(a)

1	0	x
$\pi/2$	0	0

Beta function: The function denoted by $B(m,n)$, where m and n are positive values $[m,n>0]$ and if defined by $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ is known as Beta function. It is also called the first Eulerian integral.

Gamma function: The function is denoted by Γn and if defined by $\Gamma n = \int_0^\alpha e^{-x}x^{n-1}dx$ where $n>0$ is known as gamma function. It is also known as second Eulerian integral.

Soln: We know that

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Let $x = \sin^2\theta$

x	0	1
θ	0	$\pi/2$

$dx = 2\sin\theta\cos\theta\,d\theta$

Putting this value in the equ (1) we get,

$$B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\,2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta\cos^{2n-2}\theta\,\sin\theta\cos\theta\,d\theta$$

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

$$\frac{B(m,n)}{2} = \int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

Let, $2m-1 = p$ and $2n-1 = q$

$m = \frac{p+1}{2}$ $\qquad$ $n = \frac{q+1}{2}$

And we know,

$$B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

Therefore

$$\frac{\Gamma m\,\Gamma n}{2(m+n)} = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)} = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$$\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+1}{2}} = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$$\therefore \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}} = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

(b) $$\int_0^{\alpha} \frac{x^{n-1}}{1+x} \, dx$$

x	0	α
θ	0	$\pi/2$

Let, $x = \tan^2\theta$

$dx = 2\tan\theta \sec^2\theta \, d\theta$

Now, $$\int_0^{\alpha} \frac{x^{n-1}}{1+x} \, dx$$

$$= \int_0^{\pi/2} \frac{(\tan^2\theta)^{n-1}}{1+\tan^2\theta} \, 2\tan\theta \sec^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\tan^{2n-2}}{\sec^2\theta} \cdot \tan\theta \sec^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \tan^{2n-1}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sin^{2n-1}\theta}{\cos^{2n-1}\theta} \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n-1} \cdot \cos^{(2n-1)}\theta \, d\theta$$

$$= \frac{2\,\Gamma\left(\frac{2n-1+1}{2}\right)\Gamma\left(\frac{-2n+1+1}{2}\right)}{2\,\Gamma\left(\frac{2n-1+2n+1+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{2n}{2}\right)\Gamma\left(\frac{-2n+2}{2}\right)}{\Gamma\frac{2}{2}}$$

$$= \frac{\Gamma n \, \Gamma 1-n}{\Gamma 1}$$

$$= \Gamma n \, \Gamma 1-n \quad (\text{Ans})$$

(2) $\int_0^1 x^{m-1}(1-x^a)^n\,dx$

Let $x^a = \sin^2\theta$

$x = \sin^{2/a}\theta$

$a\,x^a x^{-1}\,dx = 2\sin\theta\cos\theta\,d\theta$

$dx = \frac{2}{a}\sin^{-2}\theta\,\sin^{2/a}\theta\,\sin\theta\cos\theta\,d\theta$

Now, $\int_0^1 x^{m-1}(1-x^a)^n\,dx$

x	0	1
θ	0	$\pi/2$

$= \int_0^{\pi/2} (\sin^{2/a}\theta)^{m-1}(1-\sin^2\theta)^n \cdot \frac{2}{a}\sin^{2/a-2+1}\cos\theta\,d\theta$

$= \int_0^{\pi/2} \sin^{\frac{2m}{a}-\frac{2}{a}}\theta\,\cos^{2n}\theta\;\frac{2}{a}\sin^{2/a-1}\theta\cos\theta\,d\theta$

$= \frac{2}{a}\int_0^{\pi/2}\sin^{\frac{2m}{a}-\frac{2}{a}+\frac{2}{a}-1}\cos^{2n+1}\theta\,d\theta$

$= \frac{2}{a}\int_0^{\pi/2}\sin^{\frac{2m}{a}-1}\theta\,\cos^{2n+1}\theta\,d\theta$

$= \frac{2}{a}\times\frac{\Gamma\left(\frac{2m/a-1+1}{2}\right)\Gamma\left(\frac{2n+1+1}{2}\right)}{2\,\Gamma\left(\frac{\frac{2m-1}{a}+2n+1+2}{2}\right)}$

$$= \frac{1}{a} \times \frac{\Gamma\frac{m}{a} \times n!}{\Gamma\left(\frac{m}{a}+n+1\right)}$$

