Define Beta and Gama function.

Definition:

Beta function:

$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ denoted by $B(m,n)$ where $m,n>0$ is called the first Eulerian integral or Beta function.

Gama function:

$\int_0^\infty e^{-x}x^{n-1}\,dx$ denoted by Γn $[n>0]$ is called the second Eulerian integral or Gama function.

$$\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

Soln: We have, $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Put, $x = \sin^2\theta$

or, $dx = 2\sin\theta\cos\theta\,d\theta$

Lt.

x	0	1
θ	0	$\pi/2$

$$\therefore B(m,n) = \int_0^{\pi/2}\sin\theta^{(2m-2)}\cos\theta^{(2n-2)}\;2\sin\theta\cos\theta\,d\theta$$

or, $$\frac{1}{2}B(m,n) = \int_0^{\pi/2} \frac{\sin^{2m}\theta}{\sin^2\theta} \cdot \frac{\cos^{2n}\theta}{\cos^2\theta} \sin\theta \cos\theta \, d\theta$$

or, $$\frac{1}{2}B(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Again, we know,

$$B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

$$\therefore \frac{\Gamma m \, \Gamma n}{2\Gamma(m+n)} = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Putting,

$2m-1 = p$ and $2n-1 = q$

ie, $m = \frac{p+1}{2}$ and $n = \frac{q+1}{2}$

we get,

$$\frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+1}{2} + \frac{q+1}{2}\right)} = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$$\therefore \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

(proved.)

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

Soln: We know,

$$\beta(m,n) = \int_0^1 z^{m-1}(1-z)^{n-1}\, dz$$

Put

$$z = \sin^2 x$$

$$\text{or, } dz = 2\sin x \cos x \, dx$$

Limit change,

z	0	1
x	0	$\frac{\pi}{2}$

we get,

$$\beta(m,n) = \int_0^{\pi/2} \sin^{2m-2} x \cos^{2n-2} x \; 2\sin x \cos x \, dx$$

$$\frac{1}{2}\beta(m,n) = \int_0^{\pi/2} \sin^{2m-2} x \cos^{2n-2} x \, dx .$$

Also we know,

$$\beta(m,n) = \frac{\Gamma m \,\Gamma n}{\Gamma m+n}$$

Now we get from equ. (1)

$$\frac{1}{2}\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} = \int_0^{\pi/2} \sin^{2m-1}x \cos^{2n-1}x\, dx$$

Putting $2m-1 = p$ and $2n-1 = q$

$\therefore m = \frac{p+1}{2}$ $\therefore n = \frac{q+1}{2}$

we get,

$$\frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)} = \int_0^{\pi/2} \sin^p x \cos^q x\, dx$$

$$\Rightarrow \int_0^{\pi/2} \sin^p x \cos^q x\, dx = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\frac{p+q+2}{2}}$$

Ans.

(1) $\Gamma(n+1) = n\Gamma n = n!$

Soln:

Let us consider the definition,

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\, dx$$

Integrating by parts taking e^{-x} as second function

$$\int_0^{\infty} e^{-x} x^n\, dx = \left[-e^{-x}x^n\right]_0^{\infty} - n\int_0^{\infty} -e^{-x} x^{n-1}\, dx$$

$$= n\int_0^{\infty} e^{-x} x^{n-1}\, dx = n\,\Gamma n$$

Showed.

Put, $n = n+1$

$$\therefore \ \overline{(n+1)} = \int_0^\infty e^{-x} x^n \, dx$$

Then, we know, $\overline{n} = (n-1)!$

So,

$$n\overline{n} = n(n-1)!$$
$$= n!$$

(showed)

Q Establish the relation between Beta and Gama function,

OR, prove that, $B(m,n) = \dfrac{\overline{m}\,\overline{n}}{\Gamma(m+n)}$

Soln: We have,

$$\overline{m}\,\overline{n} = \int_0^\infty e^{-x} x^{m-1} dx \int_0^\infty e^{-y} y^{n-1} dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x+y)} x^{m-1} y^{n-1} dx\, dy$$

