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Exponents & Radicals

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1. Exponents

1.1 Formula Summary

A. Positive Exponents

Product Rule: This is useful for multiplying two numbers that have the same base. It converts multiplication into addition.

$$a^m \times a^n = a^{m+n}$$
$$Eg: 5^2 \times 5^4 = 5^{2+4} = 5^6$$

Quotient rule is useful for dividing one number by another number that has the same base. It converts division into subtraction

$$\frac{a^m}{a^n} = a^{m-n}, \qquad Eg: \frac{5^7}{5^3} = 5^{7-3} = 5^4$$

Power rule is useful for taking a power of a power exponentiation into multiplication.

$$(a^m)^n = a^{mn}, \qquad Eg: (5^2)^3 = 5^{2\times 3} = 5^6$$

Common power rule: If the powers are the same, multiply the bases.

$$a^m \times b^m = (ab)^m, \qquad Eg: 2^3 \times 5^3 = (2 \times 5)^3 = 10^3$$

Fractions

To calculate the exponent of a fraction, calculate the exponent of the numerator and the denominator separately:

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \qquad Eg: \left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3}$$

Order of operations

Exponentiation takes higher priority over multiplication.

$$3 \cdot 2^2 = 3 \cdot 4 = 12$$
$$3 \cdot 2^2 \neq 6^2 = 36$$

$$-2^2 = -(2^2) = -4$$
$$-2^2 \neq (-2)^2 = (-2)(-2) = 4$$

B. Zero and Negative Exponents

Anything raised to the power of zero is 1 (except for zero):

$$Zero\ Exponents: a^0 = 1, \qquad a \neq 0$$

Moving a term from numerator to denominator or vice versa changes the sign of the exponent (not the term):

$$Negative\ Exponents: \frac{xa^m}{y} = \frac{x}{ya^{-m}}, \qquad 5^3 = \frac{1}{5^{-3}}, \qquad 5^{-3} = \frac{1}{5^3}$$

Taking the reciprocal of a fraction changes the sign of the exponent:

$$\left(\frac{a}{b}\right)^m = \left(\frac{b}{a}\right)^{-m}$$
$$\left(\frac{1}{3}\right)^{-3} = \left(\frac{3}{1}\right)^3$$

C. Radicals/ Fractional Exponents

$$\sqrt{a} = a^{\frac{1}{2}}$$
$$\sqrt[n]{a} = a^{\frac{1}{n}}$$
$$\sqrt[n]{a^m} = a^{\frac{m}{n}}$$

D. Squares

No.	Square	No.	Square	No.	Square
1	1	11	121	21	441
2	4	12	144	22	484
3	9	13	169	23	529
4	16	14	196	24	576
5	25	15	225	25	625
6	36	16	256	26	676
7	49	17	289	27	729
8	64	18	324	28	784
9	81	19	361	29	841
10	100	20	400	30	900
				31	961
				32	1024

E. Powers

x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}
	Squares	Cubes	Fourth Power	Fifth Power	Sixth Power	Seventh Power	Eighth Power	Ninth Power	Tenth Power
2	4	8	16	32	64	128	256	512	1024
3	9	27	81	243	729				
4	16	64	256	1024	2048				
5	25	125	625						
6	36	216							
7	49	343							
8	64	512							
9	81	729							
10	100	1000							

$$2^1 = 2$$
$$2^2 = 2 \cdot 2 = 4$$
$$2^3 = 2 \cdot 2 \cdot 2 = 8$$
$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$$
$$2^5 = 32$$
$$2^6 = 64$$
$$2^7 = 128$$
$$2^8 = 256$$
$$2^9 = 512$$
$$2^{10} = 1024$$
$$2^{11} = 2048$$
$$2^{12} = 4096$$

$$3^1 = 3$$
$$3^2 = 9$$
$$3^3 = 3 \cdot 3 \cdot 3 = 27$$
$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 9 \cdot 9 = 81$$
$$3^5 = 243$$
$$3^6 = 729$$

1.2 Exponents with Positive Numbers

A. Multiples of Ten

Example 1.1

Find the squares of the following numbers:

Part A: Two Digit Numbers

A. 70
B. 50
C. 90
D. 30
E. 60
F. 20
G. 40
H. 80

Part B: Three Digit Numbers-I

I. 100
J. 300
K. 500
L. 400
M. 900
N. 700
O. 600

Part C: Three Digit Numbers-II

P. 120
Q. 160
R. 130
S. 170
T. 230
U. 270
V. 150
W. 210
X. 250
Y. 220
Z. 280

Part A

$$70 \times 70 = 7 \times 7 \times 10 \times 10 = 49 \times 100 = 4900$$
$$50^2 = 5 \times 5 \times 10 \times 10 = 2500$$
$$90^2 = 9 \times 9 \times 10 \times 10 = 8100$$
$$30^2 = 900$$
$$60^2 = 3600$$
$$20^2 = 400$$
$$40^2 = 1600$$
$$80^2 = 6400$$

Part B

$$100 \times 100 = 10{,}000$$
$$300 \times 300 = 3 \times 3 \times 100 \times 100 = 90{,}000$$
$$900 \times 900 = 9 \times 9 \times 100 \times 100 = 810{,}000$$
$$500 \times 500 = 5 \times 5 \times 100 \times 100 = 250{,}000$$

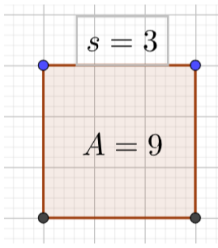
Part C

$$120^2 = 14{,}400$$

Example 1.2: Area and Perimeter

A. A square has a perimeter of 12. Find the area.
B. A square has a perimeter of 24. Find the area.
C. A square has a perimeter of 40. Find the area.
D. A square has a perimeter of 80. Find the area.
E. A square has a perimeter of 120. Find the area.

$$Side\ Length = \frac{Perimeter}{4} = \frac{12}{4} = 3 \Rightarrow Area = 3^2 = 9$$
$$Side\ Length = \frac{Perimeter}{4} = \frac{24}{4} = 6 \Rightarrow Area = 6^2 = 36$$
$$Side\ Length = \frac{Perimeter}{4} = \frac{40}{4} = 10 \Rightarrow Area = 10^2 = 100$$
$$Side\ Length = \frac{Perimeter}{4} = \frac{80}{4} = 20 \Rightarrow Area = 20^2 = 400$$
$$Side\ Length = \frac{Perimeter}{4} = \frac{120}{4} = 30 \Rightarrow Area = 30^2 = 900$$



Example 1.3

A. A square has an area of 64. Find the perimeter.
B. A square has an area of 81. Find the perimeter.
C. A square has an area of 100. Find the perimeter.
D. A square has an area of 900. Find the perimeter.
E. A square has an area of 400. Find the perimeter.

$$Area = 64 \Rightarrow Side\ Length = 8 \Rightarrow Perimeter = 32$$
$$Area = 81 \Rightarrow Side\ Length = 9 \Rightarrow Perimeter = 36$$
$$Area = 100 \Rightarrow Side\ Length = 10 \Rightarrow Perimeter = 40$$

Example 1.4

A. What is the value of $(2^2 - 2) - (3^2 - 3) + (4^2 - 4)$ **(AMC 10A 2021/1)**
B. What is the sum of the perfect squares between 15 and 25, inclusive, minus the sum of the other integers from 15 and 25, inclusive? **(AOPS Alcumus)**

Part A

$$(4 - 2) - (9 - 3) + (16 - 4) = 2 - 6 + 12 = 8$$

Part B

$$16 + 25 - 15 - 17 - 18 - 19 - 20 - 21 - 22 - 23 - 24$$

Let's rearrange:

$$= (16 - 15) - 17 - 18 - 19 - 20 - 21 - 22 - 23 + (25 - 24)$$

The expression in the brackets is easy to simplify:

$$= 2 - \left(\underbrace{17}_{20-3} + \underbrace{18}_{20-2} + \underbrace{19}_{20-1} + 20 + \underbrace{21}_{20+1} + \underbrace{22}_{20+2} + \underbrace{23}_{20+3}\right) = 2 - 140 = -138$$

Example 1.5

The English Mathematician Augustus DeMorgan lived in the nineteenth century (1801-1900). The mathematician made the following statement: I was x years old in the year x^2. What was his age?

$$40^2 = 1600, \qquad 50^2 = 2500$$
$$42^2 = 1764$$
$$43^2 = 1849$$
$$44^2 = 1936$$
$$Age = 43$$

B. Properties

1.6: Squares and Cubes of Different Ranges

The behaviour changing depends upon where x is.

	x	x^2	x^3	
$x > 1$	2	4	8	$x < x^2 < x^3$
$0 < x < 1$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$x^3 < x^2 < x$
$-1 < x < 0$	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{8}$	$x < x^3 < x^2$
$x < -1$	-2	4	-8	$x^3 < x < x^2$

Example 1.7

Consider $x = \frac{2}{3}$. Arrange in increasing order:

$$x, x^2, x^3$$

$$x^3, x^2, x$$

1.8: Positive Exponent: Definition of x^y

Exponents ask us to multiply a number by itself the number of times mentioned in the exponent.

$$x^y = \underbrace{x \times x \times \ldots \times x}_{y\ times}$$

Example 1.9

Evaluate the following. Give your answers as numbers.

A. 5^2
B. 2^3
C. 10^2
D. 3^2
E. 8^2
F. 5^3

G. 10^3
H. 2^4
I. 3^3
J. 7^3
K. 10^4
L. 2^5

M. 5^4
N. 2^6
O. 9^2
P. 2^7
Q. 3^4
R. 8^3

S. 9^3
T. 2^{10}
U. 3^5
V. 11^3
W. 17^2
X. 19^2

Y. 12^3
Z. 3^6

$$5^3 = 125$$
$$2^6 = 64$$
$$8^2 = 64$$
$$7^3 = 343$$
$$2^{10} = 1024$$
$$5^4 = 625$$
$$9^2 = 81$$
$$3^4 = 81$$
$$8^3 = 512$$
$$9^3 = 729$$

1.10: Exponent Form and Expanded Form

You can convert between expanded form and exponent form:

$$\underbrace{x^y}_{Exponent\ Form} = \underbrace{x \times x \times \ldots \times x}_{Expanded\ Form}$$

Example 1.11

Write the following in exponent form:

A. $2 \times 2 \times 2$
B. $11 \times 11 \times 11 \times 11$
C. $a \times a$

$$2^3$$
$$11 \times 11 \times 11 \times 11 = 11^4$$
$$a \times a = a^2$$

C. Fractions

1.12: Power of an exponent

You can evaluate a power of a fraction by exponentiating the numerator and the denominator separately:

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

$$\left(\frac{a}{b}\right)^m = \underbrace{\frac{a}{b} \cdot \frac{a}{b} \ldots \cdot \frac{a}{b}}_{m\ times} = \frac{a \cdot a \ldots \cdot a}{b \cdot b \ldots \cdot b} = \frac{a^m}{b^m}$$

Example 1.13: Fractions

Evaluate. Give your answer as a fraction.

Basics

1) $\frac{3^3}{2^4}$
2) $\frac{5^4}{3^6}$
3) $\frac{7^3}{9^2}$
4) $\frac{3^4}{7^2}$

Squares

5) $\left(\frac{3}{5}\right)^2$
6) $\left(\frac{2}{7}\right)^2$
7) $\left(\frac{7}{11}\right)^2$
8) $\left(\frac{8}{13}\right)^2$
9) $\left(\frac{16}{17}\right)^2$
10) $\left(\frac{19}{21}\right)^2$

Cubes

11) $\left(\frac{2}{3}\right)^3$
12) $\left(\frac{5}{3}\right)^3$
13) $\left(\frac{9}{7}\right)^3$
14) $\left(\frac{1}{6}\right)^3$
15) $\left(\frac{4}{11}\right)^3$

Higher Powers

Convert from the form $\left(\frac{a}{b}\right)^m$ to the form $\frac{a^m}{b^m}$:

16) $\left(\frac{7}{11}\right)^{12}$
17) $\left(\frac{3}{8}\right)^5$
18) $\left(\frac{9}{11}\right)^{100}$
19) $\left(\frac{x}{y}\right)^7$
20) $\left(\frac{y}{z}\right)^4$
21) $\left(\frac{3}{s}\right)^{11}$
22) $\left(\frac{r}{6}\right)^{19}$

Basics

$$\frac{3^3}{2^4} = \frac{27}{16}$$
$$\frac{5^4}{3^6} = \frac{625}{729}$$
$$\frac{7^3}{9^2} = \frac{343}{81}$$
$$\frac{3^4}{7^2} = \frac{81}{49}$$

Squares of Fractions

$$\left(\frac{3}{5}\right)^2=\frac{3}{5}\times\frac{3}{5}=\frac{3^2}{5^2}=\frac{9}{25}$$
$$\left(\frac{2}{7}\right)^2=\frac{4}{49}$$
$$\left(\frac{7}{11}\right)^2=\frac{49}{121}$$
$$\left(\frac{8}{13}\right)^2=\frac{64}{169}$$
$$\left(\frac{16}{17}\right)^2=\frac{256}{289}$$
$$\left(\frac{19}{21}\right)^2=\frac{361}{441}$$

Cubes

$$\left(\frac{2}{3}\right)^3=\frac{8}{27}$$
$$\left(\frac{9}{7}\right)^3=\frac{729}{343}$$
$$\left(\frac{1}{6}\right)^3=\frac{1}{216}$$
$$\left(\frac{4}{11}\right)^3=\frac{64}{1331}$$

Higher Powers

$$\left(\frac{7}{11}\right)^{12}=\frac{7^{12}}{11^{12}}$$
$$\left(\frac{3}{8}\right)^5=\frac{3^5}{8^5}$$
$$\left(\frac{9}{11}\right)^{100}=\frac{9^{100}}{11^{100}}$$
$$\left(\frac{x}{y}\right)^7=\frac{x^7}{y^7}$$
$$\left(\frac{y}{z}\right)^4=\frac{y^4}{z^4}$$
$$\left(\frac{3}{s}\right)^{11}=\frac{3^{11}}{s^{11}}$$
$$\left(\frac{r}{6}\right)^{19}=\frac{r^{19}}{6^{19}}$$

Example 1.14: Powers of Fractions

$$\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}$$

Simplify:

Squares

A. $\left(\frac{1}{2}\right)^2$

B. $\left(\frac{5}{7}\right)^2$

C. $\left(\frac{3}{4}\right)^2$

D. $\left(\frac{11}{13}\right)^2$

E. $\left(\frac{7}{11}\right)^2$

F. $\left(\frac{13}{19}\right)^2$

Cubes

G. $\left(\frac{3}{5}\right)^3$

H. $\left(\frac{2}{3}\right)^3$

I. $\left(\frac{5}{7}\right)^3$

J. $\left(\frac{11}{10}\right)^3$

Fourth Powers

K. $\left(\frac{2}{3}\right)^4$

L. $\left(\frac{1}{2}\right)^4$

M. $\left(\frac{4}{5}\right)^4$

Combine and then Simplify

N. $\frac{2^4}{4^4}$

O. $\frac{9^4}{27^4}$

P. $\frac{125^3}{25^3}$

Q. $\frac{6^3}{2^3}$

Variables

R. $\left(\frac{a}{b}\right)^3$

Squares

To find the square of a fraction, we multiply it by itself:

$$\left(\frac{1}{2}\right)^2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$
$$\left(\frac{5}{7}\right)^2 = \frac{5}{7} \times \frac{5}{7} = \frac{5^2}{7^2} = \frac{25}{49}$$
$$\left(\frac{3}{4}\right)^2 = \frac{9}{16}$$
$$\left(\frac{7}{11}\right)^2 = \frac{49}{121}$$
$$\left(\frac{13}{19}\right)^2 = \frac{169}{361}$$

Cubes

To find the cube of a fraction, we multiply it by itself three times:

$$\left(\frac{3}{5}\right)^3 = \frac{27}{125}$$
$$\left(\frac{2}{3}\right)^3 = \frac{8}{27}$$
$$\left(\frac{5}{7}\right)^3 = \frac{125}{343}$$

Fourth Powers

$$\left(\frac{2}{3}\right)^4 = \frac{16}{81}$$
$$\left(\frac{1}{2}\right)^4 = \frac{1}{16}$$
$$\left(\frac{4}{5}\right)^4 = \frac{4^4}{5^4} = \frac{256}{625}$$

Combining Fractions

$$\frac{2^4}{4^4} = \left(\frac{2}{4}\right)^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$
$$\frac{9^4}{27^4} = \left(\frac{9}{27}\right)^4 = \left(\frac{1}{3}\right)^4 = \frac{1}{81}$$
$$\frac{125^3}{25^3} = \left(\frac{125}{25}\right)^3 = \left(\frac{5}{1}\right)^3 = 125$$
$$\frac{6^3}{2^3} = \left(\frac{6}{2}\right)^3 = 3^3 = 27$$
$$\frac{6^3}{9^3} = \left(\frac{6}{9}\right)^3 = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$
$$\frac{12^3}{9^3} = \left(\frac{12}{9}\right)^3 = \left(\frac{4}{3}\right)^3 = \frac{64}{27}$$
$$\frac{12^3}{16^3} = \left(\frac{12}{16}\right)^3 = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$$

Variables

$$\left(\frac{a}{b}\right)^3 = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}$$

D. Simple Equations

1.15: Power of an exponent

Simple equations with exponents can be solved directly. The parity (odd/even) of the exponent decides the number of solutions[1].

If x has

- an odd exponent, it will have exactly one real solution.
- an even exponent, it will have exactly two real solutions.

$$x^2 = 1 \Rightarrow x = \pm 1$$
$$x^3 = 1 \Rightarrow x = 1$$

Example 1.16: Equations

Find the value of the variable in each part below:

A. $x^3 = 27$

[1] We are working in the real number system. When we work in complex numbers: $x^3 = 1$ *has* 3 *solutions*

B. $y^4 = 16$
C. $z^2 = 625$
D. $v^9 = 512$
E. $h^4 = 625$
F. $t^6 = 729$

$$x^3 = 27 \Rightarrow x = 3$$
$$y^4 = 16 \Rightarrow y = \pm 2$$
$$z^2 = 625 \Rightarrow z = \pm 25$$
$$v = 2$$
$$h = \pm 5$$
$$t = \pm 3$$

Example 1.17: Number Theory

A. Find the largest prime factor of $2^3 + 3^3$.
B. Find the largest divisor of $3^2 + 3^3$ that is smaller than the number.
C. Find the sum of digits of $2^{10} - 10^2$.
D. Find the sum of digits of $3^6 + 8^2$.
E. Find the smallest non-zero digit in 2^{10}.
F. Find the difference between the smallest and the largest digit in 12^3.

$$2^3 + 3^3 = 8 + 27 = 35 = 5 \times 7 \Rightarrow Largest\ Prime\ Factor = 7$$
$$3^2 + 3^3 = 9 + 27 = 36 \Rightarrow Largest\ Divisor = 18$$
$$2^{10} - 10^2 = 1024 - 100 = 924 \Rightarrow Sum\ of\ Digits = 9 + 2 + 4 = 15$$
$$3^6 + 8^2 = 729 + 64 = 793 \Rightarrow Sum\ of\ Digits = 19$$
$$2^{10} = 1024 \Rightarrow Smallest\ non - zero\ digit = 1$$
$$12^3 = 1728 \Rightarrow Difference = 8 - 1 = 7$$

E. HCF and LCM

You can use the exponent form of the prime factorization of a number to determine the HCF and LCM.

1.18: HCF and LCM

To find the

- HCF, choose the lower value of each exponent in the prime factorization.
- LCM, choose the higher value of each exponent in the prime factorization.

$$p = \underbrace{2^2}_{Lower} \times \underbrace{3^3}_{Higher}, \quad q = \underbrace{2^4}_{Higher} \times \underbrace{3}_{Lower}$$
$$HCF = 2^2 \times 3^1$$
$$LCM = 2^4 \times 3^3$$

Example 1.19: HCF and LCM Basics

p and q are written in exponent form. Find the HCF and LCM of p and q in each part below. Write your answers in exponent form.

A. $p = 5^3 \times 7^2,\ q = 5^2 \times 7^5$
B. $p = 2^2 \times 3^3 \times 5^1,\ q = 2^5 \times 3^2 \times 5^4$
C. $p = 2^{12} \times 3^4, q = 2^2 \times 3^{12}, r = 2^{20} \times 3^8$
D. $p = 2^4 \times 3^2, q = 3^4 \times 5^2$

E. $p = 2^x \times 3^{3y}, q = 2^{2x} \times 3^y$

Part A

$$HCF = 5^2 \times 7^2$$
$$LCM = 5^3 \times 7^5$$

Part B

$$HCF = 2^2 \times 3^2 \times 5^1$$
$$LCM = 2^5 \times 3^3 \times 5^4$$

Part C

$$HCF = 2^2 \times 3^4$$
$$LCM = 2^{20} \times 3^{12}$$

Part D

$$HCF = 3^2$$
$$LCM = 2^4 \times 3^4 \times 5^2$$

Part E

$$HCF = 2^x \times 3^y$$
$$LCM = 2^{2x} \times 3^{3y}$$

Example 1.20: Back Calculations

I have two numbers p and q with $p = 2^a \times 3^b$ and $q = 2^5 \times 3^7$.

A. If $HCF(p, q) = 2^3 \times 3^7$ and $LCM(p, q) = 2^5 \times 3^{11}$, what is $a + b$?
B. If $HCF(p, q) = 2^5 \times 3^7$, what values can a and b take?

Part A

$$p = 2^a \times 3^b, q = 2^5 \times 3^7$$
$$HCF\colon \min(2^a, 2^5) = 2^3 \Rightarrow a = 3$$
$$LCM\colon \max\left(3^b, 3^7\right) = 3^{11} \Rightarrow b = 11$$
$$a + b = 14$$

Part B

$$\min(2^a, 2^5) = 2^5 \Rightarrow 2^a \geq 2^5 \Rightarrow a \geq 5 \Rightarrow a \in \{5,6,7, \dots\}$$
$$\min\left(3^b, 3^7\right) = 3^7 \Rightarrow 3^b \geq 3^7 \Rightarrow b \geq 7 \Rightarrow b \in \{7,8,9, \dots\}$$

F. Powers of Zero and One

1.21: Powers of 1 and 0

One to any power is just 1

$$1^x = 1$$

Zero to any power is zero

$$0^x = 0$$

(except for 0^0)[2]

Example 1.22: Powers of One and Zero

A. 1^{45}
B. 0^4
C. x^y, if $x = 0, y = 1$
D. $1^1 + 1^2 + 1^3 + \cdots + 1^{2022}$
E. $0^0 + 0^1 + 0^2 + \cdots + 0^{2022}$

Part A, B, C

$$1^{45} = 1$$
$$0^4 = 0$$
$$0^1 = 0$$

[2] which can be either 0 or 1, depending on the context

Part D

$$\underbrace{1+1+1+\cdots+1}_{2022\,Times} = 1 \times 2022 = 2022$$

Part E

Look at the very first term. It is 0^0, which is not defined.
Hence, the entire expression is not defined.

1.3 Exponents with Negative Numbers

A. Negative Numbers

1.23: Negative Numbers

You can find exponents of negative numbers.

$$(-x)^y = \underbrace{(-x)\times(-x)\times\ldots\times(-x)}_{y\ times} = (-1)^y \times \underbrace{x\times x\times\ldots\times x}_{y\ times} = (-1)^y x^y$$

Even Power

When y is even, the minus signs are multiplied an even number of times, and hence the final answer is positive.

$$(-x)^y = x^y$$

Odd Power

When y is odd, the minus signs are multiplied an odd number of times, and hence the final answer is negative.

$$(-x)^y = -(x^y) = -x^y$$

Example 1.24

Evaluate the following. Give your answers as numbers.

Basics

A. $(-4)^3$
B. $(-2)^3$
C. $(-5)^2$
D. $(-7)^3$
E. $(-11)^2$
F. $(-2)^{10}$
G. $(-3)^5$
H. $(-4)^5$
I. $(-2)^7$

Part II:

J.

Using the property above, a negative number raised to an odd power will give a negative result:

$$(-4)^3 = -(4^3) = -(64) =$$
$$(-2)^3 = -8$$

Using the property above, a negative number raised to an even power will give a positive result:

$$(-5)^2 = 5^2 = 25$$
$$(-7)^3 = -343$$
$$(-11)^2 = 121$$
$$(-2)^{10} = 1024$$
$$(-3)^5 = -243$$
$$(-4)^5 = 1024$$
$$(-2)^7 = -128$$

B. Order of Operations

1.25: Exponentiation in Order of Operations

Exponentiation takes higher priority over multiplication in the order of precedence of operations. Hence:

$$-x^y = -(x^y)$$

In other words, when the minus sign is not inside the parentheses, the minus sign is not multiplied, and hence, the answer will always be negative.

Example 1.26

A. -2^2
B. -2^3
C. $(-2)^2$
D. $(-2)^3$

$$-2^2 = -(2^2) = -4$$
$$-2^3 = -(2^3) = -8$$
$$(-2)^2 = (-1)^2(2^2) = 1(4) = 4$$
$$(-2)^3 = (-1)^3(2^3) = -1(8) = -8$$

Example 1.27

Simplify $-5^2 + (-5)^2$

Consider order of operations:
The purple term gives higher priority to the exponent. So, the square is only applicable to the 5.
The red term has a bracket around the minus sign. So, the minus sign also gets squared.

$$-5^2 + (-5)^2 = -(5 \times 5) + (-5)(-5) = -25 + 25 = 0$$

Be careful with minus sign before exponents. The minus sign will get multiplied only if it is in brackets.

An even number of negative signs results in a positive expression. An odd number of negative signs results in a negative expression.

Example 1.28

Evaluate the following. Give your answers as numbers.

$Quasi - ve$ Numbers
A. -2^3
B. -3^4
C. -5^2
D. -7^3
E. -9^2

Mixed
F. $(-3)^2$
G. -3^2
H. $(-5)^3$
I. -5^3

Fractions
J. $\frac{(-3)^5}{-3^5}$
K. $\frac{(-5)^2}{-5^2}$
L. $\frac{(-2)^{23}}{-2^{23}}$
M. $\frac{(-7)^5}{7^5}$
N. $\frac{(-19)^{12}}{19^{12}}$

Fractions with Variables
O. $\frac{(-x)^5}{-x^5}, x > 0$
P. $\frac{(-y)^6}{-y^6}, y > 0$

$Quasi - ve$ Numbers

$$-2^3 = -(2^3) = -(8) = -8$$
$$-3^4 = -(3^4) = -81$$
$$-5^2 = -25$$
$$-7^3 = -343$$
$$-9^2 = -81$$

Mixed

$$(-3)^2 = 9$$
$$-3^2 = -9$$
$$(-5)^3 = -125$$
$$-5^3 = -125$$

Example 1.29

Fractions

$$\frac{(-3)^5}{-3^5}=\frac{(-1)^5\times\mathbf{3^5}}{(-1)\times\mathbf{3^5}}=\frac{-1}{-1}=1$$
$$\frac{(-5)^2}{-5^2}=\frac{(-1)^2\times 5^2}{(-1)\times 5^2}=\frac{1}{-1}=-1$$
$$\frac{(-2)^{23}}{-2^{23}}=\frac{(-1^{23})(\mathbf{2^{23}})}{(-1)(\mathbf{2^{23}})}=\frac{-1}{-1}=1$$

Fractions with Variables

$$\frac{(-x)^5}{-x^5}=\frac{(-1^5)(x^5)}{(-1)(x^5)}=\frac{-1}{-1}=1$$
$$\frac{(-y)^6}{-y^6}=\frac{(-1^6)(y^5)}{(-1)(y^5)}=\frac{1}{-1}=-1$$

C. Expressions

Example 1.30

Addition and Subtraction

A. $(-3)^3-3^3$
B. $5^2-(-5)^2$
C. 7^2-7^2
D. $3^2-2^3+(-2)^3-(-3)^2$
E. $5^2-(-4)^2-2^2$

Addition and Subtraction with Variables

F. x^2-x^2
G. $a^4-(-a)^4$
H. $b^2+(-b)^2$
I. $a^3+(-y)^2$
J. $x^2-(-y)^3-z^4$

Multiplication

K. $(-3)^2(-3^2)$
L. $-\{(-3)^2[-(3^2)]\}$

Multiplication with Variables

M.

Addition and Subtraction

$$(-3)^3-3^3=-27-27=-54$$
$$5^2-(-5)^2=25-25=0$$
$$7^2-7^2=0$$
$$9-8-8-9=-16$$
$$5^2-(-4)^2-2^2=25-16-4=25-20=5$$

Addition and Subtraction with Variables

$$x^2-x^2=0$$
$$a^4-(-a)^4=a^4-a^4=0$$
$$b^2+(-b)^2=b^2+b^2=2b^2$$
$$a^3+(-y)^2=a^3+y^2$$
$$x^2-(-y)^3-z^4=x^2+y^3-z^4$$

Multiplication

$$[(-3)^2](-3^2)=(9)(-9)=-81$$
$$-\{[(-3)^2][-(3^2)]\}=-\{[9][-9]\}=81$$

Example 1.31

A. $-(-1)^2(-1)^3 \ldots (-1)^{10} - [-(1)^2(1)^3 \ldots (1)^{10}]$
B. $-(-x)^2(-x)^3(-x)^4 \ldots (-x)^{10} - [-(x)^2(x)^3(x)^4 \ldots (x)^{10}]$

Part A

$$-\underbrace{(-1)^2}_{+1}\underbrace{(-1)^3}_{-1}\ldots\underbrace{(-1)^{10}}_{-1} - [-(1)^2(1)^3 \ldots (1)^{10}]$$

We don't need the positive values. We only count the negative values:

$$-(-1)^3(-1)^5(-1)^7(-1)^9 - [-1]$$
$$-1 + 1 = 0$$

Part B

$$-\underbrace{(-x)^2}_{+1}\underbrace{(-x)^3}_{-1}\ldots\underbrace{(-x)^{10}}_{-1} - [-(x)^2(x)^3 \ldots (x)^{10}]$$

We don't need the positive values. We only count the negative values:

$$-x^{2+3+\cdots+10} - [-x^{2+3+\cdots+10}]$$
$$-x^{2+3+\cdots+10} + x^{2+3+\cdots+10} = 0$$

Example 1.32

Solve for the variable.

A. $x^3 = -125$
B. $y^2 = 25$
C. $a^4 = 16$
D. $h^2 = 121$
E. $g^3 = -64$

$$x^3 = -125 \Rightarrow x = -5$$
$$y^2 = 25 \Rightarrow y = \pm 5$$
$$a^4 = 16 \Rightarrow a = \pm 2$$
$$h^2 = 121 \Rightarrow h = \pm 11$$
$$g^3 = -64 \Rightarrow g = -4$$

1.33: Square of a Real Number

The square of a real number is always nonnegative. Hence, some equations do not have solutions in the real number system.

Example 1.34

$$z^2 = -36$$

$$z = -6 \Rightarrow z^2 = (-6)^2 = 36 \textit{ does not work}$$
$$z = 6 \Rightarrow z^2 = 6^2 = 36 \textit{ still does not work}$$

No Solutions in the Real Number System

Example 1.35: Equations with Odd Power

Solve for the variable. If there is no solution, write no solutions.

A. $x^5 = 4^5$
B. $x^5 = (-4)^5$
C. $-x^5 = (4)^5$

D. $-x^5 = (-4)^5$

$$x^5 = 4^5 \Rightarrow x = 4$$
$$x^5 = (-4)^5 \Rightarrow x = -4$$
$$-x^5 = (4)^5 \Rightarrow x = -4$$
$$-x^5 = (-4)^5 \Rightarrow x = 4$$

Example 1.36: Equations with Even Power

Solve for the variable. If there is no solution, write no solutions.

A. $x^6 = 4^6$
B. $x^6 = -4^6$
C. $-x^6 = 4^6$
D. $-x^6 = (-4)^6$

$$x^6 = 4^6 \Rightarrow x = 4$$
$$x^6 = -4^6 \Rightarrow No\ Solution$$
$$-x^6 = 4^6 \Rightarrow No\ Solution$$
$$-x^6 = (-4)^6 \Rightarrow No\ Solution$$

D. Powers of Unity

1.37: Powers of Unity

Even powers of 1 are positive 1, and odd powers of 1 are negative one.

$$(-1)^x = \begin{cases} +1, & if\ x\ is\ even \\ -1, & if\ x\ is\ odd \end{cases}$$

Example 1.38: Powers of Unity

Simplify

A. $(-1)^{2020} + \{-1^{2020}\} + \{(-1)^{2021}\} + \{-1^{2021}\}$

Summary

$$(-1)^{2020} + \{-1^{2020}\} + \{(-1)^{2021}\} + \{-1^{2021}\} = 1 - 1 - 1 - 1 = 1 - 3 = -2$$

Detailed Explanation

$$(-1)^{2020} = 1$$
$$\{-1^{2020}\} = (-1) \times 1^{2020} = -1 \times 1 = -1$$
$$\{(-1)^{2021}\} = -1$$
$$\{-1^{2021}\} = (-1) \times 1^{2021} = -1 \times 1 = -1$$

Example 1.39: Comparing Powers of Positive and Negative Unity

Let $x = -1$ and $y = 1$. When is $x^n = y^n, n \in N$?

n	x^n	y^n
1	-1	1
2	1	1

3	-1	1
4	1	1

$x^n = y^n$ precisely when n is even $\Rightarrow n \in 2k, k \in N$

E. Sign of an expression

Base	Positive	Positive	Negative	Negative
Exponent	Odd	Even	Odd	Even
Answer	Positive	Positive	Negative	Positive

A quantity raised to a power is negative only if the quantity is negative AND it is raised to an odd power
Expressions will be negative only if both the base and the power are red.

Example 1.40: Determining the sign of an expression

Determine the sign of the expressions below if x, y, z are positive and p, q, r are negative.

A. $\frac{x^3p^5}{y^2r^2}$

$$\frac{x^3p^5}{y^2r^2} = \frac{(\boldsymbol{positive})^{odd}(\boldsymbol{negative})^{odd}}{(\boldsymbol{positive})^{even}(\boldsymbol{negative})^{even}} = \frac{(\boldsymbol{positive})(\boldsymbol{negative})}{(\boldsymbol{positive})(\boldsymbol{positive})} = \boldsymbol{negative}$$

F. Multiplicative Inverse

A multiplicative inverse is a number that multiplies with another number to give one.

$$5 \times \frac{1}{5} = 1$$

Hence, 5 and $\frac{1}{5}$ are multiplicative inverses.

Similarly,

$$\frac{3}{4} \times \frac{4}{3} = 1 \Rightarrow \frac{3}{4} \text{ and } \frac{4}{3} \text{ are multiplicative inverses.}$$

Multiplicative inverse is the reciprocal of the original number.

Example 1.41: Multiplicative Inverse for Exponents

Find the multiplicative inverse for:

A. 5^2
B. 3^2
C. 7^3
D. 25^7
E. 2^{19}
F. -2^{19}

Multiplicative Inverse = Reciprocal of $5^2 = \frac{1}{5^2}$

$$5^2 \times \frac{1}{5^2} = 1$$

$$\frac{1}{3^2}$$

$$\frac{1}{7^3}$$

$$\frac{1}{25^7}$$
$$\frac{1}{2^{19}}$$
$$\frac{1}{-2^{19}}$$

Example 1.42: Multiplicative Inverse of Negative Numbers

Find the multiplicative inverse of -1

To find the multiplicative inverse of a negative, you still take its reciprocal

Multiplicative Inverse $=$ Reciprocal of $-1 = \frac{1}{-1} = -1 \Rightarrow Check: (-1)(-1) = 1$

1.4 Exponent Rules

A. Basics

1.43: Definition of Exponents (Natural Number)

The left-hand side is called the expanded form or the product form. The right-hand side is called the exponent form.

$$\underbrace{a \times a \times \ldots \times a}_{m\ times} = a^m$$

Example 1.44

Write in exponent form.

