Differentiation

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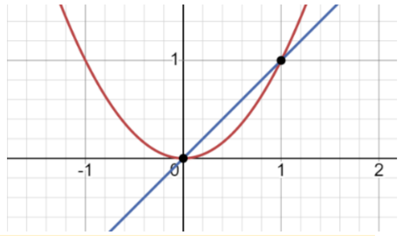
1. Finding Derivatives

1.1 Derivative Definition

A. Average Rate of Change

1.1: Secant
The secant is a line that connects two points that lie on $\boldsymbol{f(x)}$.

Example 1.2
The red graph alongside is the graph of
$$y = x^2$$
The secant line is the blue line connecting the two points on the graph.



A. Identify the points where the line intersects the parabola.
B. Identify the equation of the line.
C. What is the slope of the line?

Part A
$$(0,0)\ \&\ (1,1)$$

Part B
Substitute $Slope = m = \frac{Rise}{Run} = 1$ and $y - intercept = c = 0$ in $y = mx + c$:
$$y = x + 0 \Rightarrow y = x$$

Part C
$$Slope = m = 1$$

1.3: Average Rate of Change
The average rate of change of a function between two points $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ is given by the slope of the secant line that connects the points $(\boldsymbol{x_1}, \boldsymbol{f(x_1)})$ and $(\boldsymbol{x_2}, \boldsymbol{f(x_2)})$
$$\boldsymbol{Slope = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$

We learnt the concept of slope in coordinate geometry, at which point of time, you may or may not have seen functions.
However, the average rate of change can be defined in terms of functions.

Note that for a point (x, y) that lies on the graph of the function $y = f(x)$:
$$y = f(x)$$
Hence,
$$y_1 = f(x_1), \qquad y_2 = f(x_2)$$
Substitute the above in the definition of slope to get what we want to show:
$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example 1.4
Find the average rate of change of $y = x^2$ over the interval

A. (1,2). Use the formula $Slope = \frac{y_2 - y_1}{x_2 - x_1}$
B. (2,3). Use the formula $Slope = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

Part A

Substitute $(x_1, x_2) = (1,2) \Rightarrow (y_1, y_2) = (1,4)$ in $\frac{y_2 - y_1}{x_2 - x_1}$:

Part B

Substitute $(x_1, x_2) = (2,3) \Rightarrow f(x_1) = 4,\ f(x_2) = 9$ in $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$:

1.5: Average Rate of Change: Alternate Definition

$$\boldsymbol{Avg. Rate\ of\ Change} = \frac{\boldsymbol{f(x+h) - f(x)}}{\boldsymbol{h}}$$

where $\boldsymbol{h}$ represents the difference between $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$.

The average rate of change has an alternate definition, which is very useful later on, since we will define the derivative in terms of the alternate definition.

To get the alternate definition:
Substitute $x_2 = x_1 + h$ in:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{(x_1 + h) - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}$$

Since there is only one x variable, $x_1 = x$:

$$Avg. Rate\ of\ Change = \frac{f(x+h) - f(x)}{h}$$

Example 1.6

Find the average rate of change of $f(x) = x^3$ over the interval $(1,2)$ using the alternate definition above.

We have been given

$$(x, x+h) = (1,2) \Rightarrow h = (x+h) - x = 2 - 1 = 1$$

Also

$$f(x) = f(1) = 1$$
$$f(x+h) = f(2) = 8$$

Example 1.7

$$f(x) = x^2$$

A. Find the average rate of change of $f(x)$ over the interval $[2, 2+h]$.
B. Find the slope of the tangent to $f(x)$ at the point $(2,4)$.

Average Rate of change

$$= \frac{f(x+h) - f(x)}{h} = \frac{f(2+h) - f(2)}{h} = \frac{(2+h)^2 - 2^2}{h} = \frac{4 + 4h + h^2 - 4}{h} = \frac{4h + h^2}{h} = 4 + h$$

As the value of h reduces, $4 + h$ becomes closer and closer to zero.
Hence, the slope of the parabola at $x = 2$ is

$$4.$$

Example 1.8

$$f(x) = x^2 - 2x - 3$$

A. Find the slope of the tangent to $f(x)$ at the point $(2, -3)$.

Example 1.9

$$f(x) = \frac{1}{x}$$

A. Find the slope of the tangent to $f(x)$ at the point $(2, -3)$.

Slope

$$= \frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{-2+h} + \frac{1}{2}}{h} = \frac{\frac{2}{(-2+h)(2)} + \frac{-2+h}{(-2+h)(2)}}{h} = \frac{h}{(-2+h)(2)} \times \frac{1}{h} = \frac{1}{-4+2h}$$

As h goes to zero, the slope becomes

$$\frac{1}{-4}$$

Example 1.10

$$y = \frac{x}{2-x}, P(4, -2)$$

$$Secant\ line\ slope = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{\left(\frac{4+h}{2-(4+h)}\right) - \left(\frac{4}{2-4}\right)}{4+h-4} = \frac{\left(\frac{4+h}{-2-h}\right) - (-2)}{h} = \frac{\left(\frac{4+h}{-2-h}\right) + 2}{h} = \frac{\frac{4+h}{-2-h} + \frac{-4-2h}{-2-h}}{h} = \frac{-h}{-2-h} \times \frac{1}{h}$$

$$= \frac{-1}{-2-h} = \frac{1}{2+h}$$

$$\frac{1}{2}$$

B. Tangent

1.11: Tangent

In geometry, the tangent to a circle is the line that touches a circle in exactly one point.
A tangent to the graph of a function is the line that gives the slope of the function at that point.

The secant is the slope of two points that lie on a function.
The tangent is the slope of the function at a point.

In the formula for average rate of change which is the slope of a secant:

$$\frac{f(x+h) - f(x)}{h}$$

as h approaches zero, the distance between the two points approaches zero, and the secant approaches the tangent.

Example 1.12

Consider the function $f(x) = x^2$.

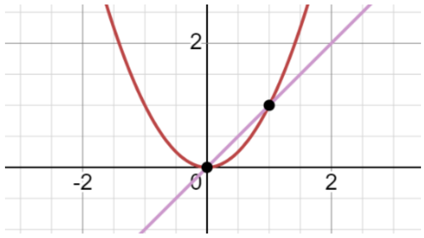
A. Find the average rate of change over the interval $(0,1)$
B. Find the average rate of change over the interval $(0,0.5)$

C. Find the average rate of change over the interval (0,0.1)
D. Hence, guess what the tangent at the point (0,0) will be.

Part A

Secant Line connecting (0,0) and (1,1):

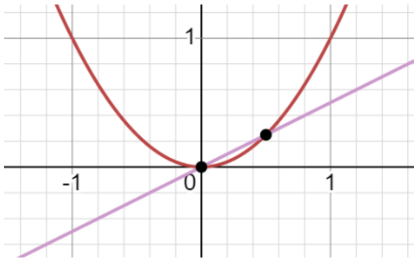
$$Slope = 1$$



Part B

Secant Line connecting (0,0) and (0.5,0.25):

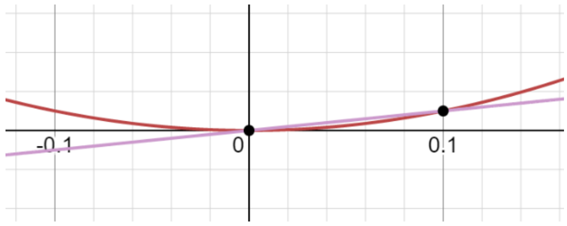
$$Slope = 0.5$$



Part C

Secant Line connecting (0,0) and (0.1,0.01):

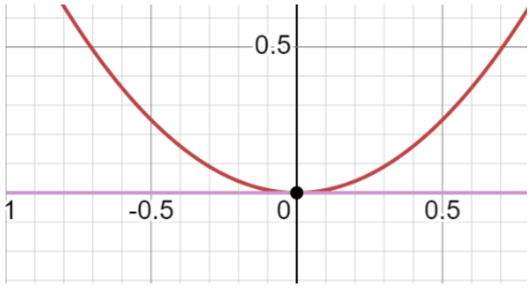
$$Slope = 0.1$$



Part D

Visually, the slope is getting closer and closer to zero.
Hence, we can take a guess, and suppose that at

$$x = 0 \Rightarrow Slope = 0$$



1.13: Tangent

In the formula for average rate of change which is the slope of a secant:

$$\frac{f(x+h) - f(x)}{h}$$

as h approaches zero, the distance between the two points approaches zero, and the secant approaches the tangent.

$$Slope\ of\ the\ tangent = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

1.14: Interpretations of the Derivative

The derivative can be interpreted:

- Graphical: Slope of the tangent line at the point of evaluation
- Algebraic: Rate of change of the function at a point

C. Derivatives from first principles

1.15: Derivative of a Function

The derivative of the function $f(x)$ is given by the limit:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

If it exists.

1.16: Tangents and Normals

- The slope of the tangent line to $f(x)$ is given by $f'(x)$.
- To find the equation of the tangent, we need one more point, given by $\big(x, f(x)\big)$
- The normal to a function is the line which is perpendicular to the tangent line.

The slope of the tangent line to a function is given by evaluating the derivative of the function at that point.

Example 1.17

Find the tangent and the normal to the function $f(x) = x^2 + x$ at $x = 4$

Find the derivative

The definition of the derivative is:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{(x+h)^2+(x+h)-(x^2+x)}{h} \\ &= \lim_{h\to 0}\frac{x^2+2xh+h^2+x+h-x^2-x}{h} \\ &= \lim_{h\to 0}\frac{2xh+h^2+h}{h} \\ &= \lim_{h\to 0}\frac{h(2x+h+1)}{h} \\ &= \lim_{h\to 0} 2x+h+1 \\ &= 2x+1 \end{aligned}$$

Evaluate the derivative

$$f'(x) = 2x+1 \Rightarrow f'(4) = 2(4)+1 = 9$$

The slope of the tangent line to $f(x)$ at 4 is 9.

Find the y-intercept

Substitute $x = 4, y = 20$:

$$y = 9x + c \Rightarrow c = -16$$
$$m = 9, c = -16 \Rightarrow y = 9x - 16$$

Normal

$$Slope\ of\ Normal: m_n \times 9 = -1 \Rightarrow m_n = -\frac{1}{9}$$

Substitute $x = 4, y = 20$ in

$$y = -\frac{1}{9}x + c \Rightarrow 20 = -\frac{1}{9}(4) + c \Rightarrow c = \frac{184}{9}$$
$$y = -\frac{1}{9}x + \frac{184}{9}$$

1.2 Differentiation

A. Basic Rules

1.18: Notation for Derivative

$$y = f(x)$$

The derivative of the y with respect to x is written:

$$Derivative = \underbrace{y'}_{\substack{Prime\\Notation}} = \underbrace{f'(x)}_{\substack{Prime\\Notation}} = \underbrace{\frac{dy}{dx}}_{\substack{Leibniz\\Notation}}$$

1.19: Tangents and Normals

- The slope of the tangent line to $f(x)$ is given by $f'(x)$.

1.20: Constant Function Rule

$$Constant\ Function\ Rule: f(x) = c \Rightarrow f' = 0$$

- A constant function represents a horizontal line.
- A horizontal line has slope zero (from coordinate geometry).

Example 1.21

A. $y = 4$
B. $y = \pi$

$$\frac{dy}{dx} = \frac{d}{dx}(4) = 0$$
$$\frac{dy}{dx} = \frac{d}{dx}(4) = 0$$

1.22: Power Rule

$$\frac{d}{dx}x^n = nx^{n-1}, \qquad n \neq 0$$

- This is very important for calculating the derivatives of polynomial functions.
- x is a variable, and n is a constant.

Example 1.23

Find $\frac{dy}{dx}$:

A. $y = x^5$
B. $y = \sqrt{x}$
C. $y = \frac{1}{x^2}$
D. $y = \frac{1}{\sqrt{x}}$
E. $y = \sqrt[3]{x}$

Part A

$$\frac{dy}{dx} = \frac{d}{dx}x^5 = 5x^4$$

We can also do in this in function notation:

$$f(x) = x^5 \Rightarrow f'(x) = 5x^4$$

We can also write this in y' notation:

$$y = x^5 \Rightarrow y' = 5x^4$$

Part B

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Part C

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}x^{-2} = (-2)x^{-3} = -\frac{2}{x^3}$$

Part D

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}x^{-\frac{1}{2}} = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x^{\frac{3}{2}}}$$

Part E

$$\frac{d}{dx}(\sqrt[3]{x}) = \frac{d}{dx}x^{\frac{1}{3}} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$$

1.24:Constant Multiple Rules

If c is a constant and $f(x)$ is a differentiable function, then

$$[cf(x)]' = c \times f'(x)$$

We can justify the constant multiple rule using transformations.

- $cf(x)$ is a vertical scaling of $f(x)$.
- Hence, its slope (and derivative) is c times the original function $f(x)$.

Example 1.25

Differentiate:

A. $3x^4$
B. $\frac{1}{2}x^5$
C. $\frac{\sqrt{x}}{3}$

$$(3x^4)' = 3(x^4)' = 3(4x^3) = 12x^3$$
$$\left(\frac{1}{2}x^5\right) = \frac{1}{2}(x^5)' = \frac{1}{2}(5x^4) = \frac{5}{2}x^4$$
$$\left(\frac{\sqrt{x}}{3}\right)' = \frac{1}{3}(\sqrt{x})' = \frac{1}{3}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{6\sqrt{x}}$$

1.26: Derivative of $\sqrt{x}$

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

Example 1.27

A. $y = \frac{\sqrt{x}}{3}$

$$y' = \frac{dy}{dx} = \frac{1}{6\sqrt{x}}$$

1.28: Second and Further Derivatives

If you take the derivative of $f'(x)$, that is written:

$$f''(x)$$

Example 1.29

Find all derivatives of
A. $f(x) = 2x^4$
B. $f(x) = x^5$

Part A

$$f'(x) = (2x^4)' = 2(x^4)' = 2(4x^3) = 8x^3$$
$$f''(x) = 24x^2$$
$$f'''(x) = 48x$$
$$f^{(4)}(x) = 48$$
$$f^{(n)} = 0, \qquad n \geq 5, \qquad n \in \mathbb{N}$$

Part B

$$\frac{dy}{dx} = 5x^4$$
$$\frac{d^2y}{dx^2} = 20x^3$$
$$\frac{d^3y}{dx^3} = 60x^2$$
$$\frac{d^4y}{dx^4} = 120x$$
$$\frac{d^5y}{dx^5} = 120$$
$$\frac{d^ny}{dx^n} = 0, n \geq 6$$

B. Sum, Difference, Product and Quotient Rules

Finding the derivative of a sum or a difference is as simple as finding the derivative of the individual terms. In other words, we can distribute the operation of taking a derivative.

1.30: Sum and Difference Rule

$$\frac{d}{dx}(a \pm b) = \frac{d}{dx}a \pm \frac{d}{dx}b$$

- The sum and difference is very useful when differentiating a polynomial, since it lets us differentiate "term by term".
- In other words, you can distribute the *derivative operator* over the brackets.

Just as:

$$y(a+b) = ya + yb$$

Similarly:

$$\frac{d}{dx}(a \pm b) = \frac{d}{dx}a \pm \frac{d}{dx}b$$

Example 1.31

Find the derivative of:

Polynomials

A. $x^2 + x$
B. $t^2 + \pi t + e$
C. $x^5 + \frac{x^4}{2} - \frac{x^3}{3} + \frac{x^2}{4} + \frac{x}{5} + 6$
D. $5\theta^{2022} - 3\theta^{1011} + 2022$

Negative Exponents

E. $x^3 + \frac{1}{x^2} - 5x + 2$
F. $\frac{1}{x} - \frac{2}{5x^{\frac{2}{3}}} + \frac{\sqrt{3}}{5x^{\frac{5}{3}}} + \pi^e x - e^\pi$

Radicals

Polynomials

Part A

$$\frac{d}{dx}(x^2+x)=\frac{d}{dx}x^2+\frac{d}{dx}x=2x+1$$

Part B

Negative Exponents

Part F

Move the variable x to the numerator in all terms:

$$x^{-1}-\frac{2x^{-\frac{2}{3}}}{5}+\frac{\sqrt{3}x^{-\frac{5}{3}}}{5}+\pi^e x-e^\pi$$

Differentiate term by term:

$$(-1)x^{-2}-\left(\frac{2}{5}\right)\left(-\frac{2}{3}\right)x^{-\frac{5}{3}}+\left(\frac{\sqrt{3}}{5}\right)\left(-\frac{5}{3}\right)x^{-\frac{8}{3}}+\pi^e$$

Simplify:

$$-\frac{1}{x^2}+\left(\frac{4}{15x^{\frac{5}{3}}}\right)-\left(\frac{\sqrt{3}}{3x^{\frac{8}{3}}}\right)+\pi^e$$

Example 1.32: Radicals

Find the derivative of:

A. $\frac{2}{3}\sqrt{\theta}+\frac{3}{5}\sqrt[3]{\theta^2}+\frac{5}{9}\sqrt[6]{\theta^7}$

B. $2\left(\frac{1}{\sqrt{\beta}}+\sqrt{\beta}\right)$

Part G

Rewrite using fractional exponents:

$$y=\frac{2}{3}\theta^{\frac{1}{2}}+\frac{3}{5}\theta^{\frac{2}{3}}+\frac{5}{9}\theta^{\frac{7}{6}}$$

Differentiate term by term:

$$\frac{dy}{dx}=\frac{2}{3}\left(\frac{1}{2}\right)\theta^{-\frac{1}{2}}+\frac{3}{5}\left(\frac{2}{3}\right)\theta^{-\frac{1}{3}}+\frac{5}{9}\left(\frac{7}{6}\right)\theta^{\frac{1}{6}}$$

Rewrite:

$$=\left(\frac{1}{3}\right)\theta^{-\frac{1}{2}}+\left(\frac{2}{5\theta^{\frac{1}{3}}}\right)+\left(\frac{35}{54}\right)\theta^{\frac{1}{6}}$$

Part H

$$\frac{d}{dx}\left[2\left(\frac{1}{\sqrt{\beta}}+\sqrt{\beta}\right)\right]=2\left[\frac{d}{dx}\left(\beta^{-\frac{1}{2}}+\beta^{\frac{1}{2}}\right)\right]=2\left[-\frac{1}{2}\beta^{-\frac{3}{2}}+\frac{1}{2}\beta^{-\frac{1}{2}}\right]=2\left[-\frac{1}{2\beta^{\frac{3}{2}}}+\frac{1}{2\sqrt{\beta}}\right]$$

1.33: Exponential and Logarithmic Derivatives

$$\frac{d}{dx}e^x=e^x$$

$$\frac{d}{dx}e^{-x}=-e^x$$

$$\frac{d}{dx}\ln x=\frac{1}{x}$$

- e^x is the *only* function that is its own derivative.
- e is a constant and x is a variable

Example 1.34: Exponential Derivatives

A. $2e^\theta$
B. $e \cdot e^z$
C. e^{z+2}
D. e^π
E. $4 \cdot e^y$
F. e^{z-1}

$$\left(2e^\theta\right)' = 2 \times \left(e^\theta\right)' = 2 \times e^\theta = 2e^\theta$$
$$(e \cdot e^z)' = e(e^z)' = e \cdot e^z = e^{z+1}$$
$$(e^2 \times e^z)' = e^2 \times (e^z)' = e^2 \times e^z = e^{z+2}$$
$$4 \cdot e^y$$
$$e \cdot e^z$$
$$e^{z+2}$$
$$e^{z-1}$$

Example 1.35

A. $\ln(x^4) + x^2 + 2$
B. $e^x + \pi + e$
C. $e^z + z^e$
D. $e^{r+2} - r^{e+2}$

$$[4 \ln x]' + (2x)' + (2)' = \frac{4}{x} + 2x$$
$$e^x$$
$$e^z + ez^{e-1}$$
$$e^{r+2} - (e+2)r^{e+1}$$

1.36: Product Rule

$$(fg)' = f'g + fg' = \underbrace{f'}_{\substack{Derivative\\ of\ f}} \underbrace{g}_{\substack{No\ change\\ to\ g}} + \underbrace{f}_{\substack{No\ change\\ to\ f}} \underbrace{g'}_{\substack{Derivative\\ of\ g}}$$

To find the derivative of the product of two functions find the sum of:
- The product of the derivative of the first function and the second function.
- The product of the derivative of the second function by the first function.

Example 1.37

$$y = (3x+1)(x^2-1)$$

$$y' = (3)(x^2-1) + (3x+1)(2x) = 3x^2 - 3 + 6x^2 + 2x = 9x^2 + 2x - 3$$

Example 1.38

Find the derivative of:
A. $x^3(x+1)$
B. xe^x
C. $e^x \cdot \ln x$
D. $x^2 e^x$
E. $\sqrt{x}e^x$

$$y = x^3(x+1) = x^4 + x^3 \Rightarrow y' = 4x^3 + 3x^2$$

$$y = \underbrace{x^3}_{f}\underbrace{(x+1)}_{g} \Rightarrow y' = f'g + g'f = (3x^2)(x+1) + (x^3)(1) = 3x^3 + 3x^2 + x^3 = 4x^3 + 3x^2$$

Product Rule

$$x \times (e^x)' + (x)' \times e^x = xe^x + e^x = e^x(1+x)$$

$$e^x \times (\ln x)' + (e^x)' \times \ln x = \frac{e^x}{x} + e^x \ln x = e^x\left(\frac{1}{x} + \ln x\right)$$

$$x^2 \times (e^x)' + (x^2)' \times e^x = x^2 e^x + 2xe^x = e^x(x^2 + 2x)$$

$$\sqrt{x} \times (e^x)' + \left(\sqrt{x}\right)' \times e^x = \sqrt{x}e^x + \frac{e^x}{2\sqrt{x}} = e^x\left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right)$$

1.39: Quotient Rule

For $g \neq 0$:

$$\left(\frac{f}{g}\right)' = \frac{\boldsymbol{g}f' - g'f}{\boldsymbol{g^2}}$$

Example 1.40

Find the derivative of:
A. $\frac{x}{2x+1}$
B. $\frac{3x^3}{x^2+x}$
C. $\frac{\sqrt{x}-1}{\sqrt{x}+1}$
D. $\frac{5x+1}{2\sqrt{x}}$
E. $\frac{2t}{2+\sqrt{t}}$

Part A

$$f = x \Rightarrow f' = 1$$
$$g = 2x+1 \Rightarrow g' = 2$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - g'f}{g^2} = \frac{(2x+1)\cdot 1 - 2\cdot x}{(2x+1)^2} = \frac{1}{(2x+1)^2}$$

Part B

$$f = 3x^3 \Rightarrow f' = 9x^2$$
$$g = x^2 + x \Rightarrow g' = 2x+1$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - g'f}{g^2} = \frac{(x^2+x)\cdot 9x^2 - (2x+1)\cdot(3x^3)}{(x^2+x)^2} = \frac{9x^4 + 9x^3 - 6x^4 - 3x^3}{x^4 + 2x^3 + x^2} = \frac{3x^4 + 6x^3}{x^4 + 2x^3 + x^2}$$

Divide the numerator and denominator by x^2:

$$\frac{3x^2+6x}{x^2+2x+1}$$

Shortcut:

$$\frac{3x^3}{x^2+x}=\frac{3x^2}{x+1}$$

Part C

$$\frac{dy}{dx}=\frac{(\sqrt{x}+1)\left(\frac{1}{2\sqrt{x}}\right)-\left(\frac{1}{2\sqrt{x}}\right)(\sqrt{x}-1)}{(\sqrt{x}+1)^2}=\frac{\frac{1}{2}+\frac{1}{2\sqrt{x}}-\frac{1}{2}+\frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2}=\frac{\frac{1}{\sqrt{x}}}{(\sqrt{x}+1)^2}=\frac{1}{\sqrt{x}(\sqrt{x}+1)^2}$$

Part D

$$\frac{dy}{dx}=\frac{(2\sqrt{x})(5)-\left(\frac{1}{\sqrt{x}}\right)(5x+1)}{4x}=\frac{10\sqrt{x}-5\sqrt{x}-\frac{1}{\sqrt{x}}}{4x}=\frac{5\sqrt{x}-\frac{1}{\sqrt{x}}}{4x}=\frac{\frac{5x-1}{\sqrt{x}}}{4x}=\frac{5x-1}{4x^{\frac{3}{2}}}$$

Part E

$$\frac{dy}{dt}=\frac{(2+\sqrt{t})(2)-\left(\frac{1}{2\sqrt{t}}\right)(2t)}{(2+\sqrt{t})^2}=\frac{4+2\sqrt{t}-\sqrt{t}}{(2+\sqrt{t})^2}=\frac{4+\sqrt{t}}{(2+\sqrt{t})^2}$$

Example 1.41

Find the first four derivatives of $f(x)=\frac{2+x}{1-x}$

$$\left(\frac{f}{g}\right)'=\frac{\boldsymbol{g}f'-g'f}{\boldsymbol{g^2}}$$

$$f'(x)=\frac{(1-x)(1)-(-1)(2+x)}{(1-x)^2}=\frac{3}{(1-x)^2}$$

$$f''(x)=\frac{d}{dx}\left(\frac{3}{(1-x)^2}\right)=\frac{d}{dx}3(1-x)^{-2}=6(1-x)^{-3}$$

$$f'''(x)=\frac{d}{dx}6(1-x)^{-3}=18(1-x)^{-4}$$

Example 1.42

A. $\frac{e^x}{x}$

B.

C. $\frac{e^x}{\sqrt{x}}$

D. $\frac{A}{B+Ce^x}$

$$\frac{x\times(e^x)'-e^x(x)'}{x^2}=\frac{xe^x-e^x}{x^2}=\frac{e^x(x-1)}{x^2}$$

$$\left(\frac{e^x}{\sqrt{x}}\right)' = \frac{(\sqrt{x})(e^x) - (e^x)\left(\frac{1}{2\sqrt{x}}\right)}{x} = \frac{\frac{2xe^x - e^x}{2\sqrt{x}}}{x} = \frac{e^x(2x-1)}{2x^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{(B + Ce^x) \times 0 - A(Ce^x)}{(B + Ce^x)^2} = \frac{-A(Ce^x)}{(B + Ce^x)^2}$$

Example 1.43

$$y = \frac{1 - xe^x}{x + e^x}$$

$$(1 - xe^x)' = -(x \times e^x + 1 \times e^x) = -(xe^x + e^x)$$

$$\frac{dy}{dx} = \frac{-(x + e^x)(xe^x + e^x) - (1 - xe^x)(1 + e^x)}{(x + e^x)}$$

1.44: Identities

$$(a+b)(a-b) = a^2 - b^2$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Example 1.45

Find the derivatives of the following:

A. $y = (x-1)(x+1)$
B. $y = (x-1)(x+1)(x^2+1)$
C. $y = (x-1)(x+1)(x^2+1)(x^4+1)$
D. $y = \frac{(x-1)(x^2+x+1)}{x^3}$

Part A

$$\frac{d}{dx}(x+1)(x-1) = \frac{d}{dx}(x^2-1) = 2x$$

Part B

$$y = (x+1)(x-1)(x^2+1) = (x^2-1)(x^2+1) = x^4 - 1$$
$$\frac{dy}{dx} = 4x^3$$

Part C

$$y = x^8 - 1 \Rightarrow \frac{dy}{dx} = 8x^7$$

Part D

$$y = \frac{x^3 - 1}{x^3} = \frac{x^3}{x^3} - \frac{1}{x^3} = 1 - \frac{1}{x^3} = 1 - x^{-3}$$
$$\frac{dy}{dx} = \frac{3}{x^4}$$

C. Further Topics

Example 1.46
Let $P(x)$ be a polynomial of degree n.

A. Show that the first derivative of $P(x)$ is a polynomial of degree $n-1$, with $n-1$ terms
B. Hence, show that every successive derivative reduces the degree of $P(x)$ by 1.
C. Find the n^{th} derivative of $P(x)$.

Part A

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

If we use

$$P' to\ represent\ first\ derivative\ of\ P(x)$$
$$P'(x) = a_n(nx^{n-1}) + a_{n-1}\left((n-1)x^{n-2}\right) + \cdots + a_1 + 0$$

And the above has $n-1$ terms.

Part B

In Part A, we proved $\frac{d}{dx}P(x) = P'(x)\ has\ degree\ n-1$, which is true for all polynomials. Hence,

$$P''(x)\ has\ degree\ n-2$$
$$P'''(x)\ has\ degree\ n-3$$

Part C

Use $P^n(x)$ to represent the n^{th}derivative:

$$P^n(x)\ is\ a\ polynomial\ of\ degree\ n-n = degree\ 0$$

Which has value

1.47: Sum and Difference Rule: Function Variant

$$\boldsymbol{y = f(x) + h(x) \Rightarrow \frac{d}{dx}y = \frac{d}{dx}[f(x) + h(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}h(x)}$$

Example 1.48
$f(x)$ and $h(x)$ are functions differentiable on their domain:

$$f(2) = 3, h(2) = -1, f'(2) = 4, h'(2) = 1$$

A. Find $g'(2)$ if $g(x) = 2f(x) + 3h(x)$
B. Find $p'(2)$ if $p(x) = f(x)g(x)$

Part A

$$\frac{d}{dx}[2f(x) + 3h(x)] = 2\frac{d}{dx}f(x) + 3\frac{d}{dx}h(x) = 2f'(x) + 3h'(x) = 2(4) + 3(1) = 11$$

Part B

$$g(2) = 2f(2) + 3h(2) = (2)(3) + 3(-1) = 3$$
$$p'(x) = [f'(x)]g(x) + f(x)g'(x) = (4)(3) + (3)(11) = 12 + 33 = 45$$

Challenge 1.49
Consider the function

$$y = ax^n$$

A. Find the first, second and third derivatives.
B. See the pattern in part A, and use that to write a formula for the k^{th} derivative, $k \in \mathbb{N}$.
C. Find the n^{th} derivative

D. Find the $(n+1)^{st}$ derivative

Part A

$$\frac{dy}{dx} = a(nx^{n-1})$$
$$\frac{d^2y}{dx^2} = a[n(n-1)x^{n-2}]$$
$$\frac{d^3y}{dx^3} = a[n(n-1)(n-2)x^{n-3}]$$

Part B

$$\frac{d^ky}{dx^k} = a\big[(n)(n-1)(n-2)\ldots(n-k+1)\big(x^{n-k}\big)\big]$$

$$(n)(n-1)(n-2)\ldots(n-k+1)$$
$$= \frac{(n)(n-1)\ldots(n-k+1)\boldsymbol{(n-k)(n-k-1)(n-k-2)\ldots(2)(1)}}{\boldsymbol{(n-k)(n-k-1)(n-k-2)\ldots(2)(1)}}$$
$$= a\big[\big(x^{n-k}\big)\big]$$
$$a\left[\left(\frac{n!}{(n-k)!}\right)\big(x^{n-k}\big)\right]$$

$$5! = 1\times 2\times 3\times 4\times 5 = 120$$
$$5\times 4 = \frac{5\times 4\times 3\times 2\times 1}{3\times 2\times 1} = \frac{5!}{3!}$$
$$(1)(2)(3)\ldots(499)(500)$$

Part C

$$\frac{d^ny}{dx^n} = a[n(n-1)(n-2)\ldots(1)x^{n-n}]$$

$$\frac{d^ny}{dx^n} = a[n!] = a(n!)$$

Part C

$$\frac{d^{n+1}y}{dx^{n+1}} = \frac{d}{dx}a(n!) = 0$$

Challenge 1.50

Find $\frac{d}{dx}(y\cdot y_0\cdot y_1\cdot\ldots\cdot y_n)$ given that:

$$y = x-1, \qquad y_n = x^{(2^n)}+1$$

$$z = y\cdot y_0\cdot y_1\cdot\ldots\cdot y_n$$
$$z = (x-1)(x+1)(x^2+1)(x^4+1)(x^8+1)\ldots\big(x^{(2^n)}+1\big)$$
$$= (x^2-1)(x^2+1)(x^4+1)(x^8+1)\ldots\big(x^{(2^n)}+1\big)$$
$$= (x^4-1)(x^4+1)(x^8+1)\ldots\big(x^{(2^n)}+1\big)$$
$$= (x^8-1)(x^8+1)\ldots\big(x^{(2^n)}+1\big)$$
$$= (x^{16}-1)\ldots\big(x^{(2^n)}+1\big)$$
$$= \big(x^{(2^n)}-1\big)\big(x^{(2^n)}+1\big)$$
$$= \big(x^{(2^n+2^n)}-1\big)$$

$$= x^{\left(2^{n+1}\right)} - 1$$

$$\frac{dz}{dx} = 2^{n+1}\left(x^{\left(2^{n+1}\right)-1}\right)$$

D. Abstract Differentiation

Example 1.51

$$H(x) = 2f(x) + 3g(x) + 4h(x)$$
$$f'(3) = 7$$
$$g'(3) = 12$$
$$h'(3) = -14$$

Using the information given, evaluate:

A. $H'(3)$

$$H'(x) = [2f(x) + 3g(x) + 4h(x)]'$$

Apply the sum rule:

$$H'(x) = [2f(x)]' + [3g(x)]' + [4h(x)]'$$

Apply the constant multiple rule:

$$H'(x) = 2f'(x) + 3g'(x) + 4h'(x)$$
$$H'(3) = 2f'(3) + 3g'(3) + 4h'(3)$$
$$H'^{(3)} = 2(7) + 3(12) + 4(-14) = 14 + 36 - 56 = -6$$

1.3 Rates of Change

A. Rate of Change

Example 1.52

Find the rate of change in the:

A. area of a square with respect to a change in its side length when its side length is 3?
B. area of a circle with respect to a change in its radius when its radius is 4?
C. volume of a sphere with respect to a change in its radius when its radius is 3?

Part B

$$A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r \Rightarrow \left.\frac{dA}{dr}\right|_{r=4} = 2\pi r|_{r=4} = 2\pi(4) = 8\pi\frac{m^2}{m}$$

Part C

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 2\pi r \Rightarrow \left.\frac{dA}{dr}\right|_{r=4} = 2\pi r|_{r=4} = 2\pi(4) = 8\pi\frac{m^2}{m}$$

Example 1.53

Economics
Biology
Other

B. Kinematics

Normally, we consider the coordinate plane with variables:

$$x \text{ on the horizontal axis}$$
$$y \text{ on the vertical axis}$$

We can also consider other variables. For example, we could plot:

$$Time \text{ on the Horizontal Axis}$$
$$Position \text{ of the object on the vertical axis}$$

1.54: Position at time *t*

Consider an object moving along a coordinate line.
The position of a body at time t is given by s.

$$s = f(t)$$

Example 1.55

A body moves along a coordinate line, with its position given by $s = t^3 - 4$. Find its position when

A. $t = 0$
B. $t = 2$

1.56: Distance Travelled

Total length of the path taken by an object from the start time to the end time is the distance.

1.57: Speed

$$Average\ Speed = \frac{Total\ Distance}{Total\ Time}$$

1.58: Distance and Speed are Scalars

Distance and Speed are scalar quantities. They do not have a direction associated with them.

Example 1.59

You start your morning walk at $t_1 = 6{:}05\ am$. You complete it at $t_2 = 6{:}55\ am$. What is Δt. (Δ) is used to represent a change in a quantity.

$$\Delta t = t_2 - t_1 = 6{:}55 - 6{:}05 = 50\ minutes$$

1.60: Displacement

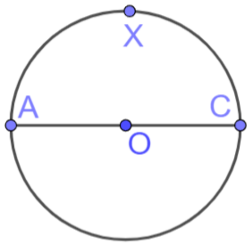
Displacement is the straight-line distance (shortest path) from initial to final position

$$Change\ in\ position = \Delta s = \underbrace{s_2}_{\substack{New\\Position}} - \underbrace{s_1}_{\substack{Old\\Position}} = \underbrace{f(t_2)}_{\substack{New\\Position}} - \underbrace{f(t_1)}_{\substack{Old\\Position}}$$

Example 1.61

In the circle drawn alongside, an object at point A moves to C via the path at X. $AC = O = 5\ meter$. Determine the:

A. Distance Travelled
B. Displacement



Example 1.62

An object 30 meters from the origin moves 20 meter to the left and then 10 m to the right.

Determine:

A. Distance Travelled
B. Displacement

Example 1.63

The position of a object along a coordinate line is given by $s = 2t + 5$.

A. Determine the displacement between $t = 3$ and $t = 5$
B. Let $f(t) = \Delta s$. Find a closed form solution for $f(t)$ in terms of $t_1 = Start\ Time$ and $t_2 = End\ Time$

$$\Delta s = s(t_2) - s(t_1) = s(5) - s(3) = 2(5) + 5 - [2(3) + 5] = 15 - 11 = 4$$
$$\Delta s = s(t_2) - s(t_1) = 2t_2 + 5 - (2t_1 + 5) = 2(t_2 - t_1)$$

C. Velocity

1.64: Velocity

Velocity is the rate of change of position.

1.65: Average Velocity

$$Average\ Velocity = v_{avg} = \frac{s(t_1 + \Delta t) - s(t_1)}{\Delta t}$$

Where

$$s(t_1) = Displacement\ at\ time\ t_1$$
$$s(t_1 + \Delta t)$$

$$Average\ Velocity = \frac{Displacement}{Time\ Taken} = \frac{Final\ Position - Initial\ Position}{Time\ Taken}$$

Substitute $s(t_2) = Final\ Position$ and $s(t_1) = Initial\ Position$:

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{s(t_1 + \Delta t) - s(t_1)}{t_1 + \Delta t - t_1} = \frac{s(t_1 + \Delta t) - s(t_1)}{\Delta t}$$

Example 1.66

The position of an object along a coordinate line is given by $s = 2t + 5$. Determine the average velocity between $t = 3$ and $t = 5$.

The average velocity is:

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{s(5) - s(3)}{5 - 3} = \frac{2(5) + 5 - [2(3) + 5]}{2} = \frac{15 - 11}{2} = \frac{4}{2} = 2$$

1.67: Average Velocity: Alternate Definition

$$Average\ Velocity = \frac{Displacement}{Time\ Taken} = \frac{Change\ in\ Position}{Change\ in\ Time} = \frac{\Delta s}{\Delta t}$$

Example 1.68

1.69: Instantaneous Velocity as a Derivative
Instantaneous velocity is the derivative of position with respect to time.
$$Instantaneous\ \ Velocity = v(t) = \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t} = \frac{ds}{dt}$$

We can find the average velocity for any time period, ranging from a millisecond to a year. However, if we find the average velocity for smaller and smaller time periods, then we get closer and closer to finding the velocity at a particular point in time.
This is made precise in the definition of instantaneous velocity.
Note that instantaneous velocity is written as a function, since if we find the limit, that lets us find the velocity as a function of time.

Example 1.70
The position of an object along a coordinate line is given by $s = 2t + 5$.

A. Determine the velocity.
B. Show that the velocity is a constant.

$$Velocity = v(t) = \frac{ds}{dt} = \frac{d}{dt}(2t+5) = 2$$

The velocity function is:
$$v(t) = 2$$
Which is a constant function.
Hence, the velocity is constant.

D. Acceleration

1.71: Acceleration
Acceleration is the rate of change of velocity.

1.72: Average Acceleration
$$Average\ Acceleration = \frac{v(t_1 + \Delta t) - v(t_1)}{\Delta t}$$

The average acceleration is given by:
$$\frac{Final\ Velocity - Initial\ Velocity}{Time\ Taken} = \frac{v(t_2) - v(t_1)}{t_2 - t_1} = \frac{v(t_1 + \Delta t) - v(t_1)}{t_1 + \Delta t - t_1} = \frac{v(t_1 + \Delta t) - v(t_1)}{\Delta t}$$

Example 1.73
The position of an object along a coordinate line is given by $s = 2t + 5$. Determine the average acceleration between $t = 3$ and $t = 5$.

The average acceleration is:

$$v(t) = \frac{ds}{dt} = \frac{d}{dt}(2t+5) = 2$$
$$\frac{v(t_2) - v(t_1)}{t_2 - t_1} =$$

1.74: Instantaneous Acceleration as a Derivative

Instantaneous acceleration is the derivative of velocity with respect to time.

$$Instantaneous\ Acceleration = \frac{dv}{dt} = \frac{d\left(\frac{ds}{dt}\right)}{dt} == \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t} = \frac{d^2 s}{dt^2}$$

We can find the average velocity for any time period, ranging from a millisecond to a year. However, if we find the average velocity for smaller and smaller time periods, then we get closer and closer to finding the velocity at a particular point in time.
This is made precise in the definition of instantaneous velocity.

The derivative of displacement with respect to time is velocity.
The derivative of velocity with respect to time is acceleration.
The derivative of acceleration with respect to time is jerk.

The second derivative of displacement with respect to time is acceleration.

1.4 Trigonometric Derivatives

A. Basics

In Calculus, we state the trigonometric functions in terms of radians, unless explicitly stated otherwise.

1.75: Derivative of $\sin x$ and $\cos x$

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$

- The derivative of a trigonometric function will always be a trigonometric function.

