3D GEOMETRY

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REVISION: 889

AZIZ MANVA

AZIZMANVA@GMAIL.COM

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1. VISUALIZATION

1.1 General

> **Example 1.1**
> One can holds 12 ounces of soda. What is the minimum number of cans needed to provide a gallon (128 ounces) of soda? (AMC 10A 2009/1)

> **Example 1.2**
> A small bottle of shampoo can hold 35 milliliters of shampoo, whereas a large bottle can hold 500 milliliters of shampoo. Jasmine wants to buy the minimum number of small bottles necessary to completely fill a large bottle. How many bottles must she buy? (AMC 10A 2011/2)

> **Example 1.3**
> A child's wading pool contains 200 gallons of water. If water evaporates at the rate of 0.5 gallons per day and no other water is added or removed, how many gallons of water will be in the pool after 30 days? (AMC 8 1998/8)

1.2 Toroid

A. Definition

B. Volume and Surface Area

$$V = 2\pi RA$$
$$S = 2\pi RP$$

where

$$R = Radius\ of\ Revolution$$
$$A = Area\ of\ Cross\ Section$$
$$P = Perimeter\ of\ the\ cross - section$$

C. Torus (Circular Toroid)
When a circle is rotated around a straight line outside the circle, we get a circular toroid, also called a torus. If we take
$$r = radius\ of\ cross - section\ of\ circle$$
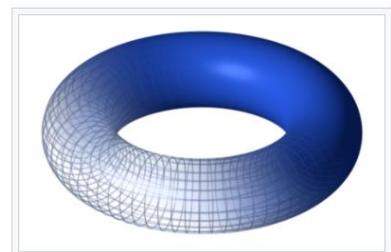
Volume
Substituting $A = \pi r^2$ in gives:
$$V = 2\pi RA = 2\pi R(\pi r^2) = 2\pi^2 r^2 R$$

Surface Area
Substituting $P = 2\pi r$ in gives:
$$V = V = 2\pi RP = 2\pi R(2\pi r) = 4\pi^2 rR$$

D. Square Toroid

$$V = 2\pi RA$$
$$S = 2\pi RP$$

Example 1.4
A square of side 1 cm is revolved about a line which is parallel to one of its diagonals. If the shortest distance of a vertex from this line is 1 cm, then the volume of the solid thus generated is: (JMET 2011/66)

First, note that the diagonal of the square is
$$\sqrt{2}$$
And hence the line of rotation is outside the square.

$$V = 2\pi RA$$

$$A = Area\ of\ Square = s^2 = 1^2 = 1$$

We need to find radius of revolution(R), which will be
$$= Average(Min\ Distance\ from\ Vertex, Max\ Distance\ from\ Vertex)$$
$$= Average\left(1,1 + \sqrt{2}\right) = \frac{1 + 1 + \sqrt{2}}{2} = \frac{2 + \sqrt{2}}{2}$$

$$V = 2\pi RA = 2\pi \left(\frac{2 + \sqrt{2}}{2}\right)(1) = \pi\left(2 + \sqrt{2}\right)$$

1.3 Cubes

## A. Cutting into Cubes
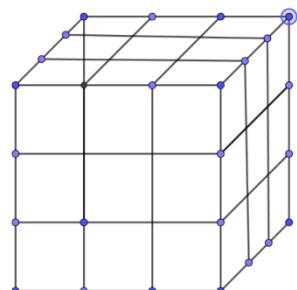
Example 1.5
A cube has side length 3 cm. It is divided into smaller cubes by making two cuts lengthwise, two cuts widthwise and two cuts height wise. Determine the number of smaller cubes formed, and the size of each cube.

The number of cubes is:
$$\underbrace{9}_{\substack{Top \\ Layer}} + \underbrace{9}_{\substack{Middle \\ Layer}} + \underbrace{9}_{\substack{Bottom \\ Layer}} = 3 \times 9 = 27$$

The size of each cube
$$1\ cm \times 1\ cm \times 1\ cm$$
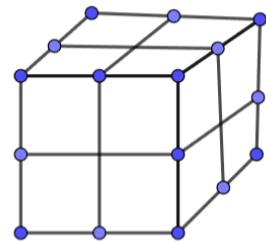
Example 1.6
A cube has side length 2 cm. It is divided into smaller cubes by making one cut lengthwise, one cuts widthwise and one cut height wise. Determine the number of smaller cubes formed, and the size of each cube.

$$2 \times 2 \times 2 = 8 \; cubes$$
$$1 \; cm \; side \; length$$

Example 1.7
A cube has side length $n \; cm$. It is divided into smaller cubes by making n cuts lengthwise, n cuts widthwise and n cuts height wise. Determine the number of smaller cubes formed, and the size of each cube.
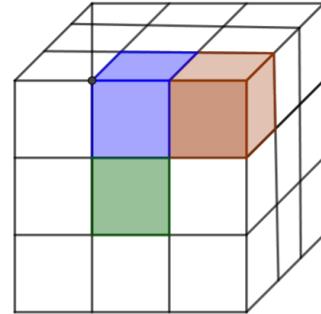
$$(n + 1)(n + 1)(n + 1) = (n + 1)^3 \; cubes$$
$$1 \; cm \; side \; length$$

Example 1.8
The diagram alongside has a larger cube divided into 27 smaller cubes.
- A. The top right smaller cube has three faces exposed, which are colored brown. In total, how many smaller cubes have three faces exposed.
- B. The middle smaller cube has two faces exposed, which are colored blue. In total, how many smaller cubes have two faces exposed.
- C. The middle smaller cube in the middle of the front face has a single face exposed, colored green. In total, how many smaller cubes have a single face exposed.
- D. How many smaller cubes have no faces exposed?
- E. Verify that the answers to Parts A-D add up to 27.

The cubes that have three faces exposed will be the cubes at the corners:
$$8$$

The cubes that two faces exposed will be between two corner cubes. There
$$4 + 4 + 4 = 12$$

Each face has one cube with a single face exposed
$$1 \times 6 = 6$$

There will be a middle cube inside the larger cube which no face exposed.
$$= 1$$

$$Total = 8 + 12 + 6 + 1 = 27$$

Example 1.9
A larger cube is divided into 64 equal sized smaller cubes by making three cuts each lengthwise, widthwise and height wise. How many smaller cubes have
- A. three faces exposed.
- B. two faces exposed.
- C. have a single face exposed.
- D. have no faces exposed?
Verify that the answers to Parts A-D add up to 64.
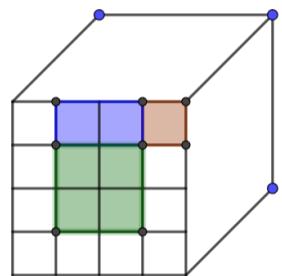
The cubes that have three faces exposed will be the cubes at the corners:
$$8$$

The cubes that have two faces exposed will be between two corner cubes. There
$$8 + 8 + 8 = 24$$

Each face has one cube with a single face exposed
$$4 \times 6 = 24$$
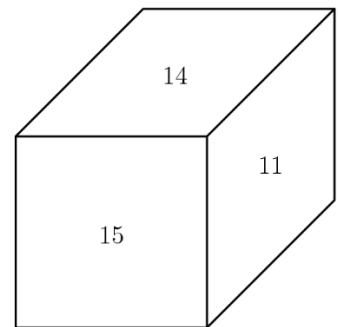
There will be a middle cube inside the larger cube which no face exposed.
$$= 2 \times 2 \times 2 = 8$$

$$Total = 8 + 24 + 24 + 8 = 64$$

Example 1.10

The numbers on the faces of this cube are consecutive whole numbers. The sums of the two numbers on each of the three pairs of opposite faces are equal. The sum of the six numbers on this cube is: (AMC 8 1990/11)

Case I: 10,11,12,13,14,15
Make pairs of the numbers to get the same total:
$$10 + 15 = 25, \qquad 11 + 14 = 25, \qquad 12 + 13 = 25$$

But $11 + 14$ is not possible since they are on adjacent faces, not opposite faces.

Case II: Numbers are 11,12,13,14,15,16
Make pairs of the numbers to get the same total:
$$11 + 16 = 27, \qquad 12 + 15 = 27, \qquad 13 + 14 = 27 \Rightarrow Works$$
Hence, the sum is

$$27 \times 3 = 81$$

Example 1.11

The eight vertices of a cube are randomly labelled with the integers from 1 to 8 inclusive. Judith looks at the labels of the four vertices of one of the faces of the cube. She lists these four labels in increasing order. After doing this for all six faces, she gets the following six lists: (1; 2; 5; 8); (3; 4; 6; 7); (2; 4; 5; 7); (1; 3; 6; 8); (2; 3; 7; 8), and (1; 4; 5; 6). The label of the vertex of the cube that is farthest away from the vertex labelled 2 is (Gauss Grade 7 2016/22)

Faces 1 and 2:
(1,2,5,8) and (3,4,6,7) have no vertex in common.
Consider (1,2,5,8) to be the front, (3,4,6,7) must be the back.

Face 3: Right Face
(2,4,5,7) shares 2 and 5 with the front.
(2,4,5,7) shares 4 and 7 with the back.
Place (2,4,5,7) as the right face with the front reading (1,2,5,8) in clockwise order, and the back reading (3,4,7,6) in clockwise order.

Face 4: Left Face

(1,3,6,8) shares 1 and 8 with the front
(1,3,6,8) shares 3 and 6 with the front
Place (2,4,5,7) as the left face of the cube.

Face 5:
(2,3,7,8) shares 2 and 8 with the front.
(2,3,7,8) shares 3 and 7 with the back.
The arrangement that we have does not work.
Swap the position of the 1 and 8, giving us (8,2,5,1), and swap the position of the 4 and the 7, giving us (3,7,4,6) then face 5 can be the top face.

Face 6:
(1,4,5,6) shares 1 and 5 with the front.
(1,4,5,6) shares 4 and 6 with the front.
And this can be the bottom face.

And hence the faces we have are:
Front: (8,2,5,1)
Back: (3,7,4,6)
Right: (2,7,4,5)
Left: (8,1,6,3)
Top: (3,7,2,8)
Bottom: (1,5,4,6)
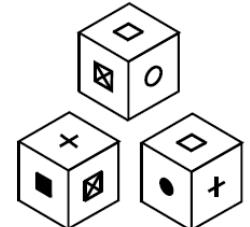
The vertex labelled 2 is top right on the front face. Hence, we are looking for vertex that is bottom left on back face, which is 6.

<div style="background:#fdf0d0;">

Example 1.12
Three different views of the same cube are shown. The symbol on the face opposite ● is
(Gauss 7 2020/19, Gauss 8 2020/18)

</div>

Bottom left view:
Plus symbol on the top.
Filled square, and patterned square adjacent to it.

Bottom Right view:
Plus symbol on the right.
Filled circle, and empty square adjacent to it.

But, there only four faces adjacent to any face on a cube, and hence, these four symbols must be adjacent to the plus sign:
Filled Square
Patterned Square
Filled Circle
Empty Circle

From the top cube, the empty square is adjacent to the patterned square.
From the bottom left cube, the filled circle is the shape that can be opposite the patterned square.
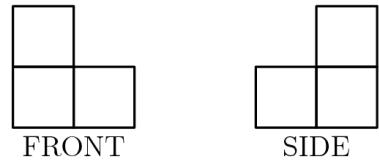
Hence, the shape opposite the filled circle is the patterned square.
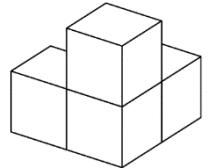
B. Nets

Example 1.13
A figure is constructed from unit cubes. Each cube shares at least one face with another cube. What is the minimum number of cubes needed to build a figure with the front and side views shown? (AMC 8 2003/15)

FRONT   SIDE

We need a minimum of 3 from the front view, and a minimum of 3 from the side view. Combining the two, we can achieve this in 4 cubes.
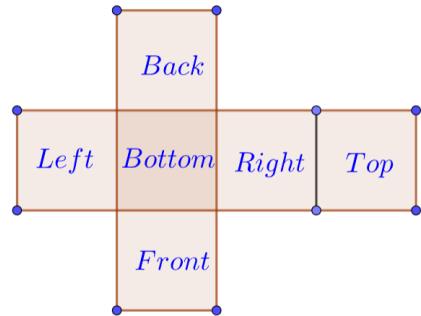
C. Nets of a Cube

1.14: Net
A net is a two-dimensional shape generated by "unfolding" a three dimensional shape.
A shape can have more than one net.

Back

Left *Bottom* *Right* *Top*

Front
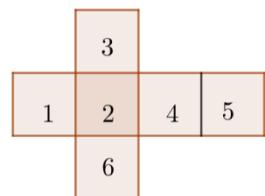
1.15: Net of a Cube
A cube has 11 distinct possible nets.

Example 1.16
The shape alongside is folded to make a cube.
  A. Determine which faces are opposite which faces.
  B. Let M be the largest sum of two opposite faces, and m be the smallest sum of two opposite faces. Determine $\frac{M+m}{M-n}$.

	3		
1	2	4	5
	6		

Part A

$$1(Left) \ is \ opp \ 4(Right)$$
$$2(Bottom) \ is \ opp \ 5(Top)$$
$$3(Back) \ is \ opp \ 6(Front)$$
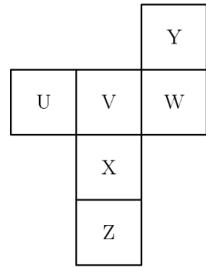
Part B

$$1 + 4 = 5 = m$$
$$2 + 5 = 7$$
$$3 + 6 = 9 = M$$

$$\frac{M+m}{M-m} = \frac{9+5}{9-5} = \frac{14}{4} = \frac{7}{2}$$

Example 1.17
A piece of paper containing six joined squares labeled as shown in the diagram is folded along the edges of the squares to form a cube. The label of the face opposite the face labeled X is (AMC 8 1985/11)

X will get folded in front.
Z will get folded on top.
U will get folded on the left.
W will get on the right
Y will get folded on the back
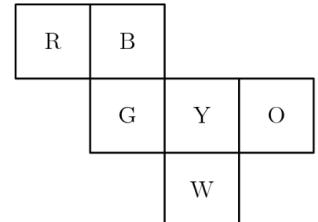
Y is opposite to X.

Example 1.18
Six squares are colored, front and back, (R = red, B = blue, O = orange, Y = yellow, G = green, and W = white). They are hinged together as shown, then folded to form a cube. The face opposite the white face is (AMC 8 1999/8)

Case I

$$Green = Base$$
$$Yellow\ face\ is\ the\ Right\ face$$
$$White = Front$$
$$Orange = Top$$
$$Blue = Back$$
$$Red = Left$$

Case II

$$Yellow = Base$$
$$Orange = Right$$
$$White = Front$$
$$Green = Left$$
$$Blue = Back$$
$$Red = Top$$
$$White\ is\ opposite\ to\ Blue$$
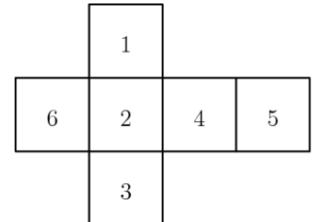
Example 1.19
The figure may be folded along the lines shown to form a number cube. Three number faces come together at each corner of the cube. What is the largest sum of three numbers whose faces come together at a corner? (AMC 8 1989/20)

$$6 + 5 + 4$$
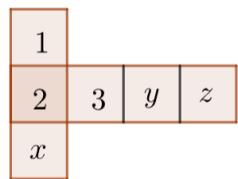
This is not possible because 6 and 4 are opposite.

$$6 + 5 + 3 = 14:\ Possible$$

Example 1.20

The net on the right is folded to form a cube. The sum of the numbers on opposite faces is equal. If x, y and z are positive integers less than 100, find the
- A. number of possible triplets (x, y, z).
- B. sum of the possible triplets (x, y, z).

	1		
2	3	y	z
x			

Part A

$$x + 1 = y + 2 = z + 3$$
$$x = y + 1 = z + 2 = k$$

$$x = k, y = k - 1, z = k - 2$$

From here, we know that z is the smallest.

$$(x, y, z) = (3, 2, \mathbf{1})$$
$$(x, y, z) = (4, 3, \mathbf{2})$$
$$.$$
$$.$$
$$.$$
$$(x, y, z) = (99, 98, \mathbf{97})$$

Part B

$$(3, 2, \mathbf{1}) \rightarrow Total = 6$$
$$(4, 3, \mathbf{2}) \rightarrow Total = 9$$
$$(5, 4, \mathbf{3}) \rightarrow Total = 12$$
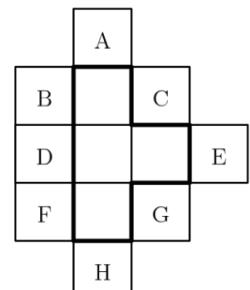
$$6 + 9 + 12 + \cdots + 294$$

$$48.5(6 + 294) = 48.5(300) =$$

Example 1.21

Suppose one of the eight lettered identical squares is included with the four squares in the T-shaped figure outlined. How many of the resulting figures can be folded into a topless cubical box? (AMC 8 1986/21)

	A		
B		C	
D			E
F		G	
	H		

Consider the central figure as the base. The remaining three squares can be folded to give a front, a back and a right side. The missing sides are the left and the top.
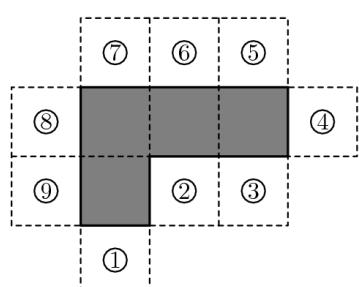
$$D, E, B, F \Rightarrow Left \Rightarrow 4\ Squares$$
$$A, H \Rightarrow Top \Rightarrow 2\ Squares$$

A cube with five sides is a topless cube.

$$4 + 2 = 6\ Squares = 6\ Figures$$

Example 1.22

The polygon enclosed by the solid lines in the figure consists of 4 congruent

squares joined edge-to-edge. One more congruent square is attached to an edge at one of the nine positions indicated. How many of the nine resulting polygons can be folded to form a cube with one face missing? (AMC 10A 2003/10)
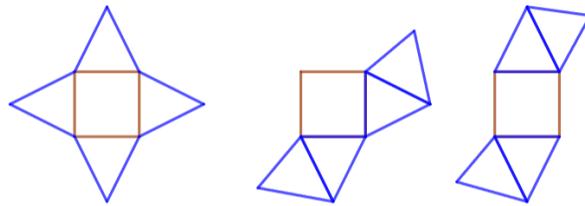
Take the top left square as the base. This gives
$$Base, Front, Right, Top$$

$$1 = Top \Rightarrow Not\ Valid$$
$$2 = Front \Rightarrow Not\ Valid$$
$$3 = Front \Rightarrow Not\ Valid$$
$$4 = Left \Rightarrow Valid$$
$$5 = Front \Rightarrow Valid$$
$$6 = Front \Rightarrow Valid$$
$$7 = Front \Rightarrow Valid$$
$$8 = Left\ \Rightarrow Valid$$
$$9 = Left\ \Rightarrow Valid$$

$$6\ valid\ choices$$
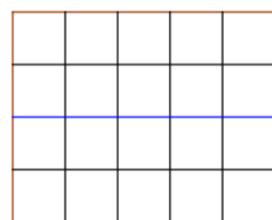
1.23: Net of a Square Pyramid

$$1 + 1 + 1 + 1$$
$$2 + 1 + 1$$
$$2 + 2\ (2\ Shapes)$$
$$3 + 1$$
$$4$$

Example 1.24
The rectangle alongside comprises 20 squares, each of size 1 $unit$. Of the 10 squares above the blue line, 4 are red, 4 are blue and the rest are white. When the upper half is folded down over the blue line, 2 pairs of red triangles coincide, as do 3 pairs of blue triangles. There is 1 red-white pair. Find the number of blue-red pairs.