A. 9×9
B. $3 \times 3 \times 3$
C. 27×27
D. $7 \times 7 \times 7 \times 7$

Write as a product. Do not evaluate.

S. 3^2
T. 7^3

$$9^2$$
$$3^3$$
$$27^2$$
$$7^4$$
$$3 \times 3$$
$$7 \times 7 \times 7$$

1.45: Product Rule

Exponents convert multiplication into addition if the bases are the same:

$$a^m \times a^n = a^{m+n}$$

Numbers

$$5^3 \times 5^2 = \underbrace{5 \times 5 \times 5}_{5^3} \times \underbrace{5 \times 5}_{5^2} = \underbrace{5 \times 5 \times 5 \times 5 \times 5}_{=5^{3+2}=5^5} = 5^5$$

Variables

The laws of exponents work the same for variables as they do for numbers.

- If the bases are the same, we can multiply the numbers by adding the exponents
- If the bases are not the same, we cannot multiply.

$$x^2 \times x^5 = x^{2+5} = x^7$$

Like Terms

Variables that can be added together are like terms.

$$\underbrace{4x}_{Like\ Term} + \underbrace{3x}_{Like\ Term} = 7x$$

$$\underbrace{4x}_{Unlike\ Term} + \underbrace{3y}_{Unlike\ Term} \neq 7x \neq 7y$$

Example 1.46

Answers in exponent form, not as a number.

Numbers

Evaluate.

1) $11^{12} \times 11$
2) 17×17^6
3) $3^2 \times 3$
4) $7^3 \times 7$
5) $19^2 \times 19$
6) $2^3 \times 4$
7) $4^3 \times 4^7$
8) $7^3 \times 7^{12}$

Variables

Simplify:

9) $y^7 \times y^3$
10) $t^3 \times t^9$

Like Terms

Simplify

11) $y^3 \times z^2 \times y \times z$
12) $2^2 \times x^3 \times 2^6 \times x$
13) $3^4 \times y^7 \times 3^7 \times y^5$
14) $z^4 \times n^2$
15) $a^2 \times b^3 \times c^4$

Back Calculations

Solve:

16) $5^7 \times x^2 = 5^9$
17) $7^{11} \times y^3 = 7^{14}$
18) $8^7 \times z^4 = 8^7$
19) $9^7 \times s^4 = 0$
20) $5^6 \times 5^x = 5^{10}$
21) $7^8 \times 7^y = 7^9$

Fractions

22) $\left(-\frac{2}{3}\right)^3 \times \left(-\frac{2}{3}\right)^7$
23) $\left(\frac{3}{4}\right)^9 \times \frac{3}{4}$
24) $\left(\frac{2}{9}\right)^{12} \times \frac{2^5}{7^5}$
25) $\left(-\frac{x}{y}\right)^7 \times \frac{-x^4}{y^5}$

Numbers

$$11^{12} \times 11 = 11^{13}$$
$$17 \times 17^6 = 17^7$$
$$3^2 \times 3 = 3^3$$
$$7^3 \times 7 = 7^4$$
$$19^2 \times 19 = 19^3$$
$$2^3 \times 2^2 = 2^5$$
$$4^3 \times 4^7 = 4^{10}$$
$$7^3 \times 7^{12} = 7^{15}$$

Variables

$$y^7 \times y^3 = y^{10}$$
$$t^3 \times t^9 = t^{12}$$

Like Terms

We can combine like terms together and then multiply them:

$$\underbrace{y^3 \times y}_{Multiply\ Like\ Terms} \times \underbrace{z^2 \times z}_{Multiply\ Like\ Terms} = y^4 \times z^3$$

The y term and the z term will remain separate, since they cannot be combined.

$$2^2 \times x^3 \times 2^6 \times x = 2^8 \times x^4$$
$$3^4 \times y^7 \times 3^7 \times y^5 = 3^{11} \times y^{12}$$

$$z^4 \times n^2 = z^4 n^2$$

$$\underbrace{a^2 \times b^3 \times c^4}_{You\ can't\ multiply\ these\ together} = a^2 b^3 c^4$$

Back Calculations

$$5^7 \times 5^2 = 5^9 \Rightarrow x = 5$$
$$7^{11} \times 7^3 = 7^{14} \Rightarrow y = 7$$
$$5^6 \times 5^x = 5^{10} \Rightarrow 5^{6+x} = 5^{10} \Rightarrow 6 + x = 10 \Rightarrow x = 4$$
$$7^8 \times 7^y = 7^9 \Rightarrow$$

Fractions

$$\left(-\frac{2}{3}\right)^3 \times \left(-\frac{2}{3}\right)^7 = \left(-\frac{2}{3}\right)^{3+7} = \frac{2^{10}}{3^{10}}$$
$$\left(\frac{3}{4}\right)^9 \times \frac{3}{4} = \left(\frac{3}{4}\right)^9 \times \left(\frac{3}{4}\right)^1 = \left(\frac{3}{4}\right)^{10}$$

$$\left(\frac{2}{9}\right)^{12} \times \frac{2^5}{7^5} = \frac{2^{12}}{9^{12}} \times \frac{2^5}{7^5} = \frac{2^{17}}{9^{12} \times 7^5}$$
$$\left(-\frac{x}{y}\right)^7 \times \frac{-x^4}{y^5} = \frac{-x^7}{y^7} \times \frac{-x^4}{y^5} = \frac{x^{11}}{y^{12}}$$

1.47: Quotient Rule

Exponents convert division into subtraction if the bases are the same:

$$\frac{a^m}{a^n} = a^{m-n}$$

Numbers

$$5^3 \div 5^2 = \frac{5^3}{5^2} = \frac{5 \times \mathbf{5 \times 5}}{\mathbf{5 \times 5}} = 5$$

The same result can be achieved by subtracting the exponent of the second number from the exponent of the first number.

$$5^3 \div 5^2 = 5^{3-2} = 5^1 = 5$$

Variables

Division with variables proceeds the same way as division with numbers. The exponent of the divisor is subtracted from the exponent of the dividend.

$$\underbrace{h^5}_{Dividend} \div \underbrace{h^2}_{Divisor} = h^{5-2} = h^3$$

Example 1.48

Simplify. Write your final answer in exponent form, not as numbers.

Numbers

1) $\frac{4^5}{4^3}$
2) $\frac{7^{12}}{7^2}$
3) $\frac{101^{97}}{101^{59}}$

Variables

4) $x^4 \div x^2$
5) $y^7 \div y^3$
6) $r^{12} \div r^1$
7) $\frac{t^{15}}{t^5}$
8) $\frac{z^{12}}{z^7}$

Multi-Variable Expressions

9) $a^4 \times b^3 \div a^2 \times b^2$
10) $\frac{x^5}{x^3} + \frac{y^3}{y^5}$

Fractions

11) $\left(\frac{3}{4}\right)^{12} \div \left(\frac{3}{4}\right)^7$
12) $\left(\frac{y}{z}\right)^y \div \left(\frac{y}{z}\right)^z$

Numbers

$$\frac{4^5}{4^3} = 4^{5-3} = 4^2$$
$$\frac{7^{12}}{7^2} = 7^{12-2} = 7^{10}$$
$$\frac{101^{97}}{101^{59}} = 101^{97-59} = 101^{38}$$

Variables

$$x^4 \div x^2 = x^{4-2} = x^2$$
$$y^7 \div y^3 = y^{7-3} = y^4$$
$$r^{12} \div r^1 = r^{12-1} = r^{11}$$
$$x^4 \div x^2 = x^2$$
$$y^7 \div y^3 = y^4$$

$$r^{12} \div r^1 = r^{11}$$
$$\frac{t^{15}}{t^5} = t^{15} \div t^5 = t^{10}$$
$$\frac{z^{12}}{z^7} = z^{12} \div z^7 = z^5$$

Multi-Variable Expressions

$$a^4 \times b^3 \div a^2 \times b^2 = a^{4-2} \times b^{3+2} = a^2 b^5$$
$$\frac{x^5}{x^3} + \frac{y^3}{y^5} = x^{5-3} + \frac{1}{y^{5-3}} = x^2 + \frac{1}{y^2}$$

Fractions

$$\left(\frac{3}{4}\right)^{12} \div \left(\frac{3}{4}\right)^7 = \left(\frac{3}{4}\right)^{12-7} = \left(\frac{3}{4}\right)^5$$
$$\left(\frac{y}{z}\right)^y \div \left(\frac{y}{z}\right)^z = \left(\frac{y}{z}\right)^{y-z}$$

Example 1.49

$$\frac{10^7}{5 \times 10^4} = \text{(AMC 8 1985/3)}$$

$$\frac{10^7}{5\times 10^4}=\frac{10^{7-4}}{5}=\frac{10^3}{5}=\frac{1000}{5}=200$$

1.50: Powers of Powers Rule

$$(a^m)^n=a^{mn}$$

Exponentiation converts the process of taking a power into multiplication.

$$(5^2)^3=5^2\times 5^2\times 5^2=5^{2+2+2}=5^{2\times 3}=5^6$$

$$(a^m)^n=\underbrace{a^m\times a^m\times \dots.\times a^m}_{n\ times}=\underbrace{a^{m+m+\dots+m}}_{m\ occurs\ n\ times}=a^{mn}$$

Example 1.51

Simplify. Write your answer in terms of exponents.

Numbers

A. $(7^2)^3$
B. $(3^3)^3$
C. $(11^5)^4$

Fractions

D. $\left[\left(\frac{1}{2}\right)^3\right]^3$
E. $\left[\left(\frac{1}{3}\right)^2\right]^3$
F. $\left[\left(\frac{2}{5}\right)^2\right]^2$
G. $\left[\left(\frac{3}{4}\right)^2\right]^3$

Variables

H. $(x^2)^3$
I. $(z^5)^2$
J. $(y^3)^6$
K. $(x^y)^z$

Variables with Fractions

L. $\left[\left(\frac{x}{y}\right)^3\right]^4$
M. $\left[\left(\frac{a}{b}\right)^2\right]^7$
N. $\left[\left(\frac{p}{q}\right)^3\right]^1$
O. $\left[\left(\frac{m}{n}\right)^{43}\right]^0, m,n\neq 0$

Variables Expressions

P. $\left(\frac{x}{y}\right)^2+\left(\frac{a^2}{b^3}\right)^4$

Numbers

$$(7^2)^3=7^{2\times 3}=7^6$$
$$(3^3)^3=3^{3\times 3}=3^9$$
$$(11^5)^4=11^{3\times 3}=11^{20}$$

Fractions

$$\left[\left(\frac{1}{2}\right)^3\right]^3=\left(\frac{1}{2}\right)^{3\times 3}=\left(\frac{1}{2}\right)^9=\frac{1^9}{2^9}=\frac{1}{2^9}$$
$$\left[\left(\frac{1}{3}\right)^2\right]^3=\left(\frac{1}{3}\right)^{2\times 3}=\left(\frac{1}{3}\right)^6=\frac{1^6}{3^6}=\frac{1}{729}$$
$$\left[\left(\frac{2}{5}\right)^2\right]^2=\left(\frac{2}{5}\right)^{2\times 2}=\left(\frac{2}{5}\right)^4=\frac{2^4}{5^4}$$
$$\left[\left(\frac{3}{4}\right)^2\right]^3=\left(\frac{3}{2^2}\right)^{2\times 3}=\left(\frac{3}{2^2}\right)^6=\frac{3^6}{2^{2\times 6}}=\frac{3^6}{2^{12}}$$

Variables

$$(x^2)^3=x^2\times x^2\times x^2=x^{2+2+2}=x^{2\times 3}=x^6$$
$$(z^5)^2=z^{10}$$
$$(y^3)^6=y^{18}$$
$$(x^y)^z=x^{yz}$$

Variables with Fractions

$$\left[\left(\frac{x}{y}\right)^3\right]^4=\left(\frac{x}{y}\right)^{3\times 4}=\left(\frac{x}{y}\right)^{12}=\frac{x^{12}}{y^{12}}$$
$$\left[\left(\frac{a}{b}\right)^2\right]^7=\frac{a^{14}}{b^{14}}$$
$$\left[\left(\frac{p}{q}\right)^3\right]^1=\frac{p^3}{q^3}$$
$$\left[\left(\frac{m}{n}\right)^{43}\right]^0=1$$

Variables Expressions

$$\left(\frac{x}{y}\right)^2+\left(\frac{a^2}{b^3}\right)^4=\frac{x^2}{y^2}+\frac{a^{2\times 4}}{b^{3\times 4}}=\frac{x^2}{y^2}+\frac{a^8}{b^{12}}$$

Example 1.52: Equations

Find the values of x, y, m and n, in each case below, given that x and y are prime numbers, and m, and n are integers:

Basics

1) $\left[\left(\frac{2}{5}\right)^3\right]^7 = \frac{x^m}{y^n}$
2) $\left(\frac{9}{8}\right)^6 = \frac{x^m}{y^n}$
3) $\left(\frac{27}{32}\right)^4 = \frac{x^m}{y^n}$
4) $\left(\frac{625}{49}\right)^9 = \frac{x^m}{y^n}$
5) $\left(\frac{1331}{256}\right)^{17} = \frac{x^m}{y^n}$

Power Rule

6) $\left[\left(\frac{4}{5}\right)^6\right]^7 = \frac{x^m}{y^n}$
7) $\left[\left(\frac{243}{256}\right)^3\right]^4 = \frac{x^{2m}}{y^{3n}}$

Basics

$$\left[\left(\frac{2}{5}\right)^3\right]^7 = \left(\frac{2}{5}\right)^{21} = \frac{2^{21}}{5^{21}} = \frac{x^m}{y^n} \Rightarrow x = 2, y = 5, m = n = 21$$

$$\left(\frac{9}{8}\right)^6 = \left(\frac{3^2}{2^3}\right)^6 = \frac{(3^2)^6}{(2^3)^6} = \frac{3^{2\times 6}}{2^{3\times 6}} = \frac{3^{12}}{2^{18}} = \frac{x^m}{y^n} \Rightarrow x = 3, y = 2, m = 12, n = 18$$

$$\left(\frac{27}{32}\right)^4 = \left(\frac{3^3}{2^5}\right)^4 = \frac{(3^3)^4}{(2^5)^4} = \frac{3^{3\times 4}}{2^{5\times 4}} = \frac{3^{12}}{2^{20}} = \frac{x^m}{y^n} \Rightarrow x = 3, y = 2, m = 12, n = 20$$

$$\left(\frac{625}{49}\right)^9 = \frac{5^{36}}{7^{18}} = \frac{x^m}{y^n} \Rightarrow x = 5, y = 7, m = 36, n = 18$$

$$\left(\frac{1331}{256}\right)^{17} = \frac{11^{3\times 17}}{2^{8\times 17}} = \frac{11^{51}}{2^{136}} = \frac{x^m}{y^n} \Rightarrow x = 11, y = 2, m = 51, n = 136$$

Power Rule

$$\left[\left(\frac{4}{5}\right)^6\right]^7 = \frac{x^m}{y^n} \Rightarrow \frac{2^{84}}{5^{42}} = \frac{x^m}{y^n} \Rightarrow x = 2, y = 5, m = 84, n = 42$$

Part C

$$\left[\left(\frac{243}{256}\right)^3\right]^4 = \left[\left(\frac{x^m}{y^n}\right)^2\right]^3 \Rightarrow \left[\left(\frac{3^5}{2^8}\right)^3\right]^4 = \frac{x^{6m}}{y^{6n}} \Rightarrow \frac{3^{60}}{2^{96}} = \frac{x^{6m}}{y^{6n}}$$

$$x = 3, y = 2$$
$$6m = 60 \Rightarrow m = 10$$
$$6n = 96 \Rightarrow n = 16$$

Example 1.53

Rewrite each expression as a power of a prime number.

Numbers

A. 4^3
B. 8^4
C. 25^4
D. 49^3
E. 121^7
F. 16^5
G. 81^{10}
H. 125^{12}
I. 27^{10}
J. 625^{12}

Expressions

K. $4^5 \times 8^3$
L. $25^7 \times 125^4$
M. $16^5 \times 32^4$
N. $9^3 \times 27^2$
O. $49^4 \times 343^{10}$
P. $11^1 \times 121^2 \times 1331^3$
Q. $17^6 \times 289^5$
R. $2^2 \times 4^4 \times 8^8$

Variables

S. 4^x
T. 9^x
U. 125^y
V. 81^z
W. 1024^q

Numbers

$$4^3 = (4)^3 = (2^2)^3 = 2^6$$
$$8^4 = (2^3)^4 = 2^{12}$$
$$25^4 = (5^2)^4 = 5^8$$
$$49^3 = 7^6$$
$$121^7 = 11^{14}$$
$$16^5 = 2^{20}$$
$$81^{10} = (3^4)^{10} = 3^{40}$$

$$125^{12} = 5^{36}$$
$$27^{10} = 3^{30}$$
$$625^{12} = 5^{48}$$

Expressions

$$4^5 \times 8^3 = (2^2)^5 \times (2^3)^3 = 2^{10} \times 2^9 = 2^{19}$$
$$25^7 \times 125^4 = 5^{14} \times 5^{12} = 5^{26}$$
$$16^5 \times 32^4 = (2^4)^5 \times (2^5)^4 = 2^{20} \times 2^{20} = 2^{40}$$

Variables

$$(2^2)^x = 2^{2\times x} = 2^{2x}$$
$$9^x = (3^2)^x = 3^{2x}$$

Challenge 1.54: Series

A. Simplify: $\frac{2^1\times4^2\times8^3\times16^4\times32^5}{2^1\times2^2\times\ldots\times2^{10}}$
B. Find a general formula for $3^1 \times 9^2 \times 27^3 \times \ldots \times (3^n)^n$, and evaluate it for $n = 7$.

Part A

$$Numerator = 2^1 \times (2^2)^2 \times (2^3)^3 \times (2^4)^4 \times (2^5)^5 = 2^1 \times 2^{2\times2} \times 2^{3\times3} \times 2^{4\times4} \times 2^{5\times5} = 2^{1+2^2+3^2+4^2+5^2} = 2^{55}$$
$$Denominator = 2^{1+2+\cdots+10} = 2^{55}$$

Part B

$$3^1 \times (3^2)^2 \times (3^3)^3 \times \ldots \times (3^n)^n = 3^1 \times 3^{2\times2} \times 3^{3\times3} \times \ldots \times 3^{n\times n}$$

We get the squares of the first n natural numbers, and we make use of the formula for that:

$$3^{1+2^2+3^2+\cdots+n^2} = 3^{\frac{n(n+1)(2n+1)}{6}} \Rightarrow n = 7 \Rightarrow \frac{7(8)(15)}{6} = 140$$

B. Exponential Equations

Example 1.33

Solve the equations below and find the values of the variable(s):

A. $(3^2)^3 = 3^x$
B. $(5^4)^6 = (5^3)^x$
C. $\frac{8^x\times4^2\times16^3}{32\times64^5} = 128$
D. $\frac{9^x\times3^5\times27^3}{3\times81^4} = 81$
E. $\frac{125^x\times25^{2x}\times5^{-3x}}{25^4\times125^5} = 25^{4x}$
F. $\frac{2^6\times4^3\times8^{-2x}}{2^{-3}\times4^6} = \frac{32^x\times16^4}{8^{-2x}\times64^{3x}}$
G. $\frac{3^x\times2^y}{27^2\times32^3} = 6$
H. $\frac{2^x\times3^y\times5^z}{32\times81\times125} = \frac{8^x\times9^{2y}\times125^{3z}}{16^3\times9^5\times25^4}$

Part A

$$3^6 = 3^x \Rightarrow x = 6$$

Part B

$$5^{24} = 5^{3x} \Rightarrow 3x = 24 \Rightarrow x = 8$$

Part C

$$\frac{2^{3x} \times 2^4 \times 2^{12}}{2^5 \times 2^{30}} = 2^7$$
$$2^{3x+4+12-5-30} = 2^7$$
$$2^{3x-19} = 2^7 \Rightarrow 3x - 19 = 7 \Rightarrow$$
$$x = \frac{26}{3}$$

Part D

Convert all numbers to have a base of 3:

$$\frac{3^{2x} \times 3^5 \times 3^9}{3 \times 3^{16}} = 3^4$$
$$3^{2x+5+9-1-16} = 3^4$$
$$3^{2x-3} = 3^4$$
$$2x - 3 = 4$$
$$x = \frac{7}{2}$$

Part E

$$\frac{5^{3x} \times 5^{4x} \times 5^{-3x}}{5^8 \times 5^{15}} = 5^{8x}$$
$$5^{3x+4x-3x-8-15} = 5^{8x}$$
$$4x - 23 = 8x$$
$$-23 = 4x$$

$$-\frac{23}{4}=x$$

Part F

$$\frac{2^6\times 2^6\times 2^{-6x}}{2^{-3}\times 2^{12}}=\frac{2^{5x}\times 2^{16}}{2^{-6x}\times 2^{18x}}$$
$$2^{6+6-6x+3-12}=2^{5x+16+6x-18x}$$
$$2^{3-6x}=2^{-7x+16}$$
$$3-6x=-7x+16$$
$$x=13$$

Part G

$$\frac{3^x\times 2^y}{3^6\times 2^{15}}=2\times 3$$
$$3^{x-6}\times 2^{y-15}=2^1\times 3^1$$
$$x-6=1\Rightarrow x=7$$
$$y-15=1\Rightarrow y=16$$

Part H

$$\frac{2^x\times 3^y\times 5^z}{2^5\times 3^4\times 5^3}=\frac{2^{3x}\times 3^{4y}\times 5^{9z}}{2^{12}\times 3^{10}\times 5^8}$$
$$2^{x-5}\times 3^{y-4}\times 5^{z-3}=2^{3x-12}\times 3^{4y-10}\times 5^{9z-8}$$
$$x-5=3x-12\Rightarrow 7=2x\Rightarrow x=\frac{7}{2}$$
$$y-4=4y-10\Rightarrow 6=3y\Rightarrow y=2$$
$$z-3=9z-8\Rightarrow 5=8z\Rightarrow z=\frac{5}{8}$$

Example 1.33

Find the value of x in each case:

A. $64+64+64+64=2^x$

$$4\times 64=2^x$$
$$2^2\times 2^6=2^x$$
$$2^8=2^x$$
$$x=8$$

Example 1.33

Find the value of x in each case:

A. $3(2^{x+1})=48$

B. $4\times 3^{x-1}+2\times 3^{x+1}=594$

$$4\times 3^{x-1}+2\times 3^{x-1+2}=594$$
$$4\times 3^{x-1}+2\times 3^{x-1}\times 3^2=594$$
$$4\times 3^{x-1}+18\times 3^{x-1}=594$$
$$22\times 3^{x-1}=594$$
$$3^{x-1}=\frac{594}{22}=27=3^3$$
$$x-1=3$$
$$x=4$$

C. Distributing Exponents

1.55: Distributing Exponents

$$(ab)^m=a^m\times b^m$$

$$(ab)^m=\underbrace{(ab)(ab)\dots(ab)}_{m\ times}=\underbrace{a\times a\times\dots\times a}_{m\ times}\times\underbrace{b\times b\times\dots\times b}_{m\ times}=a^m\times b^m$$

This rule is easier to use when converting from the RHS to the LHS. It is more difficult to realise that this rule is applicable when we have the LHS.

In general, remember that in order to simplify, the standard way is to convert everything to prime number bases.

Example 1.56

Open the parentheses and simplify. Write your answer in terms of exponents. Do *not* multiply numbers inside the parentheses.

A. $(3 \times 7)^2$
B. $(5 \times 7)^3$
C. $(3 \times 4)^7$
D. $(2 \times 7 \times 11)^{11}$
E. $(2^2 \times 3)^2$
F. $\left(\frac{1}{2} \times \frac{3}{5}\right)^4$
G. $\left(\frac{2}{5} \times \frac{7}{11}\right)^{12}$
H. $(ab)^3$
I. $(a^2c^5)^2$
J. $\left(\frac{x}{y}\right)^6$
K. $\left(\frac{w^2}{x^3x^4}\right)^9$

Part A

$$(3 \times 7)^2 = (3 \times 7)(3 \times 7) = (3 \times 3)(7 \times 7) = 3^2 \times 7^2$$

Shortcut

$$(3 \times 7)^2 = 3^2 \times 7^2$$
$$5^3 \times 7^3$$
$$3^7 \times 4^7$$
$$2^{11} \times 7^{11} \times 11^{11}$$
$$2^4 \times 3^2$$
$$\left(\frac{1}{2}\right)^4 \times \left(\frac{3}{5}\right)^4$$
$$\left(\frac{2}{5}\right)^{12} \times \left(\frac{7}{11}\right)^{12}$$
$$a^3b^3$$
$$a^4c^{10}$$
$$\frac{x^6}{y^6}$$
$$\frac{w^{18}}{x^{63}}$$

Example 1.57

Write each number as the product of powers of prime numbers:

A. 15^3
B. 14^3
C. 55^{22}
D. 39^{17}
E. 12^{15}
F. 30^7
G. 1001^3
H. 72^{11}
I. 110^{110}
J. 48^4
K. 144^6
L. 1000^{12}
M. $1{,}000{,}000^4$
N. 108^7
O. $6^7 \times 15^6$
P. $12^{15} \times 18^{12}$

$$15^3 = (3 \times 5)^3 = 3^3 \times 5^3$$
$$14^3 = (2 \times 7)^3 = 2^3 \times 7^3$$
$$55^{22} = (5 \times 11)^{22} = 5^3 \times 11^{22}$$
$$39^{17} = 3^{17} \times 13^{17}$$
$$12^{15} = (2^2 \times 3)^{15} = (2^2)^{15} \times 3^{15} = (2^{30} \times 3^{15})$$
$$30^7 = 2^7 \times 3^7 \times 5^7$$
$$1001^3 = 7^3 \times 11^3 \times 13^3$$
$$72^{11} = (2^3 \times 3^2)^{11} = 2^{33} \times 3^{22}$$
$$48^4 = (2^4 \times 3)^4 = 2^{16} \times 3^4$$
$$144^6 = (12^2)^6 = ((2^2 \times 3)^2)^6 = (2^4 \times 3^2)^6 = 2^{24} \times 3^{12}$$
$$1000^{12} = (10^3)^{12} = 10^{36} = 2^{36} \times 5^{36}$$
$$1{,}000{,}000^4 = (10^6)^4 = 10^{24} = 2^{24} \times 5^{24}$$
$$6^7 \times 15^6 = (2^7 \times 3^7) \times (3^6 \times 5^6) = 2^7 \times 3^{13} \times 5^6$$
$$12^{15} \times 18^{12} = (2^2 \times 3)^{15} \times (2 \times 3^2)^{12} = (2^{30} \times 3^{15}) \times (2^{12} \times 3^{24}) = 2^{42} \times 3^{39}$$

Example 1.58

Simplify the fractions below:

A. $\frac{10^{12}}{5^{12}}$
B. $\frac{12^{12}}{2^{12}}$
C. $\frac{21^7}{3^7}$
D. $\frac{10^{23}}{6^{32}}$
E. $\frac{12^{14}}{15^{12}}$

$$\frac{10^{12}}{5^{12}} = \frac{(5 \times 2)^{12}}{5^{12}} = \frac{\mathbf{5^{12}} \times 2^{12}}{\mathbf{5^{12}}} = 2^{12}$$
$$\frac{12^{12}}{2^2} = \frac{6^{12} \times 2^{12}}{2^{12}} = 6^{12}$$
$$\frac{10^{23}}{6^{32}} = \frac{2^{23} \times 5^{23}}{2^{32} \times 3^{32}} = \frac{\mathbf{2^{23}} \times 5^{23}}{\mathbf{2^{23}} \times 2^9 \times 3^{32}} = \frac{5^{23}}{2^9 \times 3^{32}}$$
$$\frac{12^{14}}{15^{12}} = \frac{(2^2 \times 3)^{14}}{(3 \times 5)^{12}} = \frac{2^{28} \times 3^{14}}{3^{12} \times 5^{12}} = \frac{2^{28} \times 3^2}{5^{12}}$$

Challenge 1.59: Breaking up Larger Numbers

Simplify each fraction below:

A. $\frac{24^{14} \times 18^{16} \times 10^{13}}{36^{15} \times 144 \times 15^{12}}$
B. $\frac{48^4 \times 72^3 \times 36^{12} \times 30^9}{216^6 \times 20^{13} \times 15^4 \times 30^7}$

Part A

Prime factorize each base:

$$= \frac{(2^3 \times 3)^{14} \times (2 \times 3^2)^{16} \times (2 \times 5)^{13}}{(2^2 \times 3^2)^{15} \times (2^4 \times 3^2) \times (3 \times 5)^{12}}$$

Use the power of power rules to simplify:

$$\frac{(2^{42} \times 3^{14}) \times (2^{16} \times 3^{32}) \times (2^{13} \times 5^{13})}{(2^{30} \times 3^{30}) \times (2^4 \times 3^2) \times (3^{12} \times 5^{12})}$$

Combine all powers:

$$= 2^{42+16+13-30-4} \times 3^{14+32-30-2-12} \times 5^{13-12}$$

Simplify:

$$2^{37} \times 3^2 \times 5^1$$

Part B

D. Evaluating Exponents

Example 1.60

$$\frac{25 \times 4^4}{8^2} = \frac{25 \times (2^2)^4}{(2^3)^2} = \frac{25 \times 2^8}{2^6} = 25 \times 4 = 100$$

$$4^{x+1} = 2$$

E. Expressions

Example 1.61

A. Find the value of 5^{y+1} in terms of k given that $5^y = k$.
B. Find the value of 3^{z+2} in terms of a given that $3^z = a$.
C. Find the value of 49^{z-2} in terms of b given that $7^z = b$.

Part A

$$5^{y+1} = 5^y \times 5^1 = 5 \times 5^y = 5 \times k = 5k$$

Part B

$$3^{z+2} = 9 \times 3^z = 9a$$

Part C

$$49^{z-2} = \frac{49^z}{49^2} = \frac{(7^2)^z}{(7^2)^2} = \frac{(7^z)^2}{7^4} = \frac{b^2}{2401}$$

1.62: Summary of Exponent Rules

Product Rule: Converts multiplication to addition:

$$a^m \times a^n = a^{m+n}$$

Quotient Rule: Converts division to subtraction:

$$\frac{a^m}{a^n} = a^{m-n}$$

Power Rule: Converts exponentiation to multiplication:

$$(a^m)^n = a^{mn}$$

Distribution of exponents:

$$(ab)^m = a^m \times b^m$$

F. Factoring Exponents

Example 1.63
Factor:

A. $3^5 + 3^6$

$$3^5 + 3^6 = 3^5(1+3) = 4 \cdot 3^5$$

Example 1.64
The ratio $\frac{10^{2000}+10^{2002}}{10^{2001}+10^{2001}}$ is closest to which integer? **(AMC 10A 2002/1)**

$$\frac{10^{2000}(1+10^2)}{10^{2000}(10+10)} = \frac{101}{20} \approx 5$$

1.5 Zero and Negative Exponents

A. Zero Exponent

1.65: Zero Exponents

$$a^0 = 1, \qquad a \neq 0$$

$$a^0 = a^{m-m} = \frac{a^m}{a^m} = 1, \qquad m \neq 0$$

Any number raised to the power of zero is zero.
(Except 0^0).

Example 1.66

A. Simplify $\frac{5^6}{5^6}$
B. Show that $1 = \frac{5^6}{5^6}$. Hence, show that $5^0 = 1$.
C. Find 7^0.
D. 19^0
E. 503^0
F. $x^0, x \neq 0$
G. $x^0, x = 0$
H. $\left(\frac{x^2}{y^3} + \frac{y^2}{z^3} + \frac{z^2}{x^3}\right)^0$, $x = \frac{11}{19}, y = \frac{23}{29}, z = \frac{59}{97}$

$$\frac{5^6}{5^6} = 5^{6-6} = 5^0$$

$$1 = \frac{1}{1} = \frac{1}{1} \times \frac{5^6}{5^6} = \frac{5^6}{5^6} = 5^{6-6} = 5^0$$

$$7^0 = 7^{1-1} = \frac{7^1}{7^1} = 1$$

$$19^0 = 1$$

$$503^0 = 1$$

$$x^0 = 1$$
$$0^0 \text{ is not defined}$$
$$\left(\frac{x^2}{y^3} + \frac{y^2}{z^3} + \frac{z^2}{x^3}\right)^0 = 1$$

B. Exponential Equations

Example 1.67

Solve for the variable in each part below

Basics

A. $x + 5 = 5^0$
B. $y - 3 = 7^0$
C. $(4^0)z = 11^0 + 1$

Variable Exponents

D. $45^x = 1$
E. $34^{2x-6} = 1$
F. $23^{\frac{1}{2}x+\frac{2}{7}} = 1$

Variables in the Base

G. $(x+2)^{(x-6)} = 1$
H. $\left(\frac{1}{2}x + \frac{4}{3}\right)^{3x+4} = 1$

Basics

$$x + 5 = 5^0 \Rightarrow x + 5 = 1 \Rightarrow x = -4$$
$$y - 3 = 7^0 \Rightarrow y - 3 = 1 \Rightarrow y = 4$$
$$(4^0)z = 11^0 + 1 \Rightarrow z = 2$$

Variable Exponents

$$45^x = 1 \Rightarrow x = 0$$
$$34^{2x-6} = 1 \Rightarrow 2x - 6 = 0 \Rightarrow x = \frac{6}{2} = 3$$
$$23^{\frac{1}{2}x+\frac{2}{7}} = 1 \Rightarrow \frac{1}{2}x + \frac{2}{7} = 0 \Rightarrow \frac{1}{2}x = -\frac{2}{7} \Rightarrow x = -\frac{4}{7}$$

Variables in the Base

$$(x+2)^{(x-6)} = 1 \Rightarrow x - 6 = 0 \Rightarrow x = 6$$
$$x + 2 = 6 + 2 \neq 0 \neq$$

C. Negative Exponents

Consider the following pattern:

$125 = 25 \times 5$	5^3	125	
$25 = 5 \times 5$	5^2	25	$25 = \frac{125}{5}$
$5 = 1 \times 5$	5^1	5	$5 = \frac{25}{5}$
	5^0	1	$1 = \frac{5}{5}$
	5^{-1}	$\frac{1}{5}$	Type equation here.
	5^{-2}	$\frac{1}{25} = \frac{1}{5^2}$	
	5^{-3}	$\frac{1}{125} = \frac{1}{5^3}$	

As the exponent reduces, the number is divided by 5.

1.68: Negative Exponents Rules

$$a^{-m} = \frac{1}{a^m}$$

$$(a^m)^n = \underbrace{a^m \times a^m \times \dots \times a^m}_{n\ times} = a^{mn}$$
$$\left(\frac{a}{b}\right)^{-x} = \left(\frac{b}{a}\right)^{x}$$

When moving an expression from numerator to denominator, or from denominator to numerator change the sign of the exponent.

$$\left(\frac{a}{b}\right)^{-x} = \frac{a^{-x}}{b^{-x}} = \frac{b^x}{a^x} = \left(\frac{b}{a}\right)^{x}$$

If you have a fraction with a negative exponent, you can make exponent positive by taking the reciprocal of the fraction.

