Previous year questions solve

complex variable

2021

1.

(a) Define beta and gamma function, Establish the relation.

(b) $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

(b) Evaluate the integrals (i) $\int_0^{\alpha} \dfrac{x^{n-1}}{1+x} dx$

(ii) $\int_0^1 x^{m-1}(1-x^a)^n dx$ and $\int_0^1 \dfrac{3\pi}{32} \dfrac{x^3}{\sqrt{1-x}} dx$

2. (a) Define Laplace transformation and explain its properties.

(b) Find Laplace transform of cosht and cosat.

(c) Evaluate $L(\sin 2t + 2\cos 3t)$

3.

(a) Define hypergeometric function. Derive confluent hypergeometric function from hypergeometric equation.

(b) Define Legendre polynomials, Express $f(x) = x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Legendre polynomials.

(c) Show that $P_n(-x) = (-1)^n P_n(x)$

4. (a) Derive the cauchy-Reimann condition for the complex function to be analytic. Are the conditions sufficient?

(b) Evaluate the following integrals

(i) $\int_0^\alpha \frac{\sin x}{x} dx$ (ii) $\int_{-\alpha}^{\alpha} \frac{dx}{1+x^2}$

5. state and prove poisson's integral formula for a circle.

(b) state and prove residue theorem.

(c) Find the Residues of $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}$ at all its poles in the finite plane.

6. (a) What is tensor? why it is important.

(b) What do you mean by spaces of N dimensions.

(c) Define transformation for the tensor

(i) $A^{i}_{j}{}^{k}$, (ii) B^{mn}_{ijk} (iii) C^{m}

(d) Define transformation of co-ordinates from one frame of referes to another.

2020

1. Define odd and even function with example

(b) What is Fourier series? show that an even function can have no sine term in its Fourier series

(c) Find the Fourier series of $f(x) = \begin{cases} -\cos x & -\pi < x < 0 \\ \cos x & 0 < x < \pi \end{cases}$

2. (a) Define Bessel's function, Laguerre polynomials and hypergeometric function.

(b) $_2F_1(\alpha, \beta, \gamma, x)_{x=0} = 1$ (ii) $\frac{d}{dx}\left[{}_2F_1(\alpha, \beta, \gamma; x)\right]_{x=0} = \frac{\alpha\beta}{\gamma}$ and (iii) $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

(c) State and prove parseval's identity

4. Find the Laplace transform of $\sin^3 2t$

5. (a) Define complex number perform the following operation both analytically

and graphy cally.

(i) $(2+3i)+(4-5i)$

(ii) $(6-2i) \div (2-5i)$

(b) what is harmonic function. prove that function $u = \cos x \cosh y$ is harmonic

(c) show that $\frac{d\bar{z}}{dz}$ does not exist anywhere.

6. state and prove Rouche's theorem.

(b) what is residue? How can you calculate it?

(c) Find the residues of $f(z) = \frac{z^2+4}{z^4+2z^2+2z}$ at all its poles.

7. ~~Def~~ (b) Define (i) covariant tensor (ii) contravariant tensor (iii) mixed tensor (iv) summation convention (v) Dummy index (vi) Free index (vii) symmetric tensor.

(c) show that (i) $\frac{\partial x^p}{\partial x^q} = \delta^p_q$ and

(ii) $\frac{\partial x^p}{\partial x^q} \cdot \frac{\partial x^q}{\partial x^n} = \delta^p_n$

2019

1. (b) prove the following identities

i) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (ii) $\int_0^\alpha e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

2. (b) write down the Legendre polynomial $P_n(x)$ and hence evoluate $P_3(x)$

(c) Establish the Rodrigues formula

$$P_n(x) = \frac{2^n d^n}{n! \, dx^n} (x^2-1)^n$$

(3) solve the following differential equation with Laplace transform

$d^2y/dx^2 + 9y = \cos 2t \quad y(0) = 1 \quad y(\pi/2) = -1$

(c) Evaluate $L^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}$

4(a) Define analytic function. Find the values of a, b, c and d such that the function $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$ is analytic

(b) Show that the function $u = e^{-x}(x\sin y - y\cos y)$ is harmonic. Find its harmonic conjugate v such that $u+iv$ is harmonic

(c) Prove that $\lim_{z\to 0} \frac{d\bar{z}}{dz}$ does not exist.

6. (a) State and prove Cauchy's theorem.

(b) Use Cauchy's theorem to evaluate

$$\oint_C \frac{z+4}{z^2+2z+5}\, dz \quad \text{where } C: |z+1| = 1$$

(c) State Cauchy's integral. Evaluate $\oint_C \frac{e^{2z}}{(z+1)^5}\, dz$ where C is the curve $|z| = 4$

8. (a) Find the covariant derivative of $A^j_k B^{lm}_n$ with respect to x^q.

(b) Prove that $A_{p,qr} - A_{p,rq} = R^n_{pqr} A_n$

Where A_p is an arbitrary covariant

(c) Show that the inner product of two tensors A^{pq}_r and B^s_t is a tensor of rank three.

2018

1. (a) Show that $\Gamma(n+1) = n\Gamma(n)$

(c) Find the value of $\Gamma(\frac{1}{2})$

2. (b) Express $f(x) = x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Legendre polynomials.

4. $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$ is

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2018

1. (a) Show that $\Gamma(n+1) = n\Gamma(n)$

(c) Find the value of $\Gamma(\frac{1}{2})$

2. (b) Express $f(x) = x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Legendre polynominals.

4. $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$ is analytic.

Previous year question solution

complex variable

Mathematical method in Physics – II

2021

1. (a)

Beta function: The function denoted by $B(m,n)$, where m and n are positive values $[m,n>0]$ and if defined by $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ is known as Beta function. It is also called the first Eulerian integral.