		Red & Blue Pairs		Red-White Pairs		Rest
Red	4	2	2	1	1	
Blue	4	3	1		1	
White	2		2	1	1	
	10					

$Red - red$ pairs are already counted.

Red − white pairs are also already counted.
The remaining red must pair with blue.
$$Blue - red\ pairs = 1$$
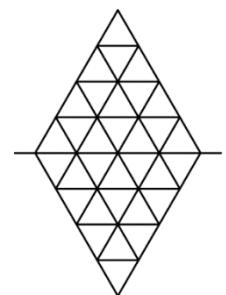
Example 1.25
Each half of this figure is composed of 3 red triangles, 5 blue triangles and 8 white triangles. When the upper half is folded down over the centerline, 2 pairs of red triangles coincide, as do 3 pairs of blue triangles. There are 2 red-white pairs. How many white pairs coincide? (AMC 8 2001/24)
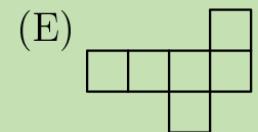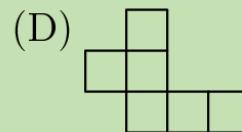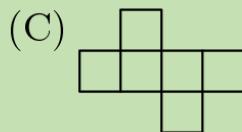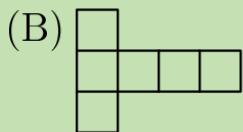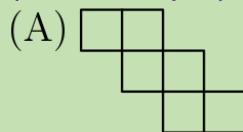
		Red & Blue Pairs		Red-White Pairs		Rest	
Red	3	2	1	1	0		
Blue	5	3	2		2	2	
White	8		8	1	7	2 + 5
						Blue White
	16						

Example 1.26
Which pattern of identical squares could NOT be folded along the lines shown to form a cube?
(AMC 8 1992/20)

(A)   (B)   (C)   (D)   (E)
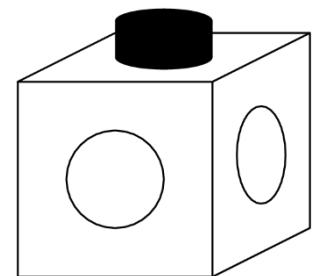
Example 1.27
A fly trapped inside a cubical box with side length 1 meter decides to relieve its boredom by visiting each corner of the box. It will begin and end in the same corner and visit each of the other corners exactly once. To get from a corner to any other corner, it will either fly or crawl in a straight line. What is the maximum possible length, in meters, of its path? (AMC 10A 2010/20)

D. Visualization

Example 1.28
A plastic snap-together cube has a protruding snap on one side and receptacle holes on the other five sides as shown. What is the smallest number of these cubes that can be snapped together so that only receptacle holes are showing? (AMC 8 1995/21)
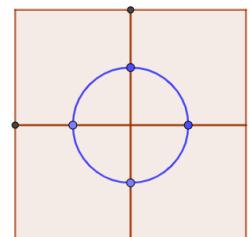
Every cube has
$$1\ snap, and\ 5\ holes$$

The minimum cubes that we can imagine is
$$2\ cubes \Rightarrow 1\ snap\ inserted\ into\ the\ other\ cube$$
But this leaves the other cube with snap showing.
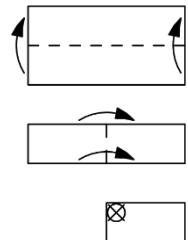
If you arrange cubes one after the other in a "circular" fashion then we can achieve this with four cubes.
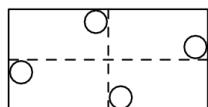
E. Unfolding
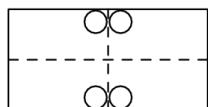
Example 1.29
As indicated by the diagram, a rectangular piece of paper is folded bottom to top, then left to right, and finally, a hole is punched at X. What does the paper look like when unfolded? (AMC 8 1998/18)
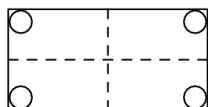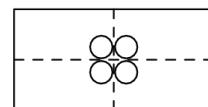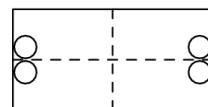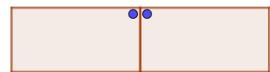
(A)   (B)   (C)   (D)   (E)
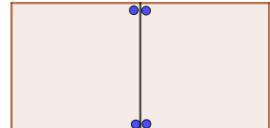
We can do this by following the process in reverse.
Unfold left to right to get the first diagram. The holes are at the top.

Unfold top to bottom to get the second diagram. The top holes move to the bottom when unfolded.

Option B
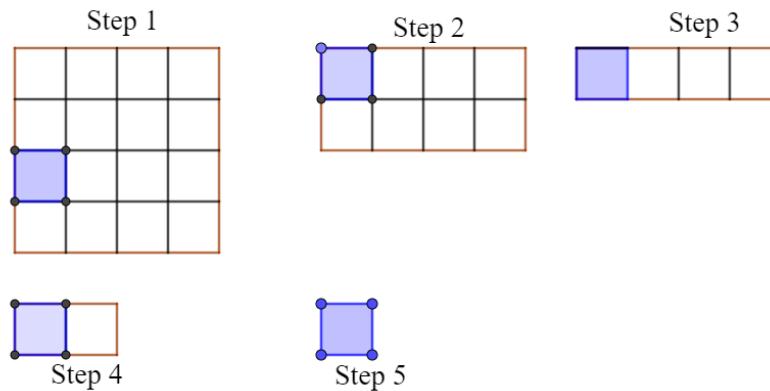
Example 1.30
The 16 squares on a piece of paper are numbered as shown in the diagram. While lying on a table, the paper is folded in half four times in the following sequence:
(1) fold the top half over the bottom half
(2) fold the bottom half over the top half
(3) fold the right half over the left half
(4) fold the left half over the right half.
Which numbered square is on top after step 4? (AMC 8 1991/16)

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Step 1

Step 2

Step 3

Step 4

Step 5

9

Example 1.31
A 3x3x3 cube is made of 27 normal dice. Each die's opposite sides sum to 7. What is the smallest possible sum of all of the values visible on the 6 faces of the large cube? (AMC 10A 2002/18)

Example 1.32
Figure 1 is called a "stack map." The numbers tell how many cubes are stacked in each position. Fig. 2 shows these cubes, and Fig. 3 shows the view of the stacked cubes as seen from the front.
Which of the following is the front view for the stack map in Fig. 4?(AMC 8 1999/20)

3	4
2	1

Figure 1
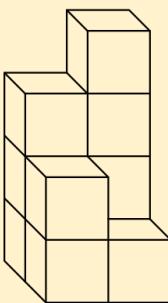
Figure 2
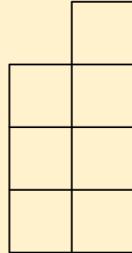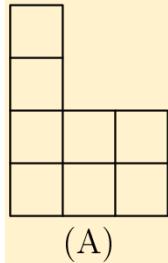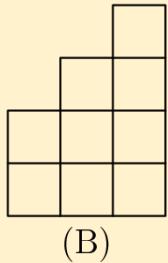
Figure 3

2	2	4
1	3	1
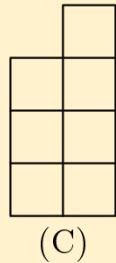
Figure 4
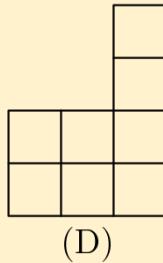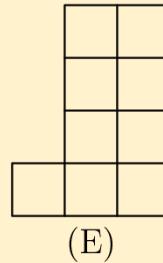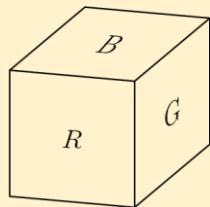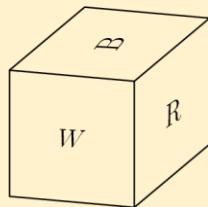
(A)　　(B)　　(C)　　(D)　　(E)

Example 1.33
The faces of a cube are painted in six different colors: red (R), white (W), green (G), brown (B), aqua (A), and purple (P). Three views of the cube are shown below. What is the color of the face opposite the aqua face? (AMC 8 2019/12)
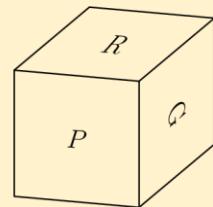
(A)red　　(B)white　　(C)green　　(D)brown　　(E)purple

2. Cubes and Cuboids

2.1 Volume and Surface Area

A. Volume

2.1: Volume of a Cube

A cube with side length s has
$$Volume = V = s^3$$

Example 2.2

    A. What is the volume of a cube with side length 3 inches?

    B. Carl has 5 cubes each having side length 1, and Kate has 5 cubes each having side length 2. What is the total volume of these 10 cubes? (AMC 10B 2020/2)

Part A
$$V = s^3 = 3^3 \text{ cubic inches} = 27 \text{ cubic inches}.$$
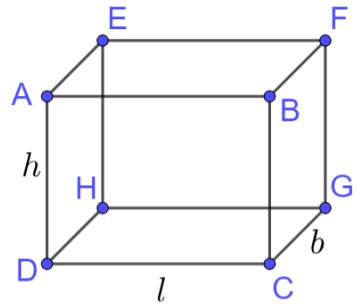
Part B
$$5 \times \underbrace{1^3}_{\substack{Smaller \\ Cube}} + 5 \times \underbrace{2^3}_{\substack{Larger \\ Cube}} = 5 \times 1 + 5 \times 8 = 5(9) = 45 \text{ units}^3$$

2.3: Volume of a Cuboid
$$Volume = lbh$$

Example 2.4

    A. Find the volume of a cubical fish tank that has length 2 feet, width 3 feet and height of 6 inches.

    B. A cubical container which holds 500 ml of air has width 5 cm, and length 10 cm. What is its height?

    C.

$$V = lbh = 2 \times 3 \times \frac{1}{2} = 3 \text{ feet}^3$$

$$(5)(10)b = 500$$
$$b = 10 \text{ cm}$$

2.5: Capacity

The capacity of a three-dimensional figure (such as a cube or cuboid) is given by its volume.
$$Capacity = Volume$$

2.6: Volume of a 3D Figure
$$Base\ Area \times Height$$

Example 2.7

Find the water capacity in cubic feet of a kids splash pool with base area 30 feet and depth 6 inches.

$$30 \times \frac{1}{2} = 15 \, ft^3$$

B. Surface Area

2.8: Surface Area

$$Surface \; Area \; of \; a \; Cube = 6s^2, s = side \; length$$
$$Surface \; Area \; of \; a \; Cuboid = 2(lb + bh + hl), h = height, l = length, b = breadth$$

The front is $ABCD$, and the back is $EFGH$. They each have area:
$$lh$$
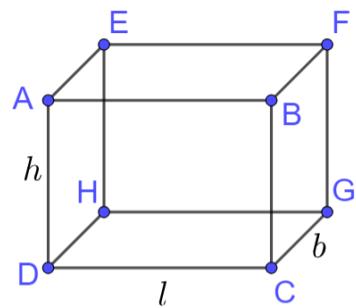
The right side is $BFGC$. The left side is $AEHD$. They each have area:
$$bh$$

The top is $AEBF$. The bottom is $DHGC$. They each have area:
$$lb$$

Example 2.9
A cuboidal milk carton with $\underset{l}{\underline{length \; 4 \; cm}}$, $\underset{w}{\underline{breadth \; \; 3 \; cm}}$ and $\underset{h}{\underline{height \; 5 \; cm}}$ is oriented so that the length (4 cm) faces you. Find the surface area.

$$Top = Base = lb = 3 \times 4 = 12$$
$$Left \; Pane = Right \; Pane = bh = 5 \times 3 = 15$$
$$Front = Back = hl = 5 \times 4 = 20$$

Total Area
$$= 2(12 + 20 + 15) = 2 \times 47 = 94 \; \text{cm}^2$$

Example 2.10
Find the surface area of the following cuboids in square feet:
   A. Length 5 feet, breadth 3 feet and height 7 feet
   B. Length, breadth and height: 3 inches
   C. Length 4 inches, breadth 3 inches and height 2 inches
   D. Length 0.5 feet, breadth 1/3 feet and height double of its breadth

Part A
$$2(5 \times 3 + 5 \times 7 + 7 \times 3) = \; 142 \; ft^2$$

Part B

$$6s^2 = 6 \times 3^2 = 54 \; \text{in}^2 = \frac{54}{144} \text{ft}^2 = \frac{3}{8} \text{ft}^2{}^{[1]}$$
$$6s^2 = 6 \times \left(\frac{3}{12} \text{ft}\right)^2 = 6 \times \frac{1}{16} = \frac{3}{8} \text{ft}^2$$

Part C

---

[1] Common Mistake: Dividing by 12 to convert from inches to feet is wrong. You need to divide by $12^2 = 144$.

$$\underbrace{2(4 \times 3 + 4 \times 2 + 3 \times 2)}_{\substack{Get\ the\ area\ first.\\ Then\ convert\ to\ feet}} = 52\ in^2 = \frac{52}{144} ft^2 = \frac{13}{36} ft^2$$

Part D

0.5 feet = 6 inches, 1/3 feet = 4 inches, Height = 8 inches
2(6 * 4 + 6 * 8 + 4 * 8) = 208 square inches = 208/144 square feet = 13/9 square feet

Example 2.11
An interstellar spaceship has a polished titanium cubical hyperdrive with edge length 4 feet.
    A.   What is the surface area of the hyperdrive.
    B.   How much space does it occupy?

$$6s^2 = 6 \times 4^2 = 6 \times 16 = 96\ ft^2$$
$$Space\ Occupied = Volume = s^3 = 4^3 = 64\ ft^3$$

Example 2.12
Isabella's house has 3 bedrooms. Each bedroom is 12 feet long, 10 feet wide, and 8 feet high. Isabella must paint the walls of all the bedrooms. Doorways and windows, which will not be painted, occupy 60 square feet in each bedroom. How many square feet of walls must be painted? (AMC 10B 2007/1)

Note that walls does not include floor or ceiling. The painting required per

$$= 3[2(hb + hl) - 60]$$
$$= 3[2(8 \cdot 10 + 8 \cdot 12) - 60]$$
$$= 3[2(8 \cdot 22) - 60]$$
$$= 3[2(176) - 60]$$
$$= 3[352 - 60]$$
$$= 3[292]$$
$$= 876\ ft^2$$

C. Conversions

2.13: Liter
$$1\ Liter = 1000\ cm^3$$
$$1\ ml = 1\ cm^3$$
$$1m = 100\ cm$$

Example 2.14
A container has length, width and height 1 m.
    A.   Find its volume in cubic meters.
    B.   Find its volume in liters.

Part A
$$V = s^3 = 1 \times 1 \times 1 = 1\ m^3$$

Part B
First convert the volume from cubic meters to cubic centimeters:

$$V = 1 \, m^3 = (1 \, m)^3$$

Substitute $1m = 100c$

$$= (100 \, cm)^3 = 100 \times 100 \times 100 \times cm^3 = 1,000,000 \, cm^3 = 1000 \, Litres$$

2.15: Liters to Cubic Meters
$$1000 \, Liters = 1 \, m^3$$

Example 2.16
Find the capacity, in liters, of a water tank, that has width 1 meter, length 2 meters, and height 3 meters.

$$Capacity = V = lbh = 1 \times 2 \times 3 = 6 \, m^3 = 6000 \, Litres$$

Example 2.17
A rectangular water tank is 20 meters by 10 meters by 35 meters. What is its surface area. Find its capacity in both cubic meters, and also litres (Conversion Factor: 1 m = 100 cm, and 1 litre = 1000 cm³)

$$2(lb + bh + lh) = 2(20 \times 10 + 10 \times 35 + 20 \times 35) = 2500 \, m^2$$

$$V = lbh = 20 \times 10 \times 35 = 7000 \, m^3$$

$$Litres = 7000 \times 100^3 \times \left(\frac{1}{1000}\right) = 7,000,000 \, litres$$

2.18: Length
$$1 \, Yard = 3 \, Feet$$
$$1 \, Feet = 12 \, Inches$$

Example 2.19
A straight concrete sidewalk is to be 3 feet wide, 60 feet long, and 3 inches thick. How many cubic yards of concrete must a contractor order for the sidewalk if concrete must be ordered in a whole number of cubic yards? (AMC 8 1990/17)

Convert feet to yards by dividing by three.
Volume of concrete needed for the sidewalk

$$= \frac{3}{3} \times \frac{60}{3} \times \frac{3}{36} = 1 \times 20 \times \frac{1}{12} = \frac{5}{3} = 1\frac{2}{3} \approx 2 \, cubic \, yards$$

D. Back Calculations

Example 2.20
A. A narrow passageway has a volume of 306 ft^3. It has a height of 9 feet, and a width of 2 feet. Find its length.
B. The volume of a cuboidal tank is 6500 units. If the length and breadth are 25 units, and 13 units respectively, find the height.
C. The surface area of a cube is 294 square meters. What is its edge length?

