# Polynomials

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# Table of contents

# 1. Polynomials

## A. Learning Note

There are many definitions in this Note. These definitions are not difficult, but need to be understood and remember to solve numerical questions

## 1.1 Basics

### A. Polynomial

An algebraic expression made of more than one term is called a polynomial. The powers in a polynomial must be:

$$\textit{Natural Numbers}: n \in \mathbb{N}$$

Some examples of polynomials are:

$$4x^2 + 3x - 6$$
$$x^3 - 12$$
$$x + 5$$
$$3x$$
$$12$$

**1.1: Polynomial**

An expression of the form:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Is a polynomial in the variable $x$ with degree $n$

With

$$\textit{Terms are with the variable}: a_n x^n, a_{n-1} x^{n-1}, \dots, a_1 x, a_0$$
$$\textit{Coefficients are without the variable}:$$
$$a_n \neq 0$$

$n$ must be a nonnegative integer.

**Example 1.2**

Is $y = x^{4.5}$

$4.5 = \frac{9}{2}$ is a rational number, but not a nonnegative integer.

Hence, the given expression is not a polynomial.

**1.3: Terms**

The terms in a polynomial are separated by plus or minus signs. The terms in the polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad n \in \mathbb{N}$$

Are

$$: a_n x^n, a_{n-1} x^{n-1}, \dots, a_1 x, a_0$$

Terms are stated with the variable, if they have one.

**1.4: Coefficients**

The numbers which multiply with the variables in a polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad n \in \mathbb{N}$$

are the coefficients.

$$a_n, a_{n-1}, \dots, a_0$$

Coefficients are without the variable.

As we will see below, polynomials with real coefficients can have complex roots. But, we will not consider polynomials with complex coefficients (for this chapter).

### Example 1.5

For each polynomial, identify the terms, the coefficients, and the number of terms

A. $4x^2 + 3x - 6$
B. $5x^2 - \frac{2}{77}x + 31$
C. $ax^3 + bx^2 + cx + d$

#### Part A

$$First\ Term = 4x^2$$
$$Second\ Term = 3x$$
$$Third\ Term = -6$$

$$Coefficients: 4,3,-6$$

#### Part B

$$5x^2, -\frac{2}{77}x, 31$$
$$Coefficients: 5, -\frac{2}{77}, 31$$

#### Part C

$$ax^3, bx^2, cx, d$$

### Example 1.6

Is the expression

$$3x^2 + 2ax + 4a^2$$

A. A polynomial in $x$
B. A polynomial in $a$
C. A polynomial in either $a$, or $x$, depending upon the interpretation
D. A polynomial in neither $a$, nor $x$

Option C

Rewrite $3x^2 + 2ax + 4a^2$ as a polynomial in $a$ in standard form

$$4a^2 + 2ax + 3x^2$$

### 1.7: Constant Term in a Polynomial

The *term independent of* $x$ in the polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad n \in \mathbb{N}$$

Is called the constant term

When we say independent of $x$, we mean that as the value of $x$ changes, the value of the term does not change.

**Example 1.8**

$$P(x) = 4x^2 + 3x - 6$$

A. If $x = 0$, what is the value of the last term?
B. If $x = 23$, what is the value of the last term?

$$x = 0 \Rightarrow P(x) = 4(0)^2 + 3(0) - 6 \Rightarrow Last\ Term = -6$$
$$x = 23 \Rightarrow P(x) = 4(23)^2 + 3(23) - 6 \Rightarrow Last\ Term = -6$$

**Example 1.9**

For each polynomial, identify the constant term:

A. $4x^2 + 3x - 6$
B. $5x^2 - \frac{2}{77}x + 31$
C. $ax^3 + bx^2 + cx + d$

## B. Special Cases

**1.10: Monomial**

A monomial is a polynomial with a single term:

- A Variable $(eg: y, x^2)$
- A number $(eg: 7, \frac{1}{2})$
- Or a product of a number and a variable $(\frac{1}{2}x^2, 7y)$

**1.11: Binomial**

A binomial is a polynomial with two terms:

- $(a + b)$
- $(7x^3 + 3y^2)$

**1.12: Trinomial**

A trinomial is a polynomial with three terms:

- $a^2 + ab + b^2$

## C. Standard Form

**1.13: Standard Form**

- In standard form, a polynomial is written in descending powers of the variable.
- Like terms have been added together

**Example 1.14**

Write the following in standard form:

A. $5 + x + x^2$
B. $x^4 + x^7 - 9$

C. $5 + x + x^2 + 3$

$$x^2 + x + 5$$
$$x^7 + x^4 - 9$$
$$x^2 + x + 8$$

### Example 1.15

Decide whether the following are in standard form:

A. $3x^2 + 2x^2 + 5$

### Example 1.16

*The sum of the roots of the equation* $4x^2 + 5 - 8x = 0$ *is equal to:* **(AHSME 1950/3)**

The sum of the roots of the quadratic $ax^2 + bx + c = 0$ is given by:

$$-\frac{b}{a}$$

Explain the "trick" in this question, and why $-\frac{5}{4}$ is not the answer.

$$ax^2 + bx + c = 0$$

Is written in standard form.

But

$$4x^2 + 5 - 8x = 0$$

Is not written in standard.

## D. Leading Coefficient

### 1.17: Leading Coefficient

The first coefficient when the polynomial is written in standard form is called the leading coefficient.

### Example 1.18

What is the leading coefficient in the following polynomials:

A. $3x^4 + 7x^5$
B. $0x^5 + 3x^4 + 7$
C. $0x^7 - 3x^2 + 4x^6$

$$A: 7x^5 + 3x^4 \to 7$$
$$B: 3x^4 + 7 \to 3$$

### 1.19: Monic Polynomial

A monic polynomial is a polynomial where the leading coefficient is 1.

### Example 1.20

Consider the equation below

$$3x^3 + 27x + 81x^2 + 243 = 0$$

where the left hand is not a monic polynomial. Rewrite the equation so that the left side is a monic polynomial, and write the polynomial in standard form.

Divide both sides by 3:

$$x^3 + 27x^2 + 9x + 81 = 0$$

### Example 1.21

Consider the equation below

$$\frac{1}{64}x^3 + \frac{1}{16}x + \frac{1}{128}x^2 + \frac{3}{256} = 0$$

where the left hand is not a monic polynomial. Rewrite the equation so that the left side is a monic polynomial, and write the polynomial in standard form.

$$x^3 + \frac{1}{2}x^2 + 4x + \frac{3}{4} = 0$$

## E. Coefficients

The numbers that multiply with the variables are called coefficients.

### Example 1.22

Consider

$$P(x) = 3x^2 + 5x + 4$$

A. What is the sum of the coefficients?
B. What is the product of the coefficients?

#### Part A

$$P(x) = 3x^2 + 5x + 4x^0$$

Hence, the last term (4) is also a coefficient.

To find the sum of the coefficients, you substitute $x = 1$:

$$Sum = 3 + 5 + 4 = 12$$

#### Part B

$$Product = 3 \times 5 \times 4 = 60$$

### Multiple Choice Multiple Correct 1.23

The quadratic polynomial which has a non-zero constant term is:

$$P(x) = ax^2 + bx + c$$

Has product of its coefficients zero. Then, which of the following can be true:

A. $a = 0$
B. $b = 0$
C. $ab = 0$
D. $ac = 0$
E. $bc = 0$

Product of coefficients is zero

$$abc = 0 \Rightarrow a = 0\ OR\ b = 0\ OR\ c = 0$$

Polynomial is quadratic

$$a \neq 0$$

And the question tells us

$$c \neq 0$$

This means that

$$b = 0$$

Hence, the right answer is:

$$Option\ B, Option\ C, Option\ E$$

### Multiple Choice Multiple Correct 1.24

The quadratic polynomial which has a non-zero constant term

$$P(x) = ax^2 + bx + c$$

Has sum of its coefficients zero. Then, classify each of the following as:

1. Can be true
2. Must be true
3. Must be false

A. $a = 0$
B. $b = 0$
C. $ab = 0$
D. $ac = 0$
E. $bc = 0$

$$a = 0 \Rightarrow False$$
$$b = 0 \Rightarrow Can\ be\ true$$
$$ab = 0 \Rightarrow Can\ be\ true$$
$$ac = 0 \Rightarrow False$$
$$bc = 0 \Rightarrow Can\ be\ true$$

## F. Degree

### 1.25: Degree

The degree of a polynomial is the value of the highest power in the polynomial.

### Example 1.26

Is the condition that $a_n \neq 0$ required for a polynomial? If yes, explain why

Otherwise, you can change the degree of the polynomial to any degree that you wish.

### Example 1.27

Determine the degree of the following expressions:

A. $4x^2 + 5x + 6$
B. $7x^5$
C. $ax^4 + bx^5 + c$
D. $ax^{5!} + bx^{4!} + c$

$$A: 2$$
$$B: 5$$
$$C: 5$$
$$D: 120$$

### Example 1.28
Determine the degree and the maximum number of terms of:
A. A Quadratic Polynomial
B. A Cubic Polynomial

$$Quadratic: ax^2 + bx + c \to Degree\ 2, \quad Terms\ 3$$
$$Cubic: ax^3 + bx^2 + cx + d \to Degree\ 3, \quad Terms\ 4$$

### Example 1.29
In a polynomial of degree $n$, what is the:
A. Maximum number of non-zero terms?
B. Minimum number of non-zero terms?

$$P(x) = a_n x^n + a_n x^{n-1} + \cdots + a_1 x + a_0$$
$$Max\ Terms: n + 1$$
$$Min\ Terms: 0$$

## G. Roots

### 1.30: Roots of a Polynomial
The roots of a polynomial $P(x)$ are the values that satisfy
$$P(x) = 0$$

### Example 1.31
Determine the roots of the polynomial:
$$3x + 5$$

$$3x + 5 = 0 \Rightarrow x = -\frac{5}{3} \Rightarrow Root\ is\ -\frac{5}{3}$$

### Example 1.32
Determine the roots of the polynomial:
$$\left(\frac{2}{3}x + \frac{4}{5}\right)\left(-\frac{3}{4}x + \frac{7}{2}\right)$$

$$\left(\frac{2}{3}x + \frac{4}{5}\right)\left(-\frac{3}{4}x + \frac{7}{2}\right) = 0$$
Apply the zero-product property:
$$\frac{2}{3}x + \frac{4}{5} = 0 \Rightarrow \frac{2}{3}x = -\frac{4}{5} \Rightarrow x = -\frac{4}{5} \times \frac{3}{2} = -\frac{6}{5}$$
$$-\frac{3}{4}x + \frac{7}{2} = 0 \Rightarrow -\frac{3}{4}x = -\frac{7}{2} \Rightarrow x = \frac{7}{2} \times \frac{4}{3} = \frac{14}{3}$$
$$Roots\ are\ \left\{-\frac{6}{5}, \frac{14}{3}\right\}$$

### Example 1.33
Determine the roots of the polynomial:
$$P(x) = cx^2 + bx + a$$

Substitute $a = c, b = b, c = a$ in the quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}$$

### Example 1.34

Determine the number of roots and the degree of the polynomial

A. $(x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$
B. $(x - \alpha_1)(x - \alpha_2)(x - \alpha_n)$

$$A: n\ roots, degree\ n$$
$$A: 3\ roots, degree\ 3$$

## H. Number and Type of Roots

### 1.35: Number of Roots

A polynomial of degree $n$ has exactly $n$ roots.

- Roots may be real or complex
- Roots may be repeated

We will not prove this here (though it can be proven).

### Example 1.36

A polynomial of degree 12 has exactly 4 real roots. How many complex roots does it have?

$$12 - 4 = 8$$

### Example 1.37

What is the number of roots of the polynomial:

$$\frac{27}{26}x^2 + \frac{26}{27}x + \frac{12}{19}$$

$$Quadratic: 2\ roots$$

### Example 1.38

A polynomial is of degree 35. What is the number of roots it has?

$$35$$

### 1.39: Multiplicity of Roots

If a root of a polynomial occurs more than once, it is said to have the corresponding multiplicity.

### Example 1.40

What is the multiplicity of the root $-5$ in the following polynomials:

A. $x + 5$
B. $(x + 5)^2(x - 5)$

**Part A**

$$x + 5 = 0 \Rightarrow x = -5 \Rightarrow Multiplicity\ is\ One$$

### Part B

$$x + 5 = 0 \Rightarrow x = -5 \Rightarrow Multiplicity\ is\ One$$
$$(x+5)^2 \Rightarrow x = -5 \Rightarrow Multiplicity\ is\ Two$$

## 1.41: Odd and Even Multiplicity of Roots

- A root that occurs an odd number of times has odd multiplicity.
- A root that occurs an even number of times has even multiplicity.

## Example 1.42

$$(3x-4)^3\left(\frac{6}{7}x-\frac{2}{5}\right)^4\left(\frac{2}{3}x-\frac{7}{9}\right)^{2020}\left(\frac{2020}{2019}x-\frac{2019}{2020}\right)^{2021}$$

A. What is the degree of the polynomial?
B. What is the leading coefficient of the polynomial? Give your answer in unsimplified form.
C. State each root of the polynomial.
D. Determine for each root whether it has odd or even multiplicity.

### Parts A and B

$$Degree = 2021 + 2020 + 4 + 3 = 4048$$
$$Leading\ Coefficient = 3^3 \times \left(\frac{6}{7}\right)^4 \times \left(\frac{2}{3}\right)^{2020} \times \left(\frac{2020}{2019}\right)^{2021}$$

### Parts C and D

To find the roots, we equate polynomial to zero:

$$(3x-4)^3\left(\frac{6}{7}x-\frac{2}{5}\right)^4\left(\frac{2}{3}x-\frac{7}{9}\right)^{2020}\left(\frac{2020}{2019}x-\frac{2019}{2020}\right)^{2021} = 0$$

Apply the zero-product property:

$$abcd = 0 \Rightarrow a = 0\ OR\ b = 0\ OR\ c = 0\ OR\ d = 0$$

### Take the first Term

$$(3x-4)^3 = 0$$

Take Cube Roots both sides:

$$3x - 4 = 0 \Rightarrow x = \frac{4}{3}, \qquad 3\ is\ odd \Rightarrow Odd\ Multiplicity$$
$$\left(\frac{6}{7}x-\frac{2}{5}\right)^4 = 0 \Rightarrow \frac{6}{7}x-\frac{2}{5} = 0 \Rightarrow x = \frac{2}{5}\times\frac{7}{6} = \frac{7}{15}, \qquad 4\ is\ even \Rightarrow Even\ Multiplicity$$
$$\left(\frac{2}{3}x-\frac{7}{9}\right)^{2020} = 0 \Rightarrow \frac{2}{3}x-\frac{7}{9} = 0 \Rightarrow x = \frac{7}{9}\times\frac{3}{2} = \frac{7}{6}, \qquad 2020\ is\ even \Rightarrow Even\ Multiplicity$$

$$\left(\frac{2020}{2019}x-\frac{2019}{2020}\right)^{2021} = 0$$
$$\frac{2020}{2019}x-\frac{2019}{2020} = 0 \Rightarrow x = \frac{2019}{2020}\times\frac{2019}{2020} = \left(\frac{2019}{2020}\right)^2, \qquad 2021\ is\ odd \Rightarrow Odd\ Multiplicity$$

## 1.43: Forming Polynomial with Given Roots

If the roots of a polynomial are

$$\alpha_1, \alpha_2, \ldots, \alpha_n$$

Then the polynomial is

$$(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$$

### Example 1.44

Form the polynomial with

$$\text{Root } 3 \text{ of Multiplicity } 2$$
$$\text{Root } -2 \text{ of Multiplicity } 3$$

$$P(x) = (x-3)^2(x+2)^3$$

## 1.45: Complex Roots occur in pairs

- Complex roots of a polynomial with real coefficients occur only in pairs.
- These pairs are called complex conjugates of each other.

Every complex root has associated with it, one more root, which is its complex conjugate. For instance, in a quadratic, the roots are:

$$\frac{-b+\sqrt{b^2-4ac}}{2a}, \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Complex roots are roots which have not only real parts but also imaginary parts. Complex numbers are written in the form

$$a+bi, \qquad i=\sqrt{-1}$$

In general, complex conjugates are of the form:

$$a+bi, \qquad a-bi$$

### Example 1.46

Consider the polynomial with roots 4 and $5i$. Does it violate the condition that complex roots occur only in pairs?

$$(x-4)(x-5i) = x^2-4x-5ix+2i$$

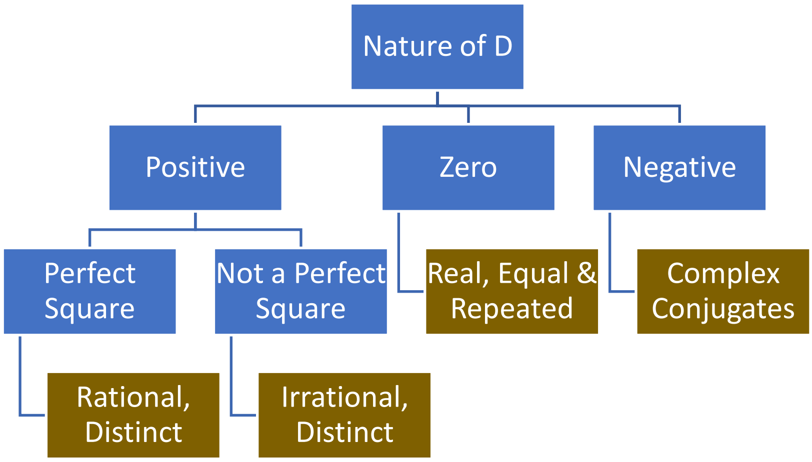
And as we can see above, the coefficients are complex.
Hence, the condition is not violated.

### Example 1.47

Is it possible that a polynomial of degree 12 with real coefficients has exactly 5 real roots?

$$No.\,of\ Complex\ Roots = 12-5 = 7 \Rightarrow Odd \Rightarrow Not\ possible$$

## I. Revision: Nature of Roots of A Quadratic




**1.48: Discriminant of a Quadratic**

The discriminant of the quadratic $y = ax^2 + bx + c$ is

$$b^2 - 4ac$$

**Example 1.49**

List the possible cases for the roots of a quadratic in terms of whether they are real or complex.

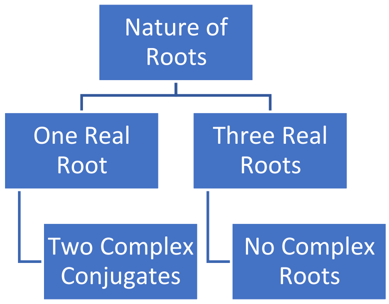
**Example 1.50**

List the possible cases for the roots of a quadratic with real distinct roots in terms of whether they are irrational or real

## J. Nature of Roots of a Cubic

**Example 1.51**

Explain why a cubic cannot have either no real roots, or two real roots.




The number of roots that a cubic has is

$$3$$

Recall that complex roots occur in pairs. Hence, the only possible values for the number of complex roots:

$$0 \textit{ and } 2$$

The corresponding value for the number of real roots is:

$$3 \textit{ and } 1$$

**Example 1.52**

List the possible cases for the roots of a cubic in terms of whether they are real or complex.

**1.53: Number of Real Roots of a Polynomial**

- A polynomial with even degree must have an even number of real roots.
- A polynomial with odd degree must have an odd number of real roots.

## K. Zeroes of a Function

**1.54: Zeroes of a Polynomial Function**

The zeroes of a polynomial function are the roots of the corresponding polynomial equation.

**Example 1.55**

**1.56: Cross versus Bounce**

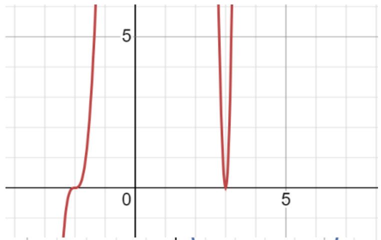
The graph of a polynomial will

- Bounce at the $x$-axis at a zero, if the zero has even multiplicity
- Cross at the $x$-axis at a zero, if the zero has odd multiplicity

### Example 1.57
Use a graphing calculator to sketch the polynomial with roots

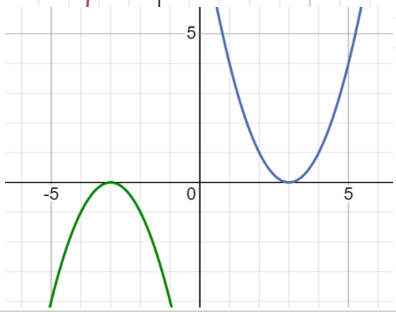
$$Root\ 3\ of\ Multiplicity\ 2$$
$$Root\ -2\ of\ Multplicity\ 3$$

$$P(x) = (x-3)^2(x+2)^3$$




## L. End Behavior of a Polynomial
The leading coefficient determines the behavior of the polynomial for very large values ($\infty$) and very negative values ($-\infty$).

**1.58: Polynomials of even degree**
If the leading coefficient is
- positive, then the polynomial is positive at $\pm\infty$
- negative, then the polynomial is negative at $\pm\infty$




**1.59: Polynomials of odd degree**
If the leading coefficient is
- positive, then the polynomial is positive at $\infty$, and negative at $-\infty$
- negative, then the polynomial is negative at $\infty$, and positive at $-\infty$

# 1.2 Addition, Subtraction and Multiplication
## A. Addition and Subtraction

**1.60: Adding and Subtracting Polynomials**
We add polynomials term by term. Terms with like coefficients can be added.

### Example 1.61
Given that

$$P(x) = 3x^2 - 5x + 3$$
$$Q(x) = 2x^2 - 4x + 7$$

Determine the value of:
A. $P(x) + Q(x)$
B. $P(x) - Q(x)$

$$P(x) + Q(x) = 5x^2 - 9x + 10$$
$$P(x) - Q(x) = x^2 - x - 4$$

### Example 1.62
The expressions below are polynomials written in standard form:

$$P(x) = x^{999} + 2x^{996} + 3x^{993} + \cdots + a$$
$$Q(x) = x^{998} + 2x^{996} + 3x^{994} + \cdots + b$$

A. Determine the value of $a$.
B. Determine the value of $b$.
C. Determine the first six terms of $P(x) + Q(x)$.

D. Determine the number of terms in $P(x) + Q(x)$.

**Part A**

$$P(x) = \underbrace{1x^{999}}_{1st\ Term} + \underbrace{2x^{996}}_{2nd\ Term} + \underbrace{3x^{993}}_{3rd\ Term} + \cdots + \underbrace{a}_{a^{th}\ Term}$$

Note that the coefficients have a pattern

$$Value\ of\ coefficient = Term\ Number\ from\ the\ Lefy$$

Look at the exponents:

$$999, 996, 993 = \{3 \times 333, 3 \times 332, \dots, 3 \times 0\} \Rightarrow a = 334$$

**Part B**

$$P(x) = x^{999} + 2x^{996} + 3x^{993} + \cdots + a$$
$$Q(x) = x^{998} + 2x^{996} + 3x^{994} + \cdots + b$$

$$P(x) + Q(x) = x^{999} + x^{998} + 4x^{996} + 3x^{994} + 3x^{993} + 4x^{992} + \cdots$$
$$P(x): \{999, 996, \dots, 0\} \Rightarrow 334\ terms$$
$$Q(x): \{998, 996, \dots, 0\} \Rightarrow 500\ terms$$
$$P(x)\ \&$$

Total terms

$$= 33 + 4$$

## 1.63: Addition is Negative Subtraction

$$x - y = x + (-y)$$

### Example 1.64

Find $R(x)$ given that:

$$P(x) = 3x^2 + 4x + 7$$
$$Q(x) = 2x^2 - 4x + 3$$
$$S(x) = P(x) - Q(x) = P(x) + R(x)$$

$$S(x) = P(x) - Q(x) = P(x) + [-Q(x)]$$

Hence, by comparing, we know that:

$$R(x) = -Q(x) = -(2x^2 - 4x + 3) = -2x^2 + 4x - 3$$

# B. Degree

## 1.65: Degree Function

For the sake of simplifying notation, introduce a function $d$ that takes a polynomial as an input and gives the degree of the polynomial as an output.

$$d[P(x)] = Degree\ of\ Polynomial\ P(x)$$

### Example 1.66

Evaluate

A. $d(2x^2 + 5x - 7)$
B. $d(7x^{99} + 5x - 7)$
C. $d(x - 7)$
D. $d(-7)$

$$d(2x^2 + 5x - 7) = 2$$

$$d(7x^{99} + 5x - 7) = 99$$
$$d(x - 7) = 1$$
$$d(-7) = 0$$

### Example 1.67

Write the degrees of the following functions:

A. Quadratic
B. Cubic
C. Linear
D. Constant

$$Quadratic \Rightarrow Degree\ 2$$
$$Cubic \Rightarrow Degree\ 3$$
$$Linear \Rightarrow Degree\ 1$$
$$Constant \Rightarrow Degree\ 0$$

### 1.68: Maximum Function

The maximum function returns the greatest of its inputs.

$$Max(a, b, c) = Greatest\ Value\ among\ a, b, c$$

### 1.69: Minimum Function

The minimum function returns the smallest of its inputs.

$$Min(a, b, c) = Smallest\ Value\ among\ a, b, c$$

### Example 1.70

Find

A. $Max(\pi^2, 10)$
B. $Min\left(1.5, \frac{\pi}{2}\right)$
C. $Min(-3, -2)$

$$Max(\pi^2, 10) = Max(\approx 9.85, 10) = 10$$
$$Min\left(1.5, \frac{\pi}{2}\right) = Min\left(1.5, \approx \frac{3.14}{2}\right) Min\left(1.5, \approx \frac{1.57}{2}\right) = 1.5$$
$$Min(-3, -2) = -3$$

### 1.71: Degree of Result

Consider two polynomials $P(x)$ and $Q(x)$ with:

$$d(P(x)) = p, \qquad d(Q(x)) = q,$$

Case I: $p \neq q$:

$$Max(d[P(x) + Q(x)]) = Min(d[P(x) + Q(x)]) = Max(p, q)$$

Case II: $p = q$:

$$Max(d[P(x) + Q(x)]) = \mathrm{p} = \mathrm{q}, \qquad Min(d[P(x) + Q(x)]) = 0$$

### Example 1.72

Determine the degree of the answer in each case:

A. The sum of a fifth-degree polynomial and a third-degree polynomial

B. The sum of a fifth-degree polynomial and a fifth-degree polynomial

$$\textit{Part A}: 5$$
$$\textit{Part B}: \textit{Anything from } 0 \textit{ to } 5$$

### 1.73: Degree is Zero

If

$$d[P(x)+Q(x)]=0$$

Then if you *remove the constant term*

$$P(x)=-Q(x)$$

### Example 1.74

$$P(x)=3x+5$$
$$Q(x)=-3x+7$$

$$P(x)+Q(x)=3x+5(-3x+7)=12$$

### Example 1.75

$$d[P(x)+Q(x)]=0$$

What is the relation between $P(x)$ and $Q(x)$

$$d[P(x)+Q(x)]=0$$

This means that the degree of their sum is zero. Hence, for some constant c, we must have:

$$P(x)+Q(x)=c \Rightarrow P(x)=c-Q(x)$$

Hence, the two polynomials only differ by a constant.

