The quadratic eigenvalue problem

Math 5524: Matrix Theory

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1 Introduction

In this report, we introduce and demonstrate some basic aspects of the quadratic eigenvalue problem and see a few basic applications. This report is mostly based off of [1].

A general m-th degree λ-dependent polynomial matrix is of the form

$$\boldsymbol{P}(\lambda) = \boldsymbol{A}_m\lambda^m + \ldots + \boldsymbol{A}_1\lambda + \boldsymbol{A}_0,$$

where $\lambda \in \mathbb{C}$, and the *coefficient matrices* $\boldsymbol{A}_j$ (j ranges from 1 to m) are complex-valued and of size $n \times n$. The goal of the polynomial eigenvalue problem (or PEP for short) is to find eigenvalues λ and left/right eigenvectors $\boldsymbol{y}$ and $\boldsymbol{x}$ such that

$$\boldsymbol{P}(\lambda)\boldsymbol{x} = \boldsymbol{0}, \quad \boldsymbol{y}^*\boldsymbol{P}(\lambda) = \boldsymbol{0}.$$

The standard eigenvalue problem (SEP) and generalized eigenvalue problem (GEP),

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}, \quad (\boldsymbol{A} - \lambda\boldsymbol{B})\boldsymbol{x} = \boldsymbol{0},$$

as well as the quadratic eigenvalue problems (QEP), are all just special cases of the PEP [2]. In this paper, we will specifically look at properties of the QEP. Throughout the paper, we will write a generic quadratic λ-matrix as

$$\boldsymbol{Q}(\lambda) = \lambda^2\boldsymbol{M} + \lambda\boldsymbol{C} + \boldsymbol{K}. \tag{1}$$

The goal of the QEP is then to find scalars λ and vectors $\boldsymbol{y}$ and $\boldsymbol{x}$ such that

$$\boldsymbol{Q}(\lambda)\boldsymbol{x} = \boldsymbol{0}, \quad \boldsymbol{y}^*\boldsymbol{Q}(\lambda) = \boldsymbol{0}. \tag{2}$$

When we say "eigenvector", we will usually be referring to right eigenvectors. We will soon see that the eigenvalues which satisfy (2) will have important consequences for the qualitative behavior of the solutions to systems of constant coefficient second order differential equations. In the next section, we introduce some basic notions associated with QEP. In the section after that, we see how QEPs arise from studying differential equations with damping terms.

2 Spectral theory

2.1 The basics

One important difference between the QEP and the GEP is that the QEP has $2n$ eigenvalues, with up to $2n$ right and $2n$ left eigenvectors. Of course, if there are more than n eigenvectors, then they cannot form a linearly independent set. Consider the trivial $n = 1$ case with $\boldsymbol{Q}(\lambda) = \lambda^2 + 3\lambda + 2$, so that the $2 \cdot 1$ eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 1$, and the corresponding normalized eigenvectors are just 1 and 1. Just like in the GEP, we can have "infinite eigenvalues", which correspond to the zero eigenvalues of $\boldsymbol{M}$. The next example taken from [1] illustrates many of these basic concepts more clearly. Let the coefficient matrices be

$$\boldsymbol{K}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad \boldsymbol{C}=\begin{bmatrix}1&-6&0\\2&-7&0\\0&0&0\end{bmatrix},\quad \boldsymbol{M}=\begin{bmatrix}0&6&0\\0&6&0\\0&0&1\end{bmatrix},$$

so that the quadratic matrix polynomial $\boldsymbol{Q}(\lambda)$ looks like

$$\boldsymbol{Q}(\lambda)=\begin{bmatrix}\lambda+1&6\lambda^2-6\lambda&0\\2\lambda&6\lambda^2-7\lambda+1&0\\0&0&\lambda^2+1\end{bmatrix}.$$

Then the eigenvalues are just the roots of the characteristic polynomial obtained by setting the determinant equal to zero:

$$\det(\boldsymbol{Q}(\lambda))=(\lambda^2+1)(1-6\lambda+11\lambda^2-6\lambda^3)=0.$$

Since $\boldsymbol{M}$ has a single zero eigenvalue, we also consider ∞ to be an eigenvalue for this QEP, so the spectrum for this QEP is $\Lambda(\boldsymbol{Q})=\{1/3,1/2,1,-i,i,\infty\}$. It can be easily checked that the corresponding eigenvectors $\boldsymbol{x}_j$ (j ranges from 1 to 6), which are listed below, satisfy $\boldsymbol{Q}(\lambda)\boldsymbol{x}_j=\boldsymbol{0}$:

$$\boldsymbol{Q}(1/3)\boldsymbol{x}_1=\begin{bmatrix}\frac{4}{3}&-\frac{4}{3}&0\\\frac{2}{3}&-\frac{2}{3}&0\\0&0&\frac{1}{9}\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}=\boldsymbol{0},\quad \boldsymbol{Q}(1/2)\boldsymbol{x}_2=\begin{bmatrix}\frac{3}{2}&-\frac{3}{2}&0\\1&-1&0\\0&0&\frac{5}{4}\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}=\boldsymbol{0},$$
$$\boldsymbol{Q}(1)\boldsymbol{x}_3=\begin{bmatrix}2&0&0\\2&0&0\\0&0&2\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\boldsymbol{0},\quad \boldsymbol{Q}(-i)\boldsymbol{x}_4=\begin{bmatrix}1-i&-6+6i&0\\-2i&-5+7i&0\\0&0&0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\boldsymbol{0},$$
$$\boldsymbol{Q}(i)\boldsymbol{x}_5=\begin{bmatrix}1+i&-6-6i&0\\2i&-5-7i&0\\0&0&0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\boldsymbol{0},\quad \boldsymbol{Q}(\infty)\boldsymbol{x}_6=\boldsymbol{M}\begin{bmatrix}1\\0\\0\end{bmatrix}=\boldsymbol{0}.$$

This example illustrates the nontrivial fact that if we have $2n$ distinct eigenvalues, then we get a full set of n linearly independent eigenvectors ($2n$ eigenvectors total), and that two distinct eigenvalues can share a common eigenvector. Since the coefficient matrices in this example are real, the (finite) eigenvalues of $\boldsymbol{Q}(\lambda)$ are real or come in complex conjugate pairs.

