Basic Number Theory

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Aziz Manva

azizmanva@gmail.com

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Part I: Primes Numbers, Factors and Multiples

1. Types of Numbers

1.1 Prime Numbers

Prime Number

Numbers which have exactly two factors are called prime numbers.

- Numbers which are not prime are called composite numbers.
- Note that 1 is neither prime nor composite. 1 is the only number which is neither prime nor composite.

We will talk about prime numbers only in the context of numbers such as:

$$1,2,3,4\ldots. \Rightarrow Natural\ Numbers$$

We do not say that zero or a fraction is prime or composite.

Example 1.1

Find the factors of 2. Hence, decide if 2 is prime.

$$Factors\ of\ 2 = \{1,2\} \Rightarrow 2\ Factors \Rightarrow \boldsymbol{Prime}$$

A number which has *exactly* 2 factors is prime.
2 has 2 factors. Hence, it is prime.

Example 1.2

Find the factors of 4. Hence, decide if 4 is prime.

$$Factors\ of\ 4 = \{1,2,4\} \Rightarrow 3\ Factors \Rightarrow \boldsymbol{Not\ Prime}$$

A number which has *exactly* 2 factors is prime.
4 has 3 factors. Hence, it is not prime.

Example 1.3

Find the factors of 6. Hence, decide if 6 is prime.

$$Factors\ of\ 6 = \{1,2,3,6\} \Rightarrow 4\ Factors \Rightarrow \boldsymbol{Not\ Prime}$$

A number which has *exactly* 2 factors is prime.
6 has 3 factors. Hence, it is not prime.

Example 1.4

Find the factors of 5. Hence, decide if 5 is prime.

$$Factors\ of\ 5 = \{1,5\} \Rightarrow 2\ Factors \Rightarrow \boldsymbol{Prime}$$

A number which has *exactly* 2 factors is prime.
5 has 2 factors. Hence, it is prime.

Example 1.5

Find the factors of 8. Hence, decide if 8 is prime.

$$Factors\ of\ 8 = \{1,2,4,8\} \Rightarrow 4\ Factors \Rightarrow \textbf{Not Prime}$$

A number which has *exactly* 2 factors is prime.
8 has 4 factors. Hence, it is not prime.

Example 1.6
Find the factors of 11. Hence, decide if 11 is prime.

$$Factors\ of\ 11 = \{1,11\} \Rightarrow 2\ Factors \Rightarrow \textbf{Prime}$$

A number which has *exactly* 2 factors is prime.
11 has 2 factors. Hence, it is prime.

Example 1.7
Find the factors of 15. Hence, decide if 15 is prime.

$$Factors\ of\ 15 = \{1,3,5,15\} \Rightarrow 2\ Factors \Rightarrow \textbf{Prime}$$

A number which has *exactly* 2 factors is prime.
11 has 2 factors. Hence, it is prime.

Example 1.8
Find the factors of 91. Hence, decide if 91 is prime.

$$Factors\ of\ 91 = \{1,7,13,91\} \Rightarrow\ 4\ Factors \Rightarrow \textbf{Not Prime}$$

A number which has *exactly* 2 factors is prime.
91 has 4 factors. Hence, it is not prime.

Properties and Table of Prime Numbers
The smallest prime number is 2.
One is not a prime number.

Prime Numbers from 2 to 100									
2	11	23	31	41	53	61	71	83	97
3	13	29	37	43	59	67	73	89	
5	17			47			79		
7	19								

Tips to help you differentiate a prime number from a composite number
- Not every number that ends in 1 is prime.
- Be careful of multiples of 3. Learn the test of divisibility of 3, and use it.

Sieve of Eratosthenes[1]
Eratosthenes was a Greek Mathematician, who wrote about about a way to find the primes till any number. We

[1] Click for an animated version of the sieve.

will follow his example and find all primes less than 49.
The method goes as follows:

Remove all the numbers which are multiples of any number

- To get primes till 48, remove all the multiples of primes $\boldsymbol{P} = \{\mathbf{2}, \mathbf{3}, \mathbf{5}\}$

Multiples of 2							Multiples of 3							Multiples of 5					
1	2	3	4	5	6		1	2	3	4	5	6		1	2	3	4	5	6
7	8	9	10	11	12		7	8	9	10	11	12		7	8	9	10	11	12
13	14	15	16	17	18		13	14	15	16	17	18		13	14	15	16	17	18
19	20	21	22	23	24		19	20	21	22	23	24		19	20	21	22	23	24
25	26	27	28	29	30		25	26	27	28	29	30		25	26	27	28	29	30
31	32	33	34	35	36		31	32	33	34	35	36		31	32	33	34	35	36
37	38	39	40	41	42		37	38	39	40	41	42		37	38	39	40	41	42
43	44	45	46	47	48		43	44	45	46	47	48		43	44	45	46	47	48

Problem Solving

Example 1.9: Before and After

A. Find the prime number that comes just before 100.
B. Find the prime number that comes after 50.

97
53

Example 1.10: Largest and Smallest

A. Find the smallest two-digit prime.
B. Find the largest two-digit prime.

11
97

Example 1.11: Listing

A. List the primes less than 100 that have units digit one.
B. Which are the primes less than 100 that have tens digit seven?

11, 31, 41, 61, 71
71, 73, 79

Example 1.12: Counting

A. How many prime numbers are less than 100?
B. How many two-digit prime numbers are there?
C. How many two-digit prime numbers have a units digit of 2?
D. How many two-digit prime numbers have a units digit of one?
E. How many prime numbers less than 100 end in 9?

$$25 \textit{ Prime Numbers less than } 100$$
$$25 - 4 = 21 \textit{ Two Digit Prime Numbers}$$
$$0: \textit{All two digits numbers that end in two are composite}$$
$$5 \textit{ Numbers}: 11, 31, 41, 61, 71$$
$$5 \textit{ Numbers}: 19, 29, 59, 79, 89$$

Example 1.13: Sum and Difference

A. What is the sum of all single digit primes?
B. What is the difference between the largest two-digit prime, and the smallest two-digit prime?
C. What is the sum of the smallest two-digit prime and the largest two-digit prime?

$$2 + 3 + 5 + 7 = 17$$
$$97 - 11 = 86$$
$$97 + 11 = 108$$

Example 1.14: Factoring

A. Write 91 as the product of two primes.
B. Write 51 as the product of two primes.
C. Write 30 as the product of three distinct primes.

$$91 = 7 \times 13$$
$$51 = 3 \times 17$$
$$30 = 2 \times 3 \times 5$$

Example 1.15: Even and Odd

A. How many even primes are there?
B. What is the largest even prime number?
C. How many primes are multiples of seven?
D. The sum of two primes is odd. Then, one of the primes must be?

The first part is based on a very important property, which is used a lot in more difficult question as well:

$$2 \textit{ is the only even prime}$$

Since 2 is the only prime number, it is also the largest even prime number:

$$2 \textit{ is the largest even prime number}$$

Consider the multiples o f 7: $(7, 14, 21, \ldots.)$. Out of this, only one number is prime, which is

$$7 \textit{ itself}$$
$$\textit{One of the primes must be } 2$$

The sum of two odd numbers is even.
The sum of two even numbers is also even.
Hence, if the sum of two numbers is odd, then one of the numbers must be even.
And since there is only one even prime, that number must be 2.

Twin Primes

Twin primes are odd numbers that have a difference of two. The first few twin primes are:

$$3 \;\&\; 5, 5 \;\&\; 7, \ldots$$

Example 1.16

The number of people at a party is one less than a prime, and also one more than a prime. If the number of people is less than hundred, find the number of people who could be at the party?

The possible values are

$$4,6,12,18,30,42,60,72$$

Prime Gap

The difference between two primes is called a prime gap. The prime gap is odd only if one of the primes is 2.

Example 1.17

The difference between two primes is called a prime gap. For this question, consider numbers between 20 and 90 only. Find the prime gap between the largest prime ending in 7, and the smallest prime ending in 7.

$$67 - 37 = 30$$

Example 1.18

The number of students in the fifth grade at Mystic Falls High is a prime number. The number of students in the sixth grade at Mystic Falls High is also a prime number, and four more than the number in fifth grade. Mystic Falls does not allow more than a hundred students in a grade. What are the possible values of the students?

The prime numbers that will work are:

$$3 \ \& \ 7$$
$$7 \ \& \ 11$$
$$13 \ \& \ 17$$
$$19 \ \& \ 23$$
$$37 \ \& \ 41$$

1.2 Consecutive Numbers

Consecutive Numbers

Numbers that keep increasing by one are consecutive numbers.

