MOMC IOQM Mock Blueberry 2

Instructions:

- All answers are in the integer range of $00-99$. Although there is a non-zero chance of an intentional bonus.
- Problems $1-10$ are 2 Markers, $11-20$ are 3 Markers and $21-30$ are 5 Markers.
- Total time is 3 Hours.
- The test begins from the next page. So only proceed to the next page if you are starting the test!
- Answer keys and sources are on the last two pages.
- Mock Compiled by **Agamjeet Singh**
- The problems are credited to their respective sources.
- Good luck!

1. Suppose a, b, and c are relatively prime integers such that

$$\frac{a}{b+c} = 2 \quad \text{and} \quad \frac{b}{a+c} = 3.$$

What is the value of $|c|$?

2. Let ABC be a triangle. Point P lies in the interior of ABC such that $\angle ABP = 20^\circ$ and $\angle ACP = 15^\circ$. Compute $\angle BPC - \angle BAC$.

3. A total of 30 students go on a road trip. They take two cars, each of which seats 15 people. Call two students friendly if they sat together in the same car going to the trip and in the same car going back home. What is the smallest possible number of friendly pairs of students on the trip?

4. Compute the number of ways to rearrange nine white cubes and eighteen black cubes into a $3 \times 3 \times 3$ cube such that each $1 \times 1 \times 3$ row or column contains exactly one white cube. Note that rotations are considered distinct.

5. Let ABC be a triangle with side lengths $5, 4\sqrt{2}$, and 7. Let the area of the triangle with side lengths $\sin A, \sin B$, and $\sin C$ be $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m + n$.

6. Michelle is at the bottom-left corner of a 6×6 lattice grid, at $(0, 0)$. The grid also contains a pair of onetime-use teleportation devices located at $(2, 2)$ and $(3, 3)$; the first time Michelle moves to one of these points she is instantly teleported to the other point, and the devices disappear. If she can only move up or to the right in unit increments, find the last two digits of the number of ways in which can she reach the point $(5, 5)$.

7. Fifteen integers $a_1, a_2, a_3, \ldots, a_{15}$ are arranged in order on a number line. The integers are equally spaced and have the property that

$$1 \leq a_1 \leq 10, \ 13 \leq a_2 \leq 20, \ 241 \leq a_{15} \leq 250.$$

What is the sum of digits of a_{14}?

8. Let $ABCD$ be a unit square, and suppose that E and F are on $\overline{AD}$ and $\overline{AB}$ such that $AE = AF = \frac{2}{3}$. Let $\overline{CE}$ and $\overline{DF}$ intersect at G. If the area of $\triangle CFG$ can be expressed as simplified fraction $\frac{p}{q}$, find $p + q$.

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9. Jan rolls a fair six-sided die and calls the result r. Then, he picks real numbers a and b between 0 and 1 uniformly at random and independently. If the probability that the polynomial $f(x) = \frac{x^2}{r} - x\sqrt{a} + b$ has a real root can be expressed as simplified fraction $\frac{p}{q}$, find $|q - 2p|$.

10. How many ordered triples (a, b, c) of integers satisfy the inequality

$$a^2 + b^2 + c^2 \leq a + b + c + 2?$$

11. Let $a > 1$ be a positive integer. The sequence of natural numbers $\{a_n\}$ is defined as follows: $a_1 = a$ and for all $n \geq 1, a_{n+1}$ is the largest prime factor of $a_n^2 - 1$. Determine the smallest possible value of a such that the numbers $a_1, a_2, \ldots, a_7$ are all distinct.

12. Let $ABCD$ be a square of side length 1, and let P be a variable point on $\overline{CD}$. Denote by Q the intersection point of the angle bisector of $\angle APB$ with $\overline{AB}$. The set of possible locations for Q as P varies along $\overline{CD}$ is a line segment. If the length of this segment is $a - \sqrt{b}$ where a, b are positive integers then find $a + b$.

13. Suppose that a and b are non-negative integers satisfying $a + b + ab + a^b = 42$. Find the largest prime divisor of the sum of all possible values of $a + b$.

14. 2018 little ducklings numbered 1 through 2018 are standing in a line, with each holding a slip of paper with a nonnegative number on it; it is given that ducklings 1 and 2018 have the number zero. At some point, ducklings 2 through 2017 change their number to equal the average of the numbers of the ducklings to their left and right. Suppose the new numbers on the ducklings sum to 1000. What is the number of positive integer factors of the maximum possible sum of the original numbers on all 2018 slips?

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15. Find the sum of all integers n for which $(n-1)\cdot 2^n+1$ is a perfect square.

16. Suppose a, b, and c are nonzero real numbers such that

$$bc+\frac{1}{a}=ca+\frac{2}{b}=ab+\frac{7}{c}=\frac{1}{a+b+c}.$$

If $(a+b+c)^3=\frac{m}{n}$ where m, n are relatively prime integers. Find $m+n$.

