Logarithms

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Table of contents

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0.1 Formula Summary

A. Cheat Sheet

Definition

$$\underbrace{\log_y x = z}_{\textit{Logarthmic Form}} \quad \Leftrightarrow \quad \underbrace{y^z = x}_{\textit{Exponential Form}} \qquad , x > 0, y > 0, y \neq 1$$

$$\textit{Common Logs}: \log_x x = 1, \log_x 1 = 0$$

Basic Rules

$$\textit{Product Rule}: \log_x mn = \log_x m + \log_x n$$
$$\textit{Quotient Rule}: \log_x \frac{m}{n} = \log_x m - \log_x n$$
$$\textit{Power Rule}: \log_x a^n = n \log_x a$$
$$\textit{Power Rule Extension}: \log_{a^n} x^m = \log_a x^{\frac{m}{n}} = \frac{m}{n} \log_a x$$

Change of Base

$$\log_x m = \frac{\log m}{\log x} = \frac{1}{\frac{\log x}{\log m}} = \frac{1}{\log_m x}$$

Where log is log in any valid base.

Exponentiation

$$\textit{Exponentiation}: a^{\log_a x} = x$$
$$e^{\ln x} = x$$
$$e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$$

Common logs to the Base 10

$\log 2$	$\log 3$	$\log 5$	$\log 7$
$0.3010 \approx 0.3$	0.4771	0.7	0.84

1. Algebra of Logarithms

Example 1.1
Professor Gamble buys a lottery ticket which requires that he pick six different integers from 1 through 46, inclusive. He chooses his numbers so that the sum of the base-10 logarithms of his six numbers is an integer. It so happens that the integers on the winning ticket have the same property-the sum of the base-ten logarithms is an integer. What is the probability that Professor Gamble holds the winning ticket? **(AMC 12 2000/23)**

The sum of the base-10 logarithms of the six numbers is an integer.

Let the numbers be a, b, c, d, e, f:

$$\log_{10} a + \log_{10} b + \log_{10} c + \log_{10} d + \log_{10} e + \log_{10} f = n \in \mathbb{Z}$$
$$\log_{10} abcdef = n \in \mathbb{Z}$$
$$abcdef = 10^n, \qquad n \in \mathbb{Z}$$
$$abcdef = 2^n \cdot 5^n, \qquad n \in \mathbb{Z}$$

From the above, the only possible numbers that can be valid are numbers whose prime factorization only includes powers of 2 and 5.

$$1 = 2^0 \cdot 5^0$$
$$2 = 2^1 \cdot 5^0$$
$$5 = 2^0 \cdot 5^1$$
$$8 = 2^3 \cdot 5^0$$
$$10 = 2^1 \cdot 5^1$$
$$16 = 2^4 \cdot 5^0$$
$$20 = 2^2 \cdot 5^1$$
$$25 = 2^0 \cdot 5^2$$
$$25 = 2^0 \cdot 5^2$$
$$40 = 2^3 \cdot 5^1$$

1.1 Evaluating Logarithms

A. Log *Base* 10

When working with exponents, we learn properties like:

$$a^m \times a^n = a^{m+n}$$

Exponents are very common in some situations, and writing them out in superscripts is not very comfortable. In these kinds of situation, it is useful to have a mathematical technique which will you work with the exponents directly.

This is the base of the chapter logarithms.

1.2: Definition

$$\underbrace{a^x = n}_{\text{Exponential Form}} \Leftrightarrow \underbrace{\log_a n = x}_{\text{Logarithmic Form}}, a > 0, a \neq 1, n > 0$$

We look at some examples:

$$10^{\mathbf{2}} = 100 \Rightarrow \log_{10} 100 = \mathbf{2}$$
$$2^{\mathbf{3}} = 8 \Rightarrow \log_2 8 = \mathbf{3}$$

Example 1.3: Base 10

Evaluate:

Integers

A. $\log_{10} 1{,}000$
B. $\log_{10} 10{,}000$
C. $\log_{10} 10^a \times 100$
D. $\log_{10} 100^b \times 10$

Decimals

E. $\log_{10} 0.001$
F. $\log_{10} \frac{1}{1{,}000{,}000}$
G. $\log_{10} \frac{100}{0.001}$

H. $\log_{10} \frac{1000}{0.01}$
I. $\log_{10} \frac{0.01^a}{0.001^b}$

Integers

$$\log 1{,}000 = \log 10^3 = 3$$
$$\log 10{,}000 = \log 10^4 = 4$$

Decimals

$$\log 0.001 = \log 10^{-3} = -3$$

$$\log \frac{1}{1{,}000{,}000} = \log \frac{1}{10^6} = \log 10^{-6} = -6$$
$$\log \frac{100}{0.001} = \log \frac{10^2}{10^{-3}} = \log 10^{2-(-3)} = \log 10^5 = 5$$
$$\log_{10} \frac{0.01^a}{0.001^\mathrm{b}} = \log_{10} \frac{(10^{-2})^a}{(10^{-3})^\mathrm{b}} = -2a + 3b$$

Example 1.4: Base 10 with Radicals

Radicals in the Number

A. $\log_{10} \sqrt{10}$
B. $\log_{10} \sqrt{\frac{0.01}{1000}}$
C. $\log_{10} \sqrt[n]{\frac{0.1^a}{100^b}}$

Radicals in the Base

D. $\log_{\sqrt{10}} 10$
E. $\log_{\sqrt[3]{10}} \sqrt{10}$
F. $\log_{\sqrt[a]{10}} \sqrt{10}$

Radicals in the Number

$$\log_{10} \sqrt{10} = \log_{10} 10^{\frac{1}{2}} = \frac{1}{2}$$
$$\log_{10} \sqrt{\frac{0.01}{1000}} = \log_{10} \sqrt{10^{-5}} = \log_{10} 10^{-\frac{5}{2}} = -\frac{5}{2}$$
$$\log_{10} \sqrt[n]{\frac{0.1^a}{100^b}} = \log_{10} \sqrt[n]{\frac{(10^{-1})^a}{(10^2)^b}} = \log_{10} \sqrt[n]{10^{-a-2b}} = \log_{10} 10^{\frac{-a-2b}{n}} = -\frac{a+2b}{n}$$

Radicals in the Base

$$\log_{\sqrt{10}} 10 = \log_{10^{\frac{1}{2}}} 10 = 2$$
$$\log_{\sqrt[3]{10}} \sqrt{10} = \log_{10^{\frac{1}{3}}} \sqrt{10} = 3$$
$$\log_{\sqrt[a]{10}} \sqrt{10} = \log_{10^{\frac{1}{a}}} \sqrt{10} = a$$

Example 1.5: Integer Bases

Evaluate:

Integers

A. $\log_2 16$
B. $\log_3 81$

C. $\log_7 49$
D. $\log_5 125$
E. $\log_8 512$

F. $\log_9 81$
G. $\log_3 729$
H. $\log_2 2048$
I. $\log_5 15625$

Multi-Step

J. $\log_5 \frac{(125)(625)}{25}$ **(AHSME 1950/25)**
K. If $x = (\log_8 2)^{\log_2 8}$, then $\log_3 x$ equals: **(AHSME 1966/9)**

Decimals

L. $\log_2 0.125$
M. $\log_2 0.0625$ **(AHSME 1959/11)**

Radicals

N. $\log_3 27 \sqrt[4]{9}\sqrt[3]{9}$ **(AHSME 1953/22)**

Integers

$$\log_2 16 = 4$$
$$\log_3 81 = 4$$
$$\log_7 49 = 2$$
$$\log_5 125 = 3$$
$$\log_8 512 = 3$$
$$\log_9 81 = 2$$
$$\log_3 729 = 6$$
$$\log_2 2048 = 11$$
$$\log_5 15625 = 6$$

Multi-Step

Part J

$$\log_5 \frac{(125)(625)}{25} = \log_5 (5)(625)$$
$$= \log_5 5 \cdot 5^4 = \log_5 5^5 = 5$$

Part K

$$x = (\log_8 2)^{\log_2 8} = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$
$$\log_3 x = \log_3 \frac{1}{27} = \log_3 3^{-3} = -3$$

Decimals

First convert the expression from decimals to exponents:

Part L

$$0.125 = \frac{125}{1000} = \frac{1}{8}$$
$$\log_2 0.125 = \log_2 \frac{1}{8} = \log_2 2^{-3} = -3$$

Part M

$$0.0625 = \frac{625}{10000} = \frac{5^4}{2^4 \times 5^4} = \frac{1}{2^4} = 2^{-4}$$
$$\log_2 0.0625 = \log_2 2^{-4} = -4$$

Radicals

$$\log_3 3^3\, 3^{\frac{2}{4}} 3^{\frac{2}{3}} = \log_3 3^{3+\frac{1}{2}+\frac{2}{3}} = \log_3 3^{4\frac{1}{6}} = 4\frac{1}{6}$$

Example 1.6: Fractional and Radical Bases

- If we have fractions in either the number, or the base, we need to "move" the exponent from the denominator to the numerator. This results in negative numbers.
- If, both the number and its base are fractions, then we do not need to move, and the answer remains positive.
- If a number is not a perfect power, then we may still attempt to get a fractional answer. In this case, reduce all numbers to powers of prime numbers.

Evaluate all parts below.

Fractional Bases

A. $\log_3 \frac{1}{3}$
B. $\log_{\frac{1}{2}} 4$
C. $\log_{\frac{1}{3}} 81$
D. $\log_{\frac{1}{9}} \frac{1}{729}$
E. $\log_{\frac{1}{5}} 625$

Fractional Answers

F. $\log_8 4$
G. $\log_9 27$
H. $\log_{16} 8$
I. $\log_{81} 27$
J. $\log_{512} 128$
K. $\log_{25} 125$
L. $\log_{121} 1331$
M. $\log_{64} 2048$

Radical Bases

N. $\log_{\sqrt{2}} 2$
O. $\log_3 \sqrt{3}$
P. $\log_7 \sqrt[3]{7}$
Q. $\log_{\sqrt{11}} 11^2$
R. $\log_{2\sqrt{3}} 144$

Fractional Bases

$$\log_3 \frac{1}{3} = \log_3 3^{-1} = -1$$

$$\underbrace{\log_{\frac{1}{2}} 4 = x}_{\substack{Logarithmic\\Form}} \Rightarrow \underbrace{\left(\frac{1}{2}\right)^x = 4}_{\substack{Exponential\\Form}} \Rightarrow 2^{-x} = 2^2 \Rightarrow x = -2$$

$$\log_{\frac{1}{3}} 81 = -4$$

$$\log_{\frac{1}{9}} \frac{1}{729} = 3$$

$$\log_{\frac{1}{5}} 625 = -4$$

Fractional Answers

Part F

Observe that

$$2^3 = 8, \qquad 2^2 = 4$$

Hence, if we find the cube root of 8, and then square it, we will get 4.

$$\log_8 4 = \frac{2}{3}$$

$$\log_8 4 = x \Rightarrow 8^x = 4 \Rightarrow 2^{3x} = 2^2 \Rightarrow 3x = 2 \Rightarrow x = \frac{2}{3}$$

Parts G-M

$$\log_9 27 = \frac{3}{2}$$

$$\log_{16} 8 = \frac{3}{4}$$

$$\log_{81} 27 = \frac{3}{4}$$

$$\log_{512} 128 = \frac{7}{9}$$

$$\log_{25} 125 = \frac{3}{2}$$

$$\log_{121} 1331 = \frac{3}{2}$$

$$\log_{64} 2048 = \frac{11}{6}$$

Radical Bases

Part N

$$\log_{\sqrt{2}} 2 \Rightarrow \left(\sqrt{2}\right)^x = 2 \Rightarrow 2^{\frac{x}{2}} = 2^1 \Rightarrow \frac{x}{2} = 1 \Rightarrow x = 2$$

Parts O-Q

$$\log_3 \sqrt{3} = \frac{1}{2}$$

$$\log_7 \sqrt[3]{7} = \frac{1}{3}$$

$$\log_{\sqrt{11}} 11^2 = 4$$

Part R

$$\log_{2\sqrt{3}} 144 = \log_{2\sqrt{3}} 16 \times 9 = \log_{2\sqrt{3}} 2^4 \times \left(\sqrt{3}\right)^4$$
$$= \log_{2\sqrt{3}} \left(2\sqrt{3}\right)^4 = 4$$

If you can directly observe the answer, it will save you many steps.

$$\left(2\sqrt{3}\right)^2 = 12 \Rightarrow \left(2\sqrt{3}\right)^4 = 12^2 = 144 \Rightarrow \log_{2\sqrt{3}} 144 = 4$$

1.7: Important Values

$\log 2$	$\log 3$	$\log 5$	$\log 7$
$0.3010 \approx 0.3$	0.4771	0.7	0.84

B. Natural Logarithms

1.8: Natural Logarithm (Log Base e)

The number e has great importance in mathematics.

$$e \approx 2.71$$

The log of a number x to the base e is written:

$$\log_e x = \ln x$$

e is a constant like π

Example 1.9: Irrational Bases

Evaluate:

Basics

A. $\ln e^2$
B. $\ln e^5$
C. $\ln e^{\frac{1}{2}}$
D. $\log_\pi \pi^2$

Radicals

E. $\ln \sqrt{e}$
F. $\ln \sqrt[3]{e}$
G. $\ln \sqrt[\pi]{e^e}$

Basics

$$\log_e e^2 = 2$$
$$\log_e e^5 = 5$$
$$\log_e e^{\frac{1}{2}} = \frac{1}{2}$$
$$\log_\pi \pi^2 = 2$$

Radicals

$$\log_e \sqrt{e} = \log_e e^{\frac{1}{2}} = \frac{1}{2}$$
$$\log_e \sqrt[3]{e} = \frac{1}{3}$$
$$\ln \sqrt[\pi]{e^e} = \frac{e}{\pi}$$

1.2 Exponential and Logarithmic Form

A. Basics

1.10: Exponential and Logarithmic Form

$$\underbrace{a^x = n}_{\substack{\text{Exponential}\\ \text{Form}}} \Leftrightarrow \underbrace{\log_a n = x}_{\substack{\text{Logarithmic}\\ \text{Form}}}, a > 0, a \neq 1, n > 0$$

Every exponential expression has a logarithmic version, and vice versa. With practice, it is possible to move from one to the another. This kind of fluency is important in being able to simplify complicated equations.

$$\underbrace{\begin{matrix} 4^3 = 64 \\ 2^5 = 32 \end{matrix}}_{\text{Exponential Form}} \Leftrightarrow \underbrace{\begin{matrix} \log_4 64 = 3 \\ \log_2 32 = 5 \end{matrix}}_{\text{Logarithmic Form}}$$

Example 1.11:

Convert as indicated below.

Convert to logarithmic form

A. $4^2 = 16$
B. $2^{11} = 2048$
C. $6^y = 216$
D. $y^4 = 81$
E. $a^b = c$
F. $x = e^3$
G. $e^{2x} = 21$

Convert to exponential form

H. $\log_2 64 = 6$
I. $\log_x 64 = 6$
J. $\log_2 x = 6$
K. $\log_2 64 = x$
L. $\log_a b = c$
M. $\ln(2x) = 3.1$

Convert to logarithmic form

$$\log_4 16 = 2$$
$$\log_2 2048 = 11$$
$$\log_6 216 = y$$
$$\log_y 81 = 4$$
$$\log_a c = b$$
$$\ln x = 3$$
$$2x = \ln 21$$

Convert to exponential form

$$2^6 = 64$$
$$x^6 = 64$$
$$2^6 = x$$
$$2^x = 64$$
$$a^c = b$$

$$2x = e^{3.1}$$

Example 1.12: Simple Equations

Find the value of x in each case below:

Integer Values

A. $\log_x 128 = 7$
B. $\log_x 81 = 2$
C. $\log_x 729 = 6$
D. $\log_x 729 = 3$
E. $\log_x 729 = 2$
F. $\log_x 12 = 2$
G. $\log_x 27 = 2$
H. $\log_x 1024 = 3$

Fractional Values

I. $\log_x 4 = \frac{1}{2}$
J. $\log_{81} x = \frac{3}{4}$
K. $\log_{64} x = \frac{3}{2}$
L. $\log_x 6 = \log_5 \frac{1}{\sqrt{5}}$

Natural Logarithm Related

M. $\log_x e^3 = 4$

Integer Values

Part A

$$\log_x 128 = 7 \Rightarrow x^7 = 128 \Rightarrow x^7 = 2^7 \Rightarrow x = 2$$

Part B

$$\log_x 81 = 2 \Rightarrow x^2 = 81 \Rightarrow x = \pm\sqrt{81} = \pm 9$$

However, since x is the base of the logarithm,

$$x > 0 \Rightarrow Reject\ x = -9 \Rightarrow x = 9$$

Parts C,D,E

$$\log_x 729 = 6 \Rightarrow x = 3$$
$$\log_x 729 = 3 \Rightarrow x = 9$$
$$\log_x 729 = 2 \Rightarrow x = 27$$

Parts F,G,H

$$\log_x 12 = 2 \Rightarrow x^2 = 12 \Rightarrow x = \sqrt{12} = 2\sqrt{3}$$
$$\log_x 27 = 2 \Rightarrow x^2 = 27 \Rightarrow x = \sqrt{27} = 3\sqrt{3}$$
$$\log_x 1024 = 3 \Rightarrow x^3 = 1024 = 2^{10}$$
$$x = (2^{10})^{\frac{1}{3}} = 2^{\frac{10}{3}} = 2^{3\frac{1}{3}} = 2^3 \times 2^{\frac{1}{3}} = 8\sqrt[3]{2}$$

Fractional Values

$$4 = x^{\frac{1}{2}} \Rightarrow x = 16$$
$$x = 81^{\frac{3}{4}} = 3^3 = 27$$
$$x = 64^{\frac{3}{2}} = 8^3 = 512$$
$$\log_x 6 = -\frac{1}{2} \Rightarrow x^{-\frac{1}{2}} = 6 \Rightarrow \left(x^{-\frac{1}{2}}\right)^{-2} = 6^{-2}$$
$$x = 6^{-2} = \frac{1}{36}$$

Natural Logarithm Related

Part M

$$\log_x e^3 = 4 \Rightarrow x^4 = e^3 \Rightarrow x = \pm e^{\frac{3}{4}}$$

But

$$x > 0 \Rightarrow x = e^{\frac{3}{4}}$$

Example 1.13: Inequalities

A. $\log_2 x > 3$
B. $\log_x 2 > 3$
C. $\log_2(x + 2) > 7$

Part A

$$\underbrace{\log_2 x > 3}_{\substack{\textit{Logarithmic} \\ \textit{Form}}} \Rightarrow \underbrace{x > 2^3}_{\textit{Exponent Form}} \Rightarrow x > 8 \Rightarrow x \in (8, \infty)$$

Part B

$$\underbrace{\log_x 2 > 3}_{\substack{\textit{Logarithmic} \\ \textit{Form}}} \Rightarrow \underbrace{2 > x^3}_{\textit{Exponent Form}} \Rightarrow x^3 < 2 \Rightarrow x < \sqrt[3]{2} \Rightarrow x \in (0,1) \cup \left(1, \sqrt[3]{2}\right)$$

Part C

$$x + 2 > 2^7 \Rightarrow x > 126$$

Example 1.14

Find the value of x in the form $p^a q^b$, where p and q are prime numbers:

$$\log_{\sqrt{x}} 72 = \log_{3\sqrt{3}} \frac{\sqrt[4]{3}}{\sqrt[5]{3}}$$

Work with the RHS first:

$$\log_{3\sqrt[3]{3}}\frac{3^{\frac{1}{4}}}{3^{\frac{1}{5}}}=\log_{3\sqrt[3]{3}}3^{\frac{1}{4}-\frac{1}{5}}=\underbrace{\log_{3^{\frac{1}{3}}}3^{\frac{1}{20}}=y}_{Logarithmic\ Form}$$

$$\underbrace{\left(3^{\frac{1}{3}}\right)^{y}=3^{\frac{1}{20}}}_{Exponential\ Form}\Rightarrow 3^{\frac{y}{3}}=3^{\frac{1}{20}}\Rightarrow\frac{y}{3}=\frac{1}{20}\Rightarrow y=\frac{3}{20}$$

$$\underbrace{\log_{\sqrt{x}}72=\frac{3}{20}}_{\substack{Logarithmic\\ Form}}\Rightarrow\underbrace{\left(x^{\frac{1}{2}}\right)^{\frac{3}{20}}=72}_{\substack{Exponential\\ Form}}\Rightarrow x^{\frac{3}{40}}=72\Rightarrow\left(x^{\frac{3}{40}}\right)^{\frac{40}{3}}=72^{\frac{40}{3}}\Rightarrow x^{\frac{3}{40}\times\frac{40}{3}}=72^{\frac{40}{3}}$$

$$x=\left(2^3\times3^2\right)^{\frac{40}{3}}=2^{40}\times3^{\frac{80}{3}}$$

Example 1.15

If $\log_{2x}216 = x$, where x is real, then x is:

A. A non-square, non-cube integer
B. A non-square, non-cube, non-integral rational number
C. An irrational number
D. A perfect square
E. A perfect cube **(AHSME 1960/24)**

Convert from logarithmic to exponential form:

$$(2x)^x=216$$
$$2^x x^x=6^3=2^3\times3^3$$
$$x=3$$

Example 1.16: Change of Subject

Make x the subject in the following equations:

A. $y=\log_5(x+4)+3$
B. $y=4\ln\left(\sqrt{x}+3\right)$
C. $y=\sqrt{\ln(x^2+2)-2}$

Part A

Subtract 3 from both sides:

$$y-3=\log_5(x+4)$$

Convert to exponential form:

$$5^{y-3}=x+4$$

Subtract 4 from both sides:

$$5^{y-3}-4=x$$

Part B

$$\frac{y}{4}=\ln\left(\sqrt{x}+3\right)$$
$$e^{\frac{y}{4}}=\sqrt{x}+3$$
$$\left(e^{\frac{y}{4}}-3\right)^2=x$$

Part C

$$y^2=\ln(x^2+2)-2$$
$$y^2+2=\ln(x^2+2)$$
$$e^{y^2+2}=x^2+2$$
$$e^{y^2+2}-2=x^2$$
$$\pm\sqrt{e^{y^2+2}-2}=x$$

Example 1.17

A. Find the domain of $y=\sqrt{\ln(x^2+2)-2}$

The expression for which we are finding the logarithm must be positive:

$$x^2+2>0\Rightarrow Always\ holds$$

The expression inside the square root must be non-negative:

$$\ln(x^2+2)-2 \ge 0$$
$$\ln(x^2+2) \ge 2$$

Convert to exponential form:

$$x^2+2 \ge e^2$$

Subtract 2 from both sides:

$$x^2 \ge e^2-2$$

Use the property that $x^2 \ge a \Rightarrow x > \sqrt{a}\ OR\ x < -\sqrt{a}$:

$$x \ge \sqrt{e^2-2}\ OR\ x \le -\sqrt{e^2-2}$$

Example 1.18

A. $(Answer\ in\ simplest\ radical\ form)$ If $\log_6 x = 2.5$, the value of x is: (AHSME 1953/5)
B. $(Quadratics)$ The values of a in the equation: $\log_{10}(a^2-15a) = 2$ are: (AHSME 1951/22)
C. If $\log_{10} x^2 - 3x + 6 = 1$, the value(s) of x is/are: (AHSME 1953/21)

Part A

Convert to exponential form:

$$x = 6^{2.5} = 6^2 \times 6^{0.5} = 36\sqrt{6}$$

Part B

Convert to exponential form:

$$10^2 = a^2 - 15a$$
$$(a-20)(a+5) = 0$$
$$a \in \{-5,20\}$$

Part C

Convert to exponential form:

$$x^2 - 3x + 6 = 10$$
$$x^2 - 3x - 4 = 0$$
$$(x-4)(x+1) = 0$$
$$x \in \{-1,4\}$$

Example 1.19: Equations Reducible to Quadratics

A. Find the value(s) of x that satisfy $(\log x)^2 - 11\log x + 10 = 0$
B. $(Challenge)$ Find the last three digits of the product of the positive roots of $\sqrt{1995}x^{\log_{1995} x} = x^2$. (AIME 1995/2)

Part A

Change of variable.

