Combinations

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1. Overcounting

1.1 Handshakes and Pairs

A. Handshakes

Example 1.1: Enumeration

A group of five friends (A, B, C, D, E) goes for a picnic. At the end of the picnic, they all shake each other's hands. List the handshakes that will take place. How many handshakes take place?

Consider the handshakes that A will get into, which is with everybody:

$$AB, AC, AD, AE$$

When we get to B, we do not look back, since BA is the same as AB, so we only have 3 additional handshakes:

$$BC, BD, BE$$

Similarly for C and D, we have:

$$CD, CE$$
$$DE$$

And there are no handshakes for E that have not already been counted.

We can arrange the above handshakes into a beautiful triangular pattern:

$$\begin{gathered} AB, AC, AD, AE \\ BC, BD, BE \\ CD, CE \\ DE \end{gathered}$$

Example 1.2: Logical Counting

A group of five friends goes for a picnic. At the end of the picnic, they all shake each other's hands. How many handshakes take place?

Counting Decreasing Handshakes

First person will shake hands with 4 people
Second person cannot shake hands with the first person anymore, since that has already been counted. He shakes with only three people.
Third person shakes hands with two people
Fourth person shakes hand with one person

$$Total\ Handshakes = \underbrace{4}_{\substack{First\\Person}} + \underbrace{3}_{\substack{Second\\Person}} + \underbrace{2}_{\substack{Third\\Person}} + \underbrace{1}_{\substack{Fourth\\Person}} + \underbrace{0}_{\substack{Fifth\\Person}} = 10$$

Example 1.3

There are 20 cities in a certain country. Every pair of cities is connected by an air route. How many air routes are there? (NMTC Primary-Final, 2004/3)

Method I

Name the cities

$$A_1, A_2, A_3, \dots, A_{20}$$

You can take

$$19\ Flights\ from\ the\ A_1$$

From the second city, you can also 19 flights. But the flight A_2A_1 was already counted above. Hence, we will get

$$18\ additional\ flights\ from\ A_2$$

Continuing this way, we get:

$$1+2+\cdots+19=\frac{19\times 20}{2}=190$$

Method II

From each city, you have 19 $flights$. Hence, the total number of air routes should be:

$$19\times 20$$

However, a flight from A_1 to A_2 operates on the same air route as the flight from A_2 to A_1:

$$\frac{19\times 20}{2}=190$$

Example 1.4

Consider the first five natural numbers 1,2,3,4,5. This set of five numbers is divided into two sets A and B, where A contains two numbers, and B contains the other three numbers. One example is $A=\{2,4\}$, and $B=\{1,3,5\}$. How many such pairs of A, B are there? **(NMTC Primary-Final, 2004/7)**

Method I

Once we select A, the numbers in B are automatically selected. Hence, we only count the possible sets that you can make for A:

$$\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}$$

Method II

$$\frac{4\times 5}{2}=10$$

B. Handshakes Formula

When you shake hands, it's very important to note that

$$\underbrace{A\ shaking\ hands\ with\ B}_{\textbf{Event X}} \Rightarrow \underbrace{B\ shaking\ hands\ with\ A}_{\textbf{Event Y}}$$

Event X and Event Y are actually the same event. So, if we want to count the number of handshakes, we need to find

$$\frac{No.of\ People\times No.of\ Hands\ shaken\ by\ each\ person}{2}$$

Example 1.5

A group of five friends goes for a picnic. At the end of the picnic, they all shake each other's hands. How many handshakes take place?

Counting Total Handshakes and Correcting Overcounting

We can count the total number of handshakes that happen in the group.

$$Each\ person\ shakes\ hands\ with\ 4\ people$$
$$Total\ Handshakes=5\times 4=20$$

But, the above calculation overcounts the actual number of handshakes by a factor of two, since A shaking hands with B, is the same as B shaking hands with A.

Hence, to find the actual number of handshakes, we need to divide by 2:

$$\frac{20}{2} = 10$$

1.6: Counting Handshakes among n people

The number of handshakes when $\boldsymbol{n}$ people all shake hands with each other is:

$$\frac{\boldsymbol{n(n-1)}}{\boldsymbol{2}}$$

Add up Handshakes

$$\underbrace{(n-1)}_{\substack{First\\Person}} + \underbrace{(n-2)}_{\substack{Second\\Person}} + \underbrace{(n-3)}_{\substack{Third\\Person}} + \cdots + \underbrace{1}_{\substack{(n-1)^{th}\\Person}} + \underbrace{0}_{\substack{n^{th}\\Person}} = \frac{n(n-1)}{2}$$

Correct for Overcounting

There are n people in the group. Each person shakes hand with the remaining $n-1$ people, giving a total of

$$\underbrace{n}_{People} \times \underbrace{(n-1)}_{Handshakes} = n(n-1)$$

Since this overcounts the number of handshakes by a factor of two, the actual number of handshakes is:

$$\frac{n(n-1)}{2}$$

Example 1.7

How many handshakes will there be in a party with 14 people, if everyone shakes hands with everyone else?

Each person will shake hands with 13 other people. Since, Person A shaking hands with Person B, is the same as Person B shaking hands with Person A, we are overcounting by a factor of two.

$$\frac{14 \times 13}{2} = 91$$

Example 1.8: Multiplication Principle

Tyler has entered a buffet line in which he chooses one kind of meat, two different vegetables and one dessert. If the order of food items is not important, how many different meals might he choose?
Meat: beef, chicken, pork
Vegetables: baked beans, corn, potatoes, tomatoes
Dessert: brownies, chocolate cake, chocolate pudding, ice cream **(AMC 8 2001/14)**

Method I: Handshake Formula

Use the multiplication principle.

$$Number\ of\ Meals = \underbrace{3}_{\substack{Choices\\for\ Meat}} \times \underbrace{\frac{4 \times 3}{2}}_{\substack{Choices\ for\\Vegetables}} \times \underbrace{4}_{\substack{Choices\\for\ Dessert}} = 3 \times 6 \times 4 = 72$$

Method II: Combinations

$$Choose\ 1\ Meat\ out\ of\ 3 = \binom{3}{1}$$

$$Choose\ 2\ Vegetables\ out\ of\ 4 = \binom{4}{2}$$

$$Choose\ 1\ Dessert\ \ out\ of\ 4 = \binom{4}{1}$$

And since we need to make all the choices independently, we multiply them all:

$$\binom{3}{1}\times\binom{4}{2}\times\binom{4}{1}=\frac{3!}{1!\,2!}\times\frac{4!}{2!\,2!}\times\frac{4!}{1!\,3!}=3\times6\times4=72$$

Example 1.9: Multiplication Principle versus Overcounting

The chess club wants to send its two best players to represent the club at the district tourney. It has identified star players to choose from, of which there are seven. In how many ways can the coach make the selection:

A. If the players have to be identified as $first\ seed$, and $second\ seed$.
B. If only the two best players have to be identified.

Part A

Since the player are identified as first seed and second seed, order is important, and hence, we will use the multiplication principle.
For example

$$\left(\underbrace{Player\ A}_{First\ Seed},\underbrace{Player\ B}_{Second\ Seed}\right)\neq\left(\underbrace{Player\ B}_{First\ Seed},\underbrace{Player\ A}_{Second\ Seed}\right)$$

Using the multiplication principle, the number of ways to select the players if they need to be identified:

$$\underbrace{7}_{\substack{First\\Seed}}\times\underbrace{6}_{\substack{Second\\Seed}}=42$$

Part B

If we don't identify the players by seeds, we are overcounting by the number of ways that two players can be arranged among themselves, which is 2.
For example:

$$\left(\underbrace{Player\ A}_{First\ Seed},\underbrace{Player\ B}_{Second\ Seed}\right)=\left(\underbrace{Player\ B}_{First\ Seed},\underbrace{Player\ A}_{Second\ Seed}\right)$$

Hence, the number of ways to select two players from seven is:

$$\frac{7\times6}{2}=21$$

C. Back Calculations

It is possible to calculate the number of people, given the number of handshakes, by substituting it into the formula. Rather, than solving an equation a guess and check method is usually faster.

Example 1.10

A. In the $BIG\ N$, a middle school football conference, each team plays every other team exactly once. If a total of 21 conference games were played during the 2012 season, how many teams were members of the $BIG\ N$ conference? **(AMC 8 2012/14)**
B. At the end of a match between two teams, the players, the two coaches, and the two referees all shook hands with each other. If the total number of handshakes was 55, what is the number of players?

Part A

The number of games between two teams is the same as the number of pairs that can be formed.
To form a pair, suppose that at the beginning of each match, the two opposing captains shake hands.

Then, the number of games is the same as the number of handshakes that can take place.

Let there be n teams. Then, the number of games is:

$$\frac{n(n-1)}{2}=21\Rightarrow n(n-1)=42=6\times7$$

This is a quadratic. But it is better to solve by factoring the RHS:

$$n(n-1) = 42 = 6 \times 7 \Rightarrow n = 7$$

Part B

We first find the number of people:

$$\frac{n(n-1)}{2} = 55 \Rightarrow n = No.of\ People = 11$$

From the people, we subtract the non-players (two coaches, and two referees = 4):

$$No.of\ Players = n - 4 = 11 - 4 = 7$$

D. Forming Pairs

Again, the concept of forming pairs is directly related to the concept of handshakes, since we can imagine that each pair that is formed shakes hands once.

More difficult questions can provide a real scenario life that will require careful reading to

- Identify the parts of the question that matter
- Ignore the parts of the question that extra information and/or flavor text

Further, the question can also make minor variations on what is being asked to ensure that you have read it carefully.

Example 1.11: Dinner in Pairs

A group of nine friends decides to celebrate by going out for dinner. They only go out for dinner in pairs. How many dinners will be needed to ensure that everyone has dinner in a pair with everyone else possible?

Assume that at the start of every dinner, the pair going for the dinner shakes hands.

$$No.of\ Dinners = No.of\ Handshakes = \frac{9 \times 8}{2} = 36$$

Example 1.12: Study Pairs

A school teacher forms study pairs in a history class of 31 students. Every pair has two students. What is the number of distinct study pairs that can be formed?

$$No.of\ Study\ Pairs = No.of\ Handshakes = \frac{31 \times 30}{2} = 465$$

Example 1.13: Hugs

8 long-lost friends meet and hug each other. How many hugs will take place?

If a group of friends hugs each other, the number of hugs will be the same as the number of handshakes, since we are still forming pairs.

Imagine that the friends shake hands before exchanging hugs. Then:

$$No.of\ Hugs = No.of\ Handshakes = \frac{8 \times 7}{2} = 28$$

*Example 1.14: Standing guard

Frodo Baggins, Samwise Gamgee, Merry Brandybuck, and Pippin Took are four hobbits on a quest in dangerous lands to rid the world from the *One Ring*. They travel by day and set up camp by night. One of them stands guard the first half, and another the second half of the night. For this question, the order of standing guard in a single night does not matter.

A. Which is the first night where the pair standing guard must repeat?

B. The *Fellowship of the Ring* consists of the four aforementioned hobbits, Gandalf, Aragorn, the dwarf Gimli, the elf Legolas, and Boromir. Answer the same question as before.

Imagine that at the change of guard, the outgoing guard shakes the hand of the incoming guard.

Part A

$$\text{Four Hobbits}: \text{No. of Guard Pairs} = \text{No. of Handshakes} = \frac{4 \times 3}{2} = 6$$

We have six unique pairs, which can stand guard for six consecutive nights.
On the seventh night, the guard pair must repeat.

Part B

$$\text{Nine Members}: \text{No. of Guard Pairs} = \text{No. of Handshakes} = \frac{9 \times 8}{2} = 36$$

We have thirty-six unique pairs, which can stand guard for thirty-six consecutive nights.
On the thirty seventh night, the guard pair must repeat.

*Example 1.15

How many different pairs of positive whole numbers have a greatest common factor of 4 and a lowest common multiple of 4620? **(CEMC Gauss Grade 7 2021/24)**

Since the GCF of the two numbers is 4, both numbers must be a multiple of 4. Let the numbers be:

$$(4x, 4y), x \text{ and } y \text{ are co} - \text{prime}$$

Since the numbers are co-prime

$$LCM(x, y) = xy = \frac{4620}{4} = 1{,}155 = 3 \times 5 \times 7 \times 11$$

For each prime factor, we have two choices: factor of x, or factor of y.

$$\underbrace{2}_{\substack{Choices\\ for\ 3}} \times \underbrace{2}_{\substack{Choices\ for\\ for\ 5}} \times \underbrace{2}_{\substack{Choices\\ for\ 7}} \times \underbrace{2}_{\substack{Choices\\ for\ 11}} = 2^4 = 16$$

However, the pair (x, y) is the same as the pair (y, x) since the pairs are not ordered.

Hence, we need to divide 16 by 2 to prevent overcounting. Hence, the final answer is:

$$\frac{16}{2} = 8$$

*Example 1.16

In *Secret Santa*, each person in the office receives a gift from someone unknown to them, and they also give a gift back. If *Secret Santa* is held every year, after how many years must the pairs who give gifts to each other start repeating if:

A. The office has 22 people
B. The office has n people
C. Are there any restrictions on the value of n?

$$\text{No. of Years} = \frac{\text{Total No. of Pairs}}{\text{Pairs per Year}} = \frac{\frac{22 \times 21}{2}}{\frac{22}{2}} = \frac{22 \times 21}{22} = 21$$

$$No.of\ Years = \frac{Total\ No.of\ Pairs}{Pairs\ per\ Year} = \frac{\frac{n(n-1)}{2}}{\frac{n}{2}} = \frac{n(n-1)}{n} = n-1$$

n must be an even number. Else, one person will be left out of *Secret Santa*.

E. Selective Handshakes

To the base formula for handshakes, we can make the question a little more difficult by adding conditions that increase or decrease the number of handshakes.

If certain people do not shake hands among themselves, then those handshakes will need to be subtracted in order to find the actual count of handshakes.

$$Handshakes_{Actual} = Handshakes_{Max\ Possible} - Handshakes_{which\ did\ not\ happen}$$

The formula is based on the concept of complementary counting:

$$Total - Unwanted = Wanted$$

Example 1.17

The *Dance India Dance* competition has six dancing pairs. At the end of the competition the contestants shake hands among themselves. However, they do not shake hands with their respective partners. Find the number of handshakes that take place.

There are six dancing pairs, which gives the number of contestants as

$$12$$

Complementary Counting

Find the number of handshakes with no restrictions, and then subtract the handshakes that would have happened among the dancing pairs:

$$\frac{12 \times 11}{2} - 6 = 66 - 6 = 60$$

Direct Method

12 contestants will each shake hands with everyone, except their partner.
This means, every contestant will shake hands with 10 other people.

$$\underbrace{12}_{No.of\ People} \times \underbrace{10}_{No.of\ Handshakes} \times \underbrace{\frac{1}{2}}_{\substack{Overcounting \\ Factor}} = 60$$

Example 1.18

A party has fourteen people. Everyone at the party shakes hands with everyone else, except their spouse. Find the number of handshakes if:

A. The party has seven husband-wife pairs.
B. The party has five husband-wife pairs, and the rest are singles.

Since there 14 people, the number of handshakes with no restrictions

$$= \frac{14 \times 13}{2} = 7 \times 13 = 91$$

Part A

If there are seven pairs which do not shake hands among themselves, then the number of handshakes

$$91 - 7 = 84$$

Part B

If there are five pairs which do not shake hands among themselves, then the number of handshakes

$$91 - 5 = 86$$

F. Location Based Restrictions

Example 1.19

At the beginning of the month, four senior citizens decide to sit on a park bench in a line every morning, and hug the person next to them. They start on day 1. On day n everyone has hugged everyone else. What is the smallest possible value of n?

Count Number of Hugs

Count the number for everyone to hug each other.

$$No.of\ Hugs = No.of\ Handshakes = \frac{4 \times 3}{2} = 6$$

We not need to count the number of handshakes/hugs, which is 6. We also need to see how many days it will take. In particular, there is a restriction on location.

How many days does it take?

Four people sitting in a row gives us gives us 3 gaps, which means 3 hugs every day.

$$\text{Minimum number of days} = \frac{6}{3} = 2$$

Day 1	A	B	C	D
	AB, BC, CD			
Day 2	C	A	D	B
	CA, AD, DB			

But, we don't know whether this is achievable till we actually make them sit and hug each other.

$$\underbrace{\{A, B, C, D\}}_{People} \Rightarrow \underbrace{AB, AC, AD, BC, BD, CD}_{Distinct\ Hugs}$$

*Example 1.20

If the number of senior citizens increases to five, what is the answer?

Count Number of Hugs

Count the number for everyone to hug each other.

$$No.of\ Hugs = No.of\ Handshakes = \frac{5 \times 4}{2} = 10$$

How many days does it take?

