Mirror Symmetry of the Quintic Threefold

A report summarizing the work completed as an undergraduate student research assistant at the University of British Columbia under the supervision of Sébastien Picard.

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Abstract. The following report aims to understand a specific family of Calabi-Yau manifolds and its mirror, given by an equation for a quintic threefold in complex projective space. It follows certain existing papers on the subject in order to provide more explicit derivations and explanations for existing calculations.

Contents

1 Overview

Calabi-Yau manifolds are an interesting class of complex manifolds with specific properties which make it an object of interest for both theoretical physicists and mathematicians. Put very simply, the “mirror symmetry conjecture” related to these manifolds stipulates that every Calabi-Yau manifold has a “mirror” such that there is a certain connection between the geometry of the two manifolds. There are multiple points of view with which one can approach this topic, including a more algebraic and homological viewpoint, as well as a more geometric approach relying on motivations from string theory. An in-depth review of this conjecture and its history was written by Gross [5].

An important paper in theoretical physics by Candelas, De La Ossa, Green, and Parkes [2] examined a specific family M_ψ of threefolds, given as the zero set of the polynomial

$$p = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4$$

in complex projective space, and its moduli space, determined by ψ. The paper further looked at the behavior of the mirror family, thus lending support to the mirror symmetry conjecture. It found interesting parallels between calculations performed on the original manifold and integrals on its mirror, and many of the coefficients and values were, in fact, predicted by string theory.

The goal of the following report will be to understand and expand on some of the concepts and calculations featured in this paper by Candelas et al. [2]. It will assume some background knowledge in complex differential geometry and complex analysis. The report will be structured a follows: in section 2 a brief overview of relevant concepts will be given, in order to establish specific definitions and notions for the remainder of the paper. The following section **??** will focus in on the paper by Candelas et al., beginning by introducing the quintic three-fold of interest in 3.1, and then focusing on some of the calculations of interest in the subsequent subsections. Finally, the conclusion will address some of the relevant next steps.

2 Preliminaries

The goal of this section will be to establish some relevant concepts in order to define and understand Calabi-Yau manifolds. Some relevant references include [9] and [3].

Let us establish some notation. For the remainder of this section, let M be a compact Kähler manifold with complex structure J and complex dimension n with a hermitian metric h and Kähler form ω so that $h = g - i\omega$ for a Riemannian metric g.

While there are many definitions of a Calabi-Yau manifold, we will be using the follwing one. Recall that a **Kähler** manifold is one which as a closed Kähler form.

The following is the standard definition for a Calabi-Yau manifold in mathematics:

Definition 1 (Calabi-Yau manifold)
A **Calabi-Yau** manifold is a compact Kähler manifold of complex dimension n such that it has vanishing first Chern class and is Ricci flat.

However for the purposes of this report we will make some additional assumptions about the manifold. Thus, we will rely on the following, more restrictive definition:

Definition 2 (Calabi-Yau manifold)
A **Calabi-Yau** manifold is a compact Kähler manifold of complex dimension n such that it has a holomorphic volume form and a trivial canonical bundle.

In addition, we will be working in the simply connected case for the remainder of this report. An additional property of interest is that Calabi-Yau manifolds have a local holonomy contained in $SU(n)$.

The characteristics which define such manifolds will be touched upon below.

2.1 Ricci Curvature

The **Ricci tensor** helps us compare a manifold to Euclidean space by measuring the extent to which a volume deforms along a geodesic [7]. A manifold is then **Ricci-flat** if its Ricci tensor is zero everywhere. Note that this implies that the overall change in volume of a shape transported along a geodesic is zero; it does not mean that the volume does not change at all, and as such the manifold may still have curvature.

2.2 Hodge Numbers

Before considering the complex case, suppose X is a real manifold of dimension n and recall that the k^{th} **exterior derivative** d is given as the map

$$d : A^k(X) \to A^{k+1}(X)$$

where $A^k(X)$ is the space of all differential k-forms on X. Notice that $d \circ d = 0$ and so one can form a chain complex

$$A^0(X) \xrightarrow{d} A^1(X) \xrightarrow{d} .... \xrightarrow{d} A^n(X)$$

This complex has a **de Rham cohomology group** given as the quotient group

$$H^k(X) = \frac{\{\alpha \in A^k(X) : d\alpha = 0\}}{\{\beta \in A^{k+1}(X) : \beta = d\alpha, \alpha \in A^k(X)\}} = \frac{\text{closed k-forms}}{\text{exact k-forms}}$$

Then, the k^{th} **Betti number** is given as

$$b_k = dim_{\mathbb{F}} H^k(X)$$

and the **Euler characteristic** of the manifold is

$$\chi = \sum_k (-1)^k b_k$$

Note that in our case we will be assuming that our manifold in question has a holomorphic volume form, which implies that the manifold is orientable. This allows us to use the Poincaré duality, which states that for an orientable manifold M,

$$H^k(M) \cong H^{n-k}(M)$$

for cohomology groups H_{n-k} and H^k.

Now, consider the complex case of the manifold M with coordinates z. Recall that a differential k-form α is of **type $(\mathbf{p}, \mathbf{q})$** if it can be locally written as

$$\alpha = \psi_{I,J} dz_I \wedge d\overline{z}_J$$

where I, J are multi-indices of length p, q, respectively, and $p + q = k$. Denote the space of all such (p, q)-forms on M as $A^{p,q}(M)$.

Then, one can decompose the exterior derivative into two operators ∂ and $\overline{\partial}$, where the latter is called the **Dolbeault operator**. This can be done by writing $d = \partial + \overline{\partial}$ so that

$$\overline{\partial} : A^{p,q}(M) \to A^{p,q+1}(M)$$
$$\partial : A^{p,q}(M) \to A^{p+1,q}(M)$$

and where $\overline{\partial}^2 = d^2 = \partial^2 = 0$ so that this new operator also forms a complex. The complex formed by $\overline{\partial}$ is called the **Dolbeault complex** and it gives rise to the **Dolbeault cohomology group**, wich is the quotient group

$$H^{p,q} = \frac{\{\omega \in A^{p,q} : \overline{\partial}\omega = 0\}}{\{\omega \in A^{p,q+1} : \omega = \overline{\partial}\alpha, \alpha \in A^{p,q}\}}$$

Then, the Hodge numbers $h^{p,q}$ are the complex dimensions of these cohomology groups; more concretely, these are given as

$$h^{p,q} = dim_{\mathbb{C}} H^{p,q}$$

These numbers have the property that for a manifold of complex dimenion n, $h^{p,q} = h^{n-p,n-q}$. As well, if the manifold is Kähler, then

1. $h^{p,q} = h^{q,p}$
2. $h^{p,p} \geqslant 1$ for all $1 \leqslant p \leqslant n$
3. $b_k = \sum_{p+q=k} h^{p,q}$ where b_k is the k^{th} Betti number

Note that there is an exchange of Hodge numbers between a Calabi-Yau manifold and its mirror, so that $h^{p,q}$ on the original manifold is equal to $h^{n-p,q}$ of the mirror.

2.3 Holonomy

Recall that the parallel transport of a vector in the tangent space along a curved loop may result in the rotation of such a vector upon return to the starting position. The **holonomy group** of a manifold at a point x is the group of all possible rotations. If one limits onesself to contractible loops only then this is called the **restricted** holonomy group.

Recall that an important aspect of Kähler manifolds is that $[T_x^{1,0}(M), T_x^{1,0}(M)] \subset T_x^{1,0}(M)$ and this makes the manifold more rigid; for this reason, the holonomy group of a Kähler manifold is smaller than a simple complex one, and thus is contained in $\mathcal{U}(n)$. If one further restricts to the case when the manifold is (compact) Calabi-Yau, then in fact the holonomy group becomes a subset of $SU(n)$.

3 The Specific Mirror Pair

Now that some relevant definitions and concepts have been established, let us focus in on the paper of interest by Candelas et al [2]. To begin we must set the scene by understanding the quintic threefold and how its mirror manifold arises. Some relevant references which have been consulted for this section are [4], [1], and [6].

3.1 Understanding the Quintic Threefold

Let M_ψ denote the family of quintic three-folds defined as the zero set of the polynomial

$$p_\psi = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4$$

in complex projective space so that $M_\psi \subset \mathbb{CP}^4$. This manifold has complex dimension three. Note that the ψ may be dropped at times from M_ψ, p_ψ for the sake of brevity.

This manifold is Calabi-Yau threefold with holomorphic volume form

$$\Omega_{i,j} = \frac{\bigwedge_{k=0,k\neq i,j}^{4} dx_k}{\frac{\partial p}{\partial x_j}}$$

on an affine subset $U_i = \{\overline{x} \in M_\psi : x_i = 1\}$ and where $i \neq j$ [4].

3.1.1 The Action on the Manifold

Consider the group $(\mathbb{Z}/5\mathbb{Z})^5$ which acts on the points in projective space by multiplication of fifth roots of unity, and consider further the subgroup with the additional condition that if $a = (a_0, a_1, a_2, a_3, a_4)$ is a element of the group then

$|a| = \sum_{i=0}^{4} a_i \cong 0 \mod 5$. This subgroup acts on M; indeed, if a is a point in the subgroup as before, and $x = (x_0, x_1, x_2, x_3, x_4)$ a point on the manifold then

$$a \cdot x = (\omega_5^{a_0} x_0, \omega_5^{a_1} x_1, \omega_5^{a_2} x_2, \omega_5^{a_3} x_3, \omega_5^{a_4} x_4)$$

where ω_5 is a fifth root of unity. Further,

$$\begin{aligned} p_\psi(a \cdot x) &= \sum_{i=0}^{4} (\omega_5^{a_i} x_i)^5 - 5\psi \prod_{i=0}^{4} \omega_5^{a_i} x_i \\ &= \sum_{i=0}^{4} x_i^5 - 5\psi \omega_5^{\sum_{i=0}^{4} a_i} \prod_{i=0}^{4} x_i \\ &= p_\psi(x) = 0 \end{aligned}$$

Note that there is some redundancy in this subgroup, as the diagonal elements of the group preserve the point as we are working in projective space, and so the ratio between points remain the same.

