Definition of BCH codes

BCH codes are cyclic codes over $\mathrm{GF}(q)$ (the *channel alphabet*) that are defined by a $(d-1) \times n$ check matrix over $\mathrm{GF}(q^m)$ (the *decoder alphabet*):

$$H = \begin{bmatrix} 1 & \alpha^b & \alpha^{2b} & \cdots & \alpha^{(n-1)b} \\ 1 & \alpha^{b+1} & \alpha^{2(b+1)} & \cdots & \alpha^{(n-1)(b+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{b+d-2} & \alpha^{2(b+d-2)} & \cdots & \alpha^{(n-1)(b+d-2)} \end{bmatrix}$$

Design parameters:

- α is an element of $\mathrm{GF}(q^m)$ of order n
- b is any integer ($0 \leq b < n$ is sufficient)
- d is an integer with $2 \leq d \leq n$ ($d = 1$ and $d = n + 1$ are trivial cases)

Rows of H are the first n powers of *consecutive* powers of α.

Special cases of BCH codes

A *primitive BCH code* is a BCH code defined using a primitive element α.

If α is a primitive element of $\mathrm{GF}(q^m)$, then the blocklength is $n = q^m - 1$.

This is the maximum possible blocklength for decoder alphabet $\mathrm{GF}(q^m)$.

A *narrow-sense BCH code* is a BCH code with $b = 1$.

Some decoding formulas simplify when $b = 1$. However, $b \neq 1$ is usually used.

The parity-check matrix for a t-error-correcting primitive narrow-sense BCH code is

$$\begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{(n-1)} \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{2t} & \alpha^{4t} & \cdots & \alpha^{2t(n-1)} \end{bmatrix},$$

where $n = q^m - 1$ and α is an n-th root of unity in $\mathrm{GF}(q^m)$.

Each row of H is a row of the finite field Fourier transform matrix of size n.
Codewords are n-tuples whose spectra have 0's at $2t$ consecutive frequencies.

Reed-Solomon codes

Reed-Solomon codes are BCH codes where decoder alphabet = channel alphabet.

Minimal polynomials over $\mathrm{GF}(Q)$ of elements of $\mathrm{GF}(Q)$ have degree 1.

Thus the generator polynomial of a t-error-correcting Reed-Solomon code is

$$\begin{aligned} g(x) &= (x-\alpha^b)(x-\alpha^{b+1})\cdots(x-\alpha^{b+2t-1}) \\ &= g_0 + g_1 x + \cdots + g_{2t-1}x^{2t-1} + x^{2t}\,, \end{aligned}$$

where $g_0, g_1 \ldots, g_{2t-1}$ are elements of $\mathrm{GF}(Q)$.

The minimum distance is $2t+1$, independent of the choice of α and b.

Usually α is chosen to be primitive in order to maximize blocklength.

The base exponent b can be chosen to reduce encoder/decoder complexity.

Reed-Solomon code: example

Audio compact discs and CD-ROMs use 2EC Reed-Solomon codes over $\mathrm{GF}(2^8)$.

The primitive polynomial used to define field arithmetic is $x^8 + x^4 + x^3 + x^2 + 1$.

The base exponent is $b = 0$.

The generator polynomial of the (255,251) Reed-Solomon code is

$$\begin{aligned} g(x) &= (x+1)(x+\alpha)(x+\alpha^2)(x+\alpha^3) \\ &= x^4 + \alpha^{75}x^3 + \alpha^{249}x^2 + \alpha^{78}x + \alpha^6 \\ &= x^4 + \mathtt{0f}x^3 + \mathtt{36}x^2 + \mathtt{78}x + \mathtt{40}\,. \end{aligned}$$

In the hexadecimal representations of the coefficients, the most significant bit is on the *left*; that is, $1 = \mathtt{01}$, $\alpha = \mathtt{02}$, $\alpha^2 = \mathtt{04}$, and so on.

There are no prime trinomials of degree 8; in fact, there are no prime trinomials of degree $8m$ for any m.

(15,7,5) BCH code: parity-check matrix

Parity-check matrix of $(15, 7, 5)$ binary BCH code:

$$H = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \cdots & \alpha^{14} \\ 1 & \alpha^3 & \alpha^6 & \alpha^9 & \cdots & \alpha^{42} \end{bmatrix} = \left[\begin{array}{cccccccccccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

Since rows of H are linearly independent, there are 2^8 syndromes. There are

$$1 + \binom{15}{1} + \binom{15}{2} = 121 < 2^7 < 2^8$$

error patterns of weight 2. This code does *not* achieve the Hamming bound.

A systematic parity-check matrix can be found using the generator polynomial.

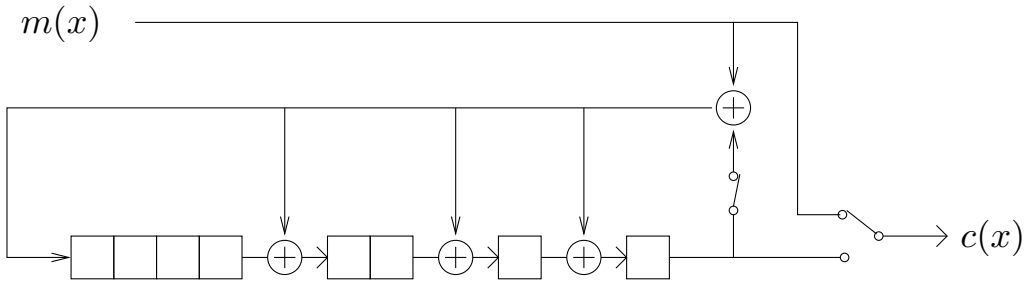
There is no (15,8) binary linear block code with minimum distance 5.

(15,7,5) BCH code: generator polynomial

Generator polynomial is LCM of minimal polynomials of α, α^2, α^3, α^4 :

$$\begin{aligned} g(x) = f_1(x) f_3(x) &= (x^4 + x + 1)(x^4 + x^3 + x^2 + x + 1) \\ &= x^8 + x^7 + x^6 + x^4 + 1 = 1 + x^4 + x^6 + x^7 + x^8 \end{aligned}$$

BCH codes are cyclic, hence have shift register encoders and syndrome circuits:



Modified error trapping can be used for $(15, 7, 5)$ binary BCH code.

Any 2-bit error pattern can be rotated into the 8 check positions. However, two error trapping passes may be needed.

(15,5,7) BCH code

The generator polynomial of a 3EC BCH code is defined by zeroes α, α^3, α^5:

$$g(x) = f_1(x)f_3(x)f_5(x) = (x^4+x+1)(x^4+x^3+x^2+x+1)(x^2+x+1)$$
$$= x^{10}+x^8+x^5+x^4+x^2+x+1$$

Parity-check matrix is 3×15 over $\mathrm{GF}(2^4)$ or 12×15 over $\mathrm{GF}(2)$:

$$H = \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{14} \\ 1 & \alpha^3 & \alpha^6 & \cdots & \alpha^{42} \\ 1 & \alpha^5 & \alpha^{10} & \cdots & \alpha^{70} \end{bmatrix} = \left[\begin{array}{cccccccccccccc} 1&0&0&0&1&0&0&1&1&0&1&0&1&1&1 \\ 0&1&0&0&1&1&0&1&0&1&1&1&1&0&0 \\ 0&0&1&0&0&1&1&0&1&0&1&1&1&1&0 \\ 0&0&0&1&0&0&1&1&0&1&0&1&1&1&1 \\ \hline 1&0&0&0&1&1&0&0&0&1&1&0&0&0&1 \\ 0&0&0&1&1&0&0&0&1&1&0&0&0&1&1 \\ 0&0&1&0&1&0&0&1&0&1&0&0&1&0&1 \\ 0&1&1&1&1&0&1&1&1&1&0&1&1&1&1 \\ \hline 1&0&1&1&0&1&1&0&1&1&0&1&1&0&1 \\ 0&1&1&0&1&1&0&1&1&0&1&1&0&1&1 \\ 0&1&1&0&1&1&0&1&1&0&1&1&0&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right]$$

The last two rows are linearly redundant $\Rightarrow$ 10 check equations $\Rightarrow k = 5$.

(15,5,7) BCH code: redundant rows

H has two redundant rows:

- bottom row is zero
- next to last row is same as previous row

H has 10 independent rows and defines a $(15, 5)$ binary cyclic code.

Parity-check polynomial $h(x) = (x^4+x^3+1)(x+1)$ includes all the prime divisors of $x^{15}-1$ that are *not* included in $g(x)$.

The dual of this BCH code is a $(15, 10)$ expurgated Hamming code with $d^* = 4$.

The $(15, 5)$ BCH code is obtained from the $(15, 4)$ maximum-length code by augmentation — including the complements of the original codewords.

The weight enumerator is $A(x) = 1 + 15x^7 + 15x^8 + x^{15}$.

(31,16,7) BCH code

Zeroes of codewords are α, α^3, α^5 in $\mathrm{GF}(2^5)$. Parity-check matrix:

$$H = \left[\begin{array}{ccccccccccccccccccccccccccccc}
1&0&0&0&0&1&0&0&1&0&1&1&0&0&1&1&1&1&1&0&0&0&1&1&0&1&1&1&0&1&0\\
0&1&0&0&0&0&1&0&0&1&0&1&1&0&0&1&1&1&1&1&0&0&0&1&1&0&1&1&1&0&1\\
0&0&1&0&0&1&0&1&1&0&0&1&1&1&1&1&0&0&0&1&1&0&1&1&1&0&1&0&1&0&0\\
0&0&0&1&0&0&1&0&1&1&0&0&1&1&1&1&1&0&0&0&1&1&0&1&1&1&0&1&0&1&0\\
0&0&0&0&1&0&0&1&0&1&1&0&0&1&1&1&1&1&0&0&0&1&1&0&1&1&1&0&1&0&1\\
\hline
1&0&0&0&0&1&1&0&0&1&0&0&1&1&1&1&1&0&1&1&1&0&0&0&1&0&1&0&1&1&0\\
0&0&1&1&1&1&1&0&1&1&1&0&0&0&1&0&1&0&1&1&0&1&0&0&0&0&1&1&0&0&1\\
0&0&0&0&1&1&0&0&1&0&0&1&1&1&1&1&0&1&1&1&0&0&0&1&0&1&0&1&1&0&1\\
0&1&1&1&1&1&0&1&1&1&0&0&0&1&0&1&0&1&1&0&1&0&0&0&0&1&1&0&0&1&0\\
0&0&0&1&0&1&0&1&1&0&1&0&0&0&0&1&1&0&0&1&0&0&1&1&1&1&1&0&1&1&1\\
\hline
1&1&1&1&0&1&0&0&0&1&0&0&1&0&1&0&1&1&0&0&0&0&1&1&1&0&0&1&1&0&1\\
0&0&0&1&0&0&1&0&1&0&1&1&0&0&0&0&1&1&1&0&0&1&1&0&1&1&1&1&1&0&1\\
0&1&0&1&1&0&0&0&0&1&1&1&0&0&1&1&0&1&1&1&1&1&0&1&0&0&0&1&0&0&1\\
0&0&0&1&1&1&0&0&1&1&0&1&1&1&1&1&0&1&0&0&0&1&0&0&1&0&1&0&1&1&0\\
0&0&1&1&0&1&1&1&1&1&0&1&0&0&0&1&0&0&1&0&1&0&1&1&0&0&0&0&1&1&1
\end{array}\right]$$

Generator polynomial: $x^{15} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^5 + x^3 + x^2 + x + 1$.

It is *not* obvious that every set of 6 columns of H is linearly independent!

For blocklength 31, all binary BCH codes with $d^* = 7$ have 15 check bits.

The expanded code with $(n, k, d^*) = (32, 16, 8)$ is a Reed-Muller code.

BCH codes with decoder alphabet GF(16)

Suppose that α is primitive in $\mathrm{GF}(16)$.

The following parity-check matrix defines a primitive, narrow-sense BCH code over each channel alphabet that is a subfield of $\mathrm{GF}(16)$.

$$H = \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{14} \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{28} \\ 1 & \alpha^3 & \alpha^6 & \cdots & \alpha^{42} \\ 1 & \alpha^4 & \alpha^8 & \cdots & \alpha^{56} \end{bmatrix}$$

The three possible channel alphabets are $\mathrm{GF}(2)$, $\mathrm{GF}(2^2)$, and $\mathrm{GF}(2^4)$:

The BCH codes corresponding to these channels alphabets are

- (15,7) binary BCH code over $\mathrm{GF}(2)$ (presented earlier in lecture)
- (15,9) BCH code over $\mathrm{GF}(4)$
- (15,11) Reed-Solomon code over $\mathrm{GF}(16)$

The blocklengths *in symbols* are 15; blocklengths *in bits* are 15, 30, and 60.

Channel alphabet GF(16)

The four rows of H are linearly independent over $\mathrm{GF}(2^4)$ (BCH bound, later).

Thus H defines a $(15, 11)$ code over $\mathrm{GF}(2^4)$ with minimum distance 5.

This code is a $(15, 11)$ 2EC *Reed-Solomon code* over $\mathrm{GF}(16)$.

Using table of powers of α (Blahut p. 86), we can find generator polynomial:

$$\begin{aligned} g(x) &= (x+\alpha)(x+\alpha^2)(x+\alpha^3)(x+\alpha^4) \\ &= \alpha^{10} + \alpha^3 x + \alpha^6 x^2 + \alpha^{13} x^3 + x^4 \;=\; \mathtt{7} + \mathtt{8}\,x + \mathtt{E}\,x^2 + \mathtt{D}\,x^3 + x^4 \end{aligned}$$

Coefficients of generator polynomial are computed using $\mathrm{GF}(2^4)$ arithmetic.

Coefficients can be expressed either in exponential or binary representation.

- exponential notation simplifies multiplication (add exponents mod 15)
- binary notation simplifies addition (exclusive-or of 4-bit values)

Hardware implementations use bit vectors and often log/antilog tables.

Channel alphabet GF(4)

Let $\mathrm{GF}(4)$ be $\{0, 1, \beta, \delta\}$, where $\delta = \beta + 1 = \beta^2$.

$\mathrm{GF}(16)$ consists of 2-tuples over $\mathrm{GF}(4)$ using primitive polynomial $x^2 + x + \beta$.

H defines a BCH code over subfield $\mathrm{GF}(4)$.

$$H_{[2^2]} = \left[\begin{array}{cccccccccccccc} 1 & 0 & \beta & \beta & 1 & \beta & 0 & \delta & \delta & \beta & \delta & 0 & 1 & 1 & \delta \\ 0 & 1 & 1 & \delta & 1 & 0 & \beta & \beta & 1 & \beta & 0 & \delta & \delta & \beta & \delta \\ \hline 1 & \beta & 1 & 0 & \delta & \delta & 1 & \delta & 0 & \beta & \beta & \delta & \beta & 0 & 1 \\ 0 & 1 & 1 & \beta & 1 & 0 & \delta & \delta & 1 & \delta & 0 & \beta & \beta & \delta & \beta \\ \hline 1 & \beta & 0 & \beta & 1 & 1 & \beta & 0 & \beta & 1 & 1 & \beta & 0 & \beta & 1 \\ 0 & \delta & \beta & \beta & \delta & 0 & \delta & \beta & \beta & \delta & 0 & \delta & \beta & \beta & \delta \\ \hline 1 & 1 & \delta & 1 & 0 & \beta & \beta & 1 & \beta & 0 & \delta & \delta & \beta & \delta & 0 \\ 0 & 1 & 1 & \delta & 1 & 0 & \beta & \beta & 1 & \beta & 0 & \delta & \delta & \beta & \delta \end{array}\right]$$

The final two rows, corresponding to conjugate α^4 over $\mathrm{GF}(4)$, are redundant. (Row 7 equals row 1 + row 2, while row 8 equals row 2).

