# Continuous Geometric Combinatorics

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# 1. Continuous Geometry

## 1.1 Area as Probability

### A. Probability as Area

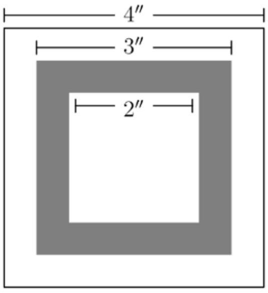
A formula for probability that you might have seen before is:

$$P = \frac{Successful\ Outcomes}{Total\ Outcomes}$$

The above formula is useful when the sample space has discrete outcomes. But it does not work for continuous outcomes.

**Example 1.1**

A dart is thrown at the square target shown. Assuming the dart hits the target at a random location, what is the probability that it will be in the shaded region? Express your answer as a common fraction. **(MathCounts 1991 Chapter Sprint)**




Can we apply

$$P = \frac{Successful\ Outcomes}{Total\ Outcomes}$$

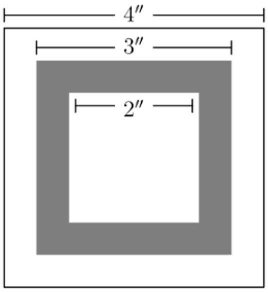
If we try to calculate the number of total outcomes, the dart can hit infinitely many points in the target. Hence, this formula does not work.

**1.2: Probability as Area**

$$Probability = \frac{Successful\ Area}{Total\ Area}$$

**Example 1.3**

A dart is thrown at the square target shown. Assuming the dart hits the target at a random location, what is the probability that it will be in the shaded region? Express your answer as a common fraction. **(MathCounts 1991 Chapter Sprint)**




The probability that the dart hits the shaded region

$$= \frac{Successful\ Area}{Total\ Area} = \frac{3^2 - 2^2}{4^2} = \frac{9-4}{16} = \frac{5}{16}$$

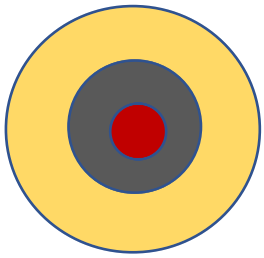
**Example 1.4**

On a checkerboard composed of 64 unit-squares, what is the probability that a randomly chosen unit square does not touch the outer edge of the board? **(AMC 8 2009/10)**

$$P = \frac{Successful\ Area}{Total\ Area} = \frac{6^2}{8^2} = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

### Example 1.5

A dart is thrown at the circular dartboard shown alongside. The circles are concentric, with the innermost circle having a radius of two units, the middle circle a radius of four units, and the largest circle a radius of six units. Find the probability of hitting each color.



**Total Area**

$$A = \pi r^2 = 36\pi$$

**Colored Areas**

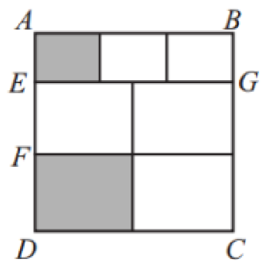
$$Red\ Area = \pi r^2 = 4\pi$$
$$Gray\ Area = 16\pi - 4\pi = 12\pi$$
$$Yellow\ Area = 36\pi - 16\pi = 20\pi$$

**Probability**

$$P(Red) = \frac{4\pi}{36\pi} = \frac{1}{9}$$
$$P(Gray) = \frac{12\pi}{36\pi} = \frac{1}{3}$$
$$P(Yellow) = \frac{20\pi}{36\pi} = \frac{5}{9}$$

### Example 1.6

In the figure below, $ABCD$ is a square with an enclosed area of 576. Line segment $AE$ is one fourth of $AD$, and $ABGE$ is divided into three equal rectangles. Line segment $FD$ is one-half of $ED$, and $EGCD$ is divided into 4 equal rectangles. If a point is randomly chosen from within square $ABCD$, what is the probability that the point will be from a shaded region? ($MA\theta$ 2016/Open/Prob&Combi/8)




We can solve in terms of ratios.

$$[Smaller\ Shaded\ Region] = AE \cdot \frac{AB}{3} = \frac{AD}{4} \cdot \frac{AB}{3} = \frac{[ABCD]}{12}$$

$$[Larger\ Shaded\ Region] = FD \cdot \frac{AB}{2} = \frac{ED}{2} \cdot \frac{AB}{2} = \frac{AD \cdot \frac{3}{4}}{2} \cdot \frac{AB}{2} = \frac{3AD}{8} \cdot \frac{AB}{2} = \frac{3[ABCD]}{16}$$

$$P = \frac{Successful\ Area}{Total\ Area} = \frac{\frac{[ABCD]}{12} + \frac{3[ABCD]}{16}}{[ABCD]} = \frac{\frac{1}{12} + \frac{3}{16}}{1} = \frac{4+9}{48} = \frac{13}{48}$$

**Strategy:** The value of 576 is not required to solve the question.

## B. Geometric Conditions

### 1.7: Distance From

We may be interested in a region that satisfies certain constraints. A simple constraint is distance from a point.

### Example 1.8

What is the probability that a random point in the interior of a circle of radius 5 is more than 3 units from the center? ($MA\theta$ 2016/Open/Prob&Combi/3)

$$P(point\ in\ outer\ ring) = \frac{25\pi - 9\pi}{25\pi} = \frac{16}{25}$$

### Example 1.9

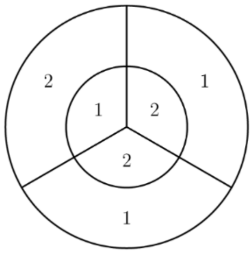
A point $P$ is randomly selected from the square region with vertices at $(\pm 2, \pm 2)$. What is the probability that $P$ is within one unit of the origin. (MathCounts 2004 Warm-up 11)

$$P = \frac{Successful\ Area}{Total\ Area} = \frac{\pi r^2}{16} = \frac{\pi \cdot 1^2}{16} = \frac{\pi}{16}$$

## C. Probability Rules

### Example 1.10

On the dart board shown in the figure below, the outer circle has radius 6 and the inner circle has radius 3. Three radii divide each circle into three congruent regions, with point values shown. The probability that a dart will hit a given region is proportional to the area of the region. When two darts hit this board, the score is the sum of the point values in the regions. What is the probability that the score is odd? (AMC 8 2007/25)




The probability of getting a "2":

$$= \frac{9\pi \cdot \frac{2}{3} + \frac{27\pi}{3}}{36\pi} = \frac{6\pi + 9\pi}{36\pi} = \frac{15\pi}{36\pi} = \frac{5}{12}$$

The probability of getting a "1":

$$= 1 - \frac{5}{12} = \frac{7}{12}$$

The final probability

$$P(1)P(2) + P(2)P(1) = \frac{5}{12} \cdot \frac{7}{12} + \frac{7}{12} \cdot \frac{5}{12} = 2 \cdot \frac{35}{144} = \frac{35}{72}$$

### 1.11: Sum of Probabilities

The sum of probabilities for mutually exclusive events that cover the entire sample space is 1:

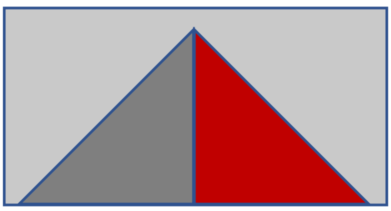
### Example 1.12

Kunal throws a dart at the dartboard in the diagram. If the dart has equal probability of any part of the dartboard, find the probability of hitting each colored region. (Note: Areas that look equal are equal).

$$p + p = 1 \Rightarrow 2p = 1 \Rightarrow p = \frac{1}{2}$$

### Example 1.13

A surveyor releases a survey probe from a balloon at a mountain (side view shown in the diagram). If it is known that the probe hits the mountain, and it has equal probability of reaching any part of the diagram, find the probability of hitting each colored region.

