MOMC IOQM Mock Apple 3

Instructions:

- All answers are in the integer range of $00 - 99$. Although there is a non-zero chance of an intentional bonus.
- Problems $1 - 10$ are 2 Markers, $11 - 20$ are 3 Markers and $21 - 30$ are 5 Markers.
- Total time is 3 Hours.
- The test begins from the next page. So only proceed to the next page if you are starting the test!
- Answer keys and sources are on the last two pages.
- Mock Compiled by **Agamjeet Singh**
- The problems are credited to their respective sources.
- Good luck!

1. A teacher gives a multiple choice test to 15 students and that each student answered each question. Each question had 5 choices, but remarkably, no pair of students had more than 2 answers in common. What is the maximum number of questions that could have been on the quiz?

2. Suppose in your sock drawer of 14 socks there are 5 different colors and 3 different lengths present. One day, you decide you want to wear two socks that have both different colors and different lengths. Given only this information, what is the maximum number of choices you might have?

3. A farmer learns that he will die at the end of the year (day 365, where today is day 0) and that he has a number of sheep. He decides that his utility is given by ab where a is the money he makes by selling his sheep (which always have a fixed price) and b is the number of days he has left to enjoy the profit; i.e., $365 - k$ where k is the day. If every day his sheep breed and multiply their numbers by $103/101$ (yes, there are small, fractional sheep), then find the sum of digits of the day on which he should sell them all.

4. Line segements $\overline{AB}$ and $\overline{AC}$ are tangent to a convex arc $\widehat{BC}$ and $\angle BAC = \frac{\pi}{3}$. If $\overline{AB} = \overline{AC} = 3\sqrt{3}$, find the integer nearest to 10 times the length of minor arc $\widehat{BC}$.

5. A secant line incident to a circle at points A and C intersects the circle's diameter at point B with a 45° angle. If the length of AB is 1 and the length of BC is 7, then what is the circle's radius?

SPACE FOR ROUGH WORK

6. Find the number of positive integer solutions to the equation $\lfloor \sqrt[3]{x} \rfloor + \lfloor \sqrt[4]{x} \rfloor = 4$.

7. Let $\frac{a}{b}$ be a fraction such that a and b are positive integers and the first three digits of its decimal expansion are 0.527. What is the smallest possible value of $a + b$?

8. Let $\overline{AD}$ be a diameter of a circle. Let point B be on the circle, point C be on $\overline{AD}$ such that A, B, C form a right triangle with right angle at C. The value of the hypotenuse of the triangle is 4 times the square root of its area. If $\overline{BC}$ has length 20.23 , what is the integer nearest to the length of the radius of the circle?

9. You have four fair 6-sided dice, each numbered from 1 to 6 (inclusive). If all four dice are rolled, the probability that the product of the rolled numbers is prime can be written as $\frac{a}{b}$, where a, b are relatively prime. What is $a + b$?

10. Suppose that you start with the number 8 and always have two legal moves:

- Square the number
- Add one if the number is divisible by 8 or multiply by 4 otherwise

How many sequences of 4 moves are there that return to a multiple of 8 ?

SPACE FOR ROUGH WORK

11. Define a function given the following 2 rules:

- for prime p, $f(p) = p + 1$
- for positive integers a and b, $f(ab) = f(a) \cdot f(b)$.

For how many positive integers $n \leq 100$ is $f(n)$ divisible by 3 ?

12. Let $0 \leq a, b, c, d \leq 10$. Suppose there are N ordered quadruples (a, b, c, d) such that $ad - bc$ is a multiple of 11. Find the sum of square of digits of N.

13. For how many positive integers n less than 2018 does n^2 have the same remainder when divided by $7, 11$, and 13?

SPACE FOR ROUGH WORK

14. Let ABC be a triangle with side lengths $13, 14, 15$. The points on the interior of ABC with distance at least 1 from each side are shaded. Find the integer nearest to the area of the shaded region.

15. Some number of regular polygons meet at a point on the plane, so that the polygons' interiors do not overlap, but the polygons fully surround the point (i.e. a sufficiently small circle centered at the point would be contained in the union of the polygons). What is the largest possible number of sides in any of the polygons?

16. Let triangle $\triangle ABC$ have $AB = 90$ and $AC = 66$. Suppose that the line IG is perpendicular to side BC, where I and G are the incenter and centroid, respectively. Find the length of BC.

SPACE FOR ROUGH WORK

17. If a and b are positive integers such that $3\sqrt{2+\sqrt{2+\sqrt{3}}} = a\cos\frac{\pi}{b}$, find $a+b$.

18. Let $b_1 = 1$ and $b_{n+1} = 1 + \frac{1}{n(n+1)b_1 b_2 \ldots b_n}$ for $n \geq 1$. Let $b_i = \frac{m_i}{n_i}$ where m_i, n_i are relatively prime positive integers. Find the smallest possible value of k such that $m_k + n_k > 1000$.

