. 6, 2, 28, 30, 16, 24, 45, 34

45, 34,
26, 12.

Chapter - 3
Recurrence Relation.

~~Fibonacci Series: The Sequence~~

Recurrence Relation.

The Sequence which can be defined in terms of its previous elements is known as recurrence relation.

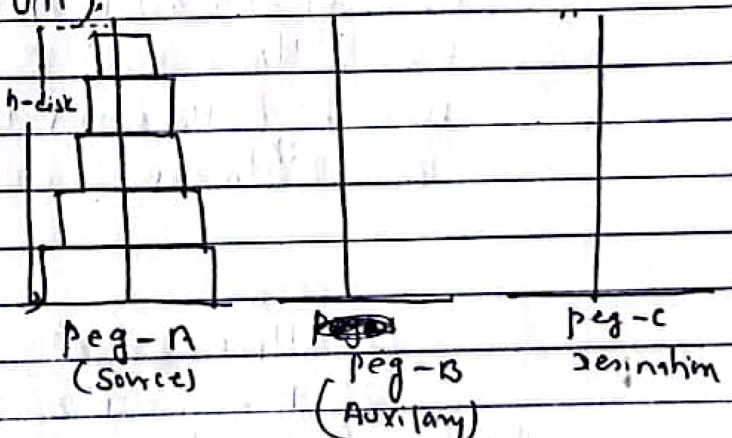
① Fibonacci Series:

eg:

$a_n = a_{n-1} + a_{n-2}$ where $a_0 = 0$
$a_1 = 1$.

Tower of Hanoi (TOH).

Tower of Hanoi is a mathematical puzzle where we have to move n disk from peg-A to peg-C by obeying following rules.



1) larger disk cannot be placed over Smaller-disk.
2) only top most disk can be moved.

3) only one disk can be moved at a time

let H_n be the total move required

1) we need to move (n-1) disk from peg-A to peg-B with the help of peg-C.

This requires H_{n-1} moves

2) The largest disk from peg-A is moved to peg-C.
This requires 1 move.

3.) Finally (n-1) disk from peg-B is moved peg-C with the help of peg-A
This requires H_{n-1} moves.

total moves,

$$H_n = H_{n-1} + 1 + H_{n-1}$$

$$\therefore \boxed{H_n = 2H_{n-1} + 1}$$ This is required recurrence relation/

$$H_n = 2[2H_{n-2} + 1] + 1$$

$$H_n = 2^2 H_{n-2} + 2^1 + 2^0$$

$$H_n = 2^2 [2H_{n-3} + 1] + 2^1 + 2^0$$

$$H_n = 2^3 H_{n-3} + 2^2 + 2^1 + 2^0.$$

$\vdots$

$$= 2^{n-1} H_1 + \ldots + 2^2 + 2^1 + 2^0$$

$$= 2^{n-1} + \ldots + 2^2 + 2^1 + 2^0$$

This is Geometric series

common ratio $(r) = \frac{2^1}{2^0} = 2.$

$$H_n = \frac{a[r^n - 1]}{r-1} ; r \neq 1$$

$$H_n = \frac{1(2^n - 1)}{2-1}$$

$$= 2^n - 1.$$

$$\therefore \boxed{H_n = 2^n - 1}$$

* Suppose who deposited rupees 1000 at an interest rate of 5% compounded annually. What is the value of investment at the end of 4 year. Find recurrence relation.

Solution

Here,

Initial investment (A_0) = Rs 1000

$r = 5\%$ p.a.

$A_4 = ?$

① After 1 year (A_1): $A_0 + 5\% \cdot A_0$

$= A_0 + 0.05 A_0$

$= (1+0.05) A_0$

~~② After 2 years (A_2) = 1.05 A_0 + 5% of A_1~~

② After 2 years (A_2) = $A_1 + 5\%$ of A_1

$A_2 = (1+0.05) A_1$

$\vdots$

$A_n = (1+0.05) A_{n-1}$.

* Suppose that a person invest ₹2000 at 14% compounded annually

(a) Find the recurrence relation

(b) Find initial condition

(c) Find $a_1, a_2,$ and a_3

(d) Find explicit formula.

(e) How long will it take for a person to double the initial investment.

