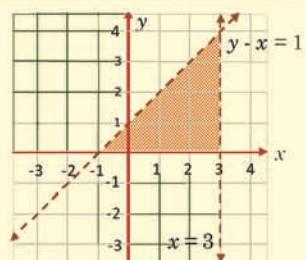
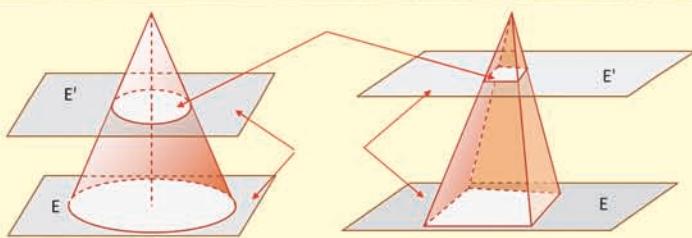




MATHEMATICS

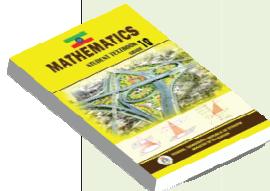
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GRADE **10**



FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA
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MATHEMATICS

STUDENT TEXTBOOK

GRADE 10

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FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA

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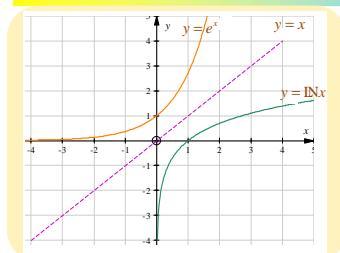


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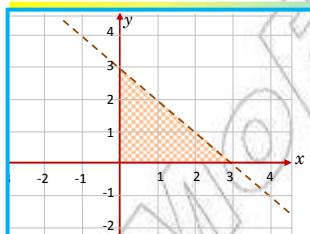
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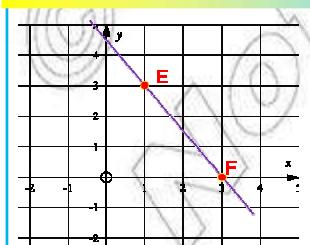
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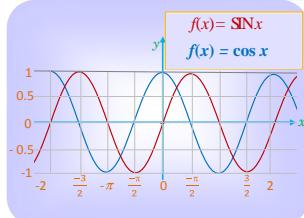


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1

Unit



POLYNOMIAL FUNCTIONS ARE THE MOST WIDELY USED FUNCTIONS IN MATHEMATICS. THEY ARSE NATURALLY IN MANY APPLICATIONS. ESSENTIALLY, THE GRAPH OF A POLYNOMIAL FUNCTION HAS NO BREAKS AND GAPS. IT DESCRIBES SMOOTH CURVES AS SHOWN IN THE FIGURE ABOVE.

POLYNOMIAL FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- ⊕ define polynomial functions.
- ⊕ perform the four fundamental operations on polynomials.
- ⊕ apply theorems on polynomials to solve related problems.
- ⊕ determine the number of rational and irrational zeros of a polynomial.
- ⊕ sketch and analyse the graphs of polynomial functions.

Main Contents

- 1.1 Introduction to polynomial functions**
- 1.2 Theorems on polynomials**
- 1.3 Zeros of polynomial functions**
- 1.4 Graphs of polynomial functions**

Key Terms

Summary

Review Exercises

INTRODUCTION

THERE IS AN EXTREMELY IMPORTANT FAMILY OF FUNCTIONS IN MATHEMATICS CALLED POLYNOMIAL FUNCTIONS.

STATED QUITE SIMPLY, POLYNOMIAL FUNCTIONS ARE FUNCTIONS OF ONE VARIABLE, CONSISTING OF THE SUM OF SEVERAL TERMS, EACH TERM IS A PRODUCT OF TWO FACTORS, THE FIRST BEING A REAL NUMBER COEFFICIENT AND THE SECOND BEING A NON-NEGATIVE INTEGER POWER.

IN THIS UNIT YOU WILL BE LOOKING AT THE DIFFERENT COMPONENTS OF POLYNOMIAL FUNCTIONS. THESE ARE THEOREMS ON POLYNOMIAL FUNCTIONS; ZEROS OF A POLYNOMIAL FUNCTION, GRAPHS OF POLYNOMIAL FUNCTIONS. BASICALLY THE GRAPH OF A POLYNOMIAL FUNCTION IS A SMOOTH AND CONTINUOUS CURVE. HOWEVER, YOU WILL BE GOING OVER HOW TO USE THE DEGREE (EVEN OR ODD) AND THE LEADING COEFFICIENT TO DETERMINE THE END BEHAVIOUR OF THE FUNCTION.

1.1

INTRODUCTION TO POLYNOMIAL FUNCTIONS



OPENING PROBLEM

OBVIOUSLY, THE VOLUME OF WATER IN ANY DAM FLUCTUATES FROM SEASON TO SEASON. AN ENGINEER SUGGESTS THAT THE VOLUME OF THE WATER (IN GIGA LITRES) IN A CERTAIN DAM OVER t -MONTHS (STARTING ST SEPTEMBER) IS DESCRIBED BY THE MODEL:

$$v(t) = 450 - 170t + 22t^2 - 0.6t^3$$

ELECTRIC POWER CORPORATION RULES THAT IF THE VOLUME FALLS BELOW 200 GIGA LITRES, THE WISE PROJECT, “IRRIGATION”, IS PROHIBITED. DURING WHICH MONTHS, IF ANY, WAS THE PROJECT PROHIBITED IN THE LAST 12 MONTHS?

RECALL THAT **function f** IS A RELATION IN WHICH NO TWO ORDERED PAIRS HAVE THE SAME ELEMENT, WHICH MEANS THAT FOR ANY x IN THE DOMAIN THERE IS A UNIQUE PAIR

(x, y) BELONGING TO THE FUNCTION f .

IN **UNIT 4** OF **GRADE 10** MATHEMATICS, YOU HAVE DISCUSSED FUNCTIONS SUCH AS:

$$f(x) = \frac{2}{3}x + \frac{1}{2}, \quad g(x) = 5 - 3x, \quad h(x) = 8x \text{ AND } j(x) = -\sqrt{3x} + 2.7.$$

SUCH FUNCTIONS ARE **linear functions**.

A FUNCTION f IS A **linear function**, IF IT CAN BE WRITTEN IN THE FORM

$$f(x) = ax + b, \quad a \neq 0,$$

WHERE a AND b ARE REAL NUMBERS.

THE **domain** OF IS THE SET OF ALL REAL NUMBERS ~~range~~ AND THE SET OF ALL REAL NUMBERS.

IF $a = 0$, THEN IS CALLED A **constant function**. IN THIS CASE,

$$f(x) = b.$$

THIS FUNCTION HAS THE SET OF ALL REAL NUMBERS AS ITS ~~domain~~ AND $\{b\}$ AS ITS ~~range~~. ALSO RECALL WHAT YOU STUDIED ABOUT ~~quadratic functions~~. EACH OF THE FOLLOWING FUNCTIONS IS A QUADRATIC FUNCTION.

$$f(x) = x^2 + 7x - 12, \quad g(x) = 9 + \frac{1}{4}x^2, \quad h(x) = -x^2 + , \quad k(x) = x^2,$$

$$l(x) = 2(x - 1)^2 + 3, \quad m(x) = (x + 2)(1 - x)$$

IF a , b , AND c ARE REAL NUMBERS ~~WITHIN~~ THE FUNCTION

$$f(x) = ax^2 + bx + c$$
 IS A **quadratic function**.

SINCE THE EXPRESSION $bx + c$ REPRESENTS A REAL NUMBER FOR ANY ~~REAL NUMBER~~ domain OF A QUADRATIC FUNCTION IS THE SET OF ALL REAL NUMBERS ~~AND THE RANGE OF~~ FUNCTION DEPENDS ON THE VALUES OF a , b AND

Exercise 1.1

- 1** IN EACH OF THE FOLLOWING CASES, CLASSIFY THE FUNCTION ~~AS~~ CONSTANT, LINEAR OR NONE OF THESE:

A $f(x) = 1 - x^2$

B $h(x) = \sqrt{2x-1}$

C $h(x) = 3 + 2^x$

D $g(x) = 5 - \frac{4}{5}x$

E $f(x) = 2\sqrt{3}$

F $f(x) = \left(\frac{2}{3}\right)^{-1}$

G $h(x) = 1 - |x|$

H $f(x) = (1 - \sqrt{2}x)(1 + \sqrt{2}x)$

I $k(x) = \frac{3}{4}(12 + 8x)$

J $f(x) = 12x^{-1}$

K $l(x) = \frac{(x + 1)(x - 2)}{x - 2}$

L $f(x) = x^4 - x + 1$

- 2** FOR WHAT VALUES OF ~~AND~~ a IS $f(x) = ax^2 + bx + c$ A CONSTANT, A LINEAR OR A QUADRATIC FUNCTION?

1.1.1**Definition of a Polynomial Function**

CONSTANT, LINEAR AND QUADRATIC FUNCTIONS ARE ALL SPECIAL CASES OF A WIDE CLASS OF FUNCTIONS CALLED **polynomial functions**.

Definition 1.1

Let n be a non-negative integer and let $a_n, a_{n-1}, \dots, a_1, a_0$ be real numbers with $a_n \neq 0$. The function

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

is called a **polynomial function in variable x of degree n** .

NOTE THAT IN THE DEFINITION OF A POLYNOMIAL FUNCTION

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

- I $a_n, a_{n-1}, a_{n-2}, \dots, a_1, a_0$ ARE CALLED COEFFICIENTS OF THE POLYNOMIAL FUNCTION (OR SIMPLY THE POLYNOMIAL).
- II THE NUMBER a_n IS CALLED THE LEADING COEFFICIENT OF THE POLYNOMIAL FUNCTION AND $a_n x^n$ IS THE LEADING TERM.
- III THE NUMBER a_0 IS CALLED THE CONSTANT TERM OF THE POLYNOMIAL.
- IV THE NUMBER n IS THE EXPONENT OF THE HIGHEST POWER OF x OR DEGREE OF THE POLYNOMIAL.

NOTE THAT THE DOMAIN OF A POLYNOMIAL FUNCTION IS

EXAMPLE 1 WHICH OF THE FOLLOWING ARE POLYNOMIAL FUNCTIONS? FOR THOSE WHICH ARE POLYNOMIALS, FIND THE DEGREE, LEADING COEFFICIENT, AND CONSTANT TERM.

A $f(x) = \frac{2}{3}x^4 - 12x^2 + x + \frac{7}{8}$

B $f(x) = \frac{x}{x}$

C $g(x) = \sqrt{(x+1)^2}$

D $f(x) = 2x^4 + x^2 + 8x + 1$

E $k(x) = \frac{x^2+1}{x^2+1}$

F $g(x) = \frac{8}{5}x^{15}$

G $f(x) = (1 - \sqrt{2}x)(1 + \sqrt{2}x)$

H $k(y) = \frac{6}{y}$

SOLUTION:

A IT IS A POLYNOMIAL FUNCTION OF DEGREE 4 WITH LEADING COEFFICIENT $\frac{2}{3}$

CONSTANT TERM $\frac{7}{8}$

B IT IS NOT A POLYNOMIAL FUNCTION BECAUSE ITS DOMAIN IS NOT \mathbb{R}

- C** $g(x) = \sqrt{(x+1)^2} = |x+1|$, SO IT IS NOT A POLYNOMIAL FUNCTION BECAUSE IT CANNOT BE WRITTEN IN THE FORM $+a_{n-1}x^{n-1} + \dots + a_1x + a_0$
- D** IT IS NOT A POLYNOMIAL FUNCTION BECAUSE ONE OF ITS TERMS HAS A NEGATIVE EXPONENT.
- E** $k(x) = \frac{x^2+1}{x^2+1} = 1$, SO IT IS A POLYNOMIAL FUNCTION OF DEGREE 0 WITH LEADING COEFFICIENT 1 AND CONSTANT TERM 1.
- F** IT IS A POLYNOMIAL FUNCTION OF DEGREE $\frac{8}{5}$ WITH THE ENDING CONSTANT TERM 0.
- G** IT IS A POLYNOMIAL FUNCTION OF DEGREE 2 WHICH HAS NO CONSTANT TERM 1.
- H** IT IS NOT A POLYNOMIAL FUNCTION BECAUSE ITS DOMAIN IS NOT \mathbb{R}

A POLYNOMIAL EXPRESSION ~~AND~~ AN EXPRESSION OF THE FORM

$$a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$$

WHERE n IS A NON NEGATIVE INTEGER ~~AND~~ AN INDIVIDUAL EXPRESSION MAKING UP THE POLYNOMIAL IS CALLED A **term**.

ACTIVITY 1.1



- 1** FOR THE POLYNOMIAL EXPRESSION $\frac{x^2+3-6x^4}{4} + \frac{7}{8}x - x^3$,
- A** WHAT IS THE DEGREE? **B** WHAT IS THE LEADING COEFFICIENT?
- C** WHAT IS THE COEFFICIENT OF x^3 ? **D** WHAT IS THE CONSTANT TERM?
- 2** A MATCH BOX HAS LENGTH ~~1 CM~~ WIDTH x CM AND HEIGHT 3 CM,
- A** EXPRESS ITS SURFACE AREA AS A FUNCTION OF x
- B** WHAT IS THE DEGREE AND THE CONSTANT TERM OBTAINED ABOVE?

WE CAN RESTATE THE DEFINITION ~~AND~~ **quadratic functions** USING THE TERMINOLOGY FOR POLYNOMIALS. LINEAR FUNCTIONS ARE POLYNOMIALS OF ~~degree 1~~. SIMILARLY, **constant functions** ARE POLYNOMIAL FUNCTIONS OF DEGREE ~~0~~. **quadratic functions** ARE POLYNOMIAL FUNCTIONS OF DEGREE 2. THE ~~ZERO FUNCTION~~, CONSIDERED TO BE A POLYNOMIAL FUNCTION BUT IS NOT ASSIGNED A DEGREE AT THIS POINT.

NOTE THAT IN EXPRESSING A POLYNOMIAL, WE USUALLY OMIT ALL TERMS WHICH APPARENTLY HAVE ZERO COEFFICIENTS AND WRITE OTHERS IN DECREASING ORDER, OR INCREASING ORDER, OF THE EXPONENTS.

EXAMPLE 2 FOR THE POLYNOMIAL FUNCTION $p(x) = \frac{x^2 - 2x^5 + 8}{4} + \frac{7}{8}x - x^3$,

- A WHAT IS ITS DEGREE? B FIND a_n, a_{n-1}, a_{n-2} AND a_0
 C WHAT IS THE CONSTANT TERM? D WHAT IS THE COEFFICIENT OF

SOLUTION:
$$\begin{aligned} p(x) &= \frac{x^2 - 2x^5 + 8}{4} + \frac{7}{8}x - x^3 = \frac{x^2}{4} - \frac{2}{4}x^5 + \frac{8}{4} + \frac{7}{8}x - x^3 \\ &= -\frac{1}{2}x^5 - x^3 + \frac{1}{4}x^2 + \frac{7}{8}x + 2 \end{aligned}$$

- A THE DEGREE IS 5.
 B $a_n = a_5 = -\frac{1}{2}, a_{n-1} = a_4 = 0, a_{n-2} = a_3 = -1$ AND $a_0 = \frac{1}{4}$.
 C THE CONSTANT TERM IS 2.
 D THE COEFFICIENTS OF $\frac{7}{8}$

ALTHOUGH THE domain OF A POLYNOMIAL FUNCTION IS THE SET OF ALL REAL NUMBERS, WE HAVE SET A RESTRICTION ON THE DOMAIN BECAUSE OF OTHER CIRCUMSTANCES. FOR A GEOMETRICAL APPLICATION, IF A RECTANGLE IS LONG AND NARROW, THE AREA OF THE RECTANGLE, THE DOMAIN OF THE FUNCTION p IS THE SET OF POSITIVE REAL NUMBERS. IN A POPULATION FUNCTION, THE DOMAIN IS THE SET OF POSITIVE INTEGERS.

Based on the types of coefficients it has, a polynomial function p is said to be:

- ✓ APOLYNOMIAL FUNCTION over integers, IF THE COEFFICIENTS ARE ALL INTEGERS.
- ✓ APOLYNOMIAL FUNCTION over rational numbers, IF THE COEFFICIENTS ARE ALL RATIONAL NUMBERS.
- ✓ APOLYNOMIAL FUNCTION over real numbers, IF THE COEFFICIENTS ARE ALL REAL NUMBERS.

Remark: EVERY POLYNOMIAL FUNCTION THAT WE WILL CONSIDER IN THIS UNIT IS A POLYNOMIAL FUNCTION OVER THE REAL NUMBERS.

FOR EXAMPLE, IF $p(x) = \frac{2}{3}x^4 - 13x^2 + \frac{7}{8}$, THEN p IS A POLYNOMIAL FUNCTION OVER RATIONAL AND REAL NUMBERS, BUT NOT OVER INTEGERS.

IF $p(x)$ CAN BE WRITTEN IN THE FORM $a_{n-1}x^{n-1} + \dots + a_1x + a_0$ THEN DIFFERENT EXPRESSIONS CAN DEFINE THE SAME POLYNOMIAL FUNCTION.

FOR EXAMPLE, THE FOLLOWING EXPRESSIONS ALL DEFINE THE SAME POLYNOMIAL

$$\frac{1}{2}x^2 - x.$$

- A** $\frac{x^2 - 2x}{2}$ **B** $-x + \frac{1}{2}x^2$ **C** $\frac{1}{2}(x^2 - 2x)$ **D** $x\left(\frac{1}{2}x - 1\right)$

ANY EXPRESSION WHICH DEFINES A POLYNOMIAL FUNCTION IS CALLED A **polynomial expression**.

EXAMPLE 3 FOR THE POLYNOMIAL EXPRESSION $\frac{(x+3)(x-1)}{x-1}$,

- A** WHAT IS THE DEGREE? **B** WHAT IS THE COEFFICIENT OF x^3 ?
C WHAT IS THE LEADING COEFFICIENT? WHAT IS THE CONSTANT TERM?

SOLUTION:

- A** THE DEGREE IS 5. **B** THE COEFFICIENT IS 1.
C THE LEADING COEFFICIENT IS -1. THE CONSTANT TERM IS 1.

CONSIDER THE FUNCTIONS $\frac{(x+3)(x-1)}{x-1}$ AND $g(x) = x + 3$.

WHEN IS SIMPLIFIED IT GIVES $x + 3$, WHERE $x \neq 1$. AS THE DOMAIN IS NOT THE SET OF ALL REAL NUMBERS, IT IS NOT A POLYNOMIAL FUNCTION. BUT THIS DOMAIN IS NOT THE SET OF ALL REAL NUMBERS. THE FUNCTIONS HAVE DIFFERENT DOMAINS AND YOU CAN CONCLUDE THAT f AND g ARE NOT THE SAME FUNCTIONS.

WHEN YOU ARE TESTING AN EXPRESSION TO CHECK WHETHER OR NOT IT DEFINES A FUNCTION, YOU MUST BE CAREFUL AND WATCH THE **domain** OF THE FUNCTION DEFINED.

Exercise 1.2

1 WHICH OF THE FOLLOWING ARE POLYNOMIAL FUNCTIONS?

- | | |
|---|---|
| A $f(x) = 3x^4 - 2x^3 + x^2 + 7x - 9$ | B $f(x) = x^{25} + 1$ |
| C $f(x) = 3x^{-3} + 2x^{-2} + x + 4$ | D $f(y) = \frac{1}{3}y^2 + \frac{2}{3}y + 1$ |
| E $f(t) = \frac{3}{t} + \frac{2}{t^2}$ | F $f(y) = 108 - 95y$ |
| G $f(x) = 312x^6$ | H $f(x) = \sqrt{3}x^2 - x^3 + \sqrt{2}$ |

I $f(x) = \sqrt{3x} + x + 3$

J $f(x) = \frac{4x^2 - 5x^3 + 6}{8}$

K $f(x) = \frac{3}{6+x}$

L $f(y) = \frac{18}{y}$

M $f(a) = \frac{a}{2a}$

N $f(x) = \frac{x}{12}$

O $f(x) = 0$

P $f(a) = a^{\frac{1}{2}} + 3a + a^2$

Q $f(x) = \frac{9}{17}x^{83} + \sqrt{54}x^{97} +$

R $f(t) = \frac{4}{7} - 2$

S $f(x) = (1-x)(x+2)$

T $g(x) = \left(x - \frac{2}{3}\right)\left(x + \frac{3}{4}\right)$

2 GIVE THE DEGREE, THE LEADING COEFFICIENT AND THE CONSTANT TERM OF EACH FUNCTION IN QUESTIONS ABOVE.

3 WHICH OF THE POLYNOMIAL FUNCTIONS ARE POLYNOMIALS?

A POLYNOMIAL FUNCTIONS OVER INTEGERS?

B POLYNOMIAL FUNCTIONS OVER RATIONAL NUMBERS?

C POLYNOMIAL FUNCTIONS OVER REAL NUMBERS?

4 WHICH OF THE FOLLOWING ARE POLYNOMIAL EXPRESSIONS?

A $2\sqrt{3}-x$

B $y(y-2)$

C $\frac{(x+3)^2}{x+3}$

D $\sqrt{y^2+3} + 2 - 3y^3$

E $\frac{(y-3)(y-1)}{2}$

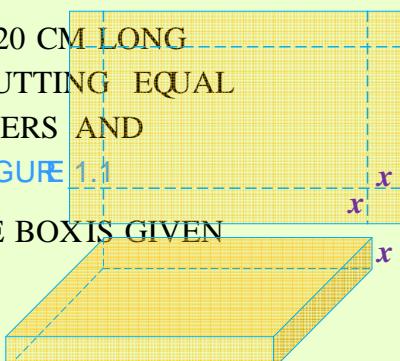
F $\frac{(t-5)(t-1)}{t-1}$

G $\frac{(x-3)(x^2+1)}{x^2+1}$

H $y+2y-3y$

I $\frac{x^2+4}{x^2+4}$

5 AN OPEN BOX IS TO BE MADE FROM A 20 CM LONG SQUARE PIECE OF MATERIAL, BY CUTTING EQUAL SQUARES OF LENGTH x CM FROM THE CORNERS AND TURNING UP THE SIDES AS SHOWN IN FIGURE 1.1.



A VERIFY THAT THE VOLUME OF THE BOX IS GIVEN BY THE FUNCTION $V(x) = 80x^3 - 80x^2 + 400x$.

B DETERMINE THE DOMAIN OF v .

Figure 1.1

1.1.2 Operations on Polynomial Functions

RECALL THAT, IN ALGEBRA, THE FUNDAMENTAL OPERATIONS ARE ADDITION, MULTIPLICATION AND DIVISION. THE FIRST STEP IN PERFORMING OPERATIONS ON FUNCTIONS IS TO USE THE COMMUTATIVE, ASSOCIATIVE AND DISTRIBUTIVE LAWS TO COMBINE LIKE TERMS TOGETHER.

ACTIVITY 1.2



- 1** WHAT ARE LIKE TERMS? GIVE AN EXAMPLE.
- 2** ARE $8a^2$, $2a^3$ AND $5a$ LIKE TERMS? EXPLAIN.
- 3** FOR ANY THREE REAL NUMBERS, DETERMINE WHETHER EACH OF THE FOLLOWING STATEMENTS IS TRUE OR FALSE. GIVE REASONS FOR YOUR ANSWERS.

A $a - (b + c) = a - b + c$	B $a + (b - c) = a + b - c$
C $a - (b - c) = a - b + c$	D $a - (b - c) = a - b - c$
- 4** VERIFY EACH OF THE FOLLOWING STATEMENTS:

A $(4x + a) + (2a - x) = 3(a + x)$	B $5x^2y + 2xy^2 - (x^2y - xy^2) = 4x^2y + 3xy^2$
C $8a - (b + 9a) = -(a + b)$	D $2x - 4(x - y) + (y - x) = 5y - 3x$
- 5** IF $f(x) = x^3 - 2x^2 + 1$ AND $g(x) = x^2 - x - 1$, THEN WHICH OF THE FOLLOWING STATEMENTS ARE TRUE?

A $f(x) + g(x) = x^3 + x^2 - x$	B $f(x) - g(x) = x^3 - 3x^2 + x + 2$
C $g(x) - f(x) = 3x^2 + x^3 - x - 2$	D $f(x) - g(x) \neq g(x) - f(x)$.
- 6** IF f AND g ARE POLYNOMIAL FUNCTIONS OF DEGREE 3, THEN WHICH OF THE FOLLOWING STATEMENTS ARE NECESSARILY TRUE?

A $f + g$ IS OF DEGREE 3.	B $f + g$ IS OF DEGREE 6.
C $2f$ IS OF DEGREE 3.	D fg IS OF DEGREE 6.

Addition of polynomial functions

YOU CAN ADD POLYNOMIAL FUNCTIONS IN THE SAME WAY AS YOU ADD REAL NUMBERS. ADD THE LIKE TERMS BY ADDING THEIR COEFFICIENTS. NOTE THAT LIKE TERMS ARE TERMS WITH THE SAME VARIABLES TO THE SAME POWERS BUT POSSIBLY DIFFERENT COEFFICIENTS.

FOR EXAMPLE, IF $f(x) = 5x^4 - x^3 + 8x - 2$ AND $g(x) = 4x^3 - x^2 - 3x + 5$, THEN THE SUM OF $f(x)$ AND $g(x)$ IS THE POLYNOMIAL FUNCTION:

$$\begin{aligned} f(x) + g(x) &= (5x^4 - x^3 + 8x - 2) + (4x^3 - x^2 - 3x + 5) \\ &= 5x^4 + (-x^3 + 4x^3) - x^2 + (8x - 3x) + (-2 + 5) \dots \text{(grouping like terms)} \\ &= 5x^4 + (4 - 1)x^3 - x^2 + (8 - 3)x + (5 - 2) \dots \text{(adding their coefficients)} \\ &= 5x^4 + 3x^3 - x^2 + 5x + 3 \dots \text{(combining like terms).} \end{aligned}$$

THEREFORE, THE SUM $f(x) + g(x) = 5x^4 + 3x^3 - x^2 + 5x + 3$ IS A POLYNOMIAL OF DEGREE 4.

THE sum OF TWO POLYNOMIAL FUNCTIONS, WRITTEN AS, AND IS DEFINED AS:

$$f + g : (f + g)(x) = f(x) + g(x), \text{ FOR ALL } x.$$

EXAMPLE 4 IN EACH OF THE FOLLOWING, FIND THE SUM OF f

A $f(x) = x^3 + \frac{2}{3}x^2 - \frac{1}{2}x + 3$ AND $g(x) = -x^3 + \frac{1}{3}x^2 + x - 4$.

B $f(x) = 2x^5 + 3x^4 - 2\sqrt{2}x^3 + x - 5$ AND $g(x) = x^4 + \sqrt{2}x^3 + x^2 + 6x + 8$.

SOLUTION:

$$\begin{aligned} \text{A } f(x) + g(x) &= \left(x^3 + \frac{2}{3}x^2 - \frac{1}{2}x + 3\right) + \left(-x^3 + \frac{1}{3}x^2 + x - 4\right) \\ &= (x^3 - x^3) + \left(\frac{2}{3}x^2 + \frac{1}{3}x^2\right) + \left(-\frac{1}{2}x + x\right) + (3 - 4) \dots \text{(grouping like terms)} \\ &= (1 - 1)x^3 + \left(\frac{2}{3} + \frac{1}{3}\right)x^2 + \left(1 - \frac{1}{2}\right)x + (3 - 4) \dots \text{(adding their coefficients)} \\ &= x^2 + \frac{1}{2}x - 1 \dots \text{(combining like terms)} \end{aligned}$$

SO, $f(x) + g(x) = x^2 + \frac{1}{2}x - 1$, WHICH IS A POLYNOMIAL OF DEGREE 2.

$$\begin{aligned} \text{B } f(x) + g(x) &= (2x^5 + 3x^4 - 2\sqrt{2}x^3 + x - 5) + (x^4 + \sqrt{2}x^3 + x^2 + 6x + 8) \\ &= 2x^5 + (3x^4 + x^4) + (-2\sqrt{2}x^3 + \sqrt{2}x^3) + x^2 + (x + 6x) + (-5 + 8) \\ &= 2x^5 + (3 + 1)x^4 + (-2\sqrt{2} + \sqrt{2})x^3 + x^2 + (1 + 6)x + (8 - 5) \\ &= 2x^5 + 4x^4 - \sqrt{2}x^3 + x^2 + 7x + 3 \end{aligned}$$

SO, $f(x) + g(x) = 2x^5 + 4x^4 - \sqrt{2}x^3 + x^2 + 7x + 3$, WHICH IS A POLYNOMIAL FUNCTION OF DEGREE 5.

ACTIVITY 1.3



- 1** WHAT DO YOU OBSERVE IN EXAMPLE 4 ABOUT THE DEGREES OF $f(x)$ AND $g(x)$?
- 2** IS THE DEGREE OF $f(x)$ EQUAL TO THE DEGREE OF $g(x)$, WHICHEVER HAS THE HIGHEST DEGREE?
- 3** IF $f(x)$ AND $g(x)$ HAVE SAME DEGREE, THEN THE DEGREE MIGHT BE LOWER THAN THE DEGREE OR THE DEGREE WHICH PART OF EXAMPLE 4 ILLUSTRATES THIS SITUATION? WHY DOES THIS HAPPEN?
- 4** WHAT IS THE DOMAIN OF $f(x)$?

Subtraction of polynomial functions

TO SUBTRACT A POLYNOMIAL FROM A POLYNOMIAL, SUBTRACT THE COEFFICIENTS OF CORRESPONDING LIKE TERMS. SO, WHICHEVER TERM IS TO BE SUBTRACTED, ITS SIGN IS CHANGED AND THEN THE TERMS ARE ADDED.

FOR EXAMPLE, IF $f(x) = 2x^3 - 5x^2 + x - 7$ AND $g(x) = 8x^2 - x^3 + 4x + 5$, THEN THE DIFFERENCE OF $f(x)$ AND $g(x)$ IS THE POLYNOMIAL FUNCTION:

$$\begin{aligned}
 f(x) - g(x) &= (2x^3 - 5x^2 + x - 7) - (8x^2 - x^3 + 4x + 5) \\
 &= 2x^3 - 5x^2 + x - 7 - 8x^2 + x^3 - 4x - 5 \dots \dots \dots \text{(removing brackets)} \\
 &= (2 + 1)x^3 + (-5 - 8)x^2 + (1 - 4)x + (-7 - 5) \dots \dots \dots \text{(adding coefficients of like terms)} \\
 &= 3x^3 - 13x^2 - 3x - 12 \dots \dots \dots \text{(combining like terms)}
 \end{aligned}$$

THE difference OF TWO POLYNOMIAL FUNCTIONS IS WRITTEN AS AND IS DEFINED AS:

$$(f - g) : (f - g)(x) = f(x) - g(x), \text{ FOR ALL } x \in \mathbb{R}.$$

EXAMPLE 5 IN EACH OF THE FOLLOWING, FIND $f - g$

- A** $f(x) = x^4 + 3x^3 - x^2 + 4$ AND $g(x) = x^4 - x^3 + 5x^2 + 6x$
B $f(x) = x^5 + 2x^3 - 8x + 1$ AND $g(x) = x^3 + 2x^2 + 6x - 9$

SOLUTION:

$$\begin{aligned}
 \textbf{A} \quad f(x) - g(x) &= (x^4 + 3x^3 - x^2 + 4) - (x^4 - x^3 + 5x^2 + 6x) \\
 &= x^4 + 3x^3 - x^2 + 4 - x^4 + x^3 - 5x^2 - 6x \dots \dots \dots \text{(removing brackets)} \\
 &= (1 - 1)x^4 + (3 + 1)x^3 + (-1 - 5)x^2 - 6x + 4 \dots \dots \dots \text{(adding their coefficients)} \\
 &= 4x^3 - 6x^2 - 6x + 4 \dots \dots \dots \text{(combining like terms)}
 \end{aligned}$$

THEREFORE, THE DIFFERENCE IS A POLYNOMIAL FUNCTION OF DEGREE 3,

$$f(x) - g(x) = 4x^3 - 6x^2 - 6x + 4$$

B

$$\begin{aligned}
 f(x) - g(x) &= (x^5 + 2x^3 - 8x + 1) - (x^3 + 2x^2 + 6x - 9) \\
 &= x^5 + 2x^3 - 8x + 1 - x^3 - 2x^2 - 6x + 9 \\
 &= x^5 + (2x^3 - x^3) - 2x^2 + (-8x - 6x) + (1 + 9) \\
 &= x^5 + (2 - 1)x^3 - 2x^2 + (-8 - 6)x + (1 + 9) \\
 &= x^5 + x^3 - 2x^2 - 14x + 10
 \end{aligned}$$

THEREFORE THE DIFFERENCE $= x^5 + x^3 - 2x^2 - 14x + 10$, WHICH IS A POLYNOMIAL FUNCTION OF DEGREE 5.

NOTE THAT IF THE DEGREES OF f AND g ARE EQUAL TO THE DEGREE OF f , THEN THE DEGREE OF $f + g$ IS THE SAME AS THE DEGREE OF f OR THE DEGREE OF g , WHICHEVER HAS THE HIGHEST DEGREE. IF THEY HAVE THE SAME DEGREE, HOWEVER, THE DEGREE OF $f + g$ MIGHT BE LOWER THAN THIS COMMON DEGREE WHEN THEY HAVE THE SAME LEADING COEFFICIENT. **EXAMPLE 5** ILLUSTRATED IN

Multiplication of polynomial functions

TO MULTIPLY TWO POLYNOMIAL FUNCTIONS, MULTIPLY EACH TERM OF ONE BY EACH TERM OF THE OTHER, AND COLLECT LIKE TERMS.

FOR EXAMPLE, LET $f(x) = 2x^3 - x^2 + 3x - 2$ AND $g(x) = x^2 - 2x + 3$. THEN THE PRODUCT OF $f(x)$ AND $g(x)$ IS A POLYNOMIAL FUNCTION:

$$\begin{aligned}
 f(x) \cdot g(x) &= (2x^3 - x^2 + 3x - 2) \cdot (x^2 - 2x + 3) \\
 &= 2x^3(x^2 - 2x + 3) - x^2(x^2 - 2x + 3) + 3x(x^2 - 2x + 3) - 2(x^2 - 2x + 3) \\
 &= 2x^5 - 4x^4 + 6x^3 - x^4 + 2x^3 - 3x^2 + 3x^3 - 6x^2 + 9x - 2x^2 + 4x - 6 \\
 &= 2x^5 + (-4x^4 - x^4) + (6x^3 + 2x^3 + 3x^3) + (-3x^2 - 6x^2 - 2x^2) + (9x + 4x) - 6 \\
 &= 2x^5 - 5x^4 + 11x^3 - 11x^2 + 13x - 6
 \end{aligned}$$

THE product OF TWO POLYNOMIAL FUNCTIONS IS WRITTEN AS AND IS DEFINED AS:

$$f \cdot g : (f \cdot g)(x) = f(x) \cdot g(x), \text{ FOR ALL } \mathbb{R}.$$

EXAMPLE 6 IN EACH OF THE FOLLOWING, FIND AND GIVE THE DEGREE OF $f \cdot g$:

A

$$f(x) = \frac{3}{4}x^2 + \frac{9}{2}, \quad g(x) = 4x$$

B

$$f(x) = x^2 + 2x, \quad g(x) = x^5 + 4x^2 - 2$$

SOLUTION: A

$$f(x) \cdot g(x) = \left(\frac{3}{4}x^2 + \frac{9}{2} \right) \cdot (4x) = 3x^3 + 18x$$

SO, THE PRODUCT $f(x) \cdot g(x) = 3x^3 + 18x$ HAS DEGREE 3.

$$\begin{aligned}
 \mathbf{B} \quad f(x) \cdot g(x) &= (x^2 + 2x) \cdot (x^5 + 4x^2 - 2) \\
 &= x^2(x^5 + 4x^2 - 2) + 2x(x^5 + 4x^2 - 2) \\
 &= x^7 + 2x^6 + 4x^4 + 8x^3 - 2x^2 - 4x
 \end{aligned}$$

SO, THE PRODUCT $f(x)g(x) = x^7 + 2x^6 + 4x^4 + 8x^3 - 2x^2 - 4x$ HAS DEGREE 7.

IN EXAMPLE 6, YOU CAN SEE THAT THE DEGREE OF THE SUM OF THE DEGREES OF THE TWO POLYNOMIAL FUNCTIONS f AND g .

TO FIND THE PRODUCT OF TWO POLYNOMIAL FUNCTIONS, WE CAN ALSO USE A VERTICAL FOR MULTIPLICATION.

EXAMPLE 7 LET $f(x) = 3x^2 - 2x^3 + x^5 - 8x + 1$ AND $g(x) = 5 + 2x^2 + 8x$. FIND $f(x) \cdot g(x)$ AND THE DEGREE OF THE PRODUCT.

SOLUTION: TO FIND THE PRODUCT FIRST REARRANGE EACH POLYNOMIAL IN DESCENDING POWERS OF x AS FOLLOWS:

$$\begin{array}{r}
 x^5 - 2x^3 + 3x^2 - 8x + 1 \\
 2x^2 + 8x + 5 \\
 \hline
 5x^5 + 0x^4 - 10x^3 + 15x^2 - 40x + 5 \dots \dots \dots \text{(multiplying by 5)} \\
 8x^6 + 0x^5 - 16x^4 + 24x^3 - 64x^2 + 8x \dots \dots \dots \text{(multiplying by 8x)} \\
 2x^7 + 0x^6 - 4x^5 + 6x^4 - 16x^3 + 2x^2 \dots \dots \dots \text{(multiplying by } 2x^2\text{)} \\
 \hline
 2x^7 + 8x^6 + x^5 - 10x^4 - 2x^3 - 47x^2 - 32x + 5 \dots \dots \dots \text{(adding vertically.)}
 \end{array}$$

*Like terms are written
in the same column.*

THUS $f(x) \cdot g(x) = 2x^7 + 8x^6 + x^5 - 10x^4 - 2x^3 - 47x^2 - 32x + 5$ AND HENCE THE DEGREE IS 7.

ACTIVITY 1.4



- FOR ANY NON-ZERO POLYNOMIAL FUNCTION, IF THE DEGREE OF f IS m AND THE DEGREE OF g IS n , THEN WHAT IS THE DEGREE OF $f \cdot g$?
- IF EITHER f OR g IS THE ZERO POLYNOMIAL, WHAT IS THE DEGREE OF $f \cdot g$?
- IS THE PRODUCT OF TWO OR MORE POLYNOMIALS ALWAYS A POLYNOMIAL?

EXAMPLE 8 (*Application of polynomial functions*)

A PERSON WANTS TO MAKE AN OPEN BOX BY CUTTING EQUAL SQUARES FROM THE CORNERS OF A PIECE OF METAL 160 CM BY 240 CM AS SHOWN. IF THE EDGE OF EACH CUTOUT SQUARE IS x CM, FIND THE VOLUME OF THE BOX WHEN $x = 3$.

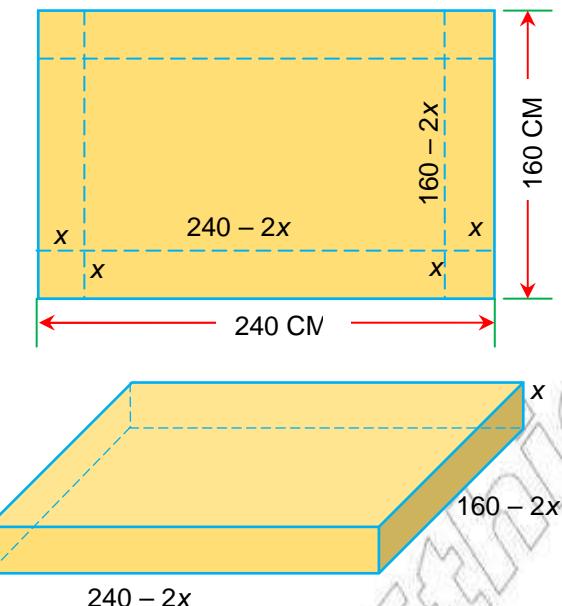


FIGURE 1.2

SOLUTION: THE VOLUME OF A RECTANGULAR BOX IS EQUAL TO THE PRODUCT OF ITS LENGTH, WIDTH AND HEIGHT. FROM THE FIGURE, THE LENGTH IS $240 - 2x$, THE WIDTH IS $160 - 2x$, AND THE HEIGHT IS x . THE VOLUME OF THE BOX IS

$$\begin{aligned} v(x) &= (240 - 2x)(160 - 2x)(x) \\ &= (38400 - 800x + 4x^2)(x) \\ &= 38400x - 800x^2 + 4x^3 \end{aligned}$$

(A POLYNOMIAL OF DEGREE 3)

WHEN $x = 1$, THE VOLUME OF THE BOX IS $38400 - 800 + 4 = 37604 \text{ CM}^3$

WHEN $x = 3$, THE VOLUME OF THE BOX IS

$$v(3) = 38400(3) - 800(3)^2 + 4(3)^3 = 115200 - 7200 + 108 = 108,108 \text{ CM}^3$$

Division of polynomial functions

IT IS POSSIBLE TO DIVIDE A POLYNOMIAL BY A POLYNOMIAL. THIS PROCESS IS SIMILAR TO THAT USED IN ARITHMETIC.

LOOK AT THE CALCULATIONS BELOW, WHERE 939 IS BEING DIVIDED BY 12.

$$\begin{array}{r} 7 \\ 12 \overline{)939} \\ 84 \\ \hline 99 \\ 84 \\ \hline 15 \\ 12 \\ \hline 3 \end{array}$$

THE SECOND DIVISION CAN BE EXPRESSED BY AN EQUATION WHICH SAYS NOTHING ABOUT THE DIVISION.

$$939 = (78 \times 12) + 3. \text{ OBSERVE THAT, } 939 = 78 + (3 \div 12) \text{ OR } \frac{939}{12} = 78 + \frac{3}{12}.$$

HERE 939 IS THE DIVIDEND, 12 IS THE DIVISOR, 78 IS THE QUOTIENT AND 3 IS THE REMAINDER. WHAT WE ACTUALLY DID IN THE ABOVE CALCULATION WAS TO CONTINUE THE DIVISION AS LONG AS THE QUOTIENT AND THE REMAINDER ARE INTEGERS AND THE REMAINDER IS LESS THAN THE DIVISOR.

ACTIVITY 1.5



- 1 CONSIDER THE FOLLOWING: $\frac{x^2 - x + 2}{x - 2} = x + 1 + \frac{4}{x - 2}$. WHICH POLYNOMIALS DO YOU THINK WE SHOULD CALL THE DIVISOR, DIVIDEND, QUOTIENT AND REMAINDER?
- 2 DIVIDE $x^3 + 1$ BY $x + 1$. (YOU SHOULD SEE THAT THE REMAINDER IS 0)
- 3 WHEN DO WE SAY THE DIVISION IS EXACT?
- 4 WHAT MUST BE TRUE ABOUT THE DEGREES OF THE DIVIDEND AND THE DIVISOR BEFORE WE CAN TRY TO DIVIDE POLYNOMIALS?
- 5 SUPPOSE THE DEGREE OF THE DIVIDEND IS n AND THE DEGREE OF THE DIVISOR IS m , $n > m$, THEN WHAT WILL BE THE DEGREE OF THE QUOTIENT?

WHEN SHOULD WE STOP DIVIDING ONE POLYNOMIAL BY ANOTHER? LOOK AT THE CALCULATIONS BELOW:

$$\begin{array}{r} x \\ \hline x+1 \end{array} \left[\begin{array}{r} x^2 + 3x + 5 \\ x^2 + x \\ \hline 2x + 5 \end{array} \right] \quad \begin{array}{r} x+2 \\ \hline x+1 \end{array} \left[\begin{array}{r} x^2 + 3x + 5 \\ x^2 + x \\ \hline 2x + 5 \\ 2x + 2 \\ \hline 3 \end{array} \right] \quad \begin{array}{r} x+2 + \frac{3}{x+1} \\ \hline x+1 \end{array} \left[\begin{array}{r} x^2 + 3x + 5 \\ x^2 + x \\ \hline 2x + 5 \\ 2x + 2 \\ \hline 3 \\ 3 \\ \hline 0 \end{array} \right]$$

THE FIRST DIVISION ABOVE TELLS US THAT

$$x^2 + 3x + 5 = x(x+1) + 2x + 5.$$

IT HOLDS TRUE FOR ALL VALUES OF x . MIDDLE ONE OF THE THREE DIVISIONS, YOU CONTINUED AS LONG AS YOU GOT A QUOTIENT AND REMAINDER WHICH ARE BOTH POLYNOMIALS.

WHEN YOU ARE ASKED TO DIVIDE ONE POLYNOMIAL BY ANOTHER, STOP THE DIVISION WHEN YOU GET A QUOTIENT AND REMAINDER THAT ARE POLYNOMIALS AND THE DEGREE OF THE REMAINDER IS LESS THAN THE DEGREE OF THE DIVISOR.

STUDY THE EXAMPLE BELOW TO DIVIDE $x^3 - 2x^2 + 4x + 7$ BY $x - 2$.

$$\begin{array}{r}
 \text{Divisor} \longrightarrow x - 2 \\
 \text{Quotient} \\
 \text{Dividend} \\
 \text{Subtract} \\
 \text{Multiply} \\
 \text{Subtract} \\
 \text{Multiply} \\
 \text{Subtract} \\
 \text{Remainder} \longrightarrow 19
 \end{array}
 \begin{array}{l}
 \text{Think } \frac{x^2}{x} = x \\
 \text{Think } \frac{6x}{x} = 6
 \end{array}$$

$$\begin{array}{r}
 \boxed{2x^3 - 3x^2 + 4x + 7} \\
 \underline{- (2x^3 - 4x^2)} \\
 \hline
 x^2 + 4x + 7 \\
 \underline{- (x^2 - 2x)} \\
 \hline
 6x + 7 \\
 \underline{- (6x - 12)} \\
 \hline
 19
 \end{array}$$

Note: Red annotations explain the steps:

- Divisor: $x - 2$
- Quotient: $\frac{x^2}{x} = x$
- Dividend: $2x^3 - 3x^2 + 4x + 7$
- Multiply: $2x^2(x - 2)$
- Subtract: $x^2 + 4x + 7 - (x^2 - 2x)$
- Multiply: $x(x - 2)$
- Subtract: $6x + 7 - (6x - 12)$
- Remainder: 19

SO, DIVIDING $2x^3 - 3x^2 + 4x + 7$ BY $x - 2$ GIVES A QUOTIENT OF $x^2 + 6$ AND A

$$\text{REMAINDER OF } 19. \text{ THAT IS, } \frac{2x^3 - 3x^2 + 4x + 7}{x - 2} = 2x^2 + x + 6 + \frac{19}{x - 2}$$

THE **quotient (division) OF TWO POLYNOMIAL FUNCTIONS** IS WRITTEN AS, AND IS DEFINED AS:

$$f \div g : (f \div g)(x) = f(x) \div g(x), \text{ PROVIDED THAT } g \neq 0, \text{ FOR ALL } x.$$

EXAMPLE 9 DIVIDE $x^4 - 3x^3 + 5$ BY $x - 3$

SOLUTION:

$$\begin{array}{r}
 2x^2 + 3x + 3 \\
 2x - 3 \overline{)4x^3 + 0x^2 - 3x + 5} \\
 \underline{- (4x^3 - 6x^2)} \\
 \hline
 6x^2 - 3x + 5 \\
 \underline{- (6x^2 - 9x)} \\
 \hline
 6x + 5 \\
 \underline{- (6x - 9)} \\
 \hline
 \text{REMAINDER} \longrightarrow 14
 \end{array}$$

$$\text{THEREFORE, } x^4 - 3x^3 + 5 = (2x^2 + 3x + 3)(x - 3) + 14$$

Arrange the dividend and the divisor in descending powers of x .

Insert (with 0 coefficients) for missing terms.

Divide the first term of the dividend by the first term of the divisor.

Multiply the divisor by $2x^2$, line up like terms and, subtract

Repeat the process until the degree of the remainder is less than that of the divisor.

EXAMPLE 10 FIND THE QUOTIENT AND REMAINDER WHEN $x^5 + 4x^3 - 6x^2 - 8$ IS DIVIDED BY $3x + 2$.

SOLUTION:

$$\begin{array}{r} x^3 - 3x^2 + 11x - 33 \\ \underline{x^2 + 3x + 2} \quad | \quad x^5 + 0x^4 + 4x^3 - 6x^2 + 0x - 8 \\ x^5 + 3x^4 + 2x^3 \\ \hline -3x^4 + 2x^3 - 6x^2 + 0x - 8 \\ -3x^4 - 9x^3 - 6x^2 \\ \hline 11x^3 + 0x^2 + 0x - 8 \\ 11x^3 + 33x^2 + 22x \\ \hline -33x^2 - 22x - 8 \\ -33x^2 - 99x - 66 \\ \hline 77x + 58 \end{array}$$

THEREFORE THE QUOTIENT IS $x^3 - 3x^2 + 11x - 33$ AND THE REMAINDER IS $77x + 58$.

WE CAN WRITE THE RESULT AS $\frac{x^5 + 4x^3 - 6x^2 - 8}{x^2 + 3x + 2} = x^3 - 3x^2 + 11x - 33 + \frac{77x + 58}{x^2 + 3x + 2}$.

Group Work 1.1



- 1 FIND TWO POLYNOMIAL FUNCTIONS BOTH OF DEGREE WITH $f(x) + g(x)$ OF DEGREE ONE. WHAT RELATIONS DO YOU NOTICE BETWEEN THE LEADING COEFFICIENTS OF $f(x)$ AND $g(x)$?
- 2 GIVEN $f(x) = x + 2$ AND $g(x) = ax + b$, FIND ALL VALUES OF a AND b SO THAT $f(x)$ IS A POLYNOMIAL FUNCTION.
- 3 GIVEN POLYNOMIAL FUNCTIONS $f(x) = x^2 + 3$, $q(x) = x^2 - 5$ AND $r(x) = 2x + 1$, FIND A FUNCTION $g(x)$ SUCH THAT $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$.

Exercise 1.3

- 1 WRITE EACH OF THE FOLLOWING EXPRESSIONS AS A POLYNOMIAL IN THE FORM $a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$.

A $(x^2 - x - 6) - (x + 2)$	B $(x^2 - x - 6)(x + 2)$
C $(x+2) - (x^2-x-6)$	D $\frac{x^2 - x - 6}{x + 2}$
E $\frac{x + 2}{x^2 - x - 6}$	F $(x^2 - x - 6)^2$
G $2^{x-3} + 2^3 - x$	H $(2x+3)^2$
I $(x^2 - x + 1)(x^2 - 3x + 5)$	J $(x^3 - x^4 + 2x + 1) - (x^4 + x^3 - 2x^2 + 8)$

- 2** LET f AND g BE POLYNOMIAL FUNCTIONS SUCH THAT $f(x) = x^2 - x + 3$. WHICH OF THE FOLLOWING FUNCTIONS ARE ALSO POLYNOMIAL FUNCTIONS?
- A** $f + g$ **B** $g - f$ **C** fg **D** $\frac{f}{g}$
E $f^2 - g$ **F** $2f + 3g$ **G** $\sqrt{f^2}$
- 3** IF f AND g ARE ANY TWO POLYNOMIAL FUNCTIONS, WHICH OF THE FOLLOWING WILL A POLYNOMIAL FUNCTION?
- A** $f + g$ **B** $f - g$ **C** fg **D** $\frac{f}{g}$
E f^2 **F** $\frac{3}{4}g - \frac{1}{3}f$ **G** $\frac{f - g}{f + g}$
- 4** IN EACH OF THE FOLLOWING, FIND f AND g AND GIVE THE DEGREE OF g , THE DEGREE OF f AND THE DEGREE OF $f -$
- A** $f(x) = 3x - \frac{2}{3}; g(x) = 2x + 5$
B $f(x) = -7x^2 + x - 8; g(x) = 2x^2 - x + 1$
C $f(x) = 1 - x^3 + 6x^2 - 8x; g(x) = x^3 + 10$
- 5** IN EACH OF THE FOLLOWING,
- I** FIND THE FUNCTION
II GIVE THE DEGREE OF THE DEGREE OF
III GIVE THE DEGREE OF f .
- A** $f(x) = 2x + 1; g(x) = 3x - 5$
B $f(x) = x^2 - 3x + 5; g(x) = 5x + 3$
C $f(x) = 2x^3 - x - 7; g(x) = x^2 + 2x$
D $f(x) = 0; g(x) = x^3 - 8x^2 + 9$
- 6** IN EACH OF THE FOLLOWING, DIVIDE THE FIRST BY THE SECOND:
- A** $x^3 - 1; x - 1$ **B** $x^3 + 1; x^2 - x + 1$
C $x^4 - 1; x^2 + 1$ **D** $x^5 + 1; x + 1$
E $2x^5 - x^6 + 2x^3 + 6; x^3 - x - 2$
- 7** FOR EACH OF THE FOLLOWING, FIND THE QUOTIENT AND REMAINDER:
- A** $(5 - 6x + 8x^2) \div (x - 1)$ **B** $(x^3 - 1) \div (x - 1)$
C $(3y - y^2 + 2y^3 - 1) \div (y^2 + 1)$ **D** $(3x^4 + 2x^3 - 4x - 1) \div (x + 3)$
E $(3x^3 - x^2 + x + 2) \div \left(x + \frac{2}{3}\right)$



1.2**THEOREMS ON POLYNOMIALS****1.2.1 Polynomial Division Theorem**

RECALL THAT, WHEN WE DIVIDED ONE POLYNOMIAL BY ANOTHER, WE APPLY THE LONG DIVISION PROCEDURE, UNTIL THE REMAINDER WAS EITHER THE ZERO POLYNOMIAL OR A POLYNOMIAL OF DEGREE THAN THE DIVISOR.

FOR EXAMPLE, IF WE DIVIDE $x^2 + 3x + 7$ BY $x + 1$, WE OBTAIN THE FOLLOWING.

$$\begin{array}{r}
 & x+2 \leftarrow \text{quotient} \\
 \text{Divisor} \longrightarrow & x+1 \overline{)x^2 + 3x + 7} \leftarrow \text{dividend} \\
 & x^2 + x \\
 \hline
 & 2x + 7 \\
 & 2x + 2 \\
 \hline
 & 5 \leftarrow \text{remainder}
 \end{array}$$

IN FRACTIONAL FORM, WE CAN WRITE THIS RESULT AS FOLLO

$$\frac{\text{dividend}}{\text{divisor}} = \frac{x^2 + 3x + 7}{x+1} = x+2 + \frac{5}{x+1} \quad \begin{matrix} \text{quotient} \\ \downarrow \\ \text{divisor} \end{matrix} \quad \begin{matrix} \text{remainder} \\ \downarrow \\ \text{divisor} \end{matrix}$$

THIS IMPLIES THAT $x^2 + 3x + 7 = (x + 1)(x + 2) + 5$ WHICH ILLUSTRATES THE THEOREM CALLED THE **polynomial division theorem**.

ACTIVITY 1.6

- 1** FOR EACH OF THE FOLLOWING PAIRS OF POLYNOMIALS, FIND THE QUOTIENT $q(x)$ AND THE REMAINDER $r(x)$ THAT SATISFY $f(x) = q(x)d(x) + r(x)$.
 - A** $f(x) = x^2 + x - 7$; $d(x) = x - 3$
 - B** $f(x) = x^3 - x^2 + 8$; $d(x) = x + 2$
 - C** $f(x) = x^4 - x^3 + x - 1$; $d(x) = x - 1$
- 2** IN QUESTION 1, WHAT DID YOU OBSERVE ABOUT THE DEGREES OF THE POLYNOMIALS $f(x)$ AND $d(x)$?
- 3** IN QUESTION 1, THE FRACTIONAL EXPRESSION $\frac{f(x)}{d(x)}$ IS IMPROPER. WHY?
- 4** IS $\frac{r(x)}{d(x)}$ PROPER OR IMPROPER? WHAT CAN YOU SAY ABOUT THE DEGREE OF



Theorem 1.1 Polynomial division theorem

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x) q(x) + r(x)$$

Dividend Quotient Divisor Remainder

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, $f(x)$ divides exactly into $d(x)$.

Proof:-

I Existence of the polynomials $q(x)$ and $r(x)$

SINCE (x) AND (x) ARE POLYNOMIALS, LONG DIVISION WILL GIVE A QUOTIENT AND REMAINDER WITH DEGREE OF DEGREE OF $(x) = 0$.

II The uniqueness of $q(x)$ and $r(x)$

TO SHOW THE UNIQUENESS OF (x) , SUPPOSE THAT

$$f(x) = d(x)q_1(x) + r_1(x) \text{ AND ALSO}$$

$$f(x) = d(x)q_2(x) + r_2(x) \text{ WITH } \deg r_1 < \deg d(x) \text{ AND } \deg r_2 < \deg d(x).$$

$$\text{THEN } r_2(x) = f(x) - d(x)q_2(x) \text{ AND } r_1(x) = f(x) - d(x)q_1(x)$$

$$\Rightarrow r_2(x) - r_1(x) = d(x)[q_1(x) - q_2(x)]$$

THEREFORE (x) IS A FACTOR OF $-r_1(x)$

AS $\deg [(x) - r_1(x)] \leq \max\{\deg q_1(x), \deg r_1(x)\} < \deg d(x)$ IT FOLLOWS THAT,

$$r_2(x) - r_1(x) = 0$$

AS A RESULT $r_2(x) = r_1(x)$ AND $q_1(x) = q_2(x)$.

THEREFORE (x) AND (x) ARE UNIQUE POLYNOMIAL FUNCTIONS.

EXAMPLE 1 IN EACH OF THE FOLLOWING PAIRS OF POLYNOMIALS, FIND
A POLYNOMIAL $q(x)$ AND A REMAINDER $r(x)$ SUCH THAT $f(x) = d(x)q(x) + r(x)$.

- A $f(x) = 2x^3 - 3x + 1; d(x) = x + 2$
- B $f(x) = x^3 - 2x^2 + x + 5; d(x) = x^2 + 1$
- C $f(x) = x^4 + x^2 - 2; d(x) = x^2 - x + 3$

SOLUTION:

A $\frac{f(x)}{d(x)} = \frac{2x^3 - 3x + 1}{x + 2} = 2x^2 - 4x + 5 - \frac{9}{x + 2}$

$$\Rightarrow 2x^3 - 3x + 1 = (x+2)(2x^2 - 4x + 5) - 9$$

THEREFORE $q(x) = 2x^2 - 4x + 5$ AND $r(x) = -9$.

B $\frac{f(x)}{d(x)} = \frac{x^3 - 2x^2 + x + 5}{x^2 + 1} = x - 2 + \frac{7}{x^2 + 1}$

$$\Rightarrow x^3 - 2x^2 + x + 5 = (x^2 + 1)(x - 2) + 7$$

THEREFORE $q(x) = x - 2$ AND $r(x) = 7$.

C $\frac{f(x)}{d(x)} = \frac{x^4 + x^2 - 2}{x^2 - x + 3} = x^2 + x - 1 + \frac{-4x + 1}{x^2 - x + 3}$

$$\Rightarrow x^4 + x^2 - 2 = (x^2 - x + 3)(x^2 + x - 1) + (-4x + 1)$$

GIVING $q(x) = x^2 + x - 1$ AND $r(x) = -4x + 1$.

Exercise 1.4

- 1 FOR EACH OF THE FOLLOWING PAIRS OF POLYNOMIALS, FIND
THE QUOTIENT AND
REMAINDER THAT SATISFY THE REQUIREMENTS OF THE POLYNOMIAL DIVISION THEOREM.

 - A $f(x) = x^2 - x + 7; d(x) = x + 1$
 - B $f(x) = x^3 + 2x^2 - 5x + 3; d(x) = x^2 + x - 1$
 - C $f(x) = x^2 + 8x - 12; d(x) = 2$

- 2 IN EACH OF THE FOLLOWING, EXPRESS THE FUNCTION $f(x) = (x - c)q(x) + r(x)$ FOR THE GIVEN NUMBER c

$$f(x) = (x - c)q(x) + r(x) \text{ FOR THE GIVEN NUMBER } c$$

A $f(x) = x^3 - 5x^2 - x + 8; c = -2$ B $f(x) = x^3 + 2x^2 - 2x - 14; c = \frac{1}{2}$

- 3 PERFORM THE FOLLOWING DIVISIONS, ASSUMING THAT a IS AN INTEGER:

A
$$\frac{x^{3n} + 5x^{2n} + 12x^n + 18}{x^n + 3}$$

B
$$\frac{x^{3n} - x^{2n} + 3x^n - 10}{x^n - 2}$$

1.2.2 Remainder Theorem

THE EQUALITY $f(x) = d(x)q(x) + r(x)$ EXPRESSES THE FACT THAT

Dividend = (divisor) (quotient) + remainder.

ACTIVITY 1.7

- 1 LET $f(x) = x^4 - x^3 - x^2 - x - 2$.
 - A FIND $f(-2)$ AND $f(2)$.
 - B WHAT IS THE REMAINDER DIVIDED BY 2?
 - C IS THE REMAINDER EQUAL TO $f(-2)$?
 - D WHAT IS THE REMAINDER DIVIDED BY 2?
 - E IS THE REMAINDER EQUAL TO $f(2)$?
- 2 IN EACH OF THE FOLLOWING, FIND THE REMAINDER WHEN THE POLYNOMIAL IS DIVIDED BY THE POLYNOMIAL. FOR THE GIVEN NUMBER, FIND $f(c)$.

A $f(x) = 2x^2 + 3x + 1; c = -1$	B $f(x) = x^6 + 1; c = -1, 1$
C $f(x) = 3x^3 - x^4 + 2; c = 2$	D $f(x) = x^3 - x + 1; c = -1, 1$



Theorem 1.2 Remainder theorem

Let $f(x)$ be a polynomial of degree greater than or equal to 1 and let c be any real number. If $f(x)$ is divided by the linear polynomial $(x - c)$, then the remainder is $f(c)$.

Proof:-

WHEN $(x - c)$ IS DIVIDED BY c , THE REMAINDER IS ALWAYS A CONSTANT. WHY?
BY THE POLYNOMIAL DIVISION THEOREM

$$f(x) = (x - c)q(x) + k$$

WHERE k IS CONSTANT. THIS EQUATION HOLDS FOR EVERY REAL NUMBERS WHEN $x = c$.

IN PARTICULAR, IF YOU OBSERVE A VERY INTERESTING AND USEFUL RELATIONSHIP

$$\begin{aligned} f(c) &= (c - c)q(c) + k \\ &= 0 \cdot q(c) + k \\ &= 0 + k = k \end{aligned}$$

IT FOLLOWS THAT THE VALUE OF THE POLYNOMIAL IS THE SAME AS THE REMAINDER OBTAINED WHEN YOU DIVIDE c .

EXAMPLE 2 FIND THE REMAINDER BY DIVIDING IN EACH OF THE FOLLOWING PAIRS OF POLYNOMIALS, USING THE POLYNOMIAL DIVISION AND THE REMAINDER THEOREM

A $f(x) = x^3 - x^2 + 8x - 1; d(x) = x + 2$

B $f(x) = x^4 + x^2 + 2x + 5; d(x) = x - 1$

SOLUTION:

A Polynomial division theorem

$$\begin{array}{r} x^3 - x^2 + 8x - 1 \\ \hline x + 2 \\ = x^2 - 3x + 14 - \frac{29}{x+2} \end{array}$$

Remainder theorem

$$\begin{aligned} f(-2) &= (-2)^3 - (-2)^2 + 8(-2) - 1, \\ &= -8 - 4 - 16 - 1 = -29 \end{aligned}$$

THEREFORE, THE REMAINDER IS -29.

B Polynomial division theorem

$$\begin{array}{r} x^4 + x^2 + 2x + 5 \\ \hline x - 1 \\ = x^3 + x^2 + 2x + 4 + \frac{9}{x-1} \end{array}$$

Remainder theorem

$$\begin{aligned} f(1) &= (1)^4 + (1)^2 + 2(1) + 5 \\ &= 1 + 1 + 2 + 5 = 9 \end{aligned}$$

THEREFORE, THE REMAINDER IS 9.

EXAMPLE 3 WHEN $x^3 - 2x^2 + 3bx + 10$ IS DIVIDED BY $3x$ THE REMAINDER IS 37. FIND THE VALUE OF

SOLUTION: LET $f(x) = x^3 - 2x^2 + 3bx + 10$.

$$f(3) = 37. \text{ (BY THE REMAINDER THEOREM)}$$

$$\Rightarrow (3)^3 - 2(3)^2 + 3b(3) + 10 = 37$$

$$27 - 18 + 9b + 10 = 37 \Rightarrow 9b + 19 = 37 \Rightarrow b = 2.$$

Exercise 1.5

1 IN EACH OF THE FOLLOWING, EXPRESS THE EQUATION IN

$$f(x) = (x - c) q(x) + r(x)$$

FOR THE GIVEN NUMBER SHOW THAT r IS THE REMAINDER.

A $f(x) = x^3 - x^2 + 7x + 11; c = 2$

B $f(x) = 1 - x^5 + 2x^3 + x; c = -1$

C $f(x) = x^4 + 2x^3 + 5x^2 + 1; c = -\frac{2}{3}$

- 2** IN EACH OF THE FOLLOWING, USE THE REMAINDER THEOREM TO FIND THE REMAINDER WHEN THE POLYNOMIAL IS DIVIDED BY c FOR THE GIVEN NUMBER c

A $f(x) = x^{17} - 1; c = 1$

B $f(x) = 2x^2 + 3x + 1; c = -\frac{1}{2}$

C $f(x) = x^{23} + 1; c = -1$

- 3** WHEN $f(x) = 3x^7 - ax^6 + 5x^3 - x + 11$ IS DIVIDED BY 1, THE REMAINDER IS 15. WHAT IS THE VALUE OF a

- 4** WHEN THE POLYNOMIAL $ax^3 + bx^2 - 2x + 8$ IS DIVIDED BY 1 AND + 1 THE REMAINDERS ARE 3 AND 5 RESPECTIVELY. FIND THE VALUES OF

1.2.3 Factor Theorem

RECALL THAT **factoring** A POLYNOMIAL MEANS WRITING IT AS A PRODUCT OF TWO OR POLYNOMIALS. YOU WILL DISCUSS BELOW AN INTERESTING THEOREM, KNOWN AS **Factor theorem**, WHICH IS HELPFUL IN CHECKING WHETHER A LINEAR EXPRESSION IS A FACTOR OF A GIVEN POLYNOMIAL OR NOT.

ACTIVITY 1.8



- 1** LET $f(x) = x^3 - 5x^2 + 2x + 8$.
 - A** FIND $f(2)$.
 - B** WHAT IS THE REMAINDER WHEN $f(x)$ IS DIVIDED BY 2?
 - C** IS $x - 2$ A FACTOR OF $f(x)$?
 - D** FIND $f(-1)$ AND $f(1)$.
 - E** EXPRESS $f(x)$ AS $f(x) = (x - C)q(x)$ WHERE $q(x)$ IS THE QUOTIENT.
- 2** LET $f(x) = x^3 - 3x^2 - x + 3$.
 - A** WHAT ARE THE VALUES OF $f(1)$ AND $f(3)$?
 - B** WHAT DOES THIS TELL US ABOUT THE REMAINDER WHEN $f(x)$ IS DIVIDED BY $x - 1$ AND $x - 3$?
 - C** HOW CAN THIS HELP US IN FACTORIZING?

Theorem 1.3 Factor theorem

Let $f(x)$ be a polynomial of degree greater than or equal to one, and let c be any real number, then

- I** $x - c$ is a factor of $f(x)$, if $f(c) = 0$, and
- II** $f(c) = 0$, if $x - c$ is a factor of $f(x)$.

TRY TO DEVELOP A PROOF OF THIS THEOREM USING THE REMAINDER THEOREM

Group Work 1.2



- 1 LET $f(x) = 4x^4 - 5x^2 + 1$.
 - A FIND -1 AND SHOW THAT IS A FACTOR OF f
 - B SHOW THAT 2 IS A FACTOR OF f
 - C TRY TO COMPLETELY FACTORIZE LINEAR FACTORS.
- 2 GIVE THE PROOF OF THE FACTOR THEOREM

Hint: YOU HAVE TO PROVE THAT

- I IF $f(c) = 0$, THEN $x - c$ IS A FACTOR OF $f(x)$
- II IF $x - c$ IS A FACTOR OF $f(x)$, THEN $f(c) = 0$

USE THE POLYNOMIAL DIVISION THEOREM WITH FACTOR $(x - c)$ TO EXPRESS $f(x)$ AS

$$f(x) = d(x)q(x) + r(x), \text{ WHERE } d(x) = x - c.$$

USE THE REMAINDER THEOREM $(x) = k = f(c)$, GIVING YOU

$$f(x) = (x - c)q(x) + f(c)$$

WHERE $q(x)$ IS A POLYNOMIAL OF DEGREE LESS THAN THE DEGREE OF $f(x)$. IF $f(c) = 0$, THEN WHAT WILL $r(x)$ BE? COMPLETE THE PROOF.

EXAMPLE 4 LET $f(x) = x^3 + 2x^2 - 5x - 6$. USE THE FACTOR THEOREM TO DETERMINE WHETHER:

- A $x + 1$ IS A FACTOR OF f B $x + 2$ IS A FACTOR OF f

SOLUTION:

- A SINCE $x + 1 = x - (-1)$, IT HAS THE FORM WITH $c = -1$.

$$f(-1) = (-1)^3 + 2(-1)^2 - 5(-1) - 6 = -1 + 2 + 5 - 6 = 0.$$

SO, BY THE FACTOR THEOREM $x + 1$ IS A FACTOR OF f .

- B $f(-2) = (-2)^3 + 2(-2)^2 - 5(-2) - 6 = -8 + 8 + 10 - 6 = 4 \neq 0$.

BY THE FACTOR THEOREM $x + 2$ IS NOT A FACTOR OF f .

EXAMPLE 5 SHOW THAT $x - 2$ AND $x + 3$ ARE FACTORS AND NOT A FACTOR OF $f(x) = x^4 + x^3 - 7x^2 - x + 6$.

SOLUTION: $f(-3) = (-3)^4 + (-3)^3 - 7(-3)^2 - (-3) + 6 = 81 - 27 - 63 + 3 + 6 = 0$.

HENCE $x + 3$ IS A FACTOR OF f .

$$f(2) = 2^4 + (2)^3 - 7(2)^2 - 2 + 6 = 16 + 8 - 28 - 2 + 6 = 0.$$

HENCE -2 IS A FACTOR OF $f(x)$

$$f(-1) = (-1)^4 + (-1)^3 - 7(-1)^2 - (-1) + 6 = 1 - 1 - 7 + 1 + 6 = 0$$

HENCE $+1$ IS A FACTOR OF $f(x)$

$$f(-2) = (-2)^4 + (-2)^3 - 7(-2)^2 - (-2) + 6 = 16 - 8 - 28 + 2 + 6 = -12 \neq 0$$

HENCE $+2$ IS NOT A FACTOR OF $f(x)$

Exercise 1.6

- 1** IN EACH OF THE FOLLOWING, USE THE TEST TO DETERMINE WHETHER OR NOT $x - 2$ IS A FACTOR OF $f(x)$.

A $g(x) = x + 1; f(x) = x^5 + 1$

B $g(x) = x - 1; f(x) = x^7 + x - 1$

C $g(x) = x - \frac{3}{2}; f(x) = 6x^2 + x - 1$

D $g(x) = x + 2; f(x) = x^3 - 3x^2 - 4x - 12$

- 2** IN EACH OF THE FOLLOWING, FIND A NUMBER k THAT SATISFIES THE GIVEN CONDITION:

A $x - 2$ IS A FACTOR OF $8x^2 - kx + 6$

B $x + 3$ IS A FACTOR OF $x^4 - 6x^3 - x^2 + 4x + 29$

C $3x - 2$ IS A FACTOR OF $4x^2 + kx - k$

- 3** FIND NUMBERS a AND k SO THAT 2 IS A FACTOR OF $f(x) = x^4 - 2ax^3 + ax^2 - x + k$ AND $f(-1) = 3$.

- 4** FIND A POLYNOMIAL FUNCTION OF DEGREE 3 SUCH THAT, x AND $+2$ ARE FACTORS OF THE POLYNOMIAL.

- 5** LET a BE A REAL NUMBER A POSITIVE INTEGER. SHOW THAT $x - a$ IS A FACTOR OF $x^3 - 2ax^2 + a^2x - a^3$.

- 6** SHOW THAT k AND $+1$ ARE FACTORS AND NOT A FACTOR OF $2x^2 + 2x + 1$.

- 7** IN EACH OF THE FOLLOWING, FIND THE CONDS THAT THE DENOMINATOR WILL DIVIDE THE NUMERATOR EXACTLY:

A $\frac{x^3 + 3x^2 - 3x + c}{x - 3}$

B $\frac{x^3 - 2x^2 + x + c}{x + 2}$

- 8** THE AREA OF A RECTANGLE IS $3\sqrt{13}$ FEET. HOW MUCH LONGER IS THE LENGTH THAN THE WIDTH OF THE RECTANGLE?

1.3**ZEROS OF A POLYNOMIAL FUNCTION**

IN THIS SECTION, YOU WILL DISCUSS AN INTERESTING CONCEPT KNOWN AS CONSIDER THE POLYNOMIAL FUNCTION

WHAT IS IT? NOTE THAT $f(1) = 1 - 1 = 0$.

AS $f(1) = 0$, WE SAY THAT 1 IS THE ZERO OF THE POLYNOMIAL FUNCTION

TO FIND THE ZERO OF A LINEAR (FIRST DEGREE POLYNOMIAL) FUNCTION OF THE FORM $a \neq 0$, WE FIND THE NUMBER WHICH $b = 0$.

NOTE THAT EVERY LINEAR FUNCTION HAS EXACTLY ONE ZERO

$$ax + b = 0 \Rightarrow ax = -b \quad \dots \dots \text{ Subtracting } b \text{ from both sides}$$

$$\Rightarrow x = -\frac{b}{a} \quad \dots \dots \text{ Dividing both sides by } a, \text{ since } a \neq 0.$$

THEREFORE, $-\frac{b}{a}$ IS THE ONLY ZERO OF THE LINEAR FUNCTION.

EXAMPLE 1 FIND THE ZEROS OF THE POLYNOMIAL $\frac{2x-1}{3} - \frac{x+2}{3} - 2$.

SOLUTION: $f(x) = 0 \Rightarrow \frac{2x-1}{3} - \frac{x+2}{3} = 2$

$$2x - 1 - (x + 2) = 6 \Rightarrow 2x - 1 - x - 2 = 6 \Rightarrow x = 9.$$

SO, 9 IS THE ZERO OF

SIMILARLY, TO FIND THE ZEROS OF A QUADRATIC FUNCTION (SECOND DEGREE POLYNOMIAL) $f(x) = ax^2 + bx + c$, $a \neq 0$, WE FIND THE NUMBER WHICH

$$ax^2 + bx + c = 0, a \neq 0.$$

ACTIVITY 1.9

- 1 FIND THE ZEROS OF EACH OF THE FOLLOWING FUNCTIONS

A $h(x) = 1 - \frac{3}{5}(x + 2)$	B $k(x) = 2 - (x^2 - 4) + x^2 - 4x$
C $f(x) = 4x^2 - 25$	D $f(x) = x^2 + x - 12$
E $f(x) = x^3 - 2x^2 + x$	F $g(x) = x^3 + x^2 - x - 1$
- 2 HOW MANY ZEROS CAN A QUADRATIC FUNCTION HAVE?
- 3 STATE TECHNIQUES FOR FINDING ZEROS OF A QUADRATIC FUNCTION
- 4 HOW MANY ZEROS CAN A POLYNOMIAL FUNCTION HAVE ABOUT DEGREE 4?

EXAMPLE 2 FIND THE ZEROS OF EACH OF THE FOLLOWING QUADRATIC FUNCTIONS

A $f(x) = x^2 - 16$ **B** $g(x) = x^2 - x - 6$ **C** $h(x) = 4x^2 - 7x + 3$

SOLUTION:

A $f(x) = 0 \Rightarrow x^2 - 16 = 0 \Rightarrow x^2 - 4^2 = 0 \Rightarrow (x - 4)(x + 4) = 0$
 $\Rightarrow x - 4 = 0 \text{ OR } x + 4 = 0 \Rightarrow x = 4 \text{ OR } x = -4$

THEREFORE, -4 AND 4 ARE THE ZEROS OF

B $g(x) = 0 \Rightarrow x^2 - x - 6 = 0$

FIND TWO NUMBERS WHOSE SUM IS -1 AND WHOSE PRODUCT IS -6 . THESE ARE -3 AND 2
 $x^2 - 3x + 2x - 6 = 0 \Rightarrow x(x - 3) + 2(x - 3) = 0 \Rightarrow (x + 2)(x - 3) = 0$

$$\Rightarrow x + 2 = 0 \text{ OR } x - 3 = 0 \Rightarrow x = -2 \text{ OR } x = 3$$

THEREFORE, -2 AND 3 ARE THE ZEROS OF g

C $h(x) = 0 \Rightarrow 4x^2 - 7x + 3 = 0$

FIND TWO NUMBERS WHOSE SUM IS -7 AND WHOSE PRODUCT IS 12 . THESE ARE -4 AND -3
HENCE, $4x^2 - 7x + 3 = 0 \Rightarrow 4x^2 - 4x - 3x + 3 = 0 \Rightarrow 4x(x - 1) - 3(x - 1) = 0$

$$\Rightarrow (4x - 3)(x - 1) = 0 \Rightarrow 4x - 3 = 0 \text{ OR } x - 1 = 0 \Rightarrow x = \frac{3}{4} \text{ OR } x = 1.$$

THEREFORE, $\frac{3}{4}$ AND 1 ARE THE ZEROS OF h

Definition 1.2

For a polynomial function f and a real number c , if

$f(c) = 0$, then c is a **zero** of f .

NOTE THAT IF $x - c$ IS A FACTOR OF $f(x)$, THEN c IS A ZERO OF $f(x)$.

EXAMPLE 3

- A USE THE FACTOR THEOREM TO SHOW THAT $x + 1$ IS A FACTOR OF $f(x) = x^{25} + 1$.
- B WHAT ARE THE ZEROS OF $f(x) = (x - 5)(x + 2)(x - 1)$?
- C WHAT ARE THE REAL ZEROS OF $f(x) = x^4 - 3x^2 + 1$?
- D DETERMINE THE ZEROS OF $f(x) = x^4 - 3x^2 + 1$.

SOLUTION:

- A SINCE $x + 1 = x - (-1)$, WE HAVE -1 AS A ZERO OF $f(x)$.

$$f(-1) = (-1)^{25} + 1 = -1 + 1 = 0$$

HENCE, -1 IS A ZERO OF $f(x) = x^{25} + 1$, BY THE FACTOR THEOREM.

SO $x - (-1) = x + 1$ IS A FACTOR OF $f(x)$.

- B SINCE $x - 5$, $x + 2$ AND $x - 1$ ARE ALL FACTORS OF $f(x)$, -5 , -2 AND 1 ARE THE ZEROS OF $f(x)$.

C FACTORISING THE LEFT SIDE, WE HAVE

$$x^4 - 1 = 0 \Rightarrow (x^2 - 1)(x^2 + 1) = 0 \Rightarrow (x - 1)(x + 1)(x^2 + 1) = 0$$

SQ THE REAL ZEROS OF $x^4 - 1$ ARE -1 AND 1 .

D $f(x) = 0 \Rightarrow 2x^4 - 3x^2 + 1 = 0 \Rightarrow 2(x^2)^2 - 3x^2 + 1 = 0$

LET $y = x^2$. THEN $2y^2 - 3y + 1 = 0 \Rightarrow 2y^2 - 3y + 1 = 0 \Rightarrow (2y - 1)(y - 1) = 0$
 $\Rightarrow 2y - 1 = 0 \text{ OR } y - 1 = 0$

HENCE $y = \frac{1}{2}$ OR $y = 1$

SINCE $y = x^2$, WE HAVE $x^2 = \frac{1}{2}$ OR $x^2 = 1$.

THEREFORE $\pm \sqrt{\frac{1}{2}}$ OR $x = \pm 1$. (Note that $\sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$.)

HENCE, $\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}, -1$ AND 1 ARE ZEROS OF

A POLYNOMIAL FUNCTION CANNOT HAVE MORE ZEROS THAN ITS DEGREE.

1.3.1 Zeros and Their Multiplicities

IF $f(x)$ IS A POLYNOMIAL FUNCTION OF DEGREE n , **THEOREM** OF THE EQUATION

$f(x) = 0$ IS CALLED **ZERO** OF f .

BY THE **FACTOR THEOREM**, EACH ZERO OF A POLYNOMIAL FUNCTION GENERATES A FIRST DEGREE FACTOR (c) OF $f(x)$. WHEN $f(x)$ IS FACTORIZED COMPLETELY, THE SAME FACTOR (c) OCCUR MORE THAN ONCE, IN WHICH CASE IT IS **REPEATED** OR **multiple zero** OF $f(x)$. IF $x - c$ OCCURS ONLY ONCE, IT IS **simple zero** OF $f(x)$.

Definition 1.3

If $(x - c)^k$ is a factor of $f(x)$, but $(x - c)^{k+1}$ is not, then c is said to be a **zero of multiplicity k** of f .

EXAMPLE 4 GIVEN THAT -1 AND 2 ARE ZEROS OF $x^3 - 3x^2 - 5x - 2$, DETERMINE THEIR MULTIPLICITY.

SOLUTION: BY THE **FACTOR THEOREM**, $(x + 1)$ AND $(x - 2)$ ARE FACTORS OF

HENCE, $f(x)$ CAN BE DIVIDED BY $((x - 2))$, GIVING YOU

$$f(x) = (x^2 - x - 2)(x^2 + 2x + 1) = (x + 1)(x - 2)(x + 1)^2 = (x + 1)^3(x - 2)$$

THEREFORE, -1 IS A ZERO OF MULTIPLICITY 3 AND 2 IS A ZERO OF MULTIPLICITY 1 .

Exercise 1.7

1 FIND THE ZEROS OF EACH OF THE FOLLOWING FUNCTIONS:

A $f(x) = 1 - \frac{3}{5}x$

B $f(x) = \frac{1}{4}(1 - 2x) - (x + 3)$

C $g(x) = \frac{2}{3}(2 - 3x)(x - 2)(x + 1)$

D $h(x) = x^4 + 7x^2 + 12$

E $g(x) = x^3 + x^2 - 2$

F $f(t) = t^3 - 7t + 6$

G $f(y) = y^5 - 2y^3 + y$

H $f(x) = 6x^4 - 7x^2 - 3$

2 FOR EACH OF THE FOLLOWING, LIST THE ZEROS OF THE FUNCTION AND STATE THE MULTIPLICITY OF EACH ZERO.

A $f(x) = x^{12}\left(x - \frac{2}{3}\right)$

B $g(x) = 3(x - \sqrt{2})^2(x + 1)$

C $h(x) = 3x^6(-x)^5(x - (-+1))^3$

D $f(x) = 2(x - \sqrt{3})^5(x + 5)^9(1 - 3x)$

E $f(x) = x^3 - 3x^2 + 3x - 1$

3 FIND A POLYNOMIAL FUNCTION OF DEGREE 3 SUCH THAT 17 AND THE ZEROS OF ARE 0, 5 AND 8.

4 IN EACH OF THE FOLLOWING, THE INDICATE THE NUMBER OF POLYNOMIAL FUNCTION $f(x)$. DETERMINE THE MULTIPLICITY OF THIS ZERO.

A 1; $f(x) = x^3 + x^2 - 5x + 3$

B -1; $f(x) = x^4 + 3x^3 + 3x^2 + x$

C $\frac{1}{2}$; $f(x) = 4x^3 - 4x^2 + x$.

5 SHOW THAT $x - 3$ IS A FACTOR OF SOME POLYNOMIAL $f(x)$ WHICH HAS A ZERO OF $\frac{4}{3}$.

6 IN EACH OF THE FOLLOWING, FIND A POLYNOMIAL THAT HAS THE GIVEN ZEROS SATISFYING THE GIVEN CONDITION.

A 0, 3, 4 AND $f(1) = 5$

B -1, $1 + \sqrt{2}$, $1 - \sqrt{2}$ AND $f(0) = 0$

7 A POLYNOMIAL FUNCTION OF DEGREE 3 HAS ZEROS AND $\frac{1}{2}$ AND ITS LEADING COEFFICIENT IS NEGATIVE. WRITE AN EXPRESSION FOR DIFFERENT POLYNOMIAL FUNCTIONS ARE POSSIBLE FOR

- 8 IF $p(x)$ IS A POLYNOMIAL OF DEGREE 3 WITH $p(-1) = 0$ AND $p(2) = 6$, THEN
- SHOW THAT $x = -p(x)$.
 - FIND THE INTERVAL IN WHICH $p(x)$ IS POSITIVE.
- 9 FIND THE VALUES OF p IF $x - 1$ IS A COMMON FACTOR OF $f(x) = x^4 - px^3 + 7qx + 1$, AND $g(x) = x^6 - 4x^3 + px^2 + qx - 3$.
- 10 THE HEIGHT ABOVE GROUND LEVEL OF A MISSILE LAUNCHED VERTICALLY, IS GIVEN BY $h(t) = -16t^3 + 100t$. AT WHAT TIME IS THE MISSILE 72 M ABOVE GROUND LEVEL [IN SECONDS].

1.3.2 Location Theorem

A POLYNOMIAL FUNCTION WITH RATIONAL COEFFICIENTS MAY HAVE NO RATIONAL ZEROS. FOR EXAMPLE, THE ZEROS OF THE POLYNOMIAL FUNCTION:

$$f(x) = x^2 - 4x - 2 \text{ ARE ALL IRRATIONAL.}$$

CAN YOU WORKOUT WHAT THE ZEROS ARE? THE POLYNOMIAL FUNCTION HAS RATIONAL AND IRRATIONAL ZEROS, CAN YOU CHECK THIS?

ACTIVITY 1.10



- IN EACH OF THE FOLLOWING, DETERMINE WHETHER THE CORRESPONDING FUNCTION ARE RATIONAL, IRRATIONAL OR TRANSCENDENTAL.

 - $f(x) = x^2 + 2x + 2$
 - $f(x) = x^3 + x^2 - 2x - 2$
 - $f(x) = (x + 1)(2x^2 + x - 3)$
 - $f(x) = x^4 - 5x^2 + 6$

- FOR EACH OF THE FOLLOWING POLYNOMIALS MAKE A TABLE:

 - $f(x) = 3x^3 + x^2 + x - 2$
 - $f(x) = x^4 - 6x^3 + x^2 + 12x - 6$

MOST OF THE STANDARD METHODS FOR FINDING THE IRRATIONAL ZEROS OF A POLYNOMIAL INVOLVE A TECHNIQUE OF SUCCESSIVE APPROXIMATION. ONE OF THE METHODS IS BASED ON THE IDEA OF CHANGE OF SIGN OF A FUNCTION. CONSEQUENTLY, THE FOLLOWING THEOREM IS

Theorem 1.4 Location theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a)$ and $f(b)$ have opposite signs, then there is at least one zero of f between a and b .

THIS THEOREM HELPS US TO LOCATE THE REAL ZEROS OF A POLYNOMIAL FUNCTION. IT IS POSSIBLE TO ESTIMATE THE ZEROS OF A POLYNOMIAL FUNCTION FROM A TABLE OF VALUES.

EXAMPLE 5 LET $f(x) = x^4 - 6x^3 + x^2 + 12x - 6$. CONSTRUCT A TABLE OF VALUES AND USE THE LOCATION THEOREM TO LOCATE THE ZEROS BETWEEN SUCCESSIVE INTEGERS.

SOLUTION: CONSTRUCT A TABLE AND LOOK FOR CHANGES IN SIGN AS FOLLO

x	-3	-2	-1	0	1	2	3	4	5	6
$f(x)$	210	38	-10	-6	2	-10	-42	-70	-44	102

SINCE $f(-2) = 38 > 0$ AND $f(-1) = -10 < 0$, WE SEE THAT THE VALUE CHANGES FROM POSITIVE TO NEGATIVE BETWEEN -2 AND -1 . THEREFORE, THERE IS A ZERO BETWEEN -2 AND -1 .

SINCE $f(0) = -6 < 0$ AND $f(1) = 2 > 0$, THERE IS ALSO ONE ZERO BETWEEN 0 AND 1 .

SIMILARLY, THERE ARE ZEROS BETWEEN 1 AND 2 AND BETWEEN 2 AND 3 .

EXAMPLE 6 USING THE LOCATION THEOREM, SHOW THAT THE POLYNOMIAL

$f(x) = x^5 - 2x^2 - 1$ HAS A ZERO BETWEEN 1 AND 2 .

SOLUTION: $f(1) = (1)^5 - 2(1)^2 - 1 = 1 - 2 - 1 = -2 < 0$.

$f(2) = (2)^5 - 2(2)^2 - 1 = 32 - 8 - 1 = 23 > 0$.

HERE $f(1)$ IS NEGATIVE AND $f(2)$ IS POSITIVE. THEREFORE, THERE IS A ZERO BETWEEN 1 AND 2 .

Exercise 1.8

- 1 IN EACH OF THE FOLLOWING, USE THE TABLE OF THE POLYNOMIAL FUNCTION TO LOCATE ZEROS OF:

A

x	-5	-3	-1	0	2	5
$f(x)$	7	4	2	-1	3	-6

B

x	-6	-5	-4	-3	-2	-1	0	1	2
$f(x)$	-21	-10	8	-1	-5	6	4	-3	18

- 2 USE THE LOCATION THEOREM TO VERIFY THAT IT HAS A ZERO BETWEEN:

A $f(x) = 3x^3 + 7x^2 + 3x + 7$; $a = -3$, $b = -2$

B $f(x) = 4x^4 + 7x^3 - 11x^2 + 7x - 15$; $a = 1$, $b = \frac{3}{2}$

C $f(x) = -x^4 + x^3 + 1$; $a = -1$, $b = 1$

D $f(x) = x^5 - 2x^3 - 1$; $a = 1$, $b = 2$

- 3** IN EACH OF THE FOLLOWING, USE THE LOCATION TEST FOR EACH REAL ZERO OF BETWEEN SUCCESSIVE INTEGERS:
- $f(x) = x^3 - 9x^2 + 23x - 14$; FOR $-4 \leq x \leq 6$
 - $f(x) = x^3 - 12x^2 + x + 2$; FOR $-8 \leq x \leq 8$
 - $f(x) = x^4 - x^2 + x - 1$; FOR $-3 \leq x \leq 3$
 - $f(x) = x^4 + x^3 - x^2 - 11x + 3$; FOR $-3 \leq x \leq 3$
- 4** IN EACH OF THE FOLLOWING, FIND ALL THE ZEROS OF THE FUNCTION, FOR $-4 \leq x \leq 4$:
- $f(x) = x^4 - 5x^3 + \frac{15}{2}x^2 - 2x - 2$
 - $f(x) = x^5 - 2x^4 - 3x^3 + 6x^2 + 2x - 4$
 - $f(x) = x^4 + x^3 - 4x^2 - 2x + 4$
 - $f(x) = 2x^4 + x^3 - 10x^2 - 5x$
- 5** IN QUESTION 1 OF EXERCISE 1.7, AT WHAT TIME IS THE MISSILE 50 M ABOVE THE GROUND LEVEL?
- 6** IS IT POSSIBLE FOR A POLYNOMIAL FUNCTION WITH DEGREES COEFFICIENT TO HAVE NO REAL ZEROS? EXPLAIN YOUR ANSWER.

1.3.3 Rational Root Test

THE rational root test relates the possible rational zeros of a polynomial to the leading coefficient and to the constant term of the polynomial.

Theorem 1.5 Rational root test

IF THE RATIONAL NUMBERS LOWEST TERMS, IS A ZERO OF THE POLYNOMIAL

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

WITH INTEGER COEFFICIENTS, THEN A FACTOR OF a_0 MUST BE A FACTOR OF

ACTIVITY 1.11

- WHAT SHOULD YOU DO FIRST TO USE THE TEST?
- WHAT MUST THE LEADING COEFFICIENT BE FOR ALL THE ZEROS TO BE FACTORS OF THE CONSTANT TERM?
- SUPPOSE THAT ALL OF THE COEFFICIENTS ARE RATIONAL. WHAT COULD BE DONE TO CHANGE THE POLYNOMIAL INTO ONE WITH INTEGER COEFFICIENTS? DOES THE RESULTANT POLYNOMIAL HAVE THE SAME ZEROS AS THE ORIGINAL?
- THERE IS AT LEAST ONE RATIONAL ZERO WHOSE CONSTANT TERM IS ZERO. WHAT IS THIS NUMBER?

EXAMPLE 7 IN EACH OF THE FOLLOWING, FIND ALL THE RATIONAL ZEROS OF THE POLYNOMIAL:

A $f(x) = x^3 - x + 1$

B $g(x) = 2x^3 + 9x^2 + 7x - 6$

C $g(x) = \frac{1}{2}x^4 - 2x^3 - \frac{1}{2}x^2 + 2x$

SOLUTION:

A THE LEADING COEFFICIENT IS 1 AND THE CONSTANT TERM, THE POSSIBLE RATIONAL ZEROS ARE USING THE REMAINDER THEOREM TEST THESE POSSIBLE ZEROS.

$$f(1) = (1)^3 - 1 + 1 = 1 - 1 + 1 = 1$$

$$f(-1) = (-1)^3 - (-1) + 1 = -1 + 1 + 1 = 1$$

SO WE CAN CONCLUDE THAT THE GIVEN POLYNOMIAL HAS NO RATIONAL ZEROS.

B $a_n = a_3 = 2$ AND $a_0 = -6$

POSSIBLE VALUES ARE FACTORS OF -6 . THESE ARE $\pm 1, \pm 2, \pm 3, \pm 6$.

POSSIBLE VALUES ARE FACTORS OF 2 . THESE ARE

THE POSSIBLE RATIONAL ZEROS $\frac{p}{q} \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$.

OF THESE 12 POSSIBLE RATIONAL ZEROS, AT MOST 3 CAN BE THE ZEROS OF

CHECK THAT $f(3) = 0, f(-2) = 0$ AND $f\left(\frac{1}{2}\right) = 0$.

USING THE FACTOR THEOREM, WE CAN FACTOR $f(x)$ AS:

$$2x^3 + 9x^2 + 7x - 6 = (x + 3)(x + 2)(2x - 1). \text{ SO, } g(x) = 0 \text{ AT}$$

$$x = -3, x = -2 \text{ AND AT } \frac{1}{2}.$$

THEREFORE $-3, -2, \frac{1}{2}$ ARE THE ONLY (RATIONAL) ZEROS OF

C LET $h(x) = 2g(x)$. THUS $h(x)$ WILL HAVE THE SAME ZEROS, BUT HAS INTEGER COEFFICIENTS.

$$h(x) = x^4 - 4x^3 - x^2 + 4x$$

x IS A FACTOR SO $= x(x^3 - 4x^2 - x + 4) = xk(x)$

$k(x)$ HAS A CONSTANT TERM OF 4 AND LEADING COEFFICIENT OF 1. THE POSSIBLE ZEROS ARE $\pm 2, \pm 4$.

USING THE REMAINDER THEOREM $k(1) = 0$, $k(-1) = 0$ AND $k(4) = 0$

SO, BY THE FACTOR THEOREM $k(x) = (x - 1)(x + 1)(x - 4)$.

HENCE $h(x) = x k(x) = x(x - 1)(x + 1)(x - 4)$ AND

$$g(x) = \frac{1}{2}h(x) = \frac{1}{2}x(x - 1)(x + 1)(x - 4).$$

THEREFORE, THE ZEROS ARE 0 ± 1 AND 4 .

Exercise 1.9

- 1** IN EACH OF THE FOLLOWING, FIND THE ZEROS AND MULTICITY OF EACH ZERO. WHAT IS THE DEGREE OF THE POLYNOMIAL?
- A** $f(x) = (x + 6)(x - 3)^2$ **B** $f(x) = 3(x + 2)^3(x - 1)^2(x + 3)$
- C** $f(x) = \frac{1}{2}(x - 2)^4(x + 3)^3(1 - x)$ **D** $f(x) = x^4 - 5x^3 + 9x^2 - 7x + 2$
- E** $f(x) = x^4 - 4x^3 + 7x^2 - 12x + 12$
- 2** FOR EACH OF THE FOLLOWING POLYNOMIALS, FIND ALL POSSIBLE RATIONAL ZEROS:
- A** $p(x) = x^3 - 2x^2 - 5x + 6$ **B** $p(x) = x^3 - 3x^2 + 6x + 8$
- C** $p(x) = 3x^3 - 11x^2 + 8x + 4$ **D** $p(x) = 2x^3 + x^2 - 4x - 3$
- E** $p(x) = 12x^3 - 16x^2 - 5x + 3$
- 3** IN EACH OF THE FOLLOWING, FIND A LITERAL EQUATION, AND EXPRESS THE POLYNOMIAL IN FACTORIZED FORM:
- A** $f(x) = x^3 - 5x^2 - x + 5$ **B** $g(x) = 3x^3 + 3x^2 - x - 1$
- C** $p(t) = t^4 - t^3 - t^2 - t - 2$
- 4** IN EACH OF THE FOLLOWING, FIND ALL RATIONAL ZEROS:
- A** $p(y) = y^3 + \frac{11}{6}y^2 - \frac{1}{2}y - \frac{1}{3}$ **B** $p(x) = x^4 - \frac{25}{4}x^2 + 9$
- C** $h(x) = x^4 - \frac{21}{10}x^2 + \frac{3}{5}x$ **D** $p(x) = x^4 + \frac{7}{6}x^3 - \frac{7}{3}x^2 - \frac{5}{2}x$
- 5** FOR EACH OF THE FOLLOWING, FIND A LITERAL EQUATION:
- A** $2x^3 - 5x^2 + 1 = 0$ **B** $4x^4 + 4x^3 - 9x^2 - x + 2 = 0$
- C** $2x^5 - 3x^4 - 2x + 3 = 0$

1.4**GRAPHS OF POLYNOMIAL FUNCTIONS**

IN YOUR PREVIOUS GRADES, YOU HAVE DISCUSSED HOW TO DRAW GRAPHS OF FUNCTIONS OF DEGREE ZERO, ONE AND TWO. IN THE PRESENT SECTION, YOU WILL LEARN ABOUT GRAPHS OF FUNCTIONS OF DEGREE GREATER THAN TWO.

TO UNDERSTAND PROPERTIES OF POLYNOMIAL FUNCTIONS, TRY THE FOLLOWING

ACTIVITY 1.12

- 1** SKETCH THE GRAPH OF EACH OF THE FOLLOWING FUNCTIONS

A $f(x) = 3$

B $f(x) = -2.5$

C $g(x) = x - 2$

D $g(x) = -3x + 1$

- 2** LET $f(x) = x^2 - 4x + 5$

- A COPY AND COMPLETE THE TABLE OF VALUES GIVEN BELOW.

x	-2	-1	0	1	2	3	4
$f(x) = x^2 - 4x + 5$							



- B PLOT THE POINTS WITH COORDINATES $(x, f(x))$ ON THE COORDINATE PLANE.

- C JOIN THE POINTS ABOVE BY A SMOOTH CURVE TO GET THE GRAPH OF $y = f(x)$. YOU CALL THE GRAPH OF THE DOMAIN AND RANGE OF f .

- 3** CONSTRUCT A TABLE OF VALUES FOR EACH OF THE FOLLOWING FUNCTIONS AND SKETCH THE GRAPH:

A $f(x) = x^2 - 3$

B $g(x) = -x^2 - 2x + 1$

C $h(x) = x^3$

D $p(x) = 1 - x^4$

WE SHALL DISCUSS SKETCHING THE GRAPHS OF HIGHER DEGREE POLYNOMIAL FUNCTIONS IN THE FOLLOWING EXAMPLES.

EXAMPLE 1 LET US CONSIDER THE FUNCTION $y = 3x^3 - 4$.

THIS FUNCTION CAN BE WRITTEN AS $y = 3x^3 - 4$

COPY AND COMPLETE THE TABLE OF VALUES BELOW.

x	-3	-2	-1	0	1	2	3
y		-6	-2		-6		14

OTHER POINTS BETWEEN INTEGERS MAY HELP YOU TO DETERMINE THE SHAPE OF THE GRAPH BETTER.

FOR INSTANCE, FOR $x = \frac{1}{2}$

$$y = p\left(\frac{1}{2}\right) = -\frac{43}{8}$$

THEREFORE, THE POINT $\left(\frac{1}{2}, -\frac{43}{8}\right)$ IS ON THE GRAPH. SIMILARLY, FOR

$$x = \frac{5}{2}, y = p\left(\frac{5}{2}\right) = \frac{33}{8}.$$

SO, $\left(\frac{5}{2}, \frac{33}{8}\right)$ IS ALSO ON THE GRAPH OF

PLOT THE POINTS WITH COORDINATES FROM THE TABLE AS SHOWN IN 3A
NOW JOIN THESE POINTS BY A SMOOTH CURVE TO GET THIS GRAPH ON FIGURE 1.3B

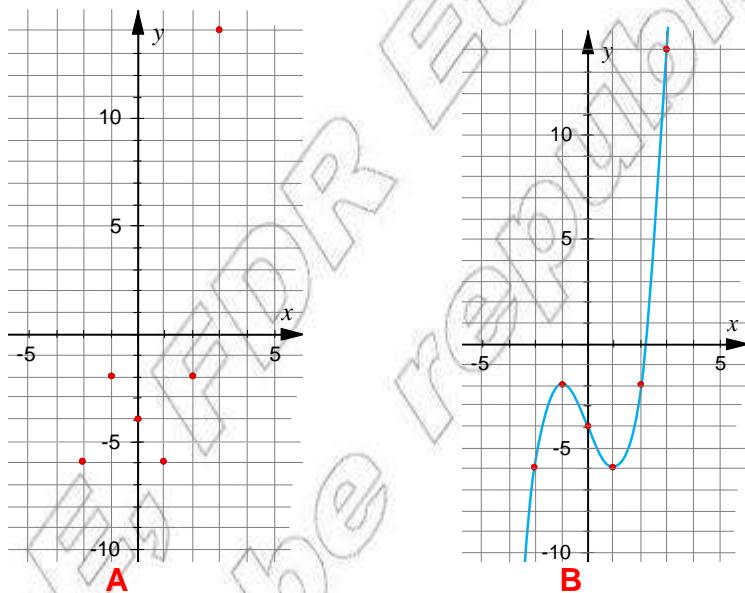


Figure 1.3 Graph of $p(x) = x^3 - 3x - 4$

EXAMPLE 2 SKETCH THE GRAPH OF $-x^4 + 2x^2 + 1$

SOLUTION: TO SKETCH THE GRAPH, FIND POINTS ON THE GRAPH USING A TABLE OF VALUES

x	-2	-1	0	1	2
$y = -x^4 + 2x^2 + 1$	-7	2	1	2	-7

PLOT THE POINTS WITH COORDINATES FROM THIS TABLE AND JOIN THEM BY A SMOOTH CURVE FOR INCREASING X VALUES, AS SHOWN IN FIGURE 1.4

FROM THE GRAPH, FIND THE DOMAIN AND THE RANGE. OBSERVE THAT THE GRAPH IS DOWNWARD.

AS OBSERVED FROM THE ABOVE TWO EXAMPLES, THE GRAPH OF A POLYNOMIAL FUNCTION HAS NO JUMPS, GAPS AND HOLES. IT HAS NO SHARP CORNERS. THE GRAPH OF A POLYNOMIAL FUNCTION IS A SMOOTH AND CONTINUOUS CURVE WHICH MEANS THERE IS NO BREAK ANYWHERE ON THE GRAPH.

THE GRAPH ALSO SHOWS THAT FOR EVERY VALUE OF x IN THE DOMAIN OF A POLYNOMIAL FUNCTION $p(x)$, THERE IS EXACTLY ONE VALUE WHERE $y = p(x)$.

THE FOLLOWING ARE NOT GRAPHS OF POLYNOMIAL FUNCTIONS.

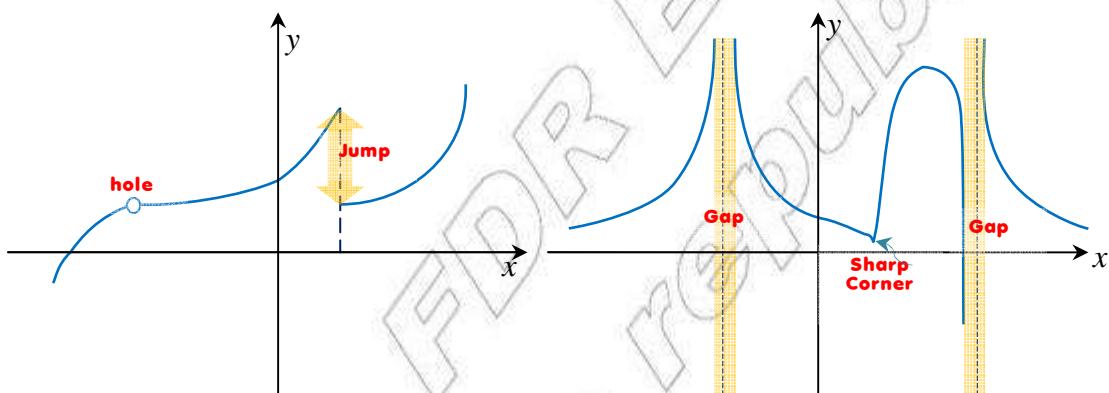


Figure 1.5

FUNCTIONS WITH GRAPHS THAT ARE NOT POLYNOMIAL FUNCTIONS.

LOOK AT THE GRAPH OF THE FUNCTION GIVEN IN FIGURE 1.6. IT HAS A SHARP CORNER AT THE POINT $(0, 0)$ AND HENCE $|x|$ IS NOT A POLYNOMIAL FUNCTION.

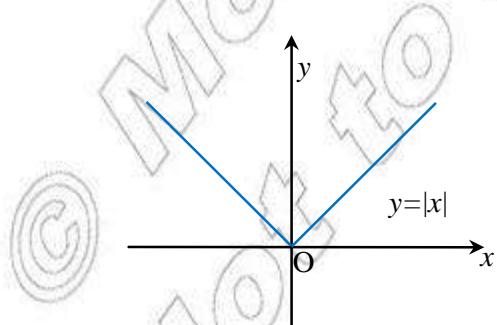


Figure 1.6

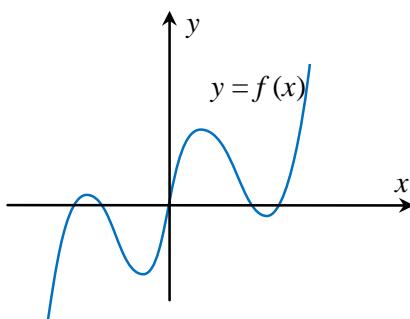


Figure 1.7

Is the function $f(x) = |x - 2|$ a polynomial function? Give reasons for your answer.

THE GRAPH OF THE FUNCTION $f(x) = |x - 2|$ IS A SMOOTH CURVE. HENCE IT REPRESENTS A POLYNOMIAL FUNCTION. OBSERVE THAT THE RANGE OF

THE POINTS AT WHICH THE GRAPH OF A FUNCTION CROSSES (MEETS) THE COORDINATE AXES IS IMPORTANT TO NOTE.

IF THE GRAPH OF A FUNCTION CROSSES THE X-AXIS AT $(x_1, 0)$, THEN x_1 IS THE INTERCEPT OF THE GRAPH. IF THE GRAPH CROSSES THE Y-AXIS AT THE POINT $(0, y_1)$, THEN y_1 IS THE INTERCEPT OF THE GRAPH.

How do we determine the x-intercept and the y-intercept?

