Semi-analytical finite element method for guided wave propagation

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Motivation: Nondestructive testing and structural health onitoring

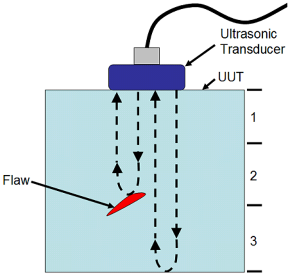
- One application of elastic wave propagation is for use in damage detection and health assessment of structures (such as pipeline).

Figure: A pipeline technician performing a guided wave test on the left. The Trans-Alaska crude oil pipeline on the right. Images taken from wikipedia.

Conventional ultrasonic testing

Ultrasonic waves are sent into an elastic structure by transducer and read back by receiver. The measured amplitude and arrival time can be used to determine the existence of flaws. Downsides:

1. Only inspects small region of structure
2. Transducers must be placed directly on structure



Guided wave inspection

If the structure can act as a waveguide we can instead transmit waves, which are guided by structural boundaries, down the structure.

1. Inspects large regions of structure from single test site
2. Can test physically inaccessible regions of structure

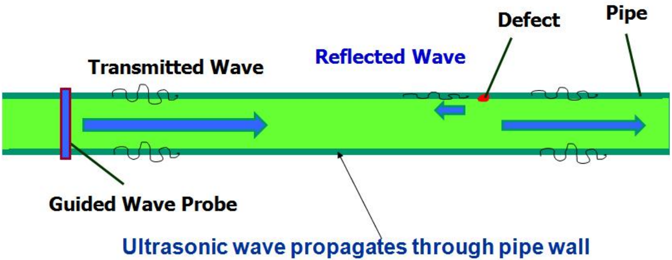


Figure: Image taken from https://guidedwavetesting.com/guided_wave_testing.html

Practical issues with guided wave inspection

- Multiple modes can exist at a given frequency
- Excitation of unwanted modes leads to recieval of unwanted modes

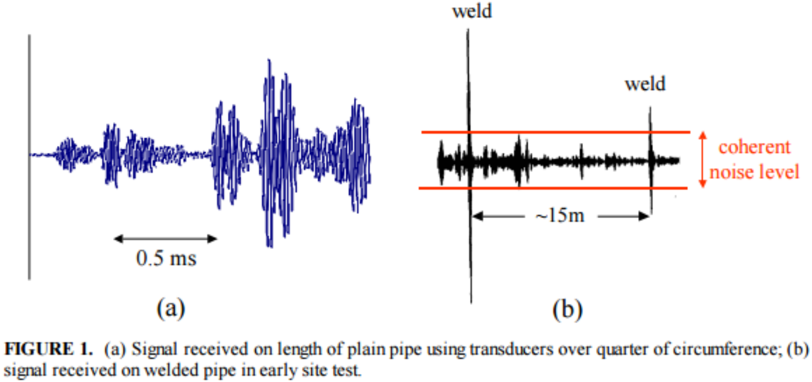


Figure: Unwanted echoes in (a) and coherent noise in (b) are caused by the excitation and reception of unwanted modes (Cawley et al. 2003).

Mode selection by use of dispersion curves

- “The key to controlling coherent noise is therefore to excite and receive a single mode...” - Peter Crawley, Practical Long Range Guided Wave Inspection
- “Wave propagation possibilities must come from points on the dispersion curves and not points in between.” - Rose, Dispersion Curves in GWT

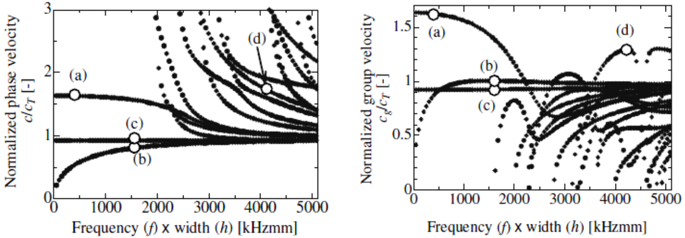


Figure: Dispersion curves for a square rod. Image taken from (Hayashi et al. 2004).

Dispersion curves

- Wave-propagation velocity depends on frequency (f) (dispersion).
- Each wave has a phase-velocity (v_p) and a group-velocity (v_g).
- We plot v_p or v_g against f to get a dispersion curve. To get the values of v_p and v_g we need two parameters: angular frequency ω and wavenumber ξ (inverse of wavelength).

