Counting Basics

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1. Counting Strategies

1.1 Enumeration

A. Basics

- Enumeration requires counting by listing. Some questions in this section are easy, but some are hard. You should come back to the ones you find hard later.
- Note that we will introduce formulas for counting later. But some exam questions require enumeration, and some require enumerating cases, combined with formulas.

Example 1.1

Enumerate the number of ways in which three people (A, B, C) can be seated in a row.

Start with A	Start with B	Start with C
ABC	BAC	CAB
ACB	BCA	CBA

Example 1.2

Enumerate the number of ways in which four people (P, Q, R, S) can be seated.

Start with P			Start with Q			Start with R			Start with S		
PQ	PR	PS	QP	QR	QS	RP	RQ	RS	SP	SQ	SR
PQRS	PRQS	PSQR	QPRS	QRPS	QSPR	RPQS	RQPS	RSPQ	SPQR	SQPR	SRPQ
PQSR	PRSQ	PSRQ	QPSR	QRSP	QSRP	RPSQ	RQSP	RSQP	SPRQ	SQRP	SRQP

Example 1.3

Carlos is reading a book with 160 pages. How many page numbers in the book contain the digit zero?

The digit zero could be in the Units Place:

$$\underbrace{10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160}_{\text{Zero in the Units place}} = 16 \text{ Numbers}$$

The digit zero could be in the Ten's Place:

$$\underbrace{100, 101, 102, 103, 104, 105, 106, 107, 108, 109}_{\text{Zero in the Tens place}} = 10 \text{ Numbers}$$

But the number 100 gets used in both.

$$Total = 16 + 10 - 1 = 25\ Numbers$$

Example 1.4

What is the number of ways in which we can arrange the letters of:

A. JEE
B. IIIT

We can only choose the location of the letter that is different, giving us:

A. JEE, EJE, EEJ
B. TIII, ITII, IITI, IIIT

Example 1.5

A. Ms. Hamilton's eighth-grade class wants to participate in the annual three-person-team basketball tournament. Lance, Sally, Joy, and Fred are chosen for the team. In how many ways can the three starters be chosen? (AMC 8 2004/4)

B. I need to choose nine friends to invite to my birthday party from my ten best friends. In how many ways can I do this?

C. A bag contains four pieces of paper, each labeled with one of the digits 1, 2, 3 or 4, with no repeats. Three of these pieces are drawn, one at a time without replacement, to construct a three-digit number. What is the probability that the three-digit number is a multiple of 3? (AMC 8 2007/24)

Part A
Choosing three starters is the same as rejecting one person.
Since there are four people in the team, we can do this in four ways.

Part B

$$10\ Ways$$

Part C

$$123 \to 6$$
$$124 \to 7$$
$$134 \to 8$$
$$234 \to 9$$
$$P = \frac{2}{4} = \frac{1}{2}$$

Example 1.6

Find the number of handshakes if in a room, every person shakes hand with everyone else, and there are:

A. Two People
B. Three People
C. Four People
D. Five People

A shaking hands with B is the same as B shaking hands with A
If there are n people in the room, any one person can shake hands with the remaining $(n - 1)$ people

Two People: A, B	**Three People: A, B, C**	Four People: A, B, C, D	Five People: A, B, C, D, E
A: B	A: B, C B: C	A: B, C, D B: C, D C: D	A: B, C, D, E B: C, D, E C: D, E D: E

Example 1.7

How many triangles are created by drawing n lines from n collinear points to a point Z not on the line defined by the n collinear points?

Two Points: A, B	Three Points: A, B, C	Four Points: A, B, C, D	Five Points: A, B, C, D, E
AZB	AZB, AZC BZC	AZB, AZC, AZD BZC, BZD CZD	AZB, AZC, AZD, AZE BZC, BZD, BZE CZD, CZE DZE

Example 1.8

Victor has a quarter, a dime, a nickel and a penny. He wants to buy an item from the shop and pay with exact change. How many different ways can he pay the shopkeeper?

Suppose Victor pays with exactly one coin. He has four options:
25 cents
10 cents
5 cents
1 cent

Suppose Victor pays with exactly two coins. This means he has to make pairs of the coins. He has six options:
25+10=35 cents
25+5=30 cents
25+1=26 cents
10+5=15 cents
10+1=11 cents
5+1=6 cents

Suppose Victor pays with exactly three coins. Choosing three coins is the same as not choosing one coin. He can do it in four ways:
25+10+5=40 Cents
25+10+1=36 Cents
25+5+1= 31 Cents
10+5+1=16 Cents

He can pay with all four coins, which can be done in 1 Way:
25+10+5+1=46 Cents

Suppose you want to check whether you have missed any option. Note that there are four coins. Each coin can be used, or not used. There are two choices for each coin.
Total Choices=2^4=16

Out of which the choice where you use no coins is not applicable, since you are not paying any money to the shopkeeper. Hence, we have 15 valid choices.

B. Number of Power Sets

A well-defined collection of elements is called a set. A subset of a set A is a set that does not contain any element not in the set A:

$$A = \{2, 3, 4\} \text{ is a subset of } B = \{x: x \text{ is a natural number}\}$$

1.9: Power Set

The power set of A is the set of all subsets of A.
Null set means the set has no elements. The symbol ϕ is used to indicate this.

Example 1.10

What is the number of elements of the power set of the following sets: $\{\phi\}, \{1\}, \{1, 2\}, \{1, 2, 3\}$

No. of Elements	{ϕ}	{1}	{1, 2}	{1, 2, 3}
0	{ϕ}	{ϕ}	{ϕ}	{ϕ}
1		{1}	{1}, {2}	{1}, {2}, {3}
2			{1, 2}	{1, 2}, {1, 3}, {2, 3}

3				{1, 2, 3}
Total	1	2	4	8
	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$

C. Casework

Breaking into cases builds on enumeration.
Here, not only must each choice be listed, but also its choices must be counted.

Many aspects need to be kept in mind when deciding the cases (particularly for difficult questions):

- Comprehensiveness: Cases cover all possible events
- Overcounting: None of the cases, or parts of the cases, cover events twice. If they do, then accounting for that.
- Approach: The “right” approach can make work much easier. This comes only from experience and practice.
- Formula: Choosing the right formula to apply to each case.
- Handling edge cases: Cases that need special treatment because of their special situation are handled appropriately.

In situations where we need to count, we may need to break down the possibilities into cases.
The number of times that something happens is then the sum of all cases.

Example 1.11

You have two coins of 1 paise, two coins of 5 paise, two coins of 10 paise and one coin of 20 paise. You purchase an item using exactly two coins, and getting no change back. How many different values can the item have?

	1	5	10	20
1	2	6	11	21
5		10	15	25
10			20	30

Example 1.12

I roll two standard six-sixed dice. What are possible values of the sum?

	1	2	3	4	5	6
1	2	3	4	5	6	7
2						8
3						9
4						10
5						11
6						12

Example 1.13

My clock chimes two times 15 minutes after the hour, four times 30 minutes after the hour and six times 45 minutes after the hour. The clock also chimes eight times on each hour in addition to chiming the number of times equal to the hour. (So, at 2:00 p.m., the clock chimes $8 + 2 = 10$ times.) Starting at 12:05 a.m., how many times does the clock chime in a 24-hour period? (Mathcounts 2007 Warm-Up 14)

We use casework
Case I: Chimes that don't change when the hour changes.

$$24\left(\underbrace{2}_{15\ Minutes}+\underbrace{4}_{30\ Minutes}+\underbrace{6}_{45\ Minutes}+\underbrace{8}_{On\ the\ Hour}\right)=24(20)=480$$

Case II: Chimes that change when the hour changes.

$$2\times\underbrace{(1+2+3+\cdots+12)}_{Times\ Equal\ to\ the\ Hour}=2\left(\frac{12\times 13}{2}\right)=(12\times 13)=156$$

Add the two cases:

$$480+156=636$$

Example 1.14

The number 64 has the property that it is divisible by its units digit. How many whole numbers between 10 and 50 have this property? (AMC 8 2000)

We break into cases based on the units digit. Each case must be considered separately.

$$\underbrace{4}_{11,21,31,41}+\underbrace{4}_{12,22,32,42}+\underbrace{1}_{33}+\underbrace{2}_{24,44}+\underbrace{4}_{15,25,35,45}+\underbrace{1}_{36}+\underbrace{0}_{7:None}+\underbrace{1}_{48}+\underbrace{0}_{9:None}=17$$

Example 1.15

An ascending integer is one in which each digit is greater than any other digit which precedes it (Example: 359). How many ascending integers are there between 200 and 300? (NMTC Primary/Final/2010/8)

All numbers between 200 and 300 will have hundreds digit 2. The lowest value of the ten's digit will be 3.

Ten's Digit	One's Digit	Number
3	4,5,6,7,8,9	6
4	5,6,7,8,9	5
5	6,7,8,9	4
6	7,8,9	3
7	8,9	2
8	9	1
9		0
Total		21

Example 1.16

For how many three-digit whole numbers does the sum of the digits equal 25? (AMC 8 1994/8)

If the three digits have maximum value, then the sum will be

$$9+9+9=27$$

Sum 25 can be accomplished in two ways:

- Reduce two from a single digit, giving {7,9,9}, which can be arranged to get
  - ✓ 997, 979, 799
- Reduce one from two digits, giving {8,8,9}
  - ✓ 889, 898, 988

In all, we get 6 numbers.

Shortcut

We can also visualize the number of ways to arrange $\{7,9,9\}$ by saying that we have three places for the 7, and the remaining numbers give us no choice.
Similarly, there are 3 ways to arrange the 9 in $\{8,8,9\}$

$$Total\ Ways = 3 + 3 = 6$$

Example 1.17
How many different four-digit numbers can be formed be rearranging the four digits in 2004? **(AMC 8 2004/2)**

There are exactly two choices for the first digit:

$$2\ and\ 4$$

Once we choose the first digit, the second non-zero digit can occupy any of the remaining three places, giving us numbers:

$$\{2004,2040,2400\}, \qquad \{4002,4020,4200\}$$

This gives us a total of six numbers.

Example 1.18
What is the number of ways of arranging the letters of the word CAT?

$$\{CAT, CTA, TAC, TCA, ATC, ACT\}$$

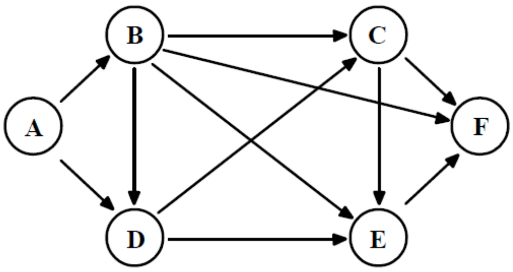
Example 1.19
What is the number of ways of arranging the letters of the word EXAM?

From the previous question, the number of ways of arranging three letters is 6.
I have 4 choices for the first letter: $\{E, X, A, M\}$
With choice of each first letter, I can make six three-letter arrangements of the remaining three letters.

So, total number of choices:

$$\underbrace{6}_{\substack{First\\Letter:E}} + \underbrace{6}_{\substack{First\\Letter:X}} + \underbrace{6}_{\substack{First\\Letter:A}} + \underbrace{6}_{\substack{First\\Letter:M}} = \underbrace{6}_{\substack{Choices\ for\\First\ Letter}} \times \underbrace{4}_{\substack{Arrangments\ for\\each\ letter}} = 24$$

Example 1.20
The figure shows the network connecting cities A, B, C, D, E and F. The arrows indicate the permissible direction of travel. What is the number of distinct paths from A to F. **(CAT 2001/47)**



Via D	Via B
ADEF	ABF
ADCF	ABEF
ADCEF	ABCF
	ABCEF
	ABDCF
	ABDEF
	ABDCEF

3 *Ways*	7 *Ways*

$$Total = 3 + 7 = 10$$

Example 1.21

Assign a numerical code to the letters of the English alphabet so that $A = 1, B = 2, \dots, Z = 26$. What is the number of ways to pick three different letters from the English alphabet such the largest code is greater than the product of the other two codes?

Consider cases for the smallest number.

Case I: Smallest number is 1

$$1 \times 2 = 2 \Rightarrow 3,4,\dots,26 \Rightarrow 26 - 3 + 1 \Rightarrow 24 \ choices$$
$$1 \times 3 = 3 \Rightarrow 4,\dots,26 \Rightarrow 26 - 4 + 1 \Rightarrow 23 \ choices$$
$$\vdots$$
$$1 \times 25 = 25 \Rightarrow 26 \Rightarrow 26 \Rightarrow 1 \ choice$$

The total from this case is:

$$1 + 2 + \cdots + 24 = \frac{24(25)}{2} = 12 \times 25 = 300$$

Case II: Smallest number is 2

$$2 \times 3 = 6 \Rightarrow 7,8,\dots,26 \Rightarrow 26 - 7 + 1 = 20 \ choices$$
$$2 \times 4 = 8 \Rightarrow 9,10,\dots,26 \Rightarrow 26 - 9 + 1 = 18 \ choices$$
$$\vdots$$
$$2 \times 12 = 24 \Rightarrow 25,26 \Rightarrow 2 \ choices$$

The total from this case is:

$$2 + 4 + \cdots + 20 = 2(1 + 2 + \cdots + 10) = 2 \cdot \frac{10(11)}{2} = 110$$

And we can calculate the cases:

$$Smallest\ number\ is\ 3$$
$$Smallest\ number\ is\ 4$$

Similarly, and add all to get the final answer.

1.2 Counting with Number Theory

A. Divisibility Rules

1.22: Divisibility by 2

A number is divisible by 2, its last digit is divisible by 2. That is, it is from:

$$\{0,2,4,6,8\}$$

Example 1.23
A three-digit number has the form $76A$, where A is a single digit. Determine the number of such numbers that are divisible by 2.

$$760,762,764,766,768 \Rightarrow 5\ Numbers$$

1.24: Divisibility by 3
A number is divisible by 3 if the sum of its digits is divisible by 3.

Example 1.25
$x = 5B76$, where B is a single digit. If the number is divisible by 3, what are the possible values of B?

The sum of digits must be a multiple of 3:
$$5 + B + 7 + 6 = 18 + B$$

$$B \in \{0,3,6,9\}$$

Example 1.26
$x = 5BA6$, where B and A are each single digits. If the number is divisible by 3, how many possibles values can x take?

The sum of the digits:
$$5 + B + A + 6 = 11 + A + B$$

We consider cases based on the sum of digits:
$$11 + A + B = 12 \Rightarrow A + B = 1 \Rightarrow (0,1),(1,0) \Rightarrow 2\ Solutions$$
$$11 + A + B = 15 \Rightarrow A + B = 4 \Rightarrow (0,4),(1,3),(2,2),(3,1),(4,0) \Rightarrow 5\ Solutions$$
$$11 + A + B = 18 \Rightarrow A + B = 7 \Rightarrow (0,7),(1,6),\dots,(7,0) \Rightarrow 8\ Solutions$$
$$11 + A + B = 21 \Rightarrow A + B = 10 \Rightarrow (1,9),(2,8),\dots,(9,1) \Rightarrow 9\ Solutions$$
$$11 + A + B = 24 \Rightarrow A + B = 13 \Rightarrow (4,9),(5,8),\dots,(9,4) \Rightarrow 6\ Solutions$$
$$11 + A + B = 27 \Rightarrow A + B = 16 \Rightarrow (7,9),(8,8),(9,7) \Rightarrow 3\ Solutions$$
$$11 + A + B = 30 \Rightarrow A + B = 19 \Rightarrow 0\ Solutions$$

Total number of solutions:
$$2 + 5 + 8 + 9 + 6 + 3 = 33\ Solutions$$

Example 1.27
$x = B8A3$ is a four-digit number, where B and A are each distinct digits. If the number is divisible by 3, how many possibles values can x take?

The sum of the digits:
$$B + 8 + A + 3 = 11 + A + B$$

We consider cases based on the sum of digits:
$$11 + A + B = 12 \Rightarrow A + B = 1 \Rightarrow (A,B) = (0,1) \Rightarrow 1\ Solutions$$
$$11 + A + B = 15 \Rightarrow A + B = 4 \Rightarrow (A,B) = (0,4),(1,3),(3,1) \Rightarrow 3\ Solutions$$
$$11 + A + B = 18 \Rightarrow A + B = 7 \Rightarrow (A,B) = (0,7),(1,6),(2,5),(3,4),(4,3),(5,2),(6,1) \Rightarrow 7\ Solutions$$

$$11 + A + B = 21 \Rightarrow A + B = 10 \Rightarrow (1,9),(2,8),\dots,(9,1) \Rightarrow 8\ Solutions\ (without\ (5,5)$$
$$11 + A + B = 24 \Rightarrow A + B = 13 \Rightarrow (4,9),(5,8),\dots,(9,4) \Rightarrow 6\ Solutions$$
$$11 + A + B = 27 \Rightarrow A + B = 16 \Rightarrow (7,9),(9,7) \Rightarrow 2\ Solutions$$

Total number of solutions:

$$1 + 3 + 7 + 8 + 6 + 2 = 27\ Solutions$$

1.28: Divisibility by 11

A number is divisible by 11 if

$$Odd\ Sum - Even\ Sum = Multiple\ of\ 11$$

Where

$$Odd\ sum\ is\ sum\ of\ digits\ in\ odd\ positions$$
$$Even\ sum\ is\ sum\ of\ digits\ in\ even\ positions$$

Example 1.29

A five-digit number is given by $x = 3A7B4$. If the number is divisible by 11, find the number of values that x can take.

Consider cases for the difference:
Case I:

$$3 + 7 + 4 - A - B = 0$$
$$A + B = 14 \Rightarrow (5,9),(6,8),\dots,(9,5) \Rightarrow 9 - 5 + 1 = 5\ solutions$$

Case II:

$$3 + 7 + 4 - A - B = 11$$
$$A + B = 3 \Rightarrow (0,3),(1,2),(2,1),(3,0) \Rightarrow 4\ Solutions$$

$$5 + 4 = 9$$

Example 1.30

A five-digit number is given by $x = 3AB74$. If the number is divisible by 11, find the number of values that x can take.

$$3 + B + 4 - A - 7 = 0$$
$$B - A = 0$$
$$B = A \Rightarrow 10\ Solutions$$

$$3 + B + 4 - A - 7 = 0$$
$$B - A = 11$$
$$Zero\ Solutions$$

$$10\ Solutions$$

1.31: Divisibility by 4

A number is divisible by 4 if and only if the last two digits are divisible by 4.

1.32: Divisibility by Composite Numbers

If x has prime factorization

$$x = p^a q^b, \qquad p, q\ are\ prime$$

Then a number is divisible by x if and only if

$$p^a \textit{ divides the number}$$
$$q^b \textit{ divides the number}$$

In other words, check for each prime factor separately.

- To check divisibility by 6, check for 2 and 3 separately.
- To check divisibility by 18, check for 2 and 9 separately.

Example 1.33

A five-digit number $x = 347AB$ is divisible by 12. How many possible values can x take?

$$3 + 4 + 7 + A + B = 14 + A + B$$

$$14 + A + B = 15 \Rightarrow A + B = 1 \Rightarrow (A, B) = (0,1)(1,0) \Rightarrow 0\ Solutions$$
$$14 + A + B = 18 \Rightarrow A + B = 4 \Rightarrow (A, B) = (0,4), (1,3), (2,2)(3,1), (4,0) \Rightarrow 2\ Solutions$$
$$14 + A + B = 21 \Rightarrow A + B = 7 \Rightarrow (A, B) = (0,7), (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) \Rightarrow 2\ Solutions$$
$$14 + A + B = 24 \Rightarrow A + B = 10 \Rightarrow (1,9), (2,8), (3,7), (4,6)(5,5), (6,4), (7,3)(8,2), (9,1) \Rightarrow 2\ Solutions$$
$$14 + A + B = 27 \Rightarrow A + B = 13 \Rightarrow (4,9), (5,8), (6,7), (7,6), (8,5)(9,4) \Rightarrow 1\ Solutions$$
$$14 + A + B = 30 \Rightarrow A + B = 16 \Rightarrow (7,9), (8,8), (9,7) \Rightarrow 1\ Solutions$$

$$2 + 2 + 2 + 1 + 1 = 8\ Solutions$$

1.3 Complementary Counting

A. Complementary Counting

In complementary counting, we count the items that we don't and subtract those from the total.

1.34: Number of Prime Numbers

There are 25 primes less than 100.

$$n\{2,3,5,7, \ldots, 97\} = 100$$

Example 1.35

What is the number of positive composite numbers less than 100?

There are 25 prime numbers less than 100. Also, the number 1 is neither prime nor composite.

$$99 - \underbrace{25}_{\substack{\textit{Prime}\\ \textit{Numbers}}} - \underbrace{1}_{\substack{\textit{Neither Prime}\\ \textit{and Composite}}}$$

Example 1.36

How many positive integers less than 10000 do not have exactly three factors?

Consider the numbers which have three factors:

$$2^2 = 4\ has\ factors\ \{1,2,4\}$$
$$3^2 = 9\ has\ factors\ \{1,3,9\}$$
$$5^2 = 25\ has\ factors\ \{1,5,25\}$$

In general:

$$p^2 has\ factors\ \{1, p, p^2\}$$

Hence, we want all numbers that are not of the form p^2, where p is a prime.

$$10000 = 100^2$$

There are 25 primes less than 100.

Hence, the numbers that meet this condition are:

$$9999 - \underbrace{25}_{\substack{\text{Numbers with}\\ \text{three factors}}} = 9975$$

Example 1.37

How many positive integers less than or equal to 100 have a prime factor that is greater than 4? **(MathCounts 2004 Workout 9)**

A number that only has prime factors greater than 4 must not have prime factors of 2 or 3. We classify on the basis of powers of 3:

Powers of 3		Numbers
3^0	$2^0, 2^1, \dots, 2^6$	7
3^1	$3 \times 2^0, 3 \times 2^1, \dots, 3 \times 2^5$	6
3^2	$3^2 \times 2^0, 3^2 \times 2^1, \dots, 3^2 \times 2^3$	4
3^3	$3^2 \times 2^0, 3^2 \times 2^1$	2
3^4	3^4	1
	Total	20

The numbers we want are:

$$100 - 20 = 80$$

Instead of listing out in exponent form, we can also list out in number form:

									No.
	1	2	4	8	16	32	64		7
	3	6	12	24	48	96			6
	9	18	36	72					4
	27	54							2
	81								1

1.4 Reframing the Question / Clever Logic

A. Reframing the Question

There are several ways to approach a question. The approach to a question can make it easier or more difficult. In the questions below, some strategies work for all questions, while some are easy to apply for specific types of questions only. Choosing the right strategy is important in arriving at the answer quickly and correctly.

Rearranging

A standard, powerful, mathematical technique used to prove many results (including important ones) is to:

- Show that a complicated expression is equivalent to another, simpler expression
- Calculate the simpler expression

Example 1.38
Town A and B are 60 km apart, and are connected by a straight-line track. Train X starts from town A and goes towards town B. At the same time, train Y starts from town B, and goes toward town A. Also, at the same time, a crow sitting at the top of train X starts flying towards train Y. As soon as it touches train Y, it turns back and flies towards train X. It continues this process till the trains meet. (Make simplifying assumptions: the distance between the two tracks is negligible, the turning time required for the bird is negligible, etc). Assume that each train has a speed of 60 $\frac{km}{hr}$, and the bird has a speed of 30 $\frac{km}{hr}$. Find the total distance travelled by the bird.

Long Method

$$D = 60\ km \Rightarrow Train = 40\ km, Bird = 20\ km$$
$$D = 20\ km \Rightarrow Train = \frac{40}{3}\ km, Bird = \frac{20}{3}\ km$$

$$20 + \frac{20}{3} + \frac{20}{3^2} + \frac{20}{3^3} + \cdots$$

Short Method
$$Time_{Bird} = Time_{Train} = \frac{D}{S} = \frac{60}{60+60} = \frac{60}{120} = \frac{1}{2} hr \Rightarrow Distance_{Bird} = 15\ km$$

Pure Logic Method
Trains meet in the middle since they have the same speed. Hence, each train covers 30 km.

Speed of Bird is half the speed of the train, but it travels for same time as the trains.
$$\therefore Distance_{Bird} = \frac{30}{2} = 15\ km$$

B. Single Elimination Tournaments
Unless otherwise specified:
- Matches are played between two competitors or teams

Single Elimination Format
Matches are played between two competitors. Winners move on to play winners of other matches.
Round Robin Format
All competitors play all other competitors once.

Example 1.39
Ms. Hamilton's eighth-grade class wants to participate in the annual three-person-team basketball tournament. The losing team of each game is eliminated from the tournament. If sixteen teams compete, how many games will be played to determine the winner? (AMC 8 2004)

Enumeration:
$$\underbrace{8}_{Pre-Quarter-Final} + \underbrace{4}_{Quarter-Final} + \underbrace{2}_{Semi-Final} + \underbrace{1}_{Final} = 15$$
Logic
Every team which leaves the tournament must lose a game. 15 teams will lose, and leave the game. One team will have a string of wins, and become the winner.
$$Total\ Games = Total\ No.of\ Losers = 15$$

Example 1.40

Repeat the previous example, except that instead of 16 teams, suppose that there are 12 teams.

