# Rotational Mechanics

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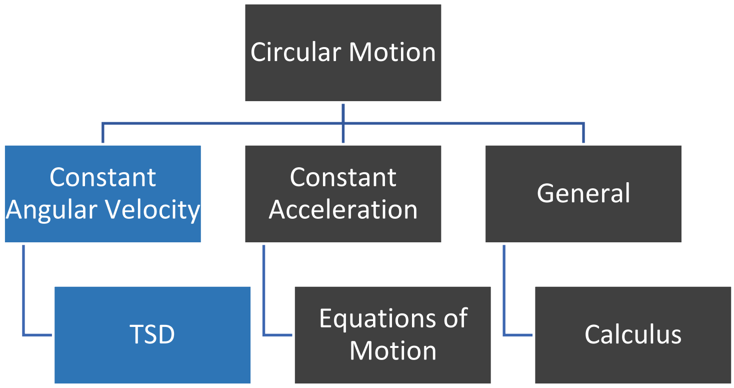
# 1. Circular Motion

## 1.1 Uniform Circular Motion

### A. Circular Motion




**1.1: Uniform Circular Motion**
An object in uniform circular motion travels on a circle (or arc of a circle) at constant speed

- A ball attached to a string being spun around.
- A pail full of water being spun around

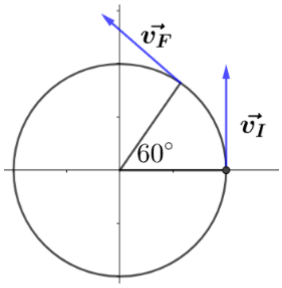
**1.2: Speed**

$$v = Speed = \frac{Distance}{Time}$$

**Example 1.3**
Explain why when an object moves in uniform circular motion, it does not have constant velocity

Because the direction of motion is constantly changing.

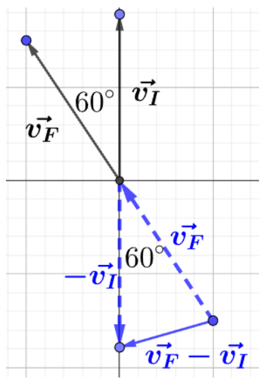
**Example 1.4**
A particle is moving along a circular path with a constant speed of $10\frac{m}{s}$. What is the magnitude of the change in velocity of the particle when it moves through an angle of 60° around the center of the circle? (JEE-M 2015)




**Vectors on a circle**
Draw the initial vector $\vec{v_I}$ and final vector $\boldsymbol{v_F}$ on a circle.

**Change Vector**
Since we want to find magnitude of change, we focus on

$$|\Delta v| = |\vec{v_F} - \vec{v_I}|$$




- Reflect $\vec{v_I}$ across the $x$ axis to get $-\vec{v_I}$
- Move $\vec{v_F}$ so that its tail matches with the tail of $\vec{v_I}$. Note that the angle between the two vectors is 60°.
- To add the two vectors arrange them tip to tail. Move $\vec{v_F}$ parallel to itself so that its tip is at the tail of $-\vec{v_I}$
- Since we moved $\vec{v_F}$ parallel to itself, by vertically opposite angles, the angle between the two sides of the triangle is 60°.
- Since $|\vec{v_I}| = |\vec{v_F}|$, the triangle is isosceles. Hence, the base angles are 60°. Hence, the triangle is equilateral.
- Hence, the third side of the triangle, which is

$$|\Delta v| = |v_F - v_I|$$

Has magnitude same as

$$|\vec{v_I}| = 10$$

## B. Angular Velocity and Angular Speed

### 1.5: Angular Frequency / Angular Speed

$$\omega = \frac{Rotations}{Time} \times 2\pi$$

Unit of angular velocity is $\frac{rad}{s}$ or $\frac{degrees}{s}$

$$1\ Rotation\ per\ Second \Leftrightarrow \omega = 2\pi \frac{rad}{s}$$

### Example 1.6

An object has an angular speed of $\pi \frac{rad}{s}$. Interpret this speed in terms of rotations.

It makes

$$\frac{1}{2}\ Rotation\ per\ second$$

### Example 1.7

A. A car wheel rotates 15 times a minute. Determine the angular speed of the wheel in $\frac{rad}{s}$.
B. A fan rotates 3 times every second. Calculate its angular speed in $\frac{rad}{min}$.
C. The angular speed of a flywheel making 120 revolutions/minute is: (NEET 1995)

**Part A**

$$\omega = \frac{Rotations}{Unit\ time} \times 2\pi = 15 \frac{Rotations}{min} \times 2\pi = 30\pi \frac{rad}{min} = \frac{\pi}{2} \frac{rad}{s}$$

**Part B**

$$\omega = 3 \times (2\pi) = 6\pi \frac{rad}{s} = 360\pi \frac{rad}{min}$$

**Part C**

$$\omega = \frac{Rotations}{Unit\ time} \times 2\pi = 2 \frac{Rotations}{s} \times 2\pi = 4\pi \frac{rad}{s}$$

### Example 1.8

A. How many revolutions does a wheel with angular speed $88 \frac{rad}{s}$ make in one second (take $\pi = \frac{22}{7}$)? (EAMCET, 21 Sep 2020, Shift-II)
B. A fan has an angular speed of $3 \frac{rad}{s}$. Calculate the number of rotations per minute.

**Part A**

$$Rotations = \frac{\omega}{2\pi} = \frac{88}{2\pi} = \frac{44}{\frac{22}{7}} = 44 \times \frac{7}{22} = 14 \frac{Rotations}{sec}$$

**Part B**

$$Rotations = \frac{\omega}{2\pi} = \frac{3}{2\pi} \frac{Rotations}{sec} = \frac{90}{\pi} \frac{rotations}{min}$$

### 1.9: Speed versus Angular Speed

Speed is not the same as angular speed.

### Example 1.10

A circle with radius $2m$ has an object at its rightmost position. It travels along the circle to its bottommost position in $75$ *seconds*. Calculate, for the object:

A. Speed
B. Angular Speed

The distance travelled by the particle on the circular path will be:

$$\frac{3}{4} \times 2\pi(2) = 3\pi$$

**Part A**

$$Speed = \frac{Distance}{Time} = \frac{3\pi}{75} = \frac{\pi}{25}\frac{m}{s}$$

**Part B**

$$Angular\ Speed = \frac{\frac{3}{4} \times 2\pi}{75} = \frac{\pi}{50}\frac{rad}{s}$$

### 1.11: Angular Speed

Angular speed has an alternate formula, given by:

$$\omega = \frac{v}{r}$$

Where

$$v = linear\ speed = \frac{Distance}{Time}$$

### Example 1.12

A. A circle of radius $r$ units is such that the numerical value of the angular speed of a particle in $\frac{rad}{s}$ is the same as the numerical value of the speed of the particle in $\frac{units}{s}$. Determine the value of $r$.
B. A horse in a merry go around is $5m$ away from the center. The horse travels $23\frac{m}{min}$. Calculate the angular speed of the horse.

**Part A**

$$\omega = \frac{\omega}{r} \Rightarrow 1 = \frac{1}{r} \Rightarrow r = 1$$

**Part B**

$$\omega = \frac{v}{r} = \frac{23}{5}\frac{rad}{min}$$

### 1.13: Angular Velocity (Signed Scalar)

Angular velocity has the same magnitude as angular speed. If the movement is in the

- counterclockwise direction, angular velocity is positive.
- clockwise direction, angular velocity is negative.

- This is the same as the trigonometry convention for positive and negative angles.

## C. Period of Revolution

### 1.14: Period of Revolution

This is the time taken to complete one cycle around the circle by an object in uniform circular motion.

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$$

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\frac{v}{r}} = \frac{2\pi}{\omega}$$

### Example 1.15

Two racing cars of masses $m_1$ and $m_2$ are moving in circles of radii $r_1$ and $r_2$ respectively. Their speeds are such that each makes a complete circle in the same time $t$. The ratio of the angular speeds of the first to the second car is: **(NEET 2019; NEET 2001; NEET 1999; EAMCET 22 Sep 2020, Shift-I)**

$$\omega_A = \omega_B = \frac{2\pi}{T} \Rightarrow Ratio = 1:1$$

## D. Centripetal Acceleration

### Example 1.16

An object is in uniform circular motion. Explain why its

A. velocity is not constant.
B. acceleration is non-zero

Velocity is not constant. The magnitude remains the same, but the direction is always changing.
Hence, acceleration is non-zero.

### 1.17: Centripetal Acceleration

Centripetal acceleration is the acceleration of an object in uniform circular motion.

- The direction of centripetal acceleration is towards the center of the circle.
- The velocity vector and the acceleration vector are at right angles to each other.

### 1.18: Centripetal Acceleration

There are two versions of the formula for centripetal acceleration. The first is in terms of linear speed, and the second in terms of angular speed.

$$a_c = \frac{v^2}{r} = \omega^2 r$$

$$a = \frac{v^2}{r} = \frac{v^2}{r^2} \cdot r = \omega^2 r$$

### Example 1.19

A. A particle moves in a circle of radius 5 $cm$ with constant speed and time period $0.2\pi$ $s$. The acceleration of the particle is: **(NEET 2011)**
B. A stone tied to the end of a string of 1 $m$ long is whirled in a horizontal circle with a constant speed. If the stone makes 22 revolutions in 44 seconds, what is the magnitude and direction of acceleration of the stone? **(NEET 2005)**

$$A: a = \omega^2 r = \left(\frac{2\pi}{T}\right) r = \left(\frac{2\pi}{0.2\pi}\right)^2 \cdot \frac{5}{100} = 5\frac{m}{s^2}$$
$$B: a = \omega^2 r = \left(\frac{2\pi}{T}\right) r = \left(\frac{2\pi}{0.5}\right)^2 \cdot 1 = 4\pi\frac{m}{s^2}$$

### Example 1.20

A stone tied to 180 $cm$ long string at its end is making 28 revolutions in horizontal circle in every minute. The magnitude of acceleration of stone is $\frac{1936}{x}\frac{m}{s^2}$. The value of $x$ is: (take $\pi = \frac{22}{7}$) **(JEE-M 2023)**

The angular velocity is:

$$\omega = \frac{Rev}{Time} \cdot 2\pi = \frac{28\ Rev}{Min} \cdot 2\pi = \frac{28\ Rev}{60\ s} \cdot 2\left(\frac{22}{7}\right) = \frac{44}{15}\frac{rad}{s}$$

Since the acceleration is uniform, it is only centripetal:

$$a_c = \omega^2 r = \left(\frac{44}{15}\right)^2\left(\frac{180}{100}\right) = \frac{1936}{15 \times 15} \times \frac{9}{5} = \frac{1936}{125}$$
$$x = 125$$

### Example 1.21

A clock has a continuously moving second's hand of 0.1 $m$ length. The average acceleration of the tip of the hand is of the order of:

A. $10^{-3}$
B. $10^{-4}$
C. $10^{-2}$
D. $10^{-1}$ **(JEE-M 2020)**

$$\omega = \frac{Rev}{s} \cdot 2\pi = \frac{1}{30} \cdot \pi = \frac{1}{30} \cdot \frac{22}{7} = \frac{11}{105} > \frac{11}{110} = 0.1$$

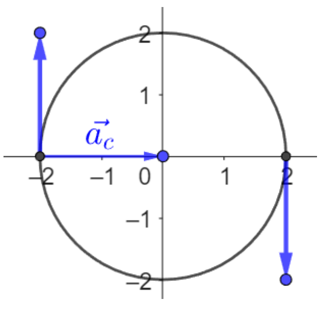
The acceleration is then

$$a_c = \omega^2 r > (0.1)^2(0.1) = 0.001 = 10^{-3}$$
*Option A*

### Example 1.22

An object moves at a constant speed along a circular path in a horizontal plane with center at the origin. When the object is at $x = +2m$, its velocity is $-4\hat{\boldsymbol{j}}\frac{m}{s}$. The object's velocity vector and acceleration vector at $x = -2\ m$ will be: **(JEE-M 2023)**

$$\vec{\boldsymbol{v}} = 4\hat{\boldsymbol{j}}\frac{m}{s}$$
$$a_c = \frac{v^2}{r} = \frac{16}{2} = 8\frac{m}{s} \Rightarrow \vec{\boldsymbol{a_c}} = 8\hat{\boldsymbol{\imath}}\frac{m}{s^2}$$




## E. TSD Questions

### Example 1.23

A. A car wheel has an angular speed of $1\frac{rad}{s}$ and the axle of the car, which is connected to the center of the wheel is 25 $cm$ above the ground. Determine the speed of the car.

B. The car is on a circular road with radius $100m$. Determine the angular speed of the car in $\frac{rad}{s}$.

**Part A**

The distance travelled (arc length) in 1 second is:

$$\theta_1 = \frac{l_1}{r_1} \Rightarrow l_1 = r_1\theta_1 = (25\ cm)(1) = 25\ cm$$

$$Speed = 25\frac{cm}{s} = 0.25\frac{m}{s}$$

**Part B**

$$\theta_2 = \frac{l_2}{r_2} = \frac{0.25\frac{m}{s}}{100\ m} = 0.0025\frac{rad}{s}$$

## 1.24: Angular Speed

When a circle with radius $r_1$ rotates with angular speed $\theta_1$ on a larger circle with radius $r_2$, and the smaller circle has angular speed $\theta_2$ around the larger circle, then the relation between them is:

$$\theta_1 r_1 = \theta_2 r_2$$

$$\theta_2 = \frac{l_2}{r_2}$$

Note that $l_2$ which is the length travelled by the car is also the length travelled by the wheel:

$$\theta_2 = \frac{l_1}{r_2} = \frac{r_1\theta_1}{r_2}$$

Rearrange to get:

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} \Rightarrow \theta_2 r_2 = \theta_1 r_1$$

## Example 1.25

A. A car wheel has an angular speed of $1\frac{rad}{s}$ and the car axle (connected to the center of the wheel) is $25\ cm$ above the ground. The car is on a circular road with radius $100m$. Determine the angular speed of the car with reference to the center of the circle in $\frac{rad}{s}$.

B. A truck has tires with a diameter of $1m$. The tires have an angular speed of $\pi\frac{rad}{s}$. The truck is travelling on a semi-circular route with a diameter of 1 km. Determine the angular speed of the truck with reference to the center of the semicircle.

**Part A**

$$\theta_2 = \frac{r_1\theta_1}{r_2} = \frac{(25\ cm)(1)}{100\ m} = 0.0025\ \frac{rad}{s}$$

**Part B**

$$\theta_2 = \frac{r_1\theta_1}{r_2} = \frac{D_1\theta_1}{D_2} = \frac{(1m)(\pi)}{1000\ m} = \frac{\pi}{1000}\ \frac{rad}{s}$$

## Example 1.26: Relative Angular Speed

Two sportsmen are on a circular track at the same position. A has an angular speed of $2\frac{rad}{min}$, while $B$ has an angular speed of $3\frac{rad}{min}$. They start at the same time. If they run

A. in the same direction, when will $B$ first overtake $A$ from behind.

B. In opposite directions, when will they encounter each other first?

**Part A**
Time

$$=\frac{Distance}{Relative\ Speed}=\frac{2\pi}{3-2}=\frac{2\pi}{1}=2\pi\ min$$

**Part B**
Time

$$=\frac{Distance}{Relative\ Speed}=\frac{2\pi}{3+2}=\frac{2\pi}{5}=\frac{2}{5}\pi\ min$$

### Example 1.27

97: Two particles A and B are moving on two concentric circles of radii $R_1$ and $R_2$ with equal angular speed $\omega$. At $t=0$, their positions and direction of motion are shown in the figure. The relative velocity $\overrightarrow{\boldsymbol{v_A}}-\overrightarrow{\boldsymbol{v_B}}$ at $t=\frac{\pi}{2\omega}$ is given by: **(JEE-M 2019)**

$$D=S\times T$$
$$\theta=\omega t$$

$$\theta_A=\omega\cdot\frac{\pi}{2\omega}=\frac{\pi}{2}\Rightarrow Quarter\ Circle$$

$$\overrightarrow{\boldsymbol{v_A}}-\overrightarrow{\boldsymbol{v_B}}=-\omega R_1\hat{\boldsymbol{\imath}}-(-\omega R_2\hat{\boldsymbol{\imath}})=\omega(R_2-R_1)\hat{\boldsymbol{\imath}}$$

## F. Parametrizations

### 1.28: Parametrization

$$x=a\sin\omega t\,,\qquad y=a\cos\omega t$$

Is a parametrization of uniform circular motion with

$$Position\ Vector=\vec{\boldsymbol{s}}=a(\sin\omega t\,,\cos\omega t)$$
$$Velocity\ vector=\vec{\boldsymbol{v}}=a\omega(\cos\omega t\,,-\sin\omega t)$$
$$Acceleration\ vector=\vec{\boldsymbol{a}}=a\omega^2(-\sin\omega t\,,-\cos\omega t)$$

### Example 1.29

*Mark all correct options*

A particle moves in $x-y$ plane according to rule $x=a\sin\omega t$ and $y=a\cos\omega t$. The particle follows

A. an elliptical path
B. a circular path
C. a parabolic path
D. a straight-line path inclined equally to x and y-axes **(NEET 2010, Adapted)**

The parametrization is of uniform circular motion. A circle is a special case of an ellipse.

*Option A, B*

### 1.30: Angular Frequency

$$x=a\sin\omega t\,,y=a\cos\omega t$$

The angular frequency is

$$\omega$$

### Example 1.31

*Mark all correct options*

The position vector of a particle $\vec{R}$ as a function of time is given by $\vec{R}(t) = 4\sin(2\pi t)\,\hat{\imath} + 4\cos(2\pi t)\,\hat{\jmath}$ where $\vec{R}$ is in meters, $t$ is in seconds and $\hat{\imath}$ and $\hat{\jmath}$ and denote unit vectors along $x$-and $y$-directions, respectively.

A. Magnitude of the velocity of particle is $8\,\frac{m}{s}$.
B. Path of the particle is a circle of radius 4 meter.
C. Acceleration vector is along $-\vec{R}$.
D. Magnitude of acceleration vector is $\frac{|\vec{v}|^2}{|\vec{R}|}$, where $\vec{v}$ is the velocity of particle. **(NEET 2015, Adapted)**

**Option A**

$$\omega = \frac{v}{r} \Rightarrow v = \omega r = 2\pi(4) = 8\pi \Rightarrow Not\ correct$$

**Option B**

$$Option\ B\ is\ correct$$

**Option C**

The acceleration vector for uniform circular is in the opposite direction of the position vector.

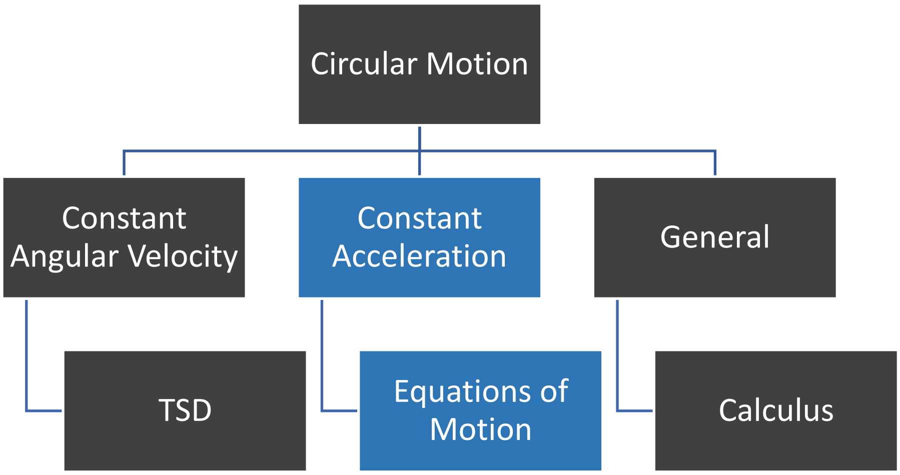
$$Option\ C\ is\ correct$$

**Option D**

$$a = \frac{v^2}{R} \Rightarrow Option\ D\ is\ correct$$

# 1.2 Equations of Motion: Circular

## A. Circular Motion with Constant Acceleration




### 1.32: Circular Motion with Constant Acceleration

The simplest case of non-uniform circular motion is when an object undergoes constant acceleration. In such a scenario, centripetal acceleration is not the only kind of acceleration.

$$\vec{a} = \vec{a_c} + \vec{a_t}$$

Where

$$\vec{a} = acceleration$$

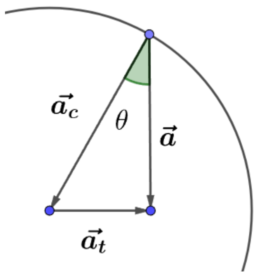
$$\vec{a_c} = centripetal\ acceleration$$
$$\vec{a_t} = tangential\ acceleration$$

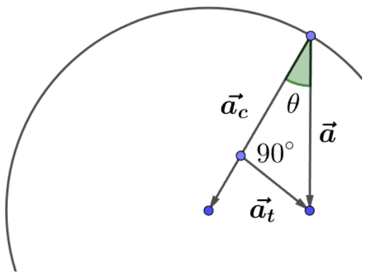
Note: We will discuss tangential acceleration in further detail shortly.

## Example 1.33

Two diagrams illustrate the vector for acceleration as the sum of centripetal and tangential acceleration.

A. Determine the value of magnitude of acceleration in each diagram.
B. Using Part A determine which diagram is correct and which diagram is wrong.
C. Without using Part A, determine which diagram is correct.




Left diagram:

$$cos\ \theta = \frac{a}{a_c} \Rightarrow a_c = \frac{a}{\cos\theta}$$

Right diagram:

$$\cos\theta = \frac{a_c}{a} \Rightarrow a = \frac{a_c}{\cos\theta}$$

Left diagram is wrong because

$$a_c = \frac{a}{\cos\theta} > a$$

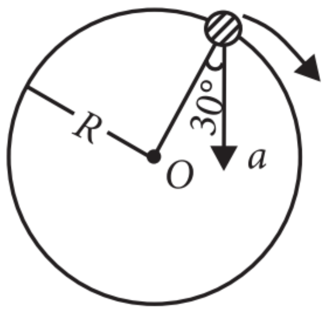
Left diagram is also wrong because the direction of the tangential vector is wrong.

Right diagram, and expression from right diagram is correct.

## Example 1.34

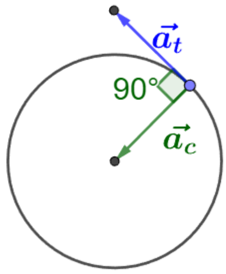
In the given figure $a = 15\frac{m}{s^2}$ represents the total acceleration of a particle moving in the clockwise direction in a circle of radius $R = 2.5\ m$ at a given instant of time. The speed of the particle is:

A. $4.5\frac{m}{s}$
B. $5.0\frac{m}{s}$
C. $5.7\frac{m}{s}$
D. $6.2\frac{m}{s}$ **(NEET 2016)**




$$\frac{v^2}{r} = a_c \Rightarrow v = \sqrt{a_c r} = \sqrt{(a\cos 30°)r} = \sqrt{15\cdot\frac{\sqrt{3}}{2}\cdot 2.5} = \sqrt{18.75\cdot\sqrt{3}} \approx \sqrt{19\cdot 1.7} = \sqrt{32.3}$$

*Option C*




### 1.35: Tangential Acceleration

An object which is in non-uniform motion also has a component of acceleration along its velocity.
This component of acceleration is called tangential acceleration.

$$a_t = r\alpha$$

### Example 1.36

What is the value of the dot product of the centripetal acceleration vector and the tangential acceleration vector?