Example 1.19

Find the next three consecutive numbers:

A. after 7?
B. After y
C. After $x + 2$
D. After $x - 2$

Part A

$$8,9,10$$

Part B

When we add 1 to y we get:

$$y + 1$$

When we add 1 to the above, we get:

$$y + 2$$

When we add 1 to the above, we get:

$$y + 3$$

Part C

$$x + 3, x + 4, x + 5$$

Part C

$$x - 2 + 1 = x - 1$$
$$x - 2 + 2 = x$$
$$x + 1$$

Example 1.20

Find the four consecutive numbers:

A. before 23?
B. Before p?
C. Before $p + 2$

Part A

$$22,21,20,19$$

Part B

$$p - 1$$
$$p - 2$$
$$p - 3$$
$$p - 4$$

Part C

$$p + 1$$
$$p$$
$$p - 1$$
$$p - 2$$

1.3 Even and Odd Numbers

Even Numbers

Even numbers are numbers which are divisible by two. The first few even natural numbers are:

$$2,4,6,8,10 \ldots$$

Is Zero an Even Number

This is an important concept. Is zero an even number. To answer this question, let us apply the test of an even number: it should be divisible by 2.
Since zero is divisible by 2, it is an even number.
(In fact, zero is divisible by any number, except zero itself).

Example 1.21

If I have zero apples and I share them between two people, how many apples will each person get?

$$0 \text{ divided by } 2 = 0 \div 2 = \frac{0}{2} = 0$$

Odd Numbers

Odd Numbers are numbers that are not divisible by 2. The first few odd natural numbers are:

$$1,3,5,7,9 \ldots$$

Is Zero Odd

Odd numbers are those numbers which are not divisible by 2. Since zero is divisible by 2, it is not odd.

Test of Even and Odd Numbers

To check whether a number is even or odd, it is sufficient to check the last digit. If the last digit is even, then the number itself is even.

Example 1.22

Decide whether the following numbers are odd or even:

Two Digit Numbers

A. 12
B. 75
C. 84
D. 63
E. 21
F. 66

Three Digit Numbers

G. 451
H. 687
I. 914
J. 603

Four Digit Numbers

K. 4,021
L. 4,388
M. 3,592
N. 7,415
O. 8,618

Five Digit Numbers

P. 12,819
Q. 89,710
R. 67,111

Six Digit Numbers

S. 1,53,789
T. 5,20,742

Numbers before and After

Example 1.23

Find the:

A. Third even number after 8
B. Second even number after 12
C. Fourth even number after 7
D. Second even number before 10
E. Fifth even number before 24

$$8 \rightarrow 10 \rightarrow 12 \rightarrow 14$$
$$12 \rightarrow 14 \rightarrow 16$$
$$7 \rightarrow \underbrace{8}_{First} \rightarrow \underbrace{10}_{Second} \rightarrow \underbrace{12}_{Third} \rightarrow \underbrace{14}_{Fourth}$$
$$24 \rightarrow 22 \rightarrow 20 \rightarrow 18 \rightarrow 16$$

Example 1.24

Find the:

A. Second odd number after 7
B. Third odd number after 15
C. Second odd number after 9
D. Second odd number before 29

E. Second odd number before 18

$$15 \rightarrow 17 \rightarrow 19 \rightarrow 21$$

Variables

Instead of using numbers, we can use letters to represent numbers. We do not need to know what number the letter stands for in order to work with it.

Example 1.25

Suppose x is an even number. Answer the following questions in terms of x:

A. What is the even number that comes just after x?
B. What is the even number that comes just before x?
C. What is the odd number that comes just after x?
D. What is the odd number that comes just before x?

If you have an even, adding two will give you the next even number. Hence, the next number after x is:

$$x + 2$$
$$x - 2$$
$$x + 1$$
$$x - 1$$

Parity and Basic Applications

Example 1.26

Shubham plays football and cricket on alternate days. That is, on one day he plays football, and the next day he plays cricket.

A. On the 5th of January in a particular year, he played football. What sport will he play on the last day of the month?
B. On the 2nd of February in a particular year, he played cricket. What sport will he play on the last day of the same month?
C. On the 1st of January in a year which was not a leap year, he played cricket. What sport will he play on 1st of January of the next year?

Part A

Note the pattern

$$Day\ 5: Football$$
$$Day\ 6: Cricket$$
$$Day\ 7: Football$$
$$Day\ 8: Cricket$$

Football on odd-numbered days, and cricket on even-numbered days.

Part B

If it is a leap year, the last day will be 29 February, and hence the sport will be:

$$Football$$

If it is not a leap year, the last day will be 28 February, and hence the sport will be:

$$Cricket$$

Hence, we cannot decide which sport he will play till we know whether the year is a leap year or not.

Part C
A non-leap year has 365 days. So, from 1st Jan of one year, to 1st Jan of the next year is 365 days.

$$1+365=366th\ \ Day \to Even \to Football$$

1.4 Patterns

Days of the Week

A week has seven days. After every seven days, the days of the week will repeat.

$$Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday.$$

Example 1.27

Find the day of the week in the following cases:

A. Today is Wednesday. What is the day of the week five days from now?
B. Today is Friday. What was the day of the week three days prior?
C. Today is Saturday. What is the day of the week seven days from now? Fourteen days from now? Twenty-one days from now? Seven hundred days from now?
D. Today is Sunday, the 1st of January. What is the day of the week of the end of the month?
E. Today is Thursday, the first day of the month. What are the possible days that the end of the month can be?

Part A

$$Wed+5=Monday$$

Part B

$$Friday-3=Tuesday$$

Part C

$$Sat+7=Sat+14=Sat+700=Sat$$

Part D

$$Sun+31=Sun+28+3=Sun+3=Wed$$

Part E

$$Thu+28=Thu$$
$$Thu+29=Fri$$
$$Thu+30=Sat$$
$$Thu+31=Sun$$

Months of the Year

Example 1.28

A. Currently, it is March. What month will it be five months from now?
B. Currently, it is October. What month will it be eleven months from now?
C. Currently, it is June. What month will it be twelve months from now? Twenty-four months from now? One hundred and twenty months from now?

$$March + 5 = August$$
$$October + 11 = September$$
$$June + 12 = June + 24 = June + 120 = June$$

Example 1.29
A lightbulb used in a decoration for a wedding is set to

2. Sums and Averages

2.1 Sums

Consecutive Numbers

Example 2.1

Find the sum of the three consecutive numbers starting from 7.

We want the numbers starting from 7. So, we write:

$$7 + 8 + 9$$

We can rewrite the 9 as $8 + 1$ to give us:

$$7 + 8 + (8 + 1)$$

And we can add that extra to the 7:

$$8 + 8 + 8 = 24$$

Example 2.2

Find the sum of the three consecutive numbers starting after 11.

Here, we want the numbers after 11, and hence we start with 12.

$$12 + 13 + 14 = 13 + 13 + 13 = 39$$

Example 2.3

Find the sum of the three consecutive numbers before 15.

$$14 + 13 + 12 = 13 + 13 + 13 = 39$$

Example 2.4

Find the sum of three consecutive numbers, going backward, and starting from 15.

$$15 + 14 + 13 = 14 + 14 + 14 = 42$$

Example 2.5

Find the sum of four consecutive numbers:

A. Starting from 10
B. Starting after 10

$$10 + 11 + 12 + 13 = 46$$
$$11 + 12 + 13 + 14 = 50$$

We can do Part A in a shorter way after having Part A like this:

$$11 + 12 + 13 + 14 = 11 + 12 + 13 + 10 + 4 = 46 + 4 = 50$$

2.2 Averages

Find Averages

To find the average of two numbers, add the numbers, and divide the sum by 2.

Example 2.6

Calculate the average of the numbers below

A. $8,22$
B. $6 \text{ and } 22$
C. $4 \text{ and } 12$
D. $11 \text{ and } 15$
E. $100 \text{ and } 200$
F. 5 and 6

If there are two numbers, you add the numbers and divide by two:

$$\frac{8+22}{2}=\frac{30}{2}=15$$
$$\frac{5+6}{2}=\frac{11}{2}=5\frac{1}{2}$$

Example 2.7

Find the average of the numbers below

A. $7, 12, 15$

If there are three numbers, you add the numbers, and divide by three:

$$\frac{7+12+15}{3}=\frac{34}{3}=11\frac{1}{3}$$

Back Calculations

Example 2.8

The average of three integers is fifteen.

A. Find them if they are consecutive.
B. Find them if they are consecutive and odd.
C. Is it possible for the three integers to be consecutive and even.

Part A

Start with numbers that have average (but may not meet all the conditions):

$$15=15\times 1=15\times\frac{3}{3}=\frac{15+15+15}{3}\Rightarrow Avg\ of\ (15,15,15)=15$$

And we can convert this into three consecutive numbers that still have average 15:

$$14,15,16$$

Part B

$$15, 15, 15$$

$$13,15,17$$

Part C

Not Possible

Example 2.9

The sum of three consecutive integers is 21. Find the numbers.

$$\frac{21}{3} = 7 \Rightarrow Average = 7 \Rightarrow 7,7,7 \rightarrow 6,7,8$$

Example 2.10

The sum of two consecutive integers is 21. Find the numbers.

$$\frac{21}{2} = 10.5 \rightarrow 10.5, 10.5 \rightarrow 10,11$$

3. Factors: Basics

3.1 Factors

Revision: Division

Example 3.1

Find the division in the following cases:

A. $18 \div 3$ and $20 \div 3$
B. $8 \div 2$ and $11 \div 2$

$$18 \div 3 = 6, \qquad 20 \div 3 = Quotient\ 6\ Remainder\ 2$$
$$8 \div 2 = 4, \qquad 11 \div 2 = Quotient\ 5\ Remainder\ 1$$

In each case, the first pair does not leave a remainder, and the second pair leaves a remainder.
So, if there is no remainder when we complete a division, then the number which divides is called a factor.

