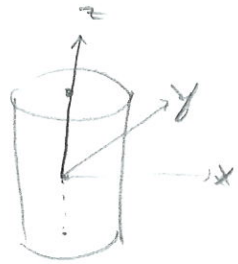
Ahmet Burak Koç
N20152984

CMP 755 Robotics
HW #4

1) $\hat{p} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \rightarrow -\hat{p}\hat{p} = \begin{bmatrix} z^2+y^2 & -xy & -xz \\ -xy & z^2+x^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}$

density $\rightarrow \rho(x,y,z) = \rho = \dfrac{m}{\pi R^2 H}$



$$J = \int_V \rho(x,y,z) \cdot (-\hat{p}\hat{p}) \cdot dx\,dy\,dz$$

$$= \int_{-\frac{H}{2}}^{H/2} \int_{-R}^{R} \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \frac{m}{\pi R^2 H} \begin{bmatrix} z^2+y^2 & -xy & -xz \\ -xy & z^2+x^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dx\,dy\,dz$$

All elements except I_{xx}, I_{yy} and I_{zz} contains odd order of x, y or z. These terms results in 0 after integration.

$I_{xx} = \dfrac{m}{\pi R^2 H} \int_V (z^2+y^2)\,dx\,dy\,dz$ which can be converted into polar coordinates as $I_{xx} = \dfrac{m}{\pi R^2 H} \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \int_{z=-H/2}^{H/2} (z^2 + r^2 \sin^2\theta) \cdot r\,dz\,dr\,d\theta$ where $y = r\sin\theta$

$$= \frac{m}{\pi R^2 H} \int_{\theta=0}^{2\pi}\int_{r=0}^{R} \left(\frac{H^3}{12} r + H r^3 \sin^2\theta\right) dr\,d\theta = \frac{m}{\pi R^2 H} \int_{\theta=0}^{2\pi} \left(\frac{H^3 R^2}{24} + \frac{H R^4 \sin^2\theta}{4}\right) d\theta$$

$$= \frac{m}{\pi R^2 H} \cdot \left(2\pi \cdot \frac{H^3 R^2}{24} + \pi \frac{H R^4}{4}\right) = m\left(\frac{H^2}{12} + \frac{R^2}{4}\right)$$

$I_{yy} = I_{xx}$ by symmetry.

$$I_{zz} = \frac{m}{\pi R^2 H} \int_V (x^2+y^2)\, dx\,dy\,dz$$ convert into polar coordinates.

$$= \frac{m}{\pi R^2 H} \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \int_{z=-H/2}^{H/2} r^2 \cdot r \, dz\, dr\, d\theta$$

$$= \frac{m}{\pi R^2 H} \underbrace{\int_{\theta=0}^{2\pi} d\theta}_{2\pi} \cdot \underbrace{\int_{r=0}^{R} r^3\, dr}_{\frac{R^4}{4}} \cdot \underbrace{\int_{z=-H/2}^{H/2} dz}_{H} = \frac{m}{\pi R^2 H} \cdot \frac{2\pi R^4 H}{4} = \frac{mR^2}{2}$$

$$I = \begin{bmatrix} \frac{mH^2}{12} + \frac{mR^2}{4} & 0 & 0 \\ 0 & \frac{mH^2}{12} + \frac{mR^2}{4} & 0 \\ 0 & 0 & \frac{mR^2}{2} \end{bmatrix}$$

where m, H, R are mass of cylinder, heigh of cylinder and radius of cylinder respectively.

2) 1 joint be in 24^3 different state due to 24 different values of position, velocity and acceleration of joint.

Since there is 3 joint, there is $(24^3)^3$ different combination of 3 joints' states.

