MOMC IOQM Mock Apple 2

Instructions:

- All answers are in the integer range of $00 - 99$. Although there is a non-zero chance of a intentional bonus.
- Problems $1 - 10$ are 2 Markers, $11 - 20$ are 3 Markers and $21 - 30$ are 5 Markers.
- Total time is 3 Hours.
- The test begins from the next page. So only proceed to the next page if you are starting the test!
- Answer keys and sources are on the last two pages.
- Mock Compiled by **Agamjeet Singh**
- The problems are credited to their respective sources.
- There are a few bonus problems on the last two pages.
- Good luck!

1. Consider the set $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many distinct 3-element subsets are there such that the sum of the elements in each subset is divisible by 3 ?

2. Lisa has a 2D rectangular box that is 6 units long and 21 units wide. She shines a laser beam into the box through one of the corners such that the beam is at a $45°$ angle with respect to the sides of the box. Whenever the laser beam hits a side of the box, it is reflected perfectly, again at a $45°$ angle. Find the integer nearest to the distance that the laser beam travels until it hits one of the four corners of the box.

3. Let A and B be fixed points on a 2-dimensional plane with distance $AB = 1$. An ant walks on a straight line from point A to some point C on the same plane and finds that the distance from itself to B always decreases at any time during this walk. Let $\frac{m}{n}\pi$ be the area of the locus of points where point C could possibly be located, where m, n are relatively prime positive integers. Find $m + n$.

4. In a standard game of Rock-Paper-Scissors, two players repeatedly choose between rock, paper, and scissors, until they choose different options. Rock beats scissors, scissors beats paper, and paper beats rock. Nathan knows that on each turn, Richard randomly chooses paper with probability 33%, scissors with probability 44%, and rock with probability 23%. If Nathan plays optimally against Richard, the probability that Nathan wins is expressible as a/b where a and b are coprime positive integers. Find $a + b$.

5. Let S be the sum of all positive integers n whose digits (in decimal representation) add up to $n/57$. Find the integer nearest to S.

SPACE FOR ROUGH WORK

6. Find the sum of all positive integers k such that there exists a positive integer a such that $7k^2 = a^3 + a! + 2767$

7. How many ways can you divide a heptagon into five non-overlapping triangles such that the vertices of the triangles are vertices of the heptagon?

8. Let S be the sum of all positive integers n with at most three digits that satisfy $n = (a+b)\cdot(b+c)$ when n is written in base 10 as $\underline{abc}$. Find the remainder when S is divided by 100. Note: The integer n can have leading zeroes.

9. Let f be the cubic polynomial that passes through the points $(1, 30), (2, 15), (3, 10)$, and $(5, 6)$. Compute the product of the roots of f

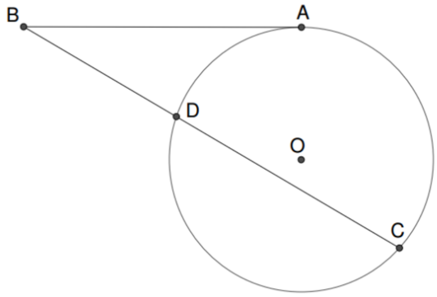
10. Consider the equation

$$1 - \frac{1}{d} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

with a, b, c, and d being positive integers. What is the largest value for d ?

SPACE FOR ROUGH WORK

11. In the diagram below, point A lies on the circle centered at $O.AB$ is tangent to circle O with $\overline{AB} = 6$. Point C is $\frac{2\pi}{3}$ radians away from point A on the circle, with BC intersecting circle O at point D. The length of BD is 3. Let the radius of the circle be $\sqrt{m} - \sqrt{n}$ where m, n are distinct positive integers. Find $m + n$.



12. Two unit squares are stacked on top of one another to form a 1×2 rectangle. Each of the seven edges is colored either red or blue. How many ways are there to color the edges in this way such that there is exactly one path along all-blue edges from the bottom-left corner to the top-right corner?

13. Consider a triangle where the sum of the three side lengths is equal to the product of the three side lengths. If the circumcircle has 25 times the area of the incircle, the distance between the incenter and the circumcenter can be expressed in the form $\frac{\sqrt{x}}{y}$, for integers x and y, with x square-free. Find $x + y$.

SPACE FOR ROUGH WORK

14. Assuming real values for p, q, r, and s, the equation

$$x^4 + px^3 + qx^2 + rx + s$$

has four non-real roots. The sum of two of these roots is $4 + 7i$, and the product of the other two roots is $3 - 4i$. Find q.