Definition: Factors

If a number divides another number completely, then the first number is a factor of the second number.

Example 3.2: Checking for factors

In each case, find whether the first number is a factor of the second number:

A. Is 4 a factor of 12?
B. Is 3 a factor of 16?
C. Is 5 a factor of 95?
D. Is 7 a factor of 42?
E. Is 9 a factor of 36?
F. Is 4 is a factor of 62?
G. Is 11 a factor of 77?
H. Is 6 a factor of 98?
I. Is 4 a factor of 82?
J. Is 3 a factor of 82?
K. Is 7 a factor of 48?
L. Is 5 a factor of 125?
M. Is 3 a factor of 54?
N. Is 4 a factor of 54?
O. Is 5 a factor of 54?

Example 3.3

Numbers are given below in pairs. For each pair, check whether the first number is a factor of the second number.

$$\{(4,12), (6,28)\ (8,72), (3,10), (12,36), (20,200)\}$$

Properties of Factors

There are two numbers that will definitely be a factor of every number.

> One is a factor of every number.
> Every number is a factor of itself.

Example 3.4

Identify the factors of 11.

The factors of 11 are:

$$1\ and\ 11$$

Example 3.5

I have a natural number greater than one. What is the minimum of factors that the number must have?

The minimum is two:

- One is a factor of every number.

- ✓ In fact, it is the smallest factor of every number
- ➢ Every number is a factor of itself.
  - ✓ In fact, the number itself is its largest factor.

Example 3.6

I have a natural number. What is the minimum of factors that the number must have?

- ➢ One is a factor of every number
- ➢ Every number is a factor of itself.

But the number could be 1. In this case, 1 and the number itself are the same.
Hence, if the number is 1, it will have only 1 factor.

Example 3.7

A. Consider the largest positive factor and the smallest positive factor of 18.
   I. Find their sum.
   II. Find their difference.

B. Consider the largest positive factor and the smallest positive factor of 785.
   I. Find their sum.
   II. Find their difference.

$$\underbrace{18}_{\substack{Largest\\Factor}} + \underbrace{1}_{\substack{Smallest\\Factor}} = 18 + 1 = 19$$

$$\underbrace{785}_{\substack{Largest\\Factor}} + \underbrace{1}_{\substack{Smallest\\Factor}} = 785 + 1 = 786$$

Finding Factors

We usually write the factors of a number in ascending order. This makes it easy to use the factors in some analysis.

Example 3.8: Factors

Find the factors of the number 4.

Division Method

We check all the numbers less than and upto the number itself:

$$\underbrace{4 \div 1 = 4\ R0}_{Factor}, \quad \underbrace{4 \div 2 = 2\ R0}_{Factor}, \quad \underbrace{4 \div 3 = 1\ R1}_{Not\ a\ Factor}, \quad \underbrace{4 \div 4 = 1\ R0}_{Factor} \Rightarrow Factors\ are\ \{1,2,4\}$$

Skip Counting Method

In this method, we check if the second number is included when skip count using the first number.

$$\underbrace{\{1,2,3,\mathbf{4},5,6\}}_{1\ is\ a\ factor}, \quad \underbrace{\{2,\mathbf{4},6\}}_{2\ is\ a\ factor}, \quad \underbrace{\{3,6,9\}}_{3\ is\ not\ a\ factor}, \quad \underbrace{\{\mathbf{4},8\}}_{4\ is\ a\ factor}$$

Example 3.9: Factors

Find the factors of the following numbers:

A. 6	D. 8	G. 15	J. 28
B. 9	E. 12	H. 18	K. 25
C. 10	F. 16	I. 24	L. 21

M. 33
N. 44
O. 35
P. 48
Q. 60

$$Factors\ of\ 6 \in \{1,2,3,6\}$$
$$Factors\ of\ 8 \in \{1,2,4,8\}$$
$$Factors\ of\ 9 \in \{1,3,9\}$$

Prime and Composite Numbers

Prime Number

If a number has exactly two factors: one and itself, then the number is called a prime number.

Composite Numbers

If a number has more than two factors, then the number is composite.

One

The number is neither prime nor composite. It has a very special position among numbers.

The smallest prime number is 2.
One is not a prime number.

Prime Numbers from 2 to 100									
2	11	23	31	41	53	61	71	83	97
3	13	29	37	43	59	67	73	89	
5	17			47			79		
7	19								

Tips to help you differentiate a prime number from a composite number

- Not every number that ends in 1 is prime.
- Be careful of multiples of 3. Learn the test of divisibility of 3, and use it.

Example 3.10: Identifying Prime Numbers

Out of the two numbers, 7 and 8, one is a prime and the other is not. Identify which one is which, and why?

3.2 Factor Pairs

Definition

A factor pair for a number x is two numbers that multiply together to give x. Factors pair are one method of finding all the factors of a number.

Consider the number 6:

$$6 = \underbrace{1 \times 6}_{1st\ Factor\ Pair} = \underbrace{2 \times 3}_{2nd\ Factor\ Pair} \Rightarrow \text{Factors of } 6 = \underbrace{\{1,2,3,6\}}_{4\ Factors}$$

Trends in the Factors

The second number in the factor pair keeps getting smaller, and the first number in the factor pair keeps getting bigger.

1	12		1	60	
2 ↑	6 ↓		2↑	30↓	
3↑	4↓		3↑	20↓	
			4↑	15↓	
			5↑	12↓	

			6↑	10↓	
			10↑	6↓	Repeating
			12↑	5↓	
			20↑	3↓	
			30↑	2↓	
			60↑	1↓	

When to stop finding factors

If we try to go beyond 2×3, we get 3×2, which is a repetition of a factor pair that we have already found. That means that you can stop when the numbers become equal, or will start repeating.

Prime Numbers have only one factor pair

For a prime p, the only factor pair is

$$1 \times p$$

Example 3.11

Find the factor pairs of 12

One is always a factor of every number

- Hence, every number other than 1 has at least one factor pair, which is one and the number itself. Write this factor pair first.
- 1 does not have any factor pairs.

$$12 = \underbrace{1 \times 12}_{\textit{Factor Pair 1}}$$

Write the factor pairs with the smaller number first

$$\underbrace{\underbrace{2 \times 6 = 12}_{2<6}}_{\textit{Smaller Number First}} \quad , \quad \underbrace{\underbrace{6 \times 2 = 12}_{6>2}}_{\textit{Larger Number First}}$$

Write more factors pairs by increasing the left-side number

$$12 = \underbrace{1 \times 12}_{\textit{Factor Pair 1}} = \underbrace{2 \times 6}_{\textit{Factor Pair 2}} = \underbrace{3 \times 4}_{\textit{Factor Pair 3}}$$

Write all the factors in increasing order by reading from left to right, and then right to left

$$12 = \mathbf{1} \times \mathbf{12} = \mathbf{2} \times \mathbf{6} = \mathbf{3} \times \mathbf{4} \Rightarrow \text{Factors of } 12 = \underbrace{\{1,2,3,4,6,12\}}_{\textit{6 Factors}}$$

Practice 3.12

Find factor pairs of the following numbers:

Basics

A. 25
B. 18
C. 49
D. 24
E. 45
F. 14
G. 21
H. 15

Many Factors

I. 60
J. 48
K. 36
L. 72

Larger Numbers

M. 50
N. 100
O. 121

Larger Primes

P. 65
Q. 95
R. 76
S. 46

Perfect Powers

T. 16
U. 27
V. 125
W. 32
X. 81
Y. 64

Miscellaneous

Z. 144
AA. 216

$$18 = \underbrace{1 \times 18}_{Factor\ Pair\ 1} = \underbrace{2 \times 9}_{Factor\ Pair\ 2} = \underbrace{3 \times 6}_{Factor\ Pair\ 3} \Rightarrow \text{Factors of 12} = \underbrace{\{1,2,3,6,9,12\}}_{6\ Factors}$$

$$24 = \underbrace{1 \times 24}_{Factor\ Pair\ 1} = \underbrace{2 \times 12}_{Factor\ Pair\ 2} = \underbrace{3 \times 8}_{Factor\ Pair\ 3} = \underbrace{4 \times 6}_{Factor\ Pair\ 3} \Rightarrow \text{Factors of 24} = \underbrace{\{1,2,3,4,6,8,12,24\}}_{6\ Factors}$$

$$45 = \underbrace{1 \times 45}_{Factor\ Pair\ 1} = \underbrace{3 \times 15}_{Factor\ Pair\ 2} = \underbrace{5 \times 9}_{Factor\ Pair\ 3} \Rightarrow \underbrace{\{1,3,5,9,15,45\}}_{6\ Factors}$$

$$48 = \underbrace{1 \times 48}_{Factor\ Pair\ 1} = \underbrace{2 \times 24}_{Factor\ Pair\ 2} = \underbrace{3 \times 16}_{Factor\ Pair\ 3} = \underbrace{4 \times 12}_{Factor\ Pair\ 4} = \underbrace{6 \times 8}_{Factor\ Pair\ 4} \Rightarrow \underbrace{\{1,2,3,4,6,8,12,16,24,48\}}_{6\ Factors}$$

Concept 3.13

A. Riddhi is listing down all the counting numbers from 1 to 1000, that do not have one as a factor. How many numbers will she list?

