Special Function (MMP–II)

Define Beta and Gama function:

Definition:

Beta function: $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ denoted by $\beta(m,n)$ where $m,n>0$ is called the first Eulerian integral or Beta function.

Gama function:

$\int_0^\infty e^{-x}x^{n-1}\,dx$ denoted by Γn $[n>0]$ is called the second Eulerian integral or Gama function.

Prove the following expression:

(1) $$\frac{\Gamma n}{\lambda^n} = \int_0^\infty e^{-\lambda y}\, y^{n-1}\, dy$$

Soln: We have, $\Gamma n = \int_0^\infty e^{-x}x^{n-1}\,dx$; $n>0$

put, $x = \lambda y$

or $dx = \lambda\, dy$

$$\therefore \Gamma n = \int_0^\infty e^{-\lambda y}(\lambda y)^{n-1}\lambda\, dy$$

$\lambda^{n-1+1}\, y^{n-1}\, \lambda y$

$$\Rightarrow \frac{\Gamma n}{\lambda^n} = \int_0^\infty e^{-\lambda y}\, y^{n-1}\, dy$$

(proved)

(ii) $\Gamma n = \int_0^1 \log\left(\frac{1}{y}\right)^{n-1} dy$

Soln: We have, $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \;;\; n>0$

put, $e^{-x} = z$

or, $-e^{-x} dx = dz$

or $dx = -\frac{1}{e^{-x}} dz$

$= -\frac{1}{z} dz$

Let :

x	0	∞
z	1	0

$\therefore \Gamma n = -\int_1^0 z \left(\log \frac{1}{z}\right)^{n-1} \cdot \frac{1}{z} dz \quad \left[\because x = \log\frac{1}{z}\right]$

$= \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy$ (proved)

(iii) $\Gamma n = (m+1)^n \int_0^1 y^m \left(\log \frac{1}{y}\right)^{n-1} dy$

Soln: We have,

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \;;\; n>0$$

put, $x = -(m+1)\log y$

or, $dx = -(m+1)\frac{1}{y} dy$

Let ;

x	0	∞
y	1	0

$$\therefore \Gamma n = \int_1^0 e^{(m+1)\log y}\left[-(m+1)\log y\right]^{n-1} - (m+1)\, {}^{1}/{}_{y}\, dy$$

$$= \int_0^1 e^{(m+1)\log y}\left[-(m+1)\log y\right]^{n-1} (m+1)\, {}^{1}/{}_{y}\, dy$$

$$= (m+1)^n \int_0^1 e^{(m+1)\log y} (-1)^{n-1} (\log y)^{n-1} \frac{1}{y}\, dy$$

$$= (m+1)^n \int_0^1 y^{(m+1)} (-\log y)^{n-1} \frac{1}{y}\, dy$$

$$= (m+1)^n \int_0^1 y^m \cdot y\, (-\log y)^{n-1} \cdot \frac{1}{y} \cdot dy$$

$$= (m+1)^n \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy \quad \text{(proved)}$$

$\boxed{}$ Deduce wallis Formula or proved that :

① $$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\frac{q+1}{2}}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

Sol^n: We have, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

put, $x = \sin^2\theta$

or $dx = 2\sin\theta \cos\theta \, d\theta$

Lt :

x	0	1
θ	0	$\pi/2$

$$\therefore \beta(m,n) = \int_0^{\pi/2} \sin^{(2m-2)}\theta \cos^{(2n-2)}\theta \; 2\sin\theta\cos\theta d\theta$$

$$\text{or, } \frac{1}{2}\beta(m,n) = \int_0^{\pi/2} \frac{\sin^{2m}\theta}{\sin^2\theta} \cdot \frac{\cos^{2n}\theta}{\cos^2\theta} \cdot \sin\theta\cos\theta d\theta$$

$$\text{or, } \frac{1}{2}\beta(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$$

Again we know,

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

$$\therefore \frac{\Gamma m \cdot \Gamma n}{2\Gamma(m+n)} = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

putting, $2m-1 = p$ and $2n-1 = q$

that means $m = \frac{p+1}{2}$ and $n = \frac{q+1}{2}$

we get,

$$\frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+1}{2} + \frac{q+1}{2}\right)} = \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$$\therefore \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$ (proved)

④ Establish the relation and hence show that :