Let, $u = x+y$ and $v = \dfrac{x}{x+y}$

or, $x = uv$ and $y = u(1-v)$

and $dy\, dx = |J|\, du\, dv$

As x and y range 0 to ∞, u ranges 0 to ∞ and v ranges 0 to 1

Now,

$$|J| = \left\| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right\|$$

$$= \left\| \begin{matrix} v & u \\ 1-v & -u \end{matrix} \right\|$$

$$= |-uv - (u-uv)|$$

$$= |-uv - u + uv|$$

$$= u$$

$[\because |J| = u]$

$$\therefore \Gamma m \, \Gamma n = \int_{u=0}^{\infty} \int_{v=0}^{1} e^{-u} (uv)^{m-1} \{u(1-v)\}^{n-1} u \, du \, dv$$

$$= \int_0^{\infty} e^{-u} u^{m-1+n-1+1} du \int_0^1 v^{m-1} (1-v)^{n-1} dv$$

$$= \Gamma(m+n) \, \beta(m,n)$$

$$\therefore \beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

(Proved).

প্র Define (i) even function and (ii) odd function

Definitions:

(i) Even function: A function $f(x)$ is said to be an even function if $f(x) = f(x)$.

e.g. $f(x) = x^2$, $f(x) = \cos x$

(ii) odd function: A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$

e.g. $f(x) = x$, $f(x) = \sin x$

প্র $\Gamma\frac{1}{2} = \sqrt{\pi}$

Proof:

$$B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

Put, $m = n = \frac{1}{2}$

$$\therefore B\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \frac{\Gamma\frac{1}{2} \, \Gamma\frac{1}{2}}{\Gamma 1}$$

$$= \Gamma\tfrac{1}{2} \, \Gamma\tfrac{1}{2} \quad \text{------} \quad (1)$$

But $B(m,n) = 2 \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Put $m = n = \frac{1}{2}$

$$\therefore B\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = 2 \int_0^{\pi/2} \sin^0\theta \cos^0\theta \, d\theta$$

$$= 2 \int_0^{\pi/2} d\theta$$

$$= 2\,[\theta]_0^{\pi/2}$$

$$= \pi \quad \text{-----------} (2)$$

From (i) & (ii) we get,

$$\Gamma\tfrac{1}{2}\,\Gamma\tfrac{1}{2} = \pi$$

$$\text{or, } \left(\Gamma\tfrac{1}{2}\right)^2 = \pi$$

$$\text{or, } \Gamma\tfrac{1}{2} = \sqrt{\pi}$$ proved.

Prove that,

$$\int_0^1 x^{m-1}(1-x^a)^n\,dx = \frac{1}{a}\,\frac{n!\,\Gamma(m/a)}{\Gamma(m/a+n+1)}$$

Put $x^a = \sin^2\theta$

$$\therefore x = \sin^{2/a}\theta$$

$$\therefore dx = 2/a\,\sin^{(2/a-1)}\theta\,\cos\theta\,d\theta$$

$$\therefore I = \frac{2}{a}\int_0^{\pi/2}\sin^{2(m-1)/a}\theta\,\cos^{2n}\theta\,\sin^{(2/a-1)}\theta\,\cos\theta\,d\theta$$

$$= \frac{2}{a}\int_0^{\pi/2}\sin^{2m/a}\,\sin^{-2/a}\,\cos^{2n+1}\theta\,\sin^{2/a-1}\,d\theta$$

$$= \frac{2}{a}\int_0^{\pi/2}\sin^{2m/a-1}\theta\,\cos^{2n+1}\theta\,d\theta$$

$$= \frac{2}{a}\,\frac{\Gamma m/a\;\Gamma n+1}{2\,\Gamma(m/a+n+1)}$$

$$= \frac{2}{a} \quad \frac{\Gamma m/a \quad n!}{2\,\Gamma(m/a + n+1)}$$

$$= \frac{1}{a} \quad \frac{n!\,\Gamma(m/a)}{\Gamma(m/a+n+1)}$$

proved:

