MOMC IOQM Mock Blueberry 3

Instructions:

- All answers are in the integer range of $00 - 99$. Although there is a non-zero chance of an intentional bonus.
- Problems $1 - 10$ are 2 Markers, $11 - 20$ are 3 Markers and $21 - 30$ are 5 Markers.
- Total time is 3 Hours.
- The test begins from the next page. So only proceed to the next page if you are starting the test!
- Answer keys and sources are on the last two pages.
- Mock Compiled by **Agamjeet Singh**
- The problems are credited to their respective sources.
- Good luck!

1. Given that there exists a unique 3 digit number $N = \underline{ABC}$, whose digits (A, B, C) are all nonzero, with the property that the product $P = \underline{ABC} \times \underline{AB} \times \underline{A}$ is divisible by 1000, find $\underline{BC}$.

2. How many multiples of 12 divide $12!$ and have exactly 12 divisors?

3. Triangle ABC has a right angle at $A, AB = 20$, and $AC = 21$. Circles ω_A, ω_B, and ω_C are centered at A, B, and C respectively and pass through the midpoint M of $\overline{BC}$. ω_A and ω_B intersect at $X \neq M$, and ω_A and ω_C intersect at $Y \neq M$. Find XY.

4. The incircle of $\triangle ABC$ is tangent to sides $\overline{BC}, \overline{AC}$, and $\overline{AB}$ at D, E, and F, respectively. Point G is the intersection of lines AC and DF as shown. The sides of $\triangle ABC$ have lengths $AB = 73, BC = 123$, and $AC = 120$. Find the length EG.

5. We have a 9 by 9 chessboard with 9 kings (which can move to any of 8 adjacent squares) in the bottom row. What is the minimum number of moves, if two pieces cannot occupy the same square at the same time, to move all the kings into an X shape (a 5×5 region where there are 5 kings along each diagonal of the X, as shown below)?

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O       O
  O   O
    O
  O   O
O       O
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SPACE FOR ROUGH WORK

6. Dilhan has objects of 3 types, A, B, and C, and 6 functions

$$f_{A,B}, f_{A,C}, f_{B,A}, f_{B,C}, f_{C,A}, f_{C,B}$$

where $f_{X,Y}$ takes in an object of type X and outputs an object of type Y. Dilhan wants to compose his 6 functions, without repeats, such that the resulting expression is well-typed, meaning an object can be taken in by the first function, and the resulting output can then be taken in by the second function, and so on. In how many orders can he compose his 6 functions, satisfying this constraint?

7. Given a trapezoid with bases AB and CD, there exists a point E on CD such that drawing the segments AE and BE partitions the trapezoid into 3 similar isosceles triangles, each with long side twice the short side. What is the integer nearest to the sum of all possible values of $\frac{CD}{AB}$?

8. Consider all ordered pairs (m, n) of positive integers satisfying $59m - 68n = mn$. Find the largest prime factor of the sum of all the possible values of n in these ordered pairs.

9. Pam lists the four smallest positive prime numbers in increasing order. When she divides the positive integer N by the first prime, the remainder is 1. When she divides N by the second prime, the remainder is 2. When she divides N by the third prime, the remainder is 3. When she divides N by the fourth prime, the remainder is 4. Find the least possible value for N.

10. The side lengths of a scalene triangle are roots of the polynomial

$$x^3 - 20x^2 + 131x - 281.3.$$

Find the integer nearest to the area of the triangle.

SPACE FOR ROUGH WORK

11. What is the largest prime factor of the 101^{st} smallest integer which can represented in the form $3^a + 3^b + 3^c$, where a, b, and c are integers?

12. Let a and b be complex numbers such that $(a+1)(b+1) = 2$ and $\left(a^2+1\right)\left(b^2+1\right) = 32$. Let the sum of all possible values of $\left(a^4+1\right)\left(b^4+1\right)$ be S. Find the integer nearest to $\sqrt{S}$.

13. Points A, B, and C lie on a line, in that order, with $AB = 8$ and $BC = 2.B$ is rotated 20° counterclockwise about A to a point B', tracing out an arc R_1. C is then rotated 20° clockwise about A to a point C', tracing out an $\operatorname{arc} R_2$. Let the area of the region bounded by $\operatorname{arc} R_1$, segment $B'C$, arc R_2, and segment $C'B$ be $\mathcal{A}$. Find the integer nearest to $10\mathcal{A}$.

14. Consider trapezoid $[ABCD]$ which has $AB \| CD$ with $AB = 5$ and $CD = 9$. Moreover, $\angle C = 15^\circ$ and $\angle D = 75^\circ$. Let M_1 be the midpoint of AB and M_2 be the midpoint of CD. What is the distance M_1M_2?