SINCE $(x_1, 0)$ LIES ON THE GRAPH, $f(x_1) = 0$. SO x_1 IS A ZERO OF

SIMILARLY, y_1 LIES ON THE GRAPH, SO $f(0) = y_1$.

CONSIDER THE FUNCTION

$$f(x) = ax + b, a \neq 0$$

What is the x-intercept and the y-intercept?

$$f(x_1) = ax_1 + b = 0. \text{ SOLVING FOR } x_1 \text{ GIVES } x_1 = -\frac{b}{a}$$

SO $-\frac{b}{a}$ IS THE INTERCEPT OF THE GRAPH OF

AGAIN, $f(0) = a \cdot 0 + b = b$. THE NUMBER b IS THE INTERCEPT.

TRY TO FIND THE X-INTERCEPT AND THE Y-INTERCEPT OF $f(x) = -3x + 5$.

THE ABOVE METHOD CAN ALSO BE APPLIED TO A QUADRATIC FUNCTION. CONSIDER THE EXAMPLE.

EXAMPLE 3 FIND THE X-INTERCEPTS AND THE Y-INTERCEPT OF THE GRAPH OF

$$f(x) = x^2 - 4x + 3$$

SOLUTION: $f(x_1) = x_1^2 - 4x_1 + 3 = 0 \Rightarrow (x_1 - 1)(x_1 - 3) = 0 \quad \therefore x_1 = 1 \text{ OR } x_1 = 3$

THEREFORE, THE GRAPHS HAS TWO X-INTERCEPTS, 1 AND 3.

NEXT $f(0) = 0^2 - 4 \cdot 0 + 3 = 3$. HERE $y_1 = 3$ IS THE Y-INTERCEPT.

THE GRAPH CROSSES THE X-AXIS AT $(1, 0)$ AND $(3, 0)$. IT CROSSES THE Y-AXIS AT $(0, 3)$.

THE GRAPH OPENS UPWARD AND TURNING POINT $(2, 1)$ IS THE VERTEX OR TURNING POINT OF THE GRAPH. THE MINIMUM VALUE OF THE GRAPH IS 1. THE RANGE OF $\{y : y \geq 1\}$.

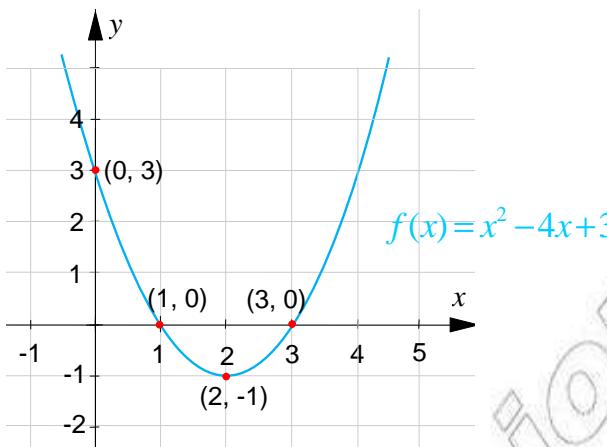


Figure 1.8

NOTE THAT THE GRAPH OF ANY QUADRATIC FUNCTION HAS AT MOST TWO x -INTERCEPTS AND EXACTLY ONE y -INTERCEPT. TRY TO FIND THE REASON.

AS SEEN FROM FIGURE 1.8, $a = 1$ IS POSITIVE AND THE PARABOLA OPENS UPWARD.

What can be stated about the graph of $g(x) = -2x^2 + 4x$?

Does the graph open upward?

The coefficient of x^2 is negative. What is the range of g ?

TO STUDY SOME PROPERTIES OF POLYNOMIALS, LOOK AT THE GRAPHS OF SOME POLYNOMIAL FUNCTIONS OF HIGHER DEGREES ON THE FORM $f(x) = ax^n + b$.

EXAMPLE 4 BY SKETCHING THE GRAPHS OF $g(x) = x^3 + 1$ AND $h(x) = -2x^3 + 1$, OBSERVE THEIR BEHAVIOURS AND GENERAL TRENDS.

SOLUTION: PLOT THE POINTS OF THE GRAPHS OF

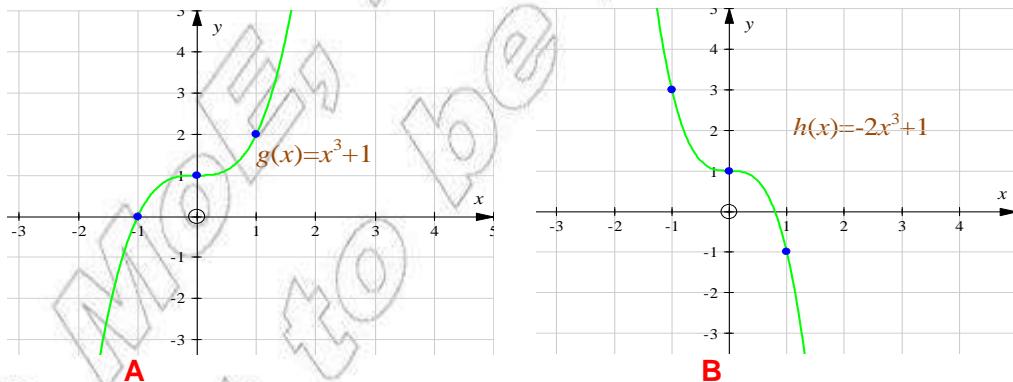


Figure 1.9

AS SHOWN IN FIGURE 1.9A, WHEN BECOMES LARGE IN ABSOLUTE VALUE AND IS NEGATIVE BUT LARGE IN ABSOLUTE VALUE (down). WHEN x TAKES LARGE POSITIVE VALUES BECOMES LARGE POSITIVE.

IN FIGURE 1.9B, THE COEFFICIENT OF THE LEADING TERM IS -2 WHICH IS NEGATIVE AS A RESULT, WHEN x BECOMES LARGE IN ABSOLUTE VALUE, $|x|$ BECOMES LARGE POSITIVE WHEN x TAKES LARGE POSITIVE VALUES, $f(x)$ BECOMES NEGATIVE BUT LARGE IN ABSOLUTE VALUE.

THE GRAPH OF $f(x) = a_n x^n + b$ SHOWS THE SAME BEHAVIOUR WHEN n IS LARGE AS THE GRAPH OF g FOR $a > 0$ AND AS THE GRAPH OF h FOR $a < 0$ AND n ODD.

EXAMPLE 5 BY SKETCHING THE GRAPHS OF $g(x) = 2x^4$ AND $h(x) = -x^4$, OBSERVE THEIR BEHAVIOUR AND GENERALIZE FOR EVEN n WHEN n IS LARGE.

SOLUTION: THE SKETCHES OF THE GRAPHS OF g AND h ARE AS FOLLOWS.

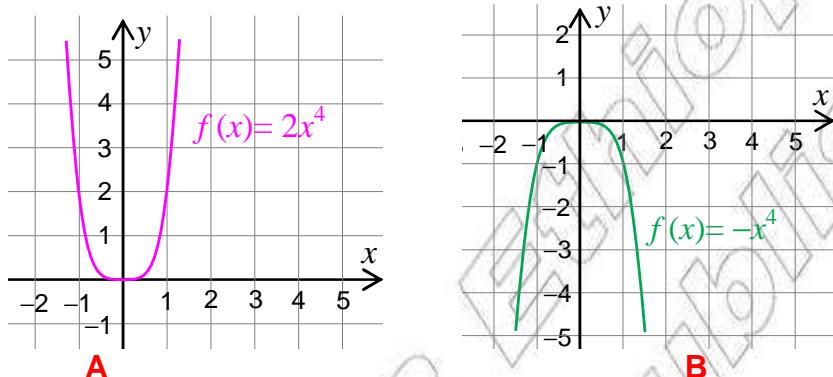


Figure 1.10

FROM FIGURE 1.10A, WHEN $|x|$ TAKES LARGE VALUES, y BECOMES LARGE POSITIVE. ON THE OTHER HAND, FROM FIGURE 1.10B, WHEN $|x|$ TAKES LARGE VALUES, $f(x)$ BECOMES NEGATIVE BUT LARGE IN ABSOLUTE VALUE AND THE GRAPH OPENS DOWNWARD. WHEN n IS EVEN THE GRAPH OF f OPENS UPWARD FOR $a > 0$ AND OPENS DOWNWARD FOR $a < 0$. DRAW AND OBSERVE THE GRAPHS OF $g(x) = 2(x-1)^4$ AND $h(x) = -(x-1)^4$.

ACTIVITY 1.13

1 CONSIDER THE FOLLOWING GRAPHS:

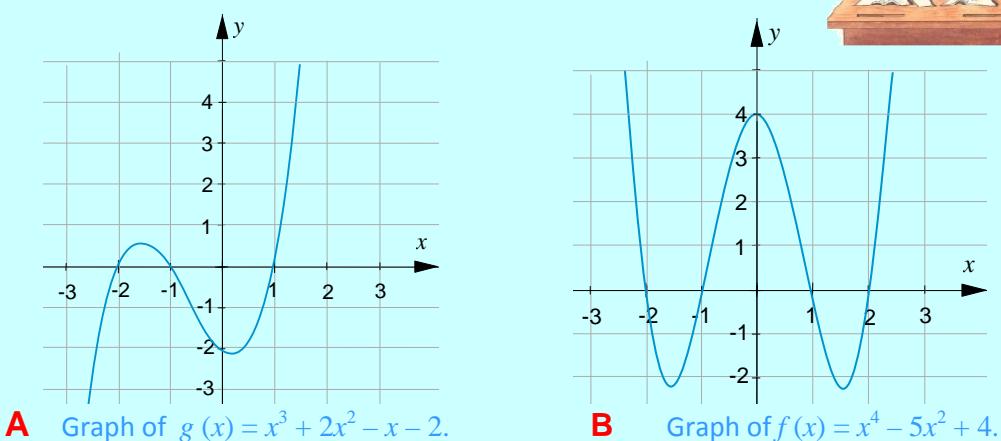


Figure 1.11

- A** WHAT ARE THE DOMAIN OF
- B** WHAT CAN BE SAID ABOUT THE VALUES OF WHEN x IS LARGE AND POSITIVE, OR LARGE AND NEGATIVE?
- C** IF $x = 2^{10}$, WILL THE TERM $g(x)$ AND $f(x)$ BE POSITIVE OR WILL THEY BE NEGATIVE? WHAT HAPPENS WHEN $x = 2^{-10}$?
- 2 A** DO YOU THINK THAT THE RANGE OF EVERY POLYNOMIAL FUNCTION IS THE SET OF ALL REAL NUMBERS?
- B** WILL THE GRAPH OF EVERY POLYNOMIAL FUNCTION CROSS THE ONE POINT? WHY?

Group Work 1.3



- 1** ON THE GRAPH $f(x) = x^4 - 5x^2 + 4$
- A** WHAT ARE THE VALUES OF THE POINTS WHERE THE GRAPH CROSSES THE x -axis? AT HOW MANY POINTS DOES THE GRAPH CROSS THE x -axis?
- B** WHAT IS THE VALUE OF EACH OF THESE POINTS OBTAINED IN (A)?
- C** WHAT IS THE TRUTH SET OF THE EQUATION $x^4 - 5x^2 + 4 = 0$?
- 2** CONSIDER THE FUNCTION $f(x) = (x + 2)(x + 1)(x - 1)(x - 2)$
- A** ON THE GRAPH OF THE FUNCTION ARE THE COORDINATES OF THE POINTS WHERE THE GRAPH CROSSES THE x -axis?
- B** DO YOU THINK THAT QUESTION 1 ABOVE ARE THE SAME FUNCTION?
- 3** AS SHOWN IN FIGURE 1.11, THE GRAPH OF THE POLYNOMIAL FUNCTION DEFINED BY
- $f(x) = x^4 - 5x^2 + 4$ CROSSES THE x -axis FOUR TIMES AND THE GRAPH OF $g(x) = x^3 + 2x^2 - x - 2$ CROSSES THE x -axis THREE TIMES.
- IN A SIMILAR WAY, HOW MANY TIMES DOES THE GRAPH OF EACH OF THE FOLLOWING FUNCTIONS INTERSECT THE x -axis?
- | | |
|---------------------------|---|
| A $p(x) = 2x + 1$ | B $p(x) = x^2 + 4$ |
| C $p(x) = x^2 - 8$ | D $f(x) = (x - 2)(x - 1)(x^2 + 4)$. |
- 4** DO YOU THINK THAT THE GRAPH OF EVERY POLYNOMIAL FUNCTION CROSSES THE x -axis FOUR TIMES?

NOTE THAT THE GRAPH OF A POLYNOMIAL FUNCTION OF EVEN DEGREE IS BOUNDED AT BOTH ENDS, WHILE THE GRAPH OF A POLYNOMIAL FUNCTION OF ODD DEGREE IS UNBOUNDED AT BOTH ENDS. SO (AS STATED PREVIOUSLY), EVERY POLYNOMIAL FUNCTION OF DEGREE ZERO IS A CONSTANT FUNCTION.

IN GENERAL, THE BEHAVIOUR OF THE GRAPH OF A POLYNOMIAL FUNCTION AS X APPROACHES POSITIVE INFINITY DEPENDS ON THE DEGREE (EVEN OR ODD) AND BY ITS LEADING COEFFICIENT.

THE GRAPH OF THE POLYNOMIAL FUNCTION $f(x) = a_n x^{n-1} + \dots + a_1 x + a_0$ EVENTUALLY RISES OR FALLS. OBSERVE THE EXAMPLES GIVEN BELOW.

EXAMPLE 6 DESCRIBE THE BEHAVIOUR OF THE GRAPH OF $f(x) = -x^3 + x$ AS X DECREASES TO THE LEFT AND INCREASES TO THE RIGHT.

SOLUTION: BECAUSE THE DEGREE IS ODD AND THE LEADING COEFFICIENT IS NEGATIVE, THE GRAPH RISES TO THE LEFT AND FALLS TO THE RIGHT AS SHOWN IN

A AND B ARE THE TURNING POINTS OF THE GRAPH OF

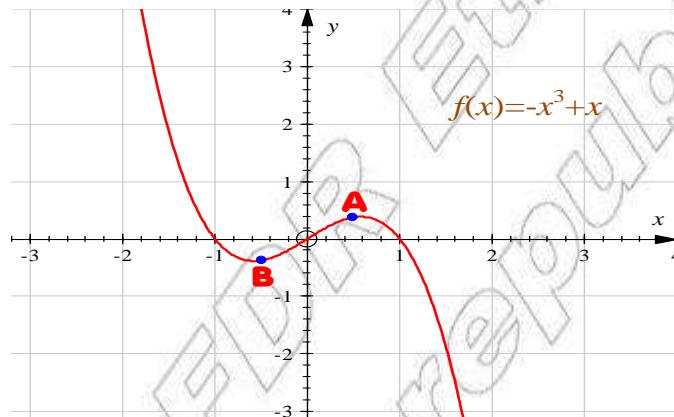


Figure 1.12

FIGURE 1.13 SHOWS AN EXAMPLE OF A POLYNOMIAL FUNCTION WHOSE GRAPH HAS PEAKS AND VALLEYS. THE TERM PEAK REFERS TO A LOCAL MAXIMUM AND THE TERM VALLEY REFERS TO A LOCAL MINIMUM. SUCH POINTS ARE OFTEN CALLED TURNING POINTS OF THE GRAPH.

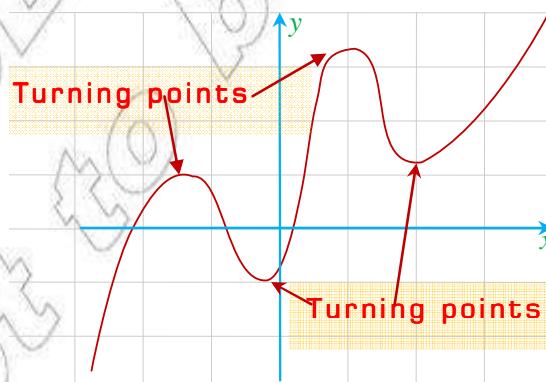


Figure 1.13

A POINT THAT IS EITHER A MAXIMUM POINT OR MINIMUM POINT ON ITS DOMAIN IS **local extremum point** OF.

NOTE THAT THE GRAPH OF A POLYNOMIAL FUNCTION HAS NO TURNING POINTS.

EXAMPLE 7 CONSIDER THE POLYNOMIAL

$$f(x) = x(x - 2)^2(x + 2)^4.$$

THE FUNCTION HAS A SIMPLE ZERO AT 0, A ZERO OF MULTIPLICITY 2 AT 2 AND A ZERO OF MULTIPLICITY 4 AT -2, AS SHOWN IN **FIGURE 1.14**, IT HAS A LOCAL MAXIMUM AT $x = -2$ AND DOES NOT CHANGE SIGN AT $x = -2$. ALSO, f HAS A RELATIVE (LOCAL) MINIMUM AT $x = 2$ AND DOES NOT CHANGE SIGN HERE. BOTH $x = -2$ AND $x = 2$ ARE ZEROS OF EVEN MULTIPLICITY.

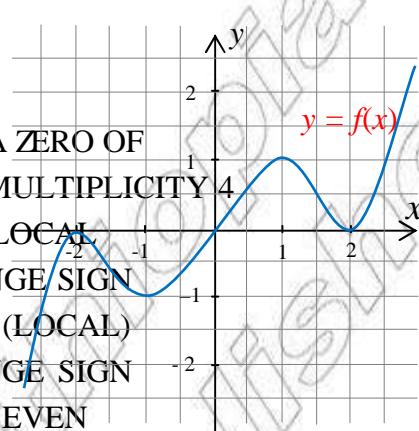


Figure 1.14

ON THE OTHER HANDS A ZERO OF ODD MULTIPLICITY CHANGES SIGN AT, AND DOES NOT HAVE A TURNING POINT AT

EXAMPLE 8 TAKE THE POLYNOMIAL $f(x) = 3x^4 + 4x^3$. IT CAN BE EXPRESSED AS

$$f(x) = x^3(3x + 4).$$

THE DEGREE IS EVEN AND THE LEADING COEFFICIENT IS POSITIVE. HENCE, THE FUNCTION RISES UP AS $|x|$ BECOMES LARGE.

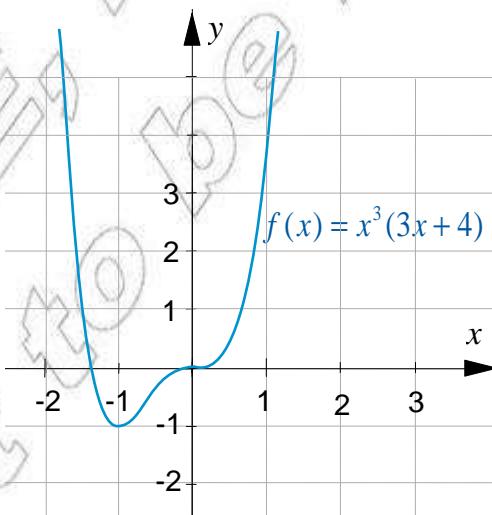


Figure 1.15

THE FUNCTION HAS A SIMPLE ZERO AT $x = \frac{4}{3}$ AND CHANGES SIGN AT POINT $\left(\frac{4}{3}\right)$.

THE GRAPH HAS A LOCAL MINIMUM AT POINT $(-1, -1)$.

ALSO HAS A ZERO AT 0 AND CHANGES SIGN HERE. SO, 0 IS OF ODD MULTIPLICITY.

THERE IS NO LOCAL MINIMUM OR MAXIMUM AT $(0, 0)$.

The above observations can be generalized as follows:

- 1 IF c IS A ZERO OF ODD MULTIPLICITY OF n , THEN THE GRAPH OF THE FUNCTION CROSSES THE x -AXIS AT $x = c$ AND DOES NOT HAVE A RELATIVE EXTREMUM AT $x = c$.
- 2 IF c IS A ZERO OF EVEN MULTIPLICITY, THEN THE GRAPH OF THE FUNCTION TOUCHES THE x -AXIS AT $x = c$ AND HAS A LOCAL EXTREMUM AT $x = c$.

Group Work 1.4



- 1 GIVE SOME EXAMPLES OF POLYNOMIAL FUNCTIONS. DISCUSS THE BEHAVIOUR OF THEIR GRAPHS AS x INCREASES WITHOUT BOUND (THEIR ABSOLUTE VALUE).

DID YOU NOTE THAT FOR $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, $a_n \neq 0$ IF $a_n > 0$ AND

IS ODD, $f(x)$ BECOMES LARGE POSITIVE AS x BECOMES LARGE IN ABSOLUTE VALUE AND BECOMES NEGATIVE BUT LARGE IN ABSOLUTE VALUE AS x BECOMES LARGE FOR NEGATIVE?

DISCUSS THE CASES WHERE:

- | | | | |
|------------|-----------------------|-----------|-----------------------|
| I | $a_n > 0$ AND IS EVEN | II | $a_n < 0$ AND IS EVEN |
| III | $a_n < 0$ AND IS ODD | IV | $a_n > 0$ AND IS ODD |

- 2 ANSWER THE FOLLOWING QUESTIONS:

- A** WHAT IS THE LEAST NUMBER OF TURNING POINTS A POLYNOMIAL FUNCTION CAN HAVE? WHAT ABOUT AN EVEN DEGREE POLYNOMIAL FUNCTION?
- B** WHAT IS THE MAXIMUM NUMBER OF TURNING POINTS THE GRAPH OF A POLYNOMIAL FUNCTION OF DEGREE n CAN HAVE?
- C** WHAT IS THE MAXIMUM NUMBER OF REAL ZEROS A POLYNOMIAL FUNCTION OF DEGREE n CAN HAVE?
- D** WHAT IS THE LEAST NUMBER OF TURNING POINTS THE GRAPH OF A POLYNOMIAL FUNCTION OF ODD DEGREE/EVEN DEGREE CAN HAVE?

Exercise 1.10

- 1** MAKE A TABLE OF VALUES AND DRAW THE GRAPH FOR EACH OF THE FOLLOWING POLYNOMIAL FUNCTIONS:
- | | |
|----------------------------------|---|
| A $f(x) = 4x^2 - 11x + 3$ | B $f(x) = -1 - x^2$ |
| C $f(x) = 8 - x^3$ | D $f(x) = x^3 + x^2 - 6x - 10$ |
| E $f(x) = 2x^2 - 2x^4$ | F $f(x) = \frac{1}{4}(x-2)^2(x+2)^2$. |
- 2** WITHOUT DRAWING THE GRAPHS OF THE FOLLOWING POLYNOMIALS, STATE FOR EACH, AS MUCH AS YOU CAN, ABOUT:
- I** THE BEHAVIOUR OF THE GRAPH AS VALUES FAR TO THE RIGHT AND FAR TO THE LEFT.
 - II** THE NUMBER OF INTERSECTIONS OF THE GRAPH WITH THE X-AXIS.
 - III** THE DEGREE OF THE FUNCTION AND WHETHER THE DEGREE IS EVEN OR ODD.
 - IV** THE LEADING COEFFICIENT AND WHETHER IT IS POSITIVE OR NEGATIVE.
- | | |
|--|--------------------------------------|
| A $f(x) = (x-1)(x-1)$ | B $f(x) = x^2 + 3x + 2$ |
| C $f(x) = 16 - 2x^3$ | D $f(x) = x^3 - 2x^2 - x + 1$ |
| E $f(x) = 5x - x^3 - 2$ | F $f(x) = (x-2)(x-2)(x-3)$ |
| G $f(x) = 2x^5 + 2x^2 - 5x + 1$ | |
- 3** FOR THE GRAPHS OF EACH OF THE FUNCTIONS GIVEN ABOVE:
- I** DISCUSS THE BEHAVIOUR OF THE GRAPH AS VALUES FAR TO THE RIGHT AND FAR TO THE LEFT.
 - II** GIVE THE NUMBER OF TIMES THE GRAPH INTERSECTS THE X-AXIS.
 - III** FIND THE VALUE OF THE FUNCTION WHERE THE GRAPH CROSSSES THE X-AXIS.
 - IV** GIVE THE NUMBER OF TURNING POINTS.
- 4** IN EACH OF THE FOLLOWING, DECIDE WHETHER THE GRAPH COULD POSSIBLY BE THE GRAPH OF A POLYNOMIAL FUNCTION:

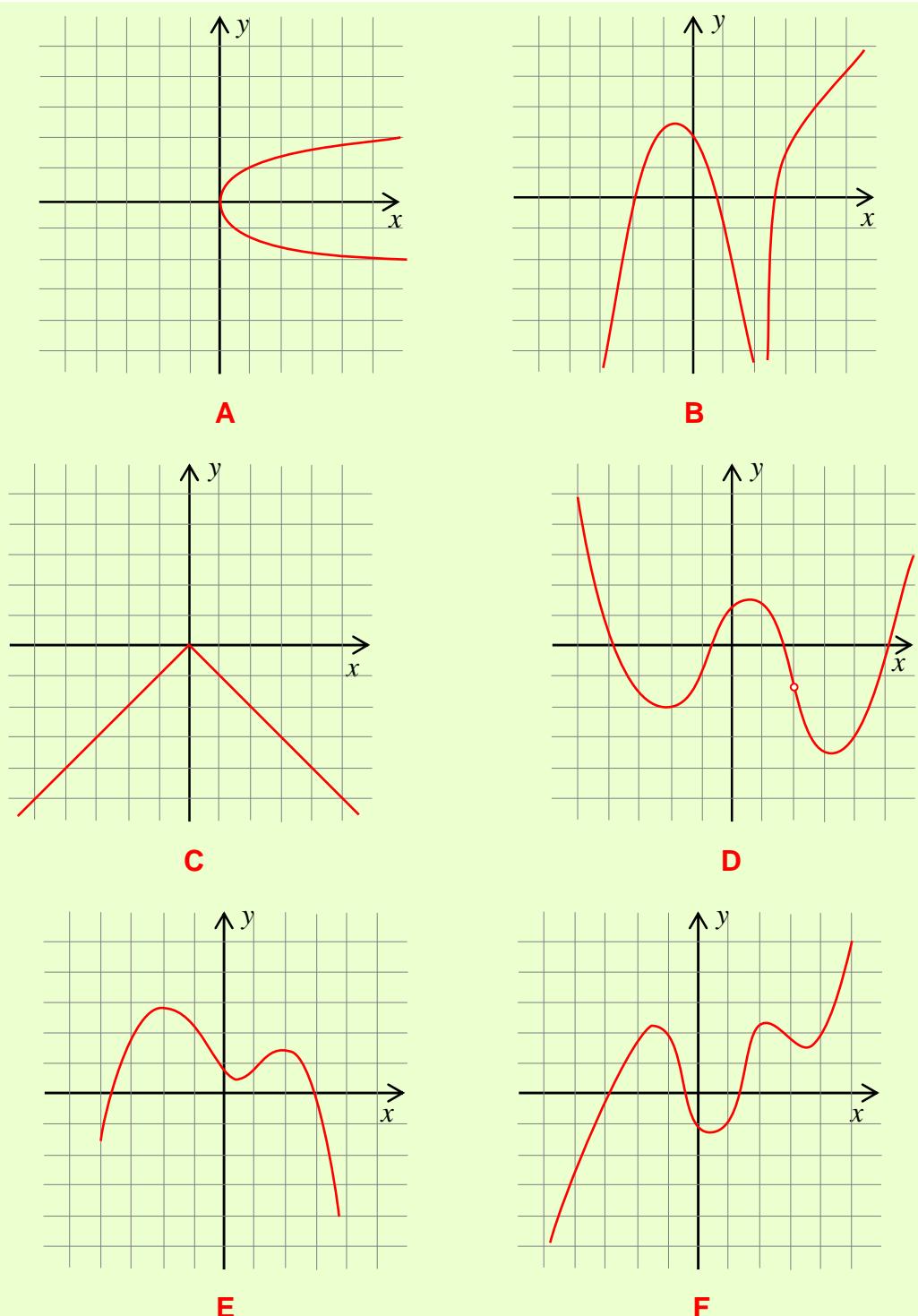
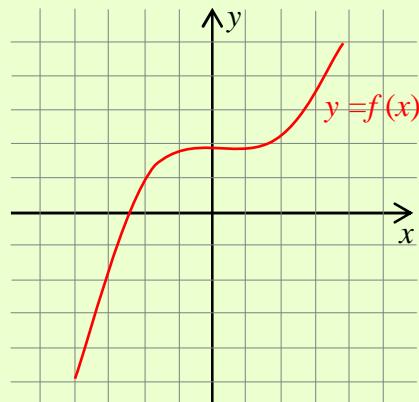
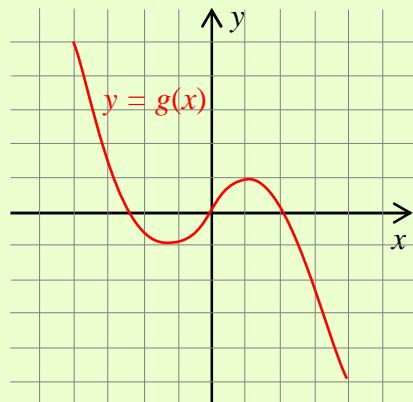
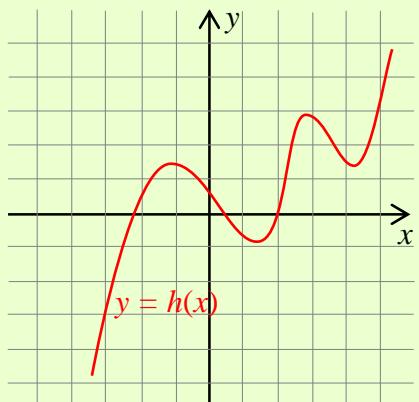
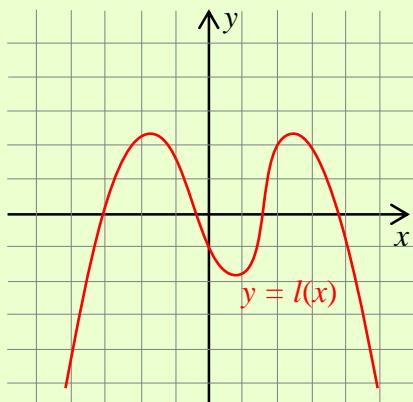
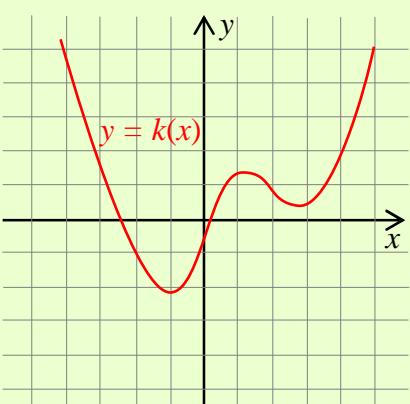
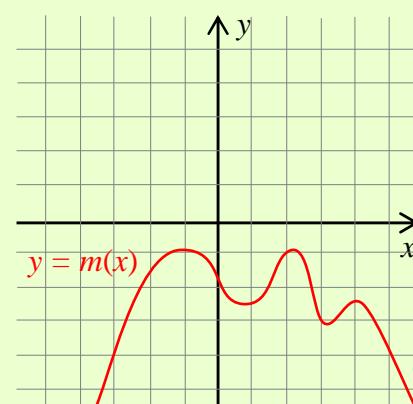


Figure 1.16

- 5** GRAPHS OF SOME POLYNOMIAL FUNCTIONS ARE BASED ON THE FOLLOWING:
- I IDENTIFY THE SIGN OF THE LEADING COEFFICIENT.

- II** IDENTIFY THE POSSIBLE DEGREE OF EACH FUNCTION, AND STATE WHETHER THE DEGREE IS EVEN OR ODD.
- III** DETERMINE THE NUMBER OF TURNING POINTS.

**A****B****C****D****E****F**

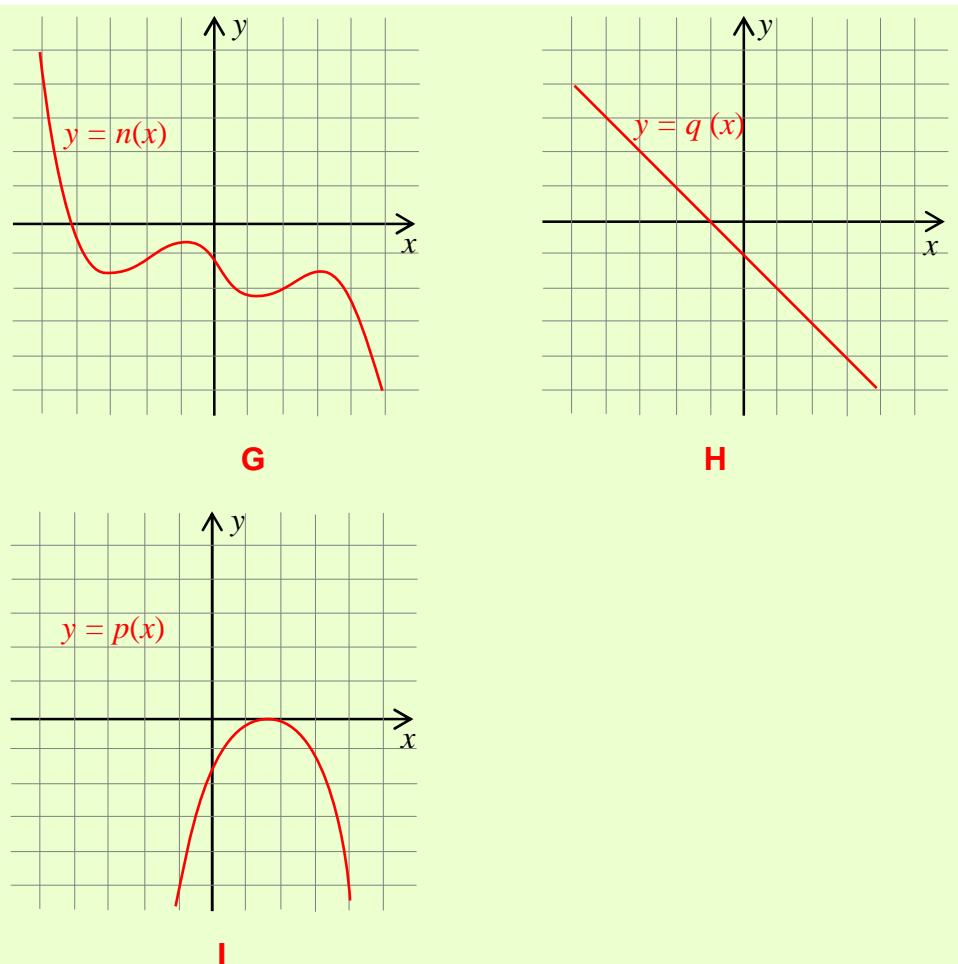


Figure 1.17

6 DETERMINE WHETHER EACH OF THE FOLLOWING STATEMENTS JUSTIFY YOUR ANSWER:

- A** A POLYNOMIAL FUNCTION OF DEGREE 6 CAN HAVE 6 TURNING POINTS.
- B** IT IS POSSIBLE FOR A POLYNOMIAL FUNCTION TO INTERCEPT THE x-AXIS AT ONE POINT.



Key Terms

constant function	linear function	rational root
constant term	local extremum	remainder theorem
degree	location theorem	turning points
domain	multiplicity	x-intercept
factor theorem	polynomial division theorem	y-intercept
leading coefficient	polynomial function	zero(s) of a polynomial
leading term	quadratic function	



Summary

- 1 A **linear function** IS GIVEN $f(x) = ax + b; a \neq 0$.
- 2 A **quadratic function** IS GIVEN $f(x) = ax^2 + bx + c; a \neq 0$
- 3 LET n BE A NON-NEGATIVE INTEGER AND LET a_0 BE REAL NUMBERS, WITH THE FUNCTION $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ IS CALLED **a polynomial function in x of degree n** .
- 4 A POLYNOMIAL FUNCTION IS OVER INTEGERS IF ITS ELEMENTS ARE ALL INTEGERS.
- 5 A POLYNOMIAL FUNCTION IS OVER RATIONALS IF ITS ELEMENTS ARE ALL RATIONAL NUMBERS.
- 6 A POLYNOMIAL FUNCTION IS OVER REAL NUMBERS IF ITS ELEMENTS ARE ALL REAL NUMBERS.
- 7 OPERATIONS ON POLYNOMIAL FUNCTIONS:
 - I Sum: $(f + g)(x) = f(x) + g(x)$
 - II Difference: $(f - g)(x) = f(x) - g(x)$
 - III Product: $(fg)(x) = f(x) \cdot g(x)$
 - IV Quotient: $(f \div g)(x) = f(x) \div g(x)$, IF $g(x) \neq 0$
- 8 IF $f(x)$ AND $g(x)$ ARE POLYNOMIALS SUCH THAT THE DEGREE OF $f(x)$ IS LESS THAN OR EQUAL TO THE DEGREE OF $g(x)$ THERE EXIST UNIQUE POLYNOMIALS $q(x)$ AND $r(x)$ SUCH THAT $f(x) = q(x)g(x) + r(x)$, WHERE $r(x) = 0$ OR THE DEGREE OF $r(x)$ IS LESS THAN THE DEGREE OF $g(x)$
- 9 IF A POLYNOMIAL IS DIVIDED BY A FIRST DEGREE POLYNOMIAL OF THE FORM $x - a$ THE **remainder** IS THE NUMBER $f(a)$

10 GIVEN THE POLYNOMIAL FUNCTION

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

IF $p(c) = 0$, THEN IS A zero of the polynomial AND root OF THE EQUATION 0. FURTHERMORE IS A factor OF THE POLYNOMIAL.

11 FOR EVERY POLYNOMIAL, IF c IS A NUMBER, $f(c) = 0$, THEN $= c$ IS A ZERO OF THE POLYNOMIAL FUNCTION**12** IF $(x-c)^k$ IS A FACTOR OF $f(x)$ BUT $x(-c)^{k+1}$ IS NOT, WE SAY c IS A ZERO OF multiplicity k of f .**13** IF THE RATIONAL NUMBERS IN THE LOWEST TERM, IS A ZERO OF THE POLYNOMIAL

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ WITH INTEGER COEFFICIENTS, THEN AN INTEGER FACTOR MUST BE AN INTEGER FACTOR OF

14 LET a AND b BE REAL NUMBERS SUCH THAT $f(x)$ IS A POLYNOMIAL FUNCTION SUCH THAT (a) AND (b) HAVE OPPOSITE SIGNS, THEN THERE IS AT LEAST ONE ZERO OF BETWEEN a AND b .**15** THE GRAPH OF A POLYNOMIAL FUNCTION HAS AT MOST turning points AND INTERSECTS THE X-AXIS AT MOST TIMES.**16** THE GRAPH OF EVERY POLYNOMIAL FUNCTION HAS NO SHARP CORNERS, SO IS A SMOOTH AND CONTINUOUS CURVE.

Review Exercises on Unit 1

1 IN EACH OF THE FOLLOWING, FIND THE QUOTIENT AND REMAINDER WHEN THE FIRST POLYNOMIAL IS DIVIDED BY THE SECOND:

A $x^3 + 7x^2 - 6x - 5$; $x + 1$

B $3x^3 - 2x^2 - 4x + 4$; $x + 1$

C $3x^4 + 16x^3 + 6x^2 - 2x - 13$; $x + 5$

D $2x^3 + 3x^2 - 6x + 1$; $x - 1$

E $2x^5 + 5x^4 - 4x^3 + 8x^2 + 1$; $2x^2 - x + 1$

F $6x^3 - 4x^2 + 3x - 2$; $2x^2 + 1$

2 PROVE THAT WHEN A POLYNOMIAL IS DIVIDED BY A FIRST DEGREE POLYNOMIAL THE REMAINDER IS $\frac{b}{a}$.**3** PROVE THAT IS A FACTOR OF WHERE N IS AN ODD POSITIVE INTEGER.**4** SHOW THAT $\sqrt{2}$ IS AN IRRATIONAL NUMBER.

Hint: $\sqrt{2}$ IS A ROOT OF $x^2 - 2$. DOES THIS POLYNOMIAL HAVE ANY RATIONAL ROOTS?

5 FIND ALL THE RATIONAL ZEROS OF:

A $f(x) = x^5 + 8x^4 + 20x^3 + 9x^2 - 27x - 27$

B $f(x) = (x - 1)(x(x + 1) + 2x)$

6 FIND THE VALUE OF k SO THAT:

- A** $2x^3 - 3x^2 - kx - 17$ DIVIDED BY 3 HAS A REMAINDER OF -2.
- B** $x - 1$ IS A FACTOR OF $x^3 + 2kx - 3$.
- C** $5x - 2$ IS A FACTOR OF $x^2 + kx + 15$.

7 SKETCH THE GRAPH OF EACH OF THE FOLLOWING:

- A** $f(x) = x^3 - 7x + 6; -4 \leq x \leq 3$
- B** $f(x) = x^4 - x^3 - 4x^2 + x + 1; -2 \leq x \leq 3$
- C** $f(x) = x^3 - 3x^2 + 4$
- D** $f(x) = \frac{1}{4}(1-x)(1+x^2)(x-2)$

8 SKETCH THE GRAPH OF THE FUNCTION EXPLAIN FOR EACH OF THE FOLLOWING CASES HOW THE GRAPHS DIFFER FROM THE GRADE DETERMINE WHETHER G IS ODD, EVEN OR NEITHER.

- | | |
|----------------------------|----------------------------|
| A $g(x) = f(x) + 3$ | B $g(x) = f(-x)$ |
| C $g(x) = -f(x)$ | D $g(x) = f(x + 3)$ |

9 THE POLYNOMIAL $P(x) = A(x-1)^2 + B(x+2)^2$ IS DIVIDED BY 1 AND -2. THE REMAINDERS ARE 3 AND -15 RESPECTIVELY. FIND THE VALUES OF

10 IF $x^2 + (c-2)x - c^2 - 3c + 5$ IS DIVIDED BY c , THE REMAINDER IS -1. FIND THE VALUE OF

11 IF $x - 2$ IS A COMMON FACTOR OF THE EXPRESSIONS $(m+n)x - n$ AND $x^2 + mx - 1$, FIND THE VALUES OF m AND n .

12 FACTORIZE FULLY:

- A** $x^3 - 4x^2 - 7x + 10$
- B** $2x^5 + 6x^4 + 7x^3 + 21x^2 + 5x + 15$.

13 A PSYCHOLOGIST FINDS THAT THE RESPONSES IN MICROSECONDS WITH AGE GROUP ACCORDING TO

$$R = y^4 + 2y^3 - 4y^2 - 5y + 14,$$

WHERE R IS RESPONSE IN MICROSECONDS AND y IS AGE GROUP IN YEARS. FOR WHAT AGE GROUP IS THE RESPONSE EQUAL TO 8 MICROSECONDS?

14 THE PROFIT OF A FOOTBALL CLUB AFTER A TAKEOVER IS MOD

$$p(t) = t^3 - 14t^2 + 20t + 120,$$

WHERE t IS THE NUMBER OF YEARS AFTER THE TAKEOVER. IN WHICH YEARS WAS MAKING A LOSS?

Unit

2

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- understand the laws of exponents for real exponents.
- know specific facts about logarithms.
- know basic concepts about exponential and logarithmic functions.
- solve mathematical problems involving exponents and logarithms.

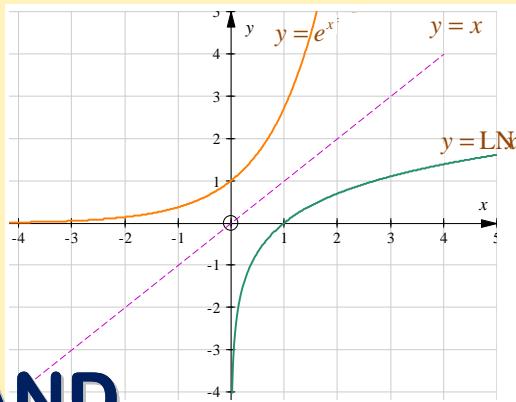
Main Contents

- 2.1 Exponents and logarithms**
- 2.2 Exponential functions and their graphs**
- 2.3 Logarithmic functions and their graphs**
- 2.4 Equations involving exponents and logarithms**
- 2.5 Applications of exponential and logarithmic functions**

Key Terms

Summary

Review Exercises



INTRODUCTION

EXPONENTIAL AND LOGARITHMIC FUNCTIONS COME INTO PLAY WHEN A VARIABLE IS RAISED TO AN EXPONENT, FOR EXAMPLE, IN AN EXPRESSION SUCH AS 2^6 . THESE EXPRESSIONS ARISE IN MANY APPLICATIONS AND ARE POWERFUL MATHEMATICAL TOOLS FOR SOLVING REAL LIFE PROBLEMS. THEY ARE USEFUL IN ANALYZING GROWTH OF POPULATIONS OF PEOPLE, ANIMALS, AND BACTERIA; DECAY OF SUBSTANCES; GROWTH OF MONEY AT COMPOUND INTEREST; ABSORPTION OF LIGHT BY AIR, WATER, OR GLASS, ETC.

IN THIS UNIT, YOU WILL STUDY THE VARIOUS PROPERTIES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS AND LEARN HOW THEY CAN BE USED IN SOLVING REAL LIFE PROBLEMS.

2.1 EXPONENTS AND LOGARITHMS

2.1.1 Exponents

WHILE SOLVING MATHEMATICAL PROBLEMS, THERE ARE OCCASIONS, YOU NEED TO MULTIPLY A NUMBER BY ITSELF. FOR EXAMPLE,

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64.$$

MATHEMATICIANS USE THE **exponent** TO REPRESENT A PRODUCT INVOLVING THE SAME FACTOR. FOR EXAMPLE,

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6.$$

EXPONENTS ARE FREQUENTLY USED IN MANY AREAS OF PHYSICS, ENGINEERING, FINANCE, AND COMPUTER SCIENCE, ETC., TO REPRESENT SITUATIONS WHERE QUANTITIES INCREASE OR DECREASE OVER TIME.



OPENING PROBLEM

ETHIOPIA HAS A POPULATION OF AROUND 80 MILLION PEOPLE AND IT IS ESTIMATED THAT THE POPULATION GROWS EVERY YEAR AT AN AVERAGE GROWTH RATE OF 2.3%. IF THE POPULATION CONTINUES AT THE SAME RATE,

- A** WHAT WILL BE THE POPULATION AFTER
 - I 10 YEARS? II 20 YEARS?
- B** HOW MANY YEARS WILL IT TAKE FOR THE POPULATION TO DOUBLE?
- C** WHAT WILL THE GRAPH OF THE NUMBER OF PEOPLE PLOTTED AGAINST TIME LOOK LIKE?

IT IS HOPED THAT AFTER STUDYING THE CONCEPTS DISCUSSED IN THIS CHAPTER, YOU WILL BE ABLE TO SOLVE PROBLEMS LIKE THE ONE GIVEN ABOVE.

Exponent notation

THE PRODUCT $2 \times 2 \times 2 \times 2$ IS WRITTEN⁶ AS 2^6 (READ “two to the power of six”) SIMILARLY $3 \times 3 \times 3 \times 3$ AND $4 \times 4 \times 4 \times 4 \times 4$.

IF n IS A POSITIVE INTEGER, a^n IS THE PRODUCT OF n FACTORS OF a .

I.E. $a^n = \underbrace{a \times a \times a \times \dots \times a}_{n \text{ FACTORS}}$

IN a^n , a IS CALLED **base**, n IS CALLED THE **exponent** AND a^n IS THE **power** OF a .

ACTIVITY 2.1

- 1 IDENTIFY THE BASE AND THE EXPONENT AND FIND THE VALUE OF EACH OF THE FOLLOWING POWERS:

A 4^3 B $(-2)^8$ C $\left(\frac{2}{7}\right)^4$ D $-(-1)^{23}$ E $(5T)^4$



- 2 FIND THE VALUES OF THE FOLLOWING POWERS:

A $(-1)^1$	B $(-1)^2$	C $(-1)^3$	D $(-1)^4$	E $(-1)^5$
F $(-1)^6$	G $(-2)^1$	H $(-2)^2$	I $(-2)^3$	J $(-2)^4$
K $(-2)^5$	L $(-2)^6$			

- 3 WHICH ONES GIVE YOU A NEGATIVE VALUE: A NEGATIVE BASE WITH AN ODD EXPONENT OR A NEGATIVE BASE RAISED TO AN EVEN EXPONENT?

EXAMPLE 1 EVALUATE:

A $(-3)^4$ B -3^4 C $(-3)^5$ D $-(-3)^5$

SOLUTION:

A $(-3)^4 = -3 \times -3 \times -3 \times -3 = 81$
B $-3^4 = -1 \times 3^4 = -1 \times 3 \times 3 \times 3 \times 3 = -81$
C $(-3)^5 = -3 \times -3 \times -3 \times -3 \times -3 = -243$
D $-(-3)^5 = -1 \times (-3)^5 = -1 \times -243 = 243$

REMEMBER THAT, IN $(-3)^4$ THE BASE IS -3 BUT 4 BY 3 IS THE BASE.

WHAT IS THE BASE IN $(-4t)^3$? THE BASE IS $-4t$ AND $(-4t)^3 = (-4t) \times (-4t) \times (-4t) = -64t^3$

TO WHAT BASE DOES THE EXPONENT 3 REFER IN $(-4t)^3$? IT REFERS TO $t \times t \times t$. THEREFORE THE EXPONENT 3 IN $(-4t)^3$ REFERS TO THE BASE t .

Laws of exponents

THE FOLLOWING GROUP WORK WILL HELP YOU RECALL THE LAWS OF EXPONENTS DISCUSSED IN GRADE 9.

Group Work 2.1

1 SIMPLIFY EACH OF THE FOLLOWING:

A $2^3 \times 2^5$

B $4^3 \times 4^8$

C $\frac{2^7}{2^3}$

D $\frac{2^{-5}}{2^{-9}}$

E $(2 \times 3)^3$

F $5^{-2} \times 3^{-2}$

G $(3^2)^5$

H $\left(\frac{2}{3}\right)^3$

I $a^c \times a^d$

2 WHICH LAW OF EXPONENTS DID YOU APPLY TO SIMPLIFY THE ABOVE EXPRESSIONS? (DISCUSS WITH YOUR FRIENDS).

IF THE BASES AND ARE NON-ZERO REAL NUMBERS AND THE EXPONENTS GERS, THEN,

1 $a^m \times a^n = a^{m+n}$

TO MULTIPLY POWERS OF THE SAME BASE, KEEP THE BASE AND ADD THE EXPONENTS.

2 $\frac{a^m}{a^n} = a^{m-n}$

TO DIVIDE POWERS OF THE SAME BASE, KEEP THE BASE AND SUBTRACT THE EXPONENTS.

3 $(a^m)^n = a^{m \times n} = a^{mn}$

TO TAKE A POWER OF A POWER, KEEP THE BASE AND MULTIPLY THE EXPONENTS.

4 $(a \times b)^n = a^n \times b^n$

THE POWER OF A PRODUCT IS THE PRODUCT OF THE POWERS.

5 $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

THE POWER OF A QUOTIENT IS THE QUOTIENT OF THE POWERS.

EXAMPLE 2 SIMPLIFY EACH OF THE FOLLOWING:

A $(4t)^2 \times (4t)^7$

B $r^8 \times r^{-3}$

C $\frac{10^3}{10^5}$

D $(x^2)^m$

E 16×4^{3t}

F $\left(\frac{2y}{25}\right)^2$

SOLUTION:

A $(4t)^2 \times (4t)^7 = (4t)^{2+7} = (4t)^9$

B $r^8 \times r^{-3} = r^{8+(-3)} = r^5$

C $\frac{10^3}{10^5} = 10^{3-5} = 10^{-2}$

D $(x^2)^m = x^{2 \times m} = x^{2m}$

E $16 \times 4^{3t} = 2^4 \times (2^2)^{3t} = 2^4 \times 2^{6t} = 2^{4+6t}$

F $\left(\frac{2y}{25}\right)^2 = \frac{2^2 \times y^2}{25^2} = \frac{4y^2}{625}$

ACTIVITY 2.2

- 1** EVALUATE EACH OF THE FOLLOWING USING THE LAW $\frac{a^m}{a^n} = a^{m-n}$

A $\frac{2^3}{2^3}$; IS 2^0 EQUAL TO WHY?

B $\frac{10^5}{10^5}$; IS 10^0 EQUAL TO WHY?

C $\frac{(-8)^3}{(-8)^3}$; IS $(-8)^0$ EQUAL TO WHY?



- 2** FROM YOUR ANSWERS, CAN YOU SUGGEST WHAT ~~EVER NONZERO~~ IS?

ANY NON-ZERO NUMBER RAISED TO ZERO IS ONE.

THAT IS, $a^0 = 1$, IF $a \neq 0$

EXAMPLE 3

A $8^0 = 1$

B $(-100)^0 = 1$

C $\left(\frac{3}{5}\right)^0 = 1$

D $(\sqrt{23})^0 = 1$

E $(0.153)^0 = 1$

Group Work 2.2

OBSERVE THE FOLLOWING:



- $\frac{2^2}{2^5} = \frac{2 \times 2}{2 \times 2 \times 2 \times 2 \times 2} = \frac{1}{2 \times 2 \times 2} = \frac{1}{2^3}$
- IF WE USE THE RULE $a^{m-n} = \frac{a^m}{a^n}$ $\frac{2^2}{2^5} = 2^{2-5} = 2^{-3}$

A USING THE ABOVE TWO STEPS TRY $\frac{3^5}{3^7}$ TO SIMPLIFY

B DISCUSS THE RELATIONSHIP BETWEEN:

I $\frac{1}{2^3}$ AND 2^3 II $\frac{1}{3^2}$ AND 3^2

C WHAT CAN YOU CONCLUDE ABOUT a^{-n}

FOR $a \neq 0$ AND $n > 0$

$$a^{-n} = \frac{1}{a^n}$$

ANY NON-ZERO NUMBER RAISED TO A NEGATIVE EXPONENT IS THE RECIPROCAL OF THE SAME POWER WITH POSITIVE EXPONENT.

EXAMPLE 4 SIMPLIFY AND WRITE YOUR ANSWER AS A NONNEGATIVE EXPONENTIAL

A 2^{-3}

B $\frac{2^4}{2^9}$

C $\left(\frac{3}{2}\right)^{-3}$

SOLUTION:

A $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

B $\frac{2^4}{2^9} = 2^{(4-9)} = 2^{-5} = \frac{1}{2^5} = \frac{1}{32}$

C $\left(\frac{3}{2}\right)^{-3} = \frac{1}{\left(\frac{3}{2}\right)^3} = \frac{1}{\left(\frac{3^3}{2^3}\right)} = 1 \times \frac{2^3}{3^3} = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

IN EXAMPLE 4C ABOVE YOU HAVE SEEN THAT $\left(\frac{3}{2}\right)^{-3} = \left(\frac{2}{3}\right)^3$. USE THIS TECHNIQUE TO SIMPLIFY THE FOLLOWING:

EXAMPLE 5

A $\left(\frac{4}{5}\right)^{-1}$

B $\left(\frac{2}{5}\right)^{-4}$

C $\left(\frac{3}{10}\right)^{-2}$

SOLUTION:

A $\left(\frac{4}{5}\right)^{-1} = \frac{5}{4}$

B $\left(\frac{2}{5}\right)^{-4} = \left(\frac{5}{2}\right)^4 = \frac{625}{16}$

C $\left(\frac{3}{10}\right)^{-2} = \left(\frac{10}{3}\right)^2 = \frac{100}{9}$

Note: FOR $a \neq 0$, $a^{-1} = \frac{1}{a}$

THE ABOVE EXAMPLES LEAD YOU TO THE FOLLOWING FACT:

IF a AND b ARE NON-ZERO REAL NUMBERS THEN IT IS ALWAYS TRUE THAT FOR

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

Exercise 2.1

- 1** USE THE LAWS OF EXPONENTS TO SIMPLIFY THE EXPONENTIAL EXPRESSIONS:

A $t^2 \times t$ B $t^3 \times t \times t^5$ C $r \times r^4 \times r^5 \times r$ D $a^3 \times a \times a^{-5}$

E $\frac{7^6}{7^4}$ F $\frac{(-3y)^2}{(-3y)^5}$ G $\frac{(2x)^7}{(2x)^8}$ H $b^{2x} \div b$

I $(5^5)^{2n}$ J $(b^y)^x$ K $(7^3)^{-2}$ L $(a^{3x})^2$

- 2** WRITE EACH OF THE FOLLOWING WITH A PRIME NUMBER AS THE

A 81 B $\frac{16^{2x+3}}{16^{2x-3}}$ C $\frac{49^x}{7^y}$ D $64^a \times 4^a$

3 REMOVE THE BRACKETS FROM EACH OF THE FOLLOWING EXPRESSIONS.

A $(xyz)^2$ B $(2ab^2)^5$ C $\left(\frac{9}{3}\right)^2$ D $\left(-\frac{2}{2n}\right)^6$

4 SIMPLIFY AND GIVE YOUR ANSWERS IN SIMPLEST RATIONAL FORM.

A $\left(\frac{3}{2}\right)^0$ B $\left(\frac{8}{3}\right)^{-2}$ C $\left(\frac{1}{4^{-3}}\right)^{-1}$ D $(-2)^{-5}$ E $(3x^2)^{-3}$

Rational exponents

SO FAR WE HAVE CONSIDERED EXPRESSIONS WHERE THE EXPONENT IS AN INTEGER. NOW WE KNOW WHAT 3^0 AND 7^0 MEAN. BUT WHAT DO EXPRESSIONS $3^{1/2}$ AND $7^{2/3}$ MEAN? WE NOW EXTEND THE LAWS OF EXPONENTS TO RATIONAL NUMBERS.

ACTIVITY 2.3

USING THE LAW $a^n = a^{m+n}$, DO THE FOLLOWING:



1 A SIMPLIFY

I $6^{\frac{1}{2}} \times 6^{\frac{1}{2}}$ II $\sqrt{6} \times \sqrt{6}$

B COMPARE THE RESULTS WITH THE RESULTS. WHAT DO YOU NOTICE?

2 A SIMPLIFY

I $6^{\frac{1}{3}} \times 6^{\frac{1}{3}} \times 6^{\frac{1}{3}}$ II $\sqrt[3]{6} \times \sqrt[3]{6} \times \sqrt[3]{6}$

B COMPARE THE RESULTS WITH THE RESULTS. WHAT DO YOU NOTICE?

3 A SIMPLIFY

I $2^{\frac{1}{4}} \times 2^{\frac{1}{4}} \times 2^{\frac{1}{4}} \times 2^{\frac{1}{4}}$ II $\sqrt[4]{2} \times \sqrt[4]{2} \times \sqrt[4]{2} \times \sqrt[4]{2}$.

B COMPARE THE RESULTS WITH THE RESULTS. WHAT DO YOU NOTICE?

4 IN GENERAL, WHAT DO YOU THINK IS TRUE ABOUT

IF $a \geq 0$ AND n IS AN INTEGER, WITH $a^{\frac{1}{n}} = \sqrt[n]{a}$. THIS ALSO HOLDS WHEN n IS ODD. (READ AS "THE n TH ROOT OF")

EXAMPLE 6 EXPRESS EACH OF THE FOLLOWING IN THE FORM $\frac{1}{\sqrt[n]{a}}$

A $\sqrt[4]{3}$ B $\sqrt[5]{64}$ C $\frac{1}{\sqrt[9]{9}}$ D $\frac{\left(\sqrt[3]{32}\right)^2}{4^{\frac{5}{3}}}$

SOLUTION:

A $\sqrt[4]{3} = 3^{\frac{1}{4}}$ B $\sqrt[5]{64} = 64^{\frac{1}{5}}$ C $\frac{1}{\sqrt{9}} = \frac{1}{9^{\frac{1}{2}}} = \frac{1}{(3^2)^{\frac{1}{2}}} = \frac{1}{3} = 3^{-1}$

D $\frac{\left(\sqrt[3]{32}\right)^2}{4^{\frac{5}{3}}} = \frac{\left(32^{\frac{1}{3}}\right)^2}{\left(2^2\right)^{\frac{5}{3}}} = \frac{32^{\frac{2}{3}}}{2^{\frac{10}{3}}} = \frac{\left(2^5\right)^{\frac{2}{3}}}{2^{\frac{10}{3}}} = \frac{2^{\frac{10}{3}}}{2^{\frac{10}{3}}} = 2^{\left(\frac{10}{3} - \frac{10}{3}\right)} = 2^0 = 1$

WHAT IS THE RESULT FOR $6^{\frac{2}{3}}$?

$$6^{\frac{2}{3}} \times 6^{\frac{2}{3}} \times 6^{\frac{2}{3}} = 6^{\frac{2}{3} + \frac{2}{3} + \frac{2}{3}} = 6^{\frac{6}{3}} = 6^2$$

ALSO $6^{\frac{2}{3}} \times 6^{\frac{2}{3}} \times 6^{\frac{2}{3}} = \left(6^{\frac{2}{3}}\right)^3 = 6^2$ using the law $(a^m)^n = a^{m \times n}$

THEREFORE $= 6 \left(6^2\right)^{\frac{1}{3}} = \sqrt[3]{6^2}$

IN GENERAL, IF AND n ARE INTEGERS, THEN $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$.

EXAMPLE 7 EXPRESS IN THE FORM $\frac{m}{n}$, WITH m BEING A PRIME NUMBER.

A $\sqrt[5]{64}$

B $\sqrt[3]{16}$

C $\sqrt[8]{27}$

SOLUTION:

A $\sqrt[5]{64} = 64^{\frac{1}{5}} = (2^6)^{\frac{1}{5}} = 2^{\frac{6}{5}}$

B $\sqrt[3]{16} = 16^{\frac{1}{3}} = (2^4)^{\frac{1}{3}} = 2^{\frac{4}{3}}$

C $\sqrt[8]{27} = 27^{\frac{1}{8}} = (3^3)^{\frac{1}{8}} = 3^{\frac{3}{8}}$

REMEMBER THATS NOT A REAL NUMBER NEGATIVE AND AN EVEN NATURAL NUMBER.

HOWEVER \sqrt{a} IS A REAL NUMBER NEGATIVE AND AN ODD NATURAL NUMBER.

FOR EXAMPLE, $\sqrt{-4}$, $\sqrt[4]{-5}$, $\sqrt[6]{-9}$, $\sqrt[8]{-8}$, ETC, ARE NOT REAL NUMBERS, WHEREAS, $\sqrt{-32}$, $\sqrt[3]{-8}$, $\sqrt[9]{-81}$, ETC, ARE REAL NUMBERS.

EXAMPLE 8 SIMPLIFY EACH OF THE FOLLOWING:

A $\sqrt[3]{-27}$

B $\sqrt[7]{-128}$

C $\frac{\sqrt[5]{-32}}{\sqrt[3]{-64}}$

SOLUTION:

A $\sqrt[3]{-27} = \sqrt[3]{(-3) \times (-3) \times (-3)} = -3$

B $\sqrt[7]{-128} = \sqrt[7]{(-2)^7} = (-2^7)^{\frac{1}{7}} = -2$

C $\frac{\sqrt[5]{-32}}{\sqrt[3]{-64}} = \frac{\sqrt[5]{(-2)^5}}{\sqrt[3]{(-4)^3}} = \frac{-2}{-4} = \frac{1}{2}$

WE CONCLUDE OUR DISCUSSION OF RATIONAL EXPONENTS BY THE FOLLOWING REMARK:
ALL RULES FOR INTEGRAL EXPONENTS DISCUSSED EARLIER ALSO HOLD TRUE FOR RATIONAL EXPONENTS.

Irrational exponents

NOW CONSIDER EXPRESSIONS WITH IRRATIONAL EXPONENTS, SUCH AS $2^{\sqrt{5}}$.

EXAMPLE 9 WHICH NUMBER IS THE LARGEST, $2^{\sqrt{2}}$, $2^{\sqrt{3}}$, OR $2^{\sqrt{5}}$?

SOLUTION: THE ANSWER WILL NOT BE SIMPLE BECAUSE WE DO NOT KNOW THE EXACT VALUE OF $2^{\sqrt{5}}$.

TO APPROXIMATE THE NUMBER $2^{\sqrt{5}}$, CONSIDER THE FOLLOWING TABLE FOR 2^x :

x	-4	-3	-2	-1	0	1	2	3	4	5
2^x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32

FROM THE TABLE WE SEE THAT FOR ANY VALUES OF x_1 AND x_2 , WHERE $x_1 < x_2$, THEN $2^{x_1} < 2^{x_2}$.

THEREFORE, SINCE $\sqrt{2} < 2.3$, WE HAVE $2^{\sqrt{2}} < 2^{\sqrt{5}} < 2^{2.3}$.

LET US NOW TAKE CLOSER APPROXIMATIONS USING A CALCULATOR.

$$\begin{aligned} 2^{2.2} &< 2^{\sqrt{5}} < 2^{2.3} \\ 2^{2.23} &< 2^{\sqrt{5}} < 2^{2.24} \\ 2^{2.236} &< 2^{\sqrt{5}} < 2^{2.237} \\ 2^{2.2360} &< 2^{\sqrt{5}} < 2^{2.2361} \\ 2^{2.23606} &< 2^{\sqrt{5}} < 2^{2.23607} \end{aligned}$$



AS WE CAN SEE FROM THE ABOVE LIST, THE NUMBERS $2^{2.23606}$ AND $2^{2.23607}$ APPROACH TO THE SAME NUMBER. SIMILARLY, THE NUMBERS $2^{2.2360}$, $2^{2.2361}$, ... ALSO APPROACH TO THE SAME NUMBER. SO $2^{\sqrt{5}}$ IS BOUNDED BY TERMS OF CONVERGING RATIONAL APPROXIMATIONS. USING A CALCULATOR, WE FIND THAT $2^{\sqrt{5}} \approx 4.7111$, TO FOUR DECIMAL PLACES; IT IS A NUMBER BETWEEN 4.7 AND 4.8. SO THE LARGEST OF THE NUMBERS MUST BE $2^{\sqrt{5}}$.

EXAMPLE 10 GIVE AN APPROXIMATION TO**SOLUTION:** RECALL THAT $\pi \approx 3.1415926$. A CALCULATOR GIVES THE ROUNDED VALUES:

$$\begin{aligned}3^{3.1} &\approx 30.1353 \\3^{3.14} &\approx 31.4891 \\3^{3.141} &\approx 31.5237 \\2^{3.1415} &\approx 31.5411 \\3^{3.14159} &\approx 31.5442 \\3^{3.141592} &\approx 31.5443 \\3^{3.1415926} &\approx 31.5443\end{aligned}$$



HENCE $3^{\pi} \approx 31.5443$, ROUNDED TO FOUR DECIMAL PLACES. A TEN-PLACE CALCULATOR APPROXIMATES $3^{3.141592654} \approx 31.5442807002$.

THE ABOVE TWO EXAMPLES SUGGEST THE FOLLOWING:

IF x IS AN IRRATIONAL NUMBER, AND x_1 AND x_2 ARE REAL NUMBERS BETWEEN x_1 AND x_2 , THEN FOR ALL POSSIBLE CHOICES OF RATIONAL NUMBERS $x_1 < x < x_2$,

THE ABOVE STATEMENT ABOUT IRRATIONAL EXPONENTS IS TRUE FOR EXPRESSIONS DEFINED NOT ONLY FOR INTEGRAL AND RATIONAL EXPONENTS BUT ALSO FOR IRRATIONAL EXPONENTS.

EXAMPLE 11 SIMPLIFY EACH OF THE FOLLOWING:

A $4^{\sqrt{3}} \times 4^{\sqrt{12}}$

B $\frac{2^{\sqrt{5}} \times 2^{\sqrt{20}}}{8^{\sqrt{5}}}$

C $\frac{3^{\sqrt{2}} \times 3^{-\sqrt{2}} \times 27^{\sqrt{2}}}{3^{\sqrt{8}}}$.

SOLUTION:

A $4^{\sqrt{3}} \times 4^{\sqrt{12}} = 4^{\sqrt{3}} \times 4^{2\sqrt{3}} = 4^{\sqrt{3} + 2\sqrt{3}} = 4^{3\sqrt{3}} = (4^3)^{\sqrt{3}} = 64^{\sqrt{3}}$

B $\frac{2^{\sqrt{5}} \times 2^{\sqrt{20}}}{8^{\sqrt{5}}} = \frac{2^{\sqrt{5}+2\sqrt{5}}}{8^{\sqrt{5}}} = \frac{2^{3\sqrt{5}}}{8^{\sqrt{5}}} = \frac{(2^3)^{\sqrt{5}}}{8^{\sqrt{5}}} = \frac{8^{\sqrt{5}}}{8^{\sqrt{5}}} = 1$

C $\frac{3^{\sqrt{2}} \times 3^{-\sqrt{2}} \times 27^{\sqrt{2}}}{3^{\sqrt{8}}} = \frac{3^0 \times 3^{3\sqrt{2}}}{3^{\sqrt{8}}} = \frac{3^{3\sqrt{2}}}{3^{\sqrt{8}}} = \frac{3^{3\sqrt{2}}}{3^{2\sqrt{2}}} = 3^{(3\sqrt{2}-2\sqrt{2})} = 3^{\sqrt{2}}$

THE LAWS OF EXPONENTS DISCUSSED EARLIER FOR INTEGRAL AND RATIONAL EXPONENTS HOLD TRUE FOR IRRATIONAL EXPONENTS.

IN GENERAL, IF a AND b ARE POSITIVE NUMBERS AND r AND s ARE REAL NUMBERS, THEN

1 $a^r \times a^s = a^{r+s}$

2 $\frac{a^r}{a^s} = a^{r-s}$

3 $(a^r)^s = a^{rs}$

4 $(a \times b)^s = a^s \times b^s$

5 $\left(\frac{a}{b}\right)^s = \frac{a^s}{b^s}$

Group Work 2.3

DISCUSS IN GROUPS AND ANSWER EACH OF THE FOLLOWING QUESTIONS.



1 A Is $24 > 23$; Is $24^2 > 23^2$?

B Is $81 > 16$; Is $81^{\frac{1}{4}} > 16^{\frac{1}{4}}$?

C Is $20 > 10$; Is $20^{-2} > 10^{-2}$?

D Is $\frac{1}{100} < \frac{1}{10}$; Is $\left(\frac{1}{100}\right)^2 < \left(\frac{1}{10}\right)^2$?

E Is $\frac{1}{100} < \frac{1}{10}$; Is $\left(\frac{1}{100}\right)^{-2} < \left(\frac{1}{10}\right)^{-2}$?

2 A Let $a > b > 1$.

Is $a^x > b^x$, for $x > 0$?

Is $a^x > b^x$, for $x < 0$?

B Let $0 < a < b < 1$.

Is $a^x < b^x$, for $x > 0$?

Is $a^x < b^x$, for $x < 0$?

Exercise 2.2

SIMPLIFY EACH OF THE FOLLOWING EXPRESSIONS USING ONE OR MORE OF THE LAWS OF EXPONENTS.

A $a^2 \times a \times a^3$

B $(2^{-3} + 3^{-2})^{-1}$

C $(\sqrt[3]{343})^{-2}$

D $(2a^{-3} \times b^2)^{-2}$

E $\frac{(3a)^4}{(3a)^3}$

F $\left(\frac{a^2}{b}\right)^3$

G $\left(\frac{a^3}{b^5}\right)^{-2}$

H $\frac{(n^2)^4 \times (n^3)^{-2}}{n^{-1}}$

I $\left(\frac{m^{-3}m^3}{n^{-2}}\right)^{-2}$

J $\left(\frac{m^{\frac{-2}{3}}}{n^{\frac{-1}{2}}}\right)^{-6}$

K $\left(\frac{a^{\frac{-1}{3}}b^{\frac{1}{2}}}{a^{\frac{-1}{4}}b^{\frac{1}{3}}}\right)^6$

L $\frac{\left(3^{\sqrt{2}}\right)^2 \times 9^{-\sqrt{3}}}{3^{-\sqrt{12}}}$

M $\left(2^{\sqrt{3}}\right)^2 \div \left(4^{\sqrt{3}}\right)^{-2}$

N $\left(\frac{2^{\sqrt{5}} \times 2^{-\sqrt{5}}}{\sqrt{2}}\right)^2$

O $\frac{2^{\sqrt{2}} \times 2^{-\sqrt{2}} \times 32^{\sqrt{2}}}{4^{\sqrt{8}}}$

P $\sqrt[6]{64a^6b^{-2}}$

2.1.2 Logarithms

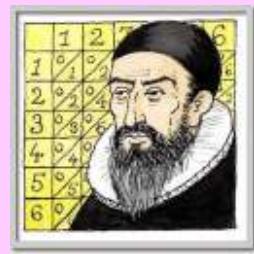
Logarithms CAN BE THOUGHT "OF AS INVERSE" OF EXPONENTS.

FOR EXAMPLE, WE KNOW THAT THE FOLLOWING EXPONENTIAL EQUATION IS TRUE: $3^2 = 9$.
IN THIS CASE, **base** IS 3 AND **exponent** IS 2. WE WRITE THIS EQUATION IN **LOGARITHM form** (WITH IDENTICAL MEANING) AS

WE READ THIS AS "THE LOGARITHM OF 9 TO THE BASE 3 IS 2".

HISTORICAL NOTE:

LOGARITHMS were developed in the 17th century by the Scottish mathematician, John Napier (1550-1617). They were clever methods of reducing long multiplications into much simpler additions and reducing divisions into subtractions. While he was young, Napier had to help his father, who was a tax collector. John got sick of multiplying and dividing large numbers all day and devised logarithms to make his life easier!



SINCE $2^4 = 16$, WE CAN SAY THAT $\log_2 16 = 4$.

AS $10^3 = 1000$, $3 = \log_{10} 1000$.

THE FOLLOWING ACTIVITY WILL HELP YOU LEARN HOW TO CONVERT EXPONENTIAL STATEMENTS INTO LOGARITHMIC STATEMENTS AND VICE VERSA.

ACTIVITY 2.4

COMPLETE THE FOLLOWING TABLE:

Exponential statement	Logarithmic statement
$2^3 = 8$	$\log_2 8$
$2^5 = 32$	
$2^6 = 64$	
	$\log_{10} 1000$
$2^x = y$	



IN GENERAL,

FOR A FIXED **positive** NUMBER $b \neq 1$, AND FOR EACH
 $b^c = a$, IF AND ONLY IF $\log_b a = c$.

OBSERVE FROM THE ABOVE NOTE THAT EVERY LOGARITHMIC STATEMENT CAN BE TRA
EXPONENTIAL STATEMENT AND VICE VERSA.

Note: THE VALUE OF $\log_b x$ IS THE ANSWER TO THE QUESTION: "TO WHAT POWER MUST b BE RAISED TO PRODUCE x ?"

EXAMPLE 1 WRITE AN EQUIVALENT LOGARITHMIC STATEMENT FOR:

A $3^4 = 81$

B $4^3 = 64$

C $8^{\frac{1}{3}} = 2$

SOLUTION:

A FROM $3^4 = 81$, WE DEDUCE $\log_3 81 = 4$

B FROM $4^3 = 64$, WE HAVE $\log_4 64 = 3$

C SINCE $8^{\frac{1}{3}} = 2$, $\log_8 2 = \frac{1}{3}$

EXAMPLE 2 WRITE AN EQUIVALENT EXPONENTIAL STATEMENT FOR:

A $\log_2 144$

B $\log_{\frac{1}{64}} -$

C $\log_{10} \sqrt{10} = \frac{1}{2}$

SOLUTION:

A FROM $\log_2 144$, WE DEDUCE THAT $2^x = 144$.

B $\log_{\frac{1}{64}} -$ IS THE SAME AS SAYING $\frac{1}{64} = -$

C $\log_{10} \sqrt{10} = \frac{1}{2}$ CAN BE WRITTEN IN EXPONENTIAL FORM AS

EXAMPLE 3 FIND:

A $\log_2 6$

B $\log_3 \frac{1}{9}$

C $\log_{1000} 1$

SOLUTION:

A TO FIND $\log_2 6$, YOU ASK "what power must 2 be raised to get 64?"

AS $2^6 = 64$, $\log_2 64$ OR FROM THE EXPONENTIAL EQUATIONS DISCUSSED IN GRADE 9, YOU CAN FORM THE EQUATION

SOLVING THIS GIVES $2^6 \Rightarrow x = 6$.

... remember that $b^x = b^y$, if and only if $x = y$, for $b > 0, b \neq 1$.

B TO FIND $\log_3 \frac{1}{9}$, WE ASK "to what power must 3 be raised to get $\frac{1}{9}$?"

AS $3^{-2} = \frac{1}{9}$, $\log_3 \frac{1}{9} = -2$ OR $3^x = \frac{1}{9} \Rightarrow 3^x = 3^{-2} \Rightarrow x = -2$.

C TO FIND $\log_{1000} 1$, WE ASK "to what POWER must 1000 be raised to get 10?"
AS $1000^{\frac{1}{3}} = 10$, $\log_{1000} 10 = \frac{1}{3}$ OR $1000^x = 10 \Rightarrow 10^{3x} = 10^1 \Rightarrow 3x = 1$
 $\Rightarrow x = \frac{1}{3}$.

Exercise 2.3

1 WRITE AN EQUIVALENT LOGARITHMIC STATEMENT FOR:

A $100^2 = 10000$ **B** $2^{-5} = \frac{1}{32}$ **C** $125^{\frac{1}{3}} = 5$ **D** $8^{\frac{-2}{3}} = \frac{1}{4}$

2 WRITE AN EQUIVALENT EXPONENTIAL STATEMENT FOR:

A $\log_{10} 10000 = 4$ **B** $\log_{\sqrt{49}} 1 =$
C $\log_{10} 0.1 = -1$ **D** $\log_2 \frac{1}{4} = -$

3 FIND:

A \log_2 **B** $\log_8 8$
C $\log_{100} 1000$ **D** \log_{49}

Laws of logarithms

THE FOLLOWING GROUP WORK WILL HELP YOU OBSERVE DIFFERENT LAWS WHILE USING LOG

Group Work 2.4



1 FIND:

A $\log_8 + \log_2$; COMPARE THE RESULTS $(\text{WITH } 8)$
B $\log_{10} 100 + \log_{10} 1000$; COMPARE THE RESULTS $(\text{WITH } 100 \times 1000)$
C $\log_3 + \log_3 \left(\frac{1}{27} \right)$; COMPARE THE RESULTS $(\text{WITH } \frac{1}{27})$

FROM YOUR ANSWERS, CAN YOU SUGGEST A POSSIBLE LOGICAL APPLICATION FOR

2 FIND:

A $\log_8 - \log_2$; COMPARE THE RESULTS $(\text{WITH } \frac{8}{2})$
B $\log_{10} 100 - \log_{10} 1000$; COMPARE THE RESULTS $(\text{WITH } \frac{100}{1000})$.
C $\log_3 - \log_3 \frac{1}{27}$; COMPARE THE RESULTS $(\text{WITH } \frac{1}{27})$

FROM YOUR ANSWERS, CAN YOU SUGGEST A POSSIBLE LOGICAL APPLICATION FOR

3 FIND:

- A $\log_3 3$; COMPARE THE RESULT WITH
- B $\log_{10} 100$; COMPARE THE RESULT WITH
- C $\frac{1}{2} \log_2 1$; COMPARE THE RESULT WITH $\left(\frac{1}{2}\right)^2$

FROM YOUR ANSWERS, CAN YOU SUGGEST A POSSIBLE SIMPLIFICATION FOR

4 FIND:

- A $\log_3 3$
- B \log_3
- C $\log_{10} 100$
- D $\log_{\frac{1}{3}} \frac{1}{3}$

FROM YOUR ANSWERS, CAN YOU SUGGEST A POSSIBLE SIMPLIFICATION FOR
AND $b \neq 1$?

5 FIND:

- A $\log_3 1$
- B \log_1
- C $\log_{\frac{1}{3}} 1$
- D $\log_{1000} 1$

FROM YOUR ANSWERS, CAN YOU SUGGEST A POSSIBLE SIMPLIFICATION FOR
AND $b \neq 1$?

THE FOLLOWING ARE LAWS OF LOGARITHMS:

IF b, x AND y ARE POSITIVE NUMBERS AND $b \neq 1$,

- I $\log_b xy = \log_b x + \log_b y$
- II $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$
- III FOR ANY REAL NUMBER $x^k = k \log_b x$

Note: IF $b > 0$ AND $b \neq 1$, THEN

- I $\log_b b = 1$
- II $\log_b 1 = 0$

EXAMPLE 4 USE THE LAWS OF LOGARITHMS TO FIND:

- A $\log_2 16 - \log_2$
- B $\log_4 \sqrt{16} - \log_4 4$
- C $2(\log_{10} 100)$
- D $\log_{10} \sqrt[4]{0.01}$

SOLUTION:

A $\log_2 16 - \log_2 = \log_2 (16 \leftarrow \log_2 16 = 6)$

... using the law $\log_b xy = \log_b x + \log_b y$

B $\log_4 \sqrt{16} - \log_4 4 = \log_4 \frac{\sqrt{16}}{4} = \log_4 \frac{4}{4} = \log_4 1 = 0$

... using the law $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$

C $2(\log_{10} 100) = 2(\log_{10} 100 - \log_{10} 10) = 2 \log_{10} \left(\frac{100}{10} \right) = 2 \log_{10} 10 = 2$

... using the law $\log_b \left(\frac{x}{y} \right) = \log_b x - \log_b y$

D $\log_{10} \sqrt[4]{0.01} = \log_{10} (0.01)^{\frac{1}{4}} = \log_{10} \left(\frac{1}{100} \right)^{\frac{1}{4}} = \log_{10} (10^2)^{-\frac{1}{4}} = 10^{-\frac{2}{4}} \log_{10}$

$$= \frac{-2}{4} \log_{10} 10 = \frac{-2}{4} \times 1 = \frac{-2}{4} = \frac{-1}{2}$$

... using the law $\log_b (x^k) = k \log_b x$

Two additional laws of logarithms

IF a, b AND c ARE POSITIVE REAL NUMBERS, $b \neq 1$, THEN

I $\log_a c = \frac{\log_c}{\log_a}$ ("CHANGE OF BASE LAW") $b^{\log_b c} = c$

EXAMPLE 5 USING THE ABOVE TWO LAWS FIND

A $\log_{10} 6$

B $\log_3 6$ (GIVEN THAT $\log_{10} 2 = 0.3010$ AND $\log_{10} 3 = 0.4771$)

C $10^{\log_6 7}$

SOLUTION

A $\log_{10} 6 = \frac{\log 6}{\log 10} = \frac{6}{4} = \frac{3}{2}$ OR

$$\log_{10} 64 = \frac{\log_{10} 64}{\log_{10} 10} = \frac{\log_{10} 4}{\log_{10} 2} = \frac{3 \log_{10} 4}{3 \log_{10} 2} = \frac{\log_{10} 4}{\log_{10} 2} = \frac{2}{1} = 2$$

... you can use any base $b > 0$, $b \neq 1$

B $\log_3 6 = \frac{\log_{10} 6}{\log_{10} 3} = \frac{0.3010}{0.4771} = 0.6309$

C $10^{\log_6 7} = 7$

Exercise 2.4

1 FIND:

A $\log_{10} 121$ **B** $\log_6 6$ **C** $\log_{10} 100000$ **D** $\log_5 25$

E $\log_{\sqrt{3}} 3$ **F** $\log_3 3$ **G** $\log_{10} \sqrt[5]{100}$ **H** $\log_{\frac{1}{5}} 125$

2 SIMPLIFY:

A $\log_2(64 \cdot 1024)$

B $\log_{\frac{256}{128}} \frac{32}{64}$

C $\log_{512} 3$

D $\log_{10} 2 \times 10^{-3}$

E $\log_2 \left(\frac{128 \times 64}{512} \right)$

F $\log_3 + \log_{\frac{1}{27}}$

G $\log_2 64 \div \log_2 7$

3 USING THE LAW $\log_a c = \frac{\log_b c}{\log_b a}$ OR $b^{\log_b c} = c$ FIND:

A $\log_{\left(\frac{1}{3}\right)} 8$

B $\log_{\left(\frac{1}{2}\right)} \frac{1}{8}$

C $\log_{\frac{1}{3}} \frac{1}{27}$

D $5 \log_5 3$

E $6 \log_6 0$

4 If $\log_{10} 2 = 0.3010$ AND $\log_{10} 3 = 0.4771$, THEN FIND:

A $\log_{10} \sqrt{3}$

B $\log_{\frac{1}{2}} 5$

C $\log_{\frac{1}{3}} 0.00$

Logarithms in base 10 (common logarithms)

OUR DECIMAL SYSTEM IS BASED ON NUMBERS 1, 10, 100, 1000, 10000, 100000, 1000000, ... OR HEXADECIMAL 1, 10, 100, 1000, 10000, 100000, 1000000, ...

$10000 = 10^4$

$0.0001 = 10^{-4}$

$1000 = 10^3$

$0.001 = 10^{-3}$

$100 = 10^2$

$0.01 = 10^{-2}$

$10 = 10^1$

$0.1 = 10^{-1}$

$1 = 10^0$

ALSO NUMBERS $\sqrt{10}$, $\sqrt[3]{10}$, $10\sqrt{10}$ AND $\frac{1}{\sqrt[5]{10}}$ CAN BE WRITTEN AS

$10^{\frac{1}{2}}, 100^{\frac{1}{2}}, 10^1 \times 10^{\frac{1}{2}} = 10^{\frac{3}{2}}$ AND $10^{-\frac{1}{5}}$ RESPECTIVELY.

IN FACT, ALL POSITIVE NUMBERS CAN BE WRITTEN IN THE FORM $a \times 10^n$ WHERE a IS A NUMBER BETWEEN 1 AND 10, AND n IS AN INTEGER. THIS IS KNOWN AS SCIENTIFIC NOTATION.

INTRODUCING THE CONCEPT OF LOGARITHMS. THE LOGARITHM OF A POSITIVE NUMBER TO BASE 10 IS CALLED A COMMON LOGARITHM.

THE COMMON LOGARITHM IS USUALLY THE MOST CONVENIENT ONE TO USE FOR COMPUTATIONS DUE TO THE EXTENSIVE USAGE OF VARIOUS ADVANCED CALCULATORS, OF THE USAGE OF LOGARITHMS AT PRESENT IS NOT AS IT WAS IN THE PAST. HOWEVER, THE OPERATIONS LIKE DIVISION, MULTIPLICATION, ROOTS AND POWERS CAN BE EASILY PERFORMED USING COMMON LOGARITHMS.

ONE IMPORTANT USAGE OF COMMON LOGARITHMS IS IN THEIR USE IN SIMPLIFYING COMPUTATIONS. DUE TO THE EXTENSIVE USAGE OF VARIOUS ADVANCED CALCULATORS, OF THE USAGE OF LOGARITHMS AT PRESENT IS NOT AS IT WAS IN THE PAST. HOWEVER, THE OPERATIONS LIKE DIVISION, MULTIPLICATION, ROOTS AND POWERS CAN BE EASILY PERFORMED USING COMMON LOGARITHMS.

THIS IS DUE TO THE FACT THAT ANY LOGARITHM TO BASE OTHER THAN 10 CAN BE EXpressed AS A COMMON LOGARITHM SO THAT ONE CAN USE THE TABLE OF COMMON LOGARITHM FOR STANDARD BOOKS AND MATHEMATICAL TABLES.

A COMMON LOGARITHM IS USUALLY WRITTEN WITHOUT INDICATING ITS BASE. FOR EXAMPLE, $\log 10^x$ IS SIMPLY DENOTED BY \log . SO IF A LOGARITHM IS GIVEN WITH NO BASE, WE TAKE IT TO BE BASE 10.

ACTIVITY 2.5

FIND THE FOLLOWING COMMON LOGARITHMS:



A $\log \sqrt[5]{1}$ **B** $\log 0.000$ **C** $\log 1$ **D** $\log \left(\frac{10}{10^n}\right)$

EXAMPLE 6 FIND THE FOLLOWING COMMON LOGARITHMS:

A $\log 100,00$ **B** $\log \sqrt[3]{100}$ **C** $\log 0.001$

SOLUTION

A $\log 100,00 = 5$ BECAUSE $10^5 = 100,000$ OR $\log 100,00 = \log 10^5 = 5 \log 10$

B $\log \sqrt[3]{100} = \frac{2}{3}$ BECAUSE $\sqrt[3]{100} = \sqrt[3]{10^2} = 10^{\frac{2}{3}}$ OR

$$\log \sqrt[3]{100} = \log 10^{\frac{2}{3}} = \frac{2}{3} \log 10 = \frac{2}{3} \cdot 1 = \frac{2}{3}$$

C $\log 0.001 = -3$ BECAUSE $0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$ OR

$$\log 0.001 = \log \frac{1}{10^3} = -\log 10^3 = -3 \log 10$$

EXAMPLE 7 FIND THE COMMON LOGARITHM OF 526.

SOLUTION: $\log 526 = \log(5 \times 10^2) = \log 5.26 + \log 10^2$... by $\log_b xy = \log_b x + \log_b y$
 $= \log 5.26 + 2 \log 10$. NOW WE STILL NEED TO FIND

SINCE $\log 10 = 1$, WE KNOW THAT $\log 5.26$.

SO, THE COMMON LOGARITHM OF A NUMBER BETWEEN 1 AND 10 IS A NUMBER BETWEEN 0 AND 1. THE SPECIFIC COMMON LOGARITHMIC VALUES FOR NUMBERS BETWEEN 1 AND 10 ARE GIVEN IN WHAT IS CALLED A TABLE OF COMMON LOGARITHMS.

A COPY OF THE TABLE IS ATTACHED AT THE END OF THIS BOOK

FROM THE COMMON LOGARITHM TABLE, WE READ THAT $\log 526 = 2.7210$

(It should be noted that this value is only an approximate value.)

HENCE,

$$\log 526 = \log(5 \times 10^2) = \log 5.26 + \log 10^2 = \log 5.26 + 2 = 0.7210 + 2 = 2.7210$$

Mantissa
Characteristic
M
C

IF WE WRITE A NUMBER $m \times 10^c$, $0 \leq m < 10$, THEN THE LOGARITHM CAN BE READ FROM A COMMON LOGARITHM TABLE. THE ~~LOG IS RELATED TO THE~~ ~~mantissa~~ OF THE LOGARITHM OF THE NUMBER IS CALLED ~~THE~~ ~~CHARACTERISTIC~~ OF THE LOGARITHM. THEREFORE, THE COMMON LOGARITHM OF A NUMBER IS ~~CHARACTERISTIC~~ PLUS ITS ~~mantissa~~.

EXAMPLE 8 IDENTIFY THE CHARACTERISTIC AND MANTISSA OF THE COMMON LOGARITHMS:

A $\log 0.000415$ B $\log 239$ C $\log 0.001$

SOLUTION:

A $0.000415 = 4.15 \times 10^{-4}$

THEREFORE, THE CHARACTERISTIC IS -4 AND THE MANTISSA IS

B $239 = 2.39 \times 10^2$

THEREFORE, THE CHARACTERISTIC IS 2 AND THE MANTISSA IS $\log 2.39$.

C $0.001 = 1 \times 10^{-3}$

THEREFORE, THE CHARACTERISTIC IS -3 AND THE MANTISSA IS $\log 1 = 0$.

Using the logarithm table

THE LOGARITHM OF ANY TWO DECIMAL PLACE NUMBER BETWEEN 1 AND 10 CAN BE READ DIRECTLY FROM THE COMMON LOGARITHM TABLE (A PART OF THE TABLE IS GIVEN BELOW FOR REFERENCE).

x	0	1	2	...	9
1.0	0.0000	0.0043	0.0086	...	0.0374
1.1	0.0414	0.0453	0.0492	...	0.0755
1.2	0.0792	0.0828	0.0864	...	0.1106
1.3	0.1139	0.1173	0.1206	...	0.1430
.
.
.
1.9	0.2788	0.2810	0.2833	...	0.2989
2.0	0.3010	0.3032	0.3054	...	0.3201
2.1	0.3222	0.3243	0.3263	...	0.3404
2.2	0.3424	0.3444	0.3464	...	0.3598
.
.
.
9.9	0.9956	0.9961	0.9965	...	0.9996

EXAMPLE 9 USE THE TABLE OF LOGARITHMS TO FIND:

- A LOG 2.29 B LOG 1.21 C LOG 1.386 D LOG 21,200

SOLUTION:

A READ THE NUMBER AT THE INTERSECTION OF ROW 2.2 AND COLUMN 9. READING THE NUMBER IN ROW 2.2 UNDER COLUMN 9, WE GET 0.3598.

$$\therefore \log 2.29 = 0.3598 .$$

B READING THE NUMBER AT THE INTERSECTION OF ROW 1.2 AND COLUMN 9. WE GET 0.0828

$$\therefore \log 1.21 = 0.0828 .$$

C 1.386 IS BETWEEN 1.38 AND 1.39.

SQ ROUND (TO 2 DECIMAL PLACES) LOG 1.386 AS LOG 1.39. READING IN ROW 1.3 UNDER COLUMN 9, WE GET 0.143. LOG 1.386 \approx 0.1430.

D FIRST WRITE 21,200 AS 2.1×10^4

$$\begin{aligned}\therefore \log 21,200 &= \log(2.1 \times 10^4) = \log 2.1 + \log 10^4 \\ &= 0.3263 + 4 = 4.3263.\end{aligned}$$

Note: NUMBERS GREATER THAN 10 HAVE LOGARITHMS GREATER THAN 1.

Antilogarithms

SUPPOSE $\log x = 0.6665$. WHAT IS THE VALUE OF x ?

IN SUCH CASES, WE APPLY WHAT IS CALLED THE *antilogarithm of the logarithm of x* , WRITTEN AS *antilog (LOG)x* OR *ANTILOG (LOG)0.6665*.

WE HAVE TO SEARCH THROUGH THE LOGARITHM TABLE, FOR THE VALUE 0.6665. WE LOCATE THE NUMBER LOCATED WHERE THE ROW WITH HEADING 4.6 MEETS THE COLUMN WITH HEADING 4. THEREFORE $\log 4.64 = 0.6665$, AND WE HAVE

In general, $\text{Antilog}(\log c) = c$.

EXAMPLE 10 FIND:

- | | |
|-------------------------|-------------------------|
| A ANTILOG 0.7348 | B ANTILOG 0.9335 |
| C ANTILOG 0.8175 | D ANTILOG 2.4771 |

SOLUTION:

A THE NUMBER 0.7348 IS FOUND IN THE TABLE WHERE ROW 7 AND COLUMN 3 MEET.

$$\therefore \text{ANTILOG } 0.7348 = 5.43 .$$

B THE NUMBER 0.9335 IS FOUND IN THE TABLE WHERE ROW 9 AND COLUMN 8 MEET.

$$\therefore \text{ANTILOG } 0.9335 = 8.58 .$$

- C** THE NUMBER 0.8175 DOES NOT APPEAR IN THE TABLE. THE CLOSEST VALUE IS AND $0.8176 = \log 6.57$.

$\therefore \text{ANTILOG } 0.8175 \text{ CAN BE APPROXIMATED BY } 6.57$.

- D** $\text{ANTILOG } 2.4771 = \text{ANTILOG } (0.4771 + 2) \times 10^2 \approx 300$

(The antilogarithm of the decimal part 0.4771 is found using the table of logarithms and equals 3. The antilogarithm of 2 is 10^2 because $\log 10^2 = 2$.)

EXAMPLE 11 FIND:

- A** $\text{ANTILOG } 3.9058$ **B** $\text{ANTILOG } 5.9586$. **C** $\text{ANTILOG } (-1.0150)$

SOLUTION:

- A** $\text{ANTILOG } 3.9058 = \text{ANTILOG } (0.9058 + 3) \times 10^3 = 8050$.
- B** $\text{ANTILOG } 5.9586 = \text{ANTILOG } (0.9586 + 5) \times 10^5 = 909000$.
- C** $\text{ANTILOG } (-1.0150) = \text{ANTILOG } (2 - 1.0150 - 2) = 0.0966$
- $$= 9.66 \times 10^{-2} = 0.0966.$$

Note: DO NOT WRITE -1.0150 AS 0.0150 ! THE ARITHMETIC IS NOT CORRECT!

Computation with logarithms

IN THIS SECTION YOU WILL SEE HOW LOGARITHMS ARE USED.

FOR INSTANCE, TO FIND THE PRODUCT OF 32 AND 128 USING LOGARITHM TO THE BASE 2 IT AS FOLLOWS:

LET $x = 32 \times 128$

$$\log_2 x = \log_2 (32 \times 128) \dots \text{WHY?}$$

$$\log_2 x = \log_2 32 + \log_2 128 \dots \text{WHY?}$$

$$\log_2 x = 5 \Rightarrow \log_2 x = 10 \dots \text{WHY?}$$

$$\therefore x = 2^{10}$$

IN THE NEXT EXAMPLES YOU WILL SEE HOW COMMON LOGARITHMS ARE USED IN MATH COMPUTATIONS:

REMEMBER THAT $\text{ANTILOG} = \log$

In order to compute c you can perform the following two steps:

Step 1 FIND $\log c$ USING THE LAWS OF LOGARITHMS.

Step 2 FIND THE ANTILOGARITHM OF $\log c$

EXAMPLE 12 COMPUTE $\frac{354 \times 605}{8450}$ USING LOGARITHMS.

SOLUTION:

Step 1 LET $x = \frac{354 \times 605}{8450}$

$$\log x = \log \frac{354 \times 605}{8450}$$

$$\log x = \log (354 \times 605) - \log 8450$$

$$\log x = \log 354 + \log 605 - \log 8450$$

$$\log x = (0.5490 + 2 + 0.7818 + 2) - (0.9269 + 3)$$

$$\log x = 0.4039 + 1$$

$$\text{SO } x = \text{ANTILOG}(0.4039 + 1) \Rightarrow x \approx 2.53 \times 10 \approx 25.3$$

$$\therefore \frac{354 \times 605}{8450} \approx 25.3$$

EXAMPLE 13 COMPUTE $\sqrt{35}$ USING LOGARITHMS.**SOLUTION:** LET $x = \sqrt{35}$

$$\log x = \log \sqrt{35} \Rightarrow \log x = \log 35^{\frac{1}{2}} \Rightarrow \log x = \frac{1}{2} [\log 35 - 1]$$

$$\log x = \frac{1}{2} [0.5441 + 1] \Rightarrow \log x \approx 0.77205 ; \log x \approx 0.7721$$

$$\text{SO } x = \text{ANTILOG}(0.7721) \Rightarrow x \approx 5.92$$

$$\therefore \sqrt{35} \approx 5.92$$

EXAMPLE 14 COMPUTE $80^{\frac{1}{3}}$ USING LOGARITHMS.**SOLUTION:** LET $x = 80^{\frac{1}{3}}$

$$\log x = \log 80^{\frac{1}{3}} ; \log x = \frac{1}{3} [\log 80 - 1] ; \log x = \frac{1}{3} [0.5798 + 2] ;$$

$$\log x \approx 0.8599 \quad \text{SO } x = \text{ANTILOG}(0.8599) \Rightarrow x \approx 7.24 \quad \therefore 80^{\frac{1}{3}} \approx 7.24$$

Group Work 2.5**DISCUSS**

- 1** WHICH BASE IS PREFERABLE FOR MATHEMATICAL COMPUTATIONS? WHY? PRESENT YOUR FINDINGS TO YOUR GROUP.
- 2** APPROXIMATE $\sqrt[3]{380}$ USING LOGARITHM.
- 3** USE YOUR RESULT IN 2 TO COMPARE YOUR RESULTS. WHAT DIFFERENCES DO YOU GET?



Exercise 2.5

1 FIND EACH OF THE FOLLOWING COMMON LOGARITHMS:

A $\log(10\sqrt[4]{10})$ B $\log_{\frac{1}{\sqrt{10}}} 100$ C $\log_{\sqrt[4]{10}} \frac{1}{10}$ D $\log_{\frac{10^m}{10^n}}$

2 IDENTIFY THE CHARACTERISTIC AND MANTISSA OF THE LOG OF THE FOLLOWING:

A 0.000402	B 203	C 5.5	D 2190
E $\frac{1}{4}$	F 8	G 23	H 35.902

3 USE THE TABLE OF LOGARITHMS TO FIND:

A $\log 3.12$	B $\log 1.99$	C $\log 7.2$	D $\log 5.436$
E $\log 0.12$	F $\log 9.99$	G $\log 0.00007$	H $\log 300$

4 FIND:

A ANTILOG 0.8998	B ANTILOG 0.8	C ANTILOG 1.3010
D ANTILOG 0.9953	E ANTILOG 5.721	F ANTILOG 1.9999
G ANTILOG (-6)	H ANTILOG(-0.2)	

5 COMPUTE USING LOGARITHMS:

A 6.24×37.5	B $\sqrt[9]{125}$	C $2^{1.42}$
D $(2.4)^{1.3} \times (0.12)^{4.1}$	E $\frac{37.9\sqrt{488}}{(1.28)^3}$	F $\sqrt[5]{0.0641}$

2.2 THE EXPONENTIAL FUNCTIONS AND THEIR GRAPHS

IN THIS SECTION YOU WILL DRAW GRAPHS AND ANSWER PROPERTIES OF FUNCTIONS OF THE FORM $f(x) = 2^x$, $f(x) = 10^x$, $f(x) = 3^{-x}$, $f(x) = (0.5)^x$, ETC.

ACTIVITY 2.6

SUPPOSE AN AMOEBA CELL DIVIDES ITSELF INTO TWO AFTER ONE HOUR.

- A** CALCULATE THE NUMBER OF CELLS CREATED ONE HOUR, TWO, THREE, FOUR, FIVE AND SIX.
- B** COMPLETE THE FOLLOWING TABLE.

Time in hour (t)	0	1	2	3	4	5	...	t
Number of cells created (y)	1							

- C** WRITE A FORMULA TO CALCULATE THE NUMBER OF HOURS. CREA

THE FUNCTION $f(x) = b^x$, $b > 0$ AND $\neq 1$ DEFINES AN EXPONENTIAL FUNCTION.

THE FOLLOWING FUNCTIONS ARE ALL EXPONENTIAL:

A $f(x) = 2^x$

B $g(x) = \left(\frac{3}{2}\right)^x$

C $h(x) = 3^x$

D $k(x) = 10^x$

E $f(x) = \left(\frac{1}{10}\right)^x$

F $g(x) = \left(\frac{1}{3}\right)^x$

G $h(x) = \left(\frac{1}{2}\right)^x$

H $k(x) = \left(\frac{2}{3}\right)^x$

2.2.1 Graphs of Exponential Functions

LET US NOW CONSIDER THE GRAPHS OF SOME OF THE ABOVE EXPONENTIAL FUNCTIONS. CAN EXPLORE SOME OF THEIR PROPERTIES.

EXAMPLE 1 DRAW THE GRAPH OF 2^x .

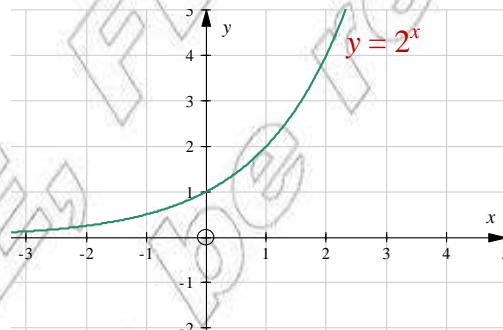
SOLUTION: EVALUATE 2^x FOR SOME INTEGRAL VALUES OF x . PREPARE A TABLE OF VALUES.

FOR EXAMPLE: $f(-3) = 2^{-3} = \frac{1}{8}$; $f(-2) = 2^{-2} = \frac{1}{4}$; $f(-1) = 2^{-1} = \frac{1}{2}$;

$f(0) = 2^0 = 1$; $f(1) = 2^1 = 2$; $f(2) = 2^2 = 4$; $f(3) = 2^3 = 8$.

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

NOW PLOT THESE POINTS ON THE CO-ORDINATE SYSTEM AND JOIN THEM BY A SMOOTH CURVE. OBTAIN THE GRAPH OF $y = 2^x$.



Graph of $f(x) = 2^x$

Figure 2.1

ACTIVITY 2.7

- 1 WHAT IS THE DOMAIN OF THE FUNCTION?
- 2 FOR WHAT VALUES OF x IS THE FUNCTION NEGATIVE?
- 3 CAN THE FUNCTION EVER BE 0?
- 4 WHAT IS THE RANGE OF THE FUNCTION?
- 5 WHAT IS THE INTERCEPT OF $y = 2^x$?



- 6 FOR WHICH VALUES OF x IS 2^x GREATER THAN 1?
- 7 WHAT CAN YOU SAY ABOUT THE VALUE OF 2^{-x} ?
- 8 DOES 2^x INCREASE AS x INCREASES?
- 9 WHAT HAPPENS TO THE GRAPH WHEN x TAKE LARGER AND LARGER POSITIVE VALUES OF x ?
- 10 WHAT HAPPENS TO THE GRAPH OF 2^x AS x BECOME MORE AND MORE NEGATIVE?
- 11 DOES THE GRAPH CROSS THE x -axis?
- 12 WHAT IS THE ASYMPTOTE OF THE GRAPH OF 2^x ?

EXAMPLE 2 DRAW THE GRAPH OF $\left(\frac{3}{2}\right)^x$

SOLUTION:

x	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{3}{2}\right)^x$	$\frac{8}{27}$	$\frac{4}{9}$	$\frac{2}{3}$	1	$\frac{3}{2}$	$\frac{9}{4}$	$\frac{27}{8}$

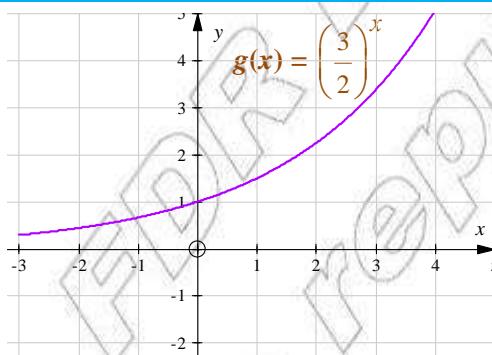


Figure 2.2 Graph of $g(x) = \left(\frac{3}{2}\right)^x$

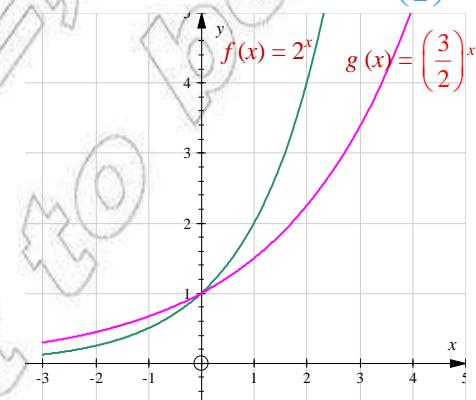
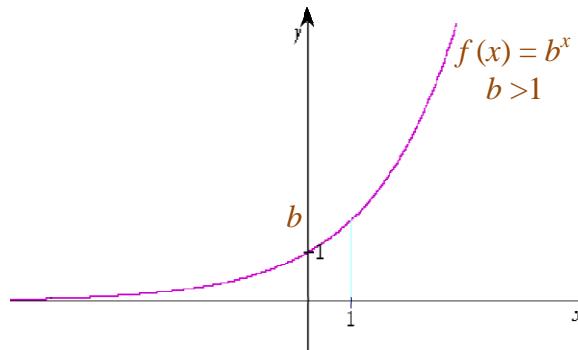


Figure 2.3 Graphs of $f(x) = 2^x$ and $g(x) = \left(\frac{3}{2}\right)^x$ drawn using the same co-ordinate system

IN GENERAL, THE GRAPH OF FOR ANY $b > 1$ HAS SIMILAR SHAPE AS THE GRAPHS OF AND $y = \left(\frac{3}{2}\right)^x$.