15. If a, b, c are positive reals such that $abc = 64$ and $3a^2 + 2b^3 + c^6 = 384$, compute maximum value of $a + b + c$.

16. In $\triangle ABC$, let $\angle CAB = 45°$, and $|AB| = \sqrt{2}, |AC| = 6$. Let M be the midpoint of side BC. The line AM intersects the circumcircle of $\triangle ABC$ at P. The circle centered at M with radius MP intersects the circumcircle of ABC again at $Q \neq P$. Suppose the tangent to the circumcircle of $\triangle ABC$ at B intersects AQ at T. Find TC^2.

SPACE FOR ROUGH WORK

17. Suppose that the number of ways in which we can form a group with an odd number of members (plural) from 99 people total is $a^b + c$ where a is as small as possible and $|c| < 1000$ (a, b, c are integers). Find the remainder when $a + b + c$ is divided by 100.

18. A robot starts in the bottom left corner of a 4×4 grid of squares. How many ways can it travel to each square exactly once and then return to its start if it is only allowed to move to an adjacent (not diagonal) square at each step?

19. Let σ be a permutation of the numbers $1, 2, 3, 4$. If

$$\sigma(a) \cdot \sigma^2(a) \cdot \sigma^3(a) \cdot \sigma^4(a) + 1$$

is divisible by 5 for all $a \in \{1, 2, 3, 4\}$, compute the number of possible σ.

20. Let f be a function over the natural numbers so that

- $f(1) = 1$
- If $n = p_1^{e_1} \dots p_k^{e_k}$ where $p_1, \cdots, p_k$ are distinct primes, and $e_1, \cdots e_k$ are non-negative integers, then $f(n) = (-1)^{e_1 + .. + e_k}$

Find $\sum_{i=1}^{2019} \sum_{d|i} f(d)$.

SPACE FOR ROUGH WORK

21. Compute

$$\left\lfloor \sum_{n=0}^{49} \sin\left(\frac{\pi n}{100}\right) \right\rfloor$$

22. Consider the triangle with vertices $(0,0), (1,0)$, and $(0,1)$. Let A, B, and C denote the distances from a given point to each of the three vertices. Denote the distance from the point that minimizes $A+B+C$ to the point that minimizes $A^2+B^2+C^2$ by d. If d is written as $\frac{\sqrt{a}-\sqrt{b}}{c}$ where a and b are square free, find $a+b+c$.

23. In quadrilateral $ABCD$, angles A, B, C, D form an increasing arithmetic sequence. Also, $\angle ACB = 90°$. If $CD = 7$ and the length of the altitude from C to AB is $\frac{9}{2}$, compute the area of $ABCD$.

SPACE FOR ROUGH WORK

24. Let Q be a quadratic polynomial. If the sum of the roots of $Q^{100}(x)$ (where $Q^i(x)$ is defined by $Q^1(x) = Q(x), Q^i(x) = Q\left(Q^{i-1}(x)\right)$ for integers $i \geq 2$) is 8 and the sum of the roots of Q is S, find $|\log_2(S)|$.

25. Compute the lowest positive integer k such that none of the numbers in the sequence $\{1, 1+\ k, 1+k+k^2, 1+k+k^2+k^3, \cdots\}$ are prime.

26. Consider the first set of 38 consecutive positive integers who all have sum of their digits not divisible by 11 . Find the sum of digits of the smallest integer in this set.

SPACE FOR ROUGH WORK

27. Let n be the smallest positive integer which can be expressed as a sum of multiple (at least two) consecutive integers in precisely 2019 ways. Then n is the product of k not necessarily distinct primes. Find the sum of digits of k.

28. For a positive integer n, let $f(n) = \sum_{i=1}^{n} \lfloor \log_2 n \rfloor$. Find the largest two digit positive integer n such that $n \mid f(n)$.

29. A 7×7 grid of unit-length squares is given. Twenty-four 1×2 dominoes are placed in the grid, each covering two whole squares and in total leaving one empty space. It is allowed to take a domino adjacent to the empty square and slide it lengthwise to fill the whole square, leaving a new one empty and resulting in a different configuration of dominoes. Given an initial configuration of dominoes for which the maximum possible number of distinct configurations can be reached through any number of slides, find the maximum number of distinct configurations.

30. Alice wants to paint each face of an octahedron either red or blue. She can paint any number of faces a particular color, including zero. Compute the number of ways in which she can do this. Two ways of painting the octahedron are considered the same if you can rotate the octahedron to get from one to the other.

SPACE FOR ROUGH WORK