Five people sitting in a row gives us gives us 4 gaps, which means 4 hugs every day.

$$\text{Minimum number of days} = \frac{10}{4} = 2.5 \Rightarrow Round\ Up = 3$$

$$\underbrace{\{A, B, C, D, E\}}_{People} \Rightarrow \underbrace{AB, AC, AD, AE, BC, BD, BE, CD, CE, DE}_{Distinct\ Hugs}$$

Day 1	A	B	C	D	E
	AB, BC, CD, DE				
Day 2	C	A	D	B	E
	CA, AD, DB, BE				
Day 3	A	E	C	B	D
	AE, CE				

G. Differential Handshakes

Example 1.21

I go to a fund-raising party, which is attended by 30 representatives from companies, 40 representatives from angel investors and thirty-one representatives from venture capitalists. Find the number of handshakes if all the $\underbrace{company\ representatives}_{CR}$ shake hands with all the $\underbrace{angel\ investors}_{AI}$ once, and all the $\underbrace{venture\ capitalists}_{VC}$ twice. All the angel investors shake hands with the venture capitalists thrice.

$$\underbrace{30 \times 40}_{CR\ with\ AI} + \underbrace{30 \times 31 \times 2}_{CR\ with\ VC} + \underbrace{40 \times 31 \times 3}_{AI\ with\ VC} = 1200 + 1860 + 3720 = 6780$$

Note that in the above calculation, we did not count:

- Angel Investors shaking hands with Company Representatives
- Venture Capitalists shaking hands with Company Representatives
- Venture Capitalists shaking hands with Angel Investors

Hence each handshake is counted only once, not twice.
And hence, we do not need to divide by two.

Example 1.22

The Little Twelve Basketball Conference has two divisions, with six teams in each division. Each team plays each of the other teams in its own division twice and every team in the other division once. How many conference games are scheduled? **(AMC 8 2005/14)**

Method I

First, calculate the number of games that each team will play.
Each division has 6 teams. So, within its division, a team has five other teams to play with. And with each of those teams, it will play two games each, giving:

$$\underbrace{5}_{\substack{No.of\\Teams}} \times \underbrace{2}_{\substack{Games\\per\ Team}} = 10$$

Outside the division, there are six teams, and each team will play with each of those six teams, giving:

$$\underbrace{6}_{\substack{No.of\\Teams}} \times \underbrace{1}_{\substack{Games\\per\ Team}} = 6$$

Hence, the total number of games that each team will play is:

$$10 + 6 = 16$$

Finally, the total number of games across all teams is:

$$\frac{16 \times 12}{2} = 8 \times 12 = 96$$

(where we divided by 2, because like handshakes, Team A playing Team B is the same as Team B playing against Team A).

Method II

Games within the first division:

$$2 \times \frac{6 \times 5}{2} = 30$$

Games within the second division:

$$2 \times \frac{6 \times 5}{2} = 30$$

Games between the divisions:

$$\frac{6 \times 6}{2} \times 2 = 36$$

Example 1.23

At a carrom tournament, people compete in teams of two, and pairs in a team shake hands with their partner once, and with everyone else three times each. How many handshakes will happen at the:

> A. Final
> B. Semi-Final
> C. Quarter Final

Part A: Final

Count the number of handshakes for each person:

$$\text{Handshakes within the team: } \underbrace{1}_{\substack{No.of\\People}} \times \underbrace{1}_{\substack{Handshakes\\per\ Person}} = 1$$

$$\text{Handshakes outside the team: } \underbrace{4-2}_{\substack{No.of\\People}} \times \underbrace{3}_{\substack{Handshakes\\per\ Person}} = 2 \times 3 = 6$$

Total handshakes per person

$$1 + 6 = 7$$

Total Handshakes in All

$$= \frac{7 \times 4}{2} = 14$$

Part B: Semi-Final

Count the number of handshakes for each person:

Handshakes within the team:

$$\underbrace{1}_{\substack{No.of\\People}} \times \underbrace{1}_{\substack{Handshakes\\per\ Person}} = 1$$

Handshakes outside the team:

$$\underbrace{8-2}_{\substack{No.of\\People}} \times \underbrace{3}_{\substack{Handshakes\\per\ Person}} = 6 \times 3 = 18$$

Total handshakes per person:

$$1 + 18 = 19$$

Total Handshakes in All

$$= \frac{19 \times 8}{2} = 76$$

Part C: Quarter-Final

Count the number of handshakes for each person:

Handshakes within the team:

$$\underbrace{1}_{\substack{No.of\\People}} \times \underbrace{1}_{\substack{Handshakes\\per\ Person}} = 1$$

$$\text{Handshakes outside the team: } \underbrace{16-2}_{\substack{No.of\\People}} \times \underbrace{3}_{\substack{Handshakes\\per\ Person}} = 14 \times 3 = 42$$

Total handshakes per person:

$$1 + 42 = 43$$

Total Handshakes in All:

$$= \frac{43 \times 16}{2} = 344$$

H. Ratios

At a formal western dance party, dances will happen in pairs, with every dance having a male partner and a female partner. These are often referred to in questions on dances.

Example 1.24

At a party, there were ten men and five women. Each man danced twice. Each woman had the same number of dances. Dances happened only in man-woman pairs. Find:

A. The total number of dances danced by the men.
B. The total number of dances danced by the women.
C. The number of dances that each woman had.
D. The ratio of the number of dances that each woman had to the number of dances that each man had. Compare this to the ratio of the number of women to the number of men.

Total Dances for Men

We use the multiplication principle:

$$Total\ Dances = \underbrace{10}_{\substack{No.of\\ Men}} \times \underbrace{2}_{\substack{Dances\\ per\ Man}} = 20$$

Total Dances for Women

Since we only have men-women dancing pairs, the total number of dances for the women must equal the total number of dances for the men.

$$Dances\ (Women) = Dances(Men) = 20$$

Total Dances

We use the multiplication principle, and solve for the dances per woman

$$Dances\ per\ Woman = \frac{Total\ Dances}{No.of\ Women} = \frac{20}{5} = 4$$

Ratio

$$\frac{Dances\ per\ Woman}{Dances\ per\ Man} = \frac{4}{2} = \frac{2}{1}$$

$$\frac{No.of\ Women}{No.of\ Men} = \frac{5}{10} = \frac{1}{2}$$

Note that the first ratio is the reciprocal of the second ratio. This is a general property.

I. Back Calculations

Example 1.25

Team A consists of 7 boys and n girls, and Team B has 4 boys and 6 girls. If a total of 52 single matches can be arranged between these two teams when a boy plays against a boy and a girl plays against a girl, then n is equal to: (JEE-M 2021)

Example 1.26

Two women and some men participated in a chess tournament in which every participant played two games with each of the other participants. If the number of games that the men played among themselves exceeds the

Example 1.27

At a party, each man danced with exactly three women and each woman danced with exactly two men. Twelve men attended the party. How many women attended the party? (AMC 10 2004)

Direct Method

$$No.of\ Dance(Men) = \underbrace{12}_{\substack{No.of\ Men\\ at\ the\ Party}} \times \underbrace{3}_{\substack{No.of\ Dances\\ by\ each\ Man}} = 36$$

No. of dances for men has to be same as total no. of dances for women

$$No.of\ Dance(Women) = 36$$

$$No.of\ women = \frac{No.of\ Dances}{Dances\ per\ Woman} = \frac{36}{2} = 18$$

Ratio Method

$$\frac{Dances\ per\ Man}{Dances\ per\ Woman} = \frac{3}{2}$$

$$\frac{No.of\ Men}{No.of\ Women} = \frac{2}{3} = \frac{12}{18} \Rightarrow No.of\ Women = 18$$

*Example 1.28

At a party, each man danced with exactly three women and each woman danced with the same number of men. Twelve men attended the party. Let n be the number of women at the party. Find the number of values that n can take? (AMC 10 2004, Adapted)

As in the earlier example,

$$No.of\ Dance(Men) = \underbrace{12}_{\substack{No.of\ Men\\ at\ the\ Party}} \times \underbrace{3}_{\substack{No.of\ Dances\\ by\ each\ Man}}$$

Again, as in the earlier example,

$$No.of\ women = \frac{No.of\ Dances}{Dances\ per\ Woman} = \frac{36}{Dances\ per\ Woman} = n, n \in N$$

$Dances\ per\ Woman$ can take the values of the positive factors of 36:

$$36 = 2^2 \times 3^2 \Rightarrow (2+1)(2+1) = 3 \times 3 = 9$$

But $n \geq 3$:

$$No.of\ Values\ of\ n = 9 - 2 = 7$$

J. Mixing Concepts / Extra Information

Example 1.29

What is the positive difference in the number of matches that two tournaments with n competitors (the first one being round-robin, the second one being single elimination) have?

$$\frac{n(n-1)}{2} - (n-1) = \frac{n(n-3)+2}{2}$$

K. Review and Challenge

Example 1.30

A teacher has made ten statements for a True-False test. Four statements are true and six are false. How many distinct answer keys could there be for the test? (MathCounts 2003 Counting/Combinatorics Stretch)

Overcounting

We need to decide the position of the True answers among the ten answers. For this, we have:

$$\underbrace{10}_{\substack{First\\Answer}} \times \underbrace{9}_{\substack{Second\\Answer}} \times \underbrace{8}_{\substack{Third\\Answer}} \times \underbrace{7}_{\substack{Third\\Answer}} \quad ways$$

However, we are not interested in the way that these four answers can be arranged among themselves, since we cannot distinguish one False answer from another. Hence, we are overcounting by a factor of 4!;

Combinations

Imagine that True is represented by T, and False is represented by F.
We need arrange the T's and the F's in the answer key. Once we choose the position of the F's, there is no choice left for the T's.

Hence, we need to count the number of ways to select the position of four F's among ten places:

$$\binom{10}{4} = \frac{10!}{4!\,6!} = \frac{10 \times 9 \times 8 \times 7}{4!}\frac{10 \times 9 \times 8 \times 7}{4!} = 210$$

*Example 1.31

A mathematical organization is producing a set of commemorative license plates. Each plate contains a sequence of five characters chosen from the four letters in AIME and the four digits in 2007. No character may appear in a sequence more times than it appears among the four letters in AIME or the four digits in 2007. A set of plates in which each possible sequence appears exactly once contains N license plates. Find $\frac{N}{10}$. **(AIME 2007/II/1)**

We will do this using casework. There are two possible cases:

- Either no or one zero is used in the license plate
  - ✓ This can be done using the multiplication principle
- Two zeros are used. In this case, we will place the zeros first.

Case I: No zero or One Zero is Used

We have seven alphanumeric digits to choose from, out of which we want to arrange five digits, which can be done, using the multiplication principle in

$$\underbrace{7}_{\substack{First\\Digit}} \times \underbrace{6}_{\substack{Second\\Digit}} \times \underbrace{5}_{\substack{Second\\Digit}} \times \underbrace{4}_{\substack{Second\\Digit}} \times \underbrace{3}_{\substack{Second\\Digit}} = 2{,}520\ Ways$$

Case II: Two Zeros Used

There are five spots

$$\underbrace{}\ \underbrace{}\ \underbrace{}\ \underbrace{}\ \underbrace{}$$

Of which we need to pick a pair for placing our two zeroes. This can be done in

$$\frac{5 \times 4}{2} = 10\ Ways$$

The remaining three digits can be filled in

$$6 \times 5 \times 4 = 120\ Ways$$

Giving us a total of

$$10 \times 120 = 1200\ Ways$$

Final Answer

The total ways for both the cases are:

$$N = 2520 + 1200 = 3720$$

$$\frac{N}{10} = \frac{3720}{10} = 372$$

1.2 Rotational Symmetry

A. Shaking hands with yourself

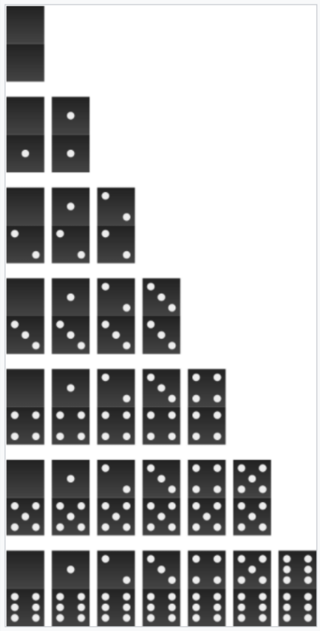
In this section, we will count the number of tiles in a domino set. A domino question is similar to the handshake problem, since you have to make pairs of numbers. But, it is also different since in a domino you can pair a number with itself, whereas in handshakes, a person will not shake hands with himself.

Example 1.32

You go to a party with 10 people, including yourself. Everyone shakes hands with everyone else (they are highly sociable). They also shake hands with themselves (you are very confused, and you *do not* shake hands with yourself). What is the number of handshakes that take place?

$$\frac{10 \times 9}{2} + 10 - 1 = 45 + 10 - 1 = 54$$

B. Background Information: Dominos


Complete double-six set

What is a domino

Dominoes is a game played with two squares, joined end to end. We will not be interested in the rules of the game. Rather, we will be interested in counting the number of tiles needed to form a complete set.
For our purposes, we will consider that that the back of the looks the same for all tiles. That is, the dominoes are identical when viewed from the back.

Dots on Dominos

Each of the squares on a domino has a number of dots on it. The number of dots starts at 0, and goes up to different numbers in different sets. A standard *Sino European* set of dominoes has 0 to 6 dots, and 28 dominoes. The illustration alongside shows all 28 dominoes in the set.
The next example provides a method of counting the number of tiles in a set of dominos.

Rotational Symmetry

Dominoes show rotational symmetry, as can be seen form the illustration on the right.

Example 1.33

What is the order of rotational symmetry for a domino?

C. Sino European Set

Example 1.34: Tabulation Method

The illustration alongside shows the dominoes in a traditional Sino European set.

A. How many dominoes does it have?
B. Tabulate the values as well.

First	Valid Pairs	No. of

Dot		Pairs
6	(6,6)(6,5)(6,4)(6,3)(6,2)(6,1)(6,0)	7
5	(5,5)(5,4)(5,3)(5,2)(5,1)(5,0)	6
4	(4,4)(4,3)(4,2)(4,1)(4,0)	5
3	(3,3)(3,2)(3,1)(3,0)	4
2	(2,2)(2,1)(2,0)	3
1	(1,1)(1,0)	2
0	(0,0)	1

Example 1.35: Forming Pairs

In a traditional Sino-European domino set, each square in the front of the domino has a number of dots ranging from zero to six. It is valid to pair a number on the left square with the same number on the right square. Find the number of distinct dominos in the set.

Since dominos have rotational symmetry, the dominoes

$$(0,7) \text{ and } (7,0)$$

Are the same, by rotation.

Hence, we need to find the number of ways that we can make pairs of the numbers from

$$\{0,1,2,3,4,5,6\} \Rightarrow 7 \text{ Numbers} \Rightarrow \frac{7 \times 6}{2} = 21 \text{ Dominos}$$

But, doubles are also allowed in dominoes:

$$(0,0),(1,1),\dots(6,6) \Rightarrow 7 \text{ Dominoes}$$

And hence the total is

$$21 + 7 = 28$$

Example 1.36: Alternate Solution

We redo the example above using a table method that helps visualize the number of dominoes.

If we had to count just handshakes, we would get:

$$\frac{7 \times 6}{2} = 21$$

$$Shortcut: \frac{7 \times 6}{2} + 7 = 21 + 7 = 28 \text{ Ways}$$

	0	1	2	3	4	5	6	Handshakes	Dominoes
0								0	1
1								1	2
2								2	3
3								3	4
4								4	5
5								5	6
6								6	7
							Total	$\frac{7 \times 6}{2} = 21$	28

1.37: Number of Dominoes

The number of distinct dominoes where the dots are from 0 to n is:

$$\frac{(n+1)(n+2)}{2}$$

$$\frac{(n+1)(n)}{2}+n=\frac{n^2-n+2n}{2}=\frac{n^2+n}{2}=\frac{n(n+1)}{2}$$

D. Other Sets

Example 1.38

Domino sets that have all possible combinations of dots from 0 to 9 are commercially available. What is the number of distinct dominos in such sets?

The number of possible dots are:

$$0,1,\ldots 9 \Rightarrow 10\ Options$$

The number of pairs of distinct number of dots that can be formed is

$$No.of\ Pairs=\frac{10\times 9}{2}=45\ Tiles$$

And, the number of doubles is:

$$(0,0)(1,1),\ldots(9,9) \Rightarrow 10\ Tiles$$

And, hence the total number of tiles is:

$$45+10=55$$

Example 1.39

Like Sino European dominoes, Chinese dominoes also have two adjacent identical squares. Find the number of tiles in a Chinese domino set given that:

A. Sino European dominos have 0 to 6 dots on a square. Chinese dominoes have dots from 1 to 6. Dominoes exist for every pair/combination that can be made using the dots from 1 to 6. (Order is not important).
B. 11 tiles are repeated (occur twice) in a Chinese set. No repetitions occur in a Sino European set.

The number of possible dots are:

$$1,2,\ldots,6 \Rightarrow 6\ Options$$

The number of pairs of distinct number of dots that can be formed is

$$No.of\ Pairs=\frac{6\times 5}{2}=15\ Tiles$$

And, the number of doubles is:

$$(1,1),(2,2)\ldots(6,6) \Rightarrow 6\ Tiles$$

And, hence the total number of tiles should be:

$$15+6=21$$

But, we have 11 repeated tiles, and hence the final answer is:

$$21+11=32$$

E. Using the Multiplication Principle

Example 1.40

A domino consists of two squares of the same size joined end to end, each of which has numbers from zero to nine. The dots on the domino come in three colours (black, green and red). All dots representing a number are the same color. The back of the domino is either striped or plain. A set of dominos consists of all the possible distinct dominoes. Each domino costs $1, and the seller wants to make a 20% profit on his cost. Find the selling price of a set of dominoes.

$$\underbrace{2}_{\substack{\text{Options}\\ \text{for Back}}} \times \underbrace{3}_{\substack{\text{No.of}\\ \text{Colours}}} \times \underbrace{\frac{9 \times 10}{2} + 11}_{\substack{\text{Number of}\\ \text{Dominoes}}} = 6 \times 55 = 330$$

$$\textit{Selling Price} = \underbrace{330}_{\substack{\text{No.of}\\ \text{Dominoes}}} \times \underbrace{1}_{\substack{\text{Cost per}\\ \text{Dominoes}}} \times \underbrace{1.2}_{\text{Add 20\%}} = 396$$

F. Trominoes

Example 1.41

2. Combinations

2.1 Basics

A. Basics

2.1: Factorial

$n!$ (read n factorial) is the product of the first n natural numbers.

$$n! = 1 \times 2 \times \ldots \times n, n \geq 1$$
$$n! = 1, n = 0$$

2.2: Combination Formula: Choosing r objects out of n objects

The number of ways to choose r objects out of n objects is given by:

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

$$\binom{n}{r} \text{ is read } \boldsymbol{n} \text{ choose } \boldsymbol{r}$$

The number of ways of arranging r distinct objects in n positions is:

$$n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) = \frac{n!}{(n-r)!}$$

If we are arranging r identical objects in n positions, then we are only choosing the positions, not the objects that go into the positions.

