FROM ISING MODEL TO SCHELLING MODEL

WHY WE NEED ABM?

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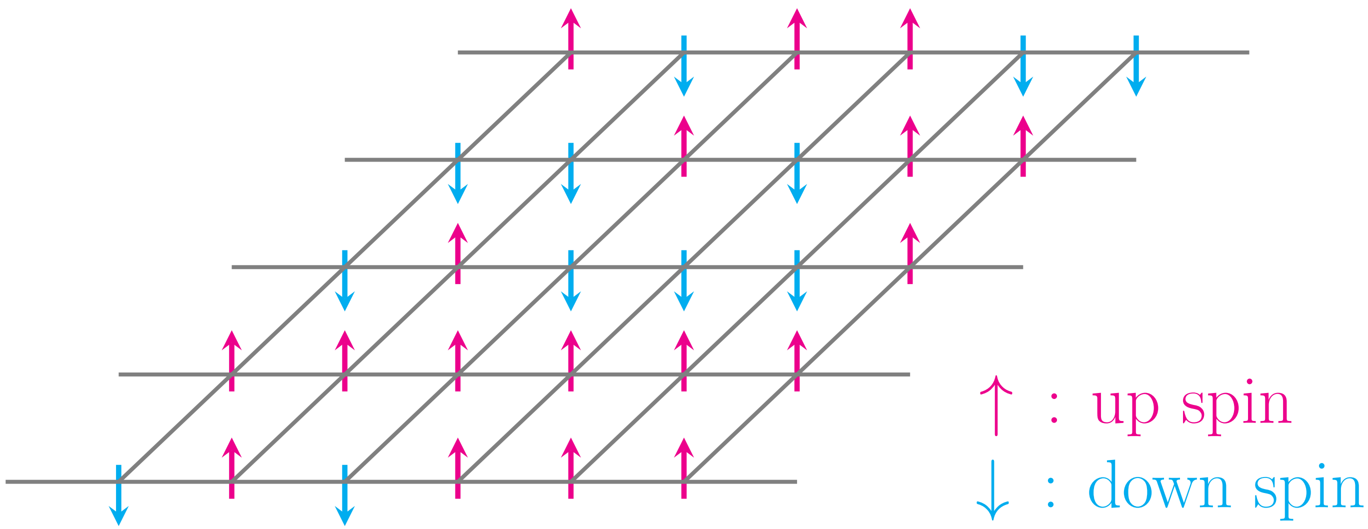
WHY WE NEED ABM?

ISING MODEL: BACKGROUND

CURIE POINT HEAT ENGINE

- 宏观
  - 铁磁体加热到居里温度会失去磁性。
  - **相变** (Phase Transition): 有磁性 → 无磁性。
- 微观
  - 金属粒子在晶格上排列 (Crystal Lattice)。
  - 只有粒子自旋方向会改变。
  - 只与相邻粒子相互作用。

ISING MODEL

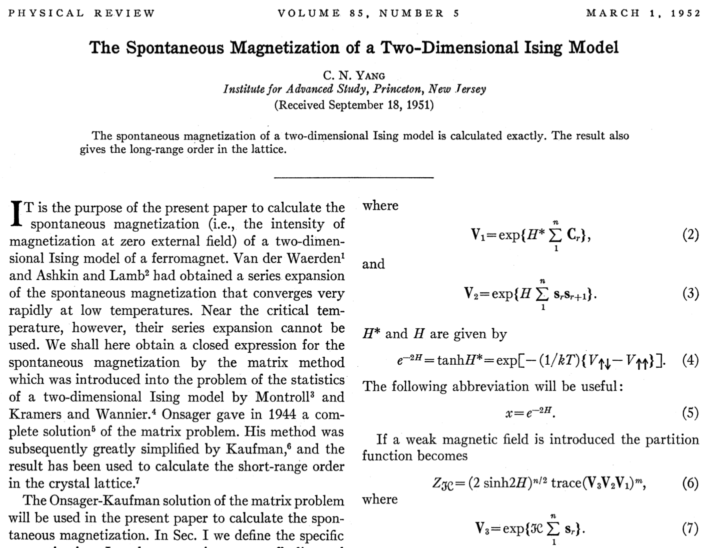
- Wilhelm Lenz (1920) 和 Ernst Ising (1925)
  - 

- 状态：每个格子上的原子有两个自旋状态：$+1$（向上）或 -1（向下）。
  - 当所有原子自旋一致时，系统表现出宏观磁性。（有序状态）
  - 当自旋随机分布时，宏观系统无磁性。（无序状态）
- 交互：相邻原子倾向于**自旋一致**以降低系统能量。$E = \sum s_i s_j$
- 动作：metropolis algorithm
  - 如果 $\Delta E < 0$，翻转
  - 如果 $\Delta E \geq 0$，以概率 $e^{-\Delta E/kT}$ 翻转。
- 随着温度 T 升高，热扰动增加。**这个模型能够模拟相变吗？相变点在哪？**
- Netlogo 实现: Ising Model

