Logic

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1. Logic

A. Summary of Truth Tables

		OR	AND	XOR
A	B	$A \vee B$	$A \wedge B$	$(A \vee B) \wedge \sim(A \wedge B)$
T	T	T	T	F
F	T	T	F	T
T	F	T	F	T
F	F	F	F	F

1.1 Statements & AND

A. Statements

1.1: Statement
A statement is a sentence or a mathematical expression that is definitely true, or definitely false.

Example 1.2
Decide whether each of the following is a statement or not. If it is a statement, decide if it is true or false.

A. Every even number is a multiple of 2.
B. The area of every circle is 3.14 times the square of its radius.
C. Divide x by 5.
D. $\pi \in \mathbb{Q}$

Part A

$$A \text{ is True}$$

Part B

$$Area\ of\ a\ circle = \pi r^2, \pi \approx 3.14., \pi \neq 3.14 \Rightarrow B\ is\ false$$

Part C

This is a mathematical instruction. It is neither true, nor false.
And it is not a mathematical statement.

Part D

$$D\ is\ false$$

B. Negation

1.3: Notation for Negation

$$\sim$$

Is the symbol for negation.

$$\boldsymbol{A}\ is\ true \Rightarrow \sim\boldsymbol{A}\ is\ false$$
$$\boldsymbol{A}\ is\ false \Rightarrow \sim\boldsymbol{A}\ is\ true$$

Example 1.4
Consider the statement **X** given below. In each part, find the negation of **X**, written $\sim\boldsymbol{X}$. Answer each part separately.

A. $X =$ At least five students are going to the picnic.
B. $X =$ The number 6 is a prime number.

C. $X =$ Boston is a city in USA.

Part A

$$\sim\boldsymbol{X}: At\ most\ four\ students\ are\ going\ to\ the\ picnic$$

Part B

$$\sim\boldsymbol{X}: The\ number\ 6\ is\ not\ a\ prime\ number$$

Part C

$$\sim\boldsymbol{X}: Boston\ is\ not\ a\ city\ in\ USA$$

C. Writing Statements using NOT conditions

Example 1.5

$$4 \neq 5$$

$$\boldsymbol{P} = 4\ is\ equal\ to\ 5$$
$$\sim\boldsymbol{P}$$

Example 1.6

The matrix $\boldsymbol{X}$ is not invertible

$$\boldsymbol{P} = The\ matrix\ \boldsymbol{X}\ is\ invertible.$$
$$\sim\boldsymbol{P}$$

Example 1.7

$$x < y$$

$$\boldsymbol{P} = x\ is\ greater\ than\ or\ equal\ to\ y$$
$$\sim\boldsymbol{P}$$

D. Truth Values

1.8: True or False

Any statement in logic is either true or false. It cannot be both at the same time.

Example 1.9

Is

$$At\ least\ five\ students\ are\ not\ going\ to\ the\ picnic.$$

the negation of

$$At\ least\ five\ students\ are\ going\ to\ the\ picnic.$$

Suppose there are 20 students.

$$\boldsymbol{X}\ is\ true \Rightarrow 5\ students\ are\ going, 15\ students\ are\ not\ going$$

Suppose $\sim X$ is represented by $At\ least\ 5\ students\ are\ not\ going\ to\ the\ picnic$. We can then have the following situation:

$$5\ students\ not\ going, 15\ students\ going \Rightarrow \boldsymbol{X}\ is\ true$$

Then we have a situation where

$$\sim\boldsymbol{X} \Rightarrow \boldsymbol{X} \Rightarrow Contradiction$$

1.10: Notation for True and False

$$\boldsymbol{A} \rightarrow A \text{ is true}$$
$$\sim\boldsymbol{A} \rightarrow A \text{ is not true}$$

E. AND

1.11: Notation for AND

$$\wedge$$

For an AND condition to be true, both parts of the condition must be true.

A	B	$\boldsymbol{A} \wedge \boldsymbol{B}$
T	F	F
F	T	F
T	T	T
F	F	F

Example 1.12

$$You\ can\ go\ to\ London\ if\ \underbrace{you\ have\ a\ valid\ flight\ ticket}_{A},and\ \underbrace{a\ valid\ visa}_{B}.$$

The statement above is true.

A. I have a valid flight ticket, but I cannot go to London. What can you conclude about my visa?
B. I cannot go to London, hence I do not have a valid flight ticket. Is this conclusion correct?

Part A

$$A \text{ is true}, \qquad \boldsymbol{A} \wedge \boldsymbol{B} \text{ is } false \Rightarrow Row\ 1 \text{ is applicable}$$

Hence, you do not have a valid visa.

A	B	$\boldsymbol{A} \wedge \boldsymbol{B}$
T	**F**	$\boldsymbol{F}$
F	T	F
T	**T**	$\boldsymbol{T}$
F	F	F

Part A

$$\boldsymbol{A}\quad \boldsymbol{B} is\ false \Rightarrow Row\ 1\ and\ Row\ 4\ are\ possible$$

I do not have a valid flight ticket is equivalent to:

$$4A\ is\ false \Rightarrow Correct\ in\ Row\ 4, Not\ Correct\ in\ Row\ 1 \Rightarrow Not\ True\ in\ general$$

A	B	$\boldsymbol{A} \wedge \boldsymbol{B}$
T	F	F
F	T	F
T	T	T
F	F	F

F. Maths

Example 1.13

Decide whether each of the following statements is true or false. If the statement is true, prove the statement. If the statement is false, show that it is false.

A. If a number is divisible by 8, then it is divisible by 2 and by 4.
B. If a number is divisible by 6 and 4, then it is divisible by 24.

Part A

Any number divisible by 8 can be written in the form:

$$8x, \qquad x \in \mathbb{Z}$$

$$8x = \underbrace{2(4x)}_{\substack{Divisible \\ by\ 2}} = \underbrace{4(2x)}_{\substack{Divisble \\ by\ 4}}$$

Part B

This statement is false. We can show that it is false, by giving a counterexample.

6 divides 12, and 4 also divides 12.
However, 24 does not divide 12.
This can be expressed in mathematical notation as:

$$6 \mid 12, \qquad 4 \mid 12, \qquad 24 \nmid 12$$

Hence, the statement is false.

G. Writing Statements

Example 1.14

$$If\ a\ number\ is\ divisible\ by\ 8, then\ it\ is\ divisible\ by\ 2\ and\ by\ 4.$$

Write the above statement in the form **P ^ Q**. State what is $\boldsymbol{P}$, and what is $\boldsymbol{Q}$.

Let

$$\boldsymbol{P} = If\ a\ number\ is\ divisible\ by\ 8, then\ it\ is\ divisible\ by\ 2.$$
$$\boldsymbol{Q} = If\ a\ number\ is\ divisible\ by\ 8, then\ it\ is\ divisible\ by\ 4.$$

$\boldsymbol{P}$ ^ $\boldsymbol{Q}$ is true.

Example 1.15

$$If\ a\ number\ is\ divisible\ by\ 6\ and\ 4, then\ it\ is\ divisible\ by\ 24.$$

$$\boldsymbol{P} = A\ number\ is\ divisible\ by\ 6$$
$$\boldsymbol{Q} = A\ number\ is\ divisible\ by\ 4$$

$$\boldsymbol{Z} = A\ number\ is\ divisible\ by\ 24$$

P, Q and Z refer to the same number.

$$\boldsymbol{P} \quad \boldsymbol{Q} \Rightarrow \boldsymbol{Z}$$

Example 1.16

$$The\ number\ 8\ is\ both\ even\ and\ a\ power\ of\ 2.$$

$$\boldsymbol{P} = The\ number\ 8\ is\ even$$
$$\boldsymbol{Q} = The\ number\ 8\ is\ a\ power\ of\ 2$$

$$\boldsymbol{P\ AND\ Q}\ is\ true.$$

Example 1.17

$$4 = 5$$

Using an AND Condition

$$\boldsymbol{P} = 4\ is\ greater\ than\ 5$$

$$\boldsymbol{Q} = 4 \text{ is less than } 5$$

$$\sim\boldsymbol{P} \textbf{ AND } \sim\boldsymbol{Q}$$

1.2 OR

A. OR condition

Example 1.18

IF $\underbrace{have\ a\ perfect\ SAT\ score}_{A}$, OR $\underbrace{you\ are\ good\ at\ extracurriculars}_{B}$, only then you will go to Harvard.

In an OR condition, we are promised that at least one of the conditions which is given is true.

			Go to Harvard
Person 1	Perfect SAT Score	Not Good at extracurriculars	Yes
Person 2	Bad SAT Score	Good at Extracurriculars	Yes
Person 3	Perfect SAT Score	Good at Extracurriculars	Yes
Person 4	Bad SAT Score	Not Good at extracurriculars	No

1.19: Truth Table for OR

OR is represented using the symbol

$$\vee$$

A	B	$\boldsymbol{A \vee B}$
T	F	T
F	T	T
T	T	T
F	F	F

- An OR condition is true, if one or more of the underlying conditions is true.
- Conversely, if an OR condition is true, at least one of the underlying conditions is true.
- An OR condition is false $if\ and\ only\ if$ all of the underlying conditions are false.
- Hence, if an OR statement is true, at least one of the underlying conditions must be true.

Example 1.20

If you $\underbrace{have\ a\ perfect\ SAT\ score}_{A}$, OR $\underbrace{you\ are\ good\ at\ extracurriculars}_{B}$, only then you will go to Harvard.

Person X has a perfect SAT score, and he went to Harvard. Then:

A. He is good at extracurriculars
B. He is not good at extracurriculars
C. We cannot conclude anything.

			Go to Harvard
Person 1	**Perfect SAT Score**	**Not Good at extracurriculars**	**Yes**
Person 2	Bad SAT Score	Good at Extracurriculars	Yes
Person 3	**Perfect SAT Score**	**Good at Extracurriculars**	**Yes**
Person 4	Bad SAT Score	Not Good at extracurriculars	No

We cannot conclude anything since there are two cases where he has a perfect SAT score.
Person X may or may not be good at extracurriculars.

Option C is correct.

Example 1.21

IF $\underbrace{have\ a\ perfect\ SAT\ score}_{A}$, OR $\underbrace{you\ are\ good\ at\ extracurriculars}_{B}$, only then you will go to Harvard.

Person Y has is bad at extracurriculars and he went to Harvard. Then:

A. He has a perfect SAT score
B. Does not have a perfect SAT score
C. We cannot conclude anything.

			Go to Harvard
Person 1	**Perfect SAT Score**	**Not Good at extracurriculars**	**Yes**
Person 2	Bad SAT Score	Good at Extracurriculars	Yes
Person 3	Perfect SAT Score	Good at Extracurriculars	Yes
Person 4	Bad SAT Score	Not Good at extracurriculars	No

There is only one possibility, and hence, we know for sure that Person Y must have a perfect SAT score.
Option A is correct.

Example 1.22

IF $\underbrace{have\ a\ perfect\ SAT\ score}_{A}$, OR $\underbrace{you\ are\ good\ at\ extracurriculars}_{B}$, only then you will go to Harvard.

Person Z has a bad SAT score and he went to Harvard. Then:

A. He is good at extracurriculars.
B. He is not good at extracurriculars.
C. We cannot conclude anything.

			Go to Harvard
Person 1	Perfect SAT Score	Not Good at extracurriculars	Yes
Person 2	**Bad SAT Score**	**Good at Extracurriculars**	**Yes**
Person 3	Perfect SAT Score	Good at Extracurriculars	Yes
Person 4	Bad SAT Score	Not Good at extracurriculars	No

There is only one possibility, and hence, we know for sure that Person Z must be good at extracurriculars.
Option A is correct.

Example 1.23

IF $\underbrace{have\ a\ perfect\ SAT\ score}_{A}$, OR $\underbrace{you\ are\ good\ at\ extracurriculars}_{B}$, only then you will go to Harvard.

Person P is good at extracurriculars and he went to Harvard. Then:

A. He has a perfect SAT score
B. Does not have a perfect SAT score
C. We cannot conclude anything.

			Go to Harvard
Person 1	Perfect SAT Score	Not Good at extracurriculars	Yes
Person 2	**Bad SAT Score**	**Good at Extracurriculars**	**Yes**
Person 3	**Perfect SAT Score**	**Good at Extracurriculars**	**Yes**
Person 4	Bad SAT Score	Not Good at extracurriculars	No

We cannot conclude anything since there are two cases where he is good at extracurriculars.
Person P may or may not have a good SAT score.

Option C is correct.

1.24: Conclusion from $A \vee B$

- A or B
  - ✓ A implies nothing
  - ✓ B implies nothing
  - ✓ ~A implies B
  - ✓ ~B implies A

Example 1.25

If you $\underbrace{\textit{have a perfect SAT score}}_{X}$, OR $\underbrace{\textit{you are good at extracurriculars}}_{Y}$, only then you will go to Harvard.

Determine what, if anything, you can conclude given that you are going to Harvard, and

A. $\boldsymbol{X}$ is true
B. $\boldsymbol{X}$ is not true
C. $\boldsymbol{Y}$ is true
D. $\sim\boldsymbol{Y}$ is true

$$\boldsymbol{X}\ is\ true \Rightarrow No\ Conclusion$$
$$\boldsymbol{X}\ is\ not\ true \Rightarrow \boldsymbol{Y}\ is\ true$$
$$\boldsymbol{Y}\ is\ true \Rightarrow No\ Conclusion$$
$$\sim\boldsymbol{Y}\ is\ true \Rightarrow \boldsymbol{Y}\ is\ false \Rightarrow \boldsymbol{X}\ is\ true$$

B. Using OR Conditions

Example 1.26

$$I\ play\ football\ or\ hockey$$

The statement above is true. If we have the following additional information, what, if anything can we conclude?

A. I play football.
B. I play hockey.
C. I do not play hockey.
D. I do not play football.

- We cannot conclude anything about hockey.
- I play hockey. We cannot conclude anything about football.
- I do not play hockey. Therefore, I play football
- I do not play football. Therefore, I play hockey.

Football	Hockey	Overall
Play	Don't play	Valid
Play	Play	Valid
Don't play	Play	Valid
Don't Play	Don't play	Invalid

Example 1.27

$$You\ are\ charming, or\ you\ don't\ have\ a\ job$$

The statement above is true. If we have the following additional information, what, if anything can we conclude?

A. You are not charming.
B. You have a job.
C. You don't have a job.
D. You are charming.

Charming	**No Job**	Overall
Charming	Job	Valid
Charming	No Job	Valid

Not Charming	No Job	Valid
Not Charming	Job	Invalid

Part A

$$Row\ 3 \Rightarrow No\ Job$$

Part B

$$Row\ 1 \Rightarrow Charming$$

Part C

$$Row\ 2, 3 \Rightarrow Inconclusive$$

Part D

$$Row\ 1{,}2 \Rightarrow Inconclusive$$

Example 1.28

$$I\ will\ not\ go\ to\ England, or\ I\ will\ not\ go\ to\ Europe$$

The statement above is true. If we have the following additional information, what, if anything can we conclude?

A. I am going to England.
B. I am not going to England.
C. I am going to Europe
D. I am not going to Europe.

$$I\ am\ going\ to\ England \Rightarrow I\ will\ not\ go\ to\ Europe$$
$$I\ am\ not\ going\ to\ England \Rightarrow No\ Conclusion$$
$$I\ am\ going\ to\ Europe \Rightarrow I\ will\ not\ go\ to\ England.$$
$$I\ am\ not\ going\ to\ Europe \Rightarrow No\ Conclusion.$$

C. Three Conditions

Example 1.29

$$I\ passed\ one\ of\ out\ of\ Physics, Chemistry\ and\ Maths$$

The statement above is true. If we have the following additional information, what, if anything can we conclude?

A. I passed Physics.
B. I passed Chemistry.
C. I passed Maths.
D. I did not pass Physics.
E. I did not pass Chemistry.
F. I did not pass Maths.
G. I passed Physics and Chemistry.
H. I passed Chemistry and Maths.
I. I passed Physics and Maths.
J. I failed both Physics and Chemistry.
K. I failed both Chemistry and Maths.
L. I failed both Physics and Maths.
M. I failed all three of Physics, Maths and Chemistry.

	Physics	*Chemistry*	*Maths*	Validity
	Did not pass	*Did not pass*	*Did not pass*	*Invalid*
	Passed	*Did not pass*	*Did not pass*	*Valid*
	Did not pass	*Passed*	*Did not pass*	*Valid*
	Did not pass	*Did not pass*	*Passed*	*Valid*
	Passed	*Passed*	*Did not pass*	*Valid*
	Did not pass	*Passed*	*Passed*	*Valid*
	Passed	*Did not pass*	*Passed*	*Valid*

	$Passed$	$Passed$	$Passed$	$Valid$

Parts A, B, C, G, H and I
For each situation, there are multiple rows in the table that correspond to it.
Hence, we cannot draw any conclusion.

Part D, E, F
In case you did not pass one subject.
You passed one or more out of the remaining two subjects.

Parts J, K and L
$$Failed\ Physics\ and\ Chemistry \Rightarrow Passed\ Maths$$
$$Failed\ Chemistry\ and\ Maths \Rightarrow Passed\ Physics$$
$$Failed\ Physics\ and\ Maths \Rightarrow Passed\ Chemistry$$

Parts M
$$Failed\ Physics, Maths\ and\ Chemistry \Rightarrow \boldsymbol{Contradicton}$$

D. Existential Quantifier

1.30: Existential Quantifier
The existential quantifier has the symbol
$$\exists \Rightarrow there\ exists$$
$$\nexists = there\ does\ not\ exist$$

Example 1.31
$$\mathbb{P}\ is\ the\ set\ of\ prime\ numbers.$$
$$\mathbb{C}\ is\ the\ set\ of\ composite\ numbers.$$
$$\mathbb{N}\ is\ the\ set\ of\ natural\ numbers$$
The notation above is used in the statements below. Write each statement in English, without using mathematical symbols. Then, decide whether they are true or false.

A. $\exists p \in \mathbb{P}, p\ is\ even$
B. $\exists p \in \mathbb{P}, p\ is\ odd$
C. $\exists p \in \mathbb{N}, p \notin \mathbb{P}, p \notin \mathbb{C}$
D. M_n is the set of all possible number of days in a month in year n. $\exists m \in M_{2001}, m = 29$.

Part A
There is at least one even prime number.
There is one even prime number.
Part B
There is at least one odd prime number.
There is one odd prime number.
Part D
2001 is not a leap year. Hence,
$$M_{2001} = \{28,30,31\} \Rightarrow 29 \notin M_{2001} \Rightarrow Statement\ is\ false$$

1.32: Existential Quantifier using OR conditions

Example 1.33

A. $\mathbb{P}\ is\ the\ set\ of\ prime\ numbers. \exists p \in \mathbb{P}, p\ is\ even$
B. Type equation here.

$$\mathbb{P} = \{p_1, p_2, p_3, \dots\}$$

$\exists p \in \mathbb{P}, p\ is\ even$ can be written using OR conditions as:

$$p_1\ is\ even\ OR\ p_2\ is\ even\ OR\ p_3\ is\ even\ OR\ \dots$$

E. OR using Maths

Example 1.34

Write the following using an OR statement:

A. $x \neq y,\ x \in \mathbb{R}$
B. I want you to visit London to prospect for new clients, and not New York since the US since it is not a good market.
C. A man should look for what is, and not for what he thinks should be.

Part A

$$\boldsymbol{P}: x > y$$
$$\boldsymbol{Q}: x < y$$
$$\boldsymbol{P}\ Or\ \boldsymbol{Q}$$

Part B

$$\boldsymbol{P} = I\ want\ you\ to\ visit\ London$$
$$\boldsymbol{Q} = I\ do\ not\ want\ you\ to\ visit\ New\ York$$
$$\boldsymbol{P}\ AND\ \boldsymbol{Q}$$

Part C

$$\boldsymbol{P} = A\ man\ should\ look\ for\ what\ is.$$
$$\boldsymbol{Q} = A\ man\ should\ look\ for\ he\ thinks\ should\ be.$$
$$\boldsymbol{P}\ AND\ (\sim\boldsymbol{Q})$$

F. Exclusive OR

We now look at one variation of OR, called exclusive OR. Here, we are promised that at *exactly* one of the conditions is true.

1.35: Truth Table for Exclusive OR

		Exclusive OR
A	B	Either A OR B
T	F	Applies
F	T	Applies
T	T	Does not apply
F	F	Does not apply

Example 1.36

$$Sam\ is\ either\ black\ or\ white.$$

If the above statement is true, what, if anything can you conclude given the following information:

A. Sam is white. Sam is not black
B. Sam is black. Sam is not white

C. Sam is not white. Sam is black
D. Sam is not black. Sam is white

Example 1.37

$$Either\ A\ or\ B$$

If the above statement is true, what, if anything can you conclude given the following information:

A. **A**
B. **B**
C. **~A**
D. **~B**

$$\boldsymbol{A}\ implies\ \sim\boldsymbol{B}$$
$$\boldsymbol{B}\ implies\ \sim\boldsymbol{A}$$
$$\sim\boldsymbol{A}\ implies\ \boldsymbol{B}$$
$$\sim\boldsymbol{B}\ implies\ \boldsymbol{A}$$

Example 1.38

To be eligible for a scholarship, you must meet exactly one of the two conditions below:

$$You\ must\ be\ in\ eighth\ grade, XOR\ you\ must\ live\ in\ Hawaii.$$

If the following further information is available, what can you conclude about whether you are eligible for a scholarship:

A. You are in eighth grade.
B. You are not in eighth grade.
C. You live in Hawaii.
D. You do not live in Hawaii.
E. You are in eighth grade, and you live in Hawaii
F. You are in eighth grade, and you do not live in Hawaii
G. You are not in eighth grade, and you do not live in Hawaii
H. You are not in eighth grade, and you live in Hawaii

Parts A, B, C and D

In each of these parts, we have information on only one aspect. Since exactly one condition must be true, without information on the other aspect, we cannot reach any conclusion.

Eighth Grade	Hawaii	Scholarship
Yes	Yes	No
No	No	No
Yes	No	Yes
No	Yes	Yes

Parts E, F, G, H

$$Part\ E: Both\ Conditions\ are\ true \Rightarrow No\ Scholarship$$
$$Part\ F: Exactly\ One\ Condition\ True \Rightarrow Scholarship$$
$$Part\ G: No\ Condition\ True \Rightarrow No\ Scholarship$$
$$Part\ H: Exactly\ One\ Condition\ True \Rightarrow Scholarship$$

Example 1.39

$$Either\ you\ go\ to\ town\ or\ you\ don't\ buy\ bread$$

If the above statement is true, what, if anything can you conclude given the following information:

A. You go to town implies you bought bread
B. You don't buy bread implies you didn't go to town
C. You don't go to town implies you didn't buy bread
D. You bought bread implies you went to town

G. Maths

Example 1.40: Exponents

If x and y are real numbers such that $x^y \in \{0,1\}$ then what can we conclude if (answer each separately):

A. $x = 0$
B. $x \neq 0$
C. $y = 0$
D. $y \neq 0$

Part A

Substitute $x = 0$:

$$0^y = 0 \Rightarrow y \in \mathbb{R} - \{0\}$$

Part B

$$x \neq 0 \Rightarrow x^y = 0 \Rightarrow No\ Solutions$$
$$x \neq 0 \Rightarrow x^y = 1 \Rightarrow y = 0$$

Part C

Substitute $y = 0$:

$$x^0 = 0 \Rightarrow No\ Solutions$$
$$x^0 = 1 \Rightarrow x \in \mathbb{R} - \{0\}$$

Part D

$$y \neq 0 \Rightarrow x^y = 0 \Rightarrow x = 0$$
$$y \neq 0 \Rightarrow x^y = 1 \Rightarrow x = 1, y = \mathbb{R} - \{0\}$$

Example 1.41: Set Theory / Number Theory

All elements of the set $X = \{p_1, p_2, \dots, p_n\}$ are prime numbers.