Hence, we are overcounting by the number of ways to arrange the r objects among themselves. And hence, we need to divide by $r!$

$$\frac{n!}{(n-r)!} \div r! = \frac{n!}{(n-r)!} \times \frac{1}{r!} = \frac{n!}{r!\,(n-r)!}$$

2.3: Choosing r objects out of n objects

$$^{n}C_r = \binom{n}{r}$$

Example 2.4

Calculate:

A. $\binom{4}{2}$
B. $\binom{6}{4}, \binom{6}{2}$
C. $\binom{7}{5}$
D. $\binom{10}{3}$
E. $\binom{8}{5}$
F. $\binom{5}{2}$
G. $\binom{7}{3}$
H. $\binom{9}{6}$

Part A

Substitute $n = 4, r = 2$ in $\binom{n}{r} = \frac{n!}{r!(n-r)!}$:

$$\binom{4}{2} = \frac{4!}{2!\,(4-2)!} = \frac{4!}{2!\,2!}$$

Expand and simplify:

$$\frac{4 \times 3 \times 2 \times 1}{(2 \times 1)(2 \times 1)} = \frac{4 \times 3}{2} = 2 \times 3 = 6$$

Part B

$$\binom{6}{4} = \frac{6!}{4!\,2!} = \frac{6 \times 5 \times 4!}{4!\,2!} = \frac{6 \times 5}{2} = 15$$

$$\binom{6}{2} = \frac{6!}{2!\,4!} = \binom{6}{4} = 15$$

Part C

$$\binom{7}{5} = \frac{7!}{5!\,2!} = \frac{7 \times 6 \times 5!}{5!\,2!} = \frac{7 \times 6}{2 \times 1} = 21$$

Part D

$$\binom{10}{3} = \frac{10!}{3!\,7!} = \frac{10 \times 9 \times 8 \times 7!}{3!\,7!} = \frac{10 \times 9 \times 8}{3!} = 120$$

Part E

$$\binom{8}{5} = \frac{8!}{5!\,3!} = \frac{8 \times 7 \times 6}{3!} = 56$$

Part F

$$\binom{5}{2} = \frac{5!}{2!\,3!} = \frac{5 \times 4}{2} = 10$$

Part G

$$\binom{7}{3} = \frac{7!}{3!\,4!} = \frac{7 \times 6 \times 5}{6} = 35$$

Part H

$$\binom{9}{6} = \frac{9!}{3!\,6!} = \frac{9 \times 8 \times 7}{2 \times 3} = 84$$

2.5: Interpreting n choose r

Given $\binom{n}{r}$ we can interpret it as choosing r objects out of n objects.

This may seem quite trivial, but is very useful in some difficult questions.

Example 2.6

Interpret the following as choosing $\boldsymbol{r}$ objects out of $\boldsymbol{n}$.

A. $\binom{10}{4}$
B. $\binom{7}{5}$
C. $\binom{45}{23}$

Part A

The total number of objects is 10.
The number of objects that we are choosing is 4.
Hence, $\binom{10}{4}$ is the number of ways of choosing 4 objects out of 10.

Part B
Number of ways of choosing 5 objects out of 7.

Part K
Number of ways of choosing 23 objects out of 45.

2.7: Properties of Combinations
The individual properties are not very difficult. But recognizing which property is applicable is a key skill that is needed to solve difficult questions. Each of the questions below uses a property from the ones just discussed. Identify which one, and use it.

$$\binom{n}{0} = \binom{n}{n} = 1$$
$$\binom{n}{1} = n$$
$$\binom{n}{2} = \frac{n(n-1)}{2}$$

Choosing zero objects
Number of ways of choosing zero objects out of n objects is:

$$\binom{n}{0} = \frac{n!}{0!\,(n)!} = \frac{n!}{1 \times n!} = \frac{n!}{n!} = 1$$

Recall that, by definition, zero factorial is one. This helps in simplifying a lot of calculations.
Logically speaking, there is only one way of not choosing n objects, which is to not choose all of them.

Choosing all objects
Number of ways of choosing n objects out of n objects is:

$$\binom{n}{n} = \frac{n!}{0!\,(n)!} = \frac{n!}{1 \times n!} = \frac{n!}{n!} = 1$$

Choosing/Rejecting one object out of n
Choosing one out of n objects will be:

$$\binom{n}{1} = \frac{n!}{1!\,(n-1)!} = n$$

Logically speaking if you have n objects, and you want to choose one of them, you can do it in n ways.

Choosing/Rejecting two objects out of n
Choosing two out of n objects will be

$$\binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n \times (n-1) \times (n-2)!}{2!\,(\boldsymbol{n-2})!} = \frac{n(n-1)}{2}$$

Example 2.8
A. I have five Chinese restaurants in my locality, and my family has decided to go to one of them for dinner. I don't like Chinese, and hence I will not visit any of the restaurants. In how many ways can I do this?
B. I have nine favorite *Hindi* action movies. Today is the last day of vacation, and I want to maximize my day. I have decided to watch one movie out of the nine. In how many ways can I do this?
C. I have nine favorite *Hindi* action movies. Every movie last exactly three hours. Today is the last day of vacation, and I want to maximize my day. I have twenty-four hours to watch the maximum number of movies that I can. In how many ways can I do this?
D. I have a fancy-dress party. I want to select exactly one mystery pair in the party: both of whom will play

the role of the heroes. If there are 4 people in the party, in how many ways can I do this?

E. If the above question is changed so that one person in the pair plays the role of hero, and the other becomes a villain, then how many pairs can I make?

Part A
I can do it in 1 way only. Not go to any of them.

Part B
Logically, you are going to watch one movie out of 9, hence, you can do this in

$$9\ Ways$$

$$\binom{9}{1}=\frac{9!}{1!\,8!}=9\ Ways$$

Part C
Number of movies that he can watch is:

$$\frac{24}{3}=8\ Movies$$

Logically, you are going to watch eight movies out of 9. Choosing 8 movies is the same as rejecting 1 movie. Hence, you can do this in

$$9\ Ways$$

Choosing 8 Movies out of 9:

$$\binom{9}{8}=\frac{9!}{8!\,1!}=9\ Ways$$

Choosing 8 movies is the same as rejecting 1 movie. Hence, we need to decide 1 movie to reject which can be done in:

$$\binom{9}{1}=\frac{9!}{1!\,8!}=9\ Ways$$

Part D
Method I: Enumeration

Suppose the people in the party are

$$ABCD$$

Then the number of ways of choosing the mystery pair is:

$$AB, AC, AD$$
$$BC, BD$$
$$CD$$

Note that choosing AB as your mystery pair is the same as choosing BA as your mystery, because both of these are going to play the role of hero.

Method II: Formula

There are four people in the party, of whom we want to choose 2.
Hence, the number of ways of doing this is:

$$4\ choose\ 2=\binom{4}{2}=\frac{4!}{2!\,2!}=\frac{4\times 3}{2}=6$$

Part E
Since order is important here the number of pairs will exactly double, giving us:

$$6 \times 2 = 12$$

Example 2.9

I am planning a trip to Europe. I have shortlisted ten cities to visit. If the order in which I visit the cities is not important, what is the number of ways in which I can visit (answer each question independently):

A. All of them
B. One of them
C. Two of them
D. Eight of them
E. Nine of them
F. None of them

Since the order of visiting the cities is not important, we are choosing, not arranging. Hence, we need to use combinations.

Part A

There are ten cities, and we are choosing all ten. This can be done in:

$$\binom{10}{10} = 1\ Way$$

Part B

There are ten cities, and we are choosing one of them to visit. This can be done in:

$$\binom{10}{1} = 10\ Ways$$

Part C

There are ten cities, and we are choosing two to visit. This can be done in:

$$\binom{10}{2} = \frac{10 \times 9}{2} = 45\ Ways$$

Part D

There are ten cities, and we are choosing eight to visit. This is the same as rejecting two cities to visit, and hence it can be done in:

$$\binom{10}{8} = \binom{10}{2} = \frac{10 \times 9}{2} = 45\ Ways$$

Part E

There are ten cities, and we are choosing nine of them to visit. This is the same as rejecting a single city, and this can be done in:

$$\binom{10}{9} = \binom{10}{1} = 10\ Ways$$

Part F

There are ten cities, and we are choosing zero cities, which is the same as rejecting all ten cities. This can be done in:

$$\binom{10}{0} = \binom{10}{10} = 1\ Way$$

2.10: Symmetry in Combinations

There is a powerful symmetry in the formula for combinations, which can be seen in the following relation.

$$\binom{n}{r} = \binom{n}{n-r}$$

Note that the expression below is an identity since:

- the numerator of the LHS is equal to the numerator of the RHS.
- The denominator has the same terms, but in a different order, and since the terms are multiplied, the value is the same.

$$\frac{n!}{r!\,(n-r)!}=\frac{n!}{(n-r)!\,r!}$$

But note that

$$LHS=\frac{n!}{r!\,(n-r)!}=\binom{n}{r}$$
$$RHS=\frac{n!}{(n-r)!\,r!}=\binom{n}{n-r}$$

Hence

$$\binom{n}{r}=\binom{n}{n-r}$$

Example 2.11

Determine the value of x in each case:

A. $\binom{17}{3}=\binom{17}{x}, x\neq 3$
B. $\binom{12}{5}=\binom{12}{x}, x\neq 5$
C. $\binom{14}{0}=\binom{14}{x}, x\neq 0$

$$\binom{17}{3}=\binom{17}{14}$$
$$\binom{12}{5}=\binom{12}{7}$$
$$\binom{14}{0}=\binom{14}{14}$$

2.12: Examples of Symmetry in Combinations

The most use of symmetry is encountered by substituting the first three positive values of r:

$$r=0:\ \binom{n}{0}=\binom{n}{n}=1$$
$$r=1:\ \binom{n}{1}=\binom{n}{n-1}=n$$
$$r=2:\ \binom{n}{2}=\binom{n}{n-2}=\frac{n(n-1)}{2}$$

While symmetry is very powerful, the above three examples of symmetry are encountered very often. Hence, it is important to recognize and recall them instantly.

Example 2.13

Calculate $\binom{20}{18}$

$$\binom{20}{18}=\binom{20}{2}=\frac{20\times 19}{2}=10\times 19=190$$

B. Scenarios

Example 2.14

A. I have a team of eleven basketball players, of whom I must choose the starting five. What is the number of ways in which I can do this?

B. I have a team of soccer players that has ten people who practice. I need to select three of the best players for extra practice. What is the number of ways in which I can make the selection? What if I want to rank the three best players, not just select them?

C. I must choose four out of six friends to invite for a party. In how many ways can I do this?

D. I am applying for college to the US, and need to list my three favorite books. I have made a shortlist of six books. In how many ways can I do this, if the order does not matter? If the order does matter?

E. There are nine applicants applying for three positions of Software Engineer. If the candidates are equally capable, in how many ways can the selection be made?

F. Saloni has shortlisted seven activities that she wants to do during the vacation. If she only has time for four of them, in how many ways can she choose the activities?

G. Election for Officers are taking place at my favorite fraternity. There are 9 candidates of whom six are to be selected. In how many ways can the winners be selected?

Part A

Number of ways to arrange five players out of eleven players is:

$$11 \times 10 \times 9 \times 8 \times 7 = \frac{11!}{6!}$$

We want to choose, not arrange. So, we need to divide by the number of ways of arranging five players among themselves.

Hence, the final number of ways is:

$$\binom{11}{5} = \frac{11!}{5!\,6!}$$

Part B

Formula

Number of ways to choose three players out of ten:

$$= \binom{10}{3} = \frac{10!}{3!\,7!} = \frac{10 \times 9 \times 8}{2 \times 3} = 10 \times 3 \times 4 = 120$$

Method II: Selecting using Permutations

Number of ways of selecting and arranging three players out of ten is:

$$\underbrace{10}_{\substack{First\\Player}} \times \underbrace{9}_{\substack{Second\\Player}} \times \underbrace{8}_{\substack{Third\\Player}} = 10 \times 9 \times 8$$

However, you wish to only select, and not arrange. Hence, we are overcounting by the number of ways that three players can be arranged in a row.

$$\therefore No. of\ ways\ to\ select = \frac{10 \times 9 \times 8}{3!} = 120$$

Ranking

$$\underbrace{10}_{\substack{First\\Player}} \times \underbrace{9}_{\substack{Second\\Player}} \times \underbrace{8}_{\substack{Third\\Player}} = 10 \times 9 \times 8$$

Part C

Using Combinations

We need to choose 4 objects out of 6 objects, which can be done is

$$\binom{6}{4} = \frac{6!}{4!\,2!} = \frac{6 \times 5}{2} = 15$$

Using Overcounting

The number of ways to arrange four friends out of six is:

$$6 \times 5 \times 4 \times 3$$

We need to only select the friends, not arrange them, and hence we need to divide by the number of ways in which four friends can be arranged

$$\frac{6\times 5\times \mathbf{4}\times \mathbf{3}}{\mathbf{4}\times \mathbf{3}\times 2\times 1}=\frac{6\times 5}{2}=15$$

Part D

Using Combinations

We need to choose 3 objects out of 6 objects, which can be done is

$$\binom{6}{3}=\frac{6!}{3!\,3!}=\frac{6\times 5\times 4}{6}=20$$

Using Overcounting

The number of ways to arrange three objects out of six is:

$$6\times 5\times 4$$

We need to only select the objects, and not arrange them, and hence we need to divide by the number of ways in which three objects can be arranged:

$$\frac{6\times 5\times 4}{3\times 2\times 1}=20$$

Part E

We need to choose 3 objects out of 9 objects, which can be done is

$$\binom{9}{3}=\frac{9!}{3!\,6!}=\frac{9\times 8\times 7}{6}=84$$

Part F

We need to choose 4 objects out of 7 objects, which can be done is

$$\binom{7}{4}=\frac{7!}{4!\,3!}=\frac{7\times 6\times 5}{6}=35$$

Part G

We need to choose 6 objects out of 9 objects, which can be done in

$$\binom{9}{6}=\frac{9!}{6!\,3!}=\frac{9\times 8\times 7}{6}=84\,Ways$$

Example 2.15

I am planning a hiking trail. I have eight populated towns that are near enough to forested areas to be visited as part of the hike. I need to choose three towns for this purpose. In how many ways can I do this, if:

A. the order of visiting the towns is not important?
B. the order of visiting the towns is important?

Using Combinations

We need to choose 3 objects out of 8 objects, which can be done is

$$\binom{8}{3}=\frac{8!}{3!\,5!}=\frac{8\times 7\times 6}{6}=56$$

Using Overcounting

The number of ways to arrange three objects out of eight is:

$$8\times 7\times 6$$

We need to only select the objects, and not arrange them, and hence we need to divide by the number of ways in which three objects can be arranged:

$$\frac{8\times 7\times 6}{6}=56$$

Part B

We can visit the towns in

$$8P3=\frac{8!}{5!}=8\times 7\times 6=336$$

C. Choosing between Permutations and Combinations

2.16: Choices with Repetition

If I have to make r independent choices, for each of which I have n options, then the number of ways in which I can do it is:

$$\underbrace{n \times n \times \dots \times n}_{r\ times} = n^r$$

2.17: Examples of Choices with Repetitions

- Ten people each of whom will pass or fail a course $= 2^{10} \neq 2 \times 10$
- Ordering Continental, Indian or Chinese cuisine for each day from Monday to Friday, with repetition allowed $(= 3^5 \neq 3 \times 5)$

2.18: Examples of Arrangements (Permutations)

Some of the common/classic situations asked in exam questions which $often$[1] lead to permutations are:

- Arranging books in a row
- Seating people in a row
- Selecting top three ranks in a class to receive gold, silver, bronze medals
- Arranging letters in a license plate
- Forming options to break a code or a lock
- Ranking a team of players

In all of the above situations, order is important. Remember that arrangements make use of permutations where

$$\text{No. of ways of arranging } r \text{ out of } n \text{ distinct objects is:} \frac{n!}{(n-r)!}$$

2.19: Examples of Selections (Combinations)

- Selecting books to read
- Choosing people to sit
- Selecting three people in a class to receive identical achievement medals
- Forming a team of players
- Forming a committee from a set of people
- Forming Subsets

In all of the above situations, order is *not* important. Remember that selections make use of combinations where

$$\text{No. of ways of choosing } r \text{ out of } n \text{ distinct objects is:} \frac{n!}{r!\,(n-r)!}$$

Example 2.20

In how many ways can I order pizza over four days, if

A. every day I will order a different topping out of tomatoes, bell peppers, chicken, and green chillies?

---

[1] The emphasis is on often, since you should not memorize keywords. It is possible to create questions that have keywords associated with certain concepts, but because of the phrasing of the question, use a different concept.