B. Identify whether the following numbers are prime or composite?
   I. It has exactly three factors.
   II. It has exactly two factors.
   III. It has exactly one factor.
   IV. A number has two factor pairs.
   V. A number has one factor pair. Can the number be neither prime nor composite?

C. Which is the smallest number that has three factors?

A. Every number has one as a factor. So, there are no numbers that Riddhi can list.

B. Composite

Rectangular Arrangements

A number can be arranged as a rectangle in as many ways as the number of its factor pairs.
To be specific, a prime number can be arranged in only two ways.

Example 3.14

What are the factor pairs of 6?

$$Factor\ Pairs = (1,6)(2,3)(3,2)(6,1)$$

(Continuation) Example 3.15

Shoba has 6 coins. She wants to arrange them in the shape of a rectangle. In how many ways can she do this?

They can be arranged in four ways:

- In a single row, and six columns

O	O	O	O	O	O

- In two rows and three columns

O	O	O
O	O	O

- In three rows and two columns

O	O
O	O
O	O

- In one column and six rows

0
0
0
0
0
0

Example 3.16

Rishi has five coins, which he wants to arrange in the shape of a rectangle. In how many ways can he do this?

They can be arranged in two ways:

- In five rows, and a single column

0	0	0	0	0	0

- In five columns and one row

0
0
0
0
0

Example 3.17

Rishi has eight coins, which he wants to arrange in the shape of a rectangle. In how many ways can he do this?

They can be arranged in four ways:

- In eight rows, and a single column
- In two rows and four columns
- In four rows and two columns
- In eight columns and one row

0	0	0	0
0	0	0	0

0	0
0	0
0	0
0	0

0	0	0	0	0	0	0	0

0
0
0
0
0
0
0
0

Factor Pairs of Perfect Squares

When we find the factor pairs of perfect squares, one number will always be repeated twice. This number is the square root of the number.

Example 3.18: Factor Pairs of Perfect Squares

Find the factor pairs for each of the following numbers

$$\{4,9,16,25\} = \{2\times 2, 3\times 3, 4\times 4, 5\times 5\} = \{2^2, 3^2, 4^2, 5^2\}$$

$$4 = \underbrace{1\times 4}_{\textit{Factor Pair 1}} = \underbrace{2\times 2}_{\textit{Factor Pair 2}} \Rightarrow \text{Factors of } 4 = \underbrace{\{1,2,4\}}_{\textit{3 Factors}}$$

$$9 = \underbrace{1\times 9}_{\textit{Factor Pair 1}} = \underbrace{3\times 3}_{\textit{Factor Pair 2}} \Rightarrow \text{Factors of } 9 = \underbrace{\{1,3,9\}}_{\textit{3 Factors}}$$

$$16 = \underbrace{1\times 16}_{\textit{Factor Pair 1}} = \underbrace{2\times 8}_{\textit{Factor Pair 2}} = \underbrace{4\times 4}_{\textit{Factor Pair 3}} \Rightarrow \text{Factors of } 16 = \underbrace{\{1,2,4,8,16\}}_{\textit{5 Factors}}$$

$$25 = \underbrace{1\times 25}_{\textit{Factor Pair 1}} = \underbrace{5\times 5}_{\textit{Factor Pair 2}} \Rightarrow \text{Factors of } 25 = \underbrace{\{1,5,25\}}_{\textit{3 Factors}}$$

Perfect Square Factors

We found out the number of factors that a few perfect squares have.

$$4 \to \underbrace{3}_{Odd}\text{ Factors},\quad 9 \to \underbrace{3}_{Odd}\text{ Factors},\quad 16 \to \underbrace{5}_{Odd}\text{ Factors},\quad 25 \to \underbrace{3}_{Odd}\text{ Factors}$$

Property: A perfect square has an odd number of factors.

Example 3.19

Rishi has four coins, which he wants to arrange in the shape of a rectangle. In how many ways can he do this?

They can be arranged in three ways:

- In four rows, and a single column
- In two rows and two columns
- In one row and four columns
- In eight columns and one row

0	*0*
0	*0*

0	*0*
0	*0*

0	*0*	*0*	*0*

3.3 Prime Factorization

Definition

Writing a number as the product of its prime factors is called prime factorization.

For example, consider the number 6:

$$6 = 2 \times 3 \Rightarrow \textit{Prime Factorization} \ (\quad)$$

- 2 and 3 are both prime numbers
- The product of the numbers gives 6.

$$6 = 1 \times 6 \Rightarrow \textit{Not the Prime Factorization}$$

- 1 is not a prime number.
- Only prime numbers can be used in prime factorization.

$$6 = 2 \times 4 \Rightarrow \textit{Not the Prime Factorization}$$

- $2 \times 4 = 8$, which is not 6.
- Hence, this is not the correct prime factorization.

There are a number of ways of finding the prime factorization of a number.
For the ladder method, we write the number in a table, and keep dividing by prime numbers till we reach one.

Factor Tree Method

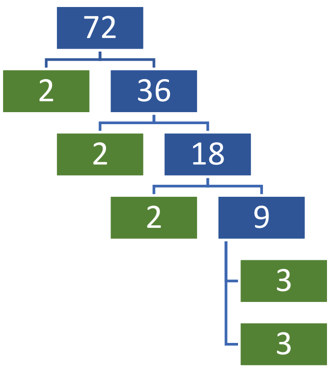
The factor tree method has the same numbers as the ladder method, but the presentation is different.

Concept Example 3.20

Find the prime factorization of 72 using the:

A. Factor Tree Method
B. Ladder Method

$$72 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$$

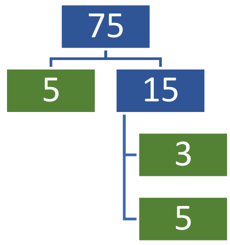


Ladder Method

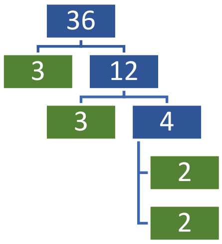
Prime Factor	Number
2	72
2	36
2	18
3	9
3	3
	1

Concept Example 3.21

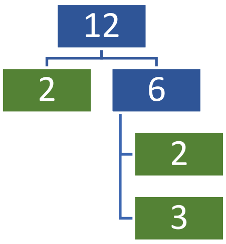
Find the prime factorization of the following numbers using the factor tree method {12,30,66,75}.



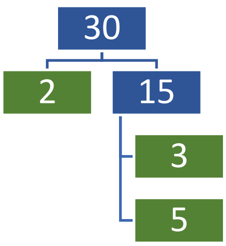
Example 3.22



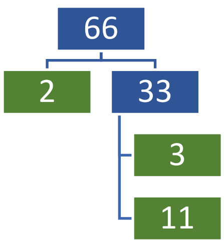
Example 3.23



Example 3.24



Example 3.25



Practice 3.26

Find the prime factorization of the following numbers:

Basics

A. 16
B. 12
C. 25
D. 50
E. 24
F. 80

Perfect Squares

G. 64
H. 100

Perfect Cubes

I. 125
J. 216

Mixed Review

K. 72
L. 82
M. 69
N. 65
O. 85
P. 78
Q. 95
R. 63
S. 60

Larger Numbers

T. 155
U. 210
V. 230
W. 180
X. 500
Y. 400

Part A

$$16 = 2 \times 2 \times 2 \times 2$$

2	16
2	8
2	4
2	2
	1

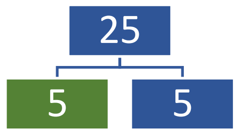
Part B

$$12 = 2 \times 2 \times 3$$

2	12
2	6
3	3
	1

Part C

$$25 = 5 \times 5$$



Part D

$$80 = 2 \times 2 \times 2 \times 2 \times 5$$

2	80
2	40
2	20
2	10
5	5
	1

Part

$$125 = 5 \times 5 \times 5$$

5	125
5	25
5	5
	1

2	216
2	108
2	54
3	27
3	9
3	3

Part

	1

$$216 = 2 \times 2 \times 2 \times 3 \times 3 \times 3$$

Part

$$100 = 2 \times 2 \times 5 \times 5$$

2	100
2	50
5	25
5	5
	1

Writing Prime Factorization

When finding the prime factorization, we use the ladder method, or the factor tree method. But the final answer needs to be written as a product of prime numbers that multiply together to give the number.

Concept Example 3.27: Prime Factorization in Product Form

In the earlier examples, you found the prime factorization of a few numbers. Write the prime factorization of these numbers in product form

$$72 = 2 \times 2 \times 2 \times 3 \times 3$$
$$36 = 2 \times 2 \times 3 \times 3$$
$$12 = 2 \times 2 \times 3$$
$$30 = 2 \times 3 \times 5$$
$$66 = 2 \times 3 \times 11$$
$$75 = 3 \times 5 \times 5$$

Concept Example 3.28: Ladder Method

Find the prime factorization of the following numbers using the ladder method: {81,65,36,75}, and write the answer in product form.