Enumeration:

The enumeration method requires a little more work:

$$\underbrace{6}_{Pre-Quarter-Final} + \underbrace{3}_{Quarter-Final}$$

Now we have three teams, which is an odd number of teams, so one team will have to sit out, and get a bye. We will play:

$$\underbrace{1}_{Semi-Final}$$

This will give one winner from the 3rd round, and one team which got a bye, making a total of two teams. These two teams will play one game to decide the winner:

$$\underbrace{1}_{Final}$$

$$Total = 6 + 3 + 1 + 1 = 11$$

Logic

The logic doesn't change and is easy to apply:

$$Total\ Games = Total\ No.of\ Losers = 12 - 1 = 11$$

Example 1.41

Two tournaments A and B are played in a single elimination format, with no draws. How many matches will there be if there are:

A. 64 competitors?
B. 37 competitors?

63, 36

Example 1.42

116 people participated in a singles tennis tournament of knock out format. The players are paired up in the first round, the winners of the first round are paired up in second round, and so on till the final is played between two players. If after any round, there is odd number of players, one player is given a bye, i.e. he skips that round and plays the next round with the winners. Find the total number of matches played in the tournament. (CAT 1990/72)

Shortcut

$$116 - 1 = 115$$

Long Method (Not Recommended)

$$116\ participants \Rightarrow 58\ Matches, 58\ Winners$$
$$58\ participants \Rightarrow 29\ Matches, 29\ Winners$$
$$29\ participants \Rightarrow 14\ Matches, 14\ Winners + 1\ Bye = 15$$
$$15\ participants \Rightarrow 7\ Matches, 7\ Winners + 1\ Bye = 8$$
$$8\ participants \Rightarrow 4\ Matches, 4\ Winners$$
$$4\ participants \Rightarrow 2\ Matches, 2\ Winners$$
$$2\ participants \Rightarrow 1\ Matches, 1\ Winners$$

Total

$$= 58 + 29 + 14 + 7 + 4 + 2 + 1 = 115$$

Example 1.43
CAT 2008 – Data Sufficiency

C. Round Robin Tournaments
All competitors play all other competitors once.

D. Self-Referential Counting

Example 1.44
There are _____ vowels in this short sentence.

A. Twelve
B. Thirteen
C. Fourteen
D. Fifteen
E. Sixteen

$$Twelve: 12 + 2 = 14$$
$$Thirteen: 12 + 3 = 15$$
$$Fourteen: 12 + 4 = 16$$
$$Fifteen: 12 + 3 = 15$$
$$Sixteen: 12 + 3 = 15$$

$$Option\ D$$

2. Counting Rules

2.1 Addition and Multiplication Rules

A. Addition Rule

The addition and multiplication rules are key building blocks in counting problems.

2.1: Addition Rule

Suppose $\boldsymbol{A}$ can happen in $\boldsymbol{n}$ ways. And $\boldsymbol{B}$ can happen in $\boldsymbol{m}$ ways. Then, the number of ways in which either $\boldsymbol{A}$ or $\boldsymbol{B}$ can happen is:

$$\boldsymbol{n+m}$$

Exactly *one* of the events happens.

The recommended keyword to look for is OR, but this is not necessary, since questions can be phrased in a number of ways.

Example 2.2

What is the number of ways of picking a single element from any one of the sets?

$$A = \{a, b, c, d, \dots, N\}, \qquad B = \{1,2,3, \dots, M\}$$

Note: N is a variable that represents the N^{th} letter in the set.

$$\underbrace{\{a, b, c, d, \dots, N\}}_{\substack{\text{Number of Ways of picking} \\ \text{one element from A=N}}} + \underbrace{\{1,2,3, \dots, M\}}_{\substack{\text{Number of ways of picking} \\ \text{one element from B=M}}} \Rightarrow N + M$$

Example 2.3

At Euclid Middle School the mathematics teachers are Mrs. Germain, Mr. Newton, and Mrs. Young. There are 11 students in Mrs. Germain's class, 8 students in Mr. Newton's class, and 9 students in Mrs. Young's class taking the AMC 8 this year. How many mathematics students at Euclid Middle School are taking the contest? (AMC 8 2010/1)

This is a direct application of the addition rule:

$$\underbrace{11}_{\substack{Germain's \\ Class}} + \underbrace{8}_{\substack{Newton's \\ Class}} + \underbrace{9}_{\substack{Young's \\ Class}} = 28$$

Example 2.4

How many choices do I have in the following situations?

A. Choose a pair of shoes out of four pairs.
B. Travel from Andheri to Churchgate via one of Metro, Bus, or Car.
C. Choose a captain from a team of eleven people.
D. Go from City A to City B, using one of five roads that go from one city to the other.
E. Choose a fruit from Apple, Watermelon, and Chickoo.
F. Wear either a cap or a bandana, if I have 10 distinct caps, and 5 distinct bandanas
G. Enter a room that has four doors, and five windows, if I can enter via either a door or a window?
H. Assign a single alphanumeric character identifying a book. (An alphanumeric character can be from the English alphabet, or a digit from the decimal system).

Part A

1 out of the 4 pairs = 4 choices

Part B

Metro / Bus / Car = 3 choices

Part C

Choosing a captain from a team of 11 people = 11 Choices

Part D

One of 5 Roads = 5 Choices

Part E

Choosing one of three fruits = 3 choices

Part F

Cap = 10 Choices
Bandana = 5 Choices
Total = 10 + 5 = 15

Part G

Door = 4 choices
Window = 5 choices
Door or Window = 9 choices

Part H

Number = 10 choices
Letter = 26 choices
Number or Letter = 36 choices

B. Multiplication Rule

2.5: Multiplication Rule

Suppose $\boldsymbol{A}$ can happen in $\boldsymbol{m}$ ways. And $\boldsymbol{B}$ can happen in $\boldsymbol{n}$ ways. The number of ways in which $\boldsymbol{A}$ and $\boldsymbol{B}$ can happen is:

$$\boldsymbol{m} \times \boldsymbol{n}$$

Both events A and B are going to happen. This contrasts with the addition rule, where *exactly one* of A and B is going to happen.
Hence, any option for event A can be mixed with any option for event B.

Example 2.6

What is the number of ways of picking a single element from A, and a single element from B?

$$A = \{a, b, c, d, \dots, N\}, \qquad B = \{1,2,3, \dots, M\}$$

Note: N is a variable that represents the N^{th} letter in the set.

$$N \times M$$

Example 2.7

How many distinct outfits consisting of a shirt and a pair of jeans can I make from four shirts and three pairs of jeans?

$$\begin{gathered} First\ Shirt: 3\ Choices\ for\ Jeans \\ Second\ Shirt: 3\ Choices\ for\ Jeans \\ Third\ Shirt: 3\ Choices\ for\ Jeans \end{gathered}$$

$$\textit{Fourth Shirt}: 3\ \textit{Choices for Jeans}$$

$$\textit{Total Choices} = 3 + 3 + 3 + 3 = 3 \times 4 = 12$$

C. Deciding between Addition and Multiplication

Example 2.8

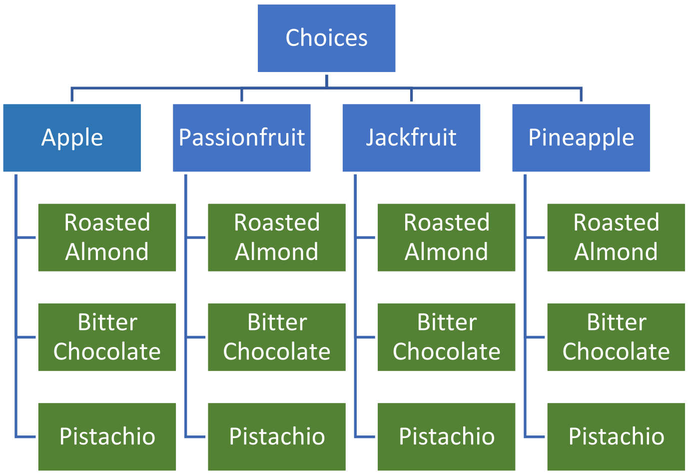
Answer each part separately

Shivansh has three flavors of ice-cream: Roasted Almond, Bitter Chocolate, and Pistachio. He has four fruits: apple, passionfruit, jackfruit, and pineapple. How many choices does he have if he is going to eat:

A. One ice-cream and one fruit
B. Either an ice cream or a fruit, but not both

Part A

Shivansh can first choose his fruit (4 choices), and then choose his ice-cream (3 choices).
This corresponds to choosing one of the blue options on the chart alongside, and then choosing one of the green options.



In all, there are:

$$\underbrace{3}_{\substack{\textit{Ways of picking}\\ \textit{one ice-cream}}} \times \underbrace{4}_{\substack{\textit{Ways of picking}\\ \textit{a fruit}=4}} \Rightarrow 3 \times 4 = 12$$

Part B

$$\underbrace{\{\text{Roasted Almond, Bitter Chocolate, Pistachio}\}}_{\substack{\textit{Number of Ways of picking}\\ \textit{one ice-cream}=3}} + \underbrace{\{\text{Apple, Passionfruit, Jackfruit, Pineapple}\}}_{\substack{\textit{Number of ways of picking}\\ \textit{a fruit}=4}} \Rightarrow 3 + 4 = 7$$

Example 2.9

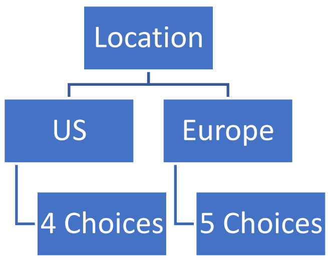
If Vedika visits the US, she wants to go to one of four cities: Seattle, New York, Philadelphia, or Denver. If she decides to visit Europe, she wants to go one of five cities: Rome, Milan, London, Istanbul, or Madrid. How many choices does she have if she is going to visit:

A. A single city
B. A city in the USA, and a city in Europe, and the order of visiting the cities does not matter.

Part A

$$\underbrace{\{\text{Seattle, New York, Philadelphia or Denver}\}}_{\substack{\textit{Number of Ways of picking}\\ \textit{one city from USA}=4}}$$

$$+ \underbrace{\{\text{Rome, Milan, London, Istanbul or Madrid}\}}_{\substack{\textit{Number of ways of picking}\\ \textit{a city from Europe}=5}} \Rightarrow 4 + 5 = 9$$



Part B

		Europe

		Rome	Milan	London	Istanbul	Madrid
USA	Seattle	$(Seattle, Rome)$	$(Seattle, Milan)$			
	New York					
	Philadelphia					
	Denver					

$$\underbrace{\{\text{Seattle, New York, Philadelphia or Denver}\}}_{\substack{\text{Number of Ways of picking}\\ \text{one city from USA}=4}} \times \underbrace{\{\text{Rome, Milan, London, Istanbul or Madrid}\}}_{\substack{\text{Number of ways of picking}\\ \text{a city from Europe}=5}} \Rightarrow 4 \times 5 = 20$$

D. "Nothing" as an option

Not doing anything is sometimes also a valid option. This needs to be considered and added to the number of valid choices.

2.10: Nothing as an Option

If I must choose from 1 of n options, and I also the option to not choose, it is a hidden option, and hence the number of choices is

$$n + 1$$

Example 2.11

I need to choose my shoes and my socks for my dinner outing. I have shoes in three colours: black, brown, and white. I have socks in four colours: gray, blue, yellow and red. In how many ways can I choose if:

A. I always wear shoes and socks.
B. I always wear shoes, but also have the option of not wearing socks.

Part A

I need to choose

- Shoes
- Socks

And I can combine them in whichever way I wish. The table shows the options.

		Sock Color			
		Gray	Blue	Yellow	Red
Shoe Color	Black				
	Brown		Brown Shoe, Blue Sock		
	White				

Each cell in the table corresponds to a choice of shoe and a choice of socks. For example, the cell in the second row and second column corresponds to a brown shoe paired with a blue sock.

The total number of choices is:

$$\underbrace{3}_{\substack{No.of\\ Shoes}} \times \underbrace{4}_{\substack{No.of\\ Socks}} = 12$$

Part B

If I also have the option of not wearing socks, my number of options for socks increases by one, making the final answer:

$$\underbrace{3}_{\substack{No.of\\ Shoes}} \times \underbrace{5}_{\substack{No.of\ options\\ for\ Socks}} = 15$$

		Sock Color				
		Gray	Blue	Yellow	Red	No Socks
Shoe Color	Black					
	Brown		Brown Shoe, Blue Sock			
	White					

Example 2.12

I need to choose my jacket, and my tie. I have a red jacket, a blue jacket, and a green jacket. I have a polka-dotted tie, and a striped tie. I also have the option of not wearing one or more of the above two items. In how many ways can I choose my outfit?

$$Choices\ for\ Jacket: Red, Blue, Green, No\ Jacket \Rightarrow 4\ Choices$$
$$Choices\ for\ Tie: Polka\ Dotted, Striped, No\ Tie \Rightarrow 3\ Choices$$

$$4 \times 3 = 12$$

E. Routes between Cities

(Important) Example 2.13

Mumbai is connected to Alibaugh via road, ferry, and train. A person will travel using exactly one of these methods for a journey from one city to another. The return journey can be via a different method. In how many ways can a person:

A. Go to Alibaugh from Mumbai
B. Go to Mumbai from Alibaugh
C. Go to Alibaugh from Mumbai and come back given that:
   a. There are no restrictions
   b. You come back via the same route that you went.
   c. You come back via a different way from the one that you went by.

Part A and B

$$\underbrace{1}_{Road} + \underbrace{1}_{Ferry} + \underbrace{1}_{Train} = 3$$

	Road	Ferry	Train
Road			
Ferry			
Train			

Part C

If there are no restrictions:

$$a: \underbrace{3}_{Going} \times \underbrace{3}_{Coming\ Back} = 9$$

If you come back via the same route that you went.

$$b: \underbrace{3}_{Going} \times \underbrace{1}_{Coming\ Back} = 3$$

If you come back via a different way from the one that you went by.

$$c: \underbrace{3}_{Going} \times \underbrace{2}_{Coming\ Back} = 6$$

Example 2.14

Answer the same questions as above, except that now there are three roads, two ferries and five trains. (Assume that the roads, ferries, and trains are all distinct, and if you come back a different way from the one that you went by, you can still repeat the same type of travel. For example, you can go by Road 1,but come back by Road 2.)

Parts A and B:

$$\underbrace{3}_{Road} + \underbrace{2}_{Ferry} + \underbrace{5}_{Train} = 10$$

Part C:

$$I: \underbrace{3+2+5}_{Going} \times \underbrace{3+2+5}_{Coming\ Back} = 10 \times 10 = 100$$

$$II: \underbrace{3+2+5}_{Going} \times \underbrace{1}_{Coming\ Back} = 10 \times 1 = 10$$

If you can repeat the type of travel, then you get:

$$III: \underbrace{3+2+5}_{Going} \times \underbrace{3+2+5-1}_{Coming\ Back} = 10 \times 9 = 90$$

Example 2.15

There are three highways that go from Denver to Philadelphia, and six flights that go from Philadelphia to Washington and *vice versa*. Due to a snowstorm, the only route between Denver to Washington is via Philadelphia.

A. What is the number of ways to go from Denver to Washington?
B. What is the number of ways to go from Denver to Washington and then come back if:
   a. You come back the same way that you went
   b. You come back via any way that you want
   c. You come back, but not via the exact same way that you took to reach Washington. (If you went by Highway 1, followed by Flight 2, coming back via anything except Flight 2, followed by Highway 1 is acceptable.)

Part A:

$$\underbrace{3}_{\substack{Ways\ to\ go\ from\\ Denver\ to\ Philadelphia}} \times \underbrace{6}_{\substack{Ways\ to\ go\ from\\ Philadelphia\ to\ Washingtona}} \Rightarrow 3 \times 6 = 18$$

Part B

Sub-Part a

$$\underbrace{18}_{\substack{Ways\ to\ go\ from\\ Denver\ to\ Washington}} \times \underbrace{1}_{\substack{Ways\ to\ go\ from\\ Washington\ to\ Denver}} = 18$$

Sub-Part b

$$\underbrace{18}_{\substack{Ways\ to\ go\ from\\ Denver\ to\ Washington}} \times \underbrace{18}_{\substack{Ways\ to\ go\ from\\ Washington\ to\ Denver}} \Rightarrow 18 \times 18 = 18^2 = 324$$

Sub-Part c

$$\underbrace{18}_{\substack{Ways\ to\ go\ from\\ Denver\ to\ Washington}} \times \underbrace{17}_{\substack{Ways\ to\ go\ from\\ Washington\ to\ Denver}} = 306$$

F. Deciding between addition and multiplication

Example 2.16: Travelling by road or by flight

Han wants to reach Toronto from Ontario. He can take any one of five roads to reach Toronto. Or, he can pick one of three flights to reach Toronto. What is the number of ways in which he can reach Toronto?

$$\underbrace{Roads}_{5\ Choices} + \underbrace{Flights}_{3\ Choices} \Rightarrow 5 + 3 = 8$$

Example 2.17: Exam Scenario

Answer each part separately

An exam with seven subjective (free response) and three objective (multiple choice) questions can be attempted in one of four different ways. The details are, by answering exactly:

A. one subjective question
B. one objective question
C. one subjective and one objective question
D. one subjective or one objective question

What is the number of ways in which the paper can be attempted under the different options?

Option A

There are seven subjective questions, of which exactly one must be attempted. Hence, I have

$$7\ Questions = 7\ Ways$$

Option B

There are three objective questions, of which exactly one must be attempted. Hence, I have

$$3\ Questions = 3\ Ways$$

Option C

We must combine one subjective question with one objective question. By the multiplication principle, this can be done in

$$\underbrace{7}_{\substack{Subjective\\Questions}} \times \underbrace{3}_{\substack{Objective\\Questions}} = 21\ Ways$$

Option D

We must answer either a subjective question or an objective question. By the addition principle, this can be done in

$$\underbrace{7}_{\substack{Subjective\\Questions}} + \underbrace{3}_{\substack{Objective\\Questions}} = 10\ Ways$$

2.2 Using the Rules

A. Extending the Multiplication Principle

The multiplication principle can be extended from two making a choice that has two parts to many parts. In addition, it is important to recognize when not doing something is also a choice.

2.18: Extending the Multiplication Principle

If I can do event A in a ways, event B in b ways, and event C in c ways, and event D in d ways, then I can do all of them together in

$$abcd\ ways$$

Example 2.19

A. I am going to a party, and I need to choose my shoes, socks, jacket, and my tie. I have an ochre jacket, a maroon jacket, and a green jacket. I have a polka dotted tie, and a striped tie. I have shoes in three colours: black, brown, and white. I have socks in four colours: gray, blue, yellow and red. In how many ways can I choose my outfit if I always wear shoes but have the option of not wearing one or more of my jackets, my ties, and my socks.
B. I have two jackets, four shirts, four pairs of trousers, two pairs of shoes, and only one tie. How many outfits can I make consisting of a shirt, a pair of trousers, a pair of shoes, an (optional) tie, and an (optional) jacket?
C. Items at a restaurant are available in any of three bases (unless specified otherwise): $Vegetarian$, $Chicken$ and $Seafood$. In main course, I can order $Sichuan$ or $Manchurian$, while for soup, I can

choose between four options: *Hot & Sour*, *Winter Melon*, *Wonton* and *Slow Cooked*. In how many ways can I order a soup and a main course, given that I can (optionally) order dry noodles as a side dish (dry noodles are available in only one variety – no choice of base)

Part A

$$\underbrace{3}_{Shoes} \times \underbrace{5}_{Socks} \times \underbrace{4}_{Jackets} \times \underbrace{3}_{Ties} = 180$$

Part B

$$\underbrace{3}_{Jacket} \times \underbrace{4}_{Shirt} \times \underbrace{4}_{Trousers} \times \underbrace{2}_{Shoes} \times \underbrace{2}_{Ties} = 192$$

Part C

$$\underbrace{\text{Soup}}_{4\ Varieties \times 3\ Bases = 12} \times \underbrace{Main\ Course}_{2\ Varieties \times 3\ Bases = 6} \times \underbrace{Side\ Dish}_{\substack{2\ Choices \\ Order\ or\ don't\ order}} = 12 \times 6 \times 2 = 144$$

Example 2.20: Double Decker Sandwich

A sandwich has three layers of bread with two fillings (Bread / Filling / Bread / Filling / Bread). The types of fillings and types of butter are mentioned in the table. The location of the filling matters, and bread in touch with a filling is buttered.

Fillings		Butter
Veg	Non-veg	
Aloo Salad Peas	Chicken Fish	Plain Garlic

A. For each sandwich, how many sides will I butter?
B. What is the number of distinct sandwiches that can be made if one filling must be veg, and the other filling must be non-veg, and butter in a single sandwich is the same for all sides? (Note: Two sandwiches which can be flipped over to make them the same are still considered distinct).
C. How many sides will I butter to make each distinct sandwich once if:
   a) Fillings can be repeated, and butter in a single sandwich is the same for all sides.
   b) Fillings cannot be repeated, but butter in a single sandwich does not have to be the same for all sides, but any one side uses a single type of butter.

Part A

Three layers of bread will have

$$6\ Sides$$

Out of the 6 sides, all except the topmost and the bottommost will be buttered. Hence, the number of sides to be buttered

$$= Total\ Sides - Sides\ Not\ Buttered = 6 - 2 = 4$$

Top Layer	Top Side **Bottom Side**
	Filling
Middle Layer	**Top Side** **Bottom Side**
	Filling
Bottom Layer	**Top Side** Bottom Side

Part B

Number of Distinct Sandwiches:

$$\underbrace{3}_{\substack{Veg \\ Filling}} \times \underbrace{2}_{\substack{Non-Veg \\ Filling}} \times \underbrace{2}_{Butter} = 12$$

Location of Filling:

We also have a choice of deciding whether the topmost filling is veg, or the bottommost filling is veg, giving us two choices:

Bread	*Choice I*	*Choice II*
Filling	Veg	Non-Veg
Bread		
Filling	Non-Veg	Veg
Bread		

Hence the final answer is:

$$12 \times 2 = 24$$

Part C-I: Fillings can be repeated

Number of Distinct Sandwiches

$$\underbrace{5}_{\substack{\text{First}\\\text{Filling}}} \times \underbrace{5}_{\substack{\text{Second}\\\text{Filling}}} \times \underbrace{2}_{\text{Butter}} = 50$$

Total Sides Buttered:

$$Total\ Sides\ Buttered = 50 \times 4 = 200$$

Part C-II: Fillings cannot be repeated

Count the number of options when buttering the sides:

$$\underbrace{2}_{\text{Side 1}} \times \underbrace{2}_{\text{Side 1}} \times \underbrace{2}_{\text{Side 1}} \times \underbrace{2}_{\text{Side 1}} = 16$$

Number of Distinct Sandwiches

$$\underbrace{5}_{\substack{\text{First}\\\text{Filling}}} \times \underbrace{4}_{\substack{\text{Second}\\\text{Filling}}} \times \underbrace{16}_{\text{Butter}} = 320$$

Total Sides Buttered

$$Total\ Sides\ Buttered = 320 \times 4 = 1{,}280$$

Example 2.21: Filling Posts

A MUN committee has 10 representatives from Asia, 8 from Europe, and 7 from Africa. If a representative is eligible for a maximum of one post, in how many ways can a $\underbrace{Secretary\ General}_{SG}$, a $\underbrace{Deputy\ Secretary\ General}_{DSG}$ and a $\underbrace{President}_{P}$ be chosen for the committee if:

A. There are no further restrictions
B. The Secretary General must be from Asia, the Deputy Secretary General must from Europe and the President must be from Africa
C. They must each be from different continents
D. They must not all be from the same continent
E. Exactly two of them must be from the same continent

Total Number of Representatives

$$\underbrace{10}_{Asia} + \underbrace{8}_{Europe} + \underbrace{7}_{Africa} = 25$$

Part A

$$\underbrace{25}_{SG} \times \underbrace{24}_{DSG} \times \underbrace{23}_{P} = 13{,}800$$

Part B

$$\underbrace{10}_{SG} \times \underbrace{8}_{DSG} \times \underbrace{7}_{P} = 560$$

Part C

We first need to allocate the continent to the post. This can be done in:

$$\underbrace{3\ Continents}_{SG} \times \underbrace{2\ continents}_{DSG} \times \underbrace{1\ continent}_{P} = 6$$

Then, whichever continent we have allocated to the post, we need to select the representative who will fill that post:

$$\underbrace{10}_{Asia} \times \underbrace{8}_{Europe} \times \underbrace{7}_{Africa} = 560$$

Finally, the two choices above can be combined to get:

$$560 \times 6 = 3{,}360$$

Part D

Counting all the different cases where they are not all from the same continent will be a lot of casework. Instead, we use complementary counting.

All posts from the same continent

$$All\ from\ Asia: 10 \times 9 \times 8 = 720$$
$$All\ from\ Europe: 8 \times 7 \times 6 = 336$$
$$All\ from\ Africa: 7 \times 6 \times 5 = 210$$
$$Total: 720 + 336 + 210 = 1{,}266$$

$$\underbrace{13{,}800}_{Total\ Ways} - \underbrace{1{,}266}_{\substack{All\ from\ same\\ Continent}} = 12{,}534$$

Part E: Exactly two must be from the same continent

$$\underbrace{13{,}800}_{Total\ Ways} - \underbrace{1{,}266}_{\substack{All\ from\ same\\ Continent}} - \underbrace{3360}_{\substack{All\ three\\ different\ continents}} = 9174$$

$$3 + 2x < x\ AND\ 2 + x > 8$$

B. Back Calculations

Example 2.22

A restaurant offers three desserts, and exactly twice as many appetizers as main courses. A dinner consists of an appetizer, a main course, and a dessert. What is the least number of main courses that a restaurant should offer so that a customer could have a different dinner each night in the year 2003? **(AMC 10B 2003/16)**

2003 is a non-leap year. Hence, the number of Days in 2003 is

$$365$$

Let the number of main courses be x. Then, the number of different dinners is:

$$= \underbrace{3}_{Desserts} \times \underbrace{2x}_{Appetizers} \times \underbrace{x}_{\substack{Main\\ Courses}} = 6x^2$$

But the number of dinners has to be greater than or equal to 365.

Method I: Solve using Guess and Check

If you don't know inequalities, the values are small enough for you to get via trial and error. 7 is too small, and hence 8 is the smallest number that satisfies the condition.

$$6 \times 7^2 = 294 < 365, \qquad 6 \times 8^2 = 384 > 365$$

Method II: Solve using Inequalities

$$6x^2 \ge 365 \Rightarrow x^2 \ge \frac{365}{6} = 60\frac{5}{6} \Rightarrow 7^2 = 49 < 60\frac{5}{6} < 64 = 8^2 \Rightarrow Smallest\ x\ is\ 8$$

C. Creating Positions

Many times, when we want to count the number of ways, something can be arranged, we look at the positions. Sometimes, it may be useful to create positions to make our counting easy.
The following examples show this.