Gamma function: The function is denoted by Γn and if defined by $\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$ where $n>0$ is known as gamma function. It is also known as second Eulerian integral.

soln: we know that

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Let $x = \sin^2\theta$

x	0	1
θ	0	$\pi/2$

$dx = 2\sin\theta\cos\theta\,d\theta$

Putting this value in the equ (1) we get,

$$B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\,2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta\cos^{2n-2}\theta\cdot\sin\theta\cos\theta\,d\theta$$

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

$$\frac{B(m,n)}{2} = \int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

Let, $2m-1 = p$ and $2n-1 = q$

$m = \frac{p+1}{2}$ $\quad$ $n = \frac{q+1}{2}$

And we know,

$$B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

Therefore

$$\frac{\Gamma m \Gamma n}{2\Gamma(m+n)} = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)} = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$$\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+1}{2}} = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$$\therefore \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}} = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

(b) $\displaystyle\int_0^{\alpha} \frac{x^{n-1}}{1+x}\,dx$

x	0	α
θ	0	$\pi/2$

Let, $x = \tan^2\theta$

$dx = 2\tan\theta \sec^2\theta \, d\theta$

Now, $\displaystyle\int_0^{\alpha} \frac{x^{n-1}}{1+x}\,dx$

$$= \int_0^{\pi/2} \frac{(\tan^2\theta)^{n-1}}{1+\tan^2\theta} \, 2\tan\theta \sec^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\tan^{2n-2}}{\sec^2\theta} \cdot \tan\theta \sec^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \tan^{2n-1}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sin^{2n-1}\theta}{\cos^{2n-1}\theta} \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n-1} \cdot \cos^{(2n-1)}\theta \, d\theta$$

$$= \frac{2\,\Gamma\left(\frac{2n-1+1}{2}\right)\Gamma\left(\frac{-2n+1+1}{2}\right)}{2\,\Gamma\left(\frac{2n-1-2n+1+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{2n}{2}\right)\Gamma\left(\frac{-2n+2}{2}\right)}{\Gamma\left(2/2\right)}$$

$$= \frac{\Gamma(n)\,\Gamma(1-n)}{\Gamma(1)}$$

$$= \Gamma(n)\,\Gamma(1-n) \quad \text{(Ans)}$$

$$= 2\int_0^{\pi/2} \frac{\tan^{2n-2}}{\sec^2\theta} \cdot \tan\theta \sec^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \tan^{2n-1}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sin^{2n-1}\theta}{\cos^{2n-1}\theta} \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n-1} \cdot \cos^{(2n-1)}\theta \, d\theta$$

$$= \frac{2\,\Gamma\left(\frac{2n-1+1}{2}\right)\Gamma\left(\frac{-2n+1+1}{2}\right)}{2\,\Gamma\left(\frac{2n-1-2n+1+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{2n}{2}\right)\Gamma\left(\frac{-2n+2}{2}\right)}{\Gamma\left(\frac{2}{2}\right)}$$

$$= \frac{\Gamma(n)\,\Gamma(1-n)}{\Gamma(1)}$$

$$= \Gamma(n)\,\Gamma(1-n) \quad \text{(Ans)}$$

Q) $\int_0^1 x^{m-1}(1-x^a)^n dx$

Let $x^a = \sin^2\theta$

$x = \sin^{2/a}\theta$

$a\, x^a x^{-1} dx = 2\sin\theta\cos\theta\, d\theta$

$dx = \frac{2}{a}\sin^{-2}\theta \sin^{2/a}\theta \sin\theta\cos\theta\, d\theta$

Now, $\int_0^1 x^{m-1}(1-x^a)^n dx$

x	0	1
θ	0	$\pi/2$

$= \int_0^{\pi/2} (\sin^{2/a}\theta)^{m-1}(1-\sin^2\theta)^n \cdot \frac{2}{a}\sin^{2/a-2+1}\cos\theta\, d\theta$

$= \int_0^{\pi/2} \sin^{\frac{2m}{a}-\frac{2}{a}}\theta \cos^{2n}\theta \; \frac{2}{a}\sin^{2/a-1}\theta\cos\theta\, d\theta$

$= \frac{2}{a}\int_0^{\pi/2} \sin^{\frac{2m}{a}-\frac{2}{a}+\frac{2}{a}-1} \cos^{2n+1}\theta\, d\theta$

$= \frac{2}{a}\int_0^{\pi/2} \sin^{\frac{2m}{a}-1}\theta \cos^{2n+1}\theta\, d\theta$

$= \frac{2}{a} \times \frac{\Gamma\left(\frac{2m/a-1+1}{2}\right)\Gamma\left(\frac{2n+1+1}{2}\right)}{2\,\Gamma\left(\frac{2m-1+2n+1+2}{\ldots}\right)}$

$$= \frac{1}{a} \times \frac{\Gamma\frac{m}{a} \times n!}{\Gamma\left(\frac{m}{a}+n+1\right)}$$

(Ans)

$$\int_0^1 \frac{35}{32} \frac{x^3}{\sqrt{1-x}}\, dx$$

Let

$$x = \sin^2\theta$$

$$dx = 2\sin\theta\cos\theta\, d\theta$$

Limit

x	0	1
θ	0	$\pi/2$

$$\therefore \int_0^1 \frac{35}{32} \frac{x^3}{\sqrt{1-x}}\, dx$$

$$= \int_0^{\pi/2} \frac{35}{32} \frac{(\sin^2\theta)^3}{\sqrt{1-\sin^2\theta}}\, 2\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{35}{32} \frac{\sin^6\theta}{\sqrt{\cos^2\theta}}\, 2\sin\theta\cos\theta\, d\theta$$

$$= \frac{35}{16}\int_0^{\pi/2} \frac{\sin^7\theta}{\cos\theta}\cos\theta\, d\theta$$

$$= \frac{35}{16}\int_0^{\pi/2} \sin^7\theta \cdot \cos^0\theta\, d\theta$$

$$= \frac{35}{16} \frac{\Gamma\frac{7+1}{2}\,\Gamma\frac{0+1}{2}}{2\,\Gamma\frac{7+0+2}{2}}$$

$$\frac{35}{16}\ \frac{\Gamma 4\ \ \Gamma\frac{1}{2}}{2\ \Gamma\frac{9}{2}}$$

$$= \frac{35}{32} \times \frac{3!\ \Gamma\frac{1}{2}}{\Gamma\left(\frac{7}{2}+1\right)}$$

$$= \frac{35}{32} \times \frac{3!\ \Gamma\frac{1}{2}}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}}$$

$$= \frac{35}{16} \times \frac{3\times2\times1}{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}}$$

$$= \frac{16}{16}$$

$= 1$ (Ans)

