PÓLYA THEORY, MUSICAL SCALES, AND QUATRO

Griffin Paddock, Alex Klee, Jiancheng Lao, Fulong Gan

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1 Polya Theory Background

Pólya theory is a powerful method used to count distinct arrangements of a set, using the results found from Burnside's Lemma about the number of orbits of a group action. This theory was popularized by George Pólya, who applied this result to many counting problems, in particular to the enumeration of chemical compounds. Pólya theory is very useful because it gives a systematic and efficient way to count distinct arrangements of any sort of set, even with a huge size, while allowing for symmetrical transformations, simplifying computationally complex combinatorial problems.

Cycle Types

To fully understand Pólya theory, its helpful to break down the symmetries of a set using **cycle types**. A cycle type generalizes how elements are permuted under the actions of a permutation group. Consider the symmetric group S_3, which consists of all permutations of a set with three elements, for example $\{1,2,3\}$. The group S_3 includes the following six permutations on the set $\{1,2,3\}$:

e: (1)(2)(3) (no change)

(12): Swaps elements 1 and 2, resulting in (2)(1)(3)

(13): Swaps elements 1 and 3, resulting in (3)(2)(1)

(23): Swaps elements 2 and 3, resulting in (1)(3)(2)

(123): Rotates the elements, resulting in (2)(3)(1)

(132): Rotates in the opposite direction, resulting in (3)(1)(2)

The cycle type described as how many cycles of each legnth are present in each permutation. For example, in the group S_3, the identity permuation does not move anything, so it has 3 1-cycles, and thus its cycle type is $(1,1,1)$. For the rest of the group S_3:

(12), (13), and (23) all have one 2-cycles and one 1-cycle, so their cycle type is $(1,1,0)$

(123) and (132) are 3 cycles, so they both have cycle type $(0,0,1)$

Cycle Index Polynomial

The **cycle index polynomial** is a compact representation of these cycle types. For a group G with n elements, the cycle index $Z(G)$ is given by:

$$Z(G) = \frac{1}{|G|} \sum_{\sigma \in G} x_1^{c_1(\sigma)} x_2^{c_2(\sigma)} \cdots x_n^{c_n(\sigma)}$$

where $c_k(\sigma)$ is the number of fixed points of σ (the number of cycles of length k in the permutation σ), and $|G|$ is the order of the group. For the symmetric group S_3, using the cycle types computed above, the cycle index is computed as:

$$Z(S_3) = \frac{1}{6}\left(x_1^3 + 3x_2x_1 + 2x_3\right)$$

Unweighted Polya Theory

Now that we have the definition for the cycle index polynomial, we can introduce unweighted Polya theory. Unweighted Polya theory directly applies this cycle index polynomial and allows us to count different coloring's of a set.

Unweighted Polya Enumeration Theorem:
Let $G \curvearrowright X$ be a group action and Y a set with size $\#Y = N$. Plugging N into the cycle index Z_G of the group action gives the number of orbits in $\mathrm{Hom}(X, Y)$ (the set of all functions from $X \to Y$):

$$\#\mathrm{Hom}(X,Y)/G = Z_G(N,N,\ldots,N) = \frac{1}{\#G}\sum_{g\in G} N^{\sum_i c_i(g)}$$

Effectively, we have figured out the amount of different ways to "color" the set X under the action of G, using the N different "colors" from the set Y. A great example of this is the action of the dihedral group D_8, the symmetries of a square. Let $D_8 \curvearrowright [4]$ be a group action where $[4] = \{1,2,3,4\}$, which can be thought of as labels for the vertices of a square. Computing the cycle index of the permutation representation of this action and plugging in N as shown above will give the number of possible coloring's of the square with N colors, no matter how large or small N is.

Weighted Polya Theory

Weighted Polya Theorem is likewise a powerful tool to simplify combinatorial calculations, and this time each color is allowed to have a *weight*. Instead of simply counting how many arrangements are possible, the weighted version gives a more refined count, usually generating functions that can track how weights are distributed across different configurations. This is especially useful when we have a similar problem counting arrangements of a set, but different colors have different values (weights) associated with them.

