Rank counting and maximum subsets of $\mathbb{F}_q^n$ containing no right angles

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Abstract

Recently, Croot, Lev, and Pach (Ann. of Math., 185:331–337, 2017.) and Ellenberg and Gijswijt (Ann. of Math., 185:339–443, 2017.) developed a new polynomial method and used it to prove upper bounds for three-term arithmetic progression free sets in $\mathbb{Z}_4^n$ and $\mathbb{F}_3^n$, respectively. Their approach was summarized by Tao as a rank counting argument of functions and hypermatrices. In this paper, we first present a variant of Tao's counting formula, then apply it to obtain an upper bound for the cardinality of maximal subsets of $\mathbb{F}_q^n$ containing no right angles.

To be more precise, we prove that if q is a fixed odd prime power, then the maximal cardinality of a subset A of $\mathbb{F}_q^n$ with no three distinct elements $x, y, z \in A$ satisfying $< z - x, y - x > = 0$ is at most $\binom{n+q}{q-1} + 3$. Our bound substantially improves the known upper bound $\mathcal{O}(q^{\frac{n+2}{3}})$. Moreover, it is tight up to a constant depending only on q. Interestingly, our result is quite different from the previous ones obtained by the newly developed polynomial method, i.e., our upper bound is of magnitude $\text{poly}(n)$ instead of $\exp(n)$.

Keywords: extremal problem, polynomial method, right angle over finite field
Mathematics subject classifications: 05D05, 15A03, 15A69

1 Introduction

1.1 The polynomial method

The recent breakthrough papers of Croot, Lev, and Pach [2] and Ellenberg and Gijswijt [4] showed respectively that three-term arithmetic progression free sets in $\mathbb{Z}_4^n$ and $\mathbb{F}_3^n$ are exponentially small. The core idea involved in their proofs was the application of a novel polynomial method. Later this method was summarized in Tao's blog post [14] as a principle which counts the rank of certain functions and diagonal hypermatrices.

This problem can also be viewed as an extremal problem over finite field which forbids the existence of some given configuration. In order to deal with such a problem using the polynomial method, the basic strategy is to characterize the forbidden configuration by an appropriate polynomial. For example, let A be a subset of $\mathbb{F}_3^n$ with no three-term arithmetic progressions. Then it is equivalent to say that for three elements $x, y, z \in A$, $x + y + z = 0^n$ if and only if $x = y = z$. In [14], this observation is described as an identity between two polynomials defined on $A \times A \times A$ to $\mathbb{F}_3$:

$$\delta_{0^n}(x + y + z) = \sum_{a \in A} \delta_a(x)\delta_a(y)\delta_a(z), \tag{1}$$

where $\delta_a(x)$ is the Kronecker function such that $\delta_a(x) = 1$ if $x = a$, and $\delta_a(x) = 0$ otherwise. Therefore, if we can define an appropriate rank for multivariable functions over $\mathbb{F}_3^n$, then we can possibly bound the cardinality of A from the upper by computing the rank of the functions in (1) in two ways.

We call such extremal problems "symmetric" in the sense that some elements of the target subset form a forbidden configuration if and only if they are all equal. For example, for arbitrary elements $a, b, c \in \mathbb{F}_q$ satisfying $a + b + c = 0$, the problem to determine the maximal cardinality of subsets of $\mathbb{F}_q^n$ with no nontrivial solutions (a solution is trivial if $x = y = z$) to $ax + by + cz = 0^n$ is symmetric. The tri-colored sum-free sets [10] and the sunflower-free sets [11] are both symmetric, too. All of these symmetric extremal problems can be handled by rank counting arguments which are essentially similar to (1). However, in the literature, a lot of "asymmetric" extremal problems are also of great interest. For instance, a family $\mathcal{F} \subseteq 2^{[n]}$ is called 2-cover-free [5] if for arbitrary $A, B, C \in \mathcal{F}$, $A \subseteq B \cup C$ if and only if $A = B$ or $A = C$. Obviously, by our notion this problem is an asymmetric one. The $\{1, 2\}$-separating hash family defined in [12] is also an asymmetric extremal problem.