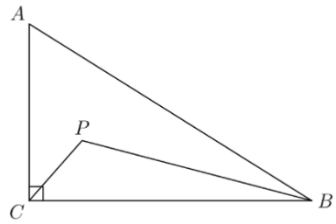


$$p + p = 1 \Rightarrow 2p = 1 \Rightarrow p = \frac{1}{2}$$

## D. Area Properties

### Example 1.14

A point $P$ is randomly placed in the interior of the right triangle below. What is the probability that the area of triangle $PBC$ is less than half the area of triangle $ABC$. (MathCounts 2004 National Team)




We must have:

$$[PBC] < \frac{[ABC]}{2} \Rightarrow \frac{1}{2} \cdot h \cdot CB < \frac{\frac{1}{2} \cdot AC \cdot CB}{2} \Rightarrow h < \frac{AC}{2}$$

Let $D$ and $E$ be midpoints of $AC$ and $AB$. Draw $DE$.
The probability will be:

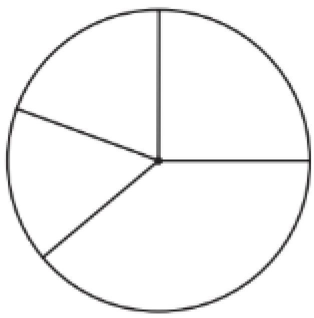
$$P = \frac{Successful\ Area}{Total\ Area} = \frac{[DEBC]}{[ACB]} = 1 - \frac{[ADE]}{[ACB]}$$

Since $\Delta ADE \sim \Delta ACB \Rightarrow \frac{[ADE]}{[ACB]} = \left(\frac{AD}{AC}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

### Example 1.15

A circular dartboard is divided into regions with various central angles, as shown. The probability of a dart randomly landing in a particular region is $\frac{1}{6}$. What is the corresponding measure, in degrees, of the central angle of the central angle of this section of the dartboard? (MathCounts 2009 School Sprint)



$$P = \frac{Successful\ Area}{Total\ Area} = \frac{\frac{\pi r^2 \cdot \theta}{360}}{\pi r^2} = \frac{\theta}{360} = \frac{1}{6} \Rightarrow \theta = 60°$$

## E. Distance

### Example 1.16

A garden has grass planted in a square shape with area 16 units. An earthen walkway is built adjacent (but outside) the grass so that it has width 1. A ball falls at a random point in the garden. What is the probability that it is within one unit of the border separating the grass from the grass from the earth?

The side length of the grass garden

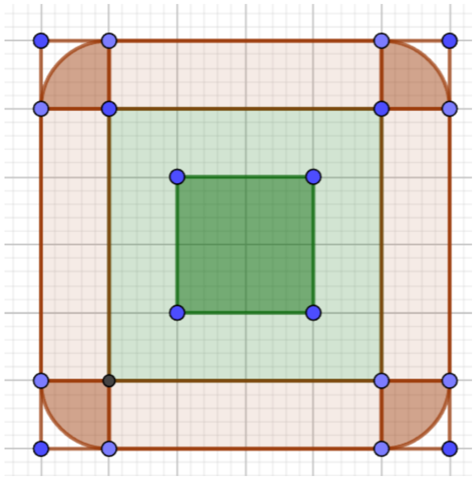
$$= \sqrt{16} = 4$$

The area of the grass that is not within one unit of the border is

$$(4-2)^2 = 2^2 = 4$$

The area of the grass garden within 1 unit of the border is:

$$16 - 4 = 12$$



The area of the earthen walkway that is within one unit of the border is:

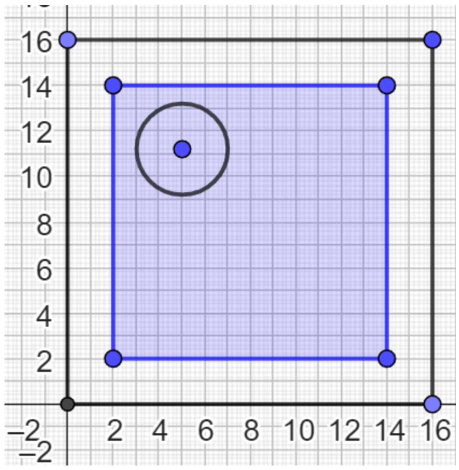
$$4(1 \cdot 4) + 4\left(\frac{\pi \cdot 1^2}{4}\right) = 16 + \pi$$

Hence, the final probability is:

$$\frac{12 + (16 + \pi)}{6^2} = \frac{28 + \pi}{36}$$

### Example 1.17

A square grid is composed of 100 small squares, each with side length of 16. A circular disk of diameter 4 is tossed at the grid, and the disk's center lands on the grid. What is the probability that the disk does not cover any part of any side of any of the 100 small squares within the grid? (*MAΘ*, 2021, Alpha, Prob&Combi/10; 2016, Open, Prob&Combi/22)




The circle has

$$Radius = \frac{Diameter}{2} = \frac{4}{2} = 2$$

We can solve this for 1 square instead of 100, and the answer is the same for each square, and hence the final answer remains the same by symmetry.

Within each square, there is a smaller square where the center of the disk can land without the disk touching the square's side. The smaller square's length

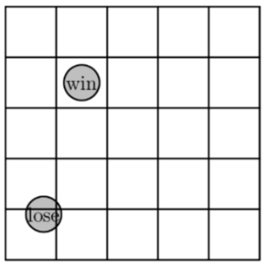
$$16 - 2 \cdot 2 = 12$$

Thus,

$$P(not\ touching) = \frac{12^2}{16^2} = \frac{9}{16}$$

### Example 1.18

A coin that is 8 cm in diameter is tossed on a 5 by 5 grid of squares each having side length 10 cm. A coin is in a winning position if no part of it touches or crosses a grid line. Otherwise, it is in a losing position. Given that the coin lands in a random position so that no part of it is off the grid, what is the probability that it is in a winning position? (CEMC 2010 Cayley)

