATTERN DUE TO A STRAIGHT EDGE

Let S be narrow slit illuminated by a source of monochromatic light of wavelength λ. The length of the slit is perpendicular to the plane of the paper. AD is the straight edge and the length of the edge is parallel to the length of the slit (Fig. 9.13) XY is the incident cylindrical wavefront. P is a point on the screen and SAP is perpendicular to the screen. The screen is perpendicular to the plane of the paper. Below the point P is the geometrical shadow and above P is the illuminated portion. Let the

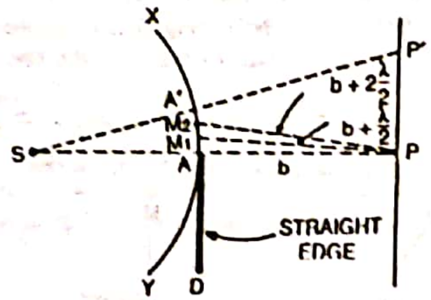


Fig. 9.13

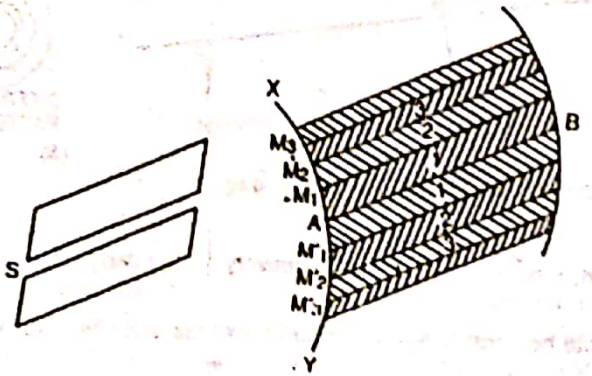


Fig. 9.14

distance AP be b. With reference to the point P, the wavefront can be divided into a number of a half period strips as shown in Fig. 9.14. XY

is the wavefront, A is the pole of the wavefront and AM_1, M_1M_2, M_2M_3 etc. measure the thickness of the 1 st, 2 nd, 3 rd etc. half period strips. With the increase in the order of the strip, the area of the strip decreases (Fig. 9.14).

In Fig. 9.13,

$$AP = b, \qquad PM_1 = b + \frac{\lambda}{2}$$

$$PM_2 = b + \frac{2\lambda}{2} \text{ etc.}$$

Let P' be a point on the screen in the illuminated portion (Fig. 9.15). To calculate the resultant effect at P' due to the wavefront XY, join S to P'. This line meets the wavefront at B. B is the pole of the wavefront with reference to the point P' and the intensity at P' will depend mainly on the number of half period strips enclosed between the points A and

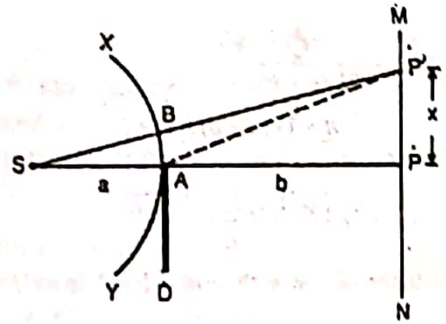


Fig. 9.15

B. The effect at P' due to the wavefront above B is the same at all points on the screen whereas it is different at different points due to the wavefront between B and A. The point P' will be of maximum intensity, if the number of half period strips enclosed between B and A is odd and the intensity at P' will be minimum if the number of half period strips enclosed between B and A is even.

9.13 POSITIONS OF MAXIMUM AND MINIMUM INTENSITY

Let the distance between the slit and the straight edge be a and the distance between the straight edge and the screen be b (Fig. 9.15). Let PP' be x.

The path difference,

$$\delta = AP' - BP'$$

$$= (b^2+x^2)^{1/2} - [SP' - SB]$$

$$= (b^2+x^2)^{1/2} - [\sqrt{(a+b)^2+x^2} - a]$$

$$= b\left[1+\frac{x^2}{2b^2}\right] - (a+b)\left[1+\frac{x^2}{2(a+b)^2}\right] + a$$

$$= b + \frac{x^2}{2b} - a - b - \frac{x^2}{2(a+b)} + a$$

$$= \frac{x^2}{2}\left(\frac{1}{b} - \frac{1}{a+b}\right)$$

$$= \frac{x^2}{2}\left(\frac{a+b-b}{b(a+b)}\right)$$

$$\delta = \frac{x^2}{2} \cdot \frac{a}{b(a+b)}$$

The point P' will be of maximum intensity if $\delta = (2n+1)\frac{\lambda}{2}$

$$\therefore \quad (2n+1)\frac{\lambda}{2} = \frac{ax_n^2}{2b(a+b)}$$

$$x_n^2 = \frac{(2n+1)(a+b)b\lambda}{a}$$

or

$$x_n = \sqrt{\frac{(2n+1)(a+b)b\lambda}{a}} \qquad ...(i)$$

where x_n is the distance of the n th bright band from P.

Similarly, P' will be of minimum intensity if $\delta = 2n\frac{\lambda}{2}$.

$$\therefore \quad 2n\frac{\lambda}{2} = \frac{ax_n^2}{2b(a+b)}$$

$$x_n^2 = \frac{2n(a+b)b\lambda}{a}$$

$$x_n = \sqrt{\frac{2n(n+b)b\lambda}{a}}$$

where x_n is the distance of the n th dark band from P. thus, diffraction bands of varying intensity (roughly corresponding to maxima and minima) are observed above the geometrical shadow *i.e.*, above P and the bands disappear and uniform illumination occurs if P' is far away from P.

9.14 INTENSITY AT A POINT INSIDE THE GEOMETRICAL SHADOW (STRAIGHT EDGE)

If P' is a point below P (Fig. 9.16) and B is the new pole of the wavefront with reference to the point P', then the half period strips below B are cut off by the obstacle and only the uncovered half period strips above B will be effective in producing the illumination at P'. As P' moves

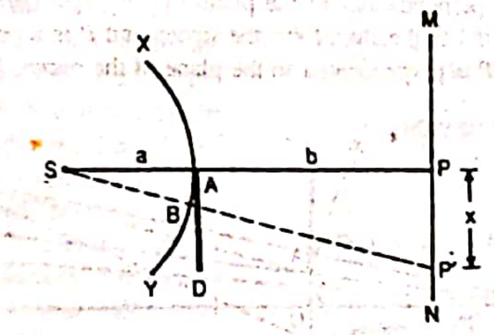


Fig. 9.16

farther from P, more number of half period strips above B are also cut off and the intensity gradually falls. Thus with in the geometrical shadow, the intensity gradually falls off depending on the position of P' with respect to P.

The intensity distribution on the screen due to a straight edge is shown in Fig. 9.17 S is the source, AD is the straight edge and MN is the screen. In the illuminated portion PM, alternate bright and dark bands of gradually diminishing intensity will be observed and the intensity falls off gradually in the region of the geometrical shadow. Thus according to the wave theory, the shadows cast by obstacles in the path of light are not sharp and hence rectilinear propagation of light is only approximately true. In general, there is gradual fading of intensity in the region of the

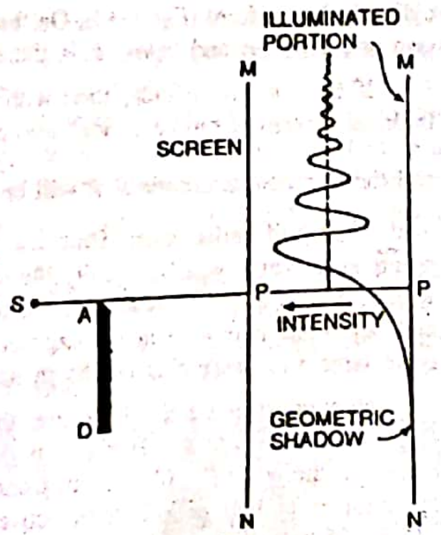


Fig. 9.17

geometrical shadow and with monochromatic light, bright and dark bands (diffraction bands) are observed in the illuminated portion of the screen. However, with white light coloured bands will be observed and the bands of shorter wavelength are nearer the point P.

9.15 DIFFRACTION PATTERN DUE TO A NARROW SLIT

S is a narrow slit illuminated by monochromatic light. The length of the slit is perpendicular to the plane of the paper. AB is a rectangular aperture parallel to the slit, MN is the screen and P is a point on the screen such that SOP is perpendicular to the plane of the paper, XY is the incident

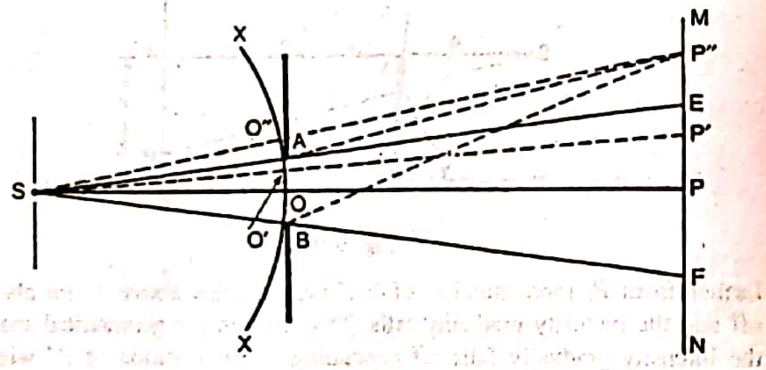


Fig. 9.18

cylindrical wavefront (Fig. 9.18). On the screen, EF is the illuminated portion and above E and below F is the region of the geometrical shadow.

If the slit AB is wide, then with reference to the point P, the cylindrical wavefront can be divided into a large number of half period strips and the resultant amplitude at P will be $\frac{m_1}{2}$ where m_1 is the amplitude due to the first half period strip. Thus, the point P will be illuminated. Even points very near P will be equally illuminated. If the wavefront is divided with reference to points nearer P, the number of half period strips above and below the new pole in the exposed portion of the wavefront will be quite large and hence this results in uniform illumination.

Now consider a point P' nearer the edge of the geometrical shadow (Fig. 9.18). Join S to P'. Here O' is the pole of the wavefront with reference to the point P'. If the wavefront is divided into half period strips, the number of half period strips between O' and B will be quite large and the illumination at P' due to the lower portion of the wavefront will be the same at all points near the edge of the geometrical shadow. But the intensity at P due to the exposed portion of the wavefront between

A and O' will depend on the number of half period strips present. If the number of half period elements is odd, the point P' will be of maximum intensity and if it is even the point will be of minimum intensity.

Let P'' be a point in the region of the geometrical shadow. Join S to P''. Here O'' is the pole of the wavefront with reference to the point P''. If the wavefront is divided into half period elements, then the upper half of the wavefront between X and O'' is cut off by the obstacle and only a portion between A and B is exposed to the point P''. If the number of half period elements exposed by AB is odd, then P'' will be of maximum intensity and if it is even it will be of minimum intensity. But as the most effective central half period strips between O'' and A are cut off, the intensity falls off rapidly in the region of the geometrical shadow and maxima and minima cannot be distinguished. The intensity distribution due to a wide aperture is shown in Fig. 9.19 (*b*).

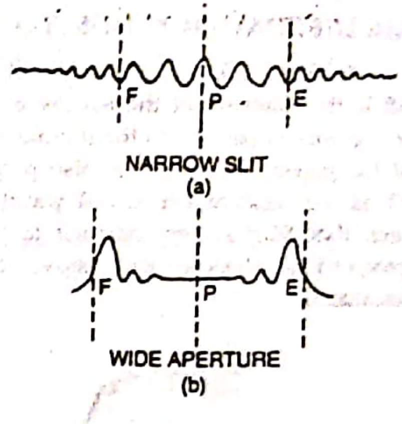


Fig. 9.19

On the other hand, if the slit is narrow, the intensity at the point P will depend on the number of half period strips that can be constructed on the exposed wavefront between A and B. If the number of half period strips is odd, the intensity at P will be maximum and if it is even the intensity at P will be minimum (Fig. 9.18) Thus, the point P can be bright or dark. If we consider a point P' in the illuminated portion EF of the screen, the intensity at P will depend on the number of half period strips that can be constructed between A and O' where O' is the pole of the wavefront with reference to the point P'. If the number of half period strips between A and O' is odd, P' will be a point of maximum intensity. Thus, between E and F alternate bright and dark bands will be observed and the point P may be bright or dark.

Now consider a point P'' in the region of the geometrical shadow (Fig. 9.18). O'' is the pole of the wavefront with reference to the point P'' and the intensity at P'' will depend on the number of half period strips exposed by the slit AB. The upper half of the wavefront above O'' is obstructed by the obstacle and even the most effective central half period strips between O'' and A are cut off by the obstacle. Thus, the intensity at P'' is far away from E, the maxima and minima become indistinguishable. There is no marked transition between the diffraction bands observed in the geometrical shadow and the illuminated portion. The intensity distribution on the screen due to a narrow slit (say less than the wavelength of light) a broad central maximum will be observed in the illuminated portion and the intensity variation cannot be distinguished. The intensity gradually falls off in the region of the geometrical shadow.

9.16 DIFFRACTION DUE TO NARROW WIRE

In Fig. 9.20, S is a narrow slit illuminated by monochromatic light. AB is the diameter of the narrow wire and MN is the screen. The length of the wire is parallel to the illuminated slit and perpendicular to the plane of the paper. The screen is also perpendicular to the plane of the paper. XY is the incident cylindrical wavefront and P is a point on the screen such that SOP is perpendicular to the screen. EF is the region of the geometrical shadow and above E and below F, the screen is illuminated.

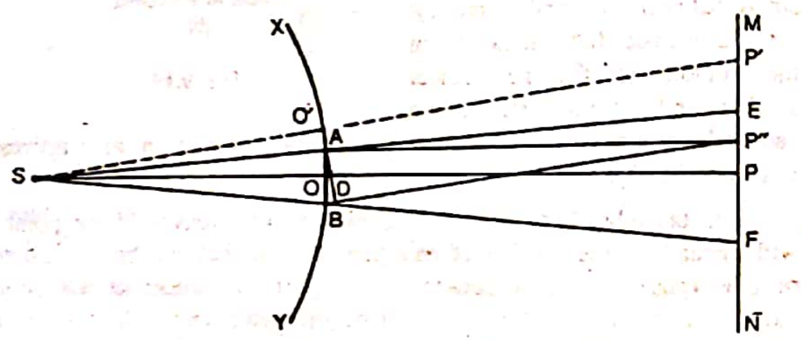


Fig. 9.20

Now consider a point P' on the screen in the illuminated portion. Join S meeting the wavefront at O'. O' is the pole of the wavefront with reference to P'. The intensity at P' due to the wavefront above O' is the same at all points and the effect due to the wavefront BY is negligible. The intensity at P' will be maximum or minimum depending on whether the number of half period strips between O' and A is odd or even. Thus, in the illuminated portion of the screen, diffraction bands of gradually diminishing intensity will be observed. The distinction between maxima and minima will become less if P' is far away from the edge E of the geometrical shadow. Maxima and minima cannot be distinguished if the wire is very narrow, because in that case the portion BY of the wavefront also produces illumination at P.

Next consider a point P'' in the region of the geometrical shadow. Interference bands of equal width will be observed in this region due to the fact the points A and B of the incident wavefront, are similar to two coherent sources. The point P'' will be of maximum or minimum intensity, depending on whether the path difference $(BP'' - AP'')$ is equal to even or odd multiples of $\lambda/2$. The fringe width β is given by

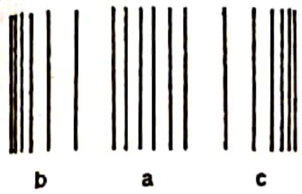


Fig 9.21

$$\beta = \frac{D\lambda}{d}$$

where D is the distance between the wire and the screen, λ is the wavelength of light and d is the distance between the two coherent sources. In this case, $d = 2r$ where $2r$ is the diameter of the wire ($AB = 2r$).

$$\therefore \quad \beta = \frac{D\lambda}{2r} \quad \text{...(i)}$$

$$\therefore \quad r = \frac{D\lambda}{2\beta} \quad \text{...(ii)}$$

or
$$\lambda = \frac{2r\beta}{D} \quad \text{...(iii)}$$

Here, β the fringe width corresponds to the distance between any two consecutive maxima. Thus, from equations (*ii*) and (*iii*), knowing the values of r or λ ; λ or r can be determined. In Fig. 9.21, the bands marked "a" represent the interference bands in the region of the geometrical shadow and the bands marked b, b represent the diffraction bands in the illuminated portion. The intensity distribution due to a narrow wire is shown in Fig. 9.22 (*a*). The centre of the geometrical shadow is bright.

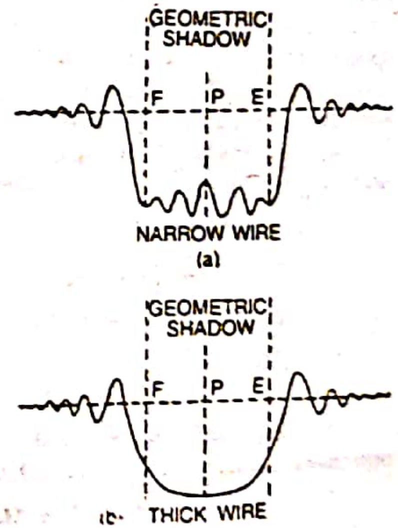


Fig 9.22

On the other hand, if the wire is very thick, the interference bands cannot be noticed.

From equation (*i*),

$$\beta = \frac{D\lambda}{2r}$$

where β is the fringe width. As the diameter of the wire increases the fringe width decreases and if the wire is sufficiently thick, the width of the interference fringes decreases considerably and they cannot be distinguished. The intensity falls off rapidly in the geometrical shadow. The diffraction pattern in the illuminated portion will be similar to that of a thin wire [Fig. 9.22 (*b*)]. Coloured fringes will be observed with white light.

9.17 CORNU'S SPIRAL

To find the effect at a point due to an incident wavefront Fresnel's method consists in dividing the wavefront into half period strips or half period zones. the path difference between the secondary waves from two corresponding points of neighbouring zones is equal to $\frac{\lambda}{2}$.

In Fig. 9.23, S is a point source of light and XY is the incident spherical wavefront. With reference to the point P, O is the pole of the wavefront. Let a and b be the distances of the points S and P from the pole of the wavefront. With P as centre and radius b draw a sphere touching the incident wavefront at O. The path difference between the waves travelling in the directions SAP and SOP is given by

$$d = SA + AP - SOP$$
$$= SA + AP - (SO + OP)$$
$$= a + AB + b - (a + b)$$
$$= AB$$

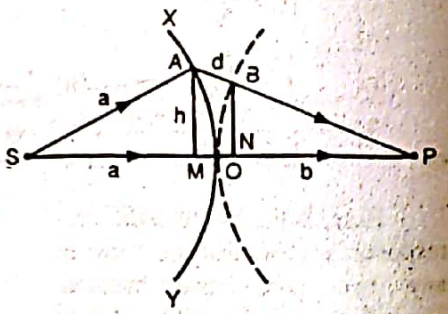


Fig. 9.23

For large distances of a and b, AM and BN can be taken to be approximately equal and the path difference d can be written as

$$d = AB = MO + ON$$

But, from the property of a circle,

$$MO = \frac{AM^2}{2\,SO} = \frac{h^2}{2a}$$

approximately

and
$$ON = \frac{BN^2}{2\,OP} = \frac{h^2}{2b}$$

$$\therefore \quad d = \frac{h^2}{2a} + \frac{h^2}{2b} = \frac{h^2(a+b)}{2ab} \quad \text{...(i)}$$

If AM happens to be the radius of the n th half period zone, then this path difference is equal to $\frac{n\lambda}{2}$ according to Fresnel's method of constructing the half period zones.

$$\therefore \quad \frac{h^2(a+b)}{2ab} = \frac{n\lambda}{2} \quad \text{...(ii)}$$

The resultant amplitude at an external point due to the wavefront can be obtained by the following method. Let the first half period strip of the Fresnel's zones be divided into eight substrips and these vectors are represented from O to M_1 (Fig. 9.24). The continuous phase change is due to the continuous increase in the obliquity factor from O to M_1. The resultant amplitude at the external point due to the first half period strip is given by OM_1 ($= m_1$). Similarly if the process is continued, we obtain the vibration curve M_1M_2. The portion M_1M_2 corresponds to the second half period strip. The resultant amplitude at the point due to the first two half period strip is given by OM_2 ($= A$). If instead of eight substrips, each half period zone is divided into substrips of infinitesimal width, a smooth curve will be obtained. The complete vibration curve for the whole wavefront will be a spiral as shown in Fig. 9.23. X and Y correspond to the two extremities of the wavefront and M_1, M_2 etc. refer to the edge of the first, second, etc. half period strips. Similarly M_1', M_2' etc. refer to the edge of the first, second etc. half period strips of the lower portion of the wavefront. this is called Cornu's spiral. The characteristic of this curve is that for any point P on the curve, the phase lag δ is directly proportional to the square of the distance v. The distance is measured along the length of the curve from the point O. For a path difference of λ the phase difference is 2π. Hence, for a path difference of d, the phase difference δ is given by

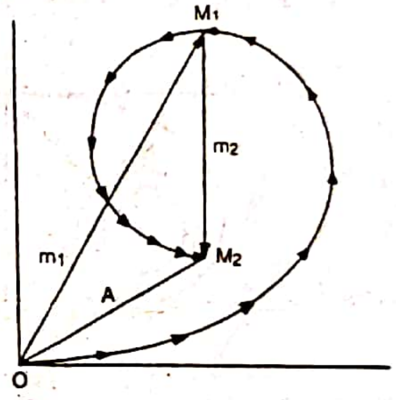


Fig. 9.24

$$\delta = \frac{2\pi}{\lambda} \cdot d$$

Substituting the value of d from equation (*ii*)

$$\delta = \left[\frac{2\pi}{\lambda}\frac{h^2(a+b)}{2ab}\right] \qquad ...(iii)$$

$$\delta = \frac{\pi}{2}\left[\frac{2h^2(a+b)}{ab\lambda}\right]$$

$$\delta = \frac{\pi}{2}.v^2 \qquad ...(iv)$$

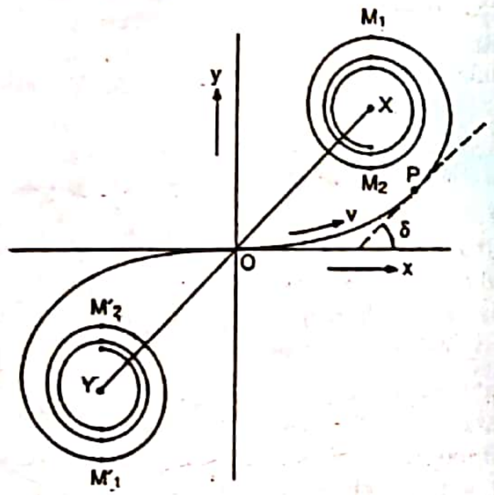


Fig. 9.25

Now, a new variable v (which is dimensionless) is introduced and the value of v is given by

$$v^2 = \frac{2h^2(a+b)}{ab\lambda}$$

or

$$v = h\sqrt{\frac{2(a+b)}{ab\lambda}} \qquad ...(v)$$

Cornu's spiral can be used for any diffraction problem irrespective of the values of a, b and λ.

9.18 FRESNEL'S INTEGRALS

For any point on the Cornu's spiral, the x and y co-ordinates are given by two integrals known as Fresnel's integrals. Consider the point P on the spiral (Fig. 9.25). The distance of the point P along the curve

from the origin is v. The tangent to the curve at P makes and angle δ with the x axis. δ corresponds to the phase change from O to P. For a small displacement dv of the point along the curve, let the corresponding changes in the co-ordinates of the point be dx and dy.

Then, $$dx = dv \cos\delta$$

and $$dy = dv \sin\delta$$

Substituting the value of δ from equation (iv) of article 9.17

$$dx = \cos\left(\frac{\pi v^2}{2}\right)dv \qquad ...(i)$$

and $$dy = \sin\left(\frac{\pi v^2}{2}\right)dv \qquad ...(ii)$$

The co-ordinates x and y of the Cornu's spiral are given by

$$x = \int dx = \int_0^v \cos\left(\frac{\pi v^2}{2}\right)dv \qquad ...(iii)$$

and $$y = \int dy = \int_0^v \sin\left(\frac{\pi v^2}{2}\right)dv \qquad ...(iv)$$

These are called Fresnel' s integrals.

9.19 MAXIMA AND MINIMA IN DIFFRACTION PATTERNS (CORNU'S SPIRAL)

The various diffraction patterns discussed in the earlier articles and the positions of maxima and minima can be easily explained with the help of Cornu's spiral.

In Fig. 9.26, O is the origin of co-ordinates, OX is the vibration curve for the upper half of the wavefront OY refers to the vibration curve for the lower half of the wavefront. If the whole wavefront is unobstructed, the resultant amplitude at a point is given by XY.

If a cylindrical wavefront is incident on a straight edge, the amplitude at a point P on the edge of the geometrical shadow (refer to the discussion on diffraction at a straight edge) is given by XY. If points above P in the illuminated portion are considered, gradually more of the lower half of the wavefront is also exposed to the screen and the amplitude vector passes through maxima and minima. Xb', Xd' etc. refer to maximum amplitudes and Xc' refers to the minimum amplitude. Thus,in the illuminated portion alternate bright and dark bands parallel to the length of the slit are observed on the screen. If points below P and in the region of the geometrical shadow are considered) the lower half of the wavefront and a portion of the upper half of the wavefront are cut off and the tail of the amplitude vector moves to the right of O. The amplitude gradually decreases and becomes zero when the tail approaches X. Thus, in the

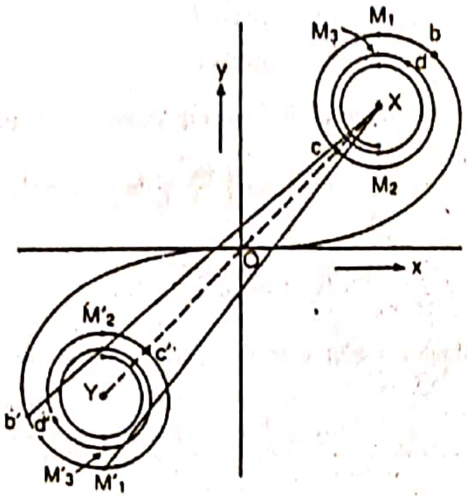


Fig. 9.26

region of the geometrical shadow the intensity falls off gradually. Quantitative values of intensity for different points on the screen can be obtained by finding the amplitude A for different values of v. The square of the amplitude measures the intensity at the point. The points M_1, M_2 etc. correspond to the edges of the first, second etc. half period strips in the upper half of the wavefront and the points M'_1, M'_2 etc. refer to the lower half of the wavefront. The points b', d' etc. on the spiral corresponding to maximum intensity, occur a little before the points M'_1, M'_3 etc. are reached.

The coordinates x and y of the Cornu's spiral are given by

$$x = \int dx = \int_0^v \cos\left(\frac{\pi v^2}{2}\right) dv \qquad ...(i)$$

$$y = \int dy = \int_0^v \sin\left(\frac{\pi v^2}{2}\right) dv \qquad ...(ii)$$

The values of these integrals can be calculated for different values of v. The graph is as shown in Fig. 9.26. The two integrals represent the horizontal and vertical components of the resultant amplitude. The intensity is proportional to the square of the resultant amplitude

$$I_p = k\,[x^2 + y^2] \qquad ...(iii)$$

When the whole of the wavefront is exposed to the point,

$$v \rightarrow \infty$$

and the values of the integrals will be

$$x = \int_0^\infty \cos\left(\frac{\pi v^2}{2}\right) dv = \frac{1}{2}$$

and

$$y = \int_0^\infty \sin\left(\frac{\pi v^2}{2}\right) dv = \frac{1}{2}$$

Thus, for the point X in Fig. 9.26, the x and y coordinates are $\left(\frac{1}{2}, \frac{1}{2}\right)$. Similarly for the point Y on the lower half of the spiral the coordinates are $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

At the origin *i.e.* when $v = 0$, $x = 0$ and $y = 0$. The spiral passes through the origin and it is symmetrical with the origin. At any point on the spiral, the tangent to the curve makes an angle ϕ with the x axis and

$$\tan\phi = \frac{dy}{dx}$$

$$\tan\phi = \frac{\sin\left(\frac{\pi v^2}{2}\right) dv}{\cos\left(\frac{\pi v^2}{2}\right) dv}$$

or

$$\tan\phi = \tan\left(\frac{\pi v^2}{2}\right)$$

or

$$\phi = \frac{\pi v^2}{2} \qquad ...(iv)$$

when $v = 0$, $\phi = 0$

It means the curve is parallel to the x axis at the origin.

The element of length dv along the spiral in given by

$$dv = \sqrt{(dx)^2 + (dy)^2} \qquad ...(v)$$

$\therefore$ Differentiating equation (iv)

$$d\phi = \frac{2\pi v}{2}\,dv$$

or

$$\frac{dv}{d\phi} = \frac{1}{\pi v} \qquad ...(vi)$$

Here $\frac{dv}{d\phi}$ measures the radius of curvature of the spiral at the point under consideration.

From equation (*vi*)

$$\frac{dv}{d\phi} \propto \frac{1}{v}$$

It shows that with the increase in the value of v, the radius of curvature of the curve gradually decreases and takes the shape of a spiral. Finally for $v \to \infty$, the curve ends in a point (X or Y).