Example 1.69

Write with positive exponents

1. 9^{-2}
2. 11^{-1}
3. 2^{-3}
4. 7^{-3}
5. 19^{-2}
6. 17^{-3}
7. 5^{-7}
8. 4^{-12}

Write with negative exponents

9. $\frac{1}{5^2}$
10. $\frac{1}{7^3}$
11. $\frac{1}{9^4}$
12. 3^5
13. 5^7
14. 11^6

Evaluate

15. 9^{-2}

Simplify. Write with positive exponents.

16. $(3^{-3})^2$
17. $(4^5)^{-3}$
18. $(5^4)^{-2}$
19. $(7^3)^5$
20. $(11^{-5})^7$
21. $(17^{-3})^4$
22. $(15^{-2})^9$

Write as a power with a prime number base

23. 4^{-2}
24. $(25)^{-3}$
25. $(8)^{-2}$
26. $(125)^{-5}$

Evaluate:

27. $\left(\frac{2}{3}\right)^{-3}$
28. $\left(\frac{4}{5}\right)^{-2}$
29. $\left(\frac{3}{2}\right)^{-2}$
30. $\left(\frac{7}{9}\right)^{-2}$
31. $\left(\frac{1}{3}\right)^{-2}$
32. $\left(\frac{2}{3}\right)^{-1} \times \left(\frac{4}{5}\right)^{-1}$
33. $\left(\frac{5}{7}\right)^{-2} \times \frac{5}{7}$
34. $\left(\frac{2}{3}\right)^{-10} \times \left(\frac{9}{4}\right)^{-3}$

Carry out the multiplication

35. 9×6^{-3}
36. $(36^{-1} \times 6^5) \times 6^{-1}$
37. $4^{-1} \times 8^3 \times 16^{-2}$

Carry out the division

38. $\frac{3^2}{3^5}$
39. $\frac{5^{-2}}{5^3}$
40. $\frac{7^4}{7^2}$
41. $\frac{11^4}{11^{-2}}$
42. $\frac{13^3}{13^{-3}}$
43. $2^5 \div 2^4$
44. $7^{11} \div 7^{13}$
45. $7^5 \div 7^{-9}$

Simplify these compound fractions

46. $\frac{3^3 \times 2^5}{3^4 \times 2^{-5}}$
47. $\frac{3^5 \times 2^7}{3^9 \times 2^4}$
48. $\frac{(3^2)^{-3} \times (2^3)^4}{(3^3)^4 \times (2^{-2})^7}$

Write with positive exponents

$$9^{-2} = \frac{1}{9^2} = \frac{1}{81}$$
$$11^{-1} = \frac{1}{11}$$
$$2^{-3} = \frac{1}{8}$$
$$7^{-3} = \frac{1}{343}$$
$$19^{-2} = \frac{1}{19^2}$$
$$17^{-3} = \frac{1}{17^3}$$
$$5^{-7} = \frac{1}{5^7}$$
$$4^{-12} = \frac{1}{4^{12}}$$

Write with negative exponents

$$\frac{1}{5^2} = 5^{-2}$$

$$3^5 = \frac{1}{3^{-5}}$$

Simplify. Write with positive exponents.

$$(3^{-3})^2 = \left(\frac{1}{3^3}\right)^2 = \frac{1}{(3^3)^2} = \frac{1}{3^{3\times 2}} = \frac{1}{3^6}$$
$$(4^5)^{-3} = \frac{1}{(4^5)^3} = \frac{1}{4^{15}}$$
$$(5^4)^{-2} = 5^{-8} = \frac{1}{5^8}$$
$$(7^3)^{-5} = 7^{3\times(-5)} = 7^{-15} = \frac{1}{7^{15}}$$
$$(11^5)^7 = 11^{-35} = \frac{1}{11^{35}}$$

Write as a power with a prime number base

$$4^{-2} = (2^2)^{-2} = 2^{-4}$$
$$(25)^{-3} = (5^2)^{-3} = 5^{-6}$$
$$(8)^{-2} = (2^3)^{-2} = 2^{-6}$$
$$(125)^{-5} = (5^3)^{-5}$$

Simplify these fractions

$$\left(\frac{2}{3}\right)^{-3} = \frac{2^{-3}}{3^{-3}} = \frac{3^3}{2^3} = \frac{27}{8}$$
$$\left(\frac{4}{5}\right)^{-2} = \left(\frac{5}{4}\right)^2 = \frac{25}{16}$$
$$\left(\frac{3}{2}\right)^{-2} = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$
$$\left(\frac{7}{9}\right)^{-2} = \left(\frac{9}{7}\right)^2 = \frac{81}{49}$$
$$\left(\frac{1}{3}\right)^{-2} = 9$$
$$\left(\frac{2}{3}\right)^{-1} \times \left(\frac{4}{5}\right)^{-1} = \frac{3}{2} \times \frac{5}{4} = \frac{15}{8}$$
$$\left(\frac{5}{7}\right)^{-2} \times \frac{5}{7} = \frac{7^2}{5^2} \times \frac{5}{7} = \frac{7}{5}$$
$$\left(\frac{2}{3}\right)^{-10} \times \left(\frac{9}{4}\right)^{-3} = \frac{3^{10}}{2^{10}} \times \frac{4^3}{9^3} = \frac{3^{10}}{2^{10}} \times \frac{2^6}{3^6} = \frac{3^4}{2^4} = \frac{81}{16}$$

Multiplication

$$(36^{-1} \times 6^{-5}) \times 6^1 = (6^2)^{-1} \times 6^{-5} \times 6^1$$
$$= 6^{-2} \times 6^{-5} \times 6^1 = 6^{-2-5+1} = 6^{-6}$$
$$4^{-1} \times 8^3 \times 16^{-2} = 2^{-2} \times 2^9 \times 2^{-8} = 2^{-10}$$
$$9 \times 6^{-3} = \frac{9}{6^3} = \frac{9}{2^3 \cdot 3^3} = \frac{9}{8 \times 27} = \frac{1}{24}$$

Division

$$\frac{13^3}{13^{-3}} = 13^{3-(-3)} = 13^{3+3} = 13^6$$
$$\frac{3^3 \times 2^5}{3^4 \times 2^{-5}} = \frac{2^{10}}{3^1}$$
$$\frac{3^5 \times 2^7}{3^9 \times 2^4} = 3^{-4} \times 2^3$$
$$\frac{(3^2)^{-3} \times (2^3)^4}{(3^3)^4 \times (2^{-2})^7} = \frac{3^{-6} \times 2^{12}}{3^{12} \times 2^{-14}} = 3^{-18} \times 2^{26}$$

1.70: Negative Base

$$(-x)^n = -x^n, for\ odd\ values\ of\ n$$
$$(-x)^n = x^n, for\ even\ values\ of\ n$$

- If you raise a positive number to *any* power, the final answer is positive.
- If you raise a negative number to an odd power, the final answer is negative.
- If you raise a negative number to an even power, the final answer is positive.

Example 1.71

Simplify. Write your answers as a number, if possible. If the calculations are too complicated, write your answerin in terms of positive exponents.

Basics

A. $(-2)^3$
B. $(-3)^3$
C. $(-3)^2$

Negative Exponents

D. $(-2)^{-3}$
E. $(-3)^{-3}$
F. $(-5)^{-2}$

Powers of 1

G. $(-1)^5$
H. $(-1)^8$
I. $(-1)^{2020}$
J. $(-1)^{2021}$
K. $(-1)^{-3}$
L. $(-1)^{-8}$

Expressions

M. $(-2)^5 \times 2^3$
N. $(-3)^4 \times 3^7$
O. $(-5)^3 \times (-5)^2$
P. $(-2)^2 \times (-2)^6$
Q. $(-7)^3 \times (-7)^7$
R. $(-3)^{-3} \times (-3)^5$
S. $(-5)^{-2} \times 5^{-3}$
T. $(-17)^{-17} \times 17^{17}$

Variables

U. $ab^{-1} + a^{-1}b + (ab)^{-1}$

Basics

$$(-2)^3 = (-1 \times 2)^3 = (-1)^3 \times 2^3 = -2^3 = -8$$
$$(-3)^3 = -3^3 = -27$$
$$(-1 \times 3)^2 = (-1)^2 \times 3^2 = 1 \times 9 = 9$$

Negative Exponents

$$(-2)^{-3} = \frac{1}{(-2)^3} = -\frac{1}{8}$$
$$(-3)^{-3} = \frac{1}{(-3)^3} = \frac{1}{-3^3} = -\frac{1}{27}$$
$$(-5)^{-2} = \frac{1}{(-5)^2} = \frac{1}{25}$$

Powers of 1

$$(-1)^{-3} \times 1^3 = \frac{1}{(-1)^3} \times 1 = \frac{1}{-1} = -1$$

Expressions

$$(-2)^5 \times 2^3 = -2^5 \times 2^3 = -2^8 = -256$$
$$(-3)^4 \times 3^7 = 3^4 \times 3^7 = 3^{11}$$
$$(-5)^3 \times (-5)^2 = (-5)^5 = -5^5$$
$$(-2)^2 \times (-2)^6 = (-2)^{2+6} = (-2)^8 = 2^8 = 256$$
$$(-7)^3 \times (-7)^7 = (-7)^{10} = 7^{10}$$
$$(-3)^{-3} \times (-3)^5 = (-3)^2 = 3^2 = 9$$
$$(-5)^{-2} \times 5^{-3} = 5^{-2} \times 5^{-3} = 5^{-5} = \frac{1}{5^5}$$
$$(-17)^{-17} \times 17^{17} = \frac{17^{17}}{(-17)^{17}} = \frac{17^{17}}{-17^{17}} = -1$$
$$ab^{-1} + a^{-1}b + (ab)^{-1} = \frac{a}{b} + \frac{b}{a} + \frac{1}{ab} = \frac{a^2 + b^2 + 1}{ab}$$

Example 1.72

$$\left[\left(\frac{3}{2}\right)^{-18} \div \left(\frac{1}{\frac{16}{81}}\right)^{-6}\right]^{-\frac{2}{3}}$$

$$\left[\left(\frac{3}{2}\right)^{-18} \div \left(\frac{81}{16}\right)^{-6}\right]^{-\frac{2}{3}}$$
$$\left[\left(\frac{3}{2}\right)^{-18} \div \left[\left(\frac{3}{2}\right)^{4}\right]^{-6}\right]^{-\frac{2}{3}}$$
$$\left[\left(\frac{3}{2}\right)^{-18} \div \left(\frac{3}{2}\right)^{-24}\right]^{-\frac{2}{3}}$$

$$\left[\left(\frac{3}{2}\right)^{-18-(-24)}\right]^{-\frac{2}{3}}$$
$$\left[\left(\frac{3}{2}\right)^{6}\right]^{-\frac{2}{3}}$$
$$\left(\frac{3}{2}\right)^{-4}$$
$$\left(\frac{2}{3}\right)^{4}$$
$$\frac{16}{81}$$

Example 1.73

$$\frac{1}{1+x^{(y-z)}}+\frac{1}{1+x^{(z-y)}}$$

Use a change of variable. Let

$$t=x^{(y-z)}\Rightarrow\frac{1}{t}=\frac{1}{x^{(y-z)}}=x^{(z-y)}$$

Then we substitute to get:

$$\frac{1}{1+t}+\frac{1}{1+\frac{1}{t}}=\frac{1}{1+t}+\frac{1}{\frac{t+1}{t}}=\frac{1}{1+t}+\frac{t}{t+1}=\frac{1+t}{t+1}=1$$

Example 1.74: Equations

Find the value of n in each case below:

A. $x^2\times x^5=x^n$
B. $x^3\times x^n=x^7$
C. $x^4\times x^n=x^1$
D. $x^4\times x^3=n^7$

$$x^7=x^n\Rightarrow n=7$$
$$x^{3+n}=x^7\Rightarrow 3+n=7\Rightarrow n=4$$
$$x^4\times x^n=x^1\Rightarrow 4+n=1\Rightarrow n=-3$$
$$x^4\times x^3=n^7\Rightarrow x^7=n^7\Rightarrow n=x$$

Example 1.75

By what number should $(-18)^{-1}$ be divided so that the quotient is $\left(\frac{3}{5}\right)^{-1}$

$$(-18)^{-1}=\frac{1}{(-18)^1}=-\frac{1}{18}$$
$$\left(\frac{3}{5}\right)^{-1}=\left(\frac{5}{3}\right)^{1}=\frac{5}{3}$$

The first number divided by some number x gives the second number:

$$-\frac{1}{18}\div x=\frac{5}{3}$$
$$-\frac{1}{6x}=\frac{5}{1}$$

Take the reciprocal on both sides:

$$-6x = \frac{1}{5}$$

Divide both sides by -6:

$$x = -\frac{1}{30}$$

1.76: Equations with 1

$$a^b = 1 \Rightarrow b = 0, a \neq 0 \; OR \; a = 1$$

Example 1.77

Solve for x:

A. $(x+5)^{(x+2)} = 1$
B. $(x+5)^{(x+2)} = 0$
C. $(2x+7)^{3x+1} = 1$

Part A

If the base is 1:

$$x + 5 = 1 \Rightarrow x = -4$$
$$\Rightarrow Power = x + 2 = -2 \Rightarrow Valid$$

If the power is zero:

$$x + 2 = 0 \Rightarrow x = -2 \Rightarrow (-2+5)^{(-2+2)} = (3)^{(0)} = 1$$

Part C

If the power is zero:

$$3x + 1 = 0 \Rightarrow x = -\frac{1}{3}$$

$$2\left(-\frac{1}{3}\right) + 7 = -\frac{2}{3} + 7 \neq 0$$

If the base is 1:

$$2x + 7 = 1 \Rightarrow x = 3$$
$$3(3) + 1 = 10$$

1.78: Equations with zero

$$x^y = 0 \Rightarrow No \; Solutions$$

Example 1.79

Solve for x:

A. $(x+5)^{(x+2)} = 0$

Part B

Does not have solutions.

Example 1.80

Simplify and write with positive exponents

A. $\frac{27p^2q^{-3}}{9^0pq^{-8}}$
B. $\frac{(3x^{-2}y^8)^4}{(9x^4y^{-3})^2}$
C. $\left(\frac{x^{-5}y^2}{x^{-3}y^5}\right)^{-2}$

D. $\left(\frac{ab^{-3}c^{-2}}{a^{-3}b^0c^{-5}}\right)^{-1}$

E. $\left(\frac{5x^{-3}(y^5z^{-2})^2}{2x^{-3}y^{-10}z}\right)^{-3}$

F. $\frac{3a}{b^{-1}} \times \frac{2b}{a^{-1}}$

$$\frac{27p^2q^{-3}}{9^0pq^{-8}} = 27pq^5$$

$$\frac{(3x^{-2}y^8)^4}{(9x^4y^{-3})^2} = \frac{3^4x^{-8}y^{32}}{3^4x^8y^{-6}} = \frac{y^{38}}{x^{16}}$$

$$\left(\frac{x^{-5}y^2}{x^{-3}y^5}\right)^{-2} = (x^{-2}y^{-3})^{-2} = x^4y^6$$

$$\left(\frac{ab^{-3}c^{-2}}{a^{-3}b^0c^{-5}}\right)^{-1} = (a^4b^{-3}c^3)^{-1} = a^{-4}b^3c^{-3} = \frac{b^3}{a^4c^3}$$

$$\left(\frac{5x^{-3}(y^5z^{-2})^2}{2x^{-3}y^{-10}z}\right)^{-3} = \left(\frac{5y^{10}z^{-4}}{2y^{-10}z}\right)^{-3} = \left(\frac{5y^{20}z^{-5}}{2}\right)^{-3} = \frac{5^{-3}y^{-60}z^{15}}{2^{-3}} = \frac{8z^{15}}{125y^{60}}$$

$$\frac{3a}{b^{-1}} \times \frac{2b}{a^{-1}} = \frac{6ab}{b^{-1}a^{-1}} = 6a^2b^2$$

Example 1.81

$$x^{m+2} \times x^2$$

$$x^{m+2} \times x^2 = x^{m+2+2} = x^{m+4}$$

1.82: LCM with exponents

If n and m are integers, then

$$LCM(x^n, x^m) = x^{Higher\ of\ (n,m)}$$

Example 1.83: Expressions

A. $\frac{5x^{-4}y^{-1}}{12} - \frac{x^{-2}y^{-2}}{5}$

Convert the variables to positive exponents by moving them to the denominator:

$$\frac{5}{12x^4y} - \frac{1}{5x^2y^2}$$

LCM is $60x^4y^2$:

$$\frac{5(5y) - 1(12x^2)}{60x^4y^2} = \frac{25y - 12x^2}{60x^4y^2}$$

Example 1.84

$$\frac{\frac{a}{b}}{\frac{c}{d}}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Example 1.85

$$\frac{3m^3 - 3m}{3m^4} \times \frac{2m^5}{m+1}$$

$$Factor \rightarrow Cancel \rightarrow Multiply$$

Factor:

$$\frac{3m(m^2-1)}{3} \times \frac{2m}{m+1}$$
$$\frac{m(m+1)(m-1)}{1} \times \frac{4m}{m+1}$$

Cancel:

$$\frac{m(m-1)}{1} \times \frac{4m}{1}$$

Multiply:

$$4m^2(m-1)$$

Example 1.86

$$\frac{\frac{b^2-100}{b^2}}{\frac{3b^2-31b+10}{2b}}$$

$$3b^2 - 31b + 10 = 3b^2 - 30b - b + 10 = 3b(b-10) - 1(b-10) = (3b-1)(b-10)$$

$$\frac{(b+10)(b-10)}{b^2} \times \frac{2b}{(3b-1)(b-10)} = \frac{2(b+10)}{b\,(3b-1)}$$

D. Factoring

Example 1.87

A. $2^{x-6} + 2^x$

$$2^x \cdot 2^6 + 2^x = 2^x(2^{-6}+1) = 2^x\left(\frac{1}{2^6} + 1\right) =$$

E. Reciprocals

Example 1.88

Find the reciprocal of

$$(a+b)^{-1}(a^{-1}+b^{-1})$$

The given expression is:

$$\left(\frac{1}{a+b}\right)\left(\frac{1}{a}+\frac{1}{b}\right)=\left(\frac{1}{a+b}\right)\left(\frac{a+b}{ab}\right)=\frac{1}{ab}$$

And its reciprocal is

$$ab$$

F. Reciprocals

Example 1.89

Use the rule $\frac{a^m}{a^n}=a^{m-n}$:

$$\frac{a^5}{a^3}=a^{5-3}=a^2$$
$$\frac{b^7}{b^4}=b^{7-4}=b^3$$
$$\frac{c^5}{c^6}=c^{5-6}=c^{-1}$$
$$\frac{d^{12}}{d^{18}}=d^{12-18}=d^{-6}$$

$$\frac{e^{-7}}{e^{-2}}=e^{-7+2}=e^{-5}$$

$$\frac{f^{-4}}{f^{-3}}=f^{-4+3}=f^{-1}$$

Example 1.90

Use the rule $\frac{a^m}{a^n}=a^{m-n}$:

$$\frac{a^7\times b^2}{a^5\times b^8}=a^2b^{-6}$$
$$\frac{a^{12}\times b^5}{a^{15}\times b^9}=a^{-3}b^{-4}$$
$$\frac{a^7\times b^{-12}}{a^{-3}\times b^4}=a^{7-(-3)}b^{-12-4}=a^{10}b^{-16}$$
$$\frac{a^{-10}\times b^{11}}{a^{-7}\times b^{-13}}=a^{-3}b^{24}$$

Example 1.91

Use the rule $\frac{a^m}{a^n}=a^{m-n}$:

$$\frac{16a^3}{24a^4}=\frac{2a^{-1}}{3}$$

$$\frac{18b^6}{27b^{-3}} = \frac{18}{27} \times \frac{b^6}{b^{-3}} = \frac{2}{3}b^9$$
$$\frac{12c^{-9}}{32c^{-5}} = \frac{12}{32} \times \frac{c^{-9}}{c^{-5}} = \frac{3c^{-9+5}}{8} = \frac{3c^{-4}}{8}$$
$$\frac{63d^{72}}{72d^{66}} = \frac{7d^6}{8}$$

Example 1.92

$$\frac{22a^6 \times 54b^{-4}}{18a^{-2} \times 33b^5}$$

$$\frac{22a^6 \times 54b^{-4}}{18a^{-2} \times 33b^5} = \frac{11a^8 \times 18b^{-9}}{9 \times 11} = 2a^8b^{-9}$$

Example 1.93

$$\frac{27a^9 \times 18b^5 \times 4c^2}{18a^{-4} \times 12b^{-2} \times 2c^{-1}}$$

$$\frac{3a^{13} \times 3b^7 \times 2c^3}{2 \times 2} = \frac{9a^{13}3b^7c^3}{2}$$

Example 1.94

$$\frac{10^5 \times 216^4 \times 18^3}{25^4 \times 51^4}$$

$$\frac{(2 \times 5)^5 \times (6^3)^4 \times (2 \times 3^2)^3}{(5^2)^4 \times (3 \times 17)^4}$$

Use $a^m \times b^m = (ab)^m$

$$= \frac{2^5 \times 5^5 \times (2^3 \times 3^3)^4 \times 2^3 \times 3^6}{5^8 \times 3^4 \times 17^4}$$
$$= \frac{2^5 \times 5^5 \times 2^{12} \times 3^{12} \times 2^3 \times 3^6}{5^8 \times 3^4 \times 17^4}$$
$$= \frac{2^{5+12+3} \times 3^{12+6-4} \times 3^6}{5^{8-5} \times 17^4}$$

$$= \frac{2^{20} \times 3^{14} \times 3^6}{5^3 \times 17^4}$$

1.6 Scientific Notation

A. Converting to Scientific Notation

Scientific notation is useful in all kinds of science (physics, chemistry, astronomy) because they often deal with the very large, or the very small, and hence regular numbers are not the best way to represent the quantities involved. Large numbers will result in positive exponents. Small numbers will result in negative exponents. However, the exponents will be easier to work with, as compared to numbers with a lot of zeroes.

1.95: Scientific Notation

There is exactly one non-zero digit to the left of the decimal point.

There can be one or more digits to the right of the decimal point.

Example 1.96

Convert the following numbers into scientific notation:

A. 35,000
B. 8,100
C. 72,000
D. 450,000
E. 1,200,000
F. 52,300
G. 128,000
H. 891
I. 7,819
J. 34005.87

Part A

First, get rid of all the zeros:

$$35000 = 35 \times 1000$$

Then, make it shorter by writing:

$$35 \times 1000 = 35 \times 10^3$$

But, we also need to meet the condition that there should be exactly one non-zero digit to the left of the decimal point. Hence, we write it like this:

$$35 \times 10^3 = \frac{35}{10} \times 10 \times 10^3 = 3.5 \times 10^4$$

Shortcut

We want 1 digit before the decimal point. Count the number of digits that are after the decimal point, and that is the power of ten

$$35000 = \underbrace{3.5000 \times Something}_{5000 \to 4\ digits} = 3.5 \times 10^4$$

Other Parts

$$8100 = \underbrace{8.100 \times Something}_{100 \to 3\ digits} = 8.1 \times 10^3$$

$$72000 = 7.2 \times 10^4$$

$$450000 = 4.5 \times 10^5$$

$$1,200,000 = 1.200000 = 1.2 \times 10^6$$

$$52300 = 5.23 \times 10^4$$

$$128000 = 1.28 \times 10^5$$

$$8.91 \times 10^2$$

$$7.819 \times 10^3$$

There are currently five digits to the left of the decimal point. We want one digit. Hence, we get:

$$3.400587 \times 10^4$$

Example 1.97: Small Numbers

Convert the following numbers into scientific notation:

Decimals

A. 0.00035
B. 0.91
C. 0.045
D. 0.0067
E. 0.000031
F. 0.00453

Decimals in Words

G. 5 *Tenths*
H. 7 *Hundredths*
I. 21 *Hundredths*
J. 9 *Thousandths*
K. 72 *Thousandths*
L. 153 *Thousandths*

Decimals

First, write it as a decimal fraction:

$$0.00035 = \frac{35}{100,000}$$

Then, convert the denominator into a power of ten:

$$\frac{35}{100,000} = \frac{35}{10^5}$$

Move the term with ten in it to the numerator:

$$\frac{35}{10^5} = 35 \times 10^{-5}$$

But, we also need to meet the condition that there should be exactly one non-zero digit to the left of the decimal point. Hence, we write it like this:

$$35 \times 10^{-5} = \frac{35}{10} \times 10 \times 10^{-5} = 3.5 \times 10^{-4}$$

$$0.00035 \Rightarrow 3.5 \times 10^{-4}$$

Other Parts

$$0.91 = \frac{9.1}{10} = \frac{9.1}{10^1} = 9.1 \times 10^{-1}$$

$$0.045 = 4.5 \times 10^{-2}$$

$$0.0067 = 6.7 \times 10^{-3}$$
$$0.000031 = 3.1 \times 10^{-5}$$
$$0.00453 = 4.53 \times 10^{-3}$$

Decimals in Words

$$5\ Tenths = 0.5 = 5 \times 10^{-1}$$
$$7\ Hundredths = \frac{7}{100} = 0.07 = 7 \times 10^{-2}$$
$$21\ Hundredths = \frac{21}{100} = 0.21 = 2.1 \times 10^{-1}$$
$$9\ Thousandths = \frac{9}{1000} = 0.009 = 9 \times 10^{-3}$$
$$72\ Thousandths = \frac{72}{1000} = 0.072 = 7.2 \times 10^{-2}$$
$$153\ Thousandths = \frac{153}{1000} = 0.153 = 1.53 \times 10^{-1}$$

1.98: Standard Form

Numbers written the usual way are called numbers in standard form.

Example 1.99

Convert the following numbers from scientific notation into decimal notation.

Large Numbers

A. 5.9×10^2
B. 2.3×10^2
C. 7.1×10^3
D. 5.41×10^3
E. 4.71×10^4
F. 2.5×10^4
G. 1.9×10^5
H. 2.852×10^5
I. 1.2×10^6
J. 8.21×10^6
K. 4.3×10^7
L. 2.71×10^8
M. 5.3×10^9

Small Numbers

N. 6.4×10^{-2}
O. 3.2×10^{-3}
P. 8.1×10^{-4}
Q. 9.77×10^{-2}
R. 1.26×10^{-3}

Large Numbers

To convert into standard form, carry out the multiplication:

$$5.9 \times 10^2 = 5.9 \times 100 = 590$$
$$7.1 \times 10^3 = 7.1 \times 1000 = 7100$$
$$5.41 \times 10^3 = 5.41 \times 1000 = 5410$$
$$4.71 \times 10^4$$

Small Numbers

This is similar to the previous kind, but the exponents here are negative. Write it out using positive exponents, and convert the denominator to a regular first:

$$6.4 \times 10^{-2} = \frac{6.4}{100} = 0.064$$
$$3.2 \times 10^{-3} = \frac{3.2}{1{,}000} = 0.0032$$
$$8.1 \times 10^{-4} = \frac{8.1}{10{,}000} = 0.00081$$
$$9.77 \times 10^{-2} = \frac{9.77}{100} =$$

Example 1.100: Addition and Subtraction

- If the exponents are the same, you can add directly.
- If the exponents are not the same, first make the exponents the same, and then add them.
- When you add or subtract, the final answer may no longer remain in scientific notation. If that is the case, you need to rewrite it to ensure that there is one non-zero digit to the left of the decimal point.

A. $3.4 \times 10^{-14} + 2.3 \times 10^{-14}$
B. $2.7 \times 10^5 - 1.3 \times 10^5$
C. $3.4 \times 10^{12} - 2.9 \times 10^{12}$
D. $4.71 \times 10^{17} - 4.65 \times 10^{17}$
E. $5.5 \times 10^{12} + 6.6 \times 10^{12}$
F. $4.1 \times 10^5 + 3.7 \times 10^4$

G. $1.2 \times 10^5 - 9.9 \times 10^4$

Parts A and B

The exponents in the power of 10 in both numbers are equal, so we can directly add the numbers:

$$3.4 \times \mathbf{10^{-14}} + 2.3 \times \mathbf{10^{-14}} = (3.4 + 2.3) \times \mathbf{10^{-14}}$$
$$= 5.7 \times 10^{-14}$$
$$2.7 \times \mathbf{10^{5}} - 1.3 \times \mathbf{10^{5}} = (2.7 - 1.3) \times \mathbf{10^{5}}$$
$$= 1.4 \times 10^5$$

Part C

Take 10^{12} common:
$$(3.4 - 2.9) \times 10^{12}$$
Perform the subtraction:
$$0.5 \times 10^{12}$$
Rewrite it into scientific notation:
$$0.5 \times 10^{12} = 0.5 \times 10 \times 10^{11} = 5.0 \times 10^{11}$$

Part D

$$(4.71 - 4.65) \times 10^{17} = 0.06 \times 10^{17}$$
Rewrite it into scientific notation:
$$6 \times 10^{-2} \times 10^{17} = 6 \times 10^{15}$$

Part E

Take 10^{12} common:
$$(5.5 + 6.6) \times 10^{12} = 12.1 \times 10^{12}$$
Rewrite it into scientific notation:
$$12.1 \times 10^{12} = \frac{12.1}{10} \times 10 \times 10^{12} = 1.21 \times 10^{13}$$

Part F

Make the exponents the same. Write the second number with an exponent of 10^5:
$$3.7 \times 10^4 = \frac{3.7}{10} \times 10 \times 10^4 = 0.37 \times 10^5$$
Do the addition:
$$(4.1 + 0.37) \times 10^5 = 4.47 \times 10^5$$

Part G

Make the exponents the same. Write the first number with an exponent of 10^4:
$$1.2 \times 10^5 = 12 \times 10^4$$
Do the addition:
$$= (12 - 9.9) \times 10^4 = 2.1 \times 10^4$$

B. Multiplication and Division

$$A = 1.1 \times 10^4, \qquad B = 3.1 \times 10^5$$

Rearrange to write the exponents together, and the number together:
$$AB = 1.1 \times 3.1 \times 10^4 \times 10^5 = 3.41 \times 10^9$$

Example 1.101

$$p = 1.2 \times 10^3, \qquad q = 3.0 \times 10^4, \qquad r = 2.3 \times 10^5, \qquad s = 4.0 \times 10^6$$
Given the values above, calculate the expressions below and write your answers in scientific notation:

Multiplication

A. pq
B. rq
C. ps
D. rs
E. pr

Division

F. $\frac{p}{q}$
G. $\frac{q}{s}$
H. $\frac{p}{s}$
I. $\frac{r}{s}$

Multiplication

$$pq = 1.2 \times 3.0 \times 10^3 \times 10^4 = 3.6 \times 10^7$$
$$rq = 6.9 \times 10^9$$
$$ps = 4.8 \times 10^9$$
$$rs = 9.2 \times 10^{11}$$
$$pr = 2.76 \times 10^8$$

Division

$$\frac{p}{q} = \frac{1.2 \times 10^3}{3.0 \times 10^4} = 0.4 \times 10^{-1} = 4.0 \times 10^{-2}$$

$$\frac{q}{s}=\frac{3.0\times 10^4}{4.0\times 10^6}=\frac{3}{4}\times 10^{-2}=0.75\times 10^{-2}=7.5\times 10^{-3}$$
$$\frac{p}{s}=\frac{1.2\times 10^3}{4.0\times 10^6}=0.3\times 10^{-3}=3.0\times 10^{-4}$$
$$\frac{r}{s}=\frac{2.3\times 10^5}{4.0\times 10^6}=5.75\times 10^{-1}$$

Example 1.102

A. $(2\times 10^3)\times(8\times 10^7)$

B. $\frac{0.00065}{5\times 10^{-2}}$

$$14\times 10^{10}=14\times\frac{1}{10}\times 10\times 10^{10}=1.4\times 10^{11}$$
$$\frac{65\times 10^{-5}}{5\times 10^{-2}}=13\times 10^{-5-(-2)}=13\times 10^{-3}=1.3\times 10^{-2}$$

Example 1.103

The distance light travels in one year is approximately 5,870,000,000,000 miles. The distance light travels in 100 years is: (write your answer in scientific notation): (AHSME 1956/3)

1.7 Clever Manipulations

A. Equations

Example 1.104

$$3^{9+x}=x^x$$

Split the LHS:

$$3^9\cdot 3^x=x^x$$

Take all x terms to the RHS:

$$3^9=\frac{x^x}{3^x}$$

Use $\frac{a^m}{b^m}=\left(\frac{a}{b}\right)^m$ on the RHS, and substitute $3=\frac{9}{3}$ in the LHS:

$$\left(\frac{9}{3}\right)^9=\left(\frac{x}{3}\right)^x$$

By observation:

$$x=9$$

And verify this in the given equation

$$LHS=3^{9+9}=3^{18}=9^9=RHS$$

2. Roots and Radicals

2.1 Square Roots of Numbers

A. Roots of Numbers

2.1: Square Root of a Number

The square root of a number is a number that, when squared, gives us the original number. The symbol for square root is $\sqrt{}$

Specifically, if

$$4^2 = 16 \Rightarrow \sqrt{16} = 4$$

Example 2.2

A. $\sqrt{16}$
B. $\sqrt{36}$
C. $\sqrt{4}$

$$\sqrt{16} = 4$$
$$\sqrt{36} = 6$$
$$\sqrt{4} = 2$$

Example 2.3

Find the square roots below:

A. $\sqrt{196}$
B. $\sqrt{576}$
C. $\sqrt{1024}$
D. $\sqrt{324}$
E. $\sqrt{484}$
F. $\sqrt{121}$
G. $\sqrt{361}$
H. $\sqrt{676}$
I. $\sqrt{225}$

$$\sqrt{196} = 14$$
$$\sqrt{576} = 24$$
$$\sqrt{1024} = 32$$
$$\sqrt{324} = 18$$
$$\sqrt{484} = 22$$
$$\sqrt{121} = 11$$
$$\sqrt{361} = 19$$
$$\sqrt{676} = 26$$
$$\sqrt{225} = 15$$

2.4: Square Root of Negative Quantities

Both positive and negative quantities become positive when squared. Hence, there are no solutions for

$\underbrace{negative\ square\ roots}_{\text{in the real number system}}.$

$2^2 = 4, (-2)^2 = 4 \Rightarrow x = \sqrt{-4}$ has no solutions in the real number system

Example 2.5

2.6: Square Root of a Fraction

If you want to take the square root of a fraction, you can take the square root of the numerator and the denominator separately.

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Example 2.7

Use the property $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ to simplify each part below:

A. $\sqrt{\frac{64}{121}}$

B. $\sqrt{\frac{49}{144}}$

C. $\sqrt{\frac{36}{81}}$

D. $\sqrt{\frac{100}{169}}$

$$\sqrt{\frac{64}{121}} = \frac{\sqrt{64}}{\sqrt{121}} = \frac{8}{11}$$

$$\sqrt{\frac{49}{144}} = \frac{7}{12}$$

$$\sqrt{\frac{36}{81}} = \frac{6}{9} = \frac{2}{3}$$

$$\sqrt{\frac{100}{169}} = \frac{10}{13}$$

2.8: Simplifying a Fraction

If needed, you can simplify a fraction before taking the square root.

18 does not have a "nice" square root, and neither does 72. But we can simplify the fraction inside the square root:

$$\sqrt{\frac{18}{72}} = \sqrt{\frac{9}{36}} = \frac{\sqrt{9}}{\sqrt{36}} = \frac{3}{6} = \frac{1}{2}$$

$$\sqrt{\frac{18}{72}} = \sqrt{\frac{1}{4}} = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2}$$

Example 2.9

Simplify the fractions below and then use the property $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ to arrive at an answer without radicals.