Thus $g(x) = f_1(x) f_2(x) f_3(x)$ has degree $6 \;\Rightarrow\; (15, 9, 5)$ code over $\mathrm{GF}(4)$.

Channel alphabet GF(2)

This 16×15 matrix has redundant rows over $\mathrm{GF}(2)$. Every row in the second and fourth blocks is a linear combination of the first four rows.

$$H_{[2^1]} = \left[\begin{array}{ccccccccccccccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
\hline
1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1
\end{array}\right]$$

When we delete the redundant rows of H, we obtain the parity-check matrix of the $(15, 7)$ 2EC BCH code over $\mathrm{GF}(2)$ shown earlier.

Vandermonde matrix

Definition: The $\mu \times \mu$ Vandermonde matrix $V(X_1, \ldots, X_\mu)$ is

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_1 & X_2 & \cdots & X_\mu \\ \vdots & \vdots & \ddots & \vdots \\ X_1^{\mu-1} & X_2^{\mu-1} & \cdots & X_\mu^{\mu-1} \end{bmatrix}.$$

One application of Vandermonde matrices is for polynomial interpolation.

Given values of $f(x)$ of degree $\mu - 1$ at μ distinct points $X_1, \ldots, X_\mu$,

$$Y_i = f(X_i) = f_0 + f_1 X_i + \cdots + f_{\mu-1} X^{\mu-1} \quad (i = 1, \ldots, \mu)$$

the coefficients of $f(x)$ can be found by solving the matrix equation

$$\begin{bmatrix} Y_1 & Y_2 & \ldots & Y_\mu \end{bmatrix} = \begin{bmatrix} f_0 & f_1 & \ldots & f_{\mu-1} \end{bmatrix} V(X_1, \ldots, X_\mu).$$

Nonsingular Vandermonde matrix

Lemma: The Vandermonde matrix $V(X_1, \ldots, X_\mu)$ is nonsingular if and only if the parameters $X_1, \ldots, X_\mu$ are distinct. In fact,

$$\det V(X_1, \ldots, X_\mu) = \prod_{i>j}(X_i - X_j) = \prod_{i=1}^{\mu}\prod_{j=1}^{i-1}(X_i - X_j).$$

Proof: The determinant is a polynomial in μ variables, $X_1, \ldots, X_\mu$.

As a polynomial in X_i, its zeroes are X_j for $j \neq i$.

Thus $X_i - X_j$ is a factor of the determinant for every pair (i, j) with $i > j$.

These are *all* the factors because the degree of $\det V(X_1, \ldots, X_\mu)$ is

$$0 + 1 + \cdots + (\mu - 2) + (\mu - 1) = \frac{\mu(\mu-1)}{2} = \binom{\mu}{2}.$$

The coefficient of the main diagonal monomial

$$\prod_{i=1}^{\mu} X_i^{i-1} = 1 \cdot X_2 \cdot X_3^2 \cdot \ldots \cdot X_\mu^{\mu-1}$$

equals 1 in both the determinant and the above formula for the determinant.

BCH bound

Theorem: A BCH code whose parity-check matrix has $d-1$ rows has $d_{\min} \geq d$.

Proof: Every set of $d-1$ columns of H is linearly independent over $\mathrm{GF}(q^m)$.

To see this, consider a submatrix consisting of columns $i_1, \ldots, i_{d-1}$.

$$\det \begin{bmatrix} \alpha^{i_1 b} & \cdots & \alpha^{i_{d-1} b} \\ \alpha^{i_1 (b+1)} & \cdots & \alpha^{i_{d-1}(b+1)} \\ \vdots & \ddots & \vdots \\ \alpha^{i_1(b+d-2)} & \cdots & \alpha^{i_{d-1}(b+d-2)} \end{bmatrix} =$$

$$(\alpha^{i_1 b} \cdots \alpha^{i_{d-1} b}) \det \begin{bmatrix} 1 & \cdots & 1 \\ \alpha^{i_1} & \cdots & \alpha^{i_{d-1}} \\ \vdots & \ddots & \vdots \\ \alpha^{(d-2)i_1} & \cdots & \alpha^{(d-2)i_{d-1}} \end{bmatrix} \neq 0$$

This determinant is nonzero because $\alpha^{i_1} \neq 0, \ldots, \alpha^{i_{d-1}} \neq 0$ and the second matrix is a *Vandermonde* matrix with distinct columns.

Design of BCH codes

Codewords of BCH code have zeroes that are $d-1$ consecutive powers of α.

Conjugates over channel alphabet $\mathrm{GF}(q)$ are also zeroes.

The degree of the generator polynomial is the total number of conjugates.

Example: Channel alphabet $\mathrm{GF}(2)$, decoder alphabet $\mathrm{GF}(2^6)$.

The first six conjugacy classes, represented by exponents:

$$\{0\} \quad \{1,2,4,8,16,32\} \quad \{3,6,12,24,48,33\}$$
$$\{5,10,20,40,17,34\} \quad \{7,14,28,56,49,35\} \quad \{9,18,36\}$$

- $d^* = 5$ requires 4 powers. Exponents $\{1,2,3,4\} \Rightarrow 12$ conjugates.
- $d^* = 9$ requires 8 powers. Exponents $\{1,\ldots,8\} \Rightarrow 24$ conjugates.
- $d^* = 11$ requires 10 powers. Exponents $\{1,\ldots,10\} \Rightarrow 27$ conjugates.
- $d^* = 4$ requires 3 powers.
  - Exponents $\{1,2,3\} \Rightarrow 12$ conjugates.
  - Better: $\{0,1,2\} \Rightarrow 7$ conjugates (expurgated code)

GF(256): powers of primitive element

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	01	02	04	08	10	20	40	80	1D	3A	74	E8	CD	87	13	26
16	4C	98	2D	5A	B4	75	EA	C9	8F	03	06	0C	18	30	60	C0
32	9D	27	4E	9C	25	4A	94	35	6A	D4	B5	77	EE	C1	9F	23
48	46	8C	05	0A	14	28	50	A0	5D	BA	69	D2	B9	6F	DE	A1
64	5F	BE	61	C2	99	2F	5E	BC	65	CA	89	0F	1E	3C	78	F0
80	FD	E7	D3	BB	6B	D6	B1	7F	FE	E1	DF	A3	5B	B6	71	E2
96	D9	AF	43	86	11	22	44	88	0D	1A	34	68	D0	BD	67	CE
112	81	1F	3E	7C	F8	ED	C7	93	3B	76	EC	C5	97	33	66	CC
128	85	17	2E	5C	B8	6D	DA	A9	4F	9E	21	42	84	15	2A	54
144	A8	4D	9A	29	52	A4	55	AA	49	92	39	72	E4	D5	B7	73
160	E6	D1	BF	63	C6	91	3F	7E	FC	E5	D7	B3	7B	F6	F1	FF
176	E3	DB	AB	4B	96	31	62	C4	95	37	6E	DC	A5	57	AE	41
192	82	19	32	64	C8	8D	07	0E	1C	38	70	E0	DD	A7	53	A6
208	51	A2	59	B2	79	F2	F9	EF	C3	9B	2B	56	AC	45	8A	09
224	12	24	48	90	3D	7A	F4	F5	F7	F3	FB	EB	CB	8B	0B	16
240	2C	58	B0	7D	FA	E9	CF	83	1B	36	6C	D8	AD	47	8E	01

Decoder alphabet GF(256)

Narrow-sense primitive 2EC BCH codes over $\mathrm{GF}(2)$, $\mathrm{GF}(2^2)$, $\mathrm{GF}(2^4)$, $\mathrm{GF}(2^8)$ can be defined by the same parity-check matrix:

$$H = \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{254} \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{508} \\ 1 & \alpha^3 & \alpha^6 & \cdots & \alpha^{752} \\ 1 & \alpha^4 & \alpha^8 & \cdots & \alpha^{1016} \end{bmatrix}$$

Generator polynomials $\mathrm{LCM}(f_1(x), \ldots, f_4(x))$ have coefficients from subfields.

$$\mathrm{GF}(2^2) = \{0, 1, \alpha^{85}, \alpha^{170}\} = \{\texttt{00}, \texttt{01}, \texttt{D6}, \texttt{D7}\}$$

$$\mathrm{GF}(2^4) = \{0, 1, \alpha^{17}, \ldots, \alpha^{238}\} = \mathrm{span}\{\texttt{01}, \texttt{0B}, \texttt{98}, \texttt{D6}\}$$

Subfield	Degree	Polynomial coefficients
$\mathrm{GF}(2^8)$	4	`01 1E D8 E7 74`
$\mathrm{GF}(2^4)$	8	`01 D6 01 DD 0B 98 98 98 D7`
$\mathrm{GF}(2^2)$	12	`01 01 00 D7 00 00 00 D6 D7 D7 01 D7 01`
$\mathrm{GF}(2)$	16	`1 0 1 1 0 1 1 1 1 0 1 1 0 0 0 1 1`

Encoding and syndrome circuits

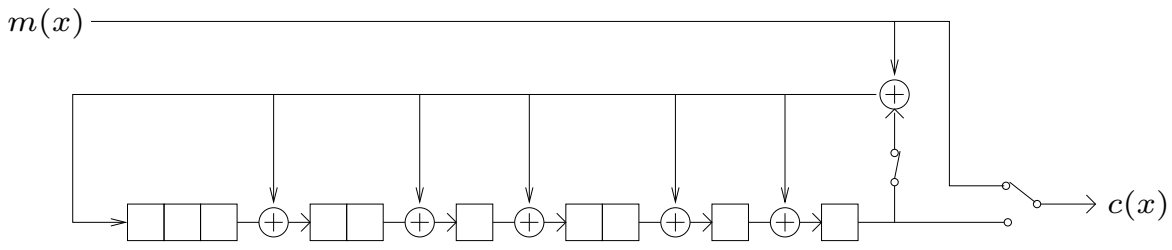
Binary BCH codes are defined using $\mathrm{GF}(2^m)$ but are still cyclic over $\mathrm{GF}(2)$.

Shift registers can be used for encoding and for syndrome computation.

The $(31, 21)$ binary primitive 2EC BCH code with generator polynomial

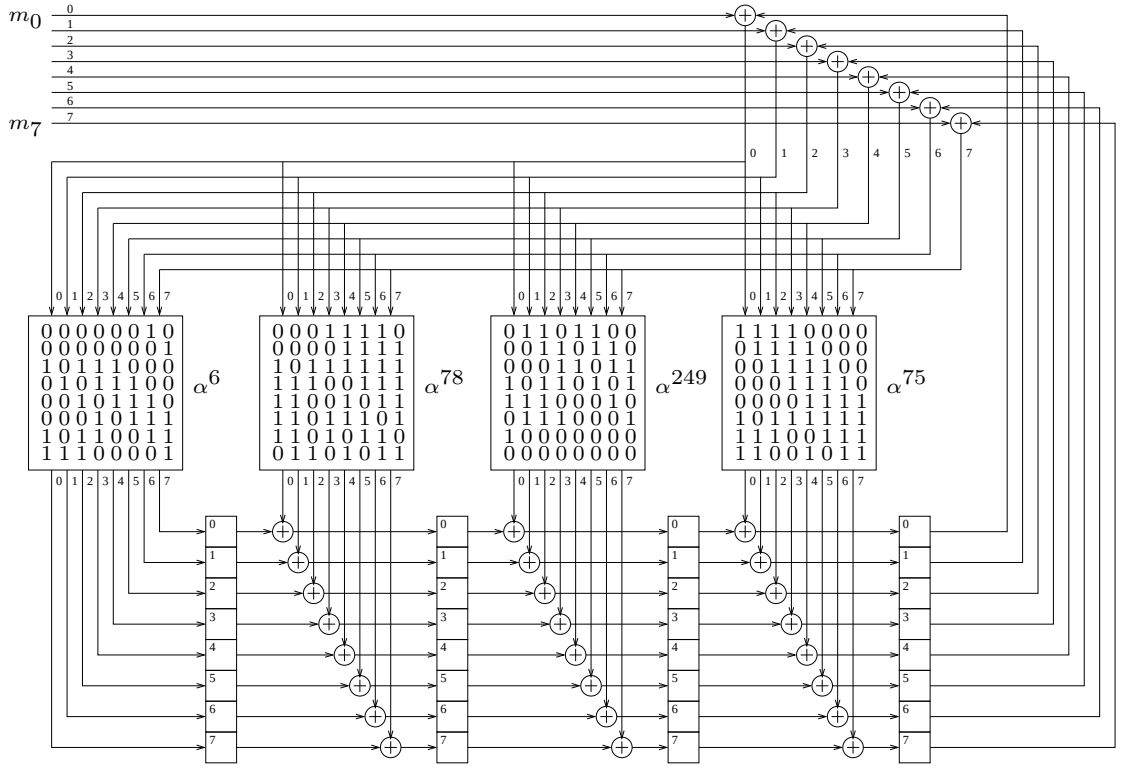
$$(x^5 + x^2 + 1)(x^5 + x^4 + x^3 + x^2 + 1) = 1 + x^3 + x^5 + x^6 + x^8 + x^9 + x^{10}$$

has the following shift register encoder.



Syndrome $s(x) \bmod g(x)$ used for error detection has a similar circuit.