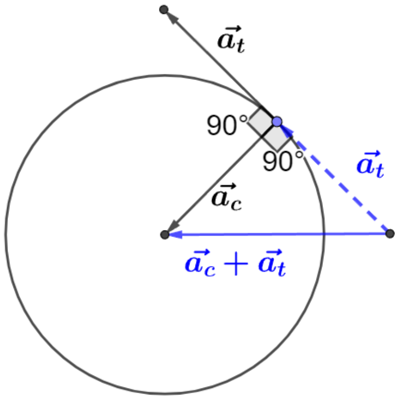
The two vectors are perpendicular to each other. Hence, their dot product is always zero.

### 1.37: Total Acceleration

$$\vec{a} = \vec{a_c} + \vec{a_t}$$

The magnitude of the total acceleration is:

$$a = \sqrt{a_t^2 + a_c^2}$$




Shift $\vec{a_t}$ so that its tip is at the tail of $\vec{a_c}$.

Using Pythagoras Theorem, the magnitude of the total acceleration is:

$$a = \sqrt{a_t^2 + a_c^2}$$

## B. Equations of Motion

### 1.38: Average Angular Acceleration

The angular acceleration of an object over time $\Delta t$ is:

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{a}{r}$$

Recall that acceleration is the change in velocity over the change in time:

$$Acceleration = a = \frac{\Delta v}{\Delta t}$$

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\frac{\Delta v}{r}}{\Delta t} = \frac{1}{r} \cdot \frac{\Delta v}{\Delta t} = \frac{a}{r}$$

### Example 1.39

A particle starting from rest, moves in a circle of radius $r$. It attains a velocity of $V_0 \frac{m}{s}$ in the $n^{th}$ round. Its angular acceleration will be: **(NEET 2019)**

Substitute $v = V_0, u = 0, s = (2\pi r)n$ in $v^2 = u^2 + 2as$:

$$V_0^2 = 0 + 2a(2\pi r)n \Rightarrow a = \frac{V_0^2}{4\pi nr} \Rightarrow \alpha = \frac{a}{r} = \frac{V_0^2}{4\pi nr^2}$$

## 1.40: Variables in Circular Motion

| | Linear | Angular | |
|---|---|---|---|
| Position | $x$ | $\theta$ | |
| Displacement | $s$ | $\Delta\theta$ | |
| Velocity | $v$ | $\omega$ | $\omega = \frac{v}{r}$ |
| Acceleration | $a$ | $\alpha$ | $\alpha = \frac{a}{r}$ |

## 1.41: Equations of Motion

Assumptions:

- Constant Acceleration

| | Linear | Angular |
|---|---|---|
| | | |
| | $v = u + at$ | $\omega_F = \omega_I + \alpha t$ |
| | $s = s_0 + ut + \frac{1}{2}at^2$ | $\theta_F = \theta_I + \omega_I t + \frac{1}{2}\alpha t^2$ |
| | $s = s_0 + vt - \frac{1}{2}at^2$ | $\theta_F = \theta_I + \omega_F t - \frac{1}{2}\alpha t^2$ |
| | $v^2 = u^2 + 2as$ | $\omega_F^2 = \omega_I^2 + 2\alpha(\Delta\theta)$ |
| | $v_{avg} = \frac{v+u}{2}$ | $\omega_{avg} = \frac{\omega_F + \omega_I}{2}$ |
| Displacement in the $n^{th}$ second | $s_n = u + \frac{1}{2}a(2n-1)$ | $(\Delta\theta)_{n^{th}} = \omega_I + \frac{1}{2}\alpha(2n-1)$ |

### Example 1.42

A particle starting from reset, moves in a circle of radius $r$. It attains a velocity of $V_0 \frac{m}{s}$ in the $n^{th}$ round. Its angular acceleration will be: **(NEET 2019)**

Substitute $\omega_F^2 = \left(\frac{V_0}{r}\right)^2 = \frac{V_0^2}{r^2}, \omega_I^2 = 0, \Delta\theta = 2\pi n$ in $\omega_F^2 = \omega_I^2 + 2\alpha(\Delta\theta)$

$$\frac{V_0^2}{r^2} = 0 + 2\alpha(2\pi n) \Rightarrow \alpha = \frac{V_0^2}{4\pi n r^2}$$

### Example 1.43

A flywheel starts from rest and rotates at a constant acceleration of $2\frac{rad}{s^2}$. The number of revolutions that it makes in first 10 $s$ is nearest which integer? **(EAMCET, 17 Sep 2020, Shift-I)**

Substitute $\theta_I = 0, \omega_I = 0, \alpha = 2$ in

$$\theta_F = \theta_I + \omega_I t + \frac{1}{2}\alpha t^2 = 0 + 0 + \frac{1}{2}(2)(10^2) = 100\ rad$$

The number of revolutions is:

$$\frac{100}{2\pi} = \frac{50}{\pi} \approx \frac{50}{\frac{22}{7}} = \frac{350}{22} = \frac{175}{11} \approx 16$$

# 1.3 Centripetal/Centrifugal Forces

## A. Frame of Reference

### 1.44: Inertial vs. Non-Inertial Frame

An inertial frame of reference is one which is not undergoing any acceleration.

- A train at rest is an inertial frame of reference
- An accelerating train is not an inertial frame of reference

### Example 1.45

Consider a glass on a smooth table on a rough carpet in the dining carriage of a train at rest. The train accelerates sharply. Describe the horizontal position of the glass in the frame of reference of an observer

A. on the ground
B. in the train.

**Part A**

Since the table is smooth, the friction is negligible.
Hence, when the table moves, the glass remains in position (horizontally) till the table moves away from it, and the glass drops to the ground.

**Part B**

The glass moves in the negative $x$ direction.

### 1.46: NLM apply only to Inertial Frames of Reference

Newton's Law of Motion apply only to inertial frames of reference.
In an accelerating frame of reference, you need to introduce pseudoforces to explain the behaviour of objects.

### Example 1.47

Consider the glass from the previous example. Describe the forces on the glass, and acceleration of the glass from the point of view of an observer

A. on the ground
B. In the train

**Part A**

$$F = ma$$

Since $a = 0$

$$F = m(0) = 0$$

**Part B**

$$F = ma$$

Substitute $a = -A$

$$F = -mA$$

## B. Centripetal Force

### 1.48: Centripetal Force

In an inertial frame of reference, an object undergoing uniform circular motion has a centripetal force acting on it.

$$Centripetal\ force\ is\ the\ force\ which\ causes\ centripetal\ acceleration.$$

### Example 1.49

In each case, identify the centripetal force:

A. Moon going around the Earth
B. Ball tied to a string moved around in a circle
C. Car taking a turn on a horizontal, circular road
D. A bike going around a well of death
E. A coin placed on a rotating gramophone disc

$$Centripetal\ Force: Gravity$$
$$Centripetal\ Force: Tension\ in\ the\ string$$
$$Centripetal\ Force: Friction\ from\ the\ road$$
$$Centripetal\ Force: The\ wall\ from\ the\ well\ of\ death$$
$$Centripetal\ Force: Friction\ from\ the\ gramophone\ disc$$

### 1.50: Centripetal Force

It is an actual force, not a pseudoforce.
It does not have to be introduced.

### 1.51: Centripetal Force

$$Centripetal\ Force = ma_c = \frac{mv^2}{r} = m\omega^2 r$$

### Example 1.52

A 500 $kg$ car takes a round turn of radius 50 $m$ with a velocity of $36\frac{km}{hr}$. The centripetal force is: **(NEET 1999)**

$$v = 36\frac{km}{hr} = 10\frac{m}{s}$$
$$F_{Centripetal} = \frac{mv^2}{r} = \frac{500 \times 10^2}{50} = 1000\ N$$

### Example 1.53

Find the centripetal force acting on a coin weighing 0.1 $kg$ placed 0.1 $m$ from the center on a gramophone disc rotating at 600 $rpm$. **(EAMCET 17 Sep 2020, Shift-II)**

Substitute $\omega = \frac{Rotations}{Time} \times 2\pi = \frac{600}{60\ s} \times 2\pi = 20\pi$

$$F = mr\omega^2 = (0.1)(0.1)(20\pi)^2 = 4\pi^2$$

## C. Centrifugal Force

### 1.54: Centrifugal Force

Centrifugal force is a pseudoforce that must be introduced in a rotating frame of reference to explain the behaviour of objects.
Recall that Newton's Laws of Motion are not applicable in a rotating frame of reference.

### Example 1.55

You are in a car sitting on a smooth seat moving forward. The car takes a sharp turn to the left. You get pushed to the right. Identify the force on you considering:

A. A rotating (non-inertial) frame of reference
B. An inertial frame of reference

#### Part A

You in a rotating frame of reference. The force pushing you right is a pseudoforce:

$$Outward = Centrifugal\ Force$$

#### Part B

In an inertial frame of reference, the car is moving to the left (as a part of the circular motion). There is no force acting on you, and hence you continue movement forward.
Hence, there is no force on you.

### Example 1.56

A coin is placed on a rough, round turntable about 2 feet from the center. The turntable starts spinning at a constant speed. The coin retains its relative position on the turntable. Identify the forces on the coin considering:

A. A rotating frame of reference
B. An inertial frame of reference

#### Part A

Consider a camera at the center of the table positioned with the coin in its view. The camera also rotates when the table rotates.

$$Outward\ Force = Centrifugal\ Force$$
$$Inward\ Force = Frictional\ Force$$

#### Part B

If the turntable were smooth, the coin would remain in the same position.
Since the turntable is rough, the coin moves with the table.

The direction of velocity of the coin is tangential to the movement of the coin.
It maintains its place due to friction from the table, which acts as a centripetal force.

### 1.57: Actual Force versus Pseudo Force

Centripetal force is an actual force (for example, friction).
Centrifugal force is a pseudoforce (to be introduced by you).

### Example 1.58

When milk is churned, cream gets separated due to:

A. Centripetal force
B. Centrifugal force
C. Frictional force
D. Gravitational force

$$Option\ B$$

### Example 1.59

When milk is churned in a vessel (consider an inertial frame of reference), what is the centripetal force?

$$Normal\ force\ of\ liquid\ with\ walls\ of\ the\ vessel$$

### 1.60: Analysis using Centrifugal Force

Analysis using centrifugal force (rotating frame of reference) is often easier.
Hence, for solving questions, we prefer rotating frame of reference.

## D. Magnitude of Centrifugal Force

### 1.61: Centrifugal Force

Centrifugal force is opposite in direction and equal in magnitude to centripetal force:

$$Centrifugal\ Force = Centripetal\ Force = ma_c = \frac{mv^2}{r} = m\omega^2 r$$

- Consider a person in a car which takes a sharp turn leftwards
- Person moves right and the magnitude of moving right is based on magnitude of car moving left.

### Example 1.62

A ball of mass 0.25 $kg$ attached to the end of a string of length $1.96m$ is moving in a horizontal circle. The string will break if the tension is more than $25N$. What is the maximum speed with which the ball can be moved? **(NEET 1998)**

$$\text{Centrifugal force} = \frac{mv^2}{r} = 25$$

$$v = \sqrt{\frac{25r}{m}} = \sqrt{\frac{25 \times 1.96}{\frac{1}{4}}} = \sqrt{196} = 14\frac{m}{s}$$

## E. Equilibrium Conditions: Maintaining Position

### Example 1.63

A car of mass $m$ is moving on a level circular track of radius $R$. If $\mu_s$ represents the static friction between the road and tires of the car, the maximum speed of the car in circular motion is given by: **(NEET 2012)**

Consider a rotating frame of reference.
In the equilibrium condition, the outward (centrifugal) force must equal the maximum value of the frictional force:

$$Outward\ Force = Max(Frictional\ Force)$$

$$\frac{mv^2}{R} = \mu_s mg$$

$$v = \sqrt{\mu_s gR}$$

**Example 1.64**
Find the range of $r$.
(NEET 2010)

56. A gramophone record is revolving with an angular velocity ω. A coin is placed at a distance $r$ from the centre of the record. The static coefficient of friction is μ. The coin will revolve with the record if

The outward centrifugal force should be less than the inward frictional force:

$$m\omega^2 r \le \mu mg \Rightarrow r \le \frac{\mu g}{\omega^2}$$

### F. Equilibrium Conditions: Weightlessness

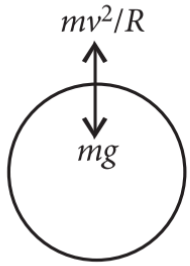
**1.65: Newton's First Law**
For an object to be in equilibrium, the net force on it must be zero.

**1.66: Weightlessness**
Force of gravity will just balance upward force.

$$v = \sqrt{rg}$$

Consider a rotating frame of reference.

$$Upward\ Force = Outward\ Force = Centrifugal\ force = \frac{mv^2}{r}$$
$$Downward\ Force = Gravity = mg$$




For weightlessness, the downward force must just equal the upward force:

$$mg = \frac{mv^2}{r}$$

**Pending**

**Example 1.67**
(NEET 2008)

57. A roller coaster is designed such that riders experience "weightlessness" as they go round the top of a hill whose radius of curvature is 20 m. The speed of the car at the top of the hill is between

$$v = \sqrt{rg} = \sqrt{20 \times 10} = \sqrt{200} = 10\sqrt{2} \approx 14.1$$

# 1.4 Calculus with Circular Motion

## A. Velocity and Acceleration

**1.68: Velocity and Acceleration Vectors**
If $\vec{s}(t)$ is the position vector for a particle, then:

$$\frac{d\vec{s}}{dt} = \vec{v}(t) = velocity\ vector$$
$$\frac{d^2\vec{s}}{dt^2} = \frac{d\vec{v}}{dt} = \vec{a}(t) = acceleration\ vector$$

**1.69: Instantaneous Angular Velocity**

$$\omega = \frac{d\theta}{dt}$$

## 1.70: Instantaneous Angular Acceleration

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \omega\frac{d\omega}{dt} = \frac{a}{r}$$

### Part A

We define angular acceleration as the instantaneous rate of change of angular velocity with respect to time.

### Part B

We show that angular acceleration is the second derivative of angular displacement.

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$$

### Part C

Use the formula/derivation from linear kinematics:

$$a = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx}$$

Using a change of variable:

$$\alpha = \omega\frac{d\omega}{dt}$$

### Part D

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{v}{r}\right) = \frac{a}{r}$$

## 1.71: Equations of Motion

$$\alpha = \frac{d\omega}{dt}, \omega = \frac{d\theta}{dt}$$

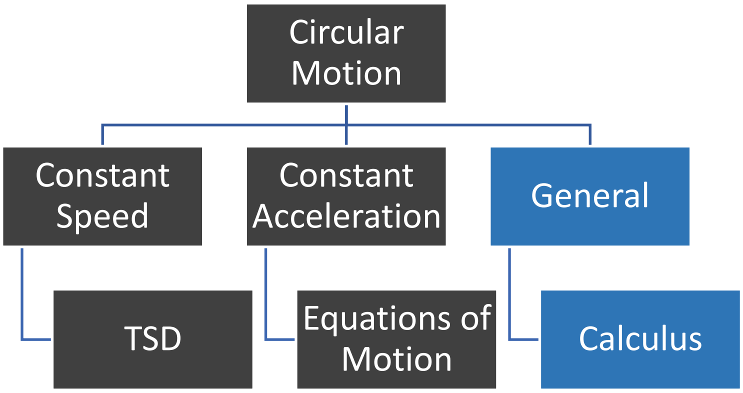
Use a change of variable. Let $a = \alpha, v = \omega$:

$$a = \frac{dv}{dt}, v = \frac{ds}{dt}$$

Using the above, we derived the equations of motion in linear kinematics, and the same process can be repeated. Once the process is repeated, switch back to the original variable.

| | Linear | Angular |
|---|---|---|
| | | |
| | $v = u + at$ | $\omega_F = \omega_I + \alpha t$ |
| | $s = s_0 + ut + \frac{1}{2}at^2$ | $\theta_F = \theta_I + \omega_I t + \frac{1}{2}\alpha t^2$ |
| | $s = s_0 + vt - \frac{1}{2}at^2$ | $\theta_F = \theta_I + \omega_F t - \frac{1}{2}\alpha t^2$ |
| | $v^2 = u^2 + 2as$ | $\omega_F^2 = \omega_I^2 + 2\alpha(\Delta\theta)$ |
| | $v_{avg} = \frac{v+u}{2}$ | $\omega_{avg} = \frac{\omega_F + \omega_I}{2}$ |
| Displacement in the $n^{th}$second | $s_n = u + \frac{1}{2}a(2n-1)$ | $(\Delta\theta)_{n^{th}} = \omega_I \frac{1}{2}\alpha(2n-1)$ |

## B. Tangential and Centripetal Acceleration




### 1.72: Tangential and Centripetal Acceleration

The acceleration of an object in circular motion can be split into tangential and centripetal components, which are perpendicular to each other.

$$\textit{Tangential Acceleration} = \vec{\boldsymbol{a}}_t$$
$$\textit{Centripetal Acceleration} = \vec{\boldsymbol{a}}_c$$

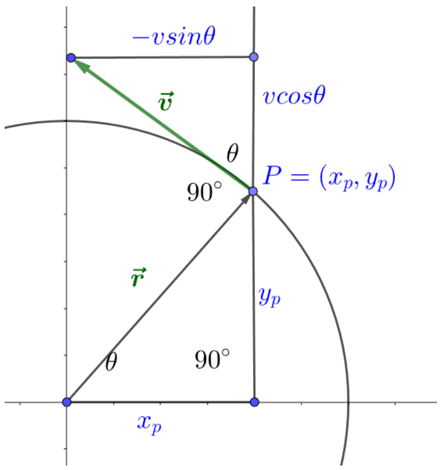
### 1.73: Tangential and Centripetal Acceleration

For an object in circular motion as shown with velocity $\vec{\boldsymbol{v}}$, the acceleration is:

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}_t + \vec{\boldsymbol{a}}_c = \frac{dv}{dt}(-\sin\theta, \cos\theta) + \frac{v^2}{r}(-\cos\theta, -\sin\theta)$$

where

$$\theta = \textit{Angle between radius of circle and x axis}$$




Write the velocity vector in component form:

$$\vec{\boldsymbol{v}} = (v_x, v_y)$$

Write the components in terms of $\theta$:

$$= (-v\sin\theta, v\cos\theta) = v(-\sin\theta, \cos\theta)$$

Substitute $\sin\theta = \frac{y_p}{r}, \cos\theta = \frac{x_p}{r}$, and factor the $v$ out:

$$= v\left(-\frac{y_p}{r}, \frac{x_p}{r}\right)$$

Differentiate velocity to get acceleration. Use the product rule of differentiation:

$$\frac{dv}{dt}\left(-\frac{y_p}{r}, \frac{x_p}{r}\right) + \frac{v}{r}\left(-\frac{dy_p}{dt}, \frac{dx_p}{dt}\right)$$

Substitute $\frac{dy_p}{dt} = v_y = v\cos\theta, \frac{dx_p}{dt} = v_x = -v\sin\theta, \frac{y_p}{r} = \sin\theta, \frac{x_p}{r} = \cos\theta$:

$$\frac{dv}{dt}(-\sin\theta, \cos\theta) + \frac{v}{r}(-v\cos\theta, -v\sin\theta)$$

Factor $v$ from the second term:

$$= \frac{dv}{dt}(-\sin\theta, \cos\theta) + \frac{v^2}{r}(-\cos\theta, -\sin\theta)$$

### 1.74: Direction of Centripetal Acceleration

The direction of centripetal acceleration is towards the center of the motion, and opposite to the position vector.

$$\vec{\boldsymbol{a}}_c = \frac{v^2}{r}(-\cos\theta, -\sin\theta)$$

Suppose:

$$\vec{\boldsymbol{a}}_c \textit{ makes angle } \phi \textit{ with the } x \textit{ axis}$$

$$\tan\phi = \frac{-\sin\theta}{-\cos\theta} = \tan\theta$$
$$\phi = 180 + \theta$$

### 1.75: Components of Acceleration Vector

The tangential and centripetal vectors are the components of the acceleration vector.
Note: Components means they are perpendicular.

If the dot product of two non-zero vectors is zero then the vectors are perpendicular:

$$\left(\frac{dv}{dt}(-\sin\theta, \cos\theta)\right) \cdot \left(\frac{v^2}{r}(-\cos\theta, -\sin\theta)\right)$$

Rearrange:

$$\frac{dv}{dt} \cdot \frac{v^2}{r}(-\sin\theta, \cos\theta) \cdot (-\cos\theta, -\sin\theta)$$

Focus on the dot product, which we will show is zero:

$$(-\sin\theta, \cos\theta) \cdot (-\cos\theta, -\sin\theta)$$
$$= \sin\theta\cos\theta - \sin\theta\cos\theta = 0$$

### 1.76: Magnitude of Tangential and Centripetal Acceleration

The formulas for the magnitudes of the vectors do not require Calculus, and hence are important for answering numerical questions asked in exams:

$$|\vec{\boldsymbol{a}}_t| = \frac{dv}{dt}$$
$$|\vec{\boldsymbol{a}}_c| = \frac{v^2}{r}$$

$$|\vec{\boldsymbol{a}}_t| = \frac{dv}{dt}\sqrt{(-\sin\theta)^2 + \cos^2\theta} = \frac{dv}{dt}\sqrt{1} = \frac{dv}{dt}$$
$$|\vec{\boldsymbol{a}}_c| = \frac{v^2}{r}\sqrt{(-\cos\theta)^2 + (-\sin\theta)^2} = \frac{v^2}{r}\sqrt{1} = \frac{v^2}{r}$$

### Example 1.77

An object is in uniform circular motion with speed $v$ around a circle with radius $r$. What are the magnitudes of its tangential and centripetal acceleration?

$$|\vec{\boldsymbol{a}}_t| = \frac{dv}{dt} = 0$$

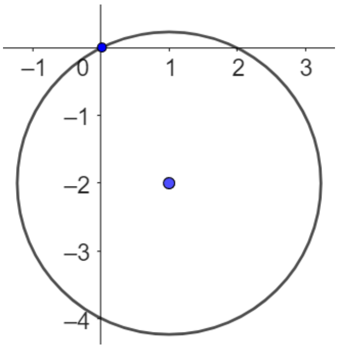
$$|\vec{a}_c| = \frac{v^2}{r}$$

### Example 1.78

*Mark all correct options*

An object is in uniform circular motion around the point $(1, -2)$. It must have non-zero:

A. Centripetal acceleration
B. Tangential acceleration
C. Acceleration
D. Velocity
E. Displacement from the origin




Because the object is in uniform circular motion, it must have magnitude of centripetal acceleration

$$a_c = \frac{v^2}{r} \neq 0 \Rightarrow Option\ A$$

Because speed is constant

$$a_t = 0 \Rightarrow Option\ B\ is\ not\ Valid$$

Because

$$a_c \neq 0 \Rightarrow a \neq 0 \Rightarrow Option\ C$$

Because the object is in uniform circular motion,

$$v \neq 0 \Rightarrow Option\ D$$

$$Options\ A, C, D$$

## C. Parametrizations

### Example 1.79

$$x = a \sin \omega t\,, y = a \cos \omega t$$

Determine the:

A. position vector $\vec{s}$. Determine the distance from the origin. Hence, show that the path represents a circle.
B. velocity vector and its magnitude. Explain why the motion is uniform circular motion.
C. acceleration vector. Calculate its direction and interpret it.

#### Part A

$$\vec{s} = (a \sin \omega t\,, a \cos \omega t)$$

The distance between the origin and the object is:

$$|\vec{s}| = \sqrt{a^2 \sin^2 \omega t + a^2 \cos^2 \omega t} = a(1) = a = Constant$$

Hence, the path is a circle with radius $a$.