Required number of memory location is 24^9 to store all states of dynamic equations in a look up table

3) $\theta_1(t) = bt + ct^2$

$\rightarrow \dot{\theta}_1(t) = b + 2ct$

$\Rightarrow \ddot{\theta}_1(t) = 2c$

$${}^1\dot{\omega}_1 = \cancel{{}^1_0R\,{}^0\dot{\omega}_0} + \cancel{{}^1_0R\,{}^0\omega_0 \times \dot{\theta}_1\,{}^1\hat{Z}_1} + \ddot{\theta}_1\,{}^1\hat{Z}_1 \quad \text{where} \quad \omega_0 = 0, \ \dot{\omega}_0 = 0$$

$$= \ddot{\theta}_1\,\hat{Z} = \begin{bmatrix} 0 \\ 0 \\ 2c \end{bmatrix}$$

$${}^1\dot{v}_{c_1} = {}^1\dot{\omega}_1 \times {}^1P_{c_1} + {}^1\omega_1 \times ({}^1\omega_1 \times {}^1P_{c_1}) + \cancel{{}^1\dot{v}_1} \quad \text{where} \quad \dot{v}_1 = 0 \ (\text{revolute joint})$$

and where,

$${}^1\omega_1 = \cancel{{}^1_0R\,{}^0\omega_0} + \dot{\theta}_1\,{}^1\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ b+2ct \end{bmatrix}$$

so,

$$\dot{v}_{c_1} = \begin{bmatrix} 0 \\ 0 \\ 2c \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b+2ct \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ 0 \\ b+2ct \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 \\ 4c \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b+2ct \end{bmatrix} \times \begin{bmatrix} 0 \\ 2(b+2ct) \\ 0 \end{bmatrix} = \begin{bmatrix} -2(b+2ct)^2 \\ 4c \\ 0 \end{bmatrix}$$

4) $\theta(0) = -5^\circ$

$\theta(4) = 80^\circ$

$\dot{\theta}(0) = 0$

$\dot{\theta}(4) = 0$

for cubic polynomial,

$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2$

previous equations are

$\theta(0) = a_0 = -5$

$\theta(4) = a_0 + 4a_1 + 16a_2 + 64a_3 = 80$

$\dot{\theta}(0) = a_1 = 0$

$\dot{\theta}(4) = a_1 + 8a_2 + 48a_3 = 0$

$\Rightarrow a_0 = -5 \quad , \quad a_1 = 0$

updated equations are

$16a_2 + 64a_3 = 85$

$8a_2 + 48a_3 = 0$

$\Rightarrow 16a_2 = -96a_3 \Rightarrow -32a_3 = 85 \Rightarrow a_3 = \frac{-85}{32}$

$a_2 = \frac{85 \times 3}{16}$

acceleration $\longrightarrow$ $\ddot{\theta}(t) = 2a_2 + 6a_3 t$

5)

$$\theta(t) = a_{10} + a_{11} t + a_{12} t^2 + a_{13} t^3$$

$$\theta(t) = a_{20} + a_{21} t + a_{22} t^2 + a_{23} t^3$$

$$\dot{\theta}(t) = a_{11} + 2a_{12} t + 3a_{13} t^2$$

$$\dot{\theta}(t) = a_{21} + 2a_{22} t + 3a_{23} t^2$$

$$\ddot{\theta}(t) = 2a_{12} + 6a_{13} t$$

$$\ddot{\theta}(t) = 2a_{22} + 6a_{23} t$$

segment 1 $\rightarrow$
$(t_f = 1)$
$(\theta_0 \rightarrow \theta_v)$

$$\theta_0 = a_{10}$$

$$\theta_v = a_{10} + a_{11} t + a_{12} t^2 + a_{13} t^3$$

$$\dot{\theta}_0 = 0$$

$$\dot{\theta}_v = a_{11} + 2a_{12} t + 3a_{13} t^2$$

$$\ddot{\theta}_v = 2a_{12} + 6a_{13} t$$

segment 2
$(t_f = 1)$
$(\theta_v \rightarrow \theta_g)$

$$\theta_v = a_{20}$$

$$\theta_g = a_{20} + a_{21} t + a_{22} t^2 + a_{23} t^3$$

$$\dot{\theta}_v = a_{21}$$

$$\dot{\theta}_g = a_{21} + 2a_{22} t + 3a_{23} t^2 = 0$$

$$\ddot{\theta}_v = 2a_{22}$$

Eight unknowns: $a_{10}, a_{11}, \ldots a_{23}$

Eight linear equations when $t_f = t_{f1} = t_{f2}$ taken;