Example 2.23: Wearing Two Rings

I am going to a party wearing a total of two rings on the five appendages of my right hand (four fingers, and one thumb, with no restrictions). I can wear more than one ring on an appendage. Also, the order of the rings matters. In how many ways can I wear two rings?

Method I: Casework

Case I: Two Rings on the same finger

$$\underbrace{\mathbf{5}}_{\substack{\textit{No.of}\\ \textit{Appendages}}} \times \underbrace{\mathbf{2}}_{\substack{\textit{Arranging}\\ \textit{Two Rings}}} = \mathbf{10}$$

Case II: Two Rings on two different fingers

$$\underbrace{5}_{\substack{\textit{First}\\ \textit{Appendage}}} \times \underbrace{4}_{\substack{\textit{Second}\\ \textit{Appendage}}} = 20$$

$$Total\ Ways = 10 + 20 = 30$$

Method II: Multiplication Principle

Wear the first ring. This can be done on any of the fingers, or the thumb, giving us

$$5\ Positions$$

Now, the second ring can go on any finger or thumb. On whichever finger the first ring was worn, it can go in two positions. So, the total number of positions for the second ring is:

$$6\ Positions$$

By the multiplication principle, the total number of ways of wearing both rings is:

$$\underbrace{5}_{\substack{\textit{First}\\ \textit{Ring}}} \times \underbrace{6}_{\substack{\textit{Second}\\ \textit{Ring}}} = 30$$

(Continuation) Example 2.24: Wearing Three Rings

For the next party, I wear three rings instead of two, order still matters, and there are no restrictions on how I can wear the rings. In how many ways can I do this?

Multiplication Principle

$$\underbrace{5}_{\substack{\textit{First}\\ \textit{Ring}}} \times \underbrace{6}_{\substack{\textit{Second}\\ \textit{Ring}}} \times \underbrace{7}_{\substack{\textit{Third}\\ \textit{Ring}}} = 210$$

Casework

Case I: All Three Rings on one Finger
Case II: Two Rings on One Finger, and One Ring on a Different Finger
Case III: Three Rings on Three Different Fingers

D. Pairs

2.25: Pair

If we put two things together, they form a pair.

For example:

$$(5,7) \Rightarrow Successive\ Odd\ Numbers$$
$$(x, y) \Rightarrow Coordinates\ in\ a\ coordinate\ system$$
$$(Delhi, India) \Rightarrow City\ and\ its\ country$$
$$(Sachin\ Tendulkar, Cricket) \Rightarrow Sportsperson\ and\ Sport$$

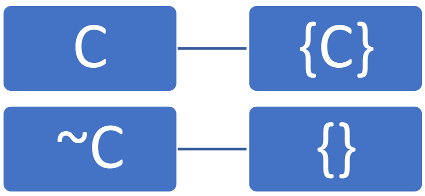
2.26: Unordered Pair
If the sequence of the pair is not important, then the pair is called an unordered pair. In such a case
$$(5,7) = (7,5)$$

2.27: Ordered Pair
On the other hand, if sequence is important, then the pair is an ordered pair. Then
$$(5,7) \neq (7,5)$$

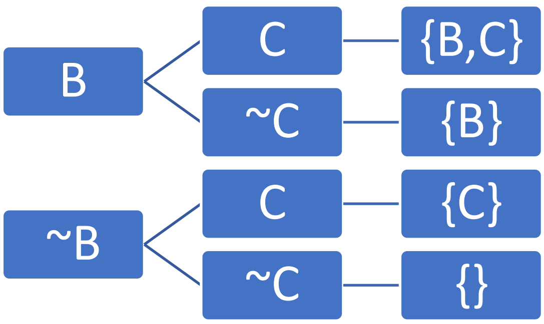
E. Number of Subsets

Example 2.28
List the subsets of the set $\{A, B, C\}$ including the set itself, and the set with no elements (null set).
Recall that the subset of a set X is a set that does not have any elements which are not present in X.

Consider only C
We will build up the solution step by step. Consider that we had a set with a single element C. We have a choice of taking that element, or not taking it.

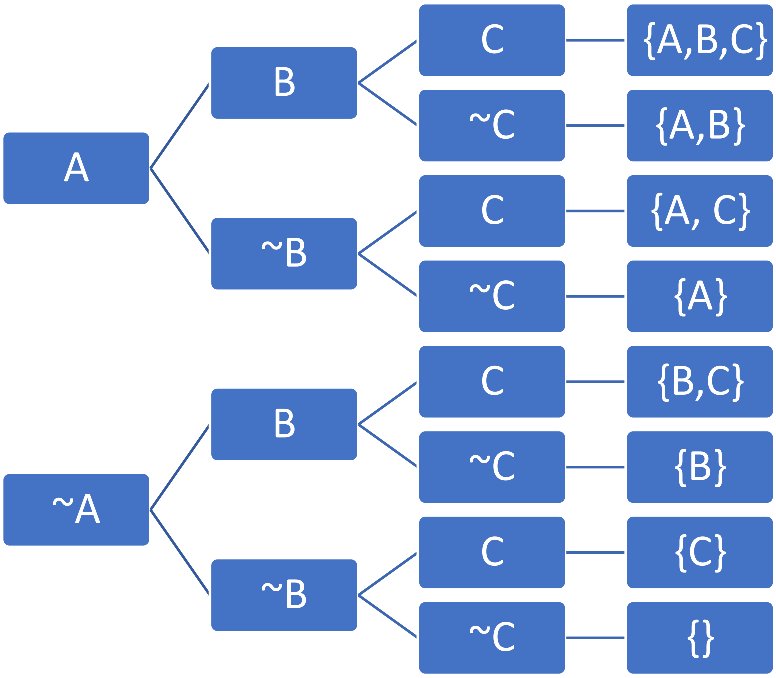


Consider B and C
With each of B and C, I have a choice of picking or not picking the element. I can combine the choices into a tree diagram.



Consider A, B and C

IF we consider all three elements, with each of these, I have a choice of picking or not picking the element. I can represent these using a much larger tree diagram, which tells me that I have 8 choices in all.



$$\{\{A, B, C\}, \{A, B\}, \{A, C\}, \{A\}, \{B, C\}, \{B\}, \{C\}, \{\phi\}\}$$

Find the number of different subsets listed above:

A. By counting
B. By using the multiplication rule

Counting

$$\textit{There are } 8 \textit{ subsets}$$

My choices are:

$$\underbrace{2}_{\substack{\textit{Two Choices for } A: \\ \textit{Take or Don't Take}}} \times \underbrace{2}_{\substack{\textit{Two Choices for } B: \\ \textit{Take or Don't Take}}} \times \underbrace{2}_{\substack{\textit{Two Choices for } C: \\ \textit{Take or Don't Take}}} = 2^3 = 8$$

2.29: Number of Subsets

The number of subsets of a set with n elements is:

$$2^n$$

Example 2.30

What is the number of subsets of a set which contains:

A. 20 elements?
B. as its elements the capital letters of the English alphabet?

$$2^{20}$$
$$2^{26}$$

Example 2.31

7 prizes, named $A, B, \dots, G$ must all be distributed among 2 boys based on their performance in a sport competition. What is the number of ways in which the prizes can be distributed?

The prizes are

$$\{A, B, C, D, E, F, G\}$$

Consider making subsets of the prizes, and let the first boy get the prizes in that subset.
Then, the number of ways of distributing the prizes is the same as the number of subsets of a set with 7 elements:

$$= 2^7 = 128$$

Alternate Solution

Prize A can be given either to the first boy, or the second boy. This gives us

$$2\ choices$$

Similarly, Prize B can be given either to the first boy, or the second boy. This also gives us:

$$2\ choices$$

We have two choices for each prize, giving us:

$$\underbrace{2}_{\substack{First\\Prize}} \times \underbrace{2}_{\substack{Second\\Prize}} \times \dots \times \underbrace{2}_{\substack{Seventh\\Prize}} = 2^7 = 128\ Choices$$

Example 2.32

Two leading scientists have been nominated for 9 awards, numbered 1, 2, … ,9. In order to not upset the competitive scientists, each scientist must receive at least one award. If they are the only ones who have been nominated for these awards, then in how many ways can the awards be distributed?

Consider making subsets of the awards, and let the first scientist get the awards in that subset.
Then, the number of ways of distributing the awards is the same as the number of subsets of a set with 9 elements:

$$= 2^9 = 512$$

But we cannot have a null set (no prizes to Scientist A), or the entire set (no prizes to Scientist B).
Hence, the final answer is

$$512 - 2 = 510$$

Alternate Solution

We use the concept of complementary counting. Imagine there are no restrictions. Then, as in the previous example, the number of ways to distribute the awards is:

$$\underbrace{2}_{\substack{First\\Award}} \times \underbrace{2}_{\substack{Second\\Award}} \times \dots \times \underbrace{2}_{\substack{Ninth\\Award}} = 2^9\ Choices$$

But, we have to give at least one award to each scientist. Hence, there are two ways in which we cannot distribute the awards

$$Way\ 1: All\ awards\ to\ first\ scientist$$
$$Way\ 2: All\ awards\ to\ second\ scientist$$

Hence, the final answer is:

$$2^9 - 2 = 512 - 2 = 510$$

Example 2.33

Two tour guides are leading six tourists. The guides decide to split up. Each tourist must choose one of the guides, but with the stipulation that each guide must take at least one tourist. How many different groupings of guides and tourists are possible? **(AMC 10A 2004/12)**

You can think about this question in the context of the powerset formulas that we just built up.
Consider that the guides are:

$$A \text{ and } B$$

And that you are considering the set of tourists that will go with A.
Then, the total number of choices you can make is the same as the number of non-null, proper subsets that you can make from the set of six tourists

$$2^6 - 1 - 1 = 64 - 2 = 62$$

Alternate Solution

Consider the number of choices for each tourist:

$$\underbrace{2}_{\substack{First\\Tourist:\\Two\ Choices}} \times \underbrace{2}_{\substack{Second\\Tourist:\\Two\ Choices}} \times \ldots \times \underbrace{2}_{\substack{Sixth\\Tourist:\\Two\ Choices}} = 2^6 = 64$$

$$64 - 2 = 62$$

A. Powerset

2.34: Powerset

The powerset consists of the set of all subsets of a set.

Example 2.35

Consider the set

$$\{A, B, C\}$$

A. What is the powerset of the set
B. What is the cardinality of the powerset?

We already did this, without using the term powerset:

$$\{\{A, B, C\}, \{A, B\}, \{A, C\}, \{A\}, \{B, C\}, \{B\}, \{C\}, \{\phi\}\}$$

And, we also calculated the cardinality of the above set, which is:

$$2^3 = 8$$

2.36: Cardinality of Powerset

Cardinality is the number of elements of a set.

The cardinality of the powerset of a set A with n elements is given by:

$$2^n$$

$$\underbrace{2}_{\substack{2\ Choices \\ for\ the \\ 1^{st}\ Element}} \times \underbrace{2}_{\substack{2\ Choices \\ for\ the \\ 2^{nd}\ Element}} \times \ldots \times \underbrace{2}_{\substack{2\ Choices \\ for\ the \\ n^{th}\ Element}} = 2^n$$

2.37: Null and Non-Null Sets

An empty set is called a null set.
A non-empty set is called a non-null set.

$$Empty\ Set = \{\phi\}$$

2.38: Proper Subset

A subset of a set, which is not the set itself is called a proper subset.

Example 2.39

A. How subsets of a set are not proper?

$$Exactly\ One$$

Example 2.40

Mary has five treasured family heirlooms to be distributed among her two children, A and B. She is fine with any distribution that does not require all of the heirlooms to be given to a single child. In how many ways can she make this distribution?

Let the heirlooms be:

$$H = \{h_1, h_2, h_3, h_4, h_5\}$$

We can make a division of the heirlooms in various ways. For example, if we decide to give heirlooms h_2 and h_5 to A, then we will get the distribution:

$$A = \{h_2, h_5\}, B = \{h_1, h_3, h_4\}$$

In other words, giving heirlooms to A is the same as making a subset of H.
The number of ways we can make subsets of H is:

$$2^5$$

However, we cannot have the following distributions:

$$A = \{\phi\}, B = \{h_1, h_2, h_3, h_4, h_5\}$$
$$A = \{h_1, h_2, h_3, h_4, h_5\}, B = \{\phi\}$$

The final answer is:

$$2^5 - 2 = 32 - 2 = 30$$

Example 2.41

Consider the set that has the letters of the English Alphabet:

$$L = \{A, B, C, \ldots, Z\}$$

A. What is the cardinality of the powerset?
B. What is the number of non-null elements of the powerset?
C. What is the number of proper subsets of L?
D. What is the number of non-null, proper subsets of L?

$$2^{26}$$
$$2^{26}-1$$
$$2^{26}-1$$
$$2^{26}-1-1=2^{26}-2$$

Repeat the above question with the set of digits in the decimal system:
$$D=\{0,1,2,\dots,9\}$$

$$2^{10}$$
$$2^{10}-1$$
$$2^{10}-1$$
$$2^{10}-1-1=2^{26}-2$$

2.3 Repetition

A. Repetition Not Allowed

The questions seen so far required picking an element from Set A, an element from Set B, and so on. We now look at questions where Set A and Set B are actually the same set.

2.42: Repetition Not Allowed

If the selection is from the same set, the question arises, whether you can select the same element twice. The choice of whether repetition is allowed may be

- Explicit: Made clear in the question itself.
- Implicit: Because the question gives a real-life scenario where repetition would not be meaningful

The examples below show questions where repetition is not allowed because the question specifically forbids it.

Example 2.43

A haunted house has six windows. In how many ways can Georgie the Ghost enter the house by one window and leave by a different window? **(AMC 8 2007/4)**

The choices for entering and the choice for leaving are both from the same set. But repetition is not allowed. Hence, the number of choices for leaving will be one less than the number of choices for entering:

$$\underbrace{Entering}_{6\ Choices}\times\underbrace{Leaving}_{6-1=5\ Choices}=6\times5=30$$

		Entering						No. of
		1	2	3	4	5	6	Choices
Leaving	1							5
	2							5
	3							5
	4							5
	5							5
	6							5
								$5\times6=30$

Example 2.44: Entering and Leaving a city

Victor is a spy entering the city of Transylvania. He has the option of entering by road, by sea, or by flight. He knows that the police will get alerted once he enters a city via a route, so he cannot leave by the same route that he enters. How many different routes can he plan to enter and exit the city?

Victor has three choices when he enters. When he leaves, he has one less choice. The choice of route that he chooses to enter does not affect this.

$$\underbrace{\{Road, Sea, Flight\}}_{\substack{\text{Number of Ways of Entering} \\ =3}} \times \underbrace{\{\quad\}}_{\substack{\text{Number of ways of leaving} \\ =2}} \Rightarrow 3 \times 2 = 6$$

Example 2.45: Routes between Cities

How many ways are there to go from City A to City B (three routes), then City B to City C (four routes) and then come back to City A via City B, but via a different way from the one used to travel to City C? (Two routes are the same, if and only if they use the same route from A to B, and also from B to C, and the other way around).

Going from City A to City C:

	City A to City B	City B to City C	
	X_1	Y_1	$\{X_1Y_1, X_1Y_2, X_1Y_3, X_1Y_4\} \to 4\ Routes$
	X_2	Y_2	$\{X_2Y_1, X_2Y_2, X_2Y_3, X_2Y_4\} \to 4\ Routes$
	X_3	Y_3	$\{X_3Y_1, X_3Y_2, X_3Y_3, X_3Y_4\} \to 4\ Routes$
		Y_4	
Total Ways	3	4	$3 \times 4 = 12$

$$\underbrace{A \to B}_{3} \times \underbrace{B \to C}_{4} \Rightarrow A \to C = 3 \times 4 = 12\ Routes$$

Going from City C to City A:

$$A \to C = 3 \times 4 = 12$$

But we cannot go back via the same route that we came from A to C. Hence, we have one route less

$$12 - 1 = 11$$

$$\underbrace{A \to C}_{12} \times \underbrace{C \to A}_{11} = 12 \times 11 = 132$$

B. Repetition Understood from the Context

In real-life scenarios, the situation may tell us whether repetition is allowed. In such cases, the question may not tell this to us directly.

Selecting two elements from a set A with n elements gives:

$$(a, b)$$

Example 2.46: Selecting Monitors

A class has 12 boys and 7 girls. Find the number of ways to choose:

A. A monitor.
B. A monitor of each gender.
C. A monitor and a co-monitor (where a student cannot be both a monitor and a co-monitor at the same time).
D. A monitor and a co-monitor, where one of them is a boy, and the other is a girl.

Part A

There are no restrictions on the gender of the monitor. Hence, the monitor can be any of the students of the class, which is:

$$\underbrace{12}_{Boys} + \underbrace{7}_{Girls} = 19\ students \Rightarrow 19\ Ways$$

Part B

One monitor must be a boy (12 Choices), and the other must be a girl (7 Choices). These two choices can be combined, hence total number of choices is:

$$\underbrace{12}_{Boy} \times \underbrace{7}_{Girl} = 84$$

$$\underbrace{Jack}_{Monitor}, \underbrace{Jill}_{Monitor} = \underbrace{Jill}_{Monitor}, \underbrace{Jack}_{Monitor}$$

Part C

There are no restrictions on the gender here. Hence, we have:

$$\underbrace{19}_{Monitor} \times \underbrace{18}_{Co-Monitor} = 342$$

Part D: Direct Counting

From Part B, we know that the number of ways to get two monitors is 84. Out of these, we need to make 1 a monitor, and the other a co-monitor.

Consider that Jack (boy) and Jill(girl) are two students in the class.

$$\underbrace{Jack}_{Monitor}, \underbrace{Jill}_{Co-monitor} \neq \underbrace{Jill}_{Monitor}, \underbrace{Jack}_{Co-monitor}$$

Hence, for every pair that was counted in Part B, we need to multiply by 2:

$$84 \times 2 = 168$$

Part D: Complementary Counting

In part C, we calculated that the number of ways to select a monitor and a co-monitor (with no restrictions)

$$= \underbrace{19}_{Monitor} \times \underbrace{18}_{Co-Monitor} = 342$$

Out of these, subtract the choices where both are boys, or both are girls:

$$342 - \underbrace{12 \times 11}_{Both\ Boys} - \underbrace{7 \times 6}_{Both\ Girls} = 342 - 132 - 42 = 168$$

C. Ordered Pairs (Optional)

In some applications, the order in which the elements are written is important. For example,

- the ranking in an exam, or a competition, or a race
- distribution of 1st, 2nd and 3rd prizes.

Such pairs are called ordered pairs.

$$\left(\underbrace{a}_{1st}, \underbrace{b}_{2nd}\right) \neq \left(\underbrace{b}_{1st}, \underbrace{a}_{2nd}\right)$$

However, if order is not important in a situation. For example, a medical study may be interested in considering whether a patient survived a medical procedure. In such a case:

$$\left(\underbrace{a}_{Survived}, \underbrace{b}_{Survived}\right) = \left(\underbrace{b}_{Survived}, \underbrace{a}_{Survived}\right)$$

By default, the multiplication principle gives us ordered pairs.

Example 2.47: Selecting Monitors

A class has 12 boys and 7 girls. Find the number of ways to choose two monitors.

This looks very similar to Part C of the previous example. There are no restrictions on the gender of the monitor

- Pick the first monitor from any of the 19 students in the class.
- For the second monitor, we have only 18 choices, since the first monitor cannot also be the second monitor.

$$\underbrace{19}_{\substack{First\\Monitor}} \times \underbrace{18}_{\substack{Second\\Monitor}} = 342$$

This overcounts the number of ways by two because the way we have counted incorporates:

$$AB\ as\ different\ from\ BA$$

Hence, the actual number of choices will be:

$$\frac{342}{2} = 171$$

Example 2.48

I need to select a team of two people from a choice of 11 students. How many ways can I do this?

$$\frac{11 \times 10}{2}$$

2.49: Repetitions Not Allowed

If repetitions are not allowed, the number of objects available for selection will keep decreasing, and this leads to an especially important pattern:

$$n \times (n-1) \times (n-2) \times \dots$$

We have so far seen questions that required us to multiply two numbers.
Now, we see an example where we have more objects.

Example 2.50

I need to select a captain, a vice-captain, and a goalkeeper from a club of eleven people. No person can have more than one role at a time. Find the number of ways in which the selection can be made.

$$\underbrace{11}_{Captain} \times \underbrace{10}_{Vice-Captain} \times \underbrace{9}_{Goalkeeper} = 990$$

D. Repetitions Allowed

The real-life scenarios given to us let us choose certain things:

- When selecting two monitors for a class, the two monitors need to be different.
- In arranging EXAM, we are arranging letters, and they cannot be repeated.

However, certain scenarios allow for repetition, and this is an important concept.

Example 2.51

How many outfits consisting of a shirt, a pair of trousers, and a pair of shoes can I make if I have five shirts, five pairs of trousers and five pairs of shoes?

$$\underbrace{5\ Choices}_{Shirt} \times \underbrace{5\ Choices}_{Trousers} \times \underbrace{5\ Choices}_{Shoes} = 5 \times 5 \times 5 = 5^3 = 125$$

2.52: Arranging r out of n objects (Repetition Allowed)

The number of ways of arranging r distinct objects out of n objects in n positions:

$$\underbrace{n}_{\substack{First\\Location}} \times \underbrace{n}_{\substack{Second\\Location}} \times \dots \times \underbrace{n}_{\substack{rth\\Location}} = n^r$$

Example 2.53: Situations leading to Repetition

What is the number of possible outcomes if I:

A. Make three-digit numbers using odd digits (if repetition is allowed)
B. Make three-digit numbers using even digits (if repetition is allowed)
C. Answer a question paper with six questions that has four options for each question, if each question must be answered using one of the given options.
D. Answer a question paper with six questions that has four options for each question, if you have the option of not attempting one or more of the questions.

Part A

We can structure this counting by thinking of the choices that we have for each digit of the number.
The odd digits are:

$$1,3,5,7,9 \Rightarrow 5\ Odd\ Digits$$

$$\underbrace{5}_{\substack{Hundred's\\Digit}} \times \underbrace{5}_{\substack{Ten's\\Digit}} \times \underbrace{5}_{\substack{Unit's\\Digit}} = 5^3 = 125$$

Part B

The even digits are:

$$0,2,4,6,8 \Rightarrow 5\ Even\ Digits$$

We have no restrictions on the units digit and the ten's digit. But zero cannot be the first digit of a three-digit number.

$$\underbrace{4}_{\substack{Hundred's\\Digit}} \times \underbrace{5}_{\substack{Ten's\\Digit}} \times \underbrace{5}_{\substack{Unit's\\Digit}} = 4 \times 5^2 = 100$$

Part C

$$\underbrace{4}_{\substack{1st\\Question}} \times \underbrace{4}_{\substack{2nd\\Question}} \times \underbrace{4}_{\substack{3rd\\Question}} \times \underbrace{4}_{\substack{4th\\Question}} \times \underbrace{4}_{\substack{5th\\Question}} \times \underbrace{4}_{\substack{6th\\Question}} = 4^6 = 2^{12} = 4096$$

Part D

$$\underbrace{5}_{\substack{1st\\Question}} \times \underbrace{5}_{\substack{2nd\\Question}} \times \underbrace{5}_{\substack{3rd\\Question}} \times \underbrace{5}_{\substack{4th\\Question}} \times \underbrace{5}_{\substack{5th\\Question}} \times \underbrace{5}_{\substack{6th\\Question}} = 5^6$$

$$5^6$$

E. Complementary Counting

2.54: Complementary Counting

When counting what we do not want is easier than counting what we do want, complementary counting is a useful technique.

- We have already used complementary counting in some of the questions above.

Example 2.55

You are arranging plates for dinner at a dinner table that has numbered seats for four people. Every person gets exactly one plate to eat from, so you will use 4 plates in all. Plates made from different materials are distinguishable, but two plates of the same material look identical. What is the number of ways of arranging plates if you have:

A. 4 ceramic and 4 melamine plates

B. 3 ceramic and 3 melamine plates

Part A

For the first person, you have a choice of giving him ceramic or melamine, giving you two choices. In fact, for every seat, you have the same number of choices.

$$\underbrace{2}_{\substack{First\\Seat}} \times \underbrace{2}_{\substack{Second\\Seat}} \times \underbrace{2}_{\substack{Third\\Seat}} \times \underbrace{2}_{\substack{Fourth\\Seat}} = 2^4 = 16$$

Part B

The only difference between Part A and Part B is that you will not be able to have:

- all four seats with ceramic plates OR
- all four seats with melamine plates

Hence, the number of choices is:

$$16 - \underbrace{1}_{\substack{All\ Four\\Ceramic\ Plates}} - \underbrace{1}_{\substack{All\ Four\\Melamine\ Plates}} = 16 - 2 = 14$$

F. Further Examples

Example 2.56

I am going to order out from Monday to Friday for lunch. I have a choice from Chinese, Japanese, Indian or Continental cuisine each day. How many ways can I do this if:

A. I cannot repeat cuisines
B. I can repeat cuisines
C. I can repeat cuisines, but I cannot have the same cuisine all five days of the week. For example,

$$Not\ Valid: Chinese + Chinese + Chinese + Chinese + Chinese$$
$$Valid: Chinese + Chinese + Chinese + Chinese + Japanese$$

Part A: No Restrictions

$$\underbrace{4}_{\substack{1st\\Day}} \times \underbrace{3}_{\substack{2nd\\Day}} \times \underbrace{2}_{\substack{3rd\\Day}} \times \underbrace{1}_{\substack{4th\\Day}} \times \underbrace{0}_{\substack{5th\\Day}} = 0$$

On the fifth day, no matter what I choose on the earlier four days, I run out of choices.
Hence, I can do this in zero ways.

