

Define Beta and Gamma function.

Definition:

Beta function:

$\int_0^1 x^{m-1} (1-x)^{n-1} dx$ denoted by $B(m,n)$ where $m, n > 0$ is called the first Eulerian integral or Beta function.

Gamma function:

$\int_0^\infty e^{-x} x^{n-1} dx$ denoted by Γ_n [$n > 0$] is called the second Eulerian integral or Gamma function.

Sol: $\int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta = \frac{\Gamma(\frac{p+1}{2}) \Gamma(\frac{q+1}{2})}{2 \Gamma(\frac{p+q+2}{2})}$

Sol: We have, $B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

Put, $x = \sin^2 \theta$

Lt.

or, $dx = 2 \sin \theta \cos \theta d\theta$

x	0	1
θ	0	$\pi/2$

$\therefore B(m,n) = \int_0^{\pi/2} \sin^m \theta \cos^n \theta 2 \sin \theta \cos \theta d\theta$

(2)

$$\text{or, } \frac{1}{2} P(m,n) = \int_0^{\pi/2} \frac{\sin^{2m} \theta}{\sin^2 \theta} \cdot \frac{\cos^{2n} \theta}{\cos^2 \theta} \sin \theta \cos \theta d\theta$$

$$\text{or, } \frac{1}{2} P(m,n) = \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

Again, we know,

$$P(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

$$\therefore \frac{\Gamma m \Gamma n}{2 \Gamma(m+n)} = \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

Putting,

$$2m-1 = p \quad \text{and} \quad 2n-1 = q$$

$$\text{i.e., } m = \frac{p+1}{2} \quad \text{and} \quad n = \frac{q+1}{2}$$

we get,

$$\frac{\Gamma(\frac{p+1}{2}) \Gamma(\frac{q+1}{2})}{2 \Gamma(\frac{p+1}{2} + \frac{q+1}{2})} = \int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta$$

$\pi/2$

$$\therefore \int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta = \frac{\Gamma(\frac{p+1}{2}) \Gamma(\frac{q+1}{2})}{2 \Gamma(\frac{p+q+2}{2})}$$

(proved.)

$$\checkmark \int_0^{\pi/2} \sin^p x \cos^q x dx = \frac{\Gamma(\frac{p+1}{2}) \Gamma(\frac{q+1}{2})}{2 \Gamma(\frac{p+q+2}{2})}$$

Soln: we know,

$$B(m,n) = \int_0^1 z^{m-1} (1-z)^{n-1} dz$$

Put

$$z = \sin^2 x$$

$$\text{or, } dz = 2 \sin x \cos x dx$$

limit change,

z	0	1
x	0	$\frac{\pi}{2}$

we get,

$$B(m,n) = \int_0^{\pi/2} \sin^{2m-2} x \cos^{2n-2} x 2 \sin x \cos x dx$$

$$\frac{1}{2} B(m,n) = \int_0^{\pi/2} \sin^{2m-2} x \cos^{2n-2} x dx$$

Also we know,

$$B(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$$

(4)

Now we get from equ. ①

$$\frac{1}{2} \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} = \int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x dx$$

Putting

$$2m-1 = p$$

$$\text{and } 2n-1 = q$$

$$\therefore m = \frac{p+1}{2}$$

$$\therefore n = \frac{q+1}{2}$$

We get,

$$\frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)} = \int_0^{\pi/2} \sin^p x \cos^q x dx$$

$$\Rightarrow \int_0^{\pi/2} \sin^p x \cos^q x dx = \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2 \Gamma\left(\frac{p+q+2}{2}\right)}$$

Ans.

$$\Gamma(n+1) = n \Gamma(n) = n!$$

Ans.

Soln:

Let us consider the definition

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

Integrating by parts taking e^{-x} as second function

$$\begin{aligned} \int_0^\infty e^{-x} x^n dx &= [-e^{-x} x^n]_0^\infty - n \int_0^\infty -e^{-x} x^{n-1} dx \\ &= n \int_0^\infty e^{-x} x^{n-1} dx = n \Gamma(n) \end{aligned}$$

Showed.