A. $\sqrt{\frac{32}{50}}$

B. $\sqrt{\frac{8}{18}}$

$$\sqrt{\frac{32}{50}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\sqrt{\frac{8}{18}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$$

2.10: Decimals

One way of finding the square root of a decimal is to rewrite it as a decimal fraction, and then take the square root of the corresponding fraction.

$$\sqrt{0.04}$$

Rewrite the decimal as a decimal fraction, and then take the square root:

$$= \sqrt{\frac{4}{100}} = \frac{\sqrt{4}}{\sqrt{100}} = \frac{2}{10} = 0.2$$

Example 2.11

Simplify. Write your answer as a decimal.

A. $\sqrt{0.81}$

B. $\sqrt{0.36}$

C. $\sqrt{0.64}$

D. $\sqrt{0.16}$

E. $\sqrt{0.49}$

F. $\sqrt{1.69}$

G. $\sqrt{1.44}$

H. $\sqrt{1.96}$

I. $\sqrt{2.25}$

J. $\sqrt{0.01}$

$$\sqrt{0.81} = \sqrt{\frac{81}{100}} = \frac{9}{10} = 0.9$$

$$\sqrt{0.36} = \sqrt{\frac{36}{100}} = \frac{6}{10} = 0.6$$

$$\sqrt{0.49} = 0.7$$

$$\sqrt{0.64} = 0.8$$

$$\sqrt{1.69} = 1.3$$

$$\sqrt{2.25}=\sqrt{\frac{225}{100}}=\frac{15}{10}=1.5$$
$$\sqrt{8.41}=2.9$$
$$\sqrt{0.01}=\sqrt{\frac{1}{100}}=\frac{1}{10}=0.1$$

2.12: Shortcut

The number of decimal places in the final answer is half of the number decimal places in the question.

$$\sqrt{5.29}$$

Note that 5.29 has 2 digits to the right of the decimal point. The final answer after the square root will have one digit to the right to the decimal point.

Without the decimal points, 5.29 would have been 529, and it has square root:

$$\sqrt{529}=23$$

Since we want one decimal point, we get:

$$\sqrt{5.29}=2.3$$

Example 2.13

A. $\sqrt{8.41}$

2.14: Decimals in Fractions

If the numerator or the denominator have a fraction with decimals, the decimals can be eliminated by multiplying by $10 \text{ or } 100 \text{ or } 1000 \text{ or } \dots$. as suitable

$$\sqrt{\frac{0.09}{0.36}}$$

Both the numerator and the denominator have two digits to the right of the decimal place.
Hence, multiply numerator and denominator by $10^2=100$ to remove the decimals:

$$\sqrt{\frac{9}{36}}=\sqrt{\frac{1}{4}}=\frac{1}{2}$$

Example 2.15

Simplify.

A. $\sqrt{\frac{0.5}{0.72}}$

B. $\sqrt{\frac{3.38}{2.88}}$

C. $\sqrt{\frac{3.43}{2.52}}$

Numerator has one digit to the right of the decimal place.
Denominator has two digits to the right of the decimal place.

We take the larger of the two, and multiply by 100:

$$\sqrt{\frac{0.5}{0.72}}=\sqrt{\frac{50}{72}}=\sqrt{\frac{25}{36}}=\frac{5}{6}$$

$$\sqrt{\frac{3.38}{2.88}}=\sqrt{\frac{338}{288}}=\sqrt{\frac{169}{144}}=\frac{13}{12}$$

$$\sqrt{\frac{3.43}{2.52}}=\sqrt{\frac{343}{252}}=\sqrt{\frac{49}{36}}=\frac{7}{6}$$

Example 2.16:Perfect Squares

$$(1000-1)^2=1{,}000{,}000-2000+1=9998001$$

2.17: Last Digit of Perfect Squares

Last Digit of			
No.	**Square**	**No.**	**Square**
0	0	5	5
1	1	6	6
2	4	7	9
3	9	8	4
4	6	9	1

Example 2.18

A. What is the last digit of 34^2
B. Identify the square root of 10404
C. Identify the square root of 6084
D. Identify the square root of 21025
E. Which are the digits which can / cannot be the last digit of a perfect square?

Part A

34 has last digit 4.
∴ From the table, a number with last digit 4 has a square with last digit 6.
We can also verify it by calculating:

$$34\times 34=1{,}156$$

Part B

10404 has last digit 4.
∴ Its square root must have last digit 2 or 8.
10404 is very close to 10,000.

Hence, try 102:

$$102^2=(100+2)^2$$

Using $(a+b)^2=a^2+2ab+b^2$:

$$100^2+2\times 2\times 100+2^2=10{,}000+400+4=10{,}404$$

Part C

6084 has last digit 4. Hence, the last digit of $\sqrt{6084}$ must be either 2 or 8.
Further, we can approximate $\sqrt{6084}$ as:

$$\sqrt{4900} < \sqrt{6084} < \sqrt{6400} \Rightarrow 70 < \sqrt{6084} < 80$$

Since 6084 is much closer to 6400 than to 4900, we try

$$78^2 = (80-2)^2$$

Using $(a-b)^2 = a^2 - 2ab + b^2$:

$$80^2 - (2)(2)(80) + 2^2 = 6400 - 320 + 4 = 6084$$

Part D

$$\sqrt{19{,}600} < \sqrt{21025} < \sqrt{22{,}500}$$
$$140 < \sqrt{21025} < 150$$

The last digit of $\sqrt{21025}$ must be a 5. Hence, we try

$$145^2 = 21025$$

Part E

From the table, the only possible values of the last digit of a perfect square are

$$\{0,1,4,5,6,9\}$$

The missing values are

$$\{2,3,7,8\}$$

B. Multiplication of Radicals

2.19: Multiplication property

We can use the multiplication property to combine quantities inside a radical sign, and then simplify:

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

For example, we can combine the two radicals below into a single radical and then simplify by taking the square root:

$$\sqrt{2} \times \sqrt{2} = \sqrt{2 \times 2} = \sqrt{4} = 2$$

Example 2.20: Multiplication

Multiply and simplify:

A. $\sqrt{3} \times \sqrt{3}$
B. $\sqrt{5} \times \sqrt{5}$
C. $\sqrt{2} \times \sqrt{18}$
D. $\sqrt{5} \times \sqrt{20}$
E. $\sqrt{8} \times \sqrt{2}$

$$\sqrt{3} \times \sqrt{3} = \sqrt{9} = 3$$
$$\sqrt{5} \times \sqrt{5} = \sqrt{25} = 5$$
$$\sqrt{2} \times \sqrt{18} = \sqrt{36} = 6$$
$$\sqrt{5} \times \sqrt{20} = \sqrt{100} = 10$$
$$\sqrt{8} \times \sqrt{2} = \sqrt{16} = 4$$

C. Distributive Property

The multiplication property of radicals can also be used the other around to split radicals. This is a less obvious than multiplication, but very useful when simplifying radicals.

2.21: Distributive property

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

One way of simplfying $\sqrt{36}$ is to recognize that $36 = 6^2$:

$$\sqrt{36} = \sqrt{6^2} = 6$$

Another way of simplifying $\sqrt{36}$ is to split 36 as the product of two perfect squares, and then find the square roots separately:

$$\sqrt{36} = \sqrt{4} \times \sqrt{9} = 2 \times 3 = 6$$

Example 2.22

Simplify

A. $\sqrt{50}$

We use the prime factorization of

$$50 = 2 \times 5^2$$

And split it so that the perfect square ($5^2 = 25$) is in one radical, and the 2 is in another radical:

$$\sqrt{50} = \underbrace{\sqrt{25}}_{\substack{Perfect\\Square}} \times \underbrace{\sqrt{2}}_{\substack{Not\ a\\Perfect\ Square}}$$

The perfect square can be simplified, and the radical with 2 remains as it is:

$$= 5 \times \sqrt{2} = 5\sqrt{2}$$

Example 2.23

Simplify:

Perfect Cubes

A. $\sqrt{27}$
B. $\sqrt{8}$
C. $\sqrt{125}$
D. $\sqrt{343}$
E. $\sqrt{216}$

Multiples of Four

F. $\sqrt{12}$
G. $\sqrt{20}$
H. $\sqrt{28}$
I. $\sqrt{44}$
J. $\sqrt{52}$

Multiples of Nine

K. $\sqrt{18}$
L. $\sqrt{45}$
M. $\sqrt{63}$

Perfect Cubes

$$\sqrt{27} = \sqrt{9} \times \sqrt{3} = 3\sqrt{3}$$
$$\sqrt{8} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$$
$$\sqrt{125} = \sqrt{25} \times \sqrt{5} = 5\sqrt{5}$$
$$\sqrt{343} = \sqrt{49} \times \sqrt{7} = 7\sqrt{7}$$

$$\sqrt{216} = \sqrt{36} \times \sqrt{6} = 6\sqrt{6}$$

Multiples of Four

Every number in this part is a multiple of 4, which is a perfect square. Hence, we separate out the 4 in each question:

$$\sqrt{12} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$$
$$\sqrt{20} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$$
$$\sqrt{28} = \sqrt{4} \times \sqrt{7} = 2\sqrt{7}$$
$$\sqrt{44} = \sqrt{4} \times \sqrt{11} = 2\sqrt{11}$$
$$\sqrt{52} = \sqrt{4} \times \sqrt{13} = 2\sqrt{13}$$

Multiples of Nine

Every number in this part is a multiple of 9, which is a perfect square. Hence, we separate out the 4 in each question:

$$\sqrt{18} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$$
$$\sqrt{45} = \sqrt{9} \times \sqrt{5} = 3\sqrt{5}$$
$$\sqrt{63} = \sqrt{9} \times \sqrt{7} = 3\sqrt{7}$$

Example 2.24

Simplify:

Double of a Perfect Square

1. $\sqrt{288}$
2. $\sqrt{200}$
3. $\sqrt{32}$
4. $\sqrt{128}$
5. $\sqrt{392}$
6. $\sqrt{162}$
7. $\sqrt{242}$
8. $\sqrt{338}$
9. $\sqrt{800}$

Powers of Ten

10. $\sqrt{100}$
11. $\sqrt{1000}$
12. $\sqrt{10{,}000}$
13. $\sqrt{100{,}000}$
14. $\sqrt{1{,}000{,}000}$

Double of a Perfect Square

Note that factoring 2 from the each number in this part results in a perfect square, which we can then take the square root of:

$$\sqrt{288} = \sqrt{144} \times \sqrt{2} = 12\sqrt{2}$$
$$\sqrt{200} = \sqrt{100} \times \sqrt{2} = 10\sqrt{2}$$
$$\sqrt{32} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$$
$$\sqrt{128} = \sqrt{64} \times \sqrt{2} = 8\sqrt{2}$$
$$\sqrt{392} = 14\sqrt{2}$$
$$\sqrt{162} = \sqrt{81} \times \sqrt{2} = 9\sqrt{2}$$
$$\sqrt{242} = \sqrt{121} \times \sqrt{2} = 11\sqrt{2}$$

$$\sqrt{800} = \sqrt{400} \times \sqrt{2} = 20\sqrt{2}$$

Powers of Ten

$$\sqrt{100} = 10$$
$$\sqrt{1000} = \sqrt{100} \times \sqrt{10} = 10\sqrt{10}$$
$$\sqrt{10{,}000} = 100$$
$$\sqrt{100{,}000} = 100\sqrt{10}$$
$$\sqrt{1{,}000{,}000} = 1000$$

Example 2.25

Simplify:

Mixed

1. $\sqrt{72}$
2. $\sqrt{28}$
3. $\sqrt{54}$
4. $\sqrt{80}$
5. $\sqrt{10{,}000}$
6. $\sqrt{99}$
7. $\sqrt{175}$
8. $\sqrt{200}$

Large Numbers

9. $\sqrt{27000}$
10. $\sqrt{216000}$
11. $\sqrt{44000}$
12. $\sqrt{12800000}$

Decimals

13. $\sqrt{0.0004}$
14. $\sqrt{0.0000027}$

Mixed

The key here is to find a factorization that a perfect square:

$$\sqrt{72} = \sqrt{36} \times \sqrt{2} = 6\sqrt{2}$$
$$\sqrt{28} = \sqrt{4} \times \sqrt{7} = 2\sqrt{7}$$
$$\sqrt{54} = \sqrt{9} \times \sqrt{6} = 3\sqrt{6}$$
$$\sqrt{80} = \sqrt{16} \times \sqrt{5} = 4\sqrt{5}$$
$$\sqrt{99} = \sqrt{9} \times \sqrt{11} = 3\sqrt{11}$$
$$\sqrt{175} = \sqrt{25} \times \sqrt{7} = 5\sqrt{7}$$

Large Numbers

We can rewrite $\sqrt{27000}$ as

$$\sqrt{27} \times \sqrt{1000}$$

Which can then be further broken up as:

$$\sqrt{9}\times\sqrt{3}\times\sqrt{100}\times\sqrt{10}$$
$$=3\sqrt{3}\times 10\sqrt{10}$$
$$=30\sqrt{30}$$

$$\sqrt{216000}=\sqrt{36}\times\sqrt{6}\times\sqrt{100}\times\sqrt{10}=60\sqrt{60}$$
$$\sqrt{44000}=\sqrt{4}\times\sqrt{11}\times\sqrt{100}\times\sqrt{10}=20\sqrt{110}$$

Break up $\sqrt{12800000}$

$$=\sqrt{64}\times\sqrt{2}\times\sqrt{10000}\times\sqrt{10}=1600\sqrt{5}$$

Small Numbers

$$\sqrt{0.0004}=\sqrt{\frac{4}{10{,}000}}=\frac{2}{100}=0.02$$

Break up $\sqrt{0.0000027}$

$$=\sqrt{\frac{27}{10{,}000{,}000}}=\frac{3\sqrt{3}}{1000\sqrt{10}}=\left(\frac{3}{1000}\right)\sqrt{\frac{3}{10}}$$

2.26: Surds

- Square roots of prime numbers are irrational.

Even numbers like

$$\sqrt{15}=\sqrt{3\times 5}$$

Are also irrational

Example 2.27

Classify the following numbers as rational or irrational:

A. $\sqrt{2}$
B. $\sqrt{5}$
C. $\sqrt{10}$
D. $\sqrt{4}$

$$\sqrt{2}\ is\ irrational$$
$$\sqrt{5}\ is\ irrational$$
$$\sqrt{10}=\sqrt{2\times 5}\ is\ irrational$$
$$\sqrt{4}=2\ is\ rational$$

2.28: Adding Radicals with the same base

If radicals have the same base, then they can be added, or subtracted:

$$a\sqrt{x}+b\sqrt{x}=(a+b)\sqrt{x}$$
$$a\sqrt{x}-b\sqrt{x}=(a-b)\sqrt{x}$$

$$3x+5x=8x$$

Substituting $x=\sqrt{11}$ in the above gives:

$$3\sqrt{11}+5\sqrt{11}=8\sqrt{11}$$

Example 2.29

Find the value of:

A. $\sqrt{7}+\sqrt{7}$
B. $\sqrt{3}+\sqrt{3}+\sqrt{3}$
C. $\sqrt{5}+\sqrt{5}$
D. $\sqrt{19}+\sqrt{19}+\sqrt{19}+\sqrt{19}$
E. $4\sqrt{2}+3\sqrt{2}$
F. $7\sqrt{3}+4\sqrt{3}$
G. $12\sqrt{5}+\sqrt{5}$
H. $3\sqrt{11}-2\sqrt{11}$
I. $7\sqrt{19}-9\sqrt{19}$

$$\sqrt{7}+\sqrt{7}=2\times\sqrt{7}=2\sqrt{7}$$
$$\sqrt{3}+\sqrt{3}+\sqrt{3}=3\sqrt{3}$$
$$\sqrt{5}+\sqrt{5}=2\sqrt{5}$$
$$\sqrt{19}+\sqrt{19}+\sqrt{19}+\sqrt{19}=4\sqrt{19}$$
$$4\sqrt{2}+3\sqrt{2}=7\sqrt{2}$$
$$7\sqrt{3}+4\sqrt{3}=11\sqrt{3}$$
$$12\sqrt{5}+\sqrt{5}=13\sqrt{5}$$
$$3\sqrt{11}-2\sqrt{11}=\sqrt{11}$$
$$7\sqrt{19}-9\sqrt{19}=-2\sqrt{19}$$

2.30: Radicals that are "different"

If radicals are different, they cannot be added. That is, for $a\neq b$

$$\sqrt{a}+\sqrt{b}\neq\sqrt{a+b}$$

Example 2.31

A. $\sqrt{3}+\sqrt{7}$
B. $7\sqrt{5}+3\sqrt{7}$
C. $5\sqrt{11}+11\sqrt{5}$

You can only add radicals that have the same value. So, none of the above questions can be simplified further.

Example 2.32: Simplifying Square Root Expressions

Simplify:

A. $\sqrt{8}+\sqrt{18}$
B. $\sqrt{72}+\sqrt{128}$
C. $3\sqrt{32}+4\sqrt{128}$
D.

$$\sqrt{8}+\sqrt{18}=\sqrt{4}\sqrt{2}+\sqrt{9}\sqrt{2}\ =2\sqrt{2}+3\sqrt{2}=5\sqrt{2}$$
$$\sqrt{72}+\sqrt{128}=\sqrt{36}\sqrt{2}+\sqrt{64}\sqrt{2}=6\sqrt{2}+8\sqrt{2}=14\sqrt{2}$$
$$3\sqrt{32}+4\sqrt{128}=3\sqrt{16}\sqrt{2}+4\sqrt{64}\sqrt{2}=12\sqrt{2}+32\sqrt{2}=44\sqrt{2}$$

Example 2.33:

$$\frac{\sqrt{243}}{27}+\frac{\sqrt{192}}{16}+\frac{\sqrt{125}}{10}+\frac{\sqrt{216}}{18}$$

Split the roots:

$$\frac{\sqrt{81}\sqrt{3}}{27}+\frac{\sqrt{64}\sqrt{3}}{16}+\frac{\sqrt{25}\sqrt{5}}{10}+\frac{\sqrt{36}\sqrt{6}}{18}$$

Simplify each square root:

$$\frac{9\sqrt{3}}{27}+\frac{8\sqrt{3}}{16}+\frac{5\sqrt{5}}{10}+\frac{6\sqrt{6}}{18}$$

Simplify each fraction:

$$=\frac{\sqrt{3}}{3}+\frac{\sqrt{3}}{2}+\frac{\sqrt{5}}{2}+\frac{\sqrt{6}}{3}$$

Add the fractions by taking a common denominator of 6:

$$=\frac{2\sqrt{3}+3\sqrt{3}+3\sqrt{5}+2\sqrt{6}}{6}$$

Add like terms:

$$=\frac{5\sqrt{3}+3\sqrt{5}+2\sqrt{6}}{6}$$

$$\sqrt{12}\times\sqrt{3}=\sqrt{36}=6$$

The square and the square root cancel, giving us just 10:

$$\sqrt{10}\times\sqrt{10}=\left(\sqrt{10}\right)^2$$

Example 2.34: Simplifying Square Root Expressions

A. $\sqrt{12}\times\sqrt{3}$
B. $\sqrt{10}\times\sqrt{10}$
C. $\sqrt{24}\times\sqrt{40}$

$$\sqrt{12}\times\sqrt{3}=\sqrt{36}=6$$

$$\sqrt{10}\times\sqrt{10}=\sqrt{10^2}=10$$

$$\sqrt{24}\times\sqrt{40}=\sqrt{(2^3\times 3)\times(2^3\times 5)}=\sqrt{2^6\times 3\times 5}=8\sqrt{15}$$

Example 2.35: Simplifying Square Root Expressions

$$\frac{\sqrt{2^3}\times\sqrt{2^4}}{\sqrt{2^5}}\times\frac{\sqrt{5^5}\times\sqrt{5^2}}{\sqrt{5^3}}$$

First Term:

Since all the terms are inside a square root, we can combine them into a single square root:

$$=\sqrt{\frac{2^3\times 2^4}{2^5}}$$

Use $a^m \cdot a^n = a^{m+n}, \frac{a^m}{a^n} = a^{m-n}$

$$= \sqrt{2^{3+4-5}} = \sqrt{2^2} = 2$$

Second Term:

$$\sqrt{\frac{5^5 \times 5^2}{5^3}} = \sqrt{5^{5+2-3}} = \sqrt{5^4} = 25$$

The final answer is

$$25 \times 2 = 50$$

Example 2.36

$$\frac{\sqrt{3^4} \times \sqrt{9^2}}{\sqrt{27^3}}$$

$$\sqrt{\frac{3^4 \times 9^2}{27^3}} = \sqrt{\frac{3^4 \times (3^2)^2}{(3^3)^3}} = \sqrt{\frac{3^4 \times 3^4}{3^9}} = \sqrt{3^{4+4-9}} = \sqrt{3^{-1}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

D. Approximating non-perfect squares

2.37: Approximating non-perfect squares

If a number is not a perfect square, we can still determine the two perfect squares between which it lies, and hence determine a range for the value.

Hence, we can say that

$$\underbrace{9}_{=\sqrt{81}} < \sqrt{82} < \underbrace{10}_{=\sqrt{100}}$$

Example 2.38

Determine the two integers between which expression lies:

A. $\sqrt{50}$
B. $\sqrt{270}$
C. $\sqrt{1000}$
D. $\sqrt{0.72}$

$$7 < \sqrt{50} < 8$$
$$16 < \sqrt{270} < 17$$
$$31 < \sqrt{1000} < 32$$
$$0.8 < \sqrt{0.72} < 0.9$$

E. Multiplication of Expressions

2.39: Distributive Property

$$a(\sqrt{x} + \sqrt{y}) = a\sqrt{x} + a\sqrt{y}$$

$$3(\sqrt{2}+\sqrt{3}) = 3\sqrt{2}+3\sqrt{3}$$

Example 2.40

A. A rectangle has length 4 and width $\sqrt{3}+\sqrt{5}$. Find its area.
B. The area of a parallelogram is the product of its base and the height. Find the area of a parallelogram with base $\sqrt{2}+\sqrt{9}$ and height 7.

$$Area(Rectangle) = 4(\sqrt{3}+\sqrt{5}) = 4\sqrt{3}+4\sqrt{5}$$
$$Area(Parallelogram) = 7(\sqrt{2}+\sqrt{9}) = 7\sqrt{2}+21$$

2.41: Distributive Property

$$(a+b)(c+d) = a(c+d)+b(c+d) = ac+ad+bc+bd$$

Example 2.42

The area of a kite is half the product of the diagonals. Find the area of a kite with diagonals $(\sqrt{2}+\sqrt{3})$ and $(\sqrt{3}+\sqrt{5})$.

$$Area\ of\ Kite = \frac{1}{2}d_1d_2$$

Substitute the values:

$$= \frac{1}{2}[(\sqrt{2}+\sqrt{3})(\sqrt{3}+\sqrt{5})]$$
$$= \frac{1}{2}[\sqrt{2}(\sqrt{3}+\sqrt{5})+\sqrt{3}(\sqrt{3}+\sqrt{5})]$$
$$= \frac{1}{2}[\sqrt{6}+\sqrt{10}+3+\sqrt{15}]$$

Example 2.43

The area of a right triangle is half the product of the legs. Find the area of a right triangle with one leg $(\sqrt{8}+\sqrt{27})$ and other leg $(\sqrt{32}+\sqrt{18})$.

$$\sqrt{8}+\sqrt{27} = 2\sqrt{2}+3\sqrt{3}$$
$$\sqrt{32}+\sqrt{18} = 4\sqrt{2}+3\sqrt{2} = 7\sqrt{2}$$

Substitute

$$Area = \frac{1}{2}l_1l_2 = \frac{1}{2}(2\sqrt{2}+3\sqrt{3})(7\sqrt{2})$$

Use the distributive property:

$$= \frac{1}{2}[(2\sqrt{2})(7\sqrt{2})+(3\sqrt{3})(7\sqrt{2})]$$
$$= \frac{1}{2}\left[14\cdot(\sqrt{2})^2+21\sqrt{6}\right]$$
$$= \frac{1}{2}[14\cdot 2+21\sqrt{6}]$$

$$= 14 + \frac{21}{2}\sqrt{6}$$

Example 2.44

The area of a triangle is half the product of its base and its height. Find the area, in square meters, of a triangle with base $(\sqrt{50} + \sqrt{18})\ cm$ and height $(\sqrt{12} + \sqrt{27})\ cm$.

$$\sqrt{50} + \sqrt{18} = \sqrt{25}\sqrt{2} + \sqrt{9}\sqrt{2} = 5\sqrt{2} + 3\sqrt{2} = 8\sqrt{2}$$
$$\sqrt{12} + \sqrt{27} = \sqrt{4}\sqrt{3} + \sqrt{9}\sqrt{3} = 2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}$$

$$\frac{1}{2}(8\sqrt{2})(5\sqrt{3}) = 20\sqrt{6}\ cm^2$$

Convert from cm^2 to m^2:

$$20\sqrt{6}\ cm \times cm = 20\sqrt{6}\left(\frac{1}{100}m\right)\left(\frac{1}{100}m\right) = \frac{\sqrt{6}}{500}m^2$$

Example 2.45

$$Area\ of\ Trapezoid = \frac{b_1 + b_2}{2}h, \qquad b = base, h = height$$
$$Volume\ of\ Cuboid = lwh, \qquad l = length, w = width, h = height$$

A. A triangle has base Type equation here. and height Type equation here. Find its area.
B. A cuboid has length Type equation here. width Type equation here. and height Type equation here. Find its volume.

2.46: Squaring

When we square a square root, the square and the square root "cancel".

$$\sqrt{a} \times \sqrt{a} = \left(\sqrt{a}\right)^2 = a$$

Example 2.47

Find the squares of the expressions below:

A. $\sqrt{5}$
B. $\sqrt{26}$
C. $\sqrt{99}$
D. $\sqrt{1771}$
E. $3\sqrt{2}$
F. $4\sqrt{5}$
G. $10\sqrt{10}$
H. $7\sqrt{3}$
I. $\frac{\sqrt{3}}{3}$
J. $\frac{\sqrt{125}}{10}$
K. $\frac{\sqrt{512}}{32}$
L. $\frac{\sqrt{1779}}{1779}$

We want to find $\left(\sqrt{5}\right)^2$ and note the square and the square root operation cancel, since they are the exact opposite operation:

$$\left(\sqrt{5}\right)^2 = 5$$
$$\left(\sqrt{26}\right)^2 = 26$$
$$\left(\sqrt{99}\right)^2 = 99$$
$$\left(\sqrt{1771}\right)^2 = 1771$$
$$\left(3\sqrt{2}\right)^2 = 3^2 \times \left(\sqrt{2}\right)^2 = 9 \times 2 = 18$$
$$\left(4\sqrt{5}\right)^2 = 4^2 \times 5 = 16 \times 5 = 80$$
$$\left(10\sqrt{10}\right)^2 = 10^2 \times 10 = 100 \times 10 = 1000$$
$$\left(7\sqrt{3}\right)^2 = 7^2 \times 3 = 49 \times 3 = 147$$
$$\left(\frac{\sqrt{3}}{3}\right)^2 = \frac{3}{3^2} = \frac{1}{3}$$

$$\left(\frac{\sqrt{125}}{10}\right)^2 = \frac{125}{100} = \frac{5}{4}$$
$$\left(\frac{\sqrt{512}}{32}\right)^2 = \frac{512}{32^2} = \frac{2^9}{2^{10}} = \frac{1}{2}$$
$$\left(\frac{\sqrt{1779}}{1779}\right)^2 = \frac{1779}{1779^2} = \frac{1}{1779}$$

2.48: Simplifying before squaring

If possible, simplifying before squaring a radical expression usually reduces the calculations.

Example 2.49

The side length of a square is $\left(\sqrt{27} + \sqrt{108}\right)$. Find its area.

The area of the square will be:

$$\left(\sqrt{27} + \sqrt{108}\right)^2 = \left(3\sqrt{3} + 6\sqrt{3}\right)^2 = \left(9\sqrt{3}\right)^2 = 81 \times 3 = 243$$

Note that using the formula $(a+b)^2 = a^2 + 2ab + b^2$ leads to much longer calcualtions:

$$\left(\sqrt{27} + \sqrt{108}\right)^2 = \left(\sqrt{27}\right)^2 + 2\left(\sqrt{27}\right)\left(\sqrt{108}\right) + \left(\sqrt{108}\right)^2$$

Subsitute $2\left(\sqrt{27}\right)\left(\sqrt{108}\right) = 2\left(\sqrt{27}\right)\left(2\sqrt{27}\right) = 4 \times 27 = 108$:

$$= 27 + 108 + 108$$
$$= 243$$

Example 2.50

The radius of a circle is $\left(\sqrt{8} + \sqrt{50}\right)$. Find its area in terms of π.

$$\pi\left(\sqrt{8} + \sqrt{50}\right)^2 = \pi\left(2\sqrt{2} + 5\sqrt{2}\right)^2 = \pi\left(7\sqrt{2}\right)^2 = 98\pi$$

Example 2.51

Find the largest prime factor of

$$\left(\sqrt{12}+\sqrt{3}+\sqrt{27}\right)^2$$

Simplify before you square:

$$\left(2\sqrt{3}+\sqrt{3}+3\sqrt{3}\right)^2=\left(6\sqrt{3}\right)^2=36\times 3=2^2\times 3^3\Rightarrow Largest\ Prime\ Factor\ is\ 3$$

Example 2.52

Find $\frac{a^2}{4}$ if:

$$a=\sqrt{8}+\sqrt{32}+\sqrt{128}+\sqrt{512}+\sqrt{2048}$$

$$a=2\sqrt{2}+4\sqrt{2}+8\sqrt{2}+16\sqrt{2}+32\sqrt{2}=62\sqrt{2}$$

Then, the required expression is:

$$\frac{a^2}{4}=\frac{\left(62\sqrt{2}\right)^2}{2^2}=\left(31\sqrt{2}\right)^2=961(2)=1922$$

2.53: Identities

$$(a+b)^2=a^2+2ab+b^2$$
$$(a-b)^2=a^2-2ab+b^2$$
$$(a+b)(a-b)=a^2-b^2$$

Example 2.54

Expand:

A. $\left(4+\sqrt{3}\right)^2$
B. $\left(4-\sqrt{3}\right)^2$
C. $\left(4+\sqrt{3}\right)\left(4-\sqrt{3}\right)$
D. $\left(\sqrt{3}+2\right)^2$
E. $\left(5-\sqrt{7}\right)^2$
F. $\left(\sqrt{2}+\sqrt{5}\right)^2$

Use the expansion $(a+b)^2=a^2+2ab+b^2$:

$$\left(4+\sqrt{3}\right)^2=4^2+2(4)\left(\sqrt{3}\right)+\left(\sqrt{3}\right)^2=16+8\sqrt{3}+3=19+8\sqrt{3}$$

Use the expansion $(a-b)^2=a^2-2ab+b^2$:

$$\left(4-\sqrt{3}\right)^2=4^2-2\times 4\times\sqrt{3}+\left(\sqrt{3}\right)^2=19-8\sqrt{3}$$

Use the expansion $(a+b)(a-b)=a^2-b^2$:

$$\left(4+\sqrt{3}\right)\left(4-\sqrt{3}\right)=4^2-\left(\sqrt{3}\right)^2=16-9=7$$

$$\left(\sqrt{3}+2\right)^2=3+4+2\left(\sqrt{3}\right)(2)=7+4\sqrt{3}$$
$$\left(5-\sqrt{7}\right)^2=25+7-2(5)\sqrt{7}=32-10\sqrt{7}$$
$$\left(\sqrt{2}+\sqrt{5}\right)^2=2+5+2\sqrt{10}=7+2\sqrt{10}$$

2.55: Identities

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc)$$

Let $x = b + c$

$$(a+x)^2 = a^2 + 2ax + x^2$$

Substitute $x = b + c$:

$$= a^2 + 2a(b+c) + (b+c)^2$$
$$= a^2 + 2ab + 2ac + b^2 + 2bc + c^2$$

Rearrange:

$$= a^2 + b^2 + c^2 + 2(ab + ac + bc)$$

Example 2.56

A. $\left(\sqrt{2}+\sqrt{5}\right)^3$
B. $\left(\sqrt{2}+\sqrt{5}+\sqrt{7}\right)^2$

$$\left(\sqrt{2}\right)^3 = \left(2^{\frac{1}{2}}\right)^3 = 2^{\frac{1}{2}\times 3} = 2^{\frac{3}{2}}$$
$$\left(\sqrt{5}\right)^3 = \left(5^{\frac{1}{2}}\right)^3 = 5^{\frac{1}{2}\times 3} = 5^{\frac{3}{2}}$$
$$\left(\sqrt{2}+\sqrt{5}\right)^3 = 2^{\frac{3}{2}} + 3\times 2\sqrt{5} + 3\times\sqrt{2}\times 5 + 5^{\frac{3}{2}} = 2\sqrt{2} + 6\sqrt{5} + 15\sqrt{2} + 5\sqrt{5} = 17\sqrt{2} + 11\sqrt{5}$$

$$2+5+7+2\left(\sqrt{10}+\sqrt{14}+\sqrt{35}\right) = 14 + 2\left(\sqrt{10}+\sqrt{14}+\sqrt{35}\right)$$

Example 2.57: Expansions

A. $\left(3 - x\sqrt{5}\right)^2$

$$\left(3 - x\sqrt{5}\right)^2 = 9 - (2)(3)\left(x\sqrt{5}\right) + 5x^2 = 9 - 6x\sqrt{5} + 5x^2$$

Example 2.58

A. The side length of a square is $\left(\sqrt{7}-7\right)$. Find its area.
B. The radius of a circle is $\left(2\sqrt{5}-3\sqrt{7}\right)$ Find its area.
C. The side length of a cube is $\left(\sqrt{7}+\sqrt{3}\right)^3$. Find its volume.

$$7 + 49 - 14\sqrt{7} = 56 - 14\sqrt{7}$$
$$\left(2\sqrt{5}-3\sqrt{7}\right)^2 = 20 - (2)\left(2\sqrt{5}\right)\left(3\sqrt{7}\right) + 63 = 83 - 12\sqrt{35}$$

$$\left(\sqrt{7}\right)^3 = \left(7^{\frac{1}{2}}\right)^3 = 7^{\frac{1}{2}\times 3} = 7^{\frac{3}{2}}$$

$$\left(\sqrt{3}\right)^3 = \left(3^{\frac{1}{2}}\right)^3 = 3^{\frac{1}{2}\times 3} = 3^{\frac{3}{2}}$$

$$\left(\sqrt{7}+\sqrt{3}\right)^3 = 7^{\frac{3}{2}} + (3)(7)\left(\sqrt{3}\right) + (3)\left(\sqrt{7}\right)(3) + 3^{\frac{3}{2}} = 7^{\frac{3}{2}} + 21\sqrt{3} + 9\sqrt{7} + 3^{\frac{3}{2}}$$

Example 2.59: Nested Square Roots

A. $\sqrt{\sqrt{144}+\sqrt{169}}$
B. $\sqrt{\sqrt{81}+7}$
C. $\sqrt{\sqrt{25}-\sqrt{9}+\sqrt{\sqrt{25}+\sqrt{16}}+\sqrt{\sqrt{121}+\sqrt{25}}}$
D. $\sqrt{\frac{\sqrt{1600}+\sqrt{1681}}{\sqrt{225}-\sqrt{36}}}$
E. $\sqrt{10+\sqrt{25+\sqrt{108+\sqrt{154+\sqrt{225}}}}}$

$$\sqrt{\sqrt{144}+\sqrt{169}} = \sqrt{12+13} = \sqrt{25} = 5$$

$$\sqrt{\sqrt{81}+7} = \sqrt{9+7} = \sqrt{16} = 4$$

$$\sqrt{\sqrt{5-3}+\sqrt{5+4}+\sqrt{11+5}} = \sqrt{2+3+4} = 3$$

$$\sqrt{\frac{\sqrt{1600}+\sqrt{1681}}{\sqrt{225}-\sqrt{36}}} = \sqrt{\frac{40+41}{15-6}} = \sqrt{\frac{81}{9}} = \sqrt{9} = 3$$

$$\sqrt{10+\sqrt{25+\sqrt{108+\sqrt{169}}}} = \sqrt{10+\sqrt{25+\sqrt{121}}} = \sqrt{10+\sqrt{36}} = \sqrt{16} = 4$$