B. every day I will order one topping out of tomatoes, bell peppers, chicken, and green chillies?

Part A

I need to select a different topping each day of the week.
Order is important, since having chicken on Tuesday is not the same as having on Wednesday.
Total Ways

$$= Ways\ to\ arrange\ four\ items = 4!$$

Part B

$$4^4 = 2^8 = 256$$

Example 2.21

In how many ways can I choose three of five main courses, two desserts of four, and add-on an (optional) entrée?

The number of ways to choose three of five main courses is:

$$\binom{5}{3} = 10\ Choices$$

The number of ways to choose two desserts out of four is:

$$\binom{4}{2} = 6\ Choices$$

With respect to the entrée, you have two choices (add, or don't add):

$$2\ Choices$$

And, by the multiplication rule (since each of the choices can be mixed in whichever way you want)

$$\underbrace{10}_{\substack{Main\\Course}} \times \underbrace{6}_{Desserts} \times \underbrace{2}_{Entree} = 120$$

Example 2.22

A. In how many ways can I get the three chemistry toppers to sit in the three front row seats, the five physics toppers to sit in the five second row seats, and the seven math toppers to sit in the seven third row seats?
B. Answer the above question if the chemistry toppers sit in the third row, the physics toppers sit in the first row, and the math toppers sit in the second row. (Only focus on seating arrangements, and keep in mind that one seat can sit a maximum of one person).

Part A

$$3! \times 5! \times 7!$$

Part B

Number of ways to seat the chemistry toppers:

$$= \binom{7}{3} \times 3!$$

Number of ways to seat the physics toppers:

$$= \binom{5}{3} \times 3!$$

Number of ways to seat the math toppers:

$$= \binom{7}{5} \times 5!$$

D. Contrasting arrangements, selections and repetition

Example 2.23

I have seven novels on my reading list. I need to read three of them. In how many ways can I do this. Answer in two different ways: if order matters, and if it doesn't. Also, answer in two different ways: books can be repeated, and books can't be.

		Order	
		Matters	Does Not Matter
Repetition	Allowed	$7 \times 7 \times 7 = 7^3$	$\binom{7}{1} + \binom{7}{2} + \binom{7}{3}$
	Not Allowed	$7 \times 6 \times 5 = 210$	$\binom{7}{3}$

If repetition is allowed and order does not matter

We consider three distinct cases. We can choose

Case I: Three distinct books

$$\binom{7}{3}$$

Case II: Two distinct books

$$\binom{7}{2}$$

Case III: Only one distinct book

$$\binom{7}{1}$$

Example 2.24

My college offers five elective subjects. I need to choose three subjects in the next semester. In how many ways can I do this if:

A. The sequence in which I take the subjects is important, since I must submit the subjects in order of priority for me.
B. The order of selection of subjects is not important.

Part A

Since the order does matter, I need to arrange three out of seven books, which can be done in

$$5 \times 4 \times 3 = 60\ Ways$$

Part B

Since the order does not matter, I need to choose three out of five electives, which can be done in

$$\binom{5}{3} = 10\ Ways$$

Example 2.25

I need to display five of seven books on a bookshelf. In how many ways can I do this if:

A. The order of the books does not matter
B. The order of the books does matter

Part A

Since the order does not matter, I need to choose five out of seven books, which can be done in

$$\binom{7}{5} = \binom{7}{2} = \frac{7 \times 6}{2} = 21\ Ways$$

Part B

Since the order does matter, I need to arrange five out of seven books, which can be done in

$$^7P_5 = \frac{7!}{2!} = \frac{5040}{2} = 2520\ Ways$$

Example 2.26

My approved list of restaurants has ten names. Four of my friends are visiting me on four distinct (fixed) days. I am making my schedule for those four days, where on each day I mention where I will go for dinner. A sample schedule is shown in the table alongside.

Date	Friend	*Restaurant*
7^{th} *July*	Ilaiyaraja	*Golden Delicacies*
8^{th} *July*	Ilanthirayan	*Ming's Paradise*
9^{th} *July*	Chinnamani	*Fastest Fork First*
10^{th} *July*	Eelampirai	*Radhakrishna*

How many distinct schedules can I make if:

A. I can repeat restaurants. The order of visiting the restaurants matters.
B. I cannot repeat restaurants. The order of visiting the restaurants matters.
C. I cannot repeat restaurants. The order of visiting the restaurants does not matter.
D. I can repeat restaurants. The order of visiting the restaurants does not matter.

		Order	
		Matters	Does Not Matter
Repetition	Allowed	$10 \times 10 \times 10 \times 10 = 10^4$ ***Part A***	$\binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \binom{10}{4}$ ***Part D***
	Not Allowed	$10 \times 9 \times 8 \times 7$ ***Part B***	$\binom{10}{4}$ ***Part C***

E. Reframing the Question

Example 2.27

100 raffle tickets for a trip to Mount Doom are available. Bilbo buys ticket 27. Five raffle winners are announced randomly. What is the probability that Bilbo has a winning ticket?

Method I: Combinations

The total number of ways to select 5 tickets from 100 is

$$\binom{100}{5}$$

Divide the 100 tickets into two types: Ticket 27, and the remaining 99 tickets.
We need to select ticket 27, so we have no choice there. From the remaining 99, we need to select 4, which can be done in

$$\binom{99}{4}\ ways$$

The probability is:

$$\frac{\binom{99}{4}}{\binom{100}{5}} = \frac{\frac{99!}{4!\,95!}}{\frac{100!}{5!\,95!}} = \frac{99!}{4!\,95!} \cdot \frac{5!\,95!}{100!} = \frac{5}{100} = \frac{1}{20}$$

Method II: Clever

Reverse the process. Let the winning tickets be decided from the 100 tickets which are sold. Then, there are:

$$Winning\ Tickets = 5, \qquad Total\ tickets = 100$$

The probability of picking a winning ticket is then:

$$\frac{5}{100}=\frac{1}{20}$$

F. Review

Example 2.28

Vedika likes four cities in the US: Seattle, New York, Philadelphia and Denver. She like five cities in Europe: Rome, Milan, London, Istanbul and Madrid. How many choices does she have if she is going to visit:

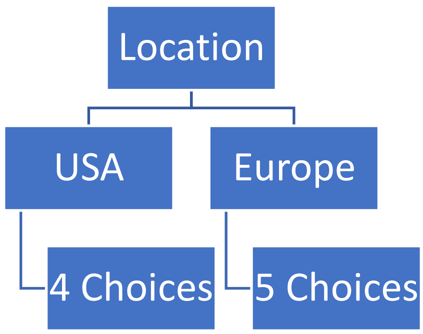
A. A single city
B. Two cities in USA
C. Two cities in Europe
D. A city in the USA, and a city in Europe
E. All the cities

Answer each question above considering

I. Order is important
II. Order is not important

Part A

$$9\ Choices$$



Part B

$$4\times 3$$
$$\binom{4}{2}$$

Part C

$$5\times 4$$
$$\binom{5}{2}$$

Part D

$$5\times 4\times 2$$
$$5\times 4$$

Part E

$$9!$$
$$1$$

True/False 2.29

I have a counting question. I know Method A will find the answer, but it is very calculation-intensive. I think a while and find the answer to the original question using Method B. Have I also found the answer to Method A?

Yes, counting in two ways is a powerful technique.
It uses the bijection principle.

True/False 2.30

A. Imposing a restriction in a combination question will always reduce the number of ways in the final answer.

B. Removing a restriction in a combination question will always increase the number of ways in the final answer.

False
False

Example 2.31

For $n > 1$, under what conditions does the following hold:

$$\binom{n}{2} > \binom{n}{1}$$

$$\frac{n(n-1)}{2} > n \Rightarrow \frac{n-1}{2} > 1 \Rightarrow n-1 > 2 \Rightarrow \ n > 3$$

Example 2.32

Is there a solution in natural numbers n and r to $\binom{n}{r} = 0$, given that $0 \le r \le n$,. Explain why or why not?

$$\min\left(\binom{n}{0}, \binom{n}{1}, \dots \binom{n}{n}\right) = 1 \Rightarrow \textit{No Solutions}$$

G. Extension

Example 2.33

What is the difference between an arrangement and a selection?

A. Explain using the definition
B. Explain using a practical example
C. Calculate the numerical difference in your practice example between the arrangement and the selection.

Example 2.34

A. Read the questions in these notes and elsewhere to see the keywords that distinguish questions that require permutations from those that require combinations?
B. Does the presence of a keyword guarantee the use of permutations or combinations?
C. Write a question which has a "permutation" keyword, but the answer uses combinations.
D. Write a question which has a "combination" keyword, but the answer uses permutation.
E. What do the above parts tell you about a keyword-based approach to questions. Is it advisable?

Example 2.35

The number of ways of arranging r objects out of n is given by nPr, while the number of ways of selecting r objects out of n is given by $\binom{n}{r}$.

A. Write the formulas for nPr and $\binom{n}{r}$. You will find them in the chapter on Permutations, and Combinations, respectively.
B. Find the ratio nPr: $\binom{n}{r}$
C. Use your answer to the previous part to determine under what conditions is $nPr = \binom{n}{r}$

Investigation 2.36: Pascal's Triangle

A. Write the values of $\binom{4}{r}$ for $r = 0$ to $r = 4$. Where is it maximum? Where is it minimum? Do you notice a pattern?
B. Repeat the above question for $\binom{5}{r}, \binom{6}{r}$ and $\binom{7}{r}$ (for suitable values of r). Does the pattern still hold?
C. What observations can you make about the pattern. If you start from $n = 1$ and proceed, you create *Pascal's Triangle*, which is an important area of mathematics, with many interesting properties and applications.
D. When is $\binom{n}{r+1}$ greater than $\binom{n}{r}$. You will find the previous investigation useful.
E. What is the maximum value of $\binom{n}{r}$. Consider two cases. First, when n is odd and second, when n is even.

2.2 Counting Numbers

A. Product of Digits

Example 2.37

Find the number of four digit numbers whose product of digits is 18.

abcd = 18 =2*3*3
Product of two factors greater than 1:
(1,18) does not work
(2,9) 1129 arranged in 4C2=6 Ways
(3,6) 1136 arranged in 4C2 = 6 Ways

Product of three factors, each greater than 1:
(2,3,3) 1233 arranged in 4C2 = 6 Ways

B. Descending and Ascending Numbers

2.38: Descending Numbers

- A descending number is a number such that each successive digit is smaller than the one that comes before it.
- A single digit number is also descending since there is only one digit, and it does not violate the condition imposed.
- A descending number cannot repeat digits

A descending number refers to the digits of a number, and not two different numbers.
For example, the numbers 23, 72, 11 when arranged in descending order are

$$\Rightarrow 72, 23, 11$$

However, in a descending number, the digits are arranged in descending order.

Some examples of descending numbers are:

$$91: 9 > 1$$
$$541: 5 > 4 > 1$$
$$870: 8 > 7 > 0$$

$$755: 7 > 5 = 5 \Rightarrow \textit{Neither Ascending nor descending}$$
$$99: 9 = 9 \Rightarrow \textit{Neither Ascending nor descending}$$

$$437: 4 > 3, \qquad 3 < 7 \Rightarrow Neither\ Ascending\ nor\ descending$$

Example 2.39

A. What is the largest descending number?
B. Hence, what is the maximum number of digits that a descending number can have?

$$9{,}876{,}543{,}210 \Rightarrow 10\ Digits$$

Example 2.40

I have a five-digit descending number. Answer each question independently.

A. What is the largest digit that can be at the ten thousand's place?
B. What is the smallest digit that can be at the ten thousand's place?
C. What is the largest digit that can be at the unit's place?
D. What is the smallest digit that can be at the unit's place?

$$98765 \Rightarrow 5$$
$$43210 \Rightarrow 0$$
$$98765 \Rightarrow 5$$
$$43210 \Rightarrow 0$$

Example 2.41

What is the number of four-digit descending numbers?

Note that there are ten digits in the decimal system:

$$0-9$$

From above, we know that digits cannot be repeated.

Hence, for a four-digit number, we must choose four distinct digits out of ten possible digits.
Once we choose the digits, they can be arranged in descending order in only one possible way.

$$\binom{10}{4} = 210\ Ways$$

Example 2.42

What is the number of n digit descending numbers?

$$n = 0: Zero\ Numbers$$
$$1 \le n \le 10 \Rightarrow \binom{10}{n}$$
$$n > 10 \Rightarrow Zero\ Numbers$$

2.43: Ascending Numbers

- An ascending number is a number such that each successive digit is greater than the one that comes before it.
- A single digit number is also ascending since there is only one digit, and it does not violate the condition imposed.

Some examples of ascending numbers are:

$$249: 2 < 4 < 9$$
$$28: 2 < 8$$

$$189: 1 < 8 < 9$$

Example 2.44
What is the number of ascending numbers which are two digits or less?

Consider choosing two digits out of ten, which can be done in

$$\binom{10}{2} = 45\ Ways$$

And then arranging them in ascending order. There are two cases:

First Digit is not Zero

This becomes a two-digit ascending number.

First Digit is Zero

This becomes a one-digit ascending number.

Number Zero

The number zero is not included in the above counting, but is nevertheless an ascending number.

Hence, final answer is:

$$45 + 1 = 46$$

Example 2.45
What is the number of ascending numbers which are three or four digits?

Consider choosing four digits out of ten, which can be done in

$$\binom{10}{4} = 210\ Ways$$

And then arranging them in ascending order. There are two cases:

First Digit is not Zero

This becomes a four-digit ascending number.

First Digit is Zero

This becomes a three-digit ascending number.

Example 2.46
What is the number of ascending numbers which are four digits or less?

For four-digit numbers, the starting digit cannot be zero. Hence, we only have nine digits to choose from:

$$Four\ Digit\ Numbers = \binom{9}{4}$$

Similarly, number of three-digit ascending numbers

$$= \binom{9}{3}$$

And, number of two-digit ascending numbers

$$\binom{9}{2}$$

However, for a single digit number, zero is an acceptable ascending number. Hence, the number of single digit ascending numbers is

$$\binom{10}{1} = 10$$

$$\binom{9}{4} + \binom{9}{3} + \binom{9}{2} + \binom{10}{1} = 256$$

2.3 Restrictions

A. Introduction

When choosing objects, we use the formula that the number of ways to choose r objects out of n objects is given by

$$n \ choose \ r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

To this choosing, we can apply one or more restrictions that complicate the situation, and reduce our choices.

Some important things to remember in complex problems:

- Apply the most restrictive condition first. (This is not a rule, but it often helps in making the rest of the counting easier).
- At no stage will a condition or a restriction increases the number of choices. At best, it will keep the number of choices the same.

B. Basic Restrictions

2.47: Never Selected

The number of ways of choosing r objects from n objects, out of which a objects are never to be selected is:

$$\binom{n-a}{r}$$

$$\underbrace{n \ objects}_{Starting\ Scenario} \rightarrow Never\ choose\ a\ objects \rightarrow \underbrace{(n-a)\ objects}_{\substack{Effective\ Number\\ of\ Objects}}$$

- If some objects are never selected, then remove them from the set of objects from which the choice is to be made.
- In more difficult questions, this kind of condition is combined with other conditions.

Example 2.48

A Go Club must choose three team members out of fifteen candidates. In how many ways can the selection be made

A. With no restrictions
B. If, out of the fifteen, a set of triplets have allergic rhinitis, and cannot go for the tournament.

Write your answer in $\binom{n}{r}$ notation.

Part A

$$\binom{15}{3}$$

Part B

There are fifteen members, but three of them are not available. Hence, we only have twelve members from which the team can be formed.

In other words, we need to choose a team of three members from twelve eligible candidates, which can be done in

$$\binom{12}{3} = \frac{12!}{3!\,9!} = \frac{12 \times 11 \times 10}{6} = 220$$

2.49: Always Selected

The number of ways of choosing r objects from n objects, out of which b objects are always to be selected is:

$$\binom{n-b}{r-b}$$

$$\underbrace{n\ objects}_{Starting\ Scenario} \rightarrow Always\ choose\ a\ objects \rightarrow \underbrace{(n-a)\ objects\ to\ choose\ from}_{\substack{Effective\ Number \\ of\ Objects}}$$

$$No.\ of\ objects\ to\ actually\ choose\ becomes\ (r-a)$$

Example 2.50

A Scrabble group has eight enthusiasts, of which four are to be selected for a Scrabble meet. Two of the participants are so good that the coach has already confirmed their selection. In how many ways can the team of four be selected?

To select 4 candidates out of 8, we would normally do:

$$\binom{8}{4}$$

Out of which two have already been selected, leaving only a pool of six enthusiasts for the remaining two slots. Hence, we need to select two more participants out of the remaining six, which can be done in

$$\binom{8-2}{4-2} = \binom{6}{2} = \frac{6!}{2!\,4!} = \frac{6 \times 5}{2} = 15$$

2.51: Combining Restrictions

The number of ways to select r objects from n objects, out of which a objects are never selected, and b objects are always selected is:

$$\binom{n-a-b}{r-b}$$

Example 2.52

Five players going for a camp must be chosen from ten Ice Hockey players. There are two good players, who must be selected, and one player is down with an injury and cannot be selected. What is the number of ways in which I can make the selection?