Figure 2.4 Graph of $f(x) = b^x$, for any $b > 1$

Basic properties

THE GRAPH OF $y = b^x$, $b > 1$ has the following basic properties:

- 1 THE DOMAIN IS THE SET OF ALL REAL NUMBERS.
- 2 THE RANGE IS THE SET OF ALL POSITIVE REAL NUMBERS.
- 3 THE GRAPH INCLUDES THE POINT $(0, 1)$. INTERCEPT IS 1.
- 4 THE FUNCTION IS INCREASING.
- 5 THE VALUES OF THE FUNCTION ARE GREATER THAN 1 FOR $x > 0$ AND LESS THAN 1 FOR $x < 0$.
- 6 THE GRAPH APPROACHES THE AS AN ASYMPTOTE ON THE LEFT AND INCREASES WITHOUT BOUND ON THE RIGHT.

WE WILL NEXT DISCUSS HOW THE GRAPH OF THE BLOCKS ONE WHEN $0 < b < 1$.

EXAMPLE 3 DRAW THE GRAPH OF EACH OF THE FOLLOWING USING:

- I DIFFERENT COORDINATE AXES THE SAME COORDINATE AXES.

A $h(x) = \left(\frac{1}{2}\right)^x$ B $k(x) = \left(\frac{2}{3}\right)^x$

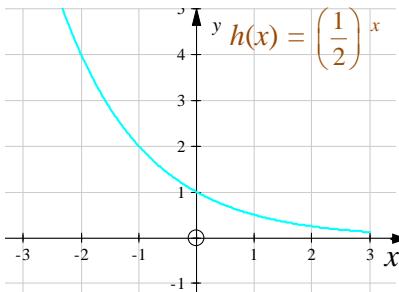
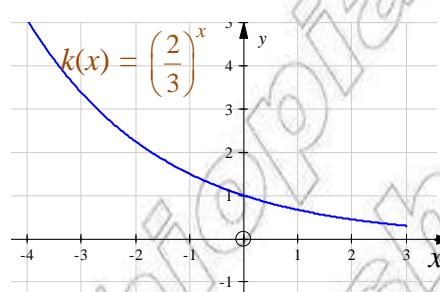
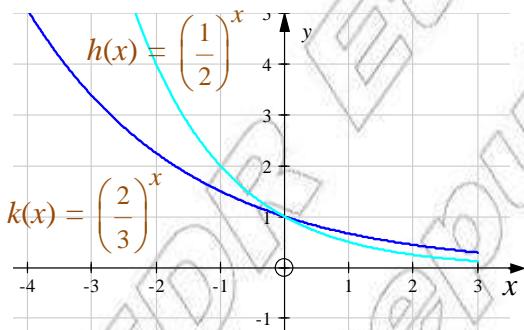
SOLUTION: AS BEFORE, CALCULATE THE VALUES OF THE EXPONENTIAL FUNCTION FOR THE VALUES OF x SHOWN IN THE TABLES BELOW. THEN PLOTTING CORRESPONDING POINTS ON THE CO-ORDINATE SYSTEM. JOIN THESE POINTS BY SMOOTH CURVES TO GET THE GRAPHS AS INDICATED BELOW.

A

x	-3	-2	-1	0	1	2	3
$H(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

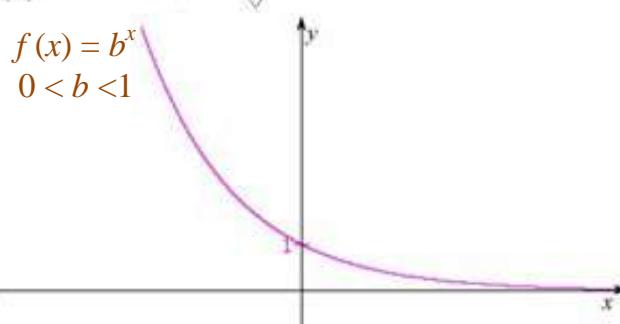
B

x	-3	-2	-1	0	1	2	3
$k(x) = \left(\frac{2}{3}\right)^x$	$\frac{27}{8}$	$\frac{9}{4}$	$\frac{3}{2}$	1	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{27}$

Figure 2.5 Graph of $h(x) = \left(\frac{1}{2}\right)^x$ Figure 2.6 Graph of $k(x) = \left(\frac{2}{3}\right)^x$ Figure 2.7 Graphs of $h(x) = \left(\frac{1}{2}\right)^x$ and $k(x) = \left(\frac{2}{3}\right)^x$ drawn using the same coordinate axes

THE GRAPH OF $f(x) = b^x$, FOR ANY $0 < b < 1$ HAS SIMILAR SHAPE TO THE GRAPH OF $y = \left(\frac{1}{2}\right)^x$.

$$y = \left(\frac{2}{3}\right)^x.$$

Figure 2.8 Graph of $f(x) = b^x$, for any $0 < b < 1$

Basic properties

THE GRAPH OF $y = b^x$, $0 < b < 1$ has the following basic properties:

- 1** THE DOMAIN IS THE SET OF ALL REAL NUMBERS.
- 2** THE RANGE IS THE SET OF ALL POSITIVE REAL NUMBERS.
- 3** THE GRAPH INCLUDES THE POINT $(0, 1)$.
INTERCEPT IS 1.
- 4** THE FUNCTION IS DECREASING.
- 5** THE VALUES OF THE FUNCTION ARE GREATER THAN 1 FOR $x > 0$ AND LESS THAN 1 FOR $x < 0$.
- 6** THE GRAPH APPROACHES THE X-AXIS AS AN ASYMPTOTE ON THE RIGHT AND INCREASES WITHOUT BOUND ON THE LEFT.

Exercise 2.6

- 1** GIVE THREE EXAMPLES OF EXPONENTIAL FUNCTIONS.
 - 2** GIVEN THE GRAPH OF $y = 2^x$ (see FIGURE 2.9) WE CAN FIND APPROXIMATE VALUES OF 2 VARIOUS VALUES FOR EXAMPLE,
- $2^{1.8} \approx 3.5$ (SEE POINT A).
- $2^{2.3} \approx 5$ (SEE POINT B).

USE THE GRAPH TO DETERMINE APPROXIMATE VALUES OF

A $2^{\frac{1}{2}}$ (I.E. $\sqrt{2}$)

B $2^{0.8}$

C $2^{1.5}$

D $2^{-1.6}$.

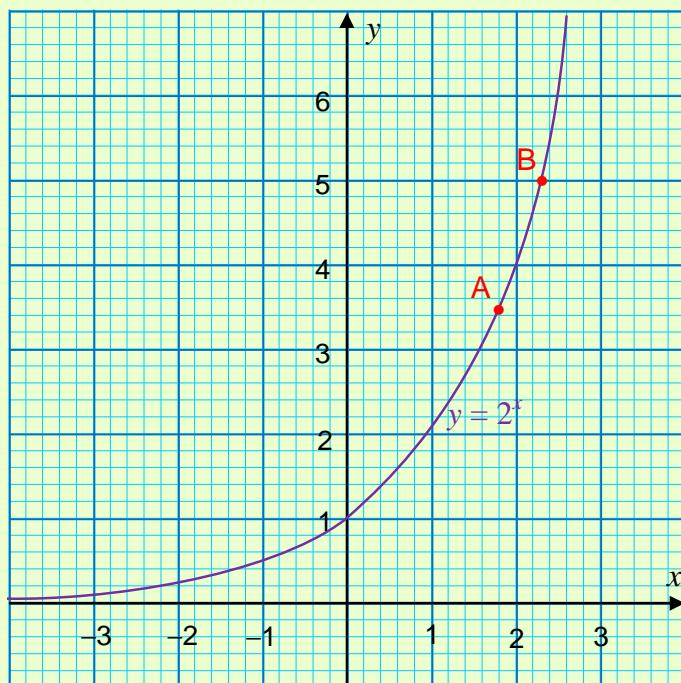


Figure 2.9

3 CONSTRUCT SUITABLE TABLES OF VALUES AND DRAW THE GRAPHS

A $h(x) = 3^x$ AND $g(x) = \left(\frac{1}{3}\right)^x$ USING THE SAME CO-ORDINATE SYSTEM.

B $k(x) = 10^x$ AND $f(x) = \left(\frac{1}{10}\right)^x$ USING THE SAME CO-ORDINATE SYSTEM.

C $f(x) = 4^x$ AND $g(x) = \left(\frac{1}{4}\right)^x$ USING THE SAME CO-ORDINATE SYSTEM.

4 REFERRING TO THE FUNCTIONS

A FIND THE DOMAIN AND THE RANGE OF EACH FUNCTION,

B WHAT IS THE INTERCEPT OF EACH FUNCTION?

C WHICH FUNCTIONS ARE INCREASING AND WHICH ARE DECREASING?

D FIND THE ASYMPTOTE FOR EACH GRAPH

The exponential function with base e

UNTIL NOW THE NUMBER ~~PI~~ PROBABLY BEEN THE MOST IMPORTANT IRRATIONAL NUMBER YOU HAVE ENCOUNTERED. NEXT, WE WILL INTRODUCE ANOTHER USEFUL IRRATIONAL NUMBER WHICH IS VERY IMPORTANT IN THE FIELD OF MATHEMATICS AND OTHER SCIENCES.

2.2.2 The Number e

DO YOU KNOW THAT SOME BANKS CALCULATE INTEREST EVERY ~~MONTHLY~~? THIS IS CALLED **continuous compounding**. OTHER BANKS EVEN ADVERTISE ~~continuous~~ COMPOUNDING. TO ILLUSTRATE THE IDEA OF ~~continuous~~ COMPOUNDING, WE WILL STUDY HOW 1 BIRR GROWS FOR 1 YEAR AT A PERCENT ANNUAL INTEREST, USING VARIOUS PERIODS OF COMPOUNDING.

IN THIS CASE, WE USE THE AMOUNT FORMULA, WHERE THE PRINCIPAL

TAKING THE ANNUAL RATE $= 1$, $i = \frac{1}{n}$ IF THERE ARE n PERIODS OF COMPOUNDING PER YEAR, THEN THE AMOUNT AFTER 1 YEAR IS GIVEN BY THE FORMULA:

$$A = \left(1 + \frac{1}{n}\right)^n$$

THE FOLLOWING TABLE GIVES THE AMOUNTS (IN BIRR) AFTER 1 YEAR USING VARIOUS PERIODS OF COMPOUNDING.

Number of compounding periods per year	Amount after one year
yearly	$\left(1 + \frac{1}{1}\right)^1 = 2$
semi-annually	$\left(1 + \frac{1}{2}\right)^2 = 2.25$
quarterly	$\left(1 + \frac{1}{4}\right)^4 = 2.44140625$
monthly	$\left(1 + \frac{1}{12}\right)^{12} \approx 2.61303529022\dots$
weekly	$\left(1 + \frac{1}{52}\right)^{52} \approx 2.69259695444\dots$
daily	$\left(1 + \frac{1}{365}\right)^{365} \approx 2.71456748202\dots$
hourly	$\left(1 + \frac{1}{8760}\right)^{8760} \approx 2.71812669063\dots$
every minute	$\left(1 + \frac{1}{525600}\right)^{525600} \approx 2.7182792154\dots$
every second	$\left(1 + \frac{1}{31536000}\right)^{31536000} = 2.7182817853\dots$

THE LAST ROW OF THE ABOVE TABLE SHOWS THE EFFECT OF COMPOUNDING APPROXIMATELY ONCE PER SECOND. THE IDEA OF CONTINUOUS COMPOUNDING IS THAT THE TABLE IS CONTINUED FOREVER. AS n INCREASES, THE LARGER VALUES APPROACH A LIMIT. THIS NUMBER APPROXIMATELY 2.718281828459... IS KNOWN AS THE NUMBER e.

THIS IRRATIONAL NUMBER IS REPRESENTED BY THE LETTER e.

$$e = 2.718281828459\dots$$

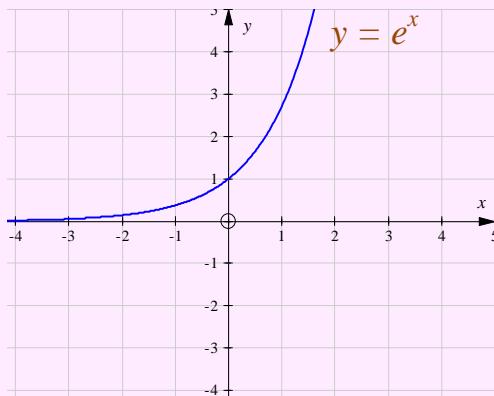
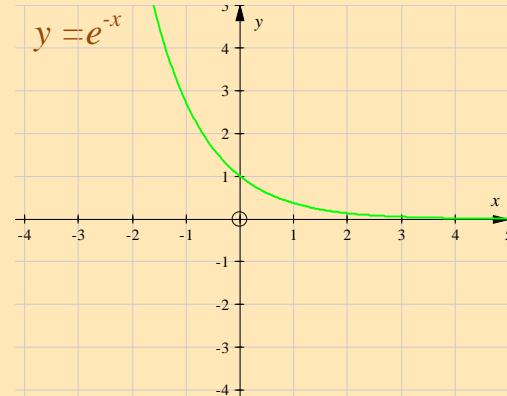
e IS THE NUMBER $\left(\frac{1}{n}\right)^n$ APPROACHED AS n APPROACHES INFINITY. WHO FIRST DISCOVERED

STILL BEING DEBATED. THE NUMBER IS NAMED AFTER THE SWISS MATHEMATICIAN LEONHARD

(1707 – 1783), WHO COMPUTED 23 DECIMAL PLACES USING

2.2.3 The Natural Exponential Function

FOR ANY REAL NUMBER FUNCTION $f(x) = e^x$ DEFINES THE EXPONENTIAL FUNCTION WITH BASE e , USUALLY CALLED THE exponential function.

Figure 2.10 The graph of $y = e^x$.Figure 2.11 The graph of $y = e^{-x}$.

THE DOMAIN OF e^x IS \mathbb{R} .

THE DOMAIN OF e^{-x} IS \mathbb{R}

THE RANGE IS $(0, \infty)$

THE RANGE IS $(0, \infty)$

$y = e^x$ IS AN INCREASING FUNCTION.

$y = e^{-x}$ IS A DECREASING FUNCTION.

THE GRAPH OF e^x INTERSECTS THE x -AXIS AT $(0, 1)$.

THE GRAPH OF e^{-x} INTERSECTS THE x -AXIS AT $(0, 1)$.

$e^x > 1$, if $x > 0$

$e^{-x} > 1$, if $x < 0$

$0 < e^x < 1$, if $x < 0$

$0 < e^{-x} < 1$, if $x > 0$

EXAMPLE 1 SKETCH THE GRAPH OF $y = e^{2x}$.

SOLUTION: WE CALCULATE AND PLOT SOME POINTS TO DRAW THE REQUIRED GRAPH, AS SHOWN IN FIGURE 2.12.

x	$y = e^{2x}$
-3	≈ 0.0025
-2	≈ 0.0183
-1	≈ 0.1353
0	= 1
1	≈ 7.7391
2	≈ 54.5981

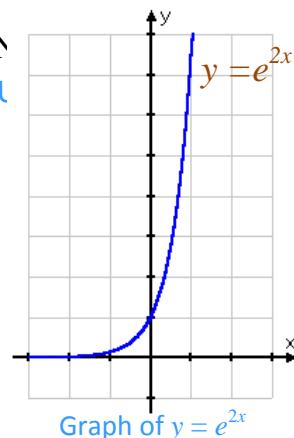


Figure 2.12

Exercise 2.7

- 1** SKETCH THE GRAPHS OF EACH OF THE FOLLOWING FUNCTIONS:
- A** $f(x) = 2^{x-1}$ **B** $g(x) = 3^{x-2}$ **C** $k(x) = 3^{2-x}$
- 2** USE THE KEY ON YOUR CALCULATOR TO EVALUATE EACH OF THE FOLLOWING EXPRESSIONS TO 7 DECIMAL PLACES:
- A** e^3 **B** $e^{\sqrt{3}}$ **C** $e^{-7.3011}$ **D** $e^{\sqrt{5}}$
- 3** CONSTRUCT TABLES OF VALUES FOR SOME FINITE INTERVALS FOR EACH OF THE FOLLOWING FUNCTIONS:
- A** $y = -e^x$ **B** $y = -e^{-x}$ **C** $y = 10e^{0.2x}$
- 4** STATE THE DOMAIN AND RANGE OF EACH OF THE FUNCTIONS I

2.3 THE LOGARITHMIC FUNCTIONS AND THEIR GRAPHS

FROM SECTION 2.1 YOU SHOULD REMEMBER THAT AND ONLY $\text{LOG}_b x = y$ ($b > 0$, $b \neq 1$ AND $x > 0$)

HENCE, THE FUNCTION $\text{LOG}_b x$, WHERE $x > 0$, $b > 0$ AND $\neq 1$ IS CALLED **LOGarithmic function with base b** .

THE FOLLOWING FUNCTIONS ARE ALL LOGARITHMIC:

- A** $f(x) = \text{LOG}_2 x$ **B** $g(x) = \text{LOG}_{\frac{1}{2}} x$ **C** $h(x) = \text{LOG}_3 x$
- D** $k(x) = \text{LOG}_{10} x$ **E** $f(x) = \text{LOG}_{\frac{1}{10}} x$ **F** $g(x) = \text{LOG}_{\frac{1}{3}} x$
- G** $h(x) = \text{LOG}_{\frac{1}{2}} x$ **H** $k(x) = \text{LOG}_{\frac{1}{3}} x$

ACTIVITY 2.8

THE CONCENTRATION OF HYDROGEN IONS IN A GIVEN SOLUTION IS NOTED BY $[\text{H}^+]$ AND IS MEASURED IN MOLES PER LITER.



FOR EXAMPLE, 10^{-5} FOR BEER AND 10^{-4} FOR WINE.

CHEMISTS DEFINE THE PH OF THE SOLUTION AS THE NUMBER OF MOLES OF HYDROGEN IONS IN A LITER OF SOLUTION. THE PH IS SAID TO BE AN ACID IF PH < 7 AND A BASE IF PH > 7 . PURE WATER HAS A PH OF 7, WHICH IS NEUTRAL.

- A** IS BEER AN ACID OR A BASE? WHAT ABOUT WINE?
- B** WHAT IS THE HYDROGEN ION CONCENTRATION IF THE PH OF EGGS IS 7.8?

2.3.1 Graphs of Logarithmic Functions

IN THIS SECTION, WE CONSIDER THE GRAPHS OF SOME LOGARITHMIC FUNCTIONS, SO EXPLORE THEIR PROPERTIES.

EXAMPLE 1 DRAW THE GRAPH OF EACH OF THE FOLLOWING USING:

- I DIFFERENT COORDINATE SYSTEMS THE SAME COORDINATE SYSTEM.

A $f(x) = \text{LOG}_2 x$

B $g(x) = \text{LOG}_{\frac{3}{2}} x$

SOLUTION: THE TABLES BELOW INDICATE SOME VALUES FOR THE CORRESPONDING POINTS ON THE CO-ORDINATE SYSTEM. JOIN THESE POINTS BY CURVES TO GET THE REQUIRED GRAPHS AS INDICATED.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$f(x) = \text{LOG}_2 x$	-2	-1	0	1	2

A

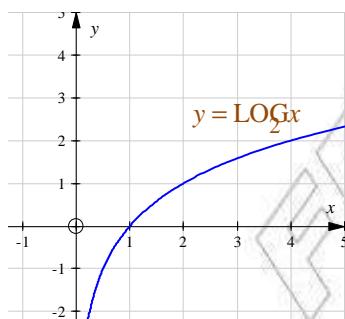


Figure 2.13 Graph of $f(x) = \text{LOG}_2 x$

x	$\frac{4}{9}$	$\frac{2}{3}$	1	$\frac{3}{2}$	$\frac{9}{4}$
$g(x) = \text{LOG}_{\frac{3}{2}} x$	-2	-1	0	1	2

B

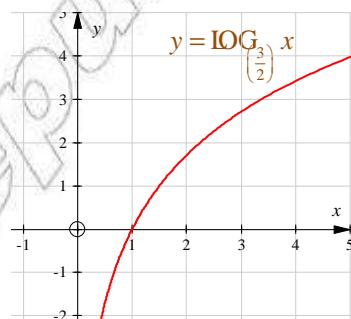


Figure 2.14 Graph of $g(x) = \text{LOG}_{\frac{3}{2}} x$

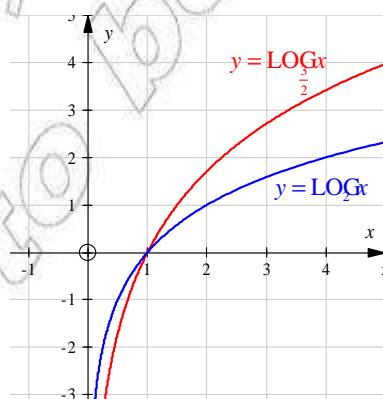


Figure 2.15 Graphs of $y = \text{LOG}_2 x$ and $y = \text{LOG}_{\frac{3}{2}} x$ drawn using the same coordinate axes

ACTIVITY 2.9



STUDY THE GRAPHS OF \log_x AND $g(x) = \log_{\left(\frac{3}{2}\right)} x$ TO ANSWER THE FOLLOWING QUESTIONS:

- 1** WHAT ARE THE DOMAINS OF \log_x ?
- 2** FOR WHICH VALUES OF \log_x NEGATIVE? POSITIVE?
- 3** FOR WHICH VALUES OF $\log_{\frac{3}{2}} x$ NEGATIVE? POSITIVE?
- 4** WHAT IS THE RANGE OF g ?
- 5** WHAT IS THE Y-INTERCEPT?
- 6** DOES \log_x INCREASE AS x INCREASES? WHAT ABOUT $\log_{\frac{3}{2}} x$?
- 7** DO THE GRAPHS CROSS THE y -AXIS?
- 8** WHAT IS THE ASYMPTOTE OF THE GRAPHS?

IN GENERAL, THE GRAPH OF $\log_b x$, FOR ANY $b > 1$ LOOKS LIKE THE ONE GIVEN BELOW.

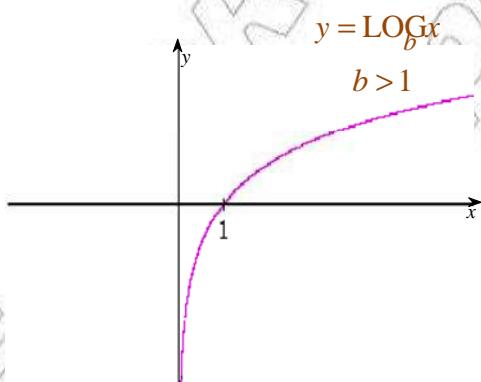


Figure 2.16 Graph of $y = \log_b x$

Basic properties

THE GRAPH OF $\log_b x$, $b > 1$, HAS THE FOLLOWING PROPERTIES.

- 1** THE DOMAIN IS THE SET OF ALL POSITIVE REAL NUMBERS.
- 2** THE RANGE IS THE SET OF ALL REAL NUMBERS.
- 3** THE GRAPH INCLUDES THE POINT $(1, 0)$. THE Y-INTERCEPT OF THE GRAPH IS 1.
- 4** THE FUNCTION INCREASES AS x INCREASES.
- 5** THE y -AXIS IS A VERTICAL ASYMPTOTE OF THE GRAPH.
- 6** THE VALUES OF THE FUNCTION ARE NEGATIVE FOR x ARE POSITIVE FOR $x < 1$.

YOU WILL NEXT DISCUSS WHAT THE GRAPH OF THE FUNCTION IS WHEN $a < 1$.

EXAMPLE 2 DRAW THE GRAPH OF EACH OF THE FOLLOWING USING:

I DIFFERENT COORDINATE SYSTEMS THE SAME COORDINATE SYSTEM.

A $h(x) = \log_{\frac{1}{2}} x$

B $k(x) = \log_{\frac{2}{3}} x$

SOLUTION: CALCULATE THE VALUES OF THE GIVEN FUNCTIONS IN THE TABLES BELOW. THEN PLOT THE CORRESPONDING POINTS ON THE CO-ORDINATE SYSTEM. JOIN THESE POINTS BY SMOOTH CURVES TO GET THE REQUIRED GRAPHS INDICATED IN FIGURE 2.1 AND 2.18

x	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$h(x) = \log_{\frac{1}{2}} x$	-3	-2	-1	0	1	2	3

x	$\frac{27}{8}$	$\frac{9}{4}$	$\frac{3}{2}$	1	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{27}$
$k(x) = \log_{\frac{2}{3}} x$	-3	-2	-1	0	1	2	3

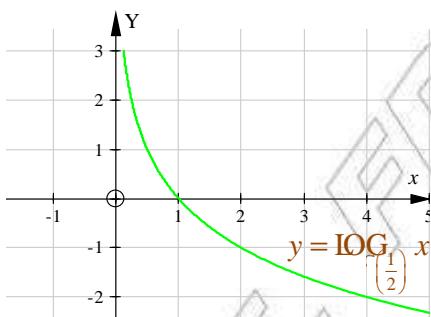


Figure 2.17 Graph of $h(x) = \log_{\frac{1}{2}} x$

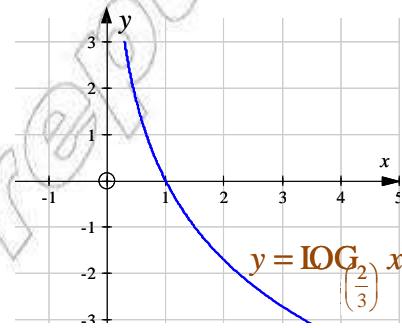


Figure 2.18 Graph of $k(x) = \log_{\frac{2}{3}} x$

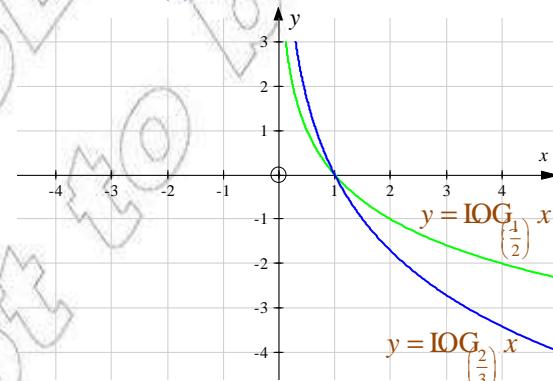


Figure 2.19 Graphs of $y = \log_{\frac{1}{2}} x$ and $y = \log_{\frac{2}{3}} x$ drawn using the same coordinate axes

IN GENERAL, THE GRAPH OF $\log_b x$ FOR $0 < b < 1$ LOOKS LIKE THE ONE GIVEN BELOW.

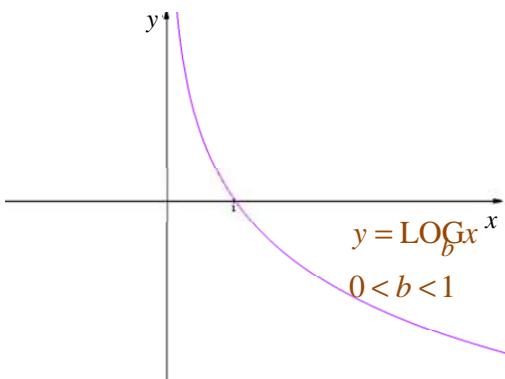


Figure 2.20

Basic properties

THE GRAPH OF $\log_b x$, ($0 < b < 1$) HAS THE FOLLOWING PROPERTIES.

- 1 THE DOMAIN IS THE SET OF ALL POSITIVE REAL NUMBERS.
- 2 THE RANGE IS THE SET OF ALL REAL NUMBERS.
- 3 THE GRAPH HAS INTERCEPT AT $(1, 0)$. INTERCEPT IS 1.
- 4 THE FUNCTION DECREASES.
- 5 THE y AXIS IS AN ASYMPTOTE OF THE GRAPH.
- 6 THE VALUES OF THE FUNCTION ARE POSITIVE WHEN x ARE NEGATIVE WHEN $x > 1$.

Exercise 2.8

- 1 DRAW THE GRAPHS OF:

- A $h(x) = \log_3 x$ AND $g(x) = \log_{\left(\frac{1}{3}\right)} x$ USING THE SAME CO-ORDINATE SYSTEM.
- B $k(x) = \log_{10} x$ AND $f(x) = \log_{\left(\frac{1}{10}\right)} x$ USING THE SAME CO-ORDINATE SYSTEM.

- 2 REFERRING TO THE FUNCTIONS IN QUESTION 1

- A WHAT ARE THE DOMAIN AND THE RANGE OF EACH FUNCTION?
- B WHAT IS THE INTERCEPT OF EACH?
- C WHICH FUNCTIONS ARE INCREASING AND WHICH ARE DECREASING?
- D FIND THE ASYMPTOTES OF THE GRAPHS OF THE FUNCTIONS.

2.3.2 The Relationship Between the Functions

$$y = b^x \text{ and } y = \log_b x \quad (b > 0, b \neq 1)$$

CONSIDER THE FOLLOWING TABLES OF VALUES FOR THE FUNCTIONS IN THE PREVIOUS SECTION FOR $y = 2^x$ AND $y = \log_2 x$.

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$f(x) = \log_2 x$	-3	-2	-1	0	1	2	3

ACTIVITY 2.10



REFER TO THE TABLES OF VALUES FOR $y = \log_2 x$ TO ANSWER THE FOLLOWING QUESTIONS:

- 1 HOW ARE THE VALUES OF X AND Y RELATED?
- 2 SKETCH THE GRAPHS OF THE TWO FUNCTIONS IN THE SAME SYSTEM.
- 3 FIND A RELATIONSHIP BETWEEN THE DOMAIN IN THE TWO FUNCTIONS.
- 4 DRAW THE LINE USING THE SAME CO-ORDINATE SYSTEM.
- 5 HOW ARE THE GRAPHS OF $y = 2^x$ AND $y = \log_2 x$ RELATED?
- 6 WHAT IS THE SIGNIFICANCE OF THE LINE $y = x$?

EXAMPLE 1 LET US CONSIDER THE FUNCTIONS $y = 10^x$ AND $y = \log_{10} x$.

THE TABLES OF VALUES FOR $y = 10^x$ AND $y = \log_{10} x$ FOR SOME INTEGRAL VALUES OF GIVEN BELOW:

x	-2	-1	0	1	2
$y = 10^x$	$\frac{1}{100}$	$\frac{1}{10}$	1	10	100

x	$\frac{1}{100}$	$\frac{1}{10}$	1	10	100
$y = \log_{10} x$	-2	-1	0	1	2

OBSERVE THAT:

THE VALUES OF x AND y ARE INTERCHANGED IN BOTH FUNCTIONS. THAT IS, THE DOMAIN OF $y = 10^x$ IS THE RANGE OF $y = \log_{10} x$ AND VICE VERSA.

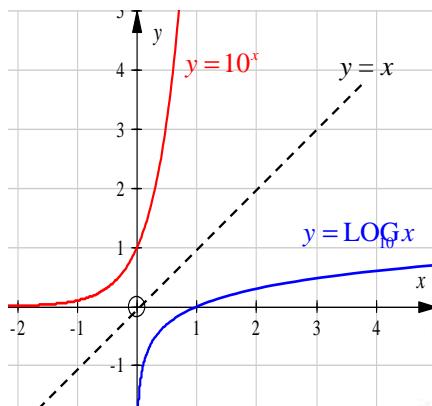


Figure 2.21

$y = 10^x$ IS OBTAINED BY REFLECTING ALONG THE LINE y

IN SUCH CASES WE SAY ONE OF THE FUNCTIONS IS THE INVERSE OF THE OTHER.

IN GENERAL, THE RELATION BETWEEN THE FUNCTIONS ($b > 1$) IS SHOWN BELOW:

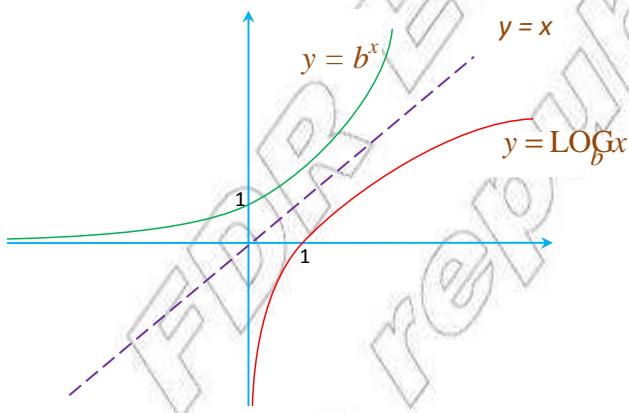


Figure 2.22

FROM THE GRAPHS ABOVE, WE OBSERVE THE FOLLOWING RELATIONSHIPS:

- 1 THE DOMAIN OF $y = b^x$ IS THE SET OF ALL REAL NUMBERS, WHICH IS THE SAME AS THE RANGE OF $y = \text{LOG}_b x$.
- 2 THE RANGE OF $y = b^x$ IS THE SET OF ALL POSITIVE REAL NUMBERS, WHICH IS THE SAME AS THE DOMAIN OF $y = \text{LOG}_b x$.
- 3 THE x -AXIS IS THE ASYMPTOTE OF $y = b^x$ WHEREAS THE y -AXIS IS THE ASYMPTOTE OF $y = \text{LOG}_b x$.
- 4 $y = b^x$ HAS A Y-INTERCEPT AT $(0, 1)$ WHILE $y = \text{LOG}_b x$ HAS AN X-INTERCEPT AT $(1, 0)$.

DOMAIN OF $y = b^x$ IS EQUAL TO THE RANGE OF $y = \text{LOG}_b x$.

RANGE OF $y = b^x$ IS EQUAL TO THE DOMAIN OF $y = \text{LOG}_b x$.

THE FUNCTIONS $y = b^x$ AND $y = \text{LOG}_b x$ ($b > 1$) ARE INVERSES OF EACH OTHER.

2.3.3 The Natural Logarithm

IF WE START WITH NATURAL EXPONENTIA AND INTERCHANGE, WE OBTAIN

$x = e^y$ WHICH IS THE SAME $\log_e x$

$y = \log_e x$ IS THE MIRROR IMAGE OF x ALONG THE LINE $y = x$.

Notation: $\ln x$ IS USED TO REPRESENT $\log_e x$

$\ln x$ IS CALLED THE NATURAL LOGARITHM OF x .

THE GRAPHS OF $y = e^x$, $y = \ln x$ AND THE LINE $y = x$ ARE SHOWN BELOW:

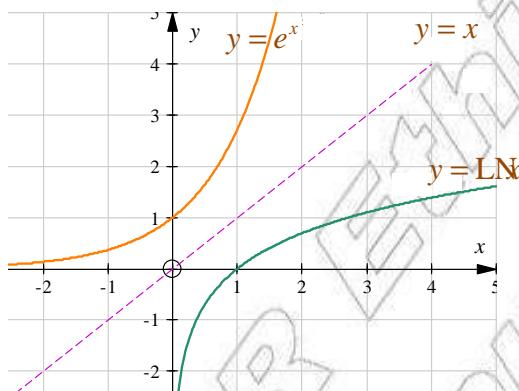


Figure 2.23

EXAMPLE 1 FIND:

- A $\ln 1$ B $\ln e$ C $\ln^2 e$ D $\ln \sqrt{e}$ E $\ln \frac{1}{e}$

SOLUTION:

- A $\ln 1 = 0$ BECAUSE $e^0 = 1$ B $\ln e = 1$ BECAUSE $e^1 = e$
 C $\ln^2 e = 2\ln e = 2 \times 1 = 2$ D $\ln \sqrt{e} = \ln e^{\frac{1}{2}} = \frac{1}{2} \ln e = \frac{1}{2}$
 E $\ln \frac{1}{e} = \ln e^{-1} = -1 \ln e = -1$

Note: IN GENERAL, $\ln x^a = a \ln x$

Exercise 2.9

1 SKETCH THE GRAPHS OF:

- A $f(x) = 4^x$, $g(x) = \log_4 x$ AND $y = x$ USING THE SAME COORDINATE SYSTEM.
 B $h(x) = \left(\frac{1}{4}\right)^x$ AND $k(x) = \log_{\frac{1}{4}} x$ USING THE SAME COORDINATE SYSTEM.

- C** HOW DO YOU COMPARE THE DOMAIN AND THE RANGE OF THE FUNCTIONS f GIVEN IN QUESTION 1A
- D** HOW DO YOU COMPARE THE DOMAIN AND THE RANGE OF THE FUNCTIONS h GIVEN IN QUESTION 1B
- 2** FIND:
- A** $\ln \sqrt[3]{e}$ **B** $\ln \frac{1}{e^2}$ **C** $\ln e^{3x}$ **D** $e^{\ln 3}$
- 3** SIMPLIFY:
- A** $\ln e$ **B** $\ln(e \times e)$ **C** $\ln(e^x \times e^y)$ **D** $\ln\left(\frac{e^x}{e^y}\right)$

2.4 EQUATIONS INVOLVING EXPONENTS AND LOGARITHMS

AN EXPONENTIAL EQUATION IS AN EQUATION WITH ONE EXPONENT.

EXAMPLES OF EXPONENTIAL EQUATIONS ARE:

$$\begin{array}{ll} 4^x = 8 & 4^x - 2^{x+1} - 8 = 0 \\ 2^{3x-2} = 5 & 9^{x^2+4x} = 3^{3x+7} \end{array}$$

A LOGARITHMIC EQUATION IS AN EQUATION THAT INVOLVES THE LOGARITHM OF AN UNKNOWN.

EXAMPLES OF LOGARITHMIC EQUATIONS ARE:

$$\begin{array}{ll} 4\log x - 5 & \log(x -) = 6 \\ \log(x + 3) + \log = 1 & \log(x +) = 2 \end{array}$$

2.4.1 Solving Exponential Equations

PROPERTIES OF EXPONENTS DISCUSSED IN THE PREVIOUS SECTIONS PLAY A MAJOR ROLE IN SOLVING EXPONENTIAL EQUATIONS. READ CAREFULLY THROUGH THE PROPERTIES BELOW, TO MEMORY!

IF a AND b ARE POSITIVE NUMBERS $\neq 1$, AND m AND n ARE REAL NUMBERS, THEN

- | | |
|--|--|
| 1 $a^m \times a^n = a^{m+n}$ | 2 $(a^m)^n = a^{mn}$ |
| 3 $(a \times b)^n = a^n \times b^n$ | 4 $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ |
| 5 $\frac{a^m}{a^n} = a^{m-n}$ | 6 $a^{-n} = \frac{1}{a^n}$ AND $\frac{1}{a^{-n}} = a^n$ |
| 7 IT IS ALWAYS TRUE THAT $\left(\frac{a}{b}\right)^{-k} = \left(\frac{b}{a}\right)^k$ | |

Additional properties:**Property of equality for exponential equations**FOR $b > 0, b \neq 1, x$ AND REAL NUMBERS,

1 $b^x = b^y$, IF AND ONLY IF $x = y$

2 $a^x = b^x$, ($x \neq 0$), IF AND ONLY IF $a = b$

EXAMPLE 1 SOLVE FOR x .

$$\text{A} \quad 3^x = 81 \quad \text{B} \quad 2^x = \frac{1}{32} \quad \text{C} \quad \left(\frac{2}{3}\right)^{2x+1} = \left(\frac{9}{4}\right)^x \quad \text{D} \quad 4^x = \left(\frac{1}{2}\right)^{x-3}$$

SOLUTION:

$$\text{A} \quad 3^x = 81 = 3^4 \quad \dots \text{look for a common base}$$

$$\Rightarrow x = 4 \quad \dots \text{property of equality of bases}$$

$$\text{B} \quad 2^x = \frac{1}{2^5} = 2^{-5} \quad \dots \text{look for a common base}$$

$$\Rightarrow x = -5 \quad \dots \text{property of equality of bases}$$

$$\text{C} \quad \left(\frac{2}{3}\right)^{2x+1} = \left(\frac{9}{4}\right)^x$$

$$\Rightarrow \left(\frac{2}{3}\right)^{2x+1} = \left(\frac{3}{2}\right)^{2x} = \left(\frac{2}{3}\right)^{-2x}$$

$$\Rightarrow 2x + 1 = -2x$$

$$\Rightarrow 2x + 2x = -1$$

$$\Rightarrow x = -\frac{1}{4}$$

$$\text{D} \quad 4^x = \left(\frac{1}{2}\right)^{x-3}$$

$$\Rightarrow 4^x = (2^{-1})^{x-3} = 2^{(-x+3)}$$

$$\Rightarrow (2^2)^x = 2^{(-x+3)}$$

$$\Rightarrow 2^{2x} = 2^{(-x+3)}$$

$$\Rightarrow 2x = -x + 3 \Rightarrow x = 1$$

IF YOU CANNOT EASILY WRITE EACH SIDE OF AN EXPONENTIAL EQUATION USING THE SAME BASE, CAN SOLVE THE EQUATION BY TAKING LOGARITHMS OF EACH SIDE.

EXAMPLE 2 SOLVE FOR x BY TAKING THE LOGARITHM OF EACH SIDE:

A $4^x = 10$ B $2^{3x-2} = 5$ C $2^{2x} = 11$

SOLUTION:

$$\text{A} \quad 4^x = 10$$

$$\text{LOG}^x = \text{LOG}10 \quad \dots \text{taking the logarithm of each side}$$

$$x \text{ LOG } 4 = 1 \quad \dots \text{since } \text{LOG}10 = 1, \text{ AND } \text{LOG}k \text{ LOG}$$

$$x = \frac{1}{\text{LOG } 4} = \frac{1}{0.6021} = 1.6609$$

$$\begin{aligned}
 \mathbf{B} \quad & 2^{3x-2} = 5 \\
 \Rightarrow & \text{LOG}^{(3x-2)} = \text{LOG } 5 \\
 \Rightarrow & (3x-2)\text{LOG } 2 = \text{LOG } 5 \\
 \Rightarrow & 3x-2 = \frac{\text{LOG}}{\text{LOG}} \\
 \Rightarrow & 3x = \frac{\text{LOG}}{\text{LOG}} + 2 \\
 \Rightarrow & x = \frac{1}{3} \left(\frac{\text{LOG}}{\text{LOG}} + 2 \right) = 1.4408
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{C} \quad & 2^{2x} = 11 \\
 \Rightarrow & \text{LOG}^{2x} = \text{LOG } 11 \\
 \Rightarrow & 2x \text{LOG } 2 = \text{LOG } 11 \\
 \Rightarrow & 2x = \frac{\text{LOG } 11}{\text{LOG } 2} \\
 \Rightarrow & x = \frac{1}{2} \left(\frac{\text{LOG } 11}{\text{LOG } 2} \right) = 1.730
 \end{aligned}$$

Exercise 2.10

1 SOLVE FOR x

A $5^x = 625$ **B** $2^{3-x} = 16$ **C** $4^{3x-8} = 2^{3x+9}$

D $\frac{1}{27} = \left(\frac{1}{9}\right)^{2x}$ **E** $3^{-x} = 81$ **F** $2^{x^2-2} = 4$

G $7^{x^2+x} = 49$ **H** $3^{6(x+2)} = 9^{x+2}$ **I** $3\left(\frac{27}{8}\right)^{\frac{2}{3}x-1} = 2\left(\frac{32}{243}\right)^{2x}$

2 SOLVE FOR y BY TAKING THE LOGARITHM OF EACH SIDE:

A $2^x = 15$ **B** $10^x = 14.3$ **C** $10^{3x+1} = 92$ **D** $1.05^x = 2$
E $6^{3x} = 5$ **F** $4^{2x} = 61$ **G** $10^{5x-2} = 348$ **H** $2^{-x} = 0.238$

2.4.2 Solving Logarithmic Equations

PROPERTIES OF LOGARITHMS DISCUSSED IN THE PREVIOUS SECTIONS PLAY A MAJOR ROLE IN SOLVING LOGARITHMIC EQUATIONS. REMEMBER THAT

IF a, b, c, x AND y ARE POSITIVE NUMBERS AND $\neq 1$, THEN

1 $\text{LOG}_b(xy) = \text{LOG}_b x + \text{LOG}_b y$

2 $\text{LOG}_b\left(\frac{x}{y}\right) = \text{LOG}_b x - \text{LOG}_b y$

3 FOR ANY REAL NUMBER x , $\text{LOG}_b(x^k) = k \text{LOG}_b x$

4 $\text{LOG}_b b = 1$

5 $\text{LOG}_b 1 = 0$

6 $\text{LOG}_a x = \frac{\text{LOG}_b x}{\text{LOG}_b a}$... change of base law

7 $b^{\text{LOG}_b x} = x$

EXAMPLE 1 SOLVE EACH OF THE FOLLOWING EQUATIONS, CHECKING THAT YOUR SOLUTIONS ARE VALID.

- A $\log(x - 3) = 5$
 C $\log(x+3) + \log y = 1$
 E $\log_{10} x + \log_{10}(x - 20) = 3$

- B $\log(5x - 1) = 3$
 D $\log(x + 1) - \log(x + 3) = 1$

SOLUTION:

A $\log(x - 3) = 5 \Rightarrow 10^5 = x - 3 \quad \dots \text{changing to exponential form}$
 HENCE, $32 = x - 3$
 THEREFORE, $x = 35$

Check!

FROM THE DEFINITION OF LOGARITHMS, WE KNOW THAT $\log(x - 3)$ IS VALID ONLY WHEN $x - 3 > 0$, I.E. WHEN $x > 3$. SO $\{x | x > 3\} = (3, \infty)$ IS KNOWN AS UNIVERSE FOR $\log(x - 3)$. SINCE $x = 35$ IS AN ELEMENT OF THE UNIVERSE, THE SOLUTION OF THE GIVEN EQUATION.

A UNIVERSE IS THE LARGEST SET FOR WHICH THE GIVEN EXPRESSION IS DEFINED.

- B $\log(5x - 1) \text{ IS VALID WHEN } 5x - 1 > 0$

$$\text{SO } x > \frac{1}{5}. \text{ THEREFORE, THE UNIVERSE } U = \left(\frac{1}{5}, \infty\right)$$

$$\begin{aligned} \log_4(5x - 1) &= 3 \\ \Rightarrow 5x - 1 &= 4^3 \\ \Rightarrow 5x &= 64 + 1 \\ \Rightarrow x &= \frac{65}{5} = 13. \text{ SINCE } 13 \in \left(\frac{1}{5}, \infty\right), x = 13 \text{ IS THE SOLUTION.} \end{aligned}$$

- C REMEMBER THAT $\log(x)$ IS VALID FOR $x > 0$ AND $\log y$ IS VALID FOR $y > 0$.
 THEREFORE $\log(x+3) + \log y$ IS VALID FOR $x > 0$ AND $y > 0$. SO $U = (0, \infty)$.

NOW $\log(x+3) + \log y = 1$

$$\begin{aligned} \Rightarrow \log(x+3) &= 1 && \dots \text{since } \log + \log = \log y \\ \Rightarrow x(x+3) &= 10^1 && \dots \text{changing to exponential form} \\ \Rightarrow x^2 + 3x - 10 &= 0 \\ \Rightarrow (x+5)(x-2) &= 0 \end{aligned}$$

Thus, $x = -5$ OR $x = 2$

BUT -5 IS NOT AN ELEMENT OF THE UNIVERSE.

SO, THE ONLY SOLUTION IS

D $\log(x+1) - \log(x+3)$ IS VALID FOR $x > 0$ AND $x+3 > 0$,

I.E. FOR $x > -1$ AND $x > -3$.

THEREFORE THE $U = (-1, \infty)$,

$$\log(x+1) - \log(x+3) = 1$$

$$\Rightarrow \log\left(\frac{x+1}{x+3}\right) = 1 \quad \dots \text{since } \log\left(\frac{x}{y}\right) = \log x - \log y$$

$$\Rightarrow \frac{x+1}{x+3} = 3^1$$

$$\Rightarrow x+1 = 3(x+3) = 3x+9$$

THEREFORE $x = 2$ AND $x = -4$.

HOWEVER, 4 IS NOT IN THE UNIVERSE. HENCE, THE SATISFYING THE GIVEN EQUATION AND THE SOLUTION SET IS THE EMPTY SET.

E $\log_8 x + \log_{(-20)} x = 3$ IS VALID FOR $x > 0$ AND $-20 > 0$; I.E. FOR $x > 0$ AND $x > 20$.

SO $U = (20, \infty)$.

NOW $\log_8 x + \log_{(-20)} x = 3$

$$\Rightarrow \log_8(x-20) = 3 \quad \dots \log_b xy = \log_b x + \log_b y$$

$$\Rightarrow 8(x-20) = 10^3 = 1000$$

$$\Rightarrow 8x^2 - 160x = 1000$$

$$\Rightarrow 8x^2 - 160x - 1000 = 0$$

$$\Rightarrow 8(x^2 - 20x - 125) = 0$$

$$\Rightarrow x^2 - 20x - 125 = 0$$

$$\Rightarrow (x-25)(x+5) = 0$$

SO $x = 25$ OR $x = -5$. BUT $x \in (20, \infty)$

SO THE ONLY SOLUTION IS

Property of equality for logarithmic equations

IF B, x , AND y ARE POSITIVE NUMBERS, WHEN

$\log_b x = \log_b y$, IF AND ONLY IF,

FOR INSTANCE, IF $\log_2 7 = \log_2 7$, THEN $x = 7$. IF $x = 7$, THEN $\log_2 x = \log_2 7$.

EXAMPLE 2 SOLVE EACH OF THE FOLLOWING FOR

A $\log_3 x - \log(2x) = 0$

B $\log(4x-7) = \log(x+5)$

C $\log(x-5) + \log(10x) = \log(-6) + \log(-1)$

SOLUTION:

A $\log_3 x$ IS VALID WHEN $x > 0$ AND $\log(2-x)$ IS VALID WHEN $x < 2$.

SO $U = (0, 2)$.

NOW $\log_3 \log(2-x) = 0$ GIVES

$$\log_3 x = \log(2-x)$$

HENCE, $x = 2 - x$... property of equality

$$\Rightarrow 3x + x = 2$$

SO $x = \frac{1}{2}$ IS THE SOLUTION IN $(0, 2)$.

B $\log_{\frac{1}{4}}(x-7)$ IS VALID WHEN $x-7 > 0$ AND $\log(x+5)$ IS VALID WHEN $x+5 > 0$.

SO $U = \left(\frac{7}{4}, \infty\right)$. NEXT $\log(x-7) = \log(x+5)$ GIVES

$$4x-7 = x+5 \Rightarrow 3x = 12. \text{ SO } x = 4 \text{ IS THE SOLUTION.}$$

C THE TERM $\log 5$ IS VALID WHEN $5 > 0$, THE TERM $\log(10)$ IS VALID WHEN $x < 10$, THE TERM $\log(6)$ IS VALID WHEN $6 > 0$, AND THE TERM $\log(-1)$ IS VALID WHEN $-1 > 0$.

IF WE RESTRICT THE UNIVERSE TO THE SET OF ~~REAL NUMBERS~~ BETWEEN NUMBERS OR $6 < x < 10$, EVERY TERM IN THE EQUATION IS VALID.

THEREFORE $(6, 10)$ IS THE UNIVERSE.

$$\log(-5) + \log(10-x) = \log(-6) + \log(-1)$$

$$\Rightarrow \log((-5)(10-x)) = \log((-6)(-1))$$

$$\Rightarrow (x-5)(10-x) = (x-6)(x-1)$$

$$\Rightarrow -x^2 + 15x - 50 = x^2 - 7x + 6$$

$$\Rightarrow 15x - 50 = 2x^2 - 7x + 6 \quad \dots \text{ adding } x^2 \text{ to both sides}$$

$$\Rightarrow -50 = 2x^2 - 22x + 6$$

$$\Rightarrow 0 = 2x^2 - 22x + 56$$

$$\Rightarrow 0 = x^2 - 11x + 28 \quad \dots \text{ dividing both sides by 2}$$

$$\Rightarrow (x-7)(x-4) = 0.$$

$$\Rightarrow x = 7 \text{ OR } x = 4, \text{ BUT ONLY } 7 \text{ IS IN THE UNIVERSE.}$$

HENCE $x = 7$ IS THE SOLUTION.

Exercise 2.11

1 STATE THE UNIVERSE AND SOLVE EACH OF THE FOLLOWING FOR X.

- | | |
|---|---|
| A $\log_3(2x - 1) = 5$
C $\log_3(x^2 - 2x) = 1$
E $\log_x(1 + \frac{1}{x}) = 3$
G $\log_{x^2}(x^2 - 121) - \log_{x^2}(+ 11) = 1$
I $\log(x + 5) - \log 3 = \log 2 - \log(x - 1)$
K $\log(x + 1) + \log(x + 3) = 1$
M $\log_x(x + 6) = 2$ | B $\log_{\sqrt{2}}x = -1$
D $\log(x^2 + 3x + 2) = 1$
F $\log(x - 1) + \log 3 = 3$
H $\log(x + 4) - \log(x - 1) = 1$
J $\log - \log 3 = \log 4 - \log 10$
L $\log + \log(x + 2) - \log(3x - 5) = 3$ |
|---|---|

2 APPLY THE PROPERTY OF EQUALITY FOR LOGARITHMS TO SOLVE THE FOLLOWING EQUATIONS (CHECK THAT YOUR SOLUTIONS ARE VALID):

- | | |
|---|--|
| A $\log_3x + \log_35 = 2$
C $\log_3x + \log_3(x + 1) = 1$
E $\log_3^{6(x+2)} - \log_3^{9^{x+2}} = 0$ | B $\log_3 25 = 2 \log_3 5$
D $\log_2 x^2 - \log_2 16 = 0$
F $\log(x^2 - 9) - \log(3 + x) = 2$ |
|---|--|

2.5

APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

AS MENTIONED AT THE START OF THIS UNIT, EXPONENTIAL AND LOGARITHMIC FUNCTIONS ARE USED IN DESCRIBING AND SOLVING A WIDE VARIETY OF REAL-LIFE PROBLEMS. IN THIS SECTION, WE WILL DISCUSS SOME OF THEIR APPLICATIONS.

EXAMPLE 1 *Population Growth*

- A** SUPPOSE THAT YOU ARE OBSERVING THE BEHAVIOUR OF CELLS IN A LABORATORY. IN AN EXPERIMENT, YOU STARTED WITH ONE CELL AND THE NUMBER OF CELLS DOUBLED EVERY MINUTE.
 - I** WRITE AN EQUATION TO DETERMINE THE NUMBER OF CELLS.
 - II** DETERMINE HOW LONG IT WOULD TAKE FOR THE NUMBER OF CELLS TO REACH 100,000.
- B** ETHIOPIA HAS A POPULATION OF AROUND 80,000,000. IT IS ESTIMATED THAT THE POPULATION GROWS EVERY YEAR AT AN AVERAGE GROWTH RATE OF 2%. IF THE POPULATION GROWTH CONTINUES AT THE SAME RATE,
 - I** WHAT WILL BE THE POPULATION AFTER
 - 10 YEARS?
 - 20 YEARS?
 - II** HOW MANY YEARS WILL IT TAKE THE POPULATION TO DOUBLE? (SEE THE OPENING PROBLEM)

SOLUTION AND EXPLANATION:

A I FIRST RECORD YOUR OBSERVATIONS BY MAKING A TABLE. ONE COLUMN FOR THE TIME AND THE OTHER FOR THE NUMBER OF CELLS. THE NUMBER DEPENDS ON THE TIME.

FOR EXAMPLE, AT THERE IS 1 CELL, AND THE CORRESPONDING POINT IS (0, 1).

AT $t = 1$, THERE ARE 2 CELLS, AND THE CORRESPONDING POINT IS (1, 2).

AT $t = 2$, THERE ARE 4 CELLS, AND THE CORRESPONDING POINT IS (2, 4).

AT $t = 3$, THERE ARE 8 CELLS, AND THE CORRESPONDING POINT IS (3, 8), ETC.

THIS RELATIONSHIP IS SUMMARIZED IN THE FOLLOWING TABLE:

Time (in min.) (t)	0	1	2	3	4	5	6
No. of cells (y)	$1 = 2^0$	$2 = 2^1$	$4 = 2^2$	$8 = 2^3$	$16 = 2^4$	$32 = 2^5$	$64 = 2^6$

THEREFORE, THE FORMULA TO ESTIMATE THE NUMBER OF CELLS IS $f(t) = 2^t$

$$f(t) = 2^t$$

DETERMINE THE NUMBER OF CELLS AFTER ONE HOUR:

CONVERT ONE HOUR TO MINUTES. (1 HR = 60 MIN)

SUBSTITUTE 60 FOR T IN THE EQUATION,

$$f(60) = 2^{60} = 1.15 \times 10^{18} = 1,150,000,000,000,000,000$$

SO THE NUMBER OF CELLS AFTER 1 HOUR WILL BE $1,150,000,000,000,000,000 = 1.15 \times 10^{18}$

B II IN THIS EXAMPLE, YOU KNOW THE NUMBER OF CELLS AT THE BEGINNING OF THE EXPERIMENT (1) AND AT THE END OF THE EXPERIMENT (100,000), BUT YOU DO NOT KNOW THE TIME. SUBSTITUTE $f(0) = 1$ AND $f(100,000) = 100,000$ IN THE EQUATION 2^t :

$$100,000 = 2^t$$

TAKE THE NATURAL LOGARITHM OF BOTH SIDES:

$$\ln(100,000) = \ln(2^t) \Rightarrow \ln(100,000) = t \ln(2)$$

DIVIDE BOTH SIDES BY $\ln(2)$:

$$t = \frac{\ln(100,000)}{\ln(2)} \Rightarrow t = 16.60964 \text{ MINUTES}$$

IT WOULD TAKE ABOUT 16.6 MINUTES, FOR THE NUMBER OF CELLS TO REACH 100,000.

B I LET P REPRESENT THE CURRENT POPULATION WHICH IS $80,000,000 = 8.0 \times 10^7$ MILLION.

LET r REPRESENT THE ANNUAL GROWTH RATE WHICH IS 2.3%;

LET t REPRESENT THE TIME IN YEARS FROM NOW.

THE TOTAL POPULATION AFTER ONE YEAR:

$$\begin{aligned} A_1 &= 80 \text{ MILLION} + 2.3\% (80 \text{ MILLION}) \\ &= 8.0 \times 10^7 (1 + 2.3\%) \end{aligned}$$

THE TOTAL POPULATION AFTER TWO YEARS:

$$\begin{aligned} A_2 &= A_1 + 2.3\% (A_1) = A_1(1 + 2.3\%) = 8.0 \times 10^7 (1 + 2.3\%)(1 + 2.3\%) \\ &= 8.0 \times 10^7 (1 + 2.3\%)^2 \end{aligned}$$

THE TOTAL POPULATION AFTER THREE YEARS:

$$\begin{aligned} A_3 &= A_2 + 2.3\% (A_2) = A_2 (1 + 2.3\%) = 8.0 \times 10^7 (1 + 2.3\%)^2 (1 + 2.3\%) \\ &= 8.0 \times 10^7 (1 + 2.3\%)^3 \end{aligned}$$

FROM THE ABOVE PATTERN WE CAN GENERALIZE:

THE TOTAL POPULATION ~~NEARLY~~ GIVEN BY THE FORMULA:

$$A_t = P (1 + r)^t$$

SO THE TOTAL POPULATION AFTER 10 YEARS WILL BE

$$A_{10} = 8.0 \times 10^7 (1 + 2.3\%)^{10} = 100,426,036.81$$

THE TOTAL POPULATION AFTER TWENTY YEARS WILL BE

$$A_{20} = 8.0 \times 10^7 [1 + 2.3\%]^{20} = 126,067,360.86$$

II WHEN WILL THE TOTAL POPULATION DOUBLE? (END OF ELLIPSIS)
THE TOTAL POPULATION ~~NEARLY~~:

$$\begin{aligned} 8.0 \times 10^7 [1 + 2.3\%]^T &= 160,000,000 \\ \Rightarrow [1 + 2.3\%]^t &= \frac{160,000,000}{80,000,000} = 2 \Rightarrow \log(1 + 2.3\%) = \log 2 \\ \Rightarrow t \log(1 + 0.023) &= 0.3010 \Rightarrow t \log(1.023) = 0.3010 \\ \text{THEREFORE, } \frac{0.3010}{\log 1.023} &\approx \frac{0.3010}{0.0099} \approx 30.40 \end{aligned}$$

THEREFORE, THE CURRENT POPULATION IS EXPECTED TO DOUBLE IN ABOUT

EXAMPLE 2 Compound Interest

IF BIRR 5000 IS INVESTED AT A RATE OF 6% ~~COMPOUNDED~~ (4 TIMES A YEAR), THEN

- A WHAT IS THE AMOUNT AT THE END OF 4 YEARS AND 10 YEAR
- B HOW LONG DOES IT TAKE TO DOUBLE THE INVESTMENT?

SOLUTION: WE USE THE FORMULA $P \left(1 + \frac{r}{n}\right)^{nt}$

HERE $P = 5000$, $r = 6\% = 0.06$

$n = 4$ (COMPOUNDED 4 TIMES)

A TO FIND THE BALANCE AT THE END OF THE 4

$$A = p \left(1 + \frac{r}{n}\right)^{nt} = 5000 \left(1 + \frac{0.06}{4}\right)^{4 \times 4} = 5000 (1 + 0.015)^{16}$$

$$= 5000 (1.015)^{16} \approx 5000 (1.2690) = \text{BIRR } 6345$$

THE BALANCE AT THE END OF 10 YEARS

$$A = p \left(1 + \frac{r}{n}\right)^{nt} = 5000 \left(1 + \frac{0.06}{4}\right)^{4 \times 10} = 5000 (1 + 0.015)^{40} = 5000 (1.015)^{40}$$

$$\approx 5000 (1.8140) = \text{BIRR } 9070$$

B IF THE INVESTMENT IS TO BE DOUBLED, $\text{5000} \rightarrow \text{10000}$

$$A = p \left(1 + \frac{r}{n}\right)^{nt}$$

$$\Rightarrow 10,000 = 5000 \left(1 + \frac{0.06}{4}\right)^{4t} = 5000 (1 + 0.015)^{4t}$$

$$\Rightarrow 10,000 = 5000 (1.015)^{4t}$$

$$2 = (1.015)^{4t} \quad \dots \text{dividing both sides by 5000}$$

$$\text{LOG}2 = \text{LOG} (1.015)^{4t} \Rightarrow t = \frac{\text{LOG}2}{\text{LOG}(1.015)} = \frac{0.3010}{0.0065} = 46.30769 \Rightarrow t = \frac{46.30769}{4} \approx 11.58 \text{ YEARS}$$

IT TAKES ABOUT 12 YEARS TO DOUBLE THE INVESTMENT.

EXAMPLE 3 Chemistry (REFER BACK TO ACTIVITY 2.8)

THE CONCENTRATION OF HYDROGEN IONS IN A SOFT DRINK IS MEASURED IN MOLES PER LITRE. FOR BEER IT IS $10^{-4.3}$ MOLES/LITRE. FOR BEER AND $[\text{H}^+] = 0.0004$ FOR WINE. CHEMISTS DEFINE THE PH OF A SOFT DRINK AS THE NUMBER $\text{PH} = -\text{LOG}[\text{H}]$. THE SOLUTION IS SAID TO BE AN ACID IF $\text{PH} < 7$ AND A BASE IF $\text{PH} > 7$. PURE WATER HAS A PH OF 7, WHICH MEANS IT IS NEUTRAL.

A IS BEER AN ACID OR A BASE? WHAT ABOUT WINE?**B** WHAT IS THE HYDROGEN ION CONCENTRATION? THE PH OF EGGS IS 7.8?**SOLUTION:****A** (TEST FOR BEER)

$$\text{PH} = -\text{LOG}[\text{H}]$$

$$\text{PH} = -\text{LOG}[0.000050] = -\text{LOG}[5 \times 10^{-5}] = [\text{LOG}5.01 + (-5)] = 4.3$$

SINCE $\text{PH} = 4.3 < 7$ BEER IS AN ACID.

(TEST FOR WINE)

$$\begin{aligned} \text{PH} &= -\log[\text{H}] = -\log[0.0004] = \log[4^{-1}] = -\log[4(-)] \\ &= -[0.6021 + (-4)] \approx 3.4 \Rightarrow \text{PH} = 3.4 < 7. \end{aligned}$$

SOWINE IS AN ACID.

B $\text{PH} = -\log[\text{H}] \Rightarrow -\log[\text{H}] = 7$

$$\begin{aligned} \Rightarrow \log[\text{H}] &= -7 \Rightarrow [\text{H}^+] = 10^{-7.8} \\ \Rightarrow [\text{H}^+] &= 1.58 \times 10^{-8} \end{aligned}$$

Group Work 2.6

NEWTON'S LAW OF COOLING STATES THAT AN OBJECT'S TEMPERATURE IS PROPORTIONAL TO THE DIFFERENCE BETWEEN THE TEMPERATURE OF THE OBJECT AND THE ROOM TEMPERATURE. THE TEMPERATURE OF THE OBJECT AT A TIME t IS GIVEN BY A FUNCTION

$$f(t) = ce^{rt} + a,$$

WHERE a = ROOM TEMPERATURE c = INITIAL DIFFERENCE IN TEMPERATURE BETWEEN THE OBJECT AND THE ROOM TEMPERATURE r = CONSTANT DETERMINED BY DATA IN THE PROBLEM

PROBLEM: SUPPOSE YOU MAKE YOURSELF A CUP OF TEA. AT 100°, THE TEA HAS COOLED TO 65°. WHEN WILL THE TEA REACH A DRINKABLE TEMPERATURE OF 40°?

Hint: ASSUME THAT THE ROOM TEMPERATURE IS 22°. FIRST SOLVE FOR c AND THEN FIND t APPLYING THE NATURAL LOGARITHM.

Exercise 2.12

1. SUPPOSE YOU ARE OBSERVING THE BEHAVIOR OF CELLS IN A LABORATORY. IN ONE EXPERIMENT, YOU START WITH ONE CELL AND THE CELL POPULATION IS TRIPLING EVERY HOUR.
 - A. WRITE A FORMULA TO DETERMINE THE NUMBER OF CELLS AFTER t HOURS.
 - B. USE YOUR FORMULA TO CALCULATE THE NUMBER OF CELLS AFTER AN HOUR.
 - C. DETERMINE HOW LONG IT WOULD TAKE THE NUMBER OF CELLS TO REACH 1,000.
2. SUPPOSE IN AN EXPERIMENT YOU STARTED WITH 100,000 CELLS AND OBSERVED THAT THE CELL POPULATION DECREASED BY ONE HALF EVERY MINUTE.
 - A. WRITE A FORMULA FOR THE NUMBER OF CELLS AFTER t MINUTES.
 - B. DETERMINE THE NUMBER OF CELLS AFTER 10 MINUTES.
 - C. DETERMINE HOW LONG IT WOULD TAKE THE CELL POPULATION TO REACH 10,000.

- 3** A BIRR 1,000 DEPOSITS IS MADE AT A BANK THAT PAYS 12% INTEREST COMPOUNDED MONTHLY. HOW MUCH WILL BE IN THE ACCOUNT AT THE END OF 10 YEARS?
- 4** IF YOU START A BIOLOGY EXPERIMENT WITH 5,000,000 CELLS AND 25% OF THEM ARE DYING EVERY MINUTE, HOW LONG WILL IT BE BEFORE THERE ARE FEWER THAN 1,000 CELLS?
- 5** **Learning curve:** IN PSYCHOLOGICAL TESTS, IT IS FOUND THAT THE NUMBER OF WORDS LEARNED BY A STUDENT IN THE NINTH HOUR OF STUDY IS APPROXIMATELY $y = 100(1 - e^{-0.03t})$, WHERE t IS THE NUMBER OF HOURS OF STUDY. FIND HOW MANY WORDS A STUDENT WOULD BE EXPECTED TO LEARN IN THE NINTH HOUR OF STUDY.
- 6** THE ENERGY RELEASED BY THE LARGEST EARTHQUAKE EVER TO RULES, IS ABOUT 100 BILLION TIMES THE ENERGY RELEASED BY A SMALL EARTHQUAKE THAT FELT. IN 1935 THE CALIFORNIA SEISMOLOGIST Charles Richter DEVISED A LOGARITHMIC SCALE THAT BEARS HIS NAME AND IS STILL WIDELY USED. ON THE RICHTER SCALE IS GIVEN AS FOLLOWS:

$$M = \frac{2}{3} \log_{10} \frac{E}{E_0} \text{ RICHTER SCALE}$$

WHERE E IS THE ENERGY RELEASED BY THE EARTHQUAKE MEASURED IN JOULES, AND E_0 IS THE ENERGY RELEASED BY A VERY SMALL REFERENCE EARTH QUAKE WHICH HAS BEEN STANDARDIZED TO BE 4.40×10^{-22} JOULES.

QUESTION

AN EARTH QUAKE IN A CERTAIN TOWN X RELEASED APPROXIMATELY 5.96×10^{16} JOULES OF ENERGY. WHAT WAS ITS MAGNITUDE ON THE RICHTER SCALE? GIVE YOUR ANSWER TO TWO DECIMAL PLACES.

- 7** **Physics:** THE BASIC UNIT OF SOUND MEASUREMENT IS A BEL. A BEL IS THE INVENTOR OF TELEPHONE, ALEXANDER GRAHAM BELL (1847-1922). THE LOUDEST SOUND THAT AVERAGE HEALTHY PERSON CAN HEAR WITHOUT DAMAGE TO THE EARDRUM HAS AN INTENSITY 10^{12} TIMES THAT OF THE SOFTEST OF SOUND. THE RELATIONSHIP OF LOUDNESS OF SOUND INTENSITIES IS GIVEN BY

$$L = 10 \log_{10} \frac{I}{I_0},$$

WHERE L IS MEASURED IN DECIBELS, I_0 IS THE INTENSITY OF THE LEAST AUDIBLE SOUND THAT AN AVERAGE HEALTHY PERSON CAN HEAR, WHICH IS 10^{-12} WATT PER SQUARE METER, AND I IS THE INTENSITY OF THE SOUND IN QUESTION.

QUESTION

FIND THE NUMBER OF DECIBELS:

- A** FROM AN ORDINARY CONVERSATION WITH SOUND INTENSITY PER SQUARE METER.
- B** FROM A ROCKMUSIC CONCERT WITH SOUND INTENSITY 10^3 WATT PER SQUARE CENTIMETRE.



Key Terms

antilogarithm	exponential expression	logarithmic expression
base	exponential function	logarithmic function
characteristics	logarithm	mantissa
common logarithm	logarithm of a number	natural logarithm
exponent	logarithmic equation	power
exponential equation		



Summary

1 IF n IS A POSITIVE INTEGER, ~~IS THE~~ PRODUCT OF ~~FACTORS~~ OF

I.E. $a^n = \underbrace{a \times a \times a \times \dots \times a}_{n \text{ FACTORS}}$

IN a^n , a IS CALLED ~~THE~~ base, n IS CALLED ~~THE~~ exponent AND n IS THE ~~THE~~ power OF a .

2 Laws of Exponents

FOR a AND b POSITIVE AND r AND s REAL NUMBERS

A $a^r \times a^s = a^{r+s}$

B $\frac{a^r}{a^s} = a^{r-s}$

C $(a^r)^s = a^{rs}$

D $(a \times b)^s = a^s \times b^s$

E $\left(\frac{a}{b}\right)^s = \frac{a^s}{b^s}$

3 ANY NON – ZERO NUMBER RAISED TO ZERO IS EQUAL TO 0.

4 FOR $a \neq 0$ AND $n > 0$, $a^{-n} = \frac{1}{a^n}$.

5 FOR $a \neq 0$, $b \neq 0$ AND $n > 0$, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

6 FOR ANY REAL NUMBER AND ANY INTEGER $a^{\frac{1}{n}} = \sqrt[n]{a}$.

$\sqrt[n]{a} \in \mathbb{R}$ IF $a \in \mathbb{R}$ AND n IS ODD $\sqrt[n]{a} \notin \mathbb{R}$ IF $a < 0$ AND n IS EVEN

- 7 IF $a > 0$ AND n ARE INTEGERS, THEN $\sqrt[n]{a^m} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$.
- 8 IF x IS AN IRRATIONAL NUMBER, THEN x IS A REAL NUMBER BETWEEN a^x_1 FOR ALL POSSIBLE CHOICES OF RATIONAL NUMBERS a^x_2 SUCH THAT $x < x_2$.
- 9 FOR A FIXED POSITIVE NUMBER b AND FOR EACH x , $b^x = a$, IF AND ONLY IF $x = \log_b a$. ($c = \log_b a$ IS READ AS THE LOGARITHM OF a TO THE BASE b)

10 Laws of logarithms

IF b, x AND y ARE POSITIVE NUMBERS AND $b \neq 1$

- A $\log_b (xy) = \log_b x + \log_b y$ B $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$
 C FOR ANY REAL NUMBER x , $\log_b x^k = k \log_b x$ D $\log_b b^x = x$
 E $\log_b 1 = 0$

- 11 LOGARITHMS TO BASE 10 ARE CALLED Common logarithms.
- 12 THE CHARACTERISTIC OF A COMMON LOGARITHM IS THE DECIMAL POINT. THE MANTISSA IS A POSITIVE DECIMAL LESS THAN 1.
- 13 IF a, b, c ARE POSITIVE REAL NUMBERS, THEN

A $\log_c x = \frac{\log_b x}{\log_b c}$ ("CHANGE OF BASE LAW") B $b^{\log_b c} = c$

- 14 $\log_e x = \ln x$ IS CALLED THE natural logarithm OF x .
- 15 THE FUNCTION $y = b^x$, $b > 0$ AND $b \neq 1$ DEFINES AN EXPONENTIAL FUNCTION.
- 16 THE FUNCTION $y = e^x$ IS CALLED THE natural exponential function.
- 17 ALL MEMBERS OF THE FAMILY

($b > 0$, $b \neq 1$) HAVE GRAPHS WHICH

- ✓ PASS THROUGH THE POINT $(0, 1)$
- ✓ ARE ABOVE THE X-AXIS FOR ALL VALUES OF x
- ✓ ARE ASYMPTOTIC TO THE X-AXIS
- ✓ HAVE DOMAIN THE SET OF ALL REAL NUMBERS.
- ✓ HAVE RANGE THE SET OF ALL POSITIVE REAL NUMBERS.

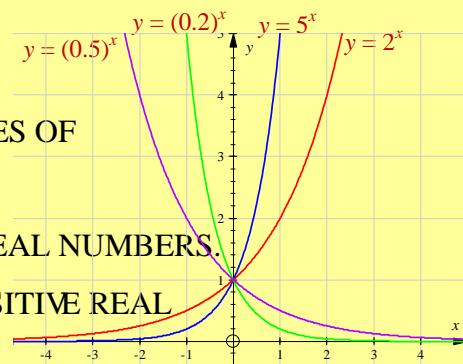


Figure 2.24

- 18** THE FUNCTION $y = \log_b x$, $b > 0$ AND $b \neq 1$ IS CALLED A LOGARITHMIC FUNCTION WITH BASE b .
- 19** THE FUNCTION $y = \log_b x = \ln x$ IS CALLED THE **natural logarithm** OF x .
- 20** ALL MEMBERS OF THE FAMILY, $(b > 0, b \neq 1)$ HAVE GRAPHS WHICH
- ✓ PASS THROUGH THE POINT $(1, 0)$
 - ✓ ARE ASYMPTOTIC TO THE x -axis
 - ✓ HAVE DOMAIN THE SET OF ALL POSITIVE REAL NUMBERS
 - ✓ HAVE RANGE THE SET OF REAL NUMBERS.

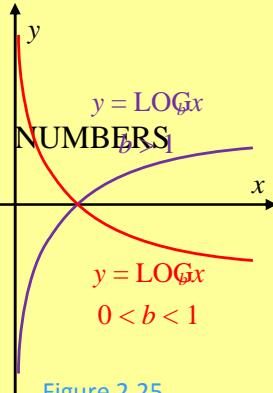


Figure 2.25



Review Exercises on Unit 2

- 1** WRITE THE SIMPLIFIED FORM OF EACH OF THE EXPRESSIONS FOLLOWING:
- | | | | |
|---------------------------------------|--|----------------------------------|--|
| A 2^5 | B -2^5 | C 2^{-5} | D -2^{-5} |
| E $\left(\frac{2}{3}\right)^2$ | F $\left(\frac{2}{3}\right)^{-2}$ | G $\frac{2^{-2}}{3^{-2}}$ | H $\left(-\frac{2}{3}\right)^2$ |
- 2** USE THE LAWS OF EXPONENTS TO SIMPLIFY EACH OF THE EXPRESSIONS:
- | | | | |
|-----------------------------|---|---|-------------------------|
| A $2^5 \times 2^2$ | B $\left(6^{\frac{1}{2}}\right)^2$ | C $\frac{64^{\frac{3}{2}}}{8^2}$ | D $a^{-3}b^{-3}$ |
| E $(4n^5)^2$ | F $\left(\frac{x}{2y}\right)^2$ | G $\frac{d^{-4}}{d^{-2}}$ | H $(x^{-3})^2$ |
| I $E^{3x-1} E^{4-x}$ | J $\frac{3^x}{3^{1-x}}$ | K $\frac{5^{x-3}}{5^{x-4}}$ | L $(2^x 3^y)^z$ |
- 3** CHANGE EACH LOGARITHMIC FORM TO AN EQUIVALENT EXPONENTIAL FORM:
- | | |
|--------------------------------------|-----------------------------------|
| A $\log_3 81 = 4$ | B $\log_5 5 = \frac{1}{2}$ |
| C $\log_{\frac{1}{4}} 1 = -2$ | D $\log_{\frac{1}{2}} 4 =$ |

4 FIND x IF:

A $\log_2 x = 5$

B $\log_{16} x = x$

C $\log_7 x = x$

D $\log_x 16 = 2$

E $\log_8 x = \frac{1}{3}$

F $\log_{\frac{1}{3}} 9 = x$

G $\log_9 \frac{1}{7} = x$

H $\log_{1000} \frac{3}{2} = x$

5 USE THE PROPERTIES OF LOGARITHMS TO ~~WRITING~~ EXPRESSIONS AS A SINGLE LOGARITHM:

A $\log 2 + \log 25$

B $\log 18 - \log 3$

C $3\log 5 - 2\log 7$

D $5\log x + 3\log y$

E $\log x^3 + \log \left(\frac{b}{\sqrt[3]{x}} \right)$

F $\ln x^3 - \ln \sqrt{x}$

6 USE THE TABLE OF COMMON LOGARITHMS TO FIND:

A $\log 4.21$

B $\log 0.99$

C $\log 8.2$

D $\log 123$

E $\log 0.34$

F $\log 8.88$

G $\log 0.00001$

H $\log 500$

7 FIND:

A ANTILOG 0.4183

B ANTILOG 0.3507

C ANTILOG 0.5428

D ANTILOG 0.8831

E ANTILOG 5.9736

F ANTILOG 1.7559

G ANTILOG(0)

H ANTILOG(3)

8 STUDY THE FOLLOWING ~~GRAPH~~ AND ANSWER THE QUESTIONS THAT FOLLOW:

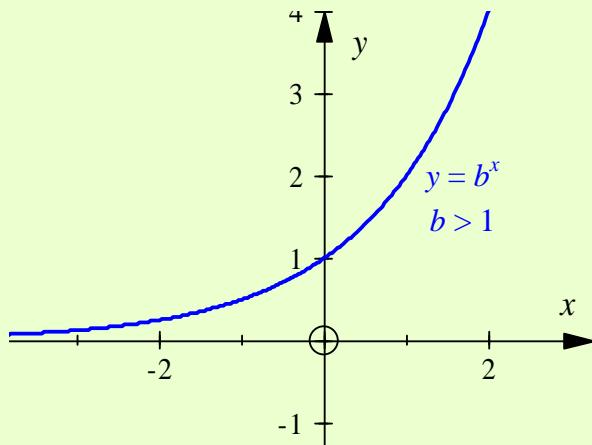


Figure 2.26

- A** GIVE THE DOMAIN AND THE RANGE OF THE FUNCTION.
- B** WHAT IS THE ASYMPTOTE OF THE GRAPH?
- C** IS THE FUNCTION INCREASING OR DECREASING?
- D** WHAT IS THE INTERCEPT?
- E** FOR WHICH VALUES OF x IS y GREATER THAN 1?
- F** WHAT CAN YOU SAY ABOUT THE VALUES OF y FOR NEGATIVE x ?
- G** FOR WHICH VALUES OF x IS y LESS THAN ZERO?
- 9** STUDY THE FOLLOWING GRAPH AND ANSWER THE QUESTIONS GIVEN BELOW.

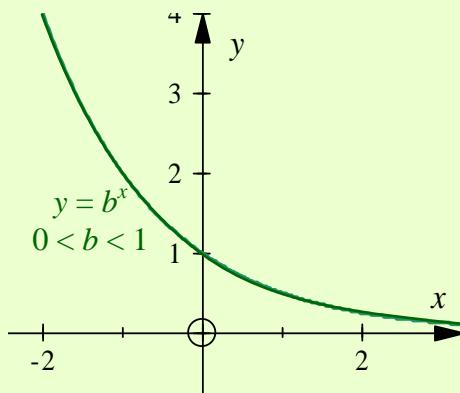


Figure 2.27

- A** GIVE THE DOMAIN AND THE RANGE OF THE FUNCTION.
- B** WHAT IS THE ASYMPTOTE OF THE GRAPH?
- C** IS THE FUNCTION INCREASING OR DECREASING?
- D** WHAT IS THE INTERCEPT?
- E** FOR WHICH VALUES OF x IS $y > 1$?
- F** WHAT IS THE VALUE OF y IF x IS POSITIVE?
- G** FOR WHICH VALUES OF x IS $y < 0$?
- 10** SKETCH THE FOLLOWING PAIRS OF FUNCTIONS IN A COORDINATE SYSTEM:
- A** $f(x) = 2^x - 3$ AND $g(x) = 2^x + 3$
- B** $f(x) = 3^x$ AND $g(x) = 3^x + 2$
- C** $f(x) = \left(\frac{3}{5}\right)^x$ AND $g(x) = \left(\frac{3}{5}\right)^{x+1}$
- D** $f(x) = 5^x$ AND $g(x) = \left(\frac{1}{5}\right)^x$

11 STUDY THE FOLLOWING GRAPH AND ANSWER THE QUESTIONS THAT FOLLOW:

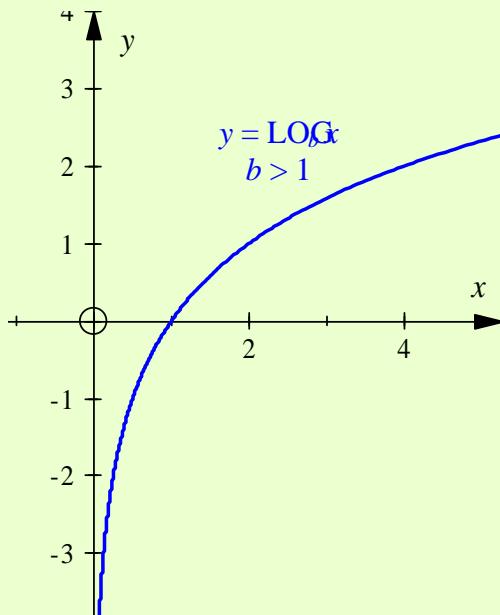


Figure 2.28

- A** GIVE THE DOMAIN AND THE RANGE OF THE FUNCTION.
 - B** WHAT IS THE ASYMPTOTE OF THE GRAPH?
 - C** IS THE FUNCTION INCREASING OR DECREASING?
 - D** WHAT IS THE INTERCEPT?
 - E** FOR WHICH VALUES OF $\text{LOG}_b x > 0$?
 - F** WHEN IS $\text{LOG}_b x < 0$?
- 12** SKETCH THE FOLLOWING PAIRS OF FUNCTIONS IN A COORDINATE SYSTEM:
- A** $f(x) = \text{LOG}_b x$ AND $g(x) = \text{LOG}_b(x-2)$
 - B** $f(x) = \text{LN}x$ AND $g(x) = \text{LN}(x+2)$
 - C** $f(x) = \text{LOG}_b x$ AND $g(x) = \text{LOG}_{\left(\frac{1}{5}\right)} x$
 - D** $f(x) = 5^x$ AND $g(x) = \text{LOG}_5 x$
- 13** STATE THE UNIVERSE FOR EACH OF THE FOLLOWING FUNCTIONS
- | | |
|---|--|
| A $f(x) = \text{LOG}_b x$ | B $g(x) = \text{LOG}_{\left(\frac{1}{3}\right)}(x+)$ |
| C $f(x) = \text{LOG}(3-x)$ | D $g(x) = \text{LOG}(7x-12)$ |
| E $f(x) = \text{LOG}(3-x) + \text{LOG}(3+x)$ | F $f(x) = \text{LOG}(x^2 - 2x)$ |

14 SOLVE EACH OF THE FOLLOWING EXPONENTIAL EQUATIONS:

A $3^x = 27$

B $2^{3-x} = 16$

C $5^{(4x-5)} = \frac{1}{25}$

D $4^{3x-8} = 2^{3x+9}$

E $36^{5x} = 6$

F $7^{x^2+x} = 49$

G $2^{6(x+2)} = 4^{x+2}$

H $2\left(\frac{243}{32}\right)^{2x} = 3\left(\frac{8}{27}\right)^{\left(\frac{2}{3}x-1\right)}$

15 SOLVE EACH OF THE FOLLOWING LOGARITHMIC EQUATIONS, CHECKING VALIDITY OF SOLUTIONS:

A $\log x = 3$

B $\log x = \frac{3}{2}$

C $\log e^5 = 5$

D $\log x^2 - \log x = 2$

E $\log - \log 3 = \log 4 \log (+4)$

F $\ln(+3) - \ln(-1) = 2 \ln 2$

G $\ln(2+1) - \ln(-1) = \ln$

H $\log x^2 - 3 = 2 \log(-1)$

I $\log(4x)^5 = 5$

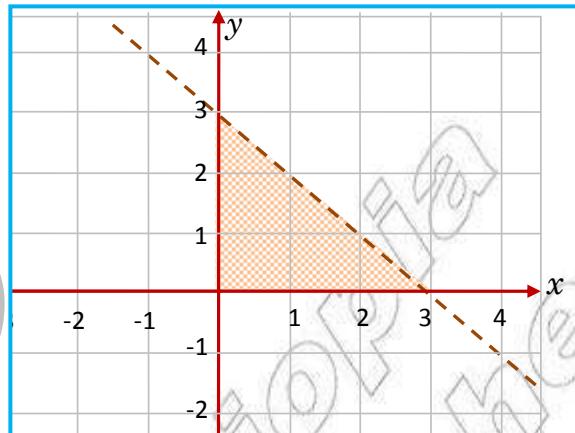
J $\log x + \log x^2 = 15$

K $\log(3+x) - \log x = 2$

16 IF 2000 BIRR IS INVESTED AT 4% INTEREST, COMPOUNDED FOR 5 YEARS, WHAT IS THE AMOUNT REALIZED AT THE END OF 5 YEARS**17** SUPPOSE THAT THE NUMBER OF BACTERIA OR AT CERTAIN POINT GROWS AT THE RATE OF 5% PER DAY. IF THERE ARE 1000 BACTERIA PRESENT INITIALLY, THEN WHAT IS THE NUMBER OF BACTERIA PRESENT AFTER:**A** 1 DAY? **B** 2 DAYS? **C** 3 DAYS? **D** 0 DAYS? **E** DAYS?**18** THE POPULATION OF COUNTRY A IS AND THAT OF COUNTRY B IS 10^8 . IF THE ANNUAL GROWTH OF POPULATION OF COUNTRIES A AND B ARE 5.2% AND 4.8% RESPECTIVELY, WHEN WILL COUNTRIES A AND B HAVE THE SAME POPULATION?**19** A CAR PURCHASED FOR 30,000 BIRR DEPRECIATES AT 3% PER YEAR. THE DEPRECIATION BEING WORKED OUT ON THE VALUE OF THE CAR AT THE BEGINNING OF EACH YEAR. FIND ITS VALUE AFTER 10 YEARS.

Hint: IF V_0 IS THE VALUE OF A CERTAIN OBJECT AT A CERTAIN TIME, AND THE RATE OF DEPRECIATION PER YEAR, THEN THE VALUE AT THE END OF YEARS IS GIVEN BY: $V_t = V_0 \left(1 - \frac{r}{100}\right)^t$, WHERE V_0 IS THE INITIAL VALUE.

Unit 3



SOLVING INEQUALITIES

Unit Outcomes:

After completing this unit, you should be able to:

- know and apply methods and procedures in solving problems on inequalities involving absolute value.
- know and apply methods for solving systems of linear inequalities.
- apply different techniques for solving quadratic inequalities.

Main Contents

- 3.1 Inequalities involving absolute value**
- 3.2 Systems of linear inequalities in two variables**
- 3.3 Quadratic inequalities**

Key Terms

Summary

Review Exercises

INTRODUCTION

RECALL THAT OPEN STATEMENTS OF THE FORM $x < 0$, $ax + b < 0$ AND $ax + b \geq 0$ FOR $a \neq 0$ ARE INEQUALITIES WITH SOLUTIONS USUALLY INVOLVING INTERVALS.

IN THIS UNIT, YOU WILL STUDY METHODS OF SOLVING INEQUALITIES INVOLVING ABSOLUTE VALUE, SYSTEM OF LINEAR INEQUALITIES IN TWO VARIABLES AND QUADRATIC INEQUALITIES. LEARN ABOUT THE APPLICATIONS OF THESE METHODS IN SOLVING PRACTICAL PROBLEMS INVOLVING INEQUALITIES.

3.1 INEQUALITIES INVOLVING ABSOLUTE VALUE

THE METHODS FREQUENTLY USED FOR DESCRIBING SETS ARE THE PARTIAL LISTING METHOD AND THE SET-BUILDER METHOD. SETS OF REAL NUMBERS CAN BE DESCRIBED BY USING THE SET-BUILDER METHOD OR INTERVALS (between any two given real numbers).

Notation: FOR REAL NUMBERS a WHERE $a < b$,

- ✓ (a, b) IS AN OPEN INTERVAL;
- ✓ $(a, b]$ AND $[a, b)$ ARE HALF CLOSED OR HALF OPEN INTERVALS; AND
- ✓ $[a, b]$ IS A CLOSED INTERVAL.

FOR EXAMPLE, $(5, 9)$ IS THE SET OF REAL NUMBERS BETWEEN 5 AND 9 AND $[5, 9]$ IS THE SET OF REAL NUMBERS BETWEEN 5 AND 9 INCLUDING 5 AND 9.

THAT IS, $(5, 9) = \{x : 5 < x < 9 \text{ AND } x \in \mathbb{R}\}$

$[5, 9] = \{x : 5 \leq x \leq 9 \text{ AND } x \in \mathbb{R}\}$

IN GENERAL, IF a AND b ARE FIXED REAL NUMBERS WITH

$$[a, b] = \{x : a \leq x \leq b \text{ AND } x \in \mathbb{R}\} \quad (a, b) = \{x : a < x < b \text{ AND } x \in \mathbb{R}\}$$

$$(-\infty, a] = \{x : x \leq a \text{ AND } x \in \mathbb{R}\} \quad (a, \infty) = \{x : x > a \text{ AND } x \in \mathbb{R}\}$$

Note: THE SYMBOL ∞ IS USED TO MEAN positive infinity AND $-\infty$ IS USED TO MEAN negative infinity.

GRAPHICALLY, $[a, b]$ IS

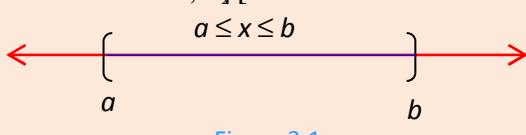


Figure 3.1

GRAPHICALLY, (a, b) IS

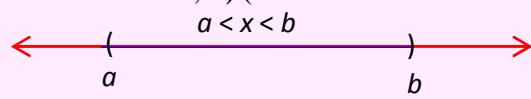


Figure 3.2

GRAPHICALLY, $[-\infty, a]$ IS

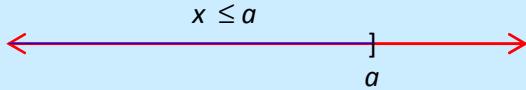


Figure 3.3

GRAPHICALLY, (a, ∞) IS

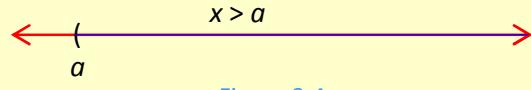


Figure 3.4

INTERVALS ARE COMMONLY USED TO EXPRESS THE SOLUTION SETS OF INEQUALITIES. FOR US FIND THE SOLUTION SET OF $2x + 4 \leq 3x - 5$.

$2x + 4 \leq 3x - 5$ IS EQUIVALENT TO $x \geq -9$.

MULTIPLYING BOTH SIDES BY -1 GIVES REMEMBER THAT, WHEN YOU MULTIPLY OR DIVIDE BY A NEGATIVE NUMBER, THE INEQUALITY CHANGES.

SO, THE SOLUTION SET IS $[9, \infty)$.

ACTIVITY 3.1



- 1 DISCUSS THE 3-METHODS OF DESCRIBING SETS: *SETS IN THE method, THE partial listing method AND THE set-builder method.*
- 2 GIVE EXAMPLES FOR EACH OF THE METHODS.
- 3 DESCRIBE EACH OF THE FOLLOWING SETS USING ANY ONE OF
 - A THE SET OF NUMBERS 2, 1, 0, 2, 3.
 - B THE SET OF ALL NEGATIVE MULTIPLES OF 2.
 - C THE SET OF NATURAL NUMBERS GREATER THAN 506 AND LESS THAN 507.
- 4 DESCRIBE EACH OF THE FOLLOWING SETS USING THE SET-BUILDER NOTATION.

A $\{ \dots, -3, -2, -1, 0, 1, 2, 3, \dots \}$	B $\{ 0, 3, 6, 9, \dots \}$
C $[-3, 5)$	D $[2, \infty)$
- 5 WRITE EACH OF THE FOLLOWING USING INTERVALS:

A $\{ x : x \in \mathbb{R} \setminus \{0\} \}$	B $\{ x : -1 \leq x \leq 2 \text{ AND } x \in \mathbb{R} \}$
C $\{ x : 0.2 < x \leq 0.8 \text{ AND } x \in \mathbb{R} \}$	D $\{ x : x \in \mathbb{R} \text{ AND } x \neq -1 \}$
- 6 FIND ALL VALUES SATISFYING THE FOLLOWING INEQUALITIES:

A $2x - 1 < 7$	B $4 \leq 1 - x < 5$
-----------------------	-----------------------------

LOOK AT THE NUMBER LINE GIVEN BELOW.

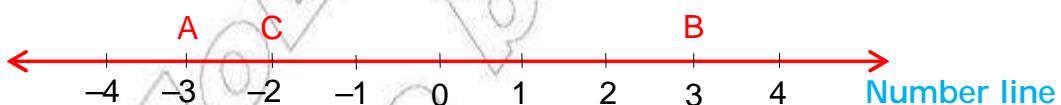


Figure 3.5

WHAT ARE THE COORDINATES OF POINTS A AND B?

WHAT IS THE DISTANCE OF POINT A FROM THE ORIGIN? WHAT ABOUT B?

THE NUMBER THAT SHOWS ONLY THE DISTANCE FROM THE POINT CORRESPONDING TO THE DIRECTION (REGARDLESS OF THE DIRECTION) IS CALLED THE **Absolute value**. FOR EXAMPLE, THE POINT C (WITH COORDINATE -2) IS 2 UNITS FROM THE POINT CORRESPONDING TO ZERO. THIS IS DENOTED BY

ON THE NUMBER LINE, THE DISTANCE BETWEEN THE POINT CORRESPONDING TO NUMBER x AND THE POINT CORRESPONDING TO ZERO, REGARDLESS OF WHETHER THE POINT IS TO THE LEFT OR TO THE RIGHT OF THE POINT CORRESPONDING TO ZERO, IS SHOWN BELOW.

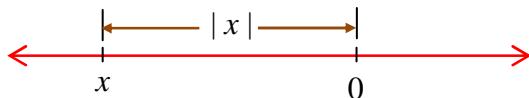


Figure 3.6

Definition 3.1

If x is a real number, then the absolute value of x , denoted by $|x|$, is defined by

$$|x| = \begin{cases} x, & \text{IF } x \geq 0 \\ -x, & \text{IF } x < 0 \end{cases}$$

EXAMPLE 1

A $|25| = 25$ BECAUSE $25 > 0$ B $\left| -\frac{4}{5} \right| = -\left(-\frac{4}{5} \right) = \frac{4}{5}$ BECAUSE $\frac{4}{5} < 0$

ACTIVITY 3.2



- 1 WHY IS IT ALWAYS TRUE THAT FOR ANY REAL NUMBER x , $|x| \geq 0$?
- 2 EVALUATE EACH OF THE FOLLOWING EXPRESSIONS:

A $ -3 $	B $ 0 $	C $ \sqrt{-5} $
D $ -3 + -2 $	E $ 1 - \sqrt{2} $	F $ \sqrt{3} - \sqrt{5} $

- 3 IF $x = -2$ AND $y = 3$, THEN EVALUATE EACH OF THE FOLLOWING:

A $ 6x + y $	B $ 6x + y $	C $ 2x - 3y $
--------------	----------------	---------------

- 4 VERIFY EACH OF THE FOLLOWING USING EXAMPLES:

A $ x - y = y - x $	B $ 2x - 3y = 3y - 2x $	C $\sqrt{x^2} = x $
D $ x y = xy $	E $\left \frac{x}{y} \right = \frac{ x }{ y }$	

GEOMETRICALLY, THE EQUATIONS THAT THE POINT WITH COORDINATE x IS n UNITS AWAY FROM THE POINT CORRESPONDING TO ZERO, ON THE NUMBER LINE. OBVIOUSLY, THE NUMBER LINE CONTAINS TWO POINTS THAT ARE n UNITS FROM THE POINT CORRESPONDING TO ZERO, ONE TO THE LEFT AND THE OTHER TO THE RIGHT. THESE TWO SOLUTIONS ARE $x = n$ AND $x = -n$.

Theorem 3.1 Solutions of the equation $|x| = a$

For any real number a , the equation $|x| = a$ has

- I two solutions $x = a$ and $x = -a$, if $a > 0$;
- II one solution, $x = 0$, if $a = 0$; and
- III no solution, if $a < 0$.

EXAMPLE 2 SOLVE EACH OF THE FOLLOWING ABSOLUTE VALUE EQUATIONS:

A $|3x + 5| = 2$

B $\left|\frac{2}{3}x + 1\right| = 0$

C $|2x - 1| = -3$

SOLUTION:

A $|3x + 5| = 2$ IS EQUIVALENT TO $3x + 5 = 2$ OR $3x + 5 = -2$

$$\Rightarrow 3x + 5 - 5 = -2 - 5 \text{ OR } 3x + 5 - 5 = 2 - 5$$

$$\Rightarrow 3x = -7 \text{ OR } 3x = -3$$

$$\Rightarrow x = -\frac{7}{3} \text{ OR } x = -1$$

THEREFORE, $-\frac{7}{3}$ AND -1 ARE THE TWO SOLUTIONS.

B WE KNOW THAT $\frac{2}{3}x + 1 \neq 0$ IF AND ONLY IF $\frac{2}{3}x + 1 = 0$. THEREFORE $\left|\frac{2}{3}x + 1\right| = 0$ IS

EQUIVALENT TO $\frac{2}{3}x + 1 = 0$. HENCE, $\frac{2}{3}x = -1$

$$\Rightarrow x = -\frac{3}{2} \text{ IS THE SOLUTION.}$$

C SINCE $|x| \geq 0$ FOR ALL $x \in \mathbb{R}$, THE GIVEN EQUATION $|x| = -3$ HAS NO SOLUTION.

AS DISCUSSED ABOVE, $|x| = 4$ MEANS $x = -4$ OR $x = 4$. HENCE, ON THE NUMBER LINE, THE POINT CORRESPONDING TO 4 UNITS AWAY FROM THE POINT CORRESPONDING TO 0 . WE SEE THAT $|x| \leq 4$, THE DISTANCE BETWEEN THE POINT CORRESPONDING TO x AND THE POINT CORRESPONDING TO 0 IS LESS THAN 4 OR EQUAL TO 4 . IT FOLLOWS THAT $|x| \leq 4$ IS EQUIVALENT TO $-4 \leq x \leq 4$. WE HAVE THE FOLLOWING GENERALIZATION.

Theorem 3.2 Solution of $|x| < a$ and $|x| \leq a$

For any real number $a > 0$,

- I the solution of the inequality $|x| < a$ is $-a < x < a$.
- II the solution of the inequality $|x| \leq a$ is $-a \leq x \leq a$.

EXAMPLE 3 SOLVE EACH OF THE FOLLOWING ABSOLUTE VALUE INEQUALITIES:

A $|2x - 5| < 3$

B $|3 - 5x| \leq 1$

SOLUTION:

$$\begin{aligned} \text{A } |2x - 5| &< 3 \text{ IS EQUIVALENT TO } -3 < 2x - 5 \text{ AND } 2x - 5 < 3 \\ &\Rightarrow -3 < 2x - 5 \text{ AND } 2x - 5 < 3 \\ &\Rightarrow -3 + 5 < 2x - 5 + 5 \text{ AND } 2x - 5 + 5 < 3 + 5 \\ &\Rightarrow 2 < 2x \text{ AND } 2 < 8 \\ &\Rightarrow 1 < x \text{ AND } x < 4 \text{ THAT IS, } 1 < x < 4 \end{aligned}$$

THEREFORE, THE SOLUTION SET IS $x \in (1, 4)$

WE CAN REPRESENT THE SOLUTION SET ON THE NUMBER LINE AS FOLLOWS:



$$\begin{aligned} \text{B } |3 - 5x| &\leq 1 \text{ IS EQUIVALENT TO } -1 \leq 3 - 5x \text{ AND } 3 - 5x \leq 1 \\ &\Rightarrow -1 \leq 3 - 5x \text{ AND } 3 - 5x \leq 1 \\ &\Rightarrow -1 - 3 \leq 3 - 3 - 5x \text{ AND } 3 - 3 - 5x \leq 1 - 3 \\ &\Rightarrow -4 \leq -5x \text{ AND } -5x \leq -2 \\ &\Rightarrow 5x \leq 4 \text{ AND } 2 \leq x \\ &\Rightarrow x \leq \frac{4}{5} \text{ AND } x \geq \frac{2}{5} \text{ THAT IS } \frac{2}{5} \leq x \leq \frac{4}{5} \end{aligned}$$

THEREFORE, THE SOLUTION SET IS $\left[\frac{2}{5}, \frac{4}{5}\right]$ **Note:** IN $|x| < a$, IF $a < 0$ THE INEQUALITY HAS NO SOLUTION.**Theorem 3.3** Solution of $|x| > a$ and $|x| \geq a$ For any real number a , if $a > 0$, then

- I the solution of the inequality $|x| > a$ is $x < -a$ or $x > a$.
- II the solution of the inequality $|x| \geq a$ is $x \leq -a$ or $x \geq a$.

EXAMPLE 4 SOLVE EACH OF THE FOLLOWING INEQUALITIES:

A $|5 + 2x| > 6$

B $\left|\frac{3}{5} - 2x\right| \geq 1$

C $|3 - x| > -2$

SOLUTION: ACCORDING TO THEOREM,

A $|5 + 2x| > 6$ IMPLIES $5 + 2x < -6$ OR $5 + 2x > 6$

$$\Rightarrow 5 - 5 + 2x < -6 - 5 \text{ OR } 5 - 5 + 2x > 6 - 5$$

$$\Rightarrow 2x < -11 \text{ OR } 2x > 1$$

$$\Rightarrow x < \frac{-11}{2} \text{ OR } x > \frac{1}{2}$$

THEREFORE, THE SOLUTION SET IS $\left\{ x \mid x < \frac{-11}{2} \text{ OR } x > \frac{1}{2} \right\}$.

(TRY TO REPRESENT THIS SOLUTION ON THE NUMBER LINE)

B $\left| \frac{3}{5} - 2x \right| \geq 1$ IMPLIES $\frac{3}{5} - 2x \leq -1$ OR $\frac{3}{5} - 2x \geq 1$

HENCE $\frac{3}{5} - 2x \leq -1$ OR $\frac{3}{5} - 2x \geq 1$ GIVES $\frac{3}{5} - \frac{3}{5} - 2x \leq -1 - \frac{3}{5}$ OR $\frac{3}{5} - \frac{3}{5} - 2x \geq 1 - \frac{3}{5}$

$$\Rightarrow -2x \leq \frac{-8}{5} \text{ OR } -2x \geq \frac{2}{5}$$

$$\Rightarrow \frac{8}{5} \leq 2x \text{ OR } -\frac{2}{5} \geq 2x$$

$$\Rightarrow x \geq \frac{4}{5} \text{ OR } x \leq -\frac{1}{5}$$

THEREFORE, THE SOLUTION SET IS $\left\{ x \mid x \leq -\frac{1}{5} \text{ OR } x \geq \frac{4}{5} \right\}$.

C BY DEFINITION $|x| = |x - 3| \geq 0$. SO, $|3 - x| > -2$ IS TRUE FOR ALL REAL NUMBERS x

THEREFORE, THE SOLUTION SET IS

Group Work 3.1



- 1 GIVEN THAT $a < b$, EXPRESS THE FOLLOWING WITHOUT VALUE.

A $|a - b|$

B $|ab - a|$

C $\left| \frac{b}{a} \right|$

- 2 FOR ANY REAL NUMBER a THAT

A $a \leq |a|$

Hint: IF $a \geq 0$, THEN $|a| = a$. SO, $a \leq |a|$.

IF $a < 0$, THEN $|a| > 0$. COMPARE a AND $|a|$

B $-|a| \leq a \leq |a|$

- 3 FOR ANY REAL NUMBERS SHOW THAT**

A $|x + y| \leq |x| + |y|$

Hint: START FROM $|x + y|^2 = (x + y)^2$ AND EXPAND. THEN USE B ABOVE.

B $|x - y| \geq |x| - |y|$

- 4 SOLVE EACH OF THE FOLLOWING**

A $\frac{3x - 1}{2} + x \leq 7 + \frac{1}{2}x$

B $|-2| \geq 8 - |4x + 6|$

C $\left| \frac{1}{4}x - 2 \right| > 1$

D $|2x - 1| < x + 3$

Exercise 3.1

- 1 SIMPLIFY AND WRITE EACH OF THE FOLLOWING USING INTERVALS:**

A $\{ x : x \in \mathbb{R} \text{ AND } x \neq -2 \}$

B $\{ x : -1 \leq x - 2 \leq 2 \}$

C $\{ x : x + 3 > 2 \}$

D $\{ x : 5x - 9 \leq 9 \}$

E $\{ x : 2x + 3 \geq -5x \}$

F $\{ x : 2x - 1 < x < 3 \}$

- 2 SOLVE EACH OF THE FOLLOWING INEQUALITIES:**

A $2x - 5 \geq 3x$

B $3x + 1 < \frac{8x - 3}{2}$

C $\frac{1}{4}t + 2 > 3(5 - t)$

- 3 A NUMBER IS 15 LARGER THAN A POSITIVE NUMBER. SUM IS NOT MORE THAN 85, WHAT ARE THE POSSIBLE VALUES OF SUCH NUMBER y**

- 4 IF $x = -\frac{2}{3}$ AND $y = \frac{1}{5}$, THEN EVALUATE THE FOLLOWING:**

A $|6x| + |5y|$

B $|3x| - |10y|$

C $|3x - 10y|$

D $\left| \frac{3x - 2y}{x + y} \right|$

- 5 SOLVE EACH OF THE FOLLOWING ABSOLUTE VALUE EQUATIONS:**

A $|3x + 6| = 7$

B $|5x - 3| = 9$

C $|x - 6| = -6$

D $|7 - 2x| = 0$

E $|6 - 3x| + 5 = 14$

F $\left| \frac{3}{4}x + \frac{1}{8} \right| = \frac{1}{2}$

- 6 SOLVE EACH OF THE FOLLOWING ABSOLUTE VALUE EQUATIONS AND EXPRESS THEIR SOLUTION SETS IN INTERVALS:**

A $|3 - 5x| \leq 1$

B $|5x| - 2 < 8$

C $\left| \frac{2}{3}x - \frac{1}{9} \right| \geq \frac{1}{3}$

D $|6 - 2x| + 3 > 8$

E $|3x + 5| \leq 0$

F $|x - 1| > -2$

- 7 FOR ANY REAL NUMBERS a AND b, SUCH THAT $a \neq 0$ AND $b \neq 0$, SOLVE EACH OF THE FOLLOWING INEQUALITIES:**

A $|ax + b| < c$

B $|ax + b| \leq c$

C $|ax + b| > c$

D $|ax + b| \geq c$

3.2**SYSTEMS OF LINEAR INEQUALITIES IN TWO VARIABLES**

RECALL THAT A FIRST DEGREE (LINEAR) EQUATION HAS THE FORM

$$ax + by = c$$

WHERE AND BOTH ARE NOT 0.