Encoder for (255,251) Reed-Solomon code



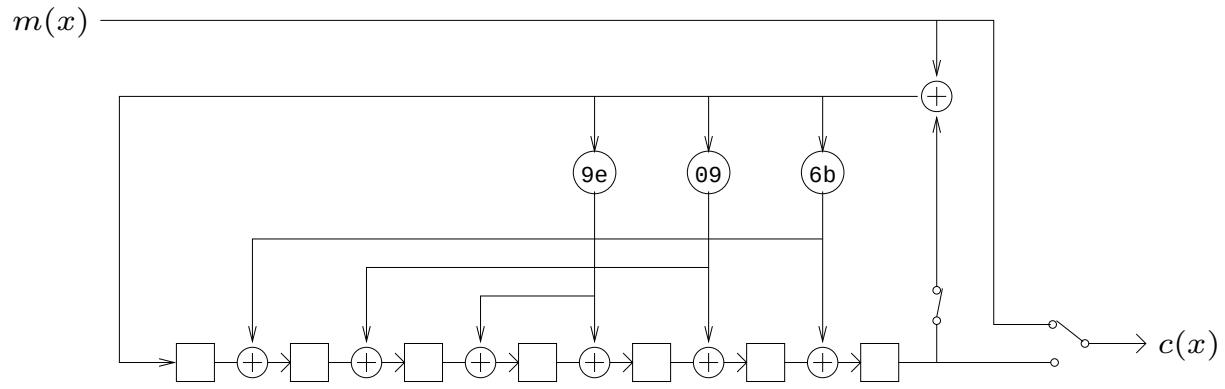
Reed-Solomon encoder

A Reed-Solomon code with $d^* = 8$ has the following generator polynomial:

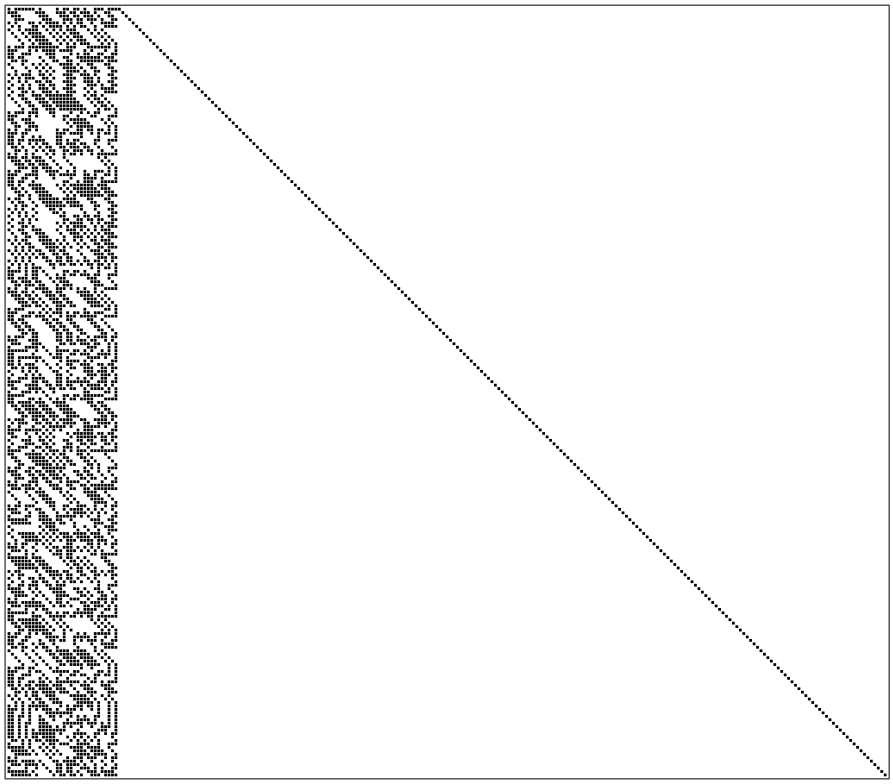
$$g(x) = (x + \alpha^{-3})(x + \alpha^{-2})(x + \alpha^{-1})(x + 1)(x + \alpha^{+1})(x + \alpha^{+2})(x + \alpha^{+3})$$
$$= x^7 + \mathsf{6b}x^6 + \mathsf{09}x^5 + \mathsf{9e}x^4 + \mathsf{9e}x^3 + \mathsf{09}x^2 + \mathsf{6b}x + 1$$

Since the reciprocals of its zeroes are also zeroes, $g(x)$ is its mirror image.

Thus the encoder corresponding to $g(x)$ has only 3 distinct scalers.



Generator matrix for (255,223) 4EC BCH code



Decoding algorithms for BCH codes

Decoding BCH and Reed-Solomon codes consists of the following major steps.

1. Compute partial syndromes $S_i = r(\alpha^i)$ for $i = b, \ldots, b+d-2$.
2. Find coefficients $\Lambda_1, \ldots, \Lambda_\nu$ of *error-locator polynomial* $\Lambda(x) = \prod_{i=1}^{\nu}(1 - xX_i)$ by solving *linear* equations with constant coefficients $S_b, \ldots, S_{b+d-2}$.
3. Find the zeroes $X_1^{-1}, \ldots, X_\nu^{-1}$ of $\Lambda(x)$. If there are ν symbol errors, they are in locations $i_1, \ldots, i_\nu$ where $X_1 = \alpha^{i_1}, \ldots, X_\nu = \alpha^{i_\nu}$.
4. Solve linear equations, whose "constant" coefficients are powers of X_i, for *error magnitudes* $Y_1, \ldots, Y_\nu$. (Not needed for channel alphabet $\mathrm{GF}(2)$.)

Efficient procedures for solving linear systems of equations in steps 2 and 4:

- Berlekamp, Massey (step 2)
- Forney (step 4)
- Sugiyama-Kasahara-Hirasawa-Namekawa ("Euclidean") (steps 2 and 4)

Error locations and magnitudes

Suppose there are $\nu \leq t$ errors in locations $i_1, \ldots, i_\nu$.

Let the error magnitudes be $e_{i_1}, \ldots, e_{i_\nu}$ (values in $\mathrm{GF}(q)$, *channel* alphabet).

The *error polynomial* is

$$e(x) = e_{i_1} x^{i_1} + \cdots + e_{i_\nu} x^{i_\nu} .$$

The senseword $r(x)$ can be written $r(x) = c(x) + e(x)$.

The partial syndromes are values in *decoder* alphabet $\mathrm{GF}(q^m)$:

$$\begin{aligned} S_j = r(\alpha^j) \; &= \; c(\alpha^j) + e(\alpha^j) \; = \; e(\alpha^j) \\ &= e_{i_1}\alpha^{j i_1} + \cdots + e_{i_\nu}\alpha^{j i_\nu} \; = \; e_{i_1}\alpha^{i_1 j} + \cdots + e_{i_\nu}\alpha^{i_\nu j} . \end{aligned}$$

Change of variables:

- *error locators*: $X_1 = \alpha^{i_1}, \ldots, X_\nu = \alpha^{i_\nu}$
- *error magnitudes*: $Y_1 = e_{i_1}, \ldots, Y_\nu = e_{i_\nu}$ (just renaming)

Syndrome equations

Error locators are elements of the *decoder* alphabet $\mathrm{GF}(q^m)$.

Error magnitudes are elements of the *channel* alphabet $\mathrm{GF}(q)$.

Important special case: $Y_i = 1$ for channel alphabet $\mathrm{GF}(2)$.

For today, assume narrow-sense BCH code ($b = 1$) with $d = 2t + 1$.

Partial syndromes are constants in system of $2t$ equations in 2ν unknowns:

$$\begin{aligned} S_1 \;&=\; Y_1 X_1 \;+\cdots+\; Y_\nu X_\nu \\ S_2 \;&=\; Y_1 X_1^2 \;+\cdots+\; Y_\nu X_\nu^2 \\ &\;\;\vdots \\ S_{2t} \;&=\; Y_1 X_1^{2t} +\cdots+\; Y_\nu X_\nu^{2t} \end{aligned}$$

This is an algebraic system of equations of degree $2t$.

Goal: reduce to one-variable polynomial equation with ν solutions.

Error-locator polynomial

The *error-locator polynomial* $\Lambda(x)$ is defined by

$$\begin{aligned}\Lambda(x) &= (1 - xX_1)(1 - xX_2)\cdots(1 - xX_\nu) \\ &= \prod_{i=1}^{\nu}(1 - xX_i) \\ &= \prod_{i=1}^{\nu}(-X_i) \cdot \prod_{i=1}^{\nu}(x - X_i^{-1}) \\ &= 1 + \Lambda_1 x + \cdots + \Lambda_\nu x^\nu \,.\end{aligned}$$

The zeroes of $\Lambda(x)$ are $X_1^{-1}, \ldots, X_\nu^{-1}$ — the *reciprocals* of error locators.

The degree of $\Lambda(x)$ is the number of errors. elegant!

The decoder must determine ν as well as the error locations.

The Peterson-Gorenstein-Zierler decoder can be used to find $\Lambda(x)$ from S_j.

PGZ is not efficient for large t but is easy to understand.

PGZ decoder example

Syndromes for 2EC narrow-sense BCH code with decoder alphabet $\mathrm{GF}(2^m)$:

$$S_j = Y_1 X_1^j + Y_2 X_2^j \,, \quad j = 1, \ldots, 4$$

Suppose two errors. Then zeroes of $\Lambda(x) = 1 + \Lambda_1 x + \Lambda_2 x^2$ are X_1^{-1}, X_2^{-1}.

$$0 = 1 + \Lambda_1 X_1^{-1} + \Lambda_2 X_1^{-2} \xrightarrow{\cdot\, Y_1 X_1^3} Y_1 X_1^3 + \Lambda_1 Y_1 X_1^2 + \Lambda_2 Y_1 X_1 = 0$$

$$0 = 1 + \Lambda_1 X_2^{-1} + \Lambda_2 X_2^{-2} \xrightarrow{\cdot\, Y_2 X_2^3} Y_2 X_2^3 + \Lambda_1 Y_2 X_2^2 + \Lambda_2 Y_2 X_2 = 0$$

$$(\underbrace{Y_1 X_1^3 + Y_2 X_2^3}_{S_3}) + \Lambda_1(\underbrace{Y_1 X_1^2 + Y_2 X_2^2}_{S_2}) + \Lambda_2(\underbrace{Y_1 X_1 + Y_2 X_2}_{S_1}) \;=\; 0$$

Similarly, multiplying by $Y_i X_i^4$ and summing gives another equation:

$$(\underbrace{Y_1 X_1^4 + Y_2 X_2^4}_{S_4}) + \Lambda_1(\underbrace{Y_1 X_1^3 + Y_2 X_2^3}_{S_3}) + \Lambda_2(\underbrace{Y_1 X_1^2 + Y_2 X_2^2}_{S_2}) \;=\; 0$$

We have obtained two linear equations in the unknowns Λ_1, Λ_2:

$$\begin{aligned} S_3 + S_2\Lambda_1 + S_1\Lambda_2 &= 0 \\ S_4 + S_3\Lambda_1 + S_3\Lambda_2 &= 0 \end{aligned} \;\Rightarrow\; \begin{bmatrix} S_1 & S_2 \\ S_2 & S_3 \end{bmatrix} \begin{bmatrix} \Lambda_2 \\ \Lambda_1 \end{bmatrix} = -\begin{bmatrix} S_3 \\ S_4 \end{bmatrix}.$$

PGZ decoder example (2)

The determinant of the coefficient matrix is:

$$S_1 S_3 - S_2^2 = Y_1 Y_2 (X_1 X_2^3 + X_1^3 X_2) = Y_1 Y_2 X_1 X_2 (X_1 + X_2)^2 \neq 0$$

because $Y_i \neq 0$, $X_i \neq 0$, and $X_1 \neq X_2$. So we can solve for Λ_1, Λ_2.

$$M_2 = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_3 \end{bmatrix} \Rightarrow M_2^{-1} = \Delta^{-1} \begin{bmatrix} S_3 & S_2 \\ S_2 & S_1 \end{bmatrix}$$

where $\Delta = \det M_2 = S_1 S_3 + S_2^2$. Coefficients of $\Lambda(x)$ are given by

$$\begin{bmatrix} \Lambda_2 \\ \Lambda_1 \end{bmatrix} = \Delta^{-1} \begin{bmatrix} S_3 & S_2 \\ S_2 & S_1 \end{bmatrix} \begin{bmatrix} S_3 \\ S_4 \end{bmatrix} = \Delta^{-1} \begin{bmatrix} S_3^2 + S_2 S_4 \\ S_2 S_3 + S_1 S_4 \end{bmatrix}$$

The error locator polynomial is $\Lambda(x) = 1 + \Lambda_1 x + \Lambda_2 x^2$, where

$$\Lambda_1 = \frac{S_2 S_3 + S_1 S_4}{S_1 S_3 + S_2^2}, \quad \Lambda_2 = \frac{S_3^2 + S_2 S_4}{S_1 S_3 + S_2^2}.$$

The common denominator $\Delta = S_1 S_3 + S_2^2$ need be computed only once.

Computation of Λ_1, Λ_2 uses 8 multiplications and one inversion in $\mathrm{GF}(2^m)$.

PGZ decoder example (3)

Next find two zeroes X_1^{-1}, X_2^{-1} of $\Lambda(x)$ (perhaps by exhaustive search).

If $\Lambda(x)$ does not have two distinct zeroes, an uncorrectable error has occurred.

Finally find the error magnitudes:

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ X_1^2 & X_2^2 \end{bmatrix}^{-1} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

$$= \frac{1}{X_1 X_2 (X_1 + X_2)} \begin{bmatrix} X_2^2 & X_2 \\ X_1^2 & X_1 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

Matrix-vector product gives error magnitudes:

$$Y_1 = \frac{X_2^2 S_1 + X_2 S_2}{X_1 X_2 (X_1 + X_2)} = \frac{X_2 S_1 + S_2}{X_1 (X_1 + X_2)}$$

$$Y_2 = \frac{X_1^2 S_1 + X_1 S_2}{X_1 X_2 (X_1 + X_2)} = \frac{X_1 S_1 + S_2}{X_2 (X_1 + X_2)}$$

Computation of Y_1, Y_2 takes about 6 multiplications and 2 reciprocals.

PGZ decoder example (4)

If M_2 is singular, that is, $S_1 S_3 + S_2^2 = 0$, then we solve the simpler equation

$$M_1 [\, \Lambda_1 \,] \;=\; [\, S_2 \,] \;\Rightarrow\; [\, S_1 \,][\, \Lambda_1 \,] \;=\; [\, S_2 \,]$$

The error locator polynomial has degree 1:

$$\Lambda(x) \;=\; 1 + \Lambda_1 x \;=\; 1 + \frac{S_2}{S_1} x \;\Rightarrow\; X_1^{-1} = \frac{S_1}{S_2}$$

If $S_2 \neq 0$ then the single error locator is the reciprocal of the zero of $\Lambda(x)$:

$$X_1 \;=\; \frac{S_2}{S_1} \;=\; \frac{Y_1 X_1^2}{Y_1 X_1}$$

Error magnitude is obtained from $S_1 = Y_1 X_1$:

$$Y_1 \;=\; \frac{S_1}{X_1} \;=\; \frac{S_1^2}{S_2} \;=\; \frac{Y_1^2 X_1^2}{Y_1 X_1^2}\,.$$

Finally we check $S_4 = Y_1 X_1^4$. If not, an uncorrectable error has been detected.