THE STRUGGLE FOR EXACT SOLUTIONS

- **The 1D Disappointment**
  - 由 Wilhelm Lenz (1920) 和 Ernst Ising (1925) 提出。
  - Ising (1925) 证明：一维情况下没有相变。如果 $T > 0$，系统总是无序的。
  - Ising 误以为二维和三维情况也一样没有相变。
- 平均场理论 (Mean-Field Theory) Pierre Curie
  - 近似方法，忽略空间关联。预测了临界温度，但不精确。
- **The 2D Breakthrough**
  - **Lars Onsager (1944)**: 推导出了二维ising的精确解，预测了自发磁化的系数。1968年诺贝尔奖。 $\frac{2}{\ln(1+\sqrt{2})} \approx 2.269$
  - **C.N. Yang, (1952)**: 计算出了精确的自发磁化强度 (M)。
- **The 3D Challenge**
  - 至今没有找到解析解 。
  - **Hugo Duminil-Copin (2022 fields prize)**:
    - 证明了三维相变的连续性和清晰度。
    - 但仍然没有2D ising的封闭公式。

> The most striking results of Duminil-Copin are for **Ising-type models in dimensions three and four**. He has established the continuity and sharpness of the phase transition in dimension three. — The Fields Medal 2022 Citation



PHYSICAL REVIEW VOLUME 85, NUMBER 5 MARCH 1, 1952

The Spontaneous Magnetization of a Two-Dimensional Ising Model

C. N. Yang
Institute for Advanced Study, Princeton, New Jersey
(Received September 18, 1951)

The spontaneous magnetization of a two-dimensional Ising model is calculated exactly. The result also gives the long-range order in the lattice.

IT is the purpose of the present paper to calculate the spontaneous magnetization (i.e., the intensity of magnetization at zero external field) of a two-dimensional Ising model of a ferromagnet. Van der Waerden[1] and Ashkin and Lamb[2] had obtained a series expansion of the spontaneous magnetization that converges very rapidly at low temperatures. Near the critical temperature, however, their series expansion cannot be used. We shall here obtain a closed expression for the spontaneous magnetization by the matrix method which was introduced into the problem of the statistics of a two-dimensional Ising model by Montroll[3] and Kramers and Wannier.[4] Onsager gave in 1944 a complete solution[5] of the matrix problem. His method was subsequently greatly simplified by Kaufman,[6] and the result has been used to calculate the short-range order in the crystal lattice.[7]

The Onsager-Kaufman solution of the matrix problem will be used in the present paper to calculate the spontaneous magnetization. In Sec. I we define the specific

where

$$\mathbf{V}_1 = \exp\{H^* \sum_1^n \mathbf{C}_r\}, \quad (2)$$

and

$$\mathbf{V}_2 = \exp\{H \sum_1^n \mathbf{s}_r \mathbf{s}_{r+1}\}. \quad (3)$$

H^* and H are given by

$$e^{-2H} = \tanh H^* = \exp[-(1/kT)\{V_{\uparrow\downarrow} - V_{\uparrow\uparrow}\}]. \quad (4)$$

The following abbreviation will be useful:

$$x = e^{-2H}. \quad (5)$$

If a weak magnetic field is introduced the partition function becomes

$$Z_{\mathfrak{H}} = (2 \sinh 2H)^{n/2} \operatorname{trace}(\mathbf{V}_3 \mathbf{V}_2 \mathbf{V}_1)^m, \quad (6)$$

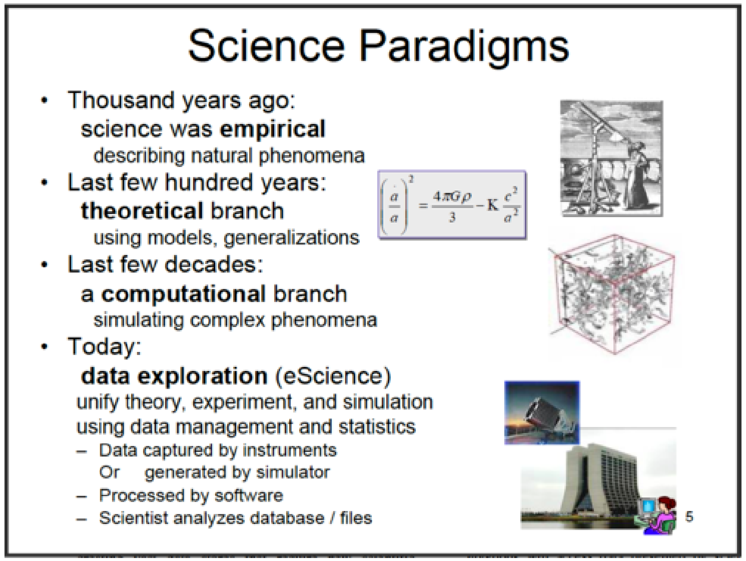
where

$$\mathbf{V}_3 = \exp\{\mathfrak{H} \sum_1^n \mathbf{s}_r\}. \quad (7)$$

> “就这样，我开始了漫长的计算，这是我职业生涯中**最长的一次计算**… 最后，**经过大约六个月断断续续的工作**，所有的碎片突然拼凑在了一起… 我看到了令人惊讶的简单的最终结果。”
>
> — Selected Papers, 1945-1980, by C.N.Yang, 1982.