(Ans)

$$\int_0^1 \frac{35}{32} \frac{x^3}{\sqrt{1-x}}\,dx$$

Let

$x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$

Limit

x	0	1
θ	0	$\pi/2$

$$\int_0^1 \frac{35}{32} \frac{x^3}{\sqrt{1-x}}\,dx$$

$$= \int_0^{\pi/2} \frac{35}{32} \frac{(\sin^2\theta)^3}{\sqrt{1-\sin\theta}}\, 2\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \frac{35}{32} \frac{\sin^6\theta}{\sqrt{\cos^2\theta}}\, 2\sin\theta\cos\theta\,d\theta$$

$$= \frac{35}{16}\int_0^{\pi/2} \frac{\sin^7\theta}{\cos\theta}\cos\theta\,d\theta$$

$$= \frac{35}{16}\int_0^{\pi/2} \sin^7\theta\cdot\cos^0\theta\,d\theta$$

$$= \frac{35}{16}\, \frac{\Gamma\frac{7+1}{2}\,\Gamma\frac{0+1}{2}}{2\,\Gamma\frac{7+0+2}{2}}$$

$$\frac{35}{16}\;\frac{\Gamma 4\;\;\Gamma \tfrac{1}{2}}{2\,\Gamma \tfrac{9}{2}}$$

$$= \frac{35}{32} \times \frac{3!\;\Gamma\tfrac{1}{2}}{\Gamma\left(\tfrac{7}{2}+1\right)}$$

$$= \frac{35}{32} \times \frac{3!\;\Gamma\tfrac{1}{2}}{\tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\cdot\Gamma\tfrac{1}{2}}$$

$$= \frac{35}{16} \times \frac{3\times2\times1}{\tfrac{7}{2}\times\tfrac{5}{2}\times\tfrac{3}{2}}$$

$$= \frac{16}{16}$$

$= 1$ (Ans)

2. (a) Laplace transformation of derivative

1st derivative:

$$L\left\{\frac{df(t)}{dt}\right\} = \int_0^\alpha e^{-st}\,\frac{df(t)}{dt}\,dt$$

$$= \left[e^{-st} f(t)\right]_0^\alpha + s\int e^{-st} f(t)\,dt$$

$$= -f(0) + s f(s)$$

$$= s f(s) - f(0)$$

2nd derivative :

$$L\left\{\frac{d^2 f(t)}{dt^2}\right\} = \int_0^\alpha e^{-st} \frac{d^2 f(t)}{dt^2}\,dt$$

$$= \left[e^{-st} \frac{d f(t)}{dt}\right]_0^\alpha + s\int e^{-st} \cdot \frac{d f(t)}{dt} \cdot dt$$

$$= -\frac{d f(0)}{dt} + s\left[-f(0) + s\,f(s)\right]$$

$$= -f(0) - s f(0) + s^2 f(s)$$

Laplace transformation: Laplace transformation of a function $f(t)$ is defined by $L\{f(t)\}$

$$L\{f(t)\} = \int_0^\alpha e^{-st} f(t)\,dt = f(s)$$

(b) using linear property of laplace transformation we can write,

$$L\{\sin 2t + 2\cos 3t\} = L\{\sin 2t\} + 2L\{\cos 3t\}$$

Here,

$$L\{\sin 2t\} = \int_0^\alpha e^{-st} \sin 2t\,dt$$

$$= \int_0^{\alpha} e^{-st} \left(\frac{e^{2ist} - e^{-2it}}{2i} \right) dt$$

$$= \frac{1}{2i} \int_0^{\alpha} e^{-st+2it} \, dt - \frac{1}{2i} \int_0^{\alpha} e^{-st-2it} \, dt$$

$$= \frac{1}{2i} \int_0^{\alpha} e^{-(s-2i)t} \, dt - \frac{1}{2i} \int_0^{\alpha} e^{-(s+2i)t} \, dt$$

$$= \frac{1}{2i} \left[\frac{e^{-(s-2i)t}}{-(s-2i)} - \frac{e^{-(s+2i)t}}{(s+2i)} \right]_0^{\alpha}$$

