CH5180 Assignment 5

By

CH13B009

CH13B016

CH13B042

CH13B069

Question 1

Problem: 1

Date: / / Solved Problems: Sub Obj

Q. No. Exercise No.

Assignment-6

Ques 1

(1) $\dot{x} = x^3 + x^2 + ax$.

In order to find steady state we put

$\dot{x} = 0$

$\Rightarrow 0 = x(x^2 + x + a)$

steady states are $x = 0$ $0 = x^2 + x + a$.

$x^2 + x + a = 0$

$$x = \frac{-1 \pm \sqrt{1 - 4a}}{2}.$$

Stability of steady states

$f' = 3x^2 + 2x + a$.

$$f'@x=0 = a \Rightarrow \begin{cases} \text{if } a < 0 \Rightarrow \text{stable steady state} \\ \text{if } a > 0 \Rightarrow \text{unstable stea[dy] state} \end{cases}$$

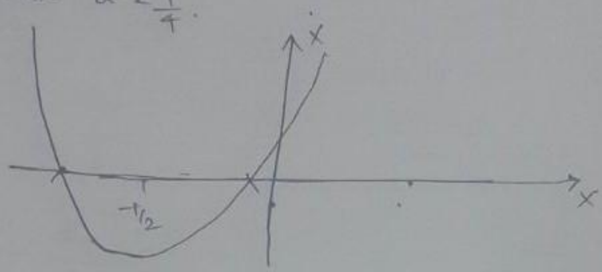
Date: ☐☐/☐☐/☐☐☐☐

ASB 15

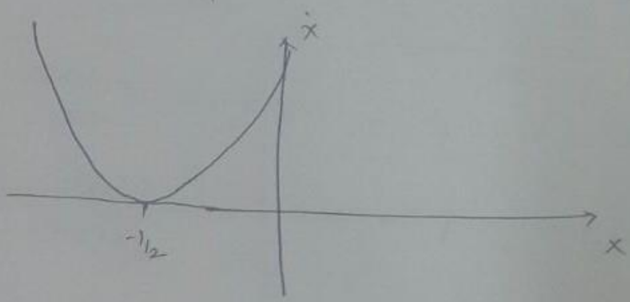
Q. No. Exercise No. ☐☐ Solved Problems: Sub ☐ Obj ☐

$f(x) = \dot{x} = x(x^2 + x + a)$

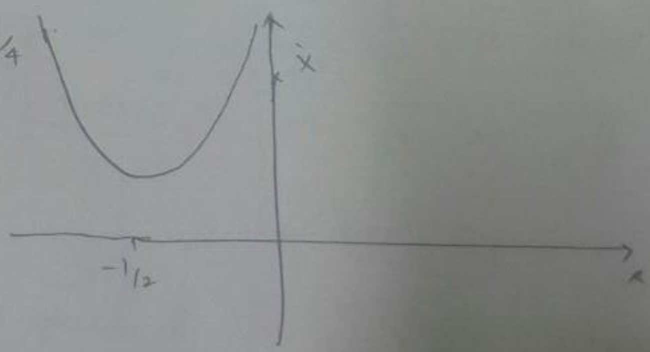
when $a < \frac{1}{4}$.



when $a = \frac{1}{4}$.



when $a > 1/4$



$f'@(x^2+x+a=0) = (x^2+x)+(x^2+x)+(x^2+a)$

$= -a \quad -a \quad -x$

$= -x-2a.$

(i) when $x = \dfrac{-1+\sqrt{1-4a}}{2}$

$f' = \dfrac{1-\sqrt{1-4a}}{2} - 2a = \dfrac{1-\sqrt{1-4a}-4a}{2}$

$= \dfrac{\sqrt{1-4a}\left(\sqrt{1-4a}-1\right)}{2}$

for $a<0$ f' is >0 $\Rightarrow$ unstable steady state.

for $0<a<\frac{1}{4}$ f' is <0 $\Rightarrow$ stable steady state.

(ii) when $x = \dfrac{-1-\sqrt{1-4a}}{2}$

$f' = \dfrac{1+\sqrt{1-4a}}{2} - 2a = \dfrac{1-4a+\sqrt{1-4a}}{2}$

$= \dfrac{\sqrt{1-4a}\left(1+\sqrt{1-4a}\right)}{2}$

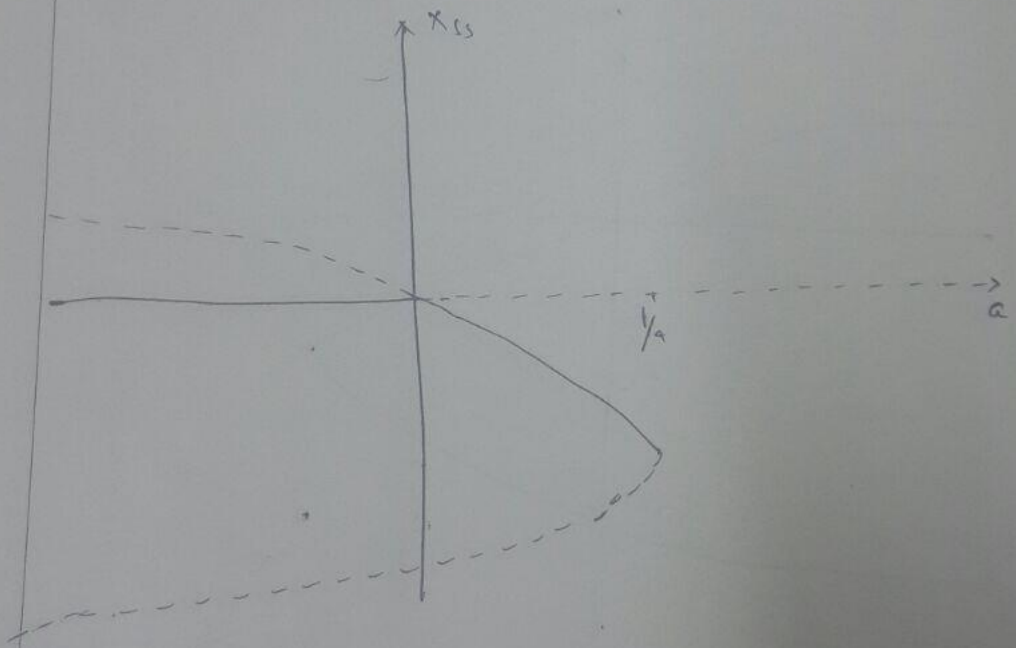
for $a<\frac{1}{4}$ f' is always >0 $\Rightarrow$ unstable steady state.

Date: ☐☐/☐☐/☐☐☐☐

ASB 17

Q. No. Exercise No. ☐☐ Solved Problems: Sub ☐ Obj ☐

Bifurcation diagram is:-



Problem: 2

Date: □□/□□/□□□□ ASB 23

Q. No. Exercise No. □□ Solved Problems: Sub □ Obj □

Ques 1

(2) $\ddot{y}_1 = a(1-y_1^2)\dot{y}_1 - y_1$

Let $\dot{y}_1 = x$

$\Rightarrow \dot{x} = a(1-y_1^2)x - y_1$

$\dot{y}_1 = x$.

Steady state $\Rightarrow$ $\dot{x} = 0$ $\dot{y}_1 = 0$.