### Example 1.76

$$P(x)=2x^2+8x-4$$
$$d[P(x)+Q(x)]=0$$

If $Q(x)$ is a polynomial with constant term 6, find $Q(x)$.

$$P(x)+Q(x)=C$$
$$Q(x)=C-P(x)=C-(2x^2+8x-4)=-2x^2-8x+4+C=-2x^2-8x+C_1$$

And we know from the question that $C_1=6$:

$$Q(x)=-2x^2-8x+6$$

### Example 1.77

$P(x)$ is a polynomial of degree $p$, and $Q(x)$ is a polynomial of degree $q$.

A. What is the range of degrees that $P(x)+Q(x)$ can take?
B. What is the product of the minimum value that the degree of $P(x)+Q(x)$ can take, and the maximum value that the degree of $P(x)+Q(x)$ can take?

$$Max(p,q)=n \Rightarrow Range\{0,1,2,\dots,n\}$$
$$Product=Min \times Max=0 \times n=0$$

### Example 1.78

Consider polynomials $P(x)$ and $Q(x)$ such that the sum of the polynomials is equal to the degree of the sum of the polynomials. That is:

$$P(x) + Q(x) = d[P(x) + Q(x)], \qquad d\ is\ the\ degree\ function$$

Then find:

$$P(x) + Q(x)$$

### Method I

For some constant c:

$$d[P(x) + Q(x)] = c$$
$$P(x) + Q(x) = c$$

But degree of c is 0. Hence

$$c = 0 = P(x) + Q(x)$$

### Method II

We do not know anything about $P(x)$ or $Q(x)$. However, we can give the range of $d[P(x) + Q(x)]$, which is:

$$Range\{0,1,2,\ldots\}$$

Suppose

$$d[P(x) + Q(x)] = 0 \Rightarrow P(x) + Q(x) = 0 \Rightarrow Works$$
$$d[P(x) + Q(x)] = 1 \Rightarrow P(x) + Q(x) = 1 \Rightarrow Does\ not\ Work$$
$$d[P(x) + Q(x)] = 2 \Rightarrow P(x) + Q(x) = 2 \Rightarrow Does\ not\ Work$$

## C. Multiplication

### Example 1.79

Multiplication Algorithms

### Example 1.80

Consider the product

$$(2x^4 - 3x^3 + 7x^2 + 3x + 6)(-5x^4 + 2x^3 + 5x^2 + 8x - 3)$$

Find the coefficient of the:

A. constant term
B. $x$ term
C. $x^2$ term
D. $x^5$ term
E. $x^7$ term

Constant Term

$$6(-3) = 18$$

$x$ term

$$3(-3) + 8(6) = -9 + 48 = 39$$

$x^2$ term

$$7(-3) + 3(8) + 6(5) = -21 + 24 + 30 = 33$$

$x^5$ term

$$2(8) - 3(5) + 7(2) + 3(-5) = 16 - 15 + 14 - 15 = 0$$

$x^7$ term

$$2(2) - 3(-5) = 4 + 15 = 19$$

### Example 1.81

A. What is the coefficient of $x^3$ when $x^4 - 3x^3 + 5x^2 - 6x + 1$ is multiplied by $2x^3 - 3x^2 + 4x + 7$ and the like terms are combined? (MathCounts 2005 Warm-Up 10)
B. If $(x^2 - k)(x + k) = x^3 + k(x^2 - x - 5)$ and $k \neq 0$, what is the value of $k$? (MathCounts 2009 Warm-up 10)

**Part A**

$$(-3)(7) + (5)(4) + (-6)(-3) + (1)(2) = -21 + 20 + 18 + 2 = 19$$

**Part B**

The given equality is true for all values of x, and hence the coefficients must be equal on both sides. Since, we have only one variable, we only need to consider the constant terms:

$$\underbrace{-k^2}_{LHS} = \underbrace{-5k}_{RHS} \Rightarrow k = 5$$

### Example 1.82: Sum of Coefficients

To find the sum of the coefficients of an expression, substitute: $x = 1$

A. If $(x + 2)(3x^2 - x + 5) = Ax^3 + Bx^2 + Cx + D$, what is the value of $A + B + C + D$?(MathCounts 1993 Chapter Sprint)
B. Find the sum of the coefficients of $(x + 1)^{50}$
C. If the product $(3x^2 - 5x + 4)(7 - 2x)$ can be written in the form $ax^3 + bx^2 + cx + d$, where $a, b, c, d$ are real numbers, then find $8a + 4b + 2c + d$. (AOPS Alcumus, Algebra, Polynomial Multiplication)

**Part A**

Substitute $x = 1$:

$$(1 + 2)(3 - 1 + 5) = (3)(7) = 21$$

**Part B**

Substitute $x = 1$:

$$(1 + 1)^{50} = 2^{50}$$

**Part C**

Substitute $x = 2$:

$$18$$

### 1.83: Degree of Product and Power

$$d[P(x)Q(x)] = d[P(x)] + d[Q(x)]$$
$$[P(x)]^n \text{ has degree} = n \times d[P(x)]$$

The highest power is what matters when deciding the degree.

$$(a_n x^n + a_{n-1}x^{n-1} + \cdots)(a_m x^m + a_{m-1}x^{m-1} + \cdots) \Rightarrow a_n a_m x^{n+m} + \cdots$$

$$(a_n x^n + a_{n-1}x^{n-1} + \cdots)^2 = (a_n x^n)^2 + \cdots$$

### Example 1.84: Degree

A. Consider the polynomials $P(x) = 3x^4 + 2x^3 - 4x^2 + 7$ and $Q(x) = 7x^3 - 2x^2 + x + 4$. What is the degree of (a) $P(x)Q(x)$ (b)$[P(x)]^2$ (c) $[P(x)]^5$
B. Consider polynomial $P(x)$, which has degree 5, and polynomial $Q(x)$, which has degree 7. What is the degree of (a) $P(x)Q(x)$ (b) $[P(x)]^2[Q(x)]^3$

**Part A**

$$Degree\ of\ [P(x)]^2 = Degree\ of\ [P(x)P(x)] = 4 + 4 = 8$$
$$Degree\ of\ [P(x)]^5 = 4 \times 5 = 20$$

**Part B**

$$5 + 7 = 12$$
$$[P(x)]^2[Q(x)]^3 = 10 + 21 = 31$$

### Example 1.85: Counting with Degree

A. Suppose that $P(x)$ and $Q(x)$ are polynomials and $P(x)Q(x)$ has degree 10. If $a = d[P(x)]$ and $b = d[Q(x)]$, then how many values can the ordered pair $(a, b)$ take?
B. In the above, if the polynomials are of distinct degree, then what is the number of ordered pairs $(a, b)$?
C. In part A, what is the number of unordered pairs $(a, b)$?

**Part A**

$$a + b = 10$$
$$(0,10), (1,9), \dots (10,0) \Rightarrow 11\ Solutions$$

**Part B**

We are not ok with (5,5). Hence, the number of solutions is
$$11 = 1 - 10$$

**Part C**

$$(0,10), (1,9), \dots (5,5) \Rightarrow 6\ Solutions$$

# 1.3 Division

## A. Division

### Example 1.86

Divide using factorization:

A. $\frac{x^2+5x+6}{x+3}$
B. $\frac{x^2+5x+8}{x+3}$

$$\frac{x^2+5x+6}{x+3} = \frac{(x+2)(x+3)}{x+3} = x + 2$$
$$\frac{x^2+5x+8}{x+3} = \frac{(x+2)(x+3)}{x+3} + \frac{2}{x+3} = x + 2 + \frac{2}{x+3}$$

### 1.87: Division

$$\frac{Dividend}{Divisor} = Quotient + \frac{Remainder}{Divisor}$$

$$\frac{42}{5} = 8 + \frac{2}{5}$$

### 1.88: Polynomial Division

$$\frac{P(x)}{x-a} = Q(x) + \frac{r}{x-a} \text{ for Quotient } Q(x), \text{and Remainder } r$$

When we divide a polynomial

$$Polynomial \rightarrow P(x)$$

by a linear expression

$$linear\ expression \rightarrow x - a$$

we get another polynomial

$$Polynomial \rightarrow Q(x)$$

And a Remainder

$$Remainder - r$$

## B. Polynomial Long Division

In the previous example, you knew the factorization, and hence were able to cancel. Suppose you do not know the factorization. You can still carry out the division.

### Example 1.89

Simplify by using Polynomial Long Division

A. $\frac{x^2+5x+8}{x+3}$

$$\begin{array}{r|l} & x\ +2 \\ \hline x+3 & x^2+5x+8 \\ & \underline{-x^2-3x} \\ & \quad +2x+8 \\ & \quad \underline{-2x-6} \\ & \qquad +2 \end{array}$$

### 1.90: Degree

In polynomial division, consider

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}, \qquad P(x) \rightarrow Dividend, \qquad D(x) \rightarrow Divisor, \qquad R(x) \rightarrow Remainder$$

Given that

$$D(x)\ has\ degree\ n$$

Then:

$$Degree\ of\ R(x)\ can\ be\ maximum\ (n-1)$$
$$Degree\ of\ D(x) + Degree\ of\ Q(x) = Degree\ of\ P(x)$$

### Example 1.91

Carry out the division

$$\frac{x^3+2x^2+5x+3}{x+2}$$

$$\frac{x^3+2x^2+5x+3}{x+2} = x^2+5-\frac{7}{x+2}$$

$$\begin{array}{r|l}
 & x^2+0x\ \ +5 \\ \hline
x+2 & x^3+2x^2+5x+3 \\
 & -x^3-2x^2 \\ \hline
 & \quad 0 \quad +5x+3 \\
 & \qquad\quad -5x-10 \\ \hline
 & \qquad\qquad -7
\end{array}$$

**Example 1.92**

Carry out the division

$$\frac{2x^3+3x+5}{x-1}$$

$$\begin{array}{r|l}
 & 2x^2+2x\ \ +5 \\ \hline
x-1 & 2x^3+0x^2+3x+5 \\
 & -2x^3+2x^2 \\ \hline
 & \quad +2x^2+3x \\
 & \quad -2x^2+2x \\ \hline
 & \qquad +5x+5 \\
 & \qquad -5x+5 \\ \hline
 & \qquad\quad +10
\end{array}$$

## C. Synthetic Division

**1.93: Synthetic Division**

Synthetic division has the same calculations as Polynomial Long Division, but with a few key differences:

- Only applicable for division by linear factors
- Writing required is less since repetitive writing of the variables is simply omitted.
- Instead of writing the variables every time, the coefficients are written, and the variables are understood from the column.

Note: Polynomial long division works for divisors of any degree.

**Example 1.94**

Divide using Synthetic Division

$$\frac{x^2+5x+8}{x+3}$$

### Clever Method

$$\frac{x^2+5x+6}{x+3}+\frac{2}{x+3}=\frac{(x+3)(x+2)}{x+3}+\frac{2}{x+3}=x+2+\frac{2}{x+3}$$

### Method II: Longer Method

$$x + 3 = 0 \Rightarrow x = -3$$

$$\begin{array}{r|l}
 & x \ \ +2 \\
\hline
x+3 & x^2+5x+8 \\
 & \underline{-x^2-3x} \\
 & \quad +2x+8 \\
 & \quad \underline{-2x-6} \\
 & \qquad +2
\end{array}$$

| Original Polynomial | $x^2$ | $x$ | $Constant$ |
|---|---|---|---|
| | 1 | 5 | 8 |
| $-3$ | | $-3$ | $-6$ |
| | | | |
| | 1 | 2 | 2 |
| Answer | $x$ | $Constant$ | Remainder |
| | Quotient | | |

## 1.95: "Missing" Terms

If the polynomial has a "missing" term, then that term is a zero coefficient, and is essential for the synthetic division.

$$2x^3 + 3x + 5 = 2x^3 + 0x^2 + 3x + 5$$

### Example 1.96

Divide using Synthetic Division

$$\frac{2x^3 + 3x + 5}{x - 1}$$

$$\begin{array}{r|l}
 & 2x^2+2x \ +5 \\
\hline
x-1 & 2x^3+0x^2+3x+5 \\
 & \underline{-2x^3+2x^2} \\
 & \quad +2x^2+3x \\
 & \quad \underline{-2x^2+2x} \\
 & \qquad +5x+5 \\
 & \qquad \underline{-5x+5} \\
 & \qquad\quad +10
\end{array}$$

$$x - 1 = 0 \Rightarrow x = 1$$
$$Dividend = 2x^3 + 0x^2 + 3x + 5$$

This time we have done the synthetic division without writing the column names:

| | | | | |
|---|---|---|---|---|
| | 2 | 0 | 3 | 5 |
| 1 | | 2 | 2 | 5 |
| | 2 | 2 | 5 | 10 |

## D. Variables

## 1.97: Determining an unknown coefficient

If a polynomial is divisible by a linear factor, we can determine the value of one unknown coefficient.

- If $x^2 + ax + 6$ is divisible by $x + 2$, then we can find the value of $a$.

### Example 1.98

If $x^2 + ax + 6$ is divisible by $x + 2$, then find the value of $a$.

### Method of Undetermined Coefficients

Because this is a quadratic, we can multiply easily to find the value of a.

$$x^2 + ax + 6 = (x+2)(x+c) = x^2 + (2+c)x + 2c$$
$$x^2 + ax + 6 = x^2 + (2+c)x + 2c$$

Match the constant terms:

$$2c = 6 \Rightarrow c = 3 \Rightarrow a = 2 + c = 2 + 3 = 5$$

### Synthetic Division

Equate the factor to zero, and find the root:

$$x + 2 = 0 \Rightarrow x = -2$$

| | $x^2$ | $x$ | *Constant* Term |
|---|---|---|---|
| $-2$ | 1 | $a$ | 6 |
| | | $-2$ | $-2a+4$ |
| | 1 | $a-2$ | $10-2a$ |

Since $x+2$ is a factor of $x^2 + ax + 6$, the remainder must be zero. Hence,

$$10 - 2a = 0$$
$$2a = 10$$
$$a = 5$$

## Example 1.99

If $x^2 + ax + 6$ is divisible by $x - 2$, then find the value of $a$ using synthetic division.

Equate the factor to zero, and find the root:

$$x - 2 = 0 \Rightarrow x = 2$$

| | $x^2$ | $x$ | *Constant* Term |
|---|---|---|---|
| 2 | 1 | $a$ | 6 |
| | | 2 | $4+2a$ |
| | 1 | $a+2$ | $10+2a$ |

However, since $x-2$ is a factor of $x^2 + ax + 6$, the remainder must be zero.
Hence,

$$10 + 2a = 0$$
$$2a = -10$$
$$a = -5$$

## Example 1.100

If $x^3 + ax^2 - 5x - 6$ is divisible by $x + 3$, then find the value of $a$ using synthetic division.

$$x + 3 = 0 \Rightarrow x = -3$$

| | | | | |
|---|---|---|---|---|
| $-3$ | 1 | $a$ | $-5$ | $-6$ |
| | | $-3$ | $9-3a$ | $9a-12$ |
| | 1 | $a-3$ | $4-3a$ | $9a-18$ |

$$9a - 18 = 0$$
$$9a = 18$$
$$a = 2$$

## 1.101: Determining an unknown coefficient

If a polynomial is divisible by two distinct linear factors (or by a linear factor repeated twice), we can determine the value of two unknown coefficients using a system of equations.

- If $x^4 + ax^3 + bx^2 - 5x + 14$ is divisible by $x^2 + 8x + 7$, then we can find the value of $a$ and $b$.

### (Calculator) Example 1.102

If $x^4 + ax^3 + bx^2 - 5x + 14$ is divisible by $x^2 + 8x + 7$, then find the value of $a$ and $b$.

Factor the given quadratic divisor into two linear factors, each of which divides the given fourth degree polynomial:

$$x^2 + 8x + 7 = (x + 7)(x + 1)$$

Using synthetic division with $x + 7 = 0 \Rightarrow x = -7$

| | | | | | |
|---|---|---|---|---|---|
| | | | | | |
| $-7$ | 1 | $a$ | $b$ | $-5$ | 14 |
| | | $-7$ | $49 - 7a$ | $-343 + 49a - 7b$ | $2436 - 343a + 49b$ |
| | 1 | $-7 + a$ | $49 - 7a + b$ | $-348 + 49a - 7b$ | $2450 - 343a + 49b$ |
| | | | | | |

Using synthetic division with $x + 1 = 0 \Rightarrow x = -1$, we get:

| | | | | | |
|---|---|---|---|---|---|
| | | | | | |
| $-1$ | 1 | a | $b$ | -5 | 14 |
| | | | | | |
| | | | | | |
| | | | | | |

We now have two equations in $a$ and $b$.

$$2450 - 343a + 49b = 0$$

Solve the system of equations:

## E. A Division Pattern

### Example 1.103

Divide

$$\frac{x^6 - 1}{x - 1}$$

We can do this using long division. It is a little tedious (and repetitive), but doable with some patience.

$$
\begin{array}{r|l}
 & x^5+x^4\ +x^3\ +x^2\ +x\ \ +1 \\
\hline
x-1 & x^6+0x^5+0x^4+0x^3+0x^2+0x-1 \\
 & \underline{-x^6+x^5} \\
 & \quad +x^5\ +0x^4 \\
 & \quad \underline{-x^5\ +x^4} \\
 & \qquad +x^4\ +0x^3 \\
 & \qquad \underline{-x^4\ +x^3} \\
 & \qquad\quad +x^3\ +0x^2 \\
 & \qquad\quad \underline{-x^3\ +x^2} \\
 & \qquad\qquad +x^2\ +0x \\
 & \qquad\qquad \underline{-x^2\ +x} \\
 & \qquad\qquad\quad +x\ -1 \\
 & \qquad\qquad\quad \underline{-x\ +1} \\
 & \qquad\qquad\qquad\ 0
\end{array}
$$

You can also do the same using synthetic division:

$$
\begin{array}{c|ccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
 & & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}
$$

$$\frac{x^6-1}{x-1}=x^5+x^4+x^3+x^2+x+1$$

## F. Formula-I

### 1.104: Division Formula-I

$$\frac{x^n-1}{x-1}=x^{n-1}+x^{n-2}+x^{n-3}+\cdots+1$$

$$x-1\equiv 0\ (mod\ x-1)$$

Add 1 to both sides:

$$x\equiv 1\ (mod\ x-1)$$

Multiply LHS by $x$, and RHS by 1 $(n-1)$ times:

$$x^n\equiv 1\ (mod\ x-1)$$

Subtract 1 from both sides:

$$x^n-1\equiv 0\ (mod\ x-1)$$

We can also prove the above property using synthetic division

$$x^n-1=x^n+0x^{n-1}+0x^{n-2}+\cdots+0x-1$$

$$x-1=0\Rightarrow x=1$$

| | $x^n$ | $x^{n-1}$ | . | . | . | $x^0$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | $-1$ |
| | | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 0 |
| Answer | $x^{n-1}$ | $x^{n-2}$ | . | . | . | Remainder |

### Example 1.105

$$\frac{x^3-1}{x-1}$$

$$\frac{x^3-1}{x-1}=x^2+x+1$$

### Example 1.106

What is the number of terms in the quotient below:

$$\frac{x^6-1}{x-1}$$

$$\frac{x^6-1}{x-1}=x^5+x^4+x^3+x^2+x+1$$
$$No.of\ Terms=6$$

### Example 1.107

What is the number of terms in the quotient when we carry out the division in the expression below:

$$\frac{x^{2022}-1}{x-1}$$

$$x^{2021}+x^{2020}+\cdots+1\Rightarrow 2022\ Terms$$

### 1.108: Number of Terms

$\frac{x^n-1}{x-1}=x^{n-1}+x^{n-2}+x^{n-3}+\cdots+1$ has

$$n\ terms$$

### Example 1.109

Divide

A. $\frac{x^7+2}{x-1}$
B. $\frac{x^7-5}{x-1}$

Some manipulation is here required here before applying the formula. Split the given fraction:

$$\frac{x^7-1}{x-1}+\frac{3}{x-1}=(x^6+x^5+\cdots+1)+\frac{3}{x-1}$$
$$\frac{x^7-1}{x-1}+\frac{-4}{x-1}=(x^6+x^5+\cdots+1)+\frac{-4}{x-1}$$

## Example 1.110

A. When $x^{13}+1$ is divided by $x-1$, the remainder is: (AHSME 1950/20)
B. What is the sum of the coefficients of the quotient?
C. What is the sum of the exponents of the terms in the quotient that have non-zero coefficient?

### Part A

$$\frac{x^{13}+1}{x-1}=\frac{x^{13}-1}{x-1}+\frac{2}{x-1}=(x^{12}+x^{11}+\cdots+1)+\frac{2}{x-1}$$

$$Quotient = x^{12}+x^{11}+\cdots+1$$
$$Remainder = 2$$

### Part B

The quotient is this:

$$x^{12}+x^{11}+\cdots+1$$

Substitute $x=1$ in the quotient to find the sum of the coefficients:

$$\underbrace{1+1+\cdots 1}_{13\ Times}=1\times 13=13$$

### Part C

All terms in the quotient have non-zero coefficient.

$$12+11+\cdots+1+0=\frac{12(13)}{2}=6\times 13=78$$[1]

## 1.111: Multiplication Formula-I

$$x^n-1=(x-1)(x^{n-1}+x^{n-2}+x^{n-3}+\cdots+1)$$

We can rearrange the division formula above to get a formula that will let us multiply.

