MATH 515: Numerical Analysis I
Homework 1

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If matrix A is *SPD*, all of its diagonal elements are positive.

Consider $\mathbf{x}^\intercal \mathbf{A}\mathbf{x}$, where $\mathbf{x} = \mathbf{e}_i$, and $\mathbf{e}_i$ is the ith column of the identity matrix $\mathbf{I}$.
Given that A is *SPD*, we have:

$$\mathbf{x}^\intercal \mathbf{A}\mathbf{x} = a_{ii} > 0, i = 1, 2, ..., n \tag{1}$$

If matrix A is *SPD*, $|a_{ij}| < \sqrt{a_{ii}a_{jj}}$ for all $i \neq j$.

As given in the hint, consider $\mathbf{x}^\intercal \mathbf{A}\mathbf{x}$, where $\mathbf{x} = \alpha\mathbf{e}_i - \mathbf{e}_j$, and $\mathbf{e}_i, \mathbf{e}_j$ are separately the ith and jth columns of the identity matrix $\mathbf{I}$. Given that A is *SPD*, we have:

$$\mathbf{x}^\intercal \mathbf{A}\mathbf{x} = (\alpha\mathbf{e}_i - \mathbf{e}_j)^\intercal \mathbf{A}(\alpha\mathbf{e}_i - \mathbf{e}_j) = \alpha^2 a_{ii} - 2\alpha a_{ij} + a_{jj} > 0 \tag{2}$$

The inequality above is in quadratic form and from the result of first question we know that $a_{ii} > 0$ (shape of the parabola is upwards). To let the above quadratic inequality to hold for any real valued α, we have:

$$\begin{array}{lcl} \Delta & = & 4a_{ij}^2 - 4a_{ii}a_{jj} < 0 \qquad (\forall i \neq j) \\ a_{ij}^2 & < & a_{ii}a_{jj} \\ |a_{ij}| & < & \sqrt{a_{ii}a_{jj}} \end{array} \tag{3}$$

What are the results from dense and sparse matrix calculation ?

As Figure 1 shows, when the dimension of the matrix is low, there are not any obvious difference. However, as the dimension of the matrix grows, the time for calculating (even the simple) linear systems increases exponentially.

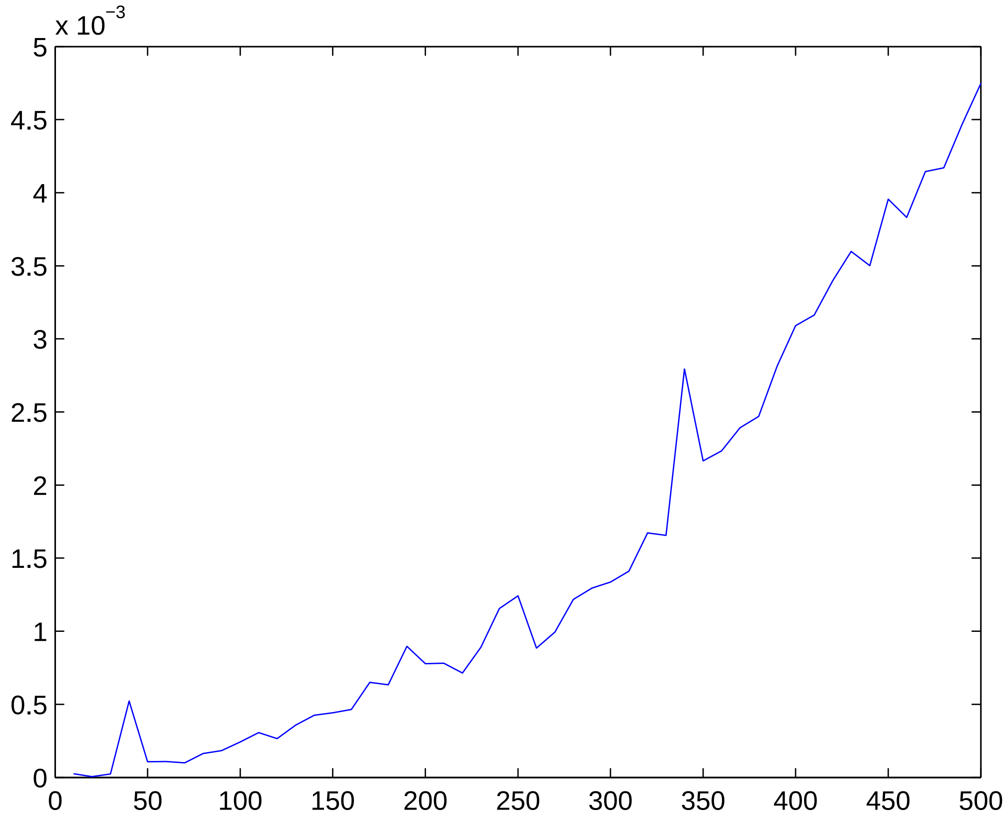

Figure 1: Efficiency comparison for dense and sparse matrix calculations