Solution

After 1 year $(A_1) = A_0 + 14\%\ A_0$

$A_1 = A_0 + 0.14)\ A_0$

$A_1 = (1 + 0.14)\ A_0$

After 2 years $(A_2) = A_1 + 14\%$ of A_1

$A_2 = (1 + 0.14)\ A_1.$

a) $A_n = (1 + 0.14)\ A_{n-1}.$

b) ₹2000

c) $A_1 = (1+0.14)\ A_0) = 2280$

$A_2 = (1+0.14)\ A_1 = 2599.2$

$A_3 = (1+0.14)\ A_2 = 2963.088$

(d) $A_n = (1+0.14)\ A_{n-1}$

$A_1 = (1+0.14)\ A_0$

$A_2 = (1+0.14)\ A_1 = (1+0.14)(1+0.14)\ A_0$

$A_2 = (1+0.14)^2\ A_0$

$A_n = (1+0.14)^n\ A_0$

(e) $A_0 =$ ₹2000

$A_n = 2A_0 = 4000$

$n = ?$

$A_n = (1+0.14)^n A_0$

$296 = (1+0.14)^n 90$

$2 = (1+0.14)^n$

$n = 5.29$

* A patient is injected with 160ml of drug. Every 6 hour 25% of the drugs passes out of her blood stream. To compensate a further 20ml of doze is given every 6 hour.

a) Find recurrence relation for amount of drugs in her blood stream.

b) Find the amount of drugs in the blood stream after 24 hours.

Initial $(V_0) = 160ml$.

After 6 hrs

$V_1 = 75\%$ of $V_0 + 20$

$V_1 = 0.75V_0 + 20$ -- (i)

after 6 hours

$V_2 = 75\%$ of $V_1 + 20$

$V_2 = 0.75 V_1 + 20$ -- (ii)

$V_n = 0.75 V_{n-1} + 20$

Linear Homogeneous Recurrence Relation.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_n a_{n-k} \quad \text{--- (i)}$$

degree = 'k'

Example,

~~$a_n = 5a_{n-1} + 8a$~~

① $a_n = 5a_{n-2} + 8a_{n-3}$ --- (a)

degree = 3

(ii) $a_n = 2a_{n-1} + 3a_{n-2}$ --- (b)

degree = 2.

Solution of linear Homogeneous Recurrence Relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \quad \text{--- (i)}$$

let $a_n = r^n$ be the solution of (i)

$$\frac{r^n}{r^{n-k}} = \frac{c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}}{r^{n-k}}$$

$$\Rightarrow r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k$$

$$\Rightarrow r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0 \quad \text{--- (ii)}$$

equation (ii) is called characteristic equation and its root is called characteristic root.

Example

$$a_n = 5a_{n-1} + 6a_{n-2}$$

Solution

Here, $a_n = r^n$ --- ① be the solution

then

$$r^n = 5r^{n-1} + 6r^{n-2}$$

Dividing on both side r^{n-2}

$$\frac{r^n}{r^{n-2}} = \frac{5r^{n-1} + 6r^{n-2}}{r^{n-2}}$$

$r^2 = 5r + 6$

$r^2 - 5r - 6 = 0 \quad -(i)$

Solving (i)

$\cancel{r^2 - 6r + 1r - 6 = 0}$

$r \qquad r_1 = 3$

$r_2 = -2$ } characteristic root

Alternative

$a_n = 5a_{n-1} + 6a_{n-2}$

$r^2 = 5r^{2-1} + 6r^{2-2}$

$r^2 = 5r + 6$

V.V.I

Theoram 1:

let $a_n = C_1 a_{n-1} + C_2 a_{n-2}$ be LHRR of degree 2.

$r^2 = C_1 r + C_2$

$r^2 - C_1 r - C_2 = 0. \quad -(ii)$

(a) If r_1 and r_2 are distinct root. Then Solution is in the form,

$\therefore \boxed{a_n = \alpha_1 r_1^n + \alpha_2 r_2^n}$

b) If r_1 and r_2 both root are same. Then solution is in the form

$\therefore \boxed{a_n = (\alpha_1 + n\alpha_2) r^n}$

(c) complex root: $(\alpha + i\beta)$

$a_n = [\alpha_1 \cos(n\theta) + \alpha_2 \sin(n\theta)] R^n$

$R = \sqrt{\alpha^2 + \beta^2}$

$\theta = \tan^{-1}(\beta/\alpha)$

(1)

(*) Find the solution of recurrence relation.

