TRIGONOMETRY

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REVISION: 4752

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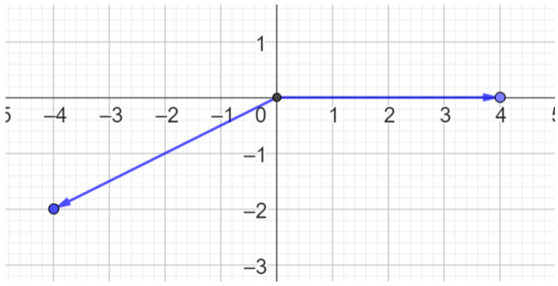
1. Unit Circle Trigonometry

1.1 Radians and the Unit Circle

A. Angles



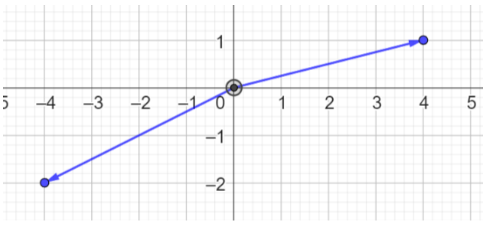
1.1: Angles in Standard Position

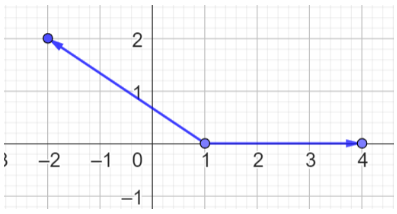
An angle is in standard position when

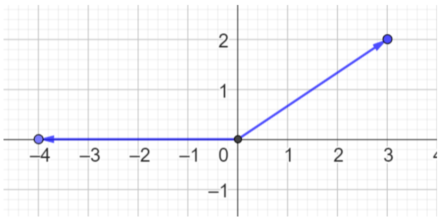
- The *vertex* is at the origin.
- One of the rays forming the angle has its endpoint on the x-axis, and extends along the x-axis in the positive direction

Example 1.2

Are the angles below in standard position. If not, explain why not?







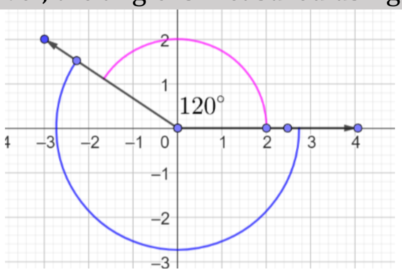
Not in standard position because

A. none of the rays extends in the positive direction of the x −axis.
B. the vertex is not at the origin.
C. The ray extends on $x - axis$, but not in the positive direction.

1.3: Rotation of Second Ray

For an angle in standard position, the terminal ray can be in any direction. However, the angle is measured using a specific convention:

- Counter-Clockwise Rotation = Positive
- Clockwise Rotation = Negative



Example 1.4

In the adjoining diagram, the angle is shown to be 120° in the positive direction. Calculate the value of the angle in the negative direction.

Going 120° in the positive direction is equivalent to going

$$360 - 120 = 240° \textit{ in the negative direction}$$

That is, the value of the angle is:

$$-240°$$

1.5: Positive and Negative Values

Any angle can be represented using both positive and negative values.

Example 1.6

Some angle values are given below. Convert them to positive if they are negative, and negative if they are positive.

A. $30°$
B. $150°$
C. $-50°$
D. $-110°$

$$-(360-30)=-330$$
$$-(360-150)=-210$$
$$360-50=310$$
$$360-110=250$$

1.7: Co-terminal Angles

If two or more angles have the same terminal side, they are said to be co-terminal.
Co-terminal angles are obtained by adding or subtracting multiples of 360.

Example 1.8

Give a positive angle between 0 and 360 for:

A. $-90°$
B. $-270°$
C. $-1000°$
D. $1000°$

$$-90°+360=270°$$
$$-270+360=90°$$
$$-1000+360(3)=80°$$
$$1000-360(2)=280°$$

Example 1.9

Give a negative angle between 0 and -360 for:

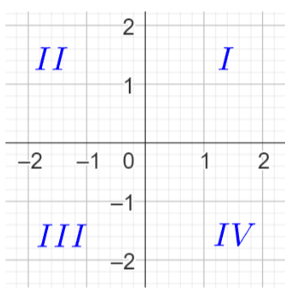
A. $80°$
B. $170°$
C. $4000°$

$$80-360=-280°$$
$$170-360=-190°$$
$$4000-360(12)=-320°$$

1.10: Quadrants and Angle Classification

The coordinate plane is divided into four quadrants (shown alongside).
An angle can be classified on the basis of the quadrant that its terminal ray lies in.

- $0<\theta<90°$: Quadrant I
- $90<\theta<180°$: Quadrant II
- $180<\theta<270°$: Quadrant III
- $270<\theta<360°$: Quadrant IV

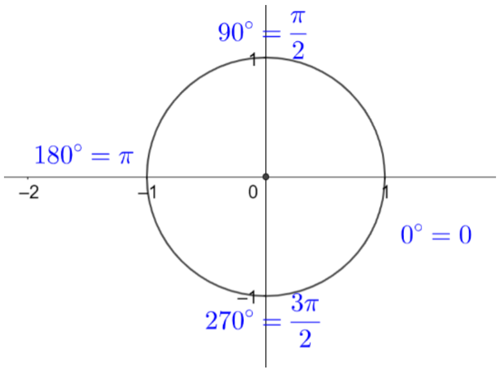


Example 1.11
Classify the following diagrams on the basis of the quadrants that the angles lie in.

Example 1.12
Classify the following angles on the basis of the quadrants that the angles lie in.

1.13: Quadrants and Angle Classification
An angle can be classified on the basis of the quadrant that its terminal ray lies in using negative angles as well
- $-360 < \theta < -270°$: Quadrant I
- $-270 < \theta < -180°$: Quadrant II
- $-180 < \theta < -90°$: Quadrant III
- $-90 < \theta < 0°$: Quadrant IV

Example 1.14
Classify the following angles on the basis of the quadrants that the angles lie in.



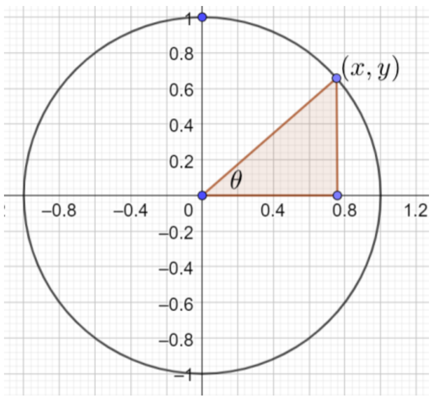
B. Unit Circle Trigonometry

1.15: Unit Circle
- A circle with radius 1 is called a unit circle.
- The circle is often drawn with its center at the origin.

1.16: Right Triangle Trigonometry
Right trigonometry defines the trigonometric ratios using a right triangle for θ between 0 and 90°. This is correct, but we will now generalize.

1.17: Trigonometric Ratios in the Unit Circle
For any point on the Unit Circle, determine the trigonometric functions by drawing a reference triangle.

- The reference triangle is always right-angled.
- The reference triangle goes in the x direction, same as the x coordinate of the point on the unit circle.
- The reference triangle goes in the y direction, to the same extent as the y coordinate of the point on the unit circle.
- The hypotenuse of the reference triangle is always drawn from the origin to the unit circle. Hence, it always has length 1.



1.18: Coordinates of a Point
The coordinates of a point on the unit circle

$$= (x, y) = (\cos\theta, \sin\theta)$$

Where

$$\theta \textit{ is the angle made by the radius with the positive } x - axis$$
$$\textit{The angle is positive in the counter} - \textit{clockwise direction}$$

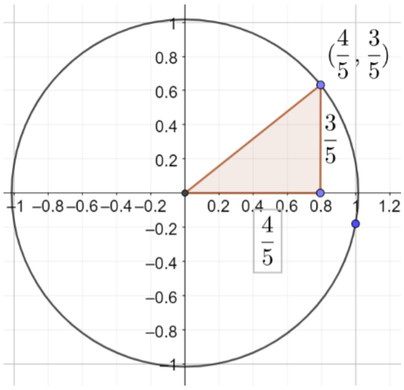
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y$$
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x$$

Example 1.19

A point has coordinates $\left(\frac{4}{5}, \frac{3}{5}\right)$ on the unit circle.

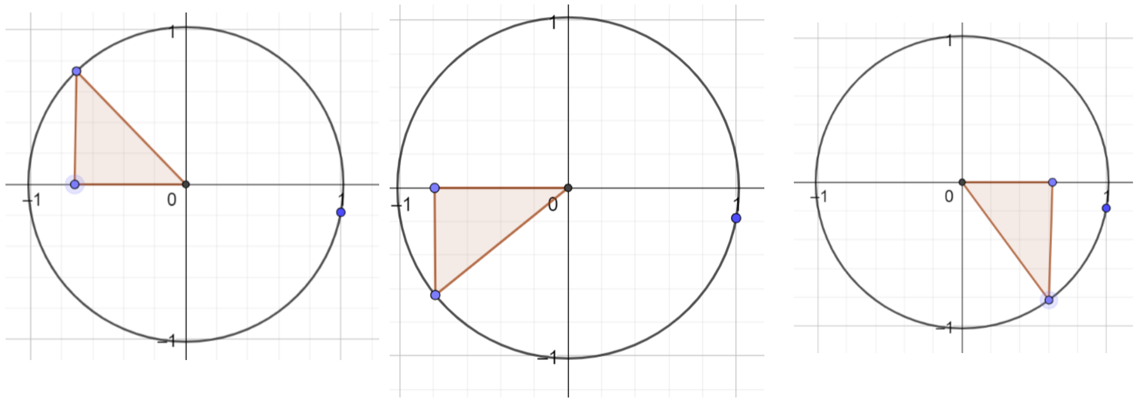
A. Draw the unit circle and the point on it.
B. Find the values of the six trigonometric functions

$$\sin\theta = y = \frac{3}{5}$$
$$\cos\theta = x = \frac{4}{5}$$
$$\tan\theta = \frac{y}{x} = \frac{0.6}{0.8} = \frac{6}{8} = \frac{3}{4}$$



1.20: Trigonometric Functions in any Quadrant

- To get the value of a trigonometric function in the first quadrant, you draw a reference triangle.
- You can draw a similar reference triangle for any quadrant that a point is in.