① $\Gamma(n+1) = n\Gamma n = n!$ ⑪ $\int_0^{\pi/2} \cos^q\theta \, d\theta = \frac{\Gamma(1/2)\,\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{q+2}{2}\right)}$

⇒ Solⁿ ⑪ same as Num. ① Math, At last let $p=0$

① Solⁿ :

Let us consider the definition,

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

Integrating by parts taking e^{-x} as second function

$$\int_0^\infty e^{-x} x^n dx = \left[-e^{-x} x^n\right]_0^\infty - n\int -e^{-x} x^{n-1} dx$$

$$= n\int_0^\infty e^{-x} x^{n-1} dx$$

$$= n\Gamma n \quad \text{(showed)}$$

put, $n = n+1$

$$\therefore \Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$$

Then we know, $\Gamma n = (n-1)!$

So, $n\Gamma n = \cancel{n(n-2)}$

$= n(n-1)!$

$= n!$ (showed)

Establish the relation between Beta and Gamma function, OR prove that $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$

Sol^n: We have,

$$\Gamma m \Gamma n = \int_0^\infty e^{-x} x^{m-1} dx \int_0^\infty e^{-y} y^{n-1} dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x+y)} x^{m-1} y^{n-1} dx \, dy$$

let, $u = x+y$ and $v = \frac{x}{x+y}$

or, $x = uv$ and $y = u(1-v)$

and $dy\,dx = |J| \, du \, dv$

As x and y range 0 to ∞, u ranges 0 to ∞ and v range 0 to 1.

Now,

$$|J| = \left\| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right\|$$

$$= \left\| \begin{matrix} v & u \\ 1-v & -u \end{matrix} \right\|$$

$$= |-uv - (u - uv)|$$

$$= |-\cancel{uv} - u + \cancel{uv}|$$

$$= |-u|$$

$$= u \qquad [\because |J| = u]$$

$$\therefore \Gamma m \Gamma n = \int_{u=0}^{\infty} \int_{v=0}^{1} e^{-u} (uv)^{m-1} \{u(1-v)^{n-1}\} u \, du \, dv$$

$$= \int_0^{\infty} e^{-u} u^{m-1+n-1+1} du \int_0^1 v^{m-1} (1-v)^{n-1} dv$$

$$= \Gamma(m+n) \; \beta(m,n) \qquad \Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$\therefore \beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

(proved)

Even function: A function $f(x)$ is said to be an even function if $f(-x) = f(x)$.

e.g. $f(x) = x^2$, $f(x) = \cos x$

Odd function: A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$

e.g. $f(x) = x$, $f(x) = \sin x$

Evaluate the following function using the Gama function:

(a) $I = \int_0^\infty \frac{x^{n-1}}{1+x} dx$

(b) $I = \int_0^1 \frac{dx}{(1-x^n)}$

(c) $I = \int_0^{\pi/2} (\tan\theta)^{1/2} d\theta$

(d) $I = \int_0^1 (1-x^n)^{1/n} dx$

(e) $I = \int_0^1 x^{m-1} (1-x^a)^n dx$

(f) $I = \int_0^1 \frac{35x^3}{32\sqrt{(1-x)}} dx$

Prove $\Gamma 1 = 1$

Soln: We know, $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

put, $n = 1$

$$\therefore \Gamma 1 = \int_0^\infty e^{-x} x^0 dx$$

$$= \int_0^\infty e^{-x} dx$$

$$= \left[-e^{-x}\right]_0^\infty$$

$$= 1 \quad \text{(proved)}$$

e^{-x} $\quad \frac{e^{mx}}{m}$ $\quad \frac{e^{\cdot\cdot}}{m}$ $\quad e^{mx}$ $\quad -e^{mx}$

Prove $\Gamma(n+1) = n\Gamma n = n!$

Soln: We know/ Let us consider the definition,

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

let, $n = n+1$

$$\therefore \Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$$

Integrating by parts taking e^{-x} as second function,

$$\int_0^\infty e^{-x} x^n \, dx$$

$$= \left[-e^{-x} x^n\right]_0^\infty - n\int_0^\infty -e^{-x} x^{n-1} \, dx$$

$$= n\int_0^\infty e^{-x} x^{n-1} \, dx$$

$$= n\Gamma n \quad \text{(proved)}$$

Again, we know,

$$\Gamma \overline{n+1} = n\Gamma n$$

When $n=1$, $\Gamma 1 = 1\Gamma 1 = 1 = 1!$

When $n=2$, $\Gamma 3 = 2\Gamma 2 = 2\times 1 = 2!$

$\Gamma 4 = \Gamma \overline{3+1} = 3\Gamma 3 = 3!$

and so on,

$$\cancel{\Gamma 1 = 1!}$$
$$\cancel{\Gamma 2 = 2!}$$
$$\cancel{\Gamma 3 = 3!}$$

$\Gamma 4 = 3!$

$\Gamma 5 = 4!$

$\Gamma 6 = 5!$

$\Gamma \overline{n+1} = n!$

$\therefore \Gamma \overline{n+1} = n!$ (Provided n is a positive integer)