Evaluate $\int_0^{\pi/2} \cos^q\theta \, d\theta$

$$= \int_0^{\pi/2} \sin^0\theta \quad \cos^q\theta \, d\theta \quad [\because \sin^0\theta = 1]$$

$$= \frac{\Gamma\frac{0+1}{2} \quad \Gamma\frac{q+1}{2}}{\Gamma\frac{0+q+2}{2}}$$

$$= \frac{\Gamma\frac{1}{2} \quad \Gamma\frac{q+1}{2}}{2\,\Gamma\frac{q+2}{2}}$$

$$= \frac{\Gamma\frac{q+1}{2} \quad \sqrt{\pi}}{2\,\Gamma\frac{q+2}{2}}$$

Answer.

Q. $I = \int_0^\infty \frac{x^{n-1}}{1+x} dx$

Soln:

Let, $x = \tan^2\theta$

$\Rightarrow dx = 2\tan\theta \sec^2\theta \, d\theta$

x	0	∞
θ	0	$\frac{\pi}{2}$

Now,

$$I = \int_0^{\pi/2} \frac{(\tan^2\theta)^{n-1}}{1+\tan^2\theta} 2\tan\theta \sec^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\tan^{2n-2}\theta}{\sec^2\theta} \tan\theta \sec^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \tan^{2n-2}\theta \tan\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \tan^{2n-1}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{\sin\theta}{\cos\theta}\right)^{2n-1} d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{1-2n}\theta \, d\theta$$

$$= 2 \cdot \frac{1}{2} \frac{\Gamma\frac{2n-1+1}{2}\,\Gamma\frac{1-2n+1}{2}}{\Gamma\frac{2n-1+1-2n+2}{2}}$$

$$= \frac{\Gamma\frac{2n}{2}\,\Gamma\frac{2-2n}{2}}{\Gamma\frac{2}{2}}$$

$$= \frac{\Gamma n\,\Gamma 1-n}{\Gamma 1}$$

$$= \Gamma n\,\Gamma 1-n$$ Answer.

(b) Show that, $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \frac{\Gamma\frac{1}{n}}{\Gamma(1/n+\frac{1}{2})}$

or, $I = \int_0^1 \frac{dx}{\sqrt{1-x^n}}$

Soln: Let, $x^n = \sin^2\theta$; $x = \sin^{2/n}\theta$

$n x^{n-1} dx = 2 \sin\theta \cos\theta \, d\theta$

L.t.

x	0	1
θ	0	$\frac{\pi}{2}$

$$\Rightarrow n\, x^n\, x^{-1}\, dx = 2 \sin\theta \cos\theta\, d\theta$$

$$\cancel{\Rightarrow n \cdot x^n x}$$

$$\Rightarrow n \cdot \sin^2\theta \, \sin^{-\frac{2}{n}} dx = 2 \sin\theta \cos\theta\, d\theta$$

$$\Rightarrow dx = \frac{2}{n} \sin^{\left(\frac{2}{n}-1\right)}\theta \cos\theta\, d\theta$$

$$L.H.S. = \int_0^1 \frac{dx}{1-x^n}$$

$$= \int_0^{\pi/2} \frac{\frac{2}{n} \sin^{\left(\frac{2}{n}-1\right)}\theta \cos\theta}{1-\sin^2\theta}\, d\theta$$

$$= \int_0^{\pi/2} \frac{2}{n} \frac{\sin^{\left(\frac{2}{n}-1\right)}\theta \cdot \cos\theta}{\cos\theta}\, d\theta$$

$$= \frac{2}{n} \int_0^{\pi/2} \sin^{\left(\frac{2}{n}-1\right)}\theta\, d\theta$$

$$= \frac{2}{n} \frac{\Gamma\left(\frac{\frac{2}{n}-1+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{\frac{2}{n}-1+2}{2}\right)}$$

$$= \frac{1}{n} \frac{\Gamma\left(\frac{1}{n}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\frac{2}{n}+1}{2}\right)}$$

$$= \frac{\sqrt{\pi}}{n} \frac{\Gamma\left(1/n\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$$

$= R.H.S.$ (Showed).