$$\theta_0 = a_{10}$$

$$\theta_v = a_{10} + a_{11} t_f + a_{12} t_f^2 + a_{13} t_f^3$$

$$\theta_v = a_{20}$$

$$\theta_g = a_{20} + a_{21} t_f + a_{22} t_f^2 + a_{23} t_f^3$$

$$0 = a_{11}$$

$$0 = a_{21} + 2a_{22} t_f + 3a_{23} t_f^2$$

$$a_{21} = a_{11} + 2a_{12} t_f + 3a_{22} t_f^2$$

$$2a_{22} = 2a_{12} + 6a_{13} t_f$$

When position, velocity and accelerations are calculated from both sides of via point and solved using constraints that all properties should be equal at via point, following solutions are found. ($t_f = t_{f1} = t_{f2} = 1$)

$$a_{10} = \theta_0 = 5^\circ$$

$$a_{11} = 0$$

$$a_{12} = \frac{12\theta_v - 3\theta_g - 9\theta_0}{4t_f^2} = \frac{12\times 15 - 3\times 40 - 9\times 5}{4\times 1^2} = \frac{45}{4}$$

$$a_{13} = \frac{-8\theta_v + 3\theta_g + 5\theta_0}{4t_f^3} = \frac{-8\times 15 + 3\times 40 + 5\times 5}{4\times 1^3} = \frac{265}{4}$$

$$a_{20} = \theta_v = 15$$

$$a_{21} = \frac{3\theta_g - 3\theta_0}{4t_f} = \frac{3\times 40 - 3\times 5}{4\times 1} = \frac{105}{4}$$

$$a_{22} = \frac{-12\theta_v + 6\theta_g + 6\theta_0}{4t_f^2} = \frac{90}{4}$$

$$a_{23} = \frac{8\theta_v - 5\theta_g - 3\theta_0}{4t_f^3} = \frac{-95}{4}$$

6)

characteristic equation is

$$2s^2+6s+4=0$$

so

$$s^2+3s+2=0$$

$$(s+2).(s+1)=0$$

roots are $s_1=-2$ and $s_2=-1$

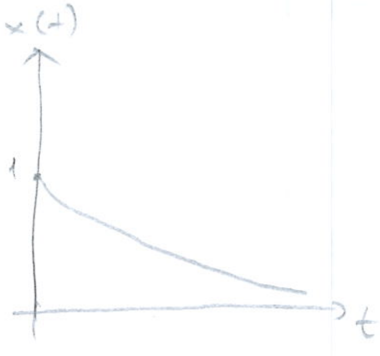
$\rightarrow x(t)= c_1 e^{-2t}+c_2 e^{-t}$

$x(0)=1$ and $\dot{x}(0)=0$ are given.

Then,

$$x(0)=c_1+c_2=1$$
$$\dot{x}(0)=-2c_1-c_2=0$$
$\Rightarrow$ $c_1=-1$, $c_2=2$

Then, $x(t)=-e^{-2t}+2e^{-t}$



7) At highest stiffness of the system, operation point is at critically damp because otherwise stiffness dominates and oscillation occurs.

$m\ddot{x} + b\dot{x}\ kx = f$ where $m=1$, $b=4$, $k=5$

$\ddot{x} + 4\dot{x} + 5x = f$

$f = -k_p x - k_v \dot{x}$ $\Rightarrow$ $\ddot{x} + (4+k_v)\dot{x} + (5+k_p)x = 0$

Laplace trans. $\Rightarrow$ $s^2 + (4+k_v)s + (5+k_p) = 0$

since we are looking for highest stiffness and $w_{res} = 6$. rad/s

$s^2 + (4+k_v)s + (5+k_p) = s^2 + 2\,w_{res}\,s + w_{res}^2 = s^2 + 12s + 36$

$\Rightarrow k_v = 12 - 4 = 8$

$k_p = 36 - 5 = 31$