2.2 The Jordan form

Let us briefly review some basic notions associated with the SEP. If λ_j is an eigenvalue of $\boldsymbol{A}\in\mathbb{C}^{n\times n}$, then the *algebraic multiplicity* a_j of λ_j is the number of times λ_j appears as a root of the characteristic polynomial of $\boldsymbol{A}$. The *geometric multiplicity* g_j of λ_j is the number of linearly independent eigenvectors of $\boldsymbol{A}$ corresponding to λ_j, i.e., $g_j=\dim(\mathrm{Ker}(\boldsymbol{A}-\lambda_j\boldsymbol{I}))$. The vectors $\boldsymbol{x}_k$ satisfying $(\boldsymbol{A}-\lambda_j\boldsymbol{I})^m\boldsymbol{x}_k=\boldsymbol{0}$ for some positive integer m are called the *generalized eigenvectors* associated with λ_j. If λ_j has algebraic multiplicity a_j, then $\dim(\mathrm{Ker}(\boldsymbol{A}-\lambda_j\boldsymbol{I})^{a_j})=a_j$, so there is a basis for $\mathbb{C}^n$ consisting of generalized eigenvectors of $\boldsymbol{A}$. If m is the smallest positive integer such that for some generalized eigenvector $\boldsymbol{x}_k$ we have $(\boldsymbol{A}-\lambda_j\boldsymbol{I})^m\boldsymbol{x}_{\boldsymbol{k}}$, then the sequence of generalized eigenvectors

$$(\boldsymbol{A}-\lambda\boldsymbol{I})^{m-1}\boldsymbol{x}_k,(\boldsymbol{A}-\lambda\boldsymbol{I})^{m-2}\boldsymbol{x}_k,\ldots,(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{x}_k,\boldsymbol{x}_k$$

is a *Jordan chain* of length m. The index i_j of λ_j is the length of the longest Jordan chain associated with λ_j.

The *Jordan decomposition* of a matrix $\boldsymbol{A}$ characterizes all of these spectrum related notions in a succinct way. If $\boldsymbol{A}$ is an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$, with algebraic multiplicities $a_1, \ldots, a_p$, geometric multiplicities $g_1, \ldots, g_p$, and indices $i_1, \ldots, i_p$, then there exists a block-diagonal matrix $\boldsymbol{J}$ containing the eigenvalues of $\boldsymbol{A}$, and an invertible matrix $\boldsymbol{X}$ containing the corresponding Jordan chains, such that

$$\boldsymbol{A} = \boldsymbol{X}\boldsymbol{J}\boldsymbol{X}^{-1} = \begin{bmatrix} \boldsymbol{X}_1 & \ldots & \boldsymbol{X}_p \end{bmatrix} \begin{bmatrix} \boldsymbol{J}_1 & & \\ & \ddots & \\ & & \boldsymbol{J}_p \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{X}}_1^* \\ \vdots \\ \widehat{\boldsymbol{X}}_p^* \end{bmatrix}$$

with $\boldsymbol{X}_j \in \mathbb{C}^{n \times a_j}$, and $\boldsymbol{J}_j \in \mathbb{C}^{a_j \times a_j}$. The matrix $\boldsymbol{J}$ is called the *Jordan form* of $\boldsymbol{A}$, and the sub-matrices $\boldsymbol{J}_j$ are of the form

$$\boldsymbol{J}_j = \begin{bmatrix} \boldsymbol{J}_{j,1} & & \\ & \ddots & \\ & & \boldsymbol{J}_{j,g_j} \end{bmatrix}, \text{ consisting of } \textit{Jordan blocks } \boldsymbol{J}_{j,k} = \begin{bmatrix} \lambda_j & 1 & & \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}.$$

Counting algebraic multiplicities, the p eigenvalues of $\boldsymbol{A}$ lie along the diagonal of the Jordan form $\boldsymbol{J}$. The geometric multiplicity g_j is the number of Jordan blocks corresponding to λ_j. The sum of the sizes of all Jordan blocks corresponding to λ_j is the algebraic multiplicity a_j. The index i_j is the size of the largest Jordan block of λ_j.

We now introduce the QEP version of all that has been said. We say that the algebraic multiplicity α of an eigenvalue λ_0 is the number of times λ_0 appears as a root of $\det(\boldsymbol{Q}(\lambda))$. The geometric multiplicity γ of λ_0 is the dimension of the null space of $\boldsymbol{Q}(\lambda_0)$. We say that $\boldsymbol{x}_1$ is a generalized eigenvector associated with λ_0 if $\boldsymbol{x}_1$ is a solution of the equation $\boldsymbol{Q}(\lambda_0)\boldsymbol{x}_1 = -\boldsymbol{Q}'(\lambda_0)\boldsymbol{x}_0$, where $\boldsymbol{x_0}$ is an eigenvector associated with λ_0, and $\boldsymbol{Q}'(\lambda_0) = 2\lambda_0\boldsymbol{M} + \boldsymbol{C}$. More generally, we say that $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{m-1}$ form a (right) Jordan chain of length m for $\boldsymbol{Q}(\lambda)$ associated with the eigenvalue λ_0 provided that the following m relations hold:

$$\begin{aligned}
&\boldsymbol{Q}(\lambda_0)\boldsymbol{x}_0 = 0, \\
&\boldsymbol{Q}(\lambda_0)\boldsymbol{x}_1 + \boldsymbol{Q}'(\lambda_0)\boldsymbol{x}_0 = 0, \\
&\boldsymbol{Q}(\lambda_0)\boldsymbol{x}_2 + \boldsymbol{Q}'(\lambda_0)\boldsymbol{x}_1 + \frac{1}{2}\boldsymbol{Q}''(\lambda_0)\boldsymbol{x}_0 = 0, \\
&\qquad\qquad\qquad\vdots \\
&\boldsymbol{Q}(\lambda_0)\boldsymbol{x}_{m-1} + \boldsymbol{Q}'(\lambda_0)\boldsymbol{x}_{m-2} + \frac{1}{2}\boldsymbol{Q}''(\lambda_0)\boldsymbol{x}_{m-3} = 0,
\end{aligned}$$

where $\boldsymbol{Q}''(\lambda_0) = 2\boldsymbol{M}$. Here, the vector $\boldsymbol{x}_0 \neq 0$ is an eigenvector, and the subsequent vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x_{m-1}}$ are generalized eigenvectors.