3	81		5	65		3	75		3	36
3	27		13	13		5	25		3	12
3	9			1		5	5		2	4
3	3						1		2	2
	1									1
$\mathbf{81 = 3 \times 3 \times 3 \times 3}$			$65 = 5 \times 13$						$36 = 2 \times 2 \times 3 \times 3$	

Distinct Prime Factors

Example 3.29

How many $\underbrace{distinct}_{different, or\ unique}$ prime factors are there in the prime factorization of each of the following numbers

A. 12
B. 30
C. 66
D. 75

E. 240

$$12 = 2 \times 2 \times 3 \Rightarrow (2,3) = 2\ distinct\ prime\ factors$$
$$30 = 2 \times 3 \times 5 \Rightarrow 3\ distinct\ prime\ factors$$
$$66 = 2 \times 3 \times 11 \Rightarrow 3\ distinct\ prime\ factors$$
$$75 = 3 \times 5 \times 5 \Rightarrow 2\ distinct\ prime\ factors$$
$$240 = 2 \times 2 \times 2 \times 2 \times 3 \times 5 \Rightarrow 3\ distinct\ prime\ factors$$

4. FACTORS-II

4.1 Generating Factors

Generating the Factors

Example 4.1

Find the factors of 6

$$6 = 2 \times 3$$

1	1	1		1	1	2
2	3	3		2	3	6

Example 4.2

Find the factors of 12.

$$12 = 2 \times 2 \times 3 = 2^2 \times 3^1$$

1	1	1		1	1	3
2	3	2		2	3	6
4		4		4		12

Example 4.3

Find the factors of 18.

$$18 = 2 \times 9 = 2 \times 3^2$$

1	1	1		1	1	2
2	3	3		2	3	6
	9	9			9	18

Example 4.4

Find the factors of 24.

$$24 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3$$

4.2 Number of Factors

Counting

Example 4.5

Find how many factors the following numbers have $\{10, 12, 15\}$

The factors are:

$$\{1,2,5,10\} \Rightarrow 4\ Factors$$

Example 4.6

Find the number of factors of 12

$$1,2,3,4,6,12 \Rightarrow 6\ Factors$$

Example 4.7

Find the number of factors of 15.

$$\{1,3,5,15\} \Rightarrow 4\ Factors$$

Example 4.8

Find the number of factors of 140 by listing factor pairs.

$$(1,140), (2,70), (4,35), (5,28), (7,20), (10,14) \Rightarrow 12\ Numbers$$

Example 4.9

Find the number of factors of 64 by listing factor pairs.

$$(1,64), (2,32), (4,16), (8,8) \Rightarrow 7\ Factors$$

Formula

Example 4.10

Find the number of factors of each of the following numbers

A. 12
B. 18

$$12 = 2 \times 2 \times 3 = 2^2 \times 3^1 \Rightarrow (2+1)(1+1) = (3)(2) = 6$$
$$18 = 2 \times 3 \times 3 = 2^1 \times 3^2 \Rightarrow (1+1)(2+1) = (2)(3) = 6$$

4.3 Sum of Factors

Perfect Powers

The sum of factors is simply the total of the factors of a number

Example 4.11

Anis finds the sum of the factors of the number 4. What is the sum that he finds?

The factors of 4 are

$$\{1,2,4\}$$

Their sum is

$$1 + 2 + 4 = 7$$

Example 4.12

Find the sum of the factors of the following numbers:

A. 49
B. 9
C. 25
D. 121
E. 8
F. 27
G. 125
H. 343
I. 81
J.

Part A
The factors of 49 are:

$$1, 7 \text{ } and \text{ } 49$$

And hence their sum is:

$$1 + 7 + 49 = 57$$

Part B
The factors of 9 are:

$$1, 3 \text{ } and \text{ } 9$$

And hence their sum is:

$$1 + 3 + 9 = 13$$

Part C
The factors of 25 are:

$$1, 5 \text{ } and \text{ } 25$$

And hence their sum is:

$$1 + 5 + 25 = 31$$

Part D
The factors of 121 are:

$$1, 11 \text{ } and \text{ } 121$$

And hence their sum is:

$$1 + 11 + 121 = 133$$

General Numbers

Example 4.13

Kripa used the following process on the number six:

- Find all the numbers that divide six
- Add all such numbers

What is the final answer that Kripa got?

The number that divide six are just the factors of 6

$$Factors\ of\ 6 \in \{1,2,3,6\}$$

The sum of the factor of 6 is:

$$Sum = 1 + 2 + 3 + 6 = 12$$

Example 4.14

Find the sum of factors of the following numbers: {12}

$$Factors\ of\ 12 \in \{1,2,3,4,6,12\} \Rightarrow Sum = 28$$

Example 4.15

Find the sum of factors of the following numbers, not including the number itself: {4,6,8,12}

$$Factors\ of\ 12 \in \{1,2,3,4,6,12\} \Rightarrow Sum = 16$$

Proper Factors

Example 4.16

What is the sum of the proper factors of 6? (Recall that a proper factor is a factor of the number, but is not the number itself)

$$Factors\ of\ 6 \in \{1,2,3,6\}$$

We do not include 6 in the sum, which is a factor, but not a proper factor.
The sum of the proper factors is then:

$$Sum = 1 + 2 + 3 = 6$$

Sum of Factors of a Perfect Power of 2

Example 4.17

Find the sum of the factors of the following numbers {2,4,8,16,32}. Do not include the number itself. Find the pattern

$$Factors\ of\ \mathbf{2} \in \{1\} \Rightarrow Sum = \mathbf{1 = 2 - 1}$$
$$Factors\ of\ \mathbf{4} \in \{1,2\} \Rightarrow Sum = \mathbf{3 = 4 - 1}$$
$$Factors\ of\ \mathbf{8} \in \{1,2,4\} \Rightarrow Sum = \mathbf{7 = 8 - 1}$$
$$Factors\ of\ \mathbf{16} \in \{1,2,4,8\} \Rightarrow Sum = \mathbf{15 = 16 - 1}$$

Example 4.18

Use the pattern from the previous example to the sum of factors (not including the number itself) of $\{64, 512, 1024\}$

Perfect, Abundant and Deficient Numbers

Earlier, we found the sum of the factors of numbers (without including the number itself).

Perfect Number: Sum of factors is equal to the number itself
Abundant Number: Sum of factors is more than the number
Deficient Number: Sum of factors is less than the number

Example 4.19

From the following numbers, classify them as perfect, abundant or deficient. You have already found the sum of the factors earlier: {4,6,8,12}

Deficient	Perfect	Abundant
$4 \in \{1,2\} \Rightarrow Sum = 3$ $8 \in \{1,2,4,8\} \Rightarrow Sum = 7$	$6 \in \{1,2,3,6\} \Rightarrow Sum = 6$	$12 \in \{1,2,3,4,6,12\} \Rightarrow Sum = 28$

Formula

Example 4.20

Find the sum of the factors of 12 using the formula.

Find the prime factorization of 12:

$$12 = \underbrace{2 \times 2}_{2\ Times} \times \underbrace{3}_{1\ Time}$$

$$(1 + 2 + 4)(1 + 3) = 1(1 + 3) + 2(1 + 3) = 4(1 + 3) = 1 + 3 + 2 + 6 + 4 + 12 = 28$$

Example 4.21

Find the sum of the factors of 28 using the formula.

Find the prime factorization of 28:

$$28 = \underbrace{2 \times 2}_{2\ Times} \times \underbrace{7}_{1\ Time}$$

$$(1 + 2 + 4)(1 + 7) = 7 \times 8 = 56$$

Listing it out:

$$\{1,2,4,7,14,28\} \Rightarrow Sum = 56$$

Example 4.22

Find the sum of the factors of 70 using the formula.

Find the prime factorization of 70:

$$70 = 2 \times 5 \times 7$$

$$(1 + 2)(1 + 5)(1 + 7) = 3 \times 6 \times 8 = 144$$

Example 4.23

Find the sum of the factors of 100 using the formula.

Find the prime factorization of 100:

$$100 = \underbrace{2 \times 2}_{2\ Times} \times \underbrace{5 \times 5}_{1\ Time}$$

$$(1 + 2 + 4)(1 + 5 + 25) = 7 \times 31 = 217$$

Example 4.24

Find the sum of the factors of 98 using the formula.

Find the prime factorization of 98:

$$98 = \underbrace{2}_{1\ Times} \times \underbrace{7 \times 7}_{1\ Time}$$

$$(1 + 2)(1 + 7 + 49) = 3 \times 57 = 171$$

Example 4.25

Find the sum of the factors of 144 using the formula.

Find the prime factorization of 144:

$$144 = 12 \times 12 = \underbrace{2 \times 2 \times 2 \times 2}_{2\ Times} \times \underbrace{3 \times 3}_{1\ Time}$$

$$(1 + 2 + 4 + 8 + 16)(1 + 3 + 9) = 31 \times 13 = 403$$

5. Multiples

5.1 Multiples

Introduction

Table of 4

$$\{4,8,12,16,20,24,\dots\} = \{4\times 1, 4\times 2, 4\times 3, 4\times 4, 4\times 5, 4\times 6,\dots\}$$

Definition

The numbers which come in the table of 4 are called multiples of 4.
In general, numbers which come in the table of a number are multiples of that number.