$a_n = 5a_{n-1} - 6a_{n-2}$ with $a_0 = 3$, $a_1 = 5$.

Solution

Given, recurrence relation,

$$a_n = 5a_{n-1} - 6a_{n-2} \quad \text{(i)}$$

The characteristic equation is,

$r^2 = 5r - 6.$

$r^2 - 5r + 6 = 0$

$r_1 = 3$

$r_2 = 2$

Since both root are distinct the general solution is given by,

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

$$\boxed{a_n = \alpha_1 3^n + \alpha_2 2^n} \quad \text{(ii)}$$

now using initial condition

For $n = 0$,

$a_0 = \alpha_1 3^0 + \alpha_2 2^0$

$3 = \alpha_1 + \alpha_2 \quad \text{(a)}$

For, $n = 1$,

$a_1 = \alpha_1 3^1 + \alpha_2 2^1$

$5 = 3\alpha_1 + 2\alpha_2 \quad \text{(b)}$

Solving (a) and (b)

$\alpha_1 = -1$

$\alpha_2 = 4.$

Now putting $\alpha_1 = -1$ and $\alpha_2 = 4$ in eqn (ii)

$$\boxed{a_n = -1(3)^n + 4(2)^n} \quad \text{(iii)}$$

This is required equation.

② Find the solution of recurrence relation.

$a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 5$, $a_1 = 7$.

Solution

Given, Recurrence relation,

$a_n = 6a_{n-1} - 9a_{n-2}$ — (i)

The characteristic equation is,

$r^2 = 6r - 9$

$r^2 - 6r + 9 = 0$

$r_1 = 3$

$r_2 = 3$

Since both are same then the general solution is given by

~~$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$~~

$a_n = (\alpha_1 + n\alpha_2) r^n$

$a_n = (\alpha_1 + n\alpha_2) 3^n$ -- (ii)

now using initial condition,

for $n = 0$

$a_n = (\alpha_1 + 0\alpha_2) 3^n$

$a_n = \alpha_1 . 3^n$.

$5 = \alpha_1 3^n$ -- (a)

$\alpha_1 = 5$

for $n = 1$,

$a_1 = (\alpha_1 + 1\alpha_2) 3^n$

$7 = (\alpha_1 + \alpha_2) 3^n$ -- (b)

~~$7 = \alpha_1 3^n + \alpha_2$~~ 3^n.

$7 = 3\alpha_1 + \alpha_2 . 3^1$

$7 = 5 \times 3 + 3\alpha_2$

$-8 = 3\alpha_2$

$\alpha_2 = -8/3$.

$\therefore \alpha_1 = 5$

$\alpha_2 = -8/3$

Now putting $\alpha_1 = 5$ and $\alpha_2 = -8/3$ in eqn (ii)

$a_n = \left(5 - \frac{8n}{3}\right) 3^n$ -- (iii)

This is required equation.

(*) Find the explicit formula for fibonacci sequence.

Sequence:

$$a_n = a_{n-1} + a_{n-2}, \quad a_0 = 0, a_1 = 1$$

Solution

Here

Given recurrence relation,

$$a_n = a_{n-1} + a_{n-2} \quad \text{--- (i)}$$

The characteristic equation is,

$$r^2 = r + 1$$

$$r^2 - r - 1 = 0$$

$$r_1 = \frac{1+\sqrt{5}}{2}$$

$$r_2 = \frac{1-\sqrt{5}}{2}.$$

Since roots are distinct,

$$a_n = (\alpha_1 r_1^n + \alpha_2 r_2^n)$$

$$a_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \text{ --- (ii)}$$

Now, using initial condition.

For $n=0$ | For $n=1$,

a_0 | a_1

Solving eqn (a) and (b)

$$r_1 = 1/\sqrt{5}$$

$$r_2 = -1/\sqrt{5}$$

Now, putting $r_1 = 1/\sqrt{5}$ and $r_2 = -1/\sqrt{5}$ in eqn (ii).

$a_n =$

V.V.V.I

(7) Find the solution of recurrence relation.
$a_n + 2a_{n-1} + 2a_{n-2} = 0$, with $a_0 = 0$, $a_1 = -1$.