Example 1.76

A. Find the first four derivatives of $y = \sin x$.
B. Find a pattern for the derivatives, and write the pattern.
C. Hence, find $y^{(2021)}$, where $y^{(n)}$ represents the n^{th} derivative of y

Part A

$$y' = \cos x$$
$$y'' = (y')' = (\cos x)' = -\sin x$$
$$y^{(3)} = (y'')' = (-\sin x)' = -(\sin x)' = -\cos x$$
$$y^{(4)} = (y^3)' = (-\cos x)' = -(\cos x)' = -(-\sin x) = \sin x$$

Part B

$n \in \mathbb{W}$	
$y^{(4n+1)}$	$\cos x$
$y^{(4n+2)}$	$-\sin x$

$y^{(4n+3)}$	$-\cos x$
$y^{(4n)}$	$\sin x$

Part C

$$y^{(2021)} = y^{(400n+1)} = \cos x$$

Example 1.77: Sum/Difference and Constant Multiple Rule

$$\textit{Sum and Difference Rule}: (f \pm g)' = f' \pm g'$$
$$\textit{Constant Multiple Rule}: (cf)' = cf', c = constant$$

A. $\sin x + \cos x$
B. $4 \sin x - 2 \cos x$

Sum and Difference Rule

$$\cos x - \sin x$$

Constant Multiple Rule

$$(4 \sin x - 2 \cos x)'$$

Using the sum and difference rule:

$$= (4 \sin x)' - (2 \cos x)'$$

Use the constant multiple rule

$$= 4(\sin x)' - 2(\cos x)'$$

Take the derivatives now:

$$= 4 \cos x - 2(-\sin x)$$
$$= 4 \cos x + 2 \sin x$$

Example 1.78: Product Rule

$$\textit{Product Rule}: (fg)' = f'g + fg'$$

A. $x^2 \sin x$
B. $(\ln x)(\sin x)$

$$\frac{d}{dx} \underbrace{x^2}_{f} \underbrace{\sin x}_{g} = \underbrace{2x}_{f'} \underbrace{\sin x}_{g} + \underbrace{x^2}_{f} \underbrace{\cos x}_{g'} = x(2 \sin x + x \cos x)$$

$$\frac{d}{dx}[(\ln x)(\sin x)] = (\ln x)'(\sin x) + (\ln x)(\sin x)' = \frac{1}{x}(\sin x) + (\ln x) \cos x$$

Example 1.79

Show that $\frac{d}{dx} \sin x \cos x = \cos 2\theta$

$$\frac{d}{dx} \underbrace{\sin x}_{f} \underbrace{\cos x}_{g} = \cos x \cos x + \sin x\,(-\sin x) = \cos^2 x - \sin^2 x = \underbrace{\cos 2\theta}_{\substack{\textit{Using Double} \\ \textit{Angle Identity}}}$$

Example 1.80

$$\textit{Quotient Rule}: \left(\frac{f}{g}\right) = \frac{\boldsymbol{g}f' - g'f}{\boldsymbol{g^2}}$$

A. $y = \frac{\sin x + \ln x}{\cos x + e^x}$
B. $y = \frac{x \sin x}{\cos x + x}$
C. $y = \frac{\sin x + \cos x}{\sin x - \cos x}$

Part A

$$\frac{dy}{dx}=\frac{(\cos x+e^x)\left(\cos x+\frac{1}{x}\right)-(\sin x+\ln x)(-\sin x+e^x)}{(\cos x+e^x)^2}$$
$$=\frac{(\cos x+e^x)\left(\cos x+\frac{1}{x}\right)+(\sin x+\ln x)(\sin x-e^x)}{(\cos x+e^x)^2}$$

Part B

$$\frac{dy}{dx}=\frac{(\cos x+x)(\sin x+x\cos x)-(x\sin x)(-\sin x+1)}{(\cos x+x)^2}$$
$$=\frac{(\cos x+x)(\sin x+x\cos x)+(x\sin x)(\sin x-1)}{(\cos x+x)^2}$$

Part C

$$\frac{dy}{dx}=\frac{\underbrace{(\sin x-\cos x)}_{g}\underbrace{(\cos x-\sin x)}_{f\prime}-\underbrace{(\cos x+\sin x)}_{g\prime}\underbrace{(\sin x+\cos x)}_{f}}{\underbrace{(\sin x-\cos x)^2}_{g^2}}$$

Factor -1 out of the first term, and note that the second term is a square:

$$=\frac{-(\sin x-\cos x)^2-(\sin x+\cos x)^2}{(\sin x-\cos x)^2}$$

Use a change of variable. If we let $a=\sin x\,, b=\cos x$, then the numerator:

$$=-(a-b)^2-(a+b)^2$$
$$=-(a^2-2ab+b^2)-(a^2+2ab+b^2)$$
$$=-a^2+2ab-b^2-a^2-2ab-b^2$$
$$=-2a^2-2b^2$$
$$=(-2)(a^2+b^2)$$

Change back to the original variables:

$$=(-2)(\sin^2 x+\cos^2 x)$$

By the Pythagorean Identity, the second term $=1$:

$$=-2$$

Hence, the final answer is:

$$=\frac{-2}{(\sin x-\cos x)^2}$$

Example 1.81

Find the derivative

A. $\tan x$
B. $\csc x$
C. $\sec x$
D. $\cot x$

$$(\tan x)'=\left(\frac{\sin x}{\cos x}\right)'=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x$$
$$(\cot x)'=\left(\frac{\cos x}{\sin x}\right)'=\frac{\sin x\,(-\sin x)-\cos x\,(\cos x)}{\sin^2 x}=\frac{-(\sin^2 x+\cos^2 x)}{\sin^2 x}=-\frac{1}{\sin^2 x}=-\csc^2 x$$

$$(\sec x)'=\left(\frac{1}{\cos x}\right)'=\frac{\cos x\times 0-1\times(-\sin x)}{\cos^2 x}=\frac{\sin x}{\cos x}\times\frac{1}{\cos x}=\sec x\tan x$$
$$(\csc x)'=\left(\frac{1}{\sin x}\right)'=\frac{\sin x\times 0-1\times\cos x}{\sin^2 x}=-\frac{\cos x}{\sin x}\times\frac{1}{\sin x}=-\csc x\cot x$$

1.82: Trigonometric Derivatives: Summary

$$\frac{d}{dx}(\sin x) = \cos x\,, \qquad \frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x\,, \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x\,, \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

- The above are the standard forms of the derivatives of the trigonometric functions.
- Note that the functions on the right (the co-functions) each have a negative sign in their derivative.

Example 1.83

Find the derivative of:

A. $y = \sec x \tan x$
B. $y = \frac{\sec t}{e^t}$

Part A

$$y' = \underbrace{\sec x \tan x}_{f'}\underbrace{\tan x}_{g} + \underbrace{\sec x}_{f}\underbrace{\sec^2 x}_{g'} = \sec x\,(\tan^2 x + \sec^2 x)$$

Part B

$$y' = \frac{\underbrace{(e^t)}_{g}\underbrace{(\sec t \tan t)}_{f'} - \underbrace{(e^t)}_{g'}\underbrace{(\sec t)}_{f}}{e^{2t}}$$

Factor $e^t \sec t$ from both terms in the numerator:

$$\frac{e^t(\sec t)(\tan t - 1)}{e^{2t}} = \frac{(\sec t)(\tan t - 1)}{e^t}$$

B. Extension

1.84: Product Rule

$$(fgh)' = fgh' + fg'h + f'gh$$
$$[f(x)g(x)h(x)]' = f(x)g(x)h'(x) + f(x)g'$$

$$(fgh)' = (fg \times h)' = (fg)' \times h + fg \times h' = (fg' + f'g)h + fgh' = fgh' + fg'h + f'gh$$

Example 1.85

$$e^x \sin x \cos x$$

$$e^x \sin x \cos x = -e^x \sin x \sin x + e^x \cos x \cos x + e^x \sin x \cos x$$
$$= e^x(\cos^2 x - \sin^2 x + \sin x \cos x)$$

1.86: Product Rule

$$(efgh)' = efgh' + efg'h + ef'gh + e'fgh$$

C. Using Identities

1.87: Identities derived from the Pythagorean Identity

$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Example 1.88

Show that the derivative of $\frac{(\sec x+\tan x)(\sec x-\tan x)}{(\csc x+\cot x)(\csc x-\cot x)}$ is zero.

$$y=\frac{\sec^2 x-\tan^2 x}{\csc^2 x-\cot^2 x}$$

Substitute $\sec^2\theta-\tan^2\theta=1$ and $\csc^2\theta-\cot^2\theta=1$:

$$y=\frac{1}{1}=1\Rightarrow\frac{dy}{dx}=0$$

D. Proving Identities

Example 1.89

Prove that

$$\sin^2\theta+\cos^2\theta=1$$

$$y=\sin^2\theta+\cos^2\theta$$

$$\frac{dy}{d\theta}=2\sin\theta\cos\theta-2\cos\theta\sin\theta=0$$

Since the derivative is zero,

$$\frac{dy}{d\theta}=0\Rightarrow y=C, for\ some\ constant\ C$$

Substitute $\theta=\frac{\pi}{4}$

$$\sin^2\left(\frac{\pi}{4}\right)+\cos^2\left(\frac{\pi}{4}\right)=\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2=\frac{1}{2}+\frac{1}{2}=1\Rightarrow C=1$$

1.5 Chain Rule

A. Basics

1.90: Chain Rule

Suppose

$$y=g(u),\qquad u=h(x)\Rightarrow y=f(x)=g(u)=g\big(h(x)\big)$$

And the derivative of $f(x)$ is given by:

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

- If you wish to find the derivative of y with respect to x, but you know y as a function of another variable u, then you can differentiate using:

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

B. Power Rule

Example 1.91

Find $\frac{dy}{dx}$:

A. $y = (2x+1)^5$
B. $y = [f(x)]^n$
C. $y = (x+b)^2$

Part A

We can expand $(2x+1)^5$ using the binomial theorem, and then differentiate the resulting polynomial, but that is lengthy.

Instead, we express y as a function of u and find the derivative of y with respect to u:

$$Let\ u = 2x+1 \Rightarrow \frac{du}{dx} = 2$$

Write the given function in terms of u and differentiate it:

$$y = (2x+1)^5 = u^5 \Rightarrow \frac{dy}{du} = 5u^4$$

$$\frac{dy}{dx} = \underbrace{5u^4}_{\frac{dy}{du}} \times \underbrace{2}_{\frac{du}{dx}} = 10u^4$$

Change back to the original variable:

$$\frac{dy}{dx} = 10(2x+1)^4$$

Part B

$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} \times f'(x)$$

Part C

$$u = x+b \Rightarrow \frac{du}{dx} = 1$$

$$y = (x+b)^2 = u^2 \Rightarrow \frac{dy}{du} = 2u$$

$$\frac{dy}{dx} = (2u)(1) = 2u = 2(x+b)$$

1.92: Outer and Inner Functions

- Differentiate the “outer function” as usual, keeping the inner function in place.
- Differentiate the “inner function”

Example 1.93

A. $y = (3x+4)^5$
B. $y = (2x-7)^3$
C. $y = (-6x+5)^8$
D. $y = \frac{1}{(5x-4)^3}$

Part A

$$y = \left(\underbrace{3x+4}_{\substack{Inner\\Function}}\right)^5 \Rightarrow \frac{dy}{dx} = 5(3x+4)^4 \times 3 = 15(3x+4)^4$$

Part B

$$\frac{dy}{dx} = 3(2x-7)^2 \times 2 = 6(2x-7)^2$$
$$u = 2x - 7 \Rightarrow y = u^3 \Rightarrow \frac{dy}{du} = 3u^2$$

Part C

$$y = -48(-6x+5)^7$$

Part D

Rewrite using exponent rules:

$$y = \frac{1}{(5x-4)^3} = (5x-4)^{-3}$$
$$\frac{dy}{dx} = -3(5x-4)^{-5}(5) = -15(5x-4)^{-5}$$

Example 1.94

Find $\frac{dy}{dx}$:

A. $y = (x^2+5)^{100}$
B. $y = \sin^2 x$
C. $y = \cos^5 x$
D. $y = \sin^{100} x$
E. $y = \sin\left(\frac{x}{2}\right)$

Part A

Write $y = f(x)$ as a composite function.

$$u = h(x) = x^2 + 5 \Rightarrow \frac{du}{dx} = 2x$$
$$y = u^{100} \Rightarrow \frac{dy}{du} = 100u^{99} = 100(x^2+5)^{99}$$

Differentiate using $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$:

$$= 100(x^2+5)^{99} \times 2x = 200x(x^2+5)^{99}$$

Part B

Expand to get:

$$y = (\sin x)(\sin x)$$

Differentiate using the product rule:

$$y' = (\sin x)(\cos x) + (\cos x)(\sin x)$$

Simplify to get:

$$= 2(\sin x)(\cos x)$$

$$u = \sin x \Rightarrow \frac{du}{dx} = \cos x$$

$$y = \sin^2 x = u^2$$
$$\frac{dy}{du} = 2u$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2u(\cos x) = 2(\sin x)(\cos x)$$

Part C

Let:

$$u = \cos x \Rightarrow y = u^5$$

Differentiate:

$$\frac{dy}{dx} = \underbrace{5u^4}_{\frac{dy}{du}} \times \underbrace{(-\sin x)}_{\frac{du}{dx}} = -5(\cos^4 x)(\sin x)$$

Part D

$$y' = 100(\sin^{99} x)(\cos x)$$

Part E

$$\cos\left(\frac{x}{2}\right) \times \frac{1}{2}$$

Example 1.95: Inverse Trig Functions

A. $\frac{d}{dx}(\sin^{-1} x)^2$
B. $\frac{d}{dx}(\tan^{-1} x)^n, n \in \mathbb{N}, n \geq 2$

Part I

$$\frac{d}{dx}(\sin^{-1} x)^2 = \frac{2\ \sin^{-1} x}{\sqrt{1-x^2}}$$

Part J

$$\frac{d}{dx}(\tan^{-1} x)^n = \frac{n\,(\tan^{-1} x)^{n-1}}{1+x^2}$$

1.96: Chain Rule

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}, \qquad \frac{d}{dx}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}} \times f'(x)$$

When differentiating square roots make use of the formula for square roots, and its chain rule version. Avoid converting into exponent form since this will increase the time taken.

Example 1.97: Square Roots

$$y = \sqrt{e^x}$$

$$y = \sqrt{e^x} = (e^x)^{\frac{1}{2}} = e^{\frac{x}{2}}$$
$$y' = e^{\frac{x}{2}} \times 2$$

Example 1.98: Square Roots

Find $\frac{dy}{dx}$ for each part below:

A. $y = \sqrt{2x}$

B. $y = \sqrt{\sin\left(\frac{x}{2}\right)}$

$$\frac{d}{dx}\sqrt{2x} = \frac{1}{2\sqrt{2x}} \times 2 = \frac{1}{\sqrt{2x}}$$
$$\frac{d}{dx}\sqrt{\sin x} = \frac{\cos x}{2\sqrt{\sin x}}$$
$$\frac{d}{dx}\sqrt{e^x} = \frac{1}{2\sqrt{e^x}} \times e^x = \frac{\sqrt{e^x}}{2}$$
$$\frac{1}{2\sqrt{\sin\left(\frac{x}{2}\right)}} \times \cos\left(\frac{x}{2}\right) \times \frac{1}{2} = \frac{\cos\left(\frac{x}{2}\right)}{4\sqrt{\sin\left(\frac{x}{2}\right)}}$$

Example 1.99: Nested Square Roots

Differentiate with respect to x:

A. $\sqrt{x+\sqrt{x}}$

B. $\sqrt{\sin x + \sqrt{\cos x}}$

C. $\sqrt{2 - \sqrt{\frac{x}{2}} + \sqrt{x+1}}$

Part A

Use the chain rule:

$$\frac{1}{2\sqrt{x+\sqrt{x}}}\times\left(1+\frac{1}{2\sqrt{x}}\right)$$

Use the distributive property:

$$=\frac{1}{2\sqrt{x+\sqrt{x}}}+\frac{1}{4\sqrt{x}\sqrt{x+\sqrt{x}}}$$

Add the two fractions by taking the LCM which is $4\sqrt{x}\sqrt{x+\sqrt{x}}$:

$$=\frac{2\sqrt{x}}{2\sqrt{x+\sqrt{x}}\left(2\sqrt{x}\right)}+\frac{1}{4\sqrt{x}\sqrt{x+\sqrt{x}}}=\frac{2\sqrt{x}+1}{4\sqrt{x}\sqrt{x+\sqrt{x}}}$$

Part B

$$\frac{1}{2\sqrt{\sin x+\sqrt{\cos x}}}\times\left(\cos x+\frac{1}{2\sqrt{\cos x}}\right)\times(-\sin x)$$

Part C

$$\frac{1}{2\sqrt{2-\sqrt{\frac{x}{2}+\sqrt{x+1}}}}\times\frac{-1}{2\sqrt{\frac{x}{2}+\sqrt{x+1}}}\times\left(\frac{1}{2}+\frac{1}{2\sqrt{x+1}}\right)$$

$$=\frac{-1}{4\sqrt{\frac{x}{2}+\sqrt{x+1}}\sqrt{2-\sqrt{\frac{x}{2}+\sqrt{x+1}}}}\left(\frac{1}{2}+\frac{1}{2\sqrt{x+1}}\right)$$

C. Exponentials

Example 1.100

A. $y=e^{-x}$

B. $y=e^{x^2}$

C. $y=e^{f(x)}$

D. $y=\pi e^{\sin x}$

E. $y=e^{\sec x}$

Part A

$$u=-x\Rightarrow\frac{dy}{du}=-1$$

$$\frac{dy}{dx}=e^{-x}\times\frac{d}{dx}(-x)=e^{-x}\times(-1)=-e^{-x}$$

Part B

$$y=e^u\Rightarrow\frac{dy}{du}=e^u,\qquad u=x^2\Rightarrow\frac{du}{dx}=2x$$

$$\frac{dy}{dx}=\underbrace{e^u}_{\frac{dy}{du}}\times\underbrace{2x}_{\frac{du}{dx}}=2xe^{x^2}$$

Shortcut

$$y'=e^{x^2}\times 2x=2xe^{x^2}$$

Part C

$$f(x) = u \Rightarrow y = e^u \Rightarrow \frac{dy}{du} = e^u$$
$$u = f(x) \Rightarrow \frac{du}{dx} = f'(x)$$
$$\frac{dy}{dx} = \underbrace{e^{f(x)}}_{\frac{dy}{du}} \times \underbrace{f'(x)}_{\frac{du}{dx}}$$

Part D

$$\frac{dy}{dx} = \pi \underbrace{e^{\sin x}}_{e^{f(x)}} \times \underbrace{\cos x}_{f'(x)} = \pi \cos x \, e^{\sin x}$$

Part E

$$\frac{dy}{dx} = \underbrace{e^{\sec x}}_{e^{f(x)}} \times \underbrace{\sec x \tan x}_{f'(x)} = \sec x \tan x \, e^{\sec x}$$

D. Trigonometric Functions

Example 1.101

A. $y = \tan 2x$
B. $y = \csc\frac{x}{2}$
C. $y = \cot\sqrt{x}$
D. $y = \sec(\ln x)$
E. $y = \sin(\cos x)$

Part A

$$u = 2x \Rightarrow \frac{du}{dx} = 2$$
$$y = \tan 2x = \tan u \Rightarrow \frac{dy}{du} = \sec^2 u$$
$$y' = \underbrace{\sec^2 u}_{\frac{dy}{du}} \times \underbrace{2}_{\frac{du}{dx}} = 2\sec^2 2x$$

Part B

$$y' = -\csc\frac{x}{2}\cot\frac{x}{2} \times \frac{1}{2}$$

Part C

$$y' = -\csc^2\sqrt{x} \times \frac{1}{2\sqrt{x}}$$

Part D

$$y' = \frac{\sec(\ln x)\tan(\ln x)}{x}$$

Part E

$$y = \cos(\cos x)\,(-\sin x)$$

Example 1.102

Find the first few derivatives of $y = \tan x$.

$$y' = \sec^2 x$$

$$y'' = 2\,(\sec x)(\sec x \tan x) = 2\sec^2 x \tan x$$

$$\begin{aligned} y''' &= 2[2\,(\sec x)(\sec x \tan x)\tan x] + 2\sec^2 x \sec^2 x \\ &= 4\sec^2 x \tan^2 x + 2\sec^4 x \\ &= 2\sec^2 x\,(2\tan^2 x + \sec^2 x) \end{aligned}$$

E. Mult-Step Chain Rule

1.103: Chain Rule with Three Functions

$$y = f(u), u = f(v), v = f(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$$

Example 1.104

$$y = e^{\cos^3 x}$$

$$\begin{aligned} &y = e^u, \qquad && u = v^3, \qquad && v = \cos x \\ &\frac{dy}{du} = e^u, && \frac{du}{dv} = 3v^2, && \frac{dv}{dx} = -\sin x \end{aligned}$$

$$\frac{dy}{dx} = (e^u)(3v^2)(-\sin x) = \left(e^{\cos^3 x}\right)(3\cos^2 x)(-\sin x)$$

Example 1.105

Find the first and the second derivative of:

$$y = 9\tan\left(\frac{x}{3}\right)$$

First derivative:

$$y = 9\tan u\,, \qquad u = \frac{x}{3}$$

$$\frac{dy}{dx} = \underbrace{9\sec^2 u}_{\frac{dy}{du}} \times \underbrace{\frac{1}{3}}_{\frac{du}{dx}} = 9\sec^2\left(\frac{x}{3}\right) \times \frac{1}{3} = 3\sec^2\left(\frac{x}{3}\right)$$

Second derivative:

$$y = 3u^2, \qquad u = \sec v\,, \qquad v = \frac{x}{3}$$

$$\frac{dy}{dx} = \underbrace{6u}_{\frac{dy}{du}} \times \underbrace{\sec v \tan v}_{\frac{du}{dv}} \times \underbrace{\frac{1}{3}}_{\frac{dv}{dx}} = 6\sec\left(\frac{x}{3}\right) \times \sec\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right) \times \frac{1}{3} = 2\sec^2\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)$$

Example 1.106

A. $y = \sin\left(\cos e^{\frac{t}{2}}\right)$

B. $y = \sqrt{\ln\left(\csc\left(\pi x - \frac{2}{x}\right)\right)}$

$$\frac{dy}{dx} = \cos\left(\cos e^{\frac{t}{2}}\right)\left(-\sin e^{\frac{t}{2}}\right)\left(e^{\frac{t}{2}}\right)\left(\frac{1}{2}\right)$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\ln\left(\csc\left(\pi x - \frac{2}{x}\right)\right)}} \times \frac{1}{\csc\left(\pi x - \frac{2}{x}\right)} \times \left(-\csc\left(\pi x - \frac{2}{x}\right)\cot\left(\pi x - \frac{2}{x}\right)\right) \times \left(\pi + \frac{2}{x^2}\right)$$

$$= -\frac{\cot\left(\pi x - \frac{2}{x}\right)\left(\pi + \frac{2}{x^2}\right)}{2\sqrt{\ln\left(\csc\left(\pi x - \frac{2}{x}\right)\right)}}$$

F. Product Rule

Example 1.107

A. $y = 2xe^{-3x} + e^{\frac{x^3}{3}}$
B. $y = (2x+3)^4 e^{-2x}$
C. $y = \sqrt{x}\sin\sqrt{x}$
D. $y = \sin^3(3x^2)$

Part C

$$u = (2x+1)^5 \Rightarrow \frac{du}{dx} = 10(2x+1)^4$$
$$v = (3x-4)^{-3} \Rightarrow \frac{dv}{dx} = -9(3x-4)^{-4}$$

And now we can find:

$$(uv)' = \underbrace{(2x+1)^5}_{u}\underbrace{[-9(3x-4)^{-4}]}_{v'} + \underbrace{[10(2x+1)^4]}_{u'}\underbrace{(3x-4)^{-3}}_{v}$$

Factor out the lowest power in each term:

$$= (2x+1)^4(3x-4)^{-4}[-9(2x+1) + 10(3x-4)]$$
$$= (2x+1)^4(3x-4)^{-4}[-18x - 9 + 30x - 40]$$
$$= \frac{(2x+1)^4(12x-49)}{(3x-4)^4}$$

Part D

$$\left(e^{x^3}\right)' = 3x^2\left(e^{x^3}\right)$$
$$(xe^{-x})' = -(x)(e^{-x}) + (1)(e^{-x}) = e^{-x} - xe^{-x}$$
$$y' = e^{-x} - xe^{-x} + 3x^2\left(e^{x^3}\right)$$

Part E

$$y = -2(2x+3)^4(e^{-2x}) + 8(2x+3)^3(e^{-2x})$$
$$= (e^{-2x})(2x+3)^3[-2(2x+3)^1 + 8]$$
$$= (e^{-2x})(2x+3)^3[-4x - 6 + 8]$$
$$= (e^{-2x})(2x+3)^3[-4x + 2]$$

Part F

$$y = \sqrt{x}\left(\frac{\cos\sqrt{x}}{2\sqrt{x}}\right) + \frac{\sin\sqrt{x}}{2\sqrt{x}} = \frac{\cos\sqrt{x}}{2} + \frac{\sin\sqrt{x}}{2\sqrt{x}}$$

Part G

$$y' = 3\sin^2(3x^2) \times \cos(3x^2) \times 6x = 18x\sin^2(3x^2)\cos(3x^2)$$
$$u = \sin(3x^2) \Rightarrow y = u^3 \Rightarrow \frac{dy}{du} = 3u^2$$

G. Quotient Rule

Example 1.108

A. $y = \frac{(2x+3)^7}{(4x-1)^3}$

B. $y = \sqrt{7 + x\sec x}$

Part A

$$y = \frac{(2x+3)^7}{(4x-1)^3}$$
$$f = (2x+3)^7 \Rightarrow f' = 7(2x+3)^6 \times 2 = 14(2x+3)^6$$
$$g = (4x-1)^3 \Rightarrow g' = 3(4x-1)^2 \times 4 = 12(4x-1)^2$$
$$\frac{dy}{dx} = \frac{14(4x-1)^3(2x+3)^6 - 12(2x+3)^7(4x-1)^2}{(4x-1)^6}$$

Divide numerator and denominator by $(4x-1)^2$:

$$= \frac{14(4x-1)(2x+3)^6 - 12(2x+3)^7}{(4x-1)^4}$$

Factor out $(2x+3)^6$ in the numerator:

$$= \frac{(2x+3)^6[14(4x-1) - 12(2x+3)]}{(4x-1)^4}$$

Simplify:

$$= \frac{(2x+3)^6[56x - 14 - 24x - 36]}{(4x-1)^4}$$
$$= \frac{(2x+3)^6[32x - 50]}{(4x-1)^4}$$

Part B

$$y = \sqrt{7 + x\sec x}$$
$$u = 7 + x\sec x \Rightarrow \frac{du}{dx} = x\sec x\tan x + (1)\sec x = \sec x\,(x\tan x + \sec x)$$
$$y = \sqrt{u} \Rightarrow \frac{dy}{du} = \frac{1}{2\sqrt{u}}$$

Differentiate the "square root" using $\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{u}} \times \sec x\,(x\tan x + \sec x) = \frac{\sec x\,(x\tan x + \sec x)}{2\sqrt{7 + x\sec x}}$$

Example 1.109

$$y = \cot^{-1}\sqrt{t-1}$$

$$\frac{dy}{dx} = \frac{-1}{1 + \left(\sqrt{t-1}\right)^2} \times \frac{1}{2\sqrt{t-1}} = \frac{-1}{2t\sqrt{t-1}}$$

Example 1.110

$$y = \cot^{-1}\frac{1}{x} - \tan^{-1}x$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\left(\cot^{-1}\frac{1}{x}\right) = \frac{-1}{1+\left(\frac{1}{x}\right)^2}\times\left(-\frac{1}{x^2}\right) = \frac{1}{\frac{x^2+1}{x^2}}\times\left(\frac{1}{x^2}\right) = \frac{x^2}{x^2+1}\times\left(\frac{1}{x^2}\right) = \frac{1}{x^2+1}$$
$$\frac{d}{dx}\left(\cot^{-1}\frac{1}{x} - \tan^{-1}x\right) = \frac{1}{x^2+1} - \frac{1}{1+x^2} = 0$$

H. Second Derivatives

Example 1.111

Find the second derivative of:

A. $y = \left(1+\frac{1}{x}\right)^3$

B. $y = (1-\sqrt{x})^{-1}$

C. $y = \sin(x^2 e^x)$

Part A

$$\frac{dy}{dx} = 3\left(1+\frac{1}{x}\right)^2\left(-\frac{1}{x^2}\right) = -3\left(1+\frac{1}{x}\right)^2\left(\frac{1}{x^2}\right)$$

$$\frac{d^2y}{dx^2} = -3\left[\left(1+\frac{1}{x}\right)^2\left(-\frac{2}{x^3}\right) + 2\left(1+\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)\left(\frac{1}{x^2}\right)\right] =$$

Factor out $(-2)\left(1+\frac{1}{x}\right)\left(\frac{1}{x^3}\right)$:

$$= 6\left(1+\frac{1}{x}\right)\left(\frac{1}{x^3}\right)\left[\left(1+\frac{1}{x}\right)+\frac{1}{x}\right]$$

Simplify:

$$= 6\left(1+\frac{1}{x}\right)\left(\frac{1}{x^3}\right)\left[\left(1+\frac{2}{x}\right)\right]$$

Part B

Find the first derivative:

$$y' = -(1-\sqrt{x})^{-2}\left(-\frac{1}{2\sqrt{x}}\right) = \left(\frac{1}{2}\right)(1-\sqrt{x})^{-2}\left(x^{-\frac{1}{2}}\right)$$

Find the second derivative using the product rule:

$$y'' = \left(\frac{1}{2}\right)\left[(1-\sqrt{x})^{-2}\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) + \left(-2(1-\sqrt{x})^{-3}\right)\left(-\frac{1}{2\sqrt{x}}\right)\left(x^{-\frac{1}{2}}\right)\right]$$
$$= \left(\frac{1}{2}\right)\left[\frac{1}{(1-\sqrt{x})^2}\left(-\frac{1}{2x^{\frac{3}{2}}}\right) + \left(\frac{1}{(1-\sqrt{x})^3}\right)\left(\frac{1}{x}\right)\right]$$

Factor out $\frac{1}{(1-\sqrt{x})^2 x}$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{(1-\sqrt{x})^2}\right)\left(\frac{1}{x}\right)\left[\left(-\frac{1}{2\sqrt{x}}\right) + \left(\frac{1}{1-\sqrt{x}}\right)\right]$$

Simplify:

$$= \left(\frac{1}{2}\right)\left(\frac{1}{(1-\sqrt{x})^2}\right)\left(\frac{1}{x}\right)\left[\frac{1}{1-\sqrt{x}} - \frac{1}{2\sqrt{x}}\right]$$

Part C

$$y' = \cos(x^2e^x)\,(2xe^x + x^2e^x) = \cos(x^2e^x)\,[e^x(2x + x^2)] =$$
$$(\cos(x^2e^x))' = -\sin(x^2e^x)\,(2xe^x + x^2e^x)$$
$$e^x(2x + x^2) = e^x(2 + 2x) + e^x(2x + x^2)$$

$$y'' = \cos(x^2e^x)\,[e^x(2 + 2x) + e^x(2x + x^2)] - \sin(x^2e^x)\,(2xe^x + x^2e^x)[e^x(2x + x^2)]$$
$$= \cos(x^2e^x)\,[e^x(2 + 4x + x^2)] - \sin(x^2e^x)\,e^{2x}(2x + x^2)^2$$

Example 1.112
Find the solution sets to $y' = 0$ and $y'' = 0$ given that $y = x(x-4)^3$

Find the first derivative, and equate it to zero:
$$y' = (1)(x-4)^3 + 3x(x-4)^2 = (x-4)^2(x - 4 + 3x) = (x-4)^2(4x-4)$$
$$y' = 4(x-4)^2(x-1) = 0 \Rightarrow x \in \{1,4\}$$
Find the second derivative, and equate it to zero:
$$y'' = 4[2(x-4)(x-1) + (x-4)^2] = 4(x-4)[2x - 2 + (x-4)]$$
$$y'' = 12(x-4)[x-2] = 0 \Rightarrow x \in \{2,4\}$$

Example 1.113: n^{th} Derivative
The n^{th} derivative of $y = a\sin\left(\frac{x}{2}\right), a > 0$ is $y = \sin\left(\frac{x}{2}\right)$. If $n > 10$, find the minimum value of a.

$$\frac{dy}{dx} = \frac{a}{2}\cos\left(\frac{x}{2}\right)$$
$$\frac{d^2y}{dx^2} = -\frac{a}{2^2}\sin\left(\frac{x}{2}\right)$$
$$\frac{d^3y}{dx^3} = -\frac{a}{2^3}\cos\left(\frac{x}{2}\right)$$
$$\frac{d^4y}{dx^4} = \frac{a}{2^4}\sin\left(\frac{x}{2}\right)$$

$$\frac{d^{12}y}{dx^{12}} = \frac{a}{2^{12}}\sin\left(\frac{x}{2}\right) \Rightarrow Min(a) = 2^{12} = 4096$$

I. Chain Rule: Alternate Version

1.114: Chain Rule (Alternate Version)
Let $f(x)$ be a composite function such that
$$f(x) = g\big(h(x)\big)$$
Then:
$$f'(x) = g'\big(h(x)\big) \times h'(x)$$

Example 1.115
Given $f(x) = (x^2 + 5)^{100}$ find $f'(x)$

We want to write $f(x)$ in the form $g\big(h(x)\big)$. Let"
$$h(x) = x^2 + 5 \Rightarrow h'(x) = 2x$$
$$g(x) = x^{100} \Rightarrow g'(x) = 100x^{99}$$

Now, start with the formula:

$$f'(x) = g'\big(h(x)\big) \times h'(x)$$

Substitute $h(x) = x^2 + 5, h(x) = x^2 + 5 \Rightarrow h'(x) = 2x$

$$= g'(x^2 + 5) \times 2x$$

Apply $g'(x) = 100x^{99}$:

$$= 100(x^2 + 5)^{99} \times 2x = 200x(x^2 + 5)^{100}$$

J. Applications (Optional)

Example 1.116: Maximum Slope

A. Determine the maximum and minimum slope of $y = \sin x$.
B. Also determine where these occur.

Part A
Slope of any function is given by its derivative.

$$y' = \cos x$$

$\cos x$ has

$$Max = 1, \qquad Min = -1$$

Part B

$$Max\ occurs\ at \cos x = 1 \Rightarrow x = 2\pi k, k \in \mathbb{Z}$$
$$Min\ occurs\ at \cos x = -1 \Rightarrow x = \pi + 2\pi k, k \in \mathbb{Z}$$

Example 1.117: Maximum slope

A. Determine the maximum and minimum slope of $y = \sin mx$.
B. Also determine where these occur.

Part A
Slope of any function is given by its derivative.

$$y' = m \cos mx$$

We need to determine maximum and minimum of the quantity:

$$m \cos mx$$

Consider Cases
Case I: $m = 0$

$$m \cos mx = 0 \Rightarrow Max = min = 0$$

Case II: $m > 0$

$$-1 \le \cos mx \le 1$$
$$-m \le m \cos mx \le m$$

$$Max = m, Min = -m$$

Case III: $m < 0$

$$-1 \le \cos mx \le 1$$
$$-m \ge m \cos mx \ge m$$
$$m \le m \cos mx \le -m$$

Max is

$$Max = -m, Min = m$$

Combining Case I, II and III:

$$Max = |m|, Min = -|m|$$

1.118: Velocity and Acceleration

$$Velocity = Rate\ of\ change\ of\ position = v = \frac{dx}{dt}$$
$$Acceleration = Rate\ of\ change\ of\ velocity = a = \frac{d^2x}{dt^2}$$

Example 1.119

$$x = e^{2x}$$

$$v = \frac{dx}{dt} = 2e^{2x}$$
$$a = \frac{d^2x}{dt^2} = 4e^{2x}$$

Evaluate at $x = 2$

$$v = 2e^{2x} = 2e^{2\cdot 2} = 2e^4$$
$$a = 4e^{2x} = 4e^{2\cdot 2} = 4e^4$$

Example 1.120: Simple Harmonic Motion

Assume that displacement is given by

$$x = A\sin[\omega t + \phi] + C, \qquad A, \omega, \phi\ and\ C\ are\ constants$$

Determine what happens to the acceleration when you double ω

$$v = \frac{dx}{dt} = A\omega\cos[\omega t + \phi]$$
$$a = \frac{dx^2}{dt^2} = -A\omega^2\sin[\omega t + \phi]$$

If you double ω

$$a \propto \omega^2$$

1.6 Implicit Differentiation

A. Basics

1.121: Chain Rule

If we differentiate $y = f(u)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

- If you are differentiating with respect to x, and you introduce another variable, then you have to differentiate that variable as well.

Example 1.122

$$y = u^2$$
$$u = 2x$$

Differentiate both sides of $u = 2x$ with respect to x:

$$\frac{du}{dx} = 2$$

Differentiate both sides of $y = u^2$ with respect to u:

$$\frac{dy}{du} = \frac{d}{du}u^2 = 2u$$

Differentiate both sides of $y = u^2$ with respect to x:

$$\frac{dy}{dx} = \frac{d}{dx}u^2 = \frac{dy}{du} \times \frac{du}{dx} = \underbrace{2u}_{\frac{dy}{du}} \times \frac{du}{dx}$$

Example 1.123

$$y = (1 + x^2)^2, \qquad r = 1 + x^2$$

$$y = r^2$$
$$\frac{dy}{dx} = 2r \times \frac{dr}{dx}$$

1.124: Chain Rule Concept

If r is a function of x, then:

$$\frac{d}{dx}(r^2) = 2r\frac{dr}{dx}$$

If r is not a function of x, then we cannot find:

$$\frac{d}{dx}(r^2)$$

Example 1.125

$$\frac{d}{dx}y^2$$

$$z = y^2$$

$$\frac{d}{dx}z = \frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}$$

Example 1.126

Find the given derivatives, if they exist.

A. $\frac{d}{dx}\sqrt{x}$

B. $\frac{d}{dx}\sqrt{r}, r = f(x)$

C. $\frac{d}{dx}\sqrt{r}, r \neq f(x)$

$$\frac{1}{2\sqrt{x}}$$

$$\frac{1}{2\sqrt{r}} \times \frac{dr}{dx}$$

$$Derivative\ does\ not\ exist$$

B. Implicit Differentiation with Expressions

1.127: Implicit Differentiation

$$z = f(y) \Rightarrow \frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Example 1.128: Chain Rule

Differentiate each expression below with respect to x. You may assume that $y = f(x)$.

A. y^2
B. $\sin y$
C. $\ln y$
D. e^y
E. $\tan y$
F. $\sec y$
G. $\cot y$
H. $\csc y$

$$\frac{d}{dx} y^2 = \frac{dz}{dy} \times \frac{dy}{dx} = \underbrace{2y}_{\frac{dz}{dy}} \frac{dy}{dx}$$

$$\frac{d}{dx} \sin y = \cos y \frac{dy}{dx}$$

$$\frac{d}{dx} \ln y = \frac{1}{y} \cdot \frac{dy}{dx}$$

$$\frac{d}{dx} e^y = e^y \frac{dy}{dx}$$

$$\frac{d}{dx} \tan y = \sec^2 y \frac{dy}{dx}$$

$$\frac{d}{dx} \sec y = \sec y \ tan\ y \frac{dy}{dx}$$

$$\frac{d}{dx} \cot y = -\csc^2 y \frac{dy}{dx}$$

$$\frac{d}{dx} \csc y = -\csc y \cot y \frac{dy}{dx}$$

Example 1.129: Multi-Step Chain Rule

Differentiate each expression below with respect to x:

A. $\cos^2 y$

B. e^{y^5}
C. $e^{\sin y}$
D. $e^{\sec\frac{y}{2}}$
E. $\cot^n(\ln y)\,, n\ is\ a\ constant, n \neq 1$

$$\frac{d}{dx}\cos^2 y = 2\cos y\ (-\sin y)\left(\frac{dy}{dx}\right)$$
$$\frac{d}{dx}e^{y^5} = \left(e^{y^5}\right)(5y^4)\left(\frac{dy}{dx}\right)$$
$$\frac{d}{dx}e^{\sin y} = \left(e^{\sin y}\right)(\cos y)\left(\frac{dy}{dx}\right)$$
$$\frac{d}{dx} = \left(e^{\sec\frac{y}{2}}\right)\left(\sec\frac{y}{2}\tan\frac{y}{2}\right)\left(\frac{1}{2}\frac{dy}{dx}\right)$$
$$\frac{d}{dx}\cot^n(\ln y) = n\cot^{n-1}(\ln y)\left(-\csc^2(\ln y)\right)\frac{1}{y}\frac{dy}{dx}$$

Example 1.130: Multi-Step Chain Rule

Differentiate each expression below with respect to x:

$$e^{\sec^2\left(\ln\frac{y}{2}\right)}$$

$$\frac{d}{dx}e^{\sec^2\ln\frac{y}{2}} = \left(e^{\sec^2\ln\frac{y}{2}}\right)\left(2\sec\left(\ln\frac{y}{2}\right)\right)\left(\sec\left(\ln\frac{y}{2}\right)\tan\left(\ln\frac{y}{2}\right)\right)\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right)$$

Note that:

$$\ln\frac{y}{2} = \ln y - \ln 2 \Rightarrow \frac{d}{dx}\ln y - \underbrace{\ln 2}_{Constant} = \frac{1}{y} - 0 = \frac{1}{y}$$

C. Implicit Differentiation with Equations

Example 1.131

The equation of a circle with center at the origin and radius r is given by $x^2 + y^2 = r^2$, where r is a constant.

A. Solve the equation for y, and hence find $\frac{dy}{dx}$.
B. Without solving for y, differentiate both sides of the above equation with respect to x, and hence find $\frac{dy}{dx}$.

Part A

$$y^2 = r^2 - x^2 \Rightarrow y = \pm\sqrt{r^2 - x^2}$$

Here, we get two functions, which together make up the circle. We can find their derivatives separately:

$$\frac{d}{dx}\sqrt{r^2 - x^2} = \frac{-2x}{2\sqrt{r^2 - x^2}} = \frac{-x}{\sqrt{r^2 - x^2}}$$
$$\frac{d}{dx}\left(-\sqrt{r^2 - x^2}\right) = -\frac{-2x}{2\sqrt{r^2 - x^2}} = \frac{x}{\sqrt{r^2 - x^2}}$$

And we can combine the two to get:

$$\frac{dy}{dx} = \pm\frac{x}{\sqrt{r^2 - x^2}}$$

Part B

$$x^2 + y^2 = r^2$$

Differentiate both sides with respect to x:

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}r^2$$
$$2x + 2y \cdot \frac{dy}{dx} = 0$$

Now, we have an equation in terms of $\frac{dy}{dx}$. Treat $\frac{dy}{dx}$ as a regular variable and solve the above equation for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y} = \pm\frac{x}{\sqrt{r^2 - x^2}}$$

1.132: Implicit Functions

When y is given implicitly as a function of x, rather than explicitly, implicit differentiation is very useful.