2\. (a) Laplace transformation of derivative

1st derivative:

$$L\left\{\frac{df(t)}{dt}\right\} = \int_0^\alpha e^{-st}\ \frac{df(t)}{dt}\,.dt$$

$$= \left[e^{st} f(t)\right]_0^\alpha + s\int e^{-st} f(t)\,dt$$

$$= -f(0) + Sf(s)$$

$$= Sf(s) - f(0)$$

2nd derivative :

$$L\left\{\frac{d^2 f(t)}{dt^2}\right\} = \int_0^\alpha e^{-st} \frac{d^2 f(t)}{dt^2} \cdot dt$$

$$= \left[e^{-st} \frac{df(t)}{dt}\right]_0^\alpha + s\int e^{-st} \cdot \frac{df(t)}{dt} \cdot dt$$

$$= -\frac{df(0)}{dt} + s\left[-f(0) + s\, f(s)\right]$$

$$= -f(0) - sf(0) + s^2 f(s)$$

Laplace transformation: Laplace transformation of a function $f(t)$ is defined by $L\{f(t)\}$

$$L\{f(t)\} = \int_0^\alpha e^{-st} f(t)\, dt = f(s)$$

(b) using linear property of laplace transformation we can write,

$$L\{\sin 2t + 2\cos 3t\} = L\{\sin 2t\} + 2L\{\cos 3t\}$$

Here,

$$L\{\sin 2t\} = \int_0^\alpha e^{-st} \sin 2t\, dt$$

$$= \int_0^\infty e^{-st} \left(\frac{e^{2ist} - e^{-2it}}{2i} \right) dt$$

$$= \frac{1}{2i} \int_0^\infty e^{-st+2it} \, dt - \frac{1}{2i} \int_0^\infty e^{-st-2it} \, dt$$

$$= \frac{1}{2i} \int_0^\infty e^{-(s-2i)t} \, dt - \frac{1}{2i} \int_0^\infty e^{-(s+2i)t} \, dt$$

$$= \frac{1}{2i} \left[\frac{e^{-(s-2i)t}}{-(s-2i)} - \frac{e^{-(s+2i)t}}{(s+2i)} \right]_0^\infty$$

$$= \frac{1}{2i} \left[\frac{1}{(s-2i)} - \frac{1}{s+2i} \right]$$

$$= \frac{1}{2i} \frac{4i}{s^2+2^2}$$

$$= \frac{2}{s^2+2^2} = \frac{2}{s^2+4} = \frac{2}{s^2+4} \text{ (Ans)}$$

(b) Laplace tranform of $\cosh t$

$f(t) = \cosh t$

$$L\{f(t)\} = \int_0^\infty e^{-st} \cosh t \, dt \qquad \left[\because \cosh t = \frac{e^t + e^{-t}}{2} \right]$$

$$= \int_0^\infty e^{-st} \left(\frac{e^t + e^{-t}}{2} \right) dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-1)t}\,dt + \frac{1}{2}\int_0^\infty e^{-(s+1)t}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^\infty + \frac{1}{2}\left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{0-1}{-(s-1)}\right] + \frac{1}{2}\left[\frac{-1}{-(s+1)}\right]$$

$$= \frac{1}{2(s-1)} + \frac{1}{2(s+1)}$$

$$= \frac{(s+1)+(s-1)}{2(s+1)(s-1)}$$

$$= \frac{s+2+s-2}{2(s-1)(s+1)}$$

$$= \frac{2s}{2(s^2-1)} = \frac{s}{s^2-1} \quad (\text{Ans})$$

$f(t) = \cos at$

$$L\{f(t)\} = f(s) = \int_0^\infty e^{-st}\cos at\,dt$$

$$= \int_0^\infty e^{-st}\,\frac{1}{2}\cdot 2\cos at\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st}\left(e^{iat}+e^{-iat}\right)dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st+iat}\,dt + \frac{1}{2}\int_0^\infty e^{-st-iat}\,dt$$

$$= \frac{1}{2}\int_0^{\alpha} e^{-(s-ia)t}\,dt + \frac{1}{2}\int_0^{\alpha} e^{-(s+ia)t}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-ia)t}}{-(s-ia)} + \frac{e^{-(s+ia)t}}{-(s+ia)}\right]_0^{\alpha}$$

$$= \frac{1}{2}\left[\frac{1}{(s-ia)} + \frac{1}{(s+ia)}\right]$$

$$= \frac{1}{2}\left[\frac{(s+ia)+(s-ia)}{s^2-i^2a^2}\right]$$

$$= \frac{1}{2}\,\frac{2s}{s^2+a^2}$$

$$= \frac{s}{s^2+a^2} \quad \text{(Ans)}$$

3. (a) The series $1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\cdots$ (i)

is called Hypergeometric series.

The generation of this series is $1+x+x^2+\cdots$

And the Hypergeometric equation of this Hypergeometric series is

$$x(1-x)y''+[\gamma-(\alpha+\beta+1)x]y'-\alpha\beta y=0$$

we may write this series in this form;

$$x(1-x)y'' + [\gamma + (\alpha+\beta+1)x]y' - \alpha\beta y = 0$$

$$_2F_1(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\alpha} \frac{(\alpha)_r (\beta)_r}{r! \, (\gamma)_r} x^r$$

The Hypergeometric equation

$$x(1-x)\frac{d^2y}{dx^2} + [\gamma - (\alpha+\beta+1)x]\frac{dy}{dx} - \alpha\beta y = 0$$

putting $x = \alpha/\beta$

$$x(1-\alpha/\beta)\frac{d^2y}{dx^2} + [\gamma - (1+\frac{\alpha+1}{\beta}x]\frac{dy}{dx} - \alpha y = 0$$

The solⁿ of which represented by the function $_2F_1(\alpha, \beta; \gamma; \alpha/\beta)$ Letting $\beta \to \alpha$ the function

becomes,

$$\lim_{\beta\to\alpha} {_2F_1}(\alpha, \beta; \gamma; x/\beta)$$ and reduces it;

$$x\, d^2y/dx^2 + (\gamma - \alpha)\, dy/dx - xy = 0$$

This 2nd order D.E (ii) in which α and γ are constant is called confluent hypergeometric function.