$\Rightarrow a(1-y_1^2)x = y_1$ $\qquad x = \dfrac{y_1}{a(1-y_1^2)}$.

$\dot{y}_1 = 0 \rightarrow x = 0$

$\rightarrow$ Steady state is $x = 0,\ y_1 = 0$.

$\dfrac{\partial f_1}{\partial x} = a(1-y_1^2) \qquad \dfrac{\partial f_1}{\partial y_1} = ax(-2y_1) - 1$

$\dfrac{\partial f_2}{\partial x} = 1 \qquad \dfrac{\partial f_2}{\partial y_1} = 0.$

$$J = \begin{bmatrix} a(1-y_1^2) & ax(-2y_1) - 1 \\ 1 & 0 \end{bmatrix}$$

Date: __/__/____ ASB 24

Q. No. Exercise No. Solved Problems: Sub Obj

$J @ (x=0, y_1=0) \rightarrow \begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix}$

The necessary and sufficient condition for stability is $tr(J) < 0$ & $\det(J) > 0$

$tr(J) = a \qquad \det(J) = 1$

For stability $\quad a < 0$.

$$\begin{vmatrix} a-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0. \qquad (a-\lambda)(-\lambda)+1 = 0.$$

$$-a\lambda + \lambda^2 + 1 = 0.$$

$$\lambda^2 - a\lambda + 1 = 0.$$

$$\lambda = \frac{a \pm \sqrt{a^2-4}}{2}.$$

<u>Nature of eigen-values</u>

(i) when $a < 0$ λ's are complex conjugate, and $Re(\lambda) < 0 \Rightarrow$ stable focus.

(ii) when $0 < a < 2$ λ's are complex conjugate, and $Re(\lambda) > 0 \rightarrow$ Limit cycle.

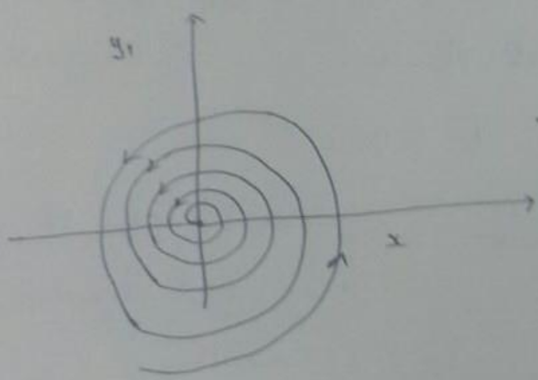
Date: __/__/____

Q. No. Exercise No. __ Solved Problems: Sub ☐ Obj ☐

(iii) When $a > 2$, λ's are Real, and both the λ's are > 0 $\Rightarrow$ unstable steady state.

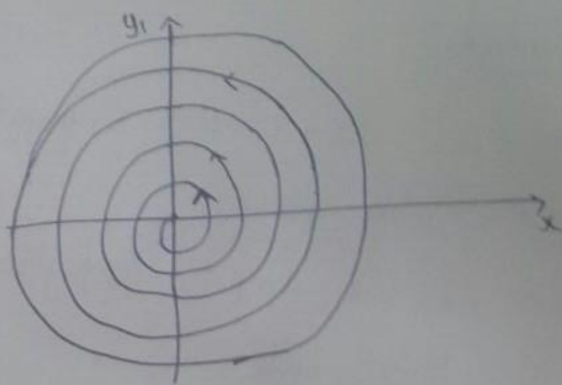
Phase-planes

(i) when $a < 0$.



$\rightarrow$ stable focus

(ii) when $0 < a < 2$



$\Rightarrow$ Limit cycle.

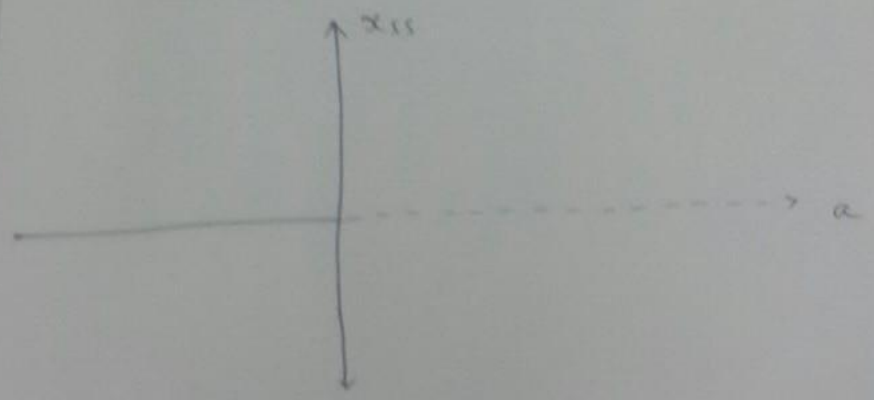
Date: □□/□□/□□□□ ASR 25

Q. No. Exercise No. □□ Solved Problems: Sub □ Obj □

(iii) When $a > 2$,

(Diverging away)
Unstable node

Bifurcation diagram



Problem: 5

Date: ☐☐/☐☐/☐☐☐☐ Solved Problems: Sub ☐ Obj ☐

Q. No. Exercise No. ☐☐

Ques 5

$\dot{x} = y + ax.$

$\dot{y} = -x + ay - x^2 y.$

In order to get the steady state,

$\dot{x} = 0, \quad \dot{y} = 0$

$\dot{x} = 0 \Rightarrow y = -ax.$

$\dot{y} = 0 \Rightarrow -x + ay - x^2 y = 0.$

Sub. $y = -ax$ in the above expression, we get

$x(ax^2 + -a^2 - 1) = 0.$

the steady state is $x = 0$, $x = \pm\sqrt{\frac{a^2+1}{a}}$

Corresponding y's are $y = 0$ $\quad y = \mp\sqrt{a(a^2+1)}$

$\therefore$

$$\boxed{\begin{array}{l} x = 0,\ y = 0 \\ x = \sqrt{\frac{a^2+1}{a}},\ y = -\sqrt{a(a^2+1)} \\ x = -\sqrt{\frac{a^2+1}{a}} \quad y = \sqrt{a(a^2+1)} \end{array}}$$

$\Rightarrow$ steady state solutions.

Date: ☐☐/☐☐/☐☐☐☐ ASB 28

Q. No. Exercise No. ☐☐ Solved Problems: Sub ☐ Obj ☐

$\frac{\partial f_1}{\partial x} = a \qquad \frac{\partial f_1}{\partial y} = 1$

$\frac{\partial f_2}{\partial x} = -1 - 2xy \qquad \frac{\partial f_2}{\partial y} = a - x^2$

$$J = \begin{bmatrix} a & 1 \\ -1-2xy & a-x^2 \end{bmatrix}$$

$$J @_{x=0, y=0} = \begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix}.$$

For stability $tr(J) < 0 \Rightarrow 2a < 0$.

$\Rightarrow a < 0$ & $\det(J) > 0$ $\det(J) = a^2+1$ (which is always >0).

$\lambda = a \pm i$.

when $a < 0$, @ $x=0, y=0$ there will be a stable focus.

when $a > 0$ @ $x=0, y=0$ there will be a unstable focus.