To develop a generalization of the Jordan form to quadratic matrix polynomials, we will begin by assuming the coefficient matrix $\boldsymbol{M}$ is nonsingular. Then we have a Jordan matrix $\boldsymbol{J} = \operatorname{diag}(\boldsymbol{J}_1, \ldots, \boldsymbol{J}_t)$ of size $2n \times 2n$, where $\boldsymbol{J}_k$ (k ranges from 1 to t) is a Jordan block of size m_k, and $m_1 + \ldots + m_t = 2n$. Let the $n \times 2n$ matrix $\boldsymbol{X}$ contain the corresponding Jordan chains, and partition $\boldsymbol{X}$ to comform naturally with $\boldsymbol{J}$, i.e., $\boldsymbol{X} = \begin{bmatrix}\boldsymbol{X}_1, \ldots, \boldsymbol{X}_t\end{bmatrix}$, where $\boldsymbol{X}_k$ (k ranges from 1 to t) is $n \times m_k$, and the columns $\boldsymbol{X}_k = \begin{bmatrix}\boldsymbol{x}_0^k, \ldots, \boldsymbol{x}_{m_k-1}^k\end{bmatrix}$ form a Jordan chain of length m_k corresponding to the eigenvalue λ_k. The pair $(\boldsymbol{X}, \boldsymbol{J})$ is called a *Jordan pair* of $\boldsymbol{Q}(\lambda)$, and is such that $\begin{bmatrix}\boldsymbol{X} \\ \boldsymbol{XJ}\end{bmatrix}$ is nonsingular and

$$\boldsymbol{MXJ}^2 + \boldsymbol{CXJ} + \boldsymbol{KX} = \boldsymbol{0}. \tag{3}$$

The left Jordan chains can be obtained from the $2n \times n$ matrix $\boldsymbol{Y}$ defined by

$$\boldsymbol{Y} = \begin{bmatrix}\boldsymbol{X} \\ \boldsymbol{XJ}\end{bmatrix}^{-1} \begin{bmatrix}\boldsymbol{0} \\ \boldsymbol{I}\end{bmatrix} \boldsymbol{M}^{-1}.$$

When all the eigenvalues have equal geometric and algebraic multiplicities, the conjugates of the rows of $\boldsymbol{Y}$ form the left eigenvectors of $\boldsymbol{Q}(\lambda)$. We can then call $(\boldsymbol{X}, \boldsymbol{J}, \boldsymbol{Y})$ a *Jordan triple* of $\boldsymbol{Q}(\lambda)$. The Jordan chain matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ satisfy the biorthogonality condition

$$\begin{bmatrix}\boldsymbol{Y} & \boldsymbol{JY}\end{bmatrix} \begin{bmatrix}\boldsymbol{C} & \boldsymbol{M} \\ \boldsymbol{M} & \boldsymbol{0}\end{bmatrix} \begin{bmatrix}\boldsymbol{X} \\ \boldsymbol{XJ}\end{bmatrix} = \boldsymbol{I}, \text{ or equivalently, } \boldsymbol{YCX} + \boldsymbol{YMXJ} + \boldsymbol{JYMX} = \boldsymbol{I}, \tag{4}$$

and we also have the follow identities:

$$\boldsymbol{XYM} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{XJYM} = \boldsymbol{I}. \tag{5}$$

A remarkable fact is that we can express the coefficient matrices $\boldsymbol{M}$, $\boldsymbol{C}$, and $\boldsymbol{K}$ of $\boldsymbol{Q}(\lambda)$ directly in terms of the Jordan triple $(\boldsymbol{X}, \boldsymbol{J}, \boldsymbol{Y})$:

$$\boldsymbol{M} = (\boldsymbol{XJY})^{-1}, \quad \begin{bmatrix}\boldsymbol{K} & \boldsymbol{C}\end{bmatrix} = -\boldsymbol{MXJ}^2 \begin{bmatrix}\boldsymbol{X} \\ \boldsymbol{XJ}\end{bmatrix}^{-1}. \tag{6}$$

To illustrate this, consider the following example. Let the quadratic matrix polynomial be

$$\boldsymbol{Q}(\lambda) = \lambda^2 \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} + \lambda \begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix} + \begin{bmatrix}-1 & 0 \\ 0 & -1\end{bmatrix} = \begin{bmatrix}\lambda^2 - 1 & \lambda \\ 0 & \lambda^2 - 1\end{bmatrix},$$

so that the eigenvalues are the roots of $\det \boldsymbol{Q}(\lambda) = (\lambda^2 - 1)^2$, which are $\lambda_1 = 1$ (with $a_1 = 2$) and $\lambda_2 = -1$ (with $a_2 = 2$). Then

$$\boldsymbol{Q}(1)\boldsymbol{x}_1 = \begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix} \begin{bmatrix}1 \\ 0\end{bmatrix} = \boldsymbol{0}, \quad \boldsymbol{Q}(-1)\boldsymbol{x}_2 = \begin{bmatrix}0 & -1 \\ 0 & 0\end{bmatrix} \begin{bmatrix}1 \\ 0\end{bmatrix} = \boldsymbol{0},$$

so $\boldsymbol{x}_1 = \begin{bmatrix}1 & 0\end{bmatrix}^T$ and $\boldsymbol{x}_2 = \begin{bmatrix}1 & 0\end{bmatrix}^T$ are the eigenvalues associated with λ_1 and λ_2. We can then compute the generalized eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ by solving the systems

$$\begin{aligned}
&\boldsymbol{Q}(1)\boldsymbol{v}_1 = -\boldsymbol{Q}'(1)\boldsymbol{x}_1, \quad \boldsymbol{Q}(-1)\boldsymbol{v}_2 = -\boldsymbol{Q}'(-1)\boldsymbol{x}_2, \\
&\begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix} \boldsymbol{v}_1 = \begin{bmatrix}-2 & -1 \\ 0 & -2\end{bmatrix} \begin{bmatrix}1 \\ 0\end{bmatrix} \implies \boldsymbol{v}_1 = \begin{bmatrix}\text{anything} \\ -2\end{bmatrix}, \\
&\begin{bmatrix}0 & -1 \\ 0 & 0\end{bmatrix} \boldsymbol{v}_2 = \begin{bmatrix}2 & -1 \\ 0 & 2\end{bmatrix} \begin{bmatrix}1 \\ 0\end{bmatrix} \implies \boldsymbol{v}_2 = \begin{bmatrix}\text{anything} \\ -2\end{bmatrix}.
\end{aligned}$$