A. p_3 is even. What can we say about the parity of the remaining members of X.
B. p_1 is odd. What can we say about the parity of the remaining members of X.

Part A

2 is the only even number. All other prime numbers are odd. Hence, the remaining elements of the set X are odd.

Part B

p_1 is odd. We cannot say anything about the remaining elements.

Example 1.42

A natural number x greater than one is either $\underbrace{prime}_{P}$ or $\underbrace{composite}_{C}$.

Language	Variables	Examples
$x\ is\ prime \Rightarrow x\ is\ not\ composite$	$P \Rightarrow \sim C$	$5\ is\ prime \Rightarrow 5\ is\ not\ composite$
$x\ is\ composite \Rightarrow x\ is\ not\ prime$	$C \Rightarrow \sim P$	$6\ is\ composite \Rightarrow 6\ is\ not\ prime$
$x\ is\ not\ prime \Rightarrow x\ is\ composite$	$\sim C \Rightarrow P$	$4\ is\ not\ prime \Rightarrow 4\ is\ composite$
$x\ is\ not\ composite \Rightarrow x\ is\ prime$	$\sim P \Rightarrow C$	$3\ is\ not\ composite \Rightarrow 3\ is\ prime$

H. Writing OR Statements

Example 1.43

An isosceles triangle is equilateral, or it is not.

$$\boldsymbol{P} = An\ isoceles\ triangle\ is\ equilateral$$
$$\sim\boldsymbol{P} = An\ isoceles\ triangle\ is\ not\ equilateral$$
$$\boldsymbol{P\ OR \sim P}$$

Example 1.44

A. I have an interview on Wednesday or Friday.
B. I have an interview on either Wednesday or Friday.

$$\boldsymbol{P} = I\ have\ an\ interview\ on\ Wednesday.$$
$$\boldsymbol{Q} = I\ have\ an\ interview\ on\ Friday.$$

$$\boldsymbol{P\ OR\ Q}$$
$$\boldsymbol{P\ XOR\ Q}$$

Example 1.45

I am going to either Harvard or Yale.

$$\boldsymbol{P} = I\ am\ going\ to\ Harvard$$
$$\boldsymbol{Q} = I\ am\ going\ to\ Yale$$

$$\boldsymbol{P\ XOR\ Q}$$

1.3 Conditionals, Converses and Biconditionals

A. Conditionals

1.46: Conditional Statements

A statement written in $If - Then$ is a conditional statement.

If the statement $\boldsymbol{P}$ is true, then the statement $\boldsymbol{Q}$ is true.

$$If\ \boldsymbol{P},\qquad then\ \boldsymbol{Q}$$

This can also be written as

$$\boldsymbol{P \Rightarrow Q}$$

$\boldsymbol{P}$	$\boldsymbol{Q}$	$\boldsymbol{P \Rightarrow Q}$
T	T	T
T	F	F
F	F	T
F	T	T

B. Writing Properties as Conditionals

Example 1.47

A.
B. If two rays meet at a vertex, $then$ they form an angle.
C. If a triangle is a right-angled triangle, $then$ the square of the length of the hypotenuse is equal to the sum of the squares of the other two sides.

$$\boldsymbol{P} = \text{a triangle is a right} - \text{angled triangle}$$
$$\boldsymbol{Q} = \text{the square of the length of the hypotenuse is equal to the sum of the squares of the other two sides}$$

$$If\ \boldsymbol{P},\qquad then\ \boldsymbol{Q}$$

Example 1.48

A. An isosceles triangle has two equal sides.
B. The sum of the measures of the angles of a n-sided polygon is $180(n-2)°$.
C. You can't divide by zero.
D. Binomial Theorem: $(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0 y^n$
E. Permutations: $\frac{n!}{(n-r)!} = {}^nP_r$
F. Permutations with repeated objects: $\frac{n!}{(n-r)!} = {}^nP_r$
G. Combinations: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
H. A quadratic with no real roots must have negative discriminant.

Part A

If a triangle has two equal sides, $then$ it is isosceles.

Part B

If a polygon has n sides, then the sum of the measures of its angles $180(n-2)°$.

Part C

A rational number is a number of the form $\frac{p}{q}$, where p, q are integers, and $q \neq 0$. If $q = 0$, then the denominator is zero, and the result of the operation $\frac{p}{q}$ is not defined.

Part D

If x, y are expressions, and n is an integer, then:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0 y^n$$

Part E

If I have n distinct objects, then the number of ways of arranging r out of these objects in a row is given by

$${}^nP_r = \frac{n!}{(n-r)!}$$

Part F

If I have n objects, out of which certain objects are repeated p times, certain objects are repeated q times, and the remaining objects are repeated r times, then the number of ways of arranging these objects in a row is given by:

$$\frac{n!}{p!\,q!\,r!}$$

Part G

If I have n objects, then I can choose r objects, $0 \leq r \leq n$ out of them in $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ Ways.

Part H

If a quadratic has no real roots, then its discriminant is negative.

C. Sufficient and Necessary

1.49: Sufficient Condition

$$\boldsymbol{P}\ is\ a\ sufficient\ condition\ for\ \boldsymbol{Q}$$

The above tells us that if **P** holds, then **Q** must hold. In other words:

$$\boldsymbol{P} \Rightarrow \boldsymbol{Q}$$

1.50: Necessary Condition

$\boldsymbol{Q}$ *is a necessary condition for* $\boldsymbol{P}$

Q cannot happen without **P**.
Then, if **P** is true, **Q** must be true.
In other words:

$$\boldsymbol{P} \Rightarrow \boldsymbol{Q}$$

Example 1.51

$$If\ a\ triangle\ has\ two\ equal\ sides, then\ it\ is\ isosceles.$$

- A triangle is isosceles if it has two equal sides.
- A triangle is isosceles whenever it has two equal sides.
- A triangle is isosceles provided that it has two equal sides.
- Whenever a triangle has two equal sides, then it is also isosceles.
- It is sufficient for a triangle to have two equal sides for it to be an isosceles triangle.
- For a triangle to be isosceles, it is sufficient that it has two equal sides.
- For a triangle to be isosceles, it is necessary that it has two equal sides.
- For a triangle to have two equal sides, it is necessary that it should be isosceles.

D. Language Questions

Example 1.52

$$If\ a\ polygon\ is\ a\ square, then\ it\ is\ a\ rectangle.$$

What if anything, can you conclude about the polygons below:

A. I have polygon $\boldsymbol{A}$, and it is a square.
B. I have polygon $\boldsymbol{B}$, and it is not a square.
C. I have polygon $\boldsymbol{C}$, and it is a rectangle.
D. I have polygon $\boldsymbol{D}$, and it is not a rectangle.
E. I have polygon $\boldsymbol{E}$, and it is a square, but not a rectangle.

$$\boldsymbol{P} = Polygon\ is\ a\ square$$
$$\boldsymbol{Q} = Polygon\ is\ a\ rectangle$$

$\boldsymbol{P}$	$\boldsymbol{Q}$	$\boldsymbol{P \Rightarrow Q}$
T	T	T
T	F	F
F	F	T
F	T	T

Part A

$$Row\ 1: \boldsymbol{P}\ is\ true \Rightarrow \boldsymbol{Q}\ is\ true$$

Part B

$$Polygon\ B\ may\ or\ may\ not\ be\ a\ rectangle.$$
$$Row\ 3,4: \boldsymbol{P}\ is\ false \Rightarrow \boldsymbol{Q}\ may\ be\ true, or\ may\ be\ false$$

Part C

$$The\ polygon\ may\ or\ may\ not\ be\ a\ square.$$
$$\boldsymbol{Q}\ is\ true \Rightarrow \boldsymbol{P}\ may\ be\ true, or\ may\ be\ false$$

Part D

$$\boldsymbol{Q}\ is\ false \Rightarrow \boldsymbol{P}\ is\ false$$

Part E

$$Row\ 2: \boldsymbol{P}\ is\ true, \boldsymbol{Q}\ is\ false \Rightarrow \boldsymbol{Contradiction}$$

Example 1.53

Hasan said, "If I do well in my *Sociology* exam, I will distribute sweets in my building the day after the exam."

A. Hasan was found distributing sweets in his building the day after the exam. Hence, his mother concluded that his Sociology exam went well. Is her conclusion valid?
B. Hasan was in a great mood, and played video games the entire day the day after the exam. He definitely did not leave his house. What, if anything, can we conclude from this information?
C. Hasan's Sociology exam did not go well. Can we conclude that he did not distribute sweets in his building the day after the exam?

$$P = Hasan\ does\ well\ in\ his\ Sociology\ exam\ on\ 21st\ Jan.$$
$$Q = Hasan\ distributes\ sweets\ in\ his\ locality\ on\ 22nd\ Jan.$$

$\boldsymbol{P}$	$\boldsymbol{Q}$	$\boldsymbol{P \Rightarrow Q}$
T	T	T
T	F	$\boldsymbol{F}$
F	F	T
F	T	T

This can be written in the language of logic as:

$$If\ \boldsymbol{P}, then\ \boldsymbol{Q}$$

Part A

The information given tells us that

$$\boldsymbol{Q}\ is\ true \Rightarrow Row\ 1,4$$
$$P\ is\ true\ in\ Row\ 1$$
$$P\ is\ false\ in\ Row\ 4$$

In general, we cannot conclude anything about P since we do not know whether Row 1 is applicable, or Row 4 is applicable.
Hence, Hasan's mother conclusion is not valid.

Part B

$$\boldsymbol{Q}\ is\ false \Rightarrow Row\ 2, 3$$

In Row 2, $\boldsymbol{P} \Rightarrow \boldsymbol{Q}$ is false, which the statement being made is false.
However, we have to conduct the analysis assuming that the statement is true.

Hence, the only possibility is Row 3.

$$Q\ is\ false \Rightarrow P\ is\ false \Rightarrow Hasan\ did\ not\ do\ well.$$

Part C

$$\boldsymbol{P}\ is\ false \Rightarrow Row\ 3, 4$$

We cannot conclude anything about **Q**.
Hence, we do not know whether he did, or did not distribute sweets.

Example 1.54

If it rains, I carry an umbrella to protect myself from the rain. On 24^{th} June 2021, it was a very windy day, and I carried an umbrella. Was it raining?

$$\boldsymbol{P} = It\ rains$$
$$\boldsymbol{Q} = I\ carry\ an\ Umbrella$$
$$\boldsymbol{Q}\ is\ true \Rightarrow Row\ 1, 4 \Rightarrow No\ Conclusion\ about\ \boldsymbol{P}$$
$$I\ don't\ know\ whether\ it\ was\ raining.$$

$\boldsymbol{P}$	$\boldsymbol{Q}$	$\boldsymbol{P \Rightarrow Q}$
T	T	T
T	F	$\boldsymbol{F}$
F	F	T
F	T	T

Example 1.55

Mark the Best Option

If my dog sees the color purple, it barks. I see my dog barking. Did it:

A. See the color purple.

B. Run after a ball.
C. See its dog trainer coming
D. Could be any of the above three.

$$\boldsymbol{P} = My\ dog\ sees\ the\ color\ purple$$
$$\boldsymbol{Q} = My\ dog\ barks$$

$$\boldsymbol{Q}\ is\ true \Rightarrow Row\ 1,4 \Rightarrow No\ Conclusion\ about\ \boldsymbol{P}$$

Option D is correct.

$\boldsymbol{P}$	$\boldsymbol{Q}$	$\boldsymbol{P \Rightarrow Q}$
T	T	T
T	F	$\boldsymbol{F}$
F	F	T
F	T	T

Example 1.56

Mark the Best Option

If I see aliens in my sleep, I will prove the four-color theorem. I did not prove the four-color theorem. Then:

A. I went to Mars in my dreams.
B. I did not see aliens in my sleep.
C. I saw aliens in my sleep.
D. None of the above.

$$\boldsymbol{P} = I\ saw\ aliens\ in\ my\ sleep.$$
$$\boldsymbol{Q} = I\ proved\ the\ four\ color\ theorem$$

$$\boldsymbol{Q}\ is\ false \Rightarrow Row\ 3 \Rightarrow \boldsymbol{P}\ is\ false \Rightarrow Option\ B\ is\ correct.$$

$\boldsymbol{P}$	$\boldsymbol{Q}$	$\boldsymbol{P \Rightarrow Q}$
T	T	T
T	F	$\boldsymbol{F}$
F	F	T
F	T	T

Example 1.57

If you get more than 70% marks in your final exam, you get a job.

A. You got more than 70% marks in your final exam.
B. You got less than 70% marks in your final exam.

$$P = You\ get\ more\ than\ 70\%\ marks\ in\ your\ final\ exam$$
$$Q = You\ get\ a\ job$$

Part A

$$P\ is\ true \Rightarrow Q\ is\ true$$

Part B

$$P\ is\ false \Rightarrow Q\ may\ be\ true, or\ may\ be\ false$$

Example 1.58

If you get a JMET rank of better than 500, then you will get an admission in your preferred institute.
If the statement above is true, which of the following must also be true?

A. If you do not get a JMET rank of better than 500, then you will not get an admission in your preferred institute.
B. If you get an admission in your preferred institute, then you must have got a JMET rank of better than 500.
C. If you did not get an admission in your preferred institute, then you did not get a JMET rank of better than 500. (JMET 2008/43)

$\boldsymbol{P}$	$\boldsymbol{Q}$	$\boldsymbol{P \Rightarrow Q}$

$$P = Get\ JMET\ Rank\ of\ better\ than\ 500$$
$$Q = Admission\ in\ your\ preferred\ institute$$

T	T	T
T	F	F
F	F	T
F	T	T

$$Statement\ A: Row\ 3\ and\ 4$$
$$Statement\ B: Row\ 1\ and\ 4$$
$$Statement\ C: Row\ 3: Must\ be\ true$$

$$Correct\ Answer: Statement\ C$$

Example 1.59

X when/if/along with Y

- X when/if/along with Y
  - ✓ ~Y implies nothing
  - ✓ X implies nothing
  - ✓ Y implies X
  - ✓ ~X implies ~Y

Example 1.60

- Tata Motors gives a truck(X) along with a Nano(Y)
  - ✓ No Nano implies nothing
  - ✓ Got a truck implies nothing
  - ✓ Nano implies truck
  - ✓ No truck implies no Nano

Example 1.61

- You will add more value to the brand (X) if strategic planning is done(Y)
  - ✓ Strategic planning was not done implies nothing
  - ✓ You added more value to the brand implies nothing
  - ✓ Strategic planning was done implies you added more value to the brand
  - ✓ You did not add more value to the brand implies strategic planning was not done

Example 1.62

- It does not snow(X) when the sun shines (Y)
  - ✓ Sun does not shine implies nothing
  - ✓ Does not snow implies nothing
  - ✓ Sun shines implies does not snow
  - ✓ Snows implies sun does not shine

Example 1.63

- I eat a samosa (X) when I am hungry(Y)
  - ✓ I am not hungry. No Conclusions
  - ✓ I ate a samosa. No Conclusion
  - ✓ I am hungry. I ate a samosa

- ✓ I did not eat a samosa. I am not hungry

Example 1.64

- ➢ You get a scholarship(X) alongwith a medal(Y)
  - ✓ You get a scholarship implies nothing
  - ✓ You did not get a medal implies nothing
  - ✓ You got a medal. You got a scholarship
  - ✓ You did not get a scholarship. You did not get a medal.

X only when/if/alongwith Y

Example 1.65

- ➢ X only when/if/alongwith Y
  - ✓ ~Y implies nothing
  - ✓ X implies nothing
  - ✓ Y implies X
  - ✓ ~X implies ~Y

Example 1.66

- ➢ It is gold only if it glitters
  - ✓ It glitters implies nothing
  - ✓ It is gold. It glitters
  - ✓ It does not glitter. It is not gold.
  - ✓ It is not gold. It does not glitter.

E. Checking for Truthfulness

Example 1.67

Five cards are lying on a table as shown. Each card has a letter on one side and a whole number on the other side. Jane said, "If a vowel is on one side of any card, then an even number is on the other side." Mary showed Jane was wrong by turning over one card. Which card did Mary turn over? **(AMC 8 1985/25)**

	P	Q
3	4	6

If $\underbrace{a\ vowel\ is\ on\ one\ side\ of\ any\ card}_{P}$, then $\underbrace{an\ even\ number\ is\ on\ the\ other\ side}_{Q}$.

$$Jane\ said\ "If\ \boldsymbol{P}, then\ \boldsymbol{Q}"$$
$$Mary\ showed\ Jane\ was\ wrong \Rightarrow Row\ 2$$

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	F	T
F	T	T

In Row 2:

$$\underbrace{Vowel}_{\boldsymbol{P}\ is\ true}\ AND\ \underbrace{Odd\ Number}_{\boldsymbol{Q}\ is\ false}$$

Example 1.68

Each of the following 15 cards has a letter on one side and a positive integer on

e	17	57	60	D
43	E	3	7	13
31	88	G	H	21

the other side. What is the minimum number of cards that need to be turned over to check if the following statement is true?
"If a card has a lower-case letter on one side, then it has an odd integer on the other side." (CEMC Pascal 9 2020/19)

If the card shows an upper-case letter, then it does not meet the if condition. We do not need to check the $then$. If the card shows an odd number, then it meets the $then$ condition. We do not need to check the if.

We only need to check:

- Cards which show a lower-case letter. These meet the if, and we need to check that they meet the $then$.
  - ✓ One card: e
- Cards which show an even number. These violate the $then$, and we need to check that the if is not met.
  - ✓ Two cards: 60,88

$$Total\ Cards = \underbrace{1}_{e} + \underbrace{2}_{60,88} = 3$$

F. Converses

1.69: Statement

A statement is usually of the form:

$$If \underbrace{A}_{Requirement}, then \underbrace{B}_{Promised}$$
$$A \Rightarrow B$$

If $\underbrace{a\ quadrilateral\ is\ a\ square}_{A}$, than $\underbrace{it\ is\ a\ rectangle}_{B}$.

$$A \to A\ quadrilateral\ is\ a\ square$$
$$B \to A\ quadrilateral\ is\ a\ rectangle$$

1.70: Converse

A statement which reverses the direction of causality is the converse of another statement.

$$If \underbrace{B}_{Requirement}, then \underbrace{A}_{Promised}$$
$$A \Leftarrow B$$

If $\underbrace{a\ quadrilateral\ is\ a\ rectangle}_{A}$, then $\underbrace{it\ is\ a\ square}_{B}$.

Example 1.71

Find the converses of the following statements:

A. Every citizen must fight for his country.
B. It is a truth universally acknowledged that a single man in possession of a good fortune, must be in want of a wife. ($Opening\ line\ of\ Pride\ and\ Prejudice, Jane\ Austen$)
C. The determinant of an invertible matrix is non-zero.
D. If you knew what a conflict goes on in the business mind, you would be amused. ($A\ Tale\ of\ Two\ Cities, Charles\ Dickens$)

Part A

If a person is a citizen of a country, then he must fight for his country.
If a person fights for his country, then he is a citizen of his country.

Part B
It is a truth universally acknowledged that a single man in want of a wife, must be in possession of a good fortune.

Part C
If a matrix is invertible, then its determinant is non-zero.
Converse: If the determinant of a matrix is non-zero, then it is invertible.

Part D
If you would be amused, then you would know what a conflict goes on in the business mid.

Example 1.72
Are the following statements true or false? Find their converses. Are their converses true or false?

A. If $\underbrace{a\ polygon\ is\ a\ square}_{P}$, then $\underbrace{it\ is\ a\ rectangle}_{Q}$.
B. If a statement is true, its converse is true.
C.

Part A

$$Original\ Statement: True$$
$$Converse: If\ a\ polygon\ is\ a\ rectangle, then\ it\ is\ a\ square.$$
$$Converse: False$$

Part B

$$Original\ Statement: False$$
$$If\ the\ converse\ of\ a\ statement\ is\ true, then\ the\ statement\ is\ true.$$
$$Converse: False$$

G. Biconditionals

1.73: Bidirectional Causality
If we combine $A \Rightarrow B$ with $B \Rightarrow A$, and we claim that both of these are true, then the direction of causality is two way.

$$\boldsymbol{A \Leftrightarrow B}$$
$$A\ if\ and\ only\ if\ B \Rightarrow A\ iff\ B$$

1.74: Necessary and Sufficient

$$\boldsymbol{P}\ is\ a\ sufficient\ condition\ for\ \boldsymbol{Q}$$
$$\boldsymbol{P}\ is\ a\ necessary\ condition\ for\ \boldsymbol{Q}$$

$$\boldsymbol{P}\ is\ a\ sufficient\ condition\ for\ \boldsymbol{Q}: \boldsymbol{P \Rightarrow Q}$$
$$\boldsymbol{P}\ is\ a\ necessary\ condition\ for\ \boldsymbol{Q}: \boldsymbol{Q \Rightarrow P}$$

Combine the two, and we get:

$$\boldsymbol{P} \Leftrightarrow \boldsymbol{Q}$$

Example 1.75

A. Every rectangle is a square.
B. If a number is even, it is divisible by 2.

Part A

If a polygon is a square, then it is a rectangle.

$$\boldsymbol{P} = Polygon\ is\ \ a\ square$$
$$\boldsymbol{Q} = Polygon\ is\ \ a\ rectangle$$

$$\boldsymbol{P} \Rightarrow \boldsymbol{Q}$$

If a polygon is a rectangle, then it is a square.

$$\boldsymbol{Q} \Rightarrow \boldsymbol{P}$$

Part B

$$\boldsymbol{P} = A\ number\ is\ even$$
$$\boldsymbol{Q} = A\ number\ is\ divisible\ by\ 2$$

$$\boldsymbol{P} \Leftrightarrow \boldsymbol{Q}$$

Example 1.76

A. If a triangle is a right-angled triangle, $then$ the square of the length of the hypotenuse is equal to the sum of the squares of the other two sides.

$$\boldsymbol{P} = \text{a triangle is a right} - \text{angled triangle}$$
$$\boldsymbol{Q} = \text{the square of the length of the hypotenuse is equal to the sum of the squares of the other two sides}$$

$$If\ \boldsymbol{P}, \qquad then\ \boldsymbol{Q}$$
$$If\ \boldsymbol{Q}, \qquad then\ \boldsymbol{P}$$

If the square of the square of the length of the hypotenuse is equal to the sum of the squares of the other two sides in a triangle, then the triangle is a right-angled triangle.

Example 1.77

A triangle is isosceles if and only if it has two equal sides.

$$P = Triangle\ is\ isoceles$$
$$Q = it\ has\ two\ equal\ sides$$

$$\text{A } \underbrace{\text{triangle is isosceles}}_{P}\ if\ \underbrace{\text{it has two equal sides}}_{Q} : \boldsymbol{Q} \Rightarrow \boldsymbol{P}$$
$$\text{A } \underbrace{\text{triangle is isosceles}}_{P}\ only\ if\ \underbrace{\text{it has two equal sides}}_{Q} : \boldsymbol{P} \Rightarrow \boldsymbol{Q}$$

$$\text{A } \underbrace{\text{triangle is isosceles}}_{P}\ if\ and\ only\ if\ \underbrace{\text{it has two equal sides}}_{Q} : \boldsymbol{Q} \Leftrightarrow \boldsymbol{P}$$

1.4 Truth Tables

A. Basics

Example 1.78

Give the truth table for:

A. $P\ OR\ Q\ OR\ R$
B. $P\ AND\ Q\ AND\ R$

P	Q	R	$P\ OR\ Q\ OR\ R$	$P\ AND\ Q\ AND\ R$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	F
T	F	F	T	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

Example 1.79

Consider the truth table for the statement $P_1\ OR\ P_2\ OR \dots OR\ P_n$.

A. How many rows will the truth table for the following statement contain (not including the title row)?
B. How many of those rows will contain $True$ in the final column? How many will contain $False$ in the final column?
C. Answer the above two questions for $P_1\ AND\ P_2\ AND \dots AND\ P_n$.