Put, $n = n+1$

$$\therefore (n+1) = \int_0^{\infty} e^{-x} x^n dx$$

Then, we know, $\Gamma n = (n-1)!$

So,

$$n\Gamma n = n(n-1)! \\ = n! \\ (\underline{\text{showed}})$$

Q Establish the relation between Beta and Gamma function,

OR, prove that, $B(m, n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$

Solⁿ: We have,

$$\begin{aligned} \Gamma m \Gamma n &= \int_0^{\infty} e^{-x} x^{m-1} dx \int_0^{\infty} e^{-y} y^{n-1} dy \\ &= \int_0^{\infty} \int_0^{\infty} e^{-(x+y)} x^{m-1} y^{n-1} dx dy \end{aligned}$$

Let,

$$u = x+y \quad \text{and} \quad v = \frac{x}{x+y}$$

or, $x = uv \quad \text{and} \quad y = u(1-v)$

and $dy dx = |J| du dv$

As x and y range 0 to ∞ , u ranges 0 to ∞
and v ranges 0 to 1

Now,

$$|J| = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right|$$

$$= \left| \begin{array}{cc} v & u \\ 1-v & -u \end{array} \right|$$

$$= | -uv - (u - uv) |$$

$$= | -uv - u + uv |$$

$$= u$$

$$\therefore |J| = u$$

$$\therefore \Gamma_m \Gamma_n = \int_{u=0}^{\infty} \int_{v=0}^1 e^{-u} (uv)^{m-1} \{u(1-v)\}^{n-1} u du dv$$

$$= \int_0^{\infty} e^{-u} u^{m-1+n-1+1} du \int_0^1 v^{m-1} (1-v)^{n-1} dv$$

$$= \Gamma(m+n) \beta(m, n)$$

$$\therefore \beta(m, n) = \frac{\Gamma_m \Gamma_n}{\Gamma(m+n)}$$

(proved).

Q1 Define (i) even function and (ii) odd function.

Definitions

(i) Even function: A function $f(x)$ is said to be an even function if $f(-x) = f(x)$.

$$\text{e.g. } f(x) = x^2, f(x) = \cos x$$

(ii) Odd function: A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$

$$\text{e.g. } f(x) = x, f(x) = \sin x$$

Q2 $\Gamma_{1/2} = \sqrt{\pi}$

Proof:

$$B(m, n) = \frac{\Gamma_m \Gamma_n}{\Gamma(m+n)}$$

$$\text{Put, } m = n = \frac{1}{2}$$

$$\therefore B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma_{1/2} \Gamma_{1/2}}{\Gamma_1} \\ = \Gamma_{1/2} \Gamma_{1/2} \quad \dots \dots \dots (1)$$

$$\text{But } B(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

$$\text{Put } m = n = \frac{1}{2}$$

$$\therefore B\left(\frac{1}{2}, \frac{1}{2}\right) = 2 \int_0^{\pi/2} \sin^\circ \theta \cos^\circ \theta d\theta \\ = 2 \int_0^{\pi/2} d\theta$$

$$= 2 [\theta]_0^{2/2}$$

$$= \pi \quad \dots \quad \dots \quad (2)$$

From ① & ⑪ we get,

$$\Gamma_{1/2} \Gamma_{1/2} = \pi$$

$$\text{or, } (\Gamma_{1/2})^2 = \pi$$

$$\text{or, } \Gamma_{1/2} = \sqrt{\pi} \quad \underline{\text{proved}}$$



Prove that,

$$\int_0^1 x^{m-1} (1-x^a)^n dx = \frac{1}{a} \frac{n! \sqrt{(m/a)}}{\Gamma(m/a + n+1)}$$

$$\text{Put } x^a = \sin^2 \theta$$

$$\therefore x = \sin^{2/a} \theta$$

$$\therefore dx = \frac{2}{a} \sin^{(2/a)-1} \theta \cos \theta d\theta$$

$$\therefore I = \frac{2}{a} \int_0^{\pi/2} \sin^{2(m-1)/a} \theta \cos^{2n} \theta \sin^{(2/a-1)} \theta \cos \theta d\theta$$