## Example 1.112

$$(x-1)(x^{2021}+x^{2020}+\cdots+1)$$

$$\frac{x^{2022}-1}{x-1}=x^{2021}+x^{2020}+\cdots+1$$

Multiply both sides by $x-1$:

$$x^{2022}-1=(x-1)(x^{2021}+x^{2020}+\cdots+1)$$

# G. Odd Values of $n$

## 1.113: Division Formula-II

$$\frac{x^n+1}{x+1}=x^{n-1}-x^{n-2}+x^{n-3}-\cdots+1, \qquad n\ is\ odd\ natural\ number$$

Note the signs alternate.

$$x+1=0\Rightarrow x=-1$$

[1] Using the formula $n+(n-1)+(n-2)+\cdots+1=\frac{n(n+1)}{2}$

| | Odd | Even | Odd | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $x^n$ | $x^{n-1}$ | $x^{n-2}$ | . | . | . | $x^2$ | $x^1$ | $x^0$ |
| $-1$ | 1 | 0 | 0 | | | | 0 | 0 | 1 |
| | | $-1$ | 1 | | | | $-1$ | 1 | $-1$ |
| | 1 | $-1$ | 1 | | | | $-1$ | 1 | 0 |
| | | | | | | | | | |
| Answer | $x^{n-1}$ | | | | | | | | |
| | Even | Odd | Even | | | | | | |

If $n$ is even, then this does not work

| | Even | Odd | Even | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $x^n$ | $x^{n-1}$ | $x^{n-2}$ | . | . | . | $x^2$ | $x^1$ | $x^0$ |
| $-1$ | 1 | 0 | 0 | | | | 0 | 0 | 1 |
| | | $-1$ | 1 | | | | 1 | $-1$ | 1 |
| | 1 | $-1$ | 1 | | | | 1 | $-1$ | 2 |
| Answer | $x^{n-1}$ | | | | | | | | |
| | Odd | Even | Odd | | | | | | |

### Example 1.114

$$\frac{x^2+1}{x+1}$$

$$\frac{x^2-1}{x+1}+\frac{2}{x+1}=x-1+\frac{2}{x+1}$$

### Example 1.115

Divide

$$\frac{x^7+2}{x+1}$$

$$\frac{x^7+1}{x+1}+\frac{1}{x+1}=(x^6-x^5+x^4-x^3+\cdots+1)+\frac{1}{x+1}$$

| | $x^7$ | $x^6$ | $x^5$ | $x^4$ | $x^3$ | $x^2$ | $x^1$ | $x^0$ |
|---|---|---|---|---|---|---|---|---|
| $-1$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| | | $-1$ | 1 | $-1$ | 1 | $-1$ | 1 | $-1$ |
| | 1 | $-1$ | 1 | $-1$ | 1 | $-1$ | 1 | 0 |
| Quotient | $x^6$ | $x^5$ | $x^4$ | $x^3$ | $x^2$ | $x^1$ | $x^0$ | $Remainder$ |

### Example 1.116

Divide

$$\frac{x^7-1}{x+1}$$

$$\frac{x^7+1}{x+1}+\frac{-2}{x+1}=(x^6-x^5+x^4-x^3+\cdots+1)+\frac{-2}{x+1}$$

What is the sum of the coefficients of the quotient?

$$x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$$

Substitute $x = 1$:

$$\underbrace{1-1}_{Pair} + \underbrace{1-1}_{Pair} + \underbrace{1-1}_{Pair} + 1 = 0 + 0 + 0 + 1 = 1$$

## H. Even Values of $n$

### 1.117: Division Formula-III

$$\frac{x^n - 1}{x+1} = x^{n-1} - x^{n-2} + x^{n-3} - \cdots - 1, \qquad n \text{ is an even natural number}$$

$$x^n + 1 = x^n + 0x^{n-1} + 0x^{n-2} + \cdots + 0x + 1$$

If $n$ is even:

- All the even powers have coefficient 1
- All the odd powers have coefficient $-1$

| | Even | Odd | Even | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Dividend | $x^n$ | $x^{n-1}$ | $x^{n-2}$ | . | . | . | $x^2$ | $x^1$ | $x^0$ |
| $-1$ | 1 | 0 | 0 | | | | 0 | 0 | $-1$ |
| | | $-1$ | 1 | | | | 1 | $-1$ | 0 |
| *Quotient* | 1 | $-1$ | 1 | | | | 1 | $-1$ | 0 |
| Quotient | Odd | Even | Odd | | | | | | Remainder |

### Example 1.118

Divide

$$\frac{x^6 + 1}{x+1}$$

$$\frac{x^6 + 1}{x+1} = \frac{x^6 - 1}{x+1} + \frac{2}{x+1} = (x^5 - x^4 + x^3 - x^2 + x - 1) + \frac{2}{x+1}$$

## I. Sum of Coefficients

### Example 1.119

Find the sum of the coefficients of the quotient when we carry out the division below:

$$\frac{x^{2022} + 2022}{x+1}$$

$$\frac{x^{2022} + 2022}{x+1} = \frac{x^{2022} - 1}{x+1} + \frac{2023}{x+1} = (x^{2021} - x^{2020} + x^{2019} - x^{2018} + \cdots + x - 1) + \frac{2023}{x+1}$$

We only need to focus on the quotient here:

$$x^{2021} - x^{2020} + x^{2019} - x^{2018} + \cdots + x - 1$$

To find sum of coefficients, substitute $x = 1$, and form 1010 pairs, each of which adds up to zero:

$$\underbrace{1-1}_{Pair} + \underbrace{1-1}_{Pair} + \cdots + \underbrace{1-1}_{Pair} = 0 + 0 + \cdots + 0 = 0$$

**Example 1.120**

$$For\ some\ positive\ Integer\ n: \frac{x^n+n}{x+1} = Q_1(x) + \frac{R_1(x)}{x+1}$$

Where

$$Q_1(x) is\ the\ quotient\ of\ the\ first\ division$$

Answer each part independently:

A. If the sum of the coefficients of $Q_1(x)$ is 0. Determine whether $n$ is even or odd?
B. If the sum of the coefficients of $Q_1(x)$ is 1. Determine whether $n$ is even or odd?

$$Part\ A: Even$$
$$Part\ B: Odd$$

## J. Multiplication Formula

**1.121:**

$$x^n + 1 = (x+1)(x^{n-1} - x^{n-2} + x^{n-3} - \cdots + 1), \qquad n\ is\ odd\ natural\ number$$
$$x^n - 1 = (x+1)(x^{n-1} - x^{n-2} + x^{n-3} - \cdots - 1), \qquad n\ is\ an\ even\ natural\ number$$

**Example 1.122**

$$(x^2+1)(x^{20} - x^{18} + x^{16} + \cdots + 1)$$

Use a change of variable. Let $y = x^2$:

$$y^{11} + 1 = (y+1)(y^{10} - y^9 + x^8 + \cdots + 1)$$

Change the variable back:

$$x^{22} + 1$$

**1.123:**

$$\frac{x^{3n}-1}{x^3-1} = x^{3(n-1)} + x^{3(n-2)} + x^{3(n-3)} + \cdots + 1$$

## K. Applications

**Example 1.124**

What is the greatest integer value of $x$ such that $\frac{x^2+2x+5}{x-3}$ is an integer? **(MathCounts 2002 National Sprint)**

$$\frac{x^2+2x+5}{x-3} = \underbrace{x+5}_{Integer} + \frac{20}{x-3}$$

Hence, we need $\frac{20}{x-3}$ to always be an integer, which will happen only when $x-3$ is a factor of 20.
The greatest factor of 20, is 20 itself.

$$x - 3 = 20 \Rightarrow x = 23$$

Hence, the greatest of $x$ is 23.

What are the number of values that $x$ can take in the above question?

The number of positive factors of 20:

$$20 = 4 \times 5 = 2^2 \times 5^1 \Rightarrow No.of\ Factors = (2+1)(1+1) = 3 \times 2 = 6$$

Hence, the number of integer factors is double of the positive factor:

$$Total\ Factors = 2 \times 6 = 12 \Rightarrow 12\ values\ of\ x$$

What are the possible integer values that $x$ can take in the above question?

$x$ can take values such that $x - 3$ is a factor of 20.

| Factor | $x-3$ | 20 | 10 | 5 | 4 | 2 | 1 | -1 | -2 | -4 | -5 | -10 | -20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Value | $x$ | 23 | 13 | 8 | 7 | 5 | 4 | 2 | 1 | -1 | -2 | -7 | -17 |

### Example 1.125

What is the value of $c$ such that $\frac{x^2+2x+c}{x-3}$ is an integer for integer values of $x$?

$$\frac{x^2+2x+c}{x-3} = \underbrace{x+5}_{Integer} + \frac{15+c}{x-3}$$

The first term is an integer.
Hence, the second term must be an integer for all integer values of $x$.
The only way this will happen is if the numerator of the expression $\frac{15+c}{x-3}$ is zero.

$$15 + c = 0 \Rightarrow c = -15$$

# 1.4 Remainder-Factor Theorem

## A. Remainder Theorem

### 1.126: Division Property

$$\frac{Dividend}{Divisor} = Quotient + \frac{Remainder}{Divisor}$$

$$Alternate\ Form: Dividend = Divisor \times Quotient + Remainder$$

**Example with Numbers:**

$$\underbrace{43}_{Dividend} \div \underbrace{7}_{Divisor}$$

$$\frac{43}{7} = 6 + \frac{1}{7}$$

Multiply both sides by 7:

$$\underbrace{43}_{Dividend} = \underbrace{7}_{Divisor} \times \underbrace{6}_{Quotient} + \underbrace{1}_{Remainder}$$

**Example with Variables:**

$$\frac{x^2+7x+12}{x+4} = \frac{(x+3)(x+4)}{x+4} = x + 3$$

$$\frac{x^2+7x+14}{x+4}=\frac{(x+3)(x+4)+2}{x+4}=x+3+\frac{2}{x+4}$$

Multiply both sides by $x+4$:

$$\underbrace{x^2+7x+14}_{Dividend}=\underbrace{(x+3)}_{Divisor}\underbrace{(x+4)}_{Quotient}+\underbrace{2}_{Remainder}$$

## 1.127: Remainder Theorem

The remainder when a polynomial function $f(x)$ is divided by a linear expression $x-a$ is the same as the value of the function at the root of the expression.

$$\frac{f(x)}{x-a}=\underbrace{P(x)}_{Quotient}+\frac{r}{x-a}\Rightarrow r=Remainder=f(a)$$

Multiply both sides of the equality by $x-a$ to get:

$$f(x)=\underbrace{(x-a)}_{Divisor}\underbrace{P(x)}_{Quotient}+\underbrace{r}_{Remainder}$$

Substitute $x=a$:

$$f(a)=(a-a)P(x)+r=(0)P(x)+r=0+r=r$$

## Example 1.128

Find the remainder when:

A. $P(x)=3x^3+2x^2+4x-2$ is divided by $x-1$.
B. $P(x)=2x^3+6x^2-11x+4$ is divided by $x+1$.
C. $x^{13}+1$ is divided by $x-1$ **(AHSME 1950/20)**
D. $x^{2020}$ is divided by $x+1$
E. $x^{35}$ is divided by $x-1$
F. $x^{1949}$ is divided by $x+c$
G. $P_1(x)=x^{2020}+x^{2019}+\cdots+x+1$ is divided by $x-1$
H. $P_1(x)=x^{2020}+x^{2019}+\cdots+x+1$ is divided by $x+1$

### Part A

$$x-1=0\Rightarrow x=1\Rightarrow P(1)=3+2+4-2=7$$

### Part B

$$x+1=0\Rightarrow x=-1\Rightarrow P(-1)=-2+6+11+4=19$$

### Part C

Find the root of $x-1$:

$$x-1=0\Rightarrow x=1$$

Evaluate the polynomial at the root:

$$x=1\Rightarrow x^{13}+1=1^{13}+1=1+1=2$$

### Part D

$$x+1=0\Rightarrow x=-1\Rightarrow x^{2020}=(-1)^{2020}=1$$

### Part E

$$x-1=0\Rightarrow x=1\Rightarrow x^{35}=1^{35}=1$$

### Part F

$$x+c=0\Rightarrow x=-c\Rightarrow x^{1949}=(-c)^{1949}=-c^{1949}$$

### Part G

$$x-1=0\Rightarrow x=1\Rightarrow P_1(1)=\underbrace{1+1+\cdots+1}_{2021\ times}=2021$$

**Part H**

$$x-1=0 \Rightarrow x=-1 \Rightarrow P_1(1)=\underbrace{-1+1-1+1}_{\frac{2020}{2}\,Pairs}+1=1$$

## B. Factor Theorem

The factor theorem is a special case of the remainder theorem. Factor theorem is a very important application of the remainder theorem.

### Example 1.129

A. Find the remainder when $y=f(x)=4x^2-8x+3$ is divided by $2x-1$.
B. Hence, explain why $2x-1$ is a factor of $4x^2-8x+3$.

Find the root of the divisor:

$$2x-1=0 \Rightarrow x=\frac{1}{2}$$

Evaluate the function at the root:

$$f\left(\frac{1}{2}\right)=4\left(\frac{1}{2}\right)^2-8\left(\frac{1}{2}\right)+3=1-4+3=0$$

The Remainder is zero.

Since the remainder is zero, $2x-1$ is a factor of $f(x)$.

### 1.130: Factor Theorem

If $P(\boldsymbol{a})=\mathbf{0}$ for a polynomial $\boldsymbol{P(x)}$, then $\boldsymbol{x-a}$ is a factor of $\boldsymbol{P(x)}$.

$$f(x)=Q(x)(x-a)+R$$

Let $f(a)=0$:

$$f(a)=0=Q(0)(a-a)+R$$
$$0=R$$

Which tells us the Remainder when dividing $f(x)$ by $(x-a)$ is zero.

### Example 1.131

Check if $x+2$ is a factor of $f(x)=x^3+2x^2-7x+4$

$$x+2=0 \Rightarrow x=-2$$
$$f(-2)=(-2)^3+2(-2)^2-7(-2)+4=\underbrace{-8+8}_{Zero}\underbrace{+14+4}_{+ve}=18$$
$$\Rightarrow x+2\ is\ not\ a\ factor$$

| | $x^3$ | $x^2$ | $x$ | $Constant$ |
|---|---|---|---|---|
| $-2$ | 1 | 2 | -7 | 4 |
| | | $-2$ | 0 | 14 |
| | 1 | 0 | $-7$ | 18 |

### 1.132: Cyclicity of Units Digit

| | | | | Last Digit of No. | Last Digit of Square | Cube | 4th Power | Pattern | Cyclicity |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 4 | 8 | 16 | 2 | 4 | 8 | 6 | 2, 4, 8, 6 | 4 |
| 3 | 9 | 27 | 81 | 3 | 9 | 7 | 1 | 3, 9, 7, 1 | 4 |
| 4 | 16 | 64 | 256 | 4 | 6 | 4 | 6 | 4, 6 | 2 |
| 5 | 25 | 125 | 625 | 5 | 5 | 5 | 5 | 5 | 1 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 6 | 36 | 216 | 1296 | 6 | 6 | 6 | 6 | 6 | 1 |
| 7 | 49 | 343 | 2401 | 7 | 9 | 3 | 1 | 7, 9, 3, 1 | 4 |
| 8 | 64 | 512 | 4096 | 8 | 4 | 2 | 6 | 8, 4, 2, 6 | 4 |
| 9 | 81 | 729 | 6561 | 9 | 1 | 9 | 1 | 9, 1 | 2 |

### Example 1.133: Units Digit

A. Find the units digits of the remainder when $x^{1949}$ is divided by $x+2$.
B. When $x^n, n \in \mathbb{N}$ is divided by $x-6$, what is the units digit of the remainder?
C. When $x^n, n \in \mathbb{N}$ is divided by $x-a$, where $a$ is a single digit integer, the units digits of the remainder is 6. What can be the possible values of $a$?

**Part A**

$$x+2=0 \Rightarrow x=-2 \Rightarrow x^{1949}=(-2)^{1949}$$
$$1949 \equiv 49 \equiv 1 \ (mod\ 4)$$

$$Last\ Digit = 2$$

| $n(mod\ 4)$ | $2^n$ | $x$ | $2^n$ | $x$ | Last Digit |
|---|---|---|---|---|---|
| 1 | $2^1$ | 2 | $2^5$ | 32 | 2 |
| 2 | $2^2$ | 4 | $2^6$ | 64 | 4 |
| 3 | $2^3$ | 8 | $2^7$ | 128 | 8 |
| 0 | $2^4$ | 16 | $2^8$ | 256 | 6 |

**Part B**

$$x-6=0 \Rightarrow x^n=6^n$$
$$6, 36, 216, 1296 \Rightarrow Last\ Digit\ is\ always\ 6$$
$$Last\ Digit\ of\ 6^n = 6$$

**Part C**

$$x-a=0 \Rightarrow x=a \Rightarrow x^n=a^n$$
$$Last\ Digit\ of\ a^n = 6 \Rightarrow a \in \{2,4,6,8\}$$

## C. Division Properties

### 1.134: Property-I

$x^n-1, n \in \mathbb{N}$ is divisible by $x-1$

$$\frac{x^n-1}{x-1} = 1+x+x^2+\cdots+x^{n-1}$$

Consider the root:

$$x-1=0 \Rightarrow x=1$$

Use the remainder theorem to evaluate $x^n-1$ at $x=1$:

$$x=1 \Rightarrow x^n-1=1^n-1=0$$

Since the value is 0, $x-1$ is a factor of $x^n-1$. To find the quotient, use synthetic division:

| | $x^n$ | $x^{n-1}$ | $x^{n-2}$ | . | . | . | Constant |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | $-1$ |
| | | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 0 |

### 1.135: Property-II

$x^n+1$ is divisible by $x+1$ for odd values of $n$.
$x^n+1$ has remainder two when divided by $x+1$ for even values of $n$.

$$\frac{x^n+1}{x+1} = x^{n-1}-x^{n-2}+\cdots-x+1$$

$$x+1=0 \Rightarrow x=-1$$
$$x^n+1=(-1)^n+1=-1+1$$

$$n \text{ is even} \Rightarrow (-1)^n+1=1+1=2$$
$$n \text{ is odd} \Rightarrow (-1)^n+1=-1+1=0$$

### Example 1.136

$$\frac{x^3+1}{x+1}=x^2-x+1$$
$$\frac{x^5+1}{x+1}=x^4-x^3+x^2-x+1$$

### 1.137: Property-III

$x^n-1$ is divisible by $x+1$ for even values of $n$.
$x^n-1$ has remainder $-2$ when divided by $x+1$ for odd values of $n$.

$$\frac{x^n-1}{x+1}=x^{n-1}-x^{n-2}+\cdots+x-1$$

$$x+1=0 \Rightarrow x=-1$$
$$x^n-1=(-1)^n-1$$

$$n \text{ is even} \Rightarrow (-1)^n+1=1-1=0$$
$$n \text{ is odd} \Rightarrow (-1)^n+1=-1-1=-2$$

### Example 1.138

$$\frac{x^2-1}{x+1}=x-1$$
$$\frac{x^4-1}{x+1}=x^3-x^2+x-1$$

## D. Evaluating a Function (Synthetic Substitution)

We look at an interesting application of the remainder theorem that let us use synthetic division in a different way.

By the remainder theorem, we know that

- if $P(a)=R$, then $R$ is the remainder when $P(x)$ is divided by $x-a$.

Also, we know that

- synthetic division gives us the remainder when $P(x)$ is divided by $x-a$.

We can combine the above two to evaluate a function at a point.

### Example 1.139

A. Given that $f(x)=x^2+5x+8$, evaluate $f(-3)$ by using synthetic division.
B. Evaluate $f(x)=-3x^2+2x-4$ at $x=-3$.
C. Evaluate $f(x)=4x^3-5x^2+7$ at $x=2$.

**Part A**

$$f(-3) = (-3)^2 + 5(-3) + 8 = 9 - 15 + 8 = 2$$

$$\frac{x^2 + 5x + 8}{x + 3} = x + 2 + \frac{2}{x + 3}$$

| | | | |
|---|---|---|---|
| | 1 | 5 | 8 |
| $-3$ | | $-3$ | $-6$ |
| | 1 | 2 | 2 |

The remainder when $f(x)$ is divided by $x + 3$ is the same as $f(-3)$.
We can also get the remainder using polynomial long division.
And synthetic division is just a shortcut to perform polynomial long division.

**Part B**

| | $x^2$ | $x$ | $Constant$ |
|---|---|---|---|
| $-3$ | $-3$ | 2 | $-4$ |
| | 0 | 9 | $-33$ |
| | $-3$ | 11 | $-37$ |

$$f(-3) = -3(9) + 2(-3) - 4 = -27 - 6 - 4 = -37$$

**Part C**

| | $x^3$ | $x^2$ | $x$ | $Constant$ |
|---|---|---|---|---|
| 2 | 4 | $-5$ | 0 | 7 |
| | | 8 | 6 | 12 |
| | 4 | 3 | 6 | 19 |

$$f(2) = 4(8) - 5(4) + 7 = 32 - 20 + 7 = 19$$

## E. Back Calculations

### Example 1.140: Basics

Find the value of $a$ if:

A. $x^2 + ax + 6$ is divisible by $x + 2$
B. $x^2 + ax + 6$ is divisible by $x - 2$
C. $x^3 + ax^2 - 5x - 6$ is divisible by $x + 3$

**Part A**

$$x + 2 = 0 \Rightarrow x = -2$$
$$(-2)^2 + a(-2) + 6 = 0 \Rightarrow -2a + 10 = 0 \Rightarrow a = 5$$

| | $x^2$ | $x$ | $Constant$ |
|---|---|---|---|
| $-2$ | 1 | $a$ | 6 |
| | | $-2$ | $-2a + 4$ |
| | 1 | $a - 2$ | $-2a + 10$ |

$Alternate\ Method: Undetermined\ Coefficients$

$$x^2 + ax + 6 = (x + 2)(x + c) = x^2 + 2x + cx + 2c$$
$$2c = 6 \Rightarrow c = 3$$

| | $x^2$ | $x$ | $Constant$ |
|---|---|---|---|
| 2 | 1 | $a$ | 6 |
| | | 2 | $2a + 4$ |
| | 1 | $a + 2$ | $2a + 10$ |

**Part B**

$$x - 2 = 0 \Rightarrow x = 2$$
$$(2)^2 + a(2) + 6 = 0 \Rightarrow 2a + 10 = 0 \Rightarrow a = -5$$

**Part C**

$$x + 3 = 0 \Rightarrow x = -3$$

$$(-3)^3 + a(-3)^2 - 5(-3) - 6 = 0$$
$$-27 + 9a + 15 - 6 = 0 \Rightarrow 9a - 18 = 0 \Rightarrow 9a = 18 \Rightarrow a = 2$$

### Example 1.141

A. When $f(x) = x^3 + kx^2 - 7x + 3$ is divided by $x + 1$, the remainder is seven times the remainder that is found when the same expression is divided by $x+2$. Find the value of $k$.

B. $f(x) = 2x^3 - 7x^2 + 7ax + 16$ is divisible by $x - a$. Find the remainder when $f(x)$ is divided by $2x + 1$.

C. $x - 1$ is a factor of $f(x) = x^3 - 6x^2 + ax + b$. Show that the remainder when $f(x)$ is divided by $x - 3$ is twice the remainder when $f(x)$ is divided by $x - 2$.

D. When $f(x) = 3x^5 - ax + b$ is divided by $x - 1$ and $x + 1$, the remainders are equal. Find the values that $a$ and $b$ can take.