To understand weighted Polya Theory, it is helpful to first define a few things. A **weighting** is a function $w : Y \to \mathbb{N}$, usually from the set of "colors" to the natural numbers. The weight of a function $f : X \to Y$ is simply the sum of all of the weights of the values of f. Consider the quotient $\#\mathrm{Hom}_{w-n}(X,Y)/G$, the number of different "colorings" f of a given weight $\overline{w}(f) = n$. To answer this question, weighted Polya Theory states that if $G \curvearrowright X$ and $w : Y \to \mathbb{N}$ is a weighting, and G and X are both finite, then we can construct a generating function to model the number of different colorings of each weight:

$$p(t) = \sum_{k\geq 0} p_k t^k = Z_G(s_w(t), s_w(t^2), \ldots, s_w(t^n)), \quad p_k = \#\mathrm{Hom}(X,Y)_{\overline{w}=k}(X,Y)/G.$$

Note that each function $s(t)$ is an ordinary generating function for the number of elements of Y that have a given weight:

$$s_w(t) = s(t) = \sum_{k\geq 0} \#w^{-1}(k)\, t^k$$

To make the calculation, the cycle index Z_G is evaluated at the generating function for the number of colors of a given weight, effectively creating a generating function $p(t)$ of the number of different colorings of each weight.

2 Musical Applications

2.1 Equivalent Scales

Western music operates on a 12 note system. In music, this note scale is generally labeled with the letters A-G (although German music includes H) and modifiers known as sharps and flats to make up tones in-between these natural tones. Mathematically, we can label this 12 note system using $0, 1, \ldots, 11$, which naturally gives rise to viewing the system mathematically as the quotient group $\frac{\mathbb{Z}}{12\mathbb{Z}}$ as the 13^{th} note would be equivalent to the 0^{th}. In the first part of this application, we describe how Polya's enumeration theory (or Polya theory in short) allows for determination of the number of unique scales. Much of this is an adaptation of [1].

To begin formally counting the number of unique scales possible, we must first quantify what makes a scale unique. In music, two scales are considered to be the same, even if their note position may differ, if there is a transposition that shifts the notes while keeping the intervals the same. Most musicians would refer to this colloquially as playing an octave up or down. Formally, a **transposition** of a scale is a uniform transposition that replaces each note x in a scale by $x + a(\mod 12)$, where a is constant. Since notes in a scale have an intuitive repeating order, we can represent a scale as a cyclic structure. In this graphical structure, a transposition corresponds to a rotation, as the ordering of notes must be kept the same. Figure 1, taken from [1], shows 3 arbitrary scales as a graph structure. S1 and S2 are equivalent due to being able to be rotated into one another.

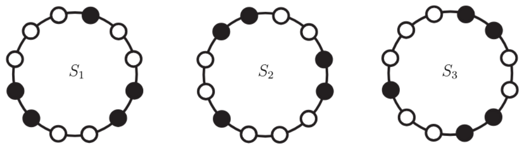

Figure 1: Adapted from [1], this figure shows the cyclic structure of scales as a graph. S1 and S2 are equivalent under rotation, while S3 is not, showing what makes up an "equivalent" scale.

To use Polya theory, we must decide the formal group acting on our scales as shown in 1. The subgroup of D_{24} consisting of all rotations is an excellent choice, as it a group that satisfies the structure restrictions imposed by the definition of a musical scale. We will denote this group as $T_{12} = \{\tau, \ldots, \tau^n : n \in [12]\}$, where each τ is a rotation. This could also be thought of as the cyclic group generated by $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$. The first 6 cycles of this group are shown below (excluding the identity) with the rest being calculated as the inverse:

$$
\begin{aligned}
&\tau : (0,1,2,3,4,5,6,7,8,9,10,11) \\
&\tau^2 : (0,2,4,6,8,10)(1,3,5,7,9,11) \\
&\tau^3 : (0,3,6,9)(2,5,8,11)(4,7,10,1) \\
&\tau^4 : (0,4,8)(2,6,10)(3,7,11)(1,5,9) \\
&\tau^5 : (0,5,10,3,8,1,6,11,4,9,2,7) \\
&\tau^6 (0,6)(5,11)(10,4)(3,9)(8,2)(1,7) \\
&\ldots
\end{aligned}
$$