PGZ in general

By definition of the error locator polynomial, $\Lambda(X_i^{-1}) = 0$:

$$1 + \Lambda_1 X_i^{-1} + \cdots + \Lambda_\nu X_i^{-\nu} = 0 \qquad (i = 1, \ldots, \nu)$$

Multiply this equation by $Y_i X_i^{j+\nu}$ for any $j \geq 1$:

$$Y_i X_i^{j+\nu} + \Lambda_1 Y_i X_i^{j+\nu-1} + \cdots + \Lambda_\nu Y_i X_i^{j} = 0$$

This equation has only positive powers of X_i. Now sum over i:

$$\sum_{i=1}^{\nu} Y_i X_i^{j+\nu} \;+\; \Lambda_1 \sum_{i=1}^{\nu} Y_i X_i^{j+\nu-1} \;+\; \cdots \;+\; \Lambda_\nu \sum_{i=1}^{\nu} Y_i X_i^{j} \;\; = 0$$

Thus if $j \geq 1$ and $j + \nu \leq 2t$, that is, $1 \leq j \leq 2t - \nu$,

$$S_{j+\nu} + \Lambda_1 S_{j+\nu-1} + \cdots + \Lambda_\nu S_j = 0$$

We have obtained $2t - \nu \geq \nu$ linear equations in ν unknowns $\Lambda_1, \ldots, \Lambda_2$:

$$S_j \Lambda_\nu + S_{j+1} \Lambda_{\nu-1} + \cdots + S_{j+\nu-1} \Lambda_1 = -S_{j+\nu}$$

Linear equations for $\Lambda_1, \ldots, \Lambda_\nu$

The first ν linear equations for $\Lambda_1, \ldots, \Lambda_\nu$ have a $\nu \times \nu$ coefficient matrix:

$$\begin{bmatrix} S_1 & S_2 & \cdots & S_\nu \\ S_2 & S_3 & \cdots & S_{\nu+1} \\ \vdots & \vdots & \ddots & \vdots \\ S_\nu & S_{\nu+1} & \cdots & S_{2\nu-1} \end{bmatrix} \begin{bmatrix} \Lambda_\nu \\ \Lambda_{\nu-1} \\ \vdots \\ \Lambda_1 \end{bmatrix} = \begin{bmatrix} -S_{\nu+1} \\ -S_{\nu+2} \\ \vdots \\ -S_{2\nu} \end{bmatrix}$$

For any $\mu = 1, 2, \ldots, t$, let M_μ be the matrix

$$M_\mu = \begin{bmatrix} S_1 & S_2 & \cdots & S_\mu \\ S_2 & S_3 & \cdots & S_{\mu+1} \\ \vdots & \vdots & \ddots & \vdots \\ S_\mu & S_{\mu+1} & \cdots & S_{2\mu-1} \end{bmatrix}.$$

Lemma: Suppose that there are $\nu \leq t$ symbol errors. Then M_ν is nonsingular, but M_μ is singular for $\mu > \nu$.

Matrices that are constant along anti-diagonals are called *Hankel* matrices.

Determining number of errors ν

Proof: Syndrome equations are satisfied if we define $X_i = 0$ when $\nu < i \leq t$.

$$M_\mu = \begin{bmatrix} S_1 & \cdots & S_\mu \\ \vdots & \ddots & \vdots \\ S_\mu & \cdots & S_{2\mu-1} \end{bmatrix} = \begin{bmatrix} \sum_1^\mu Y_i X_i^1 & \cdots & \sum_1^\mu Y_i X_i^\mu \\ \sum_1^\mu Y_i X_i^2 & \cdots & \sum_1^\mu Y_i X_i^{\mu+1} \\ \vdots & \ddots & \vdots \\ \sum_1^\mu Y_i X_i^\mu & \cdots & \sum_1^\mu Y_i X_i^{2\mu-1} \end{bmatrix}$$

$$= \begin{bmatrix} Y_1 X_1 & \cdots & Y_\mu X_\mu \\ \vdots & \ddots & \vdots \\ Y_1 X_1^\mu & \cdots & Y_\mu X_\mu^\mu \end{bmatrix} \cdot \begin{bmatrix} 1 & \cdots & 1 \\ X_1 & \cdots & X_\mu \\ \vdots & \ddots & \vdots \\ X_1^{\mu-1} & \cdots & X_\mu^{\mu-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ X_1^{\mu-1} & \cdots & X_\mu^{\mu-1} \end{bmatrix} \cdot \begin{bmatrix} Y_1 X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Y_\mu X_\mu \end{bmatrix} \cdot \begin{bmatrix} 1 & \cdots & X_1^{\mu-1} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & X_\mu^{\mu-1} \end{bmatrix}$$

If $i \leq \nu$ then $X_i \neq 0$ and $Y_i \neq 0$. Therefore M_ν is the product of nonsingular Vandermonde and diagonal matrices and is nonsingular.

But if $\mu > \nu$ the middle matrix has a zero element $Y_\mu X_\mu$ on its diagonal. The middle matrix is singular for $\mu > \nu$ and therefore M_μ is singular.

Peterson-Gorenstein-Zierler (PGZ) decoder: summary

1. Compute partial syndromes $S_j = r(\alpha^j)$.
2. Find largest $\nu \leq t$ such that $\det M_\nu \neq 0$.
3. Solve the following *linear* system for the coefficients of $\Lambda(x)$.
$$M_\nu \left[\Lambda_\nu, \ldots, \Lambda_1 \right]^T = \left[-S_{\nu+1}, \ldots, -S_{2\nu} \right]^T$$
4. Find $X_1^{-1}, \ldots, X_\nu^{-1}$, the zeroes of $\Lambda(x)$, in $\mathrm{GF}(q^m)$, the decoder alphabet. If $\Lambda(x)$ has $< \nu$ distinct zeroes, an uncorrectable error has occurred.
5. Solve following system of linear equations for error magnitudes $Y_1, \ldots, Y_\nu$.
$$\begin{aligned} Y_1 X_1 &+ \cdots + Y_\nu X_\nu &&= S_1 \\ Y_1 X_1^2 &+ \cdots + Y_\nu X_\nu^2 &&= S_2 \\ &\vdots && \vdots \\ Y_1 X_1^\nu &+ \cdots + Y_\nu X_\nu^\nu &&= S_\nu \\ &\vdots && \vdots \\ Y_1 X_1^{2t} &+ \cdots + Y_\nu X_\nu^{2t} &&= S_{2t} \end{aligned}$$

The Forney algorithm, $Y_i = -\frac{\Omega(X_i^{-1})}{\Lambda'(X_i^{-1})}$, is a "closed" form solution for step 5.

3EC Reed-Solomon code

Narrow-sense BCH codes usually do not have the simplest generator polynomial or parity-check matrices.

For that reason, Reed-Solomon codes are usually defined using $b = 0$.

The following matrix defines a three error correcting Reed-Solomon code:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \cdots & \alpha^{n-1} \\ 1 & \alpha^2 & \alpha^4 & \alpha^6 & \cdots & \alpha^{2(n-1)} \\ 1 & \alpha^3 & \alpha^6 & \alpha^9 & \cdots & \alpha^{3(n-1)} \\ 1 & \alpha^4 & \alpha^8 & \alpha^{12} & \cdots & \alpha^{4(n-1)} \\ 1 & \alpha^5 & \alpha^{10} & \alpha^{15} & \cdots & \alpha^{5(n-1)} \end{bmatrix}$$

The generator polynomial is

$$g(x) = (x+1)(x+\alpha)(x+\alpha^2)(x+\alpha^3)(x+\alpha^4)(x+\alpha^5)$$

Another trick to reduce encoder complexity is to choose b so that the generator polynomial is reversible — inverses of zeroes are also zeroes — so half as many scalers are needed in the encoding circuit.

3EC Reed-Solomon decoding (1)

The partial syndromes defined by $S_j = r(\alpha^j)$ for $j = 0, \ldots, 5$ satisfy the equations

$$\begin{aligned} S_0 &= Y_1 + Y_2 + Y_3 \\ S_1 &= Y_1X_1 + Y_2X_2 + Y_3X_3 \\ S_2 &= Y_1X_1^2 + Y_2X_2^2 + Y_3X_3^2 \\ S_3 &= Y_1X_1^3 + Y_2X_2^3 + Y_3X_3^3 \\ S_4 &= Y_1X_1^4 + Y_2X_2^4 + Y_3X_3^4 \\ S_5 &= Y_1X_1^5 + Y_2X_2^5 + Y_3X_3^5 \end{aligned}$$

where X_1, X_2, X_3 are error location numbers and Y_1, Y_2, Y_3 are error magnitudes.

The coefficients of the error locator polynomial $\Lambda(x)$ satisfy the linear equations:

$$\begin{bmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{bmatrix} \begin{bmatrix} \Lambda_3 \\ \Lambda_2 \\ \Lambda_1 \end{bmatrix} = \begin{bmatrix} S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

3EC Reed-Solomon decoding (2)

If there are three errors, then the solutions can be found using Cramer's rule:

$$\begin{aligned} \Lambda_0 &= S_2(S_1S_3 + S_2S_2) + S_3(S_0S_3 + S_1S_2) + S_4(S_0S_2 + S_1S_1) \\ \Lambda_1 &= S_3(S_1S_3 + S_2S_2) + S_4(S_0S_3 + S_1S_2) + S_5(S_0S_2 + S_1S_1) \\ \Lambda_2 &= S_3(S_1S_4 + S_2S_3) + S_4(S_0S_4 + S_2S_2) + S_5(S_0S_3 + S_1S_2) \\ \Lambda_3 &= S_3(S_2S_4 + S_3S_3) + S_4(S_1S_4 + S_2S_3) + S_5(S_1S_3 + S_2S_2) \end{aligned}$$

Note that we can choose $\Lambda_0 = 1$ by dividing the other coefficients by Λ_0.

Let X_1, X_2, X_3 be the zeroes of

$$\Lambda(x) = \Lambda_0 + \Lambda_1 x + \Lambda_2 x^2 + \Lambda_3 x^3 \,.$$

The location of the incorrect symbols are i_1, i_2, i_3, where

$$X_1 = \alpha^{i_1} \,, \quad X_2 = \alpha^{i_2} \,, \quad X_3 = \alpha^{i_3} \,.$$

3EC Reed-Solomon decoding (3)

Finally, the error magnitudes Y_1, Y_2, Y_3 can be found by solving equations that use the first three syndrome components, S_0, S_1, S_2:

$$Y_1 = \frac{S_2 + S_1(X_2 + X_3) + S_0 X_2 X_3}{(X_1 + X_2)(X_1 + X_3)}$$

$$Y_2 = \frac{S_2 + S_1(X_1 + X_3) + S_0 X_1 X_3}{(X_2 + X_1)(X_2 + X_3)}$$

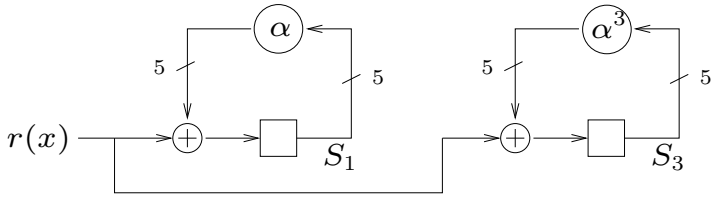
$$Y_3 = \frac{S_2 + S_1(X_1 + X_2) + S_0 X_1 X_2}{(X_3 + X_1)(X_3 + X_2)}$$

Starting from the partial syndromes $S_0, S_1, \ldots, S_5$, approximately 30 Galois field multiplications and 3 Galois field divisions are needed to perform decoding.

This estimate does not count the effort needed to find the zeroes of $\Lambda(x)$.

Partial syndrome circuits for GF(32)

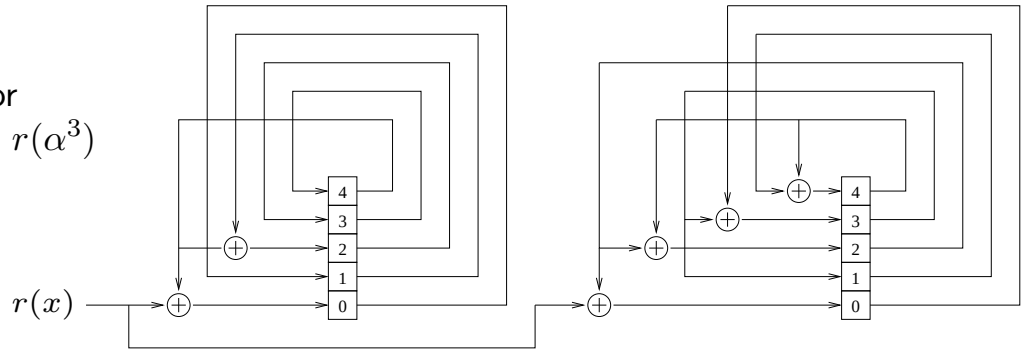
Horner's method:



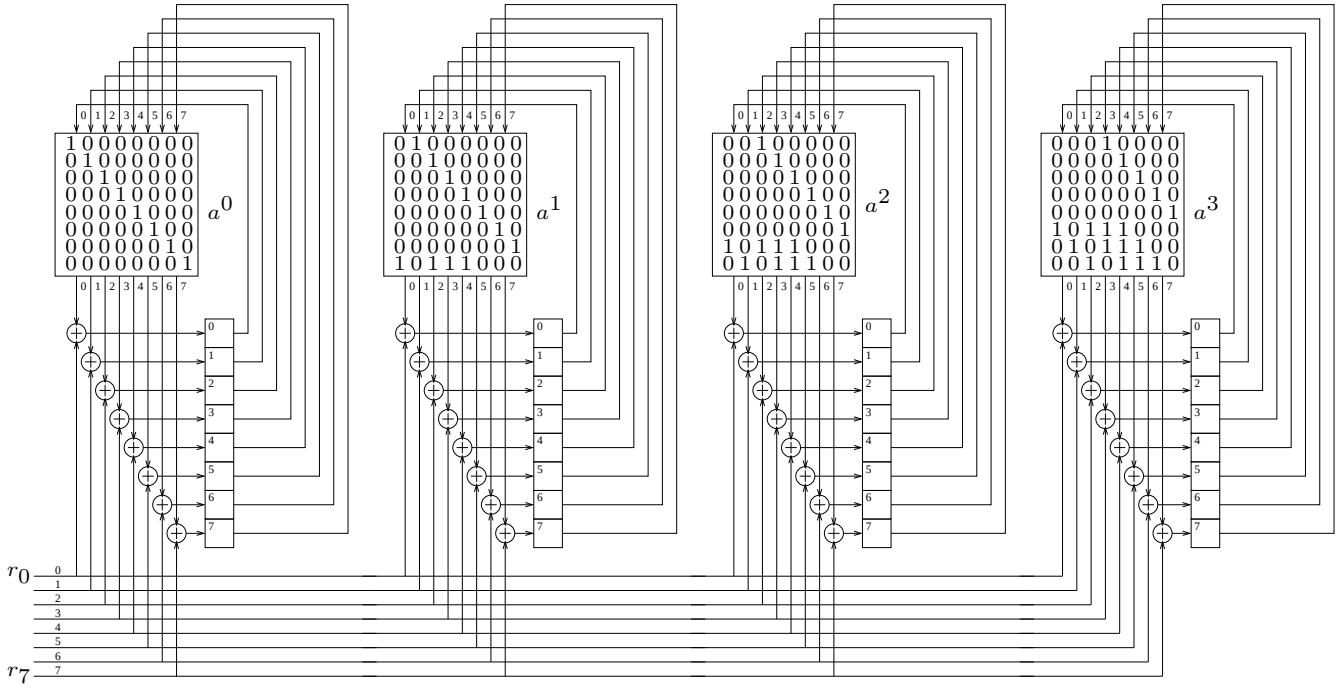
Multiplication by α, α^3 uses matrices: $(\alpha^5 + \alpha^2 + 1 = 0)$

$$M_\alpha = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad M_{\alpha^3} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Circuit for $r(\alpha)$ and $r(\alpha^3)$



Partial syndrome circuit for (255,251) R-S code



Chien search

The Chien search is a clever method for finding zeroes of the error locator polynomial by brute force.