9.20 CORNU'S SPIRAL (ALTERNATIVE METHOD)

A narrow slit illuminated by light gives rise to a cylindrical wavefront. Let S be a narrow slit perpendicular to the plane of the paper and illuminated by monochromatic light of wavelength λ [Fig. 9.27]

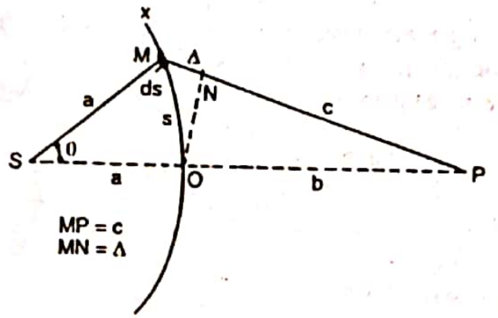


Fig. 9.27

For the cylindrical wavefront XY, the slit is the axis of the cylindrical surface. The effect of the wavefront at a point P in the plane is the same at all points along a line passing through the point P and parallel to the length of the slit. Let y be the displacement at all points on the wavefront XY at time t such that

$$y = a \sin \omega t = a \sin \frac{2\pi t}{T}$$

According to Huygen's Principle, every point on the primary wavefront is a source of secondary disturbance and the resultant intensity at P can be obtained by combining the effect of all the secondary wavelets. Consider a small element ds of the wavefront at M. The distance of the element measured along the curve is s. For the secondary disturbances from O to P *i.e.* for a distance $OP = b$, the phase difference is $\frac{2\pi b}{\lambda}$. The disturbance at P due to a small element ds is given by

$$dy = K \sin 2\pi\left[\frac{t}{T} - \frac{b}{\lambda}\right] ds \qquad \text{...}(i)$$

Similarly, the disturbance at P due to a small element at M is given by

$$dy = K \sin 2\pi\left[\frac{t}{T} - \frac{c}{\lambda}\right] ds \qquad \text{...(2)}$$

where $MP = c = b + \Delta$

$$dy = K \sin 2\pi\left[\frac{t}{T} - \frac{b}{\lambda} - \frac{\Delta}{\lambda}\right] ds \qquad \text{...(3)}$$

$$dy = K \sin\left[2\pi\left(\frac{t}{T} - \frac{b}{\lambda}\right) - \frac{2\pi\Delta}{\lambda}\right] ds$$

$$= K\left[\sin 2\pi\left(\frac{t}{T} - \frac{b}{\lambda}\right)\cos\frac{2\pi\Delta}{\lambda}\right] ds$$

$$- K\left[\cos 2\pi\left(\frac{t}{T} - \frac{b}{\lambda}\right)\sin\frac{2\pi\Delta}{\lambda}\right] ds \qquad \text{...(4)}$$

Integrating between the limits O and s

$$y = K \sin 2\pi\left(\frac{t}{\lambda} - \frac{b}{\lambda}\right)\int_0^s \cos\frac{2\pi\Delta}{\lambda} ds$$

$$- K \cos 2\pi\left(\frac{t}{T} - \frac{b}{\lambda}\right)\int_0^s \sin\frac{2\pi\Delta}{\lambda} ds \qquad \text{...(5)}$$

Let $$K\int_0^s \cos\frac{2\pi\Delta}{\lambda} ds = R\cos\phi$$

and $$K\int_0^s \sin\frac{2\pi\Delta}{\lambda} ds = R\sin\phi$$

Substituting these values in equation (5)

$$y = R \sin 2\pi\left(\frac{t}{T} - \frac{b}{\lambda}\right)\cos\phi - R\cos 2\pi\left(\frac{t}{T} - \frac{b}{\lambda}\right)\sin\phi$$

$$= R \sin\left[2\pi\left(\frac{t}{T} - \frac{b}{\lambda}\right) - \phi\right]$$

Also $$R^2 = R^2\sin^2\phi + R^2\cos^2\phi$$

$$= K^2\left[\left\{\int_0^s \sin\frac{2\pi\Delta}{\lambda}ds\right\}^2 + \left\{\int_0^s \cos\frac{2\pi\Delta}{\lambda}ds\right\}^2\right]$$

From the ΔSMP

$$c^2 = (a+b)^2 + a^2 - 2a(a+b)\cos\theta$$

$$= (a+b)^2 + a^2 - 2a(a+b)\left[1 - \frac{\theta^2}{2!}\right]$$

$$= b^2 + a(a+b)\theta^2$$

But $\theta = \frac{s}{a}$

$$\therefore \quad c^2 = b^2 + \frac{a(a+b)s^2}{a^2}$$

$$= b^2 + \frac{(a+b)}{a}.s^2$$

$$= b^2\left[1 + \frac{(a+b)}{a}.\frac{s^2}{b^2}\right]$$

or

$$c = b\left[1 + \frac{(a+b)}{a}.\frac{s^2}{b^2}\right]^{1/2}$$

$$= b + \frac{(a+b)}{2ab}.s^2$$

$$MP - NP = \Delta = c - b$$

$$= b + \frac{(a+b)}{2ab}s^2 - b$$

or

$$\Delta = \frac{(a+b)s^2}{2ab} \quad ...(6)$$

Let $\frac{2\pi\Delta}{\lambda} = \frac{\pi v^2}{2}$ where v is a new variable depending on the values of a, b, λ and s. Substituting the value of Δ from equation (6)

$$\frac{2\pi}{\lambda}\left[\frac{(a+b)s^2}{2ab}\right] = \frac{\pi v^2}{2}$$

$$s^2 = v^2\left[\frac{ab\lambda}{2(a+b)}\right]$$

$$s = v\sqrt{\frac{ab\lambda}{2(a+b)}} \quad ...(7)$$

$$v = s\sqrt{\frac{2(a+b)}{ab\lambda}} \quad ...(8)$$

From equation (7)

$$ds = dv\sqrt{\frac{ab\lambda}{2(a+b)}}$$

Substituting the values of ds and $\frac{2\pi\Delta}{\lambda}$

$$R^2 = K^2\left[\left\{\int_0^s \cos\frac{2\pi\Delta}{\lambda}ds\right\}^2 + \left\{\int_0^s \sin\frac{2\pi\Delta}{\lambda}\right\}^2\right]$$

We get

$$R^2 = \frac{K^2 ab\lambda}{2(a+b)}\left[\left(\int_0^v \cos\frac{\pi v^2}{2}dv\right)^2 + \left(\int_0^v \sin\frac{\pi v^2}{2}dv\right)^2\right]$$

or $R^2 = K_1[X^2 + Y^2]$

where

$$X = \int_0^v \cos\frac{\pi v^2}{2}dv.$$

and

$$Y = \int_0^v \sin\frac{\pi v^2}{2}dv.$$

$$\int_0^v \cos\left(\frac{\pi v^2}{2}\right)dv = M\cos\left(\frac{\pi v^2}{2}\right) + N\sin\left(\frac{\pi v^2}{2}\right)$$

and

$$\int_0^v \sin\left(\frac{\pi v^2}{2}\right)dv = M\sin\left(\frac{\tau v^2}{2}\right) - N\sin\frac{\pi v^2}{2}$$

Here

$$M = \frac{\pi^0 v}{1} - \frac{\pi^2 v^5}{1\cdot3\cdot5} + \frac{\pi^4 v^9}{1\cdot3\cdot5\cdot7\cdot9}$$

and
$$N = \frac{\pi v^3}{1\cdot 3} - \frac{\pi^3 v^7}{1\cdot 3\cdot 5\cdot 7} + \frac{\pi^5 v^{11}}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}$$

For a given value of s (the extent of the wavefront contributing for the intensity at a point on the screen) the corresponding value of v is evaluated from the relation

$$v = \sqrt{\frac{2(a+b)}{ab\lambda}}$$

From the value of v thus obtained, the values of X and Y corresponding to the X and Y coordinates of a point on the spiral are obtained from Fresnel's integrals. For the properties of Cornu's spiral refer to Article 9.19 (page 437). Cornu's spiral is useful in understanding the Fresnel's diffraction patterns due to obstacles such as straight edge, thin wire, thick wire, narrow slit, wide aperture etc. placed in the path of light.

Application of Cornu's spiral to Fresnel's diffraction at a straight edge is given in Article 9.21. Proceeding in the same way, the intensity distribution on the screen due to Fresnel's diffraction at a narrow wire, narrow slit, thick wire, wide aperture etc. can also be obtained.

9.21 DIFFRACTION AT A STRAIGHT EDGE

S is a narrow slit illuminated by monochromatic light of wavelength λ. The length of the slit is perpendicular to the plane of the paper. AD is a straight edge and the length of the edge is parallel to the length of the slit (Fig. 9.28) XY is the incident cylindrical wavefront. Cornu's spiral helps to obtain qualitatively the intensity distribution at a point on the screen.

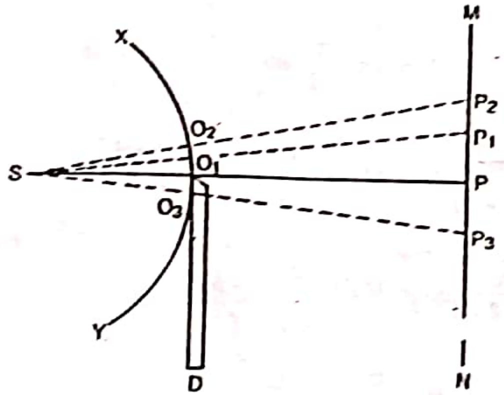


Fig. 9.28

Consider the points P, P_1 and P_2 in the illuminated portion and point P_3 in the geometrical shadow region of the screen MN. A, O_1, O_2 and O_3 are the poles of the wavefront for the points P, P_1, P_2 and P_3 respectively.

Intensity at the point P

The pole of the wavefront is the origin of coordinates. The lower half of the wavefront is cut off by the obstacle and the intensity at P is due to the upper half of the wavefront between A and X. This intensity is equal to $\frac{I_0}{4}$ where I_0 is the intensity due to the whole wavefront (Fig. 9.29)

$$I = K_1 [X^2 + Y^2]$$

$$I = K_1\left[\left(\int_{-\infty}^{+\infty} \cos\frac{\pi v^2}{2}\, dv\right)^2 + \left(\int_{-\infty}^{+\infty} \sin\frac{\pi v^2}{2}\, dv\right)^2\right]$$

$$= K_1 [1 + 1] = 2K_1$$

Let I_0 be the intensity due to the whole wavefront.

Then $$I_0 = 2K_1$$

or $$K_1 = \frac{I_0}{2}$$

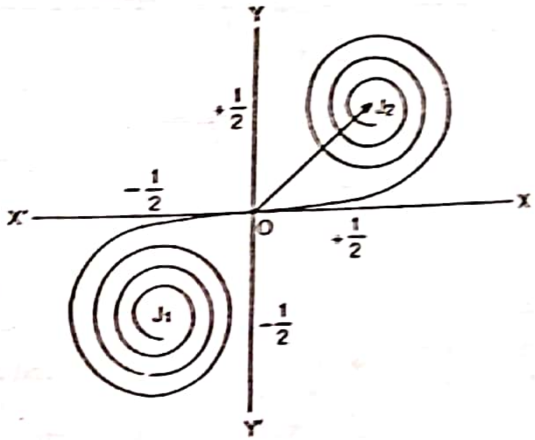


Fig. 9.29

The intensity due to the upper half of the wavefront alone is given by

$$I = \frac{I_0}{2}\left[\left(\int_0^{-\infty} \cos\frac{\pi v^2}{2}\,dv\right)^2 + \left(\int_0^{-\infty} \sin\frac{\pi v^2}{2}\,dv\right)^2\right]$$

$$= \frac{I_0}{2}\left[\frac{1}{4}+\frac{1}{4}\right]$$

$$= \frac{I_0}{4}$$

This intensity is proportional to the square of the amplitude vector OJ_2.

Intensity at the point P_1

For the point P_1, O_1 is the pole of the wavefront and let the exposed portion of the wavefront between A and O_1, correspond to the spiral OM_1 (Fig. 9.30). The intensity in this case is proportional to the square of the vector M_1J_2. Thus, the point P_1 will be of maximum intensity.

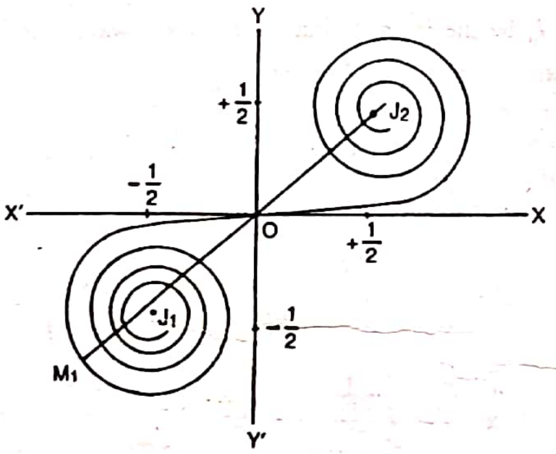


Fig. 9.30

Intensity at the point P_2

For the point P_2, O_2 is the pole of the wave front and let the exposed portion of the wavefront between A and O_2 correspond to the length of the spiral OM_2 (Fig. 9.31). The intensity *at* P_2 is proportional to the square of the amplitude vector M_2J_2 which is a minimum.

Thus, when the point shifts away from P, the resultant intensity is proportional to the square of the amplitude vector whose magnitude passes through maxima and minima. When the point M shifts very near J_1, the

difference in intensity between maxima and minima is very small and this results in uniform illumination.

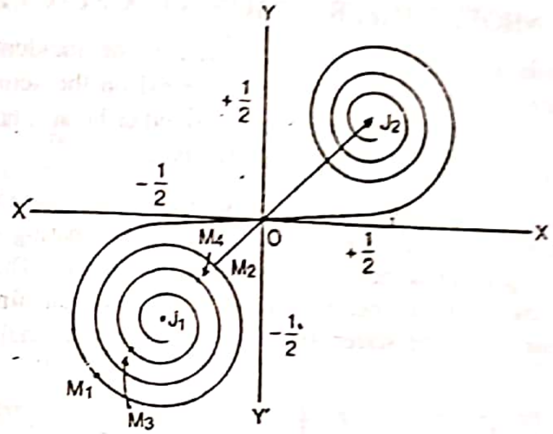


Fig. 9.31

Intensity at the point P_3

For the point P_3, O_3 is the pole of the wavefront. The lower half of the wavefront is cut off by the obstacle and the portion of the wavefront between A and O_3 is also obstructed corresponding to the length of the spiral OM (Fig. 9.32). The intensity at P_3 is proportional to the square of the amplitude MJ_2.

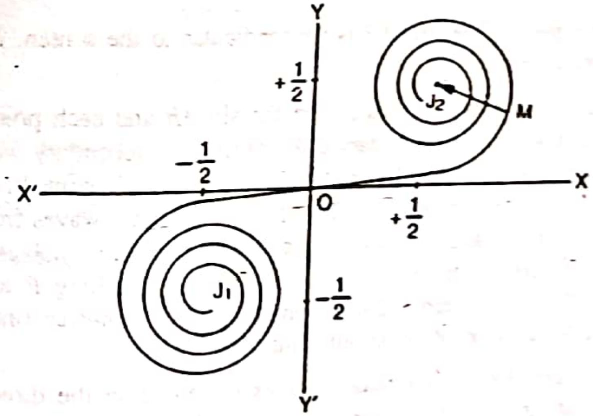


Fig. 9.32

As the point shifts more and more into the region of the geometrical shadow, the point M shifts more and more towards J_2. Thus, the magnitude of MJ_2 gradually decreases. In other words the intensity falls off gradually in the region of the geometrical shadow. The intensity distribution due to Fresnel's diffraction at a straight edge is given in Fig. 9.17 on page 429.

9.22 FRAUNHOFER DIFFRACTION AT A SINGLE SLIT

To obtain a Fraunhofer diffraction pattern, the incident wavefront must be plane and the diffracted light is collected on the screen with the help of a lens. Thus, the source of light should either be at a large distance from the slit or a collimating lens must be used.

In Fig. 9.33, S is a narrow slit perpendicular to the plane of the paper and illuminated by monochromatic light. L_1 is the collimating lens and AB is a slit of width a. XY is the incident spherical wavefront. The light passing through the slit AB is incident on the lens L_2 and the final refracted beam is observed on the screen MN. The screen is perpendicular to the

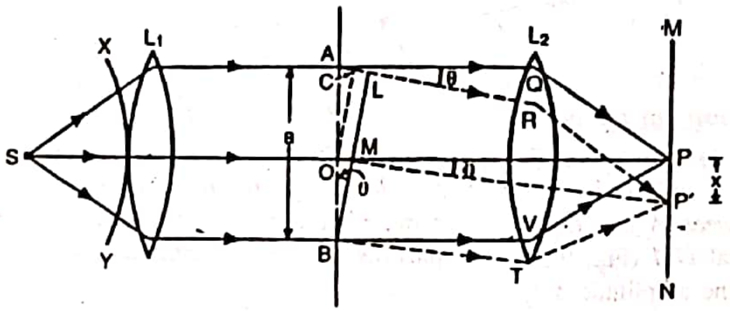


Fig. 9.33

plane of the paper. The line SP is perpendicular to the screen. L_1 and L_2 are achromatic lenses.

A plane wavefront is incident on the slit AB and each point on this wavefront is a source of secondary disturbance. The secondary waves travelling in the direction parallel to OP *viz.* AQ and BV come to focus at P and a bright central image is observed. The secondary waves from points equidistant from O and situated in the upper and lower halves OA and OB of the waterfront travel the same distance in reaching P and hence the path difference is zero. The secondary waves reinforce one another and P will be a point of maximum intensity.

Now, consider the secondary waves travelling in the direction AR, inclined at an angle θ to the direction OP. All the secondary wave travelling in this direction reach the point P' on the screen. The point P' will be of maximum or minimum intensity depending on the path difference between the secondary waves originating from the corresponding points of the wavefront. Draw OC and BL perpendicular to AR.

Then, in the ΔABL

$$\sin\theta = \frac{AL}{AB} = \frac{AL}{a}$$

or

$$AL = a\sin\theta$$

where a is the width of the slit and AL is the path difference between the secondary waves originating from A and B. If this path difference is equal to λ the wavelength of light used, then P' will be a point of minimum intensity. The whole wavefront can be considered to be of two halves OA and OB and if the path difference between the secondary waves from A and B is λ, then the path difference between the secondary waves from A and O will be $\frac{\lambda}{2}$. Similarly for every point in the upper half OA, there is a corresponding point in the lower half OB, and the path difference between the secondary waves from these points is $\frac{\lambda}{2}$. Thus, destructive interference takes place and the point P' will be of minimum intensity. If the direction of the secondary waves is such that $AL = 2\lambda$, then also the point where they meet the screen will be of minimum intensity. This is so, because the secondary waves from the corresponding points of the lower half, differ in path by $\frac{\lambda}{2}$ and this again gives the position of minimum intensity. In general

$$a\sin\theta_n = n\lambda$$

$$\sin\theta_n = \frac{n\lambda}{a}$$

where θ_n gives the direction of the n th minimum. Here n is an integer. If, however, the path difference is odd multiples of $\frac{\lambda}{2}$, the directions of the secondary maxima can be obtained. In this case,

$$a\sin\theta_n = (2n+1)\frac{\lambda}{2}$$

or

$$\sin\theta_n = \frac{(2n+1)\lambda}{2a}$$

where

$$n = 1, 2, 3 \text{ etc.}$$

Thus, the diffraction pattern due to a single slit consists of a central bright maximum at P followed by secondary maxima and minima on both the sides. The intensity distribution on the screen is given in Fig. 9.34.

P corresponds to the position of the central bright maximum and the points on the screen for which the path difference between the points A and B

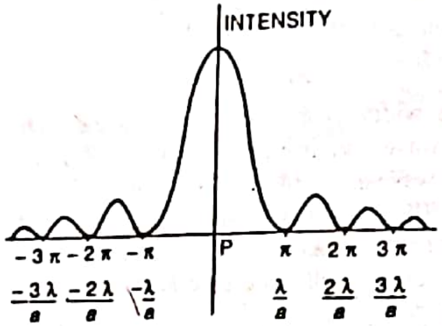


Fig. 9.34

is λ, 2λ etc., correspond to the positions of secondary minima. The secondary maxima are of much less intensity. The intensity falls off rapidly from the point P outwards.

If the lens L_2 is very near the slit or the screen is far away from the lens L_2, then

$$\sin\theta = \frac{x}{f} \quad ...(i)$$

where f is the focal length of the lens L_2

But, $$\sin\theta = \frac{\lambda}{a} \quad ...(ii)$$

$$\therefore \quad \frac{x}{f} = \frac{\lambda}{a}$$

or $$x = \frac{f\lambda}{a}$$

where x is the distance of the secondary minimum from the point P. Thus, the width of the central maximum = $2x$.

or $$2x = \frac{2f\lambda}{a} \quad ...(iii)$$

The width of the central maximum is proportional to λ, the wavelength of light. With red light (longer wavelength), the width of the central maximum is more than with violet light (shorter wavelength). With a narrow slit, the width of the central maximum is more. The diffraction pattern consists of alternate bright and dark bands with monochromatic light. With white light, the central maximum is white and the rest of the diffraction

bands are coloured. From equation (*ii*), if the width a of the slit is large, $\sin\theta$ is small and hence θ is small. The maxima and minima are very close to the central maximum at P. But with a narrow slit, a is small and hence θ is large. This results a distinct diffraction maxima and minima on both the sides of P.

Example 9.9. *Find the half angular width of the central bright maximum in the Fraunhofer diffraction pattern of a slit of width 12×10^{-5} cm when the slit is illuminated by monochromatic light of wavelength 6000 Å.*

Here $$\sin\theta = \frac{\lambda}{a}$$

where θ is half angular width of the central maximum.

$$a = 12 \times 10^{-5}\ \text{cm},\ \lambda = 6000\ \text{Å} = 6 \times 10^{-5}\ \text{cm}.$$

$$\therefore \quad \sin\theta = \frac{\lambda}{a} = \frac{6 \times 10^{-5}}{12 \times 10^{-5}} = 0.50$$

or $$\theta = 30°$$

Example 9.10. *In Fraunhofer diffraction due to a narrow slit a screen is placed 2 m away from the lens to obtain the pattern. If the slit width is 0.2 mm and the first minima lie 5 mm on either side of the central maximum, find the wavelength of light.* [*Delhi (Sub) 1977*]

In the case of Fraunhofer diffraction at a narrow rectangular aperture,

$$a\sin\theta = n\lambda$$

$$n = 1$$

$$\therefore \quad a\sin\theta = \lambda$$

$$\sin\theta = \frac{x}{D}$$

$$\therefore \quad \frac{ax}{D} = \lambda$$

$$\lambda = \frac{ax}{D}$$

Here $$a = 0.2\ \text{mm} = 0.02\ \text{cm}$$

$$x = 5\ \text{mm} = 0.5\ \text{cm}$$

$$D = 2\ \text{m} = 200\ \text{cm}$$

$$\therefore \quad \lambda = \frac{0.02 \times 0.5}{200}$$

$$\lambda = 5 \times 10^{-5}\ \text{cm}$$

$$\lambda = 5000\ \text{Å}$$

Example 9.11. *Light of wavelength 6000 Å is incident on a slit of width 0.30 mm. The screen is placed 2 m from the slit. Find (a) the position of the first dark fringe and (b) the width of the central bright fringe.*

The first dark fringe is on either side of the central bright fringe.

Here $n = \pm 1,\ D = 2\,\text{m}$

$$\lambda = 6000\,\text{Å} = 6\times10^{-7}\,\text{m}$$

$$\sin\theta = \frac{x}{D}$$

$$a = 0.30\ \text{mm} = 3\times10^{-4}\,\text{m}$$

$$a\sin\theta = n\lambda$$

$$\frac{ax}{D} = n\lambda$$

(a)
$$x = \frac{n\lambda D}{a}$$

$$x = \pm\left[\frac{1\times6\times10^{-7}\times2}{3\times10^{-4}}\right]$$

$$x = \pm4\times10^{-3}\,\text{m}$$

The positive and negative signs correspond to the dark fringes on either side of the central bright fringe.

(b) The width of the central bright fringe,

$$y = 2x$$
$$= 2\times4\times10^{-3}$$
$$= 8\times10^{-3}\,\text{m}$$
$$= 8\,\text{mm}$$

Example 9.12 *A single slit of width 0.14 mm is illuminated normally by monochromatic light and diffraction bands are observed on a screen 2 m away. If the centre of the second dark band is 1.6 cm from the middle of the central bright band, deduce the wavelength of light used.*

(IAS, 1990)

In the case of Fraunhofer diffraction at a narrow rectangular slit,

$$a\sin\theta = n\lambda$$

Here θ gives the directions of the minimum

$$n = 2$$
$$\lambda = ?$$

$$a = 0.14\ \text{mm} = 0.14\times10^{-3}\,\text{m}$$
$$D = 2\,\text{m}$$
$$x = 1.6\,\text{cm} = 1.6\times10^{-2}\,\text{m}$$
$$\sin\theta = \frac{x}{D} = \frac{n\lambda}{a}$$

$$\therefore\quad \lambda = \frac{xa}{nD}$$
$$= \frac{1.6\times10^{-2}\times0.14\times10^{-3}}{2\times2}$$
$$= 5.6\times10^{-7}\,\text{m}$$
$$= 5600\,\text{Å}$$

Example 9.13. *A screen is placed 2 m away from a narrow slit which is illuminated with light of wavelength 6000 Å. If the first minimum lies 5 mm on either side of the central maximum, calculate the slit width.*

(Delhi, 1990)

In the case of Fraunhofer diffraction at a narrow slit,

$$a\sin\theta = n\lambda$$
$$\sin\theta = \frac{x}{D}$$
$$\therefore\quad \frac{ax}{D} = n\lambda$$

Here, width of the slit $= a = ?$

$$x = 5\,\text{mm} = 5\times10^{-3}\,\text{m}$$
$$D = 2\,\text{m}$$
$$\lambda = 6000\,\text{Å} = 6\times10^{-7}\,\text{m}$$
$$n = 1$$
$$a = \left(\frac{n\lambda D}{x}\right)$$
$$a = \left(\frac{1\times6\times10^{-7}\times2}{5\times10^{-3}}\right)$$
$$a = 2.4\times10^{-4}\,\text{m}$$
$$a = 0.24\,\text{mm}$$

Example 9.14. *Find the angular width of the central bright maximum in the Fraunhofer diffraction pattern of a slit of width 12×10^{-5} cm when the slit is illuminated by monochromatic light of wavelength 6000 Å.*

(Kanpur, 1990)

Here $$\sin\theta = \frac{\lambda}{a}$$

where θ is the half angular width of the central maximum

$$a = 12 \times 10^{-5}\,\text{cm} = 12 \times 10^{-7}\,\text{m}$$

$$\lambda = 6000\,\text{Å} = 6 \times 10^{-7}\,\text{m}$$

$$\sin\theta = \frac{6 \times 10^{-7}}{12 \times 10^{-7}} = 0.5$$

$$\theta = 30^\circ$$

Angular width of the central maximum,

$$2\theta = 60^\circ$$

Example 9.15. *Diffraction pattern of a single slit of width 0.5 cm is formed by a lens of focal length 40 cm. Calculate the distance between the first dark and the next bright fringe from the axis. Wavelength = 4890 Å.* *[Kanpur, 1991]*

For minimum intensity

$$a \sin\theta_n = n\lambda$$

$$\sin\theta_n = \frac{x_1}{f}, \quad n = 1$$

$$\frac{x_1}{f} = \frac{\lambda}{a}$$

Here $$\lambda = 4890\,\text{Å} = 4890 \times 10^{-10}\,\text{m}$$

$$a = 0.5\,\text{cm} = 5 \times 10^{-3}\,\text{m}$$

$$f = 40\,\text{cm} = 0.4\,\text{m}$$

$$x_1 = \frac{f\lambda}{a}$$

$$x_1 = \frac{0.4 \times 4890 \times 10^{-10}}{5 \times 10^{-3}}$$

$$x_1 = 3.912 \times 10^{-5}\,\text{m}$$

For secondary maximum

$$a \sin\theta_n = \frac{(2n+1)\lambda}{2}$$

For the first secondary maximum

$$n = 1$$

$$\sin\theta_n = \frac{x_2}{f}$$

$$\frac{x_2}{f} = \frac{3\lambda}{2a}$$

$$x_2 = \frac{3\lambda f}{2a}$$

$$x_2 = \frac{3 \times 4890 \times 10^{-10} \times 0.4}{2 \times 5 \times 10^{-3}}$$

$$x_2 = 5.868 \times 10^{-5}\,\text{m}$$

Difference, $$x_2 - x_1 = 5.868 \times 10^{-5} - 3.912 \times 10^{-5}$$

$$= 1.956 \times 10^{-5}\,\text{m}$$

$$= 1.596 \times 10^{-2}\,\text{mm}$$

9.23 INTENSITY DISTRIBUTION IN THE DIFFRACTION PATTERN DUE TO A SINGLE SLIT

The intensity variation in the diffraction pattern due to a single slit can be investigated as follows. The incident plane wavefront on the slit *AB* (Fig. 9.33) can be imagined to be divided into a large number of infinitesimally small strips. The path difference between the secondary waves emanating from the extreme points *A* and *B* is $a \sin\theta$ where a is the width of the slit and $\angle ABL = \theta$. For a parallel beam of incident light, the amplitude of vibration of the waves from each strip can be taken to be the same. As one considers the secondary waves in a direction inclined

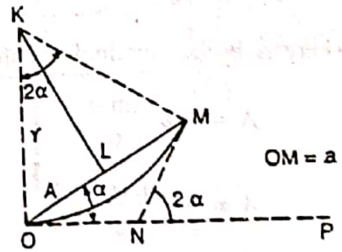


Fig. 9.35

at an angle θ from the point B upwards, the path difference changes and hence the phase difference also increases. Let α be the phase difference between the secondary waves from the points B and A of the slit (Fig. 9.27). As the wavefront is divided into a large number of strips,the resultant amplitude due to all the individual small strips can be obtained by the vector polygon method. Here, the amplitudes are small and the phase difference increases by infinetesimally small amounts from strip to strip. Thus, the vibration polygon coincides with the circular arc OM (Fig. 9.35). OP gives the direction of the initial vector and NM the direction of the final vector due to the secondary waves from A. K is the centre of the circular arc.

$$\angle MNP = 2\alpha$$

$$\therefore \quad \angle OKM = 2\alpha$$

In the Δ OKL

$$\sin\alpha = \frac{OL}{r}\ ;\ OL = r\sin\alpha$$

where r is the radius of the circular arc

$$\therefore \quad \text{Chord } OM = 2\,OL = 2r\sin\alpha \qquad \ldots (i)$$

The length of the arc OM is proportional to the width of the slit.