(proved)

* $0! = \Gamma(0+1) = \Gamma 1 = 1$

* Gauss's pi function, $\pi(n) = \Gamma(n+1)$

☐ prove $\beta(m,n) = \beta(n,m)$

Proof: By definition $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

If, we let, $x = 1-y$

$$\beta(m,n) = \int_0^1 (1-y)^{m-1} y^{n-1} dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1} dy$$

$$= \beta(n,m)$$

Hence $\beta(m,n) = \beta(n,m)$ (proved)

☐ $\Gamma\frac{1}{2} = \sqrt{\pi}$

Proof: $\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$

put, $m = n = \frac{1}{2}$

$$\therefore \beta(1/2, 1/2) = \frac{\Gamma 1/2 \, \Gamma 1/2}{\Gamma 1}$$

$$= \Gamma 1/2 \, \Gamma 1/2 \quad \text{——— (1)}$$

But, $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

put, $m = n = \frac{1}{2}$

$\therefore \beta(1/2, 1/2) = 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta \, d\theta$

$= 2\int_0^{\pi/2} 1 \, d\theta$

$= 2[\theta]_0^{\pi/2}$

$= \pi$ ——— (II)

From (I) and (II) we get,

$\Gamma(1/2)\,\Gamma(1/2) = \pi$

or, $(\Gamma(1/2))^2 = \pi$

or, $\Gamma(1/2) = \sqrt{\pi}$ (proved)

4. Evaluate, $\Gamma\left(\frac{1}{2}\right)$ and $\Gamma\left(-\frac{3}{2}\right)$

<u>Sol^n</u> : We know, $n\Gamma(n) = \Gamma(n+1)$

put, $n = -\frac{1}{2}$

$$\therefore \Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma(1/2)}{-1/2}$$

$$= -2\,\Gamma(1/2)$$

$$= -2\sqrt{\pi}$$

put, $n = -3/2$

$$-3/2\;\Gamma(-3/2) = \Gamma(-3/2+1)$$

or, $$\Gamma(-3/2) = \frac{\Gamma(-1/2)}{-3/2}$$

or, $$\Gamma(-3/2) = -\frac{2}{3}(-2\sqrt{\pi}) \qquad \left[\because \Gamma(-1/2) = -2\sqrt{\pi}\right]$$

or, $$\Gamma(-3/2) = \frac{4}{3}\cancel{2}\sqrt{\pi}$$ <u>Ans.</u>

A particle is a attracted toward a fixed point with a force inversely proportional to it's instantaneous distance from O. Appling the Knowledge of gama function, show that if the particle is released from rest the time for it to reach O is given by $T = a\sqrt{\frac{\pi m}{2K}}$

Proof: At time $t=0$ let the particle be located on the x-axis at $x = a > 0$ and let O be the origin. Then according to the Newton's law,

$$m\frac{d^2x}{dt^2} = -K/x \quad \text{——— (I)}$$

or, $m\frac{d^2x}{dt^2} + K/x = 0$

Where m is the mass and K is the proportional constant.

multiplying (I) by $2\frac{dx}{dt}dt$,

$$2m\frac{d^2x}{dt^2}\frac{dx}{dt}dt = -\frac{2K}{x}dx \quad \text{——— (II)}$$

let $v = \frac{dx}{dt}$ be the velocity of the particle.