Thus, the group of interest is $G = \{a \in (\mathbb{Z}/5\mathbb{Z})^5/\langle(0,0,0,0,0)\rangle : |a| \cong 0 \mod 5\} \cong (\mathbb{Z}/5\mathbb{Z})^3$. As stated in [2] this group of order 5^3 has generators

$$g_0 = (1,0,0,0,4), g_1 = (0,1,0,0,4), g_2 = (0,0,1,0,4), g_3 = (0,0,0,1,4)$$

where a choice of any three of the two will generate the whole group.

In general, the quotient of a manifold by a group which acts freely on the manifold is again a manifold. Unfortunately the action of G is not free, and as such the quotient $W_\psi = M_\psi/G$ is an orbifold. The fixed points of G will be discussed more in depth in section 3.1.2. Moving forward, we will call the quotient W_ψ the "mirror manifold".

3.1.2 Fixed Points and the Euler Characteristic

Note that the Euler characteristic of M is known to be -200. This section will discuss how to calculate the Euler characteristic of its mirror, W.

First, recall that $W = M/G$ where $G \cong (\mathbb{Z}/5\mathbb{Z})^3$ as discussed in section 3.1.1, and use the same notation introduced there. To determine the Euler characteristic of an orbifold, one can use the formula

$$\chi(W) = \frac{1}{|G|} \sum_{g,h \in G : gh = hg} \chi(M^{(g,h)})$$

where $M^{(g,h)}$ is the submanifold stabilized by both g and h, so that $M^{(g,h)} = \{m \in M : g \cdot m = h \cdot m = m\}$

Note that in our case, G is abelian, so the above amounts to a double sum over the elements of G.

Denote the three generators of G as g_0, g_1, g_2, g_3 as before and the identity as e. Instead of looking at the elements in G separately, consider its subgroups of interest. These are:

- The trivial subgroup, e
- Three subgroups of order 5, $\langle g_i \rangle$
- $\binom{3}{2} = 3$ subgroups of order 25, $\langle g_i, g_j \rangle, i \neq j$ and
- The whole group

Next, one must determine the general form of the submanifolds of M which are stabilized by two elements in these subgroup. Using the above order, these submanifolds are:

- The whole manifold, M
- The curves $C_{ijk} = x_i^5 + x_j^5 + x_k^5 = 0$ for $i \neq j \neq k$
- The points $P_{ij} = x_i^5 + x_j^5 = 0$, $i \neq j$
- Nothing

Note that there is an "inclusion" between each submanifold above; this must be taken into account when calculating the Euler numbers. Let us now calculate the Euler characteristic of the orbifold by determining the Euler number of each submanifold, as well as its overall contribution to the sum, which depends on how many elements of the group it is stabilized by.

Let us begin with the points P_{ij}. The Euler characteristic of a point is simply 1, and because the equation for P_{ij} describes a manifold in $\mathbb{P}^1$, since one must have either $x_i \neq 0$ or $x_j \neq 0$, and so the equation can be rewritten as $X^5 + 1 = 0$. There are five solutions to this equation, and thus the total Euler characteristic is $\chi(P_{ij}) = 5$. There are, in total, $\binom{5}{2} = 10$ points, and each point will show up in the double sum $|\langle g_i, g_j \rangle|^2 = 25^2$ times. Thus the total contribution to the sum will be $25^2 \times 10 \times 5$.

Next, consider the curve $C_{ijk} \subset \mathbb{P}^2$. For curves in the projective plane, the equation for its genus is given as $g = \frac{(d-2)(d-1)}{2}$ and so te genus of C_{ijk} is 6. Since $\chi = 2 - 2g$, we get that $\chi(C_{ijk}) = 2 - 2(6) = -10$.

However, the Euler characteristic for points P_{ij} has already been taken into account, and each curve C_{ijk} contains $\binom{3}{2} = 3$ such points. Thus, these must be removed from the curves; since the intersection of these points is empty, we find that

$$\begin{aligned}\chi(C_{ijk} \backslash P_{ij} \cup P_{jk} \cup P_{ik}) &= \chi(C_{ijk}) - \chi(P_{ij} \cup P_{jk} \cup P_{ik}) + \chi(C_{ijk} \cap (P_{ij} \cup P_{jk} \cup P_{ik})) \\ &= -10 - (3 \times 5) + 0 \\ &= -25\end{aligned}$$

Since there are $\binom{5}{3} = 10$ such curves, each stabilized by a subgroup of order 5, the total contribution to the sum will be $5^2 \times 10 \times (-25)$

Finally, the Euler characteristic of M is known to be -200. However, the contribution of the curves C_{ijk} and the points P_{ij} must be removed. Thus,

$$\begin{aligned}\chi(M \backslash \{allC_{ijk}\} \cup \{allP_{ij}\}) &= \chi(M) - \chi(\{allC_{ijk}\}) - \chi(\{allP_{ij}\}) \\ &= -200 - 10 \times (-25) - 10 \times 5\end{aligned}$$

where intersections do not appear in the sum explicitly since they were taken into account for the Euler numbers.

This final Euler number will only appear in the sum one time. Thus, returning to the Euler characteristic formula for an orbifold, we find that

$$\begin{aligned}\chi(W) &= \frac{1}{5^3}\left(-200 - 10 \times (-25) - 10 \times 5 + 5^2 \times 10 \times (-25) + 25^2 \times 10 \times 5\right) \\ &= \frac{1}{5^3}\left(0 + 5^3 \times 10 \times (-5) + 5^4 \times 10 \times 5\right) \\ &= -50 + 5^3 \times 2 \\ &= 200\end{aligned}$$

3.1.3 Singularities

We want M_ψ to be a smooth manifold, and so it helps to understand the values for which this is no longer the case; these singularities, which may depend on the value of ψ, will be of importance later. To begin, consider the derivatives of p_ψ, given as

$$\frac{\partial p}{\partial x_k} = 5x_k^4 - 5\psi \prod_{j=0, j\neq k}^{4} x_j$$

Suppose $\overline{x}$ is a singular so that all such derivatives vanish. Then, we get that

$$x_k^4 = \psi \prod_{j=0, j\neq k}^{4} x_j \implies x_k^5 = \psi \prod_{j=0}^{4} x_j$$

and so, in fact, $x_0^5 = ... = x_4^5 = \psi x_0x_1x_2x_3x_4$. Multiplying the equations together one gets that $\prod_{i=0}^4 x_i^5 = \psi^5 \prod_{i=0}^4 x_i^5$.

Suppose $\psi = 0$. Then, it follow that each $x_i = 0$, which cannot be since this is a projective manifold. Similarly, if *any* of the $x_i = 0$, then *all* of the $x_i = 0$. Thus, none of the x_i can be zero, and so one finds that $\psi^5 = 1$, and so it is a fifth root of unity, say $\psi = \omega_5^\alpha$.

In fact, as stated above, $x_0^5 = ... = x_4^5$, and so after reducing (as we are working in projective space) one finds that each each x_i must be a fifth root of unity, so $x_i = \omega_5^{\alpha_i}$ for some mod-five integer α_i

Then, the equality

$$1 = \psi\omega_5^{\sum_{i=0}^4 \alpha_i} = \omega_5^{\alpha+\sum_{i=0}^4 \alpha_i}$$

must be satisfied, so $\alpha + \sum_{i=0}^4 \alpha_i \equiv 0 \mod 5$.

Let

$$W = \{\overline{x} \in M : \overline{x} = (\omega_5^{\alpha_0}, \omega_5^{\alpha_1}, \omega_5^{\alpha_2}, \omega_5^{\alpha_3}, \omega_5^{\alpha_4}), \alpha + \sum_{i=0}^4 \alpha_i \equiv 0 \mod 5\}$$

Proposition 3.1.3.1
Let $\psi = 1$ so that $\alpha \equiv 0 \mod 5$. Let $w = (1,1,1,1,1)$. Then, $G \cdot w = W$ and so it suffices to only consider this point w when studying the singularity at $\psi = 1$ of the mirror manifold W_ψ.

Proof. The inclusion $G \cdot w \subset W$ is obvious; note that any element $g \in G$ acts via fifth roots of unity and are automorphisms of p. In fact, by definition of G, the sum of powers of any $g \in G$ is $0 \mod 5$, since $G = \langle (1,0,0,0,4), (0,1,0,0,4), (0,0,1,0,4) \rangle$

On the other hand, suppose let $v \in W$ be any element, so $v = (\omega_5^{\alpha_0}, \omega_5^{\alpha_1}, \omega_5^{\alpha_2}, \omega_5^{\alpha_3}, \omega_5^{\alpha_4})$ for some integers α_i. Let $g_0 = (1,0,0,0,4), g_1 = (0,1,0,0,4), g_2 = (0,0,1,0,4), g_3 = (0,0,0,1,4)$, which are all elements of G. Then,

$$g_0^{\alpha_0} g_1^{\alpha_1} g_2^{\alpha_2} g_3^{\alpha_3} w = (\omega_5^{\alpha_0}, \omega_5^{\alpha_1}, \omega_5^{\alpha_2}, \omega_5^{\alpha_3}, \omega_5^{4\sum_{i=0}^3 \alpha_i})$$

But, by the requirements placed on W we know that $\sum_{i=0}^4 \alpha_i \equiv 0 \mod 5$ and so $\sum_{i=0}^3 \alpha_i \equiv -\alpha_4 \mod 5$, which implies that

$$4\sum_{i=0}^4 \alpha_i \equiv -4\alpha_4 \mod 5 \equiv \alpha_4 \mod 5$$

Thus,

$$v = g_0^{\alpha_0} g_1^{\alpha_1} g_2^{\alpha_2} g_3^{\alpha_3} w$$

and $v \in W$ was chosen arbitrarily. Thus, $W \subset G \cdot w$. $\square$

This proposition implies two things: first, that only considering w when examining the singularity is sufficient, as the other points are all identified under the action of G, and that $|W| = |G \cdot w| = |G| = 125$, and so there are 125 identified points in projective space for which there is a singularity when $\psi = 1$.