Finding Multiples

Example 5.1: Largest/Smallest multiple of n digits

Find the:

A. Largest two-digit multiple of 4.
B. Largest two digit multiple of 7
C. Smallest two-digit multiple of 3
D. Smallest three-digit multiple of 12
E. Largest two-digit multiple of 8

Part A: $4\times 10 = 40, 4\times 20 = 80, 4\times 24 = 96$

$$\text{Shortcut: } 4\times 25 = \underbrace{100}_{3\ Digits} \Rightarrow 100 - 4 = \underbrace{96}_{Largest\ Two\ Digit\ Multiple} = 4\times 24$$

Part B: $7\times 15 = \underbrace{105}_{3\ Digits} \Rightarrow 105 - 7 = \underbrace{98}_{2\ Digits}$

Example 5.2: Largest/Smallest multiple near a number

Find the:

A. Largest multiple of 7 that is less than 34
B. Largest multiple of 12 that is less than 85
C. Smallest multiple of 13 greater than 91
D. Smallest multiple of 4 greater than 500

The number 1001

You should know that

$$7\times 11\times 13 = 1001$$

Hence, 1001 is a multiple of 7, 11 and 13.
This is quite useful in a number of calculations.

Challenge 5.3: Largest/Smallest multiple of n digits

Find the:

A. Largest three digit multiple of 7
B. Largest three digit multiple of 11
C. Largest three digit multiple of 13

We start with the smallest three digit multiple of 7, which we know is 1001, and then we subtract 7 to find the multiple of 7 prior to 1001:

$$\underbrace{1001}_{Multiple\ of\ 7} \Rightarrow \underbrace{1001-7}_{Multiple\ of\ 7} = 994$$

Similarly, we start with 1001, and subtract 11 to find the multiple of 11 prior to 1001:

$$7 \times 11 \times 13 = \underbrace{1001}_{Multiple\ of\ 11} \Rightarrow \underbrace{1001-11}_{Multiple\ of\ 11} = 990$$

We follow the same process as before

$$7 \times 11 \times 13 = \underbrace{1001}_{Multiple\ of\ 11} \Rightarrow \underbrace{1001-13}_{Multiple\ of\ 11} = 988$$

Example 5.4: Difference/Sum in Multiples

A. Find the difference between the smallest and the largest two-digit multiple of 4.
B. Find the sum of the largest two-digit multiple of 7, and the largest two-digit multiple of 8.

PART II: OTHER STUFF

6. TESTS OF DIVISIBILITY

6.1 Divisibility by 2

Even and Odd Numbers

Even numbers are numbers like

$$0, \quad 2, \quad 4, \quad 6, \quad 8, \quad 10, \ldots$$

These numbers can be divided into two equal parts.
0 is also an even number.

Odd numbers are numbers that are one more or one less than an even number

$$1, \quad 3, \quad 5, \quad 7, \quad 9, \ldots$$

Test of Divisibility by 2

A number is divisible by 2, if the last digit of the number is any out of the following digits:

$$0, \quad 2, \quad 4, \quad 6, \quad 8$$

If the last digit of the number is not from the digits above, then the number is odd.

Example 6.1

State whether the following numbers are even or odd:

A. 23
B. 70
C. 88
D. 99
E. 145
F. 614
G. 677
H. 815
I. 9102
J. 3572

Example 6.2

For this question, the numbers under consideration are the counting numbers, starting from zero:

$$0, \quad 1, \quad 2, \quad 3, \quad 4, \ldots$$

Find the answers to the following:

A. What is the smallest
   I. even number
   II. two-digit odd number
B. State the number of:
   I. even primes
   II. odd primes
   III. single digit even numbers
   IV. single digit odd numbers
C. What is the largest
   I. even prime number

  II. single digit even number
  III. two-digit odd number
  IV. two digit even number smaller than fifty

D. What is the smallest number that remains even after being divided by:
  I. Two
  II. Three

E. Find the product of
  I. the smallest even number and the smallest odd number

F. Find the sum of
  I. the largest single digit odd number and the smallest two digit even number.

7. HCF

7.1 Common Factors

Common Factor

A factor is a common factor of two numbers, if it is a factor of each number separately.

Example 7.1
Find the common factors of each pair of numbers below.

Basics
A. 6 and 9
B. 6 and 8
C. 12 and 16
D. 8 and 12
E. 18 and 24
F. 25 and 35
G. 28 and 36 (Devansh)

Many Factors
H. 30 and 36
I. 24 and 36
J. 18 and 48

Larger Primes
K. 62 and 64
L. 57 and 95

Common Factors of 6 and 9

$$Factors\ of\ 6 = \{1,2,3,6\}$$
$$Factors\ of\ 9 = \{1,3,9\}$$
$$Common\ Factors = \{1,3\}$$

Common Factors of 6 and 8

$$Factors\ of\ 6 = \{\mathbf{1},\mathbf{2},3,6\}$$
$$Factors\ of\ 8 = \{\mathbf{1},\mathbf{2},4,8\}$$
$$Common\ Factors = \{1,2\}$$

Common Factors of 12 and 16

$$Factors\ of\ 12 = \{1,2,3,4,6,12\}$$
$$Factors\ of\ 16 = \{1,2,4,8,16\}$$
$$Common\ Factors = \{1,2,4\}$$

Common Factors of 8 and 12

$$Factors\ of\ 8 = \{1,2,4,8\}$$
$$Factors\ of\ 12 = \{1,2,3,4,6,12\}$$
$$Common\ Factors = \{1,2,4\}$$

Common Factors of 18 and 24

$$Factors\ of\ 18 = \{1,2,3,6,9,18\}$$
$$Factors\ of\ 24 = \{1,2,3,4,6,12,24\}$$
$$Common\ Factors = \{1,2,3,6\}$$

Common Factors of 25 and 35

$$Factors\ of\ 25 = \{1,5,25\}$$
$$Factors\ of\ 35 = \{1,5,7,35\}$$
$$Common\ Factors = \{1,5\}$$

28= 1 2 4 7 14 28
36= 1 2 3 4 6 8 12 18 36

7.2 HCF

HCF – Highest Common Factor

We have already seen how to find the factors of a number. Now, we look at using this in more than one number. One common property of two numbers that is of great interest is to the find the

$$Highest\ common\ factor$$

This number is the largest number that is a factor of both numbers.

Listing Method

Concept 7.2

Find the highest common factor of 6 and 9.

Find the factors of the two numbers:

$$\text{Factors of } 6 = \{1,2,3,6\}, \qquad \text{Factors of } 9 = \{1,3,9\}$$

Color the common factors **red**. These are factors of 6, which are also factors of 9.

$$\text{Factors of } 6 = \{\mathbf{1}, 2, \mathbf{3}, 6\}, \qquad \text{Factors of } 9 = \{\mathbf{1}, \mathbf{3}, 9\}$$

Write the common factors separately, and choose the factor which is the highest among them.

$$\text{Common Factors} = \{1,3\} \Rightarrow \text{Highest Common Factor} = HCF = 3$$

Practice 7.3

Find the HCF of each pair of numbers by listing out the factors:

A. 16 and 24
B. 12 and 16
C. 8 and 12
D. 10 and 15
E. 24 and 36
F. 14 and 98
G. 27 and 18

Ladder Method

In the ladder method, we find the HCF by dividing by numbers that divide all numbers that we want to find the HCF for.

Example 7.4

Find the HCF of 12 and 16 using ladder method.

		12	16
2 divides both 12 and 16	2	6	8
2 divides both 6 and 8	2	3	4
No number divides both 3 and 4. Stop dividing			

HCF = Product of the numbers dividing both the numbers = $2 \times 2 = 4$

Practice 7.5

Find the HCF of the following pairs using the ladder method

Two Digit Numbers

1. 18 and 24
2. 35 and 45
3. 30 and 36
4. 16 and 44
5. 75 and 90
6. 52 and 60
7. 78 and 68
8. 27 and 63
9. 44 and 54
10. 26 and 54
11. 44 and 84
12. 99 and 69

Prime Numbers

13. 23 and 67
14. 61 and 97
15. 73 and 47
16. 59 and 19
17. 89 and 67
18. 31 and 37
19. 71 and 29
20. 101 and 53

More Two Digit Numbers

21. 26 and 91
22. 51 and 85
23. 64 and 80
24. 34 and 68
25. 38 and 76
26. 39 and 78

Co-Prime Numbers

27. 12 and 25
28. 16 and 27
29. 30 and 55

Mixed Bag

30. 58 and 88
31. 38 and 95

Larger Numbers

32. 132 and 126
33. 576 and 676

HCF of 18 and 24

	18	24
2	9	12
3	3	4
$HCF = 2 \times 3 = 6$		

HCF of 35 and 45

	35	45
5	7	9
$HCF = 5$		

HCF of 75 and 90

	75	90
5	15	18
3	5	6
$HCF = 3 \times 5 = 15$		

HCF of 44 and 54

	44	54
2	22	27
$HCF = 2$		

HCF of 27 and 63

	27	63

3	9	21
3	3	7
$HCF = 3 \times 3 = 9$		

		52	60			78	68			132	126		
	2	26	30		2	39	34		2	66	63		
	2	13	15						3	22	21		
	$HCF = 2 \times 2 = 4$				$HCF = 2$				$HCF = 2 \times 3 = 6$				

Example 6: Larger Numbers

Ladder Method: Three or More Numbers

The ladder method can be extended to three or more numbers. We need to find numbers that only divide all three numbers

Example 7.7

Find the HCF of the following pairs using the ladder method

A. 28, 36 and 44
B. 21, 28 and 35

HCF of 28, 36 and 44

		28	36	44
	2	14	18	22
	2	7	9	11
	$HCF = 2 \times 2 = 4$			

Word Problems

Example 8

Prime Factorization Method

8. LCM: Least Common Multiple

8.1 Common Multiples

A. Listing Method

In the listing method, we list the multiples, and identify the smallest multiple that comes in both the lists.