### Part A

$$f(-1) = (-1)^3 + k(-1)^2 - 7(-1) + 3 = k + 9$$
$$f(-2) = (-2)^3 + k(-2)^2 - 7(-2) + 3 = -8 + 4k + 14 + 3 = 4k + 9$$

$$f(-1) = 7f(-2) \Rightarrow k + 9 = 7(4k + 9) \Rightarrow k + 9 = 28k + 63 \Rightarrow -54 = 27k \Rightarrow k = -2$$

### Part B

$$2a^3 - 7a^2 + 7a^2 + 16 = 0$$
$$2a^3 + 16 = 0$$
$$a^3 = -8$$
$$a = -2$$

| | $x^3$ | $x^2$ | $x$ | $Constant$ |
|---|---|---|---|---|
| $a$ | 2 | $-7$ | $7a$ | 16 |
| | | $2a$ | $2a^2 - 7a$ | $2a^3$ |
| | 2 | $2a - 7$ | $2a^2$ | $2a^3 + 16$ |

$$f(x) = 2x^3 - 7x^2 - 14x + 16$$
$$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$$
$$f\left(-\frac{1}{2}\right) = 21$$

### Part C

$$f(1) = 1^3 - 6 \cdot 1^2 + a \cdot 1 + b = 0 \Rightarrow -5 + a + b = 0 \Rightarrow b = 5 - a$$
$$f(x) = x^3 - 6x^2 + ax + 5 - a$$
$$f(2) = 2^3 - 6 \cdot 2^2 + a \cdot 2 + 5 - a = 8 - 24 + 2a + 5 - a = a - 11$$
$$f(3) = 3^3 - 6 \cdot 3^2 + a \cdot 3 + 5 - a = 27 - 54 + 3a + b = 2a - 22 = 2(a - 11)$$

### Part D

$$f(1) = f(-1) \Rightarrow 3 - a + b = -3 + a + b \Rightarrow 2a = 6 \Rightarrow a = 3, b \in \mathbb{R}$$

## Example 1.142: Simultaneous Equations

A. When the polynomial $3x^3 + ax + b$ is divided by $x - 2$, the remainder is 2, and when divided by $x + 1$, it is 5. Find the value of $a$, and the value of $b$.

B. The cubic polynomial $3x^3 + px^2 + qx - 2$ has a factor $(x + 2)$, and leaves a remainder 4 when divided by $x + 1$. Find the value of $p$ and $q$.

C. $2x - 1$ is a factor of $f(x) = ax^3 + 4x^2 + bx - 2$. If $\frac{f(x)}{x-2}$ has remainder twice of the remainder of $\frac{f(x)}{x+1}$, find the value of $a$ and $b$.

### Part A

$$f(2) = 24 + 2a + b = 2 \Rightarrow \underbrace{2a + b = -22}_{Equation\ I}$$
$$f(-1) = -3 - a + b = 5 \Rightarrow \underbrace{-a + b = 8}_{Equation\ II}$$

Subtract Equation II from Equation I, and then Substitute $a = -10$ in Equation II:

$$3a = -30 \Rightarrow a = -10$$
$$10 + b = 8 \Rightarrow b = -2$$

**Part B**

$$f(-2) = 0 \Rightarrow -24 + 4p - 2q - 2 = 0 \Rightarrow \underbrace{2p - q = 13}_{\text{Equation I}}$$

$$f(-1) = 4 \Rightarrow -3 + p - q - 2 = 4 \Rightarrow \underbrace{p - q = 9}_{\text{Equation II}}$$

Subtract Equation II from Equation I:

$$p = 4 \Rightarrow q = -5$$

**Part C**

$$f\left(\frac{1}{2}\right) = \frac{a}{8} + 1 + \frac{b}{2} - 2 = 0 \Rightarrow \underbrace{a + 4b = 8}_{\text{Equation I}}$$

$$f(2) = 2f(-1) \Rightarrow 8a + 16 + 2b - 2 = 2(-a + 4 - b - 2) \Rightarrow \underbrace{10a + 4b = -10}_{\text{Equation II}}$$

$$a = -2, b = \frac{5}{2}$$

## Example 1.143: Quadratics

A. The same remainder is found when $f(x) = 2x^3 + kx^2 + 6x + 32$ and $g(x) = x^4 - 6x^2 - k^2x + 9$ are divided by $x + 1$. Find the possible values of $k$.

B. When $x^2 + 4x - b$ is divided by $x - a$, the remainder is $a$. Find the smallest possible value of $b$.

C. When $x^2 + 4x - b$ is divided by $x - a$, the remainder is 2. Find the smallest possible value of $b$.

D. The remainder when $f(x) = x^3 + px^2 + p^2x + 21$ is divided by $x + 3$ is positive. What can be the value of $p$?

**Part A**

$$f(-1) = g(-1) \Rightarrow -2 + k - 6 + 32 = 1 - 6 + k^2 + 9$$

Collate all terms on side to get a quadratic and solve:

$$k^2 - k - 20 = 0 \Rightarrow (k + 5)(k - 4) = 0 \Rightarrow k \in \{-4,5\}$$

**Part B**

$$f(a) = a^2 + 4a - b = a$$

$$b = a^2 + 3a = a^2 + 3a + \frac{9}{4} - \frac{9}{4} = \left(a + \frac{3}{2}\right)^2 - \frac{9}{4}$$

**Part C**

$$x - a = 0 \Rightarrow x = a$$

$$f(a) = a^2 + 4a - b = 2$$

$$b = a^2 + 4a - 2 = a^2 + 4a + 4 - 4 - 2 = (a + 2)^2 - 6$$

**Part D**

$$f(-3) = -27 + 9p - 3p^2 + 21 = 9p - 3p^2 - 6$$

$$9p - 3p^2 - 6 > 0 \Rightarrow p^2 - 3p + 2 < 0 \Rightarrow (p - 1)(p - 2) < 0 \Rightarrow p \in (1,2)$$

# F. Back Calculations: Three Variables

## Example 1.144

The polynomial $f(x) = x^4 + px^3 + qx^2 + rx + 6$ is exactly divisible by each of $(x - 1)$, $(x - 2)$ and $(x - 3)$. Find the values of $p$, $q$ and $r$.

$$x - 1 = 0 \Rightarrow x = 1 \Rightarrow f(1) = 1 + p + q + r + 6 = 0 \Rightarrow \underbrace{p + q + r = -7}_{\text{Equation I}}$$

$$x - 2 = 0 \Rightarrow x = 2 \Rightarrow f(2) = 16 + 8p + 4q + 2r + 6 = 0 \Rightarrow \underbrace{4p + 2q + r = -11}_{\text{Equation II}}$$

$$x-3=0 \Rightarrow x=3 \Rightarrow f(3)=81+27p+9q+3r+6=0 \Rightarrow \underbrace{9p+3q+r=-29}_{Equation\ III}$$

Subtract Equation I from Equation II:

$$\underbrace{3p+q=-4}_{Equation\ IV}$$

Subtract Equation I from Equation III:

$$8p+2q=-22 \Rightarrow \underbrace{4p+q=-11}_{Equation\ V}$$

Subtract Equation IV from Equation V:

$$p=-7$$

Substitute $p=-7$ in Equation IV:

$$3p+q=-4 \Rightarrow -21+q=-4 \Rightarrow q=17$$

Substitute $p=-7, q=17$ in Equation I:

$$p+q+r=-7 \Rightarrow -7+17+r=-7 \Rightarrow r=-17$$

$$(p,q,r)=(-7,17,-17)$$

Alternate Method

$$x^4+px^3+qx^2+rx+6=(x-a)(x-1)(x-2)(x-3)$$

$$6=(-a)(-1)(-2)(-3) \Rightarrow 6=6a \Rightarrow a=1$$

$$f(x)=(x-1)(x-1)(x-2)(x-3)=$$

## G. Back Calculations: Further Examples

### Example 1.145

Let P(x) be the unique polynomial of minimal degree with the following properties:
P(x) has a leading coefficient 1,
1 is a root of P(x)-1,
2 is a root of P(x-2),
3 is a root of $P(3x), and
4 is a root of 4P(x).
The roots of P(x) are integers, with one exception. The root that is not an integer can be written as m/n where m and n are relatively prime integers. What is m+n? (AMC 2023 10A/21)

3 is a root of P(3x)
Hence, substituting x=3 must give zero.
P(3x) = P(3\bullet3)=P(9)=0
Hence, 9 is a root of P(x)
Hence (x-9) is a factor of P(x)

2 is a root of P(x-2). Substituting x=2 must evaluate to zero:
P(x-2) = P(2-2)=P(0)=0
Hence, x is a factor of P(x)

4 is a root of 4P(x)

4P(4)=0
P(4)=0
(x-4) is a factor of P(x)

Combining the above three, and assuming the roots that is not an integer as a:
P(x)=x(x-4)(x-9)(x-a)

P(x)-1 has root 1:
P(1)-1=0\RightarrowP(1)=1

Hence,
1(1-4)(1-9)(1-a) =1
1(-3)(-8)(1-a) =1
24(1-a) =1
1-a =1/24
a = 1 - 1/24 = 23/24
m+n = 23+24=47

# 1.5 Higher Degree Divisors

## A. Exactly Divisible

### 1.146: Applying Factor Theorem to Quadratic Expressions

If $Q(x) = (x - \alpha)(x - \beta)$ divides a polynomial $P(x)$ then

$$P(\alpha) = 0$$
$$P(\beta) = 0$$

This property tells us that if a quadratic expression divides a polynomial, then each of its factors also divides the polynomial.

### Example 1.147

A. Find $ab$ if $3x^6 + ax^5 + bx^4 - x^3 + 12x^2 - x - 7$ is divisible by $x^2 - 1$.
B. $f(x) = Ax^3 + Bx^2 + x + 6$ is divisible by $(x^2 - x - 2)$. Find the value of $A$ and $B$.
C. If $x^4 + ax^3 + bx^2 - 5x + 14$ is divisible by $x^2 + 8x + 7$, then find the value of $a$ and $b$. $(Calculator\ Allowed)$

**Part A**

$$x - 1 = 0 \Rightarrow x = 1 \Rightarrow 3 + a + b - 1 + 12 - 1 - 7 = 0 \Rightarrow \underbrace{a + b + 6 = 0}_{Equation\ I}$$
$$x + 1 = 0 \Rightarrow x = -1 \Rightarrow 3 - a + b + 1 + 12 + 1 - 7 = 0 \Rightarrow \underbrace{-a + b + 10 = 0}_{Equation\ II}$$

Add Equation I and II:

$$2b + 16 = 0 \Rightarrow b = -8$$

Substitute $b = -8$ in Equation I:

$$a - 8 + 6 = 0 \Rightarrow a = 2 \Rightarrow ab = (2)(-8) = -16$$

**Part B**

$$x^2 - x - 2 = 0 \Rightarrow (x + 1)(x - 2) = 0 \Rightarrow x \in \{-1,2\}$$

$$f(-1) = -\mathrm{A} + \mathrm{B} - 1 + 6 = 0 \Rightarrow \underbrace{\mathrm{A} - \mathrm{B} = 5}_{Equation\ I}$$

$$f(2) = 8A + 4B + 2 + 6 = 0 \Rightarrow \underbrace{2A + B = -2}_{\textit{Equation II}}$$

Add Equations I and II:

$$3A = 3 \Rightarrow A = 1 \Rightarrow B = -4$$

**Part C**

$$x^2 + 8x + 7 = (x+7)(x+1) \Rightarrow x \in \{-1, -7\}$$

$$(-7)^4 + a(-7)^3 + b(-7)^2 - 5(-7) + 14 = 0$$
$$2401 - 343a + 49b + 35 + 14 = 0$$
$$\underbrace{-343a + 49b = -2450}_{\textit{Equation I}}$$

$$1 - a + b + 5 + 14 = 0 \Rightarrow \ -a + b = -20$$

Multiply by $-49$:

$$\underbrace{49a - 49b = 980}_{\textit{Equation II}}$$

Add Equations I and II:

$$-294a = -1470 \Rightarrow a = -\frac{1470}{-294} = 5$$

$$-a + b = -20 \Rightarrow -5 + b = -20 \Rightarrow b = -15$$

## B. Multi-Step Division

### Example 1.148

If $q_1(x)$ and $r_1$ are the quotient and remainder, respectively, when the polynomial $x^8$ is divided by $x + \frac{1}{2}$, and if $q_2(x)$ and $r_2$ are the quotient and remainder, respectively, when $q_1(x)$ is divided by $x + \frac{1}{2}$, then $r_2$ equals:
(AHSME 1979/25)

Divide $x^8$ by $x + \frac{1}{2}$ using synthetic division:

$$x + \frac{1}{2} = 0 \Rightarrow x = -\frac{1}{2}$$

| | $x^8$ | $x^7$ | $x^6$ | $x^5$ | $x^4$ | $x^3$ | $x^2$ | $x^1$ | $x^0$ |
|---|---|---|---|---|---|---|---|---|---|
| $-\frac{1}{2}$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | | $-\frac{1}{2}$ | $\frac{1}{4}$ | $-\frac{1}{8}$ | $\frac{1}{16}$ | $-\frac{1}{32}$ | $\frac{1}{64}$ | $-\frac{1}{128}$ | $\frac{1}{256}$ |
| | 1 | $-\frac{1}{2}$ | $\frac{1}{4}$ | $-\frac{1}{8}$ | $\frac{1}{16}$ | $-\frac{1}{32}$ | $\frac{1}{64}$ | $-\frac{1}{128}$ | $\frac{1}{256}$ |
| Quotient | $x^7$ | $x^6$ | $x^5$ | $x^4$ | $x^3$ | $x^2$ | $x^1$ | $x^0$ | Remainder |

Note that because of the zeroes the values in the middle row exactly match the values in the bottom row.

$$q_1(x) = x^7 - \frac{1}{2}x^6 + \frac{1}{4}x^5 - \frac{1}{8}x^4 + \frac{1}{16}x^3 - \frac{1}{32}x^2 + \frac{1}{64}x - \frac{1}{128}$$

To find the remainder when the $q_1$ is divided by $x + \frac{1}{2}$, we use the remainder theorem and evaluate $q_1\left(-\frac{1}{2}\right)$:

$$q_1\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^7 - \frac{1}{2}\left(-\frac{1}{2}\right)^6 + \frac{1}{4}\left(-\frac{1}{2}\right)^5 - \frac{1}{8}\left(-\frac{1}{2}\right)^4 + \frac{1}{16}\left(-\frac{1}{2}\right)^3 - \frac{1}{32}\left(-\frac{1}{2}\right)^2 + \frac{1}{64}\left(-\frac{1}{2}\right) - \frac{1}{128}$$

Each of the terms in the above expression evaluates to $-\frac{1}{2^7}$:

$$= -\frac{1}{2^7} - \frac{1}{2}\left(\frac{1}{2^6}\right) - \frac{1}{2^2}\left(\frac{1}{2^5}\right) - \frac{1}{2^3}\left(\frac{1}{2^4}\right) - \frac{1}{2^4}\left(\frac{1}{2^3}\right) - \frac{1}{2^5}\left(\frac{1}{2^2}\right) - \frac{1}{2^6}\left(\frac{1}{2^1}\right) - \frac{1}{2^7}$$

$$= \underbrace{-\frac{1}{2^7} - \frac{1}{2^7} - \cdots - \frac{1}{2^7}}_{8\ Times}$$

$$= -\frac{8}{2^7} = -\frac{2^3}{2^7} = -\frac{1}{2^4} = -\frac{1}{16}$$

| | $x^7$ | $x^6$ | $x^5$ | $x^4$ | $x^3$ | $x^2$ | $x^1$ | $x^0$ |
|---|---|---|---|---|---|---|---|---|
| $-\frac{1}{2}$ | 1 | $-\frac{1}{2}$ | $\frac{1}{4}$ | $-\frac{1}{8}$ | $\frac{1}{16}$ | $-\frac{1}{32}$ | $\frac{1}{64}$ | $-\frac{1}{128}$ |
| | | $-\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{3}{8}$ | $\frac{4}{16}$ | $-\frac{5}{32}$ | $\frac{6}{64}$ | $-\frac{7}{128}$ |
| | 1 | $-1$ | $\frac{3}{4}$ | $-\frac{1}{2}$ | $\frac{5}{16}$ | $-\frac{6}{32}$ | $\frac{7}{64}$ | $-\frac{8}{128} = -\frac{1}{16}$ |
| Quotient | | | | | | | | Remainder |

## Example 1.149

Find the remainder when $f(x) = x^n - nx + n - 1, n > 3$ is divided by $x^3 - 3x^2 + 3x - 1$

**Step I**

Factor the divisor

$$x^3 - 3x^2 + 3x - 1 = (x-1)^3$$

**Step II**

We can check the remainder when $x^n - nx + n - 1$ is divided by $x - 1$:

$$x - 1 = 0 \Rightarrow x = 1 \Rightarrow f(1) = 1 - n + n - 1 = 0 \Rightarrow Divisible$$

Now that we know it is divisible, carry out the division by splitting the terms:

$$\frac{x^n - nx + n - 1}{(x-1)^3} = \frac{x^n - 1}{(x-1)^3} - \frac{n(x-1)}{(x-1)^3} = \frac{x^{n-1} + x^{n-2} + \cdots + 1}{(x-1)^2} - \frac{n}{(x-1)^2}$$

**Step III**

Divide by $x - 1$ one more time by using synthetic division:

| | $x^{n-1}$ | $x^{n-2}$ | $x^{n-3}$ | . | . | . | $x^1$ | $x^0$ |
|---|---|---|---|---|---|---|---|---|
| | 1 | 1 | 1 | . | . | . | 1 | $1-n$ |
| 1 | | 1 | 2 | | | | $n-2$ | $n-1$ |
| | 1 | 2 | 3 | . | . | . | $n-1$ | 0 |
| | $x^{n-2}$ | $x^{n-3}$ | $x^{n-4}$ | . | . | . | $x^0$ | *Remainder* |

$$Quotient = x^{n-2} + 2x^{n-3} + 3x^{n-4} + \cdots + (n-1)$$

**Step III**

Divide by $x - 1$ one last time by using synthetic division:

| | $x^{n-2}$ | $x^{n-3}$ | $x^{n-4}$ | $x^{n-5}$ | . | . | . | $x^0$ |
|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | . | . | . | $n-1$ |
| 1 | | 1 | 3 | 6 | . | . | . | |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 3 | 6 | 10 | . | . | . | $\frac{(n-1)n}{2}$ |
| | $x^{n-3}$ | $x^{n-4}$ | $x^{n-5}$ | $x^{n-6}$ | . | . | . | $Remainder$ |

### Example 1.150
Verify that the formula from the previous example works for $n = 4$

$$\frac{x^2+4x-3}{(x-1)^3} = \frac{x^3+x^2+x-3}{(x-1)^2} = \frac{x^2+2x+3}{x-1} = x+3+\frac{6}{x-1}$$

## C. Degree of Remainder

### 1.151: Degree of Remainder
When a polynomial function is divided by another polynomial function the degree of Remainder is at max, 1 less than degree of Divisor

So,

$$Degree[R(x)] \le Degree[P(x)] - 1$$

where

$$\frac{P_1(x)}{P_2(x)} = Q(x) + R(x)$$
$$P_1(x) = Dividend$$
$$P_2(x) = Divisor$$
$$Q(x) = Quotient$$
$$R(x) = Remainder$$

### Example 1.152
$P(x)$ is divided by a quadratic expression. What are the possible values that the degree of the remainder can be?

$$Degree \in \{0,1\}$$

## D. Finding Linear Remainders

### 1.153: Dividing by Linear Expression

$$P(x) = \underbrace{(x-a)}_{Divisor}\ \underbrace{Q(x)}_{Quotient} + \underbrace{r}_{Remainder}$$

### 1.154: Degree of Remainder
Suppose

- $P(x)$ is a polynomial of degree greater than two.
- $D(x) = (x-\alpha)(x-\beta)$ is a quadratic expression.
- $r = x - \gamma$ is the remainder when $P(x)$ is divided by $D(x)$. $r$ is a linear polynomial.
- $Q(x)$ is the quotient when $P(x)$ is divided by $Q(x)$

Then, we can write:

$$\underbrace{P(x)}_{Dividend} = \underbrace{(x-\alpha)(x-\beta)}_{Divisor}\ \underbrace{Q(x)}_{Quotient} + \underbrace{r}_{Remainder}$$

### Example 1.155

Determine the remainder of
$$\frac{x^{2021}}{x^2+3x+2}$$

**Step I: Factor the Divisor**
Note that
$$x^2+3x+2=(x+2)(x+1)$$

**Step II: Find Equations for the Remainder**
$$\underbrace{x^{2021}}_{Dividend}=\underbrace{(x+2)(x+1)}_{Divisor}\underbrace{Q(x)}_{Quotient}+\underbrace{ax+b}_{Remainder}\quad for\ some\ a\ and\ b$$

Note that if any term of the Divisor becomes zero, it multiplies with the quotient, and we are only left with the remainder term.
Substitute $x+1=0\Rightarrow x=-1$:
$$(-1)^{2021}=(x+2)(-1+1)Q(x)+a(-1)+b$$

The term with the quotient is multiplied with zero, and hence it "vanishes":
$$-1=-a+b\Rightarrow\underbrace{1=a-b}_{Equation\ I}$$

Substitute $x+2=0\Rightarrow x=-2$:
$$(-2)^{2021}=(-2+2)(-2+1)Q(x)+a(-2)+b$$
The term with the quotient is multiplied with zero, and hence it "vanishes":
$$\underbrace{(-2)^{2021}=-2a+b}_{Equation\ II}$$

**Step III: Solve the Equations to find the values of *a* and *b***
Add Equations I and II:
$$(-2)^{2021}+1=-a$$
$$a=2^{2021}-1$$

Substitute $a=2^{2021}-1$ in Equation I:
$$1=2^{2021}-1-b$$
$$b=2^{2021}-2$$

When divided $x^{2021}$ by $x^2+3x+2$:
$$Remainder=ax+b=(2^{2021}-1)x+2^{2021}-2$$

## Example 1.156
The remainder $R$ obtained by dividing $x^{100}$ by $x^2-3x+2$ is a polynomial of degree less than 2. Then $R$ may be written as: (AHSME 1969/34)

**Step I: Factor the Divisor**
Note that
$$x^2-3x+2=(x-2)(x-1)$$

**Step II: Find Equations for the Remainder**
$$\underbrace{x^{100}}_{Dividend}=\underbrace{(x-2)(x-1)}_{Divisor}\underbrace{Q(x)}_{Quotient}+\underbrace{ax+b}_{Remainder}\quad for\ some\ a\ and\ b$$

Note that if any term of the Divisor becomes zero, it multiplies with the quotient, and we are only left with the remainder term.
Substitute $x-1=0 \Rightarrow x=1$:

$$1^{100}=(x-2)(1-1)Q(x)+a(1)+b$$

The term with the quotient is multiplied with zero, and hence it "vanishes":

$$\underbrace{1=a+b}_{Equation\ I}$$

Substitute $x-2=0 \Rightarrow x=2$:

$$2^{100}=(2-2)(2-1)Q(x)+a(2)+b$$

The term with the quotient is multiplied with zero, and hence it "vanishes":

$$\underbrace{2^{100}=2a+b}_{Equation\ II}$$

**Step III: Solve the Equations to find the values of $a$ and $b$**
Subtract Equation I from Equation II:

$$2^{100}-1=a$$

Substitute $a=2^{100}-1$ in Equation I:

$$1=2^{100}-1+b$$
$$b=2-2^{100}$$

When divided $x^{100}$ by $x^2-3x+2$, the remainder is:

$$Remainder=ax+b=(2^{100}-1)x+2-2^{100}$$

## E. Division by $x^2+x+1$

### Example 1.157

## F. Properties of $x^2+x+1=0$

$x^2+x+1$ has three terms. We can, perhaps, counter-intuitively, simplify it by multiplying.

### 1.158: Simplifying

$$x^2+x+1=0 \Rightarrow x^3=1$$

Multiply both sides by $x-1$:

$$x^3=1$$

### Example 1.159

Show by reduction that if $x^2+x+1=0$, then $x^3=1$.

$$x^2=-x-1$$
$$x^3=x\cdot x^2=x(-x-1)=-x^2-x=-(-x-1)-x=x+1-x=1$$

### 1.160: Reduction Technique

The reduction technique lowers the degree of a polynomial by substituting an equivalent value of lower degree.

## G. Remainders

We will do the next question in multiple ways. The underlying concept remains the same, but it gets used in different ways. Make sure you understand both the concept and the technique.
We do this using division first.