$\therefore$ Length of the arc $OM = Ka$

where K is a constant and a is the width of the slit.

$$\text{Also,} \qquad 2\alpha = \frac{\text{Arc } OM}{\text{radius}} = \frac{Ka}{r}$$

$$\text{or} \qquad 2r = \frac{Ka}{\alpha} \qquad \ldots (ii)$$

Substituting this value of $2r$ in equation (i)

$$\text{Chord } OM = \frac{Ka}{\alpha}\cdot\sin\alpha$$

But, $OM = A$ where A is the amplitude of the resultant.

$$\therefore \quad A = (Ka)\frac{\sin\alpha}{\alpha}$$

$$A = A_0\frac{\sin\alpha}{\alpha} \qquad \ldots (iii)$$

Thus, the resultant amplitude of vibration at a point on the screen is given by $A_0\dfrac{\sin\alpha}{\alpha}$ and the intensity I at the point is given by

$$I = A^2 = A_0^2\frac{\sin^2\alpha}{\alpha^2} = I_0\cdot\left(\frac{\sin\alpha}{\alpha}\right)^2 \qquad \ldots (iv)$$

The intensity at any point on the screen is proportional to $\left(\dfrac{\sin\alpha}{\alpha}\right)^2$. A phase difference of 2π corresponds to a path difference of λ. Therefore a phase difference of 2α is given by

$$2\alpha = \frac{2\pi}{\lambda}\cdot a\sin\theta \qquad \ldots (iv)$$

where $a \sin\theta$ is the path difference between the secondary waves from A and B (Fig. 9.35).

$$\alpha = \frac{\pi}{\lambda}\cdot a\sin\theta \qquad \ldots (v)$$

Thus, the value of α depends on the angle of diffraction θ. The value of $\dfrac{\sin^2\alpha}{\alpha^2}$ for different values of θ gives the intensity at the point under consideration. Fig. 9.34 represents the intensity distribution. It is a graph of $\dfrac{\sin^2\alpha}{\alpha^2}$ (along the Y-axis), as a function of α or $\sin\theta$ (along the X-axis).

9.24 FRAUNHOFER DIFFRACTION AT A SINGLE SLIT (CALCULUS METHOD)

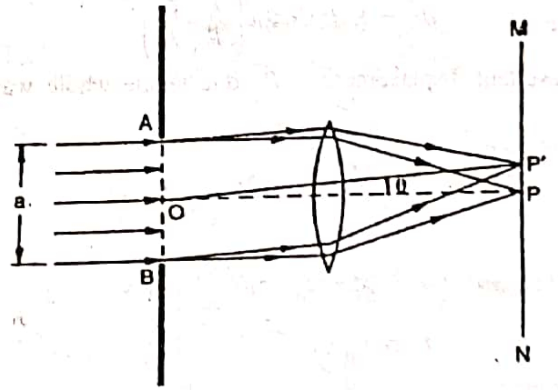


Fig. 9.36

Let a monochromatic parallel beam of light be incident on the slit AB of width a. The secondary waves travelling in the same direction as the incident light come to focus at the point P. The secondary waves travelling at an angle θ come to focus at P' (Fig. 9.36).

Consider the screen to be at a distance r from the slit. The centre of the slit O is the origin of coordinates. Consider a small element dz of the wavefront with coordinates (o, z). The coordinates of the point P' are (x_0, z_0) [Fig. 9.37]. The distance of the element from the point P' is ρ.

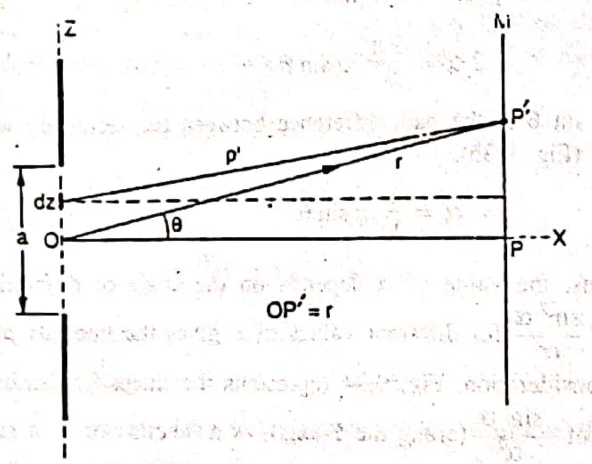


Fig. 9.37

The displacement at the point P' due to the element dz at any instant is given by,

$$dy = K\,dz \sin 2\pi\left(\frac{t}{T} - \frac{\rho}{\lambda}\right) \qquad ...(i)$$

The resultant displacement at P' due to the whole wavefront,

$$y = K\int_{-\frac{a}{2}}^{+\frac{a}{2}} \sin 2\pi\left(\frac{t}{T} - \frac{\rho}{\lambda}\right) dz \qquad ...(ii)$$

Also $$\rho^2 = x_0^2 + (z_0 - z)^2 \qquad ...(iii)$$

$$r^2 = x_0^2 + z_0^2$$

or $$x_0^2 = r^2 - z_0^2$$

Substituting the value of x_0^2 in equation (iii)

$$\rho^2 = r^2 - z_0^2 + (z_0 - z)^2$$

$$\rho^2 = r^2\left[1 - \frac{2zz_0}{r^2} + \frac{z^2}{r^2}\right] \qquad ...(iv)$$

In the case of Fraunhofer diffraction, the screen is at a very large distance from the slit, therefore $r >> z$ and $\frac{z^2}{r^2}$ is negligible.

$$\therefore \quad \rho^2 = r^2\left[1 - \frac{2zz_0}{r^2}\right]$$

$$\rho = r\left[1 - \frac{zz_0}{r^2}\right]$$

$$\rho = r - \frac{zz_0}{r}$$

But, $$\frac{z_0}{r} = \sin\theta$$

$$\therefore \quad \rho = r - z\sin\theta \qquad ...(v)$$

Substituting this value of ρ in equation (ii)

$$y = K\int_{-\frac{a}{2}}^{+\frac{a}{2}} \sin\left[2\pi\left(\frac{t}{T} - \frac{r}{\lambda} + \frac{z\sin\theta}{\lambda}\right)\right] dz$$

$$y = -\frac{K\lambda}{2\pi\sin\theta}\left[\cos 2\pi\left(\frac{t}{T} - \frac{r}{\lambda} + \frac{a\sin\theta}{2\lambda}\right) - \cos 2\pi\left(\frac{t}{T} - \frac{r}{\lambda} - \frac{a\sin\theta}{2\lambda}\right)\right]$$

$$y = -\frac{K\lambda}{2\pi\sin\theta}\left[2\sin 2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right)\sin 2\pi\left(-\frac{a\sin\theta}{2\lambda}\right)\right]$$

$$y = \frac{K\lambda}{\pi\sin\theta}\left[\sin 2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right)\sin\left(\frac{\pi a\sin\theta}{\lambda}\right)\right]$$

Let $$\frac{\pi a\sin\theta}{\lambda} = \alpha$$

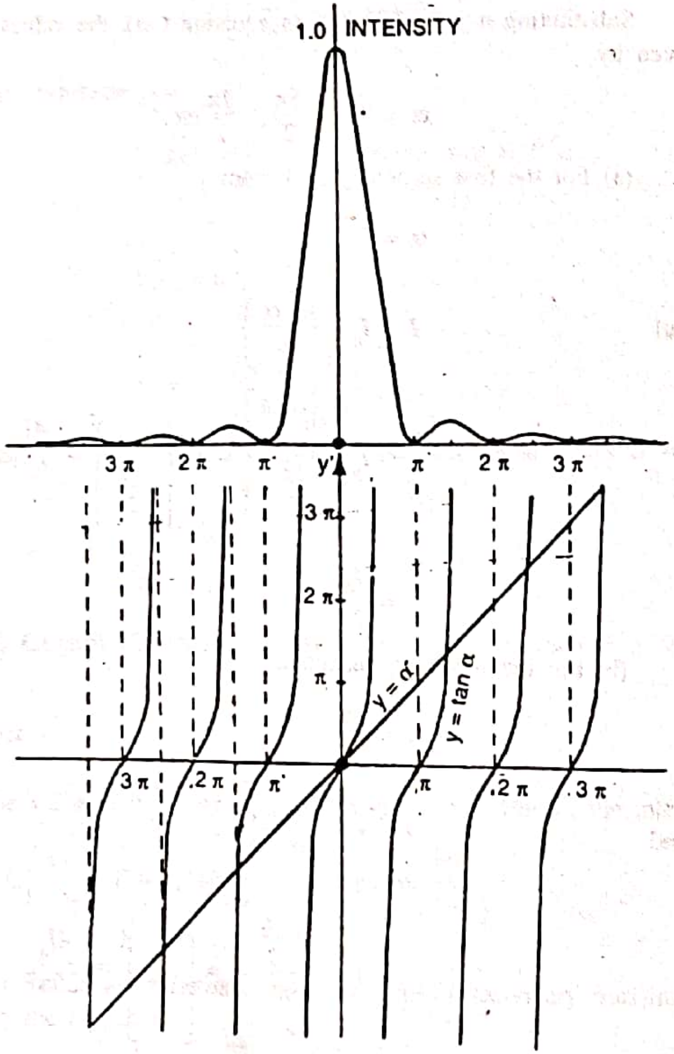


Fig. 9.38

$$dI' = I_0\left[\frac{\alpha^2\, 2\sin\alpha\cos\alpha - (\sin^2\alpha)\, 2\alpha}{\alpha^4}\right]d\alpha$$

For I' to be maximum

$$\frac{dI'}{d\alpha} = 0$$

$$\therefore \quad \alpha^2\,(2\sin\alpha\cos\alpha) - (\sin^2\alpha)\,2\alpha = 0$$

$$\tan\alpha = \alpha$$

If graphs are plotted for $y = \alpha$ and $y = \tan\alpha$ it will be found that the secondary maxima are not exactly midway between two minima. The positions of the secondary maxima are slightly towards the central maximum (Fig. 9.38).

(*iii*) **Secondary Minima.** The directions of the secondary minima are given by the equation

$$a\sin\theta = n\lambda$$

Substituting the value of $a\sin\theta$ in equation (v), (page 457)

$$\alpha = \frac{\pi}{\lambda}\cdot n\lambda = n\pi \qquad \text{...}(ix)$$

Substituting $\quad n = 1, 2, 3$ etc. in equation (ix),

$$\alpha = \pi, 2\pi, 3\pi \text{ etc.}$$

When these values of α are substituted in the equation for the intensity viz.

$$I = I_0\left(\frac{\sin\alpha}{\alpha}\right)^2; \quad I = 0$$

In Fig. 9.34, the positions of the secondary minima are shown for the values of

$$\alpha = \pi, 2\pi, 3\pi \text{ etc.}$$

$\frac{\lambda}{a}, \frac{2\lambda}{a}, \frac{3\lambda}{a}$ etc. refer to the values of $\sin\theta$ for these positions.

9.25 FRAUNHOFER DIFFRACTION AT A CIRCULAR APERTURE

In Fig. 9.39, AB is a circular aperture of diameter d. C is the centre

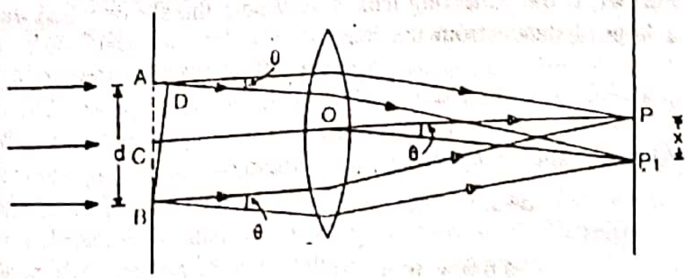


Fig. 9.39

of the aperture and P is a point on the screen. CP is perpendicular to the screen. The screen is perpendicular to the plane of the paper. A plane wavefront is incident on the circular aperture. The secondary waves travelling in the direction CO come to focus at P. Therefore, P corresponds to the position of the central maximum. Here, all the secondary waves emanating from points equidistant from O travel the same distance before reaching P and hence they all reinforce one another. Now consider the secondary waves travelling in a direction inclined at an angle θ with the direction CP. All these secondary waves meet at P_1 on the screen. Let the distance PP_1 be x. The path difference between the secondary waves emanating from the points B and A (extremities of a diameter) is AD.

From the ΔABD,

$$AD = d \sin\theta$$

Arguing as in Article 9.22, the point P_1 will be of minimum intensity if this path difference is equal to integral multiples of λ *i.e.*,

$$d \sin\theta_n = n\lambda \qquad \text{...(i)}$$

The point P_1 will be of maximum intensity if the path difference is equal to odd multiples of $\frac{\lambda}{2}$ *i.e.*,

$$d \sin\theta_n = \frac{(2n+1)\lambda}{2} \qquad \text{...(ii)}$$

If P_1 is a point of minimum intensity, then all the points at the same distance from P as P_1 and lying on a circle of radius x will be of minimum intensity. Thus, the diffraction pattern due to a circular aperture consists of a central bright disc called the Airy's disc, surrounded by alternate dark and bright concentric rings called the Airy's rings. The intensity of the dark rings is zero and that of the bright rings decreases gradually outwards from P.

Further, if the collecting lens is very near the slit or when the screen is at a large distance from the lens,

$$\sin\theta = \theta = \frac{x}{f} \qquad \text{...(iii)}$$

Also, for the first secondary minimum,

$$d \sin\theta = \lambda$$

$$\sin\theta = \theta = \frac{\lambda}{d} \qquad \text{...(iv)}$$

From equations (*iii*) and (*iv*)

$$\frac{x}{f} = \frac{\lambda}{d}$$

or

$$x = \frac{f\lambda}{d} \qquad \text{...(v)}$$

where x is the radius of the Airy's disc. But actually, the radius of the first dark ring is slightly more than that given by equation (*v*). According to Airy, it is given by

$$x = \frac{1.22 f\lambda}{d} \qquad \text{...(vi)}$$

The discussion of the intensity distribution of the bright and dark rings is similar to the one given for a rectangular slit. With increase in the diameter of the aperture, the radius of the central bright ring decreases.

Example 9.16. *In Fraunhofer diffraction pattern due to a single slit, the screen is at a distance of 100 cm from the slit and the slit is illuminated by monochromatic light of wavelength 5893 Å. The width of the slit is 0.1 mm. Calculate the separation between the central maximum and the first secondary minimum.* *(Mysore)*

For a rectangular slit,

$$x = \frac{f\lambda}{d}$$

Here $f = 100$ cm, $\lambda = 5893$ Å $= 5893 \times 10^{-8}$ cm,

$d = 0.1$ mm $= 0.01$ cm, $x = ?$

$$\therefore \quad x = \frac{100 \times 5893 \times 10^{-8}}{0.01} = \mathbf{0.5893\ cm}$$

9.26 FRAUNHOFER DIFFRACTION AT DOUBLE SLIT

In Fig. 9.40, AB and CD are two rectangular slits parallel to one another and perpendicular to the plane of the paper. The width of each slit is a and the width of the opaque portion is b. L is a collecting lens and MN is a screen perpendicular to the plane of the paper. P is a point on the screen such that OP is perpendicular to the screen. Let a plane wavefront be incident on the surface of XY. All the secondary waves travelling in a direction parallel to OP come to focus at P. Therefore, P corresponds to the position of the central bright maximum.

In this case, the diffraction pattern has to be considered in two parts; (*i*) the interference phenomenon due to the secondary waves emanating from th corresponding points of the two slits and (*ii*) the diffraction pattern due to the secondary waves from the two slits individually. For calculating

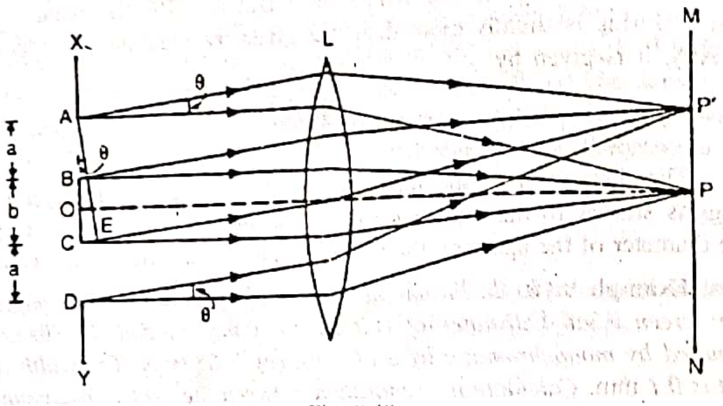


Fig. 9.40

the positions of interference maxima and minima, the diffracting angle is denoted as θ and for the diffraction maxima and minima it is denoted as ϕ. Both the angles θ and ϕ refer to the angle between the direction of the secondary waves and the initial direction of the incident light.

(*i*) **Interference maxima and minima.** Consider the secondary waves travelling in a direction inclined at an angle θ with the initial direction.

In the Δ *ACN* (Fig. 9.41)

$$\sin\theta = \frac{CN}{AC} = \frac{CN}{a+b}$$

or

$$CN = (a+b)\sin\theta$$

If this path difference is equal to odd multiples of $\frac{\lambda}{2}$, θ gives the direction of minima due to interference of the secondary waves from the two slits.

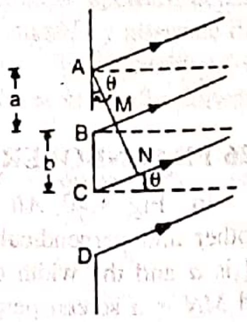


Fig. 9.41

$$\therefore \quad CN = (a+b)\sin\theta_n = (2n+1)\frac{\lambda}{2} \quad ...(i)$$

Putting n = 1, 2, 3 etc., the values of $\theta_1, \theta_2, \theta_3$ etc., corresponding of minima can be obtained.

From equation (*i*)

$$\sin\theta_n = \frac{(2n+1)\lambda}{2(a+b)} \quad ...(ii)$$

On the other hand, if the secondary waves travel in a direction θ' such that the path difference is even multiples of $\frac{\lambda}{2}$, then θ' gives the direction of the maxima due to interference of light waves emanating from the two slits.

$$\therefore \quad CN = (a+b)\sin\theta'_n = 2n.\frac{\lambda}{2}$$

or

$$\sin\theta'_n = \frac{n\lambda}{(a+b)} \quad ...(iii)$$

Putting n = 1, 2, 3 etc., the values $\theta'_1, \theta'_2, \theta'_3$ etc., corresponding to the directions of the maxima can be obtained.

From equation (*ii*)

$$\sin\theta_1 = \frac{3\lambda}{2(a+b)}$$

and

$$\sin\theta_2 = \frac{5\lambda}{2(a+b)}$$

$$\therefore \quad \sin\theta_2 - \sin\theta_1 = \frac{\lambda}{a+b} \quad ...(iv)$$

Thus, the angular separation between any two consecutive minima (or maxima) is equal to $\frac{\lambda}{a+b}$. The angular separation is inversely proportional to $(a+b)$, the distance between the two slits.

(*ii*) **Diffraction maxima and minima.** Consider the secondary waves travelling in a direction inclined at an angle ϕ with the initial direction of the incident light.

If the path difference *BM* is equal to λ the wavelength of light used then ϕ will give the direction of diffraction minimum (Fig. 9.41). That is, the path difference between the secondary waves emanating from the extremities of a slit (*i.e.*, points *A* and *B*) is equal to λ. Considering the wavefront on *AB* to be made up of two halves, the path difference between the corresponding points of the upper and the lower halves is equal to $\frac{\lambda}{2}$. The effect at *P'* due to the wavefront incident on *AB* is zero. Similarly for the same direction of the secondary waves, the effect at P' due to the wavefront incident on th slit CD is also zero. In general,

$$a \sin \phi_n = n\lambda \qquad ...(v)$$

Putting $n = 1, 2$ etc., the values of ϕ_1, ϕ_2 etc., corresponding to the directions of diffraction minima can be obtained.

9.27 FRAUNHOFER DIFFRACTION AT DOUBLE SLIT CALCULUS METHOD)

The intensity distribution due to Fraunhofer diffraction at double slit (two parallel slits) can be obtained by integrating the expression for dy (vide single slit) for both the slits.

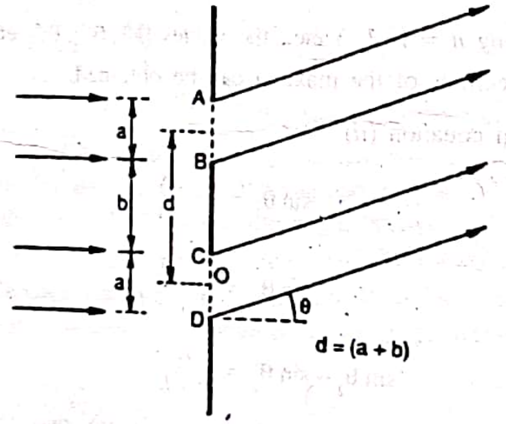


Fig. 9.42

Here

$$y = K\left[\int_{-\frac{a}{2}}^{+\frac{a}{2}} \sin\left[2\pi\left(\frac{t}{T} - \frac{r}{\lambda} + \frac{z \sin\theta}{\lambda}\right)\right] dz + \int_{d-\frac{a}{2}}^{d+\frac{a}{2}} \sin\left[2\pi\left(\frac{t}{T} - \frac{r}{\lambda} + \frac{z\sin\theta}{\lambda}\right)\right] dz\right]$$

$$\therefore \quad y = Ka\left(\frac{\sin\alpha}{\alpha}\right)\left[\sin 2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right)\right.$$

$$\left. - \frac{K\lambda}{2\pi \sin\theta}\left[\cos 2\pi\left(\frac{t}{T} - \frac{r}{\lambda} + \frac{z\sin\theta}{\lambda}\right)\right]_{d-\frac{a}{2}}^{d+\frac{a}{2}}\right]$$

$$y = Ka\left(\frac{\sin\alpha}{\alpha}\right)\sin 2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right) - \frac{K\lambda}{2\pi\sin\theta}\left[\cos 2\pi\left(\frac{t}{T} - \frac{r}{\lambda} + \frac{d\sin\theta}{\lambda} + \frac{a\sin\theta}{2\lambda}\right)\right.$$

$$\left. - \cos 2\pi\left(\frac{t}{T} - \frac{r}{\lambda} + \frac{d\sin\theta}{\lambda} - \frac{a\sin\theta}{2\lambda}\right)\right]$$

$$y = Ka\left(\frac{\sin\alpha}{\alpha}\right)\sin 2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right)$$

$$+ \frac{K\lambda}{\pi\sin\theta}\left[\sin 2\pi\left(\frac{t}{T} - \frac{r}{\lambda} + \frac{d\sin\theta}{\lambda}\right)\sin\left(\frac{\pi a\sin\theta}{\lambda}\right)\right]$$

But $\alpha = \dfrac{\pi a \sin\theta}{\lambda}$

$$\therefore \quad y = Ka\left(\frac{\sin\alpha}{\alpha}\right)\left[\sin 2\pi\left(\frac{t}{T} - \frac{r}{\lambda}\right) + \sin 2\pi\left(\frac{t}{T} - \frac{r}{\lambda} + \frac{d\sin\theta}{\lambda}\right)\right]$$

$$y = 2Ka\left(\frac{\sin\alpha}{\alpha}\right)\sin 2\pi\left(\frac{t}{T} - \frac{r}{\lambda} + \frac{d\sin\theta}{2\lambda}\right)\cos\frac{\pi d\sin\theta}{\lambda}$$

Let $\dfrac{\pi d\sin\theta}{\lambda} = \beta$

$$\therefore \quad y = 2Ka\left(\frac{\sin\alpha}{\alpha}\right)\cos\beta \sin 2\pi\left[\frac{t}{T} - \frac{r}{\lambda} + \frac{d\sin\theta}{2\lambda}\right]$$

The intensity at a point P' is given by

$$I = 4K^2a^2\left(\frac{\sin^2\alpha}{\alpha^2}\right)\cos^2\beta$$

$$\therefore \quad I_0 = K^2a^2$$

$$\therefore \quad I = 4I_0\left(\frac{\sin^2\alpha}{\alpha^2}\right)\cos^2\beta$$

The intensity of the central maximum = $4I_0$ when $\alpha = 0$ and $\beta = 0$.

In Fig. 9.43, the dotted curve represents the intensity distribution due to diffraction pattern due to double slit and the thick line curve represents the intensity distribution due to interference between the light from both the slits. The pattern consists of diffraction maxima within each diffraction maximum.

The intensity distribution due to Fraunhofer diffraction at two parallel slits is shown in Fig. 9.43. The full line represents equally spaced interference maxima and minima and the dotted curve represents the diffraction maxima and minima. In the region originally occupied by the

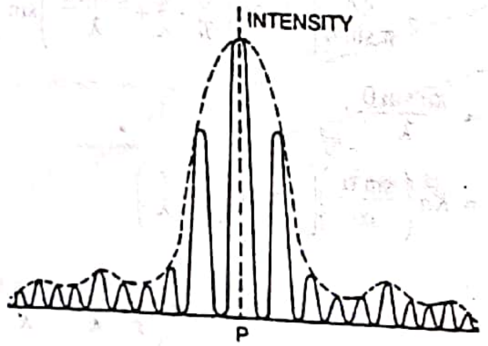


Fig. 9.43

central maximum of the single slit diffraction pattern, equally spaced interference maxima and minima are observed. The intensity of the central interference maximum is four times the intensity of the central maximum of the single slit diffraction pattern. The intensity of the other interference maxima on the two sides of the central maximum of the single slit diffraction pattern. The intensity of the other interference maxima on the two sides of the central maximum gradually decreases. In the region of the secondary maxima due to diffraction at a single slit, equally spaced interference maxima of low intensity are observed. The intensity

distribution shown in Fig. 9.43 corresponds to $2a = b$ where a is the width of each slit and b is the opaque spacing between the two slits. Thus, the pattern due to diffraction at a double slit consists of a diffraction pattern due to the individual slits of width a each and the interference maxima and minima of equal spacing. The spacing of the interference maxima and minima is dependent on the values of a and b.

9.28 DISTINCTION BETWEEN SINGLE SLIT AND DOUBLE SLIT DIFFRACTION PATTERNS

The single slit diffraction pattern consists of a central bright maximum with secondary maxima and minima of gradually decreasing intensity. The double slit diffraction pattern consists of equally spaced interference maxima and minima within the central maximum. The intensity of the central maximum in the diffraction pattern due to a double slit is four times that of the central maximum due to diffraction at a single slit. In the above arrangement, if one of the slits is covered with an opaque screen, the pattern observed is similar to the one observed with a single slit. The spacing of the diffraction maxima and minima depends on a, the width of the slit and the spacing of the interference maxima and minima depends on the value of a and b where b is the opaque spacing between the two slits. The intensities of the interference maxima are not constant but decrease to zero on either side of the central maximum. These maxima reappear two or three times before the intensity becomes too low to be observed.

9.29 MISSING ORDERS IN A DOUBLE SLIT DIFFRACTION PATTERN

In the diffraction pattern due to a double slit discussed earlier, the slit width is taken as a and the separation between the slits as b. If the slit width a is kept constant, the diffraction pattern remains the same. Keeping a constant, if the spacing b is altered the spacing between the interference maxima changes. Depending on the relative values of a and b certain orders of interference maxima will be missing in the resultant pattern.

The directions of interference maxima are given by the equation

$$(a+b)\sin\theta = n\lambda \qquad ...(i)$$

The direction of diffraction minima are given by the equation,

$$a\sin\theta = p\lambda \qquad ...(ii)$$

In equations (i) and (ii) n and p are integers. If the values of a and b are such that both the equations are satisfied simultaneously for the same values of θ, then the positions of certain interference maxima correspond to the diffraction minima at the same position on the screen.

(i) Let $a = b$

Then $2a \sin\theta = n\lambda$

and $a \sin\theta = p\lambda$

$\therefore \frac{n}{p} = 2$

or $n = 2p$

If $p = 1, 2, 3$ etc.,

then $n = 2, 4, 6$ etc.

Thus, the orders 2, 4, 6 etc. of the interference maxima will be missing in the diffraction pattern. There will be three interference maxima in the central diffraction maximum

(ii) if $2a = b$

then $3a \sin\theta = n\lambda$

and $a \sin\theta = p\lambda$

$\therefore \frac{n}{p} = 3$

or $n = 3p$

If $p = 1, 2, 3$ etc.

$n = 3, 6, 9$ etc.

Thus the orders 3, 6, 9 etc. of the interference maxima will be missing in the diffraction pattern. On both sides of the central maximum, the number of interference maxima is 2 and hence there will be five interference maxima in the central diffraction maximum. The position of the third interference maximum also corresponds to the first diffraction minimum.

(iii) If $a + b = a$

i.e., if $b = 0$

The two slits join and all the orders of the interference maxima will be missing. The diffraction pattern observed on the screen is similar to that due to a single slit of width equal to $2a$.

Example 9.17. *Deduce the missing orders for a double slit Fraunhofer diffraction pattern, if the slit widths are 0.16 mm and they are 0.8 mm apart.* [*Berhampur (Hons.)*]

The direction of interference maxima are given by the equation,

$$(a+b)\sin\theta = n\lambda \quad ...(i)$$

The directions of diffraction minima are given by

$$a \sin\theta = p\lambda \quad ...(ii)$$

$$\therefore \frac{(a+b)}{a} = \frac{n}{p}$$

Here $a = 0.16$ mm $= 0.016$ cm

$b = 0.8$ mm $= 0.080$ cm

$$\therefore \frac{0.016 + 0.080}{0.016} = \frac{n}{p}$$

$$\therefore \frac{n}{p} = 6$$

$n = 6p$

For values of $p = 1, 2, 3$ etc.

$n = 6, 12, 18$ etc.

Thus the orders 6, 12, 18 etc. of the interference maxima will be missing in the diffraction pattern.