Example 1.133

Find $\frac{dy}{dx}$ in terms of x:

$$x^3 + y^2 = 2x + 5$$

$$y = \pm\sqrt{2x + 5 - x^3}$$

Differentiate both sides of the given equation with respect to x:

$$3x^2 + 2y\frac{dy}{dx} = 2$$
$$\frac{dy}{dx} = \frac{2 - 3x^2}{2y} = \pm\frac{2 - 3x^2}{2\sqrt{2x + 5 - x^3}}$$

Example 1.134

Find $\frac{dy}{dx}$:

$$3x^2 + y^3 = y^2$$

$$3x^2 = y^2 - y^3$$

Differentiate both sides of the given equation with respect to x:

$$6x = 2y\frac{dy}{dx} - 3y^2\frac{dy}{dx}$$

Factor out $\frac{dy}{dx}$:

$$\frac{dy}{dx}(3y^2 - 2y) = -6x$$

Divide both sides by $(3y^2 - 2y)$:

$$\frac{dy}{dx} = \frac{-6x}{3y^2 - 2y} = \frac{6x}{2y - 3y^2}$$

Example 1.135

Find $\frac{dy}{dx}$:

$$\sin y = \sin x$$

$$\cos y \frac{dy}{dx} = \cos x$$
$$\frac{dy}{dx} = \frac{\cos x}{\cos y}$$

Example 1.136

Find $\frac{dy}{dx}$:

A. $e^{2y} = e^x$
B. $e^{\cos y} + e^{\sin y} = e^{\tan x}$

Part A

$$2e^{2y}\frac{dy}{dx} = e^x$$
$$\frac{dy}{dx} = \frac{e^x}{2e^{2y}}$$

Part B

$$e^{\cos y}(-\sin y)\frac{dy}{dx} + e^{\sin y}(\cos y)\frac{dy}{dx} = e^{\tan x}(\sec^2 x)$$
$$\frac{dy}{dx}\left(e^{\sin y}(\cos y) - e^{\cos y}(\sin y)\right) = e^{\tan x}(\sec^2 x)$$
$$\frac{dy}{dx} = \frac{e^{\tan x}(\sec^2 x)}{e^{\sin y}(\cos y) - e^{\cos y}(\sin y)}$$

D. Product Rule

1.137: Product Rule

$$(fg)' = f'g + fg'$$

Example 1.138:

Differentiate with respect to x:

$$\cos y \sin y$$

$$\left(-\sin y\frac{dy}{dx}\right)(\sin y) + \cos y\left(\cos y\frac{dy}{dx}\right) = \cos^2 y\frac{dy}{dx} - \sin^2 y\frac{dy}{dx} = \frac{dy}{dx}(\cos^2 y - \sin^2 y)$$

Example 1.139:

Find $\frac{dy}{dx}$:

$$xy^2 = yx^2 + x$$

Differentiate both sides of the given equation with respect to x:

$$\underbrace{x}_{f}\underbrace{(2y)\left(\frac{dy}{dx}\right)}_{g'} + \underbrace{(1)}_{f'}\underbrace{y^2}_{g} = \underbrace{y}_{f}\underbrace{(2x)}_{g'} + \underbrace{\frac{dy}{dx}}_{f'}\underbrace{x^2}_{g} + 1$$

Collate all $\frac{dy}{dx}$ terms on the LHS, and all other terms on the RHS:

$$\left(\frac{dy}{dx}\right)2xy - \left(\frac{dy}{dx}\right)x^2 = 2xy - y^2 + 1$$

Factor $\frac{dy}{dx}$ on the LHS:

$$\left(\frac{dy}{dx}\right)(2xy - x^2) = 2xy - y^2 + 1$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{2xy - y^2 + 1}{2xy - x^2}$$

Example 1.140

Find $\frac{dy}{dx}$:

$$y^3 = \sin x \cos y$$

Differentiate both sides of the given equation with respect to x:

$$3y^2\frac{dy}{dx} = \underbrace{\cos x}_{f'}\underbrace{\cos y}_{g} + \underbrace{\sin x}_{f}\underbrace{(-\sin y)\frac{dy}{dx}}_{g'}$$

Collate all $\frac{dy}{dx}$ terms on one side:

$$3y^2\frac{dy}{dx} + \sin x \sin y\frac{dy}{dx} = \cos x \cos y$$

Factor out $\frac{dy}{dx}$:

$$\frac{dy}{dx}(3y^2 + \sin x \sin y) = \cos x \cos y$$

Divide both sides by $3y^2 + \sin x \sin y$:

$$\frac{dy}{dx} = \frac{\cos x \cos y}{3y^2 + \sin x \sin y}$$

Example 1.141

Find $\frac{dy}{dx}$:

$$\sec(x + y) = \csc(xy)$$

$$\sec(x + y)\tan(x + y)\left(1 + \frac{dy}{dx}\right) = -\csc(xy)\cot(xy)\left(x\frac{dy}{dx} + y\right)$$

Use a change of variable. Let:

$$\sec(x + y)\tan(x + y) = \boldsymbol{A}$$
$$-\csc(xy)\cot(xy) = \boldsymbol{B}$$

Then, we get:

$$A\left(1 + \frac{dy}{dx}\right) = B\left(x\frac{dy}{dx} + y\right)$$

Use the distributive property:

$$A + A\frac{dy}{dx} = Bx\frac{dy}{dx} + By$$

Collate $\frac{dy}{dx}$ terms on one side:

$$A\frac{dy}{dx} - Bx\frac{dy}{dx} = By - A$$

Factor $\frac{dy}{dx}$:

$$\frac{dy}{dx}(A - Bx) = By - A$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{By - A}{A - Bx}$$

Change back to the original variable:

$$\frac{dy}{dx} = -\frac{\csc(xy)\cot(xy)\,y + \sec(x+y)\tan(x+y)}{\sec(x+y)\tan(x+y) + \csc(xy)\cot(xy)\,x}$$

E. Second Derivatives/Quotient Rule

Example 1.142

Evaluate $\frac{d^2y}{dx^2}$ at the point $\left(\sqrt{2}, \sqrt{2}\right)$ given that $x^2 + y^2 = r^2, where\ r\ is\ a\ constant$

Differentiate both sides of the given equation with respect to x:

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \underbrace{\frac{dy}{dx} = -\frac{x}{y}}_{Equation\ I}$$

Differentiate both sides of Equation I with respect to x using the Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$ in the RHS:

$$\frac{d^2y}{dx^2} = -\frac{(y)(1) - (x)\left(\frac{dy}{dx}\right)}{y^2} = \frac{x\frac{dy}{dx} - y}{y^2}$$

Substitute $\frac{dy}{dx} = -\frac{x}{y}$

$$= \frac{x\left(-\frac{x}{y}\right) - y}{y^2} = \frac{-x^2 - y^2}{y^3}$$

Substitute $x^2 = r^2 - y^2$:

$$= \frac{-(r^2 - y^2) - y^2}{y^3} = -\frac{r^2}{y^3}$$

Substitute $(x, y) = \left(\sqrt{2}, \sqrt{2}\right)$ in the above to evaluate the second derivative at the given point:

$$= -\frac{r^2}{\left(\sqrt{2}\right)^3} = -\frac{r^2}{2\sqrt{2}}$$

F. Tangents, Slopes and Normals

Example 1.143

$$x^2 + y^2 = r^2, where\ r\ is\ a\ constant$$

A. If the point $(-1,1)$ lies on the curve above, find the value of r.

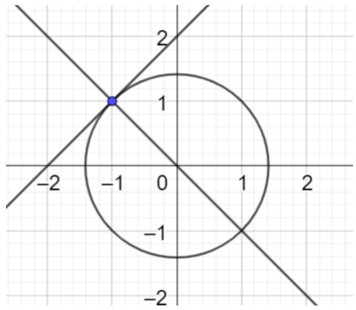
> B. Find the tangent and the normal to the curve above at the point $(-1,1)$ for the value of r that you calculated.

Part A

Substitute $(x, y) = (-1,1)$ in $x^2 + y^2 = r^2$:

$$(-1)^2 + 1^2 = r^2$$
$$r^2 = 2$$
$$r = \pm\sqrt{2}$$

Part B



$$x^2 + y^2 = r^2$$

Differentiate both sides of the given equation with respect to x:

$$2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

Tangent

Slope at $(-1,1)$:

$$Slope = -\frac{-1}{1} = 1$$

Substitute $(x_1, y_1) = (-1,1), m = 1$ into the point-slope form of the equation of a line:

$$y - y_1 = m(x - x_1)$$
$$y - 1 = 1(x + 1)$$
$$y = x + 2$$

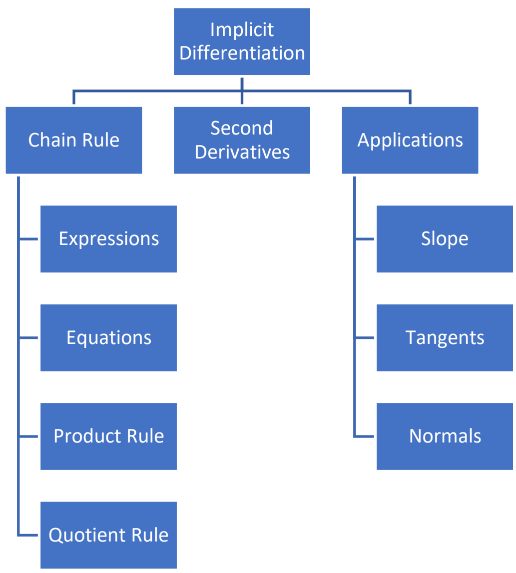
Normal

Slope of normal is the negative reciprocal of the slope of the tangent:

$$= -1$$

Substitute $(x_1, y_1) = (-1,1), m = -1$ into the point-slope form of the equation of a line:

$$y - y_1 = m(x - x_1)$$
$$y - 1 = -1(x + 1)$$
$$y = -x$$



1.7 Inverse and Logarithmic Derivatives

A. Derivative of an Inverse Function

1.144: Derivative of an Inverse Function

$$(f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Provided that

- $f(x)$ is invertible and differentiable
- $f'(x) \neq 0$ at any point in its domain

By definition, a function and its inverse "cancel" each other. Hence:

$$f\left(f^{-1}(x)\right) = x$$

Differentiate both sides of the above using the chain rule:

$$f'\left(f^{-1}(x)\right) \times (f^{-1})'(x) = 1 \Rightarrow (f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Example 1.145

$$y = f(x) = \ln x$$

A. Find f' using exponentials
B. Find f' using the formula for the derivative of an inverse function

Part A

$$e^{\ln x} = x$$

Differentiate both sides with respect to x:

$$e^{\ln x} \times (\ln x)' = 1$$

Solve for $(\ln x)'$:

$$(\ln x)' = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Part B

$$g(x) = e^x \Rightarrow (g^{-1})(x) = \ln x\,, g'(x) = e^x$$

$$(g^{-1})'(x) = \frac{1}{g'\left(g^{-1}(x)\right)}$$

$$(\ln x)'(x) = \frac{1}{g'(\ln x)} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Example 1.146

$$y = f(x) = \log_a x$$

A. Find f' using exponentials
B. Find f' using the formula for the derivative of an inverse function

Part B

$$a^{\log_a x} = x$$

Differentiate both sides with respect to x:

$$a^{\log_a x} \times \ln_a x \times (\log_a x)' = 1$$

Solve for $(\log_a x)'$:

$$(\log_a x)' = \frac{1}{a^{\log_a x} \times \ln_a x} = \frac{1}{x\ \ln_a x}$$

B. Natural Logarithm

1.147: Derivative of the Natural Logarithm

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Example 1.148: Log Rules

Some derivatives can be calculated conveniently by making use of the laws of logarithms.

A. $\ln x^\pi$
B. $\ln \theta^e$
C. $\ln 2x$
D. $\ln x^2$

Use the power rule for logarithms: $\log x^n = n \log x$

$$\frac{d}{dx}\ln x^\pi = \frac{d}{dx}\pi \ln x = \pi \frac{d}{dx}\ln x = \pi\left(\frac{1}{x}\right) = \frac{\pi}{x}$$

$$(\ln \theta^e)' = (e \ln \theta)' = e(\ln \theta)' = e\left(\frac{1}{\theta}\right) = \frac{e}{\theta}$$

$$\frac{d}{dx}(\ln 2x) = \frac{d}{dx}(\ln 2 + \ln x) = \frac{d}{dx}\ln 2 + \frac{d}{dx}\ln x = 0 + \frac{1}{x} = \frac{1}{x}$$

$$\frac{d}{dx}(\ln x^2) = \frac{d}{dx}(2\ln x) = 2\frac{d}{dx}\ln x = \frac{2}{x}$$

Example 1.149: Log Rules

A. $\frac{d}{dy}e^{\ln y}$
B. $\frac{d}{dx}\ln e^x$

$$e^{\ln y} = y \Rightarrow \frac{d}{dy}y = 1$$

$$\ln e^x = x \Rightarrow Derivative\ is\ 1$$

Example 1.150

$$\frac{d}{dx}(3\ln 2x^4)$$

Simplify the function that we wish to differentiate:

$$3\ln 2x^4 = 3(\ln 2 + \ln x^4) = 3(\ln 2 + 4\ln x) = 3\ln 2 + 12\ln x$$

Substitute the simplified version and find the derivative:

$$\frac{d}{dx}3\ln 2 + 12\ln x = 0 + \frac{12}{x} = \frac{12}{x}$$

Example 1.151: Product Rule

$$(fg) = f'g + fg'$$

A. $x \cdot \ln x$
B. $\sqrt{x} \cdot \ln x$

$$x \times (\ln x)' + (x)' \times \ln x = 1 + \ln x$$

$$\left(\sqrt{x}\right)' \ln x + \sqrt{x}(\ln x)' = \frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x}$$

Example 1.152: Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{\boldsymbol{g}f' - g'f}{\boldsymbol{g^2}}$$

A. $\frac{\ln x}{x}$
B. $\frac{\ln x}{\sqrt{x}}$

$$\frac{x \times (\ln x)' - \ln x\,(x)'}{x^2} = \frac{1 - \ln x}{x^2}$$

$$\frac{d}{dx}\left(\frac{\ln x}{\sqrt{x}}\right) = \frac{\left(\sqrt{x}\right)\left(\frac{1}{x}\right) - \left(\frac{1}{2\sqrt{x}}\right)(\ln x)}{\left(\sqrt{x}\right)^2} = \frac{\left(\frac{1}{\sqrt{x}}\right) - (\ln x)\left(\frac{1}{2\sqrt{x}}\right)}{x} = \frac{\frac{2 - \ln x}{2\sqrt{x}}}{x} = \frac{2 - \ln x}{2x^{\frac{3}{2}}}$$

1.153: Three Term Product Rule

$$(fgh)' = fgh' + fg'h + f'gh$$

$$(fgh)' = (fg \times h)' = (fg)' \times h + fg \times h' = (fg' + f'g)h + fgh' = fgh' + fg'h + f'gh$$

Example 1.154

Find the derivative with respect to suitable variables:

$$xe^x \ln x$$

Method I

$$(x)'e^x \ln x + x(e^x)' \ln x + xe^x(\ln x)'$$

Differentiate:

$$= e^x \ln x + xe^x \ln x + e^x$$

Factor out e^x:

$$= e^x(\ln x + x\ln x + 1)$$

Factor $\ln x$ from the first two terms:

$$= e^x[(\ln x)(1 + x) + 1]$$

Method II

C. Exponentials to Any Base

1.155: Derivative of a Power of any Base

$$\frac{d}{dx}a^x = a^x \ln a$$

Example 1.156

A. 3^x
B. b^x
C. e^{-x}

$$\frac{d}{dx}3^x = 3^x \ln 3$$
$$\frac{d}{dx}b^x = b^x \ln b$$
$$\frac{d}{dx}e^{-x} = \frac{d}{dx}(e^{-1})^x = (e^{-1})^x \ln(e^{-1}) = -e^{-x}\ln e = -e^{-x}$$

1.157: Derivative of a Power of any Base

If $u = f(x)$:

$$\frac{d}{dx}a^u = a^u \ln a \times \frac{du}{dx}$$

Example 1.158

A. 3^{2x}
B. $7^{\frac{x}{3}}$
C. 5^{x^2}
D. $7^{\frac{x}{x+1}}$

$$\frac{d}{dx}(3^{2x}) = (3^x \ln 3)(2)$$
$$\frac{d}{dx}\left(7^{\frac{x}{3}}\right) = 7^{\frac{x}{3}}$$
$$\frac{d}{dx}\left(5^{x^2}\right) = (5x^2 \ln x)(2x)$$
$$\frac{d}{dx}\left(7^{\frac{x}{x+1}}\right) = \left(7^{\frac{x}{x+1}}\ln 7\right)\left(\frac{(x+1)(1)-(x)(1)}{(x+1)^2}\right) = \left(7^{\frac{x}{x+1}}\ln 7\right)\left(\frac{1}{(x+1)^2}\right)$$

1.159: Derivative of Log to any Base

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$$

$$\frac{d}{dx}\log_a x = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a}\cdot\frac{d}{dx}\ln x = \frac{1}{\ln a}\times\frac{1}{x} = \frac{1}{x\ln a}$$

Example 1.160

Use the formula for log of any base to find:

A. $\frac{d}{dx}\ln x$
B. $\frac{d}{dx}e^x$

$$\frac{d}{dx}\ln x = \frac{d}{dx}\log_e x = \frac{1}{x\ln e} = \frac{1}{x\times 1} = \frac{1}{x}$$

$$\frac{d}{dx}e^x = e^x \ln e = e^x \times 1 = e^x$$

Example 1.161

Find the derivative with respect to x:

A. $\log_3 x$
B. $\log_5(x+1)$
C. $\log_b x$
D. c^x
E. $\pi \times c^x$
F. $e \times \log_z x$
G. c^{x+1}

$$\frac{d}{dx}\log_3 x = \frac{1}{x \ln 3}$$
$$\frac{d}{dx}\log_b x = \frac{1}{x \ln b}$$
$$\frac{d}{dx}c^x = c^x \ln c$$
$$\frac{d}{dx}\pi \times c^x = \pi c^x \ln c$$
$$\frac{d}{dx}e \times \log_z x = \frac{e}{x \ln z}$$
$$\frac{d}{dx}c^{x+1} = c^{x+1} \ln c$$

Example 1.162

$$y = \log_{\pi^3} e^x - \log_\pi \sqrt[3]{x}$$

Use log properties to simplify before we differentiate.
Use the change of base rule:

$$y = \frac{\log_e e^x}{\log_e \pi^3} - \frac{\log_e \sqrt[3]{x}}{\log_e \pi} = \frac{\ln e^x}{\ln \pi^3} - \frac{\ln \sqrt[3]{x}}{\ln \pi} = \frac{x}{3\ln \pi} - \frac{\ln x}{3 \ln \pi} = \left(\frac{1}{3 \ln \pi}\right)(x - \ln x)$$

$$\frac{dy}{dx} = \left(\frac{1}{3\ln \pi}\right)\left(1 - \frac{1}{x}\right) = \left(\frac{1}{3 \ln \pi}\right)\left(\frac{x-1}{x}\right) = \left(\frac{x-1}{x(3\ln \pi)}\right)$$

Example 1.163: Chain Rule

A. $\frac{d}{dx}5^{\log_2 x}$

$$\frac{d}{dx}5^{\log_2 x} = \left(5^{\log_2 x}\right)(\ln 5)\left(\frac{1}{x \ln 2}\right) = \frac{5^{\log_2 x} \times \log_2 5}{x}$$

D. Chain Rule

Example 1.164

Compare the two functions below. Are they the same? Are their derivatives the same? Are their derivatives found using the same properties?

A. $f = \ln\sqrt{x}, g = \sqrt{\ln x}$
B. $f = \ln x^2, g = (\ln x)^2$
C.

Part A

$$f' = \ln\sqrt{x} = \ln x^{\frac{1}{2}} = \frac{1}{2}\ln x \Rightarrow \frac{d}{dx}\frac{1}{2}\ln x = \frac{1}{2x}$$
$$g' = \frac{1}{2\sqrt{\ln x}} \times \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

Part B

$$f' = \frac{d}{dx}(\ln x^2) = \frac{d}{dx}(2\ln x) = 2\frac{d}{dx}\ln x = \frac{2}{x}$$
$$\frac{d}{dx}(\ln x)^2 = 2(\ln x) \times \frac{1}{x} = \frac{2\ln x}{x}$$

Example 1.165

A. $\frac{d}{dx}\ln(x^2 + x)$
B. $\sqrt[3]{\ln x}$
C. $(\ln x)^3$

$$\frac{d}{dx}[\ln(x^2 + x)] = \frac{1}{x^2 + x} \times (2x + 1) = \frac{2x + 1}{x^2 + x}$$
$$\frac{d}{dx}\sqrt[3]{\ln x} = \frac{d}{dx}(\ln x)^{\frac{1}{3}} = \frac{1}{3(\ln x)^{\frac{2}{3}}} \times \frac{1}{x} = \frac{1}{3x(\ln x)^{\frac{2}{3}}}$$
$$\frac{d}{dx}(\ln x)^3 = \frac{3(\ln x)^2}{x}$$

Example 1.166

A. $\ln(\sin x)$
B. $\ln\left(\tan\frac{x}{2}\right)$

$$\frac{d}{dx}\ln(\sin x) = \frac{\cos x}{\sin x} = \cot x$$
$$\frac{d}{dx}\ln\left(\tan\frac{x}{2}\right) = \frac{1}{\tan\frac{x}{2}} \times \sec^2\frac{x}{2} \times \frac{1}{2} = \frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}} \times \frac{1}{\cos^2\frac{x}{2}} = \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

Using $\sin 2\theta = 2\sin\theta\cos\theta$:

$$= \frac{1}{\sin x}$$

Example 1.167: Product Rule

$$\frac{d}{dx}\underbrace{\sin x}_{f}\underbrace{\ln(\sec x)}_{g}$$

$$\frac{d}{dx} = \underbrace{\cos x}_{f'}\underbrace{\ln(\sec x)}_{g} + \underbrace{\sin x}_{f} \cdot \underbrace{\frac{1}{\sec x} \times \sec x\tan x}_{g'} = \cos x\ln(\sec x) + \sin x\tan x$$

Example 1.168: Multi Step Chain Rule

$$\frac{d}{dx}e^{\cot\left(\ln\left(\csc\frac{x}{2}\right)\right)}$$

$$e^{\cot\left(\ln\left(\csc\frac{x}{2}\right)\right)}\cdot\left(-\csc^2\ln\left(\csc\frac{x}{2}\right)\right)\cdot\frac{1}{\csc\frac{x}{2}}\cdot\left(-\csc\frac{x}{2}\cot\frac{x}{2}\right)\cdot\frac{1}{2}$$
$$=\frac{1}{2}e^{\cot\left(\ln\left(\csc\frac{x}{2}\right)\right)}\cdot\left(\csc^2\ln\left(\csc\frac{x}{2}\right)\right)\left(\cot\frac{x}{2}\right)$$

Example 1.169

Find y' given that $y=\ln\left(\frac{e^x}{1+e^x}\right)$

Simplify the expression using $\log\frac{a}{b}=\log a-\log b$:

$$y=\ln e^x-\ln(1+e^x)=x\ln e-\ln(1+e^x)=x-\ln(1+e^x)$$

First, find the derivative of $z=\ln(1+e^x)$:

$$u=1+e^x\Rightarrow\frac{du}{dx}=e^x$$
$$\frac{dz}{du}=\frac{d}{du}\ln(1+e^x)=\frac{d}{dx}\ln u=\frac{1}{u}$$
$$\frac{dz}{dx}=\frac{dz}{du}\times\frac{du}{dx}=\frac{1}{u}\times e^x=\frac{e^x}{1+e^x}$$

Bringing it all together:

$$y'=1-\frac{e^x}{1+e^x}=\frac{1+e^x-e^x}{1+e^x}=\frac{1}{1+e^x}$$

Example 1.170

Find

$$\left.\frac{dy}{dx}\right|_{x=e^2}\ if\ y=\ln(x^3+x)$$

$$\frac{dy}{dx}=\frac{3x^2}{x^3+x}=\frac{3x}{x^2+1}$$
$$\left.\frac{dy}{dx}\right|_{x=e^2}=\frac{3e^2}{(e^2)^2+1}=\frac{3e^2}{e^4+1}$$

E. Logarithmic Differentiation

Example 1.171

Find the derivative of $y=x(x+1)$

A. Using the product rule
B. Using implicit differentiation

Part A

$$y=x(1)+(1)(x+1)=2x+1$$

Part B

Take the natural log of both sides of the given equality:

$$\ln y = \ln x + \ln(x+1)$$

Differentiate both sides with respect to x:

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{1}{x} + \frac{1}{x+1}$$

Solve the above for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y\left(\frac{1}{x} + \frac{1}{x+1}\right)$$

Substitute $y = x(x+1)$:

$$\frac{dy}{dx} = x(x+1)\left(\frac{1}{x} + \frac{1}{x+1}\right) = \frac{x(x+1)}{x} + \frac{x(x+1)}{x+1} = x+1+x = 2x+1$$

Example 1.172

$$\text{Find } \frac{dy}{dx} \text{ if } y = \sqrt{\frac{(x^2-1)^5}{(x^3+2)^3}}$$

Convert the square root into an exponent and combine:

$$y = \frac{(x^2-1)^{\frac{5}{2}}}{(x^3+2)^{\frac{3}{2}}}$$

Take the natural log of both sides:

$$\ln y = \ln\left[\frac{(x^2-1)^{\frac{5}{2}}}{(x^3+2)^{\frac{3}{2}}}\right]$$

Use the quotient rule from logarithms:

$$\ln y = \ln(x^2-1)^{\frac{5}{2}} - \ln(x^3+2)^{\frac{3}{2}}$$

Use the power rule from logarithms:

$$\ln y = \frac{5}{2}\ln(x^2-1) - \frac{3}{2}\ln(x^3+2)$$

Differentiate both sides implicitly with respect to x:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \left(\frac{5}{2}\right)\left(\frac{2x}{x^2-1}\right) - \frac{3}{2}\left(\frac{3x^2}{x^3+2}\right)$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y\left[\left(\frac{5x}{x^2-1}\right) - \frac{9x^2}{2(x^3+2)}\right]$$

Substitute $y = \sqrt{\frac{(x^2-1)^5}{(x^3+2)^3}}$

$$\frac{dy}{dx} = \sqrt{\frac{(x^2-1)^5}{(x^3+2)^3}}\left[\left(\frac{5x}{x^2-1}\right) - \frac{9x^2}{2(x^3+2)}\right]$$

Example 1.173

Can you find the derivative of $y = x^x$ using the

A. Power Rule
B. Log Rule

The power rule is

$$\frac{d}{dx}x^n = nx^{n-1}, n\ is\ a\ constant$$

Since x is not a constant, the power rule is not applicable.

$$\frac{d}{dx}a^x = a^x \ln a\,, a\ is\ a\ constant$$

Since x is not a constant, the log rule is not applicable.

Example 1.174

Find the derivative of $y = x^x$.

$$(x \ln x)' = x \times (\ln x)' + (x)' \times \ln x = 1 + \ln x$$

Part A

$$y = x^x = e^{\ln x^x} = e^{x \ln x}$$

Hence, we get:

$$y = e^{x \ln x}$$

Differentiate both sides, and use the chain rule on the RHS:

$$\frac{dy}{dx} = \left(e^{x \ln x}\right)(x \ln x)' = (x^x)(1 + \ln x)$$

Part B: Using Implicit Differentiation

$$y = x^x$$

Take the natural log of both sides:

$$\ln y = \underbrace{x}_{f}\underbrace{\ln x}_{g}$$

Differentiate both sides implicitly with respect to x:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln x + 1$$

$$\frac{dy}{dx} = y(\ln x + 1) = (x^x)(\ln x + 1)$$

Example 1.175

A. $y = x^{x^x}$
B. $y = (2x + 3a)^{3x+2a}, a\ is\ constant$

Part A

Take the natural log of both sides:

$$\ln y = \underbrace{x^x}_{f}\underbrace{\ln x}_{g}$$

Differentiate both sides with respect to x:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \underbrace{(x^x)(\ln x + 1)}_{f'}\underbrace{\ln x}_{g} + \underbrace{x^x}_{f}\underbrace{\frac{1}{x}}_{g'}$$

Factor out x^x:

$$\frac{dy}{dx} = y(x^x)\left[(\ln x + 1)\ln x + \frac{1}{x}\right]$$

Substitute $y = x^{x^x}$

$$\frac{dy}{dx} = \left(x^{x^x}\right)(x^x)\left[(\ln x)^2 + \ln x + \frac{1}{x}\right]$$

Part B

$$y = (2x + 3a)^{3x+2a}$$

Take the natural log of both sides:

$$\ln y = (3x + 2a)\ln(2x + 3a)$$

Differentiate both sides with respect to x:

$$\frac{1}{y}\cdot\frac{dy}{dx} = (3)\ln(2x + 3a) + \frac{2(3x + 2a)}{2x + 3a}$$

$$\frac{dy}{dx} = y\left[(3)\ln(2x + 3a) + \frac{2(3x + 2a)}{2x + 3a}\right]$$

$$\frac{dy}{dx} = \left((2x + 3a)^{3x+2a}\right)\left[(3)\ln(2x + 3a) + \frac{2(3x + 2a)}{2x + 3a}\right]$$

Example 1.176

Find $\frac{dy}{dx}$ given that

$$x^y = y^x$$

Take the natural log of both sides:

$$y\ln x = x\ln y$$

Differentiate both sides with respect to x:

$$y\left(\frac{1}{x}\right) + \left(\frac{dy}{dx}\right)(\ln x) = x\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right) + \ln y$$

Collate all $\frac{dy}{dx}$ terms on the LHS:

$$\frac{dy}{dx}\left[\ln x - \frac{x}{y}\right] = \ln y - \frac{y}{x}$$

Add the fractions:

$$\frac{dy}{dx}\left[\frac{y(\ln x) - x}{y}\right] = \frac{x(\ln y) - y}{x}$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)\left(\frac{x(\ln y) - y}{y(\ln x) - x}\right)$$

$$\frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}} = \frac{\frac{x\ln y - y}{x}}{\frac{y\ln x - x}{y}} = \frac{x\ln y - y}{x} \times \frac{y}{y\ln x - x} =$$

Example 1.177

$$y = x^{\ln x}$$

$$\ln y = \ln x^{\ln x}$$
$$\ln y = \ln x \cdot \ln x$$

$$\ln y = (\ln x)^2$$

Example 1.178

$$y = \ln(x^2 - 16)$$

$$y = \underbrace{\ln\left(\underbrace{x^2 - 16}_{Inner\ f}\right)}_{Outer\ f}$$

$$y = \frac{1}{x^2 - 16} \times (x^2 - 16)' = \frac{2x}{x^2 - 16}$$

1.8 Inverse Trigonometric Derivatives

A. Derivative of $sin^{-1}x$ and $cos^{-1}x$

The derivatives of the inverse trigonometric functions can be calculated using a variety of methods. An important thing to note is that the derivative of an inverse trigonometric function does not involve trigonometry, but has only algebraic terms.

1.179: Derivative of $\sin^{-1} x$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

Method I: Implicit Differentiation

Let

$$y = \sin^{-1} x \Rightarrow \sin y = x$$

Differentiate both sides implicitly with respect to x:

$$\cos y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$

Method II: Using the Chain Rule

Differentiate both sides of the identity $\sin(\sin^{-1} x) = x$ using the chain rule:

$$\cos(\sin^{-1} x)\frac{d}{dx}(\sin^{-1} x) = 1 \Rightarrow \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\cos(\sin^{-1} x)}$$

Method III: Using the formula for the derivative of an inverse function

Make the substitutions $f(x) = \sin x \Rightarrow f'(x) = \cos x, f^{-1}(x) = \sin^{-1} x$ in the formula:

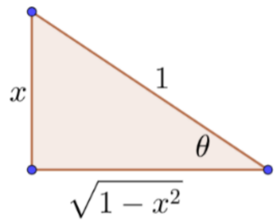
$$(f^{-1})'(x) = \frac{1}{f'\big(f^{-1}(x)\big)} = \frac{1}{\cos(\sin^{-1} x)}$$

In all cases above, we need to find the value of the denominator, which is a trigonometric function of an inverse trigonometric function. We accomplish this using a combination of two techniques:

- a change of variable
- the introduction of a reference triangle.

Start with the change of variable. Let the angle given by $\sin^{-1} x$ be θ. That is:

$$\theta = \sin^{-1} x \Rightarrow \sin\theta = \frac{x}{1} = \frac{opp}{hyp}$$



Draw a reference triangle, and note that by the Pythagorean theorem, the side opposite θ:

$$Adj = \sqrt{Hyp^2 - Opp^2} = \sqrt{1-x^2}$$

Then:

$$\cos\theta = \frac{adj}{hyp} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

And finally:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-x^2}}$$

1.180: Chain Rule with $\sin^{-1} x$

If $u = f(x)$, then

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \times \frac{du}{dx}$$

Example 1.181

A. $\frac{d}{dx}\sin^{-1}(2x^3)$
B. $\frac{d}{dx}\sin^{-1}(\sqrt{x})$
C. $\frac{d}{dx}\ln(\sin^{-1} 2x)$

Part A

$$\frac{d}{dx} sin^{-1}(2x^3) = \frac{1}{\sqrt{1-(2x^3)^2}} \times 6x^2 = \frac{6x^2}{\sqrt{1-4x^6}}$$

Part B

$$\frac{d}{dx}\sin^{-1}(\sqrt{x}) = \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(1-x)}}$$

Part C

$$\frac{d}{dx}\ln(\sin^{-1} 2x) = \frac{1}{\sin^{-1} 2x} \times \frac{1}{\sqrt{1-4x^2}} \times 2 = \frac{2}{(\sin^{-1} 2x)\left(\sqrt{1-4x^2}\right)}$$

Example 1.182

$$\frac{1}{2}t\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t$$

Using the product rule

$$\frac{d}{dt}\left(\frac{1}{2}t\sqrt{1-t^2}\right) = \frac{1}{2}\sqrt{1-t^2} + \frac{1}{2}t \cdot \frac{-2t}{2\sqrt{1-t^2}} = \frac{1}{2}\sqrt{1-t^2} + \frac{-t^2}{2\sqrt{1-t^2}} = \frac{1-2t^2}{2\sqrt{1-t^2}}$$

$$\frac{d}{dt}\left(\frac{1}{2}\sin^{-1} t\right) = \frac{1}{2\sqrt{1-t^2}}$$

$$\frac{1-2t^2}{2\sqrt{1-t^2}}+\frac{1}{2\sqrt{1-t^2}}=\frac{2-2t^2}{2\sqrt{1-t^2}}=\frac{1-t^2}{\sqrt{1-t^2}}=\sqrt{1-t^2}$$

1.183: Inverse Trig Identity

$$\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x$$

1.184: Derivative of $\cos^{-1}x$

$$\frac{d}{dx}\cos^{-1}x=-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1}x=\frac{d}{dx}\left(\frac{\pi}{2}-\sin^{-1}x\right)=-\frac{1}{\sqrt{1-x^2}}$$

Example 1.185

A. $\frac{d}{dx}\cos^{-1}(\ln x)$

$$\frac{d}{dx}\cos^{-1}(\ln x)=-\frac{1}{\sqrt{1-(\ln x)^2}}\times\frac{1}{x}=-\frac{1}{x\sqrt{1-(\ln x)^2}}$$

B. Derivatives of $tan^{-1}x$ and $cot^{-1}x$

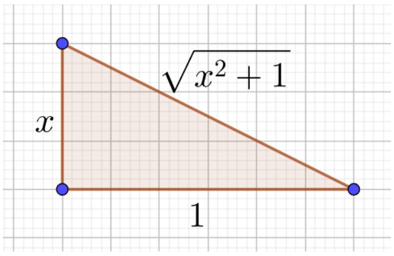
1.186: Derivative of $\tan^{-1}x$

$$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$$
$$\frac{d}{dx}(\tan^{-1}u)=\frac{1}{1+u^2}\times\frac{du}{dx}$$

$$\tan y=x$$
$$\frac{dy}{dx}=\frac{1}{\sec^2 y}$$

From the reference triangle:

$$\tan y=\frac{x}{1}\Rightarrow\frac{1}{\sec^2 y}=\cos^2 y=\left(\frac{1}{\sqrt{x^2+1}}\right)^2=\frac{1}{1+x^2}$$



Example 1.187

A. $\frac{d}{dx}\tan^{-1}(2\sqrt{x})$

B. $\frac{d}{dx}\tan^{-1}\left(e^{\frac{x}{2}}\right)$

$$\frac{d}{dx}\tan^{-1}(2\sqrt{x})=\frac{1}{1+4x}\times\frac{1}{\sqrt{x}}=\frac{1}{\sqrt{x}(1+4x)}$$
$$\frac{d}{dx}\tan^{-1}\left(e^{\frac{x}{2}}\right)=\frac{1}{1+e^x}\times e^{\frac{x}{2}}\times\frac{1}{2}=\frac{e^{\frac{x}{2}}}{2(1+e^x)}$$

1.188: Inverse Trig Identity

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

1.189: Derivative of $\cot^{-1} x$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}(\cot^{-1} u) = -\frac{1}{1+u^2} \times \frac{du}{dx}$$

$$\frac{d}{dx}(\cot^{-1} x) = \frac{d}{dx}\left(\frac{\pi}{2} - \tan^{-1} x\right) = -\frac{1}{1+x^2}$$

Example 1.190

A. $\frac{d}{dt}\cot^{-1}\left(e^{\sqrt{t}}\right)$
B. $\frac{d}{dx}\cot^{-1}(x \cdot \ln x)$

Part A

$$\left(e^{\sqrt{t}}\right)^2 = e^{2\sqrt{t}}$$
$$\frac{d}{dt}\cot^{-1}\left(e^{\sqrt{t}}\right) = -\frac{1}{1+e^{2\sqrt{t}}} \times e^{\sqrt{t}} \times \frac{1}{2\sqrt{t}} = -\frac{e^{\sqrt{t}}}{2\sqrt{t}\left(1+e^{2\sqrt{t}}\right)}$$

Part B

$$\frac{d}{dx}\cot^{-1}(x \cdot \ln x) = -\frac{1}{1+(x \cdot \ln x)^2} \times (1+\ln x) = -\frac{1+\ln x}{1+(x \cdot \ln x)^2}$$

C. Derivatives of $sec^{-1}x$ and $csc^{-1} x$

1.191: Derivative of $\sec^{-1} x$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}} \times \frac{du}{dx}$$

Example 1.192

A. $\frac{d}{dx}\sec^{-1}\frac{x}{2}$
B. $\frac{d}{dx}\sec^{-1} x^2$

$$\frac{d}{dx}\sec^{-1}\frac{x}{2} = \frac{1}{2\left|\frac{x}{2}\right|\sqrt{\frac{x^2}{4}-1}}$$
$$\frac{d}{dx}\sec^{-1} x^2 = \frac{1}{|x^2|\sqrt{x^4-1}} \times 2x = \frac{2}{x\sqrt{x^4-1}}$$

1.193: Inverse Trig Identity

$$\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$$

1.194: Derivative of $\csc^{-1} x$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\csc^{-1} x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sec^{-1} x\right) = -\frac{1}{|x|\sqrt{x^2-1}}$$

Example 1.195

A. $\frac{d}{dx}\left(\csc^{-1}\frac{e}{2}\right)$

$$\frac{d}{dx}\left(\csc^{-1}\frac{e}{2}\right) = 0$$

D. Further Examples

1.196: Summary of Inverse Trigonometric Derivatives

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}, \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

- All of the cofunction derivatives have a minus sign in front, and the expression is otherwise the same as the corresponding function.

Example 1.197: Quotient Rule

Find the derivative of:

$$\frac{d}{dx}\sin^{-1}\left(\frac{x-1}{x+1}\right)$$

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}$$

$$\left(\sin^{-1}\frac{x-1}{x+1}\right)' = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} \times \left(\frac{x-1}{x+1}\right)'$$

$$u = \left(\frac{x-1}{x+1}\right)' = \frac{(x+1)(1)-(x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$v = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} = \frac{1}{\sqrt{1-\frac{x^2-2x+1}{x^2+2x+1}}} = \frac{1}{\sqrt{\frac{4x}{(x+1)^2}}} = \frac{1}{\frac{2\sqrt{x}}{|x+1|}} = \frac{|x+1|}{2\sqrt{x}}$$

$$uv = \frac{2}{(x+1)^2} \cdot \frac{|x+1|}{2\sqrt{x}} = \frac{1}{\sqrt{x}(x+1)}$$

1.9 Graphical Transformations

A. Shifts

Recall that, by definition, the derivative of $f(x)$ is $f'(x)$. That is:

$$\frac{d}{dx}f(x) = f'(x)$$

1.198: Vertical Shift

$$\frac{d}{dx}f(x) + c \Rightarrow f'(x)$$

Algebraic Method

Use the sum rule:

$$\frac{d}{dx}f(x) + \frac{d}{dx}c = f'(x) + 0 = f'(x)$$

Graphical Method

Moving a function up or down does not change the slope at any point of the function.
Hence, the derivative remains unchanged.

Example 1.199

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}e^x + 2 = e^x$$

1.200: Vertical Scale

$$\frac{d}{dx}kf(x) = k\frac{d}{dx}f(x)$$

Algebraic Method

This is true by the constant multiple rule.

Graphical Method

Scaling a function vertically by a factor of k also changes the slope by a factor of k, and hence the property is true.

1.201: Horizontal Scaling

$$f(kx) = k\frac{d}{dx}f(x)$$

Graphical Method

Compressing a function horizontally by a factor of k, increases the rate of change, and hence the slope by a factor of k.

1.202: Horizontal Shift

$$f(x+k)$$

A horizontal shift is equivalent to replacing x with $x+k$. In general, the derivative does not remain the same.

Example 1.203

A. Show by counterexample that

$$f(x) = x^2 \Rightarrow f'(x) = 2x$$

Shift $f(x)$ by 1 to the right to get $g(x)$

$$g(x) = (x-1)^2 = x^2 - 2x + 1 \Rightarrow g'(x) = 2x - 2$$

Example 1.204

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}e^{x+c} = e^{x+c}$$

Example 1.205

$$y = \sin(\theta) \Rightarrow y = \cos\theta$$

$$y = \sin(2\pi + \theta) = \sin(\theta) \Rightarrow \frac{dy}{dx} = \cos\theta$$

1.10 Differentiation w.r.t. another function

A. Basics

1.206: Differentiation with respect to another function

$$\frac{dy}{du} = \frac{\frac{dy}{dx}}{\frac{du}{dx}}$$

From the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Which can be rearranged to:

$$\frac{dy}{du} = \frac{\frac{dy}{dx}}{\frac{du}{dx}}$$

Example 1.207
Find the derivative of $f(x) = x$ with respect to $g(x) = x^2$.

Substitute $y = x, u = x^2$ in the formula:

$$\frac{dy}{du} = \frac{\frac{dy}{dx}}{\frac{du}{dx}} = \frac{d}{d(x^2)}x = \frac{\frac{d}{dx}x}{\frac{d(x^2)}{dx}} = \frac{1}{2x}$$

1.208: Manipulation

Example 1.209
Find the derivative of $f(x) = x$ with respect to $g(x) = x^2$.