### (Algebraic Manipulation Method) Challenge 1.161

Determine the remainder when $(x^4-1)(x^2-1)$ is divided by $1+x+x^2$. (HMMT 2000, Guts Round/13)

Note that:

$$\underbrace{(1+x+x^2)(x-1)=x^3-1}_{\textit{Equation I}}$$
$$\underbrace{(1+x+x^2)(x-1)x=(x^3-1)x=x^4-x}_{\textit{Equation II}}$$

Start with the division:

$$\frac{(x^4-1)(x^2-1)}{1+x+x^2}=\frac{(x^4-x+x-1)(x^2-1)}{1+x+x^2}=\underbrace{\frac{(x^4-x)(x^2-1)}{1+x+x^2}}_{\textit{Term A}}+\underbrace{\frac{(x-1)(x^2-1)}{1+x+x^2}}_{\textit{Term B}}$$

Term A is now completely divisible by using Equation II:

$$\frac{(x^4-x)(x^2-1)}{1+x+x^2}=\frac{[(1+x+x^2)(x-1)x][x^2-1]}{1+x+x^2}=[(x-1)x][x^2-1]$$

Hence, we only need to consider Term B:

$$\frac{(x-1)(x^2-1)}{1+x+x^2}=\frac{x^3-x^2-x+1}{1+x+x^2}=\underbrace{\frac{x^3-1}{1+x+x^2}}_{\textit{Term C}}-\underbrace{\frac{x^2+x-2}{1+x+x^2}}_{\textit{Term D}}$$

By Equation I, Term C is also completely divisible. Hence, we only need to consider Term D:

$$-\frac{x^2+x+1}{1+x+x^2}+\frac{3}{1+x+x^2}=-1+\frac{3}{1+x+x^2}$$

Hence, the remainder is 3.

### (Alternate: Remainder Theorem Method) Challenge 1.162

Determine the remainder when $(x^4-1)(x^2-1)$ is divided by $1+x+x^2$. (HMMT 2000, Guts Round/13)

A. State the division in the form $Divisor = Dividend \times Quotient + Remainder$
B. Multiply both sides by $x-1$. How does this help?
C. We now want to make the term with the Quotient vanish. Substituting $x=1$ will do that. But why is it not a good idea?
D. Show that $x^3=1x\neq 1 \Rightarrow x^2+x+1=0$
E. Instead, make the substitution $x^3=1, x\neq 1$.

#### Part A

We can get the same result that we obtained above by stating the division in terms of what the remainder theorem tells us.

$$\underbrace{(x^4-1)(x^2-1)}_{Divisor}=\underbrace{(1+x+x^2)}_{Dividend}\underbrace{Q(x)}_{Quotient}+\underbrace{R(x)}_{Remainder}$$

### Part B
Note that $x-1$ multiplied by the dividend gives us $x^3-1$:
$$(x-1)\underbrace{(x^4-1)(x^2-1)}_{Divisor}=(x^3-1)\underbrace{Q(x)}_{Quotient}+\underbrace{R(x)}_{Remainder}(x-1)$$

### Part C
Because then the Remainder term becomes
$$\underbrace{R(x)}_{Remainder}(x-1)=R(x)\times 0=0$$
And if you are not careful, this can lead you to conclude that the remainder is zero.

### Part D
$$x^3-1=0\Rightarrow(x-1)(x^2+x+1)=0\Rightarrow x^2+x+1=0$$

### Part E
Substitute $x^3=1, x\neq 1$:
The term with the quotient vanishes.
Also, since $x^4=x\cdot x^3=x\cdot 1=x$:
$$(x-1)(x-1)(x^2-1)=\underbrace{R(x)}_{Remainder}(x-1)$$
Since $x\neq 1$, divide both sides by $x-1$:
$$(x-1)(x^2-1)=\underbrace{R(x)}_{Remainder}$$

We might feel like stopping here, but the expression can be further simplified:
$$R(x)=x^3-x^2-x+1$$
Substitute $x^3=1$:
$$R(x)=-x^2-x+2=-(x^2+x+1)+3$$
Substitute $x^2+x+1=0$:
$$R(x)=3$$

## (Shorter Method) Challenge 1.163
Determine the remainder when $(x^4-1)(x^2-1)$ is divided by $1+x+x^2$. (HMMT 2000, Guts Round/13)

$$\underbrace{(x^4-1)(x^2-1)}_{Divisor}=\underbrace{(1+x+x^2)}_{Dividend}\underbrace{Q(x)}_{Quotient}+\underbrace{R(x)}_{Remainder}$$
Substitute $x^3=1, x\neq 1\Rightarrow x^2+x+1=0$:
$$(x-1)(x^2-1)=(0)\underbrace{Q(x)}+\underbrace{R(x)}_{Remainder}$$
$$R(x)=\underbrace{x^3}_{=0}-x^2-x+1=-\underbrace{(x^2+x+1)}_{=0}+3=3$$

## H. Basics

## Example 1.164
Find the remainder when $x^{2022}$ is divided by $x^2+x+1$

$$x^{2022}=(x^2+x+1)Q(x)+R(x)$$
Substitute $x^2+x+1=0\Rightarrow x^3=1$:
$$R(x)=x^{2022}$$
But note that 2022 has sum of digits $2+0+2+2=6\Rightarrow 2022$ is divisible by 3. Hence, for some natural number

$n$:

$$R(x) = x^{2022} = (x^3)^n = 1^n = 1$$

### Example 1.165

Find the remainder when $x^{2020}$ is divided by $x^2 + x + 1$

$$x^{2020} = (x^2 + x + 1)Q(x) + R(x)$$

Substitute $x^2 + x + 1 = 0 \Rightarrow x^3 = 1$:

$$R(x) = x^{2020}$$

But note that 2022 has sum of digits $2 + 0 + 2 + 0 = 4 \Rightarrow 2022$ has remainder 1 when divided by 3:

Hence, for some natural number $n$:

$$R(x) = x^{3n+1} = x^{3n} \times x = 1 \times x = x$$

### Example 1.166

Find the remainder when $x^{2021}$ is divided by $x^2 + x + 1$

$$x^{2021} = (x^2 + x + 1)Q(x) + R(x)$$

Substitute $x^2 + x + 1 = 0 \Rightarrow x^3 = 1$:

$$R(x) = x^{2021}$$

But note that 2021 has sum of digits $2 + 0 + 2 + 1 = 5 \Rightarrow 2021$ has remainder 2 when divided by 3:

Hence, for some natural number $n$:

$$R(x) = x^{3n+2} = x^{3n} \times x^2 = 1 \times x^2 = x^2$$

However, when dividing by a quadratic the highest possible degree of the remainder is linear. Hence, we need to reduce the expression above still further.

Substitute $x^2 + x + 1 = 0 \Rightarrow x^2 = -x - 1$:

$$R(x) = x^2 = -x - 1$$

## I. Applications

### Example 1.167

Find the remainder when $2020^{2021} + 2020$ is divided by $2020^2 + 2021$

Let $x = 2020$:

$$x^{2021} + x = (x^2 + x + 1)Q(x) + R(x)$$

Substitute $x^2 + x + 1 = 0 \Rightarrow x^3 = 1, x \neq 1$:

$$R(x) = x^{2021} + x = x^2 + x$$

Substitute $x^2 = -x - 1$:

$$R(x) = -x - 1 + x = -1$$

Interpret the negative remainder:

$$R(x) = 2020^2 + 2021 - 1 = 2020^2 + 2020 = 2020(2021)$$

# 1.6 Expressions and Equations

## A. Expressions

### Example 1.168
If $x = 3 + \sqrt{2}$, find the value of $x^3 - 6x^2 + 9x - 2\sqrt{2}$

$$x(x^2 - 6x + 9) - 2\sqrt{2} = x(x-3)^2 - 2\sqrt{2}$$

Substitute $x = 3 + \sqrt{2} \Rightarrow x - 3 = \sqrt{2} \Rightarrow x - 3 = 2$:

$$2x - 2\sqrt{2} = 2(x - \sqrt{2})$$

Substitute $x = 3 + \sqrt{2} \Rightarrow x - \sqrt{2} = 3$:

$$2 \times 3 = 6$$

## B. Equations and Functions
We have seen in the Note on Quadratics how to solve, graph and analyze quadratic equations, or equations which can be reduced to Quadratics.
Here, we look at equations of higher degree.

### 1.169: Cubic and Quartic Polynomials
A cubic polynomial is a third-degree polynomial. That is, it has the form:

$$ax^3 + bx^2 + cx + d, \qquad a \neq 0$$

A quartic polynomial is a fourth-degree polynomial. That is, it has the form:

$$ax^4 + bx^3 + cx^2 + dx + e, \qquad a \neq 0$$

There are two key points here
- $a \neq 0$
- The term with highest power is of degree 3 (for cubic) or degree 4 (for quartic).

### 1.170: Cubic and Quartic Equations
A cubic equation is an equation of the form

$$ax^3 + bx^2 + cx + d = 0, \qquad a \neq 0$$

A quartic equation is an equation of the form

$$ax^4 + bx^3 + cx^2 + dx + e, \qquad a \neq 0$$

### Example 1.171
What is the degree of
- A. A cubic polynomial
- B. A quartic polynomial

3
4

### 1.172: Coefficients

$$ax^3 + bx^2 + cx + d \Rightarrow Coefficients\ are\ a, b, c, d$$
$$ax^4 + bx^3 + cx^2 + dx + e \Rightarrow Coefficients\ are\ a, b, c, d, e$$

### 1.173: Real Values for Coefficients
Unless otherwise stated, we will consider only real values for coefficients.
We can have complex values for solutions.

### Example 1.174

What are the coefficients of:

$$4x^3 + 3x^2 - 2x + 7 = 0$$

$$a = 4, \qquad b = 3, \qquad c = -2, \qquad d = 7$$

### Example 1.175: Sum of Coefficients

The sum of the coefficients of an equation or expression are found by substituting

$$x = 1$$

A. Find the sum of the coefficients of: $\frac{1}{2}x^3 + \frac{3}{4}x^2 - \frac{5}{3}x + 2 = 0$

Substitute $x = 1$:

$$\frac{1}{2} + \frac{3}{4} - \frac{5}{3} + 2 =$$

### Example 1.176: Finding Roots

The roots of $(x^2 - 3x + 2)(x)(x - 4) = 0$ are: (AHSME 1950/13)

$$0,1,2,4$$

### Example 1.177: Trivial Cubics

Trivial cubics are equations that have degree three, but do not require sophisticated methods to solve them. They look like third degree equations, but can be solved much faster.

Solve:

A. $x^3 = 27$
B. $x^3 = -27$
C. $x^3 = 16$

$$x^3 = 27 \Rightarrow x = \sqrt[3]{27} = 3$$
$$x^3 = -27 \Rightarrow x = \sqrt[3]{-27} = -3$$
$$x^3 = 16 \Rightarrow x = \sqrt[3]{16} = \sqrt[3]{8} \times \sqrt[3]{2} = 2\sqrt[3]{2}$$

### 1.178: Cube Root of a Number

$$x^3 = a \Rightarrow x = \sqrt[3]{a} = a^{\frac{1}{3}}$$

Note two differences from the equation below:

$$x^2 = a \Rightarrow x = \pm\sqrt{a}$$

- The cubic equation has only one real solution, not two.
- The cubic equation always has a real solution, unlike the square equation, which only has a solution when $a \geq 0$.

### 1.179: Equations through the Origin

When the graph of an equation passes through the origin, zero is a root of the equation, and this can be easily factored.

### Example 1.180

Find the mean of all solutions for $x$ when $x^3 + 3x^2 - 10x = 0$. (MathCounts 1995 Warm-Up 10)

$$x(x^2 + 3x - 10) = 0$$
$$x(x + 5)(x - 2) = 0$$

Roots are

$$0, 2, -5$$

Mean of Roots

$$= \frac{0 + 2 - 5}{3} = -\frac{3}{3} = -1$$

### 1.181: Forming Equations with Given Roots

Given the roots $\alpha, \beta, \gamma$, all equations with these roots must be of the form:

$$a(x - \alpha)(x - \beta)(x - \gamma) = 0, \qquad a \neq 0$$

Where $a$ is a scaling factor.

### Example 1.182

Find the general cubic equation that has roots $1, -1, 2$. State your answer as a third-degree polynomial in standard form.

We know that the roots are

$$x = 1 \Rightarrow x - 1 = 0$$
$$x = -1 \Rightarrow x + 1 = 0$$
$$x = 2 \Rightarrow x - 2 = 0$$

Multiply the above three:

$$a(x - 1)(x + 1)(x - 2) = 0$$
$$a(x^2 - 1)(x - 2) = 0$$
$$a(x^3 - 2x^2 - x + 2) = 0, \qquad a \neq 0$$

### Example 1.183

Find all cubic equations with roots $1, -1$. State your answer as a third-degree polynomial in standard form.

**Case I: 1 is a repeated root**

$$a(x - 1)^2(x + 1) = 0$$

**Case I: $-1$ is a repeated root**

$$a(x + 1)^2(x - 1) = 0$$

### Example 1.184: Factoring by Grouping

A. $x^3 - 3x^2 + 5x - 15 = 0$
B. $x^3 - 4x^2 + 2x - 8 = 0$
C. $x^2 + 2x^2 + 7x + 14 = 0$
D. $3x^3 - 12x^2 + 2x - 8 = 0$
E. $5x^3 + 5x^2 + x + 1 = 0$

**Part A**

$$x^2(x - 3) + 5(x - 3) = 0$$
$$(x - 3)(x^2 + 5) = 0$$
$$x - 3 = 0 \Rightarrow x = 3$$
$$x^2 + 5 = 0 \Rightarrow x = \pm i\sqrt{5}$$

**Part B**

$$x^2(x - 4) + 2(x - 4) = 0$$
$$(x - 4)(x^2 + 2) = 0$$
$$x \in \{4, \pm i\sqrt{2}\}$$

**Part C**

$$x^2(x+2)+7(x+2)=0$$
$$(x^2+7)(x+2)=0$$
$$x \in \{-7, \pm i\sqrt{2}\}$$

**Part D**

$$3x^2(x-4)+2(x-4)=0$$
$$(3x^2+2)(x-4)=0$$

$$x \in \left\{4, \pm i\sqrt{\frac{2}{3}}\right\}$$

**Part E**

$$5x^2(x+1)+1(x+1)=0$$
$$(5x^2+1)(x+1)=0$$

## C. Rational Roots Theorem[2]

The rational roots theorem does not guarantee finding any roots. What it does do is give a list of potential roots to check using the remainder theorem. If the equations do not have rational roots, then the theorem will not be able to find the roots. Exam questions are meant to be solved in some way, and the rational roots theorem is a primary method of attack.

### 1.185: Rational Roots Theorem

Any rational root of a polynomial must belong to the set of potential roots generated by the following formula

$$\pm\frac{Factor\ of\ constant\ term}{Factor\ of\ leading\ coefficient}$$

The theorem only talks about rational roots. It does not:

- Guarantee the existence of rational roots
- Allow you to find complex roots

### Example 1.186: Finding Potential Roots

Use the rational roots theorem to find the potential roots of the polynomial:

$$2x^3+3x^2-11x-6$$

The constant term is $-6$:

$$Factors\ of\ -6\ are\ \{1,2,3,6\}$$

The leading coefficient is 2:

$$Factors\ of\ 2\ are\ \{1,2\}$$

We use the formula $\pm\frac{Factor\ of\ constant\ term}{Factor\ of\ leading\ coefficient}$:

$$\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{6}{1}, \pm\frac{1}{2}, \pm\frac{2}{2}, \pm\frac{3}{2}, \pm\frac{6}{2}$$

But some of these are repeated. In particular:

$$\frac{6}{2}=\frac{3}{1}, \frac{2}{2}=\frac{1}{1}$$

Hence, the final set is:

$$\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{6}{1}, \pm\frac{1}{2}, \pm\frac{3}{2}$$

### Example 1.187

[2] You can watch this video to see an example of the technique in action. While the rational roots theorem is not mentioned, it is the underlying technique used to get the roots.

Solve

$$2x^3 + 3x^2 - 32x + 15 = 0$$

Make a list of potential factors to check:

$$Factors\ of\ 15 = \pm\{1,3,5,15\}$$
$$Factors\ of\ 2 = \pm\{1,2\}$$

The factors to check are:

$$\pm\frac{1}{1},\frac{3}{1},\frac{5}{1},\frac{15}{1},\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{15}{2}$$

Let $P(x) = 2x^3 + 3x^2 - 32x + 15$

Check the values from the list, beginning with the integers first:

$$P(1) = 2 + 3 - 32 + 15 = -12$$
$$P(-1) = -2 + 3 + 32 + 15 \neq 0$$
$$P(3) = 54 + 27 - 96 + 15 = 0$$

Since

$$P(3) = 0 \Rightarrow x = 3 \Rightarrow x - 3\ is\ a\ factor$$

| | $x^3$ | $x^2$ | $x$ | $Constant$ |
|---|---|---|---|---|
| 3 | 2 | 3 | $-32$ | 15 |
| | | 6 | 27 | $-15$ |
| | 2 | 9 | $-5$ | 0 |
| | $x^2$ | $x$ | $Constant$ | Remainder |

Use synthetic division to get:

$$2x^2 + 9x - 5$$
$$P = -10 = (10)(-1), \qquad S = 9$$

$$2x^2 + 10x - x - 5 = 2x(x - 5) - 1(x + 5) = (2x - 1)(x + 5)$$

$$2x^3 + 3x^2 - 32x + 15 = 0$$
$$(x - 3)(2x - 1)(x + 5) = 0$$

Apply the zero-product property:

$$x - 3 = 0 \Rightarrow x = 3$$
$$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$$
$$x + 5 = 0 \Rightarrow x = -5$$

And we can combine the three solutions to get:

$$x \in \left\{-5, \frac{1}{2}, 3\right\}$$

## Example 1.188

Solve

$$3x(x^2 + 6) = 8 - 17x^2$$

$$3x^3 + 17x^2 + 18x - 8 = 0$$
$$Factors\ of\ 8:\ \pm 1,2,4,8$$

$$P(1) = 3 + 17 + 18 - 8 \neq 0$$
$$P(-1) = -3 + 17 - 18 - 8 \neq 0$$
$$P(2) = 24 + 68 + 36 - 8 \neq 0$$

$$P(-2) = -24 + 68 - 36 - 8 = 0 \Rightarrow x = -2 \Rightarrow x + 2 = 0$$

$$\begin{aligned} 3x^2 + 11x - 4 = 3x^2 + 12x - x - 4 &= 3x(x+4) - 1(x+4) \\ &= (3x-1)(x+4) \\ P = -12, Sum = 11 = 12 - 1 \end{aligned}$$

| | $x^3$ | $x^2$ | $x$ | $Constant$ |
|---|---|---|---|---|
| $-2$ | $3$ | $17$ | $18$ | $-8$ |
| | | $-6$ | $-22$ | $8$ |
| | $3$ | $11$ | $-4$ | $0$ |
| | $x^2$ | $x$ | $Constant$ | |

$$(x+2)(3x-1)(x+4) = 0$$
$$x + 2 = 0 \Rightarrow x = -2$$
$$3x - 1 = 0 \Rightarrow x = \frac{1}{3}$$
$$x + 4 = 0 \Rightarrow x = -4$$

$$x \in \left\{-4, -2, \frac{1}{3}\right\}$$

## Example 1.189

Solve

$$2x^3 - 3x^2 - 11x + 6 = 0$$

$$Factors\ of\ 6: \pm 1,2,3,6$$
$$P(1) = 2 - 3 - 11 + 6 = -6$$
$$P(-1) = -2 - 3 + 11 + 6 = 12$$
$$P(2) = 16 - 12 - 22 + 6 = -12$$
$$P(-2) = -16 - 12 + 22 + 6 = 0$$

$$2x^2 - 7x + 3$$

x=-2,3,1/2

## Example 1.190

Solve

$$2x^3 - 11x^2 - 20x - 7 = 0$$

| | | | | |
|---|---|---|---|---|
| 7 | 2 | -11 | -20 | -7 |
| | | 14 | 21 | 7 |
| | 2 | 3 | 1 | 0 |

$$2x^2 + 3x + 1 = 0$$

(x+1)(x-7)(2x+1)

## Example 1.191

Solve

$$3x^3 + 7x^2 - 22x - 8 = 0$$

(x-2)(x+4)(3x+1)

$$x \in \left\{-4, -\frac{1}{3}, 2\right\}$$

## Example 1.192

Show that $2x^3 - 5x^2 + 10x - 4 = 0$ has only one real root.

$$Factors\ of\ 4:\ \pm 1,2,4$$
$$Factors\ of\ 2: \pm 1,2$$
$$\pm\left\{\frac{1}{1},\frac{2}{1},\frac{4}{1},\frac{1}{2},\frac{2}{2},\frac{4}{2}\right\} = \pm\left\{1,2,4,\frac{1}{2},\mathbf{1},\mathbf{2}\right\} = \pm\left\{1,2,4,\frac{1}{2}\right\}$$

$$P(1) =$$
$$P(-1) =$$
$$P(2) =$$
$$P(-2) =$$
$$P(4) =$$

| | | | | |
|---|---|---|---|---|
| $\frac{1}{2}$ | 2 | -5 | 10 | -4 |
| | | 1 | -2 | 4 |
| | 2 | -4 | 8 | 0 |

$$P(-4) =$$

$$P\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - 5\left(\frac{1}{2}\right)^2 + 10\left(\frac{1}{2}\right) - 4 = \frac{1}{4} - \frac{5}{4} + 5 - 4 = 0$$

$$\left(x - \frac{1}{2}\right)(2x^2 - 4x + 8) = 0$$

Calculate the discriminant of the quadratic by substituting $a = 2, b = -4, c = 8$:
$$b^2 - 4ac = (-4)^2 - (4)(2)(8) = 16 - 64 = -48 < 0$$
Hence, the quadratic has two complex roots.

Hence, the cubic has one real solution, given by:
$$x - \frac{1}{2} = 0 \Rightarrow x = \frac{1}{2}$$
And two complex solutions.

Hence, it has only one real root.

## Example 1.193

Find all primes $p, q$ such that $2p^3 - q^2 = 2(p+q)^2$

Apply Parity:
$$\underbrace{2(p+q)^2}_{Even} \Rightarrow \underbrace{2p^3 - q^2}_{Even} \Rightarrow \underbrace{2p^3}_{Even} - \underbrace{q^2}_{Even} \Rightarrow \underbrace{q}_{Even} \Rightarrow \underbrace{q = 2}_{Only\ Even\ Prime}$$
Substitute $q = 2$ in the original equation:
$$2p^3 - 4 = 2(p+2)^2$$
Expand and simplify the RHS:
$$2p^3 - 4 = 2p^2 + 8p + 8$$
Collate all terms on one side:
$$2p^3 - 2p^2 - 8p - 12 = 0$$
Divide both side by two:
$$p^3 - p^2 - 4p - 6 = 0$$

Because we are solving for primes, we need only integer solutions.
By the rational roots theorem, any rational solutions will be given by $p = \{\pm 1, \pm 2, \pm 3, \pm 6\}$:
$$p = 3 \text{ gives us a zero value for the equation} \Rightarrow (p-3) \text{ is a root.}$$

Factor the cubic expression using polynomial long division:

$$(p-3)(p^2+2p+2)=0 \Rightarrow p=3\ OR\ \underbrace{p^2+2p+2}_{\substack{D=2^2-4(1)(2)=-4<0\\ No\ Real\ Solutions}}=0$$

The only solution for $(p,q)$ is:

$$(p,q)=(3,2)$$

## D. Remainder Theorem

### Example 1.194

$4x^3+2ax-7a$ leaves a remainder of $-10$ when divided by $x-a$. Find the value of $a$.

$$x-a=0 \Rightarrow x=a \Rightarrow f(a)=4a^3+2a^2-7a=-10$$
$$4a^3+2a^2-7a+10=0$$

### Example 1.195

The polynomial $3x^3+ax+5a$ leaves a remainder of $-7$ when divided by $x-a$. Find all values of $a$.

$$f(a)\ =\ 3a^3+a^2+5a+7$$
$$(a+1)(3a^2-2a+7)=0$$

Discriminant of Quadratic:
b^2-4ac=4-(4)(3)(7)=4-72=-68
No Solutions

## E. Fourth Degree Equations

## F. HCF and LCM

Since we are now (theoretically) in a position to find the rational roots of any polynomial, we can factor it. A direct application is to find the $HCF$ of two polynomials.