17. Let ABC be a triangle with $BC=30, AC=50$, and $AB=60$. Circle ω_B is the circle passing through A and B tangent to BC at B; ω_C is defined similarly. Suppose the tangent to $\odot(ABC)$ at A intersects ω_B and ω_C for the second time at X and Y respectively. Compute XY.

18. For some positive integer k, a circle is drawn tangent to the coordinate axes such that the lines $x+y=k^2, x+y=(k+1)^2, \ldots, x+y=(k+61)^2$ all pass through it. What is the minimum possible value of k ?

19. Let CMU be a triangle with $CM=13, MU=14$, and $UC=15$. Rectangle $WEAN$ is inscribed in $\triangle CMU$ with points W and E on $\overline{MU}$, point A on $\overline{CU}$, and point N on $\overline{CM}$. If the area of $WEAN$ is 32, what is the largest integer less than the largest possible value for its perimeter?

20. Let $\triangle ABC$ be a triangle with $AB=3$ and $AC=5$. Select points D, E, and F on $\overline{BC}$ in that order such that $\overline{AD}\perp\overline{BC}, \angle BAE=\angle CAE$, and $\overline{BF}=\overline{CF}$. If E is the midpoint of segment $\overline{DF}$, what is BC^2 ?

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21. It is given that there exists a unique triple of positive primes (p, q, r) such that $p < q < r$ and

$$\frac{p^3 + q^3 + r^3}{p + q + r} = 249.$$

Find r.

22. It is given that there exist unique integers $m_1, \ldots, m_{100}$ such that

$$0 \leq m_1 < m_2 < \cdots < m_{100} \text{ and } 2018 = \binom{m_1}{1} + \binom{m_2}{2} + \cdots + \binom{m_{100}}{100}$$

If $m_1 + m_2 + \cdots + m_{100} = 100a + b$ where $a, b < 100$ are positive integers, find $a + b$.

23. Suppose $\overline{AB}$ is a segment of unit length in the plane. Let $f(X)$ and $g(X)$ be functions of the plane such that f corresponds to rotation about A 60° counterclockwise and g corresponds to rotation about B 90° clockwise. Let P be a point with $g(f(P)) = P$. Let the sum of all possible distances from P to line AB be $\frac{a+\sqrt{b}}{c}$ where a, b, c are positive integers. Find $a + b + c$.

24. Let $p, q,$ and r be the roots of the polynomial $f(t) = t^3 - 2022t^2 + 2022t - 337$. Given

$$x = (q-1)\left(\frac{2022-q}{r-1} + \frac{2022-r}{p-1}\right)$$

$$y = (r-1)\left(\frac{2022-r}{p-1} + \frac{2022-p}{q-1}\right)$$

$$z = (p-1)\left(\frac{2022-p}{q-1} + \frac{2022-q}{r-1}\right)$$

Find the remainder when $|xyz - qrx - rpy - pqz|$ is divided by 100.

25. Let ABC be a triangle with circumradius 17, inradius 4, circumcircle Γ and A-excircle Ω. Suppose the reflection of Ω over line BC is internally tangent to Γ. Compute the last two digits of the area of $\triangle ABC$.

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26. Let ABC be a triangle with $AB = 10, AC = 11$, and circumradius 6. Points D and E are located on the circumcircle of $\triangle ABC$ such that $\triangle ADE$ is equilateral. Line segments $\overline{DE}$ and $\overline{BC}$ intersect at X. Let $\frac{BX}{XC} = \frac{a}{b}$ where a, b are relatively prime positive integers. Find $a + b$.

27. Nine distinct light bulbs are placed in a circle. Each light bulb can be on or off. In order to properly light up the room, in each group of four adjacent light bulbs, at least one must be turned on. Find the sum of square of digits of the total number of configurations that are there.

28. Define an integer $n \geq 0$ to be two-far if there exist integers a and b such that a, b, and $n + a + b$ are all powers of two. If N is the number of two-far integers less than 2048, find the remainder when N is divided by 100.

29. Richard rolls a fair six-sided die repeatedly until he rolls his twentieth prime number or his second even number. Let the probability that his last roll is prime be $\frac{a \cdot 2^b}{5^c}$ where a, b, c are positive integers such that $2, 5 \nmid a$. Find the last two digits of $a + b + c$.

30. We call $\overline{a_n \ldots a_2}$ the Fibonacci representation of a positive integer k if

$$k = \sum_{i=2}^{n} a_i F_i,$$

where $a_i \in \{0, 1\}$ for all $i, a_n = 1$, and F_i denotes the i^{th} Fibonacci number ($F_0 = 0, F_1 = 1$, and $F_i = F_{i-1} + F_{i-2}$ for all $i \geq 2$). This representation is said to be minimal if it has fewer 1 s than any other Fibonacci representation of k. Find the sum of digits of the smallest positive integer that has eight ones in its minimal Fibonacci representation.

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