Q (c)

$$I = \int_0^{\pi/2} (\tan\theta)^{1/2}\, d\theta$$

Soln:

$$\int_0^{\pi/2} (\tan\theta)^{1/2}\, d\theta$$

$$= \int_0^{\pi/2} \sin\theta^{1/2} \cos\theta^{-1/2}\, d\theta$$

$$= \frac{\Gamma\frac{1/2+1}{2}\ \Gamma\frac{-1/2+1}{2}}{2\,\Gamma\frac{1/2-1/2+2}{2}}$$

$$= \frac{\Gamma 3/4\ \Gamma 1/4}{2}$$

$$= \frac{\Gamma 1/4\ \Gamma 1-1/4}{2}$$

$$= \frac{1}{2}\ \frac{\pi}{\sin \pi/4}$$

$$= \frac{\sqrt{2}}{2}\pi$$

$$= \frac{1}{\sqrt{2}}\pi$$ Answer.

(d)

$$I = \int_0^1 (1-x^n)^{1/n}\, dx$$

Let,

$$x = \sin^2\theta$$

$$dx = 2\sin\theta\cos\theta\, d\theta$$

x	0	1
θ	0	$\pi/2$

Now,

$$\int_0^{\pi/2} \left\{1-(\sin^n\theta)^2\right\}^{1/n} 2\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} (\cos^{2n}\theta)^{1/n}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \cos^3\theta\sin\theta\, d\theta$$

$$= 2\,\frac{\Gamma\frac{1+1}{2}\,\Gamma\frac{3+1}{2}}{2\,\Gamma\frac{3+1+2}{2}}$$

$$= 2\,\frac{\Gamma 1\,\Gamma 2}{2\,\Gamma 3}$$

$$= \frac{1}{2}$$

Answer.

prove that, $\int_0^1 x^{m-1}(1-x^a)^n\,dx = \frac{1}{a}\frac{n!\,\Gamma\left(\frac{m}{a}\right)}{\Gamma\left(\frac{m}{a}+n+1\right)}$

Soln:

$$x^a = \sin^2\theta$$

$$x = \sin^{2/a}\theta$$

$$\Rightarrow dx = \frac{2}{a}\sin^{(2/a-1)}\theta\cos\theta\,d\theta$$

x	0	1
θ	0	$\pi/2$

Now, $\int_0^{\pi/2}\sin^{\frac{2}{a}(m-1)}\theta\,\cos^{2n}\theta\cdot\frac{2}{a}\sin^{\left(\frac{2}{a}-1\right)}\theta\cos\theta\,d\theta$

$$= \frac{2}{a}\int_0^{\pi/2}\sin^{2m/a}\theta\,\sin^{-\frac{2}{a}}\theta\,\cos^{2n+1}\theta\,\sin^{\frac{2}{a}-1}\theta\,d\theta$$

$$= \frac{2}{a}\int_0^{\pi/2}\sin^{\frac{2m}{a}-\frac{2}{a}+\frac{2}{a}-1}\theta\,\cos^{2n+1}\theta\,d\theta$$

$$= \frac{2}{a}\int_0^{\pi/2}\sin^{\frac{2m}{a}-1}\theta\,\cos^{2n+1}\theta\,d\theta$$

$$= \frac{2}{a}\,\frac{\Gamma\left(\frac{\frac{2m}{a}-1+1}{2}\right)\Gamma\left(\frac{2n+1+1}{2}\right)}{2\,\Gamma\left(\frac{\frac{2m}{a}-1+2n+1+2}{2}\right)}$$

$$= \frac{2}{a} \frac{\Gamma\frac{m}{a}\ \Gamma(n+1)}{2\,\Gamma\left(\frac{m}{a}+n+1\right)}$$

$$= \frac{1}{a} \frac{n!\ \Gamma\frac{m}{a}}{\Gamma\left(\frac{m}{a}+n+1\right)}$$

proved.