Substitute $y = \log x$:

$$y^2 - 11y + 10 = 0 \Rightarrow y = \{1,10\}$$

Change back to the original variable

$$\log x = \{1,10\} \Rightarrow x = \{10, 10^{10}\}$$

Part B

Change of variable.

Let $\underbrace{y = \log_{1995} x}_{Logarithmic\ Form} \Rightarrow \underbrace{1995^y = x}_{Exponential\ Form}$. Then

$$\sqrt{1995}(1995^y)^y = (1995^y)^2$$

Combine using properties of exponents:

$$1995^{\frac{1}{2}+y^2}=1995^{2y}$$

Bases are same, the exponents must be same:

$$\frac{1}{2}+y^2=2y\Rightarrow y^2-2y+\frac{1}{2}=0$$

Apply the quadratic formula: $a=1,b=-2,c=\frac{1}{2}$:

$$y=\frac{2\pm\sqrt{4-(4)(1)\left(\frac{1}{2}\right)}}{2}=\frac{2\pm\sqrt{2}}{2}$$

Change back to the original variable

$$\underbrace{\log_{1995}x=\frac{2\pm\sqrt{2}}{2}}_{Logarthmic\ Form}\Rightarrow\underbrace{x=\left\{1995^{\frac{2\pm\sqrt{2}}{2}}\right\}}_{Exponential\ Form}$$

Both solutions are positive. Product is:

$$1995^{\frac{2+\sqrt{2}}{2}}\times1995^{\frac{2-\sqrt{2}}{2}}=1995^{\frac{4}{2}}=1995^2$$

Last three digits are the remainder when 1995^2 is divided by 1000:

$$1995^2\equiv995^2\equiv(-5)^2\equiv25\ (mod\ 1000)$$

Alternately, you can get the last three digits as:

$$1995^2=(2000-5)^2$$
$$=2000^2-(2)(-5)(2000)+25$$

The first two terms do not matter:

$$Final\ Answer=25$$

1.20: Nested Logs

Nested logs are best solved inside out. They look intimidating but can be solved with practice.

Example 1.21: Nested Logs

For each part below, find the value (if it is an expression), or solve for the variable (if it is an equation).

Double Nesting

A. $\log_5[\log_{10}x]=0$
B. $\log_3[\log_4 x]=0$
C. $[\log_{10}(5\log_{10}100)]^2$ **(AHSME 1964/1)**

Triple Nesting

D. $\log_3\{\log_2[\log_{10}10^{512}]\}$
E. $\log_4[\log_2(\log_3 x)]=0$
F. $\log_{\frac{1}{2}}\left[\log_{\frac{1}{3}}(\log_{27}x)\right]=0$
G. $\log_{\frac{1}{3}}\left[\log_{\frac{1}{4}}\left(\log_{\frac{1}{5}}x\right)\right]=0$

Quadruple Nesting

H. $\log_{\pi^2+e}\left\{\log_{\frac{1}{2}}[\log_{\sqrt[4]{2}}(\log_7 2401)]+4\right\}$

Double Nesting

$$\log_{10}x=1\Rightarrow x=10$$
$$\log_4 x=1\Rightarrow x=4$$
$$[\log_{10}(5\times2)]^2=[\log_{10}(10)]^2=[1]^2=1$$

Triple Nesting

$$\log_3\{\log_2[512]\} = \log_3 9 = 2$$
$$\log_2(\log_3 x) = 1 \Rightarrow \log_3 x = 2 \Rightarrow x = 9$$
$$\log_{\frac{1}{3}}(\log_{27} x) = 1 \Rightarrow \log_{27} x = \frac{1}{3} \Rightarrow x = 27^{\frac{1}{3}} = 3$$
$$\log_{\frac{1}{4}}\left(\log_{\frac{1}{5}} x\right) = 1 \Rightarrow \log_{\frac{1}{5}} x = \frac{1}{4} \Rightarrow x = \left(\frac{1}{5}\right)^{\frac{1}{4}} = 5^{-\frac{1}{4}}$$

Quadruple Nesting

$$\log_{\pi^2+e}\left\{\log_{\frac{1}{2}}\left[\log_{\sqrt[4]{2}}(4)\right] + 4\right\} = \log_{\pi^2+e}\left\{\log_{\frac{1}{2}}[8] + 4\right\} = \log_{\pi^2+e}\{1\} = 0$$

Example 1.22

A. If $\log_2(\log_2(\log_2 x)) = 2$, then how many digits are in the base-ten representation for x? **(AHSME 1993/11)**
B. If $\log_7(\log_3(\log_2 x)) = 0$, then $x^{-\frac{1}{2}}$ equals **(AHSME 1983/12)**
C. If $\log_2(\log_3(\log_4 x)) = \log_3(\log_4(\log_2 y)) = \log_4(\log_2(\log_3 z)) = 0$, then the sum $x + y + z$ is equal to **(AHSME 1971/21)**

Part A

$$\log_2(\log_2 x) = 2^2 = 4$$
$$\log_2 x = 2^4 = 16$$
$$x = 2^{16} = 2^{10} \cdot 2^6 = 1024 \cdot 64 \approx 64000$$
$$5\ Digits$$

Part B

$$\log_3(\log_2 x) = 1$$
$$\log_2 x = 3$$
$$x = 8$$
$$x^{-\frac{1}{2}} = 8^{-\frac{1}{2}} = \frac{1}{8^{\frac{1}{2}}} = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$$

Part C

$$\log_3(\log_4 x) = \log_4(\log_2 y) = \log_2(\log_3 z) = 1$$
$$x = 4^3 = 64$$
$$y = 2^4 = 16$$
$$z = 3^2 = 9$$
$$x + y + z = 64 + 16 + 9 = 89$$

Example 1.23

How many different prime numbers are factors of N if
$$\log_2(\log_3(\log_5(\log_7 N))) = 11$$
(AHSME 1998/12)

$$N = 7^{5^{3^{2^{11}}}} = 7^n$$

The only prime factor is
$$7 \Rightarrow 1\ prime$$

Example 1.24

A. $\log_7|2\log_5 x| = 0$

$$|2\log_5 x| = 1 \Rightarrow 2\log_5 x = \pm 1 \Rightarrow \log_5 x = \pm\frac{1}{2} \Rightarrow x \in \left\{\frac{1}{\sqrt{5}}, \sqrt{5}\right\}$$

B. Simultaneous Equations

Example 1.25

Solve the system of equations:

$$5x - 3\ln y = 2$$
$$x + \ln y = 1$$

Use a change of variable. Let $z = \ln y$. Then the equations become:

$$5x - 3z = 2$$
$$x + z = 1$$

Solve the first equation for z:

$$5x - 2 = 3z \Rightarrow z = \frac{5x - 2}{3}$$

Solve the second equation for $\ln y$:

$$z = 1 - x$$

Equate the above two:

$$\frac{5x - 2}{3} = 1 - x$$
$$5x - 2 = 3 - 3x$$
$$8x = 5$$

Substitute $x = \frac{5}{8}$ into the first equation:

$$z = \ln y = 1 - x = 1 - \frac{5}{8} = \frac{3}{8}$$
$$y = e^{\frac{3}{8}}$$

1.3 Domain

A. Uniqueness of Log Function

1.26: Uniqueness of Logarithm

$$\log_a x = n$$

For a specific base a, x is unique for unique values of n.

1.27: Using Uniqueness of Logarithm

We can use the uniqueness of logarithm to conclude:

$$\log a = \log b \Leftrightarrow a = b$$

Example 1.28: No Solutions

Find the value(s) of x that satisfy $\log(x + 1) - \log x = 0$

$$\log(x + 1) = \log x$$
$$x + 1 = x \Rightarrow 1 = 0 \Rightarrow x \in \phi$$

B. Variables in the Base

1.29: Restrictions on Base

$$\log_a n = x, \qquad a > 0, a \neq 1, n > 0$$

a is the base of the logarithm
$a > 0$ because for a negative number, the answer alternates between positive and negative as the power to which it is raised is even or odd.
$a \neq 1$ since 1 to any power is 1 itself.

1.30: Restrictions on Number

$$\log_a n = x, \qquad a > 0, a \neq 1, n > 0$$

Logarithms of negative numbers are not defined.

Why is n not allowed to be negative? As an example, consider:

$$\log_{10} -5 \Rightarrow 10^x = -5 \Rightarrow \textit{Has no } \textbf{real} \textit{ solution}$$

We have to check two things:

- Base should be positive, and not equal to 1.
- Number of which we are taking the base should be positive.

Example 1.31

Are the following logarithms defined:

A. $\log_3 5$
B. $\log_1 1$
C. $\log_{\frac{1}{3}} \frac{1}{2}$
D. $\log_4(-2)$
E. $\log_{(-2)}(4)$

Part A

$$No. = 5 > 0 \Rightarrow Allowed$$
$$Base = 3 > 0, 3 \neq 1 \Rightarrow Allowed$$
$$Defined$$

Part B

$$No. = 1 \Rightarrow Allowed$$
$$Base = 1 \Rightarrow Not\ Allowed$$
$$Not\ Defined$$

Part C

$$No. = \frac{1}{2} > 0 \Rightarrow Allowed$$
$$Base = \frac{1}{3} > \frac{0,1}{3} \neq 1 \Rightarrow Allowed$$
$$Defined$$

Part D

$$Number = -2 < 0 \Rightarrow Not\ Valid$$

Part E

$$Base = -2 < 0 \Rightarrow Not\ Valid$$

Example 1.32: Finding Domain

Find the domain of the following logarithms:

Basics

A. $\log_5 x$
B. $\log_x 5$

Linear Expressions

C. $\log_{x-2}(x+5)$
D. $\log_{\frac{2}{7}x+\sqrt{2}}\left(-\frac{3}{5}x + \frac{2}{3}\right)$

Basics
Part A
$$x > 0 \Rightarrow x \in (0, \infty)$$
Part B
$$x > 0, x \neq 1 \Rightarrow (0,1) \cup (1, \infty)$$
Linear Expressions
Part C
$$Number: x + 5 > 0 \Rightarrow \underbrace{x > -5}_{\substack{Valid\ Domain \\ for\ Number}}$$
$$Base: x - 2 > 0 \Rightarrow \underbrace{x > 2}_{\substack{Valid\ Domain \\ for\ Base}}$$
$$Base: x - 2 = 1 \Rightarrow \underbrace{x = 3}_{\substack{Not\ Allowed \\ for\ Base}}$$
Combine the three conditions:
$$x > 2, x \neq 3$$
$$x \in (2,3) \cup (3, \infty)$$
$$\{x | x > 2, x \neq 3\}$$
Part D
The number must be greater than zero:
$$-\frac{3}{5}x + \frac{2}{3} > 0 \Rightarrow -\frac{3}{5}x > -\frac{2}{3} \Rightarrow x < \frac{2}{3} \times \frac{5}{3} \Rightarrow x < \frac{10}{9}$$
The base must be greater than zero:
$$\frac{2}{7}x + \sqrt{2} > 0 \Rightarrow \frac{2}{7}x > -\sqrt{2} \Rightarrow x > -\frac{7}{2}\sqrt{2}$$
The base must not be equal to 1:
$$\frac{2}{7}x + \sqrt{2} = 1 \Rightarrow \frac{2}{7}x = 1 - \sqrt{2} \Rightarrow x = \frac{7}{2}(1 - \sqrt{2})$$
Which we can estimate as:
$$\approx -\frac{7}{2}(0.41)$$
Combine the three conditions:
$$\left(-\frac{7}{2}\sqrt{2}, -\frac{7}{2}(1 - \sqrt{2})\right) \cup \left(-\frac{7}{2}(1 - \sqrt{2}), \frac{10}{9}\right)$$

Example 1.33: Finding Domain of Quadratic Expressions
Find the domain of $\log_{x^2+5x+6}(x^2 - 7x + 12)$

The number must be greater than zero:
$$x^2 - 7x + 12 > 0 \Rightarrow (x-3)(x-4) > 0 \Rightarrow x \in \mathbb{R} \setminus [3,4]$$
The base must be greater than zero:
$$x^2 + 5x + 6 > 0 \Rightarrow (x+2)(x+3) > 0 \Rightarrow x \in \mathbb{R} \setminus [-3,-2]$$
The base must not be equal to 1:
$$x^2 + 5x + 6 = 1 \Rightarrow x^2 + 5x + 5 = 0 \Rightarrow x = \frac{-5 \pm \sqrt{5^2 - (4)(1)(5)}}{2(1)} = \frac{-5 \pm \sqrt{5}}{2}$$
$$x \in \mathbb{R} \setminus \left\{ [-3,-2] \cup [3,4] \cup \left\{ \frac{-5 \pm \sqrt{5}}{2} \right\} \right\}$$

Example 1.34: Equations
Solve the following:

A. $\log_x(3x^2 + 10x) = 3$
B. $\log_{x+2}(3x^2 + 4x - 14) = 2$

Part A
Convert from logarithmic form to exponential form:
$$3x^2 + 10x = x^3$$
$$x^3 - 3x^2 - 10x = 0$$
$$x(x^2 - 3x - 10) = 0$$
$$x(x-5)(x+2) = 0$$
$$x = \{\mathbf{-2}, \mathbf{0}, 5\}$$
But the base must be $x > 0, x \neq 1$. Hence,
$$x = 5$$
Substitute $x = 5$ in the number $3x^2 + 10x$:
$$3(5^2) + 10(5) = +ve$$

Part B
Convert from logarithmic form to exponential form:
$$3x^2 + 4x - 14 = (x+2)^2$$
$$2x^2 - 18 = 0$$
$$x^2 = 9$$
$$x = \{-3, 3\}$$
Check -3:
$$x + 2 = -3 + 2 = -1 \Rightarrow Reject - 3$$
Check 3:
$$x + 2 = 3 + 2 = 5 \Rightarrow Valid$$
$$3(3)^2 + 4(3) - 14 = 27 + 12 - 14 \Rightarrow +ve$$

$$x = 3$$

Example 1.35: Domain

Number

A. Linear
B. Radical
C. Quadratic
D. Polynomial
E. Exponential

Base

A. Linear
B. Radical
C. Quadratic
D. Polynomial
E. Exponential

C. Review

Example 1.36

Is the solution below correct?

$$\log_x e^2 = 4 \Rightarrow x^4 = e^2 \Rightarrow x = \pm(e^2)^{\frac{1}{4}} = \pm\sqrt{e}$$

$$x \text{ is base} \Rightarrow x > 0 \Rightarrow x \neq -\sqrt{e} \Rightarrow Sol \text{ is incorrect}.$$

1.4 Product and Quotient Rules

A. Product Rule

Exponentiation lets us convert multiplication into addition:

$$\underbrace{x^p \times x^q}_{Multiplication} = \underbrace{x^{p+q}}_{Addition}$$

Since logarithms are defined based on exponents, every exponent property has a corresponding logarithm property. We start by looking the logarithm property corresponding to the exponent property above.

$$\log 10 + \log 100 = 1 + 2 = 3 = \log 1000$$

1.37: Product Rule: Log of a product equals the sum of logs

Log of a product equals the sum of the logs:

$$\underbrace{\log_a xy}_{Multiplication} = \underbrace{\log_a x + \log_a y}_{Addition}$$

Applies to any valid base

Let:

$$\log_a x = m \Rightarrow a^m = x, \qquad \log_a y = n \Rightarrow a^n = y$$

Then:

$$RHS = \log_a x + \log_a y = m + n$$

Substitute $a^m = x$ and $a^n = y$:

$$LHS = \log_a xy = \log_a a^m a^n = \log_a a^{m+n} = m + n = RHS$$

Shorter Version

Let $x = a^m$, and $y = a^n$ for some m and n.

$$\log_a xy = \log_a a^m a^n = \log_a a^{m+n} = m + n = \log_a a^m + \log_a a^n = \log_a x + \log_a y$$

Example 1.38: Expressions

Combine into a single term:

A. $\log_{20} 10 + \log_{20} 2$
B. $\log_6 12 + \log_6 3$
C. $\log_{10} 125 + \log_{10} 8$
D. $1 + \log_5 4$
E. $2 + \log_2 5$

Use the product rule to expand:

F. $\log_2 20$
G. $\log_{10} 125 + \log_{10} 4$

Combine into a single term:

Parts A, B and C

$$\log_{20} 10 + \log_{20} 2 = \log_{20} 10 \times 2 = \log_{20} 20 = 1$$
$$\log_6 12 + \log_6 3 = \log_6 36 = 2$$
$$\log_{10} 125 + \log_{10} 8 = \log_{10} 1000 = 3$$

Parts D and E

Here, some parts of the expression are not in logarithmic form. So, we need to convert them into logarithmic form before we can apply the property.

$$1 + \log_5 4 = \log_5 5 + \log_5 4 = \log_5 20$$
$$2 + \log_2 5 = \log_2 4 + \log_2 5 = \log_2 20$$

Use the product rule to expand:

$$\log_2 20 = \log_2(5 \times 4) = \log_2 5 + \log_2 4 = \log_2 5 + 2$$
$$\log_{10} 125 + \log_{10} 4 = \log_{10} 500$$
$$= \log_{10} 100 + \log_{10} 5 = 2 + \log_{10} 5$$

Example 1.39

For all positive integers n, let $f(n) = \log_{2002} n^2$. Let $N = f(11) + f(13) + f(14)$. Which of the following relations is true? (AMC 12A 2002/14)

A. $N < 1$
B. $N = 1$
C. $1 < N < 2$
D. $N = 2$
E. $N > 2$

$$N = f(11) + f(13) + f(14) = \log_{2002} 11^2 + \log_{2002} 13^2 + \log_{2002} 14^2$$

Using the product rule:

$$= \log_{2002} 11^2 \cdot 13^2 \cdot 14^2$$

Rearrange:

$$= \log_{2002} 2^2 \cdot 7^2 \cdot 11^2 \cdot 13^2$$

Substitute $7 \cdot 11 \cdot 13 = 1001$:

$$= \log_{2002} 2002^2$$
$$= 2$$

Option D

Example 1.40

A. Find the sum of $\log 1 + \log 2 + \cdots + \log n$
B. Find the sum of eight terms of $\ln x + \ln x^2 y + \ln x^3 y^2 + \cdots$
C. Simplify $X_a = \log_a \frac{1}{2} + \log_a \frac{2}{3} + \log_a \frac{3}{4} + \cdots + \log_a \frac{1946}{1947}, a > 0, a \neq 1$

Part A

$$\log 1 + \log 2 + \cdots + \log n = \log(1 \times 2 \times \ldots \times n) \log n!$$

Part B

Use the property that $\ln a + \ln b = \ln ab$

$$\ln \underbrace{x \times x^2 y \times x^3 y^2 \times \ldots}_{8\,Terms}$$
$$\ln(x^{1+2+\cdots+8} \times y^{0+1+\cdots+7})$$
$$\ln\left(x^{\frac{8\times 9}{2}} \times y^{\frac{7\times 8}{2}}\right)$$
$$\ln(x^{36}y^{28})$$

Part C

This is a telescoping series, which we will able to collapse once we combine:

$$X_a = \log_a \frac{1}{2} \times \frac{2}{3} \times \ldots \times \frac{1946}{1947}$$

Telescope:

$$X_a = \log_a \frac{1}{1947}$$

Example 1.41: Equations

Solve for x:

$$\frac{1}{2}(2\log 4 + 2\log 2) = \log x$$

$$\log 4 + \log 2 = \log x$$
$$\log 8 = \log x$$
$$8 = x$$

Example 1.42: Equations

A. $\log(x+5) = \log 3 + \log 5$

B. If $\log(x) + \log 2 = \log\frac{1}{2} + \log 1024$, then find $\log_2 x$.

C. $\log\left(\frac{x}{2}\right) + \log\left(\frac{x}{3}\right) = \log(x) + \log 3$

D. If $\log_{10} m = b - \log_{10} n$, then $m =$ **(AHSME 1950/26)**

E. If $\log_{10} a + \log_{10} b = c - \log_{10} d$, then $a =$

F. If all the logarithms are real numbers, the equality $\log(x+3) + \log(x-1) = \log(x^2 - 2x - 3)$ is satisfied for: **(AHSME 1968/23)**

Part A

$$\log(x+5) = \log 15 \Rightarrow x + 5 = 15 \Rightarrow x = 10$$

Part B

$$\log(2x) = \log 512$$
$$2x = 512$$
$$x = 256$$
$$\log_2 256 = 8$$

Part C

$$\log\left(\frac{x^2}{6}\right) = \log(3x)$$

$$\frac{x^2}{6} = 3x$$
$$x^2 = 18x$$

Since $x \neq 0$:

$$x = 18$$

Part D

$$\log_{10} m + \log_{10} n = b$$
$$\log_{10} mn = b$$

Convert from logarithmic form to exponential form:

$$mn = 10^b$$

$$m = \frac{10^b}{n}$$

Part E

$$\log_{10} abd = c$$

Convert from logarithmic form to exponential form:

$$abd = 10^c$$

$$a = \frac{10^c}{bd}$$

Part F

Using the product rule:

$$\log(x+3)(x-1) = \log(x^2 - 2x - 3)$$

$$\log(x^2 + 2x - 3) = \log(x^2 - 2x - 3)$$

$$x^2 + 2x - 3 = x^2 - 2x - 3$$

$$4x = 0$$

$$x = 0$$

However:

$$\log(x-1) = \log(0-1) = \log -1$$

Hence, there are no real solutions.

$$x \in \phi$$

Example 1.43: Freshman's Dream

Mark the correct option

$\log p + \log q = \log(p+q)$ only if:

A. $p = q = zero$
B. $p = \frac{q^2}{1-q}$
C. $p = q = 1$
D. $p = \frac{q}{q-1}$
E. $p = \frac{q}{q+1}$ (AHSME 1952/18)

$$\log pq = \log(p+q)$$

$$pq = p + q$$

$$pq - p = q$$

$$p(q-1) = q$$

$$p = \frac{q}{q-1}$$

Example 1.44: Inequalities

A. $\log_2 x + \log_2 5 > 3$
B.

Part A

$$\log_2 5x > 3$$

Convert from logarithmic form to exponential form:

$$5x > 2^3$$

$$5x > 8$$

$$x > \frac{8}{5}$$

$$x \in \left(\frac{8}{5}, \infty\right)$$

Example 1.45: System of Equations

If $\log(xy^3) = 1$ and $\log(x^2y) = 1$, what is $\log(xy)$? (AMC 12B 2003/17)

Convert the given equations to exponential form:

$$\log_b(xy^3) = 1 \Rightarrow xy^3 = b$$

$$\log_b(x^2y) = 1 \Rightarrow x^2y = b$$

$$xy^3 = x^2y \Rightarrow x = y^2$$

$$xy^3 = y^2y^3 = y^5 = b \Rightarrow y = b^{\frac{1}{5}}$$

$$\log_b(xy) = \log_b(y^2y) = \log_b(y^3) = \log_b\left(\left(b^{\frac{1}{5}}\right)^3\right) = \log_b b^{\frac{3}{5}} = \frac{3}{5}$$

Note:

1. The answer is independent of b. If you assume any value of b, say 10, you will still get the same answer.

Example 1.46: Vieta's Formulas

For some real numbers a and b, the equation $8x^3 + 4ax^2 + 2bx + a = 0$ has three distinct positive roots. If the sum of the base-2 logarithms of the roots is 5, what is the value of a? **(AMC 12B 2004/17)**

Since the sum of the roots is 5:

$$\log_2 \alpha + \log_2 \beta + \log_2 \gamma = 5, \qquad \alpha, \beta, \gamma \ are\ roots$$

Use the product rule for logarithms:

$$\log_2 \alpha\beta\gamma = 5$$

Convert to exponential form:

$$\alpha\beta\gamma = 2^5 = 32$$

From Vieta's Formulas, the product of roots of the cubic equation above is:

$$-\frac{a}{8} = 32 \Rightarrow a = -256$$

B. Quotient Rule

First, look at the exponent rule for division.

$$\frac{x^m}{\underbrace{x^n}_{Division}} = \underbrace{x^m \div x^n}_{Division} = \underbrace{x^m \times x^{-n}}_{Multiplication} = \underbrace{x^{m-n}}_{Subtraction}$$

1.47: Quotient Rule

Log of a quotient equals the difference of the logs.

$$\underbrace{\log \frac{x}{y}}_{Division} = \underbrace{\log x - \log y}_{Subtraction}$$

Let $x = a^m, y = a^n$:

$$\log_a \frac{x}{y} = \log_a \frac{a^m}{a^n} = \log_a a^{m-n} = m - n = \log_a a^m - \log_a a^n = \log x - \log y$$

True or False 1.48:

$$\log x - \log y = \frac{\log x}{\log y}$$

False

Example 1.49: Expressions

Simplify:

A. $\log_2 \frac{5}{4}$
B. $\log_2 20 - \log_2 5$
C. $\log_7 686 - \log_7 2$
D. $\log \frac{a}{b} + \log \frac{b}{c} + \log \frac{c}{d} - \log \frac{ay}{dx}$ **(AHSME 1957/5)**
E. $\log_z \frac{a}{b} + \log_z \frac{b}{c} + \cdots + \log_z \frac{y}{z} - \log_z a$

Parts A-C

$$\log_2 \frac{5}{4} = \log_2 5 - \log_2 4 = \log_2(5) - 2$$
$$\log_2 20 - \log_2 5 = \log_2 \frac{20}{5} = \log_2 4 = 2$$
$$\log_7 686 - \log_7 2 = \log_7 \frac{686}{2} = \log_7 343 = 3$$

Parts D-E (Telescoping)

$$\log \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \times \frac{dx}{ay} = \log \frac{x}{y}$$
$$\log_z \frac{a}{b} \times \frac{b}{c} \times \ldots \times \frac{y}{z} \times \frac{1}{a} = \log_z \frac{1}{z} = \log_z z^{-1} = -1$$

Example 1.50

$$\log_b 10 = X, \log_b 5 = Y, \log_b 3 = Z$$

Write the following expressions in terms of X, Y and Z, if possible. Explain, for the expressions which cannot be written in terms of X, Y and Z why they cannot be so written.