Part A

For every statement $P_1, P_2, \dots P_n$, there are two options. Each of those can be combined with any option for any other statement. Hence, the number of rows is:

$$\underbrace{2}_{\substack{Choices\\ for\ P_1}} \times \underbrace{2}_{\substack{Choices\\ for\ P_2}} \times \dots \times \underbrace{2}_{\substack{Choices\\ for\ P_n}} = 2^n$$

Part B

$$True: 2^n - 1$$
$$False: 1$$

Part C

$$No.of\ Rows = 2^n$$
$$False: 2^n - 1$$
$$True: 1$$

Example 1.80

A. Construct a truth table for the statement $(A \vee B) \wedge \sim(A \wedge B)$.
B. Compare this truth table with the truth table for Exclusive OR (XOR.)
C. Hence, conclude the relations between $(A \vee B) \wedge \sim(A \wedge B)$ and XOR

		OR	AND		XOR
A	B	$\boldsymbol{A \vee B}$	$\boldsymbol{A \wedge B}$	$\sim(\boldsymbol{A \wedge B})$	$(\boldsymbol{A \vee B}) \wedge \sim(\boldsymbol{A \wedge B})$
T	T	T	T	F	F
F	T	T	F	T	T
T	F	T	F	T	T

F	F	F	F	T	F

$$(A \lor B) \land \sim(A \land B) \Leftrightarrow \text{XOR}$$

Example 1.81

$$ab\ is\ even\ if\ at\ least\ one\ of\ A\ and\ B\ is\ even$$

Make a truth table for the statement above.

$$\underbrace{(ab\ is\ even)}_{X} \Leftrightarrow \left(\underbrace{a\ is\ even}_{A}\ OR\ \underbrace{B\ is\ even}_{B}\right)$$

X	A	B	$A\ OR\ B$	$X \Leftrightarrow (A\ OR\ B)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
F	F	F	F	T
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
T	F	F	F	F

B. DeMorgan's Laws

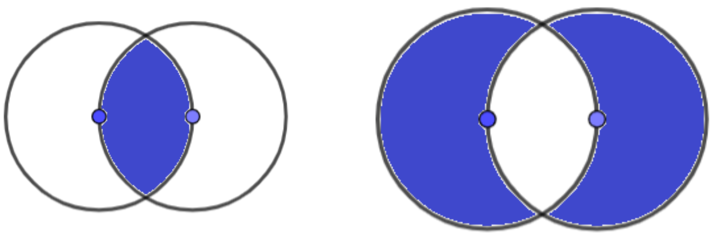
Example 1.82

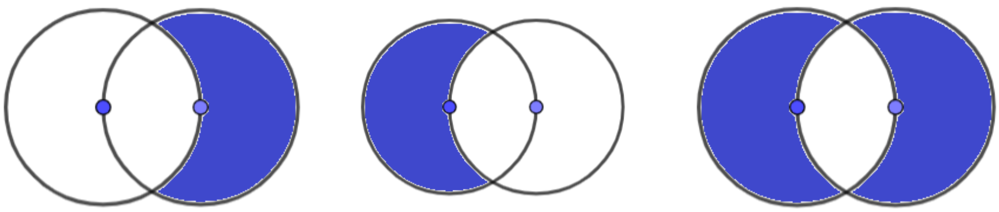
A. Show using a truth table that $\sim(P\ AND\ Q) = (\sim P)\ OR\ (\sim Q)$
B. Show using Venn Diagrams that $(P \cap Q)' = (P' \cup Q')$

Part A

P	Q	$\sim P$	$\sim Q$	$P\ AND\ Q$	$(\sim)(P\ AND\ Q)$	$(\sim P)\ OR (\sim Q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Part B





Example 1.83

A. Show using a truth table that $\sim(P\ OR\ Q) = (\sim P) AND (\sim Q)$
B. Show using Venn Diagrams that $(P \cup Q)' = (P' \cap Q')$

Part A

P	Q	$\sim P$	$\sim Q$	$(\sim P) AND (\sim Q)$	$P\ OR\ Q$	$\sim(P\ OR\ Q)$
T	T	F	F	F	T	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	T	F	T

C. More Equivalences

Example 1.84: Contrapositive Law

$$P \Rightarrow Q = (\sim Q) \Rightarrow (\sim P)$$

$\boldsymbol{P}$	$\boldsymbol{Q}$	$\boldsymbol{P \Rightarrow Q}$	$\sim Q$	$\sim P$	$(\sim Q) \Rightarrow (\sim P)$
T	T	T	F	F	T
T	F	F	T	F	F
F	F	T	F	T	T
F	T	T	T	T	T

Example 1.85

$$\boldsymbol{P \Rightarrow Q} = (\sim P)\ AND\ Q$$

$\boldsymbol{P}$	$\boldsymbol{Q}$	$\boldsymbol{P \Rightarrow Q}$	$\sim P$	$(\sim P)\ OR\ Q$
T	T	T	F	T
T	F	F	F	F
F	F	T	T	T
F	T	T	T	T

Example 1.86

$$P \Rightarrow\ Q = (P\ AND \sim Q) \Rightarrow (Q\ AND \sim Q)$$

$\boldsymbol{P}$	$\boldsymbol{Q}$	$\boldsymbol{P \Rightarrow Q}$	$\sim Q$	$\sim P$	$P\ AND \sim Q$	$Q\ AND \sim Q$	$(P\ AND \sim Q) \Rightarrow (Q\ AND \sim Q)$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	F	F
F	F	T	F	T	F	F	T
F	T	T	T	T	F	F	T

Example 1.87

$$P\ OR\ Q, \sim(\sim P\ OR \sim Q)$$

P	Q	P AND Q	$\sim P$	$\sim Q$	$\sim P\ OR \sim Q$	$\sim(\sim P\ OR \sim Q$
T	T	T	F	F	F	T

T	F	F	F	T	T	F
F	F	F	T	F	T	F
F	T	F	T	T	T	F

2. Proof

2.1 Direct Proof

A. Number Theory

Example 2.1
The test of divisibility of 4 says that a number is divisible by 4 $if\ and\ only\ if$ the last the two digits are divisible by 4.

We need to prove:
Statement: If the last two digits are divisible by 4, the number is divisible by 4.
Converse: If the number is divisible by 4, the last two digits are divisible by 4

Statement
Consider a number with digits from left to right as $d_1, d_2, \dots d_n$. We can write the number as:

$$d_1 d_2 \dots d_{n-2} d_{n-1} d_n$$

We can expand the number as:

$$d_1 d_2 \dots d_{n-2} \times 100 + d_{n-1} d_n$$

Factor out 4 from the first term:

$$4(d_1 d_2 \dots d_{n-2} \times 25) + d_{n-1} d_n$$

Hence, the first term is divisible by 4. Hence, the entire number will be divisible by 4 if:

$$d_{n-1} d_n \ is\ divisible\ by\ 4$$

Converse
Consider a number which is divisible by 4. Hence, it must of the form

$$4n, n \in \mathbb{Z}$$

From above, we know that we can write a number as:

$$4(d_1 d_2 \dots d_{n-2} \times 25) + d_{n-1} d_n$$

In our case, the number is 4n. Hence:

$$4n = 4(d_1 d_2 \dots d_{n-2} \times 25) + d_{n-1} d_n$$

Divide by 4 both sides:

$$n = (d_1 d_2 \dots d_{n-2} \times 25) + \frac{d_{n-1} d_n}{4}$$

$$LHS \in \mathbb{Z} \Rightarrow RHS \in \mathbb{Z}$$
$$(d_1 d_2 \dots d_{n-2} \times 25) \in \mathbb{Z}$$

Hence:

$$\frac{d_{n-1} d_n}{4} \in \mathbb{Z}$$

B. Direct Proof

Components of a proof:

- Given: Facts/properties which are assumed at the beginning of a proof
- Logical sequence of steps, which connect the given statements to the conclusion
- The final conclusion, which says that if the conditions of the proof are met, then the conclusion holds.

For example, consider the quadratic

$$x^2 + 7x + 10 = 0 \Rightarrow x \in \{-2, -5\}$$

The above is an example where we are solving for specific values. In a proof, we often to solve the general case.

Example 2.2: Equations and Properties

Consider a quadratic in standard form, given by $ax^2 + bx + c = 0,\ \ a \neq 0,\ \ a, b, c \in \mathbb{R}$.

A. Show that the roots are $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.
B. Prove Vieta's Formulas for a quadratic.

Part A

Divide throughout by a to make the leading coefficient one:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Add and subtract the square of half the second term in preparation to complete the square:

$$\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right] - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

Rewrite the terms inside the square brackets as a perfect square.

$$\left[x + \frac{b}{2a}\right]^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

Isolate the perfect square term on the LHS:

$$\left[x + \frac{b}{2a}\right]^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

Add the two fractions in the RHS by taking the LCM, and then take the square root on both sides:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Simplify the denominator in the RHS, and isolate x on the LHS to arrive at the standard quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Part B

$$Sum\ of\ Roots = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{2b}{2a} = -\frac{b}{a}$$

$$Product\ of\ Roots = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

Example 2.3: Identities

$$(a + b)(a - b) = a^2 - b^2$$

A. Prove the above identity algebraically.
B. Prove the above identity geometrically

Part A: Algebraic

Apply the Distributive Property:

$$(a + b)(a - b) = a(a - b) + b(a - b) = a^2 \boldsymbol{- ab + ab} - b^2 = a^2 - b^2$$

Part B: Geometric

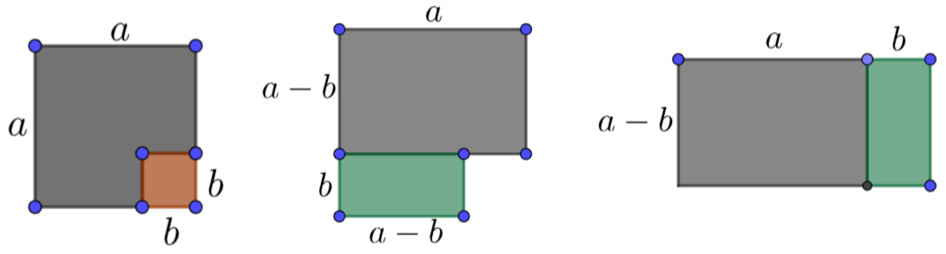
Step I: Start with a square of side a. Choose $b < a$, and remove an area equal to b^2 from the original square.

Step II: Divide the remaining area of the original square into two parts. Note the dimensions:

$$\underbrace{Length = a, Width = a - b}_{Grey\ Rectangle}, \quad \underbrace{Length = a - b, Width = b}_{Green\ Rectangle}$$

Step III: Rotate the green rectangle, and place it alongside the grey rectangle, giving us:

$$Larger\ Rectangle: Length = a + b, Width = a - b$$



The area of the $\underbrace{polygon\ in\ Step\ I}_{a^2-b^2}$ must match the area of the $\underbrace{rectangle\ in\ Step\ III}_{(a+b)(a-b)}$. Hence:

$$(a + b)(a - b) = a^2 - b^2$$

2.4: Transitive Property of Equality

If two things are equal to the same thing, then they are equal to each other.

$$\because a = b, b = c \Rightarrow a = c$$

2.5: Counting the Elements of a Set Twice

If, for a set S, when counted in one way $n(S) = a$, and when counted in a different way, $n(S) = b$, then by the transitive property of equality

$$a = n(S) = b \Rightarrow a = b$$

Example 2.6

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

A. Prove the identity above by using counting arguments.
B. Prove the identity above by using the binomial theorem.

Part A

I want to select zero or more out of n distinguishable objects. For each object, I have two choices (to pick or not pick):

$$\underbrace{2}_{\substack{First\\Object}} \times \underbrace{2}_{\substack{Second\\Object}} \times \ldots \times \underbrace{2}_{\substack{N^{th}\\Object}} \Rightarrow 2^n = LHS$$

I can also pick zero or more out of n distinguishable objects by picking $0\ Objects, 1\ Object, \ldots, n\ objects$
And the number of ways to do this is:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = RHS$$

But the number of ways to choose the objects has to be the same, irrespective of which method we use to count it. Hence, by the transitive property of equality:

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

Part B

By the binomial theorem:

$$n \in \mathbb{N} \Rightarrow (x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} x^0 y^n$$

Substitute $x = 1, y = 1$ in the above:

$$LHS = (x+y)^n = (1+1)^n = 2^n$$
$$RHS = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

And we know that:

$$LHS = RHS \Rightarrow 2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

2.7: Casework Approach

You may need to break the property that you want to prove into different cases, and then handle each separately.

Example 2.8: Inequalities

A. Prove the trivial inequality: $a^2 \geq 0$

B. Prove the AM-GM Inequality: $\frac{a+b}{2} \geq \sqrt{ab}, a, b \in \mathbb{R}^+$

C. Prove Cauchy-Schwarz, which says given two sets of numbers $A = \{a_1, a_2, \dots, a_n\}, B = \{b_1, b_2, \dots, b_n\}$: $(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$

Part A

We have three cases

$$a = 0 \Rightarrow Satisfies$$
$$a > 0 \Rightarrow a^2 > 0$$
$$a < 0 \Rightarrow a = (-x), x > 0 \Rightarrow a^2 = (-x)^2 = x^2 > 0 \Rightarrow a^2 > 0$$

Part B

Substitute the definitions, clear fractions, square, move all terms to RHS and then factor to get:

$$AM \geq GM \Leftrightarrow \frac{a+b}{2} \geq \sqrt{ab} \Leftrightarrow a + b \geq 2\sqrt{ab} \Leftrightarrow a^2 + 2ab + b^2 \geq 4ab \Leftrightarrow (a-b)^2 \geq 0$$

The last always holds by the trivial inequality.

Note the use of biconditionals throughout. This is what lets us move "forward", since the steps are reversible. In general, we cannot begin with something that needs to be proved and manipulate it.

Part C

Consider the expression:

$$X = (a_1 x - b_1)^2 + (a_2 x - b_2)^2 + \cdots + (a_n x - b_n)^2$$

Expand:

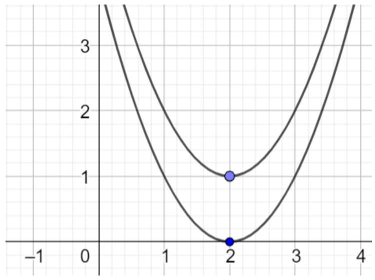
$$(a_1^2 x^2 - 2x a_1 b_1 + b_1^2) + (a_2^2 x^2 - 2x a_2 b_2 + b_2^2) + \cdots + (a_n^2 x^2 - 2x a_n b_n + b_n^2)$$

Collate all like terms together:

$$(a_1^2 x^2 + a_2^2 x^2 + \cdots + a_n^2 x^2) + (-2x a_1 b_1 - 2x a_2 b_2 - \cdots - 2x a_n b_n) + (b_1^2 + b_2^2 + \cdots + b_n^2)$$

Factor:

$$x^2(a_1^2 + a_2^2 + \cdots + a_n^2) - 2x(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) + (b_1^2 + b_2^2 + \cdots + b_n^2)$$



The above is now a quadratic in x.

Since X is the sum of perfect squares, we must have:

$$X \geq 0 \Rightarrow Zero\ x\ intercept, or\ single\ x\ intercept$$
$$\Rightarrow Discriminant \leq 0 \Rightarrow b^2 - 4ac \leq 0$$

Substitute the values from the quadratic above:

$$[-2(a_1b_1 + a_2b_2 + \cdots + a_nb_n)]^2 \leq 4(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$$

Simplify and divide both sides by 4:

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$$

C. Warning

Example 2.9

A student came up with the following proof that $6 = 5$. Explain what is wrong:

$$6 = 5$$

Multiply both sides by zero:

$$6 \times 0 = 5 \times 0$$

Simplify:

$$0 = 0$$

You cannot start by proving what you wish to prove.

D. Summation and Applications

Example 2.10: Arithmetic Series and Geometric Series

Show that the sum to n terms of

A. an arithmetic series $S = a + a + d + \cdots + a + (n-1)d$ with first term a, and common difference d, is $S_n = \frac{n}{2}[2a + (n-1)d]$.

B. a geometric series $S = a + ar + \cdots + ar^{n-1}$ with first term a, and common ratio r, is $S_n = \frac{a(1-r^n)}{1-r}$

Part A

We write the sum twice, first in regular order, and then back to front.

$$S = \underbrace{a}_{\substack{1st\\Term}} + \underbrace{a+d}_{\substack{2nd\\Term}} + \cdots + \underbrace{a+(n-1)d}_{\substack{n^{th}\\Term}}$$

$$S = \underbrace{a+(n-1)d}_{\substack{n^{th}\\Term}} + \underbrace{a+(n-2)d}_{\substack{(n-1)^{st}\\Term}} + \cdots + \underbrace{a}_{\substack{1st\\Term}}$$

Add the above two.

The left-hand side is simply:

$$2S$$

The right hand is a little more complicated. But we add term by term:

$$1^{st}\ Term = a + a + (n-1)d = 2a + (n-1)d$$
$$2^{nd}\ Term = a + d + a + (n-2)d = 2a + d(1 + n - 2) = 2a + (n-1)d$$
$$.$$
$$.$$
$$.$$
$$Last\ Term = a + (n-1)d + a = 2a + (n-1)d$$

Now, each term is the same, and we have n terms. Hence, the total is:

$$n[2a+(n-1)d]$$

And, now we bring the left-hand side and the right-hand side together:

$$2S = n[2a+(n-1)d] \Rightarrow S = \frac{n}{2}[2a+(n-1)d]$$

Part B

Multiply both sides of the given series by r:

$$rS = \qquad ar + ar^2 + \cdots + ar^{n-1} + ar^n$$
$$S \;=\; a + ar + ar^2 + \cdots + ar^{n-1}$$

$$rS - S = ar^n - a \Rightarrow S(r-1) = a(r^n - 1) \Rightarrow S = \frac{a(r^n-1)}{r-1} = \frac{a(1-r^n)}{1-r}$$

Example 2.11

For $-1 < r < 1$:

$$n \to \infty \Rightarrow r^n \to 0$$

Hence, for an infinite geometric series, the formula for the sum simplifies to:

$$\frac{a(1-r^n)}{1-r} = \frac{a(1-0)}{1-r} = \frac{a(1)}{1-r} = a$$

Example 2.12

Show that the remainder when a polynomial function $f(x)$ is divided by a linear expression $x - a$ is the same as the value of the function at the root of the expression.

$$\frac{f(x)}{x-a} = \underbrace{P(x)}_{Quotient} + \frac{r}{x-a} \Rightarrow r = Remainder = f(a)$$

$f(x) = 7, P(x) = 1, r = 2, Divisor = 5$:

$$\frac{7}{5} = 1 + \frac{2}{5} \Rightarrow 7 = (5)(1) + 2$$

$$\frac{f(x)}{x-a} = \underbrace{P(x)}_{Quotient} + \frac{r}{x-a}$$

The above is an identity since it is always true for some value of $P(x)$, and some value of r. Multiply both sides of the above identity by equality by $x - a$ to get:

$$f(x) = \underbrace{(x-a)}_{Divisor}\underbrace{P(x)}_{Quotient} + \underbrace{r}_{Remainder}$$

Substitute $x = a$:

$$f(a) = (a-a)P(x) + r = (0)P(x) + r = 0 + r = r$$

Example 2.13

$$p^n - q^n = (p-q)(p^{n-1} + p^{n-2}q + \cdots + q^{n-1}), n \in \mathbb{N}$$

A. Show that the above is true for $n = 2, n = 3$
B. Use the Remainder Theorem to show that the LHS is divisible by $p - q$.
C. Use the formula for the sum of a geometric series to show that the above is an identity.

Part A

$$p^2 - q^2 = (p-q)(p+q)$$
$$p^4 - q^4 = (p-q)(p^3q^0 + p^2q + pq^2 + p^0q^3)$$

Part B

Consider this as a polynomial in p.

$$p - q = 0 \Rightarrow p = q$$

Substitute $p = q$ in $p^n - q^n$:

$$q^n - q^n = 0$$

Part C

Note that the expression in the second bracket of the RHS is a geometric series with first term p^{n-1} and common ratio $\frac{q}{p}$. Substitute $First\ Term = a = p^{n-1}, Common\ Ratio = r = \frac{q}{p}$ in the formula for the sum of a geometric series:

$$S = \frac{a(1-r^n)}{1-r} = \frac{p^{n-1}\left[1-\left(\frac{q}{p}\right)^n\right]}{1-\frac{q}{p}} = \frac{p^{n-1}\left[\frac{p^n-q^n}{p^n}\right]}{\frac{p-q}{p}}$$

Dividing by a fraction is the same as multiplying by its reciprocal:

$$p^{n-1}\left[\frac{p^n-q^n}{p^n}\right] \times \frac{p}{p-q} = \left[\frac{p^n-q^n}{p^n}\right] \times \frac{p^n}{p-q} = \left[\frac{p^n-q^n}{p-q}\right]$$

Hence, the RHS simplifies to:

$$(p-q)(p^{n-1} + p^{n-2}q + \cdots + q^{n-1}) = (p-q)\left[\frac{p^n-q^n}{p-q}\right] = p^n - q^n$$

Repeat the above question for:

$$p^n + q^n = (p+q)(p^{n-1} - p^{n-2}q + \cdots + q^{n-1}), \qquad n = 2m+1, \qquad m \in \mathbb{N}$$

Part A

$$p^2 + q^2$$
$$p^3 + q^3 = (p+q)(p^2 - pq + q^2)$$

Part B

Consider this as a polynomial in p.

$$p + q = 0 \Rightarrow p = -q$$

Substitute $p = -q$ in $p^n + q^n$:

$$(-q)^n + q^n = 0 \Rightarrow -q^n + q^n = 0$$

Part C

Note that the expression in the second bracket of the RHS is a geometric series with first term p^{n-1} and common ratio $\left(-\frac{q}{p}\right)$. Substitute $First\ Term = a = p^{n-1}, Common\ Ratio = r = -\frac{q}{p}$ in the formula for the sum of a geometric series:

$$S = \frac{a(1-r^n)}{1-r} = \frac{p^{n-1}\left[1-\left(-\frac{q}{p}\right)^n\right]}{1-\left(-\frac{q}{p}\right)} = \frac{p^{n-1}\left[1-\left(-\frac{q^n}{p^n}\right)\right]}{1-\left(-\frac{q}{p}\right)} = \frac{p^{n-1}\left[\frac{p^n+q^n}{p^n}\right]}{\frac{p+q}{p}}$$

Dividing by a fraction is the same as multiplying by its reciprocal:

$$p^{n-1}\left[\frac{p^n+q^n}{p^n}\right] \times \frac{p}{p+q} = \left[\frac{p^n+q^n}{p^n}\right] \times \frac{p^n}{p+q} = \left[\frac{p^n+q^n}{p+q}\right]$$

Hence, the RHS simplifies to:

$$(p+q)(p^{n-1} - p^{n-2}q + \cdots + q^{n-1}) = (p+q)\left[\frac{p^n+q^n}{p+q}\right] = p^n + q^n$$

Example 2.14: Arithmetico-Geometric Series

$$a+(a+d)r+(a+2d)r^2+\cdots+[a+(n-1)d]r^{n-1}$$

Find the value of $2^{\frac{1}{4}}\cdot 4^{\frac{1}{8}}\cdot 8^{\frac{1}{16}}\cdot\ldots\infty$ (JEE Main 2002)

Convert each term to have a prime factor base:

$$(2^1)^{\frac{1}{4}}\cdot(2^2)^{\frac{1}{8}}\cdot(2^3)^{\frac{1}{16}}\cdot\ldots\infty$$

Use the exponent law $(a^m)^n=a^{mn}$:

$$2^{\frac{1}{4}}\cdot 2^{\frac{2}{8}}\cdot 2^{\frac{3}{16}}\cdot\ldots\infty$$

Use the exponent law $a^{x_1}\cdot a^{x_2}\cdot a^{x_3}\cdot\ldots=a^{x_1+x_2+x_3+\cdots}$

$$2^{\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\cdots}$$

$$\textit{Numerators form an arithmetic series}: 1,2,3,\ldots$$

$$\textit{Denominators form a geometric series}: 4,8,16,\ldots$$

We want to find the value of the exponent:

$$\underbrace{Let\ Sum=S=\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\cdots}_{\textit{Equation I}}$$

Multiply both sides of the above by $\frac{1}{2}$:

$$\underbrace{\frac{1}{2}S=0+\frac{1}{8}+\frac{2}{16}+\frac{3}{32}+\cdots}_{\textit{Equation II}}$$

Subtract Equation II from Equation I:

$$\frac{1}{2}S=\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

The above is a geometric series with

$$First\ Term=a=\frac{1}{4},\qquad Common\ Ratio=\frac{1}{2}$$

Substitute the above in the formula for the sum of an infinite geometric series:

$$\frac{1}{2}S=\frac{a}{1-r}=\frac{\frac{1}{4}}{1-\frac{1}{2}}=\frac{\frac{1}{4}}{\frac{1}{2}}=\frac{1}{4}\times 2=\frac{1}{2}\Rightarrow S=1$$

Hence,

$$2^{\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\cdots}=2^S=2^1=2$$

E. Axioms

2.15: Geometry

- A straight line can be drawn from one point to any other point
- A terminated line can be further produced indefinitely
- A circle can be drawn with any center and radius
- All right angles are equal to one another

2.16: 5Th Axiom

- If you have a line l, and a point P not on the line, you can draw exactly one line through P parallel to l.
- If a straight line falling on two other straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines if produced indefinitely meet the side on which the sum of angles is less than two right angles.