There are ten players. But one is injured, so we only have

$$10 - 1 = 9\ Fit\ Players$$

Out of the fit players, the already selected players block two spots:

$$\underbrace{Player\ 1}_{Very\ Good\ Player}, \underbrace{Player\ 2}_{Very\ Good\ Player}, \underbrace{Player\ 3}_{Slot\ Open}, \underbrace{Player\ 4}_{Slot\ Open}, \underbrace{Player\ 5}_{Slot\ Open}$$

Hence, we must fill three slots from remaining $9 - 2 = 7$ players:

$$= \binom{9-2}{3} = \binom{7}{3} = \frac{7!}{4!\,3!} = \frac{7 \times 6 \times 5}{6} = 35$$

Shortcut

$$\binom{10-1-2}{3} = \binom{7}{3} = 35$$

C. Always selected together

In certain scenarios, an object can only be selected when other objects in the group are also selected.

Calculating the number of ways this can be done usually requires casework.

We can have a condition where a pair gets selected together, or does not get selected at all. There may be various real-life scenarios for this:

- Two players who play really well together on a team, and hence must be chosen together, or not at all.
- A parent who must always stay with a child.

Such kind of restrictions usually require casework.

2.53: Always Selected Together

When two or more objects are always selected together, the number of ways to do so can be calculated using casework.

Example 2.54

I must select four students out of 9 for my team math contest. The Williams triplets do math fabulously together, but are miserable without each other, so I will choose them all together, or not at all. In how many ways can I make the selection?

There are two cases to consider:

Case I: You pick the triplets

Making use of the triplets "always selected" logic:

$$\binom{9-3}{4-3} = \binom{6}{1} = 6$$

Case II: You don't pick the triplets

Making use of the triplets "never selected" logic:

$$\binom{9-3}{4} = \binom{6}{4} = \binom{6}{2} = \frac{6 \times 5}{2} = 15$$

And, hence the final number of ways to select is:

$$= 6 + 15 = 21$$

Example 2.55

I must select four team members for my team math contest. The Irving twins do math fabulously together, but are miserable without each other, so I have to choose between taking both of them, or neither of them. If there are ten club members to choose from, in how many ways can I make the selection?

There are two cases to consider:

Case I: Twins Always Selected

There are ten people to choose from, out of which since you have already picked the twins, you are left with

$$10 - 2 = 8 \ Members\ to\ choose\ from$$

There are four team members to be chosen, out which you have already chosen the twins, hence, you are left with

$$4 - 2 = 2 \ Members\ to\ be\ chosen$$

Hence, we need to chose the remaining two members out of the remaining 8 members, which is:

$$\binom{10-2}{4-2} = \binom{8}{2} = 28$$

Case II: Twins Never Selected

There are ten people to choose from, out of which since you not picking the twins, you are left with

$$10 - 2 = 8 \; Members\ to\ choose\ from$$

There are four team members to be chosen, on which there are no restrictions.
Hence, the number of ways to form a team while not picking the twins is:

$$\binom{10-2}{4} = \binom{8}{4} = 70$$

Adding the Cases

And, hence the final number of ways to select is:

$$28 + 70 = 98$$

Example 2.56

I must select a team of four for the math contest from ten students. The Irving twins do great math together, but are miserable alone, so I must take both, or neither of them. Similarly, for the Jones' twins I need to choose both, or neither of them. In how many ways can I make the selection?

We have multiple cases, and we need to handle them carefully.

		Irving Twins	
		Yes	No
Jones Twins	Yes	1	15
	No	15	15

Case I: You pick neither the Irving twins, nor the Jones twins

$$\binom{10-4}{4} = \binom{6}{4} = \frac{6 \times 5}{2} = 15$$

Case II: You pick the Irving twins, but not the Jones twins

You already picked the Irving twins, so you can't pick them again. You decided not to pick the Jones twins, so you can't do that do either. Hence, the number of choices is:

$$\binom{10-2-2}{2} = \binom{6}{2} = 15$$

Case III: You pick the Jones twins, but not the Irving twins

This case, by symmetry, is the same as the one above, and hence the answer is also:

$$15$$

Case IV: You pick both the Jones twins, and the Irving twins

This can be done in 1 way.
And, hence the final number of ways to select is:

$$15 + 15 + 15 + 1 = 46$$

Example 2.57

A mother-toddler club with ten mothers has six spots booked on a bus for a picnic. Two of the mothers have twins, and the rest have a single child each. Toddlers only go for the picnic with their mother. Similarly, a mother must take all her children along. Everyone occupies exactly one spot. In how many ways can the members going for the picnic be selected?

Note: A spot will be un-utilized if a single set of twins goes for the picnic. In all other cases, all spots will be utilized.

Mother	Mother	Mother	Mother	Mother	Mother	Mother	Mother	Mother	Mother
Twin A	Twin X	Toddler	Toddler	Toddler	Toddler	Toddler	Toddler	Toddler	Toddler
Twin B	Twin Y								

We do this according to casework. We can identify three distinct cases:

- Both sets of twins go for the picnic
- One set of twins goes for the picnic
- No twins go for the picnic

Case I: Two sets of twins go for the picnic

$$\underbrace{Mother\ 1}\underbrace{Twin\ A}\underbrace{Twin\ B}\underbrace{Mother\ 2}\underbrace{Twin\ X}\underbrace{Twin\ Y} \Rightarrow All\ Tickets\ Utilized$$

There is no choice here. Every eligible set of twins gets selected. So, we have only

$$1\ Way$$

Case II: One set of twins goes for the picnic

$$\underbrace{\underbrace{Mother}\underbrace{Twin}\underbrace{Twin}}_{Three\ Spots}\underbrace{\underbrace{Mother}\underbrace{Toddler}}_{Two\ Spots}\underbrace{Empty\ Spot} \Rightarrow One\ Ticket\ Un-Utilized$$

Choose the mother with the twins, which can be done in

$$2\ Ways$$

and the mother with one child, which can be done in:

$$8\ ways$$

Total Ways are

$$2 \times 8 = 16$$

Case III: None of the twins go for the picnic

$$\underbrace{\underbrace{Mother}\underbrace{Toddler}}_{Two\ Spots}\underbrace{\underbrace{Mother}\underbrace{Toddler}}_{Two\ Spots}\underbrace{\underbrace{Mother}\underbrace{Toddler}}_{Two\ Spots} \Rightarrow All\ Tickets\ Utilized$$

Since the twins don't go, we are left with

$$10 - 2 = 8\ Mothers$$

From these 8 mothers, we have to choose 3 mothers (toddlers automatically get chosen), which can be done in:

$$\binom{8}{3} = 56$$

Total Number of Ways

$$56 + 16 + 1 = 73$$

D. At Least One Selected

If you have multiple types of objects, and from each type of object, you must select at least one object, then we have the

$$At\ Least\ One\ Selected$$

Condition.

(Calculator) Example 2.58

A movie club is selecting four movies for its festival from ten proposed by the 9^{th} grade, and an equal number proposed by the 10^{th} grade. At least one movie must be selected from each grade's proposals. In how many ways can the selections be made?

There are two methods to do this. The first one is direct counting. Consider the number of tenth grade movies that can be chosen, without any restrictions:

$$\underbrace{0}_{\substack{Not\\Possible}},1,2,3,\underbrace{4}_{\substack{Not\\Possible}}$$

Hence, we are left with three cases to consider:

Case I: Choose 1 10th Grade Movie (and hence 3 9th Grade Movies)

$$\binom{10}{1}\times\binom{10}{3}=10\times 120=1200$$

Case II: Choose 3 10th Grade Movies (and hence 1 9th Grade Movie)

By symmetry, this is also

$$\binom{10}{1}\times\binom{10}{3}=10\times 120=1200$$

Case III: Choose 2 Movies of each type

$$\binom{10}{2}\times\binom{10}{2}=\left[\binom{10}{2}\right]^2=\left[\frac{10\times 9}{2}\right]^2=45^2=2025$$

And hence the final answer is:

$$1200+1200+2025=4425$$

2.59: At least one selected

Consider n objects of Type 1, and m objects of Type 2, resulting in a total of $n+m$ objects.

The number of ways of selecting r objects, such that at least one object of each type is selected:

$$\underbrace{\binom{n+m}{r}}_{Total}-\underbrace{\binom{m}{r}}_{\substack{All\ of\\Type\ 1}}-\underbrace{\binom{n}{r}}_{\substack{All\ of\\Type\ 2}}$$

If we want to select at least one of each type from a pool of objects, we can find the ways to make the selection using complementary counting.

$$Ways_{At\ Least\ One}=Ways_{Total}-Ways_{Only\ single\ type\ is\ selected}$$

(Calculator) Example 2.60

A school movie club is selecting four movies to showcase at its film festival from ten movies proposed by the ninth grade, and ten movies proposed by the tenth grade. To not hurt anyone feelings at least one movie must be selected from the tenth-grade proposals, and at least one movie must be selected from the ninth-grade proposals. In how many ways can the selections be made?

If there were no restrictions, then we needed to choose 4 movies out of 20 movies available, which can be done in:

$$\binom{20}{4}=\frac{20\times 19\times 18\times 17}{4!}=4845$$

But we cannot choose all 4 movies to be 10th grade, or all four movies to be 9th grade, which can be done in:

$$\binom{10}{4} \times 2 = 210 \times 2 = 420$$

And hence, the final answer is:

$$4845 - 420 = 4425$$

(Calculator) Example 2.61

In the question above, both of the methods took similar time and calculations. Which of the methods is preferable if we needed to choose six movies instead of four, and other conditions remained the same? Do the calculations and confirm.

$$\binom{20}{6} - \binom{10}{6} - \binom{10}{6} = \binom{20}{6} - \binom{10}{6} \times 2 = 38340$$

$$\binom{10}{1}\binom{10}{5} + \binom{10}{2}\binom{10}{4} + \binom{10}{3}\binom{10}{3} + \binom{10}{4}\binom{10}{2} + \binom{10}{5}\binom{10}{1}$$

Example 2.62

A school must send three of its best programmers for a mixed-gender programming competition, which specifies that at least one member of the team must be male, and one must be female. It must select its team from a pool of five boys and four girls. In how many ways can the selection be made?

We can do this using complementary counting.

- Count the number of ways without restrictions
- And subtract both $all - boy$ and $all - girl$ teams from the total

Set up the Logic

Without Restrictions

If there are no restrictions, the selection can be made in:

$$\binom{9}{3} = 84\ Ways$$

Ways to Select All Boys

An all-boy team can be selected in

$$\binom{5}{3} = 10\ Ways$$

Ways to Select All Girls

An all-girl team can be selected in

$$\binom{4}{3} = 4\ Ways$$

Complementary Counting

Total Number of Ways

$$= \underbrace{84}_{\substack{No\\Restrictions}} - \underbrace{10}_{All\ Boys} - \underbrace{4}_{All\ Girls} = 70$$

(Alternate Solution) A school must send three of its best programmers for a mixed-gender programming competition, which specifies that at least one member of the team must be male, and one must be female. It must select its team from a pool of five boys and four girls. In how many ways can the selection be made?

There are only two cases possible:

- Two boys and one girl
- Two girls and one boy

$$\underbrace{\binom{5}{2}\binom{4}{1}}_{\substack{Two\ Boys\\ and\ one\ Girl}} + \underbrace{\binom{5}{1}\binom{4}{2}}_{\substack{Two\ Girls\\ and\ One\ Boy}} = 10 \times 4 + 5 \times 6 = 40 + 30 = 70$$

Example 2.63

A syllabus committee consisting of three people must be formed in a college. There are four Sociology professors and three Anthropology professors eligible for the committee. In how many ways can this be done if the committee consists of at least one professor from each field.

Strategy

Counting all the ways in which it is possible will result in a lot of nasty casework.
Instead, count the total number of ways, and subtract the number of ways where the condition of one professor from each field is not met.

Total Ways

I want to select three professors from seven professors, which can be done in

$$= \binom{7}{3} = \frac{7!}{4!\,3!} = \frac{7 \times 6 \times 5}{6} = 35$$

Single Field Committees

A committee consisting of only Anthropology professors can be done in only

$$= \binom{3}{3} = 1$$

A committee consisting of only Sociology professors can be done in:

$$= \binom{4}{3} = \binom{4}{1} = 4$$

Complementary Counting

The number of ways we want is

$$\underbrace{35}_{Total\ Ways} - \underbrace{1}_{All\ Anthropology} - \underbrace{4}_{All\ Sociology} = 30$$

Example 2.64

Out of 6 ruling and 5 opposition party members, 4 are to be selected for a delegation. In how many ways can this be done so as to include at least one opposition member. (JMET 2011/69)

$$\underbrace{\binom{11}{4}}_{\substack{Total\\ ways}} - \underbrace{\binom{6}{4}}_{\substack{No\ Opposition\\ Member}} = 330 - 15 = 315$$

Example 2.65

The planning committee at school has 10 members. Exactly four of these members are teachers. A four-person subcommittee with at least one member who is a teacher must be formed from the members of the planning committee. How many distinct subcommittees are possible? (MathCounts 2007 National Sprint)

The number of four people subcommittees is the number of ways to choose four people out of ten:

$$\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{2 \cdot 3 \cdot 4} = 210$$

Out of the above, we do not want the subcommittees where no teacher is present. This can be done by selecting four people from the six non-teachers in:

$$\binom{6}{4} = \frac{6 \cdot 5}{2} = 15$$

Hence, the number of subcommittees with at least one teacher is:

$$210 - 15 = 195$$

E. Always in a particular order

Example 2.66

Find the number of ways in which four people (P, Q, R, S) can be seated such that R is always to the right of P.

Enumeration

We list out the 24 possible ways in which four people can be seated. And we highlight and count the ways in which R is to the right of P, which gives us:

$$6 + 3 + 3 = 12\ Ways$$

Start with P			Start with Q			Start with R			Start with S		
PQRS	PRQS	PSQR	QPRS	QRPS	QSPR	RPQS	RQPS	RSPQ	SPQR	SQPR	SRPQ
PQSR	PRSQ	PSRQ	QPSR	QRSP	QSRP	RPSQ	RQSP	RSQP	SPRQ	SQRP	SRQP
6			3			0			3		

Constructive Counting

We can choose the positions of Q and S in

$$\binom{4}{2}\ ways$$

And arrange them among themselves in

$$2!\ ways$$

Finally, R and P can only be arranged in 1 way.
Hence, the final answer is:

$$\binom{4}{2} \times 2 = 12$$

Symmetry

The number of ways that four people can be arranged without restrictions

$$4! = 24$$

But, note that, for every arrangement where R is to the right of P, there is exactly one arrangement where P is to the right of R (obtained simply by interchanging the positions of P and R).

Hence, the number of required ways is simply:

$$\frac{4!}{2} = 12$$

Example 2.67

In how many ways can the letters in *case* be arranged so that the vowels are all in alphabetic order.

This is the same question as part A.

$$12$$

Example 2.68

In how many ways can the letters in $facetious$ be arranged so that the vowels are all in alphabetic order.

Shortcut
Arrange without restrictions in 9! Ways and divide by the number of ways in which a, e, i, o, u can be arranged among themselves:

$$\frac{9!}{5!}$$

Long Method
We can choose the positions of $\{f, c, t, s\}$ in

$$\binom{9}{4} = \frac{9!}{4!\,5!}\ ways$$

Then the vowels can be arranged in only 1 way.
Finally, $\{f, c, t, s\}$ can be arranged among themselves in

$$4!\, ways$$

Hence, the final answer is:

$$\frac{9!}{4!\,5!} \times 4! = \frac{9!}{5!}$$

2.4 Restrictions: Add and Multiply Rules

A. Addition Rule (Casework)

2.69: Addition Rule
If there are distinct cases, and the number of possible ways in each case can be counted using combinations, then you will get combinations with the addition rule.

Count ABC:

- Case I: Has $\binom{n_1}{k_1}$ $ways$
- Case II: Has $\binom{n_2}{k_2}$ $ways$
- .
- .
- .
- Case x: Has $\binom{n_x}{k_x}$ $ways$

The total number of ways will be:

$$\binom{n_1}{k_1} + \binom{n_2}{k_2} + \cdots + \binom{n_x}{k_x} = \sum_{i=1}^{i=x} \binom{n_i}{k_i}$$

Example 2.70
Bill is sent to a donut shop to purchase exactly six donuts. If the shop has four kinds of donuts and Bill is to get at least one of each kind, how many combinations will satisfy Bill's order requirements? (MathCounts 2008 State Target)

Consider that the donuts are of Type

$$A, \quad B, \quad C, \quad D$$

We need to purchase at least one donut of each type. Hence, even though we are purchasing six donuts, we do have a choice in the first four donuts. They must be, in some order:

$$A, \quad B, \quad C, \quad D$$

So, the choice only lies in the last two donuts to be purchased.

$$A, \quad B, \quad C, \quad D, \quad ??, \quad ??$$

Now, we can break the choice of purchasing the last two donuts in two ways:

Case I: Both donuts are the same

Here, we can only choose the type of donut, which can be done in:

$$4\ Ways$$

$$\begin{matrix} A, & B, & C, & D, & A, & A \\ A, & B, & C, & D, & B, & B \\ A, & B, & C, & D, & C, & C \\ A, & B, & C, & D, & D, & D \end{matrix}$$

Case II: Both donuts are different

Here, we need to choose two donuts out of 4, which can be done in:

$$\binom{4}{2} = \frac{4 \times 3}{2} = 6$$

$$Total\ Ways = 4 + 6 = 10$$

(Alternate Solution)

The question above can be solved using Diophantine Equations, which we do next.