$$= \frac{1}{2i} \left[\frac{1}{(s-2i)} - \frac{1}{s+2i} \right]$$

$$= \frac{1}{2i} \cdot \frac{4i}{s^2+2^2}$$

$$= \frac{2}{s^2+2^2} = \frac{2}{s^2+4} = \frac{2}{s^2+4} \text{ (Ans)}$$

(b) Laplace transform of $\cosh t$

$f(t) = \cosh t$

$$L\{f(t)\} = \int_0^{\alpha} e^{-st} \cosh t \, dt \qquad \left[\because \cosh t = \frac{e^t + e^{-t}}{2} \right]$$

$$= \int_0^{\alpha} e^{-st} \left(\frac{e^t + e^{-t}}{2} \right) dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{-(s-1)t}\,dt + \frac{1}{2}\int_0^{\infty} e^{-(s+1)t}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^{\infty} + \frac{1}{2}\left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[\frac{0-1}{-(s-1)}\right] + \frac{1}{2}\left[\frac{-1}{-(s+1)}\right]$$

$$= \frac{1}{2(s-1)} + \frac{1}{2(s+1)}$$

$$= \frac{(s+1)+(s-1)}{2(s+1)(s-1)}$$

$$= \frac{s+2+s-2}{2(s-1)(s+1)}$$

$$= \frac{2s}{2(s^2-1)} = \frac{s}{s^2-1} \quad \text{(Ans)}$$

$f(t) = \cos at$

$$L\{f(t)\} = f(s) = \int_0^{\infty} e^{-st}\cos at\,dt$$

$$= \int_0^{\infty} e^{-st}\,\frac{1}{2}\cdot 2\cos at\,dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{-st}\left(e^{iat} + e^{-iat}\right)dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{-st+iat}\,dt + \frac{1}{2}\int_0^{\infty} e^{-st-iat}\,dt$$

$$= \frac{1}{2}\int_0^{\alpha} e^{-(s-1)t}\,dt + \frac{1}{2}\int_0^{\alpha} e^{-(s+1)t}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^{\alpha} + \frac{1}{2}\left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^{\alpha}$$

$$= \frac{1}{2}\left[\frac{0-1}{-(s-1)}\right] + \frac{1}{2}\left[\frac{-1}{-(s+1)}\right]$$

$$= \frac{1}{2(s-1)} + \frac{1}{2(s+1)}$$

$$= \frac{(s+1)+(s-1)}{2(s+1)(s-1)}$$

$$= \frac{s+2+s-2}{2(s-1)(s+1)}$$

$$= \frac{2s}{2(s^2-1)} = \frac{s}{s^2-1} \quad \text{(Ans)}$$

$f(t) = \cos at$

$$L\{f(t)\} = f(s) = \int_0^{\alpha} e^{-st}\cos at\,dt$$

$$= \int_0^{\alpha} e^{-st}\,\frac{1}{2}\cdot 2\cos at\,dt$$

$$= \frac{1}{2}\int_0^{\alpha} e^{-st}\left(e^{iat} + e^{-iat}\right)dt$$

$$= \frac{1}{2}\int_0^{\alpha} e^{-st+iat}\,dt + \frac{1}{2}\int_0^{\alpha} e^{-st-iat}\,dt$$

$$\frac{1}{2}\int_0^{\alpha} e^{-(s-ia)t}\,dt + \frac{1}{2}\int_0^{\alpha} e^{-(s+ia)t}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-ia)t}}{-(s-ia)} + \frac{e^{-(s+ia)t}}{-(s+ia)}\right]_0^{\alpha}$$

$$= \frac{1}{2}\left[\frac{1}{(s-ia)} + \frac{1}{(s+ia)}\right]$$

$$= \frac{1}{2}\left[\frac{(s+ia)+(s-ia)}{s^2-i^2a^2}\right]$$

$$= \frac{1}{2}\,\frac{2s}{s^2+a^2}$$

$$= \frac{s}{s^2+a^2} \quad (\text{Ans})$$

3. (a) The series $1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2$ ------ is called Hypergeometric series.