The goal of this paper is to extend the polynomial method developed by the previous researchers [2, 4, 13, 14] to deal with the asymmetric extremal problems, with the aid of a variant of Tao's counting formula (see Lemma 3 of this paper). Our new counting formula is also a sum of Kronecker functions and it allows us to use a strategy similar to that of [14] to obtain a new upper bound for an extremal problem over finite field, which will be introduced in the next subsection.

1.2 Right angles over $\mathbb{F}_q^n$

Let q be a prime power and $V := \mathbb{F}_q^n$ be an n-dimensional vector space over some finite field $\mathbb{F}_q$. In this paper we will investigate an extremal property of V. We are interested in the maximal cardinality of subsets of V, in which no right angles are contained. A set $A \subseteq V$ is said to contain a right angle if there exist three distinct elements x, y, z of A such that $< z - x, y - x >= 0$, where $< \cdot, \cdot >$ denotes the inner product over $\mathbb{F}_q$. This problem is a finite field version of the Erdős-Flaconer type problem. It is a natural analog of the one in the setting of Euclidean space [6, 7, 8], which asks for given n and α, the smallest d for which any compact set $A \subseteq \mathbb{R}^n$ with Hausdorff dimension larger than d contains three points forming an angle α. The reader is referred to [9] for more finite field analogs of problems in Euclidean space.

Note that if A is free of right angles, then for arbitrary three (not necessarily distinct) elements x, y, z of A, $< z - x, y - x >= 0$ if and only if $z = x$ or $y = x$, or $z = y$ and $< y - x, y - x >= 0$. It is easy to see that this problem is asymmetric and the previous formula (1) (and the methods of [2, 3, 4]) can not be applied to it.

Denote by $R(n, q)$ the maximal cardinality of subsets of $\mathbb{F}_q^n$ with no right angles. In [1], Bennett showed that $R(n, q) \leq \mathcal{O}(q^{\frac{n+2}{3}})$. In this paper, for odd prime power q, it is shown that $R(n, q) \leq \binom{n+q}{q-1} + 3$. One can see that our result substantially improves the previous one for fixed q. For example, when $q = 3$, $R(n, 3)$ is improved from $\mathcal{O}(3^{\frac{n+2}{3}})$ to $(n + 3)^2 + 3$. Our result is more surprising compared to another result of [1], which states that the maximum cardinality of subsets of $\mathbb{F}_q^n$ containing no right angles with the vertex at the all zero vector is $\Theta(q^{\frac{n+2}{2}})$. Our new upper bound is indeed a polynomial function of n. This is very interesting since the bounds obtained using a similar polynomial method in [2, 4, 10, 11] are all exponential functions of n.

We also present a constructive result which shows that $R(n, q) \geq \binom{n}{q-1}$ for every prime power q. Combing the upper and lower bounds, it is easy to see $R(n, q) = \Theta(n^{q-1})$ for odd q. It seems that the following conjecture is reasonable.

Conjecture 1. $R(n, q) = \Theta(n^{q-1})$ holds for all prime power q.

The rest of this paper is organised as follows. In Section 2, we first review the necessary terminologies introduced in [14] and then present our new counting formula. In Section 3, we use this new formula to obtain an upper bound for $R(n, q)$. A simple but nearly optimal construction is also presented in this section. Section 4 consists of some concluding remarks.

2　A variant of Tao's counting formula

We begin with some necessary terminologies introduced in [14]. Let $\mathbb{F}$ be a field and A be a finite set. A function $T : A^k \longrightarrow \mathbb{F}$ defined on k variables $x_1, \ldots, x_k$ is said to be rank one if it is nonzero and of the form $T(x_1, \ldots, x_k) = f(x_i)g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$ for some $1 \leq i \leq k$ and some functions $f : A \longrightarrow \mathbb{F}$ and $g : A^{k-1} \longrightarrow \mathbb{F}$. The rank of a general function $T : A^k \longrightarrow \mathbb{F}$ is the least number of rank one functions needed to express T as a linear combination. For example, when $k = 2$, a rank one function is of the form $T(x, y) = f(x)g(y)$ for some $f, g : A \longrightarrow \mathbb{F}$. When $k = 3$, the rank one functions take the form $T_1(x, y, z) = f_1(x)g_1(y, z)$, $T_2(x, y, z) = f_2(y)g_2(x, z)$ and $T_3(x, y, z) = f_3(z)g_3(x, y)$ for some $f_i : A \longrightarrow \mathbb{F}$ and $g_i : A^2 \longrightarrow \mathbb{F}$, $1 \leq i \leq 3$. Note that the linear combination of r rank one functions will give a function of rank at most r.