Then we have

$$\boldsymbol{J} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -2 \end{bmatrix}.$$

We can compute $\boldsymbol{Y}$, and confirm equations (3), (4), (5), and (6) in MATLAB by running the following script:

```
M = eye(2,2); C = [0 1; 0 0]; K = -eye(2,2);
J = [1 1 0 0; 0 1 0 0; 0 0 -1 1; 0 0 0 -1];
X = [[1;0], [0;-2], [1;0], [0;-2]];
eq3 = M*X*J*J + C*X*J + K*X
Y = inv([X;X*J])*[zeros(2,2);eye(2,2)]*inv(M)
eq3 = M*X*J*J + C*X*J + K*X
eq4 = Y*C*X + Y*M*X*J + J*Y*M*X
eq5part1 = X*Y*M
eq5part2 = X*J*Y*M
eq6M = inv(X*J*Y)
eq6KC = -M*X*J^2 *inv([X;X*J])
```

the formatted output is:

```
eq3 =

     0     0     0     0
     0     0     0     0


Y =

    0.5000         0
         0   -0.2500
   -0.5000         0
         0    0.2500

eq3 =

     0     0     0     0
     0     0     0     0


eq4 =

     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1

eq5part1 =

     0     0
     0     0


eq5part2 =

     1     0
     0     1


eq6M =

     1     0
     0     1


eq6KC =

    -1     0     0     1
     0    -1     0     0
```

In the case that $\boldsymbol{M}$ is singular, we cannot form a matrix of left Jordan chains $\boldsymbol{Y}$. When $\boldsymbol{M}$ is singular, we have infinite eigenvalues, and we decompose the Jordan pair $(\boldsymbol{X}, \boldsymbol{J})$ into a finite Jordan pair $(\boldsymbol{X}_F, \boldsymbol{J}_F)$ corresponding to the finite eigenvalues and an infinite Jordan pair $(\boldsymbol{X}_\infty, \boldsymbol{J}_\infty)$ corresponding to the infinite eigenvalues. Here, $\boldsymbol{J}_\infty$ is a Jordan matrix formed by Jordan blocks with eigenvalue $\lambda = 0$. If $\boldsymbol{Q}(\lambda)$ has r finite eigenvalues, then $\boldsymbol{X}_F \in \mathbb{C}^{n\times r}$, $\boldsymbol{J}_F \in \mathbb{C}^{r\times r}$, $\boldsymbol{X}_\infty \in \boldsymbol{C}^{n\times(2n-r)}$, and $\boldsymbol{J}_\infty \in \mathbb{C}^{(2n-r)\times(2n-r)}$.

To illustrate this, consider the following example from [1]. Let the quadratic matrix polynomial be

$$\boldsymbol{Q}(\lambda) = \lambda^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

for which $\det \boldsymbol{Q}(\lambda) = (\lambda - 1)^3(\lambda + 1)$, so the eigenvalues are $\lambda_1 = -1$ ($a_1 = 1$), $\lambda_2 = 1$ ($a_2 = 3$), and $\lambda_3 = \infty$ ($a_3 = 2$). Then we have

$$\boldsymbol{X}_F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{J}_F = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{X}_\infty = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{J}_\infty = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

3 Applications to differential equations

3.1 Stability

Consider the constant-coefficient second-order linear differential equation

$$\boldsymbol{M}\boldsymbol{q}''(t) + \boldsymbol{C}\boldsymbol{q}'(t) + \boldsymbol{K}\boldsymbol{q}(t) = \boldsymbol{f}(t). \tag{7}$$

If $\boldsymbol{M}$ is nonsingular, we have a Jordan triple $(\boldsymbol{X}, \boldsymbol{J}, \boldsymbol{Y})$ for $\boldsymbol{Q}(\lambda)$. It is easy to verify that verify that the general solution to the the homogeneous ($\boldsymbol{f}(t) = 0$) differential equation (7) has the form

$$\boldsymbol{q}(t) = \boldsymbol{X} e^{\boldsymbol{J}t} \boldsymbol{a}, \tag{8}$$

where $\boldsymbol{a} \in \mathbb{C}^{2n}$ is a vector of arbitrary constants. We can do so by computing the first and second derivatives of $\boldsymbol{q}(t)$, substituting these into (7), and using (3):

$$\begin{aligned} &\boldsymbol{q}'(t) = \boldsymbol{X}\boldsymbol{J}e^{\boldsymbol{J}t}\boldsymbol{a}, \quad \boldsymbol{q}''(t) = \boldsymbol{X}\boldsymbol{J}^2 e^{\boldsymbol{J}t}\boldsymbol{a}, \text{ so (7) becomes} \\ &(\boldsymbol{M}\boldsymbol{X}\boldsymbol{J}^2 + \boldsymbol{C}\boldsymbol{X}\boldsymbol{J} + \boldsymbol{K}\boldsymbol{X})(e^{\boldsymbol{J}t}\boldsymbol{a}) = \boldsymbol{0}. \end{aligned}$$

Notice that (8) says that the solutions to (7) with $\boldsymbol{f}(t) = 0$ are basically determined by the eigenvalues and eigenvectors of the QEP. If all of the eigenvalues of $\boldsymbol{Q}(\lambda)$ associated with (7) have negative real part, then the entries in $\boldsymbol{q}(t)$ must decay to zero, i.e., $\|\boldsymbol{q}(t)\| \to 0$ as $t \to \infty$. We then say that (7) is *asymptotically stable* if all eigenvalues of $\boldsymbol{Q}(\lambda)$ lie in the left-half of the complex plane. If at least one of the eigenvalues of $\boldsymbol{Q}(\lambda)$ is positive, then for some initial conditions, we may have $\|\boldsymbol{q}(t)\| \to \infty$ as $t \to \infty$, and (7) is said to be *asymptotically unstable.* If all the eigenvalues have real part less than or equal to zero, but those equal to zero have index 1 (i.e., have 1×1 Jordan blocks) then (7) is said to be *weakly-stable* since $\|\boldsymbol{q}(t)\|$ is bounded as $t \to \infty$ for all initial conditions.