$$= \frac{2}{a} \int_0^{\pi/2} \sin^{2m/a} \theta \sin^{-2/a} \theta \cos^{2n+1} \theta \sin^{2/a-1} \theta \cos \theta d\theta$$

$$= \frac{2}{a} \int_0^{\pi/2} \sin^{2m/a-1} \theta \cos^{2n+1} \theta d\theta$$

$$= \frac{2}{a} \frac{\Gamma(m/a) \Gamma(n+1)}{2 \Gamma(m/a + n+1)}$$

$$= \frac{2}{\alpha} \cdot \frac{\Gamma(m/\alpha - n)}{2 \Gamma(m/\alpha + n + 1)}$$

$$= \frac{1}{\alpha} \cdot \frac{n! \Gamma(m/\alpha)}{\Gamma(m/\alpha + n + 1)} \quad \underline{\text{proved.}}$$

 Evaluate $\int_0^{\pi/2} \cos^2 \theta d\theta$

$$= \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta \quad [\because \sin^2 \theta = 1]$$

$$= \frac{\frac{1}{2} \Gamma(\frac{0+1}{2}) \Gamma(\frac{q+1}{2})}{\Gamma(\frac{0+q+2}{2})}$$

$$= \frac{\frac{1}{2} \Gamma(\frac{q+1}{2})}{2 \Gamma(\frac{q+2}{2})}$$

$$= \frac{\sqrt{\frac{q+1}{2}} \sqrt{\pi}}{2 \sqrt{\frac{q+2}{2}}}$$

Answer.

$$\text{Q} \quad I = \int_0^\infty \frac{x^{n-1}}{1+x} dx$$

Soln:

$$\text{Let, } x = \tan^2 \theta$$

$$\Rightarrow dx = 2 \tan \theta \sec^2 \theta d\theta$$

x	0	∞
θ	0	$\frac{\pi}{2}$

Now,

$$I = \int_0^{\pi/2} \frac{(\tan^2 \theta)^{n-1}}{1+\tan^2 \theta} 2 \tan \theta \sec^2 \theta d\theta$$

$$= 2 \int_0^{\pi/2} \frac{\tan^{2n-2} \theta}{\sec^2 \theta} \tan \theta \sec^2 \theta d\theta$$

$$= 2 \int_0^{\pi/2} \tan^{2n-2} \theta \tan \theta d\theta$$

$$= 2 \int_0^{\pi/2} \tan^{2n-1} \theta d\theta$$

$$= 2 \int_0^{\pi/2} \left(\frac{\sin \theta}{\cos \theta} \right)^{2n-1} d\theta$$

$$= 2 \int_0^{\pi/2} \sin^{2n-1} \theta \cos^{1-2n} \theta d\theta$$

$$= 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{\frac{2n-1+1}{2}} \sqrt{\frac{1-2n+1}{2}}}{\sqrt{\frac{2n-1+1-2n+2}{2}}}$$

$$= \frac{\sqrt{\frac{2n}{2}} \sqrt{\frac{2-2n}{2}}}{\sqrt{\frac{2}{2}}}$$

$$= \frac{\sqrt{n} \sqrt{1-n}}{\sqrt{1}}$$

$$= \sqrt{n} \sqrt{1-n}$$

Answer.

~~(12)~~ Show that, $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \frac{\sqrt{\frac{1}{n}}}{\Gamma(\frac{1}{n} + \frac{1}{2})}$

or, $I = \int_0^1 \frac{dx}{\sqrt{1-x^n}}$

Sol:

Let, $x^n = \sin^2 \theta ; x = \sin^{2n} \theta$

$$nx^{n-1} dx = 2 \sin \theta \cos \theta d\theta$$

L.t.