Then, $\dfrac{dv}{dt} = \dfrac{dx^2}{dt^2}$

$\therefore \dfrac{d^2x}{dt^2} = \dfrac{dv}{dt}$

$= \dfrac{dv}{dx}\dfrac{dx}{dt}$

$= v\dfrac{dv}{dx}$

Then equation (ii) becomes,

$$2mv\frac{dv}{dx}\,dx = -\frac{2K}{x}\,dx$$

or, $mv\,dv = -K/x\,dx$ ——— (iii)

Integrating equ. (iii) we get,

$$\frac{mv^2}{2} = -K\ln x + c$$

When $x = a$, $v = \dfrac{dx}{dt} = 0$

$$\therefore c = K\ln a$$

$$\therefore \frac{mv^2}{2} = -K\ln x + K\ln a$$

or, $\dfrac{mv^2}{2} = K\ln\dfrac{a}{x}$

or, $v = \sqrt{\frac{2K}{m}} \sqrt{\ln \frac{a}{x}}$

$\therefore \frac{dx}{dt} = \sqrt{\frac{2K}{m}} \sqrt{\ln \frac{a}{x}}$

Now,

$$\int_0^T dt = \sqrt{\frac{m}{2K}} \int_0^a \left(\ln a/x\right)^{-\frac{1}{2}} dx$$

or, $T = \sqrt{\frac{m}{2K}} \int_0^a (\ln a/x)^{-\frac{1}{2}} dx$

$\therefore T = -a\sqrt{\frac{m}{2K}} \int_\infty^0 (P)^{-\frac{1}{2}} p\, e^{-P} dP$

$= a\sqrt{\frac{m}{2K}} \int_0^\infty e^{-P} p^{-\frac{1}{2}+1} dp$

$= a\sqrt{\frac{m}{2K}} \int_0^\infty e^{-P} p^{1-\frac{1}{2}} dp$

$= a\sqrt{\frac{m}{2K}} \Gamma(1/2)$

$= a\sqrt{\frac{m}{2K}} \sqrt{\pi}$

let $\ln a/x = p$

or $x = ae^{-P}$

or $dx = -ape^{-P} dp$

x	0	a
P	∞	0

Hence, $T = a\sqrt{\frac{\pi m}{2K}}$ (Proved)

$\boxed{}$ $\Gamma m \, \Gamma(1-m) = \dfrac{\pi}{\sin m\pi}$

$\boxed{}$ Evaluate $\int_0^{\pi/2} \cos^q \theta \, d\theta$ and $\int_0^{\pi/2} \sin^p \theta \, d\theta$

$\boxed{}$ Prove that, $\int_0^1 x^4 (1-x)^3 dx = \dfrac{1}{280}$

<u>Sol</u>

Soln: $L.H.S = \int_0^1 x^{5-1} (1-x)^{4-1} dx$

$= \beta(5,4)$

$= \dfrac{\Gamma 5 \, \Gamma 4}{\Gamma(5+4)} = \dfrac{4! \, 3!}{8!} = \dfrac{1}{280} = R.H.S$ (Proved)

$\boxed{}$ Evaluate $\int_0^{\pi/2} \sin^6 x \, dx$; Ans: $\dfrac{5\pi}{32}$

$\boxed{}$ Evaluate $\int_0^{\pi/2} \cos^5 x \sin^4 x \, dx$; Ans: $\dfrac{8}{315}$

$\boxed{}$ Evaluate $\int_0^1 x^6 \sqrt{1-x^2} \, dx$; Ans: $\dfrac{5\pi}{256}$

$\boxed{}$ Evaluate $\int_0^1 x^3 (1-x)^3 dx$; Ans: $\dfrac{1}{140}$

$\boxed{}$ Evaluate $\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta$; Ans $\dfrac{\sqrt{2}}{2}\pi \Big/ \dfrac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{2}}{2}$

☐ Evaluate, $\int_0^1 \frac{35x^3\,dx}{32\sqrt{1-x}}$

<u>Sol^n</u>:

let, $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$

Lt:

x	0	1
θ	0	$\pi/2$

$$I = \int_0^{\pi/2} \frac{35\sin^6\theta\; 2\sin\theta\cdot\cos\theta}{32\cos\theta}\,d\theta$$

$$= \int_0^{\pi/2} \frac{35}{16}\sin^7\theta\,d\theta$$

$$= \frac{35}{16}\,\frac{\Gamma\left(\frac{7+1}{2}\right)}{\Gamma\left(\frac{7+2}{2}\right)}\;\frac{\sqrt{\pi}}{2}$$

$$= \frac{35}{16}\;\frac{3\cdot 2\cdot 1\cdot\sqrt{\pi}}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot 2}$$

$$= \frac{\cancel{35}}{\cancel{16}}\;\frac{\cancel{3}\cdot\cancel{2}\cdot\cancel{16}}{\cancel{7}\cdot\cancel{5}\cdot\cancel{3}\cdot\cancel{2}}$$

$= 1$ <u>Ans</u>.