Part B: No Restrictions

If there are no restrictions

$$\underbrace{4}_{Monday} \times \underbrace{4}_{Tuesday} \times \underbrace{4}_{Wednesday} \times \underbrace{4}_{Thursday} \times \underbrace{4}_{Friday} = 4^5 = 2^{10} = 1024$$

Part C

We use complementary counting. The number of ways to order (with no restrictions) is

$$4^5 = 1024$$

The number of ways where the same cuisine gets repeated all five days of the week is:

$$\underbrace{1}_{\substack{All\ 5\ Days\\Chinese}} + \underbrace{1}_{\substack{All\ 5\ Days\\Japanese}} + \underbrace{1}_{\substack{All\ 5\ Days\\Punjabi}} + \underbrace{1}_{\substack{All\ 5\ Days\\Continental}} = 4\ Ways$$

Valid number of ways

$$1024 - 4 = 1020$$

Example 2.57

Sorting Hat needs to sort 12 twelve students. Number of ways to sort into:

A. Two Houses
B. Three Houses
C. Four Houses

$$2^{12}$$
$$3^{12}$$
$$4^{12}$$

Example 2.58

Sorting Hat needs to sort 12 twelve students such that each house must have at least one student. Determine the number of ways to sort into Two Houses

The number of ways without restriction is:

$$2^{12}$$

The number of ways to not meet the condition means that at least one house has no students. This can be done in:

$$House\ 1\ has\ no\ students: 1\ way$$
$$House\ 2\ has\ no\ students: 1\ way$$
$$Total = 2\ Ways$$

By complementary counting, the answer we want is:

$$2^{12} - 2$$

Example 2.59

Sorting Hat needs to sort 12 twelve students such that each house must have at least one student. Determine the number of ways to sort into Three Houses

The number of ways without restriction is:

$$3^{12}$$

The number of ways to not meet the condition means that at least one house has no students. This can be done in:

$$Case\ I: House\ 1\ has\ no\ student, House\ 2\ and\ 3\ no\ restrictions \Rightarrow 1 \times 2^{12} = 2^{12}$$
$$Case\ II: House\ 2\ has\ no\ student, House\ 1\ and\ 3\ no\ restrictions \Rightarrow 1 \times 2^{12} = 2^{12}$$
$$Case\ III: House\ 3\ has\ no\ student, House\ 1\ and\ 2\ no\ restrictions \Rightarrow 1 \times 2^{12} = 2^{12}$$

Hence, the number of ways to not meet the condition is

$$2^{12} + 2^{12} + 2^{12} = 3 \cdot 2^{12}$$

However, note that

$$All\ students\ in\ House\ 3 \Rightarrow Counted\ in\ Case\ I, Counted\ in\ Case\ II$$
$$All\ students\ in\ House\ 2 \Rightarrow Counted\ in\ Case\ I, Counted\ in\ Case\ III$$
$$All\ students\ in\ House\ 1 \Rightarrow Counted\ in\ Case\ II, Counted\ in\ Case\ III$$

Hence, these three values are double counted and must be subtracted:

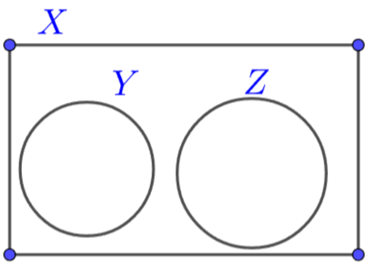
$$3 \cdot 2^{12} - 3$$

The final answer is:

$$3^{12} - (3 \cdot 2^{12} - 3)$$

Example 2.60

Let $X = \{1,2,3,4,5\}$. The number of different ordered pairs (Y, Z) that can be formed such that $Y \subseteq X, Z \subseteq X$, and $Y \cap Z$ is empty is: **(JEE Main 2012)**



Draw a Venn Diagram, as shown alongside. Since

$$(Y \cap Z) = \phi \Rightarrow Y \text{ and } Z \text{ have no overlap}$$

Hence, for each number in $X = \{1,2,3,4,5\}$ we have exactly three choices:

$$\text{It belongs to } Y \subseteq X$$
$$\text{It belongs to } Z \subseteq X$$
$$\text{It belongs to } X \text{ but not in } Y \text{ or } Z: X \cap (Y \cup Z)$$

Hence, the total number of choices is:

$$= 3^5 = 243$$

G. Contrasting arrangements with and without Repetition

Whether repetition is allowed needs to be determined from the question. It can be given either directly, or indirectly. The number of arrangements with repetition, and without repetition, is, in general, not the same, as the next example shows.

Example 2.61: Selecting People for Posts

In how many ways can a college committee with twelve professors select a chairperson and a co-chairperson. Answer the question assuming that a single person can occupy multiple posts. Then, answer again assuming that a single person can occupy only a single post.

Multiple Posts

Here, one person is allowed to hold multiple posts, and hence, the same person can hold more than one post. This means that repetition is allowed.

$$\underbrace{\underbrace{12}_{Chairperson} \times \underbrace{12}_{Co-Chairperson}}_{Multiple\ Posts\ (Repetition\ Allowed)} = 144$$

Single Post

Here, one person is allowed to hold only a single post, and hence, the same person cannot hold more than one post. This means that repetition is not allowed, and as you choose people for a particular post, the choice of people will keep reducing.

$$\underbrace{\underbrace{12}_{Chairperson} \times \underbrace{11}_{Co-Chairperson}}_{Distinct\ People\ (Repetition\ Not\ Allowed)} = 132$$

Example 2.62: Selecting People for Posts

In how many ways can a Reader's Club with seven members select a President, a Vice-President and a Treasurer. Answer the question assuming that a single person can occupy multiple posts. Then, answer again assuming that a single person can occupy only a single post.

Multiple Posts

$$\underbrace{\underbrace{7}_{President} \times \underbrace{7}_{Vice-President} \times \underbrace{7}_{Treasurer}}_{Multiple\ Posts\ (Repetition\ Allowed)} = 343$$

Single Post

$$\underbrace{\underbrace{7}_{President} \times \underbrace{6}_{Vice-President} \times \underbrace{5}_{Treasurer}}_{Distinct\ People\ (Repetition\ Not\ Allowed)} = 210$$

Example 2.63: Arranging Flags

In calculating the number of ways in which red, blue and green flags can be arranged in a line of three, Rohan did not repeat colours, while Rahul assumed no restrictions. Find the positive difference between Rahul's answer and Rohan's Answer.

$$\underbrace{\underbrace{3}_{\substack{First\\Flag}} \times \underbrace{3}_{\substack{Second\\Flag}} \times \underbrace{3}_{\substack{Third\\Flag}}}_{Rahul's\ Answer} - \underbrace{\underbrace{3}_{\substack{First\\Flag}} \times \underbrace{2}_{\substack{Second\\Flag}} \times \underbrace{1}_{\substack{Third\\Flag}}}_{Rohan's\ Answer} = 3^3 - 6 = 27 - 6 = 21$$

Example 2.64: Arrangements with Repetition Allowed

Using only the digits 4, 5, 7 and 2, what is the positive difference between the number of three-digit numbers that can be created if repetition is allowed, and the number of four-digit numbers that can be created if repetition is not allowed.

$$Three\ Digit\ Numbers: 4 \times 4 \times 4 = 4^3 = 64$$
$$Four\ Digit\ Numbers: 4 \times 3 \times 2 \times 1 = 24$$
$$64 - 24 = 40$$

H. Answering Question Papers

Example 2.65

In how many ways can you attempt the following if you must enter a response:

A. 3 True/False Questions
B. 4 True/False Questions
C. 7 True/False Questions

$$2^3 = 8$$
$$2^4 = 16$$
$$2^7 = 128$$

Answer the above if not entering a response is also an option?

$$3^3 = 27$$
$$3^4 = 81$$
$$3^7$$

Example 2.66

Multiple Choice Questions (MCQs) need a single option to be marked correct. In how many ways can you attempt:

A. Questions 1-4 in AMC 12, given that each question has five options

B. Questions 1-5 in the CFA Level I paper, given that each question has 3 options
C. Questions 1-3 in the CAT, given that each question has 4 options

$$AMC\ 12: 5 \times 5 \times 5 \times 5 = 5^4 = 625$$
$$CFA: 3 \times 3 \times 3 \times 3 \times 3 = 3^5 = 243$$
$$CAT: 4 \times 4 \times 4 = 4^3 = 64$$

Example 2.67

An exam which has three questions, with five answer options each, one of which is to be marked correct for each question. In how many ways can you

A. Attempt all questions of the paper
B. Attempt two questions of the paper
C. Attempt a single question in the paper
D. Attempt one or more questions of the paper
   I. By adding the answers to the earlier three parts
   II. By using complementary counting

Part A

$$\underbrace{5\ Choices}_{First\ Question} \times \underbrace{5\ Choices}_{Second\ Question} \times \underbrace{5\ Choices}_{Third\ Question} = 5 \times 5 \times 5 = 5^3 = 125$$

First Question	Second Question	Third Question	
A	A	A	
B	B	B	
C	C	C	
D	D	D	
E	E	E	
5	5	5	$5^3 = 125$

Part B

There are two parts:

- Choice of question
  - ✓ Choosing two questions is the same as rejecting one question. Hence, number of ways to choose questions is 3.
- Choice of option within the question

$$\underbrace{3}_{Choice\ of\ Question} \times \underbrace{5 \times 5}_{Choice\ of\ Options} = 3 \times 25 = 75$$

Part C

There are two parts:

- Choice of question
- Choice of option within the question

$$3 \times 5 = 15$$

Part D-I

$$\underbrace{125}_{\substack{Attempt\\3\ Questions}} + \underbrace{75}_{\substack{Attempt\\2\ Questions}} + \underbrace{15}_{\substack{Attempt\\1\ Question}} = 215$$

Part D-II

Total choices in attempting $0-3$ Questions

Not attempting a question gives an additional choice. Hence, instead of five choices for each question, Pushpak now has six choices.

Hence, number of Ways to attempt zero or more questions is:

$$\underbrace{6}_{\substack{Question\\One}} \times \underbrace{6}_{\substack{Question\\Two}} \times \underbrace{6}_{\substack{Question\\Three}} = 216$$

Complementary Counting

However, this also includes the "attempt", where no questions have no choice selected, which we do not want to count.

Hence, the final answer is

$$216-1=215$$

Example 2.68

A Multiple-Correct, Multiple-Correct question in a JEE paper has question text and four options(A, B, C, D) for each question. How many different ways are there in which a multiple-correct question can be answered (keeping in mind that the correct options need to be marked and one or more option(s) is/are correct)? Two ways of answering the question are $\{A, D\}\{A, B, C, D\}$.

Multiplication Principle with Repetition

$$\underbrace{\{\text{Option A}\}}_{\substack{2\ Choices:\\Correct\ or\ Not\ Correct}} \times \underbrace{\{\text{Option B}\}}_{\substack{2\ Choices:\\Correct\ or\ Not\ Correct}} \times \underbrace{\{\text{Option C}\}}_{\substack{2\ Choices:\\Correct\ or\ Not\ Correct}} \times \underbrace{\{\text{Option D}\}}_{\substack{2\ Choices:\\Correct\ or\ Not\ Correct}} = 2^4 = 16$$

$$16-1=15$$

Enumeration

$$\left\{\underbrace{A, B, C, D}_{Single\ Correct\ Option}, \underbrace{AB, AC, AD, BC, BD, CD}_{Two\ Correct\ Options}, \underbrace{ABC, ABD, ACD, BCD}_{Three\ Correct\ Options}, \underbrace{ABCD}_{Four\ Correct\ Options}\right\}$$

Multiplication Principle with Repetition

Think of the multiple choice, multiple correct question as four true\false questions combined, which can be answered in

$$2^4 = 16\ ways$$
$$16-1=15$$

Challenge 2.69[1]

A Matrix-Match question in a JEE paper looks the Match the Column given. It is a multiple-choice, multiple correct question. A statement can have zero or more correct answers, upto the maximum of four mentioned in the question. What is the number of ways in which the Matrix-Match question above can be answered?

Options	Correct Answers
Statement A	Correct Answer I
Statement B	Correct Answer II
Statement C	Correct Answer III
Statement D	Correct Answer IV

[1] This also explains why a true matrix match question is very difficult. To get it fully correct, you have to mark the correct option out of the available 2^{16} choices.

From the previous question, has

$$\text{Statement A: } 2^4 = 16 \textit{ choices}$$
$$\text{Statement B: } 2^4 = 16 \textit{ choices}$$
$$\text{Statement C: } 2^4 = 16 \textit{ choices}$$
$$\text{Statement D: } 2^4 = 16 \textit{ choices}$$

Each statement is to be answered independently, therefore total number of choices is

$$2^4 \times 2^4 \times 2^4 \times 2^4 = 2^{16}$$

Example 2.70

Virat is appearing for his midterm exam, which has 9 True/False questions. After he submits the paper, his teacher informs him that his marks for the True/False questions are neither zero, nor maximum. In how many ways could Virat have attempted the questions, if (answer each separately):

A. He guessed the answers to all of the questions.
B. He marked the third question as True, and guessed the rest.

Part A

The total number of ways that Virat can attempt the paper is:

$$2^9$$

Two cases are not possible: All correct and all false. Subtract them:

$$2^9 - 2 = 512 - 2 = 510$$

Part B

The total number of ways that Virat can attempt the paper is:

$$2^8 = 256$$

If Viraat gets the third question correct, he can attempt the rest so they are incorrect.
And if Viraat gets the third question incorrect, he can attempt the rest so they are correct.

Example 2.71

A school asks three types of questions in its question papers, with Type I and Type II multiple choice and Type III as True/False.

	No. of Options		No. of Questions		True/False
	Type I	Type II	Type I	Type II	Type III
Biology	3	5	4	4	5

Multiple choice questions require a single option to be marked correct. Find the number of ways in which all the questions of the papers can be attempted.

$$3^4 \times 5^4 \times 2^5 = 81 \times 10{,}000 \times 2 = 1{,}620{,}000$$

2.4 Counting Numbers

A. Idea

Competitions love to ask questions that involve counting numbers. It ticks a number of checkboxes:

- Checks understanding of numbers, which students are expected to have, but may not be fluent in practice.
- Allows mixing of concepts from number theory (prime/composite, odd/even, multiples, etc) with counting concepts. The individual concepts are not very difficult, but the larger the number of concepts involved, the more difficult the question becomes.
- It allows asking of Counting questions in exams in lower grades where it is not officially a part of the

syllabus.

Hence, it is important to practice these questions, till a high degree of comfort is achieved.

B. Two Digit Numbers

Two-digit numbers are very simple. Yet, they show a lot of underlying structure and properties that can be used to connect with them with very powerful ideas.
Some of the questions in the example below may be quite easy to solve using one method. Ensure that you are able to solve them using all the methods given, so that you can use those methods in questions where they are most suitable.

Example 2.72

Find the number of two-digit numbers with no restrictions.

Multiplication Rule Method

$$\underbrace{Ten's\ Digit}_{\{1,2,\dots,9\}:\ 9\ Choices} \times \underbrace{Unit's\ Digit}_{\{0,1,2,\dots,9\}:\ 10\ Choices} = 9 \times 10 = 90\ Numbers$$

Using Lists

$$\underbrace{99}_{\substack{Largest\\ Two\ Digit\\ Number}} - \underbrace{10}_{\substack{Smallest\\ Two\ Digit\\ Number}} + 1 = 90$$

Example 2.73

Find the number of two-digit numbers which are odd.

Method I: Pair the numbers as follows:

$$\left\{\underbrace{(10,11)}_{First\ Pair}, \underbrace{(12,13)}_{Second\ Pair}, \underbrace{(14,15)}_{Third\ Pair}, \dots, \underbrace{(98,99)}_{Forty\ Fifth\ Pair}\right\}$$

Every even number is followed by exactly one odd number.

Hence, the number of odd numbers must be exactly half that of the total two-digit numbers.

$$= \frac{90}{2} = 45$$

Method II: Multiplication Principle

There is no restriction on the ten's digit.
But the units digit of an odd number must be odd.

$$\underbrace{Ten's\ Digit}_{\{1,2,\dots,9\}:\ 9\ Choices} \times \underbrace{Unit's\ Digit}_{\{1,3,5,7,9\}:5\ Choices} = 9 \times 5 = 45\ Numbers$$

Example 2.74

Find the number of two-digit numbers which are even.

Complementary Counting

Since there are 90 two-digit numbers, and 45 of them are odd, the remaining must be even:

$$90 - 45 = 45\ even\ numbers$$

Multiplication Principle

$$\underbrace{Ten's\ Digit}_{\{1,2,\dots,9\}:\ 9\ Choices} \times \underbrace{Unit's\ Digit}_{\{0,2,4,6,8\}:5\ Choices} = 9 \times 5 = 45\ Numbers$$

2.75: Zero cannot be the first digit: Common Mistake

0 cannot be the first digit of a number, which is two digits or more.
Keep this in mind.

- For example, in the previous question, the number of choices for the ten's digit is 9 and not 10.

Example 2.76

Find the number of two-digit numbers where both the digits are odd.

The odd digits are:

$$1,3,5,7,9 \Rightarrow 5\ Digits$$

$$\underbrace{Ten's\ Digit}_{\{1,3,5,7,9\}:\ 5\ Choices} \times \underbrace{Unit's\ Digit}_{\{1,3,5,7,9\}:5\ Choices} = 5 \times 5 = 25\ Numbers$$

Example 2.77

Find the number of two-digit numbers with exactly two even digits.

The even digits are:

$$0,2,4,6,8 \Rightarrow 5\ Digits$$

But we cannot have zero as the first digit, since it will then not be a two-digit number. This reduces the number of choices for ten's digit to only 4.

$$\underbrace{Ten's\ Digit}_{\{2,4,6,8\}:\ 4\ Choices} \times \underbrace{Unit's\ Digit}_{\{0,2,4,6,8\}:5\ Choices} = 4 \times 5 = 20\ Numbers$$

Example 2.78

Find the number of two-digit numbers where the digits must be, with repetition allowed, from the set {3,4,5,6,7,0}.

$$\underbrace{Ten's\ Digit}_{\{3,4,5,6,7\}:\ 5\ Choices} \times \underbrace{Unit's\ Digit}_{\{3,4,5,6,7,0\}:6\ Choices} = 5 \times 6 = 30\ Numbers$$

Example 2.79

Find the number of two-digit numbers with odd digits, and repetition of digits is not allowed.

$$\underbrace{Ten's\ Digit}_{\{1,3,5,7,9\}:\ 5\ Choices} \times \underbrace{Unit's\ Digit}_{(5-1)=4\ Choices} = 5 \times 4 = 20\ Numbers$$

Example 2.80

Find the number of two-digit numbers with even digits, and repetition of digits, is not allowed.

$$\underbrace{Ten's\ Digit}_{\{2,4,6,8\}:\ 4\ Choices} \times \underbrace{Unit's\ Digit}_{\{0,2,4,6,8\}-1=4\ Choices} = 4 \times 4 = 16\ Numbers$$

Example 2.81

Find the number of two-digit numbers where the tens digit is greater than three, and the units digit is greater than five.

$$\underbrace{Ten's\ Digit}_{\{4,5,6,7,8,9\}:\ 6\ Choices} \times \underbrace{Unit's\ Digit}_{\{6,7,8,9\}:\ 4\ Choices} = 6 \times 4 = 24\ Numbers$$

2.82: Off by 1 (Common Mistake)

When counting, it is quite easy to count incorrectly, and get your answer different from the actual answer by a difference of 1.

- For example, in the previous question, you could get

$$7 \times 5 = 35$$

Example 2.83

Find the number of two-digit numbers with only prime digits.

$$\underbrace{Ten's\ Digit}_{\{2,3,5,7\}:\ 4\ Choices} \times \underbrace{Unit's\ Digit}_{\{2,3,5,7\}:\ 4\ Choices} = 4 \times 4 = 16\ Numbers$$

C. Choosing the Right Tool

The above set of questions provides a lot of variety regarding different types of two-digit numbers. It also gives an idea of the power of the multiplication principle.
But this power can also create the impression that this tool should be applied to every question.
In certain cases, it become better or more useful to fall back on a simple enumeration strategy.
Choosing the right method of attack for a question is very important in being able to solve the question. This comes with practice.
The conditions given can be complicated enough that the final answer is a small number. In such scenarios, it is often better to use enumeration.

Example 2.84

How many two-digit numbers have ten's digit exactly two more than the unit's digit?

Bogus Solution

$$\underbrace{Ten's\ Digit}_{9\ Choices} \times \underbrace{Unit's\ Digit}_{8\ Choices} = 9 \times 8 = 72$$

This overcounts the numbers because once you fix the Ten's Digit, with each choice of ten's digit, there is only one acceptable choice of Unit's Digit.

Valid Solution

Pick one number at a time each number as the Ten's Digit, which makes the number

$$0A \rightarrow Not\ a\ two\ digit\ number$$

$$\begin{gathered} 1A \rightarrow No\ Choices \\ 2A \rightarrow 20 \\ 3A \rightarrow 31 \\ 4A \rightarrow 42 \\ 5A \rightarrow 53 \\ 6A \rightarrow 64 \\ 7A \rightarrow 75 \\ 8A \rightarrow 86 \\ 9A \rightarrow 97 \end{gathered}$$

Shortcut

This can be done using the multiplication principle, creating an elegant (but perhaps not necessarily exam-suitable) solution:

$$\underbrace{Ten's\ Digit}_{\substack{\text{Only One Choice}\\ \text{Automatically Fixed}}} \times \underbrace{Unit's\ Digit}_{\{0,1,2,3,4,5,6,7\}=8\ Choices} = 1 \times 8 = 8 \Rightarrow \{20,31,42,53,64,75,86,97\}$$

Example 2.85: Listing Numbers

How many two-digit numbers have an even tens digit, and the units digit is double of the ten's digit?

$$\underbrace{Ten's\ Digit}_{\{2,4\}:\ 2\ Choices} \times \underbrace{Unit's\ Digit}_{No\ Choice=1\ Choice} = 2 \times 1 = 2 \Rightarrow \{24,48\}$$

D. Three Digit Numbers

Example 2.86

Find the number of three-digit numbers which:

A. Have no restrictions
B. Have no zeros (NMTC Primary/Final 2005/02)
C. Are Odd
D. Are multiples of two
E. Have only Odd Digits
F. Have only Even Digits
G. Have only prime digits
H. Have only odd prime digits
I. Have only prime digits and the numbers are odd

Part A

The Hundreds Digit cannot be zero:

$$\underbrace{Hundred's\ Digit}_{\{1,2,\dots,9\}:\ 9\ Choices} \times \underbrace{Ten's\ Digit}_{\{0,1,2,\dots,9\}:\ 10\ Choices} \times \underbrace{Unit's\ Digit}_{\{0,1,2,\dots,9\}:\ 10\ Choices} = 9 \times 10 \times 10 = 900\ Numbers$$

Part B

If we cannot use zero as a digit, then we are left with exactly nine choices for each digit:

$$\underbrace{Hundred's\ \ Digit}_{\{1,2,\dots,9\}:\ 9\ Choices} \times \underbrace{Ten's\ Digit}_{\{1,2,\dots,9\}:\ 9\ Choices} \times \underbrace{Unit's\ Digit}_{\{1,2,\dots,9\}:\ 9\ Choices} = 9 \times 9 \times 9 = 729\ Numbers$$

Part C

$$\frac{900}{2}$$

$$\underbrace{First\ Digit}_{\{1,2,\dots,9\}:\ 9\ Choices} \times \underbrace{Second\ Digit}_{\{0,1,2,\dots,9\}:\ 10\ Choices} \times \underbrace{Third\ Digit}_{\{1,3,5,7,9\}:\ 5\ Choices} = 9 \times 10 \times 5 = 450\ Numbers$$

Part D

$$\underbrace{First\ Digit}_{\{1,2,\dots,9\}:\ 9\ Choices} \times \underbrace{Second\ Digit}_{\{0,1,2,\dots,9\}:\ 10\ Choices} \times \underbrace{Third\ Digit}_{\{0,2,4,6,8\}:\ 5\ Choices} = 9 \times 10 \times 5 = 450\ Numbers$$

Part E

$$\underbrace{First\ Digit}_{\{1,3,5,7,9\}:\ 5\ Choices} \times \underbrace{Second\ Digit}_{\{1,3,5,7,9\}:\ 5\ Choices} \times \underbrace{Third\ Digit}_{\{1,3,5,7,9\}:\ 5\ Choices} = 5 \times 5 \times 5 = 125\ Numbers$$

Part F

$$\underbrace{First\ Digit}_{\{2,4,6,8\}:\ 4\ Choices} \times \underbrace{Second\ Digit}_{\{0,2,4,6,8\}:\ 5\ Choices} \times \underbrace{Third\ Digit}_{\{0,2,4,6,8\}:\ 5\ Choices} = 4 \times 5 \times 5 = 100\ Numbers$$

Part G
Prime digits are:

$$2,3,5,7 \Rightarrow 4\ Digits$$
$$\underbrace{First\ Digit}_{\{2,3,5,7\}:\ 4\ Choices} \times \underbrace{Second\ Digit}_{\{2,3,5,7\}:\ 4\ Choices} \times \underbrace{Third\ Digit}_{\{2,3,5,7\}:\ 4\ Choices} = 4 \times 4 \times 4 = 64\ Numbers$$

Part H
Odd Prime digits are:

$$3,5,7 \Rightarrow 3\ Digits$$
$$\underbrace{First\ Digit}_{\{3,5,7\}:\ 3\ Choices} \times \underbrace{Second\ Digit}_{\{3,5,7\}:\ 3\ Choices} \times \underbrace{Third\ Digit}_{\{3,5,7\}:\ 3\ Choices} = 3 \times 3 \times 3 = 27\ Numbers$$

Part I
Prime digits are:

$$2,3,5,7 \Rightarrow 4\ Digits$$
$$\underbrace{First\ Digit}_{\{2,3,5,7\}:\ 4\ Choices} \times \underbrace{Second\ Digit}_{\{2,3,5,7\}:\ 4\ Choices} \times \underbrace{Third\ Digit}_{\{3,5,7\}:\ 3\ Choices} = 4 \times 4 \times 3 = 48\ Numbers$$

Example 2.87

Find the number of three-digit numbers where the product of the digits is 18.