#### Part B

$$\vec{v} = \frac{d\vec{s}}{dt} = (a\omega \cos \omega t\,, -a\omega \sin \omega t)$$

$$|\vec{v}| = \sqrt{a^2\omega^2 \cos^2 \omega t + a^2\omega^2 \sin^2 \omega t} = \sqrt{a^2} = |a\omega| = constant$$

Hence, the motion is uniform circular motion because the magnitude is a constant.

### Part C

$$\vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{v}}}{dt} = (-a\omega^2 \sin \omega t\,, -a\omega^2 \cos \omega t) = -a\omega^2(\sin \omega t\,, \cos \omega t) = -\omega^2 \vec{\boldsymbol{s}}$$

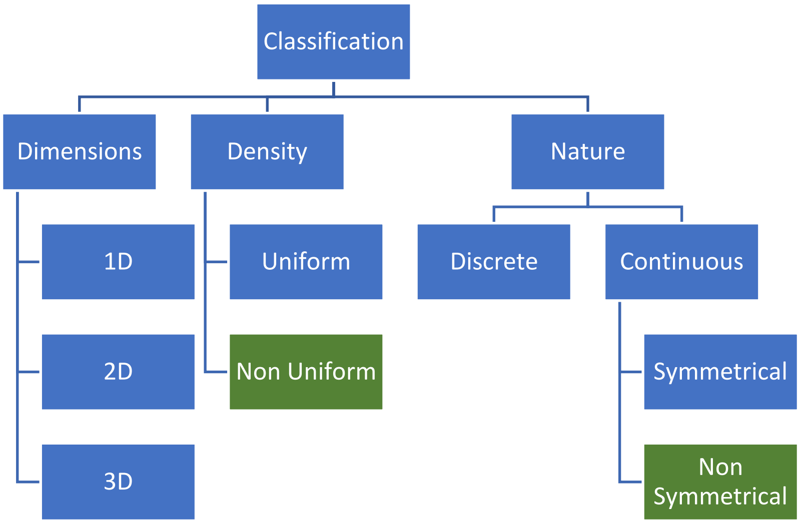
Hence, the acceleration vector has direction which is diametrically opposite to the position vector.
Hence, the acceleration vector always points inwards.

# 2. Center of Mass

## 2.1 Center of Mass: 1D

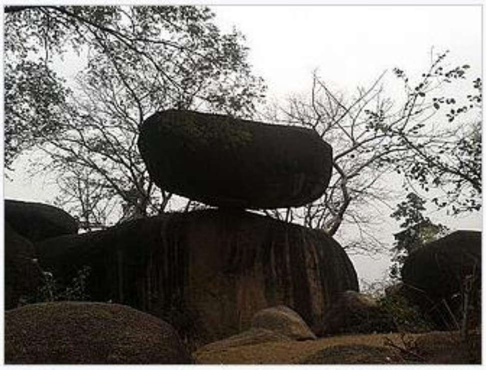
### A. Introduction

**2.1: Center of Mass (COM)**
Center of mass is a single point where you can imagine the mass to be concentrated as a point object.




- The two types colored green must be analyzed using Calculus.
- Hence, a large object can be considered as a point object with the location of the point at its center of mass.
- This simplification is useful in analyzing a range of situations.

**2.2: Interpretations**
There are several interpretations of center of mass. While all are valid, different interpretations will be useful in in analyzing different scenarios.


Balancing rock near Madan Mahal Fort, Jabalpur, Madhya Pradesh, India

**2.3: Balancing Point (Geometry)**
Center of mass is the point at which an object is balanced.

**Examples**

- Placing a rock such the bottom point is exactly below the center of mass keeps the rock balanced.
- If a force is applied to the center of mass of an object, it will result in linear acceleration. If the force is applied at a point other than the center of mass, it will result in angular acceleration.
- In the chapter on Newton's Laws of Motion, we had objects which were pulled or pushed. It is assumed that the forces are applied at the center of mass.

**Applications**

- For an object of uniform density, this is a geometrical interpretation. You can apply geometric properties of symmetry to determine the center of mass.

- If you need to balance an object, you should do so at using its center of mass to not have to apply force on it.

## 2.4: Point Object from a Large Object (Simplification)

Center of mass is the single point to be considered as the location of the point when considering a large object as a point mass.
This is useful when analyzing forces

### Examples

- A dancer jumps through the air. The hands and legs of the dancer do not follow a parabolic trajectory. But the center of mass does.

### Applications

- You need to make simplifying assumptions that the force is applied to the center of mass when applying Newton's Law of Motion. Else, you need to consider that the object will turn. This is, in fact, the whole topic of rotational mechanics.

## 2.5: Mass Weighted Average (Math/Stats)

The center of mass is the $mass - weighted\ average$ of the $positions$ of the elements that make up the object.

### Applications

- You can make use of properties of averages from math/statistics to work with the center of mass.

### Example 2.6

*Choose the correct option*

The center of mass of system of particles does not depend on:

A. Position of the particles
B. Relative distances between the particles
C. Masses of the particles
D. Force acting on the particle **(NEET 1997)**

*Option D*

### Example 2.7

*Choose the correct option*

The center of gravity of the solar system is most near:

A. The Earth
B. Venus
C. Jupiter
D. The Sun

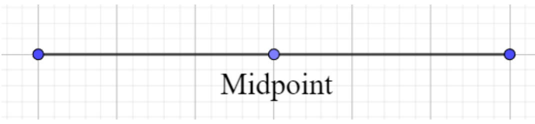
The sun makes up $\approx 99.8\%$ of the mass of the Solar system.

*Option D*

## B. Uniform Density

### 2.8: Linear Object

The center of mass of a linear object of uniform density is the midpoint of the object




### Example 2.9

A tube of length $L$ is filled completely with an incompressible liquid of mass $M$ and closed at both the ends. The tube is then rotated in a horizontal plane about one of its ends with a uniform angular velocity $\omega$. The force exerted by the liquid at the other end is: **(NEET 2006; JEE-A 1992)**

$$F_{Centrifugal} = m\omega^2 r$$

Center of mass of the fluid is at the center of the tube. Replace the fluid with an equivalent point object halfway down the tube.

Substitute $r = \frac{L}{2}, m = M$:

$$= \frac{M\omega^2 L}{2}$$

## C. Point/Discrete Objects

### 2.10: COM: Mass-Weighted Average

The center of mass of a system of discrete particles in a single dimension is the mass weighted average of the positions of the particles:

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \cdots m_nx_n}{m_1 + m_2 + \cdots m_n} = \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i}$$

Where

$$Weights\ are\ given\ by\ the\ masses$$

### Example 2.11

A. Two particles of mass 5 $kg$ and 10 $kg$ respectively are attached to the two ends of a rigid rod of length 1 $m$ with negligible mass. The center of mass of the system from the 5 kg particle is at a distance of: (Express your answer to the nearest cm.) **(NEET 2020)**

B. Three masses are placed on the $x$ axis: 300$g$ at origin, 500 $g$ at $x = 40\ cm$ and 400 $g$ at $x = 70$ cm. The distance of the center of mass from the origin is: **(NEET 2012)**

#### Part A

Introduce a coordinate system with origin at the 5 kg mass:

$$\sum_{i=1}^{n} \frac{m_ix_i}{m_i} = \frac{5(0) + 10(1)}{15} = \frac{10}{15} = \frac{2}{3} \approx 0.666\ m = 66.6\ cm \approx 67\ cm$$

#### Part B

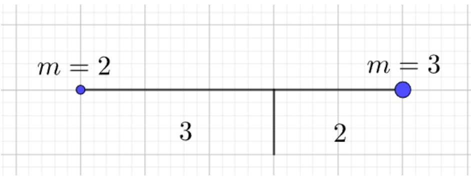
Convert the grams into $kg$ to reduce the numbers of zeroes that we must deal with:

$$\sum_{i=1}^{n} \frac{m_ix_i}{m_i} = \frac{0.3(0) + 0.5(40) + 0.4(70)}{0.3 + 0.5 + 0.4} = \frac{20 + 28}{1.2} = \frac{48}{1.2} = \frac{480}{12} = 40\ cm$$

### 2.12: COM as Balance

The center of mass is the point at which an object will balance.

### 2.13: Seesaw




- If weights on both sides of a seesaw are same, and the arrangement is symmetrical, the seesaw will balance.
- If weights are not the same, the heavier mass must be moved closer to the support.

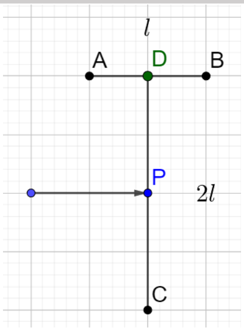
## D. Compound Objects with Uniform Density

### 2.14: Compound Object

A compound object is an object made of multiple shapes, or parts with different densities. Each of its parts must be treated individually.

### 2.15: Replacing with a point object

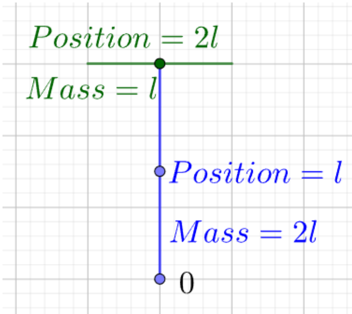
When calculating the center of mass of a compound object, each part of the object can be replaced with a point object of same mass at the location of center of mass of that part.




**Example 2.16**

A $T$ shaped object with dimensions shown in the figure is lying on a smooth floor. A force $\vec{F}$ is applied at the point $P$ parallel to the horizontal part of the object, such that the object has only the translational motion without rotation. Find the location of $P$ with respect to $C$ (in terms of $l$). (JEE-M 2005)

Note: The horizontal length is $l$ and the vertical length is $2l$.




Point $P$ must be the center of mass to ensure only translational motion with no rotation.

Introduce an origin at point $C$. Let the mass of the object be 1 per unit length. The center of mass is:

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = \frac{(\boldsymbol{l})(2\boldsymbol{l}) + (2\boldsymbol{l})(\boldsymbol{l})}{l + 2l} = \frac{4l^2}{3l} = \frac{4}{3}l$$

### 2.17: Back Calculations

You can calculate the center of mass of some part of the object given information on other parts of the object.

## E. 2 Body Systems

### 2.18: Internal Forces on COM

Forces internal to a system do not change the center of mass of the system.

- This is a very general property.

- It can be applied to any situation where there are no forces external to the system.

## Example 2.19

Two particles which are initially at rest move towards each other under the action of their internal attraction. If their speeds are $v$ and $2v$ at any instant, then the speed of center of mass of the system will be: **(NEET 2010)**

There is no external force. Hence, velocity, or speed will be zero.

## Example 2.20

*Choose the correct option*

A body $A$ of mass $M$ while falling vertically downwards under gravity breaks into two parts: a body $B$ of mass $\frac{1}{3}M$ and a body $C$ of mass $\frac{2}{3}M$. The center of mass of bodies $B$ and $C$ taken together shifts compared to that of body A towards:

A. Does not shift
B. Depends on height of breaking
C. Body $B$
D. Body $C$ **(JEE-M 2005)**

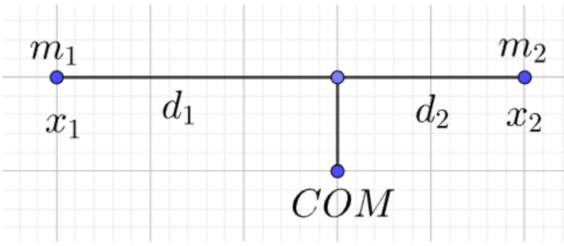
There is no external force. Hence, center of mass continues on its original path. Hence it does not shift.

*Option A*

## 2.21: 2 Body System: Relationship between Mass and Distance

If mass $m_1$ is located at the origin, and $m_2$ is located at a length $l$, then:

$$l_1 = COM = \frac{m_2}{m_1 + m_2} l$$
$$l_2 = \frac{m_1}{m_1 + m_2} l$$




$$l = x_2$$

The center of mass is the mass weighted average of the positions of the objects:

$$l_1 = COM = \frac{m_1(0) + m_2 l}{m_1 + m_2} = \frac{m_2}{m_1 + m_2} l$$

The distance between the second object and the center of mass is:

$$l_2 = l - l_1 = l - \frac{m_2}{m_1 + m_2} l = \frac{m_1}{m_1 + m_2} l$$

## 2.22: 2 Body System: Relationship between Mass and Distance

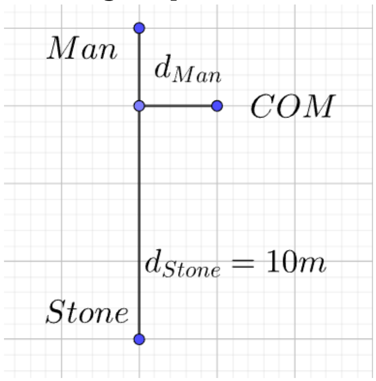
$$l_1 m_1 = l_2 m_2$$

$$\frac{l_1}{l_2} = \frac{\frac{m_2}{m_1 + m_2} l}{\frac{m_1}{m_1 + m_2} l} = \frac{m_2}{m_1}$$

## Example 2.23

A man of 50 $kg$ mass is standing in a gravity free space at a height of 10 m above the floor. He throws a stone of 0.5 $kg$ mass downwards with a speed of $2\frac{m}{s}$. When the stone reaches the floor, the distance of the man above the floor will be: **(NEET 2010)**

- The initial center of mass of the man-stone system is exactly where the man and stone both are.
- Forces internal to a system do not change the center of mass of the system.
- Hence, even when the stone reaches the floor, the center of mass remains at the original point.




Hence, using the formula $d_1 m_1 = d_2 m_2$:

$$d_{man} m_{man} = d_{stone} m_{stone}$$
$$d_{man}(50) = (10)(0.5)$$
$$d_{man} = 0.1$$

Final distance

$$= 10 + 0.1 = 10.1\ m$$

## F. Relative Motion

### 2.24: Relative Motion in a 2 Body System

To reduce calculations, let the origin be the position of $m_1$, and the mass of the system be 1.

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_2 x_2}{1}$$

The new center of mass must equal the old center of mass:

$$m_1 d + m_2(x_2 - D) = m_2 x_2$$
$$m_1 d + m_2 x_2 - m_2 D = m_2 x_2$$
$$m_1 d - m_2 D = 0$$
$$D = \frac{m_1}{m_2} d$$

Relative to the first object, if the second object is:

- heavy, then $m_2$ is greater than $m_1$, and the distance moved is small.
- light, then $m_1$ is less than $m_2$, and the distance moved is large.

### Example 2.25

Consider a system of two particles having masses $m_1$ and $m_2$. If the particle of mass $m_1$ is pushed towards the center of mass of the particles through a distance $d$, by what distance would particle of mass $m_2$ move to keep the center of mass of the particles at the original position. **(NEET 2004; JEE-M 2006)**

$$D = \frac{m_1}{m_2} d$$

### 2.26: Change of Center of Mass

- When considering change of center of mass, we only need to consider positions that have changed.
- This is analogous to change to change in average where we only need to consider data points that have changed.

$$\bar{X} = New\ Center\ of\ Mass$$

$$\bar{x} = Old\ Center\ of\ Mass$$

The difference between the two is:

$$\bar{X} - \bar{x} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nX_n}{M} - \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{M}$$
$$= \frac{m_n(X_n - x_n)}{M} = \frac{m_n\Delta x}{M}$$

Where

$$M = Mass\ of\ System$$
$$X_n - x_n = \Delta x = change\ in\ position$$

### Example 2.27

Two persons of masses 55 $kg$ and 65 $kg$ respectively are at the opposite ends of a boat. The length of the boat is 3.0 $m$ and weighs 100 $kg$. The 55 $kg$ man walks up to the 65 $kg$ man and sits with him. If the boat is in still water, the center of mass of the system shifts by: **(NEET 2012)**

#### Short Method

The 55 $kg$ man shifts position by 3 $m$, and hence the change in the center of mass is:

$$= \frac{m_{65\ kg}(\Delta x)}{M} = \frac{55(3)}{220} = \frac{165}{220} = \frac{3}{4}m$$

#### Long Method

We need to work out the center of mass of the boat. Suppose we assume that the mass of the boat is uniformly distributed. Then the center of mass of the boat is the midpoint

$$= 1.5\ m$$

$$\bar{x} = \frac{55(0) + 100(1.5) + 65(3)}{220}$$
$$\bar{X} = \frac{100(1.5) + 65(3) + 55(3)}{220}$$
$$\bar{X} - \bar{x} = \frac{55(3)}{220} = \frac{3}{4}m$$

### Example 2.28

Two persons of masses 55 $kg$ and 65 $kg$ respectively are at the opposite ends of a boat. The length of the boat is 3.0 $m$ and weighs 100 $kg$. The density of the boat increases exponentially from the left endpoint to the right. The 55 $kg$ man walks up to the 65 $kg$ man and sits with him. If the boat is in still water, the center of mass of the system shifts by:

You cannot use the long method here, but the short method still works, and we get the same answer:

$$\Delta\bar{x} = \frac{m_{65\ kg}(\Delta x)}{M} = \frac{55(3)}{220} = \frac{165}{220} = \frac{3}{4}m$$

## G. Velocity of COM

### 2.29: Velocity of COM

$$\vec{\boldsymbol{v}}_{com} = \frac{\sum_{i=1}^{n} m_i\vec{\boldsymbol{v}}_i}{\sum_{i=1}^{n} m_i}$$

**2.30: Acceleration of COM**

$$\vec{\boldsymbol{a}}_{com} = \frac{\sum_{i=1}^{n} m_i \vec{\boldsymbol{a}}_i}{\sum_{i=1}^{n} m_i}$$

# 2.2 Center of Mass: 2D/3D

## A. Point/Discrete Objects

**2.31: COM in 3D**

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}$$

$$\bar{z} = \frac{m_1 z_1 + m_2 z_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i}$$

**2.32: COM in 3D: Vector Version**

$$\vec{\boldsymbol{r}}_{com} = (\bar{x}, \bar{y}, \bar{z}) = \frac{\sum_{i=1}^{n} m_i \vec{\boldsymbol{r}}_i}{\sum_{i=1}^{n} m_i}$$

### Example 2.33

Two bodies of mass 1 $kg$ and 3 $kg$ have position vectors $\hat{\imath} + 2\hat{\jmath} + \hat{k}$ and $-3\hat{\imath} - 2\hat{\jmath} + \hat{k}$ respectively. Determine the magnitude of the position vector for the center of mass of the system. **(NEET 2009; JEE-M 2022)**

$$\vec{\boldsymbol{r}}_{com} = \sum_{i=1}^{n} \frac{m_i \vec{\boldsymbol{r}_i}}{m_i} = \frac{1(1,2,1) + 3(-3,-2,1)}{1+3} = \frac{(1,2,1) + (-9,-6,3)}{4} = \frac{(-8,-4,4)}{4} = (-2,-1,1)$$

Magnitude

$$= \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$$

**2.34: Equal Masses**

If the masses are equal, the center of mass is only the average of the positions, and does not depend on the masses being heavier or lighter.

- Hence, we can take equal masses of any magnitude.
- Simple numbers like 1 lead to easier calculations.

$$\frac{\sum_{i=1}^{n} m_i \vec{\boldsymbol{r}}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m \vec{\boldsymbol{r}}_i}{\sum_{i=1}^{n} m} = \frac{m \sum_{i=1}^{n} \vec{\boldsymbol{r}}_i}{mi} = \frac{\sum_{i=1}^{n} \vec{\boldsymbol{r}}_i}{i}$$

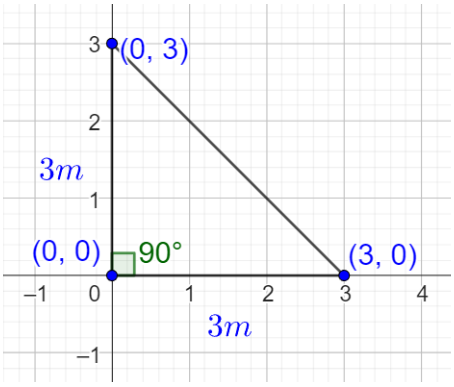
### Example 2.35

Three identical spheres each of mass $M$ are placed at the corners of a right-angled triangle with mutually perpendicular sides equal to 3m each. Taking point of intersection of mutually perpendicular sides as origin, the magnitude of position vector of center of the mass will be $\sqrt{x}\ m$. The value of $x$ is: **(JEE-M 2022)**

Take the bottom left corner of the triangle as the origin, and let the mutually perpendicular sides be aligned with $x$ and $y$ axes.

We can replace $M$ with any value and the answer will remain the same. Take $M = 1$

$$\vec{r}_{com} = \frac{\sum_{i=1}^{n} \vec{r}_i}{i} = \frac{(0,3) + (3,0) + (0,0)}{3} = \frac{(3,3)}{3} = (1,1)$$




The magnitude of the position vector is

$$= \sqrt{1^2 + 1^2} = \sqrt{2} \Rightarrow x = 2$$

## B. Uniform Density and Symmetry

### 2.36: Center of Mass

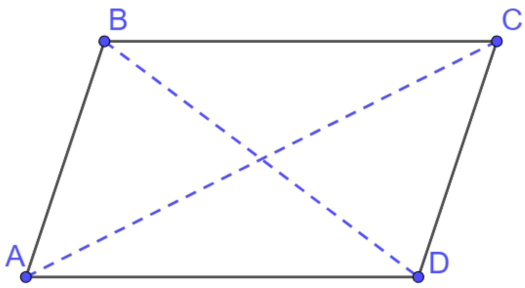
The center of mass of a 2D object with uniform density is the point of intersection of the lines of symmetry of the object.

- If an object is not symmetrical, it still has a center of mass, but the center of mass must be found using some other method.

### 2.37: Parallelogram

The diagonals of a parallelogram divide it into two parts of equal mass. Hence, the center of mass lies on these diagonals.

Hence, the point of intersection of the diagonals is its center of mass (assuming uniform density).




- Squares and rectangles are special cases of parallelograms. Hence, their center of mass is also the intersection of their diagonals.

### 2.38: Circle

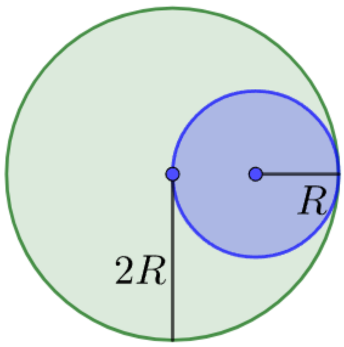
The center of mass of a circle of uniform density is its geometrical center.

- Circle has infinite lines of symmetry, which intersect at its center.
- Center of mass is also the center of the circle.

### Example 2.39

A circular disc of radius $R$ is removed from a bigger circular disc of radius $2R$ such that the circumferences of the discs coincide. The center of mass of the new disc is $\alpha R$ from the center of the bigger disc, where $\alpha R$ represents distance. The value of $\alpha$ is: **(JEE-M 2007)**

Draw a diagram. In the absence of information, let the circular disc removed be at the rightmost position of the circle.




The center of mass of the original circular disc would also be the center of the original circle.
Introduce an origin at the center of the original circle.

From the diagram, the mass above the origin is equal to the mass below the origin. (This is also true because the shape removed has equal mass above and below the origin).

Hence, the $y$ coordinate of the center of mass remains unchanged.