### Example 1.196: HCF

Find the $HCF$ of $3x^3+x^2-5x+21$ and $6x^3+29x^2+26x-21$

**First Polynomial: $P(x)=6x^3+29x^2+26x-21$**

$$\underbrace{\pm\{1,3,7,21\}}_{Denominator\ 1},\quad \underbrace{\pm\left\{\frac{1}{6},\frac{3}{6},\frac{7}{6},\frac{21}{6}\right\}}_{Denominator\ 6}=\underbrace{\pm\left\{\frac{1}{6},\mathbf{1},\frac{7}{6},\mathbf{7}\right\}}_{1\ and\ 7\ are\ repeated}$$

Try the roots starting from the simplest.

$x=\pm1$ does not work

$$x=1 \Rightarrow P(x)=6+29+26-21=40\neq0 \Rightarrow Not\ a\ factor$$
$$x=-1 \Rightarrow P(x)=-6+29-26-21=-24\neq0 \Rightarrow Not\ a\ factor$$

$x=3$ does not work

$$x=3 \Rightarrow P(x)=6(27)+29(9)+26(3)-21\neq0 \Rightarrow Not\ a\ factor$$

$x=-3$ does work

$$6(-27)+29(9)+26(-3)-21=-162+261-78-21=0 \Rightarrow \underbrace{x+3}_{Is\ a\ factor}$$

| | | | | |
|---|---|---|---|---|
| $-3$ | 6 | 29 | 26 | $-21$ |
| | | $-18$ | $-33$ | 21 |
| | 6 | 11 | $-7$ | 0 |

$$6x^3+29x^2+26x-21=(6x^2+11x-7)(x+3)=(2x-1)(3x+7)(x+3)$$

We want to factor $6x^2+11x-7$

$$Product=-7\times 6=-42=(-1)(42)=(-2)(21)=(-3)(14)$$
$$Sum=11$$
$$6x^2+14x-3x-7=2x(3x+7)-1(3x+7)=(2x-1)(3x+7)$$

### Second Polynomial

$$Constant\ Term=21\Rightarrow Factors\ of\ 21=\pm\{1,3,7,21\}$$
$$Leading\ Coefficient=3\Rightarrow Factors\ of\ 3=\pm\{1,3\}$$

Combinations from $\underbrace{\{1,3,7,21\}}_{Numerator}$ and $\underbrace{\{1,3\}}_{Denominator}$ :

$$\underbrace{\pm\{1,3,7,21\}}_{Denominator\ 1},\quad \underbrace{\pm\left\{\frac{1}{3},\frac{3}{3},\frac{7}{3},\frac{21}{3}\right\}}_{Denominator\ 3}=\underbrace{\pm\left\{\frac{1}{3},1,\frac{7}{3},7\right\}}_{1\ and\ 7\ are\ repeated}$$

$$Final\ Set\ of\ Potential\ Roots\ to\ Check=\pm\left\{1,3,7,21,\frac{1}{3},\frac{7}{3}\right\}$$

## Example 1.197: Multiple Variables

A. Find the value(s) of $x$ such that $8xy-12y+2x-3=0$ is true for all values of $y$. (AMC 10B 2002/13). Then, find the value(s) of y such that the equation holds for all values of $x$.

B. Find the value(s) of $x$ such that the $(y+2)x^2+(7y+14)x+12y$ is $-24$ for all values of $y$. Then, find the value(s) of $y$ such that the above expression is $-24$ for all values of $x$.

### Part A

Factor the equation:

$$(4y+1)(2x-3)=0$$

If the second term is zero, then the equality holds for all values of $y$:

$$2x-3=0\Rightarrow x=\frac{3}{2}$$

If the first term is zero, then the equality holds for all values of $x$.

$$4y+1=0\Rightarrow y=-\frac{1}{4}$$

### Part B

$$(y+2)x^2+(7y+14)x+12y=-24$$
$$x^2y+7xy+12y+2x^2+14x+24=0$$
$$y(x^2+7x+12)+2\big((x^2+7x+12)\big)=0$$
$$(y+2)(x^2+7x+12)=0$$

$$x^2+7x+12=0\Rightarrow x\in\{-3,-4\}$$
$$y+2=0\Rightarrow y=-2$$

## Example 1.198: Solving with unknown constants

If $x^3 + x^2 + x + 1 = 0$ and $x^2 + px - q = 0$, then:

A. Find the value of x as a rational function of $p$ and $q$. (Recall that a rational function will not include square roots).
B. Show that $q \neq p^2 + p - 1$

**Part A**

$$x^2 = q - px$$
$$x(q - px) + (q - px) + x + 1 = 0$$
$$xq - p(q - px) + (q - px) + x + 1 = 0$$
$$xq - pq - p^2x + q - px + x + 1 = 0$$
$$xq - p^2x - px + x = pq - q - 1$$
$$x(q - p^2 - p + 1) = pq - q - 1$$
$$x = \frac{pq - q - 1}{q - p^2 - p + 1}$$

**Part B**

$$q - p^2 - p + 1 = 0 \Rightarrow q = p^2 + p - 1$$

## G. Graphs

### 1.199: y-intercept

The y-intercept of a cubic equation $ax^3 + bx^2 + cx + d = 0$ is its constant term
$$d$$

At the y-axis, $x = 0$, which we substitute in the given equation:
$$x = 0 \Rightarrow a(0^3) + b(0^2) + c(0) + d = d$$

### Example 1.200: Roots

*Mark the correct option*

The adjoining graph is that of a cubic polynomial function. We can see two roots. The third root:

A. Is real
B. Is complex
C. Can be either complex or real

The graph intersects the x-axis twice, which means there are at least two real roots.
Complex roots only occur in pairs, so the third root must be real as well.

### Example 1.201: Repeated Roots

The adjoining graph is that of a cubic polynomial function. We can see two roots. The third root:

A. Is real
B. Is complex
C. Can be either complex or real

The graph bounces at the x-axis. That means the root has multiplicity two.
Complex roots only occur in pairs, so the third root must be real as well.

### Example 1.202: End Behavior

### Example 1.203: Number of Turning Points

An equation of degree $n$ has exactly $(n-1)$ turning points.
How many turning points will a quartic $(degree\ 4)$ equation have?

$$4-1=3\ Turning\ Points$$

## H. System of Equations

### Example 1.204: Systems of Equations

For the system of equations $x^2+x^2y^2+x^2y^4=525$ and $x+xy+xy^2=35$, the sum of the real $y$ values that satisfy the equations is: **(CEMC Cayley 2000/25)**

We want to connect the first equation with the second. Factor the first equation by adding and subtracting $x^2y^2$:

$$x^2+2x^2y^2+x^2y^4-x^2y^2=(x+xy^2)^2-x^2y^2=525$$

Factor the above using a difference of squares:

$$(x+xy^2+xy)(x+xy^2-xy)=525$$

Substitute $\underbrace{x+xy+xy^2=35}_{Equation\ I}$:

$$35(x+xy^2-xy)=525$$
$$\underbrace{x+xy^2-xy=15}_{Equation\ II}$$

Subtract Equation II from Equation I:

$$2xy=20\Rightarrow xy=10\Rightarrow \underbrace{x=\frac{10}{y}}_{Equation\ III}$$

Now that we have the relation between x and y, substitute Equation I in Equation II:

$$\frac{10}{y}+\left(\frac{10}{y}\right)y^2-\left(\frac{10}{y}\right)y=15$$
$$\frac{10}{y}+10y-10=15$$

Multiply both sides by y to eliminate fractions:

$$10+10y^2-10y=15y$$
$$10y^2-25y+10=0$$
$$2y^2-5y+2=0$$
$$(2y-1)(y-2)=0$$
$$y\in\left\{\frac{1}{2},2\right\}$$

Sum of real $y$ values is

$$\frac{1}{2}+2=\frac{5}{2}$$

# 1.7 Roots

## A. Fundamental Theorem of Algebra

### 1.205: Fundamental Theorem of Algebra[3]
Any non-constant polynomial with complex coefficients will have all its roots in the complex plane.

## B. Vieta's Formulas

### 1.206: Relation between Roots
For a quadratic equation, $ax^2+bx+c=0, a\neq 0$, the roots $\alpha$ and $\beta$, satisfy the relations:

$$\alpha+\beta=-\frac{b}{a},\qquad \alpha\beta=\frac{c}{a}$$

The Note on Quadratics covers the relation between roots for a quadratic in some degree of detail. You should revise it now, if needed.

### 1.207: Relation between Roots
For a cubic equation, $ax^3+bx^2+cx+d=0$, the roots $\alpha,\beta,\gamma$, satisfy the relations:

$$Sum\ of\ Roots=\alpha+\beta+\gamma=-\frac{b}{a}$$

$$Sum\ of\ Product\ of\ Roots\ (2\ at\ a\ time): \alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$$

$$Product\ of\ Roots=\alpha\beta\gamma=-\frac{d}{a}$$

### Example 1.208
A. For the equation $\frac{1}{2}x^3+\frac{3}{4}x^2-\frac{5}{3}x+2=0$, find (a) the sum of the roots, (b) the product of the roots, (c) the sum of the products of the roots taken two at a time

B. Find the mean of all solutions for $x$ when $x^3+3x^2-10x=0$. (MathCounts 1995 Warm-Up 10)

**Part A**

$$\alpha+\beta+\gamma=-\frac{b}{a}=-\frac{3}{4}\div\frac{1}{2}=-\frac{3}{4}\times 2=-\frac{3}{2}$$

$$\alpha\beta\gamma=-\frac{d}{a}=-2\div\frac{1}{2}=-4$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}=-\frac{5}{3}\div\frac{1}{2}=-\frac{5}{3}\times 2=-\frac{10}{3}$$

**Part B**

$$Mean\ of\ Roots=\frac{Sum\ of\ Roots}{No.of\ Roots}=\frac{-\frac{b}{a}}{3}=\frac{-\frac{3}{1}}{3}=-1$$

### Example 1.209
A. $(Zero\ Root)$ One of the roots of $ax^3+bx^2+cx+d=0$ is zero. Find the product of the other two roots.

B. The polynomial $P(x)=x^3+ax^2+bx+c$ has the property that the mean of its zeros, the product of its zeros, and the sum of its coefficients are all equal. If the $y$-intercept of the graph of $y=P(x)$ is 2, what is $b$? (AMC 12 2001/19)

**Part A**

[3] An equivalent statement is that any non-constant polynomial has at least one complex root. The given statement then holds by induction.

Substitute $x = 0$:

$$ax^3 + bx^2 + cx + d = 0 \Rightarrow d = 0$$

Substitute $d = 0$:

$$ax^3 + bx^2 + cx = 0 \Rightarrow x(ax^2 + bx + c) = 0 \Rightarrow x = 0, \alpha\beta \Rightarrow \alpha\beta = \frac{c}{a}$$

We can do this another way as well. Without loss of generality, let $\gamma = 0$. Then:

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \Rightarrow \alpha\beta + \beta \times 0 + 0 \times \alpha = \frac{c}{a} \Rightarrow \alpha\beta = \frac{c}{a}$$

### Part B

$y$-intercept gives us the constant term

$$c = 2$$
$$Product\ of\ Roots = -c = -2$$
$$Mean\ of\ roots = \frac{Sum\ of\ Roots}{3} = -\frac{a}{3} = -2 \Rightarrow a = 6$$

$$Sum\ of\ Coefficients = -2$$
$$1 + a + b + c = -2$$
$$1 + 6 + b + 2 = -2$$
$$b = -11$$

## 1.210: Relation between Roots

The fourth-degree equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0\ with\ Roots \in \{\alpha, \beta, \gamma, \delta\}$$

$$Sum\ of\ Roots = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$$
$$2\ Roots\ at\ a\ time = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$
$$3\ Roots\ at\ a\ time = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$$
$$4\ Roots\ at\ a\ time = \alpha\beta\gamma\delta = \frac{e}{a}$$

Some patterns can be used to remember the formulas better:

- The coefficients $b, c, d, \ldots$ occur in succession.
- The signs alternate between positive and negative (starting with negative)
- The leading coefficient is always in the denominator.

## Example 1.211

A. The fourth-degree polynomial equation $x^4 - 7x^3 + 4x^2 + 7x - 4 = 0$ has four real roots, $a, b, c$ and $d$. What is the value of the sum $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$? Express your answer as a common fraction. (MathCounts 2014 State Sprint)

B. Suppose 1,2,3 are the roots of the equation $x^4 + ax^2 + bx = c$. Find the value of $c$. (PRMO 2017/19)

### Part A

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{abc + abd + acd + bcd}{abcd} = -\frac{7}{-4} = \frac{7}{4}$$

### Part B

Rewrite the given equation:

$$x^4 + 0x^3 + ax^2 + bx - c = 0$$

A fourth-degree equation has four roots. If the fourth root is $\alpha$, then

$$Sum\ of\ Roots = -\frac{0}{1} = 0 \Rightarrow 1 + 2 + 3 + \alpha = 0 \Rightarrow \alpha = -6$$
$$Product\ of\ Roots = -\frac{c}{1} = -c$$
$$Product\ of\ Roots = (1)(2)(3)(-6) = -36$$
$$-c = -36 \Rightarrow c = 36$$

### 1.212: Relation between Roots

$$a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \cdots + a_{n+1} = 0$$
$$Sum\ of\ Roots = Roots\ taken\ 1\ at\ a\ time = -\frac{a_2}{a_1}$$
$$Roots\ taken\ 2\ at\ a\ time = \frac{a_3}{a_1}$$
$$Roots\ taken\ 3\ at\ a\ time = -\frac{a_4}{a_1}$$

.

.

.

$$Product\ of\ Roots = (-1)^n\frac{a_{n+1}}{a_1}$$

## C. Vieta's Formulas: Applications

### Example 1.213

A. Find $(\alpha+\beta-\gamma)(\alpha+\gamma-\beta)(\alpha+\beta-\gamma)$ given that $x^3+3x^2-6x-8=0,\ x \in \{\alpha,\beta,\gamma\}$.

B. For a cubic equation with roots $\{\alpha,\beta,\gamma\}$ let $s = sum\ of\ roots = \alpha+\beta+\gamma$. Show that $(\alpha+\beta-\gamma)(\alpha+\gamma-\beta)(\beta+\gamma-\alpha) = 8\left(\frac{s}{2}-\gamma\right)\left(\frac{s}{2}-\beta\right)\left(\frac{s}{2}-\alpha\right)$

C. Use the property from Part B to redo Part A.

**Part A**

Roots are

$$x \in \{-1,2,4\}$$
$$(\alpha+\beta-\gamma)(\alpha+\gamma-\beta)(\alpha+\beta-\gamma) = 35$$

**Part B**

In the first term, add and subtract $\gamma$:

$$\alpha+\beta\underbrace{+\gamma-\gamma}_{Add\ and\ Subtract}-\gamma = \alpha+\beta+\gamma-2\gamma = s-2\gamma$$
$$\alpha+\beta+\gamma-\beta-\beta = \alpha+\beta+\gamma-2\beta = s-2\beta$$
$$\beta+\gamma+\alpha-\alpha-\alpha = \beta+\gamma+\alpha-2\alpha = s-2\alpha$$

Hence:

$$(\alpha+\beta-\gamma)(\alpha+\gamma-\beta)(\beta+\gamma-\alpha) = (s-2\gamma)(s-2\beta)(s-2\alpha)$$

Factor out 2 from each bracket:

$$8\left(\frac{s}{2}-\gamma\right)\left(\frac{s}{2}-\beta\right)\left(\frac{s}{2}-\alpha\right)$$

**Part C**

### Example 1.214: Sum of Squares of Roots

Using the identity $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$ find the sum of the squares of the roots of the following:

A. $2x^3 + 3x^2 - 5x + 4 = 0$
B. $x^3 + x^2 + x + 1 = 0$

**Part A**

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \left(-\frac{3}{2}\right)^2 - 2\left(-\frac{5}{2}\right) = \frac{9}{4} + 5 = \frac{29}{4}$$

**Part B**

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \left(-\frac{1}{1}\right)^2 - 2\left(\frac{1}{1}\right) = 1 - 2 = -1$$

## Example 1.215

$$\textit{Find the flaw in this solution to the above question}$$

Since $\alpha$ is a root of the equation, it must satisfy the equation, and hence substitute $\alpha$ in the original equation:

$$\alpha^3 + \alpha^2 + \alpha + 1 = 0$$

Factor out $\alpha^2$ from the first two terms:

$$\alpha^2(\alpha + 1) + \alpha + 1 = 0$$

Begin to isolate $\alpha^2$:

$$\alpha^2(\alpha + 1) = -(\alpha + 1)$$

Isolate $\alpha^2$ by dividing both sides by $\alpha + 1$:

$$\alpha^2 = -\frac{\alpha + 1}{\alpha + 1} = -1$$

Similarly,

$$\beta^2 = -1, \gamma^2 = -1$$

Hence,

$$\alpha^2 + \beta^2 + \gamma^2 = -1 - 1 - 1 = -3$$

When you arrive at a solution using algebraic manipulation, rather than standard methods you need to be extra careful. Each step of the solution should be valid.

When dividing by $\alpha + 1$, we need to add the condition that $\alpha + 1 \neq 0 \Rightarrow \alpha \neq -1$.

When check, we see that $a = -1$ does satisfy the equation.
Also, if $\alpha \neq -1$, then

$$\alpha^2 = -1 \Rightarrow \alpha = \sqrt{-1}$$

Hence, the equation has one real root and two complex roots:

$$\alpha = -1, \beta^2 = \gamma^2 = -1$$

And hence

$$\alpha^2 + \beta^2 + \gamma^2 = (-1)^2 - 1 - 1 = 1 - 1 - 1 = -1$$

## Example 1.216

If $\alpha, \beta, \gamma$ are the roots of the equation $x^3 + x^2 - x + 1 = 0$, then

A. Show that $\alpha^2 = \frac{\alpha - 1}{\alpha + 1},\ \beta^2 = \frac{\beta - 1}{\beta + 1},\ \gamma^2 = \frac{\gamma - 1}{\gamma + 1}$
B. $\frac{\alpha - 1}{\alpha + 1} + \frac{\beta - 1}{\beta + 1} + \frac{\gamma - 1}{\gamma + 1}$

**Part A**

$$\alpha^3 + \alpha^2 - \alpha + 1 = 0$$
$$\alpha^2(\alpha + 1) = \alpha - 1$$

For $\alpha \neq -1$:

$$\alpha^2 = \frac{\alpha - 1}{\alpha + 1}$$

Similarly,

$$\beta^2 = \frac{\beta - 1}{\beta + 1}, \qquad \gamma^2 = \frac{\gamma - 1}{\gamma + 1}$$

**Part B**

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \left(-\frac{1}{1}\right)^2 - 2\left(-\frac{1}{1}\right) = 1 + 2 = 3$$

### Example 1.217: Minimum

The polynomial $x^3 - ax^2 + bx - 2010$ has three positive integer roots. What is the smallest possible value of $a$? (AMC 10A 2010/21)

The sum of the roots is

$$-\frac{-a}{1} = a$$

And the product of the roots is

$$-\frac{-2010}{1} = 2010 = 2 \times 3 \times 5 \times 67$$

To get three roots, any two of the prime factors must be multiplied together.
To make the sum minimum, the difference between the roots should be the least.
Hence, we get the roots as

$$5, \quad \underbrace{6}_{2\times 3}, \quad 67$$

And the value of $a$ is:

$$5 + 6 + 67 = 78$$

### Example 1.218: Variables

$ax^3 + bx^2 + cx + d = 0$ has one root $-1$. Find the product of the other roots in terms of the coefficients.

Without loss of generality, let $\gamma = -1$. Then:

$$\alpha + \beta + \gamma = -\frac{b}{a} \Rightarrow \alpha + \beta - 1 = -\frac{b}{a} \Rightarrow \alpha + \beta = -\frac{b}{a} + 1$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \Rightarrow \alpha\beta + \beta(-1) + (-1)\alpha = \frac{c}{a} \Rightarrow \alpha\beta = \frac{c}{a} + (\alpha + \beta) = \frac{c}{a} - \frac{b}{a} + 1 = \frac{c - b + a}{a}$$

### Example 1.219: Equations with Related Roots

## D. Roots meeting conditions

### Example 1.220: Integer Roots

Let $a, b$ be integers such that all the roots of the equation $(x^2 + ax + 20)(x^2 + 17x + b) = 0$ are negative integers. What is the smallest possible value of $a + b$? (IOQM 2017/4)

**First Quadratic: $x^2 + ax + 20$**

$$Product\ of\ Roots = \alpha\beta = 20$$
$$Sum\ of\ Roots = \alpha + \beta = -a$$

| | | | |
|---|---|---|---|
| | $-1$ | $-2$ | $-4$ |
| | $-20$ | $-10$ | $-5$ |
| $-a$ | $-21$ | $-12$ | $-9$ |
| $a$ | 21 | 12 | 9 |
| | | | Smallest |

**Second Quadratic: $x^2 + 17x + b$**

$$Product\ of\ Roots = \alpha\beta = b$$
$$Sum\ of\ Roots = a + \beta = -17$$

| | | | | | | |
|---|---|---|---|---|---|---|
| | $-16$ | $-15$ | . | . | . | $-9$ |
| | $-1$ | $-2$ | . | . | . | $-8$ |
| $b$ | 16 | 30 | . | . | . | 72 |
| | Smallest | | | | | Largest |

## Example 1.221: Consecutive Integers

A. $x^3 + 15x^2 + 71x + 105$ has roots which are consecutive odd integers. Find the roots.
B. Consecutive Odd Integers
C. Consecutive Even Integers

Let the middle root be $\alpha$. Then, the three roots are:

$$\alpha - 2, \alpha, \alpha + 2$$

Also, we have:

$$Sum\ of\ Roots = -15$$
$$(\alpha - 2) + \alpha + (\alpha + 2) = -15$$
$$3\alpha = -15 \Rightarrow \alpha = -5$$

Roots are:

$$-3, -5, -7$$

# E. Roots in Arithmetic Sequence/ Geometric Sequence

## Example 1.222

An arithmetic sequence is defined as a sequence where the difference between two terms is constant. It is given by:

$$\dots, \alpha - 2d, \alpha - d, \alpha, \alpha + d, \alpha + 2d, \dots$$

$ax^3 - 6x^2 + bx - c$ has three distinct real roots in arithmetic progression with common difference $d$. (You can use information from previous parts of the question in the following parts).

A. Find the middle root in terms of $a$. (Middle root is the root that is neither the largest nor the smallest).
B. Given that $a = 1$, find the value of the middle root.
C. Given that $d = 1$, find the other two roots.
D. Find the values of $b$ and $c$.

