MATH 1102 Winter 2020

February 25, 2020

Recall: A $n \times n$, $A\mathbf{v} = \lambda\mathbf{v}$, $\mathbf{v} \neq \mathbf{0}$ $det(A - \lambda I) = 0$ and $E_\lambda = ker(A - \lambda I)$

Def 1. *Suppose $A \in M_{nn}(F)$ has eigenvalue λ_1, and $p_A = (\lambda - \lambda_1)^{\alpha_A(\lambda_1)}$ then $q_A(\lambda_1)$ is the algebraic multiplicity of λ_1. The geometric multiplicity $\gamma_A(\lambda_1)$ of λ_1 is equal to $dim(E_{\lambda_1})$. If A is clear from context, we write $\alpha_A(\lambda_1) = \alpha(\lambda_1)$, $\gamma_A(\lambda_1) = \gamma(\lambda_1)$*

Theorem 1. *Suppose $A \in M_{nn}(F)$ has eigenvalue λ_1. Then, $1 \leq \gamma(\lambda_1) \leq \alpha(\lambda_1) \leq n$*

Proof. $(\lambda - \lambda_1)^{\alpha(\lambda)}$ is a factor of $p_A(\lambda)$, a degree n polynomial, Thus $\alpha_A(\lambda_1) \leq n$. Since λ_1 is an eigenvalue, there is some $\mathbf{v} \neq \mathbf{0}$ in E_{λ_1}. Thus $E_{\lambda_1} \neq \{\mathbf{0}\}$, and so $\alpha(\lambda_1) = dim(E_{\lambda_1}) \geq 1$.

Let $g = \alpha(\lambda_1) = dim(E_{\lambda_1})$, and $\mathbf{v_1} \ldots \mathbf{v_g}$ be a basis of E_{λ_1}. By the linear independence to basis theorem, we can find $\mathbf{u_1}, \ldots, \mathbf{u_{n-g}} \in F^n$ so that $\mathbf{v_1}, \ldots, \mathbf{v_g}, \mathbf{u_1}, \ldots, \mathbf{u_{n-g}}$ is a basis of F^n.

Let $P = [\mathbf{v_1} \ldots \mathbf{v_g}\ \mathbf{u_1} \ldots \mathbf{u_{n-g}}]$. Since the columns of P are a basis of F^n, P is nonsingular and hence invertible. Now that $\mathbf{e_1} \ldots \mathbf{e_n} = I = P^{-1}P = P^{-1}[\mathbf{v_1} \ldots \mathbf{v_g}\ \mathbf{u_1} \ldots \mathbf{u_{n-g}}]$, and so for $1 \leq i \leq g$, $P^{-1}\mathbf{v_i} = \mathbf{e_i}$.

Let $B = P^{-1}AP$. Since A and B are similar matrices, $p_A(\lambda) = p_B(\lambda)$. Since $B = P^{-1}AP = P^{-1}A[\mathbf{v_1} \ldots \mathbf{v_g}\ \mathbf{u_1} \ldots \mathbf{u_{n-g}}]$, for $1 \leq i \leq g$, the i^{th} column of B is equal to

$$\begin{aligned} P^{-1}A\mathbf{v_i} &= P^{-1}(\lambda_1\mathbf{v_i}) && \text{(since } \mathbf{v_i} \in E_{\lambda_1}\text{)} \\ &= \lambda_1(P^{-1}\mathbf{v_i}) \\ &= \lambda_1\mathbf{e_i} \end{aligned}$$

Thus,

$$
\begin{aligned}
P_A(\lambda) &= P_B(\lambda) \\
&= det(B-\lambda I) \\
&= det \begin{pmatrix}
\lambda_1-\lambda & 0 & \cdots & 0 & c_{11} & \cdots & \cdots & \cdots & \cdots & c_{1(n-g)} \\
0 & \lambda_1-\lambda & \cdots & 0 & \vdots & \ddots & & & & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & & & \vdots \\
\vdots & \vdots & & \lambda_1-\lambda & \vdots & & & \ddots & & \vdots \\
\vdots & \vdots & & \vdots & \vdots & & & & \ddots & \vdots \\
0 & 0 & \cdots & 0 & c_{n1} & \cdots & \cdots & \cdots & \cdots & c_{n(n-g)}
\end{pmatrix} \\
&= (\lambda_1-\lambda)^g det(M) \qquad\qquad \text{(by expanding along the first column, repeated g times)} \\
&= (-1)^g(\lambda-\lambda_1)^g det(M)
\end{aligned}
$$

M is the $(n-g)\times(n-g)$ matrix made up of the lower right corner of the matrix above. Since det(M) is a polynomial in λ, we have shown that $(\lambda-\lambda_1)^g$ is a factor of $p_(A)(\lambda)$.

Since $\alpha(\lambda_1)$ is the greatest power of $(\lambda-\lambda_1)$ that divides $p_(A)(\lambda)$. Note that $\mathbf{u}\in\ker(B)$ which means that $B\mathbf{u}=\mathbf{0}$. Thus, we have

$$\gamma(\lambda_1) = g \leq \alpha(\lambda_1)$$

□