Part A
Substitute $V = 306, h = 9, b = 2$ in $V = lbh$:
$$306 = l \times 2 \times 9 \Rightarrow 17$$

Part B

$$lbh = V \Rightarrow 25 \times 13 \times h = 6500 \Rightarrow h = \frac{6500}{13 \times 25} = 20$$

Part C

$$6s^2 = 294 \Rightarrow s^2 = \frac{294}{6} = 49 \Rightarrow s = 7$$

Part D

Example 2.21
A painter needs 1 liter of paint to paint 32 square meters of surface. He gets a cube to be installed for an event and needs 12 liters of paint for it.
    A.   What is the volume of the cube?
    B.   If the paint is applied evenly, what is the thickness of the paint in mm?

Part A
The surface area of the cube

$$= 32 \times 12 = 384 \ m^2$$

Each face of the cube has surface area

$$= \frac{384}{6} = 64 \ \text{m}^2$$

Each side of the cube has length

$$= \sqrt{64} = 8$$

Volume of the cube

$$= 8^3 = 512$$

Part B

$$32 \ m^2 = 32 \times 100^2 = 320{,}000 \ cm^2$$

$$V = lbh$$
$$1000 \ cm^3 = (320{,}000 \ cm^2)h$$
$$\frac{1}{320} \ cm = h$$
$$h = \frac{1}{32} \ mm$$

Example 2.22
The surface area of four walls and the ceiling of a water tank (with length, breadth and height the same) is 180 square inches. What is twice the length, plus thrice the breadth, plus four times the height (in feet)?

Side = Root (180/5) = Root (36) = 6
2 * 6 + 3 *6 + 4 * 6 = 6(2 + 3 + 4) = 6 * 9 = 54 inches = 5.5 feet

Example 2.23
The surface area of a cube is 378 square feet. The top of the cube has surface area $t \ ft^2$, the edge of the cube has

length e ft units, and the capacity of the cube is c ft^3. Find $t + e + c$.

6 * Side² = 378
Side² = 63

$$Side = \sqrt{63} = \sqrt{9} \times \sqrt{7} = 3\sqrt{7}$$
$$Capacity = Volume = \left(3\sqrt{7}\right)^3 = 3^3 \times \left(\sqrt{7}\right)^3 = 27 \times 7 \times \sqrt{7}$$
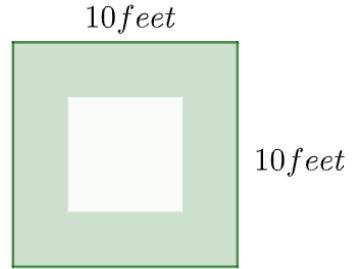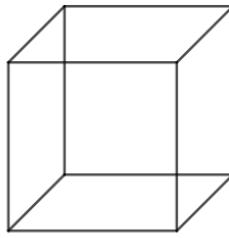
Example 2.24

Marla has a large white cube that has an edge of 10 feet. She also has enough green paint to cover 300 square feet. Marla uses all the paint to create a white square centered on each face, surrounded by a green border. What is the perimeter of one of the white squares, in yards? (AMC 8 2012/21, Adapted)

In the left diagram
A cube has six faces. The paint needs to be equally divided among the six faces.
$$= \frac{300}{6} = 50 \ ft^2$$

In the right diagram
Area of each face

$$= 10^2 = 100$$
$$A(White \ Square) = 100 - 50 = 50 \ ft^2$$
$$S(White \ Square) = \sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2} \ ft$$
$$P(White \ Square) = 4s = 4\left(5\sqrt{2}\right) = 20\sqrt{2} \ ft = \frac{20}{3}\sqrt{2} \ yards$$

E. Ratios

Example 2.25

Mark the correct option

A rectangular box has integer side lengths in the ratio $1: 3: 4$. Which of the following could be the volume of the box?

  A. 48
  B. 56
  C. 64
  D. 96
  E. 144 (AMC 10A 2016/5)

The side lengths are in the ratio
$$1: 3: 4 = x: 3x: 4x$$
$$V = (x)(3x)(4x) = 12x^3 \Rightarrow Multiple \ of \ 12$$

The option which is a multiple of 12 is:
$$Option \ D$$

F. Systems of Equations

Example 2.26
A. The volume of a rectangular solid each of whose side, front, and bottom faces are $12\ in^2, 8in^2$, and $6\ in^2$ respectively is: (AHSME 1950/21)
B. Find the lengths of the sides of the cuboid.

Part A
Let the sides of the solid be x, y, z. Then:
$$\underbrace{Side\ Face = xy = 12}_{Equation\ I}$$
$$\underbrace{Front\ Face = yz = 8}_{Equation\ II}$$
$$\underbrace{Bottom\ Face = zx = 6}_{Equation\ III}$$

Notice the symmetry in the above: each variable occurs twice. Multiply Equations I, II and III:
$$x^2y^2z^2 = 12 \times 8 \times 6$$

Take Square Roots:
$$\underbrace{Volume = xyz = 24}_{Equation\ IV}$$

Part B
Divide Equation IV by Equation I:
$$\frac{xyz}{xy} = \frac{24}{12} \Rightarrow z = 2$$

Divide Equation IV by Equation II
$$\frac{xyz}{zx} = \frac{24}{6} \Rightarrow y = 2$$

Divide Equation IV by Equation III
$$\frac{xyz}{yz} = \frac{24}{8} \Rightarrow x = 3$$

Example 2.27
A three-dimensional rectangular box with dimensions X, Y, and Z has faces whose surface areas are 24, 24, 48, 48, 72, and 72 square units. What is $X + Y + Z$? (AMC 10B 2018/4)

Let the sides of the solid be X, Y, Z. Then:
$$\underbrace{XY = 24}_{Equation\ I}, \qquad \underbrace{YZ = 48}_{Equation\ II}, \qquad \underbrace{ZX = 72}_{Equation\ III}$$

Notice the symmetry in the above: each variable occurs twice. Multiply Equations I, II and III:
$$X^2Y^2Z^2 = 24 \times 48 \times 72 = 24 \times 3 \times 16 \times 2 \times 36 = 144 \times 16 \times 36$$

Take Square Roots:
$$\underbrace{Volume = XYZ = 48 \times 6}_{Equation\ IV}$$

Divide Equation IV by Equation I:

$$\frac{XYZ}{XY} = \frac{48 \times 6}{24} \Rightarrow Z = 12$$
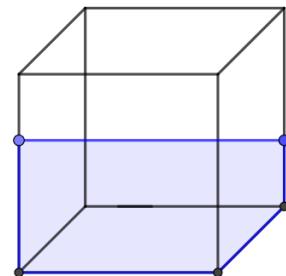
Divide Equation IV by Equation II

$$\frac{XYZ}{ZX} = \frac{48 \times 6}{72} \Rightarrow Y = 4$$

Divide Equation IV by Equation III

$$\frac{XYZ}{YZ} = \frac{48 \times 6}{48} \Rightarrow X = 6$$

$$X + Y + Z = 12 + 4 + 6 = 22$$

G. Immersions

2.28: Immersion
When you immerse an object in a liquid completely, the height of the liquid increases by the volume of the object.

Example 2.29
A tank of water has a capacity of $30L$. It is filled with water upto nine-tenths of its capacity. How many bricks of volume $400 \, ml$ can be immersed without the water spilling, if bricks can only be immersed in whole number values?

Current water in the tank

$$= \frac{9}{10} \times 30 = 27L$$

Extra capacity in the tank

$$= 30L - 27L = 3L = 3000 \, ml$$

The number of bricks

$$= \frac{3000}{400} = \frac{30}{4} = 7.5$$

7 bricks

2.30: Volume by Object
The volume of an object which is immersed is equal to the difference in the volume before and after immersion.
$$Volume \ of \ Object = Volume_{After \ immersion} - Volume_{Before \ Immersion}$$

Example 2.31
In an experiment to measure the volume of a rock, a beaker was filled with $1 \, Liter$ of water. The rock was immersed completely in the beaker. The volume reading for the beaker now read $1100 \, ml$. Find the volume of the rock.

Volume of the rock

$$= 1100 \, ml - 1000 \, ml = 100ml$$

2.32: Increase in Height
The increase in height on immersion of an object in a liquid
$$= \frac{Volume \ of \ Object}{Area \ of \ Base}$$

Example 2.33
An aquarium has a rectangular base that measures 100 cm by 40 cm and has a height of 50 cm. The aquarium is filled with water to a depth of 37 cm. A rock with volume 1000 cm^3 is then placed in the aquarium and completely submerged. By how many centimeters does the water level rise? (AMC 8 2006/21)

Increase in height of the water level
$$= \frac{Volume\ of\ Object}{Area\ of\ Base} = \frac{1000}{00 \times 40} = \frac{1000}{4000} = \frac{1}{4} = 0.25\ cm$$

Example 2.34
An aquarium has a rectangular base that measures 100 cm by 40 cm and has a height of 50 cm. It is filled with water to a height of 40 cm. A brick with a rectangular base that measures 40 cm by 20 cm and a height of 10 cm is placed in the aquarium. By how many centimeters does the water rise? (AMC 10A 2007/3)

Increase in height of the water level
$$= \frac{Volume\ of\ Object}{Area\ of\ Base} = \frac{40 \times 20 \times 10}{100 \times 40} = 2\ cm$$

H. Density

2.35: Density
$$\rho = \frac{m}{V} \Rightarrow m = V\rho$$

Example 2.36
A spaceship has a storage container in the form of a cube, with side length 3 $units$. One cubic unit of air weights $\frac{5}{3}\ kg$. The cost of air is $\frac{4}{3}\ credits$ per kg.
    A. Find the volume of the container.
    B. Find the mass of the air in the container.
    C. Find the cost of the air in the container.

$$Volume = s^3 = 3^3 = 27\ units^3$$
$$m = V\rho = 27\ units^3 \times \frac{5}{3}\frac{kg}{unit^3} = 45\ kg$$
$$C = 45\ kg \times \frac{4}{3}\frac{credits}{kg} = 60\ Credits$$

Example 2.37
Cuboid A is twice as dense, twice as long, twice as broad, and twice as high as Cuboid B. Find the ratio of the mass of Cuboid A to Cuboid B.

The ratio of the masses is:
$$m_A : m_B$$

Use the formula for mass $m = V\rho$:
$$= V_A\rho_A : V_B\rho_B$$

Take some values to make the calculations easier
$$Cuboid\ B: l = b = h =, \rho = 1 \Rightarrow Cuboid\ A: l = b = h = 2, \rho = 2$$

Substitute to get:
$$(2 \times 2 \times 2)(2): (1 \times 1 \times 1)(1) = 16:1$$

Example 2.38

A pound of *admantium* costs 70 gold coins, and one cubic unit of admantium is $\frac{1}{3}rd$ of a pound. What is the cost of a cube of admantium that has side length 2 units.

$$Volume\ of\ the\ Cube = s^3 = 2^3 = 8\ Units$$
$$Mass\ of\ the\ Cube = V\rho = \underbrace{8}_{Volume} \times \underbrace{\frac{1}{3}}_{Density} = \underbrace{\frac{8}{3}}_{Mass}\ Pounds$$
$$Cost = \underbrace{\frac{8}{3}}_{Mass} \times \underbrace{70}_{Cost\ per\ Pound} = \frac{560}{3}\ gold\ coins$$

Example 2.39

A two-inch cube $(2 \times 2 \times 2)$ of silver weighs 3 pounds and is worth \$200. How much is a three-inch cube of silver worth? (AMC 8 1997/22)

Method I

$$Cost\ per\ cubic\ inch = \frac{Total\ Cost}{Volume} = \frac{200}{8} = 25$$
$$Cost\ of\ 3 - inch\ Cube = \underbrace{25}_{=\frac{Cost}{Inch^3}} \times \underbrace{27}_{\substack{No.of \\ Cubic\ Inches}} = \$675$$

Method II

$$Value\ of\ 3 - inch\ cube = 200 \times \underbrace{\frac{1}{2^3}}_{\substack{Vol\ of \\ 2-inch\ cube}} \times \underbrace{3^3}_{\substack{Vol\ of \\ 3-inch\ Cube}} = 200 \times \frac{1}{8} \times 27 = \$675$$

I. Costs

Example 2.40: Costs with Surface Area

A. A toy is packed into a cubical wooden container of side 4 units. Find the cost of packing material needed to cover the container at a cost of Rs. 2 per square unit.
B. A cuboid has length 1 meter, width 2 meters, and height 3 meters. Find the cost of painting it at Rs. 40 per square meter.

Part A

$$Cost = 2(SA) = 2(6s^2) = 12s^2 = 12 \times 4^2 = 12 \times 16 = 192\ Rs.$$

Part B

$$SA = 2(1 \times 2 + 2 \times 3 + 3 \times 1) = 2(2 + 6 + 3) = 2(11) = 22\ m^2$$
$$Cost = 22 \times 40 = 880\ Rs.$$

Example 2.41: Costs with Volume
The maximum amount of a gas that can be stored in a tank is 100 kg per cubic feet. The cost of gas is 5000 rupees per ton. Find the value of gas stored in a tank that is 3 feet by 5 feet by 4 feet, if the tank is half full.

$$V = lbh = 3 \times 5 \times 4 = 60 \ feet^3$$
$$Weight \ of \ Gas \ in \ Full \ Tank = 6000 \ kg = 6 \ Tons$$
$$Weight \ of \ Gas \ in \ Half \ Tank = 3 \ Tons$$
$$Cost \ of \ Gas = 3 \times 5000 = 15,000 \ Rs.$$

Example 2.42
What is the cost of building a 3-foot high, and 1-foot-wide retaining wall on ($not \ surrounding$) a 31-feet by 36-feet rectangular roof at a cost of Rs. 150 per cubic foot?
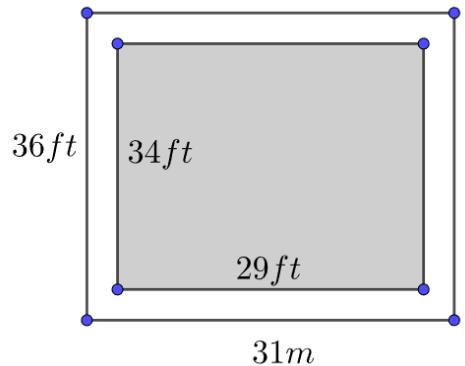
The area of the wall
$$= Outer \ Area - Inner \ Area$$
$$= (31 \cdot 36) - (29 \cdot 34) = 1116 - 986 = 130 \ ft^2$$

Volume
$$= 130 \ ft^2 \times 3 = 390 \ ft^3$$

Cost
$$= 150 \times 390 = 58,500 \ Rs.$$

2.2 Diagonals

A. Diagonals

2.1: Diagonals in a Cube
The shortest diagonal is $\sqrt{2}$ times the side length
The longest diagonal is $\sqrt{3}$ times the side length

Without loss of generality[2], consider a cube with side length 1.

Shortest Diagonal is a face diagonal (all faces are congruent).
Consider face $HGCD$, which is a square,
By Pythagoras in right ΔDCG, shortest Diagonal:
$$= DG = \sqrt{3^2 + 3^2} = \sqrt{9 \times 2} = 3\sqrt{2}$$

Longest Diagonal connects a top vertex with a bottom vertex.
There are four longest diagonals, one of which is
$$AG$$
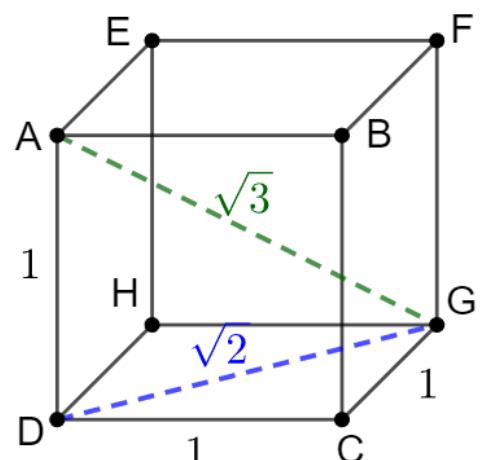
By Pythagoras in right ΔADG, longest Diagonal:
$$AG = \sqrt{DG^2 + AD^2}$$

Substitute:

---

[2] Because all cubes are similar, we can take a cube of any side length. Here, we take a cube of side length 1.

$$= \sqrt{\left(3\sqrt{2}\right)^2 + 3^2} = \sqrt{3^2 \times 2 + 3^2} = \sqrt{3^2 \times 3} = 3\sqrt{3}$$

Example 2.43: Fractions

A. Find the length of the shortest and longest diagonal of a cube with length $\frac{5}{7}$ units
B. Find the length of the shortest and longest diagonal of a cube with length $\sqrt{2}$ units
C. Find the length of the longest diagonal of a cube with surface area 150 square units.