Example 2.60

A. One leg of a right triangle has length $\sqrt[4]{9}$ and the other leg has length $\sqrt[4]{16}$. Find the length of the hypotenuse.
B. The fourth power of $\sqrt{1+\sqrt{1+\sqrt{1}}}$ is **(AHSME 1970/1)**
C. Find the number of ordered pairs of integers (a, b) that satisfy: $\sqrt{\sqrt{a}+\sqrt{b}} = 10$

Part A

$$h^2 = \sqrt{9}+\sqrt{16} = 3+4 \Rightarrow h = \sqrt{7}$$

Part B

We simplify the innermost square root and then square:

$$\left(\sqrt{1+\sqrt{2}}\right)^4 = \left(1+\sqrt{2}\right)^2 = 3+2\sqrt{2}$$

Part C

$$\sqrt{a}+\sqrt{b} = 100$$

$$a = \{0^2, 1^2, 2^2, \dots, 100^2\}$$
$$b = \{100^2, \dots, 2^2, 1^2, 0^2\}$$

From the above, we get 101 pairs.

$$(a, b) = (0{,}10{,}000) \Rightarrow \sqrt{a} + \sqrt{b} = \sqrt{0} + \sqrt{10{,}000} = 100$$
$$(a, b) = (1, 99^2) \Rightarrow \sqrt{1} + \sqrt{99^2} = 1 + 99 = 100$$

F. Fractional Exponents

Example 2.61

Write in the form a^b, where a is a prime number.

$$\frac{\sqrt[5]{27}}{\sqrt[3]{81}}$$

$$\frac{(3^3)^{\frac{1}{5}}}{(3^4)^{\frac{1}{3}}} = \frac{3^{\frac{3}{5}}}{3^{\frac{4}{3}}} = 3^{\frac{3}{5}-\frac{4}{3}} = 3^{\frac{9}{15}-\frac{20}{15}} = 3^{-\frac{11}{15}}$$

Example 2.62

Write in the form a^b, where a is a prime number.

$$\frac{\sqrt[4]{8}}{\sqrt[2]{32}}$$

$$\frac{2^{\frac{3}{4}}}{2^{\frac{5}{2}}} = 2^{\frac{3}{4}-\frac{5}{2}} = 2^{\frac{3}{4}-\frac{10}{4}} = 2^{-\frac{7}{4}}$$

Example 2.63

Simplify and write as a single expression without a denominator:

$$\frac{\sqrt[x]{48}}{\sqrt[y]{45}}$$

$$\frac{(2^4 \cdot 3)^{\frac{1}{x}}}{(3^2 \cdot 5)^{\frac{1}{y}}} = \frac{2^{\frac{4}{x}} \cdot 3^{\frac{1}{x}}}{3^{\frac{2}{y}} \cdot 5^{\frac{1}{y}}} = 2^{\frac{4}{x}} \cdot 3^{\frac{1}{x}-\frac{2}{y}} \cdot 5^{-\frac{1}{y}}$$

Example 2.64: Fractional Exponents

Simplify

A. $\sqrt{\sqrt[3]{\sqrt{\frac{1}{4096}}}}$

B. $\sqrt{\frac{1}{2}\times\sqrt[3]{\frac{1}{2}\times\sqrt{\frac{1}{4096}}}}$

C. $\sqrt{\frac{1}{x}\times\sqrt[3]{\frac{1}{x}\times\sqrt{\frac{1}{x}}}}$

$$\left\{\left[\left(\frac{1}{2^{12}}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}\right\}^{\frac{1}{2}}=\left(\frac{1}{2^{12}}\right)^{\frac{1}{2}\times\frac{1}{3}\times\frac{1}{2}}=\left(\frac{1}{2^{12}}\right)^{\frac{1}{12}}=\frac{1}{2}$$

$$\left\{\frac{1}{2}\left[\frac{1}{2}\times\left(\frac{1}{2^{12}}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}\right\}^{\frac{1}{2}}=\left\{\frac{1}{2}\left[\frac{1}{2}\times\frac{1}{2^{6}}\right]^{\frac{1}{3}}\right\}^{\frac{1}{2}}=\left\{\frac{1}{2}\left[\frac{1}{2^{7}}\right]^{\frac{1}{3}}\right\}^{\frac{1}{2}}=\left\{\frac{1}{2}\times\frac{1}{2^{\frac{7}{3}}}\right\}^{\frac{1}{2}}=\left\{\frac{1}{2^{\frac{10}{3}}}\right\}^{\frac{1}{2}}=\frac{1}{2^{\frac{10}{6}}}=\frac{1}{2^{\frac{5}{3}}}$$

$$\left(\frac{1}{x}\left(\frac{1}{x}\left(\frac{1}{x}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}=\left(\frac{1}{x}\left(\frac{1}{x^{\frac{3}{2}}}\right)^{\frac{1}{3}}\right)^{\frac{1}{2}}=\left(\frac{1}{x}\times\frac{1}{x^{\frac{1}{2}}}\right)^{\frac{1}{2}}=\left(\frac{1}{x^{\frac{3}{2}}}\right)^{\frac{1}{2}}=\frac{1}{x^{\frac{3}{4}}}$$

Example 2.65: Square Roots of Radical Expressions

Find the square root of $9+2\sqrt{18}$

Simplify the radical to get:

$$9+6\sqrt{2}$$

Factor:

$$3(3+2\sqrt{2})$$

Focus on the expression inside the brackets. We want:

$$3+2\sqrt{2}=\left(a+b\sqrt{2}\right)^2$$

Expand the expression on the RHS:

$$3+2\sqrt{2}=a^2+2b^2+2\sqrt{2}ab$$

Equate coefficients.
The radical term on the left must equal the radical term on the right:

$$2\sqrt{2}=2\sqrt{2}ab$$
$$1=ab$$

Also, the whole number term on the left must equal the whole number term on the right:

$$3=a^2+2b^2$$

By observation, we see that:

$$a=b=1\ works$$

Hence, we get:

$$3(3+2\sqrt{2})=3(1+\sqrt{2})^2$$

And if we want the 3 inside the square sign, then we do:

$$(\sqrt{3})^2(1+\sqrt{2})^2=[(\sqrt{3})(1+\sqrt{2})]^2=[\sqrt{3}+\sqrt{6}]$$

Example 2.66: Square Roots of Radical Expressions

Find the square root of $15 + 6\sqrt{6}$ using integers.

$$15 + 6\sqrt{6} = \left(a + b\sqrt{6}\right)^2$$
$$15 + 6\sqrt{6} = a^2 + 2ab\sqrt{6} + 6b^2$$

$$2ab\sqrt{6} = 6\sqrt{6} \Rightarrow ab = 3$$

Try

$$a = 1, b = 3: \left(a + b\sqrt{6}\right)^2 = \left(1 + 3\sqrt{6}\right)^2 = 1 + 6\sqrt{6} + 9 \cdot 6 = 55 + 6\sqrt{6}$$
$$a = 3, b = 1: \left(a + b\sqrt{6}\right)^2 = \left(3 + \sqrt{6}\right)^2 = 9 + 6\sqrt{6} + 6 = 15 + 6\sqrt{6}$$

Example 2.67

If $x = 9 - 4\sqrt{5}$, then find $\sqrt{x} + \frac{1}{\sqrt{x}}$

$$\frac{1}{x} = \frac{1}{9 - 4\sqrt{5}} = \frac{1}{9 - 4\sqrt{5}} \cdot \frac{9 + 4\sqrt{5}}{9 + 4\sqrt{5}} = \frac{9 + 4\sqrt{5}}{81 - 80} = 9 + 4\sqrt{5}$$

$$x + \frac{1}{x} + 2 = 9 - 4\sqrt{5} + 9 + 4\sqrt{5} + 2 = 20$$
$$\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 = \sqrt{x + \frac{1}{x} + 2} = \sqrt{20} = 2\sqrt{5}$$

G. Nested Fractions

Example 2.68

$$\frac{\frac{3\sqrt{x}}{2x+5}}{\frac{\sqrt{2x+5}}{5x}}$$

Convert the fraction into division:

$$\frac{3\sqrt{x}}{2x+5} \div \frac{\sqrt{2x+5}}{5x}$$

Convert the division into multiplication by taking the reciprocal of the second term:

$$\frac{3\sqrt{x}}{2x+5} \times \frac{5x}{\sqrt{2x+5}}$$

Simplify:

$$\frac{15x^{\frac{3}{2}}}{(2x+5)^{\frac{3}{2}}} = 15\left(\frac{x}{2x+5}\right)^{\frac{3}{2}}$$

H. Even Powers and Cube Roots

Example 2.69

$$\sqrt[3]{(x-3)^8}$$

$$\frac{8}{3}=2\frac{2}{3}$$

$$=\sqrt[3]{(x-3)^6}\times\sqrt[3]{(x-3)^2}=(x-3)^2\cdot\sqrt[3]{(x-3)^2}$$

2.2 Square Roots of Variables

A. Roots of Variables

2.70: Square Roots with Variables

If you have a square root for a variable, the exponent becomes half

$$\sqrt{x^2}=x, x>0$$

Example 2.71

- Finding the square root of an expression halves the exponent. Hence $\sqrt{x^a}=x^{\frac{a}{2}}$

Roots with Variables

A. $\sqrt{x^2}$
B. $\sqrt{x^6}$
C. $\sqrt{x^4}$
D. $\sqrt{x^{10}}$
E. $\sqrt{x^{1000}}$
F. $\sqrt{x^y}$
G. $\sqrt{x^4y^6}$
H. $\sqrt{x^{10}y^{20}}$
I. $\sqrt{x^{100}y^{200}}$
J. $\sqrt{16x^6}$
K. $\sqrt{144x^8}$
L. $\sqrt{81x^{12}}$
M. $\sqrt{49x^{20}}$
N. $\sqrt{81x^6y^{10}}$
O. $\sqrt{144x^{58}y^{12}z^{24}}$

$$\sqrt{x^2}$$

Here, we have to create a square:

$$\sqrt{x^6}=\sqrt{(x^3)^2}=x^3$$

Again, we create a square:

$$\sqrt{x^4}=x^2$$

$$\sqrt{x^{10}}=\sqrt{(x^5)^2}=x^5$$

$$\sqrt{x^{1000}}=\sqrt{(x^{500})^2}=x^{500}$$

$$\sqrt{x^y}=\sqrt{\left(x^{\frac{y}{2}}\right)^2}=x^{\frac{y}{2}}$$

$$\sqrt{x^4y^6}=\sqrt{x^2y^3}$$

$$\sqrt{x^{10}y^{20}}=\sqrt{x^5y^{10}}$$

$$\sqrt{x^{50}y^{100}}$$

$$\sqrt{16x^6}=\sqrt{4x^3}$$

$$\sqrt{144x^8}=\sqrt{12x^4}$$

$$\sqrt{81x^{12}}=\sqrt{9x^6}$$

$$\sqrt{49x^{20}} = \sqrt{7x^{10}}$$
$$\sqrt{81x^6y^{10}} = \sqrt{9x^3y^5}$$
$$\sqrt{144x^{58}y^{12}z^{24}} = \sqrt{12x^{29}y^6z^{12}}$$

Example 2.72

A. $\sqrt{\frac{x^6}{y^{12}}}$
B. $\sqrt{\frac{x^{18}}{y^{48}}}$
C. $\sqrt{\frac{x^{1000}}{y^{2000}}}$
D. $\sqrt{\frac{a^{28}b^{58}}{c^{78}d^{98}}}$
E. $\sqrt{\frac{a^{76}b^{38}}{c^{96}d^{148}}}$
F. $\sqrt{\frac{49p^{52}}{121q^{78}}}$
G. $\sqrt{\frac{64p^{26}}{289q^{92}}}$
H. $\sqrt{\frac{196p^{1000}}{100q^{10000}}}$

$$\sqrt{\frac{169a^{26}b^{98}c^{44}}{289q^{78}r^{52}s^{92}}}$$

$$\sqrt{\frac{x^6}{y^{12}}} = \frac{x^3}{y^6}$$

$$\sqrt{\frac{49p^{52}}{121q^{78}}} = \frac{7p^{26}}{11q^{39}}$$

$$\sqrt{\frac{64p^{26}}{289q^{92}}}$$
$$\sqrt{\frac{196p^{1000}}{100q^{10000}}}$$
$$\sqrt{\frac{169a^{26}b^{98}c^{44}}{289q^{78}r^{52}s^{92}}} = -$$

Example 2.73: Variables

If a variable has:

- an even exponent, then you can obtain an integer square root by dividing the exponent by two.

$$\sqrt{y^{12}} = y^6$$

- a odd exponent, then you can rewrite the exponent to have an even exponent plus one, and then take the square root of the even exponent, while leaving the single one as a radical

$$\sqrt{x^5} = \sqrt{x^5} = \sqrt{x^4 \times x} = \sqrt{x^4} \times \sqrt{x} = x^2\sqrt{x}$$

A. $\sqrt{x^3}$
B. $\sqrt{x^{11}}$
C. $\sqrt{x^{15}}$
D. $\sqrt{y^{31}}$
E. $\sqrt{x^{101}}$
F. $\sqrt{x^{1001}}$
G. $\sqrt{x^{15}y^{31}}$
H. $\sqrt{x^{49}y^{99}}$
I. $\sqrt{za^{11}b^{101}c^{1001}}$
J. $\sqrt{72x^3}$
K. $\sqrt{27x^7y^8z}$
L. $\sqrt{\frac{81x^9y}{169z^{11}a}}$

$$\sqrt{x^3} = \sqrt{x^2 \times x} = \sqrt{x^2} \times \sqrt{x} = |x|\sqrt{x}$$
$$\sqrt{x^{11}} = \sqrt{x^{10}} \times \sqrt{x} = |x^5|\sqrt{x}$$
$$\sqrt{x^{15}} = \sqrt{x^{14}} \times \sqrt{x} = |x^7|\sqrt{x}$$
$$\sqrt{y^{31}} = \sqrt{y^{30}} \times \sqrt{y} = y^{15}\sqrt{x}$$
$$\sqrt{x^{101}} = \sqrt{x^{100}} \times \sqrt{x} = x^{50}\sqrt{x}$$
$$\sqrt{x^{1001}} = \sqrt{x^{1000}} \times \sqrt{x} = x^{500}\sqrt{x}$$
$$\sqrt{x^{15}y^{31}} = \sqrt{x^{15}} \times \sqrt{y^{31}} = x^7\sqrt{x} \times y^{15}\sqrt{y} = x^7y^{15}\sqrt{xy}$$
$$\sqrt{x^{49}y^{99}} = \sqrt{x^{49}} \times \sqrt{y^{99}} = x^{24}\sqrt{x} \times y^{49}\sqrt{y} = x^{24}y^{49}\sqrt{xy}$$
$$\sqrt{za^{11}b^{101}c^{1001}} = a^5b^{50}c^{500}\sqrt{zabc}$$

Separate out the numbers and the variables:

$$\sqrt{72x^3} = \sqrt{72} \times \sqrt{x^3} = \sqrt{36} \times \sqrt{2} \times \sqrt{x^2} \times \sqrt{x} = 6\sqrt{2} \times x\sqrt{x} = 6x\sqrt{2x}$$
$$\sqrt{27x^7y^8z} = \sqrt{9} \times \sqrt{3} \times \sqrt{x^6} \times \sqrt{x} \times \sqrt{y^8} \times \sqrt{z} = 3x^3y^4\sqrt{3xz}$$
$$\sqrt{\frac{81x^9y}{169z^{11}a}} = \frac{\sqrt{81} \times \sqrt{x^8} \times \sqrt{x} \times \sqrt{y}}{\sqrt{169} \times \sqrt{z^{10}} \times \sqrt{z} \times a} = \frac{9x^4\sqrt{xy}}{13z^5\sqrt{za}}$$

Example 2.74

A. $\sqrt[3]{16x^7y^8}$
B.

$$\sqrt[3]{(8 \times 2)(x^6 \times x)(y^6 \times y^2)} = 2x^2y^2\sqrt[3]{2xy^2}$$

Example 2.75: Simplifying Square Root Expressions

A. $-8\sqrt{90x^5} - 3x\sqrt{10x^3}$
B.

$$-8\sqrt{9\times 10\times x^4\times x}-3x\sqrt{10x^2\times x}$$
$$-24x^2\sqrt{10x}-3x^2\sqrt{10x}$$
$$-27x^2\sqrt{10x}$$

2.3 Rationalization

A. Rationalizing Radicals

When writing answers, we often require that the denominator of a fraction should not have radicals.

2.76: Why Rationalize

Example 2.77

In each parts, does the denominator have a radical.

A. $\frac{3}{4}$
B. $\frac{3}{\sqrt{16}}$

$$No$$
$$Yes$$

2.78: Rationalization of Radicals

The process of eliminating radicals is called rationalization. As discussed earlier, rationalization is often needed for denominators.

Given a radical $\sqrt{a}$ in the denominator, the rationalizing factor is also $\sqrt{a}$.

$$\frac{1}{\sqrt{a}}\times\frac{\sqrt{a}}{\sqrt{a}}=\frac{\sqrt{a}}{a}$$

If we do not want a radical in the denominator of a fraction, we multiply the fraction with a form of one that will rationalize the denominator.

Here, $\frac{\sqrt{a}}{\sqrt{a}}$ is the called rationalizing factor.

$$\frac{3}{\sqrt{3}}\times=\frac{3\sqrt{3}}{3}=\sqrt{3}$$

Example 2.79

A. $\frac{5}{\sqrt{5}}$
B. $\frac{21}{\sqrt{7}}$
C. $\frac{12}{\sqrt{3}}$
D. $\frac{343}{\sqrt{7}}$

$$\frac{5}{\sqrt{5}}=\frac{5}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}=\sqrt{5}$$

$$\frac{21}{\sqrt{7}}\cdot\frac{\sqrt{7}}{\sqrt{7}}=\frac{21\sqrt{7}}{7}=3\sqrt{7}$$
$$\frac{12}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{12\sqrt{3}}{3}=4\sqrt{3}$$
$$\frac{343}{\sqrt{7}}=\frac{7^3\sqrt{7}}{7}=7^2\sqrt{7}=49\sqrt{7}$$

Example 2.80

In each part below, simplify and rationalize the denominator:

A. $\sqrt{\frac{81}{7}}$
B. $\sqrt{\frac{\sqrt{81}}{7}}$
C. $\sqrt{\frac{\sqrt{81}}{\sqrt{25}}}$
D. $\sqrt{\frac{\sqrt{3}\times\sqrt{12}}{\sqrt{5}\times\sqrt{20}}}$
E. $\frac{1}{\sqrt{27}+\sqrt{12}+\sqrt{75}}$

$$\sqrt{\frac{81}{7}}=\frac{\sqrt{81}}{\sqrt{7}}=\frac{9}{\sqrt{7}}\cdot\frac{\sqrt{7}}{\sqrt{7}}=\frac{9\sqrt{7}}{7}$$
$$\sqrt{\frac{\sqrt{81}}{7}}=\frac{\sqrt{9}}{\sqrt{7}}=\frac{3\sqrt{7}}{7}$$
$$\sqrt{\frac{\sqrt{81}}{\sqrt{25}}}=\sqrt{\frac{9}{5}}=\frac{3}{\sqrt{5}}=\frac{3\sqrt{5}}{5}$$
$$\sqrt{\frac{\sqrt{3}\times\sqrt{12}}{\sqrt{5}\times\sqrt{20}}}=\sqrt{\frac{\sqrt{36}}{\sqrt{100}}}=\sqrt{\frac{6}{10}}=\sqrt{\frac{3}{5}}=\frac{\sqrt{3}}{\sqrt{5}}=\frac{\sqrt{15}}{5}$$
$$\sqrt{\frac{\sqrt{3}\times\sqrt{12}}{\sqrt{5}\times\sqrt{20}}}=\sqrt{\frac{\sqrt{3}\times\sqrt{3}}{\sqrt{5}\times\sqrt{5}}}=\sqrt{\frac{3}{5}}=\frac{\sqrt{3}}{\sqrt{5}}=\frac{\sqrt{15}}{5}$$
$$\frac{1}{\sqrt{27}+\sqrt{12}+\sqrt{75}}=\frac{1}{3\sqrt{3}+2\sqrt{3}+5\sqrt{3}}=\frac{1}{10\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{3}}{30}$$

Example 2.81

A. Rationalize $\frac{2}{\sqrt{3}}$ and write your answer in the form $\frac{a\sqrt{b}}{c}$ where $HCF(a,c)=1$ and $\sqrt{b}$ is a radical written in simplified form. Find $\frac{a^b}{c}$
B. The ratio of the area of two squares is $1:3$. Find the ratio of their side lengths as a fraction.
C. The ratio of the area of two squares is $36:5$. Find the ratio of their side lengths as a fraction.

Part A

$$\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}\Rightarrow a=3,b=3,c=3\Rightarrow\frac{a^b}{c}=\frac{3^3}{3}=3^2=9$$

Part B

To find the ratio of side lengths, we take the square root of the fraction $\frac{1}{3}$:

$$=\frac{1}{\sqrt{3}}=\frac{1}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$

Part C

To find the ratio of side lengths, we take the square root of the fraction $\frac{36}{5}$:

$$=\sqrt{\frac{36}{5}}=\frac{\sqrt{36}}{\sqrt{5}}=\frac{6}{\sqrt{5}}=\frac{6\sqrt{5}}{5}$$

B. Conjugate Surds

The technique of conjugate surds is especially useful when rationalizing an expression with two terms, one or more of which have radicals. It is useful in many other chapters as well.

Example 2.82

Carry out the multiplication in each part below:

A. $(\sqrt{2}+\sqrt{5})(\sqrt{2}-\sqrt{5})$
B. $(\sqrt{5}+\sqrt{7})(\sqrt{5}-\sqrt{7})$
C. $(\sqrt{13}+\sqrt{3})(\sqrt{13}-\sqrt{3})$

Part A

Use $(a+b)(a-b)=a^2-b^2$:

$$\left(\underbrace{\sqrt{2}}_{a}+\underbrace{\sqrt{5}}_{b}\right)(\sqrt{2}-\sqrt{5})=\underbrace{\left(\sqrt{2}\right)^2}_{a^2}-\underbrace{\left(\sqrt{5}\right)^2}_{b^2}=2-5=-3$$

Parts B-C

$$(\sqrt{5}+\sqrt{7})(\sqrt{5}-\sqrt{7})=\left(\sqrt{5}\right)^2-\left(\sqrt{7}\right)^2=5-7=-2$$
$$(\sqrt{13}+\sqrt{3})(\sqrt{13}-\sqrt{3})=13-3=10$$

2.83: Conjugate Surds for Rationalization

When we have two surds, we rationalize by multiplying by its conjugate surd:

$$\frac{1}{\sqrt{a}+\sqrt{b}}=\frac{1}{\sqrt{a}+\sqrt{b}}\times\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{\sqrt{a}-\sqrt{b}}{a-b}$$

Example 2.84

Rationalize the denominator:

A. $\frac{1}{\sqrt{3}+\sqrt{7}}$
B. $\frac{1}{\sqrt{5}-\sqrt{2}}$
C. $\frac{2}{\sqrt{7}-\sqrt{5}}$
D. $\frac{\sqrt{2}}{\sqrt{7}+\sqrt{2}}$
E. $\frac{3\sqrt{5}}{2\sqrt{5}-3\sqrt{7}}$
F. $\frac{1}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{5})}$
G. $\frac{1}{\sqrt{27}+\sqrt{45}+\sqrt{80}+\sqrt{75}}$

Part A

We want to write the larger number first, so that after rationalization we get a positive number in both the numerator and the denominator:

$$\frac{1}{\sqrt{7}+\sqrt{3}}\times\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}-\sqrt{3}}=\frac{\sqrt{7}-\sqrt{3}}{7-3}=\frac{\sqrt{7}-\sqrt{3}}{4}$$

Part B

This is like the first one, except the denominator has a minus sign. So, the conjugate will have a plus sign.

$$\frac{1}{\sqrt{5}-\sqrt{2}}\times\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}=\frac{\sqrt{5}+\sqrt{2}}{5-2}=\frac{\sqrt{5}+\sqrt{2}}{3}$$

Part C

$$\frac{2}{\sqrt{7}-\sqrt{5}}\times\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}+\sqrt{5}}=\frac{2\left(\sqrt{7}+\sqrt{5}\right)}{7-5}=\frac{2\left(\sqrt{7}+\sqrt{5}\right)}{2}=\sqrt{7}+\sqrt{5}$$

Part D

$$\frac{\sqrt{2}}{\sqrt{7}+\sqrt{2}}\times\frac{\sqrt{7}-\sqrt{2}}{\sqrt{7}-\sqrt{2}}=\frac{\sqrt{14}-2}{7-2}=\frac{\sqrt{14}-2}{5}$$

Part E

$$\frac{3\sqrt{5}}{2\sqrt{5}-3\sqrt{7}}\times\frac{2\sqrt{5}+3\sqrt{7}}{2\sqrt{5}+3\sqrt{7}}=\frac{30+9\sqrt{35}}{20-63}=-\frac{30+9\sqrt{35}}{43}$$

Part F

It is not a good idea to multiply the expressions in the denominator. That will give a radical with more than two terms, and those are more difficult to rationalize.

$$\frac{1}{\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{5}\right)}\times\frac{\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{5}\right)}{\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{5}\right)}=\frac{3+\sqrt{15}-\sqrt{6}-\sqrt{10}}{(3-2)(3-5)}=\frac{\sqrt{6}+\sqrt{10}-\sqrt{15}-3}{2}$$

Part G

First, we simplify the denominator:

$$\frac{1}{\sqrt{27}+\sqrt{45}+\sqrt{80}+\sqrt{75}}=\frac{1}{3\sqrt{3}+3\sqrt{5}+4\sqrt{5}+5\sqrt{3}}=\frac{1}{8\sqrt{3}+7\sqrt{5}}$$

Multiply by the conjugate of the denominator:

$$\frac{1}{8\sqrt{3}+7\sqrt{5}}\times\frac{8\sqrt{3}-7\sqrt{5}}{8\sqrt{3}-7\sqrt{5}}=\frac{8\sqrt{3}-7\sqrt{5}}{64\times 3-49\times 5}=\frac{8\sqrt{3}-7\sqrt{5}}{192-245}=\frac{8\sqrt{3}-7\sqrt{5}}{-53}=\frac{7\sqrt{5}-8\sqrt{3}}{53}$$

Example 2.85

The area of a triangle is $\sqrt{7}+\sqrt{5}$, and its height is given by $2\sqrt{7}-2\sqrt{5}$. Find the length of its base.

$$\frac{1}{2}hb=A\Rightarrow\frac{1}{2}\left(2\sqrt{7}-2\sqrt{5}\right)b=\sqrt{7}+\sqrt{5}\Rightarrow b=\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}}$$

We need to rationalize the denominator:

$$\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}}\times\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}+\sqrt{5}}=\frac{7+(2)\left(\sqrt{5}\right)\left(\sqrt{7}\right)+5}{7-5}=\frac{12+2\sqrt{35}}{2}=6+\sqrt{35}$$

C. Variables

Example 2.86

Rationalize:

A. $\frac{1}{\sqrt{p}+\sqrt{q}}$

B. $\frac{1}{\sqrt{x}-\sqrt{y}}$
C. $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$
D. $\frac{1}{\sqrt{a+1}-\sqrt{a}}$
E. $\frac{1}{\sqrt{a+k}-\sqrt{a+k-1}}$

Part A

$$\frac{1}{\sqrt{p}+\sqrt{q}} \times \frac{\sqrt{p}-\sqrt{q}}{\sqrt{p}-\sqrt{q}} = \frac{\sqrt{p}-\sqrt{q}}{p-q}$$

Part B

$$\frac{1}{\sqrt{x}-\sqrt{y}} \times \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \frac{\sqrt{x}+\sqrt{y}}{x-y}$$

Part C

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} \times \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{a+b+2\sqrt{ab}}{a-b}$$

Part D

$$\frac{1}{\sqrt{a+1}-\sqrt{a}} \times \frac{\sqrt{a+1}+\sqrt{a}}{\sqrt{a+1}+\sqrt{a}} = \frac{\sqrt{a+1}+\sqrt{a}}{(a+1)-a} = \sqrt{a+1}+\sqrt{a}$$

Part E

$$\frac{1}{\sqrt{a+k}-\sqrt{a+k-1}} \times \frac{\sqrt{a+k}+\sqrt{a+k-1}}{\sqrt{a+k}+\sqrt{a+k-1}} = \frac{\sqrt{a+k}+\sqrt{a+k-1}}{(a+k)-(a+k-1)} = \sqrt{a+k}+\sqrt{a+k-1}$$

D. Rationalizing the Numerator

The same techniques of rationalizing the denominator will work rationalizing the numerator.

Example 2.87

After rationalizing the numerator of $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}}$, the denominator in simplest form is: **(AHSME 1950/10)**

$$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}} = \frac{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}{\sqrt{3}(\sqrt{3}+\sqrt{2})} = \frac{1}{3+\sqrt{6}} \Rightarrow Denominator = 3+\sqrt{6}$$

Note: It was not necessary to work out the numerator, since only the denominator was asked.

E. Three Part Expressions

Example 2.88

Rationalize:

$$\frac{1}{1+\sqrt{2}-\sqrt{5}}$$

$$\frac{1}{1+\sqrt{2}-\sqrt{5}} \times \frac{1+\sqrt{2}+\sqrt{5}}{1+\sqrt{2}+\sqrt{5}}$$

The denominator of the above is given by:

$$\left(1+\sqrt{2}\right)^2 - \left(\sqrt{5}\right)^2 = 1+2+2\sqrt{2}-5 = 2\sqrt{2}-2$$

Hence, we get:

$$\frac{1+\sqrt{2}+\sqrt{5}}{2\sqrt{2}-2}$$

This is now the standard type that we already know how to rationalize.:

$$\frac{1+\sqrt{2}+\sqrt{5}}{2\sqrt{2}-2}\times\frac{2\sqrt{2}+2}{2\sqrt{2}+2}$$

$$\frac{(1+\sqrt{2}+\sqrt{5})(2\sqrt{2}+2)}{8-4}=\frac{(1+\sqrt{2}+\sqrt{5})(\sqrt{2}+1)}{2}$$

F. Products

If you need to find a product, and the denominator is not rational, you can sometimes reduce your work by rationalizing the denominator before multiplying. This is a matter of judgement and comes only after practice.

Example 2.89

A car travels at a speed of $\frac{1}{\sqrt{3}-\sqrt{2}}$ miles per hour for $\frac{1}{\sqrt{3}+\sqrt{11}}$ hours. Find the distance travelled.

$$Speed=S=\frac{1}{\sqrt{3}-\sqrt{2}}\times\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\frac{\sqrt{3}+\sqrt{2}}{3-2}=\sqrt{3}+\sqrt{2}$$

$$Time=T=\frac{1}{\sqrt{3}+\sqrt{11}}\times\frac{\sqrt{3}-\sqrt{11}}{\sqrt{3}-\sqrt{11}}=\frac{\sqrt{3}-\sqrt{11}}{3-11}=\frac{\sqrt{11}-\sqrt{3}}{8}$$

And we can now find the distance as

$$D=\underbrace{(\sqrt{3}+\sqrt{2})}_{Speed}\underbrace{\left(\frac{\sqrt{11}-\sqrt{3}}{8}\right)}_{Time}=\frac{\sqrt{33}-3+\sqrt{22}-\sqrt{6}}{8}=\frac{\sqrt{33}+\sqrt{22}-\sqrt{6}-3}{8}$$

Example 2.90

$$x=\frac{\sqrt{2}+\sqrt{5}}{\sqrt{2}-\sqrt{5}},\qquad y=\frac{\sqrt{3}-\sqrt{11}}{\sqrt{3}+\sqrt{11}}$$

A. Rationalize the denominator of x.
B. Rationalize the denominator of y.
C. Find xy, and state your final answer so that the denominator is rational. Is it better to use the rationalized versions, or the original versions?

Part A

$$\frac{\sqrt{2}+\sqrt{5}}{\sqrt{2}-\sqrt{5}}\times\frac{\sqrt{2}+\sqrt{5}}{\sqrt{2}+\sqrt{5}}=\frac{2+2\sqrt{10}+5}{2-5}=-\frac{7+2\sqrt{10}}{3}$$

Part B

$$\frac{\sqrt{3}-\sqrt{11}}{\sqrt{3}+\sqrt{11}}\times\frac{\sqrt{3}-\sqrt{11}}{\sqrt{3}-\sqrt{11}}=\frac{3-2\sqrt{33}+11}{3-11}=\frac{14-2\sqrt{33}}{-8}=\frac{\sqrt{33}-7}{4}$$

Part C

Multiply the two fractions:

$$=\frac{98-14\sqrt{33}+28\sqrt{10}-4\sqrt{330}}{24}=\frac{49-7\sqrt{33}+7\sqrt{10}-2\sqrt{330}}{24}$$

Example 2.91

$$x=\frac{\sqrt{a}+\sqrt{a+1}}{\sqrt{a}-\sqrt{a+1}},\qquad y=\frac{\sqrt{a+1}+\sqrt{a+2}}{\sqrt{a+1}-\sqrt{a+2}}$$

A. Rationalize the denominator of x.
B. Rationalize the denominator of y.
C. Find xy, and state your final answer so that the denominator is rational. Is it better to use the rationalized versions, or the original versions?

$$x=\frac{\sqrt{a}+\sqrt{a+1}}{\sqrt{a}-\sqrt{a+1}}\times\frac{\sqrt{a}+\sqrt{a+1}}{\sqrt{a}+\sqrt{a+1}}=\frac{a+(a+1)+2\sqrt{a(a+1)}}{a-(a+1)}=-\left[(2a+1)+2\sqrt{a(a+1)}\right]$$

$$y=\frac{\sqrt{a+1}+\sqrt{a+2}}{\sqrt{a+1}-\sqrt{a+2}}\times\frac{\sqrt{a+1}+\sqrt{a+2}}{\sqrt{a+1}+\sqrt{a+2}}=\frac{(a+1)+(a+2)+2\sqrt{(a+1)(a+2)}}{(a+1)-(a+2)}$$
$$=-\left[(2a+3)+2\sqrt{(a+1)(a+2)}\right]$$

$$xy=(2a+1)(2a+3)+2(2a+1)\sqrt{(a+1)(a+2)}+2(2a+3)\sqrt{a(a+1)}+4\sqrt{a(a+1)^2(a+2)}$$

G. Sums

Example 2.92

Simplify:

$$\frac{1}{\sqrt{2}-\sqrt{1}}+\frac{1}{\sqrt{4}-\sqrt{3}}$$

$$=\frac{\sqrt{2}+\sqrt{1}}{2-1}+\frac{\sqrt{4}+\sqrt{3}}{4-3}=\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}$$

Example 2.93

A. Tap A can fill a tank in 5 hours. Tap B can fill the same tank in 7 hours. If both taps are opened at the same time, in how much time will they fill the tank?
B. Tap A can fill a tank in $\sqrt{7}+\sqrt{10}$ hours. Tap B can fill the same tank in $\sqrt{10}+\sqrt{13}$ hours. If both taps are opened at the same time, in how much time will they fill the tank?