(f)

$$I = \int_0^1 \frac{35x^3}{32\sqrt{(1-x)}}\, dx$$

Soln:

Let, $x = \sin^2\theta$

$\Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

Lt.

x	0	1
θ	0	$\pi/2$

Now,

$$\int_0^{\pi/2} \frac{35\sin^6\theta}{32\sqrt{1-\sin^2\theta}} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= \frac{35}{16}\int_0^{\pi/2} \sin^7\theta\, d\theta$$

$$= \frac{35}{16} \frac{\Gamma\frac{7+1}{2}\ \Gamma\frac{1}{2}}{2\,\Gamma\frac{7+2}{2}}$$

$$= \frac{35}{16} \cdot \frac{3.2.1\ \Gamma\frac{1}{2}}{2\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}}$$

$$= \frac{16}{16} = 1$$ Ans.

Q. $0! = \Gamma(0+1) = \Gamma 1 = 1$

Soln:

we Know, $n! = \Gamma n+1$

So, $0! = \Gamma 0+1$

$= \Gamma 1$

$= 1$ Answer. $\because [\Gamma 1 = 1]$

Q. Evaluate $\int_0^{\pi/2} \sin^6 x \, dx$

Soln:

$$I = \int_0^{\pi/2} \sin^6 x \, dx$$

$$= \frac{\Gamma \frac{6+1}{2}}{\Gamma \frac{6+2}{2}} \frac{\sqrt{\pi}}{2}$$

$$= \frac{\Gamma \frac{7}{2} \cdot \sqrt{\pi}}{\Gamma 4 \cdot 2}$$

$$= \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \cdot \sqrt{\pi}}{3 \cdot 2 \cdot 1 \cdot 2}$$

$$= \frac{5 \cdot 3 \cdot 1 \cdot \pi}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 8} = \frac{5\pi}{32}$$ Answer.

Q. Evaluate $\int_0^{\pi/2} \cos^5 x \sin^4 x \, dx$

Solⁿ:

$$I = \int_0^{\pi/2} \cos^5 x \sin^4 x \, dx$$

$$= \frac{\Gamma\left(\frac{5+1}{2}\right) \Gamma\left(\frac{4+1}{2}\right)}{2\,\Gamma\left(\frac{5+4+2}{2}\right)}$$

$$= \frac{\Gamma 3 \; \Gamma 5/2}{2\,\Gamma 11/2}$$

$$= \frac{2 \cdot 1 \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}}{2 \cdot \frac{9}{2} \cdot 7/2 \cdot 5/2 \cdot 3/2 \cdot 1/2 \cdot \sqrt{\pi}}$$

$$= \frac{3 \cdot 32}{9 \cdot 7 \cdot 5 \cdot 3 \cdot 4}$$

$$= \frac{8}{315}$$ Ans

④ $\int_0^1 x^6 \sqrt{1-x^2}\,dx$

Soln:

Let, $x^2 = \sin^2\theta$

$\Rightarrow x = \sin\theta$

$\Rightarrow 2x\,dx = 2\sin\theta\cos\theta\,d\theta$

$\Rightarrow 2\sin\theta\,dx = 2\sin\theta\cos\theta\,d\theta$

$\therefore dx = \cos\theta\,d\theta$

Lt,

x	0	1
θ	0	$\pi/2$

Now, $\int_0^{\pi/2} \sin^6\theta \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$

$= \int_0^{\pi/2} \sin^6\theta\cos^2\theta\,d\theta$

$= \dfrac{\Gamma\frac{6+1}{2}\,\Gamma\frac{2+1}{2}}{2\,\Gamma\frac{6+2+2}{2}}$

$= \dfrac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}}{2\cdot4\cdot3\cdot2\cdot1}$

$= \dfrac{5}{256}\pi$ Answer.

Q) Evaluate $\int_0^1 x^3(1-x)^3 dx$

Soln:

$$I = \int_0^1 x^{4-1}(1-x)^{4-1} dx$$

$$= \beta(4,4)$$

$$= \frac{\Gamma 4 \, \Gamma 4}{\Gamma 4+4}$$

$$= \frac{3.2.1.3.2.1}{7.6.5.4.3.2.1}$$

$$= \frac{1}{140}$$

Answer.