The sol of (ii) is represented by,

$$\lim_{\beta \to \alpha} {}_2F_1(\alpha, \beta; \gamma; x/\beta) = \lim_{\beta \to \alpha} \sum_{k=0}^{\alpha} \frac{(\alpha)_k (\beta)_k}{k!\,(\gamma)_k} \cdot \frac{x^k}{\beta k}$$

(b) $f(x) = x^4 + 2x^3 + 2x^2 - x - 3$

we know from Legendre polynomials,

$$P_4(x) = \frac{1}{2^4 4!} \times \frac{d^4}{dx^4} (x^2-1)$$

$= P_4(x) = \frac{1}{8} \times (35x^4 - 30x^2 + 3)$

$= 8P_4(x) = 35x^4 - 30x^2 + 3$

$35x^4 = 8P_4(x) + 30x^2 - 3$

$x^4 = \frac{8}{35} P_4(x) + \frac{6}{7} x^2 - \frac{3}{35}$

And,

$P_3(x) = \frac{1}{2} \times (5x^3 - 3x)$

$\Rightarrow 2P_3(x) = 5x^3 - 3x$

$5x^3 = 2P_3(x) + 3x$

Also, $P_2(x) = \frac{1}{2}(3x^2 - 1)$

$2P_2(x) = 3x^2 - 1$

$3x^2 = 2P_2(x) + 1$

$x^2 = \frac{2}{3} P_2(x) + \frac{1}{3}$

and also, $P_1(x) = x$

$x = P_1(x)$

also, $P_0(x) = 1$

$1 = P_0(x)$

putting the values of x^4, x^3, x^2 and x we get,

$f(x) = \frac{8}{35} P_4(x) + \frac{4}{5} P_3(x) + \frac{40}{21} P_2(x) + \frac{1}{5} P_1(x) -$

$\frac{224}{105} P_0(x).$

(a) we know that,

$$P_n(x) = \frac{(2n)!}{2^n(n!)^2}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \cdots\right]$$

therefore,

$$P_n(-x) = \frac{(2n)!}{2^n(n!)^2}\left[(-x)^n - \frac{n(n-1)}{2(2n-1)}(-x)^{n-2} + \cdots\right]$$

$$= \frac{(2n)!}{2^n(n!)^2}\left[(-1)^n x^n - \frac{n(n-1)}{2(2n-1)} \cdot (-1)^{n-2}(x)^{n-2} + \cdots\right]$$

$$= \frac{(2n)!}{2^n(n!)^2}\left[(-1)^n x^n - \frac{n(n-1)}{2(2n-1)} \cdot \frac{(-1)^n}{(-1)^2} x^{n-2} + \cdots\right]$$

$$= \frac{(2n)!}{2^n(n!)^2} \times (-1)^n \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \cdots\right]$$

$$= (-1)^n \frac{(2n)!}{2^n n!} \times \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \cdots\right]$$

$P_n(-x) = (-1)^n P_n(x)$ [showed]

4. If $f(z)$ is single valued in some Region R of the z plane, the derivative of $f(z)$ is defined as, $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

Analytic function:

If the derivative $f'(z)$ exists at all points of z of a region R, then $f(z)$ is said to be analytic in R and is referred to as an analytic function in R or a function analytic in R.

(a) Let $f(z)$ be analytic inside and on the circle C defined by $|z| = R$, then is $z = re^{i\theta}$ is any point inside C then

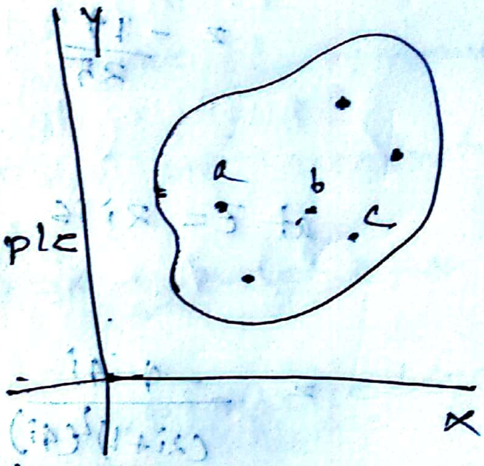
we have $$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})}{R^2 - 2Rr\cos(\theta - \phi) + r^2} d\phi$$

If $u(r,\theta)$ and $v(r,\theta)$ are the real and imaginary parts of $f(re^{i\theta})$ while $u(R,\phi)$ and $v(R,\phi)$ and the real and imaginary parts of $f(Re^{i\theta})$ then,

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\,u(R,\varphi)}{R^2-2Rr\cos(\theta-\varphi)+r^2}\,d\varphi$$

$$v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\,v(R,\varphi)}{R^2-2Rr\cos(\theta-\varphi)+r^2}\,d\varphi$$

(b) The Residue theorem:



Let $f(z)$ be single-valued and analytic inside and on a simple closed residues given by a_{-1}, b_{-1}, c_{-1}, Then the residue theorem states that,

$$\oint_c f(z)\,dz = 2\pi i\,(a_{-1}+b_{-1}+c_{-1}+\cdots)$$

i.e the integral of $f(z)$ around c is $2\pi i$ times the sum of the residues of $f(z)$ at the singularities enclosed by c.

(c) $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+4)}$

1. $\displaystyle\lim_{z\to -1} \frac{1}{1!}\frac{d}{dz}\left\{(z+1)^2 \frac{z^2-2z}{(z+1)^2(z^2+4)}\right\}$

$\displaystyle = \lim_{z\to -1} \frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2}$

$= -\dfrac{14}{25}$

at $z = 2i$ is $\displaystyle\lim_{z\to 2i} \left\{(z-2i)\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right.$

$= \dfrac{-4-4i}{(2i+1)^2(4i)} = \dfrac{7+i}{25}$

Residue at $z=-2i$ is

$\displaystyle\lim_{z\to -2i} \left\{(z+2i)\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right.$

$= \dfrac{-4+4i}{(-2i+1)^2(-4i)} = \dfrac{7-i}{25}$ (Ans)

6.(a) A study of the consequences of this re- mechanics, elasticity, hydrodynamics, electromagnetic theory and numerous other fields of science and engineering.