Show that, $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \frac{\Gamma(1/n)}{\Gamma(1/n+1/2)}$

Let, $x^n = \sin^2\theta$ $\therefore x = \sin^{2/n}\theta$

$$dx = \frac{2}{n}\sin^{(2/n-1)}\theta\cos\theta\, d\theta$$

Thus,

$$I = \frac{2}{n}\int_0^{\pi/2} \frac{\sin^{(2/n-1)}\theta \cos\theta}{\cos\theta}\, d\theta$$

$$= \frac{2}{n}\int_0^{\pi/2} \sin^{\left(\frac{2-n}{n}\right)}\theta\, d\theta$$

$$= \frac{2}{n} \frac{\Gamma\left(\frac{\frac{2-n}{n}+1}{2}\right)}{\Gamma\left(\frac{\frac{2-n}{n}+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2}$$

$$= \frac{2}{n} \frac{\Gamma(1/n)}{\Gamma\left(1/n+\frac{1}{2}\right)} \cdot \frac{\sqrt{\pi}}{2}$$

$$= \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma(1/n)}{\Gamma\left(1/n+\frac{1}{2}\right)}$$ (showed)

Evaluate $\int_0^1 (1-x^n)^{1/n}\, dx$

put, $x^n = \sin^2\theta$ i.e. $x = \sin^{2/n}\theta$

$\therefore dx = \frac{2}{n}\sin^{2/n-1}\theta\cos\theta\, d\theta$

x	0	1
θ	0	$\pi/2$

$$I = \int_0^{\pi/2} (\cos^2\theta)^{1/n}\cdot\frac{2}{n}\sin^{2/n-1}\theta\cos\theta\, d\theta$$

$$= \frac{2}{n}\int_0^{\pi/2}\cos^{2/n+1}\theta\,\sin^{2/n-1}\theta\, d\theta$$

$$= \frac{2}{n}\,\frac{\Gamma\frac{1}{2}\left(2/n+1+1\right)\,\Gamma\frac{1}{2}\left(2/n-1+1\right)}{2\,\Gamma\frac{1}{2}\left(2/n+1+2/n-1+2\right)}$$

$$= \frac{2}{n}\,\frac{\Gamma(1/n+1)\,\Gamma 1/n}{2\,\Gamma(2/n+1)}$$

$$= \frac{1}{n}\,\frac{\frac{1}{n}\,\Gamma 1/n\;\Gamma 1/n}{2/n\;\Gamma 2/n}$$

$$= \frac{1}{n}\,\frac{\left\{\Gamma 1/n\right\}^2}{2\,\Gamma 2/n}$$

$$= \frac{1}{n}\,\frac{\left[\Gamma 1/n\right]^2}{2\,\Gamma 2/n}$$ Ans.

Prove that, $\int_0^1 x^{m-1}(1-x^a)^n \, dx = \frac{1}{a} \frac{n!\,\Gamma(m/a)}{\Gamma(m/a+n+1)}$

put, $x^a = \sin^2\theta$ $\therefore x = \sin^{2/a}\theta$

$\therefore dx = \frac{2}{a}\sin^{(2/a-1)}\theta \cos\theta \, d\theta$

$$I = \frac{2}{a}\int_0^{\pi/2} \sin^{2(m-1)/a}\theta \cos^{2n}\theta \sin^{(2/a-1)}\theta \cos\theta \, d\theta$$

$$= \frac{2}{a}\int_0^{\pi/2} \sin^{2m/a} \sin^{-2/a} \cos^{2n+1} \sin^{2/a-1} d\theta$$

$$\cancel{= \frac{2}{a}}$$

$$= \frac{2}{a}\int_0^{\pi/2} \sin^{2m/a-1} \cos^{2n+1} d\theta \qquad \left[\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}\right.$$

$$= \frac{2}{a} \frac{\Gamma(m/a)\,\Gamma(n+1)}{2\,(m/a+n+1)}$$

$$= \frac{2}{a} \frac{\Gamma(m/a)\, n!}{2\,\Gamma(m/a+n+1)}$$

$$= \frac{1}{a} \frac{n!\,\Gamma(m/a)}{\Gamma(m/a+n+1)}$$ (Proved)