Example 1:

One place where linear second-order differential equations arise is in the study mechanical oscillations. Consider a simple mass-spring system illustrated in Figure 1 (taken from [1]). Here, the ith mass has weight m_i and is connected to its $(i+1)$st neighbor by a spring with constant k_i and a damper with constant d_i. The ith mass is also connected to the ground by a spring with constant κ_i and damper with constant τ_i. The complete vibration of this system is governed by the differential equation

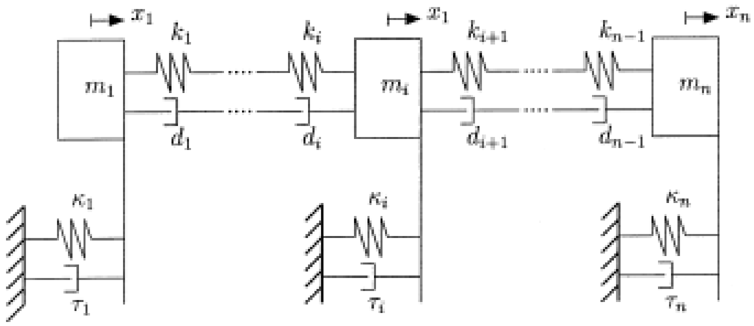

Fig. 2. An *n* degree of freedom damped mass-spring system.

Figure 1: A mass-spring system with n degrees of freedom [1].

$$\boldsymbol{M}\frac{\partial^2}{\partial t^2}\boldsymbol{x}(t) + \boldsymbol{C}\frac{\partial}{\partial t}\boldsymbol{x}(t) + \boldsymbol{K}\boldsymbol{x}(t) = \boldsymbol{0}, \tag{9}$$

where the mass matrix $\boldsymbol{M} = \mathrm{diag}(m_1, \ldots, m_n)$ is diagonal, and the damping matrix $\boldsymbol{C}$ and stiffness matrix $\boldsymbol{K}$ are symmetric tridiagonal and have the form

$$\boldsymbol{C} = \underbrace{\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}}_{\boldsymbol{P}} \begin{bmatrix} d_1 & & & \\ & \ddots & & \\ & & d_{n-1} & \\ & & & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -1 & & \\ & 1 & \ddots & \\ & & \ddots & -1 \\ & & & 1 \end{bmatrix}}_{\boldsymbol{P}^T} + \begin{bmatrix} \tau_1 & & & \\ & \tau_2 & & \\ & & \ddots & \\ & & & \tau_n \end{bmatrix},$$

$$\boldsymbol{K} = \underbrace{\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}}_{\boldsymbol{P}} \begin{bmatrix} k_1 & & & \\ & \ddots & & \\ & & k_{n-1} & \\ & & & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -1 & & \\ & 1 & \ddots & \\ & & \ddots & - \\ & & & 1 \end{bmatrix}}_{\boldsymbol{P}^T} + \begin{bmatrix} \kappa_1 & & & \\ & \kappa_2 & & \\ & & \ddots & \\ & & & \kappa_n \end{bmatrix}.$$

This damped spring-mass model is available in the NLEVP software package [3] under the name spring.m. Taking $n = 2$ masses, with each $m_i = 1$, and each spring constant $k_i = \kappa_i = 5$, we vary the damping coefficients and solve (9) with initial conditions $x_1(0) = -1, x_2(0) = 1$, by solving the corresponding QEP using MATLABs polyeig command. Figures (2), (3), (4), and (5) show the solution with different damping constants, with eigenvalues listed in the caption (figure (2) showing the weakly stable case, and figures (3), (4), and (5) showing stable cases). The damping coefficients are provided in the MATLAB code at the end of the report.

Example 2:

Consider another example that comes from the linearized stability analysis from a Whipple bicycle model. The basis of this model is two coupled second-order constant-coefficient ordinary differential equations associated with the lean and steer of a bicycle. This system of differential equations can be written as

$$\boldsymbol{M}\boldsymbol{q}''(t) + v\boldsymbol{C}_1\boldsymbol{q}'(t) + [g\boldsymbol{K}_0 + v^2\boldsymbol{K}_2]\boldsymbol{q} = \boldsymbol{f}(t). \tag{10}$$