Solution.

Here

Given Recurrence relation

$a_n + 2a_{n-1} + 2a_{n-2} = 0$ --- (1)

The characteristic equation is

$r^2 + 2r + 2 = 0$

$r_1 = -1 + i$

$r_2 = -1 - i$

Since the roots are in complex form. So the solution is given by

$a_n = [\alpha_1 \cos(n\theta) + \alpha_2 \sin(n\theta)] R^n$

$R = \sqrt{\alpha^2 + \beta^2}$

$\theta = \tan^{-1}(\beta/\alpha)$

$r = -1 \pm i$

$\alpha = -1$, $\beta = 1$

$R = \sqrt{\alpha^2 + \beta^2} = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\theta = \tan^{-1}(1/-1) = \tan^{-1}(-1) = 3\pi/4$

$a_n = [\alpha_1 \cos(n \frac{3\pi}{4}) + \alpha_2 \sin(n \frac{3\pi}{4})] (\sqrt{2})^n$ --- (11)

Using initial condⁿ

a) $n=0$, $a_0 = 0$

$a_0 = [\alpha_1 \cos(0 \times \frac{3\pi}{4}) + \sin(0 \times \frac{3\pi}{4})] (\sqrt{2})^0$

$\alpha_1 = 0$

b) $n=1$, $a_1 = -1$

$a_1 = [\alpha_1 \cos(1 \times \frac{3\pi}{4}) + \sin(1 \times \frac{3\pi}{4})] (\sqrt{2})^1$

$\alpha_2 = -1$, substituting in (11), we get

$a_n = -[\sin(n \frac{3\pi}{4})] \cdot (\sqrt{2})^n$

V.V.I

Theoram II : let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3}$ be a linear homogenous recurrence Relation of degree of 3

The characteristic roots are:

$r^3 = c_1 r^2 + c_2 r + c_3$

~~$r^3 - c_1 r^2$~~

$r^3 - c_1 r^2 - c_2 r - c_3 = 0$

(r_1, r_2, r_3)

a) If all roots are distinct.

$\boxed{a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n}$

b) If two roots are Same.

$\boxed{a_n = (\alpha_1 + n\alpha_2) r^n + \alpha_3 r_2^n}$

c) If all roots are Same.

$\boxed{a_n = (\alpha_1 + n\alpha_2 + n^2\alpha_3) r^n}$

(*) Find the solution of:

$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3},\ a_0 = 2,\ a_1 = 5, a_2 = 15$

Solution

Given recurrence relation,

$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$

The characteristic equation.

$r^3 = 6r^2 - 11r + 6$

$r^3 - 6r^2 + 11r - 6 = 0$

$\therefore r_1 = 1$

$r_2 = 3$

$r_3 = 2$

Since all roots are distinct.

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n$

$\boxed{a_n = \alpha_1 + \alpha_2 3^n + \alpha_3 \cdot 2^n}$ (ii)

For $n=0$,

$a_0 = \alpha_1 + \alpha_2 3^0 + \alpha_3 2^0$

$\boxed{\alpha_1 + \alpha_2 + \alpha_3 = 2}$ -- (a)

For $n=1$,

$a_1 = \alpha_1 + \alpha_2 \cdot 3^1 + \alpha_3 \cdot 2^1$

$\alpha_1 + 3\alpha_2 + 2\alpha_3 = 5$ -- (b)

For $n=2$

$\alpha_1 = \alpha_1 + \alpha_2 \cdot 3^2 + \alpha_3 \cdot 2^2$

$\alpha_1 + 9\alpha_2 + 4\alpha_3 = 15$ -- (c).

Solving eqn (a), (b), (c), we get

$\alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha_3 = -1.$

$\therefore a_n = 1 + 2 \cdot 3^n - 2^n$

$\therefore$ This is required solution.

V.IMP

Linear Non-homogenous Recurrence Relation.

let

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + C_3 a_{n-3} + \dots + C_k a_{n-k} + f(n) \quad \text{(i)}$$

Eqn (i) is known as linear non-homogeneous where $f(n)$ is non-homogeneous part.

* The solution of homogeneous part : $a_n = C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k}$ is known as homogeneous solution : $a_n(h)$ -- (a)
* The solution of non-homogeneous part $f(n)$ is known as particular solution.