Let's say the number of donuts purchases of different varieties is:

$$d_1 = No.of\ Donuts\ of\ First\ Type\ Purchased$$
$$d_2 = No.of\ Donuts\ of\ Second\ Type\ Purchased$$
$$d_3 = No.of\ Donuts\ of\ Third\ Type\ Purchased$$
$$d_4 = No.of\ Donuts\ of\ Four\ Type\ Purchased$$

We need to purchase six donuts, and hence:

$$d_1 + d_2 + d_3 + d_4 = 6, \quad d_n > 0, \quad d_n \in \mathbb{Z}$$

But, we need to purchase one of each kind. So, we do not have any choice there. What we are left with is

$$D_1 + D_2 + D_3 + D_4 = 2, \quad D_n \geq 0, \quad d_n \in \mathbb{Z}$$

There are two ways we can do the allocation:

- Assign the value of two to a single variable. This can be done in **four ways.**

$$2 + 0 + 0 + 0 = 2$$
$$0 + 2 + 0 + 0 = 2$$
$$0 + 0 + 2 + 0 = 2$$
$$0 + 0 + 0 + 2 = 2$$

- Assign a value of one each to two of the four variables. This amounts to choosing two out of four objects, which can be done in

$$\binom{4}{2} = 6\ Ways$$

$$Total\ Ways = 4 + 6 = 10$$

Example 2.71
A bakery sells three kinds of rolls. How many different combinations of rolls could Jack purchase if he buys a total of six rolls and includes at least one of each kind? (MathCounts 2010 National Countdown)

Let the three types be A, B and C. Then, Jack must purchase one of each, and has three empty spots that could be filled with any rolls of his choice:

$$ABC\ XXX$$

We do this using casework, and focus only on the last rolls, since that is where we have a choice
Case I: All three of a different type

$$ABC \Rightarrow 1\ Way$$

Case II: All three of the same type

$$AAA, BBB, CCC \Rightarrow 3\ Ways$$

Case III: Two of a type, and the third of a different type

$$AAB, AAC, BBA, BBC, CCA, CCB$$

Use the multiplication principle here:

$$\underbrace{3\ Choices}_{Same\ Type} \times \underbrace{2\ Choices}_{Different\ Type} = 3 \times 2 = 6$$

Total

$$= 1 + 3 + 6 = 10$$

B. Increasing and Decreasing Numbers

Example 2.72
How many even three-digit integers have the property that their digits, all read from left to right, are in strictly increasing order? (AMC 2006 12B/9)

The smallest number and largest numbers that satisfy this property are:

$$124, \qquad 678$$

The possible values of the last digit are:

$$\{4,6,8\}$$

Consider cases on the values of the last digit.

Case I: Last digit is 4
Take exactly two digits from $\{1,2,3\}$, and arrange them in increasing order, which can be done in:

$$\binom{3}{2} = 3$$

Case II: Last digit is 6
Take exactly two digits from $\{1,2,3,4,5\}$, and arrange them in increasing order, which can be done in:

$$\binom{5}{2} = 10$$

Case III: Last digit is 8

Take exactly two digits from $\{1,2,\ldots,7\}$, and arrange them in increasing order, which can be done in:

$$\binom{7}{2} = 21$$

Add the three cases to get:

$$3 + 10 + 21 = 34$$

Example 2.73

How many even three-digit integers have the property that their digits, all read from left to right, are in strictly decreasing order?

Case I: Last digit is 0

Take exactly two digits from $\{1,2,\ldots,9\}$, and arrange them in decreasing order, which can be done in:

$$\binom{9}{2} = 36$$

Case II: Last digit is 2

Take exactly two digits from $\{3,\ldots,9\}$, and arrange them in decreasing order, which can be done in:

$$\binom{7}{2} = 21$$

Case III: Last digit is 4

Take exactly two digits from $\{5,6,7,8,9\}$, and arrange them in decreasing order, which can be done in:

$$\binom{5}{2} = 10$$

Case III: Last digit is 6

Take exactly two digits from $\{7,8,9\}$, and arrange them in decreasing order, which can be done in:

$$\binom{3}{2} = 3$$

Add the three cases to get:

$$3 + 10 + 21 + 36 = 70$$

Example 2.74

How many even four-digit integers have the property that their digits, all read from left to right, are in strictly increasing order?

Case I: Last digit is 4

Take exactly three digits from $\{1,2,3\}$, and arrange them in increasing order, which can be done in:

$$\binom{3}{3} = 1$$

Case II: Last digit is 6

Take exactly three digits from $\{1,2,3,4,5\}$, and arrange them in increasing order, which can be done in:

$$\binom{5}{3} = 10$$

Case III: Last digit is 8

Take exactly three digits from $\{1,2,\ldots,7\}$, and arrange them in increasing order, which can be done in:

$$\binom{7}{3} = 35$$

Add the three cases to get:

$$1+10+35=46$$

Example 2.75

How many even three-digit integers (written in Hexadecimal) have the property that their digits, all read from left to right, are in strictly increasing order?

Note: Hexadecimal is a number system where place value becomes 16 times (unlike the decimal system where it becomes ten times), and digits can be

$$0,1,2,\ldots,9,A=10,B=11,C=12,D=13,E=14,F=15$$

The possible values of the last digit are:

$$\{4,6,8,A=10,C=12,E=14\}$$

Consider cases on the values of the last digit.

$$\underbrace{\binom{3}{2}}_{\substack{Last\\ Digit:4}}+\underbrace{\binom{5}{2}}_{\substack{Last\\ Digit:6}}+\binom{7}{2}+\binom{9}{2}+\binom{11}{2}+\binom{13}{2}=3+10+21+36+55+78=203$$

Example 2.76

How many even three-digit integers have the property that their digits, all read from left to right, are in non-decreasing order?

Build on the solution from the previous example. We already counted the numbers where the digits were increasing. Hence, if we add the case where the numbers are non-decreasing, then we have all the numbers we want.

Case I: Last digit is 2

$$\underbrace{0}_{Increasing}+\underbrace{3}_{Non\ decreasing}=3$$

$$112,\quad 122,\quad 222$$

Case II: Last digit is 4

$$\underbrace{\binom{3}{2}}_{Increasing}+\underbrace{4}_{Non\ decreasing}=3+4=7$$

$$114,\quad 224,\quad 334,\quad 444$$

Case III: Last digit is 6

$$\underbrace{\binom{5}{2}}_{Increasing}+\underbrace{6}_{Non\ decreasing}=10+6=16$$

$$xx6,\quad 1\le x\le 6,\quad x\in\mathbb{Z}$$

Case III: Last digit is 8

$$\underbrace{\binom{7}{2}}_{Increasing}+\underbrace{8}_{Non\ decreasing}=21+8=29$$

$$xx6,\quad 1\le x\le 8,\quad x\in\mathbb{Z}$$

Add the values from the cases:

$$3+7+16+29=55$$

C. Multiplication Rule

2.77: Multiplication Rule

If there are distinct parts to the problem; each part can be done independently; and all parts are to be done together we use the multiplication rule.

Count ABC:

- Part I: Has $\binom{n_1}{k_1}$ $ways$
- Part II: Has $\binom{n_2}{k_2}$ $ways$
- .
- .
- .
- Part x: Has $\binom{n_x}{k_x}$ $ways$

To achieve ABC, we need to do each of I, II,…,x, and each of the parts is independent, then the total number of ways will be:

$$\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_x}{k_x}=\prod_{i=1}^{i=x}\binom{n_i}{k_i}$$

Example 2.78

A class has 12 boys and 9 girls. In how many ways can you pick two monitors, such that each is of a different gender.

Without using Combinations

We have done this question without combinations (by using the multiplication rule) as follows:

$$\underbrace{12}_{\substack{\text{Choices for a}\\ \text{male monitor}}}\times\underbrace{9}_{\substack{\text{Choices for a}\\ \text{female monitor}}}=108$$

With Combinations

The number of choices for the male monitor and the female monitor can be written in the language of combinations, without changing the final answer:

$$\underbrace{\binom{12}{1}}_{\substack{\text{Choices for a}\\ \text{male monitor}}}\times\underbrace{\binom{9}{1}}_{\substack{\text{Choices for a}\\ \text{female monitor}}}=12\times 9=108$$

Example 2.79

Out of 6 ruling and 5 opposition party members, 4 are to be selected for a delegation. In how many ways can this be done so as to include exactly one ruling party member. (JMET 2011/68)

$$\underbrace{\binom{6}{1}}_{Ruling} \times \underbrace{\binom{5}{3}}_{Opposition} = 6 \times 10 = 60$$

Example 2.80

I have five books and seven movies. I am planning to watch two movies and read two books. In how many ways can I do this?

$$\underbrace{\binom{5}{2}}_{\substack{Choosing\\Books}} \times \underbrace{\binom{7}{2}}_{\substack{Choosing\\Movies}} = \frac{5 \times 4}{2} \times \frac{7 \times 6}{2} = 10 \times 21 = 210$$

Example 2.81

Let $S = \{1,2,3,\ldots,9\}$. For $k = 1,2,\ldots 5$, let N_k be the number of subsets of S, each containing 5 elements out of which exactly k are odd. Then $N_1 + N_2 + N_3 + N_4 + N_5 =$ **(JEE-A 2017)**

Method I: Casework

Consider cases for the value of k:

$$0\ Even, 5\ Odd \Rightarrow \binom{4}{0}\binom{5}{5} = 1$$
$$1\ Even, 4\ Odd \Rightarrow \binom{4}{1}\binom{5}{4} = 20$$
$$2\ Even, 3\ Odd \Rightarrow \binom{4}{2}\binom{5}{3} = 60$$
$$3\ Even, 2\ Odd \Rightarrow \binom{4}{3}\binom{5}{2} = 40$$
$$4\ Even, 1\ Odd \Rightarrow \binom{4}{4}\binom{5}{1} = 5$$

Total

$$= 1 + 20 + 60 + 40 + 5 = 126$$

Method II: Shortcut

Since $k \in \{1,2,3,4,5\}$, the question requires us to calculate all possible ways in which we can choose 5 digits out of 9, and hence the answer is simply:

$$= \binom{9}{5} = \frac{9!}{4!\,5!} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{24} = 126$$

Example 2.82

Two subsets of the set $S = \{a, b, c, d, e\}$ are to be chosen so that their union is S and their intersection contains exactly two elements. In how many ways can this be done, assuming that the order in which the subsets are chosen does not matter? **(AMC 2008 10B/23)**

Let the sets be S_1 and S_2. We can choose the elements of $S_1 \cap S_2$ (the intersection of the two sets) in

$$\binom{5}{2} = \frac{5 \times 4}{2} = 10\ Ways$$

The remaining three elements can be assigned to one of the two sets in

$$2^3 = 8\ Ways$$

The total number of ways in which this can be done, by the multiplication principle is:

$$10 \times 8 = 80\ Ways$$

But note that since the order of the subsets does not matter, the following two are the same:

$$S_1 = \{a, b, c, d, e\}, \qquad S_2 = \{d, e\}$$
$$S_2 = \{a, b, c, d, e\}, \qquad S_1 = \{d, e\}$$

And hence, to account for the overcounting, we divide by 2:

$$\frac{80}{2} = 40\ Ways$$

2.83: Selecting from distinct types

Sometimes, when selecting we may have multiple groups, and specific conditions attached to each group.
If specific seats are reserved for specific groups, then the question becomes:
Select x objects from group y

$$\binom{y}{x}\ Ways$$

Select p objects from group q

$$\binom{q}{p}\ Ways$$

Combine the two using the multiplication principle.

$$\binom{y}{x} \times \binom{q}{p}$$

Example 2.84

A college has seven Maths professors, three Statistics professors, two Computer Science (CS) professors. What is the number of ways in which a committee can be formed consisting of:

A. six professors
B. six professors with an equal number of professors of each background

Part A

We have

$$7 + 3 + 2 = 12\ Professors$$

out of which we need to choose six professors for the committee.

This can be done in:

$$\binom{12}{6} = \frac{12!}{6!\,6!} = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{6 \times 5 \times 4 \times 3 \times 2} = 11 \times 2 \times 6 \times 7 = 924$$

Part B

Since, we need to choose an equal number of each background, we need

$$2 + 2 + 2 = 6\ Professors$$

And, the choice of professors is independent of each other. Hence, by the multiplication rule:

$$\underbrace{\binom{7}{2}}_{Maths} \times \underbrace{\binom{3}{2}}_{Stats} \times \underbrace{\binom{2}{2}}_{CS} = \frac{7!}{2!\,5!} \times \frac{3!}{2!\,1!} \times \frac{2!}{2!\,0!} = (7 \times 3) \times 3 \times 1 = 63$$

Example 2.85

Amit has 11 friends: 7 boys and 4 girls. In how many ways, can Amit invite them, if there have to be exactly 4 boys in the invitees? (JMET 2011/81)

Ways of inviting exactly 4 boys out of 7

$$=\binom{7}{4}=35$$

For each girl, we have a choice of inviting or not inviting, which us total choices as:

$$2^4=16$$

Total Ways

$$=16\times 35=560$$

Example 2.86

Robert has 4 indistinguishable gold coins and 4 indistinguishable silver coins. Each coin has an engraving of one face on one side, but not on the other. He wants to stack the eight coins on a table into a single stack so that no two adjacent coins are face to face. Find the number of possible distinguishable arrangements of the 8 coins. **(AIME 2005/I/5)**

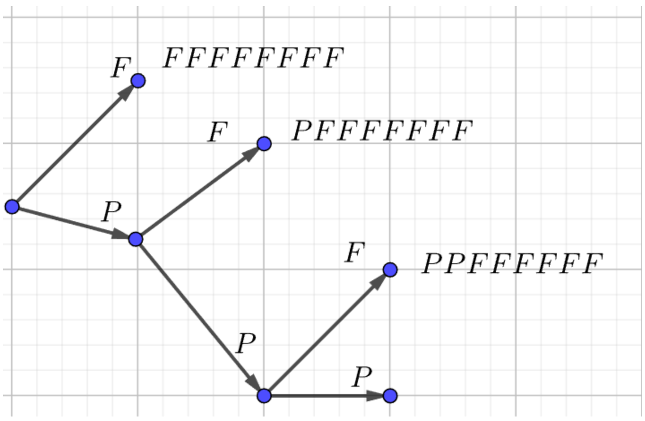
Imagine that we are stacking the coins left to right.

There are two parts to the question:
- Orientation of the coin: $Engraved\ face(F)$ to the right, or $plain\ surface(P)$ to the right
- Nature of the coin: Gold or Silver

Step I: Orientation

$$FFFFFFFF$$
$$PFFFFFFF$$
$$PPFFFFFF$$
$$PPPFFFFF$$
$$PPPPFFFF$$
$$PPPPPFFF$$
$$PPPPPPFF$$
$$PPPPPPPF$$
$$PPPPPPPP$$

$$9\ possible\ orientations$$



Step II: Nature

We must arrange 4 gold coins and 4 silver coins in 8 positions. Arranging the gold coins can be done in:

$$\binom{8}{4}=\frac{8\cdot 7\cdot 6\cdot 5}{24}=70$$

By the multiplication principle:

$$9\times 70=630$$

2.5 Distinguishability

A. Distinguishability

When objects cannot be identified, the number of choices will reduce.
If objects can be identified, the number of choices will increase.

In both cases above, in some special cases, the number of choices can remain same.

Example 2.87

I have four identical red balls, and three identical black balls. In how many ways can I place two red balls and two black balls in a decorative vase, if the order in which I place the balls:

A. Is not important
B. Is important

Since the balls are identical having more balls than the required number does not increase the number of ways.

Part A

Since I cannot distinguish between the balls, there is only one way to do it.

Part B

If order is important, then I need to choose the positions of the red balls among the four balls.

$$\underbrace{\quad}_{Ball\ 1}\ \underbrace{\quad}_{Ball\ 2}\ \underbrace{\quad}_{Ball\ 3}\ \underbrace{\quad}_{Ball\ 4}$$

And hence the number of ways in which I can place the balls is:

$$\binom{4}{2} = 6$$

(Important) Example 2.88

I have four red balls, numbered one to four, and three black balls, numbered one to three. In how many ways can I place two red balls and two black balls in a decorative vase, if the order in which I place the balls:

A. Is not important
B. Is important

Strategy

Because the balls are numbered, therefore they are distinct or identifiable. Hence, we have multiple ways to choose among them.

Part A

Here, we only need to select the balls, since the order of placing them in the vase is not important. However, we need to break into parts.

Red Balls

Out of the four red balls, we want to select two, which can be done in

$$\binom{4}{2} = \frac{4!}{2!\,2!} = 6\ Ways$$

Black Balls

Out of the three black balls, we want to select one, which can be done in

$$\binom{3}{2} = \frac{3!}{1!\,2!} = 3\ Ways$$

Multiplication Principle

By the multiplication principle, since choosing the black balls is independent of the choosing the red balls, the total number of ways of choosing the black and red balls:

$$\underbrace{\binom{4}{2}}_{\substack{Selecting\\ Red\ Balls}} \times \underbrace{\binom{3}{2}}_{\substack{Selecting\\ Black\ Balls}} = \frac{4!}{2!\,2!} \times \frac{3!}{1!\,2!} = 6 \times 3 = 18$$

Part B

Break the process into two parts:

- Select the balls $= 18\ Ways$

- Arrange the balls $= 4! = 24\ Ways$

Then, the total ways to do the entire process is:

$$\underbrace{18}_{\substack{Select \\ Four\ Balls}} \times \underbrace{24}_{\substack{Arrange \\ Four\ Balls}} = 432$$

B. Complementary Counting

Example 2.89

A team needs to choose the colours of its uniform, which consists of a jersey and shorts. The coach has chosen turquoise as one of the colors, and the players need to choose the second color out of fifteen colours (of which turquoise is one color). The color of the jersey can be different or same as the color of the shorts. In how many ways can this selection be done?