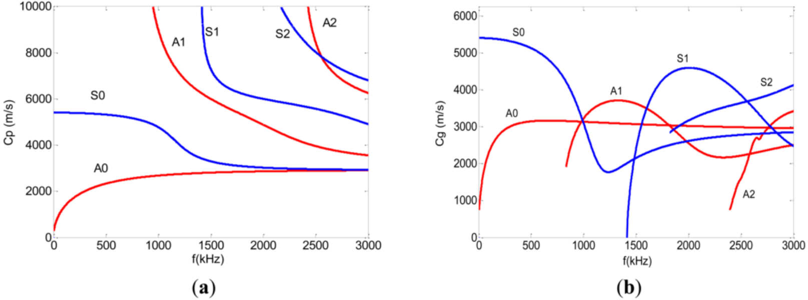


Figure: Phase-velocity (a) and group-velocity (b) curve for 2mm thick aluminum pipe (Wan et al. 2014).

Solid mechanics and linear elasticity

- Strain (ϵ) is the measure of the deformation of a material relative to its reference state; $\epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$.
- Stress ($\boldsymbol{\sigma}$) is a quantity expressing the internal forces within a continuous material.

Fundamental physical assumptions

We are dealing with a homogeneous isotropic linearly elastic material, so:

- Deformation is small (strain is infinitesimal).
- Experienced strain varies linearly with applied stress.
- The stress-strain stiffness tensor can be expressed in terms of two material-dependent elastic constants (Lamé constants λ and μ).

These assumptions lead to the final stress-strain relation:

$$\boldsymbol{\sigma} = \lambda \operatorname{Tr}(\epsilon)\boldsymbol{I} + 2\mu\epsilon \tag{1}$$

Waves in elastic media

The governing equations for elastic wave propagation may be written as

$$\rho \ddot{\boldsymbol{u}} = \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b}. \tag{2}$$

This is a set of three partial differential equations. Applying our final stress-strain relation to equation (2) leads to the Navier-Lamé equation,

$$\rho \ddot{\boldsymbol{u}} = \mu \nabla^2 \boldsymbol{u} + (\lambda + \mu) \nabla (\nabla \cdot \boldsymbol{u}). \tag{3}$$

One can impose boundary conditions (structure-dependent) to get a boundary value problem for **u**. This BVP can be posed in weak form, allowing us to compute an approximation to **u** using a finite element method.

Semi-analytical finite element method (SAFEM)

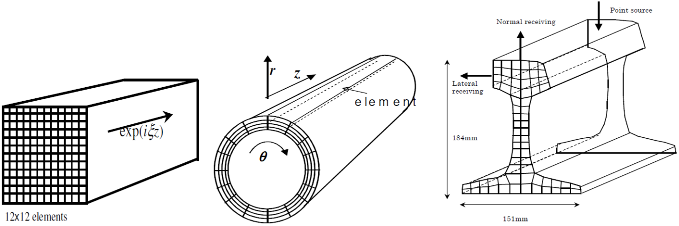
For constant cross-section waveguides, we make the following simplification:

Semi-analytic assumption

The displacement field in the propagation (z) direction has the form $u(z,t) = e^{i(\xi z - \omega t)}$, so we can factorize the displacement field:

$$\mathbf{u} = \mathbf{U}(x,y)e^{i(\xi z - \omega t)}$$

Advantage: we only need mesh a 2D cross section of the structure.



(a) Square rod (b) Straight pipe (c) Rail

SAFEM for computing dispersion curves

This semi-analytical finite element formulation of our problem provides an efficient way to plot a dispersion curve:

1. Substitute $\mathbf{u} = \mathbf{U}(x, y)e^{i(\xi z - \omega t)}$ into the Navier-Lamé equation.
2. Discretizing the boundary value problem allows us to form n by n matrices $\boldsymbol{K_1}$, $\boldsymbol{K_2}$, $\boldsymbol{K_3}$, and $\mathbf{M}$ which satisfy

$$(\omega_m^2 \mathbf{M})\boldsymbol{u_m} = (\boldsymbol{K_1} + i\xi_m \boldsymbol{K_2} + \xi_m^2 \boldsymbol{K_3})\boldsymbol{u_m}$$

3. Input ω and find ξ ... or input ξ and find ω.
4. Use each (ω, ξ) pair to compute v_p or v_g and plot the results.

It is possible to obtain other physically meaningful information as well (attenuation curves, displacement distribution).