A. $\log_b \left(\frac{50}{9}\right)$
B. $\log_b \left(\frac{20}{3}\right)$
C. $\log_b 2$
D. $\log_b 17$

Part A

$$\log_b \left(\frac{50}{9}\right) = \log_b \left(\frac{5 \times 10}{3^2}\right) = \log_b 5 + \log_b 10 - \log_b(3^2) = Y + X - 2Z$$

Where in the last step we made use of:

$$\log_b 3 = Z \Rightarrow b^Z = 3 \Rightarrow b^{2Z} = 3^2 \Rightarrow \log_b(3^2) = 2Z$$

Part B

$$\log_b \left(\frac{20}{3}\right) = \log_b \left(\frac{100}{15}\right) = \log_b \left(\frac{10^2}{3 \times 5}\right) = \log_b 10^2 - \log_b 3 - \log_b 5 = 2X - Z - Y$$

Part C

$$\log_b 2 = \log_b \frac{10}{5} = \log_b 10 - \log_b 5 = X - Y$$

Part D

Note that

$$\log_b 3 = Z \Rightarrow 3 \text{ is prime}$$
$$\log_b 5 = Y \Rightarrow 5 \text{ is prime}$$

$$\log_b 2 \times 5 = X \Rightarrow 2{,}5 \text{ are prime}$$

The prime numbers at our disposal are:

$$\{2,3,5\}$$

17 is a prime number not in the above set. Hence $\log_b 17$ cannot be written in terms of only X, Y and Z.

Example 1.51: Equations

A. Solve for x: $\log_3 x - \log_3 4 = \log_3 12$
B. Solve for x:$\log_3 x - \log_3 \frac{1}{10} = 2 + \log_3 5$
C. If $P = \frac{s}{(1+k)^n}$ then n equals: **(AHSME 1958/12)**

Part A

$$\log_3 \frac{x}{4} = \log_3 12 \Rightarrow \frac{x}{4} = 12 \Rightarrow x = 48$$

Part B

$$\log_3 \frac{x}{0.1} = \log_3 45 \Rightarrow \frac{x}{0.1} = 45 \Rightarrow x = 4.5$$

Part C

$$(1+k)^n = \frac{s}{P}$$

Take logs to the base $1 + k$:

$$\log_{1+k}(1+k)^n = \log_{1+k}\frac{s}{P}$$
$$n = \log_{1+k}\frac{s}{P}$$

$$(1+k)^n = \frac{s}{P}$$
$$\log(1+k)^n = \log\frac{s}{P}$$
$$n \log 1 + k = \log\frac{s}{P}$$
$$n = \frac{\log\frac{s}{P}}{\log(1+k)} = \frac{\log s - \log P}{\log(1+k)}$$

Example 1.52: Concept Check

Of the following:

1. $a(x-y) = ax - ay$
2. $a^{x-y} = a^x - a^y$
3. $\log(x-y) = \log x - \log y$
4. $\frac{\log x}{\log y} = \log x - \log y$
5. $a(xy) = ax \times ay$

A. Only 1 and 4 are true
B. Only 1 and 5 are true
C. Only 1 and 3 are true
D. Only 1 and 2 are true
E. Only 1 is true **(AHSME 1950/18)**

$$2(x+y) = 2x + 2y$$

$$a(xy) = ax \times ay = a^2xy$$

$$2(3 \times 4) = 2(12) = 24$$
$$2(3 \times 4) = (2 \times 3) \times (2 \times 4) = 6 \times 8 = 48$$
$$3(xy) = 3x \times 3y = 9xy \Rightarrow Wrong$$

Option E.
Only 1 is true.

1.5 Power Rule

A. Power Rule

The power rule in exponentiation tells us that we can convert exponentiation into multiplication of exponents.

$$\underbrace{(x^m)^n}_{Exponentiation} = \underbrace{x^{mn}}_{Multiplication}$$

The same rule is applicable for logarithms. Exponents can be converted into multiplication of logs.

1.53: Power Rule

Log of the n^{th} power of a quantity is n times the log of the quantity:

$$\underbrace{\log x^n}_{Exponentiation} = \underbrace{n \log x}_{Multiplication}$$

Let

$$m = \log_b x$$

Convert to exponential form:

$$b^m = x$$

Raise both sides to power n

$$b^{nm} = x^n$$

Take log both sides:

$$nm = \log x^n$$

Substitute $m = \log_b x$:

$$n(\log_b x) = \log x^n$$

Which is what we wished to prove

Example 1.54

Evaluate

A. $\log_a a^3, a > 0, a \neq 1$
B. $\log_9 27^3$
C. $\log_8 2^9$
D. $\log 8 \div \log \frac{1}{8}$ (AHSME 1961/6)
E. $\frac{\log 27}{\log 9}$
F. $\log 0.25 \div \log 8$

Combine into a single logarithm

G. $2 \log 3 + 3 \log 2$
H. $\frac{1}{2} \log \frac{1}{49} + \frac{1}{3} \log 125$
I. $2 \log 3 + 3 \log 5 - \frac{1}{2} \log 125 - \frac{1}{3} \log 9$

Evaluate

$$\log_a a^3 = 3$$
$$\log_9 27^3 = \log_9(3^3)^3 = \log_9 3^9 = \frac{9}{2}$$
$$\log_8 2^9 = \log_8(2^3)^3 = \log_8(8)^3 = 3$$

$$\log 8 \div \log\frac{1}{8} = \frac{\log 8}{\log 8^{-1}} = \frac{\log 8}{-\log 8} = -1$$
$$\frac{\log 3^3}{\log 3^2} = \frac{3\log 3}{2\log 3} = \frac{3}{2}$$

$$\frac{\log 0.25}{\log 8} = \frac{\log\frac{1}{4}}{\log 2^3} = \frac{\log\ 2^{-2}}{\log 2^3} = \frac{-2\log 2}{3\log 2} = -\frac{2}{3}$$

Combine into a single logarithm

$$2\log 3 + 3\log 2 = \log 9 + \log 8 = \log 72$$
$$\frac{1}{2}\log\frac{1}{49} + \frac{1}{3}\log 125 = \log\frac{1}{7} + \log 5 = \log\frac{5}{7}$$
$$\log 3^2\ + \log 5^3 - \log 5^{\frac{3}{2}} - \log 3^{\frac{2}{3}}$$
$$= \log\frac{3^2 \times 5^3}{3^{\frac{2}{3}} \times 5^{\frac{3}{2}}} = \log 3^{\frac{4}{3}} \times 5^{\frac{3}{2}}$$

1.55: Approximation

- Currently, logarithms are important for their application in Calculus and other areas of math (modelling, etc).
- But, historically, logarithms were used to carry out calculations for which the values of logarithms from a log table were important.
- Even now, exam questions can still check your ability to work with logarithms

Example 1.56

If you are given $\log 8 \approx 0.9031$ and $\log 9 \approx 0.9542$, then the only logarithm that cannot be found without the use of tables is:

A. $\log 17$
B. $\log\frac{5}{4}$
C. $\log 15$
D. $\log 600$
E. $\log .4$ (AHSME 1951/45)

Convert the values given to have a prime number:

$$\log 8 \approx 0.9031 \Rightarrow \log 2^3 \approx 0.9031 \Rightarrow 3\log 2 \approx 0.9031 \Rightarrow \log 2 \approx \frac{0.9031}{3}$$
$$\log 9 \approx 0.9542 \Rightarrow \log 3 = \frac{0.9542}{3}$$

Note that you can calculate:

$$\log\frac{5}{4} = \log\frac{10}{8} = \log 10 - \log 8 = 1 - \log 8$$
$$\log 15 = \log 3 + \log 5 = \log 3 + \log\frac{10}{2} = \log 3 + \log 10 - \log 2 = \log 3 + 1 - \log 2$$
$$\log 600 = \log 6 \times 100 = \log 2 + \log 3 + \log 100 = \log 2 + \log 3 + 2$$
$$\log 0.4 = \log\frac{4}{10} = \log 4 - \log 10 = (2\log 2) - 1$$

However

$$\log 17 \Rightarrow 17\ \textit{is a prime that we do not know how to handle}$$

Option A

Example 1.57: (Calculator allowed, but do not use the log keys)

If $\log 2 = .3010$ and $\log 3 =\ .4771$, the value of x when $3^{x+3} = 135$ is approximately (AHSME 1954/38)

$$27 \cdot 3^x = 135$$

Divide both sides by 27:

$$3^x = 5$$

Take logs both sides:

$$\log 3^x = \log 5$$

Use the power rule:

$$x \log 3 = \log 5$$

Solve for x:

$$x = \frac{\log 5}{\log 3} = \frac{\log \frac{10}{2}}{\log 3} = \frac{\log 10 - \log 2}{\log 3} = \frac{1 - \log 2}{\log 3} = \frac{1 - 0.3010}{0.4771} = \frac{0.699}{0.4771} = 1.47$$

Example 1.58

A sum of money, when invested in a bank, doubles in 20 years under compound interest. Determine, to the closest year, when will it treble.

Hint: Make use of the values of $\log 2$ and $\log 3$

Substitute $A = 2P, n = 20$ in $A = P\left(1 + \frac{r}{100}\right)^n$:

$$2P = P\left(1 + \frac{r}{100}\right)^{20} \Rightarrow 2 = \left(1 + \frac{r}{100}\right)^{20}$$

Use a change of variable. Let $x = 1 + \frac{r}{100}$:

$$x^{20} = 2 \Rightarrow 20 \log x = \log 2 \Rightarrow \underbrace{\log x = \frac{\log 2}{20}}_{Equation\ I}$$

Substitute $A = 3P$ in $A = P\left(1 + \frac{r}{100}\right)^n$:

$$3P = P\left(1 + \frac{r}{100}\right)^{20} \Rightarrow 3 = x^n \Rightarrow n \log x = \log 3$$

Solving for n, and substituting Equation I:

$$n = \frac{\log 3}{\log x} = \frac{\log 3}{\frac{\log 2}{20}} = \frac{\log 3 \times 20}{\log 2} = \frac{0.477 \times 20}{0.3} \approx 31.8 \approx 32\ years$$

Example 1.59

$\log 125$ equals:

A. $100 \log 1.25$
B. $5 \log 3$
C. $3 \log 25$
D. $3 - 3 \log 2$
E. $(\log 25)(\log 5)$ (AHSME 1954/15)

$$\log 125 = \log \frac{1000}{8} = \log 1000 - \log 8 = \log 1000 - \log 2^3 = 3 - 3 \log 2$$

Option C

Example 1.60

Let $F = \log\frac{1+x}{1-x}$. Find a new function G by replacing each x in F by $\frac{3x+x^3}{1+3x^2}$, and simplifying. The simplified expression G, in terms of F, is equal to: **(AHSME 1963/30 Adapted, AHSME 1972/29 Adapted)**

$$G = \log\frac{1+\frac{3x+x^3}{1+3x^2}}{1-\frac{3x+x^3}{1+3x^2}}$$

Find the LCM to add fractions:

$$= \log\frac{\frac{1+3x^2}{1+3x^2}+\frac{3x+x^3}{1+3x^2}}{\frac{1+3x^2}{1+3x^2}-\frac{3x+x^3}{1+3x^2}}$$

Add and rearrange:

$$= \log\frac{\frac{1+3x+3x^2+x^3}{1+3x^2}}{\frac{1-3x+3x^2-x^3}{1+3x^2}}$$

Since the denominator is the same in both the fractions, we can cancel:

$$= \log\frac{1+3x+3x^2+x^3}{1-3x+3x^2-x^3}$$

Note that the numerator is the expansion of $(1+x)^3$, and the denominator is the expansion of $(1-x)^3$:

$$= \log\frac{(1+x)^3}{(1-x)^3} = \log\left(\frac{1+x}{1-x}\right)^3 = 3\log\frac{1+x}{1-x} = 3F$$

B. Equations

Example 1.61: Equations

Find x in terms of $\log 2$ and $\log 5$:

$$8^{x-2} = 5^x$$

Substitute $8 = 2^3$:

$$2^{3x-6} = 5^x$$

Rearrange:

$$\frac{2^{3x}}{2^6} = 5^x$$

Take log both sides:

$$\log 2^{3x} - \log 2^6 = \log 5^x$$

Use the power rule:

$$3x\log 2 - 6\log 2 = x\log 5$$

Collate all variables on the LHS, and numbers on the RHS:

$$3x\log 2 - x\log 5 = 6\log 2$$

Factor x:

$$x(3\log 2 - \log 5) = 6\log 2$$

Solve for x:

$$x = \frac{6\log 2}{3\log 2 - \log 5}$$

Example 1.62

There exist positive integers A, B, and C, with no common factor greater than 1, such that

$$A\log_{200} 5 + B\log_{200} 2 = C$$

What is $A+B+C$? **(AHSME 1995/24)**

Using the power rule:

$$\log_{200} 5^A + \log_{200} 2^B = C$$

Using the product rule:

$$\log_{200} 5^A 2^B = C$$

Convert to exponential form:

$$5^A 2^B = 200^C$$
$$5^A 2^B = 5^{2C} 2^{3C}$$

Since A, B, and C have no common factors:

$$A = 2C \Rightarrow C = 1, A = 2$$
$$B = 3C \Rightarrow C = 1, B = 3$$

$$A + B + C = 2 + 3 + 1 = 6$$

Example 1.63: Equations

Solve for x:

$$2\log_a 3 + \log_a(x-4) = \log_a(x+8)$$

Use the power rule:

$$\log_a 3^2 + \log_a(x-4) = \log_a(x+8)$$

Use the product rule:

$$\log_a(9x - 36) = \log_a(x+8)$$

Exponentiate both sides:

$$9x - 36 = x + 8$$
$$8x = 44$$
$$x = \frac{44}{8} = \frac{11}{2} = 5.5$$

Example 1.64

Consider the graphs of $y = 2\log x$ and $y = \log 2x$. We may say that:

A. They do not intersect
B. They intersect at 1 point only
C. They intersect at 2 points only
D. They intersect at a finite number of points but greater than 2
E. They coincide **(AHSME 1961/19)**

The graphs will coincide when the y values are equal:

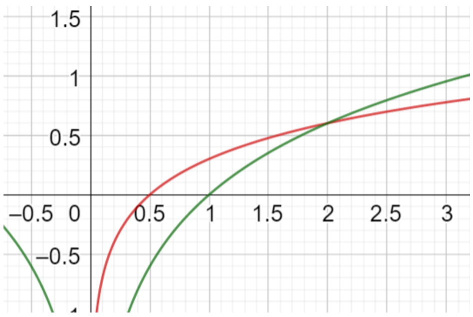
$$2\log x = \log 2x$$
$$\log x^2 = \log 2x$$

Exponentiate both sides:

$$x^2 = 2x$$

Since $x \neq 0$, divide by x both sides:

$$x = 2$$



We get one solution for the system of equations.

Option B

Example 1.65

If $\log_{10} x - 5\log_{10} 3 = -2$, then x equals: **(AHSME 1955/17)**

$$\log x - \log 3^5 = -2$$
$$\log \frac{x}{243} = -2$$
$$\frac{x}{243} = 10^{-2} = \frac{1}{100}$$
$$x = \frac{243}{100} = 2.43$$

Example 1.66

Solve for x

$$x^{\ln x} = e^{(\ln x)^3}$$

Take the natural log of both sides:

$$\ln\left(x^{\ln x}\right) = \ln\left(e^{(\ln x)^3}\right)$$

Use the power rule for logarithms:

$$(\ln x)(\ln x) = (\ln x)^3 \ln e$$

Simplify:

$$(\ln x)^2 = (\ln x)^3$$

If $\ln x = 0$, equation is satisfied. And hence:

$$\ln x = 0 \Rightarrow x = 1$$

If $\ln x \neq 0$, divide both sides by $(\ln x)^2$

$$1 = \ln x \Rightarrow x = e$$
$$x \in \{1, e\}$$

Example 1.67

If $m, n \in \mathbb{R}^+$, find the value of m in terms of n given that

$$\log\left(\frac{1}{2\sqrt{2}}(m + 2n)\right) = \frac{1}{2}(\log m + \log n)$$

Use the product rule on the RHS:

$$\log\left(\frac{m + 2n}{2\sqrt{2}}\right) = \frac{1}{2}(\log mn)$$

Use the power rule on the RHS:

$$\log\left(\frac{m + 2n}{2\sqrt{2}}\right) = \log\sqrt{mn}$$

Exponentiate both sides:

$$\frac{m + 2n}{2\sqrt{2}} = \sqrt{mn}$$

Clear fractions:

$$m + 2n = 2\sqrt{2}\sqrt{mn}$$

Square both sides both to remove the radical:

$$m^2 + 4mn + 4n^2 = 8mn$$

Collate all terms on one side

$$m^2 - 4mn + 4n^2 = 0$$

Factor:

$$(m - 2n)^2 = 0$$

Take square roots, and solve for m:

$$m = 2n$$

Example 1.68

Solve for x in terms of $\log_2$ given that $2^{2x+1} = 3 \times 2^{1-2x}$

Method I:

$$2 \cdot 2^{2x} = \frac{3 \times 2}{2^{2x}}$$
$$2^{4x} = 3$$
$$4x = \log_2 3$$
$$x = \frac{\log_2(3)}{4}$$

Method II:

Take log to the base 2 both sides:

$$\log_2(2^{2x+1}) = \log_2(3 \times 2^{1-2x})$$

Use log rules on both sides:

$$(2x+1)\log_2 2 = \log_2(3) + (1-2x)\log_2(2)$$

Substitute $\log_2 2 = 1$ and simplify:

$$(2x+1) = \log_2(3) + (1-2x)$$
$$4x = \log_2(3)$$
$$x = \frac{\log_2(3)}{4}$$

Example 1.69

Given $\frac{\log a}{p} = \frac{\log b}{q} = \frac{\log c}{r} = \log x$, all logarithms to the same base and $x \neq 1$. If $\frac{b^2}{ac} = x^y$, then y, in terms of p, q and r is: **(AHSME 1967/4)**

Rewrite:

$$\frac{1}{p}\log a = \frac{1}{q}\log b = \frac{1}{r}\log c = \log x$$

Use the power rule:

$$\log a^{\frac{1}{p}} = \log b^{\frac{1}{q}} = \log c^{\frac{1}{r}} = \log x$$

Exponentiate throughout:

$$a^{\frac{1}{p}} = b^{\frac{1}{q}} = c^{\frac{1}{r}} = x$$

Solve for a, b, c in terms of x:

$$a^{\frac{1}{p}} = x \Rightarrow a = x^p$$
$$b^{\frac{1}{q}} = x \Rightarrow b = x^q$$
$$c^{\frac{1}{r}} = x \Rightarrow c = x^r$$

Substitute the above:

$$x^y = \frac{b^2}{ac} = \frac{(x^q)^2}{x^p x^r} = x^{2q-p-r}$$

If bases are same, exponents are also same:

$$y = 2q - p - r$$

Example 1.70
Make x the subject of the equation:

$$y = \frac{1}{1 + e^{-x}}$$

Move the x term to the LHS, and the y term to the RHS:

$$1 + e^{-x} = \frac{1}{y}$$

Solve for e^{-x}

$$e^{-x} = \frac{1}{y} - 1 = \frac{1-y}{y}$$

Take the natural log of both sides:

$$x = -\ln\left(\frac{1-y}{y}\right) = \ln\left(\frac{y}{1-y}\right)$$

C. Exponential Equations leading to Logs

Example 1.71
Solve $5^x = 7$ in terms of:

A. Log to the base 10
B. Log to the base e

Take log to the base 10:

$$x \log 5 = \log 7 \Rightarrow x = \frac{\log 7}{\log 5}$$

Take log to the base e:

$$x \ln 5 = \ln 7 \Rightarrow x = \frac{\ln 7}{\ln 5}$$

Example 1.72: Solving Exponential Equations via Logarithms
Find xy given that $3 = 5^x, 5 = 3^y$

$$3 = 5^x \Rightarrow x = \frac{\log 3}{\log 5}$$
$$5 = 3^y \Rightarrow y = \frac{\log 5}{\log 3}$$
$$xy = \frac{\log 3}{\log 5} \times \frac{\log 5}{\log 3} = 1$$

Example 1.73: Solving Exponential Equations via Logarithms
In an exponential equation, it may be necessary to take logarithms in order get the variable down from the exponent.

A. Solve for z in terms of log to the base 10: $5 \times 10^{\left(\frac{z}{4}\right)} = 32$
B. Solve for x: $\frac{3^x}{3^x+3} = \frac{1}{3}$
C. Solve for x in terms of $\log 2$ and $\log 3$: $8^{x-1} = 6^{3x}$
D. Find the value of x in terms of $\ln 2$: $\ln 2^{4x-1} = \ln 8^{x+5} + \log_2 16^{1-2x}$

Part A

Divide by 5 both sides:

$$10^{\left(\frac{z}{4}\right)} = \frac{32}{5}$$

Take log to the base 10 both sides:

$$\log 10^{\left(\frac{z}{4}\right)} = \log \frac{32}{5}$$

Use the power rule on the LHS:

$$\frac{z}{4} = \log \frac{32}{5} \Rightarrow z = 4 \log \frac{32}{5}$$

Part B

Cross-multiply to eliminate fractions:

$$3 \cdot 3^x = 3^x + 3$$

Collate all x terms on the LHS:

$$3 \cdot 3^x - 3^x = 3$$

Factor 3^x:

$$3^x(3-1) = 3$$

$$3^x = \frac{3}{2}$$

$$x = \log_3 \frac{3}{2}$$

Part C

$$2^{3x-3} = 2^{3x} \cdot 3^{3x}$$

$$2^{-3} = 3^{3x}$$

$$2^{-1} = 3^x$$

$$-\log 2 = x \log 3$$

$$x = -\frac{\log 2}{\log 3}$$

Part D

$$\ln \frac{2^{4x-1}}{2^{3x+15}} = (1-2x)\log_2 2^4$$

$$\ln 2^{x-16} = 4 - 8x$$

$$x \ln 2 - 16 \ln 2 = 4 - 8x$$

$$x \ln 2 + 8x = 4 + 16 \ln 2$$

$$x = \frac{16 \ln 2 + 4}{\ln 2 + 8}$$

Example 1.74: Reducible to Quadratic

The equation $2^{2x} - 8 \cdot 2^x + 12 = 0$ is satisfied by:

A. $\log 3$
B. $\frac{1}{2}\log 6$
C. $1 + \log \frac{3}{2}$
D. $1 + \frac{\log 3}{\log 2}$
E. none of these (AHSME 1969/17)

Use a change of variable. Let $y = 2^x$:

$$y^2 - 8y + 12 = 0$$

$$(y-6)(y-2) = 0$$

$$y = 2^x \in \{2,6\}$$

Consider Case I:

$$2^x = 2^1 \Rightarrow x = 1$$

Consider Case II:

$$2^x = 6$$

$$\log 2^x = \log 6$$

$$x \log 2 = \log 2 + \log 3$$

$$x = 1 + \frac{\log 3}{\log 2}$$

Hence,

$$\textit{Option D is correct}$$

D. More Equations

Example 1.75: Simulataneous Equations

Solve each system of equations. First treat them as simultaneous equations, and then solve by converting them to exponential equations.