Let the three-digit number be abc. Then

$$Product\ of\ digits = abc = 18 = 2 \times 3 \times 3$$

We consider cases.

Case I: One of the digits is 1.
Then the other two digits must multiply to 18:
(1,18) does not work
(2,9): Digits are 1,2,9 which can be arranged in $3! = 6$ Ways
(3,6) Digits are 1,3,6 arranged in $3! = 6$ Ways

Case II: None of the digits is 1.
Product of three factors, each greater than 1:

$$(2,3,3)\ Arranged\ 3\ Ways$$

$$Total = 6 + 6 + 3 = 15$$

Example 2.88

How many 3-digit numbers contain the digit 7 exactly once? **(PUMaC 2008-9/Combi/2)**

Consider Cases.
7 in the Hundred's Place

$$1 \times 9 \times 9 = 81$$

7 in the Ten's Place

$$8 \times 1 \times 9 = 72$$

7 in the Unit's Place

$$8 \times 9 \times 1 = 72$$

Total

$$= 81 + 72 + 72 = 225$$

E. Complementary Counting

Example 2.89

The number of 3-digit numbers that contain 7 as at least one of the digits is: (NMTC Primary/Screening/2016/12)

We will use complementary counting.
Total number of 3-digit numbers is:

$$900$$

The number of three digits numbers with no 7 is:

$$\underbrace{8}_{\substack{Hundred's\\Digit}} \times \underbrace{9}_{\substack{Ten's\\Digit}} \times \underbrace{9}_{\substack{Ten's\\Digit}} = 648$$

By complementary counting, the number of numbers with at least one 7

$$= 900 - 648 = 252$$

Example 2.90

A. How many whole numbers between 1 and 1000 do not contain the digit 1? (AMC 8 2009/22)
B. How many whole numbers less than 1000 have the digit 1 at least once?

Part A

We use all digits other the digit 1.

$$\{0,2,3,4,5,6,7,8,9\} \Rightarrow 9\ Digits$$

We are left 9 choices for each place. We would normally exclude zero from the counting, but here we are counting 1-digit, 2-digit and 3-digit numbers, which means we can include zero:

$$\underbrace{Hundred's\ Digit}_{\{0,2,\dots,9\}:\ 9\ Choices} \times \underbrace{Ten's\ Digit}_{\{0,2,\dots,9\}:\ 9\ Choices} \times \underbrace{Unit's\ Digit}_{\{0,2,\dots,9\}:\ 9\ Choices} = 9 \times 9 \times 9 = 729\ Numbers$$

But the 729 number also include 0, which we don't want to count. Hence, the final answer is:

$$729 - 1 = 728$$

Part B

There are 999 numbers in the range

$$1,2, \dots, 999$$

Of these, 728 do not contain the digit 1. Hence, the number of numbers which do contain the digit 1 are

$$999 - 728 = 271$$

F. Exponents

When the number of choices that we get repeats, then the expression that we get when counting is best expressed using exponents.

Example 2.91

Assume every 7-digit whole number is a possible telephone number except those that begin with 0 or 1. What fraction of telephone numbers begin with 9 and end with 0? **(AMC 8 1985/22)**

Total Telephone Numbers:
First Digit cannot be zero or one. Hence, we are left with 8 digits for the first digit.

$$\underbrace{8}_{\text{First Digit}} \times \underbrace{10}_{\text{Second Digit}} \times \underbrace{10}_{\text{Third Digit}} \times \underbrace{10}_{\text{Fourth Digit}} \times \underbrace{10}_{\text{Fifth Digit}} \times \underbrace{10}_{\text{Sixth Digit}} \times \underbrace{10}_{\text{Seventh Digit}} = 8 \times 10^6$$

Eligible Telephone Numbers
Number of telephone numbers that begin with 9, and end with zero:

$$\underbrace{1}_{\text{First Digit}} \times \underbrace{10}_{\text{Second Digit}} \times \underbrace{10}_{\text{Third Digit}} \times \underbrace{10}_{\text{Fourth Digit}} \times \underbrace{10}_{\text{Fifth Digit}} \times \underbrace{10}_{\text{Sixth Digit}} \times \underbrace{1}_{\text{Seventh Digit}} = 10^5$$

$$\text{Required Fraction} = \frac{10^5}{8 \times 10^6} = \frac{1}{80}$$

Shortcut
The middle digits have the same choice in both the fractions. Hence, the comparison is only in the first digit and the last digit, where we have:

$$\frac{1 \times 1}{8 \times 10} = \frac{1}{80}$$

2.5 Counting Words

A. Forming Words

Example 2.92
How many passwords can I frame consisting of three parts such that the first part is a digit from the decimal system, the second part is a letter from the English alphabet, and the third part is a special character from the set $\{!, @, \#, \$, \%, \&, *\}$.

$$\underbrace{10}_{\text{Digit}} \times \underbrace{26}_{\text{Letter}} \times \underbrace{7}_{\text{Special Character}} = 1820$$

Example 2.93[2]
I want to create a secret language. The language uses the letters $\{A, B, C\}$, and every word in the language is a three-letter word. Find the number of possible words if:

A. Repetition is allowed.
B. Repetition is not allowed.
C. Repetition is allowed but the same letter cannot be repeated thrice.

$$3 \times 3 \times 3 = 27$$
$$3 \times 2 \times 1 = 6$$

Complementary Counting
The words where the letters are repeated thrice are:

$$AAA, BBB, CCC \rightarrow 3\ Options$$
$$27 - 3 = 24$$

Example 2.94: Morse Code

[2] This is a comparative example. Parts A, B and C denote different conditions that can be applied on the question. Make you understand the difference.

Morse code is used in telecommunication and sends signals by means of combination of "dots" and "dashes". If you want to encode the letters of the English alphabet, and the digits in the decimal system using a word made of upto n dots/dashes, what is the minimum value of n?

$$Total\ No.of\ values = 26\ Letters + 10\ Digits = 36$$

The words I can make with length 1 are:

$$\{Dot, Dash\} = 2^1 = 2$$

The words I can make with length 2 are:

$$\{Dot - Dot, Dash - Dash, Dot - Dash, Dash - Dot\} = 2^2 = 4$$

The words I can make with length 3 are:

$$2^3 = 8$$

The words I can make with length 4 are:

$$2^4 = 16$$

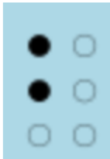
The words I can make with length 5 are:

$$2^5 = 32$$

The total is:

$$2 + 4 + 8 + 16 + 32 = 62$$

Example 2.95: Braille Language



A. Braille is a tactile language, which has six dots in a 3×2 pattern (three rows and two columns). Every dot is either raised, or "non-raised". All letters consists of all six dots. For example, the diagram alongside represents two raised dots as black, and four regular dots as white. How many "letters" can you make in the Braille language using the above pattern?
B. The language on which Braille was based used 12 dots. If the same raised/non-raised pattern is used, how many letters can you make using 12 dots?

Part A

$$\underbrace{2}_{\substack{First\\Dot}} \times \underbrace{2}_{\substack{Second\\Dot}} \times \ldots \times \underbrace{2}_{\substack{Sixth\\Dot}} = 2^6 = 64$$

Part B

$$2^{12} = 4096$$

Example 2.96

Answer each part separately

A code maker uses the Greek Letters, Alpha(α), Beta(β), Gamma(γ), Delta (Δ), and Chi(χ) to write a code that consists of 4 letters. Find the number of possibles codes if:

A. There are no restrictions.
B. Every letter in the code must be unique.
C. The same letter cannot be repeated four times.

Part A

$$\underbrace{5}_{\substack{First\\Letter}} \times \underbrace{5}_{\substack{Second\\Letter}} \times \underbrace{5}_{\substack{Third\\Letter}} \times \underbrace{5}_{\substack{Fourth\\Letter}} = 5^4 = 625$$

Part B

$$\underbrace{5}_{\substack{First\\Letter}} \times \underbrace{4}_{\substack{Second\\Letter}} \times \underbrace{3}_{\substack{Third\\Letter}} \times \underbrace{2}_{\substack{Fourth\\Letter}} = 20 \times 6 = 120$$

Part C

$$\underbrace{625}_{\substack{Total\\Choices}} - \underbrace{5}_{\substack{Repeated\\Four\ Times}} = 620$$

Example 2.97

Answer each part separately

A language has exactly four words kay, kya, aky, and yka. Each sentence in the language consists of four words. Find the number of possible sentences if:

A. the same word cannot be immediately repeated. For example, $kay\ kya\ kay\ kya$ is a valid sentence but $kay\ kay\ kya\ yka$ is not.
B. Sentences must be written using words in either ascending order (alphabetically), or descending order (alphabetically).
C. If every word can be written in either capital letters (KAY) or small letters (kay), and the two are considered different.
D. If every letter can be written in either capital letters (K) or small letters (k), and kAy is different from kay.

Part A

$$\underbrace{4}_{\substack{First\\Word}} \times \underbrace{3}_{\substack{Second\\Word}} \times \underbrace{3}_{\substack{Third\\Word}} \times \underbrace{3}_{\substack{Fourth\\Word}} = 108\ Sentences$$

Part B

If you arrange the words in ascending order, you get one sentence:

$$aky, kay, kya, yka$$

If you arrange the words in descending order, you get one sentence:

$$yka, kya, kay, aky$$

The total number is:

$$1 + 1 = 2$$

Part C

You get 8 choices for each word:

$$\underbrace{8}_{\substack{First\\Word}} \times \underbrace{8}_{\substack{Second\\Word}} \times \underbrace{8}_{\substack{Third\\Word}} \times \underbrace{8}_{\substack{Fourth\\Word}} = 8^4 = 2^{12} = 4096$$

Part D

Each word can be written in

$$\underbrace{2}_{\substack{First\\Letter}} \times \underbrace{2}_{\substack{Second\\Letter}} \times \underbrace{2}_{\substack{Third\\Letter}} = 8\ Ways$$

Since every word given in the question can be written in 8 ways, the number of "words" that we have is:

$$4 \times 8 = 32\ Words$$

Hence, the number of sentences is:

$$32 \times 32 \times 32 \times 32 = 32^4 = (2^5)^4 = 2^{20}$$

Example 2.98

Answer each part separately

How many 5-words sentences can I make in a language that consists of exactly three words: "brillig", "slithy", and "toves" if:

A. There are no restrictions.
B. A sentence cannot consist of the same word repeated five times.

Part A

$$\underbrace{3}_{\substack{First\\ Word}} \times \underbrace{3}_{\substack{Second\\ Word}} \times \underbrace{3}_{\substack{Third\\ Word}} \times \underbrace{3}_{\substack{Fourth\\ Word}} \times \underbrace{3}_{\substack{Five\\ Word}} = 3^5 = 243$$

Part B

The number of sentences with the same word repeated five times are:

$$brillig, brillig, brillig, brillig, brillig$$
$$slithy, slithy, slithy, slithy, slithy$$
$$toves, toves, toves, toves, toves$$

Using complementary counting, the number of possible sentences is:

$$\underbrace{243}_{\substack{Total\\ Sentences}} - \underbrace{3}_{\substack{Repeated\\ Five\ Times}} = 240$$

Example 2.99

Find the number of words (not necessarily meaningful) made from letters in the English Alphabet if:

A. Each word has two letters, and repetition is allowed
B. Each word has two letters, and repetition is not allowed
C. Each word has two letters, of which one is a vowel, and the other is a consonant.

Part A

I have twenty-six choices for the first letter. Similarly, I have twenty-six choices for the second letter. The choices are independent. That is, I can mix the choices in any way I want.

Hence, the number of words I can make is:

$$\underbrace{26}_{First\ Letter} \times \underbrace{26}_{Second\ Letter} = 676$$

Part B

I have twenty-six choices for the first letter. Similarly, I have twenty-six choices for the second letter, but I cannot the repeat the letter which was used first, and hence the number of choices comes down to 25.

The choices are independent. That is, I can mix the choices in any way I want

$$\underbrace{26}_{First\ Letter} \times \underbrace{25}_{Second\ Letter} = 650$$

Part C

First, we need to decide the order of the vowel and the consonant. There are two choices:

- Vowel followed by consonant
- Consonant followed by vowel

And we need to find the number of choices in each:

$$\underbrace{5 \times 21}_{Vowel+Consonant} + \underbrace{21 \times 5}_{Consonant+Vowel} = 105 + 105 = 210$$

Example 2.100

How many two-letters words in the English language made of consonants have at least one B. (For this question, consider Y to be a vowel).

Complementary Counting

$$Total\ Words - Words\ with\ No\ B = 20^2 - 19^2 = 400 - 361 = 39$$

Direct Counting

First Letter is a B

$$1 \times 20 = 20$$

Second Letter is a B

$$20 \times 1 = 20$$

Both Letters are B

$$1 \times 1 = 1$$

$$20 + 20 - 1 = 40 - 1 = 39$$

Example 2.101

In a certain land, they really like the number four. The alphabet has only four letters, and their words have a maximum length of four. Since they have so few choices, they make their letters work really hard. They use all the words that can be made. How many words can be made?

We can divide this into four cases:

- Words of Length 1
- Words of Length 2
- Words of Length 3
- Words of Length 3

Add up the number of words in each to get the total number of words.

$$\underbrace{4}_{Length:1} + \underbrace{4^2}_{Length:2} + \underbrace{4^3}_{Length:3} + \underbrace{4^4}_{Length:4} = 4 + 16 + 64 + 256 = 340$$

Example 2.102

Phi(ϕ), Tau(τ), and Chi(χ) are three letters of the Greek alphabet. How many (not necessarily meaningful) words of maximum length four can you make which use the letter ϕ at least once?

Complementary Counting

We will count the total number of words, and from that We will subtract the words which do not use the letter ϕ even once.

However, the question asks for words of maximum length four, and hence, we need to break the problem down into cases.

Casework with Complementary Counting

$$Words\ with\ \phi = Total\ Words - Words\ with\ no\ \phi$$

Length of Word	Total Words	Words with No ϕ	Words with at least one ϕ
1	3	2	1
2	$3^2 = 9$	$2^2 = 4$	5
3	$3^3 = 27$	$2^3 = 8$	19

4	$3^4 = 81$	$2^4 = 16$	65
	120	30	90

B. License Plates and Codes

Example 2.103: License Plates

Bicycle license plates in Flatville each contain three letters. The first is chosen from the set {C,H,L,P,R}, the second from {A,I,O}, and the third from {D,M,N,T}. When Flatville needed more license plates, they added two new letters. The new letters may both be added to one set or one letter may be added to one set and one to another set. What is the largest possible number of ADDITIONAL license plates that can be made by adding two letters? **(AMC 8 1999/8)**

$$Current\ No. of\ License\ Plates = 5 \times 3 \times 4 = 60$$

Now, to maximize the increase in the new license scheme, we need to make numbers as close as possible. Hence, we get

$$\{5,3,4\} \to \{5,4,4\} \to \{5,5,4\}$$

Hence, the number of license plates in the new scheme

$$5 \times 5 \times 4 = 100$$

Additional license plates

$$= 100 - 60 = 40$$

Example 2.104: Codes

An online meeting code has the form $\underbrace{XXX - XXXX - XXX}_{Example:\ yqk-wkfq-rsc}$, where each X is a non-capitalized letter from the English alphabet, and the hyphens are found in all codes. $\underbrace{Letters\ can\ be\ repeated}_{Repetition\ Allowed}$. An alternate version of the code has the format $XXXXX$, where X can also be a digit. Find in simplest form, the ratio of the number of codes possible in the base version and the alternate version.

Main Version: $\underbrace{XX \dots X}_{10\ X's:\ 26\ Choices\ each} = \underbrace{26 \times 26 \times \dots \times 26}_{10\ times} = 26^{10}$

Alternate Version: $\underbrace{XX \dots X}_{5\ X's:\ 36\ Choices\ each} = \underbrace{36 \times 36 \times \dots \times 36}_{5\ times} = 36^5$

$$\frac{26^{10}}{36^5} = \frac{2^{10} \times 13^{10}}{4^5 \times 9^5} = \frac{2^{10} \times 13^{10}}{2^{10} \times 9^5} = \frac{13^{10}}{9^5}$$

2.6 Difficult Questions on Counting

A. Multiple Restrictions

Certain questions will impose multiple conditions on what is to be counted. Such questions have to be handled carefully. They may be more difficult when done one way, and much easier when done another way. The best way to learn how to do such questions is by practice.

2.105: Multiple Restrictions

When encountering multiple restrictions, it often makes sense to handle the most restrictive condition first.

- This is a suggestion, not a rule.

Example 2.106

How many four-digit whole numbers are there such that the leftmost digit is odd, the second digit is even, and all four digits are different? **(AMC 8 1995/23)**

$$Leftmost\ digit = Odd = 5\ Numbers$$
$$Second\ Digit\ = Even = 5\ Numbers$$
$$Third\ Digit = 10 - 2 = 8$$
$$4^{th}\ Digit = 10 - 3 = 8$$
$$\underbrace{5}_{\substack{Thousands\\Digit}} \times \underbrace{5}_{\substack{Hundreds\\Digit}} \times \underbrace{8}_{\substack{Tens\\Digit}} \times \underbrace{7}_{\substack{Units\\Digit}} = 200 \times 7 = 1400$$

Example 2.107

A play has two male roles, two female roles and two roles that can be either gender. Only a man can be assigned to a male role, and only a woman can be assigned to a female role. If five men and six women audition, in how many ways can the six roles be assigned? **(MathCounts Chapter Sprint 2005/29)**

If we choose the gender-neutral roles first, there will be cases to consider based on number of males and females chosen. Hence, we pick the gender specific roles first, and avoid the nasty casework.

The number of ways to pick the two male roles from five available male auditioners is:

$$5 \times 4 = 20$$

The number of ways to pick the two female roles from six available female auditioners is:

$$6 \times 5 = 30$$

The number of ways to pick two gender neutral roles from remaining $(5-2)+(6-2)=3+4=7$ auditioners is

$$7 \times 6 = 42$$

Hence, the final answer is:

$$20 \times 30 \times 42 = 25200\ ways$$

Example 2.108

How many ways are there to split the integers 1 through 14 into 7 pairs such that in each pair, the greater number is at least 2 times the lesser number? **(AMC 10A 2022/14)**

Numbers from 8 through 14 cannot be paired with each other.
Numbers from 1 through 6 must pair with numbers from 8 through 14.

We do this from most restrictive to least restrictive.

Step I: 7 must pair with 14.

$$Exactly\ 1\ way$$

Step II: 6 can pair with 12 or 13

$$2\ ways$$

Step III: 5 can pair with {10,11,12,13,14}. But 14 is not available. And one of 12 or 13 is also not available.

$$(5-2) = 3\ ways$$

Step IV: {1,2,3,4} can pair with {8,9,10,11,12,13,14}. But 14, and two others are not available. Hence, the numbers available are

$$7-3=4$$

Since there are no restrictions, you can pair {1,2,3,4} with four numbers from {8,9,10,11,12,13,14} in

$$4!\ ways$$

The final answer, by the multiplication principle, is:

$$1 \cdot 2 \cdot 3 \cdot 4! = 144$$

B. Complementary Counting

Challenge 2.109

$\boxed{36}$ $\boxed{3}$ $\boxed{1}$ and $\boxed{61}$ are four physical blocks. Using these four blocks, how many six-digit numbers can be formed? **(NMTC Final/Primary 2005/4)**

We can arrange these four blocks in:

$$\underbrace{4}_{First\ Block} \times \underbrace{3}_{Second\ Block} \times \underbrace{2}_{Third\ Block} \times \underbrace{1}_{Fourth\ Block} = 24\ ways$$

Out of the 24 arrangements, two arrangements will give the same number:

$$\boxed{36}\ \boxed{1}\boxed{3}\ \boxed{61} \quad \text{and} \quad \boxed{3}\boxed{61}\boxed{36}\boxed{1}$$

Hence, the total number of six-digit numbers is:

$$24-1=23$$

C. Casework

In general, questions which need to be handled using cases require far more effort and care then regular questions. Some things to take care of:

- Cases need to be identified carefully
- Cases should not overlap (*mutually exclusive*).
- Cases should together cover the entire requirement that you want to count (*collectively exhaustive*).

Example 2.110

How many three-digit even numbers greater than 300, with distinct digits, can be formed from the set {0,1,3,5,8}.

Step I: First Digit

Since the number is greater than 300, the first digit cannot be zero or one. Hence, the first digit must be one out of

$$3, 5, or\ 8$$

Step II: Last Digit

The possible numbers for the last digit are 0 and 8.
At this stage, it is important to recognize that if 8 is the first digit, then it cannot be the last digit.
Hence, we need to consider two different cases:

Case I: First Digit is 8
Last digit must be 0.
3 choices for middle digit: 1,3,5

$$810,830,850$$

Case II: First Digit is Odd
Last digit can be 0 or 8: 2 Choices
Middle Digit: 3 Choices

$$\text{Total Choices} = \underbrace{2}_{\substack{\text{First}\\\text{Digit}}} \times \underbrace{3}_{\substack{\text{Middle}\\\text{Digit}}} \times \underbrace{2}_{\substack{\text{Last}\\\text{Digit}}} = 12$$

The total number of choices is

$$= 12 + 3 = 15$$

Example 2.111

How many whole numbers between 99 and 999 contain exactly one 0? (AMC 8 2002/19)

Method I

Numbers greater than 99 and less than 999 must be three digits numbers. The zero cannot be in the hundred's place. Hence, we consider two cases:

$$\text{Zero in Units Digit}: \underbrace{9}_{\substack{\text{Hundreds}\\\text{Digit}}} \times \underbrace{9}_{\substack{\text{Tens}\\\text{Digit}}} = 81$$

$$\text{Zero in Tens Digit}: \underbrace{9}_{\substack{\text{Hundreds}\\\text{Digit}}} \times \underbrace{9}_{\substack{\text{Units}\\\text{Digit}}} = 81$$

$$\text{Total} = 81 + 81 = 162$$

Method II

We can also subtract the numbers with no zeros, and the numbers with two zeros from the total number of three-digit numbers:

$$\underbrace{900}_{\substack{\text{Three}\\\text{Digit Numbers}}} - \underbrace{9^3}_{\substack{\text{No}\\\text{Zeroes}}} - \underbrace{9}_{\substack{\text{Two}\\\text{Zeroes}}} = 900 - 729 - 9 = 162$$

Note that the number of numbers between 99 and 999 is actually:

$$998 - 100 + 1 = 898 + 1 = 899$$

While we took 900, but this does not affect our answer since we subtracted out the 999 when removing the complement.

Example 2.112

The number of nonnegative integers which are less than 1000, and end with only one zero is: (NMTC Sub-Junior/Screening 2004/8)

$$\text{Ten's Digit is Not Zero}: \underbrace{10}_{\substack{\text{Hundred}\\\text{Digit}}} \times \underbrace{9}_{\substack{\text{Ten's}\\\text{Digit}}} \times \underbrace{1}_{\substack{\text{Units}\\\text{Digit}}} = 10 \times 9 \times 1 = 90$$

$$\text{Ten's Digit is Zero}: \text{Only One Number}: 0$$

$$\text{Total} = 90 + 1 = 91$$

Complementary Counting

We want to the last digit to be a zero. Consider the numbers less than 1000 that end with a zero:

$$0,10,20,30, \dots, 990$$

If we consider only the first two digits we get:

$$0,1,2,\ldots,99 \Rightarrow 100\ Numbers$$

But we do not want the numbers with first two digits:

$$10,20,\ldots,90 \Rightarrow 9\ Numbers$$

And hence the final answer is

$$100 - 9 = 91$$

Example 2.113

(MathCounts Chapter Sprint 2019/26)

In the grid shown, how many ways are there to spell the word "QUEUE" by moving one square at a time either horizontally or vertically, and provided squares may be revisited?

E	U	E	U	E
U	E	U	E	U
E	U	Q	U	E
U	E	U	E	U
E	U	E	U	E

Q can be achieved only by starting at the center of the grid. Hence,
QU can be achieved in
4 ways

Because of the symmetry, we consider only the top U, and multiply with the above.
Up: 3 choices for QUEU, then 3 choices for E. Total choices: 3\bullet3=9
Left: 4 choices for QUEU, then 3 choices for E. Total choices: 4\bullet3=12
Right: (Same as Left above) Choices: 12

Adding up the cases:
=9+12+12=33

The final answer is:
=4(33)=132

D. Divisibility

Example 2.114

The total number of possible proper three-digit integers that can be formed using 0,1,3,4 and 5 without repetition such that they are divisible by 5 are: (JMET 2011/69)

Since the number must be divisible by 5, the last digit must be either 5 or 0.

Case I: The last digit is 0:

$$4 \times 3 \times 1 = 12$$

Case II: The last digit is 5:

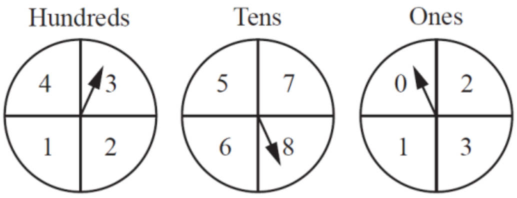
$$3 \times 3 \times 1 = 9$$

Total

$$= 12 + 9 = 21$$

Example 2.115

Three spinners are shown. The spinners are used to determine the hundreds, tens and ones digits of a three-digit number. How many possible three-digit numbers that can be formed in this way are divisible by 6? (Gauss 8 2020/22)



Number is divisible by 6 if and only if it is divisible by 2 and 3. Valid one's digits are only 0 and 2.

