Category Theory
Spring 2017

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Assignment 5
Hand in date: November 06, 2018

Exercise 1. *Show that any preorder is equivalent (as a category) to a poset.*

Exercise 2. *Show that a cartesian closed category with a* zero object *(recall this is an object which is both initial and terminal) is* equivalent *to the category with exactly one object and one arrow, the identity.*

Exercise 3. *Let $\mathbb{C}$ be a cartesian closed category. Show the following properties.*

- *Let $f : A \times B \to C$ and $g : C \to D$ be morphisms. Recall that using exponential transposes and evaluation we can define the morphism $g^B : C^B \to D^B$. Show*

  $$g^B \circ \widetilde{f} = \widetilde{g \circ f}$$

  as morphisms $A \to D^B$.

- *Show that for any $f : A \times B \to C$ and any $h : A' \to A$ we have*

  $$\widetilde{f} \circ h = \widetilde{f \circ (h \times id)}$$

  as morphism $A' \to C^B$.

Exercise 4. *Let $\mathbb{C}$ be a cartesian closed category. Recall that in such a category the mapping $f \mapsto \widetilde{f}$ is an isomorphism (bijection) of hom-sets*

$$\mathbf{Hom}_{\mathbb{C}}(A \times B, C) \to \mathbf{Hom}_{\mathbb{C}}\left(A, C^B\right).$$

Let us call this isomorphism $\Lambda_{A,B,C}$. Show that it is natural in C and A.

Concretely this means that you must show that the following diagram commutes for any morphism $g : C \to D$

$$\begin{tikzcd}
\mathbf{Hom}_{\mathbb{C}}(A \times B, C) \arrow[r, "\Lambda_{A,B,C}"] \arrow[d, "\mathbf{Hom}_{\mathbb{C}}(A \times B, g)"'] & \mathbf{Hom}_{\mathbb{C}}\left(A, C^B\right) \arrow[d, "\mathbf{Hom}_{\mathbb{C}}(A, g^B)"] \\
\mathbf{Hom}_{\mathbb{C}}(A \times B, D) \arrow[r, "\Lambda_{A,B,D}"] & \mathbf{Hom}_{\mathbb{C}}\left(A, D^B\right)
\end{tikzcd}$$

and that the following diagram commutes for any morphism $h : A' \to A$.

$$\begin{tikzcd}
\mathbf{Hom}_{\mathbb{C}}(A \times B, C) \arrow[r, "\Lambda_{A,B,C}"] \arrow[d, "\mathbf{Hom}_{\mathbb{C}}(h \times id, C)"'] & \mathbf{Hom}_{\mathbb{C}}\left(A, C^B\right) \arrow[d, "\mathbf{Hom}_{\mathbb{C}}(h, C^B)"] \\
\mathbf{Hom}_{\mathbb{C}}(A' \times B, C) \arrow[r, "\Lambda_{A',B,C}"] & \mathbf{Hom}_{\mathbb{C}}\left(A', C^B\right)
\end{tikzcd}$$

Assignment continues on the next page.

Remark 1. *In brief, this shows that the two functors*

$$(A, C) \mapsto \mathbf{Hom}_{\mathbb{C}}(A \times B, C)$$

and

$$(A, C) \mapsto \mathbf{Hom}_{\mathbb{C}}\left(A, C^B\right)$$

are isomorphic as functors $\mathbb{C}^{op} \times \mathbb{C} \to \mathbf{Sets}$. *Later on we shall see that this means precisely that the functor* $A \mapsto A \times B$ *is* left adjoint *to the functor* $C \mapsto C^B$.

Exercise 5. *Do exercise* 3.3 *of the notes on categorical logic.*