Number of Ways of Choosing the second color is:

$$\binom{15}{1} = 15$$

We also need to choose which color is the jersey, and which color are the shorts.

$$\underbrace{Jersey}_{Turquoise} + \underbrace{Shorts}_{2nd\ Colour},\ \underbrace{Shorts}_{Turquoise} + \underbrace{Jersey}_{2nd\ Colour} \Rightarrow 2\ Ways$$

Total Ways

$$15 \times 2 = 30$$

However, turquoise is one of the 15 colours. So, if you pick turquoise as one of the colors, then you have only one choice, not two.
Hence, the final answer is

$$30 - 1 = 29$$

Example 2.90

I have eight boxes in a row. In how many ways can I place two identical red balls, one per box, in the boxes so that no two adjacent boxes have red balls.

Method I: Complementary Counting

If there are no restrictions, then the number of ways to place two red balls, one per box in eight boxes, is

$$\binom{8}{2} = \frac{8 \times 7}{2} = 28$$

Number the boxes 1 to 8. The two red balls can go in box numbers given below if they are adjacent:

$$\{12, \quad 23, \quad 34, \quad 45, \quad 56, \quad 67, \quad 78\} \Rightarrow 7\ Ways$$

The final answer is:

$$28 - 7 = 21\ Ways$$

Method II: Direct Counting

Number the balls. Consider cases based on the position of the first ball

Case I: Ball is at an end:

$$1st\ Ball = 2\ choices$$

$$2nd\ Ball = 6\ Choices$$

Total choices, by the multiplication principle

$$= 2 \cdot 6 = 12$$

Case II: Ball is not an end:

$$1st\ Ball = 6\ choices$$
$$2nd\ Ball = 5\ Choices$$

Total choices, by the multiplication principle

$$= 6 \cdot 5 = 30$$

Adding up the cases gives

$$12 + 30 = 42$$

However, since the balls are not numbered, we need to divide by the number of ways in which the balls can be arranged among themselves. Hence, the final answer is:

$$= \frac{42}{2!} = 21$$

Example 2.91

I have eight boxes in a row. In how many ways can I place three identical red balls, one per box, in eight boxes so that no two adjacent boxes have red balls.

Total Ways

We can place three identical balls in eight boxes in a row in:

$$\binom{8}{3} = 56\ ways$$

Balls Together

Two Balls Together

Number the boxes 1 to 8. The number of ways to put two red balls in eight boxes so that they are adjacent is:

$$\{12, \quad 23, \quad 34, \quad 45, \quad 56, \quad 67, \quad 78\} \Rightarrow 7\ Ways$$

For each arrangement above, the third ball can go in any of the remaining six positions:

$$6\ Ways$$

The final number of ways to have at least two red balls together is:

$$7 \times 6 = 42\ Ways$$

But the above arrangement counts each way that three red balls can be together twice:

$$12\ with\ the\ third\ ball\ in\ the\ third\ position: 12\mathbf{3}$$
$$23\ with\ the\ third\ ball\ in\ the\ first\ position: \mathbf{1}23$$

Three Balls Together

$$123, \quad 234, \quad 345, \quad 456, \quad 567, \quad 678 \Rightarrow 6\ Ways$$

Subtract the number of ways three balls can be together from our count of two balls together.

$$42 - 6 = 36$$

Complementary Counting

The number of valid ways is

$$56 - 36 = 20$$

C. Forming Two Teams

If order is not important, then we are selecting, not arranging.

Example 2.92

In a football tournament, we want to make teams play against each. There are four countries participating: France, Spain, Italy, and Norway. If all possible pairs are made, how many games will be played?

$$\binom{4}{2} = 6$$

$$\begin{gathered} France\ vs\ Spain \\ France\ vs\ Italy \\ France\ vs\ Norway \\ Spain\ vs\ Italy \\ Spain\ vs\ Norway \\ Italy\ vs\ Norway \end{gathered}$$

Example 2.93

Alice, Betty, Cathy, and Diana are playing doubles tennis. They need to be divided into the green team (two players), and the blue team(two players), who will then play against each other. How many ways can we form the teams?

Blue	AB	AC	$\boldsymbol{AD}$	$\boldsymbol{BC}$	BD	CD
Green	CD	BD	$\boldsymbol{BC}$	$\boldsymbol{AD}$	AC	AB

Note that the teams are colored, and hence distinguishable.

$$\boldsymbol{AD}\ vs\ \boldsymbol{BC} \neq \boldsymbol{BC}\ vs\ \boldsymbol{AD}$$

Example 2.94

Alice, Betty, Cathy, and Diana are playing doubles tennis, and two teams of two each must be formed from the four of them who will play against each other. How many ways can we form the teams?

Blue	AB	AC	AD	BC	BD	CD
Green	CD	BD	BC	AD	AC	AB

Note that the teams are colored, and hence distinguishable.

$$AD\ vs\ BC = BC\ vs\ AD$$

Hence, we need to divide the answer from the previous example by 2, giving us:

$$\frac{\binom{4}{2}}{2} = \frac{6}{2} = 3$$

Example 2.95

Five people (Alice, Betty, Cathy, Diana, and Xavier) are playing a sport. Two teams (one with two people and one with three people) are to be formed. How many ways can we form the teams?

We can select two people (in which case the other three are automatically selected) to get:

$$\binom{5}{2} = 10$$

We can also select three people (in which case the other two are automatically selected) to get:

$$\binom{5}{3} = \binom{5}{2} = 10$$

Note: Since one team is two people, and the other team is three people, the teams are automatically distinguishable.

Example 2.96

Two teams are to be formed from n people. How many ways can we form the teams if:

A. Each team is assigned a color
B. The teams are not assigned a color

Note: If n is odd, then one team will have one person more than the other team. If n is even, the teams will be equal.

Case I: n is even

$$Part\ A: \binom{n}{\left(\frac{n}{2}\right)}$$

$$Part\ B: \frac{\binom{n}{\left(\frac{n}{2}\right)}}{2}$$

Case II: n is odd

$$Part\ A: \binom{n}{\left(\frac{n}{2} - 1\right)} = \binom{n}{\left(\frac{n}{2} + 1\right)}$$

$$Part\ B: Answer\ remains\ same$$

D. Forming Multiple Teams

Example 2.97

Six people are playing doubles tennis, and three teams of two each must be formed. How many ways can we form the teams, if

A. Each team is assigned a color
B. The teams are not distinguished.

Part A: Distinguishable Teams

Select the first team by choosing two people from six:

$$\binom{6}{2}\ ways$$

Select the second team by choosing two people from four:

$$\binom{4}{2}\ way$$

There is only one way to choose the third team, since the remaining two people form the third team:

$$1\ way$$

The total number of ways is:

$$\binom{6}{2}\binom{4}{2}\cdot 1 = 15\cdot 6 = 90$$

Part B: Indistinguishable Teams

We will overcount by the number of ways in which the teams can be assigned colors after being made, which is:

$$3! = 6$$

Hence, we will need to divide by 6, giving:

$$= \frac{90}{6} = 15$$

Example 2.98: Making Opposing Teams

I have ten basketball players, whom I wish to get to practice in teams of five versus five. In how many ways can I do this if:

A. I have a Blue team and a Red Team
B. I just want two teams, without distinguishing between the teams.

Part A

We need to break the players into a Blue team and a Red Team. Suppose we select players for the Blue Team. Then, the players for the Red Team are automatically decided.

$$\underbrace{b}_{\substack{\text{Choices for}\\\text{Blue Team}}} \times \underbrace{1}_{\substack{\text{Choices for}\\\text{Red Team}}} = \underbrace{b}_{\substack{\text{Total}\\\text{Choices}}}$$

The number of ways to pick five players out of ten for the Blue Team is

$$\binom{10}{5} = \frac{10!}{5!\,5!} = \frac{10\times 9\times 8\times 7\times 6}{2\times 3\times 4\times 5} = 252\ ways$$

Part B

Suppose that the players are:

$$A, B, C, D, E, F, G, H, I, J$$

Here, we want to divide ten players into two teams of five each, but the teams are not distinguishable. Hence, start by assuming, that the teams are called Blue, and Red, in which case, we can divide the players in:

$$\binom{10}{5} = \frac{10!}{5!\,5!} = \frac{10\times 9\times 8\times 7\times 6}{2\times 3\times 4\times 5} = 252\ ways$$

But, since the teams are indistinguishable, the following arrangements are the same.

$$\left(\underbrace{\{A,B,C,D,E\}}_{Team\ 1},\underbrace{\{F,G,H,I,J\}}_{Team\ 2}\right) = \left(\underbrace{\{A,B,C,D,E\}}_{Team\ 2},\underbrace{\{F,G,H,I,J\}}_{Team\ 1}\right)$$

And hence, we need to divide the answer that we got above by the number of ways that two teams can be arranged among themselves:

$$\frac{252}{2} = 126\ Ways$$

Example 2.99: Opposing Teams with some people left out

I have eleven basketball players, whom I wish to get to practice in a Blue team (with five players) and a Red Team (with five players). One player will not in any team, and will be benched. In how many ways can I do this?

Benched Player

The benched player is one out of 11 players, and he can be chosen in

$$11\ Ways$$

Selecting Blue and Red Teams

Having chosen the benched player, the question reduces to dividing ten players into a Blue Team and a Red Team and that can be done (which we have seen above) in:

$$\binom{10}{5}\ Ways$$

Multiplication Rule

By the multiplication rule, the final number of ways is:

$$11 \times \binom{10}{5} = 11 \times 252 = 2{,}772$$

Imagine that we go about our selection team by team:

- First the Blue Team
- Then the Red Team
- Finally, the benched player

Blue Team

There are 11 players, of which we want to choose five, which can be done in:

$$\underbrace{\binom{11}{5}}_{\substack{Blue\\Team}} = \frac{11!}{6!\,5!}$$

Red Team

Once we have chosen the five players for the Blue Team, we are only left with six players. Of these players, we want to choose five players, which can be done in

$$\underbrace{\binom{6}{5}}_{\substack{Red\\Team}} = \frac{6!}{1!\,5!}$$

Benched Player

Once we have chosen the Blue and the Red Team, we don't have a choice for the benched player. He must be the player remaining, and hence, he can only be chosen in

$$1\ way$$

Multiplication Rule

We need to choose

$$(Blue\ Team)\ \&\ (Red\ Team)\ \&\ (Benched\ Player)$$

And this can be done in

$$\frac{11!}{\underbrace{6!\,5!}_{Blue\ Team}} \times \frac{6!}{\underbrace{1!\,5!}_{Red\ Team}} \times \underbrace{1}_{Benched\ Player} = \frac{11!}{5!\,5!} = 11 \times \frac{10!}{5!\,5!} = 11 \times 252 = 2{,}772$$

Example 2.100

The basketball coach wants to have a four versus four match among his players. In how many ways can he do this, if, he has:

A. Eight players
B. Nine Players

Part A

We need to choose four from the eight players available to form the first team, giving us:

$$8\ Choose\ 4 = \binom{8}{4} = \frac{8!}{4!\,4!} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2} = 70$$

Now, having chosen four players, we are left, by default, with the four players who must form our second team, giving us

$$\binom{4}{4} = 1\ Way$$

But, we are interested in the match, and the teams are not identifiable. Hence, just as we divide by 2 to count the

number of handshakes, we must divide by 2 here as well.
Hence, the total number of ways is:

$$\frac{70 \times 1}{2} = 35$$

Part B

We need to choose four from the nine players available to form the first team, giving us:

$$9 \text{ Choose } 4 = \binom{9}{4} = \frac{9!}{4!\,4!} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2} = 126$$

Now, having chosen four players, we are left, by default, with the four players who must form our second team, giving us

$$\binom{5}{4} = 5 \text{ Ways}$$

But, we are interested in the match, and the teams are not identifiable. Hence, just as we divide by 2 to count the number of handshakes, we must divide by 2 here as well.
Hence, the total number of ways is:

$$\frac{126 \times 5}{2} = 315$$

Example 2.101: Restrictions

The sports team coach has a first-string team and a second-string team, each consisting of five players. The coach has identified three superstars among the ten players, two of whom must be selected in the first-string team, leaving only one for the second-string team. In how many ways can the coach form:

A. The Second-String Team
B. The First-string Team
C. Both the teams

Strategy

The coach has divided the players into two types:

- Superstars
- Regular Player

These two are distinct, so a player falls in exactly one of these categories:

$$\underbrace{3}_{\text{Superstars}} + \underbrace{7}_{\text{Regular Players}} = 10$$

Part A: Second-String Team

The second-string team has five players, out of which exactly one is a superstar. We must choose

- exactly one super star out of the three available superstars
$$= \binom{3}{1} = \frac{3!}{1!\,2!} = 3$$
- exactly four regular players from the seven available regular players
$$= \binom{7}{4} = \frac{7!}{4!\,3!} = \frac{7 \times 6 \times 5}{6} = 35$$

The total number of ways to select the team is:

$$\underbrace{3}_{\substack{\text{Selecting} \\ \text{Superstar}}} \times \underbrace{35}_{\substack{\text{Selecting} \\ \text{Regular Players}}} = 105$$

Part B: First-String Team

The first-string team also has five players, out of which exactly two are superstars. We must choose

- exactly two superstars out of the three available superstars:

$$=\binom{3}{2}=\frac{3!}{1!\,2!}=3$$

- exactly three regular players from the seven available regular players

$$=\binom{7}{3}=\frac{7!}{4!\,3!}=\frac{7\times 6\times 5}{6}=35$$

The total number of ways to select the team is:

$$\underbrace{3}_{\substack{Selecting\\ Superstar}}\times\underbrace{35}_{\substack{Selecting\\ Regular\ Players}}=105$$

The number of ways to select the first-string team is the same as the number of ways to select the second-string team. Is this surprising?

No. Because

$$\binom{n}{r}=\binom{n}{n-r}$$

Part C: Both the Teams

Number of ways to select the

- First Team $=105$
- Second Team $=105$

Bogus Solution

You might want to multiply the two numbers giving you

$$105^2=11{,}025$$

But this would be wrong.

Why?

Because, once you select

- five players for the first-string team, the remaining five players are automatically the ones that make up the second-string team
- OR five players for the second-string team, the remaining five players are automatically the ones that make up the first-string team

Hence, the total number of ways

$$First-String\ Team=Second-String\ Team=Both\ Teams=105$$

2.6 Further Questions

A. Consecutive

Example 2.102

In how many ways can five friends of different heights stand in a line so that no three consecutive people have increasing height?

Total Arrangements

If there are no restrictions, the number of ways to arrange five people is:

$$5!=120$$

Complementary Arrangements

Name the events that we do not want:

$$A = people\ at\ spots\ 123\ are\ in\ increasing\ height\ order$$
$$B = people\ at\ spots\ 234\ are\ in\ increasing\ height\ order$$
$$C = people\ at\ spots\ 345\ are\ in\ increasing\ height\ order$$

One at a Time

For A, we need to choose 3 people out of 5, which can be done in

$$\binom{5}{3}\ ways$$

We can arrange them in height order in exactly one way. The remaining two spots can be filled in

$$2 \times 1 = 2\ ways$$

Total number of ways that we can achieve A is

$$\binom{5}{3} \times 2 = 10 \times 2 = 20\ Ways$$

Similarly:

$$n(A) = n(B) = n(C) = 20\ Ways$$

Two at a Time

$A \cap B$ is the event where spots 1234 are in height order, which can be done in

$$= \binom{5}{4} 1! = 5\ Ways$$

$B \cap C$ is the event where spots 2345 are in height order, which can also be done in

$$= \binom{5}{4} 1! = 5\ Ways$$

$A \cap C$ is the event where spots 12345 are in height order, which can be done in

$$= 1\ Way\ only$$

Three at a Time

$A \cap B \cap C$ is also the event where spots 12345 are in height order, and this can again be done in

$$1\ Way$$

$$n(A \cup B \cup C) = \underbrace{n(A) + n(B) + n(C)}_{One\ at\ at\ a\ Time} - \underbrace{[n(A \cap B) + n(B \cap C) + n(A \cap C)]}_{Two\ at\ a\ Time} + \underbrace{n(A \cap B \cap C)}_{Three\ at\ a\ time}$$
$$= 20 + 20 + 20 - (5 + 5 + 1) + 1 = 60 - 11 + 1 = 50$$

Complementary Counting

The number of valid arrangements is

$$5! - 50 = 120 - 50 = 70$$

B. Gap Method

Example 2.103

What is the number of ways to pick three integers out of the numbers 1 to 10 such that no two are consecutive.