Riemannian

Example 2.17
Sum of Angles of a Triangle

2.18: Sets
A set, informally, is a well-defined collection of objects.

We do not define a set since trying to define a set results in circular logic.

Example 2.19: Clock Arithmetic
Consider a mathematical system where the numbers are:
$$\{0,1,2,3,\dots,24\}$$
And also
$$24 = 0$$

Currently it is 20: 00 hours. What is the time 20 hours from now?

$$20 + 20 = 20 + 4 + 16 = 24 + 16 = 0 + 16 = 16$$

In regular natural numbers:
$$25 \neq 1$$

However, in the system we just defined above:
$$25 \equiv 1$$

2.2 Mathematical Induction-I: Algebra

A. Summation Formulas

Many summation formulas of different types can be proved using mathematical induction. We consider a variety below.

2.20: Proof by Induction
Base Case: Show that a property is true for some natural number n.
Inductive Case: Show that if it is true for some $n, n \in \mathbb{N}$, then it is true for $n + 1$.

Example 2.21: Basics
Prove the following formulas for the sum of a series using mathematical induction:

A. The sum of the first n natural numbers gives us the n^{th} $triangular\ numbers$. Show that the n^{th} $triangular\ number$ is given by $1 + 2 + \dots + n = \frac{n(n+1)}{2}, n \in \mathbb{N}$
B. The sum of the first n odd numbers gives us the n^{th} square number. Show that the n^{th} $square\ number$ is given by $1 + 3 + \dots + (2n - 1) = n^2, n \in \mathbb{N}$

Part A
Base Case: $n = 1$:
$$LHS = 1 + 2 + \dots + n = 1$$
$$RHS = \frac{n(n+1)}{2} = \frac{1(2)}{2} = 1$$
$$LHS = RHS \Rightarrow Base\ Case\ is\ valid$$

Inductive Case:
Let the statement be true for k:
$$1 + 2 + \dots + k = \frac{k(k+1)}{2}$$
Add $k + 1$ to both sides:
$$\text{LHS} = 1 + 2 + \dots + k + k + 1$$

$$RHS = \frac{k^2 + k + 2k + 2}{2}$$
$$= \frac{(k+1)(k+2)}{2}$$
$$= \frac{(k+1)\big((k+1)+1\big)}{2}$$

Hence, it is true for $k+1$. This completes the proof.

Part B

Base Case: $n = 1$:

$$LHS = 1 + 3 + \cdots + (2n-1) = 1$$
$$RHS = n^2 = 1$$
$$LHS = RHS \Rightarrow Base\ Case\ is\ valid$$

Let the statement be true for k:

$$1 + 3 + \cdots + (2k-1) = k^2$$

Add $(2(k+1)-1) = 2k+1$ to both sides:

$$\text{LHS} = 1 + 3 + \cdots + (2k-1) + (2k+1)$$
$$RHS = k^2 + 2k + 1 = (k+1)^2$$

Hence, it is true for $k+1$. This completes the proof.

Example 2.22: Sums of Perfect Powers

The squares, cubes and so on of the natural numbers have a set of formulas which can be proved using mathematical induction. The key skills needed here are factoring and manipulation.

A. $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}, n \in \mathbb{N}$

B. $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Part A

Base Case: $n = 1$:

$$LHS = 1^2 + 2^2 + \cdots + n^2 = 1$$
$$RHS = \frac{n(n+1)(2n+1)}{6} = \frac{1(2)(3)}{6} = 1$$
$$LHS = RHS \Rightarrow Base\ Case\ is\ valid$$

Inductive Case:

Let the statement be true for k:

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Add $(k+1)^2$ to both sides:

$$LHS = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2$$
$$RHS = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

Factor out $(k+1)$:

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$
$$= \frac{(k+1)[2k^2 + k + 6k + 6]}{6}$$
$$= \frac{(k+1)[2k^2 + 7k + 6]}{6}$$
$$= \frac{(k+1)[2k^2 + 3k + 4k + 6]}{6}$$
$$= \frac{(k+1)[k(2k+3) + 2(2k+3)]}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}$$
$$= \frac{(k+1)\big((k+1)+1\big)(2(k+1)+1)}{6}$$

Hence, it is true for $k+1$. This completes the proof.

Part B

Base Case: $\boldsymbol{n=1}$:

$$LHS = 1^3 + 2^3 + \cdots + n^3 = 1$$
$$RHS = \left[\frac{n(n+1)}{2}\right]^2 = \left[\frac{1(2)}{2}\right]^2 = 1^2 = 1$$
$$LHS = RHS \Rightarrow Base\ Case\ is\ valid$$

Inductive Case:

Let the statement be true for k:

$$1^3 + 2^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$$

Add $(k+1)^3$ to both sides:

$$LHS = 1^3 + 2^3 + \cdots + k^3 + (k+1)^3$$
$$RHS = \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$$
$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$
$$= \frac{(k+1)^2[k^2 + 4(k+1)]}{4}$$
$$= \frac{(k+1)^2(k+2)^2}{4}$$
$$= \left[\frac{(k+1)(k+2)}{2}\right]^2$$

Hence, it is true for $k+1$. This completes the proof.

Example 2.23: Multiplication

In some cases, factoring may not be easy. However, we can still achieve the desired result through multiplication.

Prove using mathematical induction that the sum of the squares of the first n odd natural numbers is equal to $\frac{n(4n^2-1)}{3}$

Write out the verbal statement in mathematical form.

We need to prove

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$$

Base Case: $\boldsymbol{n=1}$:

$$LHS = 1^2 + 3^2 + \cdots + (2n-1)^2 = 1$$
$$RHS = \frac{n(4n^2-1)}{3} = \frac{1[(4(1)^2-1)]}{3} = \frac{3}{3} = 1$$
$$LHS = RHS \Rightarrow Base\ Case\ is\ valid$$

Inductive Case:

Let the statement be true for $n=k$:

$$1^2 + 3^2 + \cdots + (2k-1)^2 = \frac{k(4k^2-1)}{3}$$

Add $(2k+1)^2$ to both sides:

The left-hand side becomes:

$$1^2 + 3^2 + \cdots + (2k-1)^2 + (2k+1)^2$$

The right-hand side becomes

$$\frac{k(4k^2-1)}{3} + (2k+1)^2 = \frac{4k^3 + 12k^2 + 11k + 3}{3}$$

We need to show that the RHS is equal to $\frac{(k+1)[(4(k+1)^2-1)]}{3}$. If we have done the math correctly, $k+1$ will be a factor of the numerator, and we could divide using either polynomial long division, or synthetic division and then proceed.

However, we instead show that $\frac{(k+1)[(4(k+1)^2-1)]}{3}$

$$=\frac{(k+1)[((4k^2+8k+3))]}{3}$$

$$=\frac{4k^3+8k^2+3k+4k^2+8k+3}{3}$$

$$=\frac{4k^3+12k^2+11k+3}{3}=RHS$$

Which is what we wished to prove.

2.24: Induction starting from zero

- It is not necessary that the induction always goes for n from 1 to infinity.
- In the next example, we see a case where induction starts from $n=0$.
- In general, induction can start from any value of n.

Example 2.25: Exponents

If the variable n is in the exponent, then it is logical that we will need to work with laws of exponents. We start with a simple (but important) property that should be committed to memory.

$$2^0+2^1+\cdots+2^n=2^{n+1}-1, n\ge 0, n\in\mathbb{W}$$

A. Prove the above using the formula for the sum of a geometric series.
B. Prove the above using mathematical induction.

Part A

The sum of n terms of a geometric series with first term a, and common ratio r is given by

$$\frac{a(r^n-1)}{r-1}$$

Substitute $a=1, r=2, n=n+1$ in the above formula:

$$\frac{1(2^{n+1}-1)}{2-1}=2^{n+1}-1$$

Part B

Base Case: $n=0$:

$$LHS=2^0+2^1+\cdots+2^n=1$$

$$RHS=2^{n+1}-1=2-1=1$$

$$LHS=RHS\Rightarrow Base\ Case\ is\ valid$$

Inductive Case:

Let the statement be true for k:

$$2^0+2^1+\cdots+2^k=2^{k+1}-1$$

Add 2^{k+1} both sides:

$$LHS=2^0+2^1+\cdots+2^k+2^{k+1}$$

$$RHS=2^{k+1}-1+2^{k+1}$$

$$=2\cdot 2^{k+1}-1$$

$$=2^{k+2}-1$$

Example 2.26: Arithmetic Series and Geometric Series

Show, using mathematical induction that the sum to n terms of:

A. a geometric series $S=a+ar+\cdots+ar^{n-1}$ with first term a, and common ratio r, is $S_n=\frac{a(1-r^n)}{1-r}$
B. an arithmetic series $S=a+a+d+\cdots+a+(n-1)d$ with first term a, and common difference d, is $S_n=\frac{n}{2}[2a+(n-1)d]$.

Part A

Base Case: $n=1$:

$$LHS=a+ar+\cdots+ar^{n-1}=a$$

$$RHS=\frac{a(1-r^n)}{1-r}=\frac{a(1-r^1)}{1-r}=a$$

$$LHS=RHS\Rightarrow Base\ Case\ is\ valid$$

Inductive Case:

Let the statement be true for k:

$$a+ar+\cdots+ar^{k-1}=\frac{a(1-r^k)}{1-r}$$

Add ar^k to both sides:

$$LHS=a+ar+\cdots+ar^{k-1}+ar^k$$

$$RHS=\frac{a(1-r^k)}{1-r}+ar^k$$

$$=\frac{a-ar^k+ar^k-ar^{k+1}}{1-r}$$

$$=\frac{a-ar^{k+1}}{1-r}$$

$$=\frac{a(1-r^{k+1})}{1-r}$$

Part B

Base Case: $n = 1$:

$$LHS = a + a + d + \cdots + a + (n-1)d \ = a$$
$$RHS = \frac{n}{2}[2a + (n-1)d] = \frac{1}{2}[2a + (1-1)d] = a$$
$$LHS = RHS \Rightarrow Base\ Case\ is\ valid$$

Inductive Case:

Let the statement be true for k:

$$a + d + \cdots + a + (k-1)d = \frac{k}{2}[2a + (k-1)d]$$

Add $a + kd$ to both sides:

$$LHS = a + d + \cdots + a + (k-1)d + a + kd$$

$$RHS = \frac{k}{2}[2a + (k-1)d] + a + kd$$
$$= \frac{2ak + k^2d - kd + 2a + 2kd}{2}$$
$$= \frac{2ak + 2a + k^2d + kd}{2}$$
$$= \frac{2a(k+1) + kd(k+1)}{2}$$
$$= \frac{(k+1)}{2}[2a + kd]$$
$$= \frac{(k+1)}{2}[2a + (k+1-1)d]$$

Hence, it is true for $k + 1$. This completes the proof.

Example 2.27: Expressions

Sometimes the individual terms in a summation formula can be expressions. Prove the below properties using mathematical induction:

A. $(1)(1) + (2)(3) + (3)(5) + \cdots + (n)(2n-1) = \frac{n(n+1)(4n-1)}{6}$
B. $(1)(1) + (3)(2) + (5)(4) + \cdots + (2n-1)(2^{n-1}) = 3 + 2^n(2n-3)$

Part A

Base Case: $n = 1$:

$$LHS = (1)(1) + (2)(3) + (3)(5) + \cdots + (n)(2n-1) = 1$$
$$RHS = \frac{n(n+1)(4n-1)}{6} = \frac{1(2)(3)}{6} = 1$$
$$LHS = RHS \Rightarrow Base\ Case\ is\ valid$$

Inductive Case:

Let the statement be true for k:

$$(1)(1) + (2)(3) + (3)(5) + \cdots + (k)(2k-1) = \frac{k(k+1)(4k-1)}{6}$$

Add $(k+1)(2(k+1)-1) = (k+1)(2k+1)$ to both sides:

$$(1)(1) + (2)(3) + (3)(5) + \cdots\ + (k)(2k-1) + (k+1)(2k+1) = \frac{k(k+1)(4k-1)}{6} + (k+1)(2k+1)$$

Work only with the RHS. Factor $k + 1$ from the two terms and add:

$$= (k+1)\left[\frac{4k^2 - k + 12k + 6}{6}\right] = (k+1)\left[\frac{4k^2 + 11k + 6}{6}\right]$$

Factor the numerator using grouping:

$$4k^2 + 8k + 3k + 6 = 4k(k+2) + (k+2) = (k+2)(4k+3) = (k+2)(4(k+1)-1)$$

Substitute the factored numerator back in the RHS:

$$(1)(1) + (2)(3) + (3)(5) + \cdots + (k)(2k-1) + (k+1)(2k+1) = \frac{(k+1)(k+2)(4(k+1)-1)}{6}$$

Hence, it is true for $k + 1$. This completes the proof.

Part B

Base Case: $n = 1$:

$$LHS = (1)(1) + (3)(2) + (5)(4) + \cdots + (2n-1)(2^{n-1}) = 1$$
$$RHS = 3 + 2^n(2n-3) = 3 + 2^1(2(1)-3) = 1$$
$$LHS = RHS \Rightarrow Base\ Case\ is\ valid$$

Inductive Case:

Let the statement be true for k:

$$(1)(1)+(3)(2)+(5)(4)+\cdots+(2k-1)\left(2^{k-1}\right)=3+2^k(2k-3)$$

Add $(2k+1)\left(2^k\right)$ to both sides:

$$(1)(1)+(3)(2)+(5)(4)+\cdots+(2k-1)\left(2^{k-1}\right)+(2k+1)\left(2^k\right)=3+2^k(2k-3)+(2k+1)\left(2^k\right)$$

Work only with the RHS. Factor 2^k from the second and the third terms:

$$=3+2^k(2k-3+2k+1)$$
$$=3+2^k(4k-2)$$
$$=3+2^k(4(k+1)-6)$$
$$=3+2^{k+1}(2(k+1)-3)$$

Hence, it is true for $k+1$. This completes the proof.

Example 2.28: Rational Expressions

Working with rational expressions can be trickier. Prove the below properties using mathematical induction:

A. $\frac{1}{2(3)}+\frac{1}{3(4)}+\cdots+\frac{1}{(n+1)(n+2)}=\frac{n}{2(n+2)}$
B. $\frac{1}{1(3)}+\frac{1}{2(4)}+\cdots+\frac{1}{n(n+2)}=\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}$

Part A

Base Case: $n=1$:

$$LHS=\frac{1}{2(3)}+\frac{1}{3(4)}+\cdots+\frac{1}{(n+1)(n+2)}=\frac{1}{2(3)}=\frac{1}{6}$$
$$RHS=\frac{n}{2(n+2)}=\frac{1}{2(3)}=\frac{1}{6}$$
$$LHS=RHS\Rightarrow Base\ Case\ is\ valid$$

Inductive Case:

Let the statement be true for k:

$$\frac{1}{2(3)}+\frac{1}{3(4)}+\cdots+\frac{1}{(k+1)(k+2)}=\frac{k}{2(k+2)}$$

Add $\frac{1}{(k+2)(k+3)}$ to both sides:

$$\frac{1}{2(3)}+\frac{1}{3(4)}+\cdots+\frac{1}{(k+1)(k+2)}+\frac{1}{(k+2)(k+3)}=\frac{k}{2(k+2)}+\frac{1}{(k+2)(k+3)}$$

Work only with the RHS. Add the fractions:

$$\frac{k(k+3)+2}{2(k+2)(k+3)}=\frac{k^2+3k+2}{2(k+2)(k+3)}=\frac{(k+1)(k+2)}{2(k+2)(k+3)}=\frac{k+1}{2(k+3)}$$

Hence, it is true for $k+1$. This completes the proof.

Part B

Base Case: $n=1$:

$$LHS=\frac{1}{1(3)}+\frac{1}{2(4)}+\cdots+\frac{1}{n(n+2)}=\frac{1}{3}$$
$$RHS=\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}=\frac{3}{4}-\frac{2+3}{2(2)(3)}=\frac{3}{4}-\frac{5}{12}=\frac{4}{12}=\frac{1}{3}$$
$$LHS=RHS\Rightarrow Base\ Case\ is\ valid$$

Inductive Case:

Let the statement be true for k:

$$\frac{1}{1(3)}+\frac{1}{2(4)}+\cdots+\frac{1}{k(k+2)}=\frac{3}{4}-\frac{2k+3}{2(k+1)(k+2)}$$

Add $\frac{1}{(k+1)(k+3)}$ to both sides:

$$\frac{1}{1(3)}+\frac{1}{2(4)}+\cdots+\frac{1}{k(k+2)}+\frac{1}{(k+1)(k+3)}=\frac{3}{4}-\frac{2k+3}{2(k+1)(k+2)}+\frac{1}{(k+1)(k+3)}$$

Work only with the RHS. Add the fractions:

$$\frac{3}{4}-\frac{[2k+3](k+3)-2(k+2)}{2(k+1)(k+2)(k+3)}$$
$$=\frac{3}{4}-\frac{(2k^2+9k+9)-2(k+2)}{2(k+1)(k+2)(k+3)}$$
$$=\frac{3}{4}-\frac{2k^2+7k+5}{2(k+1)(k+2)(k+3)}$$

Factor the numerator:

$$=\frac{3}{4}-\frac{(2k+5)(k+1)}{2(k+1)(k+2)(k+3)}$$
$$=\frac{3}{4}-\frac{(2k+5)}{2(k+2)(k+3)}$$
$$=\frac{3}{4}-\frac{2k+2+3}{2(k+2)(k+3)}$$
$$=\frac{3}{4}-\frac{2(k+1)+3}{2(k+2)(k+3)}$$

Hence, it is true for $k+1$. This completes the proof.

B. Divisibility

Divisibility properties are often easier to prove without using mathematical induction than with. You can decide your preference after comparing both the methods. However, keep in mind that induction is a powerful technique that should be in your toolkit, and even if induction is not your preferred method, you should still be able to solve divisibility questions using induction.

Example 2.29: Basics

$$n^3-n, n\in\mathbb{Z} \text{ is divisible by } 6.$$

A. Prove the above property using factorization and remainders.
B. Prove the above property using induction. You may combine induction with other techniques.

Hint 1: Find a way to connect the integers to natural numbers.
Hint 2: $(a+b)^3=a^3+3a^2b+3ab^2+b^3$

Part A: Proof using Remainders

$$n^3-n=n(n^2-1)=n(n+1)(n-1)$$

Divisibility by 3:
The last expression is the product of three consecutive integers which will have three distinct remainders when divided by 3.
Since the set of remainders when a number is divided by 3 is $\{0,1,2\}$, at least one of the three remainders is zero.
The number with remainder zero is then a multiple of 3.
Which makes the entire expression a multiple of 3.
Divisibility by 2:
Out of two consecutive numbers, at least one of them must be even. Hence, the expression is divisible by 2.
If an expression is divisible by 2, and it is divisible by 3, then it is divisible by:

$$lcm(2,3)=6$$

Part B: Proof using Induction

Before we do the induction, consider

$$n\le 0, n\in\mathbb{Z}\Rightarrow n=-m, m\in\mathbb{N}$$

Then:

$$n^3-n=(-m)^3-(-m)=-m^3+m$$
$$=-(m^3-m)$$

Hence, if

$$m^3-m \textit{ is divisible by } 3\Rightarrow n^3-n \textit{ is divisible by } 3$$

Hence, it is sufficient to prove the property using induction for

$$n \in \{0,1,2,3,\dots\}$$

Base Case: $n = 0$:

$$n^3 - n = 0 - 0 = 0$$
$$Base\ Case\ is\ Valid$$

Inductive Case:

Let the statement be true for k.

$$k^3 - k\ is\ divisible\ by\ 6$$

Consider the next number after k, which is $k+1$.
We must then show that $(k+1)^3 - (k+1)$ is divisible by 3.

Expanding:

$$k^3 + 3k^2 + 3k + 1 - k - 1$$

Simplifying, rearranging and factoring:

$$= \underbrace{k^3 - k}_{\substack{Divisible \\ by\ 6}} + \underbrace{3k(k+1)}_{\substack{Divisible \\ by\ 3}}$$

Out of k and $k+1$, exactly one number is divisible by 2. Hence, $3k(k+1)$ is divisible by 6.
Since each term is divisible by 6, the expression is divisible by 6.
Hence, it is true for $k+1$. This completes the proof.

Example 2.30: Binomial Expansion

$$n^5 - n, n \in \mathbb{Z} \text{ is divisible by 30.}$$

Prove the above property using induction. You may combine induction with other techniques.

Hint 1: Find a way to connect the integers to natural numbers.
Hint 2: $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

To prove that a number is divisible, it has to be divisible by 5 *and* 6

$$n^5 - n = n(n^4 - 1) = n(n^2 - 1)(n^2 + 1) = (n^3 - n)(n^2 + 1)$$

As proved earlier, $n^3 - n$ is divisible by 6. Hence, we only need to show that $n^5 - n$ is divisible by 5.

Part A: Factorization

$$n(n-1)(n+1)(n^2+1)$$
$$= n(n-1)(n+1)(n^2 - 4 + 5)$$
$$= n(n-1)(n+1)[(n+2)(n-2) + 5]$$

Apply the distributive property:

$$= \underbrace{n(n-1)(n+1)(n+2)(n-2)}_{\substack{Five\ Consecutive\ Integers \\ Divisible\ by\ 5}} + \underbrace{5n(n-1)(n+1)}_{Divisible\ by\ 5}$$

Part B: Induction

Before we do the induction, consider

$$n \le 0, n \in \mathbb{Z} \Rightarrow n = -m, m \in \mathbb{N}$$

Then:

$$n^5 - n = (-m)^5 - (-m) = -m^5 + m = -(m^5 - m)$$

Hence, if

$$m^5 - m\ is\ divisible\ by\ 3 \Rightarrow n^3 - n\ is\ divisible\ by\ 3$$

Hence, it is sufficient to prove the property using induction for

$$n \in \{0,1,2,3,\dots\}$$

Base Case:$n = 0$

$$n^5 - n = 0 - 0 = 0 \Rightarrow Divisible\ by\ 5$$

Inductive Case

Let the statement be true for k.

$$k^5 - k\ is\ divisible\ by\ 5$$

Consider the next number after k, which is $k+1$. We must then show that $(k+1)^5 - (k+1)$ is divisible by 5.

$$k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$$
$$\underbrace{k^5 - k}_{\substack{Divisible \\ by\ 5}} + \underbrace{5(k^4 + 2k^3 + 2k^2 + k)}_{\substack{Divisible \\ by\ 5}}$$

2.31: Parity

Parity refers to whether a number is even or odd. A property that is not generally true, can be true of integers of a specific parity.

