Intro to Network Science: Homework #1

Due on Feb 08 2016 by 5:00pm

Intro to Network Science · Z. Toroczkai · Spring 2016

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February 7, 2016

Problem 1

Solution:

From the so called Handshaking Lemma: In any graph, the sum of all the vertex-degree is equal to twice the number of edges. In this case n=26 and m=58, therefore given the initial vertex-degree set (or sequence)

$$\{5 : deg(v) = 4, 6 : deg(v) = 5, 7 : deg(v) = 6\}$$

which adds up to 18 vertices, leaving us with 8 vertices, and from the First Theorem of Graph Theory we have that if G is a graph of orer n and size m, then

$$\sum_{i=1}^{n} deg\ v_i = 2m$$

means that the remining vertices (8 of them) will each have to be of $(2*m) - (5*4+6*5+7*6) = 24/8 = 3$. Three vertices each of the 8 remaining vertices will be their vertex-degree.

Problem 2

Solution

By observation, graph G_1 and G_2 are of the same order and size, their respeive vertex degree are the same and consisten with Theorem 2 (Lecture 3: Fundamentals of graph theory II.), thus these graphs are isomorphic.

$$n = 7$$
$$m = 10$$

$$G_1 \simeq G_2$$

By the same process of observation and analysis,

$$H_1 \simeq H_2$$

Problem 3

Given graph G with order n = 3k + 3 for some positive integer k. Every vertex of G has degree k + 1, k+2 and k+3. Prove that G has at least k+3 vertices of degree k+1 or at least k+1 vertices of degree k+2 or at least k + 2 vertices of degree k + 3.

Problem 3 continued on next page...

Solution:

It is possible if and only if n and r (the degree of each vertex) are not both odd integers.

Proof that G has at least k+3 vertices of degree k+1

Given the initial conditions for graph G, we are dealing with r-regular graphs where if we restrict r to even integers, then to show that G has at least k+3 vertices of degree k+1, let can say that: Let $k = 2$, then $n = 3k + 3 = 9$, $k + 3$ vertices totaling 5 may each have degree 3, $5 * 3 = 15$. From Corollary 1.5 [Charttrand et al.] $0 < \delta(3) \leq n - 1 = 8$, this requirement is maintained and we can have the remainder of the nodes with the same degree will still add up to a total number of edges to be within $n - 1$.

Proof that G has at least k+1 vertices of degree k+2.

A graph G of order n, where n = 3k+3, if we let n be an odd integer and again let k = 2, then the number of vertices is an odd integer.
Out of $n = 3k + 3 = 3(2) + 3 = 9$ vertices, $(k + 1) = 2 + 1 = 3$ nodes of degree $k + 2 = 2 + 2 = 4$ results in a total of 12 edges.
With the remaining 6 vertices, each of degree 4 we end up with a total of 36 connected stubs, which results in 18 edges well below the n(n-1)/2 max.

Proof:

Let k = 0, with n = 3k+3, n = 3
For (k=0)+2 = 2 vertices each of degree (k=0) + 3 = 3 (i.e., r)
NB: Notice both n and r are both odd.
Leaves us with 1 node of degree 3, but since n=3, the following would be violated: $0 < \delta(v) = 3 \leq (n - 1) = 2$.
On the other hand, if we let k = 1, we end with n = 6, which is even.
Where k+2 (or 3) vertices each of degree k+3 = 4 results in deg(v) being even, in this case Theorem 1.7 [1] holds.

Problem 4

Show that any graph contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$, if $\delta(G) \geq 2$.

Solution:

First, the minimum degree of G is denoted by $\delta(G)$.

Problem 4 continued on next page...

Pick an arbitrary vertex and label it v_1.
Next, select a neighbor and label it v_2.
Then select any of its other neighbors (different from v_1) and label it v_3. Now we have a path of length $\delta(G)$ corresponding to length of 1. If we continue like this, we can always select a path of length $\delta(G)$
To show that this graph can have a cyle of length at least $\delta(G) + 1$, which in this case it would be a path of length 2 and abide with the rule that $\delta(G) \geq 2$,then we connect v_3 to v_1 results in a graph with a path of maximal length. Now, each node is of degree 2, which is also the minimal node degree and the cycle is of length $\delta(G) + 1$. Note that if we do not connect it back to v_1 the graph does not result in graph with $\delta(G) = 2$ in this case.

Problem 5

Show that for every finite set S of positive integers, there exists a positive integer k such that the sequence obtained by listing each element of S k times is graphical. Find the smallest such k for $S = \{2, 6, 7\}$.

Solution:

To show that any degree sequence is graphical, the following conditions will be met:

1. The degree of any vertex v_i shall be less than or equal to n-1, for $i(1 \leq i \leq n)$.
2. The Sum of the sum of all deg v_i, for i 1 through n, shall be even.
3. G must also have even number of of vertices (thus in the example below there has of be a k of deg v = 7 which has to be even.)
4. In addition, the sequence must hold up to the Havel-Hakimi Theorem [Chartrand et al.]. To illustrate the theorem using the sequence obtained by listing each element of S k times: if k = 2

Problem 5 continued on next page. . .

$$s : 2, 2, 6, 6, 7, 7$$

Reordering the sequence:

$$s : 7, 7, 6, 6, 2, 2$$

After one application of the theorem, e.g., deleting 7 and subtracting 1 from the next 7 items, which we lack, then we change k to 4 and try again.

$$s : 2, 2, 2, 2, 6, 6, 6, 6, 7, 7, 7, 7$$

Reordering the sequence:

$$s : 7, 7, 7, 7, 6, 6, 6, 6, 2, 2, 2, 2$$

Applying the Havel-Hakimi Theorem:

$$\begin{aligned}
s'_1 &: 6, 6, 6, 5, 5, 5, 5, 2, 2, 2, 2 \\
s'_2 &: 5, 5, 4, 4, 4, 4, 2, 2, 2, 2 \\
s'_3 &: 4, 3, 3, 3, 3, 2, 2, 2, 2 \\
s'_4 &: 2, 2, 2, 2, 2, 2, 2, 2 \\
s'_5 &: 1, 1, 2, 2, 2, 2, 2 \\
s'_6 &: 0, 1, 2, 2, 2, 2 \\
s'_7 &: 1, 2, 2, 2, 2, 0 \\
s'_8 &: 1, 2, 2, 2, 0 \\
s'_9 &: 1, 2, 2, 0 \\
s'_A &: 1, 2, 0 \\
s'_B &: 1, 0 \\
s'_C &: 0
\end{aligned}$$

Therefor, s is graphical. the smallest $k = 4$.

Problem 6

Solution:

References

Chartrand, Gary and Lesniak, Linda and Zhang, Ping; *Graphs & Digraphs*, Fifth Edition; 2010; Chapman & Hall/CRC