Obtain 10 counters, but do not number them.

$$X, X, \dots, X$$

Since we are going to pick three numbers, remove three counters for now, leaving us

$$7\ Counters$$

Arrange the 7 counters in a row, giving us 8 gaps between them:

$$-X-X-X-X-X-X-X-$$

Choose any three gaps in

$$\binom{8}{3} = 56\ ways$$

Now, we claim that the number of ways to choose the gaps is the same as the number of ways to pick three integers from 1 to 10, such that no two are consecutive.
For example, suppose you pick the first, second and eighth gap:

$$\boldsymbol{X}\ X\ X\ X-X-X-X-X-X\ X$$

Now, you can number the counters:

$$X_1\ X_2\ X_3\ X_4-X_5-X_6-X_7-X_8-X_9\ X_{10}$$

Note that the numbering is to be done after the gaps after the gaps are picked. For example, if you pick the second, third and seventh gaps, the numbering for the first seven counters changes.

$$-X\ \boldsymbol{X}\ X\ \boldsymbol{X}\ X-X-X-X\ \boldsymbol{X}\ X\ -$$

$$-X_1\ \boldsymbol{X_2}\ X_3\ \boldsymbol{X_4}\ X_5-X_6-X_7-X_8\ \boldsymbol{X_9}\ X_{10}\ -$$

C. Repetition

Example 2.104

How many 4-digit numbers greater than 1000 are there that use the four digits of 2012? **(AMC 8 2012/10)**

Ans = 9

Consider two cases.

First Digit is 2

Remaining digits are 0,1,2 which can be arranged in

$$3!\ ways = 6$$

First Digit is 1

Remaining digits are 0,2,2. We can only choose the position of the zero, which can be done in

$$\binom{3}{1} Ways = 6$$

$$Total = 6 + 3 = 9$$

Example 2.105

A mathematical organization is producing a set of commemorative license plates. Each plate contains a sequence of five characters chosen from the four letters in $AIME$ and the four digits in 2007. No character may appear in a sequence more times than it appears among the four letters in $AIME$ or the four digits in 2007. A set of plates in which each possible sequence appears exactly once contains N license plates. Find $\frac{N}{10}$. **(AIME 2007/II/1)**

We want to choose five digits out of eight, and then we want to arrange them.

Choosing five digits out of eight can be done in:

$$\binom{8}{5} = \frac{8 \times 7 \times 6}{6} = 56\ Ways$$

But the number of ways of arranging these five digits will depend upon whether the digits are repeated or not. Hence, we proceed based on casework.

Case I: No zero or One Zero is Used

If there is a maximum of one zero, then the five digits can be chosen in

$$\binom{7}{5} = 21\ Ways$$

And they can be arranged in

$$5! = 120\ Ways$$

Giving a total of

$$21 \times 120 = 2520$$

Case II: Two Zeros Used

If there are two zeros, then the two zeroes are always selected, which can be done in only

$$1\ Way$$

The remaining three digits can be chosen in

$$\binom{6}{3} = 20\ Ways$$

And they can be arranged in

$$\frac{5!}{2!} = 60\ Ways$$

Giving a total of

$$20 \times 60 = 1200\ Ways$$

Final Answer

The total ways for both the cases are:

$$N = 2520 + 1200 = 3720 \Rightarrow \frac{N}{10} = \frac{3720}{10} = 372$$

Example 2.106

A mathematical organization is producing a set of commemorative license plates. Each plate contains a sequence of five characters chosen from:

$$AIME\ \boldsymbol{II}\ 2007$$

Since the two Roman Characters for ***II*** are written in a different font/colour, they are distinguishable from the I used in AIME. No character may appear in a sequence more times than it appears above. Find the number of distinct license plates. **(AIME 2007/II/1, Adapted)**

	Zeros Not Repeated	Zeros Repeated	
I Not Repeated	$A = 6720$	$C = 2100$	
I Repeated	$B = 2100$	$D = 180$	
	8820	2280	11100

Case A

Here we only have 8 digits to pick from (since we cannot repeat the **I** or the zero).

$$\binom{8}{5} \times 5! = 6720$$

Case B

The two **I's** can be selected in only

$$1\ Way$$

The remaining three digits can be chosen in

$$\binom{7}{3} = 35\ Ways$$

And they can be arranged in

$$\frac{5!}{2} = 60\ Ways$$

Giving a total of

$$20 \times 60 = 2100\ Ways$$

Case C

The two zeroes can be selected in only

$$1\ Way$$

The remaining three digits can be chosen in

$$\binom{7}{3} = 35\ Ways$$

And they can be arranged in

$$\frac{5!}{2} = 60\ Ways$$

Giving a total of

$$20 \times 60 = 2100\ Ways$$

Case D

The two zeroes and the two **I's** can be selected in only

$$1\ Way$$

The remaining one digit can be chosen in

$$\binom{6}{1} = 6\ Ways$$

And they can be arranged in

$$\frac{5!}{2!\,2!} = 30\ Ways$$

Giving a total of

$$6 \times 30 = 180\ Ways$$

Example 2.107

Stones are numbered 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. Three groups of stones can be selected so that the sum of each group is 11. For example, one arrangement is {1, 10}, {2, 3, 6}, {4, 7}. Including the example, how many arrangements are possible? **(CEMC Grade 8 2012-24)**

We need to partition the set {1,2,3,4,5,6,7,8,9,10} into four parts with the three of the parts being used, and one part containing the numbers which are not used.

To handle this via cases:

Case I: All three sets have exactly two elements

$$1 + 10 = 2 + 9 = 3 + 8 = 4 + 7 = 5 + 6$$

We have five sets. Out of the five sets, we have to select any three.

$$\binom{5}{3} = \frac{5!}{2!\,3!} = \frac{5 \times 4}{2} = 10$$

Case II: One set has three elements, and other two sets have two elements

{1,2,8}{4,7}{5,6}
{1,3,7}{2,9}{5,6}
{1,4,6}{2,9}{3,8}
{2,3,6}{1,10}{4,7}
{2,4,5}{1,10}{3,8}

From the above, we get

$$5\ cases$$

Remaining Cases
Two or more groups of size 3 or more are not possible.

Hence, the final answer is:
$$10 + 5 = 15$$

2.7 Algebraic Applications

A. Tossing Coins

Example 2.108

I have a 1794 Flowing Hair Silver Dollar, a highly valuable coin among the first ones issued by the US Mint. The obverse side has the Bust of Liberty engraved on it. The reverse side has an eagle surrounded by a wreath. I toss the coin thrice. Count the number of outcomes with:

A. Zero Eagles
B. One Eagles
C. Two Eagles
D. Three Eagles

For the sake of simplicity, we just count Heads and Tails.

Strategy
We have three coin tosses:

$$\underbrace{}_{Toss\ 1}\underbrace{}_{Toss\ 2}\underbrace{}_{Toss\ 3}$$

Zero Heads
Enumeration:
Zero Heads means that all the tosses must be Tails, and this can be done in just one way:

$$\underbrace{Tail}_{Toss\ 1}\ \underbrace{Tail}_{Toss\ 2}\ \underbrace{Tail}_{Toss\ 3}$$

Combinations:
This is equivalent to choosing zero out of three objects, which can be done in

$$\binom{3}{0} = \frac{3!}{0!\,3!} = 1\ Way$$

One Head
Enumeration:
One Head means that exactly one of the tosses must be Heads and this can be done in three ways:

$$\underbrace{Head}_{Toss\ 1}\ \underbrace{}_{Toss\ 2}\ \underbrace{}_{Toss\ 3}\quad OR\quad \underbrace{}_{Toss\ 1}\ \underbrace{Head}_{Toss\ 2}\ \underbrace{}_{Toss\ 3}\quad OR\quad \underbrace{}_{Toss\ 1}\ \underbrace{}_{Toss\ 2}\ \underbrace{Head}_{Toss\ 3} = 3\ Ways$$

Combinations:
From the diagram above, of the three tosses we must choose exactly one to be Heads, which is the same as selecting one object of three, which can be done in:

$$\binom{3}{1} = \frac{3!}{1!\,2!} = 3\ Ways$$

Two Heads
Enumeration:
Two Heads means that exactly one of the tosses must be Tails and this can be done in three ways:

$$\underbrace{Tails}_{Toss\ 1}\ \underbrace{}_{Toss\ 2}\ \underbrace{}_{Toss\ 3}\quad OR\quad \underbrace{}_{Toss\ 1}\ \underbrace{Tails}_{Toss\ 2}\ \underbrace{}_{Toss\ 3}\quad OR\quad \underbrace{}_{Toss\ 1}\ \underbrace{}_{Toss\ 2}\ \underbrace{Tails}_{Toss\ 3}$$

Combinations:
From the diagram above, of the three tosses we must choose exactly two to be Heads, which is the same as

selecting two objects out of three, which can be done in:

$$\binom{3}{2} = \binom{3}{1} = \frac{3!}{2!\,1!} = 3\ Ways$$

Three Heads
Enumeration:
Three Heads means that all the tosses must be Heads, and this can be done in just one way:

$$\underbrace{Heads}_{Toss\ 1}\underbrace{Heads}_{Toss\ 2}\underbrace{Heads}_{Toss\ 3}$$

Combinations:
This is equivalent to choosing three out of three objects, which can be done in

$$\binom{3}{3} = \frac{3!}{0!\,3!} = 1\ Way$$

Example 2.109
In the previous example, we calculated the number of outcomes with zero, one, two and three heads when tossing a coin thrice. Tabulate the outcomes you calculated in the previous example.

No. of Heads	
Zero	$\binom{3}{0} = \frac{3!}{0!\,3!} = 1$
One	$\binom{3}{1} = \frac{3!}{1!\,2!} = 3$
Two	$\binom{3}{2} = \frac{3!}{2!\,1!} = 3$
Three	$\binom{3}{3} = \frac{3!}{3!\,0!} = 1$
Total	8 Outcomes

Example 2.110
I toss a coin four times, and record each outcome as it happens. What is the:

A. Total Number of Outcomes
B. Outcomes with Zero, One, Two, Three and Four Heads respectively.

For total outcomes, we use the idea of objects with repetition. We have four tosses, and in each toss, we have two choices:

$$Total\ Outcomes = \underbrace{2}_{Toss\ 1} \times \underbrace{2}_{Toss\ 2} \times \underbrace{2}_{Toss\ 3} \times \underbrace{2}_{Toss\ 4} = 2^4 = 16$$

No. of Heads	
Zero	$\binom{4}{0} = \frac{4!}{0!\,4!} = 1$
One	$\binom{4}{1} = \frac{4!}{1!\,3!} = 4$
Two	$\binom{4}{2} = \frac{4!}{2!\,2!} = 6$
Three	$\binom{4}{3} = \frac{4!}{3!\,1!} = 4$
Four	$\binom{4}{4} = \frac{4!}{4!\,0!} = 1$
Total	16 Outcomes

Example 2.111

I toss a coin five times, and record each outcome as it happens. What is the:

A. Total Number of Outcomes
B. Outcomes with Zero, One, Two, Three, Four and Five Heads respectively.

For total outcomes, we use the idea of objects with repetition. We have four tosses, and in each toss, we have two choices:

$$Total\ Outcomes = 2^5 = 32$$

No. of Heads	
Zero	$\binom{5}{0} = \frac{5!}{0!\,5!} = 1$
One	$\binom{5}{1} = \frac{5!}{1!\,4!} = 5$
Two	$\binom{5}{2} = \frac{5!}{2!\,3!} = 10$
Three	$\binom{5}{3} = \frac{5!}{3!\,2!} = 10$
Four	$\binom{5}{4} = \frac{5!}{4!\,1!} = 5$
Five	$\binom{5}{5} = \frac{5!}{5!\,0!} = 1$
Total	32 Outcomes

2.112: Tossing A Coin: Number of Heads and Total Outcomes

I toss a coin n times:

$$Total\ outcomes = 2^n$$

$$Outcomes\ with\ m\ heads: \binom{n}{m}$$

Of course, we need to put the condition that:

$$0 \le m \le n$$

Else, the number of outcomes will be zero.

B. Rolling Dice

Example 2.113

I toss five eight-sided dice. In how many ways can exactly four of the dice show a two.

I want four of the dice to show a two.

$$\underbrace{Two}_{Die\ 1}, \quad \underbrace{Two}_{Die\ 2}, \quad \underbrace{Two}_{Die\ 3}, \quad \underbrace{Two}_{Die\ 4}, \quad \underbrace{Not\ Two}_{Die\ 5} \Rightarrow 1\ Way$$

However, the roll which is not two, could be in any position.
Hence, we need to multiply by

$$5$$

Further, the number which is not two, can be anything from

$$\{1,3,4,5,6,7,8\} \Rightarrow 7 \; Choices$$

Hence, we need to multiply by

$$7$$

Hence, the final answer

$$1 \times 5 \times 7 = 35$$

(Combinations) Example 2.114

I toss five eight-sided dice. In how many ways can exactly four of the dice show a two.

$$\binom{5}{4} \times \binom{7}{1} = 5 \times 7 = 35$$

Example 2.115

I toss nine eight-sided dice. In how many ways can exactly seven of the dice show a six.

We can choose seven dice out of nine to show the six in

$$\binom{9}{7} = \frac{9 \cdot 8}{7} = 36 \; ways$$

The remaining two dice then can show a non-six in:

$$7 \cdot 7 = 49 \; ways$$

By the multiplication principle, the total number of ways is:

$$36 \times 49 = 1764$$

C. Choosing Cards

Example 2.116

A. I draw three cards from a standard pack of cards. In how many ways can exactly two of the cards be Red.
B. I draw four cards from a standard pack of cards. In how many ways can exactly one of them be an Ace.
C. I draw five cards from a standard pack of cards. In how many ways can exactly two of them be Black
D. I draw ten cards from a standard pack of cards. In how many ways can exactly three of them be Spades.

$$\binom{3}{2} = \frac{3!}{1!\,2!} = 3$$

$$\binom{4}{1} = 4$$

$$\binom{5}{2} = 10$$

$$\binom{10}{3} = \frac{10 \times 9 \times 8}{6} = 120$$

Example 2.117

I draw ten cards from a standard pack of cards. In how many ways can exactly six of the cards be Kings if the cards are drawn:

A. Without replacement
B. With replacement

Part A: Without Replacement

There are only four kings.
And we want six.

Hence, without replacement, we can do this in

$$Zero\ Ways$$

Part B: With Replacement

$$\binom{10}{6} = \frac{10!}{4!\,6!} = \frac{10 \times 9 \times 8 \times 7}{24} = 210$$

D. Set Theory

Example 2.118

(Formula): A set has n elements. What is the number of subsets that have r elements?

By definition, order is not important in a set.

The number of subsets with r elements
= Number of ways of selecting r elements out of n elements

$$= {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

Example 2.119: Subsets of a Set

$A = \{x: 0 < x < 10, x \in N\}$

List the elements of X in roster form and then find the number of subsets of A that contain

A. Three elements (which are all odd)
B. Three elements (which are all even)
C. Two elements (which are both prime)
D. Five numbers

A = {1, 2, 3, 4, 5, 6, 7, 8, 9}

A. Selecting three elements $\underbrace{\text{out of five}}_{\{1,3,5,7,9\}} = {}^{5}C_{3} = \frac{5!}{2!\,(3)!} = \frac{5 \times 4}{2} = 10$

B. Selecting three even numbers $\underbrace{\text{out of four}}_{\{2,4,6\ 8\}} = {}^{4}C_{3} = \frac{4!}{1!\,(3)!} = 4$

C. Selecting two prime numbers $\underbrace{\text{out of four}}_{\{2,3,5,7\}} = {}^{4}C_{2} = \frac{4!}{2!\,(2)!} = \frac{4 \times 3}{2} = 6$

$$D.\text{ Selecting five numbers out of all nine numbers} = {}^{9}C_{5} = \frac{9!}{5!\,(4)!} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2} = 126$$

2.120: Sum of Combinations: Combinatorial

$$\boldsymbol{{}^{n}C_{0} + {}^{n}C_{1} + \cdots + {}^{n}C_{n} = 2^{n}}$$

Count the number of ways of selecting zero or more balls from n balls in two ways.

$$\text{Combinations: } \underbrace{{}^{n}C_{0}}_{\substack{Select\ 0\\Balls}} + \underbrace{{}^{n}C_{1}}_{\substack{Select\ 1\\Ball}} + \cdots + \underbrace{{}^{n}C_{n}}_{\substack{Select\ n\\Balls}} = LHS$$

$$\text{Multiplication Principle: } \underbrace{2}_{\substack{First\\Ball}} \times \underbrace{2}_{\substack{Second\\Ball}} \times \ldots \times \underbrace{2}_{\substack{n^{th}\\Ball}} = 2^{n} = RHS$$

But, the number of ways counted both ways must be the same.

$$\therefore LHS = RHS$$

E. Sums of Subsets

Example 2.121

The set $\{1,2,3,4\}$ has n subsets. Let s_m be the sum of the elements of the m^{th} subset. Find $n + s_1 + s_2 + \cdots + s_n$.

$$n = 2^4 = 16$$

We do this using cases.

0 Element Sets

$$\{\phi\} \Rightarrow Sum = 0$$

1 Element Sets

$$\{1\},\{2\},\{3\},\{4\} \Rightarrow Sum = 1 + 2 + 3 + 4 = 10$$

2 Element Sets

$$\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}$$

Note that each element of the original set occurs twice. Hence, the sum is:

$$2(1 + 2 + 3 + 4) = 2(10) = 20$$

3 Element Sets

$$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}$$

Compare this with:

$$\{1,2,3,\mathbf{4}\},\{1,2,\mathbf{3},4\},\{1,\mathbf{2},3,4\},\{\mathbf{1},2,3,4\}$$

And hence the sum we want is equal to:

$$4(1 + 2 + 3 + 4) - 1(1 + 2 + 3 + 4) = 30$$

4 Element Sets

$$\{1,2,3,4\} \Rightarrow Sum = 10$$

And hence, the final answer is:

$$0 + 10 + 20 + 30 + 10 = 70$$

2.8 Equations/Further Topics

A. Equations

122 Examples