Example 9.18. *A diffraction phenomenon is observed using a double slit (illuminated with light of wavelength 5000 Å. The slit width is 0.02 mm and spacing between the two slits is 0.10 mm. The distance of the screen from the slits where the observation is made is 100 cm. Calculate (i) the distance between the central maximum and the first minimum of the fringe envelope and (ii) the distance between any two consecutive double slit dark fringes.* [*IAS.*]

Here $a = 0.02$ mm $= 2 \times 10^{-5}$ m

$b = 0.1$ mm $= 10^{-4}$ m

$(a + b) = 1.2 \times 10^{-4}$ m

$\lambda = 5000$ Å $= 5 \times 10^{-7}$ m

$d = 100$ cm $= 1$ m

(i) The angular separation between the central maximum and the first minimum is

$$\sin\theta_1 = \theta_1 = \frac{\lambda}{2(a+b)}$$

and $$\theta_1 = \frac{x_1}{D}$$

$$\frac{x_1}{D} = \frac{\lambda}{2(a+b)}$$

$$x_1 = \frac{\lambda D}{2(a+b)}$$

$$x_1 = \frac{5\times10^{-7}\times1}{2(1.2\times10^{-4})}$$

$$x_1 = 2.08\times10^{-3}\ \text{m}$$

$$x_1 = 2.08\ \text{mm}$$

The distance between the central maximum and the first minimum is **2.08 mm,**

(*ii*) The angular separation between two consecutive dark fringes,

$$\sin\theta_2 - \sin\theta_1 = \theta_2 - \theta_1 = \theta = \frac{3\lambda}{2(a+b)} - \frac{\lambda}{2(a+b)}$$

$$\theta = \frac{\lambda}{(a+b)}$$

Also
$$\theta = \frac{x_2}{D} = \frac{\lambda}{(a+b)}$$

$$x_2 = \frac{\lambda D}{(a+b)}$$

$$x_2 = \frac{5\times10^{-7}\times1}{1.2\times10^{-4}}$$

$$x_2 = 4.16\times10^{-3}\ \text{m}$$

$$x_2 = 4.16\ \text{mm}$$

Example 9.19. *In double slit Fraunhofer diffraction, calculate the fringe spacing on a screen 50 cm away from the slits; if they are illuminated with blue light (λ = 4800 Å). Slit separation b = 0.1 mm and slit width a = 0.020 mm.*

What is the linear distance from the central maximum to the first minimum of the fringe envelope ? [*IAS, 1989*]

Here
$$a = 0.02\ \text{mm} = 2\times10^{-5}\ \text{m}$$
$$b = 0.1\ \text{mm} = 10^{-4}\ \text{m}$$
$$(a+b) = 1.2\times10^{-4}\ \text{m}$$
$$\lambda = 4800\ \text{Å} = 4.8\times10^{-7}\ \text{m}$$
$$D = 50\ \text{cm} = 0.5\ \text{m}$$

(*i*) The angular separation between two consecutive fringes

$$\sin\theta_2 - \sin\theta_1 = \theta_2 - \theta_1 = \theta = \frac{3\lambda}{2(a+b)} - \frac{\lambda}{2(a+b)}$$

$$\theta = \frac{\lambda}{(a+b)}$$

Also
$$\theta = \frac{x}{D} = \frac{\lambda}{(a+b)}$$

$$x = \frac{\lambda D}{(a+b)}$$

$$x = \frac{4.8\times10^{-7}\times0.5}{1.2\times10^{-4}}$$

$$x = 2\times10^{-3}\ \text{m}$$

$$x = 2\ \text{mm}$$

Fringe spacing on the screen = 2mm

(*ii*) The angular separation between the central maximum and the first minimum is

$$\sin\theta_1 = \theta = \frac{\lambda}{2(a+b)}$$

$$\theta = \frac{x_1}{D} = \frac{\lambda}{2(a+b)}$$

$$x_1 = \frac{D\lambda}{2(a+b)}$$

$$x_1 = \frac{0.5\times4.8\times10^{-7}}{2\times1.2\times10^{-4}}$$

$$x_1 = 10^{-3}\ \text{m}$$

$$x_1 = 1\ \text{mm}$$

The distance between the central maximum and the first minimum is **1 mm.**

9.30 INTERFERENCE AND DIFFRACTION

It is clear from the double slit diffraction pattern that interference takes place between the secondary waves originating from the corresponding points of the two slits and also that the intensity of the interference maxima and minima is controlled by the amount of light reaching the screen due to diffraction at the individual slits. The resultant intensity at any point on the screen is obtained by multiplying the intensity function for the interference and the intensity function for the diffraction at the two slits. The values of the intensity functions are taken for the same direction of the secondary waves. But the interference of all the secondary waves originating from the whole wavefront is termed as diffraction. Hence the pattern obtained on the screen may be called an interference pattern or a diffraction pattern. The term interference may be used for those cases in which the resultant amplitude at a point is obtained by the superimposition of two or more beams. Diffraction can be defined as the phenomenon in which the resultant amplitude at a point on the screen is obtained by integrating the effect of infinitesimally small number of elements into which the whole wavefront can be divided. Thus, the resultant diffraction pattern obtained with a double slit can be taken as a combination of the effect of both interference and diffraction.

9.31 FRAUNHOFER DIFFRACTION AT *N* SLITS

Fraunhofer diffraction at two slits consists of diffraction maxima and minima governed by

$$\frac{\sin^2 \alpha}{\alpha^2}$$

and sharp interference maxima and minima, in each diffraction maximum governed by the $\cos^2 \beta$ term.

To derive an expression for the intensity distribution due to diffraction at N slits, the expression for dy has to be integrated for N slits.

For a single slit,

$$dy = K\int_{-\frac{a}{2}}^{+\frac{a}{2}} \sin\left[2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{z\sin\theta}{\lambda}\right)\right]dz$$

Let $\quad \sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{z\sin\theta}{\lambda}\right)$ be equal to $\phi(z)$ (*i.e.* function of z)

For N slits

$$dy = K\left[\int_{-\frac{a}{2}}^{+\frac{a}{2}} \phi(z)\,dz + \int_{d-\frac{a}{2}}^{d+\frac{a}{2}} \phi(z)\,dz + \int_{2d-\frac{a}{2}}^{2d+\frac{a}{2}} \phi(z)\,dz + \ldots + \int_{(N-1)d-\frac{a}{2}}^{(N-1)d+\frac{a}{2}} \phi(z)\,dz\right]$$

On simplification

$$y = Ka\frac{\sin\alpha}{\alpha}\left[\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}\right)+\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{d\sin\theta}{\lambda}\right)\right.$$
$$+\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{2d\sin\theta}{\lambda}\right)+\ldots\ldots$$
$$\left.\ldots\ldots+\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{(N-1)\,d\sin\theta}{\lambda}\right)\right]$$

Here $\quad \alpha = \frac{\pi a\sin\theta}{\lambda}$

For a general trigonometric summation

$$\sum_{p=0}^{p=n}\sin(x+pm) = \frac{\sin\left(x+\frac{nm}{2}\right)\sin\left[\left(\frac{n+1}{2}\right)m\right]}{\sin\left(\frac{m}{2}\right)}$$

Here $\quad x = 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}\right)$

and $\quad m = \frac{2\pi d\sin\theta}{\lambda}$

$$= 2\left[\frac{\pi d\sin\theta}{\lambda}\right] = 2\beta$$

$$\beta = \frac{\pi d\sin\theta}{\lambda}$$

$$n = (N-1)$$

$$y = \frac{Ka\left(\frac{\sin\alpha}{\alpha}\right)\left[\sin\left(x+\frac{(N-1)m}{2}\right)\sin\left(\frac{Nm}{2}\right)\right]}{\sin\left(\frac{m}{2}\right)}$$

$$y = Ka\left(\frac{\sin\alpha}{\alpha}\right)\frac{\sin\left(\frac{Nm}{2}\right)}{\sin\left(\frac{m}{2}\right)}\left[\sin\left(x+\frac{(N-1)\,m}{2}\right)\right]$$

$$y = Ka\left(\frac{\sin\alpha}{\alpha}\right)\frac{(\sin N\beta)}{\sin\beta}$$

$$\left[\sin 2\pi\left(\frac{t}{T}-\frac{r}{\lambda}+\frac{(N-1)\,d\sin\theta}{2\lambda}\right)\right] \quad ...(i)$$

The intensity at a point P'

$$I = K^2a^2\left(\frac{\sin^2\alpha}{\alpha^2}\right)\left(\frac{\sin^2 N\beta}{\sin^2\beta}\right) \quad ...(ii)$$

The maximum intensity, when $\alpha = 0$ and $\beta = 0$

$$I_0 = K^2a^2$$

$$\therefore \quad I = I_0\left(\frac{\sin^2\alpha}{\alpha^2}\right)\left(\frac{\sin^2 N\beta}{\sin^2\beta}\right) \quad ...(iii)$$

Since the expression $\frac{\sin^2\alpha}{\alpha^2}$ represents the diffraction pattern due to a single slit. The additional factor $\frac{\sin^2 N\beta}{\sin^2\beta}$ represents the interference effects due to the secondary waves from the N slits.

The numerator will be zero when

$$N\beta = 0, \pi, 2\pi, 3\pi \ldots \text{etc} = k\pi$$

The denominator is also zero when

$$\beta = 0, \pi, 2\pi, 3\pi \ldots \text{etc.}$$

Since the quotient $\frac{0}{0}$ is indeterminate, therefore $N\beta = k\pi$ gives the condition for minimum intensity for all values of k other than

$$k = 0, N, 2N, 3N \text{ etc.}$$

The directions of principal maxima correspond to the values of $k = 0, N, 2N$ etc.

$$\therefore \quad N\beta = \frac{N\pi d\sin\theta}{\lambda}$$

or

$$k\pi = \frac{N\pi d\sin\theta}{\lambda}$$

For the directions of principal maxima,

$$k = 0, 1N, 2N, 3N \text{ etc.} = nN$$

When $\quad n = 0, 1, 2, \ldots 3$ etc.

$$nN\pi = \frac{N\pi d\sin\theta}{\lambda}$$

$$d\sin\theta = n\lambda$$

Here $\quad n = 0, 1, 2, 3$ etc.

If the width of the slit is a and the width of the opaque spacing is b.

$$d = (a+b)$$

$$(a+b)\sin\theta = n\lambda$$

Putting $n = 1, 2, 3$ etc., the directions of principal maxima $\theta_1, \theta_2, \theta_3 \ldots\ldots$ etc. can be determined.

For values of k in between 0 and N, between N and $2N$, etc., there are $(N - 1)$ secondary minima and $(N - 2)$ secondary maxima.

The intensity distribution due to diffraction and N slits is shown in Fig. 9.46.

9.32 INTENSITY OF PRINCIPAL MAXIMA

In a diffraction grating there are about 6000 narrow slits in one cm. For values of $\beta = k\pi$ and $\beta = 0, \pi, 2\pi$ etc.

$$\frac{\sin N\beta}{\sin\beta} = \frac{0}{0}$$

It is indeterminate.

To find the value of this limit, the numerator and the denominator are differentiated

$$\therefore \quad \underset{\beta\to n\pi}{\mathrm{Lt}}\frac{\sin N\beta}{\sin\beta} = \underset{\beta\to n\pi}{\mathrm{Lt}}\frac{\frac{d}{d\beta}(\sin N\beta)}{\frac{d}{d\beta}(\sin\beta)}$$

$$= \underset{\beta\to n\pi}{\mathrm{Lt}}\frac{N\cos N\beta}{\cos\beta} = \pm N$$

Thus, the resultant amplitude is proportional to N and resultant intensity is proportional to N^2.

$$\therefore \qquad I = N^2 I_0 \left(\frac{\sin^2 \alpha}{\alpha^2}\right)$$

These maxima are intense and are called principal maxima.

9.33 PLANE DIFFRACTION GRATING

A diffraction grating is an extremely useful device and in one of its forms it consists of a very large number of narrow slits side by side. The slits are separated by opaque spaces. When a wavefront is incident on a grating surface, light is transmitted through the slits and obstructed by the opaque portions. Such a grating is called a transmission grating. The secondary waves from the positions of the slits interfere with one another, similar to the interference of waves in Young's experiment. Joseph Fraunhofer used the first gratings which consisted of a large number of parallel fine wires stretched on a frame. Now, gratings are prepared by ruling equidistant parallel lines on a glass surface. The lines are drawn with a fine diamond point. The space in between any two lines is transparent to light and the lined portion is opaque to light. Such surfaces act as transmission gratings. If, on the other hand, the lines are drawn on a silvered surface (plane or concave) then light is reflected from the positions of the mirror in between any two lines and such surfaces act as reflection gratings.

If the spacing between the lines is of the order of the wave length of light, then an appreciable deviation of the light is produced. Gratings used for the study of the visible region of the spectrum contain 10,000 lines per cm. Gratings, with originally ruled surfaces are only few. For practical purposes, replicas of the original grating are prepared. On the original grating surface a thin layer of collodion solution is poured and the solution is allowed to harden. Then, the film of collodion is removed from the grating surface and then fixed between two glass plates. This serves as a plane transmission grating. A large number of replicas are prepared in this way from a single original ruled surface.

9.34 THEORY OF THE PLANE TRANSMISSION GRATING

In Fig. 9.44, XY is the grating surface and MN is the screen, both perpendicular to the plane of the paper. The slits are all parallel to one another and perpendicular to the plane of the paper. Here AB is a slit and BC is an opaque portion. The width of each slit is a and the opaque spacing between any two consecutive slits is b. Let a plane wavefront be incident on the grating surface. Then all the secondary waves travelling in the same direction as that of the incident light will come to focus at the

point P on the screen. The screen is placed at the focal plane of the collecting lens. The point P where all the secondary waves reinforce one another corresponds to the position of the central bright maximum.

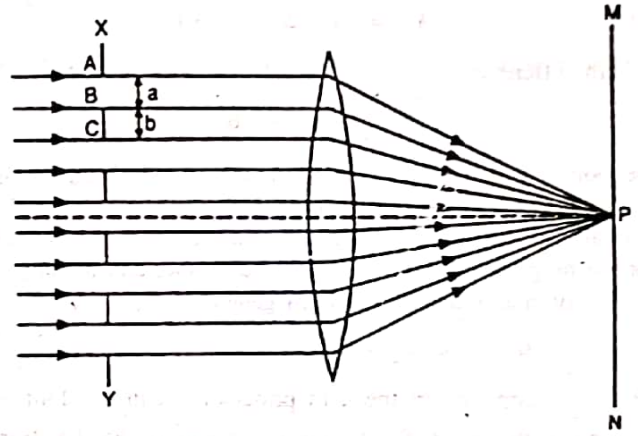


Fig. 9.44

Now, consider the secondary waves travelling in a direction inclined at an angle θ with the direction of the incident light (Fig. 9.45). The collecting lens also is suitably rotated such that the axis of the lens is

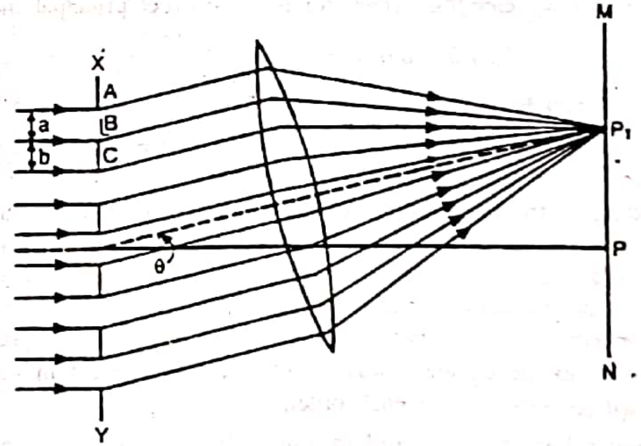


Fig. 9.45

parallel to the direction of the secondary waves. These secondary waves come to focus at the point P_1 on the screen. The intensity at P_1 will depend on the path difference between the secondary waves originating from the corresponding points A and C of two neighbouring slits. In Fig. 9.45, $AB = a$ and $BC = b$. The path difference between the secondary waves starting from A and C is equal to $AC \sin\theta$. (This will be clear from Fig. 9.41).

But $\quad AC = AB + BC = a + b$

$\therefore$ Path difference $= AC \sin\theta$

$= (a+b)\sin\theta$

The point P_1 will be of maximum intensity if this path difference is equal to integral multiples of λ where λ is the wavelength of light. In this case, all the secondary waves originating from the corresponding points of the neighbouring slits reinforce one another and the angle θ gives the direction of maximum intensity. In general

$$(a+b)\sin\theta_n = n\lambda \qquad \text{...}(i)$$

where θ_n is the direction of the n th principal maximum. Putting $n = 1$, 2, 3 etc., the angles $\theta_1, \theta_2, \theta_3$ etc. corresponding to the directions of the principal maxima can be obtained.

If the incident light consists of more than one wavelength, the beam gets dispersed and the angles of diffraction for different wavelengths will be different. Let λ and $\lambda + d\lambda$ be two nearby wavelengths present in the incident light and θ and $(\theta + d\theta)$ be the angles of diffraction corresponding to these two wavelengths. Then, for the first order principal maxima

$$(a+b)\sin\theta = \lambda$$

and $$(a+b)\sin(\theta + d\theta) = \lambda + d\lambda$$

Thus, is any order, the number of principal maxima corresponds to the number of wavelengths present. A number of parallel slit images corresponding to the different wavelengths will be observed on the screen. In equation (i), $n = 1$ gives the direction of the first order image, $n = 2$ gives the direction of the second order image and so on. When white light is used, the diffraction pattern on the screen consists of a white central bright maximum and on both the sides of this maximum a spectrum corresponding to the different wavelengths of light present in the incident beam will be observed in each order.

Secondary maxima and minima. The angle of diffracting θ_n corresponding to the direction of the n th principal maximum is given by the equation

$$(a+b)\sin\theta_n = n\lambda$$

In this equation, $(a+b)$ is called the **grating element.** Here a is the width of the slit and b is the width of the opaque portion. For a grating with 15,000 lines per inch the value of

$$(a+b) = \frac{2.54}{15000} \text{ cm}$$

Now, let the angle of diffraction be increased by a small amount $d\theta$ such that the path difference between the secondary waves from the points A and C (Fig. 9.45) increases by $\frac{\lambda}{N}$. Here N is the total number of lines on the grating surface. Then, the path difference between the secondary waves from the extreme points of the grating surface will be $\frac{\lambda}{N} \cdot N = \lambda$. Assuming the whole wavefront to be divided into two halves, the path difference between the corresponding points of the two halves will be $\frac{\lambda}{2}$ and all the secondary waves cancel one another's effect. Thus, $(\theta_n + d\theta)$ will give the direction of the first secondary minimum after the

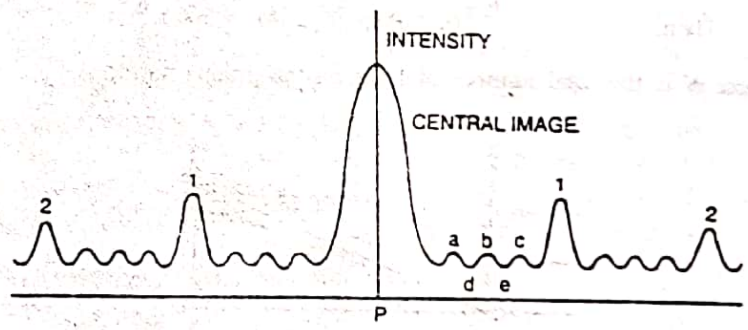


Fig. 9.46

n th primary maximum. Similarly, if the path difference between the secondary waves from the points A and C is $\frac{2\lambda}{N}, \frac{3\lambda}{N}$ etc. for gradually increasing values of $d\theta$, these angles correspond to the directions of 2 nd, 3 rd etc. secondary minima after the n th primary maximum. If the value is $\frac{2\lambda}{N}$, then the path difference between the secondary waves from the extreme points of the grating surface is $\frac{2\lambda}{N} \times N = 2\lambda$ and considering the wavefront to be divided into 4 portions, the concept of the 2 nd secondary minimum can be understood. The number of secondary minima in between any two primary maxima is $N - 1$ and the number of secondary maxima is $N - 2$.

The intensity distribution of the screen is shown in Fig. 9.46. P corresponds to the position of the central maxima and 1, 2 etc. on the two sides of P represent the 1st, 2nd etc. principal maxima. a, b, c etc. are secondary maxima and d, e etc. are the secondary minima. The intensity as well as the angular spacing of the secondary maxima and minima are so small in comparison to the principal maxima that they cannot be observed. It results in uniform darkness between any two principal maxima.

9.35 WIDTH OF PRINCIPAL MAXIMA

The direction of the n th principal maximum is given by

$$(a+b)\sin\theta_n = n\lambda \qquad ...(i)$$

Let $\theta_n + d\theta$ and $\theta_n - d\theta$ give the directions of the first secondary minima on the two sides of the n th primary maxima (Fig. 9.47).

Then.
$$(a+b)\sin[\theta_n \pm d\theta] = n\lambda \pm \frac{\lambda}{N} \qquad ...(ii)$$

where N is the total number of lines on the grating surface.

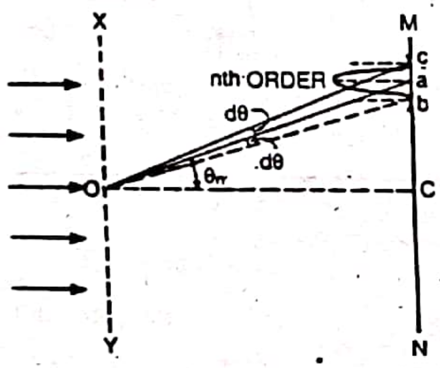


Fig. 9.47

Dividing (ii) by (i)

$$\frac{(a+b)\sin(\theta_n \pm d\theta)}{(a+b)\sin\theta_n} = \frac{n\lambda \pm \frac{\lambda}{N}}{n\lambda}$$

$$\frac{\sin(\theta_n \pm d\theta)}{\sin\theta_n} = 1 \pm \frac{1}{Nn}$$

Expanding this equation

$$\frac{\sin\theta_n \cdot \cos d\theta \pm \cos\theta_n \sin d\theta}{\sin\theta_n} = 1 \pm \frac{1}{Nn} \qquad ...(iii)$$

For small values of $d\theta$; $\cos d\theta = 1$ and $\sin d\theta = d\theta$.

$$\therefore \quad 1 \pm \cot\theta_n \, d\theta = 1 \pm \frac{1}{Nn} \qquad ...(iv)$$

$$\cot\theta_n \, d\theta = \frac{1}{Nn}$$

$$d\theta = \frac{1}{Nn\cot\theta_n}$$

In equation (iv), $d\theta$ refers to half the angular width of the principal maximum. The half width $d\theta$ is (i) inversely proportional to N the total number of lines and (ii) inversely proportional to $n\cot\theta_n$. The value of $n\cot\theta_n$ is more for higher orders because the increase in the value of $\cot\theta_n$ is less than the increase in the order. Thus, the half width of the principal maximum is less for higher orders. Also, the larger the number of lines on the grating surface, the smaller is the value of $d\theta$. Further, the value of θ_n is higher for longer wavelengths and hence the spectral lines are more sharp towards the violet than the red end of the spectrum.

9.36 OBLIQUE INCIDENCE

Let a parallel beam of light be incident obliquely on the grating surface at an angle of incidence i (Fig. 9.48).

Path difference between the secondary waves passing through the points A and $C = FC + CE$.

Here, $AB = a$, the width of the slit and $BC = b$, the width of the opaque portion.

From the $\Delta\, AFC$

$FC = (a+b)\sin i$ and from $\Delta\, ACE$

$$CE = (a+b)\sin\theta$$

$$\therefore \quad FC + CE = (a+b)(\sin\theta + \sin i) \qquad ...(i)$$

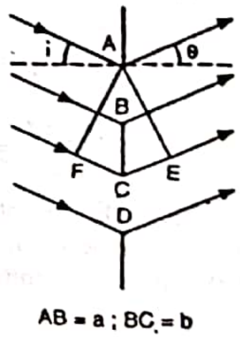


Fig. 9.48

Equation (*i*) holds good if the beam is diffracted upwards. Fig. 9.49 illustrates the diffraction of the beam downwards. In this case the path difference

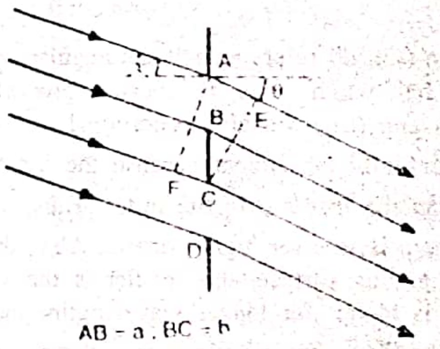
$$= (a+b)\,[\sin\theta - \sin i] \qquad ...(ii)$$

For the n th primary maximum

$$(a+b)\,[\sin\theta_n + \sin i] = n\lambda \qquad ...(iii)$$

or
$$(a+b)\left[2\cdot\sin\frac{\theta_n+i}{2}\cdot\cos\frac{\theta_n-i}{2}\right] = n\lambda$$

or
$$\sin\frac{\theta_n+i}{2} = \frac{n\lambda}{2\,(a+b)\cos\dfrac{\theta_n-i}{2}} \qquad ...(iv)$$



AB = a, BC = b

Fig. 9.49

The deviation of the diffraction beam $= \theta_n + i$

For the deviation $(\theta_n + i)$ to be minimum, $\sin\dfrac{\theta_n+i}{2}$ must be minimum. This is possible if the value of $\cos\dfrac{\theta_n-i}{2}$ is maximum *i.e.*,

$$\frac{\theta_n-i}{2} = 0 \quad \text{or} \quad \theta_n = i$$

Thus, the deviation produced in the diffracted beam is minimum when the angle of incidence is equal to the angle of diffraction. Let D_m be the angle of minimum deviation

Then $D_m = \theta_n + i$

But $\theta_n = i$

$$\therefore \quad \theta_n = \frac{D_m}{2} \quad \text{and} \quad i = \frac{D_m}{2}$$

$$\therefore \quad (a+b)\left(\sin\frac{D_m}{2} + \sin\frac{D_m}{2}\right) = n\lambda$$

or
$$2\,(a+b)\sin\frac{D_m}{2} = n\lambda \qquad ...(v)$$

Equation (*v*) refers to the principal maximum of the n th order for a wavelength λ.

Example 9.20. *A parallel beam of light of wavelength 5460 Å is incident at an angle of 30° on a plane transmission grating which has 6000 lines/cm. Find the highest order spectrum that can be observed.*

[*Delhi (Hons.) 1984*]

$$(a+b)\,[\sin\theta_n + \sin i] = n\lambda$$

Here $\theta_n = i$

$$(a+b)\,(2\sin i) = n\lambda$$

Here $(a+b) = \left(\dfrac{1}{6\times10^5}\right)$ m

$$\lambda = 5460\text{ Å} = 5460\times10^{-10}\text{ m}$$

$$\sin 30 = 0.5$$

$$\therefore \quad n = \frac{(a+b)\,(2\sin i)}{\lambda} = \frac{1}{6\times10^5\times5460\times10^{-10}}$$

$$n = 3.05$$

or $n = 3$

9.37 ABSENT SPECTRA WITH A DIFFRACTION GRATING

In the equation $(a+b)\sin\theta = \lambda$, if $(a+b) < \lambda$ then $\sin\theta > 1$. But this is not possible. Hence the first order spectrum is absent. Similarly, the second, the third etc. order spectra will be absent if $(a+b) < 2\lambda$, $(a+b) < 3\lambda$ etc. In general, if $(a+b) < n\lambda$, then the n th order spectrum will be absent.

The condition for absent spectra can be obtained from the following consideration. For the n th order principal maximum

$$(a+b)\sin\theta_n = n\lambda \qquad ...(i)$$

Further, if the value of a and θ_n are such that

$$a \sin \theta_n = \lambda \quad ...(ii)$$

then, the effect of the wavefront from any particular slit will be zero. Considering each slit to be made up of two halves, the path difference between the secondary waves from the corresponding points will be $\frac{\lambda}{2}$ and they cancel one another's effect. If the two conditions given by equations (i) and (ii) are simultaneously satisfied, then dividing (i) by (ii)

$$\frac{(a+b)\sin\theta_n}{a\sin\theta_n} = \frac{n\lambda}{\lambda}$$

or

$$\frac{a+b}{a} = n \quad ...(iii)$$

In equation (iii), the values of $n = 1, 2, 3$ etc. refer to the order of the principal maxima that are absent in the diffraction pattern.

(*i*) if $\frac{a+b}{a} = 1;\ b = 0$

In this case, the first order spectrum will be absent and the resultant diffraction pattern is similar to that due to single slit.

(*ii*) if $\frac{a+b}{a} = 2;\ a = b$

i.e., the width of the slit is equal to the width of the opaque spacing between any two consecutive slits. It this case, the second order spectrum will be absent.

9.38 OVERLAPPING OF SPECTRAL LINES

If the light incident on the grating surface consists of a large range of wavelengths, then the spectral lines of shorter wavelength and of higher order overlap on the spectral lines of longer wavelength and of lower order. Let the angle of diffraction θ be the same for (*i*) the spectral line of wavelength λ_1 in the first order, (*ii*) the spectral line of wavelength λ_2 in the second order and (*iii*) the spectral line of wavelength λ_3 in the third order. Then

$$(a+b)\sin\theta = 1\cdot\lambda_1 = 2\lambda_2 = 3\lambda_3 = \ldots$$

The red line of wavelength 7000 Å in the third order, the green line of wavelength 5250 Å in the fourth order and the violet line of wavelength 4200 Å in the fifth order are all formed at the same position of the screen because,

$$(a+b)\sin\theta = 3\times 7000\times 10^{-8}$$
$$= 4\times 5250\times 10^{-8}$$
$$= 5\times 4200\times 10^{-8}$$

Here $(a + b)$ is expressed in cm.

For the visible region of the spectrum, there is no overlapping of the spectral lines. The range of wavelengths for the visible part of the spectrum is 4000 Å to 7200 Å. Thus, the diffracting angle for the red end of the spectrum in the first order is less than the diffracting angle for the violet end of the spectrum in the second order. If, however, the observations are made with a photographic plate, the spectrum recorded may extend up to 2000 Å in the ultraviolet region. In this case, the spectral line corresponding to a wavelength of 4000 Å in the first order and a spectral line of wavelength 2000 Å in the second order overlap. Suitable fillers are used to absorb those wavelengths of the incident light which will overlap with the spectral lines in the region under investigation.

