An iterative algorithm for computing the dominant eigenpair of an elliptic matrix

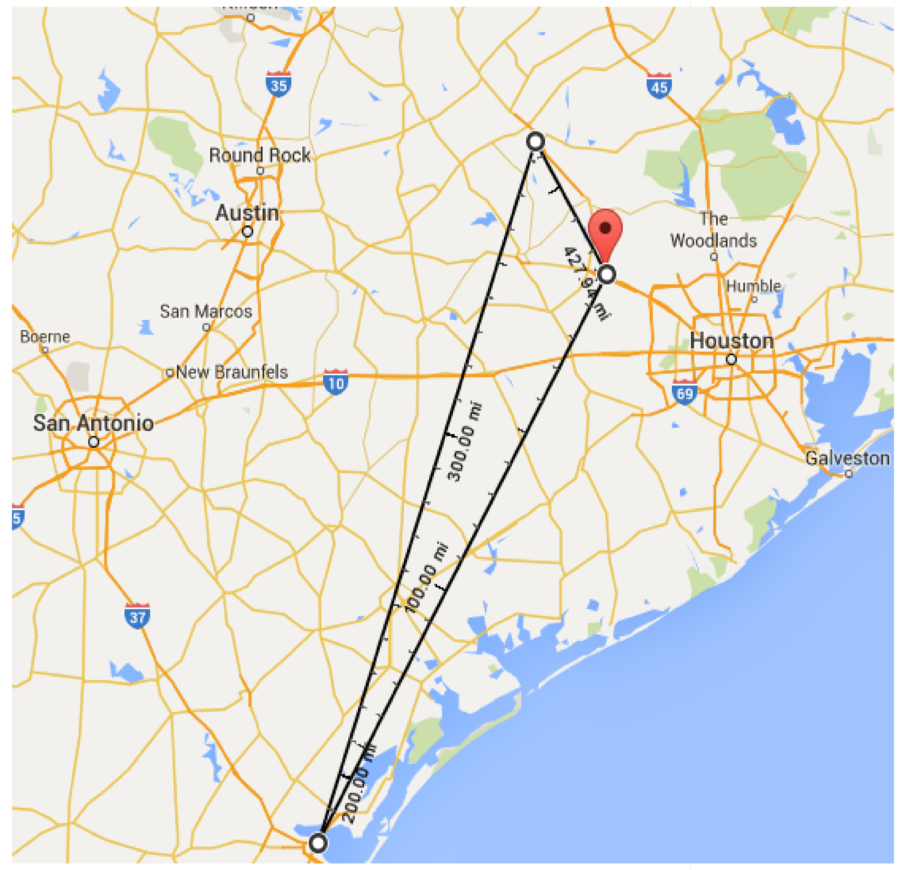
LSAMP THE TEXAS A&M UNIVERSITY SYSTEM
Louis Stokes Alliance for Minority Participation

Angelo Marney
Department of Mathematics & Statistics
Texas A&M University-Corpus Christi

TEXAS A&M UNIVERSITY CORPUS CHRISTI

Abstract

An elliptic matrix is a symmetric matrix with exactly one positive simple eigenvalue. For the set of elliptic matrices, the remaining (n – 1) eigenvalues are zero or negative, so multiplying a matrix in this set by itself gives a new matrix with non-negative eigenvalues. Taking advantage of this eigenvalue structure, we develop an algorithm for computing the dominant eigenpair for these matrices. This algorithm proceeds similarly to the Power Method, in that iterations are performed by taking matrix-vector products, making it a viable option for large matrices. Numerical evidence demonstrates that this algorithm generally outperforms the Power Method for the set of elliptic matrices.



$$
\begin{array}{cccc}
 & \text{College Station} & \text{Corpus Christi} & \text{Prairie View} \\
 & \left[\begin{array}{c} 0 \\ 206^2 \\ 42^2 \end{array}\right. & \begin{array}{c} 206^2 \\ 0 \\ 180^2 \end{array} & \left.\begin{array}{c} 42^2 \\ 180^2 \\ 0 \end{array}\right]
\end{array}
$$

Figure 1. Euclidean distance matrices are an important subset of the elliptic matrices.