WHY WE NEED ABM?

- 解析方法的困境
  - 回顾 Ising: **最简单的状态，最简单的交互，最简单的规则，求解如此困难！**
  - 当系统包含异质性个体（Heterogeneity）和复杂交互时，解析解通常不存在（Intractable）。
  - 社会系统比 Ising 模型更复杂，无法用 Equation based modelling (EBM) 还原。

- Jim Gray, 2007
  - 

- 科学研究的四种范式 (Four Paradigms of Scientific Research)
  - 实验方法 (Experimental Methods)： 经验主义。
  - 解析方法 (Theoretical Methods)： 解析推导, 如麦克斯韦方程组, 杨振宁的解
  - 计算模拟方法 (Computational Methods) : 计算, 如 **ABM**, 蒙特卡洛
  - 数据驱动方法 (Data-Driven Methods) : 大数据, 机器学习

- **范式转移** (The Paradigm Shift)
  1. 实验范式的困境
     - 社会科学难以进行大规模受控实验。
     - 伦理限制：不能随意操纵人类社会。历史不可重复：社会现象具有独特性和时效性。

- 相关性不等于因果性：大数据只能揭示模式，无法解释机制。
- 预测 vs 解释：数据驱动方法擅长预测结果，但难以解释过程。

3. 计算模拟的独特性
    - 处理异质性：每个个体可以有不同属性和行为。
    - 捕捉非线性：复杂交互和反馈机制。
    - 揭示涌现：宏观模式如何从微观规则中产生。

CRITICAL PHENOMENA

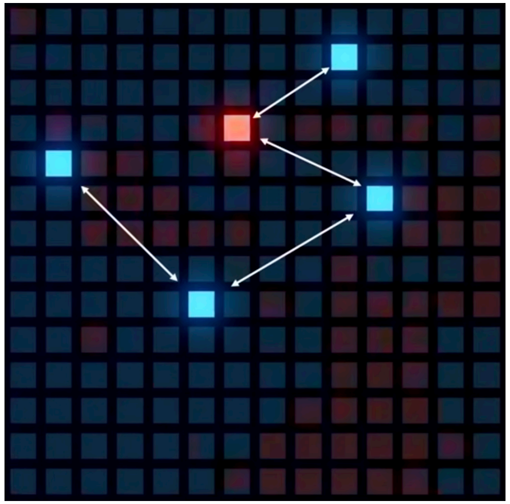
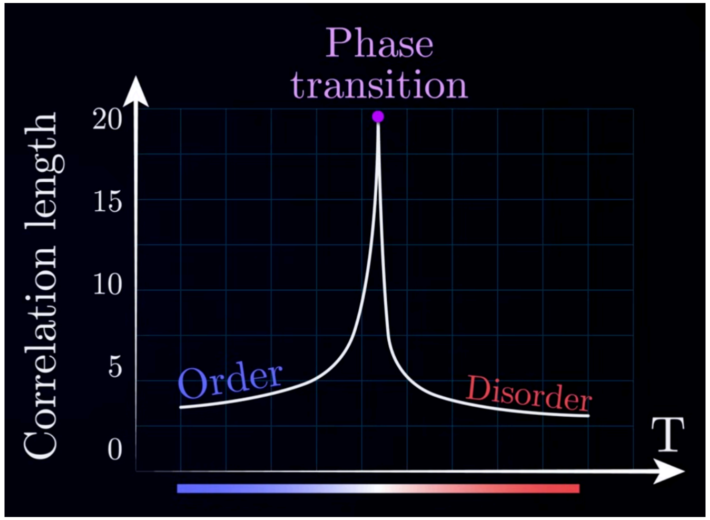
SCALE-FREE PROPERTIES

- 在临界点附近，系统表现出无特征尺度 (Scale-free)。

CRITICAL PHENOMENA

CORRELATION LENGTH

- 系统中两个点之间的关联强度。

- 
- 

- Critical Phenomena - 共形不变性
  - 在临界点附近，系统表现出**共形不变性 (Conformal Invariance)**。
  - Smirnov (2010 fields prize)