Part A

Every hour, tap A fills

$$\frac{1}{5}th\ of\ the\ tank$$

Every hour, tap B fills

$$\frac{1}{7}th\ of\ the\ tank$$

When both are opened at the same time, then in one hour they will fill:

$$\frac{1}{5}+\frac{1}{7}=\frac{7+5}{35}=\frac{12}{35}$$

And hence, they will need:

$$\frac{35}{12} hours\ to\ fill\ the\ tank$$

Part B

Every hour, tap A fills

$$\frac{1}{\sqrt{7}+\sqrt{10}} th\ of\ the\ tank$$

Every hour, tap B fills

$$\frac{1}{\sqrt{10}+\sqrt{13}} th\ of\ the\ tank$$

When both are opened at the same time, then in one hour they will fill:

$$\frac{1}{\sqrt{7}+\sqrt{10}}+\frac{1}{\sqrt{10}+\sqrt{13}}=\frac{\sqrt{10}-\sqrt{7}}{3}+\frac{\sqrt{13}-\sqrt{10}}{3}=\frac{\sqrt{13}-\sqrt{7}}{3}$$

And hence, they will need:

$$\frac{3}{\sqrt{13}-\sqrt{7}}=\frac{3\left(\sqrt{13}+\sqrt{7}\right)}{13-7}=\frac{\sqrt{13}+\sqrt{7}}{2}\ hours\ to\ fill\ the\ tank$$

Example 2.94

$$x=\frac{1}{\sqrt{a+1}+\sqrt{a}},\qquad y=\frac{1}{\sqrt{a+2}+\sqrt{a+1}}$$

A. Rationalize the denominator of x.
B. Rationalize the denominator of y.
C. Find $x+y$

H. LCM

Example 2.95

$$\frac{1}{\sqrt{3}+\sqrt{5}}+\frac{1}{\sqrt{3}-\sqrt{5}}$$

$$\frac{1}{\sqrt{3}+\sqrt{5}}\times\frac{\sqrt{3}-\sqrt{5}}{\sqrt{3}-\sqrt{5}}+\frac{1}{\sqrt{3}-\sqrt{5}}\times\frac{\sqrt{3}+\sqrt{5}}{\sqrt{3}+\sqrt{5}}$$

$$\frac{\sqrt{3}-\sqrt{5}}{3-5}+\frac{\sqrt{3}+\sqrt{5}}{3-5}=\frac{2\sqrt{3}}{-2}=-\sqrt{3}$$

Example 2.96

Show that:

$$\frac{1}{\sqrt{a}+\sqrt{b}}+\frac{1}{\sqrt{a}-\sqrt{b}}=\frac{2\sqrt{a}}{a-b}$$

$$\frac{1}{\sqrt{a}+\sqrt{b}}\times\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}+\frac{1}{\sqrt{a}-\sqrt{b}}\times\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}$$
$$\frac{\sqrt{a}-\sqrt{b}}{a-b}+\frac{\sqrt{a}+\sqrt{b}}{a-b}=\frac{2\sqrt{a}}{a-b}$$

I. Complex Numbers

2.97: Definition

A complex number is a number of the form

$$a+bi,\qquad i=\sqrt{-1},\qquad i^2=-1$$

Example 2.98

Show, using multiplication by conjugate of the denominator that:

$$\frac{1}{a+bi}=\frac{a-bi}{a^2+b^2}$$

$$\frac{1}{a+bi}\times\frac{a-bi}{a-bi}=\frac{a-bi}{a^2-b^2i^2}=\frac{a-bi}{a^2+b^2}$$

J. Series

Example 2.99

$$x=\frac{1}{\sqrt{2}-\sqrt{1}},\qquad y=\frac{1}{\sqrt{4}-\sqrt{3}},\qquad z=\frac{1}{\sqrt{6}-\sqrt{5}}$$

If x, y and z are the sides of a triangle then determine the semi-perimeter of the triangle.

$$\frac{1}{\sqrt{2}-\sqrt{1}}=\frac{\sqrt{2}+\sqrt{1}}{2-1}=\sqrt{2}+\sqrt{1}$$
$$\frac{1}{\sqrt{4}-\sqrt{3}}=\frac{\sqrt{4}+\sqrt{3}}{4-3}=\sqrt{4}+\sqrt{3}$$
$$\frac{1}{\sqrt{6}-\sqrt{5}}=\frac{\sqrt{6}+\sqrt{5}}{6-5}=\sqrt{6}+\sqrt{5}$$

$$\frac{1}{2}\left(\frac{1}{\sqrt{2}-\sqrt{1}}+\frac{1}{\sqrt{4}-\sqrt{3}}+\frac{1}{\sqrt{6}-\sqrt{5}}\right)=\frac{1}{2}\left(\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{5}+\sqrt{6}\right)$$

Example 2.100

Find the sum to n terms of:

$$\frac{1}{\sqrt{2}-\sqrt{1}}+\frac{1}{\sqrt{4}-\sqrt{3}}+\frac{1}{\sqrt{6}-\sqrt{5}}+\cdots$$

$$\underbrace{\frac{1}{\sqrt{2}-\sqrt{1}}}_{1st\ Term}+\underbrace{\frac{1}{\sqrt{4}-\sqrt{3}}}_{2nd\ Term}+\underbrace{\frac{1}{\sqrt{6}-\sqrt{5}}}_{3rd\ Term}+\cdots+\underbrace{\frac{1}{\sqrt{2n}-\sqrt{2n-1}}}_{n^{th}\ Term}$$

Rationalizing each term in the above expression:

$$\frac{1}{\sqrt{2}-\sqrt{1}} = \frac{\sqrt{2}+\sqrt{1}}{2-1} = \sqrt{2}+\sqrt{1}$$
$$\frac{1}{\sqrt{4}-\sqrt{3}} = \frac{\sqrt{4}+\sqrt{3}}{4-3} = \sqrt{4}+\sqrt{3}$$
$$\frac{1}{\sqrt{6}-\sqrt{5}} = \frac{\sqrt{6}+\sqrt{5}}{6-5} = \sqrt{6}+\sqrt{5}$$
$$\vdots$$
$$\frac{1}{\sqrt{2n}\ -\sqrt{2n-1}} = \frac{\sqrt{2n}+\sqrt{2n-1}}{(2n)-(2n-1)} = \sqrt{2n}+\sqrt{2n-1}$$

Bring it all together to get:

$$= \sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{2n}$$

Example 2.101

$$\frac{1}{\sqrt{2}-\sqrt{1}}+\frac{1}{\sqrt{4}-\sqrt{3}}+\frac{1}{\sqrt{6}-\sqrt{5}}+\cdots$$

Find the value of n given that 50 terms of the expression above are added, the denominator is rationalized, and the expression is written in the form

$$a\sqrt{b}+c\sqrt{d}+\cdots+n, n \in \mathbb{N}$$

$$= \sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{100} \Rightarrow 100\ Terms$$

All perfect squares will be replaced by numbers.

$$\sqrt{1} = 1, \sqrt{4} = 2, \dots, \sqrt{100} = 10$$

The value of n will be:

$$1+2+3+\cdots+10 = \frac{10 \times 11}{2} = 55$$

2.102: Telescoping

Telescoping is where the middle terms "cancel", and we are left with the first and the last term.[3]

Example 2.103

Simplify:

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{8}+\sqrt{9}}$$

Multiply each term by the *conjugate surd* of the denominator:

$$\frac{1}{\sqrt{1}+\sqrt{2}}\cdot\frac{\sqrt{2}-\sqrt{1}}{\sqrt{2}-\sqrt{1}}+\frac{1}{\sqrt{2}+\sqrt{3}}\cdot\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}+\cdots+\frac{1}{\sqrt{8}+\sqrt{9}}\cdot\frac{\sqrt{9}-\sqrt{8}}{\sqrt{9}-\sqrt{8}}$$

Use the formula $(a+b)(a-b) = a^2-b^2$ in the denominator, and then telescope:

[3] For greater detail on the concept of Telescoping, and how we can use it to add different kinds of series, you can see the Note on Sequences and Series.

$$\frac{\sqrt{2}-\sqrt{1}}{1}+\frac{\sqrt{3}-\sqrt{2}}{1}+\frac{\sqrt{4}-\sqrt{3}}{1}+\cdots+\frac{\sqrt{9}-\sqrt{8}}{1}=\frac{\sqrt{9}-\sqrt{1}}{1}=\frac{3-1}{1}=\frac{2}{1}=2$$

Example 2.104

Find the reciprocal of:

$$\frac{1}{\sqrt{4}+\sqrt{7}}+\frac{1}{\sqrt{7}+\sqrt{10}}+\frac{1}{\sqrt{10}+\sqrt{13}}+\frac{1}{\sqrt{13}+\sqrt{16}}$$

$$=\frac{\sqrt{7}-\sqrt{4}}{3}+\frac{\sqrt{10}-\sqrt{7}}{3}+\frac{\sqrt{13}-\sqrt{10}}{3}+\frac{\sqrt{16}-\sqrt{13}}{3}=\frac{\sqrt{16}-\sqrt{4}}{3}=\frac{4-2}{3}=\frac{2}{3}$$

The reciprocal is

$$\frac{3}{2}$$

Example 2.105

Find the sum to n terms of

$$\frac{1}{\sqrt{7}+\sqrt{10}}+\frac{1}{\sqrt{10}+\sqrt{13}}+\frac{1}{\sqrt{13}+\sqrt{16}}+\cdots$$

Rewrite the series to include the n^{th} term:

$$\frac{1}{\sqrt{7}+\sqrt{10}}+\frac{1}{\sqrt{10}+\sqrt{13}}+\frac{1}{\sqrt{13}+\sqrt{16}}+\cdots+\frac{1}{\sqrt{4+3n}+\sqrt{7+3n}}$$

Rationalize the denominators and then telescope it:

$$=\frac{\sqrt{10}-\sqrt{7}}{3}+\frac{\sqrt{13}-\sqrt{10}}{3}+\frac{\sqrt{16}-\sqrt{13}}{3}+\cdots+\frac{\sqrt{7+3n}-\sqrt{4+3n}}{3}=\frac{\sqrt{7+3n}-\sqrt{7}}{3}$$

Example 2.106

$$\frac{1}{\sqrt{a+1}+\sqrt{a}}+\frac{1}{\sqrt{a+2}+\sqrt{a+1}}+\frac{1}{\sqrt{a+3}+\sqrt{a+2}}$$

Rationalize the denominators:

$$\frac{\sqrt{a+1}-\sqrt{a}}{(a+1)-a}+\frac{\sqrt{a+2}-\sqrt{a+1}}{(a+2)-(a+1)}+\frac{\sqrt{a+3}-\sqrt{a+2}}{(a+3)-(a+2)}$$

Telescope:

$$=\sqrt{a+3}-\sqrt{a}$$

Example 2.107

Find the sum to n terms of:

$$\frac{1}{\sqrt{a+1}+\sqrt{a}}+\frac{1}{\sqrt{a+2}+\sqrt{a+1}}+\cdots$$

Write the series to include the n^{th} term:

$$\frac{1}{\sqrt{a+1}+\sqrt{a}}+\frac{1}{\sqrt{a+2}+\sqrt{a+1}}+\cdots+\frac{1}{\sqrt{a+n}+\sqrt{a+n-1}}$$

Rationalize the denominators:

$$\frac{\sqrt{a+1}-\sqrt{a}}{(a+1)-a}+\frac{\sqrt{a+2}-\sqrt{a+1}}{(a+2)-(a+1)}+\cdots+\frac{\sqrt{a+n}-\sqrt{a+n-1}}{(a+n)-(a+n-1)}$$

Telescope:

$$=\sqrt{a+n}-\sqrt{a}$$

K. Rationalizing n^{th} Roots

2.108: Rationalizing Factor

If we rationalize $\frac{1}{\sqrt{3}}$, we multiply the fraction by $\frac{\sqrt{3}}{\sqrt{3}}$. This number is called the rationalizing factor.

2.109: Rationalizing cube roots

Example 2.110

In all parts below, simplify and rationalize the denominator:

A. $\frac{1}{\sqrt[3]{9}}$
B. $\frac{1}{\sqrt[3]{x}}$
C. $\frac{5}{\sqrt[3]{5}}$
D. $\frac{14}{\sqrt[3]{7}}$
E. $\frac{y}{\sqrt[3]{y}}$
F. $\frac{1}{\sqrt[3]{5^2}}$
G. $\frac{1}{\sqrt[3]{7^4}}$
H. $\frac{1}{\sqrt[3]{x^5}}$

2.111: Rationalizing n^{th} roots

The rationalizing factor for

$$\sqrt[n]{x}=x^{\frac{1}{n}}$$

Is

$$x^{\frac{n-1}{n}}=\sqrt[n]{x^{n-1}}$$

L. Addition and Subtraction

Example 2.112

$$\frac{1}{\sqrt[3]{4}}+\frac{1}{\sqrt[3]{7}}$$

$$\frac{1}{\sqrt[3]{4}}=\frac{\sqrt[3]{4^2}}{\sqrt[3]{4}\times\sqrt[3]{4^2}}=\frac{\sqrt[3]{16}}{4}=\frac{2\sqrt[3]{2}}{4}=\frac{\sqrt[3]{2}}{2}$$

$$\frac{1}{\sqrt[3]{7}}=\frac{\sqrt[3]{7^2}}{\sqrt[3]{7}\times\sqrt[3]{7^2}}=\frac{\sqrt[3]{49}}{7}=\frac{\sqrt[3]{49}}{7}$$

$$\frac{\sqrt[3]{2}}{2}+\frac{\sqrt[3]{49}}{7}=\frac{7\sqrt[3]{2}+2\sqrt[3]{49}}{14}$$

2.4 Radical Equations and Inequalities

A. Basics

Suppose you are given the equation $x^2 = 4$ and you solve it by taking square roots both sides to get

$$x^2 = 4 \Rightarrow x = \sqrt{4} \Rightarrow x = 2$$

This is correct.

But notice, that

$$(-2)^2 = 4$$

Is also a valid solution.

In order to not miss solutions, you need to consider both positive and negative values when taking the square root.

2.113: Square Root Sign

The square root operator always gives a result that is greater than or equal to zero.
For any real number x:

$$x > 0 \Rightarrow \sqrt{x} > 0$$
$$x = 0 \Rightarrow \sqrt{x} = 0$$
$$x < 0 \Rightarrow \sqrt{x} \text{ is not defined}$$

2.114: Square Root Sign

In general, for all values for which $\sqrt{x}$ is defined:

$$\sqrt{x} \geq 0$$

$\sqrt{-4}$ is not defined in the real number system, though it is defined in the complex number system.

Example 2.115

Solve:

A. $y^2 = 9$
B. $y = \sqrt{9}$

Explain why Part A has two solutions but Part B has only one solution.

$$y^2 = 9 \Rightarrow y = \pm\sqrt{9} = \pm 3$$
$$y = \sqrt{9} = 3$$

Example 2.116

Under what conditions is it true that:

$$y = \sqrt{y^2}$$

We can get a counterexample easily:

$$y = -1 \Rightarrow y^2 = 1 \Rightarrow \sqrt{y^2} = \sqrt{1} = 1 \neq -1 = y$$

And some thinking tells us that this counterexample will apply to all negative numbers.

$$y < 0 \Rightarrow \sqrt{y^2} = -y$$

Positive numbers will work:

$$\sqrt{y^2} = y$$

And zero will also work.

$$\sqrt{0^2} = \sqrt{0} = 0$$

Hence, the final answer is that y must be a *nonnegative* real number:

$$y \geq 0 \Leftrightarrow \ y \in [0, \infty) \Leftrightarrow y \in \mathbb{R}^+ + \{0\}$$

2.117: Square Root Sign

For a real number x, the square roots of x cannot be negative. Hence, radicals which lead to negative values have no solution.

$$\sqrt{x} < 0 \Rightarrow x \in \phi$$

$$\sqrt{4} = 2, \qquad \pm\sqrt{4} = \pm 2$$

By convention, the square root sign only refers to the positive root. If you want the negative root as well, you need to put the

$$\pm$$

Sign before the square root symbol.

Example 2.118

Find all real solutions:

A. $\sqrt{x} < 0$
B. $3\sqrt{x-2} + 5 = 3$
C. $2\sqrt{x-2} + 3 = 5 + 3\sqrt{x-2}$

$$No\ Real\ Solutions$$

$$3\sqrt{x-2} = -2 \Rightarrow \sqrt{x-2} = -\frac{2}{3} \Rightarrow No\ Real\ Solutions$$

$$-\sqrt{x-2} = 2 \Rightarrow \sqrt{x-2} = -2 \Rightarrow No\ Real\ Solutions$$

Example 2.119

A. $3x - x\sqrt{7} = 2$
B. $\sqrt{3x-1} - 3 = 0$

Part A

$$x(3-\sqrt{7}) = 2$$

$$x(3-\sqrt{7})(3+\sqrt{7}) = 2(3+\sqrt{7})$$

$$x = 3 + \sqrt{7}$$

Part B

$$\sqrt{3x-1} - 3 = 0$$

$$\sqrt{3x-1} = 3$$

$$3x - 1 = 9$$

$$3x = 10$$

$$x = \frac{10}{3}$$

2.120: Cube Root Sign

The cube operator gives a result that is greater than zero for positive numbers, and less than zero for negative numbers. or equal to zero.

$$x > 0 \Rightarrow \sqrt[3]{x} > 0$$
$$x < 0 \Rightarrow \sqrt[3]{x} < 0$$
$$x = 0 \Rightarrow \sqrt[3]{x} = 0$$

Example 2.121

Under what conditions is it true that:

$$y = \sqrt[3]{y^3}$$

The above equation is true for all real numbers.

$$x = 1 \Rightarrow x^3 = 1 \Rightarrow \sqrt[3]{x} = \sqrt[3]{1} = 1$$
$$x = -1 \Rightarrow x^3 = -1 \Rightarrow \sqrt[3]{x} = \sqrt[3]{-1} = -1$$

Example 2.122

A. $\sqrt[3]{x+3} = 2$

$$x + 3 = 8 \Rightarrow x = 5$$

Example 2.123: Radical Coefficients

Solve. Write your answer so that the denominator (if any) is free of radicals.

A. $2x - 3\sqrt{3} = 1 - x\sqrt{3}$
B. $x + 2\sqrt{5} = 3 + x\sqrt{20}$

Part A

Collate all the x terms on the LHS, and all the other terms on the RHS:

$$2x + x\sqrt{3} = 1 + 3\sqrt{3}$$

Factor out x:

$$x\left(2 + \sqrt{3}\right) = 1 + 3\sqrt{3}$$

Divide both sides by $2 + \sqrt{3}$, and then rationalize the denominator:

$$x = \frac{1 + 3\sqrt{3}}{2 + \sqrt{3}} = \frac{1 + 3\sqrt{3}}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}} = \frac{2 - \sqrt{3} + 6\sqrt{3} - 9}{4 - 3} = -7 + 5\sqrt{3}$$

Part B

$$x - 2\sqrt{5}x = 3 - 2\sqrt{5}$$
$$x\left(1 - 2\sqrt{5}\right) = 3 - 2\sqrt{5}$$
$$x = \frac{3 - 2\sqrt{5}}{1 - 2\sqrt{5}} \times \frac{1 + 2\sqrt{5}}{1 + 2\sqrt{5}} = \frac{3 + 6\sqrt{5} - 2\sqrt{5} - 20}{1 - 20} = \frac{-17 + 4\sqrt{5}}{-19} = \frac{17 - 4\sqrt{5}}{19}$$

Example 2.124: Variables inside the Radical

A. $\sqrt{3x - 6} - 3 = 5$
B. $\sqrt{2x - 1} - 3 = 0$
C. $\sqrt{1 - 3x} + 3 = 1$

Part A

Isolate the radical on one side of the equation:

$$\sqrt{3x-6}=8$$

Square both sides:

$$3x-6=64 \Rightarrow 3x=70 \Rightarrow x=\frac{70}{3}$$

Part B

$$\sqrt{2x-1}=3$$
$$2x-1=9$$
$$x=5$$

Part C

$$\underbrace{\sqrt{1-3x}}_{Nonnegative} = \underbrace{-2}_{Negative} \Rightarrow No\ Solutions$$

Example 2.125: No Solutions

$$\sqrt{3x-6}+6=5$$

A. Show that there are no solutions to the above equation.

B. Identify the flaw, if any, in the following solution: $\sqrt{3x-6}=-1 \Rightarrow 3x-6=1 \Rightarrow 3x=7 \Rightarrow x=\frac{7}{3}$

Part A

Isolate the radical:

$$\underbrace{\sqrt{3x-6}}_{\substack{Non-Negative\\ Quantity}} = \underbrace{-1}_{\substack{Negative\\ Quantity}}$$

A negative quantity can never equal a non-negative quantity.

Hence, the equation has no solutions.

Part B

Check the solution that you have found by substituting in the original equation:

$$\sqrt{3\left(\frac{7}{3}\right)-6}=\sqrt{7-6}=\sqrt{1}=1\neq -1$$

The flaw is at the step where you square.

B. Checking for Solutions

Example 2.126

Find all solutions to the following equations:

A. $\sqrt{2x+1}+\sqrt{x+\frac{1}{2}}=0$

B. $\sqrt{x}+\sqrt{2x+1}+\sqrt{x+\frac{1}{2}}=0$

C. $\sqrt{x^2+6x+8}+\sqrt{x^2+5x+6}+\sqrt{x^2-x-6}=0$

Part A

$$\sqrt{2}\sqrt{x+\frac{1}{2}}+\sqrt{x+\frac{1}{2}}=0$$

$$(\sqrt{2}+1)\left(\sqrt{x+\frac{1}{2}}\right)=0$$

$$\sqrt{x+\frac{1}{2}}=0$$

$$x+\frac{1}{2}=0$$

$$x = -\frac{1}{2}$$

Shortcut:

For all values for which $\sqrt{x}$ is defined:

$$\sqrt{x} \geq 0$$

Hence, each term on the LHS must be greater than or equal to zero.

However, the RHS is exactly zero.

Since negative quantities are not allowed, each term on the LHS must be *individually* zero.

$$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$$

$$x + \frac{1}{2} = 0 \Rightarrow x = -\frac{1}{2}$$

Part B

As in part A, each term on the LHS must be *individually* zero.

$$\sqrt{x} = 0 \Rightarrow x = 0$$

$$\sqrt{2x+1} = 0 \Rightarrow 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$$

$$\sqrt{x + \frac{1}{2}} = 0 \Rightarrow \ x + \frac{1}{2} = 0 \Rightarrow x = -\frac{1}{2}$$

x cannot be 0 and $-\frac{1}{2}$ at the same time.

Hence, there are no solutions to this equation.

Part C

Each quantity on the LHS must be individually equal to zero:

$$x^2 + 6x + 8 = 0 \Rightarrow x \in \{-4, -2\}$$

$$x^2 + 5x + 6 = 0 \Rightarrow x \in \{-3, -2\}$$

$$x^2 - x - 6 = 0 \Rightarrow x \in \{-2, 3\}$$

We need to meet the conditions for each square term.

The only solution that works is

$$-2$$

Example 2.127: Variables outside the Radical

Example 2.128: Nested Radicals

Solve:

A. $\sqrt{1+\sqrt{x+2}} = 3$

B. $\sqrt{1+\sqrt{x-1}} - \sqrt{x} = 0$

C. $\sqrt{4+\sqrt{8+4c}} + \sqrt{2+\sqrt{2+c}} = 2 + 2\sqrt{2}$ (Alcumus, Algebra, Radical Expressions and Functions)

Part A

Square both sides, and then isolate the radical:

$$1 + \sqrt{x+2} = 9 \Rightarrow \sqrt{x+2} = 8$$

Again, square both sides, and solve for x:

$$x + 2 = 64 \Rightarrow x = 62$$

Check whether the solutions work by substitution:

$$\sqrt{1+\sqrt{x+2}} = \sqrt{1+\sqrt{62+2}} = \sqrt{1+8} = 3 \Rightarrow Works$$

Part B

We only want one radical on side of the equation. So, take the second term to the RHS, and then square:

$$\sqrt{1+\sqrt{x-1}} = \sqrt{x} \Rightarrow 1 + \sqrt{x-1} = x$$

Solve for the inner radical, and then square again:

$$\sqrt{x-1} = x - 1 \Rightarrow x - 1 = (x-1)^2$$

Case I: $x - 1 = 0$

$$x = 1$$

Case I: $x - 1 \neq 0$

Divide both sides by $x - 1$:

$$1 = x - 1 \Rightarrow x = 2$$

Check whether the solution works:

$$\sqrt{1+\sqrt{1-1}}-\sqrt{1}=\sqrt{1}-\sqrt{1}=0 \Rightarrow Works$$

$$\sqrt{1+\sqrt{2-1}}-\sqrt{2}=\sqrt{1+1}-\sqrt{2}=\sqrt{2}-\sqrt{2}=0 \Rightarrow Works$$

Alternate Solution

$$x-1=x^2-2x+1 \Rightarrow 0=x^2-3x+2 \Rightarrow 0=(x-2)(x-1) \Rightarrow x \in \{1,2\}$$

Part C

$$\sqrt{4+\sqrt{8+4c}}=\sqrt{4+2\sqrt{2+c}}=\sqrt{2}\sqrt{2+\sqrt{2+c}}$$

$$\sqrt{2}\sqrt{2+\sqrt{2+c}}+\sqrt{2+\sqrt{2+c}}=2\sqrt{2}+2$$

$$(\sqrt{2}+1)\left(\sqrt{2+\sqrt{2+c}}\right)=2(\sqrt{2}+1)$$

$$\sqrt{2+\sqrt{2+c}}=2$$

$$2+\sqrt{2+c}=4$$

$$\sqrt{2+c}=2$$

$$2+c=4$$

$$c=2$$

Example 2.129: Equations Leading to Quadratics

Solve:

A. $\sqrt{x-3}=x+2$
B. $(7+\sqrt{3})x^2+(5-2\sqrt{3})x-5=0$
C. What is the product of the real roots of the equation $x^2+18x+30=2\sqrt{x^2+18x+45}$? **(AIME 1983/3)**

Part A

Square both sides:

$$x-3=x^2+4x+4$$

$$x^2+3x+7=0$$

Substitute $a=1, b=3, c=7$ the discriminant to get:

$$b^2-4ac=3^2-(4)(1)(7)=9-28 \Rightarrow Negative \Rightarrow No\ Real\ Solutions$$

Part B

This is a quadratic equation $ax^2+bx+c=0$ with $a=7+\sqrt{3}, b=(5-2\sqrt{3}), c=-5$

Substitute the above into the quadratic formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=-$$

Part C

Use a change of variable. Let $y=x^2+18x+30$:

$$y=2\sqrt{y+15} \Rightarrow y^2=4(y+15) \Rightarrow (y-10)(y+6)=0 \Rightarrow y \in \{-6,10\}$$

But note that:

$$RHS=2\sqrt{y+15}>0 \Rightarrow y>0 \Rightarrow y=10$$

Hence:

$$x^2+18x+30=10 \Rightarrow x^2+18x+20=0$$

$$Product\ of\ roots = \frac{c}{a} = \frac{20}{1} = 20$$

C. Using Identities

Example 2.130
https://www.youtube.com/watch?v=1M1TqqhT630

D. Squaring and "Extraneous" Roots

2.131: Squaring
In order to eliminate radicals, we can square them. However, in this process, we may introduce "extra" roots into the equation.

Example 2.132

$$\sqrt{x} = 2 \Rightarrow x^2 = 4 \Rightarrow x = \pm\sqrt{4} = \pm 2$$

Identify the flaw, if any.

We squared the equation that we started with. Hence, before writing the final solution set, we need to check whether the solutions satisfy the original equation.

$$2\ satisfies\ the\ original\ equation.$$
$$-2\ does\ not\ satisfy\ the\ original\ equation.$$

Example 2.133
The equation $x + \sqrt{x-2} = 4$ has:

A. 2 real roots
B. 1 real and 1 imaginary root
C. 2 imaginary roots
D. no roots
E. 1 real root **(AHSME 1950/24)**

$$\sqrt{x-2} = 4 - x$$
$$x - 2 = 16 - 8x + x^2$$
$$x^2 - 9x + 18 = 0$$
$$x \in \{3,6\}$$

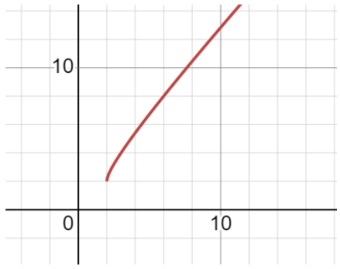
Check in the original equation:

$$3 + \sqrt{3-2} = 3 + \sqrt{1} = 3 + 1 = 4 \Rightarrow Works$$
$$6 + \sqrt{6-2} = 6 + \sqrt{4} = 6 + 2 = 8 \neq 4 \Rightarrow Does\ not\ Work$$

Alternate Solution 2.134
This is a "clever" solution. It is important to understand so that you can apply this kind of logic to questions where the standard method is too difficult, or just not possible to apply.

By observation, 3 works:

$$3+\sqrt{3-2}=3+\sqrt{1}=3+1=4$$



From the graph, we can see that $x+\sqrt{x-2}$ is an increasing function:

$$As\ x\ increases \Rightarrow \underbrace{x}_{Increases}+\underbrace{\sqrt{x-2}}_{Increases}$$

Hence, the only real solution is 3.

Example 2.135

Under what conditions does $\sqrt{a^2+b^2}=a+b$ hold? **(AMC 12A 2021/2)**

Note that

$$LHS \geq 0 \Rightarrow RHS \geq 0 \Rightarrow \underbrace{a+b\geq 0}_{Condition\ I}$$

Square both sides of the given equality:

$$a^2+b^2=a^2+2ab+b^2 \Rightarrow 2ab=0 \Rightarrow \underbrace{ab=0}_{Condition\ II}$$

Example 2.136: Pythagorean Theorem

A. Points A, B, C and D lie on a line, in that order, with $AB=CD\ and\ BC=12$. Point E is not on the line, and $BE=CE=10$. The perimeter of ΔAED is twice the perimeter of ΔBEC. Find AB. **(AMC 10A 2002/23)**

B. Let ΔXOY be a right-angled triangle with $m\angle XOY=90°$. Let M and N be the midpoints of the legs OX and OY, respectively. Given $XN=19$ and $YM=22$, find XY. **(AMC 10B 2002/22)**

Part A

Shortcut

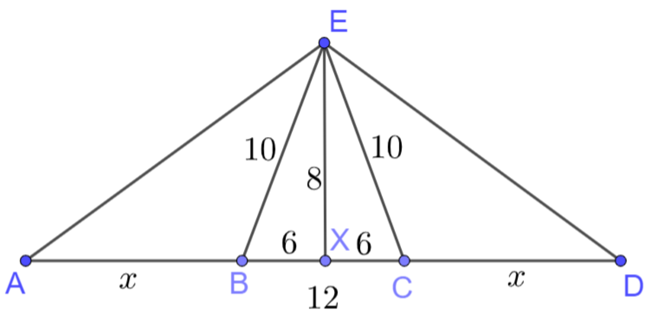
The only primitive Pythagorean Triplet with 8 in it is

$$(8,15,17)$$

We try this and it works:

$$P(AED)=2\times 17+2\times 15=64=2P(BEC)$$

$$AB=15-6=9$$



Algebraic Method

$$P(\Delta BEC)=10+10+12=32$$

$$P(AED)=2\times 32=64$$

By Pythagorean Theorem in ΔAEX, AE^2

$$=AX^2+EX^2=(x+6)^2+8^2=x^2+12x+100 \Rightarrow AE=\sqrt{x^2+12x+100}$$

$$P(\Delta AED)=2\times AE+AD=2\sqrt{x^2+12x+100}+2x+12=64$$

Isolate the radical:

$$2\sqrt{x^2+12x+100}=52-2x$$

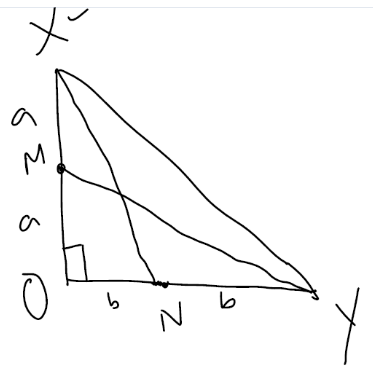
$$\sqrt{x^2+12x+100}=26-x$$

Square both sides:

$$x^2+12x+100=676-52x+x^2$$

$$64x=576$$

$$x = \frac{576}{64} = \frac{24^2}{64} = \frac{8^2 \times 3^2}{8^2} = 9$$



Part B

By the Pythagorean Theorem in ΔXOY:

$$XY = \sqrt{(2a)^2 + (2b)^2} = \sqrt{4a^2 + 4b^2} = \sqrt{4(a^2 + b^2)} = 2\sqrt{a^2 + b^2}$$

In right ΔXON, by Pythagorean Theorem:

$$(2a)^2 + b^2 = 19^2 \Rightarrow \underbrace{4a^2 + b^2 = 361}_{Equation\ I}$$

In right ΔMOY, by Pythagorean Theorem:

$$a^2 + (2b)^2 = 22^2 \Rightarrow \underbrace{a^2 + 4b^2 = 484}_{Equation\ II}$$

Add Equations I and II:

$$5a^2 + 5b^2 = 845 \Rightarrow a^2 + b^2 = 169 \Rightarrow 2\sqrt{a^2 + b^2} = 26$$

E. Radical Inequalities

Example 2.137: Domain

Find the domain, and then solve the inequality.

A. $\frac{2}{5}\sqrt{10x + 6} < 12$

B. $\sqrt{2x - 1} \le \sqrt{x + 4}$

Part A

$$Domain: 10x + 6 \ge 0 \Rightarrow 10x \ge -6 \Rightarrow x \ge -\frac{6}{10} \Rightarrow x \ge -\frac{3}{5} \Rightarrow \underbrace{x \in \left[-\frac{3}{5}, \infty\right)}_{Condition\ I}$$

$$\sqrt{10x + 6} < 30 \Rightarrow 10x + 6 < 900 \Rightarrow \underbrace{x < 89.4}_{Condition\ II}$$

$$x \in \left[-\frac{3}{5}, 89.4\right)$$

Part B

Find the domain for each radical:

$$2x - 1 \ge 0 \Rightarrow \underbrace{x \ge \frac{1}{2}}_{Condition\ I}, \qquad x + 4 \ge 0 \Rightarrow \underbrace{x \ge -4}_{Condition\ II}$$

Find the intersection of the two conditions:

$$x \ge \frac{1}{2} \Rightarrow \underbrace{x \in \left[\frac{1}{2}, \infty\right)}_{Domain}$$

Solve the inequality:

$$2x - 1 \le x + 4$$
$$\underbrace{x \le 5}_{Condition\ III}$$

Find the intersection of the domain and Condition III to find the valid solution set:

$$x \in \left[\frac{1}{2}, 5\right]$$

Example 2.138: Domain

Find the domain, and then solve the inequality.

