1 Section 3.2

Recall: Uniform sample space S.

$$Pr(w) = \frac{1}{|S|}\text{for each w} \in \text{S}$$
$$Pr(A) = \sum_{w \in A} Pr(w) = \frac{|A|}{|S|}$$

2 Birthday Paradox

- There are d = 365 days in a year.
- There are n people with birthdays $b_1, b_2, ..., b_n \in 1, .., 365$
- Assumption: Uniform sample spaces
- $|S| = d^n$ (for this class 365^{120})
- A = "at least two people have the same birthday".
- A = "$b_i = b_j$ for some $i \neq j$."
- $A^{\complement}$ ="everyone has a different birthday"
- $A^{\complement} = b_i \neq b_j$ for any $i \neq j$."

(insert drawing)
- $|A^{\complement}| = \#$ of one-to-one functions P(people) to D(birthdays)
- $d \cdot (d-1) \cdot (d-2) \ldots (d-n) = \frac{d!}{(d-n)!}$

$$Pr(A^{\complement}) = \frac{A^{\complement}}{|S|} = \frac{d!}{(d-n)!} \setminus d^n$$
$$Pr(A^{\complement}) = 1 - \frac{d!}{(d-n)! \cdot d^n}$$

Example 1. *if $d = 365$, $n = 22$, $Pr(A) = 0.476$*
if $d = 365$, $n = 23$, $Pr(A) = 0.507$

Theorem 1. *If we throw n balls uniformly at random into d buckets(bins) then the probability that at least one bin contains at least 2 balls is*

$$1 - \frac{d!}{(d-n)!d^n}$$

3 The Big Box Problem

- I (secretly) pick two integers $x < y$, $x, y \in \{0, \ldots, 100\}$
- I (secretly) do one of these two things:

1. put \$x in the left box and \$y in the right box; OR
2. put \$y in the left box and \$x in the right box;

- I choose both both boxes

A. You open one of the boxes and look inside

B. You decide to keep it or pick the other box

- You win if you pick the box with \$y.

  - Pick a random z. - $z \in \{\frac{1}{2}, 1 \cdot \frac{1}{2}, 2 \cdot \frac{1}{2}, \ldots, 99 \cdot \frac{1}{2}\}$ - Pick $z \in \{0.5, \ldots, 99.5\}$
- Open the box picked at random and we see some money
- If $\$a < z$, then take the other box.
- If $\$a > z$, then keep the box.
- $|S| = \{0.5, \ldots, 99.5\} \times \{\$x, \$y\}$ $|S| = 200$
- $Pr(Win)$, lets say W = "I win the game"
- W_x = "I open the box containing \$x and $z > x$".
- W_y = "I open the box containing \$y and $z < y$".
- $Pr(W) = Pr(W_x) + Pr(W_y)$

We start by finding $Pr(w_x)$

- $w_x = \{(z, \$x) : z \in \{x + 0.5, x + 1.5, \ldots, 99.5\}\}$ $|W_x| = 100 - x$

We start by finding $Pr(w_y)$

- $w_y = \{(z, \$y) : z \in \{0.5, 1.5, \ldots, y - 0.5\}\}$ $|W_y| = y$

$$\begin{aligned} Pr(W) &= Pr(W_x) + Pr(W_y) \\ &= \frac{W_x}{|S|} + \frac{W_y}{|S|} \\ &= \frac{100 - x + y}{200} \\ &= \frac{100 + y - x}{200} \geq \frac{101}{200} \end{aligned}$$

$$
\begin{aligned}
&q = 1 - p \\
&q = \frac{d!}{(d-n)!d^n} = \frac{d(d-1)(d-2)\dots(d-(n-1))}{d^n} \\
&q = \frac{d}{d} \cdot \frac{d-1}{d} \cdot \dots \cdot \frac{d-(n-1)}{d} \\
&q = (1 - \frac{1}{d}) \cdot (1 - \frac{2}{d}) \cdot \dots \cdot (1 - \frac{(n-1)}{d}) \\
&\quad \leq e^0 \cdot e^{-1/d} \cdot e^{-2/d} \cdot e^{-3/d} \dots e^{-(n-1)/d} \\
&\quad = e^{0-(1/d)-(2/d)-\dots-(n-1)/d} \\
&\quad = e^{-n(n-1)/2d} = d
\end{aligned}
$$