9.39 DETERMINATION OF WAVELENGTH OF A SPECTRAL LINE USING PLANE TRANSMISSION GRATING

In the laboratory, the grating spectrum of a given source of light is obtained by using a spectrometer. Initially all the adjustments of the spectrometer are made and it is adjusted for parallel rays by Schuster's method. The slit of the collimator is illuminated by monochromatic light (say light from a sodium lamp) and the position of the telescope is

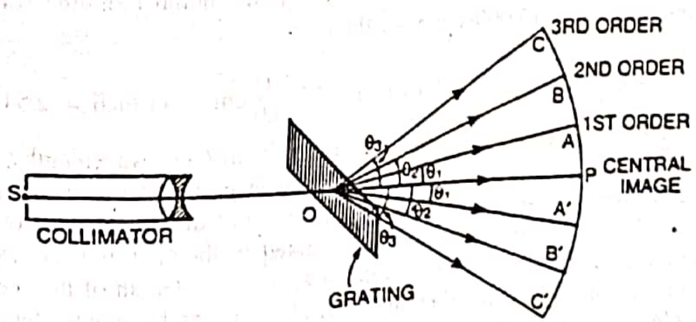


Fig. 9.50

adjusted such that the image of the slit is obtained at the position of the vertical cross-wire in the field of view of the telescope. Now the axes of the collimator and the telescope are in the same line. The position of the telescope is noted on the circular scale and 90° is added to this reading. The telescope is turned to this position. In this position the axis of the telescope is perpendicular to the axis of the collimator. The position of the telescope is fixed. The given transmission grating is mounted at the centre of the prism table such that the grating surface is perpendicular to the prism table. The prism table is suitably rotated such that the image of the slit reflected from the grating surface is obtained in the centre of the field of view of the telescope. This means that the parallel rays of light from the collimator are incident at an angle of 45° on the grating surface because the axes of the collimator and the telescope are perpendicular to each other. The reading of the prism table is noted and adding 45° to this reading the prism table is suitably rotated to the new position so that the grating surface is normal to the incident light.

If the wavelength of sodium light is to be determined, then the angles of diffraction θ_1 and θ_2 corresponding to the first and the second order principal maxima are determined (Fig. 9.50). OA, OB etc., give the directions of the telescope corresponding to the first and second order images. A', B', etc. refer to the positions of these images towards the left of the central maximum. The angles AOA' and BOB' are measured and half of these angles measure θ_1 and θ_2. Then

$$(a+b)\sin\theta_1 = 1\lambda \qquad ...(i)$$

and

$$(a+b)\sin\theta_2 = 2\lambda \qquad ...(ii)$$

Then the value of λ is calculated from equations (*i*) and (*ii*) and the mean value is taken. $(a + b)$ is the grating element and it is equal to the reciprocal of the number of lines per cm. If the number of lines on the grating surface is 15,000 per inch then

$$(a+b) = \frac{2.54}{15000}\text{ cm} \qquad (1\text{ inch} = 2.54\text{ cm})$$

If the source of light emits radiations of different wavelengths, then the beam gets dispersed by the grating and in each order a spectrum of the constituent wavelengths is observed. To find the wavelength of any spectral line, the diffracting angles are noted in the first and the second orders and using the equations given above, the wavelength of the spectral line can be calculated. Overlapping spectral orders can be avoided by using suitable colour filters so that the wavelengths beyond the range of study are eliminated.

With a diffraction grating, the wavelength of a spectral line can be determined very accurately. The method involves only the accurate measurement of the angles of diffraction.

Take $\lambda = 6000\text{ Å} = 6000 \times 10^{-8}$ cm

and $(a+b) = \frac{2.54}{15000}$

Then from the equations,

$$(a+b)\sin\theta_1 = 1\lambda$$

and

$$(a+b)\sin\theta_2 = 2\lambda$$

$$\theta_1 = 20° - 45'$$

and

$$\theta_2 = 45° - 7'$$

As the angles are large they can be measured accurately with a properly calibrated spectrometer. The number of lines per inch (or cm), is given on the grating by the manufacturing company and hence $(a + b)$ can be calculated. As the method does not involve measurements of very small distances (as in the case of interference experiments) an accurate value of λ can be obtained.

Example 9.21. *Light is incident normally on a grating 0.5 cm wide with 2500 lines. Find the angles of diffraction for the principal maxima of the two sodium lines in the first order spectrum.*

$$\lambda_1 = 5890\text{ Å} \quad \textit{and} \quad \lambda_2 = 5896\text{ Å}.$$

Are the two lines resolved ? (*Punjab*)

$$(a+b)\sin\theta_n = n\lambda$$

$$\text{Width} = 0.5\text{ cm}$$

$$\text{Total number of lines} = N = 2500\text{ lines}$$

$$\text{Number of lines / cm} = N' = \frac{2500}{0.5} = 5000\text{ lines / cm}$$

$$\text{Grating element } (a+b) = \frac{1}{N'} = \frac{1}{5000}\text{ cm}$$

(1) For $\lambda_1 = 5890 \times 10^{-8}$ cm, $n = 1$

$$(a+b)\sin\theta_1 = 1 \times \lambda_1$$

$$\frac{1}{5000}\sin\theta_1 = 5890 \times 10^{-8}$$

$$\sin\theta_1 = 0.2945$$

$$\theta_1 = 17.1°$$

(2) For $\lambda_2 = 5896 \times 10^{-8}$ cm, $n = 1$

$$(a+b)\sin\theta'_1 = 1 \times \lambda_2$$

$$\left(\frac{1}{5000}\right)\sin\theta'_1 = 5896 \times 10^{-8}$$

$$\sin\theta'_1 = 0.2948$$

$$\theta_1 = 17.2°$$

(3) The condition for just resolution is

$$\frac{\lambda}{d\lambda} = nN$$

Here $\lambda = 5890 \times 10^{-8}$ cm

$d\lambda = 6 \times 10^{-8}$ cm

$n = 1$

$N = ?$

$$\frac{5890 \times 10^{-8}}{6 \times 10^{-8}} = 1 \times N$$

$$N = 982$$

As the total number of lines on the grating is 2500, the two lines will appear well resolved.

Example 9.22. *A parallel beam of monochromatic light is allowed to be incident normally on a plane grating having 1250 lines per cm and a second order spectral line is observed to be deviated through 30°. Calculate the wavelength of the spectral line.* *(Agra)*

$$(a+b)\sin\theta = n\lambda$$

Here $(a+b) = \frac{1}{1250}$, $\theta = 30°$, $\sin 30 = \frac{1}{2}$

$n = 2$, $\lambda = ?$

$$\lambda = \frac{(a+b)\sin\theta}{n} = \frac{1 \times 1}{1250 \times 2 \times 2}$$

$$= 2 \times 10^{-4} \text{ cm}$$

Example 9.23. *What is the highest order spectrum, which may be seen with monochromatic light of wavelength 6000 Å by means of a diffraction grating with 5000 lines/cm.* *[Delhi B.Sc. (Hons)]*

Here $(a+b)\sin\theta_n = n\lambda$

The maximum possible value of

$$\sin\theta_n = 1$$

$$(a+b) = n\lambda$$

Here $(a+b) = \frac{1}{5000}$ cm

$\lambda = 6000 \times 10^{-8}$ cm, $n = ?$

$$\frac{1}{5000} = n \times 6000 \times 10^{-8}$$

or $n = 3.33$

The highest order of the spectrum that can be seen is 3.

E **Example 9.14.** *A plane grating has 15000 lines per inch. Find the angle of separation of the 5048 Å and 5016 Å lines of helium in the second order spectrum.* *[Delhi B.Sc. (Hons)]*

Here $\lambda_1 = 5016$ Å $= 5016 \times 10^{-8}$ cm

$\lambda_2 = 5048$ Å $= 5048 \times 10^{-8}$ cm

$n = 2$

$$(a+b) = \frac{2.54}{15000} \text{ cm}$$

Let θ_1 and θ_2 be the angles of diffraction for the second order for the wavelengths λ_1 and λ_2 respectively.

$\therefore$ $(a+b)\sin\theta_1 = 2 \times \lambda_1$; $(a+b)\sin\theta_2 = 2 \times \lambda_2$

$$\sin\theta_1 = \frac{2\lambda_1}{(a+b)} = \frac{2 \times 5016 \times 10^{-8} \times 15000}{2.54}$$

$$= 0.5924$$

or $\theta_1 = 36° - 20'$

$$\sin\theta_2 = \frac{2\lambda_2}{a+b} = \frac{2 \times 5048 \times 10^{-8} \times 15000}{2.54}$$

$$= 0.5962$$

or $\theta_2 = 36° - 36'$

$\therefore$ Angle of separation, $(\theta_2 - \theta_1) = 16'$

Example 9.25. *A plane transmission grating having 6000 lines/cm is used to obtain a spectrum of light from a sodium lamp in the second order. Calculate the angular separation between the two sodium lines whose wavelengths are 5890 Å and 5896 Å.* [Bombay]

$$(a+b)\sin\theta_1 = 2\lambda_1$$

$$(a+b)\sin\theta_2 = 2\lambda_2$$

Here

$$a+b = \frac{1}{6000}$$

$$\lambda_1 = 5890 \times 10^{-8} \text{ cm}$$

$$\lambda_2 = 5896 \times 10^{-8} \text{ cm}$$

$$\therefore \quad \sin\theta_1 = 2 \times 5890 \times 10^{-8} \times 6000 = 0.7068$$

or

$$\theta_1 = 44° - 58'$$

and

$$\sin\theta_2 = 2 \times 5896 \times 10^{-8} \times 6000 = 0.7075$$

or

$$\theta_2 = 45° - 2'$$

The angular separation $= \theta_2 - \theta_1$

= 4 minutes of an arc.

Example 9.26. *Light which is a mixture of two wavelengths 5000 Å and 3200 Å is incident normally on a plane transmission grating having 10,000 lines per cm. A lens of focal length 150 cm is used to observe the spectrum on a screen. Calculate the separation in cm of the two lines in the first order spectrum.*

Here

$$a+b = \frac{1}{10000} = 10^{-4} \text{ cm}$$

$$\lambda_1 = 5000 \text{ Å} = 5.0 \times 10^{-5} \text{ cm}$$

$$\lambda_2 = 5200 \text{ Å} = 5.2 \times 10^{-5} \text{ cm}$$

$$n = 1$$

$$(a+b)\sin\theta_1 = n\lambda_1$$

$$\sin\theta_1 = \frac{n\lambda_1}{a+b} = \frac{1 \times 5.0 \times 10^{-5}}{10^{-4}} = 0.5$$

$$\theta_1 = 30°$$

Similarly

$$\sin\theta_2 = \frac{n\lambda_2}{a+b} = \frac{1 \times 5.2 \times 10^{-5}}{10^{-4}} = 0.52$$

or

$$\theta_2 = 31.3°$$

Further

$$\tan\theta_1 = \frac{x_1}{f}$$

and

$$\tan\theta_2 = \frac{x_2}{f}$$

$$\therefore \quad (x_2 - x_1) = f\,[\tan\theta_2 - \tan\theta_1]$$

$$= 150\,[0.6087 - 0.5774] = 150 \times 0.0313$$

$$= \mathbf{4.695 \text{ cm}}$$

Example 9.27. *Light of wavelength 5000 Å is incident normally on a plane transmission grating. Find the difference in the angles of deviation in the first and third order spectra. The number of lines per cm on the grating surface is 6000.*

Here

$$\lambda = 5000 \text{ Å} = 5 \times 10^{-5} \text{ cm}$$

$$(a+b) = \frac{1}{6000}$$

For the first order,

$$(a+b)\sin\theta_1 = 1 \times \lambda$$

$$\sin\theta_1 = \frac{\lambda}{a+b} = 5 \times 10^{-5} \times 6000 = 0.30$$

or

$$\theta_1 = 17.5°$$

For the third order,

$$(a+b)\sin\theta_3 = 3\lambda$$

$$\sin\theta_3 = \frac{3\lambda}{a+b} = 3 \times 5 \times 10^{-5} \times 6000 = 0.90$$

$$\theta_3 = 64.2°$$

$$\theta_3 - \theta_1 = 64.2 - 17.5 = \mathbf{46.7°}$$

Example 9.28. *In a plane transmission grating the angle of diffraction for the second order principal maximum for the wavelength 5×10^{-5} cm is 30°. Calculate the number of lines in one cm of the grating surface.*

(Delhi)

$$(a+b)\sin\theta_n = n\lambda$$

Here $n = 2, \quad \lambda = 5\times10^{-5}\,\text{cm}, \quad \theta_2 = 30°$

$$\sin 30 = 0.5$$

$$(a+b) = \frac{n\lambda}{\sin\theta_2}$$

$$= \frac{2\times5\times10^{-5}}{0.5} = 10^{-3}\,\text{cm}$$

$\therefore$ Number of lines per cm $= N'$

$$= \frac{1}{(a+b)} = \mathbf{1000}$$

Example 9.29. *How many orders will be visible if the wavelength of the incident radiation is 5000 Å and the number of lines on the grating is 2620 in one inch.* [Delhi, 1978]

Here $(a+b)\sin\theta_n = n\lambda$

The maximum possible value of

$$\sin\theta_n = 1$$

$\therefore$ $(a+b) = n\lambda$

Here, $(a+b) = \dfrac{2.54}{2620}$ cm

$$\lambda = 5000\ \text{Å} = 5\times10^{-5}\,\text{cm}$$

$$n = ?$$

$$n = \frac{(a+b)}{\lambda}$$

$$n = \frac{2.54}{2620\times5\times10^{-5}}$$

$$n > 19$$

The highest order of the spectrum that can be seen is **19**.

Example 9.30. *A parallel beam of monochromatic light is allowed to be incident normally on a plane transmission grating having 5000 lines/cm and the second order spectral line is found to be diffracted through 30°. Calculate the wavelength of light.* [Delhi (Sub) 1978]

$$(a+b)\sin\theta = n\lambda$$

Here $(a+b) = \dfrac{1}{5000}$ cm

$$\theta = 30°, \quad \sin 30° = 0.5$$

$$n = 2$$

$$\lambda = ?$$

$$\lambda = \frac{(a+b)\sin\theta}{n}$$

$$= \frac{0.5}{5000\times2}$$

$$\lambda = \mathbf{5\times10^{-5}\,cm}$$

$$\lambda = \mathbf{5000\ \text{Å}}$$

Example 9.31. *A diffraction grating used at normal incidence gives a line, λ_1 = 6000 Å in a certain order superimposed on another line λ_2 = 4500 Å of the next higher order. If the angle of diffraction is 30°, how many lines are there in a cm in the grating.* [Delhi (Hons) 1977]

Here, $\lambda_1 = 6000\ \text{Å} = 6\times10^{-5}\,\text{cm}$

$$\lambda_2 = 4500\ \text{Å} = 4.5\times10^{-5}\,\text{cm}$$

$$\theta_1 = \theta_2 = 30°$$

$\therefore$ $\sin\theta_1 = \sin\theta_2 = 0.5$

$$(a+b)\sin\theta_1 = n_1\lambda_1 \quad ...(i)$$

$$(a+b)\sin\theta_2 = (n_1+1)\lambda_2 \quad ...(ii)$$

Dividing *(ii)* by *(i)*

$$1 = \frac{(n_1+1)}{n_1}\times\frac{4.5\times10^{-5}}{6\times10^{-5}}$$

$\therefore$ $n_1 = 3$

From equation *(i)*

$$(a+b)\times0.5 = 3\times6\times10^{-5}$$

$$(a+b) = 36\times10^{-5}\,\text{cm}$$

The number of lines / cm

$$= \frac{1}{(a+b)}$$

$$= \frac{1}{36 \times 10^{-3}}$$

$$= \mathbf{2778} \text{ lines / cm}$$

Example 9.32. *Light is incident normally on a grating of total ruled width 5×10^{-3} m with 2500 lines in all. Find the angular separation of the sodium lines in the first order spectrum. Wavelengths of lines are 589 and 589.6 nm. Can they be seen distinctly ?* [IAS, 1983]

Number of lines per metre

$$= \frac{2500}{5 \times 10^{-3}} = 5 \times 10^5$$

For $n = 1$

(i) $(a+b) \sin\theta_1 = n\lambda_1$

$$\frac{\sin\theta_1}{5 \times 10^5} = 1 \times 589 \times 10^{-9}$$

$$\sin\theta_1 = 0.2945$$

$$\theta_1 = 17.1275°$$

(ii) $(a+b) \sin\theta_2 = \lambda_2$

$$\therefore \frac{\sin\theta_2}{5 \times 10^5} = 589.6 \times 10^{-9}$$

$$\sin\theta_2 = 0.2948$$

$$\theta_2 = 17.1455$$

$$d\theta = \theta_2 - \theta_1 = 17.1455 - 17.1275 = 0.0180$$

But, as the separation is very small, the two lines cannot be seen distinctly.

Example 9.33. *Parallel been of light is incident normally on a diffraction grating having 6000 line/cm. Find the angular separation between the maxima for wavelengths 5890 Å and 5896 Å in the second order.* [IAS, 1986]

Here $n = 2$

$$\lambda_1 = 5890 \text{ Å} = 5890 \times 10^{-10} \text{ m}$$

$$\lambda_2 = 5896 \text{ Å} = 5896 \times 10^{-10} \text{ m}$$

Number of li[illegible] $= 6000$ per cm

$$= 6 \times 10^5 \text{ per metre}$$

$$\therefore \quad (a+b) = \frac{1}{6 \times 10^5} \text{ metre}$$

(i) $(a+b) \sin\theta_1 = n\lambda_1$

$$\sin\theta_1 = \frac{n\lambda_1}{(a+b)}$$

$$\sin\theta_1 = \frac{2 \times 5890 \times 10^{-10} \times 6 \times 10^5}{1}$$

$$\sin\theta_1 = 0.707$$

$$\theta_1 = \mathbf{45°}$$

(ii) $(a+b) \sin\theta_2 = n\lambda_2$

$$\sin\theta_2 = \frac{n\lambda_2}{(a+b)}$$

$$\sin\theta_2 = \frac{2 \times 5896 \times 10^{-10} \times 6 \times 10^5}{1}$$

$$\sin\theta_2 = 0.7075$$

$$\theta_2 = \mathbf{45° - 2'}$$

Angular separation

$$d\theta = \theta_2 - \theta_1 = \mathbf{2'} \text{ minutes of an arc}$$

Example 9.34. *Monochromatic light of wavelength 6.56×10^{-7} m falls normally on a grating 2 cm wide. The first order spectrum is produced at an angle of $18° - 15'$ from the normal. Deduce the total number of lines on the grating.* [IAS, 1987]

Here $\lambda = 6.56 \times 10^{-7}$ m

$$\text{width} = 2 \text{ cm} = 2 \times 10^{-2} \text{ m}$$

$$n = 1$$

$$\theta_1 = 18° - 15', \ \sin\theta_1 = 0.3131$$

$$(a+b) \sin\theta_1 = n\lambda$$

$$(a+b) = \frac{n\lambda}{\sin\theta_1}$$

$$(a+b) = \frac{1 \times 6.56 \times 10^{-7}}{0.3131}$$

$$(a+b) = 20.95 \times 10^{-7} \text{ m}$$

Number of lines per metre

$$= \frac{1}{(a+b)}$$

$$= \frac{1}{20.95 \times 10^{-7}}$$

$$= 4.77 \times 10^{5} \text{ per metre}$$

Total number of lines,

$$N = 4.773 \times 10^{5} \times 2 \times 10^{-2}$$

$$N = 9.546 \times 10^{3}$$

$$N = 9546$$

Example 9.35. *How many orders will be visible if the wavelength of light is 5000 Å and the number of lines per inch on the grating is 2620 ?* [*Lucknow, 1990*]

Here $(a+b)\sin\theta_n = n\lambda$

The maximum possible value of

$$\sin\theta_n = 1$$

$\therefore$ $(a+b) = n\lambda$

Here $(a+b) = \frac{2.54}{2620}$ cm

$$(a+b) = 9.694 \times 10^{-4} \text{ cm}$$

$$(a+b) = 9.694 \times 10^{-6} \text{ m}$$

$$\lambda = 5000 \text{ Å} = 5 \times 10^{-7} \text{ m}$$

$$n = ?$$

$$n = \frac{(a+b)}{\lambda}$$

$$n = \frac{9.694 \times 10^{-6}}{5 \times 10^{-7}} = 19.388$$

$$n > 19$$

Number of orders visible in the spectrum = 19

Example 9.36. *How many orders will be observed by a grating having 4000 lines/cm, if it is stimulated by visible light is the range 4000 Å to 7000 Å.* [*Kanpur, 1991*]

Here $(a+b) = \left(\frac{1}{4000}\right)$cm $= 2.5 \times 10^{-4}$ cm

$$(a+b) = 2.5 \times 10^{-6} \text{ m}$$

(1) $\lambda_1 = 4000 \text{ Å} = 4 \times 10^{-7}$ m

$$\sin\theta = 1$$

$$(a+b)\sin\theta = n_1\lambda_1$$

$$n_1 = \frac{(a+b)}{\lambda_1}$$

$$n_1 = \frac{2.5 \times 10^{-6}}{4 \times 10^{-7}} = 6.25$$

(2) $\lambda_2 = 7000 \text{ Å} = 7 \times 10^{-7}$ m

$$(a+b)\sin\theta = n_2\lambda_2$$

$$\sin\theta = 1$$

$$n_2 = \frac{(a+b)}{\lambda_2} = \frac{2.5 \times 10^{-6}}{7 \times 10^{-7}}$$

$$n_2 = 3.57$$

The order of the spectrum varies from 3 to 6 depending upon the wavelength of the visible range.

Example 9.37. *A diffraction grating used at normal incidence gives a green line, λ = 5400 Å in a certain order superimposed on the violet line, λ = 4500 Å of the next higher order. If the angle of diffraction is 10°, how many lines are there per centimetre in the grating ?* [*Delhi (Hons) 1992*]

Here, grating element = $(a + b)$

$$(a+b)\sin\theta = n\lambda_1 \quad ...(i)$$

$$(a+b)\sin\theta = (n+1)\lambda_2 \quad ...(ii)$$

$\therefore$ $n\lambda_1 = (n+1)\lambda_2$

$$\lambda_1 = 5400 \text{ Å} = 5.4 \times 10^{-7} \text{ m}$$

$$\lambda_2 = 4050 \text{ Å} = 4.05 \times 10^{-7} \text{ m}$$

$\therefore$ $n \times 5.4 \times 10^{-7} = (n+1) \times 4.05 \times 10^{-7}$

$$n = 3$$

Also $\theta = 30°, \sin 30° = 0.5$

$$(a+b)\sin\theta = n\lambda_1$$

$$(a+b)\times 0.5 = 3\times 5.4\times 10^{-7}$$

$$(a+b) = 3.24\times 10^{-6}\text{ m}$$

$$(a+b) = 3.24\times 10^{-4}\text{ cm}$$

Number of lines per cm,

$$N = \frac{1}{(a+b)}$$

$$N = \frac{1}{3.24\times 10^{-4}}$$

$$N = 3086 \text{ lines / cm}$$

9.40 DISPERSIVE POWER OF A GRATING

Dispersive power of a grating is defined as the ratio of the difference in the angle of diffraction of any two neighbouring spectral lines to the difference in wavelength between the two spectral lines. It can also be defined as the difference in the angle of diffraction per unit change in wavelength. The diffraction of the n th order principal maximum for a wavelength λ, is given by the equation,

$$(a+b)\sin\theta = n\lambda \quad ...(i)$$

Differentiating this equation with respect to θ and λ [$(a+b)$ is constant and n is constant in a given order]

$$(a+b)\cos\theta\, d\theta = n\, d\lambda$$

or

$$\frac{d\theta}{d\lambda} = \frac{n}{(a+b)\cos\theta}$$

or

$$\frac{d\theta}{d\lambda} = \frac{nN'}{\cos\theta} \quad ...(ii)$$

In equation (*ii*) $\frac{d\theta}{d\lambda}$ is the dispersive power, n is the order of the spectrum, N' is the number of lines per cm of the grating surface and θ is the angle of diffraction for the n tn order principal maximum of wavelength λ

From equation (*ii*), it is clear, that the dispersive power of the grating is (I) directly proportional to the order of the spectrum, (2) directly proportional to the number of lines per cm and (3) inversely proportional to $\cos\theta$. Thus, the angular spacing of any two spectral lines is double in the second order spectrum in comparison to the first order.

Secondly, the angular dispersion of the lines is more with a grating having larger number of lines per cm. Thirdly, the angular dispersion is minimum when $\theta = 0$. If the value of θ is not large the value of $\cos\theta$ can be taken as unity approximately and the influence of the factor $\cos\theta$ in the equation (*ii*) can be neglected.

Neglecting the influence of $\cos\theta$, it is clear that the angular dispersion of any two spectral lines (in a particular order) is directly proportional to the difference in wavelength between the two spectral lines. A spectrum of this type is called a normal spectrum.

If the linear spacing of two spectral lines of wavelengths λ and $\lambda + d\lambda$ is dx in the focal plane of the telescope objective or the photographic plate, then

$$dx = f\, d\theta$$

where f is the focal length of the objective. The linear dispersion

$$\frac{dx}{d\lambda} = f\frac{d\theta}{d\lambda} = \frac{f\cdot nN'}{\cos\theta} \quad ...(iii)$$

or

$$dx = \frac{fnN'}{\cos\theta}\cdot d\lambda$$

The linear dispersion is useful in studying the photographs of a spectrum.

9.41 PRISM AND GRATING SPECTRA

For dispersing a given beam of light and for studying the resultant spectrum, a diffraction grating is mostly used instead of a prism.

The following points give broadly the distinction between the spectra obtained with a grating and a prism.

(*i*) With a grating, a number of spectra of different orders can be obtained on the two sides of the central maximum whereas with a prism only one spectrum can be obtained.

(*ii*) The spectra obtained with a grating are comparatively pure than those with a prism.

(*iii*) Knowing the grating element $(a + b)$ and measuring the diffracting single, the wavelength of any spectral line can be measured accurately. But in the case of a prism the angles of deviation are not directly related to the wavelength of the spectral line. The angles of deviation are dependent on the refractive index of the material of the prism, which depends on the wavelength of light.

(*iv*) With a grating, the diffracting angle for violet end of the spectrum is less than for red. In Fig. 9.51 $V_1 R_1$ and $V_1' R_1'$ refer to the first

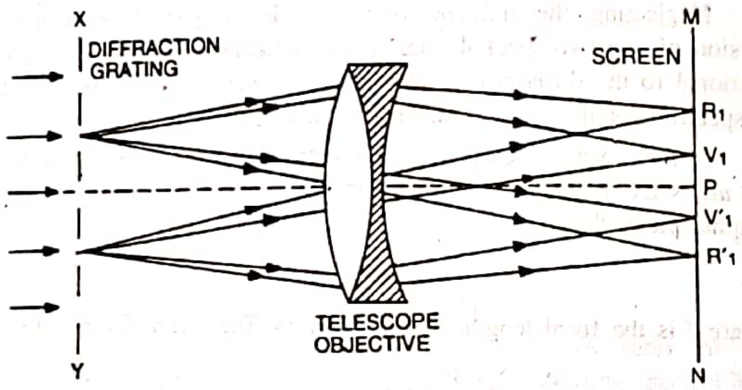


Fig. 9.51

order spectra on the two sides of the central maximum *P*. With a prism (Fig. 9.52), the angle of deviation for the violet rays of light is more than for the red rays of light.

(*v*) The intensities of the spectral lines with a grating are much less than with a prism. In a grating spectrum, most of the incident light energy is associated with the undispersed central bright maximum and the rest of the energy is distributed in the different order spectra on the two sides of the central maximum. But in a prism, most of the incident light energy is distributed in a single spectrum and hence brighter spectral lines are obtained.

(*vi*) The dispersive power of a grating is given by

$$\frac{d\theta}{d\lambda} = \frac{nN'}{\cos\theta}$$

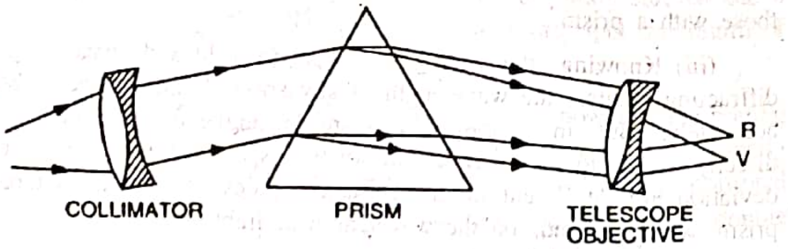


Fig. 9.52

and this is constant for a particular order. Thus, the spectral lines are evenly distributed Hence, the spectrum obtained with a grating is said to be rational (Fig. 9.53). The refractive index of the material of a prism changes more rapidly at the violet end than at the red end of the spectrum. The dispersive

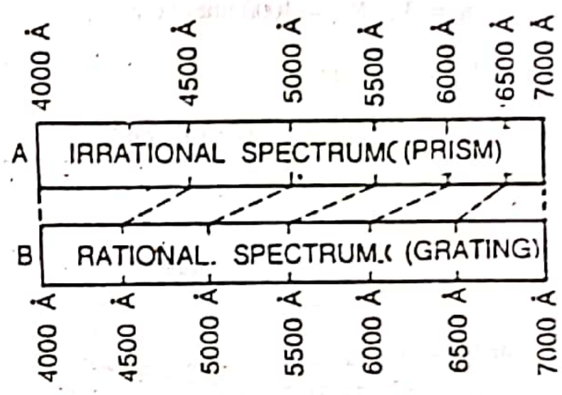


Fig. 9.53

power of a prism is given by $\frac{d\mu}{\mu - 1}$ and this has higher value in the violet region of the spectrum than in the red region. Hence, there will be more spreading of the spectral lines towards the violet and the spectrum obtained with a prism is said to be irrational (Fig. 9.53)

(*vii*) The resolving power of a grating is given by nN where n is the order of the spectrum and N is the total number of lines on the grating surface. The resolving power of a prism is given by $t\frac{d\mu}{d\lambda}$ where t is the base of the prism and $\frac{d\mu}{d\lambda}$ is the rate of change of refractive index with wavelength. The resolving power of a grating is much higher than that of a prism. Hence the same two nearby spectral lines appear better resolved with a grating than with a prism.