Now it remains to further classify the singularity at $\psi = 1$. Consider the definition of a nodal singularity:

Definition 3
A point m on a conifold M is a **nodal singularity** if the Jacobian of M at m is *singular*, but its Hessian matrix of second derivatives is nonsingular.

From the discussion above we know that the Jacobian at w is singular. Consider now the second derivatives of p, with a particular choice of coordinates, say on the subset $U_0 = \{\overline{x} \in M : x_0 = 1\}$ so one can set $Z_i = \frac{x_i}{x_0}$. The result is the same in the other coordinate patches as well. Then, for $1 \leqslant i, j \leqslant 4$,

$$\frac{\partial^2 p}{\partial z_j \partial z_i} = \begin{cases} 20z_i^3 & \text{if } i = j \\ -5\psi \prod_{k \neq i,j} z_k & \text{else} \end{cases}$$

which gives the Hessian matrix at the point w

$$\begin{bmatrix} 20 & -5 & -5 & -5 \\ -5 & 20 & -5 & -5 \\ -5 & -5 & 20 & -5 \\ -5 & -5 & -5 & 20 \end{bmatrix}$$

with eigenvalues 25 with multiplicity 3 and 5 with multiplicity 1. Thus, the matrix has positive determinant and so is non-singular. In fact, since it is a positive definite matrix, it follows that the point w is a local minimum.

Consider the following lemma:

Lemma 3.1.3.1 (Morse lemma [8])
Let $f : M \to \mathbb{R}$ be a holomorphic function, where M is a smooth manifold, and suppose $m \in M$ is a nondegenerate critical point of f. Then, there exists some neighborhood $V_m \subset M$ of m given by coordinates $\{y_1, ..., y_n\}$ such that $y_i(m) = 0$ for all i and $f(q) = f(p) - \sum_{i=1}^{k} y_i(q)^2 + \sum_{i=k+1}^{n} y_k(q)^2$ for all $q \in V_m$, and where k is the index of f at the point m.

"Translating" the above lemma into the case described above, we have the function $p : U_0 \subset M \to 0$, since $M = \{p \equiv 0\}$ by definition, and where U_0 is the particular coordinate choice as defined above. Next, the point $m = (1, 1, 1, 1) \in U_0$ is the non-degenerate critical point, and, in fact, $p(m) = 0$.

Note that since the Hessian matrix of p at m has no negative eigenvalues, the index of the function is zero. Thus, applying the Morse lemma one can conclude that within some neighborhood $V_m \subset U_0 \subset M$ of m, there are coordinates $\{y_1, y_2, y_3, y_4\}$ such that

$$p = y_1^2 + y_2^2 + y_3^2 + y_4^2$$

gives the equation of the conifold near the critical point. Now, in order to determine the behavior of the manifold near this point, let $z = (z_1, z_2, z_3, z_4)$ be a point such that $p(z) = z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$ and it is a distance of $r + \epsilon$ away from the point m so that $|z| = r$.

Since $z \in \mathbb{C}^4$, let $z_j = a_j + ib_j$ for $a_j, b_j \in \mathbb{R}$ and $1 \leqslant j \leqslant 4$, so that the constraint on $z = a + ib$ for $a, b \in \mathbb{R}^{\ngeq}$ become

$$p(z) = z^2 = a^2 - b^2 + 2iab = 0 \implies a^2 = b^2 \text{ and } ab = 0$$

and

$$|z|^2 = |a|^2 + |b|^2 = r^2$$

and these two imply that $|a|^2 = |b|^2 = \frac{r^2}{2}$ Since $|a|^2 = \frac{r^2}{2} = \sum_{i=1}^4 a_i^2$ it follow that a must lie on a three-dimensional sphere S^3.

Then, since $ab = 0$, it follows that b must be in the orthogonal complement of a. Since a spans a vector space of dimension one, the orthogonal complement of this space, in which b resides, must have dimension three, and must have the property that all vectors in the space have two-norm $\frac{r}{\sqrt{2}}$. Thus, these vectors must reside on the two-sphere so that $b \in S^2$.

Thus, in general, the base of this singularity is $S^3 \times S^2$.

Now, in the discussion above we assumed that $\psi = 1$. This begs the question, how may things change if $\psi = \omega_5^\alpha$ with $\alpha \not\equiv 0 \mod 5$? In fact, this does not change much at all; consider the proposition below, which is nearly identical to the one proved above:

Proposition 3.1.3.2
Let $\psi = \omega_5^\alpha$ with $\alpha \not\equiv 0 \mod 5$. Let $w = (\omega_5^{-\alpha}, 1, 1, 1, 1)$, so that $w \in W$ Then, $G \cdot w = W$ and so it suffices to only consider this point w when studying the singularity at ψ.

Proof. The inclusion $G \cdot w \subset W$ is obvious; note that any element $g \in G$ acts via fifth roots of unity and are automorphisms of p. In fact, by definition of G, the sum of powers of any $g \in G$ is $0 \mod 5$, since $G = \langle (1,0,0,0,4), (0,1,0,0,4), (0,0,1,0,4) \rangle$

On the other hand, suppose let $v \in W$ be any element, so $v = (\omega_5^{\alpha_0}, \omega_5^{\alpha_1}, \omega_5^{\alpha_2}, \omega_5^{\alpha_3}, \omega_5^{\alpha_4})$ for some integers α_i. Let $g_0 = (1,0,0,0,4), g_1 = (0,1,0,0,4), g_2 = (0,0,1,0,4), g_3 = (0,0,0,1,4)$, which are all elements of G. Then,

$$g_0^{\alpha_0+\alpha} g_1^{\alpha_1} g_2^{\alpha_2} g_3^{\alpha_3} w = (\omega_5^{\alpha_0}, \omega_5^{\alpha_1}, \omega_5^{\alpha_2}, \omega_5^{\alpha_3}, \omega_5^{4\alpha + 4\sum_{i=0}^3 \alpha_i})$$

But, by the requirements placed on W we know that $\alpha + \sum_{i=0}^{4} \alpha_i \equiv 0 \mod 5$ and so $4\alpha + 4\sum_{i=0}^{3} \alpha_i \equiv -4\alpha_4 \mod 5$, which implies that

$$g_0^{\alpha_0+\alpha} g_1^{\alpha_1} g_2^{\alpha_2} g_3^{\alpha_3} w = (\omega_5^{\alpha_0}, \omega_5^{\alpha_1}, \omega_5^{\alpha_2}, \omega_5^{\alpha_3}, \omega_5^{\alpha_4})$$

Thus,

$$v = g_0^{\alpha_0+\alpha} g_1^{\alpha_1} g_2^{\alpha_2} g_3^{\alpha_3} w$$

and $v \in W$ was chosen arbitrarily. Thus, $W \subset G \cdot w$. $\square$

Thus it suffices to consider the point $w_\alpha = (\omega_5^{-\alpha}, 1, 1, 1, 1)$. As per the previous discussion one can see that at w_α the Hessian matrix is positive definite so that this point gives a nodal singularity, and so the discussion above applies for any $\psi = \omega_5^{\alpha}$. As such, it is often easier to simply consider the case when $\psi = 1$ and then generalize the results.

Finally, if $\psi = \infty$ then the product $x_0 x_1 x_2 x_3 x_4 x_5 = 0$; this is a sort of limit singularity of the manifold.

3.2 Periods and Monodromy

The goal of this subsection is to compute the periods of forms of the manifold. To begin, we must define what a period is, and then understand why it is important. This section relies heavily on the reference [6], as well as [4] and [1].

Recall that we are working with a family of manifolds, so that for each ψ in the parameter space we have a manifold M_ψ and its mirror W_ψ. Let $\mathcal{M} = \{M_\psi : \psi \in \mathbb{C}\}$ be the vector bundle over this parameter space so that the fiber of each point $\psi \in \mathbb{C}$ is a manifold M_ψ.

Definition 4
Let $\gamma_0, ..., \gamma_{r-1}$ be topological n-cycles; that is, closed n-cycles forming a basis for the n^{th} homology group of M_ψ and let ω be a holomorphic n-form defined locally on M_ψ, so that it is locally holomorphic. Then, the **periods** of ω are the integrals

$$\int_{\gamma_0} \omega, \ldots, \int_{\gamma_{r-1}} \omega$$

Both the n-cycles and the n-form can be extended to all of $\mathcal{M}$; this can be done by letting ω have simple poles such that it becomes a meromorphic function, and having each $\gamma_i := \gamma_i(\psi)$ so that changes as one traverses the $\mathcal{M}$ so that it is a section.

Thus, one can look at such local concepts globally, however this introduces the issue of monodromy, which can be thought of as analogous to the rotation that occurs due to parallel transport. Recall that *monodromy* refers to the behavior of a function near a singularity; specifically, what happens when integrating around a singularity.

Since $\gamma_i(\psi)$ is a collection of homology classes which is locally constant at ψ, it is a patchwork of local functions which are extended globally, effectively "glued together". As such, monodromy becomes an issue; transport along a closed path requires a linear map $T = (T_{ij})$ so that the transport is homologous to $\sum_j T_{ij}\gamma_j$. The same thing can be said for the periods of $\omega(\psi)$.

Our goal is to find these periods. This will be done by determining one of the periods by direct integration, which will then be used to help find an ordinary differential equation (ODE) which must be satisfied by all of the periods. This ODE will be used to find the second period.