Date: / / Solved Problems

Q. No. Exercise No. Q. No.

$J @ x = \pm\sqrt{\frac{a^2+1}{a}}$, $y = \mp\sqrt{a(a^2+1)}$ $= \begin{bmatrix} a & 1 \\ -1+2(a^2+1) & a-\left(\frac{a^2+1}{a}\right) \end{bmatrix}$

$$= \begin{bmatrix} a & 1 \\ 2a^2+1 & \frac{-1}{a} \end{bmatrix}$$

$tr(J) = a - \frac{1}{a}$.

$det(J) = -1-2a^2-1 = -2a^2-2$ (This is always < 0).

$\Rightarrow$ when $x = \pm\sqrt{\frac{a^2+1}{a}}$ & $y = \mp\sqrt{a(a^2+1)}$

~~Steady~~ stability is not possible.

$$\lambda^2 - \left(a - \frac{1}{a}\right)\lambda - 2a^2 - 2 = 0.$$

$$\lambda = \frac{\left(a - \frac{1}{a}\right) \pm \sqrt{\left(a - \frac{1}{a}\right)^2 - 4(-2a^2-2)}}{2}$$

$$\lambda = \frac{\left(a - \frac{1}{a}\right) \pm \left(3a + \frac{1}{a}\right)}{2}$$

ASB 30

Date: ☐☐/☐☐/☐☐☐☐

Solved Problems: Sub ☐ Obj ☐

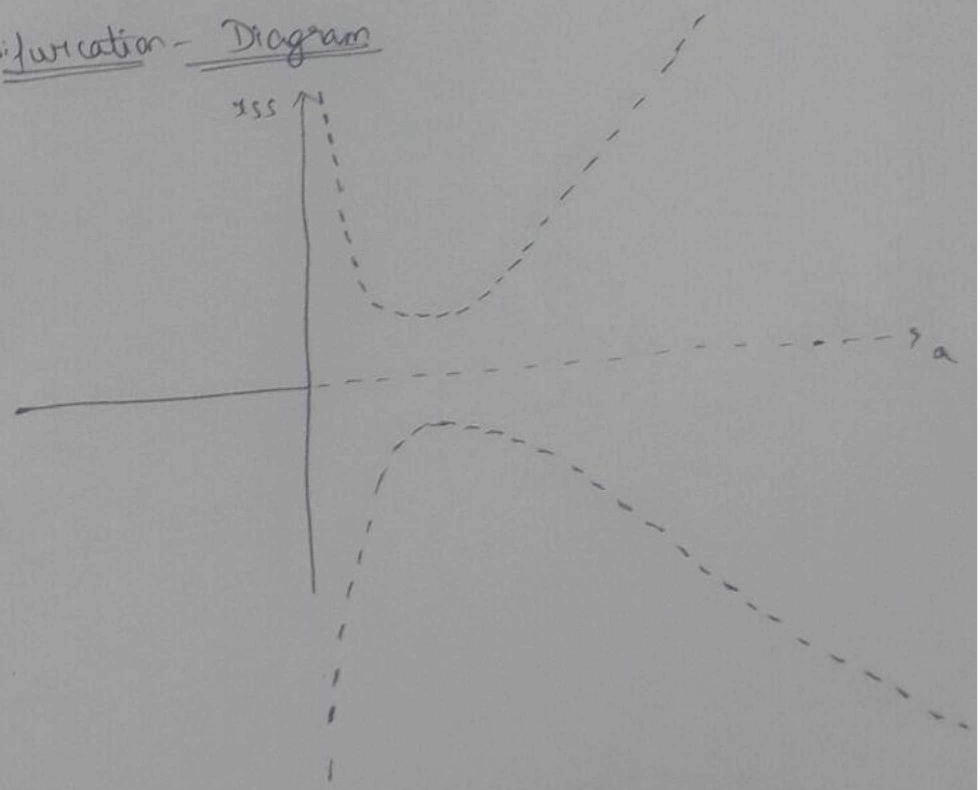
Q. No. Exercise No. ☐☐

$\lambda_1 = 2a \qquad \lambda_2 = -a - \frac{1}{a}$.

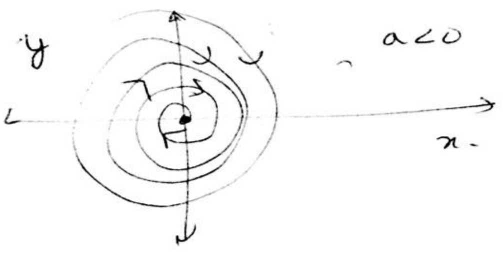
λ_1 & λ_2 are real and are of opposite signs for all values of a.

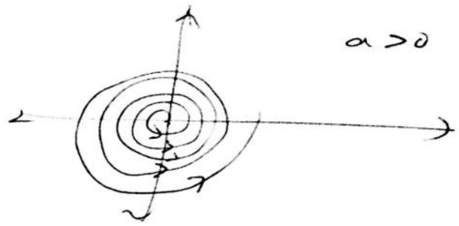
$\Rightarrow \quad x = \pm\sqrt{\frac{a^2+1}{a}}$ & $y = \mp\sqrt{a(a^2+1)}$ is a saddle point.

Bifurcation - Diagram



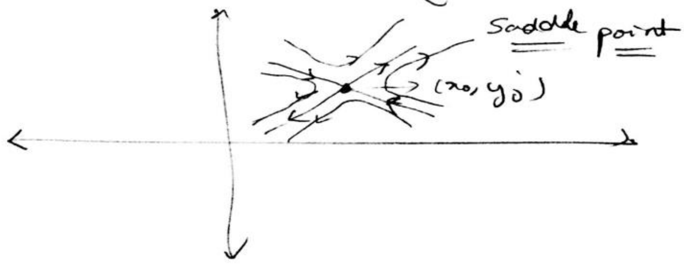
Phase plane for $(0,0)$.





Phase plane for

$$(x_0, y_0) = \left(\mp \sqrt{a(a^2+1)} \,,\, \pm \sqrt{\frac{a^2+1}{a}} \right)$$



Problem: 6

(6) $\dot{x} = y + ax - x^2$

$\dot{y} = -x + ay + 2x^2$

$\dot{x} = 0$; $y = x^2 - ax$.

$\dot{y} = 0$; $ay = x - 2x^2$.

$a(x^2 - ax) = x - 2x^2$.

$a(x - a) = \cancel{x}\, 1 - 2x$.

$x(a+2) = a^2 + 1$.

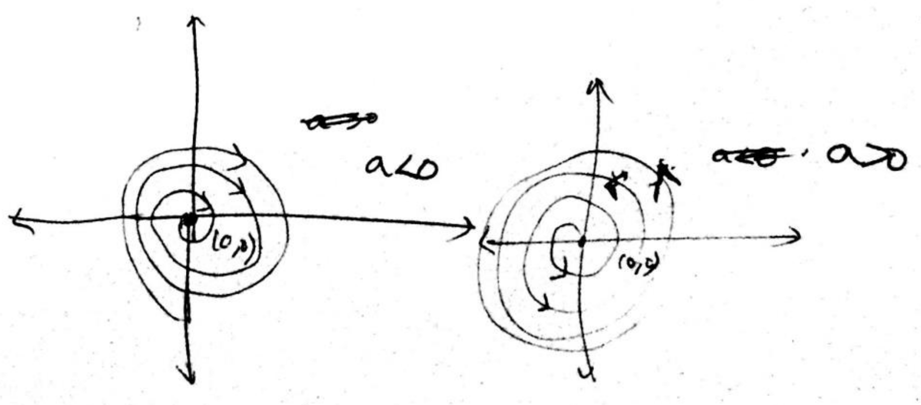
$x = \dfrac{a^2+1}{a+2}$

$y = \dfrac{(a^2+1)(1-2a)}{(a+2)^2}$.