(b) In three dimensional space a point is a set of three numbers called or frame of reference. For example (x, y, z), (ρ, ϕ, z), (r, θ, ϕ) are coordinates of a point in rectangular by analogy in set of N Numbers denoted by $(x^1, x^2, \ldots x^N)$ where 1, 2 ... N are taken not as exponents but as superscripts, a policy which will prove useful.

(c) co-ordinates transformation: Let $(x^1, x^2 \ldots x^N)$ and $(\bar{x}^1, \bar{x}^2 \ldots \bar{x}^N)$ be co-ordinates of a point in two different frames of reference.

$$x^{-1} = x^{-1}(x^1, x^2 \ldots x^N)$$

$$x^{-2} = x^{-2}(x^1, x^2 \ldots x^N)$$

$$x^{-N} = x^{-N}(x^1, x^2 \ldots x^N)$$

$$x^{-R} = x^{-R}(x^1, x^2 \ldots x^N) \quad R = 1, 2 \ldots N$$

6. (c) A^i_{jk}

According to the addition

A^i_{jk} is a mixed tensor

Addition, ~~cont~~ outer multiplication and contraction

2020

1. (a) <u>Even function:</u> (A function $f(x)$ is said to be an even function if $f(-x) = f(x)$)

For Example, $f(x) = x^2$ $f(x) = \cos x$

Odd function: A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$

(b) **Fourier series:** It is a mathematical way to represent non-trigonometric periodic function as an infinite sum of trigonometric function

A series whose each and every term is either of sine or cosine.

$= a_0 \cos 0x + a_1 \cos_1 x + a_2 \cos 2x + a_3 \cos 3x + \cdots + b_0 \sin 0x$

$+ b_1 \sin_1 x + b_2 \sin 2x + b_3 \sin 3x + \cdots$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

An even function can have no sine term in it's Fourier series;

If the function $f(x)$ is even, i.e. $f(-x) = f(x)$

then $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$\therefore a_0 = I_1 + I_2 \longrightarrow \text{(ii)}$$

Now, $I_1 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx$

putting $x = -y$

$dx = -dy$

x	0	$-\pi$
y	0	π

$$\therefore I_1 = \frac{1}{\pi}\int_{\pi}^{0} f(-y)\cdot -dy$$

$$= -\frac{1}{\pi}\int_{\pi}^{0} f(y)\cdot dy$$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(x)\,dx$$

From eqn (ii)

$$a_0 = 2\times\frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

Again, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= I_3 + I_4$$

putting, $x = -y$

$dx = -dy$

x	0	$-\pi$
y	0	π

Now, $I_3 = \frac{1}{\pi} \int_{-\pi}^{0} f(x) \cos nx \, dx$

$$= \frac{1}{\pi} \int_{\pi}^{0} f(-y) \cos n(-y) (-dy)$$

$$= -\frac{1}{\pi} \int_{\pi}^{0} f(y) \cos ny \, dy$$

$$= \frac{1}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dy$$

Therefore, $a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx$

And, $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$

$$= \frac{1}{\pi} \int_{-\pi}^{0} f(x) \sin nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx$$

$$= I_5 + I_6$$

Now, $I_5 = \frac{1}{\pi} \int_{-\pi}^{0} f(x) \sin nx \, dx$

$$= \frac{1}{\pi} \int_{-\pi}^{0} f(-y) \sin(-y) (-dy)$$

$$= \frac{1}{\pi} \int_{\pi}^{0} f(y) \sin ny \, dy$$

$$= -\frac{1}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx = -I_6$$

Therefore, $b_n = I_5 + I_6$

$$= -I_6 + I_6$$

$\therefore b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{1}{2} \times \frac{2}{\pi}\int_0^{\pi} f(x)\,dx + \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx\right]\cos nx + 0$$

$$= \frac{1}{\pi}\int_0^{\pi} f(x)\,dx + \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx\right]\cos nx$$

(c) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{\pi}^{0} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\cos x\,dx + \frac{1}{\pi}\int_{\pi}^{0} \cos x\,dx$$

$$= \frac{1}{\pi}\left[-\sin x\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\sin x\right]_{0}^{\pi}$$

$$= 0 + 0 = 0$$

Now,

$$a_n = \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 2\cos nx\cos nx\,dx + \frac{1}{2\pi}\int_0^{\pi} 2\cos nx\cos x\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0}\{\cos(nx+x)+\cos(nx-x)\}dx+\frac{1}{2\pi}\int_{0}^{\pi}\{\cos(nx+x)+\cos(nx-x)\}dx$$

$$= -\frac{1}{2\pi}\left[\frac{\sin(nx+x)}{(n+1)}+\frac{\sin(nx-x)}{(n-1)}\right]_{-\pi}^{0}+\frac{1}{2\pi}\left[\frac{\sin(nx+x)}{(n+1)}+\frac{\sin(nx-x)}{(n-1)}\right]_{0}^{\pi}$$

$$= -\frac{1}{2\pi}\times 0+\frac{1}{2\pi}\times 0 = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx+\frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-\cos x)\sin nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}\cos x\sin nx\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0} 2\sin nx\cos x\,dx+\frac{1}{2\pi}\int_{0}^{\pi} 2\sin nx\cos x\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0}\{\sin(nx+x)+\sin(nx-x)\}dx+\frac{1}{2\pi}\int_{0}^{\pi}\{\sin(nx+x)+\sin(nx-x)\}dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{(n+1)}+\frac{-\cos(n-1)x}{(n-1)}\right]_{-\pi}^{0}+\frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{(n+1)}+\frac{-\cos(n-1)x}{(n-1)}\right.$$

$$= -\frac{1}{2\pi}\left[\left(\frac{-\cos 0}{n+1} + \frac{-\cos(n-1)\cdot 0}{n-1}\right) - \left(\frac{-\cos(n+1)(-\pi)}{n+1} + \frac{-\cos(n-1)(-\pi)}{(n-1)}\right)\right]$$