WHEN TWO OR MORE LINEAR EQUATIONS INVOLVE THE SAME VARIABLES, THEY ARE CALLED **linear equations**. AN ORDERED PAIR THAT SATISFIES ALL EQUATIONS IS CALLED **solution of the system**. FOR INSTANCE

$$\begin{cases} 2x - y = 7 \\ x + 5y = -2 \end{cases}$$

IS A SYSTEM OF TWO LINEAR EQUATIONS. WHAT IS ITS SOLUTION?

ACTIVITY 3.3

- 1** WHAT CAN YOU SAY ABOUT THE SOLUTION IF THEIR GRAPHS DO NOT INTERSECT?
- 2** FIND THE SOLUTIONS OF EACH OF THE FOLLOWING SYSTEMS GRAPHICALLY:

A	B	C
$\begin{cases} x - y = -2 \\ x + y = 6 \end{cases}$	$\begin{cases} x + y = 2 \\ 2x + 2y = 8 \end{cases}$	$\begin{cases} x + 2y = 4 \\ 2x + 4y = 8 \end{cases}$
- 3** FIND THREE DIFFERENT ORDERED PAIRS WHERE R LONG TO R
 $R = \{(x, y) : y \leq x + 1\}$.
- 4** DRAW THE GRAPH OF R GIVEN IN **ABOVE** **N3**
- 5** DRAW THE GRAPHS OF EACH OF THE FOLLOWING RELATIONS:

A $R = \{(x, y) : x \geq y \text{ AND } y \geq x - 1\}$	B $R = \{(x, y) : y \leq x + 1 \text{ AND } y \geq 1 - x\}$
--	--
- 6** SOLVE EACH OF THE FOLLOWING SYSTEMS **IN YOUR OWN WORDS** ANSWER IN INTERVAL NOTATION:

$$\begin{cases} x \geq -1 \\ x \leq 3 \\ y \geq 0 \end{cases}$$

$$\begin{cases} x - y < 3 \\ x \geq 2 \end{cases}$$

A SYSTEM OF TWO LINEAR EQUATIONS IN TWO VARIABLES OFTEN INVOLVES A PAIR OF IN THE PLANE. THE SOLUTION SET OF SUCH A SYSTEM OF EQUATIONS CAN BE DETERMINED BY GRAPH AND IS THE SET OF ALL ORDERED PAIRS OF COORDINATES OF POINTS WHICH LIE

EXAMPLE 1 FIND THE SOLUTION SET OF THE SYSTEM OF EQUATIONS

SOLUTION: FIRST DRAW THE GRAPHS OF $x + 2y = 0$ AND $x - y = 3$ AS SHOWN BELOW.

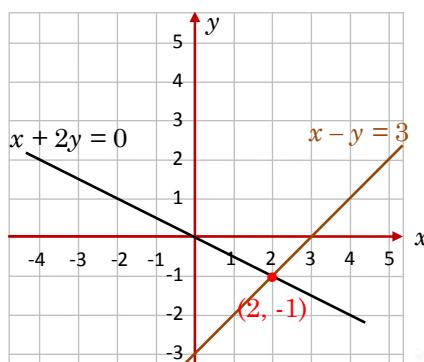


Figure 3.7

THE TWO LINES INTERSECT AT $(2, -1)$.

THEFORE, THE SOLUTION SET OF THE SYSTEM IS $\{(2, -1)\}$.

IN A SYSTEM OF EQUATIONS, IF “=” IS REPLACED BY “ \geq ” OR “ \leq ”, THE SYSTEM BECOMES A SYSTEM OF LINEAR INEQUALITIES.

EXAMPLE 2 FIND THE SOLUTION OF THE FOLLOWING SYSTEM GRAPHICALLY:

$$\begin{cases} y \geq -3x + 2 \\ y < x - 2 \end{cases}$$

SOLUTION: FIRST DRAW THE GRAPH OF ONE OF THE BOUNDARY LINES, y CORRESPONDING TO THE FIRST INEQUALITY.

THE GRAPH OF $y \geq -3x + 2$ CONSISTS OF POINTS ON OR ABOVE THE LINE y SHOWN IN FIGURE 3.8A. THIS IS OBTAINED BY TAKING A TEST POINT SAY $(2, 0)$, CHECKING THAT $0 \geq -3(2) + 2 = -4$ IS TRUE. NEXT, DRAW THE GRAPH OF THE OTHER BOUNDARY LINE, $y < x - 2$, CORRESPONDING TO THE SECOND INEQUALITY. THE GRAPH OF $y < x - 2$ CONSISTS OF POINTS BELOW THE LINE. POINTS ON THE LINE ARE EXCLUDED AS SHOWN IN FIGURE 3.8B.

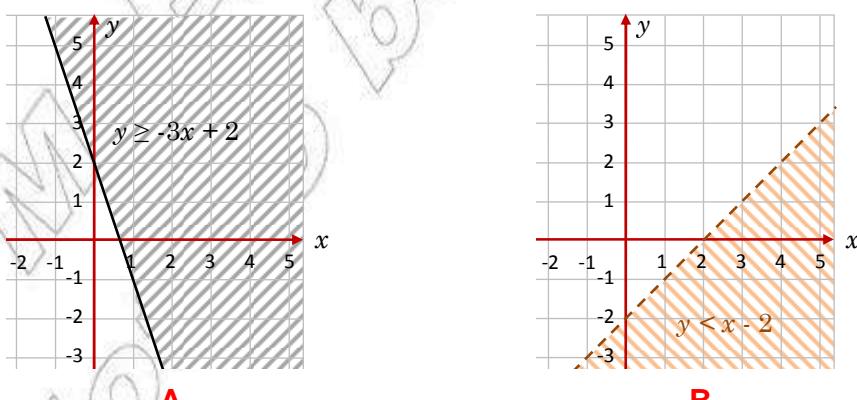


Figure 3.8

THESE GRAPHS HAVE BEEN DRAWN USING DIFFERENT COORDINATE SYSTEMS IN THEM SEPARATELY. NOW, DRAW THEM USING THE SAME COORDINATE SYSTEM. THE COORDINATE SYSTEM MARKED WITH BOTH TYPES OF SHADING IS THE SOLUTION SET AS SHOWN [FIGURE 3.9B](#)

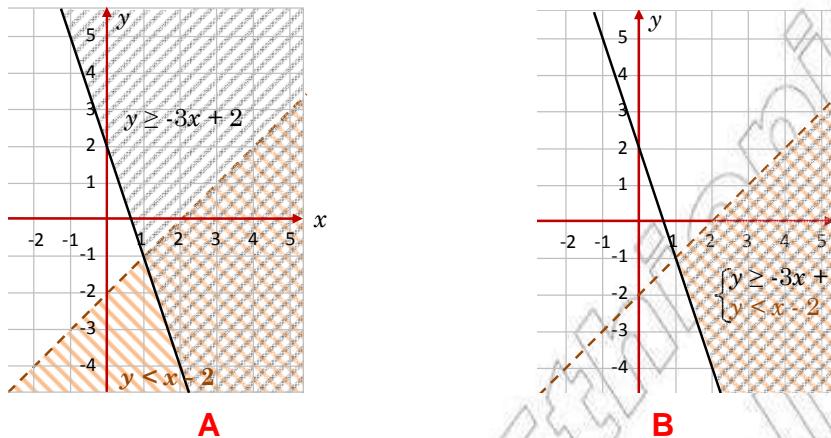


Figure 3.9

THE SOLUTION SET OF $\begin{cases} y \geq -3x + 2 \\ y < x - 2 \end{cases}$ IS SHOWN BY THE CROSS-SHADED REGION IN THE DIAGRAM.

SOLVING $\begin{cases} y = -3x + 2 \\ y = x - 2 \end{cases}$, WE GET $-3x + 2 = x - 2$

THEREFORE, $x = 1$ AND $y = -1$

SO, $x > 1$, $-3x + 2 \leq y < x - 2$

HENCE, THE SOLUTION SET OF THE SYSTEM IS EXPRESSED AS

$$\{(x, y) : -3x + 2 \leq y < x - 2 \text{ AND } x < \infty\}$$

EXAMPLE 3 FIND THE SOLUTION OF EACH OF THE FOLLOWING SYSTEMS OF INEQUALITIES, GRAPHICALLY:

A $\begin{cases} x + y < 3 \\ x \geq 0 \\ y \geq 0 \end{cases}$

B $\begin{cases} y + x > 0 \\ y - x \leq 1 \\ x \leq 2 \end{cases}$

SOLUTION:

A HERE, OUR OBJECTIVE IS TO DETERMINE THE SETS OF COORDINATES THAT SATISFY ALL THREE OF THESE CONDITIONS. TO DO SO, LET US DRAW EACH BORDER AS SHOWN BELOW. THE POINTS SATISFYING THE CONDITIONS LYING TO THE RIGHT OF THIS ARE SHOWN AS SHOWN IN [FIGURE 3.10A](#).

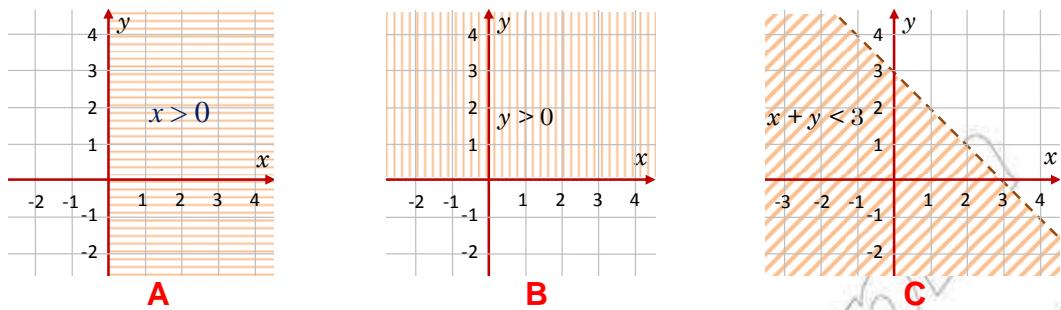


Figure 3.10

THE POINTS (y) WITH $y > 0$ ARE THE POINTS THAT LIE ABOVE THE SHOWN IN FIGURE 3.10B. THE POINTS (x, y) WITH $x + y < 3$ IS THE SET OF POINTS LYING BELOW THE LINE $x + y = 3$. POINTS ON THE LINE ARE EXCLUDED.

NOW, DRAW THE GRAPH OF THE THREE INEQUALITIES $y < 3$, USING THE SAME COORDINATE SYSTEM, TAKING ONLY THE INTERSECTION OF THE THREE REGIONS.

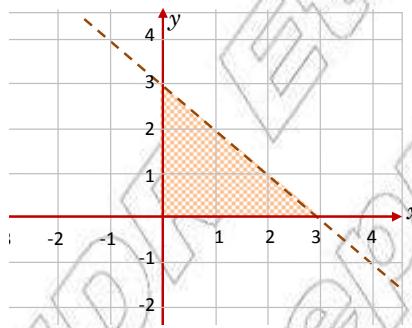


Figure 3.11

THE POINTS SATISFYING THE SYSTEM OF INEQUALITIES ARE SATISFY ALL THE THREE INEQUALITIES. THE CORRESPONDING REGION IS THE TRIANGULAR REGION FIGURE 3.11 THAT IS, THE SET OF SUCH THAT $0 < x < 3$ AND $0 < y < 3 - x$

B FIRST, DRAW THE GRAPH OF THE BOUNDARY LINE ($y = x$) FOR THE FIRST INEQUALITY. THE GRAPH OF CONSISTS OF POINTS ABOVE THE LINE.

POINTS ON THE LINE ARE EXCLUDED AS SHOWN IN FIGURE 3.12A

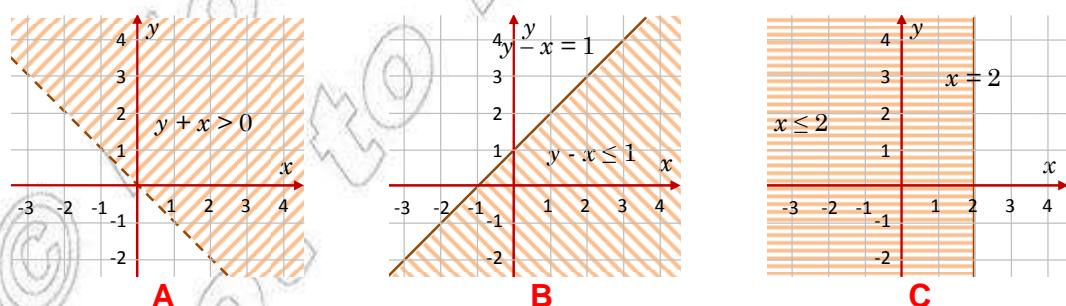


Figure 3.12

NEXT, DRAW THE GRAPH OF THE BOUNDARY FOR THE SECOND INEQUALITY. THE GRAPH OF $x \leq 1$ CONSISTS OF POINTS ON AND BELOW THE LINE SHOWN IN FIGURE 3.12B.

FINALLY, DRAW THE GRAPH OF THE BOUNDARY FOR THE THIRD INEQUALITY. THE POINTS (x, y) SATISFYING THE CONDITION ARE THOSE LYING ON AND TO THE LEFT OF THE LINE $x = 2$ AS SHOWN IN FIGURE 3.12C.

NOW, DRAW THE GRAPH OF THE THREE INEQUALITIES USING THE SAME COORDINATE SYSTEM AS SHOWN IN FIGURE 3.13A.

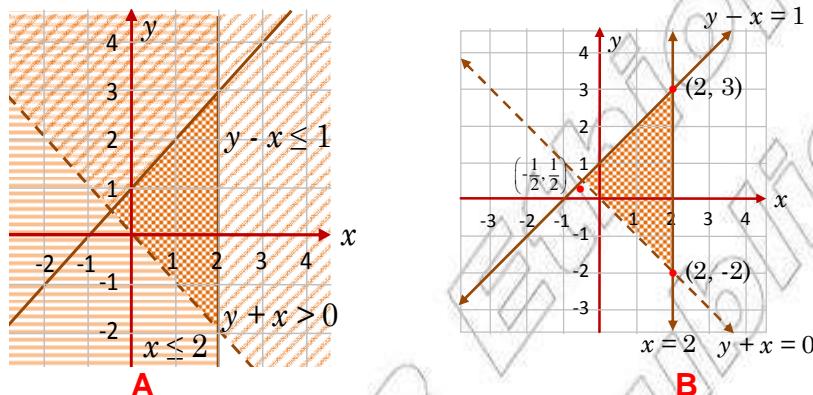


Figure 3.13

BECAUSE THERE ARE INFINITE SOLUTIONS TO THE SYSTEM, THEY CANNOT BE LISTED. BUT THE GRAPH IS EASY TO DESCRIBE. THE SOLUTION IS THE TRIANGULAR REGION WITH VERTICES $(-\frac{1}{2}, \frac{1}{2})$, $(2, 3)$ AND $(2, -2)$, EXCEPT THOSE POINTS ON THE LINES SHOWN IN FIGURE 3.13B.

ACTIVITY 3.4

- 1 BY OBSERVING THE GRAPH OF THE INEQUALITIES GIVEN IN FIGURE 3.13A, NAME AT LEAST 10 ORDERED PAIRS THAT SATISFY THE SYSTEM.
- 2 IF $R = \{(x, y) : y + x > 0, y - x \leq 1 \text{ AND } x \leq 2\}$, WHAT IS THE DOMAIN AND RANGE OF R?

WE SHALL NOW CONSIDER AN EXAMPLE INVOLVING AN APPLICATION OF A SYSTEM OF LINEAR INEQUALITIES.

EXAMPLE 4 A FURNITURE COMPANY MAKES TABLES AND CHAIRS. IT PRODUCES A TABLE ON MACHINE A IN 2 HRS AND A CHAIR ON MACHINE B IN 4 HRS. IT PRODUCES A CHAIR ON MACHINE A IN 3 HRS AND A TABLE ON MACHINE B IN 2 HRS. MACHINE A CAN OPERATE AT MOST 12 HRS A DAY AND MACHINE B CAN OPERATE AT MOST 10 HRS A DAY. IF THE COMPANY MAKES A PROFIT OF BIRR 12 ON A TABLE AND BIRR 10 ON A CHAIR, HOW MANY OF EACH SHOULD BE PRODUCED TO MAXIMIZE ITS PROFIT?

SOLUTION: LET x BE THE NUMBER OF TABLES TO BE PRODUCED AND y BE THE NUMBER OF CHAIRS TO BE PRODUCED.

THEN, IF A TABLE IS PRODUCED IN 2 HRS ON MACHINE A, $3y \leq 12$ SIMILARLY, CHAIRS REQUIRE $4x$ HRS ON MACHINE B. ON MACHINE A, 4 TABLES REQUIRE 16 HRS AND CHAIRS REQUIRE $2y$ HRS SINCE MACHINES A AND B CAN OPERATE AT MOST 12 HRS AND 16 HRS, RESPECTIVELY. WE HAVE THE FOLLOWING SYSTEM OF LINEAR INEQUALITIES:

$$\text{FROM MACHINE A: } 3y \leq 12$$

$$\text{FROM MACHINE B: } 4x \leq 16$$

ALSO, $x \geq 0$ AND $y \geq 0$ SINCE x AND y ARE NUMBERS OF TABLES AND CHAIRS.

HENCE, YOU OBTAIN A SYSTEM OF LINEAR INEQUALITIES GIVEN AS FOLLOWS:

$$\begin{cases} 2x + 3y \leq 12 \\ 4x + 2y \leq 16 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

SINCE THE INEQUALITIES INVOLVED IN THE SYSTEM ARE ALL LINEAR, THE BOUNDARY LINES OF THE GRAPH OF THE SYSTEM ARE STRAIGHT LINES. THE REGION CONTAINING THE SOLUTION OF THE SYSTEM IS THE QUADRILATERAL SHOWN BELOW.

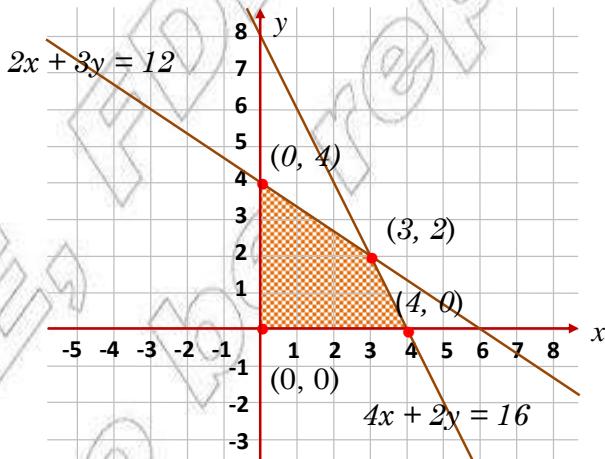


Figure 3.14

THE PROFIT MADE IS BIRR 12 ON A TABLE, ~~SO BIRR 12~~ AND BIRR 10 ON A CHAIR, SO BIRR $12x + 10y$ CHAIRS. THE PROFIT FUNCTION P IS GIVEN BY $P = 12x + 10y$.

THE VALUES ~~AND~~ WHICH MAXIMIZE OR MINIMIZE THE PROFIT FUNCTION ON SUCH A SYSTEM ARE USUALLY FOUND AT VERTICES OF THE SOLUTION REGION.

HENCE, FROM THE GRAPH, YOU HAVE THE COORDINATES OF EACH VERTEX AS **FIGURE 3.14**.

THE PROFIT: $P = 12x + 10y$ AT EACH VERTEX IS FOUND TO BE:

$$\text{AT } (0, 0), P = 12(0) + 10(0) = 0$$

$$\text{AT } (0, 4), P = 12(0) + 10(4) = 40$$

$$\text{AT } (3, 2), P = 12(3) + 10(2) = 56$$

$$\text{AT } (4, 0), P = 12(4) + 10(0) = 48$$

THEREFORE, THE PROFIT IS MAXIMUM AT THE VERTEX $(3, 2)$, SO THE COMPANY PRODUCE 3 TABLES AND 2 CHAIRS PER DAY TO GET THE MAXIMUM PROFIT OF BIRR

Group Work 3.2



- 1** FIND THE SOLUTIONS OF EACH OF THE FOLLOWING INEQUALITIES GRAPHICALLY:

A
$$\begin{cases} y + x \geq 0 \\ y - x \geq 0 \\ y \leq 3 \end{cases}$$

B
$$\begin{cases} x + y < 1 \\ 2x - y > -1 \\ y - 3x \geq -3 \end{cases}$$

- 2** LET $R = \{(x, y) : y \geq x, y \geq -x \text{ AND } y \leq 3\}$ AND

$$r = \{(x, y) : x + y < 1, 2x - y > -1 \text{ AND } y - 3x \geq -3\}$$

USING QUESTION ABOVE, FIND THE DOMAIN AND RANGE OF THE RELATIONS R

Exercise 3.2

- 1** DRAW THE GRAPHS OF EACH OF THE FOLLOWING RELATIONS:

A $R = \{(x, y) : x - y \geq 1 \text{ AND } 2x + y < 3\}$

B $R = \{(x, y) : x \leq y - 1 \text{ AND } 2x > 2\}$

C $R = \{(x, y) : x > y ; x > 0 \text{ AND } y < 1\}$

D $R = \{(x, y) : x + y \geq 0 ; y \geq 0 \text{ AND } y < 1\}$

- 2** SOLVE EACH OF THE FOLLOWING SYSTEM OF INEQUALITIES GRAPHICALLY:

A
$$\begin{cases} y \leq 2x + 3 \\ y - x \geq 0 \\ y > 0 \end{cases}$$

B
$$\begin{cases} 3x + y < 5 \\ x > 0 \\ x + y < 6 \end{cases}$$

C
$$\begin{cases} y \leq 1 - x \\ y > x + 2 \\ y > 0 \end{cases}$$

D
$$\begin{cases} x \geq -1 \\ y \leq 2 \\ y \geq x - 1 \end{cases}$$

E
$$\begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{cases}$$

F
$$\begin{cases} x > 0 \\ y > 0 \\ x + y < 4 \end{cases}$$

3 DESCRIBE EACH OF THE FOLLOWING SHADINGS PERTAINING TO LINEAR INEQUALITIES:

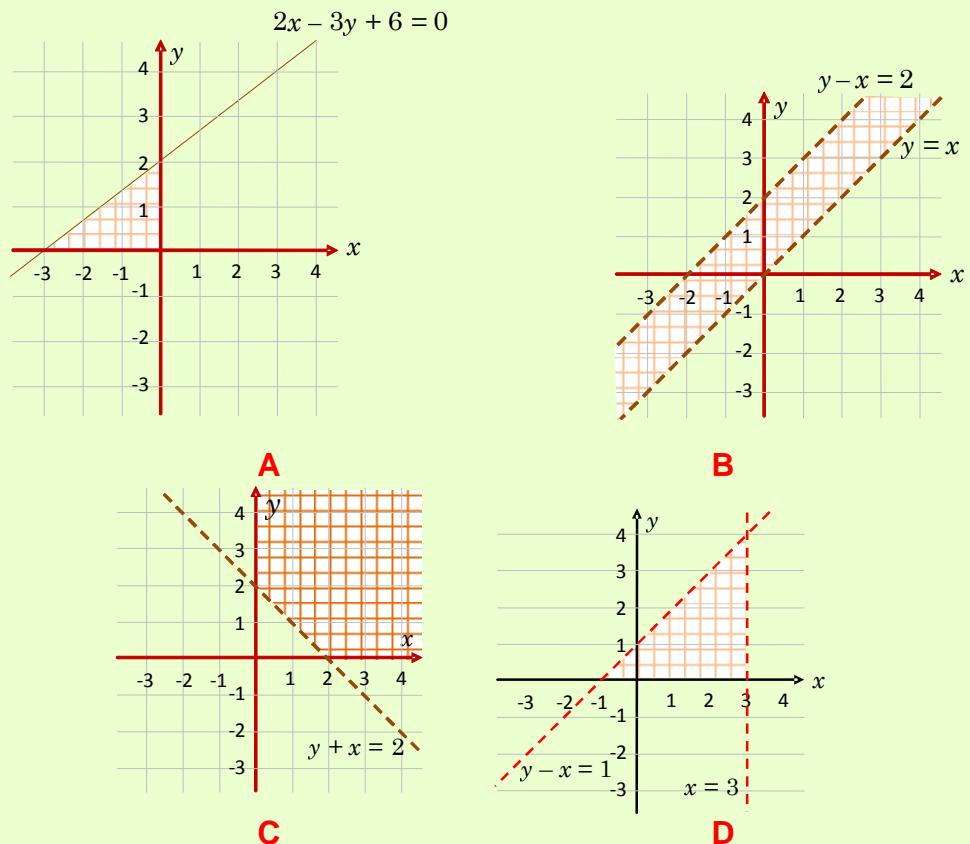


Figure 3.15

- 4 GIVE A PAIR OF LINEAR INEQUALITIES THAT DESCRIBES POINTS IN THE FIRST QUADRANT.**
- 5 GIVE A SYSTEM OF LINEAR INEQUALITIES WHOSE SOLUTION POINTS INSIDE A RECTANGLE.**
- 6 SUPPOSE THE SUM OF TWO POSITIVE AND LESS THAN 10 AND GREATER THAN 5. SHOW ALL POSSIBLE VALUES GRAPHICALLY.**
- 7 SUPPOSE A SHOE FACTORY PRODUCES BOTH LOW-GRADE AND HIGH-GRADE SHOES. THE FACTORY PRODUCES AT LEAST TWICE AS MANY LOW-GRADE AS HIGH-GRADE SHOES. THE MAXIMUM POSSIBLE PRODUCTION IS 500 PAIRS OF SHOES. A DEALER CALLS FOR DELIVERY AT LEAST 100 HIGH-GRADE PAIRS OF SHOES PER DAY. SUPPOSE THE OPERATION HAS A PROFIT OF BIRR 2.00 PER A PAIR OF SHOES ON HIGH-GRADE SHOES AND BIRR 1.00 PER PAIR OF SHOES ON LOW-GRADE SHOES. HOW MANY PAIRS OF SHOES OF EACH TYPE SHOULD BE PRODUCED FOR MAXIMUM PROFIT?**

Hint: LET x DENOTE THE NUMBER OF HIGH-GRADE SHOES.
LET y DENOTE THE NUMBER OF LOW-GRADE SHOES.

3.3**QUADRATIC INEQUALITIES**

IN **UNIT 2 OF GRADE 9** MATHEMATICS, YOU HAVE LEARNT HOW TO SOLVE QUADRATIC EQUATIONS. (RECALL THAT EQUATIONS OF THE FORM $ax^2 + bx + c = 0$ ARE QUADRATIC EQUATIONS.)

Can similar methods be used to solve quadratic inequalities?

Definition 3.2

An **inequality** that can be reduced to any one of the following forms:

$$ax^2 + bx + c \leq 0 \text{ or } ax^2 + bx + c < 0,$$

$$ax^2 + bx + c \geq 0 \text{ or } ax^2 + bx + c > 0,$$

where a , b and c are constants and $a \neq 0$, is called a **quadratic inequality**.

FOR EXAMPLE $x^2 - 3x + 2 < 0$, $x^2 + 1 \geq 0$, $x^2 + x \leq 0$ AND $x^2 - 4 > 0$ ARE ALL QUADRATIC INEQUALITIES.

THE FOLLOWING ACTIVITY WILL HELP YOU TO RECALL WHAT YOU HAVE LEARNED ABOUT QUADRATIC EQUATIONS IN **GRADE 9**.

ACTIVITY 3.5

- 1 WHICH OF THE FOLLOWING ARE QUADRATIC EQUATIONS?**

A $x - 2 = x^2 + 2x$	B $x^2 - 2x = x^2 + 3x + 6$
C $2(x - 4) - (x - 2) = (x + 2)(x - 4)$	D $x^3 - 3 = 1 + 4x + x^2$
E $(x - 1)(x + 2) \geq 0$	F $x(x - 1)(x + 1) = 0$
- 2 WHICH OF THE FOLLOWING ARE QUADRATIC INEQUALITIES?**

A $2x^2 \leq 5x + x^2 - 3$	B $2x^2 > 2x + x^2 + 8$
C $x(1 - x) \leq (x + 2)(1 - x)$	D $3x^2 + 5x + 6 > 0$
E $5 - 2(x^2 + x) < 6x - 2x^2$	F $(x - 2)(x + 1) \geq 2 - 2x$
G $-1 > (x^2 + 1)(x + 2)$	
- 3 IF THE PRODUCT OF TWO REAL NUMBERS IS ZERO, YOUNG STUDENTS CAN SAY ABOUT THE TWO NUMBERS?**
- 4 FACTORIZE EACH OF THE FOLLOWING IF POSSIBLE:**

A $x^2 + 6x$	B $35x - 28x^2$	C $\frac{1}{16} - 25x^2$	D $4x^2 + 7x + 3$
E $x^2 - x + 3$	F $x^2 + 2x - 3$	G $3x^2 - 11x - 4$	H $x^2 + 4x + 4$
- 5 GIVEN A QUADRATIC EQUATION $ax^2 + bx + c = 0$,**
 - A** WHAT IS ITS DISCRIMINANT?
 - B** STATE WHAT MUST BE TRUE ABOUT THE DISCRIMINANT IF THE EQUATION HAS ONE REAL ROOT, TWO DISTINCT REAL ROOTS, AND NO REAL ROOT.

3.3.1 Solving Quadratic Inequalities Using Product Properties

SUPPOSE YOU WANT TO SOLVE THE QUADRATIC INEQUALITY

$$(x - 2)(x + 3) > 0.$$

CHECK THAT 3 MAKES THE STATEMENT TRUE ~~MAKES~~ IT FALSE. HOW DO YOU FIND THE SOLUTION SET OF THE GIVEN INEQUALITY? OBSERVE THAT THE LEFT HAND SIDE OF THE PRODUCT OF TWO REAL NUMBERS IS POSITIVE, IF AND ONLY IF EITHER BOTH ARE POSITIVE OR BOTH ARE NEGATIVE. THIS FACT CAN BE USED TO SOLVE THE GIVEN INEQUALITY.

Product properties:

- | |
|---|
| 1 $m \cdot n > 0$, if and only if
I $m > 0$ and $n > 0$ or II $m < 0$ and $n < 0$. |
| 2 $m \cdot n < 0$, if and only if
I $m > 0$ and $n < 0$ or II $m < 0$ and $n > 0$. |

EXAMPLE 1 SOLVE EACH OF THE FOLLOWING INEQUALITIES:

- | | |
|-------------------------------|-------------------------------|
| A $(x + 1)(x - 3) > 0$ | B $3x^2 - 2x \geq 0$ |
| C $-2x^2 + 9x + 5 < 0$ | D $x^2 - x - 2 \leq 0$ |

SOLUTION:

- A** BY **PRODUCT PROPERTY**, $(x + 1)(x - 3)$ IS POSITIVE IF EITHER BOTH THE FACTORS ARE POSITIVE OR BOTH ARE NEGATIVE.

NOW, CONSIDER CASE BY CASE AS FOLLOWS:

Case i WHEN BOTH THE FACTORS ARE POSITIVE

$$x + 1 > 0 \text{ AND } x - 3 > 0$$

$$x > -1 \text{ AND } x > 3$$

THE INTERSECTION OF $x > -1$ AND $x > 3$ IS $x > 3$. THIS CAN BE ILLUSTRATED ON THE NUMBER LINE AS SHOWN IN FIGURE 3.16 BELOW.

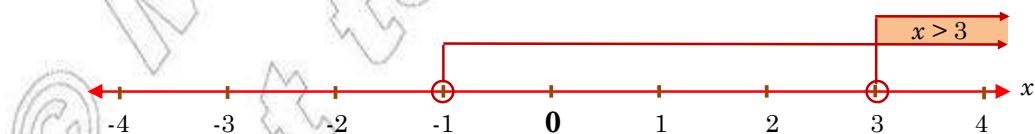


Figure 3.16

THE SOLUTION SET FOR THIS FIRST CASE IS $(3, \infty)$.

Case ii WHEN BOTH THE FACTORS ARE NEGATIVE

$$x + 1 < 0 \text{ AND } x - 3 < 0$$

$$x < -1 \text{ AND } x < 3$$

THE INTERSECTION OF AND $x < 3$ IS $x < -1$.

THIS CAN BE ILLUSTRATED ON THE NUMBER LINE AS SHOWN BELOW IN FIGURE 3.17

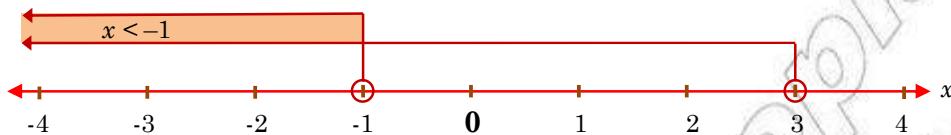


Figure 3.17

THE SOLUTION SET FOR THIS SECOND CASE IS $S_2 = (-\infty, -1)$.

THEREFORE, THE SOLUTION SET OF $(x^2 - 2x) \geq 0$ IS:

$$S_1 \cup S_2 = \{x : x < -1 \text{ OR } x \geq 3\} = (-\infty, -1) \cup (3, \infty)$$

B FIRST, FACTORIZE AS $x(3x - 2) \geq 0$

SO, $3x^2 - 2x \geq 0$ MEANS $(3x - 2) \geq 0$ EQUIVALENTLY.

I $x \geq 0$ AND $3x - 2 \geq 0$ OR

II $x \leq 0$ AND $3x - 2 \leq 0$

Case i WHEN $x \geq 0$ AND $3x - 2 \geq 0$

$$x \geq 0 \text{ AND } x \geq \frac{2}{3}$$

THE INTERSECTION OF AND $x \geq \frac{2}{3}$ IS $x \geq \frac{2}{3}$. GRAPHICALLY,

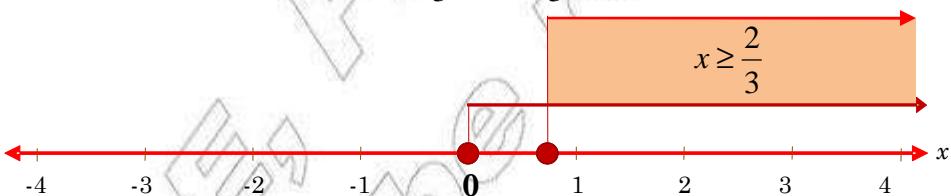


Figure 3.18

$$\text{SO, } S_1 = \{x : x \geq \frac{2}{3}\} = [\frac{2}{3}, \infty)$$

Case ii WHEN $x \leq 0$ AND $3x - 2 \leq 0$ THAT IS $x \leq 0$ AND $x \leq \frac{2}{3}$

THE INTERSECTION OF AND $x \leq \frac{2}{3}$ IS $x \leq 0$. GRAPHICALLY,

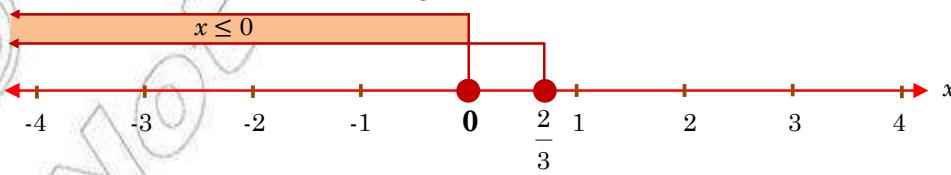


Figure 3.19

SO, $S_2 = \{x: x \leq 0\} = (-\infty, 0]$

THEREFORE, THE SOLUTION SET FOR $3x$

$$S_1 \cup S_2 = \{x: x \leq 0 \text{ OR } x > \frac{2}{3}\} = (-\infty, 0] \cup [\frac{2}{3}, \infty)$$

C $-2x^2 + 9x + 5 = (-2x - 1)(x - 5) < 0$

BY PRODUCT PROPERTY, $(2x - 1)(x - 5)$ IS NEGATIVE IF ONE OF THE FACTORS IS NEGATIVE AND THE OTHER IS POSITIVE.

AS BEFORE, CONSIDER CASE BY CASE AS FOLLOWS:

Case i WHEN $-2x - 1 > 0$ AND $x - 5 < 0$

$$x < -\frac{1}{2} \text{ AND } x < 5$$

THE INTERSECTION OF $x < -\frac{1}{2}$ AND $x < 5$ IS $x < -\frac{1}{2}$. GRAPHICALLY,

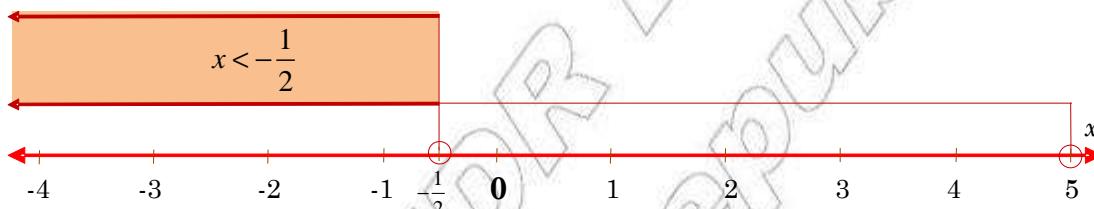


Figure 3.20

$$\text{SO, } S_1 = \{x: x < -\frac{1}{2}\} = (-\infty, -\frac{1}{2})$$

Case ii WHEN $-2x - 1 < 0$ AND $x - 5 > 0$

$$x > -\frac{1}{2} \text{ AND } x > 5$$

THE INTERSECTION OF $x > -\frac{1}{2}$ AND $x > 5$ IS $x > 5$. GRAPHICALLY,

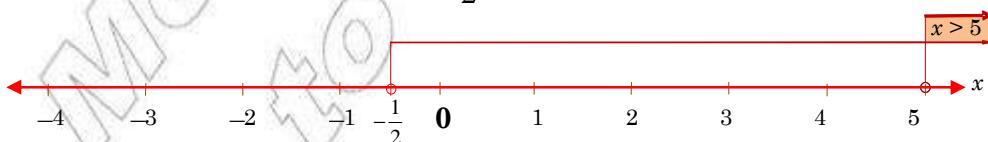


Figure 3.21

$$\text{SO, } S_2 = \{x: x > 5\} = (5, \infty)$$

THEREFORE, THE SOLUTION SET FOR $3x$ IS

$$S_1 \cup S_2 = \{x: x < -\frac{1}{2} \text{ OR } x > 5\} = (-\infty, -\frac{1}{2}) \cup (5, \infty)$$

D $x^2 - x - 2 = (x + 1)(x - 2)$

SO, $x^2 - x - 2 \leq 0$ MEANS $(x + 1)(x - 2) \leq 0$

BY PRODUCT PROPERTY, $(x + 1)(x - 2)$ IS NEGATIVE IF ONE OF THE FACTORS IS NEGATIVE AND THE OTHER IS POSITIVE. TO SOLVE $(x + 1)(x - 2) \leq 0$, CONSIDER CASE BY CASE AS FOLLOWS:

Case i $x + 1 \geq 0$ AND $x - 2 \leq 0$

$$x \geq -1 \text{ AND } x \leq 2$$

THE INTERSECTION OF $x \geq -1$ AND $x \leq 2$ IS $-1 \leq x \leq 2$. GRAPHICALLY,

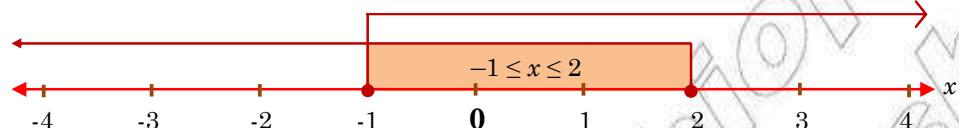


Figure 3.22

$$\text{SO, } S_1 = \{x: -1 \leq x \leq 2\} = [-1, 2]$$

Case ii $x + 1 \leq 0$ AND $x - 2 \geq 0$

$$x \leq -1 \text{ AND } x \geq 2$$

THERE IS NO INTERSECTION OF $x \leq -1$ AND $x \geq 2$. GRAPHICALLY,



Figure 3.23

$$\text{SO, } S_2 = \emptyset$$

THEREFORE, THE SOLUTION SET FOR S1 AND S2 IS

$$S_1 \cup S_2 = \{x: -1 \leq x \leq 2\} \cup \emptyset = \{x: -1 \leq x \leq 2\} = [-1, 2]$$

Exercise 3.3

1 SOLVE EACH OF THE FOLLOWING INEQUALITIES BY USING THE PRODUCT PROPERTY.

A $x(x + 5) > 0$

B $(x - 1)^2 \leq 0$

C $(4+x)(x-4) > 0$

D $(5x-3)(x+7) < 0$

E $(1+x)(3-2x) \geq 0$

F $(5-x)\left(1-\frac{1}{3}x\right) \leq 0$

2 FACTORIZE AND SOLVE EACH OF THE FOLLOWING INEQUALITIES BY USING THE QUOTIENT PROPERTY:

A $x^2 + 5x + 4 < 0$

B $x^2 - 4 > 0$

C $x^2 + 5x + 6 \geq 0$

D $x^2 - 2x + 1 \leq 0$

E $3x^2 + 4x + 1 \geq 0$

F $2x^2 - 7x + 3 < 0$

G $25x^2 - \frac{1}{16} < 0$

H $x^2 + 4x + 4 > 0$.

- 3 A** FIND THE SOLUTION SET OF THE INEQUALITY $x^2 < 25$
- B** WHY IS $\{x | x < 5\}$ NOT THE SOLUTION SET OF $x^2 < 25$?
- 4** IF $x < y$, DOES IT FOLLOW THAT $x^2 < y^2$?
- 5** IF A BALL IS THROWN UPWARD FROM GROUND LEVEL WITH AN INITIAL SPEED OF 24 M/S, ITS HEIGHT H IN METRES AFTER T SECONDS IS GIVEN BY $H = -5T^2 + 24T$. WHEN WILL THE BALL BE AT A HEIGHT OF MORE THAN 8 METRES?

3.3.2 Solving Quadratic Inequalities Using the Sign Chart Method

SUPPOSE YOU NEED TO SOLVE THE QUADRATIC INEQUALITY

$$x^2 + 3x - 4 < 0.$$

CONSIDER HOW THE SIGN OF $x^2 + 3x - 4$ CHANGES AS YOU VARY THE VALUES OF THE UNKNOWN x . AS x IS MOVED ALONG THE NUMBER LINE, THE QUANTITY $x^2 + 3x - 4$ IS SOMETIMES POSITIVE, SOMETIMES ZERO, AND SOMETIMES NEGATIVE. TO SOLVE THE INEQUALITY, YOU MUST LOCATE THE VALUES OF x FOR WHICH $x^2 + 3x - 4$ IS NEGATIVE. INTERVALS WHERE $x^2 + 3x - 4$ IS POSITIVE ARE SEPARATED FROM INTERVALS WHERE IT IS NEGATIVE BY THE VALUES WHERE $x^2 + 3x - 4 = 0$. TO LOCATE THESE VALUES, SOLVE THE EQUATION $x^2 + 3x - 4 = 0$.

FACTORIZE $x^2 + 3x - 4$ AND FIND THE TWO ROOTS (-4 AND 1). DIVIDE THE NUMBER LINE INTO THREE OPEN INTERVALS. THE EXPRESSION $x^2 + 3x - 4$ WILL HAVE THE SAME SIGN IN EACH OF THESE INTERVALS, $(-\infty, -4)$, $(-4, 1)$ AND $(1, \infty)$.

THE “SIGN CHART” METHOD ALLOWS YOU TO FIND THE SIGN OF $x^2 + 3x - 4$ IN EACH OF THESE INTERVALS.

Step 1 FACTORIZE $x^2 + 3x - 4 = (x + 4)(x - 1)$

Step 2 DRAW A SIGN CHART, NOTING THE SIGN OF EACH TERM IN THE EXPRESSION AS SHOWN BELOW.

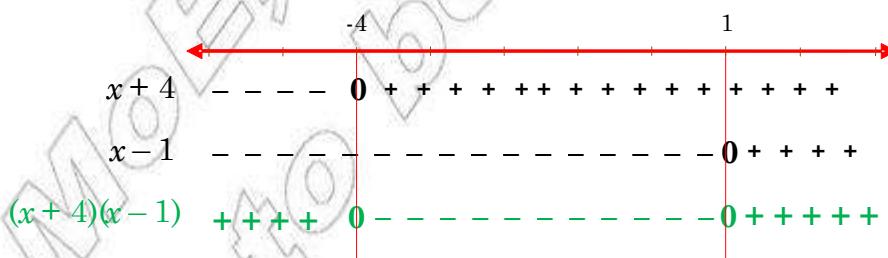


Figure 3.24

Step 3 READ THE SOLUTION FROM THE LAST LINE OF THE SIGN CHART

$$x^2 + 3x - 4 < 0 \text{ FOR } x \in (-4, 1)$$

THEREFORE, THE SOLUTION SET IS THE INTERVAL $(-4, 1)$

EXAMPLE 2 SOLVE EACH OF THE FOLLOWING INEQUALITIES USING THE SIGN CHART METHOD:

A $6 + x - x^2 \leq 0$

B $2x^2 + 3x - 2 \geq 0$.

SOLUTION:**A** FACTORIZE $6 + x - x^2$ SO THAT $6 + x - x^2 = (x + 2)(3 - x) \leq 0$.WE MAY IDENTIFY THE SIGNS OF $x + 2$ AND $3 - x$ AS FOLLOWS. $x + 2 < 0$ FOR EACH $x < -2$, $x + 2 = 0$ AT $x = -2$ AND $x + 2 > 0$ FOR EACH $x > -2$.SIMILARLY, $3 - x < 0$ FOR EACH $x > 3$, $3 - x = 0$ AT $x = 3$ AND $3 - x > 0$ FOR EACH $x < 3$.

THEREFORE, THE ABOVE RESULTS ARE SHOWN IN THE SIGN CHART GIVEN BELOW IN

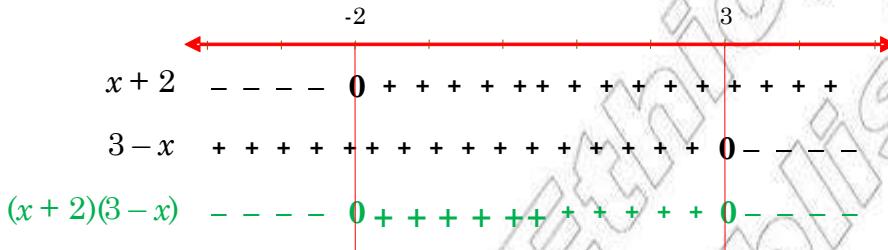


Figure 3.25

FROM THE SIGN CHART, YOU CAN IMMEDIATELY READ THE FOLLOWING:

- I** THE SOLUTION SET OF $(x+2)(3-x) < 0$ IS $\{x: x < -2 \text{ OR } x > 3\} = (-\infty, -2) \cup (3, \infty)$.
- II** THE SOLUTION SET OF $(x+2)(3-x) \geq 0$ IS $\{x: -2 \leq x \leq 3\} = [-2, 3]$.
- III** THE SOLUTION SET OF $(x+2) = 0$ IS $\{-2\}$.
- IV** THE SOLUTION SET OF $(3-x) = 0$ IS $\{3\}$.

THEREFORE, THE SOLUTION SET IS $(-\infty, -2] \cup [3, \infty)$.

B $2x^2 + 3x - 2 = (2x - 1)(x + 2) \geq 0$.

 $2x - 1 < 0$ FOR EACH $x < \frac{1}{2}$, $2x - 1 = 0$ AT $x = \frac{1}{2}$, AND $2x - 1 > 0$ FOR EACH $x > \frac{1}{2}$.SIMILARLY, $x + 2 < 0$ FOR EACH $x < -2$, $x + 2 = 0$ AT $x = -2$ AND $x + 2 > 0$ FOR EACH $x > -2$.

THE ABOVE RESULTS ARE SHOWN IN THE SIGN CHART GIVEN BELOW:

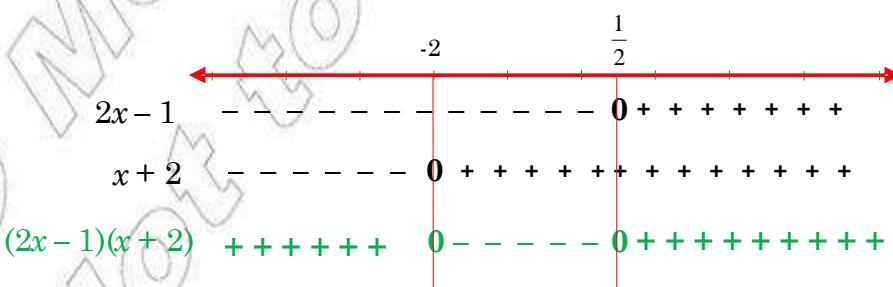


Figure 3.26

FROM THE SIGN CHART, YOU CAN CONCLUDE THAT

$(2x - 1)(x + 2) \geq 0$ FOR EACH $x \in (-\infty, -2] \cup \left[\frac{1}{2}, \infty\right)$ AND

$(2x - 1)(x + 2) < 0$ FOR EACH $x \in \left(-2, \frac{1}{2}\right)$.

THEREFORE, THE SOLUTION SET OF $kx^2 - 2x + k = 0$ IS $(-\infty, -2] \cup \left[\frac{1}{2}, \infty\right)$

EXAMPLE 3 FOR WHAT VALUE(S) OF k THE QUADRATIC EQUATION $kx^2 - 2x + k = 0$ HAS

- I ONLY ONE REAL ROOT? II TWO DISTINCT REAL ROOTS?
- III NO REAL ROOTS?

SOLUTION: THE QUADRATIC EQUATION $kx^2 - 2x + k = 0$ IS EQUIVALENT TO THE QUADRATIC EQUATION $kx^2 + bx + c = 0$ WITH $a = k$, $b = -2$ AND $c = k$

THE GIVEN QUADRATIC EQUATION HAS

- I ONE REAL ROOT WHEN $\Delta = 0$

$$\text{SO, } (-2)^2 - 4(k)(k) = 0$$

$$4 - 4k^2 = 0 \text{ EQUIVALENTLY } 2(2 - 2k^2) = 0$$

$$2 - 2k = 0 \text{ OR } 2 + 2k = 0$$

$$k = 1 \text{ OR } k = -1$$

THEREFORE, $kx^2 - 2x + k = 0$ HAS ONLY ONE REAL ROOT IF $k = 1$.

- II TWO DISTINCT REAL ROOTS WHEN $\Delta > 0$

IT FOLLOWS THAT, $4 > 4k$

$$(2 - 2k)(2 + 2k) > 0 \Rightarrow 4(1 - k)(1 + k) > 0$$

NOW, USE THE SIGN CHART SHOWN BELOW:

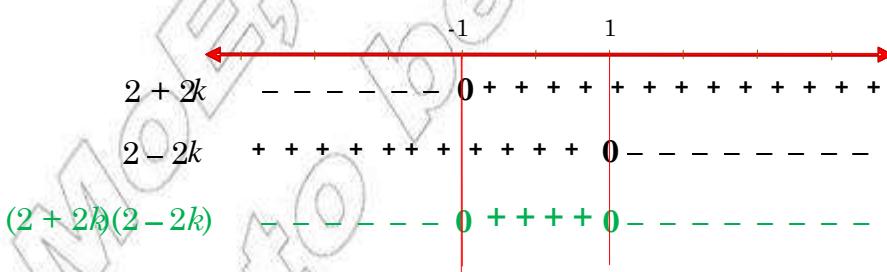


Figure 3.27

THEREFORE, FOR EACH $k \neq 1$, THE GIVEN QUADRATIC EQUATION HAS TWO DISTINCT REAL ROOTS.

- III $kx^2 - 2x + k = 0$ HAS NO REAL ROOT FOR EACH $k \in (1, \infty)$ WHERE $b^2 - 4ac < 0$

What do you do if $ax^2 + bx + c$, $a \neq 0$ is not factorizable into linear factors?

THAT IS, THERE ARE NO REAL NUMBERS SUCH THAT $bx + c = a(x - x_1)(x - x_2)$.

IN THIS CASE, EITHER $bx + c > 0$ FOR ALL VALUES OF x OR $bx + c < 0$ FOR ALL VALUES OF x . AS A RESULT, THE SOLUTION SET OF $ax^2 + bx + c \geq 0$ IS EITHER $(-\infty, \infty)$ OR $\{ \}$.

TAKE A TEST POINT AND SUBSTITUTE, IN ORDER TO DECIDE WHICH IS THE CASE.

EXAMPLE 4 SOLVE EACH OF THE FOLLOWING QUADRATIC INEQUALITIES:

A $x^2 - 2x + 5 \geq 0$

B $-3x^2 + x - 1 \geq 0$.

SOLUTION:

A FOR $x^2 - 2x + 5 \geq 0$

$$a = 1, b = -2, c = 5 \text{ AND } b^2 - 4ac = (-2)^2 - 4(1)(5) = -16 < 0.$$

HENCE, $x^2 - 2x + 5$ CANNOT BE FACTORIZED.

TAKE A TEST POINT, SAY THEN, $0^2 - 2(0) + 5 = 5 > 0$

SO, $x^2 - 2x + 5 > 0$ FOR ALL $(-\infty, \infty)$

THEREFORE, THE SOLUTION SET S = (

B FOR $-3x^2 + x - 1 \geq 0$

$$b^2 - 4ac = (1)^2 - 4(-3)(-1) = 1 - 12 = -11 < 0$$

HENCE, $-3x^2 + x - 1$ CANNOT BE FACTORIZED. TAKE A TEST POINT, SAY $x = 0$

$$-3(0)^2 + 0 - 1 = -1 < 0. \text{ HENCE, } -3x^2 + x - 1 \geq 0 \text{ IS FALSE.}$$

THEREFORE, S = { }

Group Work 3.3



1 SOLVE EACH OF THE FOLLOWING INEQUALITIES USING

I PRODUCT PROPERTIES II SIGN CHARTS:

A $x^2 - \frac{2}{3}x < 0$ B $2x^2 + 5x > 3$

C $(x-1)^2 \geq 2x^2 - 2x$ D $(2x-1)(x+1) \leq x(x-3) + 4$

2 WHAT MUST BE THE VALUE(S) OF k FOR $x^2 + 2kx - 1 = 0$ HAS

I TWO DISTINCT REAL ROOTS? ONE REAL ROOT? NO REAL ROOTS?

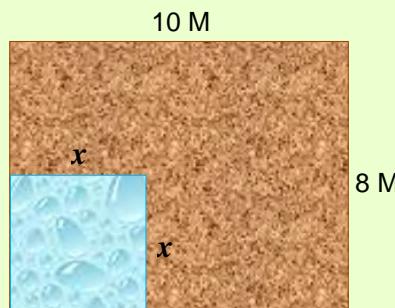
3 A MANUFACTURER DETERMINES THAT THE PROFIT OBAINED FROM CERTAIN ITEM IN BIRR IS $P = 10x - 0.002x^2$

A HOW MANY UNITS MUST BE PRODUCED TO SECURE PROFIT?

B IN THE PROCESS OF PRODUCTION, AT HOW MANY UNITS IS THERE NO PROFIT AND NO LOSS?

Exercise 3.4

- 1** SOLVE EACH OF THE FOLLOWING QUADRATIC INEQUALITIES
- A** $x(x+5) > 0$ **B** $(x-3)^2 \geq 0$
C $(4+x)(4-x) < 0$ **D** $\left(1 + \frac{x}{3}\right)(5-x) < 0$
E $3-x-2x^2 > 0$ **F** $-6x^2 + 2 \leq x$
G $2x^2 \geq -3-5x$ **H** $4x^2 - x - 8 < 3x^2 - 4x + 2$
I $-x^2 + 3x < 4.$
- 2** SOLVE EACH OF THE FOLLOWING QUADRATIC INEQUALITIES BY PRODUCT PROPERTIES OR SIGN CHARTS:
- A** $x^2 + x - 12 > 0$ **B** $x^2 - 6x + 9 > 0$ **C** $x^2 - 3x - 4 \leq 0$
D $5x - x^2 < 6$ **E** $x^2 + 2x < -1$ **F** $x - 1 \leq x^2 + 2$
- 3** FOR WHAT VALUE(S) OF k EACH OF THE FOLLOWING QUADRATIC EQUATIONS HAVE
I ONE REAL ROOT? **T**WO DISTINCT REAL ROOTS? NO REAL ROOT?
- A** $(k+2)x^2 - (k+2)x - 1 = 0$
B $x^2 + (5-k)x + 9 = 0$
- 4** FOR WHAT VALUE(S) OF k
A $kx^2 + 6x + 1 > 0$ FOR EACH REAL NUMBER
B $x^2 - 9x + k < 0$ ONLY FOR ($k=2, 11$)?
- 5** A ROCKET IS FIRED STRAIGHT UPWARD FROM GROUND LEVEL WITH VELOCITY OF 480 KM/HR. AFTER t SECONDS, ITS DISTANCE ABOVE THE GROUND LEVEL IS GIVEN BY $480t - 16t^2$.
FOR WHAT TIME INTERVAL IS THE ROCKET MORE THAN 3200 KM ABOVE GROUND LEVEL?
- 6** A FARMER HAS 8M BY 10M PLOT OF LAND. HE NEEDS TO CONSTRUCT A RESERVOIR AT ONE CORNER OF THE PLOT WITH EQUAL LENGTH AND WIDTH AS SHOWN BELOW.



FOR WHAT VALUES OF x THE AREA OF THE REMAINING PART LESS THAN THE AREA NEEDED FOR THE RESERVOIR?

3.3.3 Solving Quadratic Inequalities Graphically

IN ORDER TO USE GRAPHS TO SOLVE QUADRATIC INEQUALITIES, IT IS NECESSARY TO UNDERSTAND THE NATURE OF QUADRATIC FUNCTIONS AND THEIR GRAPHS.

- I IF $a > 0$, THEN THE GRAPH OF THE QUADRATIC FUNCTION

$f(x) = ax^2 + bx + c$ IS AN **upward parabola**.

- II IF $a < 0$, THEN THE GRAPH OF THE QUADRATIC FUNCTION

$f(x) = ax^2 + bx + c$ IS A **downward parabola**.

ACTIVITY 3.6

- 1 FOR A QUADRATIC FUNCTION $f(x) = ax^2 + bx + c$, FIND THE POINT AT WHICH THE GRAPH TURNS UPWARD OR DOWNWARD. WHAT DO YOU CALL THIS TURNING POINT?
- 2 SKETCH THE GRAPH AND FIND THE TURNING POINT OF:

A $f(x) = x^2 - 1$	B $f(x) = 4 - x^2$
---------------------------	---------------------------
- 3 WHAT IS THE CONDITION FOR THE QUADRATIC FUNCTION TO HAVE A MAXIMUM VALUE? WHEN WILL IT HAVE A MINIMUM VALUE?
- 4 WHAT IS THE VALUE AT WHICH THE QUADRATIC FUNCTION $f(x) = ax^2 + bx + c$ ATTAINS ITS MAXIMUM OR MINIMUM VALUE?

THE GRAPH OF A QUADRATIC FUNCTION HAS BOTH ITS ENDS GOING UPWARD OR DOWNWARD DEPENDING ON WHETHER a IS POSITIVE OR NEGATIVE. FROM DIFFERENT GRAPHS YOU CAN OBSERVE THAT THE GRAPH OF A QUADRATIC FUNCTION

$$f(x) = ax^2 + bx + c$$

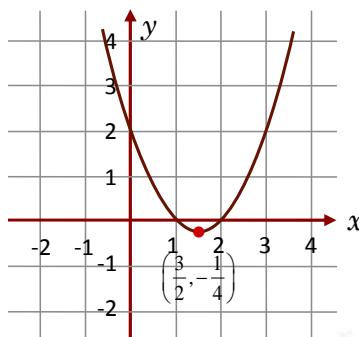
- I CROSSES THE ~~X-AXIS~~ TWICE, IF $-b^2 - 4ac > 0$.
- II TOUCHES THE ~~X-AXIS~~ AT A POINT, IF $-b^2 - 4ac = 0$.
- III DOES NOT TOUCH THE ~~X-AXIS~~ AT ALL, IF $-b^2 - 4ac < 0$.

TO SOLVE A QUADRATIC INEQUALITY GRAPHICALLY, FOR WHICH THE PART OF THE GRAPH OF THE CORRESPONDING QUADRATIC FUNCTION BE ABOVE THE ~~X-AXIS~~ OR ON THE ~~X-AXIS~~. CONSIDER THE FOLLOWING EXAMPLES.

EXAMPLE 5 SOLVE THE QUADRATIC INEQUALITY $x^2 - 3x + 2 \geq 0$; GRAPHICALLY.

SOLUTION: BEGIN BY DRAWING THE GRAPH OF $y = x^2 - 3x + 2$. SOME VALUES x FOR WHICH $y(x)$ ARE GIVEN IN THE TABLE BELOW AND THE CORRESPONDING GRAPHS ARE GIVEN IN FIGURE 3.28 COMPLETE THE TABLE FIRST.

x	-3	-2	-1	0	1	2	3
$f(x)$		12		2		0	

Figure 3.28 Graph of $f(x) = x^2 - 3x + 2$

FROM THE GRAPH, $f(x) > 0$ WHEN $x = 1$ AND WHEN $x = 2$. ON THE OTHER HAND, $f(x) < 0$ WHEN $x < 1$ AND WHEN $x > 2$ AND $f(x) = 0$ WHEN x LIES BETWEEN 1 AND 2.

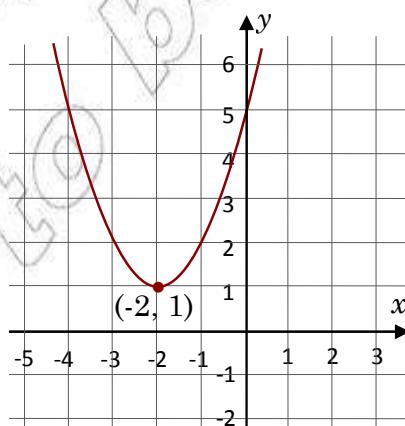
THIS INEQUALITY COULD BE TESTED BY SETTING $\left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 2 < 0$. SO $f\left(\frac{3}{2}\right) < 0$.

IT FOLLOWS THAT THE SOLUTION SET QF0 CONSISTS OF ALL REAL NUMBERS GREATER THAN 1 AND LESS THAN 2. THAT IS, $S = \{x \mid 1 < x < 2\}$.

EXAMPLE 6 SOLVE THE INEQUALITY $x^2 + 5 > 0$, GRAPHICALLY.

SOLUTION: MAKE A TABLE OF VALUES AND COMPLETE THIS TABLE OF VALUES OF x AND $f(x)$ AS IN THE TABLE BELOW AND SKETCH THE CORRESPONDING GRAPH.

x	-3	-2	-1	0	1	2	3
$f(x)$	2		2		10		

Figure 3.29 Graph of $f(x) = x^2 + 4x + 5$

AS SHOWN IN **FIGURE 3.29** ABOVE, THE GRAPH OF $x^2 + 4x + 5$ DOES NOT CROSS THE x -AXIS BUT LIES ABOVE IT. THUS, THE SOLUTION SET OF THIS INEQUALITY CONSISTS OF ALL REAL NUMBERS. $\therefore S.S = \{x \mid x \in \mathbb{R}\}$

NOTE THAT, IF YOU USE THE PROCESS OF COMPLETING THE SQUARE, YOU OBTAIN

$$x^2 + 4x + 5 > 0 \Rightarrow x^2 + 4x > -5$$

$$x^2 + 4x + 4 > -5 + 4$$

$$(x + 2)^2 > -1$$

SINCE THE SQUARE OF ANY REAL NUMBERS IS **NON-NEGATIVE** FOR ALL REAL NUMBERS

BASED ON THE ABOVE INFORMATION, COULD YOU SHOW THAT THE SOLUTION SET OF INEQUALITY $x^2 + 4x + 5 < 0$ IS THE EMPTY SET? WHY?

EXAMPLE 7 SOLVE THE INEQUALITY $x^2 + 2x + 3 < 0$, GRAPHICALLY.

SOLUTION: MAKE A TABLE OF SELECTED VALUES. THE GRAPH PASSES THROUGH $(0, 3)$ AND $(-1, 0)$ AS SHOWN IN **FIGURE 3.30**.

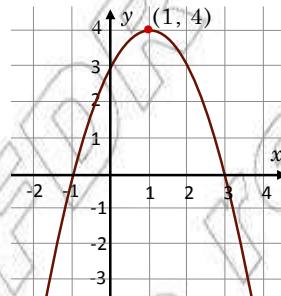


Figure 3.30 Graph of $f(x) = -x^2 + 2x + 3$

THE GRAPH OF $f(x) = -x^2 + 2x + 3$ CROSSES THE x -AXIS AT $x = -1$ AND $x = 3$. SO, THE SOLUTION SET OF THIS INEQUALITY IS

$$S.S = \{x \mid x < -1 \text{ OR } x > 3\}.$$

IF THE QUADRATIC EQUATION $ax^2 + bx + c = 0$, $a \neq 0$ HAS DISCRIMINANT $b^2 - 4ac < 0$, THEN THE EQUATION HAS NO REAL ROOTS. MOREOVER,

- I THE SOLUTION SET OF $ax^2 + bx + c \geq 0$ IS THE SET OF ALL REAL NUMBERS IF $a > 0$ AND IS EMPTY SET IF $a < 0$.
- II THE SOLUTION SET OF $ax^2 + bx + c \leq 0$ IS THE SET OF ALL REAL NUMBERS IF $a < 0$ AND IS EMPTY SET IF $a > 0$.

Exercise 3.5

1 SOLVE EACH OF THE FOLLOWING QUADRATIC INEQUALITIES, G

A $x^2 + 6x + 5 \geq 0$

B $x^2 + 6x + 5 < 0$

C $x^2 + 8x + 16 < 0$

D $x^2 + 2x + 3 \geq 0$

E $3x - x^2 + 2 < 0$

F $4x^2 - x \leq 3x^2 + 2$

G $x(x - 2) < 0$

H $(x + 1)(x - 2) > 0$

I $3x^2 + 4x + 1 > 0$

J $x^2 + 3x + 3 < 0$

K $3x^2 + 22x + 35 \geq 0$

L $6x^2 + 1 \geq 5x$

2 SUPPOSE THE SOLUTION SET OF $2x^2 + kx + 2 > 0$ CONSISTS OF THE SET OF ALL REAL NUMBERS. FIND ALL POSSIBLE VALUES OF k .



Key Terms

absolute value

linear inequality

quadratic equation

closed intervals

open downward

quadratic function

complete listing

open intervals

quadratic inequality

discriminant

open upward

sign chart

infinity

partial listing

solution set

linear equation

product property



Summary

- 1** THE OPEN INTERVAL WITH END-POINTS a AND b IS THE SET OF ALL REAL NUMBERS x SUCH THAT $a < b$.
- 2** THE CLOSED INTERVAL WITH END-POINTS a AND b IS THE SET OF ALL REAL NUMBERS x SUCH THAT $a \leq x \leq b$.
- 3** THE HALF-OPEN INTERVAL OR HALF-CLOSED INTERVAL WITH END-POINTS a AND b IS THE SET OF ALL REAL NUMBERS x SUCH THAT $a \leq x < b$.
- 4** IF x IS A REAL NUMBER, THEN THE ABSOLUTE VALUE DEFINED BY

$$|x| = \begin{cases} x, & \text{IF } x \geq 0 \\ -x, & \text{IF } x < 0 \end{cases}$$

- 5** FOR ANY POSITIVE REAL NUMBER a , THE SOLUTION SET OF:
- THE EQUATION $|x| = a$ IS $x = a$ OR $x = -a$;
 - THE INEQUALITY $|x| < a$ IS $-a < x < a$ AND
 - THE INEQUALITY $|x| > a$ IS $x < -a$ OR $x > a$.
- 6** WHEN TWO OR MORE LINEAR EQUATIONS INVOLVING THE SAME VARIABLE ARE CALLED A **system of linear equations**.
- 7** AN INEQUALITY THAT CAN BE REDUCED TO EITHER $ax^2 + bx + c < 0$, $ax^2 + bx + c \geq 0$ OR $ax^2 + bx + c > 0$, WHERE a AND b ARE CONSTANTS AND c IS A CONSTANT, ARE CALLED **quadratic inequality**.
- 8** GIVEN ANY QUADRATIC EQUATION $ax^2 + bx + c = 0$,
- IF $b^2 - 4ac > 0$, IT HAS TWO DISTINCT REAL ROOTS.
 - IF $b^2 - 4ac = 0$, IT HAS ONLY ONE REAL ROOT.
 - IF $b^2 - 4ac < 0$, IT HAS NO REAL ROOT.
- 9** WHEN THE DISCRIMINANT $\Delta = b^2 - 4ac$ IS:
- THE SOLUTION SET OF $ax^2 + bx + c > 0$ IS THE SET OF ALL REAL NUMBERS, IF $a > 0$ AND EMPTY SET IF $a < 0$.
 - THE SOLUTION SET OF $ax^2 + bx + c < 0$ IS THE SET OF ALL REAL NUMBERS, IF $a < 0$ AND EMPTY SET IF $a > 0$.
- 10** PRODUCT PROPERTY:
- $mn > 0$, IF AND ONLY IF $m > 0$ AND $n > 0$ OR $m < 0$ AND $n < 0$.
 - $mn < 0$, IF AND ONLY IF $m < 0$ AND $n > 0$ OR $m > 0$ AND $n < 0$.



Review Exercises on Unit 3

- 1** SOLVE EACH OF THE FOLLOWING INEQUALITIES PROBLEMS BY THE PRODUCT PROPERTY:
- | | | | |
|----------|------------------------------------|----------|--|
| A | $(x + 1)(x - 3) < 0$ | B | $\left(\frac{2}{3}x + 3\right)(x - 1) < 0$ |
| C | $(x - \sqrt{3})(x + \sqrt{2}) > 0$ | D | $x^2 > x$ |
| E | $x^2 + 5x + 4 \geq 0$ | F | $(x - 2)^2 \leq 2 - x$ |
| G | $1 - 2x \geq (1+x)^2$ | H | $3x^2 - 6x + 5 < x^2 - 2x + 3$. |

2 SOLVE EACH OF THE FOLLOWING INEQUALITIES USING SIGN CHARTS:

- A** $(1 - x)(5 - x) > 0$ **B** $x^2 \leq 9$ **C** $(x + 2)^2 < 25$
D $1 - x \geq 2x^2$ **E** $6t^2 + 1 < 5t$ **F** $2t^2 + 3t \leq 5.$

3 SOLVE EACH OF THE FOLLOWING INEQUALITIES GRAPHICALLY:

- A** $x^2 - x + 1 > 0$ **B** $x^2 > x + 6$ **C** $x^2 - 4x - 1 > 0$
D $x^2 + 25 \geq 10x$ **E** $x^2 + 32 \geq 12x + 6$ **F** $x(6x - 13) > -6$
G $x(10 - 3x) < 8$ **H** $(x - 3)^2 \leq 1$

4 SOLVE EACH OF THE FOLLOWING QUADRATIC INEQUALITIES USING ANY CONVENIENT METHOD:

- A** $2x^2 < x + 2$ **B** $-2x^2 + 6x + 15 \leq 0$
C $\frac{1}{2}x^2 + \frac{25}{2} \geq 5x$ **D** $6x^2 - x + 3 < 5x^2 + 5x - 5$
E $x(10x + 19) \leq 15$ **F** $(x + 2)^2 > (3x + 1)^2.$

5 WHAT MUST THE VALUE(S) OF x BE SO THAT:

- A** $kx^2 - 10x - 5 \leq 0$ FOR ALL x ?
B $2x^2 + (k - 3)x + k - 5 = 0$ HAS ONE REAL ROOT? TWO REAL ROOTS? NO REAL ROOT?

6 THE SUM OF A NON-NEGATIVE NUMBER AND ITS SQUARE IS LESS THAN 12. WHAT COULD THE NUMBER BE?

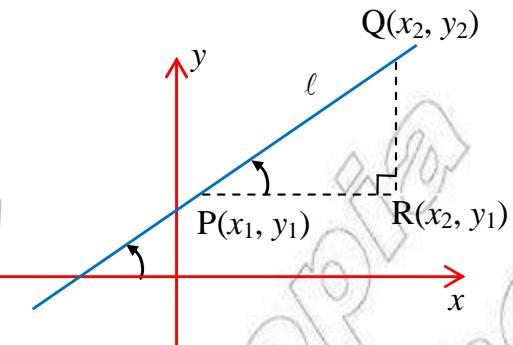
7 THE SUM OF A NUMBER AND TWICE ANOTHER IS 20. IF THE PRODUCT OF THESE NUMBERS IS NOT MORE THAN 48, WHAT ARE ALL POSSIBLE VALUES OF

8 THE PROFIT OF A CERTAIN COMPANY IS GIVEN BY $y(x) = 10,000 + 350x - \frac{1}{2}x^2$

WHERE x IS THE AMOUNT (BIRR IN TENS) SPENT ON ADVERTISING. WHAT AMOUNT GIVES A PROFIT OF MORE THAN BIRR 40,000?

Unit

4



COORDINATE GEOMETRY

Unit Outcomes:

After completing this unit, you should be able to:

- apply the distance formula to find the distance between any two given points in the coordinate plane.
- formulate and apply the section formula to find a point that divides a given line segment in a given ratio.
- write different forms of equations of a line and understand related terms.
- describe parallel or perpendicular lines in terms of their slopes.

Main Contents

- 4.1 Distance between two points**
- 4.2 Division of a line segment**
- 4.3 Equation of a line**
- 4.4 Parallel and perpendicular lines**

Key Terms

Summary

Review Exercises

INTRODUCTION

IN UNT3, YOU HAVE SEEN AN IMPORTANT CONNECTION BETWEEN ALGEBRA AND GEOMETRY. THE GREAT DISCOVERIES IN MATHEMATICS WAS THE SUBJECT OF geometry. IT IS OFTEN REFERRED TO AS CARTESIAN GEOMETRY, WHICH IS A METHOD OF STUDYING GEOMETRY BY MEANS OF A COORDINATE SYSTEM AND ASSOCIATED ALGEBRA. IN ANALYTIC GEOMETRY, WE DESCRIBE PROPERTIES OF GEOMETRIC FIGURES SUCH AS CIRCLES, ETC., IN TERMS OF ORDERED PAIRS AND EQUATIONS.

4.1

DISTANCE BETWEEN TWO POINTS

IN GRADE 9, YOU HAVE DISCUSSED THE NUMBER LINE AND YOU HAVE SEEN THAT THERE IS A ONE-TO-ONE CORRESPONDENCE BETWEEN THE SET OF REAL NUMBERS AND THE SET OF POINTS ON A NUMBER LINE. YOU HAVE ALSO SEEN HOW TO LOCATE A POINT IN THE COORDINATE PLANE. REMEMBER THE FACT THAT THERE IS A ONE-TO-ONE CORRESPONDENCE BETWEEN THE SET OF ALL ORDERED PAIRS OF REAL NUMBERS AND THE SET OF ALL POINTS IN THE PLANE AND THE SET OF ALL ORDERED PAIRS OF REAL NUMBERS?

THE FOLLOWING ACTIVITY WILL HELP YOU TO REVIEW THE FACTS YOU DISCUSSED IN GRADE 9.

ACTIVITY 4.1

- 1** CONSIDER THE NUMBER LINE GIVEN IN FIGURE 4.1

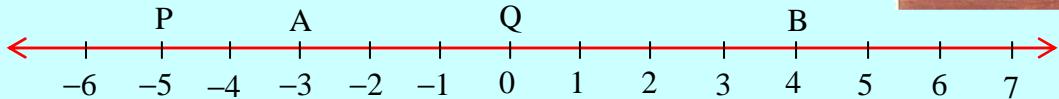


Figure 4.1

- A** FIND THE COORDINATES OF P, Q AND B.
- B** FIND THE DISTANCE BETWEEN POINTS
 - I** P AND Q **II** Q AND B **III** P AND B
- 2** ON A NUMBER LINE, THE TWO POINTS HAVE COORDINATES x_1 AND x_2 .
 - A** FIND THE DISTANCE BETWEEN P AND Q.
 - B** FIND THE DISTANCE BETWEEN Q AND B.
 - C** DISCUSS THE RELATIONSHIP BETWEEN YOUR ANSWERS IN A AND B.
 - D** DISCUSS THE RELATIONSHIP BETWEEN $|x_2 - x_1|$.
- 3** HOW DO YOU PLOT THE COORDINATES OF POINTS IN THE COORDINATE PLANE?
- 4** WHAT ARE THE COORDINATES OF THE ORIGIN OF THE xy -PLANE?
- 5** DRAW A COORDINATE PLANE AND PLOT THE FOLLOWING POINTS.
P (3, -4), Q (-3, -2), R (-2, 0), S (4, 0), T (2, 3), U (-4, 5) AND V (0, 0).

- 6** THE POSITION OF EACH POINT ON THE COORDINATE PLANE IS DETERMINED BY ITS PAIR OF NUMBERS.
- A** WHAT IS THE COORDINATE OF A POINT ON SPHE y
B WHAT IS THE COORDINATE OF A POINT ON SPHE x
- 7** LET P (2, 3) AND Q (2, 8) BE POINTS ON THE COORDINATE PLANE.
- A** PLOT THE POINTS P AND Q .
B IS THE LINE THROUGH POINTS P AND Q VERTICAL OR HORIZONTAL?
C WHAT IS THE DISTANCE BETWEEN POINTS P AND Q ?
- 8** LET R (-2, 4) AND T (5, 4) BE POINTS ON THE COORDINATE PLANE.
- A** PLOT THE POINTS R AND T .
B IS THE LINE THROUGH POINTS R AND T VERTICAL OR HORIZONTAL?
C WHAT IS THE DISTANCE BETWEEN POINTS R AND T ?

Distance between points in a plane

SUPPOSE $P(x_1, y_1)$ AND $Q(x_2, y_2)$ ARE TWO DISTINCT POINTS ON COORDINATE PLANE. WE CAN FIND THE DISTANCE BETWEEN THEM BY CONSIDERING THREE CASES.

Case i WHEN P AND Q ARE ON A LINE PARALLEL TO y -AXIS (THAT IS,

\overline{PQ} IS A HORIZONTAL SEGMENT) AS IN FIGURE 4.2

SINCE THE TWO POINTS HAVE THE SAME COORDINATE (THAT IS, THE DISTANCE BETWEEN P AND Q IS

$$PQ = |x_2 - x_1|$$

Case ii WHEN P AND Q ARE ON A LINE PARALLEL TO x -AXIS (THAT IS,

\overline{PQ} IS A VERTICAL SEGMENT) AS IN FIGURE 4.3

SINCE THE TWO POINTS HAVE THE SAME x -COORDINATE (THAT IS, THE DISTANCE BETWEEN P AND Q IS

$$PQ = |y_2 - y_1|$$

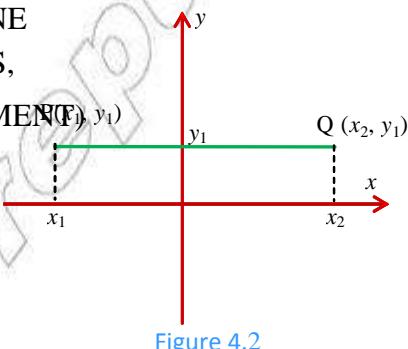


Figure 4.2

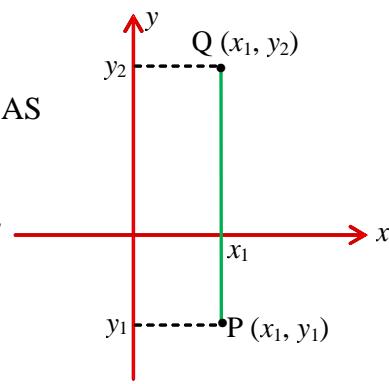


Figure 4.3

Case iii WHEN \overline{PQ} IS NEITHER VERTICAL NOR HORIZONTAL (THE GENERAL CASE).

TO FIND THE DISTANCE BETWEEN THE POINTS P AND Q , DRAW A LINE PASSING THROUGH PARALLEL TO x -AXIS AND DRAW A LINE PASSING THROUGH PARALLEL TO y -AXIS. THE HORIZONTAL LINE AND THE VERTICAL LINE INTERSECT AT $R(x_1, y_1)$.

USING CASE I AND CASE II, WE HAVE

$$PR = |x_2 - x_1| \text{ AND } RQ = |y_2 - y_1|$$

SINCE $\triangle PRQ$ IS A RIGHT ANGLED TRIANGLE, WE CAN USE Pythagoras' Theorem TO FIND THE DISTANCE BETWEEN POINTS P AND Q AS FOLLOWS:

$$PQ^2 = PR^2 + RQ^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\text{ THEREFORE } PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

THE RADICAL HAS POSITIVE SIGN (WHY?).

IN GENERAL, THE DISTANCE BETWEEN ANY TWO POINTS $P(x_1, y_1)$ AND $Q(x_2, y_2)$ IS GIVEN BY

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

THIS IS CALLED THE **distance formula**.

EXAMPLE 1 FIND THE DISTANCE BETWEEN THE GIVEN POINTS.

A $A(1, \sqrt{2})$ AND $B(1, -\sqrt{2})$

B $P\left(\frac{17}{4}, -2\right)$ AND $Q\left(\frac{1}{4}, -2\right)$

C $R(-\sqrt{2}, -1)$ AND $S(\sqrt{2}, -\sqrt{2})$

D $A(a, -b)$ AND $B(-b, a)$

SOLUTION:

$$\begin{aligned} A \quad AB &= d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(1-1)^2 + (-\sqrt{2} - \sqrt{2})^2} \\ &= \sqrt{(0)^2 + (-2\sqrt{2})^2} = 2\sqrt{2}. \end{aligned}$$

OR, MORE SIMPLY
 $AB = |y_2 - y_1| = |- \sqrt{2} - \sqrt{2}| = 2\sqrt{2}$ UNIT

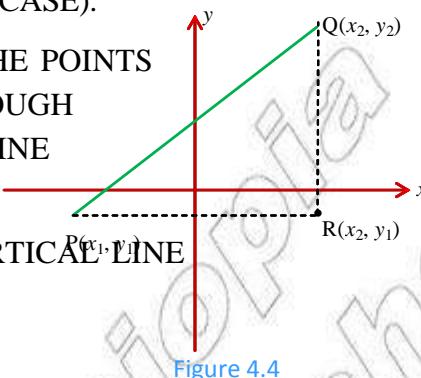


Figure 4.4

$$\begin{aligned}
 \mathbf{B} \quad PQ = d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
 &= \sqrt{\left(\frac{1}{4} - \frac{17}{4}\right)^2 + (-2 - (-2))^2} \\
 &= \sqrt{\left(\frac{-16}{4}\right)^2 + (0)^2} = \sqrt{(-4)^2} = \sqrt{16} = 4 \text{ UNIT}
 \end{aligned}$$

OR, MORE SIMPLY
 $PQ = |x_2 - x_1| = \left|\frac{1}{4} - \frac{17}{4}\right| = 4 \text{ UNITS}$

$$\begin{aligned}
 \mathbf{C} \quad RS = d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(\sqrt{2} - (-\sqrt{2}))^2 + (-\sqrt{2} - (-1))^2} \\
 &= \sqrt{(2\sqrt{2})^2 + (1 - \sqrt{2})^2} = \sqrt{11 - 2\sqrt{2}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{D} \quad AB = d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-b - a)^2 + (a - (-b))^2} \\
 &= \sqrt{(b + a)^2 + (a + b)^2} = \sqrt{2(a + b)^2} = \sqrt{2}|a + b| \text{ UNIT}
 \end{aligned}$$

Exercise 4.1

- 1** IN EACH OF THE FOLLOWING, FIND THE DISTANCE BETWEEN THE TWO GIVEN POINTS.
- | | |
|--|---|
| A A (1, -5) AND B (7, 3)
C E($\sqrt{2}$, 1) AND F($-\sqrt{6}$, $\sqrt{3}$)
E THE ORIGIN AND K($\frac{\sqrt{2}}{2}$, $-\frac{\sqrt{2}}{2}$)
G P($\sqrt{2}$, $\sqrt{3}$) AND Q($\sqrt{2}$, $-\sqrt{3}$) | B C $\left(-2, \frac{1}{2}\right)$ AND D $\left(\frac{1}{2}, 2\right)$
D G (a, -b) AND H (-a, b)
F L($\sqrt{2}$, 1) AND M($-\sqrt{2}$, 1)
H R($\sqrt{2}a$, c) AND T($-\sqrt{2}a$, c) |
|--|---|
- 2** USING THE DISTANCE FORMULA, SHOW THAT THE DISTANCE BETWEEN P AND Q IS EQUAL TO $|x_2 - x_1|$, WHEN PQ IS HORIZONTAL AND $|y_2 - y_1|$, WHEN PQ IS VERTICAL.
- 3** LET A (3, -7) AND B (-1, 4) BE TWO ADJACENT VERTICES OF A SQUARE. CALCULATE THE LENGTH OF THE SIDE OF THE SQUARE.
- 4** P (3, 5) AND Q (1, -3) ARE TWO OPPOSITE VERTICES OF A SQUARE. FIND ITS AREA.
- 5** SHOW THAT THE PLANE FIGURE WITH VERTICES:
- | | |
|--|---|
| A A (5, -1), B (2, 3) AND C (1, 1) IS A RIGHT ANGLED TRIANGLE.
B A (-4, 3), B (4, -3) AND C ($3\sqrt{3}$, $4\sqrt{3}$) IS AN EQUILATERAL TRIANGLE.
C A (2, 3), B (6, 8), C (7, -1) IS AN ISOSCELES TRIANGLE. | 6 AN EQUILATERAL TRIANGLE HAS TWO VERTICES AT A (-4, 0) AND B (4, 0). WHAT COULD BE THE COORDINATES OF THE THIRD VERTEX? |
|--|---|
- 7** WHAT ARE THE POSSIBLE VALUES OF POINT b IF POINT $(4, 0)$ IS 10 UNITS AWAY FROM B (0, -2)?

4.2**DIVISION OF A LINE SEGMENT**

RECALL THAT, A LINE SEGMENT PASSING THROUGH TWO POINTS A AND B IS **horizontal** IF THE TWO POINTS HAVE THE SAME ~~y -coordinate~~. I.E., A LINE SEGMENT WHOSE END-POINTS ARE ~~A AND~~

$B(x_2, y)$ IS A HORIZONTAL LINE SEGMENT AS SHOWN IN **FIGURE 4.5**.

What is the mid-point of \overline{AB} ?

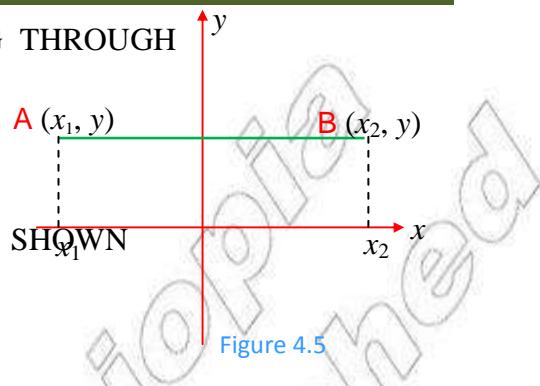


Figure 4.5

ACTIVITY 4.2

- 1 DEFINE THE RATIO OF TWO QUANTITIES.
- 2 WHAT IS MEANT BY THE RATIO OF THE LENGTH OF TWO LINE SEGMENTS?
- 3 IN **FIGURE 4.6** FIND THE RATIO OF THE LENGTHS OF

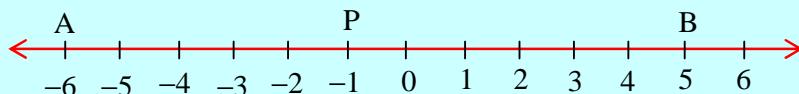


Figure 4.6

- 4 WHAT IS MEANT BY A POINT DIVIDES A LINE SEGMENT INTERNALLY?
 - 5 PLOT THE FOLLOWING POINTS ON THE COORDINATE PLANE AND FIND THE MID-POINT OF THE LINE SEGMENT JOINING THE POINTS.
- A** $A(2, -1)$ AND $B(2, 5)$ **B** $C(-3, 3)$ AND $D(3, 3)$ **C** $E(2, 0)$ AND $F(-2, 4)$.

CONSIDER THE HORIZONTAL LINE SEGMENT WITH ENDS POINTS A AND B AS SHOWN IN **FIGURE 4.7** IN TERMS OF THE COORDINATES, DETERMINE THE COORDINATES OF THE POINT $P(x_0, y_0)$ THAT DIVIDES INTERNALY IN THE RATIO

CLEARLY, THE RATIO OF THE LINE SEGMENT AP AS SEGMENT GIVEN BY $\frac{AP}{PB}$

THE DISTANCE BETWEEN A AND P IS $AP = x_0 - x_1$.

THE DISTANCE BETWEEN P AND B IS $PB = x_2 - x_0$

$$\text{THEREFORE, } \frac{AP}{PB} = \frac{m}{n} \text{ I.E., } \frac{x_0 - x_1}{x_2 - x_0} = \frac{m}{n}.$$

SOLVING THIS EQUATION FOR

$$\Rightarrow n(x_0 - x_1) = m(x_2 - x_0)$$

$$\Rightarrow nx_0 - nx_1 = mx_2 - mx_0$$

$$\Rightarrow nx_0 + mx_0 = nx_1 + mx_2$$

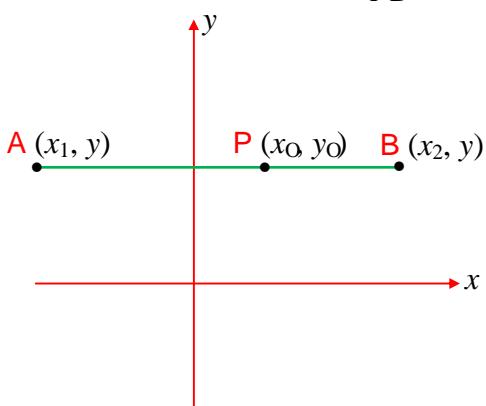


Figure 4.7

$$\Rightarrow x_0(n+m) = nx_1 + mx_2$$

$$\Rightarrow x_0 = \frac{nx_1 + mx_2}{n+m}$$

SINCE \overline{AB} IS PARALLEL TO $\overline{x_0y_0}$ (\overline{AB} IS A HORIZONTAL LINE SEGMENT) AND OBVIOUSLY, $y_0 = y$, THEREFORE, THE POINT O IS $\left(\frac{nx_1 + mx_2}{n+m}, y\right)$.

GIVEN A LINE SEGMENT WITH END POINT COORDINATES $P(x_1, y_1)$ AND $Q(x_2, y_2)$, LET US FIND THE COORDINATES OF THE POINT DIVIDING THE LINE SEGMENT INTERNAL IN THE RATIO I.E., $\frac{PR}{RQ} = \frac{m}{n}$, WHERE m AND n ARE GIVEN POSITIVE REAL NUMBERS.

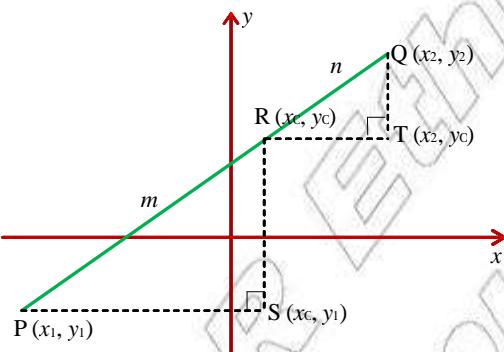


Figure 4.8

LET THE COORDINATES (x_0, y_0) OF O . ASSUME THAT x_2 AND $y_1 \neq y_2$.

IF YOU DRAW LINES THROUGH THE POINTS PARALLEL TO THE AXES AS SHOWN IN FIGURE 4.8 THE POINTS S AND T HAVE THE COORDINATES (x_c, y_1) AND (x_c, y_0) , RESPECTIVELY.

$$PS = x_0 - x_1, RT = x_2 - x_0, SR = y_0 - y_1 \text{ AND } TQ = y_2 - y_0$$

SINCE TRIANGLES PSR AND TQ ARE SIMILAR (WHY?),

$$\frac{PS}{RT} = \frac{PR}{RQ} \text{ AND } \frac{SR}{TQ} = \frac{PR}{RQ}$$

$$\frac{x_0 - x_1}{x_2 - x_0} = \frac{m}{n} \text{ AND } \frac{y_0 - y_1}{y_2 - y_0} = \frac{m}{n}$$

SOLVING FOR x_0 AND y_0

$$\Rightarrow n(x_0 - x_1) = m(x_2 - x_0) \text{ AND } ny_0 - ny_1 = m(y_2 - y_0)$$

$$\Rightarrow nx_0 - nx_1 = mx_2 - mx_0 \text{ AND } ny_0 - ny_1 = my_2 - my_0$$

$$\Rightarrow nx_0 + mx_0 = nx_1 + mx_2 \text{ AND } ny_0 + my_0 = ny_1 + my_2$$

$$\Rightarrow x_0(n+m) = nx_1 + mx_2 \text{ AND } y_0(n+m) = ny_1 + my_2$$

$$\Rightarrow x_0 = \frac{nx_1 + mx_2}{n+m} \text{ AND } y_0 = \frac{ny_1 + my_2}{n+m}$$

THE POINT $R(x_0, y_0)$ DIVIDING THE LINE SEGMENT INTERNALLY IN THE RATIO GIVEN BY

$$R(x_0, y_0) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m} \right)$$

THIS IS CALLED **SECTION formula**.

EXAMPLE 1 FIND THE COORDINATES OF THE POINTS DIVIDES THE LINE SEGMENT WITH END-POINTS A (6, 2) AND B (1, -4) IN THE RATIO 2:3.

SOLUTION: PUT $x_1, y_1 = (6, 2)$, $(x_2, y_2) = (1, -4)$, $m = 2$ AND $n = 3$. USING THE SECTION FORMULA, YOU HAVE

$$\begin{aligned} R(x_0, y_0) &= \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m} \right) = \left(\frac{3 \times 6 + 2 \times 1}{3+2}, \frac{3 \times 2 + 2 \times (-4)}{3+2} \right) \\ &= \left(\frac{18+2}{5}, \frac{6-8}{5} \right) = \left(4, -\frac{2}{5} \right) \\ \text{THEREFORE, } R &= \left(4, -\frac{2}{5} \right). \end{aligned}$$

EXAMPLE 2 A LINE SEGMENT HAS END-POINTS (-2, -3) AND (7, 12) AND IT IS DIVIDED INTO THREE EQUAL PARTS. FIND THE COORDINATES OF THE POINTS THAT TRISECT THE SEGMENT.

SOLUTION: THE FIRST POINT DIVIDES THE LINE SEGMENT IN THE RATIO 1:2, AND HENCE

$$\begin{aligned} x_0 &= \frac{nx_1 + mx_2}{n+m} \text{ AND } y_0 = \frac{ny_1 + my_2}{n+m} \\ \text{SO, } x_0 &= \frac{2 \times (-2) + 1 \times 7}{1+2} \text{ AND } y_0 = \frac{2 \times (-3) + 1 \times 12}{1+2} \\ \Rightarrow x_0 &= \frac{-4+7}{3} \text{ AND } y_0 = \frac{-6+12}{3} \Rightarrow x_0 = 1 \text{ AND } y_0 = \end{aligned}$$

THEREFORE, THE FIRST POINT IS (1, 2).

THE SECOND POINT DIVIDES THE LINE SEGMENT IN THE RATIO 2:1. THUS,

$$\begin{aligned} x_0 &= \frac{nx_1 + mx_2}{n+m} \text{ AND } y_0 = \frac{ny_1 + my_2}{n+m} \\ \text{SO, } x_0 &= \frac{1 \times (-2) + 2 \times 7}{1+2} \text{ AND } y_0 = \frac{1 \times (-3) + 2 \times 12}{1+2} \\ \Rightarrow x_0 &= \frac{-2+14}{3} \text{ AND } y_0 = \frac{-3+24}{3} \\ \Rightarrow x_0 &= 4 \text{ AND } y_0 = 7. \end{aligned}$$

THEREFORE, THE SECOND POINT IS (4, 7).

The mid-point formula

A POINT THAT DIVIDES A LINE SEGMENT INTO TWO EQUAL PARTS IS THE MID-POINT OF THE LINE SEGMENT.

ACTIVITY 4.3

- 1 CONSIDER THE POINTS P (2, 1) AND Q (12, 1).
 - A FIND THE DISTANCE BETWEEN P AND Q.
 - B IF R IS A POINT WITH COORDINATES (7, 1),
 - I FIND PR
 - II FIND RQ
 - III IS PR EQUAL TO RQ?
 - IV WHAT IS THE MID-POINT OF PR?
 - C DIVIDE PQ IN THE RATIO 1:1.
- 2 FIND THE COORDINATES OF THE MID-POINT OF EACH OF THE FOLLOWING LINE SEGMENTS, GIVING THE END-POINTS:
 - I P (x_1, y_1) AND Q (x_2, y_2). II R (x_1, y_1) AND S (x_2, y_1).

WHICH OF THE ABOVE SEGMENTS ARE HORIZONTAL?



LET P (x_1, y_1) AND Q (x_2, y_2) BE THE END-POINTS OF THE LINE SEGMENT PQ.

IF PR = RQ (THE CASE WHERE m = n), THEN R IS THE MID-POINT OF THE LINE SEGMENT PQ.

NOW LET US DERIVE THE MID-POINT FORMULA.

$$\begin{aligned}
 R(x_0, y_0) &= \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m} \right) \\
 &= \left(\frac{nx_1 + nx_2}{n+n}, \frac{ny_1 + ny_2}{n+n} \right) = \left(\frac{n(x_1 + x_2)}{2n}, \frac{n(y_1 + y_2)}{2n} \right) (\text{AS } m = n) \\
 &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
 \end{aligned}$$

THIS IS THE FORMULA USED TO FIND THE MID-POINT OF THE LINE SEGMENT WHOSE END POINTS ARE P (x_1, y_1) AND Q (x_2, y_2).

THE mid-point OF THE LINE SEGMENT JOINING THE POINTS (x_1, y_1) AND (x_2, y_2) IS GIVEN BY

$$M(x_0, y_0) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

EXAMPLE 3 FIND THE COORDINATES OF THE MID-POINT OF THE LINE SEGMENT WHOSE END POINTS ARE

- A P (-3, 2) AND Q (5, -4)
- B P $(3 - \sqrt{2}, 3 + \sqrt{2})$ AND Q $(1 + \sqrt{2}, 3 - \sqrt{2})$.

SOLUTION:

A $M(x_0, y_0) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

$$x_0 = \frac{x_1 + x_2}{2} \text{ AND } y_0 = \frac{y_1 + y_2}{2}$$

$$x_0 = \frac{-3+5}{2} = 1 \text{ AND } y_0 = \frac{2-4}{2} = -$$

THEREFORE $M(x_0, y_0) = (1, -1)$.

B $M(x_0, y_0) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

$$x_0 = \frac{x_1 + x_2}{2} \text{ AND } y_0 = \frac{y_1 + y_2}{2}$$

$$x_0 = \frac{3-\sqrt{2}+1+\sqrt{2}}{2} \text{ AND } y_0 = \frac{3+\sqrt{2}+3-\sqrt{2}}{2}$$

$$x_0 = \frac{4}{2} = 2 \quad \text{AND} \quad y_0 = \frac{6}{2} = 3$$

THEREFORE $M(x_0, y_0) = (2, 3)$.

Group Work 4.1

- 1 A LINE SEGMENT HAS END-POINTS P (-3, 1) AND Q (2, 1).
 A WHAT IS THE LENGTH OF THE LINE SEGMENT?
 B FIND THE COORDINATES OF THE MID-POINT OF THE SEGMENT.
- 2 A LINE SEGMENT HAS ONE END-POINT AT A (4, 3). IF ITS MID-POINT IS AT M (1, -1), WHERE IS THE OTHER END-POINT?
- 3 FIND THE POINTS THAT DIVIDE THE LINE SEGMENT WITH END-POINTS AT P (-6, 7) AND Q (-6, 7) INTO THREE EQUAL PARTS.
- 4 LET A (-2, -1), B (6, -1), C (6, 3) AND D (-2, 3) BE VERTICES OF A RECTANGLE. SUPPOSE P, Q, R AND S ARE MID-POINTS OF THE SIDES OF THE RECTANGLE.
 I WHAT IS THE AREA OF RECTANGLE AB
 II WHAT IS THE AREA OF QUADRILATERAL PQRS
 III GIVE THE RATIO OF THE AREAS IN I : II : III



Exercise 4.2

- 1** FIND THE COORDINATES OF THE MID-POINT OF THE LINE SEGMENTS JOINING THE POINTS
A A (1, 4) AND B (-2, 2) **B** (a, b) AND THE ORIGIN
C M (p, q) AND N (q, p) **D** $\left(1\frac{1}{2}, -1\right)$ AND $\left(-\frac{5}{2}, 1\right)$
E E($1+\sqrt{2}$, $\sqrt{2}$) AND F($2\sqrt{2}$, $2\sqrt{8}$) **F** G($\sqrt{5}$, $1-\sqrt{3}$) AND H($\sqrt{5}$, $4+\sqrt{3}$)
- 2** THE MID-POINT OF A LINE SEGMENT IS (2, 1). ONE END-POINT OF THE SEGMENT IS P (1, -3). FIND THE COORDINATES OF THE OTHER END-POINT.
- 3** FIND THE COORDINATES OF THE POINT WHICH DIVIDES THE LINE SEGMENT JOINING THE POINTS A (1, 3) AND B (-4, -3) IN THE RATIO 2:3.
- 4** A LINE SEGMENT HAS END-POINTS A (2, 3) AND Q (5, 2). FIND THE COORDINATES OF THE POINTS THAT TRISECT THE SEGMENT.
- 5** FIND THE MID-POINTS OF THE SIDES OF THE TRIANGLE WITH VERTICES A (2, 3), B (-4, -3) AND C (3, -1).

4.3 EQUATION OF A LINE

4.3.1 Gradient (slope) of a Line

FROM YOUR EVERYDAY EXPERIENCE, YOU MIGHT BEFAMILIAR WITH THE IDEA OF GRADIENT (SLOPE).

A hill MAY BE ~~steep~~ OR MAY RISE VERY SLOWLY. THE NUMBER THAT DESCRIBES THE ~~steepness~~ OF A HILL IS CALLED ~~gradient~~ (slope) OF THE HILL.

WE MEASURE THE GRADIENT OF A HILL BY THE RATIO OF THE ~~vertical rise~~ TO THE ~~horizontal run~~.

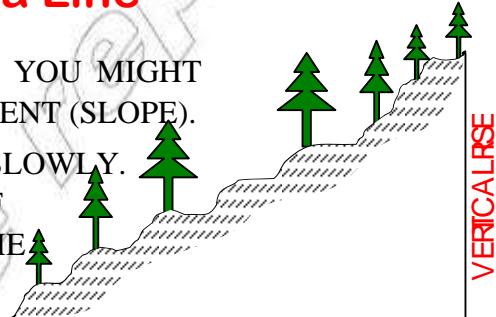


Figure 4.9

ACTIVITY 4.4

GIVEN POINTS P (1, 2), Q (-4, 1), R (0, -1) AND S (3, 8)

A FIND THE VALUE OF $\frac{y_2 - y_1}{x_2 - x_1}$ TAKING

I P AND Q **II** P AND R **III** Q AND R **IV** R AND S

B ARE THE VALUES OBTAINED EQUAL? WHAT DO YOU CALL THESE VALUES?



IN COORDINATE GEOMETRY, THE GRADIENT OF A NON-VERTICAL STRAIGHT LINE IS THE RATIO OF "CHANGE IN y -COORDINATES" TO THE CORRESPONDING "CHANGE IN x -COORDINATES". THAT IS, THE SLOPE OF A LINE THROUGH P AND Q IS THE RATIO OF THE VERTICAL DISTANCE FROM P TO Q TO THE HORIZONTAL DISTANCE FROM P TO Q

IF WE DENOTE THE GRADIENT OF A LINE BY THE LETTER m , THEN

$$m = \frac{\text{change in } y\text{-coordinates}}{\text{change in } x\text{-coordinates}} = \frac{y_2 - y_1}{x_2 - x_1}; x_1 \neq x_2$$

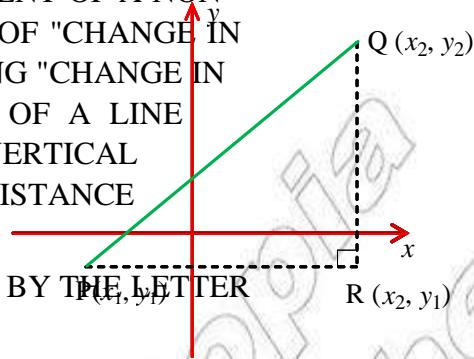


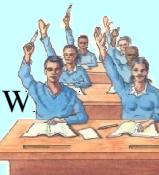
Figure 4.10

Definition 4.1

If (x_1, y_1) and (x_2, y_2) are points on a line with $x_1 \neq x_2$, then the **gradient** of the line, denoted by m , is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

ACTIVITY 4.5



- 1** IF $A(x_1, y_1)$ AND $B(x_2, y_2)$ ARE DISTINCT POINTS ON A LINE WITH $x_1 = x_2$, THEN WHAT CAN BE SAID ABOUT THE GRADIENT? IS THE LINE VERTICAL OR HORIZONTAL?
- 2** WHAT IS THE GRADIENT OF ANY HORIZONTAL LINE?
- 3** CONSIDER THE LINE WITH EQUATION $y = mx + c$. TAKE THREE DISTINCT POINTS P_1, P_2 AND P_3 ON THE LINE.
 - A** FIND THE GRADIENT m OF P_1 .
 - B** FIND THE GRADIENT m OF P_2 .
 - C** WHAT DO YOU OBSERVE FROM **A** AND **B**?
- 4** LET P_1, P_2, P_3 AND P_4 BE POINTS ON A NON-VERTICAL STRAIGHT LINE WITH COORDINATES $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ AND (x_4, y_4) RESPECTIVELY. FIND:
 - A** THE GRADIENT OF THE LINE AP_1 PASSING P_1 .
 - B** THE GRADIENT OF THE LINE AP_2 PASSING P_2 .
 - C** ARE THE RATIOS $\frac{y_2 - y_1}{x_2 - x_1}$ AND $\frac{y_4 - y_3}{x_4 - x_3}$ EQUAL?
 - D** COULD YOU CONCLUDE THAT THE GRADIENT OF A LINE DOES NOT DEPEND ON THE CHOICE OF POINTS ON THE LINE?

EXAMPLE 1 FIND THE GRADIENT OF THE LINE PASSING THROUGH EACH OF THE FOLLOWING POINTS:

A P (-7, 2) AND Q (4, 3)

B A $(\sqrt{2}, 1)$ AND B $(-\sqrt{2}, -3)$

C P (2, -3) AND Q (5, -3)

D A $\left(-\frac{1}{2}, -2\right)$ AND B $\left(\frac{1}{2}, \frac{3}{2}\right)$

SOLUTION:

A $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 2}{4 - (-7)} = \frac{1}{11}$

B $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 1}{-\sqrt{2} - \sqrt{2}} = \frac{-4}{-2\sqrt{2}} = \frac{2}{\sqrt{2}}$

C $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - (-3)}{5 - 2} = \frac{-3 + 3}{3} = \frac{0}{3} = 0$

SO, $m = 0$. IS THE LINE HORIZONTAL? WHAT IS ITS EQUATION?