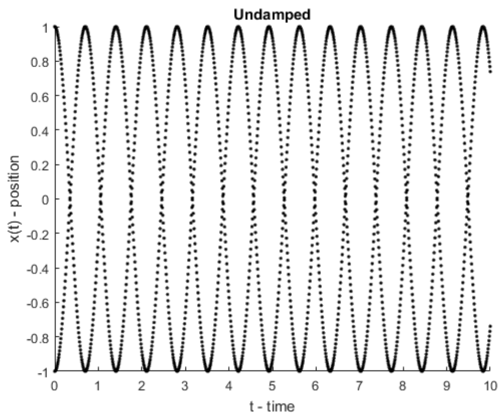

Figure 2: Re(λ) $\approx$ 0, 0, 0, 0

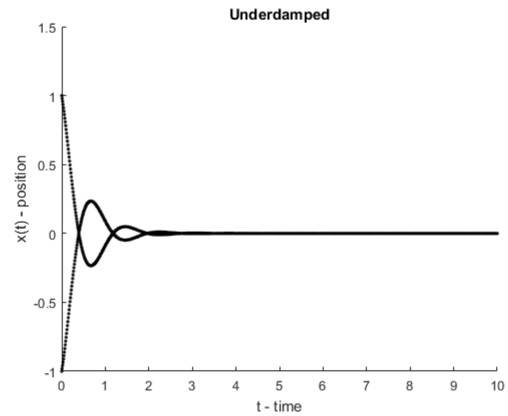

Figure 3: Re(λ) $\approx$ -2, -2, -1, -1

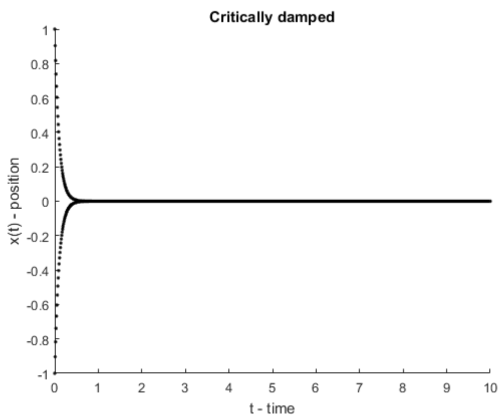

Figure 4: Re(λ) $\approx$ -1.98, -3, -3, -10.10

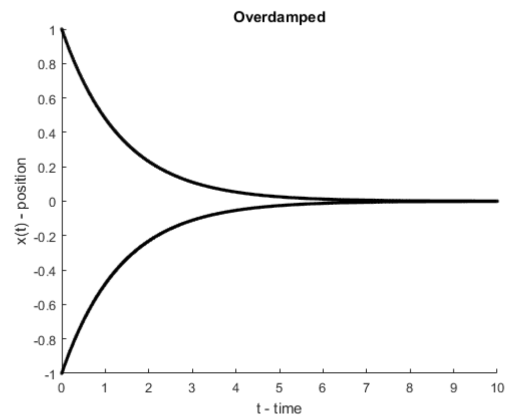

Figure 5: $\lambda \approx$ -0.74, -0.76, -13.25, -27.27

The constant matrices $\boldsymbol{M}, \boldsymbol{C}_1, \boldsymbol{K}_0, \boldsymbol{K}_2$ are defined in terms of design parameters of the bicycle. The forward velocity at which the bike moves is v, and g is the gravitational constant. A set of design parameters for a particular bicycle is taken from [4], which yields the matrices

$$\boldsymbol{M} = \begin{bmatrix} 80.81722 & 2.31941332208709 \\ 2.31941332208709 & 0.29784188199686 \end{bmatrix},$$

$$\boldsymbol{C}_1 = \begin{bmatrix} 0 & 33.86641391492494 \\ 0.85035641456978 & 1.68540397397560 \end{bmatrix},$$

$$\boldsymbol{K}_2 = \begin{bmatrix} 0 & 76.59734589573222 \\ 0 & 2.65431523794604 \end{bmatrix}, \quad \boldsymbol{K}_0 = \begin{bmatrix} 80.95 & 2.59951685249872 \\ 2.59951685249872 & 0.80329488458618 \end{bmatrix}.$$

This damped bicycle model is also available in the NLEVP software package [3] under the name bike.m. The solution to (10) with the above matrices tells us the motion (steer and lean) of a particular bicycle at a given forward velocity v. The self-stability of the bicycle at a particular velocity can then can then be determined by the real part of the eigenvalues of the QEP. For this bicycle, we plot the real part of the eigenvalues over a range of velocities to obtain figure (6). From this plot, it can be seen that the bicycle is self-stable when it is moving at a forward velocity between about 4.3 m/s and 6 m/s.

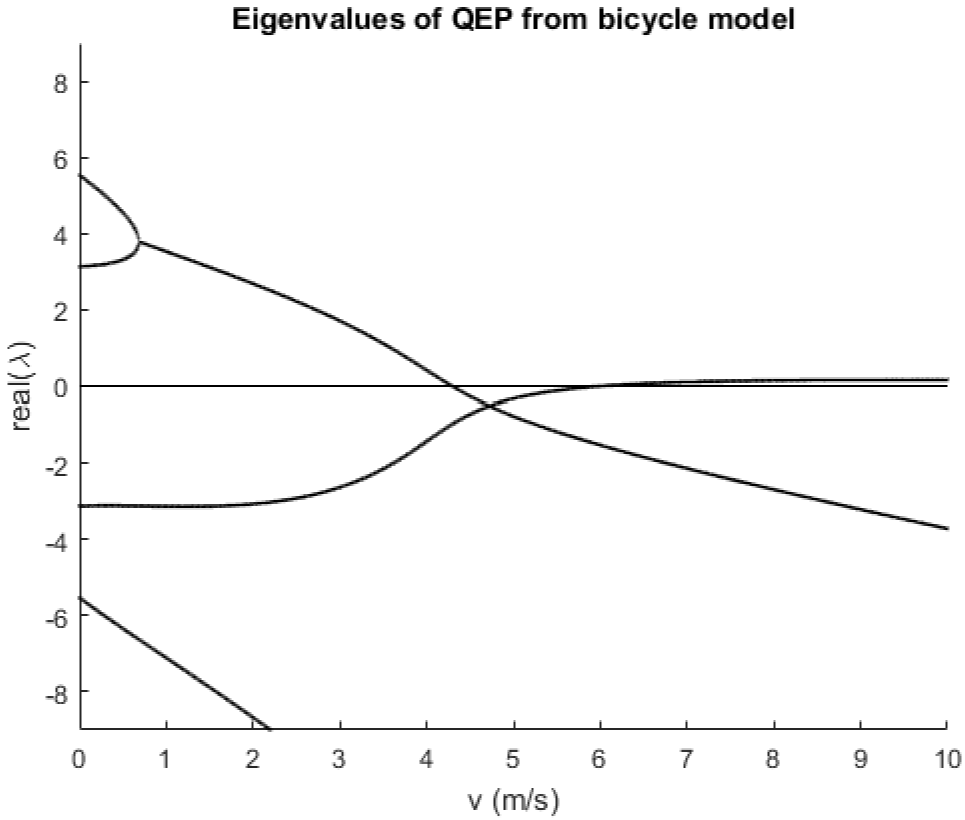

Figure 6: $\mathrm{Re}(\lambda) < 0$ when $v \in [4.3, 6]$.