x	0	1
θ	0	$\frac{\pi}{2}$

$$\Rightarrow n x^n x^{-1} dx = 2 \sin \theta \cos \theta d\theta$$

$$\Rightarrow n \cdot x^n x$$

$$\Rightarrow n \cdot \sin^2 \theta \sin^{-\frac{2}{n}} dx = 2 \sin \theta \cos \theta d\theta$$

$$\Rightarrow dx = \frac{2}{n} \sin\left(\frac{2}{n}-1\right) \theta \cos \theta d\theta$$

$$\text{L.H.S.} = \int_0^1 \frac{dx}{1-x^n}$$

$$= \int_0^{\pi/2} \frac{\frac{2}{n} \sin\left(\frac{2}{n}-1\right) \theta \cos \theta}{1 - \sin^2 \theta} d\theta$$

$$= \int_0^{\pi/2} \frac{\frac{2}{n} \sin\left(\frac{2}{n}-1\right) \theta \cdot \cos \theta}{\cos \theta} d\theta$$

$$= \frac{2}{n} \int_0^{\pi/2} \sin\left(\frac{2}{n}-1\right) \theta d\theta$$

$$= \frac{2}{n} \frac{\frac{\sqrt{\left(\frac{2}{n}-1+1\right)}}{2} \sqrt{\frac{1}{2}}}{2 \sqrt{\frac{\frac{2}{n}-1+2}{2}}}$$

$$= \frac{1}{n} \frac{\sqrt{\frac{1}{n}} \cdot \sqrt{\frac{1}{2}}}{\sqrt{\frac{\frac{2}{n}+1}{2}}}$$

$$= \frac{\sqrt{n}}{n} \frac{\sqrt{\frac{1}{n}}}{\sqrt{\frac{\frac{1}{n}+\frac{1}{2}}{2}}}$$

= R.H.S. (Showed).

$$\text{Q. } \underline{\underline{I = \int_0^{\pi/2} (\tan \theta)^{1/2} d\theta}}$$

Soln:

$$\int_0^{\pi/2} (\tan \theta)^{1/2} d\theta$$

$$= \int_0^{\pi/2} \sin \theta^{1/2} \cos \theta^{-1/2} d\theta$$

$$= \frac{\frac{1/2+1}{2} \quad \frac{-1/2+1}{2}}{2 \sqrt{\frac{1/2}{2} - \frac{1/2+1}{2}}}$$

$$= \frac{\sqrt{3/4} \quad \sqrt{1/4}}{2}$$

$$= \frac{\sqrt{3/4} \quad \sqrt{1-1/4}}{2}$$

$$= \frac{1}{2} \cdot \frac{\pi}{\sin \pi/4}$$

$$= \frac{\sqrt{2}}{2} \pi$$

$$= \frac{1}{\sqrt{2}} \pi \quad \underline{\text{Answer.}}$$

(d)

$$I = \int_0^1 (1-x^n)^{1/n} dx$$

Let,

$$x = \sin^2 \theta$$

$$dx = 2 \sin \theta \cos \theta d\theta$$

x	0	1
θ	0	$\pi/2$

Now,

$$\begin{aligned} & \int_0^{\pi/2} \left\{ 1 - (\sin^n \theta)^2 \right\}^{1/n} 2 \sin \theta \cos \theta d\theta \\ &= \int_0^{\pi/2} (\cos^{2n} \theta)^{1/n} 2 \sin \theta \cos \theta d\theta \\ &= 2 \int_0^{\pi/2} \cos^{3n} \theta \sin \theta d\theta \\ &= 2 \frac{\sqrt{\frac{1+1}{2}} \quad \sqrt{\frac{3+1}{2}}}{2 \sqrt{\frac{3+1+2}{2}}} \\ &= 2 \frac{\sqrt{1} \quad \sqrt{2}}{2 \sqrt{3}} \\ &= \frac{1}{2} \end{aligned}$$

Answer.

 prove that, $\int_0^1 x^{m-1} (1-x^\alpha)^n dx = \frac{1}{\alpha} \frac{n! \sqrt{\frac{m}{\alpha}}}{\Gamma(\frac{m}{\alpha} + n + 1)}$