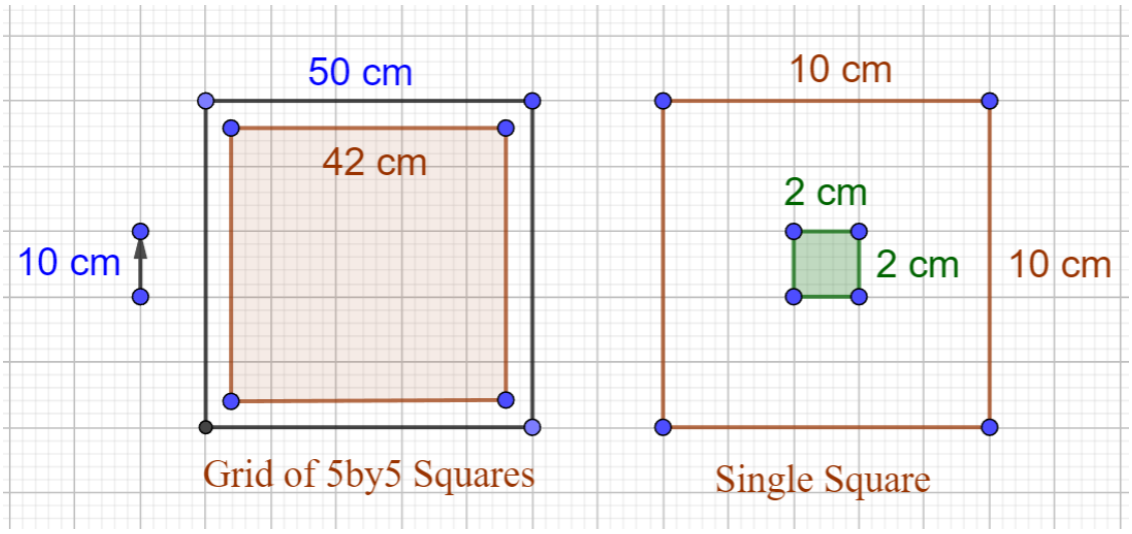



The possible landing area for the coin

$$= (50 - 4 - 4)^2 = 42^2 = 1764$$

The area for a successful outcome is

$$= \underbrace{25}_{\substack{No.\ of \\ Squares}} \cdot \underbrace{(10 - 2 \cdot 4)^2}_{\substack{Valid\ Area \\ for\ each\ square}} = 25 \cdot 2^2 = 100$$




### Example 1.19

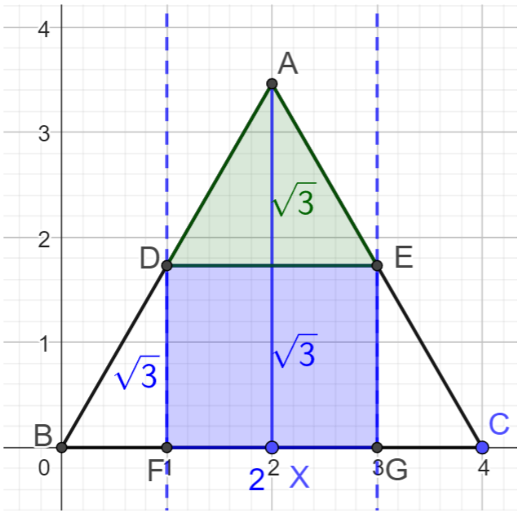
A point is randomly chosen in equilateral $\Delta ABC$ with side length 4 cm. Determine the probability that it is within 1 cm of the angle bisector from $A$.

The required probability is:

$$\frac{[ADE]+[DEGF]}{[ABC]}=\frac{\frac{\sqrt{3}}{4}\cdot DE^2+DE\cdot EG}{\frac{\sqrt{3}}{4}\cdot BC^2}$$

Simplifying gives:

$$=\frac{\frac{\sqrt{3}}{4}\cdot 2^2+2\cdot\sqrt{3}}{\frac{\sqrt{3}}{4}\cdot 4^2}=\frac{3\sqrt{3}}{4\sqrt{3}}=\frac{3}{4}$$

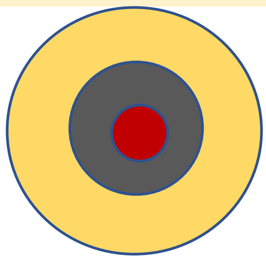



## F. Similarity

### Example 1.20

A dart is thrown at the circular dartboard shown alongside. The circles are concentric. The probability of hitting the gray region is half, and the probability of hitting the yellow region is seven-sixteenth. Find

A. The probability of hitting the red region.
B. The ratio of the radii of the respective circular regions.



#### Part A

We use complementary probability

$$P(Red)=1-P(Yellow)-P(Gray)=1-\frac{7}{16}-\frac{1}{2}=\frac{1}{16}$$

#### Part B

$$P(Red) = \frac{1}{16} \Rightarrow \frac{Area(Red)}{Total\ Area} = \frac{1}{16}$$

Recall that radius is a linear measure, while area is squared. Hence, the radii of the circles will be proportional to the square root of the areas of the circles.
Hence, take square roots to find the ratio of the radii:

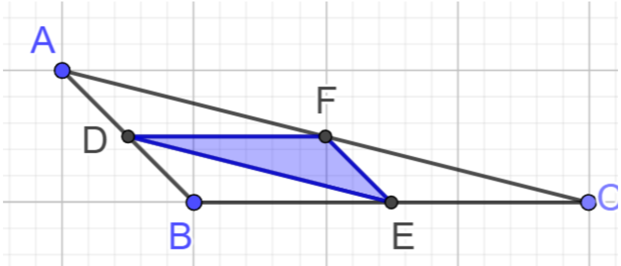
$$\frac{Radius(Red)}{Radius(Largest\ Circle)} = \sqrt{\frac{1}{16}} = \frac{1}{4}$$

### Example 1.21

What is the probability that a point selected at random inside $\Delta ABC$ lies inside the region formed by joining the midpoints of $AB, BC$ and $CA$?

Connect the midpoints of $AB, BC$ and $CA$ to form a triangle, and let them be

$$X, Y\ and\ Z\ respectively$$




By the Midpoint Theorem:

$$DF = \frac{1}{2}BC, \qquad DE = \frac{1}{2}CA, \qquad FE = \frac{1}{2}AB \Rightarrow \Delta EDF \sim \Delta ABC$$

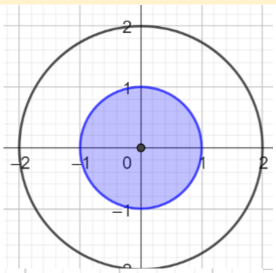
By similarity, the ratio of the areas are proportional to the square of the ratios of the sides, and this is the same as the probability of the point being inside the region:

$$\frac{P(\Delta\text{EDF})}{P(\Delta ABC)} = \frac{A(\Delta\text{EDF})}{A(\Delta ABC)} = \frac{\left(\frac{1}{2}\right)^2}{1} = \frac{1}{4}$$

## G. Closer to

### Example 1.22

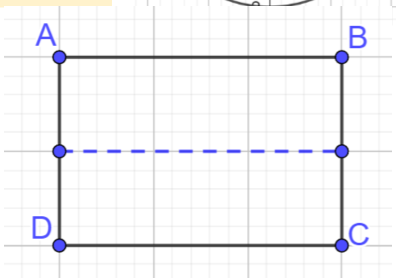
A point is chosen at random from within a circular region. What is the probability that the point is closer to the center of the region than it is to the boundary of the region? **(AMC 8 1996/25)**



$$\frac{\pi r^2}{\pi R^2} = \frac{1^2}{2^2} = \frac{1}{4}$$

### Example 1.23

A point is picked at random in rectangle $ABCD$. What is the probability that it is closer to $AB$ than to $CD$.




Draw a line through the midpoint of the left and right sides of the rectangle. The point will be closer to $AB$ in the top half and closer to $BC$ in the bottom half. Hence:

$$p = \frac{1}{2}$$

### Example 1.24

A point is picked at random in equilateral $\Delta ABC$. What is the probability that it is closer to:

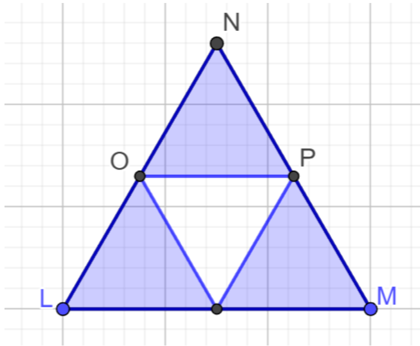
A. vertex $A$ than to the base $BC$.
B. a vertex than to the opposite base.