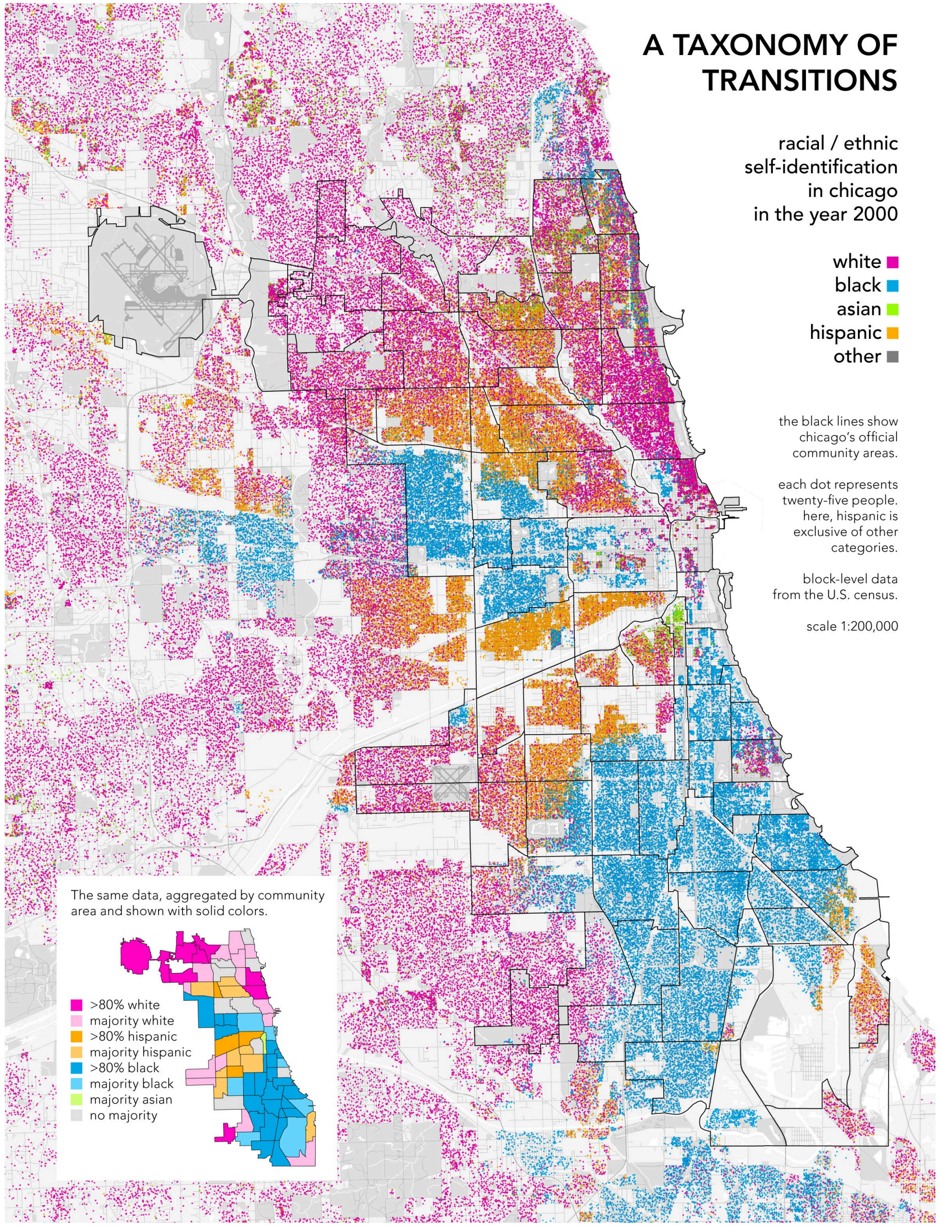
> For his work on the mathematical foundations of statistical physics, proving the **conformal invariance of the Ising model**.
>
> — The Fields Medal 2010 Citation

EXTENSIONS

- **Ising Model ⟶ Hopfield network (1982)**
  - John Hopfield 将 Ising model 用于理解记忆，创建了第一个神经网络。
  - 损失函数 $E = \sum w_{ij} s_i s_j$
    - 连接权重 w_{ij} 对应于 Ising 模型中的相互作用强度。
    - 存储记忆模式作为能量最小化的稳定态。
- 重整化, 重整化群, 重整化流 (Renormalization, Renormalization Group, Renormalization Flow)
  - Kenneth G. Wilson
- Spin glass model - Giorgio Parisi
- Junyi Gao, Joerg Heintz, Christina Mack, Lucas Glass, Adam Cross, Jimeng Sun (2023). Evidence-driven spatiotemporal COVID-19 hospitalization prediction with **Ising dynamics**. *Nature Communications*.