3.2 Perturbations

In eigenvalue perturbation theory, we are interested to see if small changes in the coefficient matrices greatly affect the eigenvalues of a matrix. One reason we might be interested in this is that it may be possible that highly sensitive eigenvalues jump over to the right-half of the complex plane when subject to small perturbations. There are many ways to see how the eigenvalues change under small perturbations to the entries of a matrix, one such way is to study the pseudospectra of a matrix. For the SEP, we recall the following definition for the ϵ-pseudospectrum of the matrix $\boldsymbol{A}$:

$$\Lambda_\epsilon(\boldsymbol{A}) = \{\lambda \in \mathbb{C} : \overbrace{(\boldsymbol{A} + \boldsymbol{\Delta A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0} \text{ for some } \boldsymbol{x} \neq 0}^{\lambda \in \Lambda(\boldsymbol{A}+\boldsymbol{\Delta A})} \text{ and } \boldsymbol{\Delta A} \text{ with } \|\boldsymbol{\Delta A}\| < \epsilon\},$$

We can extend the first definition to quadratic matrix polynomials as follows:

$$\begin{aligned}\Lambda_\epsilon(\boldsymbol{Q}(\lambda)) = \{\lambda \in \mathbb{C} : \overbrace{(\boldsymbol{Q}(\lambda) + \boldsymbol{\Delta Q}(\lambda))\boldsymbol{x} = \boldsymbol{0} \text{ for some } \boldsymbol{x} \neq 0}^{\lambda \in \Lambda(\boldsymbol{Q}+\boldsymbol{\Delta Q})} &\text{ and } \boldsymbol{\Delta Q}(\lambda) \\ \text{with } \|\boldsymbol{\Delta M}\| \leq \epsilon\alpha_2, \|\boldsymbol{\Delta C}\| \leq \epsilon\alpha_1, \|\boldsymbol{\Delta K}\| &\leq \epsilon\alpha_0\},\end{aligned}$$

where the notation $\Delta\boldsymbol{Q}(\lambda)$ just means

$$\Delta\boldsymbol{Q}(\lambda) = \lambda^2\boldsymbol{\Delta M} + \lambda\boldsymbol{\Delta C} + \boldsymbol{\Delta K}.$$

The values $\alpha_{1,2,3}$ are just non-negative weights that give us freedom in how the perturbations are measured.

Example 3:

We conclude with one final example. Consider the following quadratic matrix polynomial from [2]

$$\boldsymbol{Q}(\lambda) = \lambda^2 \begin{bmatrix} 17.6 & 1.28 & 2.89 \\ 1.28 & 0.824 & 0.413 \\ 2.89 & 0.413 & 0.725 \end{bmatrix} + \lambda \begin{bmatrix} 7.66 & 2.45 & 2.1 \\ 0.23 & 1.04 & 0.223 \\ 0.6 & 0.756 & 0.658 \end{bmatrix} + \begin{bmatrix} 121 & 18.9 & 15.9 \\ 0 & 2.7 & 0.145 \\ 11.9 & 3.64 & 15.5 \end{bmatrix}.$$

This $\boldsymbol{Q}(\lambda)$ arose from the analysis of the oscillations of a wing in an airstream, and is also available in the NLEVP software package [3] under the name wing.m. In each trial, we add normally distributed random perturbation matrices to $\boldsymbol{M}$, $\boldsymbol{C}$, and $\boldsymbol{K}$, with each random perturbation matrix having norm equal to $\epsilon = 10^{-0.8}$. Plotting the results for 1000 trials gives us rough estimate of the ϵ-pseudospectra for this QEP. Figure (7) shows that the eigenvalues λ_1 = -0.8848 + 8.4415i and λ_2 = -0.8848 - 8.4415i are highly sensitive to perturbations (even going so far as to jump over to the right-half of the complex plane), while the other eigenvalues are not too sensitive.

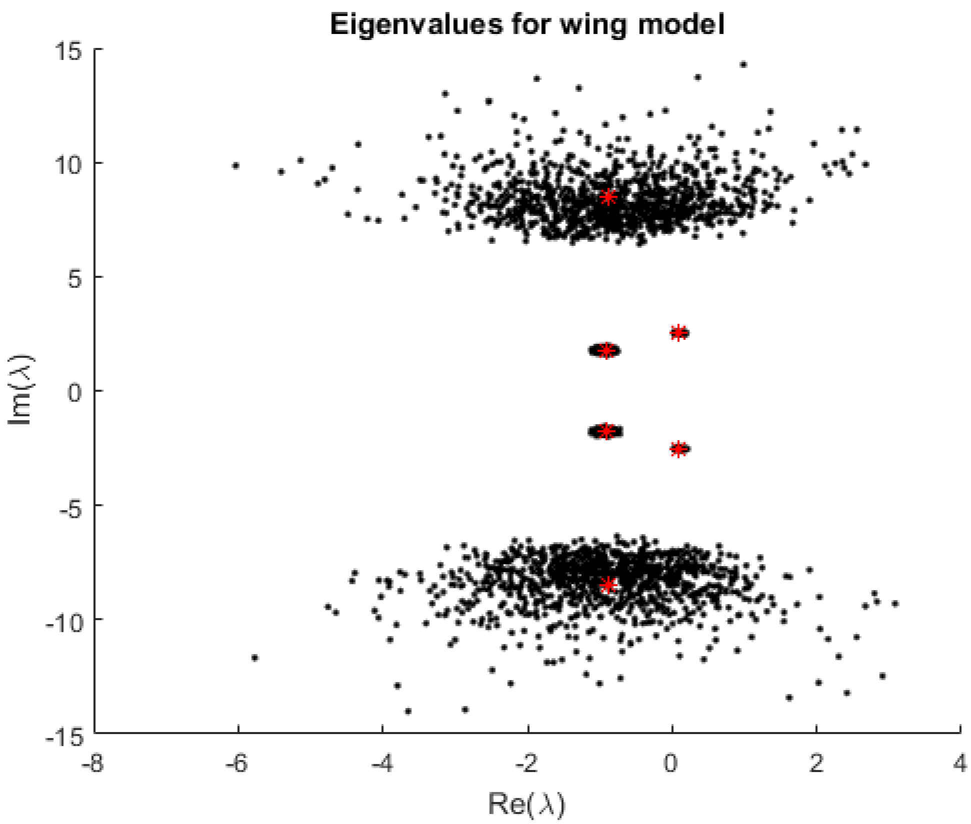

Figure 7: The eigenvalues are plotted as red stars. The ϵ-pseudospectral values plotted as black dots.