D $x_1 = -\frac{1}{2}$ AND $x_2 = -\frac{1}{2}$

THE LINE IS VERTICAL. SO IT HAS NO MEASURABLE GRADIENT.

THE EQUATION OF THE LINE IS $x_2 = -\frac{1}{2}$ OR SIMPLY $x = -\frac{1}{2}$

Note: GRADIENT FOR A VERTICAL LINE IS NOT DEFINED.

EXAMPLE 2 CHECK THAT THE LINE THROUGH P (0, 1) AND Q (-1, 4) AND THROUGH

R $\left(\frac{2}{3}, 0\right)$ AND T (1, -1) HAVE SAME GRADIENTS. ARE THE LINES PARALLEL?

SOLUTION: FOR ℓ , $m_1 = \frac{4 - 1}{-1 - 0} = \frac{3}{-1} = -3$. FOR ℓ , $m_2 = \frac{-1 - 0}{1 - \frac{2}{3}} = \frac{-1}{\frac{1}{3}} = -3$.

HERE $m_1 = m_2$. DRAW THE LINES AND SEE IF THEY ARE PARALLEL TO ℓ

Exercise 4.3

1 FIND THE GRADIENTS OF THE LINES PASSING THROUGH THE FOLLOWING POINTS:

A A (4, 3) AND B (8, 11)

B P (3, 7) AND Q (1, 9)

C C $(\sqrt{2}, -9)$ AND D $(-\sqrt{2}, -7)$

D R (-5, -2) AND S (7, -8)

E E (5, 8) AND F (-2, 8)

F H (1, 7) AND K (1, -6)

G R (1, b) AND S (a, 1), $b \neq 1$.

- 2 A (2, -3), B (7, 5) AND C (-2, 9) ARE THE VERTICES OF TRIANGLE ABC. FIND THE GRADIENT OF EACH OF THE SIDES OF THE TRIANGLE.
- 3 GIVEN THREE POINTS P(-5), Q (1, -2) AND R (5, 4), FIND THE GRADIENT OF AND \overline{QR} . WHAT DO YOU CONCLUDE FROM YOUR RESULT?
- 4 USE GRADIENTS TO SHOW THAT THE POINTS P(-7, 12) AND Q(-7, 0) ARE COLLINEAR, I.E., ALL LIE ON THE SAME STRAIGHT LINE.
- 5 SHOW THAT THE LINE PASSING THROUGH THE POINTS $A\left(0, \frac{3}{2}\right)$ ALSO PASSES THROUGH THE POINT Q).

4.3.2 Slope of a Line in Terms of Angle of Inclination

THE ANGLE MEASURED FROM THE ~~AXIS~~ TO A LINE, IN ANTICLOCKWISE DIRECTION, IS CALLED THE **inclination of the line** OR THE ANGLE OF INCLINATION OF THE LINE.

THIS ANGLE IS ALWAYS LESS THAN 180°

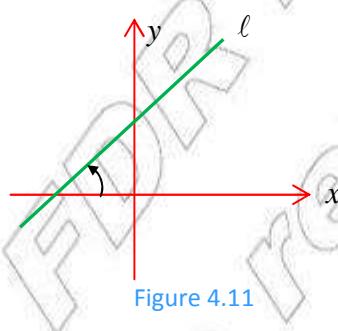


Figure 4.11

Group Work 4.2

CONSIDER THE RIGHT ANGLED TRIANGLE 4.12

- 1 HOW LONG IS THE HYPOTENUSE
- 2 WHAT IS TANGENT OF ANGLE
- 3 WHAT IS MEASURE OF ANGLE BO
- 4 WHAT IS THE ANGLE OF INCLINATION OF LINE ℓ
- 5 WHAT IS THE TANGENT OF THE ANGLE OF INCLINATION?
- 6 BY FINDING THE COORDINATES, CALCULATE THE SLOPE OF LINE ℓ
- 7 WHAT RELATIONSHIP DO YOU SEE BETWEEN YOUR ANSWERS ABOVE? QUESTIONS 5

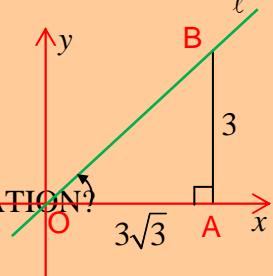


Figure 4.12

THE ABOVE GROUP WORK WILL HELP YOU TO UNDERSTAND THE RELATIONSHIP BETWEEN SLOPE AND ANGLE OF INCLINATION.

FOR A NON-VERTICAL LINE, THE TAN θ IS THE SLOPE OF THE LINE. OBSERVE THE FOLLOWING.

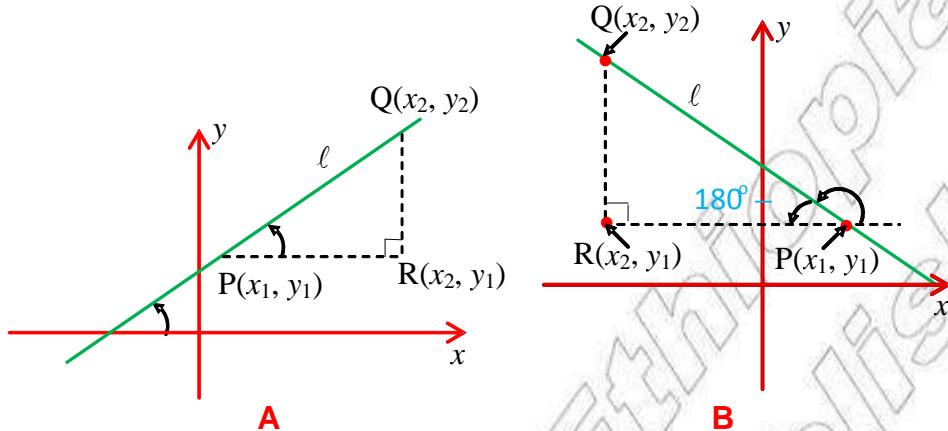


Figure 4.13

IN FIGURE 4.13A ABOVE, y_1 AS y_1 REPRESENTS THE DISTANCE – x_1 REPRESENTS THE DISTANCE PR, THE SLOPE OF THE STRAIGHT LINE ℓ ARE PRESENTED BY THE RATIO

$$m = \frac{RQ}{PR} = \frac{y_2 - y_1}{x_2 - x_1} = \tan(\angle RPO)$$

$$\therefore m = \tan \theta$$

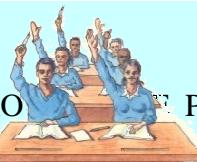
A LINE MAKING AN ACUTE ANGLE OF WITH THE POSITIVE DIRECTION OF THE x HAS POSITIVE SLOPE.

SIMILARLY, A LINE WITH OBTUSE ANGLE, HAS NEGATIVE SLOPE.

$$\text{SLOPE OF } \ell = \frac{RQ}{PR} = \frac{y_2 - y_1}{x_1 - x_2} = -\frac{y_2 - y_1}{x_2 - x_1} = -\tan(180^\circ - \theta) = -(-\tan \theta) = \tan \theta$$

(In Unit 5, this will be clarified)

ACTIVITY 4.6



- 1 HOW WOULD YOU DESCRIBE THE LINE PASSING THROUGH POINTS WITH COORDINATES (x_1, y_1) AND (x_2, y_2) ? IS IT PERPENDICULAR TO THE x -AXIS OR THE y -AXIS? WHAT IS THE TANGENT OF THE ANGLE BETWEEN THIS LINE AND THE x -AXIS?
- 2 SUPPOSE A LINE PASSES THROUGH THE POINTS WITH COORDINATES (x_1, y_1) AND (x_2, y_2) . FIND THE TANGENT OF THE ANGLE FORMED BY THE LINE AND THE x -AXIS.
- 3 WHAT IS THE ANGLE OF INCLINATION OF THE LINE $y = mx + c$?



IN GENERAL, THE SLOPE OF A LINE MAY BE EXPRESSED IN TERMS OF THE COORDINATES (x_1, y_1) AND (x_2, y_2) ON THE LINE AS FOLLOWS:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan \theta, x_2 \neq x_1$$

WHERE θ IS THE ANTICLOCKWISE ANGLE BETWEEN THE X-AXIS AND THE LINE.

EXAMPLE 3 FIND THE SLOPE OF A LINE, IF ITS INCLINATION IS:

A 60°

B 135°

SOLUTION:

A SLOPE : $m = \tan 60^\circ = \sqrt{3}$

B SLOPE : $m = \tan 135^\circ = \tan (180^\circ - 45^\circ) = -\tan 45^\circ = -1$

Note: IF θ IS AN OBTUSE ANGLE, THEN $\tan (180^\circ - \theta)$.

EXAMPLE 4 FIND THE ANGLE OF INCLINATION OF THE LINE

A CONTAINING THE POINTS A(3, -3) AND B(-1, 1)

B CONTAINING THE POINTS C(0, 5) AND D(4, 5).

SOLUTION:

A $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - (-3)}{-1 - 3} = -1$. SO $\tan \theta = -1$ AND HENCE $\theta = 135^\circ$.

B $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 5}{4 - 0} = 0$, $\tan \theta = 0$. SO $\theta = 0^\circ$.

Note: LET m BE THE SLOPE OF A NON-VERTICAL LINE.

- I IF $m > 0$, THEN THE LINE RISES FROM LEFT TO RIGHT AS SHOWN IN FIGURE 4.14A.
- II IF $m < 0$, THEN THE LINE FALLS FROM LEFT TO RIGHT AS SHOWN IN FIGURE 4.14B.
- III IF $m = 0$, THEN THE LINE IS HORIZONTAL AS IN FIGURE 4.14C.

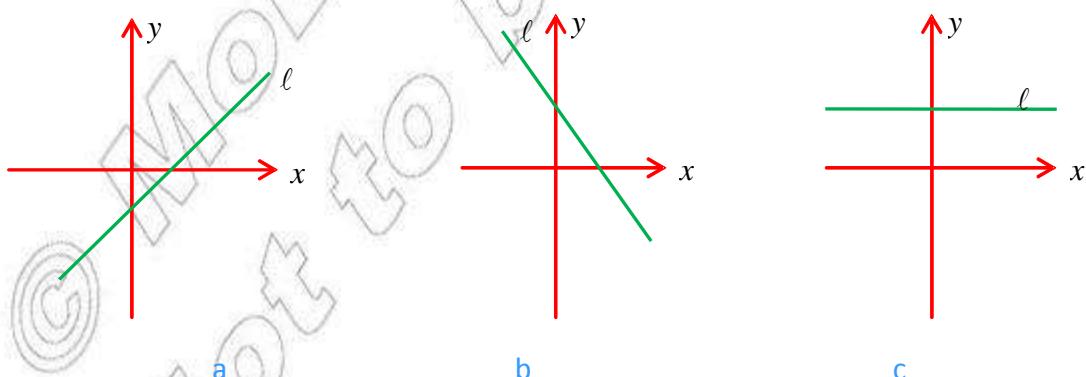


Figure 4.14

Exercise 4.4

- 1** FIND THE SLOPE OF THE LINE WHOSE ANGLE OF INCLINATION IS:
A 30° **B** 75° **C** 150° **D** 90° **E** 0° .
- 2** FIND THE ANGLE OF INCLINATION OF THE LINE IF ITS SLOPE IS:
A $-\sqrt{3}$ **B** $\frac{-\sqrt{3}}{3}$ **C** 1 **D** $\frac{1}{\sqrt{3}}$ **E** 0.
- 3** THE POINTS A(2, 0), B (0, 2) AND C (2, 0) ARE VERTICES OF A TRIANGLE. FIND THE MEASURE OF THE THREE ANGLES OF ~~TRIANGLE~~ OF TRIANGLE IS IT?

4.3.3 Different Forms of Equations of a Line

FROM EUCLIDEAN GEOMETRY, YOU MAY RECALL THAT THERE IS A UNIQUE LINE PASSING THROUGH TWO DISTINCT POINTS. THE EQUATION OF A LINE WHICH IS SATISFIED BY THE COORDINATES OF EVERY POINT ON THE LINE BUT NOT BY THE COORDINATES OF ANY POINT NOT ON THE LINE.

THE EQUATION OF A STRAIGHT LINE CAN BE EXPRESSED IN DIFFERENT FORMS. SOME OF THESE ARE THE POINT-SLOPE FORM, THE SLOPE-INTERCEPT FORM AND THE TWO-POINT FORM.

ACTIVITY 4.7

- 1** SHOW THAT THE GRAPH OF THE LINE $y = 2x$ CONTAINS POINTS
A (2, 0), **B** (2, -1), **C** (2, 2) AND **D** $\left(2, \frac{1}{3}\right)$.
- 2** CONSIDER THE GRAPH OF THE STRAIGHT LINE $y = 2x + 1$. DETERMINE WHICH OF THE FOLLOWING POINTS LIE ON THE LINE.
A (3, -1), **B** (-1, 0), **C** $\left(\frac{-1}{2}, \frac{3}{2}\right)$, **D** (0, 1), **E** $\left(\frac{-1}{2}, 1\right)$, **F** (-2, -1) AND **G** $(\frac{1}{2}, 2)$.
- 3** WHICH OF THE FOLLOWING POINTS LIE ON THE LINE $y = 2x + 1$?
A (-1, 9), **B** (-2, 12), **C** (0, 4), **D** $\left(\frac{2}{5}, 2\right)$, **E** (3, -10).
- 4** WHAT DO YOU CALL THE NUMBER THAT INTERSECTS THE Y-AXIS AT POINT P (0, b)?
- 5** CONSIDER THE GRAPH OF THE STRAIGHT LINE $y = 2x + 1$. DETERMINE ITS Y-INTERCEPT AND X-INTERCEPT.
- 6** GIVE THE EQUATIONS OF THE LINES THROUGH THE POINTS:
A P(-1, 3) AND Q(1, 3) **B** R (-1, 1) AND S(1, -1).



The point-slope form of equation of a line

WE NORMALLY USE THIS FORM OF THE EQUATION OF A LINE WHEN THE SLOPE AND COORDINATES OF A POINT ON IT ARE GIVEN.

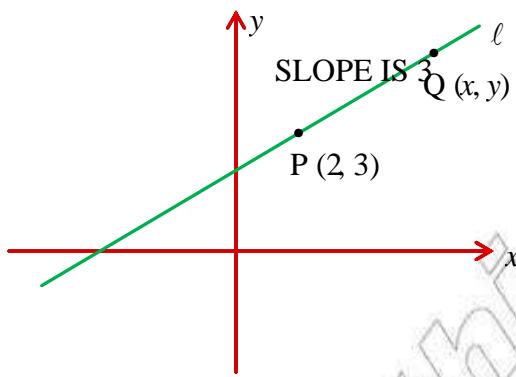


Figure 4.15

SUPPOSE YOU ARE ASKED TO FIND THE EQUATION OF THE STRAIGHT LINE WITH SLOPE 3 THROUGH THE POINT WITH COORDINATE (2, 3).

TAKE P TO BE THE POINT (2, 3) AND, LET Q ANY OTHER POINT ON THE LINE AS SHOWN IN FIGURE 4.15. WHAT IS THE SLOPE OF THE STRAIGHT LINE JOINING THE POINTS WITH COORDINATES (x_1, y_1) AND (x_2, y_2) ?

WHAT IS THE SLOPE \overline{PQ} ? YOU ARE GIVEN THAT THE SLOPE OF THIS LINE IS 3. IF YOU HAVE ANSWERED CORRECTLY, YOU SHOULD OBTAIN

$$y = 3x - 3;$$

WHICH IS THE REQUIRED EQUATION OF THE STRAIGHT LINE.

IN GENERAL, SUPPOSE YOU WANT TO FIND THE EQUATION OF THE STRAIGHT LINE THROUGH THE POINT WITH COORDINATES WHICH HAS SLOPE n , LET THE POINT WITH GIVEN COORDINATES BE A TAKE ANY OTHER POINT ON THE LINE, SAY COORDINATES AS SHOWN IN FIGURE 4.16

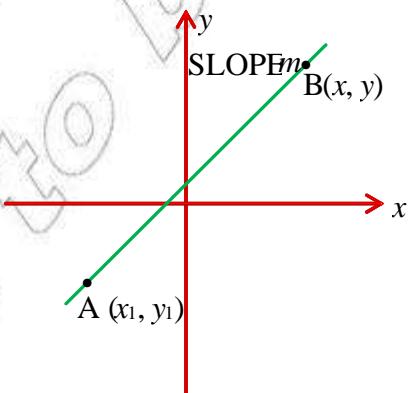


Figure 4.16

THEN THE SLOPE IS $\frac{y - y_1}{x - x_1}$

$\Rightarrow y - y_1 = m(x - x_1)$ WHICH IS THE SAME AS $y = m(x - x_1)$.

THIS EQUATION IS CALLED THE **point-slope form** of the equation of a line

EXAMPLE 5 FIND THE EQUATION OF THE STRAIGHT LINE AND THE SLOPE PASSES THROUGH THE POINT $(-3, 2)$.

SOLUTION: ASSUME THAT THE POINT IS ANY POINT ON THE LINE OTHER THAN $(-3, 2)$.
THUS, USING THE EQUATION $y = m(x - x_1)$

$$\Rightarrow y - 2 = \frac{-3}{2}(x + 3)$$

$$\Rightarrow y = -\frac{3}{2}x - \frac{5}{2} \text{ OR } 2y + 3x + 5 = 0.$$

The slope-intercept form of equation of a line

CONSIDER THE EQUATION $y = b$. WHEN $b = 0$, $y = b$. ALSO, WHEN $b \neq 0$, $y = b$ AS SHOWN IN FIGURE 4.17

YOU CAN SEE THAT $P(0, b)$ IS THE POINT WHERE THE LINE WITH EQUATION $y = b$ CROSSES THE y -AXIS. b IS CALLED THE **y -intercept** OF THE LINE.
LET Q BE $(1, m+b)$.

USING THE COORDINATES OF POINTS P AND Q , SHOW THAT THE SLOPE OF THE STRAIGHT LINE PASSING THROUGH P AND Q IS m .

WRITING THE EQUATION OF THIS LINE THROUGH THE POINT $(0, b)$ WITH SLOPE m USING THE POINT-SLOPE FORM, GIVES

$$y - b = m(x - 0) \Rightarrow y = mx + b$$

WHERE m IS SLOPE OF THE LINE AND b IS y -INTERCEPT OF THE LINE.

THIS EQUATION IS CALLED THE **slope-intercept form** OF THE EQUATION OF A LINE.

Note: THE SLOPE-INTERCEPT FORM OF EQUATION OF A LINE ENABLES US TO FIND THE y -INTERCEPT, ONCE THE EQUATION IS GIVEN.

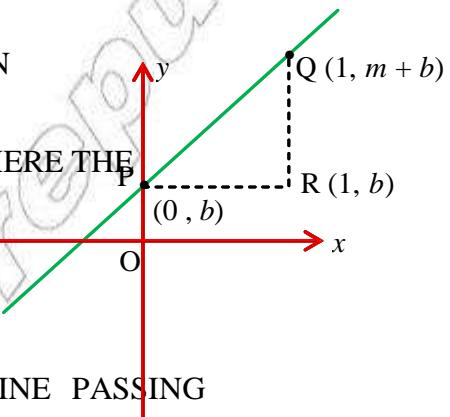


Figure 4.17

EXAMPLE 6 FIND THE EQUATION OF THE LINE $\frac{2}{3}x + y = 3$.

SOLUTION: HERE, $m = \frac{-2}{3}$ AND THE Y-INTERCEPT IS 3.

THEREFORE, THE EQUATION OF THE LINE IS $y = -\frac{2}{3}x + 3$.

The two-point form of equation of a line

FINALLY, LET US LOOK AT THE SITUATION WHERE THE SLOPE IS NOT GIVEN BUT TWO POINTS ON THE LINE ARE GIVEN.

CONSIDER A STRAIGHT LINE WHICH PASSES THROUGH THE POINTS IF $P(x_1, y_1)$ AND $Q(x_2, y_2)$. IF $R(x, y)$ IS ANY POINT ON THE LINE OTHER THAN $Q(x_2, y_2)$, THEN THE SLOPE OF

$$m = \frac{y - y_1}{x - x_1}, x \neq x_1$$

AND THE SLOPE OF

$$m = \frac{y_2 - y_1}{x_2 - x_1}, x_1 \neq x_2$$

BUT THE SLOPE OF THE SLOPE OF

$$\therefore \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

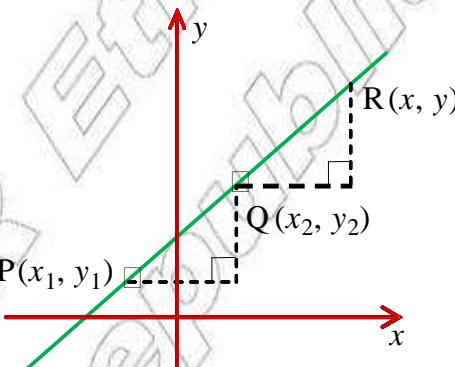


Figure 4.18

THIS EQUATION IS CALLED THE **two-point form** of the equation of a line.

EXAMPLE 7 FIND THE EQUATION OF THE LINE PASSING THROUGH THE POINTS $P(-1, 5)$ AND $Q(3, 13)$.

SOLUTION: TAKING $(-1, 5)$ AS (x_1, y_1) AND $(3, 13)$ AS (x_2, y_2) , USE THE TWO-POINT FORM TO GET THE EQUATION OF THE LINE TO BE

$$y - 5 = \frac{13 - 5}{3 + 1}(x + 1) = 2x + 2 \text{ WHICH IMPLIES } 2x - y + 7 = 0$$

The general equation of a line

A FIRST DEGREE (LINEAR) EQUATION IS AN EQUATION OF THE FORM;

$$Ax + By + C = 0$$

WHERE A AND C ARE FIXED REAL NUMBERS SUCH THAT $A \neq 0$

ALL THE DIFFERENT FORMS OF EQUATIONS OF LINES DISCUSSED ABOVE CAN BE EXPRESSED IN THE FORM

$$Ax + By + C = 0$$

CONVERSELY, ONE CAN SHOW THAT ANY LINEAR EQUATION OF A LINE SUPPOSE A LINEAR EQUATION IS GIVEN AS

$$Ax + By + C = 0.$$

IF $B \neq 0$, THEN THE EQUATION MAY BE SOLVED AS FOLLOWS:

$$Ax + By + C = 0$$

$$By = -Ax - C$$

$$y = \frac{-A}{B}x - \frac{C}{B}$$

THIS EQUATION IS OF THE FORM $y = mx + b$, AND THEREFORE REPRESENTS A STRAIGHT LINE WITH SLOPE $m = -\frac{A}{B}$ AND INTERCEPT $b = -\frac{C}{B}$.

WHAT WILL BE THE EQUATION $Ax + By + C = 0$, IF $B \neq 0$ AND $A \neq 0$?

EXAMPLE 8 FIND THE SLOPE AND INTERCEPT OF THE LINE WHOSE GENERAL EQUATION IS $3x - 6y - 4 = 0$.

SOLUTION: SOLVING FOR y IN THE EQUATION $3x - 6y - 4 = 0$ GIVES,

$$-6y = -3x + 4 \Rightarrow y = \frac{-3x}{-6} + \frac{4}{-6} = \frac{1}{2}x - \frac{2}{3}$$

SO, THE SLOPE IS $\frac{1}{2}$ AND THE INTERCEPT IS $-\frac{2}{3}$.

EXAMPLE 9 WHAT IS THE EQUATION OF THE LINE PASSING THROUGH $(-2, 0)$ AND $(0, 5)$?

SOLUTION: USING TWO-POINT FORM:

$$y - 0 = \frac{5 - 0}{0 - (-2)}(x + 2)$$

WHICH GIVES US, $2y + 10 = 0$ AS THE EQUATION OF THE LINE.

Exercise 4.5

1 FIND THE EQUATION OF THE LINE PASSING THROUGH THE GIVEN POINTS.

- | | | | |
|----------|------------------------------|----------|---|
| A | A $(-2, -4)$ AND B $(1, 5)$ | B | C $(2, -4)$ AND D $(1, 5)$ |
| C | E $(3, 7)$ AND F $(8, 7)$ | D | G $(1, 1)$ AND H $(1 + \sqrt{2}, 1 - \sqrt{2})$ |
| E | P $(-1, 0)$ AND THE ORIGIN | F | Q $(4, -1)$ AND R $(4, -4)$ |
| G | M $(\ , \)$ AND N $(3, -5)$ | H | T $\left(1\frac{1}{2}, -\frac{5}{2}\right)$ AND S $\left(-\frac{3}{2}, \ , \right)$ |

2 FIND THE EQUATION OF THE LINE PASSING THROUGH THE GIVEN POINT

A $m = \frac{3}{2}$; P (0, -6) B $m = 0$; P $\left(\frac{1}{2}, \frac{-1}{4}\right)$

C $m = 1\frac{2}{3}$; P (1, 1) D $m = -$; P (0, 0)

E $m = \sqrt{2}$; P $(\sqrt{2}, -\sqrt{2})$ F $m = -1$; P $\left(\frac{1}{3}, \frac{3}{2}\right)$.

3 FIND THE EQUATION OF THE LINE WITH SLOPE b .

A $m = 0.1$; $b = 0$ B $m = -\sqrt{2}$; $b = -1$ C $m = -$; $b = 2$

D $m = 1\frac{1}{3}$; $b = \frac{-5}{3}$ E $m = \frac{-1}{4}$; $b = 5$ F $m = \frac{2}{3}$; $b = 1.5$

4 SUPPOSE A LINE HAS INTERCEPT AND INTERCEPT FOR, $b \neq 0$; SHOW THAT THE EQUATION OF THE LINE IS $\frac{x}{a} + \frac{y}{b} = 1$.

5 FOR EACH OF THE FOLLOWING EQUATIONS, FIND THE SLOPE AND y -INTERCEPT.

A $\frac{3}{5}x - \frac{4}{5}y + 8 = 0$ B $-y + 2 = 0$ C $2x - 3y + 5 = 0$

D $x + \frac{1}{2}y - 2 = 0$ E $y + 2 = 2(x - 3y + 1)$.

6 A LINE PASSES THROUGH THE POINTS A (5, -1) AND B (-3, 3). FIND:

A THE POINT-SLOPE FORM OF THE EQUATION OF THE LINE.

B THE SLOPE-INTERCEPT FORM OF THE EQUATION OF THE LINE.

C THE TWO-POINT FORM OF THE EQUATION OF THE LINE. WHAT IS ITS GENERAL FORM?

7 FIND THE SLOPE AND y -INTERCEPT, IF THE EQUATION OF THE LINE IS:

A $\frac{1}{3}x - \frac{2}{3}y + 1 = y + x$ B $3(y - 2x) = y + \frac{1}{2}(1 - 2x)$.

8 A TRIANGLE HAS VERTICES AT A (-1, 1), B (1, 3) AND C (3, 1).

A FIND THE EQUATIONS OF THE LINES CONTAINING THE SIDES OF THE TRIANGLE.

B IS THE TRIANGLE A RIGHT-ANGLED TRIANGLE?

C WHAT ARE THE INTERCEPTS OF THE LINE PASSING THROUGH B

4.4**PARALLEL AND PERPENDICULAR LINES**

SLOPES CAN BE USED TO SEE WHETHER TWO NON-VERTICAL LINES IN A PLANE ARE PARALLEL, PERPENDICULAR, OR NEITHER.

FOR INSTANCE, THE LINES $y = x + 3$ ARE PARALLEL AND THE LINES $y = -x$ ARE PERPENDICULAR. HOW ARE THE SLOPES RELATED?

ACTIVITY 4.8

- 1** WHAT IS MEANT BY TWO LINES BEING PARALLEL? PERPENDICULAR?
- 2** IN FIGURE 4.19, ℓ_1 AND ℓ_2 ARE PARALLEL.
 - A** CALCULATE THE SLOPE OF EACH LINE. FIND THE EQUATION OF EACH LINE.
 - C** DISCUSS HOW THEIR SLOPES ARE RELATED.

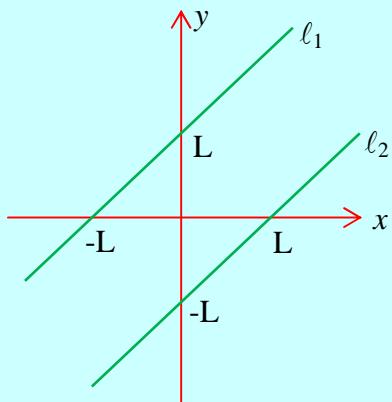


Figure 4.19

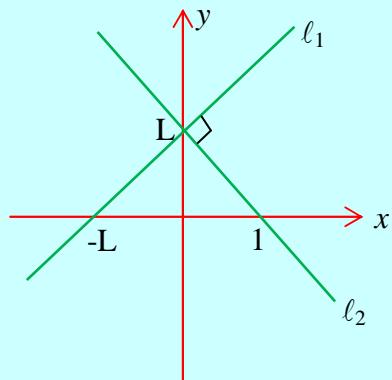


Figure 4.20

- 3** IN FIGURE 4.20 ABOVE, ℓ_1 AND ℓ_2 ARE PERPENDICULAR.
 - A** CALCULATE THE SLOPE OF EACH LINE. FIND THE EQUATION OF EACH LINE.
 - C** DISCUSS HOW THEIR SLOPES ARE RELATED.

Theorem 4.1

If two non-vertical lines ℓ_1 and ℓ_2 are parallel to each other, then they have the same slope.

SUPPOSE YOU HAVE TWO NON-VERTICAL LINES ℓ_1 AND ℓ_2 WITH SLOPES m_1 AND m_2 , RESPECTIVELY AS SHOWN IN FIGURE 4.21

IF ℓ IS PARALLEL, THEN = (WHY?)

CONSEQUENTLY, $\tan = \tan = m_2$

State and prove the converse of the above theorem.

What can be stated for two vertical lines? Are they parallel?

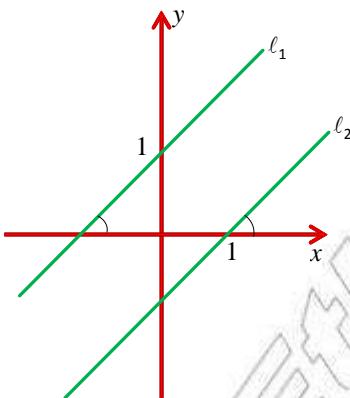


Figure 4.21

EXAMPLE 1 SHOW THAT THE LINE PASSING THROUGH A (2, -3) AND B (2, -1) IS PARALLEL TO THE LINE PASSING THROUGH P (3, -2) AND Q (3, -6).

SOLUTION: SLOPE OF $\overleftrightarrow{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - (-1)}{2 - (-1)} = \frac{-3 + 1}{2 + 1} = -\frac{2}{3}$

SLOPE OF $\overleftrightarrow{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6 - (-2)}{3 - (-3)} = \frac{-6 + 2}{3 + 3} = -\frac{2}{3}$

SINCE \overleftrightarrow{AB} AND \overleftrightarrow{PQ} HAVE THE SAME SLOPES, PARALLEL TOE. $\overleftrightarrow{AB} \parallel \overleftrightarrow{PQ}$

RECALL THAT TWO LINES ARE PERPENDICULAR, IF THEY FORM A RIGHT-ANGLE AT INTERSECTION.

Theorem 4.2

Two non-vertical lines having slopes m_1 and m_2 are perpendicular, if and only if $m_1 \cdot m_2 = -1$.

Proof: SUPPOSE IS PERPENDICULAR TO ℓ

Note: IF ONE OF THE LINES IS A VERTICAL LINE, THEN THE OTHER ONE MUST BE WHICH HAS SLOPE ZERO. SO, ASSUME THAT NEITHER LINE IS VERTICAL.

LET m_1 AND m_2 BE THE SLOPES OF ℓ_1 AND ℓ_2 RESPECTIVELY.

LET R(x_0, y_0) BE THE POINT OF INTERSECTION AND CHOOSE P(x_1, y_1) ON ℓ_1 AND Q(x_2, y_2) ON ℓ_2 , RESPECTIVELY.

DRAW TRIANGLE $\triangle QSR$ AND $\triangle RTP$ AS SHOWN IN FIGURE 4.22.

$\triangle QSR$ AND $\triangle RTP$ ARE SIMILAR, (WHY?)

$$\frac{PT}{RT} = \frac{RS}{QS} \quad (\text{WHY?})$$

$$\frac{y_1 - y_o}{x_1 - x_o} = \frac{x_o - x_2}{y_2 - y_o} = -\left(\frac{x_2 - x_o}{y_2 - y_o}\right)$$

$$\frac{y_1 - y_o}{x_1 - x_o} = \frac{-1}{\frac{y_2 - y_o}{x_2 - x_o}}$$

$$m_1 = -\frac{1}{m_2} \quad \text{OR} \quad m_1 m_2 = -1$$

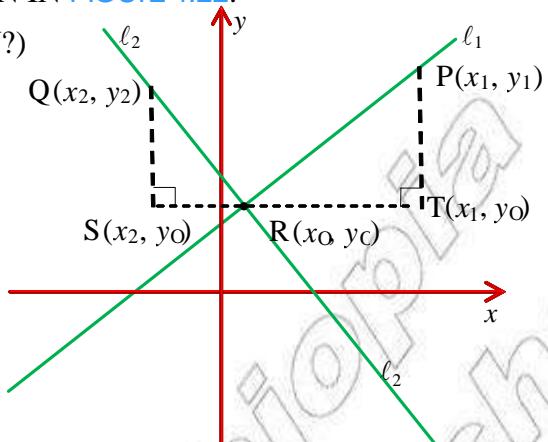


Figure 4.22

AS AN EXERCISE, START WITH $\frac{QS}{RS} = \frac{RT}{PT}$ AND CONCLUDE THAT $\frac{1}{m_1} m_2 = -1$

CONVERSELY, YOU COULD SHOW THAT IF TWO LINES HAVE SLOPES $= -1$, THEN THE LINES ARE PERPENDICULAR. THIS CAN BE DONE BY REVERSING THE ABOVE CONCLUDING THAT THE TWO TRIANGLES ARE SIMILAR. COMPLETE THE PROOF.

EXAMPLE 2 SUPPOSE A LINE l PASSES THROUGH $P(-1, -3)$ AND $Q(2, 6)$. FIND THE SLOPE OF ANY LINE l' THAT IS:

A PARALLEL TO

B PERPENDICULAR TO

SOLUTION: THE SLOPE OF l IS

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - (-3)}{2 - (-1)} = \frac{9}{3} = 3. \text{ SO,}$$

A THE SLOPE OF l' PARALLEL TO l IS $m_1 = 3$

B THE SLOPE OF l' PERPENDICULAR TO l IS $m_2 = -\frac{1}{m_1} = -\frac{1}{3}$

EXAMPLE 3 FIND THE EQUATION OF THE LINE PASSING THROUGH AND PERPENDICULAR TO THE LINE $-3y = -7$.

SOLUTION: FROM $-3y = -7$, $y = \frac{1}{3}x + \frac{7}{3}$ SO, $m_1 = \frac{1}{3}$

LET THE SLOPE OF THE REQUIRED LINE BE $m_2 = -1$ GIVES $m_2 = -\frac{1}{m_1} = -3$

THEREFORE THE REQUIRED EQUATION OF THE LINE IS 5) I.E. $y = -3x + 14$.

Exercise 4.6

- 1** IN EACH OF THE FOLLOWING, DETERMINE WHETHER THE LINE IS PARALLEL TO OR PERPENDICULAR TO THE LINE THROUGH
A A (-1, 3) AND B (2, -2) **B** A (-3, 5) AND B (2, -5)
 P (1, 4) AND Q (-2, 9) P (-1, 4) AND Q (1, 5).
- 2** FIND THE SLOPE OF THE LINE THAT IS PERPENDICULAR TO THE LINE PASSING THROUGH P (-3, -2) AND Q (-3, -2).
- 3** USE SLOPE TO SHOW THAT THE QUADRILATERAL WITH VERTICES A (-2), B (-3, 1), C (3, 0) AND D (1, -3) IS A PARALLELOGRAM.
- 4** LET ℓ BE THE LINE WITH EQUATION $2x + 6 = 0$. FIND THE SLOPE-INTERCEPT FORM OF THE EQUATION OF THE LINE THAT PASSES THROUGH AND IS INT P (2, 3).
A PARALLEL TO **B** PERPENDICULAR TO
- 5** FIND THE EQUATION OF A LINE PASSING THROUGH P (-1, 2) AND IS PARALLEL TO THE LINE
A $\ell: 2x - 5y - 4 = 0$; P (-1, 2) **B** $\ell: 3x + 6 = 0$; P (4, -6).
- 6** DETERMINE WHICH OF THE FOLLOWING PAIRS OF EQUATIONS GIVEN ARE PERPENDICULAR OR PARALLEL OR NEITHER:
A $3x - y + 5 = 0$ AND $x + 3y - 1 = 0$
B $3x - 4y + 1 = 0$ AND $4x - 3y + 1 = 0$
C $4x - 10y + 8 = 0$ AND $10x + 6y - 3 = 0$
D $2x + 2y = 4$ AND $x + y = 10$.
- 7** FIND THE EQUATION OF THE LINE PASSING THROUGH POINTS A (3, 2) AND B (1, 4).
A PARALLEL TO THE LINE PASSING THROUGH POINTS A (3, 2) AND B (1, 4)
B PARALLEL TO THE LINE JOINING THE POINTS A (3, 2) AND B (1, 4)
C PERPENDICULAR TO THE LINE JOINING THE POINTS A (3, 2) AND B (1, 4)
D PERPENDICULAR TO THE LINE JOINING THE POINTS A (3, 2) AND B (1, 4).
- 8** DETERMINE THAT THE LINE WITH EQUATION WILL BE:
A PARALLEL TO THE LINE WITH EQUATION
B PERPENDICULAR TO THE LINE WITH EQUATION
- 9** SHOW THAT THE PLANE FIGURE WITH VERTICES:
A A (6, 1), B (5, 6), C (-4, 3) AND D (-3, -2) IS A PARALLELOGRAM
B A (2, 4), B (1, 5), C (-2, 2) AND D (-1, 1) IS A RECTANGLE.
- 10** THE VERTICES OF A TRIANGLE ARE A (8, 2), B (6, -4) AND C (6, -2). SHOW THAT THE LINE JOINING THE MID-POINTS OF SIDES BC IS PARALLEL TO AND ONE-HALF THE LENGTH OF SIDE AC.



Key Terms

analytic geometry	general equation of a line	point-slope form
angle of inclination	horizontal line	slope (gradient)
coordinate geometry	inclination of a line	slope-intercept form
coordinates	mid-point	steepness
equation of a line	non-vertical line	two-point form



Summary

- 1 IF A POINT HAS COORDINATES, THEN THE NUMBER CALLED **x-coordinate** OR **abscissa** OF P AND IS CALLED **y-coordinate** OR **ordinate** OF P.
- 2 THE **distance d** BETWEEN POINTS P₁ AND Q₂, (x_2, y_2) IS GIVEN BY THE FORMULA

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- 3 THE POINT R₀(y₀) DIVIDING THE LINE SEGMENT **internally**, in the ratio **m:n** IS GIVEN BY

$$R(x_0, y_0) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m} \right),$$

WHERE P₁(y₁) AND Q₂, (x_2, y_2) ARE THE END-POINTS.

- 4 THE **mid-point** OF A LINE SEGMENT WHOSE END-POINTS ARE Q₂, (x_2, y_2) IS GIVEN BY

$$M(x_0, y_0) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

- 5 IF P₁, (x_1, y_1) AND Q₂, (x_2, y_2) ARE POINTS ON A LINE, THEN THE **slope** (gradient) OF THE LINE IS GIVEN BY

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- 6 IF θ IS THE ANGLE BETWEEN THE **axis** AND THE LINE PASSING THROUGH THE POINTS P₁, (x_1, y_1) AND Q₂, (x_2, y_2) , $x_1 \neq x_2$, THEN THE **slope** OF THE LINE IS GIVEN BY

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan \theta$$

- 7 THE GRAPH OF THE EQUATION IS THE **vertical line** THROUGH P₀ AND HAS NO SLOPE.

- 8 THE **equation of the line** WITH SLOPE m AND PASSING THROUGH THE POINTS P₁ (GIVEN BY

$$y - y_1 = m(x - x_1)$$

9 THE EQUATION OF THE LINE WITH SLOPE m AND INTERCEPT b IS GIVEN BY

$$y = mx + b$$

10 THE EQUATION OF THE LINE PASSING THROUGH POINTS $P(x_1, y_1)$ AND $Q(x_2, y_2)$ IS GIVEN BY

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1), x_1 \neq x_2$$

11 THE GRAPH OF EVERY FIRST DEGREE (LINEAR) EQUATION $C = 0$, $A, B \neq 0$ IS A **straight line** AND EVERY STRAIGHT LINE IS A GRAPH OF A FIRST DEGREE EQUATION

12 TWO NON VERTICAL LINES ARE **parallel** AND ONLY IF THEY HAVE THE SAME SLOPE

13 LET ℓ_1 BE A LINE WITH SLOPE m AND ℓ_2 BE A LINE WITH SLOPE n . THEN ℓ_1 AND ℓ_2 ARE **perpendicular** LINES IF AND ONLY IF $m \cdot n = -1$.



Review Exercises on Unit 4

1 SHOW THAT THE POINTS $A(-1)$, $B (-1, 1)$ AND $C(\sqrt{3}, \sqrt{3})$ ARE THE VERTICES OF AN EQUILATERAL TRIANGLE

2 FIND THE COORDINATES OF THE THREE POINTS THAT DIVIDE THE LINE SEGMENT $P(-4, 7)$ AND $Q(10, -9)$ INTO FOUR PARTS OF EQUAL LENGTH

3 FIND THE EQUATION OF THE LINE WHICH PASSES THROUGH $P(-4, -2)$ AND $Q(3, 6)$.

4 FIND THE EQUATION OF THE LINE

A WITH SLOPE -3 THAT PASSES THROUGH $P(8, 3)$.

B WITH SLOPE $\frac{1}{2}$ THAT PASSES THROUGH $Q(5, 0)$.

5 IN EACH OF THE FOLLOWING SHOW THAT THE THREE POINTS ARE VERTICES OF A RIGHT ANGLE TRIANGLE

A $A(0, 0)$, $B(1, 1)$, $C(2, 0)$ **B** $P(3, 1)$, $Q(-3, 4)$, $R(-3, 1)$.

6 FIND THE SLOPE AND INTERCEPT OF THE LINE WITH THE FOLLOWING EQUATIONS:

A $2x - 3y = 4$ **B** $2y - 5x - 2 = 0$

C $5y + 6x - 4 = 0$ **D** $3y = 7x + 1$.

7 FIND THE EQUATION OF THE STRAIGHT LINE PASSING THROUGH

A PARALLEL TO THE LINE WITH THE EQUATION $2x$

B PERPENDICULAR TO THE LINE WITH THE EQUATION 50 .

8 LET ℓ BE THE LINE THROUGH $A(4, 5)$ AND $B(3, t)$ THAT IS PERPENDICULAR TO THE LINE THROUGH $P(1, 3)$ AND $Q(-4, 2)$. FIND THE VALUE OF t

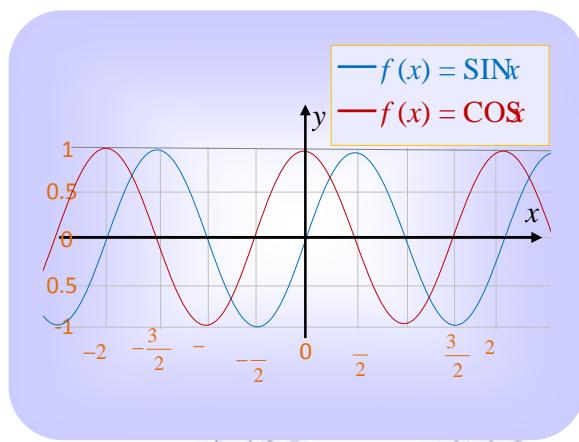
9 LET ℓ BE THE LINE THROUGH $A(3)$ AND $B(t, -2)$ THAT IS PARALLEL TO THE LINE THROUGH $P(-2, 4)$ AND $Q(4, -1)$. FIND THE VALUE OF t

10 PROVE THAT THE CONDITION FOR LINES $Ax + by + c = 0$ AND $ax + by + c = 0$ TO BE PERPENDICULAR MAY BE WRITTEN IN THE FORM

$$Aa + Bb = 0, \text{ WHERE } B \neq 0.$$

5

Unit



TRIGONOMETRIC FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- know principles and methods for sketching graphs of basic trigonometric functions.
- understand important facts about reciprocals of basic trigonometric functions.
- identify trigonometric identities.
- solve real life problems involving trigonometric functions.

Main Contents

- 5.1 Basic trigonometric functions**
- 5.2 The reciprocals of the basic trigonometric functions**
- 5.3 Simple trigonometric identities**
- 5.4 Real life application problems**

Key Terms

Summary

Review Exercises

INTRODUCTION

IN MATHEMATICS, THE trigonometric functions (ALSO CALLED CIRCULAR FUNCTIONS) ARE FUNCTIONS OF ANGLES. THEY WERE ORIGINALLY USED TO RELATE THE ANGLES OF A TRIANGLE TO THE LENGTHS OF THE SIDES OF A TRIANGLE. LOOSELY, TRIGONOMETRY MEANS triangle measure. TRIGONOMETRIC FUNCTIONS ARE HIGHLY USEFUL IN STUDYING MANY DIFFERENT PHENOMENA IN REAL LIFE.

THE MOST FAMILIAR TRIGONOMETRIC FUNCTIONS ARE tangent. IN THIS UNIT, YOU WILL BE STUDYING THE PROPERTIES OF THESE FUNCTIONS IN DETAIL, INCLUDING AND SOME PRACTICAL APPLICATIONS. ALSO, YOU WILL EXTEND YOUR STUDY WITH ANOTHER THREE MORE TRIGONOMETRIC FUNCTIONS.

5.1

BASIC TRIGONOMETRIC FUNCTIONS

HISTORICAL NOTE:

Astronomy led to the development of trigonometry. The Greek astronomer Hipparchus (140 BC) is credited for being the originator of trigonometry. To aid his calculations regarding astronomy, he produced a table of numbers in which the lengths of chords of a circle were related to the length of the radius.

Ptolemy, another great Greek astronomer of the time, extended this table in his major published work

Hipparchus (190-120 BC)

Almagest which was used by astronomers for the next 1000 years. In fact much of Hipparchus' work is known through the writings of Ptolemy. These writings found their way to Hindu and Arab scholars.

Aryabhata, a Hindu mathematician in the 6th century AD, drew up a table of the lengths of half-chords of a circle with radius one unit. Aryabhata actually drew up the first table of sine values.

In the late 16th century, Rhaeticus produced a comprehensive and remarkably accurate table of all the six trigonometric functions. These involved a tremendous number of tedious calculations, all without the aid of calculators or computers.



OPENING PROBLEM

FROM AN OBSERVER O, THE ANGLES OF ELEVATION OF THE BOTTOM AND THE TOP OF A FLAGPOLE ARE 36° AND 38° RESPECTIVELY. FIND THE HEIGHT OF THE FLAGPOLE.

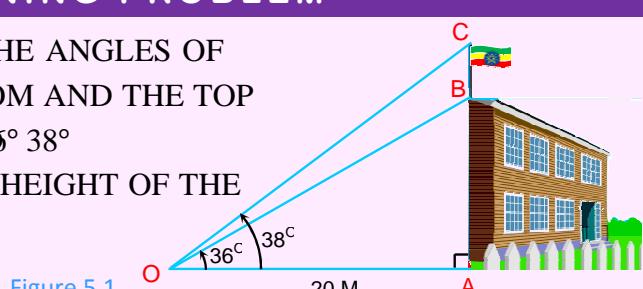


Figure 5.1

5.1.1 The Sine, Cosine and Tangent Functions

Basic terminologies

IF A GIVEN RAY WRITTEN AS ROTATES AROUND A POINT O FROM ITS INITIAL POSITION NEW POSITION, IT FORMS ANS AS SHOWN BELOW.

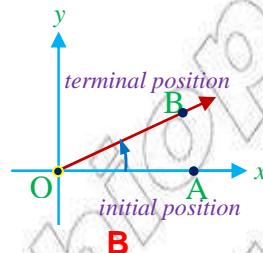
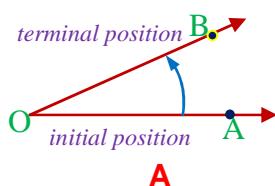


Figure 5.2

\overrightarrow{OA} (INITIAL POSITION) IS CALLED THE ~~INITIAL SIDE~~ OF .

\overrightarrow{OB} (TERMINAL POSITION) IS CALLED THE ~~TERMINAL SIDE~~ OF .

THE ANGLE FORMED BY A RAY ROTATING ANTICLOCKWISE IS TAKEN TO BE A POSITIVE ANGLE. AN ANGLE FORMED BY A RAY ROTATING CLOCKWISE IS TAKEN TO BE A NEGATIVE ANGLE.

EXAMPLE 1

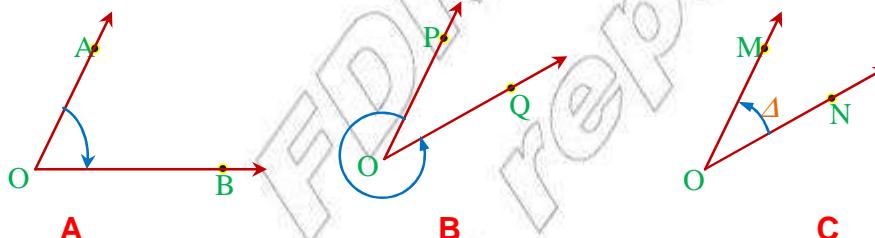


Figure 5.3

- ✓ ANGLE IN FIGURE 5.3A IS A NEGATIVE ANGLE WITH ~~INITIAL SIDE~~ AND TERMINAL SIDE \overrightarrow{OB}
- ✓ ANGLE IN FIGURE 5.3B IS A POSITIVE ANGLE WITH ~~INITIAL SIDE~~ AND TERMINAL SIDE \overrightarrow{OQ}
- ✓ ANGLE IN FIGURE 5.3C IS A POSITIVE ANGLE WITH ~~INITIAL SIDE~~ AND TERMINAL SIDE \overrightarrow{OM}

Angles in standard position

AN ANGLE IN THE COORDINATE PLANE IS SAID TO BE IN ~~STANDARD~~ POSITION

- 1 ITS VERTEX IS AT THE ORIGIN, AND
- 2 ITS INITIAL SIDE LIES ON THE ~~P~~SITIVE x

EXAMPLE 2 THE FOLLOWING ANGLES ARE ALL IN STANDARD POSITION:

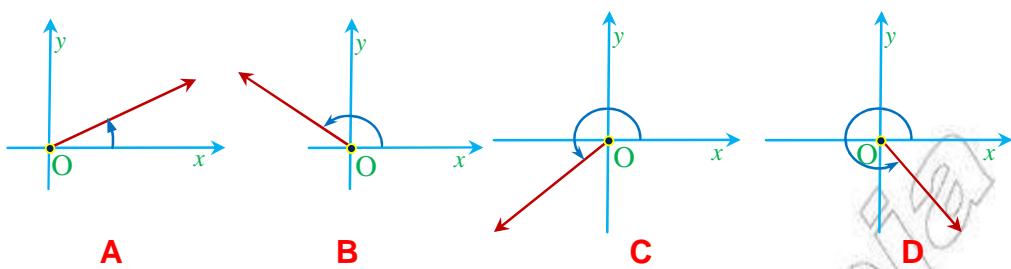


Figure 5.4

First, second, third and fourth quadrant angles

- IF THE TERMINAL SIDE OF AN ANGLE IN STANDARD POSITION LIES IN THE FIRST QUADRANT, THEN IT IS CALLED A **first quadrant angle**
- IF THE TERMINAL SIDE OF AN ANGLE IN STANDARD POSITION LIES IN THE SECOND QUADRANT, THEN IT IS CALLED A **second quadrant angle**
- IF THE TERMINAL SIDE OF AN ANGLE IN STANDARD POSITION LIES IN THE THIRD QUADRANT, THEN IT IS CALLED A **third quadrant angle**.
- IF THE TERMINAL SIDE OF AN ANGLE IN STANDARD POSITION LIES IN THE FOURTH QUADRANT, THEN IT IS CALLED A **fourth quadrant angle**

EXAMPLE 3 THE FOLLOWING ARE ANGLES IN DIFFERENT QUADRANTS:

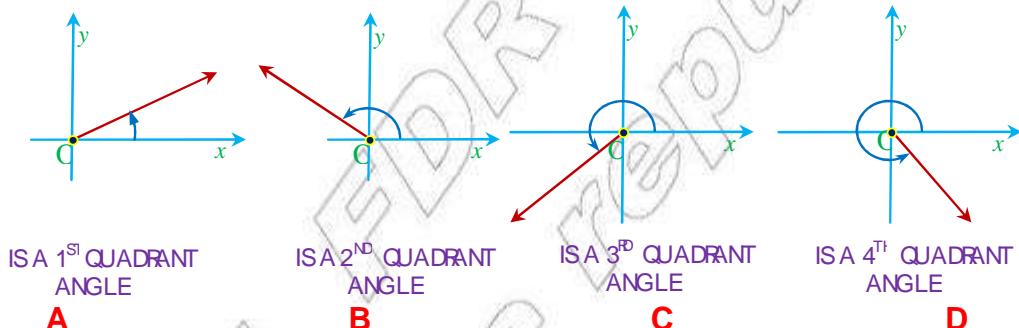
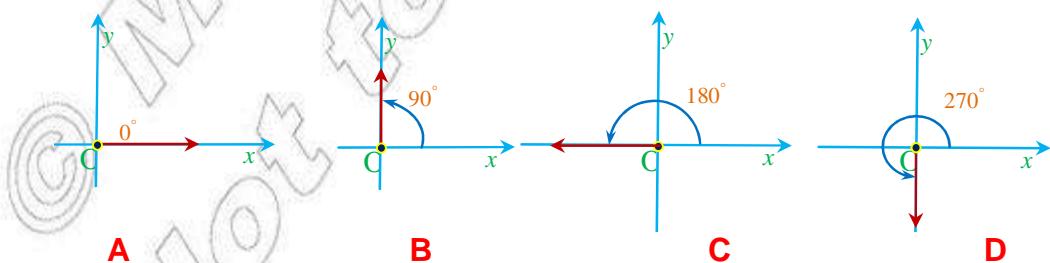


Figure 5.5

Quadrantal angles

IF THE TERMINAL SIDE OF AN ANGLE IN STANDARD POSITION COINCIDES WITH ONE OF THE COORDINATE AXES, THEN THE ANGLE IS **quadrantal angle**.

EXAMPLE 4 THE FOLLOWING ARE ALL QUADRANTAL ANGLES.



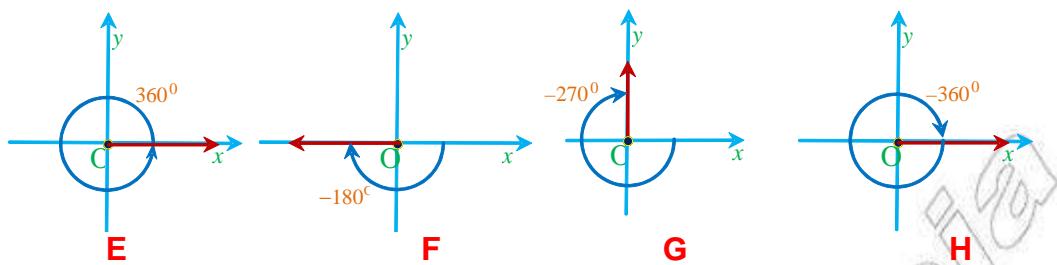


Figure 5.6

ANGLES WITH MEASURES $0^\circ, 90^\circ, 180^\circ, -90^\circ, 0^\circ, 90^\circ, 180^\circ, 270^\circ, 360^\circ$ ARE EXAMPLES OF QUADRANTAL ANGLES BECAUSE THEIR TERMINAL AXIS IS ON THE X-AXIS.

EXAMPLE 5 THE FOLLOWING ARE MEASURES OF DIFFERENT ANGLES IN STANDARD POSITION AND INDICATE TO WHICH QUADRANT THEY BELONG:

A 200°

B 1125°

C -900°

SOLUTION:

A $200^\circ = 180^\circ + 20^\circ$

\therefore AN ANGLE WITH MEASURE OF A THIRD QUADRANT ANGLE.

B $1125^\circ = 3(360^\circ) + 45^\circ$

1125° IS A MEASURE OF A FIRST QUADRANT ANGLE.

C $-900^\circ = 2(-360^\circ) + (-180^\circ)$

-900° IS A MEASURE OF A QUADRANTAL ANGLE.

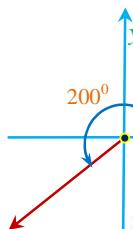


Figure 5.7

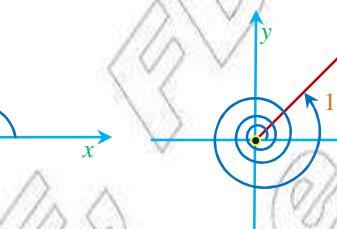


Figure 5.8

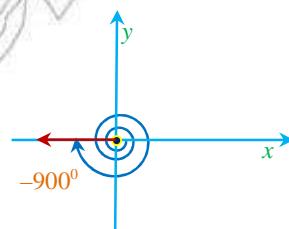


Figure 5.9

Exercise 5.1

THE FOLLOWING ARE MEASURES OF DIFFERENT ANGLES. PUT THE ANGLES IN STANDARD POSITION AND INDICATE TO WHICH QUADRANT THEY BELONG:

A 240°

B 350°

C 620°

D 666°

E -350°

F -480°

G 550°

H -1080°

Radian measure of angles

SO FAR WE HAVE MEASURED ANGLES IN DEGREES. HOWEVER, BECAUSE RADIAN MEASURE IS USED IN SCIENCE, ENGINEERING, AND MATHEMATICS, SCIENTISTS, ENGINEERS, AND MATHEMATICIANS USUALLY WORK WITH ANGLES IN RADIANS.

Group Work 5.1

- 1 DRAW A CIRCLE OF RADIUS 5 CM ON A SHEET OF PAPER.
- 2 USING A THREAD MEASURE THE CIRCUMFERENCE OF THE CIRCLE AND RECORD YOUR RESULT IN CENTIMETRES.
- 3 DIVIDE THE RESULT OBTAINED (LENGTH OF DIAMETER OF THE CIRCLE) AND GIVE YOUR ANSWER IN CENTIMETRES.
- 4 COMPARE THE ANSWER YOU OBTAINED IN 3 WITH THE VALUE OF π .
- 5 USING A THREAD, MEASURE AN ARC LENGTH OF 5 CM ON THE CIRCUMFERENCE OF THE CIRCLE AND NAME THE END POINTS A AND B AS SHOWN IN FIGURE 5.10
- 6 USING YOUR PROTRACTOR MEASURE ANGLE A
- 7 IF YOU REPRESENT THE MEASURE OF THE ANGLE AT THE CENTRE SUBTENDED BY AN ARC EQUAL IN LENGTH TO THE RADIUS AS 1 RADIAN, WHAT WILL BE THE APPROXIMATE 1 RADIAN IN DEGREES?
- 8 CAN YOU APPROXIMATE π IN RADIANS?
- 9 DISCUSS YOUR FINDINGS AND FIND A FORMULA THAT CONVERTS DEGREE MEASURE.

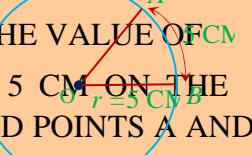


Figure 5.10

THE ANGLE SUBTENDED AT THE CENTRE OF A CIRCLE BY AN ARC EQUAL IN LENGTH TO THE RADIUS IS

1 radian. THAT IS $\frac{r}{r} = 1$ radian. (See FIGURE 5.11A)



Figure 5.11

IN GENERAL, IF THE LENGTH OF THE ARC IS THE RADIUS, THEN $\frac{s}{r}$ RADIANS

(See FIGURE 5.11B) THIS INDICATES THAT THE SIZE OF THE ANGLE IS THE RATIO OF THE ARC LENGTH TO THE LENGTH OF THE RADIUS.

EXAMPLE 6 IF $s = 3$ CM AND $r = 2$ CM, CALCULATE RADIANS.

SOLUTION: $= \frac{s}{r} = \frac{3}{2} = 1.5$ RADIAN

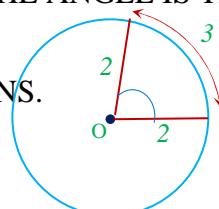


Figure 5.12

EXAMPLE 7 CONVERT 360° RADIANS.

SOLUTION: A CIRCLE WITH RADIUS r UNITS HAS CIRCUMFERENCE $2\pi r$

$$\text{IN THIS CASE } \frac{s}{r} \text{ BECOMES } = \frac{2\pi r}{r} \Rightarrow \pi = 2$$

$$\text{I.E., } 360^\circ = 2\pi \text{ RADIANS.}$$

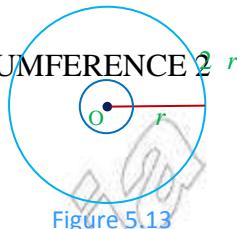


Figure 5.13

EXAMPLE 8 CAN YOU CONVERT 180° TO RADIAN MEASURE?

SOLUTION: SINCE $360^\circ = 2\pi$ RADIANS, $180^\circ = \pi$ RAD ... because $\pi = \frac{360^\circ}{2}$

$$\text{IT FOLLOWS THAT } 1\text{ RAD} = 57.3^\circ$$

Rule 1

TO CONVERT DEGREES TO RADIANS, MULTIPLY BY $\frac{180^\circ}{180^\circ}$

$$\text{I.E., radians} = \text{degrees} \times \frac{180^\circ}{180^\circ}$$

EXAMPLE 9

A CONVERT 30° RADIANS.

B CONVERT 240° RADIANS.

SOLUTION:

$$\text{A } 30^\circ = 30^\circ \times \frac{180^\circ}{180^\circ} = \frac{1}{6} \text{ RADIA.} \quad \text{B } 240^\circ = 240^\circ \times \frac{180^\circ}{180^\circ} = \frac{4}{3} \text{ RADIANS.}$$

Rule 2

TO CONVERT RADIANS TO DEGREES, MULTIPLY BY $\frac{180^\circ}{180^\circ}$

$$\text{I.E., degrees} = \text{radians} \times \frac{180^\circ}{180^\circ}$$

EXAMPLE 10

$$\text{A } -\frac{\pi}{2} \text{ RAD} = -\frac{\pi}{2} \times \frac{180^\circ}{180^\circ} = -90^\circ$$

$$\text{B } -4\pi \text{ RAD} = -4\pi \times \frac{180^\circ}{180^\circ} = -720^\circ$$

Exercise 5.2

1 CONVERT EACH OF THE FOLLOWING radians

$$\text{A } 60^\circ \quad \text{B } 45^\circ \quad \text{C } -150^\circ \quad \text{D } 90^\circ \quad \text{E } -270^\circ \quad \text{F } 135^\circ$$

2 CONVERT EACH OF THE FOLLOWING degrees

$$\text{A } -\frac{\pi}{12} \quad \text{B } -\frac{\pi}{6} \quad \text{C } \frac{2\pi}{3} \quad \text{D } \frac{5\pi}{6} \quad \text{E } -\frac{10\pi}{3} \quad \text{F } 3^\circ$$

Definition of the sine, cosine and tangent functions

THE Sine, Cosine and Tangent Functions ARE THE THREE trigonometric functions.

TRIGONOMETRIC FUNCTIONS ORIGINALLY USED TO RELATE THE ANGLES OF A TRIANGLE TO THE LENGTHS OF THE SIDES OF A TRIANGLE. IT IS FROM THIS PRACTICE OF MEASURING THE ANGLES OF A TRIANGLE WITH THE HELP OF ITS SIDES (OR VICE VERSA) THAT THE NAME TRIGONOMETRY

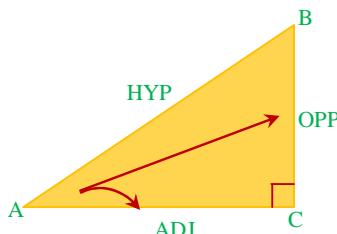


Figure 5.14

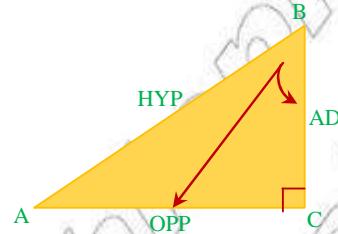


Figure 5.15

LET US CONSIDER THE RIGHT ANGLED TRIANGLE IN FIGURE 5.14.

YOU ALREADY KNOW THAT, FOR A GIVEN RIGHT ANGLED TRIANGLE, THE hypotenuse (HYP) IS THE SIDE WHICH IS OPPOSITE THE RIGHT ANGLE AND IS THE LONGEST SIDE OF THE TRIANGLE.

FOR THE ANGLE MARKED BY FIGURE 5.14

- ✓ \overline{BC} IS THE SIDE opposite (OPP) ANGLE
- ✓ \overline{AC} IS THE SIDE adjacent (ADJ) ANGLE

SIMILARLY, FOR THE ANGLE MARKED BY FIGURE 5.15

- ✓ \overline{AC} IS THE SIDE opposite (OPP) ANGLE
- ✓ \overline{BC} IS THE SIDE adjacent (ADJ) ANGLE

Definition 5.1

If θ is an angle in standard position and $P(a,b)$ is a point on the terminal side of θ , other than the origin $O(0, 0)$, and r is the distance of point P from the origin O , then

$$\sin \theta = \frac{OPP}{HYP} = \frac{b}{r}$$

$$\cos \theta = \frac{ADJ}{HYP} = \frac{a}{r}$$

$$\tan \theta = \frac{OPP}{ADJ} = \frac{b}{a}$$

REMEMBER THAT OQP IS A RIGHT ANGLE TRIANGLE.

(BY THE PYTHAGORASTHEOREM, $r^2 = a^2 + b^2$)

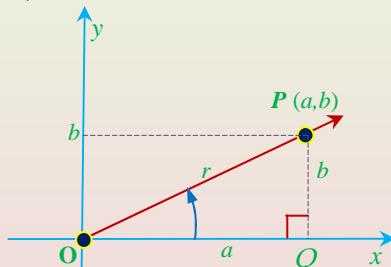


Figure 5.16

(SIN, COS AND TAN ARE ABBREVIATIONS OF SINE AND TANGENT, RESPECTIVELY.)

TRIGONOMETRIC FUNCTIONS CAN BE CONSIDERED IN THE SAME WAY AS ANY GENERAL LINEAR, QUADRATIC, EXPONENTIAL OR LOGARITHMIC.

THE INPUT VALUE FOR A TRIGONOMETRIC FUNCTION IS AN **angle** WHICH COULD BE MEASURED IN DEGREES OR RADIANS. THE OUTPUT VALUE FOR A TRIGONOMETRIC FUNCTION IS A **real number** WITH NO UNIT.

EXAMPLE 11 IF θ IS AN ANGLE IN STANDARD POSITION AND

P (3, 4) IS A POINT ON THE TERMINAL SIDE OF θ , THEN EVALUATE THE SINE, COSINE AND TANGENT OF θ .

SOLUTION: THE DISTANCE $\sqrt{3^2 + 4^2} = 5$ UNITS

$$\text{SO } \sin \theta = \frac{\text{Opp}}{\text{Hyp}} = \frac{4}{5} \quad \cos \theta = \frac{\text{Adj}}{\text{Hyp}} = \frac{3}{5} \text{ AND}$$

$$\tan \theta = \frac{\text{Opp}}{\text{Adj}} = \frac{4}{3}.$$

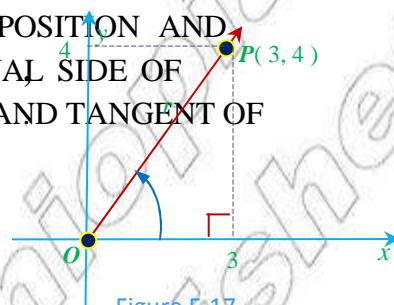


Figure 5.17

Exercise 5.3

EVALUATE THE SINE, COSINE AND TANGENT FUNCTIONS OF AN ANGLE IN STANDARD POSITION AND ITS TERMINAL SIDE CONTAINS THE GIVEN POINT P

A $P(3, -4)$

B $P(-6, -8)$

C $P(1, -1)$

D $P\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

E $P(4\sqrt{5}, -2\sqrt{5})$

F $P(1, 0)$

The unit circle

THE CIRCLE WITH CENTRE AT $(0, 0)$ AND RADIUS 1 UNIT IS CALLED THE **unit circle**

CONSIDER A POINT P(x, y) ON THE CIRCLE [FIGURE 5.18]

SINCE $OP = R$, THEN $\sqrt{(x - 0)^2 + (y - 0)^2} = R$... by distance formula

$$\therefore x^2 + y^2 = R^2 \dots \text{ squaring both sides}$$

WE SAY THAT $x^2 + y^2 = R^2$ IS THE EQUATION OF A CIRCLE WITH CENTRE $(0, 0)$ AND RADIUS R. ACCORDINGLY, THE EQUATION OF THE **unit circle** IS $x^2 + y^2 = 1$. (AS $r = 1$)

LET THE TERMINAL SIDE OF AN ANGLE θ INTERSECT THE **unit circle** AT A POINT P(x, y). SINCE $x^2 + y^2 = 1$, THE **sine**, **cosine** AND **tangent** FUNCTIONS ARE GIVEN AS FOLLOWS:

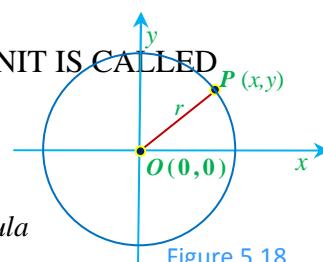


Figure 5.18

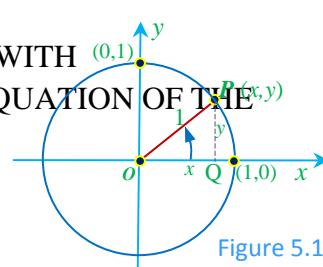


Figure 5.19

$$\sin = \frac{OPP}{HYP} = \frac{y}{r} = \frac{y}{1} = y \quad \dots \text{the } y\text{-coordinate of } P$$

$$\cos = \frac{ADJ}{HYP} = \frac{x}{r} = \frac{x}{1} = x \quad \dots \text{the } x\text{-coordinate of } P$$

$$\tan = \frac{OPP}{ADJ} = \frac{y}{x}$$

EXAMPLE 12 USING THE UNIT CIRCLE, FIND THE VALUES OF SINE, COSINE AND TANGENT OF ;
IF $= 90^\circ, 180^\circ, 270^\circ$.

SOLUTION: AS SHOWN IN FIGURE 5.20, THE TERMINAL SIDE OF ANGLE 90° INTERSECTS THE UNIT CIRCLE AT $(0, 1)$ SO $(0, 1)$.

HENCE, $\sin 90^\circ = 1$, $\cos 90^\circ = 0$ AND $\tan 90^\circ$ UNDEFINED SINCE $\frac{y}{x} = \frac{1}{0}$

THE TERMINAL SIDE OF ANGLE 180° INTERSECTS THE UNIT CIRCLE AT $(-1, 0)$.

(See FIGURE 5.21) SO, $(x, y) = (-1, 0)$.

HENCE, $\sin 180^\circ = 0$, $\cos 180^\circ = -1$ AND $\tan 180^\circ = \frac{y}{x} = \frac{0}{-1} = 0$.

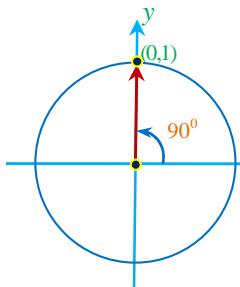


Figure 5.20

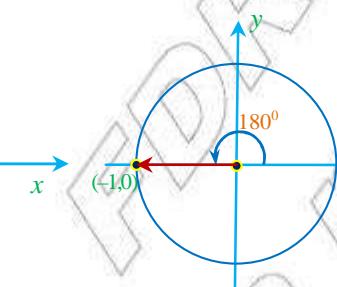


Figure 5.21

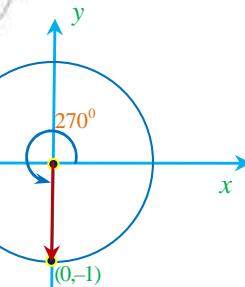


Figure 5.22

THE TERMINAL SIDE OF ANGLE 270° INTERSECTS THE CIRCLE AT $(0, -1)$. SEE FIGURE 5.22
SO $(x, y) = (0, -1)$. HENCE, $\sin 270^\circ = -1$, $\cos 270^\circ = 0$ AND $\tan 270^\circ$ UNDEFINED SINCE $\frac{y}{x} = \frac{-1}{0}$.

Exercise 5.4

- 1 USING THE UNIT CIRCLE, FIND THE VALUES OF THE SINE, COSINE AND TANGENT FOR THE FOLLOWING QUADRANTAL ANGLES:

- | | | |
|----------------------|----------------------|----------------------|
| A 0° | B 360° | C 450° |
| D 540° | E 630° | |

Trigonometric values of 30° , 45° and 60°

THE FOLLOWING GROUP WORK WILL HELP YOU TO FIND THE TRIGONOMETRIC VALUES OF THE SPECIAL ANGLE 45°

Group Work 5.2

CONSIDER THE ISOSCELES RIGHT ANGLE TRIANGLE IN FIGURE 5.23

- A CALCULATE THE LENGTH OF THE HYPOTENUSE A
- B FROM THE PROPERTIES OF AN ISOSCELES RIGHT ANGLE TRIANGLE WHAT IS THE MEASURE OF ANGLE A
- C ARE THE ANGLES A AND B CONGRUENT?
- D WHICH SIDE IS OPPOSITE TO ANGLE A
WHICH SIDE IS ADJACENT TO ANGLE A
- E FIND SIN, COS AND TAN A.

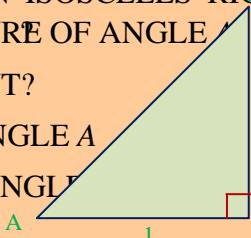


Figure 5.23

FROM GROUP WORK 2 YOU HAVE FOUND THE VALUES SIN, COS AND TAN 45°

ANOTHER WAY OF FINDING THE TRIGONOMETRIC VALUES OF ANGLE IN STANDARD POSITION AS SHOWN IN FIGURE 5.24

WHEN WE PLACE THE ANGLE IN STANDARD POSITION, ITS TERMINAL SIDE INTERSECTS THE CIRCLE AT P.

TO CALCULATE THE COORDINATES OF POINT P, THE TRIANGLE $\triangle OPD$ IS AN ISOSCELES RIGHT ANGLE TRIANGLE.

BY PYTHAGORAS, $(OD)^2 + (PD)^2 = (OP)^2$

SINCE $OD = PD$, $(OD)^2 + (PD)^2 = (OP)^2$.

THAT IS $y^2 = 1^2 \Rightarrow 2y^2 = 1 \Rightarrow y^2 = \frac{1}{2}$

$$\Rightarrow y = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

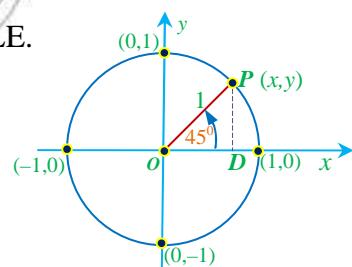


Figure 5.24

SINCE THE TRIANGLE IS ISOSCELES, THE COORDINATES ARE THE SAME.

THEREFORE THE TERMINAL SIDE OF THE ANGLE INTERSECTS THE UNIT CIRCLE AT $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

HENCE, $\sin 45^\circ = \frac{\sqrt{2}}{2}$; $\cos 45^\circ = \frac{\sqrt{2}}{2}$ AND $\tan 45^\circ = \frac{y}{x} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$

Trigonometric values for 30° and 60°

CONSIDER THE EQUILATERAL TRIANGLE WITH SIDE LENGTH 2. THE ALTITUDE \overline{BD} BISECTS $\angle B$ AS WELL AS \overline{AC} . HENCE $\angle ABD = 30^\circ$ AND $AD = 1$ (HALF OF THE LENGTH OF AC).

BY PYTHAGORAS THEOREM, THE LENGTH OF THE ALTITUDE IS h . WHERE

$$h^2 + 1^2 = 2^2 \Rightarrow h^2 = 4 - 1 = 3 \Rightarrow h = \sqrt{3}$$

NOW IN THE RIGHT-ANGLED TRIANGLE ABD,

$$\begin{array}{ll} \sin 30^\circ = \frac{1}{2} = 0.5 & \sin 60^\circ = \frac{\sqrt{3}}{2} \\ \cos 30^\circ = \frac{\sqrt{3}}{2} & \cos 60^\circ = \frac{1}{2} = 0.5 \\ \tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} & \tan 60^\circ = \frac{\sqrt{3}}{1} = \sqrt{3} \end{array}$$

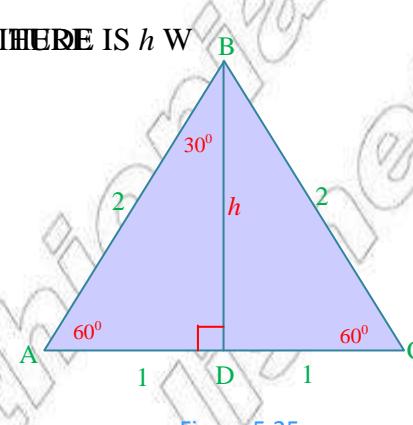


Figure 5.25

Trigonometric values of negative angles

Remember that AN ANGLE IS positive IF MEASURED ANTICLOCKWISE AND negative CLOCKWISE.

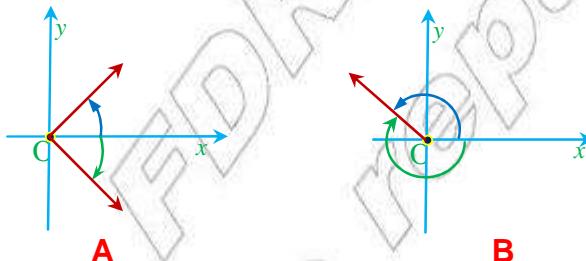


Figure 5.26

IS A POSITIVE ANGLE WHEREAS NEGATIVE ANGLE.

EXAMPLE 13 USING THE UNIT CIRCLE, FIND THE VALUES OF SINE, COSINE AND TANGENT FUNCTIONS OF WHEN $= -180^\circ$.

THE TERMINAL SIDE OF INTERSECTS THE UNIT CIRCLE AT $(-1, 0)$, SO $\theta = -180^\circ$.

HENCE, $\sin(-180^\circ) = y = 0$,

$$\cos(-180^\circ) = x = -1$$

$$\text{AND } \tan(-180^\circ) = \frac{y}{x} = \frac{0}{-1} = 0.$$

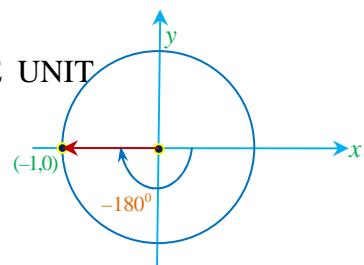


Figure 5.27

EXAMPLE 14 USING THE UNIT CIRCLE, FIND THE VALUES OF THE SINE, COSINE AND TANGENT FUNCTIONS OF WHEN $= -45^\circ$.

SOLUTION: PLACE THE ${}^{\circ} 45$ ANGLE IN STANDARD POSITION. ITS TERMINAL SIDE INTERSECTS THE UNIT CIRCLE AT Q.

TO DETERMINE THE COORDINATES OF Q
PARALLEL TO THE y

$\triangle OQL$ IS AN ISOSCELES RIGHT TRIANGLE.

BY PYTHAGORAS THEOREM, $(OL)^2 + (QL)^2 = (OQ)^2$

SINCE $OL = QL$, $(QL)^2 + (QL)^2 = (OQ)^2$.

THAT IS, $y^2 = 1^2 \Rightarrow 2y^2 = 1 \Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \pm \sqrt{\frac{1}{2}}$

$$\therefore y = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}. \dots \text{Remember that } y \text{ is negative in the fourth quadrant}$$

SINCE THE TRIANGLE IS ISOSCELES, $x = \frac{\sqrt{2}}{2}$

THEREFORE, COORDINATE OF Q IS $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$... Note that x is positive in the fourth quadrant

SO, THE TERMINAL SIDE OF ANGLE ${}^{\circ} 45$ INTERSECTS THE UNIT CIRCLE AT P($\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$)

$$\text{i.e., } (x, y) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right)$$

$$\text{HENCE, } \sin (-{}^{\circ} 45) = \frac{\sqrt{2}}{2}; \cos (-{}^{\circ} 45) = \frac{\sqrt{2}}{2} \text{ AND } \tan (-{}^{\circ} 45) = \frac{y}{x} = \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1.$$

OBSERVE THAT FROM THE TRIGONOMETRIC VALUES OF 45°

$\sin (-45^{\circ}) = -\sin 45^{\circ}$, $\cos (-45^{\circ}) = \cos 45^{\circ}$ AND $\tan(-45^{\circ}) = -\tan 45^{\circ}$.

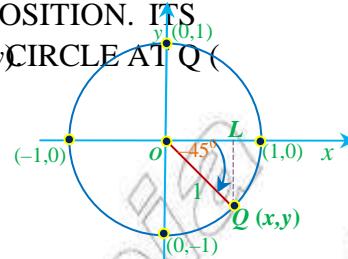


Figure 5.28

ACTIVITY 5.1

- 1 FIND THE VALUES OF THE SINE, COSINE AND TANGENT FUNCTIONS OF COMPLETE THE FOLLOWING TWO TABLES:
(USE A DASH “-” IF IT IS UNDEFINED).



	0°	30°	45°	60°	90°	180°	270°	360°
sin	0				1		-1	
cos						-1		
tan					-			

	-30°	-45°	-60°	-90°	-180°	-270°	-360°
SIN	$-\frac{1}{2}$		$-\frac{\sqrt{3}}{2}$				
COS	$\frac{\sqrt{3}}{2}$		$\frac{1}{2}$	0			
TAN	$-\frac{\sqrt{3}}{3}$		$-\sqrt{3}$	-			



2 WHICH OF THE FOLLOWING PAIRS OF VALUES ARE EQUAL?

- | | |
|--|--|
| A SIN(-30°) AND SIN(30°) | B COS(-30°) AND COS(30°) |
| C TAN(-30°) AND TAN(30°) | D SIN(-45°) AND SIN(45°) |
| E COS(-45°) AND COS(45°) | F TAN(-45°) AND TAN(45°) |
| G SIN(-60°) AND SIN(60°) | H COS(-60°) AND COS(60°) |
| I TAN(-60°) AND TAN(60°) | |

3 HOW DO YOU COMPARE THE VALUES OF:

- | | |
|----------------------------------|--------------------------------|
| A SIN ($-$) AND SIN θ | B COS ($-$) AND COS θ |
| C TAN ($-$) AND $-\tan \theta$ | |

FROM ACTIVITY 5 YOU CONCLUDE THE FOLLOWING:

IF θ IS ANY ANGLE, THEN $\sin(-\theta) = -\sin \theta$, $\cos(-\theta) = \cos \theta$ and $\tan(-\theta) = -\tan \theta$.

LET US REFER TO FIGURE 5.29 TO JUSTIFY THE ABOVE.

$$\sin \theta = \frac{y}{r}, \sin(-\theta) = \frac{-y}{r} = -\left(\frac{y}{r}\right) \therefore \sin(-\theta) = -\sin \theta$$

$$\cos \theta = \frac{x}{r}, \cos(-\theta) = \frac{x}{r} \therefore \cos(-\theta) = \cos \theta$$

$$\tan \theta = \frac{y}{x}, \tan(-\theta) = \frac{-y}{x} = -\left(\frac{y}{x}\right) \therefore \tan(-\theta) = -\tan \theta$$

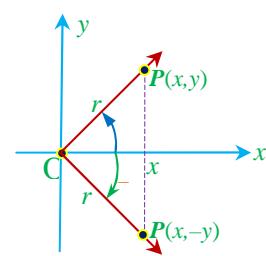


Figure 5.29

5.1.2 Values of Trigonometric Functions for Related Angles

The signs of sine, cosine and tangent functions

IN THIS SUB-SECTION YOU WILL CONSIDER WHETHER THE SIGN OR MAGNITUDE OF THE FUNCTIONS OF AN ANGLE IS POSITIVE OR NEGATIVE.

THE SIGN (WHETHER COS AND TAN ARE POSITIVE OR NEGATIVE) DEPENDS ON THE QUADRANT TO WHICH θ BELONGS.

EXAMPLE 1 CONSIDER AN ANGLE IN THE FIRST AND SECOND QUADRANTS.IF θ IS A FIRST QUADRANT ANGLE, THEN THE SIGN OF

$$\sin = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r} \text{ IS POSITIVE}$$

$$\cos = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r} \text{ IS POSITIVE}$$

$$\tan = \frac{\text{opp}}{\text{adj}} = \frac{y}{x} \text{ IS POSITIVE}$$

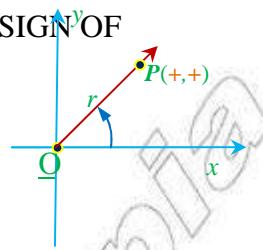


Figure 5.30

IF θ IS A SECOND QUADRANT ANGLE THEN, THE SIGN OF

$$\sin = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r} \text{ IS POSITIVE}$$

$$\cos = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r} \text{ IS NEGATIVE SINCE } x \text{ IS NEGATIVE}$$

$$\tan = \frac{\text{opp}}{\text{adj}} = \frac{y}{x} \text{ IS NEGATIVE}$$

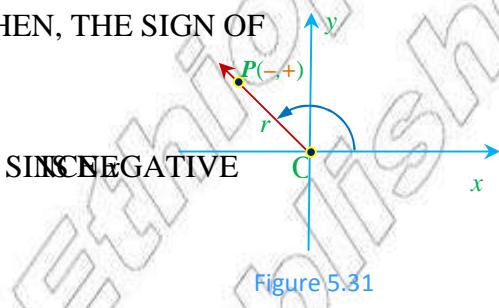


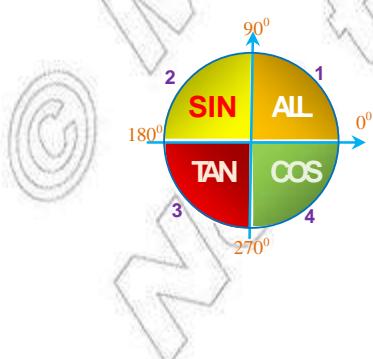
Figure 5.31

ACTIVITY 5.2

- 1** DETERMINE WHETHER THE SIGNS OF SIN, COS, AND TAN ARE POSITIVE OR NEGATIVE:
 - A** IF θ IS A THIRD QUADRANT ANGLE IF θ IS A FOURTH QUADRANT ANGLE
- 2** DECIDE WHETHER THE THREE TRIGONOMETRIC FUNCTIONS ARE POSITIVE OR NEGATIVE. COMPLETE THE FOLLOWING TABLE:

	has terminal side in quadrant			
	I	II	III	IV
sin	+			-
cos		-		
tan			+	

IN GENERAL, THE SIGNS OF THE SINE, COSINE AND TANGENT FUNCTIONS IN ALL OF THE QUADRANTS CAN BE SUMMARIZED AS BELOW:



(x, y): (-, +)	(x, y): (+, +)
SIN IS +	SIN IS +
COS IS -	COS IS +
TAN IS -	TAN IS +

SIN IS -	COS IS +
TAN IS +	TAN IS -

(x, y): (-, -) (x, y): (+, -)

- IN THE FIRST QUADRANT ~~all trigonometric~~ TRIGONOMETRIC FUNCTIONS ARE POSITIVE.
- IN THE SECOND QUADRANT ~~is positive~~ IS POSITIVE.
- IN THE THIRD QUADRANT ~~tangent only~~ TANGENT ONLY IS POSITIVE.
- IN THE FOURTH QUADRANT ~~cosine only~~ COSINE ONLY IS POSITIVE.

Do you want an easy way to remember this? KEEP IN MIND THE FOLLOWING STATEMENT:



TAKING THE FIRST LETTER OF EACH WORD WE HAVE

All	All are positive
Students	Sine is positive
Take	Tangent is positive
Chemistry	Cosine is positive

EXAMPLE 2 DETERMINE THE SIGN OF:

A $\sin 195^\circ$ B $\tan 336^\circ$ C $\cos 895^\circ$

SOLUTION:

- A OBSERVE THAT $180^\circ < 195^\circ < 270^\circ$. SO ANGLE 195° IS A THIRD QUADRANT ANGLE. IN THE THIRD QUADRANT THE SINE FUNCTION IS NEGATIVE.
 $\therefore \sin 195^\circ$ IS NEGATIVE
- B SINCE $270^\circ < 336^\circ < 360^\circ$, THE ANGLE WHOSE MEASURE IS 336° IS A FOURTH QUADRANT ANGLE. IN THE FOURTH QUADRANT THE TANGENT FUNCTION IS NEGATIVE.
HENCE $\tan 336^\circ$ IS NEGATIVE.
- C SINCE $2(360^\circ) < 895^\circ < 2(360^\circ) + 180^\circ$, THE ANGLE WHOSE MEASURE IS 895° IS A SECOND QUADRANT ANGLE. IN THE SECOND QUADRANT THE COSINE FUNCTION IS NEGATIVE.
HENCE $\cos 895^\circ$ IS NEGATIVE.

Group Work 5.3

- DISCUSS AND ANSWER EACH OF THE FOLLOWING:
 - IF $\tan > 0$ AND $\cos < 0$, THEN IS IN QUADRANT _____.
 - IF $\sin > 0$ AND $\cos < 0$, THEN IS IN QUADRANT _____.
 - IF $\cos > 0$ AND $\tan < 0$, THEN IS IN QUADRANT _____.
 - IF $\sin < 0$ AND $\tan < 0$, THEN IS IN QUADRANT _____.
- DETERMINE THE SIGN OF:
 - $\cos 267^\circ$
 - $\tan (-280^\circ)$
 - $\sin (-815^\circ)$
- DETERMINE THE SIGNS OF \cos AND \tan IF P IS AN ANGLE IN STANDARD POSITION AND $P(2, 5)$ IS A POINT ON ITS TERMINAL SIDE.



Complementary angles

ANY TWO ANGLES ARE SAID TO BE COMPLEMENTARY IF THE SUM OF THEIR MEASURES IS EQUAL TO 90° .

EXAMPLE 3 ANGLES WITH MEASURES 30° AND 60° , 20° AND 70° , 40° AND 50° , 45° AND 45° , 10° AND 80° ARE EXAMPLES OF COMPLEMENTARY ANGLES.

ACTIVITY 5.3

1 REFERRING TO FIGURE 5.32

- A FIND $\sin 30^\circ$, $\cos 30^\circ$, $\tan 30^\circ$, $\sin 60^\circ$, $\cos 60^\circ$, $\tan 60^\circ$
- B I COMPARE THE RESULTS $\sin 30^\circ$ AND $\cos 60^\circ$
- II COMPARE THE RESULTS $\sin 30^\circ$ AND $\cos 30^\circ$
- III COMPARE THE RESULTS $\tan 30^\circ$ AND $\tan 60^\circ$

2 REFER TO FIGURE 5.33 ON THE RIGHT AND FIND

- A \sin , \cos , \tan , \sin , \cos AND \tan
- B I COMPARE THE RESULTS $\sin 30^\circ$ AND $\cos 60^\circ$
- II COMPARE THE RESULTS $\sin 30^\circ$ AND $\cos 30^\circ$
- III COMPARE THE RESULTS $\tan 30^\circ$ AND $\tan 60^\circ$
- C WHAT DO YOU CONCLUDE FROM YOUR FINDINGS?



Figure 5.32

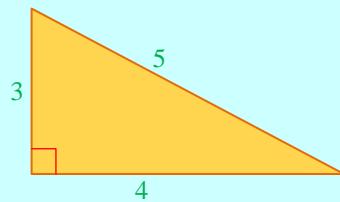


Figure 5.33

FROM ACTIVITY 5.3, THE FOLLOWING RELATIONSHIPS CAN BE CONCLUDED:

IF α AND β ARE COMPLEMENTARY ANGLES, THAT IS,

$(\alpha + \beta = 90^\circ)$ (See FIGURE 5.34), THEN WE HAVE,

$$\sin \alpha = \frac{a}{c} \quad \cos \alpha = \frac{a}{c} \quad \tan \alpha = \frac{b}{a}$$

$$\sin \beta = \frac{b}{c} \quad \cos \beta = \frac{b}{c} \quad \tan \beta = \frac{a}{b} = \frac{1}{\left(\frac{b}{a}\right)}$$

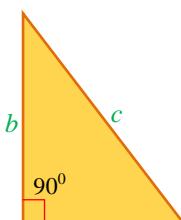


Figure 5.34

HENCE, FOR COMPLEMENTARY ANGLES

$$\sin \alpha = \cos \beta, \cos \alpha = \sin \beta \text{ AND } \tan \alpha = \frac{1}{\tan \beta}.$$

Exercise 5.5

ANSWER EACH OF THE FOLLOWING QUESTIONS:

- A** IF $\sin 31^\circ = 0.5150$, THEN WHAT IS $\cos 59^\circ$
- B** IF $\sin \theta = \frac{3}{5}$, THEN WHAT IS $\cos(90^\circ - \theta)$
- C** IF $\cos \theta = \frac{4}{5}$, THEN WHAT IS $\sin(90^\circ - \theta)$
- D** IF $\sin \theta = k$, THEN WHAT IS $\cos(90^\circ - \theta)$
- E** IF $\cos \theta = r$, THEN WHAT IS $\sin(90^\circ - \theta)$
- F** IF $\tan \theta = \frac{m}{n}$, THEN WHAT IS $\frac{1}{\tan(90^\circ - \theta)}$?

Reference angle(θ_r)

IF θ IS AN ANGLE IN STANDARD POSITION WHOSE TERMINAL SIDE DOES NOT LIE ON A COORDINATE AXIS, THEN A **reference angle** θ_r FOR θ IS THE **acute angle** FORMED BY THE TERMINAL SIDE AND THE X-AXIS AS SHOWN IN THE FOLLOWING FIGURES:

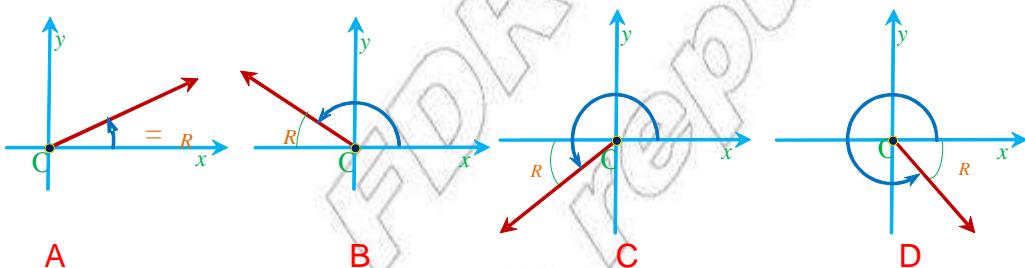


Figure 5.35

EXAMPLE 4 FIND THE REFERENCE ANGLE IF:

$$\text{A} \quad \theta = 110^\circ \quad \text{B} \quad \theta = 212^\circ \quad \text{C} \quad \theta = 280^\circ$$

SOLUTION:

A SINCE $\theta = 110^\circ$ IS A SECOND QUADRANT ANGLE,

$$\theta_r = 180^\circ - 110^\circ = 70^\circ$$

B SINCE $\theta = 212^\circ$ IS A THIRD QUADRANT ANGLE,

$$\theta_r = 212^\circ - 180^\circ = 32^\circ$$

C SINCE $\theta = 280^\circ$ IS A FOURTH QUADRANT ANGLE,

$$\theta_r = 360^\circ - 280^\circ = 80^\circ$$

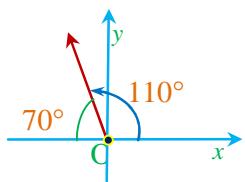


Figure 5.36

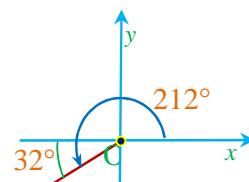


Figure 5.37

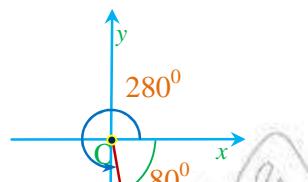


Figure 5.38

Exercise 5.6

FIND THE REFERENCE ANGLE:

- | | | | | | | | |
|----------|---------------|----------|---------------|----------|---------------|----------|---------------|
| A | = 150° | B | = 170° | C | = 240° | D | = 320° |
| E | = 99° | F | = 225° | G | = 315° | H | = 840° |

Values of the trigonometric functions of θ and its reference angle R

LET US CONSIDER A SECOND QUADRANT ANGLE IN STANDARD POSITION AS SHOWN IN THE FIGURE 5.39 AND LET $P(x, y)$ BE A POINT ON ITS TERMINAL SIDE. AS AN AXIS OF SYMMETRY, REFLECT THROUGH y -AXIS. THIS WILL GIVE YOU ANOTHER POINT P' WHICH IS THE IMAGE OF THE TERMINAL SIDE.

THIS IMPLIES THAT $OP = OP' = \sqrt{x^2 + y^2} = r$

$$\begin{aligned} \text{HENCE, } \sin \theta &= \frac{y}{r}, \quad \sin R = \frac{y}{r} \Rightarrow \sin \theta = \sin R \\ \cos \theta &= \frac{-x}{r}, \quad \cos R = \frac{x}{r} \Rightarrow \cos \theta = -\cos R \\ \tan \theta &= \frac{y}{-x} = -\frac{y}{x}, \quad \tan R = \frac{y}{x} \Rightarrow \tan \theta = -\tan R \end{aligned}$$

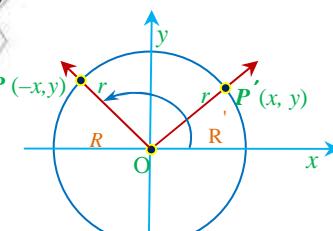


Figure 5.39

THE VALUES OF THE TRIGONOMETRIC FUNCTION OF ANGLES OF THE CORRESPONDING TRIGONOMETRIC FUNCTIONS OF THE REFERENCE ANGLE ABSOLUTE VALUE BUT THEY MAY DIFFER IN SIGN

EXAMPLE 5 EXPRESS THE SINE, COSINE AND TANGENT FUNCTIONS OF ITS REFERENCE ANGLE.

SOLUTION: Remember that an angle with measure θ is a second quadrant angle. In quadrant II, only sine is positive.

THE REFERENCE ANGLE $180^\circ - 160^\circ = 20^\circ$

THEREFORE, $\sin \theta = \sin 160^\circ$, $\cos \theta = -\cos 20^\circ$ AND $\tan \theta = -\tan 20^\circ$

Supplementary angles

TWO ANGLES ARE SAID TO BE **supplementary**, IF THE SUM OF THEIR MEASURES IS EQUAL TO 180°.

EXAMPLE 6 PAIRS OF ANGLES WITH MEASURES OF 120° AND 60° , 45° AND 135° , 75° AND 105° AND 10° AND 170° ARE EXAMPLES OF SUPPLEMENTARY ANGLES.

EXAMPLE 7 FIND THE VALUES OF $\sin 50^\circ$ AND $\tan 50^\circ$

SOLUTION: THE REFERENCE ANGLE $180^\circ - 150^\circ = 30^\circ$

$$\text{ THEREFORE, } \sin 50^\circ = \sin 30^\circ = \frac{1}{2}, \quad \cos 50^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

$$\text{ AND } \tan 50^\circ = -\tan 30^\circ = -\frac{\sqrt{3}}{3}.$$

EXAMPLE 8 FIND THE VALUES OF $\sin 140^\circ$ AND $\tan 240^\circ$

SOLUTION: THE REFERENCE ANGLE $180^\circ - 140^\circ = 60^\circ$

$$\sin 140^\circ = -\sin 60^\circ = -\frac{\sqrt{3}}{2}, \quad \cos 140^\circ = -\cos 60^\circ = -\frac{1}{2} \text{ AND}$$

$$\tan 140^\circ = \tan 60^\circ = \sqrt{3}.$$

... remember that in quadrant III only tangent is positive.

IN GENERAL,

IF IS A SECOND QUADRANT ANGLE, THEN ITS REFERENCE ANGLE WILL BE $(180^\circ - \theta)$

$$\sin \theta = \sin(180^\circ - \theta), \quad \cos \theta = -\cos(180^\circ - \theta), \quad \tan \theta = -\tan(180^\circ - \theta)$$

IF IS A THIRD QUADRANT ANGLE, ITS REFERENCE ANGLE WILL BE $(180^\circ + \theta)$

$$\text{HENCE } \sin \theta = -\sin(180^\circ + \theta), \quad \cos \theta = -\cos(180^\circ + \theta) \text{ AND } \tan \theta = \tan(180^\circ + \theta).$$

Exercise 5.7

1 EXPRESS THE SINE, COSINE AND TANGENT NOTIONS OF THE FOLLOWING ANGLE MEASURES IN TERMS OF THEIR REFERENCE ANGLE:

- | | | |
|----------------|----------------|---------------|
| A 105° | B 175° | C 220° |
| D -260° | E -300° | F 380° |

2 FIND THE VALUES OF:

- | | |
|--|--|
| A $\sin 135^\circ$, $\cos 135^\circ$ AND $\tan 135^\circ$ | B $\cos 143^\circ$ IF $\cos 37^\circ = 0.7986$ |
| C $\tan 138^\circ$ IF $\tan 42^\circ = 0.9004$ | D $\sin 113^\circ$, IF $\sin 65^\circ = 0.9063$ |
| E $\tan 159^\circ$ IF $\tan 21^\circ = 0.3839$ | F $\cos 24^\circ$ IF $\cos 156^\circ = -0.9135$ |

Co-terminal angles

Co-terminal angles ARE ANGLES IN STANDARD POSITION THAT HAVE A COMMON TERMINAL SIDE

EXAMPLE 9

- A THE THREE ANGLES WITH MEASURES 30° , 330° AND 390° ARE CO-TERMINAL ANGLES.
(See FIGURE 5.40)

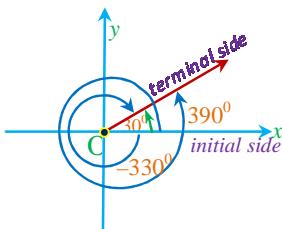


Figure 5.40

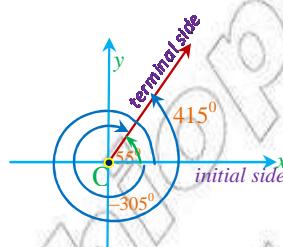


Figure 5.41

- B THE THREE ANGLES WITH MEASURES 55° , 305° AND 415° ARE ALSO CO-TERMINAL.
(See FIGURE 5.41)

ACTIVITY 5.4

- 1 WITH THE HELP OF THE FOLLOWING TABLE FIND ANGLES WHICH ARE CO-TERMINAL WITH 60°



Angles which are co-terminal with 60°	
$60^\circ + 1(360^\circ) = 420^\circ$	$60^\circ - 1(360^\circ) = -300^\circ$
$60^\circ + 2(360^\circ) = 780^\circ$	$60^\circ - 2(360^\circ) = -660^\circ$
_____	_____
_____	_____
_____	_____
$60^\circ + 6(360^\circ) = 2220^\circ$	$60^\circ - 6(360^\circ) = -2100^\circ$
.	.
.	.

- 2 GIVE A FORMULA TO FIND ALL ANGLES WHICH ARE CO-TERMINAL WITH 60°

GIVEN AN ANGLE , ALL ANGLES WHICH ARE CO-TERMINAL WITH IT ARE GIVEN BY THE FORMULA

$$\pm n (360^\circ), \text{ WHERE } n = 1, 2, 3, \dots$$

- EXAMPLE 10 FIND A POSITIVE AND A NEGATIVE ANGLE CO-TERMINAL WITH 75° .

SOLUTION: TO FIND A POSITIVE AND A NEGATIVE ANGLE CO-TERMINAL WITH 75° , YOU CAN ADD OR SUBTRACT 360° . HENCE, $75^\circ - 360^\circ = -285^\circ$; $75^\circ + 360^\circ = 435^\circ$.



THEREFORE, -285° AND 435° ARE CO-TERMINAL WITH 75° .

THERE ARE AN INFINITE NUMBER OF OTHER ANGLES CO-TERMINAL WITH 75° . THEY BY $75^\circ \pm n$ (360°), $n = 1, 2, 3, \dots$

Exercise 5.8

FIND ANY TWO CO-TERMINAL ANGLES (ONE OF THEM POSITIVE AND THE OTHER NEGATIVE) OF THE FOLLOWING ANGLE MEASURES:

- | | | | | | | | |
|----------|-------------|----------|-------------|----------|-------------|----------|-------------|
| A | 70° | B | 110° | C | 220° | D | 270° |
| E | -90° | F | -37° | G | -60° | H | -70° |

Trigonometric values of co-terminal angles

ACTIVITY 5.5

CONSIDER FIGURE 5.42 AND FIND THE TRIGONOMETRIC VALUES OF $P(x, y)$ IS A POINT ON THE TERMINAL SIDE OF BOTH ANGLES.

ANSWER EACH OF THE FOLLOWING QUESTIONS:

- A** ARE AND CO-TERMINAL ANGLES? WHY?
- B** WHICH ANGLE IS POSITIVE? WHICH ANGLE IS NEGATIVE?
- C** FIND THE VALUES OF SIN, TAN IN TERMS OF, r .
- D** FIND THE VALUES OF COS, TAN IN TERMS OF, r .
- E** IS $\sin = \sin$? IS $\cos = \cos$? IS $\tan = \tan$?
- F** WHAT CAN YOU CONCLUDE ABOUT THE TRIGONOMETRIC VALUES?

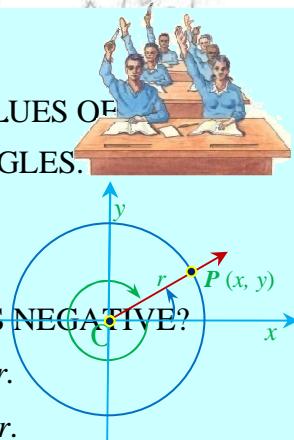


Figure 5.42

CO-TERMINAL ANGLES HAVE THE SAME TRIGONOMETRIC VALUES.

EXAMPLE 11 FIND THE TRIGONOMETRIC VALUES OF

- A** -330° AND 30° **B** 120° AND -240°

SOLUTION:

- A** OBSERVE THAT BOTH ANGLES ARE CO-TERMINAL. THIS IS BECAUSE THE FIRST QUADRANT (FIGURE 5.43).