Let $f(x) = y = x \Rightarrow y^2 = x^2$. Then:

$$\frac{d(y^2)}{d(x^2)} = \frac{d(x^2)}{d(x^2)}$$

Differentiating implicitly:

$$2y \cdot \frac{dy}{d(x^2)} = 1 \Rightarrow \frac{dy}{d(x^2)} = \frac{1}{2y}$$

Change back to $f(x)$ and $g(x)$:

$$\frac{d[f(x)]}{d\big(g(x)\big)} = \frac{1}{2x}$$

Example 1.210

B. Further Properties

1.211: Getting the same function back

2. Using Derivatives

2.1 Slope, Tangents and Normals

A. Slope

2.1: Slope Interpretation of Derivative
The derivative of a function at a point represents the slope of the function at that point.

$$f'(x) \text{ is the slope of } f(x) \text{ at } x$$

Example 2.2
Find the interval where the function $f(x) = 2x^3 + 15x^2 + 36x$ has positive slope.

$$f' = 6x^2 + 30x + 36$$

$$6(x^2 + 5x + 6)$$
$$6(x+2)(x+3)$$

$$(-\infty, -3) \cup (-2, \infty)$$

Example 2.3
Find the values where the slope of $y = \frac{1}{4}x^2(x^2 - 1)$ is positive.

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{4}x^2(x^2-1)\right] = \frac{1}{4}(4x^3 - 2x)$$

The slope of y is given by $\frac{dy}{dx}$. Hence, we need the slope to be positive:

$$\frac{dy}{dx} > 0$$

Substitute $\frac{dy}{dx}$ as calculated above:

$$\frac{1}{4}(4x^3 - 2x) > 0$$
$$4x^3 - 2x > 0$$
$$x(2x^2 - 1) > 0$$

To find the critical points, equate the LHS to zero:

$$x(2x^2 - 1) = 0 \Rightarrow x = 0 \text{ } OR \text{ } 2x^2 - 1 = 0 \Rightarrow x = \pm\sqrt{\frac{1}{2}}$$

$$Critical\ Points: x \in \left\{-\sqrt{\frac{1}{2}}, 0, \sqrt{\frac{1}{2}}\right\}$$

The critical points divide the real number into three distinct parts:

$\left(-\infty, -\sqrt{\frac{1}{2}}\right)$	$\left(-\sqrt{\frac{1}{2}}, 0\right)$	$\left(0, \sqrt{\frac{1}{2}}\right)$
$-ve$	$+ve$	$-ve$

Example 2.4

$$y = ax + \frac{b}{x^2}$$

Find (a, b) if the graph of y has slope 3 at the point (2,0).

Use the value of the curve at the known point:

$$ax + \frac{b}{x^2}\bigg|_{x=2} = 0 \Rightarrow 2a + \frac{b}{4} = 0 \Rightarrow \underbrace{8a + b = 0}_{\textit{Equation I}}$$

Use the value of the derivative of the curve at the known point:

$$\frac{dy}{dx}\bigg|_{x=2} = 3 \Rightarrow a - \frac{2b}{x^3}\bigg|_{x=2} = 3 \Rightarrow a - \frac{2b}{8} = 3 \Rightarrow 4a - b = 12 \Rightarrow \underbrace{b = 4a - 12}_{\textit{Equation II}}$$

Substitute the value of b from Equation II in Equation I:

$$8a + 4a - 12 = 0 \Rightarrow 12a = 12 \Rightarrow a = 1$$

Substitute $a = 1$ in Equation I:

$$8a + b = 0 \Rightarrow 8 + b = 0 \Rightarrow b = -8$$

$$(a, b) = (-1,8)$$

B. Tangents

Example 2.5

$$y = 3x^3 - 2x^2 + 2x$$

Find the coordinates of the point where the tangent to the curve at $x = 1$ meets the curve again.

$$Slope\ of\ Tangent = \frac{dy}{dx}\bigg|_{x=2} = (9x^2 - 4x + 2)_{x=2} = 7$$

$$y(1) = 3(1)^3 - 2(1)^2 + 2(1) = 3$$

Substitute $m = 7, (x, y) = (1,3)$ in the slope point form $y - y_1 = m(x - x_1)$ of the equation of a line:

$$y - 3 = 7(x - 1)$$

$$y = 7x - 4$$

To find where the tangent to the curve intersects the curve again, equate the two:

$$3x^3 - 2x^2 + 2x = 7x - 4$$

Use the Remainder Theorem:

$$P(1) = 3x^3 - 2x^2 - 5x + 4 = 3(1)^3 - 2(1)^2 - 5(1) + 4 = 0 \Rightarrow x - 1\ is\ \ a\ factor$$

Factor:

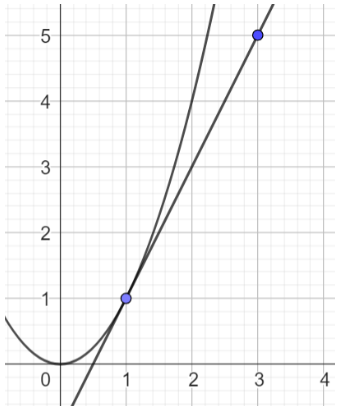
$$(x - 1)(3x^2 + x - 4) = 0$$

$$(x - 1)(3x + 4)(x - 1) = 0$$

$$x = 1 \; Or \; x = -\frac{4}{3}$$

C. Multiple Tangents

Example 2.6
Find the tangents to the parabola $y = x^2$ from the point (3,5).



The equation of a line with slope m that passes through (3,5) is

$$y - 5 = m(x - 3)$$

Since the tangent passes through a point on the parabola, its slope must match the slope of the parabola at that point.

Substitute $y = x^2 \Rightarrow Slope = m = \frac{dy}{dx} = 2x$

$$y - 5 = 2x(x - 3)$$
$$y = 2x^2 - 6x + 5$$

Since the parabola and the equation pass through the same point, we must have

$$2x^2 - 6x + 5 = x^2$$
$$x^2 - 6x + 5 = 0$$
$$(x - 5)(x - 1) = 0$$
$$x \in \{1,5\}$$
$$m \in \{2,10\}$$

Equations are:

$$y - 5 = 2(x - 3)$$
$$y - 5 = 10(x - 3)$$

Example 2.7: Multiple Tangents
Find the equation of tangents to the curve $y = x^3 + 2x - 4$ which are perpendicular to the line $x + 14y - 3 = 0$. **(CBSE 2016)**

$$x + 14y - 3 = 0$$
$$14y = -x + 3$$
$$y = -\frac{x}{14} + \frac{3}{14}$$
$$m_1 = -\frac{1}{14}$$

Differentiate

$$m_2 = y' = 3x^2 + 2$$

Since the tangent is perpendicular to the line, the product of the slopes must be -1:

$$m_1 m_2 = -1$$
$$\left(-\frac{1}{14}\right)(3x^2 + 2) = -1$$
$$3x^2 + 2 = 14$$
$$3x^2 = 12$$
$$x^2 = 4$$

$$x = \pm 2$$

When $x = 2$

$$y(2) = 2^3 + (2)2 - 4 = 8 \Rightarrow (x_1, y_1) = (2,8)$$
$$y(-2) = (-2)^3 + (-2)2 - 4 = -16 \Rightarrow (x_2, y_2) = (-2,16)$$

$$y'(2) = 3(2)^2 + 4 = 14$$
$$y'(-2) = 3(-2)^2 + 4 = 14$$

$$y - y_1 = m(x - x_1)$$
$$y - 8 = 14(x - 2)$$
$$y - 16 = 14(x + 2)$$

Example 2.8

Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point. (CBSE 2011)

$$y' = y$$
$$3x^2 = x^3$$
$$x^3 - 3x^2 = 0$$
$$x^2(x - 3) = 0$$
$$x \in \{0,3\}$$

$$x = 0 \Rightarrow y = 0 \Rightarrow (0,0)$$
$$x = 3 \Rightarrow y = 27 \Rightarrow (3,27)$$

D. Normals

2.9: Normal at a point

The line perpendicular to the tangent at a point is called the normal.

Example 2.10

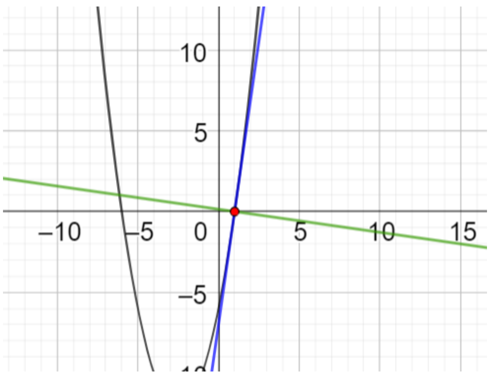
Given $f(x) = x^2 + 5x - 6$, find the

A. equation of the tangent to the function at $x = 1$.
B. equation of the normal to the function at $x = 1$.
C. find the coordinates of the point(s) where the tangent is horizontal.

Part A

We can find the coordinates of a point on the line by evaluating the function at $x = 1$:

$$f(1) = 1^2 + 5 - 6 = 0 \Rightarrow (x, y) = (1,0)$$



Find the derivative and use it to calculate the slope at $x = 1$:

$$f' = 2x + 5 \Rightarrow m = f'(1) = 2 + 5 = 7$$

Since know $(x, y) = (1,0)$ and $m = 7$, we can find the $y - intercept$ by substituting in the equation of a line:

$$y = mx + c \Rightarrow 0 = 7 + c \Rightarrow c = -7$$

Finally, we can substitute $m = 7, c = -7$ in $y = mx + c$:

$$y = 7x - 7$$

Part B

From the above, we know:

$$Slope\ of\ Tangent = 7 \Rightarrow Slope\ of\ Normal = -\frac{1}{7}$$

Substitute $(x, y) = (1,0)$ and $m = -\frac{1}{7}$ in the equation of a line:

$$y = mx + c \Rightarrow 0 = -\frac{1}{7} + c \Rightarrow c = \frac{1}{7}$$

Substitute $m = -\frac{1}{7}, c = \frac{1}{7}$ in $y = mx + c$:

$$y = -\frac{1}{7}x + \frac{1}{7}$$

Part C

$$2x + 5 = 0 \Rightarrow x = -\frac{5}{2}$$

2.11: Vertex of a Parabola

The minimum/maximum of a parabola $y = ax^2 + bx + c$ occurs when the tangent to the parabola is horizontal, and is given by:

$$-\frac{b}{2a}$$

The slope function of the parabola is given by:

$$y' = 2ax + b$$

For the tangent to be horizontal, the slope must be zero:

$$2ax + b = 0 \Rightarrow x = -\frac{b}{2a}$$

Example 2.12:

Is it possible for a parabola to not have a horizontal tangent?

Horizontal Tangents

Tangents from a point

No Tangents from a point

Example 2.13

Find the equation of the normal to the curve $y = x^3 - 8$ when $x = 2$.

Find the point on the curve when $x = 2$:

$$y(2) = 2^3 - 8 = 8 - 8 = 0$$

The derivative gives you the slope of the curve at a point. Hence, evaluate the derivative at $x = 2$:

$$Slope\ of\ Tangent = \left.\frac{dy}{dx}\right|_{x=2} = 3x^2|_{x=2} = 3(2)^2 = 12$$

Since the normal and the tangent are perpendicular to each other, their slopes are negative reciprocals of each other.

$$Slope\ of\ Normal = -\frac{1}{Slope\ of\ Tangent} = -\frac{1}{12}$$

Substitute $m = -\frac{1}{12}, (x, y) = (2,0)$ in the slope point form $y - y_1 = m(x - x_1)$ of the equation of a line:

$$y - 0 = -\frac{1}{12}(x - 2)$$

$$y = -\frac{1}{12}x + \frac{1}{6}$$

Example 2.14

1: Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$. Also, write the equation of the normal to the curve at the point of contact. **(CBSE 2019)**

Rearrange the line in slope intercept form to identify the slope:

$$4x - 2y + 5 = 0 \Rightarrow y = \frac{4}{2}x + \frac{5}{2} \Rightarrow Slope = m = \frac{4}{2} = 2$$

Calculate the derivative, and note the derivative is also the slope:

$$Slope = \frac{dy}{dx} = \frac{3}{2\sqrt{3x-2}}$$

At the point of contact of the tangent, the two values of the slope must be equal:

$$\frac{3}{2\sqrt{3x_1 - 2}} = 2 = 2$$

$$3 = 4\sqrt{3x_1 - 2}$$

$$9 = 16(3x_1 - 2)$$

$$x_1 = \frac{41}{48}$$

$$y_1 = \sqrt{3x_1 - 2} = \sqrt{3\left(\frac{41}{48}\right) - 2} = \frac{3}{4}$$

$$(x_1, y_1) = \left(\frac{41}{48}, \frac{3}{4}\right)$$

Tangent:

$$48x - 24y = 23$$

$$m_2 = -\frac{1}{2}$$

Equation of the Normal:

$$96y + 48x - 113 = 0$$

Example 2.15: Implicit Differentiation

10: Find the equations of the tangent and the normal to the curves $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(\sqrt{2}a, b)$. **(CBSE 2014)**

E. Trigonometric Functions

Example 2.16

Find the tangent to the curve $y = \sec x$ at $(x, y) = \left(\frac{\pi}{3}, 2\right)$

$$y' = \sec x \tan x$$

$$m = \sec x \tan x = \sec\frac{\pi}{3}\tan\frac{\pi}{3} = (2)(\sqrt{3}) = 2\sqrt{3}$$

$$y - y_1 = m(x - x_1)$$
$$y - 2 = 2\sqrt{3}\left(x - \frac{\pi}{3}\right)$$
$$y = 2\sqrt{3}x + \frac{6 - 2\pi\sqrt{3}}{3}$$

Example 2.17

$$f(x) = \sin x$$

A. Find the equation of the tangent when $x = 0$.
B. Find the values of x for which the function has a horizontal tangent.

Part A

$$x = 0 \Rightarrow f(0) = \sin(0) = 0$$

Hence, the coordinates of the point (x, y) on the function at $x = 0$ are

$$(x, y) = (0,0) \Rightarrow y - intercept = 0$$

$$f' = \cos x$$
$$f'(0) = \cos(0) = 1$$

Substitute $m = 1, c = 0$ in $y = mx + c$

$$y = x$$

Part B

Slope of a horizontal line is zero. Hence:

$$f'(0) = 0$$
$$\cos x = 0$$
$$x \in \left\{\frac{\pi}{2} + 2\pi k, k \in \mathbb{Z}\right\}$$

Example 2.18

$$f(x) = \sin x + \cos x$$

A. Find the equation of the tangent when $x = \frac{\pi}{2}$.
B. Find the values of x for which the function has a horizontal tangent.

Part A

$$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1 + 0 = 1$$

A point that lies on the line is:

$$(x, y) = \left(\frac{\pi}{2}, 1\right)$$

$$f' = \cos x - \sin x$$

$$m = f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} - \sin\frac{\pi}{2} = 0 - 1 = -1$$

Substitute $m = -1, (x, y) = \left(\frac{\pi}{2}, 1\right)$ in $y = mx + c$:

$$1 = (-1)\left(\frac{\pi}{2}\right) + c$$

$$1 + \frac{\pi}{2} = c$$

Hence, the equation of the tangent is:

$$y = -x + 1 + \frac{\pi}{2}$$

Part B

$$f' = \cos x - \sin x = 0$$

$$\cos x = \sin x$$

$$x \in \left\{\frac{\pi}{4} + \pi k, k \in \mathbb{Z}\right\}$$

F. Logarithmic Functions

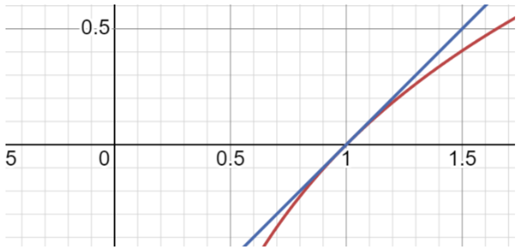
Example 2.19

Given $f(x) = \ln x$ find the equation of the

A. Tangent when $x = 1$.

B. Normal when $x = 1$.

Tangent:



$$f(1) = \ln 1 = 0 \Rightarrow (x, y) = (1,0)$$

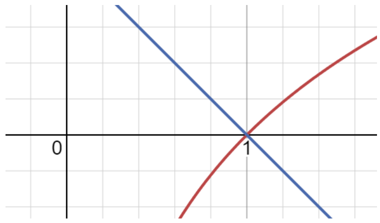
$$\frac{dy}{dx} = \frac{1}{x}$$

$$m = \left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{x} = \frac{1}{1} = 1$$

$$0 = 1 + c \Rightarrow c = -1$$

$$Equation\ of\ Tangent: y = x - 1$$

Normal:



$$y = -x + 1$$

Example 2.20

Find the area of ΔPQR if P is the point on the curve $y = 1 + \tan 3x$ with x value $\frac{\pi}{12}$, Q is the point where $x = \frac{\pi}{12}$ meets the x −axis, and R is the point where the normal from P intersects the x −axis.

$$P = \left(\frac{\pi}{12}, 2\right), \qquad Q = \left(\frac{\pi}{12}, 0\right)$$

$$Slope\ of\ Tangent = \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{12}} = 3\sec^2 3x\big|_{x=\frac{\pi}{12}} = 3\sec^2\frac{\pi}{4} = 3\left(\sqrt{2}\right)^2 = 6$$

$$Slope\ of\ Normal = -\frac{1}{6}$$

Equation of the normal $y - y_1 = m(x - x_1)$:

$$y - 2 = -\frac{1}{6}\left(x - \frac{\pi}{12}\right)$$

Substitute $y = 0$ to find the $x - intercept$:

$$-2 = -\frac{1}{6}\left(x - \frac{\pi}{12}\right) \Rightarrow 12 = x - \frac{\pi}{12} \Rightarrow x = 12 + \frac{\pi}{12} \Rightarrow R = \left(12 + \frac{\pi}{12}, 0\right)$$

$$Area = \frac{1}{2}(PQ)(QR) = \frac{1}{2} \times 2 \times 12 = 12$$

G. Integration

2.2 Linearization and Differentials

A. Linearization

Linearization uses the tangent line to a function to approximate the function at a point.

- The approximation is better near the point of intersection of the tangent and the function, and usually becomes worse as we move away.

2.21: Linearization[1]

The linearization of the function $f(x)$ at the point a is given by

$$L(x) = f(a) + f'(a)(x - a)$$

Consider a function $f(x)$ and the tangent that passes through the point $\big(a, f(a)\big)$.
The point-slope form of the equation of the tangent will be:

$$y - f(a) = m(x - a)$$

Solve for y:

$$y = f(a) + m(x - a)$$

Substitute $m = f'(a)$:

$$y = f(a) + f'(a)(x - a)$$

Example 2.22

Approximate $\sqrt[3]{63}$ using a suitable function.

Consider the function

$$f(x) = \sqrt[3]{x}$$

The linearization of $f(x)$ is:

$$y = f(a) + f'(a)(x - a) = f(a) + \frac{1}{3a^{\frac{2}{3}}}(x - a)$$

Substitute $a = 64, x = 63$:

$$y = \sqrt[3]{64} + \frac{1}{3 \times 64^{\frac{2}{3}}}(63 - 64) = 4 + \frac{1}{48}(-1) = 3\frac{47}{48}$$

Example 2.23

Find approximate values for both $g(2.1)$ and $g(1.85)$, given that $g(2.0) = -3.5$ and
(a) $g'(2.0) = 10$
(b) $g'(2.0) = -4$ (Phillips Exeter Math-4 2022/330)

$$L(x) = f(a) + f'(a)(x - a)$$

Use the tangent line.

Part A

Substitute $Slope = 10, (x_1, y_1) = (2, -3.5)$:

$$y = -3.5 + 10(x - 2) = 10x - 23.5$$

Substitute

$$x = 2.1 \Rightarrow y = 10(2.1) - 23.5$$
$$x = 1.85 \Rightarrow y = 10(1.85) - 23.5$$

Part B

Substitute $Slope = -4.2, (x_1, y_1) = (2, -3.5)$:

[1] Linearization is a special case of Taylor series.

$$y + 3.5 = -4.2(x - 2)$$
$$y = -4.2x + 4.9$$

Substitute:

$$x = 2.1 \Rightarrow y = -4.2(2.1) + 4.9$$
$$x = 1.85 \Rightarrow y = -4.2(1.85) + 4.9$$

2.3 Related Rates: 2D Geometry

A. Basics

Related rates questions give a rate of change, and ask to find another rate of change. For example, given the rate of change of volume of a sphere with respect to time $\left(= \frac{dV}{dt}\right)$, find the rate of change of radius $\left(= \frac{dr}{dt}\right)$, or the rate of change of surface area $\left(= \frac{dS}{dt}\right)$. Hence, they involve problems from geometry or coordinate geometry. Related rates of change are important in applied subjects such as physics, or biology.

Example 2.24: Differentiation[2]

Find the value of $\frac{dy}{dt}$ if:

A. $y = 3x$, when $\frac{dx}{dt} = \frac{2}{3}$.
B. $y = 5x^2$, given that $x = 4$ and $\frac{dx}{dt} = \frac{2}{3}$.
C. $y = p^2 + q^2 + r^2, \frac{dp}{dt} = 3, \frac{dq}{dt} = 4, \frac{dr}{dt} = 9, p = 1, q = 2, r = 3$

Part A

Differentiate both sides of the above equation with respect to time:

$$\frac{dy}{dt} = 3 \cdot \frac{dx}{dt} = 3\left(\frac{2}{3}\right) = 2$$

Part B

$$y = 5x^2 \Rightarrow \frac{dy}{dt} = 10x \cdot \frac{dx}{dt} = 10(4) \cdot \left(\frac{2}{3}\right) = \frac{80}{3}$$

Part C

$$\frac{dy}{dt} = 2p\frac{dp}{dt} + 2q\frac{dq}{dt} + 2r\frac{dr}{dt} = 2(1)3 + 2(2)4 + 2(3)9 =$$

Example 2.25: Product Rule

A. Given that $y = ab, a = f(t), b = f(t)$, find $\frac{dy}{dt}$ if for a particular value of t, $a = 2, b = 3, \frac{da}{dt} = \frac{1}{2}, \frac{db}{dt} = 7$.
B. If $V = lwh$ and $\frac{dh}{dt} = 2\frac{ft}{s}, \frac{dw}{dt} = -3\frac{ft}{s}, \frac{dl}{dt} = 1\frac{ft}{s}$ when $l = 5, w = 7, h = 3$, find $\frac{dV}{dt}$.

Part A

$$\frac{dy}{dt} = a \cdot \frac{db}{dt} + b \times \frac{da}{dt} = (2)(7) + (3)\left(\frac{1}{2}\right) = 14 + \frac{3}{2} = 15.5$$

Part B

$$\frac{dV}{dt} = lw \cdot \frac{dh}{dt} + lh \cdot \frac{dw}{dt} + wh \cdot \frac{dl}{dt}$$

Substitute

$$= (5)(7)(2) + (5)(3)(-3) + (7)(3)(1) = 70 - 45 + 21 = 46$$

[2] Related rates of change questions make use of implicit differentiation. If you need a refresher, look up the section on implicit differentiation.

B. 2D Geometry: Basics

We look at some basic geometry questions.

2.26: Area of a Circle

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$
$$C = 2\pi r \Rightarrow \frac{dC}{dt} = 2\pi \cdot \frac{dr}{dt}$$

Example 2.27: Circles

A. A circle is expanding so that its radius increases 0.02 units every second. Find the rate of change of area when the radius is 60 units. Answer in terms of π.

B. A *Battle Royale* game uses a circular island whose radius shrinks at the rate of $1\,\frac{m}{s}$. Find the rate of decrease, in terms of πm in the area of the island in km^2 when the radius is 6000 m.

Part A

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt} = 2\pi(60) \cdot (0.02) = \frac{12}{5}\pi\ units^2\ per\ second$$

Part B

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt} = 2\pi(6000) \cdot (-1) = 12000\pi\ m^2 = -0.012\pi \frac{km^2}{s}$$

Hence,

$$Rate\ of\ decrease = -\frac{dA}{dt} = 0.012\pi\ \frac{km^2}{s}$$

2.28: Warning: Premature Substitution

In a related rates question, it is important to make the substitution at the right time, and not before. Substituting beforehand can result in not being able to solve the question.

Example 2.29: Circles

Find the mistake in the following "solution" to the question "A circle is expanding so that its radius increases 0.02 units every second. Find the rate of change of area when the radius is 60 units."

The area of a circle is:

$$A = \pi r^2$$

Substitute $r = 60$:

$$A = \pi(60)^2$$
$$A = 3600\pi$$

Differentiate:

$$\frac{dA}{dt} = 0$$

You need to identify the relation between the variables in terms of their derivatives before you substitute the values.
Since this was not done, the answer is incorrect.

2.30: Converting Word Expressions into Rates

The key idea in a related rates problem is to be able to convert the rate of change given in the question into data of the form $\frac{dx}{dt}$, etc.

Example 2.31

A. Rate of change of volume with respect to time is $-3\frac{m^3}{s}$.
B. The surface area of a balloon is increasing $5\ cm^3$ every minute.

$$\frac{dV}{dt} = -3\ \frac{m^3}{s}$$
$$\frac{dS}{dt} = 5\frac{cm^3}{min}$$

2.32: Area of a Square

$$A = s^2 \Rightarrow \frac{dA}{dt} = 2s \cdot \frac{ds}{dt}$$

Example 2.33

The side of a square is increasing at the rate of $1\frac{cm}{s}$. Find the rate of change of area when the area of the square is $36\ cm^2$.

$$A = 36 \Rightarrow s = 6\ cm$$

Substitute $s = 6$ in

$$A = s^2 \Rightarrow \frac{dA}{dt} = 2s \cdot \frac{ds}{dt} = 2(6)(1) = 12\frac{cm^2}{s}$$

2.34: Area and Perimeter of a Rectangle

$$P = 2(x+y) \Rightarrow \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$
$$A = xy \Rightarrow \frac{dA}{dt} = x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt}$$

Example 2.35

The length x of a rectangle is decreasing at the rate of $5\frac{cm}{min}$, and the width y is increasing at the rate of $4\frac{cm}{min}$. When $x = 8\ cm$, and $y = 6\ cm$, find the rate of change of the perimeter, and the area of the rectangle. **(CBSE 2017)**

$$x = 8, \qquad y = 6, \qquad \frac{dx}{dt} = -5\ cm, \qquad \frac{dy}{dt} = 4\ cm$$

Perimeter

$$P = 2(x+y) \Rightarrow \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-5+4) = 2(-1) = -2\frac{cm}{min}$$

Area

$$A = xy \Rightarrow \frac{dA}{dt} = x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt} = (8)(4) + (6)(-5) = 32 - 30 = 2\frac{cm^2}{min}$$

2.36: Drawing Diagrams

In geometry questions, it is usually very helpful to draw a diagram.

2.37: Area of an Equilateral Triangle

$$A = \frac{\sqrt{3}}{4}s^2 \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{2}s \cdot \frac{ds}{dt}$$

Example 2.38

The side of an equilateral triangle is increasing at the rate of $2\,\frac{cm}{s}$. At what rate is its area increasing, when the side of the triangle is 20 cm? **(CBSE 2015)**

$$A = \frac{\sqrt{3}}{4}s^2 \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{2}s \cdot \frac{ds}{dt} = \frac{\sqrt{3}}{2} \cdot 20 \cdot 2 = 20\sqrt{3}\ cm^2\ per\ second$$

2.39: Pythagorean Theorem

Differentiating the Pythagorean Theorem gives:

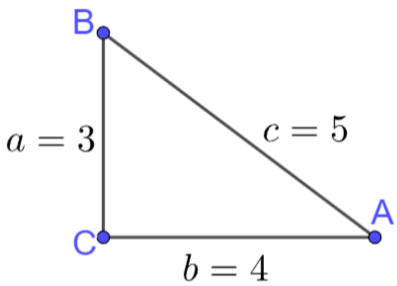
$$a^2 + b^2 = c^2 \Rightarrow 2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

If c is a constant then:

$$a^2 + b^2 = c^2 \Rightarrow 2a\frac{da}{dt} + 2b\frac{db}{dt} = 0$$

Example 2.40

A ladder $5m$ long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of $2\,\frac{m}{s}$. How fast is the height on the wall decreasing when the foot of ladder is $4m$ away from the wall? **(CBSE 2012, Type ISC 2019)**



Since c is a constant differentiate $a^2 + b^2 = c^2$ to get:

$$2a \cdot \frac{da}{dt} + 2b \cdot \frac{db}{dt} = 0$$

Solve for $\frac{da}{dt}$:

$$\frac{da}{dt} = -\frac{2b}{2a} \cdot \frac{db}{dt} = -\frac{b}{a} \cdot \frac{db}{dt}$$

Substitute $\frac{db}{dt} = 2, b = 4, a = \sqrt{5^2 - 4^2} = \sqrt{25 - 16} = \sqrt{9} = 3$:

$$-\frac{4}{3} \cdot 2 = -\frac{8}{3}\frac{m}{s}$$

Example 2.41

Changing length of sides of a triangle

2.42: Area of a Triangle

$$A = \frac{1}{2}hb$$

Differentiating both sides, and using the product rule:

$$\frac{dA}{dt} = \frac{1}{2}h \cdot \frac{db}{dt} + \frac{1}{2}b \cdot \frac{dh}{dt}$$

Example 2.43

In a right triangle, leg x is increasing at the rate of $2\frac{m}{s}$, while leg y is decreasing so that the area of the triangle is always equal to $6\ m^2$. How fast is the hypotenuse changing when $x = 3m$?

$$Area = \frac{1}{2}xy = 6 \Rightarrow xy = 12$$

When:

$$x = 3 \Rightarrow 3y = 12 \Rightarrow y = 4$$

Also, differentiate $xy = 12$:

$$x\frac{dy}{dt} + y\frac{dx}{dt} = 0$$

Substitute $x = 3, y = 4, \frac{dx}{dt} = 2$:

$$3\left(\frac{dy}{dt}\right) + 4(2) = 0$$

$$\frac{dy}{dt} = -\frac{8}{3}$$

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$2(5)\frac{dz}{dt} = 2(3)(2) + 2(4)\left(-\frac{8}{3}\right)$$

$$10\frac{dz}{dt} = 12 - \frac{64}{3}$$

$$\frac{dz}{dt} = -\frac{28}{30} = -\frac{14}{15}$$

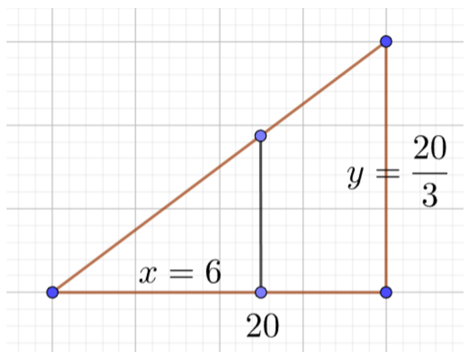
C. 2D Geometry: Similarity[3]

Example 2.44

A light is placed on the ground 20 m away from a building. It shines on the building. A man 2 m tall walks directly towards the building at $1\frac{m}{s}$. How fast is the length of his shadow on the building changing when he is 14 m from the building.

Define

$$x = Length\ of\ man\ from\ shadow$$
$$y = Height\ of\ shadow$$



From the question

$$\frac{dx}{dt} = 1\frac{m}{s} \Rightarrow \frac{dy}{dt} = ?$$

Hence, set up an equation that relates x and y using similarity of triangles:

$$\frac{x}{20} = \frac{2}{y} \Rightarrow xy = 40$$

---

[3] Questions on similarity are generally more difficult than other questions. You can come back to these questions later, if required.

Differentiate the above to get:

$$\underbrace{x\frac{dy}{dt}+y\frac{dx}{dt}=0}_{Equation\ I}$$

Using the relation that we found above, determine the value of y when $x=3$:

$$\frac{x}{20}=\frac{2}{y}\Rightarrow y=\frac{40}{x}=\frac{40}{6}=\frac{20}{3}$$

Substitute $x=6, \frac{dx}{dt}=1\frac{m}{s}, y=\frac{20}{3}$:

$$6\frac{dy}{dt}+\left(\frac{20}{3}\right)(1)=0\Rightarrow 6\frac{dy}{dt}=-\frac{20}{3}\Rightarrow\frac{dy}{dt}=-\frac{20}{18}=-\frac{10}{9}\frac{m}{s}$$

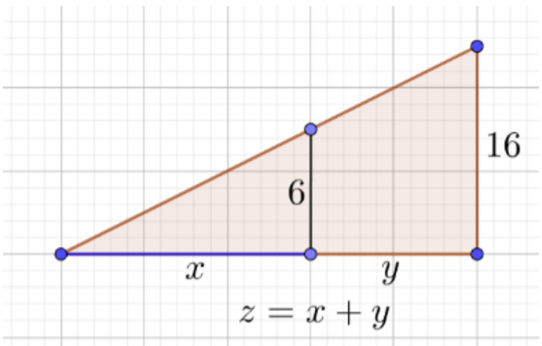
Example 2.45

A 6 feet tall man walks at $5\frac{ft}{sec}$ toward a streetlight with a height of 16 feet. When the base of the light is 10 feet away, at what rate is the:

A. length of his shadow changing?
B. the tip of his shadow moving?

Part A

Let the length of the shadow be x. Let the distance between the man and the streetlight be y. (See adjoining diagram).



From the diagram, and using the fact that the two triangles are similar, we can set up the ratio:

$$\frac{16}{x+y}=\frac{6}{x}\Rightarrow 5x=3y\Rightarrow x=\frac{3}{5}y$$

Differentiate the above with respect to $time=t$, and substitute $\frac{dy}{dt}=-5\frac{m}{s}$:

$$\frac{dx}{dt}=\frac{3}{5}\cdot\frac{dy}{dt}=\frac{3}{5}\cdot(-5)=-3\frac{m}{s}$$

The rate of change of the length of his shadow is:

$$\frac{dx}{dt}=-3\frac{m}{s}$$

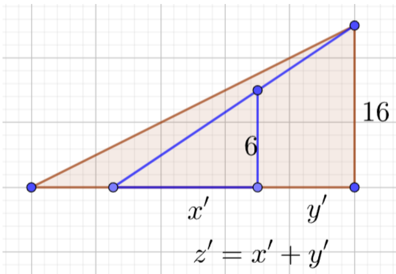
Note that the rate is negative since the length of the shadow is decreasing.

Part B

The length of the base of the triangle is:

$$z=x+y$$

The rate of change of the tip of the shadow is equal to the rate of change of the length of the base of the triangle:



$$\frac{dz}{dt}=\frac{dx}{dt}+\frac{dy}{dt}=-3-5=-8\frac{m}{s}$$

Note that the tip of the shadow is moving at a speed of $8\frac{m}{s}$ in the direction of due north.

D. 2D Geometry: Trigonometry

2.46: Area of a Triangle

The trigonometric formula for the area of a triangle is:

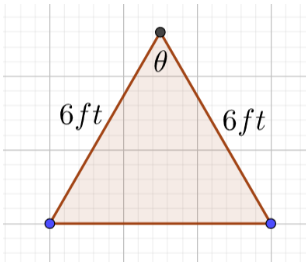
$$A = \frac{1}{2}ab\sin\theta$$

We can differentiate both sides of the area formula with respect to *time* and make use of the three-term product rule:

$$\frac{dA}{dt} = \frac{1}{2}\left(b\sin\theta\frac{da}{dt} + a\sin\theta\frac{db}{dt} + ab\cos\theta\frac{d\theta}{dt}\right)$$

Example 2.47

Jennifer works at a construction company. She props up two ladders, each 6 feet tall, on the floor, and leans them against one other at the top forming an isosceles triangle with the ground. She begins to pull the bases of the ladders such that the top of the triangle is coming closer to the ground. When the ladders form an equilateral triangle, she notices that the angle between them is increasing at $\frac{1}{4}$ radians per second. At what rate is the area of the triangle increasing in that exact moment? **(MAΘ, Mu Area and Volume 2022/7)**



The formula that connects the area of the triangle with the angle between the two ladders is:

$$A = \frac{1}{2}ab\sin\theta$$

Substitute $a = b = 6$ in the above formula:

$$A = \frac{1}{2}ab\sin\theta = \frac{1}{2}(6)(6)(\sin\theta) = 18\sin\theta$$

Differentiate both sides with respect to time:

$$\frac{dA}{dt} = 18\cos\theta\frac{d\theta}{dt}$$

To find the rate of change of area when the triangle is equilateral, substitute $\theta = 60° = \frac{\pi}{3}, \frac{d\theta}{dt} = \frac{1}{4}\frac{radians}{sec}$:

$$= 18\left(\cos\frac{\pi}{3}\right)\cdot\frac{1}{4} = 18\cdot\frac{1}{2}\times\frac{1}{4} = \frac{9}{4} = 2.25\frac{ft}{sec}$$

Example 2.48

A. A boat approaches a lighthouse 600 meters above sea level at 250 meters per minute. When the boat is 800 meters away, at what rate is the angle between the ship and the top of the lighthouse changing?
B. Plane Flying
C. Kite Flying

Part B

$$\frac{d\theta}{dt}$$

2.4 Related Rates: 3D Geometry, and Others

A. Single Parameter

2.49: Volume of a Cube

$$V = x^3 \Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

$$S = 6x^2 \Rightarrow \frac{dS}{dt} = 12x\frac{dx}{dt}$$

Example 2.50: Cubes

The volume of a cube is increasing at the rate of 8 $\frac{cm^3}{s}$. How fast is the surface area increasing when the length of its edge is 12 cm. (CBSE 2019)

Let the edge length of the cube be x cm.

$$V = x^3 \Rightarrow \frac{dV}{dt} = 3x^2\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{dV}{dt} \times \frac{1}{3x^2} = \frac{8}{3x^2}$$

$$S = 6x^2 \Rightarrow \frac{dS}{dt} = 12x\frac{dx}{dt} = 12x \cdot \frac{8}{3x^2} = \frac{4 \times 8}{x} = \frac{4 \times 8}{12} = \frac{8}{3}\frac{cm^2}{s}$$

2.51: Volume of a Sphere

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

$$\frac{dV}{dt} = \left(\frac{4}{3}\pi\right)(3r^2)\left(\frac{dr}{dt}\right) = 4\pi r^2\frac{dr}{dt}$$

Example 2.52

A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 $\frac{cm^3}{min}$. When the thickness of ice is 5 cm, then the rate at which the thickness of ice decreases is (Answer in terms of π): (JEE Main 2005)

When the ice melts, the volume decreases. Hence, rate of decrease of volume of ice with respect to time:

$$= \frac{dV}{dt} = 50\frac{cm^3}{min}$$

Rate of decrease of thickness of ice is the rate of decrease of radius of sphere (including the ice):

$$= \frac{dr}{dt}$$

The radius of the sphere (which includes the iron ball and the ice) when thickness of ice is 5 cm

$$= r = 10 + 5 = 15\ cm$$

Substitute $\frac{dV}{dt} = 50\frac{cm^3}{min}$ and $r = 15\ cm$:

$$\frac{dr}{dt} = \frac{dV}{dt} \times \frac{1}{4\pi r^2} = 50\frac{cm^3}{min} \times \frac{1}{4\pi(15)^2 cm^2} = \frac{1}{18\pi}\frac{cm}{min}$$

Example 2.53

A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is: (JEE Main 2012)

Volume after 49 minutes:

$$= 4500\pi - 49(72\pi) = 9\pi[500 - 49(8)] = 9\pi[500 - 392] = 972\pi$$

Radius at 49 minutes:

$$\frac{4}{3}\pi r^3 = 972\pi \Rightarrow r^3 = 729 \Rightarrow r = 9$$

Substitute $\frac{dV}{dt} = 72\pi$, and $r = 9$ in Equation I:

$$\frac{dr}{dt} = \frac{dV}{dt} \times \frac{1}{4\pi r^2} = 72\pi \times \frac{1}{4\pi(9)^2} = \frac{2}{9}\frac{m}{min}$$

2.54: Surface Area of a Sphere

$$S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt}$$

Example 2.55

The volume of a sphere is increasing at the rate of 8 $\frac{cm^3}{s}$. Find the rate at which its surface area is increasing when the radius is 12 cm. **(CBSE 2017)**

Substitute $\frac{dV}{dt} = 8$:

$$\frac{dr}{dt} = \frac{dV}{dt} \times \frac{1}{4\pi r^2} = 8 \times \frac{1}{4\pi r^2} = \frac{2}{\pi r^2}$$

Substitute $r = 12$:

$$\frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt} = 8\pi r \cdot \frac{2}{\pi r^2} = \frac{16}{r} = \frac{16}{12} = \frac{4}{3}\frac{cm^2}{s}$$

B. Multiple Parameters

Example 2.56

A right circular cone shaped block of ice is melting in such a way that its height and radius are both decreasing at 1 $\frac{cm}{hr}$. How fast is the volume decreasing when $r = h = 10\ cm$.

The question has given us the relation

$$r = h \Leftrightarrow \frac{dr}{dt} = \frac{dh}{dt} = -1\frac{cm}{hr}$$

Substitute $r = h$ in the formula for the volume of a cone and differentiate

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \pi r^2\frac{dr}{dt} = \pi(10)^2(-1) = -100\pi\frac{cm^3}{hr}$$

Example 2.57

Sand is pouring from a pipe at a rate of 12 $\frac{cm^3}{s}$. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of sand cone increasing when the height is 4 cm? **(CBSE 2011)**

Substitute $h = \frac{1}{6}r \Rightarrow r = 6h$ in the formula for the volume of a cone:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(6h)^2 h = 12\pi h^3 \Rightarrow \frac{dV}{dt} = 36\pi h^2 \cdot \frac{dh}{dt}$$

Solve for $\frac{dh}{dt}$ and substitute $\frac{dV}{dt} = -12\frac{cm^3}{s},\ h = 4\ cm$:

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{1}{36\pi h^2} = 12 \times \frac{1}{36\pi(4)^2} = \frac{1}{48\pi}\frac{cm}{s}$$

Example 2.58

A conical cup is 4 cm across and 6 cm deep. Water leaks out of the bottom at the rate of $2\frac{cm^3}{s}$. How fast is the water level dropping when the height of the water is 3 cm?