As a trivial example, all integers are not divisible by 2, but all even integers are.

Example 2.32

Let $x_1, x_2, \dots, x_n$ denote some permutation of the numbers $1,2, \dots n$. For how many odd integers $n \le 2022$ does there exist such a permutation $x_1, x_2, \dots, x_n$ such that the product $(x_1 - 1)(x_2 - 2) \dots (x_n - n)$ is odd? (JHMMC 2022 R1/19)

We prove that this does not work in general using mathematical induction.

Base Case:

For $n = 1$:

$$1 - 1 = 0 \Rightarrow Not\ possible$$

For $n = 3$: 1,2,3

$$(2-1)(3-2)(1-3)$$
$$(E-1)(O-2)(E-3)$$

Inductive Case:

Assume that for n=k, it does not work.

We need to show that it does not work for n=k+2 if it is true for $n = k$

When n increases to its next possible number, the number of odd numbers increases by 1, and the number of even numbers increases also by 1.

Hence, the issue where the number of even numbers required is more than what is available remains, and hence this is not possible.

Since we have proved the inductive case, it is not possible for any odd number.

Example 2.33

Prove that $n^2 - 1$ is divisible by 8 for odd values of n

A. Using Factorization
B. Using Mathematical Induction

Part A: Using Factorization

$$n^2 - 1 = (n+1)(n-1)$$
$$n\ is\ odd \Rightarrow n+1\ is\ even$$
$$n\ is\ odd \Rightarrow n-1\ is\ even$$

Further $n - 1$ and $n + 1$ are consecutive even numbers.

Hence, one of them is a multiple of 4.

$$n^2 - 1 = \underbrace{(n+1)(n-1)}_{2\times 4=8}$$

Hence, $n^2 - 1$ is divisible by 8.

Part B: Using Mathematical Induction

For $n < 0 \Rightarrow n = -m, m > 0$:

$$n^2 - 1 = (-m)^2 - 1 = m^2 - 1$$

Hence, we only do the induction for positive values of n.

Base Case: $n = 1$:

$$n^2 - 1 = 1 - 1 = 0 \Rightarrow Divisible\ by\ 8$$
$$Base\ Case\ is\ Valid$$

Inductive Case:

Let the statement be true for k.

Consider the next odd number after k.

$$(k+2)^2 - 1 = k^2 + 4k + 4 - 1 = \underbrace{k^2 - 1}_{\substack{Divisible \\ by\ 8}} + 4k + 4$$

It remains to show that $4k + 4$ is divisible by 8.

k is of the form $2j + 1, j \in \mathbb{N}$

$$4k + 4 = 4(2j+1) + 4 = 8j + 8 = 8(j+1)$$

Which is a multiple of 8.

Hence, it is true for $k+1$. This completes the proof.

Example 2.34

Prove that n^4-1 is divisible by 16 for odd values of n

A. Using Factorization
B. Using Mathematical Induction

Hint: $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Part A: Using Factorization

$$n^4 - 1 = (n^2+1)(n^2-1)$$
$$n \text{ is odd} \Rightarrow n^2 + 1 \text{ is even}$$
$$n \text{ is odd} \Rightarrow n^2 - 1 \text{ is divisible by } 8$$

Hence, n^4-1 is divisible by 16.

Part B: Using Mathematical Induction

For $n<0 \Rightarrow n=-m, m>0$:

$$n^4 - 1 = (-m)^4 - 1 = m^4 - 1$$

Hence, we only do the induction for positive values of n.

Base Case: $n=1$:

$$n^4 - 1 = 1 - 1 = 0 \Rightarrow \text{Divisible by } 16$$
$$\text{Base Case is Valid}$$

Inductive Case:

Let the statement be true for k. Consider the next odd number after k.

$$(k+2)^4 - 1$$
$$= k^4 + (4)(k^3)(2) + (6)(k^2)(2^2) + (4)(a)(2^3) + 2^4 - 1$$
$$= k^4 + 8k^3 + 24k^2 + 32k + 16 - 1$$
$$= \underbrace{k^4 - 1}_{\substack{\text{Divisible}\\ \text{by } 16}} + \underbrace{8k^2}_{\substack{\text{Divisible}\\ \text{by } 8}}(k+3) + \underbrace{16(2k+1)}_{\substack{\text{Divisible}\\ \text{by } 16}}$$

It remains to show that $k+3$ is divisible by 2.

$$k \text{ is odd} \Rightarrow k+3 \text{ is even} \Rightarrow \text{Divisible by } 2$$

Hence, it is true for $k+1$. This completes the proof.

Example 2.35: Exponents

$$5^n + 2^n \text{ is divisible by } 7 \text{ for positive odd values of } n$$

A. Prove the above using mod arithmetic.
B. Prove the above using induction.

Part A: Mod Arithmetic

This is elegant:

$$(-2)^n + 2^n \equiv -2^n + 2^n \equiv 0 \pmod 7$$

Part B: Induction

Base Case: $n=1$

$$5^1 + 2^1 = 5 + 2 = 7 \Rightarrow \text{Divisible by } 7$$
$$\text{Base Case is true}$$

Inductive Case

Let the statement be true for some odd n.

$$5^n + 2^n \text{ is divisible by } 7.$$

Consider the next odd number after n, which is $n+2$:

$$5^{n+2} + 2^{n+2}$$

Rewrite the above expression in terms of the exponents in the base case:

$$= 5^n \cdot 5^2 + 2^n \cdot 2^2 = 25 \cdot 5^n + 4 \cdot 2^n$$

Since the last term has $4 \cdot 2^n$, split the $25 \cdot 5^n$ into two parts:

$$= 21 \cdot 5^n + 4 \cdot 5^n + 4 \cdot 2^n$$

And now note that each part is individually divisible by 7:

$$= \underbrace{3 \cdot 7 \cdot 5^n}_{\text{Divisible by 7}} + \underbrace{4(5^n + 2^n)}_{\text{Divisible by 7}}$$

Example 2.36: Exponents

$$2^n + (-1)^{n-1} \textit{ is divisible by } 3, \qquad n \in \mathbb{N}$$

A. Prove the above using mod arithmetic
B. Prove the above using the binomial theorem
C. Prove the above using mathematical induction

Part A: Mod Arithmetic

$$2^n + (-1)^{n-1} \equiv (-1)^n + (-1)^{n-1}$$

But note that the above terms are equal in magnitude, but opposite in parity. Hence, their sum is:

$$\equiv 0 \ (mod\ 3)$$

Part B: Binomial Theorem

$$2^n = (3-1)^n$$

Expand the above using $(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} x^0 y^n$

$$(3-1)^n + (-1)^{n-1} = \underbrace{\binom{n}{0} 3^n (-1)^0 + \binom{n}{1} 3^{n-1} (-1)^1 + \binom{n}{2} 3^{n-2} (-1)^2 + \cdots +}_{\text{Divisible by 3}} \binom{n}{n} 3^0 (-1)^n + (-1)^{n-1}$$

All the terms except the last term have an integer power of 3. Hence, they are divisible by three, and can be ignored.

Hence, we are left with:

$$\binom{n}{n} 3^0 (-1)^n + (-1)^{n-1} = (-1)^n + (-1)^{n-1}$$

But note that the above terms are equal in magnitude, but opposite in parity. Hence, their sum is:

$$0$$

Part C: Induction

Base Case: $n = 1$

$$2^n + (-1)^{n-1} = 2^1 + (-1)^0 = 2 + 1 = 3$$

Inductive Case

Let the statement be true for n. Then:

$$2^n + (-1)^{n-1} \textit{ is divisible by } 3$$

Then:

$$2^{n+1} + (-1)^n$$
$$= 2 \cdot 2^n + (-1)^n$$
$$= 3 \cdot 2^n - 2^n - (-1)^{n-1}$$
$$= \underbrace{3 \cdot 2^n}_{\substack{\text{Divisible} \\ \text{by 3}}} - \underbrace{[2^n + (-1)^{n-1}]}_{\text{Divisible by 3}}$$

C. Inequalities

Example 2.37

A. $3^k \geq 3k$,

Part A
Base Case:
$\boldsymbol{k = 0}$:
$\boldsymbol{k = 1}$:
Inductive Case:

2.3 Mathematical Induction-II: Geometry, Counting and Recursion

A. Geometry

The applications of mathematical induction to geometry are not always obvious. Along with the Algebra, you will often need to combine geometrical arguments.

Example 2.38: Geometry

The sum of angles of a polygon with n vertices is $(n-2)\cdot 180$.

A. Prove the above statement using a geometrical argument.
B. Prove the above statement using mathematical induction.

Hint: The minimum number of sides of a polygon is 3. Hence, the base case for the induction will be $n = 3$.

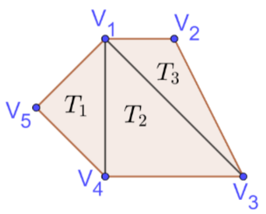
Part A: Using a geometrical argument

Consider a pentagon. Connect V_1 to other four vertices. This gives us four line segments, of which 2 are edges and 2 are diagonals.
These 2 diagonals divide the pentagon into 3 triangles, which have sum of angles

$$3 \times 180 = 540$$

Generalize the argument above to a polygon with n vertices:

$$\{V_1, V_2, \dots V_n\}$$



Without loss of generality, choose a vertex, say V_1.
Connect V_1 to the remaining $(n-1)$ vertices, resulting in $n-1$ line segments.
Two of the line segments will be edges and the remaining will be diagonals.

$$n - 1 - 2 = n - 3\ Diagonals$$

These diagonals will divide the polygon into $n-2$ triangles.
Each triangle will have sum of angles 180°.
Hence, the sum of angles of the polygon:

$$= \underbrace{(n-2)}_{\substack{No.of\\Triangles}} \times \underbrace{180}_{\substack{Sum\ of\ Angles\\of\ each\ Triangle}}$$

Part B: Using mathematical induction

Base Case: $\boldsymbol{n = 3}$:
The minimum number of sides of a polygon is 3. Hence, our base case is 3.
If $n = 3$, the polygon is a triangle, and the property gives us:

$$(n-2)\cdot 180 = (3-2)\cdot 180 = 180$$

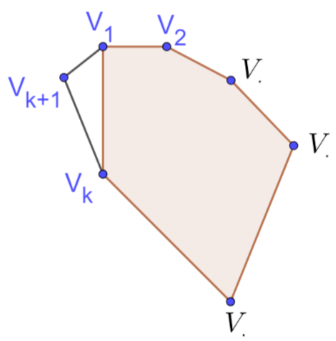
Which is the sum of the angles of a triangle.

$$Base\ Case\ is\ Valid$$

Inductive Case:
Let the statement be true for $n = k$.

$$(k-2)\cdot 180$$

Consider a polygon with k vertices. (See diagram below).
Add the $(k+1)^{st}$ vertex. Note the sum of angles of



the polygon increases by the angles of a triangle, which is 180°.
Hence, the sum of angles of the polygon with $k+1$ vertices is:

$$(k-2)\cdot 180 + 180$$

$$= 180(k - 2 + 1)$$
$$= 180(k - 1)$$

Which is what we wanted to prove.
This completes the proof.

Example 2.39: Geometry

The number of diagonals of an n-sided polygon is $\frac{n(n-3)}{2}$.

C. Prove the above statement using a geometrical argument.
D. Prove the above statement using mathematical induction.

Hint: The minimum number of sides of a polygon is 3. Hence, the base case for the induction will be $n = 3$.

Part A: Using a geometrical/counting argument

Consider a polygon with n vertices:

$$\{V_1, V_2, \dots V_n\}$$

Without loss of generality, choose a vertex, say V_1.
Connect V_1 to the remaining $(n-1)$ vertices, resulting in $n-1$ line segments.
Two of the line segments will be edges and the remaining will be diagonals.

$$n - 1 - 2 = n - 3 \; Diagonals$$

But this applies to every vertex:

$$\{V_1, V_2, \dots V_n\}$$

Hence, the total number of diagonals should be:

$$\underbrace{n}_{\substack{No.of \\ Vertices}} \times \underbrace{(n-3)}_{\substack{No.of\ Diagonals \\ per\ vertex}}$$

But this overcounts the diagonals, since the diagonal from A to B is the same diagonal as the one from B to A.
Hence, we need to divide by an overcounting factor of 2.
Hence, the number of diagonals is:

$$\frac{n(n-3)}{2}$$

Part B: Using mathematical induction

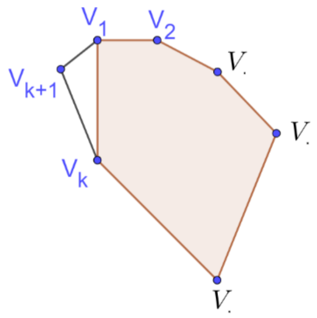
Let the number of diagonals of a polygon P_k with k sides be D_k.

Lemma: $D_{k+1} = D_k + (k-1)$

P_k has vertices $V_1, V_2, \dots, V_k$. Add a vertex V_{k+1} between V_1 and V_k.

- It connects to each of the existing vertices $V_1, V_2, \dots, V_k$, which gives k line segments. Of these $V_{k+1}V_1$ and $V_{k+1}V_k$ are edges. And rest $k-2$ are diagonals.
- The edge V_1V_n now becomes a diagonal.

Hence, the total number of diagonals increases by



$$\underbrace{k-2}_{\substack{New \\ Diagonals}} + \underbrace{1}_{\substack{Edge\ Converted \\ To\ Diagonal}} = k - 1$$

This is demonstrated in the diagram alongside for a specific case. But, the logical argument given applies to the general case.

Base Case: $n = 3$:

Number of Diagonals

$$= \frac{n(n-3)}{2} = \frac{3(3-3)}{2} = 0$$

Which is true since a polygon with three sides is a triangle, and it has no diagonals.

$$Base\ Case\ is\ Valid$$

Inductive Case:

Let the statement be true for k:

$$D_k = \frac{k(k-3)}{2}$$

By the

$$D_{k+1} = D_k + (k-1)$$
$$= \frac{k(k-3)}{2} + (k-1)$$
$$= \frac{k^2 - 3k + 2k - 2}{2}$$
$$= \frac{k^2 - k - 2}{2}$$
$$= \frac{(k+1)(k-2)}{2}$$

Which is what we wanted to prove.
This completes the proof.

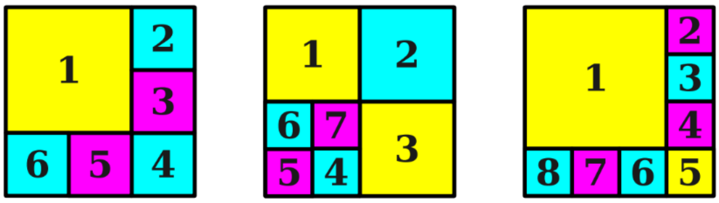
2.40: Bigger Steps

- It is not necessary in induction that the "step" should always be 1.
- We can move from something being true for k, to being true for $k+m$.

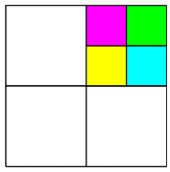
Example 2.41: Bigger Steps

Prove the following statement using mathematical induction.
Any square can be divided into $n > 6, n \in \mathbb{N}$ squares. Here dividing means that the squares do not overlap, and together cover the entire original square.

The diagram shows that the statement holds for $n \in \{6,7,8\}$



It is always possible to increase the number of squares by 3.



Hence, if the statement holds for k, it holds for $k+3$.
Hence, we get three separate cases:

$$\{6,9,12,\dots\}$$
$$\{7,10,13,\dots\}$$
$$\{8,11,14,\dots\}$$

And the above three cases together cover all natural numbers from 6 onwards.
This completes the proof.

B. Counting Arguments

Example 2.42: Factorials

Factorials are highly useful when counting arrangements and selecting of objects. They are defined as below:

$$Explicit\ Definition: n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$$
$$Recursive\ Definition: n! = n(n-1)!$$

Prove the property below using math

A. $1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1$

Using Telescoping

Add and subtract 1:

$$1 + 1 \cdot 1! + 2 \cdot 2! + \dots + n \cdot n! - 1 = (n+1)!$$

Collapse the series one term at a time using factoring:

$$\underbrace{2!}_{0!+1!=2!} + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n! - 1$$

Take 2! common in the second and third terms:

$$\underbrace{2!\,(1+2)}_{=2!\times 3=3!} + 3 \cdot 3! + \dots + n \cdot n! - 1$$

Simplify inside the bracket and use the formula $n! = n(n-1)!$:

$$3! + 3 \cdot 3! + \dots + n \cdot n!$$
$$= 3!\,(1+3) \dots + n \cdot n! - 1$$

Repeat the above process until we finally get:

$$= (n+1)! - 1$$

Using Mathematical Induction

Base Case: $n = 1$:

$$LHS = 1(1!) + 2(2!) + \cdots + n(n!) = 1$$
$$RHS = (n+1)! - 1 = 2! - 1 = 2 - 1 = 1$$
$$LHS = RHS \Rightarrow Base\ Case\ is\ valid$$

Inductive Case:

Let the statement be true for k:

$$1(1!) + 2(2!) + \cdots + k(k!) = (k+1)! - 1$$

Add $(k+1)(k+1)!$ to both sides:

$$1(1!) + 2(2!) + \cdots + k(k!) + (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)!$$

Work only with the RHS. Factor $(k+1)!$ from the first and third term:

$$(k+1)!\,(k+2) - 1 = (k+2)! - 1$$

Hence, it is true for $k+1$. This completes the proof.

Example 2.43

Prove each of the below properties twice. Once using a counting argument, and once using mathematical induction.

A. Number of handshakes at a party with n people if everyone shakes hands with each other is $\frac{n(n-1)}{2}$.
B. Number of ways to arrange n distinct objects in a row is $n!$.

Part A

Base Case: $n = 1$:

$$\frac{n(n-1)}{2} = \frac{1 \times 0}{2} = 0$$

$$\textit{Base Case is valid}$$

Inductive Case:

Let the statement be true for k.

If the number of people increases from k to $k+1$, then the new person will shake hands with each of the remaining $k-1$ people.

$$\frac{k(k-1)}{2} + k = \frac{k^2 - k + 2k}{2} = \frac{(k+1)(k)}{2}$$

Hence, it is true for $k+1$. This completes the proof.

Part B

Base Case: $n = 1$:

$$n! = 1! = 1$$

Inductive Case:

Let the statement be true for k.

If the number of objects increases from k to $k+1$, then, for any existing arrangement, the new object can be inserted in any of the $k+1$ places.

Hence, the number of arrangements is:

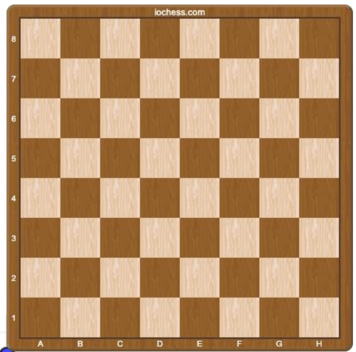
$$k! \times (k+1) = (k+1)!$$

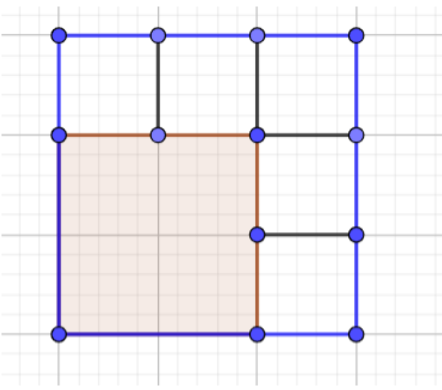
Hence, it is true for $k+1$. This completes the proof.

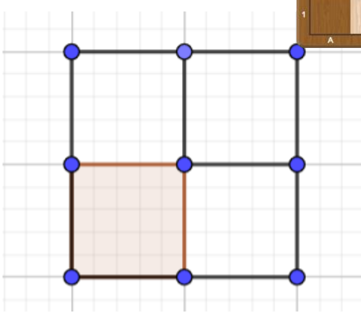
Example 2.44

Prove each of the below properties twice. Once using a counting argument, and once using mathematical induction.

A. The number of squares on a $n \times n$ chessboard having integer side length is $\frac{n(n+1)(2n+1)}{6}$. (See the diagram for a chessboard of size 8×8).







$$\textit{Additional Squares of Size } k+1: 1$$

$$\textit{Additional Squares of Size } k: 3$$

$$Additional\ Squares\ of\ Size\ k-1: 5$$

Part C

Counting Argument

$$Squares\ of\ size\ n: 1$$
$$Squares\ of\ size\ n-1: 2^2 = 4$$
$$Squares\ of\ size\ n-2: 3^2 = 9$$
$$.$$
$$.$$
$$.$$
$$Squares\ of\ size\ 1: n^2$$
$$Total = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Using Mathematical Induction

Base Case: $n = 1$:

Inductive Case:

Example:

Consider increasing the size of the chessboard from 8 to 9:

$$Squares\ of\ Size\ 9 = 1$$
$$Squares\ of\ Size\ 8: 4$$

Let the statement be true for k.

If you increase the size of the chessboard from k to $k+1$, the number of additional squares generated are:

$$Squares\ of\ Size\ (k+1): 1$$
$$Squares\ of\ Size\ k: 3$$
$$Squares\ of\ Size\ k-1: 5$$
$$.$$
$$.$$
$$.$$
$$Squares\ of\ Size\ 1: 2k+1$$

$$1\ + 3 + \cdots + (2k+1) = (k+1)^2$$
$$1^2 + 2^2 + \cdots + (k+1)^2$$
$$= \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

Hence, it is true for $k+1$. This completes the proof.

C. Recurrence Relations

Example 2.45

The Fibonacci sequence is defined by:

$$Base\ Case: f_1 = 1, f_2 = 1$$
$$Recursive\ Case: f_n = f_{n-1} + f_{n-2}, n \geq 3$$

A. Find the first few terms of the Fibonacci sequence.

1,1,2,3,5,8,13,21,34,55,89

Example 2.46

A. $f_1 + f_2 + f_3 + \cdots + f_n = f_{n+2} - 1$
B. $f_2 + f_4 + f_6 + \cdots + f_{2n} = f_{2n+1} - 1$
C. $f_1 - f_2 + f_3 + \cdots + (-1)^n f_{n+1} = (-1)^n f_n + 1$

Part A

Base Case: $n = 1$

$$LHS = f_1 = 1$$
$$RHS = f_{n+2} - 1 = f_3 - 1 = 2 - 1 = 1$$
$$LHS = RHS$$
$$Base\ Case\ is\ valid$$

Inductive Case:

Suppose for some k:

$$f_1 + f_2 + f_3 + \cdots + f_k = f_{k+2} - 1$$

Add f_{k+1} to both sides:

$$LHS = f_1 + f_2 + f_3 + \cdots + f_k + f_{k+1}$$
$$RHS = f_{k+2} - 1 + f_{k+1} = f_{k+3} - 1$$

This is what we need to show. This completes the proof.

Part B

Base Case: $n = 1$

$$LHS = f_2 = 1$$
$$RHS = f_{2n+1} - 1 = f_3 - 1 = 2 - 1 = 1$$
$$LHS = RHS$$
$$Base\ Case\ is\ valid$$

Inductive Case:

Suppose for some k:

$$f_2 + f_4 + f_6 + \cdots + f_{2k} = f_{2k+1} - 1$$

Add $f_{2(k+1)} = f_{2k+2}$ to both sides:

$$LHS = f_2 + f_4 + f_6 + \cdots + f_{2k} + f_{2k+2}$$
$$RHS = f_{2k+1} - 1 + f_{2k+2}$$
$$= f_{2k+3} - 1$$
$$= f_{2(k+1)+1} - 1$$

This is what we need to show. This completes the proof.