A. $\log \frac{x}{y} = 2, \log xy = 1$
B. $\ln \frac{x}{y} = 2,\ \ln xy = 1$

Part A

Standard Method

$$\underbrace{\log x - \log y = 2}_{\textit{Equation I}}, \qquad \underbrace{\log x + \log y = 1}_{\textit{Equation II}}$$

Add the two equations:

$$2\log x = 3$$
$$\log x^2 = 3$$
$$x^2 = 1000$$
$$x = \pm 10\sqrt{10}$$

Substitute the value of $\log x$ in the equation given in the original question:

$$\log xy = 1$$
$$\log\left(\pm 10\sqrt{10}\right)y = 1$$
$$\left(\pm 10\sqrt{10}\right)y = 10$$
$$y = \pm\frac{1}{\sqrt{10}}$$

Convert to exponential form

Convert to exponential form:

$$\log\frac{x}{y} = 2 \Rightarrow \frac{x}{y} = 10^2 \Rightarrow \underbrace{x = 100y}_{\textit{Equation III}}$$
$$\log xy = 1 \Rightarrow xy = 10$$

Substitute x from Equation VI in the above:

$$(100y)(y) = 10 \Rightarrow y^2 = \frac{1}{10} \Rightarrow y = \pm\frac{1}{\sqrt{10}}$$

Substitute the value of y from above in Equation III:

$$x = 100y = 100 \times \pm\frac{1}{\sqrt{10}} = \pm 10\sqrt{10}$$

Hence, the final answer is:

$$(x, y) = \left\{\left(10\sqrt{10}, \frac{1}{\sqrt{10}}\right), \left(-10\sqrt{10}, -\frac{1}{\sqrt{10}}\right)\right\}$$

Note: Both the negative and the positive solutions work (as you can check by substitution), since

$$(-ve)(-ve) = +ve$$

Part B

Standard Method

$$\underbrace{\ln x - \ln y = 2}_{\textit{Equation IV}}, \qquad \underbrace{\ln x + \ln y = 1}_{\textit{Equation V}}$$

Add the two equations:

$$2\ln x = 3$$
$$\ln x^2 = 3$$
$$x^2 = e^3$$
$$x = \pm e^{\frac{3}{2}}$$

Substitute the value of x in the question given in the original equation:

$$\ln xy = 1$$
$$\ln\left[\left(\pm e^{\frac{3}{2}}\right)y\right] = 1$$
$$\left(\pm e^{\frac{3}{2}}\right)y = e$$
$$y = \pm\frac{1}{\sqrt{e}}$$

Convert to exponential form

Convert to exponential form:

$$\ln\frac{x}{y}=2\Rightarrow\frac{x}{y}=e^2\Rightarrow\underbrace{x=e^2y}_{Equation\ VI}$$

Convert to exponential form:

$$\ln xy=1\Rightarrow xy=e$$

Substitute x from Equation VI in the above:

$$(e^2y)(y)=e\Rightarrow y^2=\frac{1}{e}\Rightarrow y=\pm\frac{1}{\sqrt{e}}$$

Substitute the value of y in Equation VI:

$$x=e^2y=e^2\times\pm\frac{1}{\sqrt{e}}=\pm e^{\frac{3}{2}}$$

Hence, the final answer is:

$$(x,y)=\left\{\left(e^{\frac{3}{2}},\frac{1}{\sqrt{e}}\right),\left(-e^{\frac{3}{2}},-\frac{1}{\sqrt{e}}\right)\right\}$$

Example 1.76

A. The product of all real roots of the equation $x^{\log_{10}x}=10$ is **(AHSME 1984/14)**

B. The product of all positive real values of x satisfying the equation $x^{\left(16(\log_5 x)^3-68\log_5 x\right)}=5^{-16}$ is **(JEE Advanced, 2022/Paper-II/4)**

C. Are the answers to both Part A and B the same? Why?

Part A

Take logs to the base 10 both sides:

$$\log_{10}\left(x^{\log_{10}x}\right)=\log_{10}10$$

Use the power rule:

$$(\log_{10}x)(\log_{10}x)=1$$
$$(\log_{10}x)^2=1$$
$$\log_{10}x=\pm1$$
$$x=10^{-1}\ OR\ x=10^1$$

The product of the real roots is:

$$\mathbf{10^{-1}\cdot10^1=10^{1-1}=10^0=1}$$

Part B

Use a change of variable. Let $y=\log_5 x$:

$$x^{16y^3-68y}=5^{-16}$$

Take log to the base 5 both sides:

$$\log_5\left(x^{16y^3-68y}\right)=\log_5(5^{-16})$$

Use the power rule:

$$(16y^3-68y)\log_5 x=-16\ \log_5(5)$$
$$(16y^3-68y)y=-16$$

Expand and collate terms on LHS:

$$16y^4-68y^2+16=0$$

Divide by 4 both sides:

$$4y^4-17y^2+4=0$$

This is a disguised quadratic. Again, use a change of variable. Let $z=y^2$:

$$4z^2-17z+4=0$$

Factor. $Product=16=(-16)(-1), Sum=-17=-16-1$:

$$4z^2-16z-z+4=0$$
$$4z(z-4)-1(z-4)=0$$
$$(4z-1)(z-4)=0$$

Use the zero-product property:

$$4z-1=0\Rightarrow z=y^2=\frac{1}{4}\Rightarrow y=\pm\frac{1}{2}$$
$$z-4=0\Rightarrow z=y^2=4\Rightarrow y=\pm2$$

The four solutions that we get for y are:

$$y=\log_5 x\in\left\{-\frac{1}{2},\frac{1}{2},-2,2\right\}$$

Exponentiate the four solutions above to get the value of x:

$$x\in\left\{5^{-\frac{1}{2}},5^{\frac{1}{2}},5^2,5^{-2}\right\}$$

Finally, the product of all positive real values of x is:

$$5^{-\frac{1}{2}+\frac{1}{2}+2-2}=5^0=1$$

Part C

The equations are of the form

$$(\log_b x)^2=a^2$$
$$\log_b x=\pm a$$

Resulting in answer pairs of the form

$$x=b^a\ OR\ x=b^{-a}$$

And their product is:

$$b^ab^{-a}=1$$

E. Inequalities

Example 1.77: Linear Inequalities

Solve the inequalities below:

A. $\log x \geq \log 2 + \frac{1}{2}\log x$ **(AHSME 1958/17)**
B. $-6 < 4\log_5(x) - 10 < -2$

Part A

Subtract $\frac{1}{2}\log x$ from both sides:

$$\frac{1}{2}\log x \geq \log 2$$
$$\log x \geq 2\log 2 = \log 2^2 = \log 4$$
$$x \geq 4$$

Part B

Add 10:

$$4 < 4\log_5(x) < 8$$

Divide by 4:

$$1 < \log_5(x) < 2$$

Exponentiate to the base 5:

$$5 < x < 25$$
$$x \in (5,25)$$

Example 1.78: Quadratic Inequalities

Solve the inequalities below:

A. $(\ln x)^2 - (\ln 2)(\ln x) < 2(\ln 2)^2$

Use a change of variable. Substitute $y = \ln x$, $z = \ln 2$:

$$y^2 - zy - 2z^2 < 0$$

Use $Sum = -2z + z = -z, Product = (-2z)(z) = -2z^2$:

$$(y+z)(y-2z) < 0$$

$(y+z)(y-2z)$ is an upward parabola with roots:

$$y + z = 0 \Rightarrow y = -z \Rightarrow \ln x = -\ln 2 \Rightarrow \ln x = \ln\frac{1}{2} \Rightarrow x = \frac{1}{2}$$
$$y - 2z = 0 \Rightarrow y = 2z \Rightarrow \ln x = 2\ln 2 \Rightarrow \ln x = \ln 4 \Rightarrow x = 4$$

Negative region will be between the roots.

$$\frac{1}{2} < x < 4$$

Example 1.79: Range

If one uses only the tabular information $10^3 = 1000$, $10^4 = 10{,}000$, $2^{10} = 1024$, $2^{11} = 2048$, $2^{12} = 4096$, $2^{13} = 8192$, then the strongest statement one can make for $\log_{10} 2$ is that it lies between:

A. $\frac{3}{10}$ *and* $\frac{4}{11}$
B. $\frac{3}{10}$ *and* $\frac{4}{12}$
C. $\frac{3}{10}$ *and* $\frac{4}{13}$
D. $\frac{3}{10}$ *and* $\frac{40}{132}$
E. $\frac{3}{11}$ *and* $\frac{40}{132}$ **(AHSME 1967/26)**

We establish a lower and upper bound.
First, note that:

$$1000 < 1024$$
$$\log_{10} 1000 < \log_{10} 1024$$
$$\log_{10} 10^3 < \log_{10} 2^{10}$$
$$3 < 10\log_{10} 2$$
$$\frac{3}{10} < \log_{10} 2$$

This gives us a lower bound.
Then:

$$8192 < 10{,}000$$
$$\log_{10} 8192 < \log_{10} 10{,}000$$

$$\log_{10} 2^{13} < \log_{10} 10^4$$
$$13 \log_{10} 2 < 4$$
$$\log_{10} 2 < \frac{4}{13}$$

This gives us an upper bound.
Combine the two to get:

$$\frac{3}{10} < \log_{10} 2 < \frac{4}{13}$$

1.80: Interchanging log and exponentiation (not valid)

In general:

$$\log_n x^n \neq (\log_n x)^n$$

In general, the order in which we perform the operations is going to matter.

In $\log_n x^n$, we need to perform the exponentiation first, and take the log next.
In $(\log_n x)^n$, we need to take the log first and perform the exponentiation next.

This also forms a basis of a set of questions in relation to interchanging the operations actually does not matter. For what value of x does this hold. We look at these questions next.

Example 1.81

Solve $\log_n(x^n) = (\log_n x)^n$ for $n = 2,3,4, \ldots$

$x = 1 \Rightarrow \log_n x = 0$ is a solution for all $n > 0, n \neq 1$

Case II: $x \neq 1, n = 2$

Use the property that $\log x^n = n \log x$:

$$2 \log_2 x = (\log_2 x)^2$$

Divide both sides by $\log_2 x$:

$$2 = \log_2 x \Rightarrow x = 2^2 = 4$$

Case III: $x \neq 1, n = 3$

$$3 \log_3 x = (\log_3 x)^3$$

Divide both sides by $\log_3 x$:

$$3 = (\log_3 x)^2$$

Take the square root both sides:

$$\pm\sqrt{3} = \log_3 x \Rightarrow x = 3^{\pm\sqrt{3}}$$

Case IV: $x \neq 1, n = 4$

$$4 \log_4 x = (\log_4 x)^4$$

Divide both sides by $\log_4 x$:

$$4 = (\log_4 x)^3 \Rightarrow \sqrt[3]{4} = \log_4 x \Rightarrow x = 4^{\sqrt[3]{4}}$$

General Case

The above solutions show the pattern: even n results in an odd root, while odd n results in two solutions.

$$n \log_n x = (\log_n x)^n \Rightarrow n = (\log_n x)^{n-1}$$

$$\text{Even } n\text{: } \sqrt[n-1]{n} = \log_n x \Rightarrow x = n^{\sqrt[n-1]{n}}$$

$$\text{Odd } n\text{: } \pm\sqrt[n-1]{n} = \log_n x \Rightarrow x = n^{\pm\sqrt[n-1]{n}}$$

F. Power Rule Extension

The power rule has a supercharged version that is useful.

1.82: Shortcut

$$\log_{a^n} x^m = \log_a x^{\frac{m}{n}} = \frac{m}{n} \log_a x$$

We can prove the power rule extension with using the change of base of rule (which is introduced later):

$$\log_{a^n} x^m = \frac{\log x^m}{\log a^n} = \frac{m \log x}{n \log a} = \frac{m}{n} \log_a x$$

Many questions can be solved faster with this shortcut.

Example 1.83

For any positive integer n, let

$$f(n) = \begin{cases} \log_8 n, & \text{if } \log_8 n \text{ is rational} \\ 0, & \text{otherwise} \end{cases}$$

What is $\sum_{n=1}^{1997} f(n)$? (AHSME 1997/21)

$$\log_8 n = \log_{2^3} n = \frac{1}{3}\log_2 n$$

This will be rational when n is a power of 2:

$$n \in \{2^0, 2^1, 2^2, \dots, 2^{10}\}$$

Then

$$\frac{1}{3}(1+2+3+\cdots 10) = \frac{1}{3}\left(\frac{10 \cdot 11}{2}\right) = \frac{55}{3}$$

Example 1.84: Basics

A. Simplify $\log_3 x^4 - 4\log_9 x$
B. If $a = \log_8 225$ and $b = \log_2 15$, then find the relation between a and b (AHSME 1962/17, AHSME 1970/8)
C. Solve for N: $\log_3 N + \log_9 N = 4\frac{1}{2}$

Part A

$$\log_3 x^4 - 4\log_{3^2} x$$

Use the power rule extension in the second term:

$$4\log_3 x - \frac{4}{2}\log_3 x$$
$$4\log_3 x - 2\log_3 x$$
$$2\log_3 x$$

Part B

$$a = \log_{2^3} 15^2 = \frac{2}{3}\log_2 15 = \frac{2}{3}b$$

Part C

$$\log_3 N + \frac{1}{2}\log_3 N = \frac{9}{2}$$
$$\frac{3}{2}\log_3 N = \frac{9}{2}$$
$$\log_3 N = 3$$
$$N = 3^3 = 27$$

Example 1.85: Expressions

A. Simplify $\log_{81} 9^3 + \log_{\sqrt{x}} x^3 + \log_{\sqrt[22]{\pi}} \pi^9$
B. Evaluate $\log_{\sqrt{2}} 2 \times \log_{\sqrt{\sqrt{2}}} 4 \times \log_{\sqrt{\sqrt{\sqrt{2}}}} 8$

Part A

Each term follows a pattern: get $\log_a a$, and then replace it with 1:

$$\log_{3^4} 3^6 = \frac{6}{4}\log_3 3 = \frac{3}{2}$$

$$\log_{x^{\left(\frac{1}{2}\right)}} x^3 = \frac{3}{1/2}\log_x x = 6$$

$$\log_{\pi^{\left(\frac{1}{22}\right)}} \pi^9 = \frac{9}{1/22}\log_\pi \pi = 198$$

Hence, the final answer is:

$$\frac{3}{2}+6+198=205.5$$

Part B

$$\log_{2^{\frac{1}{2}}} 2 \times \log_{2^{\frac{1}{4}}} 2^2 \times \log_{2^{\frac{1}{8}}} 2^3$$
$$2\log_2 2 \times 8\log_2 2 \times 24\log_2 2$$
$$= 2 \times 8 \times 24 = 384$$

Example 1.86: Equations

A. $\log_{x^2}\sqrt{7} = \log_{\sqrt[3]{5}}\frac{1}{\sqrt{5}}$
B. $2-\log_3(x+6) = \log_{\frac{1}{9}} 4x^2$
C. $\log_m 324 - \log_{\sqrt{m}} 2m = 2$
D. $\log_2(x-2) = \log_4(2x^2-6x-4)$
E. $\log_a(x+2) + \log_{\sqrt{a}}\sqrt{x+2} = \log_{a^2} 900$
F. $1-\log_{\sqrt{3}}(x+6) = \log_{\frac{1}{3}}(-x)$

Part A

$$\log_{x^2} 7^{\frac{1}{2}} = \log_{5^{\frac{1}{3}}} 5^{-\frac{1}{2}}$$
$$\frac{1}{4}\log_x 7 = -\frac{3}{2}\log_5 5$$
$$\log_x 7 = -6$$
$$x = 7^{-\frac{1}{6}} = \frac{1}{\sqrt[6]{7}}$$

Part B

$$\log_{\frac{1}{9}} 4x^2 + \log_3(x+6) = 2$$
$$\log_{3^{-2}}(2x)^2 + \log_3(x+6) = 2$$
$$-\log_3 2x + \log_3(x+6) = 2$$
$$\log_3\left(\frac{x+6}{2x}\right) = 2$$
$$\frac{x+6}{2x} = 9$$
$$x+6 = 18x$$
$$x = \frac{6}{17}$$

Part C

$$\log_m 324 - \log_m(2m)^2 = 2$$
$$\log_m \frac{324}{4m^2} = 2$$
$$\frac{324}{4m^2} = m^2 \Rightarrow m^4 = 81 \Rightarrow m = \pm 3$$

$$m=-3 \Rightarrow \log_{\sqrt{m}} 2m = \log_{\sqrt{-3}} -6 \Rightarrow \textit{Not Valid}$$
$$m = 3$$

Part D

$$\log_2(x-2) = \frac{1}{2}\log_2(2x^2-6x-4)$$
$$2\log_2(x-2) = \log_2(2x^2-6x-4)$$
$$\log_2(x-2)^2 = \log_2(2x^2-6x-4)$$
$$x^2-4x+4 = 2x^2-6x-4$$
$$x^2-2x-8=0$$
$$(x-4)(x+2)=0$$
$$x\in\{-2,4\}$$

$$-2 \textit{ does not work}$$
$$4 \textit{ does work}$$

Part E

$$\log_a[(x+2)(x+2)] = \log_a 30$$
$$(x+2)^2 = 30$$
$$x+2 = \pm\sqrt{30}$$
$$x = \pm\sqrt{30}-2$$
$$\sqrt{30}-2 \textit{ works}$$
$$-\sqrt{30}-2 \textit{ does not work}$$

Part F

$$\log_{3^{\frac{1}{2}}}(x+6) + \log_{3^{-1}}(-x) = 1$$
$$2\log_3(x+6) - \log_3(-x) = 1$$
$$\log_3(x+6)^2 - \log_3(-x) = 1$$
$$\log_3\left[\frac{(x+6)^2}{-x}\right] = 1$$
$$\frac{(x+6)^2}{-x} = 3$$
$$x^2+12x+36 = -3x$$
$$(x+3)(x+12) = 0$$
$$x\in\{-12,-3\}$$

Check -12:
$$\text{LHS} = 1-\log_{\sqrt{3}}(-12+6) = 1-\log_{\sqrt{3}}(-6)$$
$$\textit{Not Valid}$$

Check -3:
$$\text{LHS} = 1-\log_{\sqrt{3}}(-3+6) = 1-\log_{\sqrt{3}} 3 = 1-2$$
$$=-1$$
$$RHS = \log_{\frac{1}{3}}(-(-3)) = \log_{\frac{1}{3}} 3 = 1 = LHS$$
$$\textit{Valid Solution}$$

Example 1.87

$$0.162 = -0.06 \log_{10}\left[\frac{(2x)^{\frac{1}{3}}}{0.1}\right], \log_{10} 2 = 0.3$$

If $x = 10^m, m \in \mathbb{R}$ and x can be approximated as $4 \times 10^n, n \in \mathbb{Z}$, then find $m + n$.

Divide both sides by 0.06 and use the quotient rule on the RHS:

$$-\frac{0.162}{0.06} = \log_{10}\left[(2x)^{\frac{1}{3}}\right] - \log_{10}[0.1]$$

Simplify the LHS, and use the power rule on the LHS:

$$-2.7 = \frac{1}{3}\log_{10}[2x] + 1$$

Subtract 1 from both sides and substitute $\log_{10} 2 = 0.3$:

$$-3.7 = \frac{1}{3}(0.3 + \log_{10} x)$$
$$-11.4 = \log_{10} x$$

Exponentiate both sides:

$$x = 10^{-11.4} \approx 4 \times 10^{-12}$$

$$m + n = -11.4 - 12 = -23.4$$[1]

Challenge 1.88: Advanced Systems

Determine the value of ab if $\log_8 a + \log_4 b^2 = 5$ and $\log_8 b + \log_4 a^2 = 7$. (AIME 1984/5)

The two equations are cyclic. This gives us the idea of adding them:

$$\log_8 a + \log_4 b^2 + \log_8 b + \log_4 a^2 = 5 + 7$$

Two of the terms have a base of 8, and the other two have a base of 4. We want to have a common base, so use the property that $\log_{a^n} x = \frac{1}{n}\log_a x$:

$$\frac{1}{3}\log_2 a + \frac{2}{2}\log_2 b + \frac{1}{3}\log_2 b + \frac{2}{2}\log_2 a = 12$$

Add like terms:

$$\frac{4}{3}\log_2 a + \frac{4}{3}\log_2 b = 12$$

Factor $\frac{4}{3}$ and use $\log a + \log b = \log ab$

$$\frac{4}{3}\log_2 ab = 12 \Rightarrow \log_2 ab = 9$$

Take anti-logs both sides:

$$ab = 2^9 = 512$$

Example 1.89

Find $(\log_2 x)^2$ if $\log_2(\log_8 x) = \log_8(\log_2 x)$(AIME 1988/3)

Convert all bases to 2:

[1] Equations of this type are encountered in electrochemistry.

$$\log_2\left(\frac{1}{3}\log_2 x\right) = \log_2(\log_2 x)^{\frac{1}{3}}$$

Take Anti-Log both sides:

$$\frac{1}{3}\log_2 x = (\log_2 x)^{\frac{1}{3}}$$

Use a change of variable. Substitute $y = \log_2 x$:

$$\frac{1}{3}y = y^{\frac{1}{3}} \Rightarrow \frac{y}{y^{\frac{1}{3}}} = 3 \Rightarrow y^{\frac{2}{3}} = 3 \Rightarrow y^2 = 27$$

Change back to the original variable:

$$(\log_2 x)^2 = 27$$

Example 1.90: Inequalities

A. Solve: $5 < 6 + \log_{0.25}(x) < 7$

Subtract 6:

$$-1 < \log_{4^{-1}}(x) < 1$$
$$-1 < -\log_4(x) < 1$$

Multiply by -1:

$$1 > \log_4(x) > -1$$

Exponentiate to the base 4:

$$4 > x > \frac{1}{4}$$
$$x \in \left(\frac{1}{4}, 4\right)$$

G. Power to a Logarithm

1.91: Logs in the exponent cancel with the base

$$a^{\log_a x} = x$$

Start with $\log_a x$, and then recognize that $x = a^m$ for some m:

$$\log_a x = \log_a a^m = m\log_a a = m \Rightarrow a^{\log_a x} = a^m = x$$

$$2 \times \frac{1}{2} = 1$$
$$\left(\sqrt{2}\right)^2 = 2$$

Example 1.92

$$5^{2\log_5(3x^2) + \log_5 1}$$

Use the power rule:

$$5^{\log_5(3x^2)^2 + \log_5 1}$$
$$= 5^{\log_5(9x^4) + \log_5 1}$$
$$= 5^{\log_5(9x^4)}$$
$$= 9x^4$$

Example 1.93

Simplify:

A. $10^{\log_{10} 7}$ (AHSME 1951/34)
B. $25^{\log_5 7}$
C. $7^{\log_{\sqrt{7}} 12}$

D. $e^{\ln x+\ln y+\ln z}$

$$10^{\log_{10} 7} = 7$$
$$25^{\log_5 7} = 5^{2\log_5 7} = 5^{\log_5 7^2} = 7^2 = 49$$
$$7^{\log_{\sqrt{7}} 12} = 7^{2\log_7 12} = 7^{\log_7 12^2} = 144$$
$$e^{\ln x+\ln y+\ln z} = e^{\ln xyz} = xyz$$

Example 1.94

Find the sum of the last three non-zero digits of the decimal representation of $\left(9^{\log_3 \sqrt{17}}\right)^2$.

$$\left(9^{\log_3 \sqrt{17}}\right)^2 = \left((3^2)^{\log_3 \sqrt{17}}\right)^2 = \left(3^{2\times\log_3 \sqrt{17}}\right)^2 = \left(3^{\log_3 17}\right)^2 = (17)^2 = 289$$

Hence, the sum of the last three digits is:

$$2+8+9 = 19$$

Example 1.95

Eliminate Logs

$$\log_{10} y = a(\log_{10} x) + b$$

Exponentiate both sides to the power 10:

$$10^{\log_{10} y} = 10^{a(\log_{10} x)+b}$$

Split

$$y = 10^{(\log x^a)} \times 10^b = 10^b x^a$$
$$y = 10^b x^a$$

Example 1.96

If x is a real number that is not an integer, find the value of e^x given that:

$$4^{x+1} - 2^{x+3} = -3$$

Substitute $4 = 2^2$, and collate all terms on one side:

$$2^{2x+2} - 2^{x+3} + 3 = 0$$

Rewrite using exponent laws:

$$4 \cdot 2^{2x} - 8 \cdot 2^x + 3 = 0$$

Recognize this as a disguised quadratic. Substitute $t = 2^x$:

$$4t^2 - 8t + 3 = 0$$
$$4t^2 - 6t - 2t + 3 = 0$$
$$2t(2t-3) - 1(2t-3) = 0$$
$$(2t-3)(2t-1) = 0$$

Consider the first solution:

$$2^x = t = \frac{1}{2} = 2^{-1} \Rightarrow x = -1 \Rightarrow Reject\ because\ x\ is\ not\ an\ integer$$

Consider the second solution from the quadratic:

$$2^x = t = \frac{3}{2}$$

Substitute $2^x = \left(e^{\ln 2}\right)^x = (e^x)^{\ln 2}$

$$(e^x)^{\ln 2} = \frac{3}{2}$$

Take the power $\frac{1}{\ln 2}$ of both sides:

$$e^x = \left(\frac{3}{2}\right)^{\frac{1}{\ln 2}}$$

1.97: Interchanging

$$x^{\log_a y} = y^{\log_a x}$$

Let $x = a^m, y = a^n$:

$$x^{\log_a y} = (a^m)^{\log_a a^n} = a^{mn \log_a a} = (a^n)^{\log_a a^m} = y^{\log_a x}$$

Example 1.98

Evaluate $x^{\log_y z} - z^{\log_y x}$

$$x^{\log_y z} - z^{\log_y x} = z^{\log_y x} - z^{\log_y x} = 0$$

Example 1.99

$$x^{\ln x} = e^{(\ln x)^3}$$

Substitute $x = e^{\ln x}$:

$$\left(e^{\ln x}\right)^{\ln x} = e^{(\ln x)^3}$$
$$e^{(\ln x)^2} = e^{(\ln x)^3}$$
$$(\ln x)^2 = (\ln x)^3$$

If $\ln x = 0$, equation is satisfied. And hence:

$$\ln x = 0 \Rightarrow x = 1$$

If $\ln x \neq 0$, divide both sides by $(\ln x)^2$:

$$1 = \ln x \Rightarrow x = e$$

Final Solution Set:

$$x \in \{1, e\}$$

1.6 Change of Base

A. Basics

The change of base rule lets you split a logarithm and let you take it to any base you want.