1.21: All Silver Tea Cups (Mnemonic)

In the first quadrant, ALL functions are positive.
In the second quadrant, only (SILVER) ($\sin\theta$) is positive.
In the third quadrant, only (TEA) $\tan\theta$ is positive.
In the fourth quadrant, only (CUPS) $\cos\theta$ is positive.

Silver QII	**All** QI
Tea *QIII*	**Cups** QIV

$$(x, y) = (\cos\theta, \sin\theta)$$

- This mnemonic is useful for remembering the signs of the trigonometric functions based on the quadrant they are in.

Example 1.22

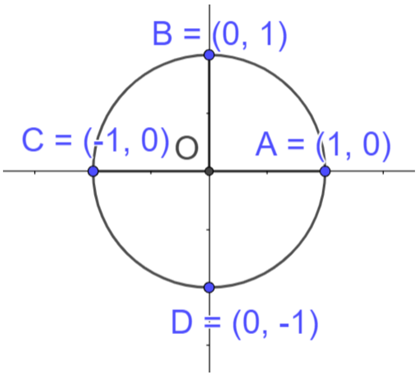
A. In the second quadrant, which functions are positive?

B. $\cos\theta$ is positive in which quadrants?

$$\sin\theta$$
$$I^{st} and\ IV^{th}$$

1.23: Angles on the axes

$$\cos 0 = 1, \sin 0 = 0$$
$$\cos 90° = 0, \sin 90° = 1$$
$$\cos 180° = -1, \sin 180° = 0$$
$$\cos 270° = 0, \sin 270° = -1$$



- All Silver Tea Cups will not give you values for angles whose terminal points lie on the axes.
- They have to picked up using the unit circle diagram alongside.

$$\angle AOA = \theta_1 = 0°$$
$$(x, y) = (\cos\theta_1, \sin\theta_1) = (\cos 0°, \sin 0°) = (1,0)$$
$$\cos 0° = 1, \sin 0° = 0$$

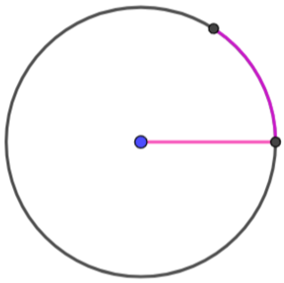
$$\angle BOA = \theta_2 = 90°$$
$$B_{(x,y)} = (\cos\theta_2, \sin\theta_2) = (\cos 90°, \sin 90°) = (0,1)$$

$$\angle COA = \theta_3 = 180°$$
$$C_{(x,y)} = (\cos\theta_3, \sin\theta_3) = (\cos 180°, \sin 180°) = (-1,0)$$

$$\angle AOD = \theta_4 = 270°$$
$$D_{(x,y)} = (\cos\theta_4, \sin\theta_4) = (\cos 270°, \sin 270°) = (0,-1)$$

C. Radians

Radians are a way of measuring angles. They are a dimensionless measure.



1.24: Radian Measure

Radians measure the number of times a radius will go around the arc of the degree measure of the angle in a circle.

$$C = 2\pi r$$

Measure of Angle	Degrees	Radians	Measure of Angle	Degrees	Radians
Full Circle	360	2π	One-Sixth of a Circle	60	$\frac{\pi}{3}$
Half Circle	180	π	One-Eighth of a Circle	45	$\frac{\pi}{4}$
Quarter Circle	90	$\frac{\pi}{2}$	One-Twelfth of a Circle	30	$\frac{\pi}{6}$

Example 1.25

A. What unit are radians written in?

B. Is it necessary that radians be in terms of π.

Part A

Radians are a dimensionless measure. Hence, they do not have any units, and the word radian is usually not written.
Unlike degrees, which are written using the ° symbol.

Part B

No.

$$1\ Radian = \frac{180}{\pi} \approx 57.32\,°$$

1.26: Converting from Radians to Degrees

$$Degrees\ to\ Radians: Multiply\ by\ \frac{\pi}{180}$$
$$Radians\ to\ Degrees: Multiply\ by\ \frac{180}{\pi}$$

Deriving the Conversion Factor

$$One\ Circle = 2\pi\ Radians \Rightarrow 360° = 2\pi\ Radians \Rightarrow 1\ Radian = \frac{360}{2\pi} = \frac{180}{\pi}$$

Formula

$$\underbrace{x\ Radians = \left(x \times \frac{180}{\pi}\right)^{\circ}}_{Conversion\ Factor=\frac{180}{\pi}} \Rightarrow \underbrace{y° = y \times \frac{\pi}{180} Radians}_{Conversion\ Factor=\frac{\pi}{180}}$$

The conversion factor for converting from degrees to radians is simply the reciprocal of the conversion factor from radians to degrees.

Example 1.27

Convert as directed:

From Radians to Degrees

A. π
B. 2π
C. $\frac{\pi}{2}$
D. $\frac{\pi}{3}$
E. $\frac{\pi}{4}$
F. $\frac{\pi}{6}$
G. $\frac{2\pi}{3}$
H. $\frac{5\pi}{6}$
I. $\frac{3\pi}{4}$

From Degrees to Radians

J. 180°
K. 360°
L. 30°
M. 90°
N. 60°
O. 45°
P. 120°
Q. 150°
R. 140°
S. 170°

From Radians to Degrees

$$\pi \times \frac{180}{\pi} = 180°$$
$$2\pi \times \frac{180}{\pi} = 360°$$
$$\frac{\pi}{2} \times \frac{180}{\pi} = 90°$$
$$\frac{\pi}{3} \times \frac{180}{\pi} = 60°$$
$$\frac{\pi}{4} \times \frac{180}{\pi} = 45°$$
$$\frac{\pi}{6} \times \frac{180}{\pi} = 30°$$
$$\frac{2\pi}{3} \times \frac{180}{\pi} = 120°$$
$$\frac{5\pi}{6} \times \frac{180}{\pi} = 150°$$
$$\frac{3\pi}{4} \times \frac{180}{\pi} = 135°$$

From Degrees to Radians

$$180° = \pi$$
$$360° = 2\pi$$
$$30° = \frac{\pi}{6}$$
$$90° = \frac{\pi}{2}$$
$$60° \times \frac{\pi}{180} = \frac{\pi}{3}\ Radians$$

$$45° = \frac{\pi}{4}$$
$$120° = \frac{2\pi}{3}$$
$$150° = \frac{5\pi}{6}$$
$$140° \times \frac{\pi}{180} = \frac{7}{9}\pi$$
$$170° \times \frac{\pi}{180} = \frac{17}{18}\pi$$

1.28: Adding and Subtracting 2π

You can add or subtract a full circle which is 2π any number of times. This can be used to change the value of an angle into an equivalent angle.

- This is mentioned before

Example 1.29

Convert to an angle in the range $[0,2\pi)$.

A. 3π
B. $-\frac{\pi}{4}$
C. $\frac{7\pi}{2}$
D. $-\frac{5\pi}{3}$
E. $\frac{13\pi}{2}$

Part A

We can convert to degrees, find the angle we are looking for, and convert back to radians:

$$3\pi = 3\pi \times \frac{180}{\pi} = 540° = 180° = \pi$$

Or we can directly work in radians:

$$3\pi - 2\pi = \pi$$

The second method is so much shorter! Try to work directly in radians as much as possible.

Part B

$$-\frac{\pi}{4} + 2\pi = -\frac{\pi}{4} + \frac{8\pi}{4} = \frac{7\pi}{4}$$
$$\frac{7\pi}{2} - 2\pi = \frac{7\pi}{2} - \frac{4\pi}{2} = \frac{3\pi}{2}$$
$$-\frac{5\pi}{3} + 2\pi = -\frac{5\pi}{3} + \frac{6\pi}{3} = \frac{\pi}{3}$$
$$\frac{13\pi}{2} = 6.5\pi = 6.5\pi - 6\pi = 0.5\pi = \frac{\pi}{2}$$

Example 1.30

Supplementary Angles

Find the supplementary angles for:

A. $\frac{\pi}{3}$
B. $\frac{\pi}{2}$
C. $\frac{\pi}{4}$
D. $\frac{\pi}{5}$

Complementary Angles

Find the complementary angles for:

E. $\frac{\pi}{6}$

F. $\frac{3\pi}{8}$

Supplementary

$$\pi - \frac{\pi}{3} = \frac{2\pi}{3}$$
$$\pi - \frac{\pi}{2} = \frac{\pi}{2}$$
$$\pi - \frac{\pi}{4} = \frac{3\pi}{4}$$
$$\pi - \frac{\pi}{5} = \frac{4\pi}{5}$$

Complementary Angles

$$\frac{\pi}{2} - \frac{\pi}{6} = \frac{3\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$$
$$\frac{\pi}{2} - \frac{3\pi}{8} = \frac{4\pi}{8} - \frac{3\pi}{8} = \frac{\pi}{8}$$

D. Geometry in Radians

1.31: Area of a Sector

The area of a sector with radius r and angle θ

$$= \frac{1}{2}r^2\theta$$

$$\pi r^2 \cdot \frac{\theta}{360} = \pi r^2 \cdot \frac{\theta}{2\pi} = \frac{1}{2}r^2\theta$$

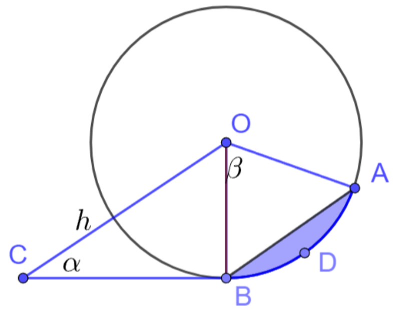
Example 1.32

1.33: Area of a Segment

The area of a segment of a circle with radius r and angle θ

$$= \frac{1}{2}r^2(\theta - \sin\theta)$$

$$= \underbrace{\frac{1}{2}r^2\theta}_{Sector} - \underbrace{\frac{1}{2}r^2\sin\theta}_{Triangle}$$
$$= \frac{1}{2}r^2(\theta - \sin\theta)$$



Example 1.34

Triangle OBC has hypotenuse h and $\angle OBC = \frac{\pi}{2}$. $\angle OCB$ is α. O is the center of the given circle. β is the angle subtended by arc BDA. Find the area of minor segment BDA in terms of the given variables.