$a_n(p)$ ---- (b)

The total solution is,

$a_n = a_n(h) + a_n(p)$.

IMP

* Method to find particular solution.

$$f(n) = [\phi(n)] \cdot b^n \quad \text{---(a)}$$

where,

b = constant.

$\phi(n)$ = polynomial of degree r.

Case-I : when b is not characteristic root

$$a_n(p) = [A_0 + A_1 n + A_2 n^2 + \dots A_r n^r] b^n \quad \text{---(a)}$$

Case-II: when b is characteristic root with multiplicity 'm'

$$a_n(p) = n^m [A_0 + A_1 n + A_2 n^2 + \dots + A_r n^r] b^n \quad \text{--- (b)}$$

Q. Write a solution of :

$$a_n - 3a_{n-1} + 2a_{n-2} = 2^n, \; a_0 = 0, a_1 = 1$$

Solution

Here

The homogenous part,

$$a_n - 3a_{n-1} + 2a_{n-2} = 0 \quad \text{--- (i)}$$

The characteristic equation,

$$r^2 - 3r + 2 = 0$$

$$r_1 = 1, \; r_2 = 2$$

Since, the roots are distinct

$$a_n(h) = \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$a_n(h) = \alpha_1 + \alpha_2 \cdot 2^n \quad \text{--- (ii)}$$

Now nonhomogeneous part,

$$f(a) = 2^n = 1 \cdot 2^n$$

$$\phi(n) = 1 \text{ (degree 0)}$$

$$b^n = 2^n$$

Here, b is characteristic root, with multiplicity, $m = 1$

$a_n(p) = n^1 [A_0] 2^n.$

$a_n(p) = n A_0 2^n$ --(iii).

from (iii) and (a)

$$\frac{n A_0 2^n - 3(n-1) A_0 2^{n-1} + 2(n-2) A_0 2^{n-2}}{2^n} = \frac{2^n}{2^n}$$

$$n A_0 - 3(n-1) A_0 2^{-1} + 2(n-2) A_0 2^{-2} = 1$$

$$n A_0 - \frac{3(n-1) A_0}{2^1} + \frac{2(n-2) A_0}{2^2} = 1.$$

$$\cancel{4n(A_0)} - \cancel{6nA_0} + 6A_0 + \cancel{2nA_0} - 4A_0 = 4$$

$$2A_0 = 4$$

$$\therefore A_0 = 2$$

Substituting $A_0 = 2$ in eqn (iii)

$$a_n(p) = n \cdot 2 \cdot 2^n$$

$$a_n(p) = n \cdot 2^{n+1} \text{ (iv)}$$

Therefore the final equation is

$$a_n = a_n(h) + a_n(p)$$

$$\boxed{a_n = \alpha_1 + 2^n \alpha_2 + n 2^{n+1}} \text{ --(v)}$$

For $n = 0$,

$a_0 = \alpha_1 + \alpha_2$

$\alpha_1 + \alpha_2 = 0$ -(a)

For $n = 1$,

$a_1 = \alpha_1 + 2^1 \alpha_2 + 1 \cdot 2^{1+1}$

$a_1 = \alpha_1 + 2\alpha_2 + 4$

$\alpha_1 + 2\alpha_2 + 4 = 1$

$\alpha_1 + 2\alpha_2 = -3$ -(b).

from (a) and (b)

$\alpha_1 = 3$

$\alpha_2 = -3$

putting α_1 and α_2 in eqn (v)

$$\boxed{a_n = 3 - 3 \cdot 2^n + n \cdot 2^{n+1}}$$

$a_n - 6a_{n-1} + 8a_{n-2} = 3$; $a_0 = 10$, $a_1 = 25$

Solution

Here:

The homogeneous part.

$a_n - 6a_{n-1} + 8a_{n-2} = 0$ --- (i)

The characteristic equation,

$r^2 - 6r + 8 = 0$

we get,

$r_1 = 2,$

$r_2 = 4.$

Since, the roots are distinct.

$a_n(h) = \alpha_1 r_1^n + \alpha_2 r_2^n$

$a_n(h) = \alpha_1 \cdot 2^n + \alpha_2 \cdot 4^n$ --- (ii)

now homogeneous part,

$f(a) = 3$