To determine the $x$ coordinate, consider the center of mass of the complete circle as the mass weighted average of the blue disc (removed circle) and green portion (remaining portion). The formula is:

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Since the center of mass is at the origin $\bar{x} = 0$.

$$\underbrace{0 = m_1 x_1 + m_2 x_2}_{\textit{Equation I}}$$

Replace each of the blue and green parts with a point object at their respective centers of mass:

$$Blue: Mass = \pi R^2, COM = R$$
$$Green: Mass = 4\pi R^2 - \pi R^2 = 3\pi r^2, COM = x$$

Substitute the above into Equation I:

$$0 = (\pi R^2)(R) + (3\pi R^2)(x)$$

Divide both sides by $\pi R$:

$$0 = R + 3x \Rightarrow x = -\frac{R}{3} \Rightarrow Distance = \alpha = \frac{1}{3}$$

## 2.40: Triangle (Centroid)

The centroid of a triangle is the point of intersection of the three medians.

Given a triangular shape of uniform density with vertices $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3)$ the centroid and center of mass are given by:

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

## 2.41: Center of Mass: 3D

The center of mass of a 3D object with uniform density is the point of intersection of the planes of symmetry of the object.

## 2.42: Sphere

The center of mass of a sphere is its geometric center.

### Example 2.43

Three identical metal balls, each of radius $r$ are placed touching each other on a horizontal surface such that an equilateral triangle is formed when centers of three balls are joined. The center of the mass of the system is located at:

A. Line joining centers of any two balls
B. Center of one of the balls
C. Horizontal surface
D. Point of intersection of the medians

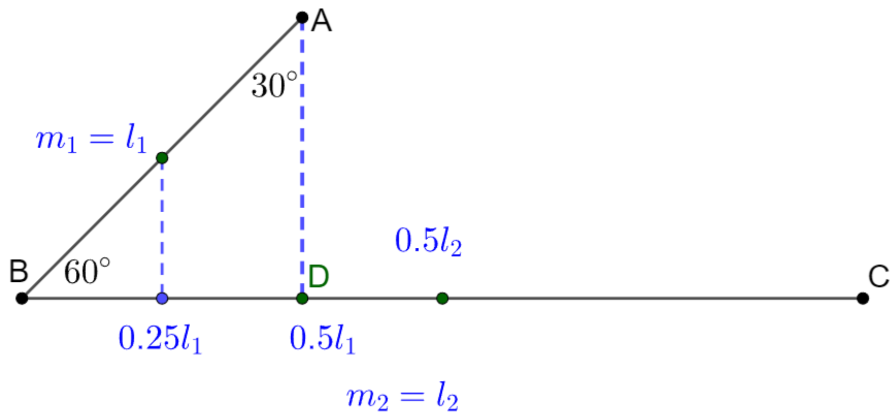
Center of mass of each sphere will be geometrical center of the individual spheres.
These form an equilateral triangle, and the center of mass of the system will be centroid of the triangle, given by point of intersection of the medians.

*Option D*

## C. Uniform Density

**Example 2.44**
In the figure shown $ABC$ is a uniform wire. $\angle ABC$ is $60°$. If center of mass of mass of wire lies vertically below point A, then $\frac{BC}{AC}$ is close to:
**(JEE-M 2016)**


$$x_{COM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

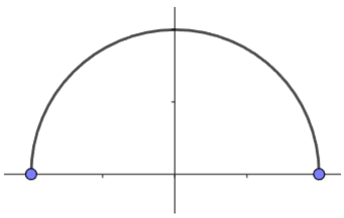



Substitute $x_{COM} = 0.5l_1 = \frac{l_1}{2}, m_1 = m_{AB} = x_1, x_1 = 0.25l_1 = \frac{l_1}{4}, m_2 = m_{BC} = l_2, x_2 = \frac{0.5l_2}{2}$:

$$\frac{l_1}{2} = \frac{l_1\left(\frac{l_1}{4}\right) + (l_2)\left(\frac{l_2}{2}\right)}{l_1 + l_2}$$
$$\frac{l_1^2 + l_1l_2}{2} = \frac{l_1^2 + 2l_2^2}{4}$$
$$\frac{l_1^2 + l_1l_2}{1} = \frac{l_1^2 + 2l_2^2}{2}$$
$$2l_1^2 + 2l_1l_2 = l_1^2 + 2l_2^2$$
$$l_1^2 + 2l_1l_2 = +2l_2^2$$
$$l_1^2 + 2l_1l_2 = 2l_2^2$$

Divide both sides by $l_1^2$:

## 2.45: Semi-Circle



- By symmetry, the $x$ coordinate of a semi-circle is the same as the $x$ coordinate of the center of the semi-circle.

### 2.46: Quarter-Circle

### 2.47: Hemisphere

### 2.48: Cone

# 2.3 Calculus with Center of Mass

## A. Velocity and Acceleration

### 2.49: Velocity of COM

$$\vec{\boldsymbol{v}}_{com}=\frac{\sum_{i=1}^{n}m_i\vec{\boldsymbol{v}}_i}{\sum_{i=1}^{n}m_i}$$

$$\vec{\boldsymbol{r}}_{com}=\frac{\sum_{i=1}^{n}m_i\vec{\boldsymbol{r}}_i}{\sum_{i=1}^{n}m_i}$$

Differentiate both sides of the above:

$$\frac{d}{dt}\vec{\boldsymbol{r}}_{com}=\frac{d}{dt}\left(\frac{\sum_{i=1}^{n}m_i\vec{\boldsymbol{r}}_i}{\sum_{i=1}^{n}m_i}\right)$$

Use the constant rule on the rule on the RHS to move the differentiating operator into the numerator:

$$\vec{\boldsymbol{v}}_{com}=\frac{\frac{d}{dt}\sum_{i=1}^{n}m_i\vec{\boldsymbol{r}}_i}{\sum_{i=1}^{n}m_i}$$

Use the sum rule to interchange differentiation and summation:

$$\vec{\boldsymbol{v}}_{com}=\frac{\sum_{i=1}^{n}\frac{d}{dt}(m_i\vec{\boldsymbol{r}}_i)}{\sum_{i=1}^{n}m_i}$$

Use the constant rule one more time:

$$\vec{\boldsymbol{v}}_{com}=\frac{\sum_{i=1}^{n}m_i\left(\frac{d}{dt}\vec{\boldsymbol{r}}_i\right)}{\sum_{i=1}^{n}m_i}$$

The derivative of each position vector is the corresponding velocity vector:

$$\vec{\boldsymbol{v}}_{com}=\frac{\sum_{i=1}^{n}m_i\vec{\boldsymbol{v}}_i}{\sum_{i=1}^{n}m_i}$$

### 2.50: Acceleration of COM

$$\vec{\boldsymbol{a}}_{com}=\frac{\sum_{i=1}^{n}m_i\vec{\boldsymbol{a}}_i}{\sum_{i=1}^{n}m_i}$$

Differentiate both sides of

$$\vec{\boldsymbol{v}}_{com}=\frac{\sum_{i=1}^{n}m_i\vec{\boldsymbol{v}}_i}{\sum_{i=1}^{n}m_i}$$

And the steps are the same as for the earlier differentiation of position.

## B. Solid Shapes with Uniform Density

### 2.51: Area as Mass

For objects of uniform density, we can consider the area of the object as proportional to the mass of the object.

$$\sum_{i=1}^{n} \frac{\underbrace{m_i}_{Mass} \underbrace{x_i}_{Position}}{\underbrace{m_i}_{Mass}}$$

### 2.52: Center of Mass

$$\bar{x} = \frac{\int \tilde{x}\, dA}{\int dA}, \qquad \bar{y} = \frac{\int \tilde{y}\, dA}{\int dA}$$

Where

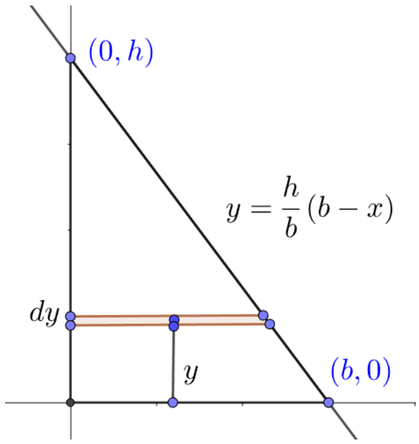
$$differential\ element\ (small\ change\ in\ area)$$

### Example 2.53

Using integration, find the centroid of the triangle formed by the intersection of the axes with the line

$$y = \frac{h}{b}(b - x), h > 0, b > 0$$

Substitute $y = \frac{h}{b}(b - x) \Rightarrow x = b - \frac{by}{h} = \frac{b}{h}(h - y), dA = \underbrace{\frac{b}{h}(h - y)}_{Width} \underbrace{dy}_{Height}$ in




$$\bar{y} = \frac{\int y\, dA}{\int dA} = \frac{\int y\left[\frac{b}{h}(h - y)\, dy\right]}{\int \frac{b}{h}(h - y)\, dy}$$

Integration is with respect to $y$. Hence, $\frac{b}{h}$ is a constant:

$$= \frac{\frac{b}{h}\int y[(h - y)\, dy]}{\frac{b}{h}\int (h - y)\, dy} = \frac{\int y[(h - y)\, dy]}{\int (h - y)\, dy} = \frac{\int (yh - y^2)\, dy}{\int (h - y)\, dy}$$

Evaluate and substitute the limits of integration:

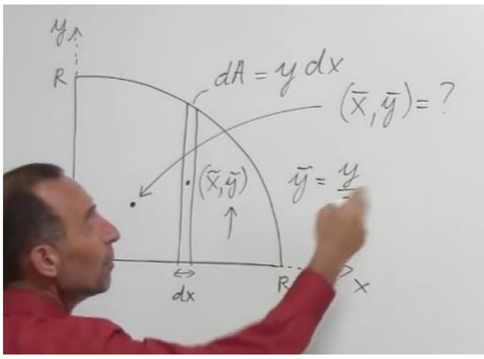
$$= \frac{\left[\frac{y^2 h}{2} - \frac{y^3}{3}\right]_0^h}{\left[yh - \frac{y^2}{2}\right]_0^h} = \frac{\frac{h^3}{2} - \frac{h^3}{3}}{h^2 - \frac{h^2}{2}} = \frac{\frac{h^3}{6}}{\frac{h^2}{2}} = \frac{h^3}{6} \cdot \frac{2}{h^2} = \frac{h}{3}$$

### 2.54: Centroid

$$\bar{x} = \frac{1}{A}\int \tilde{x}\, dA, \qquad \bar{y} = \frac{1}{A}\int \tilde{y}\, dA$$

### Example 2.55

Determine the centroid of a quarter circle with center at the origin and radius $R$.




$$\bar{y} = \frac{1}{A}\int \tilde{y}\, dA$$

Substitute $\tilde{y} = \frac{y}{2}, dA = y\,dx, A = \frac{\pi r^2}{4}$:

$$= \frac{1}{\frac{\pi R^2}{4}}\int \left(\frac{y}{2}\right) y\, dx = \frac{2}{\pi R^2}\int y^2\, dx$$

Substitute $x^2 + y^2 = R^2 \Rightarrow y^2 = R^2 - x^2$, and integrate:

$$= \frac{2}{\pi R^2}\int (R^2 - x^2)\, dx = \frac{2}{\pi R^2}\left[R^2 x - \frac{x^3}{3}\right]_0^R$$

Substitute the limits of integration, and simplify:

$$= \frac{2}{\pi R^2}\left(R^3 - \frac{R^3}{3}\right) = \frac{2}{\pi R^2}\left(\frac{2R^3}{3}\right) = \frac{4R}{3\pi}$$

## C. Wires of Uniform Density in 2D

**2.56: Center of Mass**

Center of mass of a wire of uniform density:

$$\bar{x} = \frac{\int \tilde{x}\, dL}{\int dL}, \qquad \bar{y} = \frac{\int \tilde{y}\, dL}{\int dL}$$

Where $dL = differential\ of\ length$. It can be considered in:

$$Cartesian\ Coordinates: dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$Polar\ Coordinates: dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$Parametric: dL = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

In the discrete case, the center of mass is:

$$\vec{\boldsymbol{r}}_{com} = \frac{\sum_{i=1}^{n} m_i \vec{\boldsymbol{r}}_i}{\sum_{i=1}^{n} m_i}$$

In the case of a wire, if the wire is of uniform density, then:

$$Length\ of\ wire \propto Mass$$

| | | |
|---|---|---|
| Numerator | $\sum_{i=1}^{n} m_i \vec{\boldsymbol{r}}_i$<br>$\vec{\boldsymbol{r}}_i = Position\ of\ i^{th} object$<br>$m_i = Mass\ of\ i^{th}\ object$ | $\int \tilde{x}\, dL$<br>$\tilde{x} = Position\ of\ COM\ of\ small\ element$<br>$dL = Length\ of\ small\ element \propto Mass$ |

| | Mass Weighted Average of Position | |
|---|---|---|
| Denominator | $\sum_{i=1}^{n} m_i = Sum\ of\ all\ masses$ | $\int dL = Length\ of\ the\ wire \propto Mass$ |

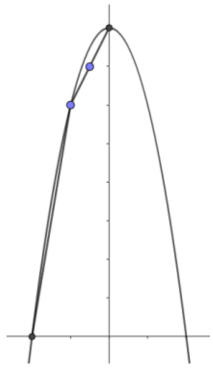
### Example 2.57

Setup, but do not evaluate, an integral for the centroid $C = (\bar{x}, \bar{y})$ of the parabolic arc $y = 16 - x^2$ over $[-4,4]$.

#### $x$ coordinate

By symmetry, the $x$ coordinate of the center of mass of the parabolic arc

$$= x\ coordinate\ of\ vertex = 0$$



#### $y$ coordinate

In Cartesian coordinates:

$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + (-2x)^2} dx = \sqrt{1 + 4x^2}\, dx$$

Approximate the arc with line segments. The centroid of the line segment is the midpoint of the line segment. As the number of line segments becomes very large, the midpoint of the line segment is the point on the arc

$$COM\ of\ Small\ Element = \tilde{y} = y = 16 - x^2$$

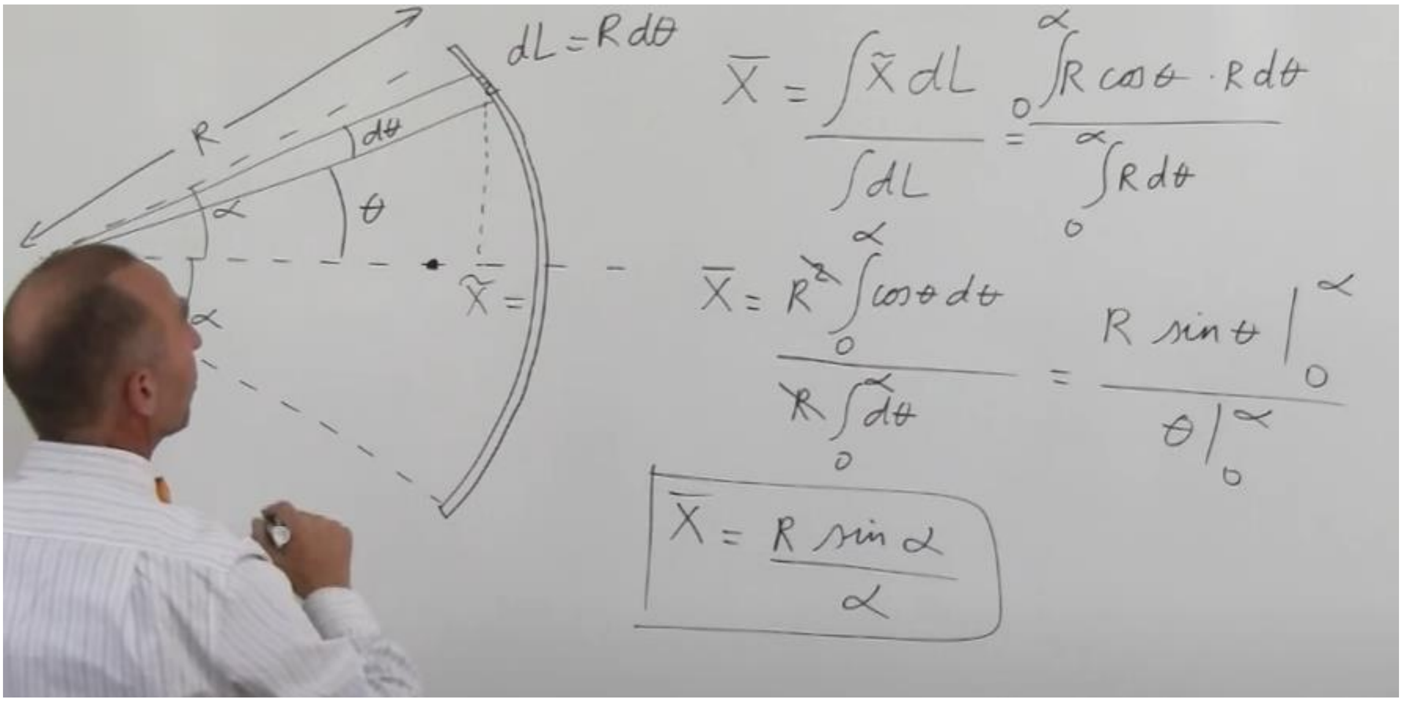
Substitute $\tilde{y} = 16 - x^2, dL = \sqrt{1 + 4x^2}\, dx, limits\ of\ integration\ as - 4\ and\ 4$ in

$$\bar{y} = \frac{\int \tilde{y}\, dL}{\int dL} = \frac{\int_{-4}^{4} (16 - x^2)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}{\int_{-4}^{4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx} = \frac{\int_{-4}^{4} (16 - x^2)\sqrt{1 + 4x^2}\, dx}{\int_{-4}^{4} \sqrt{1 + 4x^2}\, dx}$$

## Pending

### Example 2.58

Find the centroid of an arc of a circular wire.




The formula for arc length of a circular length:

$$L = R\theta$$

Convert into differentials:

$$dL = R\ d\theta$$

$$\bar{x} = \frac{\int \tilde{x}\ dL}{\int dL}$$

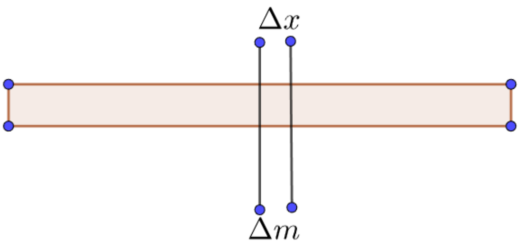
## D. Rod of Variable Density in 1D

### 2.59: Center of Mass

$$\bar{x} = \frac{\int x\ dm}{\int dm}$$

The formula in the discrete case for center of mass is:

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots}{m_1 + m_2 + \cdots}$$




Now, in a continuous mass, there is a continuous variation in the mass.

$$\bar{x} = \frac{\Delta m_1(\tilde{x}_1) + \Delta m_2(\tilde{x}_2) + \cdots}{\Delta m_1 + \Delta m_1 + \cdots}$$

As the number of divisions increases and each mass $\Delta m$
becomes very small, we take the limit as the number of masses goes to infinity and the magnitude of each mass goes to zero.
This process gives us an integral:

$$\bar{x} = \frac{\int \tilde{x}\, dm}{\int dm}$$

Since we are working in one dimension

$$\tilde{x} = x$$

Note that

$$\int dm = Sum\ of\ all\ Masses = M$$

## 2.60: Density

Questions that require integration for variable density will often provide the density of the object as a *function of position*.

$$Density = \rho(x) = A\ Function\ of\ x$$

## 2.61: Connecting density with $dm$

If density is a function of position, then:

$$dm = \rho(x)\, dx$$

Rate of change of mass with respect to change in position is the instantaneous density given by

$$\frac{dm}{dx} = \rho(x)$$

Converting to differentials gives:

$$dm = \rho(x)\, dx$$

Hence, if $m$ is know as a function of position, then the above is useful.

Here we are assuming that

$$V \propto Length$$

And hence density is a function of length.

## Example 2.62

A rod of length $L$ has non uniform linear mass density given by $\rho(x) = a + b\left(\frac{x}{L}\right)^2$, where $a$ and $b$ are constants and $0 \le x \le L$. The value of $x$ for the center of mass of the rod is at: **(JEE-M 2020, 2015)**

Since the question mentions *linear* mass density, the change is in one dimension only, and we can integrate with respect to $x$ to get the center of mass.

The mass of the rod is:

$$\int dm = \int_0^L \rho(x)\, dx = \int_0^L a + b\left(\frac{x}{L}\right)^2\, dx = \left[ax + \frac{bx^3}{3L^2}\right]_0^L = aL + \frac{bL}{3} = \frac{L(3a+b)}{3}$$

The sum of mass-weighted positions is:

$$\int x\, dm = \int_0^L x\,\rho(x)\, dx = \int_0^L ax + bx\left(\frac{x}{L}\right)^2\, dx = \left[\frac{ax^2}{2} + \frac{bx^4}{4L^2}\right]_0^L = \frac{aL^2}{2} + \frac{bL^2}{4} = \frac{L^2(2a+b)}{4}$$

And the center of mass is the average obtained by divided the mass-weighted positions by the mass (sum of weights):

$$\bar{x} = \int x\,dm \times \frac{1}{\int dm} = \frac{L^2(2a+b)}{4} \times \frac{3}{L(3a+b)} = \frac{3L}{4} \cdot \frac{2a+b}{3a+b}$$

## Example 2.63

The distances of center of mass from end $A$ of a one-dimensional rod having mass density $\lambda = \lambda_0 \left(1 - \frac{x^2}{L^2}\right) \frac{kg}{m}$ and length $L$ (in meter) is $\frac{3L}{\alpha}$ $m$. The value of $\alpha$ is: (where $x$ is the distance from end $A$). **(JEE-M 2022)**

Since the rod is *one dimensional*, the change in density is in one dimension only, and we can integrate with respect to $x$ to get the center of mass:

Note that although the question does not explicitly say so:

$$\lambda = \lambda(x)\ is\ a\ function\ of\ x$$
$$\lambda_0 = Constant$$

The mass of the rod is:

$$\int dm = \int_0^L \lambda\,dx = \int_0^L \lambda_0 \left(1 - \frac{x^2}{L^2}\right) dx = \lambda_0 \left[x - \frac{x^3}{3L^2}\right]_0^L = \lambda_0 \left(L - \frac{L}{3}\right) = \frac{2L\lambda_0}{3}$$

$$\int x\,dm = \int_0^L x\lambda\,dx = \int_0^L x\lambda_0 \left(1 - \frac{x^2}{L^2}\right) dx = \lambda_0 \left[\frac{x^2}{2} - \frac{x^4}{4L^2}\right]_0^L = \frac{L^2\lambda_0}{4}$$

$$\bar{x} = \int x\,dm \times \frac{1}{\int dm} = \frac{L^2\lambda_0}{4} \times \frac{3}{2L\lambda_0} = \frac{3L}{8}$$
$$\alpha = 8$$

# 3. Moment of Inertia

## 3.1 Rotational Inertia

### A. Definition and Basics

When we were studying translational motion, the kinetic energy of a body was given by:

$$KE = \frac{1}{2}mv^2$$

While this formula is correct for a body undergoing rotational motion, it is not easy hence to calculate. Hence, we derive an equivalent expression that is more useful for rotational motion. This makes use of rotational inertia (also called moment of inertia).