Part A

$$\text{Shortest Diagonal} = \frac{5}{7} \times \sqrt{2} = \frac{5\sqrt{2}}{7}$$
$$\text{Longest Diagonal} = \frac{5}{7} \times \sqrt{3} = \frac{5\sqrt{3}}{7}$$

Part B

$$\text{Shortest Diagonal} = \sqrt{2} \times \sqrt{2} = 2$$
$$\text{Longest Diagonal} = \sqrt{2} \times \sqrt{3} = \sqrt{6}$$

Part C

$$6s^2 = 150 \Rightarrow s^2 = 25 \Rightarrow s = 5 \Rightarrow Longest\ Diagonal = 5\sqrt{3}$$

Example 2.44

A cube has side length $\sqrt{2}$ units. Find the sum of the lengths of all the diagonals.

$$Face\ Diagonal = \sqrt{2} \times \sqrt{2} = 2$$
$$No.\ of\ Face\ Diagonals = 6 \times 2 = 12$$

$$Space\ Diagonal = \sqrt{2} \times \sqrt{3} = \sqrt{6}$$
$$No.\ of\ Space\ Diagonals = 4$$

Total Length

$$= 2 \cdot 12 + \sqrt{6} \cdot 4 = 24 + 4\sqrt{6}$$

2.45: Shortest and Longest Diagonal

A cuboid with length l, width w and height h has:
➢ Shortest Diagonal $= \sqrt{l^2 + w^2}$
➢ Longest Diagonal $= \sqrt{l^2 + w^2 + h^2}$

Shortest Diagonal is a face diagonal (all faces are congruent).
Consider face $HGCD$, which is a square,

By Pythagoras in right ΔDCG, shortest Diagonal:
$$= \text{DG} = \sqrt{l^2 + w^2}$$
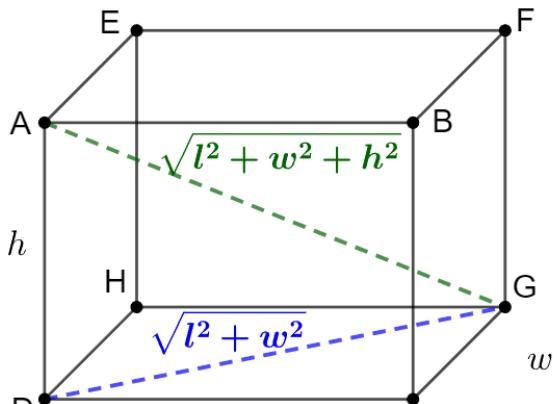
Longest Diagonal connects a top vertex with a bottom vertex.
There are four longest diagonals, one of which is
$$AG$$

By Pythagoras in right ΔADG, longest Diagonal:
$$AG = \sqrt{DG^2 + AD^2}$$
$$= \sqrt{l^2 + w^2 + h^2}$$

Example 2.46
Find the sum of the lengths of the diagonals of a cuboid with length 3 units, width 4 units and height 5 units.

$$Shortest\ Diagonal = \sqrt{3^2 + 4^2} = \sqrt{16 + 25} = \sqrt{41}$$
$$Longest\ Diagonal = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{65}$$

Sum of the lengths
$$= 12\sqrt{41} + 4\sqrt{65}$$

2.3 Cutting, Slicing and Changing Objects
A. Cutting

Example 2.47
A solid box is 15 cm by 10 cm by 8 cm. A new solid is formed by removing a cube 3 cm on a side from each corner of this box. What percent of the original volume is removed? (AMC 10A 2003/3)

We can calculate the percentage directly, which lets us cancel:
$$\frac{Removed\ Volume}{Total\ Volume} = \frac{3^3 \times 8}{15 \times 10 \times 8} = \frac{9}{50} = \frac{18}{100} = 18\%$$
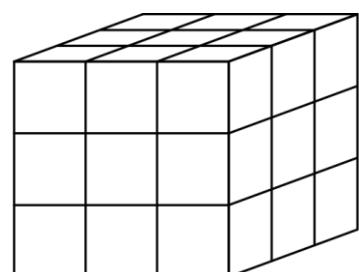
Example 2.48
A one-foot cube is to be divided completely into 6-inch cubes. All these 6-inch cubes are to be placed one on top of another. How many feet high is it from the bottom to the top?

$$No.\ of\ Cubes = 2 \times 2 \times 2 = 8$$
$$Height = \frac{8}{2} = 4\ Feet$$

Example 2.49
Each corner cube is removed from this 3 cm × 3 cm × 3 cm cube. The surface area of the remaining figure is (AMC 8 1997/21)

Removing the cubes does not change the surface area. Hence, the surface area
$$= 6s^2 = 6(3^2) = 6(9) = 54$$

Example 2.50

A solid cube has side length 3 inches. A 2-inch by 2-inch square hole is cut into the center of each face. The edges of each cut are parallel to the edges of the cube, and each hole goes all the way through the cube. What is the volume, in cubic inches, of the remaining solid? (AMC 10A 2010/17)

Direct Counting

First cut will remove
$$2 \times 2 \times 3 = 12$$

Second cut will remove
$$2 \times 2 \times 1 = 4$$

Third cut will remove
$$2 \times 2 \times 1 = 4$$

Total volume removed
$$= 12 + 4 + 4 = 20$$

Total volume remaining
$$= 3^3 - 20 = 27 - 20 = 7$$
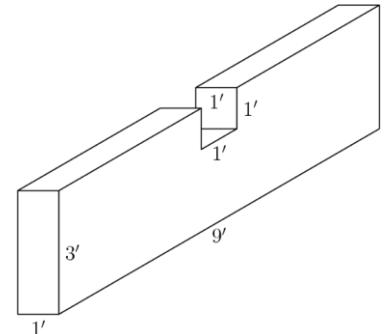
Venn Diagrams
$$3^3 - 3(2 \times 2 \times 3) + 2(2 \times 2 \times 2) = 27 - 36 + 16 = 7$$

Example 2.51

All six sides of a rectangular solid were rectangles. A one-foot cube was cut out of the rectangular solid as shown. The total number of square feet in the surface of the new solid is how many more or less than that of the original solid? (AMC 8 1991/15)
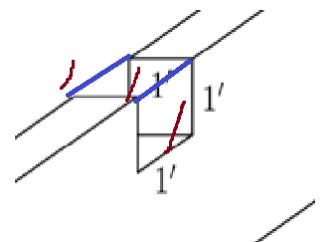
The surface which was removed has an area of
$$1 \ on \ the \ right$$
$$1 \ on \ the \ left$$
$$1 \ on \ the \ top$$

When the cube is cut out, new surface area gets exposed, which is:
$$1 \ on \ the \ bottom$$
$$1 \ on \ the \ front$$
$$1 \ on \ the \ back$$

The difference is:
$$(1 + 1 + 1) - (1 + 1 + 1) = 3 - 3 = 0$$

B. Changing Dimensions

Example 2.52

One dimension of a cube is increased by 1, another is decreased by 1, and the third is left unchanged. The volume of the new rectangular solid is 5 less than that of the cube. What was the volume of the cube? (AMC 10A

$$Cube: Side\ length = s \Rightarrow V = s^3$$
$$New\ Dimensions: s+1, s-1, s$$
$$V_{New} = (s+1)(s-1)(s) = (s^2 - 1)(s) = s^3 - s$$

From the condition given in the question:
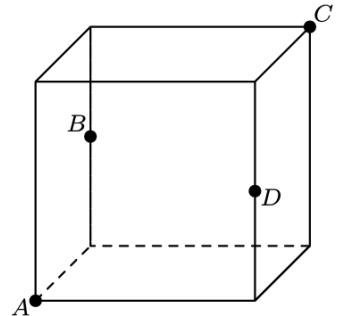
$$V - 5 = V_{New}$$
$$s^3 - 5 = s^3 - s$$
$$s = 5^3$$

C. Slicing

Example 2.53

A cube with side length 1 is sliced by a plane that passes through two diagonally opposite vertices A and C and the midpoints B and D of two opposite edges not containing A or C, as shown. What is the area of quadrilateral $ABCD$? (AMC 10A 2008/21)

By the Pythagorean Theorem in ΔAXD

$$AD = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \sqrt{\frac{5}{4}}$$

Similarly,
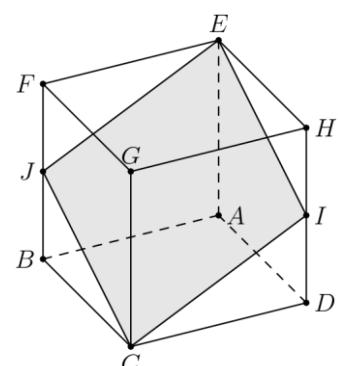
$$CD = BC = AB = \sqrt{\frac{5}{4}}$$

Since the sides are equal, the quadrilateral is a rhombus with

$$Area = \frac{d_1 d_2}{2} = \frac{(AC)(BD)}{2} = \frac{\sqrt{3}\sqrt{2}}{2} = \frac{\sqrt{6}}{2}$$

Example 2.54

In the cube $ABCDEFGH$ with opposite vertices C and E, J and I are the midpoints of edges $\overline{FB}$ and $\overline{HD}$, respectively. Let R be the ratio of the area of the cross-section $EJCI$ to the area of one of the faces of the cube. What is R^2? (AMC 8 2018/24)

Let the side length of the cube be 1.As per previous example:
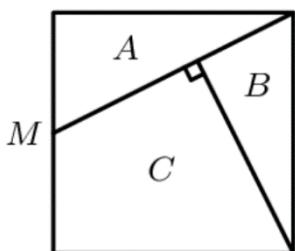
$$Area\ of\ Rhombus = \frac{\sqrt{6}}{2}$$
$$Area\ of\ a\ Face = 1 \times 1 = 1$$

$$R = \frac{\sqrt{6}}{2}:1 = \frac{\sqrt{6}}{2} \Rightarrow R^2 = \frac{6}{4} = \frac{3}{2}$$

Example 2.55

A cubical cake with edge length 2 inches is iced on the sides and the top. It is cut

---

[3] This equation requires setting and solving a cubic equation (which is not difficult to solve once set up). But most questions that we do not require cubic: linear and quadratic equations are far more common.

vertically into three pieces as shown in this top view, where M is the midpoint of a top edge. The piece whose top is triangle B contains c cubic inches of cake and s square inches of icing. What is $c + s$? (AMC 10B 2009/22)

By the Pythagorean Theorem in ΔMSR and ΔMPQ:
$$MR = MQ = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$[QMR] = [PQRS] - [MPQ] - [MSR]$$
$$= 2^2 - \left(\frac{1}{2} \cdot 1 \cdot 2\right) - \left(\frac{1}{2} \cdot 1 \cdot 2\right) = 4 - 1 - 1 = 2$$

We can find RN by calculating the area of ΔQMR as half the product of its base and height:
$$[QMR] = \frac{1}{2}(QM)(RN) \Rightarrow 2 = \frac{1}{2}(\sqrt{5})(RN) \Rightarrow RN = \frac{4}{\sqrt{5}}$$

In right ΔMNR:
$$MN = \sqrt{MR^2 - NR^2} = \sqrt{(\sqrt{5})^2 - \left(\frac{4}{\sqrt{5}}\right)^2} = \sqrt{5 - \frac{16}{5}} = \sqrt{\frac{25-16}{5}} = \sqrt{\frac{9}{5}} = \frac{3}{\sqrt{5}}$$

$$NQ = MQ - MN = \sqrt{5} - \frac{3}{\sqrt{5}} = \frac{5-3}{\sqrt{5}} = \frac{2}{\sqrt{5}}$$

We can find the area of ΔQNR now that we know its base and height:
$$[QNR] = \frac{1}{2}(NQ)(RN) = \frac{1}{2}\left(\frac{2}{\sqrt{5}}\right)\left(\frac{4}{\sqrt{5}}\right) = \frac{4}{5}$$

The quantities required in the question are:
$$c = 2[QNR] = 2 \cdot \frac{4}{5} = \frac{8}{5}$$
$$s = [QNR] + QR^2 = \frac{4}{5} + 4 = \frac{24}{5}$$
$$c + s = \frac{8}{5} + \frac{24}{5} = \frac{32}{5}$$
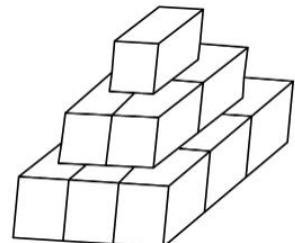
2.4 Composite Figures and Hollow Objects

A. Composite Figures

Example 2.56
An artist has 14 cubes, each with an edge of 1 meter. She stands them on the ground to form a sculpture as shown. She then paints the exposed surface of the sculpture. How many square meters does she paint? (AMC 8 1989/23)

Calculate the area of the sides:
$$Top\ Layer = 1 \times 4 = 4$$
$$Middle\ Layer = 2 \times 4 = 8$$
$$Bottom\ Layer = 3 \times 4 = 12$$
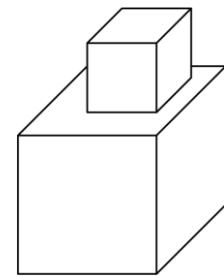
Look at a top down perspective:
$$Top\ of\ the\ cubes = 3 \times 3 = 9$$

$$4 + 8 + 12 + 9 = 33\ m^2$$

Example 2.57
A cube has edge length 2. Suppose that we glue a cube of edge length 1 on top of the big cube so that one of its faces rests entirely on the top face of the larger cube. The percent increase in the surface area (sides, top, and bottom) from the original cube to the new solid formed is:(AMC 8 2000/22)

Surface area of original solid
$$= \underbrace{6}_{Faces} \times \underbrace{2^2}_{Sides} = 24$$

Surface area of smaller solid
$$= \underbrace{6}_{Faces} \times \underbrace{1^2}_{Sides} = 6$$

Increase in area due to small cube:
$$= 6 - \underbrace{1}_{\substack{New \\ Solid}} - \underbrace{1}_{\substack{Old \\ Solid}} = 4$$
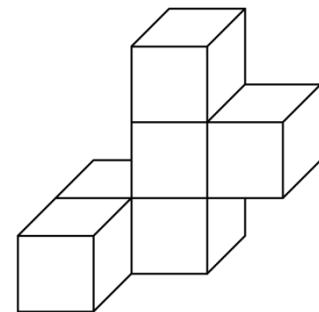
Percentage Increase
$$= \frac{4}{24} = \frac{1}{6} = 16\frac{2}{3}\%$$

Example 2.58
Six cubes, each an inch on an edge, are fastened together, as shown. Find the total surface area in square inches. Include the top, bottom and sides. (AMC 8 2002/22)
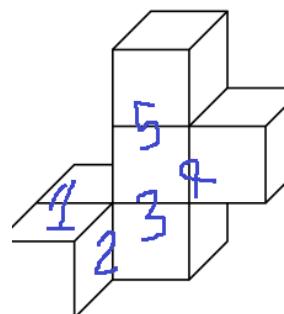
The total number of faces of six cubes would be
$$6 \times 6 = 36$$

But, as seen in the diagram alongside, there are 5 places where a face meets a face. Each face meeting a face reduces the area by two faces.

Hence, the final area
$$= 36 - 2(5) = 36 - 10 = 26$$
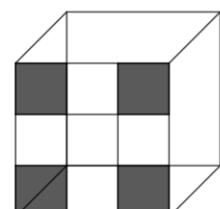
Example 2.59
A cube with 3-inch edges is made using 27 cubes with 1-inch edges. Nineteen of the smaller cubes are white and eight are black. If the eight black cubes are placed at the corners of the larger cube, what fraction of the surface area of the larger cube is white? (AMC 8 2006/18)

The total surface
$$6s^2 = 6(3^2) = 54$$
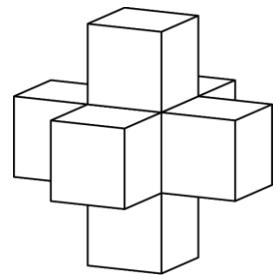
The black surface area
$$= 8(3) = 24$$

Alternatively, we can draw a diagram and note that each face looks exactly the same, and has white fraction
$$= \frac{5}{9}$$

Example 2.60
A shape is created by joining seven unit cubes, as shown. What is the ratio of the volume in cubic units to the surface area in square units? (AMC 8 2008/16)

$$Volume = 7s^3 = 7(1^3) = 7$$

$$Surface\ Area = 7(6s^2) - 7(2) = 42 - 14 = 30$$

$$Ratio = 7:30$$

Example 2.61
A cube with 3-inch edges is to be constructed from 27 smaller cubes with 1-inch edges. Twenty-one of the cubes are colored red and 6 are colored white. If the 3-inch cube is constructed to have the smallest possible white surface area showing, what fraction of the surface area is white? (AMC 8 2014/19)

We can put one cube in the middle row, middle column, middle layer, which means it will have no face exposed.

The remaining 5 cubes can be put in the middle of each face, which means there will be 1 face exposed for each cube.

$$Total = \frac{5}{6s^2} = \frac{5}{6(3^2)} = \frac{5}{54}$$

Example 2.62
Seven cubes, whose volumes are 1, 8, 27, 64, 125, 216, and 343 cubic units, are stacked vertically to form a tower in which the volumes of the cubes decrease from bottom to top. Except for the bottom cube, the bottom face of each cube lies completely on top of the cube below it. What is the total surface area of the tower (including the bottom) in square units? (AMC 10A 2020/10)

The cubes have edge lengths
$$1,2,3,4,5,6,7$$

Their original surface area would be:
$$6(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2)$$
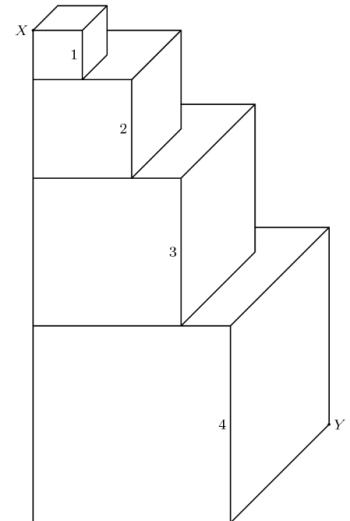
Using the formula $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
$$= 6\frac{7(8)(15)}{6} = 840$$

The area common to two cubes which is getting hidden:
$$= 2(1^2 + 2^2 + \cdots + 6^2) = 2\frac{6(7)(13)}{6} = 182$$

$$840 - 182 = 658$$

Example 2.63

Four cubes with edge lengths 1, 2, 3, and 4 are stacked as shown. What is the length of the portion of $\overline{XY}$ contained in the cube with edge length 3? (AMC 10A 2014/19)
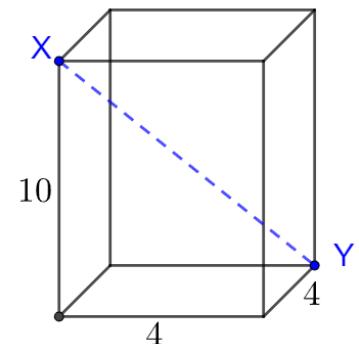
Note: X is at the top left in the diagram and Y is at the bottom right.

