MOMC IOQM Mock Blueberry 1

Instructions:

- All answers are in the integer range of $00-99$. Although there is a non-zero chance of an intentional bonus.
- Problems $1-10$ are 2 Markers, $11-20$ are 3 Markers and $21-30$ are 5 Markers.
- Total time is 3 Hours.
- The test begins from the next page. So only proceed to the next page if you are starting the test!
- Answer keys and sources are on the last two pages.
- Mock Compiled by **Agamjeet Singh**
- The problems are credited to their respective sources.
- Good luck!

1. Let n be a two-digit integer, and let m be the result when we reverse the digits of n. If $n-m$ and $n+m$ are both perfect squares, find n.

2. Compute the number of ordered triples (p, q, r) of primes, each at most 30, such that

$$p+q+r=p^2+4$$

3. Let $0° < \theta < 90°$ be an angle. If

$$\log_{\sin\theta}\cos\theta, \quad \log_{\cos\theta}\tan\theta, \quad \log_{\tan\theta}\sin\theta$$

form a geometric progression in that order, let $\sin\theta = \frac{\sqrt{a}-b}{c}$ where a, b, c are relatively prime positive integers. Find $a+b+c$.

4. Inside a bag containing geometric shapes, there are twice as many equilateral triangles with side l as squares with side 3. If the sum of the areas of all the squares is equal to the sum of the areas of all the equilateral triangles, find the integer nearest to l.

5. Let $\triangle ABC$ be an equilateral triangle, and let D be a point such that B is the midpoint of AD. If $CD = 12$, the area of $\triangle ABC$ can be expressed in the form $a\sqrt{b}$ for some integers a, b, where b is not divisible by the square of any prime. Find $a+b$.

6. There exists unique digits $a \neq 0$ and $b \neq a$ such that the fourdigit number $\overline{aabb}$ is a perfect square. Compute $a+b$.

7. How many ordered pairs (x, y) of integers satisfy

$$x^4+y^4+2x^2y^2+1023=1024x^2+1024y^2?$$

8. Let p, q, and r be distinct prime numbers such that p^2+1, q^2+2, and r^2+3, form an arithmetic sequence in that order. What is the minimum possible value of $p+q+r$?

9. Arrange the numbers $0, 1, 2, \ldots, 23$ in a circle. Let the expected number of (unordered) pairs of adjacent numbers that sum to 23 be $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m+n$.

10. Three triplets, Tam, Tom, and Tim, work together to build a treehouse. Tam works twice as fast as Tom, but three times slower than Tim. After working together for 10 days, Tim decides to quit since he is tired. After another 10 days, Tom also decides to quit. It takes Tam another 6 days to complete the treehouse. The number of days it would had taken them to complete the treehouse if none of them had quit can be written as $\frac{m}{n}$ for relatively prime positive integers m and n. What is $m+n$?

SPACE FOR ROUGH WORK

11. Let $ABCD$ be a parallelogram with area 2020 such that $AB/BC = 66/65$. The bisectors of $\angle DAB, \angle ABC, \angle BCD, \angle CDA$ form a quadrilateral. Let the area of this quadrilateral be $\frac{m}{n}$ where m, n are relatively prime positive integers. Find the integer nearest to $(m+n)/10$

12. Let ABC be a triangle with incircle ω. Points E, F lie on $\overline{AB}, \overline{AC}$ such that $\overline{EF} \| \overline{BC}$ and $\overline{EF}$ is tangent to ω. If $EF = 6$ and $BC = 7$, find $AB + AC$.

13. Define $f(n)$ to be the sum of the positive integer factors of n. If $f(n) = 360$ and $f(3n) = 1170$, what is the sum of square of digits of the smallest possible value of $f(2n)$?

14. Call a positive integer super-even if the last two digits of all its multiples are always even. If S is the sum of all super-even integers $1 \leq n \leq 2018$, what is the number of factors of S?

15. How many integers $1 \leq m \leq 100$ are there such that there exists an integer n which has the property that exactly $m\%$ of the positive integer divisors of n are perfect squares?

SPACE FOR ROUGH WORK

16. Let $ABCD$ be a quadrilateral with side lengths $AB = 2, BC = 5, CD = 3$, and suppose $\angle B = \angle C = 90°$. Let M be the midpoint of $\overline{AD}$ and let P be a point on $\overline{BC}$ so that quadrilaterals $ABPM$ and $DCPM$ have equal areas. Let PM be $\frac{\sqrt{a}}{b}$ where a, b are positive integers such that a is square-free. Find $a + b$.

17. There is a unique nondecreasing sequence of positive integers $a_1, a_2, \ldots, a_n$ such that

$$\left(a_1 + \frac{1}{a_1}\right)\left(a_2 + \frac{1}{a_2}\right)\cdots\left(a_n + \frac{1}{a_n}\right) = 2020.$$

Compute $a_1 + a_2 + \cdots + a_n$.