1.100: Change of Base Rule

$$\log_a x = \frac{\log_b x}{\log_b a}, where\ b > 0, b \neq 1$$

For some number k, let:

$$k = \log_a x$$

Convert to exponential form:

$$a^k = x$$

Take *log* base b both sides:

$$\log_b a^k = \log_b x$$

Use the power rule:

$$k \log_b a = \log_b x$$

Solve for k:

$$k = \frac{\log_b x}{\log_b a}$$

Substitute the original value of k:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Example 1.101

$$\log_{\frac{1}{3}} \sqrt[4]{27}$$

$$\frac{\log \sqrt[4]{27}}{\log \frac{1}{3}} = \frac{\log (3^3)^{\frac{1}{4}}}{\log 3^{-1}} = \frac{\log 3^{\frac{3}{4}}}{\log 3^{-1}} = \frac{\frac{3}{4}\log 3}{-\log 3} = -\frac{3}{4}$$

Example 1.102

Evaluate:

A. $\frac{\ln 27}{\ln 3}$
B. $\frac{\ln 8}{\ln 4}$
C. $\frac{\ln 144}{\ln 2\sqrt{3}}$
D. $\log_{81} \frac{1}{27}$

Write as a Fraction using Change of Base

E. $\log_{12} 7$

Parts A-C

$$\frac{\ln 27}{\ln 3} = \frac{\log_e 27}{\log_e 3} = \log_3 27 = 3$$

$$\frac{\ln 8}{\ln 4} = \log_4 8 = \frac{3}{2}$$

$$\frac{\ln 144}{\ln 2\sqrt{3}} = \log_{2\sqrt{3}} 2^4 3^2 = \log_{2\sqrt{3}} \left(2\sqrt{3}\right)^4 = 4$$

Part D

$$\frac{\log \frac{1}{27}}{\log 81} = \frac{\log 3^{-3}}{\log 3^4} = \frac{-3\log 3}{4\ \log 3} = -\frac{3}{4}$$

We could also have done this using the Power Rule Extension that we already learnt. The method below is usually faster:

$$\log_{81} \frac{1}{27} = \log_{3^4} 3^{-3} = -\frac{3}{4}\log_3 3 = -\frac{3}{4}$$

Part E

$$\log_{12} 7 = \frac{\log 7}{\log 12}$$

Example 1.103: Calculator

Calculators usually have a *log* button which gives logarithms to the base 10. And they also have a *ln* button, which gives the logarithm to the base e. However, calculators will not directly calculate the logarithm to an arbitrary base. The change of base rule is useful here.

Use the change of base to find the approximate value of the following on a calculator.

A. $\log_7 3$

B. $\log_3 45.2$

$$\log_7 3 = \frac{\log_{10} 3}{log_{10} 7} =$$
$$\log_3 45.2 = \frac{\log_{10} 45.2}{log_{10} 3} =$$

Example 1.104

If $\log_{10} 2 = a$ and $\log_{10} 3 = b$, then

A. $\log_5 12$ in terms of a and b =? (AHSME 1961/30)
B. then $\log_{125} 24$ in terms of a and b =? (AHSME 1961/30)

Part A

Use change of base:

$$= \frac{\log 12}{\log 5}$$

Use the product rule

$$= \frac{\log 4 + \log 3}{\log 10 - \log 2}$$

Substitute:

$$= \frac{2a + b}{1 - a}$$

Part B

$$= \frac{\log 24}{\log 125} = \frac{\log 8 + \log 3}{\log 1000 - \log 8} = \frac{3a + b}{3 - 3a}$$

Example 1.105

$$x = \log_a y$$

A. Solve for y
B. Solve for a

$$y = a^x$$

Raise both sides to the power $\frac{1}{x}$:

$$y^{\frac{1}{x}} = a$$

Example 1.106

If $a > 1, b > 1$, and $p = \frac{\log_b(\log_b a)}{\log_b a}$, then a^p equals (AHSME 1982/13)

$$a^p = a^{\frac{\log_b(\log_b a)}{\log_b a}}$$

Use change of base:

$$= a^{\log_a(\log_b a)}$$

The power and the log cancel:

$$= \log_b a$$

1.107: Reciprocal of a Log

$$x + \frac{1}{x} = c \Rightarrow Quadratic$$

If one term in an equation is the reciprocal of another term, use a change of variable, and this leads to a quadratic.

Example 1.108: Golden Ratio

Suppose that p and q are positive numbers for which $\log_9 p = \log_{12} q = \log_{16}(p+q)$. What is the value of $\frac{q}{p}$?
(AHSME 1988/26)

Use the change of base rule:

$$\frac{\log p}{\log 9} = \frac{\log q}{\log 12} = \frac{\log(p+q)}{\log 16}$$

Cross-multiply in the first and the second equality:

$$\underbrace{\log 12 \log p = 2 \log 3 \log q}_{Equation\ I}$$

Cross-multiply in the second and the third equality:

$$\underbrace{\log 12 \log(p+q) = 2 \log 4 \log q}_{Equation\ II}$$

Add Equation I and II:

$$\log 12 \log[p(p+q)] = 2 \log 12 \log q$$

Divide both sides by $\log 12$:

$$\log[p(p+q)] = \log q^2$$

Exponentiate both sides:

$$p(p+q) = q^2$$

Divide both sides by pq:

$$\frac{p}{q} + 1 = \frac{q}{p}$$

Use a change of variable. Let $\frac{q}{p} = x$:

$$\frac{1}{x} + 1 = x$$

Multiply both sides by x:

$$1 + x = x^2$$

Collate all terms on one side:

$$x^2 - x - 1 = 0$$

Use the quadratic formula:

$$x = \frac{1+\sqrt{5}}{2}$$

B. Cancelling Bases

1.109: Cancelling Bases

$$\log_b x \cdot \log_a b = \frac{\log x}{\log b} \times \frac{\log b}{\log a} = \frac{\log x}{\log a} = \log_a x$$

If we have:

- A term with b in the base

- A term where we are finding the logarithm of b

Then we can use the change of base rule to eliminate b from the expression.

Example 1.110: Expressions

Simplify

A. $\log_4 5 \cdot \log_5 4$
B. $\log_a b \cdot \log_b a$ (AHSME 1953/39)
C. $\log_{\sqrt{2}} 2 \times \log_{\sqrt{\sqrt{2}}} \sqrt{2} \times \log_{\sqrt{\sqrt{\sqrt{2}}}} \sqrt{\sqrt{2}}$
D. $\log_2 3 \times \log_3 4 \times \ldots \times \log_{127} 128$

Part A

$$\log_4 5 \cdot \log_5 4 = \frac{\log 5}{\log 4} \times \frac{\log 4}{\log 5} = \frac{\log 5}{\log 5} = 1$$

Part B

$$\frac{\log b}{\log a} \times \frac{\log a}{\log b} = 1$$

Part C

Use the change of base rule:

$$\frac{\log 2}{\log \sqrt{2}} \times \frac{\log \sqrt{2}}{\log \sqrt{\sqrt{2}}} \times \frac{\log \sqrt{\sqrt{2}}}{\log \sqrt{\sqrt{\sqrt{2}}}}$$

Cancel and simplify:

$$\frac{\log 2}{\log \sqrt{\sqrt{\sqrt{2}}}} = \frac{\log 2}{\log 2^{\frac{1}{8}}} = \frac{\log 2}{\frac{1}{8}\log 2} = 8$$

Part D

$$\log_2 128 = 7$$

Example 1.111: Expressions

If $a = \log_{16} 7$, and $b = \log_7 6$, then find $\log_3 2$ in terms of a and b.

Wrong Approach

We could add the equations, but then we don't have a way to add the logs since the bases are different:

$$a + b = \log_{16} 7 + \log_7 6$$

Correct Approach

The number in the first equation is the base in the second equation. Hence, multiply the two equations:

$$ab = \log_{16} \mathbf{7} \times \log_{\mathbf{7}} 6$$

Simplify the RHS:

$$\log_{2^4} 6$$
$$= \frac{1}{4}\log_2 6$$
$$= \frac{1}{4}(\log_2 2 + \log_2 3)$$
$$= \frac{1}{4}(1 + \log_2 3)$$

Now, solve for the expression we want:

$$ab = \frac{1}{4}(1 + \log_2 3)$$

$$\log_2 3 = 4ab - 1$$
$$\log_3 2 = \frac{1}{4ab - 1}$$

Example 1.112

$$\log_{63} 64 \times \log_{62} 63 \times \ldots \times \log_2 3$$

Use change of base:

$$\frac{\log 64}{\log 63} \times \frac{\log 63}{\log 62} \times \ldots \times \frac{\log 3}{\log 2}$$

Telescope:

$$\frac{\log 64}{\log 2}$$

Use reverse change of base:

$$\log_2 64 = 6$$

Example 1.113: Equations

A. Solve for x: $\log_k x \cdot \log_5 k = 3$ **(AHSME 1958/25)**
B. Which positive numbers x satisfy the equation $(\log_3 x)(\log_x 5) = \log_3 5$ **(AHSME 1975/19)**
C. Eliminate z from the system of equations $x^z = y$, and $z = \log_y x^4$. Your answer should not be in terms of logarithms.

Part A

$$\frac{\log x}{\log k} \times \frac{\log k}{\log 5} = 3$$
$$\frac{\log x}{\log 5} = 3$$
$$\log_5 x = 3$$
$$x = 5^3 = 125$$

Part B

$$\frac{\log x}{\log 3} \times \frac{\log 5}{\log x} = \frac{\log 5}{\log 3} = \log_3 5$$

This is valid for any valid value of x, which is:

All positive numbers except 1

Part C

$$z = \log_y x^4 = 4 \log_y x = 4\frac{\log x}{\log y}$$

Substitute the above in $x^z = y$:

$$x^{4\frac{\log x}{\log y}} = y$$

Take log both sides:

$$4\frac{\log x}{\log y} \log x = \log y$$
$$4 = \left(\frac{\log y}{\log x}\right)^2$$
$$\pm 2 = \frac{\log y}{\log x} = \log_x y$$

Convert from logarithmic to exponential form:

$$y = x^2 \;\; OR \;\; y = \frac{1}{x^2}$$

Part D

Example 1.114: Equations

Given that $2^x = 3^{x+1}$, find the value of x in the form $\log_b a$.

$$2^x = 3 \cdot 3^x$$
$$\frac{2^x}{3^x} = 3$$
$$\left(\frac{2}{3}\right)^x = 3$$

$$x = \log_{\frac{2}{3}} 3$$

Taking Logs First

Taking logs first is also valid, but far lengthier. *Not* recommended.

$$x \log_2 2 = (x+1)\log_2 3$$
$$x(1 - \log_2 3) = \log_2 3$$
$$x = \frac{\log_2 3}{1 - \log_2 3} = \frac{\log_2 3}{\log_2 2 - \log_2 3} = \frac{\log_2 3}{\log_2 \frac{2}{3}} = \log_{\frac{2}{3}} 3$$

A solution of the above equation obtained $\frac{1}{(\log_3 2) - 1}$ as a solution. Is the answer correct?

$$\frac{1}{(\log_3 2) - 1} = \frac{1}{(\log_3 2) - \log_3 3} = \frac{1}{\log_3 \frac{2}{3}} = \log_{\frac{2}{3}} 3$$ [2]

Example 1.115

A. For what values of x is $\log_{10} x = \ln x$

Use the change of base rule:

$$\underbrace{\frac{\log x}{\log 10} = \frac{\log x}{\log e}}_{\textit{Equation I}}$$

We *want* to divide both sides by $\log x$ to eliminate it. Apply casework.

Case I: $\log x = 0$

Substitute $\log x = 0$ in Equation I:

$$LHS = \frac{0}{\log 10} = 0$$
$$RHS = \frac{0}{\log e} = 0$$

Hence, $\log x = 0$ is a solution:

$$\log x = 0 \Rightarrow x = 1$$

Case II: $\log x \neq 0$

If $\log x \neq 0$, then we can divide both sides of Equation I by $\log x$:

$$\frac{1}{\log 10} = \frac{1}{\log e}$$

Take the reciprocal:

$$\log e = \log 10$$

But this is never true. Hence, there are no solutions from this case.

Hence, the final answer is:

$$x = 1$$

Example 1.116

If $a \geq b \geq 1$, what is the largest possible value of $\log_a\left(\frac{a}{b}\right) + \log_b\left(\frac{b}{a}\right)$. **(AMC 12A 2003/24)**

Explain what is wrong with the "solution" below.

[2] This step makes use of the reciprocal of a log property. See the next section for the property.

$$\log_a a - \log_a b + \log_b b - \log_b a = 2 - (\log_a b + \log_b a)$$

However, now, we use a different logarithmic rule stating that $\log_a b$ is simply equal to $\frac{\log_{10} b}{\log_{10} a}$. With this, we can rewrite our previous equation to give us

$$2 - (\log_a b + \log_b a) = 2 - \left(\frac{\log_{10} b}{\log_{10} a} + \frac{\log_{10} a}{\log_{10} b}\right)$$

.

We can now cross multiply to get that

$$2 - \left(\frac{\log_{10} b}{\log_{10} a} + \frac{\log_{10} a}{\log_{10} b}\right) = 2 - \left(\frac{2 \cdot \log_{10} b \cdot \log_{10} a}{\log_{10} b \cdot \log_{10} a}\right)$$

Finally, we cancel to get $2 - 2 = 0 \Rightarrow \boxed{\textbf{(B) } 0}$.

$$\frac{\log_{10} b}{\log_{10} a} + \frac{\log_{10} a}{\log_{10} b} = \frac{(\log_{10} b)^2 + (\log_{10} a)^2}{(\log_{10} a)(\log_{10} a)} \neq \frac{2 \cdot \log_{10} b \cdot \log_{10} a}{(\log_{10} a)(\log_{10} a)}$$

$$\frac{x}{y} + \frac{y}{x} = \frac{x^2 + y^2}{xy} \neq \frac{2xy}{xy}$$

$$b = 8, a = 2$$

$$\log_a b + \log_b a = \log_2 8 + \log_8 2 = 3\frac{1}{3}$$

1.7 Change of Base: Reciprocals

A. Expressions

1.117: Reciprocal of a Log

$$\log_a x = \frac{1}{\log_x a}$$

Apply the change of base rule to get:

$$\log_a x = \frac{\log x}{\log a} = \frac{\frac{\log x}{\log a}}{1}$$

Multiply both numerator and denominator by $\frac{\log a}{\log x}$:

$$\frac{\frac{\log x}{\log a}}{1} = \frac{\frac{\log x}{\log a} \times \frac{\log a}{\log x}}{1 \times \frac{\log a}{\log x}} = \frac{1}{\frac{\log a}{\log x}} = \frac{1}{\log_x a}$$

Example 1.118

Evaluate $\frac{1}{\log_2\left(\frac{1}{6}\right)} - \frac{1}{\log_3\left(\frac{1}{6}\right)} - \frac{1}{\log_4\left(\frac{1}{6}\right)}$ **(Purple Comet 2003/6)**

Move each term to the numerator:

$$\log_{\left(\frac{1}{6}\right)} 2 - \log_{\left(\frac{1}{6}\right)} 3 - \log_{\left(\frac{1}{6}\right)} 4$$

Combining the terms using the reverse of the quotient rule:

$$\log_{\left(\frac{1}{6}\right)}\left(\frac{2}{3\cdot 4}\right)=\log_{\left(\frac{1}{6}\right)}\left(\frac{1}{6}\right)=1$$

Example 1.119

What is the value of the expression:

$$\frac{1}{\log_2 100!}+\frac{1}{\log_3 100!}+\frac{1}{\log_4 100!}+\cdots+\frac{1}{\log_{100} 100!}$$

(AHSME 1998/22)

Using reciprocal change of base:

$$\log_{100!} 2+\log_{100!} 3+\cdots+\log_{100!} 100$$

Using product rule

$$=\log_{100!} 2\cdot 3\cdot \ldots\cdot 100=\log_{100!} 100!=1$$

Example 1.120

For all integers n greater than 1, define $a_n=\frac{1}{\log_n 2002}$. Let $b=a_2+a_3+a_4+a_5$ and$c=a_{10}+a_{11}+a_{12}+a_{13}+a_{14}$. Then $b-c$ equals **(AMC 12B 2002/22)**

Using the reciprocal change of base rule:

$$a_n=\frac{1}{\log_n 2002}=\log_{2002} n=\frac{\log n}{\log 2002}$$

$$b-c=(a_2+a_3+a_4+a_5)-(a_{10}+a_{11}+a_{12}+a_{13}+a_{14})$$

$$=\frac{\log 2+\log 3+\log 4+\log 5}{\log 2002}-\frac{\log 10+\log 11+\log 12+\log 13+\log 14}{\log 2002}$$

$$=\frac{\log\frac{2\cdot 3\cdot 4\cdot 5}{10\cdot 11\cdot 12\cdot 13\cdot 14}}{\log 2002}$$

$$=\frac{\log\frac{1}{2002}}{\log 2002}=\frac{\log 2002^{-1}}{\log 2002}=-1$$

Example 1.121

For all positive numbers x distinct from 1, $\frac{1}{\log_3 x}+\frac{1}{\log_4 x}+\frac{1}{\log_5 x}$ equals $\frac{1}{n}$. Find n. **(AHSME 1978/21, Adapted)**

Use the reciprocal rule:

$$\frac{1}{n}=\log_x 3+\log_x 4+\log_x 5$$

Use the product rule:

$$\frac{1}{n}=\log_x 60$$

Again, use the reciprocal rule:

$$n=\log_{60} x$$

Example 1.122

For what values of a, b, c, x is

$$\frac{1}{\log_a x} + \frac{1}{\log_b x} + \frac{1}{\log_c x} = \frac{1}{\log_{abc} x}$$

Use the reciprocal rule:

$$\log_x a + \log_x b + \log_x c$$

Use the product rule:

$$\log_x abc$$

Again, use the reciprocal rule:

$$\frac{1}{\log_{abc} x}$$

Hence, this is true for all valid values:

$$a > 0, b > 0, c > 0, x > 0$$
$$a \neq 1, b \neq 1, c \neq 1, x \neq 1$$

Example 1.123: Expressions

If $\log_8 3 = p$ and $\log_3 5 = q$, then, in terms of p and q, $\log_{10} 5$ equals (AHSME 1974/18)

One of the bases is already 3. We take the reciprocal of the other to make its base also 3:

$$\log_3 8 = \frac{1}{p} \Rightarrow \log_3 2 = \frac{1}{3p}$$

Now, we can use the change of base rule on the expression we want:

$$\log_{10} 5 = \frac{\log_3 5}{\log_3 10} = \frac{\log_3 5}{\log_3 2 + \log_3 5}$$

Substitute the values, and simplify:

$$\frac{q}{\frac{1}{3p} + q} = \frac{q}{\frac{1 + 3pq}{3p}} = \frac{3pq}{1 + 3pq}$$

Example 1.124: Expressions

Given that $\log_{10} 5 = p$, $\log_3 2 = q$, find $\log_3 5$ in terms of p and q.