In [14], it was shown that the rank of $\sum_{a \in A} \delta_a(x)\delta_a(y)\delta_a(z)$ is equal to $|A|$. The main purpose of this section is to prove a similar result, which states that the rank of $\sum_{a \in A} (\delta_a(y)\delta_a(z) + (1 - \delta_a(y))(1 - \delta_a(z)))\delta_a(x)$ is at least $|A| - 2$.

Lemma 2. *Let* $A = \{a_1, \ldots, a_m\}$ *be a finite set with cardinality* m *and let* $\mathbb{F}$ *be a finite field with odd characteristic. For each* $a \in A$, *let* $c_a \in \mathbb{F}$ *be a nonzero coefficient. Then the rank of the function* $T(y, z) : A \times A \longrightarrow \mathbb{F}$ *defined by*

$$T(y, z) = \sum_{a \in A} c_a \cdot (\delta_a(y)\delta_a(z) + (1 - \delta_a(y))(1 - \delta_a(z))) \tag{2}$$

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is at least $m - 2$.

Proof. Suppose that $r(T(y, z)) = s$, i.e., we have a representation $T(y, z) = \sum_{i=1}^{s} f_i(y)g_i(z)$ for some f_i and g_i. Our goal is to show $s \geq m - 2$. Denote $\sum_{a \in A} c_a = \tau$, then one can easily compute to obtain that

$$T(y, z) = \begin{cases} \tau, & y = z, \\ \tau - c_y - c_z, & y \neq z. \end{cases}$$

Consider the $m \times m$ matrix $\mathcal{P}$ formed by

$$\mathcal{P} = \sum_{i=1}^{s} \begin{pmatrix} f_i(a_1) \\ f_i(a_2) \\ \vdots \\ f_i(a_m) \end{pmatrix} \begin{pmatrix} g_i(a_1), & g_i(a_2), & \cdots, & g_i(a_m) \end{pmatrix}.$$

It is well-known that the conventional matrix rank of $\mathcal{P}$ over $\mathbb{F}$ is at most s. Then to prove our lemma it suffices to show the matrix rank of $\mathcal{P}$ is at least $m - 2$. Let us look at the element p_{jk} in the jth row and the kth column of $\mathcal{P}$, one can see that

$$p_{jk} = \sum_{i=1}^{s} f_i(a_j)g_i(a_k) = T(a_j, a_k) = \begin{cases} \tau, & j = k, \\ \tau - c_j - c_k, & j \neq k, \end{cases}$$

where we denote $c_j = c_{a_j}$ for $1 \leq j \leq m$. Thus $\mathcal{P}$ has the following representation

$$\mathcal{P} = \begin{pmatrix} \tau & & & \\ & \tau & & \tau - c_k - c_j \\ & & \ddots & \\ & \tau - c_j - c_k & & \ddots \\ & & & & \tau \end{pmatrix}.$$

By simple Gaussian eliminations (first eliminated by the first column of $\mathcal{P}$ and then by the second row of the new matrix) $\mathcal{P}$ can be translated into the following matrix

$$\begin{pmatrix} \tau & -c_1 - c_2 & -c_1 - c_3 & \cdots & \cdots & -c_1 - c_m \\ \tau - c_1 - c_2 & c_1 + c_2 & c_1 - c_3 & \cdots & \cdots & c_1 - c_m \\ c_2 - c_3 & -2c_2 & 2c_3 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ c_2 - c_m & -2c_2 & 0 & \cdots & 0 & 2c_m \end{pmatrix}.$$

Therefore, the matrix rank of $\mathcal{P}$ is at least $m - 2$, since the matrix above contains an $(m - 2) \times (m - 2)$ diagonal submatrix whose diagonal entries, $2c_3, \ldots, 2c_m$, are all nonzero elements of $\mathbb{F}$. Then the lemma follows easily from $s \geq m - 2$. $\square$

An important observation is that if we add $\delta_a(x)$ to the summation on the right hand side of (2), the rank of the new function $T(x, y, z)$ remains the same.