Part C

Base Case:$n = 1$

$$LHS = f_1 - f_2 = 1 - 1 = 0$$
$$RHS = (-1)^n f_1 + 1 = -1 + 1 = 0$$
$$LHS = RHS$$
$$Base\ Case\ is\ valid$$

Inductive Case:

Suppose for some k:

$$f_1 - f_2 + f_3 + \cdots + (-1)^k f_{k+1} = (-1)^k f_k + 1$$

Add $(-1)^{k+1} f_{k+2}$:

$$(-1)^k f_k + 1 + (-1)^{k+1} f_{k+2}$$

Factor out $(-1)^k$ from the first and the third term:

$$= (-1)^k [f_k + (-1) f_{k+2}] + 1$$

Expand f_{k+2} using the definition of the Fibonacci sequence:

$$= (-1)^k [f_k - f_{k+1} - f_k] + 1$$

Simplify:

$$= (-1)^k [-f_{k+1}] + 1$$

Combine the minus signs:

$$= (-1)^{k+1} [f_{k+1}] + 1$$

This is what we need to show. This completes the proof.

2.47: Strong Induction

- If you rely on more than one base case to establish the inductive case, it is called strong.
- It is strong in the sense, you have more "base cases", and hence more to work with.

Example 2.48

A. $f_{n+1} < \left(\frac{7}{4}\right)^n, n \geq 1$

Base Case: $n = 1, n = 2$:

$$f_{1+1} = f_2 = 1 < \left(\frac{7}{4}\right)^1$$
$$f_{2+1} = f_3 = 2 < \left(\frac{7}{4}\right)^2 = \frac{49}{16} = 3\frac{1}{16}$$

Inductive Case:

Suppose the statement holds for $n + 1$ and $n + 2$:

$$f_{n+1} < \left(\frac{7}{4}\right)^{n}, \qquad f_{n+2} < \left(\frac{7}{4}\right)^{n+1}$$

Add the two inequalities:

$$f_{n+1} + f_{n+2} < \left(\frac{7}{4}\right)^{n} + \left(\frac{7}{4}\right)^{n+1}$$

$$RHS = \left(\frac{7}{4}\right)^{n+2}\left[\left(\frac{7}{4}\right)^{-2} + \left(\frac{7}{4}\right)^{-1}\right] = \left(\frac{7}{4}\right)^{n+2}\left[\frac{16}{49} + \frac{28}{49}\right]$$

$$f_{n+3} < \left(\frac{7}{4}\right)^{n+2}\left[\frac{44}{49}\right] < \left(\frac{7}{4}\right)^{n+2}$$

$$f_{n+3} < \left(\frac{7}{4}\right)^{n+2}$$

Example 2.49

$$a = \frac{1+\sqrt{5}}{2}, b = \frac{1-\sqrt{5}}{2}$$

A. Show that $a^2 = \frac{3+\sqrt{5}}{2}, b^2 = \frac{3-\sqrt{5}}{2}, a+1 = a^2, b+1 = b^2$

B. Prove using mathematical induction that if f_n is a term in the Fibonacci sequence: $f_n = \frac{a^n - b^n}{\sqrt{5}}$

Part A

$$a^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$$

$$b^2 = \left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2}$$

$$a+1 = \frac{1+\sqrt{5}}{2} + 1 = \frac{3+\sqrt{5}}{2} = a^2$$

$$b+1 = \frac{1-\sqrt{5}}{2} + 1 = \frac{3-\sqrt{5}}{2} = b^2$$

Part B

Base Case:

We consider two base cases.

$$n = 1 \Rightarrow f_1 = \frac{a^1 - b^1}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \left(\frac{1-\sqrt{5}}{2}\right)}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}-1+\sqrt{5}}{2}}{\sqrt{5}} = \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$$

$$n = 2 \Rightarrow f_2 = \frac{a^2 - b^2}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} = \frac{\frac{3+\sqrt{5}}{2} - \frac{3-\sqrt{5}}{2}}{\sqrt{5}} = \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$$

Inductive Case:

Let it be true for some k that:

$$f_k = \frac{a^k - b^k}{\sqrt{5}}, f_{k+1} = \frac{a^{k+1} - b^{k+1}}{\sqrt{5}}$$

We need to show that:

$$f_{k+2} = \frac{a^{k+2} - b^{k+2}}{\sqrt{5}}$$

Start with the LHS, and apply the definition of the Fibonacci Sequence:

$$f_{k+2} = f_{k+1} + f_k = \frac{a^{k+1} - b^{k+1}}{\sqrt{5}} + \frac{a^k - b^k}{\sqrt{5}} = \frac{a^{k+1} + a^k - \left(b^{k+1} + b^k\right)}{\sqrt{5}}$$

Factor a^k from the first two terms, and b^k from the last two terms in the numerator:

$$\frac{a^k(a+1) - b^k(b+1)}{\sqrt{5}}$$

Substitute $a + 1 = a^2, b + 1 = b^2$

$$\frac{a^k(a^2) - b^k(b^2)}{\sqrt{5}} = \frac{a^{k+2} - b^{k+2}}{\sqrt{5}}$$

D. Recursion

Example 2.50

Let $a_0 = 1$ and $a_n = \sqrt{(n+2)a_{n-1}}$ for $n \geq 1$. Find a_{2014}. (JHMMC Grade 7 2014/38)

$$a_1 = \sqrt{(1+2)a_0 + 1} = \sqrt{(1+2)\cdot 1 + 1} = \sqrt{4} = 2$$
$$a_2 = \sqrt{(2+2)a_1 + 1} = \sqrt{4 \cdot 2 + 1} = \sqrt{9} = 3$$
$$a_3 = \sqrt{(3+2)a_2 + 1} = \sqrt{5 \cdot 3 + 1} = \sqrt{16} = 4$$

The pattern is

$$a_n = n + 1 \Rightarrow a_{2014} = 2014 + 1 = 2015$$

We can prove this using mathematical induction:

Base Case

$$a_1 = \sqrt{(1+2)a_0 + 1} = \sqrt{(1+2)\cdot 1 + 1} = \sqrt{4} = 2$$

Inductive Case

$a_n = n + 1$ is true for $n = 1$.
Assume it is for some k.

Then:

$$a_{k+1} = \sqrt{(k+1+2)a_k + 1} = \sqrt{(k+3)(k+1) + 1} = \sqrt{k^2 + 4k + 4} = \sqrt{(k+2)^2} = k + 2$$

Hence, the inductive case is proved

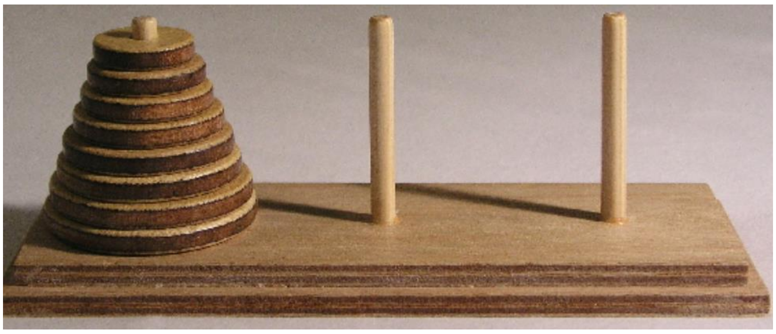
$$a_n = n + 1, n \in \mathbb{Z}^+$$

Example 2.51: Finding a Formula

The objective is to move the discs from the leftmost pole to the rightmost pole, using the middle pole as a placeholder, if necessary.

1. You can only move one disc at a time.
2. You cannot keep a disc on anything other than the three poles provided.
3. You cannot keep a larger disc on a smaller disc at any point in time on any pole.

A. State the minimum number of steps needed for one disc, two discs, and three discs and write out the steps.



B. Tabulate the number of steps needed for n discs, $n = 1,2,3, \ldots$ Formulate a conjecture about the number of steps needed for moving n discs.
C.

Call moving a disc from one pole to another an "operation".
Suppose you have only one disc.

Part A

Case I: One Disc

	$Left\ Pole$	$Middle\ Pole$	$Right\ Pole$
Start	1		
End			1

Case II: Two Discs

	$Left\ Pole$	$Middle\ Pole$	$Right\ Pole$
Start	1,2		
1	2	1	
2		1	2
End			1,2

Case III: Three Discs

	$Left\ Pole$	$Middle\ Pole$	$Right\ Pole$
Start	1,2,3		
1	2,3		1
2	3	2	1
3	3	1,2	
4		1,2	3
5	1	2	3
6			2,3
7			1,2,3

Case IV: Four Discs

	$Left$	$Middle$	$Right$	Steps
Start	1,2,3,4			
	4	1,2,3		7
		1,2,3	4	1
			1,2,3,4	7
Total				15

Part B

$$1\ Disc = 1\ Step = 2^1 - 1$$
$$2\ Discs = 3\ Steps = 2^2 - 1$$
$$3\ Discs = 7\ Steps = 2^3 - 1$$
$$4\ Discs = 15\ Steps = 2^4 - 1$$

$$n\ discs = 2^n - 1$$

Example 2.52: Proving a Formula

Prove using mathematical induction that the number of steps needed to move n discs from the leftmost pole to the rightmost pole in a Tower of Hanoi game is $2^n - 1$.

Base Case: $n = 1$
If there is one disc, then the number of steps needed is:

$$2^n - 1 = 2 - 1 = 1$$
$$Base\ Case\ is\ Valid$$

Inductive Case
Let the statement be true for n. Then, the number of steps needed for n discs is:

$$2^n - 1$$

For $n + 1$ discs:

	$Left$	$Middle$	$Right$	Steps
Start	$1,2, \ldots, n+1$			
	n	$1,2, \ldots, n$		$2^n - 1$
		$1,2, \ldots, n$	n	1
			$1,2, \ldots, n+1$	$2^n - 1$
Total				15

$$2^n - 1 + 1 + 2^n - 1$$
$$= 2 \times 2^n - 1$$
$$= 2^{n+1} - 1$$

2.4 Proof by Contradiction and Contraposition

A. Proof by Contradiction

Example 2.53

Probability - Parity
Irrationality of 2 (Number Theory)
Counting - PHP

B. Infinite Descent

2.54: Well Ordering Principle

Example 2.55: Infinite Descent

A. Show that when a perfect square is divided by 3, the remainder can be only 0 or 1.
B. Hence, or otherwise, show that the circle $x^2 + y^2 = 3$ has no rational points on it.

Part A

$$0^2 \equiv 0 \ (mod\ 3)$$
$$1^2 \equiv 1 \ (mod\ 3)$$
$$2^2 \equiv 4 \equiv 1 \ (mod\ 3)$$

$$(3n)^2 = 9n^2 \Rightarrow Remainder\ 0$$
$$(3n+1)^2 = 9n^2 + 6n + 1 \Rightarrow Remainder\ 1$$
$$(3n+2)^2 = 9n^2 + 6n + 4 \Rightarrow Remainder\ 2$$

Part B

Suppose, to the contrary, such rational points do exist. Then, for $a, b, c, d \in \mathbb{Z}$, $\gcd(a, b) = 1$, $\gcd(c, d) = 1$ we can write:

$$\left(\frac{a}{b}\right)^2 + \left(\frac{c}{d}\right)^2 = 3 \Rightarrow \frac{a^2}{b^2} + \frac{c^2}{d^2} = 3 \Rightarrow \underbrace{d^2a^2 + b^2c^2 = 3b^2d^2}_{Equation\ I}$$

Use a change of variable. For $p, r, q \in \mathbb{Z}$, let

$$p^2 = d^2a^2, q = b^2c^2, r = b^2d^2 \Rightarrow \gcd(p, q, r) = \gcd(ad, bc, bd) = 1$$

Then we can write Equation I as:

$$\underbrace{p^2 + q^2 = 3r^2}_{Equation\ II}, \qquad p, q, r \in \mathbb{Z}, \qquad \gcd(p, q, r) = 1$$

In Equation II, the LHS is a multiple of 3, since the RHS is a multiple of 3. By Part A, $2\ (mod\ 3)$ is not possible for a perfect square. If we take

$$p^2 \equiv 1 (mod\ 3) \Rightarrow 1 + 0 = 1, 1 + 1 = 2 \Rightarrow Does\ not\ match\ RHS \Rightarrow \mathbf{Contradiction}$$

Hence,

$$p^2 \equiv q^2 \equiv 0\ (mod\ 3)$$

Hence, we can write

$$(3l)^2 + (3m)^2 = 3r^2 \Rightarrow 9l^2 + 9m^2 = 3r^2 \Rightarrow 3(l^2 + m^2) = r^2$$

However, notice that the RHS is a perfect square, and hence:

$$LHS\ is\ a\ multiple\ of\ 3 \Rightarrow RHS\ is\ a\ multiple\ of\ 9$$

Hence, we can write:

$$3(l^2 + m^2) = (3n)^2 \Rightarrow l^2 + m^2 = 3n^2$$

Hence, we have found a new solution to our equation such that $(l, m, n) = \left(\frac{p}{3}, \frac{q}{3}, \frac{r}{3}\right)$. However, we had earlier shown that

$$\gcd(p, q, r) = 1 \Rightarrow \textbf{\textit{Contradiction}}$$

C. Cardinality and Infinity

2.56: Cardinality

Cardinality is the number of elements in a set.

Example 2.57

Find the cardinality of

A. $\{1,2,3, \dots ,100\}$

B. $\left\{\frac{1}{3}, \frac{2}{3}, \frac{99}{3}\right\}$

C. $\{x: x\ is\ a\ prime\ number\ less\ than\ 100\}$

2.58: Null Set / Empty Set

A set with no elements is called a null set.

ϕ, { } are ways of writing a null set
ϕ is the Greek Letter *Phi*.

Note: This is different from the Greek Letter π, which you might have already seen in Geometry.

2.59: Subsets

If every element of the set X is also an element of the set Y, then X is a subset of Y.

$$X\ is\ a\ subset\ of\ Y \Leftrightarrow X \subseteq Y$$

2.60: Proper Subsets

If X is a subset of Y, and there is at least one element in Y which is not in X, then X is a proper subset of Y.

$$X\ is\ a\ proper\ subset\ of\ Y \Leftrightarrow X \subset Y$$

2.61: Superset

If every element of the set X is also an element of the set Y, then Y is a superset of X.

$$Y\ is\ a\ superset\ of\ X \Leftrightarrow Y \supseteq X$$

2.62: Proper Superset

If Y is a superset of X, and there is at least one element in Y which is not in X, then Y is a proper superset of X.

$$Y\ is\ a\ superset\ of\ X \Leftrightarrow Y \supset X$$

2.63: Powerset

The powerset consists of the set of all subsets of a set.

Example 2.64

Consider the set

$$\{A, B, C\}$$

A. What is the powerset of the set
B. What is the cardinality of the powerset?

We already did this, without using the term powerset:

$$\{\{A, B, C\}, \{A, B\}, \{A, C\}, \{A\}, \{B, C\}, \{B\}, \{C\}, \{\phi\}\}$$

And, we also calculated the cardinality of the above set, which is:

$$2^3 = 8$$

2.65: Cardinality of Powerset

Cardinality is the number of elements of a set.

The cardinality of the powerset of a set A with n elements is given by:

$$2^n$$

$$\underbrace{2}_{\substack{2\ Choices\\ for\ the\\ 1^{st}\ Element}} \times \underbrace{2}_{\substack{2\ Choices\\ for\ the\\ 2^{nd}\ Element}} \times \ldots \times \underbrace{2}_{\substack{2\ Choices\\ for\ the\\ n^{th}\ Element}} = 2^n$$

Example 2.66

$$\mathbb{N} = \{1,2,3, \ldots\}$$

$$Cardinality\ of\ \mathbb{N} = \infty$$

$\aleph$ is often used to represent different kinds of infinity.
The infinity of the natural numbers is represented by $\aleph_0$. That is:

$$Cardinality\ of\ \mathbb{N} = \aleph_0$$

Example 2.67

Consider the bijection principle

$$Set\ A\ has\ x\ elements$$
$$Set\ B\ has\ y\ elements$$

A bijection is a rule that associates exactly one element of A with exactly one element of B.
If you are able to establish a bijection, then the number of elements in the two sets is the same.

2.68: Cardinality Notation

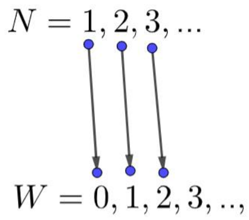
$$Cardinality\ of\ A = |A|$$

Example 2.69

$$Cardinality\ of\ \mathbb{W} = \aleph_0$$

There is a bijection between the two sets. Hence, they have the same cardinality:

$$|\mathbb{N}| = |\mathbb{W}| = \aleph_0$$



Example 2.70

$$Cardinality\ of\ X = \{100,101,102, \ldots\} = \aleph_0$$

Subtract 99 from each element of the set. This changes the individual values, but does not change the cardinality:

$$X' = \{1,2,3, \ldots\} = \aleph_0$$
$$|X'| = |X| = \aleph_0$$

Example 2.71

$$Cardinality\ of\ \mathbb{E} = \{2,4,6,8, \ldots\} = \aleph_0$$

$\mathbb{N}$	1	2	3	4	5	6	.	.	.
$\mathbb{E}$	2	4	6	8	10	12	.	.	.

Example 2.72

$$Cardinality\ of\ \mathbb{Q} = \aleph_0$$

$\mathbb{N}$	1	2	3	4	5	6	.	.	.
0	$\frac{0}{1}$	$\frac{0}{2}$	$\frac{0}{3}$	$\frac{0}{4}$	$\frac{0}{5}$	$\frac{0}{6}$			
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	.	.	.
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	.	.	.
3	.	.	.	.	.	.	.	.	.
4	.	.	.	.	.	.	.	.	.
5	.	.	.	.	.	.	.	.	.

ℕ	1	2	3	4	5	6	.	.	.
0	$\frac{0}{1}$	$\frac{0}{2}$	$\frac{0}{3}$	$\frac{0}{4}$	$\frac{0}{5}$	$\frac{0}{6}$			
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	.	.	.
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	.	.	.
3	.	.	.	.	.	.	.	.	.
4	.	.	.	.	.	.	.	.	.
5	.	.	.	.	.	.	.	.	.

Example 2.73
Between any two rational numbers lie an infinite number of rational numbers.

Example 2.74
The infinity of real numbers is infinitely more than the infinity of natural numbers.

$$Cardinality\ of\ \mathbb{R} \neq \aleph_0$$

Proof by contradiction.
Suppose there is a bijection between the natural numbers and the real numbers. Then, we can list down all the real numbers in some table, and assign a natural number to each real number.

Natural Numbers													
1		0	.	1	2	3	4	5	6	7	.	.	.
2		1	.	0	1	3	4	5	9	2	.	.	.
3		2	.	5	6	7	1	5	3	4	.	.	.
4		3	.	3	6	7	4	5	1	2	.	.	.
		.	.	.	.	.	.	.	.	.	.	.	.
		.	.	.	.	.	.	.	.	.	.	.	.

Now, chose a real number in the following manner:
- First digit does not match with the real number assigned to the natural number 1.
- Second digit does not match with the real number assigned to the natural number 2.
- Third digit does not match with the real number assigned to the natural number 3.
- .
- .
- .
- n^{th} digit does not match with the real number assigned to the natural number n.

Hence, the number so created does not match any of the numbers already assigned to the natural numbers. Hence, the infinity of real numbers is greater than the infinity of natural numbers.

Example 2.75: Diagonal Argument
Cardinality of finite and infinite sets
Incompleteness Theorems

D. Proof by Contraposition

Example 2.76

Truth Table
Proofs

3. Puzzles

3.1 Knights, Knaves, and Alternators

A. Definitions

3.1: Knights and Knaves

- Knights always speak the truth.
- Knaves always lie.

Example 3.2: Getting the same answer

A. Are you a Knight?
B. Are you a Knave?

	Knight	Knave
A.	Yes	Yes
B.	No	No

Example 3.3: Getting different Answers

I meet a Knight, and one is a Knave. Determine which one is which by asking a single question.
Note: You cannot ask questions based on eternal truths such as "Is 2 a prime number", or common knowledge such as "What time is it".

If I ask the other person standing here, "Are you a Knight" what will he respond?

Case I: Knight

$$\underbrace{Knave}_{Lie,Yes} \Rightarrow Yes$$

Case I: Knave

$$\underbrace{Knight}_{Truth,Yes} \Rightarrow No$$

Example 3.4

I am on an island. I come to a crossroads. I know that one road leads to the capital city, and the second road leads to the dragon's lair. Two people are standing at the crossroads. I know that one of them is a Knight, and one of them is a Knave. What *single* question should I ask to find which road goes to the capital city?

If I ask the other person standing here, which is the road that goes to the dragon's lair, what will he respond?

$$Knave \rightarrow \underbrace{Knight}_{Truth} \rightarrow Lie$$
$$Knight \rightarrow \underbrace{Knave}_{Lie} \rightarrow Lie$$

Example 3.5

You come across a group of three people. You ask them, "How many of you are Knaves"?
A responds, "Exactly 1 of us is a Knave".
B responds, "Exactly 2 of us are Knaves".

C responds, "Exactly 3 of us are Knaves".

Determine, if possible, how many are actually knaves, and who they are.

Note that the actual number of Knaves in the group must be 2, since out of the three statements, only 1 statement can be correct, which means the other two must be wrong.
Hence, B is a Knight, and the other two are Knaves.

Correct Statement	No. of Knaves Promised	A	B	C	Actual No. of Knaves	
A	1	Knight	Knaves	Knaves	2	Inconsistent
B	2	Knaves	Knight	Knaves	2	Consistent
C						

Example 3.6
You come across a group of three people. You ask them, "How many of you are Knight"?
A responds, "Exactly 1 of us is a Knight".
B responds, "Exactly 2 of us are Knights".
C responds, "Exactly 3 of us are Knights".

Determine, if possible, how many are actually knaves, and who they are.

Example 3.7
You come across a group of n people. You ask them, "How many of you are Knights"?
A responds, "Exactly 1 of us is a Knight".
B responds, "Exactly 2 of us are Knights".
C responds, "Exactly 3 of us are Knights".
And so on, until the n^{th} person responds
Exactly n of us are Knights

Determine the Knights.

B. Guaranteed Outcomes

Example 3.8

C. Consistency

Example 3.9
You meet A and B. A says the following:
I: B is a Knave.
II: Only one of us is a Knave.

Determine, if possible, which type are A and B.

Case I: A is a Knight $\Rightarrow$ *I and II are true*

From I, B is a Knave.
Overall, A is a Knight and B is a Knave. This is consistent with Statement II.

Case II: A is a Knave $\Rightarrow$ *I and II are lies*
From I, B is a Knight.
Overall, A is a Knave and B is a Knight.
This makes II correct.
This is not consistent, since A is a Knave.

D. Lies and Truths

Example 3.10

Peter has a collection of foxes and rabbits. He says three statements:

- The number of foxes is 20 more than the number of rabbits.
- The number of foxes is equal to the number of rabbits squared.
- The number of foxes is equal to six times the number of rabbits.

However, it is revealed that one of Peter's statements is false, while the other two are true. What is the maximum number of foxes Peter has? **(JHMMC Grade 7 2022 R2/8)**

$$F = R + 20$$
$$F = R^2$$
$$F = 6R$$

If the first statement is false:

$$R^2 = 6R \Rightarrow R = 6 \Rightarrow F = 36$$

If the second statement is false:

$$R + 20 = 6R \Rightarrow 20 = 5R \Rightarrow R = 4 < 6$$

If the third statement is false:

$$R + 20 = R^2 \Rightarrow R = 5 \text{ or } R = -4 \Rightarrow R = 5 < 6$$

We choose the largest F (which corresponds to the largest R):

$$F = 36$$

E. Alternator

Example 3.11

On Halloween 31 children walked into the principal's office asking for candy. They can be classified into three types: Some always lie; some always tell the truth; and some alternately lie and tell the truth. The alternaters arbitrarily choose their first response, either a lie or the truth, but each subsequent statement has the opposite truth value from its predecessor. The principal asked everyone the same three questions in this order.

