Communication Complexity and the Log-Rank Conjecture

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- Let X, Y be two sets of inputs.
  - For simplicity, $X = \{1, 2, \ldots, N\}, Y = \{1, 2, \ldots, M\}$.
- $f : X \times Y \to \{0, 1\}$
  - Or, equivalently, $f : X \times Y \to \{-1, 1\}$
- There are two players: Alice and Bob. **Both know** f.
- Bob knows $x \in X$, Alice knows $y \in Y$.
- Alice and Bob want to calculate $f(x, y)$.
  - They can precompute as much as they want about f.

Communication Complexity
Definitions

We can think of f as the matrix

$$M \in \{0,1\}^{X \times Y}$$

where

$$M_{xy} = f(x,y)$$

$$\begin{array}{cc} & \text{Bob: } x \in X \\ \begin{array}{c}\text{Alice:} \\ y \in Y\end{array} & \begin{array}{|c|c|c|c|c|} \hline 1 & 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 1 & 1 \\ \hline \end{array} \end{array}$$

$$f(2,4) = 1$$

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Definition

CC(f), the **Communication Complexity** of f, is the minimum required number of bits relayed between Alice and Bob to deterministically calculate $f(x, y)$.

By convention, the last bit transferred is $f(x, y)$.

$$\begin{array}{c} \text{Alice:} \\ y \in Y \end{array} \left\{ \overbrace{\begin{array}{|c|c|c|c|c|} \hline 1 & 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 1 & 1 \\ \hline \end{array}}^{\text{Bob: } x \in X} \right.$$

$$f(2, 4) = 1$$

Communication Complexity

Examples

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Theorem

Let $f_{odd} = x + y \mod 2$.
Then $CC(f_{odd}) = 2$.

- $CC(f_{odd}) \leq 2$: Alice sends $y \mod 2$.
- $CC(f_{odd}) \geq 2$: One bit for the answer. At least one bit for Bob to differentiate between $f_{odd}(x, y)$ and $f_{odd}(x, y + 1)$.

Bob: $x \in [0, N]$

Alice: $y \in [0, M]$

$$\begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 \\ \hline \end{array}$$

$f_{odd}(1, 3) = 0$

Communication Complexity

Examples

Theorem

Let $f_{eq}(x, y) = 1$ iff $x = y$.
Then $CC(f_{eq}) = \lceil \log_2(N) \rceil + 1$.

- $CC(f_{eq}) \leq \lceil \log_2(N) \rceil + 1$:
  Alice sends Bob y.
- $CC(f_{eq}) \geq \lceil \log_2(N) \rceil + 1$:
  Tricky, as no protocol can be assumed.

$$
\begin{array}{cc}
 & \overbrace{\hphantom{\begin{matrix}1&0&0&0&0\end{matrix}}}^{\text{Bob: } x \in [1, N]} \\
\begin{array}{c}\text{Alice:}\\ y \in [1, N]\end{array} &
\left\{\begin{array}{|c|c|c|c|c|}
\hline
1 & 0 & 0 & 0 & 0 \\ \hline
0 & 1 & 0 & 0 & 0 \\ \hline
0 & 0 & 1 & 0 & 0 \\ \hline
0 & 0 & 0 & 1 & 0 \\ \hline
0 & 0 & 0 & 0 & 1 \\ \hline
\end{array}\right.
\end{array}
$$

$f_{eq}(1, 3) = 0$

Communication Complexity

Examples

Definition

A monochromatic rectangle is a subset $A \times B$, where $A \subseteq X, B \subseteq Y$, and f is constant on $A \times B$.

Definition

A transcript is the ordered list of messages that Alice and Bob pass to communicate.

Bob: $x \in [1, N]$

$$
\text{Alice: } y \in [1, N] \quad \left\{ \overbrace{\begin{array}{|c|c|c|c|c|} \hline 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ \hline \end{array}}^{} \right.
$$

A monochromatic rectangle: $\{1, 3, 5\} \times \{2, 4\}$

Communication Complexity

Examples

Note that if (x, y) and (x', y') produce the same transcript, then so does (x, y').
This is because from Bob's side, this is the same case as (x, y) (He doesn't know y). For Alice, this looks like (x', y').
Since the transcript of a protocol on (x, y) contains $f(x, y)$, all transcripts of that protocol come from a monochromatic rectangle.