Lemma 3. *Let $A = \{a_1, \ldots, a_m\}$ be a finite set with cardinality m and let $\mathbb{F}$ be a finite field with odd characteristic. For each $a \in A$, let $c_a \in \mathbb{F}$ be a nonzero coefficient. Then the rank of the function $T(x, y, z) : A \times A \times A \longrightarrow \mathbb{F}$ defined by*

$$T(x, y, z) = \sum_{a \in A} c_a (\delta_a(y) \delta_a(z) + (1 - \delta_a(y))(1 - \delta_a(z))) \delta_a(x)$$

is at least $m - 2$.

Proof. Suppose that $r(T(x, y, z)) \leq m - 3$. In other words, we have an identity

$$T(x, y, z) = \sum_{\alpha \in I_1} f_\alpha(x) g_\alpha(y, z) + \sum_{\beta \in I_2} f_\beta(y) g_\beta(x, z) + \sum_{\gamma \in I_3} f_\gamma(z) g_\gamma(x, y) \quad (3)$$

for some sets I_1, I_2, I_3 with $|I_1| + |I_2| + |I_3| \leq m - 3$.

Consider the linear space over $\mathbb{F}$ of functions orthogonal to all $f_\alpha(x), \alpha \in I_1$, i.e., the space defined by

$$H = \{h : A \longrightarrow \mathbb{F} \mid \sum_{x \in A} f_\alpha(x) h(x) = 0 \text{ for all } \alpha \in I_1\}.$$

Then it holds that $d := \dim_{\mathbb{F}}(H) \geq |A| - |I_1| = m - |I_1|$. Let $\{h_1, \ldots, h_d\}$ be a basis of H. The $d \times m$ matrix generated by $h_i(a_j), 1 \leq i \leq d, 1 \leq j \leq m$, has full row rank and contains one nonsingular $d \times d$ submatrix, whose column set corresponds to some $A' \subseteq A$ with $|A'| = d$. By the nonsingularity of the columns indexed by A', it is not hard to deduce that there exits a function $h \in H$ which takes nonzero value on every element of A'.

If we multiply both sides of (3) by $h(x)$ and sum in x, it follows that

$$\sum_{x \in A} \sum_{a \in A} c_a (\delta_a(y) \delta_a(z) + (1 - \delta_a(y))(1 - \delta_a(z))) \delta_a(x) h(x)$$

$$= \sum_{a \in A} c_a h(a) (\delta_a(y) \delta_a(z) + (1 - \delta_a(y))(1 - \delta_a(z))) \quad (4)$$

$$:= T_1(y, z)$$

and

$$\sum_{x \in A} (\sum_{\alpha \in I_1} f_\alpha(x) g_\alpha(y, z) + \sum_{\beta \in I_2} f_\beta(y) g_\beta(x, z) + \sum_{\gamma \in I_3} f_\gamma(z) g_\gamma(x, y)) h(x)$$

$$= \sum_{\alpha \in I_1} g_\alpha(y, z) \sum_{x \in A} f_\alpha(x) h(x) + \sum_{\beta \in I_2} f_\beta(y) \sum_{x \in A} g_\beta(x, z) h(x)$$

$$+ \sum_{\gamma \in I_3} \sum_{x \in A} f_\gamma(z) g_\gamma(x, y) h(x) \quad (5)$$

$$= \sum_{\beta \in I_2} f_\beta(y) \sum_{x \in A} g_\beta(x, z) h(x) + \sum_{\gamma \in I_3} f_\gamma(z) \sum_{x \in A} g_\gamma(x, y) h(x)$$

$$:= T_2(x, y, z),$$

where the second equality in (5) follows from the fact that $h \in H$. Note that $c_a \neq 0$ for all $a \in A$ and $h(x)$ is nonzero on at least $m - |I_1|$ elements of A.