The generation of this series is $1+x+x^2+$ ------

And the Hypergeometric equation of this Hypergeometric series is

$$x(1-x)y''+[\gamma-(\alpha+\beta+1)x]y'-\alpha\beta y=0$$

A_j^{ik} is a mixed tensor

Addition, ~~contr~~ outer multiplication and contraction

2020

1.(a) Even function: A function $f(x)$ is said to be an even function if $f(-x) = f(x)$

For Example, $f(x) = x^2$ $f(x) = \cos x$

odd function: A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$

(b) **Fourier series:** It is a mathematical way to represent Non-trigonometric periodic function as an infinite sum of trigonometric function

A series whose each and every term is either of sine or cosine.

$= a_0 \cos 0x + a_1 \cos_1 x + a_2 \cos 2x + a_3 \cos 3x + \cdots + b_0 \sin 0x$

$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1} b_n \sin nx$

An even function can have no sine term in its Fourier series;

If the function $f(x)$ is even, i.e. $f(-x) = f(x)$

then

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$\therefore a_0 = I_1 + I_2 \text{ ——— (ii)}$$

Now, $I_1 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx$

putting $x = -y$

$dx = -dy$

x	0	$-\pi$
y	0	π

$$\therefore I_1 = \frac{1}{\pi}\int_{\pi}^{0} f(-y)\cdot -dy$$

$$= -\frac{1}{\pi}\int_{\pi}^{0} f(y)\cdot dy$$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(x)\,dx$$

From equ (ii)

$$a_0 = 2\times\frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

Again, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= I_3 + I_4$$

putting, $x = -y$

$dx = -dy$

x	0	$-\pi$
y	0	π

Now, $I_3 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx$

$= \frac{1}{\pi}\int_{\pi}^{0} f(-y)\cos n(-y)(-dy)$

$= -\frac{1}{\pi}\int_{\pi}^{0} f(y)\cos ny\,dy$

$= \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dy$

Therefore, $a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$

And, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$

$= I_5 + I_6$

Now, $I_5 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{0} f(-y)\sin(-y)(-dy)$

$= \frac{1}{\pi}\int_{\pi}^{0} f(y)\sin ny\,dy$

$= -\frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx = -I_6$

Therefore, $b_n = I_5 + I_6$

$= -I_6 + I_6$

$\therefore b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\alpha} a_n \cos nx + \sum_{n=1}^{\alpha} b_n \sin nx$$

$$= \frac{1}{2} \times \frac{2}{\pi} \int_0^{\pi} f(x)\,dx + \sum_{n=1}^{\alpha} \left[\frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx\right] \cos nx + 0$$

$$= \frac{1}{\pi} \int_0^{\pi} f(x)\,dx + \sum_{n=1}^{\alpha} \left[\frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx\right] \cos nx$$

(c) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\alpha} a_n \cos nx + \sum_{n=1}^{\alpha} b_n \sin nx$$

Now, $$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi} \int_{\pi}^{0} f(x)\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} -\cos x\,dx + \frac{1}{\pi} \int_{\pi}^{0} \cos x\,dx$$

$$= \frac{1}{\pi} \left[-\sin x\right]_{-\pi}^{0} + \frac{1}{\pi} \left[\sin x\right]_{0}^{\pi}$$

$$= 0 + 0 = 0$$

Now,
$$a_n = \frac{1}{\pi} \int_{0}^{\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\cancel{0}} f(x) \cos nx\,dx + \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} 2\cos nx \cos nx\,dx + \frac{1}{2\pi} \int_0^{\pi} 2\cos nx \cos x\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0}\{\cos(nx+x)+\cos(nx-x)\}dx + \frac{1}{2\pi}\int_{0}^{\pi}\{\cos(nx+x)+\cos(nx-x)\}dx$$

$$= -\frac{1}{2\pi}\left[\frac{\sin nx+x}{(n+1)}+\frac{\sin(nx-x)}{(n-1)}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\frac{\sin(nx+x)}{(n+1)} + \frac{\sin(nx-x)}{(n-1)}\right]_{0}^{\pi}$$