Substitute $\sin\alpha = \frac{OB}{h} \Rightarrow r = OB = h\sin\alpha$ in the formula for the area of a sector

$$= \frac{1}{2}(h\sin\alpha)^2(\beta - \sin\beta) = \frac{h^2\sin^2\alpha}{2}(\beta - \sin\beta)$$

1.35: Arc Length

The arc length of a arc of a circle sector with radius r and angle θ

$$r\theta$$

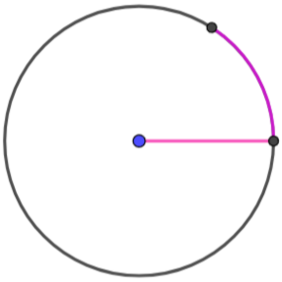
$$2\pi r\cdot\frac{\theta}{360}=2\pi r\cdot\frac{\theta}{2\pi}=r\theta$$

1.2 Unit Circle Identities

A. Unit Circle Identities

1.36: Adding/Subtracting 360°

$$\sin(360°n+\theta)=\sin\theta\,,n\in\mathbb{Z}$$
$$\cos(360°n+\theta)=\cos\theta\,,n\in\mathbb{Z}$$



Consider an angle $x°$. The angle does not change if you:
- Add a multiple of 360°
- Subtract a multiple of 360°

Example 1.37

Degrees

Find a comparable angle that measures between 0° and 360°, but represents the same angle.

A. $-90°$
B. $540°$
C. $405°$
D. $510°$
E. $-50°$
F.

Degrees

Example 1.38

Radians

Find a comparable angle that measures between 0 and 2π, but represents the same angle.

A. 3π
B. 5π
C. $\frac{7\pi}{2}$
D. $-\frac{\pi}{2}$
E. $\frac{23\pi}{6}$

Radians

$$\frac{7\pi}{2}-2\pi=\frac{3\pi}{2}$$
$$2\pi-\frac{\pi}{2}=\frac{3\pi}{2}$$
$$\frac{23\pi}{6}$$

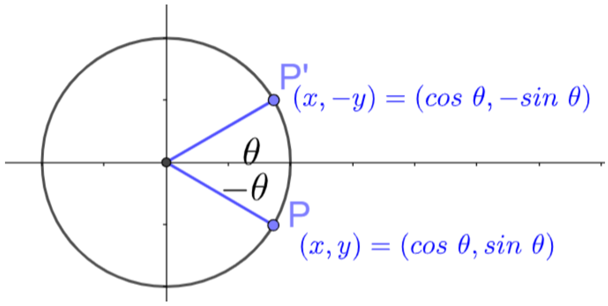
Example 1.39
Given the measure of an angle, find three positive angles, and three negative angles that represent the same angle.

A. $230°$

$$230° + 360°$$
$$230° + 720°$$
$$230° + 1080°$$
$$230° - 360°$$
$$230° - 720°$$
$$230° - 1080°$$

1.40: Reflecting across the x −axis
$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta$$

This property applies to angles in any quadrant.
But it is directly useful for converting angles in Quadrant IV into angles in Quadrant I.



$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

Example 1.41
Simplify:

Degrees

A. $\sin(-30°)$
B. $\cos(-30°)$
C. $\csc(-60°)$
D. $\sec(-45°)$
E. $\csc(-45°)$
F. $\tan(-30°)$
G. $\sin 330°$

Degrees

$$\sin -30° = -\sin 30° = -\frac{1}{2}$$
$$\cos\ -30° = \cos 30° = \frac{\sqrt{3}}{2}$$
$$\csc -60° = \frac{1}{\sin -60} = \frac{1}{-\sin 60} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}}$$
$$\sec -45° = \frac{1}{\cos -45} = \frac{1}{\cos 45} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

$$\csc -45° = \frac{1}{\sin -45} = \frac{1}{-\sin 45} = \frac{1}{-\frac{1}{\sqrt{2}}} = -\sqrt{2}$$
$$\tan -30 = -\tan 30 = -\frac{1}{\sqrt{30}}$$
$$\sin 330° = \sin -30°$$

Example 1.42
Simplify:

Radians

A. $\sin\left(-\frac{\pi}{6}\right)$
B. $\cos\left(-\frac{\pi}{4}\right)$
C. $\tan\left(-\frac{\pi}{3}\right)$
D. $\sin\left(\frac{5\pi}{3}\right)$
E. $\tan\frac{15\pi}{4}$

Radians

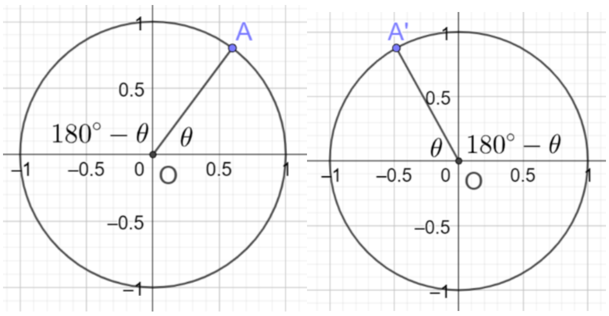
$$\sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6}$$
$$\cos\left(-\frac{\pi}{4}\right) = -\cos 45° = -\frac{1}{\sqrt{2}}$$
$$\tan\left(-\frac{\pi}{3}\right) = -\tan(60°) = -\sqrt{3}$$
$$\sin\frac{5\pi}{3} = \sin\left(-\frac{\pi}{3}\right) = -\sin(60°) = -\frac{\sqrt{3}}{2}$$
$$\tan\frac{15\pi}{4} = \tan\left(-\frac{\pi}{4}\right) = -\tan 45° = -1$$

1.43: Reflecting across the y −axis

$$\sin(180° - \theta) = \sin\theta$$
$$\cos(180° - \theta) = -\cos\theta$$
$$\tan(180 - \theta) = -\tan\theta$$

This property is directly useful for converting angles in Quadrant II into angles in Quadrant I.

Let the point A have coordinates
$$(x, y)$$
Let
$$\angle COB = \theta \Rightarrow \sin\theta = y, \cos\theta = x$$
$$\angle BOD = 180° - \theta \Rightarrow \sin(180 - \theta) = y$$
Hence,
$$\sin\theta = \sin(180 - \theta)$$
$$\cos(180° - \theta) = -\cos\theta$$
$$\tan(180 - \theta) = \frac{\sin(180 - \theta)}{\cos(180 - \theta)} = \frac{\sin\theta}{-\cos\theta} = -\tan\theta$$



Example 1.44

Simplify:

Degrees

A. $\sin 150°$
B. $\cos 120°$
C. $\sin 135°$
D. $\sin 120$
E. $\tan 150°$
F. $\sec 120°$
G. $\csc 150°$

Radians

H. $\sin\frac{2\pi}{3}$
I. $\tan\frac{5\pi}{6}$
J. $\cos\frac{3\pi}{4}$
K. $\cos\frac{17\pi}{6}$

Degrees

$$\sin 150° = \sin(180 - 30) = \sin 30° = \frac{1}{2}$$

$$\cos 120° = \cos(180 - 60) = -\cos 60° = -\frac{1}{2}$$

$$\sin 135° = \sin(180 - 45) = \sin 45 = \frac{1}{\sqrt{2}}$$

$$\sin 120° = \sin(180 - 60) = \sin 60 = \frac{\sqrt{3}}{2}$$

$$\tan 150° = -\tan 30° = -\frac{1}{\sqrt{3}}$$

$$\sec 120° = \frac{1}{\cos 120°} = \frac{1}{-\cos 60°} = \frac{1}{-\frac{1}{2}} = -2$$

$$\csc 150° = \frac{1}{\sin 150} = \frac{1}{\sin 30} = \frac{1}{\frac{1}{2}} = 2$$

$$\cot 120 = \frac{1}{\tan 120} = \frac{1}{-\tan 60} = -\frac{1}{\sqrt{3}}$$

Radians

$$\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \sin 60° = \frac{\sqrt{3}}{2}$$

$$\tan\frac{5\pi}{6} = \tan\left(\pi - \frac{5\pi}{6}\right) = \tan\frac{\pi}{6} = \tan 30° = \frac{1}{\sqrt{3}}$$

$$\cos\frac{3\pi}{4} = -\cos\left(\pi - \frac{3\pi}{4}\right) = -\cos(45°) = \frac{1}{\sqrt{2}}$$

$$\cos\frac{17\pi}{6} = \cos\frac{5\pi}{6} = -\cos\left(\pi - \frac{5\pi}{6}\right)$$

$$= -\cos\frac{\pi}{6} = -\cos 30° = -\frac{\sqrt{3}}{2}$$

Example 1.45

Let α and β be supplementary angles. Determine and prove the relationship between A and B if:

$$A = \sin\alpha\cos\beta, \qquad B = \cos\alpha\sin\beta$$

Since the angles are supplementary:

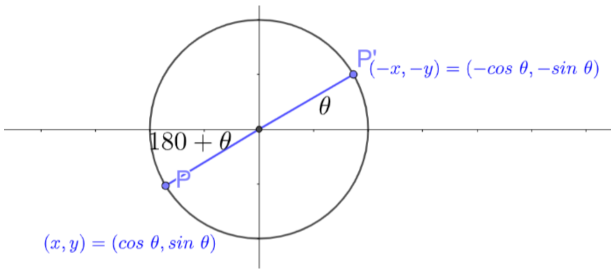
$$\alpha + \beta = \pi \Rightarrow \alpha = \pi - \beta$$

$$A = \sin(\pi - \beta)\cos\beta = \sin\beta\cos\beta$$
$$B = \cos(\pi - \beta)\sin\beta = -\cos\beta\sin\beta = -A$$