$-330^\circ = 30^\circ - 1(360^\circ)$. THIS GIVES US:

$$\sin 30^\circ = \sin (-330^\circ) = \frac{1}{2}$$

$$\cos 30^\circ = \cos (-330^\circ) = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \tan (-330^\circ) = \frac{\sqrt{3}}{3}$$

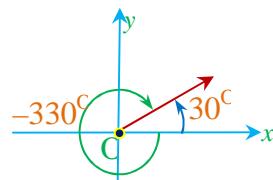


Figure 5.43

- B** BOTH 120° AND -240° ANGLES ARE CO-TERMINAL.
THEIR TERMINAL SIDE LIES IN THE SECOND QUADRANT
(See FIGURE 5.44)

$$-240^\circ = 120^\circ - 360^\circ. \text{ THUS,}$$

$$\sin 120^\circ = \sin (-240^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

... a 60° angle is the reference angle for a 120° angle

$$\cos 120^\circ = \cos (-240^\circ) = -\cos 60^\circ = -\frac{\sqrt{3}}{2}$$

... cosine is negative in quadrant II

$$\tan 120^\circ = \tan (-240^\circ) = -\tan 60^\circ = -\sqrt{3}$$

... tangent is also negative in quadrant II

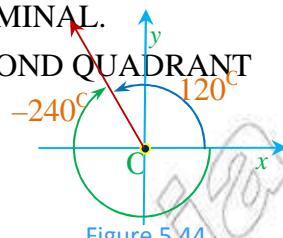


Figure 5.44

Angles larger than 360°

CONSIDER THE ANGLE

$$780^\circ = 360^\circ + 360^\circ + 60^\circ = 2(360^\circ) + 60^\circ$$

... a 60° angle is the co-terminal acute angle for a 780° angle

SINCE AN ANGLE AND ITS CO-TERMINAL HAVE TRIGONOMETRIC VALUE,

$$\sin 780^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}, \cos 780^\circ = \cos 60^\circ = \frac{1}{2}$$

AND $\tan 780^\circ = \tan 60^\circ = \sqrt{3}$.

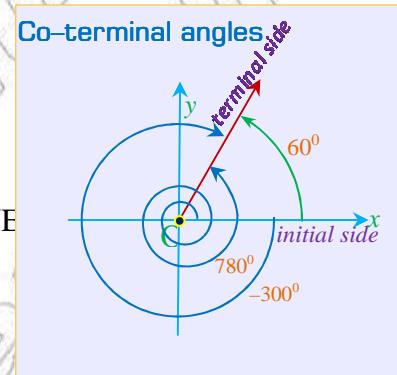


Figure 5.45

(Remember that since 780° is the measure of a first quadrant angle, all three of the functions are positive.)

EXAMPLE 12 FIND THE TRIGONOMETRIC VALUES OF 945°

$$\text{SOLUTION: } 945^\circ = 360^\circ + 360^\circ + 225^\circ = 2(360^\circ) + 225^\circ$$

THIS MEANS 945° AND 225° ARE MEASURES OF TERMINAL 3 QUADRANT ANGLES.

THE REFERENCE ANGLE FOR 225° is $225^\circ - 180^\circ = 45^\circ$.

SINCE AN ANGLE AND ITS CO-TERMINAL HAVE THE SAME TRIGONOMETRIC VALUE, IT FOLLOWS THAT

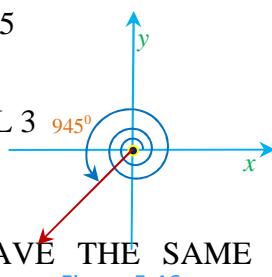


Figure 5.46

$$\sin 945^\circ = \sin 225^\circ = -\sin 45^\circ = -\frac{\sqrt{2}}{2} \quad \dots \text{sine is negative in quadrant III}$$

$$\cos 945^\circ = \cos 225^\circ = -\cos 45^\circ = -\frac{\sqrt{2}}{2} \quad \dots \text{cosine is negative in quadrant III}$$

$$\tan 945^\circ = \tan 225^\circ = \tan 45^\circ = 1 \quad \dots \text{tangent is positive in quadrant III}$$

Exercise 5.9

- 1** FIND THE VALUE OF EACH OF THE FOLLOWING:
- $\sin 390^\circ, \cos 390^\circ, \tan 390^\circ$
 - $\sin (-405^\circ), \cos (-405^\circ), \tan (-405^\circ)$
 - $\sin (-690^\circ), \cos (-690^\circ), \tan (-690^\circ)$
 - $\sin 1395^\circ, \cos 1395^\circ, \tan 1395^\circ$
- 2** EXPRESS EACH OF THE FOLLOWING AS A TRIGONOMETRIC POSITIVE ACUTE ANGLE:
- | | | | |
|----------------------------|------------------------------|------------------------------|----------------------------|
| A $\sin 138^\circ$ | B $\sin 208^\circ$ | C $\cos 168^\circ$ | D $\cos 318^\circ$ |
| E $\tan 325^\circ$ | F $\sin (-108^\circ)$ | G $\cos (-308^\circ)$ | H $\tan 415^\circ$ |
| I $\sin 1348^\circ$ | J $\tan 1125^\circ$ | K $\sin (-338^\circ)$ | L $\cos 1408^\circ$ |

5.1.3 Graphs of the Sine, Cosine and Tangent Functions

IN THIS SECTION, YOU WILL DRAW AND DISCUSS THE GRAPHS OF THE THREE TRIGONOMETRIC FUNCTIONS: SINE, COSINE AND TANGENT.

Graph of the sine function

ACTIVITY 5.6

- 1** COMPLETE THE FOLLOWING TABLE OF VALUES FOR

in deg	-360	-330	-270	-240	-180	-120	-90
$y = \sin$							



in deg	0	90	120	180	240	270	330	360
$y = \sin$								

- 2** MARK THE VALUES OF THE HORIZONTAL AXIS AND THEN THE VERTICAL AXIS AND PLOT THE POINTS YOU FIND.
3 CONNECT THESE POINTS USING A SMOOTH CURVE.
4 WHAT ARE THE DOMAIN AND THE RANGE OF

EXAMPLE 1 DRAW THE GRAPH \sin , WHERE $-360^\circ \leq 360^\circ$

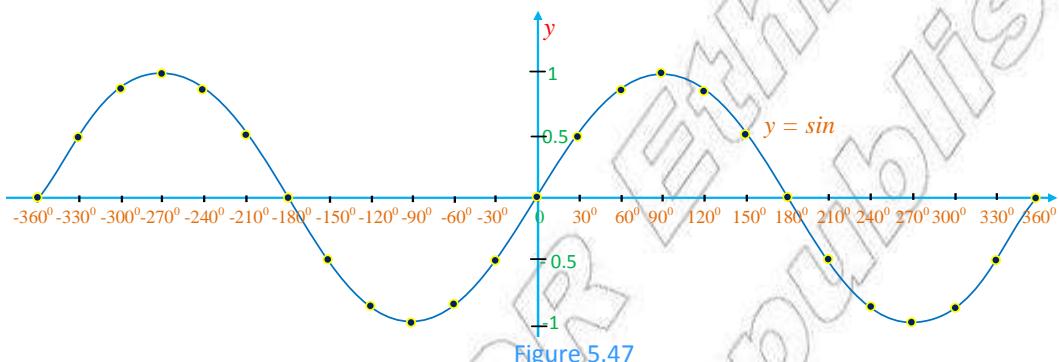
SOLUTION: TO DETERMINE THE GRAPH OF WE CONSTRUCT A TABLE OF VALUES FOR $y = \sin$, WHERE $-360^\circ \leq 360^\circ$ (WHICH IS THE SAME AS \leq IN radians.)

THE TABLES BELOW SHOW SOME OF THE SINE VALUES IN THE GIVEN INTERVAL.

in deg	-360	-330	-300	-270	-240	-210	-180	-150	-120	-90	-60	-30
in rad	-2	$-\frac{11}{6}$	$-\frac{5}{3}$	$-\frac{3}{2}$	$-\frac{4}{3}$	$-\frac{7}{6}$	$-\pi$	$-\frac{5}{6}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{6}$
y = sin	0	0.5	0.87	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5

in deg	0	30	60	90	120	150	180	210	240	270	300	330	360
in rad	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2
y = sin	0	0.5	0.87	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0

TO DRAW THE GRAPH WE MARK THESE VALUES ON THE HORIZONTAL AXIS AND THEN THE VALUES OF THE VERTICAL AXIS. THEN WE PLOT THE POINTS AND CONNECT THEM USING A SMOOTH CURVE.



AFTER A COMPLETE REVOLUTION (OR 360°) THE VALUES OF THE SINE FUNCTION REPEAT THEMSELVES. THIS MEANS

$$\sin \theta = \sin (\theta + 360^\circ) = \sin (\theta + 2(360^\circ)) = \sin (\theta + 3(360^\circ)), \text{ ETC.}$$

$$\sin 9\theta = \sin (9\theta + 360^\circ) = \sin (9\theta + 2(360^\circ)) = \sin (9\theta + 3(360^\circ)), \text{ ETC.}$$

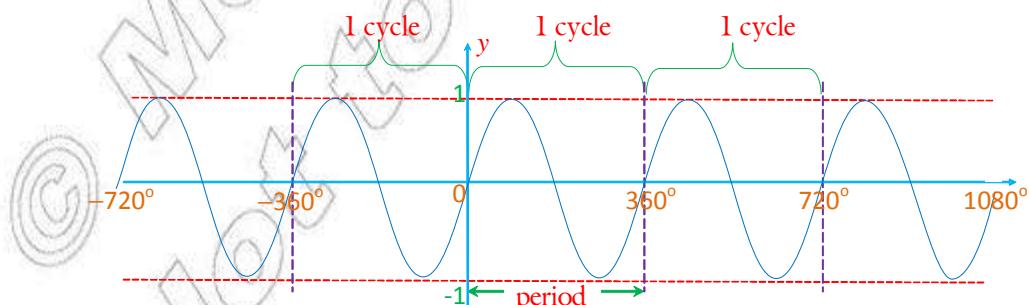
$$\sin 18\theta = \sin (18\theta + 360^\circ) = \sin (18\theta + 2(360^\circ)) = \sin (18\theta + 3(360^\circ)), \text{ ETC.}$$

IN GENERAL, $\sin(\theta + n(360^\circ))$ WHERE n IS AN INTEGER.

A FUNCTION THAT REPEATS ITS VALUES AFTER A CERTAIN PERIOD IS CALLED PERIODIC FUNCTION.

THE SINE FUNCTION REPEATS AFTER 360° OR 2π RADIAN.

THEREFORE, 360° OR 2π IS CALLED PERIOD OF THE SINE FUNCTION.



Domain and range

FOR ANY ANGLE TAKEN ON THE UNIT CIRCLE, THERE IS SOME POINT IN THE TERMINAL SIDE. SINCE $\sin = \frac{y}{1} = y$, THE FUNCTION SIN IS DEFINED FOR ANY ANGLE ON THE UNIT CIRCLE.

THEREFORE, THE DOMAIN OF THE SINE FUNCTION IS THE SET OF ALL REAL NUMBERS. ALSO, NOTE FROM THE GRAPH THAT THE VALUE OF Y IS NEVER LESS THAN -1 OR GREATER

Note: THE DOMAIN OF THE SINE FUNCTION IS \mathbb{R} .
THE RANGE OF THE SINE FUNCTION IS $\{-1\}$

Graph of the cosine function

ACTIVITY 5.7

- 1 COMPLETE THE FOLLOWING TABLES OF VALUES FOR



in deg	-360	-300	-270	-240	-180	-120	-90	-60
$y = \cos$								

in deg	0	60	90	120	180	240	270	300	360
$y = \cos$									

- 2 SKETCH THE GRAPH OF COS.
3 WHAT ARE THE DOMAIN AND THE RANGE OF
4 WHAT IS THE PERIOD OF THE COSINE FUNCTION?

FROM ACTIVITY 5.7 YOU CAN SEE THAT COS IS NEVER LESS THAN -1 OR GREATER THAN +1. JUST LIKE THE SINE FUNCTION, THE COSINE FUNCTION IS PERIODIC. EVERY 360°, THE FUNCTION REPEATS ITSELF. THEREFORE, 360° IS CALLED THE PERIOD OF THE COSINE FUNCTION.

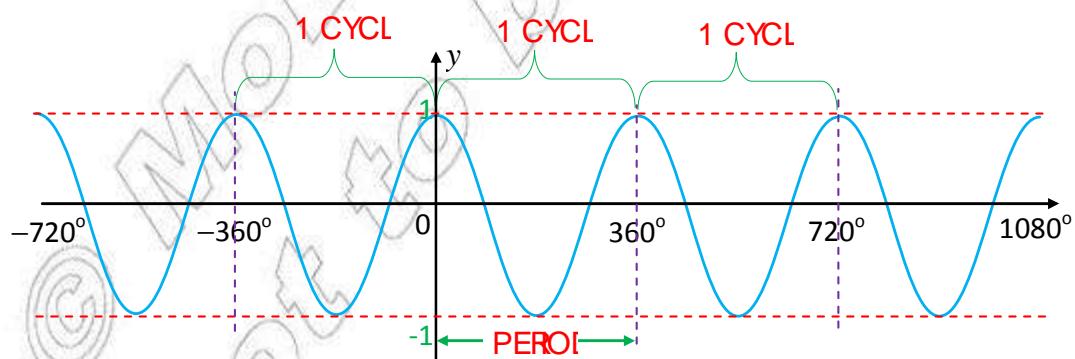


Figure 5.49 Graph of $y = \cos \theta$ for $-720^\circ \leq \theta \leq 1080^\circ$

Note: THE DOMAIN OF THE COSINE FUNCTION IS \mathbb{R} AND THE RANGE IS $[-1, 1]$.

THE RANGE OF THE COSINE FUNCTION IS $\{1\}$.

FIGURE 5.50 REPRESENTS THE SINE AND COSINE FUNCTIONS ON A COORDINATE SYSTEM.

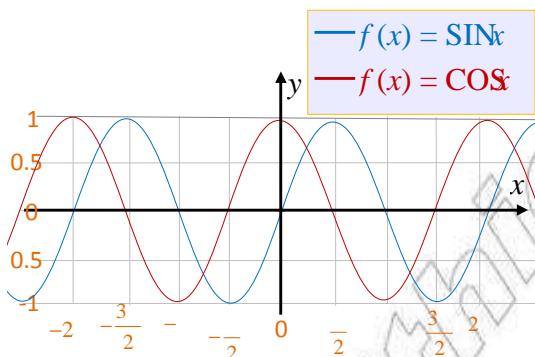


Figure 5.50

FROM THIS DIAGRAM YOU CAN SEE THAT BOTH CURVES HAVE THE SAME SHAPE. THE CURVES "FOLLOW" EACH OTHER, ALWAYS STAYING APART.

Graph of the tangent function

ACTIVITY 5.8



- 1 COMPLETE THE FOLLOWING TABLES OF VALUES FOR

in deg	-360	-315	-270	-225	-180	-135	-90	-45
$y = \tan$								

in deg	0	45	90	135	180	225	270	315	360
$y = \tan$									

- 2 USE THE TABLE YOU CONSTRUCTED ABOVE TO PLOT THE GRAPH OF $y = \tan$.
 3 FOR WHICH VALUES IS $y = \tan$ UNDEFINED?
 4 WHAT ARE THE DOMAIN AND THE RANGE OF $y = \tan$?
 5 WHAT IS THE PERIOD OF THE TANGENT FUNCTION?

THE ACTIVITY 5.8 YOU HAVE DONE ABOVE GIVES YOU A HINT ON WHAT THE GRAPH OF $y = \tan$ LOOKS LIKE. NEXT, YOU WILL SEE THE GRAPH IN DETAIL.

EXAMPLE 2 DRAW THE GRAPH OF $\tan \theta$ WHERE $-360^\circ \leq \theta \leq 360^\circ$.

SOLUTION: THE TABLES BELOW SHOW SOME OF THE VALUES OF $\tan \theta$ WHERE $-360^\circ \leq \theta \leq 360^\circ$

θ in deg	-360	-315	-270	-225	-180	-135	-90	-45	0
θ in rad	-2	$-\frac{7}{4}$	$-\frac{3}{2}$	$-\frac{5}{4}$	-	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0
$y = \tan \theta$	0	1	-	-1	0	1	-	-1	0

θ in deg	45	90	135	180	225	270	315	360
θ in rad	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$		$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2
$y = \tan \theta$	1	-	-1	0	1	-	-1	0

Remember that if θ IS IN A STANDARD POSITION AND A POINT WHERE THE TERMINAL SIDE OF θ INTERSECTS THE UNIT CIRCLE, $\tan \theta$ IS NOT DEFINED IF

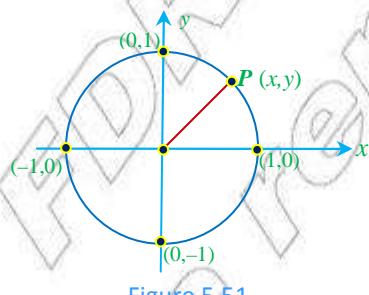


Figure 5.51

SO $\tan \theta$ IS NOT DEFINED IF

$$\theta = 90^\circ, \quad \theta = 90^\circ \pm 180^\circ, \quad \theta = 90^\circ \pm 2(180^\circ), \quad \theta = 90^\circ \pm 3(180^\circ), \text{ ETC.}$$

IN GENERAL, $\tan \theta$ IS UNDEFINED IF $90^\circ + n(180^\circ)$ OR IF $\theta = \frac{\pi}{2} + n\pi$, WHERE n IS AN INTEGER.

THE GRAPH OF $\tan \theta$ DOES NOT CROSS THE VERTICAL LINES AT $\theta = \frac{\pi}{2} + n\pi$, WHERE n IS AN INTEGER.

MOREOVER, IF WE CLOSELY INVESTIGATE THE BEHAVIOR AS θ APPROXIMATES $\frac{\pi}{2}$ FROM THE LEFT,

WE CAN SEE THAT $\tan \theta$ INCREASES FROM NEGATIVE INFINITY TO POSITIVE INFINITY (FROM $-\infty$ TO ∞). A SKETCH OF THE GRAPH FOR $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, IS SHOWN FIGURE 5.52

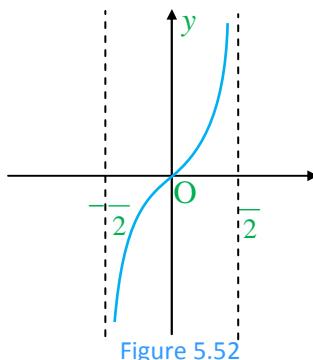


Figure 5.52

FROM THE GRAPH WE SEE THAT THE TANGENT FUNCTION IS PERIODIC WITH A PERIOD OF π . THEREFORE π IS THE PERIOD FOR THE TANGENT FUNCTION.

SINCE TAN IS PERIODIC WITH A PERIOD OF π , WE CAN EXTEND THE ABOVE GRAPH FOR AS MANY REPETITIONS (CYCLES) AS WE WANT.

FOR EXAMPLE, THE GRAPH FOR $-\pi \leq \theta \leq 2\pi$ IS SHOWN BELOW.

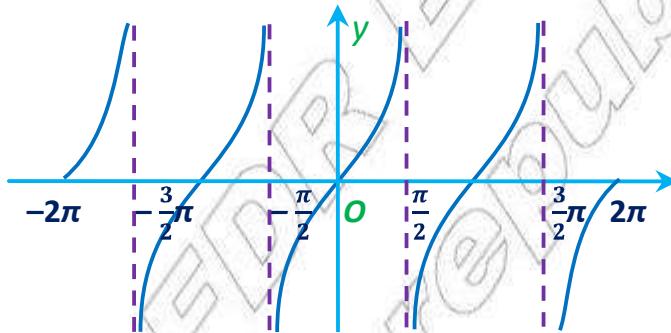


Figure 5.53

WHAT ARE THE DOMAIN AND THE RANGE OF THE TANGENT FUNCTION? FOR WHICH VALUES IS THE TAN NOT DEFINED?

USING A UNIT CIRCLE WE CAN SEE THAT THE TAN IS UNDEFINED WHENEVER THE COORDINATE ON THE UNIT CIRCLE IS 0.

THIS HAPPENS WHEN $\theta = \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \pm \frac{7\pi}{2}$, ETC. THEREFORE THE DOMAIN OF THE

TANGENT FUNCTION MUST EXCLUDE THESE ODD MULTIPLES OF $\frac{\pi}{2}$.

HENCE, THE DOMAIN OF THE TANGENT FUNCTION WHERE $\theta = \frac{(2k+1)\pi}{2}$ IS AN ODD INTEGER}.

THE RANGE OF TAN IS THE SET OF REAL NUMBERS.

Group Work 5.4

- 1 USE THE GRAPH OF THE COSINE FUNCTION TO FIND FOR WHICH θ 'S.
- 2 FROM THE GRAPH OF Y, FIND THE VALUES OF θ FOR WHICH $\sin \theta = 1$.
- 3 GRAPH THE SINE CURVE FOR THE INTERVAL $[-540^\circ, 0^\circ]$



Exercise 5.10

- 1 REFER TO THE GRAPHS OR THE TABLE OF VALUES TO DETERMINE HOW THE SINE FUNCTION BEHAVES AS θ INCREASES FROM 0° TO 360° AND ANSWER THE FOLLOWING:
 - A AS INCREASES FROM 0° TO 90° , SIN INCREASES FROM _____ 0 TO 1.
 - B AS INCREASES FROM 90° TO 180° , SIN DECREASES FROM _____ TO _____.
 - C AS INCREASES FROM 180° TO 270° , SIN DECREASES FROM _____ TO _____.
 - D AS INCREASES FROM 270° TO 360° , SIN INCREASES FROM _____ TO _____.
- 2 REFER TO THE GRAPHS OR THE TABLE OF VALUES TO DETERMINE HOW THE COSINE FUNCTION BEHAVES AS θ INCREASES FROM 0° TO 360° AND ANSWER THE FOLLOWING:
 - A AS INCREASES FROM 0° TO 90° , COS DECREASES FROM 1 TO 0.
 - B AS INCREASES FROM 90° TO 180° , COS DECREASES FROM _____ TO _____.
 - C AS INCREASES FROM 180° TO 270° , COS INCREASES FROM _____ TO _____.
 - D AS INCREASES FROM 270° TO 360° , COS INCREASES FROM _____ TO _____.
- 3 DETERMINE HOW THE TANGENT FUNCTION BEHAVES AS θ INCREASES FROM 0° TO 360° AND ANSWER THE FOLLOWING:
 - A AS INCREASES FROM 0° TO 90° , TAN INCREASES FROM COMPOSITE INFINITY (+) TO _____.
 - B AS INCREASES FROM 90° TO 180° , TAN INCREASES FROM _____ TO _____.
 - C AS INCREASES FROM 180° TO 270° , TAN INCREASES FROM _____ TO _____.
 - D AS INCREASES FROM 270° TO 360° , TAN _____ FROM ∞ TO 0.

5.2 THE RECIPROCAL FUNCTIONS OF THE BASIC TRIGONOMETRIC FUNCTIONS

IN THIS SECTION, YOU WILL LEARN ABOUT ~~THE RECIPROCAL FUNCTIONS~~, WHICH ARE CALLED THE RECIPROCALS OF THE SINE, COSINE AND TANGENT FUNCTIONS, NAMED RESPECTIVELY cosecant, secant AND cotangent FUNCTIONS.

5.2.1 The Cosecant, Secant and Cotangent Functions

Definition 5.2

If θ is an angle in standard position and $P(x, y)$ is a point on the terminal side of θ , different from the origin $O(0, 0)$, and r is the distance of point P from the origin O , then

$$\csc \theta = \frac{HYP}{OPP} = \frac{r}{y}$$

$$\sec \theta = \frac{HYP}{ADJ} = \frac{r}{x}$$

$$\cot \theta = \frac{ADJ}{OPP} = \frac{x}{y}$$

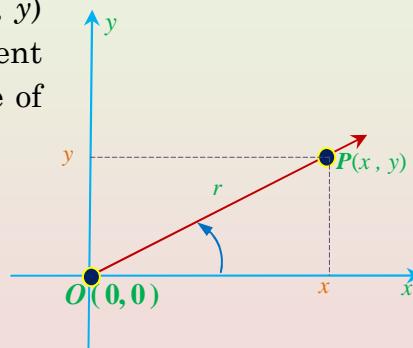


Figure 5.54

CSC, SEC AND CO~~T~~ ARE ABBREVIATIONS FOR COSECA~~M~~ AND COTANGENT RESPECTIVELY.

EXAMPLE 1 IF θ IS AN ANGLE IN STANDARD POSITION AND $P(3, 4)$ IS A POINT ON THE TERMINAL SIDE OF θ , EVALUATE THE COSECANT, SECANT AND COTANGENT FUNCTIONS.

SOLUTION: THE DISTANCE $\sqrt{R^2 + 4^2} = \sqrt{25} = 5$ UNITS

$$\text{SO, } \csc \theta = \frac{HYP}{OPP} = \frac{5}{4},$$

$$\sec \theta = \frac{HYP}{ADJ} = \frac{5}{3} \text{ AND } \cot \theta = \frac{ADJ}{OPP} = \frac{3}{4}$$

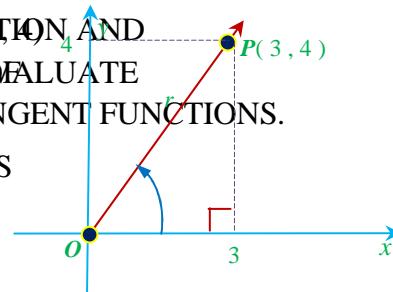


Figure 5.55

ACTIVITY 5.9

REFERRING TO FIGURE 5.55 AND:

- 1 SIN, COS AND TAN
- 2 COMPARE SIN WITH CSC, COS WITH SEC, TAN WITH COT
- 3 HOW DO THEY RELATE? ARE THEY EQUAL? ~~ARE THEY THE RECIPROCALS?~~



FROM THE RESULTS OF 5.9 YOU CAN CONCLUDE THE FOLLOWING:

$$\text{CSC} = \frac{r}{y} \quad \text{WHEREAS} \quad \sin \frac{y}{r}$$

$$\text{SEC} = \frac{r}{x} \quad \text{WHEREAS} \quad \cos \frac{x}{r}$$

$$\text{COT} = \frac{x}{y} \quad \text{WHEREAS} \quad \tan \frac{y}{x}$$

Have you noticed that one is the reciprocal of the other?

THAT IS,

$$\text{CSC} = \frac{r}{y} = \frac{1}{\frac{y}{r}} = \frac{1}{\sin}, \text{ SEC} = \frac{r}{x} = \frac{1}{\left(\frac{x}{r}\right)} = \frac{1}{\cos} \quad \text{AND}$$

$$\text{COT} = \frac{x}{y} = \frac{1}{\left(\frac{y}{x}\right)} = \frac{1}{\tan}$$

THEREFORE,

$$\text{CSC} = \frac{1}{\sin}, \text{ SEC} = \frac{1}{\cos} \quad \text{AND} \quad \text{COT} = \frac{1}{\tan}.$$

HENCE, CSC AND SIN ARE RECIPROCALS

SEC AND COS ARE RECIPROCALS

TAN AND COT ARE RECIPROCALS

EXAMPLE 2 IF $\theta = 30^\circ$, THEN FIND CSC, SEC, COT

SOLUTION:

$$\text{CSC} = \frac{1}{\sin} = \frac{1}{\left(\frac{1}{2}\right)} = 2 \quad \dots \text{remember that } \sin 30^\circ = \frac{1}{2} = 0.5$$

$$\text{SEC} = \frac{1}{\cos} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \quad \dots \text{remember that } \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\text{COT} = \frac{1}{\tan} = \frac{1}{\left(\frac{\sqrt{3}}{3}\right)} = \frac{3}{\sqrt{3}} = \sqrt{3} \quad \dots \text{remember that } \tan 30^\circ = \frac{\sqrt{3}}{3}$$

EXAMPLE 3 IF SIN IS 0.5, THEN CSC $\frac{1}{0.5} = 2$

IF COS IS -0.1035, THEN SEC $\frac{1}{-0.1035} = -9.6618$

IF TAN IS $-\frac{1}{4}$, THEN COSEC $\frac{1}{\left(-\frac{1}{4}\right)} = -4$

EXAMPLE 4 USING A UNIT CIRCLE, FIND THE VALUES OF SECANT, COSECANT AND TANGENT FUNCTIONS AT $90^\circ, 180^\circ, 270^\circ$.

SOLUTION: AS YOU CAN SEE IN THE ADJACENT FIGURE, THE TERMINAL SIDE OF ANGLES $90^\circ, 180^\circ, 270^\circ$ INTERSECTS THE UNIT CIRCLE AT $(0, 1)$

$$\text{HENCE, } \csc 90^\circ = \frac{r}{y} = \frac{1}{1} = 1$$

$$\sec 90^\circ = \frac{r}{x} = \frac{1}{0} \text{ is undefined}$$

$$\cot 90^\circ = \frac{x}{y} = \frac{0}{1} = 0$$

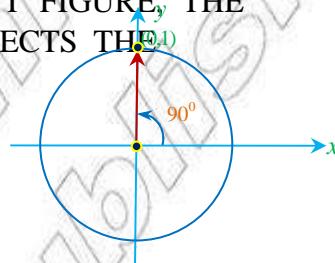


Figure 5.56

THE TERMINAL SIDE OF ANGLES 180° INTERSECTS THE UNIT CIRCLE AT $(-1, 0)$.

$$\text{HENCE, } \csc 180^\circ = \frac{r}{y} = \frac{1}{0} \text{ is undefined}$$

$$\sec 180^\circ = \frac{r}{x} = \frac{1}{-1} = -1$$

$$\cot 180^\circ = \frac{x}{y} = \frac{-1}{0} \text{ is undefined}$$

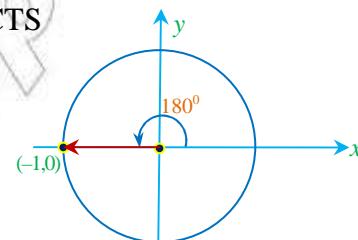


Figure 5.57

SIMILARLY THE TERMINAL SIDE OF ANGLES 270° INTERSECTS THE UNIT CIRCLE AT $(0, -1)$.

$$\text{HENCE, } \csc 270^\circ = \frac{r}{y} = \frac{1}{-1} = -1$$

$$\sec 270^\circ = \frac{r}{x} = \frac{1}{0} \text{ is undefined}$$

$$\cot 270^\circ = \frac{x}{y} = \frac{0}{-1} = 0$$

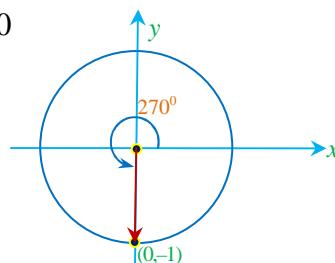


Figure 5.58

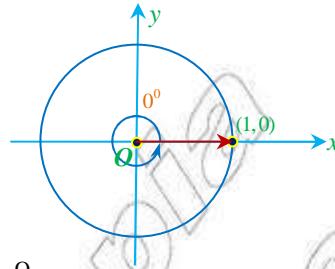
EXAMPLE 5 USING A UNIT CIRCLE, FIND THE VALUES OF SECANT, COSECANT AND TANGENT FUNCTIONS AT 60° .

SOLUTION: THE TERMINAL SIDE OF ANG θ INTERSECTS THE UNIT CIRCLE AT $(1, 0)$.

HENCE, $\csc 360^\circ = \frac{r}{y} = \frac{1}{0}$ is undefined

$$\sec 360^\circ = \frac{r}{x} = \frac{1}{1} = 1$$

$$\cot 360^\circ = \frac{x}{y} = \frac{1}{0}$$
 is undefined



Remember that THESE RESULTS ARE ALSO TRUE FOR $0^\circ, 180^\circ, 360^\circ$, ETC. Figure 5.59

WHEN DO YOU THINK THE FUNCTIONS SIN AND COT ARE UNDEFINED?

FOR EXAMPLE, $\csc \frac{r}{y}$ IS UNDEFINED WHEN THE VALUE OF y ON THE UNIT CIRCLE WILL BE 0 WHEN $\theta = 0^\circ, \pm 180^\circ, \pm 2(180^\circ), \pm 3(180^\circ), \pm 4(180^\circ)$, ETC.

IN GENERAL, \csc IS UNDEFINED FOR $\theta = 180^\circ + n(180^\circ)$, WHERE n IS AN INTEGER.

Group Work 5.5



- 1 DECIDE IF THE FOLLOWING TRIGONOMETRIC FUNCTIONS ARE POSITIVE OR NEGATIVE AND COMPLETE THE FOLLOWING TABLE.

		has terminal side in quadrant			
		I	II	III	IV
csc	+				
sec				-	
cot					-

- 2 COMPLETE THE FOLLOWING TABLE OF VALUES:

in deg	-360	-300	-270	-240	-180	-120	-90	-60	0
$y = \csc \theta$									
$y = \sec \theta$									

in deg	60	90	120	180	240	270	300	360
$y = \csc \theta$								
$y = \sec \theta$								

- 3 SKETCH THE GRAPH OF $y = \csc \theta$ AND $y = \sec \theta$ ON A SEPARATE COORDINATE SYSTEM.

- 4 CONSTRUCT A TABLE OF VALUES FOR $y = \tan \theta$ AND SKETCH THE GRAPH.

Hint: USE THE TABLE OF VALUES FOR $y = \tan \theta$

- 5 DISCUSS AND IDENTIFY THE VALUES OF θ FOR WHICH $\csc \theta$ AND $\cot \theta$ WILL BE UNDEFINED.

Exercise 5.11

- 1** SUPPOSE THE FOLLOWING POINTS LIE ON THE CIRCLE AND THE COSECANT, SECANT AND COTANGENT FUNCTIONS OF
A P (12, 5) **B** P (-8, 15) **C** P (-6, 8) **D** P (5, 3)
E P (2, 0) **F** P ($\frac{4}{5}, -\frac{3}{5}$) **G** P ($\sqrt{2}, \sqrt{5}$) **H** P ($\sqrt{6}, \sqrt{3}$)
- 2** COMPLETE EACH OF THE FOLLOWING:
A IF SIN IS -0.35, THEN CSC IS _____. **B** IF SEC IS 2.6, THEN COS IS _____.
C IF CSC IS 30.5, THEN SIN IS _____. **D** IF TAN IS 1, THEN COSS IS _____.
E IF TAN IS $\frac{\sqrt{3}}{3}$, THEN COSS IS _____. **F** IF TAN IS 0, THEN COSS IS _____.
3 FIND THE VALUES OF SEC AND COT IF IN DEGREES IS:
A 30 **B** 45 **C** 60 **D** 120
E 150 **F** 210 **G** 240 **H** 300
I -390 **J** -405 **K** -420 **L** 780.
4 IF $\cot = \frac{3}{8}$ AND IS IN THE FIRST QUADRANT, FIND THE OTHER FIVE TRIGONOMETRIC FUNCTIONS OF

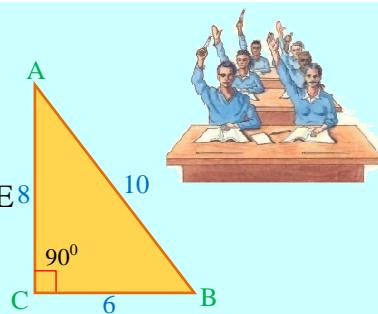
Co-functions

WHAT KINDS OF FUNCTIONS ARE CALLED CO-FUNCTIONS?

IN ORDER TO UNDERSTAND THE CONCEPT OF A CO-FUNCTION TRY THE FOLLOWING

ACTIVITY 5.10

ABC IS A RIGHT ANGLE TRIANGLE. ACUTE ANGLES. SINCE THEIR SUM IS 90° , THEY ARE **complementary angles**. FIND THE VALUES OF THE SIX TRIGONOMETRIC FUNCTIONS FOR BOTH COMPARISON. COMPARE THE RESULTS.



IDENTIFY THE FUNCTIONS THAT HAVE THE SAME VALUE.

FROM ACTIVITY 5.10, YOU MAY CONCLUDE THE FOLLOWING:

OBSERVE THAT THIS IS A RIGHT ANGLE TRIANGLE WITH

$\alpha + \beta = 90^\circ$. THIS MEANS THE ACUTE ANGLES ARE **complementary**.

HENCE WE HAVE THE FOLLOWING RELATIONSHIP:

$$\sin = \frac{a}{c} = \cos$$

$$\cos = \frac{b}{c} = \sin$$

$$\tan = \frac{a}{b} = \cot$$

$$\csc = \frac{c}{a} = \sec$$

$$\sec = \frac{c}{b} = \csc$$

$$\cot = \frac{b}{a} = \tan$$

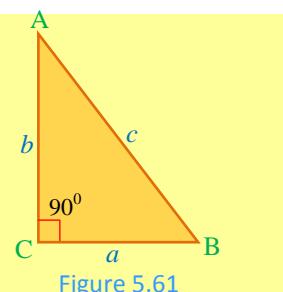


Figure 5.61

NOTE THAT, FOR THE TWO COMPLEMENTARY ANGLES

- ✓ THE SINE OF ANY ANGLE IS EQUAL TO THE CO-SINE OF ITS COMPLEMENTARY ANGLE.
- ✓ THE TANGENT OF ANY ANGLE IS EQUAL TO THE CO-TANGENT OF ITS COMPLEMENTARY ANGLE.
- ✓ THE SECANT OF ANY ANGLE IS EQUAL TO THE CO-SECANT OF ITS COMPLEMENTARY ANGLE.

THUS, THE PAIR OF FUNCTIONS sine AND cosine ARE CALLED CO-FUNCTIONS.

SIMILARLY tangent AND cotangent, secant AND cosecant ARE ALSO CO-FUNCTIONS.

ANY TRIGONOMETRIC FUNCTION OF AN ACUTE ANGLE IS EQUAL TO THE CO-FUNCTION OF ITS COMPLEMENTARY ANGLE. THAT IS, IF θ IS AN ACUTE ANGLE, THEN

$$\sin\theta = \cos(90^\circ - \theta) \quad \csc\theta = \sec(90^\circ - \theta)$$

$$\cos\theta = \sin(90^\circ - \theta) \quad \sec\theta = \csc(90^\circ - \theta)$$

$$\tan\theta = \cot(90^\circ - \theta) \quad \cot\theta = \tan(90^\circ - \theta)$$

EXAMPLE 6

A $\sin 30^\circ = \cos 60^\circ$

B $\sec 40^\circ = \csc 50^\circ$

C $\tan \frac{\pi}{3} = \cot \frac{\pi}{6}$

Exercise 5.12

1 FIND THE SIZE OF ACUTE ANGLES IN DEGREES IF:

A $\sin 2\theta = \cos$

B $\sec = \csc 80^\circ$

C $\tan 55^\circ = \cot$

D $\cos \frac{\theta}{9} = \sin$

E $\sec = \csc \frac{5}{12}$

F $\cot \theta = \tan$

2 ANSWER EACH OF THE FOLLOWING:

A IF $\cos 35^\circ = 0.8387$, THEN $\sin 55^\circ =$ _____

B IF $\sin 70^\circ = 0.9344$, THEN $\cos 20^\circ =$ _____

C IF $\tan 45^\circ = 1$, THEN $\cot 45^\circ =$ _____

D IF $\sec 15^\circ = x$, THEN $\csc 75^\circ =$ _____

E IF $\csc \theta = \frac{a}{b}$ AND $\sec \beta = \frac{a}{b}$, THEN $\sin(\theta + \beta) =$ _____

F IF $\cot 35^\circ = y$ AND $\tan 45^\circ = y$, THEN $\sec 35^\circ =$ _____

5.3 SIMPLE TRIGONOMETRIC IDENTITIES

Pythagorean identities

USING THE DEFINITIONS OF THE SIX TRIGONOMETRIC FUNCTIONS SO FAR, IT IS POSSIBLE TO FIND SPECIAL RELATIONSHIPS THAT EXIST BETWEEN THEM.

LET θ BE AN ANGLE IN STANDARD POSITION. LET $P(x, y)$ BE A POINT ON THE TERMINAL SIDE (SEE FIGURE 5.62).

FROM PYTHAGORAS THEOREM WE KNOW THAT

$$x^2 + y^2 = r^2$$

IF WE DIVIDE BOTH SIDES BY r^2 , WE HAVE

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

$$\therefore (\cos \theta)^2 + (\sin \theta)^2 = 1$$

IF WE DIVIDE BOTH SIDES OF $x^2 + y^2 = r^2$ BY x^2 , THEN WE HAVE

$$\frac{x^2}{x^2} + \frac{y^2}{x^2} = \frac{r^2}{x^2}$$

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$$

$$1 + (\tan \theta)^2 = (\sec \theta)^2$$

IF WE DIVIDE BOTH SIDES OF $x^2 + y^2 = r^2$ BY y^2 , THEN WE HAVE

$$\frac{x^2}{y^2} + \frac{y^2}{y^2} = \frac{r^2}{y^2}$$

$$\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2$$

$$(\cot \theta)^2 + 1 = (\csc \theta)^2$$

HENCE WE HAVE THE FOLLOWING RELATIONS:

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

THE ABOVE RELATIONS ARE KNOWN AS PYTHAGOREAN IDENTITIES.

Note:

$$(\sin \theta)^2 = \sin^2 \theta \quad \text{AND} \quad (\cos \theta)^2 = \cos^2 \theta, \text{ ETC.}$$

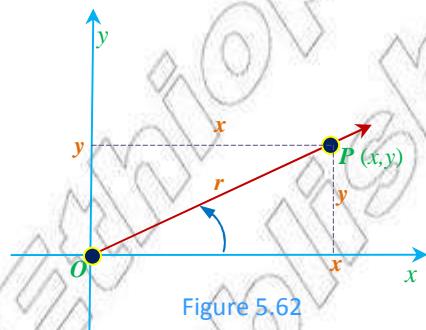


Figure 5.62

EXAMPLE 1 IF $\sin = \frac{1}{2}$ AND IS IN THE FIRST QUADRANT, FIND THE VALUES OF THE OTHER FIVE TRIGONOMETRIC FUNCTIONS OF

SOLUTION: FROM $\sin^2 + \cos^2 = 1$, WE HAVE

$$\cos^2 = 1 - \sin^2$$

$$\text{SO, } \cos = \sqrt{1 - \sin^2} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$\sec = \frac{1}{\cos} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{2}{\sqrt{3}}; \csc = \frac{1}{\sin} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

FROM $+\tan^2 = \sec^2$, WE HAVE, $\tan^2 = \sec^2 - 1$

$$\text{SO } \tan = \sqrt{\sec^2 - 1} = \sqrt{\left(\frac{2}{\sqrt{3}}\right)^2 - 1} = \sqrt{\frac{4}{3} - 1} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

FROM $\cot^2 + 1 = \csc^2$, WE HAVE $\cot^2 = \csc^2 - 1$, THIS IMPLIES THAT

$$\cot = \sqrt{\csc^2 - 1} = \sqrt{2^2 - 1} = \sqrt{4 - 1} = \sqrt{3}.$$

Exercise 5.13

1 USING THE PYTHAGOREAN IDENTITIES FIND THE OTHER THREE TRIGONOMETRIC FUNCTIONS IF:

A $\sin = \frac{15}{17}$ AND IS IN QUADRANT I.

B $\cos = \frac{-4}{5}$ AND IS IN QUADRANT II

C $\cot = \frac{7}{24}$ AND IS IN QUADRANT III.

D $\cos = \frac{24}{25}$ AND IS IN QUADRANT IV.

2 REFERRING TO THE RIGHT ANGLE TRIANGLE

(See FIGURE 5.63), FIND:

A \sin **B** \cos **C** $\sin(90^\circ - \theta)$

D $\cos(90^\circ - \theta)$ **E** $\csc(90^\circ - \theta)$ **F** $\cot(90^\circ - \theta)$

3 FILL IN THE BLANK SPACE WITH THE APPROPRIATE WORDS

A THE SINE OF AN ANGLE IS EQUAL TO THE COSINE OF

B THE COSECANT OF AN ANGLE IS EQUAL TO THE SECANT OF

C THE TANGENT OF AN ANGLE IS EQUAL TO THE COMPLEMENTARY ANGLE.

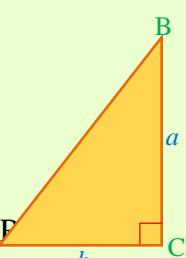


Figure 5.63

Quotient identities

THE FOLLOWING ARE ADDITIONAL RELATIONSHIPS BETWEEN THE SIX TRIGONOMETRIC FUNCTIONS:

ACTIVITY 5.11

LET θ BE AN ANGLE IN STANDARD POSITION, P BE A POINT ON THE TERMINAL SIDE OF θ (FIGURE 5.64).

THEN ANSWER THE FOLLOWING:

- A** WHAT ARE THE VALUES OF SIN, COS, TAN AND COT?
- B** HOW DO THE VALUES $\frac{\sin}{\cos}$ AND \tan COMPARE?
- C** HOW DO THE VALUES $\frac{\cos}{\sin}$ AND \cot COMPARE?

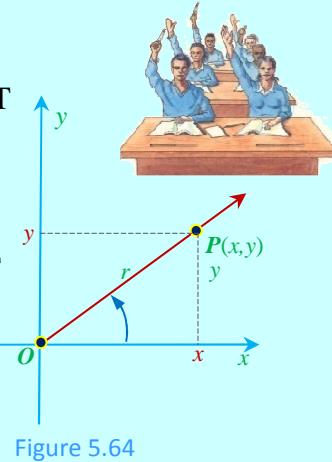


Figure 5.64

REFERRING FIGURE 5.64, WE CAN DERIVE THE FOLLOWING RELATIONSHIPS BETWEEN TRIGONOMETRIC FUNCTIONS:

$$\sin = \frac{y}{r} \text{ AND } \cos = \frac{x}{r}. \text{ FROM THIS WE HAVE, } \frac{\sin}{\cos} = \frac{\left(\frac{y}{r}\right)}{\left(\frac{x}{r}\right)} = \frac{y}{r} \times \frac{r}{x} = \frac{y}{x} = \tan$$

$$\text{SIMILARLY } \frac{\cos}{\sin} = \frac{\left(\frac{x}{r}\right)}{\left(\frac{y}{r}\right)} = \frac{x}{r} \times \frac{r}{y} = \frac{x}{y} = \cot$$

HENCE THE RELATIONS:

$$\tan = \frac{\sin}{\cos} \text{ AND } \cot = \frac{\cos}{\sin} \text{ WHICH ARE KNOWN AS QUOTIENT IDENTITIES.}$$

EXAMPLE 2 IF $\sin = \frac{4}{5}$ AND $\cos = \frac{3}{5}$, THEN FIND TAN AND COT

SOLUTION: FROM QUOTIENT IDENTITY $\frac{\sin}{\cos} = \tan$ $\frac{\left(\frac{4}{5}\right)}{\left(\frac{3}{5}\right)} = \frac{4}{3}$

$$\cot = \frac{\cos}{\sin} = \frac{\left(\frac{3}{5}\right)}{\left(\frac{4}{5}\right)} = \frac{3}{4}$$

Note: AN IDENTITY IS AN EQUATION THAT IS TRUE FOR ALL VALUES OF THE VARIABLE FOR WHICH BOTH SIDES OF THE EQUATION ARE DEFINED.

ALL IDENTITIES ARE EQUATIONS BUT NOT ALL EQUATIONS ARE IDENTITIES. THIS IS BECAUSE, UNLIKE IDENTITIES, EQUATIONS MAY NOT BE TRUE FOR SOME VALUES IN THEM. FOR EXAMPLE CONSIDER THE EQUATION $\sin \theta = \cos \theta$. FOR MOST VALUES, THIS EQUATION IS NOT TRUE, $\sin 30^\circ \neq \cos 30^\circ$. HENCE THE EXPRESSION $\sin \theta = \cos \theta$ REPRESENTS A SIMPLE TRIGONOMETRIC EQUATION, BUT NOT AN IDENTITY.

Group Work 5.6

USE THE PYTHAGOREAN AND QUOTIENT IDENTITIES TO SOLVE EACH OF THE FOLLOWING:



- 1 $\cos \alpha = -\frac{4}{5}$ AND α IS IN QUADRANT II. FIND $\tan \alpha$ AND $\cot \alpha$
- 2 $\sin \alpha = \frac{8}{17}$ AND α IS IN QUADRANT I. FIND $\tan \alpha$ AND $\cot \alpha$
- 3 $\sin 330^\circ = -\frac{1}{2}$. FIND $\tan 330^\circ$ AND $\cot 330^\circ$
- 4 $\cos 150^\circ = -\frac{\sqrt{3}}{2}$. FIND $\tan 150^\circ$ AND $\cot 150^\circ$
- 5 $\sec 60^\circ = 2$. FIND $\tan 60^\circ$ AND $\cot 60^\circ$
- 6 SUPPOSE θ IS AN ACUTE ANGLE SUCH THAT $\sin \theta = y$; FIND $\tan(90^\circ - \theta)$ AND $\cot(90^\circ - \theta)$.

Using tables of the trigonometric functions

SO FAR YOU HAVE SEEN HOW TO DETERMINE THE EXACT VALUES OF THE TRIGONOMETRIC FUNCTIONS OF SOME SPECIAL ANGLES. THE SAME METHODS CAN IN THEORY BE APPLIED TO ANY ANGLE. THE RESULTS FOUND IN THIS WAY ARE APPROXIMATIONS. THEREFORE WE USE PUBLISHED TABLES OF TRIGONOMETRIC VALUES, WHERE VALUES ARE GIVEN TO FOUR DECIMAL PLACES OF ACCURACY.

SINCE THE TRIGONOMETRIC FUNCTIONS OF A POSITIVE ANGLE ARE RELATED TO THE CO-FUNCTIONS OF THE COMPLEMENTARY ANGLE, TRIGONOMETRIC TABLES ARE OFTEN CONSTRUCTED ONLY FOR ANGLES BETWEEN 0° AND 45°.

TO FIND THE TRIGONOMETRIC FUNCTIONS OF ANGLES BETWEEN 45° AND 90°, CONSTRUCTED FOR VALUES OF ANGLES BETWEEN 0° AND 45° IS USED BY READING FROM BOTTOM UP. CORRESPONDING TO ANGLES BETWEEN 45° AND 90° LISTED IN THE LEFT HAND COLUMN, THE CO-COMPLEMENTARY ANGLES (90° - θ) ARE LISTED IN THE RIGHT HAND COLUMN. CORRESPONDING TRIGONOMETRIC FUNCTION LISTED AT THE TOP, THE CO-FUNCTION IS LISTED AT THE BOTTOM. FOR ANGLES GREATER THAN 90°, THE TRIGONOMETRIC FUNCTIONS ARE READ USING THE BOTTOM ROW AND THE RIGHT HAND

(A part of the trigonometric table is given below for your reference).

	sin	cos	tan	cot	
0°	0.0000	1.0000	0.0000	—	90°
1°	0.0175	0.9998	0.0175	57.29	89°
2°	-----	-----	-----	-----	88°
.					.
.					.
.					.
5°	0.0872	-----	0.0875	-----	85°
.					.
.					.
.					.
45°	-----	-----	-----	-----	45°
	cos	sin	cot	tan	

FOR INSTANCE, $\cos 20^\circ$ AND $\cos 50^\circ$ ARE BOTH FOUND AT THE SAME PLACE IN THE TABLE AND EQUAL APPROXIMATELY EQUAL TO 0.9397. SIMILARLY, $\tan 20^\circ$ AND $\tan 50^\circ$, ETC.

EXAMPLE 3 USE THE TABLE GIVEN AT THE END OF THE BOOK TO FIND THE VALUES OF:

A $\cos 20^\circ$ B $\cot 50^\circ$

SOLUTION:

A SINCE $20 < 45^\circ$, WE BEGIN BY LOCATING THE VERTICAL COLUMN ON THE LEFT SIDE OF THE DEGREE TABLE. THEN WE READ THE ENTRY 0.9397 UNDER THE LABELLED COS AT THE TOP.

$$\therefore \cos 20^\circ = 0.9397.$$

B SINCE $50 > 45^\circ$, WE USE THE VERTICAL COLUMN ON THE RIGHT SIDE (READ UPWARD) TO LOCATE AND READ ABOVE THE BOTTOM CAPTION "COT" TO GET 0.8391;

$$\therefore \cot 50^\circ = 0.8391.$$

EXAMPLE 4 FIND SO THAT:

A $\sec \theta = 1.624$ B $\sin \theta = 0.5831$

SOLUTION: FINDING AN ANGLE WHEN THE VALUE OF ONE SO IS GIVEN IS THE REVERSE PROCESS OF THAT ILLUSTRATED IN THE ABOVE EXAMPLE.

A GIVEN $\sec \theta = 1.624$, LOOKING UNDER THE SECANT COLUMN OR ABOVE THE SECANT COLUMN, WE FIND THE ENTRY 1.624 ABOVE THE SECANT COLUMN AND CORRESPONDING VALUE OF THEREFORE, 52° .

- B** REFERRING TO THE "SINE" COLUMNS OF THE TABLE, SIN 36° DOES NOT APPEAR THERE. THE TWO VALUES IN THE TABLE CLOSEST TO 0.5831 (ONE SMALLER AND ONE LARGER) ARE 0.5736 AND 0.5878. THESE VALUES CORRESPOND TO 35° AND 36° RESPECTIVELY. AS SHOWN BELOW, THE DIFFERENCE BETWEEN THE VALUE OF SIN 36° IS SMALLER THAN THE DIFFERENCE BETWEEN SIN 35° AND SIN 37°. THEREFORE USE THE VALUE FOR SIN 36° BECAUSE SIN 36° IS CLOSER TO SIN 35° THAN IT IS TO SIN 37°.

$$\begin{array}{ll} \text{SIN } 36^\circ = 0.5831 & \text{SIN } 37^\circ = 0.5878 \\ \text{SIN } 35^\circ = 0.5736 & \text{SIN } 36^\circ = 0.5831 \\ \text{DIFFERENCE} = 0.0095 & \text{DIFFERENCE} = 0.0047 \\ \therefore & = 36^\circ \text{ (NEAREST DEGREE).} \end{array}$$

THE FOLLOWING EXAMPLES ILLUSTRATE HOW TO DETERMINE THE VALUES OF THE FUNCTIONS FOR ANGLES MEASURED IN DEGREES (OR RADIANS) WHOSE MEASURES ARE 0° AND 90° (OR 0 AND $\frac{\pi}{2}$).

EXAMPLE 5 USE THE NUMERICAL TABLE, REFERENCE ANGLES, PERIODICITY OF THE FUNCTIONS TO DETERMINE THE VALUES OF EACH OF THE FOLLOWING:

- A** $\text{SIN } 236^\circ$ **B** $\text{COS } 69^\circ$

SOLUTION:

- A** TO FIND SIN 236° FIRST WE CONSIDER THE QUADRANT THAT THE ANGLE 236° LIES IN. THIS IS DONE BY PLACING THE ANGLE IN STANDARD POSITION AS SHOWN IN FIGURE 5.65 WE SEE THAT THE ANGLE LIES IN QUADRANT III SO THAT THE SINE VALUE IS NEGATIVE. THE REFERENCE ANGLE CORRESPONDING TO 236°

$$r = 236^\circ - 180^\circ = 56^\circ. \text{ THUS, } \text{SIN } 236^\circ = -\text{SIN } 56^\circ.$$

SINCE $56^\circ > 45^\circ$, WE LOCATE THE VERTICAL COLUMN ON THE RIGHT SIDE OF THE TRIGONOMETRIC TABLE. LOOKING IN THE VERTICAL COLUMN ABOVE THE BOTTOM CAPTION "SIN", WE SEE THAT SIN 56°

$$\text{SO SIN } 236^\circ = -\text{SIN } 56^\circ = -0.8290.$$

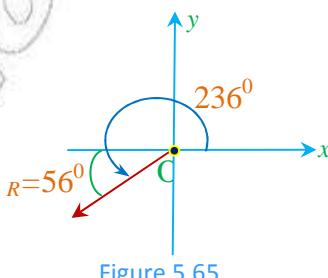


Figure 5.65

- B** TO FIND THE VALUE OF $\cos 693^\circ$ OBSERVE THAT 693° IS GREATER THAN ~~THE~~ PERIOD OF COSINE FUNCTION IS 360° . DIVIDING 693° BY 360° WE OBTAIN

$$693^\circ = 1 \times 360^\circ + 333^\circ$$

THIS MEANS THAT ~~THE~~ ANGLE IS CO TERMINAL WITH THE

333° ANGLE. I.E., $\cos 693^\circ \cos 333^\circ$

SINCE THE TERMINAL SIDE IS IN QUADRANT IV, THE

REFERENCE ANGLE $180^\circ - 333^\circ = 27^\circ$ (See FIGURE 5.66)

COSINE IS POSITIVE IN QUADRANT IV, SO $\cos 333^\circ \approx 0.8910$.

HENCE, $\cos 693^\circ \approx 0.8910$.

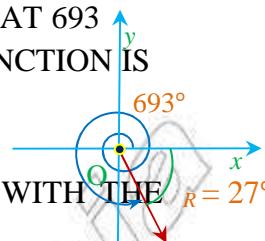


Figure 5.66

Exercise 5.14

- 1 USING TRIGONOMETRIC TABLE, FIND:

- | | | | | | | | |
|---|------------------|---|------------------|---|---------------------|---|------------------|
| A | $\sin 59^\circ$ | B | $\cos 53^\circ$ | C | $\tan 36^\circ$ | D | $\sec 162^\circ$ |
| E | $\sin 593^\circ$ | F | $\tan 593^\circ$ | G | $\cos (-143^\circ)$ | | |

- 2 IN EACH OF THE FOLLOWING PROBLEMS, FIND THE CORRECT DEGREE:

- | | | | | | |
|---|-------------------|---|-------------------|---|------------------|
| A | $\sin A = 0.5299$ | B | $\cos A = 0.6947$ | C | $\tan A = 1.540$ |
| D | $\csc A = 1.000$ | E | $\sec A = 2.000$ | F | $\cot A = 1.808$ |

5.4 REAL LIFE APPLICATION PROBLEMS

EVEN THOUGH TRIGONOMETRY WAS ORIGINALLY USED TO RELATE THE ANGLES OF A TRIANGLE TO THE LENGTHS OF THE SIDES OF A TRIANGLE, TRIGONOMETRIC FUNCTIONS ARE IMPORTANT IN THE STUDY OF TRIANGLES BUT ALSO IN MODELING MANY PERIODIC PHENOMENA IN REAL LIFE. IN THIS SECTION YOU WILL SEE SOME OF THE REAL LIFE APPLICATIONS OF TRIGONOMETRY.

Solving right-angled triangles

MANY APPLICATIONS OF TRIGONOMETRY INVOLVING TRIANGLES. A TRIANGLE HAS BASICALLY SEVEN COMPONENTS; NAMELY THREE SIDES, THREE ANGLES AND AN AREA. THUS, SOLVING A TRIANGLE MEANS TO FIND THE LENGTHS OF THE THREE SIDES, THE MEASURES OF ALL THE THREE ANGLES AND THE MEASURE OF ITS AREA.

Revision of the properties of right angle triangles

WE ALREADY KNOW THAT, FOR A GIVEN RIGHT-ANGLED TRIANGLE

THE HYPOTENUSE (HYP) IS THE SIDE WHICH IS OPPOSITE THE

RIGHT ANGLE AND IS THE LONGEST SIDE OF THE TRIANGLE.

FOR THE ANGLE MARKED IN FIGURE 5.67

- ✓ \overline{BC} IS THE SIDE Opposite (OPP) ANGLE
- ✓ \overline{AC} IS THE SIDE Adjacent (ADJ) ANGLE.

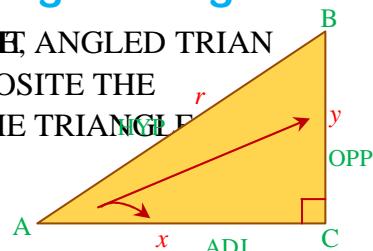


Figure 5.67

HENCE,

$1 \quad x^2 + y^2 = r^2$	$\sin = \frac{y}{r}$ $2 \quad \cos = \frac{x}{r}$ $\tan = \frac{y}{x}$	$\csc = \frac{r}{y} = \frac{1}{\sin}$ $\sec = \frac{r}{x} = \frac{1}{\cos}$ $\cot = \frac{x}{y} = \frac{1}{\tan}$
$3 \quad \sin^2 + \cos^2 = 1$ $1 + \tan^2 = \sec^2$ $1 + \cot^2 = \csc^2$	$4 \quad \tan = \frac{\sin}{\cos}$ $\cot = \frac{\cos}{\sin}$	

EXAMPLE 1 SOLVE THE RIGHT-ANGLED TRIANGLE WITH AN ACUTE ANGLE 25° AND HYPOTENUSE OF LENGTH 10 CM.

SOLUTION: IT IS REQUIRED TO FIND THE UNKNOWN ELEMENTS OF THE TRIANGLE. THESE ARE

A $m(\angle A)$

B LENGTH OF SIDE

C LENGTH OF SIDE

D THE AREA OF THE TRIANGLE

DRAW THE TRIANGLE AND LABEL ALL KNOWN PARTS (

A $M(\angle A) = 90^\circ - 25^\circ = 65^\circ$

B TO FIND, OBSERVE THAT THE SIDE OPPOSITE TO THE ANGLE, AND THE LENGTH OF THE HYPOTENUSE IS 10 CM. SO $\sin 65^\circ = \frac{a}{10}$

MULTIPLYING BOTH SIDES OF THE EQUATION BY 10, WE OBTAIN

$$a = 10 \times \sin 65^\circ$$

USING THE TRIGONOMETRIC TABLE, WE HAVE

$$a = 10 \times \sin 65^\circ \approx 10 \times 0.9063 = 9.063 \text{ CM}$$

C TO FIND, WE CAN USE THE PYTHAGOREAN THEOREM OR THE SINE FUNCTION.

$$\sin 25^\circ = \frac{b}{10}$$

MULTIPLYING BOTH SIDES BY 10 WE GET $b = 10 \sin 25^\circ$

USING TRIGONOMETRIC TABLE WE GET $\sin 25^\circ \approx 10 \times (0.4226) \approx 4.226 \text{ CM}$.

$$D \quad \text{AREA OF } \triangle ABC = \frac{1}{2}ab \approx \frac{1}{2} \times 9.063 \times 4.226 \approx 19.150 \text{ CM}^2$$

EXAMPLE 2 SOLVE THE RIGHT ANGLE TRIANGLE WHOSE HYPOTENUSE IS 17 UNITS AND THE LEGS IS 15 UNITS.

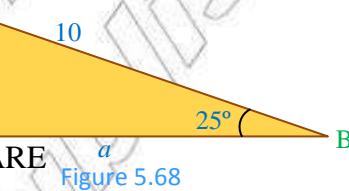


Figure 5.68

SOLUTION: THE MISSING ELEMENTS OF THE TRIANGLE ARE

A $m(\angle A)$

C LENGTH OF SIDE

B $m(\angle B)$

D THE AREA OF THE TRIANGLE

DRAW THE TRIANGLE **FIGURE 5.69.**

A SINCE THE HYPOTENUSE AND THE SIDE OPP. ARE GIVEN,

$$\sin A = \frac{17}{20} = 0.8500$$

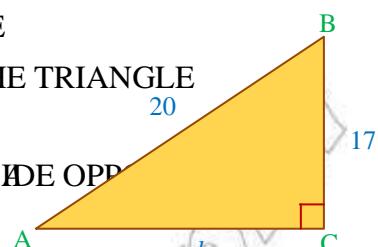


Figure 5.69

THUS, FROM THE TRIGONOMETRIC TABLE WE SEE THAT

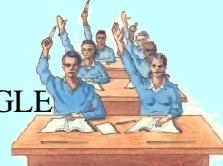
B $m(\angle B) = 90^\circ - m(\angle A) = 90^\circ - 58^\circ = 32^\circ$

C TO FIND b , USE $\cos A = \frac{b}{20}$ WHICH GIVES

$$b = 20 \cos A \approx 20 \cos 58^\circ \approx 20(0.5299) \approx 10.598$$

D AREA OF $\triangle ABC = \frac{1}{2} \times b \times 17 = \frac{1}{2} \times 10.598 \times 17 = 90.083$ UNITS²

ACTIVITY 5.12



- 1 SOLVE THE RIGHT ANGLE **TRIANGLE** THE RIGHT ANGLE $AB = 2$ CM AND $BC = 3$ CM.
- 2 SOLVE THE RIGHT ANGLE **TRIANGLE** THE RIGHT ANGLE AT $m(\angle A) = 24^\circ$ AND $AB = 20$ CM.

Angle of elevation and angle of depression

THE **line of sight** OF AN OBJECT IS THE LINE JOINING THE **EYE** AND THE **OBJECT**. IF THE OBJECT IS ABOVE THE HORIZONTAL PLANE THROUGH THE EYE OF THE OBSERVER, THEN THE ANGLE BETWEEN THE LINE OF SIGHT AND THIS HORIZONTAL PLANE IS CALLED THE **angle of elevation** (See **FIGURE 5.70**). IF THE OBJECT IS **below** THIS HORIZONTAL PLANE, THE ANGLE IS THEN CALLED THE **angle of depression**.

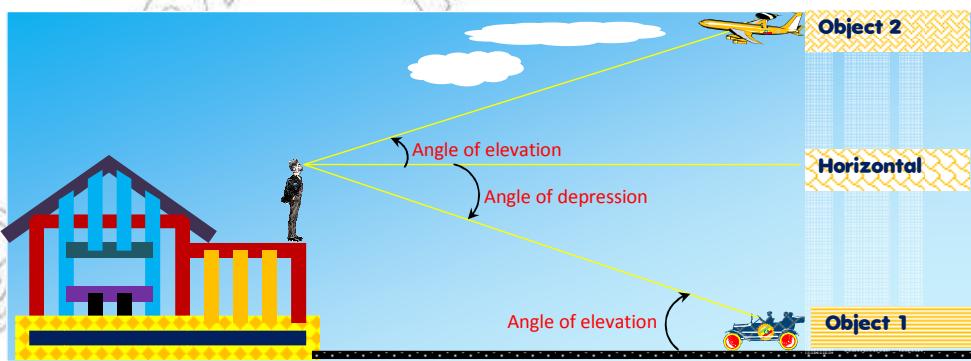


Figure 5.70

EXAMPLE 3 FIND THE HEIGHT OF A TREE WHICH CASTS A SHADOW OF 12.4 M WHEN THE ANGLE OF ELEVATION OF THE SUN IS 52°

SOLUTION: LET h BE THE HEIGHT OF THE TREE IN METRES. IN THE 52° ANGLE, THE OPPOSITE SIDE IS THE ADJACENT SIDE 12.4 M.

$$\text{THEREFORE, } \tan 52^\circ = \frac{h}{12.4}$$

$$\therefore h = 12.4 \times \tan 52^\circ = 15.9 \text{ M.}$$

THEREFORE, THE TREE IS 15.9 M HIGH.

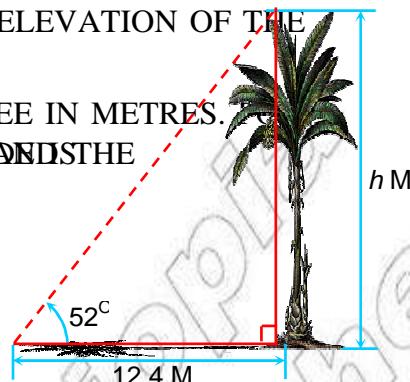


Figure 5.71

EXAMPLE 4 FROM THE TOP OF A BUILDING, THE ANGLE OF DEPRESSION ON THE GROUND 7 M AWAY FROM THE BASE OF THE BUILDING IS 60° . FIND THE HEIGHT OF THE BUILDING.

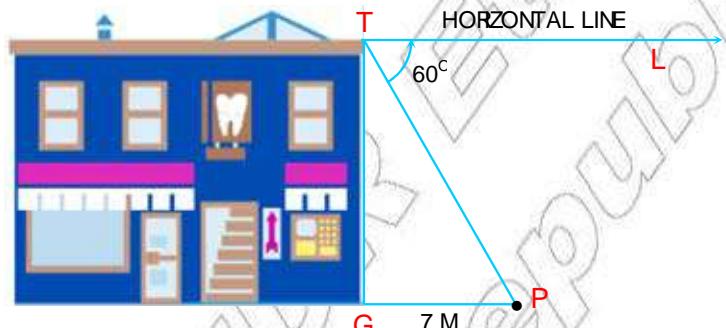


Figure 5.72

SOLUTION: IN FIGURE 5.72, T IS A POINT ON THE TOP OF THE BUILDING, G IS A POINT ON THE GROUND, AND L IS A HORIZONTAL RAY IN THE SAME PLANE AS T .

$$m(\angle GPT) = m(\angle LTP) = 60^\circ \text{ (WHY?)}$$

$$\frac{GT}{PG} = \tan(GPT) \Rightarrow \tan 60^\circ = \frac{7}{PG} \approx 7 \times 1.732 \approx 12 \text{ M.}$$

THEREFORE, THE HEIGHT OF THE BUILDING IS ABOUT 12 METRES.

EXAMPLE 5 A PERSON STANDING ON THE EDGE OF ONE BANK OF A CANAL POST ON THE EDGE OF THE OTHER BANK OF THE CANAL. THE PERSON'S EYE LEVEL IS 152 CM ABOVE THE GROUND. THE ANGLE OF ELEVATION FROM EYE LEVEL TO THE TOP OF THE LAMP POST IS 12° AND THE ANGLE OF DEPRESSION FROM EYE LEVEL TO THE BOTTOM OF THE LAMP POST IS 18° . HOW HIGH IS THE LAMP POST? HOW WIDE IS THE CANAL? (FIGURE 5.73A)

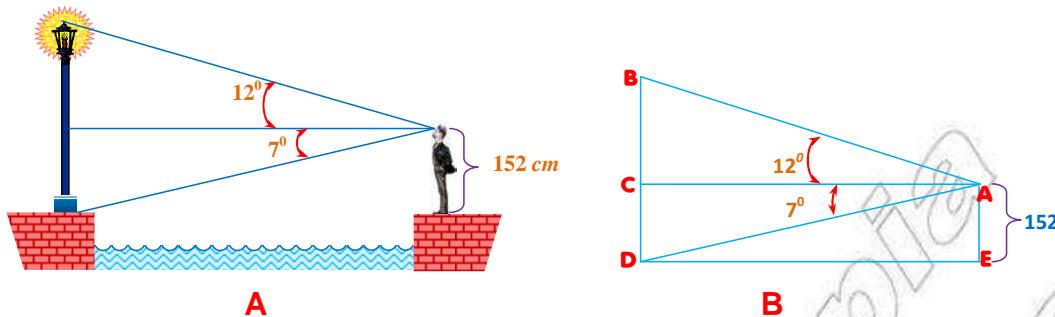


Figure 5.73

SOLUTION: CONSIDERING THE ESSENTIAL INFORMATION IN GRAMMA IN THE FIGURE 5.73B

WE WANT TO FIND THE HEIGHT OF THE LANDPOST. THE CANAL'S WIDTH IS THE DISTANCE BC. THE EYE LEVEL HEI OFF THE OBSERVER IS 152 CM SINCE ARE PARALLEL, CD ALSO HAS LENGTH 152 CM. IN THE RIGHT ANGLE TRIANGLE ABC, SINCE THAT THE SIDE CD IS OPPOSITE TO THE ANGLE OF 7°.

$$\text{SO, } \tan 7^\circ = \frac{\text{opp}}{\text{adj}} = \frac{152}{AC} \text{ GIVING } AC = \frac{152}{\tan 7^\circ}$$

$$\text{THEREFORE } AC = \frac{152}{\tan 7^\circ} = \frac{152}{0.1228} = 1237.79 \text{ CM}$$

SO THE CANAL IS APPROXIMATELY 12.4 METRES WIDE.

NOW, USING THE RIGHT TRIANGLE ACB, WE SEE THAT

$$\tan 12^\circ = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC} = \frac{BC}{1237.79}$$

$$\text{THEREFORE } BC = 1237.79 \times \tan 12^\circ = 1237.79 \times 0.2126 = 263.15 \text{ CM.}$$

SO THE HEIGHT OF THE LANDPOST

$$BC + CD = 263.15 + 152 = 415.15 \text{ CM} \approx 4.15 \text{ M}$$

Exercise 5.15

1. IN PROBLEMS 1 TO 6, $\triangle ABC$ IS A RIGHT ANGLE TRIANGLE WITH LET a, b, c BE ITS SIDES WITH THE LENGTH OF ITS HYPOTENUSE, LENGTH OPPosite ANGLE A AND ITS SIDE LENGTH OPPosite B. USE THE INFORMATION BELOW, FIND THE MISSING ELEMENTS OF EACH RIGHT ANGLE TRIANGLE, GIVING ANSWERS CORRECT TO THE NUMBER.
- | | |
|---|---|
| A $m(\angle B) = 50^\circ$ AND $b = 20$ UNITS
C $m(\angle A) = 36^\circ$ AND $b = 8$ UNITS
E $m(\angle A) = 38^\circ$ AND $b = 20$ UNITS | B $m(\angle A) = 54^\circ$ AND $a = 12$ UNITS
D $m(\angle B) = 55^\circ$ AND $a = 10$ UNITS
F $m(\angle A) = 17^\circ$ AND $a = 14$ UNITS. |
|---|---|

- 2 A** A LADDER 6 METRES LONG LEANS AGAINST A BUILDING. FORMED BY THE LADDER AND THE GROUND, HOW FAR FROM THE BUILDING IS THE FOOT OF THE LADDER?
- B** A MONUMENT IS 50 METRES HIGH. WHAT IS THE SHADOW CAST BY THE MONUMENT IF THE ANGLE OF ELEVATION OF THE SUN IS 60° ?
- C** WHEN THE SUN IS ABOVE THE HORIZON, HOW LONG IS THE SHADOW CAST BY A BUILDING 15 METRES HIGH?
- D** FROM AN OBSERVER O, THE ANGLES OF ELEVATION OF THE TOP OF A FLAGPOLE ARE 45° AND 45° RESPECTIVELY. FIND THE HEIGHT OF THE FLAGPOLE.

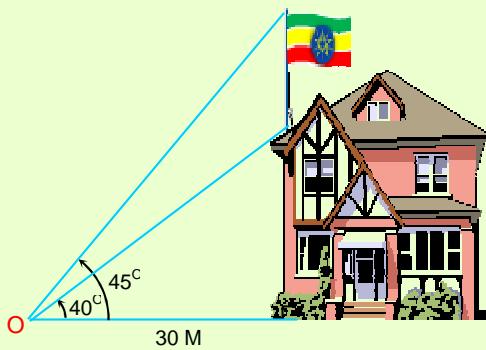


Figure 5.74

- E** FROM THE TOP OF A CLIFF 200 METRES ABOVE SEA LEVEL, THE DEPRESSION TO TWO FISHING BOATS ARE 40° AND 45° RESPECTIVELY. HOW FAR APART ARE THE BOATS?

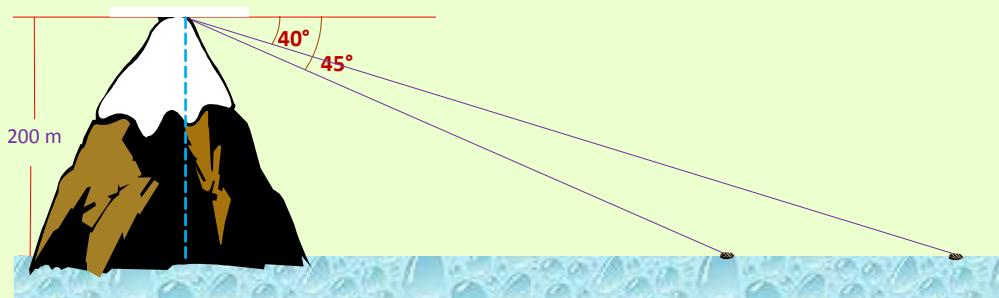


Figure 5.75

- F** A SURVEYOR STANDS IN THE MIDDLE OF A CANAL. TWO OBJECTS ARE LOCATED ON THE OPPOSITE SIDE OF THE CANAL. THE DISTANCE BETWEEN THE OBJECTS IS 120 M. IF THE ANGLE OF SIGHT BETWEEN THE SURVEYOR AND THE OBJECTS IS 37° , HOW WIDE IS THE CANAL?

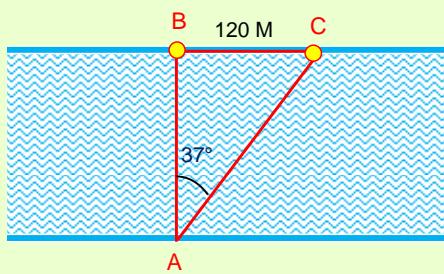


Figure 5.76



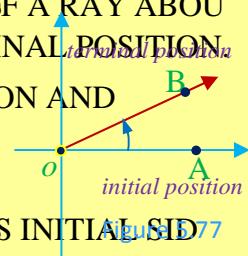
Key Terms

angle in standard position	negative angle	radian
angle of depression	period	reference angle
angle of elevation	periodic function	special angle
co-function	positive angle	supplementary angles
complementary angles	pythagorean identity	trigonometric function
co-terminal angles	quadrantal angle	trigonometry
degree	quotient identity	unit circle



Summary

- AN ANGLE IS DETERMINED BY THE ROTATION OF A RAY ABOUT A VERTEX FROM AN INITIAL POSITION TO A TERMINAL POSITION.
- AN ANGLE **positive** FOR ANTICLOCKWISE ROTATION AND **negative** FOR CLOCKWISE ROTATION.
- AN ANGLE IN THE COORDINATE PLANE IS IN **standard Position**, IF ITS VERTEX IS AT THE ORIGIN AND ITS INITIAL SIDE ALONG THE POSITIVE X-AXIS.
- RADIAN MEASURE OF ANGLES:
 $2 \text{ RADIANS} = {}^{\circ}360$ $\text{RADIANS} = {}^{\circ}180$
- TO CONVERT DEGREES TO RADIANS, MULTIPLY BY $\frac{\pi}{180^\circ}$
- TO CONVERT RADIANS TO DEGREES, MULTIPLY BY 180°
- IF θ IS AN ANGLE IN STANDARD POSITION AND IS THE DISTANCE OF POINT ON THE TERMINAL SIDE OF OTHER THAN THE ORIGIN, AND r IS THE DISTANCE OF POINT FROM THE ORIGIN, THEN



$$\sin \theta = \frac{y}{r} \quad \csc \theta = \frac{r}{y} = \frac{1}{\sin \theta}$$

$$\cos \theta = \frac{x}{r} \quad \sec \theta = \frac{r}{x} = \frac{1}{\cos \theta}$$

$$\tan \theta = \frac{y}{x} \quad \cot \theta = \frac{x}{y} = \frac{1}{\tan \theta}$$

$$r = \sqrt{x^2 + y^2} \quad (\text{PYTHAGORAS RULE})$$

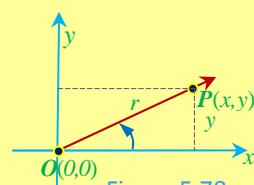


Figure 5.78

8 Signs of sine, cosine and tangent functions:

- ✓ IN THE FIRST QUADRANT ~~all the three~~ TRIGONOMETRIC FUNCTIONS ARE POSITIVE.
- ✓ IN THE SECOND QUADRANT ~~sine is~~ IS POSITIVE.
- ✓ IN THE THIRD QUADRANT ~~tangent is~~ IS POSITIVE.
- ✓ IN THE FOURTH QUADRANT ~~cosecant is~~ IS POSITIVE.

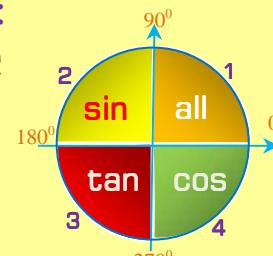


Figure 5.79

All Students Take Chemistry

9 Functions of negative angles:

IF θ IS AN ANGLE IN STANDARD POSITION, THEN

$$\sin(-\theta) = -\sin \theta \quad \cos(-\theta) = \cos \theta \quad \tan(-\theta) = -\tan \theta$$

10 Complementary angles:

TWO ANGLES ARE SAID TO BE COMPLEMENTARY, IF THEIR SUM IS EQUAL TO 90°

IF α AND β ARE ANY TWO COMPLEMENTARY ANGLES, THEN

$$\sin \alpha = \cos \beta \quad \cos \alpha = \sin \beta \quad \tan \alpha = \frac{1}{\tan \beta}$$

11 Reference angle R:

IF θ IS AN ANGLE IN STANDARD POSITION WHOSE TERMINAL SIDE DOES NOT LIE ON COORDINATE AXIS, THEN THE ~~reference angle~~ R FOR θ IS THE ~~positive acute angle~~ FORMED BY THE TERMINAL SIDE OF θ AND THE ~~positive acute angle~~ X-AXIS.

12 THE VALUES OF THE TRIGONOMETRIC FUNCTION OF A GIVEN ANGLE AND THE VALUES OF THE CORRESPONDING TRIGONOMETRIC FUNCTIONS OF THE REFERENCE ANGLE ARE THE SAME IN ABSOLUTE VALUE BUT THEY MAY DIFFER IN SIGN

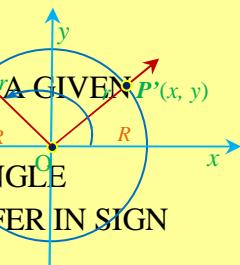


Figure 5.80

13 Supplementary angles:

TWO ANGLES ARE SAID TO BE SUPPLEMENTARY, IF THEIR SUM IS EQUAL TO 180°. IF A SECOND QUADRANT ANGLE, THEN ITS SUPPLEMENT WILL BE (180° - θ)

$$\sin \theta = \sin(180^\circ - \theta),$$

$$\cos \theta = -\cos(180^\circ - \theta),$$

$$\tan \theta = -\tan(180^\circ - \theta)$$

- 14** CO-TERMINAL ANGLES ARE ANGLES IN STANDARD POSITION (INITIAL SIDE ON THE POSITIVE AXIS) THAT HAVE A COMMON TERMINAL SIDE.
- 15** CO-TERMINAL ANGLES HAVE THE SAME TRIGONOMETRIC VALUE.
- 16** THE DOMAIN OF THE SINE FUNCTION IS THE SET OF ALL REAL NUMBERS.
- 17** THE RANGE OF THE SINE FUNCTION IS $\{-1, 1\}$.
- 18** THE GRAPH OF THE SINE FUNCTION REPEATS 360° PER PERIOD.
- 19** THE DOMAIN OF THE COSINE FUNCTION IS THE SET OF ALL REAL NUMBERS.
- 20** THE RANGE OF THE COSINE FUNCTION IS $\{-1, 1\}$.
- 21** THE GRAPH OF THE COSINE FUNCTION REPEATS 360° PER PERIOD.
- 22** THE DOMAIN OF THE TANGENT FUNCTION $\left\{ \theta \mid \theta \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \right\}$ WHERE π IS AN ODD IN.
- 23** THE RANGE OF TAN IS THE SET OF ALL REAL NUMBERS.
- 24** THE TANGENT FUNCTION HAS 90° PERIOD.
- 25** THE GRAPH OF TAN IS INCREASING FOR $\theta < \frac{\pi}{2}$.
- 26** ANY TRIGONOMETRIC FUNCTION OF AN ACUTE ANGLE IS EQUAL TO THE FUNCTION OF ITS COMPLEMENTARY ANGLE.
THAT IS, IF $0^\circ < \theta \leq 90^\circ$, THEN

$$\sin = \cos(90^\circ - \theta)$$

$$\csc = \sec(90^\circ - \theta)$$

$$\cos = \sin(90^\circ - \theta)$$

$$\sec = \csc(90^\circ - \theta)$$

$$\tan = \cot(90^\circ - \theta)$$

$$\cot = \tan(90^\circ - \theta)$$

27 Reciprocal relations:

$$\csc = \frac{1}{\sin}, \quad \sec = \frac{1}{\cos}, \quad \cot = \frac{1}{\tan}$$

28 Pythagorean identities:

$$\sin^2 + \cos^2 = 1 \quad 1 + \tan^2 = \sec^2 \quad \cot^2 + 1 = \csc^2$$

29 Quotient identities:

$$\tan = \frac{\sin}{\cos}$$

$$\cot = \frac{\cos}{\sin}$$



Review Exercises on Unit 5

- 1** INDICATE TO WHICH QUADRANT EACH OF THE FOLLOWING A
A 225° **B** 333° **C** -300° **D** 610°
E -700° **F** 900° **G** -765° **H** -1238°
I 1440° **J** 2010° .
- 2** FIND TWO CO-TERMINAL ANGLES (ONE POSITIVE AND ONE NEGATIVE) FOR EACH OF THE FOLLOWING ANGLES:
A 80° **B** 140° **C** 290° **D** 375° **E** 2900°
F -765° **G** -900° **H** -1238° **I** -1440° **J** -2010° .
- 3** CONVERT EACH OF THE FOLLOWING TO RADIANS:
A 40° **B** 75° **C** 240° **D** 330° **E** -95°
F -180° **G** -220° **H** -420° **I** -3060° .
- 4** CONVERT EACH OF THE FOLLOWING ANGLES IN RADIANS TO
A $\frac{2}{6}$ **B** $-\frac{2}{3}$ **C** $\frac{7}{18}$ **D** $\frac{43}{6}$
E $-\frac{4}{9}$ **F** 5 **G** $-\frac{3}{12}$ **H** $-\frac{1}{24}$.
- 5** USE A UNIT CIRCLE TO FIND THE VALUES OF SINE, COSINE AND TAN WHEN IS:
A 810° **B** -450° **C** 900° **D** -630°
E 990° **F** -990° **G** 1080° **H** -1170°
- 6** FIND THE VALUES OF SINE, COSINE AND TAN WHEN IS:
A $\frac{5}{6}$ **B** $\frac{7}{6}$ **C** $\frac{4}{3}$ **D** $\frac{3}{2}$
E $\frac{5}{3}$ **F** $-\frac{5}{3}$ **G** $-\frac{7}{4}$ **H** $-\frac{11}{6}$.
- 7** STATE WHETHER EACH OF THE FOLLOWING ARE FRACTIONAL OR NEGATIVE:
A $\sin 31^\circ$ **B** $\cos 220^\circ$ **C** $\cos (-220^\circ)$ **D** $\tan 785^\circ$
E $\sin (-90^\circ)$ **F** $\sec (-70^\circ)$ **G** $\tan 327^\circ$ **H** $\cot \frac{5}{3}$
I $\csc 138^\circ$ **J** $\sin \left(-\frac{11}{6}\right)$
- 8** GIVE A REFERENCE ANGLE FOR EACH OF THE FOLLOWING;
A 140° **B** 260° **C** 355° **D** 414°
E -190° **F** -336° **G** 1238° **H** -1080° .

9 REFERRING TO THE VALUES GIVEN IN THE TABLE BELOW, SKETCH THE GRAPHS OF THE SINE, COSINE AND TANGENT FUNCTIONS.

Degrees	Radians	sin	cos	tan	cot	sec	csc
0°	0	0	1	0	UNDEFINED	1	UNDEFINED
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	UNDEFINED	0	UNDEFINED	1
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	-2	$\frac{2\sqrt{3}}{3}$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	2
180°	π	0	-1	0	UNDEFINED	-1	UNDEFINED
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	-2
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	-2	$-\frac{2\sqrt{3}}{3}$
270°	$\frac{3\pi}{2}$	-1	0	UNDEFINED	0	UNDEFINED	-1
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\sqrt{3}$	2	$-\frac{2\sqrt{3}}{3}$
315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	-2
360°	2π	0	1	0	UNDEFINED	1	UNDEFINED

10 FIND THE VALUE OF EACH OF THE FOLLOWING:

- | | | | | | |
|----------|--------------------|----------|-----------------|----------|--------------------|
| A | $\sin(-120^\circ)$ | B | $\cos 60^\circ$ | C | $\tan(-30^\circ)$ |
| D | $\csc 990^\circ$ | E | $\sec 45^\circ$ | F | $\cot(-420^\circ)$ |

11 EVALUATE THE SIX TRIGONOMETRIC FUNCTIONS STANDARD POSITION AND ITS TERMINAL SIDE CONTAINS THE GIVEN POINT P (

- | | | | | | | | |
|----------|----------|----------|-----------|----------|-----------|----------|------------|
| A | P(5, 12) | B | P(-7, 24) | C | P(5, -6) | D | P(-8, -17) |
| E | P(15, 8) | F | P(1, -8) | G | P(-3, -4) | H | P(0, 1) |

12 LET θ BE AN ANGLE IN STANDARD POSITION. IDENTIFY IN WHICH QUADRANT IT LIES GIVEN THE FOLLOWING CONDITIONS:

- | | | | |
|----------|--|----------|--|
| A | IF $\sin \theta < 0$ AND $\cos \theta > 0$ | B | IF $\sin \theta > 0$ AND $\tan \theta > 0$ |
| C | IF $\sin \theta > 0$ AND $\sec \theta < 0$ | D | IF $\sec \theta > 0$ AND $\cot \theta < 0$ |
| E | IF $\cos \theta < 0$ AND $\cot \theta > 0$ | F | IF $\sec \theta < 0$ AND $\csc \theta > 0$. |

13 FIND THE ACUTE ANGLE

- | | | | | | |
|----------|---|----------|---|----------|---|
| A | $\sin 60^\circ = \frac{1}{\csc \theta}$ | B | $\sin \theta = \cos \theta$ | C | $\sin 70^\circ = \cos \theta$ |
| D | $1 = \frac{\sin \theta}{\cos 80^\circ}$ | E | $\frac{\sin \theta}{\cos \theta} = \cot 35^\circ$ | F | $\frac{\sin 70^\circ}{\cos 70^\circ} = \frac{\cos \theta}{\sin \theta}$ |

14 IF θ IS OBTUSE AND $\cos \theta = -\frac{4}{5}$, FIND:

- A** SIN **B** TAN **C** CSC **D** COT.

15 IF $-90^\circ < \theta < 0$ AND $\tan \theta = -\frac{2}{3}$, FIND COS

16 IN PROBLEMS 10 TO 15 BELOW, ABC IS A RIGHT ANGLE TRIANGLE WITH $\angle C = 90^\circ$. LET a, b, c BE ITS SIDES WHERE c IS THE HYPOTENUSE. THE SIDE OPPOSITE ANGLES ARE THE SIDE OPPOSITE ANGLES. USING THE INFORMATION BELOW, FIND THE MISSING ELEMENTS FOR EACH RIGHT TRIANGLE, ROUNDING ANSWERS CORRECT TO THE NEAREST WHOLE NUMBER.

- | | | | |
|----------|--|----------|--|
| A | $m(\angle B) = 60^\circ$ AND $b = 18$ UNITS. | B | $m(\angle A) = 45^\circ$ AND $b = 16$ UNITS. |
| C | $m(\angle A) = 22^\circ$ AND $b = 10$ UNITS. | D | $m(\angle B) = 52^\circ$ AND $b = 47$ UNITS. |

- 17** **A** FIND THE HEIGHT OF A TREE, IF THE ANGLE OF ELEVATION CHANGES FROM 25° TO 50° AS THE OBSERVER ADVANCES 15 METRES TOWARDS ITS BASE.
- B** THE ANGLE OF DEPRESSION OF THE TOP OF A POLE AND A BUILDING SEEN FROM THE TOP OF A BUILDING 145 METRES AWAY ARE 30° AND 35° RESPECTIVELY. FIND THE HEIGHTS OF THE POLE AND THE BUILDING.
- C** TO THE NEAREST DEGREE, FIND THE ANGLE OF ELEVATION WHEN A 9 METRE VERTICAL FLAGPOLE CASTS A SHADOW 3 METRES LONG.

Unit

6

PLANE GEOMETRY

Unit Outcomes:

After completing this unit, you should be able to:

- know more theorems special to triangles.
- know basic theorems specific to quadrilaterals.
- know theorems about circles and angles inside, on and outside a circle.
- solve geometrical problems involving quadrilaterals, circles and regular polygons.

Leonardo da Vinci obtained the “Mona Lisa” smile by tilting the lips so that the ends lie on a circle which touches the outer corners of the eyes.



The outline of the top of the head is the arc of another circle exactly twice as large as the first.

Main Contents

- 6.1 Theorems on triangles**
- 6.2 Special quadrilaterals**
- 6.3 More on circles**
- 6.4 Regular polygons**

Key Terms

Summary

Review Exercises

INTRODUCTION

WHY DO YOU STUDY GEOMETRY?

- ◆ GEOMETRY TEACHES YOU HOW TO THINK CLEARLY. OF ALL SUBJECTS TAUGHT AT SCHOOL LEVEL, GEOMETRY IS ONE OF THE LESSONS THAT GIVES THE BEST TRAINING AND ACCURATE METHODS OF THINKING.
- ◆ THE STUDY OF GEOMETRY HAS A PRACTICAL VALUE. IF SOMEONE WANTS TO BE A DESIGNER, A CARPENTER, A TINSMITH, A LAWYER OR A DENTIST, THE FACTS AND PRINCIPLES IN GEOMETRY ARE OF GREAT VALUE.

Abraham Lincoln BORROWED A GEOMETRY TEXT AND LEARNED THE PROOFS OF MOST PLANE GEOMETRY THEOREMS SO THAT HE COULD MAKE BETTER ARGUMENTS IN COURT.

Leonardo da Vinci OBTAINED THE “MONA LISA” SMILE BY TILTING THE LIPS SO THAT THE ENDS LIE ON A CIRCLE WHICH TOUCHES THE OUTER CORNERS OF THE EYES. THE OUTLINE OF THE TOP OF THE HEAD IS THE ARC OF ANOTHER CIRCLE EXACTLY TWICE AS LARGE AS THE FIRST. IN THE SAME ARTIST’S ‘LAST SUPPER’, THE VISIBLE PART OF CHRIST CONFORMS TO THE SHAPE OF AN EQUILATERAL TRIANGLE.

PLANE GEOMETRY (SOMETIMES CALLED EUCLIDEAN GEOMETRY) IS A BRANCH OF MATHEMATICS DEALING WITH THE PROPERTIES OF FLAT SURFACES AND PLANE FIGURES, SUCH AS QUADRILATERALS OR CIRCLES.

6.1 THEOREMS ON TRIANGLES

IN PREVIOUS GRADES, YOU HAVE LEARNT THAT A TRIANGLE IS A POLYGON WITH THREE SIDES. IT IS THE SIMPLEST TYPE OF POLYGON.

THREE OR MORE POINTS THAT LIE ON ONE LINE ARE CALLED ~~COLLINEAR POINTS~~ OR ~~THREE OR MORE~~ ~~COLLINEAR POINTS~~. LINES THAT PASS THROUGH ONE POINT ARE CALLED ~~concurrent lines~~

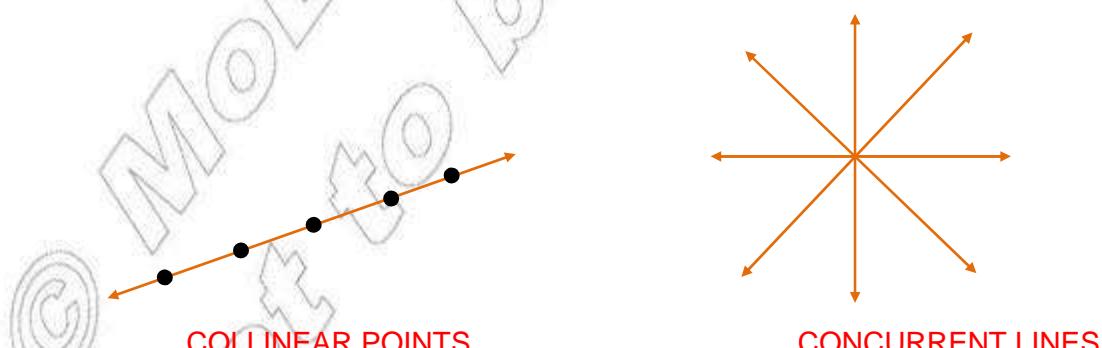


Figure 6.1

ACTIVITY 6.1



- 1 WHAT DO YOU CALL A LINE SEGMENT JOINING A VERTEX ANGLE TO THE MID-POINT OF THE OPPOSITE SIDE?
- 2 HOW MANY MEDIANES DOES A TRIANGLE HAVE?
- 3 DRAW TRIANGLE ABC WITH $\angle C = 90^\circ$, $AC = 8 \text{ CM}$ AND $BC = 6 \text{ CM}$. DRAW THE MEDIAN FROM A TO BC . HOW LONG IS THIS MEDIAN? CHECK YOUR RESULT USING THEOREM
- 4 DRAW A TRIANGLE. CONSTRUCT ALL THE THREE MEDIANES. ARE THEY CONCURRENT? THINK THAT THIS IS TRUE FOR ALL TRIANGLES? TEST THIS BY DRAWING MORE TRIANGLES.
- 5 IS IT POSSIBLE FOR THE MEDIANES OF A TRIANGLE TO MEET OUTSIDE THE TRIANGLE?

THEOREMS ABOUT COLLINEAR POINTS AND CONCURRENT LINES ARE CALLED SOME SUCH THEOREMS ARE STATED BELOW.

RECALL THAT A LINE THAT DIVIDES AN ANGLE INTO TWO CONGRUENT ANGLES IS CALLED BISECTOR OF THE ANGLE.

A LINE THAT DIVIDES A LINE SEGMENT INTO TWO CONGRUENT LINE SEGMENTS IS CALLED BISECTOR OF THE LINE SEGMENT. WHEN A BISECTOR OF A LINE SEGMENT FORMS A LINE SEGMENT, THEN IT IS CALLED THE PERPENDICULAR BISECTOR OF THE LINE SEGMENT.

Median of a triangle

A median of a triangle is a line segment drawn from a vertex of the triangle to the mid-point of the opposite side.

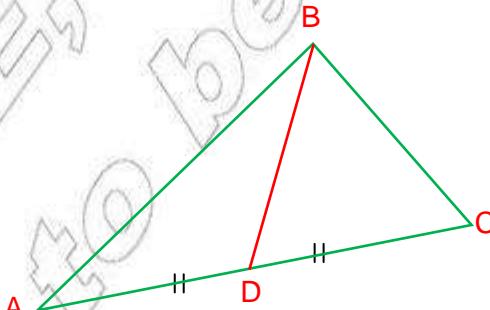


Figure 6.2

\overline{BD} is a median of triangle ABC.

ACTIVITY 6.2

COPY $\triangle ABC$ IN FIGURE 6.3

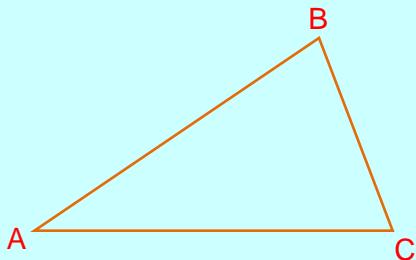


Figure 6.3

- 1** CONSTRUCT ALL THE MEDIANSCAREFULLY.
- 2**
 - I** MARK THE MID-POINT \overline{AC} AS E .
 - II** MARK THE MID-POINT \overline{BC} AS F .
 - III** MARK THE MID-POINT \overline{AB} AS D
- 3** DID THE MEDIANSSINTERSECT AT A POINT?
IF YOUR ANSWER IS YES, MARK THE POINT O.
- 4** MEASURE EACH OF THE FOLLOWING SEGMENTS AND DETERMINE THE INDICATED RATIOS

I A \overline{AO}	B \overline{OE}	$AO : OE$
II A \overline{CO}	B \overline{OD}	$CO : OD$
III A \overline{BO}	B \overline{OF}	$BO : OF$
- 5** HOW DO YOU RELATE THE RATIOS OBTAINED ABOVE?

THE ABOVE ACTIVITY HELPS YOU TO OBSERVE THE FOLLOWING THEOREM

Theorem 6.1

The medians of a triangle are concurrent at a point $\frac{2}{3}$ of the distance from each vertex to the mid-point of the opposite side.

Proof:-

SUPPOSE \overline{AE} AND \overline{DC} ARE MEDIANSAOF $\triangle ABC$ THAT ARE INTERSECTING AT POINT O.
(See FIGURE 6.4)

Statement		Reason	
1	IN $\triangle ABC$, \overline{AE} AND \overline{DC} ARE MEDIANAS INTERSECT POINT O	1	GIVEN
2	DRAW \overline{DE}	2	CONSTRUCTION
3	DRAW \overline{EG} PARALLEL TO \overline{AC} ON THE EXTENSION \overline{AC}	3	CONSTRUCTION
4	DRAW \overline{EF} PARALLEL \overline{AB} WITH F ON \overline{AC}	4	CONSTRUCTION
5	DRAW \overline{FH} PARALLEL \overline{DC} WITH H ON \overline{AB}	5	CONSTRUCTION
6	DRAW LINE PARALLEL \overline{FC} PASSING THROUGH A.	6	CONSTRUCTION
7	AFED AND CGED ARE PARALLELOGRAMS WITH SIDE \overline{DE}	7	STEPS 2 AND 4
8	THEREFORE, $AE = CG$	8	STEP 7
9	$DE = \frac{1}{2} AC = AF$	9	$\triangle ABC \sim \triangle DBE$ FROM STEP 1
10	$AF = FC = CG$	10	STEPS 8 AND 9
11	\overline{AG} IS TRISECTED BY PARALLEL LINES \overline{DE} AND \overline{EG}	11	STEPS 3, 5 AND 10
12	\overline{AE} IS TRISECTED BY \overline{HF} , \overline{DC} AND \overline{EG}	12	STEP 11 AND PROPERTY OF PARALLEL LINES

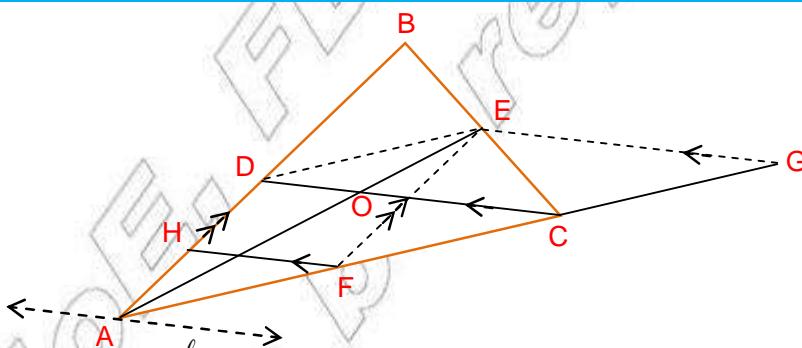


Figure 6.4

$$\text{THEREFORE, } OE = \frac{1}{3} AE, AO = \frac{2}{3} AE.$$

YOU HAVE PROVED THAT THE MEDIANAS \overline{AE} MEET AT POINT O SUCH THAT $AO = \frac{2}{3} AE$.

YOUR NEXT TASK IS TO PROVE THAT THE MEDIANAS INTERSECT AT THE SAME POINT WITH THE SAME ARGUMENT USED ABOVE, THE POINT OF INTERSECTION O AND \overline{HF}

WHOSE DISTANCE FROM AE THAT IS $= \frac{2}{3} AE$

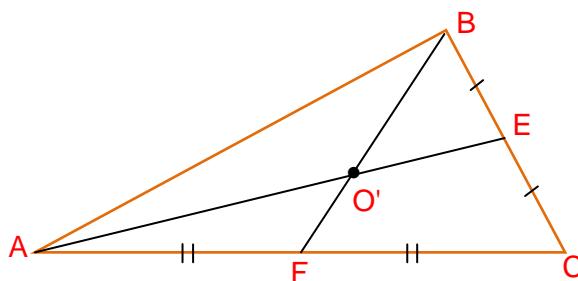


Figure 6.5

IT FOLLOWS THAT O AND HENCE O' ASO AND O ARE ONE.

THEREFORE, ALL THE THREE MEDIAN ARE CONCURRENT AT A SINGLE POINT LOCATED $\frac{2}{3}$ OF THE DISTANCE FROM EACH VERTEX TO THE MID-POINT OF THE OPPOSITE SIDE.

EXAMPLE 1 IN FIGURE 6.6, \overline{AN} , \overline{CM} AND \overline{BL} ARE MEDIAN. IF $AN = 12 \text{ CM}$, $OM = 5 \text{ CM}$ AND $BO = 6 \text{ CM}$, FIND BN AND OL .

SOLUTION:

BY THEOREM 6.1

$$BO = \frac{2}{3} BL \text{ AND } AO = \frac{2}{3} AN$$

$$\text{SUBSTITUTING } \frac{2}{3} BL \text{ AND } AO = \frac{2}{3} \times 12$$

$$\text{SO } BL = 9 \text{ CM AND } AO = 8 \text{ CM.}$$

$$\text{SINCE } BL = BO + OL,$$

$$OL = BL - BO = 9 - 6 = 3 \text{ CM.}$$

$$\text{NOW } AN = AO + ON \text{ GIVES}$$

$$ON = AN - AO = 12 - 8 = 4 \text{ CM}$$

$$\therefore BL = 9 \text{ CM}, OL = 3 \text{ CM AND } ON = 4 \text{ CM}$$

Note: THE POINT OF INTERSECTION OF THE MEDIAN IS CALLED THE CENTROID OF THE TRIANGLE.

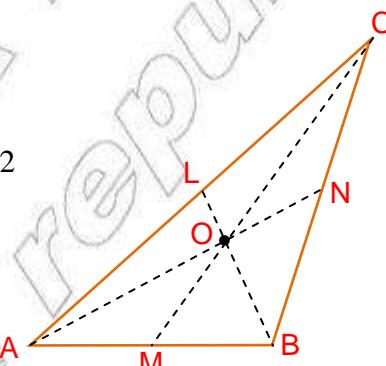


Figure 6.6

Altitude of a triangle

THE **altitude** OF A TRIANGLE IS A LINE SEGMENT DRAWN PERPENDICULARLY TO THE OPPOSITE SIDE, OR TO THE OPPOSITE SIDE PRODUCED.

THE **altitudes** THROUGH A FOR THE TRIANGLES ARE SHOWN IN

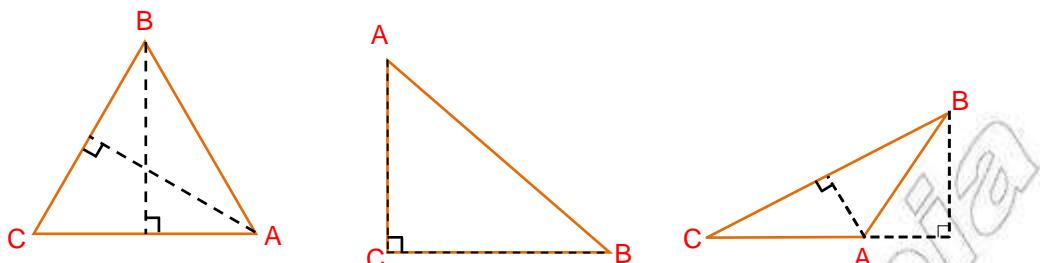


Figure 6.7

ACTIVITY 6.3



- 1 WHAT IS MEANT BY AN ANGLE BISECTOR?
- 2 ANY SIDE OF A TRIANGLE MAY BE DESIGNATED AS A BASE. HOW MANY BASES MAY A TRIANGLE HAVE?
- 3 HOW MANY ALTITUDES CAN A TRIANGLE HAVE?
- 4 BY DRAWING THE FOLLOWING TYPES OF TRIANGLES, DETERMINE WHETHER THE ALTITUDES INTERSECT INSIDE OR OUTSIDE THE TRIANGLE.
 - A AN ACUTE-ANGLED TRIANGLE;
 - B AN OBTUSE-ANGLED TRIANGLE;
 - C A RIGHT-ANGLED TRIANGLE.
- 5 DRAW THE PERPENDICULAR BISECTORS OF THE SIDES OF TRIANGLES, AND NOTE WHERE THE PERPENDICULAR BISECTORS INTERSECT.
 - A AN ACUTE-ANGLED TRIANGLE;
 - B AN OBTUSE-ANGLED TRIANGLE;
 - C A RIGHT-ANGLED TRIANGLE.
- 6 DRAW ANY $\triangle ABC$. CONSTRUCT THE PERPENDICULAR BISECTOR OF \overline{AB} AND THE SIDES \overline{BC} . LABEL THEIR INTERSECTION AS POINT O.
 - A WHY IS POINT O EQUIDISTANT FROM A AND B?
 - B WHY IS POINT O EQUIDISTANT FROM B AND C?
 - C DO YOU THINK THAT THE PERPENDICULAR BISECTORS OF THE SIDES OF $\triangle ABC$ PASS THROUGH THE POINT O? (WHY?)