$$= -\frac{1}{2\pi}\times 0 + \frac{1}{2\pi}\times 0 = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}[-\cos x]\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\cos x\sin nx\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0} 2\sin nx\cos x\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 2\sin nx\cos x\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0}\{\sin(nx+x)+\sin(nx-x)\}dx + \frac{1}{2\pi}\int_{0}^{\pi}\{\sin(nx+x)+\sin(nx-x)\}dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{(n+1)} + \frac{-\cos(n-1)x}{(n-1)}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{(n+1)} + \frac{-\cos(n-1)x}{n-1}\right]_{0}^{\pi}$$

$$= -\frac{1}{2\pi}\left[\left(\frac{-\cos 0}{n+1} + \frac{-\cos(n-1)\cdot 0}{n-1}\right) - \left(\frac{-\cos(n+1)(-\pi)}{n+1} + \frac{-\cos(n-1)(-\pi)}{(n-1)}\right)\right]$$

$$+\frac{1}{2\pi}\left[\left(\frac{-\cos(n+1)\pi}{n+1} + \frac{-\cos(n-1)\pi}{n-1}\right) - \left(\frac{-\cos(n+1)\cdot 0}{n+1} + \frac{-\cos(n+1)\cdot 0}{(n-1)}\right)\right]$$

$$= -\frac{2}{2\pi}\left(-\frac{1}{n+1} + \frac{-1}{n-1}\right) - \frac{2}{2\pi}\left(\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right)\Bigg]$$

$$= \frac{1}{\pi}\left(\frac{1}{n+1} - \frac{(-1)^{n+1}}{n+1}\right) + \frac{1}{\pi}\left(\frac{1}{n-1} - \frac{(-1)^{n-1}}{n-1}\right)$$

$$= \frac{1}{\pi}\left(\frac{1-(-1)^{n+1}}{n+1}\right) + \frac{1}{\pi}\left(\frac{1-(-1)^{n-1}}{n-1}\right)$$

Substituting the value

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty}\left[\frac{1}{\pi}\left(\frac{1-(-1)^{n+1}}{n+1}\right) + \frac{1}{\pi}\left[\frac{1-(-1)^{n-1}}{n-1}\right]\right]\sin nx$$

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{1}{\pi}\left(\frac{1-(-1)^{n+1}}{n+1}\right) + \frac{1}{\pi}\left(\frac{1-(-1)^{n-1}}{n-1}\right)\right]$$

$\sin nx$ (Ans)

2.(e) Parseval's identity:

Statement: If the Fourier series for the function $f(x)$ converges uniformly in $(-l, l)$ then,

$$\frac{1}{l}\int_{-l}^{l}\{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2);$$

where, a_0, a_n and b_n called Fourier co-efficients or constants.

Proof: The Fourier series for the function $f(x)$ in the interval $(-l, l)$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right) \text{ --- (i)}$$

where, $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$

and $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$

Multiplying both sides of equ (i) by $f(x)$ and integrating between limits $-l$ to l we get,

$$\int_{-l}^{l}\{f(x)\}^2 dx = \frac{a_0}{2}\int_{-l}^{l} f(x)\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx\right)$$

$$+ \sum_{n=1}^{\infty}\left(b_n\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\right)$$

$$= \frac{a_0}{2}\cdot a_0 l + \sum_{n=1}^{\infty} a_n \cdot a_n l + \sum_{n=1}^{\infty} b_n \cdot b_n l$$

$$= \frac{a_0^2}{2}\cdot l + \sum_{n=1}^{\infty} a_n^2 l + \sum_{n=1}^{\infty} b_n \cdot b_n l$$

$$= \frac{a_0^2}{2} l + \sum_{n=1}^{\infty} a_n^2 l + \sum_{n=1}^{\infty} b_n^2 l$$

$$\Rightarrow \int_{-l}^{l}\{f(x)\}^2 dx = \frac{a_0^2}{2} l + l\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$= \frac{1}{l}\int_{-l}^{l}\{f(x)\} = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

(Ans)

2. (ii) <u>Bessel's function:</u> The equation

$$x^2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + (x^2 - n^2)y = 0$$ is called Bessel's differential equation of order n.

By solving this equation we get,

$$J_n(x) = \sum_{r=0}^{\alpha} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r+n}}{r!\,(n+r)!}$$ and this function is called Bessel's function.