Bob: $x \in [1, N]$

Alice: $y \in [1, N]$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

A monochromatic rectangle: $\{1, 3, 5\} \times \{2, 4\}$

Communication Complexity

Examples

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Theorem

$\mathrm{CC}(f) \geq \lceil \log_2 \chi(f) \rceil + 1$
where $\chi(f)$ is the minimum number of monochromatic rectangles needed to cover f.

We have $2^{\mathrm{CC}(f)}$ possible transcripts, each one corresponding to a monochromatic rectangle.

$$\begin{array}{cc} & \text{Bob: } x \in [1, N] \\ \begin{array}{c}\text{Alice:} \\ y \in [1, N]\end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

A monochromatic rectangle: $\{1, 3, 5\} \times \{2, 4\}$

Communication Complexity

Examples

Theorem

Let $f_{eq}(x, y) = 1$ iff $x = y$.
Then $CC(f_{eq}) = \lceil \log_2(N) \rceil + 1$.

Theorem

$CC(f) \geq \lceil \log_2 \chi(f) \rceil + 1$
where $\chi(f)$ is the minimum number of monochromatic rectangles needed to cover f.

- $\chi(f_{eq}) \geq N$, so
  $CC(f_{eq}) \geq \lceil \log_2(N) \rceil + 1$.

Bob: $x \in [1, N]$

Alice: $y \in [1, N]$

$$\begin{array}{cc} & \overbrace{}^{\text{Bob: } x \in [1,N]} \\ \text{Alice: } y \in [1,N] & \left\{ \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix} \right. \end{array}$$

N monochromatic rectangles are required to cover f_{eq}'s ones.

So far, we have that in general (assuming that $\max\{|X|, |Y|\} = N$),

$$O(\log \chi(f)) \leq CC(f) \leq O(\log N).$$

In fact, we even have that

Theorem (Aho, Ullman, and Yannakakis 1983)

$CC(f) \leq O(\log^2 \chi(f))$

Is this good enough? No! We don't really understand $\chi(f)$, and we want something better!

Conjecture (Log-Rank Conjecture)

There exists a universal constant C such that

$$\mathrm{CC}(f) \leq O((\log \operatorname{rank} f)^C)$$

where $\operatorname{rank} f$ *denotes the rank of f's matrix over $\mathbb{R}$.*

Note: Rank varies by at most one between $\{0,1\}$ and $\{-1,1\}$ versions of the matrices.

Theorem

$\operatorname{rank} f \leq \chi(f)$

Let f's matrix be M_f.
Suppose that $R_1, \ldots, R_k$ are disjoint monochromatic rectangles covering all (x, y) where $f(x, y) = 1$. Then $\chi \leq \chi(f)$.
Let M_i be the indicator matrix of R_i, equal to one only in R_i.
M_i has rank 1.
Furthermore, $M_f = \sum_1^{\chi} M_i$, so

$$\operatorname{rank} M_f \leq \sum_1^{\chi} \operatorname{rank} M_i = \chi \leq \chi(f)$$

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Theorem (Lovett 2016)

Given $r = \operatorname{rank} f$,

$$\mathrm{CC}(f) \leq O(\sqrt{r} \log r)$$

- Finds large nearly monochromatic rectangles in a $\{-1, 1\}$-matrix (This is the innovation).
- Uses rank to find large fully monochromatic rectangles within.
- Establishes a protocol based on such rectangles.
- (Rothvoß 2014) gives a beautiful direct proof condensing all of the various proofs that Lovett used into a simple procedure.

- Sometimes, $\operatorname{rank}_{\mathbb{Z}_2} f = O(\log \operatorname{rank}_{\mathbb{R}} f)$ (I have examples).
  Sometimes, $\mathrm{CC}(f) = \operatorname{rank}_{\mathbb{Z}_2} f$.
  The log-rank conjecture would show these are the same times!
- Given the tight relation with monochromatic rectangles, the Log-Rank Conjecture can be restated in terms of graph chromatic number and adjacency matrix rank.

Other directions

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Theorem (Tsang et al. 2013)

For any boolean function f, let $f^{\oplus}(x, y) = f(x \oplus y)$. Then

$$\mathrm{CC}(f^{\oplus}) \leq O(\sqrt{\operatorname{rank} f^{\oplus}} \log \operatorname{rank} f^{\oplus})$$

This theorem is proved via Fourier analysis rather than hyperdimensional geometry!
Furthermore, there are randomized, undeterministic, quantum, multi-party, and I'm surely even more variations on the subject!

Bibliography

Any Questions?

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