Note that the periods must satisfy a "Pichard-Fuchs Equation" (PFE). The general steps for finding the PFE stated in [6] for parameter space $\mathcal{Z}$, are as follows:

1. Pick a local coordinate system $z \in U \subset \mathcal{Z}$ for some open subset U
2. Let
$$v_j(z) = \left[\frac{d^j}{dz^j}\int_{\gamma_0(z)} \omega(z), .., \frac{d^j}{dz^j}\int_{\gamma_{r-1}(z)} \omega(z)\right]$$
3. Let
$$d_j(z) = dim(span\{v_0(z), ..., v_j(z)\}) \leqslant r$$
so that it is bounded and constant.
4. Let s be the *smallest* index such that $d_s(z) = d_{s-1}(z)$
5. Thus, can write v_s in terms of $v_0, .., v_{s-1}$ as
$$v_s(z) = \sum_{j=0}^{s-1} C_j(z)v_j(z)$$
for coefficients $C_j(z)$ which are functions of z
6. Then, the **Picard-Fuchs equation** (PFE) satisfied *by all periods* of $\omega(z)$ is
$$\frac{d^s f}{dz^s} + \sum_{j=0}^{s-1} C_j(z)\frac{d^j f}{dz^j} = 0 \tag{1}$$

Here, $C_j(z)$ may have singularities; however, this will be at worst a regular singular point. Assuming the singularity is at $z = 0$ (choose coordinates accordingly), one can "remove" the singularity by multiplying the equation through by z^s so that one gets the **log form**

$$\left(z\frac{df}{dz}\right)^s + \sum_{j=0}^{s-1} B_j(z)\left(z\frac{df}{dz}\right)^j = 0 \tag{2}$$

Now that we know the general theory, we can return to the manifold in question.

Recall that we have the following holomorphic 3-form (on $\{x_0 = 1\} \cap M_\psi$):

$$\Omega = 5\psi \frac{dx_1 \wedge dx_2 \wedge dx_3}{\frac{dp}{dx_4}} \tag{3}$$

Note that here we have chosen a specific "branch" of the manifold (with $x_0 = 1$) but Ω could have been kept more general by including a sum in the numerator.

It is clear that Ω has no zeroes, and in fact it has a pole only if *all* of the partials of p are zero, which cannot occur for a non-singular manifold.

As such, Ω is a nowhere-vanishing holomorphic 3-form on M_ψ, and in fact descends to a holomorphic 3-form on W_ψ also, since it is invariant under G.

Now, we know that the Hodge numbers for the manifold W_ψ include the numbers $h_{3,0} = h_{0,3} = 1 = h_{1,2} = h_{2,1}$ and so the Betti number is $b_3 = 4$ and so $dimH_3(W, \mathbb{Z}) = 4$. As such, the three-forms

$$\Omega, \frac{d\Omega}{d\psi}, \frac{d^2\Omega}{d\psi^2}, \frac{d^3\Omega}{d\psi^3}, \frac{d^4\Omega}{d\psi^4}$$

are *linearly dependent* in cohomology. This dimension b_3 indicates that we have four periods of interest. The linear dependence becomes important when determining the Yukawa coupling.

3.2.1 The First Period: Direct integration

In order to determine the relative ODE, we will first perform direct integration to find the first period.

Let

$$\beta_0 = \{\overline{x} \in M_\psi : x_0 = 1, |x_1| = |x_2| = |x_3| = \delta\}$$

be the three-torus, with $x_4 \to 0$ as $\psi \to \infty$. Let γ_i denote a small loop about x_i in β_0 so that $\gamma_1 \times \gamma_2 \times \gamma_3 \subset \beta_0$. Then, to compute the period integral (including a helpful normalization):

$$\begin{aligned}\omega_0 &= \int_{\beta_0} \Omega \\ &\approx \frac{-5\psi}{(2\pi i)^3} \int_{\gamma_1 \times \gamma_2 \times \gamma_3} \Omega\end{aligned}$$

Consider

$$\int_{\gamma_4} 5\psi \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{\sum_i x_i^5 - 5\psi \prod_i x_i}$$

with a pole if $p = 0$ on the branch with $x_0 = 1$. Note that this integral depends only on x_4 so that the other variables can be treated as constants. Our goal is to somehow include this in the integral above, which depends only on x_1, x_2, x_3, as this will make it easier to integrate.

First we must ensure that there is a branch we can choose so that $x_4 \to 0$, $\psi \to \infty$, and such that this gives a simple pole. Let $x_4 = (\psi x_1 x_2 x_3)^{1/4} y$ for some y. Then, since $p(\bar{x}) = 0$,

$$\begin{aligned}p &= x_0^5 + x_1^5 + x_2^5 + x_3^5 + ((\psi x_1 x_2 x_3)^{1/4} y)^5 - 5\psi x_0 x_1 x_2 x_3 (\psi x_1 x_2 x_3)^{1/4} y) \\ &= 1 + x_1^5 + x_2^5 + x_3^5 + (\psi x_1 x_2 x_3)^{5/4} y^5 - 5(\psi x_1 x_2 x_3)^{5/4} y = 0\end{aligned}$$

and so

$$y = \frac{y^5}{5} + \frac{1 + x_1^5 + x_2^5 + x_3^5}{5(\psi x_1 x_2 x_3)^{5/4}}$$

Then, if $\psi \to \infty$ then the second term goes to zero, and so $y \to \frac{y^5}{5}$. Thus, either $y \to 0$ or $y^4 \to 5$. The first branch, $y \to 0$, implies that $x_4 \to 0$, which is what we want and what is assumed in the subset β_0, whereas if $y^4 \to 5$ then $x_4 \to \infty$. Thus, "pick" the first branch, and note that there is only one such solution which satisfies the $x_4 \to 0$ requirement, thus giving a simple pole for the $p = 0$ equation.

This is important because it helps determine the poles of $p(\overline{x})$ on the chosen branch: there is only one such pole as the equality $p = 0$ for $x_4 \to 0, \psi \to \infty$ was only satisfied if $y \to 0$, giving only one, simple pole.

Next, there may be a few ways to integrate the integral above. One method is to use residues; this is discussed below. The other is to rely on the expansion of y as a power series, which is touched on below in order to motivate the need for and simplicity of the first method.

3.2.1.1 The Difficult "Expansion Method"

Consider first the method of expanding y as a power series and trying to determine the integral that way. Ignoring normalization factors for now, rewrite 3 in terms of y, so that the integral becomes

$$\begin{aligned}
\int_{\beta_0} -5\psi \frac{dx_1 \wedge dx_2 \wedge dx_3}{\frac{dp}{dx_4}} &= \int_{\beta_0} -5\psi \frac{dx_1 \wedge dx_2 \wedge dx_3}{5x_4^4 - 5\psi x_1 x_2 x_3} \\
&= \int_{\beta_0} \frac{dx_1 \wedge dx_2 \wedge dx_3}{x_1 x_2 x_3 - \psi^{-1} x_4^4} \\
&= \int_{\beta_0} \frac{dx_1 \wedge dx_2 \wedge dx_3}{x_1 x_2 x_3 - \psi^{-1}((\psi x_1 x_2 x_3)^{1/4} y)^4} \\
&= \int_{\beta_0} \frac{dx_1 \wedge dx_2 \wedge dx_3}{x_1 x_2 x_3 - x_1 x_2 x_3 y^4} \\
&= \int_{\beta_0} \frac{dx_1 \wedge dx_2 \wedge dx_3}{x_1 x_2 x_3} \frac{1}{(1 - y^4)} \\
&= \int_{\beta_0} \frac{dx_1 \wedge dx_2 \wedge dx_3}{x_1 x_2 x_3} \sum_{n=0}^{\infty} y^{4n}
\end{aligned}$$

Let $r = \frac{1+x_1^5+x_2^5+x_3^5}{(\psi x_1 x_2 x_3)^{5/4}}$ and consider the expansion of the power series of y^4 in the integral above:

$$\begin{aligned}
\sum_{n=0}^{\infty} y^{4n} &= \sum_{n=0}^{\infty} (\frac{r}{5} + \frac{y^5}{5})^{4n} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{4n} \binom{4n}{k} \frac{1}{5^{4n}} r^k y^{5(4n-k)}
\end{aligned}$$

Recall that for a complex number z and the unit circle c centered at 0,

$$\int_c z^n dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{else} \end{cases}$$

and so the only terms which contribute to the integral are the ones where $y^{4n} = a_n$ for some constant, in which case $\int_{\beta_0} \frac{dx_1 \wedge dx_2 \wedge dx_3}{x_1 x_2 x_3} a_n = (2\pi i)^3 a_n$. The coefficients in this case will correspond to the coefficients of the r^{4n} terms.