Then the number of nonzeros in $\{c_a h(a) : a \in A\}$ is at least $m - |I_1|$. By Lemma 2, $r(T_1(y, z)) \geq m - |I_1| - 2$. On the other hand, it is obvious to see that $r(T_2(x, y, z)) \leq |I_2| + |I_3| \leq m - 3 - |I_1|$. Since $T_1(y, z) = \sum_{x \in A} T(x, y, z) h(x) = T_2(x, y, z)$, one can conclude that $m - |I_1| - 2 \leq r(\sum_{x \in A} T(x, y, z) h(x)) \leq m - 3 - |I_1|$, which is a contradiction. Therefore, it holds that $r(T(x, y, z)) \geq m - 2$. $\quad\square$

Remark 4. *Lemmas 2 and 3 are the "asymmetric" versions of Lemma 1 in [14], which can be used to deal with the asymmetric extremal problems defined in Section 1. Note that Lemma 2 only works for the case where the underlying finite field has an odd characteristic. This is because that in our problem setting $< z - x, y - x >$ possibly takes value zero when $z = y$ and $< y - x, y - x >= 0$. For other applications as x, y, z satisfy some property "P" if and only if $x = y$ or $x = z$ (but not under the condition when $y = z$), we can remove the odd characteristic restriction and the right hand side of (2) should be $\sum_{a \in A} c_a \cdot (1 - \delta_a(y))(1 - \delta_a(z))$. This argument can also be extended to the more general case when some elements $x_0, x_1, \ldots, x_k$ satisfy some property "P" if and only if $x_0 = x_i$ for some $1 \leq i \leq k$. The proof is very similar to those of Lemmas 2 and 3.*

3 Large subsets of $\mathbb{F}_q^n$ containing no right angles

In this section, we use the rank counting lemma established in the previous section to prove the main result of this paper.

Theorem 5. *Assume that q is an odd prime power. Let A be a subset of $\mathbb{F}_q^n$ such that there exist no three distinct elements $x, y, z \in A$ satisfying $< z - x, y - x >= 0$, then $|A| \leq \binom{n+q}{q-1} + 3$.*

Proof. Define a function $f : A \times A \times A \longrightarrow \mathbb{F}_q$ by

$$f(x, y, z) = \sum_{a \in A} \delta_a(y) \delta_a(z) + (1 - \sum_{a \in A} \delta_a(y) \delta_a(z)) < z - x, y - x >^{q-1}. \qquad (6)$$

Since A contains no right angles, $< y - x, z - x > \neq 0$ for distinct $x, y, z \in A$. So $< y - x, z - x >^{q-1} = 1$ if x, y, z are distinct elements of A. It is easy to check case by case that

$$f(x, y, z) = \begin{cases} 0, & y \neq z \text{ and } x = y \text{ or } x = z, \\ 1, & otherwise. \end{cases} \qquad (7)$$

Recall the function $T(x, y, z)$ defined in Lemma 3. If we take all the coefficients c_a equal to 1, and renew $T(x, y, z)$ as

$$T(x, y, z) = \sum_{a \in A} (\delta_a(y) \delta_a(z) + (1 - \delta_a(y))(1 - \delta_a(z))) \delta_a(x),$$

then it is also not hard to verify that

$$T(x, y, z) = \begin{cases} 0, & y \neq z \text{ and } x = y \text{ or } x = z, \\ 1, & otherwise. \end{cases} \qquad (8)$$