**Part A**

$$p = \frac{1}{4}$$

**Part B**

$$p = \frac{3}{4}$$




### Example 1.25

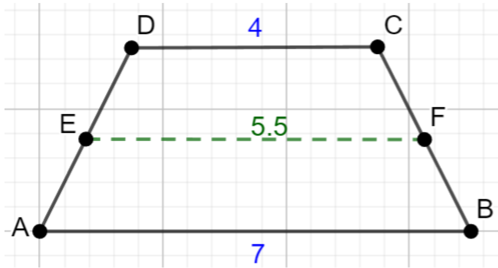
A point is picked at random in trapezoid $ABCD$ with parallel bases 4 and 7, and a height of 2. What is the probability that it is closer to the longer base than to the smaller base.

Draw $EF$ line connecting the midpoints of the sides.
Then:

$$EF = \frac{DC + AB}{2} = \frac{4+7}{2} = \frac{11}{2}$$

The required probability is:

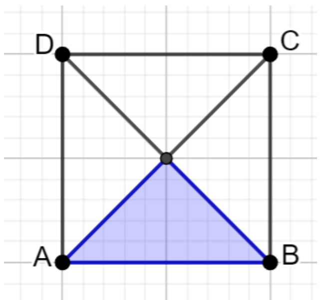
$$P = \frac{[EFBA]}{[DBCA]} = \frac{\frac{\left(\frac{11}{2}+7\right)}{2}\cdot 1}{\frac{(4+7)}{2}\cdot 2} = \frac{\frac{25}{4}}{11} = \frac{25}{44}$$




### Example 1.26

A point is picked at random in square $ABCD$. What is the probability that it is closer to $CD$ than to any of the other three sides.

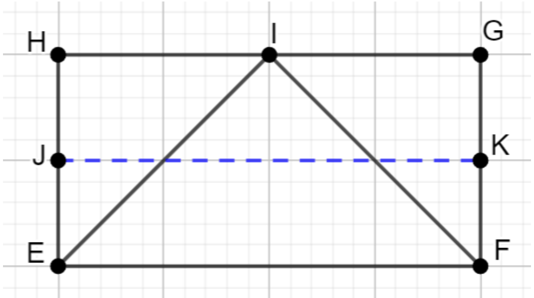
$$p = \frac{1}{4}$$



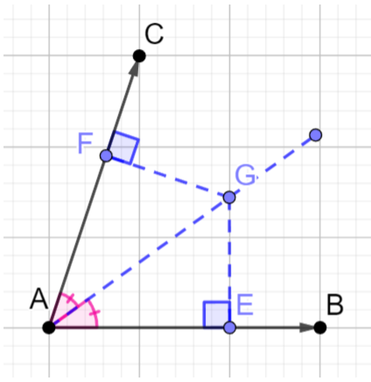

### Example 1.27

A point is picked at random in rectangle $ABCD$ with width 1 and length 2. What is the probability that it is closer to $CD$ than to any of the other three sides.

$$p = \frac{3}{8}$$




## H. Angle Bisector




**1.28: Angle Bisector**

Any point on the angle bisector of an angle is equidistant from the rays that make up the angle.

Consider $\angle CAB$ and draw the angle bisector

$$AG$$

The distance from $AG$ to $AF$ and $AB$ is measured by dropping a perpendicular from $AG$ to the respective sides.

$\Delta AGF \cong \Delta AGE$ by $AAS$ since:

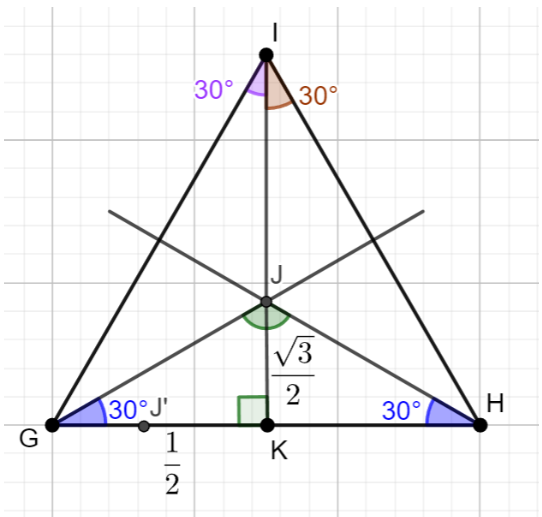
$$AG = AG \ (Common\ side)$$
$$\angle AFG = \angle AEG \ = 90°$$
$$\angle FAG = \angle GAE \ (Def.of\ angle\ bisector)$$

Therefore

$$GF = GE \ (CPCTC) \Rightarrow G\ is\ equidistant\ from\ AC\ and\ AB$$

**Example 1.29**

A point is picked at random in equilateral $\Delta ABC$. What is the probability that it is closer to the base $BC$ then to the other two sides.




The dividing line for being closer to the base BC as compared to the other two sides is the

$$angle\ bisector\ GJ, angle\ bisector\ HJ$$

By symmetry, the angle bisectors divide the triangle into 3 congruent triangles.
Hence:

$$p = \frac{1}{3}$$

**Example 1.30**

A rectangle has adjacent side lengths 1 and 2. A point $P$ is picked at random inside the rectangle. What is the probability that it is closer to the base (of side length 2), then the other sides.

## I. Perpendicular Bisector

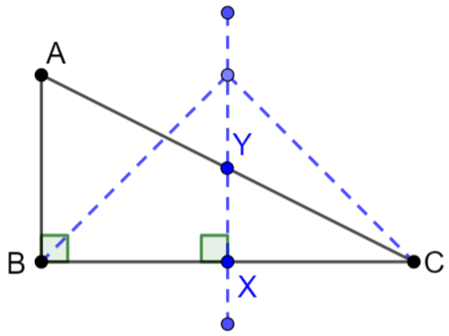
### 1.31: Perpendicular Bisector

Any point on the perpendicular bisector of a line segment is equidistant from its endpoints.

### Example 1.32

Pick a point P at random in $\Delta ABC$, right-angled at $B$. What is the probability that $P$ is closer to $B$ than to C.

Draw the perpendicular bisector of $BC$,with $YX \perp BC$.




Points to the left of $YX$ are closer to B, and points to the right of $YX$ are closer to C.