1.46: Reflecting across the origin

$$\sin(180° + \theta) = -\sin\theta$$
$$\cos(180° + \theta) = -\cos\theta$$
$$\tan(180° + \theta) = \tan\theta$$

This property is directly useful for converting angles in Quadrant I into angles in Quadrant III.



$$\tan(180° + \theta) = \frac{\sin(180° + \theta)}{\cos(180° + \theta)} = \frac{-\sin\theta}{-\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

Example 1.47

Simplify:

Degrees

A. $\sin 225°$
B. $\cos 240°$
C. $\cos 210°$

Radians

D. $\sin\frac{5\pi}{4}$

Degrees

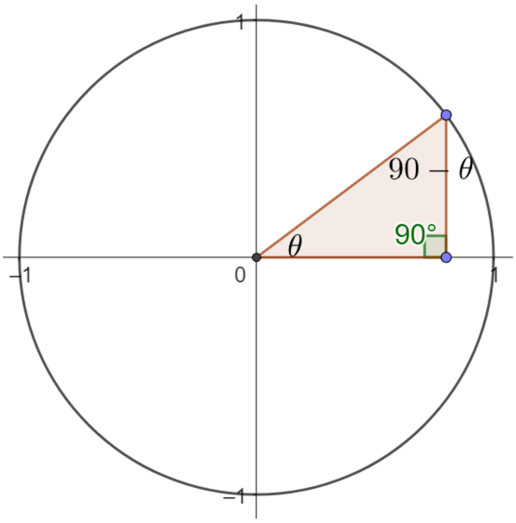
$$\sin 225° = \sin(180 + 45) = -\sin 45° = -\frac{1}{\sqrt{2}}$$
$$\cos(240) = \cos(180 + 60) = -\cos 60° =$$

Radians

$$\sin\frac{5\pi}{4} = \sin\left(\pi + \frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

1.48: Cofunction Identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$
$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$



B. Using the Identities

Example 1.49

$$\frac{-\left(\frac{\pi}{8}\right)\left(\frac{\pi}{4}\right)^{-2}\left(\frac{1}{2}\right)}{\cos\left(-\frac{7\pi}{6}\right)}$$

$$N = -\left(\frac{\pi}{16}\right)\left(\frac{16}{\pi^2}\right) = -\frac{1}{\pi}$$
$$D = \cos\left(-\frac{7\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$
$$N \times \frac{1}{D} = -\frac{1}{\pi} \times \left(-\frac{2}{\sqrt{3}}\right) = \frac{2}{\pi\sqrt{3}}$$

Example 1.50

$$\tan 15° + \frac{1}{\tan 75°} + \frac{1}{\tan 105°} + \tan 195° = 2a, \text{then } \left(a + \frac{1}{a}\right) \text{ is:}$$ **(JEE Main, Jan 30, 2023 – I)**

Write your answer as a single real number.

$$\frac{1}{\tan 105°} = \frac{1}{-\tan(180° - 105°)} = -\frac{1}{\tan 75°}$$

$$\tan 195° = \tan(180 + 15) = \tan 15°$$

$$\tan 15° + \frac{1}{\tan 75°} - \frac{1}{\tan 75°} + \tan 15° = 2a$$
$$2\tan 15° = 2a$$
$$a = \tan 15° = 2 - \sqrt{3}$$
$$\frac{1}{a} = \frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{2+\sqrt{3}}{1}$$

$$a + \frac{1}{a} = 2 - \sqrt{3} + 2 + \sqrt{3} = 4$$

Example 1.51
ISI 2023/Q1

C. More Identities

1.52: Adding multiples of π
When adding integers multiples of π to an angle, the sine and cosine are either the original angle or the negative of the original angle:

For $n \in \mathbb{Z}$:
$$\sin(n\pi + \theta) = (-1)^n \sin\theta$$
$$\cos(n\pi + \theta) = (-1)^n \cos\theta$$

We consider cases. n can be even or odd.

Case I: n is even
n is of the form $2k, k \in \mathbb{Z}$
$$LHS = \sin(2k\pi + \theta) = \sin\theta = (-1)^{2k}\sin\theta = (-1)^n \sin\theta = RHS$$

Case II: n is odd
n is of the form $2k + 1, k \in \mathbb{Z}$
$$LHS = \sin\big((2k+1)\pi + \theta\big) = \sin(2k\pi + \pi + \theta) = \sin(\pi + \theta) = -\sin\theta = (-1)^{2k+1}\sin\theta = (-1)^n \sin\theta = RHS$$

D. Applications

Example 1.53
The valve cap on a bicycle tire is a units away from the center of the wheel. The wheel has an outer radius of b units, with $a < b$. The line drawn from the center of the wheel to the valve cap exactly points in the positive x direction. The bicycle is pedaled so that wheel turns through an angle θ.

A. Show that the height of the valve cap from the ground is $b + a\sin\theta$.
B.

The left blue segment is simply:

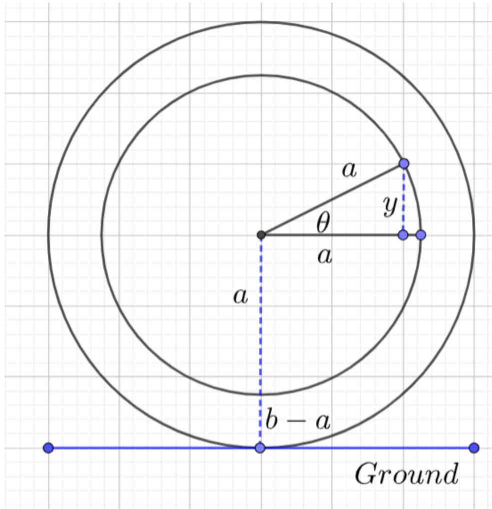
$$(b-a)+a=b$$

The right blue segment is:

$$\sin\theta=\frac{y}{a}\Rightarrow y=a\sin\theta$$

And hence, the height of the valve cap from the ground is:

$$b+a\sin\theta$$



1.54: Length of Arc

$$Circumference=2\pi r$$

$$Length\ of\ Arc=C\times\frac{\theta}{360}=2\pi r\times\frac{\theta}{360}=\pi r\times\frac{\theta}{180}$$

$$Length\ of\ Arc=C\times\frac{\theta}{2\pi}=2\pi r\times\frac{\theta}{2\pi}=r\theta$$

1.55: Area of a Sector

$$Area\ of\ a\ Sector=\underbrace{\pi r^2}_{\text{Area of a Circle}}\times\underbrace{\frac{\theta}{360}}_{\text{Proportionality Constant}}$$

$$Area\ of\ a\ Sector=\frac{r^2\theta}{2}$$

Example 1.56

Find the length of arc, and the area of a sector of a circle with radius 10, and central angle 18°.

Length of Arc

$$Length\ of\ arc=\pi r\times\frac{\theta}{180}=\pi(10)\times\frac{18°}{180}=\pi$$

$$18°=\frac{\pi}{10}\Rightarrow Length\ of\ arc=r\theta=10\times\frac{\pi}{10}=\pi$$

Area of Sector

$$Area\ of\ a\ Sector=\pi r^2\times\frac{\theta}{360}=\pi(10)^2\times\frac{18°}{360}=5\pi$$

$$Area\ of\ a\ Sector=\frac{r^2\theta}{2}=\frac{(10)^2\left(\frac{\pi}{10}\right)}{2}=5\pi$$

1.57: Angular Speed

$$Angular\ speed=\omega=\frac{\theta}{t}$$

Example 1.58

A. Ferris Wheel
B. Carousel
C. Spinning Top
D. Rotation of the Earth

1.3 Law of Sines

A. Law of Sines

The law of sines relates the sine of three angles of a triangle to their side lengths. The longest side is opposite the largest angle, and the shortest side is opposite the shortest angle.

1.59: Solution of Triangles

Solution of triangles refers to determining the lengths of the sides of a triangle, and the measures of the angles of the triangle.

A triangle can be solved using:

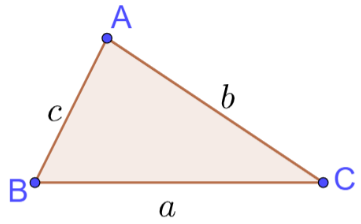
- Law of Sines
- Law of Cosines

1.60: Law of Sines

The length of the sides of a triangle are proportional to the sine of the angles of the sides.

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{Sin\ C}$$

Note: They are not proportional to the values of the angles themselves.



Calculate the area of the triangle in three different ways:

$$\frac{1}{2}ab\sin C=\frac{1}{2}ac\sin B=\frac{1}{2}bc\sin A$$

Equate the second and the third part:

$$ac\sin B=bc\sin A\Rightarrow\underbrace{\frac{a}{\sin A}=\frac{b}{\sin B}}_{\textit{Equation I}}$$

Equate the first and the second part:

$$ab\sin C=ac\sin B\Rightarrow\underbrace{\frac{b}{\sin B}=\frac{c}{\sin C}}_{\textit{Equation II}}$$

Combine Equation I and II into a tripartite equality:

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

Example 1.61

The angles of a triangle are $50°, 60°$ and $70°$. The longest side of the triangle has length 3 meters. Find a trigonometric expression for the length of the other two sides.

$$\frac{a}{\sin 50°}=\frac{b}{\frac{\sqrt{3}}{2}}=\frac{3}{Sin\ 70°}$$

$$\frac{b}{\frac{\sqrt{3}}{2}}=\frac{3}{Sin\ 70°}\Rightarrow b=\frac{3\sqrt{3}}{2\cdot Sin\ 70°}$$

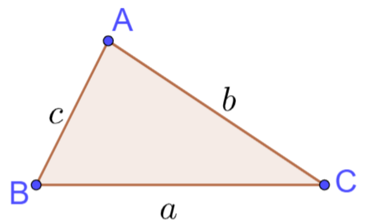
$$a = \frac{3 \sin 50°}{Sin\ 70°}$$

(Calc) Example 1.62

The angles of a triangle measure 104, 51 and 25 degrees. The perimeter of the triangle is 10m. Find, rounded to two decimal places, the length of each side of the triangle.