If the one's digit is 2, then the other two digits should be two less than a multiple of 3:

$$Total = 7 = (1,6), (2,5)$$
$$Total = 10 = (2,8), (3,7), (4,6)$$
$$5\ Numbers$$

If one's digit is 0, then the other two numbers have a total which is a multiple of 3:

$$Total = 6 = (1,5)$$
$$Total = 9 = (1,8), (2,7), (3,6), (4,5)$$
$$Total = 12 = (4,8)$$
$$6\ Numbers$$

E. Palindromes

A palindrome is a word that reads the same backwards and forwards. A palindrome can be a number also. For example:

- 22 is a two-digit palindrome
- 343 is a three-digit palindrome
- 3553 is a four-digit palindrome

Warmup 2.116

Find the:

1. Largest three-digit palindrome that does not have all digits the same
2. Smallest four-digit palindrome that does not have all digits the same

$$989$$
$$1001$$

Example 2.117

Determine the form that palindromes from one to five digits must have (answer for each separately). Use letters for digits.

$$\underbrace{a}_{\substack{\text{One Digit}\\ \text{Palindrome}}}, \underbrace{aa}_{\substack{\text{Two Digit}\\ \text{Palindrome}}}, \underbrace{aba}_{\substack{\text{Three digit}\\ \text{Palindrome}}}, \underbrace{abba}_{\substack{\text{Four digit}\\ \text{Palindrome}}}, \underbrace{abcba}_{\substack{\text{Five digit}\\ \text{Palindrome}}}$$

Where

$$a, b, c \text{ are digits}$$

Example 2.118

How many palindromes one or greater are there which are:

A. One Digit
B. Two Digits
C. Three Digits
D. Four Digits
E. Five Digits

One Digit

There is no restriction on numbers of one digit. They are all palindromes.

$$\underbrace{9}_{\substack{\text{Units}\\ \text{Digit}}} = 9$$

Two Digit

Two-digit palindromes are of the form

$$aa$$

The units digit has to be the same as the ten's digit. Hence, we only have a choice in the ten's digit.

$$\underbrace{9}_{\substack{\text{Tens}\\ \text{Digit}}} \times \underbrace{1}_{\substack{\text{Units}\\ \text{Digit}}} = 9$$

Three Digit

Three-digit palindromes are of the form

$$aba$$

Once we choose the hundred's digit, the one's digit is automatically chosen, leaving us with no choices there. We can choose any digit from 0-9 in the ten's digit, giving us ten choices.

$$\underbrace{9}_{\substack{\text{Hundreds}\\ \text{Digit}}} \times \underbrace{10}_{\substack{\text{Tens}\\ \text{Digit}}} \times \underbrace{1}_{\substack{\text{Units}\\ \text{Digit}}} = 9 \times 10 = 90$$

Four Digit

Four-digit palindromes are of the form

$$abba$$

Once we choose the thousand's digit, the one's digit is automatically chosen, leaving us with no choices there. We can choose any digit from 0-9 in the hundred's digit, which leaves us with no choices in the ten's digit.

$$\underbrace{9}_{\substack{\text{Thousands}\\ \text{Digit}}} \times \underbrace{10}_{\substack{\text{Hundreds}\\ \text{Digit}}} \times \underbrace{1}_{\substack{\text{Tens}\\ \text{Digit}}} \times \underbrace{1}_{\substack{\text{Units}\\ \text{Digit}}} = 90$$

Part E: Five Digits

$$\underbrace{9}_{\substack{\text{Ten Thousands}\\ \text{Digit}}} \times \underbrace{10}_{\substack{\text{Thousands}\\ \text{Digit}}} \times \underbrace{10}_{\substack{\text{Hundreds}\\ \text{Digit}}} \times \underbrace{1}_{\substack{\text{Tens}\\ \text{Digit}}} \times \underbrace{1}_{\substack{\text{Units}\\ \text{Digit}}} = 900$$

2.119: Number of n digit Palindromes

$$9 \times 10^{\left\lceil \frac{No.of\ Digits}{2} \right\rceil - 1}$$

Where $\lceil x \rceil$ represents the ceiling function, which is the smallest integer less than or equal to x.

Example 2.120

Find the number of

A. Five-digit palindromes
B. Six-digit palindromes
C. Nine-digit palindromes
D. 100-digit Palindromes
E. 101-Digit Palindromes

$$9 \times 10^{\left\lceil \frac{5}{2} \right\rceil - 1} = 9 \times 10^{\lceil 2.5 \rceil - 1} = 9 \times 10^{3-1} = 9 \times 10^2 = 9 \times 100 = 900$$

$$9 \times 10^{\left\lceil \frac{6}{2} \right\rceil - 1} = 9 \times 10^{\lceil 3 \rceil - 1} = 9 \times 10^{3-1} = 9 \times 10^2 = 9 \times 100 = 900$$

$$9 \times 10^{\left\lceil \frac{9}{2} \right\rceil - 1} = 9 \times 10^{\lceil 4.5 \rceil - 1} = 9 \times 10^{5-1} = 9 \times 10^4 = 9 \times 10000 = 90000$$

$$9 \times 10^{\left\lceil \frac{100}{2} \right\rceil - 1} = 9 \times 10^{\lceil 50 \rceil - 1} = 9 \times 10^{49}$$

$$9 \times 10^{\left\lceil \frac{101}{2} \right\rceil - 1} = 9 \times 10^{\lceil 50.5 \rceil - 1} = 9 \times 10^{51-1} = 9 \times 10^{50}$$

Example 2.121

How many three-digit multiples of five are palindromes?

If the number is a multiple of 5, the last digit must be

$$0 \text{ or } 5$$

But, the first digit must be the same as the last digit. Hence, the last digit cannot be 0.

Hence, the number must be of the form

$$5X5 \rightarrow \text{where } X \text{ is some digit}$$

We can put 10 digits in place of X, giving us 10 numbers:

$$505,515,\ldots,595 \Rightarrow 10\ Numbers$$

Answer the above question for four digits.

As above, the last digit has to be 5.
Also, the moment we choose the hundred's digit, the ten's digit is automatically chosen.

Hence, the numbers must be:

$$5005,5115,\ldots 5995 \Rightarrow 10\ Numbers$$

Example 2.122

What is the number of even three-digit palindromes?

If the number is a multiple of 2, the last digit must be

$$0, 2, 4, 6, 8$$

But, the first digit must be the same as the last digit. Hence, the last digit cannot be 0.

Hence, we must have:

$$\underbrace{1}_{\substack{\textit{Hundreds}\\ \textit{Digit}}} \times \underbrace{10}_{\substack{\textit{Tens}\\ \textit{Digit}}} \times \underbrace{4}_{\substack{\textit{Units}\\ \textit{Digit}}} = 10 \times 4 = 40$$

Answer the above question for four digits.

If the number is a multiple of 2, the last digit must be

$$0, 2, 4, 6, 8$$

But, the first digit must be the same as the last digit. Hence, the last digit cannot be 0.

Also, the moment we choose the hundred's digit, the ten's digit is automatically chosen.

Hence, we must have:

$$\underbrace{1}_{\substack{\textit{Thousands}\\ \textit{Digit}}} \times \underbrace{10}_{\substack{\textit{Hundreds}\\ \textit{Digit}}} \times \underbrace{1}_{\substack{\textit{Tens}\\ \textit{Digit}}} \times \underbrace{4}_{\substack{\textit{Units}\\ \textit{Digit}}} = 10 \times 4 = 40$$

Example 2.123

A palindrome is a number that reads the same forward and backwards. How many positive 3-digit palindromes are multiples of 3? **(Mathcounts 2003 Warm-up 7)**

Strategy

A three-digit palindrome must be of the form:

$$aba$$

We do this using casework on the first digit, since deciding the first digit also decides the last digit.

Casework

	Sum of first digit and last digit	Possible Middle Digits	Numbers	No. of Numbers	
1_1	2	1,4,7	111 141 171	3	
2_2	4	2,5,8	222 252 282	3	
3_3	6	0,3,6,9	303 333 363 393	4	10
4_4	8	1,4,7	414 444 474	3	
5_5	10	2,5,8	525 555 585	3	
6_6	12	0,3,6,9	606 636	4	10

			666 696		
7_7	14	1,4,7	717 747 777	3	
8_8	16	2,5,8	828 858 888	3	
9_9	18	0,3,6,9	909 939 969 999	4	10

Example 2.124

A palindrome is a number that reads the same forward and backwards. How many positive 3-digit palindromes are multiples of 3? **(MathCounts 2003 Warm-up 7)**

Based on the logic above, we can present the calculation in much shorter fashion as:

	Possible Middle Digits	No. of Numbers	No. of Cases	
1_1 4_4 5_5	1,4,7	3	3	9
2_2 7_7 8_8	2,5,8	3	3	9
3_3 6_6 9_9	0,3,6,9	4	3	12

Example 2.125

How many palindromes are there, which:

A. Have three digits, all of which are odd
B. Have four digits, all of which are prime
C. Have five digits, all digits are prime, and the number itself is odd

Part A

Choices for the hundred's digit:

$$1,3,5,7,9 \Rightarrow 5\ Choices$$

$$\underbrace{5}_{\substack{Hundreds\\Digit}} \times \underbrace{5}_{\substack{Tens\\Digit}} \times \underbrace{1}_{\substack{Units\\Digit}} = 25$$

Part B

Choices for prime digit:

$$2,3,5,7 \Rightarrow 4\ Choices$$

$$\underbrace{4}_{\substack{Thousands\\Digit}} \times \underbrace{4}_{\substack{Hundreds\\Digit}} \times \underbrace{1}_{\substack{Tens\\Digit}} \times \underbrace{1}_{\substack{Units\\Digit}} = 4 \times 4 = 16$$

Part C

Choices for odd prime digit:

$$3,5,7 \Rightarrow 3\ Choices$$

Choices for prime digit:

$$2,3,5,7 \Rightarrow 4\ Choices$$

$$\underbrace{3}_{\substack{\textit{Ten Thousands}\\ \textit{Digit}}} \times \underbrace{4}_{\substack{\textit{Thousands}\\ \textit{Digit}}} \times \underbrace{4}_{\substack{\textit{Hundreds}\\ \textit{Digit}}} \times \underbrace{1}_{\substack{\textit{Tens}\\ \textit{Digit}}} \times \underbrace{1}_{\substack{\textit{Units}\\ \textit{Digit}}} = 48$$

F. Applications of Palindromes

Palindromes in real life scenarios can have restrictions on their value.

A watch will only have certain digits used for time:

- if XY is the number of minutes on a clock, X will never 6 or greater.
- if AB is the number of hours on a clock, A will always be either zero or one.

Example 2.126

A palindrome is a whole number that reads the same forwards and backwards. If one neglects the colon, certain times displayed on a digital watch are palindromes. Three examples are: 1:01, 4:44, and 12:21. How many times during a 12-hour period will be palindromes? (AMC 8 1988/25)

The question does not specify whether the clock is a 12-hour clock, or a 24-hour clock. However, since it says 12-hour period, the answer for all 12-hours periods should be the same, and hence, we consider the time

$$from\ 00{:}00\ to\ 11{:}59$$

Case I: *aba*

The units digit(hours) can take values from 1 to 9, giving us nine choices. This automatically decides the unit's digit for the minutes.

The maximum value that the minutes can take is 59. Hence, the ten's digit can take values from 0 to 5, giving 6 choices.

We can combine the above two to get:

$$\underbrace{9}_{\substack{\textit{Units Digit}\\ \textit{Hours}}} \times \underbrace{6}_{\substack{\textit{Ten's Digit}\\ \textit{Minutes}}} \times \underbrace{1}_{\substack{\textit{Unit's Digit}\\ \textit{Minutes}}} = 9 \times 6 = 54$$

Case II: *abba*

This case has more restrictions.

The only value that the ten's digits for the hours can take is 1, giving us one choice. Automatically, the unit's digit (minutes) is chosen.

The unit's digit (hours) can take values $\{0,1,2\}$ giving us three choices.

We can combine the above to get:

$$\underbrace{1}_{\substack{\textit{Tens Digit}\\ \textit{Hours}}} \times \underbrace{3}_{\substack{\textit{Units Digit}\\ \textit{Hours}}} \times \underbrace{1}_{\substack{\textit{Ten's Digit}\\ \textit{Minutes}}} \times \underbrace{1}_{\substack{\textit{Unit's Digit}\\ \textit{Minutes}}} = 3$$

$\text{Total} = 54 + 3 = 57$

In the previous question, in Case I, explain why we did not consider Unit's Digit zero, which would give us the following five cases like

$$00{:}10, \quad 00{:}20, \quad 00{:}30, \quad 00{:}40, \quad 00{:}50$$

0010 (and the other numbers in the list above) are not palindromes.

G. Counting Digits

Example 2.127

In how many numbers does the digit 2 appear in the page numbers of the first ninety-nine pages of a book?

Method I: Enumeration

$(0-9)$: 10 *First Digits*								Total
	10	20	.	.	.	90		
1	11	21				91		
2	12	22	32	42	.	92	10	
.	.	.				.		
.	.	.				.		
9	19	29				99		
1	1	10	.	.	.	1		19

Method II: Multiplication Principle

$$No.\,of\ Digits: \underbrace{10\ Choices\ for\ Units\ Digit}_{\mathbf{2\ in\ the\ Tens\ Digit}} + \underbrace{10\ Choices\ for\ Tens\ Digit}_{\mathbf{2\ in\ the\ Units\ Digit}} = 10 + 10 = 20$$

$$No.\,of\ Numbers = \underbrace{20}_{\mathbf{No.of\ Digits}} - 1 = 19$$

2.128: Numbers versus Digits

Counting numbers is different from counting digits.

$$11 \Rightarrow Digit\ 1\ occurs\ twice$$
$$11 \Rightarrow 1\ Number\ with\ the\ digit\ 1$$

Example 2.129

How many times does the digit 2 appear in the first hundred page numbers when numbering a book?

We use the digit 2 twice in 22. So, the answer is:

$$19 + 1 = 20$$

Example 2.130

How many whole numbers between 100 and 400 contain the digit 2? **(AMC 8 1985/15)**

Method I: Casework

$$\underbrace{19}_{\mathbf{From\ 1-100}} + \underbrace{100}_{\mathbf{From\ 200\ to\ 299}} + \underbrace{19}_{\mathbf{From\ 300\ to\ 399}} = 138$$

Method II: Multiplication Principle

$$\text{Nos. from 100 till 399 without 2:} \quad \underbrace{2}_{\substack{\mathbf{Hundred's\ Digit} \\ \mathbf{1,3}}} \times \underbrace{9}_{\mathbf{Ten's\ Digit}} \times \underbrace{9}_{\mathbf{Unit's\ Digit}} = 2 \times 9 \times 9 = 162$$

Subtract 1 from the numbers from 100 till 399 because we are not counting 100:

$$162 - 1 = 161$$

Using complementary counting:

$$\text{Numbers with } 2 = 299 - 161 = 138$$

Example 2.131

Pat Peano has plenty of 0's, 1's, 3's, 4's, 5's, 6's, 7's, 8's and 9's, but he has only twenty-two 2's. How far can he number the pages of his scrapbook with these digits? (AMC 8 1993/22)

To write the numbers from $1 - 100$ will need

$$20 \; Digits$$

And we write:

$$\underbrace{102}_{21st\ 2}, \underbrace{112}_{22nd\ 2}$$

And then our 2's are over,but we can keep writing till we need a 2, which means we can write till

$$119$$

Example 2.132

How many times does the digit 1 appear when you write numbers from 1 to 399 consecutively. (NMTC Primary/Screening/2016/9)

$$1 - 99 \Rightarrow 20 \; Digits$$
$$100 - 199 \Rightarrow 100 + 20 \; Digits$$
$$200 - 299 \Rightarrow 20 \; Digits$$
$$300 - 399 \Rightarrow 20 \; Digits$$

$$Total = 20 + 120 + 20 + 20 = 180 \; Digits$$

Example 2.133

The sum of all the digits of the integers from 98 to 101 is

$$9 + 8 + 9 + 9 + 1 + 0 + 0 + 1 + 0 + 1 = 38$$

The sum of all of the digits of the integers from 1 to 2008 is (Gauss 7/2008/25)

Find the sum of the digits for $2000 - 2008$:

$$2000 + 2001 + 2002 + \cdots 2008 \Rightarrow 2(9) + (0 + 1 + 2 + \cdots + 8) = 18 + 36 = 54$$

Find the sum of the digits for $0 - 1999$:

$$Thousands \; Digit: (0 + 1)(1000) = 1000$$
$$Hundreds \; Digit: (0 + 1 + \cdots + 9)(200) = (45)(200)$$
$$Tens \; Digit: (0 + 1 + \cdots + 9)(200) = (45)(200)$$
$$Ones \; Digit: (0 + 1 + \cdots + 9)(200) = (45)(200)$$

$$Total = 54 + 1000 + (3)(45)(200) = 28054$$

H. Counting Rational Numbers

Challenge 2.134

Given a rational number, write it as a fraction in lowest terms and calculate the product of the resulting numerator and denominator. For how many rational numbers between 0 and 1 will 20! be the resulting product? (AIME 1991/5)

We want a fraction $\frac{x}{y}$ in lowest form such that:

$$x \times y = 20! = 2^a \times 3^b \times 5^c \times 7^d \times 11^e \times 13^f \times 17^g \times 19^h$$

Hence,

$$x = \frac{2^a \times 3^b \times 5^c \times 7^d \times 11^e \times 13^f \times 17^g \times 19^h}{y}$$

Note that since $\frac{x}{y}$ is in lowest terms, they cannot have any common factor.
Hence, if a single 2 is in x, then 2^a must be in x.

In other words, we need to divide the set $\{2,3,5,7,11,13,17,19\}$ into two parts. The first part will be in x, and the other part will be in y.

By the multiplication principle, this can be done in:

$$\underbrace{2}_{\substack{Choices \\ for\ 2}} \times \underbrace{2}_{\substack{Choices \\ for\ 3}} \times \ldots \times \underbrace{2}_{\substack{Choices \\ for\ 19}} = 2^8$$

But note that we also want:

$$0 < \frac{x}{y} < 1$$

And this will be true for exactly half of the 2^8 choices.
Hence, the final answer is:

$$\frac{2^8}{2} = 2^7 = 128$$

I. Cryptarithmetic

Challenge 2.135

A palindrome is a positive integer whose digits are the same when read forwards or backwards. For example, 2882 is a four-digit palindrome and 49194 is a five-digit palindrome. There are pairs of four-digit palindromes whose sum is a five-digit palindrome. One such pair is 2882 and 9339. How many such pairs are there? **(CEMC Cayley 2001/24)**

Let the four-digit palindromes be $abba$ & $cddc$. Adding them should give a five-digit palindrome (say $xyzyx$). So, we get the addition below:

	a	b	b	a
+	c	d	d	c
x	y	z	y	x

The maximum value of a is 9, and the maximum value of b is also 9, and the maximum carryforward is 1. Hence, the maximum value of

$$a + c + Carryforward = 19 \Rightarrow x = 1$$

Hence, we get:

	a	b	b	a
+	c	d	d	c
1	y	z	y	1

Value of $a + c$

Note that from the leftmost column, the tens digit of $a+c$ is 1, and from the rightmost column, the units digit of $a+c$ is also 1. Hence:

$$a+c=11 \Rightarrow (a,c)=(2,9),(3,8),(4,7),(5,6)$$

Note that after (5,6) the pairs will start to repeat, and hence we ignore them.

Value of $b+d$

$$10+y=a+c+Carryforward$$

Substitute $a+c=11$:

$$10+y=11+Carryforward$$
$$y=1+Carryforward$$

Case I: If there is no carryforward, then the only value that is works is $b=d=0 \Rightarrow z=0$

	a	0	0	a
+	c	0	0	c
1	1	0	1	1

Case II: If there is a carryforward, then $y=2$. From the ten's column

$$b+d+1=10+y$$
$$b+d+1=12$$
$$b+d=11$$

$$b+d=11 \Rightarrow (b,d)=(2,9),(3,8),(4,7),(5,6),(6,5)(7,4),(8,3),(9,2) \Rightarrow 8\ Values$$

Note that this time we take 8 pairs, since the first and the pair result in different numbers.

$$2992+9229, 2222+9999$$

The total number of values from Case I and Case II:

$$1+8=9$$

By the multiplication principle the number of palindromes

$$=\underbrace{4}_{\substack{Choices\\ for\ a+c}} \times \underbrace{9}_{\substack{Choices\\ for\ b+d}}=36$$

J. Increasing Numbers

Example 2.136

How even positive integers have their digits in strictly increasing order?

$$0123456789$$

The number 0 is not relevant since a zero before a number is not counted in the number.
The number 9 cannot be used since the number cannot end in 9.
Hence, we get

$$12345678$$

Consider cases.
Case I: The last digit is 2.

$$2, \qquad 12 \Rightarrow 2=2^1\ Numbers$$

Case II: The last digit is 4.

$$4, \quad 14, \quad 24, \quad 34, \quad 124, \quad 134, \quad 234, \quad 1234 \Rightarrow 8 = 2^3 \; Numbers$$

Recognize the pattern.
We can also count this using the multiplication principle. Each digit less than 4 can either be used, or not used. (Once it is used, it is arranged in ascending order, so it has a fixed, single position).

$$2 \times 2 \times 2 = 2^3 = 8 \; Numbers$$

Case III: The last digit is 6.
The digits before 6 are 1,2,3,4,5. Each digit can either be used or not used, giving:

$$2^5 = 32 \; Numbers$$

Case IV: The last digit is 8.
The digits before 8 are 1,2,3,4,5,6,7. Each digit can either be used or not used, giving:

$$2^7 = 128 \; Numbers$$

The final answer is:

$$2 + 8 + 32 + 128 = 170$$

K. Counting Sum of Numbers

Example 2.137

What is the sum of all the four-digit positive integers that can be written with the digits 1,2, 3 and 4 if each digit must be used exactly once in each four-digit positive integer? **(MathCounts Chapter Sprint 2003/30)**

Consider four digits a, b, c, d. We can create

$$4 \times 3 \times 2 \times 1 = 24 \; Distinct \; Numbers$$

using four digits exactly once.

Each digit occurs in each position exactly six times. For example, when a is the thousand's place, you have six numbers that can be made

$$aXYZ \Rightarrow 6 \; Numbers$$
$$XaYZ \Rightarrow 6 \; Numbers$$
$$XYaZ \Rightarrow 6 \; Numbers$$
$$XYZa \Rightarrow 6 \; Numbers$$

$$6(1000a) + 6(100a) + 6(10a) + 6a$$
$$= 6000a + 600a + 60a + 6a$$
$$= 6666a$$

By the same logic, we get:

$$6666b, 6666c, 6666d$$

Hence, the final answer is:

$$6666a + 6666b + 6666c + 6666d$$
$$= 6666(a + b + c + d)$$

Substitute $a + b + c + d = 10$

$$= 6666(10) = 66660$$

Example 2.138
(MathCounts Chapter Sprint 2004/29)

29. Camy made a list of every possible distinct five-digit positive even integer that can be formed using each of the digits 1, 3, 4, 5 and 9 exactly once in each integer. What is the sum of the integers on Camy's list?

There are 24 integers that can be formed. The unit's digits in each of the 24 integers is 4. The sum from the unit's digits is:

$$24 \cdot 4 = 96$$

The remaining four digits can be arranged to form a number of the form

$$abcd4$$

In the previous example, we saw that 24 numbers of the form $ABCD$ had total

$$6666(a + b + c + d)$$

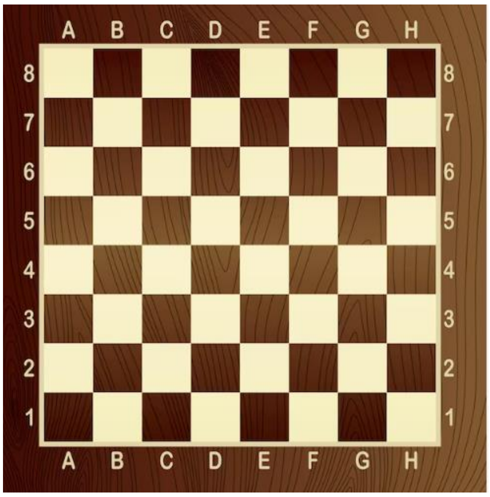
In this question, since each number is multiplied by 10, the expression is also multiplied by ten:

$$10 \cdot 6666(a + b + c + d) = 66660(1 + 3 + 5 + 9) = 66660 \cdot 18 = 1{,}199{,}880$$

The final answer is:

$$1{,}199{,}880 + 96 = 1{,}199{,}176$$

L. Rook Polynomials



Example 2.139
In how many ways can you place two rooks on an eight-by-eight chessboard so that the rooks do not attack each other.