$$+ \frac{1}{2\pi}\left[\left(\frac{-\cos(n+1)\pi}{n+1} + \frac{-\cos(n-1)\pi}{n-1}\right) - \left(\frac{-\cos(n+1)\cdot 0}{n+1} + \frac{-\cos(n+1)\cdot 0}{(n-1)}\right)\right]$$

$$= -\frac{2}{2\pi}\left(-\frac{1}{n+1} + \frac{-1}{n-1}\right) - \frac{2}{2\pi}\left(\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right)\Bigg]$$

$$= \frac{1}{\pi}\left(\frac{1}{n+1} - \frac{(-1)^{n+1}}{n+1}\right) + \frac{1}{\pi}\left(\frac{1}{n-1} - \frac{(-1)^{n-1}}{n-1}\right)$$

$$= \frac{1}{\pi}\left(\frac{1-(-1)^{n+1}}{n+1}\right) + \frac{1}{\pi}\left(\frac{1-(-1)^{n-1}}{n-1}\right)$$

substituting the value

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty}\left[\frac{1}{\pi}\left(\frac{1-(-1)^{n+1}}{n+1}\right) + \frac{1}{\pi}\left[\frac{1-(-1)^{n-1}}{n-1}\right]\right]\sin nx$$

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{1}{\pi}\left(\frac{1-(-1)^{n+1}}{n+1}\right) + \frac{1}{\pi}\left(\frac{1-(-1)^{n-1}}{n-1}\right)\right]$$

$\sin nx$ (Ans)

2i (e) parseval's identity:

statement: If the fourier series for the function $f(x)$ coverges uniformly in $(-l, l)$ then,

$$\frac{1}{l}\int_{-l}^{l}\{f(x)\}^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2);$$

where, a_0, a_n and b_n called Fourier co-efficients or constants.

proof: the Fourier series for the function $f(x)$ in the interval $(-l, l)$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \quad \text{—— (i)}$$

where, $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\, dx$$

and $b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\, dx$

multiplying both sides of equ (i) by $f(x)$ and integrating ... we get,

$$\int_{-l}^{l}\{f(x)\}^2 dx = \frac{a_0}{2}\int_{-l}^{l} f(x)\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\right)$$

$$+ \sum_{n=1}^{\infty}\left(b_n\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\right)$$

$$= \frac{a_0}{2}\cdot a_0 l + \sum_{n=1}^{\infty} a_n \cdot a_n l + \sum_{n=1}^{\infty} b_n \cdot b_n l$$

$$= \frac{a_0^2}{2}\cdot l + \sum_{n=1}^{\infty} a_n^2 l + \sum_{n=1}^{\infty} b_n \cdot b_n l$$

$$= \frac{a_0^2}{2} l + \sum_{n=1}^{\infty} a_n^2 l + \sum_{n=1}^{\infty} b_n^2 l$$

$$\Rightarrow \int_{-l}^{l}\{f(x)\}^2 dx = \frac{a_0^2}{2} l + l\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$= \frac{1}{l}\int_{-l}^{l}\{f(x)\}^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

(Ans)

2. (ii) <u>Bessel's function:</u> The equation

$$x^2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + (x^2 - n^2)y = 0$$ is called Bessel's differential equation of order n.

By solving this equation we gets

$$\int_{-l}^{l}\{f(x)\}^2\,dx = \frac{a_0}{2}\int_{-l}^{l} f(x)\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\right)$$

$$+ \sum_{n=1}^{\infty}\left(b_n\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\right)$$

$$= \frac{a_0}{2}\cdot a_0 l + \sum_{n=1}^{\infty} a_n\cdot a_n l + \sum_{n=1}^{\infty} b_n\cdot b_n l$$

$$= \frac{a_0^2}{2}\cdot l + \sum_{n=1}^{\infty} a_n^2 l + \sum_{n=1}^{\infty} b_n\cdot b_n l$$

$$= \frac{a_0^2}{2} l + \sum_{n=1}^{\infty} a_n^2 l + \sum_{n=1}^{\infty} b_n^2 l$$

$$\Rightarrow \int_{-l}^{l}\{f(x)\}^2\,dx = \frac{a_0^2}{2} l + l\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$\Rightarrow \frac{1}{l}\int_{-l}^{l}\{f(x)\}^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

(Ans)

2. (a) <u>Bessel's function:</u> The equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0$$ is called Bessel's differential equation of order n.

By solving this equation we gets

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r+n}}{r!\,(n+r)!}$$ and this function is called Bessel's function.

2. (b) We know,

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r+n}}{r!\,(n+r)!}$$

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (2r+n) (x/2)^{2r+n-1} \cdot \frac{1}{2}}{r!\,(n+r)!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2r+n) (x/2)^{2r+n}}{r!\,(n+r)!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, 2r \cdot (x/2)^{2r+n}}{r!\,(n+r)!} + n \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r+n}}{r!\,(n+r)!}$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r \frac{2}{x} (x/2)^{2r+n} \cdot r}{r!\,(n+r)!} + n J_n(x)$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{-1} (x/2)^{2r+n} \cdot r}{r!\,(n+r)!} + n J_n(x)$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)(-1)^{r-1} \left(\frac{x}{2}\right)^{2r+n-1} \cdot r}{r\,(r-1)!\,(n+r)!} + n J_n(x)$$

$$= x\sum_{r=0}^{\alpha} \frac{(-1)(-1)^{r-1}\left(\frac{x}{2}\right)^{2(r-1)+(n+1)}}{(r-1)!\,\{(n+1)+(r-1)\}!} + nJ_n(x)$$

$$= -x\sum_{r=0}^{\alpha} \frac{(-1)^{r-1}\left(\frac{x}{2}\right)^{2(r-1)+(n+1)}}{(r-1)!\,\{(n+1)+(r-1)\}!} + nJ_n(x)$$

$$= -xJ_{n+1}(x) + nJ_n(x)$$

$$= nJ_n(x) - xJ_{n+1}(x) \quad \text{proved.}$$

2.(a) Legendre polynomials: The second order differential equation,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\}y = 0$$

in which m and n are constants is known as Legendre's associated equation and its solution are called associated Legendre functions.