Hence, we want:

$$P = \frac{Successful\ Area}{Total\ Area} = \frac{[AYXB]}{[ABC]}$$

By complementary probability, the above is equal to:

$$= 1 - \frac{[YXC]}{[ABC]}$$

$\Delta YXC \sim \Delta ABC$ by AA Similarity since $\angle YXC = \angle ABC = 90°$ and $\angle YCX = \angle ACB$:

$$Ratio\ of\ Lengths = \frac{XC}{BC} = \frac{1}{2} \Rightarrow Ratio\ of\ Areas = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

And hence we get:

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

## J. Triangle Inequality

### Example 1.33

Two sides of a nondegenerate triangle measure 2" and 4" and the third side measures some whole number of inches. If a cube with faces numbered 1 through 6 is rolled, what is the probability, expressed as a common fraction, that the number showing on top could be the number of inches in the length of the third side of the triangle? (MathCounts 1995 Chapter Target)

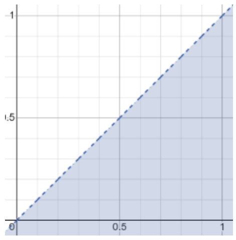
By the triangle inequality:

$$4 - 2 < x < 4 + 2 \Rightarrow 2 < x < 6 \Rightarrow x \in (3,4,5) \Rightarrow P = \frac{3}{6} = \frac{1}{2}$$

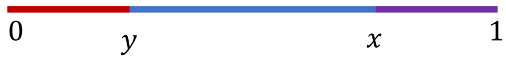
### Example 1.34

A one-meter long stick is cut at two random places. What is the probability that the three sticks so formed can be used to form a triangle?

### Total Outcomes / Area




Place a real number line along the one-meter stick (see diagram below).




Let the cuts be made at the number $x$ and the number $y$ with

$$x > y$$

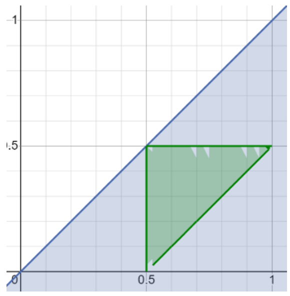
(The case where $x < y$ is symmetrical and can be ignored).

Hence, the lengths of the three sticks obtained are:

$$y, \qquad x - y, \qquad 1 - x$$

The total area that satisfies $x > y$ as the blue region shown to the right.

### Successful Outcomes / Area




The three sides of the triangle must satisfy the triangle inequality:

$$(1-x) + (x-y) > y \Rightarrow 1 - y > y \Rightarrow y < \frac{1}{2}$$

$$y + (x-y) > 1 - x \Rightarrow x > 1 - x \Rightarrow x > \frac{1}{2}$$

$$(1-x) + y > x - y \Rightarrow y > x - \frac{1}{2}$$

The region that satisfies all three inequalities is shown in the green region to the right.

### Probability

$$Probability = \frac{Green\ Area}{Blue\ Area} = \frac{\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}}{1 \times 1 \times \frac{1}{2}} = \frac{1}{4}$$

## K. Circles

### Example 1.35

Three points are chosen randomly and independently on a circle. What is the probability that all three pairwise distances between the points are less than the radius of the circle? (AMC 12 2003)

### Example 1.36

Three points are chosen at random on a circle of radius 1. What is the probability that an acute triangle is formed when they are connected? ($MA\Theta$, 2019, Mu, Combinatorics and Probability/27)

Call the 3 points chosen $A$, $B$, $C$, and place the circle centered at the origin, which we will call $O$. Without loss of generality, place $A$ at $(1, 0)$ by rotating the figure. Place $B$ in quadrants I or II by reflecting the figure, if necessary. $C$ has no restrictions when selected. Draw diameters through $A$ and $B$. For $\Delta ABC$ to be acute, $C$ must be on the sector opposite of minor arc $AB$. Otherwise, one of the three central angles is greater than $\pi$, causing the corresponding inscribed angle to be obtuse. Let's call $\angle AOB = x$, then $x$ is selected at random on $[0, \pi]$. Let $y$ be $\angle AOC$, but measured clockwise, rather than strictly the smallest angle between $OA$ and $OC$. Then $y$ is selected at random on $[0, 2\pi]$. $\Delta ABC$ is acute if $\pi - x < y < \pi$.

We can now compute the probability geometrically. $x, y$ are selected from a rectangle of dimensions $\pi$ by $2\pi$. $\Delta ABC$ is acute if the point is below $y = \pi$ and above $y = \pi - x$, which compose of $1/4$ of the rectangle.

## L. Comparison of Distances

### Example 1.37

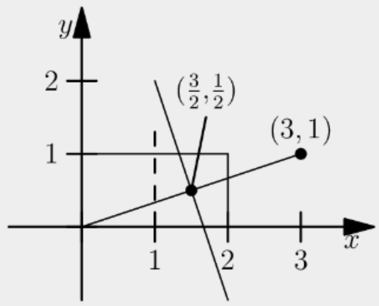
A regular hexagon and its center are drawn. Pix picks a point uniformly and at random in the interior of this hexagon. What is the probability that this point is closer to the center than it is to any of the hexagon's vertices? (MAΘ, 2021, Alpha, Combinations and Probability/27)

Drawing the perpendicular bisectors between the center and each of the vertices and taking the intersections of the regions where points are closer to the center will give a region in the center that is also a regular hexagon. The ratio of the side lengths is √3: 1 for the bigger hexagon to smaller hexagon. This makes the probability (1/√3)^2=1/3

### Example 1.38

A point P is randomly selected from the rectangular region with vertices (0,0), (2,0), (2,1) and (0,1). What is the probability that $P$ is closer to the origin than it is to the point (3,1)? (AMC 12 2002)

The area of the rectangular region is 2. Hence the probability that $P$ is closer to $(0,0)$ than it is to $(3,1)$ is half the area of the trapezoid bounded by the lines $y = 1$, the $x$- and $y$-axes, and the perpendicular bisector of the segment joining $(0,0)$ and $(3,1)$. The perpendicular bisector goes through the point $(3/2, 1/2)$, which is the center of the square whose vertices are $(1,0), (2,0), (2,1),$ and $(1,1)$. Hence, the line cuts the square into two quadrilaterals of equal area $1/2$. Thus the area of the trapezoid is $3/2$ and the probability is $\boxed{3/4}$.




## M. Discriminant

### Example 1.39

$x$ is a real number picked randomly from 1 to 5. What is the probability that the quadratic equation

$$4t^2 + 4tx + x + 2 = 0$$

has two real solutions?

$$a = 4$$
$$b = 4x$$
$$c = x + 2$$

A quadratic has two real solutions if the discriminant is positive:

$$b^2 - 4ac > 0$$

$$16x^2 - 4(4)(x+2) > 0$$
$$16x^2 - 16x + 32 > 0$$
$$x^2 - x + 2 > 0$$
$$(x+1)(x-2) > 0$$
$$x \in (-\infty, -1) \cup (2, \infty)$$

Probability

$$= \frac{5-2}{5-1} = \frac{3}{4}$$

### N. Coordinate Geometry

**Example 1.40**
https://artofproblemsolving.com/wiki/index.php/2023_AIME_II_Problems/Problem_6

# 1.2 Length/Time as Probability

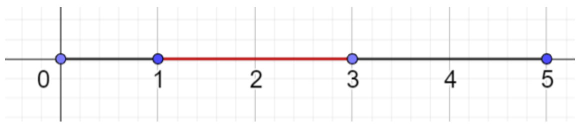
## A. Length as Probability

**1.41: Length as Probability**

$$Probability = \frac{Valid\ Length}{Total\ Length}$$

**Example 1.42**
Nick picks a real number at random from the interval (0,5). What is the probability that the number lies in the interval (1,3)?

$$Probability = \frac{Length\ of\ Valid\ Interval}{Total\ Interval} = \frac{2}{5}$$




**Example 1.43**
Britney picks a random number on the real number line between 0 and 1 and then rounds it off to the nearest integer. Find the probability of each possible integer that she can get.