(*viii*) Lastly, the spectra obtained with different gratings are identical because the dispersive power and the resolving power of a grating do not depend on the nature of the material of the grating. But the spectra obtained with different prisms are never identical because both dispersive power and resolving power of a prism depend on the nature of the material of the prism.

Example 9.38. *A diffraction grating which has 4000 lines to a cm is used at normal incidence. Calculate the dispersive power of the grating in the third order spectrum in the wavelength region 5000 Å.*

(*Delhi, 1975*)

$$\frac{d\theta}{d\lambda} = \frac{nN'}{\cos\theta_n};\quad (a+b)\sin\theta_n = n\lambda$$

Here $n = 3;\ N' = 4000$ lines / cm

$$(a+b) = \frac{1}{4000}\ \text{cm}$$

$$\lambda = 5000\ \text{Å} = 5000\times10^{-8}\ \text{cm}$$

$$\therefore\quad \sin\theta_n = \frac{n\lambda}{(a+b)}$$

$$= \frac{3\times5000\times10^{-8}\times4000}{1}$$

$$\sin\theta_n = 0.6$$

$$\therefore\quad \cos\theta_n = \sqrt{1-\sin^2\theta_n} = 0.8$$

$$\therefore\quad \frac{d\theta}{d\lambda} = \frac{3\times4000}{0.8}$$

$$\frac{d\theta}{d\lambda} = 15000$$

9.42 CONCAVE REFLECTING GRATING

The wavelength of a spectral line can be determined accurately with a plane transmission grating, knowing the grating constant $a + b$, the diffraction angle θ and the order n. From the knowledge of the wavelengths of a single line (say sodium line of wavelength 5890 Å) the wavelength of the other lines can be obtained by comparison. Use of a plane transmission grating requires two lenses, *viz.*, the collimating lens and the telescope objective. The collimating lens gives a parallel beam of light incident on the grating surface and the telescope objective focuses the diffracted beam. The use of these two lenses if they are not perfectly achromatic, make the spectrum more complex due to chromatic aberration present in the lenses. Rowland developed the *concave reflection grating*, the use of which dispenses the use of both the lenses. The rulings are made on a concave reflecting surface instead of a plane surface. The concave mirror is a highly polished metal surface and it will diffract the incident beam and also focuses it at the same time. In a concave reflection grating, the effect of chromatic aberration is completely eliminated and it can be conveniently used in those regions of the spectrum for which the glass lenses are not transparent.

In Fig. 9.54, *APB* is the surface of a concave reflecting grating in which the rulings are perpendicular to the plane of the paper. *C* is the centre of curvature of the surface *i.e.*, *CP* = *R*. The dotted circle, called the *Rowland Circle*, has a diameter *R*. The circle touches the grating

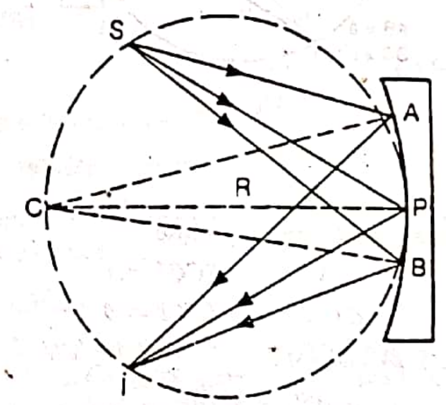


Fig. 9.54

surface at *P*. If a source of light *S* is placed at any point on the circumference of the Rowland circle *PSCI*, the dispersed spectral images of the slit are obtained at points such as *I* on the circumference of the same circle. By keeping the source of light at *C*, the spectra can be observed at other points on the circumference of the circle.

9.43 THEORY OF CONCAVE REFLECTION GRATING

In Fig. 9.55 *GG'* is the concave reflection grating and *C'* is the centre of curvature of the grating surface. *CC'* is the radius of curvature of the surface (*CC'* = *R*). The dotted circle represents the Rowland circle of diameter *CC'*. *S* is a narrow slit perpendicular to the plane of the paper and illuminated by light. *A* and *C* are two corresponding points on the grating. Here *C' A* and *C'C* are normals. *SA* and *SC* are the incident rays and AI and CI are the diffracted rays. Let the angles of incidence and diffraction be i and θ.

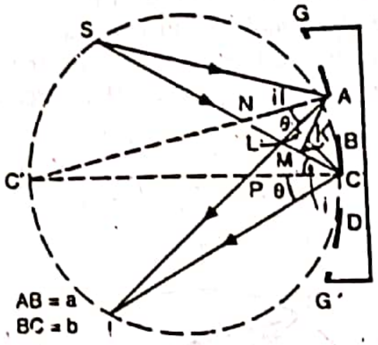


Fig. 9.55

$$\angle SAC' = i$$

and $$\angle C'AI = \theta$$

As the points A and C are very close.

$$\angle SAC' = \angle SCC' = i$$

and $$\angle C'AI = \angle C'CI = \theta$$

From the Δs SMA and IMC

$$\angle ASC = \angle AIC \quad ...(i)$$

From the Δs ASN and $CC'N$,

$$\angle ASC = \angle AC'C \quad ...(i)$$

From equations (i) and (ii)

$$\angle ASC = \angle AIC = AC'C \quad ...(iii)$$

Hence, the points A, S, C', I and C lie on the same circle.

To calculate the path difference between the secondary waves emanating from the points A and C draw perpendiculars AK and CL. As the angles at S and I are small,

$$SA = SK$$

and $$IC = IL$$

The path difference

$$= (SC + CI) - (SA + AI)$$

$$= (SK + KC + CI) - (SA + AL + LI)$$

But $$SK = SA\,;\ \ LI = CI$$

$\therefore$ $$\text{Path difference} = KC - AL \quad ...(iv)$$

In the $\angle KAC$

$$\angle KAC = i$$

and in the ΔALC, $\angle LCA = \theta$ because $C'A$ and $C'C$ are two radii with reference to the concave surface.

But $$AB = a\,:\ BC = b \text{ and } AC = a + b$$

$\therefore$ $$KC = AC \sin i = (a+b) \sin i$$

and $$AL = AC \sin\theta = (a+b)\sin\theta$$

From equation (iv)

Path difference $$= KC - AL$$

$$= (a+b)(\sin i - \sin\theta)$$

For the n th order spectrum to lie at I.

$$(a+b)(\sin i - \sin\theta) = n\lambda \quad ...(v)$$

If the point S and I lie on the same side of C', it can be shown that the path difference

$$(a+b)(\sin i + \sin\theta) = n\lambda \quad ...(vi)$$

The secondary waves from the corresponding points A and C reinforce at I and the same holds good for other pairs of corresponding points in AB and CD and also for the whole grating surface.

Equation (vi) must hold good for diffraction at every element of the grating surface. In other words, it is essential that the path difference for any pair of diffracted rays form the corresponding points of the grating surface must be constant for a particular wavelength. If this conditions is satisfied, then all the rays corresponding to a particular wavelength in a given order will come to focus at a single point on the circumference of the Rowland circle.

Differentiating equation (vi)

$$(a+b)(\cos i\, di + \cos\theta d\theta) = 0$$

or $$\cos i\, di - \cos\theta d\theta = 0 \quad ...(vii)$$

In Fig. 9.56, let

$$\angle ASC \text{ be } \alpha; \quad \angle AIC \text{ be } \beta$$

and $$\angle AC'C \text{ be } \gamma$$

Let the rays SC and SA be incident at angles i and $i + di$ and let the corresponding angles of diffraction be θ and $\theta + d\theta$. A and C are corresponding points. (Fig. 9.56)

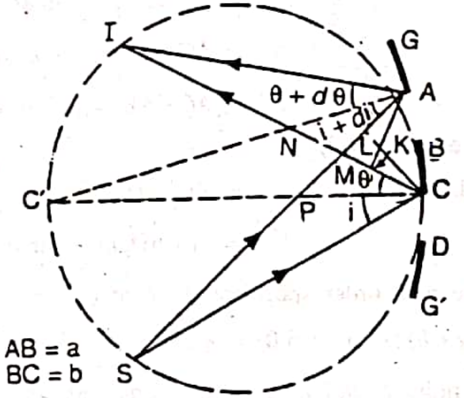


Fig. 9.56

From the Δs SPC and $C'PA$

$$i + \angle ASC = i + di + \angle AC'C$$

or
$$i + \alpha = i + di + \gamma \qquad \text{...(viii)}$$

Similarly from $\angle$s CNC' and ANI

$$\theta + \gamma = \theta + d\theta + \beta$$

or
$$d\theta = \gamma - \beta \qquad \text{...(ix)}$$

Let the radius of curvature CC' be R and let the distances SA and AI be s_1 and s_2.

$$AC = AB + BC = a + b$$

Substituting the values of di and $d\theta$ of equations (*viii*) and (*ix*) in equation (*vii*)

$$(\alpha - \gamma)\cos i - (\gamma - \beta)\cos\theta = 0 \qquad \text{...(x)}$$

CL is perpendicular to SA and AK is perpendicular to IC.

$\therefore$ $\angle ACL = i$ and $\angle KAC = \theta$ approximately.

In the Δ ACL

$$LC = AC\cos ACL = AC\cos i$$
$$= (a + b)\cos i \qquad \text{...(xi)}$$

Similarly in the Δ AKC

$$AK = AC\cos KAC = AC\cos\theta$$
$$= (a + b)\cos\theta \qquad \text{...(xii)}$$

As the points A and C are very near,

$$LC = SC\,.\,\alpha = SA.\,\alpha = s_1\alpha$$

and
$$AK = IA.\,\beta = s_2\beta$$

Substituting these values of LC and AK in equations (*xi*) and (*xii*)

$$s_1\alpha = (a + b)\cos i$$
$$s_2\beta = (a + b)\cos\theta$$

Also, $$AC = (a + b) = R\gamma$$

From the above equations

$$\alpha = \frac{(a + b)\cos i}{s_1}$$

$$\beta = \frac{(a + b)\cos\theta}{s_2}$$

and
$$\gamma = \frac{(a + b)}{R}$$

Substituting these values of α, β and γ in equation (*x*)

$$(\alpha - \gamma)\cos i - (\gamma - \beta)\cos\theta = 0$$

$$\left[\frac{(a + b)\cos i}{s_1} - \frac{(a + b)}{R}\right]\cos i - \left[\frac{(a + b)}{R} - \frac{(a + b)\cos\theta}{s_2}\right]\cos\theta = 0$$

or
$$\left(\frac{\cos i}{s_1} - \frac{1}{R}\right)\cos i - \left(\frac{1}{R} - \frac{\cos\theta}{s_2}\right)\cos\theta = 0$$

$$\frac{\cos^2 i}{s_1} - \frac{\cos i}{R} - \frac{\cos\theta}{R} + \frac{\cos^2\theta}{s_2} = 0$$

$$\frac{\cos^2\theta}{s_2} = \frac{(\cos i + \cos\theta)}{R} - \frac{\cos^2 i}{s_1}$$

$$= \frac{s_1[\cos i + \cos\theta] - R\cos^2 i}{Rs_1}$$

or
$$s_2 = \frac{Rs_1\cos^2\theta}{s_1(\cos i + \cos\theta) - R\cos^2 i} \qquad \text{...(xiii)}$$

In equation (*xiii*), if $R \cos i = s_1$, then on simplification,

$$s_2 = R \cos \theta$$

Thus, if the point S lies on the circumference of the circle of diameter R, then I also lies on the same circle.

9.44 PASCHEN MOUNTING

The common form of mounting used for a concave reflection grating is shown in Fig. 9.57. It is called the *Paschen mounting*.

In this mounting, the slit S is set on the circumference of the Rowland circle as shown. The slit is perpendicular to the plane of the paper. GG' is the concave reflection grating and OC' is the diameter of the Rowland circle. The spectra of different orders are imaged on the circumference of the circle. In the figure, C is the central image, A_1B_1 is

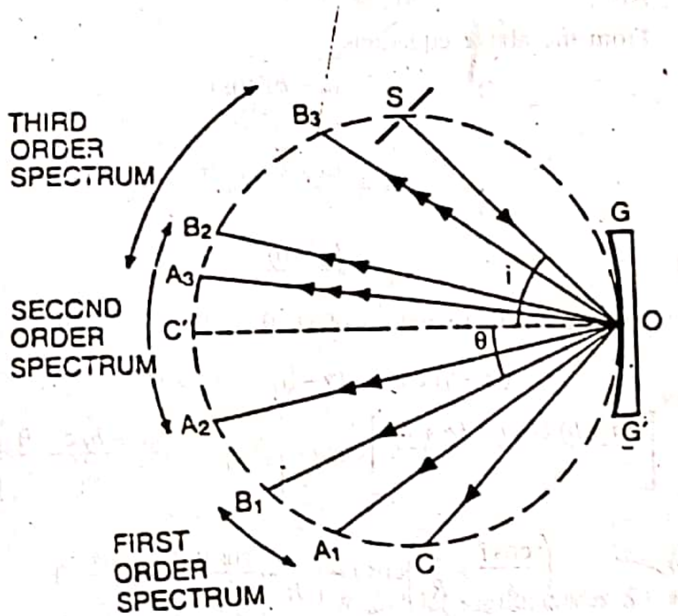


Fig. 9.57

the first order spectrum. A_2B_2 is the second order spectrum and A_3B_3 is the third order spectrum. With this mounting, several orders of the spectrum can be photographed simultaneously. The photographic plates are held in a frame which can give the plates the proper curvature coincident with the Rowland circle. For any particular order, the dispersion is minimum when $\theta = 0$. The disadvantage in this mounting is that the spacing of the

spectral lines in the different regions of the spectrum is not proportional to the difference in wavelength between the lines.

9.45 ROWLAND MOUNTING

The principle of Rowland mounting is illustrated in Fig. 9.58. G is the concave grating and P is the plate holder. The grating and the plate holder are mounted at the ends of a beam of length R equal to the radius of curvature of the grating surface. This beam GP can slide along two rails SX and SY. G, P and G', P' represent two positions of the beam.

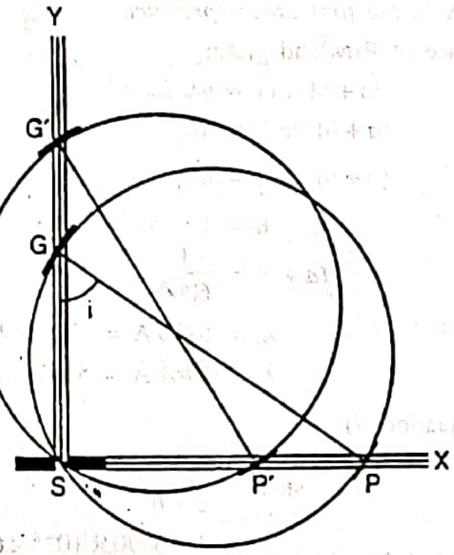


Fig. 9.58

The slit S is set at the point of intersection of the rails SX and SY. With an arrangement of this type, the region of the spectrum imaged at P can be altered by sliding the beam. Sliding the beam alters the angle of incidence i. The spectrum obtained with this arrangement is nearly normal because the angle θ is nearly zero. For any position of the beam the spectrum is imaged at P.

From the equation,

$$(a+b)(\sin i - \sin \theta) = n\lambda$$

if $$\theta = 0; \ \sin \theta = 0$$

$$\therefore \quad (a+b)\sin i = n\lambda$$

Here $(a+b)$ is a constant. For a given order

$$\sin i \propto \lambda$$

But $$\sin i \propto SP$$

$$\therefore \quad SP \propto \lambda$$

Thus with a mounting of this type, which is mostly of historical interest, it is possible to calibrate the rail SP for the wavelengths of spectral lines.

Example 9.39. *A Rowland grating has 6000 lines to a centimetre. Calculate the angular separation of two mercury lines of wavelengths 5770 Å and 5461 Å in the first order spectrum.* [Delhi 1985]

In the case of Rowland grating

$$(a+b)\sin i = n\lambda$$

$$(a+b)\sin i_1 = n\lambda_1 \quad ...(i)$$

$$(a+b)\sin i_2 = n\lambda_2 \quad ...(ii)$$

Here $$n = 1$$

$$(a+b) = \frac{1}{6000}$$

$$\lambda_1 = 5770\text{ Å} = 5770 \times 10^{-8}\text{ cm}$$

$$\lambda_2 = 5461\text{ Å} = 5461 \times 10^{-8}\text{ cm}$$

From equation (*i*)

$$\sin i_1 = \frac{n\lambda_1}{a+b}$$

$$= 1 \times 5770 \times 10^{-8} \times 6000$$

$$= 0.3462$$

$$i_1 = 20°\ 12'$$

From equation (*ii*)

$$\sin i_2 = \frac{n \times \lambda_2}{a+b}$$

$$= 1 \times 5461 \times 10^{-8} \times 6000$$

$$= 0.3277$$

$$i_2 = 19°\ 6'$$

$$i_1 - i_2 = 20°\ 12' - 19°\ 6'$$

$$= 1° - 6'$$

9.46 EAGLE MOUNTING

The Rowland and Paschen mountings have largely been replaced by the **Eagle mounting** illustrated in Fig. 9.59. In this mounting, that portion of the spectrum which is diffracted back at an angle almost equal to the angle of incidence, is focussed on the plate P. To study the different

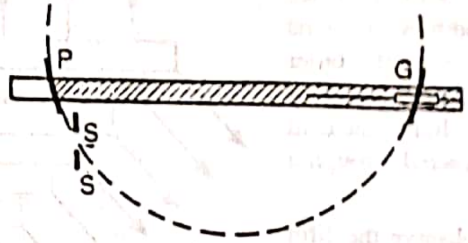


Fig. 9.59

regions of the spectrum the grating is turned about an axis perpendicular to the plane of the paper. Correspondingly, the plate holder P, which is hinged on one side of S is turned such that P and S lie on the Rowland circle. For studying spectra in the ultra-violet region, the Eagle mounting is commonly used in vacuum spectrographs.

9.47 LITTROW MOUNTING

Littrow mounting is illustrated in Fig. 9.60. Large Plane reflection gratings are mounted this way, G is a plane reflection grating, P is a photographic plate, S is a slit and L is a large achromatic lens. The principle of Littrow mounting is similar to that of Eagle mounting.

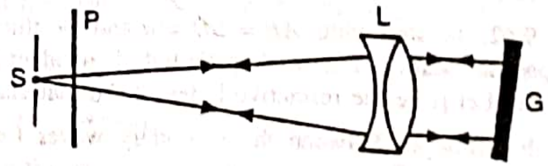


Fig. 9.60

The achromatic lens L serves two functions. It renders the incident light from the slit parallel as well as it focuses the diffracted beam on P. Thus, it serves both as a collimating lens and a telescope objective.

9.48 ECHELON GRATING

An echelon transmission grating consists of a number of optically worked glass plates arranged in the form of steps as shown in Fig. 9.61 All the plates are cut from a single optically worked glass plate. Each plate overlaps on the next by the same distance *i.e.*, the step width is the same throughout and is of the order of 1 mm. A parallel beam of monochromatic light incident normally is diffracted through a small angle.

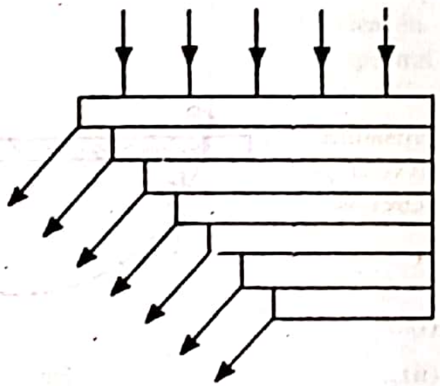


Fig. 9.61

One can observe the fifth or sixth order spectrum with a concave grating and only the second or third order spectrum with a plane diffraction grating. The resolving power of a ruled grating is dependent on the number of lines on the grating surface and the order of the spectrum. It is difficult and very tedious to draw a large number of equidistant parallel lines on a grating surface. With an echelon grating, designed by Michelson, one can observe the spectrum of a very high order and hence the resolving power of the echelon grating is very high. But, with the increase in the order of the spectrum, the intensity of the spectral lines decreases considerably. Also, the angular spacing $d\theta$ is very small and hence there will be many overlapping orders. An echelon grating, therefore, is not suitable to study the spectrum as such but essentially helps in detecting the true monochromatism of a beam of light.

9.49 THEORY OF ECHELON GRATING

In Fig. 9.62, the step width $AB = CD = a$ and the thickness of each plate $= t$. A parallel beam of monochromatic light is incident on the surface of the top plate. Let μ be the refractive index of the material of the plate.

The path difference between the secondary waves from the corresponding points A and C

$$= BC - AM$$

But $\quad AM = AQ - MQ$

$$= AQ - NP$$

$\therefore$ Path difference $= BC \pm AQ + NP$

$$BC = \mu t$$

$$AQ = t\cos\theta \text{ and}$$

$$NP = a\sin\theta$$

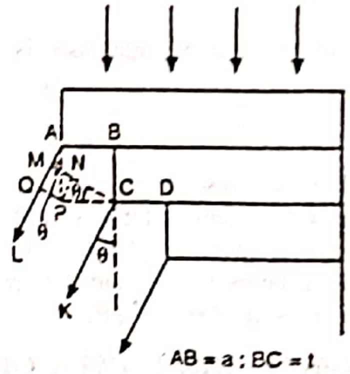


Fig. 9.62

(Here, θ is the angle between the direction of the diffracted beam and the incident direction of light).

$\therefore$ Path difference

$$= \mu t - t\cos\theta + a\sin\theta$$

If this path difference is equal to integral multiples of λ, all the secondary waves travelling in this direction reinforce with one another.

$$\therefore \quad \mu t - t\cos\theta + a\sin\theta = n\lambda \qquad ...(i)$$

For small values of θ, $\cos\theta = 1$, and $\sin\theta = \theta$

$$\therefore \quad \mu t - t + a\theta = n\lambda$$

or
$$a\theta = n\lambda - \mu t + t \qquad ...(ii)$$

or
$$\theta = \frac{1}{a}[n\lambda - \mu t + t]$$

Differentiating the equation,

$$\frac{d\theta}{d\lambda} = \frac{1}{a}\left[n - t\frac{d\mu}{d\lambda}\right] \qquad ...(iii)$$

In equation (*iii*), $\frac{d\theta}{d\lambda}$ is the dispersive power of the grating and $\frac{d\mu}{d\lambda}$ is the rate of change of refractive index of the material of the echelon with respect to wavelength.

For small values of θ, equation (*i*) can be written as

$$(\mu - 1)t = n\lambda$$

or
$$n = \frac{(\mu - 1)t}{\lambda} \qquad ...(iv)$$

In equation (iv), if $t = 1$ cm, $\lambda = 5000$ Å and $\mu = 1.5$ then $n = 10{,}000$.

If the number of plates used is 40, the resolving power of the grating

$$= nN = 10{,}000 \times 40$$

$$= 4 \times 10^5$$

Thus, the resolving power of an echelon grating is very high and if the incident beam of light is not truly monochromatic, two nearby spectral lines will appear well resolved. The high resolving power of an echelon grating helps in the study of hyperfine structure *e.g.*, the splitting of spectral lines in Zeeman effect.

9.50 RESOLVING POWER OF OPTICAL INSTRUMENTS

The magnifying power of a telescope or a microscope depends on the focal length of the lenses used. By a proper choice of the lenses, it is possible to increase the size of the image, *i.e.*, the image subtends a large angle at the eye. But, it must be remembered that increase in the size of the image, beyond a certain limit does not necessarily mean gain in detail. This is the case even if the lenses are free from all aberrations, chromatic and monochromatic. There is always a limit to the useful magnification of an optical instrument. This is due to the fact that for a wave surface, the laws of geometrical optics do not hold good. In the preceding articles, concerning diffraction of light, it has been shown that the image of a point source is not a point but it is a diffraction pattern. With a circular aperture kept in the path of incident light, the diffraction pattern of a point source of light consists of a central bright disc surrounded by alternately dark and bright diffraction rings.

If the lens diameter or the size of the aperture is large, the diffraction pattern of a point source of light is small. If there are two nearby point sources, the diffraction discs of the two patterns may overlap and the two images may not be distinguished. An optical instrument like a telescope or a microscope is said to have resolved the two point sources when the two diffraction patterns are well separated from one another or when the diffraction patterns are small so that in both the cases, the two images are seen as separate ones. **The ability of an optical instrument, expressed in numerical measure, to resolve the images of two nearby points is termed as its resolving power.**

In the case of a prism or a grating spectrograph, the term resolving power is referred to the ability of the prism or grating to resolve two nearby spectral lines so that the two lines can be viewed or photographed as separate lines.

9.51 CRITERION FOR RESOLUTION ACCORDING TO LORD RAYLEIGH

To express the resolving power of an optical instrument as a numerical value, Lord Rayleigh proposed an arbitrary criterion. **According to him, two nearby images are said to be resolved if the position of the central maximum of one coincides with the first secondary minimum of the other and vice versa. The same criterion can be conveniently applied to calculate the resolving power of a telescope, microscope, grating, prism, etc.**

In Fig. 9.63, A and B are the central maxima of the diffraction patterns of two spectral lines of wavelengths λ_1 and λ_2. The difference in

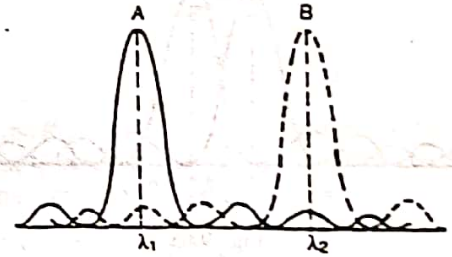


Fig. 9.63

the angle of diffraction is large and the two images can be seen as separate ones. The angle of diffraction corresponding to the central maximum of the image B is greater than the angle of diffraction corresponding to the first minimum at the right of A. Hence the two spectral lines will appear well resolved.

In Fig. 9.64 the central maxima corresponding to the wavelengths λ and $\lambda + d\lambda$ are very close. The angle of diffraction corresponding to

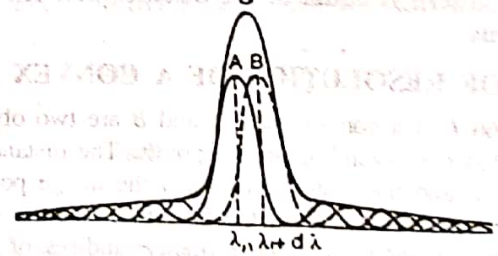


Fig. 9.64

the first minimum of A is greater than the angle of diffraction corresponding to the central maximum of B. Thus, The two images overlap and they cannot be distinguished as separate images. The resultant intensity curve gives a maximum as at C and the intensity of this maximum is higher than the individual intensities of A and B. Thus when the spectrograph is turned from A to B, the intensity increases, becomes maximum at C and then decreases. In this case, the two spectral lines are **not resolved.**

In Fig. 9.65, the position of the central maximum of A (wavelength λ) coincides with the position of the first minimum of B. (wavelength $\lambda + d\lambda$). Similarly, the position of the central maximum of B coincides with

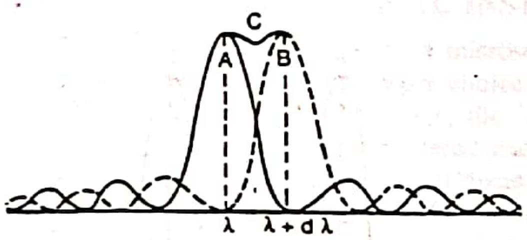


Fig. 9.65

the position of the first minimum of A. Further, the resultant intensity curve shows a **dip** at C *i.e.*, in the middle of the central maxima of A and B (Here, it is assumed that the two spectral lines are of the same intensity). The intensity at C is approximately 20% less than that at A or B. If a spectrograph is turned from the position corresponding to the central image of A to the one corresponding to the image of B, there is noticeable decrease in intensity between the two central maxima. The spectral lines can be distinguished from one another and according to Rayleigh they are said to be **just resolved. Rayleigh's condition can also be stated as follows. Two images are said to be just resolved if the radius of the central disc of either pattern is equal to the distance between the centers of the two patterns.**

9.52 LIMIT OF RESOLUTION OF A CONVEX LENS

In Fig. 9.66 L is a convex lens. A and B are two object points and A' and B' are the corresponding image points. The distance between the object points is h and the distance between the image points is h'. The distance of the object points from the lens is u and the distance of the image points is v. μ and μ' are the refractive indices of the object and image media. R is the radius of the aperture kept in front of the lens

(D is the diameter of the aperture). In the side figure, A' and B' are the centres of the central bright discs of the diffraction patterns of A and B.