Use the change of base rule to rewire the required expression:

$$\log_3 5 = \frac{\log_{10} 5}{\log_{10} 3} = \frac{p}{\log_{10} 3}$$

The numerator is directly p. To find the denominator, we need a little more work. Note that

$$q = \frac{\log_{10} 2}{\log_{10} 3}$$

Rearrange to get:

$$\log_{10} 3 = \frac{\log_{10} 2}{q} = \frac{\log_{10} \frac{10}{5}}{q} = \frac{\log_{10} 10 - \log_{10} 5}{q} = \frac{1 - p}{q}$$

Hence:

$$\frac{p}{\log_{10} 3} = \frac{p}{\frac{1 - p}{q}} = \frac{pq}{1 - p}$$

Example 1.125: Expressions

To the nearest thousandth, $\log_{10} 2 = .301$ and $\log_{10} 3$ is .477. Which of the following is the best approximation of $\log_5 10$? (AHSME 1979/18)

A. $\frac{8}{7}$
B. $\frac{9}{7}$
C. $\frac{10}{7}$
D. $\frac{11}{7}$
E. $\frac{12}{7}$

Method I

$$\log_5 10 = \log_5 5 + \log_5 2 = 1 + \log_5 2$$

Let

$$x = \log_{10} 2$$

Use the reciprocal rule:

$$\frac{1}{x} = \log_2 10 = 1 + \log_2 5$$

Solve for $\log_2 5$:

$$\log_2 5 = \frac{1}{x} - 1 = \frac{1-x}{x}$$

Use the reciprocal rule one more time:

$$\log_5 2 = \frac{x}{1-x} = \frac{0.301}{1-0.301} \approx \frac{0.3}{0.7} = \frac{3}{7}$$

$$\log_5 10 = 1 + \log_5 2 = 1 + \frac{3}{7} = \frac{10}{7}$$

Method II

$$\log_{10} 2 = \frac{\log_{10} 2}{\log_{10} 10} = \frac{\log_{10} 2}{\log_{10} 2 + \log_{10} 5} = \frac{0.301}{0.301 + \log_{10} 5} = 0.301$$

$$\frac{1}{0.301 + \log_{10} 5} = 1$$
$$1 = 0.301 + \log_{10} 5$$
$$\log_{10} 5 = 0.699$$
$$\frac{1}{\log_{10} 5} = \frac{1}{0.699}$$

Using $0.699 \approx 0.7$:

$$\log_5 10 = \frac{1}{0.7} = \frac{1}{\frac{7}{10}} = \frac{10}{7}$$

Example 1.126: Expressions

Simplify

$$\frac{1}{1+\frac{1}{x}} + \frac{1}{1+\frac{1}{y}} + \frac{1}{1+\frac{1}{z}}, \qquad x = \frac{\log_n a}{\log_n b + \log_n c}, \qquad y = \frac{\log_n b}{\log_n a + \log_n c}, \qquad z = \frac{\log_n c}{\log_n a + \log_n b}$$

Simplify x:

$$x = \frac{\log_n a}{\log_n b + \log_n c} = \frac{\log_n a}{\log_n bc} = \log_{bc} a$$

Take the reciprocal:

$$\frac{1}{x} = \log_a bc$$

Add 1 to both sides:

$$1 + \frac{1}{x} = \log_a a + \log_a bc = \log_a abc$$

Take the reciprocal:

$$\frac{1}{1 + \frac{1}{x}} = \log_{abc} a$$

Similarly:

$$\frac{1}{1 + \frac{1}{y}} = \log_{abc} b, \qquad \frac{1}{1 + \frac{1}{z}} = \log_{abc} c$$

Hence, the required expression

$$= \frac{1}{1 + \frac{1}{x}} + \frac{1}{1 + \frac{1}{y}} + \frac{1}{1 + \frac{1}{z}}$$
$$= \log_{abc} a + \log_{abc} b + \log_{abc} c$$
$$= \log_{abc} abc = 1$$

Example 1.127

If $x, y > 0$, $\log_y x + \log_x y = \frac{10}{3}$ and $xy = 144$,then $\frac{x+y}{2} =$ **(AHSME 1990/23)**

Note that $\log_y x = \frac{1}{\log_x y}$. Substitute $t = \log_y x$

$$t + \frac{1}{t} = 3\frac{1}{3} \Rightarrow t = 3 \Rightarrow \frac{\log x}{\log y} = 3$$

Cross-Multiply:

$$\log x = 3 \log y = \log y^3$$

Exponentiate to get $x = y^3$ and substitute in $xy = 144$ to get:

$$y^4 = 144 \Rightarrow y = 2\sqrt{3} \Rightarrow x = y^3 = 24\sqrt{3}$$

And finally, the answer we want is:

$$\frac{x+y}{2} = \frac{24\sqrt{3} + 2\sqrt{3}}{2} = 13\sqrt{3}$$

B. Equations

Example 1.128: Equations

If $\log_M N = \log_N M$, $M \neq N, MN > 0, M \neq 1, N \neq 1$, then MN equals: **(AHSME 1966/24)**

Use the reciprocal rule on the LHS of the given equality:

$$\frac{1}{\log_N M} = \log_N M$$

Cross-multiply:

$$1 = (\log_N M)^2$$

Take square roots both sides

$$\log_N M = \pm 1$$

Case I:

$$\log_N M = 1 \Rightarrow N = M \Rightarrow Not\ Valid$$

Case II:

$$\log_N M = -1 \Rightarrow N^{-1} = M \Rightarrow \frac{1}{N} = M \Rightarrow MN = 1$$

Example 1.129

Find all integer solutions to $\frac{1}{\log_8 n} + \frac{1}{\log_n \frac{1}{4}} = -\frac{5}{2}$ (SMT 2021 Algebra Tiebreaker/1)

Use the reciprocal rule to make all terms have n in the number:

$$\frac{1}{\log_8 n} + \log_{\frac{1}{4}} n = -\frac{5}{2}$$

Rewrite:

$$\frac{1}{\log_{2^3} n} + \log_{2^{-2}} n = -\frac{5}{2}$$

Use the power rule, and power rule extension:

$$\frac{3}{\log_2 n} - \frac{\log_2 n}{2} = -\frac{5}{2}$$

Substitute $x = \log_2 n$, converting it into a quadratic:

$$\frac{3}{x} - \frac{x}{2} = -\frac{5}{2}$$

Multiply both sides by $2x$:

$$6 - x^2 = -5x$$
$$x^2 - 5x - 6 = 0$$
$$(x-6)(x+1) = 0$$
$$x \in \{6, -1\}$$

Change back to the original variable:

$$\log_2 n = 6 \Rightarrow n = 2^6 = 64$$
$$\log_2 n = -1 \Rightarrow n = 2^{-1} = \frac{1}{2} \Rightarrow Not\ Valid$$

C. Systems of Equations

Example 1.130: Equations

Let x, y and z all exceed 1 and let w be a positive number such that $\log_x w = 24$, $\log_y w = 40$ and $\log_{xyz} w = 12$. Find $\log_z w$. (AIME 1983/1)

Note that the number in each term is the same. Hence, use $\log_a x = \frac{1}{\log_x a}$:

$$\log_x w = 24 \Rightarrow \underbrace{\log_w x = \frac{1}{24}}_{Equation\ I}$$

$$\log_y w = 40 \Rightarrow \underbrace{\log_w y = \frac{1}{40}}_{Equation\ II}$$

$$\log_{xyz} w = 12 \Rightarrow \underbrace{\log_w xyz = \frac{1}{12}}_{Equation\ III}$$

Subtract Equations I and II from Equation III:

$$\log_w xyz - \log_w x - \log_w y = \frac{1}{12} - \frac{1}{24} - \frac{1}{40}$$

$$\log_w \frac{xyz}{xy} = \frac{1}{24} - \frac{1}{40}$$
$$\log_w z = \frac{1}{60}$$

Apply $\log_a x = \frac{1}{\log_x a}$:

$$\log_z w = 60$$

Example 1.131: Manipulating an Equality

Very often the questions using the reciprocal rule require you to find an expression (rather than the value of a variable). This requires a good understanding of the way the reciprocal rule works.

If $\log_a ab = x$, then find $\log_b ab$ in terms of x.

We want to change the base from a to b. If we take the reciprocal, we will get an extra a in the base, which we don't want. Thankfully, there is a way to eliminate the a from the number:

$$x = \log_a ab = \log_a a + \log_a b = 1 + \log_a b$$

Subtract 1 from both sides:

$$x - 1 = \log_a b$$

Now we are in a position to get the base that we want. Take the reciprocal

$$\frac{1}{x-1} = \log_b a$$

And now we can add back the b in the number by adding 1 to both sides:

$$\frac{1}{x-1} + 1 = \log_b a + 1 = \log_b a + \log_b b = \log_b ab$$

D. Tripartite Equality

Example 1.132: Tripartite Equality

Given that $5^x = 0.5^y = 1000$, find the value of $\frac{x-y}{xy}$.

Simplify the required expression:

$$\frac{x-y}{xy} = \frac{x}{xy} - \frac{y}{xy} = \frac{1}{y} - \frac{1}{x}$$

Logarithms

Separate out the given information into two different equations and solve each equation for the respective variable:

$$5^x = 1000 \Rightarrow x \log 5 = \log 1000 \Rightarrow x = \frac{\log 1000}{\log 5} \Rightarrow \frac{1}{x} = \frac{\log 5}{\log 1000}$$
$$(0.5)^y = 1000 \Rightarrow y \log 0.5 = \log 1000 \Rightarrow y = \frac{\log 1000}{\log 0.5} \Rightarrow \frac{1}{y} = \frac{\log 0.5}{\log 1000}$$

$$\frac{1}{y} - \frac{1}{x} = \frac{\log 0.5}{\log 1000} - \frac{\log 5}{\log 1000} = \frac{\log \frac{0.5}{5}}{\log 1000} = \frac{\log \frac{1}{10}}{\log 1000} = \frac{\log 10^{-1}}{\log 10^3} = \frac{(-1)\log 10}{(3)\log 10} = -\frac{1}{3}$$

Exponents

$$5^x = 1000 \Rightarrow 5 = 10^{\frac{3}{x}}$$
$$0.5^y = 1000 \Rightarrow 0.5 = 10^{\frac{3}{y}} \Rightarrow 5 = 10^{\frac{3}{y}} \times 10$$

$$10^{\frac{3}{y}} \times 10 = 10^{\frac{3}{x}} \Rightarrow \frac{10^{\frac{3}{y}}}{10^{\frac{3}{x}}} = \frac{1}{10} \Rightarrow 10^{\frac{3}{y}-\frac{3}{x}} = 10^{-1} \Rightarrow \frac{3}{y} - \frac{3}{x} = -1 \Rightarrow 3\left(\frac{1}{y} - \frac{1}{x}\right) = -1 \Rightarrow \frac{1}{y} - \frac{1}{x} = -\frac{1}{3}$$

E. Reducible to Quadratic

Example 1.133

Let a, b, and x be positive real numbers distinct from one. Then $4(\log_a x)^2 + 3(\log_b x)^2 = 8(\log_a x)(\log_b x)$ for what values of a, b and x. (AHSME 1976/20, Adapted)

Use the change of base rule:

$$4\left(\frac{\log x}{\log a}\right)^2 + 3\left(\frac{\log x}{\log b}\right)^2 = 8\left(\frac{\log x}{\log a}\right)\left(\frac{\log x}{\log b}\right)$$

Combine into one term on both sides:

$$\frac{4(\log b)^2(\log x)^2 + 3(\log a)^2(\log x)^2}{(\log a \log b)^2} = 8\frac{(\log x)^2}{\log a \log b}$$

Divide both sides by $\frac{(\log x)^2}{\log a \log b}$:

$$\frac{4(\log b)^2 + 3(\log a)^2}{\log a \log b} = 8$$

Split the fraction:

$$\frac{4\log b}{\log a} + \frac{3\log a}{\log b} = 8$$

This is a disguised quadratic since the second term has the reciprocal of the first.

Use a change of variable and let $y = \frac{\log b}{\log a}$:

$$4y + \frac{3}{y} = 8 \Rightarrow 4y^2 - 8y + 3 = 0 \Rightarrow y \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$$

Change back to the original variable:

$$\frac{\log b}{\log a} = \frac{1}{2} \Rightarrow \log b^2 = \log a \Rightarrow b^2 = a$$

$$\frac{\log b}{\log a} = \frac{3}{2} \Rightarrow \log b^2 = \log a^3 \Rightarrow b^2 = a^3$$

It is already stated in the question that

$$x, a, b \text{ are positive real numbers greater than } 1$$

We do not need to apply any further restriction on x. For a, b we need the restriction that:

$$b^2 = a \text{ OR } b^2 = a^3$$

Example 1.134

What is the product of all the solutions to the equation

$$\log_{7x} 2023 \cdot \log_{289x} 2023 = \log_{2023x} 2023\,?$$

(AMC 12A 2023/19)

Take the reciprocal both sides:

$$\log_{2023} 7x \cdot \log_{2023} 289x = \log_{2023} 2023x$$

Split the logarithms using log properties:

$$(\log_{2023} 7 + \log_{2023} x)(\log_{2023} 289 + \log_{2023} x) = 1 + \log_{2023} x$$

Use a change of variable. Let $\log_{2023} x = y, \log_{2023} 7 = a, \log_{2023} 289$

$$(a+y)(b+y) = 1+y$$
$$ab + (a+b)y + y^2 = 1+y$$

Substitute $a + b = \log_{2023} 7 + \log_{2023} 289 = \log_{2023} 2023 = 1$:

$$ab + y + y^2 = 1 + y$$
$$y^2 + ab - 1 = 0$$

Let the solutions of the equation be $\alpha = \log_{2023} x_1$ and $\beta = \log_{2023} x_2$. Use Vieta's Formulas, the sum of the solutions is zero:

$$\log_{2023} x_1 + \log_{2023} x_2 = 0$$
$$\log_{2023} x_1 x_2 = 0$$

Exponentiate both sides:

$$x_1 x_2 = 1$$

And this is the required answer

F. A Factorization Identity

The reciprocal rule for change of base can be combined with the perfect square identity $(a+b)^2 = a^2 + 2ab + b^2$ in an interesting way to factor logs under a square root.

Example 1.135

A. Under what conditions is it true that $\sqrt{x} + \frac{1}{\sqrt{x}} = \sqrt{x + \frac{1}{x} + 2}$

B. Use the result from Part A to show that $\sqrt{\log_b a + \log_a b + 2} = \sqrt{\log_b a} + \sqrt{\log_a b}$

Part A

Square the LHS using $(a+b)^2 = a^2 + 2ab + b^2$:

$$\left(\sqrt{x}\right)^2 + (2)\left(\sqrt{x}\right)\left(\frac{1}{\sqrt{x}}\right) + \left(\frac{1}{\sqrt{x}}\right)^2$$

Simplify to get:

$$= x + \frac{1}{x} + 2 = RHS^2$$

Hence, this is true for all x in the domain of the expression $\sqrt{x} + \frac{1}{\sqrt{x}} = \sqrt{x + \frac{1}{x} + 2}$:

$$\textit{Square root requires nonnegative}: x \geq 0$$
$$\textit{Denominator requires nonzero}: x \neq 0$$

Combining the two gives:

$$x > 0$$

Part B

Note that the first and the second term in the LHS are reciprocals of each other.
Rewrite the second term using the reciprocal rule for logs:

$$LHS = \sqrt{\log_b a + \frac{1}{\log_b a} + 2}$$

Use a change of variable. Let $x = \log_b a$, and use the identity from Part A:

$$= \sqrt{x + \frac{1}{x} + 2} = \sqrt{x} + \frac{1}{\sqrt{x}}$$

Change back to the original variable:

$$\sqrt{\log_b a} + \frac{1}{\sqrt{\log_b a}}$$

Write the second term with a radical around the entire fraction:

$$= \sqrt{\log_b a} + \sqrt{\frac{1}{\log_b a}}$$

Again, use the reciprocal rule for logs:

$$= \sqrt{\log_b a} + \sqrt{\log_a b}$$

Challenge 1.136

Write $\sqrt{\log_2 6 + \log_3 6}$ as two separate radicals. (AMC 12B 2020/13, Adapted)

Use the product rule for logs: $\log_a xy = \log_a x + \log_a y$:

$$\sqrt{\log_2 2 + \log_2 3 + \log_3 2 + \log_3 3}$$

Substitute $\log_2 2 = \log_3 3 = 1$:

$$= \sqrt{\log_2 3 + \log_3 2 + 2}$$

Use the reciprocal rule on the second term:

$$\sqrt{\log_2 3 + \frac{1}{\log_2 3} + 2}$$

Use the identity $\sqrt{x + \frac{1}{x} + 2} = \sqrt{x} + \frac{1}{\sqrt{x}}$ to get:

$$= \sqrt{\log_2 3} + \frac{1}{\sqrt{\log_2 3}} = \sqrt{\log_2 3} + \sqrt{\log_3 2}$$

1.8 Inequalities

(Move all inequalities here)

A. Domain

Example 1.137

The set of all real numbers x for which

$$\log_{2004}(\log_{2003}(\log_{2002}(\log_{2001} x)))$$

Is defined is $\{x | x > c\}$. What is the value of c? (AMC 12A 2004/16)

$\log_b a$ is defined when:

$$a > 0, \qquad b > 0, \qquad b \neq 1$$

Consider the outermost logarithm:

$$\log_{2003}(\log_{2002}(\log_{2001} x)) > 0$$
$$\log_{2002}(\log_{2001} x) > 1$$

$$\log_{2001} x > 2002$$
$$x > 2001^{2002}$$

The inner logarithms will have values of c that are smaller than the above:

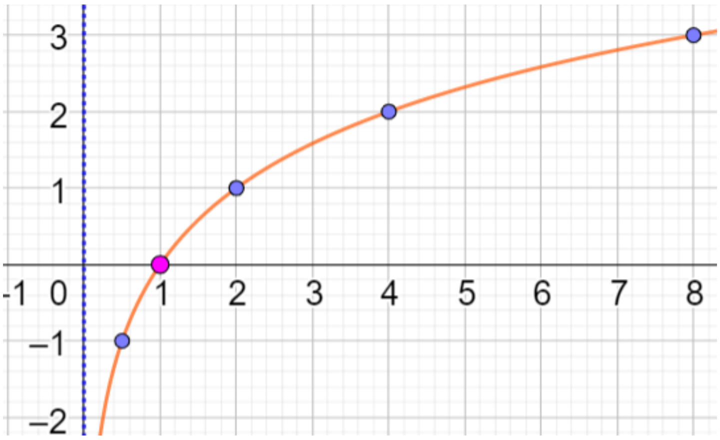
$$\log_{2001} x \Rightarrow x > 1$$
$$(\log_{2002}(\log_{2001} x)) > 0 \Rightarrow \log_{2001} x > 1 \Rightarrow x > 2001$$

2. Applications

2.1 Graphs

A. Base > 1

2.1: Graph of $y = \log_a x$
$a > 1$



Intercepts

To find the y-intercept, substitute $x = 0$:

$$y = \log x = \log 0 \Rightarrow Not\ Defined$$
$$No\ y - intercept$$

To find the x-intercept, substitute $y = 0$:

$$0 = \log x \Rightarrow x = 1$$
$$x - intercept = (1,0)$$

Asymptotes

$$Vertical\ Asymptote: x = 0$$
$$Horizontal\ Asymptote: DNE\ (Does\ Not\ Exist)$$

Positive and Negative Intervals

$$y > 0 \Rightarrow \log_a x > 0 \Rightarrow x > 1 \Leftrightarrow Graph\ is\ above\ the\ x - axis$$
$$y < 0 \Rightarrow \log_a x < 0 \Rightarrow 0 < x < 1 \Leftrightarrow Graph\ is\ below\ the\ x - axis$$

Increasing and Decreasing Intervals

The function is increasing throughout its domain:

$$(0, \infty)$$

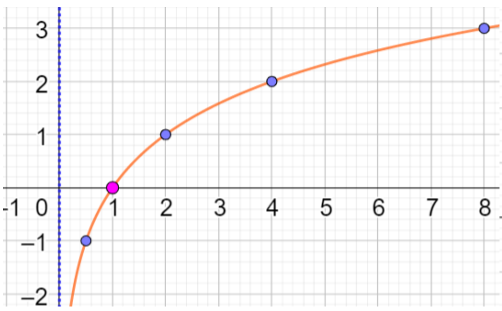
Domain and Range

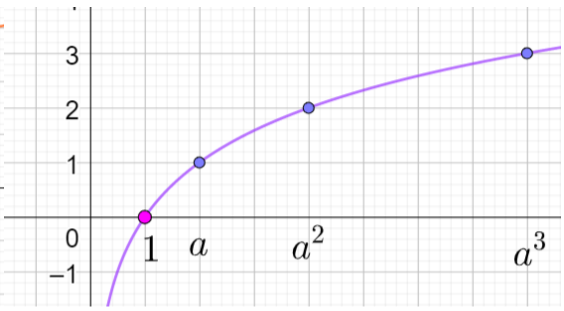
$$Domain: x > 0, a > 0, a \neq 1$$
$$Range: (-\infty, \infty)$$

Example 2.2: Property (Memorize)

Draw $f(x) = \log_2 x$ and $g(x) = \log_a x, a > 1$. Hence, for each function, determine when it is:

A. Greater than 1
B. Between 0 and 1
C. Negative



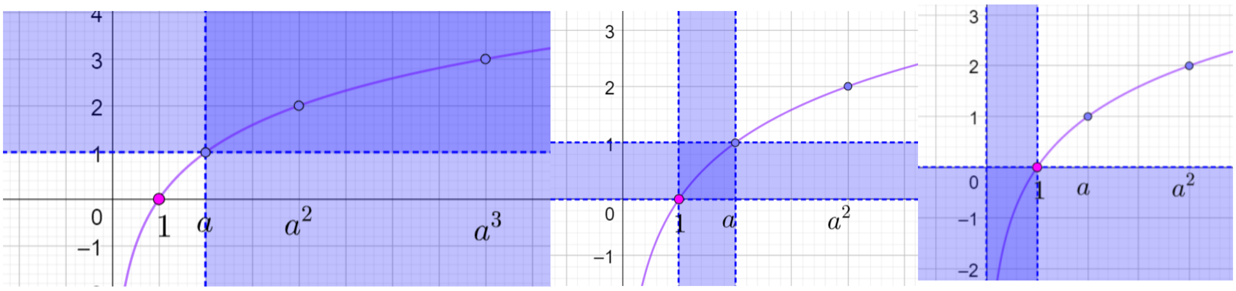


For $f(x)$

$$\log_2 x > 1 \Leftrightarrow x > 2$$
$$0 < \log_2 x < 1 \Leftrightarrow 1 < x < 2$$
$$\log_2 x < 0 \Leftrightarrow 0 < x < 1$$

For $g(x)$

$$\log_a n > 1 \Leftrightarrow n > a$$
$$0 < \log_a n < 1 \Leftrightarrow 1 < n < a$$
$$\log_a n < 0 \Leftrightarrow 0 < x < 1$$



Example 2.3: Inequalities

A. If x and $\log_{10} x$ are real numbers and $\log_{10} x < 0$, then must lie in the interval: **(AHSME 1963/5)**
B. Answer the previous question if $\log_{10} x \le 0$.

Part A

Convert from logarithmic form to exponential form:

$$x < 10^0 \Rightarrow x < 1$$

Also, the number to which we find the logarithm must be positive. Hence:

$$0 < x < 1 \Rightarrow x \in (0,1)$$

Part B

$$x \le 10^0 \Rightarrow x \le 1$$

Also, the number to which we find the logarithm must be positive. Hence,

$$0 < x \le 1 \Rightarrow x \in (0,1]$$

Example 2.4

The graph of $y = \log x$

A. Cuts the y-axis
B. Cuts all lines perpendicular to the x-axis
C. Cuts the x-axis
D. Cuts neither axis
E. Cuts all circles whose center is at the origin **(AHSME 1950/44)**

Option B

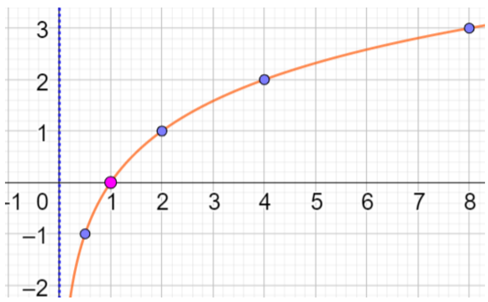
The graph cuts all lines $x = c, c > 0$, but does not cut $x = c, c \le 0$.
For example, the graph does not cut:

$$x = -1$$
$$\textit{Hence, option B is incorrect.}$$

Option C

The graph of $y = \log x$ has $x - intercept$:

$$x = 1$$
$$\textit{Option C is correct}$$



Example 2.5

A. If x is real and positive and grows beyond all bounds, then $\log_3(6x-5)-\log_3(2x+1)$ approaches: **(AHSME 1967/23)**
B. What is the horizontal asymptote of the expression in the above question?

Part A

Combine using the quotient rule:

$$\log_3\left(\frac{6x-5}{2x+1}\right)$$

We want to understand the behavior as x grows very large. Divide both numerator and denominator by the highest power of x in the denominator:

$$\log_3\left(\frac{\frac{6x}{x}-\frac{5}{x}}{\frac{2x}{x}+\frac{1}{x}}\right)=\log_3\left(\frac{6-\frac{5}{x}}{2+\frac{1}{x}}\right)$$

Note that as x becomes very large, $\frac{5}{x}$ and $\frac{1}{x}$ both become very small:

$$\approx \log_3\left(\frac{6}{2}\right)=\log_3(3)=1$$

Part B

The horizontal asymptote is the y-value when the x-value becomes very large:

$$y=1$$

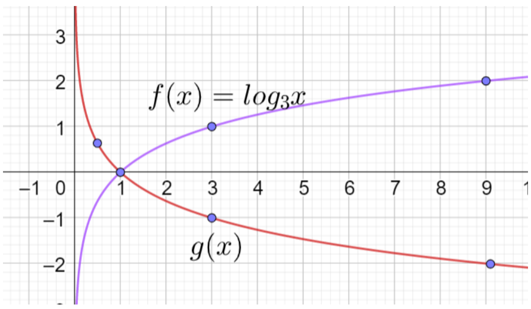
B. $0 < Base < 1$

2.6: Reflection across the $x-axis$

The graph of

$$y=\log_{\frac{1}{a}} x \quad \underbrace{\Leftrightarrow}_{\substack{Reflection\\ across\ the\ x-axis}} \quad y=\log_a x$$

- Reflecting the graph of $\log_a x$ is equivalent to taking the reciprocal of the base.
- Taking the reciprocal of the base a in $\log_a x$ is equivalent to reflecting the graph across the x-axis.