SCHELLING MODEL

BACKGROUND



- Observation
  - 极度隔离：白人(粉)、黑人(蓝)、拉美裔(橙) 界限分明。
  - 尖锐边界：几乎没有“混合区”，宛如一道无形的墙。
- 隔离的代价 (Consequences)
  - 资源不平等：美国学区房制度 → 教育隔离。
  - 贫困代际传递：医疗、治安、就业机会的空间剥夺。
- 传统社会学的解释 (Structural Explanations)
  - 宏观政策：吉姆·克劳法 (Jim Crow Laws), 红线政策 (Redlining)。
  - 核心假设：极端的宏观结果 = 极端的宏观政策 + 极端的种族歧视。
- The Puzzle
  - 法律隔离已废除 (Civil Rights Act 1964)，为何事实上的隔离依然顽固?
  - 关键问题：宏观隔离是否由微观动机产生?

SCHELLING MODEL

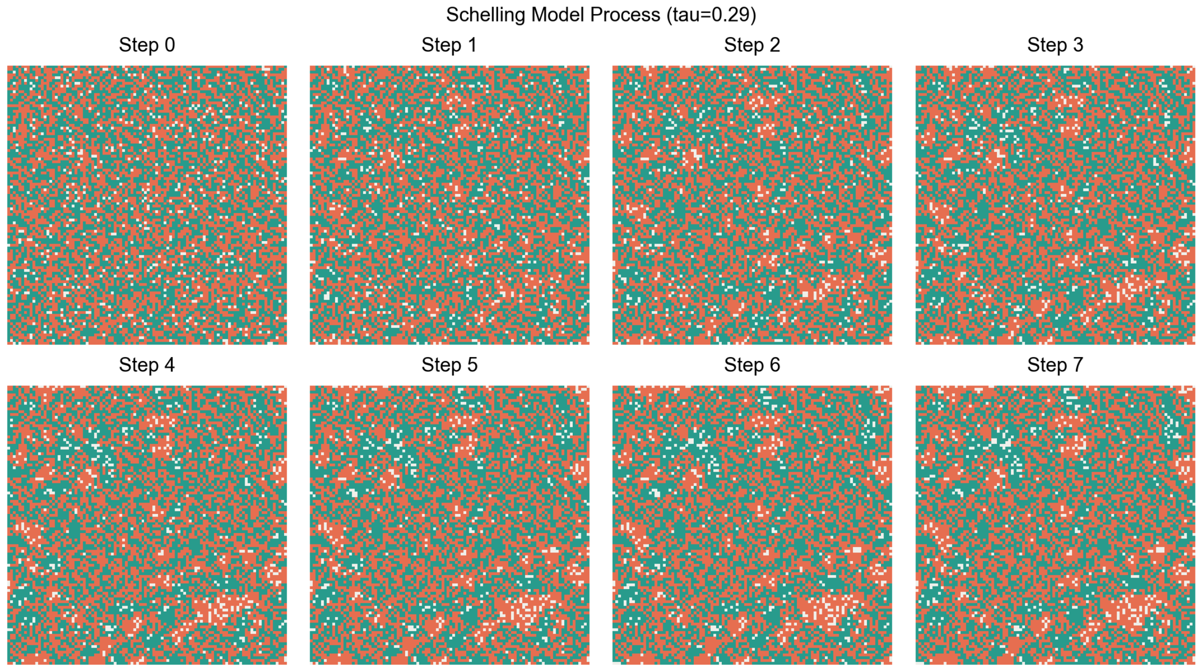
1. Historical Context

- James Sakoda (1949)
  - Dissertation: "Minidoka: An Analysis of Changing Patterns of Social Interaction".
  - 早期的棋盘模型 (Checkerboard Model)，虽早于 Schelling 20 年，但长期被忽视。
- Thomas Schelling (1971)
  - Dynamic Models of Segregation, Journal of Mathematical Sociology.
  - Micromotives and Macrobehavior (1978).
- 第一个真正意义上大规模、多变量、完全基于计算机的社会科学 ABM
  - Growing Artificial Societies: Social Science from the Bottom Up (1996) - Epstein & Axtell
  - Sugarscape 模型