$\therefore$ Steady states are $(0,0)$ $\left(\dfrac{a^2+1}{a+2}, \dfrac{(a^2+1)(1-2a)}{(a+2)^2} \right)$

$J = \begin{bmatrix} a - 2x & 1 \\ -1 + 4x & a \end{bmatrix}$

$J_{(0,0)} = \begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix} = \lambda = +a \pm i$



$$J(x_0, y_0) = \begin{bmatrix} a - 2\left(\frac{a^2+1}{a+2}\right) & 1 \\ -1 + 4\left(\frac{a^2+1}{a+2}\right) & a \end{bmatrix}$$

$$\cancel{tr = \frac{2(2a-1)}{a+2}}.$$

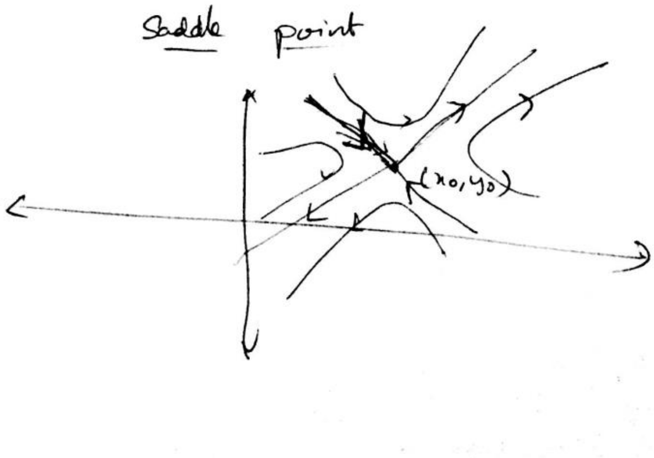
$$\lambda_{1.} = \frac{\frac{2(2a-1)}{a+2} \pm \sqrt{\frac{4(2a-1)^2}{(a+2)^2} + 4(a^2+1)}}{2}$$

$$\lambda = \frac{2a-1 \pm \sqrt{(2a-1)^2 + (a+2)^2 (a^2+1)}}{a+2}.$$

For $a < -2$ $\lambda_1 < 0$ and $\lambda_2 > 0$

For $a > -2$ $\lambda_1 > 0$ and $\lambda_2 < 0$

<u>Saddle</u> point



Problem: 4

1)

(4)

$x' = ax + y - x^3$

$y' = -x + ay + y^3$

At steady state, $x' = 0$; $y' = 0$

$\Rightarrow \quad y = x(x^2 - a) \rightarrow (1)$

$x = y(y^2 + a) \rightarrow (2)$

Plotting graphs, we understand 3 solutions (steady state) exist for all values of a.

Using Mathematica, we found the 9 solutions, out of which 6 are complex.

The x-values for 3 real solutions are

$x = 0$, $x = +\left(\frac{a + \sqrt{4+a^2}}{2}\right)^{1/2}$, $x = -\left(\frac{a + \sqrt{4+a^2}}{2}\right)^{1/2}$

1st steady state soln:

$x = 0, \ y = 0.$

$$J = \begin{bmatrix} a - 3x^2 & 1 \\ -1 & a + 3y^2 \end{bmatrix}$$

$$J_{0,0} = \begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix}$$

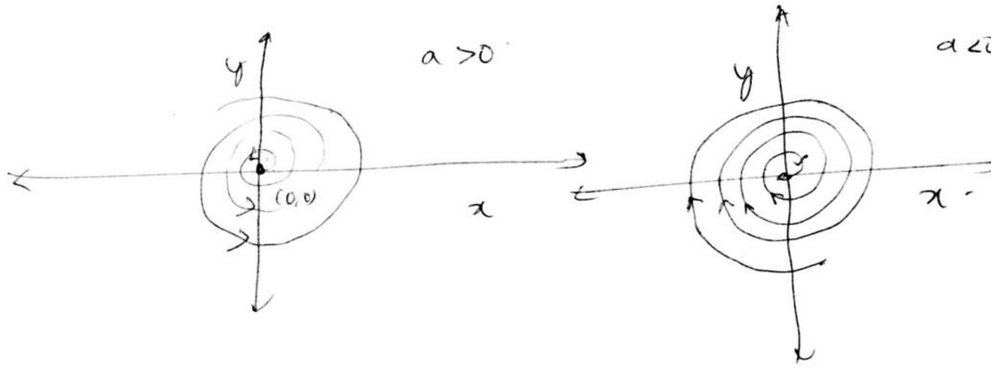
λ's of this S.S are given by $|J_{0,0}| = |\lambda I|$

$\Rightarrow \quad (a - \lambda)^2 + 1 = 0$

$\Rightarrow \quad \lambda = a \pm i$

For $a>0$, $(0,0)$ is a unstable focus as
we have $Re(\lambda)$'s positive

For $a<0$, $(0,0)$ is a stable focus as
we have $Re(\lambda)$'s negative.



2nd Steady Solution :

$$x = \left(\frac{a+\sqrt{4+a^2}}{2}\right)^{1/2}, \quad y = \left(\frac{a+\sqrt{4+a^2}}{2}\right)^{1/2}\left(\frac{\sqrt{4+a^2}-a}{2}\right)$$

Trace of Jacobian matrix $= 2a - 3(x^2 - y^2)$

$$= 2a - 3\left[\frac{a+\sqrt{4+a^2}}{2}\right]\left[1 - \frac{(\sqrt{4+a^2}-a)^2}{4}\right]$$

Determinant of Jacobian matrix

$$= (a-3x^2)(a+3y^2)+1$$

$$= \left(\frac{-a-3\sqrt{4+a^2}}{2}\right)\left(a + 3\frac{(a+\sqrt{4+a^2})}{2}\left(\frac{\sqrt{4+a^2}-a}{2}\right)^2\right)+1$$

Plotting trace vs a and determinant vs a,
we find for all values of a,
determinat <0.
This indicates that this is a saddle point ($\Delta<0$).
determinant <0

3rd steady state solution :

$$x = -\left(\frac{a+\sqrt{4+a^2}}{2}\right)^{1/2} \qquad (x_2, y_2)$$

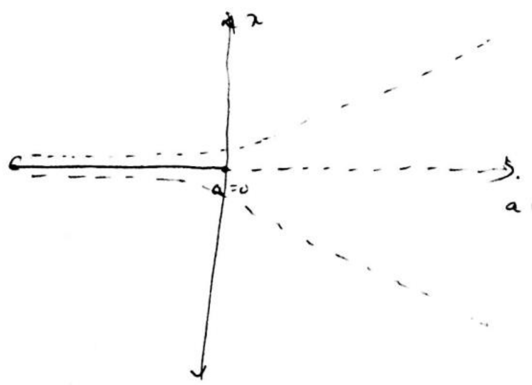
$$y = -\left(\frac{a+\sqrt{4+a^2}}{2}\right)^{1/2}\left(\frac{\sqrt{4+a^2}-a}{2}\right)$$

Nature of this steady state doesn't change as trace and determinant consists of only x^2 & y^2 terms. Hence, this is also a saddle point (determinant <0 for Jacobian matrix).