Let the sides of the triangle have lengths a, b, c. Introduce a constant k such that:

$$\frac{\sin 104}{a} = \frac{\sin 51}{b} = \frac{\sin 25}{c} = k$$



Solve for a, b, c:

$$a = \frac{\sin 104}{k}, \qquad b = \frac{\sin 51}{k}, \qquad c = \frac{\sin 25}{k}$$

Since the perimeter of the triangle is 10, we must have:

$$a + b + c = 10$$

Substitute the values of a, b and c from above:

$$\frac{\sin 104}{k} + \frac{\sin 51}{k} + \frac{\sin 25}{k} = 10$$

Using a calculator, solve for k to get:

$$k = 0.217$$

Calculate the side lengths of the triangle:

$$a = \frac{\sin 104}{k} = \frac{\sin 104}{0.217} = 4.47$$

Example 1.63: Trigonometric Equations

Triangle ABC has $AB = 2 \cdot AC$. Let D and E be on $\overline{AB}$ and $\overline{BC}$, respectively, such that $\angle BAE = \angle ACD$. Let F be the intersection of segments AE and CD, and suppose that $\triangle CFE$ is equilateral. What is $\angle ACB$? **(AMC 10A 2010/14)**

Angle Chasing

Let

$$\angle ACD = \angle BAE = x$$

By Angles in a Linear Pair:

$$\angle AFC = 180 - 60 = 120$$

By sum of angles in a triangle:

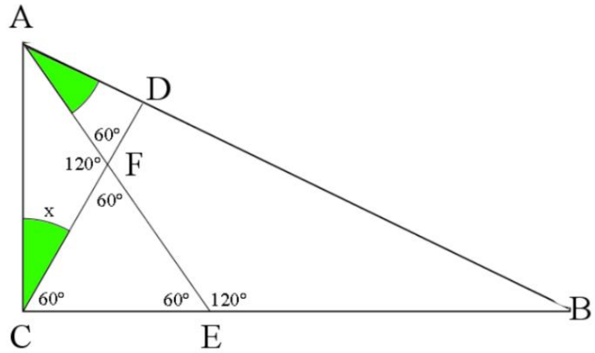
$$\angle CAF = 180 - 120 - x = 60 - x$$

By Adjacent Angles:

$$\angle CAB = \angle CAE + \angle EAB = 60 - x + x = 60$$

By sum of angles in a triangle:

$$\angle ABC = 180 - (60 + x) - 60 = 60 - x$$



Law of Sines
Trigonometric Equations

B. Bearings

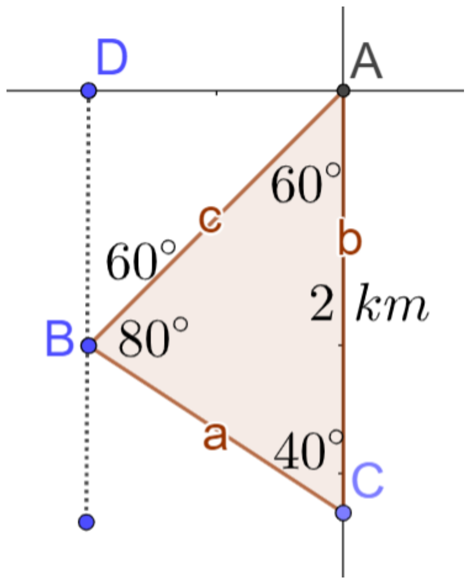
Example 1.64: True Bearings

A ship leaves port and travels on a bearing of 034° for 100 km. It then changes course, and sails for 150 km at a bearing of 015°. Find the distance and the bearing of the port from the ship.

Example 1.65: True Bearings

A hunter finds deer tracks outside his camp at a bearing of $240°\ T$. He follows the track for two hours before spotting boar tracks at a bearing of $140°\ T$ from his then position. He switches over to tracking the boar, which he does for three hours. When he finally catches up with the boar, he notes that his camp is 2 km away at a bearing of $0°\ T$. Find the difference in the distance between the hunter's return path from the boar kill, and the path to the boar kill. Also, find the ratio of speeds when tracking boar and deer respectively.

Let A be the starting point. Then B is the point where the hunter switches from tracking deer to tracking boar, and C is the point of the boar kill.



$$\angle BAC = 60°$$

Draw $DB \parallel AC$. By alternate interior angles:

$$\angle DBA = \angle BAC = 60°$$
$$\angle ABC = 140° - 60° = 80°$$
$$\angle BCA = 180 - 80 - 60 = 40°$$

By the Law of Sines, in ΔABC:

$$\frac{2}{\sin 80} = \frac{a}{\sin 60} = \frac{c}{\sin 40}$$
$$Distance\ Tracking\ Boar = a = (\sin 60)\left(\frac{2}{\sin 80}\right) = 1.75$$
$$Distance\ Tracking\ Deer = c = (\sin 40)\left(\frac{2}{\sin 80}\right) = 1.31$$
$$a + c = (\sin 60 + \sin 40)\left(\frac{2}{\sin 80}\right) = 3.06$$

Difference

$$= 3.06 - 2 = 1.06$$

We can find the ratio as:

$$Tracking\ Board: Tracking\ Deer = \frac{a}{3}:\frac{b}{2} = \frac{1.75}{3}:\frac{1.31}{2} = 10.5: 7.86 = 1050: 786 = 525: 393$$

Example 1.66: True Bearings

Long John's Silver's treasure map shows a treasure chest 1200 meters inland at a bearing of $132°\ T$ from a point on an inaccessible cliff. A beach at a bearing of $63°\ T$ from the cliff is 1700 meters from the treasure chest. Find the direction in which he must go after reaching the beach.

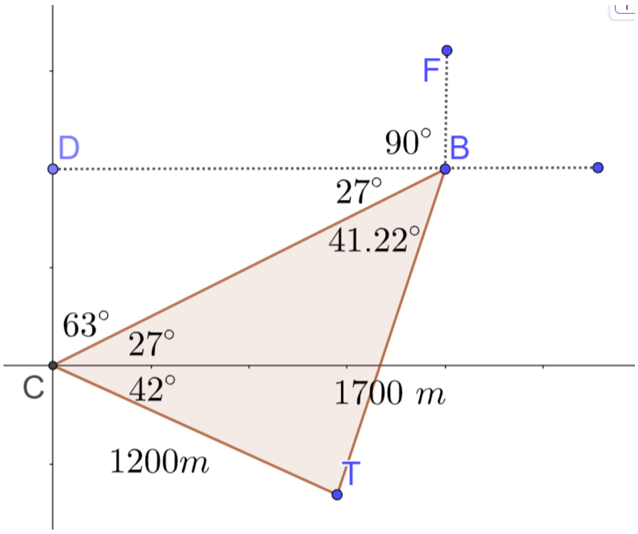
Draw a diagram with

$$C = Cliff, T = Treasure, B = Beach$$
$$\angle BCT = 27 + 42 = 69°$$

$$\frac{\sin 69}{1700} = \frac{\sin B}{1200} \Rightarrow B = \sin^{-1}\left(\frac{12 \sin 69}{17}\right) = 41.22°$$

By angles around point B:

$$\angle FBT = 360 - 90 - 27 - 41.22 = 201.78°$$



Example 1.67: Compass Bearings

C. Challenge Questions

Example 1.68
Number Theory, Min/max

D. Extension

1.69: Extended Law of Sines

$$\frac{a}{\sin A} = \frac{n}{\sin B} = \frac{c}{Sin\ C} = 2R$$

1.70: Length of Circumradius: Trigonometric Proof

$$\frac{a}{\sin A} = 2R$$

$$R = \frac{a}{2 \sin A} = \frac{abc}{2bc \sin A} = \frac{abc}{2 \times 2\left(\frac{1}{2}\ bc \sin A\right)} = \frac{abc}{4\Delta}$$

Challenge 1.71
Olympiad Problem from Putnam.
Requires some Calculus at the very end

1.4 Law of Cosines

A. Basics

The law of cosines is a generalization of the Pythagoras Theorem.

1.72: Law of Cosines

The law of cosines lets us find the length of the third side of a triangle if we know two sides and the included angle.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

- A is the angle included between sides b and c

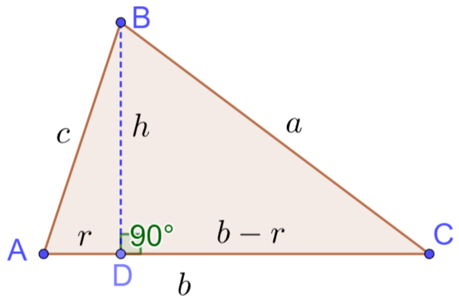
> ➢ It generalizes the Pythagorean Theorem.
> ➢ Unlike the law of Sines, it does not have an "ambiguous" case. Hence, it is preferred over the law of Sines where possible.

We consider two cases:

Case I: The altitude of the triangle is inside the base

In right ΔBDC:

$$a^2 = h^2 + (b-r)^2 = h^2 + b^2 - 2br + r^2$$



Substitute

$$\sin A = \frac{h}{c} \Rightarrow h = c\sin A \Rightarrow h^2 = c^2\sin^2 A$$
$$\cos A = \frac{r}{c} \Rightarrow r = c\cos A \Rightarrow r^2 = c^2\cos^2 A$$

$$a^2 = c^2\sin^2 A + b^2 - 2bc\cos A + c^2\cos^2 A$$

Factor c^2 from the first and last term in the RHS:

$$a^2 = c^2(\sin^2 A + \cos^2 A) + b^2 - 2bc\cos A$$

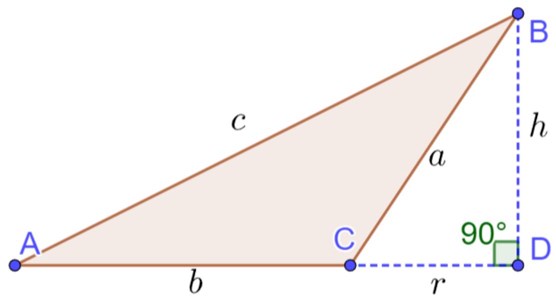
Substitute $\sin^2 A + \cos^2 A = 1$:

$$a^2 = c^2 + b^2 - 2bc\cos A$$

Case II: The altitude of the triangle is outside the base

By Pythagoras Theorem in right ΔBCD:

$$\underbrace{a^2 = r^2 + h^2}_{\textit{Equation I}}$$



We want to eliminate r and h, and introduce b and c.