Solⁿ

$$x^\alpha = \sin^2 \theta$$

$$x = \sin^{2/\alpha} \theta$$

$$\Rightarrow dx = \frac{2}{\alpha} \sin^{(2/\alpha - 1)} \theta \cos \theta d\theta$$

x	0	1
θ	0	$\pi/2$

$$\begin{aligned}
 & \text{Now, } \int_0^{\pi/2} \sin^{\frac{2}{\alpha} (m-1)} \theta \cos^{2n} \theta \cdot \frac{2}{\alpha} \sin^{(\frac{2}{\alpha} - 1)} \theta \cos \theta d\theta \\
 &= \frac{2}{\alpha} \int_0^{\pi/2} \sin^{\frac{2m}{\alpha}} \theta \sin^{\frac{2}{\alpha} \theta} \cos^{2n+1} \theta \sin^{\frac{2}{\alpha} - 1} \theta d\theta \\
 &= \frac{2}{\alpha} \int_0^{\pi/2} \sin^{\frac{2m}{\alpha}} - \frac{2}{\alpha} + \frac{2}{\alpha} - 1 \theta \cos^{2n+1} \theta d\theta \\
 &= \frac{2}{\alpha} \int_0^{\pi/2} \sin^{\frac{2m}{\alpha} - 1} \theta \cos^{2n+1} \theta d\theta \\
 &= \frac{2}{\alpha} \frac{\frac{2m}{\alpha} - 1 + 1}{2} \frac{\frac{2n+1+1}{2}}{2} \frac{2}{2}
 \end{aligned}$$

$$= \frac{2}{a} \cdot \frac{\sqrt{\frac{m}{a}} \sqrt{(n+1)}}{2 \sqrt{\frac{m}{a} + n+1}}$$

$$= \frac{1}{a} \cdot \frac{n! \sqrt{\frac{m}{a}}}{\sqrt{\frac{m}{a} + n+1}}$$

proved.

~~(Q)~~

$$I = \int_0^1 \frac{35x^3}{32\sqrt{1-x}} dx$$

Soln:

$$\text{Let, } x = \sin^2 \theta$$

$$\Rightarrow dx = 2 \sin \theta \cos \theta d\theta$$

Let,

x	0	1
θ	0	$\pi/2$

Now,

$$\int_0^{\pi/2} \frac{35 \sin^6 \theta}{32 \sqrt{1-\sin^2 \theta}} \cdot 2 \sin \theta \cos \theta d\theta$$

$$= \frac{35}{16} \int_0^{\pi/2} \sin^7 \theta d\theta$$

$$= \frac{35}{16} \cdot \frac{\sqrt{\frac{7+1}{2}} \sqrt{\frac{1}{2}}}{2 \sqrt{\frac{7+2}{2}}}$$

$$= \frac{35}{16} \cdot \frac{3 \cdot 2 \cdot 1 \sqrt{\frac{1}{2}}}{2 \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\frac{1}{2}}}$$

$$= \frac{15}{16} = 1 \quad \underline{\text{Ans.}}$$

$$\text{Q. } 0! = \sqrt{(0+1)} = \sqrt{1} = 1.$$

Soln:

$$\text{We know, } n! = \sqrt{n+1}$$

$$\text{So, } 0! = \sqrt{0+1}$$

$$= \sqrt{1}$$

$$= 1 \quad \underline{\text{Answer.}} \quad \because [\sqrt{1} = 1]$$

$$\text{Q. Evaluate } \int_0^{\pi/2} \sin^6 x \, dx$$

Soln:

$$I = \int_0^{\pi/2} \sin^6 x \, dx$$

$$= \frac{\frac{6+1}{2!}}{\frac{6+2}{2!}} \cdot \frac{\sqrt{\pi}}{2}$$

$$= \frac{\frac{7}{2} \cdot \sqrt{\pi}}{4 \cdot 2}$$

$$= \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \cdot \sqrt{\pi}}{3 \cdot 2 \cdot 1 \cdot 2}$$

$$= \frac{5 \cdot 3 \cdot 1 \cdot \pi}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 8} = \frac{5\pi}{32} \quad \underline{\text{Answer.}}$$