**3.1: Rotational KE**

$$K = \frac{1}{2}I\omega^2$$

The kinetic energy of a rotating body is the sum of the kinetic energies of the individual particles that makes up the body:

$$K = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2$$

Each particle rotates at a different (linear) speed. Hence, calculating the above expression is difficult.
Substitute $v_i = \omega r_i$

$$K = \frac{1}{2}\sum_{i=1}^{n} m_i(\omega r_i)^2 = \frac{1}{2}\sum_{i=1}^{n} m_i(\omega r_i)^2 = \frac{1}{2}\left(\sum_{i=1}^{n} m_i r_i^2\right)\omega^2$$

Use a change of variable. Let $I = \sum_{i=1}^{n} m_i r_i^2$

$$K = \frac{1}{2}I\omega^2$$

**Example 3.2**

Two bodies have their moments of inertia $I$ and $2I$ respectively about their axis of rotation. If their kinetic energies of rotation are equal, their angular velocity will be in the ratio: **(NEET 2005)**

Since their kinetic energies of rotation are equal, use the formula for kinetic energy:

$$I\omega_1^2 = (2I)\omega_2^2$$

$$\frac{\omega_1^2}{\omega_2^2} = 2$$

$$\frac{\omega_1}{\omega_2} = \frac{\sqrt{2}}{1}$$

**3.3: Rotational Inertia/Moment of Inertia**

The rotational inertia of a body is:

$$I = \sum_{i=1}^{n} m_i r_i^2$$

Where

$$m_i = mass\ of\ i^{th}\ object$$
$$r_i = distance\ of\ i^{th} object\ from\ axis\ of\ rotation$$

- Rotational inertia is also called the moment of inertia.

## 3.4: Unit of Moment of Inertia

SI unit of moment of inertia is

$$kg \cdot m^2$$

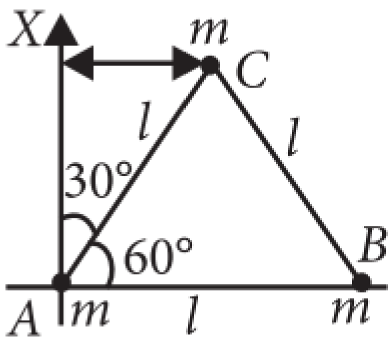
This makes dimensional sense since the formula is

$$\sum_{i=1}^{n} m_i r_i^2 = \sum (mass)\ (distance)^2$$

### Example 3.5

Three particles each of mass $m$ gram are situated at the vertices of an equilateral triangle $ABC$ of side $l$ cm. The moment of inertia of the system about a line $AX$ perpendicular to $AB$ and in the plane of $ABC$ will be: (Write your answer in SI units) **(NEET 2004)**

$$I = m_A r_A^2 + m_B r_B^2 + m_C r_C^2$$
$$= m(0) + m(l^2) + m(l \sin 30°)^2$$
$$= ml^2 + \frac{1}{4} ml^2$$
$$= \frac{5}{4} ml^2\ gram \cdot cm^2$$
$$= (1.25 \cdot 10^{-7})\ ml^2\ kg \cdot m^2$$




## 3.6: Relating inertia and rotational inertia

| | Rotational Movement | Linear Movement |
|---|---|---|
| Mass | When $\omega$ is constant $KE \propto I$ | When $v$ is constant $KE \propto m$ |
| Velocity | When $I$ is constant $KE \propto \omega^2$ | When $m$ is constant $KE \propto v^2$ |

Recall that mass is a measure of a body's inertia.

The kinetic energy of a body in:
- translational motion is based on its inertia (or mass) $m$, and its linear speed $v$.
- In rotational motion is based on its rotational inertia $I$ and its angular speed $\omega$.

### Example 3.7

*Mark all correct options*

$n$ particles, each with non-zero mass, have a combined moment of inertia $I_n$ at time $t$ at a particular point $P$. The system of $n-1$ particles obtained by not considering particle $i$ has moment of inertia $I_{n-1}$, also at time $t$ and point $P$. Which of the following can be true:

A. $I_{n-1} > I_n$

B. $I_{n-1} \geq I_n$
C. $I_{n-1} < I_n$
D. $I_{n-1} \leq I_n$
E. $I_{n-1} = I_n$

Contribution of any single particle to moment of inertia can only be positive since

$$m_i > 0, \qquad r_i^2 > 0$$
$$Option\ C$$

## B. Qualitative Comparisons

**3.8:**
As mass is distributed

- Further from the center of rotation, the moment of inertia increases
- Closer to the center of rotation, the moment of inertia decreases

### Example 3.9

*Mark the correct option*

One solid sphere A and another hollow sphere B are of same mass and same outer radii. Their moment of inertia about their diameters are respectively $I_A$ and $I_B$ such that: **(JEE-M 2004)**

A. $I_A < I_B$
B. $I_A > I_B$
C. $I_A = I_B$
D. $\frac{I_A}{I_B} = \frac{d_A}{d_B}$

The hollow sphere has mass further away than the solid sphere. Hence,

$$I_A < I_B \Rightarrow Option\ A$$

### Example 3.10

Will the answer to the above question change if sphere A and sphere B are not of uniform density.

No. Since the distance of the mass in the hollow sphere is greater than the distance in the solid sphere, and we are taking the sum over the masses, the answer remains the same.

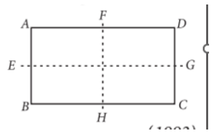
### Example 3.11

In a rectangle $ABCD$ ($BC = 2AB$) and $F, G, H$ and E are the midpoints of $AD$,$DC$, $CB$ and $BA$ respectively. The moment of inertia is minimum along axis through: **(NEET 1993)**

A. BC
B. BD
C. HF
D. EF

Mass distribution is minimum from EG.

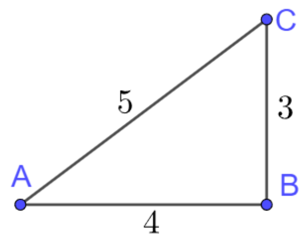
$$Option\ C$$



### Example 3.12

*Mark the correct option*

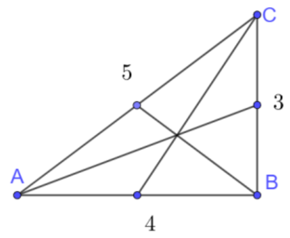
ABC is a triangular plate of uniform thickness. The sides are in the ratio shown in the figure. $I_{AB}$, $I_{BC}$ and $I_{CA}$ are the moments of inertia of the plate about $AB$, $BC$, and $CA$ respectively. Which one of the following relations is correct? **(NEET 1995, 2000)**




A. $I_{AB} + I_{BC} = I_{CA}$
B. $I_{CA}$ is maximum
C. $I_{AB} > I_{BC}$
D. $I_{BC} > I_{AB}$

The center of mass of a triangle is the point of intersection of its three medians.
C is closer to AB as compared to the distance of $B$ from BC.

*Option D*




### xample 3.13

A circular disc is to be made using iron and aluminum so that it acquires maximum moment of inertia about geometrical axis. It is possible with: **(NEET 2002)**

A. Aluminum at interior and iron surrounding it
B. Iron at interior and aluminum surrounding it
C. Using iron and aluminum layers in alternate order
D. Sheet of iron is used at both external surface and aluminum sheets as internal layers.

Iron is denser than aluminum. Hence, it should be farther away from the center.

*Option A*

## C. Discrete Masses

### 3.14: Two Body System

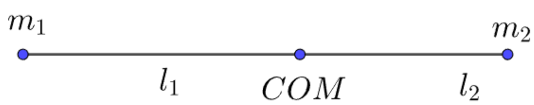
The moment of inertia of a two-body system with masses $m_1$ and $m_2$ about its center of mass is:

$$I = \frac{m_1 m_2}{m_1 + m_2} l^2$$

Where

$$l = length = distance\ between\ the\ two\ bodies$$

$$I = \sum_{i=1}^{n} m_i r_i^2 = m_1 l_1^2 + m_2 l_2^2$$




Substitute $l_1 = \left(\frac{m_2}{m_1+m_2} l\right)^2, l_2 = \left(\frac{m_1}{m_1+m_2} l\right)^2$

$$= m_1 \cdot \frac{m_2^2}{(m_1 + m_2)^2} l^2 + m_2 \frac{m_1^2}{(m_1 + m_2)^2} l^2$$

Factor $m_1 m_2 l^2$:

$$= m_1 m_2 l^2 \times \frac{m_2 + m_1}{(m_1 + m_2)^2} = \frac{m_1 m_2}{m_1 + m_2} l^2$$

**Example 3.15**

A light rod of length $l$ has two masses $m_1$ and $m_2$ attached to its two ends. The moment of inertia of the system about an axis perpendicular to the rod and passing through the center of mass is: **(NEET 2016)**

$$\frac{m_1 m_2}{(m_1 + m_2)^2} l^2$$

## D. Discs, Cylinders, and Rings

### 3.16: Moment of Inertia of a solid disc/cylinder

The rotational inertia of a solid disc/cylinder with uniform density about its central axis is:

$$I_z = \frac{1}{2} MR^2$$

Where

$$R = radius\ of\ disc$$
$$M = Mass\ of\ disc/cylinder$$

**Example 3.17**

A circular disc $X$ of radius $R$ is made from an iron plate of thickness $t$ and another disc $Y$ of radius $4R$ is made from an iron plate of thickness $\frac{t}{4}$. Then determine $\frac{I_Y}{I_X}$ (as a number) if $I_Y$ and $I_X$ are the moments of inertia of the respectively plates through their central axis. **(JEE-M 2003)**

$$\frac{I_Y}{I_X} = \frac{\frac{1}{2} M_Y R_Y^2}{\frac{1}{2} M_X R_X^2} = \frac{\frac{1}{2}\left(\pi(4R)^2 \cdot \frac{t}{4}\right)(4R)^2}{\frac{1}{2}(\pi R^2 t) R^2} = 64$$

**Example 3.18**

From a circular disc of radius $R$ and mass $9M$, a small disc of mass $M$ and radius $\frac{R}{3}$ is removed concentrically. The moment of inertia of the remaining disc about an axis perpendicular to the plane of the disc and passing through its center is: **(NEET 2010)**

The moment of inertia of the original disc is:

$$\frac{9MR^2}{2}$$

The moment of inertia of the removed disc is:

$$\frac{M\left(\frac{R}{3}\right)^2}{2} = \frac{MR^2}{18}$$

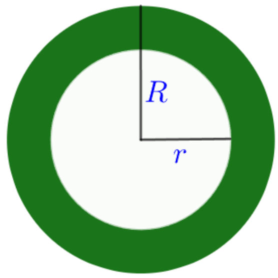
The moment of inertia of the remaining disc is

$$\frac{9MR^2}{2} - \frac{MR^2}{18} = \frac{80MR^2}{18} = \frac{40MR^2}{9}$$

### 3.19: Moment of Inertia for an annular disc/cylinder

The moment of inertia for an annular disc/cylinder with mass $M$, outer radius $R$ and inner radius $r$ about its central axis is:

$$I = \frac{1}{2}M(R^2 + r^2)$$




Where

$$R = Outer\ radius\ of\ disc$$
$$r = Inner\ radius\ of\ disc$$
$$M = Mass\ of\ disc/cylinder$$

The mass of the annular disc is

$$k^2m - m = m(k^2 - 1)$$

The moment of inertia of the original disc is:

$$0.5(k^2m)(kr)^2 = 0.5k^4mr$$

The moment of inertia of the removed disc is:

$$= 0.5mr$$

The moment of inertia of the remaining disc is

$$= 0.5k^4mr - 0.5mr$$
$$= 0.5mr(k^4 - 1)$$
$$= 0.5m(k^2 - 1)r(k^2 + 1)$$
$$= 0.5M[r^2k^2 + r^2]$$
$$= \frac{1}{2}M[R^2 + r^2]$$

### Example 3.20

For discs of uniform density, the moment of inertia through their respective central axis, the formulas are:

$$Circular\ Disc: I = \frac{1}{2}MR^2, M = mass, R = Radius$$

$$Annular\ Disc: I = \frac{1}{2}M(R^2 + r^2), M = Mass, R = Outer\ Radius, r = Inner\ Radius$$

Explain why the annular disc has a plus sign even though its moment of inertia should be less.

### 3.21: Moment of Inertia for a Ring/Hollow Cylinder

The rotational inertia of a ring/hollow cylinder with uniform density about its central axis is:

$$I_z = MR^2$$

Where

$$R = radius\ of\ disc$$
$$M = Mass\ of\ disc/cylinder$$

By substituting $r = R$ in the formula for the moment of inertia of an annular ring:

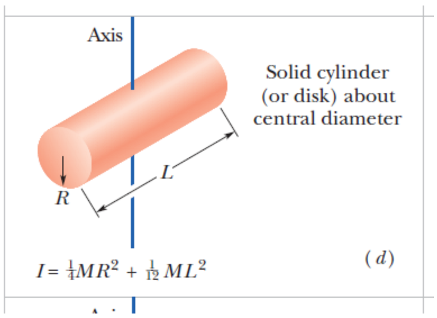
$$I = \frac{1}{2}M(R^2 + R^2) = MR^2$$

### Example 3.22
From a circular ring of mass $M$ and radius $R$ an arc corresponding to a 90° sector is removed. The moment of inertia of the remaining part of the ring about an axis passing through the center of the ring and perpendicular to the plane of the ring is $K$ times $MR^2$. Then the value of $K$ is: (NEET )

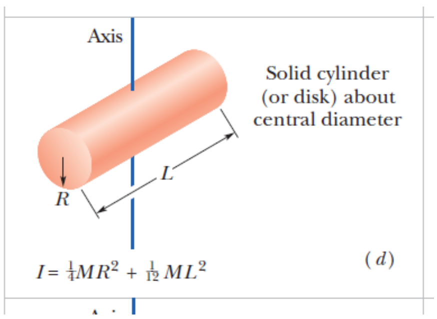
In the absence of information, we assume uniform density.
The mass of the ring is equally distributed at the same distance from the axis.

Hence
$$I \propto Angle\ of\ Sector$$
$$I = \frac{270}{360}MR^2 = \frac{3}{4}MR^2$$

### 3.23: Moment of Inertia of a Cylinder about its central diameter
The moment of inertia of a cylinder with uniform density about its central diameter is:
$$I = \frac{MR^2}{4} + \frac{ML^2}{12}$$
$$R = Radius\ of\ cylinder$$
$$L = Length\ of\ cylinder$$




### 3.24: Moment of Inertia of a Thin Rod about its central diameter
$$I = \frac{ML^2}{12}$$
$$R = Radius\ of\ cylinder$$
$$L = Length\ of\ cylinder$$




For a thin rod, the radius can be approximated as zero, and hence the moment of inertia is:
$$I = \frac{M \cdot 0^2}{4} + \frac{ML^2}{12} = \frac{ML^2}{12}$$

## E. Spheres and Spherical Shells

### 3.25: Moment of Inertia of a solid sphere
The moment of inertia of a solid sphere with uniform density about its central axis is:
$$I = \frac{2}{5}MR^2$$
Where
$$R = radius\ of\ sphere$$

$$M = Mass\ of\ sphere$$

### Example 3.26

A solid sphere of mass $m$ and radius $R$ is rotating about its diameter. A solid cylinder of the same mass and same radius is also rotating about its geometrical axis with an angular speed twice that of the sphere. The ratio of their kinetic energies of ratio $\frac{E_{Sphere}}{E_{Cylinder}}$ will be: **(NEET 2016)**

$$\frac{E_S}{E_C} = \frac{\frac{1}{2}I_S\omega_S^2}{\frac{1}{2}I_C\omega_C^2} = \frac{\frac{2}{5}MR^2\omega^2}{\frac{1}{2}MR^2(2\omega)^2} = \left(\frac{2}{5}\times\frac{2}{1}\right)\left(\frac{\omega^2}{4\omega^2}\right) = \frac{1}{5}$$

### 3.27: Moment of Inertia of a spherical shell

The moment of inertia of a spherical shell with uniform density about its central axis is:

$$I = \frac{2}{3}MR^2$$

Where

$$R = radius\ of\ spherical\ shell$$
$$M = Mass\ of\ spherical\ shell$$

### Example 3.28

*Mark the correct option*

One solid sphere $A$ and another hollow sphere $B$ are of same mass and same outer radii. Their moment of inertia about their diameters are respectively $I_A$ and $I_B$. Find $\frac{I_A}{I_B}$. **(JEE-M 2004, Adapted)**

$$\frac{I_A}{I_B} = \frac{\frac{2}{5}MR^2}{\frac{2}{3}MR^2} = \frac{2}{5}\times\frac{3}{2} = \frac{3}{5}$$

## F. Rectangles and Squares

### 3.29: Moment of Inertia of Rectangular Slab

The moment of inertia of a thin rectangular slab with dimensions $a \times b$ perpendicular to its center of mass is

$$\frac{M(a^2+b^2)}{12}$$

### 3.30: Moment of Inertia of Square Lamina

The moment of inertia of a thin square slab with side length $L$ perpendicular to its center of mass is

$$\frac{ML^2}{6}$$

$$\frac{M(a^2+b^2)}{12} = \frac{M(L^2+L^2)}{12} = \frac{ML^2}{6}$$

## G. Summary and Applications

### 3.31: Moment of Inertia

Moment of Inertia through the central axis for each of the below:

| Shape | Solid | Annular | Hollow |
|---|---|---|---|
| Cylinder/Disc | $\frac{1}{2}MR^2$<br>$R = Radius$ | $I = \frac{1}{2}M(R^2 + r^2)$<br>$R = Outer\ Radius$<br>$r = Inner\ Radius$ | $MR^2$ |
| Sphere | $\frac{2}{5}MR^2$<br>$R = radius$ | | $\frac{2}{3}MR^2$<br>$R = radius$ |

Moment of Inertia perpendicular to center of mass for the shapes below is:

| Shape | Moment of Inertia | |
|---|---|---|
| Thin Rectangular Slab | $\frac{M(a^2 + b^2)}{12}$ | |
| Thin Square Lamina | $\frac{ML^2}{6}$ | |

## 3.32: Work

$$Work = \Delta KE$$

### Example 3.33

Three objects, A: (a solid sphere), B: (a thin circular disk), and C: (a circular ring), each have the same mass $M$ and radius $R$. They all spin with the same angular speed about their own symmetry axes. Determine the ratio of the work $W_A$, $W_B$ and $W_C$ required to bring them to rest in ascending order (NEET 2018)

$$W_A : W_B : W_C$$
$$\Delta KE_A : \Delta KE_B : \Delta KE_C$$
$$\frac{1}{2}I_A\omega^2 : \frac{1}{2}I_B\omega^2 : \frac{1}{2}I_C\omega^2$$
$$I_A : I_B : I_C$$
$$\frac{2}{5}mR^2 : \frac{1}{2}mR^2 : mR^2$$
$$\frac{2}{5} : \frac{1}{2} : 1$$
$$4 : 5 : 10$$

1 Pending

### Example 3.34: Rate of Change

A solid sphere of 100 $kg$ and radius 10 $m$ moving in a space becomes a circular disc of radius 20 $m$ in one hour. Then the rate of change of moment on inertia in the process is: (EAMCET 2019)

Moment of inertia of the solid sphere,

$$I_s = \frac{2}{5} M_s R_s^2 = \frac{2}{5} \times 100 \times (10)^2 = 4000 \text{ kg-m}^2$$

Similarly,

moment of inertia of the disc, $I_c = \frac{1}{2} M_c R^2$

$$= \frac{1}{2} \times 100 \times (20)^2 = 20{,}000 \text{ kg- m}^2$$

Rate of change of moment of inertia $= \frac{I_c - I_s}{t}$

$$= \frac{20000 - 4000}{60 \times 60} = \frac{16000}{60 \times 60} = \frac{160}{36} = \frac{40}{9} \text{ kg-m}^2\text{s}^{-1}$$

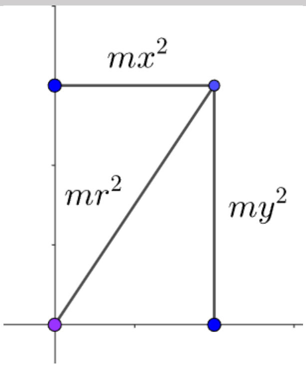
# 3.2 Parallel and Perpendicular Axes

## A. Perpendicular Axes

### 3.35: Perpendicular Axes

The moment of inertia for an axis perpendicular to a $2D$ object is the sum of the moments of inertia from perpendicular axes passing through the point of intersection of the axis with the object.

$$I_z = I_x + I_y$$




Consider an object which lies in the $xy$ plane.
We wish to calculate the moment of inertia through the $z$ axis.
For mass $m$

$$r = x^2 + y^2$$
$$mr^2 = mx^2 + my^2 = I_x + I_y$$

Warning: Theorem of perpendicular axes is only applicable to 2D objects. It is not applicable to 3D objects.

### 3.36: Disc versus Cylinder

By convention:

- Disc usually refer to a thin ($2D$) object
- Cylinder refers to an object with meaningful height ($3D$)

Check the wording of a question before deciding whether it is 2D or 3D.

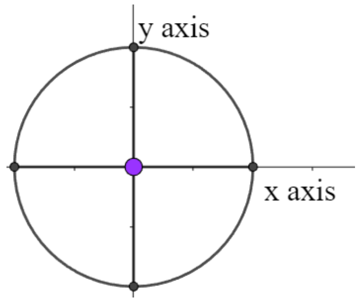
### 3.37: $I$ for a Disc about its diameter

The moment of inertia about its diameter of a thin disc with uniform thickness and uniform density

$$= I = \frac{MR^2}{4}$$

Introduce an origin (0,0,0) at the center of the disc.
By the perpendicular axes theorem, the moment of inertia for the $z$ axis

$$= I_z = I_x + I_y$$




By symmetry $I_x = I_y$:

$$I_z = 2I_x = 2I_y$$

$$I_x = I_y = \frac{I_z}{2} = \frac{\frac{MR^2}{2}}{2} = \frac{MR^2}{4}$$

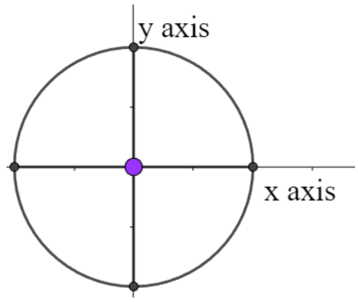
## 3.38: *I* for a Ring about its diameter

The moment of inertia about its diameter of a ring with uniform density is:

$$= I = \frac{MR^2}{2}$$

Introduce an origin (0,0,0) at the center of the disc.
By the perpendicular axes theorem, the moment of inertia for the $z$ axis

$$= I_z = I_x + I_y$$




By symmetry $I_x = I_y$:

$$I_z = 2I_x = 2I_y$$

$$I_x = I_y = \frac{I_z}{2} = \frac{MR^2}{2} = \frac{MR^2}{2}$$

## Example 3.39

Moment of inertia of a circular wire of mass $M$ and radius $R$ about its diameter is: (JEE-M 2002)

A "circular" wire means the wire is arranged in the shape of a circle with radius $R$. It has moment of inertia

$$I = \frac{MR^2}{2}$$

## Example 3.40

*Mark the correct option*

A square lamina $ABCD$ has $F$ as the midpoint of $CD$ and $E$ as the midpoint of $AB$. Then:

A. $I_{AC} = \sqrt{2}I_{EF}$
B. $\sqrt{2}I_{AC} = I_{EF}$
C. $I_{AD} = 3I_{EF}$
D. $I_{AC} = I_{EF}$ (JEE-M 2007)

Note: Lamina means thin layer.