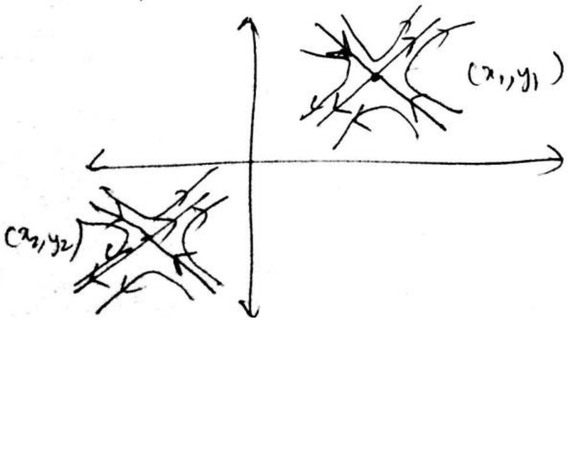
Note :

We used desmos to plot the graphs for 4th question of 1st part.

Bifurcation Diagram :-



Phase plane For other two steady states :-



Problem: 3

1) (3)

$$\dot{x} = 3(x-y)$$
$$\dot{y} = -xz + ax - y.$$
$$\dot{z} = xy - z$$

$x = y$; $z = x^2$; $-x^3 + ax - x$.

$-x^3 + ax - x = 0$ $\quad x[-x^2 + a - 1] = 0$

$(x, y, z) = (0,0,0)$ $\quad \boxed{x^2 = a-1}$; $x = \pm\sqrt{a-1}$

$(0,0,0)$ $\quad (\sqrt{a-1}, \sqrt{a-1}, a-1)$ $\quad (-\sqrt{a-1}, -\sqrt{a-1}, a-1)$

$$J = \begin{bmatrix} 3 & -3 & 0 \\ -z+a & -1 & -x \\ y & x & -1 \end{bmatrix}$$

At $(0,0,0)$

$$J = \begin{bmatrix} 3 & -3 & 0 \\ a & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\cancel{-1[-3 + 3a] = 3(1-a)}$

Finding 'λ'

$$\begin{vmatrix} 3-\lambda & -3 & 0 \\ a & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = -(1+\lambda)[3a + \lambda^2 - 2\lambda - 3]$$

$$= \cancel{\lambda^2} - \lambda^3 + \lambda^2 + \lambda[2 + (3-3a)] + (3-3a)$$

$(1+\lambda)\,[(3-\lambda)(1+\lambda) - 3a] = 0.$

$\lambda = -1$ $\quad | \quad (3 + 3\lambda - \lambda - \lambda^2 - 3a) = 0$

$\lambda = -1$; $\lambda = 1 \pm \sqrt{4-3a}.$ $\quad \lambda^2 - 2\lambda + 3a - 3 = 0.$

$\boxed{1 + \sqrt{4-3a} > 0}$

$\therefore$ Eigen values are $-$ve and $+$ve

Hence $(0,0,0)$ is a saddle point. $\forall a.$

For steady states

(x_1, y_1, z_1) $(\sqrt{a-1}, \sqrt{a-1}, a-1)$ and $(-\sqrt{a-1}, -\sqrt{a-1}, a-1)$.
(x_2, y_2, z_2)

~~We get a~~ Jacobian.

$$J_{(x_1,y_1,z_1)} = \begin{bmatrix} 3 & -3 & 0 \\ 1 & -1 & -\sqrt{a-1} \\ \sqrt{a-1} & \sqrt{a-1} & -1 \end{bmatrix} \qquad J_{(x_2,y_2,z_2)} = \begin{bmatrix} 3 & -3 & 0 \\ 1 & -1 & \sqrt{a-1} \\ -\sqrt{a-1} & \sqrt{a-1} & -1 \end{bmatrix}$$

~~Eigen v~~ We get a cubic eqn, whose roots are eigen values. The expression is very complex.

Plotting graph in desmos:-

We found that, [For both (x_1, y_1, z_1) & (x_2, y_2, z_2)]

for $a < 3.168$, we have 3 eigen values

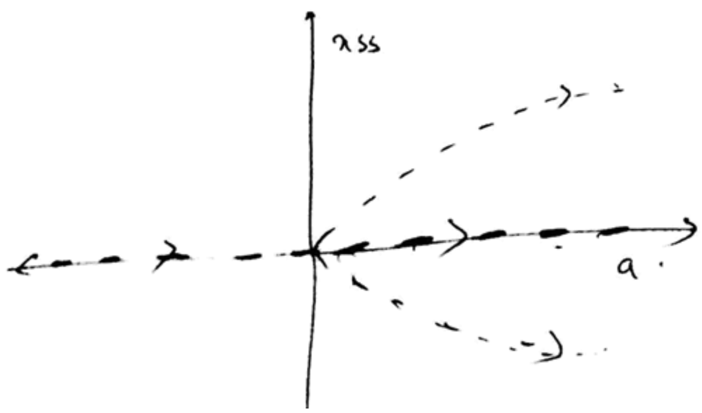
Out of which 2 are -ve and 1 +ve.

Which implies, it is not a ~~ste~~ stable steady state (saddle).

for $a > 3.168$, we have 1 eigen value which is +ve.

Implies, it is a unstable steady state.

Bifurcation diagram:-



$\boxed{\text{Steady states are Saddle points.}}$

QUESTION: 2
Problem: 1

2 (1)

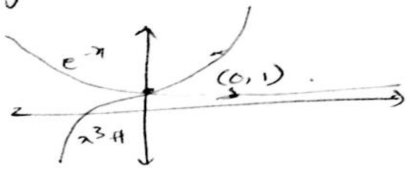
$x' = 1 + y - \exp(-x)$

$y' = x^3 - y$

$1 + y = e^{-x}$; $y = x^3$.

$1 + x^3 = e^{-x}$

$x = 0,\ y = 0.$



$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} e^{-x} & 1 \\ 3x^2 & -1 \end{bmatrix}$$

At (0,0)

$$J = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

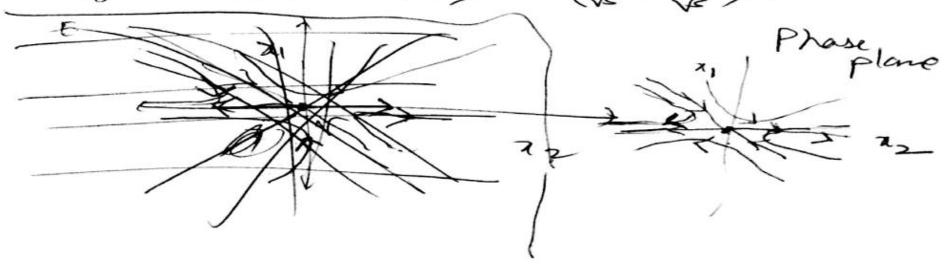
Eigen values =

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} = 0.$$

$\lambda = \pm 1.$

(0,0) is a saddle point

Eigen vectors $(1, 0)$ $\left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$.