Suppose the number Britney picks is $x$.
Apply the rounding off rules:

$$Round(x) = 1 \Rightarrow 0.5 \le x \le 1$$
$$Round(x) = 0 \Rightarrow 0 \le x \le 0.5$$

$$Probability = \frac{Length\ of\ Valid\ Interval}{Total\ Interval} = \frac{0.5}{1} = \frac{1}{2}$$

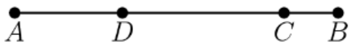
**Example 1.44**

Lee picks a random number on the real number line between 0 and 2 and then rounds it off to the nearest integer. Find the probability of each possible integer that Lee can get.


### Example 1.45

**Correct! Way to go!** In topic: Using Geometry in Probability (Counting & Probability). Source: MathCounts 1991 State Countdown

Points $A$, $B$, $C$, and $D$ are located on $\overline{AB}$ such that $AB = 3AD = 6BC$. If a point is selected at random on $\overline{AB}$, what is the probability that it is between $C$ and $D$? Express your answer as a common fraction.

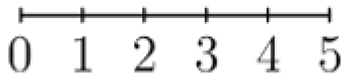



**Your Answer:** 1/2

Since $D$ and $C$ are located on segment $\overline{AB}$, if $AB = 3AD$, then $\overline{AD}$ must take up $1/3$ of line segment $\overline{AB}$. Similarly, since $AB = 6BC$, $\overline{BC}$ must take up $1/6$ of line segment $\overline{AB}$. Then, $\overline{CD}$ is the remaining segment of $\overline{AB}$ and takes up $1 - 1/3 - 1/6 = 1/2$ of the total length of $\overline{AB}$. Thus, if we were to choose a random point on segment $\overline{AB}$, there would be a $\boxed{\frac{1}{2}}$ probability that it is between points $C$ and $D$.

### Example 1.46

A point is selected at random from the portion of the number line shown here. What is the probability that the point is closed to 4 than to 0. Express your answer as a decimal to the nearest tenth. (MathCounts 2004 State Countdown)




We find the point which is equidistant from 0 and 4. Clearly, this occurs at 2. So, for all $x > 2$, $x$ is closer to 4 than 0. So, the probability is equal to the length of this region $\frac{5-2}{5} = \boxed{.6}$.

## B. 3D Probability

### Example 1.47

A cube of side length $\frac{7}{3}$ is dropped into a random place on the 3D lattice in the coordinate space (a coordinate plane with an extra dimension), with the sides of the square parallel/perpendicular to the axes. The expected number of lattice points (points with integer coordinates) in the interior of the cube is $\frac{a}{b}$ in simplest terms, what's $a + b$?(JHMMC Grade 2020 R1/31)

$$\frac{7}{3} = 2\frac{1}{3}$$

Consider the x dimension:
With probability $\frac{1}{3}$, you will have 3 points
With probability $\frac{2}{3}$, you will have 2 points

The expected number of lattice points is:

$$\frac{1}{3}\times 3 + \frac{2}{3}\times 2 = 1 + \frac{4}{3} = \frac{7}{3}$$

By symmetry, for the y dimension, the expected number of lattice points is:

$$\frac{7}{3}$$

And also for the z dimension:

$$\frac{7}{3}$$

Since the three values are independent, by the multiplication principle, the final answer is:

$$\left(\frac{7}{3}\right)^3 = \frac{343}{27}$$

## C. Time as Probability

### 1.48: Length as Probability

$$Probability = \frac{Valid\ Time}{Total\ Time}$$

### Example 1.49

Rachel and Robert run on a circular track. Rachel runs counterclockwise and completes a lap every 90 seconds, and Robert runs clockwise and completes a lap every 80 seconds. Both start from the same line at the same time. At some random time between 10 minutes and 11 minutes after they begin to run, a photographer standing inside the track takes a picture that shows one-fourth of the track, centered on the starting line. What is the probability that both Rachel and Robert are in the picture? **(AMC 10B 2009/23; 12B 2009/18)**

Since the picture is taken between 10 and 11 minutes, determine the situation after 10 minutes:

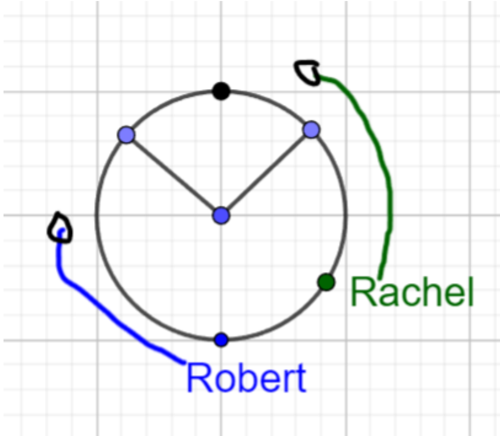
$$Rachel = \frac{600}{90} = \frac{60}{9} = \frac{20}{3} = 6\frac{2}{3}\ Laps \Rightarrow \frac{2}{3}\ along\ the\ way$$
$$Robert = \frac{600}{80} = \frac{60}{8} = \frac{30}{4} = 7\frac{1}{2}\ Laps \Rightarrow \frac{1}{2} along\ the\ way$$

The width of the picture is:

$$\frac{1}{4}^{th}\ \ of\ the\ track$$

By symmetry, the parts to the left and the right are each:

$$\frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}^{th}\ \ of\ the\ track$$




**Robert**

Robert will cross the starting point at

$$\frac{80}{2} = 40 \textit{ seconds}$$

He will be in the picture frame:

$$\left(40 - 80 \cdot \frac{1}{8}, 40 + 80 \cdot \frac{1}{8}\right) = (40 - 10, 40 + 10) = (30, 50)$$

**Rachel**

Robert will cross the starting point at

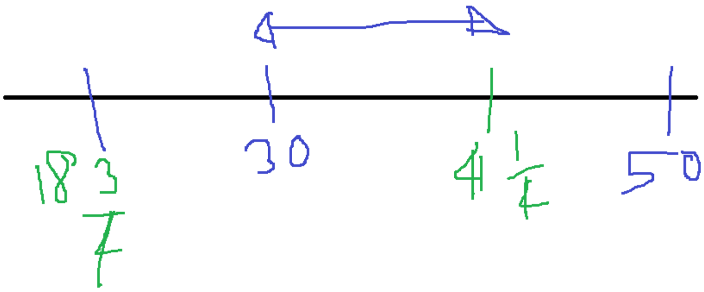
$$\frac{90}{3} = 30 \textit{ seconds}$$

He will be in the picture frame:

$$\left(30 - 90 \cdot \frac{1}{8}, 30 + 90 \cdot \frac{1}{8}\right) = \left(30 - 11\frac{1}{4}, 30 + 11\frac{1}{4}\right) = \left(18\frac{3}{4}, 41\frac{1}{4}\right)$$

We want the intersection of:

$$\left(18\frac{3}{4}, 41\frac{1}{4}\right) \textit{ AND } (30, 50)$$




The probability is:

$$= \frac{\textit{Valid Time}}{\textit{Total Time}} = \frac{\left(41\frac{1}{4} - 30\right) \textit{ seconds}}{60 \textit{ seconds}} = \frac{11\frac{1}{4}}{60} = \frac{45}{240} = \frac{3}{16}$$