The longest diagonal in a cuboid with length l, width w and height h is
$$\sqrt{l^2 + w^2 + h^2} = \sqrt{4^2 + 4^2 + 10^2} = \sqrt{132} = 2\sqrt{33}$$
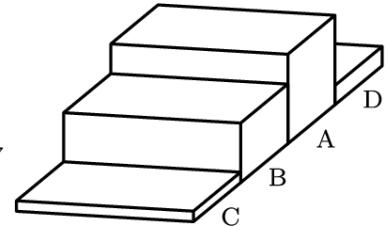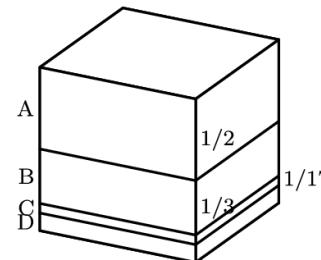
Since the diagonal is proportional to height, the length is simply:
$$2\sqrt{33} \times \frac{3}{10} = \frac{3}{5}\sqrt{33}$$

Example 2.64

A one-cubic-foot cube is cut into four pieces by three cuts parallel to the top face of the cube. The first cut is $\frac{1}{2}$ foot from the top face. The second cut is $\frac{1}{3}$ foot below the first cut, and the third cut is $\frac{1}{17}$ foot below the second cut. From the top to the bottom the pieces are labeled A, B, C, and D. The pieces are then glued together end to end as shown in the second diagram. What is the total surface area of this solid in square feet? (AMC 8 2009/25)

The top of the second solid will be:
$$1 + 1 + 1 + 1 = 4$$

The bottom of the second solid will be:
$$1 + 1 + 1 + 1 = 4$$

The right and left are:
$$1 + 1 = 2$$

The front and back are:
$$\frac{1}{2} + \frac{1}{2} = 1$$
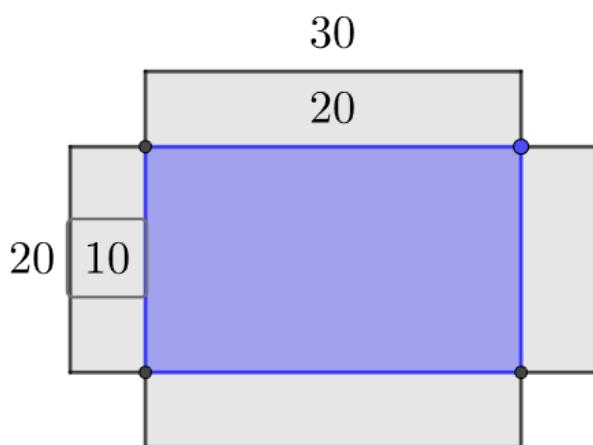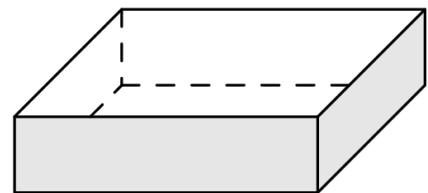
The total is:

$$4 + 4 + 2 + 1 = 11$$

B. Hollow Objects
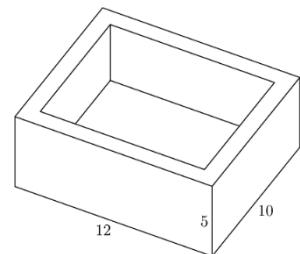
Example 2.65
Square corners, 5 units on a side, are removed from a 20 unit by 30 unit rectangular sheet of cardboard. The sides are then folded to form an open box. The surface area, in square units, of the interior of the box is (AMC 8 1993/17)

Example 2.66
Isabella uses one-foot cubical blocks to build a rectangular fort that is 12 feet long, 10 feet wide, and 5 feet high. The floor and the four walls are all one foot thick. How many blocks does the fort contain? (AMC 8 2013/18)

We use complementary counting. If the fort is filled, we get:
$$12 \times 10 \times 5 = 600$$

The empty space in the fortress:
$$10 \times 8 \times 4 = 320$$
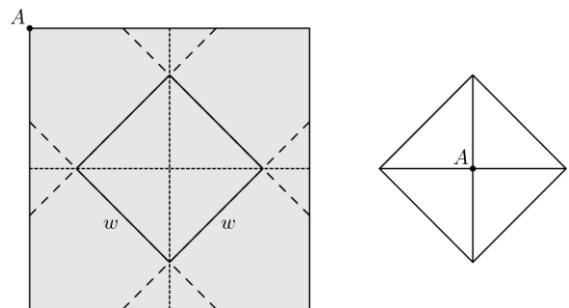
The number of blocks needed is:
$$600 - 320 = 280 \; blocks$$

Example 2.67

A closed box with a square base is to be wrapped with a square sheet of wrapping paper. The box is centered on the wrapping paper with the vertices of the base lying on the midlines of the square sheet of paper, as shown in the figure on the left. The four corners of the wrapping paper are to be folded up over the sides and brought together to meet at the center of the top of the box, point A in the figure on the right. The box has base length w and height h. What is the area of the sheet of wrapping paper? (Write your answer in terms of one or both of w and h) (AMC 10B 2018/15)
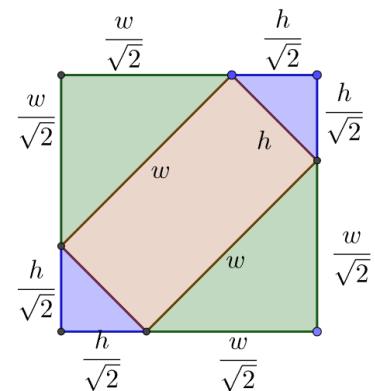
Divide the wrapping paper into four congruent rectangles. Using the property, that in a $45 - 45 - 90$ triangle, the legs are $\frac{1}{\sqrt{2}}$ of the hypotenuse, find the lengths shown.

The side length of the square formed is:

$$= s = \left(\frac{h}{\sqrt{2}} + \frac{w}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}}(h+w)^2$$

The area of the square

$$= s^2 = \frac{1}{2}(h+w)^2$$

We need four squares to form the entire wrapping paper, with area:

$$4 \times \frac{1}{2}(h+w)^2 = 2(h+w)^2$$

Example 2.68

Let B be a right rectangular prism (box) with edges lengths 1, 3, and 4, together with its interior. For real $r \geq 0$, let $S(r)$ be the set of points in 3-dimensional space that lie within a distance r of some point B. The volume of $S(r)$ can be expressed as $ar^3 + br^2 + cr + d$, where a, b, c, and d are positive real numbers. What is $\frac{bc}{ad}$? (AMC 10B 2020/20)

3. 3D SHAPES

3.1 Cylinder
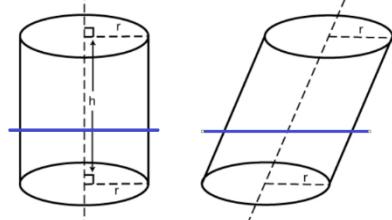
A. Basics

> ### 3.1: Volume
> The volume of a right circular cylinder with radius r and height h is
> $$V = \pi r^2 h$$

Oblique Cylinder

> ### Example 3.2
> Find, in terms of π, the volume of a cylinder with
>   A.   height 3 feet and radius 2 feet
>   B.   height 4 feet and diameter 2 feet
>   C.   height 5 feet and radius 6 inches
>   D.   height 7 feet and radius 3 inches.
>   E.   height $\dfrac{1}{6.28}$ feet and radius 4 inches

$$V_A = \pi r^2 h = \pi \times (2^2) \times 3 = 12\pi \ feet^3$$
$$V_B = \pi r^2 h = \pi \times (1^2) \times 4 = 4\pi \ feet^3$$
$$V_C = \pi r^2 h = \pi \times \left(\frac{1}{2}\right)^2 \times 5 = \frac{5}{4}\pi \ feet^3$$
$$V_D = \pi r^2 h = \frac{22}{7} \times \left(\frac{1}{4}\right)^2 \times 7 = \frac{22}{16} = \frac{11}{8} \ feet^3$$
$$V = \pi r^2 h = 3.14 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{6.28} = \frac{1}{9} \times \frac{1}{2} = \frac{1}{18} \ feet^3$$

> ### 3.3: General Cylinder
> A cylinder has a base and rises above that base.
>   ➢   A circular cylinder has a circle for a base.
>   ➢   A right cylinder has sides which make a right angle with its base.
>   ➢   An oblique cylinder has sides which do not make a right angle with its base.

> ### 3.4: Cavalieri's Principle
> Two solids between parallel planes have the same volume if every plane parallel to these two planes has a cross section area which is equal for both solids.

In the diagram alongside, the blue cross section for both cylinders is a circle with same area, and hence they have the same volume.

B. Surface Area of Base

3.5: Base Surface Area
The surface area of a right circular cylinder is:
$$2\pi r^2$$
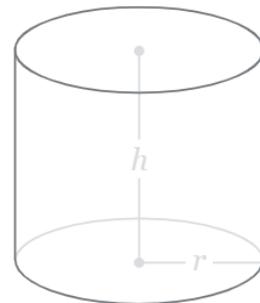
There are two bases of a cylinder:
$$Top\ Base + Bottom\ Base$$

Both the top and the bottom base are circles. Hence, apply the formula for the area of a circle:
$$BSA = Top\ Base + Bottom\ Base = \pi r^2 + \pi r^2 = 2\pi r^2$$

Example 3.6
Find, in terms of π, the base surface area of a cylinder with
  A. radius 3 feet.
  B. height 4 feet and radius 2 feet.
  C. height 97 feet and radius 6 inches.

$$SA_A = 2\pi r^2 = 2\pi \times (3^2) = 18\pi\ feet^2$$
$$SA_B = 2\pi r^2 = 2\pi \times (2^2) = 8\pi\ feet^2$$
$$= 2\pi r^2 = 2\pi \times \left(\frac{1}{2}\right)^2 = \frac{1}{2}\pi\ feet^2$$

3.7: Curved Surface Area
The curved surface area of a right circular cylinder is:
$$\pi r^2 h$$
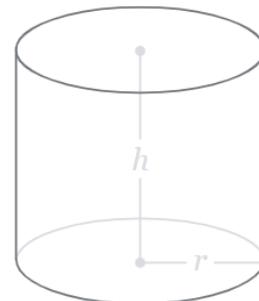
The curved surface area represents the sides of a cylinder. If you cut a cylinder along its height, and unwrap, you will get a rectangle with:
$$Width = h$$
$$Length = Circumference\ of\ Circle = 2\pi r$$
$$CSA = \underbrace{h}_{\substack{Width\ of\\Rectangle}} \times \underbrace{2\pi r}_{\substack{Length\ of\\Rectangle}} = 2\pi rh$$

Example 3.8
Find, in terms of π, the curved surface area of a cylinder that has
  A. height 4 feet and radius 3 feet.
  B. height 4 feet and radius 3 inches.
  C. height 3 feet and diameter 4 inches.

$$CSA_A = 2\pi rh = 2\pi \times 3 \times 4 = 24\pi\ feet^2$$
$$CSA_B = 2\pi rh = 2\pi \times 4 \times \frac{1}{4} = 2\pi\ feet^2$$
$$CSA_C = 2\pi rh = 2\pi \times 3 \times \frac{1}{6} = \pi\ feet^2$$

C. Using the Formula
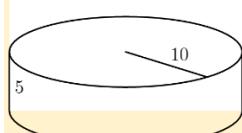
Example 3.9
Find the volume and surface area of a a cylindrical milk can with radius 7 cm and height 2 cm.
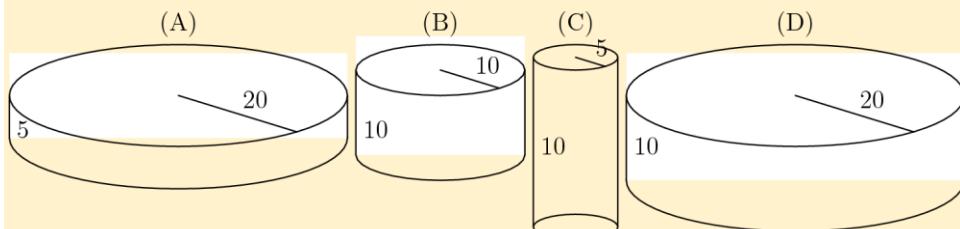
$$SA = 2\pi rh + 2\pi r^2 = 2\pi(7)(2) + 2\pi(7^2) = 126\pi \; cm^2$$
$$V = \pi r^2 h = \pi \times 7^2 \times 2 = 98\pi \; cm^3$$

Example 3.10

Which cylinder has twice the volume of the cylinder shown above? (AMC 8 1992/16)

(E) None of the above

The volume of the original cylinder is:
$$V = \pi r^2 h = \pi(10^2)(5) = 500\pi$$

The volumes of the given options are:
$$Option \; A: \pi(20^2)(5) = 2000\pi \Rightarrow Incorrect$$
$$Option \; B: \pi(10^2)(10) = 1000\pi \Rightarrow Correct$$
$$Option \; C: \pi(5^2)(10) = 250\pi \Rightarrow Incorrect$$
$$Option \; D: \pi(20^2)(10) = 4000\pi \Rightarrow Incorrect$$

D. Reverse Calculations

Example 3.11: Reverse Calculations
What is the height of a cylinder with surface area 126π and a radius 2?

$$SA = 2\pi r^2 + 2\pi rh = 2\pi(r^2 + rh)$$
$$SA = 2\pi(r^2 + rh)$$
$$126\pi = 2\pi \, (2^2 + 2h)$$
$$63 = 4 + h$$
$$h = 59$$
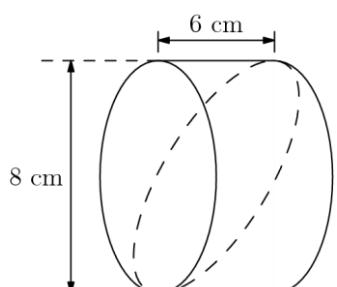
Example 3.12
Jerry cuts a wedge from a 6-cm cylinder of bologna as shown by the dashed curve.

Which answer choice is closest to the volume of his wedge in cubic centimeters? (AMC 8 2008/21)
$(A)48$ $(B)75$ $(C)151$ $(D)192$ $(E)603$

By symmetry, the volume of the wedge is half of the volume of the cylinder.

$$V_{Wedge} = \frac{1}{2}V_{Cylinder} = \frac{1}{2}\pi r^2 h = \frac{1}{2}\pi(4^2)(6) = 48\pi \approx 151 \Rightarrow Option\ C$$

Example 3.13
Alex and Felicia each have cats as pets. Alex buys cat food in cylindrical cans that are 6 cm in diameter and 12 cm high. Felicia buys cat food in cylindrical cans that are 12 cm in diameter and 6 cm high. What is the ratio of the volume of one of Alex's cans to the volume one of Felicia's cans? (AMC 8 2019/9)

$$\frac{V_A}{V_F} = \frac{\pi r^2 h}{\pi R^2 H} = \frac{3^2 \times 12}{6^2 \times 6} = \frac{3^2 \times 2}{6^2} = \frac{18}{36} = \frac{1}{2} = 1:2$$

Example 3.14
A company sells peanut butter in cylindrical jars. Marketing research suggests that using wider jars will increase sales. If the diameter of the jars is increased by 25% without altering the volume, by what percent must the height be decreased? (AMC 10A 2004/11)

Let
$$Height = h, New\ Height = H$$
$$Radius = r \Rightarrow New\ Radius = 1.25r = \frac{5}{4}r$$

The volume before the change must equal the volume after the change.
$$\pi \left(\frac{5}{4}r\right)^2 H = \pi r^2 h$$
$$\frac{25}{16}r^2 H = r^2 h$$
$$H = \frac{16}{25}h = \frac{64}{100}h = 64\%\ of\ h$$
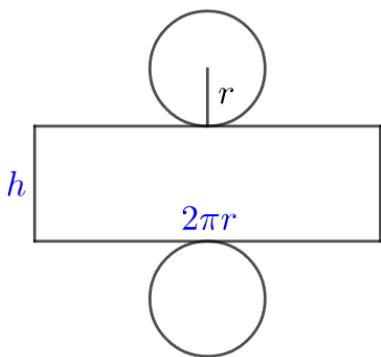
Reduction is:
$$100\% - 64\% = 36\%$$

E. Net of a Cylinder

3.15: Net of a Cylinder
The net of a cylinder is a 2D shape that can be wrapped around a 3D cylinder. The net of a cylinder consists of:
➤ A circle for the base
➤ A second circle for the top
➤ A rectangle when the cylinder is "unwrapped":
➤ One dimension of the rectangle is $2\pi r$, which comes from the circumference of the base of the cylinder.
➤ The other dimension of the rectangle is h, which comes from the

height of the cylinder.
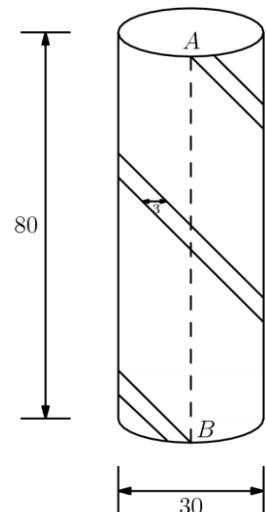
Example 3.16
What the dimensions of the shapes of the net of a cylinder with diameter $3\ m$ and height $2\ m$.

$$Height = 2m$$
$$Circumference = \pi D\ = 3\pi\ m$$

$$Dimensions = 2m \times 3\pi\ m$$

Example 3.17
A white cylindrical silo has a diameter of 30 feet and a height of 80 feet. A red stripe with a horizontal width of 3 feet is painted on the silo, as shown, making two complete revolutions around it. What is the area of the stripe in square feet? (AMC 10A 2004/19)
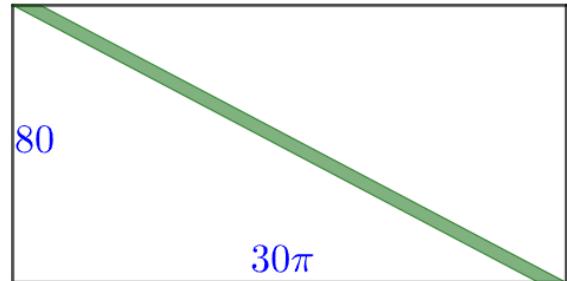
Imagine cutting along the cylinder lengthwise, and unrolling it.
The surface area of the cylinder forms a rectangle with dimensions
$$80 \times 30\pi$$