**Part A**
Let the middle root be $\alpha$. Then, the three roots are:

$$\alpha - d, \alpha, \alpha + d$$

Also, we have:

$$Sum\ of\ Roots = -\frac{(-6)}{a}$$

$$(\alpha - d) + \alpha + (\alpha + d) = \frac{6}{a}$$

$$3\alpha = \frac{6}{a} \Rightarrow \alpha = \frac{2}{a}$$

**Part B**

$$a = 2 \Rightarrow \ \alpha = \frac{2}{1} = 2$$

**Part C**

$$\alpha - d = 2 - 1 = 1$$
$$\alpha + d = 2 + 1 = 3$$

**Part D**
The roots are

$$1,2,3$$

And the equation will be

$$s(x-1)(x-2)(x-3) = s(x^3 - 6x^2 + 11x - 6) = sx^3 - 6sx^2 + 11sx - 6s$$

And we know that the equation is

$$x^3 - 6x^2 + bx - c \Rightarrow s = 1$$

Hence, the equation is:

$$x^3 - 6x^2 + 11x - 6 \Rightarrow b = 11, c = 6$$

## 1.223: Roots in Arithmetic Sequence

If the roots of $ax^3 + bx^2 + cx + d = 0$ are in arithmetic sequence, then the value of the middle root:

$$\alpha = -\frac{b}{3a}$$

Let the middle root be $\alpha$, and the common difference of the arithmetic sequence be $d$. Then, the three roots are:

$$\alpha - d, \alpha, \alpha + d$$

Sum of Roots

$$(\alpha - d) + \alpha + (\alpha + d) = -\frac{b}{a} \Rightarrow 3\alpha = -\frac{b}{a} \Rightarrow \alpha = -\frac{b}{3a}$$

## Example 1.224

The roots of $x^4 - 14x^3 + 51x^2 + ax + b = 0$ form an arithmetic sequence. Find $a\ +\ b$. (AOPS Alcumus, Intermediate Algebra, Vieta's Formulas)

$$(a - 3d) + (a - d) + (a + d) + (a + 3d) = 14 \Rightarrow a = \frac{7}{2}$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma\delta$$

Substitute the values of the roots:

$$(a-3d)(a-d+a+d+a+3d)+(a-d)(a+d+a+3d)+(a+d)(a+3d)$$

Simplify:

$$=(a-3d)(3)(a+d)+(a-d)(2a+4d)+(a+d)(a+3d)$$

Take $a+d$ common from the first and the last term:

$$=(a+d)[(3a-9d)+a+3d]+(a-d)(2a+4d)$$
$$=(a+d)[(4a-6d)]+(a-d)(2a+4d)$$

Expand:

$$=4a^2-6ad+4ad-6d^2+2a^2-2ad+4ad-4d^2$$
$$=6a^2-10d^2$$

$$6a^2-10d^2=51$$
$$10d^2=6a^2-51=6\left(\frac{49}{4}\right)-51=\frac{147}{2}-\frac{102}{2}=\frac{45}{2}$$
$$d^2=\frac{45}{20}=\frac{9}{4}$$
$$d=\pm\frac{3}{2}$$

$$(x+1)(x-2)(x-5)(x-8)=x^4-14x^3+51x^2-14x-80$$
$$a+b=-94$$

### 1.225: Roots in Geometric Sequence

An geometric sequence is defined as a sequence where the ratio between two terms is constant. It is given by:

$$\dots,\frac{\alpha}{r^2},\frac{\alpha}{r},\alpha,\alpha r,\alpha r^2,\dots$$

If the roots of $ax^3+bx^2+cx+d=0$ are in geometric sequence, then the value of the middle root:

$$\alpha=\sqrt[3]{-\frac{d}{a}}$$

Let the roots be:

$$\frac{\alpha}{r},\alpha,\alpha r$$

$$Product\ of\ Roots=-\frac{d}{a}$$

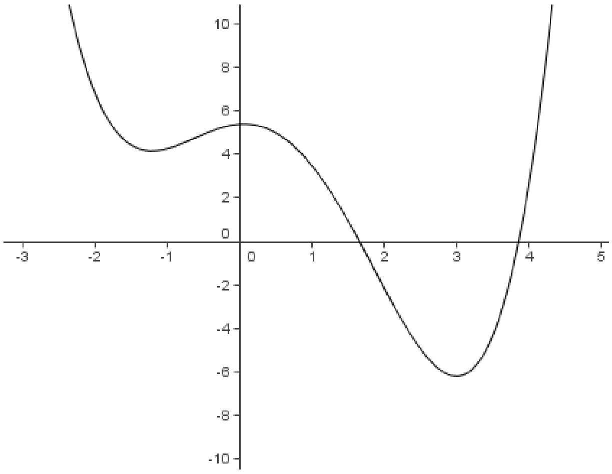
$$\frac{\alpha}{r}\times\alpha\times\ \alpha r=-\frac{d}{a}\Rightarrow\alpha^3=-\frac{d}{a}\Rightarrow\alpha=\sqrt[3]{-\frac{d}{a}}$$

## F. Graphs

### Example 1.226

The graph below shows a portion of the curve defined by the quartic polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$. Which of the following is the smallest? **(AMC 12 2000/22)**

A. $P(-1)$
B. The product of the zeroes of P
C. The product of the non-real zeroes of O
D. The sum of the coefficients of P
E. The sum of the real zeroes of P




This looks difficult, but estimating using properties is the key skill here.

**Option A**

From the graph, we can make out that:

$$P(-1) \approx 4$$

**Option B**

$$Product\ of\ roots = \frac{d}{1} = d$$

But, this is also the y-intercept

$$Product\ of\ roots \approx 5.5$$

**Option D**

Sum of coefficients

$$= P(1) \approx 3$$

**Option E**

Sum of real roots

$$\approx 1.5 + 3.5 = 5$$

**Option C**

We can see two real roots. We can also see all three turning points.
Hence, there are no more real roots, and the remaining roots are non-real (complex).

$$(Complex\ Roots)(Real\ Roots) = All\ Roots$$

$$Complex\ Roots = \frac{All\ Roots}{Real\ Roots} \approx \frac{5.5}{5} = 1.Something$$

This is clearly the lowest.

## G. Transformation of Roots

### 1.227: Roots increased/decreased by $k$

Consider the polynomial:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \ with\ zeroes \in \{x_1, x_2, \dots, x_n\}$$

For some constant $k$, the polynomial with zeroes $\{x_1 + k, x_2 + k, \dots, x_n + k\}$ is given by:

$$f(x-k) = a_n(x-k)^n + a_{n-1}(x-k)^{n-1} + \cdots + a_0$$

$$\alpha + k = t \Rightarrow \alpha = t - k$$

### Example 1.228

The roots of $2x^3 + 4x^2 - 7x + 9$ are $\alpha, \beta$ and $\gamma$.

A. Find the equation with roots $\alpha - 1, \beta - 1$, and $\gamma - 1$.

B. Use the new equation to find $\alpha+\beta+\gamma-3$
C. Use the new equation to find $\alpha\beta+\beta\gamma+\gamma\alpha-2(\alpha+\beta+\gamma)+3$
D. Use the new equation to find $(\alpha\beta-\alpha-\beta+1)(\gamma-1)$

**Part A**

Use a change of variable. Let

$$\alpha-1=t\Rightarrow\alpha=t+1$$

Substitute $x=t+1$:

$$2(t+1)^3+4(t+1)^2-7(t+1)+9=0$$
$$2(t^3+3t^2+3t+1)+4(t^2+2t+1)-7t-7+9=0$$
$$2t^3+6t^2+6t+2+4t^2+8t+4-7t-7+9=0$$
$$2t^3+10t^2+7t+8=0$$

**Part B**

$$\alpha+\beta+\gamma-3=(\alpha-1)+(\beta-1)+(\gamma-1)=Sum\ of\ Roots=-\frac{b}{a}=-\frac{10}{2}=-5$$

**Part C**

$$\alpha\beta+\beta\gamma+\gamma\alpha-2\alpha-2\beta-2\gamma+3$$
$$=\alpha\beta-\alpha-\beta+1+\beta\gamma-\beta-\gamma+1+\gamma\alpha-\gamma-\alpha+1$$
$$=(\alpha-1)(\beta-1)+(\beta-1)(\gamma-1)+(\gamma-1)(\alpha-1)$$
$$=Product\ of\ Roots\ taken\ 2\ at\ a\ time=\frac{c}{a}=\frac{7}{2}$$

**Part D**

$$(\alpha\beta-\alpha-\beta+1)(\gamma-1)$$
$$=(\alpha-1)(\beta-1)(\gamma-1)=Product\ of\ Roots\ 3\ at\ a\ time=-\frac{d}{a}=-\frac{8}{2}=-4$$

## Example 1.229

The roots of $2x^3+4x^2-7x+9$ are $\alpha,\beta$ and $\gamma$.

A. Find the equation with roots $\alpha+2$, $\beta+2$, and $\gamma+2$.

**Part A**

Use a change of variable. Let

$$\alpha+2=t\Rightarrow\alpha=t-2$$

Substitute $x=t-2$:

$$2(t-2)^3+4(t-2)^2-7(t-2)+9=0$$

### 1.230: Roots increased/decreased by $k$

Consider the polynomial:

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0\ with\ zeroes\in\{x_1,x_2,\dots,x_n\}$$

For some constant $k$, the polynomial with zeroes $\{kx_1,kx_2,\dots,kx_n\}$ is given by:

$$f(x-k)=a_n\left(\frac{x}{k}\right)^n+a_{n-1}\left(\frac{x}{k}\right)^{n-1}+\cdots+a_0$$

$$k\alpha=t\Rightarrow\alpha=\frac{t}{k}$$

### 1.231: Reciprocals

Consider the polynomial:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \text{ with zeroes} \in \{x_1, x_2, \dots, x_n\}$$

The polynomial with zeroes $\left\{\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}\right\}$ is given by reversing its coefficients:

$$f\left(\frac{1}{x}\right) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

$$f\left(\frac{1}{x}\right) = a_n\left(\frac{1}{x}\right)^n + a_{n-1}\left(\frac{1}{x}\right)^{n-1} + \cdots + a_0 = 0$$

Multiply the last two parts by $x^n$:

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

## Example 1.232

A. The roots of $2x^3 - 3x^2 + 4x + 7 = 0$ are $\alpha, \beta$ and $\gamma$. Find the equation with roots $2\alpha$, $2\beta$ and $2\gamma$.

B. The roots of $\frac{x^3}{9} - \frac{x^2}{3} + \frac{x}{27} + 4 = 0$ are $\alpha, \beta$ and $\gamma$. Find the equation with roots $\frac{\alpha}{3}, \frac{\beta}{3}$ and $\frac{\gamma}{3}$.

C. The roots of $2x^3 - 3x^2 + 4x + 7 = 0$ are $\alpha, \beta$ and $\gamma$. Find the equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$.

D. The roots of $\frac{x^3}{9} - \frac{x^2}{3} + \frac{x}{27} + 4$ are $\alpha, \beta$ and $\gamma$. Find the equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$.

### Part A

$$2\alpha = t \Rightarrow \alpha = \frac{t}{2}$$

$$2\left(\frac{t}{2}\right)^3 - 3\left(\frac{t}{2}\right)^2 + 4\left(\frac{t}{2}\right) + 7 = 0$$

$$\frac{t^3}{4} - \frac{3t}{4} + 2t + 7 = 0$$

$$t^3 - 3t + 8t + 28 = 0$$

### Part B

$$\frac{\alpha}{3} = t \Rightarrow \alpha = 3t$$

$$\frac{(3t)^3}{9} - \frac{(3t)^2}{3} + \frac{(3t)}{27} + 4 = 0$$

$$3t^3 - 3t^2 + \frac{t}{9} + 4 = 0$$

$$27t^3 - 27t^2 + t + 36 = 0$$

### Part C

$$\frac{1}{\alpha} = t \Rightarrow \alpha = \frac{1}{t}$$

$$2\left(\frac{1}{t}\right)^3 - 3\left(\frac{1}{t}\right)^2 + 4\left(\frac{1}{t}\right) + 7 = 0$$

Multiply by $t^3$:

$$2 - 3t + 4t^2 + 7t^3 = 0$$

$$7t^3 + 4t^2 - 3t + 2 = 0$$

### Part D

## Challenge 1.233

If $\alpha, \beta, \gamma$ are the roots of the equation $x^3 + x^2 - x + 1 = 0$, then show that:

$$\frac{\alpha - 1}{\alpha + 1} = t \Rightarrow \alpha = -\frac{t+1}{t-1}$$

$$\alpha - 1 = t\alpha + t$$

Collate all the $\alpha$ terms on the RHS"

$$-t - 1 = t\alpha - \alpha$$

Factor $\alpha$ on the RHS:

$$-t - 1 = \alpha(t - 1)$$

Divide both sides by $t - 1$:

$$\alpha = \frac{-t-1}{t-1} = -\frac{t+1}{t-1}$$

Find the equation with roots

$$\frac{\alpha - 1}{\alpha + 1}, \quad \frac{\beta - 1}{\beta + 1}, \quad \frac{\gamma - 1}{\gamma + 1}$$

$$\alpha^3+\alpha^2-\alpha+1=0$$

Substitute $\alpha=-\frac{t+1}{t-1}$:

$$\left(-\frac{t+1}{t-1}\right)^3+\left(-\frac{t+1}{t-1}\right)^2-\left(-\frac{t+1}{t-1}\right)+1=0$$

Multiply both sides by $(t-1)^3$:

$$[-(t+1)]^3+[-(t+1)]^2(t-1)-[-(t+1)](t-1)^2+(t-1)^3=0$$

The first and the last terms can be simplified to:

$$(-)(t^3+3t^2+3t+1)+(t^3-3t^2+3t-1)=-6t^2-2$$

The second and the third terms can be simplified to:

$$[(t+1)]^2(t-1)+[(t+1)](t-1)^2$$

Take $t+1$ common:

$$(t+1)[(t+1)(t-1)+(t-1)^2]$$
$$=(t+1)[t^2-1+t^2-2t+1]$$
$$=(t+1)[2t^2-2t]$$
$$=2t^3-2t^2+2t^2-2t$$
$$=2t^3-2t$$

Combine all the terms together:

$$2t^3-2t-6t^2-2=0$$

Divide by 2 and rewrite in standard form:

$$t^3-3t^2-t-1=0$$

Find the value of

$$\frac{\alpha-1}{\alpha+1}+\frac{\beta-1}{\beta+1}+\frac{\gamma-1}{\gamma+1}$$

$$t^3-3t^2-t-1=0$$

We want the sum of the roots, which is:

$$-\frac{b}{a}=-\frac{-3}{1}=3$$

## Challenge 1.234

If $\alpha,\beta,\gamma$ are the roots of the equation $x^3+x^2-x+1=0$, then find

$$\frac{\alpha-1}{\alpha+1}+\frac{\beta-1}{\beta+1}+\frac{\gamma-1}{\gamma+1}$$

$$x^2(x+1)=x-1$$
$$x^2=\frac{x-1}{x+1}$$

$$\frac{\alpha-1}{\alpha+1}+\frac{\beta-1}{\beta+1}+\frac{\gamma-1}{\gamma+1}=\alpha^2+\beta^2+\gamma^2$$

Use the identity $(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ and we need to find:

$$\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)=\left(-\frac{1}{1}\right)^2-2\left(-\frac{1}{1}\right)=1+2=3$$

## H. Descartes' Rule of Signs

### Example 1.235

$$2x^4 - 7x^3 + 5x^2 - 2x - 4$$

$$Total\ No.of\ Roots = 4$$
$$Sign\ Changes = 3$$

Case I: $Complex\ Roots = 0$

$$Max\ No.of\ Positive\ Zeroes = 3$$
$$No.of\ Negative\ Zeroes = 4 - 3 = 1$$

Case II: $Complex\ Roots = 2$

$$No.of\ Positive\ Zeroes = 3 - 2 = 1$$

### Example 1.236

$$3x^5 + 2x^4 - 3x^3 + 2x^2 - x + 7$$

| | Positive | Negative | Complex |
|---|---|---|---|
| | 4 | 1 | 0 |
| | 2 | 1 | 2 |
| | 0 | 1 | 4 |

### Example 1.237

| | Positive | Negative | Complex |
|---|---|---|---|
| | 2 | 2 | 0 |
| | 0 | 2 | 2 |
| | 2 | 0 | 2 |
| | 0 | 0 | 4 |

### Example 1.238

$$3x^5 + 2x^4 - 3x^3 + 2x^2 - x - 7$$

| | Positive | Negative | Complex |
|---|---|---|---|
| | 3 | 2 | 0 |
| | 3 | 0 | 2 |
| | 1 | 2 | 2 |
| | 1 | 0 | 4 |

### Example 1.239

| | Positive | Negative | Complex |
|---|---|---|---|
| | 4 | 2 | 0 |
| | 4 | 0 | 2 |
| | 2 | 2 | 2 |

| | | | |
|---|---|---|---|
| | 2 | 0 | 4 |
| | 0 | 2 | 4 |
| | 0 | 0 | 6 |

## Example 1.240

| | Positive | Negative | Complex |
|---|---|---|---|
| | 3 | 4 | 0 |
| | 3 | 2 | 2 |
| | 1 | 4 | 2 |
| | 3 | 0 | 4 |
| | 1 | 2 | 4 |
| | 1 | 0 | 6 |

# I. Relations/Identities

## Example 1.241

A. $(a+b+c)^2$
B. $(a+b+c+d)^2$

$$(a+b+c)^2=(a+b+c)(a+b+c)=a^2+b^2+c^2+2(ab+ac+bc)$$

| | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a^2$ | $ab$ | $ac$ |
| $b$ | $ba$ | $b^2$ | $bc$ |
| $c$ | $ca$ | $cb$ | $c^2$ |

$$(a+b+c+d)^2=a^2+b^2+c^2+2(ab+ac+bc)$$

## Example 1.242

$$(a_1+a_2+a_3+a_4)^2=a_1^2+a_2^2+a_3^3+a_4^4+a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4$$

$$x\ is\ odd\ multiple\ of\ seven: 7k$$

## 1.243: Square on an expression

$$(a_1+a_2+\cdots+a_n)^2=\sum_{j=1}^{n}\sum_{i=1}^{n}a_ia_j=\sum_{i=1}^{n}a_i^2+\left(2\sum_{1\le i<j\le n}a_ia_j\right)$$

Where the condition that is put on the second summation term is:

$$1\le i<j\le n$$
$$i\ ranges\ from\ 1\ to\ n-1$$
$$j\ ranges\ from\ 2\ to\ n$$

$\sum_{1\le i<j\le n}a_ia_j$ can be expanded as:

$$When\ j=2\Rightarrow i\in\{1\}\Rightarrow a_1a_2$$
$$When\ j=3\Rightarrow i\in\{1,2\}\Rightarrow a_1a_3,a_2a_3$$

$$When\ j = 4 \Rightarrow i \in \{1,2,3\} \Rightarrow a_1a_4, a_2a_4, a_3a_4$$

## Example 1.244

What is the number of terms of

$$(a_1 + a_2 + \cdots + a_n)^2$$

$$n + \binom{n}{2} = n + \frac{n(n-1)}{2} = n + \frac{n^2 - n}{2} = \frac{2n + n^2 - n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2} = \binom{n+1}{2}$$

## Example 1.245

$$(a + b + \cdots + z)^2$$

$$(a + b + \cdots + z)^2 =$$

## 1.246: Sum of Squares

The sum of squares of the roots can be calculated by substituting known values into the formulas below:

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc)$$
$$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd)$$

$$\Bigg(\underbrace{a + b + c}_{Sum\ of\ Roots}\Bigg)^2 = a^2 + b^2 + c^2 + 2\Bigg(\underbrace{ab + ac + bc}_{\substack{Sum\ of\ Roots \\ taken\ 2\ at\ a\ time}}\Bigg)$$

## Example 1.247

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{bcd + acd + abd + abc}{abcd}$$

## Example 1.248

Calculate $p(1)$ if $p(x)$ is a monic fourth degree polynomial with $y$ −intercept −2 such that the:

- sum of squares of the roots is two more than the sum of the roots
- sum of the reciprocals of the roots is four more than the sum of the roots
- sum of the roots, the sum of the squares of the roots, and the sum of the reciprocals of the roots all add up to 15.

Note: Monic means that the leading coefficient is 1.

The general fourth degree polynomial is:

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

Since the polynomial is monic, $a = 1$ and since the $y - intercept = -2 \Rightarrow e = -2$

$$x^4 + bx^3 + cx^2 + dx - 2 = 0$$

$$S_1 = Sum\ of\ the\ roots$$
$$S_2 = Sum\ of\ squares\ of\ the\ roots$$
$$S_3 = Sum\ of\ reciprocals\ of\ the\ roots$$

$$S_1 + S_2 + S_3 = 15$$
$$S_1 + (S_1 + 2) + (S_1 + 4) = 15$$
$$3S_1 = 9$$
$$S_1 = 3$$

Sum of roots is 3.

$$Sum\ of\ roots = \ -b \ = 3 \Rightarrow b = -3$$

**Step II**

$$(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd)$$
$$S_1^2 = S_2 + (Roots\ taken\ two\ at\ a\ time)$$

Substitute $Roots\ taken\ two\ at\ a\ time = \frac{c}{a} = c$

$$3^2 = 5 + c$$
$$c = 2$$

**Step III**

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{bcd + acd + abd + abc}{abcd}$$

$$Sum\ of\ reciprocal\ of\ roots = Sum\ of\ Roots + 4 = 3 + 4 = 7$$
$$\frac{Roots\ three\ at\ a\ time}{Product\ of\ Roots} = \frac{-\frac{d}{a}}{\frac{e}{a}} = \frac{-d}{e} = \frac{-d}{-2} = \frac{d}{2}$$
$$\frac{d}{2} = 7 \Rightarrow d = 14$$

$$x^4 - 3x^3 + 2x^2 + 14x - 2 = 0$$
$$p(1) = 1 - 3 + 2 + 14 - 2 = 12$$

## Example 1.249

Given that $x^4 - x^2 + 2x + 5 = 0$, determine the:

A. Sum of reciprocals of squares of the roots
B. Sum of fourth powers of the roots

The given equation can be written in terms of its roots $\alpha, \beta, \gamma, \delta$ as

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = 0$$
$$x \in \{\alpha, \beta, \gamma, \delta\}$$

Consider the equation with $\alpha^2, \beta^2, \gamma^2, \delta^2$

$$(y-\alpha^2)(y-\beta^2)(y-\gamma^2)(y-\delta^2) = 0$$
$$y \in \{\alpha^2, \beta^2, \gamma^2, \delta^2\}$$

For all values of $x$, to meet the root condition:

$$x^2 = y \Rightarrow x = \sqrt{y}$$

Substitute the above into the given equation:

$$y^2 - y + 2\sqrt{y} + 5 = 0$$

Isolate $2\sqrt{y}$ on the LHS:

$$2\sqrt{y} = -y^2 + y - 5$$

Square both sides:

$$4y = y^4 - y^3 + 5y^2 - y^3 + y^2 - 5y + 5y^2 - 5y + 25$$
$$y^4 - 2y^3 + 11y^2 - 14y + 25 = 0$$

Sum of reciprocals of squares

$$= \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} = \frac{\beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2 + \alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2}{\alpha^2\beta^2\gamma^2\delta^2} = \frac{-\frac{d}{a}}{\frac{e}{a}} = \frac{-d}{e} = \frac{-(-14)}{25} = \frac{14}{25}$$

Sum of fourth powers

$$(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)^2 = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + 2(\alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2)$$
$$2^2 = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + 2(11)$$
$$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = 4 - 22 = -18$$

# 1.8 Identities

## A. Cube of a Sum/Difference

### 1.250: Cube of a Sum/Difference

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a+b)$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 + 3ab(-a+b)$$

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)(a-b) = a^2 - b^2$$

### Example 1.251

Expand:

A. $(x+y)^3$
B. $(j-k)^3$
C. $\left(\frac{1}{2}p + \frac{1}{3}q\right)^3$
D. $\left(\frac{2}{3}m + \frac{3}{2}n\right)^3$
E. $\left(\frac{1}{5}d + 5e\right)^3$
F. $(0.3x - 0.2e)^3$
G. $\left(\frac{2}{3}x^2 - \frac{3}{4x}\right)^3$

#### Parts A-D

$$x^3 + 3x^2y + 3xy^2 + y^3$$
$$j^3 - 3j^2k + 3jk^2 - k^3$$
$$\frac{1}{8}p^3 + 3\left(\frac{1}{4}p^2\right)\left(\frac{1}{3}q\right) + 3\left(\frac{1}{2}p\right)\left(\frac{1}{9}q^2\right) + \frac{1}{27}q^3 = \frac{1}{8}p^3 + \frac{1}{4}p^2q + \frac{1}{6}pq^2 + \frac{1}{27}q^3$$
$$\frac{8}{27}m^3 + 3\left(\frac{4}{9}m^2\right)\left(\frac{3}{2}n\right) + 3\left(\frac{2}{3}m\right)\left(\frac{9}{4}n^2\right) + \frac{27}{8}n^3 = \frac{8}{27}m^3 + 2m^2n + \frac{9}{2}mn^2 + \frac{27}{8}n^3$$
$$\frac{1}{125}d^3 + 3\left(\frac{1}{25}d^2\right)(5e) + 3\left(\frac{1}{5}d\right)(25e^2) + 125e^3 = \frac{1}{125}d^3 + \frac{3}{5}d^2e + 15de^2 + 125e^3$$