By perpendicular axes theorem, the moment of inertia through the center of the square and perpendicular to the lamina is:

$$I = I_{AC} + I_{BD}$$

By symmetry, $I_{AC} = I_{BD}$

$$I_{AC} = \frac{I}{2}$$

Again, by perpendicular axes theorem (using $G$ as midpoint of $AD$, and $H$ as midpoint of $BC$):

$$I = I_{EF} + I_{GH}$$

By symmetry $I_{EF} = I_{GH}$

$$I_{EF} = \frac{I}{2}$$

Hence,

$$I_{AC} = I_{AC} = \frac{I}{2} \Rightarrow Option\ D$$

## B. Parallel Axes

### 3.41: Parallel Axes

The moment of inertia of a body through an axis parallel to another axis passing through the center of mass of the body is:

$$I = I_{COM} + Mh^2$$

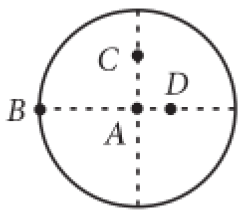
Where:

$$I_{COM} = Moment\ of\ inertia\ for\ axis\ through\ center\ of\ mass$$
$$M = Mass\ of\ body$$
$$h = perpendicular\ distance\ between\ the\ axes$$

### Example 3.42

*Mark the correct option*

The moment of inertia of a uniform circular disc is maximum about an axis perpendicular to the disc and passing through: **(NEET 2012)**




a. B
b. C
c. D
d. A

The moment of inertia at any point on the disc is:

$$I_0 + Mh^2$$

Where

$$I_0 = Moment\ of\ inertia\ for\ axis\ passing\ through\ center\ of\ mass$$

$$Center\ of\ Mass = A \Rightarrow Moment\ of\ Inertia\ is\ minimum\ at\ A$$

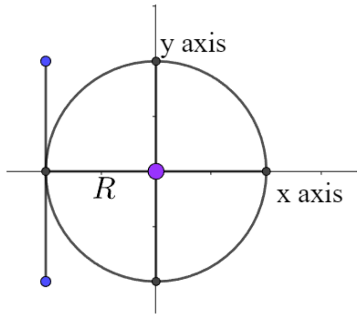
It will be maximum when

$$h\ is\ maximum \Rightarrow Point\ B \Rightarrow option\ a.$$

### Example 3.43

The moment of inertia of a disc of mass $M$ and radius $R$ about an axis, which is tangential to the circumference of the disc and parallel to its diameter is: **(NEET 1999)**

An axis which is tangential to the disc is drawn in the diagram.
Since the axis is parallel to the diameter, it must lie in the $xy$ plane.




The moment of inertia of an axis through the diameter of the disc is

$$\frac{MR^2}{4}$$

The distance between the required axis and the axis through the diameter

$$= R$$

By the parallel axis theorem, the required moment of inertia is:

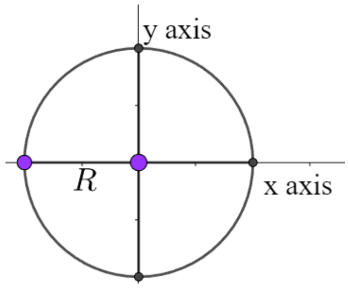
$$I = I_{Diameter-COM} + Mh^2 = \frac{MR^2}{4} + MR^2 = \frac{5}{4}MR^2$$

## Example 3.44

Moment of inertia of a uniform circular disc about a diameter is $I$. Its moment of inertia about an axis perpendicular to its plane and passing though a point on its rim will be: **(NEET 1990)**

For an axis perpendicular to the disc, and passing through its central axis:

$$I_{Z-COM} = \frac{MR^2}{2}$$




Using the parallel axis theorem:

$$I_{Rim} = I_{Z-COM} + Mh^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2$$

For an axis about its diameter:

$$I = \frac{MR^2}{4}$$

Comparing the moments of inertia about the two axes of interest:

$$I_{Rim} = \frac{3}{2}MR^2 = 6\left(\frac{MR^2}{4}\right) = 6I$$

## Example 3.45

A thin circular disc is in the $xy$ plane. The $z$ axis is perpendicular to the $xy$ plane, and passes through the center of mass of the disc. The $z'$ axis is also perpendicular to the $xy$ plane and passes through a point on the edge of the disc. Determine the ratio of moment of inertia through $z$ with the moment of inertia through $z'$. **(JEE-M 2018)**

The moment of inertia through the $z$ axis:

$$= \frac{1}{2}MR^2$$

The moment of inertia through the $z'$ axis:

$$= \frac{3}{2}MR^2$$

$$I_Z : I_{Z'} = \frac{1}{2}MR^2 : \frac{3}{2}MR^2 = 1:3 = \frac{1}{3}$$

### Example 3.46

The moment of inertia of a think uniform rod of mass $M$ and length $L$ about an axis passing through its midpoint and perpendicular to its length is $I_0$, Its moment of inertia about an axis passing through one of its ends and perpendicular to length, in terms of $I_0, M$, and $L$ is: (NEET 2011)

$I_0$ is the moment of inertia for an axis through its center of mass

$$I_0 = I_{COM}$$

The distance between the midpoint and the end:

$$= h = \frac{L}{2}$$

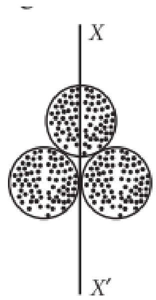
Hence, the required moment of inertia is

$$I_{COM} + Mh^2 = I_0 + M\left(\frac{L}{2}\right)^2 = I_0 + \frac{ML^2}{4}$$

## C. Multi Step Problems

### Example 3.47

Three identical spherical shells, each of mass $m$ and radius $r$ are placed as shown in the figure. Consider an axis $XX'$ which is touching two of the shells and passing through the diameter of the third shell. Moment of inertia of the system consisting of these three spherical shells about $XX'$ axis is: (NEET 2015)




The moment of inertia of the central shell about $XX'$

$$= \frac{2}{3}mr^2$$

The moment of inertia of a shell away from the center

$$= I_{COM} + Mh^2 = \frac{2}{3}mr^2 + mr^2 = \frac{5}{3}mr^2$$

The moment of inertia of both the spherical shells away from the center

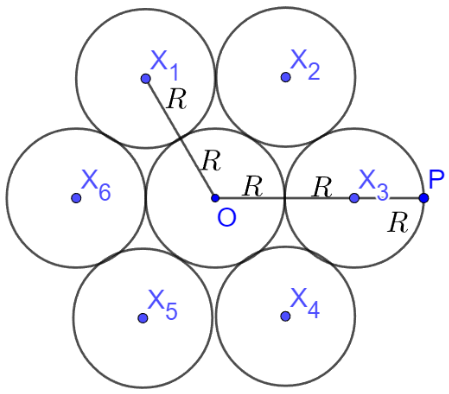
$$= 2 \cdot \frac{5}{3}mr^2 = \frac{10}{3}mr^2$$

The moment of inertia of the entire system

$$= \frac{2}{3}mr^2 + \frac{10}{3}mr^2 = \frac{12}{3}mr^2 = 4mr^2$$

### Example 3.48

Seven identical circular planar discs, each of mass $M$ and radius $R$ are welded symmetrically, such that six discs are around the central, seventh disc. The moment of inertia of the arrangement about the axis normal to the plane and passing though the edge of one of the discs on the outside is: (JEE-M 2018, Adapted)




The moment of inertia at $O$ (for the circle at $O$), perpendicular to the plane is:

$$\frac{MR^2}{2}$$

The moment of inertia at $O$ (for the circle at $X_1$), perpendicular to the plane is:

$$I_{COM} + Mh^2 = \frac{MR^2}{2} + M(2R)^2$$

The total moment of inertia at $O$ for the figure is:

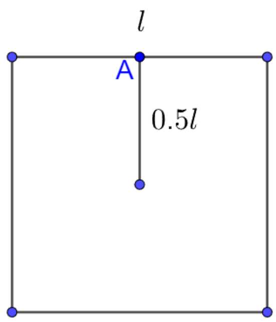
$$\frac{MR^2}{2} + 6\left(\frac{MR^2}{2} + M(2R)^2\right)$$
$$= \frac{7}{2}MR^2 + 24MR^2 = \frac{55}{2}MR^2$$

The total moment of inertia at $P$ for the figure is:

$$\frac{55}{2}MR^2 + 7M(3R)^2 = \frac{55}{2}MR^2 + 63MR^2 = \frac{181}{2}MR^2 = 90.5MR^2$$

### Example 3.49

Four identical thin rods each of mass $M$ and length $l$ form a square frame. Moment of inertia of this frame about an axis through the center of the square and perpendicular to its plane (in terms of the given variables) is: **(NEET 2009)**




The moment of inertia of the top rod at point A is

$$\frac{Ml^2}{12}$$

The moment of inertia of the top rod at the center is:

$$I_{COM} + Mh^2 = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{12} + \frac{Ml^2}{4} = \frac{4Ml^2}{12} = \frac{Ml^2}{3}$$

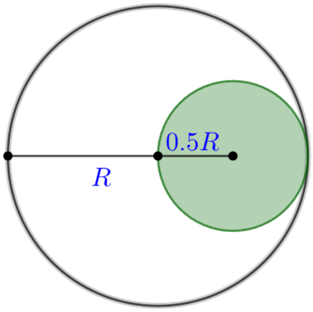
By symmetry, the moment for the other three rods is also the same. Hence, the final answer is:

$$= \frac{4}{3}Ml^2$$

## D. Removing Masses

### Example 3.50

From a disc of radius $R$ and mass $M$, a circular hole of diameter $R$, whose rim passes through the center is cut. What is the moment of inertia of the remaining part of the disc about a perpendicular axis passing through the center (in terms of the given variables) **(NEET 2016)**




The moment of inertia of the entire disc is:

$$\frac{1}{2}MR^2$$

The moment of inertia of the green disc at its center of mass

$$=\frac{1}{2}MR^2=\frac{1}{2}\left(\frac{M}{4}\right)\left(\frac{R}{2}\right)^2=\frac{MR^2}{32}$$

The moment of inertia of the green disc at the center of the white disc is:

$$I_{COM}+Mh^2=\frac{MR^2}{32}+\left(\frac{M}{4}\right)\left(\frac{R}{2}\right)^2=\frac{3MR^2}{32}$$

The moment of inertia of only the white portion of the disc is:

$$\frac{MR^2}{2}-\frac{3MR^2}{32}=\frac{13MR^2}{32}$$

# 3.3 Calculus with Moment of Inertia

## A. Definition and Basics

**3.51: Rotational Inertia**

The rotational inertia of an object is given by:

$$I=\int r^2\,dm$$

This is analogous to the discrete version of the formula for rotational inertia.

$$I=\sum_{i=1}^{n} m_i r_i^2$$

As the number of masses becomes very, and the mass of each particle becomes very small, replace

$$m_i \to dm$$

And replace the distance of the $i^{th}$ particle with a continuous variable for the distance

$$r_i^2 \to r^2$$

**3.52: Moment of Inertia of a wire**

The moment of inertia of a wire/ring about its central axis is

$$I=MR^2$$

$$I=\int r^2\,dm$$

Substitute $r=R$, and move $R$ outside since it is a constant:

$$=\int R^2\,dm=R^2\int dm$$

We can integrate over the whole mass to get $\int dm=M$:

$$=MR^2$$

## B. Cylinder and Sphere

### 3.53: Objects with uniform thickness

For a three-dimensional object with uniform density and uniform thickness:

$$dm = \rho \, dA$$

For a three-dimensional object, the differential element of mass ($dm$) is related to volume.

$$dm = \rho \, dV, \qquad \rho = density$$

But for a cylinder of uniform thickness and uniform density, the mass is proportional to the area

$$Mass \propto Area$$

And hence $dm = \rho \, dV$ simplifies to:

$$dm = \rho \, dA$$

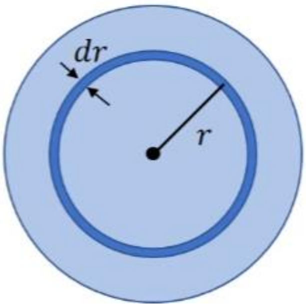
### 3.54: Moment of Inertia of a solid disc/cylinder

The rotational inertia of a solid disc/cylinder with uniform density, and of uniform thickness, around its center of mass is:

$$I = \frac{1}{2} MR^2$$

Where

$$R = radius\ of\ disc$$
$$M = Mass\ of\ disc/cylinder$$




$$I = \int r^2 \, dm$$

Write mass in terms of density. Since density and thickness are both uniform, from the previous example, substitute $dm = \rho \, dA$ in

$$= \int r^2 \, \rho \, dA$$

The change in area for an annulus of infinitesimal width $dr$ is $dA = 2\pi r \, dr$. Introduce the limits of integration from $r = 0$ to $r = R$:

$$= \int_0^R r^2 \, \rho \, (2\pi r \, dr) = 2\rho\pi \int_0^R r^3 \, dr$$

Integrate and evaluate:

$$= 2\rho\pi \left[\frac{r^4}{4}\right]_0^R = 2\rho\pi \left(\frac{R^4}{4}\right) = \frac{1}{2}\rho\pi R^4$$

Substitute $\rho = \frac{M}{\pi R^2}$:

$$\frac{1}{2}\left(\frac{M}{\pi R^2}\right)\pi R^4 = \frac{1}{2} MR^2$$

### Example 3.55

Explain the first step at which the following "proof" is incorrect. You can refer to the proof above for comparison.

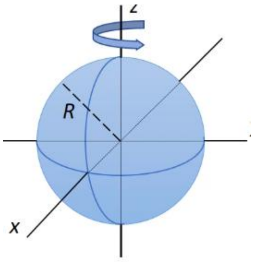
$$I = \int_0^R r^2\, dm = \int_0^R r^2\, \sigma\, dA = \int_0^R r^2\, \sigma\, (2\pi r\, dr) = \int_0^R r^2 \left(\frac{M}{\pi r^2}\right) (2\pi r\, dr) = \int_0^R 2rM\, dr = M[r^2]_0^R = MR^2$$

The substitution $\sigma = \frac{M}{\pi r^2}$ is incorrect since the density is constant where $r$ is a variable that represents distance. It should have been

$$\sigma = \frac{M}{\pi R^2}$$

## 3.56: Moment of Inertia of a solid sphere[1]

$$I = \frac{2}{5}MR^2$$




Slice the sphere horizontally into cylinders of infinitesimal height. Each cylinder has moment of inertia:

$$dI = \frac{1}{2}r^2\, dm$$

Substitute $dm = \rho\, dV$ in the above:

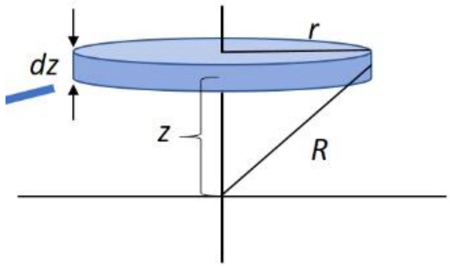
$$dI = \frac{1}{2}r^2\, \rho\, dV$$

Using volume of a cylinder $= \pi r^2 h$, substitute $dV = \pi r^2\, dz$. Integrate both sides of the above from $z = -R$ to $z = R$:

$$I = \frac{1}{2}\int_{-R}^{R} r^2\, \rho\, \pi r^2\, dz$$

Using $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$ for an even function:

$$= \rho\pi \int_0^R r^4\, dz$$




Use the Pythagorean Theorem to write $r$ in terms of $z$:

$$R^2 = z^2 + r^2 \Rightarrow r^4 = (R^2 - z^2)^2 = R^4 - 2R^2z^2 + z^4$$

The integral becomes:

$$= \rho\pi \left[R^4 z - \frac{2}{3}R^2 z^3 + \frac{z^5}{5}\right]_0^R = \rho\pi \left(R^5 - \frac{2R^5}{3} + \frac{R^5}{5}\right) = \frac{8}{15}\rho\pi R^5$$

Substitute $\rho = \frac{M}{\frac{4}{3}\pi r^3}$

$$= \frac{8}{15}\left(\frac{M}{\frac{4}{3}\pi R^3}\right)\pi R^5 = \frac{2}{5}MR^2$$

[1] Another approach is from MIT 8.01 using symmetry and shells

## C. Theorems on Perpendicular and Parallel Axes

Pending

### 3.57: Perpendicular Axes

Working in Cartesian coordinates, the moment of inertia of the planar body about the $z$ axis is given by:[3]

$$I_z = \int (x^2 + y^2)\, dm = \int x^2\, dm + \int y^2\, dm = I_y + I_x$$

Pending

### 3.58: Theorem of Parallel Axes

## D. Further Resources

**Playlist 3.59**
Micheal van Biezen on Moment of Inertia
12 Videos

# 4. Torque

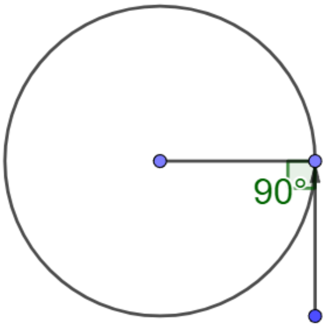
## 4.1 Torque

### A. Torque as a Scalar

**4.1: Torque: Perpendicular Force**

If a force $F$ is applied perpendicular to the radial direction of motion, then the torque is:

$$\tau = rF$$




- As the distance from the radial axis increases, the torque increases

Some everyday applications of torque are:

- Force applied to open a door
- Lifting an object by bending your elbow
- Using a wrench to open a bolt on a car tire

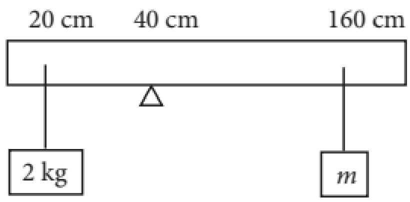
**4.2: NLM-1: Angular Version**

For a body to be in equilibrium, the net torque on it must be zero

$$\tau_{Net} = 0$$

**Example 4.3**

A uniform rod of length 200 cm and mass 500 g is balanced on a wedge placed at 40 cm mark. A mass of 2 kg is suspended from the rod at 20 cm and another unknown mass $m$ is suspended from the rod at 160 cm mark as shown in the figure. Find the value of $m$ such that the rod is in equilibrium. $\left(g = 10\frac{m}{s^2}\right)$ (NEET 2021)




Since the rod is in equilibrium the torque on the left must equal the torque on the right:

$$Torque\ due\ to\ 2\ kg\ mass: = rF = (40 - 20)(2g) = 40g$$
$$Torque\ due\ to\ Rod: rF = (100 - 40)(0.5g) = 30g$$
$$Torque\ due\ to\ unknown\ mass: rF = (160 - 40)(mg) = 120mg$$

$$\tau_L = \tau_R$$
$$40g = 30g + 120mg$$
$$40 = 30 + 120m$$
$$10 = 120m$$
$$m = \frac{1}{12}$$

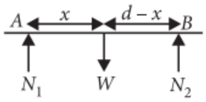
**Example 4.4**

ABC is an equilateral triangle with $O$ as its center. $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$ represent three forces acting along the sides $AB$,BC, and AC respectively. If the total torque about $O$ is zero then the magnitude of $\vec{F}_3$ (in terms of $\vec{F}_1$ and $\vec{F}_2$) is: (NEET 1998; 2012)

**30. (a) :** Let $x$ be the perpendicular distance of centre $O$ of equilateral triangle from each side.
Total torque about $O = 0$
$\Rightarrow \quad F_1x + F_2x - F_3x = 0 \quad \text{or} \quad F_3 = F_1 + F_2$

### Example 4.5
A rod of weight W is supported by two parallel knife edges A and B and in equilibrium in a horizontal position. The knives are at a distance $d$ from each other. The center of mass of the rod is at distance $x$ from A. The normal reaction on A is: (NEET 2015)

**29. (b) :** Given situation is shown in figure.
$N_1$ = Normal reaction on $A$
$N_2$ = Normal reaction on $B$
$W$ = Weight of the rod




In vertical equilibrium,
$$N_1 + N_2 = W \quad \ldots\text{(i)}$$
Torque balance about centre of mass of the rod,
$$N_1x = N_2(d - x)$$
Putting value of $N_2$ from equation (i)
$$N_1x = (W - N_1)(d - x) \Rightarrow N_1x = Wd - Wx - N_1d + N_1x$$
$$\Rightarrow \quad N_1d = W(d - x); \; \therefore \quad N_1 = \frac{W(d - x)}{d}$$

### 4.6: Torque: Trigonometric Definition
If a force $F$ is applied perpendicular to the radial direction of motion, then the torque is:
$$\tau = rF\sin\theta$$
Where
$$\tau = Magnitude\ of\ torque$$
$$r = radial\ distance\ from\ turning\ point$$
$$F = Magnitude\ of\ force$$
$$\theta = angle\ between\ r\ and\ F$$

### Example 4.7
Why are doorknobs usually placed as far from the hinges as possible?

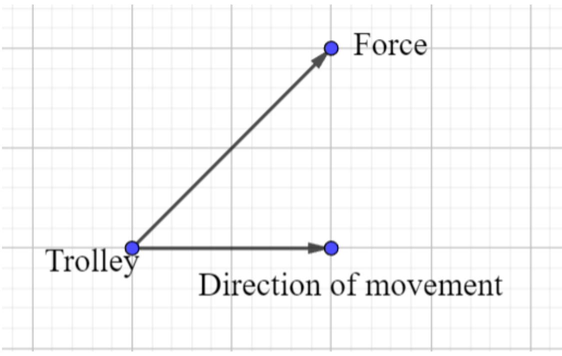
$$\tau = rF$$

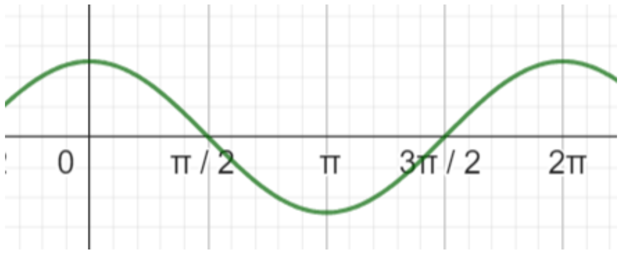
For a given force, the torque is maximum when the radial distance is maximum.
This makes doors easy to open.

### 4.8: Dot Product
- The dot product tells how much of the force is applicable in a particular direction.
- We use dot product with linear movement (and not with circular movement).
- Dot product has two vectors for input, and a scalar for output

I am pulling a trolley (see diagram) and it moves parallel to the ground. But the force is applied in a different direction to the direction of movement.

- $\theta = 0 \Rightarrow \cos\theta = 1$: All of the force is applied in the direction of movement
- $\theta = \pi \Rightarrow \cos\theta = -1$: All of the force is applied opposite to the direction of movement
- $\theta = \frac{\pi}{2} \Rightarrow \cos\theta = 0$: All of the force is applied perpendicular to the direction of movement, which means it has no effect.




## 4.9: Cross Product

- Cross product is related to circular motion (and not linear motion).
- Cross product has two vectors for input, and a vector for the output
- It tells us the moment of force (also called torque) for a turning motion.

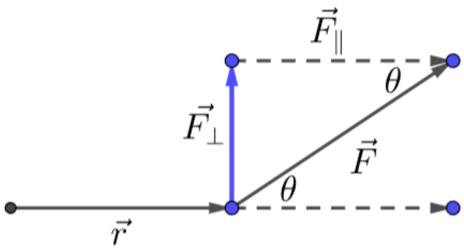
## B. Perpendicular Interpretation

## 4.10: Torque using Perpendicular Vectors

The magnitude of torque can be calculated by using the following equivalent formulas:

$$\tau = rF\sin\theta = rF_\perp = r_\perp F$$

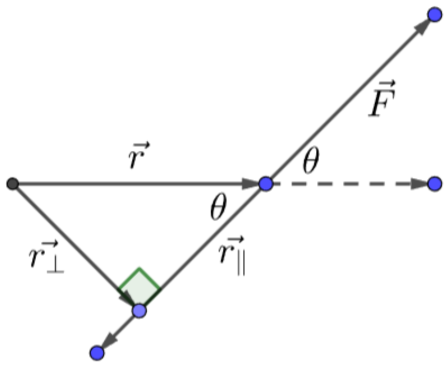
The equivalent interpretations of the torque are important both conceptually and numerically.