The Chien search evaluates $\Lambda(\alpha^i)$ for $i = 1, 2, \ldots, n$ using ν *constant* multiplications instead of ν general multiplications.

Key idea: use ν state variables $Q_1, \ldots, Q_\nu$ such that at time i

$$Q_j = \Lambda_j \alpha^{ji}, \quad j = 1, \ldots, \nu.$$

Each state variable is updated by multiplication by a constant:

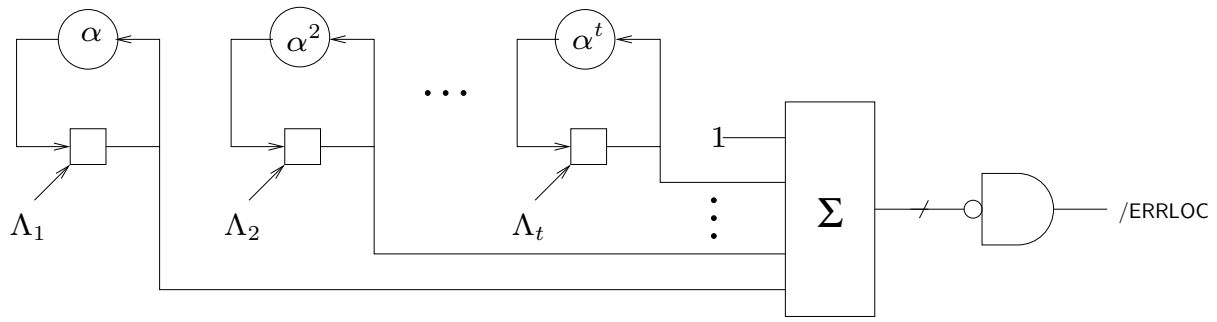
$$Q_j \rightarrow Q_j \alpha^j, \quad i = 1, \ldots, n.$$

Sum of state variables at time i is $\displaystyle\sum_{j=1}^{\nu} Q_j = \Lambda(\alpha^i) - 1.$

An error location is identified whenever this sum equals -1.

Chien search circuit #1

Memory elements are initialized with coefficients of error locator polynomial, i.e., $\Lambda_j = 0$ for $j = \nu + 1, \ldots, t$.



Output signal ERRLOC is true when $\Lambda(\alpha^i) = 0$.

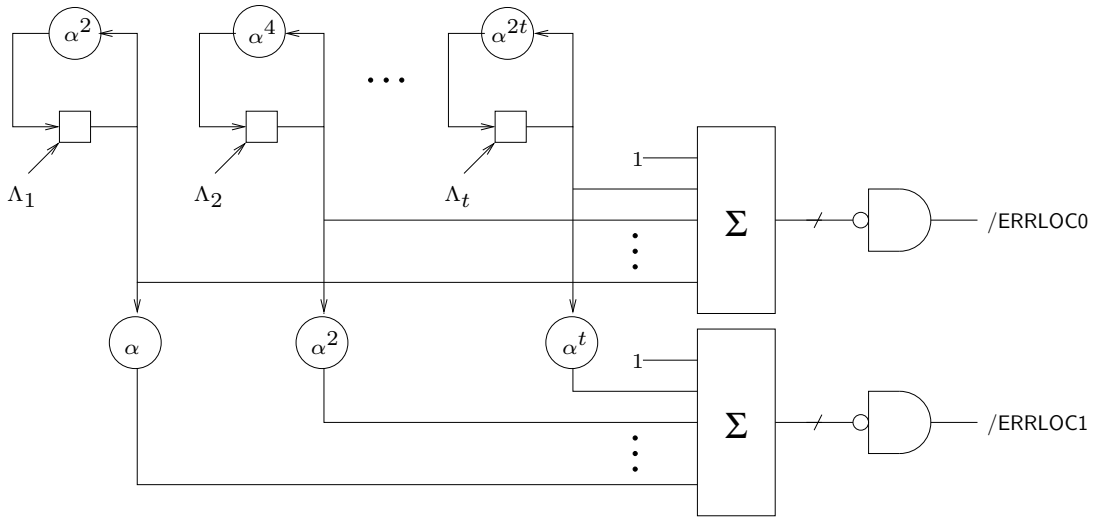
Since the zeroes of $\Lambda(x)$ are the reciprocals of the error location numbers, ERRLOC is true for values of i such that $\alpha^{-i} = \alpha^{n-i} = X_l$.

As i runs from 1 to n, error locations are detected from msb down to lsb.

Chien search can also be run backwards, using scalers for $\alpha^{-1}, \ldots, \alpha^{-t}$.

Chien search circuit #2

Double-speed Chien search: evaluate $\Lambda(\alpha^{2i})$ and $\Lambda(\alpha^{2i+1})$ at same time.



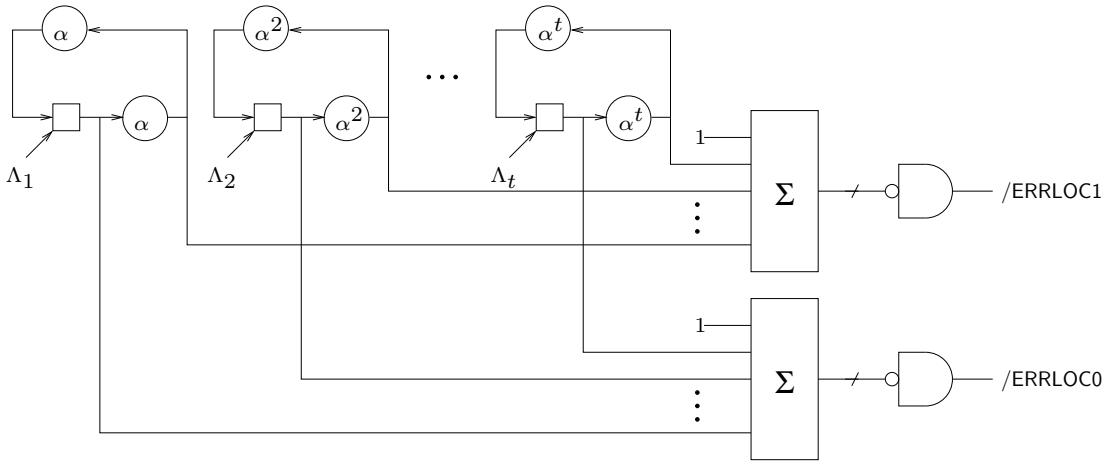
ERRLOC0 (ERRLOC1) is asserted when an even (odd) error location is found.

This circuit is more efficient than two separate copies of the Chien search engine because the memory storage elements are shared.

Chien search circuit #3

Scaler α^{2j} usually requires more gates than scaler α^j for small values of j.

We can reduce cost of double-speed Chien search by using $\alpha^{2j} = \alpha^j \cdot \alpha^j$



The cascade of two scalers for α^i may be slightly slower than one scaler for α^{2i}.

PGZ decoder: review

1. Compute partial syndromes $S_j = r(\alpha^j)$.
2. Solve a linear system of equations for coefficients of $\Lambda(x)$:

$$M_\nu \, [\, \Lambda_\nu, \ldots, \Lambda_1 \,]^T = [\, -S_{\nu+1}, \ldots, -S_{2\nu} \,]^T$$

   where ν is the largest number $\leq t$ such that $\det M_\nu \neq 0$.
3. Find the zeroes of $\Lambda(x)$ are $X_1^{-1}, \ldots, X_\nu^{-1}$, which are the reciprocals of the error locators $X_1 = \alpha^{i_1}, \ldots, X_\nu = \alpha^{i_\nu}$.
4. Solve a system of linear equations for the error magnitudes $Y_1, \ldots, Y_\nu$.

$$\begin{array}{rcl} Y_1 X_1 + \cdots + Y_\nu X_\nu & = & S_1 \\ Y_1 X_1^2 + \cdots + Y_\nu X_\nu^2 & = & S_2 \\ \vdots & & \vdots \\ Y_1 X_1^{2t} + \cdots + Y_\nu X_\nu^{2t} & = & S_{2t} \end{array}$$

The Forney algorithm (1965) is a simple closed formula for $Y_1, \ldots, Y_\nu$.

Forney Algorithm

Consider a BCH code defined by the zeroes α^b, $\alpha^{b+1}, \ldots, \alpha^{b+2t-1}$.

Forney algorithm: the error magnitude Y_i corresponding to error locator X_i is

$$Y_i = -\frac{X_i^{1-b}\Omega(X_i^{-1})}{\Lambda'(X_i^{-1})},$$

where $\Lambda'(x)$ is the *formal derivative* of the error-locator polynomial,

$$\Lambda'(x) = \sum_{i=1}^{\nu} i\Lambda_i x^{i-1},$$

and $\Omega(x)$ is the *error evaluator* polynomial, $S(x)\Lambda(x) \bmod x^{2t}$.

The Forney algorithm is slightly simpler for narrow-sense BCH codes ($b = 1$):

$$Y_i = -\frac{\Omega(X_i^{-1})}{\Lambda'(X_i^{-1})}.$$

Fact: Forney's algorithm uses $2\nu^2$ multiplications to compute all error magnitudes.

Partial syndrome polynomial

Definition: The *partial syndrome polynomial* for a narrow-sense BCH code is the *generating function* of the sequence $S_1, S_2, \ldots, S_{2t}$:

$$S(x) = S_1 + S_2x + S_3x^2 + \cdots + S_{2t}x^{2t-1}.$$

For the BCH code defined by $\alpha^b, \ldots, \alpha^{b+2t-1}$, the partial syndrome polynomial is

$$S(x) = S_b + S_{b+1}x + \cdots + S_{b+2t-1}x^{2t-1}.$$

The PGZ decoder uses linear equations for coefficients of $\Lambda(x)$, $j = 1, \ldots, 2t - \nu$.

$$S_j\Lambda_\nu + \cdots + S_{j+\nu-1}\Lambda_1 + S_{j+\nu} = 0 \qquad \text{narrow sense codes}$$

$$S_{b+j-1}\Lambda_\nu + \cdots + S_{b+j+\nu-2}\Lambda_1 + S_{b+j+\nu-1} = 0 \qquad \text{general BCH codes}$$

In both cases, the left hand side is the coefficient of $x^{\nu+j-1}$ in the polynomial product $S(x)\Lambda(x)$.

We can define partial syndromes S_i for every $i > 0$. However, the decoder can compute only the first $2t$ values.

Error evaluator polynomial

Definition: The *error evaluator polynomial* $\Omega(x)$ is defined by the *key equation*:

$$\Omega(x) \;=\; S(x)\Lambda(x) \bmod x^{2t}\,,$$

where $S(x)$ is partial syndrome polynomial and $\Lambda(x)$ is error-locator polynomial.

The coefficient of $x^{\nu+j-1}$ in $S(x)\Lambda(x)$ is 0 if $1 \le j \le 2t-\nu$ by PGZ equations.

Therefore $\deg\left(S(x)\Lambda(x) \bmod x^{2t}\right) < \nu$ if there are $\nu \le t$ errors.

The error evaluator polynomial can be computed explicitly from $\Lambda(x)$:

$$\begin{aligned}
\Omega_0 &= S_b \\
\Omega_1 &= S_{b+1} + S_b\Lambda_1 \\
\Omega_2 &= S_{b+2} + S_{b+1}\Lambda_1 + S_b\Lambda_2 \\
&\vdots \\
\Omega_{\nu-1} &= S_{b+\nu-1} + S_{b+\nu-2}\Lambda_1 + \cdots + S_b\Lambda_{\nu-1}
\end{aligned}$$

Multiply-accumulates needed: $0+1+\cdots+\nu-2 = \frac{1}{2}(\nu-1)(\nu-2) \approx \frac{1}{2}\nu^2$

Formal derivative

We can obtain a closed formula for Y_i in terms of $\Omega(x)$, $\Lambda(x)$, and X_i.

First we need the notion of the formal derivative of a polynomial.

Definition: The *formal derivative* of

$$f(x) = f_0 + f_1x + f_2x^2 + \cdots + f_nx^n$$

is the polynomial

$$f'(x) = f_1 + 2f_2x + 3f_3x^2 + \cdots + nf_nx^{n-1}$$

Most of the familiar properties of derivatives hold. In particular, product rules:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$\Big(\prod_{i=1}^{n} f_i(x)\Big)' = \sum_{i=1}^{n} f_i'(x) \prod_{j\neq i} f_j(x)$$

Formal derivatives of polynomials are defined algebraically, not by taking limits.

Properties of formal derivatives

Fact: A polynomial $f(x)$ over $\mathrm{GF}(q)$ has a repeated zero β iff $f'(x) = 0$.

Proof: If β is a zero of $f(x)$, then $x - \beta$ is a factor of $f(x)$:

$$f(x) = f_1(x)(x-\beta) \Rightarrow f'(x) = f_1(x) + f_1'(x)(x-\beta) \Rightarrow f'(\beta) = f_1(\beta)\,.$$

Thus β is a repeated zero — $f(x)$ has factor $(x-\beta)^2$ — if and only if $f'(\beta) = 0$.

Over $\mathrm{GF}(2^m)$, the formal derivative has only even powers of the indeterminant:

$$f'(x) = f_1 + 2f_2x + 3f_3x^2 + \cdots + nf_nx^{n-1} \;=\; f_1 + 3f_3x^2 + 5f_5x^4 + \cdots$$

since $2 = 1+1 = 0\,,\ 4 = 2+2 = 2(1+1) = 0\,,$ and so on.

So the formal derivative $\Lambda'(x)$ has at most $\nu/2$ nonzero coefficients.

Since $\Lambda'(x)$ is a polynomial in x^2 of degree $< \nu/2$, we can compute $\Lambda'(\beta)$ using one squaring and $\leq \nu/2$ multiply-accumulate operations.

Note that $f''(x) = 0$ for all polynomials over $\mathrm{GF}(2^m)$.

Forney algorithm: derivation (1)

We can express error evaluator $\Omega(x)$ in terms of error location numbers X_i and error magnitudes Y_i.

First we derive a closed formula for $S(x)$.

$$\begin{aligned} S(x) &= \sum_{j=0}^{2t-1} S_{b+j}x^j \\ &= \sum_{j=0}^{2t-1}\sum_{i=1}^{\nu} Y_iX_i^{b+j}x^j \\ &= \sum_{i=1}^{\nu} Y_iX_i^b \sum_{j=0}^{2t-1} X_i^jx^j = \sum_{i=1}^{\nu} Y_iX_i^b\,\frac{1-(X_ix)^{2t}}{1-X_ix} \end{aligned}$$

Next we use the definition $\Lambda(x) = \prod_{l=1}^{\nu}\left(1 - X_lx\right)$ to compute $S(x)\Lambda(x)$.