By Pythagoras Theorem in right ΔBDA:

$$(b+r)^2 + h^2 = c^2$$
$$h^2 = c^2 - (b+r)^2$$
$$h^2 = c^2 - b^2 - 2br - r^2$$
$$\underbrace{h^2 + r^2 = c^2 - b^2 - 2br}_{\textit{Equation II}}$$

From Equations I and II:

$$a^2 = c^2 - b^2 - 2br$$

Substitute $\cos A = \frac{b+r}{c} \Rightarrow b + r = c\cos A \Rightarrow r = c\cos A - b$

$$a^2 = c^2 - b^2 - 2b(c\cos A - b)$$
$$a^2 = c^2 - b^2 - 2bc\cos A + 2b^2$$
$$a^2 = c^2 + b^2 - 2bc\cos A$$

1.73: Law of Cosines: Alternate Version

Equivalently, if we know the three sides of a triangle, the law of cosines lets us find the cosine of any angle of the triangle.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Take $\cos A$ to the LHS, and the a^2 term to the RHS:

$$a^2 - b^2 - c^2 = -2bc\cos A$$

Divide both sides by $-2bc$:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Example 1.74

Show that the Pythagorean Theorem is a special case of the law of cosines.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

The Pythagorean Theorem is applicable to right triangles, which are triangles with a right angle.
Let $\angle A = 90°$ and substitute in the law of cosines

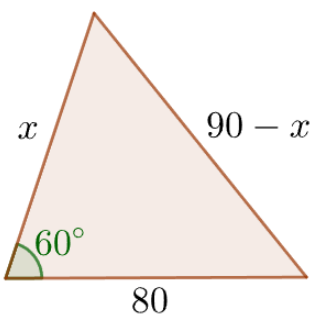
$$a^2 = b^2 + c^2 - 2bc\cos 90°$$

Substitute $\cos 90° = 0$

$$a^2 = b^2 + c^2 - 2bc \cdot 0$$
$$a^2 = b^2 + c^2$$

Example 1.75

The base of a triangle is 80, and one of the base angles is 60°. The sum of the lengths of the other two sides is 90. The shortest side is: **(AHSME 1959/36)**



Let the sides of the triangle be:

$$80, x, 90 - x$$

By the Law of Cosines:

$$(90 - x)^2 = x^2 + 80^2 - 2(x)(80)\cos 60°$$

Expand

$$8100 - 180x + x^2 = x^2 + 6400 - 160x\left(\frac{1}{2}\right)$$
$$1700 = 100x$$
$$x = 17$$
$$90 - x = 73$$

1.76: Classifying triangles using Law of Cosines

Example 1.77

AHSME

1952 AHSME Problems/Problem 15

Problem

The sides of a triangle are in the ratio $6:8:9$. Then:

$\textbf{(A)}$ the triangle is obtuse

$\textbf{(B)}$ the angles are in the ratio $6:8:9$

$\textbf{(C)}$ the triangle is acute

$\textbf{(D)}$ the angle opposite the largest side is double the angle opposite the smallest side

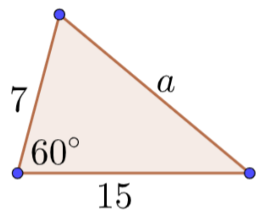
$\textbf{(E)}$ none of these

B. Working with Triangles

Example 1.78: Finding Sides

A. Find the third side of a triangle with sides 7 and 15 and included angle 60 degrees.
B. One side of a triangular garden is 10 meters long. The side adjacent to it is 5 meters long. The angle between the two sides is 45°. Find the third side of the garden.

Part A

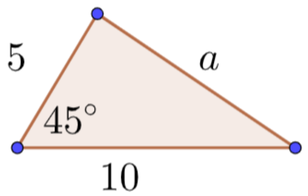


$$a^2 = b^2 + c^2 - 2bc\cos A$$

Substitute $b = 7, c = 15, A = 60°$:

$$a^2 = 7^2 + 15^2 - (2)(7)(15)(\cos 60) = 49 + 225 - 2\cdot 7\cdot 15\cdot\frac{1}{2} = 169$$
$$a = 13$$

Part B



$$a^2 = b^2 + c^2 - 2bc\cos A$$

Substitute $b = 5, c = 10, A = 45°$:

$$a^2 = 5^2 + 10^2 - (2)(5)(10)(\cos 45°) = 125 - 100\frac{\sqrt{2}}{2} = 125 - 50\sqrt{2}$$

Take the square root both sides:

$$a = \sqrt{125 - 50\sqrt{2}} = \sqrt{25(5 - 2\sqrt{2})} = 5\sqrt{5 - 2\sqrt{2}}$$

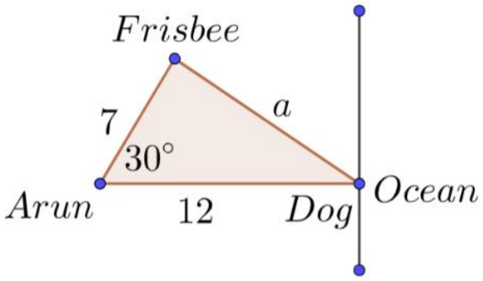
Example 1.79: Finding Sides

Arun is at the beach. He throws his frisbee and it lands 7 meters away. Arun's dog is chasing a wave at the edge of the ocean, twelve meters aways. If the angle between his dog and his frisbee is 30° for Arun, then how much does his dog have to run to get the frisbee back to Arun.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Substitute $b = 7, c = 12, A = 30°$:

$$a^2 = 7^2 + 12^2 - (2)(7)(12)(\cos 30)$$
$$= 49 + 144 - 2 \cdot 7 \cdot 12 \cdot \frac{\sqrt{3}}{2}$$
$$= 193 - 84\sqrt{3}$$
$$a = \sqrt{193 - 84\sqrt{3}}$$



The distance to be run by the dog

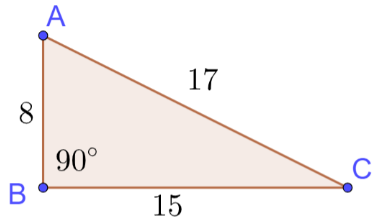
$$= 7 + a = 7 + \sqrt{193 - 84\sqrt{3}}$$

Example 1.80: Finding Angles

A. A right triangle has hypotenuse 17, and one side 15. Find the ratio of the other two angles in the triangle.

Since it is a Pythagorean Triplet, the third side of the triangle must be 8.

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$15^2 = 8^2 + 17^2 - 2(8)(17)(\cos A)$$
$$272(\cos A) = 128$$
$$A = \cos^{-1}\left(\frac{128}{272}\right)$$



(Calc) Example 1.81: Multiple Triangles

Looking down from a 62-meter-high tower, you can see point A with an angle of depression of 24.3°. By swinging the telescope 76.51° in a horizontal arc, you can see point B with an angle of depression of 30°. Calculate the distance AB if both points A and B have the same height above mean sea level.

Let

$$Base\ of\ Tower = Q, Top\ of\ Tower = P$$

Find the angles we need:

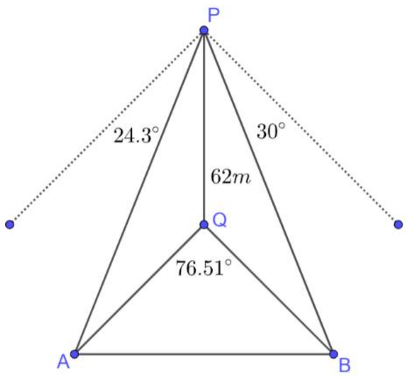
$$\angle APQ = 90 - 24.3 = 65.7°$$
$$\angle BPQ = 90 - 30 = 60°$$

In ΔAPQ:

$$\tan 65.7° = \frac{AQ}{62} \Rightarrow AQ = 62 \tan 65.7°$$

In ΔBPQ:

$$\tan 60° = \frac{BQ}{62} \Rightarrow BQ = 62 \tan 60° = 62\sqrt{3}$$

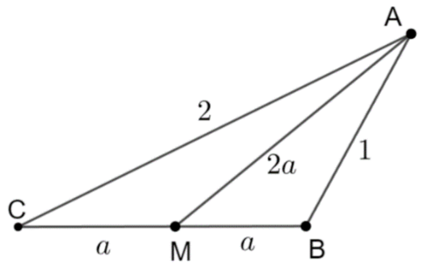


By the Law of Cosines in ΔAQB:

$$AB = \sqrt{AQ^2 + BQ^2 + 2(AQ)(BQ)(\cos 76.51°)} \approx 153.322 \approx 153$$

Example 1.82

In ΔABC, we have $AB = 1$, and $AC = 2$. Side BC and the median from A to BC have the same length. What is BC? (AMC 12 2002)



We do not know the measure of any angle in the triangle. Hence, apply the Law of Cosines to two angles which are supplementary.

By Law of Cosines in ΔAMB

$$\underbrace{AB^2 = BM^2 + AM^2 - 2(BM)(AM)(\cos\angle AMB)}_{\textit{Equation I}}$$

By Law of Cosines in ΔAMC:

$$AC^2 = CM^2 + AM^2 - 2(CM)(AM)(\cos\angle AMC)$$

Substitute $CM + BM, \cos\angle AMC = \cos(180 - \angle AMB) = -\cos\angle AMB$:

$$\underbrace{AC^2 = BM^2 + AM^2 + 2(BM)(AM)(\cos\angle AMB)}_{\textit{Equation II}}$$

Add Equations I and II:

$$AB^2 + AC^2 = 2BM^2 + 2AM^2$$

Substitute $AB^2 = 1, AC^2 = 4, BM^2 = \left(\frac{AM}{2}\right)^2 = \frac{AM^2}{4}$:

$$1 + 4 = 2\left(\frac{AM^2}{4}\right) + 2AM^2$$
$$5 = \frac{5}{2}AM^2$$
$$AM^2 = 2$$
$$AM = \sqrt{2}$$

1.83: Cevian

A line segment (or a ray) from the vertex of a triangle drawn to the opposite side (or its extension) is a cevian.