Evaluate $\int_0^{\pi/2} \cos^5 x \sin^4 x dx$

Soln:

$$I = \int_0^{\pi/2} \cos^5 x \sin^4 x dx$$

$$= \frac{\sqrt{\frac{5+1}{2}} \cdot \sqrt{\frac{4+1}{2}}}{2 \cdot \sqrt{\frac{5+4+2}{2}}}$$

$$= \frac{\sqrt{3} \cdot \sqrt{\frac{5}{2}}}{2 \cdot \sqrt{\frac{11}{2}}}$$

$$= \frac{2 \cdot 1 \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}}{2 \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}$$

$$= \frac{3 \cdot 32}{9 \cdot 7 \cdot 5 \cdot 3 \cdot 4}$$

$$= \frac{8}{315} \quad \underline{\text{Ans}}$$

$$\text{Q.} \int_0^1 x^6 \sqrt{1-x^2} dx$$

Soln:

$$\text{Let, } x^2 = \sin^2 \theta$$

$$\Rightarrow x = \sin \theta$$

$$\Rightarrow 2x dx = 2 \sin \theta \cos \theta d\theta$$

$$\Rightarrow 2 \sin \theta dx = 2 \sin \theta \cos \theta d\theta$$

$$\therefore dx = \cos \theta d\theta$$

Lt,

x	0	1
θ	0	$\pi/2$

Now,

$$\int_0^{\pi/2} \sin^6 \theta \sqrt{1-\sin^2 \theta} \cos \theta d\theta$$

$$= \int_0^{\pi/2} \sin^6 \theta \cos^2 \theta d\theta$$

$$= \frac{\sqrt{\frac{6+1}{2}} \cdot \sqrt{\frac{2+1}{2}}}{2 \cdot \sqrt{\frac{6+2+2}{2}}}$$

$$= \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{3}{2}}}{2 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \frac{5}{256} \pi$$

Answer:

Q1

$$\text{Evaluate } \int_0^1 x^3 (1-x)^3 dx$$

Solⁿ:

$$I = \int_0^1 x^{4-1} (1-x)^{4-1} dx$$

$$= B(4,4)$$

$$= \frac{\Gamma 4 \Gamma 4}{\Gamma 4+4}$$

$$= \frac{3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \frac{1}{140}$$

Answer.

Q2

$$\text{Evaluate } \int_0^{\pi/2} \sqrt{\tan \theta} d\theta$$

Solⁿ:

$$I = \int_0^{\pi/2} \sqrt{\tan \theta} d\theta \quad \text{--- (1)}$$

$$I = \int_0^{\pi/2} \sqrt{\cot \theta} d\theta \quad \text{--- (2)}$$

Adding equations (1) + (2), we get,

$$2I = \int_0^{\pi/2} (\sqrt{\tan \theta} + \sqrt{\cot \theta}) d\theta$$

$$= -\sqrt{2} \int_0^{\pi/2} \frac{\sin \theta + \cos \theta}{\sqrt{\sin 2\theta}} d\theta$$

$$\begin{aligned}
 &= -\sqrt{2} \int_0^{\pi/2} \frac{\sin \theta + \cos \theta}{\sqrt{1-(\sin \theta - \cos \theta)^2}} d\theta \\
 &= \sqrt{2} \int_{-1}^1 \frac{dt}{\sqrt{1-t^2}} \quad (\text{where } \sin \theta - \cos \theta = t) \\
 &= 2\sqrt{2} \int_0^1 \frac{dt}{\sqrt{1-t^2}} \\
 &= \sqrt{2\pi}
 \end{aligned}$$

or $I = \frac{\pi}{\sqrt{2}}$ Ans.

Q

prove:

$$\Gamma(m) \Gamma(1-m) = \frac{\pi}{\sin m\pi}$$

We know,

$$B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$\text{and } B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

$$\therefore \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$m = 1-n$$

$$m > 0, \quad 1-x > 0$$

$$m > 0, \quad 0 < n < 1$$

$$\therefore \frac{\Gamma(1-n)}{\Gamma(1-n+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{1-n+n}} dx$$

$$\Rightarrow \Gamma(1-n) = \int_0^\infty \frac{x^{n-1}}{1+x} dx \quad \text{--- (1)}$$

We know,

$$\int_0^\infty \frac{x^{m-1}}{1+x^n} = \frac{1}{n} \frac{\pi}{\sin(\frac{\pi m}{n})}$$

Hence,

$$n=1 \quad \text{and} \quad m=n$$

So, from equ (1)

$$\int_0^\infty \frac{x^{n-1}}{1+x} = \frac{\pi}{\sin n\pi}$$

$$\therefore \Gamma(n) \Gamma(1-n) = \frac{\pi}{\sin n\pi} \quad | \because m=n$$

(proved).