Q) Evaluate $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

Soln:

$$I = \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta \quad \text{———} (1)$$

$$I = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta \quad \text{———} (2)$$

Adding equations (1) & (2), we get,

$$2I = \int_0^{\pi/2} (\sqrt{\tan\theta} + \sqrt{\cot\theta})\, d\theta$$

$$= -\sqrt{2} \int_0^{\pi/2} \frac{\sin\theta + \cos\theta}{\sqrt{\sin 2\theta}}\, d\theta$$

$$= -\sqrt{2}\int_0^{\pi/2} \frac{\sin\theta + \cos\theta}{\sqrt{1-(\sin\theta - \cos\theta)^2}}\, d\theta$$

$$= \sqrt{2}\int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} \quad (\text{where } \sin\theta - \cos\theta = t)$$

$$= 2\sqrt{2}\int_0^1 \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\pi$$

or, $I = \dfrac{\pi}{\sqrt{2}}$ Answer.

প্র prove:

$$\Gamma m \,\Gamma 1-m = \frac{\pi}{\sin m\pi}$$

we know,

$$B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

and

$$B(m,n) = \frac{\Gamma m \,\Gamma n}{\Gamma(m+n)}$$

$$\therefore \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$m = 1-n$

$m>0, \quad 1-x>0$

$m>0, \quad 0<n<1$

$$\therefore \frac{\Gamma(1-n)\,\Gamma n}{\Gamma(1-n+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{1-n+n}}\,dx$$

$$\Rightarrow \Gamma(1-n)\,\Gamma n = \int_0^\infty \frac{x^{n-1}}{1+x}\,dx \qquad \text{——— (1)}$$

We Know, $\displaystyle\int_0^\infty \frac{x^{m-1}}{1+x^n} = \frac{1}{n}\,\frac{\pi}{\sin\left(\frac{\pi m}{n}\right)}$

Here, $n=1$ and $m=n$

So, from equ (i)

$$\int_0^\infty \frac{x^{n-1}}{1+x} = \frac{\pi}{\sin m\pi}$$

$$\therefore \Gamma m\,\Gamma 1-m = \frac{\pi}{\sin m\pi} \quad \Big|\ \because m=n$$

(proved).

The given function,

$$f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ \pi/2 & 0 \le x \le \pi \end{cases}$$

The fourier series for the function $f(x)$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{——— (i)}$$

Now, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= 0 + \frac{1}{\pi}\int \pi/2 \cdot dx$$

$$= \frac{1}{2}\int_0^{\pi} 1 \cdot dx$$

$$= \left[\frac{x}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

Now, $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= 0 + \frac{1}{\pi}\int_0^{\pi} \pi/2 \cdot \cos nx\,dx$$

$$= \frac{1}{2}\int_0^{\pi} \cos nx\,dx$$

$$= \left[\frac{\sin nx}{2n} \right]_0^{\pi}$$

$$= [0 - 0]$$

$$= 0$$

also,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} f(x) \sin nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx$$

$$= 0 + \frac{1}{\pi} \int_{0}^{\pi} \pi/2 \cdot \sin nx \, dx$$

$$= \frac{1}{2} \int_{0}^{\pi} \sin nx \, dx$$

$$= -\frac{1}{2n} \left[\cos nx \right]_0^{\pi}$$

$$= -\frac{1}{2n} [-1 - 1]$$

$$= \frac{1}{n}$$

Substituting this values in equ (1)

$$f(x) = \frac{\pi/2}{2} + \sum_{n=1}^{\infty} 0 \cdot \cos nx + \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

Ans.