Let us try to determine this coefficients recursively. Notice that this becomes fairly complicated quite quickly.

$$
\begin{aligned}
y^{4n} &= \sum_{k=0}^{4n} \binom{4n}{k} \frac{1}{5^{4n}} r^k y^{5(4n-k)} \\
&= \sum_{k=0}^{4n} \binom{4n}{k} \frac{1}{5^{4n}} r^k (\frac{r}{5} + \frac{y^5}{5})^{5(4n-k)} \\
&= \sum_{k=0}^{4n} \binom{4n}{k} \frac{1}{5^{4n}} r^k \left(\sum_{j=0}^{5(4n-k)} \binom{5(4n-k)}{j} \frac{1}{5^{5(4n-k)}} r^j (y^5)^{5(4n-k)-j} \right) \\
&= \sum_{k=0}^{4n} \binom{4n}{k} \frac{1}{5^{4n}} r^k \left(\sum_{j=0}^{5(4n-k)} \binom{5(4n-k)}{j} \frac{1}{5^{5(4n-k)}} r^j (\frac{r+y^5}{5})^{25(4n-k)-5j} \right) \\
&= \sum_{k=0}^{4n} \binom{4n}{k} \frac{1}{5^{24n-5k}} r^k \left(\sum_{j=0}^{5(4n-k)} \binom{5(4n-k)}{j} r^j \left(\sum_{i=0}^{5^2(4n-k)-5j} \binom{5^2(4n-k)-5j}{i} \frac{1}{5^{5^2(4n-k)-5j}} r^i (y^5)^{5^2(4n-k)-} \right.\right. \\
&= \sum_{k=0}^{4n} \binom{4n}{k} \frac{1}{5^{24n-5k}} r^k \sum_{j=0}^{5(4n-k)} \binom{5(4n-k)}{j} r^j \\
&\qquad \left(\sum_{i=0}^{5^2(4n-k)-5j} \binom{5^2(4n-k)-5j}{i} \frac{1}{5^{5^2(4n-k)-5j}} r^i (\frac{r+y^5}{5})^{5^3(4n-k)-5^2j-5i} \right) \\
&= \sum_{k=0}^{4n} \binom{4n}{k} \frac{1}{5^{24n-5k}} r^k \sum_{j=0}^{5(4n-k)} \binom{5(4n-k)}{j} r^j (\sum_{i=0}^{5^2(4n-k)-5j} \binom{5^2(4n-k)-5j}{i} \frac{1}{5^{5^2(4n-k)-5j+5^3(4n-k)-5^2j-5i}} r^{} \\
&\qquad \sum_{\ell=0}^{5^3(4n-k)-5^2j-5i} \binom{5^3(4n-k)-5^2j-5i}{\ell} r^\ell (y^5)^{5^3(4n-k)-5^2j-5i}) \\
&= \sum_{k=0}^{4n} \binom{4n}{k} r^k \sum_{j=0}^{5(4n-k)} \binom{5(4n-k)}{j} r^j (\sum_{i=0}^{5^2(4n-k)-5j} \binom{5^2(4n-k)-5j}{i} \frac{1}{5^{624n-155k-30j-5i}} r^i \\
&\qquad \sum_{\ell=0}^{5^3(4n-k)-5^2j-5i} \binom{5^3(4n-k)-5^2j-5i}{\ell} r^\ell (y^5)^{5^3(4n-k)-5^2j-5i})
\end{aligned}
$$

and so on. One can recursively substitute $y = \frac{r+y^5}{5}$ into the expansion in order to have an expansion in terms of r. As mentioned previously, r^n contains a constant term whenever $n = 4m$:

$$\begin{aligned} r^{4m} &= \left(\frac{1 + x_1^5 + x_2^5 + x_3^5}{(\psi x_1 x_2 x_3)^{5/4}}\right)^{4m} \\ &= \frac{(1 + x_1^5 + x_2^5 + x_3^5)^{4m}}{(\psi x_1 x_2 x_3)^{5m}} \\ &= \frac{b_m x_1^5 x_2^5 x_3^5 + g(x_1, x_2, x_3)}{(\psi x_1 x_2 x_3)^{5m}} \\ &= b_m + \frac{g(x_1, x_2, x_3)}{(\psi x_1 x_2 x_3)^{5m}} \end{aligned}$$

so that b_m is the constant term of interest, with the other terms contributing nothing to the integral. To find the value of b_m one can use multinomial coefficients:

$$b_m = \binom{4n}{n, n, n, n} = \frac{(4n)!}{(n!)^4}$$

Thus, the terms of interest in $\sum_{n=0}^{\infty} y^{4n}$ are the terms which contain powers of r^4. It remains to find the coefficients of these terms so that the overall contribution to the integral can be determined.

At the first iteration, have one term with coefficient $\frac{4n}{5^{4n}}$, which occurs when $k = 4n$. With a second iteration, $5(4n - k) - j = 0$ and $k + j = 0 \mod 4$ must be satisfied. Enumerating in this recursive fashion is quite complicated; instead, let us try to find a pattern for the coefficients a_{4m} of the term r^{4m} by grouping together the terms in the expansion of $\sum_{n=0}^{\infty} y^{4n}$ with the same power of r.

If $n = 0$ then $a_0 = 1$.

The power r^4 occurs in the sum only one time, when $n = 1$, with coefficient

$$a_4 = \binom{4}{4} \frac{1}{5^4} = \frac{1}{5^4}$$

.

r^8 occurs when $n = 1$ since $y^4 = \sum_{k=0}^{4} \binom{4}{k} \frac{1}{5^4} r^k (\frac{r}{5} + \frac{y^5}{5})^{5(4-k)}$, so if $k = 3$ then $\binom{4}{3} \frac{1}{5^4} r^3 (\frac{r}{5} + \frac{y^5}{5})^5$ includes the term $\frac{4}{5^9} r^8$. It also occurs when $n = 2$ so $y^8 = \sum_{k=0}^{8} \binom{8}{k} \frac{1}{5^8} r^k (\frac{r}{5} + \frac{y^5}{5})^{5(8-k)}$ and $k = 8$, so the coefficient is $\frac{1}{5^8}$, and so the total value is $a_8 = \frac{9}{5^9}$.

Continuing in this manner, r^{12} will appear in the $y^{12} = \sum_{k=0}^{12} \frac{1}{5^{12}} r^k y^{(5(12-k)}$ term when $k = 12$ so the coefficient is $\frac{1}{5^{12}}$.

If $n = 2$ so $y^8 = \sum_{k=0}^{8} \binom{8}{k} \frac{1}{5^8} r^k (\frac{r}{5} + \frac{y^5}{5})^{5(8-k)}$, notice that the smallest power of r following r^8 occurs when $k = 7$ so the power is r^{12}, with coefficient $\binom{8}{7} \frac{1}{5^8} \frac{1}{5^5} = \frac{8}{5^{13}}$.

Next, when $n = 1$ so $y^4 = \sum_{k=0}^{4} \binom{4}{k} \frac{1}{5^4} r^k (\frac{r}{5} + \frac{y^5}{5})^{5(4-k)}$, if $k = 2$ one gets the coefficient $\binom{4}{2} \frac{1}{5^4} \frac{1}{5^10} = \frac{6}{5^{14}}$.

As well, if $k = 3$ then one gets the term

$$\begin{aligned}
\binom{4}{3} \frac{1}{5^4} r^3 (\frac{r}{5} + \frac{y^5}{5})^5 &= \binom{4}{3} \frac{1}{5^4} r^3 (\frac{1}{5^5} \binom{5}{4} r^4 y) \\
&= \frac{20}{5^9} r^7 (\frac{r}{5} + \frac{y^5}{5})^5 \\
&= \frac{4}{5^13} r^12 + ...
\end{aligned}$$

Thus, the overall coefficient of r^{12} is $a_{12} = \frac{4}{5^13} + \frac{1}{5^{12}} + \frac{8}{5^{13}} + \frac{6}{5^{14}} = \frac{91}{5^{14}}$

However, obtaining the period with this method is quite laborious. Consider instead the method of using residues to transform the triple integral into a quadruple one.

3.2.1.2 A Better Residue Method

What is the residue of $p(1, x_1, x_2, x_3, x_4)$ about $x_4 = 0$?

First, note the following fact about residues: suppose $f(x) = \frac{h(x)}{g(x)}$ has a simple pole at a such that $g(a) = 0,\ g'(a) \neq 0$. Then, the residue is given as

$$\begin{aligned}
c_{-1} &= \lim_{x \to a} (x - a) f(x) \\
&= \lim_{x \to a} (x - a) \frac{h(x)}{g(x)} \\
&= \lim_{x \to a} \frac{h(x)}{\frac{g(x)}{x-a}} \\
&= \lim_{x \to a} \frac{h(x)}{\frac{g(x)-g(a)}{x-a}} \\
&= \frac{h(a)}{g'(a)}
\end{aligned}$$

Now, we know that if $\frac{\partial p}{\partial x_4}(\overline{x}) = 0$ at some point $\overline{x}$ on the manifold so that $p(\overline{x_0}) = 0$ then all of the partial derivatives must equal zero, which cannot occur by assumption, and so $\frac{\partial p}{\partial x_4}(\overline{x}) \neq 0$. Thus we can use the equation for calculating residues given above.

Recall the discussion regarding the poles at 3.2.1; p has a simple pole when $x_4 = 0$ on the β_0 subset. Focusing only on the x_4 variable, with the other variables being treated as constants, by the residue theorem and the fact given above we know that

$$\int_{\gamma_4} \frac{1}{p} dx_4 = 2\pi i Res(\frac{1}{p}; x_4 = 0) = 2\pi i \frac{1}{\frac{dp}{dx_4}}$$

and thus

$$\int_{\gamma_4} \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{\sum_i x_i^5 - 5\psi \prod_i x_i} = 2\pi i \frac{dx_1 \wedge dx_2 \wedge dx_3}{\frac{dp}{dx_4}}$$

where the left hand side is a multiple of what we had under the integral for ω_0. Thus, making this substitution, we get that

$$\omega_0 = \frac{-5\psi}{(2\pi i)^4} \int_{\gamma_1 \times \gamma_2 \times \gamma_3 \gamma_4} \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{p}$$

Next, note that the above integral does not depend on x_0 as the branch $x_0 = 1$ has been chosen. Also, $\int_{\gamma_0} dx_0 = \int_0^{2\pi i} d\theta = 2\pi i$ and this integral can be further added to the integral above, effectively including an extra differential under the integral, so that

$$\omega_0 = \frac{-5\psi}{(2\pi i)^5} \int_{\Gamma} \frac{dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{p}, \ \Gamma = \prod_{i=0}^{4} \gamma_i$$

In order for this to be helpful, one needs to rewrite p as $p = \prod_i x_i(-5\psi + \frac{\sum_i x_i^5}{\prod_i x_i})$. Then, the integral becomes

$$\begin{aligned}
\omega_0 &= \frac{1}{(2\pi i)^5} \int_{\Gamma} \frac{dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{\prod_i x_i (1 - \frac{\sum_i x_i^5}{5\psi \prod_i x_i})} \\
&= \frac{1}{(2\pi i)^5} \int_{\Gamma} \frac{dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{\prod_i x_i} \sum_{m=0}^{\infty} (\frac{\sum_i x_i^5}{5\psi \prod_i x_i})^m
\end{aligned}$$

where the last line is due to geometric series expansion.