The given function,

$$f(x) = \begin{cases} 0 & -\pi \leq x \leq 0 \\ \pi/2 & 0 \leq x \leq \pi \end{cases}$$

The Fourier series for the function $f(x)$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (i)}$$

$$\text{Now, } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^0 f(x) dx + \frac{1}{\pi} \int_0^{\pi} f(x) dx$$

$$= 0 + \frac{1}{\pi} \int \pi/2 \cdot dx$$

$$= \frac{1}{2} \int_0^{\pi} 1 \cdot dx$$

$$= \left[\frac{x}{2} \right]_0^{\pi} = \frac{\pi}{2}$$

$$\text{Now, } a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^0 f(x) \cos nx dx + \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$= 0 + \frac{1}{\pi} \int_0^{\pi} \pi/2 \cdot \cos nx dx$$

$$= \frac{1}{2} \int_0^{\pi} \cos nx dx$$

$$= \left[-\frac{\sin nx}{2n} \right]_0^\pi$$

$$= [0 - 0]$$

$$= 0$$

also,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^0 f(x) \sin nx dx + \frac{1}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

$$= 0 + \frac{1}{\pi} \int_0^{\pi} \pi/2 \cdot \sin nx dx$$

$$= \frac{1}{2} \int_0^{\pi} \sin nx dx$$

$$= -\frac{1}{2n} [\cos nx]_0^\pi$$

$$= -\frac{1}{2n} [-1 - 1]$$

$$= \frac{1}{n}$$

Substituting this value in equ ①

$$f(x) = \frac{\pi/2}{2} + \sum_{n=1}^{\infty} 0 \cdot \cos nx + \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

Ans.

The given function,

$$f(x) = \begin{cases} -\pi/2 & ; -\pi \leq x < 0 \\ \pi/2 & ; 0 \leq x \leq \pi \end{cases}$$

The Fourier Series for the function $f(x)$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

Now,

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \int_{-\pi}^0 f(x) dx + \frac{1}{\pi} \int_0^{\pi} f(x) dx \\ &= \frac{1}{\pi} \int_{-\pi}^0 -\pi/2 dx + \frac{1}{\pi} \int_0^{\pi} \pi/2 dx \\ &= -\frac{1}{2} \int_{-\pi}^0 dx + \frac{1}{2} \int_0^{\pi} dx \\ &= -\frac{1}{2} [x]_{-\pi}^0 + \frac{1}{2} [x]_0^{\pi} \\ &= -\frac{1}{2} [0 + \pi] + \frac{1}{2} [\pi - 0] \\ &= -\frac{\pi}{2} + \frac{\pi}{2} \\ &= 0 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\
 &= \frac{1}{\pi} \int_{-\pi}^0 f(x) \cos nx \, dx + \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \\
 &= \frac{1}{\pi} \int_{-\pi}^0 -\frac{\pi}{2} \cos nx \, dx + \frac{1}{\pi} \int_0^{\pi} \frac{\pi}{2} \cos nx \, dx \\
 &= -\frac{1}{2} \int_{-\pi}^0 \cos nx \, dx + \frac{1}{2} \int_0^{\pi} \cos nx \, dx \\
 &= -\frac{1}{2} \left[\frac{\sin nx}{n} \right]_{-\pi}^0 + \frac{1}{2} \left[\frac{\sin nx}{n} \right]_0^{\pi} \\
 &= -\frac{1}{2} \times 0 + \frac{1}{2} \times 0 \\
 &= 0
 \end{aligned}$$