### Part E

$$\left(\frac{3}{10}\right)^2=\frac{9}{100}=0.09,\qquad\left(\frac{3}{10}\right)^3=\frac{27}{1000}=0.027$$
$$\left(\frac{2}{10}\right)^2=\frac{4}{100}=0.04,\qquad\left(\frac{2}{10}\right)^3=\frac{8}{1000}=0.008$$
$$0.027x^3-3(0.09x^2)(0.2e)+3(0.3x)(0.04e^2)-0.008e^3$$
$$=0.027x^3-0.054x^2e+0.036xe^2-0.008e^3$$

### Part F

$$\frac{8}{27}x^6-3\left(\frac{4}{9}x^4\right)\left(\frac{3}{4x}\right)+3\left(\frac{2}{3}x^2\right)\left(\frac{9}{16x^2}\right)-\frac{27}{64x^3}$$
$$=\frac{8}{27}x^6-x^3+\frac{9}{8}-\frac{27}{64x^3}$$

## Example 1.252: Radicals

A. Expand $\left(\sqrt{a}+\sqrt{b}\right)^3$

B. Given that $\sqrt{a}+\sqrt{b}=1$, find the value of $\sqrt{ab}$ in terms of $a$ and $b$.

### Part A

$$a^{\frac{3}{2}}+3a\sqrt{b}+3\sqrt{a}b+b^{\frac{3}{2}}=a^{\frac{3}{2}}+b^{\frac{3}{2}}+3\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)$$

### Part B

$$\left(\sqrt{a}+\sqrt{b}\right)^3=1^3$$
$$a^{\frac{3}{2}}+b^{\frac{3}{2}}+3\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)=1$$
$$a^{\frac{3}{2}}+b^{\frac{3}{2}}+3\sqrt{ab}=1$$
$$3\sqrt{ab}=1-a^{\frac{3}{2}}-b^{\frac{3}{2}}$$
$$\sqrt{ab}=\frac{1-a^{\frac{3}{2}}-b^{\frac{3}{2}}}{3}$$

## Example 1.253: Simple Applications

A. Evaluate $\frac{1.331+0.363+0.0033+0.001}{1.44}$

B. Solve $x^3+3x^2+3x+1=0$

C. Find the ratio between $a$ and $b$ given that $\frac{8}{27}a^3+\frac{2}{9}a^2b+\frac{1}{6}ab+\frac{1}{8}b^3=0$

### Part A

$$\frac{1.331+0.363+0.0033+0.001}{1.44}=\frac{(1.1+0.1)^3}{1.2^2}=\frac{1.2^3}{1.2^2}=1.2$$

### Part B

$$(x+1)^3=0\Rightarrow x+1=0\Rightarrow x=-1$$

### Part C

$$\left(\frac{2}{3}a+\frac{1}{2}b\right)^3=0\Rightarrow\frac{2}{3}a+\frac{1}{2}b=0\Rightarrow\frac{2}{3}a=-\frac{1}{2}b\Rightarrow\frac{a}{b}=-\frac{3}{4}$$

## Example 1.254

$a+2b=4$, then find the value of $a^3+8b^3+30ab$

Expand $(a+2b)^3$:

$$=a^3+3(a^2)(2b)+3(a)(2b)^2+(2b)^3$$
$$a^3+6a^2b+12ab^2+8b^3$$
$$a^3+6ab(a+2b)+8b^3$$
$$a^3+30ab(5)+8b^3=(a+2b)^3=4^3=64$$

## B. Sum and Difference of Cubes

### 1.255: Sum and Difference of Cubes

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

$$(a+b)(a^2-ab+b^2)=a^3-a^2b+ab^2+a^2b-ab^2+b^3=a^3+b^3$$
$$(a-b)(a^2+ab+b^2)=a^3+a^2b+ab^2-a^2b-ab^2-b^3=a^3-b^3$$

### Example 1.256

Factorize

A. $x^3+y^3$
B. $x^3+729$
C. $40x^3+5$
D. $8p^3+27q^3$
E. $64w^3+125x^3$
F. $\frac{64}{729}r^6+\frac{216}{125}s^9$
G. $3x^3+5y^3$
H. $27a^3-8b^6$
I. $\frac{1}{8}x^3+\frac{125}{729}y^3$
J. $9a^3-27b^6$
K. $0.3^3+0.4^3$
L. $0.4^3+0.6^3$

$$x^3+y^3=(x+y)(x^2-xy+y^2)$$
$$x^3+729=(x+9)(x^2-9x+81)$$

$$40x^3+5=5(8x^3+1)=5(2x+1)(4x^2-2x+1)$$

$$a^3=8p^3\Rightarrow a=2p$$
$$b^3=27q^3\Rightarrow b=3q$$
$$(2p)^3+(3q)^3=(2p+3q)(4p^2-6pq+9q^2)$$

$$(4w)^3+(5x)^3=(4w+5x)(16w^2-20xw+25x^2)$$

$$\left(\frac{4}{9}r^2\right)^3+\left(\frac{6}{5}s^3\right)^3=\left(\frac{4}{9}r^2+\frac{6}{5}s^3\right)\left(\frac{16}{81}r^4-\frac{8}{15}r^2s^3+\frac{36}{25}s^6\right)$$

$$\left(\sqrt[3]{3}x\right)^3+\left(\sqrt[3]{5}y\right)^3=\left(\sqrt[3]{3}x+\sqrt[3]{5}y\right)\left(3^{\frac{2}{3}}x^2-\sqrt[3]{15}xy+5^{\frac{2}{3}}y^2\right)$$

$$27a^3-8b^6=(3a-2b^2)(9a^2+6ab^2+4b^4)$$
$$\frac{1}{8}x^3+\frac{125}{729}y^3=\left(\frac{1}{2}x+\frac{5}{9}y\right)\left(\frac{1}{4}x^2-\frac{5}{18}xy+\frac{25}{81}y^2\right)$$
$$9a^3-27b^6=9(a^3-3b^6)=9\left(a-\sqrt[3]{3}b^2\right)\left(a^2+\sqrt[3]{3}ab^2+3^{\frac{2}{3}}b^4\right)$$

$$0.3^3+0.4^3=(0.3+0.4)(0.3^2-0.12+0.4^2)$$
$$0.4^3+0.6^3=(0.4+0.6)(0.4^2-2.4+0.6^2)=(1)(0.16-2.4+0.36)=0.52-2.4=-1.88$$

### Example 1.257

A. $\frac{0.2^3+0.3^3}{0.5}$

B. $\frac{0.6^3+0.7^3}{1.3}$

**Part A**

$$\frac{0.2^3+0.3^3}{0.5}=\frac{(0.2+0.3)(0.2^2-0.06+0.3^2)}{0.5}=\frac{(0.5)(0.04-0.06+0.09)}{0.5}=0.07$$
$$\frac{0.2^3+0.3^3}{0.5}=\frac{0.008+0.027}{0.5}=\frac{0.035}{0.5}=0.07$$

**Part B**

$$\frac{0.6^3+0.7^3}{1.3}=\frac{(0.6+0.7)(0.6^2-0.42+0.7^2)}{1.3}=\frac{(1.3)(0.36-0.42+0.49)}{1.3}=0.43$$
$$\frac{0.6^3+0.7^3}{1.3}=\frac{0.216+0.343}{1.3}=\frac{0.559}{1.3}=0.43$$

### Example 1.258: Back Calculations

Suppose $f(x)=x^2$, and $g(x)$ is a polynomial such that $f(g(x))=4x^2+4x+1$. What are the possible values of $g(x)$? (MathCounts 1993 Chapter Countdown)

$$[g(x)]^2=4x^2+4x+1$$

Take square roots both sides:

$$g(x)=\pm\sqrt{4x^2+4x+1}=\pm\sqrt{(2x+1)^2}=\pm(2x+1)$$

### 1.259

$$a^3+\frac{1}{a^3}=\left(a+\frac{1}{a}\right)^3-3\left(a+\frac{1}{a}\right)$$
$$a^3-\frac{1}{a^3}=\left(a-\frac{1}{a}\right)^3-3\left(a-\frac{1}{a}\right)$$

### Example 1.260

Given that $a^2+\frac{1}{a^2}=18$, find $a^3-\frac{1}{a^3}$

$$a^2 + \frac{1}{a^2} = 18$$

Add 2 to both sides:

$$a^2 - 2 + \frac{1}{a^2} = 16$$

$$\left(a - \frac{1}{a}\right)^2 = 16$$

$$a - \frac{1}{a} = \pm 4$$

If $a = 4$:

$$a^3 - \frac{1}{a^3} = \left(a - \frac{1}{a}\right)^3 - 3\left(a - \frac{1}{a}\right) = 4^3 - 3(4) = 64 - 12 = 52$$

If $a = -4$:

$$a^3 - \frac{1}{a^3} = \left(a - \frac{1}{a}\right)^3 - 3\left(a - \frac{1}{a}\right) = (-4)^3 - 3(-4) = -64 + 12 = -52$$

## A. Perfect Square of a Trinomial

### 1.261: Perfect Square of a Trinomial

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + cb)$$

Group the first two terms:

$$\big((a + b) + c\big)\big((a + b) + c\big)$$

Apply $(x + y)^2 = x^2 + 2xy + y^2$:

$$= (a + b)^2 + 2(c)(a + b) + c^2$$

Expand:

$$= a^2 + 2ab + b^2 + 2ac + 2cb + c^2$$

Rearrange and Factor:

$$= a^2 + b^2 + c^2 + 2(ab + ac + cb)$$

### Example 1.262

Expand:

A. $(a + 2b + 3c)^2$

B. $\left(\frac{1}{2}x + \frac{2}{3}y - \frac{3}{4}z\right)^2$

C. $(0.1p - 0.2q + 0.3r)^2$

$$(a + 2b + 3c)^2 = a^2 + 4b^2 + 9c^2 + 2(2ab + 3ac + 6bc) = a^2 + 4b^2 + 9c^2 + 4ab + 6ac + 12bc$$

$$\left(\frac{1}{2}x + \frac{2}{3}y - \frac{3}{4}z\right)^2 = \frac{1}{4}x^2 + \frac{4}{9}y^2 + \frac{9}{16}z^2 + 2\left(\frac{1}{3}xy - \frac{3}{8}xz - \frac{1}{2}yz\right)$$

$$(0.1p - 0.2q + 0.3r)^2 = 0.01x^2 + 0.04q^2 + 0.09r^2 + 2(-0.02pq + 0.03pr - 0.06qr)$$

### Example 1.263

Given that $\frac{1}{2}x + \frac{2}{3}y - \frac{3}{4}z = 0$, we can determine the value of $36x^2 + 64y^2 + 81z^2$ in the form $axy + byz + cyz$. Find $a + b + c$.

Square both sides:

$$\frac{1}{4}x^2 + \frac{4}{9}y^2 + \frac{9}{16}z^2 + 2\left(\frac{1}{3}xy - \frac{3}{8}xz - \frac{1}{2}yz\right) = 0$$

Simplify:

$$\frac{1}{4}x^2+\frac{4}{9}y^2+\frac{9}{16}z^2+\frac{2}{3}xy-\frac{1}{4}xz-yz=0$$

Make denominators common and add:

$$\frac{36x^2+64y^2+81z^2+96xy-36xz-144yz}{144}=0$$

Multiply both sides by 144:

$$36x^2+64y^2+81z^2+96xy-36xz-144yz=0$$
$$36x^2+64y^2+81z^2=-96xy+36xz+144yz$$

$$a+b+c=-96+36+144=84$$

## 1.264: Rearranging Perfect Square of a Trinomial

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+ac+cb)$$

The identity for perfect square can be rearranged usefully to find the sum of the squares of three numbers. For example, it is useful when doing the sum and product of roots of cubic equations.

### Example 1.265

It is customary to assign the variables $\alpha,\beta,\gamma$ to the three roots of a cubic equation $ax^3+bx^2+cx+d=0$.

A. Substitute $a=\alpha, b=\beta, c=\gamma$ in the identity $a^2+b^2+c^2=(a+b+c)^2-2(ab+ac+cb)$
B. For a cubic equation, if the sum of the roots is 4, and the sum of the product of the roots, when taken two at a time, is 3, then find the sum of the squares of the roots.
C. For a cubic equation, if the sum of the roots is 2, and the sum of the product of the roots, when taken two at a time, is 4, then find the sum of the squares of the roots.
D. Find $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}$ if $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=4,\ \frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=3$
E. The sum of the reciprocals of the roots of a cubic equation is 7. The sum of the reciprocals of the product of the roots (taken 2 at a time) is 4. Find the sum of the reciprocals of the squares of the roots.

**Part A**

$$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$$

**Part B**

$$\alpha+\beta+\gamma=4$$
$$\alpha\beta+\beta\gamma+\gamma\alpha=3$$
$$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)=4^2-2(3)=16-6=10$$

**Part C**

$$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)=2^2-2(4)=4-8=-4$$
$$\alpha^2+\beta^2+\gamma^2=-4$$

Note that $\alpha^2+\beta^2+\gamma^2$ is negative, but this is possible if you consider complex solutions as well.

**Part D**

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2-2\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)=4^2-2(3)=16-6=10$$

**Part E**

Let the roots be

$$\alpha,\beta,\gamma \Rightarrow \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=7, \qquad \frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\beta\gamma}=4$$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)^2-2\left(\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\beta\gamma}\right)=7^2-2(4)=49-8=41$$

## B. Applications

There are many different applications of algebraic identities. We look at a few.

### Example 1.266: Volume

Find the value of x in meters, if a cubical tank with a capacity of 8000 liters has its volume given by the expression:

A. $x^3+3x^2+3x+1\ m^3$
B. $x^3+3x^2+3x+1\ cm^3$

$$1\ litre = 1000\ ml = 1000\ cm^3$$
$$8000\ liters = 8{,}000{,}000\ cm^3 = 8\ m^3$$
$$1\ m^3 = (1\ m)^3 = (100\ cm)^3 = 1{,}000{,}000\ cm^3$$

**Part A**

$$x^3+3x^2+3x+1\ m^3 = 8\ m^3$$
$$(x+1)^3\ m^3 = 8\ m^3$$
$$(x+1)\ m = 2\ m$$
$$x = 1\ m$$

**Part B**

$$x^3+3x^2+3x+1\ cm^3 = 8{,}000{,}000\ cm^3$$
$$(x+1)^3\ cm^3 = (200\ cm)^3$$
$$(x+1)\ cm = 200\ cm$$
$$x = 199\ cm = 1.99\ m$$

### Example 1.267: Surface Area as Limiting Value of Volume

A. Determine the volume of the space between two concentric spheres and factor it.
B. What is the significance of the trinomial and the binomial?
C. What happens to the above expression as the thickness becomes close to zero.

**Part A**

Let:

$$Outside\ sphere\ have\ radius\ R, \qquad Inside\ sphere\ radius\ r$$

$$Volume\ of\ Space = \frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(R^3-r^3) = \frac{4}{3}\pi(R^2+Rr+r^2)(R-r)$$

**Part B**

$$Trinomial \to Surface\ Area$$
$$Binomial \to Thickness\ of\ the\ space$$
$$\frac{4}{3}\pi\underbrace{(R^2+Rr+r^2)}_{Surface\ Area}\underbrace{(R-r)}_{Thickness}$$

**Part C**

Substitute $R=r+a$ in $\frac{4}{3}\pi(R^2+Rr+r^2)(R-r)$:

$$\frac{4}{3}\pi((r+a)^2+(r+a)r+r^2)(r+a-r)$$

Expand:

$$\frac{4}{3}\pi[(r^2+2ar+a^2)+r^2+ar+r^2](a)$$

Simplify:

$$\frac{4}{3}\pi\underbrace{[3r^2+3ar+a^2]}_{Surface\ Area}\underbrace{(a)}_{Thickness}$$

As $a$ becomes very small, the second expression becomes very close to the expression for surface area.
Further, as $a$ becomes very small

$$3ar \to very\ small$$
$$a^2 \to very\ small$$
$$\frac{4}{3}\pi\underbrace{[3r^2]}_{Surface\ Area}\underbrace{(a)}_{Thickness}=\underbrace{[4\pi r^2]}_{Surface\ Area}\underbrace{(a)}_{Thickness}$$

## Example 1.268: Remainders / Sequences

Consider the sequence:

$$7,63,215,511,999$$

A. Find the $n^{th}$ term.
B. Find the 218th term
C. Find the remainder when the 218th term is divided by 435.

$$T_1=7=8-1=2^3-1$$
$$T_2=63=64-1=4^3-1$$
$$T_3=215=216-1=6^3-1$$
$$T_4=511=512-1=8^3-1$$
$$T_n=(2n)^3-1$$

$$T_{218}=(2\times 218)^3-1=436^3-1$$

$$\frac{T_{218}}{435}=\frac{436^3-1}{435}=\frac{(435)(436^2+436+1)}{435}=436^2+436+1\Rightarrow Remainder=0$$

## Example 1.269: Radical Equations

Solve

$$\sqrt[3]{2x-1}+\sqrt[3]{x-1}=1$$

$$a^3+b^3+3ab(a+b)$$

Cube the given equation:

$$2x-1+x-1+3\sqrt[3]{(2x-1)(x-1)}\left(\sqrt[3]{2x-1}+\sqrt[3]{x-1}\right)=1$$
$$3x-2+3\sqrt[3]{(2x-1)(x-1)}=1$$
$$3\sqrt[3]{(2x-1)(x-1)}=3-3x$$
$$\sqrt[3]{(2x-1)(x-1)}=1-x$$

Cube the above equation again:

$$(2x-1)(x-1)=(1-x)^3$$
$$(2x-1)(-1)(1-x)=(1-x)^3$$

### Case I

$$1-x=0\Rightarrow x=1$$

Try $x = 1$ in the original equation:

$$\sqrt[3]{2(1) - 1} + \sqrt[3]{1 - 1} = 1 - 0 = 1$$

**Case II**

If $1 - x \neq 0$, then divide both sides by $1 - x$:

$$(1 - 2x) = (1 - x)^2$$
$$1 - 2x = 1 - 2x + x^2$$
$$0 = x^2$$
$$x = 0$$

Try $x = 0$ in the original equation:

$$\sqrt[3]{2(0) - 1} + \sqrt[3]{0 - 1} = Complex\ Numbers \Rightarrow Not\ Valid$$

**1.270: Sophie Germain Identity**

$$a^4 + 4b^4 = [(a + b)^2 + b^2][(a - b)^2 + b^2]$$

Add and subtract $4a^2b^2$:

$$(a^2)^2 + (2b^2)^2 + 4a^2b^2 - 4a^2b^2$$
$$= [\quad][\quad](a^2)^2 + (2b^2)^2 + 4a^2b^2 - 4a^2b^2$$

## 1.9 AMC Questions

**Example 1.271**

For certain real numbers $a$, $b$, and $c$, the polynomial $g(x) = x^3 + ax^2 + x + 10$ has three distinct roots, and each root of $g(x)$ is also a root of the polynomial $f(x) = x^4 + x^3 + bx^2 + 100x + c$. What is $f(1)$? (AMC 10A 2017/24; 12A 2017/23)

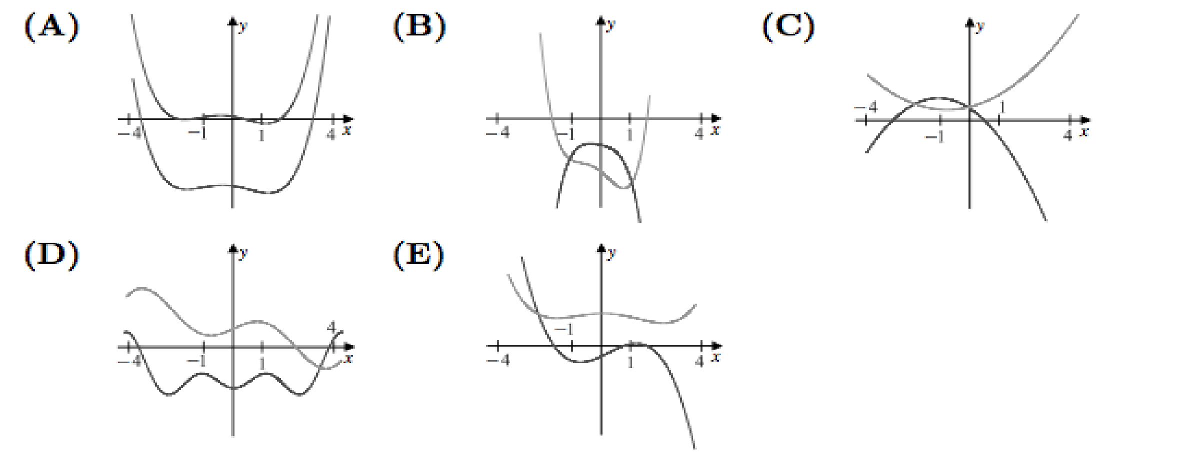
Method I: Polynomial Division

Method II: Multiplication followed by method of undetermined coefficients

Method III: Root Relation

**Example 1.272**

The nonzero coefficients of a polynomial $P$ with real coefficients are all replaced by their mean to form a polynomial $Q$. Which of the following could be a graph of $y = P(x)$ and $y = Q(x)$ over the interval $-4 \leq x \leq 4$? (AMC 12A 2002/25)




### Example 1.273
The graph of the polynomial $P(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$ has five distinct $x$-intercepts, one of which is at $(0,0)$. Which of the following coefficients cannot be zero? (AMC 12A 2003/21)

### Example 1.274
Let $P(x) = (x-1)(x-2)(x-3)$. For how many polynomials $Q(x)$ does there exist a polynomial $R(x)$ of degree 3 such that $P\big(Q(x)\big) = P(x) \cdot R(x)$? (AMC 12A 2005/24)

### Example 1.275
There is a smallest positive real number a such that there exists a positive real number b such that all the roots of the polynomial $x^3 - ax^2 + bx - a$ are real. In fact, for this value of $a$, the value of b is unique. What is the value of $b$? (AMC 12A 2016/24)

### Example 1.276
A set S is constructed as follows. To begin, $S = \{0,10\}$. Repeatedly, as long as possible, if x is an integer root of some polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ for some $n \geq 1$, all of whose coefficients $a_i$ are elements of S, then x is put into S. When no more elements can be added to S, how many elements does S have?
(AMC 12A 2017/21)

### Example 1.277
Which of the following polynomials has the greatest real root? (AMC 12A 2018/21)
$(A) x^{19} + 2018x^{11} + 1$
$(B) x^{17} + 2018x^{11} + 1$
$(C) x^{19} + 2018x^{13} + 1$
$(D) x^{17} + 2018x^{13} + 1$
$(E) 2019x + 2018$

### Example 1.278
Let $s_k$ denote the sum of the $k$th powers of the roots of the polynomial $x^3 - 5x^2 + 8x - 13$. In particular, $s_0 = 3$, $s_1 = 5$, and $s_2 = 9$. Let $a$, $b$, and $c$ be real numbers such that $s_{k+1} = as_k + bs_{k-1} + cs_{k-2}$ for $k = 2, 3, ....$ What is

$a + b + c$? (AMC 12A 2019/17)

### Example 1.279

The polynomial $x^3 - 2004x^2 + mx + n$ has integer coefficients and three distinct positive zeros. Exactly one of these is an integer, and it is the sum of the other two. How many values of $n$ are possible? (AMC 12B 2004/23)

### Example 1.280

Let $a > 0$, and let $P(x)$ be a polynomial with integer coefficients such that
$P(1) = P(3) = P(5) = P(7) = a$, and
$P(2) = P(4) = P(6) = P(8) = -a$.
What is the smallest possible value of $a$? (AMC 10A 2010/21)

### Example 1.281

Let $P$ be a cubic polynomial with $P(0) = k$, $P(1) = 2k$, and $P(-1) = 3k$. What is $P(2) + P(-2)$ ? (AMC 8 2014/16)

### Example 1.282

The graph of $y = f(x)$, where $f(x)$ is a polynomial of degree 3, contains points $A(2,4)$, $B(3,9)$, and $C(4,16)$. Lines $AB$, $AC$, and $BC$ intersect the graph again at points $D$, $E$, and $F$, respectively, and the sum of the $x$-coordinates of $D$, $E$, and $F$ is 24. What is $f(0)$? (AMC 8 2017/23)

### Example 1.283

Consider polynomials $P(x)$ of degree at most 3, each of whose coefficients is an element of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many such polynomials satisfy $P(-1) = -9$? (AMC 8 2018/22)

### 284 Examples