The stripe will form a parallelogram with
$$Base = 3, Height = 80$$

And hence the area will be:
$$= hb = 3 \times 80 = 240\ ft^2$$

Example 3.18
Two right circular cylinders have the same volume. The radius of the second cylinder is 10% more than the radius of the first. What is the ratio of the heights of the two cylinders? (AMC 10A 2014/9)

$$\pi r^2 h = \pi (1.1r)^2\ H$$
$$h = 1.21\ H$$
$$\frac{h}{H} = 1.21 = \frac{121}{100}$$

Example 3.19
A cylindrical tank with radius 4 feet and height 9 feet is lying on its side. The tank is filled with water to a depth of 2 feet. What is the volume of water, in cubic feet? (AMC 10B 2008/19)

Cross-Section
Draw a diagram of the cross section of the tank.
Let A be the center of the circle.
Let AD be the perpendicular bisector of chord BC.
$$AE = AB = AC = Radius = 4$$
$$DE = 2\ (Given)$$

$$AD = AE - AD = 4 - 2 = 2$$

Area of ΔABC:
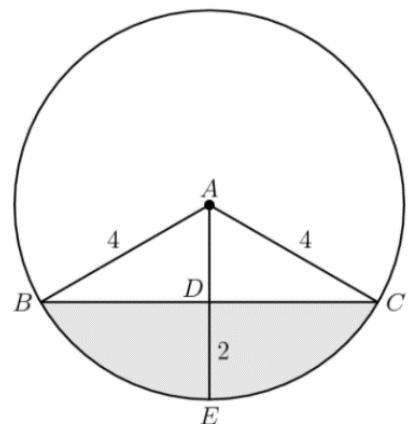
In $Right \; \Delta ADC$:

$$AD = \frac{1}{2}AC \Rightarrow \Delta ADC \; is \; a \; 30 - 60 - 90 \; triangle$$

In a 30-60-90 triangle, the side opposite 60° is $\frac{\sqrt{3}}{2}$ times the hypotenuse:

$$DC = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3} \Rightarrow BC = 4\sqrt{3}$$

$$[ABC] = \frac{1}{2}hb = \frac{1}{2} \times 2 \times 4\sqrt{3} = 4\sqrt{3}$$

Area of $Sector \; ABEC$:

$$\angle DAC = 60° \Rightarrow \angle BAC = 120°$$
$$[ABEC] = \pi r^2 \times \frac{\theta}{360} = \pi \times 4^2 \times \frac{120}{360} = \frac{16\pi}{3}$$

Area of $Segment \; BECD$:

$$[ABEC] - [ABC] = \frac{16\pi}{3} - 4\sqrt{3}$$

Volume

$$= Area \; of \; Base \times Height = 9\left(\frac{16\pi}{3} - 4\sqrt{3}\right) = 48\pi - 36\sqrt{3}$$

3.2 Sphere

A. Surface Area and Volume

3.20: Surface Area and Volume

A sphere with radius r has
$$Surface \; Area = SA = 4\pi r^2$$
$$Volume = V = \frac{4}{3}\pi r^3$$

These numbers can be justified algebraically, but a visual understanding is also important. The links show visually where the formulas come from:
  ➢ Surface Area
  ➢ Volume

Example 3.21

  A. Find the volume of a sphere with radius $\frac{1}{2}$ unit.
  B. If a cricket ball is assumed to be in the shape of a perfect sphere, then what are the volume and the surface area of a ball with radius 3 units?
  C. If the numerical value of the surface area and the volume of a sphere are the same, find the numerical value of its radius.

Part A

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{1}{2}\right)^3 = \frac{4}{3}\pi\left(\frac{1}{8}\right) = \frac{\pi}{6} \; units^3$$

Part B

$$SA = 4\pi r^2 = 4\pi \times 9 = 36\pi \ units^2$$
$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times 3^3 = 36\pi$$

Part C

$$SA = V$$
$$4\pi r^2 = \frac{4}{3}\pi r^3$$
$$r^2 = \frac{1}{3}r^3$$
$$3 = r$$

B. Reverse Calculations

Example 3.22
Find the surface area and the volume of a spherical soccer ball with circumference 10π units.

$$C = 2\pi r \Rightarrow 10\pi = 2\pi r \Rightarrow r = 5$$
$$SA = 4\pi r^2 = 4\pi \times 5^2 = 4\pi \times 25 = 100\pi$$
$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times 5^3 = \frac{500\pi}{3}$$

Example 3.23
If a sphere has surface area 49π, what is the sum of its radius, diameter and circumference?

$$49\pi = 4\pi r^2 \Rightarrow r^2 = \frac{49}{4} \Rightarrow r = \frac{7}{2} \Rightarrow D = 2r = 7$$
$$C = 2\pi r = 2\pi \times 3.5 = 7\pi$$
$$r + D + C = 3.5 + 7 + 7\pi = 10.5 + 7\pi$$

C. Ratios

Example 3.24
Logan is constructing a scaled model of his town. The city's water tower stands 40 meters high, and the top portion is a sphere that holds 100,000 liters of water. Logan's miniature water tower holds 0.1 liters. How tall, in meters, should Logan make his tower? (AMC 10A 2010/12)

There are a lot of zeroes: make sure you do the calculations carefully.
The litre is a unit of volume. So,
$$Ratio \ of \ Volumes = 0.1: 100,000 = 1: 1,000,000$$

Take the cube root to find the ratio of lengths:
$$1: 100$$

And hence, Logan's tower should be:
$$\frac{40}{100} = 0.4 \ meters$$

Example 3.25
A sphere is inscribed in a cube that has a surface area of 24 square meters. A second cube is then inscribed

within the sphere. What is the surface area in square meters of the inner cube? (AMC 10A 2007/21)

Computation

$$6s^2 = 24 \Rightarrow s^2 = 4 \Rightarrow s = 2$$

Diameter of the sphere

$$= Side\ Length\ of\ Cube = 2$$

And the longest diagonal of the inner cube will be equal to the diameter of the sphere.
Recall that the longest diagonal of the cube is $\sqrt{3}$ times the side length.
Hence,

$$Side\ length\ of\ inner\ cube = \frac{2}{\sqrt{3}}$$

And the surface will be:

$$6\left(\frac{2}{\sqrt{3}}\right)^2 = 6 \times \frac{4}{3} = 8$$

Similarity

Let the outer cube have length 1.
Longest Diagonal of Inner Cube

$$= Diameter\ of\ Sphere = Side\ of\ Outer\ Cube = 1$$

Side of Inner Cube

$$= \frac{1}{\sqrt{3}}$$

Ratio of areas will be square of ratio of lengths:

$$Side\ Lengths = 1{:}\frac{1}{\sqrt{3}} \Rightarrow Areas = 1^2{:}\left(\frac{1}{\sqrt{3}}\right)^2 = 1{:}\frac{1}{3} = 3{:}1$$

And hence, the surface area of inner cube

$$= \frac{24}{3} = 8$$

D. Others

Example 3.26
Six spheres of radius 1 are positioned so that their centers are at the vertices of a regular hexagon of side length 2. The six spheres are internally tangent to a larger sphere whose center is the center of the hexagon. An eighth sphere is externally tangent to the six smaller spheres and internally tangent to the larger sphere. What is the radius of this eighth sphere? (AMC 10A 2013/22)
Note: Write your answer in terms of π
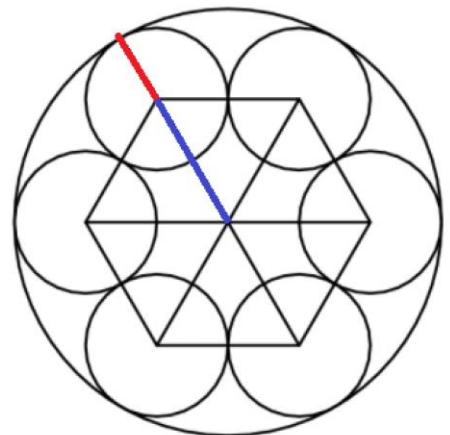
Radius of Seventh Sphere

$$Internally\ Tangent \rightarrow Inside$$

Hence, the six spheres are inside the larger sphere.
Also, we know that the six spheres are at the vertices of a regular hexagon of length 2.

Take a cross-section at the plane of the hexagon (see diagram).
Draw the diagonals of the hexagon, creating six equilateral triangles.

Radius of larger sphere
$$= \underbrace{1}_{\substack{\textbf{Radius of} \\ \textbf{smaller sphere}}} + \underbrace{2}_{\substack{\textbf{Half the diagonal} \\ \textbf{of the hexagon}}} = \textbf{3}$$

Eighth Sphere

$$External\,Tangent \rightarrow Outside$$
Since the eight sphere is externally tangent to the six spheres, its centre must be at the centre of the seventh sphere.
$$Internally\,Tangent \rightarrow Inside$$
Since the eight sphere is internally tangent to the seventh sphere, it must be above the six sphere, and not at their level.

Radius of Eighth Sphere
Consider the following three line segments
Segment 1:
The blue line in the cross-section, which has length
$$= 2$$
Segment 2:
Due to symmetry, the line segment perpendicular to the cross-section above from the centre of the seventh sphere will pass through the centre of the eighth sphere as well.
It has length
$$3 - r, \qquad r = radius\,of\,eighth\,sphere$$
Segment 3:
Connect the outer endpoint of the blue line to the centre of the seventh sphere. It will have length
$$r + 1, \qquad r = radius\,of\,eighth\,sphere$$
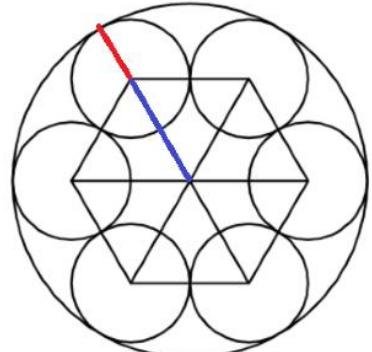
And now note that the above three form a right-angled triangle.
Apply Pythagoras Theorem to get:
$$2^2 + (3 - r)^2 = (r + 1)^2 \Rightarrow 4 + (9 - 6r + r^2) = r^2 + 2r + 1$$

This is a disguised linear equation since the square term cancels, and we get:
$$12 = 8r \Rightarrow r = \frac{3}{2}$$
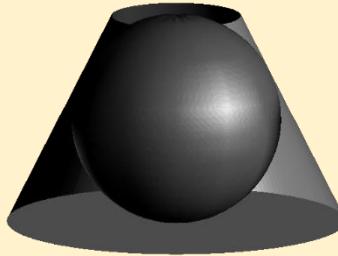
Example 3.27
A sphere with center O has radius 6. A triangle with sides of length 15, 15, and 24 is situated in space so that each of its sides are tangent to the sphere. What is the distance between O and the plane determined by the triangle? (AMC 10A 2019/21)

Example 3.28

A sphere is inscribed in a truncated right circular cone as shown. The volume of the truncated cone is twice that of the sphere. What is the ratio of the radius of the bottom base of the truncated cone to the radius of the top base of the truncated cone? **(AMC 10B 2014/23)**
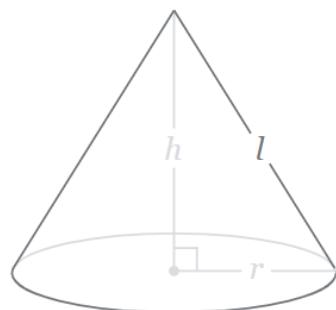
3.3 Cone Basics

A. Volume

The volume of a cone is one-third the volume of a cylinder of corresponding height.

$$V = \frac{1}{3}\pi r^2 h$$

Example 3.29

Find, in terms of π, the volume of a cone with radius 2 feet and height 3 feet.

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 2^2 \times 3 = 4\pi \ feet^3$$

Example 3.30

Find, in terms of π, the volume of a cone with radius 3 inches and height 54 feet.

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times \left(\frac{1}{4}\right)^2 \times 54 = \frac{18}{16}\pi = \frac{9}{8}pi \ feet^3$$

Example 3.31

Find the volume of a cone with radius 6 inches and height 14 feet.

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times \left(\frac{1}{2}\right)^2 \times 14 = \frac{11}{3} \ feet^3$$

B. Base Surface Area

$$BSA = \pi r^2$$

Example 3.32

C. Curved Surface Area

$$CSA = \pi r l$$

3.4 Cone Basics-II

A. Using the Formula

An ice cream cone has a radius of 3 cm, and a slant height of 4 cm. The thickness of the cone itself is assumed to be negligible. What is the
- I. Volume of ice cream required to fill the cone to the brim (without going above, so that it has a flat top)
- II. The surface area of the cone

$$SA = \pi r^2 + \pi r l = \pi \times 9 + \pi \times 3 \times 4 = 9\pi + 12\pi = 21\pi$$

B. Reverse Calculations

What is the radius of a tank with a circular base, a surface area of 122π, and a height 4 units?

$$122\pi = 2\pi \left(2r + 4\right) \Rightarrow 61 = 2r + 4 \Rightarrow r = \frac{57}{2}$$

What is the height of an art monument built in the shape of an inverted cone, with a surface area of 52π units, and radius 3 units?
SA $= \pi r (r + l)$

C. Applications
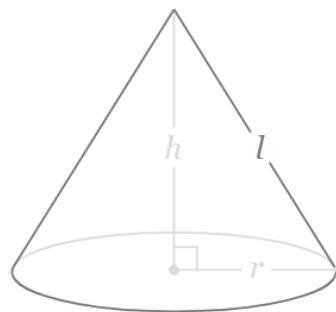
How many cones with diameter 12 inches, and height twice of their radius are needed to have a total volume of a cubic foot.

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{1}{2}\right)^2 \times 1 = \frac{1}{12} \Rightarrow 12 \text{ cones needed}$$

Find the positive difference in the volume of the two cones created by rotating:
- ➢ an isoceles right-angled triangle with legs 3 cm about a leg
- ➢ a equilateral triangle with side 6 cm about its altitude.

$$\text{Difference} = \underbrace{\frac{1}{3}\pi(3^2) \times 3\sqrt{3}}_{Equilateral\ Triangle} - \underbrace{\frac{1}{3}\pi(3^2) \times 3}_{Right\ Triangle} = 9\pi\sqrt{3} - 9\pi = 9\pi(\sqrt{3} - 1)$$

D. Slant Height

Example 3.39

Find the slant height of a cone:
with radius 5 units and height 12 units.

Dropping a perpendicular from the vertex to the base makes a right-angled triangle with:
➢ the radius (5) as one leg
➢ the height(12) as the other leg.
➢ the slant height as the hypotenuse

Applying Pythagoras in the right triangle above:
Slant height = hypotenuse

$$= \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

3.1: Slant Height of a Cone

with radius r units and height h units.

As in the previous question (see it first), dropping a perpendicular from the vertex to the base makes a right-angled triangle with:
➢ the radius (r) as one leg
➢ the height(h) as the other leg.
➢ the slant height(l) as the hypotenuse

Applying Pythagoras in the right triangle above:
$$\textbf{Slant height = hypotenuse} = \sqrt{r^2 + h^2}$$

The slant height of a right circular cone is the square root of the sum of the squares of its radius and its height.

Example 3.40

Find the slant height of a cone:
with radius 3 units and height 4 units.
with radius 9 units and height 40 units.
with radius 7 units and height 24 units.

$$\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$
$$\sqrt{9^2 + 40^2} = \sqrt{81 + 1600} = \sqrt{1681} = 41$$
$$\sqrt{7^2 + 24^2} = \sqrt{49 + 576} = \sqrt{625} = 25$$

Integer values (like the answers to the above questions) that are the sides of a right triangle are called Pythagorean triplets.
You should be familiar with Pythagorean triplets less than 100 (there are 17 of them).

Example 3.41

with radius $2h$ units and height $3r$ units.

with radius one-half units and height one-third units.
with radius 0.3 units and height 0.4 units.

$$\sqrt{(2h)^2 + (3r)^2} = \sqrt{4h^2 + 9r^2}$$
$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{9}} = \sqrt{\frac{13}{36}} = \frac{\sqrt{13}}{6}$$
$$\sqrt{0.3^2 + 0.4^2} = \sqrt{0.09 + 0.16} = \sqrt{0.25} = 0.5$$

Shortcut (Pythagorean Triplets)
(3, 4, 5) is a Pythagorean Triplet.
Divide each value above by 10.
Hence, (0.3, 0.4 and 0.5) is also a Pythagorean Triplet.