Problem: 2

② $x' = -x + x^3$

$y' = x + y$

Steady state are $(0,0)$; $(1,-1)$; $(-1,1)$

$$J = \begin{bmatrix} 3x^2 - 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \text{At } (0,0) \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

Eigen values:-

$$\begin{vmatrix} -1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = 0 \qquad \lambda^2 - 1 = 0$$

$\lambda = \pm 1.$

$(0,0)$ saddle point.

$\lambda = 1$. eigen vector $(0,1)$

$\lambda = -1$ eigen vector $\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

At $[1,-1]$ and $[-1,1]$ ~~eigen values and eigen~~ Jacobian matrix is the same. Hence eigen values and eigen vectors are the same.

At [1, -1] and [-1, 1]

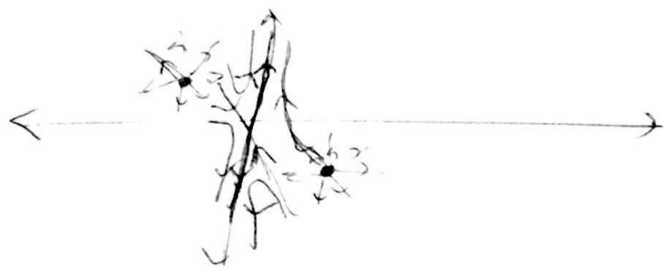
$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} - \begin{vmatrix} 2-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} \quad \lambda^2 - 3\lambda + 2 .$$

$$\boxed{\lambda = 2, 1}$$

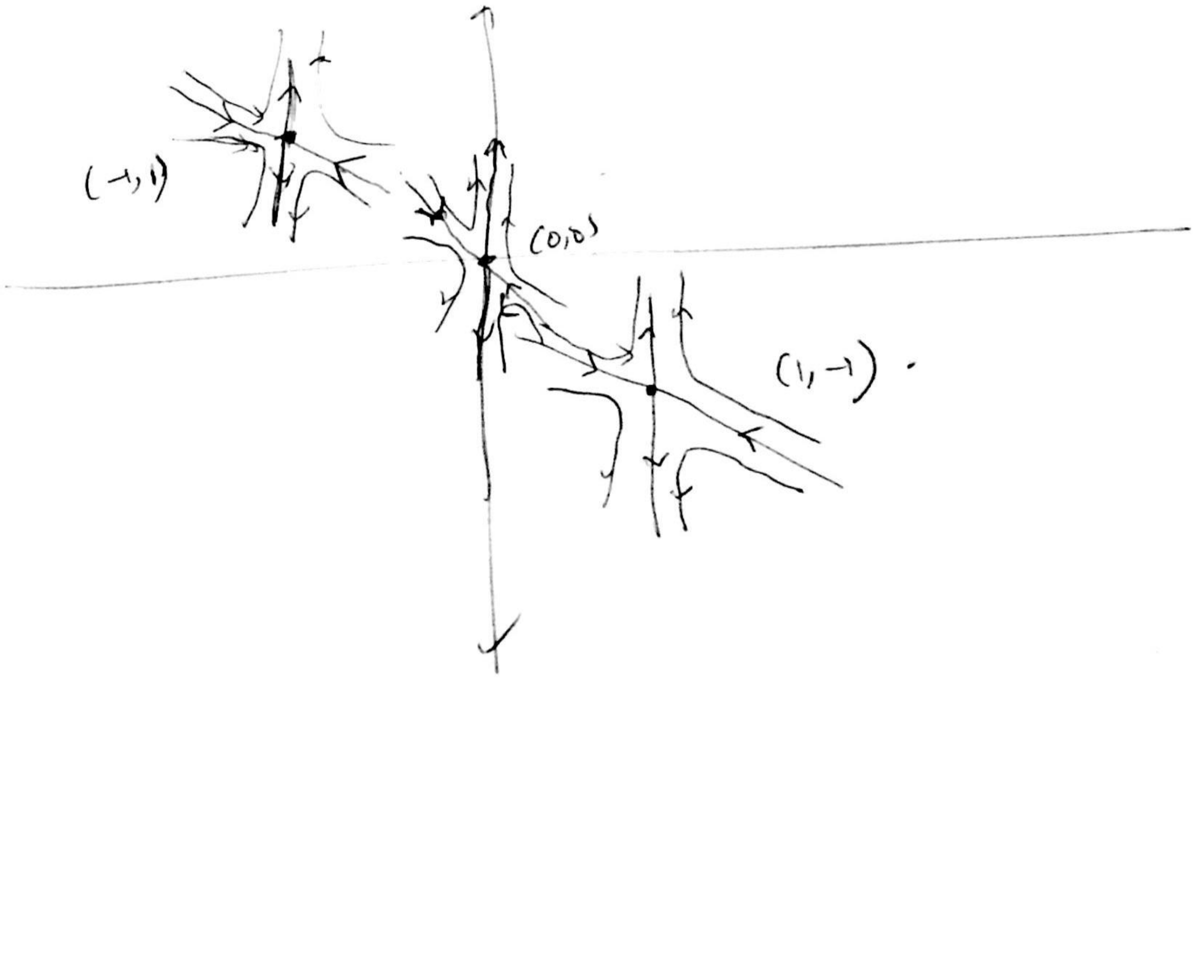
Unstable :-

$\lambda = 2$ Eigen vector is $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$\lambda = 1$ Eigen vector is $(0, 1)$.



Phase plane



Problem: 3

③
$$\dot{x} = x^2 - xy - x$$
$$y' = y^2 + xy - 2y$$

At S.S, $x(x-y-1)=0$

$y(y+x-2)=0$.

$(x,y)=(0,0)$, $(0,2)$ $(1,0)$ $(3/2, \frac{1}{2})$

$$J = \begin{bmatrix} 2x-y-1 & -x \\ y & 2y+x-2 \end{bmatrix}$$

At $(0,0)$

$$J = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$ $\lambda = -2, -1$

stable.

At $(0,2)$

$$J = \begin{bmatrix} -3 & 0 \\ 2 & 2 \end{bmatrix}$$ $(\lambda+3)(\lambda-2)$

$\lambda = -3, 2$ Saddle Point

$\lambda = -3$ Eigen vector $\left[\frac{5}{\sqrt{29}}, \frac{-2}{\sqrt{29}}\right]$

$\lambda = 2$ $[0, 1]$

At $(1,0)$

$$J = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$ $\lambda = \pm 1$ Saddle.