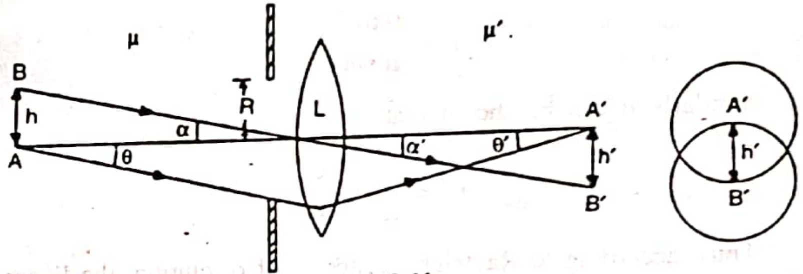


Fig. 9.66

Let λ and λ' be the wavelengths of light in the object and image media and λ the wavelength of light in vacuum. Then,

$$\lambda = \frac{\lambda_0}{\mu} \text{ and } \lambda' = \frac{\lambda_0}{\mu'}$$

According to Rayleigh, if the two images are just resolved, the distance between the centres of the two discs (h') is equal to the radius of either disc. If this condition is satisfied, then

$$\sin\alpha = \frac{1.22\lambda}{D} = \frac{1.22\lambda_0}{\mu D} = \frac{1.22\,\lambda_0}{2\mu R}$$

$$= \frac{0.61\,\lambda_0}{\mu R} \quad \text{...(}i\text{)}$$

Similarly, $$\sin\alpha' = \frac{0.61\lambda_0}{\mu' R} \quad \text{...(}ii\text{)}$$

From equation (i)

$$\mu R \sin\alpha = 0.61\lambda_0 \quad \text{...(}iii\text{)}$$

But, in Fig. 9.66 (for small angles of α and θ)

$$\sin\alpha = \tan\alpha = \frac{h}{u}$$

and $$\sin\theta = \tan\theta = \frac{R}{u}$$

or $$R = u\sin\theta$$

Substituting the values of $\sin\alpha$ and R in equation (iii)

$$\mu\, u \sin\theta \cdot \frac{h}{u} = 0.61\,\lambda_0$$

or
$$h = \frac{0.61\,\lambda_0}{\mu \sin\theta} \qquad ...(iv)$$

Similarly it can be shown that

$$h' = \frac{0.61\,\lambda_0}{\mu' \sin\theta'} \qquad ...(v)$$

Thus, according to Rayleigh's criterion of resolution, the linear distance between two just resolvable point objects is given by

$$h = \frac{0.61\,\lambda_0}{\mu \sin\theta} \qquad ...(vi)$$

and the distance between the corresponding image points is given by

$$h' = \frac{0.61\,\lambda_0}{\mu' \sin\theta'} \qquad ...(vii)$$

The quantity $\mu \sin\theta$ in equation (*iv*) is called the numerical aperture (N.A.) of the optical instrument. From equation (*iv*), the distance between two just resolvable object points is inversely proportional to the numerical aperture of the instrument,. An optical instrument with higher numerical aperture can resolve two nearer points than the one with a lower numerical aperture. The smaller the value of h, the higher is the resolving power of the instrument.

9.53 LIMIT OF RESOLUTION FO THE EYE

In Fig. 9.67, $M\,N$ is the eye lens, A and B are two object points separated by a distance h and A' and B' are the corresponding image

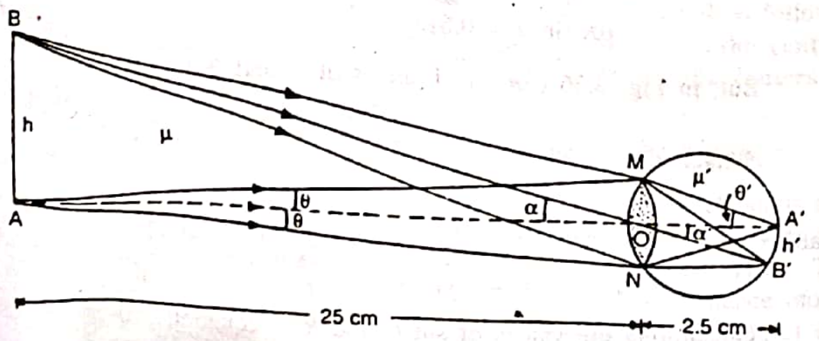


Fig. 9.67

points at a distance h' and formed on the retina. μ is the refractive index of the object medium and μ' is the refractive index of the image medium. If the object is placed in air, $\mu = 1$ and the image medium is vitreous humor whose refractive index is 1.33. If the object is situated at the least distance of distant vision, u = 25 cm = 250 mm for a normal eye. If the diameter of the eye ball is about 1 inch, then $v = 1''$ = 2.5 cm = 25 mm approximately. Taking the pupillary diameter of the eye as 2 mm, R = 1 mm.

Also, the human eye is most sensitive to a wavelength λ_0 = 5500 Å.

From the $\Delta\, AMO$, for small angles of θ.

$$\sin\theta = \tan\theta = \frac{R}{u} = \frac{1}{250}$$

$$= 0.004$$

$$\text{Numerical aperture} = \mu \sin\theta$$

$$= 1 \times 0.004 = 0.004$$

Applying Rayleigh's criterion, the minimum distance (h) between two just resolvable object points of equal intensity is given by

$$h = \frac{0.61\,\lambda_0}{\mu \sin\theta} = \frac{0.61 \times 5500 \times 10^{-8}}{0.004}$$

$$= \frac{1}{100}\text{ cm} = \frac{1}{10}\text{ mm approximately}$$

It means that, if the object is situated at the least distance of distinct vision from the eye (25 cm), the minimum linear separation between two nearby object points should be of the order of 0.1 mm. If the object points are separated by a distance larger than 0.1 mm, they are clearly visible and are well resolved.

Similarly the distance h', between the centres of the two images is given by

$$h' = \frac{0.61\,\lambda_0}{\mu' \sin\theta'}$$

$$= \frac{0.61 \times 5500 \times 10^{-8}}{1.33 \times 0.04}$$

$$\left(\sin\theta' = \frac{1}{25} = 0.04\right)$$

$$h' = \frac{1}{1000}\text{ cm} = \frac{1}{100}\text{ mm approximately}$$

Also, $$\alpha = \sin\alpha = \frac{0.61\,\lambda_0}{\mu R}$$

$$= \frac{0.61 \times 5500 \times 10^{-8}}{1 \times 0.1}$$

$$= 0.00034 \text{ radian}$$

$$= 1 \text{ minute of an are (approximately)}$$

The value of $h' (= 10^{-3}\text{ cm})$ is approximately equal to the distance between the cones in the fovea and thus the retinal structure is strikingly in accordance with the limit of resolution of the eye. Further, two point objects appear just resolved if the angle subtended by them at the eye is 1 minute of an arc. If the diameter of the pupil of the eye is smaller than 2 mm the numerical aperture decreases and hence the value of h increases, i.e., two points will appear to be just resolved if the distance between the two is larger. Thus the resolving ability of the eye is decreased.

9.54 RESOLVING POWER OF A TELESCOPE

Let a be the diameter of the objective of the telescope (Fig. 9.68). Consider the incident ray of light from two neighbouring points of a distant object. The image of each point object is a Fraunhofer diffraction pattern.

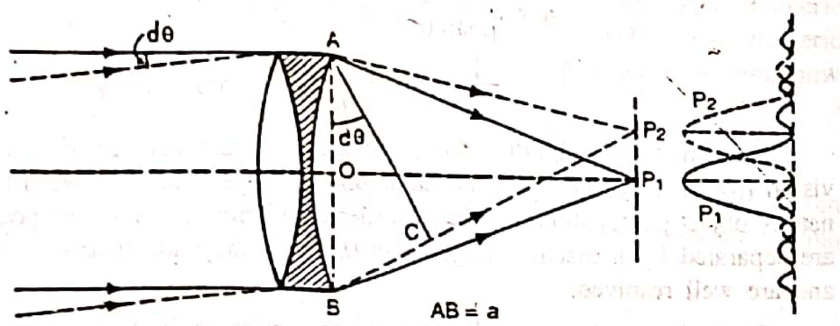


Fig. 9.68

Let P_1 and P_2 be the positions of the central maxima of the two images. According to Rayleigh, these two images are said to be resolved if the position of the central maximum of the second image coincides with the first minimum of the first image and *vice versa*. The path difference between the secondary waves travelling in the directions AP_1 and BP_1 is zero and hence they reinforce with one another at P_1. Similarly, all the secondary waves from the corresponding points between A and B will have

zero path difference. Thus, P_1 corresponds to the position of the central maximum of the first image.

The secondary waves travelling in the directions AP_2 and BP_2 will meet at P_2 on the screen. Let the angle P_2AP_1 be $d\theta$. The path difference between the secondary waves travelling in the directions BP_2 and AP_2 is equal to BC (Fig. 9.68).

From the $\Delta\, ABC$,

$$BC = AB \sin d\theta = AB.d\theta = a\,.\,d\theta$$

(for small angles)

If this path difference $a \cdot d\theta = \lambda$, the position of P_2 corresponds to the first minimum of the first image. But P_2 is also the position of the central maximum of the second image. Thus, Rayleigh's condition of resolution is satisfied if

$$a\,.\,d\theta = \lambda$$

or $$d\theta = \frac{\lambda}{a} \quad ...(i)$$

The whole aperture AB can be considered to be made up of two halves AO and OB. The path difference between the secondary waves from the corresponding points in the two halves will be $\frac{\lambda}{2}$. All the secondary waves destructively interfere with one another and hence P_2 will be the first minimum of the first image. The equation $d\theta = \frac{\lambda}{a}$ holds good for rectangular apertures. For circular apertures, this equation, according to Airy, can be written as

$$d\theta = \frac{1.22\,\lambda}{a} \quad ...(ii)$$

where λ is the wavelength of light and a is the aperture of the telescope objective. The aperture is equal to the diameter of the metal ring in which the objective lens is mounted. Here $d\theta$ refers to the **limit of resolution** of the telescope. **The reciprocal of $d\theta$ measures the resolving power of the telescope.**

$$\therefore \quad \frac{1}{d\theta} = \frac{a}{1.22\,\lambda} \quad ...(iii)$$

From equation (*iii*), it is clear that a telescope with large diameter of the objective has higher resolving power, $d\theta$ is equal to the angle subtended by the two distant object points at the objective.

Thus resolving power of a telescope can be defined as the reciprocal of the angular separation that two distant object points must have, so that their images will appear just resolved according to Rayleigh's criterion.

If f is the focal length of the telescope objective, then

$$d\theta = \frac{r}{f} = \frac{1.22\,\lambda}{a}$$

or

$$r = \frac{1.22\,f\lambda}{a} \qquad ...(iv)$$

where r is the radius of the central bright image. The diameter of the first dark ring is equal to the diameter of the central image. The central bright disc is called the Airy's disc.

From equation (*iv*), if the focal length of the objective is small, the wavelength is small and the aperture is large, then the radius of the central bright disc is small. The diffraction patterns will appear sharper and the angular separation between two just resolvable point objects will be smaller. Correspondingly, the resolving power of the telescope will be higher.

Let two distant stars subtend an angle of one second of an are at the objective of the telescope.

1 second of an are = 4.85×10^{-6} radian. Let the wavelength of light be 5500 Å. Then, the diameter of the objective required for just resolution can be calculated from the equation

$$d\theta = \frac{1.22\,\lambda}{a}$$

or

$$a = \frac{1.22\,\lambda}{d\theta} = \frac{1.22 \times 5500 \times 10^{-8}}{4.85 \times 10^{-8}}$$

$$= 13.9 \text{ cm (approximately)}$$

The resolving power of a telescope increases with increase in the diameter of the objective. With the increase in the diameter of the objective, the effect of spherical aberration becomes appreciable. So, in the case of large telescope objectives, the central portion of the objective is covered with a stop so as to minimize the effect of spherical aberration. This, however, does not affect the resolving power of the telescope.

Example 9.40. *Find the separation of two points on the moon that can be resolved by a 500 cm telescope. The distance of the moon is 3.8×10^5 km. The eye is most sensitive to light of wavelength 5500 Å.* *(Nagpur 1974)*

The limit of resolution of a telescope is given by

$$d\theta = \frac{1.22\,\lambda}{a}$$

Here $\lambda = 5500 \times 10^{-8}$ cm, $a = 500$ cm

$$\therefore \quad d\theta = \frac{1.22 \times 5500 \times 10^{-8}}{500}$$

$$\therefore \quad d\theta = 13.42 \times 10^{-8} \text{ radian}$$

Let the distance between the two points be x

$$\therefore \quad d\theta = \frac{x}{R}$$

Here $R = 3.8 \times 10^{10}$ cm

$$x = R \cdot d\theta$$

$$= 3.8 \times 10^{10} \times 13.42 \times 10^{-8}$$

$$= 50.996 \times 10^{2} \text{ cm}$$

$$= \mathbf{50.996 \text{ metres}}$$

Example 9.41. *Calculate the aperture of the objective of a telescope which may be used to resolve stars separated by 4.88×10^{-6} radian for light of wavelength 6000 Å*

Here $\lambda = 6000$ Å $= 6 \times 10^{-5}$ cm, $\theta = 4.88 \times 10^{-6}$ radian

$D = ?$

$$\theta = \frac{1.22\,\lambda}{D}$$

or

$$D = \frac{1.22\,\lambda}{\theta} = \frac{1.22 \times 6 \times 10^{-5}}{4.88 \times 10^{-6}} = \mathbf{15 \text{ cm}}$$

Example 9.42. *Two pin holes 1.5 mm apart are placed in front of a source of light of wavelength 5.5×10^{-5} cm and seen through a telescope with its objective stopped down to a diameter of 0.4 cm. Find the maximum distance from the telescope at which the pin holes can be resolved.* *[Delhi, 1977]*

Here, $\lambda = 5.5 \times 10^{-5}$ cm

$a = 0.4$ cm

$$d\theta = \frac{1.22\,\lambda}{a}$$

Also

$$d\theta = \frac{x}{d}$$

$$x = 1.5 \text{ mm} = 0.15 \text{ cm}$$

$$\therefore \quad \frac{x}{d} = \frac{1.22\lambda}{a}$$

$$d = \frac{xa}{1.22\lambda}$$

$$d = \frac{0.15 \times 0.4}{1.22 \times 5.5 \times 10^{-5}} \text{ cm}$$

$$d = 894.2 \text{ cm} = 8.942 \text{ m}$$

9.55 RELATION BETWEEN MAGNIFYING POWER AND RESOLVING POWER OF A TELESCOPE

The magnifying power of a telescope is given by

$$M = \frac{D}{d} \quad \text{...(i)}$$

where D is the diameter of the objective (entrance pupil) and d is the diameter of the exit pupil. The magnification of a telescope is said to be normal if d the diameter of the exit pupil is equal to d_e the diameter of the pupil of the eye. Therefore, the normal magnification of a telescope is given by

$$M = \frac{D}{d_e} \quad \text{...(ii)}$$

Further, the limit of resolution of a telescope is given by

$$d\theta = \frac{1.22\lambda}{D} \quad \text{...(iii)}$$

and the limit of resolution of the eye is given by

$$d\theta' = \frac{1.22\lambda}{d_e} \quad \text{...(iv)}$$

From equation (*iii*) and (*iv*),

$$\frac{d\theta'}{d\theta} = \frac{\text{Limit of resolution of the eye}}{\text{Limit of resolution of the telescope}}$$

$$= \frac{1.22\lambda}{d_e} \Big/ \frac{1.22\lambda}{D}$$

$$= \frac{D}{d_e}$$

$$= \text{Normal magnifying power of a telescope}$$

Thus, the product of the normal magnifying power of a telescope and its limit of resolution is equal to the limit of resolution of the unaided eye.

Taking the pupil diameter of the eye as 2 mm and wavelength of light as 5500 Å, the angular separation ($d\theta'$) between two distant object points resolvable by the eye is given by

$$d\theta' = \frac{1.22\lambda}{d_e}$$

$$= \frac{1.22 \times 5500 \times 10^{-8}}{0.2}$$

$$= \frac{1.22 \times 5500 \times 10^{-8}}{0.2} \times \frac{180}{\pi} \times 60 \text{ minutes}$$

$$= 1 \text{ minute of an arc approximately}$$

Similarly, the angular separation between two distant stars just resolvable by a telescope objective of diameter 254 cm is approximately 1/20th of a second of an arc.

i.e.,

$$d\theta = 1/20 \text{ th second of an arc}$$

$$d\theta' = 1 \text{ minute of an arc}$$

$$= 60 \text{ seconds of an arc}$$

Normal magnifying power of a telescope objective of diameter 254 cm

$$= \frac{d\theta'}{d\theta} = \frac{60}{\frac{1}{20}} = 1200$$

If the normal magnifying power is 1200, full advantage of the high resolving power of the telescope can be taken.

If two telescope objectives have the same focal length, the magnifying power will be the same in the two cases. But, the telescope with an objective of larger aperture has higher resolving power than the one with a smaller aperture. With increase in the diameter of the objective of a telescope, the resolving power increases. Also with a large diameter objective, the radius of the central disc of the Fraunhofer pattern is smaller and consequently the image obtained is sharp and more intense.

Example 9.43. *Calculate the useful magnifying power of a telescope of 10 cm objective, assuming that the limit of resolution of the eye is 2 minutes of an arc. Wavelength of light used is 6000 Å.*

Diameter of the objective $D = 10$ cm

$$\lambda = 6000\ \text{Å} = 6 \times 10^{-5}\ \text{cm}$$

Limit of resolution of the telescope,

$$d\theta = \frac{1.22\,\lambda}{D}$$

$$= \frac{1.22 \times 6 \times 10^{-5}}{10} = 7.32 \times 10^{-6}\ \text{radian}$$

Limit of resolution power of the eye,

$$d\theta' = 2\ \text{minutes of an arc}$$

$$= \frac{2}{60} \times \frac{22}{7 \times 180} = 582 \times 10^{-6}\ \text{radian}$$

Useful magnifying power of the telescope

$$= \frac{d\theta'}{d\theta} = \frac{582 \times 10^{-6}}{7.32 \times 10^{-6}} = 79.5$$

9.56 RESOLVING POWER OF A MICROSCOPE

In the case of a telescope, the smallest permissible angular separation between two distant objects at an unknown distance, determines the limit of resolution when the images appear just resolved. But, in the case of a microscope, the object is very near the objective of the microscope (just beyond the focus of the objective) and the objects subtend a large angle at the objective. The limit of resolution of a microscope is determined by the least permissible linear distance between the two objects so that the two images appear just resolved.

In Fig. 9.69, MN is the aperture of the objective of a microscope and A and B are two object points at a distance d apart. A' and B' are

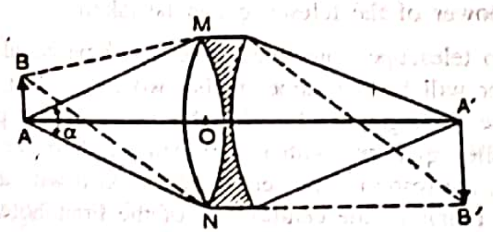


Fig. 9.69

the corresponding Fraunhofer diffraction patterns of the two images. A' ... the position of central maximum of A and B' is the position of

the central maximum of B. A' and B' are surrounded by alternate dark and bright diffraction rings. The two images are said to be just resolved if the position of the central maximum B' also corresponds to the first minimum of the image of A'.

The path difference between the extreme rays from the point B and reaching A' is given by

$$(BN + NA') - (BM + MA')$$

But $\quad NA' = MA'$

$\therefore$ Path difference $= BN - BM$

In Fig. 9.70, AD is perpendicular to DM and AC is perpendicular to BN.

$$\therefore \quad BN - BM = (BC + CN) - (DM - DB)$$

But $\quad CN = AN = AM = DM$

$\therefore$ Path difference $= BC + DB$

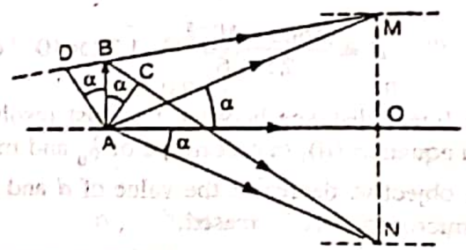


Fig. 9.70

From the Δs ACB and ADB

$$BC = AB \sin\alpha = d \sin\alpha$$

and

$$DB = AB \sin\alpha = d \sin\alpha$$

Path difference $= 2d \sin\alpha$

If this path difference $2d \sin\alpha = 1.22\lambda$, then, A' corresponds to the first minimum of the image B' and the two images appear just resolved.

$$\therefore \quad 2d \sin\alpha = 1.22\,\alpha$$

or

$$d = \frac{1.22\,\lambda}{2 \sin\alpha} \qquad \text{...(i)}$$

Equation (*i*) derived above is based on the assumption that the object points A and B are self-luminous. But actually, the objects viewed with a microscope are not self-luminous but are illuminated with light from a condenser. It is found that the resolving power depends on the mode of illumination. According to Abbe, the least distance between two just resolvable object points is given by

$$d = \frac{\lambda_0}{2\mu \sin \alpha} \qquad ...(ii)$$

where λ_0 is the wavelength of light through vacuum and μ is the refractive index of the medium between the object and the objective. The space between the object and the objective is filled with oil (cedar wood oil) in microscopes of high resolving power. This has two advantages. Firstly the loss of light by reflection at the first lens surface is decreased and secondly the resolving power of the microscope is increased. The expression $\mu \sin \alpha$ in equation (*ii*) is called the **numerical aperture** of the objective of the microscope and is a characteristic of the particular objective used. The highest value of numerical aperture obtained in practice is about 1.6. Taking the effective wavelength of white light as 5500 Å and $\mu \sin \alpha = 1.6$,

$$d = \frac{5500 \times 10^{-8}}{2 \times 1.6} = 1.72 \times 10^{-5} \text{ cm}$$

where d is the linear distance between two just resolvable object points. It is clear from equation (*ii*), that decrease of λ_0 and increase of numerical aperture of the objective decreases the value of d and hence the resolving power of the microscope is increased.

An oil immersion objective has higher numerical aperture than an ordinary objective. The resolving power of a microscope can be considerably increased by decreasing the value of λ_0. Thus, by using ultraviolet light and quartz lenses, the resolving power of the microscope can be increased further. In this case the image is photographed. Such a microscope is called an **ultra microscope.**

The magnifying power of a microscope is said to be normal if the diameter of the exit pupil is equal to the diameter of the pupil of the eye. If the magnifying power is higher than the normal, it does not correspondingly help in observing better details of the object. If the magnifying power of the microscope is less than the normal, then this means that full advantage of the available resolving power of the microscope objective is not taken.

The theory of the electron microscope is given in Chapter 12. In an electron microscope, a beam of electrons emitted from or transmitted through the different parts of the object is focussed by electric and

magnetic fields. Electrons behave like waves. The wavelength depends on the voltage through which the electron beam is accelerated. For an accelerating voltage of 10,000 volts, λ_0 is of the order of 0.12 Å. This wavelength is more than thousand times smaller than the wavelength of visible light. Hence, the resolving power of an electron microscope is much higher than an ordinary microscope However, the numerical aperture of an electron microscope is smaller than an ordinary microscope.

Example 9.44. *Sodium light of wavelength 5890 Å is used to view an object under a microscope. The aperture of the objective has a diameter of 0.9 cm*

(a) Calculate the limiting angle of resolution

(b) Using visible light, what is the maximum limit of resolution for this microscope.

(*a*) Limiting angle of resolution

$$\theta_m = 1.22\left(\frac{\lambda}{d}\right)$$

Here,

$$\lambda = 5890 \text{ Å} = 5.89 \times 10^{-7} \text{ m}$$

$$d = 0.9 \text{ cm} = 9 \times 10^{-3} \text{ m}$$

$$\theta_m = 1.22\left[\frac{5.89 \times 10^{-7}}{9 \times 10^{-3}}\right]$$

$$\theta_m = \mathbf{7.98 \times 10^{-5} \text{ radian}}$$

It means that any two points on the object subtending an angle less than 7.98×10^{-5} radian at the objective cannot be distinguished in the image.

(*b*). The wavelength of violet light in the visible spectrum is 4000 Å. The maximum limit of resolution corresponds to the smallest angle.

Here

$$\lambda = 4000 \text{ Å} = 4 \times 10^{-7} \text{ m}$$

$$\theta_m = 1.22\left(\frac{\lambda}{d}\right)$$

$$\theta_m = 1.22\left[\frac{4 \times 10^{-7}}{9 \times 10^{-3}}\right]$$

$$\theta_m = \mathbf{5.42 \times 10^{-5} \text{ radian}}$$

9.57 PHASE CONTRAST MICROSCOPE

When a beam of light passes through a *transparent object*, phase difference is produced but no change in amplitude takes place. The eye can distinguish only changes in intensity but not changes in phase. In case one wishes to see a small transparent object, say *unstained bacteria*, it is necessary to magnify it and also to convert differences in phase into differences in intensity. *Zernike* in 1935 introduced the concept of phase contrast.

Consider a beam of light passing through a transparent plate of varying thickness. The amplitude vector at the points *A, B, C* has the same

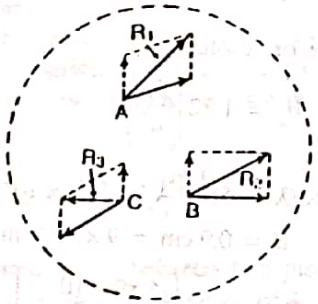


Fig. 9. 71

magnitude but is in different directions. The intensity is the same at all points but there is difference in phase between the vectors (Fig. 9.71). This

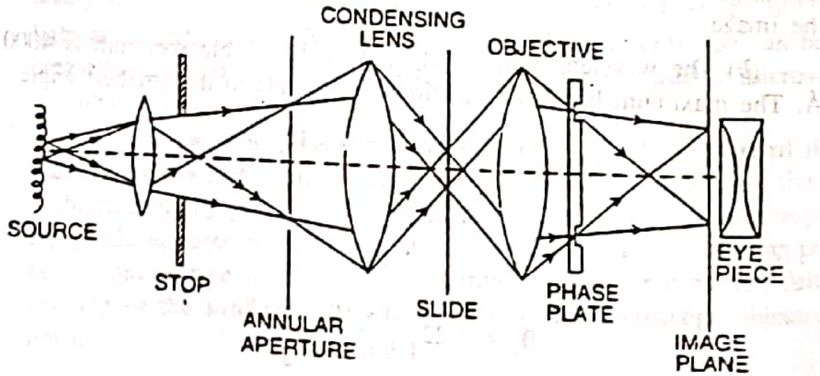


Fig. 9.72

difference in phase cannot be seen by the eye. However, if a constant phase difference is introduced with the help of a phase plate (represented by the dotted vector), the resultant amplitudes at the points *A, B,* and *C*

are R_1, R_2 and R_3 respectively. Their magnitudes are different and hence the intensities are different and can be seen by the eye. Thus, by introducing the phase plate, the object slide can be seen by phase contrast with the help of the phase plate. Phase changes due to varying thickness are converted to amplitudes of varying magnitude at different points.

A phase contrast microscope consists of a transparent object slide placed in between the condenser and the objective. An annular aperture is placed at the first focal plane of the condenser and the phase plate is kept at the second focal plane of the objective. The phase plate has an annular depression exactly conjugate to the annular aperture.

In the figure 9.72, the path of zero order light is shown. It is clear that the light passes through the thinner part of the phase plate and consequently its phase is advanced in comparison to the light diffracted through the thicker part of the plate. If t is the thickness of the depression and μ is the refractive index of the phase plate, the advance in phase of the zero order light is $\frac{2\pi}{\lambda}(\mu - 1)t$. For maximum efficiency, the image of the annular aperture should coincide exactly with the depression in the phase plate. The final image is formed at the first focal plane of the eyepiece.

Example 9.45. *A plane transmission grating has 40,000 lines per cm. These rulings are studied with an oil immersion microscope with light of wavelength 4000 Å. Calculate the numerical aperture of the microscope required to just resolve the rulings.*

The least distance between two just resolvable object points with a microscope is,

$$d = \frac{\lambda_0}{2\mu \sin\alpha}$$

The numerical aperture of the microscope,

$$\mu \sin\alpha = \frac{\lambda_0}{2d}$$

Here $\quad \lambda_0 = 4000 \times 10^{-8}$ cm

$$d = \frac{1}{40,000} \text{ cm}$$

$\therefore$ Numerical aperture

$$= \frac{4000 \times 10^{-8} \times 40,000}{2 \times 1} = 0.8$$

9.58 RESOLVING POWER OF A PRISM

The term resolving power is applied to spectrographic devices using a prism or a grating. Resolving power signifies the ability of the instrument to form separate spectral images of two neighbouring wavelengths, λ and $\lambda + d\lambda$ in the wavelength region λ.

In Fig. 9.73, S is a source of light, L_1 in a collimating lens and L_2 is the telescope objective. As the two wavelengths λ and $\lambda + d\lambda$ are very

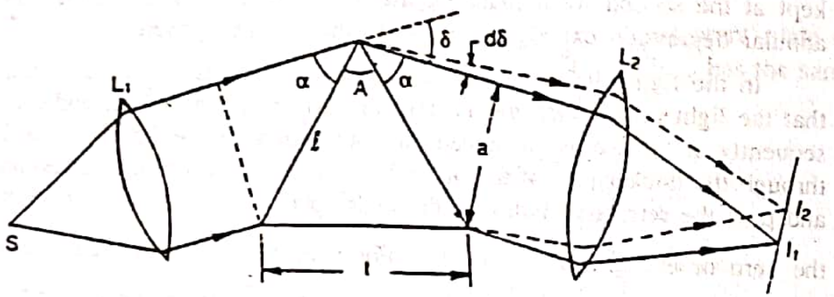


Fig. 9.73

close, if the prism is set in the minimum deviation position it would hold good for both the wavelengths. The final image I_1 corresponds to the principal maximum for wavelength λ and image I_2 corresponds to the principal maximum for wavelength $\lambda + d\lambda$. I_1 and I_2 are formed at the focal plane of the telescope objective L_2. The face of the prism limits the incident beam to a rectangular section of width a. Hence, the Rayleigh criterion can be applied in the case of a rectangular aperture.

In the case of diffraction at a rectangular aperture, the position of I_2 will correspond to the first minimum of the image I_1 for wavelength λ, provided

$$a.d\delta = \lambda$$

or

$$d\delta = \frac{\lambda}{a} \qquad \text{...}(i)$$

Here δ is the angle of minimum deviation for wavelength λ.

From the figure,

$$\alpha + A + \alpha + \delta = \pi$$

$$\alpha = \left[\left(\frac{\pi}{2}\right) - \left(\frac{A+\delta}{2}\right)\right]$$

$$\therefore \qquad \sin\alpha = \sin\left[\frac{\pi}{2} - \left(\frac{A+\delta}{2}\right)\right]$$

or

$$\sin\alpha = \cos\left(\frac{A+\delta}{2}\right)$$

But

$$\sin\alpha = \frac{a}{l}$$

$$\therefore \qquad \cos\left(\frac{A+\delta}{2}\right) = \frac{a}{l} \qquad \text{...}(iii)$$

Also

$$\sin\frac{A}{2} = \frac{t}{2l} \qquad \text{...}(iii)$$

In the case of a prism

$$\mu = \frac{\sin\dfrac{A+\delta}{2}}{\sin\dfrac{A}{2}}$$

$$\therefore \qquad \sin\frac{A+\delta}{2} = \mu\sin\frac{A}{2} \qquad \text{...}(iv)$$

Here μ and δ are dependent on wavelength of light λ.

