Category Theory	**Lars Birkedal**
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Assignment 1
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Exercise 1. *Let T be the following functor on* **Sets***. It maps the set X to its power set $\mathcal{P}(X)$ and it maps the function $f : X \to Y$ to the image function*

$$T(f) : \mathcal{P}(X) \to \mathcal{P}(Y),$$

which is defined as

$$T(f)(A) = \{f(x) \mid x \in A\}$$

- *Show that T is a functor from* **Sets** *to* **Sets**.

Exercise 2. *Define the category $\mathbb{K}$ as follows. Its objects are sets. Morphisms $X \to Y$ in $\mathbb{K}$ are morphisms $X \to T(Y)$ in* **Sets***, i.e.,*

$$\mathbf{Hom}_{\mathbb{K}}(X,Y) = \mathbf{Hom}_{\mathbf{Sets}}(X, T(Y)).$$

Composition is defined as follows: if $f : X \to Y$ and $g : Y \to Z$ are two morphisms in $\mathbb{K}$ then

$$(g \circ f)(x) = \bigcup_{y \in f(x)} g(y) = \{z \mid \exists y \in f(x), z \in g(y)\}.$$

- *Show that $\mathbb{K}$ is a category.*
- *Show that it is isomorphic to the category of sets and relations* **Rel**.
  Hint: Any subset $R \subseteq X \times Y$ can be represented as a function

  $$F(R) : X \to \mathcal{P}(Y)$$

  defined as

  $$F(R)(x) = \{y \mid (x,y) \in R\}.$$

Exercise 3. *Let $\mathbb{C}$ be a category with binary products.*

- *Is the projection $\pi_X : X \times Y \to X$ an epimorphism in general? Is it a monomorphism?*
- *Let $f : Z \to X$, $g : Z \to Y$, and $h : W \to Z$ be three morphisms. Show*

  $$\langle f, g\rangle \circ h = \langle f \circ h, g \circ h\rangle$$

  as morphisms $W \to X \times Y$.

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- *Let* $f : Z \to X$ *and* $g : W \to Y$ *be two morphisms. Show there* exists *a* unique *morphism* $u : Z \times W \to X \times Y$ *such that for all objects* A *and morphisms* $h_Z : A \to Z$ *and* $h_W : A \to W$

  $$u \circ \langle h_Z, h_W \rangle = \langle f \circ h_Z, g \circ h_W \rangle.$$

  as morphisms $A \to X \times Y$. *This unique morphism* u *is typically written as* $f \times g$.

- *Using the notation from the previous item, show that for any morphisms* $f : Z \to X, g : W \to Y$, $h : A \to Z$, *and* $k : B \to W$ *we have*

  $$(f \times g) \circ (h \times k) = (f \circ h) \times (g \circ k)$$

  as morphisms $A \times B \to X \times Y$.