The first rook can be placed without restriction.

$$8^2 = 64 \; ways$$

The second rook cannot be placed in the same column, or the same row as the first rook.

$$7^2 = 49 \; ways$$

The final answer is:

$$64 \times 49 = 3136 \; ways$$

$$16 \times 9 = 144$$

M. Overcounting

Example 2.140

How many permutations of the numbers 1, 2, 3, 4, and 5 are there such that no three consecutive numbers in the permutation form an arithmetic series? **(JHMMC Grade 7 2020 R1/27)**

Total Outcomes

The total number of ways is:

$$5! = 120\ ways$$

Complementary Outcomes

The ways in which you can form an arithmetic series are:

$$123, 234, 345, or\ 135 \Rightarrow\ 4\ ways$$

The arithmetic series can begin from any of three positions:

$$123 **, * 123 *, ** 123 \Rightarrow\ 3\ choices$$

The remaining two digits can be arranged in:

$$2 \cdot 1 = 2\ ways$$

The arithmetic series can either be increasing(123), or decreasing(321)

$$2\ ways$$

By the multiplication principle, the total number of ways is:

$$4 \cdot 3 \cdot 2 \cdot 2 = 48\ ways$$

However, we need to subtract cases that have been counted twice:

$$12345: Counted\ in\ 123, 234\ and\ 345$$
$$54321: Counted\ in\ 123, 234\ and\ 345$$

Counted thrice, so we subtract 2 for each value above:

$$2 * 2\ = 4$$

51234: Counted in 123, 234
43215: Counted in 123, 234
Counted twice, so we subtract 1 for each value above

$$1 * 2\ = 2$$

23451: Counted in 234, 345
12345 (Already listed above)
54321 (Already listed above)
15432: Counted in 234, 345
Counted twice, so we subtract 1 for each value above

$$1 * 2\ = 2$$

This gives a total of cases to be subtracted as:

$$4 + 2 + 2 = 8\ ways$$

The number of ways which are not valid is:

$$48 - 8 = 40$$

Hence, the final answer is:

$$120 - 40 = 80$$

N. Logic

Example 2.141

Call a word formed from the letters a; b; and c *mayan* if between any two a's (not necessarily adjacent) there's a b, between any two b's there's a c, and between any two c's there's an a. How many mayan words of length 2020 start with abc? **(JHMMC Grade 7 2020 R1/28)**

I cannot continue after *abc* with a c. Hence, I can have either an a or a b.
Case I: We choose b
abcb
We cannot put b next since between any two b's we need a c.
We cannot put c since between any two c's we need an a.
Hence, we must put an a:
abcba
abcbac
abcbacb

Case II: We choose a
abca

If you choose c: abcacba
If you choose b: abcab

Counting the number of solutions:
4 letters: 2 solutions
5 letters: 3 solutions
6 letters: 4 solutions
.
.
.
2020 letters: 2018 solutions

Counting the number of solutions:
abc
2 choices at abc
One of the choices has no a single solution, and hence we get 1 solution out of it.
The second choice results in two choices, one of which has a single solution, and the other results in further choices.

At each stage, we get two choices.
We have a total of 2017 stages. From the first 2016 stages, we get 2016 solutions. From the 2017th stage, we get 2 solutions.

Total
=2016+2=2018

O. Number of Factors

An important question in number theory is the number of factors of a number. Refer to the note on Divisors in the Number Theory to see how counting principles are applied in Number Theory.

P. Number Bases

Refer to the note on Number Bases in Number Theory to see how counting principles are applied in Number Theory.

2.7 Preparing for Probability

A. At Least

An at least condition means that the *minimum* value must be at least that much.

Example 2.142

Find the solution set of the following statements, and also write them as inequalities.

A. Reema rolled a standard six-sided die, and got a value, v, which was at least two.
B. Rudolph rolled a standard tetrahedral (four-sided die), and got a value, v, which was at least three.
C. In the coming week, I have my relatives visiting. So, I plan to watch a movie a day for m days. I want to watch a movie on at least three days.

Part A

The outcomes when we roll a standard six-sided dice are:

$$1, 2, 3, 4, 5, 6$$

And of the above, the outcomes which meet the conditions:

$$\{2,3,4,5,6\}$$

As an inequality, we will write:

$$2 \le v \le 6, \qquad v \in \mathbb{N}$$

Part B

The outcomes when we roll a standard four-sided dice are:

$$1, 2, 3, 4$$

And of the above, the outcomes which meet the conditions:

$$\{3,4\}$$

As an inequality, we will write:

$$3 \le v \le 4, \qquad v \in \mathbb{N}$$

Part C

$$0,1,2,3,4,5,6,7$$

$$\{3,4,5,6,7\}$$

$$3 \le x \le 7, \qquad x \in \mathbb{N}$$

B. At Most

An at most condition means that the *maximum* value can be what is given

Example 2.143

Find the solution set of the following statement:

Hari draws a card from a pack of cards. He gets a card which has a value of at most 5. (Consider the Ace to have a value of 1 for this question).

A suit of cards can be listed as below:

$$\underbrace{Ace}_{=1}, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, King, Queen$$

$$\{Ace, 2, 3, 4, 5\}$$

C. Tossing Coins

Example 2.144

How many outcomes are there if:

A. I toss a coin
B. I toss a coin twice
C. I toss a coin thrice

Part A

$$\{H, T\} \Rightarrow \ 2\ Outcomes$$

Parts B

$$\{HH, HT, TH, TT\} \Rightarrow \underbrace{2}_{\substack{First\\Toss}} \times \underbrace{2}_{\substack{Second\\Toss}} \Rightarrow 4\ Outcomes$$

Parts C

$$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$$\underbrace{2}_{\substack{First\\Toss}} \times \underbrace{2}_{\substack{Second\\Toss}} \times \underbrace{2}_{\substack{Third\\Toss}} \Rightarrow 8\ Outcomes$$

Example 2.145

What is the number of outcomes if I toss a coin n times?

This is multiplication principle with repetition. Each coin toss has 2 outcomes. Total number of outcomes is:

$$\underbrace{2}_{\substack{First\\Toss}} \times \underbrace{2}_{\substack{Second\\Toss}} \times \dots \times \underbrace{2}_{\substack{n^{th}\\Toss}} = 2^n\ Outcomes$$

Example 2.146

I toss a coin n times, and the number of outcomes is x. Find the number of outcomes when I toss it $n+1$ times.

Method I: Direct Calculation

When we toss it n times, the number of outcomes is:

$$\underbrace{2}_{\substack{First\\Toss}} \times \underbrace{2}_{\substack{Second\\Toss}} \times \dots \times \underbrace{2}_{\substack{n^{th}\\Toss}} = 2^n = x\ Outcomes$$

When we toss it $n+1$ times, the number of outcomes is:

$$\underbrace{2}_{\substack{First\\Toss}} \times \underbrace{2}_{\substack{Second\\Toss}} \times \dots \times \underbrace{2}_{\substack{(n+1)^{th}\\Toss}} = 2^{n+1}\ Outcomes$$

$$2^n = x$$
$$2 \cdot 2^n = 2x$$

$$2^{n+1} = 2x$$

Method I: Recursion

If you toss is n times, the number of outcomes is x.
The $(n+1)^{st}$ toss can be heads, or tails, which is two outcomes.

Each of heads or tails can be combined with the x outcomes from before. So, the total outcomes are:

$$2 \times x = 2x$$

2.147: Number of Outcomes when tossing a coin: Recursive Formula

If a toss a coin n times, the number of outcomes N is given by

$$N_n = 2N_{n-1}$$

Example 2.148

I toss a coin twice. What is the number of outcomes which are:

A. No Restrictions
B. All Heads
C. All Tails
D. At least one tail
E. At most one Head
F. Exactly one tail

Counting Outcomes

Let's write out the outcomes, using H for Heads, and T for Tails.

$$Both\ Heads = HH$$
$$Both\ Tails = TT$$
$$First\ Head, and\ then\ Tails = HT$$
$$First\ Tails, and\ then\ Heads = TH$$

$$\{HH, HT, TH, TT\}$$

	No Restrictions	4	HH, HT, TH, TT
	All Heads	1	HH
	All Tails	1	TT
$\geq$ Condition	At least one tail	3	HT, TH, TT
$\leq$ Condition	At most one Head	3	HT, TH, TT
	Exactly one tail	2	HT, TH

Multiplication Principle

$$\underbrace{2}_{\substack{First\\Toss}} \times \underbrace{2}_{\substack{Second\\Toss}} = 4$$

Example 2.149

I toss the same coin three times. What is the number of possible outcomes such that:

A. There are no restrictions
B. there is at least one head in the tosses.
C. there is at least one tail in the tosses
D. there is at least one tail and one head in the tosses

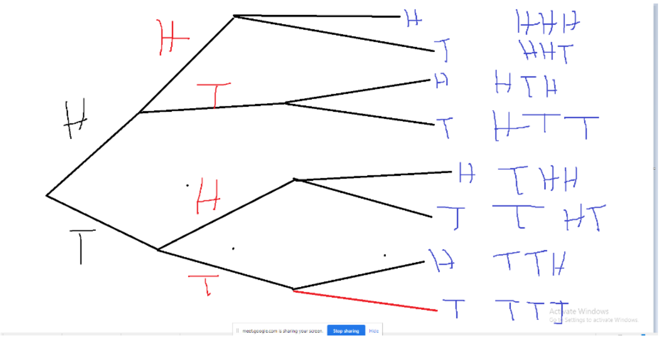
Part A
Enumeration
If we toss two coins, we get the following four outcomes:

$$HH, HT, TH, TT$$

Now imagine, that the above are the second and the third coin that we toss. The first coin can be either Heads, or Tails.

And hence, we get the following 8 outcomes:

$$\underbrace{HHH, HHT, HTH, HTT}_{\text{First Outcome:Heads}}, \quad \underbrace{THH, THT, TTH, TTT}_{\text{First Outcome:Tails}}$$



Multiplication Principle

$$\underbrace{2}_{\substack{First\\Toss}} \times \underbrace{2}_{\substack{Second\\Toss}} \times \underbrace{2}_{\substack{Third\\Toss}} = 2^3 = 8$$

Part B
The outcome with all tails is not what we want:

$$(HHH)(HHT)(HTH)(HTT)(THH)(THT)(TTH)(\boldsymbol{TTT})$$

Use Complementary Counting:

$$No.\,of\ Required\ Outcomes = \underbrace{8}_{\substack{Total\\Outcomes}} - \underbrace{1}_{\substack{Outcomes\ with\\all\ Tails}} = 7$$

Part C
Complementary Counting

$$No.\,of\ Required\ Outcomes = \underbrace{8}_{\substack{Total\\Outcomes}} - \underbrace{1}_{\substack{Outcomes\ with\\all\ Heads}} = 7$$

Part D
Complementary Counting

$$No.\,of\ Required\ Outcomes = \underbrace{8}_{\substack{Total\\Outcomes}} - \underbrace{1}_{\substack{Outcomes\ with\\all\ Heads}} - \underbrace{1}_{\substack{Outcomes\ with\\all\ Tails}} = 6$$

Example 2.150
I toss identical three coins in one toss.

A. What is the number of distinguishable outcomes?
B. Are these outcomes equally likely?

Part A
When you toss three identical coins in one toss, you cannot distinguish between the different arrangements. Therefore, the only outcome that you can measure is the number of heads

- Zero Heads
- One Head
- Two Heads
- Three Heads

Giving you four outcomes in all.

Part B
Not equally likely

Example 2.151
a penny, a nickel and a dime
in that particular order
in any order that I want, and the order of the coin tosses matters

Part A-I

$$\textit{Part } A: \underbrace{2}_{Penny} \times \underbrace{2}_{Nickel} \times \underbrace{2}_{Dime} = 2^3 = 8$$

You can list the outcomes

$$\left\{\left(\underbrace{H}_{Penny}\ \underbrace{H}_{Nickel}\ \underbrace{H}_{Dime}\right)(HHT)(HTH)(HTT)(THH)(THT)(TTH)(TTT)\right\}$$

Part A-II

In this part, I have two choices. I first need to decide the order of the coins. This I can do in six ways:

$$\{(PND)(PDN)(NPD)(NDP)(DPN)(DNP)\}$$

Or I can also get six ways using the multiplication principle:

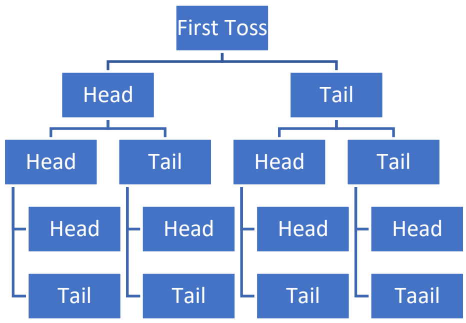
$$\underbrace{3}_{\substack{First\\Coin}} \times \underbrace{2}_{\substack{Second\\Coin}} \times \underbrace{1}_{\substack{Third\\Coin}} = 6$$

The number of sequences for heads and tails, as calculated above is

$$8$$

So, the total number of outcomes is

$$6 \times 8 = 48$$



D. Patterns

Example 2.152

I have a 1794 Flowing Hair Silver Dollar, a highly valuable coin among the first ones issued by the US Mint. The obverse side has the Bust of Liberty engraved on it. The reverse side has an eagle surrounded by a wreath. I toss the coin thrice. Count the number of outcomes with:

A. Zero Eagles
B. One Eagles
C. Two Eagles
D. Three Eagles

The language is complicated, but just think of Eagles as Heads, and Bust of Liberty as Tails.

We already know the outcomes when we toss three coins is given by:

$$(HHH)(HHT)(HTH)(HTT)(THH)(THT)(TTH)(TTT)$$

For this question, we need to classify the outcomes, which is done below:

$$A: \underbrace{TTT}_{\substack{Zero\\Heads}} \rightarrow 1\ Outcome$$

$$B: \underbrace{HTT, THT, TTH}_{One\ Head} \rightarrow 3\ outcomes$$

$$C: \underbrace{HHT, HTH, THH}_{Two\ Heads} \rightarrow 3\ outcomes$$

$$\underbrace{HHH}_{\substack{Three\\Heads}},$$

Part B and Part C have the same answer above. Explain why?

Two heads mean one tail.
And two tails mean one heads.

E. Rolling a Single Die

Example 2.153: Enumerating Outcomes
I roll a standard six-sided die once. State the outcomes.

$$\{1, 2, 3, 4, 5, 6\}$$

Example 2.154: Counting Outcomes
How many of the 6 outcomes when rolling a standard six-sided die once are:

A. Parity
  a. Even
  b. Odd

B. Prime and Composite Numbers
  a. Prime
  b. Composite

C. Ranges
  a. Greater than four
  b. Less than four

D. Ranges
  a. Greater than or equal to two
  b. Less than or equal to two

$$\{1, 2, 3, 4, 5, 6\}$$

Part A

$$\{2,4,6\} \Rightarrow Even = 3$$
$$\{1,3,5\} \Rightarrow \ Odd = 3$$

Part B

Recall that 1 is neither prime nor composite.

$$\{2,3,5\} \Rightarrow Prime = 3$$
$$\{4,6\} \Rightarrow Composite = 2$$

Part C

$$\{5,6\} \Rightarrow Outcomes\ greater\ than\ 4 = 2$$
$$\{1,2,3\} \Rightarrow Outcomes\ less\ than\ 3 = 3$$

Part D

$$\{2, 3, 4, 5, 6\} \Rightarrow Outcomes\ greater\ than\ \ or\ equal\ to\ 2 = 5$$
$$\{1,2\} \Rightarrow Outcomes\ less\ than\ or\ equal\ to\ 2 = 2$$

In each part (A, B, C, D, etc), what is the total? Does it add up to six each time? Why or why not?

Part B has “1” missing from the outcomes.
Part C has “4” missing from the outcomes.
Part D has “2” repeated in outcomes.

Hence, number of outcomes does not add up to 6.

F. Rolling Two Dice

Example 2.155: Outcomes

I roll two standard six-sided dice with faces numbered from one to six. Count the outcomes using:

A. Enumeration
B. Multiplication Principle

Enumeration

Consider the outcomes in the

- top green row as the number from the first die.
- left green column as the number from the second die.

We can combine the two to get a pair of numbers. For example

$$(1,3) \rightarrow First\ Roll = 1, \qquad Second\ \ Roll = 3$$

	Pair of Numbers					
	1	2	3	4	5	6
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
2	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
3	(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
4	(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
5	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
6	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

Multiplication Principle

Order is important here, since we can distinguish between the first roll and the second roll as two distinct objects.

$$Total\ Outcomes = \underbrace{6}_{\substack{First\\Roll}} \times \underbrace{6}_{\substack{Second\\Roll}} = 36$$

G. Restrictions: Prime And Composite Rolls

Example 2.156: Prime and Composite Numbers

I have a standard six-sided die with faces numbered one to six. I roll it twice, and record the two outcomes. What is the number of outcomes where:

A. The first roll is a prime number
B. The second roll is a composite number
C. Both rolls are prime numbers
D. Both rolls are composite numbers

$$Primes \in \{2,3,5\} = 3\ Primes$$
$$Composite \in \{4,6\} = 2\ Composite\ Numbers$$

Part A

We can also do this using the multiplication rule:

$$\underbrace{3}_{\substack{First\\Roll}} \times \underbrace{6}_{\substack{Second\\Roll}} = 18$$

This is also shaded in the table alongside.

	Pair of Numbers					
	1	2	3	4	5	6
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
2	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
3	(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
4	(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
5	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
6	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

Part B

If the second roll is a composite number, we have two options for the second roll.
And there are no restrictions on the first roll, so the total outcomes are:

$$\underbrace{6}_{\substack{First\\Roll}} \times \underbrace{2}_{\substack{Second\\Roll}} = 12$$

This is also shaded in the table alongside.

	Pair of Numbers					
	1	2	3	4	5	6
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
2	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
3	(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
4	(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
5	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
6	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

Part C

There are three possible prime numbers
- ➢ for the first roll.
- ➢ And also for the second roll

And, by the multiplication principle, the total number of outcomes is:

$$\underbrace{3}_{\substack{First\\Roll}} \times \underbrace{3}_{\substack{Second\\Roll}} = 9$$

This is also shaded alongside, where we want the intersection of the prime rows, and the prime columns.

	Pair of Numbers					
	1	2	3	4	5	6
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
2	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
3	(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
4	(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
5	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
6	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

Part D

$$\underbrace{2}_{\substack{First\\Roll}} \times \underbrace{2}_{\substack{Second\\Roll}} = 4$$

This is also shaded alongside, where we want the intersection of the composite rows, and the composite columns.

	Pair of Numbers					
	1	2	3	4	5	6
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
2	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
3	(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
4	(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
5	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
6	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

Example 2.157

I have a standard six-sided die with faces numbered one to six. I roll it twice, and record the two outcomes. What is the number of outcomes:

A. At least one roll is a prime number
B. At most one roll is a prime number
C. Exactly one roll is a prime number

Part A: Using Casework

We can do this on the basis of casework, but we have to be careful in counting our cases.
At least one roll is a prime number has three sub-cases:
- ➢ Only First Roll is a prime Number $= 3 \times 3 = 9$
- ➢ Only Second Roll is a Prime Number $= 3 \times 3 = 9$
- ➢ Both Rolls are prime Numbers $= 3 \times 3 = 9$

$$Total\ Outcomes = 9 + 9 + 9 = 27$$

Part A: Complementary Counting

At least one roll is a prime number can also be thought of as

$$Outcomes(No\ Prime\ Numbers) = \underbrace{3}_{\substack{First\\Roll}} \times \underbrace{3}_{\substack{Second\\Roll}} = 9$$

$$\underbrace{36}_{\substack{Total\\Outcomes}} - \underbrace{9}_{\substack{No\ Prime\\Numbers}} = 27$$

And the numbers where neither of the rolls are prime, are those which fall in both a shaded column, and a shaded row.

	Pair of Numbers					
	1	2	3	4	5	6
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
2	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
3	(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
4	(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
5	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
6	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

Part B

Again, use complementary counting:

$$\underbrace{36}_{\substack{Total\\Outcomes}} - \underbrace{9}_{\substack{Both\ Prime\\Numbers}} = 27$$

Part C
Complementary Counting

$$\underbrace{36}_{\substack{Total\\Outcomes}} - \underbrace{9}_{\substack{No\ Prime\\Numbers}} - \underbrace{9}_{\substack{Both\ Prime\\Numbers}} = 18$$

Direct Counting

$$\underbrace{3 \times 3}_{\substack{First\ Roll-Prime\\Second\ Roll-Not\ Prime}} + \underbrace{3 \times 3}_{\substack{First\ Roll-Not\ Prime\\Second\ Roll-Prime}} = 9 + 9 = 18$$

H. Sum

Example 2.158: Outcomes

I roll two standard six-sided dice with faces numbered from one to six. Enumerate the outcomes of the sum of the two numbers.

	Total					
	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Example 2.159

I have a standard six-side die with faces numbered one to six. I roll it twice, and record the two outcomes. What is the number of outcomes where the sum of the numbers is 7?

Table
We can also pick up the answer from the table.
See that the number seven make a diagonal, or stair-climbing pattern.

	Total					
B.	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Diophantine Equation

$$a + b = 7, \qquad 1 \le a, b \le 6, \qquad a, b \in \mathbb{N}$$

This is a Diophantine equation, which we study separately in number theory. It asks for integer solutions, with constraints, to equations.

$$a \in \{1,2,3,4,5,6\}, b \in \{6,5,4,3,2,1\} \Rightarrow 6\ Outcomes$$

Example 2.160

I have a standard six-sided die with faces numbered one to six. I roll it twice, and record the two outcomes. What is the number of outcomes where the sum of the numbers is greater than 10?

	Total	

Table
We shade the required three outcomes in the, table, forming a triangle pattern.

Algebra

$$13 > a + b > 10, \qquad 1 \le a, b \le 6, \qquad a, b \in \mathbb{Z}$$

This is a Diophantine Inequality

$$a + b = 11 \Rightarrow a \in \{5,6\}, b \in \{6,5\}$$
$$a + b = 12 \Rightarrow a \in \{6\}, b \in \{6\} \Rightarrow 3\ Outcomes$$

	1	2	3	4	5	6	Total
1	2	3	4	5	6	7	
2	3	4	5	6	7	8	
3	4	5	6	7	8	9	
4	5	6	7	8	9	10	
5	6	7	8	9	10	11	1
6	7	8	9	10	11	12	2

Example 2.161

I have a standard six-sided die with faces numbered one to six. I roll it twice, and record the two outcomes. What is the number of outcomes where the sum of the numbers is less than 6?

Fixing the First Number
We can fix the first number in the die roll.
If the first number is 1, the second number can be:

$$2, 3, 4\ or\ 5$$

Giving us 4 options in all.

And we can keep counting.
See the table to the left.

	Total						
B.	1	2	3	4	5	6	
1	2	3	4	5	6	7	4
2	3	4	5	6	7	8	3
3	4	5	6	7	8	9	2
4	5	6	7	8	9	10	1
5	6	7	8	9	10	11	
6	7	8	9	10	11	12	
							10

Fixing the Total
As a different method, we can fix the total. If the total is 5, we can get:

$$4 + 1$$
$$2 + 3$$
$$3 + 2$$
$$1 + 4$$

Giving us 4 options.
And we can keep counting. See the table to the left.

	Total						
B.	1	2	3	4	5	6	
1	2	3	4	5	6	7	1
2	3	4	5	6	7	8	2
3	4	5	6	7	8	9	3
4	5	6	7	8	9	10	4
5	6	7	8	9	10	11	
6	7	8	9	10	11	12	

And hence the final answer is:

$$1 + 2 + 3 + 4 = 10$$

I. Parity in the Sum

Parity refers to the concept of even and odd. This is a very simple, yet very powerful concept with broad application in Maths.

Example 2.162

I have a standard six-sided die with faces numbered one to six. I roll it twice, and record the two outcomes. What is the number of outcomes where the sum of the numbers is odd?

	Total					
B.	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10

Table

Shade the odd outcomes in the table, and see that we get chessboard pattern.
Hence, for every odd number, there is one even number.

5	6	7	8	9	10	11
6	7	8	9	10	11	12

And hence, the number of odd numbers is:

$$\frac{36}{2} = 18$$

Diophantine Inequality

This is a Diophantine Inequality, which can be solved by considering each of the cases that satisfy it.

$$a + b \leq 12, \qquad a + b \in 2n + 1, \qquad n \in N$$

$$a + b = \left\{ \underbrace{3}_{2\ Cases}, \underbrace{5}_{4\ Cases}, \underbrace{7}_{6\ cases}, \underbrace{9}_{4\ Cases}, \underbrace{11}_{2\ Cases} \right\} \Rightarrow 2 + 4 + 6 + 4 + 2 = 18$$

Example 2.163

I have a standard six-sided die with faces numbered one to six. I roll it twice, and record the two outcomes. What is the number of outcomes where the sum of the numbers is even?

Enumeration

$a + b$			
2	(1,1)	1	
4	(1,3)(2,2)(3,1)	3	4
6	(1,5)(2,4)(3,3)(4,2)(5,1)	5	9
8	(2,6)(3,5)(4,4)(5,3)(6,2)	5	14
10	(4,6)(5,5)(6,4)	3	17
12	(6,6)	1	18

Counting Argument

$$Odd + Odd = Even \Rightarrow 3 \times 3 = 9$$
$$Even + Even = Even \Rightarrow 3 \times 3 = 9$$

$$Total = 18$$

	Total					
B.	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

J. Product

Example 2.164: Outcomes

I roll two standard six-sided dice with faces numbered from one to six. Enumerate the product of the two numbers.

	Total					
B.	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

Example 2.165

I have a standard six-sided die with faces numbered one to six. I roll it twice, and record the two outcomes. What is the number of outcomes where the product of the numbers is 12?

Number Theory

We can determine the number of outcomes by creating factor pairs for 12.

$$12 = \underbrace{1 \times 12}_{Not\,Valid} = \underbrace{2 \times 6}_{\substack{Valid \\ (2,6),(6,2)}} = \underbrace{3 \times 4}_{\substack{Valid \\ (3,4)(4,3)}} \Rightarrow 2 + 2 = 4$$

$$(2, 6), (6, 2)$$

	Total					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

Note that out of the factor pairs, the first factor pair is not valid, since 12 is not a valid outcome when rolling a standard six-sided die.

Example 2.166

I have a standard six-sided die with faces numbered one to six. I roll it twice, and record the two outcomes.

A. What is the number of outcomes where the product of the two rolls is greater than 25?
B. If the product of the two rolls is greater than 25, than the sum of the two rolls must be greater than or equal to which number?

Part A

Product should be greater than 25:

$$(5,6)(6,5)(6,6) \Rightarrow 3\ Outcomes$$

	Total					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

Part B

Sum must be

$$\geq 11$$

	Total					
	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

K. Parity in the Product

$Odd \times Odd = Odd$	$Odd \times Even = Even$ $Even \times Odd = Even$ $Even \times Even = Even$

Example 2.167

I have a standard six-sided die with faces numbered one to six. I roll it twice, and record the two outcomes. What is the number of outcomes where the product of the numbers is odd?

Multiplication Principle
The product of two numbers is odd only when both the numbers are odd.
Hence, the number of odd products is:

$$\underbrace{3}_{Odd} \times \underbrace{3}_{Odd} = 9$$

	Total					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

Example 2.168
I have a standard six-sided die with faces numbered one to six. I roll it twice, and record the two outcomes. What is the number of outcomes where the product of the numbers is even?