It reduces to Legendre's equation when $m=0$

The Legendre differential equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0 \quad \text{———(i)}$$

And the soln of this eqn is called Legendre's function or Legendre's polynomials.

4. Laplace transform of $\sin^3 2t$

$$L\{\sin 2t\} = \int_0^\alpha e^{-st}\sin 2t\,dt$$

$$= \int_0^\alpha e^{-st}\left(\frac{e^{2ist} - e^{-2it}}{2i}\right)dt$$

$$= \frac{1}{2i}\int_0^\alpha e^{-st+2it}\,dt - \frac{1}{2i}\int_0^\alpha e^{-st-2it}\,dt$$

$$= \frac{1}{2i}\int_0^\alpha e^{-(s-2i)t}\,dt - \frac{1}{2i}\int_0^\alpha e^{-(s+2i)t}\,dt$$

$$= \frac{1}{2i}\left[\frac{e^{-(s-2i)t}}{-(s-2i)} - \frac{e^{-(s+2i)t}}{(s+2i)}\right]_0^\alpha$$

$$= \frac{1}{2i}\left[\frac{1}{(s-2i)} - \frac{1}{(s+2i)}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{(s-2i)} - \frac{1}{s+2i}\right]$$

$$= \frac{1}{2i}\left[\frac{(s+2i)-(s-2i)}{s^2-2^2 i^2}\right]$$

$$= \frac{1}{2i}\,\frac{4i}{s^2+2^2}$$

$$= \frac{2}{s^2+2^2} = \frac{2}{s^2+4} \quad \text{(Ans)}$$

5. (e) By definition,

$$\frac{d}{dz} = f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

It this limit exists independent of the manne in where $\Delta z = \Delta x + i\,\Delta y$ approaches zero,

Then, $$\frac{d\bar{z}}{dz} = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{x+iy+\Delta x+i\Delta y} - \overline{x+iy}}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x-iy+\Delta x-i\Delta y-(x-iy)}{\Delta x+i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

If $\Delta y = 0$, the required limit is,

$$\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

If $\Delta x = 0$ the required limit is

$$\lim_{\Delta y \to 0} \frac{-\Delta y}{i\Delta y} = -1$$

i,e $f(z) = \bar{z}$ is not analytic anywhere.

5. (b) $u = \cos x \cosh y$

Now $\frac{\partial u}{\partial x} = -\sin x \cosh y$ and $\frac{\partial u}{\partial y} = \cos x \sinh y$

$$\frac{\partial^2 u}{\partial x^2} = -\cos x \cosh y \qquad \frac{\partial u}{\partial y} = \cos x \cosh y$$

Now, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\cos x \cosh y + \cos x \cosh y = 0$

Therefore, $u = \cos x \cosh y$ is a harmonic function.

6. (b) Let $f(z)$ be single-valued and analytic inside and on a circle c except at the point $z=a$ chosen as the centre of c. then as we have seen $f(z)$ has a Laurent series about $z=a$ given by,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2}$$

$$a_n = \frac{1}{2\pi i} \oint_c \frac{f(z)}{(z-a)^{n+1}} dz \qquad n = 0, \pm 1, \pm 2 \cdots$$

To obtain the residue of a function $f(z)$ at $z=a$ It may appear from that the Laurent expansion of $f(z)$ about $z=a$ must be obtained. However in the case where $z=a$ is a pole of order k, there is a simple formula for a_{-1} is given by

$$a_{-1} = \lim_{z \to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \{(z-a)^k f(z)\}$$

If $k=1$ then the result is especially simple and is given by,

$$a_{-1} = \lim_{z \to a} (z-a) f(z)$$

7. (a) covariant contravariant and mixed tensor

If N^2 quantities A^{qs} in a coordinate system $(x^1, x^2 - - \cdot x^N)$ are related to N^2 other quantities in another coordinate system $(\bar{x}^1, \bar{x}^2 - - x^{-N})$ by the transformation equations

$$A^{-pr} = \sum_{s=1}^{N} \sum_{q=1}^{N} \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^{qs} \qquad p, r = 1, 2 - - N$$

$$A^{-pr} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^{qs}$$

by the adopted conventions, they are called contravariant component of a tensor,

The N^2 quantities A_{qs} are called covariant component of a tensor.

similarly the n^2 quantities A^q_s are called components of a mixed tensor.

(e) $\dfrac{\partial x^p}{\partial x^q} = \delta^p_q$

If $p = q$ $\dfrac{\partial x^p}{\partial x^q} = 1$, since $x^p = x^q$

If $p \neq q$ $\dfrac{\partial x^p}{\partial x^q} = 0$ since x^p and x^q are independent.

Then $\dfrac{\partial x^p}{\partial x^q} = \delta^p_q$

$$\frac{\partial x^p}{\partial \bar{x}^q} \frac{\partial \bar{x}^q}{\partial x^r} = \delta^p_r$$

co-ordinates x^p are functions of co-ordinates $\bar{x}^q$ which are in turn functions of co-ordinates x^r. Then by the chain rule ~~and~~ B'

$$\frac{\partial x^p}{\partial x^r} = \frac{\partial x^p}{\partial \bar{x}^q} \frac{\partial \bar{x}^q}{\partial x^r} = \delta^p_r$$

2019

1\. (a) $\Gamma\frac{1}{2} = \sqrt{\pi}$

we know that, $B(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma m+n}$ —— (i)

and $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

we can write the equation (i) and (ii)

$$\frac{\Gamma m \, \Gamma n}{\Gamma m+n} = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \quad \text{—— (iii)}$$