2. (b) We know,

$$J_n(x) = \sum_{r=0}^{\alpha} \frac{(-1)^r (x/2)^{2r+n}}{r!\,(n+r)!}$$

$$J_n'(x) = \sum_{r=0}^{\alpha} \frac{(-1)^r (2r+n)(x/2)^{2r+n-1} \cdot \frac{1}{2}}{r!\,(n+r)!}$$

$$= \sum_{r=0}^{\alpha} \frac{(-1)^r (2r+n)(x/2)^{2r+n}}{r!\,(n+r)!}$$

$$= \sum_{r=0}^{\alpha} \frac{(-1)^r\, 2r \cdot (x/2)^{2r+n}}{r!\,(n+r)!} + n \sum_{r=0}^{\alpha} \frac{(-1)^r (x/2)^{2r+n}}{r!\,(n+r)!}$$

$$= x \sum_{r=0}^{\alpha} \frac{(-1)^r \frac{2}{x} (x/2)^{2r+n}\, r}{r!\,(n+r)!} + n J_n(x)$$

$$= x \sum_{r=0}^{\alpha} \frac{(-1)^r (x/2)^{-1} (x/2)^{2r+n} \cdot r}{r!\,(n+r)!} + n J_n'(x)$$

$$= x \sum_{r=0}^{\alpha} \frac{(-1)(-1)^{r-1} \left(\frac{x}{2}\right)^{2r+n-1} \cdot r}{r(r-1)!\,(n+r)!} + n J_n(x)$$

$$= x\sum_{r=0}^{\alpha} \frac{(-1)(-1)^{r-1}\left(\frac{x}{2}\right)^{2(r-1)+(n+1)}}{(r-1)!\,\{(n+1)+(r-1)\}!} + nJ_n(x)$$

$$= -x\sum_{r=0}^{\alpha} \frac{(-1)^{r-1}\left(\frac{x}{2}\right)^{2(r-1)+(n+1)}}{(r-1)!\,\{(n+1)+(r-1)\}!} + nJ_n(x)$$

$$= -xJ_{n+1}(x) + nJ_n(x)$$

$$= nJ_n(x) - xJ_{n+1}(x) \quad \text{proved.}$$

2.(a) Legendre polynomials : The second order differential equation,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\}y = 0$$

in which m and n are constants is known as Legendre's associated equation an it's solution are called associated Legendre functions.

It reduces to Legendre's equation when $m=0$

The Legendre differential equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \text{———— (i)}$$

And the soln of this eqn is called Legendre's function or Legendre's polynomials.

4. Laplace trans form of $\sin^3 2t$

$$L\{\sin 2t\} = \int_0^\infty e^{-st} \sin 2t \, dt$$

$$= \int_0^\infty e^{-st} \left(\frac{e^{2it} - e^{-2it}}{2i} \right) dt$$

$$= \frac{1}{2i} \int_0^\infty e^{-st+2it} dt - \frac{1}{2i} \int_0^\infty e^{-st-2it} dt$$

$$= \frac{1}{2i} \int_0^\infty e^{-(s-2i)t} dt - \frac{1}{2i} \int_0^\infty e^{-(s+2i)t} dt$$

$$= \frac{1}{2i} \left[\frac{e^{-(s-2i)t}}{-(s-2i)} - \frac{e^{-(s+2i)t}}{(s+2i)} \right]_0^\infty$$

$$= \frac{1}{2i} \left[\frac{1}{(s-2i)} - \frac{1}{(s+2i)} \right]$$

$$= \frac{1}{2i} \left[\frac{1}{(s-2i)} - \frac{1}{s+2i} \right]$$

2019

1. (a) $\Gamma\frac{1}{2} = \sqrt{\pi}$

We know that,

$$B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \quad \text{———} (i)$$

and $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

We can write the equation (i) and (ii)

$$\frac{\Gamma m \, \Gamma n}{\Gamma m+n} = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \quad \text{———} (iii)$$