Direct Method

$$Odd \times Even = Even \Rightarrow \underbrace{3}_{Odd} \times \underbrace{3}_{Even} = 9$$
$$Even \times Odd = Even \Rightarrow \underbrace{3}_{Even} \times \underbrace{3}_{Odd} = 9$$
$$Even \times Even = Even \Rightarrow \underbrace{3}_{Even} \times \underbrace{3}_{Even} = 9$$

Complementary Counting

$$\underbrace{36}_{\substack{Total\\Outcomes}} - \underbrace{9}_{Even} = 27$$

	Total					
	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

Example 2.169
If you a toss a standard five-sided die twice and find the product, what is the number of even outcomes?

If you see the table you realize that:

$$No.\,of\ Odd\ Outcomes = 9$$
$$Total\ Outcomes = 25$$

$$Even\ Outcomes = 25 - 9 = 16$$

	Total					
	1	2	3	4	5	
1	1	2	3	4	5	
2	2	4	6	8	10	
3	3	6	9	12	15	
4	4	8	12	16	20	
5	5	10	15	20	25	

L. Rolling Three Dice

Example 2.170
I roll three six-sided dice (one colored red, the second colored green, and the third colored blue). What is the number of outcomes

A. With no restrictions
B. With all prime numbers
C. With all composite numbers
D. Where the product of the three outcomes is odd
E. Where the product of the three outcomes is even
F. Where all the rolls are six

Part A

$$\underbrace{6}_{Red\ Die} \times \underbrace{6}_{Green\ Die} \times \underbrace{6}_{Blue\ Die} = 6^3 = 216$$

Part B

$$Primes \in \{2,3,5\}$$
$$\underbrace{3}_{Red\ Die} \times \underbrace{3}_{Green\ Die} \times \underbrace{3}_{Blue\ Die} = 3^3 = 27$$

Part C

$$Composite\ Numbers\ \{4,6\}$$
$$\underbrace{2}_{Red\ Die} \times \underbrace{2}_{Green\ Die} \times \underbrace{2}_{Blue\ Die} = 2^3 = 8$$

Part D

$$Odd \times Odd \times Odd = Odd$$
$$\underbrace{3}_{Red\ Die} \times \underbrace{3}_{Green\ Die} \times \underbrace{3}_{Blue\ Die} = 3^3 = 27$$

Part E

We will use complementary counting and subtract the number of odd outcomes ($Part\ D$) from the total number of outcomes ($Part\ A$)

$$Total\ Outcomes - Odd\ Outcomes = 216 - 27 = 189$$

Part F

$$\underbrace{1}_{Red\ Die} \times \underbrace{1}_{Green\ Die} \times \underbrace{1}_{Blue\ Die} = 1^3 = 1$$

M. Rolling Sides Other Than Six

Example 2.171

I roll two dice, one with four sides numbered one through four, and the other with eight side numbered one through eight. What is the number of outcomes:

A. With no restrictions
B. With all prime numbers
C. With all composite numbers
D. Where the product of the two rolls is odd
E. Where the product of the two rolls is even
F. Where all the rolls are six

Part A

$$4 \times 8 = 32$$

Part B

$$Four\ Sided\ Die: \{2,3\}, Eight\ Sided\ Die: \{2,3,5,7\} \Rightarrow 2 \times 4 = 8$$

Part C

$$Four\ Sided\ Die: \{4\}, Eight\ Sided\ Die: \{4,6,8\} \Rightarrow 1 \times 3 = 3$$

Part D

The only way in which the product will be odd is if both the dice have odd outcomes

$$Odd \times Odd = Odd$$
$$Four\ Sided\ Die: \{1,3\}, Eight\ Sided\ Die: \{1,3,5,7\} \Rightarrow 2 \times 4 = 8$$

Part E

$$Total\ Outcomes - Odd\ Outcomes = 32 - 8 = 24$$

Part F

You cannot get a roll of six with a four-sided die.
Hence, zero outcomes.

N. Standard Pack of Cards

A standard pack of 52 cards has:

- Four suits (Clubs, Spades, Hearts, and Diamonds), with 13 cards in each suit.
  - ✓ Clubs and Spades are "Black" suits
  - ✓ "Hearts" and "Diamonds" are "Red" suits.
- 13 ranks in each suit
  - ✓ Ranks are: Ace, 2, 3, ...8, 9, 10, Jack, Queen and King
  - ✓ The last three are called Face Cards.
- 4 cards of each rank (one of each suit)

Example 2.172

Enumerate the outcomes of drawing a single card.

Each cell in the table represents one outcome:

		Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King	Total
Black	Clubs											Face Card	Face Card	Face Card	13
	Spades											Face Card	Face Card	Face Card	13
Red	Hearts											Face Card	Face Card	Face Card	13
	Diamonds											Face Card	Face Card	Face Card	13
		4	4	4	4	4	4	4	4	4	4	4	4	4	52

Example 2.173

I draw a single card from a pack of cards. Consider Ace as 1, and face cards as not having a number. What is the number of outcomes where the card:

A. Has no restrictions
B. is red
C. Is black
D. Is spades
E. Is Hearts
F. is a face card
G. Is a non-face card
H. Is odd numbered
I. Is prime numbered
J. Is composite numbered

K. Is Red, but not a diamond
L. Is Black, but not diamonds or hearts
M. Is either hearts or a face card
N. Is either black or not a face card
O. Is a red face card
P. Is a black non-face card
Q. Is a face card that is Spades
R. Is a non-face card that is not Spades
S. Is a black odd numbered card
T. Is an odd numbered Spades

Has no restrictions	52		
Red	26		
Black	26		
Spades	13		
Hearts	13		
Face card	12	4×3	
Not a face card	40	$52 - 12$	
Odd Numbered	20	5×4	$\{1,3,5,7,9\}$

Prime Numbered	16	4×4	$\{2,3,5,7\}$
Composite Numbered	20	5×4	$\{4,6,8,9,10\}$
Red, but not a Diamond	13	$Hearts$	
Black, but not a Diamond or a Heart	26		
Either hearts or a face card	22	$13+9$	$13+12-3$
Either black or not a face card	46	$26+20$	$52-6$
Red Face Card	6		
Black non-face card	20		
Spades face card	3		
Non-face Non-Spades Card	30		
Black Odd Numbered Card	10		
Odd Numbered Spades	5		

Is either $\underbrace{black}_{Condition\ I}$ or $\underbrace{not\ a\ face\ card}_{Condition\ II}$

When you have OR, either of the conditions being met is sufficient for the card to be included in the counting.

$$Black = 26$$

The number of non-face cards:

$$Not\ a\ Face\ Card = 4(13-3) = 4 \times 10 = 40 \Rightarrow 20\ Black, 20\ Red$$

Final Answer

$$\underbrace{26}_{Black} + \underbrace{20}_{Non-Black, Non-Face\ Cards} = 46$$

O. Drawing Two Cards

Example 2.174

I draw two cards, without replacement, from a standard pack of cards. What is the number of outcomes?

$$\underbrace{52}_{\substack{First\\Card}} \times \underbrace{51}_{\substack{Second\\Card}} = 2,652$$

Repeat the above question if the cards are drawn with replacement?

$$\underbrace{52}_{\substack{First\\Card}} \times \underbrace{52}_{\substack{Second\\Card}} = 2704$$

Example 2.175

I draw two cards, with replacement, from a standard pack of cards. What is the number of outcomes where both the cards are:

A. Red
B. Spades
C. Face cards

$$26 \times 26 = 676$$
$$13 \times 13 = 169$$
$$12 \times 12 = 144$$

Repeat the above question if the cards are drawn without replacement.

$$\underbrace{26}_{\substack{First\\Card}} \times \underbrace{25}_{\substack{Second\\Card}}$$
$$\underbrace{13}_{\substack{First\\Card}} \times \underbrace{12}_{\substack{Second\\Card}}$$
$$\underbrace{12}_{\substack{First\\Card}} \times \underbrace{11}_{\substack{Second\\Card}}$$

Example 2.176

I draw two cards, with replacement, from a standard pack of cards. What is the number of outcomes where both the cards are

D. either Clubs or an Ace

$$16 \times 16 = 256$$

Repeat the above question if the cards are drawn without replacement.

$$\underbrace{16}_{\substack{First\\Card}} \times \underbrace{15}_{\substack{Second\\Card}} = 240$$

(Important) Example 2.177

I draw two cards, without replacement, from a standard pack of cards. What is the number of outcomes where exactly one card is:

A. A club
B. Red
C. A face card

Part A: $\boldsymbol{Club = C, Non - Club = N}$

$$First\ Card\ is\ a\ Club = \underbrace{13}_{C} \times \underbrace{39}_{N}$$
$$Second\ Card\ is\ a\ Club = \underbrace{39}_{N} \times \underbrace{13}_{C}$$
$$Total\ Number\ of\ Choices = 2 \times 13 \times 39$$

Part B: $\boldsymbol{Red = \mathrm{R}, Non - \mathrm{Red} = R}$

$$First\ Card\ is\ Red = \underbrace{26}_{R} \times \underbrace{26}_{N} = 676$$
$$Second\ Card\ is\ Red = \underbrace{26}_{N} \times \underbrace{26}_{R} = 676$$
$$Total\ Number\ of\ Choices = 2 \times 676 = 1352$$

Part C: $\boldsymbol{Face = \mathrm{F}, Non - \mathrm{Face} = \mathrm{N}}$

$$First\ Card\ is\ a\ Face\ Card = \underbrace{12}_{F} \times \underbrace{40}_{N}$$
$$Second\ Card\ is\ a\ Face\ Card = \underbrace{40}_{N} \times \underbrace{12}_{C}$$
$$Total = 2 \times 12 \times 40 = 960$$

Repeat the above question if the cards are drawn with replacement.

The answers here are the same as the answers without replacement.
Because you are drawing from two different categories for each question:

- Club versus Non-Club
- Red Card versus Non-Red Card
- Face card versus Non-Face Card

Example 2.178

I draw two cards, without replacement, from a standard pack of cards. What is the number of outcomes where the cards are:

A. At most one is hearts
B. At least one is spades

This is best done with complementary counting.

At Most One is Hearts

$$\underbrace{(52\times 51)}_{All\ Outcomes}-\underbrace{(13\times 12)}_{Both\ Hearts}=2652-156=2496$$

At Least One is Spades

$$\underbrace{(52\times 51)}_{All\ Outcomes}-\underbrace{(39\times 38)}_{No\ Spades}=2652-1482=1170$$

Answer the above questions if the cards are drawn with replacement

$$52\times 52-13\times 13=52^2-13^2=2535$$
$$52\times 52-39\times 39=52^2-39^2=1183$$

(Casework) Example 2.179

I draw two cards from a standard pack of cards, without replacement. What is the number of outcomes when the first card is red, the second card is a face card?

$$26\times 12=312$$

This is not correct because the 26 red cards also include 6 face cards, and the red card that you pick could be a face card, in which case, the number of face cards available to pick is only 11.

We have to solve this question using casework:

Case I: First Card is a Red Face Card:

$$6\times 11=66$$

Case II: First is a Red Non-Face Card:

$$\underbrace{26}_{Red}-\underbrace{6}_{Red\ Face\ Cards}=20$$
$$20\times 12=240$$

$$Total=66+240=306$$

P. Drawing Three Cards

Example 2.180

I draw three cards from a standard pack of cards. What is the number of outcomes when the cards are drawn:

A. With replacement
B. Without replacement

$$52 \times 52 \times 52 = 52^3 = 140{,}608$$

Example 2.181
I draw three cards from a standard pack of cards. What is the number of outcomes when the first card is Diamonds, the second card is Hearts, and the third card is Spades?

$$13 \times 13 \times 13 = 13^3 = 2{,}197$$

Example 2.182
I draw three cards from a standard pack of cards. What is the number of outcomes where one card is Diamonds, one card is Hearts, and one card is Spades?

$$13 \times 13 \times 13 = 13^3$$

I can also arrange the cards among themselves.

$$3 \times 2 \times 1 = 6\, Ways$$
$$Total = 6 \times 13^3$$

Example 2.183
I draw three cards from a standard pack of cards, without replacement. What is the number of outcomes when the first card is red, the second card is black, and there are no restrictions on the third card?

$$26 \times 26 \times 50 = 33{,}800$$

(Casework) Example 2.184
I draw three cards from a standard pack of cards, without replacement. What is the number of outcomes when the first card is red, the second card is black, and the third card is a face card?

Q. Review

2.185: Total Outcomes
When we toss a coin, roll dice, draw cards, draw a ball from an urn, we want to know what the possible outcomes.
These will later on in probability be called total outcomes (sample space)

Example 2.186
What is the number of outcomes when we toss:

A. 3 coins
B. 5 coins
C. n coins

$$2 \times 2 \times 2 = 2^3 = 8$$
$$2^5 = 32$$

$$\underbrace{2 \times 2 \times 2 \times \ldots . \times 2}_{n \text{ times}} = 2^n$$

Example 2.187

What is the number of outcomes when we have six-sided dice and we roll:

A. Two Dice
B. Three Dice
C. Four Dice
D. n dice

$$6 \times 6 = 36$$
$$6 \times 6 \times 6 = 6^3 = 216$$
$$6 \times 6 \times 6 \times 6 = 1296$$
$$6^n$$

Example 2.188

What is the number of outcomes when we roll two:

A. Octahedral (eight-sided) dice
B. Tetrahedral (four-sided) dice
C. Ten-sided dice
D. Twenty-sided dice

$$8 \times 8 = 64$$
$$4 \times 4 = 16$$
$$10 \times 10 = 100$$
$$20 \times 20 = 400$$

Example 2.189

What is the number of outcomes when we roll:

A. Three eight-sided dice
B. Four ten-sided dice
C. Two twenty-sided dice
D. Two hundred-sided dice

$$8 \times 8 \times 8 = 512$$
$$10^4 = 10{,}000$$
$$20 \times 20 = 400$$
$$100^2 = 10{,}000$$

Example 2.190

What is the number of outcomes when I draw, with replacement, from a standard pack of cards:

A. Two Cards
B. Three Cards
C. n cards

$$52^2$$
$$52^3$$
$$52^n$$

Example 2.191

I toss six six-sided dice. What is the number of outcomes in which:

A. Every roll is odd
B. Every roll is even
C. Every roll is prime
D. Every roll is composite
E. The rolls are in increasing order
F. The rolls are in decreasing order

$$3^6 = 729$$
$$3^6 = 729$$
$$3^6 = 729$$
$$2^6 = 64$$
$$1$$
$$1$$

Example 2.192

I toss six six-sided dice. What is the number of outcomes in which every roll is a different number?

There are six dice, and, in some order, they must be the numbers:

$$1,2,3,4,5,6$$

The first die can be any number.

$$6\ choices$$

The second die cannot repeat the number from the first die

$$5\ choices$$
$$.$$
$$.$$
$$.$$

The last die has only one number:

$$1\ Choice$$

Hence, we bring the choices together:

$$6 \times 5 \times \ldots \times 1 = 6! = 720$$

Example 2.193

I toss six six-sided dice. What is the number of outcomes in which the product of the rolls is a prime?

$$Primes \in \{2,3,5\}$$

Suppose we multiply a prime with any number other than one, then the product will not be a prime. Hence, exactly one of the rolls must be prime, and all other rolls must be 1.

Consider that the prime number that we choose is 2:

$$(1,1,1,1,1,2), (1,1,1,1,2,1), (1,1,1,2,1,1), (1,1,2,1,1,1), (1,2,1,1,1,1), (2,1,1,1,1,1) \Rightarrow 6\ Outcomes$$

Similarly, we have:

$$Product = 3 \Rightarrow 6\ Outcomes$$
$$Product = 5 \Rightarrow 6\ Outcomes$$

And hence the final answer is:

$$6 + 6 + 6 = 18\ Outcomes$$

Example 2.194

What is the number of outcomes when I draw, with replacement, from a standard pack of cards:

A. Two Red Cards
B. Two Spades
C. Two Face Cards
D. Two Non-Face Cards
E. Two Kings

$$26^2 = 676$$
$$13^2 = 169$$
$$12^2 = 144$$
$$40^2 = 1600$$
$$4^2 = 16$$

2.8 Geometrical Counting

A. Number of Rays

You can use the rules of counting to count geometrical objects.
Recall from geometry that a ray has a single endpoint and extends towards infinity in the other direction.

Example 2.195: Collinear Points

Points $ABCD$ lie on a line, in that order.

A. Which are the rays that can be formed using these points only?
B. Count the number of rays that can be formed.

For the sake of simplicity assume that A, B, C and D are oriented left to right.
We can then draw a diagram like the one below:

What doesn't work

The most important thing is to realize that

$$Ray\ AB = Ray\ AC$$

Because both rays have start point A, and both rays go till infinity in the direction of B (which is the same as the direction of C).
Hence, you cannot count the above as two different rays.

Strategy

Rays starting from A:

$$AB$$

Rays starting from B:

$$BA, BC$$

Rays starting from C:

$$CB, CD$$

Rays starting from D:

$$DC$$

$$Total\ Choices = 1 + 2 + 2 + 1 = 6$$

2.196: Number of Rays on n collinear points

If $\boldsymbol{p_1, p_2, \dots p_n}$ are $\boldsymbol{n}$ collinear points, the number of rays that can be formed using these points is:

$$\boldsymbol{2(n-1)}$$

If the points are not oriented left to right, rotate them to make them left to right.

Start creating rays from left to right, getting

$$p_1p_2, p_2p_3, \dots, p_{n-1}p_n \Rightarrow n - 1\ rays$$

The last point cannot make a ray since we only have a single point.
(We can make a single ray from each of the points expect the last one, giving us $n - 1$ rays).

Repeat the process from right to left, again getting

$$n - 1\ rays$$

The total number of rays is:

$$2(n-1)$$

Example 2.197: Collinear Points

What is the number of rays that can be formed from 10 collinear points if at least two of the given points must lie on the ray?

$$2(n-1) = 2(10-1) = 2 \times 9 = 18$$

Example 2.198: Points on a Circle

Points ABC lie on a circle. Find the number of rays that can be drawn using these points if at least two of the given points must lie on the ray?

The rays that can be made are the pairs of points:

$$AB, AC, BC, BA, CA, CB \Rightarrow 6\ Pairs$$

Note that order matters here since

$$Ray\ AB \neq Ray\ BA$$

Since BA goes in the exact opposite direction as AB.

2.199: Number of Rays on a circle

$\boldsymbol{n}$ points lying on a circle can be used to form

$$\boldsymbol{n(n-1)\ Rays}$$

Each of the n points can be made into an endpoint, and it can be connected to the remaining

$$n - 1\ points$$

Using the multiplication principle, we get:

$$\underbrace{n}_{Endpoints} \times \underbrace{(n-1)}_{\substack{Points\ to\\ connect}}\ Rays$$

Example 2.200: Points on a Circle

A circle has eight distinct points spaced equally around it. Using these eight points, find the number of rays

A. Rays that can be drawn
B. Line Segments that can be drawn

Part A

$$\underbrace{8}_{Endpoints} \times \underbrace{7}_{\substack{Points\ to\\ Connect}} = 56$$

Part B

The number of line segments will be exactly half the number of rays, since in line segments direction will not matter, whereas it will matter for rays.

Hence, we can make pairs of rays that correspond to the same line segment

$$(Ray\ AB, Ray\ BA) \Leftrightarrow Line\ Segment\ AB$$

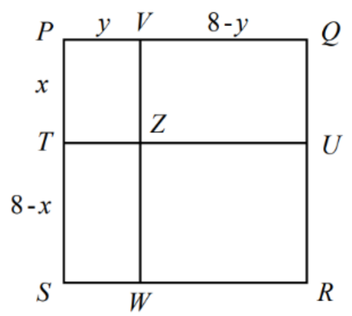
Hence, the final answer is

$$\frac{56}{2} = 28$$

B. Geometry

Example 2.201

Square $PQRS$ has sides of length 8. It is split into four rectangular regions by two line segments, one parallel to PQ and another parallel to QR. There are N ways in which these lines can be drawn so that the area of each of the four rectangular regions is a positive integer. What is the remainder when N^2 is divided by 100? **(Gauss Grade 8 2021/25)**



Consider values of x where x is written as a fraction in reduced form:

$$x = \frac{a}{b}, \qquad a, b \in \mathbb{N}, \qquad HCF(a, b) = 1$$

Note that

$$A(PQUT) = A(PVZT) + A(VQUZ) = 8x \in \mathbb{N}$$

Hence, we can ignore all possibilities where $A(PQUT)$ is not a natural number.

Let:

$$A(PQUT) = 8x = 8\left(\frac{a}{b}\right)$$

For $8\left(\frac{a}{b}\right)$ to be a natural number, b must be a factor of 8:

$$b \in \{1,2,4,8\}$$

Case I: $b = 1$

$$x \in \{1,2,\dots,7\} = 7\ Choices$$
$$y \in \{1,2,\dots,7\} = 7\ Choices$$
$$Total\ Choices = 7 \times 7 = 49\ Ways$$

(Note that y has not been taken to have fractional values. This will be considered in the cases below).

Case II: $b = 2$

$$x \in \left\{\frac{1}{2}, \frac{3}{2}, \dots, \frac{15}{2}\right\} = 8\ Choices$$
$$y \in \{2,4,6\} = 3\ Choices$$
$$Total\ Choices = 8 \times 3 = 24\ Ways$$
$$y \in \left\{\frac{1}{2}, \frac{3}{2}, \dots, \frac{15}{2}\right\} = 8\ Choices$$
$$x \in \{2,4,6\} = 3\ Choices$$
$$Total\ Choices = 8 \times 3 = 24\ Ways$$
$$24 + 24 = 48\ Ways$$

Case III: $b = 4$

$$x \in \left\{\frac{1}{4}, \frac{3}{4}, \dots, \frac{31}{4}\right\} = 16\ Choices$$
$$y \in \{4\} = 1\ Choice$$
$$Total\ Choices = 16 \times 1 = 16\ Ways$$

$$y \in \left\{\frac{1}{4}, \frac{3}{4}, \dots, \frac{31}{4}\right\} = 16\ Choices$$
$$x \in \{4\} = 1\ Choice$$
$$Total\ Choices = 16 \times 1 = 16\ Ways$$
$$16 + 16 = 32\ Ways$$

Case IV: $b = 8$

$$x \in \left\{\frac{1}{8}, \frac{3}{8}, \dots\right\}$$
$$y = 8 \Rightarrow Not\ Valid$$

Hence:

$$N = 49 + 48 + 32 = 129$$
$$N^2 = 129^2 = 16{,}641$$
$$Last\ two\ digits = 41$$

C. Coordinate Geometry: Lattice Points

We now turn our attention to counting in co-ordinate geometry, of which you should have at least a basic understanding before you start this section.

Don't memorize formulas in this section. You will most likely forget. Rather, you should be able to visualize the question.

2.202: Lattice Points

In a cartesian co-ordinate system, the location of a point P is given by:

$$P = (x, y)$$

If both x and y are integers, then P is called a lattice point.

Example 2.203: First Quadrant

Let $A = (2,3)$, $B = (7,3)$. Let p be the number of lattice points between A and B, and q be the number of lattice points from A to B. Find $2p + 3q$.

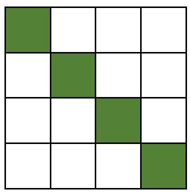
$$p = \underbrace{6}_{\substack{Start\\Point}} - \underbrace{3}_{\substack{End\\Point}} + 1 = 4, \quad q = \underbrace{7}_{\substack{Start\\Point}} - \underbrace{2}_{\substack{End\\Point}} + 1 = 6 \Rightarrow 2p + 3q = 2(4) + 6(3) = 26$$

Example 2.204: Across Multiple Quadrants

Let $A = (5, -2)$ and $B = (5,12)$. Let p be the number of lattice points between A and B, and q be the number of lattice points from A to B. Find $-pq$.

$$p = \underbrace{11}_{\substack{Start\\Point}} - \underbrace{(-1)}_{\substack{End\\Point}} + 1 = 13, \quad q = \underbrace{12}_{\substack{Start\\Point}} - \underbrace{(-2)}_{\substack{End\\Point}} + 1 = 6 \Rightarrow 2p + 3q = 2(4) + 6(3) = 26$$

2.205: Lattice Points along a Diagonal



$No.of$ Lattice $Points\ on\ the\ Diagonal = No.of\ Points$ from one corner to another $= 5$

See the square alongside with a side length of four, and convince yourself, using the bijection principle that:

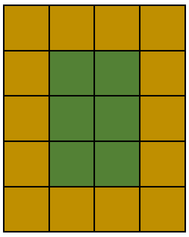
Example 2.206

Let $A = (5, -2)$ and $B = (5,12)$, $C = (19,12)$ and $D = (19, -2)$. Find the number of lattice points on the line between A and C.

$$AB = BC = CD = DA = 13 \Rightarrow ABCD\ is\ a\ square$$
$$\therefore No.of\ Lattice\ Points = No.of\ Points\ between\ A\ and\ C = 13$$

Example 2.207

$\underbrace{(2,3)}_{\substack{Bottom\\Left}}, \underbrace{(6,3)}_{\substack{Bottom\\Right}}, \underbrace{(7,3)}_{\substack{Top\\Left}}, \underbrace{(6,7)}_{\substack{Top\\Right}}$ are the coordinates of a rectangular region in the coordinate plane (see diagram). Find the number of lattice points on the boundaries of the region.



This method should remind you of the one used to calculate the area of a pathway in a rectangular region. It uses the same idea, carried over to lattice points:

$$Boundaries = \underbrace{On\ or\ Inside}_{Brown+Green=5\times 6=30} - \underbrace{Inside}_{Green=3\times 4} = 30 - 12 = 18$$

The direct method requires careful counting, and is not recommended for general use:

$$Boundaries = \underbrace{5}_{Top} + \underbrace{5}_{Bottom} + \underbrace{4}_{Right} + \underbrace{4}_{Left} = 18$$

208 Examples