Also,

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\
 &= \frac{1}{\pi} \int_{-\pi}^0 f(x) \sin nx \, dx + \frac{1}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \\
 &= \frac{1}{\pi} \int_{-\pi}^0 -\frac{\pi}{2} \sin nx \, dx + \frac{1}{\pi} \int_0^{\pi} \frac{\pi}{2} \sin nx \, dx \\
 &= -\frac{1}{2} \int_{-\pi}^0 \sin nx \, dx + \frac{1}{2} \int_0^{\pi} \sin nx \, dx
 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{2} \left[\frac{-\cos nx}{n} \right]_{-\pi}^0 + \frac{1}{2} \left[\frac{-\cos nx}{n} \right]_0^\pi \\
 &= \frac{1}{2} \left[\frac{1}{n} + \frac{1}{n} \right] - \frac{1}{2} \left[-\frac{1}{n} - \frac{1}{n} \right] \\
 &= \frac{1}{n} + \frac{1}{n} \\
 &= \frac{2}{n}
 \end{aligned}$$

Substituting these values in equ. ①

$$\begin{aligned}
 f(x) &= 0 + 0 + \sum_{n=1}^{\infty} \frac{2}{n} \sin nx \\
 \therefore f(x) &= \sum_{n=1}^{\infty} \frac{2}{n} \sin nx \quad \text{Ans.}
 \end{aligned}$$

4 Find the Fourier series which represent the function $f(x)$ defined by,

$$f(x) = \begin{cases} -\cos x & \text{for } -\pi \leq x \leq 0 \\ \cos x & \text{for } 0 \leq x \leq \pi \end{cases}$$

Soln: We know,

The Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- ①}$$

Here, $f(x)$ function is an even function so $b_n = 0$

Now,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^0 f(x) dx + \frac{1}{\pi} \int_0^\pi f(x) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^0 -\cos nx dx + \frac{1}{\pi} \int_0^\pi \cos nx dx$$

$$= \frac{1}{\pi} \left[-\sin nx \right]_{-\pi}^0 + \frac{1}{\pi} \left[\sin nx \right]_0^\pi$$

$$= 0$$

Again,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^0 f(x) \cos nx dx + \int_0^\pi f(x) \cos nx dx \right]$$

$$= \frac{1}{\pi} \left[- \int_{-\pi}^0 \cos x \cos nx dx + \int_0^\pi \cos x \cos nx dx \right]$$

$$= - \frac{1}{\pi} \left[\int_{-\pi}^0 \cos x \cos nx dx - \int_0^\pi \cos x \cos nx dx \right]$$

$$= - \frac{1}{\pi} \left[\int_{-\pi}^0 \frac{\cos(n+1)x + \cos(n-1)x}{2} dx - \int_0^\pi \frac{\cos(n+1)x + \cos(n-1)x}{2} dx \right]$$

$$= - \frac{1}{2\pi} \left[\left\{ \frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right\}_{-\pi}^0 - \right.$$

$$\left. \left\{ \frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right\}_0^\pi \right]$$

(20)

$$\begin{aligned}
 &= -\frac{1}{2\pi} \left[\left\{ \frac{\cos nx}{n+1} - \frac{\cos nx}{n-1} \right\} - \left\{ \frac{\cos nx}{n+1} + \frac{\cos nx}{n-1} \right\} \right] \\
 &= -\frac{1}{2\pi} \left[\frac{\cos nx}{n+1} - \frac{\cos nx}{n-1} - \frac{\cos nx}{n+1} - \frac{\cos nx}{n-1} \right] \\
 &= -\frac{1}{2\pi} \left[-\frac{2 \cos nx}{n-1} \right] \\
 &= \frac{2 \cos nx}{n-1} \cdot \frac{1}{2\pi} \\
 &= \frac{(-1)^n}{\pi(n-1)}
 \end{aligned}$$

Now, putting the value of a_0 and a_n in eqn.

(i) we get,

$$\begin{aligned}
 f(x) &= \frac{0}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi(n-1)} \cos nx + 0 \\
 &= \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi(n-1)} \cos nx \\
 &\quad (\text{Ans.})
 \end{aligned}$$