15. Let $p_1, p_2, p_3, p_4, p_5, p_6$ be distinct primes greater than 5. Find the minimum possible value of

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 - 6\min\left(p_1, p_2, p_3, p_4, p_5, p_6\right)$$

SPACE FOR ROUGH WORK

16. Suppose there are 160 pigeons and n holes. The 1 st pigeon flies to the 1st hole, the 2nd pigeon flies to the 4th hole, and so on, such that the i th pigeon flies to the $\left(i^2 \bmod n\right)$ th hole, where $k \bmod n$ is the remainder when k is divided by n. What is the sum of square of digits of the minimum n such that there is at most one pigeon per hole?

17. Adam has a box with 15 pool balls in it, numbered from 1 to 15, and picks out 5 of them. He then sorts them in increasing order, takes the four differences between each pair of adjacent balls, and finds exactly two of these differences are equal to 1. Suppose there are N selections of 5 balls that he could have drawn from the box. Find the sum of digits of N.

18. Bill Gates and Jeff Bezos are playing a game. Each turn, a coin is flipped, and if Bill and Jeff have $m, n > 0$ dollars, respectively, the winner of the coin toss will take $\min(m, n)$ from the loser. Given that Bill starts with 20 dollars and Jeff starts with 21 dollars, let the probability that Bill ends up with all of the money be $\frac{a}{b}$ where a, b are relatively prime positive integers. Find $a + b$.

19. Let ABC be a triangle with circumcenter O. Additionally, $\angle BAC = 20^\circ$ and $\angle BCA = 70^\circ$. Let D, E be points on side AC such that BO bisects $\angle ABD$ and BE bisects $\angle CBD$. If P and Q are points on line BC such that DP and EQ are perpendicular to AC, what is $\angle PAQ$?

20. Let

$$\sum_{i=0}^{\infty} \frac{7^i}{\left(7^i + 1\right)\left(7^i + 7\right)} = \frac{m}{n}$$

where m, n are relatively prime positive integers. Find $m + n$.

SPACE FOR ROUGH WORK

21. For each positive integer n, let $\sigma(n)$ denote the sum of the positive integer divisors of n. Suppose there exist M positive integers $n \leq 2021$ that satisfy

$$\sigma(3n) \geq \sigma(n) + \sigma(2n)$$

Find the remainder when M is divided by 100.

22. Let $f(x) = \frac{x^2}{8}$. Starting at the point $(7, 3)$, let the length of the shortest path that touches the graph of f, and then the x-axis be $a\sqrt{b} - c$ where a, b, c are positive integers such that b is square-free. Find $a + b + c$.

23. Let the remainder when

$$\left\lfloor \frac{149^{151} + 151^{149}}{22499} \right\rfloor$$

is divided by 10^4 be $100p + q$ where $p, q < 100$ are positive integers. Find $p + q$.

24. A $2\sqrt{5}$ by $4\sqrt{5}$ rectangle is rotated by an angle θ about one of its diagonals. If the total volume swept out by the rotating rectangle is 62π, find the sum of square of digits of the measure of θ in degrees.

25. Find the last two digits of the number of permutations of the string 0123456 that are there such that no contiguous substrings of lengths $1 < \ell < 7$ have a sum of digits divisible by 7?

SPACE FOR ROUGH WORK

26. As a gift, Dilhan was given the number $n = 1^1 \cdot 2^2 \cdots 2021^{2021}$, and each day, he has been dividing n by $2021!$ exactly once. One day, when he did this, he discovered that, for the first time, n was no longer an integer, but instead a reduced fraction of the form $\frac{a}{b}$. What are the last two digits of the sum of all distinct prime factors of b?

27. In convex quadrilateral $ABCD, \angle ADC = 90^\circ + \angle BAC$. Given that $AB = BC = 17$, and $CD = 16$, what are the last two digits of the largest integer less than the maximum possible area of the quadrilateral?

28. Let $P(x), Q(x)$, and $R(x)$ be three monic quadratic polynomials with only real roots, satisfying

$$P(Q(x)) = (x-1)(x-3)(x-5)(x-7)$$
$$Q(R(x)) = (x-2)(x-4)(x-6)(x-8)$$

for all real numbers x. What is $P(0) + Q(0) - R(0)$?

29. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be three circles with radii $3, 4, 9$, respectively, such that Γ_1 and Γ_2 are externally tangent at C, and Γ_3 is internally tangent to Γ_1 and Γ_2 at A and B, respectively. Suppose the tangents to Γ_3 at A and B intersect at X. The line through X and C intersect Γ_3 at two points, P and Q. Let $PQ = \frac{a\sqrt{b}}{c}$ where a, b, c are positive integers such that $a+b+c$ is minimal. Find $a+b+c$.

30. Find the sum of digits of the number of functions $f : \{1, 2, 3, \ldots, 7\} \to \{1, 2, 3, \ldots, 7\}$ that are there such that the set $\mathcal{F} = \{f(i) : i \in \{1, \ldots, 7\}\}$ has cardinality four, while the set $\mathcal{G} = \{f(f(f(i))) : i \in \{1, \ldots, 7\}\}$ consists of a single element.

SPACE FOR ROUGH WORK