- "Are you a truth-teller?" The principal gave a piece of candy to each of the 22 children who answered yes.
- "Are you an alternater?" The principal gave a piece of candy to each of the 15 children who answered yes.

➢ "Are you a liar?" The principal gave a piece of candy to each of the 9 children who answered yes. Find the number of truth-tellers, liars and alternaters. (AMC 10A 2022/13, Adapted)

	Truth Teller	Liar	Alternator	
			Case I	Case II
Are you a Truth Teller?	Yes	Yes	Yes (Lie)	No ($Truth$)
Are you an alternator?	No	Yes	Yes($Truth$)	$No(Lie)$
Are you a liar	No	No	Yes(Lie)	$No(Truth)$

Two cases are possible with Alternators. We need to decide which case is applicable in the given question.

Note that Truth Tellers and Liars both always answer "No" to Are you a Liar (Third row).
If Case II were applicable, Alternators would also answer No, and hence there would be no people answering Yes to the question.

However, 9 children answered Yes, and hence Case I is applicable. Hence, we can remove Case II from the analysis, giving us the table below:

	Truth Teller	Liar	Alternator
			Case I
Are you a Truth Teller?	Yes	Yes	Yes (Lie)
Are you an alternator?	No	Yes	Yes($Truth$)
Are you a liar	No	No	Yes(Lie)

From the last row:

$$Third\ Row: A = 9$$
$$Second\ Row: L + A = L + 9 = 15 \Rightarrow L = 6$$
$$T + L + A = 22 \Rightarrow T + 15 = 22 \Rightarrow T = 7$$

3.2 Syllogisms

A. Foundations

We can go from the general to the particular.

3.12: General to Particular

If something is true in general for a class of objects, it is true for a subclass of that object.

Example 3.13

Can you conclude Statement B from Statement A?

A. Rohan has coffee every day of the week in the morning.
B. Rohan has coffee on Sunday in the morning.

Example 3.14

Is C true given that A and B are true statements?

A. Every even integer is divisible by 2.
B. Every multiple of 4 is an even integer.
C. Therefore, every multiple of 4 is divisible by 2.

Example 3.15

Is Statement B a valid conclusion from Statement A?

A. Everyone who wears a red hat likes movies.
B. Some who wear a red hat like movies.

Statement B is a weaker version of Statement A.
Hence, it is true.

B. Existential Import

Existential Import refers to whether a statement implies that at least one of a particular set exists.

$$General\ Statement: All\ unicorns\ like\ to\ jump\ over\ the\ moon.$$

There is at one unicorn, and all unicorns like to jump over the moon.
The existence of unicorns is indicated by this statement.

$$Particular\ Statement: Some\ unicorns\ like\ to\ jump\ over\ the\ moon.$$

There is at one unicorn, and some of those unicorns like to jump over the moon.
The existence of unicorns is indicated by this statement.

Example 3.16

Can you conclude Statement B from Statement A?

A. *All unicorns like to jump over the moon.*
B. *Some unicorns like to jump over the moon.*

Yes, you can.

C. Statement Type-I

3.17: All A's are B (No A is not B)

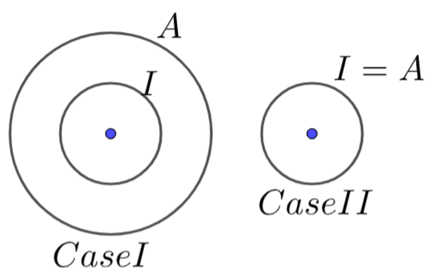
Definite Conclusion: Some B's are A's.
Possible Conclusion: Some B's are not A's

Example 3.18

$$All\ Indians\ live\ in\ Asia$$

Which of the following statements can be logically concluded from the above statement?

A. Some Asian's are Indians.
B. Some Asian's are not Indians.



Part A
Statement A is true in both cases.

Part B
Statement B is true in Case I, but not true in Case II.
Hence, it is not true in general.

Example 3.19

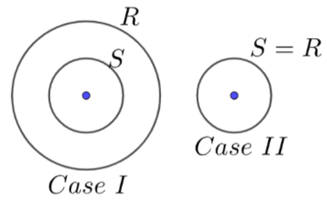
$$All\ squares\ are\ rectangles$$

Decide whether the following statements are valid conclusions from the above statement?

1. All rectangles are squares
2. Some rectangles are squares

3. Some rectangles are not squares

	Case I	Case II	Overall
All rectangles are squares	*F*	*T*	*F*
Some rectangles are squares	*T*	*T*	*T*
Some rectangles are not squares	*T*	*F*	*F*



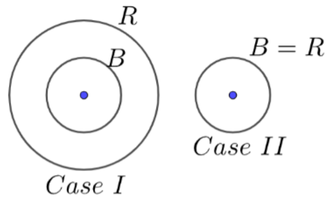
Example 3.20

Decide whether the second statement is a valid conclusion from the first?

A. There is no Blue which is not Red.
B. There is some Red, which is not Blue.

$$Case\ I \rightarrow Valid$$
$$Case\ II \rightarrow Not\ Valid$$

$$Overall: Not\ Valid$$



D. Statement Type-II

3.21: Some A's are B

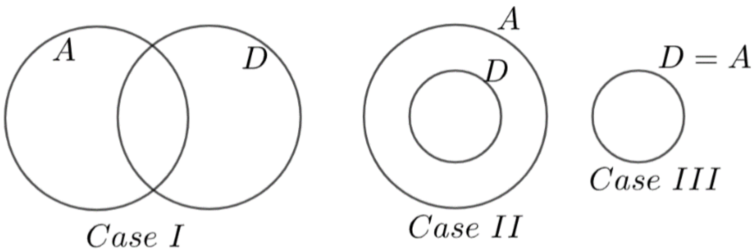
Definite Conclusion: Some B's are A's

Example 3.22

$$Some\ dogs\ are\ animals.$$

Decide whether the following statements are valid conclusions from the above statement?

A. Some dogs are not animals.
B. All dogs are animals.
C. Some animals are dogs.
D. Some animals are not dogs.



	Case I	Case II	Case III	Overall
Some dogs are not animals.	T	F	F	F
All dogs are animals.	F	T	T	F
Some animals are dogs.	T	T	T	T
Some animals are not dogs.	T	T	F	F

Example 3.23

Is the second statement a valid conclusion from the first?

A. Some rectangles are squares.
B. All rectangles are squares.

E. Statement Type-III

3.24: No A is B
Definite Conclusion: No B is A

This means that A and B are disjoint sets.

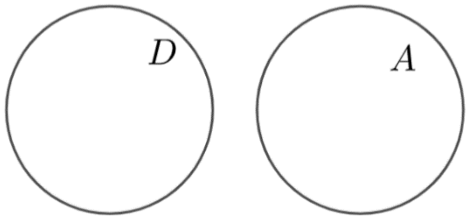
Example 3.25

No dog is an animal.

Decide whether the following statements are valid conclusions from the above statement?

A. No animal is a dog.
B. Some animals are not dogs.
C. Some animals are dogs.
D. Some dogs are not animals.
E. Some dogs are animals.

	Overall
No animal is a dog.	T
Some animals are not dogs.	
Some animals are dogs.	F
Some dogs are not animals.	
Some dogs are animals.	F



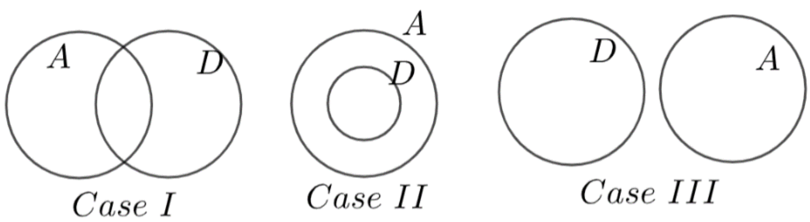
F. Statement Type-IV

3.26: Some A's are not B's

Example 3.27

Draw the Venn Diagram for the statement:

Some animals are not dogs



G. Syllogisms (3 Statement Questions)

Syllogisms were developed in classical Greece. They consist of the following format:

$$Statement\ 1$$
$$Statement\ 2$$
$$Statement\ 3$$

3.28: Valid Syllogism

A valid syllogism has three statements, out of which the third statement is a logical conclusion from ***both*** of the preceding statements.

If the third statement is a valid conclusion, but makes use of only one of the statements, then it is not a valid syllogism.

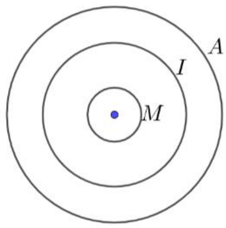
Example 3.29

$$1.\ All\ Mumbaikars\ live\ in\ India.$$
$$2. All\ those\ who\ live\ in\ India, live\ in\ Asia.$$

For each of the following statements, decide if it is logically valid. If it is logically valid, state which statement(s) are needed to make the conclusion?

A. Some Mumbaikars live in India.
B. Some Asians do not live in India
C. All Mumbaikars live in Asia
D. Some Indians live in Asia
E. Some Asians live in India

	Validity	Statements	Valid Syllogism
Some Mumbaikars live in India.	T	1	No
Some Asians do not live in India.	F		No
All Mumbaikars live in Asia.	T	1,2	Yes
Some Indians live in Asia.	T	2	No
Some Asians live in India.	T	2	No
Some Asians are Mumbaikars.	T	1,2	Yes



Example 3.30

$$All\ Dogs\ are\ Animals$$
$$All\ Animals\ are\ Vertebrates$$

For each of the following statements, decide if it is logically valid. If it is logically valid, state which statement(s) are needed to make the conclusion?

A. All Dogs are Vertebrates
B. Some Vertebrates are Animals
C. Some Dogs are Animals

	Validity	Statements	Valid Syllogism
All Dogs are Vertebrates	T	1,2	Yes
Some Vertebrates are Animals	T	2	No
Some Dogs are Animals	T	1	No

Example 3.31

Can you conclude the third statement on the basis on both the preceding statements?

A. Some mangoes are Alphonso mangoes.
B. Alphonso mangoes taste sweet.
C. Some mangoes taste sweet.

$$Yes$$

Example 3.32

Can you conclude the third statement on the basis on both the preceding statements?

A. Some children like Maths.
B. Children who like Maths also like Science
C. Children who like Science also like Maths

No

Example 3.33

No birds like rain.
No sparrows like rain.
Consider the statement below. Can we conclude any of the below statements:

A. All sparrows are birds.
B. All birds are sparrows.
C. Some birds are sparrows.
D. Some sparrows are birds.
E. No sparrows are birds.
F. No birds are sparrows.

H. Pick 3 out of 6 Type

I. Modern Logic/Existential Import

Existential Import refers to whether a statement implies that at least one of a particular set exists.

General Statement: *All unicorns like to jump over the moon.*

Any unicorns that exist like to jump over the moon.
There is *nothing being said* about the number of unicorns.

Particular Statement: *Some unicorns like to jump over the moon.*

There is *at least one unicorn*, and some of those unicorns like to jump over the moon.
The existence of unicorns is indicated by this statement.

Example 3.34

Can you conclude Statement B from Statement A?

A. *All unicorns like to jump over the moon.*
B. *Some unicorns like to jump over the moon.*

You cannot conclude Statement B from Statement A, since Statement B requires the set of unicorns to be non-null, whereas Statement A does not

J. Exam Questions-II

Example 3.35

If all alligators are ferocious creatures and some creepy crawlers are alligators, which statement(s) must be true? (AMC 10 2000/21)

I. All alligators are creepy crawlers.
II. Some ferocious creatures are creepy crawlers.

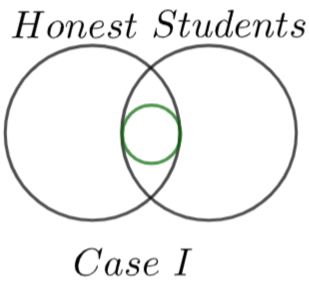
III. Some alligators are not creepy crawlers.

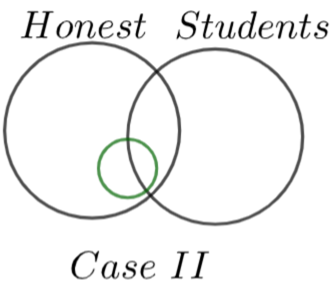
Example 3.36

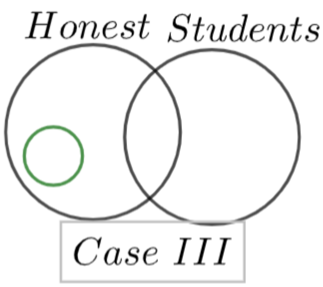
Assume that, for a certain school, it is true that
I: Some students are not honest.
II: All fraternity members are honest.
A necessary conclusion is:

A. Some students are fraternity members.
B. Some fraternity member are not students.
C. Some students are not fraternity members.
D. No fraternity member is a student.
E. No student is a fraternity member. (AHSME 1968/10)

	Case I	Case II	Case III	Overall
Some students are fraternity members.	T	T	F	F
Some fraternity member are not students.	F	T	T	F
Some students are not fraternity members.	T	T	T	T
No fraternity member is a student.	F	F	T	F
No student is a fraternity member.	F	F	T	F







Solution using Existential Import
(Official Solution)

Statements A and B require that the set of all fraternity members is non-empty. But Hypothesis I and II do not require this. Therefore, Statements A and B are not valid.

The given hypothesis would let the set of fraternity members be a non-empty set of the set of all students. But D and E do not allow this. (In other words, D and E are in conflict with Case I and II above).

Example 3.37: Group Question

Amit, Balvinder, Chetan and Deepak are employed in a company, where they have to share among themselves the work load that consists of six tasks A, B, C, D, E and F. The following statements identify their preferences for the different tasks:

- All those who like task B also like task E.
- All those who like task C also like task D.
- All those who like task E do not like task C, and vice-versa.

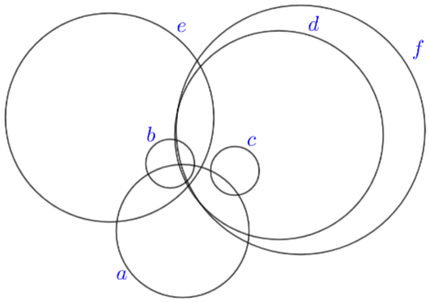
- Some of those who like task E also like task A
- Some of those who like task D also like task E
- All those who like task D also like task F. (JMET 2008)

Amit enjoys the task D. Which of the following must be true? (JMET 2008/62)
A. He may or may not like the task C.
B. He does not like the task B.
C. He likes the task A.
D. He likes the task C.

Someone who enjoys task D belongs to Set D.

- From the diagram, both cases of option A are possible.

Hence, option A is correct.



Balvinder likes the task B. He may also like any of the following tasks except: (JMET 2008/63)
A. Task A
B. Task C.
C. Task D
D. Task E

From the diagram, b potentially has a non-null intersection with A, D and E.
Hence, option B is correct (task C has no intersection with task B).

Chetan likes the task C. Which of the following must be false? (JMET 2008/64)
A. He does not like the task A.
B. He likes the task F.
C. He does not like the task B.
D. He may like the task E.

From the diagram, c and e are disjoint.
Hence, option D is correct.

Based on the information provided, which of the following statements must be true? (JMET 2008/65)
A. All those who like the task E also like the task C.
B. None of those who do not like the task F like the task A.
C. Those who like the task A may or may not like the task C.
D. None of those who like the task B do not like the task D.

e and c are disjoint. Hence, option A is false.
Those who are do not like refers to the set outside of f. However, this can be inside of a. Hence, option B is false.
The intersection between c and a can be null or non-null. Hence, option C is correct.
The people who like task b may or dislike task d. Hence, option D is incorrect.

3.3 Puzzles and Families

A. Odd Man Out

Example 3.38

I have 8 balls out of which one is heavier, and the rest have equal weights. I also have a scale with two parts, on which I can weigh the balls, and find out whether the left side is heavier, the right side is heavier or both are equal.

A. Give an algorithm that will determine the heavier ball in three steps.
B. Generalize your answer to find the number of steps required if we have n balls instead of 8.

Part A

Step I: Number the balls $1,2,3, \dots ,8$ and divide them into two sets

$$\{1,2,3,4\}\{5,6,7,8\}$$

Put one set of the left side of the scale, and the other set on the right side of the scale.
Whichever side is heavier has the culprit.

Step II:
We have four balls. Divide into sets of 2, and repeat.

Step III:
We have two balls. Put one on each side of the scale.

Part B

Let us write the number of balls from which we can find the heavier in x steps

$$Last\ Step: 2\ Balls = 2^1$$
$$Second\ Last\ Step: 4\ Balls = 2^2$$
$$Third\ Last\ Step: 8\ Balls = 2^3$$

$$2^x \le n < 2^{x+1}$$
$$No.of\ Steps = x\ if\ 2^x = n$$
$$Else\ \ No.of\ Steps = x + 1$$

Example 3.39

I have 9 balls out of which one is heavier, and the rest have equal weights. I also have a scale with two parts, on which I can weight the balls, and find out whether the left side is heavier, the right side is heavier or both are equal.

A. Give an algorithm that will determine the heavier ball in two steps.
B. Generalize your answer if we have n balls instead of 9.

Part A

Step I

This time divide the ball into three sets:

$$\{1,2,3\}\{4,5,6\}\{7,8,9\}$$

Put the first two sets on the left and right side of the scales, respectively.

- If the scale tilts to the left, the heavier ball is in the set $\{1,2,3\}$
- If the scale tilts to the right, the heavier ball is in the set $\{4,5,6\}$
- If the scales are equal, the heavier ball is in the set $\{7,8,9\}$

Step II

As before, make three groups, one with a ball in each, and repeat the process.

Part B

$$Last\ Step: 3\ Balls$$

$$Second\ Last\ Step: 9\ Balls$$
$$Third\ Last\ Step: 27\ Balls$$

$$3^x \le n < 3^{x+1}$$
$$No.of\ Steps = x\ if\ 3^x = n$$
$$Else\ \ No.of\ Steps = x + 1$$

Example 3.40: Weighing Sacks of Grain

A charity has ten truckloads of grain delivered to it. Each truck has 100 sacks, and each is supposed to have 100 kg of grain. However, one truck has sacks of grain which are only 99 kg each. The maximum number of sacks that can be weighed together is 100. Think of a strategy where the weighing needs to be done only once.

Take:

$$1\ sack\ from\ Truck\ 1$$
$$2\ sacks\ from\ Truck\ 2$$
$$.$$
$$.$$
$$.$$
$$9\ sacks\ from\ Truck\ 9$$

The total weight of the sacks, if every single sack has 100 kg is:

$$1(100) + 2(100) + \cdots + 9(100) = \frac{9 \times 10}{2}(100) = 4500\ kg$$

If truck 1 has sacks with 99 kg. Then, the weight of the sacks will be:

$$1(99) + 2(100) + 3(100) + \cdots 9(100)$$

Which is 1 less than 4500 kg.

If truck 1 has sacks with 99 kg. Then, the weight of the sacks will be:

$$1(100) + 2(99) + 3(100) + \cdots 9(100)$$

Which is 2 less than 4500 kg.

In general, the actual weight will be less than 4500 by the truck number that has sacks with less weight.

Example 3.41: Poisoned Wine Bottle

A king has 100 bottles of wine, exactly one of which is poisoned. He has rats to which he can give the wine. He wants to determine the poisoned bottle. A rat drinking poisoned wine dies with 100% probability 24 hours within after drinking the wine. Determine the valid strategy. We need to determine the poisoned bottle in 24 hours. There can be many different objectives. We consider three of them. Answer each part independently:

A. Minimize d = Number of dead rats
B. Minimize n = Number of rats
C. Minimize nd, where n and d are defined above

Part A

$$Let\ Rat\ n\ drink\ from\ Bottle\ n$$
$$Rats\ needed = 99$$
$$Dead\ Rats = 1\ OR\ Dead\ Rats = 0$$

The expected number of dead rats is:

$$1 \times \frac{99}{100} + 0 \times \frac{1}{100} = 0.99$$

Part B

Example 3.42: Poisoned Wine Bottle

Answer the above question if a rat drinking poisoned dies with 100% probability 24 hours exactly after drinking the wine.

B. State Puzzles

Example 3.43

A man needs to transport a wolf, a goat and some grass.

Example 3.44

JMET 2009 46

Three married couples on a journey come to a river where they find a boat that cannot carry more than two persons at a time. An additional condition is that a lady cannot be left on either bank where other men are present, without her husband.

(1) These people will not be able to cross the river.
(2) They will be able to cross the river in 9 steps.
(3) They will be able to cross the river in 10 steps.
(4) They will be able to cross the river in 13 steps.

C. Families

Example 3.45: Siblings

Each of my three sisters have two brothers. Each of my brothers has four sisters. What is the number of siblings in the family?

Suppose the first brother has sisters $Stephanie, Rose, Melanie,$ and $Dawn$.

Stephanie	Bill
Rose	Richard
Melanie	
Dawn	

3.4 Ranking

A. Basics

Example 3.46

AMC 8 1993/23

Five runners, P, Q, R, S, T, have a race, and P beats Q, P beats R, Q beats S, and T finishes after P and before Q. Who could NOT have finished third in the race?

$\text{(A)}\ P\ \text{and}\ Q \qquad \text{(B)}\ P\ \text{and}\ R \qquad \text{(C)}\ P\ \text{and}\ S \qquad \text{(D)}\ P\ \text{and}\ T \qquad \text{(E)}\ P, S\ \text{and}\ T$

Solution [edit]

First, note that P must beat Q, R, T, and by transitivity, S. Thus P is in first, and not in 3rd. Similarly, S is beaten by P, Q, and by transitivity, T, so S is in fourth or fifth, and not in third. All of the others can be in third, as all of the following sequences show. Each follows all of the assumptions of the problem, and they are in order from first to last: PTQRS, PTRQS, PRTQS. Thus the answer is $\boxed{\text{(C)}\ P\ \text{and}\ S}$.

I: P beats Q

$$P > Q\ (P\ is\ better\ than\ Q)$$

II: P beats R

$$P > R$$

III: Q beats S

$$Q > S$$

IV: T finishes after P

$$P > T > Q$$

V: From III and IV:

$$P > T > Q > S$$

From II: $P > R$, and hence P beats everyone.

$$P\ is\ 1st \Rightarrow P\ cannot\ be\ third.$$

P, T and Q beat S. Therefore, at best

$$S\ can\ be\ 4th\ fourth\ position.$$

Hence:

$$P\ and\ S\ cannot\ be\ in\ 3rd\ position.$$

Example 3.47

Alan, Beth, Carlos, and Diana were discussing their possible grades in mathematics class this grading period. Alan said, "If I get an A, then Beth will get an A." Beth said, "If I get an A, then Carlos will get an A." Carlos said, "If I get an A, then Diana will get an A." All of these statements were true, but only two of the students received an A. Which two received A's? (AMC 8 1986/22)

Alan said, "If I get an A, then Beth will get an A."

$$Alan \leq Beth$$

Beth said, "If I get an A, then Carlos will get an A."

$$Beth \leq Carlos$$

Carlos said, "If I get an A, then Diana will get an A."

$$Carlos \le Diana$$

$$Alan \le Beth \le Carlos \le Diana$$

The two students must be the ones that get A's. They are:

$$Carlos \le Diana$$

Example 3.48

Albert, David, Jerome and Tommy were plucking mangoes in a grove to earn some pocket money during the summer holidays. Their earnings were directly related to the number of mangoes plucked and had the following relationship:
Jerome got less money than Tommy. Jerome and Tommy together got the same amount at Albert and David taken together. Albert and Tommy together got less than David and Jerome taken together.
Rank them according to the money earned by each. **(CAT 1990/106, Adapted)**

$$J < T, \qquad \underbrace{A + T < D + J}_{Inequality\ I}, \qquad \underbrace{J + T = A + D}_{Equation\ I}$$

Subtract Equation I from both sides of Inequality I:

$$A - J < J - A \Rightarrow 2A < 2J \Rightarrow A < J$$

Rearrange Inequality I to get $T - J < D - A$ and subtract Equation I from both sides of that:

$$2T < 2D \Rightarrow T < D$$
$$A < J < T < D$$

Example 3.49

Students rank the business schools based on the following factors: Average salary of a fresh graduate, terminal degree of faculty members, and institutional facilities. In their final ranking school, A is ranked higher than school B. Which of the following will ensure that school A is ranked higher than school B? **(JMET 2008/67)**

A. The average salary of fresh graduates from school A is 60% less than the average salary of fresh graduates from school B.
B. The average salary of fresh graduates from school A is 80% more than the average salary of fresh graduates from school B.
C. All the faculty members in school A have a doctoral degree while in school B only 50% of the faculties have a doctoral degree.
D. In all the factors school A is marginally better than school B.