From this, we can determine the cycle index for T_{12} as $\frac{1}{12}(x_1^{12} + x_2^6 + 2x_3^4 + 2x_4^3 + x_6^2 + 4x_12)$. At this point, we can now invoke the unweighted Polya Theory. We define $T_{12} \curvearrowright \hom(\mathbb{Z}/12\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$, and find that the number of possible unique scales is 352 by plugging in 2 to the cycle index of T_12.

However, this evaluation also considers 2, 3, etc. note scales, which is non-standard. Thus, we naturally wish to extend our counting to consider different numbers of notes in scales. Thus, we examine weighted Polya theory. Our weighting function is as follows. Let $Y = \mathbb{Z}/2\mathbb{Z}$. Then we define a weight $w : Y \to \mathbb{N}$ by sending black to weight 1 and white to weight 0. The weight is then the number of black points in 1. Since the number of black notes (i.e. notes in the scale) implies the number of white notes, we can construct our generating function of the weight only using black notes. Thus, we evaluate $T_12(1+b, 1+b^2, \ldots, 1+b^12)$. Plugging this into the cycle index and expanding out (a rather lengthy calculation not shown here) gives:

$$b^12 + b^11 + 6b^10 + 19b^9 + 43b^8 + 66b^7 + 80b^6 + 66b^5 + 43b^4 + 19b^3 + 6b^2 + b + 1 \tag{1}$$

The coefficients of this polynomial correspond to the number of unique scales of this type. Thus, for the more typical 7-note scale (i.e. C major), there are 66 possible unique variations.

3 Quatro Application

Calculation possibilities

The Quatro board game, also known as Quarto, is a strategy board game that was developed by Swiss mathematician Blaise Müller in 1980. The game is notable for its straightforward design and intricate gameplay, making it accessible and engaging for individuals of all ages and proficiency levels. Quatro is a two-player game played on a 4x4 grid with 16 distinct pieces, each with four dichotomous attributes: height (tall or short), color (white or black), shape (circular or square), and consistency (hollow or solid). The objective of the game is to be the first player to align four pieces that share at least one of the four attributes. As an additional strategic element, your opponent will select which pieces you are permitted to play. The gameplay of Quatro is comprised of two primary actions: the placement of pieces on the board and the selection of the next piece for one's opponent. This dual action requires players to engage in both defensive and offensive strategies, constantly planning to establish a winning line while preventing their opponents from doing so. The game concludes when a player forms a qualifying line of four pieces and wins, or when a tie is reached when all the pieces have been played and there is no winner.

The analysis of Quatro's strategic elements can be conducted using the **Polya Enumeration Theorem**, a mathematical method for calculating the number of possible configurations under specific conditions. This approach offers valuable insights into the complexity of the game and the strategic possibilities it presents. To illustrate, while a 4x4 board can theoretically have $2^{16} = 65{,}536$ positional possibilities when each square has two possible states (a piece exists or it does not), However, the symmetry of the board (rotations, flips) significantly reduces the number of positional choices. For instance, a first move that places a piece in any corner is essentially the same as one that places it in the center four spaces or the outer eight spaces. Consequently, there are only three distinct initial moves. This makes manual analysis challenging. However, utilising this theorem to examine the potential board states can significantly streamline the complexity of the game, offering a more nuanced strategic experience than mere combinatorial analysis.

Apply the Polya enumeration theorem

The Pólya enumeration theorem allows us to consider different board configurations and piece properties to compute the total number of unique game states in Quarto. This section is based off the theory described in [2]. To find the overall number of unique states, we will find two different quotients using Polya theory: $|Y^X/G|$ and $|X^P/H|$. The first will represent unique rotations of the board, while the second represents unique piece configurations.