### Part I

Using $\sin\theta = \frac{F_\perp}{F} \Rightarrow F_\perp = F\sin\theta$

$$\tau = rF\sin\theta = r(F\sin\theta) = rF_\perp$$




### Part II

Using $\sin\theta = \frac{r_\perp}{F} \Rightarrow r_\perp = r\sin\theta$

$$\tau = rF\sin\theta = F(r\sin\theta) = Fr_\perp$$

## C. Torque Vector

## 4.11: Torque about the origin

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$$

- Torque is a vector.
- $\vec{r}$ is the position vector of the application of the force.
- $\vec{F}$ is the force vector
- The direction of torque is perpendicular to the plane in which the "turning" happens. That is, it is perpendicular to the plane in which $\vec{r}$ and $\vec{F}$ lie.

## Example 4.12

A. Find the torque of a force $\vec{F} = -3\hat{\imath} + \hat{\jmath} + 5\hat{k}$ acting at the point $\vec{r} = 7\hat{\imath} + 3\hat{\jmath} + \hat{k}\, m$ about origin **(NEET 1997)**

B. What is the torque of the force $\vec{F} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ acting at the point $\vec{r} = 3\hat{\imath} + 2\hat{\jmath} + 3\hat{k}\, m$ about origin **(NEET 1995)**

### Part A

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 7 & 3 & 1 \\ -3 & 1 & 5 \end{vmatrix} = [15-1]\hat{\imath} - [35+3]\hat{\jmath} + [7+9]\hat{k} = 14\hat{\imath} - 38\hat{\jmath} + 16\hat{k}$$

### Part B

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 2 & 3 \\ 2 & -3 & 4 \end{vmatrix} = 17\hat{\imath} - 6\hat{\jmath} - 13\hat{k}$$

## Example 4.13

If $\vec{F}$ is the force acting on a particle having position vector $\vec{r}$ and $\vec{\tau}$ be the torque of this force about the origin then:

A. $\vec{r} \cdot \vec{\tau} > 0$ and $\vec{F} \cdot \vec{\tau} < 0$
B. $\vec{r} \cdot \vec{\tau} = 0$ and $\vec{F} \cdot \vec{\tau} = 0$
C. $\vec{r} \cdot \vec{\tau} = 0$ and $\vec{F} \cdot \vec{\tau} \neq 0$
D. $\vec{r} \cdot \vec{\tau} \neq 0$ and $\vec{F} \cdot \vec{\tau} = 0$

Torque vector is perpendicular to both radial distance vector and force vector.
Hence, dot product of torque with both vectors is zero.

$$\textit{Option B}$$

## 4.14: Torque about a point

$$\vec{\tau} = (\vec{r} - \vec{r}_0) \times \vec{F}$$

Where

$$\vec{r} = \textit{Point at which force is applied}$$
$$\vec{r}_0 = \textit{Point at which turning is happening}$$

## Example 4.15

The moment of the force $\vec{F} = 4\hat{\imath} + 5\hat{\jmath} - 6\hat{k}$ at $(2,0,-3)$ about the point $(2,-2,-2)$ is given by **(NEET 2018)**

$$\vec{\tau} = (\vec{r} - \vec{r}_0) \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 2 & -1 \\ 4 & 5 & -6 \end{vmatrix} = 7\hat{\imath} - 4\hat{\jmath} - 8\hat{k}$$

**Example 4.16**
A force $F = \alpha\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$ is acting at a point $\vec{r} = 2\hat{\imath} - 6\hat{\jmath} - 12\hat{k}$. The value of $\alpha$ for which angular momentum about origin is conserved is: **(NEET 2015)**

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -6 & -12 \\ \alpha & 3 & 6 \end{vmatrix} = (-36+36)\hat{\imath} - (12+12\alpha)\hat{\jmath} + (6+6\alpha)\hat{k}$$

Since angular momentum is conserved, the net torque must be zero:

$$-(12+12\alpha)\hat{\jmath} + (6+6\alpha)\hat{k} = \mathbf{0}$$
$$(1+\alpha)\hat{k} = 2(1+\alpha)\hat{\jmath}$$

$$1 + \alpha = 0 \Rightarrow \alpha = -1$$

# 4.2 Linear and Angular Velocity

**4.17: Linear and Angular Velocity**

$$\vec{v} = \vec{\omega} \times \vec{r}$$
$$\vec{v} = Linear\ Velocity$$
$$\vec{\omega} = Angular\ Velocity$$
$$\vec{r} = radius$$

**Example 4.18**
What is the value of linear velocity if $\vec{r} = 3\hat{\imath} - 4\hat{\jmath} + \hat{k}$ and $\vec{\omega} = 5\hat{\imath} - 6\hat{\jmath} + 6\hat{k}$? ()

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & -6 & 6 \\ 3 & -4 & 1 \end{vmatrix} = (-6+24)\hat{\imath} - (5-18)\hat{\jmath} + (-20+18)\hat{k} = 18\hat{\imath} + 13\hat{\jmath} - 2\hat{k}$$

# 4.3 Angular Momentum

## A. Angular Momentum

**4.19: Linear Momentum**

$$P = mv$$

**4.20: Angular Momentum**
The angular momentum is:

$$L = I\omega$$

Where

$$L = Angular\ Momentum$$
$$I = Rotational\ Inertia\ (instead\ of\ m)$$
$$\omega = angular\ velocity\ (instead\ of\ v)$$

**Example 4.21**
Two rotating bodies of $A$ and $B$ of masses $m$ and $2m$ with moments of inertia $I_A$ and $I_B$ $(I_B > I_A)$ have equal

kinetic energy of rotation. If $L_A$ and $L_B$ be their angular momenta respectively, then:

A. $L_A = \frac{L_B}{2}$
B. $L_A = 2L_B$
C. $L_B > L_A$
D. $L_A > L_B$ (NEET 2016)

$$K_A = K_B$$
$$\frac{1}{2}I_A\omega_A^2 = \frac{1}{2}I_B\omega_B^2$$
$$I_A\omega_A^2 = I_B\omega_B^2$$
$$\frac{\omega_A^2}{\omega_B^2} = \frac{I_B}{I_A}$$
$$\frac{\omega_A}{\omega_B} = \sqrt{\frac{I_B}{I_A}}$$

$$\frac{L_A}{L_B} = \frac{I_A\omega_A}{I_B\omega_B} = \frac{I_A}{I_B}\sqrt{\frac{I_B}{I_A}} = \sqrt{\frac{I_A}{I_B}} < 1$$
$$L_B > L_A$$

## 4.22: Conservation of Angular Momentum

In the absence of a net force, angular momentum is conserved.
If angular momentum is conserved, then the net force (torque) is zero.

### Example 4.23

*Mark the correct option*

A particle of mass $m$ moves in the $XY$ plane with a velocity $v$ along the straight line $AB$. If the angular momentum of the particle with respect to origin $O$ is $L_A$, when it is at $A$ and $L_B$ when it is at $B$ then:

A. $L_A = L_B$
B. The relationship between $L_A$ and $L_B$ depends on the slope of the line $AB$
C. $L_A < L_B$
D. $L_A > L_B$ (NEET 2007)

There is no force on the object. Hence, the angular momentum is conserved.

*Option A*

### Example 4.24

Two discs of same moment of inertia rotating about their regular axis passing through their regular axis passing through center and perpendicular to the plane of disc have angular velocities $\omega_1$ and $\omega_2$. They are brought into contact face to face coinciding the axis of rotation. The expression for loss of energy during this process is: (NEET 2017)

According to conserv

**72. (a) :** Initial angular momentum = $I\omega_1 + I\omega_2$

Let $\omega$ be angular speed of the combined system.

Final angular momentum = $2I\omega$

∴ According to conservation of angular momentum

$$I\omega_1 + I\omega_2 = 2I\omega \quad \text{or} \quad \omega = \frac{\omega_1 + \omega_2}{2}$$

Initial rotational kinetic energy,

$$E = \frac{1}{2}I(\omega_1^2 + \omega_2^2)$$

Final rotational kinetic energy

$$E_f = \frac{1}{2}(2I)\omega^2 = \frac{1}{2}(2I)\left(\frac{\omega_1 + \omega_2}{2}\right)^2 = \frac{1}{4}I(\omega_1 + \omega_2)^2$$

∴ Loss of energy $\Delta E = E_i - E_f$

$$= \frac{I}{2}(\omega_1^2 + \omega_2^2) - \frac{I}{4}(\omega_1^2 + \omega_2^2 + 2\omega_1\omega_2)$$

$$= \frac{I}{4}\left[\omega_1^2 + \omega_2^2 - 2\omega_1\omega_2\right] = \frac{I}{4}(\omega_1 - \omega_2)^2$$

**Example 4.25**

Two discs are rotating about their axes, normal to the discs and passing through the centers of the discs. Disc $D_1$ has 2 $kg$ mass and 0.2 $m$ radius and initial angular velocity of 50 $\frac{rad}{s}$. Disc $D_2$ has 4 $kg$ mass and 0.1 $m$ radius and initial angular velocity of 200 $\frac{rad}{s}$. The two discs are brought in contact face to face, with their axes of rotation coincident. The final angular velocity in $\frac{rad}{s}$ of the system is: **(NEET 2013)**

**74. (b) :** Moment of inertia of disc $D_1$ about an axis passing through its centre and normal to its plane is

$$I_1 = \frac{MR^2}{2} = \frac{(2\text{ kg})(0.2\text{ m})^2}{2} = 0.04\text{ kg m}^2$$

Initial angular velocity of disc $D_1$, $\omega_1 = 50$ rad s$^{-1}$
Moment of inertia of disc $D_2$ about an axis passing through its centre and normal to its plane is

$$I_2 = \frac{(4\text{ kg})(0.1\text{ m})^2}{2} = 0.02\text{ kg m}^2$$

Initial angular velocity of disc $D_2$, $\omega_2 = 200$ rad s$^{-1}$
Total initial angular momentum of the two discs is

$$L_i = I_1\omega_1 + I_2\omega_2$$

When two discs are brought in contact face to face (one on the top of the other) and their axes of rotation coincide, the moment of inertia $I$ of the system is equal to the sum of their individual moment of inertia.

$$I = I_1 + I_2$$

Let $\omega$ be the final angular speed of the system. The final angular momentum of the system is

$$L_f = I\omega = (I_1 + I_2)\omega$$

According to law of conservation of angular momentum, we get

$$L_i = L_f \text{ or, } I_1\omega_1 + I_2\omega_2 = (I_1 + I_2)\omega \text{ or, } \omega = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}$$

$$= \frac{(0.04\text{ kg m}^2)(50\text{ rad s}^{-1}) + (0.02\text{ kg m}^2)(200\text{ rad s}^{-1})}{(0.04 + 0.02)\text{ kg m}^2}$$

$$= \frac{(2+4)}{0.06}\text{ rad s}^{-1} = 100\text{ rad s}^{-1}$$

### 4.26: Connecting Angular Momentum with Linear Velocity

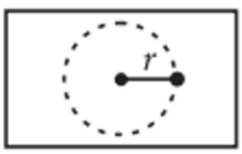
For a particle with mass $m$ in circular motion around a center of motion with radius $r$

$$L = mvr$$

$$L = I\omega$$

Substitute $I = r^2 m, \omega = \frac{v}{r}$

$$L = (r^2 m)\left(\frac{v}{r}\right) = mvr$$




### Example 4.27

A small mass attached to a string rotates on a frictionless table top as shown. If the tension in the string is increased by pulling the string causing the radius of the circular motion to decrease by a factor of 2, the kinetic energy of the mass will increase/decrease by a factor of: (NEET 2011)

By conservation of angular momentum $mv_0R_0 = mVR$:

$$v_0R_0 = V\left(\frac{R_0}{2}\right) \Rightarrow V = 2v_0$$

The final kinetic energy is:

$$K \propto v^2 \Rightarrow K = 4K_0$$
$$Increase\ by\ factor\ of\ 4$$

### Example 4.28

A mass $m$ moves in a circle on a smooth horizontal plane with velocity $v_0$ at a radius $R_0$. The mass is attached to a string which passes through a smooth hole in the plane as shown. The tension in the string is increased gradually, and finally $m$ moves in a circle of radius $\frac{R_0}{2}$. The final value of the kinetic energy (in terms of $m$ and $v_0$) is: (NEET 2015)

By conservation of angular momentum $mv_0R_0 = mVR$:

$$v_0R_0 = V\left(\frac{R_0}{2}\right) \Rightarrow V = 2v_0$$

The final kinetic energy is:

$$K = \frac{1}{2}mV^2 = \frac{1}{2}m(2v_0)^2 = 2mv_0^2$$

### 4.29: Magnitude of Angular Momentum

$$L = mvr\sin\theta$$

Where

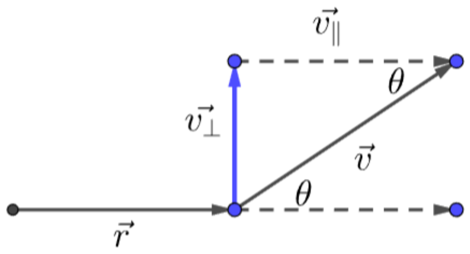
$$m = mass$$
$$v = magnitude\ of\ velocity\ vector$$
$$r = magnitude\ of\ radial\ distance\ vector$$
$$\theta = angle\ between\ velocity\ and\ radial\ distance\ vectors$$

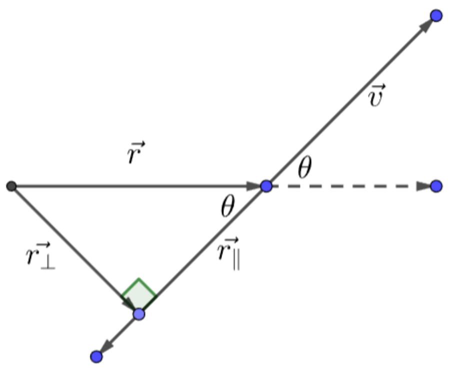
### 4.30: Angular Momentum in Perpendicular Form

$$L = mvr_\perp = mv_\perp r$$

#### Part I

Using $\sin\theta = \frac{v_\perp}{v} \Rightarrow v_\perp = v\sin\theta$



$$\tau = mvr = mv_{\perp}r$$

### Part II

Using $\sin\theta = \frac{r_{\perp}}{r} \Rightarrow r_{\perp} = r\sin\theta$

$$L = mvr\sin\theta = mvr_{\perp}$$

## Example 4.31

*Mark the correct option*

A particle of mass $m$ moves in the $XY$ plane with a velocity $v$ along the straight line $AB$. If the angular momentum of the particle with respect to origin $O$ is $L_A$, when it is at $A$ and $L_B$ when it is at $B$ then:

A. $L_A = L_B$
B. The relationship between $L_A$ and $L_B$ depends on the slope of the line $AB$
C. $L_A < L_B$
D. $L_A > L_B$ (NEET 2007)

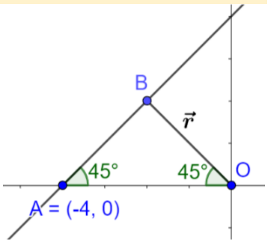
$$L = mvr_{\perp}$$

All three of the above quantities remain same.
Hence, the angular momentum is conserved.

$$Option\ A$$

## Example 4.32

A particle of mass $m = 5$ is moving with uniform speed $v = 3\sqrt{2}$ in the $XOY$ plane along the line $y = x + 4$. The magnitude of the angular momentum of the particle about the origin is: **(NEET 1991)**




$$OB = OA \times \frac{\sqrt{2}}{2} = 4 \times \frac{\sqrt{2}}{2} = 2\sqrt{2}$$
$$L = mvr_{\perp} = (5)(3\sqrt{2})(2\sqrt{2}) = 60\ units$$

## B. Angular Momentum Vector

### 4.33: Cross Product: Component Definition

If $\vec{\boldsymbol{a}} = a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{b}} = b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}$ then their cross product is:

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Expanding the determinant along the first column gives us:

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{\boldsymbol{\imath}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{\boldsymbol{\jmath}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{\boldsymbol{k}}$$

Expanding each determinant gives us:

$$= (a_2b_3 - a_3b_2)\hat{\boldsymbol{\imath}} - (a_1b_3 - a_3b_1)\hat{\boldsymbol{\jmath}} + (a_1b_2 - a_2b_1)\hat{\boldsymbol{k}}$$

### 4.34: Angular Momentum Vector

The angular momentum is:

$$\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}$$

Where

$$\vec{L} = Angular\ Momentum$$
$$\vec{r} = radial\ distance\ vector$$
$$\vec{p} = linear\ momentum\ vector$$

The direction of angular momentum vector is perpendicular to the plane of rotation. That is, it is perpendicular to:

- $\vec{r}$, which gives the radial distance
- $\vec{p}$, which gives the linear momentum

## Example 4.35

When a mass is rotating in a plane about a fixed point, its angular momentum is directed along

A. a line perpendicular to the plane of rotation
B. the line making an angle of 45° to the plane of rotation
C. the radius
D. the tangent to the orbit **(NEET 2012)**

$$Option\ A$$

## 4.36: Scalar Multiplication

The cross product is commutative with scalar multiplication. In other words, you can "factor" any scalar out of a cross product.

$$\lambda(\vec{a} \times \vec{b}) = (\lambda\vec{a}) \times \vec{b} = \vec{a} \times (\lambda\vec{b})$$

## 4.37: Angular Momentum Vector

The angular momentum in terms of the linear velocity and the mass is:

$$\vec{L} = m(\vec{r} \times \vec{v})$$

$$\vec{L} = \vec{r} \times \vec{p}$$

Substitute $Linear\ Momentum = m\vec{v}$:

$$\vec{L} = \vec{r} \times m\vec{v} = m(\vec{r} \times \vec{v})$$

## Example 4.38

A small particle of mass $m$ is projected at an angle $\theta$ with the $x$ −axis with an initial velocity $v_0$ in the $x - y$ plane. At a time $t < \frac{v_0 \sin\theta}{g}$, the angular momentum vector of the particle is: **(JEE-M 2010)**

$$\vec{v} = (v_0 \cos\theta, v_0 \sin\theta - gt, 0)$$

Using $s = ut - \frac{1}{2}at^2$ with $a = -g$:

$$\vec{r} = \left(v_0 \cos\theta t, v_0 \sin\theta\, t - \frac{1}{2}gt^2, 0\right)$$

Substitute the above into $\vec{L} = m(\vec{r} \times \vec{v})$

$$m\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_0 \cos\theta t & v_0 \sin\theta\, t - \frac{1}{2}gt^2 & 0 \\ v_0 \cos\theta & v_0 \sin\theta - gt & 0 \end{vmatrix}$$

Using the definition of the cross product, the $\hat{\imath}$ term and the $\hat{\jmath}$ term are both zero. We are only left with the $\hat{k}$

term:

$$= m\left[(v_0\, cos\, \theta t)(v_0 \sin\theta - gt) - \left(v_0 \sin\theta\, t - \frac{1}{2}gt^2\right)(v_0\, cos\, \theta)\right]\widehat{\boldsymbol{k}}$$

Carrying out the multiplication:

$$= m\left[v_0^2\, cos\, \theta \sin\theta\, t - v_0^2\, cos\, \theta g t^2 - \left(v_0^2 \cos\theta \sin\theta\, t - v_0^2\, cos\, \theta \frac{1}{2} g t^2\right)\right]\widehat{\boldsymbol{k}}$$

Simplifying:

$$= m\left[-v_0^2\, cos\, \theta g t^2 + v_0^2\, cos\, \theta \frac{1}{2} g t^2\right]\widehat{\boldsymbol{k}}$$

$$= \left(-\frac{1}{2} m g t^2 v_0^2 cos\, \theta\right)\widehat{\boldsymbol{k}}$$

# 4.4 NLM-2

## A. Newton's Second Law

### 4.39: Newton's Second Law

For linear acceleration, Newton's Second Law states that

$$\vec{\boldsymbol{F}}_{Net} = m\vec{\boldsymbol{a}}$$

where

$$F = Force$$
$$m = mass$$
$$a = linear\ acceleration$$

- Note that we mean linear acceleration in the sense that the object is not rotating.
- The force and the acceleration can be 2D or 3D.

### 4.40: Newton's Second Law: Rotational Version

For rotational acceleration:

$$\tau_{Net} = I\alpha$$

Where

$$\tau_{Net} = Net\ Rotational\ Force\ (instead\ of\ F)$$
$$I = Rotational\ Inertia\ (instead\ of\ m)$$
$$\alpha = angular\ acceleration\ (instead\ of\ a)$$

### Example 4.41

A rope is wound around a hollow cylinder of mass 3 $kg$ and radius 40 $cm$. What is the angular acceleration of the cylinder if the rope is pulled with a force of 30 $N$? **(NEET 2017)**

$$\alpha = \frac{\tau}{I} = \frac{rF}{mr^2} = \frac{F}{mr} = \frac{30}{(3)(0.4)} = \frac{100}{4} = 25\frac{rad}{s}$$

### Example 4.42

A solid cylinder of mass 2 $kg$ and radius 4 $cm$ is rotating about its axis at the rate of 3 rpm. The torque required to stop it after $2\pi$ revolutions is: **(NEET 2019)**

The moment of inertia of the solid cylinder is:

$$I = \frac{MR^2}{2} = \frac{2\left(\frac{4}{100}\right)^2}{2} = 16 \cdot 10^{-4}$$

Calculate the known values:

$$Final\ Velocity = \omega_F^2 = 0$$
$$Initial\ Velocity = \omega_I^2 = 3\ rpm = 3 \cdot \frac{2\pi}{60}\frac{rad}{s} = \frac{\pi}{10}\frac{rad}{s}$$
$$\Delta\theta = 2\pi \times 2\pi = 4\pi^2\ rad$$

Substitute the above into the equation of motion $\omega_F^2 = \omega_I^2 + 2\alpha(\Delta\theta)$:

$$0 = \left(\frac{\pi}{10}\right)^2 + 2\alpha(4\pi^2)$$
$$0 = \frac{\pi^2}{100} + 8\alpha\pi^2$$
$$\alpha = -\frac{1}{800}\frac{rad}{s^2}$$

Finally, the torque is:

$$= I\alpha = (16 \cdot 10^{-4})\left(-\frac{1}{800}\right) = -2 \cdot 10^{-6}\ Nm$$

**Example 4.43**

A uniform circular disc of radius 50 $cm$ at rest is free to turn about an axis which is perpendicular to its plane and passes through its center. It is subjected to a torque which produces a constant angular acceleration of $2.0\frac{rad}{s^2}$. The magnitude of its net acceleration in $\frac{m}{s^2}$ at the end of 2.0 $s$ is: **(NEET 2016)**

$$a_t = r\alpha = (0.5)(2) = 1\frac{m}{s^2}$$
$$a_c = \omega_F^2 r = (\omega_I + \alpha t)^2 r = (0 + 2 \cdot 2)^2(0.5) = 8\frac{m}{s^2}$$

$$a = \sqrt{a_t^2 + a_c^2} = \sqrt{1^2 + 8^2} = \sqrt{65}$$

# 4.5 Calculus with Torque

## A. NLM-2

**4.44: Newton's Second Law: Linear Form**

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$$

Force is the rate of change of linear momentum.