Forney algorithm: derivation (2)

$$\begin{aligned} S(x)\Lambda(x) &= \sum_{i=1}^{\nu} \left(Y_i X_i^b \frac{1-(X_i x)^{2t}}{1-X_i x} \right) \cdot \prod_{l=1}^{\nu} (1-X_l x) \\ &= \sum_{i=1}^{\nu} \left(Y_i X_i^b \prod_{l \neq i} (1-X_l x) \big(1-(X_i x)^{2t}\big) \right) \\ &= \sum_{i=1}^{\nu} Y_i X_i^b \prod_{l \neq i} (1-X_l x) \; - \; \sum_{i=1}^{\nu} Y_i X_i^b (X_i x)^{2t} \prod_{l \neq i} (1-X_l x) \end{aligned}$$

The second sum in the final expression is a polynomial in x of degree $\geq 2t$.

Thus the remainder modulo x^{2t} of the second sum is 0.

Therefore

$$\Omega(x) \; = \; S(x)\Lambda(x) \bmod x^{2t} \; = \; \sum_{i=1}^{\nu} Y_i X_i^b \prod_{l \neq i} (1-X_l x) \, .$$

Forney algorithm: derivation (3)

We just found $\Omega(x)$ in terms X_i and Y_i. Next use the product formula for $\Lambda'(x)$:

$$\Lambda'(x) \; = \; \left(\prod_{l=1}^{\nu} (1-X_l x) \right)' \; = \; \sum_{l=1}^{\nu} (-X_l) \prod_{j \neq l} (1-X_j x) \, .$$

When we evaluate $\Lambda'(x)$ at X_i^{-1}, only one term in the sum is nonzero:

$$\Lambda'(X_i^{-1}) \; = \; -X_i \prod_{j \neq i} (1-X_j X_i^{-1}) \, .$$

Similarly, the value of $\Omega(x)$ at X_i^{-1} includes only one term from the sum:

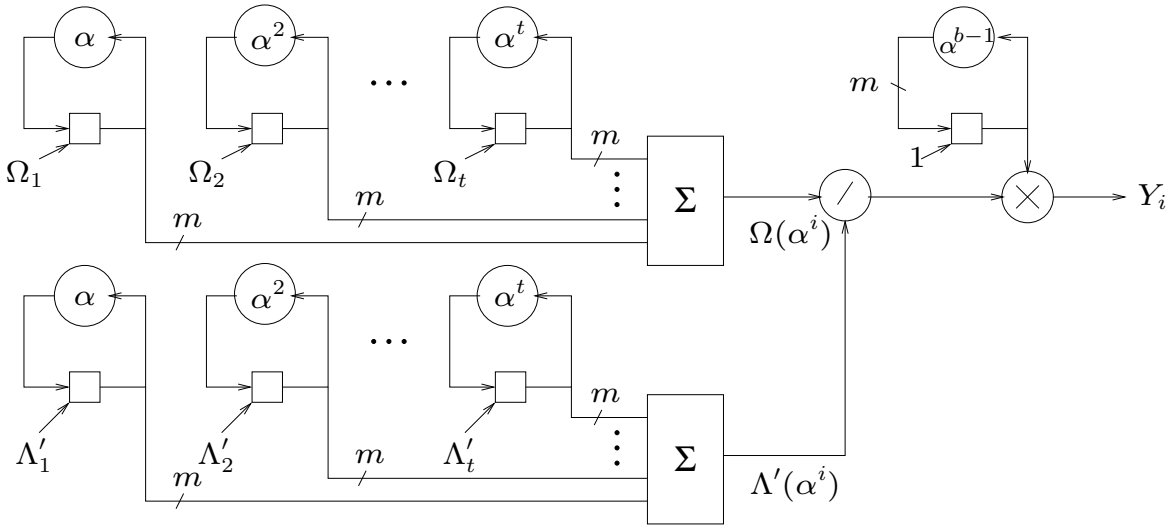
$$\Omega(X_i^{-1}) = \sum_{l=1}^{\nu} Y_l X_l^b \prod_{j \neq l} (1-X_j X_i^{-1}) = Y_i X_i^b \prod_{j \neq i} (1-X_j X_i^{-1}) \, .$$

Thus

$$\frac{\Omega(X_i^{-1})}{\Lambda'(X_i^{-1})} = \frac{Y_i X_i^b}{-X_i} \; \Rightarrow \; Y_i = -X_i^{-(b-1)} \, \frac{\Omega(X_i^{-1})}{\Lambda'(X_i^{-1})} \, .$$

Forney algorithm during Chien search

Error magnitudes Y_l can be computed by a Chien-search-like circuit:



At each time i, the values of $\Omega(\alpha^i)$ and $\Lambda'(\alpha^i)$ are available.

Thus Y_l can be computed by one division and one multiplication by $\alpha^{-i(b-1)}$.

Forney algorithm: summary

Suppose that a BCH code is defined by zeroes $\alpha^b, \alpha^{b+1}, \ldots, \alpha^{b+2t-1}$.

Suppose that the error-locator polynomial has degree ν.

The error evaluator polynomial consists of the first ν terms of $S(x)\Lambda(x)$.

The formal derivative $\Lambda'(x)$ is $\sum_{i=1}^{\nu} i\Lambda_i x^{i-1}$.

Then the error magnitude Y_i corresponding to error location number X_i is

$$Y_i = -\frac{X_i^{1-b}\Omega(X_i^{-1})}{\Lambda'(X_i^{-1})},$$

Computation of the coefficients of $\Omega(x)$ uses $\approx \nu^2/2$ multiplications.

Computation of Y_i needs $\nu + (\nu - 1) + 2 = 2\nu + 1$ multiplications + one reciprocal.

Forney's algorithm finds all ν error magnitudes using $\approx 2.5\nu^2$ multiplications.

When $\mathrm{GF}(2^m)$ is the decoder alphabet, $\Lambda'(x)$ has only $\nu/2$ nonzero coefficients, which reduces the total operation count to $2\nu^2$ multiplications.

Euclidean BCH decoding algorithm

Sugiyama, Kasahara, Hirasawa, Namekawa (1975). The key equation is

$$\Omega(x) = S(x)\Lambda(x) \bmod x^{2t} \;\Rightarrow\; \Omega(x) = S(x)\Lambda(x) + b(x)x^{2t}$$

for some polynomial $b(x)$ of degree $< \nu$.

Suppose that the extended Euclidean algorithm is used to calculate $\gcd(S(x), x^{2t})$. For $i = 1, 2, \ldots$:

$$r_i(x) = r_{i-2}(x) - Q_i(x)r_{i-1}(x) = a_i(x)S(x) + b_i(x)x^{2t}$$
$$a_i(x) = a_{i-2}(x) - Q_i(x)a_{i-1}(x)\,, \quad b_i(x) = b_{i-2}(x) - Q_i(x)b_{i-1}(x)$$

At some step i the remainder $r_i(x)$ has degree $< t$.[1]

The first such remainder is $r_i(x) = \gamma\Omega(x)$, where γ is the constant term of $a_i(x)$.

The error-locator polynomial $\Lambda(x) = \gamma^{-1}a_i(x)$ is a polynomial of least degree such that $\deg(S(x)\Lambda(x) \bmod x^{2t}) < t$.

[1] Unless $S_i = 0$ for $i \le t$, and not all $S_i = 0$, in which case an uncorrectable error has occurred.

Euclidean algorithm: pseudocode

1. Compute syndomes: $S_j = r(\alpha^j)\,,\ j = 1, \ldots, 2t$

2. Initialize:

$$s(x) \leftarrow x^{2t}\,; \quad t(x) \leftarrow \sum_{j=1}^{2t} S_j x^{j-1}\,; \quad A(x) \leftarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

3. While $\deg t(x) \ge t$

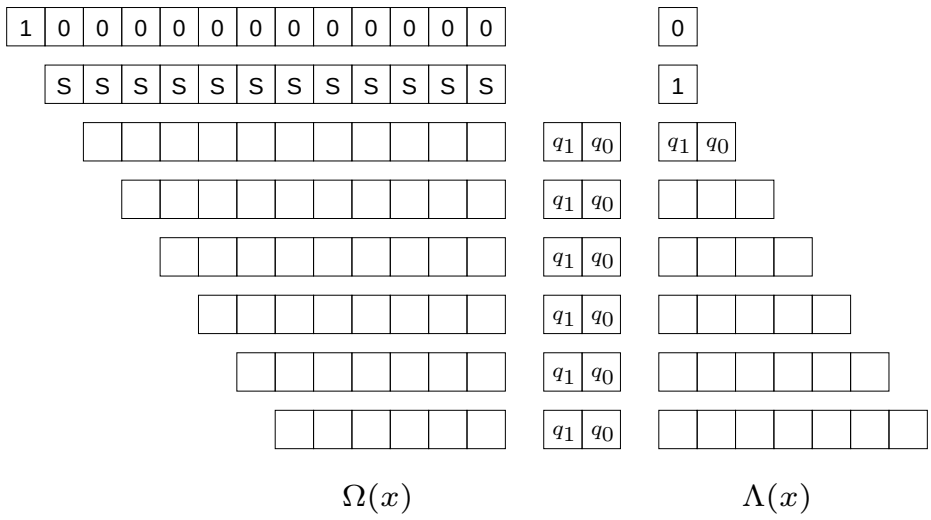
$$Q(x) \leftarrow \left\lfloor \frac{s(x)}{t(x)} \right\rfloor; \quad \begin{bmatrix} s(x) \\ t(x) \end{bmatrix} \leftarrow \begin{bmatrix} 0 & 1 \\ 1 & -Q(x) \end{bmatrix} \begin{bmatrix} s(x) \\ t(x) \end{bmatrix}; \quad A(x) \leftarrow \begin{bmatrix} 0 & 1 \\ 1 & -Q(x) \end{bmatrix} A(x)\,;$$

4. Finalize:

$$\Delta \leftarrow A_{22}(0); \quad \Lambda(x) \leftarrow \Delta^{-1}A_{22}(x); \quad \Omega(x) \leftarrow \Delta^{-1}t(x)\,;$$

The quotient $\left\lfloor \frac{s(x)}{t(x)} \right\rfloor$ is defined by $s(x) = \left\lfloor \frac{s(x)}{t(x)} \right\rfloor t(x) + r(x)\,,\ \deg r(x) < \deg t(x)\,.$

Euclidean algorithm tableau



Usually the quotient $q_i(x)$ is linear. In this case

$$r_i(x) = r_{i-2}(x) - q_{i1}xr_{i-1}(x) - q_{i0}r_{i-1}(x)$$

Coefficients of $q_i(x)$ can be found from first 2 coefficients of $r_{i-2}(x)$, $r_{i-1}(x)$.

Euclidean algorithm: example (1)

6EC Reed-Solomon code over $\mathrm{GF}(2^8)$:

One error:

$r_i(x)$	$Q_i(x)$	$a_i(x)$
00 00 00 00 00 00 00 00 00 00 00 00 01	—	00
25 2E 12 A1 D5 D8 DF 95 C9 ED B2 05	—	01
29	CE A7	CE A7

$\Omega(x) = 29/\mathtt{CE} = 25\,,\ \Lambda(x) = (\mathtt{CE\ A7})/\mathtt{CE} = \mathtt{01\ 71}$

Op count: mul $= 29\,,$ div $= 5$

Three errors:

$r_i(x)$	$Q_i(x)$	$a_i(x)$
00 00 00 00 00 00 00 00 00 00 00 00 01	—	
10 1a cf dc 28 1d d2 52 5d 19 57 ec	—	
dd 9b 4a 2f f9 3f 05 34 04 76 66	1c 6d	1c 6d
b9 54 34 fa db eb a6 87 bc 4a	6e d8	5d 97 17
b1 47 69	ae e1	d3 f2 ad 2b

$\Omega(x) = \mathtt{10\ b3\ f5}\,,\ \Lambda(x) = \mathtt{01\ 5c\ d7\ 4f}$

Op count: mul $= 91\,,$ div $= 13$

Euclidean algorithm: example (2)

6EC Reed-Solomon code over $\mathrm{GF}(2^8)$:

Six errors:

$r_i(x)$	$Q_i(x)$	$a_i(x)$
00 00 00 00 00 00 00 00 00 00 00 00 01	—	00
04 19 50 23 BB AF AE F6 41 CB 0A AB	—	01
A6 A4 36 F1 B0 C5 75 08 79 E9 6B	A7 3C	A7 3C
53 DA D8 39 2E 52 3D A6 DC 51	8B 71	DD 3C B3
B4 3E 51 46 82 BA 35 21 65	2D 5C	2D 23 06 23
70 2B 5F D7 ED F3 45 C8	0C 4F	1C 8C 5F 36 C4
94 33 CC 1F E2 84 07	69 5C	25 55 7C 53 6B D2
EC E2 0D 83 78 DB	AD 47	3B EB 8C C9 FE 8E BA

$\Omega(x) =$ 04 F4 16 CC F2 BA , $\Lambda(x) =$ 01 7C 95 B7 09 DA 82

Op count: $\mathrm{mul} = 169\,,\ \mathrm{div} = 25$

Euclidean algorithm: computational cost

The Euclidean algorithm produces remainders such that $\deg r_i(x) < \deg r_{i-1}(x)$.

- initial remainder $S(x)$ has degree $\leq 2t-1$
- final remainder $\Omega(x)$ has degree $\leq t-1$

Therefore at most t major steps are needed.

Each major step is polynomial division followed by polynomial multiplication:

$$r_{i-2}(x) = Q_i(x)r_{i-1}(x) + r_i(x)$$
$$a_i(x) = a_{i-2}(x) - Q_i(x)a_{i-1}(x)$$

At step i approximately $2 \cdot (2t-i)$ multiplications are used to find $Q_i(x)$ and $2i$ multiplications to find $a_i(x)$. Total multiplications per step $\approx 4t$.

Overall cost to find $\Lambda(x)$ and $\Omega(x)$: $4t^2$ multiplications and t reciprocals.

Solving $M_t\,[\,\Lambda_t, \ldots, \Lambda_1\,]^T = -[\,S_{t+1}, \ldots, S_{2t}\,]^T$ directly takes $\approx t^3/6$ operations.

Berlekamp decoding algorithm

Berlekamp (1967) invented an efficient iterative procedure for solving the linear equations with coefficient matrices

$$M_\mu = \begin{bmatrix} S_1 & S_2 & S_3 & \cdots & S_\mu \\ S_2 & S_3 & S_4 & \cdots & S_{\mu+1} \\ S_3 & S_4 & S_5 & \cdots & S_{\mu+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_\mu & S_{\mu+1} & S_{\mu+2} & \cdots & S_{2\mu-1} \end{bmatrix},$$

where $S_j = \sum_{i=1}^{\nu} Y_i X_i^j$ is a partial syndrome.

Each M_μ is found by using results of computations for some previous $M_{\mu-\rho}$ plus an additional $O(\mu)$ operations.

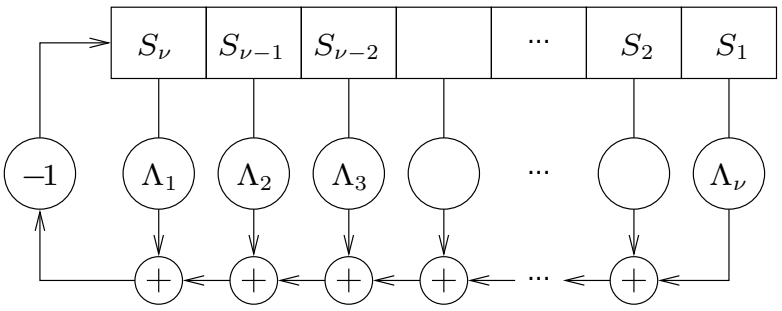
Summing over μ from 1 to ν gives total cost $O(\nu^2)$ operations.