Recall that for complex numbers, $\int_\gamma z^n dz = 2\pi i$ if $n = -1$, else it is 0. Since the above is an iterated integral, the only terms which show up at the end must have the form $\frac{\alpha dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{\prod_i x_i}$ for some constant α, with a final value $\int_\Gamma \frac{\alpha dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{\prod_i x_i} = \alpha(2\pi i)^5$. That is, the only terms which matter are the terms which are constant in the geometric series.

This requires that the numerator and denominator of the values in the geometric series have equal powers; this occurs if $m = 5n$ so that

$$(\frac{\sum_i x_i^5}{5\psi \prod_i x_i})^{5n} = (\frac{\alpha \prod_i x_i^{5n}}{(5\psi)^{5n} \prod_i x_i^{5n}}) + ...$$

where the remaining values in the sum do not matter as its integral will be zero, and thus it can be ignored.

The value of α is simply given by computing the multinomial coefficients, so that

$$\alpha = \binom{5n}{n, n, n, n, n} = \frac{(5n)!}{(n!)^5}$$

Thus, finally, the integral becomes quite simple:

$$\begin{aligned}\omega_0 &= \frac{1}{(2\pi i)^5} \int_\Gamma \frac{dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{\prod_i x_i} \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5 (5\psi)^{5n}} \\ &= \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5 (5\psi)^{5n}}\end{aligned}$$

Instead of writing the equations in terms of ψ, let $\lambda = \frac{1}{(5\psi)^5}$ so that $\omega_0(\lambda) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} \lambda^n$ is the final equation for the period.

3.2.2 Finding the ODE

Now that we know the equation for ω_0 we can use this to compute the ODE which must be satisfied by all of the periods. This can be done by finding the recurrence relation between the coefficients of powers of λ in ω_0 and working backwards to determine the ODE.

In order to find a recurrence relation for the coefficients of ω_0, let $a_n = \frac{(5n)!}{(n!)^5}$ so that

$$
\begin{aligned}
a_{n+1} &= \frac{(5(n+1))!}{((n+1)!)^5} \\
&= \frac{(5n+5)(5n+4)(5n+3)(5n+2)(5n+1)(5n!)}{(n+1)^5(n!)^5} \\
&= a_n \frac{(5n+5)(5n+4)(5n+3)(5n+2)(5n+1)}{(n+1)^5} \\
&= a_n \frac{5(5n+4)(5n+3)(5n+2)(5n+1)}{(n+1)^4}
\end{aligned}
$$

Let $\mu = \lambda \frac{d}{d\lambda}$. In order to determine the differential equation one can work backwards from this recurrence relation, by determining what derivations of a power series must have occured in order to result in this relation. Rewriting the relation as $(n+1)^4 a_{n+1} - 5(5n+4)(5n+3)(5n+2)(5n+1)a_n = 0$ one may begin by considering the left hand side of this equation, so that

$$
\begin{aligned}
\sum_{n=0}^{\infty} (n+1)^4 a_{n+1} \lambda^n &= \frac{d}{d\lambda} \sum_{n=0}^{\infty} (n+1)^3 a_{n+1} \lambda^{n+1} \\
&= \frac{d}{d\lambda} (\lambda \frac{d}{d\lambda} \sum_{n=0}^{\infty} (n+1)^2 a_{n+1} \lambda^{n+1}) \\
&= \frac{d}{d\lambda} (\lambda \frac{d}{d\lambda} (\lambda \frac{d}{d\lambda} \sum_{n=0}^{\infty} (n+1)^1 a_{n+1} \lambda^{n+1})) \\
&= \frac{d}{d\lambda} \mu^3 \sum_{n=0}^{\infty} a_{n+1} \lambda^{n+1}
\end{aligned}
$$

The rest of the terms can be rewritten as follows:

$$\begin{aligned}
-5\sum_{n=4}^{\infty} 625n^4 a_n\lambda^n &= -5\lambda\frac{d}{d\lambda}\sum_{n=3}^{\infty} 5^4 n^3 a_n\lambda^n \\
&= -5\lambda\frac{d}{d\lambda}(\lambda\frac{d}{d\lambda}\sum_{n=2}^{\infty} 5^4 n^2 a_n\lambda^n) \\
&= -5\mu^4\sum_{n=0}^{\infty} 5^4 a_n\lambda^n \\
-5\sum_{n=3}^{\infty} 1250n^3 a_n\lambda^n &= -5\mu^3\sum_{n=0}^{\infty} 1250 a_n\lambda^n \\
-5\sum_{n=2}^{\infty} 875n^2 a_n\lambda^n &= -5\mu^2\sum_{n=0}^{\infty} 875 a_n\lambda^n \\
-5\sum_{n=1}^{\infty} 250n a_n\lambda^n &= -5\mu\sum_{n=0}^{\infty} 250 a_n\lambda^n \\
-5\sum_{n=0}^{\infty} 24 a_n\lambda^n &= -5\sum_{n=0}^{\infty} 24 a_n\lambda^n
\end{aligned}$$

The above equivalences do not all start at the correct index; however, this is not an issue as $(n+1)^4 a_{n+1} - 5(5n+4)(5n+3)(5n+2)(5n+1)a_n = 0$ for all n, and so adding in additional terms would amount to adding zeroes. In order to write all derivatives in terms of μ multiply the equation through by λ; since the sum is zero this is fine; the final equation is then

$$\mu^4\sum_{n=0}^{\infty} a_{n+1}\lambda^{n+1} - 5\lambda(625\mu^4\sum_{n=0}^{\infty} a_n\lambda^n + 1250\mu^3\sum_{n=0}^{\infty} a_n\lambda^n + 875\mu^2\sum_{n=0}^{\infty} a_n\lambda^n + 250\mu\sum_{n=0}^{\infty} a_n\lambda^n + 24\sum_{n=0}^{\infty} a_n\lambda^n) = 0$$

. Thus, the differential operator becomes

$$\begin{aligned}
L &= \mu^4 - 5\lambda\prod_{j=1}^{4}(5\mu + j) \\
&= \mu^4 - 5\lambda(625\mu^4 + 1250\mu^3 + 875\mu^2 + 250\mu + 24)
\end{aligned}$$

and so $L\omega_i = 0$ for all periods ω_i.

Note then that, for some sort of function $f(\lambda)$,

$$\begin{aligned}
\mu^4 f &= (\mu^3)(\lambda\frac{df}{d\lambda}) \\
&= (\mu^2)((\lambda\frac{d}{d\lambda}\lambda)\frac{df}{d\lambda} + \lambda(\lambda\frac{d}{d\lambda}(\frac{df}{d\lambda}))) \\
&= (\mu^2)(\lambda\frac{df}{d\lambda} + \lambda^2\frac{d^2f}{d\lambda^2}) \\
&= (\mu)(\lambda\frac{df}{d\lambda} + 3\lambda^2\frac{d^2f}{d\lambda^2} + \lambda^3\frac{d^3f}{d\lambda^3}) \\
&= \lambda\frac{df}{d\lambda} + 7\lambda^2\frac{d^2f}{d\lambda^2} + 6\lambda^3\frac{d^3f}{d\lambda^3} + \lambda^4\frac{d^4f}{d\lambda^4}
\end{aligned}$$

and so,

$$\begin{aligned}
Lf(\lambda) = (\lambda\frac{df}{d\lambda} &+ 7\lambda^2\frac{d^2f}{d\lambda^2} + 6\lambda^3\frac{d^3f}{d\lambda^3} + \lambda^4\frac{d^4f}{d\lambda^4}) - 5\lambda(625(\lambda\frac{df}{d\lambda} + 7\lambda^2\frac{d^2f}{d\lambda^2} + 6\lambda^3\frac{d^3f}{d\lambda^3} + \lambda^4\frac{d^4f}{d\lambda^4}) \\
&+ 1250(\lambda\frac{df}{d\lambda} + 3\lambda^2\frac{d^2f}{d\lambda^2} + \lambda^3\frac{d^3f}{d\lambda^3}) + 875(\lambda\frac{df}{d\lambda} + \lambda^2\frac{d^2f}{d\lambda^2}) + 250(\lambda\frac{df}{d\lambda}) + 24f) \\
= (1 - 5^3 5!\lambda)\lambda\frac{df}{d\lambda} &+ (7 - 5^3 5!3\lambda)\lambda^2\frac{d^2f}{d\lambda^2} + (6 - 5^5 8\lambda)\lambda^3\frac{d^3f}{d\lambda^3} + (1 - 5^5\lambda)\lambda^4\frac{d^4f}{d\lambda^4} - 5!\lambda f
\end{aligned}$$

3.2.3 The Method of Fröbenius

The other periods are multi-valued and thus exhibit monodromy. We claim that the general form of the periods can be found using the method of Fröbenius. Indeed, since

$$Lf = (1-5^3 5!\lambda)\lambda\frac{df}{d\lambda} + (7-5^3 5!3\lambda)\lambda^2\frac{d^2f}{d\lambda^2} + (6-5^5 8\lambda)\lambda^3\frac{d^3f}{d\lambda^3} + (1-5^5\lambda)\lambda^4\frac{d^4f}{d\lambda^4} - 5!\lambda f$$

one can rewrite this as

$$Lf = \frac{d^4f}{d\lambda^4} + a(\lambda)\frac{d^3f}{d\lambda^3} + b(\lambda)\frac{d^2f}{d\lambda^2} + c(\lambda)\frac{df}{d\lambda} + d(\lambda)f$$

so that

$$\begin{aligned}
a(\lambda) &= \frac{6 - 5^5 8\lambda}{(1 - 5^5\lambda)\lambda} \\
b(\lambda) &= \frac{7 - 5^3 5!3\lambda}{(1 - 5^5\lambda)\lambda^2} \\
c(\lambda) &= \frac{1 - 5^3 5!\lambda}{(1 - 5^5\lambda)\lambda^3} \\
d(\lambda) &= \frac{-5}{(1 - 5^5\lambda)\lambda^3}
\end{aligned}$$

We wish to find a solution in a neighborhood of $\lambda = 0$; notice that $\lambda a(\lambda),\ \lambda^2 b(\lambda),\ \lambda^3 c(\lambda),\ \lambda^4 d(\lambda)$ are all holomorphic at $\lambda = 0$, and so using Fröbenius' method is valid.