A. $\sqrt{x^2 + 5} \ge x + 3$

Find the domain:

$$\underbrace{x^2}_{Nonnegative} + \underbrace{5}_{Positive} \Rightarrow Always\ Positive \Rightarrow x \in \mathbb{R}$$

$$x^2 + 5 \geq x^2 + 6x + 9$$
$$-4 \geq 6x$$
$$x \leq -\frac{4}{6}$$
$$x \leq -\frac{2}{3}$$

3. Fractional Exponents

3.1 Fractional Exponents

A. Exponent Half: $\frac{1}{2}$

3.1: Fractional Exponents

The usual laws of exponents apply even when the exponents are fractions.

$$a^m \times a^n = a^{m+n}$$
$$\frac{a^m}{a^n} = a^{m-n}$$
$$(a^m)^n = a^{mn}$$

$$2^2 \times 2^3 = 2^{2+3} = 2^5$$
$$\frac{2^3}{2^2} = 2^{3-2} = 2^1 = 2$$
$$(2^2)^3 = 2^2 \cdot 2^2 \cdot 2^2 = 2^{2+2+2} = 2^{2\times 3} = 2^6$$

Example 3.2: Half-Exponents

Simplify:

A. $17^{\left(\frac{1}{2}\right)} \times 17^{\left(\frac{1}{2}\right)}$
B. $5^{\frac{1}{2}} \times 5^{\frac{1}{2}}$
C. $8^{\frac{1}{2}} \times 8^{\frac{1}{2}}$
D. $\left(7^{\frac{1}{2}}\right)^2$

Use the property $a^m \cdot a^n = a^{m+n}$:

$$17^{\left(\frac{1}{2}\right)} \times 17^{\left(\frac{1}{2}\right)} = 17^{\frac{1}{2}+\frac{1}{2}} = 17^1 = 17$$
$$5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5^1 = 5$$
$$8^{\frac{1}{2}} \times 8^{\frac{1}{2}} = 8^{\frac{1}{2}+\frac{1}{2}} = 8^1 = 8$$

Use the property $(a^m)^n = a^{mn}$:

$$\left(7^{\frac{1}{2}}\right)^2 = 7^{\frac{1}{2}\times 2} = 7^1 = 7$$

3.3: Power "Half"

We are now in a position to connect fractional exponents with square roots. Raising a quantity to the exponent half is the same as taking the square root:

$$x^{\frac{1}{2}} = \sqrt{x}$$

We can apply the rules of exponents, and see that:

$$x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x$$
$$\sqrt{x} \times \sqrt{x} = \left(\sqrt{x}\right)^2 = x$$

In both cases, the behavior is the same. Hence, they are equivalent.

$$81^{\frac{1}{2}} = \sqrt{81} = 9$$

Example 3.4

Evaluate the numbers below, each of which is raised to the power $\frac{1}{2}$:

A. $81^{\frac{1}{2}}$
B. $100^{\frac{1}{2}}$
C. $25^{\frac{1}{2}}$
D. $36^{\frac{1}{2}}$
E. $49^{\frac{1}{2}}$
F. $121^{\frac{1}{2}}$

$$81^{\frac{1}{2}} = (9^2)^{\frac{1}{2}} = 9^{\frac{1}{2}\times 2} = 9^1 = 9 \ \ OR\ 81^{\frac{1}{2}} = \sqrt{81} = 9$$
$$100^{\frac{1}{2}} = (10^2)^{\frac{1}{2}} = 10^{\frac{1}{2}\times 2} = 10^1 = 10\ OR\ 100^{\frac{1}{2}} = \sqrt{100} = 10$$
$$25^{\frac{1}{2}} = \sqrt{25} = 5$$
$$36^{\frac{1}{2}} = \sqrt{36} = 6$$
$$49^{\frac{1}{2}} = \sqrt{49} = 7$$
$$121^{\frac{1}{2}} = \sqrt{121} = 11$$

Example 3.5

Write the following in radical form:

A. $x^{\frac{1}{2}}$
B. $y^{\frac{1}{2}}$
C.
D. $4z^{\frac{1}{2}}$

$$x^{\frac{1}{2}} = \sqrt{x}$$
$$y^{\frac{1}{2}} = \sqrt{y}$$

3.6: Order of Operations

Exponentiation takes higher priority over multiplication.

$$3z^{\frac{1}{2}} = 3 \times z^{\frac{1}{2}}$$

There are two operations in the above expression.
Exponentiation takes higher priority over multiplication.
Hence, we will not the square root of the 3.

$$(3z)^{\frac{1}{2}} = 3^{\frac{1}{2}} \times z^{\frac{1}{2}}$$

Because of the brackets, the multiplication happens first, and the exponetiation happens to both the 3 and the z.

Example 3.7

Is the equality given below, correct?

$$\underbrace{(4z)^{\frac{1}{2}}}_{LHS} = \underbrace{4z^{\frac{1}{2}}}_{RHS}$$

The left hand side

$$= (4z)^{\frac{1}{2}} = \sqrt{4z} = 2\sqrt{z}$$

The right hand side

$$= 4\sqrt{z}$$

Hence, the equality is not correct

B. Unit Fraction Exponents

3.8: Unit Fraction

A unit fraction is a fraction where the numerator is 1.
Note: There is no restriction on the denominator.

Example 3.9: Unit Fraction Exponents

Simplify:

A. $7^{\left(\frac{1}{3}\right)} \times 7^{\left(\frac{1}{3}\right)} \times 7^{\left(\frac{1}{3}\right)}$
B. $10^{\frac{1}{3}} \times 10^{\frac{1}{3}} \times 10^{\frac{1}{3}}$

$$7^{\left(\frac{1}{3}\right)} \times 7^{\left(\frac{1}{3}\right)} \times 7^{\left(\frac{1}{3}\right)} = 7^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 7^1 = 7$$
$$10^{\left(\frac{1}{3}\right)} \times 10^{\left(\frac{1}{3}\right)} \times 10^{\left(\frac{1}{3}\right)} = 10^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 10^1 = 10$$

3.10: Unit Fraction Exponent

Just as the exponent half gives a square root, a unit fraction exponent gives a n^{th} root.

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$

$$\underbrace{x^{\frac{1}{n}} \times x^{\frac{1}{n}} \times \ldots \times x^{\frac{1}{n}}}_{n\ times} = x^{\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n}} = x^{\frac{n}{n}} = x$$
$$\sqrt[n]{x} \times \sqrt[n]{x} \times \ldots \times \sqrt[n]{x} = \left(\sqrt[n]{x}\right)^n = x^{\frac{n}{n}} = x$$

$$\sqrt[3]{27} = 27^{\left(\frac{1}{3}\right)} = 3$$
$$\sqrt[4]{81} = 3$$

Example 3.11: Cube Roots

A. $27^{\frac{1}{3}}$
B. $216^{\frac{1}{3}}$
C. $125^{\frac{1}{3}}$
D. $8^{\frac{1}{3}}$
E. $1000^{\frac{1}{3}}$
F. $343^{\frac{1}{3}}$

$$27^{\frac{1}{3}} = \sqrt[3]{27} = 3$$
$$216^{\frac{1}{3}} = \sqrt[3]{216} = 6$$
$$125^{\frac{1}{3}} = \sqrt[3]{125} = 5$$
$$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$$
$$1000^{\frac{1}{3}} = \sqrt[3]{1000} = 10$$
$$343^{\frac{1}{3}} = \sqrt[3]{343} = 7$$

Example 3.12: Fourth Roots

A. $16^{\frac{1}{4}}$
B. $10000^{\frac{1}{4}}$
C. $256^{\frac{1}{4}}$
D. $81^{\frac{1}{4}}$
E. $625^{\frac{1}{4}}$

$$16^{\frac{1}{4}} = \sqrt[4]{16} = 2$$
$$10000^{\frac{1}{4}} = \sqrt[4]{10000} = 10$$
$$256^{\frac{1}{4}} = \sqrt[4]{256} = 4$$
$$81^{\frac{1}{4}} = \sqrt[4]{81} = 3$$
$$625^{\frac{1}{4}} = \sqrt[4]{625} = 5$$

Example 3.13: Fifth Roots

A. $32^{\frac{1}{5}}$
B. $1024^{\frac{1}{5}}$
C. $243^{\frac{1}{5}}$

$$32^{\frac{1}{5}} = \sqrt[5]{32} = 2$$
$$1024^{\frac{1}{5}} = \sqrt[5]{1024} = 4$$
$$243^{\frac{1}{5}} = \sqrt[5]{243} = 3$$

Example 3.14

Multiple Choice Multiple Correct

Mark the options below which are equal to 4?

A. $1024^{\frac{1}{5}}$
B. $2 \cdot 32^{\frac{1}{5}}$
C. $2 \cdot 32^{\frac{1}{3}}$
D. $4 \cdot 64^{\frac{1}{4}}$

$$1024^{\frac{1}{5}} = 4 \Rightarrow \mathit{Option\ A}$$

In option B, C and D, we need to do the exponentiation first, and we get:

$$2 \cdot 32^{\frac{1}{5}} = 2 \cdot 2 = 4 \Rightarrow Option\ B$$
$$2 \cdot 32^{\frac{1}{3}} \neq 4 \Rightarrow Option\ C$$
$$4 \cdot 64^{\frac{1}{4}} \neq 4 \Rightarrow Option\ D$$

C. General Fractional Exponents

3.15: General Fractional Exponent

We can interpret x raised to the power $\frac{m}{n}$ in the following way:

- The numerator m is the power to which the number is raised.
- The denominator n is the root which is taken for the number.

$$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m = (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$$

Use the property $a^{mn} = (a^m)^n$

$$x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}}$$

Use the property that $a^{\frac{1}{n}} = \sqrt[n]{a}$:

$$\sqrt[n]{x^m}$$

Starting again from the LHS, we have:

$$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m$$

Use the property that $a^{\frac{1}{n}} = \sqrt[n]{a}$:

$$\left(\sqrt[n]{x}\right)^m$$

Example 3.16

A. $1024^{\frac{1}{5}}$
B. $729^{\frac{1}{3}}$
C. $1024^{\frac{1}{2}}$
D. $1{,}000{,}000^{\frac{1}{3}}$

Non-Integer Answers

General Fractional Exponents

$$1024^{\frac{1}{5}} = (2^{10})^{\frac{1}{5}} = 2^{10\times\frac{1}{5}} = 2^2 = 4$$

$$\sqrt[3]{729} = 729^{\frac{1}{3}} = (9^3)^{\frac{1}{3}} = 9$$
$$1024^{\frac{1}{2}} = (2^{10})^{\frac{1}{2}} = 2^{10\times\frac{1}{2}} = 2^5 = 32$$
$$1{,}000{,}000^{\frac{1}{3}} = (10^6)^{\frac{1}{3}} = 10^{6\times\frac{1}{3}} = 10^2 = 100$$

3.17: Radical Form and Exponential Form

$$\underbrace{x^{\frac{1}{n}}}_{\substack{Exponential\\ Form}} = \underbrace{\sqrt[n]{x}}_{\substack{Radical\\ Form}}$$

$$\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$$

Example 3.18

In each part below, convert the expressions given in radical form into exponent form, and vice versa. Do not simplify.

Unit Fraction Exponents

A. $x^{\frac{1}{3}}$

B. $y^{\frac{1}{4}}$

C. $z^{\frac{1}{7}}$

Rational Exponents

D. $x^{\frac{2}{3}}$

E. $y^{\frac{3}{4}}$

F. $z^{\frac{5}{7}}$

Simple Radicals

G. $\sqrt{4}$

H. $\sqrt[3]{7}$

I. $\sqrt[4]{12}$

J. $\sqrt[5]{125}$

K. $\sqrt[2]{a}$

L. $\sqrt[5]{b}$

M. $\sqrt[6]{c}$

Radicals of Powers

N. $\sqrt[2]{a^3}$

O. $\sqrt[5]{b^4}$

P. $\sqrt[6]{c^7}$

Unit Fraction Exponents

$$x^{\frac{1}{3}}=\sqrt[3]{x}$$
$$y^{\frac{1}{4}}=\sqrt[4]{y}$$
$$z^{\frac{1}{7}}=\sqrt[7]{z}$$

Rational Exponents

$$x^{\frac{2}{3}}=(x^2)^{\frac{1}{3}}=\sqrt[3]{x^2}$$
$$y^{\frac{3}{4}}=(y^3)^{\frac{1}{4}}=\sqrt[4]{y^3}$$
$$z^{\frac{5}{7}}=(z^5)^{\frac{1}{7}}=\sqrt[7]{z^5}$$

Simple Radicals

$$\sqrt{4}=4^{\frac{1}{2}}$$
$$\sqrt[3]{7}=7^{\frac{1}{3}}$$
$$\sqrt[4]{12}=12^{\frac{1}{4}}$$
$$\sqrt[5]{125}=5^{\frac{1}{5}}$$
$$\sqrt[2]{a}=a^{\frac{1}{2}}$$
$$\sqrt[5]{b}=b^{\frac{1}{5}}$$
$$\sqrt[6]{c}=c^{\frac{1}{6}}$$

Radicals of Powers

$$\sqrt[2]{a^3}=(a^3)^{\frac{1}{2}}=a^{\frac{3}{2}}$$
$$\sqrt[5]{b^4}=b^{\frac{5}{5}}=(b^4)^{\frac{1}{5}}$$
$$\sqrt[6]{c^7}=(c^7)^{\frac{1}{6}}=c^{\frac{7}{6}}$$

Example 3.19

Evaluate.

Radicals

1) $\sqrt[3]{27}$

2) $\sqrt{81}$

3) $\sqrt[4]{16}$

4) $\sqrt[3]{125}$

5) $\sqrt{64}$

6) $\sqrt[5]{32}$

7) $\sqrt[3]{64}$

Square Roots

8) $4^{\frac{1}{2}}$

9) $9^{\frac{1}{2}}$

10) $64^{\frac{1}{2}}$

11) $25^{\frac{1}{2}}$

12) $144^{\frac{1}{2}}$

13) $81^{\frac{1}{2}}$

14) $289^{\frac{1}{2}}$

15) $256^{\frac{1}{2}}$

Cube Roots

16) $27^{\frac{1}{3}}$

17) $(-8)^{\frac{1}{3}}$

18) $64^{\frac{1}{3}}$

19) $-216^{\frac{1}{3}}$

20) $1000^{\frac{1}{3}}$

Fourth Roots

21) $81^{\frac{1}{4}}$

22) $16^{\frac{1}{4}}$

23) $625^{\frac{1}{4}}$

24) $256^{\frac{1}{4}}$

25) $1024^{\frac{1}{5}}$

26) $10000^{\frac{1}{4}}$

Fifth Roots

27) $243^{\frac{1}{5}}$

28) $-32^{\frac{1}{5}}$

29) $1{,}00{,}000^{\frac{1}{5}}$

Radicals

$$\sqrt[3]{27}=3$$
$$\sqrt{81}=9$$
$$\sqrt[4]{16}=2$$
$$\sqrt[3]{125}=5$$
$$\sqrt{64}=8$$
$$\sqrt[5]{32}=2$$

$\sqrt[3]{64} = 4$

Square Roots

$4^{\frac{1}{2}} = 2$

$9^{\frac{1}{2}} = 3$

$64^{\frac{1}{2}} = 8$

Cube Roots

$27^{\frac{1}{3}} = 3$

$(-8)^{\frac{1}{3}} = -2$

Fourth Roots

$81^{\frac{1}{4}} = 3$

$16^{\frac{1}{4}} = 2$

Fifth Roots

Example 3.20: Negative Exponents

Evaluate.

Negative Exponents

1) $4^{-\frac{1}{2}}$
2) $(-27)^{-\frac{1}{3}}$
3) $(-8)^{-\frac{1}{3}}$
4) $25^{-\frac{1}{2}}$
5) $(-216)^{-\frac{1}{3}}$
6) $(-32)^{-\frac{1}{5}}$
7) $8^{-\frac{1}{3}}$
8) $16^{-\frac{1}{4}}$
9) $(-125)^{-\frac{1}{3}}$
10) $1024^{-\frac{1}{2}}$
11) $(-343)^{-\frac{1}{3}}$
12) $64^{-\frac{1}{3}}$
13)

Negative Exponents

We need to convert the negative exponents to positive exponents by moving the term from numerator to denominator.

Recall that $a^m = \frac{1}{a^{-m}}$, and that the exponent changes sign, not the number.

$$4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

$$(-27)^{-\frac{1}{3}} = \frac{1}{(-27)^{\frac{1}{3}}} = \frac{1}{-3} = -\frac{1}{3}$$

$$(-8)^{-\frac{1}{3}} = \frac{1}{(-8)^{\frac{1}{3}}} = \frac{1}{-2} = -\frac{1}{2}$$

$$25^{-\frac{1}{2}} = \frac{1}{25^{\frac{1}{2}}} = \frac{1}{5}$$

$$(-216)^{-\frac{1}{3}} = \frac{1}{(-216)^{\frac{1}{3}}} = \frac{1}{-6} = -\frac{1}{6}$$

Example 3.21: Rational Exponents

Evaluate.

Rational Exponents

1) $4^{\frac{3}{2}}$
2) $8^{\frac{2}{3}}$
3) $16^{\frac{3}{4}}$
4) $32^{\frac{2}{5}}$
5) $256^{\frac{3}{8}}$
6) $1024^{\frac{5}{10}}$
7) $\sqrt[4]{\frac{x^2}{x^{\frac{2}{3}}}}$

Negative Rational Exponents

8) $16^{-\frac{3}{4}}$
9) $125^{-\frac{2}{3}}$
10) $64^{-\frac{5}{6}}$
11) $343^{-\frac{2}{3}}$
12) $216^{-\frac{2}{3}}$
13) $32^{-\frac{3}{5}}$
14) $121^{-\frac{3}{2}}$
15) $49^{-\frac{3}{2}}$

Negative Bases

16) $(-128)^{-\frac{4}{7}}$
17) $(-25)^{-\frac{3}{2}}$
18) $(-216)^{-\frac{2}{3}}$

Rational Exponents

Using $a^{mn} = (a^m)^n$

$$4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8$$

$$8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = 2^2 = 4$$

$$16^{\frac{3}{4}} = \left(16^{\frac{1}{4}}\right)^3 = 2^3 = 8$$

$$32^{\frac{2}{5}} = \left(32^{\frac{1}{5}}\right)^2 = 2^2$$

$$256^{\frac{3}{8}} = \left(256^{\frac{1}{8}}\right)^3 = 2^3 = 8$$

$$1024^{\frac{5}{10}} = 1024^{\frac{1}{2}} = 32$$

$$\sqrt[4]{\frac{x^2}{x^{\frac{2}{3}}}} = \sqrt[4]{x^{\frac{4}{3}}} = \left(x^{\frac{4}{3}}\right)^{\frac{1}{4}} = x^{\frac{4}{3}\times\frac{1}{4}} = x^{\frac{1}{3}}$$

Negative Rational Exponents

$$16^{-\frac{3}{4}} = \frac{1}{16^{\frac{3}{4}}} = \frac{1}{2^3} = \frac{1}{8}$$

$$125^{-\frac{2}{3}} = \frac{1}{125^{\frac{2}{3}}} = \frac{1}{5^2} = \frac{1}{25}$$

Negative Bases

$$(-128)^{-\frac{4}{7}} = \frac{1}{(-128)^{\frac{4}{7}}} = \frac{1}{(-2)^4} = \frac{1}{16}$$

Example 3.22: Mixed Review

A. 2^4
B. 2^{-4}
C. $(-2)^4$
D. $8^{\frac{1}{3}}$
E. $8^{-\frac{1}{3}}$
F. $(-8)^{\frac{1}{3}}$
G. $(-8)^{-\frac{1}{3}}$

$$8^{\frac{1}{3}} = 2$$

$$8^{-\frac{1}{3}} = \frac{1}{8^{\frac{1}{3}}} = \frac{1}{2}$$

$$(-8)^{\frac{1}{3}} = -2$$

$$(-8)^{-\frac{1}{3}} = \frac{1}{(-8)^{\frac{1}{3}}} = -\frac{1}{2}$$

$$(-8)^{-\frac{1}{3}}$$

Example 3.23: Product Rule

Use the rules above to evaluate the expressions. Write your final answer in simplified form.

Basics

$$a^m \times a^n = a^{m+n}$$

Use the product rule to evaluate the expressions below:

A. $4^{\frac{5}{4}} \times 4^{\frac{1}{4}}$
B. $169^{\frac{1}{3}} \times 169^{\frac{1}{6}}$
C. $2^{\frac{2}{3}} \times 2^{\frac{3}{4}}$
D. $x^{\frac{1}{2}} \times x^{\frac{2}{5}}$

Radicals

$$\sqrt{x} = x^{\frac{1}{2}}, \sqrt[3]{x} = x^{\frac{1}{3}}, \sqrt[n]{x^m} = x^{\frac{m}{n}}$$

Use the conversions between radicals and exponents to evaluate the expressions below.

E. $\sqrt{3^4} \times \sqrt{3^6}$
F. $\sqrt{2^3} \times \sqrt[3]{2^2}$
G. $\sqrt{x^5} \times \sqrt[4]{x^3}$
H. $\sqrt{3^2} \times \sqrt[3]{3^4} \times \sqrt[4]{3^5}$
I. $2\sqrt{x} \cdot 4x^{-\frac{5}{2}}$

Basics

$$4^{\frac{5}{4}}\times 4^{\frac{1}{4}}=4^{\frac{5}{4}+\frac{1}{4}}=4^{\frac{6}{4}}=4^{\frac{3}{2}}=\left(4^{\frac{1}{2}}\right)^3=2^3=8$$
$$169^{\frac{1}{3}}\times 169^{\frac{1}{6}}=169^{\frac{1}{3}+\frac{1}{6}}=169^{\frac{2}{6}+\frac{1}{6}}=169^{\frac{3}{6}}=169^{\frac{1}{2}}=13$$
$$2^{\frac{2}{3}}\times 2^{\frac{3}{4}}=2^{\frac{2}{3}+\frac{3}{4}}=2^{\frac{8}{12}+\frac{9}{12}}=2^{\frac{17}{12}}$$
$$x^{\frac{1}{2}}\times x^{\frac{2}{5}}=x^{\frac{1}{2}+\frac{2}{5}}=x^{\frac{5}{10}+\frac{4}{10}}=x^{\frac{9}{10}}$$

Radicals

$$\sqrt{3^4}\times\sqrt{3^6}=3^2\times 3^3=3^{2+3}=3^5=243$$
$$\sqrt{2^3}\times\sqrt[3]{2^2}=2^{\frac{3}{2}}\times 2^{\frac{2}{3}}=2^{\frac{3}{2}+\frac{2}{3}}=2^{\frac{9}{6}+\frac{4}{6}}=2^{\frac{13}{6}}$$
$$\sqrt{x^5}\times\sqrt[4]{x^3}=x^{\frac{5}{2}}\times x^{\frac{3}{4}}=x^{\frac{5}{2}+\frac{3}{4}}=x^{\frac{10}{4}+\frac{3}{4}}=x^{\frac{13}{4}}$$
$$\sqrt{3^2}\times\sqrt[3]{3^4}\times\sqrt[4]{3^5}=3\times 3^{\frac{4}{3}}\times 3^{\frac{5}{4}}=3^{1+\frac{4}{3}+\frac{5}{4}}=3^{\frac{12}{12}+\frac{16}{12}+\frac{15}{12}}=3^{\frac{43}{12}}$$
$$2\sqrt{x}\cdot 4x^{-\frac{5}{2}}=2\cdot 4\times x^{\frac{1}{2}}\cdot x^{-\frac{5}{2}}=8\times x^{\frac{1}{2}-\frac{5}{2}}=8\times x^{-\frac{4}{2}}=8x^{-2}=\frac{8}{x^2}$$

Example 3.24: Quotient Rule

$$\textit{Quotient Rule: }\frac{a^m}{a^n}=a^{m-n}$$
$$\textit{Power of a Fraction: }\left(\frac{a}{b}\right)^x=\frac{a^x}{b^x}=\frac{b^{-x}}{a^{-x}}=\left(\frac{b}{a}\right)^{-x}$$

Basics

A. $\dfrac{3^{\frac{2}{3}}}{3^{\frac{1}{3}}}$

B. $\dfrac{125^{-\frac{11}{12}}}{125^{-\frac{1}{4}}}$

C. $y^{\frac{3}{2}}\div y^{\frac{4}{5}}$

Radicals

D. $\dfrac{\sqrt{3^5}}{\sqrt[3]{3^2}}$

E. $\dfrac{\sqrt[3]{9}}{\sqrt[4]{27}}$

F. $\dfrac{\sqrt{512^3}}{\sqrt[3]{1024^2}}$

Power of a Fraction

G. $\dfrac{2^{-\frac{4}{3}}}{54^{-\frac{4}{3}}}$

H. $\dfrac{5^{-\frac{3}{4}}}{80^{-\frac{3}{4}}}$

Quotient Rule

$$\frac{3^{\frac{2}{3}}}{3^{\frac{1}{3}}}=3^{\frac{2}{3}-\frac{1}{3}}=3^{\frac{1}{3}}$$
$$\frac{125^{-\frac{11}{12}}}{125^{-\frac{1}{4}}}=125^{-\frac{11}{12}+\frac{1}{4}}=125^{-\frac{8}{12}}=125^{-\frac{2}{3}}=\frac{1}{125^{\frac{2}{3}}}=\frac{1}{5^2}=\frac{1}{25}$$
$$y^{\frac{3}{2}}\div y^{\frac{4}{5}}=y^{\frac{3}{2}-\frac{4}{5}}=y^{\frac{15}{10}-\frac{8}{10}}=y^{\frac{7}{10}}$$

Radicals

$$\frac{\sqrt{3^5}}{\sqrt[3]{3^2}}=\frac{3^{\frac{5}{2}}}{3^{\frac{2}{3}}}=3^{\frac{5}{2}-\frac{2}{3}}=3^{\frac{15}{6}-\frac{4}{6}}=3^{\frac{11}{6}}$$
$$\frac{\sqrt[3]{9}}{\sqrt[4]{27}}=\frac{\sqrt[3]{3^2}}{\sqrt[4]{3^3}}=\frac{3^{\frac{2}{3}}}{3^{\frac{3}{4}}}=3^{\frac{2}{3}-\frac{3}{4}}=3^{\frac{8}{12}-\frac{9}{12}}=3^{-\frac{1}{12}}=\frac{1}{3^{\frac{1}{12}}}$$
$$\frac{\sqrt{512^3}}{\sqrt[3]{1024^2}}=\frac{\sqrt{(2^9)^3}}{\sqrt[3]{(2^{10})^2}}=\frac{\sqrt{2^{27}}}{\sqrt[3]{2^{20}}}=\frac{2^{\frac{27}{2}}}{2^{\frac{20}{3}}}=2^{\frac{27}{2}-\frac{20}{3}}=2^{\frac{81}{6}-\frac{40}{6}}=2^{\frac{41}{6}}$$

Power of a Fraction

$$\frac{2^{-\frac{4}{3}}}{54^{-\frac{4}{3}}} = \left(\frac{2}{54}\right)^{-\frac{4}{3}} = \left(\frac{1}{27}\right)^{-\frac{4}{3}} = 27^{\frac{4}{3}} = \left(27^{\frac{1}{3}}\right)^4 = 3^4 = 81$$
$$\frac{5^{-\frac{3}{4}}}{80^{-\frac{3}{4}}} = \left(\frac{5}{80}\right)^{-\frac{3}{4}} = \left(\frac{1}{16}\right)^{-\frac{3}{4}} = 16^{\frac{3}{4}} = \left(16^{\frac{1}{4}}\right)^3 = 2^3 = 8$$

Example 3.25: Power Rule

$$(a^m)^n = a^{mn}$$
$$[(a^x)^y]^z = (a^{xy})^z = a^{xyz}$$
$$(ab)^m = a^m b^m$$

Basics

A. $\left(2^{\frac{2}{3}}\right)^{\frac{3}{4}}$

B. $\left(3^{\frac{5}{7}}\right)^{\frac{1}{2}}$

C. $\left[\left(7^{\frac{3}{7}}\right)^{\frac{2}{5}}\right]^{\frac{5}{7}}$

D. $\left[\left(x^{\frac{p}{q}}\right)^{\frac{a}{b}}\right]^{\frac{m}{n}}$

Radicals

E. $\left(\sqrt{7}\right)^{\frac{2}{3}}$

F. $\sqrt[3]{\left(\sqrt{3}\right)^{\frac{5}{3}}}$

Non-Prime Bases

G. $\left(16^{\frac{3}{5}}\right)^{\frac{2}{3}} \times \left(64^{\frac{2}{7}}\right)^{\frac{7}{3}}$

Distribution

H. $\left[\left(a^{-1}b^{\frac{1}{3}}\right)\left(a^{-\frac{4}{3}}b^2\right)\right]^2$

I. $(12x^3y^2)^{\frac{1}{2}}(6x^4y^4)^{\frac{1}{2}}$

Basics

$$\left(2^{\frac{2}{3}}\right)^{\frac{3}{4}} = 2^{\frac{2}{3}\times\frac{3}{4}} = 2^{\frac{2}{4}} = 2^{\frac{1}{2}}$$
$$\left(3^{\frac{5}{7}}\right)^{\frac{1}{2}} = 3^{\frac{5}{7}\times\frac{1}{2}} = 3^{\frac{5}{14}}$$
$$\left[\left(7^{\frac{3}{7}}\right)^{\frac{2}{5}}\right]^{\frac{5}{7}} = 7^{\frac{3}{7}\times\frac{2}{5}\times\frac{5}{7}} = 7^{\frac{6}{49}}$$

Radicals

$$\left(\sqrt{7}\right)^{\frac{2}{3}} = \left(7^{\frac{1}{2}}\right)^{\frac{2}{3}} = 7^{\frac{1}{2}\times\frac{2}{3}} = 7^{\frac{1}{3}}$$
$$\sqrt[3]{\left(\sqrt{3}\right)^{\frac{5}{3}}} = \sqrt[3]{\left(3^{\frac{1}{2}}\right)^{\frac{5}{3}}} = \sqrt[3]{3^{\frac{1}{2}\times\frac{5}{3}}} = \sqrt[3]{3^{\frac{5}{6}}} = 3^{\frac{5}{6}\times\frac{1}{3}} = 3^{\frac{5}{18}}$$

Non-Prime Bases

$$\left((2^4)^{\frac{3}{5}}\right)^{\frac{2}{3}} \times \left((2^6)^{\frac{2}{7}}\right)^{\frac{7}{3}} = 2^{4\times\frac{3}{5}\times\frac{2}{3}} \times 2^{6\times\frac{2}{7}\times\frac{7}{3}} = 2^{\frac{8}{5}} \times 2^4 = 2^{\frac{8}{5}+4} = 2^{\frac{28}{5}}$$

Distribution

$$\left[\left(a^{-1-\frac{4}{3}}b^{\frac{1}{3}+2}\right)\right]^2 = \left(a^{-\frac{7}{3}}b^{\frac{7}{3}}\right)^2 = a^{-\frac{14}{3}}b^{\frac{14}{3}} = \frac{b^{\frac{14}{3}}}{a^{\frac{14}{3}}} = \frac{a^{\frac{1}{3}}b^{\frac{14}{3}}}{a^{\frac{15}{3}}} = \frac{a^{\frac{1}{3}}b^{\frac{14}{3}}}{a^5}$$

Using the property $a^m b^m = (ab)^m$

$$(72x^7y^6)^{\frac{1}{2}} = \sqrt{72x^7y^6} = 6|x^3y^3|\sqrt{2x}$$

Example 3.26: Decimals

A. $(0.04)^{-1.5}$
B. $(0.125)^{-\frac{5}{3}}$

$$\left(\frac{4}{100}\right)^{-\frac{3}{2}} = \left(\frac{1}{25}\right)^{-\frac{3}{2}} = 25^{\frac{3}{2}} = \left(25^{\frac{1}{2}}\right)^{3} = 5^3 = 125$$
$$(0.125)^{-\frac{5}{3}} = \left(\frac{1}{8}\right)^{-\frac{5}{3}} = 8^{\frac{5}{3}} = \left(8^{\frac{1}{3}}\right)^{5} = 2^5 = 32$$

Example 3.27

Find the value of x that satisfies:

$$25^{-2} = \frac{5^{48/x}}{5^{26/x} \cdot 25^{17/x}}$$ (*AMC* 10*B* 2003/9)

Since $25 = 5^2$, rewrite all exponents to have a base of 5.

$$(5^2)^{-2} = \frac{5^{\frac{48}{x}}}{5^{\frac{26}{x}} \cdot (5^2)^{\frac{17}{x}}}$$
$$5^{-4} = \frac{5^{\frac{48}{x}}}{5^{\frac{26}{x}} \cdot 5^{\frac{34}{x}}}$$
$$5^{-4} = 5^{\frac{48}{x}-\frac{26}{x}-\frac{34}{x}}$$

Because the base is non-zero, and equal, the exponents are also equal:

$$-4 = \frac{48}{x} - \frac{26}{x} - \frac{34}{x}$$
$$-4 = -\frac{12}{x}$$
$$x = \frac{12}{4} = 3$$

3.2 Expressions and Equations

A. Expressions

Example 3.28: Distributive Property

Expand using the distributive property:

Exponents

A. $x^{\frac{1}{2}}\left(x^{\frac{1}{3}} + x^{\frac{1}{4}}\right)$
B. $x^{\frac{2}{3}}\left(x^{\frac{2}{5}} + x^{\frac{2}{7}}\right)$
C. $a^{\frac{2}{7}}\left(b^{\frac{1}{3}} + c^{\frac{1}{4}} + d^{\frac{1}{5}}\right)$

Radicals

D. $\sqrt{x}\left(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}\right)$
E. $\sqrt{a}\left(\sqrt{a} + \sqrt[3]{a} + \sqrt{b}\right)$

Exponents

Part A

$$x^{\frac{1}{2}+\frac{1}{3}} + x^{\frac{1}{2}+\frac{1}{4}} = x^{\frac{3}{6}+\frac{2}{6}} + x^{\frac{2}{4}+\frac{1}{4}} = x^{\frac{5}{6}} + x^{\frac{3}{4}}$$

Part B

$$x^{\frac{2}{3}+\frac{2}{5}} + x^{\frac{2}{3}+\frac{2}{7}} = x^{\frac{10+6}{15}} + x^{\frac{14+6}{21}} = x^{\frac{16}{15}} + x^{\frac{20}{21}}$$

Part C

$$a^{\frac{2}{7}}b^{\frac{1}{3}} + a^{\frac{2}{7}}c^{\frac{1}{4}} + a^{\frac{2}{7}}d^{\frac{1}{5}}$$

Radicals

Part D

Use the distributive property:

$$= x^{\frac{1}{2}+\frac{3}{2}} + 2x^{\frac{1}{2}+\frac{1}{2}} + 3x^{\frac{1}{2}-\frac{1}{2}}$$
$$= x^2 + 2x^1 + 3x^0$$
$$= x^2 + 2x + 3$$

Part E

$$a^{\frac{1}{2}}\left(a^{\frac{1}{2}} + a^{\frac{1}{3}} + b^{\frac{1}{2}}\right)$$
$$= a^{\frac{1}{2}+\frac{1}{2}} + a^{\frac{1}{2}+\frac{1}{3}} + a^{\frac{1}{2}}b^{\frac{1}{2}}$$
$$= a + a^{\frac{5}{6}} + a^{\frac{1}{2}}b^{\frac{1}{2}}$$

3.29: Factoring

When factoring expressions with exponents, we will factor out the lowest power of a term.