In Berlekamp's original algorithm, a table with $2t$ rows stored intermediate results.

Massey decoding algorithm: shift register synthesis

Massey (1969) showed how finding the error-locator polynomial $\Lambda(x)$ is equivalent to a *shift-register synthesis* problem:

Given a sequence $S_1, S_2, \ldots, S_{2t}$, find the shortest sequence $\Lambda_1, \Lambda_2, \ldots, \Lambda_\nu$ that generates $S_{\nu+1}, \ldots, S_{2t}$ starting from $S_1, \ldots, S_\nu$ in a shift-register of size ν.



Recall that if the number of errors is ν then

$$S_j\Lambda_\nu + S_{j+1}\Lambda_{\nu-1} + \cdots + S_{j+\nu-1}\Lambda_1 = -S_{j+\nu}$$

for $j = 1, \ldots, 2t - \nu$. The PGZ system of equations is a convolution $S * \Lambda$.

Berlekamp-Massey algorithm: overview

Partial syndromes $S_1, \ldots, S_{2t}$ are examined one at a time for $k = 1, \ldots, 2t$.

At the end of the k-th step, $\Lambda^{(k)}(x)$ of degree L satisfies first k equations:

$$S_{k-i} + S_{k-i-1}\Lambda_1^{(k)} + \cdots + S_{k-i-L}\Lambda_L^{(k)} = 0 \qquad i = 0, \ldots, k-L$$

If $\Lambda^{(k-1)}(x)$ satisfies k-th equation, then obviously

$$\Lambda^{(k)}(x) = \Lambda^{(k-1)}(x)\,.$$

The key and surprising idea: when $\Lambda^{(k-1)}(x)$ does not work (sum is $\Delta^{(k)}(x) \neq 0$), update it as follows:

$$\Lambda^{(k)}(x) = \Lambda^{(k-1)}(x) - \Delta^{(k)}T(x)\,,$$

where $T(x) = \frac{1}{\Delta^{(r)}}x^{k-r}\Lambda^{(r)}(x)$ is the last failing $\Lambda^{(r)}(x)$ shifted and scaled.

When the degree of $\Lambda(x)$ has increased, we save $\Lambda^{(k-1)}(x)$ for future steps:

$$T(x) = \frac{1}{\Delta^{(k)}}x\Lambda^{(k-1)}(x)$$

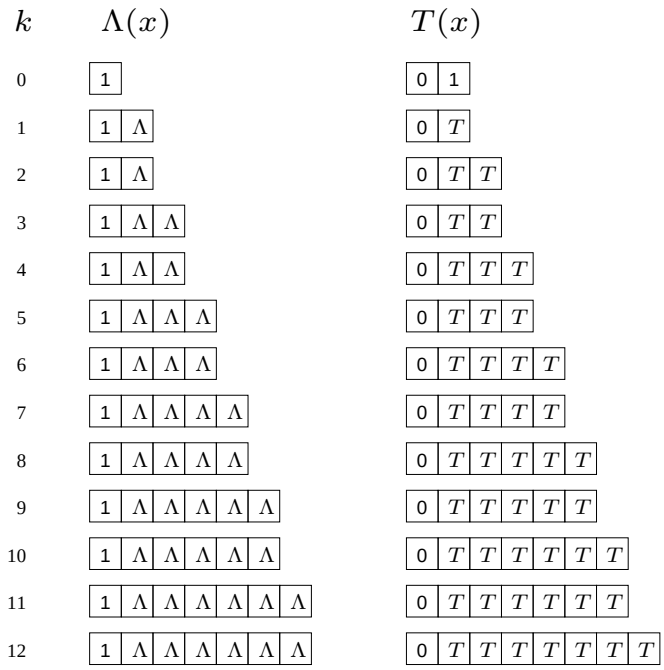
$\Delta^{(k)}(x)$ is called the *discrepancy* at step k.

Berlekamp-Massey algorithm: pseudocode

```
Λ(x) = 1;                          /* "connection polynomial" */
L = 0;                             /* L always equals deg Λ(x) */
T(x) = x;                          /* "correction polynomial" */
for (k = 1; k ≤ 2t; k++) {
  Δ = Σ_{i=0}^{L} Λ_i S_{k-i};     /* S_k + Λ_1 S_{k-1} + ··· + Λ_L S_{k-L-1} */
  if (Δ == 0) {
    N(x) = Λ(x);                   /* keep same Λ(x) if Δ == 0 */
  } else {
    N(x) = Λ(x) - ΔT(x);           /* new Λ(x) has 0 discrepancy */
    if (L < k - L) {
      L = k - L;
      T(x) = Δ^{-1}Λ(x);           /* new correction polynomial */
    }
  }
  T(x) = xT(x);                    /* shift correction polynomial */
  Λ(x) = N(x);                     /* possibly new value of Λ(x) */
}
```

Berlekamp-Massey tableau

The following figure shows typical computation for 6EC BCH code.



Berlekamp-Massey example (1)

6EC Reed-Solomon code over $\mathrm{GF}(2^8)$. One error.

$S =$ `6f 81 63 f9 74 6f 81 63 f9 74 6f 81`

k	$\Lambda^{(k)}(x)$	$T^{(k)}(x)$
1	01 6F	00 32
2	01 0A	00 00 32
3	01 0A	00 00 00 32
4	01 0A	00 00 00 00 32
5	01 0A	00 00 00 00 00 32
6	01 0A	00 00 00 00 00 00 32
7	01 0A	00 00 00 00 00 00 00 32
8	01 0A	00 00 00 00 00 00 00 00 32
9	01 0A	00 00 00 00 00 00 00 00 00 32
10	01 0A	00 00 00 00 00 00 00 00 00 00 32
11	01 0A	00 00 00 00 00 00 00 00 00 00 00 32
12	01 0A	00 00 00 00 00 00 00 00 00 00 00 00 32

Op count: mul $= 28$, div $= 1$

Berlekamp-Massey example (2)

6EC Reed-Solomon code over $\mathrm{GF}(2^8)$. Two errors.

$S =$ `b0 91 cc d1 99 26 0a 8a 70 67 96 c9`

k	$\Lambda^{(k)}(x)$	$T^{(k)}(x)$
1	01 B0	00 87
2	01 AB	00 00 87
3	01 AB A6	00 6F 16
4	01 44 87	00 00 6F 16
5	01 44 87	00 00 00 6F 16
6	01 44 87	00 00 00 00 6F 16
7	01 44 87	00 00 00 00 00 6F 16
8	01 44 87	00 00 00 00 00 00 6F 16
9	01 44 87	00 00 00 00 00 00 00 6F 16
10	01 44 87	00 00 00 00 00 00 00 00 6F 16
11	01 44 87	00 00 00 00 00 00 00 00 00 6F 16
12	01 44 87	00 00 00 00 00 00 00 00 00 00 6F 16

Op count: mul $= 45$, div $= 2$

Berlekamp-Massey example (3)

6EC Reed-Solomon code over $\mathrm{GF}(2^8)$. 6 errors.

$S =$ `bc 30 bb 24 81 74 e5 a7 bd 2b 95 34`

k	$\Lambda^{(k)}(x)$	$T^{(k)}(x)$
1	01 BC	00 95
2	01 F2	00 00 95
3	01 F2 7E	00 98 F4
4	01 D9 9D	00 00 98 F4
5	01 D9 89 74	00 AC 6F AD
6	01 88 05 58	00 00 AC 6F AD
7	01 88 88 CC 7A	00 C9 66 CA 3A
8	01 9A ED 96 23	00 00 C9 66 CA 3A
9	01 9A 06 2D 43 45	00 77 57 E6 09 DF
10	01 DF 87 96 D9 C2	00 00 77 57 E6 09 DF
11	01 DF BA 05 06 50 B4	00 5E E6 BB 6C 3F 9E
12	01 62 B4 E9 48 F7 57	00 00 5E E6 BB 6C 3F 9E

Op count: mul $= 123$, div $= 6$

Berlekamp-Massey example (4)

6EC Reed-Solomon code over $\mathrm{GF}(2^8)$. 7 errors.

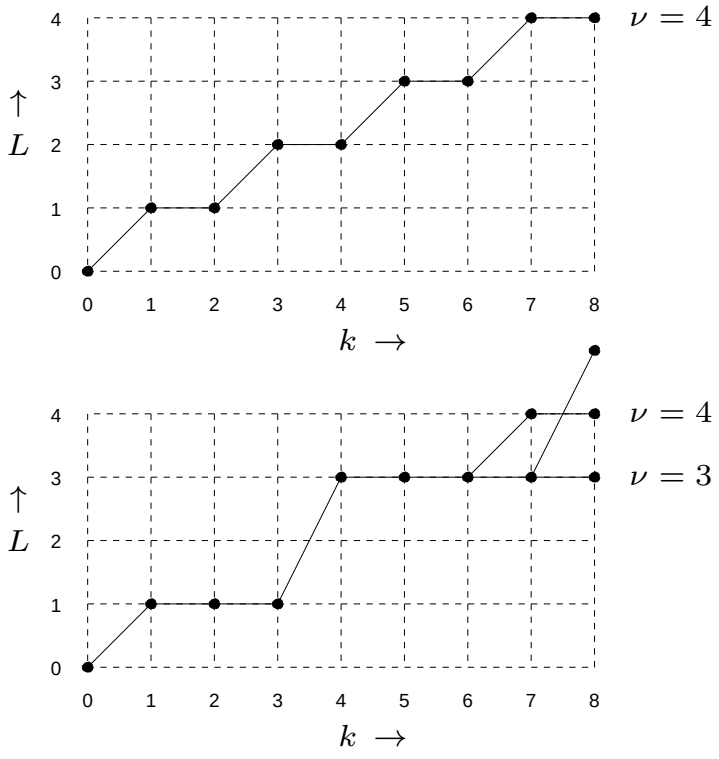
$S =$ `f1 9f 5e 6e 5c 52 b2 46 02 99 b2 17`

k	$\Lambda^{(k)}(x)$	$T^{(k)}(x)$
1	01 F1	00 E7
2	01 CC	00 00 E7
3	01 CC 24	00 CE 0B
4	01 9D D9	00 00 CE 0B
5	01 9D 0A A2	00 30 6F 33
6	01 04 1B 64	00 00 30 6F 33
7	01 04 9C A5 BD	00 4C 2D 3A B2
8	01 8F 8A 51 66	00 00 4C 2D 3A B2
9	01 8F 1E 3B A6 F3	00 F3 04 1C 6E 0D
10	01 0B D8 53 10 1B	00 00 F3 04 1C 6E 0D
11	01 0B 36 CA F8 B6 D7	00 57 7B 37 84 19 62
12	01 0F 1A 8D A9 F6 BB	00 00 57 7B 37 84 19 62

Op count: $\mathrm{mul} = 123\,,\ \mathrm{div} = 6$

If there are 7 errors, the Berlekamp-Massey algorithm usually produces a polynomial $\Lambda(x)$ of degree 6. But $\Lambda(x)$ has 6 zeroes in $\mathrm{GF}(2^m)$ with probability $1/6! =$ the conditional probability of miscorrection.

Berlekamp-Massey algorithm: program flow



Berlekamp-Massey: computational cost

The Berlekamp-Massey algorithm keeps a current estimate of

- *connection polynomial* $\Lambda(x)$
- *correction polynomial* $T(x)$.

When necessary $\Lambda(x)$ and $T(x)$ are updated by parallel assignment:

$$\begin{bmatrix}\Lambda(x)\\ T(x)\end{bmatrix} \leftarrow \begin{bmatrix}\Lambda(x) - \Delta T(x)\\ \Delta^{-1} x\Lambda(x)\end{bmatrix}$$

Storage requirements: $2t$ decoder alphabet symbols, for $\Lambda(x)$ and $T(x)$.

Worst case running time (multiply/divide):

$$2 + 4 + \cdots + 4t \approx 4t^2$$

The running time with a fixed number of multipliers is $O(t^2)$.

If t multipliers are available, the algorithm can be performed in $O(t)$ steps. Some authors refer to this as *linear* run time.

Solving error-locator polynomials: degree 2

Polynomials over $\mathrm{GF}(2^m)$ of degree ≤ 4 can be factored using linear methods.

Consider an error-locator polynomial of degree 2:

$$\Lambda(x) = 1 + \Lambda_1 x + \Lambda_2 x^2 \,.$$

Squaring is a linear transformation of $\mathrm{GF}(2^m)$ over scalar field $\mathrm{GF}(2)$.

Therefore the equation $\Lambda(x) = 0$ can be rewritten as

$$x(\Lambda_2 S + \Lambda_1 I) = 1 \,,$$

where x is the unknown m-tuple, S is the $m \times m$ matrix over $\mathrm{GF}(2)$ that represents squaring, and 1 is the m-tuple $(1, 0, \ldots, 0)$.

If there are two distinct solutions, they are the zeroes of $\Lambda(x)$.

The squaring matrix S can be precomputed, so the coefficients of the $m \times m$ matrix $A = \Lambda_2 S + \Lambda_1 I$ can be computed in $O(m^2)$ bit operations

Solving the system requires $O(m^3)$ bit operations or $O(m^2)$ word operations.

Solving error-locator polynomials faster: degree 2

Use the change of variables $x = \frac{\Lambda_1}{\Lambda_2} u$. Then $\Lambda(x) = 0$ becomes

$$\begin{aligned}\Lambda(x) &= \Lambda_2 x^2 + \Lambda_1 x + 1 \\ &= \Lambda_2 \left(\frac{\Lambda_1}{\Lambda_2} u\right)^2 + \Lambda_1 \left(\frac{\Lambda_1}{\Lambda_2} u\right) + 1 \\ &= \frac{\Lambda_1^2}{\Lambda_2} u^2 + \frac{\Lambda_1^2}{\Lambda_2} u + 1 \;=\; \frac{\Lambda_1^2}{\Lambda_2}\left(u^2 + u + \frac{\Lambda_2}{\Lambda_1^2}\right)\end{aligned}$$

The simplified equation is of the form $u^2 + u + c = 0$.

It can be solved using the precomputed pseudo-inverse of $S + I$.

If U_1 is a zero of $u^2 + u + \frac{\Lambda_2}{\Lambda_1^2}$, then $X_1 = \frac{\Lambda_1}{\Lambda_2} U_1$ is a zero of $\Lambda(x)$, as is

$$X_2 = \frac{\Lambda_1}{\Lambda_2}(U_1 + 1) = X_1 + \frac{\Lambda_1}{\Lambda_2}.$$

If 2^m is not too large, we can store a table of zeroes of $u^2 + u + c$.