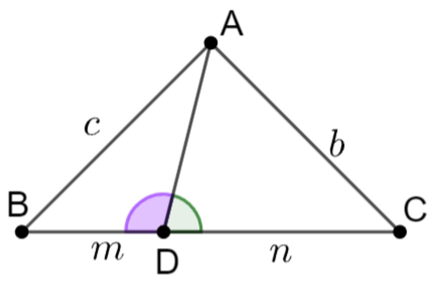
Special cases of cevian include:

- Median
- Angle Bisector
- Altitude

1.84: Stewart's Theorem

$$man + dad = bmb + cnc$$

Which can be remembered using the mnemonic:
A *man* and his *dad* put a *bomb* in the *sink*



By Law of Cosines in ΔADC:

$$b^2 = n^2 + d^2 - 2nd\cos\angle ADC$$

$$\underbrace{\cos\angle ADC = \frac{n^2 + d^2 - b^2}{2nd}}_{\textit{Identity I}}$$

By Law of Cosines in ΔADB:

$$c^2 = m^2 + d^2 - 2md\cos\angle ADB$$

Substitute $\cos\angle ADB = -\cos\angle ADC$:

$$c^2 = m^2 + d^2 + 2md\cos\angle ADC$$
$$\underbrace{\cos\angle ADC = \frac{c^2 - m^2 - d^2}{2md}}_{\textit{Identity II}}$$

Since the LHS of Identity I is the same the LHS of Identity II, their RHS must also be equal:

$$\frac{n^2 + d^2 - b^2}{2nd} = \frac{c^2 - m^2 - d^2}{2md}$$

Eliminate fractions:

$$mn^2 + md^2 - mb^2 = nc^2 - nm^2 - nd^2$$
$$mn^2 + nm^2 + md^2 + nd^2 = mb^2 + nc^2$$
$$(m+n)mn + d^2(m+n) = mb^2 + nc^2$$

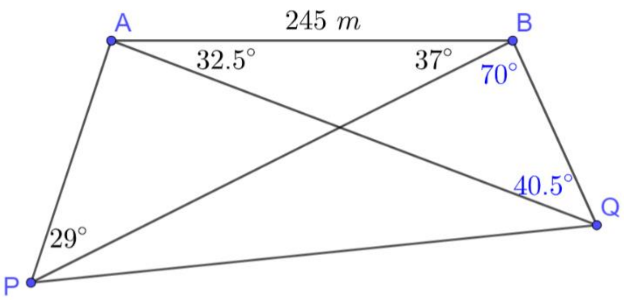
Substitute $m + n = a$:

$$amn + d^2a = mb^2 + nc^2$$
$$man + dad = bmb + cnc$$

C. Quadrilaterals

Example 1.85

Find PQ given that $AB = 245\ meter$, $\angle PAB = 114°$, $\angle QAB = 32.5°$, $\angle ABQ = 107°$, $\angle ABP = 37°$.



First, find the angles we need:

$$\angle PQB = \underbrace{107}_{\angle ABQ} - \underbrace{37}_{\angle ABP} = 70$$
$$\angle BQA = 180 - \underbrace{32.5}_{\angle QAB} - \underbrace{107}_{\angle ABQ} = 40.5$$

Apply the Law of Sines twice:

$$\Delta ABQ: \frac{245}{\sin 40.5°} = \frac{BQ}{\sin 32.5°} \Rightarrow BQ = \frac{245 \times \sin 32.5°}{\sin 40.5°}$$

$$\Delta ABP: \frac{245}{\sin 29°} = \frac{BP}{\sin 114°} \Rightarrow BP = \frac{245 \times \sin 114°}{\sin 29°}$$

By the Law of Cosines in ΔPBQ:

$$PQ = \sqrt{BP^2 + BQ^2 + 2(BP)(BP)(\cos 70°)} \approx 436.126 \approx 436$$

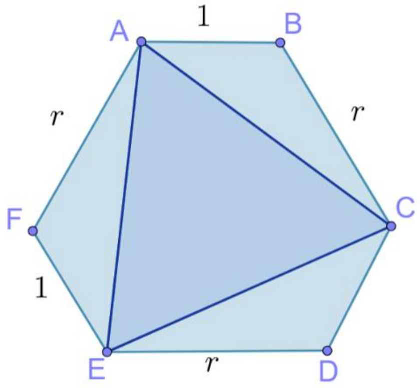
D. Polygons

Example 1.86: Hexagons

Equiangular hexagon $ABCDEF$ has side lengths $AB = CD = EF = 1$ and $BC = DE = FA = r$. The area of ΔACE is 70% of the area of the hexagon. What is the sum of all possible values of r? (AMC 10A 2010/19)[1]

[1] For a geometric solution, refer the Note on Polygons in Geometry.

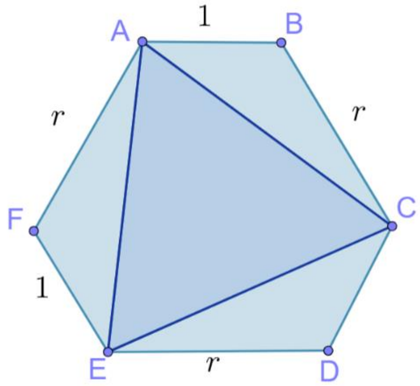
$$A(\Delta CED) + A(\Delta AFE) + A(\Delta ABC)$$
$$= \underbrace{100\%}_{A(ABCDEF)} - \underbrace{70\%}_{A(\Delta ACE)} = 30\%\ of\ A(ABCDEF)$$



Each exterior angle of $ABCDEF$
$$= \frac{360}{6} = 60$$

Hence, $\Delta CED \cong \Delta AFE \cong \Delta ABC$ by SAS Theorem since each triangle has
$$s_1 = 1, s_2 = r, Included\ angle = 180 - 60° = 120°$$

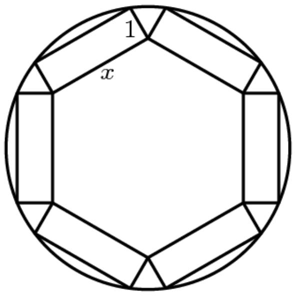
Therefore,
$$3A(\Delta CED) = 30\% \Rightarrow A(CED) = 10\%\ A(ABCDEF)$$



By CPCT:
$$AC \cong CE \cong AE \Rightarrow \Delta ACE\ is\ equilateral$$
By the trigonometric formula for area of a triangle:
$$A(CED) = \frac{1}{2}(1)(r)(\sin 120°) = \frac{\sqrt{3}}{4}r$$
By the law of cosines in ΔCED:
$$CE^2 = 1^2 + r^2 - 2(1)(r)(\cos 120°) = r^2 + r + 1$$
By the formula for area of an equilateral triangle:
$$A(ACE) = \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}(r^2 + r + 1)$$

But, $\underbrace{A(ACE)}_{70\%\ of\ A(ABCDEF)} = 7 \times \underbrace{A(CED)}_{10\%\ of\ A(ABCDEF)}$. Substituting:
$$\frac{\sqrt{3}}{4}(r^2 + r + 1) = 7\frac{\sqrt{3}}{4}r$$
$$r^2 + r + 1 = 7r$$
$$r^2 - 6r + 1 = 0$$
By Vieta's Formulas, the sum of the roots of the above quadratic is:
$$-\frac{b}{a} = -\frac{-6}{1} = 6$$

E. Circles



Example 1.87: Circles

A round table has radius 4. Six rectangular place mats are placed on the table. Each place mat has width 1 and length x as shown. They are positioned so that each mat has two corners on the edge of the table, these two corners being end points of the same side of length x. Further, the mats are positioned so that the inner corners each touch an inner corner of an adjacent mat. What is x? **(AMC 10A 2008/25)**[2]

[2] For a geometric solution, refer the Note on Polygons in Geometry.

Let the circle have center O.

$$OD = Radius = 4$$

BC forms one side of a regular hexagon.

$$\Delta BCO \text{ is equilateral} \Rightarrow BC = OC = x$$

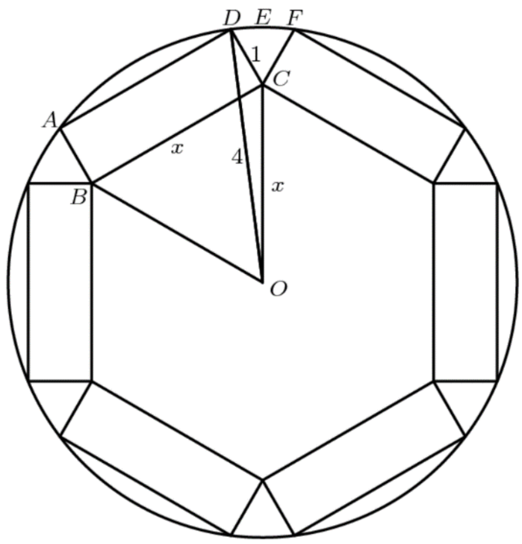
In ΔDCO:

$$\angle OCD = \underbrace{60°}_{\angle OCB} + \underbrace{90°}_{\angle BCD} = 150°$$

$$Law\ of\ Cosines \Rightarrow 4^2 = 1^2 + x^2 - 2(1)(x)(\cos 150°)$$

$$16 = 1 + x^2 - 2(x)\left(-\frac{\sqrt{3}}{2}\right)$$

$$x^2 + \sqrt{3}x - 15 = 0$$



Apply the quadratic formula with $a = 1, b = \sqrt{3}, c = -15$

$$x = \frac{-\sqrt{3} \pm \sqrt{3 - (4)(1)(-15)}}{2} = \frac{-\sqrt{3} \pm \sqrt{63}}{2} = \frac{-\sqrt{3} \pm 3\sqrt{7}}{2}$$

$$\rightarrow \underbrace{\frac{3\sqrt{7} - \sqrt{3}}{2}}_{Positive\ Value}$$

Example 1.88

Circle C with radius 2 has diameter AB. Circle D is internally tangent to circle C at A. Circle E is internally tangent to circle C, externally tangent to circle D, and tangent to AB. The radius of circle D is three times the radius of circle E, and can be written in the form $\sqrt{m} - n$, where m and n are positive integers. Find $m + n$. (AIME II 2014/8)[3]