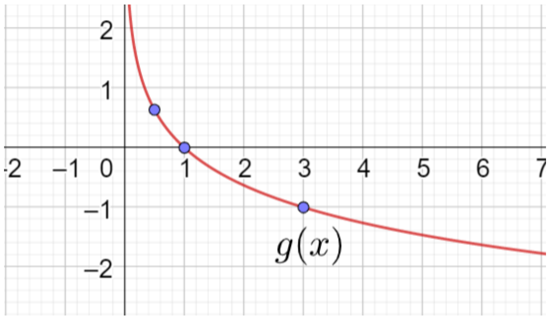
Example 2.7

A. The graph of $g(x)$ is obtained after a transformation of the graph of $f(x)=\log_3 x$. Determine $g(x)$.

$$g(x)=\log_{\frac{1}{3}} x$$

2.8: Graph of $y = \log_a x$

$0 < a < 1$



Intercepts

To find the y-intercept, substitute $x = 0$:

$$y = \log x = \log 0 \Rightarrow Not\ Defined$$
$$No\ y - intercept$$

To find the x-intercept, substitute $y = 0$:

$$0 = \log x \Rightarrow x = 1$$
$$x - intercept = (1,0)$$

Asymptotes

$$Vertical\ Asymptote: x = 0$$
$$Horizontal\ Asymptote: DNE\ (Does\ Not\ Exist)$$

Positive and Negative Intervals

$$\log_a x > 0 \Rightarrow 0 < x < 1$$
$$\log_a x < 0 \Rightarrow x > 1$$

Increasing and Decreasing Intervals

The function is decreasing throughout its domain:

$$(0, \infty)$$

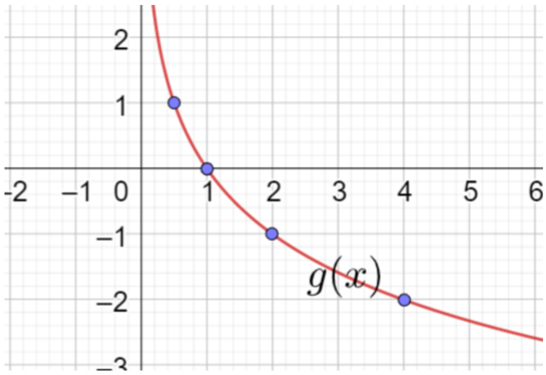
Domain and Range

$$Domain: x > 0, a > 0, a \neq 1$$
$$Range: (-\infty, \infty)$$

Example 2.9: Property (Memorize)

Draw $f(x) = \log_{\frac{1}{2}} x$ and $g(x) = \log_a x, 0 < a < 1$. Hence, for each function, determine when it is:

A. Greater than 1
B. Between 0 and 1
C. Negative



$f(x)$

$$\log_{\frac{1}{2}} x > 1 \Leftrightarrow 0 < x < \frac{1}{2}$$
$$0 < \log_{\frac{1}{2}} x < 1 \Leftrightarrow \frac{1}{2} < x < 1$$
$$\log_{\frac{1}{2}} x < 0 \Leftrightarrow x > 1$$

$g(x)$

$$\log_a x > 1 \Leftrightarrow 0 < x < a$$
$$0 < \log_a x < 1 \Leftrightarrow a < x < 1$$
$$\log_a x < 0 \Leftrightarrow x > 1$$

Example 2.10

A. Find all points on the graph of $y = \log_5 x$ at a distance of 3 units from the x −axis.
B. Without solving an equation, find all points on the graph of $y = \log_{\frac{1}{5}} x$ that are a distance of 3 units from the x-axis.
C. Without solving an equation, find all points on the graph of $y = \log_5 x$ that are a distance of 5 units from the x-axis.

Part A

For horizontal/vertical distance, recall that the absolute value of the difference of two points gives the distance between the points:

$$Distance\ from\ x-axis = 3 \Rightarrow |y-0| = 3$$
$$|y| = 3$$
$$|\log_5 x| = 3$$
$$\underbrace{\log_5 x = \pm 3}_{Logarithmic\ Form}$$
$$\underbrace{x = 5^{\pm 3}}_{Exponential\ Form}$$
$$(x,y) = \left(\frac{1}{125}, -3\right), (125,3)$$

Part B

The graph of $y = \log_{\frac{1}{5}} x$ is obtained by reflecting the graph of $y = \log_5 x$ across the x-axis.
Hence, reflect the points from Part A across the x-axis by negating their y-coordinate.

$$(x,y) = \left(\frac{1}{125}, 3\right), (125, -3)$$

Part C

$$(x,y) = \left(\frac{1}{125}, -3\right), (125,3)$$
$$(x,y) = \left(\frac{1}{125} \times \frac{1}{25}, -5\right), (125 \times 5^2, 5)$$

Example 2.11

A. Find all points on the graph of $y = \log_b x, 0 < b < 1$ at a distance of $c, c > 0$ units from the x –axis.
B. Answer the previous question if $b > 1$.
C. Find all points on the graph of $y = \log_b x, b > 1$ that are a distance of $c + 1, c > 0$ units from the x-axis.

Part A

For horizontal/vertical distance, recall that the absolute value of the difference of two points gives the distance between the points:

$$Distance\ from\ x-axis = 3 \Rightarrow |y-0| = c$$
$$|y| = c$$
$$|\log_b x| = c$$
$$\log_b x = \pm c$$
$$x = b^{\pm c}$$
$$x \in \{b^c, b^{-c}\}$$
$$(x,y) = (b^c, -c), (b^{-c}, c)$$

Part B

$$(x,y) = (b^c, c), (b^{-c}, -c)$$

Part C

$$(x,y) = (b^c \times b, c+1), \left(\frac{b^{-c}}{b}, -c-1\right)$$
$$= (b^{c+1}, c+1), (b^{-c-1}, -c-1)$$

C. Shifting and Scaling

2.12: Translation: Vertical and Horizontal Shift

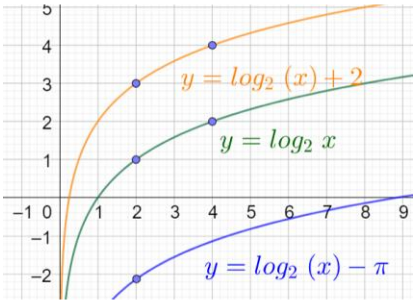
Compared to the graph of $f(x)$:

$$f(x) + k \Rightarrow Moves\ up\ for\ k > 0, Moves\ down\ for\ k < 0$$
$$f(x+k) \Rightarrow Moves\ left\ for\ k > 0, Moves\ right\ for\ k < 0$$

Example 2.13

State the transformation applied to the graph of $y = \log_2 x$ in each case:

A. $y = \log_2(x) + 2$
B. $y = \log_2(x) - \pi$
C. $y = \log_2(x) + \log_3 x$
D. $y = \log_2(x+3)$
E. $y = \log_2(x-2)$



Let

$$f(x) = \log_2 x$$

Parts A-B

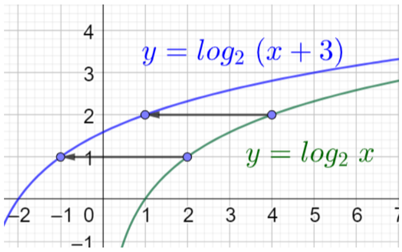
$$y = \log_2(x) + 2 = f(x) + 2 \Rightarrow Move\ up\ by\ 2\ units$$
$$y = \log_2(x) - \pi \Rightarrow Moves\ down\ by\ \pi\ units$$

Part C

$$y = \log_2(x) + \log_3 x \Rightarrow Moves\ up\ by\ \log_3 x$$

This is better written as:

$$x > 1 \Rightarrow Moves\ up\ by\ \log_3 x$$
$$0 < x < 1 \Rightarrow Moves\ down\ by\ |\log_3 x|$$



Part D

$$Shifts\ left\ by\ 3\ units$$

Part E

$$Shifts\ right\ by\ 2\ units$$

2.14: Vertical and Horizontal Scaling

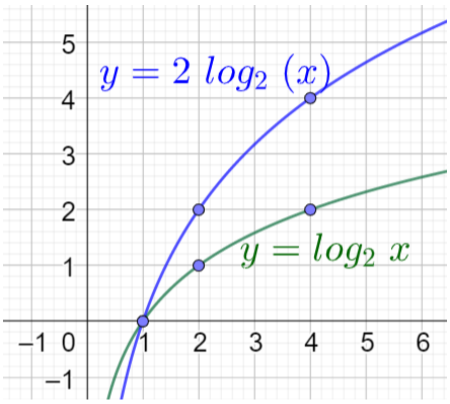
Compared to the graph of $f(x)$:

$$kf(x) \Rightarrow Scales\ the\ graph\ vertically\ by\ a\ factor\ of\ k$$
$$f(kx) \Rightarrow Scales\ the\ graph\ horizontally\ by\ a\ factor\ of\ \frac{1}{k}$$

Example 2.15

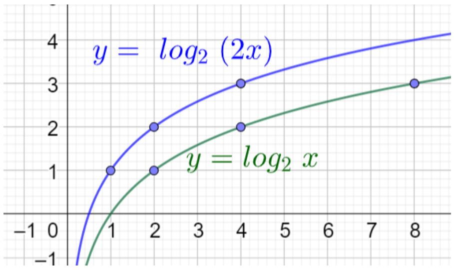
State the transformation applied to the graph of $y = \log_2 x$ in each case:

A. $y = 2\log_2 x$
B. $y = \log_2 2x$



$$Vertical\ Scaling\ by\ a\ Factor\ of\ 2$$

$$Horizontal\ Shrink\ by\ a\ factor\ of\ 2$$
$$\log_2 2x = \log_2 2 + \log_2 x = 1 + \log_2 x = f(x) + 1 \Rightarrow Move\ up\ by\ 1\ unit$$



Example 2.16: Transformations

A. Combining Transformations

2.17: Log of 1 to any valid base is zero

$$a^0 = 1, a > 0, a \neq 1 \Leftrightarrow \log_a 1 = 0$$

The condition that $a \neq 0$ is very important because:

$$0^0 \Rightarrow Not\ Defined$$

Example 2.18

Determine the value of the variable in each of the following:

A. Find $a = \log_5 1, b = \log_\pi 1, c = \log_{27.5} 1, 0 = \log_6 x, 0 = \log_y 1$

$$a = b = c = 0$$
$$x = 1$$
$$y > 0, y \neq 1$$

Example 2.19

If $\log_a x = 0$, $\log_a(y+2) = 0$ and $\log_a z^2 = 0$, then find the minimum value and the maximum value of $x + y + z$.

$$\log_a x = 0 \Rightarrow x = 1$$
$$\log_a(y+2) = 0 \Rightarrow y + 2 = 1 \Rightarrow y = -1$$
$$\log_a z^2 = 0 \Rightarrow z^2 = 1 \Rightarrow z = \pm 1$$

$$\text{Max}(x + y + z) = 1 - 1 + 1 = 1$$
$$\text{Min}(x + y + z) = (1 - 1 - 1) = -1$$

Example 2.20

If $y\ = log_a x, a > 1$, which of the following statements is incorrect?

A. If $x = 1, y = 0$
B. If $x = a, y = 1$
C. If $x = -1, y$ is imaginary (complex)
D. If $0 < x < 1, y$ is always less than 0 and decreases without limit as x approaches zero
E. Only some of the above statements are correct **(AHSME 1950/37)**

Option E

2.21: Log of a number to itself, and to its reciprocal

$$a^1 = a \Leftrightarrow log_a a = 1$$

$$a^{-1} = \frac{1}{a} \Leftrightarrow \log_{\frac{1}{a}} a = -1$$

Example 2.22

Simplify:

A. $\log_5 5$
B. $\log_{317} 317$
C. $\log_{\frac{3\pi}{e}} \frac{3\pi}{e}$
D. $\log_{\frac{1}{5}} 5$
E. $\log_5 \frac{1}{5}$
F. $\log_{\frac{4}{3}} \frac{3}{4}$
G. $\log_{\frac{e}{3\pi}} \frac{3\pi}{e}$
H. $\log_{3-\sqrt{8}} \left(\frac{1}{3+\sqrt{8}}\right)$

$$\log_5 5 = \log_{317} 317 = \log_{\frac{3\pi}{e}} \frac{3\pi}{e} = 1$$

$$\log_{\frac{1}{5}} 5 = \log_5 \frac{1}{5} = \log_{\frac{4}{3}} \frac{3}{4} = \log_{\frac{e}{3\pi}} \frac{3\pi}{e} = -1$$

$$\log_{3-\sqrt{8}} \left(\frac{1}{3+\sqrt{8}}\right) = \underbrace{\log_{3-\sqrt{8}} \left(\frac{1}{3+\sqrt{8}} \times \frac{3-\sqrt{8}}{3-\sqrt{8}}\right)}_{\textit{Rationalizing the Denominator}} = \log_{3-\sqrt{8}} \left(\frac{3-\sqrt{8}}{9-8}\right) = 1$$

D. Rotations

2.23: Rotations

$$90^\circ\ \textit{Counterclockwise}: (x, y) \to (-y, x)$$
$$90^\circ\ \textit{Clockwise}: (x, y) \to (y, -x)$$
$$180^\circ: (x, y) \to (-x, -y)$$

Example 2.24

The graph, G of $y = \log_{10} x$ is rotated 90° counter-clockwise about the origin to obtain a new graph G'. What is the equation for G'? **(AHSME 1991/24)**

Step I: Change of Variable

Introduce new variables that are equal to the transformed variables:

$$(x, y) \to (-y, x) = (X, Y)$$
$$X = -y \Rightarrow y = -X$$

Step II: Relationship between new Variables

$$\textit{Graph of } G: y = \log_{10} x$$

Substitute $Y = x,\ y = -X$:

$$-X = \log_{10} Y \Rightarrow Y = 10^{-X}$$

Step III: Write in Standard Notation

Write the final equation from Step II in terms of the usual (x, y) notation:

$$\textit{Graph of } G':\ y = 10^{-x}$$

Example 2.25

The graph, G of $y = \log_2 x$ is rotated 90° clockwise about the origin to obtain a new graph G'. What is the equation for G'?

Step I: Change of Variable

Introduce new variables that are equal to the transformed variables:

$$(x, y) \to (y, -x) = (X, Y)$$

Step II: Relationship between new Variables

$$y = \log_2 x$$

Substitute $-Y = x,\ X = y$:

$$X = \log_2(-Y)$$
$$2^X = -Y$$
$$Y = -2^X$$

Step III: Write in Standard Notation

Write the final equation from Step II in terms of the usual (x, y) notation:

$$y = -2^x$$

E. Inverses

Example 2.26

The graph of $f(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ is reflected across the line $y = x$. Determine the equation that the new graph satisfies.

Begin by substituting $y = f(x)$:

$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

A reflection across the line $y = x$ is equivalent to finding the inverse function. Swap x and y:

$$x = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$$

Now we need to solve for y. Multiply both sides by 2:

$$2x = \ln\left(\frac{1+y}{1-y}\right)$$

Convert to exponential form:

$$e^{2x} = \frac{1+y}{1-y}$$

Eliminate fractions:

$$e^{2x} - ye^{2x} = 1 + y$$

Collate all y terms on one side:

$$e^{2x} - 1 = y + ye^{2x}$$

Factor:

$$e^{2x} - 1 = y(1 + e^{2x})$$

Solve for y:

$$y = \frac{e^{2x} - 1}{1 + e^{2x}}$$

F. General Transformations

Example 2.27

When the cube root of y is plotted against the square of x on graph paper, the resulting graph is a straight line passing through (9,8) and (16,1). If the graph of the natural log of $(-x^2 + 17)$ is plotted aganist the natural log of y, determine the slope and the intercepts of the graph formed.

Use a change of variable. Let

$$Y = \sqrt[3]{y}, \qquad X = x^2$$

Then, the equation of the straight line in slope intercept form can be written:

$$Y = mX + C$$

$$m = \ Slope = \frac{1-8}{16-9} = -\frac{7}{7} = -1$$
$$C = 8 + 9 = 17$$

The equation of the line is then:

$$Y = -X + 17$$

Change back to the original variables:

$$\sqrt[3]{y} = -x^2 + 17$$
$$y = (-x^2 + 17)^3$$
$$\ln y = 3\ln(-x^2 + 17)$$

Note that $\ln(-x^2 + 17)$ $(Vertical)$ is plotted against $\ln y$ $(Horizontal)$:

$$x' = \ln y\,, \qquad y' = \ln(-x^2 + 17)$$
$$x' = 3y'$$
$$y' = \frac{1}{3}x'$$

$$Slope = \frac{1}{3}, \qquad x - intercept = y - intercept = 0$$

Example 2.28

Given that $\exp(x) = e^x$:

A. Sketch the graph of $f(x) = \exp\left(\log_\pi\left(\log_{|x|}|x|\right)\right)$
B. Determine whether the function $f(x)$ is odd, even, or neither.

Part A

The number to which we are taking the logarithm must not be zero:

$$|x| > 0 \Rightarrow x \neq 0$$

The base of the logarithm must not be greater than zero, and must not be 1.

$$|x| > 0 \Rightarrow x \neq 0$$
$$|x| = 1 \Rightarrow x \neq \pm 1$$

For $x \in \mathbb{R} - \{0, \pm 1\}$:

$$\exp\left(\log_\pi \left(\log_{|x|}|x|\right)\right) = \exp(\log_\pi 1) = \exp(0) = 1$$

Hence, we get the graph:

$$y = 1, \qquad x \in \mathbb{R} - \{0, \pm 1\}$$

Part B

Since the graph of $f(x)$ is a straight line with some points not defined, and those points are symmetrical about the y axis.
Therefore,

$$f(x)\ is\ even$$

Example 2.29

$$Mark\ all\ correct\ options$$
$$f(x) = \exp\left(\log_\pi \left(\log_{|x|}|x|\right)\right), \qquad g(x) = f(x) + b$$

Given that $\exp(x) = e^x$, and a and b are rational numbers mark all options such that $g(x)$ can be an odd function:

A. $b = 0$
B. $b < 0$
C. $b > 0$
D. $b \neq 0$

$$y = 1, \qquad x \in \mathbb{R} - \{0, \pm 1\}$$

$b = -1$ is required

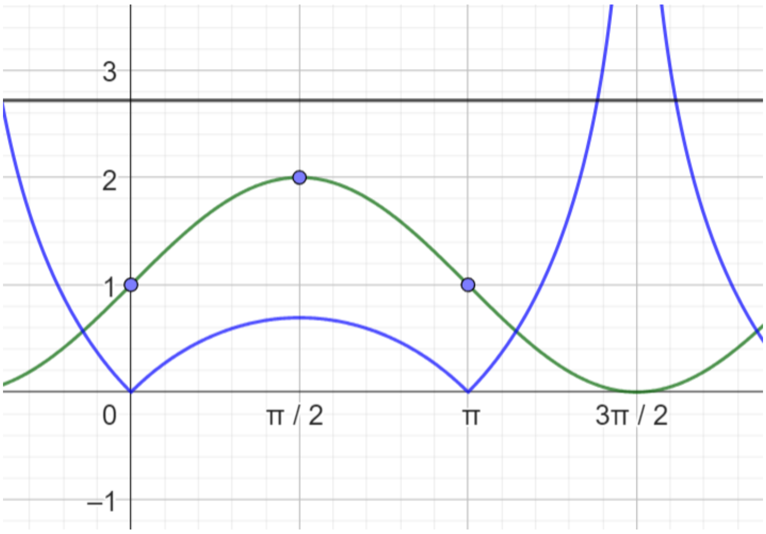
$$Options\ B, \qquad D$$

G. Absolute Value as a Reflection

Example 2.30

Sketch the graph of

$$y = |\ln(1 + \sin x)|$$



2.2 Counting and Probability

A. Approximation

If a number lies between two powers of the base, then the log of that number must lie between those powers.

2.31: Log of a number to itself, and to its reciprocal

For some valid base b, and some valid number x:

$$\underbrace{b^p < x < b^q}_{Exponential\ Form} \Leftrightarrow \underbrace{p < \log_b x < q}_{Logarithmic\ Form}$$

$$4 < 5 < 8$$
$$2^2 < 5 < 2^3$$
$$2 < \log_2 5 < 3$$

Example 2.32: Approximating logs

Find, for each term, the two integers between which it lies:

Integer Bases

A. $\log_2 9$
B. $\log_5 63$
C. $\log_3 29$
D. $\log_{10} 243$

Radical Bases

E. $\log_{\sqrt{3}} 12$

Fractional Bases

Transcendental Bases

Integer Bases

$$\log_2 8 < \log_2 9 < \log_2 16 \Rightarrow 3 < \log_2 9 < 4$$
$$2 < \log_5 63 < 3$$
$$3 < \log_3 29 < 4$$
$$2 < \log_{10} 243 < 3$$

Radical Bases

$$\left(\sqrt{3}\right)^4 = 9, \qquad \left(\sqrt{3}\right)^5 = 9\sqrt{3} \approx 15 \Rightarrow 4 < \log_{\sqrt{3}} 12 < 5$$

Example 2.33: Approximating logs

Find, for each term, the two integers between which it lies:

A. $\log_{\frac{1}{3}} 12$
B. $\log_{\frac{1}{3}} \frac{1}{2}$

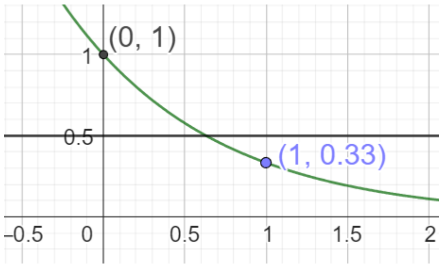
Part A

$$-3 < \log_{\frac{1}{3}} 12 < -2$$

Part B

From the graph:

$$0 < \log_{\frac{1}{3}} \frac{1}{2} < 1$$



Example 2.34: Approximating logs

Find, for each term, the two integers between which it lies:

A. $\log_\pi 2\pi$
B. $\log_e 3e$, $e \approx 2.71$

Part A

$$3 < \pi < 4 \Rightarrow \log_\pi \pi < \log_\pi 2\pi < \log_\pi \pi^2 \Rightarrow 1 < \log_\pi 2\pi < 2$$

Part B

Since $e^2 < 3e$ and $3e < e^3$:

$$\log_e e^2 < \log_e 3e < \log_e e^3$$

Hence:

$$2 < \log_e 3e < \log_e e^3$$

Example 2.35: Applying approximation to get a range

Find the sum of the values that satisfy $2 < \log_\pi n\pi < 3$, $n \in \mathbb{N}$

Solution I (Transforming Inequalities)

The given inequality has a log with the base π, but the other parts of the inequality have numbers.
Hence, convert the numbers into logs π:

$$\log_\pi \pi^2 < \log_\pi n\pi < \log_\pi \pi^3$$

Since all three parts have logs, we can "cancel" the logs:
(In other words, take anti-logs throughout the inequalities)

$$\pi^2 < n\pi < \pi^3$$

Divide throughout by π:

$$\pi < n < \pi^2$$

Convert to numerical form

$$3.14 < n < 9.86$$

Hence, the values that n can take are:

$$n \in \{4,5,6,7,8,9\}$$
$$Sum = 4 + 5 + 6 + 7 + 8 + 9 = 39$$

Solution II (Working with endpoints of the inequality)

An alternate method of solving the inequality is to solve the corresponding equations:

$$\log_\pi n\pi = 2 \Rightarrow n = \pi \approx 3.14$$
$$\log_\pi n\pi = 3 \Rightarrow n = \pi^2 \approx 9.86$$

$$\therefore 3.14 < n < 9.86 \Rightarrow n \in \{4,5,6,7,8,9\}$$

B. Counting / Number Theory

We look at some ways in which logarithms can be combined with other concepts.