Draw a diagram. Using similar triangles

$$\frac{r}{h} = \frac{2}{6} \Rightarrow r = \frac{h}{3}$$

Substitute the above in the formula for the volume of a cone:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{1}{27}\pi h^3 \Rightarrow \frac{dV}{dt} = \frac{\pi h^2}{9} \cdot \frac{dh}{dt}$$

Solve for $\frac{dh}{dt}$ and substitute $\frac{dV}{dt} = -2\frac{cm^3}{s}, h = 3\ cm$:

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{9}{\pi h^2} = (-2) \times \frac{9}{\pi(3)^2} = -\frac{2}{\pi}\frac{cm}{s}$$

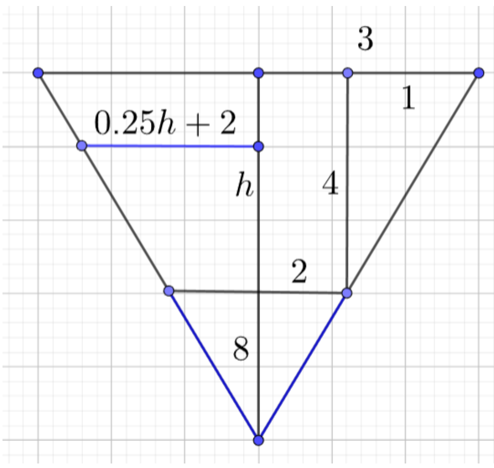
Example 2.59

Fahan is pouring apple juice into his glass that is in the shape of a right truncated cone at a constant rate of 2 cubic inches per second. The lower base is 4 inches in diameter and the upper (open) base has a diameter of 6 inches. The glass has a total height of 4 inches. At what rate, in inches per second, is the height of liquid in the glass increasing when the glass is 25% full? **(MAΘ, Mu Area and Volume 2022/23)**

Hint: The volume of the frustum of a cone (truncated cone) with height h, lower radius r, and upper radius R is $V = \frac{h\pi}{3}(R^2 + Rr + r^2)$

Substitute $R = \frac{1}{4}h + 2, r = 2$:

$$V = \frac{h\pi}{3}(R^2 + Rr + r^2)$$



C. Physics

Example 2.60: Voltage

Differentiate the below with respect to time:

$$V = IR$$

Differentiate both sides of the above with respect to $time = t$:

$$\frac{dV}{dt} = I \cdot \frac{dR}{dt} + R \cdot \frac{dI}{dt}$$

2.61: Two Parallel Resistors

The unit of resistance is $Ohms(\Omega)$. If parallel resistors have individual resistances R_1 and R_2, then the system has resistance R given by

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$$

Differentiate both sides of the above with respect to $time = t$:

$$-\frac{1}{R^2}\cdot\frac{dR}{dt}=-\frac{1}{R_1^2}\cdot\frac{dR_1}{dt}-\frac{1}{R_2^2}\cdot\frac{dR_2}{dt}$$

Solving for $\frac{dR}{dt}$:

$$\frac{dR}{dt}=R^2\left(\frac{1}{R_1^2}\cdot\frac{dR_1}{dt}+\frac{1}{R_2^2}\cdot\frac{dR_2}{dt}\right)$$

(Calculator) Example 2.62: Resistance

In a system with two resistors, R_1 is increasing by $0.5\frac{\Omega}{s}$ and R_2 is decreasing by $0.2\frac{\Omega}{s}$. Find the rate of change of total resistance when $R_1 = 30\ \Omega$, and $R_2 = 40\ \Omega$.

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}=\frac{1}{30}+\frac{1}{40}\Rightarrow R=\frac{120}{7}$$

Substitute $R=\frac{120}{7}, \frac{dR_1}{dt}=30, \frac{dR_2}{dt}=40$:

$$\frac{dR}{dt}=R^2\left(\frac{1}{R_1^2}\cdot\frac{dR_1}{dt}+\frac{1}{R_2^2}\cdot\frac{dR_2}{dt}\right)=\left(\frac{120}{7}\right)^2\left(\frac{1}{30^2}\cdot 0.5+\frac{1}{40^2}\cdot -0.2\right)=\frac{31}{245}$$

2.63: *n* Parallel Resistors

Differentiating $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\cdots+\frac{1}{R_n}$ gives us:

$$\frac{dR}{dt}=\left(\frac{R}{R_1}\right)^2\frac{dR_1}{dt}+\left(\frac{R}{R_2}\right)^2\frac{dR_2}{dt}+\cdots+\left(\frac{R}{R_n}\right)^2\frac{dR_n}{dt}$$

Differentiate both sides of the given equation:

$$-\frac{1}{R^2}\cdot\frac{dR}{dt}=-\frac{1}{R_1^2}\cdot\frac{dR_1}{dt}-\frac{1}{R_2}\cdot\frac{dR_2}{dt}-\cdots-\frac{1}{R_n}\cdot\frac{dR_n}{dt}$$

Solve for $\frac{dR}{dt}$:

$$\frac{dR}{dt}=\left(\frac{R}{R_1}\right)^2\frac{dR_1}{dt}+\left(\frac{R}{R_2}\right)^2\frac{dR_2}{dt}+\cdots+\left(\frac{R}{R_n}\right)^2\frac{dR_n}{dt}$$

Example 2.64: Kinetic Energy

Assuming that the mass remains constant, differentiate the below with respect to time:

$$K=\frac{1}{2}mv^2$$

, while the velocity is a function of time, we can differentiate both sides to get:

$$\frac{dK}{dt}=\frac{1}{2}m\cdot 2v\frac{dv}{dt}=mv\frac{dv}{dt}$$

D. Other Sciences

Example 2.65: Chemistry

$$PV = nRT$$

Example 2.66

Biology, Stewart, Section 3.9, Exercise 36

E. Coordinate Geometry and Further Resources

Example 2.67

A particle moves in the xy plane along the parabola $y = 3x^2$. Find the rate of change of the y −coordinate if the particle has x −coordinate 2, and

$$\frac{dx}{dt} = \frac{1}{9}\frac{cm}{sec}$$

$$y = 3x^2 \Rightarrow \frac{dy}{dt} = 6x \cdot \frac{dx}{dt} = 6(2) \cdot \frac{1}{9} = \frac{4}{3}\frac{cm}{sec}$$

Example 2.68

A particle moves along the curve $y = \sqrt{1 + \sin^3\left(\frac{x}{2}\right)}$. Find the rate of change of the x −coordinate when the particle is at the point $x = \frac{\pi}{2}$, and the rate of change of the y coordinate per second is 3 m.

Note: Do not approximate at any stage.

Use the chain rule to differentiate $y = \sqrt{1 + \sin^3\left(\frac{x}{2}\right)}$:

$$\frac{dy}{dt} = \frac{1}{2\sqrt{1 + \sin^3\left(\frac{x}{2}\right)}} \cdot 3\sin^2\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{2}\right) \cdot \frac{1}{2} \cdot \frac{dx}{dt}$$

Solve the above for $\frac{dx}{dt}$:

$$\frac{dx}{dt} = \frac{dy}{dt} \cdot \frac{4\sqrt{1 + \sin^3\left(\frac{x}{2}\right)}}{3\sin^2\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$$

Substitute $x = \frac{\pi}{2}, \frac{dy}{dt} = 3$:

$$\frac{dx}{dt} = 3 \cdot \frac{4\sqrt{1 + \sin^3\left(\frac{\pi}{4}\right)}}{3\sin^2\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)} = \frac{4\sqrt{1 + \left(\frac{1}{\sqrt{2}}\right)^3}}{\left(\frac{1}{\sqrt{2}}\right)^2\left(\frac{1}{\sqrt{2}}\right)} = \frac{4\sqrt{1 + \frac{1}{2\sqrt{2}}}}{\frac{1}{2\sqrt{2}}} = \frac{4\sqrt{\frac{2\sqrt{2}+1}{2\sqrt{2}}}}{\frac{1}{2\sqrt{2}}}$$

After some further simplification, we get:

$$= 4\sqrt{\frac{2\sqrt{2}+1}{2\sqrt{2}}} \times 2\sqrt{2} = 4\sqrt{2\sqrt{2}+1} \times \sqrt{2\sqrt{2}} = 4\sqrt{8+\sqrt{2}}$$

Example 2.69
Parametrization

F. Further Resources

Videos 2.70
You can find a variety of related rates questions covered in

A. This video playlist by Jake
B. This video by Blackpenredpen
C. This video by the Organic Chemistry Tutor

3. Graphing and Optimization

3.1 Extreme Values

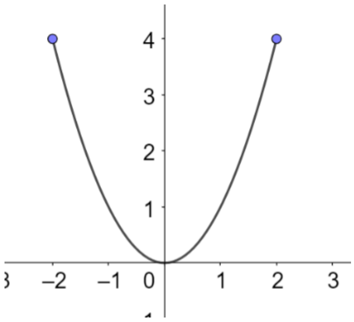
A. Maxima and Minima

JEE Advanced Subjective 17

3.1: Absolute Maximum and Minimum (Informal)

- The greatest y value on the graph of a function is its absolute maximum.
- The smallest y value on the graph of a function is its absolute minimum.

The absolute maximum (or minimum) can be attained multiple times.



Example 3.2

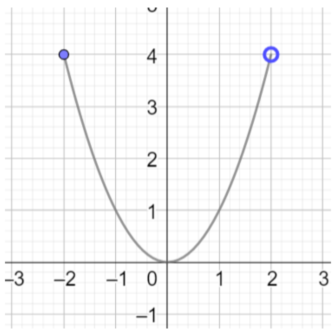
Find the coordinates $(x, y\ pairs)$ of all absolute maxima and minima for each function below:

A. The line $y = x, -3 < x < 4$

B. The graph of the parabola $y = x^2, -2 \le x < 2$.

(You should not need Calculus to do this question).

$$Absolute\ Maximum = (-2,4)$$
$$Absolute\ Minimum = (0,0)$$



3.3: Endpoints

- Checking the endpoints of a function is critical when finding maxima or minima.
- Currently, we have not introduced Calculus, but checking endpoints remains necessary even after differentiation techniques for optimization have been introduced.

Example 3.4

Find the absolute maximum and the minimum of the function $y = 3x + 4.1$

A. over the interval $[-3,4]$.

B. over the interval $(-3,4)$.

Part A

The minimum value occurs at the left endpoint:

$$y = 3(-3) + 4.1 = -9 + 4 = -4.9$$

The maximum value occurs at the right endpoint:

$$y = 3(4) + 4.1 = 12 + 4 = 16.1$$

Part B

$$\lim_{x \to -3^+} 3x + 4.1 = -4.9$$

However, -4.9 is never achieved because -3 is not included in the interval under consideration. Hence,

$$\textit{the function has no minimum over the given interval}$$

$$\lim_{x\to 4^-} 3x + 4.1 \;= 16.1$$

However, 16.1 is never achieved because 4 is not included in the interval under consideration. Hence,

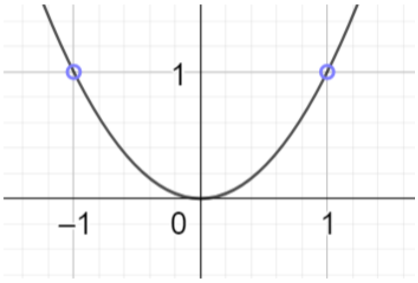
$$\textit{the function has no maximum over the given interval}$$

Example 3.5

Find the absolute maximum and minimum of the function $y = x^2$

A. over the interval $(-1,1)$
B. over the interval $[-1,1]$
C. over its natural domain

Part A



$$Minimum = (0,0)$$
$$No\ Maximum$$

Part B

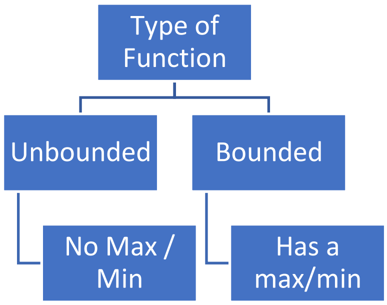
$$Minimum = (0,0)$$
$$Maximum\ at\ \{(-1,1),(1,1)\}$$

Part C

$$Minimum = (0,0)$$
$$No\ Maximum$$

3.6: Unbounded and Bounded Functions

- Functions that increase without restriction are called unbounded.
- Functions that have an absolute maximum and an absolute minimum value are called bounded.



$$\lim_{x\to\infty} -x^2 = -\infty$$

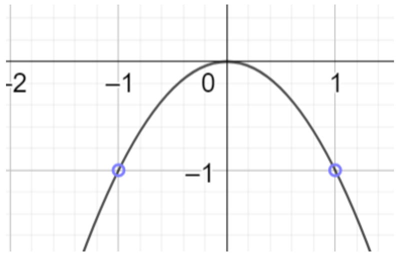
$$\lim_{x\to-\infty} -x^2 = -\infty$$

Example 3.7

Find the absolute maximum and minimum of the function $y = -x^2$

A. over the interval $(-1,1)$
B. over the interval $[-1,1]$
C. over its natural domain

Part A



$$Max = (0,0)$$
$$No\ Min$$

Part B

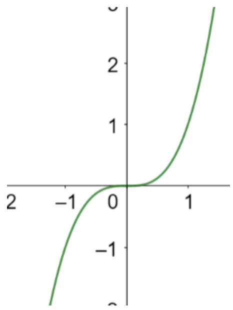
$$Max = (0,0)$$
$$Maximum\ at\ \{(-1,-1),(1,-1)\}$$

Part C

$$Max = (0,0)$$
$$No\ Min$$

Example 3.8

Does the function $f(x) = x^3$ have an absolute maximum or an absolute minimum value?



$$x \to \infty, y \to \infty$$
$$x \to -\infty, y \to -\infty$$

Therefore, $f(x)$ has neither an absolute maximum, nor an absolute minimum value. We call such a function unbounded.

3.9: Relative Maximum and Minimum

- If a function decreases both to the left and to the right of a point, that point is a *relative* maximum.
  - ✓ A relative maximum creates a "mountain" in the graph
- If a function increases both to the left and to the right of a point, that point is a *relative* minimum.
  - ✓ A relative minimum creates a "valley" in the graph

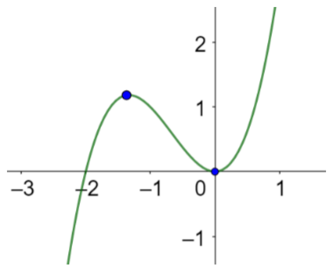
- Plural of maximum is maxima
- Plural of minimum is minima

Note:

- If the point not an endpoint, then we check both to the left, and the right of the point.
- If the point is a left endpoint, then we check only to the right of the point.
- If the point is a right endpoint, then we check only to the left of the point.

Example 3.10

Part of the graph of $y = x^3 + 2x^2, x \in \mathbb{R}$ is given alongside.



A. Does it have a relative maximum? A relative minimum? Identify the integers in the domain between which the maximum and the minimum are achieved.
B. Does it have an absolute maximum? An absolute minimum?
C. In general, if a graph does not have an absolute maximum/minimum, can it still have a relative maximum/minimum

Part A

$$Yes$$
$$Relative\ Maximum\ is\ acheived\ when\ x \in (-2,-1)$$
$$Relative\ Minimum\ is\ acheived\ when\ x = 0 \Rightarrow\ x \in (-1,1)$$

Part B

$$No$$

Part C

$$Yes, it\ can\ still\ have$$

3.11: Inclusive Interval

An interval that includes its endpoints can have a local minimum at its endpoints.

3.12: Exclusive Interval

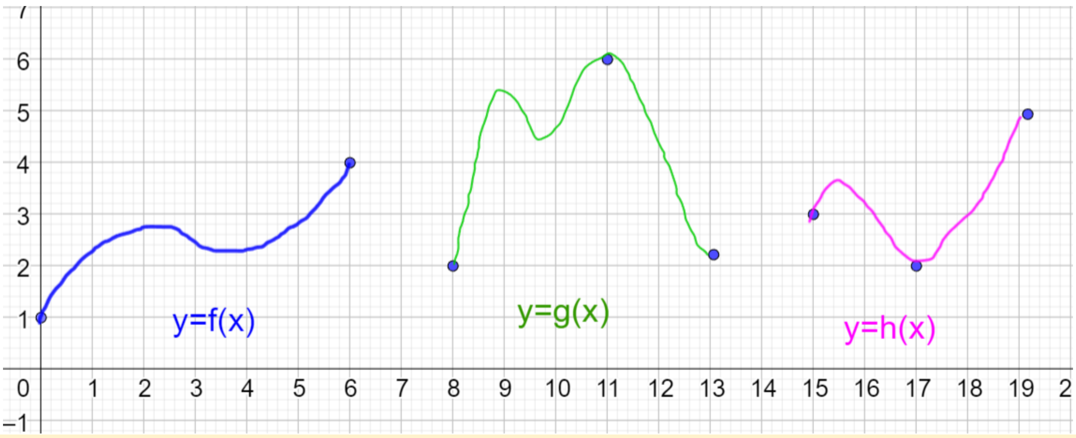
- An interval that does not its endpoints cannot have a local minimum at its endpoints.

3.13: Comparing Absolute and Relative Maxima/Minima

- Any absolute maximum/minimum is always a relative maximum.
- A relative maximum/minimum may or may not be an absolute maximum/minimum.

Example 3.14

The functions $f(x), h(x)$ and $g(x)$ are shown in the diagram. They have domains $D_f = [0,6], D_g = [8,13]$ and $D_h = [15,19]$.

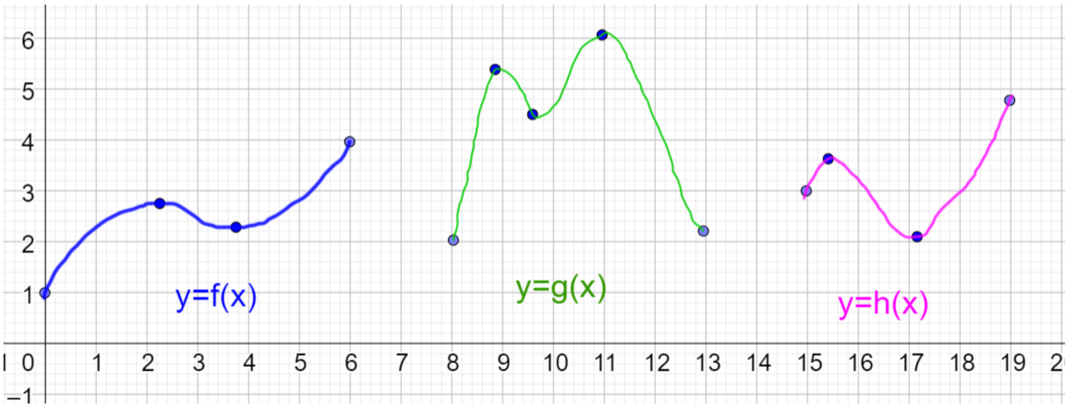


A. Identify the coordinates of the absolute maximum and the absolute minimum for each function.
B. Identify, in the diagram, the relative maxima and minima.

Part A

$$f(x): Max: (6,4), Min(0,1)$$
$$g(x): Max: (11,6), Min(8,2)$$
$$h(x): Max(19,5), Min(17,2)$$

Part B



3.15: Relative vs Absolute

- You can have more than one relative minimum or relative maximum, each of which has different values.
- You can have more than one absolute maximum or minimum, but they have the same y-value.

B. Extreme Value Theorem

3.16: Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$.

That is, there are numbers x_1 and x_2 in $[a, b]$ with

$$f(x_1) = m, f(x_2 = M) \; such \; that \; m \le f(x) \le M$$

For every other x in $[a, b]$

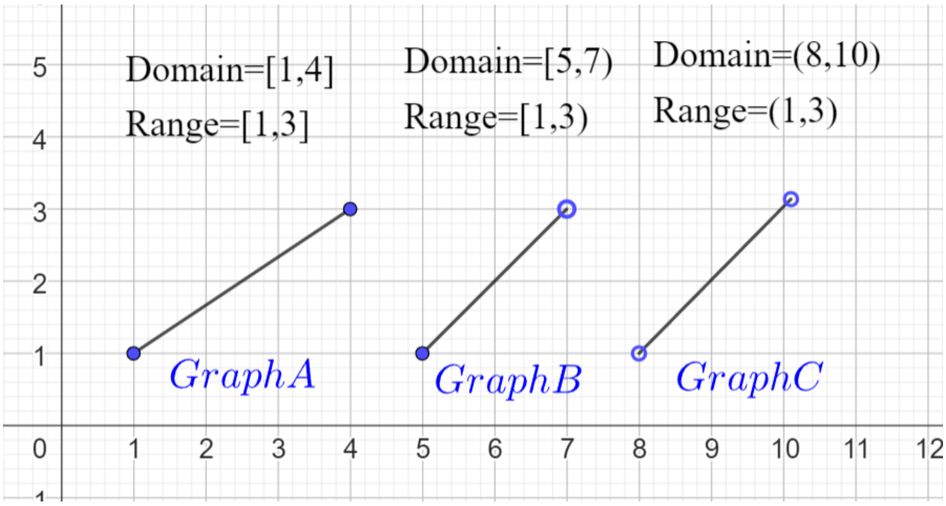
The extreme value guarantees the existence of an absolute maximum and an absolute minimum of a function over a domain on which the conditions of the theorem are satisfied.

Example 3.17

What are the conditions of the Extreme Value Theorem?

- The function f should be continuous
- The interval under considerations should be closed.

Example 3.18



Graph A satisfies the conditions of the Extreme Value Theorem since it is

$$\textit{continuous over the closed interval } [1,4]$$

It has an absolute maximum of 3, and an absolute minimum of 1.

Graph B does not satisfy the conditions of the Extreme Value Theorem since it is

$$\textit{continuous over the half open interval } [5,7)$$

It has an absolute minimum of 1, but does not have an absolute maximum.

Graph C does not satisfy the conditions of the Extreme Value Theorem since it is

$$\textit{continuous over the open interval } (8,10)$$

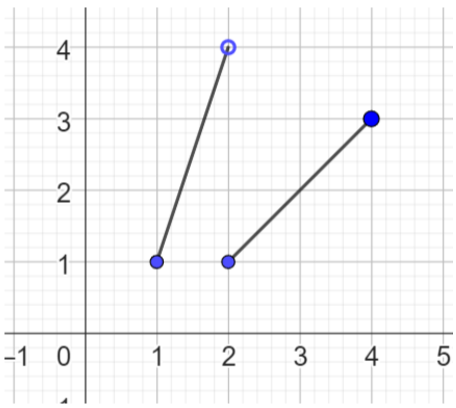
It has neither an absolute maximum, nor an absolute minimum.

Example 3.19

The graph of the function f is defined over the closed interval $[1,4]$. Does the Extreme Value Theorem apply. If yes, explain why. If no, provide a counter example.

The function need not be continuous. Hence, the conditions of the Extreme Value Theorem are not satisfied.

For example, the graph alongside over $[1,4]$ but it is not continuous. The graph has an absolute minimum, but does not have an absolute maximum.



3.20: First Derivative Test

If f has a relative maximum or a relative minimum at an interior point c of its domain, and f' is defined at c, then:

$$f'(c) = 0$$

Example 3.21

Mark all correct options

Using the first derivative test, if $f'(c)$ is zero, then the function must have:

A. A maximum
B. A minimum
C. A maximum or a minimum
D. Cannot be determined

The first derivative test does not say anything about whether a function is a maximum, a minimum or neither when the derivative is zero.

Hence:

$$Option\ D$$

3.22: Critical Point

An interior point of the domain of a function f where f' is zero or undefined is a critical point of f.

- An interior point means it lies inside the domain. Not outside, and not at the endpoints.
- Critical points are used to help in finding maximum and minimum values.

Example 3.23

$$f(x) = \sqrt{x} \Rightarrow f' = \frac{1}{2\sqrt{x}}$$

0 is a critical point of f because 0 is in the domain of f, but $f'(0)$ is not defined.

$$g(x) = \ln x \Rightarrow g' = \frac{1}{x}$$

0 is not a critical point of g because 0 is not in the domain of g.

3.24: Finding Absolute Maximum and Minimum

- Evaluate f at all critical points and endpoints.
- Take the largest and smallest of these values.

3.2 y': Monotonicity

A. Increasing and Decreasing

3.25: Increasing and Decreasing Function

An increasing function is a function where the y value *always* increases as we go from left to right:

$$f(x+h) > f(x), \qquad h > 0$$

A decreasing function is a function where the y value always *decreases* as we go from left to right:

$$f(x+h) < f(x), \qquad h > 0$$

A non-decreasing function is a function where the y value never decreases as we go from left to right:

$$f(x+h) \geq f(x), \qquad h > 0$$

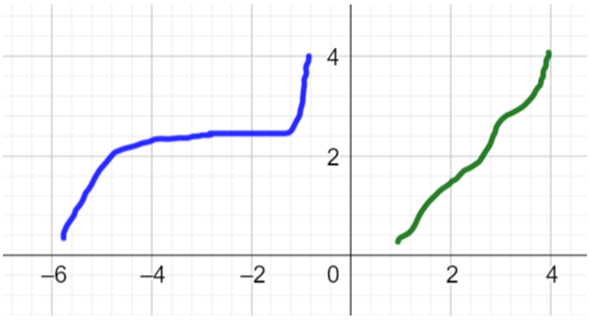
A non-increasing function is a function where the y value never increases as we go from left to right:

$$f(x+h) \le f(x), \qquad h > 0$$

Example 3.26

Classify the following functions as:

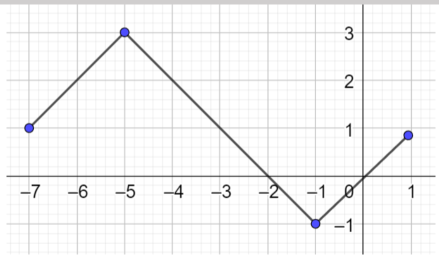
A. Increasing
B. Decreasing
C. Non-Decreasing
D. Non-Increasing
E. None of the Above



$Green$: $Increasing$
$Blue$: $Nondecreasing$

3.27: Increasing and Decreasing Function

If a function is increasing, its slope is positive.
If a function is decreasing, its slope is negative.



Example 3.28

Classify intervals in the function based on whether the

A. function is increasing or decreasing
B. the slope is positive or negative
C. the slope is increasing, decreasing, or constant
D. the function is positive or negative

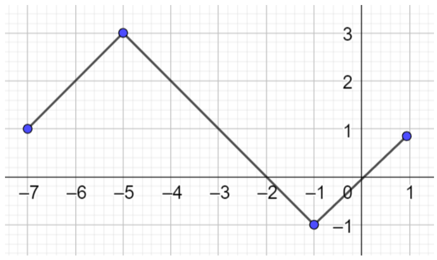
From (-7,-5):
The function is positive, and increasing.
The slope is positive, and constant.

From (-5,-2)
The function is positive, and decreasing.
The slope is negative, and constant.

From (-2,-1)
The function is negative, and decreasing.
The slope is negative, and constant.

From (-1,0)
The function is negative, and increasing.
The slope is positive, and constant.

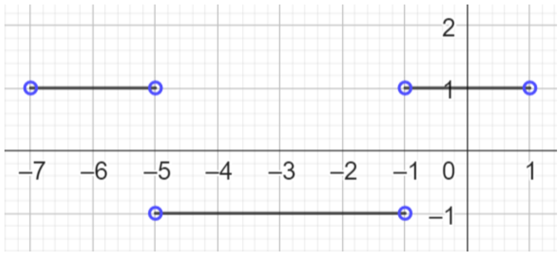
From (0,1)
The function is positive, and increasing.
The slope is positive, and constant.



Example 3.29
Graph the derivative of the function alongside.

The derivative is defined for
$$(-7,-5) \cup (-5,-1) \cup (-1,1)$$

We use the property that the derivative (where it is defined) is equal to the slope of the function at that point.



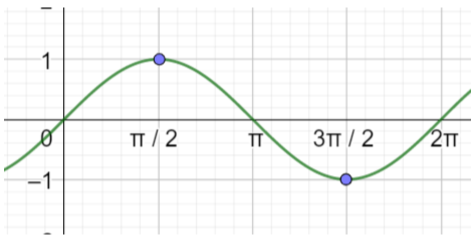
Example 3.30
Give an example of a function which does not fit of the four classifications.

$$y = \sin x$$

3.31: Increasing and Decreasing Interval
A function may not be increasing or decreasing throughout its domain. However, it may be increasing or decreasing on some interval in its domain.

Example 3.32
Without calculus, identify the intervals/points where $\sin x$ is increasing, decreasing, or neither[4].

$$Increasing: \left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right] + 2n\pi, \qquad n \in \mathbb{Z}$$
$$Decreasing: \left[\frac{\pi}{2}, \frac{3\pi}{2}\right) + 2n\pi, \qquad n \in \mathbb{Z}$$



Example 3.33
State the intervals where the graph of the parabola $f(x) = x^2 + 5x + 6$ is increasing and decreasing.

Method I: Calculus

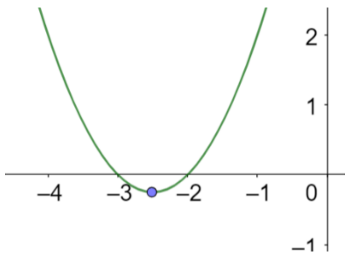
[4] We have not taken points of inflection into account for now. If you do not know what this is, you will learn when you learn concavity.

The x coordinate vertex of the parabola is:

$$= -\frac{b}{2a} = -\frac{5}{2} = -2.5$$

Hence, the interval we want are:

$$Decreasing: (-\infty, -2.5]$$
$$Increasing: [-2.5, \infty)$$



Method II: Calculus

$$f'(x) = 2x + 5$$

$$2x + 5 > 0$$
$$Decreasing: (-\infty, -2.5]$$
$$Increasing: [-2.5, \infty)$$

3.34: First Derivative Test for Increasing and Decreasing Functions

If y is continuous on an interval and:

- y' is positive on an interval, then y is increasing on that interval.
- y' is negative on an interval, then y is decreasing on that interval.
- y' is zero on an interval, then y is neither increasing nor decreasing on that interval.

$$f'(x) > 0 \Rightarrow Slope > 0 \Rightarrow Function\ is\ increasing$$
$$f'(x) < 0 \Rightarrow Slope < 0 \Rightarrow Function\ is\ decreasing$$
$$f'(x) = 0 \Rightarrow Slope = 0 \Rightarrow Function\ is\ neither\ increasing\ nor\ decreasing$$

Example 3.35

$f(x)$ is an increasing polynomial function of degree three. Determine the nature of roots of $f'(x)$.

I: $f'(x)$ is quadratic.
II: $f(x)$ is always increasing. $f'(x)$ is always positive.

From I and II:
$f'(x)$ has complex roots.

Example 3.36

For each part, find the intervals where $f(x)$ is increasing and decreasing.

A. $f(x) = e^x$
B. $f(x) = \frac{x^3}{3} - \frac{9x^2}{2} + 20x$
C. $f(x) = \frac{x^3}{3} - \frac{7x^2}{2} + 12x$
D. $f'(x) = (x-2)(x+5)(x-9)$

Part A

$$y = e^x \Rightarrow \frac{dy}{dx} = e^x > 0 \Rightarrow Always\ true \Rightarrow e^x\ is\ always\ increasing$$

Part B

$$f' > 0 \Rightarrow x^2 - 9x + 20 > 0 \Rightarrow (x-5)(x-4) > 0 \Rightarrow Zeroes\ are\ \{4,5\}$$
$$f' > 0\ over\ the\ interval\ (-\infty, 4) \cup (5, \infty) \Rightarrow f\ is\ increasing\ over\ the\ same\ interval$$
$$f' < 0\ over\ the\ interval\ (4,5) \Rightarrow f\ is\ decreasing\ over\ the\ same\ interval$$

Part C

$$f' > 0 \Rightarrow x^2 - 7x + 12 > 0 \Rightarrow (x-3)(x-4) > 0 \Rightarrow Zeroes\ are\ \{3,4\}$$
$$f' > 0\ over\ the\ interval\ (-\infty, 3) \cup (4, \infty) \Rightarrow f\ is\ increasing\ over\ the\ same\ interval$$
$$f' < 0\ over\ the\ interval\ (3,4) \Rightarrow f\ is\ decreasing\ over\ the\ same\ interval$$

Part D

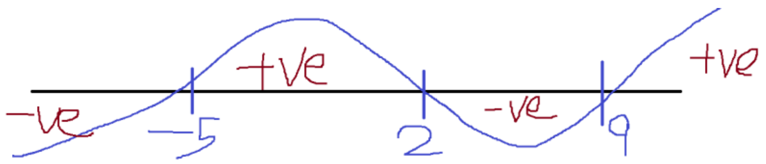
$$f'(x) > 0 \Rightarrow (x-2)(x+5)(x-9) > 0 \Rightarrow Zeroes\ are\ x \in \{-5,2,9\}$$

The intervals are:

$$(-\infty, -5) \cup (-5,2) \cup (2,9) \cup (9, \infty)$$

The leftmost interval is:

$$f = -6 \Rightarrow \underbrace{(-6-2)}_{-ve}\underbrace{(-6+5)}_{-ve}\underbrace{(-6-9)}_{-ve}$$



The final answer is the union of the positive intervals:

$$Increasing\ Intervals: (-5,2) \cup (9, \infty)$$
$$Decreasing\ Intervals: (-\infty, -5) \cup (2,9)$$

Example 3.37

For each part, find the intervals where $f(x)$ is increasing and decreasing.

A. $f(x) = e^{-x}$

B. $f(x) = \frac{2x^3}{3} - \frac{3x^2}{2} - 2x + 2023$

Part A

$$f' = -e^{-x} < 0 \Rightarrow Decreasing\ for\ all\ real\ numbers$$

Part B

$$f' = 2x^2 - 3x - 2 = (2x+1)(x-2)$$
$$f' > 0, x \in \left(-\infty, -\frac{1}{2}\right) \cup (2, \infty) \Rightarrow Function\ is\ increasing$$
$$f' > 0, x \in \left(-\frac{1}{2}, 2\right) \Rightarrow Function\ is\ decreasing$$

Example 3.38

For each part, find the intervals where $f(x)$ is increasing and decreasing.

A. $f(x) = \frac{3x+5}{5x-6}$

B. $f(x) = \frac{ax+b}{cx+d}$

Part A

Calculate the domain:

$$(5x-6)^2 = 0 \Rightarrow 5x = 6 \Rightarrow x = \frac{6}{5} \Rightarrow x \in \mathbb{R}, x \neq \frac{6}{5}$$

The value $x = \frac{6}{5}$ is a vertical asymptote for the function.

As

$$\lim_{x\to\frac{6}{5}^+} f(x) = \infty, \qquad \lim_{x\to\frac{6}{5}^-} f(x) = -\infty$$

The horizontal asymptote for the function is:

$$\lim_{x\to\infty}\frac{3x+5}{5x-6} = \lim_{x\to\infty}\frac{\frac{3x}{x}+\frac{5}{x}}{\frac{5x}{x}-\frac{6}{x}} = \lim_{x\to\infty}\frac{3+\frac{5}{x}}{5-\frac{6}{x}} = \frac{3}{5}$$

Calculate the first derivative:

$$f' = -\frac{43}{(5x-6)^2} \geq 0 \Rightarrow Never\ True$$

Since the first derivative is always negative, the function is decreasing throughout its domain:

$$f' < 0, x \in D_f$$

Example 3.39

Determine the intervals where the function is increasing and decreasing:

$$f(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Domain

Determine the domain first.

Critical points are $\{-1,1\}$

$$+ve\ between\ (-1,1)$$
$$0\ or\ negative: otherwise$$

Intervals

Using the quotient rule $\log\left(\frac{a}{b}\right) = \log a - \log b$:

$$f = \frac{1}{2}[\ln(1+x) - \ln(1-x)]$$

Differentiate:

$$\begin{aligned} f' &= \frac{1}{2}\left[\frac{1}{1+x} - \frac{1}{1-x}(-1)\right] \\ &= \frac{1}{2}\left[\frac{1}{1+x} + \frac{1}{1-x}\right] \\ &= \frac{1}{2}\left[\frac{1-x+1+x}{1-x^2}\right] \\ &= \frac{1}{2}\left[\frac{2}{1-x^2}\right] \\ &= \frac{1}{1-x^2} \end{aligned}$$

The function is increasing when the derivative is positive:

$$\frac{1}{1-x^2} > 0$$
$$1-x^2 > 0$$
$$x^2 < 1$$
$$-1 < x < 1$$

Function is increasing between $(-1,1)$, which is over its entire domain.

Example 3.40
Consider the following information, and determine the number of roots, and the interval in which they lie:

A. The function is a fifth degree polynomial
B. is increasing from (-\infty,-4) and (0,\infty).
C. It is decreasing from (-4,0)
D. The y intercept is 7
E. It is below the x-axis at x=-6

Step I: Finding a Root
y intercept is 7.
The function is decreasing from -4 to 0. Hence
f(-4) > f(0)=7
f(-4) > 7

The function is below the x-axis at x=-6.
f(-6) <0

Since the function is negative at f(-6) and positive at f(-4), by the Intermediate Value Theorem (since the function is continous because it is a polynomial), it has a root between -6 and -4.

Step II: Showing it is the only root
Function is increasing from -infty to -4. It crosses the x-axis between -6 and -4. Hence, there can only be a single root in this interval because function for the function to cross the x-axis multiple times, it must change direction multiple times.

Function is decreasing from -4 to 0. But f(0)=7>0. Hence the function does not have a root between -4 and 0.

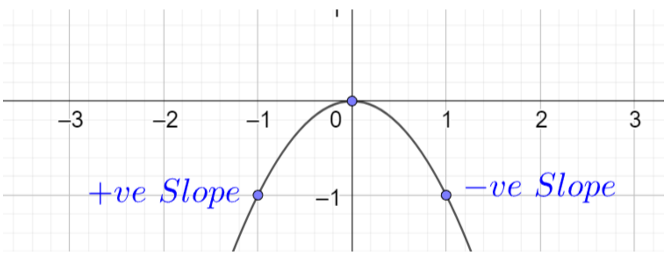
Function is increasing from 0 to \infty. f(0)=7>0. Hence, to the right of zero, the function will keep increasing.

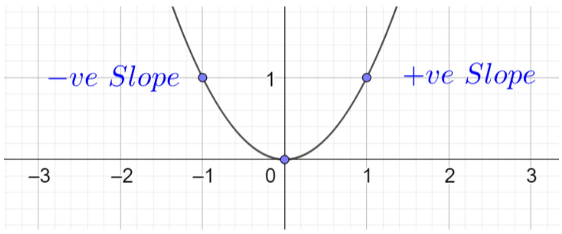
3.3 y': Maxima and Minima

A. Turning Points

3.41: Turning Point

- Note that if any maximum is not at the endpoint of the domain, it creates a "turning point".
- At a turning point, the slope changes from positive to negative, or negative to positive.





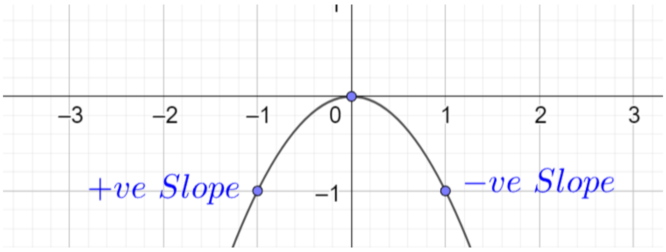
3.42: Slope at Maxima/Minima
At a maximum, or a minimum, the slope must be zero.

- The converse of the above statement is not true in general.
- If the slope is zero, this does not always mean that it is a turning point.

3.43: Maximum within the domain

If a maximum occurs within the domain of a function,

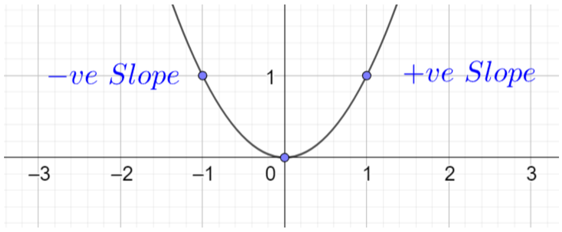
- the slope at the maximum is zero
- the slope going from left to right goes from $+ve$ to $-ve$



3.44: Minimum within the domain

If a minimum occurs within the domain of a function,

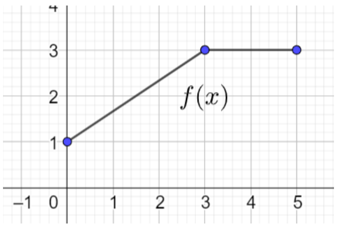
- the slope at the minimum is zero
- the slope going from right to left goes from $-ve$ to $+ve$

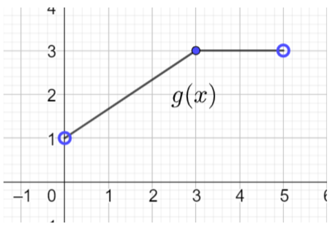


3.45: Critical Points

Critical points are points where:

- the first derivative is zero
- OR the derivative is not defined
- OR are endpoints of the domain of the function





3.46: First Derivative Test

If the first derivative changes sign at a critical point

- from positive to negative at a point, the function has a maximum at that point.
- from negative to positive at a point, the function has a minimum at that point.