ACTIVITY 6.3 CAN HELP YOU TO STATE THE FOLLOWING

Theorem 6.2

The perpendicular bisectors of the sides of any triangle are concurrent at a point which is equidistant from the vertices of the triangle.

LET $\triangle ABC$ BE GIVEN AND CONSTRUCT PERPENDICULAR BISECTORS ON ANY TWO OF THE PERPENDICULAR BISECTORS OF $\triangle ABC$ ARE SHOWN IN FIGURE 6.8A. THESE PERPENDICULAR BISECTORS INTERSECT AT O . THEY CANNOT BE PARALLEL. (WHY?)

USING A RULER, FIND THE DISTANCE AO , BO AND CO . OBSERVE THAT THE INTERSECTION POINT IS EQUIDISTANT FROM EACH VERTEX OF THE TRIANGLE.

NOTE THAT THE PERPENDICULAR BISECTOR OF THE REMAINING SIDE THROUGH THE POINT O . THEREFORE, THE POINT OF INTERSECTION OF THE THREE PERPENDICULAR BISECTORS IS EQUIDISTANT FROM THE THREE VERTICES OF

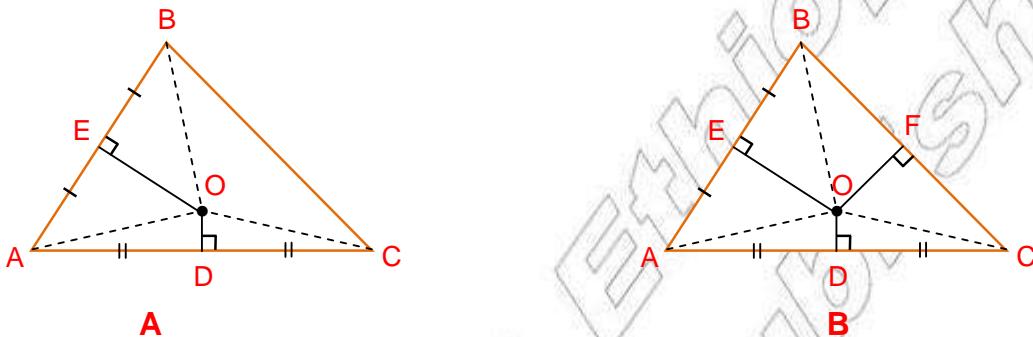


Figure 6.8

LET US TRY TO PROVE THIS RESULT.

WITH O THE POINT WHERE THE PERPENDICULAR BISECTORS MEET, AS SHOWN IN FIGURE 6.8B, $\triangle AOD \cong \triangle COD$ BY SAS AND HENCE $AO = CO$.

SIMILARLY, $\triangle BOE \cong \triangle COE$ BY SAS AND HENCE $BO = CO$.

THUS, $AO = BO = CO$. IT FOLLOWS THAT O IS EQUIDISTANT FROM THE VERTICES OF

NEXT, LET F BE THE FOOT OF THE PERPENDICULAR CAR FROM O TO THE PERPENDICULAR BISECTOR OF BC BECAUSE $\triangle BOC$ IS AN ISOSCELES TRIANGLE.

THEREFORE, THE PERPENDICULAR BISECTORS ARE CONCURRENT.

Note: THE POINT OF INTERSECTION OF THE PERPENDICULAR BISECTORS OF A TRIANGLE IS CALLED THE **circumcentre** OF THE TRIANGLE.

Theorem 6.3

The altitudes of a triangle are concurrent.

TO SHOW THAT THE THREE ALTITUDES OF A SINGLE POINT, CONSTRUCT (SHOWN IN FIGURE 6.9) SO THAT THE THREE SIDES ARE PARALLEL RESPECTIVELY TO THE THREE SIDES ABC :

LET \overline{EA} , \overline{BF} AND \overline{CD} BE THE ALTITUDES OF THE QUADRILATERAL $\triangle ABCB'$ AND $\triangle A'C'BC$ ARE PARALLELOGRAMS. (WHY?) SINCE $\triangle A'B'C$ IS A PARALLELOGRAM. (WHY?) AGAIN, SINCE $\triangle B'C$ IS A PARALLELOGRAM, $AC = BC'$. THEREFORE, $AE = BA'$ (WHY?) AND BF BISECTS $\angle C$.

ACCORDINGLY, BF IS PERPENDICULAR TO \overline{AC} AND AE IS THE PERPENDICULAR BISECTOR OF \overline{BC} . EARLIER, ONE CAN SHOW THAT \overline{AD} AND \overline{AE} ARE PERPENDICULAR BISECTORS OF \overline{AB} AND \overline{AC} RESPECTIVELY.

THEREFORE, THE ALTITUDES ARE THE SAME AS THE PERPENDICULAR BISECTORS OF SIDES $\overline{A'B'C'}$. SINCE THE PERPENDICULAR BISECTORS OF ANY TRIANGLE ARE CONCURENT (THEOREM 6.1), IT IS THEREFORE, TRUE THAT THE ALTITUDES ARE CONCURRENT.

Note: THE POINT OF INTERSECTION OF THE ALTITUDES OF A TRIANGLE IS CALLED THE ORTHOCENTRE OF THE TRIANGLE.

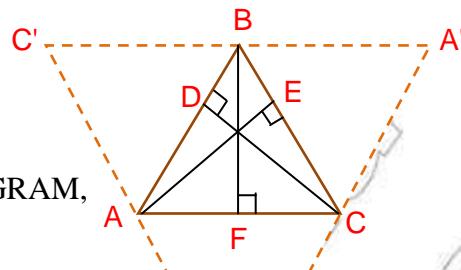


Figure 6.9

Angle bisector of a triangle

Theorem 6.4

The angle bisectors of any triangle are concurrent at a point which is equidistant from the sides of the triangle.

TO SHOW THAT THE ANGLE BISECTORS CONVERGE AT A SINGLE POINT, DRAW THE BISECTORS OF $\angle A$ AND $\angle C$, INTERSECTING EACH OTHER AT O (SEE FIGURE 6.10).

CONSTRUCT THE PERPENDICULAR BISECTORS \overline{OB} AND \overline{OC} .

DO THESE SEGMENTS HAVE THE SAME LENGTH? SHOW THAT $\triangle OBB' \cong \triangle OBA'$ AND CONCLUDE THAT

$$\angle OBB' \equiv \angle OBA'.$$

THEREFORE, THE BISECTOR OF $\angle B$ ALSO PASSES THROUGH THE POINT O .

THEREFORE, THE ANGLE BISECTORS CONVERGE AT A SINGLE POINT. ALSO THEIR POINT OF INTERSECTION IS EQUIDISTANT FROM THE THREE SIDES OF

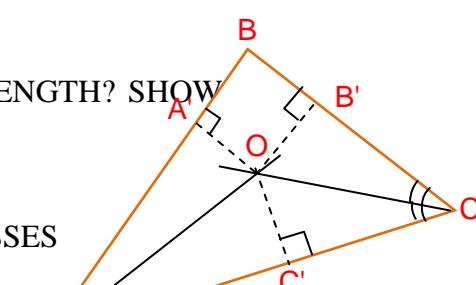


Figure 6.10

Note: THE POINT OF INTERSECTION OF THE BISECTORS OF TRIANGLE IS CALLED THE **Incentre** OF THE TRIANGLE.

EXAMPLE 2 IN A RIGHT ANGLE TRIANGLE IS A RIGHT ANGLE, 8 CM AND $CA = 6$ CM. FIND THE LENGTH OF OQ WHERE O IS THE POINT OF INTERSECTION OF THE PERPENDICULAR BISECTORS OF

SOLUTION: THE PERPENDICULAR BISECTOR OF $\angle C$ IS OB

HENCE Q IS ON OB .

THEREFORE $OQ = 4$. (BY THEOREM 6.4 $O = BO$)

BY THEOREM 6.2 Q IS EQUIDISTANT FROM C AND A

THEREFORE $OQ = AO = 4$ CM.

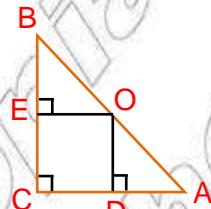


Figure 6.11

Group Work 6.1

WORK IN A SMALL GROUP ON ONE OR MORE OF THE FOLLOWING STATEMENTS. THERE WILL BE A CLASS DISCUSSION ON THESE STATEMENTS. EACH ONE SHOULD BE ATTEMPTED BY AT LEAST ONE GROUP.



Task: CHECK THAT THE FOLLOWING STATEMENTS ARE TRUE OR FALSE BY CARRYING OUT THE CONSTRUCTION CAREFULLY.

Materials required: RULER, PROTRACTOR AND COMPASSES

Method: CONSTRUCTION AND MEASUREMENT

- 1 THE MEDIAN OF ANY TRIANGLE ARE CONCURRENT.
- 2 THE MEDIAN OF A TRIANGLE ARE CONCURRENT $\frac{2}{3}$ THE DISTANCE FROM EACH VERTEX TO THE MID-POINT OF THE OPPPOSITE SIDE.
- 3 THE ALTITUDES OF ANY TRIANGLE ARE CONCURRENT.
- 4 THE PERPENDICULAR BISECTORS OF THE SIDES ARE CONCURRENT AT A POINT WHICH IS EQUIDISTANT FROM THE VERTICES OF THE TRIANGLE.
- 5 THE ANGLE BISECTORS OF ANY TRIANGLE ARE CONCURRENT AND ARE EQUIDISTANT FROM THE SIDES OF THE TRIANGLE.
- 6 GIVEN ANY TRIANGLE, EXPLAIN HOW YOU CAN FIND THE CENTRE OF
 - A INScribed IN THE TRIANGLE (INCENTRE).
 - B CIRCUMSCRIBED ABOUT THE TRIANGLE (CIRCUMCENTRE).

Altitude theorem

THE ALTITUDE THEOREM IS STATED HERE FOR A RIGHT ANGLED TRIANGLE. IT RELATES THE LENGTH OF THE ALTITUDE TO THE HYPOTENUSE OF A RIGHT ANGLED TRIANGLE, TO THE LENGTHS OF THE HYPOTENUSE.

Theorem 6.5 Altitude theorem

IN A RIGHT ANGLED TRIANGLE THE ALTITUDE ON THE HYPOTENUSE

$$\frac{AD}{DC} = \frac{CD}{DB}$$

Proof:-

CONSIDER $\triangle ABC$ AS SHOWN IN FIGURE 6.1 $\triangle ABC \sim \triangle ACD \dots$ AA SIMILARITY

SO, $\angle ABC \equiv \angle ACD$

SIMILARLY $\triangle ABC \sim \triangle CBD \dots$ AA SIMILARITY

SO, $\angle ABC \equiv \angle CBD$.

IT FOLLOWS $\angle ACD \equiv \angle CBD$.

BY AA SIMILARITY $\triangle ACD \sim \triangle CBD$.

HENCE $\frac{AD}{CD} = \frac{CD}{BD} \dots (*)$

EQUIVALENTLY, $\frac{AD}{DC} = \frac{CD}{DB}$

THE FOLLOWING ARE SOME FORMS OF THE THEOREM

FROM (*), $(CD)^2 = (AD)(BD)$

OR $(AD)(DB) = (CD)(DC)$

THIS CAN BE STATED AS:

THE SQUARE OF THE LENGTH OF THE ALTITUDE IS THE PRODUCT OF THE LENGTHS OF THE SEGMENTS OF THE HYPOTENUSE.

EXAMPLE 3 IN $\triangle ABC$, \overline{CD} IS THE ALTITUDE TO THE HYPOTENUSE \overline{AB} AND

$BD = 4$ CM HOW LONG IS THE ALTITUDE? FIGURE 6.12

SOLUTION LET $h = CD$. FROM THE ALTITUDE THEOREM, $(BD)^2 = (AD)(DC)$

SUBSTITUTING, $9 \times 4 = 36$ CM²

SQ $h = 6$ CM.

THE LENGTH OF THE ALTITUDE IS 6 CM.

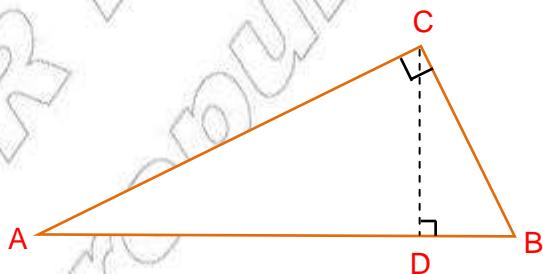


Figure 6.12

Menelaus' theorem

Menelaus' theorem WAS KNOWN TO THE ANCIENT GREEKS ALMOST TWO THOUSAND AGO. IT WAS NAMED IN HONOUR OF THE GREEK MATHEMATICIAN Menelaus ASTRONOMER (70 - 140 AD).

Theorem 6.6 Menelaus' theorem

If points D , E and F on the sides \overline{BC} , \overline{CA} and \overline{AB} respectively of $\triangle ABC$ (or their extensions) are collinear, then $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1$. Conversely, if $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1$, then the points D , E and F are collinear.

Note: 1 FOR A LINE SEGMENT WE USE THE CONVENTION BA.

2 IFF IS IN \overline{AB} , THEN $\frac{AF}{FB} = r > 0$.

IN FIGURE 6.13, \overline{D} DIVIDES \overline{BC} IN THE RATIO $\overline{D}\overline{E}$ IN THE RATIO $\overline{A}\overline{F}$ DIVIDES \overline{AB} IN THE RATIO

$$\text{i.e., } r = \frac{BD}{DC}, s = \frac{CE}{EA} \text{ AND } t = \frac{AF}{FB}.$$

WE SEE FROM THE FIGURE \overline{D} DIVIDES \overline{BC} AND \overline{D} DIVIDES \overline{CA} INTERNALLY, \overline{B} DIVIDES \overline{AB} EXTERNALLY.

ASSUME THAT E AND F ARE COLLINEAR.

DRAW \overline{AG} , \overline{BH} , \overline{CI} PERPENDICULAR TO

THEN $\triangle CEI \sim \triangle AEG$ (WHY?),

$$\text{SO, } \frac{CE}{AE} = \frac{CI}{AG} \Rightarrow \frac{CE}{EA} = \frac{CI}{AG}.$$

SIMILARLY $\triangle AFG \sim \triangle BFH$ AND $\triangle BDH \sim \triangle CDI$

$$\text{SO, } \frac{AF}{BF} = \frac{AG}{BH}, \frac{BD}{CD} = \frac{BH}{CI} \Rightarrow -\frac{AF}{FB} = \frac{AG}{BH}, -\frac{BD}{DC} = \frac{BH}{CI}.$$

$$\text{HENCE, } \left(\frac{BD}{DC} \right) \left(\frac{CE}{EA} \right) \left(\frac{AF}{FB} \right) = \left(\frac{-BH}{CI} \right) \left(\frac{-CI}{AG} \right) \left(\frac{-AG}{BH} \right) = -1$$

$$\text{THEREFORE } \left(\frac{BD}{DC} \right) \left(\frac{CE}{EA} \right) \left(\frac{AF}{FB} \right) = -1$$

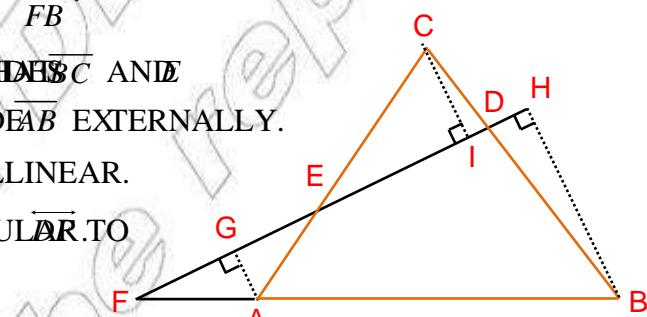


Figure 6.13

IT IS ALSO POSSIBLE FOR ALL THREE OF TO DIVIDE THEIR RESPECTIVE SIDES EXTERNALLY, AS YOU CAN SEE BY DRAWING A FIGURE. IN THIS CASE, s, t ARE ALL NEGATIVE. OTHERWISE THE PRECEDING PROOF WILL REMAIN UNCHANGED.

THEREFORE, $s = -1$ IN THIS CASE ALSO. IT IS NOT POSSIBLE TO HAVE AN EVEN NUMBER OF EXTERNAL DIVISIONS, SO $t = -1$ IN EACH OF THE POSSIBLE CASES.

TO PROVE THE CONVERSE MEET LAUS' THEOREM, ASSUME THAT $s = -1$.

EXTEND \overline{DE} UNTIL IT INTERSECTS \overline{AB} AT A POINT F . BE THE RATIO IN WHICH \overline{DE} DIVIDES \overline{AB} , THEN $t = -1$ (WHY?).

HENCE, $r = r$ (WHY?)

SINCE F IS THE ONLY POINT THAT DIVIDES \overline{AB} IN THE SAME RATIO AS E , THIS IMPLIES THAT D AND F ARE COLLINEAR.

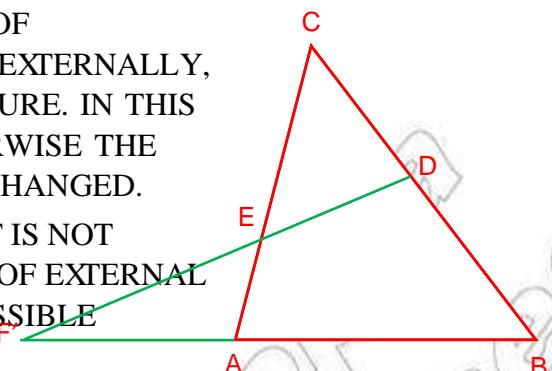


Figure 6.14

Exercise 6.1

- 1 IN FIGURE 6.15 $\overline{AD} \equiv \overline{DC}$, $\overline{AE} \equiv \overline{EB}$, F IS THE INTERSECTION OF \overline{AN} AND \overline{BD} . PROVE THAT $\overline{AF} = \frac{1}{3}\overline{EC}$.

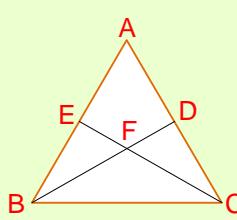


Figure 6.15

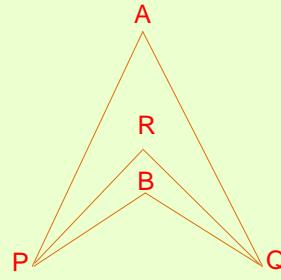


Figure 6.16

- 2 IN FIGURE 6.16 \overline{RP} AND \overline{RQ} ARE THE BISECTORS OF THE $\angle QRA$ AND $\angle PRB$ RESPECTIVELY. IF $RQ = PR$, PROVE THAT A, B LIE ON A STRAIGHT LINE.

Hint: JOIN P AND Q .

- 3 IF TWO MEDIANAS OF A TRIANGLE ARE EQUAL, THE TRIANGLE FORMED BY A SEGMENT OF EACH MEDIAN AND THE THIRD SIDE IS AN ISOSCELES TRIANGLE.
- 4 PROVE THAT THE SEGMENT JOINING THE MIDPOINTS OF A TRIANGLE IS PARALLEL TO THE THIRD SIDE AND IS HALF AS LONG AS THE THIRD SIDE.
- 5 A LET $A(0, 0)$, $B(6, 0)$ AND $C(0, 4)$ BE VERTICES OF A TRIANGLE.
I FIND THE POINT OF INTERSECTION OF THE MEDIANAS OF

II SHOW THAT THE POINT OBTAINED BY DIVIDING THE DISTANCE FROM EACH VERTEX
TO THE MID-POINT OF THE OPPOSITE SIDE.

- B** REPEAT FOR $\triangle DEF$ WHERE $D(0, 0)$, $E(4, 0)$ AND $F(2, 4)$ ARE THE VERTICES.
6 IN RIGHT ANGLED TRIANGLE SHOWN IN FIGURE 6.17, \overline{CD} IS ALTITUDE TO THE HYPOTENUSE. IF $AC = 5$ UNITS AND $AD = 4$ UNITS, FIND THE LENGTH OF

- A** \overline{BD} **B** \overline{BC}

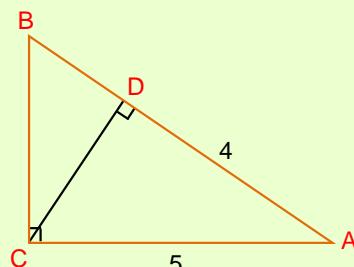


Figure 6.17

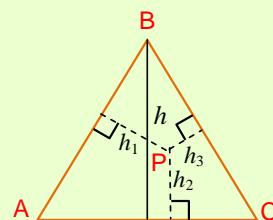


Figure 6.18

- 7** Altitude triangle for equilateral triangle: IN FIGURES 6.18, $\triangle ABC$ IS AN EQUILATERAL TRIANGLE. ALTITUDE OF LENGTHS h_1 , h_2 AND h_3 ARE DRAWN FROM THE SIDES OF THE TRIANGLE. SHOW THAT $h_1 + h_2 + h_3$.

Hint: COMPARE THE AREA OF $\triangle ABC$ WITH THE SUM OF THE AREAS OF $\triangle APC$, $\triangle APB$ AND $\triangle BPC$.

IN PROBLEMS 8 – 10, THE LETTERS A, B, C, D, E, F, R, S, T HAVE THE MEAN HAVE IN THE STATEMENT MENELAUS' THEOREM

- 8** IN FIGURE 6.19, D AND D' ARE SYMMETRICAL ABOUT THE MID-POINT OF BC. AND E' ARE ALSO SYMMETRICAL ABOUT THE MID-POINTS OF THEIR CORRESPONDING SIDES. SHOW THAT E AND E' ARE COLLINEAR AND D AND D' ARE COLLINEAR.

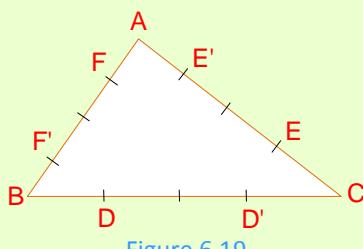


Figure 6.19

- 9** IN THE PROOF OF THE CONVERSE PART OF THE QF, ASSUME THAT \overrightarrow{DE} MEETS \overrightarrow{AB} AT SOME POINT
A PROVE THAT $\overrightarrow{DE} \parallel \overrightarrow{AB}$, THEN $\angle 1 = 1$.
B PROVE THAT IF $\angle 1 \neq 1$, THEN \overrightarrow{DE} IS NOT PARALLEL TO
C PROVE THAT IF $\angle 1 \neq 1$, THEN $\overrightarrow{DE} \parallel \overrightarrow{AB}$.

- 10** IN **FIGURE 6.20** BELOW, $\overline{D'C}$ DIVIDES \overline{AC} IN THE RATIO $1:2$, $\overline{D'B}$ DIVIDES \overline{AB} IN THE SAME RATIO. E IS THE MID-POINT OF \overline{AD} , E, F ARE COLLINEAR, AND F' ARE ALSO COLLINEAR. SHOW THAT

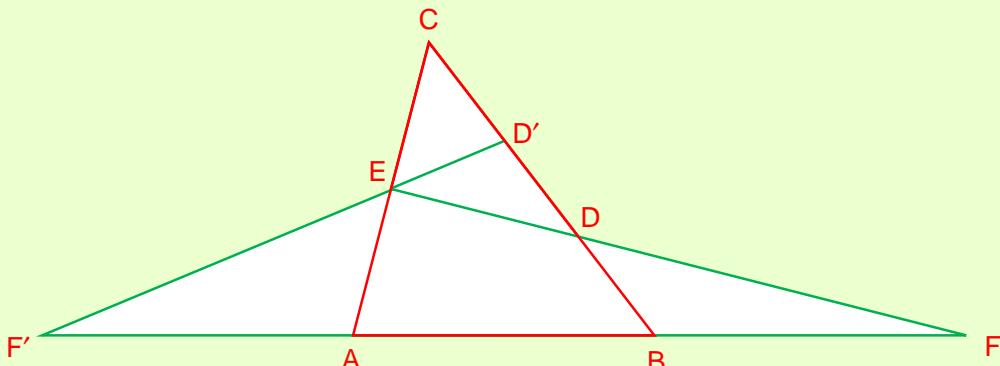


Figure 6.20

6.2 SPECIAL QUADRILATERALS

IN THIS SECTION, WE CONSIDER THE FOLLOWING SPECIAL QUADRILATERALS: **parallelogram, rectangle, rhombus AND square.**

KEEP IN MIND THE MATHEMATICAL DEFINITIONS OF EACH OF THE ABOVE QUADRILATERALS.

ACTIVITY 6.4

- 1** DISCUSS PARALLEL LINES BASED ON WHAT YOU SEE IN MY CLASSROOM.
- 2** STATE THE PARALLEL LINES POSTULATE.
- 3** DISCUSS WHAT IS MEANT BY “EQUIANGULAR QUADRILATERAL”?
- 4** DEFINE THE FOLLOWING QUADRILATERALS IN YOUR OWN TERMS.
A PARALLELOGRAM **B** RECTANGLE **C** SQUARE
- 5** WHAT IS AN ALTITUDE OF A PARALLELOGRAM?
- 6** IN **FIGURE 6.21**
 - I** INDICATE A PAIR OF ADJACENT SIDES.
 - II** INDICATE OPPOSITE VERTICES OF THE QUADRILATERAL.
 - III** JOIN TWO OPPOSITE VERTICES.
- 7** WHAT DO YOU CALL THIS LINE SEGMENT?
WHAT DOES IT HAVE TO DO WITH A PARALLELOGRAM OR RECTANGLE?

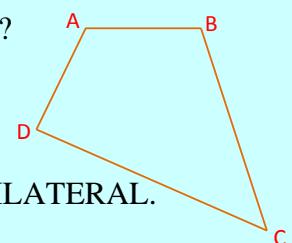


Figure 6.21

Trapezium

Definition 6.1

A **trapezium** is a quadrilateral where only two of the sides are parallel.

IN FIGURE 6.22 THE QUADRILATERAL A IS A TRAPEZIUM. THE SIDES \overline{AB} AND \overline{DC} ARE NON-PARALLEL SIDES OF THE TRAPEZIUM.

NOTE THAT IF THE SIDES \overline{AD} AND \overline{BC} OF TRAPEZIUM ABCD ARE CONGRUENT, THEN THE TRAPEZIUM IS CALLED AN **isosceles trapezium**.

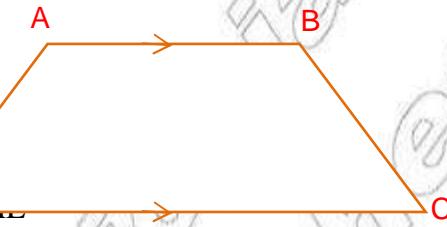


Figure 6.22

Parallelogram

Definition 6.2

A **parallelogram** is a quadrilateral in which both pairs of opposite sides are parallel.

IN FIGURE 6.23 THE QUADRILATERAL A IS A PARALLELOGRAM.

$\overline{AB} \parallel \overline{DC}$ AND $\overline{AD} \parallel \overline{BC}$

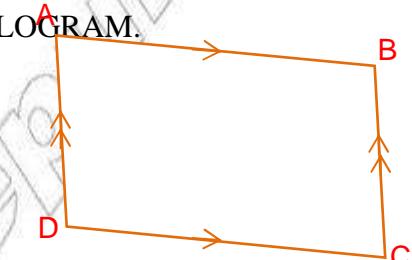


Figure 6.23

ACTIVITY 6.5



- 1 DRAW A QUADRILATERAL $PQRST$, Q, R AND S BE THE MID-POINTS OF ITS SIDES. CHECK, BY CONSTRUCTION AND MEASUREMENT, THAT $PQRS$ IS A PARALLELOGRAM.
- 2 DRAW A TRAPEZIUM $ABCD$ WITH $AB = 2$ CM, $BC = DA = 3$ CM AND $DC = 4$ CM.
 - A INDICATE AND MEASURE THE BASE ANGLES $\angle A$ AND $\angle D$. TRAPEZIUM
 - B DRAW THE DIAGONALS \overline{AC} AND \overline{BD} AND THEN MEASURE THEIR LENGTHS. ALSO, COMPARE THE LENGTHS OF THE TWO DIAGONALS.
- 3 DRAW A PARALLELOGRAM WITH $AB = 3$ CM AND $BC = 8$ CM.
 - A MARK POINTS A AND B ON THAT DIVIDE IT INTO THREE CONGRUENT PARTS. THROUGH THESE POINTS, DRAW LINES ACROSS PARALLEL TO AB . WHY DO THESE LINES DIVIDE ABCD INTO THREE SMALLER PARALLELOGRAMS?

- B** MARK POINTS ON THAT DIVIDE IT INTO FOUR CONGRUENT SEGMENTS. THROUGH THESE POINTS, DRAW LINES ACROSS PARALLEL TO HOW MANY SMALL PARALLELOGRAMS DOES THIS MAKE?
- C** DRAW THE DIAGONALS OF ALL THE SMALLER PARALLELOGRAMS. THESE DIAGONALS ALSO FORM PARALLELOGRAMS.

PROPERTIES OF A PARALLELOGRAM AND TESTS FOR A QUADRILATERAL TO BE A PARALLELOGRAM ARE STATED IN THE FOLLOWING THEOREM:

Theorem 6.7

- A** The opposite sides of a parallelogram are congruent.
- B** The opposite angles of a parallelogram are congruent.
- C** The diagonals of a parallelogram bisect each other.
- D** If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.
- E** If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.
- F** If the opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

Proof of A and B:-

Given: PARALLELOGRAM

To prove: $\overline{AB} \equiv \overline{CD}$ AND $\overline{BC} \equiv \overline{DA}$

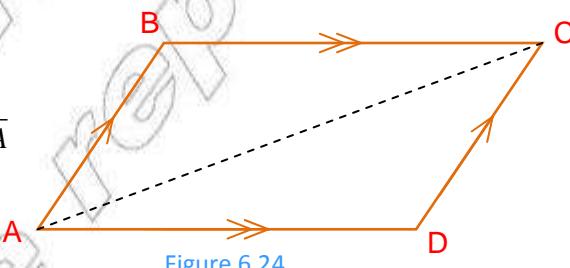


Figure 6.24

Statement		Reason
1	DRAW DIAGONAL \overline{AC}	1 THROUGH TWO POINTS THERE IS EXACTLY ONE STRAIGHT LINE.
2	$\overline{AC} \equiv \overline{CA}$	2 COMMON SIDE.
3	$\angle CAB \equiv \angle ACD$ AND $\angle ACB \equiv \angle CAD$	3 ALTERNATE INTERIOR ANGLES OF PARALLEL LINES.
4	$\Delta ABC \equiv \Delta CDA$	4 ASA POSTULATE.
5	$\overline{AB} \equiv \overline{CD}$ AND $\overline{BC} \equiv \overline{DA}$, AND $\angle ABC \equiv \angle CDA$	5 CORRESPONDING PARTS OF CONGRUENT TRIANGLES

Can you show that $\angle BAD \equiv \angle DCB$?

Proof of C:-

Given: PARALLELOGRAM WITH
DIAGONALS \overline{AC} AND \overline{BD}
INTERSECTING AT O.

To prove: $\overline{AO} \equiv \overline{OC}$ AND $\overline{BO} \equiv \overline{DO}$.

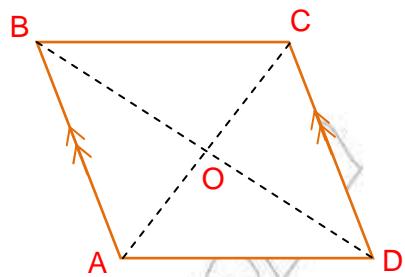


Figure 6.25

Statement		Reason	
1	$\overline{AB} \equiv \overline{CD}$	1	THEOREM 6.7A
2	$\angle CAB \equiv \angle ACD$ AND $\angle ABD \equiv \angle CDB$ HENCE, $\angle OAB \equiv \angle OCD$ AND $\angle ABO \equiv \angle CDO$	2	ALTERNATE INTERIOR ANGLES
3	$\Delta AOB \cong \Delta COD$	3	ASA POSTULATE
4	$\overline{AO} \equiv \overline{CO}$ AND $\overline{BO} \equiv \overline{DO}$	4	CORRESPONDING PARTS OF CONGRUENT TRIANGLES.

Proof of F:-

Given: A QUADRILATERAL WITH
 $\angle A \equiv \angle C$ AND $\angle B \equiv \angle D$.

To prove: ABCD IS A PARALLELOGRAM

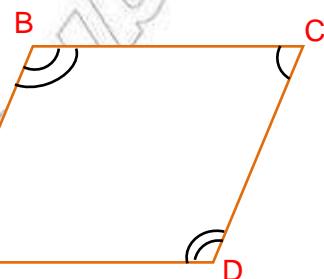


Figure 6.26

Statement		Reason	
1	$m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = 360^\circ$	1	THE SUM OF THE INTERIOR ANGLES OF A QUADRILATERAL IS 360
2	$m(\angle A) = m(\angle C)$ AND $m(\angle B) = m(\angle D)$	2	GIVEN
3	$2m(\angle A) + 2m(\angle D) = 360^\circ$	3	STEPS 1 AND 2
4	$m(\angle A) + m(\angle D) = 180^\circ$	4	SIMPLIFICATION
5	THEREFORE $\overline{AB} \parallel \overline{DC}$	5	$\angle A$ AND $\angle D$ ARE INTERIOR ANGLES ON THE SAME SIDE OF TRANSVERSAL
6	$m(\angle A) + m(\angle B) = 180^\circ$	6	STEP 2 AND 4.
7	THEREFORE $\overline{AD} \parallel \overline{BC}$	7	$\angle A$ AND $\angle B$ ARE INTERIOR ANGLES ON THE SAME SIDE OF TRANSVERSAL
8	ABCD IS A PARALLELOGRAM	8	DEFINITION OF A PARALLELOGRAM STEPS 5 AND 7.

Rectangle

Definition 6.3

A **rectangle** is a parallelogram in which one of its angles is a right angle.

IN FIGURE 6.27 THE PARALLELOGRAM IS A RECTANGLE WHOSE IS A RIGHT ANGLE. WHAT IS THE MEASURE OF EACH OF THE OTHER ANGLES OF THE RECTANGLE?

Some properties of a rectangle

- I A RECTANGLE HAS ALL PROPERTIES OF A PARALLELOGRAM.
- II EACH INTERIOR ANGLE OF A RECTANGLE IS A RIGHT ANGLE.
- III THE DIAGONALS OF A RECTANGLE ARE CONGRUENT.

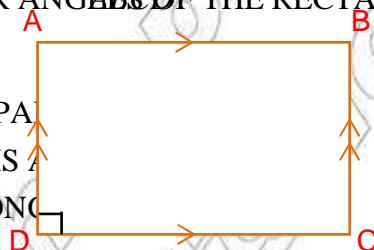


Figure 6.27

Rhombus

Definition 6.4

A **rhombus** is a parallelogram which has two congruent adjacent sides.

IN FIGURE 6.28 THE PARALLELOGRAM IS A RHOMBUS.

Some properties of a rhombus

- i A RHOMBUS HAS ALL THE PROPERTIES OF A PARALLELOGRAM.
- ii A RHOMBUS IS AN EQUILATERAL QUADRILATERAL.
- iii THE DIAGONALS OF A RHOMBUS ARE PERPENDICULAR TO EACH OTHER.
- iv THE DIAGONALS OF A RHOMBUS BISECT ITS ANGLES.

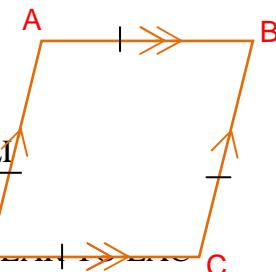


Figure 6.28

Square

Definition 6.5

A **square** is a rectangle which has congruent adjacent sides.

IN FIGURE 6.29 THE RECTANGLE IS A SQUARE.

Some properties of a square

- I A SQUARE HAS THE PROPERTIES OF A RECTANGLE.
- II A SQUARE HAS ALL THE PROPERTIES OF A RHOMBUS.

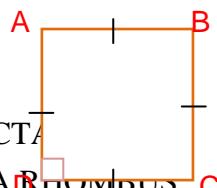


Figure 6.29

Group Work 6.2



- 1 WHAT ARE SOME SIMILARITIES AND DIFFERENCES BETWEEN A PARALLELOGRAM, A RECTANGLE AND A SQUARE?
- 2 IF $ABCD$ IS A PARALLELOGRAM WITH $\angle A = 80^\circ$, $BC = 2x + 7$ AND $CD = x + 18$, WHAT TYPE OF PARALLELOGRAM IS $ABCD$?
- 3 DISCUSS THE RELATIONSHIP AMONG THE FOUR TRIANGLES FORMED BY THE DIAGONALS OF A RHOMBUS.

Theorem 6.8

If the diagonals of a quadrilateral are congruent and are perpendicular bisectors of each other, then the quadrilateral is a square.

Proof:-

Given: $\overline{AC} \cong \overline{BD}$; \overline{AC} AND \overline{BD} ARE PERPENDICULAR BISECTORS OF EACH OTHER.

To prove: $ABCD$ IS A SQUARE.

LET O BE THE POINT OF INTERSECTION OF \overline{AC} AND \overline{BD} .

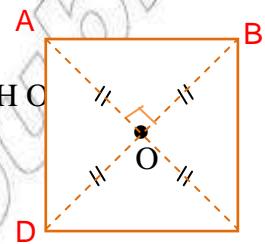


Figure 6.30

Statement		Reason	
1	$\overline{AC} \cong \overline{BD}$, \overline{AC} AND \overline{BD} ARE PERPENDICULAR BISECTORS OF EACH OTHER	1	GIVEN
2	$\overline{AO} \cong \overline{BO} \cong \overline{CO} \cong \overline{DO}$	2	STEP 1
3	$\angle AOB \cong \angle BOC \cong \angle COD \cong \angle DOA$	3	ALL RIGHT ANGLES ARE CONGRUENT
4	$\triangle AOB \cong \triangle BOC \cong \triangle COD \cong \triangle DOA$	4	SAS POSTULATE
5	$\angle CBD \cong \angle ADB$ AND $\angle DCA \cong \angle BAC$	5	CORRESPONDING ANGLES OF CONGRUENT TRIANGLES
6	$\overline{BC} \parallel \overline{AD}$ AND $\overline{AB} \parallel \overline{CD}$	6	ALTERNATE INTERIOR ANGLES ARE CONGRUENT
7	$ABCD$ IS A PARALLELOGRAM	7	DEFINITION OF A PARALLELOGRAM
8	$ABCD$ IS A RECTANGLE	8	DIAGONALS ARE CONGRUENT
9	$ABCD$ IS A SQUARE	9	DEFINITION OF A SQUARE, $\overline{AB} \cong \overline{CD}$ AND STEP 4

Exercise 6.2

- 1** ABCD IS A PARALLELOGRAM. M IS THE MID-POINT OF \overline{AB} AND Q IS THE MID-POINT OF \overline{DC} . PROVE THAT PQ IS A PARALLELOGRAM.
- 2** THE MID-POINTS OF THE SIDES OF A RECTANGLE ARE THE MID-POINTS OF THE SIDES OF A QUADRILATERAL. WHAT KIND OF QUADRILATERAL IS IT? PROVE YOUR ANSWER.
- 3** THE MID-POINTS OF THE SIDES OF A PARALLELOGRAM FORM A QUADRILATERAL. WHAT KIND OF QUADRILATERAL IS IT? PROVE YOUR ANSWER.
- 4** PROVE EACH OF THE FOLLOWING:
 - A** IF THE DIAGONALS OF A PARALLELOGRAM ARE PERPENDICULAR, THEN THE PARALLELOGRAM IS A RECTANGLE.
 - B** IF THE DIAGONALS OF A QUADRILATERAL ARE CONGRUENT, THEN THE QUADRILATERAL IS A RHOMBUS.
 - C** IF ALL THE FOUR SIDES OF A QUADRILATERAL ARE CONGRUENT, THEN THE QUADRILATERAL IS A RHOMBUS.
 - D** THE DIAGONALS OF A RHOMBUS ARE PERPENDICULAR TO EACH OTHER.
- 5** IN EACH OF THE FOLLOWING STATEMENTS, STATE WHETHER THE QUADRILATERAL IS A PARALLELOGRAM OR NOT. PROVE THIS IN EACH CASE.
 - A** IF THE OPPOSITE SIDES OF A QUADRILATERAL ARE CONGRUENT, THEN THE QUADRILATERAL IS A PARALLELOGRAM.
 - B** IF ONE PAIR OF OPPOSITE SIDES OF A QUADRILATERAL ARE PARALLEL, THEN THE QUADRILATERAL IS A PARALLELOGRAM.
 - C** IF THE DIAGONALS OF A QUADRILATERAL BISECT EACH OTHER, THEN THE QUADRILATERAL IS A PARALLELOGRAM.
- 6** DRAW A PARALLEL TO \overline{AB} EXTENDING \overline{AB} THROUGH P SO THAT $P = BP$; EXTEND \overline{AD} THROUGH Q SO THAT $Q = DQ$. PROVE THAT M AND N ALL LIE ON ONE STRAIGHT LINE. (HINT DRAW \overline{BD})
- 7** M IS THE MID-POINT OF THE SIDE \overline{AB} OF A PARALLELOGRAM $ABCD$ AND \overline{AB} PRODUCED MEET AT N. PROVE THAT
- 8** IF $ABCD$ IS A PARALLELOGRAM AND THE MID-POINTS OF \overline{DC} AND \overline{AB} RESPECTIVELY, PROVE THAT $MN \parallel CD$.
- 9** $ABCD$ IS A PARALLELOGRAM AND \overline{AD} PRODUCED AND \overline{CB} PRODUCED SO THAT $\overline{AD} \equiv \overline{CB}$. PROVE THAT CF IS A PARALLELOGRAM.

6.3**MORE ON CIRCLES**

IN THIS SECTION, YOU ARE GOING TO STUDY CIRCLES AND THE LINES AND ANGLES AROUND THEM. OF ALL SIMPLE GEOMETRIC FIGURES, A CIRCLE IS PERHAPS THE MOST APPEALING. EVER CONSIDERED HOW USEFUL A CIRCLE IS? WITHOUT CIRCLES THERE WOULD BE NO WAGONS, AUTOMOBILES, STEAMSHIPS, ELECTRICITY OR MANY OTHER MODERN CONVENiences.

RECALL THAT A CIRCLE IS A PLANE FIGURE, ALL POINTS OF WHICH ARE EQUIDISTANT FROM A GIVEN POINT CALLED THE CENTER.

AS YOU RECALL FROM GRADE 9, IN FIGURE 6.31, \overline{PQ} IS A CHORD OF

THE CIRCLE WITH CENTER O. chord \overarc{AC} (DIAMETER \overline{XYC}) IS AN arc OF THE CIRCLE.



Figure 6.31

IF A POINT C IS NOT END-POINTS OF A DIAMETER, IT IS A MINOR ARC.

$\angle BOC$ IS A central angle. \widehat{AXC} OR \widehat{ABC} IS SAID TO subtend $\angle AOC$ OR $\angle AOC$ intercepts \widehat{AXC} .

ACTIVITY 6.6

- 1 DRAW A CIRCLE AND A LINE INTERSECTING THE CIRCLE AT ONE POINT. DRAW ANOTHER LINE INTERSECTING AT ONE POINT. DRAW A LINE THAT DOES NOT INTERSECT THE CIRCLE.
- 2 IF THE LENGTH OF A RADIUS OF A CIRCLE IS 10 CM, WHAT IS THE LENGTH OF ITS DIAMETER?
- 3 REFERRING TO FIGURE 6.32, ANSWER EACH OF THE FOLLOWING QUESTIONS:
 - A NAME AT LEAST THREE CHORDS, TWO SECANTS AND TWO TANGENTS.
 - B NAME THREE ANGLES FORMED BY TWO INTERSECTING CHORDS.
 - C NAME AN ANGLE FORMED BY TWO INTERSECTING TANGENTS.
 - D NAME AN ANGLE FORMED BY TWO INTERSECTING SECANTS.

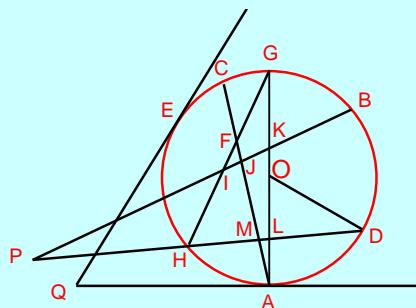


Figure 6.32

- 4 CONSTRUCT:
 - A A CENTRAL ANGLE $\angle AOB$ IN A CIRCLE.
 - B A CENTRAL ANGLE $\angle AOB$ IN A CIRCLE.

- 5 HOW LARGE IS A CENTRAL ANGLE THAT IS SUBTENDED IN A CIRCLE OF RADIUS 3 CM?
- 6 WHAT IS THE MEASURE OF A SEMI-CIRCLE AS AN ARC?
- 7 IS THE STATEMENT ‘THE MEASURE OF AN ARC IS EQUAL TO THE CORRESPONDING CENTRAL ANGLE’ TRUE OR FALSE?

6.3.1 Angles and Arcs Determined by Lines Intersecting Inside and On a Circle

WE NOW EXTEND THE DISCUSSION TO ANGLES WHOSE VERTICES DO NOT NECESSARILY LIE AT THE CENTRE OF THE CIRCLE.

IN A CIRCLE, **an inscribed angle** IS AN ANGLE WHOSE VERTEX LIES ON THE CIRCLE AND WHOSE SIDES ARE CHORDS.

IN **FIGURE 6.33**, ANG $\angle POQ$ IS INSCRIBED IN THE CIRCLE. WE ALSO SAY THAT $\angle POQ$ IS INSCRIBED IN THE $\text{arc } PQ$ AND

$\angle PRQ$ IS **subtended** BY $\text{arc } SQ$ ($\text{or } \widehat{PSQ}$).

MEASURE OF A CENTRAL ANGLE: NOTE THAT THE MEASURE OF A CENTRAL ANGLE IS THE MEASURE OF THE ARC IT INTERCEPTS.

SO, $m(\angle POQ) = m(\widehat{PQ})$.

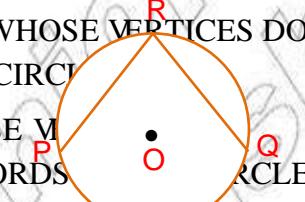


Figure 6.33

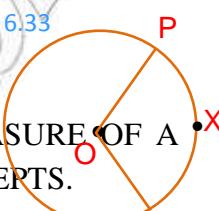


Figure 6.34

Theorem 6.9

THE MEASURE OF AN ANGLE INSCRIBED IN A CIRCLE IS HALF THE MEASURE OF THE ARC SUBTENDING IT.

Proof:-

Given: CIRCLE WITH $\angle ABC$ AN INSCRIBED ANGEL INTERCEPTING THE ARC \widehat{AC} .

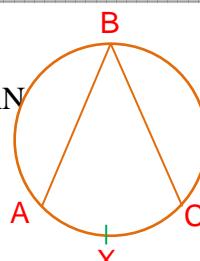


Figure 6.35

To prove: $m(\angle ABC) = \frac{1}{2} m(\widehat{AC})$, WHERE A AND C ARE POINTS ON THE CIRCLE AND X IS A POINT AS SHOWN IN **FIGURE 6.35**.

TO PROVE **THEOREM 6.9**, WE CONSIDER THREE CASES.

Case 1: SUPPOSE THAT ONE SIDE ABC IS A DIAMETER OF THE CIRCLE WITH CENTRE

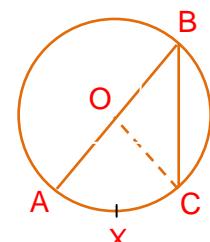


Figure 6.36

Statement		Reason	
1	DRAW RADII \overline{OB}	1	CONSTRUCTION.
2	$\overline{OC} \equiv \overline{OB}$	2	RADIIS OF THE SAME CIRCLE.
3	$\angle OBC \equiv \angle OCB$	3	BASE ANGLES OF AN ISOSCELES TRIANGLE.
4	$\angle AOC \equiv \angle OCB + \angle OBC$	4	AN EXTERIOR ANGLE OF A TRIANGLE IS EQUAL TO THE SUM OF THE TWO OPPOSITE INTERIOR ANGLES.
5	$m(\angle AOC) = 2m(\angle ABC)$	5	SUBSTITUTION.
6	BUT $m(\angle AOC) = m(\widehat{AXC})$	6	$\angle AOC$ IS A CENTRAL ANGLE.
7	$2m(\angle ABC) = m(\widehat{AXC})$	7	SUBSTITUTION.
8	$m(\angle ABC) = \frac{1}{2}m(\widehat{AXC})$	8	DIIVISION OF BOTH SIDES BY 2.

THEREFORE, $\angle ABC = \frac{1}{2}m(\widehat{AXC})$

Case 2: SUPPOSE THAT A AND C ARE ON OPPOSITE SIDES OF THE DIAMETER $THROUGH$ SHOWN IN FIGURE 6.37.

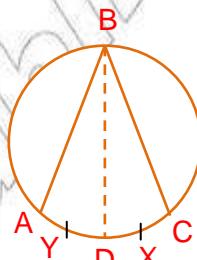


Figure 6.37

Statement		Reason	
1	$m(\angle ABD) = \frac{1}{2}m(\widehat{AYD})$	1	CASE 1
2	$m(\angle DBC) = \frac{1}{2}m(\widehat{DXC})$	2	CASE 1
3	$m(\angle ABD) + m(\angle DBC) = \frac{1}{2}m(\widehat{AYD}) + \frac{1}{2}m(\widehat{DXC})$	3	ADDITION
4	$\therefore m(\angle ABC) = \frac{1}{2}m(\widehat{AXC})$	4	SUBSTITUTION

THEREFORE, $\angle ABC = \frac{1}{2}m(\widehat{AXC})$

Case 3: SUPPOSE THAT A AND C ARE ON THE SAME SIDE OF THE DIAMETER $THROUGH$ SHOWN IN FIGURE 6.38

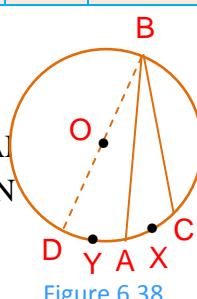


Figure 6.38

Statement		Reason	
1	$m(\angle DBC) = \frac{1}{2} m(\widehat{DAC})$	1	CASE 1
2	$m(\angle DBA) = \frac{1}{2} m(\widehat{DYA})$	2	CASE 1
3	$m(\angle DBC) - m(\angle DBA) = \frac{1}{2} m(\widehat{DAC}) - \frac{1}{2} m(\widehat{DYA})$	3	ADDITION
4	$\therefore m(\angle ABC) = \frac{1}{2} m(\widehat{AXC})$	4	SUBSTITUTION

THEREFORE $m(\angle ABC) = \frac{1}{2} m(\widehat{AXC})$ IN ALL CASES AND THE THEOREM HOLDS.

EXAMPLE 1 IN FIGURE 6.39 $m(\widehat{PXQ}) = 110^\circ$.

FIND THE MEASURE OF $\angle PRQ$.

SOLUTION: BY THEOREM 6 WE HAVE

$$m(\angle PRQ) = \frac{1}{2} m(\widehat{PXQ}) = \frac{1}{2} (110^\circ) = 55^\circ$$

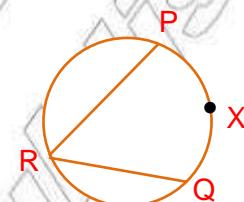


Figure 6.39

Corollary 6.9.1

An angle inscribed in a semi-circle is a right angle.

Proof:-

IN FIGURE 6.40 $\angle ABC$ IS INSCRIBED IN SEMI-CIRCLE

$\angle ABC$ IS SUBTENDED BY ARC AC WHICH IS A SEMI-CIRCLE.

THE MEASURE OF $ARC AC$ IS 180° OR π RADIAN.

$$\text{BY THEOREM 6 } m(\angle ABC) = \frac{1}{2} m(\widehat{ADC})$$

$$= \frac{1}{2} (180^\circ) = 90^\circ \text{ OR } \frac{\pi}{2} \text{ RADIAN.}$$

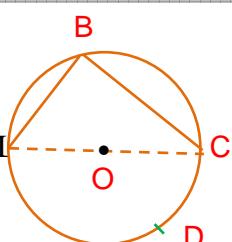


Figure 6.40

Corollary 6.9.2

An angle inscribed in an arc less than a semi-circle is obtuse.

Proof:-

$$m(\angle ABC) = \frac{1}{2} m(\widehat{ADC})$$

BUT $m(\widehat{ABC}) <$ LENGTH OF A SEMI-CIRCLE

$$m(\widehat{ABC}) < 180^\circ$$

THEREFORE $\widehat{ADC} > 180^\circ$

$$m(\angle ABC) = \frac{1}{2} m(\widehat{ADC}) > \frac{1}{2}(180^\circ)$$

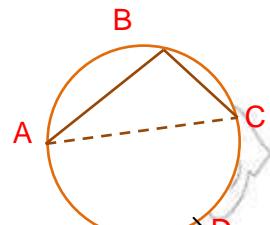
 $m(\angle ABC) > 90^\circ$. SO, $\angle ABC$ IS AN OBTUSE ANGLE.


Figure 6.41

Corollary 6.9.3An angle inscribed in an arc greater than a semi-circle is **acute**.**Theorem 6.10**

Two parallel lines intercept congruent arcs on the same circle.

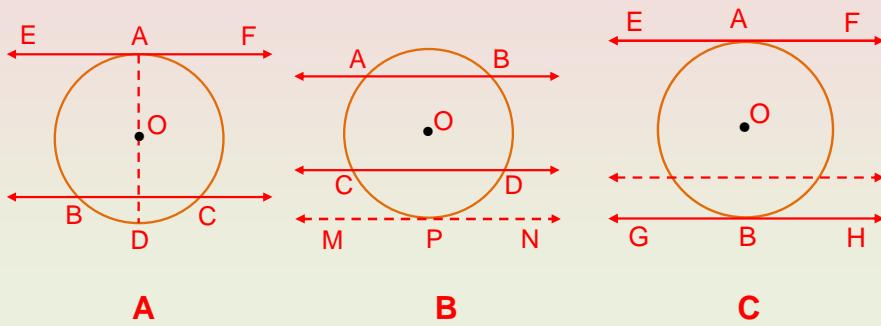


Figure 6.42

Proof:-

TO PROVE THIS FACT, YOU HAVE TO CONSIDER THREE POSSIBLE CASES:

- A** WHEN ONE OF THE PARALLEL LINES IS A TANGENT LINE AND THE OTHER IS A SECANT LINE AS SHOWN IN FIGURE 6.42A.
- B** WHEN BOTH PARALLEL LINES ARE SECANTS AS SHOWN IN FIGURE 6.42B.
- C** WHEN BOTH PARALLEL LINES ARE TANGENTS AS SHOWN IN FIGURE 6.42C.

Case a:

Given: A CIRCLE WITH CENTER O AND \overline{BC} AND \overline{EF} ARE TWO PARALLEL LINES SUCH THAT \overline{EF} IS A TANGENT TO THE CIRCLE AND \overline{BC} IS A SECANT.

To prove: $\widehat{AB} \equiv \widehat{AC}$

Statement		Reason	
1	DRAW DIAMETER \overline{AD}	1	CONSTRUCTION.
2	$\overline{AD} \perp \overline{EF}$ AND $\overline{AD} \perp \overline{BC}$	2	A TANGENT IS PERPENDICULAR TO THE DIAMETER DRAWN TO THE POINT OF TANGENCY \overline{EF} AND \overline{BC} ALSO GIVEN.
3	$\widehat{BD} \equiv \widehat{CD}$	3	ANY PERPENDICULAR FROM THE CENTRE TO A CHORD BISECTS THE CHORD AND THE ARC SUBTENDED BY IT.
4	$\widehat{AB} \equiv \widehat{AC}$	4	$\widehat{ABD} \equiv \widehat{ACD}$ (SEMICIRCLES) AND STEP 3.

PROOFS OF CASES 1 AND 2 ARE LEFT AS EXERCISES.

Theorem 6.11

An angle formed by a tangent and a chord drawn from the point of tangency is measured by half the arc it intercepts.

Given: CIRCLE WITH $\angle ABC$ FORMED BY TANGENT t AND CHORD AB THE POINT OF CONTACT.

To prove: $m(\angle ABC) = \frac{1}{2} m(\widehat{AXB})$

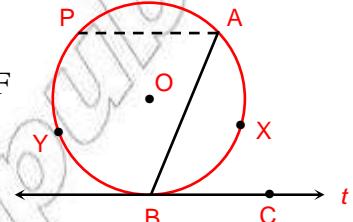


Figure 6.43

Statement		Reason	
1	DRAW \overline{AP} PARALLEL TO t	1	CONSTRUCTION.
2	$\angle PAB \equiv \angle ABC$	2	ALTERNATE INTERIOR ANGLES OF PARALLEL LINES
3	$m(\angle PAB) = \frac{1}{2} m(\widehat{PYB})$	3	THEOREM 6.9
4	$\widehat{PYB} \equiv \widehat{AXB}$	4	THEOREM 6.10
5	$\therefore m(\angle ABC) = \frac{1}{2} m(\widehat{AXB})$	5	SUBSTITUTION FROM 2 – 4

Theorem 6.12

The measure of an angle formed by two chords intersecting inside a circle is half the sum of the measures of the arc subtending the angle and its vertically opposite angle.

Proof:-

Given: TWO LINES \overleftrightarrow{AB} AND \overleftrightarrow{CD} INTERSECTING AT P INSIDE THE CIRCLE.

To prove: $m(\angle BPD) = \frac{1}{2}m(\widehat{AXC}) + \frac{1}{2}m(\widehat{BYD})$.

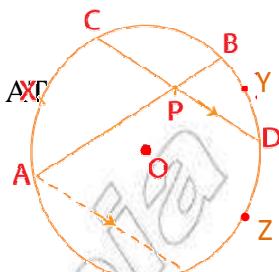


Figure 6.44

Statement		Reason	
1	DRAW A LINE THROUGH A POINT ON A CHORD	1	CONSTRUCTION
2	$m(\angle BPD) = m(\angle BAE)$	2	CORRESPONDING ANGLES FORMED BY TWO PARALLEL LINES AND A TRANSVERSAL LINE.
3	$m(\angle BAE) = \frac{1}{2}m(\widehat{BDE})$	3	THEOREM 6.9
4	$\widehat{AXC} \cong \widehat{DZE}$	4	THEOREM 6.10
5	$\therefore m(\angle BPD) = \frac{1}{2}m(\widehat{BDE})$ $= \frac{1}{2}m(\widehat{BYD}) + \frac{1}{2}m(\widehat{DZE})$	5	THEOREM 6.11
6	$m(\angle BPD) = \frac{1}{2}m(\widehat{BYD}) + \frac{1}{2}m(\widehat{AXC})$	6	SUBSTITUTION

THEREFORE $m(\angle BPD) = \frac{1}{2}[m(\widehat{AXC}) + m(\widehat{BYD})]$

EXAMPLE 2 IN FIGURE 6.45, $m(\angle MRQ) = 30^\circ$, AND $m(\angle MQR) = 40^\circ$.

WRITE DOWN THE MEASURE OF ALL THE OTHER ANGLES IN THE TWO TRIANGLES, AND $\triangle QMR$. WHAT DO YOU NOTICE ABOUT THE TWO TRIANGLES?

SOLUTION: $m(\angle QMR) = 180^\circ - (30^\circ + 40^\circ)$ (WHY?)
 $= 180^\circ - 70^\circ = 110^\circ$

$$m(\angle RQS) = \frac{1}{2}m(\widehat{RS})$$

THEREFORE, $\frac{1}{2}m(\widehat{RS})$

$$\therefore m(\widehat{RS}) = 80^\circ$$

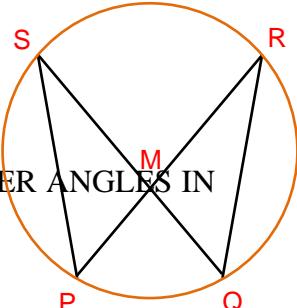


Figure 6.45

$$m(\angle PRQ) = \frac{1}{2} m(\widehat{PQ})$$

HENCE, $30^\circ = \frac{1}{2} m(\widehat{PQ})$

$$\therefore m(\widehat{PQ}) = 60^\circ$$

$$m(\angle PSQ) = \frac{1}{2} m(\widehat{PQ}) = \frac{1}{2} (60^\circ) = 30^\circ$$

$$m(\angle RPS) = \frac{1}{2} m(\widehat{RS}) = \frac{1}{2} (80^\circ) = 40^\circ$$

THE TWO TRIANGLES ARE SIMILAR BY AA SIMILARITY.

EXAMPLE 3 AN ANGLE FORMED BY TWO CHORDS INTERSECTING IN A CIRCLE MEASURES HALF THE MEASURES OF THE OTHER INTERCEPTED ARC.

SOLUTION: CONSIDER FIGURE 6.46

$$m(\angle PRB) = \frac{1}{2} m(\widehat{PB}) + \frac{1}{2} m(\widehat{AQ}) \text{ (by THEOREM 6.11)}$$

$$48^\circ = \frac{1}{2}(42^\circ) + \frac{1}{2}(m(\widehat{AQ}))$$

$$\Rightarrow 96^\circ = 42^\circ + m(\widehat{AQ})$$

$$\therefore 54^\circ = m(\widehat{AQ})$$

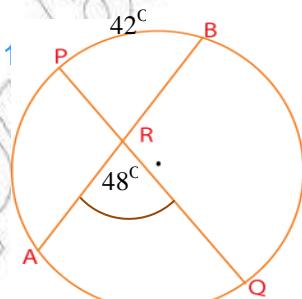


Figure 6.46

Remark: THE FOLLOWING RESULT IS SOMETIMES CALLED THE **PRODUCT OF A RECTANGLE PROPERTY OF A CIRCLE**.

IF TWO CHORDS INTERSECT IN A CIRCLE, THEN $(AP)(PB) = (XP)(PY)$.

HINT FOR PROOF:

1	$\angle XAP \equiv \angle BYP$ AND $\angle AXP \equiv \angle YBP$	(WHY?)
2	$\Delta PAX \sim \Delta PYB$	(WHY?)
3	$\frac{AP}{YP} = \frac{PX}{PB}$	(WHY?)
4	$\therefore (AP)(PB) = (YP)(PX)$	(WHY?)

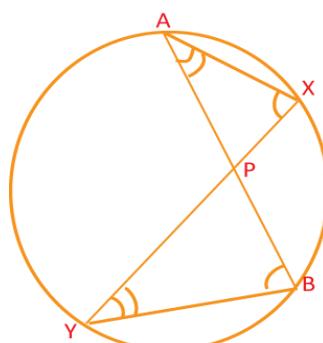


Figure 6.47

EXAMPLE 4 IN FIGURE 6.48, CALCULATE THE RADIUS OF THE CIRCLE.

SOLUTION: LET THE RADIUS OF THE CIRCLE BE r . THEN $OD = r$ AND $PD = 2r - 2$.

SINCE $A(P)(PB) = (CP)(PD)$, YOU HAVE

$$4 \times 4 = 2(2r - 2)$$

$$16 = 4r - 4$$

$$r = 5$$

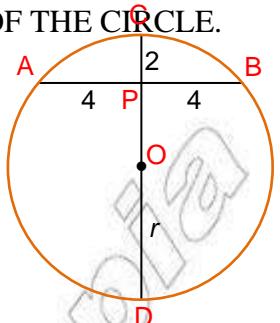


Figure 6.48

Group Work 6.3



1. IN FIGURE 6.49, \overline{AB} AND \overline{PQ} ARE PARALLEL, $\angle BQO = 70^\circ$ AND O IS THE CENTRE OF THE CIRCLE. WHAT IS THE MEASURE OF $\angle AOP$?

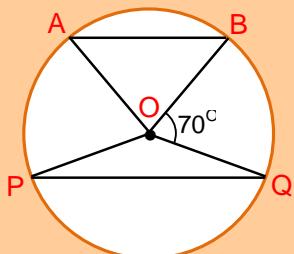


Figure 6.49

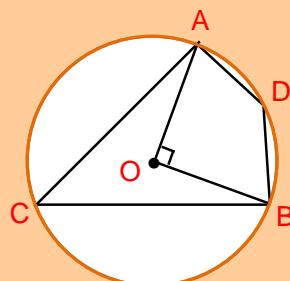


Figure 6.50

2. IN FIGURE 6.49, IF $PO = 5$ UNITS AND $\angle PQO = 120^\circ$, FIND THE LENGTH OF \overline{PH} .
 3. IN FIGURE 6.50, IF CENTRAL ANGLE COD IS A RIGHT ANGLE.
A WHAT ARE THE DEGREE MEASURES OF $\angle ADB$?
B FIND THE DEGREE MEASURE OF $m(\angle CAO) = 20^\circ$.

Exercise 6.3

1. IN FIGURE 6.51, \overline{AB} IS A DIAMETER. O IS THE CENTRE OF THE CIRCLE. $m(\angle ABD) = 60^\circ$, FIND $m(\angle OCD)$.

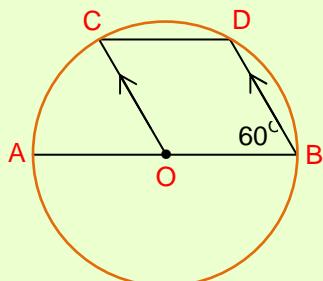


Figure 6.51

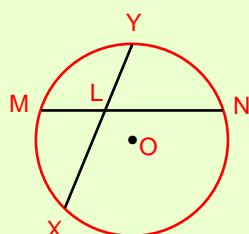


Figure 6.52

- 2** PROVE THAT IF AN ANGLE INSCRIBED IN AN ARC OF A CIRCLE IS A RIGHT ANGLE, THEN THIS IS A SEMICIRCLE.
- 3** IN FIGURE 6.52 \widehat{MX} IS AN ARC OF 28° , AND \widehat{YN} IS AN ARC OF 50° .
- WHAT IS THE DEGREE MEASURE OF $\angle YLN$?
 - IF $ML = 4$ UNITS, $LX = 5$ UNITS AND $LN = 7$ UNITS, FIND YL .
- 4** IN FIGURE 6.50 OF QUESTION 3, WOULD IT BE POSSIBLE FOR MLX TO BE A 30° ANGLE AND FOR THE MEASURE OF \widehat{MX} TO BE 40° ? IF SO, WHAT WOULD BE THE MEASURE OF \widehat{N} ?
- 5** IN FIGURE 6.53, O IS THE CENTRE OF THE CIRCLE. IF $m(\angle AOB) = 40^\circ$ AND $m(\angle COD) = 60^\circ$, FIND
- $m(\angle AQB)$
 - $m(\angle APB)$?

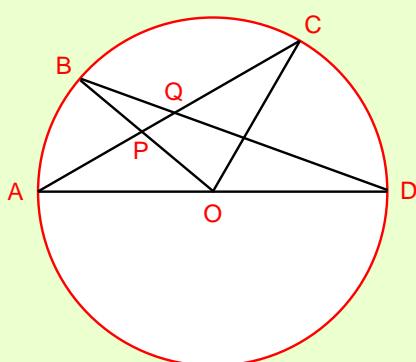


Figure 6.53

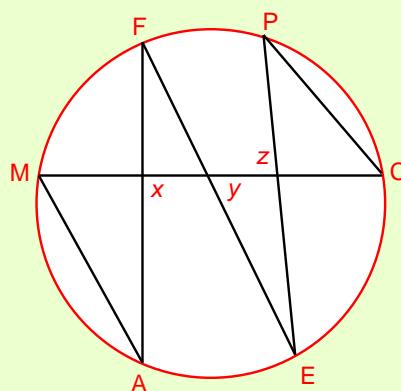


Figure 6.54

- 6** IN FIGURE 6.54, IF $m(\angle FAM) = 40^\circ$ AND $m(\angle CPE) = 50^\circ$, WHAT IS THE DEGREE MEASURE OF $\angle EYC$?
- 7** **A** IN FIGURE 6.55, THE VERTICES OF QUADRILATERAL ABCD LIE ON THE CIRCLE O. SUCH A QUADRILATERAL IS CALLED **cyclic quadrilateral**
- WHAT IS THE SUM OF THE MEASURE OF ARCS AB AND ADC ?
 - PROVE THAT OPPOSITE ANGLES OF A CYCLIC QUADRILATERAL ARE SUPPLEMENTARY.

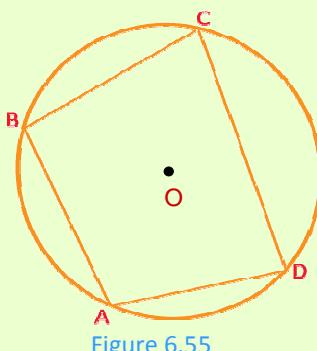


Figure 6.55

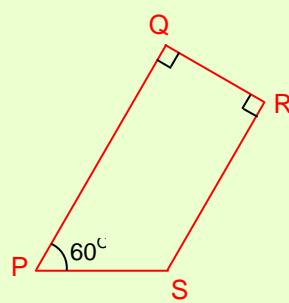


Figure 6.56

- B** IN FIGURE 6.56, IS THERE A CIRCLE CONTAINING P, QR AND S?

- 8** IN FIGURE 6.5 FIND THE VALUES OF $\angle AOC$ GIVEN THAT THE CENTRE OF THE CIRCLE AND $m(\angle AOC) = 160^\circ$

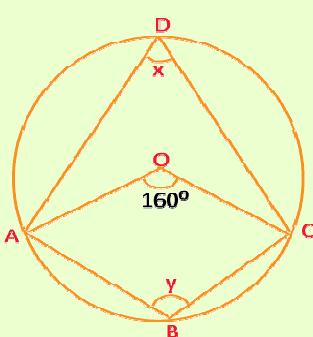


Figure 6.57

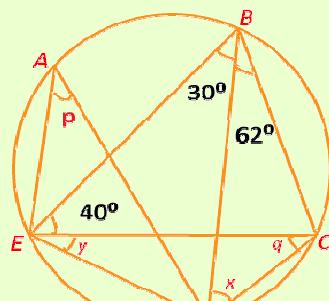


Figure 6.58

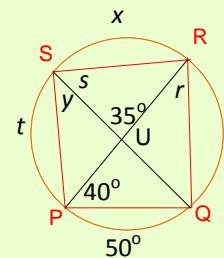


Figure 6.59

- 9** IN FIGURE 6.59 CALCULATE THE ANGLES MARKED

- 10** FIND THE VALUES OF THE ANGLES MARKED AS SHOWN IN FIGURE 6.59

6.3.2 Angles and Arcs Determined by Lines Intersecting Outside a Circle

WHAT HAPPENS IF TWO SECANT LINES INTERSECT OUTSIDE A CIRCLE? IN FIGURE 6.60 \overline{AB} AND \overline{XY} INTERSECT OUTSIDE THE CIRCLE. THEY INTERCEPT ARCS $\overset{\frown}{AX}$ AND $\overset{\frown}{BY}$. DRAW THE CHORD \overline{PQ} PARALLEL TO \overline{XY} . CAN YOU SEE THAT THE MEASURE IS HALF THE DIFFERENCE BETWEEN THE MEASURES OF THE ARCS? CAN YOU PROVE IT?

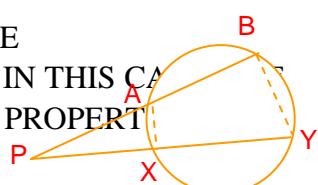
THIS IS STATED IN THEOREM 6.13.

Figure 6.60

Theorem 6.13

The measure of the angle formed by the lines of two chords intersecting outside a circle is half the difference of the measure of the arcs they intercept.

THE PRODUCT PROPERTY $(PA)(PB) = (PX)(PY)$ IS ALSO TRUE WHEN TWO CHORDS INTERSECT OUTSIDE A CIRCLE. IN THIS CASE, THE PROOF IS SIMILAR TO THE PROOF OF THE PRODUCT PROPERTY IN SECTION 6.3.1.



DRAW \overline{AX} AND \overline{BY} . TWO SIMILAR TRIANGLES ARE FORMED. BY CONSIDERING CORRESPONDING SIDES, WE SEE THAT

$$(PA)(PB) = (PX)(PY).$$

Can you point out the similar triangles, in FIGURE 6.61 and put in the other details?

Theorem 6.14

The measure of an angle formed by a tangent and a secant drawn to a circle from a point outside the circle is equal to one-half the difference of the measures of the intercepted arcs.

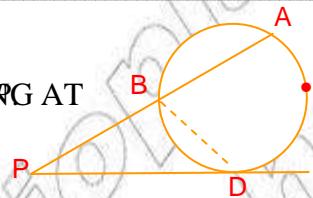
Proof:-**Given:** SECANT \overline{BA} AND TANGENT \overline{PD} INTERSECTING AT**To prove:** $m(\angle P) = \frac{1}{2}[m(\widehat{AXD}) - m(\widehat{BD})]$ 

Figure 6.62

Statement		Reason	
1	DRAW \overline{BD}	1	CONSTRUCTION.
2	$\angle ABD \equiv \angle BDP + \angle DPA$	2	AN EXTERIOR ANGLE OF A TRIANGLE IS EQUAL TO THE SUM OF THE TWO OPPOSITE INTERIOR ANGLES OF A TRIANGLE.
3	$\angle ABD - \angle BDP \equiv \angle DPA \equiv \angle P$	3	SUBTRACTION.
4	$m(\angle ABD) = \frac{1}{2}m(\widehat{AXD})$ AND $m(\angle PDB) = \frac{1}{2}m(\widehat{BD})$	4	THEOREM 6 AND THEOREM 6.11.
5	$m(\angle ABD) - m(\angle BDP)$ $= \frac{1}{2}m(\widehat{AXD}) - \frac{1}{2}m(\widehat{BD})$	5	SUBSTITUTION.
6	$\therefore m(\angle P) = \frac{1}{2}m(\widehat{AXD}) - \frac{1}{2}m(\widehat{BD})$	6	SUBSTITUTION.

Theorem 6.15

If a secant and a tangent are drawn from a point outside a circle, then the square of the length of the tangent is equal to the product of the lengths of line segments given by

$$(PA)^2 = (PB)(PC).$$



Figure 6.63

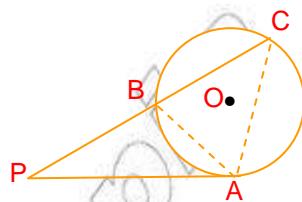
Proof:-**Given:** A CIRCLE WITH \overline{SC} AND \overline{TAN} AS IN FIGURE 6.64**To prove:** $(PA)^2 = (PB)(PC)$ DRAW \overline{AB} AND \overline{CA} . THEN $\triangle PCA \sim \triangle PAB$ (SHOW!)HENCE $\frac{PC}{PA} = \frac{PA}{PB}$ AND $(PA)^2 = (PB)(PC)$ 

Figure 6.64

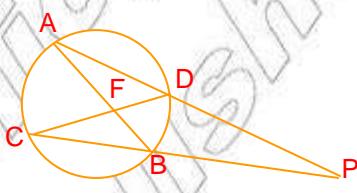
EXAMPLE 5 IN FIGURE 6.65 FROM SECANTS \overline{PA} AND \overline{PC} ARE DRAWN SO THAT $m(\angle APC) = 30^\circ$; CHORDS \overline{DB} AND \overline{CD} INTERSECT AT F SUCH THAT $m(\angle AFC) = 85^\circ$. FIND THE MEASURE OF $\angle ABC$ AND MEASURE OF $\angle BAC$.

Figure 6.65

SOLUTION: LET $m(\widehat{AC}) = x$ AND $m(\widehat{DB}) = y$

SINCE $m(\angle AFC) = \frac{1}{2}m(\widehat{AC}) + \frac{1}{2}m(\widehat{BD})$

$$85^\circ = \frac{1}{2}(x + y)$$

$$x + y = 170^\circ \dots\dots\dots (1)$$

AGAIN AS $m(\angle APC) = \frac{1}{2}m(\widehat{AC}) - \frac{1}{2}m(\widehat{BD})$

$$30^\circ = \frac{1}{2}(x - y)$$

$$x - y = 60^\circ \dots\dots\dots (2)$$

SOLVING EQUATION 1 AND EQUATION 2 SIMULTANEOUSLY, WE GET

$$\begin{cases} x + y = 170^\circ \\ x - y = 60^\circ \\ \hline 2x = 230^\circ \\ x = 115^\circ \end{cases}$$

SUBSTITUTING IN EQUATION 2,

$$115^\circ - y = 60^\circ$$

$$y = 55^\circ$$

THEREFORE $m(\widehat{AC}) = 115^\circ$ AND $m(\widehat{DB}) = 55^\circ$.

$$m(\angle ABC) = \frac{1}{2}m(\widehat{AC}) = \frac{1}{2}(115^\circ) = 57.5^\circ.$$

Group Work 6.4



- 1 **AB** IS A DIAMETER OF A CIRCLE WITH CENTER **O**. POINT **P** IS ON THE CIRCUMFERENCE. POINT **X** IS ON THE CIRCUMFERENCE SUCH THAT $\angle AOC$ BISECTS $\angle AOP$. PROVE THAT \overline{OP} IS PARALLEL TO \overline{AX} .
- 2 IN FIGURE 6.6, SUPPOSE LINES \overleftrightarrow{PA} AND \overleftrightarrow{PX} ARE TANGENTS TO A CIRCLE. PROVE THAT
 $m(\angle APX) = \frac{1}{2}(\text{MEASURE OF MAJOR ARC } AC) - \frac{1}{2}(\text{MEASURE OF MINOR ARC } AB)$
OR $m(\angle P) = \frac{1}{2}m(\widehat{ACX}) - \frac{1}{2}m(\widehat{ABX})$

Hint: DRAW A LINE THROUGH **P** PARALLEL TO \overline{AX}

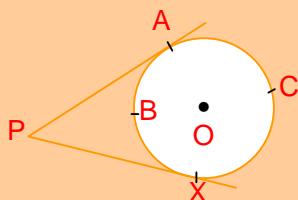


Figure 6.66

- 3 SUPPOSE A GEOSTATIONARY SATELLITE **S** ORBITS ABOVE EARTH, ROTATING SO THAT IT APPEARS TO HOVER DIRECTLY OVER THE EQUATOR. DETERMINE THE MEASURE OF THE ARC ON THE EQUATOR VISIBLE TO THIS GEOSTATIONARY SATELLITE.

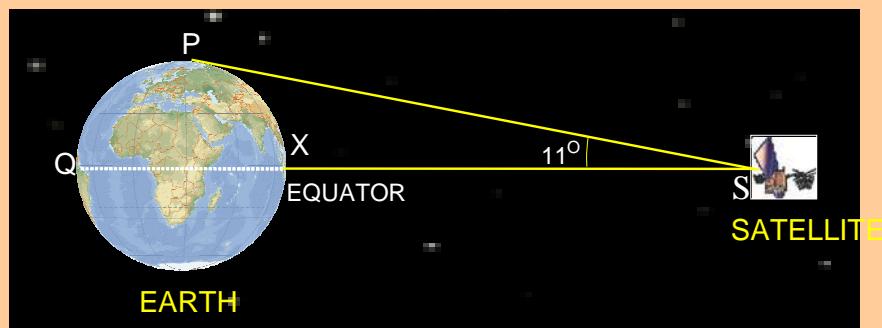


Figure 6.67

Exercise 6.4

- 1 IF THE MEASURE OF $\angle QAR$ IS 30° AND THE MEASURE OF $\angle BRF$ IS 60° , WHAT IS THE MEASURE OF $\angle P$? REFER TO FIGURE 6.68.

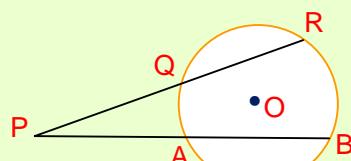


Figure 6.68

- 2 IN FIGURE 6.6 \overrightarrow{AP} IS A TANGENT TO THE CIRCLE. PROVE THAT.

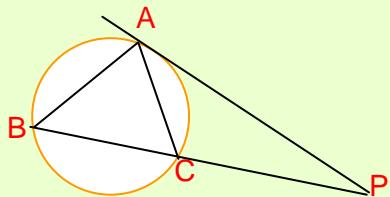


Figure 6.69

- 3 IN FIGURE 6.70 \overline{CD} IS A DIAMETER AND \overline{AB} IS BISECTED BY \overline{P} . A SQUARE WITH SIDE \overline{AP} AND A RECTANGLE WITH SIDE \overline{PD} ARE DRAWN. PROVE THAT THE AREAS OF THE SQUARE AND THE RECTANGLE ARE EQUAL.

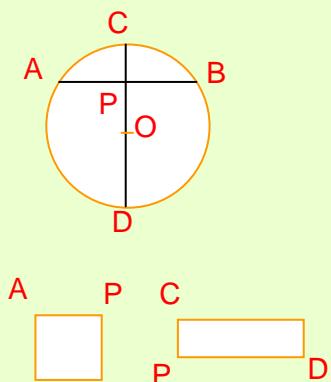


Figure 6.70

- 4 IN FIGURE 6.71 \overrightarrow{AC} , \overrightarrow{CE} AND \overrightarrow{EG} ARE TANGENTS TO THE CIRCLE WITH CENTRE O AT A , B AND F RESPECTIVELY. PROVE THAT $CA = CE$.

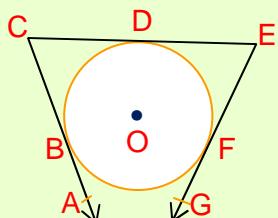


Figure 6.71

- 5 USE THE CIRCLE IN FIGURE 6.7 WITH TANGENT \overline{PT} , SECANT \overline{PS} , \overline{PC} AND CHORD \overline{BD} TO FIND THE LENGTHS OF \overline{DF} AND \overline{PT} , IF
 $CG = 4$ UNITS, $GA = 6$ UNITS, $DG = 3$ UNITS, $PF = 9$ UNITS AND $PA = 8$ UNITS.

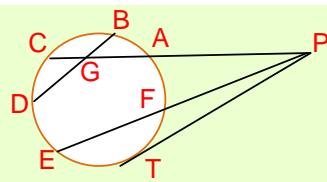


Figure 6.72

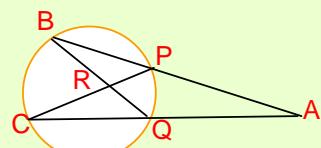


Figure 6.73

- 6 IN FIGURE 6.72 $m(\angle BPC) = 48^\circ$, $m(\angle BRC) = 68^\circ$ AND $m(\angle BCR) = 62^\circ$. CALCULATE THE MEASURES OF ANGLES OF
- 7 THE DIAGONALS AND OF THE PARALLELOGRAM OF LENGTHS 20 CM AND 12 CM RESPECTIVELY. IF THE CIRCLE IS AT F, FIND THE LENGTH OF
- 8 IN FIGURE 6.72 $BP = 6$ CM, $DC = 10$ CM AND $P = 8$ CM. CALCULATE THE LENGTHS OF THE CHORD AND THE TANGENT

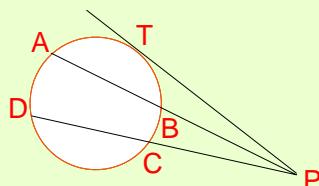


Figure 6.74

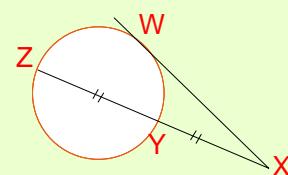


Figure 6.75

- 9 IN FIGURE 6.75 Y IS THE MID-POINT OF ZX AND WX IS TANGENT TO THE CIRCLE. FIND IN TERMS OF EXPLAIN YOUR REASONING.

6.4 REGULAR POLYGONS

A POLYGON WHOSE VERTICES ARE ON A CIRCLE IS **INScribed** IN THE CIRCLE. THE CIRCLE IS **circumscribed** ABOUT THE POLYGON.

IN FIGURE 6.76 THE POLYGON ABCDE IS INSCRIBED IN THE CIRCLE OR THE CIRCLE IS CIRCUMSCRIBED ABOUT THE POLYGON.

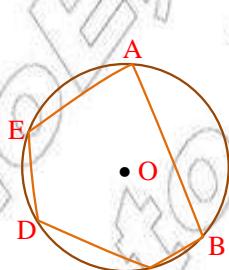


Figure 6.76

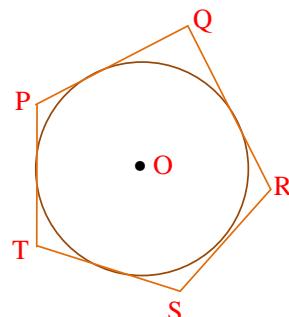


Figure 6.77

A POLYGON WHOSE SIDES ARE TANGENT TO A CIRCLE IS **CIRCumscribed** ABOUT THE CIRCLE. FIGURE 6.77 THE PENTAGON PQRST IS CIRCUMSCRIBED ABOUT THE CIRCLE. THE CIRCLE IS INSCRIBED IN THE PENTAGON.

ACTIVITY 6.7

- 1** WHAT IS A REGULAR POLYGON? GIVE EXAMPLES.
- 2** DRAW THREE CIRCLES OF RADIUS 5 CM. CIRCUMSCRIBE ABOUT THE FIRST CIRCLE, A TRIANGLE ABOUT THE SECOND, AND A 7-SIDED POLYGON ABOUT THE THIRD.
- 3** CIRCUMSCRIBE A CIRCLE ABOUT A SQUARE.
- 4** DRAW A CIRCLE SUCH THAT THREE OF THE FOUR SIDES ARE TANGENT TO IT. GIVE REASONS WHY A CIRCLE CANNOT BE INSCRIBED IN THE RECTANGLE OF UNEQUAL SIDES.
- 5** SHOW THAT A CIRCLE CAN ALWAYS BE CIRCUMSCRIBED ABOUT IT IF TWO OPPOSITE ANGLES ARE RIGHT ANGLES.
- 6** SHOW THAT, IF A CIRCLE CAN BE CIRCUMSCRIBED ABOUT IT, THEN THE PARALLELOGRAM IS A RECTANGLE.
- 7** WHAT IS THE MEASURE OF AN ANGLE BETWEEN ADJACENT SIDES OF A REGULAR POLYGON WITH 10 SIDES?
- 8** WHAT IS THE MEASURE OF AN ANGLE BETWEEN ADJACENT SIDES OF A REGULAR POLYGON WITH 3, 7, 10, ... SIDES?
- 9** DRAW A SQUARE WITH SIDE 5 CM. DRAW THE INSCRIBING CIRCLE.



6.4.1 Perimeter of a Regular Polygon

YOU HAVE STUDIED HOW TO FIND THE LENGTH OF A SIDE (S) AND PERIMETER (P) OF A REGULAR POLYGON WITH N SIDES AND THE NUMBER OF SIDES GRADE 9. THE FOLLOWING EXAMPLE IS GIVEN TO REFRESH YOUR MEMORY.

EXAMPLE 1 THE PERIMETER OF A REGULAR POLYGON WITH 9 SIDES IS

$$P = 9 \times 2r \sin \frac{180^\circ}{9} = 9d \sin \frac{180^\circ}{9}, \text{ WHERE } d = 2r \text{ IS DIAMETER}$$

$$= 9d \sin 20^\circ \approx 3.0782d$$

EXAMPLE 2 FIND THE LENGTH OF A SIDE AND THE PERIMETER OF A REGULAR QUADRILATERAL WITH RADIUS 5 UNITS.

SOLUTION:

$$s = 2r \sin \frac{180^\circ}{n}$$

$$P = 2nr \sin \frac{180^\circ}{n}$$

$$s = 2 \times 5 \sin \frac{180^\circ}{4} = 10 \sin 45^\circ$$

$$P = 2 \times 4 \times 5 \sin \frac{180^\circ}{4} = 40 \sin 45^\circ$$

$$= 10 \times \frac{\sqrt{2}}{2}$$

$$= 40 \times \frac{\sqrt{2}}{2}$$

$$\therefore s = 5\sqrt{2} \text{ UNITS.}$$

$$\therefore P = 20\sqrt{2} \text{ UNITS.}$$

6.4.2 Area of a Regular Polygon

DRAW A CIRCLE WITH CENTER O . INScribe IN IT A REGULAR POLYGON $ABC \dots M$ AS SHOWN IN FIGURE 6.78.

JOIN TO EACH VERTEX THE POLYGONAL REGION IS THEN DIVIDED INTO n TRIANGLES. $\triangle AOB$ IS ONE OF THEM.

$\angle AOB$ HAS DEGREE MEASURE $\frac{360^\circ}{n}$

RECALL THAT THE FORMULA FOR THE AREA OF A TRIANGLE WITH SIDES a AND INCLUDED BETWEEN THESE SIDES IS:

$$A = \frac{1}{2} ab \sin \angle C$$

HENCE, AREA $\triangle AOB$ IS

$$A = \frac{1}{2} r \times r \sin \angle AOB = \frac{1}{2} r^2 \sin \frac{360^\circ}{n}$$

THEREFORE, THE AREA OF THE POLYGON IS GIVEN BY

$$A = \frac{1}{2} nr^2 \sin \frac{360^\circ}{n} \quad (\text{WHY?})$$

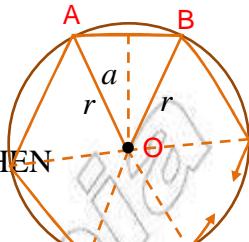


Figure 6.78

Theorem 6.16

The area A of a regular polygon with n sides and radius r is

$$A = \frac{1}{2} nr^2 \sin \frac{360^\circ}{n}.$$

THIS FORMULA FOR THE AREA OF A REGULAR POLYGON CAN BE USED TO FIND THE AREA OF A POLYGON AS ITS NUMBER OF SIDES INCREASES, THE AREA OF THE POLYGON BECOMES CLOSER TO THE AREA OF THE CIRCLE.

ACTIVITY 6.8

SQUARE $ABCD$ IS INSCRIBED IN A CIRCLE OF RADIUS r .

- A WHAT IS THE MEASURE OF ANGLES $\angle AOB$?
- B FIND THE AREA OF THE SQUARE.
- C FIND THE AREA OF THE CIRCLE.

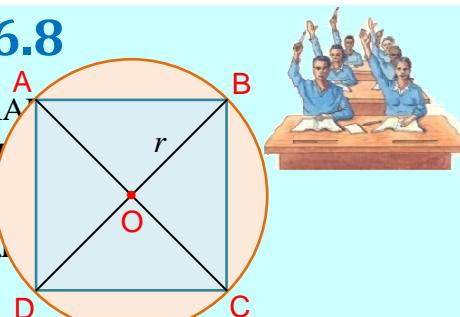


Figure 6.79

EXAMPLE 3 SHOW THAT THE AREA OF A REGULAR HEXAGON INSCRIBED IN A CIRCLE WITH

$$\text{RADIUS } r \text{ IS } \frac{3\sqrt{3}}{2} r^2.$$

SOLUTION: $A = \frac{1}{2} nr^2 \sin \frac{360^\circ}{n} = \frac{1}{2} \times 6 \times r^2 \sin \frac{360^\circ}{6} = 3r^2 \sin 60^\circ$

$$A = 3r^2 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}r^2}{2}$$

SQ UNITS.

Exercise 6.5

- 1 FIND THE AREA OF A REGULAR NINE-SIDED POLYGON WITH RADIUS r .
- 2 FIND THE AREA OF A REGULAR TWELVE-SIDED POLYGON WITH RADIUS r .
- 3 PROVE THAT THE AREA OF AN EQUILATERAL TRIANGLE INSCRIBED IN A CIRCLE WITH RADIUS r IS $\frac{3\sqrt{3}r^2}{4}$. USE THIS FORMULA TO FIND THE AREA OF AN EQUILATERAL TRIANGLE IN A CIRCLE WITH RADIUS 2 CM.
- 4 PROVE THAT THE AREA A OF A SQUARE INSCRIBED IN A CIRCLE WITH RADIUS r IS $2r^2$. USE THIS FORMULA TO FIND THE AREA OF A SQUARE INSCRIBED IN A CIRCLE WITH RADIUS 3 CM.
- 5 SHOW THAT ALL THE DISTANCES FROM THE CENTRE OF A POLYGON TO THE SIDES ARE EQUAL.

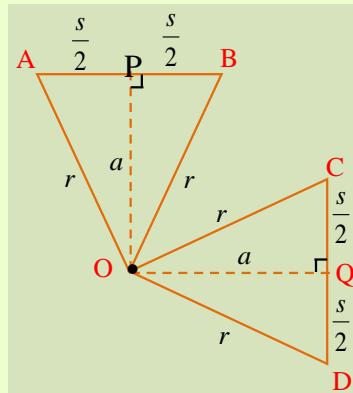


Figure 6.80

- 6 USE FIGURE 6.80 GIVEN ABOVE TO PROVE THE FORMULA FOR THE APOTHEM a :

$$a = r \cos \frac{180^\circ}{n}$$

- 7 USE THE FORMULA $a = r \cos \frac{180^\circ}{n}$ TO CALCULATE THE APOTHEMS OF THE FOLLOWING REGULAR POLYGONS INSCRIBED IN A CIRCLE OF RADIUS 12 CM:

A TRIANGLE **B** QUADRILATERAL **C** HEXAGON **D** NONAGON.

- 8** SHOW THAT A FORMULA FOR THE AREA OF A REGULAR POLYGON IS $A = \frac{1}{2} aP$.
AND PERIMETER $P = \frac{1}{2} aP$.

USE THIS FORMULA TO CALCULATE THE AREA OF A REGULAR;

A TRIANGLE **B** QUADRILATERAL **C** HEXAGON **D** OCTAGON.

GIVE YOUR ANSWER IN TERMS OF ITS RADIUS.

- 9** **A** SHOW THAT ANOTHER FORMULA FOR THE AREA OF A REGULAR POLYGON IS $A = \frac{1}{2} Pr \cos \frac{180^\circ}{n}$.

B SHOW THAT THE RATIO OF THE AREA OF TWO POLYGONS IS THE SQUARE OF THE RATIO OF THEIR RADII.

C USE THE FORMULA FOR THE AREA OF A REGULAR POLYGON $A = \frac{1}{2} Pr \cos \frac{180^\circ}{n}$ TO SHOW THAT THE RATIO OF THE AREAS OF TWO REGULAR POLYGONS WITH THE SAME NUMBER OF SIDES IS EQUAL TO THE SQUARE OF THE RATIO OF THE LENGTHS OF CORRESPONDING SIDES.

D CAN YOU PROVE THE RESULT ABOVE WITHOUT USING ANY OF THE FORMULAE OF THIS SECTION?

- 10** A CIRCULAR TIN IS PLACED ON A SQUARE. HIS SQUARE IS CONGRUENT TO THE DIAMETER OF THE TIN, CALCULATE THE PERCENTAGE OF THE SQUARE WHICH IS UNCOVERED. GIVE YOUR ANSWER CORRECT TO 2 DECIMAL PLACES.



Key Terms

altitude	concurrent lines	plane geometry
apothem	Euclidean Geometry	product property
arc	incentre	quadrilateral
bisector	incircle	rectangle
central angle	inscribed angle	regular polygon
centroid	major arc	rhombus
chord	median	semi-circle
circle	minor arc	square
circumcentre	orthocenter	trapezium
circumcircle	parallelogram	
collinear points	perpendicular	



Summary

- 1** THE MEDIAN OF A TRIANGLE ARE CONCURRENT AT A POINT WHICH IS EQUIDISTANT FROM EACH VERTEX TO THE MID-POINT OF THE OPPOSITE SIDE.
- 2** THE PERPENDICULAR BISECTORS OF THE SIDES OF A TRIANGLE ARE CONCURRENT AT A POINT CALLED CIRCUMCENTRE WHICH IS EQUIDISTANT FROM THE VERTICES OF THE TRIANGLE.
- 3** THE ALTITUDES OF A TRIANGLE ARE CONCURRENT AT THE ORTHOCENTRE OF THE TRIANGLE. IF POINTS D AND F ARE ON THE SIDES CA AND AB RESPECTIVELY, THEN $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1$. CONVERSELY, IF $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1$, THEN THE POINTS D AND F ARE COLLINEAR.
- 4** A TRAPEZIUM IS A QUADRILATERAL THAT HAS ONLY TWO SIDES PARALLEL.
- 5** A PARALLELOGRAM IS A QUADRILATERAL IN WHICH OPPOSITE SIDES ARE PARALLEL.
- 6** A RECTANGLE IS A PARALLELOGRAM IN WHICH ONE EACH CORNER IS A RIGHT ANGLE.
- 7** A RHOMBUS IS A PARALLELOGRAM WHICH HAS ALL SIDES EQUAL.
- 8** A SQUARE IS A RECTANGLE WHICH HAS CONGRUENT ADJACENT SIDES.
- 9** IN A CIRCLE, AN INSCRIBED ANGLE IS AN ANGLE WHICH IS IN THE CIRCLE AND WHOSE SIDES ARE CHORDS OF THE CIRCLE.
- 10** IN FIGURE 6.81, $m(\angle APB) = \frac{1}{2} m(\widehat{AXB})$

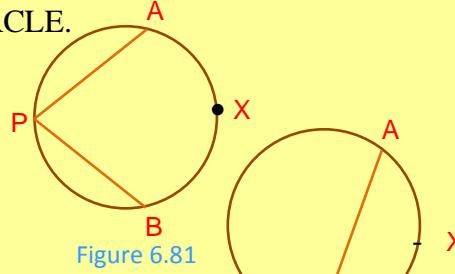


Figure 6.81

- 11** AN ANGLE INSCRIBED IN A SEMI-CIRCLE IS A RIGHT ANGLE.
- 12** AN ANGLE INSCRIBED IN AN ARC LESS THAN A SEMI-CIRCLE.
- 13** AN ANGLE INSCRIBED IN AN ARC GREATER THAN A SEMI-CIRCLE.

- 14** IN FIGURE 6.82, $m(\angle APB) = \frac{1}{2} m(\widehat{AXP})$.



Figure 6.82

- 15** IN FIGURE 6.83, $m(\angle BPD) = \frac{1}{2} m(\widehat{AXC}) + \frac{1}{2} m(\widehat{BYD})$
AND $(AP)(PB) = (CP)(PD)$

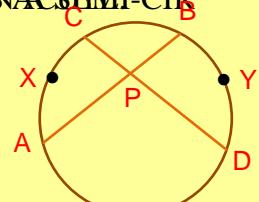


Figure 6.83

16 INFIGURE 6.84

A $m(BPD) = \frac{1}{2}m(\widehat{BD}) - \frac{1}{2}m(\widehat{AC})$

B $m(DPQ) = \frac{1}{2}m(\widehat{DQ}) - \frac{1}{2}m(\widehat{QC})$

C $(PA)(PB) = (PC)(PD)$

D $(PQ)^2 = (PC)(PD)$

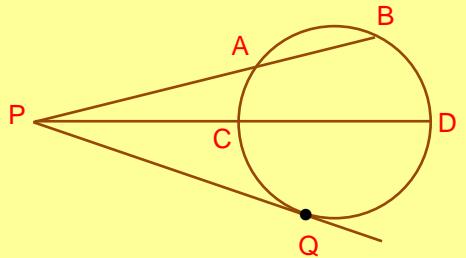


Figure 6.84

17 THE LENGTH OF A SIDE AND PERIMETER OF A REGULAR POLYGON OF N SIDES AND RADIUS r ARE:

$$s = 2r \sin \frac{180^\circ}{n} \quad P = 2n r \sin \frac{180^\circ}{n} \quad P = ns$$

18 THE AREA OF A REGULAR POLYGON OF N SIDES AND RADIUS r

$$A = \frac{1}{2}nr^2 \sin \frac{360^\circ}{n}$$



Review Exercises on Unit 6

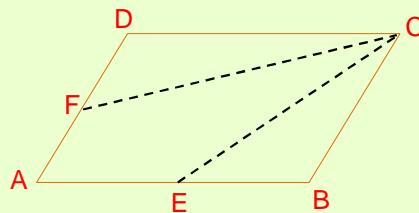
1 THE POINTS E AND F ARE THE MID-POINTS OF SIDES AD AND AB OF PARALLELOGRAM ABCD. PROVE THAT $\text{AREA}(EFC) = \frac{1}{2} \text{ AREA}(ABCD)$. (See FIGURE 6.85)

Figure 6.85

2 TWO CHORDS \overline{AB} AND \overline{CD} OF A CIRCLE INTERSECT AT RIGHT ANGLES AT A POINT INSIDE CIRCLE. IF $\angle BAC = 35^\circ$, FIND THE MEASURES OF \widehat{CB} AND \widehat{AD} .

3 INFIGURE 6.86, O IS THE CENTRE OF THE CIRCLE. CALCULATE

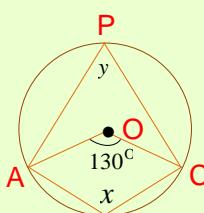


Figure 6.86

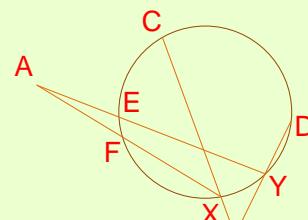


Figure 6.87

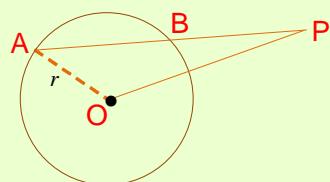


Figure 6.88

- 4** IN FIGURE 6.81 IF $m(\angle A) = 10^\circ$, $m(\widehat{EF}) = 15^\circ$ AND $m(\widehat{CD}) = 95^\circ$, FIND $m(\angle B)$.
- 5** FROM ANY POINT OUTSIDE A CIRCLE O AND P A SECANT LINE IS DRAWN CUTTING THE CIRCLE AT A AND B . PROVE THAT $(PB)^2 = (PA)(PB) - r^2$, AS SHOWN IN FIGURE 6.88
- 6** TWO CHORDS \overline{AB} AND \overline{CD} OF A CIRCLE INTERSECT WHEN PRODUCED TASTEPON THE CIRCLE O AND TANGENT FROM THE CIRCLE.
PROVE THAT $(PA)(PB) = (PC)(PD) = (PT)^2$.

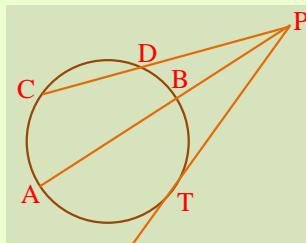


Figure 6.89

- 7** A CHORD OF A CIRCLE OF RADIUS 6 CM IS 8 CM. THE DISTANCE OF THE CHORD FROM THE CENTRE.
- 8** \overline{MN} IS A DIAMETER AND \overline{PQ} IS A CHORD OF A CIRCLE, SUCH THAT (AS SHOWN IN FIGURE 6.90) PROVE THAT $(ML)^2 = (ML).(LN)$.

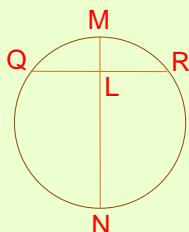


Figure 6.90

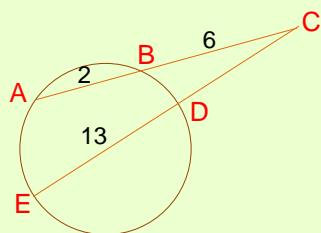


Figure 6.91

- 9** SECANTS \overline{SA} AND \overline{SC} INTERSECT A CIRCLE O AT A AND C AS SHOWN IN FIGURE 6.91 IF THE LENGTHS OF THE SEGMENTS ARE AS SHOWN, FIND THE LENGTH OF \overline{AC} .
- 10** \overrightarrow{AOB} , \overrightarrow{COD} ARE TWO STRAIGHT LINES SUCH THAT $CD = 19$ CM, $AO = 6$ CM, $CO = 7$ CM. PROVE THAT $ABCD$ IS A CYCLIC QUADRILATERAL.
- 11** $ABXY$ IS A PARALLELOGRAM OF AREA $= 18$ CM 2 , $AY = 4$ CM AND X IS A POINT ON \overline{YX} OR EXTENDED SUCH THAT $XY = 5$ CM. FIND:
- A** THE AREA OF $\triangle ABC$
 - B** THE DISTANCE FROM \overline{XY} .
 - C** THE DISTANCE FROM \overline{AB}

Unit

7



THE PYRAMIDS AT GIZA IN EGYPT ARE AMONG THE BEST KNOWN PIECES OF ARCHITECTURE IN THE WORLD. THE PYRAMID OF KHAFRE WAS BUILT AS THE FINAL RESTING PLACE OF THE PHARAOH KHAFRE AND IS ABOUT 136 M HIGH.

MEASUREMENT

Unit Outcomes:

After completing this unit, you should be able to:

- solve problems involving surface area and volume of solid figures.
- know basic facts about frustums of cones and pyramids.

Main Contents

- 7.1 Revision on Surface Areas and Volumes of Prisms and Cylinders
- 7.2 Pyramids, Cones and Spheres
- 7.3 Frustums of Pyramids and Cones
- 7.4 Surface Areas and Volumes of Composite Solids

Key Terms

Summary

Review Exercises

INTRODUCTION

RECALL THAT GEOMETRICAL FIGURES THAT HAVE THREE DIMENSIONS (LENGTH, WIDTH AND HEIGHT) ARE CALLED **Solid figures**. FOR EXAMPLE, CUBES, PRISMS, CYLINDERS, CONES AND PYRAMIDS ARE THREE DIMENSIONAL SOLID FIGURES. IN YOUR LOWER GRADES, YOU HAVE LEARNT HOW TO FIND SURFACE AREAS AND VOLUMES OF SOLID FIGURES LIKE CYLINDERS AND PRISMS. IN THIS UNIT, YOU WILL LEARN MORE ABOUT SURFACE AREAS AND VOLUMES OF OTHER SOLID FIGURES. YOU WILL STUDY ABOUT SURFACE AREAS AND VOLUMES OF COMPOSED SOLIDS AND FRUSTUMS AND CONES.



OPENING PROBLEM

ATO NIGATU DECIDED TO BUILD A GARAGE AND BEGAN BY CALCULATING THE NUMBER OF BRICKS REQUIRED. THE FLOOR OF THE GARAGE IS RECTANGULAR WITH LENGTHS 6 M AND 4 M. THE HEIGHT OF THE BUILDING IS 4 M. EACH BRICK USED TO CONSTRUCT THE BUILDING MEASURES 22 CM BY 7 CM.

- A** HOW MANY BRICKS MIGHT BE NEEDED TO CONSTRUCT THE GARAGE?
- B** FIND THE AREA OF EACH SIDE OF THE BUILDING.
- C** WHAT MORE INFORMATION DO YOU NEED TO FIND THE EXACT NUMBER OF BRICKS REQUIRED?

7.1

REVISION ON SURFACE AREAS AND VOLUMES OF PRISMS AND CYLINDERS

THERE ARE MANY THINGS AROUND US WHICH ARE OPTIMALLY SHAPED IN SHAPE. IN THIS SUB-UNIT, YOU WILL CLOSELY LOOK AT THE GEOMETRIC SOLIDS CALLED PRISMS AND CYLINDERS.

LET E_1 AND E_2 BE TWO PARALLEL PLANES, INTERSECTING BOTH PLANE AS REGION

IN E_1 . FOR EACH POINT R IN E_1 , LET P' BE THE POINT IN E_2 SUCH THAT $\overline{P'P}$ IS PARALLEL TO ℓ

THE UNION OF ALL POINTS IN REGION R' IN E_2 CORRESPONDING TO THE REGION R IN E_1 . THE UNION OF ALL THE SEGMENTS $\overline{PP'}$ IS CALLED A **solid region D**. THIS SOLID REGION IS KNOWN AS A **cylinder**. SEE FIGURE 7.2

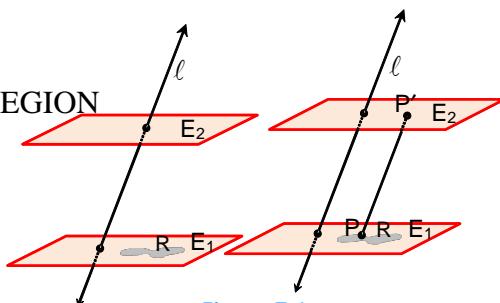


Figure 7.1

Some important terms

FOR THE CYLINDER REGIONS CALLED ITS LOWER BASE OR SIMPLY base AND AS ITS upper base.