Example 3.30: Factoring

Find the HCF, and factor it out of the expression:

Basics

A. $6x + 9x^2$
B. $12x^4 + 18x^6$
C. $x^2 + x^3$
D. $x^{22} + x^{33} + x^{44}$

Variables

E. $4x^{y+3} + 6x^y$
F. $24x^{a+2} + 16x^{a+3}$
G. $x^{b+2} + x^{b+\frac{7}{2}}$

Negative Exponents

H. $2x^5 + 3x^{-4}$
I. $4x^3 + 8x^{-2}$
J. $x^{y+3} + x^{y-2}$

Basics

Part A

Find the HCF of $6x$ and $9x^2$:

$$HCF(6x, 9x^2) = 3x$$

Factor out $3x$:

$$3x(2 + 3x)$$

Part B

HCF of $12x^4$ and $18x^6$ is $6x^4$.

Factor out x^4:

$$6x^4(2 + 3x^2)$$

Part C

Lowest power is x^2:

$$x^2 + x^3 = x^2(1 + x)$$

Part D

Lowest power is x^{22}:

$$x^{22}(1 + x^{11} + x^{22})$$

Variables

Part E

Lowest power is x^y:

$$x^y\left(\frac{x^{y+3}}{x^y} + \frac{x^y}{x^y}\right) = x^y(x^3 + 1)$$

Part F

Lowest power is $a + 2$:

$$x^{a+2}(1 + x)$$

Part G

Lowest power is x^{b+2}:

$$x^{b+2}\left(1 + x^{\frac{3}{2}}\right)$$

Negative Exponents

Part H

Lowest power is x^{-4}:

$$x^{-4}(2x^9 + 3)$$

Part G

Lowest power is x^{-2}:

$$4x^{-2}(x^5 + 2)$$

Part G

Lowest power is x^{y-2}:

$$x^{y-2}\left(\frac{x^{y+3}}{x^{y-2}} + \frac{x^{y-2}}{x^{y-2}}\right) = x^{y-2}(x^5 + 1)$$

Example 3.31: Fractions

Find the HCF, and factor it out of the expression:

A. $\left(\frac{1}{2}\right)^y + \left(\frac{1}{2}\right)^{y-4}$
B. $\left(\frac{1}{3}\right)^{z+2} + \left(\frac{1}{3}\right)^{z-1}$

Part A

Note that:

$$\left(\frac{1}{2}\right)^y + \left(\frac{1}{2}\right)^{y-4} = \frac{1}{2^y} + \frac{1}{2^{y-4}} = 2^{-y} + 2^{4-y}$$

Since $-y < 4 = y$, we factor out $2^{-y} = \left(\frac{1}{2}\right)^y$:

$$\left(\frac{1}{2}\right)^y\left[1+\left(\frac{1}{2}\right)^{-4}\right] = \left(\frac{1}{2}\right)^y[1+2^4] = \left(\frac{1}{2}\right)^y[17]$$

Part B

As before,

$$z+2 > z-1$$

However, since we are the "numbers" are in the denominators, we factor $z+2$:

$$\left(\frac{1}{3}\right)^{z+2}\left[1+\left(\frac{1}{3}\right)^{-3}\right]$$

$$\left(\frac{1}{3}\right)^{z+2}[1+3^3]$$

$$28\left(\frac{1}{3}\right)^{z+2}$$

B. Equations

3.32: Equating Exponents

In an equality, if bases are same, exponents must be same.

$$x^a = x^b \Leftrightarrow a = b$$

In the above, we are able to say

$$a = b$$

If and only if

$$\textit{The bases on each side of the equality are the same}$$

Example 3.33

Solve:

Basics

A. $17^{2x+5} = 17^{3x-2}$

B. $19^{\frac{3}{2}x-\frac{1}{4}} = 19^{\frac{5}{4}x+\frac{1}{3}}$

Radical Equations

C. $11^{\sqrt{x}+\frac{1}{2}} = 11^{\sqrt{x}+\frac{1}{3}}$

D. $21^{\sqrt{x}+\frac{1}{2}} = 21^{2\sqrt{x}+\frac{1}{3}}$

Exponent Rules

E. $3^{2x+5} = 9^{3x-5}$

Part A

Bases are same. Hence, exponents must also be same:

$$2x+5 = 3x-2 \Rightarrow x = 7$$

Part B

$$\frac{3}{2}x - \frac{1}{4} = \frac{5}{4}x + \frac{1}{3}$$

$$\frac{x}{4} = \frac{7}{12} \Rightarrow x = \frac{7}{3}$$

Part C

$$\sqrt{x} + \frac{1}{2} = \sqrt{x} + \frac{1}{3}$$

$$\frac{1}{2} = \frac{1}{3} \Rightarrow \textit{No Solutions}$$

Part D

$$\sqrt{x} + \frac{1}{2} = 2\sqrt{x} + \frac{1}{3}$$

$$\frac{1}{6} = \sqrt{x}$$

$$\frac{1}{36} = x$$

Part E

$$3^{2x+5} = (3^2)^{3x-5}$$

$$3^{2x+5} = (3^2)^{6x-10}$$

$$2x+5 = 6x-10$$

$$15 = 4x$$

$$x = \frac{15}{4}$$

3.34: Exponential Equations with Unity RHS

If the RHS of an exponential equation is 1, then the exponent on the LHS must be 0.

$$b^e = 1 \Leftrightarrow \underbrace{e = 0, b \neq 0}_{\substack{\textit{Exponent} \\ \textit{is Zero}}}\ OR\ \underbrace{b = 1}_{\textit{Base is } 1}\ OR\ \underbrace{b = -1, e \textit{ is even}}_{\textit{Base is} -1}$$

The reason for $x \neq 0$ when $a = 0$ is that:

$$a = 0, x = 0 \Rightarrow 0^0 \Rightarrow Not\ Defined$$

For example:

$$5^0 = 1, 19^0 = 1$$
$$1^5 = 1, 1^{19} = 1$$

Example 3.35

Solve for the variable:

Variables in the Exponent

A. $12^x = 1$
B. $33^{2y} = 1$
C. $56^{3x+5} = 1$
D. $\left(\frac{2}{3}\right)^{4x-2} = 1$
E. $\left(\frac{27}{19}\right)^{\frac{3}{4}x-\frac{2}{3}} = 1$

Variables in the Base

F. $x^3 = 1$
G. $(2y+1)^3 = 1$
H. $\left(\frac{3}{5}z - \frac{1}{2}\right)^6 = 1$
I. $\left(\frac{2}{5}a + \frac{7}{3}\right)^6 = 1$

Variables in the Exponent

$$12^x = 1 \Rightarrow x = 0$$
$$33^{2y} = 1 \Rightarrow 2y = 0 \Rightarrow y = 0$$
$$56^{3x+5} = 1 \Rightarrow 3x + 5 = 0 \Rightarrow x = -\frac{5}{3}$$
$$\left(\frac{2}{3}\right)^{4x-2} = 1 \Rightarrow 4x - 2 = 0 \Rightarrow x = \frac{2}{4} = \frac{1}{2}$$
$$\left(\frac{27}{19}\right)^{\frac{3}{4}x-\frac{2}{3}} = 1 \Rightarrow \frac{3}{4}x - \frac{2}{3} = 0 \Rightarrow x = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$$

Variables in the Base

Parts F-G

$$x = 1$$
$$2y + 1 = 1 \Rightarrow y = 0$$

Part H

$$\frac{3}{5}z - \frac{1}{2} = 1 \Rightarrow \frac{3}{5}z = \frac{3}{2} \Rightarrow z = \frac{5}{2}$$
$$\frac{3}{5}z - \frac{1}{2} = -1 \Rightarrow \frac{3}{5}z = -\frac{1}{2} \Rightarrow z = -\frac{5}{6}$$
$$z \in \left\{-\frac{5}{6}, \frac{5}{2}\right\}$$

Part I

$$\frac{2}{5}a + \frac{7}{3} = 1 \Rightarrow \frac{2}{5}a = -\frac{4}{3} \Rightarrow a = -\frac{10}{3}$$
$$\frac{2}{5}a + \frac{7}{3} = -1 \Rightarrow \frac{2}{5}a = -\frac{10}{3} \Rightarrow a = -\frac{25}{3}$$
$$a \in \left\{-\frac{25}{3}, -\frac{10}{3}\right\}$$

3.36: Making Bases Same

If bases are not same, convert all bases into prime numbers to make the bases same.

Example 3.37

Solve:

A. $27^{5x-2} = 9^{2x+4}$
B. $2^{3x+7} = 32^{\frac{3}{5}}$
C. $32^{\frac{2}{3}x-\frac{1}{2}} = \left(\frac{1}{128}\right)^{\frac{4}{3}}$
D. $2^{3x} \times 2^{-5x} = 2^5$
E. $2^{3x-2} \times 2^{5x-4} = 2^5$

Part A

Make the bases the same:

$$(3^3)^{5x-2} = (3^2)^{2x+4}$$

Apply the power rule:

$$3^{15x-6}=3^{4x+8}$$

Since bases are same, exponents must also be same:

$$15x-6=4x+8\Rightarrow x=\frac{14}{11}$$

Part B

Part C

Convert everything to the base 2:

$$(2^5)^{\frac{2}{3}x-\frac{1}{2}}=\left(\frac{1}{2^7}\right)^{\frac{4}{3}}\Rightarrow(2^5)^{\frac{2}{3}x-\frac{1}{2}}=(2^{-7})^{\frac{4}{3}}$$

Apply the power rule:

$$2^{5\left(\frac{2}{3}x-\frac{1}{2}\right)}=2^{\frac{-28}{3}}$$

If bases are same, exponents are also same:

$$5\left(\frac{2}{3}x-\frac{1}{2}\right)=\frac{-28}{3}\Rightarrow\frac{2}{3}x-\frac{1}{2}=\frac{-28}{15}\Rightarrow\frac{2}{3}x=-\frac{41}{30}$$

$$\Rightarrow x=-\frac{41}{30}\times\frac{3}{2}=-\frac{41}{20}$$

Part D

$$2^{3x+(-5x)}=2^5$$
$$-2x=5$$

Part E

$$2^{3x-2+(5x-4)}=2^5$$

C. Comparing Bases

Example 3.38

What is the value of A when we rewrite 15^x as $A^{\frac{x}{6}}$?

Method I

Raise both sides to the sixth power:

$$15^{6x}=A^x\Rightarrow(15^6)^x=A^x\Rightarrow A=15^6$$

Method I

$$15^x=A^{\frac{x}{6}}$$

On the LHS, the exponent is x. On the RHS, the exponent is $\frac{x}{6}$.
In order to compare, we need the exponents to be the same.
Hence, we write:

$$15^x=15^{\frac{x}{1}}=15^{\frac{x}{6}\times 6}=(15^6)^{\frac{x}{6}}$$

Now we are in a position to compare:

$$(15^6)^{\frac{x}{6}}=A^{\frac{x}{6}}$$

Because the powers are the same, the bases must be the same.

Example 3.39

Find A such that the equality is true for all values of x.

$$2^x=A^{\left(\frac{x}{12}\right)}$$

If exponents are same, then bases must be same. Then take the 12^{th} power both sides to find A.

$$2^x=\left(A^{\frac{1}{12}}\right)^x\Rightarrow 2=A^{\left(\frac{1}{12}\right)}\Rightarrow 2^{12}=\mathrm{A}$$

Example 3.40

In the equation below, A can be written in the form $2^a\times 3^b$. Find $a+b$.

$$12^x=A^{\left(\frac{x}{7}\right)}$$

$$A=12^7=(2^2\times 3)^7=2^{14}\times 3^7=2^a\times 3^b\Rightarrow a=14,b=7\Rightarrow a+b=21$$

In the equation below, A can be written in the form $2^a\times 3^b$. Find the ratio $a:b$.

$$12^x = A^{\left(\frac{x}{n}\right)}, n\ is\ a\ natural\ number$$

$$A = 12^n = (2^2 \times 3)^n = 2^{2n} \times 3^n = 2^a \times 3^b \Rightarrow a = 2n, b = n \Rightarrow a : b = 2n : n = 2 : 1$$

D. Comparing LHS and RHS

Example 3.41

Find A such that the equality is true for all values of x:

A. $4^{x+3} - 4^x = A \cdot 4^x$
B. $3^x - 3^{x-2} = A \cdot 3^x$

Part I:

Factor out the lowest power of x:

$$4^x \left(\frac{4^{x+3}}{4^x} - \frac{4^x}{4^x}\right) A \cdot 4^x \Rightarrow 4^x(4^3 - 1) = A \cdot 4^x \Rightarrow 4^3 - 1 = A \Rightarrow 63 = A$$

Part II:

$$LHS = 3^x - 3^{x-2} = 3^{x-2}\left(\frac{3^x}{3^{x-2}} - \frac{3^{x-2}}{3^{x-2}}\right) = 3^{x-2}(3^2 - 1) = 3^{x-2}(8) = 3^x \cdot \mathbf{3^{-2}(8)}$$

Compare with the RHS.

$$A = 3^{-2} \times 8 = \frac{8}{9}$$

E. Simplification

Example 3.42

$$\frac{81^{x+7}}{9^{5x-9}} = 9^{4x+1}$$

$$\frac{81^{x+7}}{9^{5x-9}} = 9^{4x+1} \Rightarrow \frac{3^{4x+28}}{3^{10x-18}} = 3^{8x+2} \Rightarrow 3^{4x+28} = 3^{8x+2+10x-18}$$

Since bases are same, powers are also same:

$$4x + 28 = 18x - 16 \Rightarrow 44 = 14x \Rightarrow x = \frac{44}{14} = \frac{22}{7}$$

F. Factoring

Example 3.43

$$e^{x-1} + e^{1-x} = 2$$

$$\frac{e^x}{e} + \frac{e^1}{e^x} - 2 = 0$$

Substitute $y = \frac{e^x}{e}$:

$$y + \frac{1}{y} - 2 = 0$$

Factor:

$$\left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^2 = 0$$

Take square roots:

$$\sqrt{y}-\frac{1}{\sqrt{y}}=0\Rightarrow\sqrt{y}=\frac{1}{\sqrt{y}}\Rightarrow y=1$$

Substitute $y=\frac{e^x}{e}$:

$$\frac{e^x}{e}=1\Rightarrow e^x=e\Rightarrow x=1$$

Example 3.44: Disguised Quadratic

Questions where one term is the square of another term can be converted into quadratics by substitution.

A. $4^x-2^x-2=0$
B. $2e^{4x}-e^{2x}-1=0$

Part A

Use a change of variables. Substitute $y=2^x$:

$$y^2-y-2=0\Rightarrow(y-2)(y+1)=0$$

Apply the zero-product property:

$$y-2=0\Rightarrow y=2\Rightarrow 2^x=2\Rightarrow x=1$$
$$y+1=0\Rightarrow y=-1\Rightarrow 2^x=-1\Rightarrow No\ Solution$$

Hence, the final answer is:

$$x=1$$

Part B

Let $a=e^{2x}$:

$$2a^2-a-1=0$$
$$2a^2+a-2a-1=0$$
$$a(2a+1)-1(2a+1)=0$$
$$(2a+1)(a-1)=0$$
$$a\in\left\{-\frac{1}{2},1\right\}$$

Change back to the original variable:

$$e^{2x}=1\Rightarrow e^{2x}=e^0\Rightarrow 2x=0\Rightarrow x=0$$
$$e^{2x}=-\frac{1}{2}\Rightarrow No\ Solutions$$

3.3 Perfect Powers

A. Perfect Powers

Example 3.45: Perfect Squares

For each number in the set $A=\{12,26,144,243\}$, find the smallest number x such that the product of x and the number is a perfect square.

$$x\times 2^2\times 3\Rightarrow x=3$$
$$x\times 26=x\times 2\times 13\Rightarrow x=26$$
$$x\times 144\Rightarrow x=1$$
$$x\times 243\Rightarrow x\times 3^5\Rightarrow x=3$$

Example 3.46: Perfect Cubes

For each number in the set $A=\{25,144,75,44\}$, find the smallest number x such that the product of x and the number is a perfect cube.

$$x \times 25 \Rightarrow x \times 5^2 = x = 5$$
$$x \times 144 \Rightarrow x \times 2^4 \times 3^2 = x = 2^2 \times 3 = 12$$
$$x \times 75 \Rightarrow x \times 3 \times 5^2 = x = 3^2 \times 5 = 45$$
$$x \times 44 \Rightarrow x \times 2^2 \times 11 = x = 2 \times 11^2 = 242$$

3.4 Exponential Functions

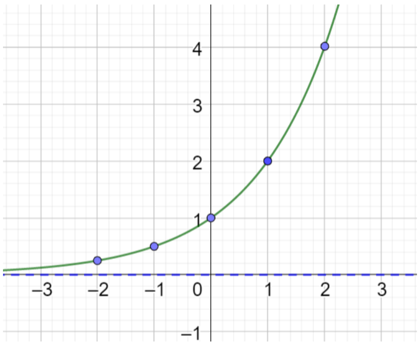
A. Basics

$$0^0 \Rightarrow Not\ Defined$$
$$f(x) = c^x, c < 0, Not\ Defined$$

3.47: Intercepts

For the general exponential function $f(x) = c^x, c > 0$:

$$y - intercept: x = 0 \Rightarrow c^x = c^0 = 1$$
$$x - intercept: 0 = c^x \Rightarrow x = \phi \Rightarrow No\ x - intercept$$



3.48: Asymptotes

$$\text{Horizontal Asymptote: } y = 0$$
$$No\ Vertical\ Asymptote$$

3.49: Domain and Range

$$Domain = D_f = \mathbb{R}$$
$$Range = R_f = (0, \infty)$$

There are no restrictions on the domain. Hence

$$Domain = All\ Real\ Numbers$$

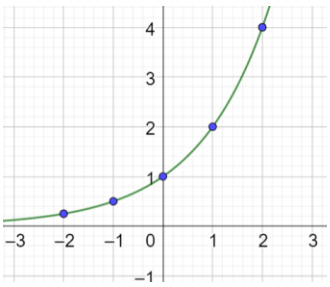
y is never negative. And y does not achieve zero. Hence

$$f(x) = c^x\ has\ no\ x - intercept$$
$$Range: y > 0$$

Example 3.50

Sketch the graph of the following, and indicate the range.

A. $y = 2^x$
B. $y = 3^x$



3.51: Reflection over the $y-axis$

$$f(x)=c^x \underbrace{\Leftrightarrow}_{\substack{Reflection\\ over\ the\ y-axis}} f(x)=\left(\frac{1}{c}\right)^x$$

- The graph of $f(-x)$ is the reflection of the graph of $f(x)$ over the y −axis.

$$f(x)=c^x \Rightarrow f(-x)=c^{-x}=\frac{1}{c^x}=\left(\frac{1}{c}\right)^x$$

- Reflecting over the y-axis does not change the range of the function $f(x)=c^x$

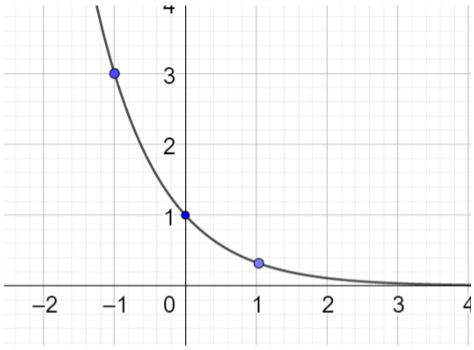
Example 3.52

Sketch the graph of the following:

A. $y=\left(\frac{1}{2}\right)^x$

B. $y=\left(\frac{1}{3}\right)^x$

$\boldsymbol{Range: y>0}$



3.53: Reflection over the $x-axis$

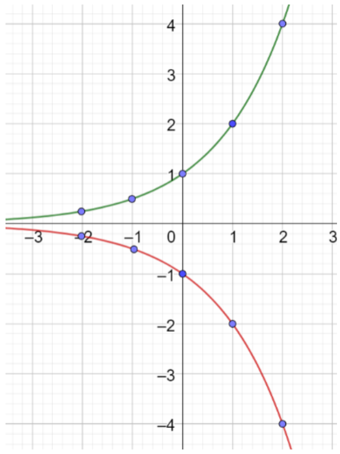
$$f(x)=c^x \underbrace{\Rightarrow}_{\substack{Reflection\\ over\ the\ x-axis}} f(x)=-c^x$$

Recall that the reflection of $f(x)$ over the x −axis is $-f(x)$.

Example 3.54

A. $y=-2^x$

$$(-2)^0 = 1$$
$$-2^0 = -1$$



$$\boldsymbol{Range: y < 0}$$

3.55: Vertical Shift

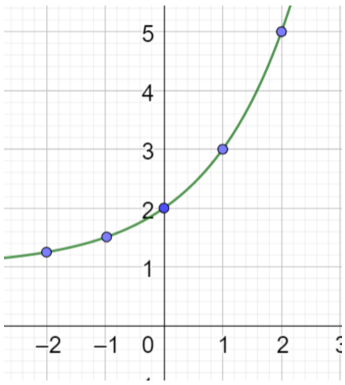
$$y = f(x) \Rightarrow y = f(x) + k$$
$$k > 0, k\ units\ up$$
$$k < 0, k\ units\ down$$

$$y = b^x$$
$$y = b^x + k\ (Moved\ up\ by\ k\ units), k > 0$$
$$y = b^x - k\ (Moved\ down\ by\ k\ units), k > 0$$

Vertical Shifts will
- not change the domain
- change the range

Example 3.56

A. $y = 2^x + 1$



3.57: Vertical Dilation

$$y = f(x) \Rightarrow y = kf(x)$$
$$k > 1: Vertical\ Stretch$$
$$0 < k < 1: Vertical\ Shrink$$

Vertical Stretch

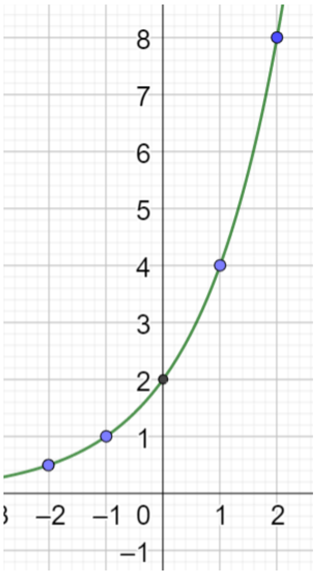
$$y = ab^x\ (Vertical\ Stretch\ by\ factor\ \boldsymbol{a}), a > 1$$

Vertical Shrink

$$y = \frac{1}{a}b^x\ (Vertical\ Shrink\ by\ factor\ \boldsymbol{a}), a > 1$$

Example 3.58

A. $y = 2(2^x)$
B. $y = \frac{1}{3}(2^x)$



3.59: Horizontal Shift

$$y = f(x) \Rightarrow y = f(x+k)$$
$$k > 0 \Rightarrow\ Shift\ to\ the\ left$$
$$k < 0 \Rightarrow\ Shift\ to\ the\ right$$

$$y = b^x$$
$$y = b^{x-h}\ (Moved\ \boldsymbol{right}\ by\ \boldsymbol{h}\ units), h > 0$$
$$y = b^{x+h}\ (Moved\ \boldsymbol{left}\ by\ \boldsymbol{h}\ units), h > 0$$

Horizontal Shifts will:

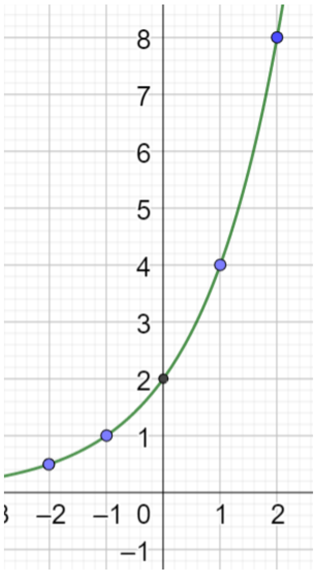
- shift the graph
- not change the domain
- but not the domain/range

Example 3.60

Graph the following and mention, asymptotes, intercepts and any other points of interest.

A. $f(x) = 2^{x+1}$

$$Consider\ g(x) = 2^x = g(x+1)$$



3.61: Horizontal Dilation

$$y = f(x) \Rightarrow y = f(kx)$$

$$k > 1: Horizontal\ Shrink\ by\ factor\ of\ \frac{1}{k}$$

$$0 < k < 1: Horizontal\ Stretch\ by\ factor\ of\ \frac{1}{k}$$

$$y = b^x$$

$$y = (cb)^x \left(Horizontal\ Stretch\ by\ factor\ \frac{1}{c}\right), c > 1$$

$$y = \left(\frac{1}{c}b\right)^x \ (Horizontal\ Shrink\ by\ factor\ c), c > 1$$

Example 3.62

Graph the following and mention, asymptotes, intercepts and any other points of interest.

A. $y = 2^{\frac{1}{2}x}$

Compare $2^{\frac{1}{2}x}$ with 2^x:

$$2^{\frac{1}{2}x} = 2^{kx} \Rightarrow k = \frac{1}{2}$$

Hence, $y = 2^{\frac{1}{2}x}$ is a horizontal stretch of $y = 2^x$ by a factor of:

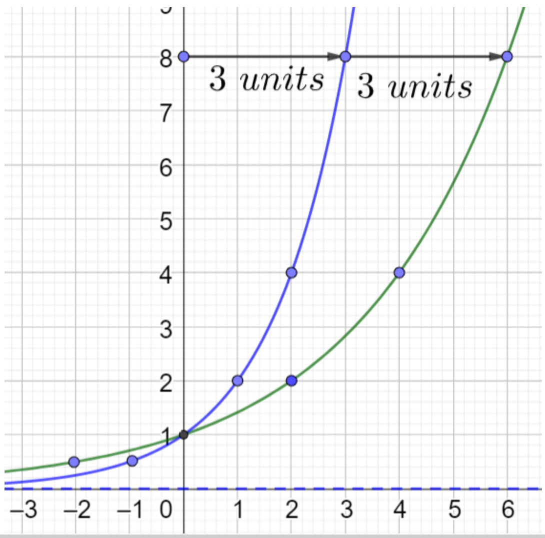
$$\frac{1}{k} = \frac{1}{\frac{1}{2}}$$

$$Horizontal\ Asymptote: y = 0$$

$$Vertical\ Asymptote: DNE$$

$$y - intercept: (0,1)$$

$$x - intercept: DNE$$



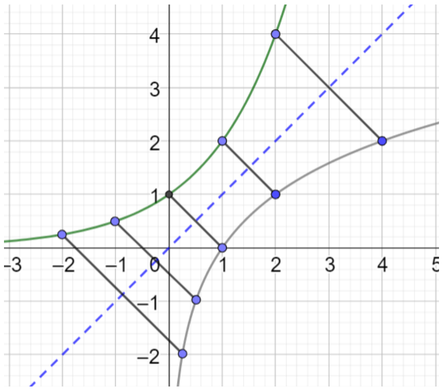
3.63: Inverse Functions

The graph of $f^{-1}(x)$ is obtained by reflecting the graph of $f(x)$ across the line $y = x$

You reflect points across the line $y = x$
- Geometrically, if you are comfortable
- Algebraically, by interchanging the x and the y coordinate.

Example 3.64

Obtain the graph of the inverse of $f(x) = 2^x$ only by using the reflection property of inverse functions.



Example 3.65

Consider the function

$$y = b^x$$

Identify the transformation applied to the following functions:

A. $y = b^x + 2$
B. $y = b^x - 1$
C. $y = b^{x-1}$
D. $y = b^{x+2}$
E. $y = 2b^x$
F. $y = \frac{1}{3}b^x$

A. Moved up by 2 units
B. Moved down by 1 unit

Example 3.66

$$y = b^x + 1$$
$$Domain = All\ Real\ Numbers$$
$$Range: y > 1$$

Example 3.67

$$y = b^x - 1$$

Compared to $y = b^x$, the graph of $y = b^x - 1$ is moved down by 1 unit.

There is no effect on the domain.

$$Domain = All\ Real\ Numbers$$

The range

$$Range: y > -1$$

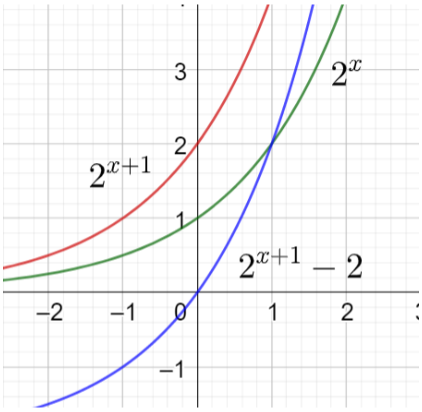
Vertical/Horizontal Scaling will not change the domain or the range.

- change the range

B. Combining Transformations

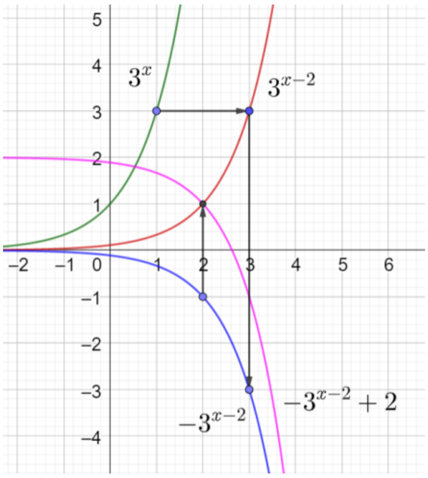
Example 3.68

$$y = 2^{x+1} - 2$$



Example 3.69

$$y = -3^{x-2} + 2$$



$$Parent\ Function: y = \mathbf{3^x}$$
$$Shift\ to\ the\ right\ by\ 2: y = \mathbf{3^{x-2}}$$
$$Reflect\ it\ across\ the\ x-axis:\ y = \mathbf{-3^{x-2}}$$
$$Move\ it\ up\ by\ 2: y = \mathbf{-3^{x-2} + 2}$$

$$Horizontal\ Asymptote: y = 2$$
$$Vertical\ Asymptote: DNE$$
$$y-intercept: \left(0, \frac{17}{9}\right)$$

$$x-intercept$$

Substitute $y = 0$

$$0 = -3^{x-2} + 2 \Rightarrow 3^{x-2} = 2 \Rightarrow x - 2 = \log_3 2 \Rightarrow x = \log_3 2 + 2$$

Example 3.70

Identify the transformation from f to g.

$$f(x) = 4^{x+1}, g(x) = 4^{x-2}$$

Horizontal Shift:

$$Move\ f\ right\ by\ 3\ units\ to\ g$$

Vertical Dilation

$$f(x) = 4^{x+1} = 4^x \times 4^1 = 4 \times 4^x$$
$$g(x) = 4^{x-2} = 4^x \times 4^{-2} = \frac{4^x}{16}$$

Vertical Dilation by factor of $\frac{1}{64}$:

$$\frac{f(x)}{64} = \frac{4 \times 4^x}{64} = \frac{4^x}{16} = g(x)$$

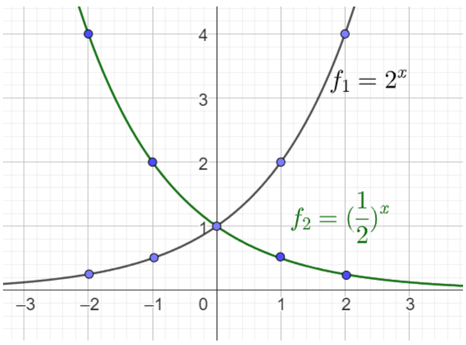
$$\frac{f(x)}{g(x)} = \frac{4^{x+1}}{4^{x-2}} = 4^{x+1-x+2} = 4^3 = 64 \Rightarrow f(x) = 64g(x) \Rightarrow g(x) = \frac{1}{64} f(x)$$

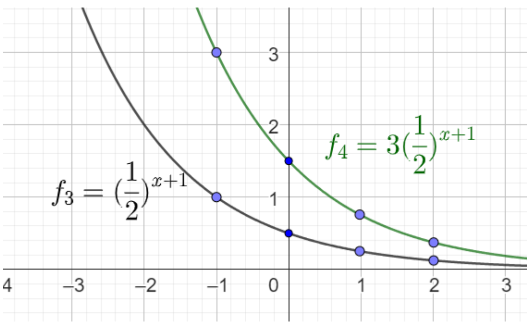
Example 3.71

$$f(x) = 3\left(\frac{1}{2}\right)^{x+1} - 2$$

$$Reflect\ f_1(x) = 2^x\ across\ the\ y\ axis: f_2(x) = f_1(-x) = 2^{-x} = \frac{1}{2^x} = \left(\frac{1}{2}\right)^x$$

$$\text{Shift to the left by 1 unit: } f_3(x) = f_2(x+1) = \left(\frac{1}{2}\right)^{x+1}$$





$$\text{Stretch by a factor of 3: } f_4 = 3f_3(x) = 3\left(\frac{1}{2}\right)^{x+1}$$

$$\text{Shift downward by 2 units: } f_5 = f_4(x) - 2 = 3\left(\frac{1}{2}\right)^{x+1} - 2$$

C. Range

Example 3.72

A. $y = 2 \times 2^x$
B. $y = -2^x$

$$Range\ of\ 2^x: y > 0$$

Part A

$$Range\ of\ 2 \times 2^x: 2y > 2 \times 0 \Rightarrow y > 0$$

Part B

$$Range\ of\ -2^x: y > 0 \Rightarrow -y > 0 \Rightarrow y < 0$$

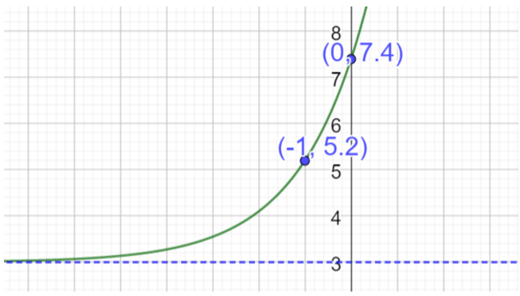
D. Finding Functions

Example 3.73

The exponential function graphed alongside is of the form

$$y = ab^x + h$$

If $b = 2$, then find the function.



Substitute $h = 3, b = 2$ in $y = ab^x + h$:

$$y = a \times 2^x + 3$$

Substitute $y = 7.4, x = 0$:

$$7.4 = a(2^0) + 3 \Rightarrow a = 4.4$$

Substitute $y = 5.2, x = -1$:

$$5.2 = a(2^{-1}) + 3 \Rightarrow a = 4.4$$

$$y = 4.4 \times 2^x + 3$$

Example 3.74
$f(x) = 2 \cdot 2^x$ is a shift of $g(x) = 2^x$ by 1 unit.
$f(x) = 3 \cdot 2^x$ is a shift of $g(x) = 2^x$ by k unit. Determine k.

$$3 \cdot 2^x = 2^{x+k}$$
$$3 \cdot 2^x = 2^x \cdot 2^k$$
$$3 = 2^k$$
$$k = \log_2 3$$

E. Domain

Example 3.75
Find the domain of

$$f(x) = \sqrt{4^x - 8^x}$$

$$4^x - 8^x \geq 0 \Rightarrow 4^x \geq 8^x \Rightarrow \frac{4^x}{8^x} \geq 1 \Rightarrow \left(\frac{4}{8}\right)^x \geq 1 \Rightarrow \left(\frac{1}{2}\right)^x \geq 1 \Rightarrow \frac{1}{2^x} > \frac{1}{2^0} \Rightarrow 2^x < 2^0 \Rightarrow x \in (-\infty, 0)$$

3.5 Factoring

Example 3.76

$$e^{x-y} + e^x + e^{-y} + 1$$

$$\frac{e^x}{e^y} + e^x + \frac{1}{e^y} + 1$$

Rearrange:

$$= \frac{e^x}{e^y} + \frac{1}{e^y} + e^x + 1$$

Factor

$$= \frac{1}{e^y}(e^x + 1) + 1(e^x + 1)$$

Factor further to get:

$$= (e^x + 1)\left(\frac{1}{e^y} + 1\right)$$

77 Examples