The linear momentum is:

$$\vec{\boldsymbol{P}} = m\vec{\boldsymbol{v}}$$

Differentiate both sides to get:

$$\frac{d}{dt}\vec{\boldsymbol{P}} = m\vec{\boldsymbol{a}}$$

Note that the RHS is just force:

$$\vec{F} = m\vec{a}$$

### 4.45: Newton's Second Law: Angular Form

$$\tau = I\alpha$$

Torque is the rate of change of angular momentum.

$$L = I\omega$$

Differentiate both sides to get:

$$\tau = I\alpha$$

## 2 Pending

### Example 4.46

When the torque acting upon a system is zero, the parameter that remains constant is: **(EAMCET 2020)**

A. Force
B. Linear momentum
C. Angular momentum
D. Linear Impulse

$$\tau = \frac{d}{dt}L = 0 \Rightarrow L = Constant$$

$$Option\ C$$

# 5. Rolling Motion

## 5.1 Rolling Motion

**5.1: Revision of Formulas**

$$Angular\ velocity = \omega = \frac{v}{r}, v = linear\ velocity, r = radius$$
$$Magnitude\ of\ Torque = rF$$

### A. Rolling Motion

**5.2: Rolling Objects**

The focus of this section is on objects that can roll.

- Objects that can roll (hollow/solid): Sphere, Cylinder/Disc
- Objects that cannot roll: Cuboid, Cube, etc

For now, we will not consider objects that "topple/fall" and move forward. For example, a cylinder with a hexagonal base.

**5.3: Pure Rolling Motion**

Rolling motion comprises

- rotational motion
- translational motion

- Rotational motion requires some force that will make the object rotate.
- Recall that $Torque = \tau = rF$

### B. Velocity

**5.4: Velocity of Center of Mass**

The velocity of the center of mass of a rolling object is:

$$v_{COM} = v = \omega r$$

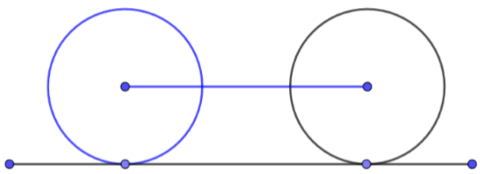
Where

$$v_{COM} = velocity\ of\ center\ of\ mass$$
$$v = Tangential\ Linear\ velocity\ of\ a\ point\ on\ edge\ undergoing\ circular\ motion$$
$$\omega = angular\ velocity$$
$$r = radius\ of\ circular\ object$$

Consider a circular object with:

$$Radius = R \Rightarrow Circumference = 2\pi r$$

$$v_{COM} = \frac{Distance}{Time} = \frac{2\pi r}{T} = \frac{2\pi}{T} r = \omega r = v$$



**Example 5.5**

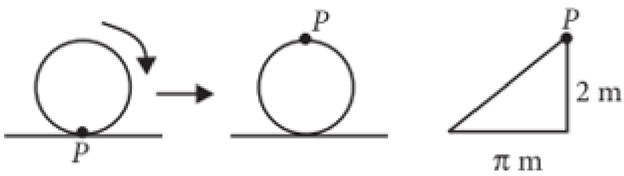
Consider a contact point $P$ of a wheel on ground which rolls on ground without slipping. Then the magnitude of displacement of point $P$ when wheel completes half of rotation is (if radius of wheel is $1\ m$) is: **(NEET 2002)**

The horizontal distance

$$=\frac{1}{2}C=\frac{1}{2}(2\pi r)=\pi r=\pi$$

The vertical distance is

$$2r=2$$




The total displacement is, by Pythagoras Theorem is

$$\sqrt{\pi^2+4}\,m$$

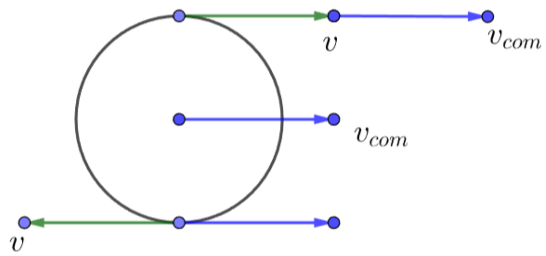
## 5.6: Velocity of Top and Bottom Point

In rolling motion, the
The bottom point of a circle in rolling motion (that is, the point touching the ground) does not move.

### Top Point in rolling motion

Moves forward because of two factors in the same direction:

- the translational motion $v_{com}=v$
- the tangential linear velocity $v$.
- Net velocity $=v+v=2v$




### Bottom Point in rolling motion

- Moves forward due to the translational motion

$$v_{com}=v$$

- Moves backward from the tangential linear velocity $v$.
- Net velocity $=v-v=0$

### Example 5.7

A disc is rolling. The velocity of its center of mass is $v_{cm}$. Determine the velocity of highest point and point of contact. (NEET 2001)

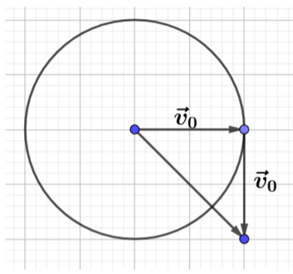
$$Velocity\ of\ Highest\ point=2v_{cm}$$
$$Velocity\ of\ point\ of\ contact=0$$

## 3 Pending

### Example 5.8

The center of a wheel rolling on a plane surface moves with a speed $v_0$. A particle on the rim of the wheel at the same level as the center will be moving at a speed: (EAMCET 2020)

$$v=\sqrt{v_0^2+v_0^2}=\sqrt{2}v_0$$



## C. Friction in Rolling

### 5.9: Work done by static friction in Rolling

$$Work = Fs\cos\theta$$

Since displacement is zero:

$$Work = F(0)\cos\theta = 0$$

### 5.10: Role of Friction in Rolling

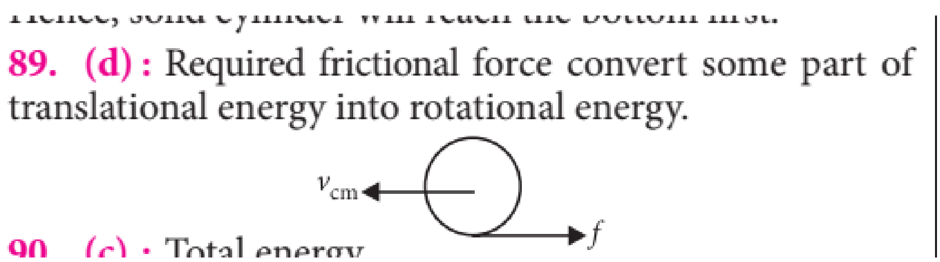
The role of static friction in rolling is to convert translational energy into rotational energy.

$$Work = \Delta KE$$

### Example 5.11

A drum of radius $R$ and mass $M$ rolls down without slipping along an inclined plane of angle $\theta$. The frictional force:

A. Dissipates energy as heat
B. Decreases the rotational motion
C. Decreases the rotational and translational motion
D. Converts translational energy to rotational energy


**89. (d) :** Required frictional force convert some part of translational energy into rotational energy.


## D. Smooth Surface

### 5.12: Slipping/Sliding

If the same point remains in contact with the surface, then the object is slipping/sliding.

### 5.13: Smooth Incline

A smooth incline has no rotational force. Hence, a body placed on a smooth incline starting from rest will not rotate. The only movement will be due to

*slipping and sliding*

Forces acting on a body on a smooth incline are:

- Parallel to the incline: $mg\sin\theta$
- Perpendicular to the incline: Normal Force

Both forces act on the center of mass of the object and hence do not generate torque.
Hence, a circular object on a smooth incline does not roll. It only moves via

*slipping and sliding*

**Example 5.14**
An object rolls is placed at the top of an incline. The top part of the incline is rough, and the bottom half is smooth. The object rolls over the top half. When it reaches the bottom half, it will:

A. Slip
B. Slide
C. Continue to roll
D. None of the above

$$Roll$$

# 5.2 Kinetic Energy

## A. Formula

### 5.15: Moment of Inertia

Moment of Inertia through the central axis for each of the below:

| Shape | Solid | Annular | Hollow |
|---|---|---|---|
| Cylinder/Disc | $\frac{1}{2}MR^2$<br>$R = Radius$ | $I = \frac{1}{2}M(R^2 + r^2)$<br>$R = Outer\ Radius$<br>$r = Inner\ Radius$ | $MR^2$ |
| Sphere | $\frac{2}{5}MR^2$<br>$R = radius$ | | $\frac{2}{3}MR^2$<br>$R = radius$ |

Moment of Inertia perpendicular to center of mass for the shapes below is:

| Shape | Moment of Inertia | |
|---|---|---|
| Thin Rectangular Slab | $\frac{M(a^2 + b^2)}{12}$ | |
| Thin Square Lamina | $\frac{ML^2}{6}$ | |

### 5.16: Kinetic Energy of Rolling Motion

The kinetic energy of rolling motion is the sum of the rotational and translational kinetic energy:

$$KE = \underbrace{\frac{1}{2}I\omega^2}_{Rotational} + \underbrace{\frac{1}{2}mv^2}_{Translational}$$

**Challenge 5.17**
A solid cylinder and a hollow cylinder, both of the same mass and same external diameter are released from the same height at the same time on an inclined plane. Both roll down without slipping. Which one will reach the bottom first? **(NEET 2010)**

A rolling object distributes its kinetic energy between its rotational part and its translational part.
From conservation of energy

$$GPE = mgh = KE$$

For a hollow cylinder, the moment of inertia is greater and hence the rotational kinetic energy is also greater. Hence, less energy is directed towards translational and hence

$$\textit{Hollow Cylinder is slower}$$

## B. Sphere

### 5.18: Kinetic Energy of a Sphere

The kinetic energy of a rolling solid sphere with radius $R$, mass $m$, velocity of center of mass $v$ is

$$\frac{7}{10}mv^2$$

$$KE = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v}{R}\right)^2 = \frac{7}{10}mv^2$$

### Example 5.19

A. A solid spherical ball rolls on a table. The ratio of its rotational kinetic energy to total kinetic energy is: **(NEET 1994)**

B. A solid sphere is in rolling motion. In rolling motion, a body possesses translational kinetic energy ($K_t$) as well as rotational kinetic energy ($K_r$) simultaneously. The ratio $K_t:(K_t + K_r)$ for the sphere is: **(NEET 2018, 1991)**

**Part A**

$$\frac{Rotational}{Total} = \frac{\frac{1}{5}mv^2}{\frac{7}{10}mv^2} = \frac{1}{5}\cdot\frac{10}{7} = \frac{2}{7}$$

**Part B**

$$\frac{K_t}{K_t + K_r} = \frac{\frac{1}{2}mv^2}{\frac{7}{10}mv^2} = \frac{1}{2}\cdot\frac{10}{7} = \frac{5}{7}$$

### Example 5.20

The speed of a homogenous solid sphere after rolling down an inclined plane of vertical height $h$ from rest without sliding is: **(NEET 1992; EAMCET 2020)**

$$mgh = \frac{7}{10}mv^2 \Rightarrow v = \sqrt{\frac{10}{7}gh}$$

## C. Cylinder/Disc

### 5.21: Kinetic Energy of a Cylinder/Disc

The kinetic energy of a rolling solid disc/cylinder with radius $R$, mass $m$, velocity of center of mass $v$ is

$$\frac{3}{4}mv^2$$

$$KE = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 = \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{v}{R}\right)^2 + \frac{1}{2}mv^2 = \frac{3}{4}mv^2$$

## 4 Pending

### Example 5.22

Two identical discs are moving with same kinetic energy. One rolls and the other slides. The ratio of their speeds is: **(EAMCET 2019)**

$$\frac{3}{4}mv_{Rolls}^2 = \frac{1}{2}mv_{Slides}^2 \Rightarrow \frac{v_{Rolls}^2}{v_{Slides}^2} = \frac{2}{3} \Rightarrow \frac{v_{Rolls}}{v_{Slides}} = \frac{\sqrt{2}}{\sqrt{3}} = \sqrt{2}:\sqrt{3}$$

### Example 5.23: Work

A disc of radius $2\ m$ and mass $100\ kg$ rolls on a horizontal floor. Its center of mass has speed of $20\frac{cm}{s}$. How much work is need to stop it? **(NEET 2019)**

$$Work = \Delta KE = \frac{3}{4}mv^2$$

Substitute $m = 100\ kg, v = 20\frac{cm}{s}$:

$$= \frac{3}{4}\cdot 100 \cdot \left(\frac{20}{100}\right)^2 = \frac{3}{4}\cdot 100 \cdot \frac{1}{25} = 3J$$

### Example 5.24: Conservation of Energy

A solid cylinder of mass $2\ kg$ and radius $50\ cm$ rolls up an inclined plane of inclination 30°. The center of mass of cylinder has a speed of $4\frac{m}{s}$. The distance travelled by the cylinder on the incline is: (take $g = 10\frac{m}{s^2}$) **(NEET 2019)**

$$GPE_{Top} = KE_{Bottom}$$

Substitute $GPE = mgh = mgs\sin 30° = \frac{mgs}{2}, KE = \frac{3}{4}mv^2$:

$$\frac{mgs}{2} = \frac{3}{4}mv^2 \Rightarrow s = \frac{3}{2}\cdot\frac{v^2}{g} = \frac{3}{2}\cdot\frac{4^2}{10} = 2.4\ m$$

### Example 5.25

A solid cylinder of mass $M$ and radius $R$ rolls without slipping down an inclined plane of length $L$ and height $h$. What is the speed of its center of mass when the cylinder reaches the bottom? **(NEET 2003, 1989; EAMCET 2020)**

$$mgh = \frac{3}{4}mv^2 \Rightarrow v = \sqrt{\frac{4}{3}gh}$$

## D. Back Calculations

### Example 5.26

A small object of uniform density rolls up a curved surface with an initial velocity $v$. It reaches upto a maximum height of $\frac{3v^2}{4g}$ with respect to the initial position. The object is: **(NEET 2013)**

A. Hollow sphere
B. Disc
C. Ring

D. Solid Sphere

The kinetic energy of the object at the bottom must equal the gravitational potential at the top

$$KE = GPE$$

Substituting the values:

$$\frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 = mgh$$

Substitute $\omega = \frac{v}{R}, h = \frac{3v^2}{4g}$:

$$\frac{1}{2}I\left(\frac{v}{R}\right)^2 + \frac{1}{2}mv^2 = mg\left(\frac{3v^2}{4g}\right)$$

$$I\frac{v^2}{2R^2} = \frac{3}{4}mv^2 - \frac{1}{2}mv^2$$

$$I\frac{v^2}{2R^2} = \frac{1}{4}mv^2$$

$$I = \frac{1}{2}mR^2$$

$$Option\ B: Disc$$

# 5.3 Kinematics

## A. Acceleration

### 5.27: Slipping and Sliding

From circular motion, we know that the relation between angular acceleration and linear acceleration is given by:

$$\alpha = \frac{a}{R}$$

If the above condition is met during rolling, then we have

$$\alpha = \frac{a}{R} \Rightarrow Pure\ Rolling$$

$$\alpha \neq \frac{a}{R} \Rightarrow Rolling\ mixed\ with\ slipping\ or\ sliding\ or\ pure\ sliding$$

### 5.28: Acceleration of a Rolling Object on a Ramp

For a circular object of uniform density that begins from rest on a rough ramp, and undergoes "pure rolling" down the ramp, the acceleration is:

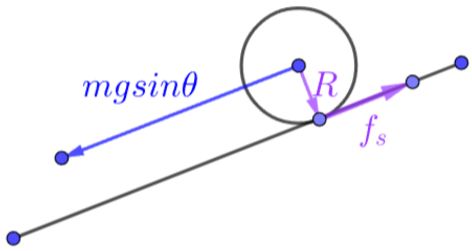
$$a = \frac{mg\sin\theta}{m + \frac{I_{COM}}{R^2}} = \frac{g\sin\theta}{1 + \frac{I}{mR^2}} = \frac{g\sin\theta}{1 + X}$$

Note that the acceleration is independent of:

$$m = Mass\ of\ the\ object$$

$$R = Radius\ of\ the\ object$$

$$\mu_s = Coefficient\ of\ static\ fraction$$




The point of contact of the object with the ramp is motionless.
The force in the direction parallel to the ramp is:

$$\underbrace{F = ma = mg\sin\theta - f_s}_{Equation\ I}$$

The torque at the point of contact of the object with the ramp is:

$$Torque = I\alpha = Rf_s$$

Substitute the condition for "pure rolling" $\alpha = \frac{a}{R}$:

$$\frac{Ia}{R} = Rf_s \Rightarrow \underbrace{f_s = \frac{Ia}{R^2}}_{Equation\ II}$$

Substitute Equation II into Equation I:

$$ma = mg\sin\theta - \frac{Ia}{R^2}$$

Collate all $a$ terms on one side:

$$ma + \frac{Ia}{R^2} = mg\sin\theta$$

Factor out $a$:

$$a\left(m + \frac{I}{R^2}\right) = mg\sin\theta$$

Solve for $a$:

$$a = \frac{mg\sin\theta}{m + \frac{I}{R^2}} = \frac{g\sin\theta}{1 + \frac{I}{mR^2}}$$

All the circular objects that we are considering all have moments of inertia of the form

$$XmR^2, \qquad X = Some\ Constant$$

We can use this to simplify the formula for the acceleration. Substitute $I = XmR^2$

$$a = \frac{mg\sin\theta}{m + \frac{I}{R^2}} = \frac{g\sin\theta}{1 + \frac{XmR^2}{mR^2}} = \frac{g\sin\theta}{1 + X}$$

## Example 5.29

The ratio of the accelerations for a solid sphere (mass $m$ and radius $R$) rolling down an incline of angle $\theta$ without slipping and slipping down the incline without rolling is: **(NEET 2014)**

Acceleration without slipping will happen only when friction is present. We assume pure rolling, which is independent of the coefficient of friction:

$$a = \frac{g\sin\theta}{1 + X} = \frac{g\sin\theta}{1 + \frac{2}{5}} = \frac{5}{7}g\sin\theta$$

Since we have no information on friction, we assume slipping happens in the absence of friction:

$$a = g\sin\theta$$

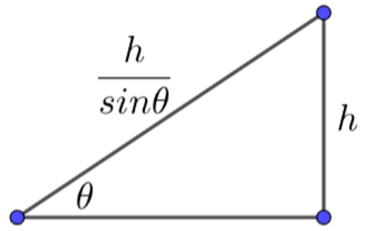
The ratio is then:

$$\frac{5}{7}g\sin\theta : g\sin\theta = \frac{5}{7}:1 = 5:7$$

## B. Velocity

### 5.30: Velocity

$$v = \sqrt{\frac{2gh}{1+X}}$$




Since the initial velocity $= u^2 = 0$ in the equation of motion:

$$v^2 = u^2 + 2as = 0^2 + 2as = 2as$$

Substitute $s = \frac{h}{\sin\theta}, a = \frac{g\sin\theta}{1+X}$:

$$v^2 = 2\left(\frac{g\sin\theta}{1+X}\right)\frac{h}{\sin\theta} = \frac{2gh}{1+X}$$

Take square roots to get:

$$v = \sqrt{\frac{2gh}{1+X}}$$

### Example 5.31

A. A solid cylinder of mass $M$ and radius $R$ rolls without slipping down an inclined plane of length $L$ and height $h$. What is the speed of its center of mass when the cylinder reaches the bottom? **(NEET 2003, 1989; EAMCET 2020)**

B. The speed of a homogenous solid sphere after rolling down an inclined plane of vertical height $h$ from rest without sliding is: **(NEET 1992; EAMCET 2020)**

$$Part\ A: v = \sqrt{\frac{2gh}{1+X}} = \sqrt{\frac{2gh}{1+\frac{1}{2}}} = \sqrt{\frac{2gh}{\frac{3}{2}}} = \sqrt{\frac{4}{3}gh}$$

$$Part\ B: v = \sqrt{\frac{2gh}{1+X}} = \sqrt{\frac{2gh}{1+\frac{2}{5}}} = \sqrt{\frac{2gh}{\frac{7}{5}}} = \sqrt{\frac{10}{7}gh}$$

## C. Time

### 5.32: Time

The time taken by a circular object to roll down an incline is

$$t = \sqrt{\frac{2h(1+X)}{g\sin^2\theta}}$$

Substitute $Initial\ Velocity = u = 0$ in

$$v = u + at = 0 + at \Rightarrow t = \frac{v}{a}$$

$$t = \frac{v}{a} = \frac{\sqrt{\frac{2gh}{1+X}}}{\frac{g\sin\theta}{1+X}} = \sqrt{\frac{2gh}{1+X} \times \frac{1+X}{g\sin\theta}} = \sqrt{\frac{2h(1+X)}{g\sin^2\theta}}$$

### Example 5.33

A disc and a sphere of same radius but different masses roll off on two inclined planes of the same altitude and length. Determine the ratio of the times in which the sphere and the disc get to the bottom of the plane? Which one reaches first? **(NEET 2016, Adapted)**

$g, \theta$ and $h$ are same.

$$\sqrt{\frac{2h\left(1+X_{Sphere}\right)}{g\sin\theta}} : \sqrt{\frac{2h(1+X_{Disc})}{g\sin\theta}}$$

$$\sqrt{1+X_{Sphere}} : \sqrt{1+X_{Disc}}$$

$$\sqrt{1+\frac{2}{5}} : \sqrt{1+\frac{1}{2}}$$

$$\sqrt{\frac{7}{5}} : \sqrt{\frac{3}{2}}$$

$$\sqrt{14} : \sqrt{15}$$

Hence,

$$Sphere\ is\ faster$$

### Example 5.34

A. A solid cylinder and a hollow cylinder, both of the same mass and same external diameter are released from the same height at the same time on an inclined plane. Both roll down without slipping. Which one will reach the bottom first? **(NEET 2010)**
B. A solid sphere, disc and solid cylinder all of the same mass and radius are allowed to roll down (from rest) on an inclined plane, then the order in which they reach the bottom is: (**NEET 1993**)

$$t = \sqrt{\frac{2h(1+X)}{g\sin^2\theta}}$$

**Part A**

Comparing $X$:

$$X_{Solid\ Cylinder} : X_{Hollow\ Cylinder} = \frac{1}{2} : 1 = 1:2 \Rightarrow Solid\ \ Cylinder\ first$$

**Part B**

$$X_{Solid\ Sphere} : X_{Disc} : X_{Solid\ Cylinder}$$

$$\frac{2}{5} : \frac{1}{2} : \frac{1}{2}$$

$$4:5:5$$

$$Solid\ sphere\ first$$

$$Disc\ and\ Cylinder\ together\ (second)$$

### D. Radius of Gyration

**5.35: Acceleration**

$$a = \frac{g \sin\theta}{1 + \frac{k^2}{R^2}}$$

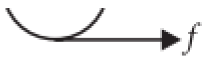
**Example 5.36**

25

A uniform solid sphere of radius $R$ and radius of gyration $k$ about an axis passing through the centre of mass is rolling without slipping. Then, the fraction of total energy associated with its rotation will be **[23 Sep. 2020, Shift-I]**

**Example 5.37**

90:



**90. (c) :** Total energy

$$= \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2 = \frac{1}{2} mv^2 (1 + K^2/R^2)$$

$$\text{Required fraction} = \frac{K^2/R^2}{1 + K^2/R^2} = \frac{K^2}{R^2 + K^2}$$

## 5.4 Further Topics

**Example 5.38**

26: A fly wheel is accelerated uniformly from rest and rotates through 5 rad in the first second. The angle rotated by the fly wheel in the next second will be: (JEE-M 2022)

**Example 5.39**

27: The angular speed of truck wheel is increased from 900 rpm to 2460 rpm in 26 seconds. The number of revolutions by the truck engine during this time is: (JEE-M 2021)

**40 Examples**