THE LINE IS CALLED directrix AND THE PERPENDICULAR

DISTANCE BETWEEN E_1 AND E_2 IS THE altitude of D . IF ℓ IS PERPENDICULAR THEN D IS CALLED right cylinder, OTHERWISE IT IS oblique cylinder. IF R IS A CIRCULAR REGION, THIS D IS CALLED circular cylinder.

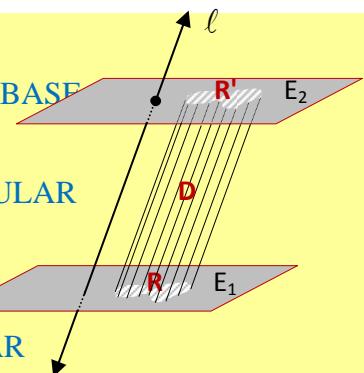


Figure 7.2

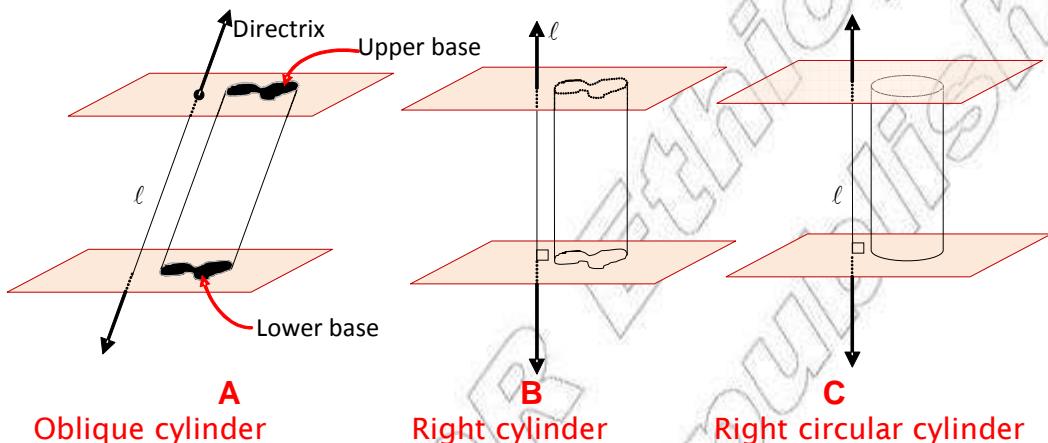


Figure 7.3

LET C BE THE BOUNDING CURVE OF THE BASE REGION. THE UNION OF ALL THE ELEMENTS WHICH BELONGS TO C IS CALLED lateral surface OF THE CYLINDER. THE total surface IS THE UNION OF THE LATERAL SURFACE AND THE BASES OF THE CYLINDER.

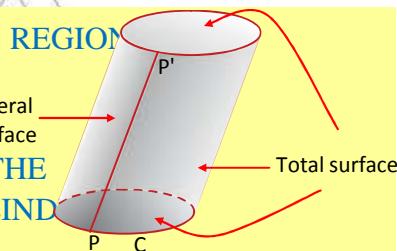


Figure 7.4

THERE ARE OTHER FAMILIAR SOLID FIGURES THAT ARE SIMILAR TO THE SOLID FIGURE DESCRIBED ABOVE IN FIGURE 7.2

Definition 1.1

If R is a polygonal region, then D is called a prism.

If R is a parallelogram region, then D is a parallelepiped.

If R is a triangular region, then D is a triangular prism.

If R is a square region, then D is a square prism.

A cube is a square right prism whose altitude is equal to the length of the edge of the base.

Note:

IN THE PRISM SHOWN IN FIGURE 7.5

- 1 \overline{AB} , \overline{BC} , \overline{CD} , \overline{DE} , \overline{EA} ARE edges OF THE UPPER BASE.
 $\overline{A'B'}$, $\overline{B'C'}$, $\overline{C'D'}$, $\overline{D'E'}$, $\overline{E'A'}$ ARE edges OF THE LOWER BASE.
- 2 $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$, $\overline{DD'}$, $\overline{EE'}$ ARE CALLED lateral edges OF THE PRISM.
- 3 THE PARALLELOGRAM REGIONS $A-BCC'B'$, $AEE'A'$, $DCC'D'$, $EDD'E$ ARE CALLED lateral faces OF THE PRISM.
- 4 THE UNION OF THE LATERAL FACES OF A PRISM IS CALLED ITS lateral surface
- 5 THE UNION OF ITS LATERAL FACES AND ITS TWO bases SURFACE ARE SIMPLY ITS surface

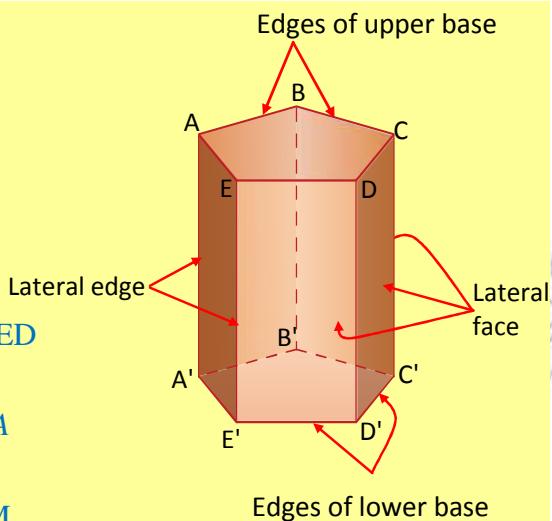


Figure 7.5

ACTIVITY 7.1

- 1 HOW MANY EDGES DOES THE BASE OF THE PRISM SHOWN HAVE? NAME THEM.
- 2 IDENTIFY EACH OF THE SOLIDS IN AS PRISM, CYLINDER, TRIANGULAR PRISM, RIGHT PRISM, PARALLELEPIPED, RECTANGULAR PARALLELEPIPED AND CUBE.

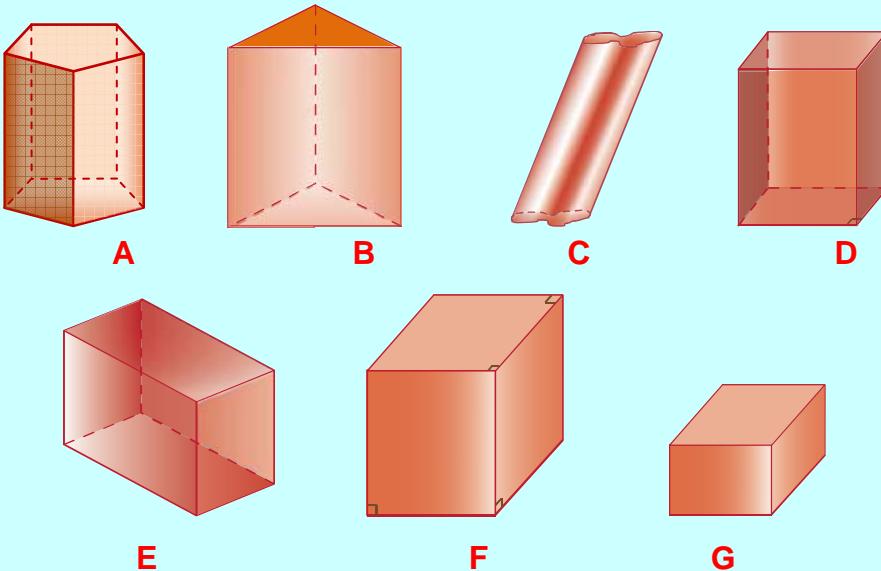


Figure 7.6

- 3 ARE THE LATERAL EDGES OF A PRISM EQUAL AND PARALLEL?

- 4** USING FIGURE 7, COMPLETE THE FOLLOWING BLANK SPACES TO MAKE TRUE STATEMENTS:

- A** THE FIGURE IS CALLED A _____.
- B** THE REGION $ABCD$ IS CALLED A _____.
- C** \overline{AE} AND \overline{CG} ARE CALLED _____.
- D** THE REGION EAD IS CALLED A _____.
- E** _____ IS THE ALTITUDE OF THE PRISM.
- F** IF $ABCD$ WERE A PARALLELOGRAM, THE _____ WOULD BE CALLED A _____.
- G** IF \overline{AE} WERE PERPENDICULAR TO THE PLANE OF THE QUADRILATERAL $ABCD$, THE PRISM WOULD BE CALLED _____.

- 5** CONSIDER A RECTANGULAR PRISM WITH DIMENSIONS AS IN FIGURE 7.7. DETERMINE:

- A** THE BASE AREA **B** LATERAL SURFACE AREA **C** TOTAL SURFACE AREA

IF WE DENOTE THE LATERAL SURFACE AREA BY A_L , THE AREA OF THE BASE BY A_B , THE ALTITUDE h AND THE TOTAL SURFACE AREA BY A_T

$$A_L = Ph; \text{ WHERE } P \text{ IS THE PERIMETER OF THE BASE AND } h \text{ IS THE HEIGHT OF THE PRISM}$$

$$A_T = 2A_B + A_L$$

EXAMPLE 1 FIND THE LATERAL SURFACE AREA OF EACH OF THE FOLLOWING RIGHT PRISMS.

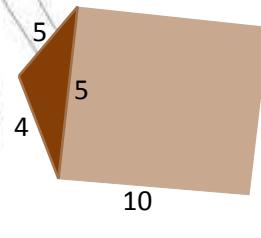
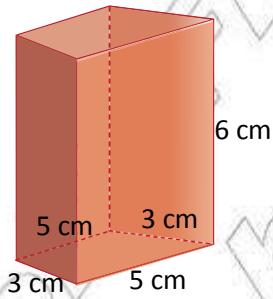


Figure 7.7

SOLUTION:

$$\text{A} \quad A_L = Ph = (3 + 5 + 3 + 5) \text{ CM} \times 6 \text{ CM} = 16 \text{ CM} \times 6 \text{ CM} = 96 \text{ CM}^2$$

$$\text{B} \quad A_L = Ph = (5 + 5 + 4) \times 10 = 14 \times 10 = 140 \text{ UNITS}^2$$

SIMILARLY, THE LATERAL SURFACE AREA OF A CIRCULAR CYLINDER IS EQUAL TO THE PRODUCT OF THE CIRCUMFERENCE OF THE BASE AND THE HEIGHT OF THE CYLINDER. THAT IS,

$$A_L = 2\pi rh, \text{ WHERE } r \text{ IS THE RADIUS OF THE BASE OF THE CYLINDER.}$$

THE TOTAL SURFACE AREA IS EQUAL TO THE SUM OF THE AREAS OF THE BASES AND THE LATERAL SURFACE AREA.

$$A_T = A_L + 2A_B$$

$$A_T = 2\pi r h + 2\pi r^2 = 2\pi r (h + r)$$

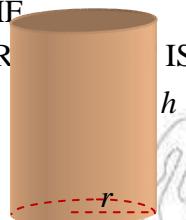


Figure 7.9

EXAMPLE 2 THE TOTAL SURFACE AREA OF A CIRCULAR CYLINDER IS 1CM. FIND THE RADIUS OF THE BASE.

SOLUTION: $A_T = 2\pi r (h + r) \Rightarrow 12 = 2\pi r (1 + r) \Rightarrow 6 = r + r^2$

$$\begin{aligned} r^2 + r - 6 &= 0 \Rightarrow (r + 3)(r - 2) = 0 \Rightarrow r + 3 = 0 \text{ OR } r - 2 = 0 \\ \Rightarrow r &= -3 \text{ OR } r = 2. \end{aligned}$$

THEREFORE, THE RADIUS OF THE BASE IS 2 CM (WHY?)

THE MEASUREMENT OF SPACE COMPLETELY ENCLOSED BY THE BOUNDING SURFACE OF A PRISM IS CALLED **VOLUME**.

THE VOLUME OF ANY PRISM EQUALS THE PRODUCT OF THE BASE AREA AND ALTITUDE THAT IS,

$$V = A_B h$$

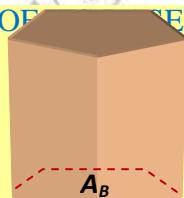


Figure 7.10

EXAMPLE 3 FIND THE TOTAL SURFACE AREA AND VOLUME OF THE FOLLOWING PRISM.

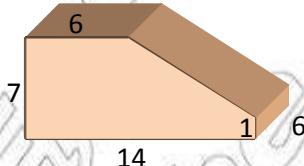


Figure 7.11

SOLUTION: TAKING THE BASE OF THE PRISM TO BE, A SHADED IN THE FOLLOWING FIGURE, WE GET

$$A_B = (7 \times 14) - \left(\frac{1}{2} \times 8 \times 6 \right)$$

$$= 98 - 24 = 74 \text{ UNITS}^2$$

$$\begin{aligned} A_L &= Ph = (7 + 6 + 10 + 14 + 1) \times 6 \\ &= 38 \times 6 = 228 \text{ UNITS}^2 \end{aligned}$$

$$A_T = A_L + 2A_B = 228 + 2 \times 74 = 376 \text{ UNITS}^2$$

$$V = A_B h = 74 \times 6 = 444 \text{ UNITS}^3$$

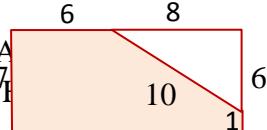


Figure 7.12

VOLUME OF A RIGHT CIRCULAR CYLINDER

THE VOLUME OF A CIRCULAR CYLINDER IS EQUAL TO THE PRODUCT OF THE BASE AREA AND ITS ALTITUDE THAT IS,

$$V = A_B h$$

$$V = r^2 h, \text{ WHERE } r \text{ IS THE RADIUS OF THE BASE.}$$

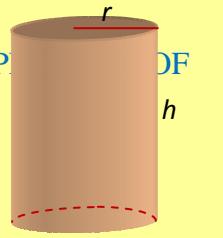


Figure 7.13

EXAMPLE 4 FIND THE VOLUME OF THE CYLINDER WHOSE BASE CIRCUMFERENCE IS 12 CM AND WHOSE LATERAL AREA IS 288 CM²

SOLUTION: $C = 2\pi r \Rightarrow 12 = 2\pi r \Rightarrow r = 6 \text{ CM}$

$$A_L = 2\pi r h$$

$$288 \text{ CM}^2 = 2\pi r h \Rightarrow 288 \text{ CM}^2 = 12 \text{ CM} \times h \Rightarrow h = 24 \text{ CM}$$

$$\text{THEREFORE, } V = r^2 h = (6 \text{ CM})^2 \times 24 \text{ CM} = 36 \text{ CM}^2 \times 24 \text{ CM} = 864 \text{ CM}^3$$

Exercise 7.1

- 1 THE ALTITUDE OF A RECTANGULAR PRISM IS 4 UNITS AND THE WIDTHS AND LENGTHS OF ITS BASE ARE 3 UNITS AND 2 UNITS RESPECTIVELY. FIND:
 - A THE LATERAL SURFACE AREA
 - B THE TOTAL SURFACE AREA
 - C THE VOLUME
- 2 THE ALTITUDE OF THE RIGHT PENTAGONAL PRISM SHOWN IS 5 UNITS AND THE LENGTHS OF THE EDGES OF ITS BASE ARE 3, 4, 5, 6 AND 4 UNITS. FIND THE LATERAL AREA OF THE PRISM.

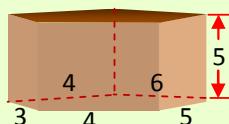


Figure 7.14

- 3 A LATERAL EDGE OF A RIGHT PRISM IS 6 CM AND THE PERIMETER OF ITS BASE IS 20 CM. FIND THE AREA OF ITS LATERAL SURFACE.
- 4 FIND THE LATERAL SURFACE AREA OF EACH OF THE SOLID FIGURES GIVEN IN FIGURE 7.15

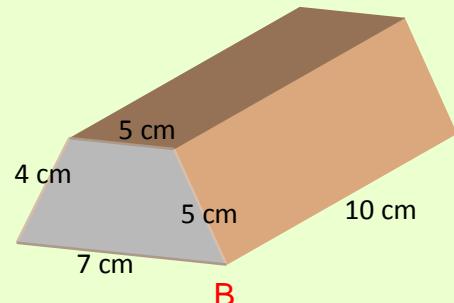
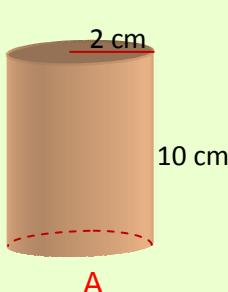


Figure 7.15

- 5** FIND THE PERIMETER OF THE BASE OF A RIGHT PRISM FOR WHICH THE AREA OF THE SURFACE IS 180 UNITS AND THE ALTITUDE IS 4 UNITS.
- 6** THE BASE OF A RIGHT PRISM IS AN EQUILATERAL TRIANGLE OF LENGTH 3 CM AND ITS SURFACES ARE RECTANGULAR REGIONS. IF ITS ALTITUDE IS 8 CM, THEN FIND:
- THE TOTAL SURFACE AREA OF THE PRISM
 - THE VOLUME OF THE PRISM.
- 7** IF THE DIMENSIONS OF A RIGHT RECTANGULAR PRISM ARE 7 CM, 9 CM AND 3 CM, THEN:
- ITS TOTAL SURFACE AREA
 - ITS VOLUME
 - THE LENGTH OF ITS DIAGONAL.
- 8** FIND THE TOTAL SURFACE AREA AND THE VOLUME OF EACH OF THE FOLLOWING SOLIDS.

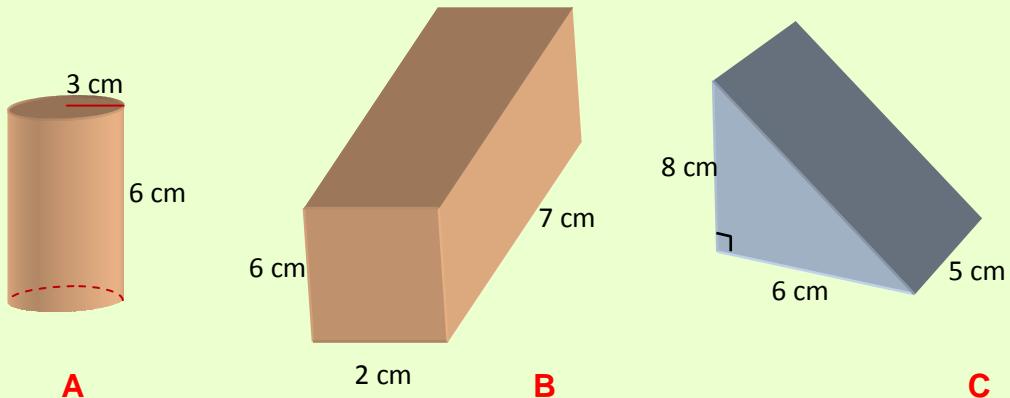


Figure 7.16

- 9** IF THE DIAGONAL OF A CYLINDER IS 10 CM, FIND THE AREA OF ITS LATERAL SURFACE.
- 10** THE RADIUS OF THE BASE OF A RIGHT CIRCULAR CYLINDER IS 2 CM AND ITS ALTITUDE IS 10 CM. FIND:
- THE AREA OF ITS LATERAL SURFACE
 - THE TOTAL SURFACE AREA
 - THE VOLUME.
- 11** SHOW THAT THE AREA OF THE LATERAL SURFACE OF A CYLINDER WHOSE CIRCULAR BASE HAS RADIUS r AND WHOSE HEIGHT IS h IS $2\pi rh$.
- 12** IMAGINE A CYLINDRICAL CONTAINER IN WHICH THE HEIGHT AND THE DIAMETER ARE EXPRESSED AS x AND y RESPECTIVELY. FIND EXPRESSIONS, IN TERMS OF ITS HEIGHT, FOR ITS
- TOTAL SURFACE AREA
 - VOLUME.
- 13** A CIRCULAR HOLE OF RADIUS 5 CM IS DRILLED THROUGH THE CENTRE OF A CYLINDER WHOSE BASE HAS RADIUS 6 CM AND WHOSE ALTITUDE IS 8 CM. FIND THE SURFACE AREA AND VOLUME OF THE RESULTING SOLID FIGURE.

7.2**PYRAMIDS, CONES AND SPHERES**

DO YOU REMEMBER WHAT YOU LEARNT ABOUT PYRAMIDS, CONES AND SPHERES IN YOUR GRADES? CAN YOU GIVE SOME EXAMPLES OF PYRAMIDS, CONES AND SPHERES FROM REAL LIFE?

Definition 7.2

A **pyramid** is a solid figure formed when each vertex of a polygon is joined to the same point not in the plane of the polygon (See [FIGURE 7.17](#)).

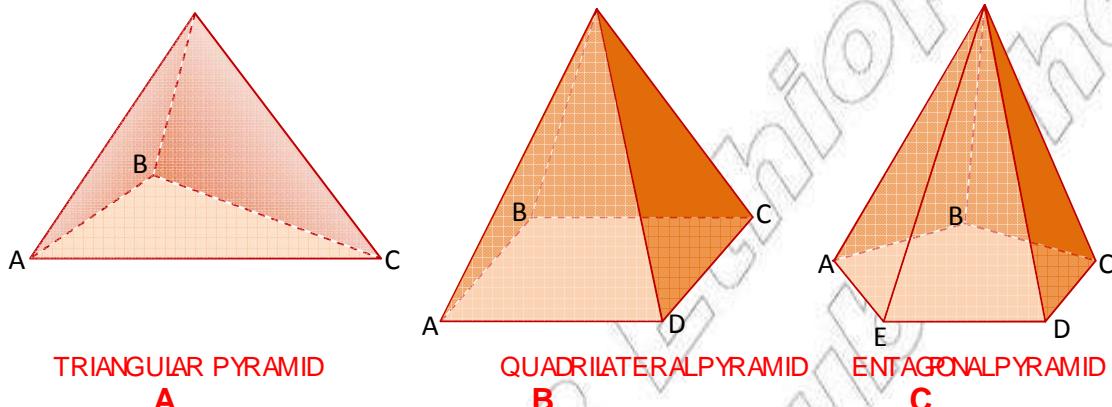


Figure 7.17

ACTIVITY 7.2

- 1 WHAT IS A REGULAR PYRAMID?
- 2 WHAT IS A TETRAHEDRON?
- 3 DETERMINE WHETHER EACH OF THE FOLLOWING STATEMENTS IS TRUE OR FALSE:
 - A THE LATERAL FACES OF A PYRAMID ARE TRIANGULAR REGIONS.
 - B THE NUMBER OF TRIANGULAR FACES OF A PYRAMID HAVING SAME VERTEXIS EQUAL TO THE NUMBER OF EDGES OF THE BASE.
 - C THE ALTITUDE OF A CONE IS THE PERPENDICULAR DISTANCE FROM THE BASE TO THE VERTEX OF THE CONE.
- 4 USING [FIGURE 7.18](#) COMPLETE THE FOLLOWING TO MAKE TRUE STATEMENTS.
 - A THE FIGURE IS CALLED A _____.
 - B THE REGION $\triangle VAD$ IS CALLED A _____.
 - C THE REGION $\triangle VDEF$ IS CALLED _____.
 - D _____ IS THE ALTITUDE OF THE PYRAMID.
 - E \overline{VE} AND \overline{VF} ARE CALLED _____.

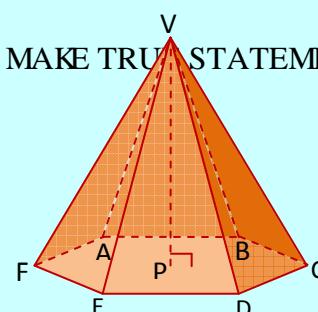


Figure 7.18

F SINCE \overline{BCDEF} IS A HEXAGONAL REGION, THE PYRAMID IS CALLED A _____.

5 DRAW A CONE AND INDICATE:

A ITS SLANT HEIGHT **B** ITS BASE **C** ITS LATERAL SURFACE.

THE **altitude** OF A PYRAMID IS THE LENGTH OF THE PERPENDICULAR FROM THE VERTEX TO THE PLANE CONTAINING THE BASE.

THE **slant height** OF A REGULAR PYRAMID IS THE ALTITUDE OF ANY OF ITS LATERAL FACES.

Definition 7.3

The solid figure formed by joining all points of a circle to a point not on the plane of the circle is called a **cone**.

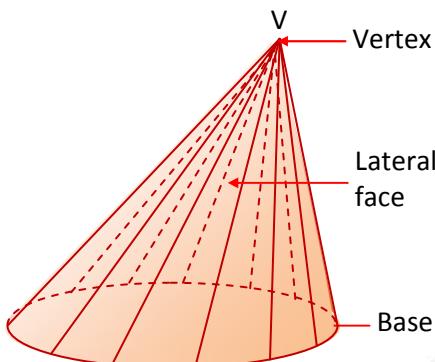


Figure 7.19

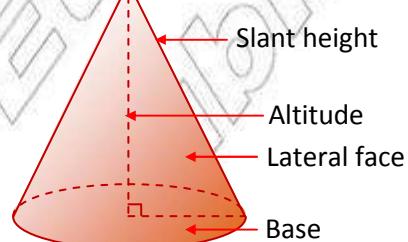


Figure 7.20

THE FIGURE SHOWING [FIGURE 7.19](#) REPRESENTS A CONE. NOTE THAT THE CURVED SURFACE IS THE **lateral surface** OF THE CONE.

A **right circular cone** (SEE [FIGURE 7.20](#)) IS A CONE WITH THE FOOT OF ITS ALTITUDE AT THE CENTRE OF THE BASE. A LINE SEGMENT FROM THE VERTEX OF A CONE TO ANY POINT ON THE BOUNDARY OF THE BASE (CIRCLE) IS CALLED THE **slant height**.

ACTIVITY 7.3



- 1 CONSIDER A REGULAR SQUARE PYRAMID WITH BASE EDGE 4 CM AND SLANT HEIGHT 5 CM.
 - A** HOW MANY LATERAL FACES DOES IT HAVE?
 - B** FIND THE AREA OF EACH LATERAL FACE.
 - C** FIND THE LATERAL SURFACE AREA.
 - D** FIND THE TOTAL SURFACE AREA.
- 2 TRY TO WRITE THE FORMULA FOR THE TOTAL SURFACE AREA OF A PYRAMID OR A CONE.

Surface area

THE LATERAL SURFACE AREA OF A REGULAR PYRAMID IS EQUAL TO HALF THE PRODUCT HEIGHT AND THE PERIMETER OF THE BASE. THAT IS,

$$A_L = \frac{1}{2} P\ell$$

WHERE A_L DENOTES THE LATERAL SURFACE AREA;
 P DENOTES THE PERIMETER OF THE BASE;
 ℓ DENOTES THE SLANT HEIGHT.

THE TOTAL SURFACE AREA OF A PYRAMID IS GIVEN BY

$$A_T = A_B + A_L = A_B + \frac{1}{2} P\ell,$$

WHERE A_B IS AREA OF THE BASE.

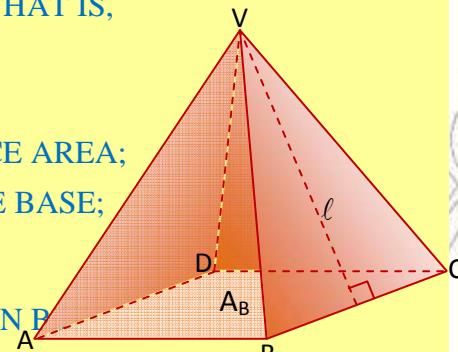


Figure 7.21

EXAMPLE 1 A REGULAR PYRAMID HAS A SQUARE BASE WHOSE SIDE IS 4 CM LONG. THE EDGES ARE 6 CM EACH.

- A** WHAT IS ITS SLANT HEIGHT? **B** WHAT IS THE LATERAL SURFACE AREA?
- C** WHAT IS THE TOTAL SURFACE AREA?

SOLUTION: CONSIDER FIGURE 7.22

A $(VE)^2 + (EC)^2 = (VC)^2$

$$\ell^2 + 2^2 = 6^2$$

$$\ell^2 = 32$$

$$\ell = 4\sqrt{2} \text{ CM}$$

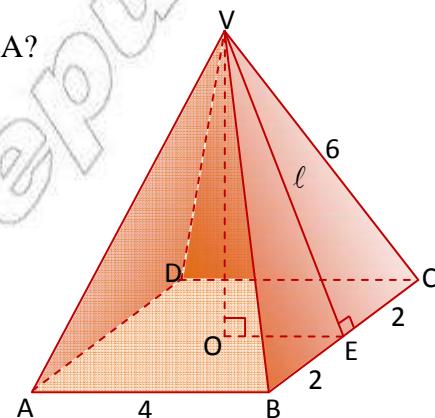


Figure 7.22

THEREFORE, THE SLANT HEIGHT IS 4

- B** THERE ARE 4 ISOSCELES TRIANGLES.

THEREFORE,

$$A_L = 4 \times \frac{1}{2} BC \times VE = 4 \left(\frac{1}{2} \times 4 \times 4\sqrt{2} \right) = 32\sqrt{2} \text{ CM}^2$$

$$\text{OR } A_L = \frac{1}{2} P\ell = \frac{1}{2} (4+4+4+4) 4\sqrt{2} = 8 \times 4\sqrt{2} = 32\sqrt{2} \text{ CM}^2$$

C $A_T = A_L + A_B = 32\sqrt{2} + 4 \times 4$
 $= 32\sqrt{2} + 16 = 16(2\sqrt{2} + 1) \text{ CM}^2$

THE LATERAL SURFACE AREA OF A RIGHT CIRCULAR CONE IS EQUAL TO HALF THE PRODUCT OF THE SLANT HEIGHT AND THE CIRCUMFERENCE OF THE BASE. THAT IS,

$$A_L = \frac{1}{2} P\ell = \frac{1}{2} (2\pi R)\ell = \pi r\ell;$$

$$\ell = \sqrt{h^2 + r^2}$$

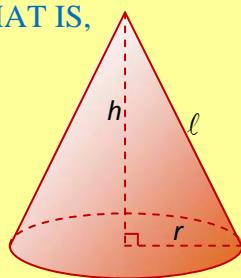


Figure 7.23

WHERE ℓ DENOTES THE LATERAL SURFACE AREA, r FOR THE BASE RADIUS, h FOR THE ALTITUDE.

THE TOTAL SURFACE AREA IS EQUAL TO THE SUM OF THE AREA OF THE BASE AND THE LATERAL SURFACE AREA. THAT IS,

$$A_T = A_L + A_B = \pi r\ell + \pi r^2 = \pi r(\ell + r)$$

EXAMPLE 2 THE ALTITUDE OF A RIGHT CIRCULAR CONE IS 8 CM. THE BASE IS 6 CM. THEN FIND ITS:

- A SLANT HEIGHT B LATERAL SURFACE AREA C TOTAL SURFACE AREA.

SOLUTION: CONSIDER FIGURE 7.24

A $\ell = \sqrt{h^2 + r^2} = \sqrt{8^2 + 6^2} = \sqrt{100}$

$$\ell = 10 \text{ CM}$$

B $A_L = \pi r\ell = \pi \times 6 \times 10 = 60 \text{ CM}^2$

C $A_T = \pi r(\ell + r) = \pi \times 6 (10 + 6) = 6 \times 16$
 $= 96 \text{ CM}^2$

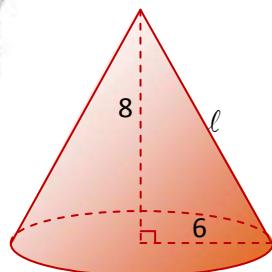


Figure 7.24

Volume

THE VOLUME OF ANY PYRAMID IS EQUAL TO ONE THIRD THE PRODUCT OF ITS BASE AREA AND ITS ALTITUDE. THAT IS,

$$V = \frac{1}{3} A_B h,$$

WHERE V DENOTES THE VOLUME, A_B THE AREA OF THE BASE AND h THE ALTITUDE.

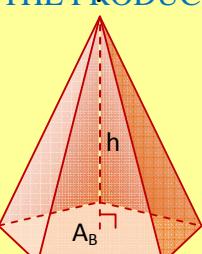


Figure 7.25

EXAMPLE 3 FIND THE VOLUME OF THE PYRAMID GIVEN ABOVE EXAMPLE 1

SOLUTION: HERE, WE NEED TO FIND THE ALTITUDE OF THE PYRAMID AS SHOWN BELOW

$$(VO)^2 + (OE)^2 = (VE)^2 \Rightarrow h^2 + 2^2 = (4\sqrt{2})^2 \\ h^2 + 4 = 32$$

$$h^2 = 28 \Rightarrow h = 2\sqrt{7} \text{ CM}$$

$$V = \frac{1}{3} A_B h = \frac{1}{3} \times (4 \times 4) \times 2\sqrt{7} = \frac{32}{3}\sqrt{7} \text{ CM}^3$$

THE VOLUME OF A CIRCULAR CONE IS EQUAL TO ONE-THIRD OF THE PRODUCT OF ITS BASE AREA AND ITS ALTITUDE. THAT IS,

$$V = \frac{1}{3} A_B h = \frac{1}{3} \pi r^2 h$$

WHERE π DENOTES THE VOLTH, r THE RADIUS OF THE BASE
AND h THE ALTITUDE

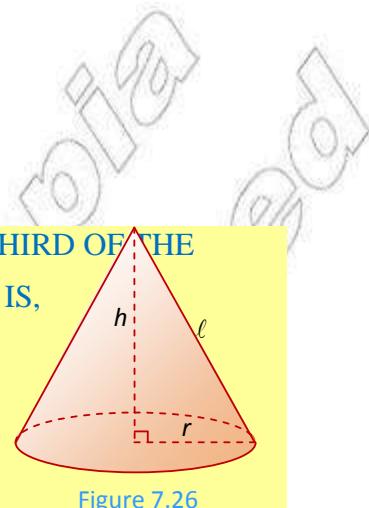


Figure 7.26

EXAMPLE 4 FIND THE VOLUME OF THE RIGHT CIRCULAR CONE GIVEN IN EXAMPLE 2

$$\text{SOLUTION: } V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi (6)^2 \times 8 = 96\pi \text{ CM}^3$$

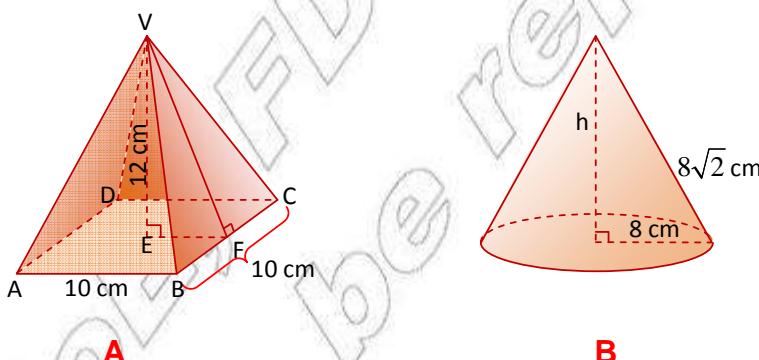
EXAMPLE 5 FIND THE LATERAL SURFACE AREA, TOTAL SURFACE AREA AND THE VOLUME OF THE FOLLOWING REGULAR PYRAMID AND RIGHT CIRCULAR CONE.

Figure 7.27

SOLUTION:

- A TO FIND THE LATERAL SURFACE AREA, WE MUST FIND THE SLANT HEIGHT ℓ IN $\triangle VEF$, WE HAVE,

$$(VE)^2 + (EF)^2 = (VF)^2 \Rightarrow 12^2 + 5^2 = (VF)^2$$

$$169 = (VF)^2 \Rightarrow VF = 13 \text{ CM}$$

THEREFORE, THE SLANT HEIGHT IS 13 CM

$$\text{NOW, } A = \frac{1}{2} P\ell = \frac{1}{2} (10 + 10 + 10 + 10)13 = 260 \text{ CM}^2$$

$$A_T = A_L + A_B = 260 \text{ CM}^2 + 100 \text{ CM}^2 = 360 \text{ CM}^2$$

$$V = \frac{1}{3} A_B h = \frac{1}{3} \times 100 \times 12 = 400 \text{ CM}^3.$$

B ALTITUDE : $\sqrt{\ell^2 - r^2} = \sqrt{(8\sqrt{2})^2 - 8^2} = \sqrt{128 - 64} = \sqrt{64} = 8 \text{ CM}$

$$A_L = r\ell = 8 \times 8\sqrt{2} = 64\sqrt{2} \text{ CM}^2$$

$$A_T = r(\ell + r) = 8(8\sqrt{2} + 8) = 64(\sqrt{2} + 1) \text{ CM}^2$$

$$V = \frac{1}{3} r^2 h = \frac{1}{3} (8)^2 \times 8 = \frac{512}{3} \text{ CM}^3$$

Surface area and volume of a sphere

THE SPHERE IS ANOTHER SOLID FIGURE YOU STUDIED IN LOWER GRADES.

Definition 7.4

A **sphere** is a closed surface, all points of which are equidistant from a point called the **centre**.

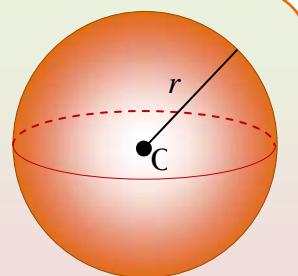


Figure 7.28

THE SURFACE AREA AND THE VOLUME OF A SPHERE OF RADIUS ARE GIVEN BY

$$A = 4\pi r^2$$

$$V = \frac{4}{3}\pi r^3$$

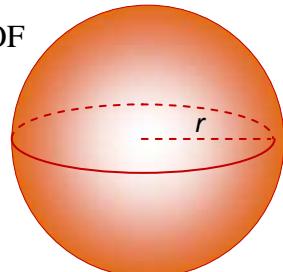


Figure 7.29

EXAMPLE 6 FIND THE SURFACE AREA AND VOLUME OF A SPHERE WITH A DIAMETER OF 10 M.

SOLUTION: WE KNOW THAT $2r = d \therefore r = \frac{d}{2} = \frac{10}{2} = 5 \text{ M}$

$$A = 4\pi r^2 = 4\pi (5)^2 = 100\pi \text{ M}^2$$

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi (5)^3 = \frac{500}{3}\pi \text{ M}^3$$

Exercise 7.2

- 1** CALCULATE THE VOLUME OF EACH OF THE FOLLOWING SOLID FIGURES:

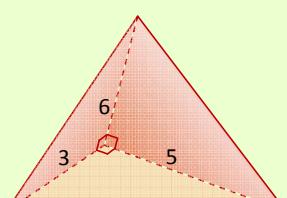
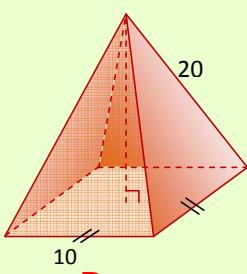
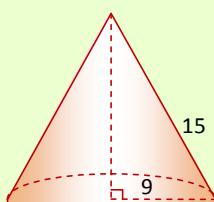
**A**

Figure 7.30

**C**

- 2** ONE EDGE OF A RIGHT SQUARE PYRAMID IS 6 CM LONG IF THE LATERAL EDGE IS 8 CM, THEN FIND:
A ITS TOTAL SURFACE AREA **B** ITS VOLUME.
- 3** THE ALTITUDE OF A RIGHT EQUILATERAL TRIANGULAR PYRAMID IS 6 CM, THE BASE IS 6 CM, THEN FIND:
A ITS TOTAL SURFACE AREA **B** ITS VOLUME.
- 4** A REGULAR SQUARE PYRAMID HAS ALL ITS EDGES 7 CM LONG. FIND:
A ITS TOTAL SURFACE AREA **B** ITS VOLUME
- 5** THE ALTITUDE AND RADIUS OF A RIGHT CIRCULAR CONE RESPECTIVELY. FIND:
A ITS TOTAL SURFACE AREA **B** ITS VOLUME.
- 6** THE VOLUME OF A PYRAMID IS 24cm^3 . THE PYRAMID HAS A RECTANGULAR BASE WITH SIDES 6 CM BY 4 CM. FIND THE ALTITUDE AND LATERAL SURFACE AREA OF THE PYRAMID HAS EQUAL LATERAL EDGES.
- 7** SHOW THAT THE VOLUME OF A REGULAR SQUARE PYRAMID IS $\frac{s^3 \sqrt{2}}{6}$ LATERAL EQUILATERAL TRIANGLES OF SIDE LENGTH s
- 8** THE LATERAL EDGE OF A REGULAR TETRAHEDRON IS 8 CM. FIND ITS ALTITUDE.
- 9** FIND THE VOLUME OF A CONE OF HEIGHT 12 CM AND SLANT HEIGHT 13 CM.
- 10** FIND THE VOLUME AND SURFACE AREA OF A SPHERICAL FOOTBALL WITH A RADIUS 10 CM.
- 11** A GLASS IS IN THE FORM OF AN INVERTED CONE WHOSE BASE DIAMETER IS 0.1 LITRES OF WATER FILLS THE GLASS COMPLETELY, FIND THE DEPTH OF WATER

$$\left(\text{TAKEN } \pi = \frac{22}{7} \right)$$
- 12** A SOLID METAL CYLINDER WITH A LENGTH OF 24 CM AND RADIUS 2 CM IS MELTED TO FORM A SPHERE. WHAT IS THE RADIUS OF THE SPHERE?

7.3**FRUSTUMS OF PYRAMIDS AND CONES**

IN THE PRECEDING SECTION, YOU HAVE STUDIED ABOUT PYRAMIDS AND CONES. YOU STUDY THE SOLID FIGURE OBTAINED WHEN A PYRAMID AND A CONE ARE CUT BY A PLANE PARALLEL TO THE BASE AS SHOWN IN [FIGURE 7.31](#)

LET π BE THE PLANE THAT CONTAINS THE BASE PLANE PARALLEL TO THE BASE THAT CUTS THE PYRAMID AND CONE.

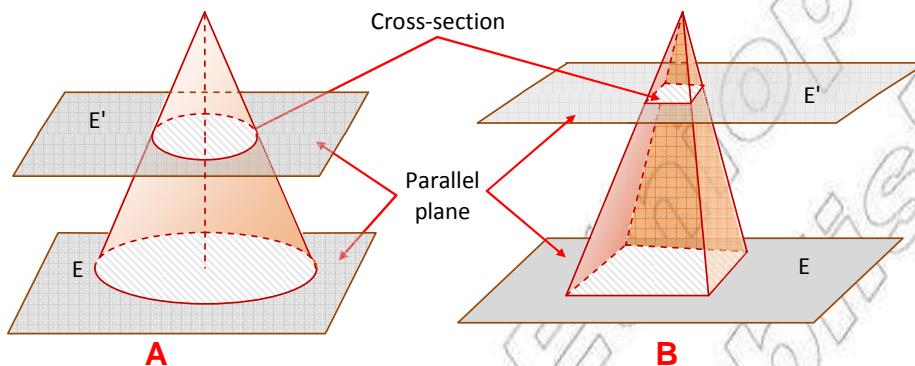


Figure 7.31

Definition 7.5

If a pyramid or a cone is cut by a plane parallel to the base, the intersection of the plane and the pyramid (or the cone) is called a **horizontal cross-section** of the pyramid (or the cone).

LET US NOW EXAMINE THE RELATIONSHIP BETWEEN THE BASE AND THE CROSS-SECTION.

LET $\triangle ABC$ BE THE BASE OF THE PYRAMID LYING IN THE PLANE π . LET h BE THE ALTITUDE OF THE PYRAMID, AND LET $\triangle A'B'C'$ BE THE CROSS-SECTION AT DISTANCE k UNITS FROM THE VERTEX.

LET D AND D' BE THE POINTS AT WHICH THE PERPENDICULAR FROM MEET π AND $\triangle A'B'C'$, RESPECTIVELY.

WE HAVE,

$$1 \quad \triangle VAD' \sim \triangle VAD.$$

THIS FOLLOWS FROM THE FACT THAT IF A PLANE INTERSECTS EACH OF TWO PARALLEL LINES, IT INTERSECTS THEM IN TWO PARALLEL LINES, AND AN APPLICATION OF THE AA SIMILARITY CRITERION. HENCE,

$$\frac{VA'}{VA} = \frac{VD'}{VD} = \frac{k}{h}$$

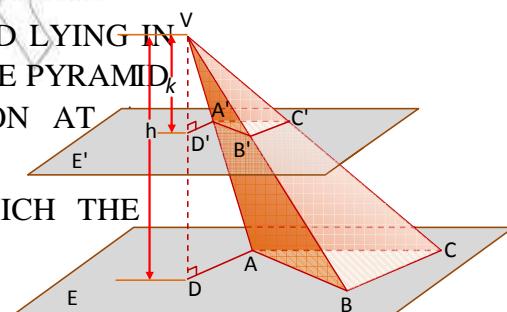


Figure 7.32

- 2 SIMILARLY, $\triangle VDB' \sim \triangle VDB$ AND HENCE,

$$\frac{VB'}{VB} = \frac{VD'}{VD} = \frac{k}{h}$$

THEN, FROM AND 2 AND THE SAS SIMILARITY THEOREM, WE GET,

- 3 $\triangle VA'B' \sim \triangle VAB$. THEREFORE, $\frac{A'B'}{AB} = \frac{VA'}{VA} = \frac{k}{h}$

BY AN ARGUMENT SIMILAR TO THAT LEADING TO (3), WE HAVE

- 4 $\frac{B'C'}{BC} = \frac{k}{h}$ AND $\frac{A'C'}{AC} = \frac{k}{h}$

HENCE, BY THE SSS SIMILARITY THEOREM,

$$\triangle ABC \sim \triangle A'B'C'$$

ACTIVITY 7.4

IN THE PYRAMID SHOWN IN FIGURE 7.33, $\triangle ABC$ IS EQUILATERAL. A PLANE PARALLEL TO THE BASE INTERSECTS THE LATERAL EDGES

THAT $VE = \frac{1}{3}EB$.

A WHAT IS $\frac{VF}{VC}$?

B WHAT IS $\frac{EF}{BC}$?

C COMPARE THE AREAS OF $\triangle VBC$ AND OF $\triangle DEF$ AND $\triangle ABC$.

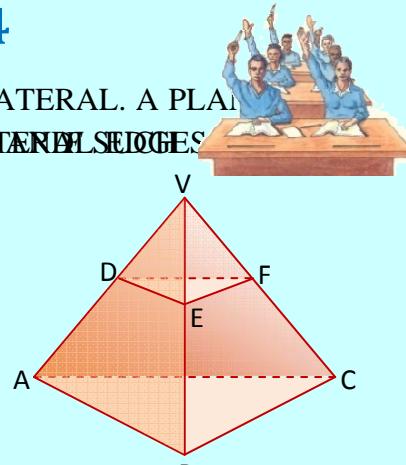
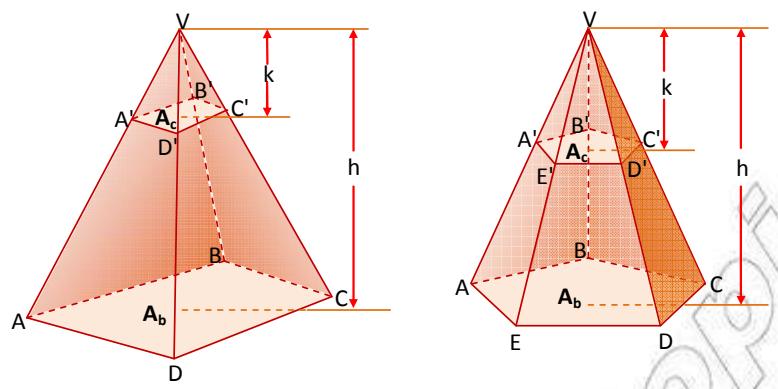


Figure 7.33

Theorem 7.1

In any pyramid, the ratio of the area of a cross-section to the area of the base is $\frac{k^2}{h^2}$ where h is the altitude of the pyramid and k is the distance from the vertex to the plane of the cross-section.



$$\frac{A_c}{A_b} = \frac{\text{AREA } A'b'c'd'}{\text{AREA } ABCD} = \frac{k^2}{h^2}$$

$$\frac{\text{AREA } A'b'c'd'e'}{\text{AREA } ABCD} = \frac{k^2}{h^2}$$

Figure 7.34

EXAMPLE 1 THE ALTITUDE OF A PYRAMID IS 12 CM. WHAT IS THE AREA OF A HORIZONTAL CROSS-SECTION 4 CM FROM THE BASE?

SOLUTION: LET A_c BE THE AREA OF THE CROSS-SECTION, A_b THE BASE AREA.

$$\text{THEN, } \frac{A_c}{A_b} = \frac{k^2}{h^2} \Rightarrow \frac{A_c}{90} = \frac{4^2}{12^2}$$

$$\therefore A_c = \frac{90 \times 16}{144} \text{ CM}^2 = 10 \text{ CM}^2$$

NOTE THAT SIMILAR PROPERTIES HOLD TRUE WHEN A CONE IS CUT BY A PLANE PARALLEL TO THE BASE. *Can you state them?*

ACTIVITY 7.5



- THE ALTITUDE OF A SQUARE PYRAMID IS 6 UNITS. THE AREA OF THE BASE IS 4 UNITS LONG. FIND THE AREA OF A HORIZONTAL CROSS-SECTION AT A DISTANCE 2 UNITS ABOVE THE BASE.
- THE AREA OF THE BASE OF A PYRAMID IS 64 UNITS. THE ALTITUDE OF THE PYRAMID IS 8 CM. WHAT IS THE AREA OF A CROSS-SECTION 2 CM FROM THE VERTEX?
- THE RADIUS OF A CROSS-SECTION OF A CONE AT A DISTANCE 5 CM FROM THE BASE IS 3 CM. THE RADIUS OF THE BASE OF THE CONE IS 3 CM, FIND ITS ALTITUDE.

WHEN A PRISM IS CUT BY A PLANE PARALLEL TO THE BASE, EACH PART OF THE PRISM IS A PRISM AS SHOWN IN FIGURE 7.35A

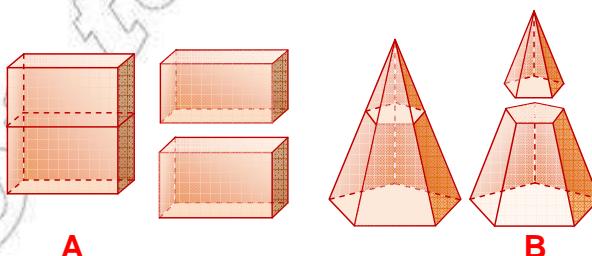


Figure 7.35

HOWEVER, WHEN A PYRAMID IS CUT BY A PLANE PARALLEL TO THE BASE, THE PART BETWEEN THE VERTEX AND THE HORIZONTAL CROSS-SECTION IS AGAIN A PYRAMID WHILE THE PART IS NOT A PYRAMID (AS SHOWN IN FIGURE 7.35B)

Frustum of a pyramid

Definition 7.6

A **frustum** of a pyramid is a part of the pyramid included between the base and a plane parallel to the base.

THE BASE OF THE PYRAMID AND THE CROSS-SECTION MADE BY THE PLANE PARALLEL TO THE **bases of the frustum**. THE OTHER FACES ARE **Lateral faces**. THE TOTAL SURFACE OF A FRUSTUM IS THE SUM OF THE LATERAL SURFACE AND THE BASES.

THE **altitude** OF A FRUSTUM OF A PYRAMID IS THE PERPENDICULAR DISTANCE BETWEEN THE BASES.

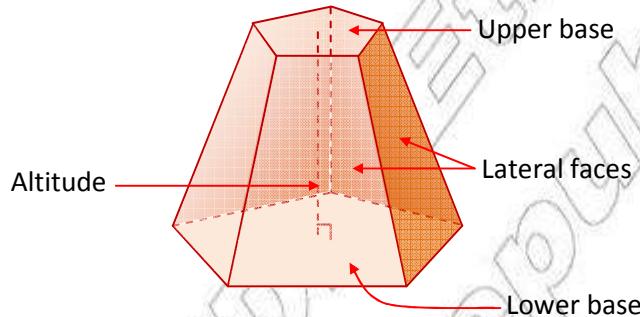


Figure 7.36

Note:

- I THE LATERAL FACES OF A FRUSTUM OF A PYRAMID ARE TRAPEZIUMS.
- II THE LATERAL FACES OF A FRUSTUM OF A REGULAR PYRAMID ARE TRAPEZIUMS.
- III THE SLANT HEIGHT OF A FRUSTUM OF A REGULAR PYRAMID IS THE ALTITUDE OF THE LATERAL FACES.
- IV THE LATERAL SURFACE AREA OF A FRUSTUM OF A PYRAMID IS SUM THE LATERAL FACES.

Frustum of a cone

Definition 7.7

A **frustum** of a cone is a part of the cone included between the base and a horizontal cross-section made by a plane parallel to the base.

FOR A FRUSTUM OF A CONE, THE BASES ARE THE TWO BASES OF THE CONE AND THE CROSS-SECTION PARALLEL TO THE BASE OF THE CONE. THE **lateral surface** IS THE CURVED SURFACE THAT MAKES UP THE FRUSTUM. THE ALTITUDE IS THE PERPENDICULAR DISTANCE BETWEEN THE BASES.

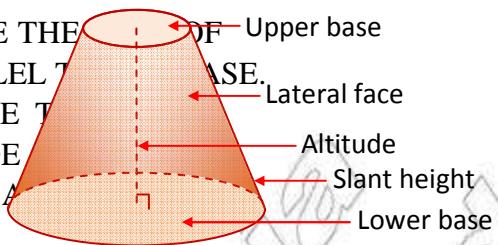


Figure 7.37

THE **slant height** OF A FRUSTUM OF A RIGHT CIRCULAR CONE IS THE PART OF THE SLANT HEIGHT OF THE CONE WHICH IS INCLUDED BETWEEN THE BASES.

CAN YOU NAME SOME OBJECTS WE USE IN REAL LIFE (AT HOME) THAT ARE FRUSTUMS? CAN YOU NAME SOME OBJECTS WE USE IN REAL LIFE (AT HOME) THAT ARE FRUSTUMS? ARE A BUCKET AND A GLASS FRUSTUM OF CONES? DISCUSS.

EXAMPLE 2 THE LOWER BASE OF THE FRUSTUM OF A REGULAR PYRAMID IS A SQUARE OF 4 CM UNITS LONG. THE UPPER BASE IS 3 CM LONG. IF THE SLANT HEIGHT IS 6 CM, FIND ITS LATERAL SURFACE AREA.

SOLUTION: AS SHOWN IN FIGURE 7.38, EACH LATERAL FACE IS A TRAPEZIUM, THE AREA OF WHICH IS $\frac{1}{2} \times h(b_1 + b_2)$. SINCE THE FOUR FACES ARE CONGRUENT ISOSCELES TRAPEZIUMS, EACH LATERAL FACE IS

$$A_L = \frac{1}{2} \times h(b_1 + b_2) = \frac{1}{2} \times 6(3 + 4) = 21 \text{ CM}^2$$

SINCE THE FOUR FACES ARE CONGRUENT ISOSCELES TRAPEZIUMS, THE LATERAL SURFACE AREA IS

$$A_L = 4 \times 21 \text{ CM}^2 = 84 \text{ CM}^2$$

EXAMPLE 3 THE LOWER BASE OF THE FRUSTUM OF A REGULAR PYRAMID IS A SQUARE OF 8 UNITS LONG. THE UPPER BASE IS 4 UNITS LONG. IF THE SLANT HEIGHT IS 6 UNITS, FIND THE LATERAL SURFACE AREA.

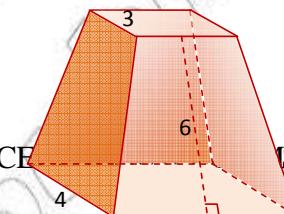


Figure 7.38

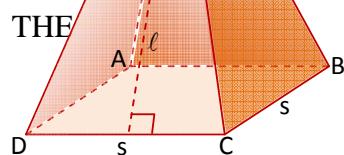


Figure 7.39

SOLUTION: FIGURE 7.39 REPRESENTS THE GIVEN PROBLEM. THE LOWER BASE IS A SQUARE OF 8 UNITS LONG. SIMILARLY, THE UPPER BASE IS A SQUARE OF 4 UNITS LONG.

LATERAL SURFACE AREA:

$$\begin{aligned} A_L &= \text{AREA}(C'D) + \text{AREA}(B'C) + \text{AREA}(B'A) + \text{AREA}(A'D) \\ &= \frac{1}{2} \ell (s + s') + \frac{1}{2} \ell (s + s') + \frac{1}{2} \ell (s + s') + \frac{1}{2} \ell (s + s') \\ A_L &= \frac{1}{2} \ell (4s + 4s') = 2\ell(s + s'). \end{aligned}$$

OBSERVE THAT $4s$ AND $4s'$ ARE THE PERIMETERS OF THE LOWER AND UPPER BASES, RESPECTIVELY. IN GENERAL, WE HAVE THE FOLLOWING THEOREM:

Theorem 7.2

The lateral surface area (A_L) of a frustum of a regular pyramid is equal to half the product of the slant height (ℓ) and the sum of the perimeter (P) of the lower base and the perimeter (P') of the upper base. That is,

$$A_L = \frac{1}{2} \ell (P + P')$$

Group Work 7.1

CONSIDER THE FOLLOWING FIGURE.

- 1 FIND THE AREAS OF THE BASES.
- 2 FIND THE CIRCUMFERENCES OF THE BASES OF THE CONE AND THE FRUSTUM.
- 3 FIND LATERAL SURFACE AREA OF THE BIGGER CONE.
- 4 FIND LATERAL SURFACE AREA OF THE SMALLER CONE.
- 5 FIND LATERAL SURFACE AREA OF THE FRUSTUM.
- 6 GIVE THE VOLUME OF THE FRUSTUM.

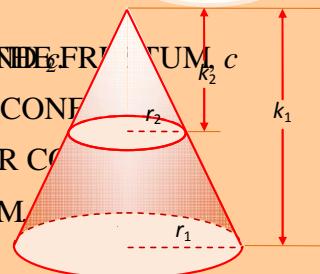


Figure 7.40

EXAMPLE 4 A FRUSTUM OF HEIGHT 4 CM IS FORMED FROM A RIGHT CIRCULAR CONE OF HEIGHT 8 CM AND BASE RADIUS 6 CM AS SHOWN IN FIGURE 7.41. CALCULATE THE LATERAL SURFACE AREA OF THE FRUSTUM.

SOLUTION: LET A , A_c AND A_b STAND FOR AREA OF THE BASE, AREA OF THE CONE, AREA OF THE CROSS-SECTION AND AREA OF THE FRUSTUM, RESPECTIVELY.

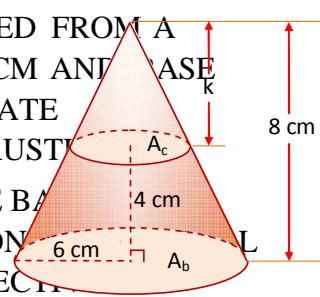


Figure 7.41

$$\frac{\text{AREA OF CROSS-SECTION}}{\text{AREA OF THE BASE}} = \left(\frac{r'}{r} \right)^2$$

$$\frac{A_c}{A_b} = \left(\frac{4}{8} \right)^2, \text{ SINCE } 8 \text{ CM} - 4 \text{ CM} = 4 \text{ CM}$$

$$\frac{A_c}{36} = \frac{1}{4} \quad (\text{AREA OF THE BASE} = \pi \times 6^2 = 36\pi)$$

$$A_c = \frac{1}{4} \times 36 = 9 \text{ CM}^2$$

$A_c = (r')^2$, WHERE r' IS RADIUS OF THE CROSS-SECTION

$$\therefore 9 = (r')^2 \Rightarrow r' = 3 \text{ CM}$$

SLANT HEIGHT OF THE BIGGER CONE IS:

$$\ell = \sqrt{h^2 + r^2} = \sqrt{8^2 + 6^2} = \sqrt{100} = 10 \text{ CM}$$

SLANT HEIGHT OF THE SMALLER CONE IS:

$$\ell' = \sqrt{k^2 + (r')^2} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5 \text{ CM}$$

NOW THE LATERAL SURFACE AREA OF:

$$\text{THE SMALLER CONE} = (3 \text{ CM}) \times 5 \text{ CM} = 15 \text{ CM}^2$$

$$\text{THE BIGGER CONE} = (6 \text{ CM}) \times 10 \text{ CM} = 60 \text{ CM}^2.$$

HENCE, THE AREA OF THE LATERAL SURFACE OF THE FRUSTUM IS

$$A_L = 60 \text{ CM}^2 - 15 \text{ CM}^2 = 45 \text{ CM}^2.$$

THE LATERAL SURFACE (CURVED SURFACE) OF A FRUSTUM OF A CIRCULAR CONE IS A TRAPEZOID. ITS PARALLEL SIDES (BASES) HAVE LENGTHS EQUAL TO THE CIRCUMFERENCE OF THE BASES AND WHOSE HEIGHT IS EQUAL TO THE HEIGHT OF THE FRUSTUM.

Theorem 7.3

For a frustum of a right circular cone with altitude h and slant height ℓ , if the circumferences of the bases are c and c' , then the lateral surface area of the frustum is given by

$$A_L = \frac{1}{2} \ell (c + c') = \frac{1}{2} \ell (2r + 2r') = \ell(r + r')$$

EXAMPLE 5 A FRUSTUM FORMED FROM A RIGHT CIRCULAR CONE HAS RADII OF 8 CM AND 12 CM AND SLANT HEIGHT OF 10 CM. FIND:

- A** THE AREA OF THE CURVED SURFACE
- B** THE AREA OF THE TOTAL SURFACE.(USE $\pi = 3.14$)

SOLUTION:

$$\begin{aligned} \mathbf{A} \quad A_L &= \ell(r + r') = 10 \text{ CM}(8 + 12) \text{ CM} = 10 \text{ CM} \times 20 \text{ CM} \\ &= 200 \text{ CM}^2 = 200 \times 3.14 \text{ CM}^2 = 628 \text{ CM}^2 \end{aligned}$$

B AREA OF BASES:

$$\begin{aligned} A_B &= A_c + A_b = (r')^2 + r^2 = (8 \text{ CM})^2 + (12 \text{ CM})^2 = 64 \text{ CM}^2 + 144 \text{ CM}^2 \\ &= 208 \text{ CM}^2 \approx 208 \times 3.14 \text{ CM}^2 \approx 653 \text{ CM}^2 \end{aligned}$$

TOTAL SURFACE AREA OF THE FRUSTUM:

$$A_T = A_L + A_B \approx 628 \text{ CM}^2 + 653 \text{ CM}^2 = 1281 \text{ CM}^2$$

EXAMPLE 6 THE AREA OF THE UPPER AND LOWER BASES OF A FRUSTUM OF A PYRAMID ARE 25 cm^2 AND 36 cm^2 RESPECTIVELY. IF ITS ALTITUDE IS 2 CM, THEN FIND k .

SOLUTION:

$$\frac{A_c}{A_b} = \left(\frac{k}{h}\right)^2 \Rightarrow \frac{25}{36} = \frac{k^2}{(2+k)^2}$$

$$\Rightarrow \frac{5}{6} = \frac{k}{2+k} \Rightarrow 6k = 5k + 10$$

$$\therefore k = 10$$

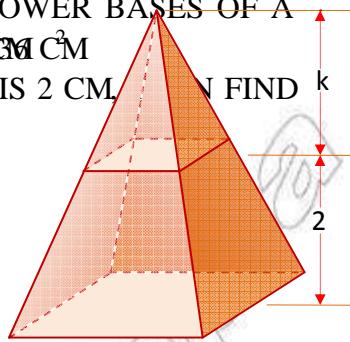


Figure 7.42

THEREFORE, THE ALTITUDE OF THE PYRAMID IS $2 \text{ cm} + 10 \text{ cm} = 12 \text{ cm}$.

NOTE THAT THE UPPER AND LOWER BASES OF THE FRUSTUM OF A PYRAMID ARE SIMILAR CIRCLES AND THAT OF A CONE ARE SIMILAR CIRCLES.

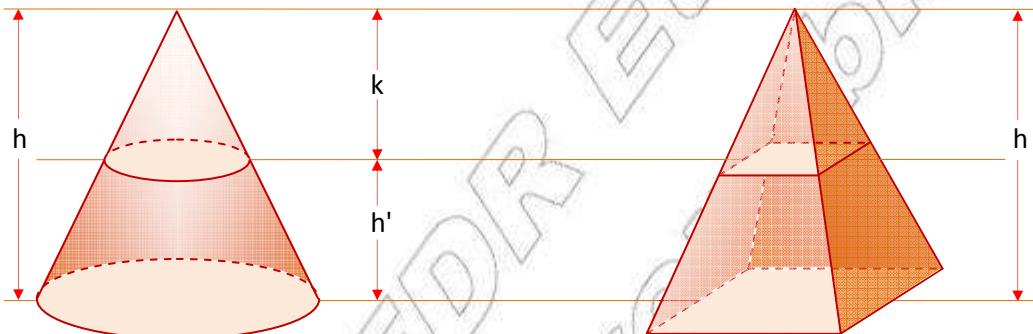


Figure 7.43

LET h = THE HEIGHT (ALTITUDE) OF THE COMPLETE CONE OR PYRAMID.

k = THE HEIGHT OF THE SMALLER CONE OR PYRAMID.

A = THE BASE AREA OF THE BIGGER CONE OR PYRAMID (LOWER BASE OF THE FRUSTUM).

A' = THE BASE AREA OF THE COMPLETING CONE OR PYRAMID (UPPER BASE OF THE FRUSTUM).

$h' = h - k$ = THE HEIGHT OF THE FRUSTUM OF THE CONE OR PYRAMID.

V = THE VOLUME OF THE BIGGER CONE OR PYRAMID.

V' = THE VOLUME OF THE SMALLER CONE OR PYRAMID (UPPER PART).

V_f = THE VOLUME OF THE FRUSTUM

$V = \frac{1}{3}Ah$ AND $V' = \frac{1}{3}A'k$, CONSEQUENTLY THE VOLUME OF THE FRUSTUM OF THE PYRAMID IS

$$V_f = V - V' = \frac{1}{3}Ah - \frac{1}{3}A'k = \frac{1}{3}(Ah - A'k)$$

USING THIS NOTION, WE SHALL GIVE THE FORMULA FOR FINDING THE VOLUME OF A CONE OR PYRAMID AS FOLLOWS:

$$V_f = \frac{h'}{3} (A + A' + \sqrt{AA'})$$

WHERE A IS THE LOWER BASE AREA, A' IS THE UPPER BASE AREA AND h' IS THE HEIGHT OF A FRUSTUM OF A CONE OR PYRAMID.

FROM THIS, WE CAN GIVE THE FORMULA FOR FINDING THE VOLUME OF A FRUSTUM IN TERMS OF r AND r' AS FOLLOWS:

$$V_f = \frac{h'}{3} (r^2 + (r')^2 + rr')$$

WHERE r IS THE RADIUS OF THE BIGGER (THE LOWER BASE OF THE FRUSTUM) CONE AND r' IS THE RADIUS OF THE SMALLER CONE (UPPER BASE OF THE FRUSTUM).

EXAMPLE 7 A FRUSTUM OF A REGULAR SQUARE PYRAMID HAS A SIDE LENGTH OF 6 CM, A SLANT HEIGHT OF 5 CM. THE UPPER BASE IS OF SIDE 2 CM AND THE LOWER BASE IS OF SIDE 6 CM. FIND THE VOLUME OF THE FRUSTUM.

SOLUTION:

SINCE THE UPPER BASE AND LOWER BASE ARE SQUARES,

$$A = (6 \text{ CM})^2 = 36 \text{ CM}^2$$

$$A' = (2 \text{ CM})^2 = 4 \text{ CM}^2$$

$$V_f = \frac{h'}{3} (A + A' + \sqrt{AA'}) = \frac{5}{3} (36 + 4 + \sqrt{36 \times 4}) \text{ CM}^3$$

$$= \frac{5}{3} (40 + 12) \text{ CM}^3 = \frac{5}{3} \times 52 \text{ CM}^3 = \frac{260}{3} \text{ CM}^3$$

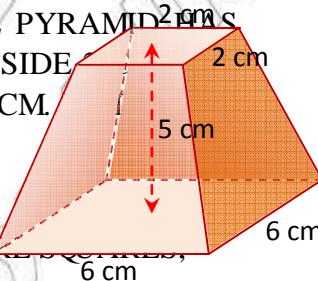


Figure 7.44

Exercise 7.3

1 THE LOWER BASE OF A FRUSTUM OF A REGULAR PYRAMID IS A SQUARE OF SIDE LENGTH 4 CM, AND THE UPPER BASE HAS SIDE LENGTH 3 CM. IF THE SLANT HEIGHT IS 8 CM, FIND:

- A** ITS LATERAL SURFACE AREA **B** ITS TOTAL SURFACE AREA.

2 A CIRCULAR CONE WITH A BASE RADIUS OF 3 CM IS CUT AT A HEIGHT OF $\frac{2}{3}$ OF THE WAY FROM THE BASE TO FORM A FRUSTUM OF A CONE. FIND THE VOLUME OF THE FRUSTUM.

3 THE AREAS OF BASES OF A FRUSTUM OF A PYRAMID ARE 4 CM^2 AND 16 CM^2 . IF ITS ALTITUDE IS 3 CM, FIND ITS VOLUME.

- 4** THE SLANT HEIGHT OF A FRUSTUM OF A CONE IS 10 CM. IF THE RADII OF THE BASES ARE 3 CM, FIND
A THE LATERAL SURFACE AREA **B** THE TOTAL SURFACE AREA
C THE VOLUME OF THE FRUSTUM.
- 5** A FRUSTUM OF A REGULAR SQUARE PYRAMID WHOSE LATERAL FACES ARE EQUILATERALS OF SIDE 10 CM HAS ALTITUDE 5 CM. CALCULATE THE VOLUME OF THE FRUSTUM.
- 6** THE ALTITUDE OF A PYRAMID IS 10 CM. THE BASE IS A SQUARE WHOSE SIDES ARE EACH 6 CM LONG. IF A PLANE PARALLEL TO THE BASE CUTS THE PYRAMID AT A DISTANCE OF 4 CM FROM THE VERTEX, THEN FIND THE VOLUME OF THE FRUSTUM FORMED.
- 7** THE BUCKET SHOWING IN FIGURE 7.41 IS IN THE FORM OF A FRUSTUM OF RIGHT CIRCULAR CONE. THE RADII OF THE BASES ARE 12 CM AND 20 CM, AND THE VOLUME IS 6000 CM³.
A HEIGHT **B** SLANT HEIGHT

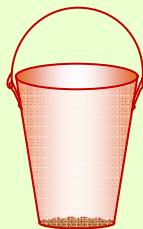


Figure 7.45

- 8** A FRUSTUM OF HEIGHT 12 CM IS FORMED FROM A RIGHT CIRCULAR CONE OF HEIGHT 18 CM AND BASE RADIUS 8 CM. CALCULATE:
A THE LATERAL SURFACE AREA OF THE FRUSTUM
B THE TOTAL SURFACE AREA OF THE FRUSTUM
C THE VOLUME OF THE FRUSTUM.
- 9** A FRUSTUM IS FORMED FROM A REGULAR PYRAMID. IF THE PERIMETER OF THE LOWER BASE IS 12 CM, THE PERIMETER OF THE UPPER BASE IS 8 CM, THE SLANT HEIGHT IS 10 CM, AND THE ALTITUDE IS 6 CM, THEN THE LATERAL SURFACE AREA OF THE FRUSTUM IS
- $$A_L = \frac{1}{2} \ell(P + P').$$
- 10** A FRUSTUM OF HEIGHT 5 CM IS FORMED FROM A RIGHT CIRCULAR CONE OF BASE RADIUS 4 CM. CALCULATE:
A THE LATERAL SURFACE AREA **B** THE VOLUME OF THE FRUSTUM.
- 11** A FRUSTUM OF A REGULAR SQUARE PYRAMID HAS HEIGHT 2 CM. THE LATERAL FACES OF THE PYRAMID ARE EQUILATERAL TRIANGLES. IF THE VOLUME OF THE FRUSTUM IS 128 CM³, THEN THE ALTITUDE OF THE PYRAMID IS

- 12** A CONTAINER IS IN THE SHAPE OF AN INVERTED FRUSTUM AS SHOWN IN FIGURE 7.46. IT HAS A CIRCULAR BOTTOM OF RADIUS 20 CM, A CIRCULAR TOP RADIUS 60 CM AND HEIGHT 40 CM. HOW MANY LITRES OF OIL COULD IT CONTAIN?

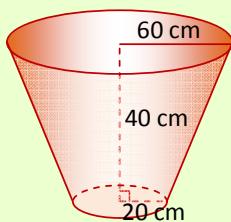


Figure 7.46

7.4**SURFACE AREAS AND VOLUMES OF COMPOSED SOLIDS**

IN THE PRECEDING SECTIONS, YOU HAVE LEARNED HOW TO FIND THE SURFACE AREA AND VOLUME OF CYLINDERS, PRISMS, CONES, PYRAMIDS, SPHERES AND FRUSTUMS. IN THIS SECTION, YOU WILL STUDY HOW TO FIND THE AREAS AND VOLUMES OF SOLIDS FORMED BY COMBINING TWO OR MORE SOLID FIGURES.

ACTIVITY 7.6

- 1** GIVE THE FORMULA USED FOR:

A FINDING THE LATERAL SURFACE AREA OF A

I CYLINDER II PRISM III CONE IV PYRAMID

V SPHERE VI FRUSTUM OF A PYRAMID VII FRUSTUM OF A CONE



B FINDING THE VOLUME OF A

I CYLINDER II PRISM III CONE IV PYRAMID

V SPHERE VI FRUSTUM OF A PYRAMID VII FRUSTUM OF A CONE

- 2** IF THE DIAMETER OF A SPHERE IS HALVED, WHAT EFFECT DOES THIS HAVE ON ITS SURFACE AREA?
3 WHAT IS THE RATIO OF THE VOLUME OF A SPHERE WHOSE RADIUS IS ONE HAVING EQUAL RADIUS AND HEIGHT $2r$ UNITS?

CONSIDER THE FOLLOWING EXAMPLES.

EXAMPLE 1 A CANDLE IS MADE IN THE FORM OF A CIRCULAR CYLINDER WITH THE RADIUS OF ITS BOTTOM AND A RIGHT CIRCULAR CONE OF ALTITUDE AS SHOWN IN FIGURE 7.47. IF THE OVERALL HEIGHT IS 12 CM, FIND THE TOTAL SURFACE AREA AND VOLUME OF THE CANDLE.

SOLUTION: SLANT HEIGHT OF THE CONE IS $\sqrt{3^2 + 4^2} = 5$ CM

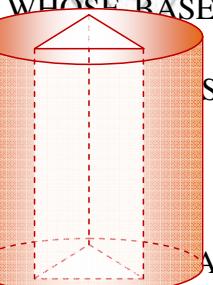
THE TOTAL SURFACE AREA OF THE CANDLE IS THE SUM OF THE LATERAL SURFACE AREAS OF THE CONE, THE CYLINDER AND THE AREA OF THE BASE OF THE CYLINDER. THAT IS,

$$\begin{aligned} A_T &= r\ell + 2rh + r^2 = (4)5 + 2(4)9 + (4)^2 \\ &= 20 + 72 + 16 = 108 \text{ CM}^2 \end{aligned}$$

THE VOLUME OF THE CANDLE IS THE SUM OF THE VOLUMES OF THE CONE AND CYLINDER.

$$\begin{aligned} V_T &= V_{cone} + V_{cylinder} = \frac{1}{3}r^2h_{co} + r^2h_{cy} \\ &= \frac{1}{3}(4)^2 \times 3 + (4)^2 \times 9 = 16 + 144 = 160 \text{ CM}^3 \end{aligned}$$

Figure 7.47



EXAMPLE 2 THROUGH A RIGHT CIRCULAR CYLINDER WHOSE BASE RADIUS IS 10 CM AND WHOSE HEIGHT IS 12 CM, A HOLE IS DRILLED A TRIANGULAR PRISM HOLE WITH ITS THREE EDGES 3 CM, 4 CM AND 5 CM AS SHOWN IN FIGURE 7.48. FIND THE TOTAL SURFACE AREA AND THE VOLUME OF THE REMAINING SOLID.

SOLUTION: THE TOTAL SURFACE AREA IS THE SUM OF THE LATERAL SURFACE AREAS OF THE CYLINDER AND PRISM, AND THE BASE AREA OF THE CYLINDER, MINUS THE BASE AREA OF THE PRISM.

$$\begin{aligned} A_T &= 2rh + ph + 2r^2 - 2\left(\frac{1}{2}ab\right) \\ &= 2(10)12 + (3+4+5)12 + 2(10)^2 - 2\left(\frac{1}{2} \times 3 \times 4\right) \\ &= 240 + 144 + 200 - 12 = (440 + 132) \text{ CM}^2 \end{aligned}$$

THE VOLUME OF THE RESULTING SOLID IS THE DIFFERENCE BETWEEN THE VOLUME OF THE CYLINDER AND PRISM.

$$\begin{aligned} V_T &= V_{cy} - V_p = r^2h - \frac{1}{2}abh = (10)^2 \times 12 - \frac{1}{2} \times 3 \times 4 \times 12 \\ &= 1200 \text{ CM}^3 - 72 \text{ CM}^3 = 24(50 - 3) \text{ CM}^3. \end{aligned}$$

EXAMPLE 3 A CONE IS CONTAINED IN A CYLINDER. THEIR BASE RADIUS AND HEIGHT ARE THE SAME AS THOSE SHOWN IN FIGURE 7.49. CALCULATE THE VOLUME OF THE SPACE INSIDE THE CYLINDER BUT OUTSIDE THE CONE.

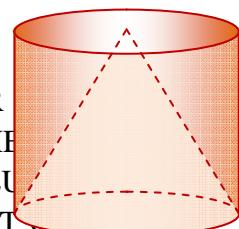


Figure 7.49

SOLUTION: THE REQUIRED VOLUME IS EQUAL TO THE DIFFERENCE BETWEEN THE VOLUME OF THE CYLINDER AND THE CONE. THAT IS,

$$V = V_{cy} - V_{co} = \pi r^2 h - \frac{1}{3} \pi r^2 h = \frac{2}{3} \pi r^2 h.$$

$$\text{AS } r=h, \text{ THEN } = \frac{2}{3} r^3.$$

Group Work 7.2



- 1 A CYLINDRICAL TIN 8 CM IN DIAMETER CONTAINS WATER TO A DEPTH OF 4 CM. IF A CYLINDRICAL WOODEN ROD 4 CM IN DIAMETER AND 6 CM LONG IS PLACED IN THE TIN IT FLOATS EXACTLY HALF SUBMERGED. WHAT IS THE NEW DEPTH OF WATER?
- 2 AN OPEN PENCIL CASE COMPRISSES A CYLINDER OF LENGTH 20 CM AND RADIUS 2 CM AND A CONE OF HEIGHT 4 CM, AS SHOWN IN FIGURE 7.50. CALCULATE THE TOTAL SURFACE AREA AND THE VOLUME OF THE PENCIL CASE.



Figure 7.50

- 3 A BALL IS PLACED INSIDE A BOX INTO WHICH IT FITS EXACTLY. IF THE RADIUS OF THE BALL IS 8 CM, CALCULATE:
 - I THE VOLUME OF THE BALL
 - II THE VOLUME OF THE BOX
- 4 AN ICE-CREAM CONSISTS OF A HEMISPHERE AND A CONE. CALCULATE ITS VOLUME AND TOTAL SURFACE AREA.

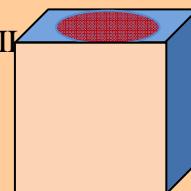


Figure 7.51

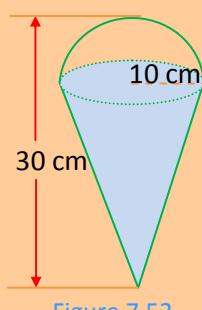


Figure 7.52

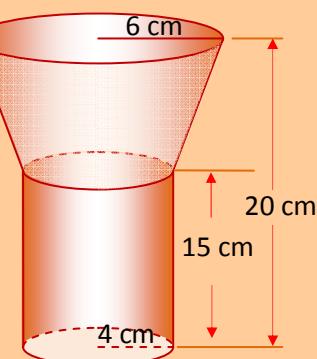


Figure 7.53

- 5 A TORCH 20 CM LONG IS IN THE FORM OF A RIGHT CIRCLE CYLINDER AND RADIUS 4 CM. JOINED TO IT IS A FRUSTUM OF A CONE OF RADIUS 6 CM. FIND THE VOLUME OF THE TORCH.

Exercise 7.4

- 1** FIND THE VOLUME OF EACH OF THE FOLLOWING.

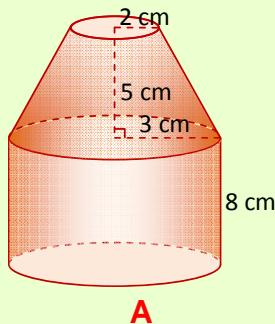
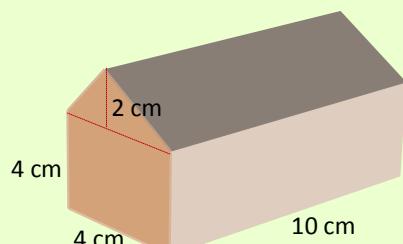
**A****B**

Figure 7.54

- 2** A STORAGE TANK IS IN THE FORM OF CYLINDER WITH ONE HEMISPHERICAL END BEING FLAT. THE DIAMETER OF THE CYLINDER IS 4 M AND THE OVERALL HEIGHT IS 9 M. WHAT IS THE CAPACITY OF THE TANK?
- 3** AN IRON BALL 5 CM IN DIAMETER IS PLACED IN A CYLINDRICAL TIN AND WATER IS POURED INTO THE TIN UNTIL ITS DEPTH IS 6 CM. IF THE BALL IS NOW REMOVED, HOW FAR DOES THE WATER LEVEL DROP?
- 4** FROM A HEMISPHERICAL SOLID OF RADIUS 8 CM, A CONEALASERISWHEM FIGURE 7.55 FIND THE VOLUME AND THE TOTAL SURFACE AREA OF THE RESULTING SOLID.

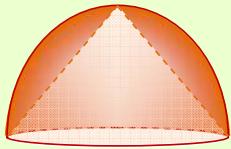


Figure 7.55

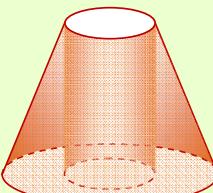


Figure 7.56

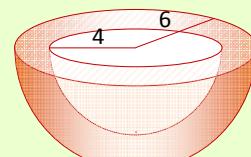


Figure 7.57

- 5** THE ALTITUDE OF A FRUSTUM OF A RIGHT CIRCULAR CONE IS 10 CM AND THE BASE IS 6 CM. A CYLINDRICAL HOLE OF DIAMETER 4 CM IS DRILLED THROUGH THE CENTRE OF THE DRILL FOLLOWING THE AXIS OF THE CONE, LEAVING A SOLID. FIGURE 7.56 FIND THE VOLUME AND THE TOTAL SURFACE AREA OF THE RESULTING SOLID.
- 6** FIGURE 7.57 SHOWS A HEMISPHERICAL SHELL. FIND THE VOLUME AND TOTAL SURFACE AREA OF THE SOLID.
- 7** A CYLINDRICAL PIECE OF WOOD OF RADIUS 8 CM HAS A CONE OF THE SAME RADIUS SCOOPED OUT TO A DEPTH OF 9 CM. FIND THE RATIO OF THE VOLUME SCOOPED OUT TO THE VOLUME OF WOOD WHICH FIGURE 7.58

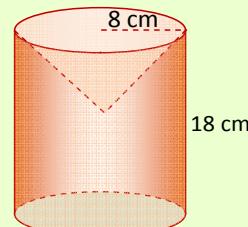


Figure 7.58



Key Terms

cone

lateral edge

regular pyramid

cross-section

lateral surface

slant height

cylinder

prism

sphere

frustum

pyramid

volume



Summary

Prism

$$A_L = Ph$$

$$A_T = 2A_b + A_L$$

$$V = A_b h$$

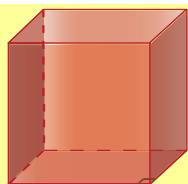


Figure 7.59

Right circular cylinder

$$A_L = 2\pi rh$$

$$A_T = 2\pi r^2 + 2\pi rh = 2\pi r(r+h)$$

$$V = \pi r^2 h$$

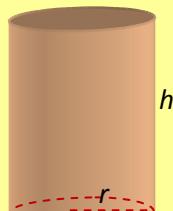


Figure 7.60

Regular pyramid

$$A_L = \frac{1}{2} P\ell$$

$$A_T = A_b + \frac{1}{2} P\ell$$

$$V = \frac{1}{3} A_b h$$

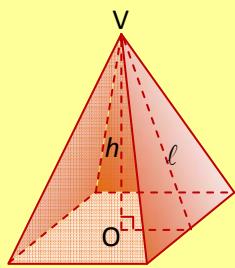


Figure 7.61

Right circular cone

$$A_L = \pi r\ell$$

$$A_T = \pi r^2 + \pi r\ell = \pi r(r+\ell)$$

$$V = \frac{1}{3} \pi r^2 h$$

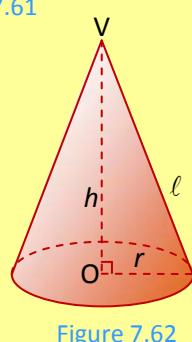


Figure 7.62

Sphere

$$A = 4\pi r^2$$

$$V = \frac{4}{3}\pi r^3$$

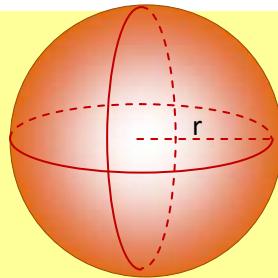


Figure 7.63

Frustum of a pyramid

$$A_L = \frac{1}{2} \ell (P + P')$$

$$A_T = \frac{1}{2} \ell (P + P') + A_b + A'_b$$

$$V = \frac{1}{3} h' (A_b + A'_b + \sqrt{A_b A'_b})$$

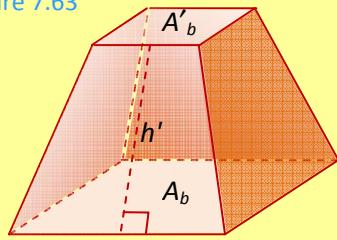


Figure 7.64

Frustum of a cone

$$A_L = \frac{1}{2} \ell (2\pi r + 2\pi r') = \ell (\pi(r + r'))$$

$$A_T = \frac{1}{2} \ell (2\pi r + 2\pi r') + \pi r^2 + \pi (r')^2 = \ell (\pi(r + r')) + (\pi r^2 + \pi (r')^2)$$

$$V = \frac{1}{3} h' (\pi r^2 + \pi (r')^2 + \pi r r')$$

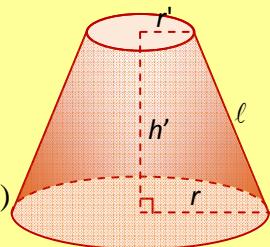


Figure 7.65

**Review Exercises on Unit 7**

- 1** FIND THE LATERAL SURFACE AREA AND VOLUME OF EACH OF THE FOLLOWING FIGURES.

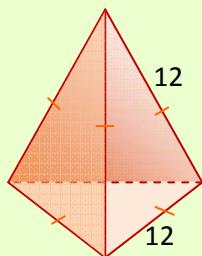
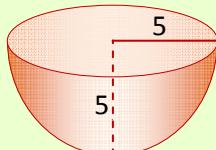
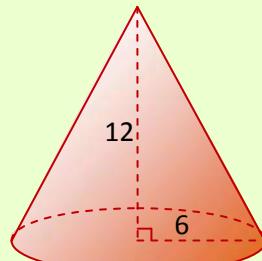
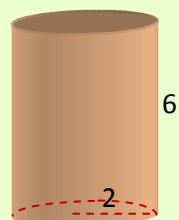
**A****B****C****D**

Figure 7.66

- 2** A LATERAL EDGE OF A RIGHT PRISM IS 6 CM AND ITS PERIMETER IS 12 CM. FIND THE AREA OF ITS LATERAL SURFACE.
- 3** THE HEIGHT OF A CIRCULAR CYLINDER IS EQUAL TO THE RADIUS OF ITS BASE. FIND ITS SURFACE AREA AND ITS VOLUME, GIVING YOUR ANSWER IN TERMS OF ITS RADIUS r .

- 4** WHAT IS THE VOLUME OF A STONE IN AN EGYPTIAN PYRAMID WITH A BASE AREA OF 100 m^2 AND A SLANT HEIGHT OF $5\sqrt{2} \text{ m}$ FOR EACH OF THE TRIANGULAR FACES.
- 5** FIND THE TOTAL SURFACE AREA OF A REGULAR HEXAGONAL PYRAMID IF THE LENGTH OF THE BASE IS 8 cm AND THE ALTITUDE IS 12 cm .
- 6** FIND THE AREA OF THE LATERALSURFACE OF A RIGHT CIRCULAR CONE IF ITS ALTITUDE IS 8 cm AND BASE RADIUS 6 cm .
- 7** FIND THE TOTAL SURFACE AREA OF A RIGHT CIRCULAR CONE IF ITS ALTITUDE IS r . (GIVE THE ANSWER IN TERMS OF π).
- 8** WHEN A STONE IS SUBMERGED IN A RECTANGULAR WATER TANK WHICH IS 25 cm BY 50 cm , THE LEVEL OF THE WATER RISES BY 1 cm . WHAT IS THE VOLUME OF THE STONE?
- 9** A FRUSTUM WHOSE UPPER AND LOWER BASES ARE CIRCULAR REGIONS OF RADII 8 cm AND 6 cm RESPECTIVELY, IS 25 cm DEEP (SEE FIGURE 7.67). FIND ITS VOLUME.

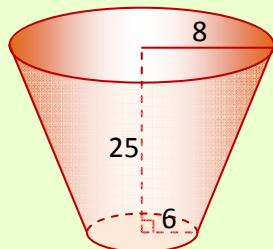


Figure 7.67

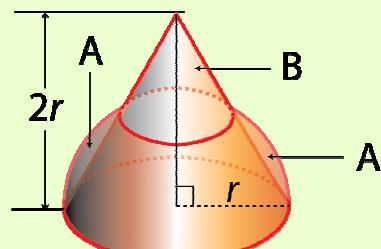


Figure 7.68

- 10** A CYLINDRICAL METAL PIPE OF OUTER DIAMETER 10 cm IS 2 cm THICK. IF THE DIAMETER OF THE HOLE? FIND THE VOLUME OF THE METAL IF THE PIPE IS 30 cm LONG.
- 11** A DRINKING CUP IN THE SHAPE OF FRUSTUM OF A CONE WITH BOTTOM DIAMETER 4 cm , TOP DIAMETER 6 cm , CAN CONTAIN A MAXIMUM OF 80 ml OF COFFEE. FIND THE HEIGHT OF THE CUP.
- 12** THE SLANT HEIGHT OF A CONE IS 16 cm AND THE RADIUS OF ITS BASE IS 6 cm . FIND THE AREA OF THE LATERALSURFACE OF THE CONE AND ITS VOLUME.
- 13** THE RADIUS OF THE BASE OF A CONE IS 12 cm AND ITS VOLUME IS $192\pi \text{ cm}^3$. FIND ITS HEIGHT, SLANT HEIGHT, AND LATERALSURFACE AREA.
- 14** IF THE RADIUS OF A SPHERE IS DOUBLED, WHAT EFFECT DOES THIS HAVE ON ITS VOLUME AND SURFACE AREA?
- 15** IN FIGURE 7.68, A CONE OF BASE RADIUS r AND ALTITUDE $2r$ AND A HEMISPHERE OF RADIUS r WHOSE BASE COINCIDES WITH THAT OF THE CONE ARE SHOWN. PART OF THE HEMISPHERE WHICH LIES OUTSIDE THE CONE AND THE PART OF THE CONE LYING OUTSIDE THE HEMISPHERE. PROVE THAT THE VOLUME IS EQUAL TO THE VOLUME OF B .

Table of Trigonometric Functions

	sin	cos	tan	cot	sec	csc	
0°	0.0000	1.0000	0.0000	1.000	90°
1°	0.0175	0.9998	0.0175	57.29	1.000	57.30	89°
2°	0.0349	0.9994	0.0349	28.64	1.001	28.65	88°
3°	0.0523	0.9986	0.0524	19.08	1.001	19.11	87°
4°	0.0698	0.9976	0.0699	14.30	1.002	14.34	86°
5°	0.0872	0.9962	0.0875	11.43	1.004	11.47	85°
6°	0.1045	0.9945	0.1051	9.514	1.006	9.567	84°
7°	0.1219	0.9925	0.1228	8.144	1.008	8.206	83°
8°	0.1392	0.9903	0.1405	7.115	1.010	7.185	82°
9°	0.1564	0.9877	0.1584	6.314	1.012	6.392	81°
10°	0.1736	0.9848	0.1763	5.671	1.015	5.759	80°
11°	0.1908	0.9816	0.1944	5.145	1.019	5.241	79°
12°	0.2079	0.9781	0.2126	4.705	1.022	4.810	78°
13°	0.2250	0.9744	0.2309	4.331	1.026	4.445	77°
14°	0.2419	0.9703	0.2493	4.011	1.031	4.134	76°
15°	0.2588	0.9659	0.2679	3.732	1.035	3.864	75°
16°	0.2756	0.9613	0.2867	3.487	1.040	3.628	74°
17°	0.2924	0.9563	0.3057	3.271	1.046	3.420	73°
18°	0.3090	0.9511	0.3249	3.078	1.051	3.236	72°
19°	0.3256	0.9455	0.3443	2.904	1.058	3.072	71°
20°	0.3420	0.9397	0.3640	2.747	1.064	2.924	70°
21°	0.3584	0.9336	0.3839	2.605	1.071	2.790	69°
22°	0.3746	0.9272	0.4040	2.475	1.079	2.669	68°
23°	0.3907	0.9205	0.4245	2.356	1.086	2.559	67°
24°	0.4067	0.9135	0.4452	2.246	1.095	2.459	66°
25°	0.4226	0.9063	0.4663	2.145	1.103	2.366	65°
26°	0.4384	0.8988	0.4877	2.050	1.113	2.281	64°
27°	0.4540	0.8910	0.5095	1.963	1.122	2.203	63°
28°	0.4695	0.8829	0.5317	1.881	1.133	2.130	62°
29°	0.4848	0.8746	0.5543	1.804	1.143	2.063	61°
30°	0.5000	0.8660	0.5774	1.732	1.155	2.000	60°
31°	0.5150	0.8572	0.6009	1.664	1.167	1.942	59°
32°	0.5299	0.8480	0.6249	1.600	1.179	1.887	58°
33°	0.5446	0.8387	0.6494	1.540	1.192	1.836	57°
34°	0.5592	0.8290	0.6745	1.483	1.206	1.788	56°
35°	0.5736	0.8192	0.7002	1.428	1.221	1.743	55°
36°	0.5878	0.8090	0.7265	1.376	1.236	1.701	54°
37°	0.6018	0.7986	0.7536	1.327	1.252	1.662	53°
38°	0.6157	0.7880	0.7813	1.280	1.269	1.624	52°
39°	0.6293	0.7771	0.8098	1.235	1.287	1.589	51°
40°	0.6428	0.7660	0.8391	1.192	1.305	1.556	50°
41°	0.6561	0.7547	0.8693	1.150	1.325	1.524	49°
42°	0.6691	0.7431	0.9004	1.111	1.346	1.494	48°
43°	0.6820	0.7314	0.9325	1.072	1.367	1.466	47°
44°	0.6947	0.7193	0.9667	1.036	1.390	1.440	46°
45°	0.7071	0.7071	1.0000	1.000	1.414	1.414	45°
	cos	sin	cot	tan	csc	sec	↔↑

Table of Common Logarithms

N	0	1	2	3	4		5	6	7	8	9
1.0	0.0000	0.0043	0.0086	0.0128	0.0170		0.0212	0.0253	0.0294	0.0334	0.0374
1.1	0.0414	0.0453	0.0492	0.0531	0.0569		0.0607	0.0645	0.0682	0.0719	0.0755
1.2	0.0792	0.0828	0.0864	0.0899	0.0934		0.0969	0.1004	0.1038	0.1072	0.1106
1.3	0.1139	0.1173	0.1206	0.1239	0.1271		0.1303	0.1335	0.1367	0.1399	0.1430
1.4	0.1461	0.1492	0.1523	0.1553	0.1584		0.1614	0.1644	0.1673	0.1703	0.1732
1.5	0.1761	0.1790	0.1818	0.1847	0.1875		0.1903	0.1931	0.1959	0.1987	0.2014
1.6	0.2041	0.2068	0.2095	0.2122	0.2148		0.2175	0.2201	0.2227	0.2253	0.2279
1.7	0.2304	0.2330	0.2355	0.2380	0.2405		0.2430	0.2455	0.2480	0.2504	0.2529
1.8	0.2553	0.2577	0.2601	0.2625	0.2648		0.2672	0.2695	0.2718	0.2742	0.2765
1.9	0.2788	0.2810	0.2833	0.2856	0.2878		0.2900	0.2923	0.2945	0.2967	0.2989
2.0	0.3010	0.3032	0.3054	0.3075	0.3096		0.3118	0.3139	0.3160	0.3181	0.3201
2.1	0.3222	0.3243	0.3263	0.3284	0.3304		0.3324	0.3345	0.3365	0.3385	0.3404
2.2	0.3424	0.3444	0.3464	0.3483	0.3502		0.3522	0.3541	0.3560	0.3579	0.3598
2.3	0.3617	0.3636	0.3655	0.3674	0.3692		0.3711	0.3729	0.3747	0.3766	0.3784
2.4	0.3802	0.3820	0.3838	0.3856	0.3874		0.3892	0.3909	0.3927	0.3945	0.3962
2.5	0.3979	0.3997	0.4014	0.4031	0.4048		0.4065	0.4082	0.4099	0.4116	0.4133
2.6	0.4150	0.4166	0.4183	0.4200	0.4216		0.4232	0.4249	0.4265	0.4281	0.4298
2.7	0.4314	0.4330	0.4346	0.4362	0.4378		0.4393	0.4409	0.4425	0.4440	0.4456
2.8	0.4472	0.4487	0.4502	0.4518	0.4533		0.4548	0.4564	0.4579	0.4594	0.4609
2.9	0.4624	0.4639	0.4654	0.4669	0.4683		0.4698	0.4713	0.4728	0.4742	0.4757
3.0	0.4771	0.4786	0.4800	0.4814	0.4829		0.4843	0.4857	0.4871	0.4886	0.4900
3.1	0.4914	0.4928	0.4942	0.4955	0.4969		0.4983	0.4997	0.5011	0.5024	0.5038
3.2	0.5051	0.5065	0.5079	0.5092	0.5105		0.5119	0.5132	0.5145	0.5159	0.5172
3.3	0.5185	0.5198	0.5211	0.5224	0.5237		0.5250	0.5263	0.5276	0.5289	0.5302
3.4	0.5315	0.5328	0.5340	0.5353	0.5366		0.5378	0.5391	0.5403	0.5416	0.5428
3.5	0.5441	0.5453	0.5465	0.5478	0.5490		0.5502	0.5514	0.5527	0.5539	0.5551
3.6	0.5563	0.5575	0.5587	0.5599	0.5611		0.5623	0.5635	0.5647	0.5658	0.5670
3.7	0.5682	0.5694	0.5705	0.5717	0.5729		0.5740	0.5752	0.5763	0.5775	0.5786
3.8	0.5798	0.5809	0.5821	0.5832	0.5843		0.5855	0.5866	0.5877	0.5888	0.5899
3.9	0.5911	0.5922	0.5933	0.5944	0.5955		0.5966	0.5977	0.5988	0.5999	0.6010
4.0	0.6021	0.6031	0.6042	0.6053	0.6064		0.6075	0.6085	0.6096	0.6107	0.6117
4.1	0.6128	0.6138	0.6149	0.6160	0.6170		0.6180	0.6191	0.6201	0.6212	0.6222
4.2	0.6232	0.6243	0.6253	0.6263	0.6274		0.6284	0.6294	0.6304	0.6314	0.6325
4.3	0.6335	0.6345	0.6355	0.6365	0.6375		0.6385	0.6395	0.6405	0.6415	0.6425
4.4	0.6435	0.6444	0.6454	0.6464	0.6474		0.6484	0.6493	0.6503	0.6513	0.6522
4.5	0.6532	0.6542	0.6551	0.6561	0.6571		0.6580	0.6590	0.6599	0.6609	0.6618
4.6	0.6628	0.6637	0.6646	0.6656	0.6665		0.6675	0.6684	0.6693	0.6702	0.6712
4.7	0.6721	0.6730	0.6739	0.6749	0.6758		0.6767	0.6776	0.6785	0.6794	0.6803
4.8	0.6812	0.6821	0.6830	0.6839	0.6848		0.6857	0.6866	0.6875	0.6884	0.6893
4.9	0.6902	0.6911	0.6920	0.6928	0.6937		0.6946	0.6955	0.6964	0.6972	0.6981
5.0	0.6990	0.6998	0.7007	0.7016	0.7024		0.7033	0.7042	0.7050	0.7059	0.7067
5.1	0.7076	0.7084	0.7093	0.7101	0.7110		0.7118	0.7126	0.7135	0.7143	0.7152
5.2	0.7160	0.7168	0.7177	0.7185	0.7193		0.7202	0.7210	0.7218	0.7226	0.7235
5.3	0.7243	0.7251	0.7259	0.7267	0.7275		0.7284	0.7292	0.7300	0.7308	0.7316
5.4	0.7324	0.7332	0.7340	0.7348	0.7356		0.7364	0.7372	0.7380	0.7388	0.7396

TABLE OF COMMON LOGARITHMS

5.5	0.7404	0.7412	0.7419	0.7427	0.7435		0.7443	0.7451	0.7459	0.7466	0.7474
5.6	0.7482	0.7490	0.7497	0.7505	0.7513		0.7520	0.7528	0.7536	0.7543	0.7551
5.7	0.7559	0.7566	0.7574	0.7582	0.7589		0.7597	0.7604	0.7612	0.7619	0.7627
5.8	0.7634	0.7642	0.7649	0.7657	0.7664		0.7672	0.7679	0.7686	0.7694	0.7701
5.9	0.7709	0.7716	0.7723	0.7731	0.7738		0.7745	0.7752	0.7760	0.7767	0.7774
6.0	0.7782	0.7789	0.7796	0.7803	0.7810		0.7818	0.7825	0.7832	0.7839	0.7846
6.1	0.7853	0.7860	0.7868	0.7875	0.7882		0.7889	0.7896	0.7903	0.7910	0.7917
6.2	0.7924	0.7931	0.7938	0.7945	0.7952		0.7959	0.7966	0.7973	0.7980	0.7987
6.3	0.7993	0.8000	0.8007	0.8014	0.8021		0.8028	0.8035	0.8041	0.8048	0.8055
6.4	0.8062	0.8069	0.8075	0.8082	0.8089		0.8096	0.8102	0.8109	0.8116	0.8122
6.5	0.8129	0.8136	0.8142	0.8149	0.8156		0.8162	0.8169	0.8176	0.8182	0.8189
6.6	0.8195	0.8202	0.8209	0.8215	0.8222		0.8228	0.8235	0.8241	0.8248	0.8254
6.7	0.8261	0.8267	0.8274	0.8280	0.8287		0.8293	0.8299	0.8306	0.8312	0.8319
6.8	0.8325	0.8331	0.8338	0.8344	0.8351		0.8357	0.8363	0.8370	0.8376	0.8382
6.9	0.8388	0.8395	0.8401	0.8407	0.8414		0.8420	0.8426	0.8432	0.8439	0.8445
7.0	0.8451	0.8457	0.8463	0.8470	0.8476		0.8482	0.8488	0.8494	0.8500	0.8506
7.1	0.8513	0.8519	0.8525	0.8531	0.8537		0.8543	0.8549	0.8555	0.8561	0.8567
7.2	0.8573	0.8579	0.8585	0.8591	0.8597		0.8603	0.8609	0.8615	0.8621	0.8627
7.3	0.8633	0.8639	0.8645	0.8651	0.8657		0.8663	0.8669	0.8675	0.8681	0.8686
7.4	0.8692	0.8698	0.8704	0.8710	0.8716		0.8722	0.8727	0.8733	0.8739	0.8745
7.5	0.8751	0.8756	0.8762	0.8768	0.8774		0.8779	0.8785	0.8791	0.8797	0.8802
7.6	0.8808	0.8814	0.8820	0.8825	0.8831		0.8837	0.8842	0.8848	0.8854	0.8859
7.7	0.8865	0.8871	0.8876	0.8882	0.8887		0.8893	0.8899	0.8904	0.8910	0.8915
7.8	0.8921	0.8927	0.8932	0.8938	0.8943		0.8949	0.8954	0.8960	0.8965	0.8971
7.9	0.8976	0.8982	0.8987	0.8993	0.8998		0.9004	0.9009	0.9015	0.9020	0.9025
8.0	0.9031	0.9036	0.9042	0.9047	0.9053		0.9058	0.9063	0.9069	0.9074	0.9079
8.1	0.9085	0.9090	0.9096	0.9101	0.9106		0.9112	0.9117	0.9122	0.9128	0.9133
8.2	0.9138	0.9143	0.9149	0.9154	0.9159		0.9165	0.9170	0.9175	0.9180	0.9186
8.3	0.9191	0.9196	0.9201	0.9206	0.9212		0.9217	0.9222	0.9227	0.9232	0.9238
8.4	0.9243	0.9248	0.9253	0.9258	0.9263		0.9269	0.9274	0.9279	0.9284	0.9289
8.5	0.9294	0.9299	0.9304	0.9309	0.9315		0.9320	0.9325	0.9330	0.9335	0.9340
8.6	0.9345	0.9350	0.9355	0.9360	0.9365		0.9370	0.9375	0.9380	0.9385	0.9390
8.7	0.9395	0.9400	0.9405	0.9410	0.9415		0.9420	0.9425	0.9430	0.9435	0.9440
8.8	0.9445	0.9450	0.9455	0.9460	0.9465		0.9469	0.9474	0.9479	0.9484	0.9489
8.9	0.9494	0.9499	0.9504	0.9509	0.9513		0.9518	0.9523	0.9528	0.9533	0.9538
9.0	0.9542	0.9547	0.9552	0.9557	0.9562		0.9566	0.9571	0.9576	0.9581	0.9586
9.1	0.9590	0.9595	0.9600	0.9605	0.9609		0.9614	0.9619	0.9624	0.9628	0.9633
9.2	0.9638	0.9643	0.9647	0.9652	0.9657		0.9661	0.9666	0.9671	0.9675	0.9680
9.3	0.9685	0.9689	0.9694	0.9699	0.9703		0.9708	0.9713	0.9717	0.9722	0.9727
9.4	0.9731	0.9736	0.9741	0.9745	0.9750		0.9754	0.9759	0.9763	0.9768	0.9773
9.5	0.9777	0.9782	0.9786	0.9791	0.9795		0.9800	0.9805	0.9809	0.9814	0.9818
9.6	0.9823	0.9827	0.9832	0.9836	0.9841		0.9845	0.9850	0.9854	0.9859	0.9863
9.7	0.9868	0.9872	0.9877	0.9881	0.9886		0.9890	0.9894	0.9899	0.9903	0.9908
9.8	0.9912	0.9917	0.9921	0.9926	0.9930		0.9934	0.9939	0.9943	0.9948	0.9952
9.9	0.9956	0.9961	0.9965	0.9969	0.9974		0.9978	0.9983	0.9987	0.9991	0.9996

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