E. Reverse Calculations – Height and Radius

Example 3.42
What is the radius of a cone with slant height 12, and height 10.

$$12 = \sqrt{r^2 + 10^2} \Rightarrow 144 = r^2 + 10^2 \Rightarrow r = \sqrt{144 - 100} = \sqrt{44} = 2\sqrt{11}$$

Example 3.43
What is the height of a cone with slant height 12, and radius 7.

$$12 = \sqrt{7^2 + h^2} \Rightarrow h = \sqrt{144 - 49} = \sqrt{95}$$

F. Making Cones from Sectors
When converting a sector of a circle with radius r and central angle θ into a cone:
$$Slant\ Height\ of\ Cone = Radius\ of\ Circle$$
$$Radius_{Cone} = Radius_{Sector} \times \frac{\theta}{360}$$

Example 3.44: Converting Sector into Cone
A circular sector of radius 2 and central angle 60° is converted into a cone. The volume of the cone can be written as $\pi \frac{\sqrt{a}}{b}$. Find $a + b$

$$r_{Sector} = 2 \Rightarrow r_{Cone} = 2 \times \frac{60}{360} = 2 \times \frac{1}{6} = \frac{1}{3}$$
$$Slant - Height_{Cone} = r_{Sector} = 2$$
$$Height_{Cone} = \sqrt{2 - \left(\frac{1}{3}\right)^2} = \sqrt{2 - \frac{1}{9}} = \sqrt{\frac{17}{9}} = \frac{\sqrt{17}}{3}$$
$$Volume_{Cone} = \pi r^2 h = \pi \left(\frac{1}{3}\right)^2 \left(\frac{\sqrt{17}}{3}\right) = \pi \frac{\sqrt{17}}{27} \Rightarrow a + b = 17 + 27 = 44$$

Example 3.45
Find the slant height, radius of the cone which can be formed from a 252° sector of a circle of radius 10 by

aligning the two straight sides? (AMC 10 2001/17, Adapted)

$$Slant\ Height\ of\ Cone = Radius\ of\ Circle = 10$$

$$Radius\ of\ Cone = 10 \times \frac{252}{360} = 7$$

The exam question did not ask for the volume, but what if it had? Find the volume of the cone above in terms of π.

$$l = \sqrt{r^2 + h^2} \Rightarrow h^2 = l^2 - r^2 = 10^2 - 7^2 = 100 - 49 = 51 \Rightarrow h = \sqrt{51}$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 49 \times \sqrt{51} = 833\pi\ units^3$$

G. Frustum of a Cone

H. Inscribed Figures

Example 3.46: Cone inscribed in a sphere

A right circular cone is inscribed in a glass sphere with surface area 400π. The distance d from the centre of the sphere to the base of the cone is 6. If the height of the cone is equal to its diameter, then what is the volume of the cone?

$$r = radius\ of\ sphere \Rightarrow 4\pi r^2 = SA \Rightarrow 4\pi r^2 = 400\pi \Rightarrow r = 10$$

The

$$R = Radius\ of\ Cone:$$

$\underset{6}{d}$, $\underset{10}{r}$ and R form a right-angled triangle. We recognize a Pythagorean triplet, and hence

$$R = 8$$

$$V_{Cone} = \pi r^2 h = \pi(8^2)(16) = \pi \times 2^6 \times 2^4 = 1024\pi$$

Example 3.47

A right circular cylinder with its diameter equal to its height is inscribed in a right circular cone. The cone has diameter 10 and altitude 12, and the axes of the cylinder and cone coincide. Find the radius of the cylinder. (AMC 10 2001/21)

Consider a cross-section of the cylinder and cone. Let the radius of the cylinder be
$$r$$
Since the height of the cylinder is equal to its diameter, we must have
$$d = 2r$$
And hence the height of the triangle atop the cylinder must be
$$12 - 2r$$
Since the top of the cylinder is parallel to the base of the cone, the angles are congruent by corresponding angles
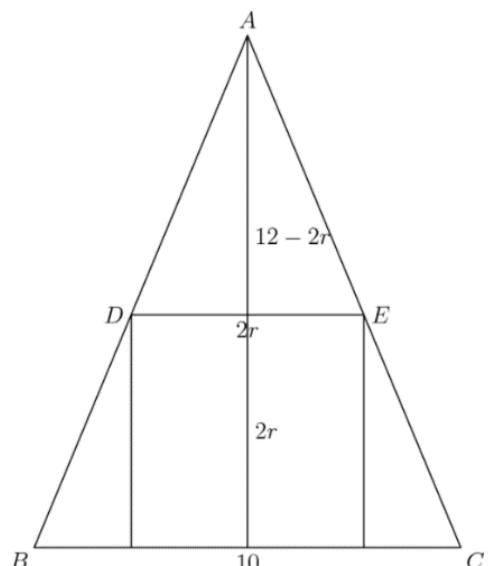
Hence, the triangle above the cylinder is similar to the entire triangle:
$$\frac{h_1}{h_2} = \frac{b_1}{b_2} \Rightarrow \frac{12 - 2r}{12} = \frac{2r}{10} \Rightarrow 120 - 20r = 24r \Rightarrow r = \frac{120}{44} = \frac{30}{11}$$
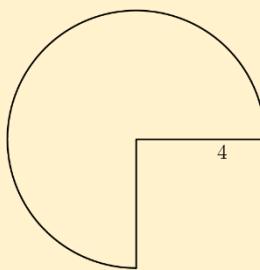
Example 3.48
Jesse cuts a circular paper disk of radius 12 along two radii to form two sectors, the smaller having a central angle of 120 degrees. He makes two circular cones, using each sector to form the lateral surface of a cone. What is the ratio of the volume of the smaller cone to that of the larger? (AMC 10B 2012/17)

Example 3.49
A three-quarter sector of a circle of radius 4 inches along with its interior is the 2-D net that forms the lateral surface area of a right circular cone by taping together along the two radii shown. What is the volume of the cone in cubic inches? (AMC 10B 2020/10)

Example 3.50
An ice cream cone consists of a sphere of vanilla ice cream and a right circular cone that has the same diameter as the sphere. If the ice cream melts, it will exactly fill the cone. Assume that the melted ice cream occupies 75% of the volume of the frozen ice cream. What is the ratio of the cone's height to its radius? (Note: a cone with radius r and height h has volume $\pi r^2 h/3$ and a sphere with radius r has volume $4\pi r^3 / 3$) (AMC 10B 2003/17)

3.5 Pyramids Basics

A. Basics

Definition
A pyramid is a polyhedron formed by connecting a polygonal apex with a base. For example, the three pyramids below have a triangle, a square and a rectangle for a base respectively.

The tip of the pyramid is called the *apex*.

Types of Pyramids
Triangular pyramids are also called **tetrahedrons**.
A square pyramid has a square for a base.
A rectangular pyramid has a rectangle for a base.
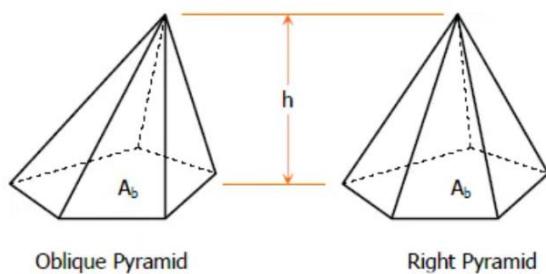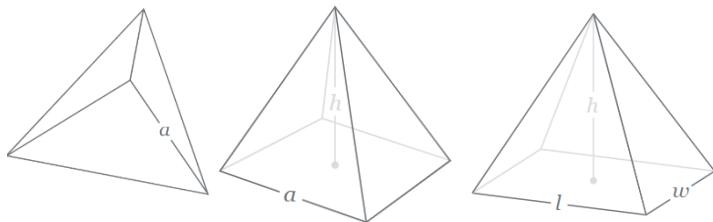
Right versus Oblique Pyramids
A *right pyramid* is one where the apex of the pyramid is above the centre[4] of the base.
An oblique pyramid is any pyramid which is not right. Informally, it looks tilted.

B. Volume: Formula
While there are many kinds of pyramids, there is one formula for the volume.

$$b = Area\ of\ base, h = height \Rightarrow V = \frac{1}{3}bh$$

This formula is applicable irrespective of:
  ➢   the nature of the base
  ➢   whether the pyramid is right or oblique

Concept 3.51
Find the volume of a:
  A.   Square pyramid with side length of base 4 feet, and height 3 feet.
  B.   Rhombic pyramid with base diagonals 3 feet and 4 feet and height 5 feet.

$$V = \frac{1}{3}bh = \frac{1}{3} \times 4^2 \times 3 = 16\ ft^3$$
$$V = \frac{1}{3}bh = \frac{1}{3} \times \frac{3 \times 4}{2} \times 5 = 10\ ft^3$$

C. Surface Area: Formula

Example 3.52
A right pyramid has a square base with side length 10 cm. Its peak is 12 cm above the center of its base. What is the total surface area of the pyramid, in square centimeters? (MathCounts 2007 Warm-Up 18)

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[4] Technically, it should be centroid, not centre. But if you do not know centroid, then just read centre.

Draw a diagram. And let the square base be ABCD.
$$[ABCD] = 10^2 = 100$$
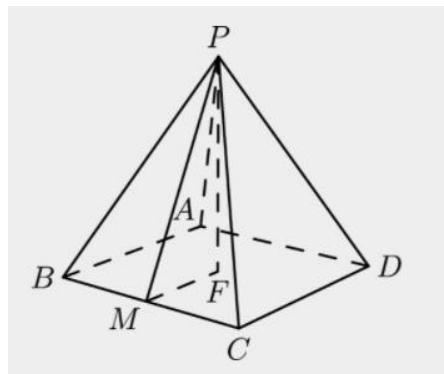
Let M be the midpoint of BC. Let F be the centre of the square. Then, in right ΔPFM:
$$PM = 12, FM = 5 \Rightarrow PM = 13$$

$$[PBC] = \frac{1}{2} \times 13 \times 10 = \frac{130}{2}$$
Surface Area of 4 triangular faces
$$= 4 \times \frac{130}{2} = 260$$

3.6 Pyramids Applications

Example 3.53
Tetrahedron $ABCD$ has $AB = 5, AC = 3, BC = 4, BD = 4, AD = 3$, and $CD = \frac{12}{5}\sqrt{2}$. What is the volume of the tetrahedron?(AMC 10A 2015/21)

Recognize that
$$AB = 5, AC = 3, BC = 4 \Rightarrow \Delta ABC\ is\ right-angled$$
$$AB = 5, AD = 3, BD = 4 \Rightarrow \Delta ABD\ is\ congruent\ to\ \Delta ABC$$
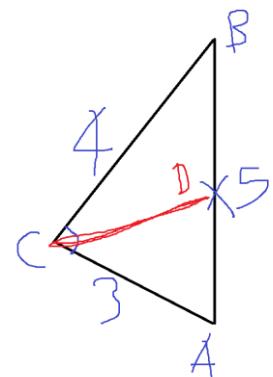
Height of perpendicular to hypotenuse $=h$
$$\frac{1}{2}b_1 h_1 = \frac{1}{2}b_2 h_2 \Rightarrow 5h_1 = 4 \times 3 \Rightarrow h_1 = \frac{12}{5}$$

$$h_{pyramid} = \frac{12}{5}$$
$$b_{pyramid} = 6$$

$$V(Tetrahedron) = \frac{1}{3}bh = \frac{1}{3} \times \frac{12}{5} \times 6 = \frac{24}{5}$$

A. Creating Pyramids

Example 3.54
Three pairwise-tangent spheres of radius 1 rest on a horizontal plane. A sphere of radius 2 rests on them. What is the distance from the plane to the top of the larger sphere? (AMC 10A 2004/25)

B. Ratios in Similar Figures
For similar figures with side lengths u and v:
The ratio of surface areas is the square of their side lengths:
$$\underbrace{u:v}_{Side\ Lengths} \leftrightarrow \underbrace{u^2:v^2}_{Areas}$$
The ratio of volumes is the cubes of their side lengths:

$$\underbrace{u:v}_{Side\ Lengths} \leftrightarrow \underbrace{u^3:v^3}_{Volumes}$$

We can combine the above two to get:

$$\underbrace{u:v}_{Side\ Lengths} \leftrightarrow \underbrace{u^2:v^2}_{Surface\ Areas} \leftrightarrow \underbrace{u^3:v^3}_{Volumes}$$

This property is applicable for any relevant comparable length:
- ➤ Height of two triangles
- ➤ Radius of a circle
- ➤ Diagonal of a quadrilateral
- ➤ Height of a pyramid

And not just side lengths

Example 3.55

A pyramid with a square base is cut by a plane that is parallel to its base and 2 units from the base. The surface area of the smaller pyramid that is cut from the top is half the surface area of the original pyramid. What is the altitude of the original pyramid? (AMC 10B 2007/23)

- A. 2
- B. $2 + \sqrt{2}$
- C. $1 + 2\sqrt{2}$
- D. 4
- E. $4 + 2\sqrt{2}$

$$Ratio\ of\ surface\ areas = \frac{2}{1} \rightarrow Ratios\ of\ altitudes = \sqrt{\frac{2}{1}} = \frac{\sqrt{2}}{1} \approx 1.41$$

Recall that options are arranged in ascending value.

$$Option\ A:\ \frac{2}{2-2} = \frac{2}{0} \rightarrow Not\ Defined$$
$$\frac{3}{3-2} = \frac{3}{1} = 3$$
$$Option\ D:\ \frac{4}{4-2} = \frac{4}{2} = 2$$
$$\frac{6}{4} = \frac{3}{2} = 1.5 > \mathbf{1.41}$$
$$\frac{7}{5} = 1.4 < \mathbf{1.41}$$

$$4 + 2\sqrt{2} \approx 4 + 2 \times 1.41 = 4 + 2.82 = 6.82 \Rightarrow \boldsymbol{Option\ E}$$

Rewrite the original question as follows and solve it: The altitude of the original pyramid can be written as $a + b\sqrt{c}$, where c does not have any perfect squares in it. Find $a + b + c$.

$$\frac{a}{a-2} = \sqrt{2}$$
$$a = a\sqrt{2} - 2\sqrt{2}$$
$$a - a\sqrt{2} = -2\sqrt{2}$$
$$a(1 - \sqrt{2}) = -2\sqrt{2}$$

$$a = \frac{-2\sqrt{2}}{1-\sqrt{2}} = \frac{-2\sqrt{2}-4}{-1} = 4 + 2\sqrt{2}$$

C. Cutting

Example 3.56
A solid tetrahedron is sliced off a wooden unit cube by a plane passing through two nonadjacent vertices on one face and one vertex on the opposite face not adjacent to either of the first two vertices. The tetrahedron is discarded and the remaining portion of the cube is placed on a table with the cut surface face down. What is the height of this object? (AMC 10B 2012/23)

D. Inscribed Figures

Example 3.57
Centers of adjacent faces of a unit cube are joined to form a regular octahedron. What is the volume of this octahedron? (AMC 10A 2006/24)

Draw a diagram.
The Octahedron is composed of two congruent pyramids.

Edge length of the octahedron
$$= e = \sqrt{2} \times \frac{1}{2} = \frac{1}{\sqrt{2}}$$
Area of Base
$$= b = e^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$
Height of pyramid
$$= h = \frac{1}{2}$$
Volume of octahedron is
$$= 2 \times V(Pyramid) = 2 \times \frac{1}{3}bh = 2 \times \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{6}$$
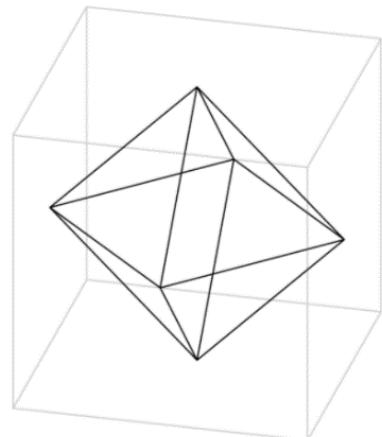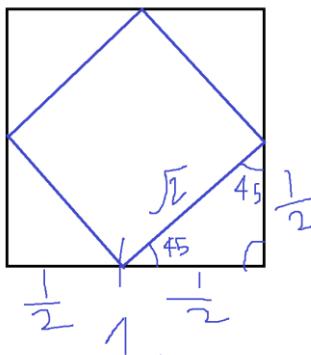
Example 3.58
When the centers of the faces of the right rectangular prism shown below are joined to create an octahedron, what is the volume of the octahedron? (AMC 10B 2015/17)
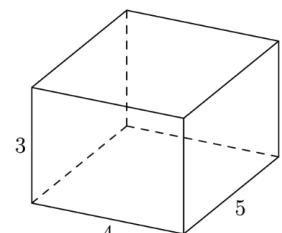
Height of the pyramid
$$= h = \frac{3}{2}$$
Area of the base
$$= \frac{4 \times 5}{2} = 10$$
Volume of octahedron is
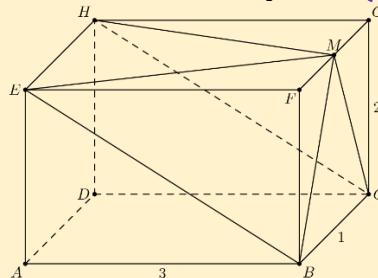
$$= 2 \times V(Pyramid) = 2 \times \frac{1}{3} \times 10 \times \frac{3}{2} = 10$$

Example 3.59

In the rectangular parallelepiped shown, $AB = 3$, $BC = 1$, and $CG = 2$. Point M is the midpoint of $\overline{FG}$. What is the volume of the rectangular pyramid with base $BCHE$ and apex M? (AMC 10B 2018/10)
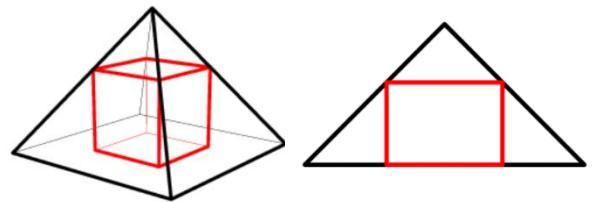
Example 3.60

A pyramid has a square base with sides of length 1 and has lateral faces that are equilateral triangles. A cube is placed within the pyramid so that one face is on the base of the pyramid and its opposite face has all its edges on the lateral faces of the pyramid. What is the volume of this cube? (AMC 10B 2011/22)

Draw a diagram and take a cross-section
  ➢ Since the edges of the cube are on the lateral faces, the vertices are all on the edges.
  ➢ Since the lateral faces are equilateral triangles, the pyramid is a right pyramid.