B. Polynomial Functions

Example 3.47

Determine the turning points for the function below and classify them as maxima or minima:

$$y = x^2 - x - 30$$

$$y' = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$$

$2x - 1$ is

$$-ve\ \left(-\infty, \frac{1}{2}\right), \qquad +ve\ \left(\frac{1}{2}, \infty\right)$$

Hence, derivative changes from negative (decreasing) to positive (increasing). Hence:

$$x = \frac{1}{2}\ is\ a\ minima$$

Example 3.48

Determine the turning points for the function below and classify them as maxima or minima:

$$y = ax^2 + bx + c$$

$$y' = 2ax + b = 0 \Rightarrow x = -\frac{b}{2a}$$

$2ax + b$ changes from negative (decreasing) to positive (increasing) if $a > 0$

$$x = -\frac{b}{2a}\ is\ a\ minima$$

$2ax + b$ changes from positive (increasing) to negative (decreasing) to if $a < 0$

$$x = -\frac{b}{2a}\ is\ a\ Maxima$$

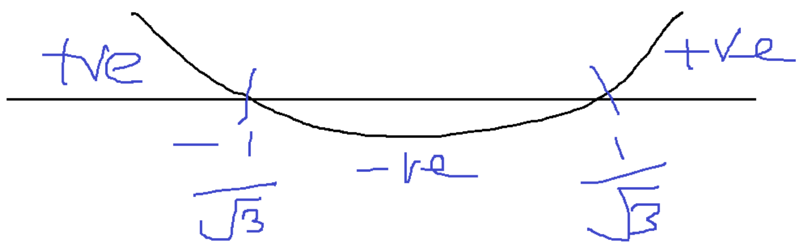
Example 3.49

Determine the turning points for the function below and classify them as maxima or minima:

$$y = x^3 - x$$

$$y' = 3x^2 - 1 = 0 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

$3x^2 - 1$ is a quadratic



Type equation here.

$$-\frac{1}{\sqrt{3}} \Rightarrow +ve\ to - ve \Rightarrow Maxima$$

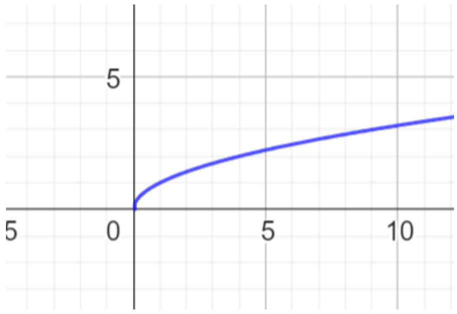
$$\frac{1}{\sqrt{3}} \Rightarrow -ve\ to + ve \Rightarrow Minima$$

C. Undefined Derivatives

3.50: Checking points where the derivative is not defined

Example 3.51

$$y = \sqrt{x}$$



Differentiate and equate to zero:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = 0 \Rightarrow No\ Solutions$$

For

$$x > 0, \frac{dy}{dx} > 0 \Rightarrow Function\ is\ increasing\ throughout\ its\ domain$$

However, this does not mean $y = \sqrt{x}$ does not have a minimum. It has a minimum where $f'(x)$ is not defined:

$$x = 0 \Rightarrow \frac{dy}{dx} \text{ is not defined}$$

Hence

$$x = 0 \Rightarrow y = \sqrt{0} = 0\ is\ a\ minimum$$

D. Exponential and Logarithmic Functions

Example 3.52

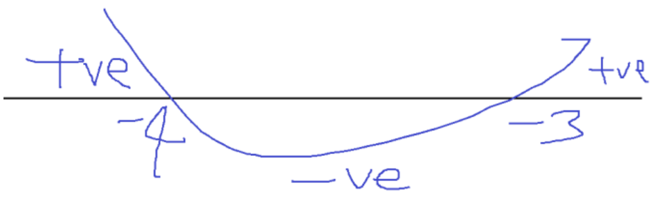
$$y = e^x(x^2 + 5x + 7)$$

Using the product rule:

$$\frac{dy}{dx} = e^x(x^2 + 7x + 12) = 0$$

$$e^x\ is\ never\ zero$$

$$x^2 + 7x + 12 = 0 \Rightarrow x = -3\ OR\ x = -4$$



$$-4 \Rightarrow +ve\ to - ve \Rightarrow Maxima$$
$$-3 \Rightarrow -ve\ to + ve \Rightarrow Minima$$

Example 3.53

$$y = e^x(\sin x)$$

$$\frac{dy}{dx} = e^x(\cos x + \sin x) = 0$$

$$\cos x + \sin x = 0$$
$$\sin x = -\cos x$$
$$\tan x = -1$$
$$x = \frac{3\pi}{4} \ or \ x = \frac{7\pi}{4}$$

$$\cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 0 + 1 = 1 > 0$$
$$\cos\pi + \sin\pi = 0 - 1 = -1 < 0$$
$$\cos 2\pi + \sin 2\pi = 1 + 0 = 1 > 0$$

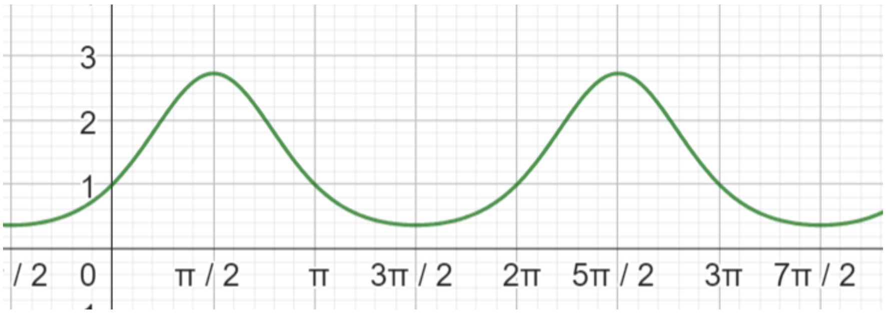
$$\text{At } x = \frac{3\pi}{4}: \frac{dy}{dx} \ goes\ from + ve\ to - ve \Rightarrow Maxima$$
$$\text{At } x = \frac{7\pi}{4}: \frac{dy}{dx} \ goes\ from - ve\ to + ve \Rightarrow Minima$$

Example 3.54

$$y = e^{\sin x}$$

Method I: Graphing

Can be done by graphing:



$$Maxima\ at\ x = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}$$

Method II: Calculus

Can also be done using Calculus:

$$\frac{dy}{dx} = (\cos x)e^{\sin x}$$
$$\cos x = 0 \Rightarrow x = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}$$

$$\cos x \ \ goes\ from\ positive\ to\ negative\ at\ the\ critical\ points \Rightarrow Maxima$$

3.4 y': Point of Inflection

A. Basics

3.55: Test for Point of Inflection

Test using

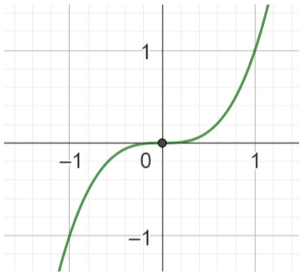
- Graph: If the slope of a graph is zero at a point, but that point is not a turning point for the graph, then

it is a point of inflection
- First derivative: If the first derivative has a turning point, then that point is a point of inflection for the graph
- First derivative: If the first derivative is zero at a point, but does not change sign before and after that point, then it is a point of inflection.

It is not necessary that the first derivative be zero at a point of inflection.

Example 3.56

$$f(x) = x^3$$



$$f'(x) = 3x^2$$

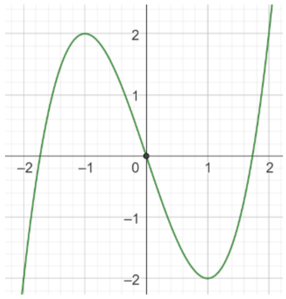
$f'(x)$ has a turning point at $x = 0$. Hence:

$$x = 0 \text{ is a point of inflection}$$

Example 3.57

$$f(x) = x^3 - 3x$$

A. Determine the points of inflection.
B. Is the first derivative zero at the point of inflection



$$f'(x) = 3x^2 - 3$$

$3x^2 - 3$ is a quadratic with vertex $(0, -3)$, and hence

$$x = 0 \text{ is a turning point of } f'(x)$$
$$x = 0 \text{ is a point of inflection of } f(x)$$

$$f'(0) = 3(0)^2 - 3 = -3 \neq 0$$

Example 3.58

$$y' = \frac{(x-7)^3(x-3)^2}{(x-1)^5(x-4)}$$

Identify the:

A. Increasing and Decreasing Intervals for y
B. x values at which y has a maximum, minimum, or points of inflection

Part A

We have already been given the derivative. Do not make the mistake of trying to differentiate it.

$$\frac{(x-7)^3(x-3)^2}{(x-1)^5(x-4)} > 0$$

To find the critical points, we equate the numerator and the denominator to zero and solve it:

$$(x-7)(x-3) = 0 \Rightarrow x \in \{3,7\}$$
$$(x-1)(x-4) = 0 \Rightarrow x \in \{1,4\}$$

The critical points are:

$$x \in \{1,3,4,7\}$$

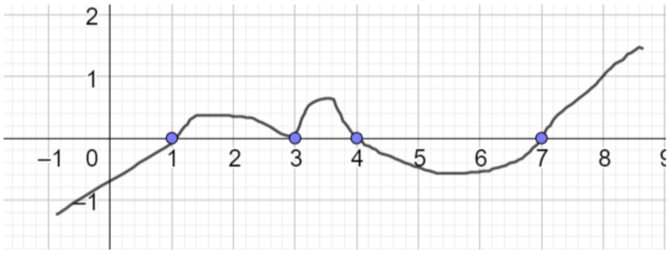
We make a sign diagram. Note that:

$$(x-7)^3 \Rightarrow Power\ is\ 3 \Rightarrow 3\ is\ odd \Rightarrow It\ will\ change\ sign$$
$$(x-3)^2 \Rightarrow Power\ is\ 2 \Rightarrow 2\ is\ even \Rightarrow It\ will\ not\ change\ sign$$
$$(x-1)^5 \Rightarrow Power\ is\ 5 \Rightarrow 5\ is\ odd \Rightarrow It\ will\ change\ sign$$
$$(x-4) \Rightarrow Power\ is\ 1 \Rightarrow 1\ is\ odd \Rightarrow It\ will\ change\ sign$$

$$x = 8, all\ the\ terms\ are\ positive \Rightarrow y' = 0$$

Since each root occurs at most once, the values alternate between positive and negative.

	$(-\infty, 1)$	$(1,3)$	$(3,4)$	$(4,7)$	$(7, \infty)$
y'	$-ve$	$+ve$	$+ve$	$-ve$	$+ve$
	$Decreasing$	$Increasing$	$Increasing$	$Decreasing$	$Increasing$



Part B

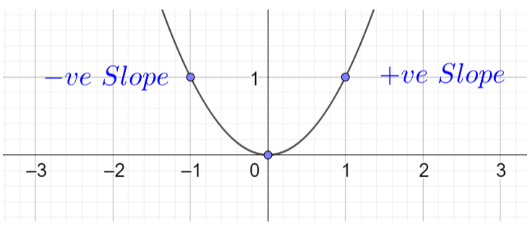
$$x = 1 \Rightarrow Minimum$$
$$x = 3 \Rightarrow Point\ of\ Inflection$$
$$x = 4 \Rightarrow Maximum$$
$$x = 7 \Rightarrow Maximum$$

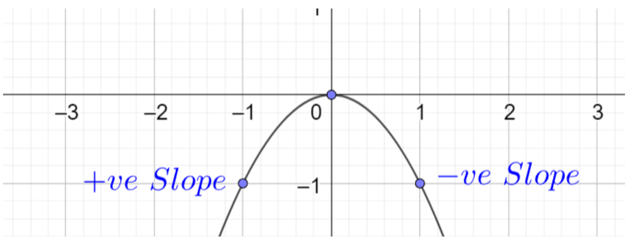
3.5 y'': Concavity and Points of Inflection

A. Concavity

3.59: Parabolas Opening Up and Down

- A parabola with positive leading coefficient is an "upward" parabola. This is a special case of a "concave up" graph.
- A parabola with negative leading coefficient is an "downward" parabola. This is a special case of a "concave down" graph.

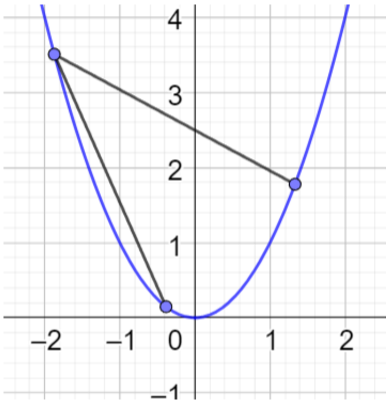




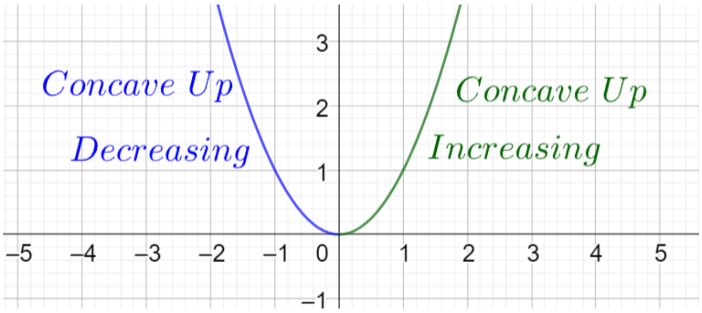
3.60: Concave Up

If the line segment joining any two points of a graph lies entirely above the graph for the entire interval of the line segment, then the graph is a concave up graph.

- Any line segment connecting two points on the blue graph always has the blue line above it (for the interval over which the line segment is drawn)

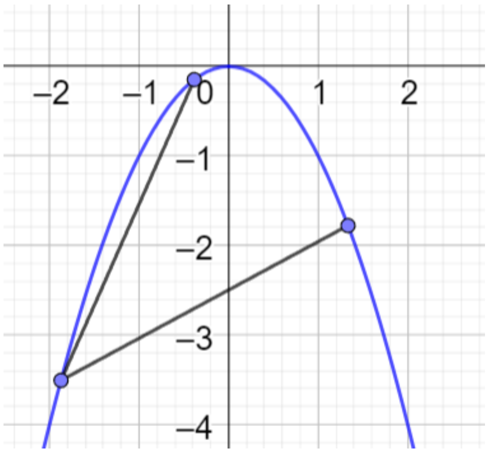


3.61: Concave Up (Increasing vs. Decreasing)

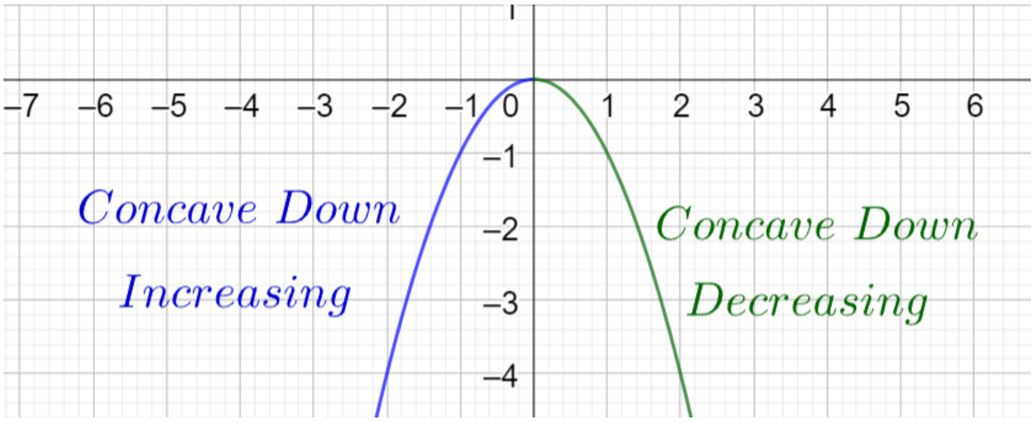


3.62: Concave Down

If the line segment joining any two points of a graph lies entirely below the graph for the entire interval of the line segment, then the graph is a concave down graph.

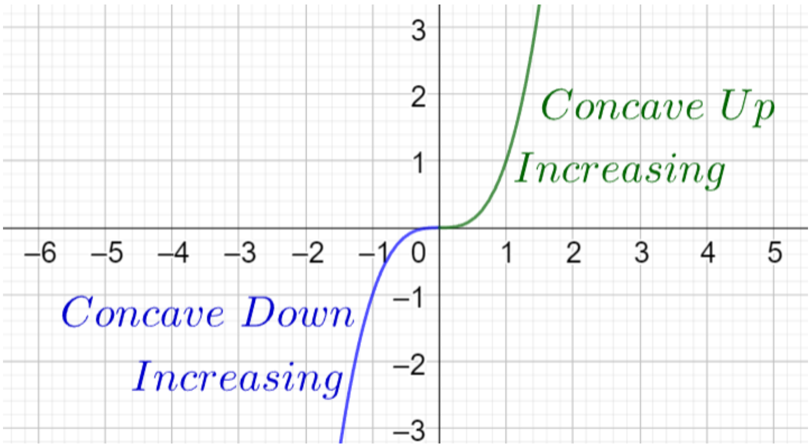


3.63: Concave Up (Increasing vs. Decreasing)



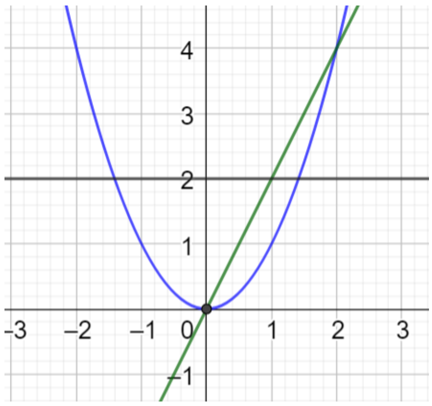
3.64: Point of Inflection

A point of inflection is a point where the graph changes concavity.



3.65: Second Derivative Test for Concavity

- If $y'' > 0$ over an interval, then the function is concave up over that interval.
- If $y'' < 0$ over an interval, then the function is concave down over that interval.



Note that

$$y'' \text{ is the derivative of } y'$$

- $y'' > 0$ means that y' has positive slope, and is increasing.
  - ✓ If y' is increasing, then the function is said to be concave up.
- $y'' < 0$ means that y' has negative slope, and is decreasing.
  - ✓ If y' is decreasing, then the function is said to be concave down.

Example 3.66

Determine the concavity of the following functions:

A. $y = x^2$
B. $y = x^3$

Part A

$$y = x^2 \Rightarrow y' = 2x \Rightarrow y'' = 2 > 0 \Rightarrow Concave\ Up\ over\ (-\infty, +\infty)$$

Part B

$$y = x^3 \Rightarrow y' = 3x^2 \Rightarrow y'' = 6x$$
$$6x > 0 \Rightarrow x > 0$$
$$Concave\ down\ for\ x \in (0, \infty)$$
$$Concave\ up\ for\ x \in (-\infty, 0)$$

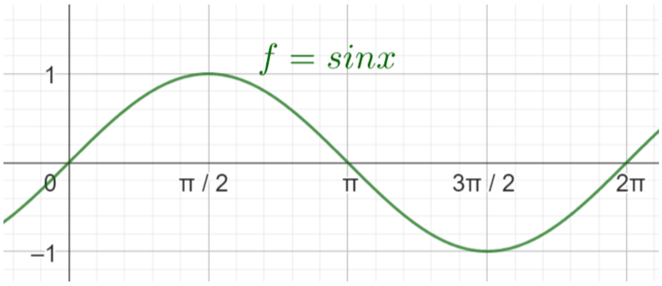
At $x = 0$:

$$function\ changes\ concavity \Rightarrow x = 0\ is\ a\ point\ of\ inflection$$

Example 3.67

Determine the concavity of $f = \sin x$ from its graph.

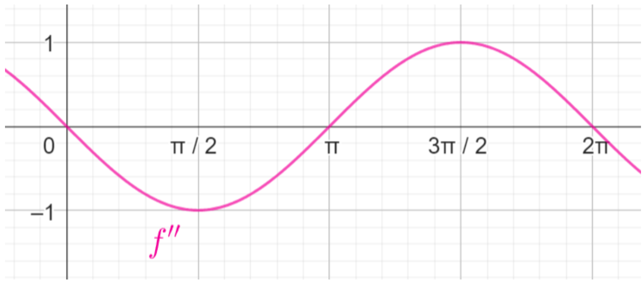
$$(0, \pi): Concave\ Up$$
$$(\pi, 2\pi): Concave\ Down$$
$$Point\ of\ Inflection: x \in \{0, \pi, 2\pi\}$$



Example 3.68

Given the graph of f'', determine over the interval $(0, 2\pi)$, the

A. concavity of f
B. points of inflection, if any



$$Between\ (0, \pi)\ f'' < 0 \Rightarrow f\ is\ concave\ down$$
$$Between\ (\pi, 2\pi)\ f'' > 0 \Rightarrow f\ is\ concave\ up$$
$$At\ \pi: f''\ changes\ sign \Rightarrow f\ has\ a\ point\ of\ inflection$$

Example 3.69

Is it necessary that f' is zero at a point of inflection?
If no, provide a counterexample.

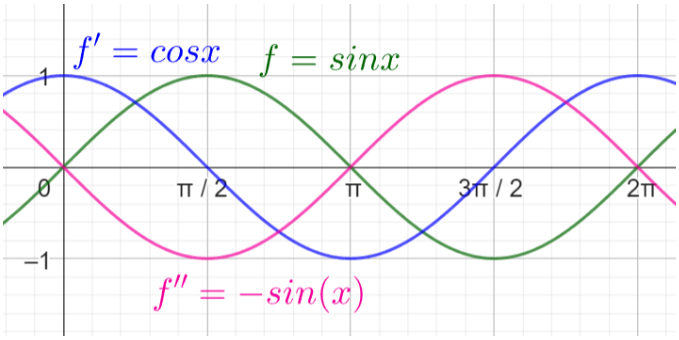
Consider

$$f = \sin x, f' = \cos x, f'' = -\sin x$$

At π:

$$f'' \text{ changes sign} \Rightarrow f \text{ has a point of inflection}$$
$$At\ \pi: f'(\pi) = \cos\pi = -1 \neq 0$$

Hence,
$$f' \text{ need not be zero at a point of inflection}$$



3.70: Second Derivative Test for Maximum and Minimum

For a function $y = f(x)$, if at a point $y' = 0$ and

- $y'' > 0$, then the function has a minimum.
- $y'' < 0$, then the function has a maximum.
- $y'' = 0$, then the function may have a maximum, a minimum or a point of inflection.

Example 3.71

Find the maxima and minima of:
$$y = x^3 - x$$

$$y' = 3x^2 - 1 = 0 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm\frac{1}{\sqrt{3}}$$
$$y'' = 6x$$

Substitute
$$y''\left(\frac{1}{\sqrt{3}}\right) = 6\left(\frac{1}{\sqrt{3}}\right) > 0 \Rightarrow Min\ at\ x = \frac{1}{\sqrt{3}}$$
$$y''\left(-\frac{1}{\sqrt{3}}\right) = 6\left(-\frac{1}{\sqrt{3}}\right) < 0 \Rightarrow Max\ at\ x = -\frac{1}{\sqrt{3}}$$

To find the points of inflection, equate the second derivative to zero:
$$y'' = 6x = 0 \Rightarrow x = 0$$
$$y''\left(\frac{1}{\sqrt{3}}\right) = 6\left(\frac{1}{\sqrt{3}}\right) > 0 \Rightarrow Concave\ Up$$
$$y''\left(-\frac{1}{\sqrt{3}}\right) = 6\left(-\frac{1}{\sqrt{3}}\right) < 0 \Rightarrow Concave\ Down$$

Example 3.72

$$y = x^3 + 3x^2 + 1$$

Find the first derivative and equate it to zero to find the critical points:
$$y' = 3x^2 + 6x = 0 \Rightarrow x(3x + 6) = 0 \Rightarrow x \in \{0, -2\}$$

Find the second derivative and use it to check the critical points:
$$y'' = 6x + 6$$
$$y''(0) = 6(0) + 6 = 6 > 0 \Rightarrow Min$$
$$y''(-2) = 6(-2) + 6 = -6 < 0 \Rightarrow Max$$

To find the points of inflection, equate the second derivative to zero:

$$6x + 6 = 0 \Rightarrow x = -1$$

Use the second derivative test for points of inflection.

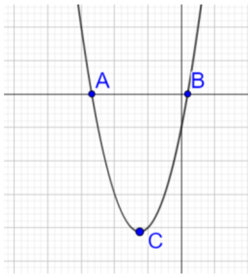
$$y''(0) = 6(0) + 6 = 6 > 0 \Rightarrow Concave\ Up$$
$$y''(-2) = 6(-2) + 6 = -6 < 0 \Rightarrow Concave\ Down$$

$x = -1$ is a point of inflection since the function changes its concavity at that point.

Example 3.73

The function $y = f(x)$ with one turning point (at C) is graphed alongside.
Determine where:

A. y is increasing/decreasing/constant.
B. y is positive/negative.
C. y' is positive/negative/zero.
D. y'' is positive/negative



Part A

We can observe from the graph that y is:

$$\underbrace{(-\infty, C)}_{Decreasing}, \underbrace{C}_{Constant}, \underbrace{(C, \infty)}_{Increasing}$$

Part B

We can observe from the graph that y is:

$$\underbrace{(-\infty, A) \cup (B, \infty)}_{Positive}, \underbrace{(A, B)}_{Negative}$$

Part C

When y is decreasing, its slope is negative, when y is increasing its slope is positive, and when y is constant, its slope is zero.
Since y' is the slope, we can directly convert the answers from Part A into the answers for Part C.

$$\underbrace{(-\infty, C)}_{Negative}, \underbrace{C}_{Zero}, \underbrace{(C, \infty)}_{Positive}$$

Part D

The function has a single turning point at C.
$y' < 0$ to the left of C, and $y' > 0$ to the right of C.
$y'' > 0$ over an interval around C.
Since the function has a single turning point, y'' cannot change its behavior.

$$\therefore y'' > 0, x \in \mathbb{R}$$

Example 3.74

The function $y = f(x)$ y is $y = x^2 + 5x + 6$. Determine where:

A. y is increasing/decreasing/constant.
B. y is positive/negative.
C. y' is positive/negative/zero.
D. y'' is positive/negative
E. y is concave up/concave down

Parts A and C

$$y' = 2x + 5 > 0 \Rightarrow x > -\frac{5}{2}, \qquad 2x + 5 = 0 \Rightarrow x = -\frac{5}{2}, \qquad 2x + 5 < 0 \Rightarrow x < -\frac{5}{2}$$

$$\underbrace{\left(-\infty,-\frac{5}{2}\right)}_{\substack{y\ is\ decreasing\\ y'\ is\ negative}},\quad \underbrace{-\frac{5}{2}}_{\substack{y\ is\ constant\\ y'=0}},\quad \underbrace{\left(-\frac{5}{2},\infty\right)}_{\substack{y\ is\ increasing\\ y'\ is\ positive}}$$

Parts D and E

$$y''=2>0\Rightarrow \underbrace{(-\infty,\infty)}_{\substack{y\ is\ positive\\ y\ is\ concave\ up}}$$

Example 3.75

For each part below, match each entry in Column I to the appropriate entry in Column II. For example, $1-A, 2-B, 3-B$ represents a answer choice.

A. The function $y=f(x)$ has a maximum at $x=c$.
B. The function $y=f(x)$ has a minimum at $x=c$.
C. The function $y=f(x)$ has a turning point at $x=c$.

Column I	Column II
1. $f(c)$	A. *Positive*
2. $f'(c)$	B. *Negative*
3. $f''(c)$	C. *Zero*
	D. *Cannot be determined*

$$1-D, 2-C, 3B$$

Example 3.76

$$y=ax^2+bx+c$$

Equate the first derivative to zero:

$$y'=2ax+b=0\Rightarrow x=-\frac{b}{2a}$$
$$y''=2a>0\Rightarrow a>0$$

If

$$a>0\Rightarrow y''>0\Rightarrow Concave\ Up\Rightarrow -\frac{b}{2a}\ is\ a\ minimum$$

$$a<0\Rightarrow y''<0\Rightarrow Concave\ Down\Rightarrow -\frac{b}{2a}\ is\ a\ maximum$$

3.77: Point of Inflection

A point of inflection is a point where the graph of a function changes shape from concave up to concave down or vice versa.

Since the graph changes shape

- the second derivative will change sign
- the first derivative will not change sign

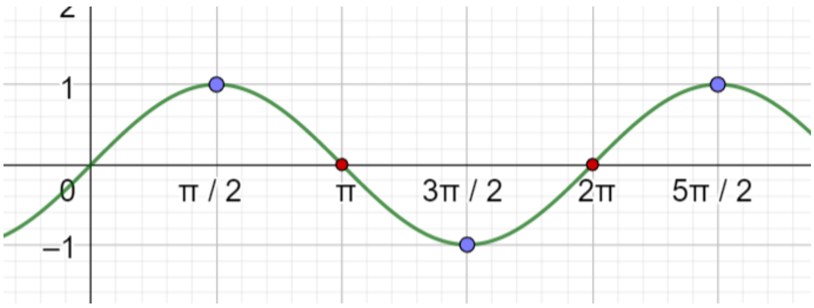
3.78: Inflection between Max and Min

Between a relative maximum and a relative minimum that are connected by a continuous graph, there exists a point of inflection.

A relative maximum has a concave down shape, and a relative minimum has a concave up shape.

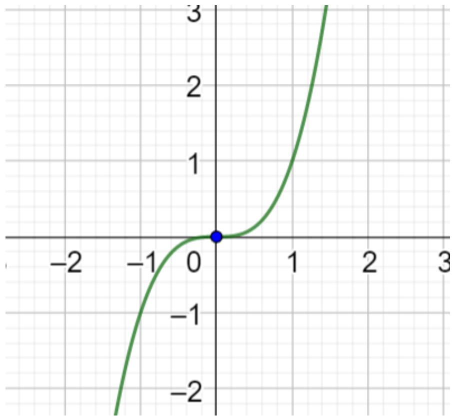
The point where the graph transitions from concave down to concave up or *vice versa* is a point of inflection.

Example 3.79
Classify each marked point in the graph alongside as a maximum, a minimum or a point of inflection.



$$\left(\frac{\pi}{2}, 1\right) \Rightarrow Max$$
$$(\pi, 0) \Rightarrow Point\ of\ Inflection$$
$$\left(\frac{3\pi}{2}, -1\right) \Rightarrow Min$$
$$(2\pi, 0) \Rightarrow Point\ of\ Inflection$$
$$\left(\frac{5\pi}{2}, 1\right) \Rightarrow Max$$

Example 3.80



$$Point\ of\ Inflection: x = 0$$

Example 3.81
Determine where the functions is increasing/decreasing, its concavity, the turning points and the points of inflection. Also, determine the vertical and horizontal asymptotes. Hence, graph the function.

A. $y = \frac{x}{x+5}$

Equate the denominator to zero to find the domain of the function

$$x + 5 = 0 \Rightarrow x = -5 \Rightarrow D_f = All\ real\ numbers\ other\ than - 5$$
$$Vertical\ Asymptote\ at\ x = -5$$

Evaluate the limit of the function at infinity to find the horizontal asymptote.

$$\lim_{x\to\infty} \frac{x}{x+5} = \lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{x}{x} + \frac{5}{x}} = = \lim_{x\to\infty} \frac{1}{1 + \frac{5}{x}} = 1$$

Find the first derivative:

$$y' = \frac{(x+5)(1) - (x)(1)}{(x+5)} = \frac{5}{(x+5)^2}$$

$\frac{5}{(x+5)^2} > 0$ for all numbers in its domain, so the function is increasing for all numbers in its domain.

Equate the first derivative to zero to find the turning points:

$$\frac{5}{(x+5)^2} = 0 \Rightarrow No\ Solutions \Rightarrow No\ turning\ points$$

Find the second derivative:

$$y'' = \frac{-10(x+5)}{(x+5)^4} = -\frac{10}{(x+5)^3}$$

The critical points are:

$$(x+5)^3 = 0 \Rightarrow x = -5$$

	$(-\infty, -5)$	$(-5, \infty)$
	$+ve$	$-ve$
Shape	$Concave\ Up$	$Concave\ Down$

The function changes from concave up to concave down at $x = -5$. This would resulted in a point of inflection, but the function is not defined at $x = -5$.
Hence, there are no points of inflection.

Example 3.82

Determine where the function is increasing/decreasing, its concavity, the turning points and the points of inflection. Also, determine the vertical and horizontal asymptotes. Hence, graph the function.

$$y = \sin x\,, 0 \le x \le \pi$$

B. Critical Points

Example 3.83

Determine where the function is increasing/decreasing, its concavity, the turning points and the points of inflection. Also, determine the vertical and horizontal asymptotes.

$$y = \frac{16}{x^2} + \frac{x^3}{3}$$

First Derivative

Find the first derivative

$$\frac{dy}{dx} = -\frac{32}{x^3} + x^2 = \frac{-32 + x^5}{x^3}$$

Equate the first derivative to zero to find the critical points:

$$\frac{-32 + x^5}{x^3} = 0 \Rightarrow x = 2$$

Set up and solve an inequality to find the increasing and decreasing intervals:

$$\frac{-32 + x^5}{x^3} > 0$$

Make a sign diagram

Second Derivative

Find the second derivative

$$\frac{d^2y}{dx^2} = \frac{96}{x^4} + 2x$$

Evaluate the second derivative at $x = 2$:

$$\left.\frac{d^2y}{dx^2}\right|_{x=2} = \frac{96}{16} + 2x = 10 > 0$$

Since the second derivative is greater than zero,

$$x = 2\ is\ a\ minimum$$

Example 3.84

A circular cylinder, open at one end, is

$$SA = \pi r^2 + 2\pi rh \Rightarrow 432\pi = \pi r(r + 2h) \Rightarrow \frac{432 - r^2}{2r} = h$$

$$V = \pi r^2 h = \pi r^2 \left(\frac{432 - r^2}{2r}\right) = \left(\frac{\pi}{2}\right)(432r - r^3) = \left(\frac{\pi}{2}\right)(432 \times 12 - 12^3)_{r=12} = 1728\pi$$

$$\frac{dV}{dr} = \left(\frac{\pi}{2}\right)(432 - 3r^2), \qquad 432 - 3r^2 = 0 \Rightarrow r = 12$$

$$\frac{d^2V}{dr^2} = \left(\frac{\pi}{2}\right)(-6r), \qquad which\ is\ -ve\ for\ positive\ r \Rightarrow Maximum$$

Example 3.85

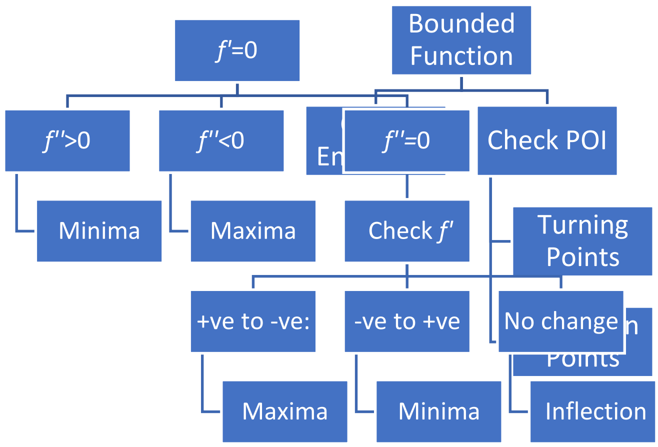
Find the area of the triangle formed by the origin, and the intercepts of the tangent to the curve $y = x^2 + \frac{24}{x}$ at the point $(2,16)$.

$$\frac{dy}{dx} = 2x - \frac{24}{x^2}, \qquad \frac{dy}{dx}_{x=2} = 2(2) - \frac{24}{2^2} = 4 - 6 = -2$$

Substitute $(2,16)$ and the slope in the slope-point form to get the equation of the tangent. Then calculate the intercept, and use the formula for area of a triangle.

$$y - y_1 = m(x - x_1) \Rightarrow y - 16 = (-2)(x - 2) \Rightarrow \underbrace{y = -2x + 20}_{\substack{x-intercept=10 \\ y-intercept=20}} \Rightarrow A(\Delta) = \frac{bh}{2} = \frac{10 \times 20}{2} = 100$$

C. Finding Maxima/Minima



Example 3.86

Find the maximum/minimum values for $f(x) = 3x^2 + 5x + 4, x \in [-4, 3]$

Find the first and second derivatives:

$$f'(x) = 6x + 5 \Rightarrow \underbrace{f'' = (6x + 5)' = 6}_{+ve \Rightarrow Minima}$$

The turning point we found is a minimum. Equate the first derivative to zero to find the value of x, at the turning point:

$$\underbrace{6x + 5 = 0 \Rightarrow x = -\frac{5}{6}}_{Equate\ f'\ to\ zero} \Rightarrow \underbrace{f\left(-\frac{5}{6}\right) = 3\left(-\frac{5}{6}\right)^2 + 5\left(-\frac{5}{6}\right) + 4 = 1.916}_{Evaluate\ f(x)\ at -\frac{5}{6}}$$

Check the value of $f(x)$ at the endpoints:

$$\underbrace{f(-4) = 3(-4)^2 + 5(-4) + 4 = 32}_{Left\ Domain\ Endpoint}, \qquad \underbrace{f(3) = 3(3)^2 + 5(3) + 4 = 46}_{Right\ Domain\ Endpoint},$$

Example 3.87

Find the turning points for $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + 99$

Find the first derivative and equate it to zero:

$$f' = x^2 + 5x + 6 = 0 \Rightarrow (x + 2)(x + 3) = 0 \Rightarrow x = (-2, -3)$$

Evaluate the second derivative at the x values where the first derivative is zero:

$$f'' = 2x + 5 \Rightarrow \underbrace{f''(-2) = 2(-2) + 5 = 1}_{+ve \Rightarrow Minima}, \quad \underbrace{f''(-3) = 2(-3) + 5 = -1}_{-ve \Rightarrow Minima}$$

Example 3.88

$$f(x) = x^3 - x$$

$$f'(x) = 3x^2 - 1 \Rightarrow 3x^2 - 1 = 0 \Rightarrow x = \frac{1}{\pm\sqrt{3}}$$

$$f''(x) = 6x \Rightarrow 6x \text{ is } + ve \text{ when } x = \frac{1}{+\sqrt{3}} \Rightarrow Minimum$$

$$6x \text{ is } - ve \text{ when } x = \frac{1}{-\sqrt{3}} \Rightarrow Maximum$$

Set the second derivative to second:

$$Point\ of\ Inflection: 6x = 0 \Rightarrow x = 0, y = 0$$
$$6x < 0 \Rightarrow x < 0 \Rightarrow Concave\ Down \in (-\infty, 0)$$
$$6x > 0 \Rightarrow x > 0 \Rightarrow Concave\ Up \in (0, \infty)$$

The graph changes from concave down to concave up at $x = 0$. Hence, this is a point of inflection.

Example 3.89

$$\mathrm{f}(x) = x^4 - 3x + 2$$
$$f'(x) = 4x^3 - 3 \Rightarrow$$

Example 3.90

Determine the concavity of $y = x(4 - x^2)^{\frac{1}{2}}$

Domain

Determine the domain:

$$4 - x^2 \geq 0 \Rightarrow x^2 \leq 4 \Rightarrow -2 \leq x \leq 2$$

First Derivative

Find the first derivative:

$$\frac{dy}{dx} = \frac{x}{2\sqrt{4 - x^2}}(-2x) + \sqrt{4 - x^2} = \frac{-x^2}{\sqrt{4 - x^2}} + \sqrt{4 - x^2} = \frac{-x^2 + 4 - x^2}{\sqrt{4 - x^2}} = \frac{-2x^2 + 4}{\sqrt{4 - x^2}}$$

The derivative is not defined for

$$\sqrt{4 - x^2} = 0 \Rightarrow 4 - x^2 = 0 \Rightarrow x = \pm 2$$

Set the first derivative equal to zero to find any other critical points:

$$\frac{-2x^2 + 4}{\sqrt{4 - x^2}} = 0 \Rightarrow -2x^2 + 4 = 0 \Rightarrow 2x^2 = 4 \Rightarrow x = \pm\sqrt{2}$$

The critical points for the function are:

$$\{\pm\sqrt{2}, \pm 2\}$$

Note that the critical points $= \pm 2$ are also the endpoints of the domain. Evaluate the values at the endpoints of the domain:

$$y(2) = x(4-(2)^2)^{\frac{1}{2}} = 2(4-4)^{\frac{1}{2}} = 2(4-4)^{\frac{1}{2}} = 2(0) = 0$$
$$y(-2) = 0$$

Increasing and Decreasing Intervals

To find the increasing intervals, check where the first derivative is greater than zero:

$$\frac{-2x^2+4}{\sqrt{4-x^2}} > 0$$

The critical points are:

$$x = \pm\sqrt{2}$$
$$\sqrt{4-x^2} \Rightarrow x = \pm 2$$

$(-2,-\sqrt{2})$	$(-\sqrt{2},\sqrt{2})$	$(\sqrt{2},2)$
$-ve$	$+ve$	$-ve$

$$At -\sqrt{2}\ y'\ goes\ from -ve\ to +ve: Min$$
$$At\ \sqrt{2}\ y'\ goes\ from +ve\ to -ve: Max$$

Second Derivative

Calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\sqrt{4-x^2}(-4x) - (-2x^2+4)\cdot\frac{-x}{\sqrt{4-x^2}}}{4-x^2}$$

$$= \frac{\sqrt{4-x^2}(-4x) + \frac{-2x^3+4x}{\sqrt{4-x^2}}}{4-x^2} = \frac{\frac{(4-x^2)(-4x)-2x^3+4x}{\sqrt{4-x^2}}}{4-x^2}$$

$$= \frac{-16x+4x^3-2x^3+4x}{(4-x^2)^{\frac{3}{2}}} = \frac{2x^3-12x}{(4-x^2)^{\frac{3}{2}}} = \frac{2x(x^2-6)}{(4-x^2)^{\frac{3}{2}}}$$

Check where the second derivative is greater than zero. The critical points:

$$2x(x^2-6) \Rightarrow x = 0, \pm\sqrt{3}$$
$$(4-x^2)^{\frac{3}{2}} = 0 \Rightarrow x = \pm 2$$

$(-2,-\sqrt{3})$	$(-\sqrt{3},0)$	$(0,\sqrt{3})$	$(\sqrt{3},2)$
$+ve$	$+ve$	$-ve$	$-ve$

$$Concave\ up\ from\ (-2,0)$$
$$Concave\ down\ from\ (0,2)$$

Example 3.91: Newton's Serpentine

The function $x^2y + a^2y - abx = 0, ab > 0$ was studied by $L'Hopital$ and $Huygens$. $Newton$ studied, named and classified it.

A. Show that it is a function.
B. Find the intercepts and asymptotes.
C. Find the critical points using the first derivative test, and determine whether they are maxima or minima.

D. Use the second derivative test to find points of inflection, if any.
E. Sketch the function.

Part A
Solve for y:

$$y(x^2 + a^2) = abx \Rightarrow y = \frac{abx}{x^2 + a^2}$$

The above is a function since it will give a single value of y for a valid value of x.