Assume the solution to Lf is of the form

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^{n+r}, \text{ for some constant} r \in \mathbb{R}$$

Substituting this into the differential equation Lf one finds that

$$\begin{aligned}
Lf &= (1 - 5^5\lambda)\lambda^4 \sum_{n=0}^{\infty} (n+r)(n+r-1)(n+r-2)(n+r-3)a_n\lambda^{n+r-4} \\
&\quad + (6 - 5^5 8\lambda)\lambda^3 \sum_{n=0}^{\infty} (n+r)(n+r-1)(n+r-2)a_n\lambda^{n+r-3} \\
&\quad + (7 - 5^3 5!3\lambda)\lambda^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n\lambda^{n+r-2} + (1 - 5^3 5!\lambda)\lambda \sum_{n=0}^{\infty} (n+r)a_n\lambda^{n+r-1} - 5!\lambda \sum_{n=0}^{\infty} a_n\lambda^{n+r} \\
&= \sum_{n=0}^{\infty} \frac{(n+r)!}{(n+r-4)!}a_n\lambda^{n+r} - 5^5 \sum_{n=0}^{\infty} \frac{(n+r)!}{(n+r-4)!}a_n\lambda^{n+r+1} + 6\sum_{n=0}^{\infty} \frac{(n+r)!}{(n+r-3)!}a_n\lambda^{n+r} \\
&\quad - 5^5 8 \sum_{n=0}^{\infty} \frac{(n+r)!}{(n+r-3)!}a_n\lambda^{n+r+1} + 7\sum_{n=0}^{\infty} \frac{(n+r)!}{(n+r-2)!}a_n\lambda^{n+r} - 5^3 5!3 \sum_{n=0}^{\infty} \frac{(n+r)!}{(n+r-2)!}a_n\lambda^{n+r+1} \\
&\quad + \sum_{n=0}^{\infty} (n+r)a_n\lambda^{n+r} - 5^3 5! \sum_{n=0}^{\infty} (n+r)a_n\lambda^{n+r+1} - 5! \sum_{n=0}^{\infty} a_n\lambda^{n+r+1} \\
&= \sum_{n=0}^{\infty} \frac{(n+r)!}{(n+r-4)!}a_n\lambda^{n+r} + 6\sum_{n=0}^{\infty} \frac{(n+r)!}{(n+r-3)!}a_n\lambda^{n+r} + 7\sum_{n=0}^{\infty} \frac{(n+r)!}{(n+r-2)!}a_n\lambda^{n+r} \\
&\quad + \sum_{n=0}^{\infty} (n+r)a_n\lambda^{n+r} - 5\sum_{n=0}^{\infty} a_n \left(\prod_{i=1}^{4} (5(n+r)+i)\right)\lambda^{n+r+1} \\
&= \sum_{n=0}^{\infty} \frac{(n+r)!}{(n+r-4)!}a_n\lambda^{n+r} + 6\sum_{n=0}^{\infty} \frac{(n+r)!}{(n+r-3)!}a_n\lambda^{n+r} + 7\sum_{n=0}^{\infty} \frac{(n+r)!}{(n+r-2)!}a_n\lambda^{n+r} \\
&\quad + \sum_{n=0}^{\infty} (n+r)a_n\lambda^{n+r} - 5\sum_{n=1}^{\infty} a_{n-1} \left(\prod_{i=1}^{4} (5(n+r-1)+i)\right)\lambda^{n+r}
\end{aligned}$$

Assume $a_0 \neq 0$. Use the coefficient of the λ^r term, which must equal zero, to solve for r as follows:

$$\begin{aligned}
\frac{r!}{(r-4)!}a_0 + 6\frac{r!}{(r-3)!}a_0 + 7\frac{r!}{(r-2)!}a_0 + ra_0 = 0 \\
\implies r^4 = 0
\end{aligned}$$

Thus, there are four identical roots. Thus, it follows that the second solution to this differential equation should be of the form

$$\omega_1 = f_0(\lambda) \log \lambda + f_1(\lambda)$$

for some power series $f_1(\lambda)$.

The general form of all the periods are given as follows in [1]:

$$\begin{aligned}
\omega_0(\lambda) &= f_0(\lambda) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} \lambda^n \\
\omega_1(\lambda) &= f_0(\lambda) \log \lambda + f_1(\lambda) \\
\omega_2(\lambda) &= f_0(\lambda) \log^2 \lambda + 2f_1(\lambda) \log \lambda + f_2(\lambda) \\
\omega_3(\lambda) &= f_0(\lambda) \log^3 \lambda + 3f_1(\lambda) \log^2 \lambda + 3f_2(\lambda) \log \lambda + f_3(\lambda)
\end{aligned}$$

where each f_i is a power series.

3.2.4 The Second Period

Now we wish to determine ω_1. While this period must satisfy $L\omega_1 = 0$, applying L directly is laborious; instead, one can simplify matters by noting the following:

Proposition 3.2.4.1 ([4])

$$\mu^n (f(\lambda) \log(\lambda)) = (\mu^n f(\lambda)) \log \lambda + n\mu^{n-1} f(\lambda)$$

for $n \geqslant 1$.

Proof. One can use induction to prove this proposition.

Starting with the base case, let $n = 1$. Then, using the product rule,

$$\begin{aligned}
\mu(f(\lambda) \log \lambda) &= \lambda \frac{d}{d\lambda}(f(\lambda) \log \lambda) \\
&= \lambda \left(\frac{df}{d\lambda} \log \lambda + f(\lambda) \frac{d}{d\lambda} \log \lambda \right) \\
&= \mu f + f(\lambda)
\end{aligned}$$

as expected, thus proving the base case.

Now, suppose this holds for all $1 \leqslant n \leqslant k$ for some integer k, and consider the case when $n = k+1$. Then,

$$\begin{aligned}
\mu^{k+1}(f(\lambda)\log\lambda) &= \mu\mu^k(f(\lambda)\log\lambda) \\
&= \mu\left(\mu^k f(\lambda))\log\lambda + k\mu^{k-1}f(\lambda)\right) \text{ by the inductive hypothesis} \\
&= \lambda\left(\frac{d}{d\lambda}(\mu^k f(\lambda))\log\lambda) + k\frac{d}{d\lambda}(\mu^{k-1}f(\lambda))\right) \\
&= \lambda\frac{d}{d\lambda}(\mu^k f(\lambda))\log\lambda) + \mu^k f(\lambda) + \lambda k\frac{d}{d\lambda}(\mu^{k-1}f(\lambda)) \\
&= \mu^{k+1}f(\lambda))\log\lambda + \mu^k f(\lambda) + k\mu^k f(\lambda) \\
&= \mu^{k+1}f(\lambda))\log\lambda + (k+1)\mu^k f(\lambda)
\end{aligned}$$

and thus the statement hold for $n = k+1$. Thus, it holds for all $n \geqslant 1$ and thus the statement is proven. □

One can use this proposition to prove the following helpful lemma:

Lemma 3.2.4.1 ([4])
Let $G(\mu) = L$ so that $G(x) = x^4 - 5\prod_{i=1}^{4}(5x+i)$ is a polynomial. Then,

$$L\omega_1 = G'(\mu)f_0(\lambda) + Lf_1(\lambda)$$

Proof. Note that $G'(\mu) = 4\mu^3 - 5\lambda(2500\mu^3 + 3750\mu^2 + 1750\mu + 250)$. Then,

$$\begin{aligned}
L\omega_1 &= L(f_0(\lambda)\log\lambda + f_1(\lambda)) \\
&= L(f_0(\lambda)\log\lambda + L(f_1(\lambda)) \\
&= (\mu^4 - 5\lambda(625\mu^4 + 1250\mu^3 + 875\mu^2 + 250\mu + 24))(f_0(\lambda)\log\lambda) + L(f_1(\lambda)) \\
&= \mu^4(f_0(\lambda)\log\lambda) - 5\lambda(625\mu^4(f_0(\lambda)\log\lambda) + 1250\mu^3(f_0(\lambda)\log\lambda) + 875\mu^2(f_0(\lambda)\log\lambda) \\
&\quad + 250\mu(f_0(\lambda)\log\lambda) + 24(f_0(\lambda)\log\lambda)) + L(f_1(\lambda)) \\
&= Lf_1(\lambda) + \mu^4 f_0(\lambda))\log\lambda + 4\mu^3 f_0(\lambda) - 5\lambda(625(\mu^4 f_0(\lambda)\log\lambda + 4\mu^3 f_0(\lambda)) \\
&\quad + 1250(\mu^3 f_0(\lambda)\log\lambda + 3\mu^2 f_0(\lambda)) + 875(\mu^2 f_0(\lambda)\log\lambda + 2\mu f_0(\lambda)) + 250(\mu f_0(\lambda)\log\lambda + f_0(\lambda))) \\
&= Lf_1(\lambda) + (Lf_0(\lambda))\log\lambda + 4\mu^3 f_0(\lambda) - 5\lambda(2500\mu^3 + 3750\mu^2 + 1750\mu + 250)f_1(\lambda) \\
&= Lf_1(\lambda) + G'(\mu)f_0(\lambda) \text{ since } Lf_0(\lambda) = 0
\end{aligned}$$