Let, $m = n = \frac{1}{2}$

putting this value in equ (3) we get,

$$\frac{\Gamma\frac{1}{2} \, \Gamma\frac{1}{2}}{\Gamma \frac{1}{2}+\frac{1}{2}} = 2\int_0^{\pi/2} \sin^{2\cdot\frac{1}{2}-1}\theta \cos^{2\cdot\frac{1}{2}-1}\theta \, d\theta$$

$$\frac{(\Gamma\frac{1}{2})^2}{(1)} = 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta \, d\theta$$

$$\frac{(\Gamma\frac{1}{2})^2}{1} = 2\int_0^{\pi/2} 1.1 \, d\theta$$

$$(\Gamma\tfrac{1}{2})^2 = 2[\theta]_0^{\pi/2}$$

$$(\Gamma\tfrac{1}{2})^2 = 2 \cdot \tfrac{\pi}{2} = \pi$$

$\therefore \sqrt{\Gamma \tfrac{1}{2}} = \sqrt{\pi}$

(ii) $\int_0^{\alpha} e^{-x^2} dx = \sqrt{\pi}/2$

Here, $x^2 = z \quad x = \sqrt{z}$

x	0	α
z	0	α

$2x\,dx = dz$

$\therefore dx = \frac{1}{2x} dz = \frac{1}{2\sqrt{z}} dz$

Now, $\int_0^{\alpha} e^{-x^2} dx = \int_0^{\alpha} e^{-z} \frac{1}{2\sqrt{z}} dz$

$= \frac{1}{2} \int_0^{\alpha} e^{-z} z^{-1/2} dz = \frac{1}{2} \int_0^{\alpha} e^{-z} z^{(\frac{1}{2}-1)} dz$

$= \frac{1}{2} \sqrt{\Gamma \tfrac{1}{2}} = \frac{1}{2}\sqrt{\pi} = \sqrt{\pi}/2$ (Ans)

2. The solution of Legendre diff eqn is

$y = c_0 \left[1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n-2)(n+1)(n+3)}{4!} x^4 \right]$

$+ c_1 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!} x^5 + \text{------} \right]$

$$P_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n r!\,(n-r)!\,(n-2r)!}\, x^{n-2r}$$

when $N = n/2$ for n even

and $N = \frac{n-1}{2}$ for n odd

$$P_n(x) = \frac{2n!}{2^n n!\,n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \cdots\right]$$

3. For $P_3(x)$

$$\sum_{r=0}^{1} (-1)^0 \frac{(2\times 3 - 2\times 0)!}{2^3 \times 0! \times (3-0)! \times (3-2\times 0)!}\, x^{3-2\times 0}$$

$$= \sum_{r=0}^{1} 1 \times \frac{6!}{8 \times 3! \times 3!}\, x^3$$

$$= \sum_{r=0}^{1} \frac{720}{280}\, x^3$$

4. (a) $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$

where, $u = x^2 + axy + by^2$ and $v = (cx^2 + dxy + y^2)$

$\frac{\partial u}{\partial x} = 2x + ay$ $\qquad \frac{\partial v}{\partial x} = 2cx + dy$

$\frac{\partial u}{\partial y} = ax + 2by$ $\qquad \frac{\partial v}{\partial y} = dx + 2y$

Since $f(z) = u + iv$ is analytic, so Cauchy-Riemann equation must be satisfied,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Therefore,

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial x}$$

$$ax + 2by = -2cx - dy \quad \text{--- (ii)}$$

Now from (i) and (ii)

$$(2-d)x + (a-2)y = 0 \quad \text{--- (iii)}$$

and $(a+2c)x + (2b+d)y = 0$ --- (iv)

(iii) and (iv) will be true when

$(2-d) = 0$ $\quad (a-2) = 0$

$a + 2c = 0$ $\quad 2b + d = 0$

$a = 2,\ d = 2$ $\quad b = -1$ and $c = -1$

Q. (b) Here

$$\frac{\partial u}{\partial x} = e^{-x}(\sin y) + (-e^{-x})(x\sin y - y\cos y)$$

$= e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$

$\frac{\partial^2 v}{\partial x^2} = -e^{-x}\sin y - (e^{-x} - xe^{-x})\sin y - ye^{-x}\cos y$

$= e^{-x}\sin y - e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y$

$\frac{\partial^2 v}{\partial x^2} = -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y$ — (i)

$\therefore \frac{\partial^2 v}{\partial y^2} = -xe^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y$ — (ii)

adding equ (i) and (ii) we get,

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Therefore $v = e^{-x}(x\sin y - y\cos y)$ is harmonic.

6.(a) A necessary condition that $w = f(z)$ = $u(x,y) + iv(x,y)$ be analytic in a region R is that, in R u and v satisfy the cauchy-Riemann equation.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ —— (1)}$$

If the partial derivatives are continuous in R then the Cauchy-Riemann equations are sufficient conditions that $f(z)$ be analytic in R.

*. 2018

1.(a) $\Gamma(n+1) = n\Gamma(n)$

We get, $\Gamma(n+1) = n\Gamma(n)$

When, $n=1$ we get,

$\Gamma 2 = 1\Gamma 1 = 1.1 = 1!$

When $n=2$ we get,

$\Gamma 3 = 2\Gamma 2 = 2.1 = 2!$

When $n=3$ we get

$\Gamma 4 = 3\Gamma 3 = 3.2.1 = 3!$

Therefore

$$\Gamma(n+1) = (n+1-1)! = n!$$

Hence

$$\Gamma(n+1) = n! = \lfloor n = n \Gamma n \text{ proved}$$

4. $P_0(x) = C_0 P_0(x) = 1$

$P_0(1) = 1$

$P_1(x) = C_1 x$

$P_1(x) = C_1 x$

so, $P_1(x) = x$

similarly,

$$P_2 x = \frac{1}{2}(3x^2 - 1)$$

$$P_3 x = \frac{1}{2}(5x^3 - 3x)$$

$$P_4 x = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_6(x) = \frac{1}{10}(123x^6 - 315x^4 + 105x^2 - 5)$$

We know from Legendre polynomials,

$$x = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$x^4 = \frac{8}{35} P_4(1) + \frac{6}{7} x^2 - \frac{3}{35}$$

$$P_3 x = \frac{1}{2}(5x^3 - 3x)$$

$$\therefore x^3 = \frac{2}{5} P_3(x) + \frac{3}{5} x$$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$x^2 = \frac{2}{3}P_2(x) + \frac{1}{2}$

$P_1(x) = x$

$x = P_1 x$

$x = P_1 x$

putting the value of x^4, x^3, x^2 and x

$f(x) = \frac{8}{35}P_4(x) + \frac{40}{21}P_2(x) + \frac{1}{5}P_1(x) - \frac{224}{105}P_0 x$

(Ans)