Introduction

For a given matrix A, if there is a nonzero vector $x \in R^n$ such that $Ax = \lambda x$ for some scalar λ, then λ is an eigenvalue of A with corresponding eigenvector x. Computing eigenpairs is of great scientific interest. We introduce an algorithm for approximating the positive (dominant) eigenvalue for matrices with exactly one positive (dominant) eigenvalue. An important subset of these matrices that appears in many applications are distance matrices. We label λ_1 as the dominant eigenvalue, and λ_2 as the second largest eigenvalue.

Results

Algorithm 1 $A + A^2$ algorithm

```
procedure FUNCTIONNAME(A, x^(0))
    Input a matrix A.
    Choose a random starting vector x^(0).
    for k = 1, 2, ... do                          ▷ while ||x^(k) − x^(k−1)|| > tol
        z^(k) = (A + A^2) x^(k−1)
        x^(k) = z^(k) / ||z^(k)||
    end for
    x_1 = x^(k)                                   ▷ The associated eigenvector is x_1
    λ_1 = (x_1^t A x_1) / (x_1^t x_1)             ▷ The dominant eigenvalue is λ_1
    return λ_1, x_1
end procedure
```

Proof of convergence:

- Simply mimic the standard proof of convergence for the Power method, but replace A with $(A + A^2)$

Rate of convergence:

- The rate of convergence depends on the speed at which $\left|\frac{\lambda_2+\lambda_2^2}{\lambda_1+\lambda_1^2}\right|^k \to 0$ as $k \to \infty$. Compare this to the Power Method quotient $\left|\frac{\lambda_2}{\lambda_1}\right|^k$. When $\lambda_1 > 0$ and $|\lambda_1| > |\lambda_2| > 0$, it can be proved that $\left|\frac{\lambda_2+\lambda_2^2}{\lambda_1+\lambda_1^2}\right| < \left|\frac{\lambda_2}{\lambda_1}\right|$ using the triangle inequality.

Computational considerations:

- We have twice the number of floating point operations per iteration in our algorithm compared to the Power Method.
- In the worst case for the class of elliptic matrices, we have around half as many iterations.
- When λ_2 is close to λ_1, the Power Method converges very slowly. In these cases, numerical experiments have shown that our algorithm takes less than half the number of iterations than what the Power Method needs.
- When $|\lambda_2| < 1$, we are adding a small negative number to a slightly smaller positive number, causing a cancellation which speeds up the algorithm by a significant amount.

Example

$$
A = \begin{bmatrix} 0 & 1.1647 & 1.9504 & 1.3570 & 1.8676 \\ 1.1647 & 0 & 1.5844 & 1.6407 & 1.2775 \\ 1.9504 & 1.5844 & 0 & 1.0321 & 0.4857 \\ 1.3570 & 1.6407 & 1.0321 & 0 & 0.2713 \\ 1.8676 & 1.2775 & 0.4857 & 0.2713 & 0 \end{bmatrix} \quad x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

A has positive eigenvalue $\lambda_1 \approx 5.1586$ with the rest negative. Rounding to fifth digit:

New Algorithm	Power Method
k = 1: $\lambda_1 \approx 5.1565$	k = 1: $\lambda_1 \approx 5.1335$
k = 2: $\lambda_1 \approx 5.1586$	k = 2: $\lambda_1 \approx 5.1513$
	k = 3: $\lambda_1 \approx 5.1565$
	k = 4: $\lambda_1 \approx 5.1580$
	k = 5: $\lambda_1 \approx 5.1585$
	k = 6: $\lambda_1 \approx 5.1586$

➢ Two iterations versus six iterations

Future Work

- Find bounds on the rate of convergence
- Explore cases outside of the elliptic matrices where similar algorithms work well
- Try different "polynomials" which minimize the rate of convergence quotient

Acknowledgements

I would like to thank NSF and LSAMP for providing funding. I would like to thank Dr. Pablo Tarazaga for his guidance throughout the semesters.

References

1. J.M. Pena, Positive symmetric matrices with exactly one positive eigenvalue, Linear Algebra Appl. (2008).
2. Miroslav Fiedler, Elliptic Matrices with zero diagonal, Linear Algebra Appl. (1994).