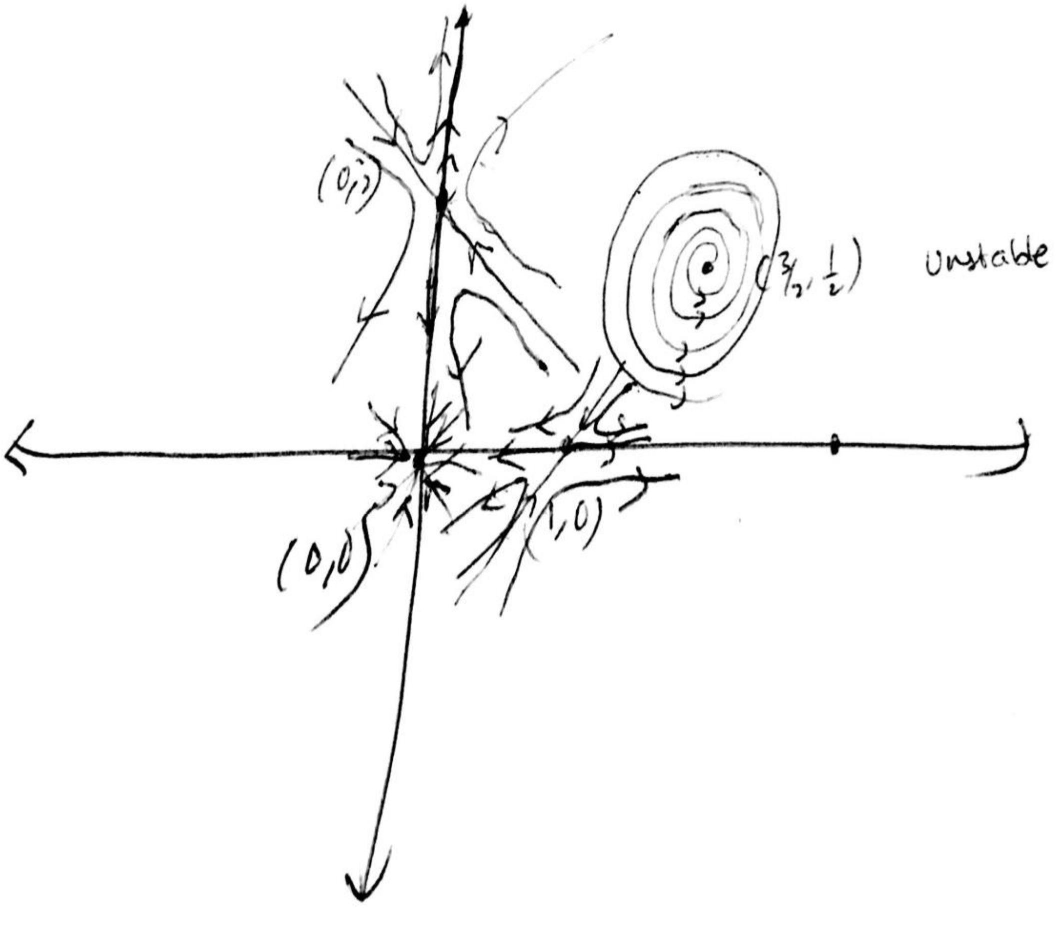
$\lambda = 1$ Eigenvector $[1, 0]$ $\lambda = -1$ $\left[\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right]$

At $(3/2, \frac{1}{2})$

$$J = \begin{bmatrix} 3/2 & -3/2 \\ \frac{1}{2} & 1/2 \end{bmatrix}$$ $$\begin{vmatrix} 3/2-\lambda & -3/2 \\ \frac{1}{2} & \frac{1}{2}-\lambda \end{vmatrix}$$ $2\lambda^2 - 4\lambda + 3 = 0$

$\lambda = 1 \pm \frac{i}{\sqrt{2}}$ Unstable.