Unique Board Configurations

First, we define the board as a 4×4 board containing 16 squares. We can define a map from the board state X to Y, where $Y = \{0, 1\}$, with 0 denoting no piece on the spot and 1 denoting a piece. To use Polya theory, we want to determine what the group G acting on the board is. The relevant symmetries of a 16x16 board form the dihedral group D_8 consisting of the identity, three rotations (90°, 180°, and 270°), and four reflections (horizontal, vertical, and two diagonal reflections).

By Polya's theorem, we know that:

$$|Y^X/G| = \frac{1}{|G|}\sum_{g\in G}|Y|^{c(g)}$$

While we know the size of G and Y, we must compute the number of cycles for each element $c(g)$. Note that this is functionally equivalent to the cycle type.

- **Identity** (e): The positions of all the pieces are kept constant, so $c(e) = 16$.
- **90° rotation**: The board is rotated 90 degrees clockwise. This transformation divides the board positions into four cycles, each containing four positions that rotate with each other, resulting in $c(r_{90}) = 4$.
- **180° rotation**: Rotates the board by 180 degrees. Each position is exchanged with its exact opposite, resulting in eight pairs of exchanged positions (eight two cycles), so $c(r_{180}) = 8$.
- **270° rotation**: Similar to 90° rotation, but in the opposite direction. Therefore, $c(r_{270}) = 4$.
- **Vertical reflection**: Reflects the board along the vertical axis. The positions on the left are exchanged with the corresponding positions on the right, resulting in $c(v) = 8$.
- **Horizontal reflection**: Reflects the board along the horizontal axis through the center. The positions in the top row are exchanged with the corresponding positions in the bottom row, so $c(h) = 8$.
- **Diagonal reflections**: Reflect the board along the two main diagonals. One diagonal is from the top-left to the bottom-right, and the other is from the top-right to the bottom-left. Each reflection results in six non-1 cycles with elements across the diaganol swapping for each. Thus, $c(d_1) = c(d_2) = 6$.

Using the Pólya enumeration theorem, we now compute the total number of unique configurations as:

$$\frac{1}{|G|}\sum_{g\in G}|Y|^{c(g)} = \frac{1}{8}\left(|Y|^{16} + 3|Y|^{8} + 2|Y|^{4} + 2|Y|^{6}\right)$$

Plugging in $|G| = 8$ and $|Y| = 2$ gives the number of unique board configurations as 8308. We therefore have finished the first qoutient.

Unique Piece Configurations

Similar to our representation of the board, we can represent each piece by a binary number, i.e. $p_0 \to 0000$.

Since all that matters are the *relationships* between pieces, and not the actual pieces themselves, by fixing an initial piece we can group pieces together by how many things in common they have with the piece. To explain this mathematically, let $p = 0000$, and let A_p, B_p, C_p, D_p be the sets of pieces with $3,2,1$ or 0 characteristics in common with P. Then:

$$\begin{aligned} A_p &= \{1000, 0100, 0010, 0001\} \\ B_p &= \{0011, 0101, 1001, 1010, 1100, 0110\} \\ C_p &= \{0111, 1011, 1110, 1101\} \\ D_p &= \{1111\} \end{aligned}$$

We have now developed an appropriate way to help us represent our pieces on the board! Now, let us map our pieces onto a board once more. We can represent this in matrix form as:

$$e = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{pmatrix}$$

Now, notice that there are "similarity bands" surrounding $p = 0000$ (the top left corner). By this, we mean that $1,2,3,4$ share 1 thing in common, $3,5,6,9,10$ share two things, etc. Although this is not discussed extensively here, this relationship is preserved under symmetries. There are four symmetry operations we can preform on this board. First R, S, which rotate or flip the entire board (matrix). However, we also have r, s which rotates one quadrant or swaps rows. Finally, we could even flip the bottom and top halves of our quadrant. One **could** now compute the cycle index by acting upon each of the permutations. However, this is outside the scope of this work, so we will conclude with setting up this computation as above for the reader. In summary, equivalent piece setups are one's that preserve the relationship of the pieces among each other under some rotation.

References

[1] Trotter W. Keller M. *Applied Combinatorics.* Self-Published, 2017.

[2] Steven Morse. Qautro, part 1 (theory), 2017.