Take a cross-section of the first diagram through the apex of the pyramid, and two non-adjacent vertices of the base.
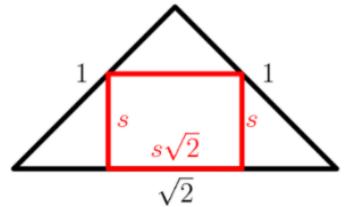
Assume Variables
The edges of the pyramid are length 1, and the apex of the pyramid forms a 90°
between non-adjacent edges. So the larger triangle in the cross section is a
$$45 - 45 - 90 \ Triangle$$
Hence,
$$Length \ of \ Base = \sqrt{2} \times side = \sqrt{2}$$

Let the edge length of the cube be s.
$$Width \ of \ Rectangle = s$$
$$Diagonal \ of \ the \ base \ of \ the \ cube = Length \ of \ Rectangle = s\sqrt{2}$$

Connecting the Dots
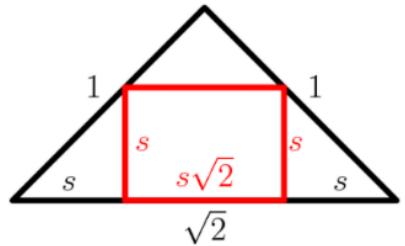
Each of the other three smaller triangles formed are also
$$45 - 45 - 90 \ Triangles$$
Hence, the sgements on the base of the triangle are also equal to the side length of the cube.
$$s + s\sqrt{2} + s = \sqrt{2} \Rightarrow s(2 + \sqrt{2}) = \sqrt{2}$$
$$s = \frac{\sqrt{2}}{2 + \sqrt{2}} = \frac{2\sqrt{2} - 2}{2} = \sqrt{2} - 1$$

The volume of the cube is:
$$s^3 = (\sqrt{2} - 1)^3 = (\sqrt{2} - 1)(2 - 2\sqrt{2} + 1) = (\sqrt{2} - 1)(3 - 2\sqrt{2}) = 3\sqrt{2} - 3 - 4 + 2\sqrt{2} = 5\sqrt{2} - 7$$

Example 3.61
Two distinct regular tetrahedra have all their vertices among the vertices of the same unit cube. What is the volume of the region formed by the intersection of the tetrahedra? (AMC 10A 2011/24)

3.7 Prisms
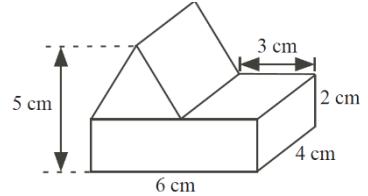
A. Pentagonal Prism

3.8 Mixing

A. Mixing

Example 3.62
A triangular prism is placed on a rectangular prism, as shown. The volume of the combined structure, in cm3, is (Gauss Grade 7 2011/20)

Volume of cuboid
$$= lwh = 6 \times 4 \times 2 = 48$$
Volume of triangular prism
$$= Area \ of \ Base \times H_{Prism} = \frac{1}{2} h_{triangle} b \times H_{Prism} = \frac{1}{2} \times 3 \times 3 \times 4 = 18$$
Hence, total area is
$$48 + 18 = 66$$

Example 3.63
A sphere, a cylinder and a cone have radius 0.5 each. The cylinder and cone have height 6. Find the surface area and the volume of each.

Example 3.64
Shaksham placed the longest possible rods in a cube (side 3 feet), a cuboid (3 feet by 4 feet by 12 feet), and a cylinder (Radius of base 2 feet, height, 3 feet). What is the difference between the length of the longest rod, and the smallest rod?

4. APPLICATIONS
4.1 Ratios
A. Single Object

Example 4.1

Find the edge length of a cube whose
 A. volume is numerically equal to its surface area.
 B. sum of edge lengths is equal to its volume
 C. sum of edge lengths is equal to its surface area

$$V = SA \Rightarrow s^3 = 6s^2 \Rightarrow s = 6$$
$$S(\text{Edge Lengths}) = V \Rightarrow 12s = s^3 \Rightarrow s = \sqrt{12}$$
$$S(\text{Edge Lengths}) = SA \Rightarrow 12s = 6s^2 \Rightarrow s = 2$$

Example 4.2

Cones and Cylinders

Example 4.3

Find the radius of a sphere whose volume is numerically equivalent to its surface area.

$$V = SA \Rightarrow \frac{4}{3}\pi r^3 = 4\pi r^2 \Rightarrow r = 3$$

Example 4.4

Find the smallest possible integer values for radius and height of a cylinder with volume numerically one-fourth times its surface area.

$$4V = SA \Rightarrow 4\pi r^2 h = 2\pi r(r + h) \Rightarrow 2rh = r + h \Rightarrow$$

Try $(1, 1)$ for r and h, which works.

Example 4.5

Find the range of values of the radius of a sphere for which the volume is greater than, equal to, and less than its surface.

$V = SA$	$V < SA$	$V > SA$
$\frac{4}{3}\pi r^3 = 4\pi r^2$	$\frac{4}{3}\pi r^3 < 4\pi r^2$	$\frac{4}{3}\pi r^3 > 4\pi r^2$
$r = 3$	$r < 3$	$r > 3$

Example 4.6

Cubes

Example 4.7

Find the ratio of volumes for a cone and a cylinder if the cone half the height of the cylinder and has the same base.

$$\frac{1}{3}\pi r^2 \frac{h}{2} : \pi r^2 h = 1:6$$

Example 4.8
Find the ratio of the surface area of a sphere with radius 5 units to the sum of the surface areas of two spheres with radius 3 units.

$$\frac{4}{3}\pi (r_1)^2 : 2 \times \frac{4}{3}\pi (r_2)^2 = (r_1)^2 : 2(r_2)^2 = 25:18$$

Example 4.9
A sphere, a cylinder and a cone each have radius r and volume V. Find the heights of the cylinder and the cone in terms of r.

$$\frac{4}{3}\pi r^3 = \pi r^2 h_{Cylinder} = \frac{1}{3}\pi r^2 h_{Cone}$$
$$\frac{4}{3}r = h_{Cylinder} = \frac{1}{3}h_{Cone}$$
$$4r = h_{Cone}$$

Example 4.10
A king asked a solid cone of gold of radius r, and height h. The goldsmith instead made a cone with height r and radius h. When the mistake was pointed out to the goldsmith, he asked his assistant to fix it. His assistant interchanged the diameter and height (of the original specifications), and made a new cone.
Find the ratio of the volumes of the three cones.

$$\boldsymbol{\pi r^2 h : \pi h^2 r : \pi \left(\frac{h}{2}\right)^2 (2r) = r^2 h : h^2 r : \frac{h^2 r}{2}}$$

4.2 Percentages

Example 4.11
Find the change in volume, and surface area, if the length, width and height of a cuboid are:
    A.   increased by 10%, 20% and 30% respectively.
    B.   increased by 10%, decreased by 20% and increased by 0% respectively.

Example 4.12
Cubes A and B (which have the same size) have their edge length increased by 10% and 20% respectively. Find, for volume and surface area,
    A.   the percentage increase for each.
    B.   the percentage point difference between the increase in A and the increase in B.
    C.   the value of B, as a percentage of A.
    D.   the ratio A:B

Example 4.13
A cylinder, cone and sphere have radius r, $2r$ and $3r$ respectively.
    A.   Find the percentage change in the volume and surface area of each, if the radius of each is increased by 10%, 20% and 30% respectively.
    B.   Find the percentage change in the volume and surface area of the cylinder and the cone, if, for each, the radius is increased by 10% and the height is increased by 20%.

Similarity

Similar figures maintain the ratio of individual parameters (length, width height, radius) that the original figure has.

Understanding the properties of similarity can greatly shorten the process of arriving at an answer.

Example 4.14

Find the height of ice-cream (filled in the usual way) occupying 25% of an ice cream cone with radius 3 cm and height 9.6 cm.

By similarity, height and radius of the ice-cream in the cone will be $\sqrt[3]{\frac{1}{4}}$ times the original ice-cream cone.

$$\text{Height} = 9.6 \times \sqrt[3]{\frac{1}{4}} = 9.6 \times \sqrt[3]{\frac{2}{8}} = \frac{9.6\sqrt[3]{2}}{2} = 4.8\sqrt[3]{2}$$

4.3 Hollow and Inscribed Objects

Hollow Objects

Questions on hollow objects require us to take into account the thickness of walls.

$Dimension_{Inside} = \text{DimensionOutside} - Wall_{Thickness}$

Example 4.15

If they are hollow with walls of thickness 1 unit, what are the capacities of a:
  A. Cuboidal tank with dimensions 3.5 feet by 5.5 feet by 7.5 feet.
  B. Sphere with radius 7 (in ml)
  C. Cylinder with radius 400 cm and height 700 cm (in cubic meters)
  D. Cone with radius 60 inches and height 144 inches (in cubic feet)

For each of the shapes in the previous question, calculate the capacity lost due to the thickness of the walls.

Example 4.16

A solid gold sphere of radius 1 cm and a solid silver sphere of radius 6 cm are melted to form a hollow sphere of radius 9 cm. Find the thickness of the sphere.

$$\underbrace{\left(\frac{4}{3}\right)\pi \times 729}_{\substack{\textbf{\textit{Outside Volume}} \\ \textbf{\textit{of Hollow Sphere}}}} = \underbrace{\left(\frac{4}{3}\right)\pi \times 1}_{\substack{\textbf{\textit{Gold Sphere}}}} + \underbrace{\left(\frac{4}{3}\right)\pi \times 216}_{\substack{\textbf{\textit{Silver Sphere}}}} + \underbrace{\left(\frac{4}{3}\right)\pi \times r^3}_{\substack{\textbf{\textit{Inner Volume}} \\ \textbf{\textit{Of Outer Sphere}}}}$$

Divide both sides by $\frac{4}{3}\pi$:

$$729 = 1 + 216 + r^3 \Rightarrow r^3 = 512 \Rightarrow r = 8$$

Thickness

$$= 9 - 8 = 1 \, cm$$

Inscribed Figures

Example 4.17

Find the volume of a cone with maximum volume that can be placed into a cube with side length 9 cm.

Height of the cone

$$= height\ of\ the\ cube = 9\ cm$$

Diameter of the base on the cone

$$= Side\ Length\ of\ the\ Cube = 9\ cm$$

Radius of the cone

$$= \frac{Diameter}{2} = \frac{9}{2} cm$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{9}{2}\right)^2 \times 9 = \frac{243\pi}{4}$$

Example 4.18

A cone of maximum volume is placed inside a cylinder of diameter 9 units, and height 12 units. What is the volume of the cylinder not occupied by the cone?

$$\pi r^2 h \times \frac{2}{3} = \pi \left(\frac{9}{2}\right)^2 \times 12 \times \frac{2}{3} = \pi \times \frac{81}{4} \times 4 \times 2 = 162\pi$$

4.4 Manipulating/Cutting 3D Objects

Equating Volumes (Melting / Converting / Filling /Cutting / Slicing)
If a 3D object is manipulated without changing its volume, this is a very useful property:
- ➢ Melting a shape, or shapes
  - ✓ Cuboid A is melted and 10 smaller spheres are made out of the resulting material.
- ➢ Filling
  - ✓ Conical tank A is filled 75% full by using 50% of the water in tank B (which was 25%) full.
  - ✓ A cuboid box is filled with smaller cuboids.
- ➢ Cutting or slicing a shape into multiple shapes
  - ✓ Cuboid A is sliced into smaller shapes by making 10 cuts along its length, width and height.

Some questions may provide additional information regarding an increase or decrease in the volume:
- ➢ Material wastage (Reduction)
- ➢ Material Increase (Addition)

Example 4.19

A rectangular water tank of dimensions 12 feet by 1 foot by 18 feet is completely emptied into a cubical tank, which is then completely filled. What must be the edge length of the cubical tank?

Cube Root (12 * 1 * 18) = Cube Root (216) = 6

Example 4.20

Rain water harvesting from a building with a roof of 200 square meters results in filling up 60% of a tank of dimensions 3 meters by 5 meters by 2 meters in two days.
   A.   How many cm of rain fell on the roof?
   B.   What is the rate (in cm/hour) of rainfall?

$$H(\text{Rain}) = \frac{V(\text{Rainfall})}{\text{Base Area}} = \frac{V(\text{Tank Filled})}{\text{Base Area}} = \frac{3m \times 5m \times 2m \times 0.6}{200m^2} = \frac{30m \times 6}{200} = \frac{3 \times 100cm \times 6}{200} = 9\ cm$$

$$\frac{9\ \text{cm}}{12\ \text{hours}} = 0.75 \frac{\text{cm}}{\text{hour}}$$

Example 4.21

A cylindrical beaker is 8 cm high and has a radius of 3 cm. How many such beakers of water will it take to fill a spherical tank of radius 6 cm?

$$\frac{V_{Sphere}}{V_{Beaker}} = \frac{4/3\,\pi \times 6^3}{\pi \times 3^2 \times 8} = \frac{4\pi \times 3^3 \times 2^3}{\pi \times 3^3 \times 2^3} = 4$$

We made the following changes in the third step:
- ➢ Moved the 3 in the numerator down to the denominator (since it is in division).
- ➢ Rewrote 6^3 as $3^3 \times 2^3$
- ➢ Rewrote 8 as 2^3

Example 4.22

If a cuboid of dimensions X, Y and Z is melted to form smaller cuboids of dimensions A, B, and C:
- A. How many cuboids will be formed?
- B. What will be the dimensions of the cube that can be formed from the material that is left over?
- C. What will be the percentage increase in surface area considering all the cubes and cuboids formed?

Example 4.23

A sphere (of diameter 4), a cylinder (of diameter 5, and height 6) and a cone (of radius 7, and height 8) are melted.
- A. If the sphere is melted into four equal cylinders of height 6, what are their radii?
- B. If the cylinder is melted into cones which are 4/5th of its height, what will be the number of complete cones that can be made?

Example 4.24
Slicing Cubes and Cuboids

Example 4.25
Slicing Spheres: Hemisphere

Example 4.26
Frustum of a Cone
Find the volume of a truncated right circular cone with radii of 4 cm and 2 cm, and height 3 cm.

$\mathbf{V} = \frac{1}{3}\pi H[r^2 + r'r + (r')^2]$

Substituting $H = 3, r = 4, r' = 2$:

$$= \frac{1}{3}\pi \times 3[4^2 + 4 \times 2 + (2)^2] = 28\pi$$

Frustum of a Pyramid

4.5 Composite Shapes

Example 4.27
Cones and Cylinders
Find the volume and surface area of a rocket. that comprises a cone on a cylinder (both with a base of 7 units). The cylinder has height 3 units, while the cone has a height one-third of that.

Example 4.28
Cones and Hemi-spheres
A chocolate ice-cream scoop in the shape of a perfect hemi-sphere (with radius 4 units) is placed on an ice-cream cone (with the same radius and double the height). Find the volume and surface area of the entire object.

Example 4.29
Find the ratio of the volumes of the three objects created by rotating a right-angled triangle with sides 3:4:5 around each side while the side is at a 90 degree angle to the base.

Example 4.30
What is the total surface area of a sphere (of radius X) perfectly balanced atop a cylinder (of radius A, and height B), resting on its base atop a cube (of side C), resting on its base atop a rectangular parallelepiped (of sides D by E by F)?

4.6 Costs

Example 4.31
How much will it cost to cover 340 boxes that are in a cuboidal shape with edge length 3 feet with packing material that costs Rs. 2/square foot.

$340 * 6 * 3^2 * 2 = $ Rs. 36,720

Example 4.32
Answer the next two questions based on the information below. You may use information from one question in other questions.
A bunker is two meters high, seven meters long, and six meters wide, with walls that are a meter thick.
   A.   How many 3-litre boxes will you need to purchase to paint the outside walls and roof of the bunker, if one liter of paint is sufficient to paint 3 square meters
   B.   How many 5-Litre boxes of paint will you need to paint the inside walls and floor of the bunker?

$$\frac{1}{3}[(2(2 \times 7 + 2 \times 6) + 7 \times 6] = \frac{1}{3}[(2(26) + 42] = \frac{94}{3} = 31\frac{1}{3} = 7 \; boxes$$

S1: [(2(1 * 6 + 1 * 5) + 6 * 5] * 1/5
= [(2(6 + 5) + 30] * 1/5
= [22 + 30] * 1/5
= 52 * 1/5
= Between 10 and 11 Boxes
= 11 Boxes

Example 4.33
How much will it cost to cover 1000 cartons of mangoes (3 feet by 4 feet by 5 feet) if jute cloth used to cover the boxes is available at Rs. (1/36) per square inch?
Cost of packing material (per square feet)
= 1/36 per square inch = 1/36 * 144 per square feet = Rs. 4 per square feet
Total Cost of Packing Material
4 * (1000 * 2[3 * 4 + 4 * 5 + 3 * 5]) * 0.305 * 0.305

$= 4 * 94000 * 0.305 * 0.305$
$= 376000 * 0.305 * 0.305$
$=$ (approx.) $376000 * 0.09$
$= 33840$

Example 4.34

Colour	Albedo (percentage of light reflected)
X	0.4
Y	0.6
Z	0.2

In a laboratory, light shining on a surface is kept at one unit per square foot. To test objects, the laboratory colours two surfaces with X, two surfaces with Y, and two surfaces with Z. A cube with side length 18 inches is colored and placed on a transparent table, while a cuboid with dimensions 24 inches, 36 inches and 48 inches is placed on an opaque table.
What is the difference in the light reflected by the cube and cuboid?

Average Albedo $=$ Avg $(0.2, 0.4, 0.6) = 0.4$
Light Reflected $= 6 * 2.25 * 0.4 = 6 * 0.9 = 5.4$ units

Example 4.35
HCF/LCM
Spheres(of radius 6m, 8m, and 10m) are being painted by three equivalent machines. How many spheres will the machines have painted when all three machines finish their current sphere at the same time?
$$Ratio\ of\ surface\ area$$
$$= 4\pi(r_1)^2 : 4\pi(r_2)^2 : 4\pi(r_3)^2 = (r_1)^2 : (r_2)^2 : (r_3)^2 = 6^2 : 8^2 : 10^2 = 36 : 64 : 100 = 9 : 16 : 25$$

4.7 Summary and Review

Example 4.36
A pizza is cut into eight congruent triangles, and the slices are separated, with a distance of 1 inch between them.
If the pizza has zero thickness, what is the percentage increase in perimeter due to the slices being cut?
If the pizza is 1-inch high, what is the increase in surface area due to the slices being cut?

5. AMC QUESTIONS

Example 5.1
The region consisting of all points in three-dimensional space within 3 units of line segment $\overline{AB}$ has volume 216π. What is the length AB? (AMC 10A 2017/11)

2 Examples