Phase plane

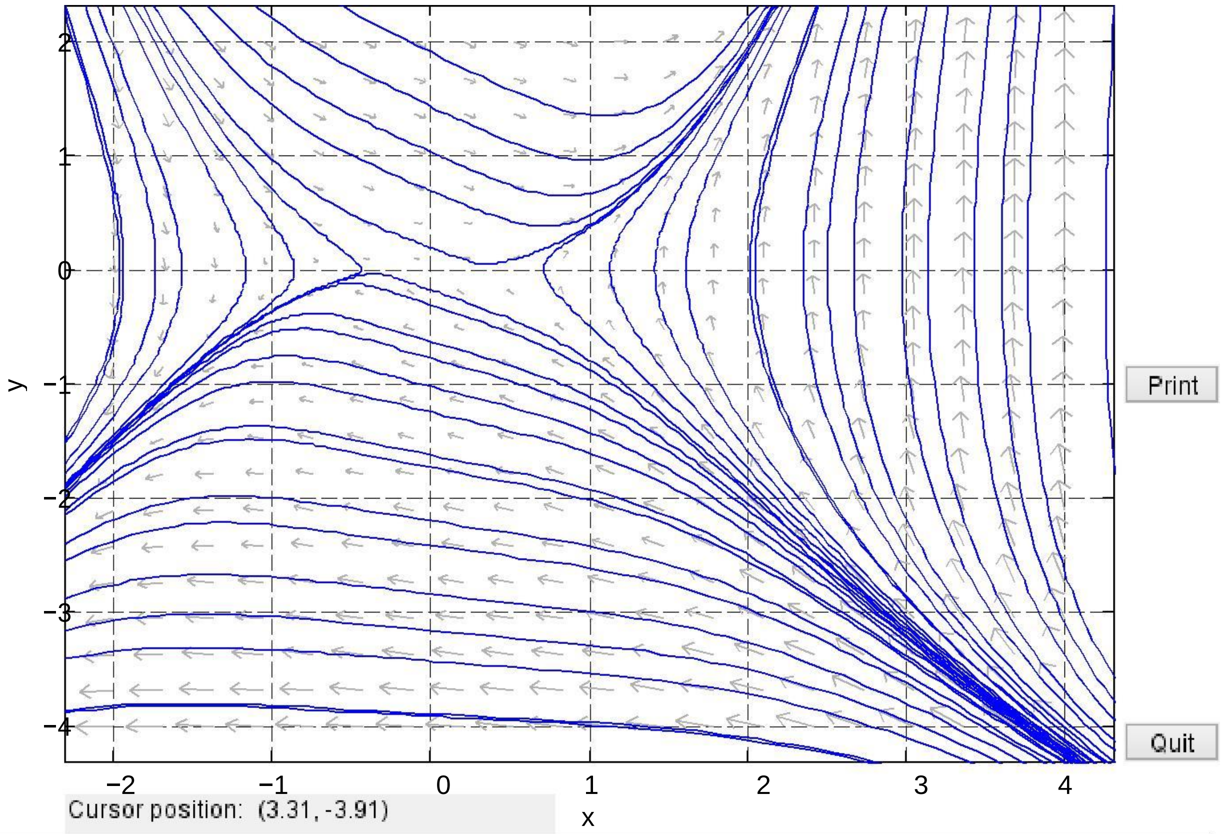


PHASE PLANES using
PPLANE7

PROB 1

$x' = 1 + y - \exp(-y)$

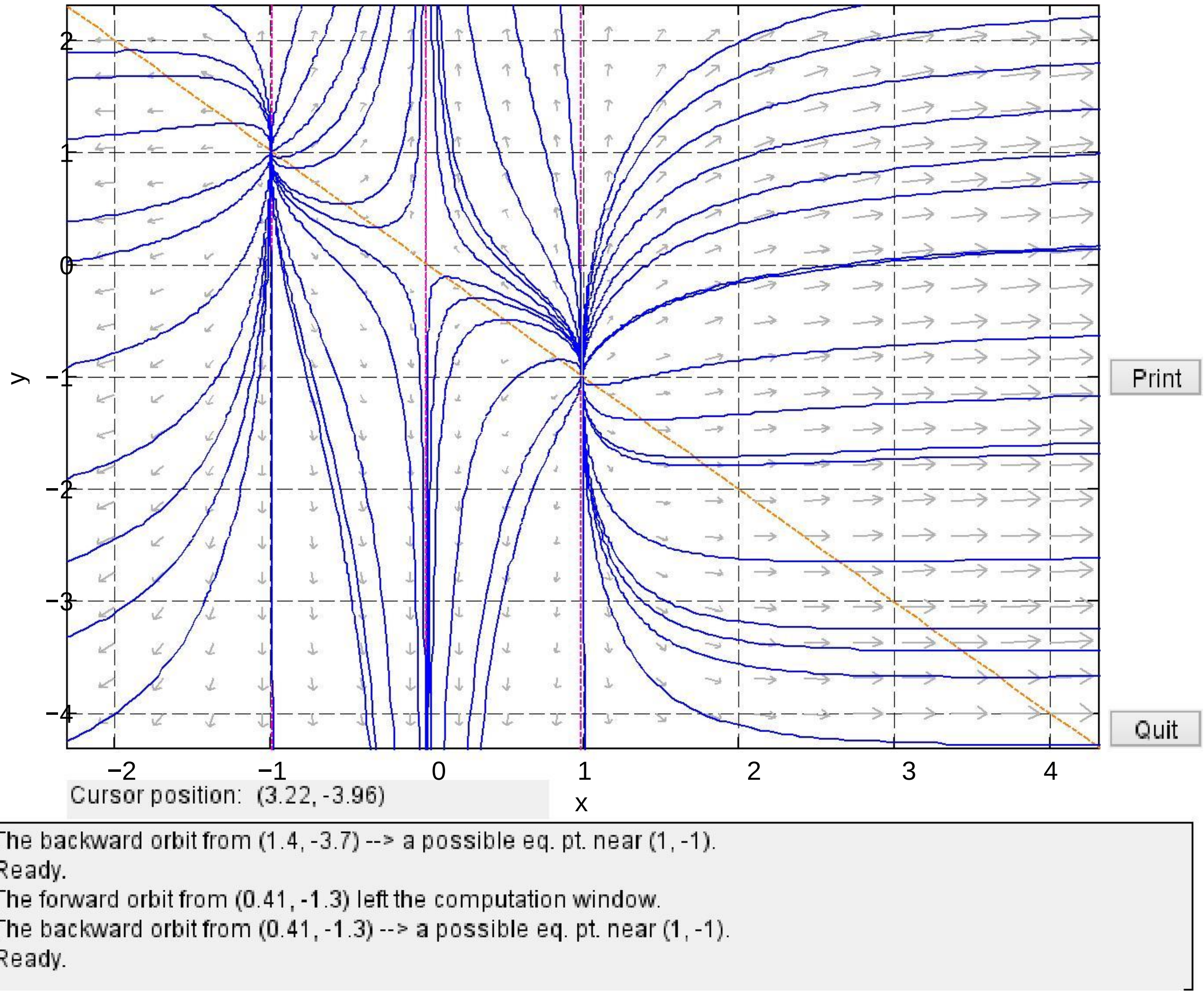
$y' = x^3 - y$



The backward orbit from (3, 0.25) left the computation window.
Ready.
The forward orbit from (2.4, 0.22) left the computation window.
The backward orbit from (2.4, 0.22) left the computation window.
Ready.

PROB 2

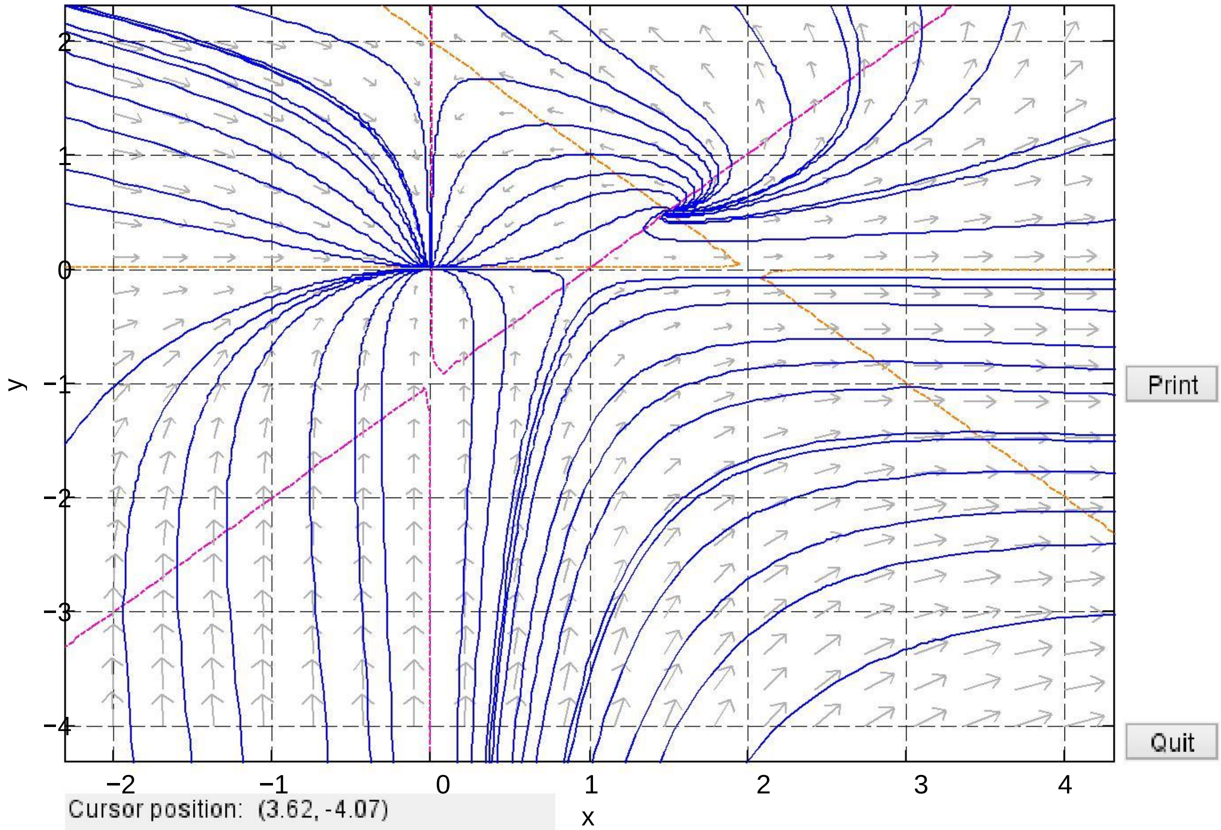
$x' = -x + x^3$
$y' = x + y$



PROB 3

$x' = x^2 - xy - x$

$y' = y^2 + xy - 2y$



The backward orbit from (0.82, 0.2) --> a possible eq. pt. near (1.5, 0.5).
Ready.
The forward orbit from (0.55, 0.59) --> a possible eq. pt. near (1.1e-13, 3.9e-17).
The backward orbit from (0.55, 0.59) --> a possible eq. pt. near (1.5, 0.5).
Ready.