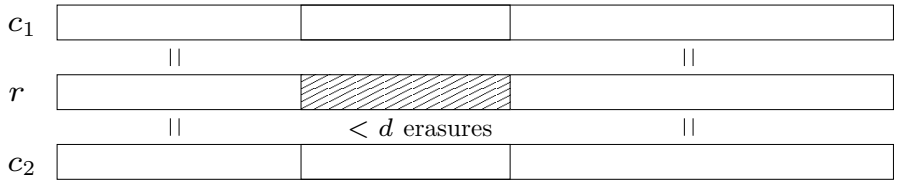
Erasure correction

Erasures are special received symbols used to represent uncertainty. Examples:

- Demodulator erases a symbol when signal quality is poor.
- Lower level decoder erases symbols of codeword that has an uncorrectable error.

Theorem: A block code can correct up to $d^* - 1$ erasures.

Proof: If the number of erasures is less than d^*, then there is only one codeword that agrees with the received sequence.



Conversely, if two codewords differ in exactly d^* symbols, the received sequence obtained by erasing the differing symbols cannot be decoded.

Fact: A block code can correct t errors and ρ erasures iff $d^* \geq 2t + \rho + 1$.

Erasure correction for linear block codes

Erasures can be corrected for *linear* block codes by solving linear equations.

The equation $\mathbf{c}H^T = 0$ gives $n-k$ equations for the ρ erasure values.

Any d^*-1 columns of H are linearly indepenent, so the equations can be solved when $\rho < d^* \leq n-k+1$.

Example: Let $\mathbf{r} = [\,0\,0\,?\,?\,0\,1\,0\,]$ be received sequence for $(7,4)$ Hamming code.

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The parity-check matrix yields three equations for the erased bits x and y:

$$\begin{aligned} 0 &= 1\cdot 0 + 0\cdot 0 + 0\cdot x + 1\cdot y + 0\cdot 0 + 1\cdot 1 + 1\cdot 0 = 1+y \\ 0 &= 0\cdot 0 + 1\cdot 0 + 0\cdot x + 1\cdot y + 1\cdot 0 + 1\cdot 1 + 0\cdot 0 = 1+y \\ 0 &= 0\cdot 0 + 0\cdot 0 + 1\cdot x + 0\cdot y + 1\cdot 0 + 1\cdot 1 + 1\cdot 0 = 1+x \end{aligned}$$

Therefore $x=1$, $y=1$ and the decoded codeword is $\mathbf{c} = [\,0\,0\,1\,1\,0\,1\,0\,]$.

Erasure correction for BCH codes (1)

Consider a BCH code defined by parameters (α, n, b, d).

Suppose $\rho < d$ erasures in locations $j_1, \ldots, j_\rho$ and no errors.

Define the *erasure locators*

$$U_l = \alpha^{j_l}, \quad l = 1, \ldots, \rho\,.$$

The syndrome equations for the erasure magnitudes are

$$\begin{aligned} S_1 &= E_1 U_1^b + E_2 U_2^b + \cdots + E_\rho U_\rho^b \\ S_2 &= E_1 U_1^{b+1} + E_2 U_2^{b+1} + \cdots + E_\rho U_\rho^{b+1} \\ &\vdots \qquad\qquad \vdots \\ S_{d-1} &= E_1 U_1^{b+d-2} + E_2 U_2^{b+d-2} + \cdots + E_\rho U_\rho^{b+d-2} \end{aligned}$$

This system of linear equations has a unique solution for $E_1, E_2, \ldots, E_\rho$ because the coefficient matrix is column-scaled Vandermonde.

The Forney algorithm provides a faster solution.

Erasure correction for BCH codes (2)

The *erasure locator polynomial* is defined by

$$\Gamma(x) \;=\; \prod_{l=1}^{\rho} (1 - U_l x) \;=\; 1 + \Gamma_1 x + \Gamma_2 x^2 + \cdots + \Gamma_\rho x^\rho$$

Unlike the error locator polynomial, the values of U_l are known.

The coefficients of $\Gamma(x)$ can be computed by polynomial multiplication.

$$\prod_{l=1}^{i} (1 - U_l x) = (1 - U_i x) \prod_{l=1}^{i-1} (1 - U_l x)$$

For each i we use $i - 1$ multiply-accumulates. Total operations: $\frac{1}{2}\rho(\rho - 1)$.

The Forney algorithm gives values of errors in erasure locations:

$$E_l = - U_l^{1-b} \frac{\Omega(U_l^{-1})}{\Gamma'(U_l^{-1})}, \quad l = 1, \ldots, \rho\,.$$

where $\Omega(X) = S(x)\Gamma(x) \bmod x^{2t}$ has degree $\leq \rho - 1$.

$$\Omega_0 = S_1, \;\; \Omega_1 = S_2 + S_1\Gamma_1, \;\; \ldots, \;\; \Omega_{p-1} = S_\rho + S_{\rho-1}\Gamma_1 + \cdots + S_1\Gamma_\rho$$

Computing ρ error magnitudes takes $\approx \frac{5}{2}\rho^2$ multiply-accumulates.

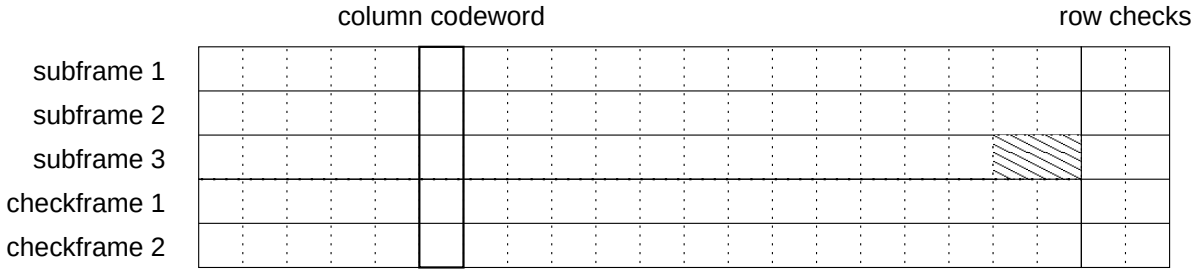
Erasure correction example: wireless network

Random bit errors, large collision rate. Maximum packet size: 600 bytes

Encoding procedure for *concatenated code*:

- Divide data into three equal rows
- Create two check rows by (5,3) shortened Reed-Solomon code on columns
- Encode rows with shortened (255,239) 2EC BCH code on *fragments* of ≤ 29 data bytes

Example: 58-byte frame. Subframes have 20 bytes and 2 BCH check bytes.



Decoding procedure for concatenated code

BCH code corrects 2 bit errors in ≤ 31 bytes, since

$$29 \cdot 8 + 16 = 232 + 16 = 248 \leq 255 = 2^8 - 1\,.$$

A subframe with 200 bytes requrires $\lceil 200/29 \rceil = 7$ fragments.

Exercise: Find probability that a frame is lost because of random errors.

Row miscorrections and burst errors are corrected using $(5, 3)$ column code. Up to two lost subframes can be replaced.

Erasure correction procedure requires solving linear equations.

We precompute the inverses of coefficient matrices for the $\binom{5}{2} = 10$ possible combinations of two lost subframes.

Each byte in missing subframe is computed using 3 Galois field multiplications.

Software correction takes 10 to 15 M68000 machine instructions per byte.

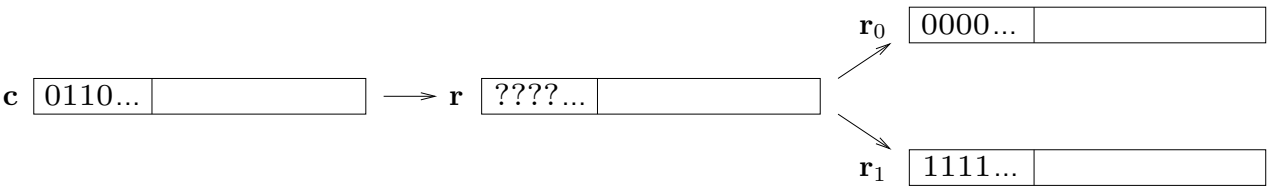
Trick to reduce time: store logarithms of precomputed matrix constants.

When only one subframe is lost, it can be replaced by XORing the other subframes.

Error and erasure decoding: binary case

If there are ρ erasures in a binary senseword $\mathbf{r}$, then $t = \lfloor \frac{1}{2}(d^* - 1 - \rho) \rfloor$ errors can be corrected using an errors-only decoder:

1. Let $\mathbf{c}_0$ and $\mathbf{c}_1$ be the codewords obtained by decoding the n-tuples $\mathbf{r}_0$ and $\mathbf{r}_1$ obtained from $\mathbf{r}$ by replacing all erasures with zeroes and ones, respectively.
2. Compare $\mathbf{c}_0$ and $\mathbf{c}_1$ with $\mathbf{r}$ and let $\hat{\mathbf{c}}$ be the one that is closer to $\mathbf{r}$. (Either $\mathbf{c}_0$ or $\mathbf{c}_1$ or both might be undefined because of decoder failure. An undefined $\mathbf{c}_i$ is ignored.)



If number of errors is $\leq \lfloor \frac{1}{2}(d^* - 1 - \rho) \rfloor$ then

$$\begin{aligned} d_H(\hat{\mathbf{c}}, \mathbf{r}) &\leq \lfloor \tfrac{1}{2}\rho \rfloor + \lfloor \tfrac{1}{2}(d^* - 1 - \rho) \rfloor \\ &\leq \tfrac{1}{2}\rho + \tfrac{1}{2}(d^* - 1 - \rho) \;=\; \tfrac{1}{2}(d^* - 1)\,. \end{aligned}$$

This shows that $\mathbf{r}$ is within the decoding sphere of $\hat{\mathbf{c}}$.

Error and erasure correction: Berlekamp-Massey

1. Compute erasure locator polynomial $\Gamma(x) = \prod_{l=1}^{\rho} (1 - U_l x)$.
2. Compute partial syndrome polynomial $S(x)$ using 0 for erased locations.
3. Compute modified syndrome polynomial $\Xi(x) = S(x)\Gamma(x) \bmod x^{2t}$. Modified syndromes are $\Xi_1, \ldots, \Xi_{2t-\rho}$.
4. Run Berlekamp-Massey algorithm with the modified syndromes $\Xi_1, \ldots, \Xi_{2t-\rho}$ to find the error-locator polynomial $\Lambda(x)$ of degree $\leq \frac{1}{2}(2t - \rho)$.
5. Use the *modified key equation* to find error evaluator polynomial $\Omega(x)$:

   $$\Omega(x) = S(x)\Lambda(x)\Gamma(x) \bmod x^{2t} = S(x)\Psi(x) \bmod x^{2t}$$

   where $\Psi(x) = \Lambda(x)\Gamma(x)$ is the *error-and-erasure locator polynomial*.
6. Use modified Forney algorithm to compute error magnitudes:

   $$Y_i = -X_i^{1-b} \frac{\Omega(X_i^{-1})}{\Psi'(X_i^{-1})}, \quad E_l = -U_l^{1-b} \frac{\Omega(U_l^{-1})}{\Psi'(U_l^{-1})}$$

   for $i = 1, \ldots, \nu$ and $l = 1, \ldots, \rho$.

Erasure correction application: variable redundancy

Some communications systems use varying amounts of error protection:

- An ECC subsystem may be customized for specific application.
- Adaptive system may increase or decrease check symbols as needed.

Obvious approach: use different Reed-Solomon code generator polynomials:

$$g_t(x) = (x + \alpha)(x + \alpha^2) \cdots (x + \alpha^{2t}), \quad t = 1, 2, \ldots, T.$$

Problem: encoders for different generator polynomials require many scalers.

Clever solution: use generator polynomial for maximum error correction but transmit only $2t$ check symbols, that is, delete $\rho = 2T - 2t$ checks symbols.

information symbols	checks	deleted

Then use errors-and-erasures decoding, where missing check symbols are erased.

Syndrome modification (1)

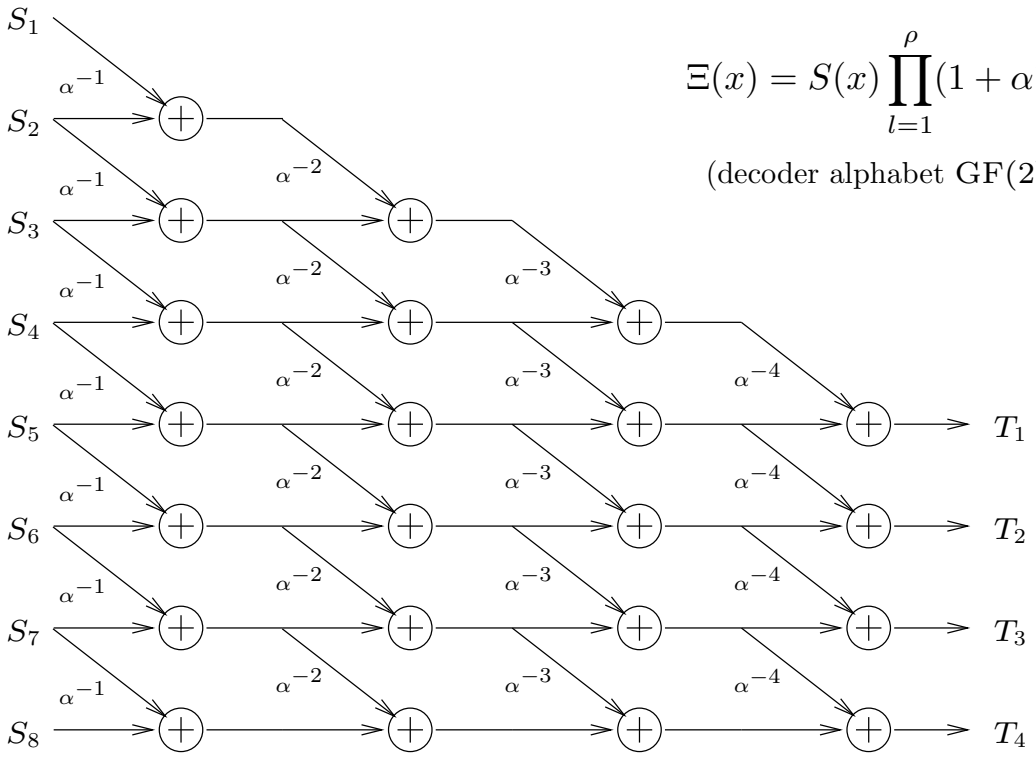
The modified syndrome polynomial for Reed-Solomon code is easy to find.

1. Deleted checks are considered to be in locations $-1, -2, \ldots, -\rho$.
2. Compute modified syndrome using circuit below.
3. Use modified syndromes $T_1, \ldots, T_{2t-\rho}$ in Berlekamp-Massey algorithm.
4. Find zeroes of $\Lambda(x)$.
5. Compute $\Psi(x) = \Gamma(x)\Lambda(x)$ and $\Omega(x) = S(x)\Gamma(x)\Lambda(x) \bmod x^{2t}$.
6. Use Forney algorithm to find error magnitudes and erasure magnitudes.

Erasure correction can be used by an encoder to generate check symbols from partial syndromes of the message symbols.

Partial syndromes may be easier to compute because the circuits are uncoupled, hence fewer long wires are needed.

Syndrome modification (2)



$$\Xi(x) = S(x) \prod_{l=1}^{\rho} (1 + \alpha^{-i} x)$$

(decoder alphabet $\mathrm{GF}(2^m)$)