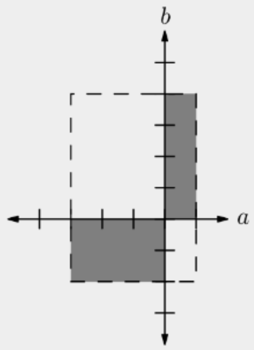
# 1.3 Coordinate Plane

## A. Regions

### Example 1.50

Given that $a$ and $b$ are real numbers such that $-3 \le a \le 1$ and $-2 \le b \le 4$, and values for $a$ and $b$ are chosen at random, what is probability that the product $a \cdot b$ is positive? Express your answer as a common fraction. (MathCounts 2004 National Target)

The probability that both $a$ and $b$ are positive is $\left(\frac{1}{4}\right)\left(\frac{2}{3}\right) = \frac{1}{6}$. The probability that $a$ and $b$ are both negative is $\left(\frac{3}{4}\right)\left(\frac{1}{3}\right) = \frac{1}{4}$. Since $ab$ is positive if and only if one of these two events occur, the probability that $ab$ is positive is $\frac{1}{6} + \frac{1}{4} = \boxed{\frac{5}{12}}$. Graphically, we can picture the possible outcomes for $(a, b)$ as a rectangle in a Cartesian plane. The shaded rectangles are the regions in which $ab > 0$.




## B. Inequalities

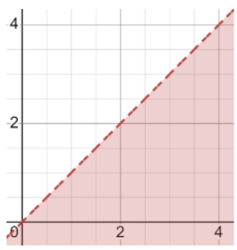
### Example 1.51

$x$ and $y$ are each random values between 0 and 2 (exclusive).

A. What is the probability that $x$ is greater than $y$?
B. If x and y are changed to be random values between 0 and 2 (inclusive), will your answer change? Your calculations?

**Part A**
**By Symmetry**

$$p + p = 1 \Rightarrow 2p = 1 \Rightarrow p = \frac{1}{2}$$



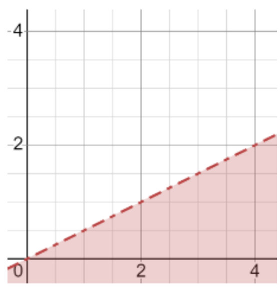

**By Coordinate Geometry**
Shade the region on the coordinate where $x$ is greater than $y$. By symmetry, the probability is

$$\frac{1}{2}$$

**Part B**
Neither answers nor calculations will change.

### Example 1.52

$x$ is a random value between 0 and 4. $y$ is also a random value between 0 and 4. What is the probability that $x$ is greater than $2y$?




Shade the region on the coordinate plane where $x$ is greater than $y$. We want to find the area of the triangle divided by the area of the square, which is given by:

$$\frac{A(Triangle)}{A(Square)} = \frac{4 \times 2 \times \frac{1}{2}}{4 \times 4} = \frac{1}{4}$$

### Example 1.53

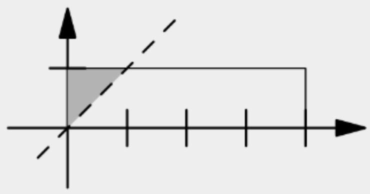
Two real positive numbers $x$ and $y$ are randomly chosen such that $x + y \leq 4$. Compute the probability that $x + y \leq 2$. (MA Theta, 2018, Open, Counting and Probability/21)

The region bounded by the coordinate axes (since the numbers must be positive) and the line $x + y = 4$ is an isosceles right triangle of area 8. The region bounded by the coordinate axes and the line $x + y = 2$ is an isosceles right triangle of area 2 within the larger triangle. Hence the probability is $2/8 = 1/4$.

## Example 1.54

A point $(x, y)$ is randomly picked from inside the rectangle with vertices $(0,0)$, $(4,0)$, $(4,1)$ and $(0,1)$. What is the probability that $x < y$? (AMC 10 2003)

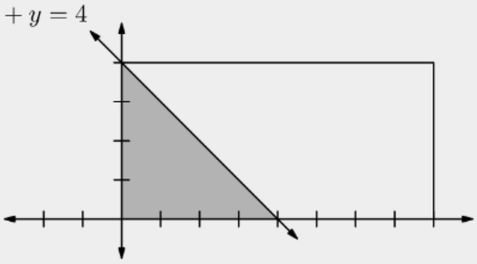
The point $(x, y)$ satisfies $x < y$ if and only if it belongs to the shaded triangle bounded by the lines $x = y$, $y = 1$, and $x = 0$, the area of which is 1/2. The ratio of the area of the triangle to the area of the rectangle is $\frac{1/2}{4} = \boxed{\frac{1}{8}}$.



## Example 1.55

A point $(x, y)$ is randomly selected such that $0 \leq x \leq 8$ and $0 \leq y \leq 4$. What is the probability that $x + y \leq 4$? Express your answer as a common fraction. (MathCounts 2001 National Sprint)

Rewrite $x + y \leq 4$ as $y \leq 4 - x$. This inequality is satisfied by the points on and under the line $y = 4 - x$. Sketching this line along with the $4 \times 8$ rectangle determined by the inequalities $0 \leq x \leq 8$ and $0 \leq y \leq 4$, we find that the points satisfying $x + y \leq 4$ are those in the shaded triangle (see figure). The area of the triangle is $\frac{1}{2}(4)(4) = 8$ square units, and the area of the rectangle is $(4)(8) = 32$ square units, so the probability that a randomly selected point would fall in the shaded triangle is $\boxed{\frac{1}{4}}$.

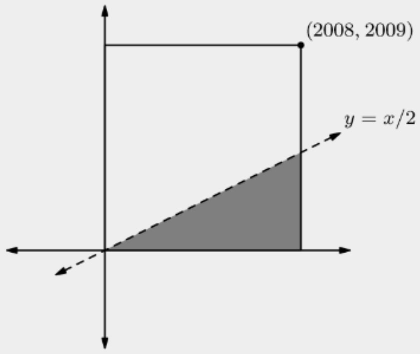



## Example 1.56

Point $(x, y)$ is randomly picked from the rectangular region with vertices at $(0,0)$, $(2008,0)$, $(2008,2009)$ and $(0,2009)$. What is the probability that  that $x > 2y$? Express your answer as a common fraction. (MathCounts 2009 National Countdown)

To see which points in the rectangle satisfy $x > 2y$, we rewrite the inequality as $y < \frac{1}{2}x$. This inequality is satisfied by the points below the line $y = \frac{1}{2}x$. Drawing a line with slope $\frac{1}{2}$ and $y$-intercept 0, we obtain the figure below. We are asked to find the ratio of the area of the shaded triangle to the area of the rectangle. The vertices of the triangle are $(0,0)$, $(2008,0)$, and $(2008, 2008/2)$, so the ratio of areas is

$$\frac{\frac{1}{2}(2008)\left(\frac{2008}{2}\right)}{2008(2009)} = \frac{2008/4}{2009} = \boxed{\frac{502}{2009}}.$$




## C. Further Types

### Example 1.57

On the Cartesian plane, a point is chosen at random within the region defined by $|x| + |y| \leq 8$. What is the probability it is within the region defined by $x^2 + y^2 \leq 8$? (MA Theta, 2017, Open, Probability and Combinatorics/4)

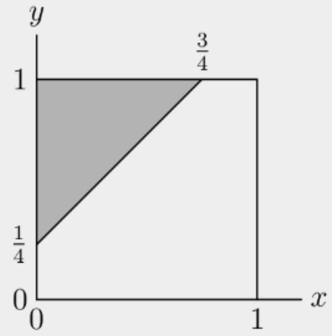
The region defined by $|x| + |y| \leq 8$ is a square with diagonal of 16. The area is 128. The region defined by $x^2 + y^2 \leq 8$ is a circle with radius $2\sqrt{2}$. The area is $8\pi$. So, the probability of the point being in the circle is $8\pi/128 = \pi/16$

## D. Difference

### Example 1.58

Two numbers between 0 and 1 on a number line are to be chosen at random. What is the probability that the second number chosen will exceed the first number by a distance greater than $\frac{1}{4}$ on the number line. Express your answer as a common fraction. (MathCounts 2010 State Team)

We want the probability that $y - x > \frac{1}{4}$. In other words, $y > x + \frac{1}{4}$. The region of points that satisfy this condition is shaded below.