Computational challenge

The major computational challenge comes from solving the family of generalized eigenvalue problems

$$(\omega^2 \mathbf{M})\boldsymbol{u_m} = (\boldsymbol{K_1} + i\xi_m \boldsymbol{K_2} + \xi_m^2 \boldsymbol{K_3})\boldsymbol{u_m}. \tag{4}$$

The matrices $\boldsymbol{K_1}$, $\boldsymbol{K_3}$, and $\boldsymbol{M}$ are symmetric, while $\boldsymbol{K_2}$ is skew-symmetric. One can either solve a generalized eigenvalue problem for ω or a quadratic eigenvalue problem for ξ. The quadratic eigenvalue problem can be reduced to a generalized eigenvalue problem with twice the size:

$$\mathbf{Aq} = \xi \mathbf{Bq}.$$

$$\mathbf{A} = \begin{bmatrix} 0 & \boldsymbol{K_1} - \omega^2 \mathbf{M} \\ \boldsymbol{K_1} - \omega^2 \boldsymbol{M} & i\boldsymbol{K_2} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \boldsymbol{K_1} - \omega^2 \mathbf{M} & 0 \\ 0 & -\boldsymbol{K_3} \end{bmatrix}, \mathbf{q} = \begin{bmatrix} \bar{\boldsymbol{u}} \\ \xi \bar{\boldsymbol{u}} \end{bmatrix}$$

We chose to input ξ and solve for smallest 30 values of ω using the "eigs" command in Matlab. This was done in a loop a lot of values of ξ.

2D FEM discretization

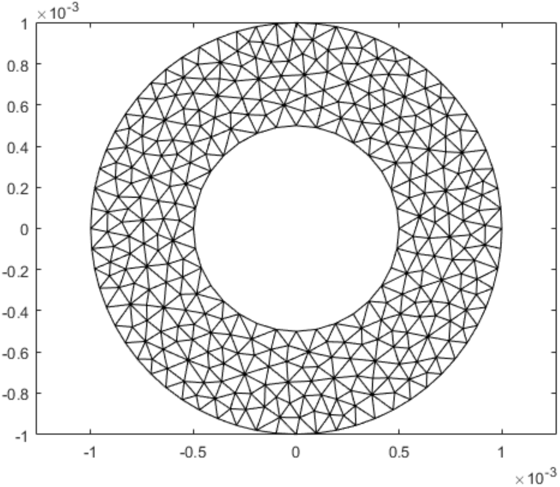
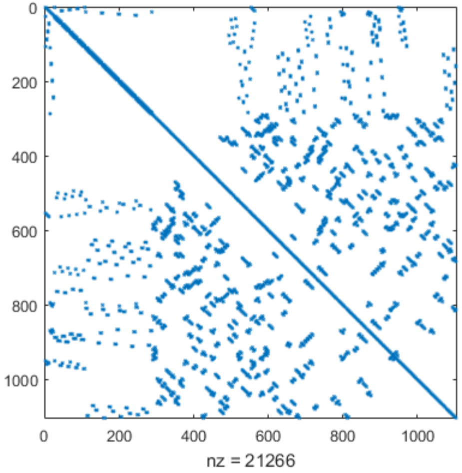


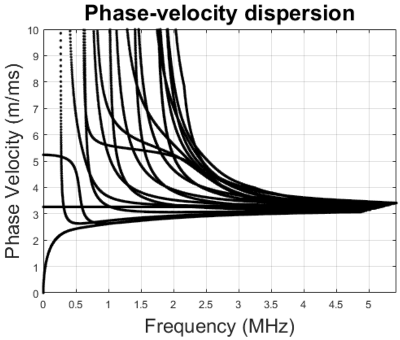
Figure: Triangular discretization of a 2D cross section of a 1mm thick steel pipe. This mesh has 369 vertices and 745 elements.

Spying on $K = K_1 + i\xi K_2 + \xi^2 K_3$

K is 1104 by 1104.



Dispersion curve on the left computed in Matlab. Dispersion curve on the right taken from (Otero et al. 2009).



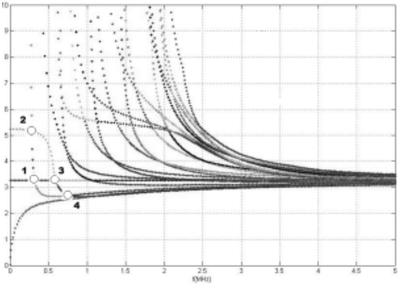


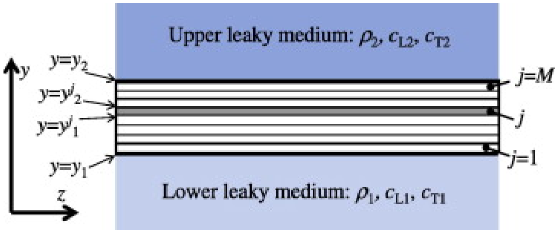
Fig. 4. Dispersion curves for a pipe with crossing points.