4 Conclusion

In this short report, we have familiarized ourselves with some of the basic aspects of the QEP. We have seen how the QEP differs from the SEP and GEP, and we have seen how to extend the Jordan form to the Jordan triple for the QEP. We have seen how this extension of the Jordan form gives us the solution to the constant-coefficient second-order linear differential equation, and have seen how the real part of the eigenvalues determine the stability of the solution. We also saw how the notion of ϵ-pseudospectra could be extended to the QEP in order to study how matrix perturbations affect the solution to the QEP. A few simple engineering examples were given to show how QEPs might show up in practice, and to demonstrate some of the basic theory.

MATLAB code:

For all of the following code, the matrices were generated using the software package NLEVP [3].

Code for generating figures (2), (3), (4), and (5):

```
n = 2;          % number of springs
m = 1;          % weight of each mass
k = 5;          % adjacent spring constants
d = 0;          % adjacent damping constants
kappa = 5;      % ground spring constants
tau = 0;        % ground damping constants
x0 = [-1 1]';   % initial positions of masses

% UNDAMPED %
[coeffs,~] = spring(n,m,tau,kappa,d,k);
M = coeffs{3}; C = coeffs{2}; K = coeffs{1};
[X,E] = polyeig(K,C,M); J = diag(E);
a = X*expm(J*0)\x0;
figure, hold on
 for t = 0:0.01:10
     q = X*expm(J*t)*a;
     plot(t*ones(size(q)),real(q),'.k')
 end
xlabel('t - time'), ylabel('x(t) - position')
title('Undamped'), hold off

% UNDERDAMPED %
d = 1; tau = 1;
[coeffs,~] = spring(n,m,tau,kappa,d,k);
M = coeffs{3}; C = coeffs{2}; K = coeffs{1};
[X,E] = polyeig(K,C,M); J = diag(E);
a = X*expm(J*0)\x0;
figure, hold on
 for t = 0:0.01:10
     q = X*expm(J*t)*a;
     plot(t*ones(size(q)),real(q),'.k')
 end
xlabel('t - time'), ylabel('x(t) - position')
title('Underdamped'), hold off

% (ALMOST) CRITICALLY DAMPED %
d = 3.02; tau = 3.02;
[coeffs,~] = spring(n,m,tau,kappa,d,k);
M = coeffs{3}; C = coeffs{2}; K = coeffs{1};
[X,E] = polyeig(K,C,M); J = diag(E);
a = X*expm(J*0)\x0;
figure, hold on
 for t = 0:0.01:10
     q = X*expm(J*t)*a;
     plot(t*ones(size(q)),real(q),'.k')
 end
xlabel('t - time'), ylabel('x(t) - position')
title('Critically damped'), hold off

% OVERDAMPED
d = 7; tau = 7;
[coeffs,~] = spring(n,m,tau,kappa,d,k);
M = coeffs{3}; C = coeffs{2}; K = coeffs{1};
```

```
[X,E] = polyeig(K,C,M); J = diag(E);
a = X*expm(J*0)\x0;
figure, hold on
 for t = 0:0.01:10
     q = X*expm(J*t)*a;
     plot(t*ones(size(q)),real(q),'.k')
 end
xlabel('t - time'), ylabel('x(t) - position')
title('Overdamped'), hold off
```

Code for generating figure 6:

```
figure, hold on
for v = 0:0.01:10
    [coeffs,fun] = bicycle(v);
    M = coeffs{3}; C = coeffs{2}; K = coeffs{1};
    val = polyeig(K,C,M);
    plot([v v v v],real(val),'.k','MarkerSize',1)
end
plot([0 10],[0 0],'k')
xlabel('v (m/s)'), ylabel('real(\lambda)')
axis([0 10 -9 9]), title('Eigenvalues of QEP from bicycle model')
hold off
```

Code for generating figure 7:

```
[coeffs,fun] = wing;
M = coeffs{3}; C = coeffs{2}; K = coeffs{1};
val = polyeig(K,C,M);
figure, hold on
for k = 1:1000
    dM = randn(3,3) + 1i*randn(3,3);
    dM = (dM/norm(dM)) * 10^(-0.8);
    dC = randn(3,3) + 1i*randn(3,3);
    dC = (dC/norm(dC)) * 10^(-0.8);
    dK = randn(3,3) + 1i*randn(3,3);
    dK = (dK/norm(dK)) * 10^(-0.8);
    DM = M + dM;
    DC = C + dC;
    DK = K + dK;
    val1 = polyeig(DK,DC,DM);
    plot(real(val1),imag(val1),'.k')
end
plot(real(val),imag(val),'*r')
title('Eigenvalues for wing model')
xlabel('Re(\lambda)'), ylabel('Im(\lambda)')
hold off
```

References

[1] Tisseur, Françoise and Meerbergen, Karl. *The quadratic eigenvalue problem.* SIAM review, 2001.

[2] Tisseur, Françoise and Higham, Nicholas J. *Structured pseudospectra for polynomial eigenvalue problems, with applications.* SIAM Journal on Matrix Analysis and Applications, 2001.

[3] Betcke, Timo and Higham, Nicholas J and Mehrmann, Volker and Schröder, Christian and Tisseur, Françoise. *NLEVP: A collection of nonlinear eigenvalue problems.* ACM Transactions on Mathematical Software (TOMS), 2013.

[4] Meijaard, Jaap P and Papadopoulos, Jim M and Ruina, Andy and Schwab, Arend L. *Linearized dynamics equations for the balance and steer of a bicycle: a benchmark and review.* Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, 2007.