The set of "possible" points is a right triangle with legs $\frac{3}{4}$ and $\frac{3}{4}$, so its area is $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{32}$.

Therefore, the probability that $y - x > \frac{1}{4}$ is

$$\frac{9/32}{1} = \boxed{\frac{9}{32}}.$$

### Example 1.59
Two numbers between 0 and 1 on a number line are to be chosen at random. What is the probability that the second number chosen will exceed the first number by a distance greater than $c$ on the number line. Express your answer as a common fraction.

Consider cases:

$$0 < c < 1: P = \frac{1}{2} \cdot (1 - c) \cdot (1 - c) = \frac{(1 - c)^2}{2}$$
$$Otherwise: P = 0$$

When $c = \frac{1}{4}$:

$$\frac{\left(1 - \frac{1}{4}\right)^2}{2} = \frac{\left(\frac{3}{4}\right)^2}{2} = \frac{\frac{9}{16}}{2} = \frac{9}{32}$$

## E. Rounding Off Function

### Example 1.60
Select numbers $a$ and $b$ between 0 and 1 independently and at random and let $c$ be their sum. Let $A, B$ and $C$ be the results when $a, b$, and $c$ are rounded to the nearest integer. What is the probability that $A + B = C$? (AMC 2004 12A/20)

$$Top\ Left: A = 0, B = 1 \Rightarrow A + B = 1$$
$$Top\ Left: A = 0, B = 1 \Rightarrow A + B = 1$$

The conditions under which $A+B=C$ are as follows.

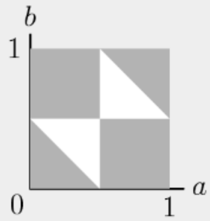
(i) If $a+b<1/2$, then $A=B=C=0$.

(ii) If $a\geq 1/2$ and $b<1/2$, then $B=0$ and $A=C=1$.

(iii) If $a<1/2$ and $b\geq 1/2$, then $A=0$ and $B=C=1$.

(iv) If $a+b\geq 3/2$, then $A=B=1$ and $C=2$.

These conditions correspond to the shaded regions of the graph shown. The combined area of those regions is 3/4, and the area of the entire square is 1, so the requested probability is $\boxed{3/4}$.




## F. Time Arrivals

### Example 1.61

Romeo and Juliet have a date at a given time, and each will arrive at the meeting place with a delay between 0 and 1 hour, with all pairs of delays being equally likely. The first to arrive will wait for 15 minutes and will leave if the other has not yet arrived. What is the probability that they will meet? (MIT OCW)

Solution

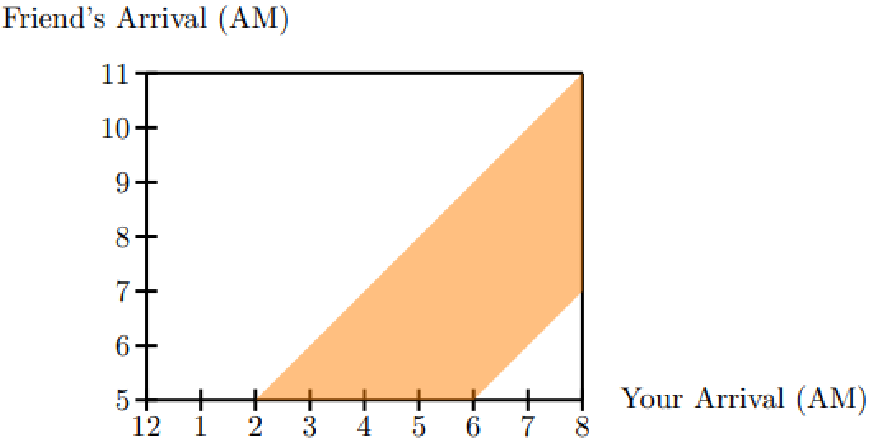
### Example 1.62

Steven and Melissa both arrive at a bus stop between 4:00PM and 6:00PM. Steven waits for 30 consecutive minutes, and Melissa waits at the bus stop for 60 consecutive minutes. Given that Steven and Melissa arrive at random times in the range independent of each other, find the probability that Steven and Melissa meet at the bus stop. ($MA\Theta$, 2019, Alpha, Combinatorics and Probability/26)

Translating the problem to geometric probability, consider the square with side length 2 in the first quadrant that is tangent to both coordinate axes. The probability that Steven and Melissa meet is the proportion of the square that is contained between the lines $y=x-0.5$ and $y=x+1$ (Steven and Melissa meet when $y=x$), which can be found by subtracting the areas of two isosceles right triangles from the square.
(4-1/2-9/8)/4 = 19/32

### Example 1.63

Every day you go to the music practice rooms at a random time from 12AM to 8AM and practice for 3 hours, while your friend goes at a random time from 5AM to 11AM and practices for 1 hour (the block of practice time need not be contained in the given time range for the arrival). What is the probability that you and your friend meet on at least 2 days in a given span of 5 days? (SMT, 2022, Discrete/3)




For a given day, the probability that you and your friend meet is $\frac{1}{3}$, which we can find by graphing the possible pairs of your and your friend's arrival times (see diagram). You can arrive no earlier than 3 hours before your friend and no later than 1 hour after your friend in order

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to meet. Then, the probability that you and your friend don't meet or only meet once in a span of 5 days is $\left(\frac{2}{3}\right)^5 + 5\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^4 = \frac{112}{243}$. Then, the probability that you meet at least twice is $1 - \frac{112}{243} = \frac{131}{243}$.

## G. Time: Back Calculations

### Example 1.64

Neel and Roshan are going to the Newark Liberty International Airport to catch separate flights. Neel plans to arrive at some random time between 5:30 am and 6:30 am, while Roshan plans to arrive at some random time between 5:40 am and 6:40 am. The two want to meet, however briefly, before going through airport security. As such, they agree that each will wait for n minutes once he arrives at the airport before going through security. What is the smallest n they can select such that they meet with at least 50% probability? The answer will be of the form a + b√c for integers a, b, and c, where c has no perfect square factor other than 1. Report a + b + c. (PUMaC 2021 Combinatorics/B/2)

Use geometric probability to see that the desired n will occur where (50−n)^2/2 +(70−n)^2 =0.5 × 3600. The larger solution obviously is not the correct one, leaving the smaller solution as the answer $(60 - 10\sqrt{17})$.

# 1.4 Further Topics

## 65 Examples