19. There exist real numbers a, b, c, d, and e such that for all positive integers n, we have

$$\sqrt{n} = \sum_{i=0}^{n-1} \sqrt[5]{\sqrt{ai^5 + bi^4 + ci^3 + di^2 + ei + 1} - \sqrt{ai^5 + bi^4 + ci^3 + di^2 + ei}}$$

Find the remainder when $a+b+c+d$ is divided by 100.

20. Let $x_0, x_1, \ldots$ be a sequence of real numbers such that $x_n = \frac{1+x_{n-1}}{x_{n-2}}$ for $n \geq 2$. Find the sum of square of digits of the number of ordered pairs of positive integers (x_0, x_1) such that the sequence gives $x_{2018} = \frac{1}{1000}$.

SPACE FOR ROUGH WORK

21. Let Δ denote the pyramid with edges of lengths $2, 3, 3, 4, 5, 5$ respectively such that it has maximal volume. If the volume of Δ can be written as $\frac{m\sqrt{n}}{p}$ where m, n, p are positive integers such that n is square-free and m, p are relatively prime, then find $m + n + p$.

22. Suppose the transformation T acts on points in the plane like this:

$$T(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right)$$

Let A be the area enclosed by the set of points of the form $T(x, y)$, where (x, y) is a point on the edge of a length-2 square centered at the origin with sides parallel to the axes. Find the integer **nearest** to $10A$.

23. Kite $ABCD$ has right angles at B and D, and $AB < BC$. Points $E \in AB$ and $F \in AD$ satisfy $AE = 4, EF = 7, FA = 5$. If $AB = 8$ and point X lies outside $ABCD$ while satisfying $XE - XF = 1$ and $XE + XF + 2XA = 23$, then XA may be written as $\frac{a - b\sqrt{c}}{d}$ for a, b, c, d. Find the remainder when $a + b + c + d$ is divided by 100.

SPACE FOR ROUGH WORK

24. Let w and h be positive integers and define $N(w,h)$ to be the number of ways of arranging wh people of distinct heights for a photoshoot in such a way that they form w columns of h people, with the people in each column sorted by height (i.e. shortest at the front, tallest at the back). Find the sum of square of digits of the largest value of $N(w,h)$ that divides 1008.

25. For a positive integer n, let $f(n)$ be the number of (not necessarily distinct) primes in the prime factorization of n. For example, $f(1) = 0, f(2) = 1$, and $f(4) = f(6) = 2$. Let $g(n)$ be the number of positive integers $k \leq n$ such that $f(k) \geq f(j)$ for all $j \leq n$. Find the sum of all distinct prime factors of $g(1) + g(2) + \ldots + g(100)$.

26. A robot is shuffling a 9 card deck. Being very well machined, it does every shuffle in exactly the same way: it splits the deck into two piles, one containing the 5 cards from the bottom of the deck and the other with the 4 cards from the top. It then interleaves the cards from the two piles, starting with a card from the bottom of the larger pile at the bottom of the new deck, and then alternating cards from the two piles while maintaining the relative order of each pile. The top card of the new deck will be the top card of the bottom pile.

The robot repeats this shuffling procedure a total of n times, and notices that the cards are in the same order as they were when it started shuffling. What is the smallest possible value of n ?

SPACE FOR ROUGH WORK

27. Suppose real numbers a, b, c, d satisfy $a + b + c + d = 17$ and $ab + bc + cd + da = 46$. If the minimum possible value of $a^2 + b^2 + c^2 + d^2$ can be expressed as a rational number $\frac{p}{q}$ in simplest form, find the remainder when $p + q$ is divided by 100.

28. Let a, b, c be non-zero real numbers that satisfy $\frac{1}{abc} + \frac{1}{a} + \frac{1}{c} = \frac{1}{b}$. The expression

$$\frac{4}{a^2+1} + \frac{4}{b^2+1} + \frac{7}{c^2+1}$$

has a maximum value M. Find the sum of the numerator and denominator of the reduced form of M.

29. There are numerous sets of 17 distinct positive integers that sum to 2018, such that each integer has the same sum of digits in base 10. Let M be the maximum possible integer that could exist in any such set. Find the last two digits of the sum of M and the number of such sets that contain M.

30. Consider a 10-dimensional $10 \times 10 \times \ldots \times 10$ cube consisting of 10^{10} unit cubes, such that one cube A is centered at the origin, and one cube B is centered at $(9, 9, 9, 9, 9, 9, 9, 9, 9, 9)$. Paint A red and remove B, leaving an empty space. Let a move consist of taking a cube adjacent to the empty space and placing it into the empty space, leaving the space originally contained by the cube empty. Let N be the minimum number of moves required to result in a configuration where the cube centered at $(9, 9, 9, 9, 9, 9, 9, 9, 9, 9)$ is red. Find the sum of square of digits of N.

SPACE FOR ROUGH WORK