Example 1.89: TSD

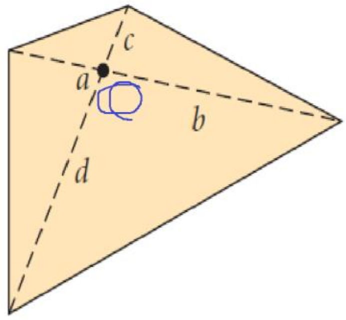
Two skaters, Allie and Billie, are at points A and B, respectively, on a flat, frozen lake. The distance between A and B is 100 meters. Allie leaves A and skates at a speed of 8 meters per second on a straight line that makes a 60-degree angle with AB. At the same time Allie leaves A, Billie leaves B at a speed of 7 meters per second and follows the straight path that produces the earliest possible meeting of the two skaters, given their speeds. How many meters does Allie skate before meeting Billie? (AIME 1989/6)

1.90: Brahmagupta's Formula: Area of a Cyclic Quadrilateral

If a cyclic quadrilateral has sides a, b, c and d, then the area of the quadrilateral is given by

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}, \qquad s = \frac{a+b+c+d}{2}$$

[3] Also solved using Descartes' Theorem in that section. And solved with the Pythagorean Theorem in the Note on Triangles.



$$\frac{1}{2}db\sin\theta + \frac{1}{2}ac\sin\theta + \frac{1}{2}ad\sin(180-\theta) + \frac{1}{2}cb\sin(180-\theta)$$

Substitute $\sin(180-\theta) = \sin\theta$:

$$\frac{1}{2}db\sin\theta + \frac{1}{2}ac\sin\theta + \frac{1}{2}ad\sin(180-\theta) + \frac{1}{2}cb\sin(180-\theta)$$

F. Comparing Law of Sines with Law of Cosines

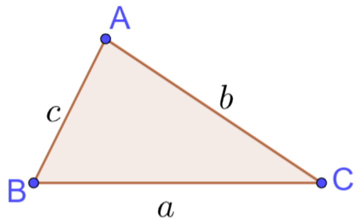
1.91: Law of Sines and Cosines

$$Law\ of\ Cosines: a^2 = b^2 + c^2 - 2bc\cos A$$
$$Law\ of\ Sines: \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{Sin\ C}$$

Example 1.92

Which law should you apply if you have

A. two sides and the included angle
B. two sides and an angle which is not included
C. the value of an angle, and the length of the side opposite it, and any other information
D. two angles and a side
E. Length of three sides



Part A

You cannot apply law of Sines.

$$\frac{a}{\mathbf{\sin A}} = \frac{\mathbf{b}}{\sin B} = \frac{\mathbf{c}}{Sin\ C}$$

This corresponds to SAS from the congruence theorems.

Part B

Law of Sines

$$\frac{a}{\sin A} = \frac{\mathbf{b}}{\mathbf{\sin B}} = \frac{\mathbf{c}}{Sin\ C}\ OR\ \frac{a}{\sin A} = \frac{\mathbf{b}}{\sin B} = \frac{\mathbf{c}}{\mathbf{Sin\ C}}$$

This corresponds to SSA, which is not a congruence theorem.

Part C

Law of Sines

$$\frac{\mathbf{a}}{\mathbf{\sin A}} = \frac{b}{\sin B} = \frac{c}{Sin\ C}$$

This corresponds to SSA, which is not a congruence theorem.

Part D

Work out the value of the third angle using the property that sum of angles is 180°

Then, you will be able to use the law of Sines.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{Sin\ C}$$

Part E

Apply law of cosines to find the value of the angles

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Use this for the first two angles, and then find the third angle using sum of angles property of a triangle.

1.93: When to use which law

SAS	Law of Cosines	Find Third Side
SSS		Find the angles
SSA	Law of Sines	Find the angle Decide whether it is acute or obtuse (or get two answers)
AAS	Law of Sines	Find the third angle, and then use law of Sines.

$$AAS$$

1.5 Geometry

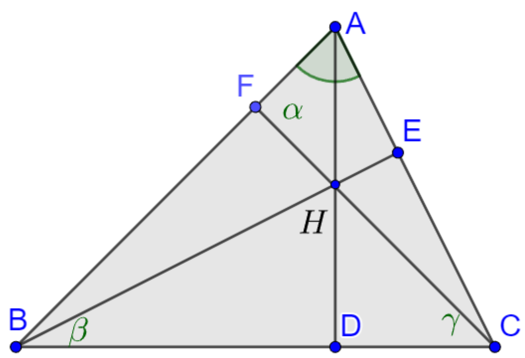
A. Altitude

1.94: Altitude Basics

- The line segment drawn from the vertex of a triangle perpendicular to the opposite side is an altitude of the triangle.
- Every triangle has three altitudes

1.95: Altitude Concurrency

The three altitudes of a triangle are concurrent at the orthocenter of the triangle.



In the diagram alongside

$$Altitudes\ are\ AD, BE, CF$$

The point of concurrency of the three altitudes

$$= Orthocenter = H$$

1.96: Orthocenter dividing altitude

In ΔABC with orthocenter H, the ratio in which the orthocenter divides the altitude is:

$$\frac{AH}{HD} = \frac{\cos\alpha}{\cos\beta\cos\gamma}$$

Step I: Angle Chasing with Geometry

In right ΔAHE:

$$\angle AHE = 90 - \angle DAC = 90 - (90 - \gamma) = \gamma$$

By vertically opposite angles:

$$\angle BHD = \angle AHE = \gamma$$

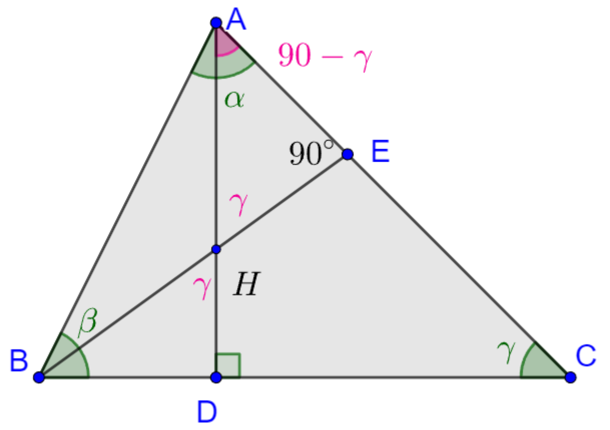
Step II: Trigonometry

Substitute ΔABE: $\cos\alpha = \frac{AE}{AB} \Rightarrow AE = AB\cos\alpha$ in:

$$\Delta AHE: \sin\gamma = \frac{AE}{AH} \Rightarrow \underbrace{AH = \frac{AE}{\sin\gamma} = \frac{AB\cos\alpha}{\sin\gamma}}_{\textit{Equation I}}$$

Substitute ΔABD: $\cos\beta = \frac{BD}{AB} \Rightarrow BD = AB\cos\beta$ in:

$$\Delta BHD\ \tan\gamma = \frac{BD}{HD} \Rightarrow \underbrace{HD = \frac{BD}{\tan\gamma} = AB\cos\beta\cot\gamma}_{\textit{Equation II}}$$



From Equations I and II:

$$\frac{AH}{HD} = \frac{\frac{AB\cos\alpha}{\sin\gamma}}{AB\cos\beta\cot\gamma} = \frac{AB\cos\alpha}{\sin\gamma} \times \frac{1}{AB\cos\beta\cot\gamma} = \frac{\cos\alpha}{\cos\beta\cdot\cos\gamma}$$

Example 1.97

The sides of a triangle are of lengths 13,14 and 15. The altitudes of the triangle meet at point H. If AD is the altitude to the side of length 14, the ratio HD: HA is **(AHSME 1964/35)**

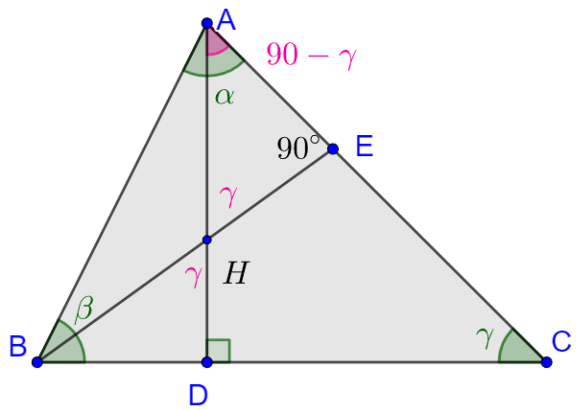
$$\frac{AH}{HD} = \frac{\cos\alpha}{\cos\beta\cos\gamma} = \frac{\frac{b^2+c^2-a^2}{2bc}}{\frac{a^2+c^2-b^2}{2ac} \times \frac{a^2+b^2-c^2}{2ab}}$$

Convert division into multiplication:

$$= \frac{b^2+c^2-a^2}{2bc} \times \frac{2ac}{a^2+c^2-b^2} \times \frac{2ab}{a^2+b^2-c^2}$$

Simplify and rearrange:

$$= \frac{2a^2(b^2+c^2-a^2)}{(a^2+c^2-b^2)(a^2+b^2-c^2)}$$



Substitute $a = 14, b = 15, c = 13$:

$$= \frac{2(14^2)(225+169-196)}{(196+169-225)(196+225-169)} = \frac{2(14^2)(198)}{(140)(252)} = \frac{2(14)(198)}{(10)(252)} = \frac{(198)}{(5)(18)} = \frac{11}{5}$$

We need:

$$\frac{HD}{AD} = \textit{Reciprocal of } \frac{AH}{HD} = \frac{5}{11}$$

98 Examples