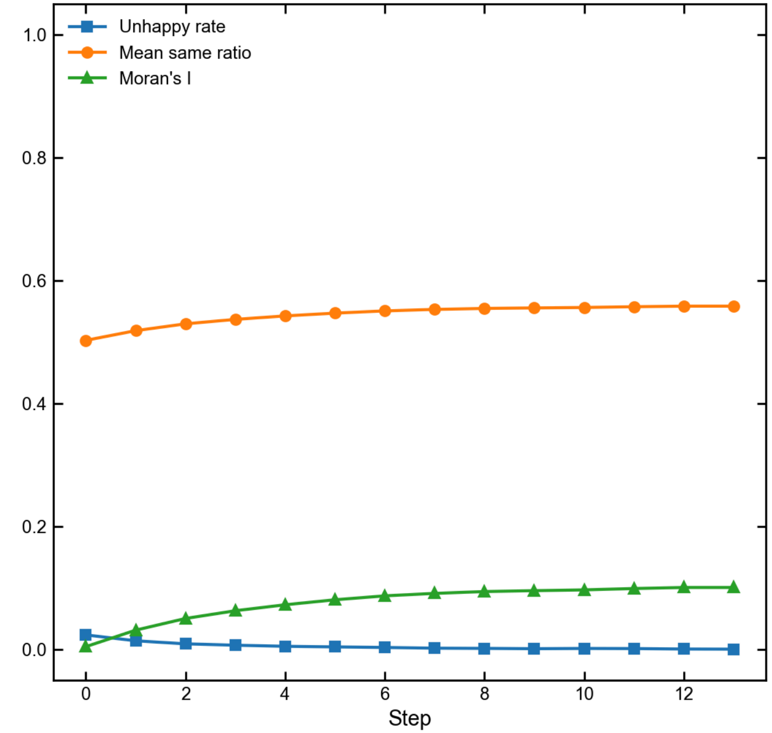
2. Formal Definition

- Environment (环境):
  - $N \times N$ 网格 (Grid)，通常为环面 (Torus)。
- Agents (智能体):
  - 两类群体：$S \in \{Red, Blue\}$。
  - 网格中包含一定比例的空地 (Empty/Void)。
- 交互:
  - Moore Neighborhood: 周围 8 个格子（上下左右+对角）。
  - Rule: Utility Function (效用规则):
    - 设定阈值 (Threshold) T
    - 计算同类比例 $f = \frac{\text{Same Color Neighbors}}{\text{Total Neighbors}}$.
  - 状态判定:
    - IF $f \geq T \longrightarrow$ Happy (Satisfied).
    - IF $f < T \longrightarrow$ Unhappy (Dissatisfied).
- 动作:
  - 每个 Time Step，随机选取 Unhappy Agent。
  - Move: 移动到网格中随机的一个空地 (通常要求该空地能使其 Happy)。
- Netlogo 实现: Schelling Model