Part B
Find the intercepts:

$$x = 0 \Rightarrow y = \frac{abx}{x^2 + a^2} = \frac{ab(0)}{0^2 + a^2} = 0 \Rightarrow y - intercept\ is\ (0,0)$$

$$y = 0 \Rightarrow 0 = \frac{abx}{x^2 + a^2} \Rightarrow 0 = abx \Rightarrow x - intercept\ is\ (0,0)$$

Find the asymptotes:

$$y = \frac{abx}{x^2 + a^2} \Rightarrow No\ vertical\ asymptote\ since\ denominator\ is\ never\ zero$$

$$y = \frac{abx}{x^2 + a^2} \Rightarrow Has\ horizontal\ asymptote\ y = 0$$

Part C
Find the first derivative:

$$y' = \frac{(ab)(a^2 - x^2)}{(x^2 + a^2)^2}$$

Equate the first derivative to zero to find the critical points:

$$\frac{(ab)(a^2 - x^2)}{(x^2 + a^2)^2} = 0 \Rightarrow (ab)(a^2 - x^2) = 0$$

Divide by ab, since $ab > 0$:

$$a^2 - x^2 = 0' \Rightarrow x = \pm a$$

Make a sign diagram for the first derivative:

$$\frac{(ab)(a^2 - x^2)}{(x^2 + a^2)^2} \to \frac{(+ve)(a^2 - x^2)}{(+ve)}$$

Two terms are positive. The sign depends on the third term:

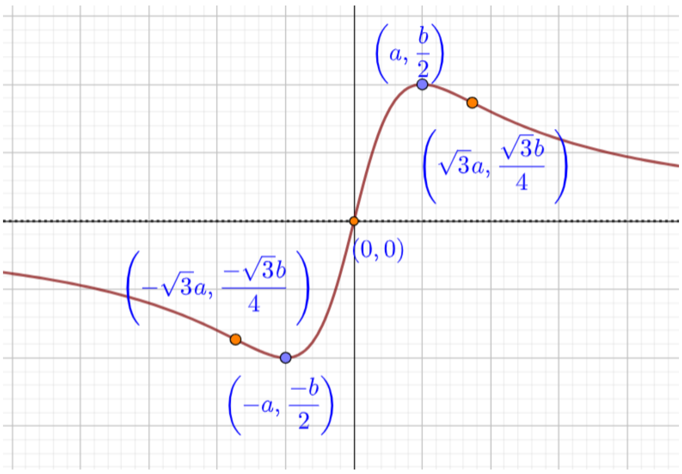
$$a^2 - x^2 > 0 \Rightarrow x^2 < a^2 \Rightarrow x \in (-a, a)$$

	$(-\infty, -a)$	$(-a, a)$	(a, ∞)
$a^2 - x^2$	$-ve$	$+ve$	$-ve$
$\frac{abx}{x^2 + a^2}$	Decreasing	Increasing	Decreasing

From the sign diagram:

$$x = -a \Rightarrow y = \frac{ab(-a)}{x^2 + a^2} = -\frac{b}{2} \text{ is a global minina}$$

$$x = a \Rightarrow y = \frac{ab(a)}{x^2 + a^2} = \frac{b}{2} \text{ is a global maxima}$$



Part D

Find the second derivative, and equate it to zero:

$$y'' = (ab)\left[\frac{(x^2 + a^2)^2(-2x) - (a^2 - x^2)2(x^2 + a^2)(2x)}{(x^2 + a^2)^4}\right]$$

Factor $(2x)(x^2 + a^2)$ from the numerator:

$$= (ab)\left[\frac{(2x)(x^2 + a^2)[-(x^2 + a^2) - (a^2 - x^2)2]}{(x^2 + a^2)^4}\right]$$

Cancel and simplify:

$$= (ab)\left[\frac{(2x)[(-x^2 - a^2) - (2a^2 - 2x^2)]}{(x^2 + a^2)^3}\right]$$

Simplify further:

$$= (ab)\left[\frac{(2x)(x^2 - 3a^2)}{(x^2 + a^2)^3}\right]$$

$$y'' = \frac{(ab)(2x)(x^2 - 3a^2)}{(x^2 + a^2)^3} = 0$$

$$x \in \{-\sqrt{3}a, 0, \sqrt{3}a\}$$

From the sign diagram for the second derivative, each of the three points $\{-\sqrt{3}a, 0, \sqrt{3}a\}$ is a point of inflection.

Example 3.92

$$y = 2e^{3x}$$

Find the first derivative and see where it is greater than zero.

$$y' = 6e^{3x} > 0 \Rightarrow e^{3x} > 0 \Rightarrow \text{Always True} \Rightarrow \text{Increasing over } \mathbb{R}$$

Find the second derivative and see where it is less:

$$y'' = 18e^{3x} < 0 \Rightarrow \text{Never True} \Rightarrow \text{Never Concave Down}$$

3.6 Curve Sketching Questions

A. Summary of Non-Derivative Rules

3.93: Intercepts
The intercepts are the places where the function intersects the x and the y axis. To find the
- x −intercepts, substitute $y = 0$
- y −intercepts, substitute $x = 0$

3.94: Domain of the Function
The domain of the function is all the numbers which are valid inputs for the function.

3.95: Continuous versus Non-Continuous Function
A continuous function is one which can be drawn without lifting your pencil. If a function is not continuous, it is discontinuous. The types of discontinuity are:

A. Jump Discontinuity
B. Removable Discontinuity
C. Infinite Discontinuity
D. Oscillating Discontinuity

3.96: Vertical Asymptotes
Identify the vertical asymptotes of a function by determining where the denominator is zero.

- We determine where the denominator is zero by setting the denominator equal to zero and solving the resulting equation.
- In an infinite discontinuity, the limit to the left, and the limit to the right at the points where the denominator equals zero approaches positive infinity, or negative infinity.
- In a removable discontinuity, the limit to the left, and the limit to the right at the points where the denominator equals zero do not approach positive infinity, or negative infinity.

3.97: Horizontal Asymptotes
To find the horizontal asymptote, you find the limit of the function at positive and negative infinity.

Some specific cases for polynomial functions, and rational functions:
- Degree of numerator is less than degree of denominator: Horizontal asymptote is the $x - axis$, or $y = 0$.
- Degree of numerator is equal to degree of denominator: Horizontal asymptote is ratio of leading coefficients when the numerator and denominator are written in standard form.
- Degree of numerator is one greater than degree of denominator: No Horizontal asymptote. Oblique asymptote, which can be found by dividing numerator by denominator.
- Degree of numerator is two greater than degree of denominator: No Horizontal asymptote. Quadratic expression which the function approaches asymptotically, which can be found by dividing numerator by denominator.

Rational functions are functions where the numerator and denominator are polynomials.

B. Summary of Derivative Rules

3.98: Increasing and Decreasing Intervals

If the first derivative is:

- Positive, function is increasing
- Negative, function is decreasing

- To find whether a quantity is increasing or decreasing, we set up an inequality.
- To solve the inequality, we find the critical points of the inequality (where the numerator or the denominator becomes zero).
- To determine the intervals where the inequality holds, we use a sign diagram and then apply. To
  - ✓ The Wavy Curve method
  - ✓ Or use test points for each interval in the sign diagram.

3.99: Turning Points

Critical points are values where:

- $f' = 0$
- OR f' is undefined
- OR It is an endpoint of the domain

If:

- f' is zero, check using the first derivative test OR the second derivative test
- f' is undefined, check whether the function is increasing or decreasing around that value using the first derivative test
- It is an endpoint, check around that value using the first derivative test

3.100: First Derivative Test

If, at a critical point the first derivative

- changes from positive to negative, the point is a max.
- changes from negative to positive, the point is a min.
- does not change sign, it is a point of inflection.

3.101: Second Derivative Test

If at a point, the second derivative is
negative, it is a max.
positive, it is a min.
zero, the test is inconclusive.

3.102: Concavity

If the second derivative is:
Positive, the function is concave up
Negative, the function is concave down

3.103: Standard Instruction for Curve Sketching

When asked to sketch a curve, pay attention to the following:

A. x and y intercepts
B. Domain

C. Discontinuities: Point/Jump/Infinite
D. Asymptotes: Horizontal / Vertical / Slant
E. Increasing and Decreasing Intervals
F. Turning Points (aka Local Maxima/Minima) and Global Max/Min
G. Concavity
H. Period, if any
I. Points of Inflection
    a. Stationary points ($f' = 0$)
    b. Non-stationary ($f' \neq 0$)
J. Intersection with other graphs
K. Other points of interest specific to the question

C. Types of Questions

Find:

A. Find and classify stationary points
    I. Where the first derivative is zero / where the tangent is horizontal
B. Find and classify points of inflections
    I. Where the second derivative is zero
    II. And it is not a turning point
C. Find local maximum and minimum / turning points
    I. Equate first derivative to zero
    II. Use second derivative test OR first derivative test to eliminate stationary points of inflection
    III. Check the endpoints as well
    IV. Check points where the derivative is not defined
D. Find global max/min
    I. Find local max/min
    II. Compare to find the global max/min
E. Find the intercepts
    I. x-intercepts: Substitute y=0
    II. y-intercepts: Substitute x=0
F. Find the asymptotes:
    I. Vertical: check where the denominator is not defined (and there must be no cancelling factor in the numerator)
    II. Horizontal: Check the behaviour at positive and negative infinity. Either logically, or by finding the limits at infinity.
G. Find the concavity
    I. Calculate the second derivative.
    II. If the second derivative is positive over an interval: concave up
    III. If the second derivative is negative over an interval: concave down

Example 3.104

$$f(x) = y = e^{\sin px}, p \geq 0$$

Identify the:

A. Increasing and Decreasing Intervals for y
B. x values at which y has a maximum or a minimum.

Case I

Note that if $p = 0$ then:

$$e^{\sin px} = e^{\sin 0x} = e^0 = 1$$

Hence,

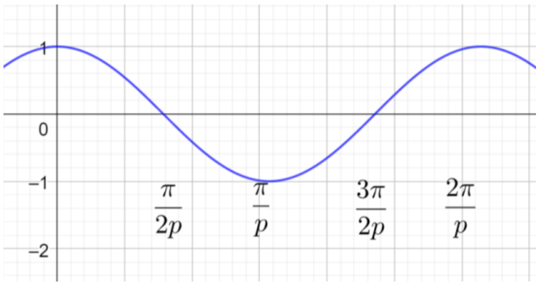
$$y \text{ is constant for all } x$$
$$\text{No increasing or decreasing intervals}$$
$$\text{No maximum or minimum}$$

Case II

If $p \neq 0$, then determine where the derivative is positive:

$$y' = p \cdot \cos px \cdot e^{\sin px} > 0$$

Note that since $p \neq 0$ *and* $e^{\sin px} > 0$, the sign is controlled by $\cos px$. Note that $\cos px$ is a horizontal stretch of $\cos x$ by a factor of $\frac{1}{p}$.



	$x = \frac{\pi}{2p} + \frac{2\pi K}{p}$	$x = \frac{3\pi}{2p} + \frac{2\pi K}{p}$
	Max	Min

$$\left(0, \frac{\pi}{2p}\right) \cup \left(\frac{3\pi}{2p}, \frac{2\pi}{p}\right)$$

But note that $\cos px$ is a period function with period $\frac{2\pi}{p}$.

Hence, we can add any integer multiple of $\frac{2\pi}{p}$ to the above solution, and it remains valid. Hence, the updated solution is:

$$\left(0 + \frac{2\pi K}{p}, \frac{\pi}{2p} + \frac{2\pi K}{p}\right) \cup \left(\frac{3\pi}{2p} + \frac{2\pi K}{p}, \frac{2\pi}{p} + \frac{2\pi K}{p}\right), K \in \mathbb{Z}$$

Example 3.105

$$y' = \cos x \cdot e^{\sin x}$$
$$y'' = \cos^2 x \cdot e^{\sin x} - \sin x \cdot e^{\sin x} = e^{\sin x}(\cos^2 x - \sin x)$$

Where is this positive:

$$e^{\sin x}(\cos^2 x - \sin x) > 0$$

Since $e^{\sin x}$ is always greater than zero, we find the critical point by equating the trigonometric term to zero:

$$\cos^2 x - \sin x = 0$$
$$1 - \sin^2 x - \sin x = 0$$

Substitute $z = \sin x$

$$1 - z^2 - z = 0$$
$$z^2 + z - 1 = 0$$
$$z \in \left\{-\frac{1 - \sqrt{5}}{2}, -\frac{1 + \sqrt{5}}{2}\right\}$$

Note that

$$z = \sin x = -\frac{1 - \sqrt{5}}{2} \approx -1.61 \Rightarrow x \in \phi$$

$$z = \sin x = -\frac{1+\sqrt{5}}{2} \Rightarrow x = \sin^{-1}\left(-\frac{1+\sqrt{5}}{2}\right)$$

Example 3.106

$$f(x) = y = e^{\sin x}$$

Identify the:

A. Concavity
B. x values at which y has a maximum or a minimum.

Type equation here.

Example 3.107

$y = ax^3 + bx^2 + cx + d$ has an inflection point at $(0,3)$ and a local maximum at $(1,5)$.

Substitute $(0,3)$ in $y = ax^3 + bx^2 + cx + d$:

$$3 = 0 + 0 + 0 + d \Rightarrow d = 3$$

Substitute $(1,5), d = 3$ in $y = ax^3 + bx^2 + cx + d$:

$$5 = a + b + c + 3 \Rightarrow \underbrace{2 = a + b + c}_{Equation\ I}$$

$$y' = 3ax^2 + 2bx + c$$

At the local maximum, y' must be zero. Substitute $(1,0)$ in $y' = 3ax^2 + 2bx + c$:

$$\underbrace{0 = 3a + 2b + c}_{Equation\ II}$$

$$y'' = 6ax + 2b$$

At the point of inflection, y'' must be zero:

$$6a(0) + 2b = 0 \Rightarrow b = 0$$

Substitute $b = 0$ in Equations I and II:

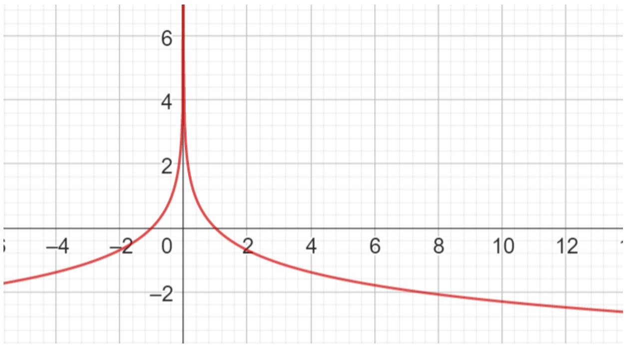
$$0 = 3a + c$$
$$2 = a + c$$

Subtract:

$$2a = -2 \Rightarrow a = -1$$
$$c = 3$$

Example 3.108

Find a function whose graph y = f(x)
A. has negative slopes, which increase as x increases;
B. has positive slopes, which decrease as x increases.



Slope is negative, we can take

$$-\frac{1}{x}, -x, \dots$$

As

$$x \text{ increases}, -x \text{ decreases}$$

$$x \text{ increases}, -\frac{1}{x} \text{ increases}$$

Hence, we want a function which when differentiated is $-\frac{1}{x}$

$$-\ln|x|$$

3.7 Graphing Derivatives

A. Basics

3.109: Using the First Derivative

We can arrive at some important conclusions related to graphs of derivatives based on properties that we already know:

- When $f(x)$ is increasing, f' is positive
- When $f(x)$ is decreasing, f' is negative
- At a turning point, f' is zero
- At a point of inflection, f' is zero

- Point of inflection is where the graph changes from concave up to concave down, or concave down to concave up. It is different from a turning point.

Example 3.110

Sketch the derivative of $f(x) = (x-2)^2 - 1$ without differentiating the function algebraically.
Note: Not using algebra is important for the case where you do not know the algebraic definition of the function.

Using the first derivative:

f has its vertex at $(2, -1)$. There is a turning point at the vertex. Hence:

$$f'(2) = 0$$

f is decreasing from $(-\infty, 2)$. Hence:

$$f' \text{ is negative over the interval } (-\infty, 2)$$

f is increasing from $(2, \infty)$. Hence:

$$f' \text{ is positive over the interval } (2, \infty)$$

Using the second derivative:

f is decreasing from $(-\infty, 2)$ at a decreasing rate. Hence, the magnitude of f' is decreasing. And even though

f' is negative, it increases from left to right.
We know that

$$f'(2) = 0$$

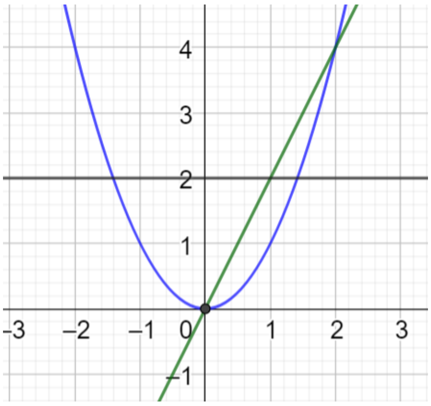
f is increasing from $(2, \infty)$ at a increasing rate. Hence, the magnitude of f' is decreasing. In particular, we know that

$$f'(2) = 0$$

And from there, it increases.

3.111: Second Derivative Test for Concavity

- If $y'' > 0$ over an interval, then the function is concave up over that interval.
- If $y'' < 0$ over an interval, then the function is concave down over that interval.



Note that

$$y'' \text{ is the derivative of } y'$$

- $y'' > 0$ means that y' has positive slope, and is increasing.
  - ✓ If y' is increasing, then the function is said to be concave up.
- $y'' < 0$ means that y' has negative slope, and is decreasing.
  - ✓ If y' is decreasing, then the function is said to be concave down.

3.112: Second Derivative: Point of Inflection

Let f have a point of inflection P
f changes concavity at P. Hence, the second derivative (which tests concavity) must change sign at P.

In general, if the second derivative changes sign at a point, that point is a point of inflection.

3.113: Second Derivative

If the second derivative is zero, the point can be a turning point, or a point of inflection.
This information is not conclusive.

3.8 Intermediate Value Theorem

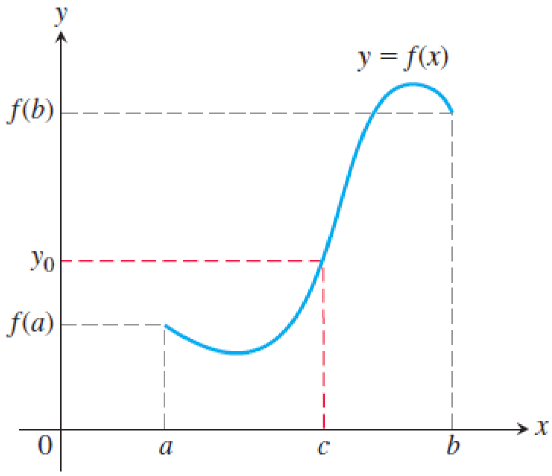
A. Basics

3.114: Intermediate Value Theorem

The conditions for the intermediate value theorem are as follows.

- The function should be defined on a closed interval $[a, b]$
- The function should be continuous over the same interval $[a, b]$

THEOREM 11—The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

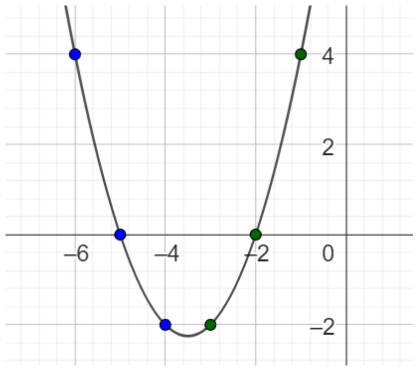


Example 3.115

Use intermediate value theorem to show that the function $y = x^2 + 7x + 10$ has

A. at least one root between -6 and -4
B. at least one root between -3 and -1

Find the roots and graph it



$$x^2 + 7x + 10 = 0$$
$$(x + 2)(x + 5) = 0$$
$$x \in \{-2, -5\}$$

Part A

The conditions for the intermediate value theorem are as follows.

- The function should be defined on a closed interval $[a, b]$
- The function should be continuous over the same interval $[a, b]$

Intermediate Value Theorem applies since:

- $f(x)$ is defined over $[-6, -4]$
- $f(x)$ is continuous over $[-6, -4]$

Hence, $f(x)$ takes on all values between

$$f(-6) = 4 \text{ and } f(-4) = -2 \Rightarrow -2 \le f(x) = y \le 4$$

In particular, it takes on the value

$$f(x) = y = 0$$

And hence $f(x)$ has a root in this interval.

Part B

$$f(-3) = -2$$
$$f(-1) = 4$$

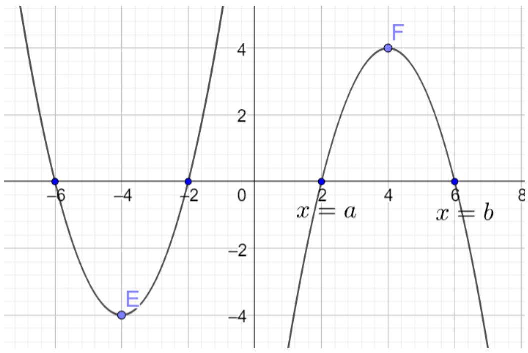
Example 3.116
Basic Example

3.9 Rolle's Theorem

A. Rolle's Theorem

3.117: Rolle's Theorem
Given a function f there is at least one number $c \in (a, b)$ at which
$$f'(c) = 0$$

Conditions to be met by f:
- $f(a) = f(b) = 0$
- f is continuous over the closed interval $[a, b]$
- f is differentiable at every point of its interior (a, b)



Example 3.118
Interpret Rolle's Theorem graphically for the diagram for $f(x)$ drawn alongside.

In the diagram, note that
$$f(-6) = f(-2) = 0$$
The function is continuous over the interval $[-6, -2]$, and it is differentiable over the same interval.
Hence, the conditions of Rolle's Theorem are met.

Hence, there is at least one point in the interval $(-6, -2)$ where:
$$f'(c) = 0$$

$f'(c)$ is the derivative. The derivative gives the slope of the function at a point. When the derivative is zero, the slope is zero, and the tangent line to the function at that point is horizontal.

3.119: Generalization of Rolle's Theorem
Rolle's Theorem can be generalized so that the condition
$$f(a) = f(b) = 0$$
Becomes simply
$$f(a) = f(b)$$

Suppose there is a function f such that it meets the other conditions of Rolle's Theorem, but
$$f(a) = f(b) = k, \qquad k \in \mathbb{R}$$
Then, define
$$g(x) = f(x) - k$$
Then, then the new function $g(x)$ meets all the conditions of Rolle's Theorem.

Hence, there is a point c at which
$$g'(c) = 0$$

Note that since $f(x)$ and $g(x)$ differ only by a constant,
$$f'(x) = g'(x)$$

Hence, the value of c at which
$$g'(c) = 0 \Rightarrow f'(c) = 0$$

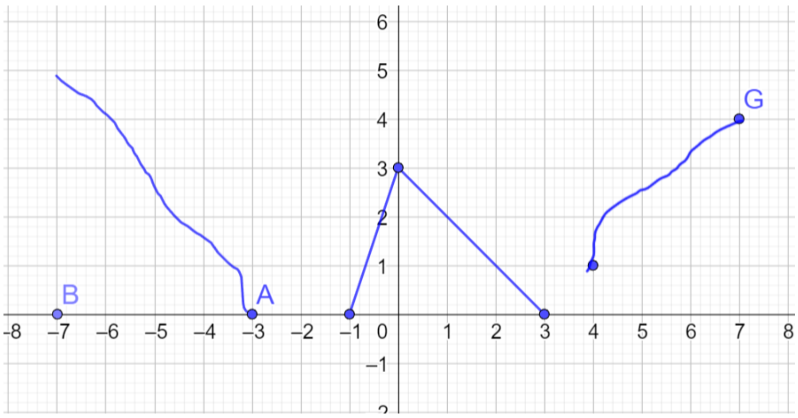
Example 3.120
Does a linear function meet the conditions for Rolle's Theorem.

A linear function will have the same value for two inputs only when it is a constant function.
In all other cases, there is no value for which
$$f(a) = f(b) = k$$

Hence, a general linear function does not meet the conditions for Rolle's Theorem.

Example 3.121
Identify the violation of the conditions in Rolle's Theorem over the given intervals.

A. Over the interval $[-7, -3]$
B. Over the interval $[-1,3]$
C. Over the interval $[4,7]$



Over the interval $[-7, -3]$
$$Function\ is\ not\ continous\ over\ [-7, -3]$$

Over the interval $[-1,3]$
$$Function\ is\ not\ differentiable\ at\ x = 0, hence\ not\ differentiable\ over\ over\ [-1,3]$$

Over the interval $[4,7]$
There are no values a and b in the interval $[4,7]$ for which $f(a) = f(b)$

Example 3.122
Determine the violation of conditions in Rolle's Theorem in the function $f(x) = \frac{1}{x}$ over the intervals -1 to 2.

$f(a) \neq f(b)$ since

$$f(-1) = -1 \neq \frac{1}{2} = f(2)$$

Example 3.123

Determine the violation of conditions in Rolle's Theorem in the function $f(x) = \left|\frac{1}{x}\right|$ over the intervals $-\frac{\pi}{e}$ to $\frac{\pi}{e}$.

$f(x)$ is not continuous at $x = 0$ since the function is not defined at $x = 0$.

Example 3.124

Determine the violation of conditions in Rolle's Theorem in the following functions over the given intervals

A. $\tan x$ over the interval $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

B. $\tan x$ over the interval $-\frac{\pi}{4}$ to $\frac{\pi}{4}$

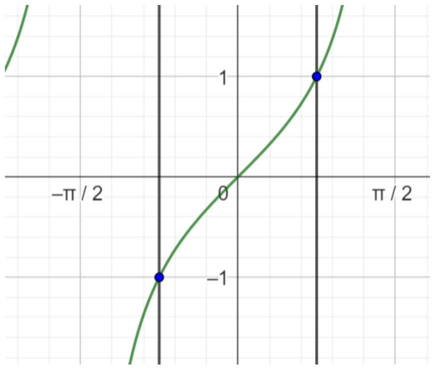
Part A

$\tan x$ is not defined at $\frac{\pi}{2}$ and $-\frac{\pi}{2}$

$\tan x$ is not continuous at

$$\frac{\pi}{2} \text{ and } -\frac{\pi}{2}$$

Hence, the functions fails to be continuous over the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and hence conditions of Rolle's Theorem are not met.



Part B

$$\tan\left(-\frac{\pi}{4}\right) = -1 \neq 1 = \tan\left(\frac{\pi}{4}\right)$$

Hence,

$$f(a) \neq f(b)$$

Example 3.125

Determine the violation of conditions in Rolle's Theorem in the following functions over the given intervals

A. $\sec x$

B. $\frac{1}{x}$

B. Verifying the Theorem

Example 3.126

Verify Rolle's Theorem for $f(x) = (x-a)^m(x-b)^n, m, n \in \mathbb{N}$ on the interval $[a, b]$

Show that the conditions hold

$$f(a) = (a-a)^m(x-b)^n = 0(x-b)^n = 0$$
$$f(b) = (x-a)^m(b-b)^n = (x-a)^m(0) = 0$$

Since f is a polynomial of degree $m+n$, it is both continuous and differentiable over $\mathbb{R}$.

Find the derivative

Use the product rule to find the derivative:

$$f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m(x-b)^{n-1}$$

Factor $(x-a)^{m-1}(x-b)^{n-1}$ from each term:
$$=(x-a)^{m-1}(x-b)^{n-1}[m(x-b)+n(x-a)]$$

Find a Value of c
Equate $f'(c)$ to zero to find the value of c:
$$f'(c)=(c-a)^{m-1}(c-b)^{n-1}[m(c-b)+n(c-a)]=0$$
Use the zero-product property:
$$(c-a)^{m-1}=0 \Rightarrow c-a=0 \Rightarrow c=a\ (Left\ endpoint)$$
$$(c-b)^{n-1}=0 \Rightarrow c-b=0 \Rightarrow c=b\ (Right\ endpoint)$$
$$m(c-b)+n(c-a)=0 \Rightarrow c(m+n)=mb+na \Rightarrow c=\frac{mb+na}{m+n}$$

Show that $c \in (a, b)$
The first two solutions give us the endpoints. Ignore them, and show that the third solution lies in the required interval.
We need:
$$\frac{mb+na}{m+n}>a \Rightarrow mb+na>am+an \Rightarrow mb>ma \Rightarrow b>a$$
$$\frac{mb+na}{m+n}<b \Rightarrow mb+na<bm+bn \Rightarrow na<nb \Rightarrow a<b$$
Since $b>a$, the conditions are always met. Hence:
$$a<\frac{mb+na}{m+n}<b \Rightarrow \frac{mb+na}{m+n}\in[a,b]$$

Example 3.127
Verify Rolle's Theorem for the function $f(x)=\ln\left[\frac{x^2+ab}{(a+b)x}\right]$ in the interval $[a,b]$ where $0\notin[a,b]$. **(ISC 1999, 2012)**

Show that the conditions hold
$$f(a)=\ln\left[\frac{a^2+ab}{(a+b)a}\right]=\ln\left[\frac{a^2+ab}{a^2+ab}\right]=\ln 1=0$$
$$f(b)=\ln\left[\frac{b^2+ab}{(a+b)b}\right]=\ln\left[\frac{b^2+ab}{ab+b^2}\right]=\ln 1=0$$
The natural log function has domain $(0,\infty)$. Since it is given that $0\notin[a,b]$,
$$a<b<0\ OR\ b>a>0$$
If $a<b<0$, then the interval $[a,b]$ lies completely outside the domain of $f(x)$.
If $b>a>0$, then the natural log function is continuous and differentiable over its domain. Hence, it is differentiable and continuous over $[a,b]$.

Find the derivative
Use the quotient log rule to simplify the function before we differentiate:
$$f(x)=\ln\left[\frac{x^2+ab}{(a+b)x}\right]=\ln[x^2+ab]-\ln x-\ln(a+b)$$
Find the derivative:
$$f'=\frac{2x}{x^2+ab}-\frac{1}{x}=\frac{2x^2-x^2-ab}{x(x^2+ab)}=\frac{x^2-ab}{x(x^2+ab)}$$

Find a Value of c
Find the value of c by equating the derivative to zero:

$$f'(c) = \frac{c^2 - ab}{c(c^2 + ab)} = 0$$
$$c^2 - ab = 0$$
$$c^2 = ab$$
$$c = \pm\sqrt{ab}$$

Since $[a, b] \in \mathbb{R}^+$, we choose the positive square root.

$$c = \sqrt{ab}$$

Show that $c \in (a, b)$

And we know that $\sqrt{ab}$ is the geometric mean of a and b. Hence

$$c = \sqrt{ab} \in [a, b]$$

Example 3.128

Verify Rolle's Theorem for the function $f(x) = e^{2x}(\sin 2x - \cos 2x)$ defined in the interval $\left[\frac{\pi}{8}, \frac{5\pi}{8}\right]$. **(ISC 2006)**

Show that the conditions hold

$$f\left(\frac{\pi}{8}\right) = e^{\frac{\pi}{4}}\left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) = e^{\frac{\pi}{4}}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = e^{\frac{\pi}{4}}(0) = 0$$
$$f\left(\frac{5\pi}{8}\right) = e^{\frac{5\pi}{4}}\left(\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right) = e^{\frac{\pi}{4}}\left(-\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right) = e^{\frac{5\pi}{4}}(0) = 0$$
$$f\left(\frac{\pi}{8}\right) = f\left(\frac{5\pi}{8}\right) = 0$$

$f(x)$ is continuous and differentiable over $\left[\frac{\pi}{8}, \frac{5\pi}{8}\right]$.
Hence, the conditions for Rolle's Theorem hold.

Find the derivative

$$f' = e^{2x}(2\cos 2x + 2\sin 2x) + 2e^{2x}(\sin 2x - \cos 2x)$$

Factor 2 in the first term:

$$= 2e^{2x}(\cos 2x + \sin 2x) + 2e^{2x}(\sin 2x - \cos 2x)$$

Factor $2e^{2x}$:

$$= 2e^{2x}(\cos 2x + \sin 2x + \sin 2x - \cos 2x)$$

Simplify:

$$= 4e^{2x}(\sin 2x)$$

Find a Value of c

$$f'(c) = 0$$
$$4e^{2c}(\sin 2c) = 0$$
$$e^{2c}(\sin 2c) = 0$$

Use the zero-product property:

$$e^{2c} = 0 \Rightarrow \mathit{No\ Solutions}$$
$$\sin 2c = 0$$

Show that $c \in (a, b)$

$$2c = 0 \Rightarrow c = 0 \notin \left[\frac{\pi}{8}, \frac{5\pi}{8}\right]$$
$$2c = \pi \Rightarrow c = \frac{\pi}{2} \in \left[\frac{\pi}{8}, \frac{5\pi}{8}\right]$$

Example 3.129

It is given that Rolle's Theorem holds good for the function $f(x) = x^3 + ax^2 + bx, x \in [1,2]$ at the point $x = \frac{4}{3}$. Find the values of a and b.

For Rolle's Theorem to hold

$$f(1) = f(2)$$

$$1^3 + a(1^2) + b(1) = 2^3 + a(2^2) + b(2) = 0$$
$$1 + a + b = 8 + 4a + 2b$$
$$3a + b = -7$$
$$b = -7 - 3a$$

$$f'(x) = 3x^2 + 2ax + b$$
$$f'(c) = 3c^2 + 2ac + b = 3\left(\frac{4}{3}\right)^2 + 2a\left(\frac{4}{3}\right) + b = \frac{16}{3} + \frac{8}{3}a + b$$

Substitute $b = -7 - 3a$:

$$\frac{16}{3} + \frac{8}{3}a - 7 - 3a = \frac{-5}{3} - \frac{1}{3}a = \frac{-5 - a}{3} = 0 \Rightarrow -5 - a = 0 \Rightarrow a = -5$$

Substitute $a = -5$ to find the value of b:

$$b = -7 - 3a = -7 - 3(-5) = -7 + 15 = 8$$

3.10 Mean Value Theorem

A. Mean Value Theorem

3.130: Mean Value Theorem

There is at least one number in c in (a, b) at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Conditions:

- f is continuous over the closed interval $[a, b]$
- f is differentiable at every point of the open interval (a, b)

Notes:

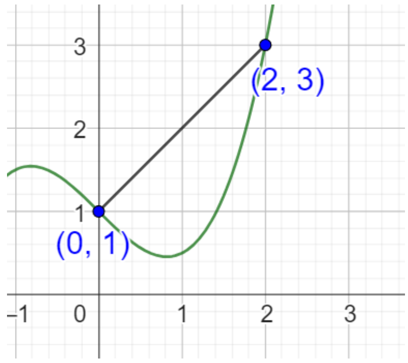
- The Mean Value Theorem is a generalization of Rolle's Theorem.

Example 3.131

Verify the Mean Value Theorem for $y = \frac{x^3}{2} - x + 1$ over the interval $(0,2)$.

The slope connecting the endpoints is:

$$a=0, b=2 \Rightarrow \frac{f(2)-f(0)}{2-0}=\frac{3-1}{2}=\frac{2}{2}=1$$



Type equation here.

$$y'=\frac{3}{2}x^2-1$$

$$\frac{3}{2}x^2-1=1$$
$$3x^2=4$$
$$x=\pm\sqrt{\frac{4}{3}}=\pm\frac{2}{\sqrt{3}}=\pm\frac{2\sqrt{3}}{3}\approx\pm1.15$$

Reject the negative value. The positive value lies in the interval

$$(0,2)$$

Hence, at $x \approx 1.15$, the slope of the function is equal to the slope at the endpoints of the interval

$$(0,2)$$

Example 3.132

A. Show that Rolle's Theorem is a special case of the Mean Value Theorem.
B. Given that Rolle's Theorem is a special case of the Mean Value Theorem, why did we mention it separately? Why not prove Rolle's Theorem using the Mean Value Theorem?

Part A

Substitute $f(b)=f(a)=0$ in the statement of the Mean Value Theorem, and we get Rolle's Theorem.

Part B

The proof of the Mean Value Theorem is established using Rolle's Theorem. Hence, you cannot prove Roller's Theorem using Mean Value Theorem unless you prove Mean Value Theorem without using Rolle's Theorem (or its equivalent).

B. Verifying the Theorem

Example 3.133

Verify Lagrange's Mean Value Theorem for the function $f(x)=\sqrt{x^2-x}$ in the interval $[1,4]$. (ISC 2013)

Show that the conditions hold

$$x^2-x\geq 0$$
$$x(x-1)\geq 0$$

Roots of the above quadratic are:

$$x\in\{0,1\}$$

This is an upward facing quadratic.

$$x^2-x\geq 0 \Rightarrow x\geq 1 \; OR \; x\leq 0$$

The function is defined throughout the given interval. It is continuous throughout $[1,4]$ and differentiable over $(1,4)$.

Find the derivative

$$f'=\frac{2x-1}{2\sqrt{x^2-x}}$$

Find a Value of c

Substitute $a=1, b=4$ in $\frac{f(b)-f(a)}{b-a}$:

$$\frac{\sqrt{4^2-4}-\sqrt{1^2-1}}{4-1}=\frac{\sqrt{12}-\sqrt{0}}{3}=\frac{2\sqrt{3}}{3}=\frac{2}{\sqrt{3}}$$

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Substitute:

$$\frac{2c-1}{2\sqrt{c^2-c}}=\frac{2}{\sqrt{3}}$$

Cross-multiply:

$$\sqrt{3}(2c-1)=4\sqrt{c^2-c}$$

Square both sides:

$$3(4c^2-4c+1)=16(c^2-c)$$

Expand:

$$12c^2-12c+3=16c^2-16c$$

Collate all terms one side:

$$4c^2-4c-3=0$$

Factor:

$$(2c+1)(2c-3)=0$$

Use the zero-product property:

$$c=-\frac{1}{2},c=\frac{3}{2}$$

Show that $c \in (a, b)$

$$\frac{3}{2}=1.5\in[1,4]$$

Example 3.134

Verify Lagrange's Mean Value Theorem for the function $f(x)=x(1-\ln x)$ and find the value of c in the interval [1,2]. (ISC 2015)

Show that the conditions hold

The function is continuous and differentiable over [1,2].

Find the derivative

$$f'=x\left(-\frac{1}{x}\right)+(1)(1-\ln x)$$
$$=-1+1-\ln x$$
$$=-\ln x$$

Find a Value of c

Substitute $a=1, b=2$ in $\frac{f(b)-f(a)}{b-a}$:

$$=\frac{2(1-\ln 2)-1(1-\ln 1)}{2-1}$$
$$=\frac{2-2\ln 2-1}{1}$$
$$=1-2\ln 2=1-\ln 4$$

Find the derivative:

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$
$$-\ln c=1-\ln 4$$
$$\ln 4-\ln c=1$$
$$\ln\frac{4}{c}=1$$
$$\frac{4}{c}=e^1$$
$$c=\frac{4}{e}\approx\frac{4}{2.71}$$

Show that $c \in (a, b)$

$$1<\frac{4}{2.71}<2\Rightarrow c\in[1,2]$$

Example 3.135

Verify Lagrange's Mean Value Theorem for the function $f(x)=2\sin x+\sin 2x$ on $[0,\pi]$. (ISC 2015)

Show that the conditions hold

The *sin* function is continuous and differentiable over $\mathbb{R}$.

Hence, it is continuous and differentiable over $[0,\pi]$.

Find the derivative

$$f'=2\cos x+2\cos 2x$$

Find a Value of c

$$f(\pi)=2\sin\pi+\sin 2\pi=0+0=0$$
$$f(0)=2\sin 0+\sin 2\times 0=0+0=0$$
$$\frac{f(b)-f(a)}{b-a}=\frac{f(\pi)-f(0)}{\pi-0}=\frac{0}{\pi}=0$$

We need to show that $f'(c)=\frac{f(b)-f(a)}{b-a}$:

$$2\cos c+2\cos 2c=0$$
$$\cos c+\cos 2c=0$$

Use the property $\cos 2c=2\cos^2 c-1$

$$2\cos^2 c+\cos c-1=0$$

Let $\cos c=k$:

$$2k^2+k-1=0$$
$$2k^2+2k-k-1=0$$
$$2k(k+1)-1(k+1)=0$$
$$(2k-1)(k+1)=0$$

Use the zero-product property:

$$\cos c=k\in\left\{\frac{1}{2},-1\right\}$$
$$c\in\left\{\frac{\pi}{3},\pi\right\}$$

Show that $c \in (a, b)$

But

$$\pi \notin (0, \pi) \Rightarrow Reject$$

$$c = \frac{\pi}{3} \in (0, \pi)$$

Show that the conditions hold
Find the derivative
Find a Value of c
Show that $c \in (a, b)$

C. Kinematics

Example 3.136

6	65.1

684. Kelly completed a 250-mile drive in exactly 5 hours — an average speed of 50 mph. The trip was not actually made at a constant speed of 50 mph, of course, for there were traffic lights, slow-moving trucks in the way, etc. Nevertheless, there must have been at least one instant during the trip when Kelly's speedometer showed exactly 50 mph. Give two explanations — one using a distance-versus-time graph, and the other using a speed-versus-time graph. Make your graphs consistent with each other!

$$(t, d) \in \{(0,0), (5,250)\}$$

By the mean value theorem, at some point in the time interval $(0,5)$, the derivative of the distance $(= Speed)$ must equal the average rate of change over the time interval:

$$Speed = d'(t) = \frac{d(5) - d(0)}{5 - 0} = \frac{250}{5} = 50$$

D. Extended Mean Value Theorem

3.137: Extended Mean Value Theorem

There is at least one number in c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Conditions:

- f and g are continuous over the closed interval $[a, b]$
- f and g are differentiable at every point of the open interval (a, b)

Notes:

- The Extended Mean Value Theorem is a generalization of the Mean Value Theorem.

Why It Works

Applying the mean theorem to f and g:

$$\underbrace{f'(c) = \frac{f(b) - f(a)}{b - a}}_{\textit{Equation I}}, \qquad \underbrace{g'(c) = \frac{g(b) - g(a)}{b - a}}_{\textit{Equation II}}$$

Divide Equation II by Equation I:

$$\frac{f'(c)}{g'(c)}=\frac{\frac{f(b)-f(a)}{b-a}}{\frac{g(b)-g(a)}{b-a}}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

Which is the result we want.

Example 3.138

3.11 Optimization

A. Numbers

Example 3.139

This video contains some easy problems to start with.

3.140: First Derivative Test

At a turning point, the slope must be zero. Hence, to find the candidates for turning points, equate the first derivative to zero.

The solutions to $\frac{dy}{dx}=0$ are the numbers to check for, to obtain the turning points.

Example 3.141

A. The product of two positive integers is 100. Find the minimum value of the sum of these numbers without using Calculus.

B. The product of two positive real numbers is 100. Find the minimum value of the sum of these numbers by using Calculus.

Part A

We find the factor pairs of 100:

$$(1,100)(2,50)(4,25)(5,20)(10,10)$$

The corresponding sums are:

$$100, 52, 29, 25, \underbrace{20}_{Minimum}$$

Part B

Let the numbers be a and b.

$$Product = ab = 100 \Rightarrow b=\frac{100}{a}$$

We wish to minimize the sum:

$$a+b=a+\frac{100}{a}$$

If there is a minimum inside the domain, the derivative at the point must be zero. Let

$$y=a+\frac{100}{a}$$

Find the derivative:

$$\frac{dy}{da}=1-\frac{100}{a^2}$$

Equate the derivative to zero:

$$1-\frac{100}{a^2}=0 \Rightarrow 1=\frac{100}{a^2} \Rightarrow a^2=100 \Rightarrow a=\pm 10$$

Reject the negative value since a is a positive number:

$$a=10$$

We know from Part A that $a<10$ and $a>10$ increase $a+b$. But how do we know this in general. To solve, we introduce the second derivative test.