Summary

1. SAFEM should be considered when solving wave propagation problems involving simple waveguides. For example, it provides an efficient way to compute dispersion curves.
2. The solution to our problem behaves in a special way in the "dominant" (propagation) direction.
3. The geometry of the problem is simple (constant cross section).
4. This allows us to reduce a 3D problem to a 2D problem by modelling the propagation direction separately.
5. If the solution to a problem behaves in a special way along a "dominant" direction and the geometry is simple, the ideas presented here may be able to be applied to other problems.

Recent applications:

Calculation of leaky Lamb waves with a semi-analytical finite element method (Hayashi et al. 2014):



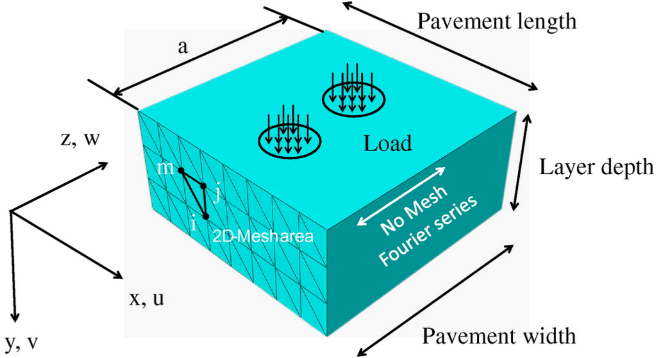
Final problem ends up being a cubic eigenvalue problem

$$(\boldsymbol{H_1} + i\xi_m \boldsymbol{H_2} + \xi_m^3 \boldsymbol{H_3})\phi_{\boldsymbol{m}} = 0$$

to compute dispersion curves, attenuation curve, displacement distribution for $m - th$ mode.

Recent applications:

Application of semi-analytical finite element method to evaluate asphalt pavement bearing capacity (Liu et al. 2015)



Recent applications:

Sensitivity analysis of leaky Lamb modes to the thickness and material properties of cortical bone with soft tissue: a semi-analytical finite element based simulation study (Tran et al. 2015)

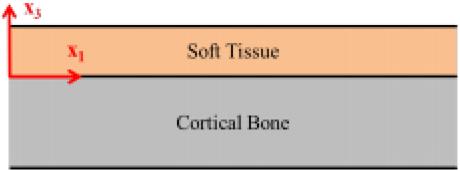
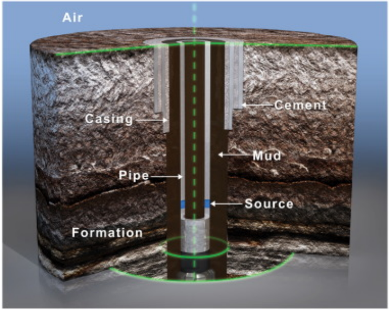


Fig. 1. A cross-sectional geometry of the layered bone model.

A. Numerical calculation of dispersion curves

Recent applications:

Rapid simulation of electromagnetic telemetry using an axisymmetric semianalytical finite element method (Chen 2017).

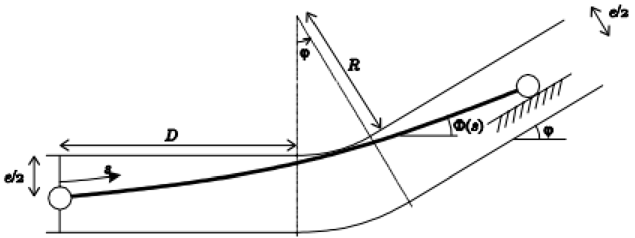


Vertical direction is taken care of by another numerical method ("Riccati equation based integration scheme").

Recent applications:

Advantages of a semi-analytical approach for the analysis of an evolving structure with contacts (Denoël. 2007)

SEMI-ANALYTICAL ANALYSIS OF AN EVOLVING STRUCTURE



Contact points can be modelled analytically, so SAFEM prevents one from having to re-mesh the full 3D structure at each point in time.

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Images sources

Other Image sources:

- http://www.ni.com/white-paper/3368/en/
- https://en.wikipedia.org/wiki/Guided_wave_testing