PROCESS VISUALIZATION

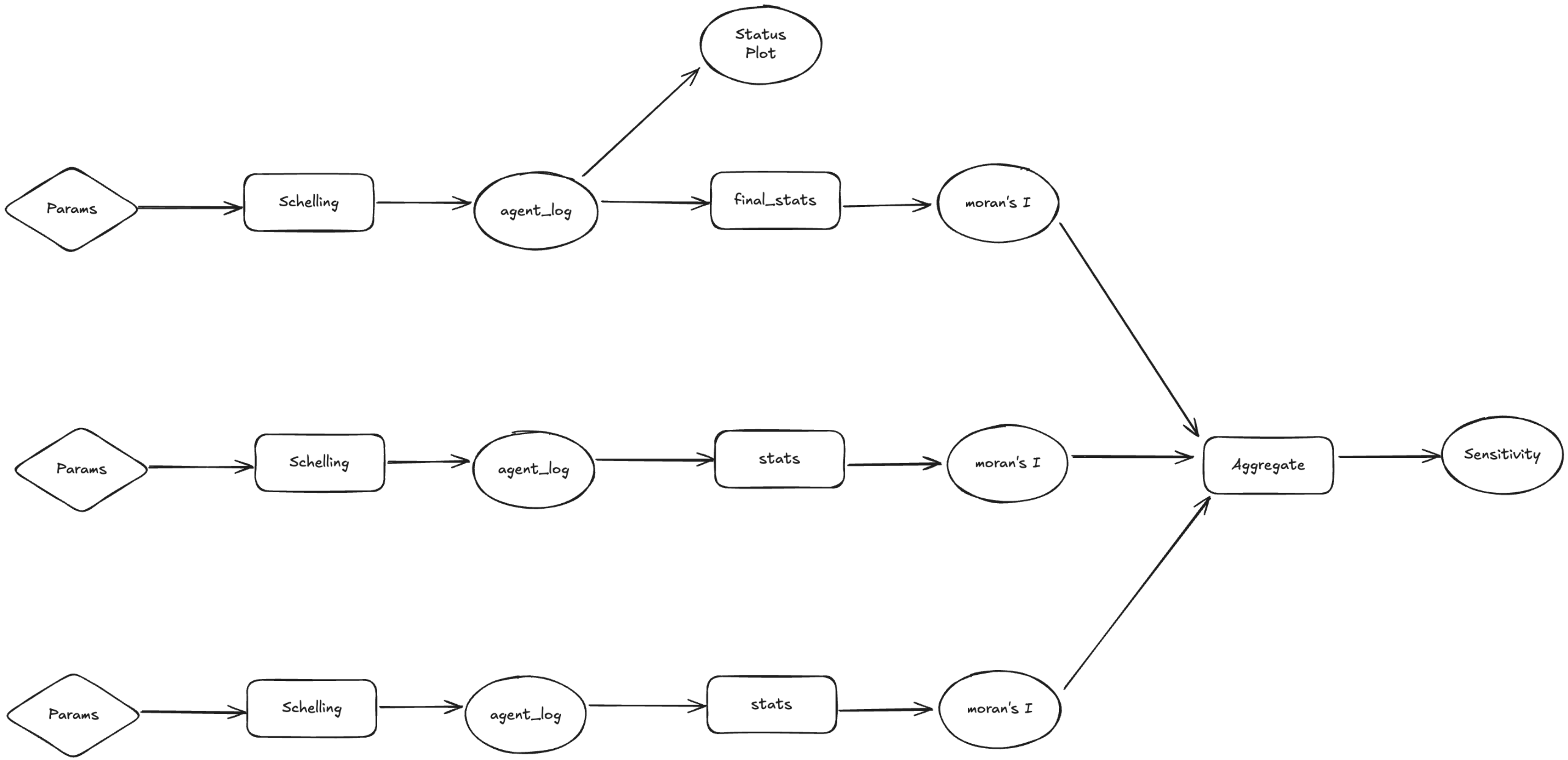
- 

Schelling Model Process (tau=0.29)
Step 0
Step 1
Step 2
Step 3
Step 4
Step 5
Step 6
Step 7

- 

Unhappy rate
Mean same ratio
Moran's I
Step

SENSITIVITY ANALYSIS



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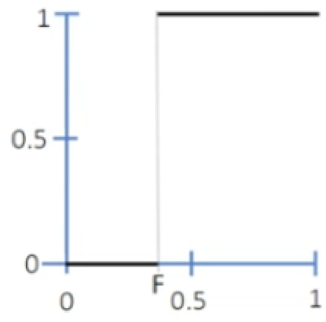
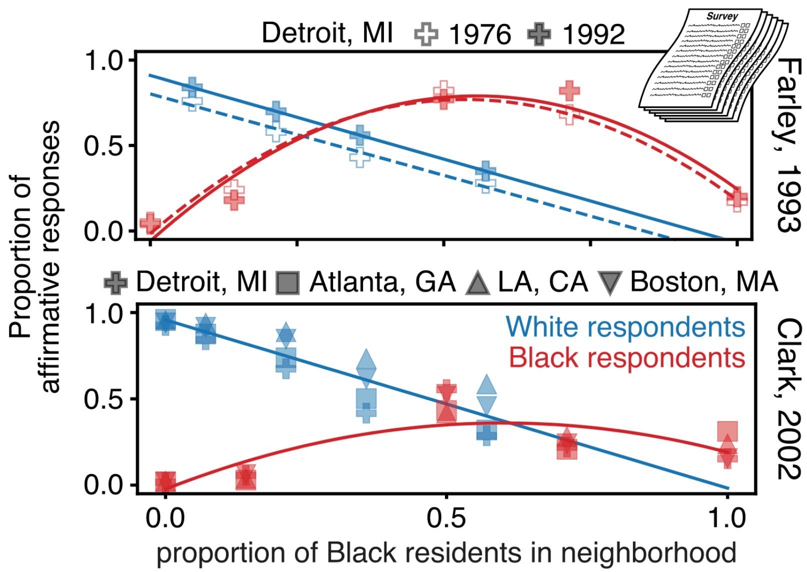
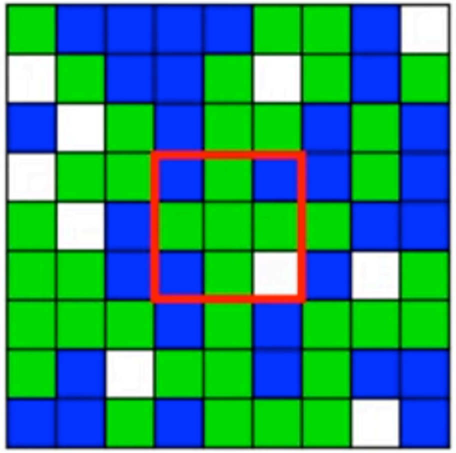
SENSITIVITY ANALYSIS

- 参数：
  - 宏观：网格大小 (Grid Size)； 人口密度 (Population Density)
  - 微观: **偏好阈值 (Preference Threshold, t)**, **邻域定义 (Neighborhood Definition)** , 移动规则 (Movement Rule)
- 偏好阈值 (t)：智能体对邻居构成比例的接受度。
- 

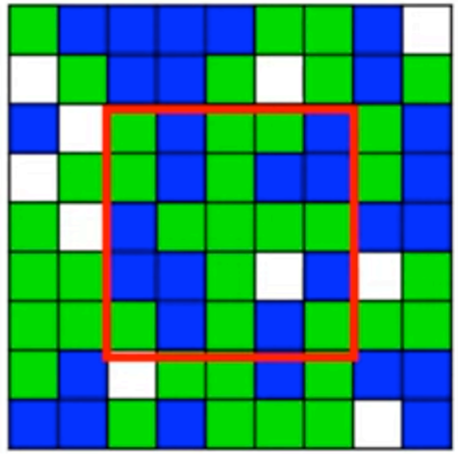


- Neigbhorhood Definition:

