Category Theory — **Spring 2017**

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Assignment 6
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Exercise 1. *Let* $\mathbb{C}$ *and* $\mathbb{D}$ *be two categories and assume* $\mathbb{D}$ has pullbacks. *Let* $F, G : \mathbb{C} \to \mathbb{D}$ *be functors.*

Show that a natural transformation $\alpha : F \to G$ *is a monomorphism* if and only if *each of the components* $\alpha_c : F(c) \to G(c)$ *is a monomorphism in* $\mathbb{D}$.

Exercise 2. *Let* **FinSets** *be the category of finite sets and functions. Let* ω *be the ordered set of natural numbers with the usual order*

$$0 \leq 1 \leq 2 \leq 3 \leq \cdots$$

1. *Show that the category* $\mathbf{FinSets}^{\omega^{op}}$ is *cartesian closed.*
   Hint: follow the general construction of exponents in $\mathbf{Sets}^{\omega^{op}}$ *and show it restricts to* $\mathbf{FinSets}^{\omega^{op}}$.
2. *Show that the category* $\mathbf{FinSets}^{\omega}$ is not *cartesian closed.*
   Hint: Consider the object N *defined as*

   $$[0] \hookrightarrow [1] \hookrightarrow [2] \hookrightarrow [3] \hookrightarrow \cdots$$

   where $[n]$ *is the set* $\{0, 1, \ldots n\}$ *and all arrows are subset inclusions. Then consider the sets* $\mathbf{Hom}(N, 2)$ *and* $\mathbf{Hom}\left(\mathbf{1}, \mathbf{2}^N\right)$ *assuming the exponential object* $\mathbf{2}^N$ *exists. The object* $\mathbf{2}$ *is as usual* $\mathbf{1} + \mathbf{1}$.

You may assume that all finite *limits exist in both* $\mathbf{FinSets}^{\omega^{op}}$ *and* $\mathbf{FinSets}^{\omega}$ *and that they are given pointwise as in* $\mathbf{Sets}^{\omega^{op}}$ *and* $\mathbf{Sets}^{\omega}$.

Remark 1. *The second item of the preceding exercise shows that even if* $\mathbb{D}$ *is cartesian closed and has all finite limits it need not be the case that* $\mathbb{D}^{\mathbb{C}}$ *is cartesian closed.*

Exercise 3. *Let* **Sets** *be the category of sets and functions and* A *a set. Does the functor*

$$\mathbf{Hom}\,(A, -) : \mathbf{Sets} \to \mathbf{Sets}$$

have a left adjoint? Does it have a right adjoint? If they exist describe them, and if not prove it.