Let, $m = n = \frac{1}{2}$

putting this value in equ (3) we get,

$$\frac{\Gamma \frac{1}{2} \, \Gamma \frac{1}{2}}{\Gamma \frac{1}{2}+\frac{1}{2}} = 2\int_0^{\pi/2} \sin^{2\cdot\frac{1}{2}-1}\theta \cos^{2\cdot\frac{1}{2}-1}\theta \, d\theta$$

$$\frac{(\Gamma \frac{1}{2})^2}{(1)} = 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta \, d\theta$$

$$\frac{(\Gamma \frac{1}{2})^2}{1} = 2\int_0^{\pi/2} 1 \cdot 1 \, d\theta$$

$$(\Gamma \tfrac{1}{2})^2 = 2[\theta]_0^{\pi/2}$$

$$(\Gamma \tfrac{1}{2})^2 = 2 \cdot \pi/2 = \pi$$

$\therefore \sqrt{\Gamma \tfrac{1}{2}} = \sqrt{\pi}$

(ii) $\int_0^{\alpha} e^{-x^2} dx = \sqrt{\pi}/2$

Here, $x^2 = z \qquad x = \sqrt{z}$

x	0	α
z	0	α

$2x\,dx = dz$

$\therefore dx = \frac{1}{2x} dz = \frac{1}{2\sqrt{z}} dz$

Now, $\int_0^{\alpha} e^{-x^2} dx = \int_0^{\alpha} e^{-z} \frac{dz}{2\sqrt{z}}$

$= \frac{1}{2} \int_0^{\alpha} e^{-z} z^{-1/2} dz = \frac{1}{2} \int_0^{\alpha} e^{-z} z^{(\frac{1}{2}-1)} dz$

$= \frac{1}{2} \sqrt{\Gamma \tfrac{1}{2}} = \frac{1}{2}\sqrt{\pi} = \sqrt{\pi}/2$ (Ans)

2. The solution of Legendre diff eqn is

$$y = c_0 \left[1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n-2)(n+1)(n+3)}{4!} x^4 \right]$$

$$+ c_1 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!} x^5 + \text{———} \right]$$

$$P_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n r!\,(n-r)!\,(n-2r)!} x^{n-2r}$$

when $N = N/2$ for n even

and $N = \frac{n-1}{2}$ for n odd

$$P_n(x) = \frac{2n!}{2^n n!\,n!}\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \cdots\right]$$

For $P_3(x)$

$$\sum_{r=0}^{1} (-1)^0 \frac{(2\times 3 - 2\times 0)!}{2^3 \times 0! \times (3-0)! \times (3-2\times 0)!} x^{3-2\times 0}$$

$$= \sum_{r=0}^{1} 1 \times \frac{6!}{8\times 3! \times 3!} x^3$$

$$= \sum_{r=0}^{1} \frac{720}{280} x^3$$

4. (a) $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$

where, $u = x^2 + axy + by^2$ and $v = cx^2 + dxy + y^2$

$\frac{\partial u}{\partial x} = 2x + ay$ $\quad\quad$ $\frac{\partial v}{\partial x} = 2cx + dy$

$\frac{\partial u}{\partial y} = ax + 2by$ $\quad\quad$ $\frac{\partial v}{\partial y} = dx + 2y$

2018

1.(a) $\Gamma(n+1) = n\Gamma(n)$

we get, $\Gamma(n+1) = n\Gamma(n)$

when, $n=1$ we get,

$\Gamma(2) = 1\Gamma(1) = 1.1 = 1!$

when $n=2$ we get,

$\Gamma(3) = 2\Gamma(2) = 2.1 = 2!$

when $n=3$ we get

$\Gamma(4) = 3\Gamma(3) = 3.2.1 = 3!$

Therefore

$$\Gamma(n+1) = (n+1-1)! = n!$$

Hence

$\Gamma(n+1) = n! = \lfloor n = n\Gamma n$ proved

4. $P_0(x) = c_0 P_0(x) = 1$

$P_0(1) = 1$

$P_1(x) = c_1 x$

$P_1(x) = c_1 x$

so, $P_1(x) = x$

similarly,

$P_2 x = \frac{1}{2}(3x^2 - 1)$

$P_3 x = \frac{1}{2}(5x^3 - 3x)$

$P_4 x = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

$P_6(x) = \frac{1}{10}(123x^6 - 315x^4 + 105x^2 - 5)$

We know from Legendre polynomials,

$x = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$x^4 = \frac{8}{35} P_4(1) + \frac{6}{7} x^2 - \frac{3}{35}$

$P_3 x = \frac{1}{2}(5x^3 - 3x)$

$\therefore x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} x$