We do not know the weights assigned to the different factors. We do not even whether the ranking is done using a weighted average, or a pure ranking method. Hence, an option that discusses an individual factor is difficult to mark true.

However, option D says that on every parameter, A is better than B. Hence, option D is correct.

Example 3.50

JMET 2011 (Ranking) 31-34
An ice-cream maker is experimenting with six chemical essences, U, V, W, X, Y and Z for developing a new flavour called "SWEETER & HEALTHIER". The details of these chemical essences are as follows:

1. U is sweeter than V and healthier than Z.
2. V is sweeter than Y and less healthy than Z.
3. W is less sweet than X and less healthy than U.
4. X is less sweet and healthier than Y.
5. Y is less sweet and healthier than U.
6. Z is sweeter than U and less healthy than W.

Consider sweetness first
U is sweeter than V:
$$V < U$$
V is sweeter than Y:
$$Y < V < U$$
W is less sweet than X
$$W < X, \qquad Y < V < U$$
X is less sweet than Y:
$$W < X < Y < V < U$$
Z is sweeter than U:
$$W < X < Y < V < U < Z$$

Consider healthiness:
1: U is healthier than Z:
$$Z < U$$
2: V is less healthy than Z:
$$V < Z < U$$
3: W is less healthy than U
$$V < Z < U$$
$$W < U$$
4: X is healthier than Y
$$V < Z < U$$
$$W < U$$
$$Y < X$$
5: Y is healthier than U
$$V < Z < U < Y < X$$
$$W < U$$
Z is less healthy than W:
$$V < Z < W < U < Y < X$$

Which is the sweetest essence?
(1) U
(2) W
(3) X
(4) Z

Which of the following essences is/are both sweeter and healthier than V?
(1) U only
(2) W only
(3) Z only
(4) U & Z only

Which of the following essences is/are sweeter than Y and healthier than W?
(1) U only
(2) V only
(3) Z only
(4) U & V only

Which is the least healthy essence?
(1) U
(2) V
(3) W
(4) Y

3.5 Arrangements

A. Linear Arrangements

Example 3.51

Abby, Bret, Carl, and Dana are seated in a row of four seats numbered #1 to #4. Joe looks at them and says:
"Bret is next to Carl."
"Abby is between Bret and Carl."
However each one of Joe's statements is false. Bret is actually sitting in seat #3. Who is sitting in seat #2? **(AMC 8 1987/17)**

We have four seats, and Bret sitting in seat #3.

1	2	3	4
		Bret	

Bret is next to Carl is $false$.
Hence, Bret is *not* next to Carl. That means, Carl cannot be in seat 2 or seat 4. Hence, Carl must be in Seat 1.

1	2	3	4
Carl		Bret	

Abby is between Bret and Carl is $false$.
Hence, Abby is not between Bret and Carl.
Hence, Abby is seat 4.

1	2	3	4
Carl		Bret	Abby

That only leaves Dana for seat 2

1	2	3	4
Carl	Dana	Bret	Abby

Example 3.52

Bridget, Cassie, and Hannah are discussing the results of their last math test. Hannah shows Bridget and Cassie her test, but Bridget and Cassie don't show theirs to anyone. Cassie says, 'I didn't get the lowest score in our class,' and Bridget adds, 'I didn't get the highest score.' What is the ranking of the three girls from highest to lowest? **(AMC 8 2013/19)**

A. Hannah, Cassie, Bridget
B. Hannah, Bridget, Cassie
C. Cassie, Bridget, Hannah
D. Cassie, Hannah, Bridget
E. Bridget, Cassie, Hannah

Cassie says, 'I didn't get the lowest score in our class,

$$Cassie > Hannah$$

Bridget adds, 'I didn't get the highest score.

$$Hannah > Bridget$$

$$Cassie > Hannah > Bridget \Rightarrow Option\ 4$$

Example 3.53

Mala, Devi, Sita, Emma, and Kala are sitting in a park. Which girl is sitting farthest to the right? **(NMTC Primary-Screening, 2006/20)**

A. Mala is not sitting on the farthest right, and Devi is not sitting on the farthest left.
B. Sita is not sitting at the farthest left or farthest right.
C. Kala is not sitting next to Sita, and Sita is not sitting next to Devi.
D. Emma is sitting to the right of Devi, but not necessarily next to her.

$$\times\ means\ Not\ Possible$$

From Clues A and B, we get:

	1	2	3	4	5
Mala					×
Devi	×				
Sita	×				×
Emma					
Kala					

From Clue D:

	1	2	3	4	5
Mala					×
Devi	×				×
Sita	×				×
Emma	×	×			
Kala					

This leaves us with only two people who can sit in the first position, Mala and Kala. Suppose, Mala is sitting in the first position, and Kala is sitting in the last position.

	1	2	3	4	5
Mala	Mala	×	×	×	×
Devi	×			×	×
Sita	×				×
Emma	×	×			×
Kala	×	×	×	×	Kala

This position results in a contradiction.

	1	2	3	4	5
Mala			Mala		×
Devi	×	Devi			×
Sita	×			Sita	×
Emma	×	×	×	×	Emma
Kala	Kala				×

Example 3.54: Redundant Information[1]

Five flags, each with a distinct symbol namely Panther, Tiger, Rose, Swan and Quail have been arranged in the following order:

1. Panther is next to Quail, and Swan is next to Rose.
2. Swan is not next to Tiger, Tiger is on the extreme left-hand side, and Rose is on the second position from the right-hand side.
3. Panther is on the right-hand side of Quail and to the right side of Tiger.
4. Panther and Rose are together.

Which of the following statements is true?

A. 3 and 4 are contradicting
B. Either 3. or 4. is redundant.
C. 3. is redundant
D. 4. is redundant (JMET 2009/49)

Analysis

The focus of the question is on Statements 3 and 4, which are either contradicting or redundant. So, we begin with Statements 1 and 2.

Statements 1 and 2

Statement 1 is difficult to make use of immediately. Start with Statement 2.

$Tiger\ is\ on\ the\ extreme\ left-hand\ side$:

	1	2	3	4	5
P					
Q					
R					
S					
T	T				

$Rose\ is\ on\ the\ second\ position\ from\ the\ right\ side$:

	1	2	3	4	5
P					
Q					
R				R	
S					
T	T				

$Swan\ is\ not\ next\ to\ Tiger$:

	1	2	3	4	5
P					
Q					
R				R	
S					
T	T				

Now, Statement 1 is useful. Panther is next to Quail means that they have to be in slots 2 and 3 (though we do not know which one is which).

Use Statement 3

$Panther\ is\ on\ the\ right-hand\ side\ of\ Quail.$

This means that Panther is in Slot 3, and Quail is Slot 2. Swan is Slot 5. Our arrangement is complete:

	1	2	3	4	5
P			P		
Q		Q			
R				R	
S					S
T	T				

Use Statement 4

Start with the position reached at the end of Statement 2.

$Panther\ and\ Rose\ are\ together.$

This tells us that Panther is in Slot 3, and Quail is Slot 2. Swan is Slot 5. Our arrangement is (again) complete, and we get the same arrangement as above.

	1	2	3	4	5
P			P		
Q		Q			
R				R	
S					S
T	T				

Hence, only one of Statement 3 and 4 is needed.

[1] This question in the exam had two other questions which were associated with it (which were quite easy to solve after the arrangement was identified).

Option B is correct.

Example 3.55: Double Linear Arrangement[2]

There are six blocks of rooms along a straight corridor in a hotel with each block containing two rooms facing other.
The following group of twelve people: J
itender, Lakshman, Mary, Narayan, Pankaj, William, Chandra, Ahmed, Balu, Ferosh, Esha and Rajender has occupied some of these rooms. There are a maximum of two people in a room, and some rooms may be empty.

Left					Right
Corridor					

A. Lakshman and his roommate stay two blocks to the right of Ahmed and his roommate Chandra.
B. Jitender stays alone, three blocks to the left of William and two blocks to the left of Esha.
C. Mary stays one block to the left of Ahmed and Chandra.
D. Narayan stays three blocks to the right of the block on which Balu and Ferosh have single rooms.
E. Rajender and Pankaj stay in single rooms two blocks to the left of Mary.

Jitender arranges to move into a room two blocks to the left, whose occupant moves into a room one block to the right. In turn, the occupant of this room moves into a room three blocks to the right, whose occupant(s) take Jitender's old room. Who is/are the new occupant(s) of Jitender's old room? **(JMET 2008/57)**

Statement A

Ahmed and Chandra ($A-C$) cannot be in the two rightmost blocks.

	1	2	3	4	5	6
A-C						

Statement B

William and Esha cannot be in the three leftmost blocks, and two leftmost blocks respectively.
Jitender cannot be in the rightmost three blocks

	1	2	3	4	5	6
A-C						
W						
E						
J						

Statement C

Combined with what we know about Ahmed and Chandra, Mary cannot be in the three rightmost blocks

	1	2	3	4	5	6
A-C						
W						
E						
J						
M						

Statement D

Narayan cannot stay in the leftmost three blocks

	1	2	3	4	5	6
A-C						
W						
E						
J						
M						
N						

Statement E

Mary cannot occupy the two leftmost rooms.

	1	2	3	4	5	6
A-C						
W						
E						
J						
M						
N						

Finally, we have a specific location. Mary must be in block 3, and Ahmed/Chandra must be in block 4:

	1	2	3	4	5	6
A-C				AC		
W						
E						
J						
M			M			
N						

[2] The exam had a total of six questions on this information. The other five were quite easy to solve once the arrangement was identified.

Also, Rajender and Pankaj must stay in block 1:

	1	2	3	4	5	6
A-C				AC		
W						
E						
J						
M			M			
N						
RP	RP					

Statement A (Revisited)
Lakshman and his roommate must be in block 6.

	1	2	3	4	5	6
A-C				AC		
W						
E						
J						
M			M			
N						
RP	RP					
L+						L+

R					
P		M	AC		L+

Statement D (Revisited)
Till now, we have ignored the double nature of the corridor. We now bring that into the picture, and make a table of the type given in the question:

Narayan is three blocks to the right of Balu and Ferosh, and they occupy single rooms. The only block they can stay is 2, and Narayan must be in Block 5

R	B			N	L+
P	F	M	AC		

Statement B (Revisited)
Since the leftmost two blocks are occupied Jitender must stay in Block 3, and Esha must stay in Block 5

R	B	J		N	LW
P	F	M	AC	E	

We are now finally in a position to answer the question asked:

- ➢ Jitender arranges to move into a room two blocks to the left (*Rajender OR Pankaj*)
- ➢ Whose occupant moves into a room one block to the right (*Balu OR Ferosh*)
- ➢ In turn, the occupant of this room moves into a room three blocks to the right whose occupant(s) take Jitender's old room. Who is/are the new occupant(s) of Jitender's old room? (*Narayan OR Esha OR both*)

Final Answer

Narayan OR Esha OR both

B. Circular Arrangements

C. Scheduling

Example 3.56

JMET 2011 (Multiple Parameters)
Directions for Questions 40 to 43: Read the following information and answer the questions.
The owner of the house has been murdered. The visitors to the house were Aditya, Vijay and Puneet. The following additional information is also given.
1. The murderer who was one of the three visitors, arrived at the house later than at least one of the other two visitors.
2. The driver of the house who was one of the three visitors, arrived at the house earlier than at least one of the two visitors.
3. The driver arrived at the house at midnight.
4. Neither Aditya nor Vijay arrived at the house after midnight.

5. Between Vijay and Puneet, the one who arrived earlier was not the driver.
6. Between Aditya and Puneet, the one who arrived later was not the murderer.

Who arrived at the house earliest?
(1) Puneet
(2) Aditya
(3) Vijay
(4) Data insufficient

Who is the murderer?
(1) Puneet
(2) Aditya
(3) Vijay
(4) Data insufficient

Who is the driver?
(1) Puneet
(2) Vijay
(3) Aditya
(4) Data insufficient

Who arrived at the house last?
(1) Puneet
(2) Aditya
(3) Vijay
(4) Data insufficient

Example 3.57

Analyze the information.

Every morning five friends Mahima, Nimisha, Omez, Parul and Quan go to the railway station and board a train that stops at six subsequent stations which are numbered 1 to 6. The train stops at Station 1 and proceeds in numerical order to Station 6.

1. Mahima gets off either at Station 1 or at Station 2.
2. Omez always gets off one station before or one station after Quan's station.
3. Parul always gets off at Station 3.
4. Quan always gets off at Station 4, 5 or 6.
5. No one re-boards the morning train after getting off.

From Statements 1,3, and 4:

	1	2	3	4	5	6
M						
N						
O						
P						
Q						

From Statement 2:

	1	2	3	4	5	6
M						
N						
O						
P						
Q						

On a morning, when no one gets off at Station 5 or 6, which of the following MUST be true? **(JMET 2011/35)**
(1) Mahima gets off at Station 2

(2) Nimisha gets off at Station 2
(3) Omez gets off at Station 4
(4) Omez and Parul get off at the same station

For this question, no one gets off at Station 5 or 6.

	1	2	3	4	5	6
M						
N						
O						
P						
Q						

This forces Quan to get off at 4, which forces Omez to get off at 3, which then makes option 4 correct.

	1	2	3	4	5	6
M						
N						
O						
P						
Q						

On a morning, when Quan gets off at Station 4 and no more than two of the friends get off at any one station, which of the following MUST be true? **(JMET 2011/36)**
(1) If Nimisha gets off at Station 2, Mahima gets off at Station I.
(2) If Nimisha gets off at Station 3, Omez gets off at Station 5.
(3) If Nimisha gets off at Station 4, Omez gets off at Station 5.
(4) If Omez gets off at Station 3, Nimisha gets off at Station 2.

Quan gets off at Station 4, and hence Omez must get off at either Station 3 or Station 5.

	1	2	3	4	5	6
M						
N						
O						
P						
Q						

Consider option 2. Nimisha gets off at Station 3. Parul always get off at Station 3. Now the two-person limit for Station 3 is reached, and the only station for Omez to get off is Station 5.

At which stations is it possible for Nimisha and Omez to be the only friends getting off the morning train? **(JMET 2011/37)**

	1	2	3	4	5	6
M						
N						
O						
P						
Q						

Omez cannot get off at Stations 1 and 2. Hence, 1 and 2 are not possible.
Parul gets off at Station 3. Hence, Nimisha and Omez cannot be the only ones getting off at Station 3.

If Omez gets off at Station 4, Quan must get off at Station 5. And Nimisha can also get off at Station 4. Hence, the required condition is met.

	1	2	3	4	5	6

M						
N						
O						
P						
Q						

If Omez gets off at Station 5, Quan must get off at Station 4 or 6. And Nimisha can also get off at Station 5. Hence, the required condition is met.

	1	2	3	4	5	6
M						
N						
O						
P						
Q						

Similarly, station 6 is also possible.
The possible stations are Stations 4, 5 and 6.

On a morning, when no one gets off at Station I and each of the five friends gets off at a different station, which of the following cannot be true? **(JMET 2011/38)**
(1) Nimisha gets off one station before Quan.
(2) Parul gets off one station before Omez.
(3) Mahima gets off at Station 2.
(4) Nimisha gets off at Station 5.

No one gets off at Station 1, leaving only Station 2 for Manisha. Also, Parul gets off at Station 3.

	1	2	3	4	5	6
M						
N						
O						
P						
Q						

Consider option 4. If Nimisha gets off at 5, Omez and Quan need to get off at 4 and 6. They are separated by two stations, which violates the condition that Omez gets off before Quan, or after Quan.

	1	2	3	4	5	6
M						
N						
O						
P						
Q						

Option 4 is correct.

D. Complex Arrangements

Example 3.58

There are five friends Amisha, Binaya, Celina, Daisy and Eshaan. Two of them play table tennis while the other three play different games, viz. football, cricket and chess. One table tennis player and the chess player stay on the same floor while the other three stay on floors 2, 4 and 5. Two of the players are

	Sports	Name	Occ	Age
2				
3	$TT, Chess$			Oldest
4				
5				

industrialists while the other three belong to different occupations viz. teaching, medicine and engineering. The chess player is the oldest while one of the table tennis players, who plays at the national level, is the youngest. The other table tennis player who plays at the regional level is between the football player and the chess player in age. Daisy is a regional player and stays on floor 2. Binaya is an engineer while Amisha is the industrialist and plays table tennis at the national level. Daisy plays cricket and football player stays on floor 4. Celina stays on floor 5. Only one of the friends is younger than Daisy.

Who all stay on floor 3? (JMET 2009/32)
(1) Amisha and Binaya
(2) Daisy and Eshaan
(3) Binaya and Daisy
(4) Celina and Daisy

What does Eshaan play? (JMET 2009/33)

Who stays on floor 4? (JMET 2009/34)

Age wise, who among the following lies between Binaya and Eshaan? (JMET 2009/35)

What is the occupation of the chess player? (JMET 2009/36)

Daisy is a regional player, stays on floor 2, and she plays cricket.
Celina stays on floor 5.
Football player stays on floor 4.

	Sports	Name	Occ	Age
2	Cricket	Daisy		
3	$TT, Chess$			Oldest
4	Football			
5		Celina		

From question 32, we know that Daisy stays on floor 2, and hence the correct option is Amisha and Binaya. Since plays TT at the national level, we can assign the correct sport. Also, Amisha is an industrialist and Binaya is an engineer
The only person left for Floor 4 is Eshaan.
The only sport left Celina is $TT - R$

	Sports	Name	Occ	Age
2	Cricket	Daisy		
3	$TT - N$ $Chess$	$Amisha$ Binaya	Ind Eng	Oldest
4	Football	Eshaan		
5	$TT - R$	Celina		

The national level TT player (Amisha) is the youngest.
Only one of the friends is younger than Daisy, which means Daisy is second youngest.
The other table tennis player who plays at the regional level ($Celina$) is between the football player ($Eshaan$) and the chess player in age ($Binaya$).

	Sports	Name	Occ	Age
2	Cricket	Daisy		$2nd\ Youngest$
3	$TT - N$ $Chess$	$Amisha$ Binaya	Ind Eng	Youngest Oldest
4	Football	Eshaan		$3rd\ Youngest$
5	$TT - R$	Celina		$4th\ Youngest$

E. Coloring

Example 3.59
We want to paint each square in the grid with colors A, B, C and D, so that neighboring squares always have different colors. (Squares which share the same corner point also count as neighboring). Some of the squares are already painted. (Austrian Kangaroo 2009/24)

A	B	A	C	D

A	B	A	C	D
D	C	D	B	A
A	B	A	C	D
D	C	D	B	A

A	B	D	C	D
D	C	A	B	A
A	B	D	C	D
D	C	A	B	A

Always A

F. Other

Example 3.60

Fact 1: A project team consisting of males and females has four members.
Fact 2: Two of the members are proficient in mathematics and the other two are proficient in computer programming.
Fact 3: Half the members are female.
If the first three statements are facts, then which of the following statements must also be a fact?
I. At least one female member is proficient in mathematics.
II. Two of the members are male.
III. The male members are proficient in computer programming. **(JMET 2008/48)**

II is correct.

Example 3.61

Fact 1: Manoj said, "Anush and I both went to a movie last night."
Fact 2: Anush said, "I was only studying last night."
Fact 3: Manoj always tells the truth, but Anush sometimes lies.
If the first three statements are facts, then which of the following statements must also be a fact?
I. Anush went to a movie last night.
II. Manoj went to a movie last night.
III. Anush was studying last night. **(JMET 2008/49)**

I and II are correct.

Example 3.62

Fact 1: Chairs cost between Rs. 200 to Rs. 2,000.
Fact 2: Some chairs are made of aluminium.

Fact 3: Some chairs are made of plastic.
If the first three statements are facts, then which of the following statements must also be a fact?
I. Aluminium chairs cost more than plastic chairs.
II. Expensive chairs last longer than cheap chairs.
III. Plastic chairs cost around Rs. 200 and aluminium chairs cost around Rs. 2000. **(JMET 2008/50)**

We cannot conclude any statement.

Example 3.63

Fact 1: All metros have ring roads.
Fact 2: Delhi is a metro.
Fact 3: Delhi has a population of more than 5 million.
If the first three statements are facts, then which of the following statements must also be a fact?
I. Delhi has a ring road.
II. All metros have a population more than 5 million.
III. All cities with a ring road are metros. **(JMET 2008/51)**

I is correct

4. Games

4.1 Games

A. Josephus Problem

Example 4.1
Ten cards 1-10 are arranged in a stack face down so that the first card is removed; the second card is put at the bottom of the stack; the third card is removed; the fourth card is put at the bottom of the stack; and so on until only one card remains. The removed cards, in order, are 1-9. The remaining card is 10. In the original stack, what was the sum of the cards adjacent to card 10? (MathCounts 2000 Chapter Team/5)

Let the cards be, in order:
ABCDEFGHIJ

1st Round:
Removed: ACEGI, Remaining:BDFHJ

2nd Round:
Removed: BFJ, Remaining:DH

3nd Round:
Removed: H, Remaining: D

Order of Removal:
ACEGIBFJD
A=1, C=2, E=3, G=4, I=5, B=6, F=7, J=8, H=9

D is next to C=2, and E=3.
Sum=2+3=5

Example 4.2
Solution for $k = 2$

Example 4.3
XAT 2011 Question
And this version

B. Nim

4.4: NIM: Counting Up

The rules of the game of Nim are:
- There are two players.
- Round 1
  - ✓ The first player says a number between 1 and n.
  - ✓ The second player adds a number between 1 and n to the first player's number, and says that number.

- Round 2 onwards
  - ✓ The first and second player each add a number to the other player's number.
  - ✓ Player 1 goes first, and player 2 goes second.
- The first player to reach X wins.

Example 4.5

Decide a winning strategy for a game of Nim where $n = 9$ and $X = 90$

A. Saying what number will guarantee that a person can say 90 on the next round?
B. Saying what number will guarantee that a person can say 90 two rounds later?
C. Generalize and work backwards

Part A

$$\begin{gathered}Winning\ Player: 80\\ Losing\ Player: 81 - 89\\ Winning\ Player: 90\end{gathered}$$

Part B

$$\begin{gathered}Winning\ Player: 70\\ Losing\ Player: 71 - 79\\ Winning\ Player: 80\end{gathered}$$

		Round Number								
		1	2	3	4	5	6	7		
Player 1	Losing Player	1-9	11-19	21-29	31-39	41-49	51-59	61-69	71-79	81-89
Player 2	Winning Player	10	20	30	40	50	60	70	80	90

Example 4.6

Decide a winning strategy for a game of Nim where $n = 10$ and $X = 100$

		Round Number								
		1	2	3	4	5	6	7		
Player 1	Winning Player	1	12	23	34	45	56	67	78	89
Player 2	Losing Player									

4.7: NIM: Counting Down

Instead of counting up, a variant of NIM requires you to count down, starting from a number X with a maximum decrease of n.

Example 4.8

Decide a winning strategy for a game of Nim where $n = 9$ and $X = 1$, and starting number for the first player is 100 (this means that the first player must say 100).

				Round Number								
				1	2	3	4	5	6	7		
Player 1	Losing		100									

Player 2	Player Winning Player		91	81	71	61	51	41	31	21	11	1

C. Subtract a Square

4.9: Definition

10 Examples