Differentiating equation (iv) with respect to λ

$$\frac{1}{2}\left(\cos\frac{A+\delta}{2}\right)\frac{d\delta}{d\lambda} = \frac{d\mu}{d\lambda}\left(\sin\frac{A}{2}\right) \qquad \text{...}(v)$$

Substituting the values from equations (ii) and (iii),

$$\frac{1}{2}\left(\frac{a}{l}\right)\frac{d\delta}{d\lambda} = \frac{d\mu}{d\lambda}\left(\frac{t}{2l}\right)$$

or

$$a.\frac{d\delta}{d\lambda} = t.\frac{d\mu}{d\lambda} \qquad \text{...}(vi)$$

Substituting the values of $d\delta$ from equation (i),

$$\frac{\lambda}{d\lambda} = t.\frac{d\mu}{d\lambda} \qquad \text{...}(vii)$$

The expression $\frac{\lambda}{d\lambda}$ measures the resolving power of the prism.

It is defined as the ratio of the wavelength of a spectral line to the difference in wavelength between this line and a neighbouring line such that the two lines appear just resolved, according to Rayleigh's criterion.

$\therefore$ Resolving power of a prism

$$= t\frac{d\mu}{d\lambda}$$

Thus, the resolving power of a prism is (*i*) directly proportional to the length of the base and (*ii*) rate of change of refractive index with respect to wavelength for that particular material. The expression for resolving power given above is applicable only to spectral lines of equal intensity. If two spectral lines are of different intensities, then the value of $d\lambda$ *i.e.*, the difference in wavelength between the two lines must be higher so that the two lines appear as separate ones.

Example 9.46. *The refractive indices of a glass prism for the C and F lines are 1.6545 and 1·6635 respectively. The wavelengths of these two lines in the solar spectrum are 6563 Å and 5270 Å respectively. Calculate the length of the base of a 60° prism which is capable of resolving sodium lines of wavelengths 5890 Å and 5896 Å.* (*Vikram University*)

Resolving power $= \frac{\lambda}{d\lambda} = t\frac{d\mu}{d\lambda}$

Here $\frac{d\mu}{d\lambda} = \frac{1.6635 - 1.6545}{(6563 - 5270)\,10^{-8}}$

$$\therefore \quad \frac{\lambda}{d\lambda} = t\left[\frac{1.6635 - 1.6545}{(6563 - 5270)\,10^{-8}}\right]$$

$$\lambda = 5893 \times 10^{-8}\text{ cm}, \quad d\lambda = (5896 - 5890)\,10^{-8}$$

$$= 6 \times 10^{-8}\text{ cm}$$

$$\therefore \quad t = \frac{5893 \times 10^{-8}}{6 \times 10^{-8}}\left[\frac{(6563 - 5270)\,10^{-8}}{1.6635 - 1.6545}\right]$$

$$= \mathbf{1.41\ cm}$$

Example 9.47. *Calculate the minimum thickness of the base of a prism which will just resolve the D_1 and D_2 lines of sodium. Given μ for wavelength 6563 Å = 1.6545 and for wavelength 5270 Å = 1.6635.* [*Bombay*]

In a prism,

Resolving power, $\frac{\lambda}{d\lambda} = t\frac{d\mu}{d\lambda}$

Here $\frac{d\mu}{d\lambda} = \left[\frac{1.6635 - 1.6545}{(6563 - 5270) \times 10^{-8}}\right] = \left(\frac{0.0090}{1293 \times 10^{-8}}\right)$

and $\frac{\lambda}{d\lambda} = \frac{5893 \times 10^{-8}}{6 \times 10^{-8}} = \frac{5893}{6}$

$$\therefore \quad t = \frac{\left(\frac{\lambda}{d\lambda}\right)}{\left(\frac{d\mu}{d\lambda}\right)}$$

$$\therefore \quad t = \left(\frac{5893 \times 1293 \times 10^{-8}}{6 \times 0.0090}\right)\text{ cm}$$

$$t = \mathbf{1.41\ cm}$$

9.59 RESOLVING POWER OF A PLANE DIFFRACTION GRATING

The resolving power of a grating is defined as the ratio of the wavelength of any spectral line to the difference in wavelength between this line and a neighbouring line such that the two lines appear to be just resolved. Thus, the resolving power of a grating appear to be just resolved. Thus, the resolving power of a grating

$$= \frac{\lambda}{d\lambda}$$

In Fig. 9.74, XY is the grating surface and MN is the field of view of the telescope, P_1 is the n th primary maximum of a spectral line of wavelength λ at an angle of diffraction θ_n. P_2 is the n th primary maximum

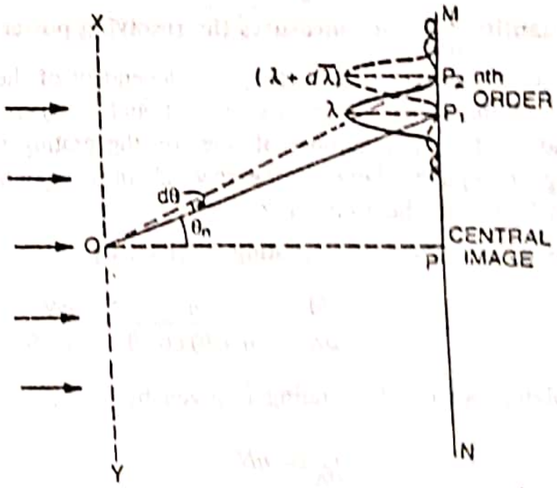


Fig. 9.74

of a second spectral line of wavelength $\lambda + d\lambda$ at a diffracting angle $\theta_n + d\theta$. P_1 and P_2 are the spectral lines in the n th order. These two spectral lines according to Rayleigh, will appear resolved if the position of P_2 also corresponds to the first minimum of P_1.

The direction of the n th primary maximum for a wavelength λ is given by

$$(a+b)\sin\theta_n = n\lambda \qquad ...(i)$$

The direction of the n th primary maximum for a wavelength $(\lambda + d\lambda)$ is given by

$$(a+b)\sin(\theta_n + d\theta) = n(\lambda + d\lambda) \qquad ...(ii)$$

The two lines will appear just resolved if the angle of diffraction $(\theta_n + d\theta)$ also corresponds to the direction of the first secondary minimum after the n th primary maximum at P_1 (corresponding to wavelength λ). This is possible if the extra path difference introduced is $\frac{\lambda}{N}$ Where N is the total number of lines on the grating surface.

$$\therefore \quad (a+b)\sin(\theta_n + d\theta) = n\lambda + \frac{\lambda}{N} \qquad ...(iii)$$

Equating the right hand sides of equations (ii) and (iii)

$$n(\lambda + d\lambda) = n\lambda + \frac{\lambda}{N}; \quad n, d\lambda = \frac{\lambda}{N}$$

$$\frac{\lambda}{d\lambda} = nN \qquad ...(iv)$$

The quantity $\frac{\lambda}{d\lambda} = nN$ measures the resolving power of a grating.

Thus, the resolving power of a grating is independent of the grating constant. The resolving power is directly proportional to (i) the order of the spectrum and (ii) the total number of lines on the grating surface. For a given grating, the spacing between the spectral lines is double in the second order than that in the first order.

The dispersive power of a grating is given by

$$\frac{d\theta}{d\lambda} = \frac{n}{(a+b)\cos\theta} = \frac{nN'}{\cos\theta}$$

and the resolving power of a grating is given by

$$\frac{\lambda}{d\lambda} = nN$$

where n is the order of the spectrum, N is the total number of lines of the grating. N' is the number of lines per cm on the grating surface. Here, θ gives the direction of the n th principal maximum corresponding to a wavelength λ. From the above equation, it is clear, that the dispersive power increases with increase in the number of lines per cm and the resolving power increases, with increase in the total number of lines on the grating surface (*i.e.*, the width of the grating surface). If N' is the same for two gratings, the dispersive power will be the same in the two cases but the one with larger width of the grating surface produces higher resolution of the spectral lines. With a grating having large width of the grating surface, the spectral lines are sharp and narrow.

High dispersive power refers to wide separation of the spectral lines whereas high resolving power refers to the ability of the instrument to show nearby spectral lines as separate ones.

Example 9.48. *What should be the minimum number of lines in a grating which will just resolve in the second order the lines whose wavelengths are 5890 Å and 5896 Å ?* (Agra)

Resolving power $= \frac{\lambda}{d\lambda} = nN$

Here, $n = 2,\ \lambda = 5890\ \text{Å},\ d\lambda = 5896 - 5890 = 6\ \text{Å}$

$$\therefore \quad \frac{5890}{6} = 2N$$

or $$N = \frac{5890}{6\times 2} = \textbf{491 approximately.}$$

Example 9.49. *Calculate the minimum number of lines in a grating which will just resolve the sodium lines in the first order spectrum. The wavelengths are 5890 Å and 5896 Å.* (Delhi)

Resolving power $= \frac{\lambda}{d\lambda} = nN$

Here, $n = 1,\ \lambda = 5890\ \text{Å},\ = 5890\times 10^{-8}\ \text{cm}$

$$d\lambda = 5896 - 5890 = 6\ \text{Å} = 6\times 10^{-8}\ \text{cm}$$

$$N = \frac{1}{n}\left[\frac{\lambda}{d\lambda}\right]$$

$$= \frac{1}{1}\left[\frac{5890}{6}\right]$$

or $$n = \textbf{982 approximately.}$$

Example 9.50. *Calculate the minimum number of lines per cm in a 2.5 cm wide grating which will just resolve the sodium lines (5890 Å and 5896 Å) in the second order spectrum.* [Delhi(Hons) 1976]

Let the total number of lines required on the grating be N.

$$\frac{\lambda}{d\lambda} = nN$$

Here $\lambda = 5890 \times 10^{-8}$ cm

$d\lambda = 6 \times 10^{-8}$ cm; $n = 2$

$N = ?$

$$N = \frac{\lambda}{n d\lambda}$$

$$N = \frac{5890 \times 10^{-8}}{2 \times 6 \times 10^{-8}} = 491$$

Width of the grating = 2.5 cm

Number of lines per cm

$$= \frac{491}{2.5} = 196.4$$

Example 9.51. *Calculate the least width of a plane diffraction grating having 500 lines/cm, which will just resolve in the second order the sodium lines of wavelengths 5890 Å and 5896 Å.*

[Berhampur (Hons) 1987]

Let the total number of lines required on the grating be N.

$$\frac{\lambda}{d\lambda} = nN$$

Here $\lambda = 5890 \times 10^{-8}$ cm, $d\lambda = 6 \times 10^{-8}$ cm

$n = 2$, $N = ?$

$$N = \frac{\lambda}{n\, d\lambda}$$

$$N = \frac{5890 \times 10^{-8}}{2 \times 6 \times 10^{-8}}$$

$$N = 491$$

Least width of the grating

$$= \frac{491}{500} = 0.982 \text{ cm}$$

Example 9.52. *Examine if two spectral lines of wavelengths 5890 Å and 5896 Å can be clearly resolved in the (i) first order and (ii) second order by a diffraction grating 2 cm wide and having 425 lines/cm.*

[Delhi B.Sc.(Hons.) 1985]

Total number of lines on the grating

$$= 2 \times 425 = 850$$

(1) For the first order

$$\frac{\lambda}{d\lambda} = nN. \quad \text{Here } n = 1$$

$$N = \frac{\lambda}{d\lambda} = \frac{5890 \times 10^{-8}}{6 \times 10^{-8}} = 982 \text{ lines}$$

As the total number of lines required for just resolution in the first order is 982 and the total number of lines on the grating is 850, the lines will not be resolved.

(2) For the second order

$$\frac{\lambda}{d\lambda} = nN, \quad n = 2$$

$$N = \frac{5890 \times 10^{-8}}{2 \times 6 \times 10^{-8}} = 491$$

As the total number of lines required is 491, and the given grating has a total of 850 lines, the lines will appear resolved in the second order.

Example 9.53. *Light is incident normally on a grating of total ruled width 5×10^{-3} m with 2500 lines in all. Calculate the angular separation of the two sodium lines in the first order spectrum. Can they be seen distinctly?* [IAS. 1983]

(i) Here, $N = 2500$

Width of the ruling $= 5 \times 10^{-3}$ m

Number of lines/metre $= \dfrac{2500}{5 \times 10^{-3}} = 5 \times 10^5$

$\therefore$ $$(a+b) = \frac{1}{5 \times 10^5} = 2 \times 10^{-6} \text{ m}$$

$$n = 1$$

For the first order $$\sin\theta_1 = \frac{n\lambda_1}{(a+b)} = \frac{\lambda_1}{(a+b)}$$

$$\sin\theta_2 = \frac{n\lambda_2}{(a+b)} = \frac{\lambda_2}{(a+b)}$$

$$\lambda_1 = 5890 \times 10^{-10} \text{ m}$$

$$\lambda_2 = 5896 \times 10^{-10} \text{ m}$$

$$\sin\theta_1 = \frac{5890 \times 10^{-10}}{2 \times 10^{-6}}$$

$$\sin\theta_1 = 2945 \times 10^{-4} = 0.2945$$

$$\theta_1 = 17^\circ - 8'$$

$$\sin\theta_2 = \frac{5896 \times 10^{-10}}{2 \times 10^{-6}}$$

$$\sin\theta_2 = 0.2948$$

$$\theta_2 = 17^\circ - 9'$$

(*ii*) The resolving power of the grating,

$$\frac{\lambda}{d\lambda} = \frac{5890 \times 10^{-10}}{6 \times 10^{-10}}$$

$$= \mathbf{982}$$

As the total number of lines on the grating is 2500 which is more than 982, the lines can be see distinctly.

Example 9.54. *The wavelengths of sodium D lines are 589.593 μm and 588.996 μm. What is the minimum number of lines that a grating must have in order to resolve these lines in the first order spectrum.*

[IAS, 1985]

Here

$$\lambda_1 = 589.593 \times 10^{-6} \text{ m}$$

$$\lambda_2 = 588.996 \times 10^{-6} \text{ m}$$

$$\Delta\lambda = \mathbf{0.597 \times 10^{-6} \text{ m}}$$

$$\frac{\lambda}{\Delta\lambda} = nN$$

Here $n = 1$

$$\therefore \quad N = \frac{589.593 \times 10^{-6}}{0.597 \times 10^{-6}} = \mathbf{988}$$

Therefore, the minimum number of lines required for just resolution in the first order is **988**.

Example 9.55 *In the second order spectrum of a plane transmission grating, a certain spectral line appears at an angle of 10° while for another wavelength which is 5×10^{-9} cm is higher, the corresponding line is observed at an angle of 10° 3''. Find the wavelength of the two lines and the maximum grating width required to resolve them.*

(sin 10° = 0.1736 and cos 10° = (1.9848).

[Delhi(Hons.1985]

Dispersive power of the grating

$$\frac{d\theta}{d\lambda} = \frac{n}{(a+b)\cos\theta}$$

$$(a+b) = \frac{n}{\cos\theta\left(\dfrac{d\theta}{d\lambda}\right)}$$

Here,

$$n = 2$$

$$\cos\theta = \cos 10^\circ = 0.9848$$

$$d\theta = 3'' = \frac{3\pi}{60 \times 60 \times 180} \text{ radian}$$

$$d\lambda = 50 \times 10^{-8} \text{ cm}$$

$$\frac{d\theta}{d\lambda} = \frac{3\pi}{50 \times 60 \times 180 \times 50 \times 10^{-8}}$$

$$= \frac{3 \times 22 \times 10^8}{7 \times 60 \times 60 \times 180 \times 50}$$

$$= \frac{11000}{378}$$

$$\therefore \quad (a+b) = \frac{2 \times 378}{0.9848 \times 11000}$$

$$= 0.06979 \text{ cm}$$

$$(a+b)\sin\theta_1 = n\lambda_1$$

$$\lambda_1 = \frac{(a+b)\sin\theta_1}{n}$$

$$(a+b) = 0.06979$$

$$\sin\theta_1 = \sin 10^\circ = 0.1736$$

$$n = 2$$

$$\therefore \quad \lambda_1 = \frac{0.06179 \times 0.1736}{2}$$

$$= 6055 \times 10^{-8} \text{ cm} = 6055 \text{ Å}$$

$$\lambda_2 = \lambda_1 + 50 \text{ Å} = 6055 + 50$$

$$= \mathbf{6105 \text{ Å}}$$

Let N be the total number of lines on the grating surface for the just resolution of the spectral lines.

$$\frac{\lambda}{d\lambda} = nN$$

$$\frac{6055 \times 10^{-8}}{50 \times 10^{-8}} = 2N$$

$$N = \frac{6055 \times 10^{-8}}{2 \times 50 \times 10^{-8}}$$

$$= 61 \text{ (approximately)}$$

Grating width $= N(a+b)$

$$= 61 \times 0.06979$$

$$= \mathbf{4.25 \text{ cm}}$$

Example 9.56. *A grating has 1000 lines ruled on it. In the region of wavelength $\lambda = 6000$ Å, find (i) the difference between two wavelengths that just appear separated in the first order and (ii) the resolving power in the second order spectrum.* *(Delhi, 1990)*

Here $N = 1000$

$$\lambda = 6000 \text{ Å} = 6 \times 10^{-7} \text{ m}$$

(i) $$\frac{\lambda}{d\lambda} = nN$$

$$n = 1$$

$$\therefore \quad d\lambda = \frac{\lambda}{nN}$$

$$= \frac{6 \times 10^{-7}}{1 \times 1000}$$

$$= 6 \times 10^{-10} \text{ m}$$

$$= 6 \text{ Å}$$

(ii) for the second order

Resolving power,

$$\frac{\lambda}{d\lambda} = nN$$

Here $n = 2$

$$\frac{\lambda}{d\lambda} = 2 \times 1000$$

$$= \mathbf{2000}$$

Example 9.57. *A transmission grating 4 cm long has 4000 lines/cm. Compute the resolving power of the grating for $\lambda = 5900$ Å in the first order spectrum. Will this grating separate the two lines of $\lambda = 5890$ Å and $\lambda = 5896$ Å which constitute the sodium line doublet ?*

[Delhi(Hons) 1990]

Resolving power of a grating

$$\frac{\lambda}{d\lambda} = nN$$

Here, $n = 1$

and $N = 4000 \times 4 = 16000$ lines

$$\therefore \quad \frac{\lambda}{d\lambda} = 1 \times 16000$$

$$= \mathbf{16000}$$

For wavelengths 5890 Å and 5896 Å

Resolving power

$$\left(\frac{\lambda}{d\lambda}\right) = nN'$$

$$N' = \frac{1}{n}\left(\frac{\lambda}{d\lambda}\right)$$

$$= 1\left(\frac{5900}{6}\right)$$

$$N' = \mathbf{983}$$

As N is greater that N', the grating will separate the two spectral lines.

Example 9.58. *Examine if two spectral lines of wavelengths 5890 Å and 5896 Å can be clearly resolved in (1) the first order and (2) the second order by a diffraction grating 2 cm wide and having 425 lines per cm.* [*Kanpur, 1990*]

Total number of lines on the grating

$$= 2 \times 425 = 850$$

Here,

$$\lambda = 5890 \text{ Å} = 5890 \times 10^{-10} \text{ m}$$

$$d\lambda = 6 \text{ Å} = 6 \times 10^{-10} \text{ m}$$

(1) For the first order,

$$\frac{\lambda}{d\lambda} = nN$$

$$n = 1$$

$$N = \frac{\lambda}{d\lambda}$$

$$N = \frac{5890 \times 10^{-10}}{6 \times 10^{-10}}$$

$$N = 982 \text{ lines}$$

As the total number of lines required for just resolution in the first order is 982, and the total number of lines on the grating is 850, the lines will not be resolved.

(2) For the second order

$$n = 2$$

$$\frac{\lambda}{d\lambda} = nN = 2N$$

$$N = \frac{5890 \times 10^{-10}}{2 \times 6 \times 10^{-10}}$$

$$N = 491$$

As the total number of lines required is 491 and the given grating has a total of 850 lines, the lines will appear resolved in the second order.

Example 9.59. *In the second order spectrum of a plane diffraction grating, a certain spectral line appears at an angle of 10°, while another line of wavelength 5×10^{-9} cm higher appears at an angle 3" more. Find the wavelengths of the lines and the minimum grating width required to resolve them. Give sin 10° = 0.1736 and cos 10° = 0.9848.*

[*Delhi(Hons) 1991*]

Dispersive power of the grating

$$\frac{d\theta}{d\lambda} = \frac{n}{(a+b)\cos\theta}$$

$$(a+b) = \frac{n}{\cos\theta\left(\dfrac{d\theta}{d\lambda}\right)}$$

Here,

$$n = 2$$

$$\cos\theta = \cos 10° = 0.9848$$

$$d\theta = 3'' = \frac{3\pi}{60 \times 60 \times 180} \text{ radian}$$

$$d\lambda = 5 \times 10^{-8} \text{ cm} = 5 \times 10^{-11} \text{ m} = 0.5 \text{ Å}$$

$$\frac{d\theta}{d\lambda} = \frac{3\pi}{60 \times 60 \times 180 \times 5 \times 10^{-11}}$$

$$= 2.91 \times 10^{5}$$

$$\therefore \quad (a+b) = \frac{2}{0.9848 \times 2.91 \times 10^{5}}$$

$$= 6.978 \times 10^{-6} \text{ m}$$

$$(a+b)\sin\theta_1 = n\lambda_1$$

$$\lambda_1 = \frac{(a+b)\sin\theta_1}{n}$$

Here,

$$(a+b) = 6.978 \times 10^{-6} \text{ m}$$

$$\sin\theta_1 = \sin 10° = 0.1736$$

$$n = 2$$

$$\lambda_1 = \frac{6.978 \times 10^{-6} \times 0.1736}{2}$$

$$= 6057 \times 10^{-10} \text{ m}$$

$$= 6057 \text{ Å}$$

$$\lambda_2 = 6057 + 0.5$$

$$= 6057.5 \text{ Å}$$

Let N be the total number of lines on the grating surface for the just resolution of the spectral lines

$$\frac{\lambda}{d\lambda} = nN$$

$$\frac{6057 \times 10^{-10}}{0.5 \times 10^{-10}} = 2 \times N$$

$$N = 6057$$

Grating width $= N(a+b)$

$= 6057 \times 6.978 \times 10^{-6}$

$= \mathbf{4.226 \times 10^{-2}\ m}$

$= \mathbf{4.226\ cm}$

Example 9.60. *Examine whether the D_1 and D_2 lines of sodium will be clearly separated in the*

(i) first order and (ii) second order by a one inch grating having 300 *lines*/cm.

[*λ for D_1 and D_2 lines are 5896 Å and 5890 Å*] [*Delhi, 1992*]

Here total number of lines on the grating

$$= 2.54 \times 300 = 762$$

(1) For the first order,

Resolving power,

$$\frac{\lambda}{d\lambda} = nN$$

Here $n = 1, \quad \lambda = 5890\ \text{Å}, \quad d\lambda = 6\ \text{Å}$

$$\therefore \quad \frac{5890}{6} = N$$

$$N = \mathbf{982}$$

As the number of lines required is more than the lines provided on the grating, the lines **will not be** resolved in the first order.

(*ii*) For the second order,

$$\frac{\lambda}{d\lambda} = nN$$

$$n = 2$$

$$\frac{5890}{6} = 2N$$

$$N = \mathbf{491}$$

As the number of lines required is less than the lines provided on the grating, the lines **will be resolved** in the second order.

9.60 MICHELSON'S STELLAR INTERFEROMETER

The smallest angular separation (θ) between two distant point sources for viewing the two images of the sources as separate with a telescope, is given by

$$\theta = \frac{1.22\,\lambda}{D} \quad ...(i)$$

where λ is the wavelength of light and D is the diameter of the objective of the telescope. Let the telescope objective be covered with a screen which is pierced with two parallel slits. Let the slit separation (d) be almost equal to the diameter of the objective of the telescope. A suitable value for $d = \frac{D}{1.22}$. Now let the telescope be directed towards a distant double star so that the line joining the two stars is perpendicular to the length of either slit. Interference fringes due to the double slit will be observed in the focal plane of the objective. The condition for the first disappearance of fringes is given by

$$\alpha = \frac{\lambda}{2d} = \frac{1.22\,\lambda}{2D} = \frac{\theta}{2} \quad ...(ii)$$

where α is the angular separation between the two stars when the first disappearance of the fringes takes place. Similarly, for values of α given by the multiples of $\frac{\lambda}{2d}$, disappearance of the fringes can be observed. If the double slit is avoided and the observations are made directly, the multiples can be ruled out. The angular separation α is half the angle θ, where θ is the minimum angle of resolution of the telescope objective.

The method employing the principle of double slit interference is used to measure the angular diameter of the disc of a star rather than the angular separation between two stars.

Michelson in 1920, successfully used this method to find the diameters of stars. The arrangement is known as Michelson's stellar interferometer (Fig. 9.75). It consists of four mirrors M_1, M_2, M_3 and M_4 arranged as shown in the figure. L is the objective of the telescope and the two slits are kept in the paths of light reflected from the mirrors M_3 and M_4. Let S_1 and S_2 be the ends of a diameter of the star. The paths of the rays of light from these two points S_1 and S_2 are shown in the figure. The mirrors M_1 and M_3 are parallel. Also mirrors M_2 and M_4 are parallel. The mirrors M_1 and M_2 are mounted on a girder and by sliding these mirrors, the distance D between the mirrors can be altered. The silvered faces of M_1 and M_2 (and M_2 and M_4) face each other. Interference fringes will be observed in the field of view of the telescope. The path difference between the rays of light from M_1 to L and M_2 to L is zero.

In the side figure, A is the point of incidence of the rays of light on the mirror M_2 and B is the point of incidence of the rays of light on the mirror M_1. The path difference between the rays travelling from S_2 (one end of the diameter of the disc of the star) and reaching A and B is equal to the distance BC.

From the $\Delta\, ABC$

$$\theta = \frac{BC}{D}$$

or $$BC = D\theta$$

For the first disappearance of the fringes, this path difference must be equal to 1.22λ.

or $$D\theta = 1.22\,\lambda$$

or $$\theta = \frac{1.22\,\lambda}{D} \quad \ldots (iii)$$

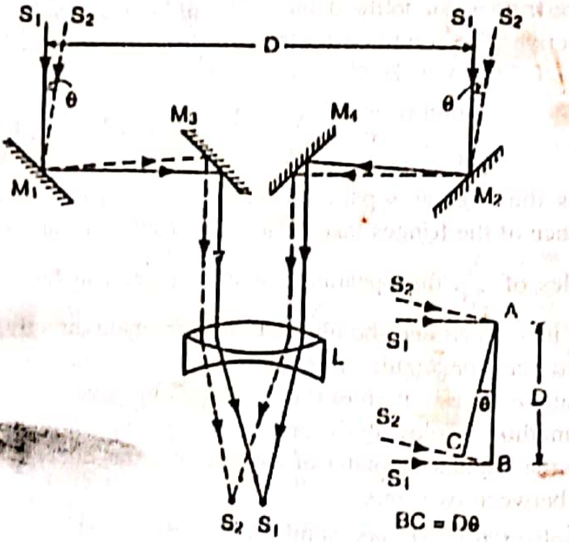


Fig. 9.75

In equation (*iii*), θ measures the angular diameter of the star.

In one of the experiments of Michelson, using a 250 cm reflecting telescope at Mount Wilson observatory, the disappearance of the fringes was observed when the distance between the mirrors M_1 and M_2 was 306.5 cm. If the average wavelength of light from the star is assumed to be 5750 Å, the angular diameter of the star can be calculated from the equation

$$\theta = 1.22\,\lambda$$

Here $\lambda = 5750\,\text{Å} = 5750 \times 10^{-2}$ cm, $D = 306.5$ cm

$$\theta = \frac{1.22 \times 5750 \times 10^{-2}}{306.5} \times \frac{180}{\pi} \times 60 \times 60 \quad \text{second of an arc}$$

$$= 0.04718 \text{ second of an arc}$$

EXERCISES IX

1. Give an account of the wave theory of light and explain the approximate rectilinear propagation of light on this theory. (*Delhi, Madras*)
2. What is a zone plate and how is it made ? Explain how a zone plate acts like a convergent lens having multiple foci. Derive an expression for its focal length. (*Lucknow*)
3. Explain the term half period zone in relation to a plane wavefront. Show that the amplitude due to a complete wavefront at a point is half of what would be caused by the first zone. (*Agra*)
4. Give an account of the phenomenon and the related theory of diffraction due to a straight edge. (*Gorakhpur, Mysore, Lucknow*)
5. Describe and explain the diffraction pattern formed by a narrow wire illuminated by monochromatic light from a narrow slit parallel to the wire. How do you use the pattern to measure the thickness of a wire ?
6. A sharp razor blade is held vertically in a beam of white light diverging from a fine aperture. Discuss the fringes obtained on a screen placed behind the blade.
7. Describe and explain the Fraunhofer diffraction pattern obtained with a narrow slit and illuminated by a parallel beam of monochromatic light. (*Mysore*)
8. Explain the construction and mode of action of a diffraction grating and derive an expression for its resolving power. (*Lucknow, Agra*)
9. Give the theory of a concave grating. Give a short account of the important methods of mounting the gratings with their respective advantages and disadvantages. (*Gorakhpur*)
10. Describe Rowland's concave grating and a method of mounting it for measuring the wavelength of light. Deduce the formula used.
11. Deduce an expression for the resolving power of a prism spectroscope.
12. What do you understand by the resolving power of a telescope ? How would you measure it experimentally ?
13. Define resolving power and dispersive power of a grating. Obtain expressions for these in the case of a plane transmission grating.
14. In what respect is an echelon grating superior to an ordinary ruled grating ? Give an account of its theory and its practical applications.
15. Derive an expression for the angular dispersion of a plane diffraction grating.
16. What is meant by resolving power of an optical instrument ? Distinguish between resolving power and magnifying power of a microscope. (*Delhi (Hons), Punjab*)