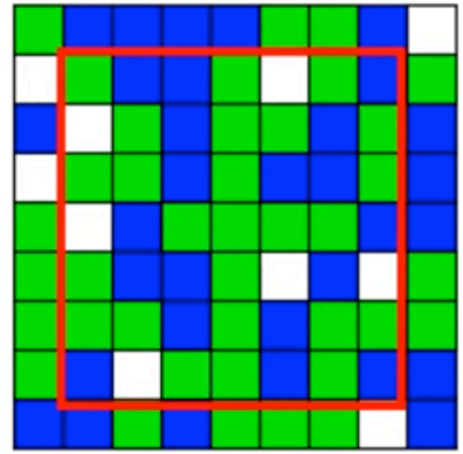
(3 by 3 neighborhood)



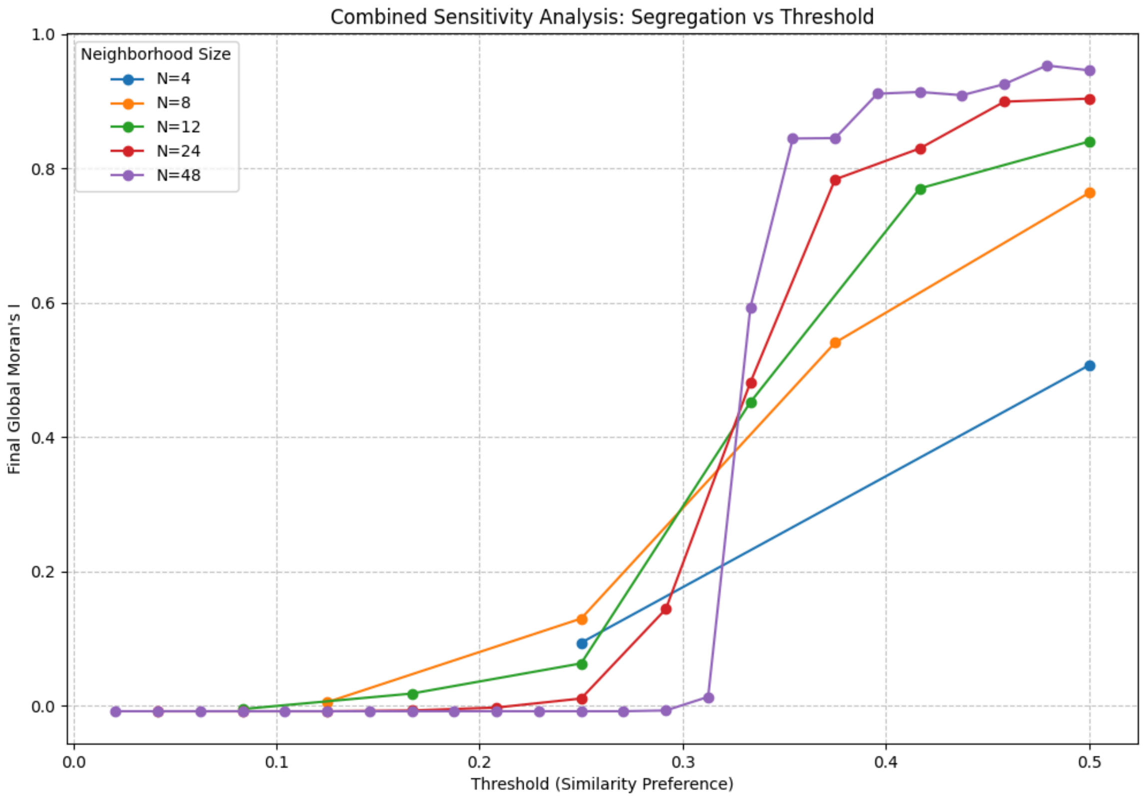
(5 by 5 neighborhood)



(7 by 7 neighborhood)



THE DYNAMICS OF SEGREGATION



- Key insights:
  - **涌现 (Emergence)**: 微观的偏好导致宏观的隔离。
  - 反直觉的 **临界点 (Tipping Point)**：非常温和的偏好 ($t = 33\%$)，就会导致高度隔离。
- Ising Model vs Schelling Model
  - Lattice (物理) ↔ Grid/City (社会)
  - Spin (自旋) ↔ Race/Class (种族/阶层)
  - Energy Minimization (能量极小) ↔ Happiness Maximization (满意度极大)
  - Temperature (温度/噪音) ↔ Tolerance/Random Move (随机搬家/容忍度)

EXTENSIONS AND MODIFICATIONS

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THANKS.

TODOS

- Level 1:
  - 对格网大小和人口密度进行敏感性分析 (Sensitivity Analysis)。
  - 比较在不同邻域定义下，格网大小和人口密度对隔离程度的影响。
- Level 2:
  - 给mesa github 提交pull request，添加schelling model中对tolerance threshold的敏感性分析。