The given function,

$$f(x) = \begin{cases} -\pi/2 & ; -\pi \le x < 0 \\ \pi/2 & ; 0 \le x \le \pi \end{cases}$$

The Fourier Series for the function f(x) is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—— (1)}$$

Now,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} f(x)\, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} -\pi/2\, dx + \frac{1}{\pi} \int_{0}^{\pi} \frac{\pi}{2}\, dx$$

$$= -\frac{1}{2} \int_{-\pi}^{0} dx + \frac{1}{2} \int_{0}^{\pi} dx$$

$$= -\frac{1}{2} [x]_{-\pi}^{0} + \frac{1}{2} [x]_{0}^{\pi}$$

$$= -\frac{1}{2} [0 + \pi] + \frac{1}{2} [\pi - 0]$$

$$= -\frac{\pi}{2} + \frac{\pi}{2}$$

$$= 0$$

Now, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x) \cos nx \, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\frac{\pi}{2} \cos nx \, dx + \frac{1}{\pi}\int_{0}^{\pi} \frac{\pi}{2} \cos nx \, dx$$

$$= -\frac{1}{2}\int_{-\pi}^{0} \cos nx \, dx + \frac{1}{2}\int_{0}^{\pi} \cos nx \, dx$$

$$= -\frac{1}{2}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{1}{2}\left[\frac{\sin nx}{n}\right]_{0}^{\pi}$$

$$= -\frac{1}{2}\times 0 + \frac{1}{2}\times 0$$

$$= 0$$

Also, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x) \sin nx \, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\frac{\pi}{2} \sin nx \, dx + \frac{1}{\pi}\int_{0}^{\pi} \frac{\pi}{2} \sin nx \, dx$$

$$= -\frac{1}{2}\int_{-\pi}^{0} \sin nx \, dx + \frac{1}{2}\int_{0}^{\pi} \sin nx \, dx$$

$$= -\frac{1}{2}\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{2}\left[-\frac{\cos nx}{n}\right]_{0}^{\pi}$$

$$= \frac{1}{2}\left[\frac{1}{n}+\frac{1}{n}\right] - \frac{1}{2}\left[-\frac{1}{n}-\frac{1}{n}\right]$$

$$= \frac{1}{n}+\frac{1}{n}$$

$$= \frac{2}{n}$$

Substituting these values in equ. (1)

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

Ans

Find the Fourier series which represent the function $f(x)$ defined by,

$$f(x) = \begin{cases} -\cos x & \text{for } -\pi \le x \le 0 \\ \cos x & \text{for } 0 \le x \le \pi \end{cases}$$

Soln: We know,

The Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (1)}$$

Here, $f(x)$ function is an even function so

$b_n = 0$

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\cos x\,dx + \frac{1}{\pi}\int_{0}^{\pi} \cos x\,dx$$

$$= \frac{1}{\pi}\left[-\sin x\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\sin x\right]_{0}^{\pi}$$

$$= 0$$

Again,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\int_{-\pi}^{0} \cos x\cos nx\,dx + \int_{0}^{\pi} \cos x\cos nx\,dx\right]$$

$$= -\frac{1}{\pi}\left[\int_{-\pi}^{0} \cos x\cos nx\,dx - \int_{0}^{\pi} \cos x\cos nx\,dx\right]$$

$$= -\frac{1}{\pi}\left[\int_{-\pi}^{0} \frac{\cos(n+1)x + \cos(n-1)x}{2}\,dx - \int_{0}^{\pi} \frac{\cos(n+1)x - \cos(n-1)x}{2}\,dx\right]$$

$$= -\frac{1}{2\pi}\left[\left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_{-\pi}^{0} - \left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_{0}^{\pi}\right]$$

$$= -\frac{1}{2\pi}\left[\left\{\frac{\cos n\pi}{n+1} - \frac{\cos n\pi}{n-1}\right\} - \left\{\frac{\cos n\pi}{n+1} + \frac{\cos n\pi}{n-1}\right\}\right]$$

$$= -\frac{1}{2\pi}\left[\frac{\cos n\pi}{n+1} - \frac{\cos n\pi}{n-1} - \frac{\cos n\pi}{n+1} - \frac{\cos n\pi}{n-1}\right]$$

$$= -\frac{1}{2\pi}\left[\frac{-2\cos n\pi}{n-1}\right]$$

$$= \frac{2\cos n\pi}{n-1} \cdot \frac{1}{2\pi}$$

$$= \frac{(-1)^n}{\pi(n-1)}$$

Now, putting the value of a_0 and a_n in equ. (i) we get,

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi(n-1)} \cos nx + 0$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi(n-1)} \cos nx$$

(Ans).