18. In $\triangle ABC, AC = 3, AB = 4, BC = 5$, and I is the incenter of the triangle. Extend BI and CI to points D and E, respectively, such that $BE \perp BI$ and $CD \perp CI$. What is the integer nearest to the ratio between the areas of $\triangle BEI$ and $\triangle CDI$?

19. A positive integer is called triangular number if it can be expressed in the form $\frac{n(n+1)}{2}$ for a positive integer n. How many ordered pairs (x, y) of triangular numbers satisfy $x - y = 31,500$?

20. Consider quadrilateral $ABCD$ with $AB = 3, BC = 4, CD = 5$, and $DA = 6$. Circle Γ passes through A, B, C and intersects DA again at X. Given that $CX = 4$, compute the area of quadrilateral $ABCD$

SPACE FOR ROUGH WORK

21. In a plane lies 14 points and all possible lines are drawn among themselves, forming 287 triangles. Suppose N quadrilaterals are formed. Find the sum of square of digits of N.

22. Each of the six boxes shown in the equation below is replaced with a distinct number chosen from $\{1, 2, 3, \ldots, 15\}$.

$$S = \frac{\square}{\square} + \frac{\square}{\square} + \frac{\square}{\square}.$$

Suppose that the order of the fractions doesn't matter. Then there is exactly one way to arrange six numbers into the boxes such that $S < 1$ and S is as large as possible. Compute the sum of the 6 numbers.

23. Compute the maximum value of n for which n cards, numbered 1 through n, can be arranged and lined up in a row such that

- it is possible to remove 20 cards from the original arrangement leaving the remaining cards in ascending order, and
- it is possible to remove 20 cards from the original arrangement leaving the remaining cards in descending order.

24. Let ABC be a triangle and let M be the midpoint of $\overline{BC}$. The lengths AB, AM, AC form a geometric sequence in that order. The side lengths of $\triangle ABC$ are 2020, 2021, x in some order. Let the sum of all possible values of x be S. Find the last two digits of the integer nearest to S.

25. Let $\lceil x \rceil$ denote the smallest integer greater than or equal to x. The sequence (a_i) is defined as follows: $a_1 = 1$, and for all $i \geq 1$,

$$a_{i+1} = \min\left\{7\left\lceil\frac{a_i + 1}{7}\right\rceil, 19\left\lceil\frac{a_i + 1}{19}\right\rceil\right\}.$$

Find the last two digits of a_{100}.

SPACE FOR ROUGH WORK

26. Miriam is trying to escape from a haunted house placed in the middle of a circular field with area 100 square feet. Each minute, she carefully moves 1 foot in any direction she wants; however, an evil ghost may decide to scare her and make her run 1 foot in the direction opposite to the one she wanted. What are the last two digits of the least integer number of feet Miriam must walk in order to escape from the field, independently from the ghost's tricks?

27. Eddie has a birthday party, in which he invited his 8, very generous, best friends. Each friend brought a certain number of gifts for Eddie. It is observed that for every group of 4 friends, at least 2 of them brought the same number of gifts. Moreover, the person with the most gifts brought 4, while the least brought 1. What is the total number of possible values of the amount of gifts Eddie received in total?

28. In $\triangle ABC, AB = 4, BC = 5, AC = 6$, and I is the incenter of the triangle. Let $\mathcal{S}$ be the locus of all points D such that the circumcenter O of $\triangle ABD$ is collinear with AI. Determine the largest integer less than or equal to the perimeter of $\mathcal{S}$.

29. The AMC 12 consists of 25 problems, where for each problem, a correct answer is worth 6 points, leaving the problem blank is worth 1.5 points, and an incorrect answer is worth 0 points. The AIME consists of 15 problems, where each problem is worth 10 points, and no partial credit is given.

Any contestant who scores at least 84 on the AMC 12 is eligible for the AIME, and the USAMO index of such a student is the sum of his AMC 12 and AIME scores. Determine the last two digits of the smallest integer $N > 300$ such that no contestant can possibly obtain a USAMO index of $\frac{1}{2}N$.

30. Let $P_0, P_1, \dots$ be points in the complex plane such that $P_0 = -1, P_1 = i$ and $P_n = i \cdot P_{n-1} + P_{n-2}$ for every $n \geq 2$. Let $\mathcal{P}$ be the polygon $P_0P_1 \dots P_{2017}$ and let P and A be its perimeter and area, respectively. If the value of $\frac{P}{A}$ can be written in the form $\frac{a\sqrt{b}+c\sqrt{d}}{e}$ for positive integers a, b, c, d, and e such that b and d are not divisible by the square of any prime and $\gcd(a, c, e) = 1$, compute the remainder when $a + b + c + d + e$ is divided by 100.

SPACE FOR ROUGH WORK