Example 8.1

List the multiples of the following numbers:

A. 7
B. 12
C. 9
D. 13
E. 19
F. 15

$$Multiples\ of\ 7 = \{7,14,21,28,35,42,49,56,63,70,77,84\}$$
$$Multiples\ of\ 12 = \{12,24,36,48,\mathbf{60},72,84,96,108,\mathbf{120}\}$$
$$Multiples\ of\ 9 = \{9,18,27,36,45,54,63,72,81,\mathbf{90},99,108\}$$
$$Multiples\ of\ 13 = \{13,26,39,52,65,78,91,104,117,130,143,156\}$$
$$Multiples\ of\ 19 = \{19,38,57,76,95,114,133,152,171,190,209,228\}$$
$$Multiples\ of\ 15 = \{15,30,45,\mathbf{60},75,\mathbf{90},105,\mathbf{120},135,150,165,180\}$$

In the lists above, identify numbers which are there in one more than one list. These are called common multiples.

$$90 = 10 \times 9, \qquad 90 = 15 \times 6$$

8.2: Common Multiples

If we have two numbers, then then numbers which are multiples of both the numbers are called common multiples.

Example 8.3

Use the listing method to identify the common multiples of 3 and 4.

Let's list the multiples of 3 and 4, and color the common multiples **red**:

$$\text{Multiples of } 3 = \{3,6,9,\mathbf{12},15,18,21,\mathbf{24},27\ldots\}, \qquad \text{Multiples of } 4 = \{4,8,\mathbf{12},16,20,\mathbf{24},28\}$$
$$\text{Common Multiples } \{12,24,36,\ldots\}$$

Example 8.4

Find the first few common multiples of 5 and 10.

$$\{5,\mathbf{10},15,\mathbf{20},25,\mathbf{30},35,\mathbf{40},45,\mathbf{50}\}, \qquad \{10,20,30,40,50\} \Rightarrow Common\ Multiples\ are\ \{10,20,30,40,50\}$$

Example 8.5

Find the first few common multiples of 6 and 8.

$$\{6,12,18,\mathbf{24},30,36,42,\mathbf{48},\ldots\}, \qquad \{8,16,\mathbf{24},32,40,\mathbf{48},\ldots\} \Rightarrow Common\ Multiples\ are\ \{24,48,\ldots\}$$

Example 8.6
Find the first few common multiples of 4 and 6.

$$\{4,8,\mathbf{12},\ldots\},\{6,\mathbf{12}\} \Rightarrow Common\ Multiples = \{12,24,36,48,60,72,\ldots\}$$

8.2 Least Common Multiple

8.7: Lowest Common Multiples
If you arrange the common multiples from least to greatest, then the multiple which is the least is the lowest common multiple.

A. Listing Method

Example 8.8
Use the listing method to identify the least common multiple of 3 and 4.

From above, we know that the common multiples of 3 and 4 are:
$$\{4,8,\mathbf{12},\ldots\},\{6,\mathbf{12}\} \Rightarrow Common\ Multiples = \{12,24,36,48,60,72,\ldots\}$$
But out of these numbers, one number is the smallest. And that number is the least common multiple, which is:
$$12$$

Example 8.9
Find the LCM of 6 and 8 using the listing method.

$$\{6,12,18,\mathbf{24},30,36\},\qquad \{8,16,\mathbf{24}\} \Rightarrow LCM = 24$$

Example 8.10
Find the LCM of 6 and 9 using the listing method.

$$Multiples\ of\ 6 = \{6,12,\mathbf{18},24,32,\ldots\}$$
$$Multiples = \{9,\mathbf{18},27,36,\ldots\}$$
$$LCM(6,9) = 18$$

Example 8.11
Find the LCM of 6 and 12 using the listing method.

$$LCM(6,12) = 12$$

Example 8.12
Find the LCM of 9 and 12 using the listing method.

$$Multiples\ of\ 6 = \{6,12,18,24,32,\ldots\}$$
$$Multiples = \{9,18,27,36,\ldots\}$$

$$LCM(9,12) = 36$$

Example 8.13
Find the LCM of 8 and 12 using the listing method.

$$LCM(8,12) = 24$$

Example 8.14
Find the LCM of 5 and 7 using the listing method.

$$LCM(5,7) = 35$$

Example 8.15
Find the LCM of 10 and 15 using the listing method.

$$LCM(10,15) = 35$$

Example 8.16
Find the LCM of 7 and 10 using the listing method.

$$LCM(7,10) = 70$$

Example 8.17
Find the LCM of 4 and 10 using the listing method.

$$LCM(4,10) = 20$$

Ladder Method

Example 8.18
Find the LCM of the following numbers using the ladder method:

Warm-Up

A. 12 and 16
B. 6 and 8
C. 8 and 12
D. 6 and 9

Basics

E. 12 and 18
F. 15 and 20
G. 16 and 20
H. 25 and 40
I. 26 and 39
J. 21 and 28
K. 25 and 35

Co-Prime Numbers

L. 12 and 25
M. 5 and 7
N. 15 and 22

Larger Primes

O. 38 and 57
P. 51 and 85

Larger Numbers

Q. 36 and 132

LCM of 12 and 16

		12	16	
	2	6	8	
	2	3	4	
	$2 \times 2 \times 3 \times 4 = 48$			

LCM of 21 and 28

		21	28	
	7	3	4	
	$7 \times 3 \times 4 = 84$			

LCM of 36 and 132

	36	132
2	18	66
2	9	33
3	3	11

$$LCM = \underbrace{2 \times 2 \times 3}_{Common\ Factors} \underbrace{\times 3 \times 11}_{Uncommon\ Factors} = 396$$

Ladder Method: Three or More Numbers

Concept 8.19

Find the LCM of the following numbers using the ladder method

A. 2, 4 and 8
B. 6,8,10
C. $6, 9$ *and* 15
D. 28, 36 and 44

Part A

	2	4	8
2	1	2	4
2	1	1	2

Part B

	6	8	10
2	3	4	5
$LCM = 2 \times 3 \times 4 \times 5 = 120$			

Part C

	28	36	44
2	14	18	22

2	7	9	11

$$LCM = 2 \times 2 \times 7 \times 9 \times 11 = 28 \times 99 = 28 \times 100 - 28 = 2772$$

Prime Factorization Method: Two Numbers

In the prime factorization method

- Write out each number as the product of its prime factors
- Mark out the factors which are common to both numbers. These are the common factors
  - ✓ Write the common factors once
- Remaining factors are non-common factors
  - ✓ Write the non-common factors once
- Multiply all the common factors, and all the non-common factors

Concept 8.20

$$36 = \underbrace{2 \times 2}_{Common} \times \underbrace{3}_{Not\ Common} \times \underbrace{3}_{Common}$$
$$48 = \underbrace{2 \times 2}_{Common} \times \underbrace{2 \times 2}_{Not\ Common} \times \underbrace{3}_{Common}$$

$$LCM = \underbrace{2 \times 2 \times 3}_{Common} \times \underbrace{2 \times 2 \times 3}_{Not\ Common} = 12 \times 12 = 144$$

Prime Factorization Method: Three or More Numbers

In the prime factorization method

- Write out each number as the product of its prime factors
- For each prime number, choose the greatest number of times that it occurs across all the numbers

Concept 8.21

Find the LCM of the following numbers

A. 2, 4 and 8
B. 28, 36 and 44

$$2 = 2 \Rightarrow 2 \text{ occurs once}$$
$$4 = 2 \times 2 \Rightarrow \ 2 \text{ occurs twice}$$
$$8 = 2 \times 2 \times 2 \Rightarrow \ 2 \text{ occurs thrice}$$

We take the maximum of times that it occurs.
Hence, the

$$LCM = 2 \times 2 \times 2 = 8$$

8.3 Word Problems

9. Applications

9.1 Dice

Hiding one Face

Two six-sided dice each have their faces numbered one to six, and are then dropped on a table. What can be the sum of the numbers on the visible faces of the die.

$$\text{Total of all faces of one die} = 1 + 2 + 3 + 4 + 5 + 6 = 21$$

One Die						
Hidden Face	1	2	3	4	5	6
Total	20	19	18	17	16	15
Two Dice	1+1	1+2	1+3	1+4	1+5	1+6
	40	39	38	37	36	35
	2+1					
	39	38	37	36	35	34
	3+1					
	38	37	36	35	34	33
	4+1					
	37	36	35	34	33	32
	5+1					
	36	35	34	33	32	31
	6+1					6+6
	35	34	33	32	31	30

10. Squares

10.1 Basics

10.1: Definition
A square of a number is the product of the number with itself.