□

Now, let $f_1(\lambda) = \sum_{n=0}^{\infty}\frac{f_1^{(n)}(0)}{n!}\lambda^n$ be the Taylor series expansion of f_1 about $\lambda = 0$. Using the fact that

$$\mu^k f_1(0) = \sum_{n=0}^{\infty}\frac{n^k f_1^{(n)}(0)}{n!}\lambda^n$$

we find that

$$
\begin{aligned}
Lf_1(\lambda) &= \mu^4 f_1 - 5\prod_{i=1}^{4}(5\mu + i)f_1 \\
&= \sum_{n=0}^{\infty} \frac{n^4 f_1^{(n)}(0)}{n!}\lambda^n - 5\sum_{n=0}^{\infty} \frac{\left(625n^4 + 1250n^3 + 875n^2 + 250n + 24\right) f_1^{(n)}(0)}{n!}\lambda^{n+1}
\end{aligned}
$$

Also, $G'(\mu) = 4\mu^3 - 5\lambda(2500\mu^3 + 3750\mu^2 + 1750\mu + 250)$ and so

$$
\begin{aligned}
G'(\mu)f_0(\lambda) &= \left(4\mu^3 - 5\lambda(2500\mu^3 + 3750\mu^2 + 1750\mu + 250)\right)\sum_{n=0}^{\infty}\frac{(5n)!}{(n!)^5}\lambda^n \\
&= 4\sum_{n=0}^{\infty}\frac{n^3(5n)!}{(n!)^5}\lambda^n - 5\sum_{n=0}^{\infty}\frac{(5n)!\left(2500n^3 + 3750n^2 + 1750n + 250\right)}{(n!)^5}\lambda^{n+1}
\end{aligned}
$$

Using the previous proposition, one can then determine f_1 as follows:

$$
\begin{aligned}
L\omega_1 &= G'(\mu)f_0(\lambda) + Lf_1(\lambda) = 0 \\
&= 4\sum_{n=0}^{\infty}\frac{n^3(5n)!}{(n!)^5}\lambda^n - 5\sum_{n=0}^{\infty}\frac{(5n)!\left(2500n^3 + 3750n^2 + 1750n + 250\right)}{(n!)^5}\lambda^{n+1} + \sum_{n=0}^{\infty}\frac{n^4 f_1^{(n)}(0)}{n!}\lambda^n \\
&\quad - \sum_{n=0}^{\infty}\frac{\left(3125n^4 + 6250n^3 + 4375n^2 + 1250n + 120\right) f_1^{(n)}(0)}{n!}\lambda^{n+1} \\
&= \sum_{n=1}^{\infty}\frac{4n^3(5n)! + (n!)^4 n^4 f_1^{(n)}(0)}{(n!)^5}\lambda^n - 5\sum_{n=1}^{\infty}\frac{(5(n-1))!\left(2500n^3 - 3750n^2 + 1750n - 250\right)}{((n-1)!)^5}\lambda^n \\
&\quad - \sum_{n=1}^{\infty}\frac{\left(3125n^4 - 6250n^3 + 4375n^2 - 1250n + 120\right) f_1^{(n-1)}(0)}{(n-1)!}\lambda^n
\end{aligned}
$$

Choosing $f_1(0) = 0$ and equating the coefficient with zero, we can calculate the first few values for $f_1^{(n)}(0)$:

$$
\begin{aligned}
f_1^1(0) &= 5 * (2500 - 3750 + 1750 - 250) - 4 * 120 = 770 \\
f_1^2(0) &= 1620450 \\
f_1^3(0) &= 7491358000
\end{aligned}
$$

and, in general, the value of $f_1^{(n)}(0)$ is:

$$\begin{aligned}
f_1^{(n)}(0) &= -\frac{n!}{n^4}\left(\frac{4n^3(5n)!}{(n!)^5} - 5\frac{(5n-5)!(2500n^3-3750n^2+1750n-250}{(n-1)!^5}\right) \\
&\quad + \frac{n!}{n^4}\left(\frac{\left(3125n^4-6250n^3+4375n^2-1250n+120\right)f_1^{(n-1)}(0)}{(n-1)!}\right) \\
&= \frac{-4(5n)!}{(n!)^4 n} + 5\frac{(5n-5)!(2500n^3-3750n^2+1750n-250}{(n-1)!^4 n^3} \\
&\quad + \frac{\left(3125n^4-6250n^3+4375n^2-1250n+120\right)f_1^{(n-1)}(0)}{(n-1)!n^3}
\end{aligned}$$

Now we wish to find a closed form for $f_1 = \sum_{n=0}^{\infty} a_n\lambda^n$ given the recurrence relation

$$\begin{aligned}
a_{n+1} &= \frac{-4(5(n+1))!}{((n+1)!)^4(n+1)} + 5\frac{(5n)!(2500n^3+3750n^2+1750n+250)}{(n!)^4(n+1)^3} \\
&\quad + \frac{\left(3125n^4+6250n^3+4375n^2+1250n+120\right)a_n}{(n+1)^3}
\end{aligned}$$

Note that this is a first order non-homogeneous recurrence relation, with variable coefficients. For the sake of brevity let $f_n = \frac{\left(3125n^4+6250n^3+4375n^2+1250n+120\right)}{(n+1)^3}$, $g_n = \frac{-4(5(n+1))!}{((n+1)!)^4(n+1)} + 5\frac{(5n)!(2500n^3+3750n^2+1750n+250}{(n!)^4(n+1)^3}$, and $A_n = \frac{a_n}{\prod_{k=0}^{n-1} f_k}$, which is valid as $f_n \neq 0$. Then, one can rewrite the relation as

$$\frac{a_{n+1}}{\prod_{k=0}^{n} f_k} - \frac{a_n}{\prod_{k=0}^{n-1} f_k} = \frac{g_n}{\prod_{k=0}^{n} f_k} = A_{n+1} - A_n$$

Next, sum over both sides of this relation:

$$\sum_{n=0}^{n-1}(A_{m+1} - A_m) = \sum_{m=0}^{n-1}\left(\frac{g_m}{\prod_{k=0}^{m} f_k}\right)$$

and notice that

$$\sum_{n=0}^{n-1}(A_{m+1} - A_m) = A_n - A_0$$

Thus, one can rearrange the equality to obtain the following:

$$a_n = \prod_{k=0}^{n-1} f_k\left(A_0 + \sum_{m=0}^{n-1}\left(\frac{g_m}{\prod_{k=0}^{m} f_k}\right)\right)$$

Now, note that $a_0 = f_1(0) = 0$ by assumption, so $A_0 = 0$. Simplifying the components of the equality above separately one obtains

$$\begin{aligned}\prod_{k=0}^{n-1} f_k &= \prod_{k=0}^{n-1} \frac{\left(3125k^4 + 6250k^3 + 4375k^2 + 1250k + 120\right)}{(k+1)^3} \\ &= \frac{(5n)!}{(n!)^4}\end{aligned}$$

$$\begin{aligned}\sum_{m=0}^{n-1} \left(\frac{g_m}{\prod_{k=0}^{m} f_k}\right) &= \sum_{m=0}^{n-1} \frac{((m+1)!)^4}{(5(m+1))!} \Big(\frac{-4(5(m+1))!}{((m+1)!)^4 (m+1)} \\ &\qquad + 5\frac{(5m)!(2500m^3 + 3750m^2 + 1750m + 250)}{(m!)^4 (m+1)^3}\Big) \\ &= \sum_{m=0}^{n-1} \left(\frac{-4}{(m+1)} + \frac{2500m^3 + 3750m^2 + 1750m + 250}{(5m+4)(5m+2)(5m+3)(5m+1)} \right) \\ &= \sum_{m=0}^{n-1} \left(\frac{-4}{m+1} + 5\left(\frac{1}{5m+4} + \frac{1}{5m+3} + \frac{1}{5m+2} + \frac{1}{5m+1} \right) \right) \\ &= 5H_{5n} - 5H_n\end{aligned}$$

where H_n is the n^{th} harmonic number, given as $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Then, notice that

$$\begin{aligned}5H_{5n} - 5H_n &= 5(\sum_{k=1}^{5n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k}) \\ &= 5 \sum_{k=n+1}^{5n} \frac{1}{k}\end{aligned}$$

Thus,

$$a_n = 5\frac{(5n)!}{(n!)^4} \sum_{k=n+1}^{5n} \frac{1}{k}$$

and so the geometric series for f_1 with the normalization $f_1(0) = 0$ is

$$f_1(\lambda) = 5 \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} \left(\sum_{k=n+1}^{5n} \frac{1}{k} \right) \lambda^n$$

4 Conclusion

There are many avenues left to explore for those interested in Calabi-Yau manifolds, mirror symmetry, or even simply the quintic threefold explored in the paper by Candelas et al. This report merely scratches the surface of the paper and topics in question; a natural continuation would be to explore Yukawa couplings in-depth and determine the canonical coordinates (and thus a q-series expansion) for it. In addition, one could continue by conducting a more in-depth exploration of the moduli space and how it relates to the manifold and its mirror, as was done in the paper.

Of course, mirror symmetry is far from being proven in the general case, however the quintic threefold explored in this report is a good starting point due to it being quite accessible and grounded in physics, thus aiding and motivating its study. The aim of this report was to understand this manifold and its mirror, and to calculate some of its periods.

While the manifold in question in this report is a very specific one, perhaps it offers some insights into the behavior of more general manifolds. Remaining within the category of submanifolds of $\mathbb{CP}^4$ with complex dimension 3, in order to align with the physical motivation of mirror symmetry, the paper by Morrison [6] looked at general Pichard-Fuchs equations for hypersurfaces. This may be a starting point for further study, by repeating many of the calculations present in this report for more general hypersurfaces. Understanding the behavior of the periods and the moduli space for such hypersurfaces may then aid in the study of Calabi-Yau manifolds in higher dimensions, which, of course, being a generalization of the three-dimensional case, would be a natural continuation of this study.

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