Category Theory
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Assignment 4
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Definition 1. *Let $F : \mathbb{C} \to \mathbb{C}$ be a functor.*

- *A* fixed point *of the functor F is an object X such that $F(X) \cong X$.*
- *An* algebra *for the functor F is a pair (X, ϕ) where X is an object of $\mathbb{C}$ and $\phi : FX \to X$ is a morphism. The object X is called the* carrier *of the algebra.*

  The algebra (L, γ) is initial *if for any other algebra (X, ϕ) there exists a* unique morphism *f such that the following diagram commutes*

$$\begin{tikzcd} F(L) \arrow[r, "F(f)"] \arrow[d, "\gamma"] & F(X) \arrow[d, "\phi"] \\ L \arrow[r, "f"] & X \end{tikzcd}$$

- *A* coalgebra *for the functor F is a pair (X, ϕ) where X is an object of $\mathbb{C}$ and $\phi : X \to FX$ is a morphism. The object X is called the* carrier *of the coalgebra.*

  The coalgebra (L, γ) is final *if for any other coalgebra (X, ϕ) there exists a* unique morphism *f such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"] & L \arrow[d, "\gamma"] \\ F(X) \arrow[r, "F(f)"] & F(L) \end{tikzcd}$$

Remark 1. *The dual concepts, of a final* algebra *and initial coalgebra, are not particularly useful.*

Exercise 1. *Show that if (L, γ) and (L', γ') are initial algebras for F then there exists a unique* isomorphism *f such that*

$$\begin{tikzcd} F(L) \arrow[r, "F(f)"] \arrow[d, "\gamma"] & F(L') \arrow[d, "\gamma'"] \\ L \arrow[r, "f"] & L' \end{tikzcd}$$

commutes, i.e., initial algebras are unique up to isomorphism.

Formulate and prove an analogous result for final coalgebras.

Exercise 2. *Show that if (L, γ) is an initial algebra (resp. final coalgebra) for the functor F then γ is an* isomorphism.
Hint: $(F(L), F(\gamma))$ is also an algebra (resp. coalgebra) for F.

Remark 2. *The previous exercise shows that initial algebras and final coalgebras of F are in particular its fixed points.*

Exercise 3. *Some functors have neither an initial algebra nor a final coalgebra.*

- *Show that the power set functor described in the first assignment has no fixed point. Recall that this functor maps the set X to its power set $\mathcal{P}(X)$ and it maps a function $f : X \to Y$ to the image function $\mathcal{P}(X) \to \mathcal{P}(Y)$.*
- *Conclude that it has neither an initial algebra nor a final coalgebra.*

Definition 2. *A* colimit of type ω *is a colimit of a diagram of type $(\mathbb{N}, \leq)$ where $(\mathbb{N}, \leq)$ is the poset of natural numbers with less than or equal relation considered as a category.*

A limit of type ω^{op} *is a limit of a diagram of type $(\mathbb{N}, \geq)$ where $(\mathbb{N}, \geq)$ is the poset of natural numbers with greater than or equal relation considered as a category.*

Exercise 4. *Suppose $\mathbb{C}$ has an initial object 0 and colimits of type ω. Suppose $F : \mathbb{C} \to \mathbb{C}$ preserves colimits of type ω. Define the sequence of objects F_n and arrows $i_n : F_n \to F_{n+1}$ as follows.*

$$F_0 = 0 \qquad\qquad i_0 = !_{F(0)}$$
$$F_{n+1} = F(F_n) \qquad\qquad i_{n+1} = F(i_n)$$

These thus define the following diagram

$$F_0 \xrightarrow{i_0} F_1 \xrightarrow{i_1} F_2 \xrightarrow{i_2} F_3 \xrightarrow{i_3} F_4 \xrightarrow{i_4} \cdots$$

Show that the colimit *L of this diagram is the carrier of the initial algebra for F. This means that you need to define a map $\gamma : F(L) \to L$ and show it satisfies the universal property described in Definition 1.*

Exercise 5. *Suppose $\mathbb{C}$ has a terminal object 1 and limits of type ω^{op}. Suppose $F : \mathbb{C} \to \mathbb{C}$ preserves limits of type ω^{op}. Define the sequence of objects F_n and arrows $p_n : F_{n+1} \to F_n$ as follows.*

$$F_0 = 1 \qquad\qquad p_0 = !_{F(0)}$$
$$F_{n+1} = F(F_n) \qquad\qquad p_{n+1} = F(p_n)$$

These thus define the following diagram

$$F_0 \xleftarrow{p_0} F_1 \xleftarrow{p_1} F_2 \xleftarrow{p_2} F_3 \xleftarrow{p_3} F_4 \xleftarrow{p_4} \cdots$$

Show that the limit *L of this diagram is the carrier of the final coalgebra algebra for F.*

Hint: Use duality.

Exercise 6. *Let A be a set and let F : **Sets** $\to$ **Sets** be the functor given on objects as*

$$F(X) = 1 + A \times X$$

and on arrows as

$$F(f) = \mathsf{id}_1 + \mathsf{id}_A \times f.$$

- *Show that the carrier of the initial algebra of F is the set of finite sequences of elements of the set A. Call this set L_A. This means that you must define a function $F(L_A) \to L_A$ and show it has the universal property stated in Definition 1.*

- *Show that the carrier of the final coalgebra of F is the set of all sequences (finite and infinite) of elements of the set A.*

- *Let A be the set of natural numbers. Use the initial algebra property of $L_{\mathbb{N}}$ to define the function $\mathsf{sum} : L_{\mathbb{N}} \to \mathbb{N}$ which maps a sequence to the sum of its elements.*

Remark 3. *For the functor F in the previous exercise, given any other algebra (X, ϕ) the unique map f making*

$$\begin{tikzcd} F(L) \arrow[r, "F(f)"] \arrow[d, "\gamma"] & F(X) \arrow[d, "\phi"] \\ L \arrow[r, "f"] & X \end{tikzcd}$$

commute is what is usually called $\mathrm{fold}\,\phi$ *in functional programming. It is the basic structural recursion operation associated with L.*