Therefore, formulas (7) and (8) show that $T(x, y, z) = f(x, y, z)$ as functions defined on $A \times A \times A \longrightarrow \mathbb{F}_q$. On one hand, by Lemma 3, we have $r(f(x, y, z)) = r(T(x, y, z)) \geq |A| - 2$. On the other hand, by (6) we can express $f(x, y, z)$ as

$$f(x, y, z) = \sum_{a \in A} \delta_a(y)\delta_a(z) + (1 - \sum_{a \in A} \delta_a(y)\delta_a(z)) \cdot$$

$$(\sum_{i=1}^{n} y_i z_i + \sum_{i=1}^{n} x_i^2 - x_1(y_1 + z_1) - \cdots - x_n(y_n + z_n))^{q-1}$$

$$= H_1(y, z) + H_2(y, z)(F_1(y, z) + F_2(x) - x_1(y_1 + z_1) - \cdots - x_n(y_n + z_n))^{q-1},$$

where $H_1(y, z) = \sum_{a \in A} \delta_a(y)\delta_a(z)$, $H_2(y, z) = 1 - \sum_{a \in A} \delta_a(y)\delta_a(z)$, $F_1(y, z) = \sum_{i=1}^{n} y_i z_i$ and $F_2(x) = \sum_{i=1}^{n} x_i^2$. One can verify that in the above expression of $f(x, y, z)$, each monomial is of the form $H_1(y, z)$ or

$$H_2(y, z)(F_1(y, z))^i (F_2(x))^j (x_1(y_1 + z_1))^{k_1} \cdots (x_n(y_n + z_n))^{k_n},$$

where $i, j, k_1, \cdots, k_n$ are nonnegative integers summing up to $q - 1$. Thus all of the monomials can be written as a rank one function $H_1(y, z)$ or

$$(F_2(x)^j x_1^{k_1} \cdots x_n^{k_n})(H_2(y, z)F_1(y, z)^i (y_1 + z_1)^{k_1} \cdots (y_n + z_n)^{k_n}).$$

Therefore, the rank of $f(x, y, z) - H_1(y, z)$ is upper bounded by the number of distinct monomials appeared in the expansion of $(X_1 + \cdots + X_{n+2})^{q-1}$, which is $\binom{n+q}{q-1}$. The theorem follows from the inequality $|A| - 2 \leq r(T(x, y, z)) = r(f(x, y, z)) \leq \binom{n+q}{q-1} + 1$. $\qquad\square$

Next we will present a simple but nearly optimal construction which indicates that $R(n, q) \geq \binom{n}{q-1}$. Our construction shows that the bound obtained in Theorem 5 is tight up to a constant depending only on q.

Construction 6. *Denote $[n] = \{1, \ldots, n\}$ and let $\binom{[n]}{q-1}$ denote the collection of all subsets of $[n]$ with $q - 1$ elements. For every subset $S \in \binom{[n]}{q-1}$, let $1^S = (1_1^S, \ldots, 1_n^S) \in \mathbb{F}_q^n$ be the vector such that $1_i^S = 1$ if $i \in S$ and $1_i^S = 0$ otherwise. We claim that $A = \{1^S \mid S \in \binom{[n]}{q-1}\}$ contains no right angles.*

Proof. Assume, to the contrary, that there exist three distinct elements $x, y, z \in A$ such that $< y - x, z - x >= 0$. Then it holds that

$$0 = <y - x, z - x> = \sum_{i=1}^{n} (y_i - x_i)(z_i - x_i)$$

$$= \sum_{i=1}^{n} y_i z_i + \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i(y_i + z_i)$$

$$= \sum_{i=1}^{n} y_i z_i + (q - 1) - 2(q - 1),$$

which implies that $\sum_{i=1}^{n} y_i z_i = q - 1$. However, this is not possible given $y \neq z$, since y and z both consist of exactly $q - 1$ 1s in $q - 1$ not totally identical coordinates. $\qquad\square$

Combing Theorem 5 and Construction 6, it is easy to obtain the following theorem.

Theorem 7. $R(n, q) = \Theta(n^{q-1})$ *for odd prime power q.*

4 Conclusions

In this paper, we first introduce a new counting technique, then apply this technique to obtain a new upper bound for an extremal problem over finite field. We think that our method is of interest, and may have new applications and further generalizations. We feel that Conjecture 1 may be solved by another trick of the polynomial method. It will be also very interesting to determine $R(n, q)$ explicitly for some small values of q.

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