3.142: Second Derivative Test

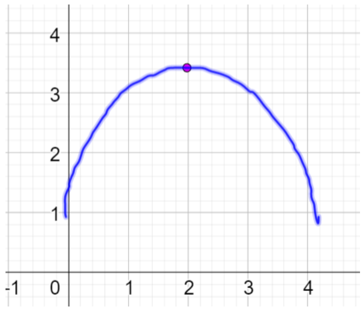
If you have a list of candidates from the first derivative test for the maximum and minimum of a function, then

$$\frac{d^2y}{dx^2}>0 \Rightarrow Point\ is\ a\ Minimum$$
$$\frac{d^2y}{dx^2}<0 \Rightarrow Point\ is\ a\ Maximum$$

Consider the maximum graphed alongside:

- To the left of the maximum, the slope (first derivative) is positive.
- To the right of the maximum, the slope (first derivative) is negative.
- At the maximum, the slope (first derivative) is zero.

At the maximum, the first derivative changes from positive to negative. Hence, at the maximum, the slope of the first derivative must be negative. Hence, the second derivative must be negative at a maximum.



Example 3.143

Continue the example from above. Use the second derivative test to determine whether $a=10$ is a maximum or a minimum.

$$\frac{dy}{da}=1-\frac{100}{a^2}$$

$$\frac{d^2y}{da^2}=\frac{dy}{da}\left(-\frac{100}{a^2}\right)=\frac{dy}{da}(-100a^{-2})=(-2)(-100)a^{-3}=\frac{200}{a^3}$$

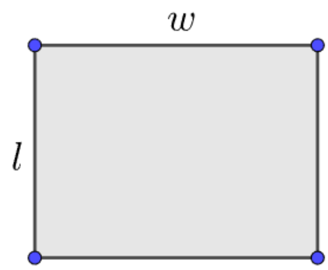
$$\left.\frac{d^2y}{da^2}\right|_{a=10}=\frac{200}{a^3}_{a=10}>0 \Rightarrow Minimum$$

B. 2D Geometry

Example 3.144

A farmer fences a rectangle using 100 m of fence. Find the dimensions that maximize the area.

$$Perimeter=P=100$$
$$2(l+w)=100$$
$$l+w=50$$
$$w=50-l$$



The area function:

$$=A=lw=l(50-l)=50l-l^2$$

Differentiate:

$$\frac{d}{dl}A = 50 - 2l = 0 \Rightarrow l = 25$$
$$w = 50 - l = 50 - 25 = 25$$

$$A = lw = 25^2 = 625\, m^2$$

$$\frac{d^2}{dl^2}A = -2 < 0 \Rightarrow Max$$

Example 3.145

Find the dimensions of a rectangle with a constant perimeter p that maximize its area. Find that area.

Find a function for the area in terms of length

Let

$$2(l + w) = p \Rightarrow w = \frac{p}{2} - l$$

Then the area is:

$$A = lw = l\left(\frac{p}{2} - l\right) = \frac{lp}{2} - l^2$$

Find the critical points and check for maximum/minimum

Differentiate the area function:

$$\frac{dA}{dl} = \frac{p}{2} - 2l,$$

Equate the derivative to zero to find the *critical points*:

$$\frac{p}{2} - 2l = 0 \Rightarrow \frac{p}{2} = 2l \Rightarrow l = \frac{p}{4}$$

Find the second derivative:

$$\frac{d^2A}{dl^2} = -2 < 0$$

Find the area for the maximum

Since the second derivative is less than zero at a critical point, it is a maximum:

$$l = w = \frac{p}{4} \Rightarrow Area = \frac{p^2}{16}$$

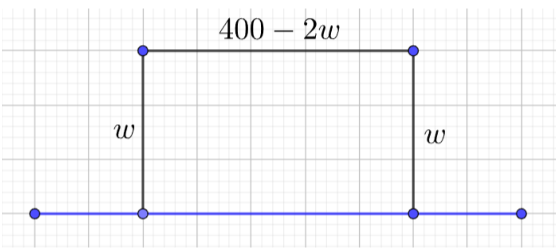
In other words, for a given perimeter, the area of a rectangle is maximized when it is a square.

Example 3.146

A farmer fences three sides of a rectangle using 400 yards of fence. (The fourth side is along a stone wall).

A. Find the dimensions of the rectangle of largest area that can be fenced.
B. The rectangle from Part A has a semicircle of maximum area planted with green grass, and the rest with purple monkshood. Find the fraction of the farm planted with purple.

Part A



Area of the rectangle is:

$$A = wl = w(400 - 2w) = 400w - 2w^2$$

Find the first derivative and equate it to zero:

$$\frac{dA}{dw} = 400 - 4w = 0 \Rightarrow 4w = 400 \Rightarrow w = 100$$

Find the second derivative:
$$\frac{d^2A}{dw^2} = \frac{d}{dw}(400 - 4w) = -4 < 0 \Rightarrow w = 100 \; is \; a \; max$$
$$400 - 2w = 400 - 200 = 200$$
$$Dimensions = (l, w) = (200,100)$$

Part B

$$\frac{20000 - 5000\pi}{5000\pi} = \frac{20000}{5000\pi} - 1 = \frac{4}{\pi} - 1$$

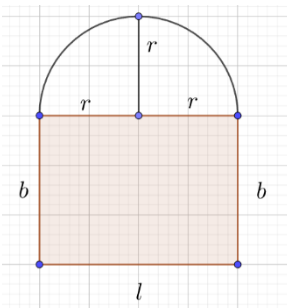
Example 3.147

A. Find the radius of the semicircle that maximizes the area of a Norman window with perimeter P. Note: A Norman window consists of a semi-circle sitting atop a rectangle.

B. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening. **(CBSE 2011,2014,2017,2018)**

Part A

Find a function for the area



From the diagram, the area of the window is:
$$A = \underbrace{\frac{\pi r^2}{2}}_{Semi-Circle} + \underbrace{lb}_{Rectangle}$$

Express the function in a single variable
$$Length = l = 2r$$
To find the breadth in terms of r, use the constraint on the perimeter:
$$2b + 2r + \pi r = P \Rightarrow b = \frac{P - 2r - \pi r}{2}$$
The area of the window (in terms of r) is:
$$A = \frac{\pi r^2}{2} + (2r)\left(\frac{P - 2r - \pi r}{2}\right)$$
$$= Pr - 2r^2 - \frac{\pi r^2}{2}$$
$$= Pr - r^2\left(2 + \frac{\pi}{2}\right)$$

Find the critical points, and their nature

Find the first derivative and equate it to zero to find the critical points:
$$\frac{dA}{dr} = P - r(4 + \pi) = 0 \Rightarrow r = \frac{P}{4 + \pi}$$
Find the second derivative and since it is negative, the critical point is a maximum:
$$\frac{d^2A}{dx^2} = -(4 + \pi)$$

Part B

The radius is:
$$r = \frac{P}{4 + \pi} = \frac{10}{4 + \pi}$$
The length is twice of the radius:
$$l = 2r = \frac{20}{4 + \pi}$$
To find the breadth, substitute $r = \frac{10}{4+\pi}$ into $b = \frac{1}{2}[P - r(\pi + 2)]$
$$= \frac{1}{2}\left[10 - \left(\frac{10}{4 + \pi}\right)(\pi + 2)\right]$$
Factor 10 out of the terms inside the bracket:
$$= 5\left[1 - \frac{\pi + 2}{4 + \pi}\right]$$
Add the fractions and simplify:
$$= 5\left[\frac{4 + \pi - \pi - 2}{4 + \pi}\right] = 5\left[\frac{2}{4 + \pi}\right] = \frac{10}{4 + \pi}$$

Example 3.148

The constraints on the domain of the function $A = Pr - r^2\left(2 + \frac{\pi}{2}\right)$ are

$$Radius > 0 \Rightarrow r > 0$$
$$Length > 0 \Rightarrow l > 0 \Rightarrow 2r > 0 \Rightarrow r > 0$$
$$Breadth > 0 \Rightarrow b > 0 \Rightarrow \frac{P - 2r - \pi r}{2} > 0 \Rightarrow P - 2r - \pi r > 0$$
$$P > 2r + \pi r$$
$$2r + \pi r < P$$
$$r(2 + \pi) < P$$
$$r < \frac{P}{2 + \pi}$$

$$r \in \left(0, \frac{P}{2 + \pi}\right)$$

Example 3.149

A wire of length 2 units is cut into two parts which are bent respectively to form a square of side $= x$ units and a circle of $radius = r$ units. If the sum of the areas of the square and the circle so formed is minimum, then, find the relation between x and r: **(JEE Main 2016-Adapted, CBSE 2010-Adapted)**

Write an expression for the length of the wire, and solve for x:

$$4x + 2\pi r = 2 \Rightarrow 2x + \pi r = 1 \Rightarrow \underbrace{x = \frac{1 - \pi r}{2}}_{Equation\ I}$$

We wish to minimize the sum of the area of the square and the circle

$$A(x) = \underbrace{x^2}_{Square} + \underbrace{\pi r^2}_{Circle}$$

Substitute $x = \frac{1-\pi r}{2}$ from Equation I:

$$= \left(\frac{1 - \pi r}{2}\right)^2 + \pi r^2 = \frac{1 - 2\pi r + r^2(\pi^2 + 4\pi)}{4}$$

We can ignore 4 since it is a constant. Find the derivative of the numerator and equate to zero:

$$A'(x) = -2\pi + 2r(\pi^2 + 4\pi) = 0 \Rightarrow r(\pi^2 + 4\pi) = \pi \Rightarrow r = \frac{1}{\pi + 4}$$

Find the second derivative:

$$A''(x) = 2(\pi^2 + 4\pi) > 0 \Rightarrow Value\ is\ a\ minimum$$

At the minimum value:

$$x = \frac{1 - \pi r}{2} = \frac{1 - \pi\left(\frac{1}{\pi + 4}\right)}{2} = \frac{\frac{\pi + 4 - \pi}{\pi + 4}}{2} = \frac{4}{\pi + 4} \times \frac{1}{2} = \frac{2}{\pi + 4}$$

And, hence

$$x = 2r$$

Example 3.150

$$4x + 2\pi r = 2$$

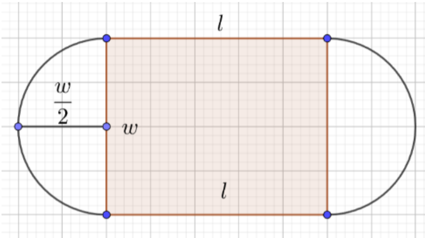
Assume that it is acceptable to form only a circle, or only a square. In that case:

$$x \geq 0, \qquad r \geq 0$$

$$x\ is\ \text{max}\ when\ r\ is\ min \Rightarrow x = \frac{1 - \pi r}{2} = \frac{1 - \pi(0)}{2} = \frac{1}{2}$$

$$r \text{ is maximum when } x \text{ is minimum} \Rightarrow r = \frac{1-2x}{\pi} = \frac{1-0}{\pi} = \frac{1}{\pi}$$
$$x = \frac{1-\pi r}{2} \Rightarrow x \text{ is minimum when } r \text{ is max}$$
$$x \in [0,2]$$

$$A(x) = x^2 + \pi r^2$$



Example 3.151

Find the dimensions of a rectangle with two semicircles on either side that maximize the area if the perimeter is 400.

$$P = 2l + 2\pi\left(\frac{w}{2}\right) = 2l + \pi w = 400 \Rightarrow w = \frac{400-2l}{\pi}$$

The area of the rectangle is:

$$A(l) = lw = l\left(\frac{400-2l}{\pi}\right) = \frac{400l - 2l^2}{\pi}$$

Find the first derivative of the area function and equate it to zero:

$$A' = \frac{400-4l}{\pi} = 0 \Rightarrow 400 - 4l = 0 \Rightarrow l = 100$$

To check whether it is a maximum or a minimum, find the second derivative:

$$A'' = -\frac{4l}{\pi} < 0$$

Since the second derivative is negative, the value that we have found is a maximum.

The width corresponding to the maximum length is:

$$w = \frac{400-2l}{\pi} = \frac{400-2(100)}{\pi} = \frac{200}{\pi}$$

3.152: Area and Arc Length of a Circle

In radians:

$$\text{Area of a circle} = \frac{1}{2}r^2\theta$$
$$\text{Arc length} = r\theta$$

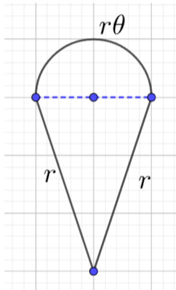
$$\text{Area of a circle} = \pi r^2 \times \frac{\theta}{360} = \pi r^2 \times \frac{\theta}{2\pi} = \frac{1}{2}r^2\theta$$
$$\text{Arc length} = 2\pi r \times \frac{\theta}{360} = 2\pi r \times \frac{\theta}{2\pi} = r\theta$$

Example 3.153

Twenty metres of wire is available for fencing off a flowerbed in the form of a circular sector. Then the maximum area (in sq. m) of the flower-bed, is: **(JEE Main 2017, Type ISC 2009, Type NDA 2005)**[5]

[5] This question is also found in the classic *Problems in Calculus of One Variable*, I. A. Maron

Find a function for the area in terms of radius



The perimeter of the sector is 20:

$$P = 20$$
$$2r + r\theta = 20$$
$$\underbrace{\theta = \frac{20 - 2r}{r}}_{Equation\ I}$$

Substitute Equation I in the formula for area of the sector:

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\frac{20 - 2r}{r}\right) = 10r - r^2$$

Find the critical points and check for maximum/minimum

Find the first derivative and equate it to zero:

$$A' = 10 - 2r = 0 \Rightarrow r = 5$$
$$A'' = -2 < 0 \Rightarrow Value\ of\ r\ is\ a\ maximum$$

Substitute the value of r in Equation I to find the corresponding value of θ:

$$\theta = \frac{20 - 2r}{r} = \frac{20 - 2(5)}{5} = \frac{10}{5} = 2$$

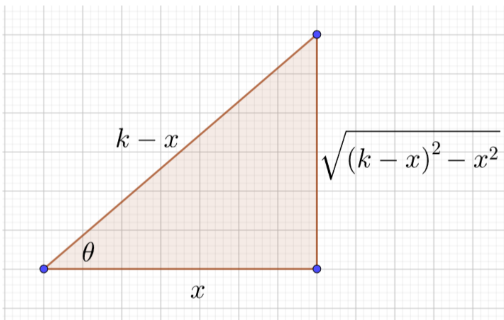
The maximum area is:

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}(5^2)(2) = 25$$

Example 3.154

If the sum of the lengths of the hypotenuse and a side of a right-angled triangle is given, find the angle between them that maximizes the area. (CBSE-Adapted 2014,2017)

Find area as a function of the base



Let

$$Side = x, Hyp + Side = k \Rightarrow Hyp = k - x$$

By the Pythagorean Theorem, the length of the third side:

$$= \sqrt{(k - x)^2 - x^2} = \sqrt{k^2 - 2kx + x^2 - x^2} = \sqrt{k^2 - 2kx}$$

Use the formula for the area of a triangle:

$$A = \frac{1}{2}x\sqrt{k^2 - 2kx}$$

To not work with square roots, we square both sides:

$$A^2 = \frac{1}{4}x^2(k^2 - 2kx) = \frac{1}{4}(x^2k^2 - 2kx^3)$$

Find the critical points

Differentiate both sides implicitly with respect to x:

$$2A \cdot \frac{dA}{dx} = \frac{1}{4}(2k^2x - 6kx^2)$$

Solve for $\frac{dA}{dx}$ and equate it to zero:

$$\frac{dA}{dx} = \frac{k^2x - 3kx^2}{4A} = 0$$
$$k^2x - 3kx^2 = 0$$
$$k^2x = 3kx^2$$
$$x = \frac{k}{3}$$

Find the nature of the critical points

Find the second derivative:

$$2\left(\frac{dA}{dx}\right)^2 + 2A\left(\frac{d^2A}{dx^2}\right) = \frac{1}{4}(2k^2 - 12kx)$$

At the critical point $\frac{dA}{dx} = 0$, and $x = \frac{k}{3}$:

$$2(0)^2 + 2A\left(\frac{d^2A}{dx^2}\right) = \frac{1}{4}\left(2k^2 - 12k\left(\frac{k}{3}\right)\right)$$

$$2A\left(\frac{d^2A}{dx^2}\right) = \frac{1}{4}(2k^2 - 4k^2)$$

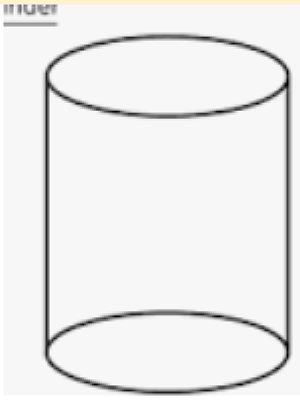
$$\frac{d^2A}{dx^2} = \frac{-k^2}{4A} < 0 \Rightarrow Maximum$$

C. 3D Geometry

Example 3.155

Minimize surface area of a cylinder given that volume of the cylinder

$$= 54\pi\ cm^3$$



$$V = 54\pi$$
$$\pi r^2 h = 54\pi$$
$$r^2 h = 54$$
$$h = \frac{54}{r^2}$$

$$SA = 2\pi r^2 + 2\pi rh$$
$$SA = 2\pi r^2 + 2\pi r \cdot \frac{54}{r^2} = 2\pi r^2 + \frac{108\pi}{r} = 2\pi r^2 + 108\pi r^{-1}$$

$$\frac{d}{dr}SA = 4\pi r + (108)(-1)\pi r^{-2}$$

Equate the first derivative to zero:

$$4\pi r + \frac{(108)(-1)\pi}{r^2} = 0$$

Multiply both sides by r^2 to eliminate fractions:

$$4\pi r^3 - 108\pi = 0$$
$$4\pi r^3 = 108\pi$$
$$r^3 = 27$$
$$r = 3$$

$$h = \frac{54}{r^2} = \frac{54}{3^2} = \frac{54}{6} = 9$$

Example 3.156

Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube. (CBSE 2017)

Finding surface area as a function of side length of base

Let the side length of the square base be x.
Let the height of the cuboid be y.

$$V = x^2y \Rightarrow y = \frac{V}{x^2}$$
$$SA = 2(x^2 + 2xy) = 2\left[x^2 + 2x\left(\frac{V}{x^2}\right)\right] = 2\left[x^2 + 2\frac{V}{x}\right]$$

Find the critical points

Ignore the 2 since it is a constant.
Find the first derivative and equate it to zero to find the critical points:

$$\frac{dS}{dx} = 2x - \frac{2V}{x^2} = 0$$
$$2x = \frac{2V}{x^2}$$
$$x^3 = V$$

Substitute the above critical point in $V = x^2y$ to find the value of y

$$x^3 = x^2y \Rightarrow x = y$$

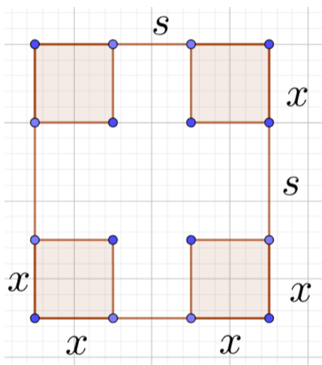
In other words, the height is the same as the side length of the base, which means the shape is a cube.

Use the second derivative test

$$\frac{d^2S}{dx^2}_{x^3=V} = 2 + \frac{4V}{x^3} = 2 + \frac{4V}{V} = 2 + 4 = 6 > 0 \Rightarrow Minimum$$

Example 3.157

A square sheet of metal of side length s has smaller squares of side length x cut at each of its corners, and the resulting "flaps" folded to form an open-top box with maximum volume. Find the value of x in terms of s. (ISC 1999, Adapted)



Find a function for the volume in terms of the cut

$$V = (s - 2x)^2(x) = (4x^2 - 4sx + s^2)(x) = 4x^3 - 4sx^2 + s^2x$$

Find the critical points and check for maximum/minimum

Differentiate both sides with respect to x, and equate the derivative to zero:

$$\frac{dV}{dx} = 12x^2 - 8sx + s^2 = 0$$

This is a quadratic in x. Apply the quadratic formula:

$$x = \frac{-(-8s) \pm \sqrt{64s^2 - (4)(12)(s^2)}}{24} = \frac{8s \pm 4s}{24} \Rightarrow x \in \left\{\frac{s}{2}, \frac{s}{6}\right\}$$

But note that:

$$s - 2x > 0 \Rightarrow s > 2x \Rightarrow x < \frac{s}{2} \Rightarrow Reject\,\frac{s}{2}$$

Find the second derivative, and evaluate it at the remaining critical point:

$$\frac{d^2V}{dx^2} = 24x - 8s \Rightarrow \left.\frac{d^2V}{dx^2}\right|_{x=\frac{s}{6}} = 24\left(\frac{s}{6}\right) - 8s = 4s - 8s = -4s < 0 \Rightarrow Maximum$$

Hence, the value in terms of s is:

$$x = \frac{s}{6}$$

Example 3.158

A. Find the height, radius, and volume of the right circular cone with maximum volume among those with slant height l units.
B. The maximum volume (in cubic meters) of the right circular cone having slant height 3 m is (JEE Main 2019, 9 Jan)

Part A

Find a function for the volume in terms of the height:

Substitute $r = \sqrt{l^2 - h^2}$ in $V = \frac{1}{3}\pi r^2 h$:

$$\frac{\pi}{3}\left(\sqrt{l^2 - h^2}\right)^2 h = \frac{\pi}{3}(l^2 h - h^3)$$

Maximize the function

Ignore $\frac{\pi}{3}$ since it is a constant. Find the first derivative of the rest and equate it to zero:

$$V' = l^2 - 3h^2 = 0 \Rightarrow h = \frac{l}{\sqrt{3}}$$

Find the second derivative:

$$V'' = -6h < 0$$
$$\Rightarrow Value\ of\ h\ above\ is\ a\ maximum$$

Substitute $h = \frac{l}{\sqrt{3}}$ in $r = \sqrt{l^2 - h^2}$ to find the radius:

$$\sqrt{l^2 - \left(\frac{l}{\sqrt{3}}\right)^2} = \sqrt{\frac{2l^2}{3}}$$

Substitute $h = \frac{l}{\sqrt{3}}, r = \sqrt{\frac{2l^2}{3}}$ in $V = \frac{1}{3}\pi r^2 h$ to find the maximum volume:

$$V = \frac{1}{3}\pi\left(\sqrt{\frac{2l^2}{3}}\right)^2 \frac{l}{\sqrt{3}} = \frac{2\pi l^3}{9\sqrt{3}}$$

Part B

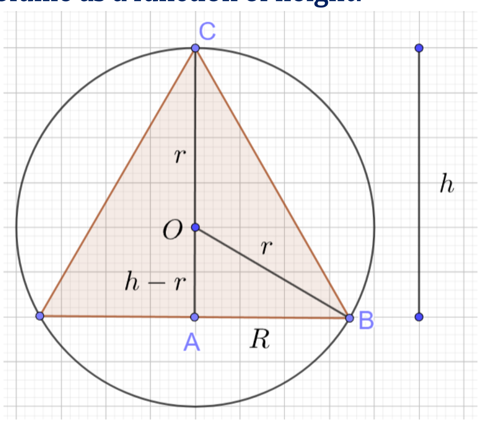
Substituting $l = 3$ in the above:

$$V = \frac{2\pi l^2}{9\sqrt{3}} = \frac{2\pi(3^3)}{9\sqrt{3}} = 2\sqrt{3}\pi$$

Example 3.159

Show that the altitude of right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also, find the maximum volume in terms of volume of the sphere. **(CBSE 2010, 2014,2016,2019)**

Find volume as a function of height:



$$OA = h - r$$

In ΔOAB, by Pythagoras Theorem:

$$AB^2 + OA^2 = OB^2$$
$$R^2 + (h - r)^2 = r^2$$
$$R^2 + h^2 - 2rh + r^2 = r^2$$
$$R^2 = 2rh - h^2$$

Substitute $R^2 = 2rh - h^2$ in the volume of a cone:

$$V = \frac{1}{3}\pi R^2 h = \frac{\pi}{3}(2rh - h^2)h = \frac{\pi}{3}(2rh^2 - h^3)$$

Find the critical points

Ignore $\frac{\pi}{3}$ since it is a constant.
Find the first derivative of the rest of the expression and equate it to zero to find the critical points:

$$\frac{dV}{dh} = 4rh - 3h^2 = 0$$
$$4rh = 3h^2$$
$$h = \frac{4r}{3}$$

Check the critical points
Find the second derivative:

$$\frac{d^2V}{dh^2} = 4r - 6h$$

Evaluate the second derivative at the critical point:

$$\left.\frac{d^2V}{dh^2}\right|_{h=\frac{4r}{3}} = 4r - 6\left(\frac{4r}{3}\right) = -4r < 0 \Rightarrow Maximum$$

Find the maximum value

Substitute $h = \frac{4r}{3}$ in $V = \frac{\pi}{3}(2rh^2 - h^3)$:

$$= \frac{\pi}{3}\left[2r\left(\frac{4r}{3}\right)^2 - \left(\frac{4r}{3}\right)^3\right]$$

Simplify:

$$\frac{\pi}{3}\left[\frac{32r^3}{9} - \frac{64r^3}{27}\right]$$

Factor $32r^3$:

$$= \frac{32\pi r^3}{3}\left[\frac{3}{27} - \frac{1}{27}\right] = \frac{32\pi r^3}{3}\left[\frac{1}{27}\right]$$
$$= \frac{4}{3}\pi r^3\left[\frac{8}{27}\right] = V_{Sphere}\left[\frac{8}{27}\right]$$

Where

$$V_{Sphere} = Volume\ of\ Sphere$$

Hence, the cone of maximum volume inscribed in a sphere of radius r has

$$h = \frac{4r}{3}, \qquad V = V_{Sphere}\left[\frac{8}{27}\right]$$

D. Coordinate Geometry

Example 3.160
Find the point on the curve $y^2 = 4x$ which is nearest to the point $(2, -8)$.

Let $P(x, y)$ be a general point on the curve $y^2 = 4x$. Using the distance formula:

$$D = \sqrt{(x-2)^2 + (y+8)^2}$$

Substitute $y^2 = 4x \Rightarrow x = \frac{y^2}{4}$:

$$D = \sqrt{\left(\frac{y^2}{4} - 2\right)^2 + (y+8)^2} = \sqrt{\frac{y^4}{16} + 16y + 68}$$

Instead of optimizing D, which has a square root, we optimize:

$$z = D^2 = \frac{y^4}{16} + 16y + 68$$

$$\frac{dz}{dy} = \frac{4y^3}{16} + 16 = \frac{y^3}{4} + 16 = 0 \Rightarrow y^3 = -64 \Rightarrow y = -4$$

$$x = \frac{(-4)^2}{4} = \frac{16}{4} = 4$$

$$\left.\frac{d^2z}{dy^2}\right|_{y=-4} = \left.\frac{3y^2}{4}\right|_{y=-4} = \frac{3(-2)^2}{4} > 0 \Rightarrow Min$$

The point is

$$(4, -4)$$

Example 3.161
Find the point on the curve $x + y^2 = 0$ that is closest to the point $(0, -3)$.

Let $P(x, y)$ be a point on the curve. Using the distance formula, the distance between P and $(0, -3)$ is:
$$d = \sqrt{(x-0)^2 + (y+3)^2}$$
Substitute $x + y^2 = 0 \Rightarrow x = -y^2$
$$D = \sqrt{(-y)^2 + (y+3)^2} = \sqrt{2y^2 + 6y + 9}$$

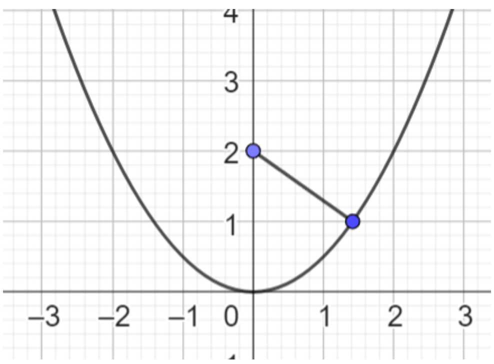
Instead of minimizing the function, we minimize its square:
$$d^2 = 2y^2 + 6y + 9$$

Let $z = D^2$. Then, find the first derivative and equate it to zero:
$$\frac{dz}{dy} = \frac{d}{dy}(2y^2 + 6y + 9) = 4y + 6 = 0 \Rightarrow y = -\frac{6}{4} = -\frac{3}{2}$$
Find the second derivative:
$$\frac{d^2z}{dy^2} = \frac{d}{dy}(4y + 6) = 4 > 0 \Rightarrow y = -\frac{3}{2}\ is\ a\ min$$

Example 3.162
Which point on the parabolic arc $y = \frac{1}{2}x^2$ for $0 \le x \le 3$ is closest to $(0, 2)$?

Let $P(x, y)$ be a point on the curve. Using the distance formula, the distance between P and $(0,2)$ is:
$$\underbrace{D = \sqrt{(x-0)^2 + (y-2)^2}}_{Equation\ I}$$
Since the above has two variables, (and we are not doing multi-variable calculus!!), reduce the above function to a single variable. Use the substitution
$$y = \frac{1}{2}x^2 \Rightarrow 2y = x^2 \Rightarrow x = \sqrt{2y}$$
in Equation I to get:
$$D = \sqrt{\left(\sqrt{2y}\right)^2 + (y-2)^2} = \sqrt{2y + y^2 - 4y + 4} = \sqrt{y^2 - 2y + 4}$$

Instead of minimizing D (which has a nasty square root), we minimize D^2, which does not have a square root:
$$D^2 = y^2 - 2y + 4$$
Let $z = D^2$. Then, find the first derivative and equate it to zero:
$$\frac{dz}{dy} = \frac{d}{dy}(y^2 - 2y + 4) = 2y - 2 = 0 \Rightarrow y = \frac{2}{2} = 1$$
Find the second derivative:
$$\frac{d^2z}{dy^2} = \frac{d}{dy}(2y - 2) = 2 > 0 \Rightarrow y = 1\ is\ a\ min$$



Substitute $y = 1$ in the formula for the arc:
$$x = \sqrt{2y} = \sqrt{2(1)} = \sqrt{2}$$
The closest point is then:
$$(\sqrt{2}, 1)$$

E. Functions

Example 3.163
Find the maximum and the minimum of
$$f(r) = Pr - r^2\left(2 + \frac{\pi}{2}\right), r \in \left(0, \frac{P}{2+\pi}\right)$$

Find the critical points, and their nature
Find the first derivative and equate it to zero to find the critical points:
$$\frac{dA}{dr} = P - r(4+\pi) = 0 \Rightarrow r = \frac{P}{4+\pi}$$
Find the second derivative and since it is negative, the critical point is a maximum:
$$\frac{d^2A}{dx^2} = -(4+\pi)$$
Check the endpoints:
$$r = 0 \Rightarrow f(r) = 0$$
$$r = \frac{P}{2+\pi} \Rightarrow f(r) = P\frac{P}{2+\pi} - \left(\frac{P}{2+\pi}\right)^2\left(2 + \frac{\pi}{2}\right) = \frac{P^2}{2+\pi} - \frac{P^2}{(2+\pi)^2}\left(\frac{4+\pi}{2}\right)$$

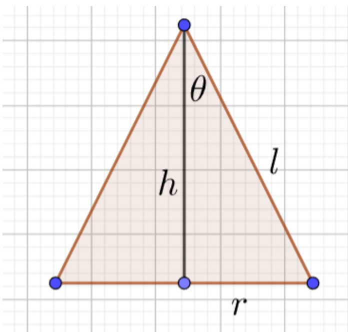
$$Max\ at\ r = \frac{P}{4+\pi}$$
$$Absolute\ Min\ at\ r = 0$$
$$Relative\ Min\ at\ r = \frac{P}{2+\pi}$$

F. Trigonometry

Example 3.164
https://www.toppr.com/ask/question/a-rectangle-is-inscribed-in-a-semicircle-of-radius-r/

Example 3.165
Determine the semi-vertical angle (θ) of a cone with given slant height and maximum volume in terms of $\cos^{-1}\theta$. **(CBSE 2016, Adapted)**
Hint: The angle at the vertex of a cone is the vertical angle. Half the angle at the vertex of a cone is the semi-vertical angle.

Draw a diagram of a cross section of the cone. Let the semi-vertical angle be θ, height be h, the radius be r, and the slant height be l.



The volume of the cone is:
$$V = \frac{1}{3}\pi r^2 h$$

Since we want to optimize in terms of θ, rewrite the above expression in those terms:
$$\sin\theta = \frac{r}{l} \Rightarrow r = l\sin\theta$$
$$\cos\theta = \frac{h}{l} \Rightarrow h = l\cos\theta$$
Substitute the above in $V = \frac{1}{3}\pi r^2 h$:

$$V = \frac{\pi}{3} r^2 h = \left(\frac{\pi}{3}\right)(l^2 \sin^2 \theta)(l \cos \theta) = \left(\frac{\pi l^3}{3}\right)(\sin^2 \theta)(\cos \theta)$$

Instead of optimizing V, we ignore the constant and optimize:

$$v = \frac{V}{\frac{\pi l^3}{3}} = (\sin^2 \theta)(\cos \theta)$$

First Derivative

Find the first derivative:

$$\frac{dv}{d\theta} = (\sin^2 \theta)(-\sin \theta) + (2 \sin \theta \cos \theta)(\cos \theta)$$

Equate the first derivative to zero:

$$(-\sin^3 \theta) + (2 \sin \theta \cos^2 \theta) = 0$$
$$\sin^3 \theta = 2 \sin \theta \cos^2 \theta$$

Since $\sin \theta \neq 0$, divide both sides by $\sin \theta$:

$$\sin^2 \theta = 2 \cos^2 \theta$$
$$\frac{\sin^2 \theta}{\cos^2 \theta} = 2$$
$$\tan^2 \theta = 2$$

Since $0 < \theta < \frac{\pi}{2}$, take the positive square root:

$$\tan \theta = \sqrt{2}$$

Second Derivative

$$\frac{d^2 v}{d\theta^2} = -3 \sin^2 \theta \cos \theta + 2 \cos \theta \cos^2 \theta + 2 \sin \theta \big(2 \cos \theta (-\sin \theta)\big)$$
$$= -3 \sin^2 \theta \cos \theta + 2 \cos^3 \theta - 4 \sin^2 \theta \cos \theta$$
$$= 2 \cos^3 \theta - 7 \sin^2 \theta \cos \theta$$

When the first derivative is zero, we must have $\sin^2 \theta = 2 \cos^2 \theta$, which we substitute in the above

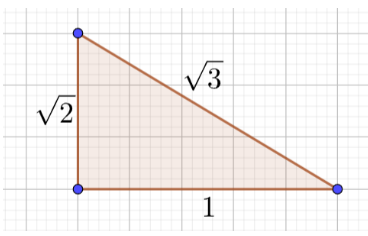
$$= 2 \cos^3 \theta - 7(2 \cos^2 \theta) \cos \theta$$
$$= 2 \cos^3 \theta - 14 \cos^3 \theta$$
$$= -12 \cos^3 \theta$$

Since θ is in the first quadrant:

$$\cos \theta > 0 \Rightarrow \cos^3 \theta > 0 \Rightarrow -12 \cos^3 \theta < 0 \Rightarrow \tan \theta = \sqrt{2} \textit{ is a maximum}$$

Draw a triangle and note that

$$Opp = \sqrt{2}, Adj = 1 \Rightarrow Hyp = \sqrt{2+1} = \sqrt{3}$$
$$\cos \theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$



G. Physics and Engineering

Example 3.166

Snell's Law

Strength of a beam

Stiffness of a beam

H. Life Sciences

Example 3.167
Branching angle for a blood vessel

I. Economics and Finance

Example 3.168: Optimal Amount of Labor
A firm has profit function $f(L) = pL^{0.5} - wL$ where p represents price of the good sold, w represents wage, L represents labor. Assume that p and w are fixed. What is the optimal level of labor to maximize profit (in terms of p and w)? (Note that at the optimal level of labour, the derivative of the profit function is zero) **(MAΘ, Limits and Derivatives, 2019/30, Adapted)**

Find the derivative of the profit function and equate it to zero:

$$f' = \frac{p}{2\sqrt{L}} - w = 0$$

Solve for L:

$$\frac{p}{2\sqrt{L}} = w \Rightarrow L = \frac{p^2}{4w^2}$$

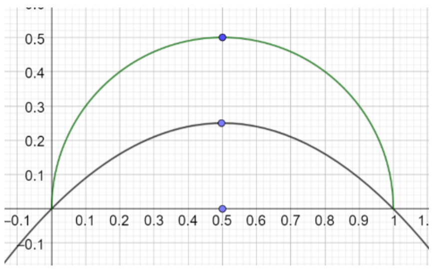
Example 3.169
EOQ
Production Quantity
Profit Maximization

J. Chemistry

K. Statistics

Example 3.170

$$y = \sqrt{x(1-x)}$$



If you maximize $y^2 = x(1-x)$, the x-value remains the same, but the y value changes.

Example 3.171
The standard deviation of a binomial distribution with n trials and probability of success p is given by

$$\sigma = \sqrt{np(1-p)}$$

A. Determine the maximum value of the standard deviation σ of a binomial distribution with n trials.
B. If n and p can both vary, determine the maximum value of the standard deviation.

Part A

As the question says, the binomial distribution has n trials. Hence, consider n as a constant, and p as an input to the function f.

$$f(p) = \sqrt{np(1-p)}$$

The domain of the function is:

$$0 \le p \le 1$$

Instead of maximizing $f(p)$, and having to deal with a square root, we maximize $[f(p)]^2$.

$$y = [f(p)]^2 = np(1-p) = n(p-p^2)$$

$$y' = n(1-2p) = 0 \Rightarrow p = \frac{1}{2}$$
$$y'' = n(-2) < 0 \Rightarrow Max$$

The maximum value of the standard deviation is:

$$\sigma\left(\frac{1}{2}\right) = \sqrt{n\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right)} = \sqrt{n\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = \frac{1}{2}\sqrt{n}$$

Part B

$$f(n,p) = \sqrt{np(1-p)}$$
$$\lim_{n\to\infty} \sigma = \lim_{n\to\infty} \frac{1}{2}\sqrt{n} = \infty \Rightarrow \sigma \text{ is unbounded}$$

3.12 Kinematics

A. Uniform Acceleration

Example 3.172

On an airless planet with gravitational constant $g = 19\frac{m}{s^2}$, a space traveler drops a rock from a height of 20m. Calculate the time for the rock to hit the surface of the planet.

$$a = g = 19\frac{m}{s^2}$$
$$v = \int a\,dt = \int 19\,dt = 19t$$
$$s = \int 19t = \frac{19t^2}{2}$$
$$20 = \frac{19t^2}{2} \Rightarrow t^2 = \frac{40}{19} = t = \sqrt{\frac{40}{19}} = 4\sqrt{\frac{10}{19}}$$

3.13 Economics

A. Elasticity

3.173: Elasticity

$$E = \frac{dQ}{dP}\cdot\frac{P}{Q}$$

Example 3.174

$$P = 400 - Q^2, Q = 12$$

$$Q = \sqrt{400-P}$$
$$\frac{dQ}{dP} = \frac{-1}{2\sqrt{400-P}} = \frac{-1}{2\sqrt{400-(400-Q^2)}} = \frac{-1}{2\sqrt{Q^2}} = \frac{-1}{2Q}$$

$$E = \frac{dQ}{dP} \cdot \frac{P}{Q} = \frac{-1}{2Q} \cdot \frac{400 - Q^2}{Q} = \frac{Q^2 - 400}{2Q^2}$$

Substitute $Q = 12$:

$$= \frac{144 - 400}{2 \cdot 144} = -\frac{256}{288} = \frac{8}{9}$$

Example 3.175

$$P = 100e^{-Q}$$

Take natural logs both sides:

$$\ln P = \ln(100e^{-Q})$$

Use the product property

$$\ln P = \ln 100 + \ln e^{-Q}$$

Use the power rule:

$$\ln P = \ln(100) - Q$$
$$Q = \ln(100) - \ln P$$
$$\frac{dQ}{dP} = -\frac{1}{P}$$

$$E = \frac{dQ}{dP} \cdot \frac{P}{Q} = -\frac{1}{P} \cdot \frac{P}{Q} = -\frac{1}{Q}$$
$$Q = 14 \Rightarrow E = -\frac{1}{14} \Rightarrow |E| = \frac{1}{14} > 1 \Rightarrow Inelastic$$

B. Optimization

Example 3.176

Quantity sold is given by $Q = 12 - \frac{1}{6}P$. Find the elasticity at the maximum revenue.

$$P = 72 - 6Q$$
$$TR = (72 - 6Q)(Q) = 72Q - 6Q^2$$

$$First\ Derivative = MR = \frac{d}{dP}(TR) = 72 - 12Q = 0 \Rightarrow Q = 6$$

$$Second\ Derivative = \frac{d}{dP}MR = \frac{d}{dP}(72 - 12Q) = -12 < 0 \Rightarrow Max$$

$$\frac{dQ}{dP} = -\frac{1}{6}$$

$$E = \frac{dQ}{dP} \cdot \frac{P}{Q} = -\frac{1}{6} \cdot \frac{72 - 6Q}{Q} = \frac{Q - 12}{Q}$$

Example 3.177

$$Price = P = 90 - 2Q - 0.1Q^2$$
$$C = 100 + 2Q + 8Q^2 - 0.1Q^3$$

A. Find the marginal revenue and marginal cost.

B. Find the maximum profit.

$$TR = 90Q - 2Q^2 - 0.1Q^3$$
$$\text{Profit} = \pi = \text{TR} - \text{C} = -100 + 88\text{Q} - 10\text{Q}^2$$
$$\pi' = 88 - 20\text{Q} = 0 \Rightarrow \text{Q} = \frac{88}{20} = 4.4$$
$$\pi'' = -20 < 0 \Rightarrow Max$$

Example 3.178

$$P = 200 - 5Q, C = 80Q + Q^2$$

$$TR = 200Q - 5Q^2$$
$$\pi = TR - C = 120Q - 6Q^2$$
$$\pi' = 120 - 12Q = 0$$

Profit is maximum when:

$$Q = \frac{120}{10} = 10$$

179 Examples