1	1		6	36		11	121		16	256		21	441		26	676
2	4		7	49		12	144		17	289		22	484		27	729
3	9		8	64		13	169		18	324		23	529		28	784
4	16		9	81		14	196		19	361		24	576		29	841
5	25		10	100		15	225		20	400		25	625		30	900

Example 10.2
What is the square of 7?

$$7 \times 7 = 49$$

Example 10.3
What is the smallest square natural number?

$$1 \textit{ is the answer}$$
$$1 \textit{ is the square of } 1$$

Example 10.4
The square of which number is 81?

Example 10.5
Find:

A. 5^2
B. 8^2
C. $1^2 + 2^2$

A. Expressions

Example 10.6
Let $a = 3, b = 5, c = 6$. Find

A. a^2
B. b^2
C. c^2
D. $a^2 + b^2$
E. $a^2 + c^2$
F. $b^2 + c^2$
G. $a^2 + b^2 + c^2$

B. Larger Numbers

Example 10.7

Find the square of the following numbers:

A. 41
B. 23
C. 62
D. 99

$$41 \times 41 = 1681$$
$$23 \times 23 = 529$$
$$62 \times 62 = 3844$$
$$99 \times 99 = 9801$$

Example 10.8

Decide whether the following numbers are perfect squares. If they are perfect squares, state which number they are the square of.

A. 289
B. 375
C. 271
D. 576
E. 410
F. 900
G. 625
H. 575
I. 121

$$289 = 17^2$$
$$375: Not\ a\ Square$$
$$271: Not\ a\ Square$$
$$576 = 24^2$$
$$410: Not\ a\ Square$$
$$900 = 30^2$$
$$625 = 25^2$$
$$575: Not\ a\ Square$$
$$121 = 11^2$$

C. Word Problems

Example 10.9

Boys in a school doing a dance rehearsal are arranged in a square pattern so that there are six rows, and six columns. What is the smallest number of boys that must leave the pattern so that the number of boys is a prime number?

$$No. of\ Boys = 6 \times 6 = 36$$
$$Prime\ Number\ before\ 36 = 31$$
$$No. of\ Boys\ to\ Leave = 36 - 31 = 5$$

10.10: Perfect Squares

Note that

$$1^2 = 1, 2^2 = 4, 3^2 = 9$$

Such numbers are called perfect squares.

Example 10.11

What is the smallest number that must be added to the smallest prime number larger than 50, so that the number becomes a perfect square?

$$\textit{Smallest prime number larger than } 50 = 53$$
$$7 \times 7 = 49 < 53$$
$$8 \times 8 = 64 > 53$$

The number to be added is:

$$64 - 53 = 11$$

10.2 Applications

A. Pythagorean Triplets

Example 10.12

$3^2 + 4^2$ is the square of some number x. Find x.

$$3^2 + 4^2 = 9 + 16 = 25 = 5^2$$

Example 10.13

$6^2 + 8^2$ is the square of some number x. Find x.

$$6^2 + 8^2 = 36 + 64 = 100 = 10^2 \Rightarrow x = 10$$

Example 10.14

$5^2 + 12^2$ is the square of some number x. Find x.

$$5^2 + 12^2 = 25 + 144 = 169 = 13^2 \Rightarrow x = 13$$

Example 10.15

$8^2 + 15^2$ is the square of some number x. Find x.

$$8^2 + 15^2 = 64 + 225 = 289 = 17^2 \Rightarrow x = 17$$

B. Last Digit of Perfect Squares

Are 12,12,201 and 12,12,202 are perfect squares. We can't check their square roots without a calculator. But, we can see which are the possible Units Digits.

$$17 \times 17 = 289$$
$$7 \times 7 = 49$$

Note that the last digit of 17^2 matches the last digit of its square.

Last Digit of Number			
No.	Square	No.	Square
0	0		
1	1	9	1
2	4	8	4
3	9	7	9
4	6	6	6
5	5		

11	121		16	256
12	144		17	289
13	169		18	324
14	196		19	361
15	225		20	400

10.16: Last Digit Candidates for Perfect Squares

Certain numbers can be the last digit of a perfect square:

$$\underbrace{\{0,1,4,5,6,9\}}_{\substack{\text{Can be the last digit}\\ \text{of a perfect square}}}$$

Certain digits can never be the last digit for a perfect square.

$$\underbrace{\{2,3,7,8\}}_{\substack{\text{Can never be the last}\\ \text{digit of a perfect square}}},$$

This test is used for rejection, not selection. If a digit falls among the set of digits on the right, it is not necessary that it is a perfect square.

Example 10.17

There are two odd numbers from $0-9$ which cannot be the last digit of a perfect square. Find the sum of these two numbers.

$$3+7=10$$

Example 10.18

There are three odd numbers from $0-9$ which can be the last digit of a perfect square. Find the sum of these three numbers.

$$1+5+9=15$$

Example 10.19

One of the following numbers is a perfect square? Which one is it?

$$p=2{,}46{,}713, \qquad q=5{,}49{,}081, \qquad r=3{,}33{,}222, \qquad s=9{,}16{,}787$$

3, 2 and 7 can never be the last digit of a perfect square.

Since one of the numbers is a perfect square, it must be 5,49,081.

11. Numbers

11.1 Basics

11.1: Definition

Write it in expanded notation
349 = 300+40+9 = 3*100 + 4*10 + 1*9

Place Value of a Digit:
Place value is the value of a digit due to its position.

Face Value of a Digit
Face value of a digit is the value of the digit by itself.

A. In the number 349, what is the place value, and what is the face value of the digit 4?
349 = 300+40+9 = 3*100 + 4*10 + 9*1
Place Value = 40
Face Value = 4

B. In the number 2396, what is the place value, and what is the face value of the digit 3?
Place Value =300
Face Value=3

Back Calculations:
C. A four digit number has the digit X in it exactly once. The difference between the place value and the face value of the digit X is 594. Determine the face value of the digit X, the position in which it occurs, and also the place value.

The possible digits that we need to choose from are:
0,1,2,...,9

Place Value(X) - Face Value(X) = 594 = 600-6
Place Value=600
Face Value =6
Position= Hundred's Place

A seven digit number has the digit Y in it exactly once. Determine the face value of the digit, the position in which it occurs, and also the place value, if the difference between the place value and the face value of the digit is
A. 1998
B. 45
C. 6993
D. 891
E. 297
F. 7992

1998 = 2000 - 2
45 = 50 - 5
6993 = 7000 - 7
891 = 900 - 9
297 = 300 - 3
7992 = 8000 - 8

A. International Place Value System

International		Million	Hundreds Thousand	Ten Thousands	Thousands	Hundred's	Ten's	One's
Indian		Ten Lakhs	Lakhs	Ten Thousands	Thousands	Hundred's	Ten's	One's

International					One Billion	Hundred Million	Ten Million	Million
Indian					Hundred Crores	Ten Crores	Crores	Ten Lakhs

Example 11.2

A. What is the number of zeroes in one million?
B. What is the number of zeroes in one lakh?
C. What is the number of zeroes in one billion?
D. What is the difference between the number of zeroes in one billion and one lakh?

$$1{,}000{,}000 \Rightarrow 6\ Zeroes$$
$$1{,}00{,}000 \Rightarrow 5\ Zeroes$$
$$1{,}000{,}000{,}000 \Rightarrow 9\ Zeroes$$
$$9 - 5 = 4$$

Example 11.3

A. What is the product of ten thousand and one thousand?
B. What is the product of one million and one lakh?
C. What is the product of three thousand and three lakh?

$$\underbrace{10{,}000}_{4\ Zeroes} \times \underbrace{1{,}000}_{3\ Zeroes} = 10{,}000{,}000 = Ten\ Million$$

$$\underbrace{One\ Million}_{6\ Zeroes} \times \underbrace{One\ Lakh}_{5\ Zeroes} = 100{,}000{,}000{,}000 = One\ Hundred\ Billion$$

$$3{,}000 \times 3{,}00{,}000 = 900{,}000{,}000 = 900\ Million$$
$$3{,}000 \times 3{,}00{,}000 = 90{,}00{,}00{,}000 = 90\ Crores$$

Example 11.4

If one US dollar is worth Rs. 80, then find the value of 3 million dollars in Indian currency? (Write your answer using Indian place value system).

$$3{,}000{,}000 \times 80 = 240{,}000{,}000 = 240\ Million\ Rupees$$

Divide by 10 to convert from millions to crore:

$$24\ Crore\ Rupees$$

11.5: Predecessor and Successor

A number which comes just before is a predecessor, and a number which comes just after is a successor.

Example 11.6

A. Which number has successor 100?
B. Which number has successor one million?
C. What is the result of adding the predecessor and the successor of one million?

$$Predecessor\ of\ One\ Million = One\ Million - 1$$
$$Sucessor\ of\ One\ Million = One\ Million + 1$$

Add the above two:

$$One\ Million + One\ Million - 1 + 1 = Two\ Million$$

B. Estimation

Example 11.7

$$399 \times 3$$

C. Counting

Example 11.8

D. Rounding