Example 2.36

Natural Numbers

Find the natural numbers that lie between:

A. $\log_2 3$ and $\log_2 2048$
B. $\log_3 \frac{1}{81}$ and $\log_3 2087$

Integers

Find the number of integers that lie between:

C. $\log_{10} 0.005$ and $\log_{10} 1{,}000{,}000{,}000$

$$\{\log_2 4, \dots, \log_2 1024\} \Rightarrow \{2, 3, \dots, 10\} \Rightarrow 10 - 2 + 1 = 9$$

$$\left\{\log_3 \frac{1}{27}, \dots, \log_3 729\right\} \Rightarrow n(-3, -2, -1, 0, 1, 2, \dots, 6) = n(1, 2, \dots, 6) = 6$$

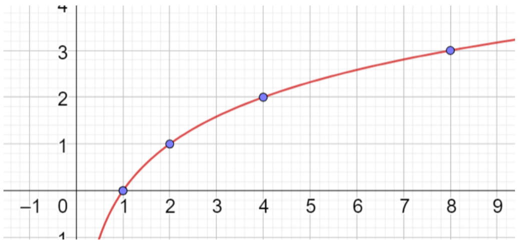
$$\{\log_{10} 0.01, \dots, \log_{10} 100{,}000{,}000\} = \{-2, -1, \dots, 8\} \Rightarrow 8 - (-2) + 1 = 11$$

Example 2.37: Natural Number Solutions

What is the sum of the values of x that satisfy:

A. $\log_2 x = c, c \in \mathbb{N}, x < 1000$
B. $\log_2 x = c, c \in \mathbb{N}, c < 1000$

Part A



In order c to be a natural number, x must be an integer power of 2.

$$\log_2 2 = 1$$
$$\log_2 4 = \log_2 2^2 = 2$$
$$\log_2 8 = \log_2 2^3 = 3$$
$$\log_2 16 = \log_2 2^4 = 4$$
$$\vdots$$
$$\log_2 512 = \log_2 2^9 = 9$$

Hence, the numbers which work are:

$$\{2,4,8,16, \dots, 512\} = \{2^1, 2^2, 2^3, 2^4, \dots, 2^9\}$$

The sum of the solutions is:

$$2^1 + 2^2 + \dots + 2^9 = \underbrace{2^0 + 2^1 + 2^2 + \dots + 2^9}_{1024-1} - 2^0 = 1022$$

Where we used the fact that:

$$1 + 2 + 2^2 + \dots 2^n = 2^{n+1} - 1$$

The above property holds because it is a geometric series with $a = 1, r = 2$:

$$S = \frac{a(r^n - 1)}{r - 1} = \frac{1(2^{n+1} - 1)}{2 - 1} = 2^{n+1} - 1$$

Part B

$$\{2^1, 2^2, 2^3, 2^4, \dots, 2^{999}\}$$
$$2^1 + 2^2 + \dots + 2^{999} = \underbrace{2^0 + 2^1 + 2^2 + \dots + 2^{999}}_{2^{1000}-1} - 2^0 = 2^{1000} - 2$$

Example 2.38: Integer Solutions

What is the sum of the values of x that satisfy:

A. $\log_3 x = c, c \in \mathbb{Z}, x < 1000$
B. $\log_3 x = c, c \in \mathbb{Z}, |c| < 1000$

Part A

$$\log_3 729 = 6$$
$$\log_3 3 = 1$$
$$\log_3 1 = 0$$
$$\log_3 \frac{1}{3} = -1$$
$$\log_3 \frac{1}{9} = -2$$
$$\vdots$$
$$\log_3 \frac{1}{3^n} = -n$$

$$729 + 243 + 81 + \cdots$$
$$3^6 + 3^5 + 3^4 + \cdots$$

This is an infinite geometric series with $a = 3^6, r = \frac{1}{3}$ which has sum:

$$S = \frac{a}{1-r} = \frac{3^6}{1-\frac{1}{3}} = \frac{3^6}{\frac{2}{3}} = \frac{729 \times 3}{2} = 1093.5$$

Part B

$$3^{999} + 3^{998} + \dots + \frac{1}{3^{999}}$$

This is a finite geometric series with $a = 3^{999}, r = \frac{1}{3}, n = 1999$:

$$S = \frac{a(1-r^n)}{1-r} = \frac{3^{999}\left[1 - \left(\frac{1}{3}\right)^{1999}\right]}{1-\frac{1}{3}}$$
$$\frac{3^{999} - 3^{999-1999}}{\frac{2}{3}}$$
$$= \frac{3^{999} - 3^{999-1999}}{\frac{2}{3}}$$
$$= \frac{3(3^{999} - 3^{-1000})}{2}$$
$$= \frac{3^{1000} - \frac{1}{3^{999}}}{2}$$

Example 2.39: Number of Solutions for the Base

Find the number of values b that satisfy each equation below (answer separately for each):

A. $\log_b 32 = n, n \in \mathbb{N}$
B. $\log_b 1024 = n, n \in \mathbb{N}$
C. $\log_b 2^{48} = n, n \in \mathbb{N}$

Part A

Since n is a natural number, b must also be a natural number.

$$\log_b 32 = n$$
$$b^n = 32 = 2^5$$
$$(b, n) = \{(2,5), (32,1)\}$$

Part B

$$\log_b 1024 = n$$
$$b^n = 1024 = 2^{10}$$
$$(2^1)^{10} = (2^{10})^1 = (2^2)^5 = (2^5)^2$$

$$(b, n) = \{(2, \mathbf{10})(4, \mathbf{5}), (32, \mathbf{2})(1024, \mathbf{1})\}$$

$$Factors\ of\ 10 = \{1,2,5,10\}$$

Part C

$$\log_b 2^{48} = n$$
$$b^n = 2^{48}$$

The number of ordered pairs (b, n) that will work is the same as the number of factors of 48:

$$(1,48)(2,24)(3,16)(4,12)(6,8) \Rightarrow 10\ Factors$$
$$10\ Solutions$$

$$(2^1)^{48}, (2^2)^{24}, (2^3)^{16}, (2^4)^{12}, (2^6)^8$$
$$(2^{48})^1, (2^{24})^2, (2^{16})^3, (2^{12})^4, (2^8)^6$$

Example 2.40

Find the number of elements in the set

$$\{(a, b) \in \mathbb{N}: 2 \le a, b \le 2023, \log_a b + 6\log_b a = 5\}\ (\boldsymbol{IOQM\ 2023/2})$$

Using the reciprocal rule:

$$\log_a b + \frac{6}{\log_a b} = 5$$

Use a change of variable. Let $t = \log_a b$

$$t + \frac{6}{t} = 5 \Rightarrow t^2 - 5t + 6 = 0 \Rightarrow t \in \{2,3\}$$

Case I: $\log_a b = 2 \Rightarrow b = a^2$

$$(a, b) = (2,4), (3,9), (4,16), \dots, (44,1936) \Rightarrow 43\ Pairs$$

Case II: $\log_a b = 3 \Rightarrow b = a^3$

$$(a, b) = (2,8), (3,27), \dots, (12,1728) \Rightarrow 11\ Pairs$$

Total number of pairs

$$= 43 + 11 = 54$$

2.41: Number of Divisors[3]

The number of divisors $\boldsymbol{\tau(x)}$ of a number $\boldsymbol{x}$ with prime factorisation $\boldsymbol{a^p b^q c^r}$ is given by:

$$\boldsymbol{x = a^p b^q c^r \Rightarrow \tau(x) = (p+1)(q+1)(r+1)}$$

Example 2.42: Number of Solutions for the Base

Find the number of values b that satisfy each equation below (answer separately for each):

A. $\log_b 5^{216} = n, n \in \mathbb{N}$
B. $\log_b 7^{5!} = n, n \in \mathbb{N}$

Part A

$$216 = 6^3 = 2^3 \times 3^3 \Rightarrow \tau(216) = (3+1)(3+1) = 16 \Rightarrow 16\ Solutions$$

Part A

$$5! = 120 = 2^3 \times 3 \times 5 \Rightarrow \tau(5!) = (3+1)(1+1)(1+1) = 16\ Solutions$$

2.43: Product of Divisors

The product of the distinct positive factors of a number N is given by:

$$N^{\left(\frac{\tau(N)}{2}\right)}$$

Example 2.44

Let S be the sum of the base 10 logarithms of the proper divisors of 1,000,000. What is the integer nearest to S? **(AIME 1986/8)**

If the divisors are $d_1, d_2, \dots d_k$, then using the product rule for logarithms:

$$\log d_1 + \log d_2 + \cdots + \log d_k = \log d_1 \times d_2 \times \dots \times d_k$$

The above is now the product of the divisors of 1,000,000, for which we have a formula:

$$d_1 \times d_2 \times \dots \times d_k = (N)^{\frac{\tau(N)}{2}} = (10^6)^{\left(\frac{49}{2}\right)} = 10^{147}$$

where:

$$N = 1{,}000{,}000 = 10^6 = 2^6 \times 5^6$$
$$\tau(N) = \text{number of divisors of } N = (6+1)(6+1) = 49$$

However, since we only need the proper divisors, we must exclude 1,000,000 from the calculations, and hence, we get:

$$= \log \frac{10^{147}}{10^6} = \log 10^{141} = 141$$

C. Probability

Example 2.45

A. Li selects an integer x from -1000 to 1000. What is the probability that $\log_3 x$ is defined?
B. Li selects an integer x between -1000 and 1000. What is the probability that $\log_3 x$ is defined?

For $\log_3 x$ to be defined:

$$x\ is\ a\ real\ number\ such\ that\ x > 0$$

[3] The details of this formula can be found in the Note on NT-Basics

Part A

The integers that meet this condition are:

$$\{1,2,3,\dots,1000\} \Rightarrow 1000\ Integers$$

The total number of integers to pick from is:

$$1000-(-1000)+1=2001$$

Hence, the probability is:

$$\frac{Successful\ Outcomes}{Total\ Outcomes}=\frac{1000}{2001}$$

Part B

The integers that meet this condition are:

$$\{1,2,3,\dots,999\} \Rightarrow 999\ Integers$$

The total number of integers to pick from is:

$$999-(-999)+1=1999$$

Hence, the probability is:

$$\frac{Successful\ Outcomes}{Total\ Outcomes}=\frac{999}{1999}$$

Example 2.46

I select a number from 1 to 1000. What is the probability that the log of the number to the base 5 is an integer?

You can select any real number from the interval

$$(1,1000) \Rightarrow Infinite\ Numbers$$

The number of numbers such that the log of the number to the base 5 is an integer will be a finite number.

$$P=\frac{Successful\ Outcomes}{Total\ Outcomes}=\frac{Finite\ Number}{\infty}=0$$

Example 2.47

I select a natural number from 1 to 1000. What is the probability that the log of the number to the base 5:

A. is an integer
B. is a natural number
C. is an even integer
D. is an odd integer
E. is a whole number

Part A

$$\log_5 x=c, c\in\mathbb{Z} \Rightarrow x\in\{1,5,25,125,625\}$$

$$P=\frac{Successful\ Outcomes}{Total\ Outcomes}=\frac{5}{1000}=\frac{1}{200}$$

Part B

$$\log_5 x=c, c\in\mathbb{N} \Rightarrow x\in\{5,25,125,625\}$$

$$P=\frac{Successful\ Outcomes}{Total\ Outcomes}=\frac{4}{1000}=\frac{1}{250}$$

Part C

$$\log_5 x=c, c\ is\ an\ even\ integer \Rightarrow x\in\{1,25,625\}$$

$$P=\frac{Successful\ Outcomes}{Total\ Outcomes}=\frac{3}{1000}$$

Part D

$$\log_5 x=c, c\ is\ an\ odd\ integer \Rightarrow x\in\{5,125\}$$

$$P=\frac{Successful\ Outcomes}{Total\ Outcomes}=\frac{2}{1000}=\frac{1}{500}$$

Part E

Same as part A.

Example 2.48

I select a natural number from 1 to 10,000. What is the probability that the log of the number to the base 2:

A. is an integer

B. is a natural number
C. is an even integer
D. is an odd integer

Part A

$$\log_5 x = c, c \in \mathbb{Z} \Rightarrow x \in \{2^0, 2^1, \dots, 2^{13}\}$$

$$P = \frac{Successful\ Outcomes}{Total\ Outcomes} = \frac{14}{10{,}000} = \frac{7}{5000}$$

Part B

$$\log_5 x = c, c \in \mathbb{Z} \Rightarrow x \in \{2^1, \dots, 2^{13}\}$$

$$P = \frac{Successful\ Outcomes}{Total\ Outcomes} = \frac{13}{10{,}000}$$

Part C

$$\log_5 x = c, c\ is\ an\ even\ integer \Rightarrow x \in \{2^0, 2^1, \dots, 2^{12}\}$$

$$P = \frac{Successful\ Outcomes}{Total\ Outcomes} = \frac{7}{10{,}000}$$

Part D

Out of 14 integers, 7 are even, so the remaining 7 are odd.

$$P = \frac{Successful\ Outcomes}{Total\ Outcomes} = \frac{7}{10{,}000}$$

Example 2.49

I select a natural number n at random from 1 to 1000. I calculate $x = \log_2 n$ and $y = \log_3 n$. What is the probability that:

A. x is an integer
B. y is an integer
C. Both x and y are integers
D. At least one of x and y is an integer
E. Exactly one of x and y is an integer.

Part A

$$\log_2 n = x, x \in \mathbb{Z}$$

$$n = \{2^0, 2^1, \dots, 2^9\} \Rightarrow 10\ Numbers$$

$$P = \frac{10}{1000} = \frac{1}{100}$$

Part B

$$\log_3 n = y, y \in \mathbb{Z}$$

$$n = \{3^0, 3^1, \dots, 3^6\} \Rightarrow 7\ Numbers$$

$$P = \frac{7}{1000}$$

Part C (Intersection of two sets)

We have already identified the values of n for which x and y are integers.

$$x \in \mathbb{Z} \Rightarrow n = \{1,2,4,8,16,32,64,128,256,512\}$$

$$y \in \mathbb{Z} \Rightarrow n = \{1,3,9,27,81,243,729\}$$

For both the numbers to be an integer, it must belong to both of the sets above. Hence, we want the intersection of the two sets, which is:

$$\{1\}$$

$$P = \frac{1}{1000}$$

Part D (Union of Two Sets)

$$\{1,2,4,8,16,32,64,128,256,512, 1, 3,9,27,81,243,729\}$$

$$\{1,2,4,8,16,32,64,128,256,512, 3,9,27,81,243,729\}$$

$$\underbrace{\frac{10}{1000}}_{Values\ of\ x} + \underbrace{\frac{7}{1000}}_{Values\ of\ y} - \underbrace{\frac{1}{1000}}_{Values\ of\ x \cap y} = \frac{16}{1000} = \frac{2}{125}$$

Part E (Exclusive OR)

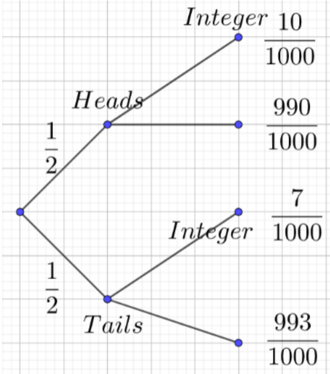
$$\{1, 2,4,8,16,32,64,128,256,512, 1, 3,9,27,81,243,729\}$$

$$\left\{\underbrace{2,4,8,16,32,64,128,256,512}_{x\ is\ an\ integer}, \underbrace{3,9,27,81,243,729}_{y\ is\ an\ integer}\right\}$$

$$\underbrace{\frac{10}{1000}}_{Values\ of\ x} + \underbrace{\frac{7}{1000}}_{Values\ of\ y} - 2\underbrace{\frac{1}{1000}}_{Values\ of\ x \cap y} = \frac{15}{1000} = \frac{3}{200}$$

Example 2.50

I select a natural number n at random from 1 to 1000. I toss a fair coin. If I get heads, I calculate $x = \log_2 n$. If the coins comes up tails, I calculate $x = \log_3 n$. What is the probability that x is an integer?



The probabilities when you toss the coin are:

$$Heads = \frac{1}{2}, \qquad Tails = \frac{1}{2}$$
$$\log_2 n = c, c \in \mathbb{Z} \Rightarrow n = \{2^0, 2^1, \dots, 2^9\} \Rightarrow 10\ Numbers$$
$$\log_3 n = c, c \in \mathbb{Z} \Rightarrow n = \{3^0, 3^1, \dots, 3^6\} \Rightarrow 7\ Numbers$$

$$\frac{1}{2} \times \frac{10}{1000} + \frac{1}{2} \times \frac{7}{1000} = \frac{1}{2}\left(\frac{10}{1000} + \frac{7}{1000}\right) = \frac{17}{2000}$$

Example 2.51

From a standard pack of playing cards, I discard the face cards and the Ace. I pick a card at random from the remaining cards and note its face value f. I then choose a natural number n at random from 1 to 1000. What is the probability that $\log_f n$ is a whole number?

I will pick a single card with face value

$$f = \{2,3, \dots ,10\}$$

$$f = 2 \Rightarrow \log_2 n = c, c \in \mathbb{Z} \Rightarrow n = \{2^0, 2^1, \dots, 2^9\} \Rightarrow 10\ Numbers$$
$$f = 3 \Rightarrow \log_3 n = c, c \in \mathbb{Z} \Rightarrow n = \{3^0, 3^1, \dots, 3^6\} \Rightarrow 7\ Numbers$$
$$f = 4 \Rightarrow \log_4 n = c, c \in \mathbb{Z} \Rightarrow n = \{4^0, \dots, 4^4\} \Rightarrow 5\ Numbers$$
$$f = 5 \Rightarrow \log_5 n = c, c \in \mathbb{Z} \Rightarrow n = \{5^0, \dots, 5^4\} \Rightarrow 5\ Numbers$$
$$f = 6 \Rightarrow \log_6 n = c, c \in \mathbb{Z} \Rightarrow n = \{6^0, \dots, 6^3\} \Rightarrow 4\ Numbers$$
$$f = 7 \Rightarrow \log_7 n = c, c \in \mathbb{Z} \Rightarrow n = \{7^0, \dots, 7^3\} \Rightarrow 4\ Numbers$$
$$f = 8 \Rightarrow \log_8 n = c, c \in \mathbb{Z} \Rightarrow n = \{8^0, \dots, 8^3\} \Rightarrow 4\ Numbers$$
$$f = 9 \Rightarrow \log_9 n = c, c \in \mathbb{Z} \Rightarrow n = \{9^0, \dots, 9^3\} \Rightarrow 4\ Numbers$$
$$f = 10 \Rightarrow \log_9 n = c, c \in \mathbb{Z} \Rightarrow n = \{10^0, \dots, 10^3\} \Rightarrow 4\ Numbers$$

$$\frac{1}{9}\left(\frac{10 + 7 + 5(2) + 4(5)}{1000}\right) = \frac{1}{9}\left(\frac{47}{1000}\right) = \frac{47}{9000}$$

Challenge 2.52

$$N = \{\log_2 n, \log_3 n, \dots\}$$

I pick a random natural number, n, from 1 to 1000, what is the probability that

A. at least one element of N is a natural number
B. at least one element of N is a natural number greater than 1.

Part A

$$\log_n n = 1 \Rightarrow P = 1$$

Part B

$$n = \{4,8,16,32,64,128,256,512\} = \{2^2, \dots, 2^9\} \Rightarrow 8$$
$$n = \{3^2, \dots, 3^6\} \Rightarrow 5\ Numbers$$
$$n\ is\ 4^x \Rightarrow 0\ Numbers$$
$$n\ is\ 5^x \Rightarrow \{25,125,625\} \Rightarrow 3\ Numbers$$
$$n\ is\ 6^x \Rightarrow \{36,216\} \Rightarrow 2\ Numbers$$
$$n\ is\ 7^x \Rightarrow \{49,343\} \Rightarrow 2\ Numbers$$
$$n\ is\ 8^x \Rightarrow 0\ Numbers$$
$$n\ is\ 9^x \Rightarrow 0\ Numbers$$
$$n\ is\ 10^x \Rightarrow 2\ Numbers$$

$$\{11^x, 12^x, 13^x, \dots, 31^2\} \Rightarrow 31 - 10 = 21\ Numbers$$
$$\{16^x, 25^x, 27^x\} \Rightarrow 3\ Numbers$$
$$21 - 3 = 18\ Numbers$$

$$8 + 5 + 3 + 2 + 2 + 2 + 18 = 40$$

1 Pending

Example 2.53

(AMC 12 2005/23)

Two distinct numbers a and b are chosen randomly from the set $\{2, 2^2, 2^3, \dots, 2^{25}\}$. What is the probability that $\log_a b$ is an integer?

$\text{(A)}\ \frac{2}{25} \quad \text{(B)}\ \frac{31}{300} \quad \text{(C)}\ \frac{13}{100} \quad \text{(D)}\ \frac{7}{50} \quad \text{(E)}\ \frac{1}{2}$

Example 2.54

x is a random number that satisfies $\frac{1}{16} < \log_x 5 < \frac{1}{2}$

A. What is the range of values that x can take?
B. What is the probability that $x = 625$, given that x is a real number?
C. What is the probability that $x = 625$, given that x is a natural number?
D. Answer Part C assuming that other things remain unchanged and $\frac{1}{16} \le \log_x 5 \le \frac{1}{2}$

Part A

Convert the inequality given in the question into exponential form:

$$x^{\frac{1}{16}} < 5 < x^{\frac{1}{2}}$$

Split the compound inequality into two inequalities:

$$x^{\frac{1}{16}} < 5 \Rightarrow x < 5^{16}$$
$$5 < x^{\frac{1}{2}} \Rightarrow x^{\frac{1}{2}} > 5 \Rightarrow x > 5^2$$

Recombine the two inequalities:

$$5^2 < x < 5^{16}$$

Part B

$$Successful\ Outcomes = 1$$
$$Total\ Outcomes = \infty$$
$$Probability = 0$$

Part C

$$Successful\ Outcomes = 1$$
$$Total\ Outcomes = (5^{16} - 1) - (5^2) = 5^{16} - 26$$
$$Probability = \frac{1}{5^{16} - 26}$$

Part D

$$Successful\ Outcomes = 1$$
$$Total\ Outcomes = (5^{16}) - (5^2) + 1 = 5^{16} - 24$$
$$Probability = \frac{1}{5^{16} - 24}$$

D. Floor Function

2.55: Floor Function

$$\lfloor x \rfloor = c \Rightarrow c \le x < c + 1$$

Example 2.56

A. $\lfloor 3.4 \rfloor$
B. $\lfloor -2.1 \rfloor$

C. $\lfloor x \rfloor = 3$

$$\lfloor 3.4 \rfloor = 3$$
$$\lfloor -2.1 \rfloor = -3$$
$$\lfloor x \rfloor = 3 \Rightarrow 3 \le x < 4$$

Example 2.57

A. $\lfloor \log_3 x \rfloor = 3$
B. $\lfloor \log_3 x \rfloor = \frac{1}{3}$
C. $\lfloor \log_b x \rfloor = k$
D. $\log_5 \lfloor x \rfloor = 3$
E. $\log_5 x = \lfloor 3 \rfloor$
F. $\log_{\lfloor x \rfloor} 25 = 2$

Part A

Use the property of the floor function:

$$3 \le \log_3 x < 4$$

Exponentiate throughout with respect to the base 3:

$$3^3 \le x < 3^4$$

Simplify:

$$27 \le x < 81$$
$$x \in [27,81)$$

Part B

$$\lfloor \log_3 x \rfloor = \frac{1}{3} \Rightarrow x \in \phi$$

Part C

If $k \notin \mathbb{Z}$:

$$x \in \phi$$

If $k \in \mathbb{Z}$:

$$k \le \log_b x < k+1$$
$$b^k \le x < b^{k+1}$$

Part D

$$\log_5 \lfloor x \rfloor = 3$$
$$\lfloor x \rfloor = 5^3 = 125$$
$$125 \le x < 126$$
$$x \in [125,126)$$

Part E

$$\log_5 x = 3$$
$$x = 5^3 = 125$$

Part F

$$\log_{\lfloor x \rfloor} 25 = 2$$
$$25 = \lfloor x \rfloor^2$$
$$25 \le x^2 < 26$$
$$5 \le x < \sqrt{26}$$

Example 2.58

$$\lfloor \log_2 x \rfloor = 5$$

A. Find the integer solutions, and the number of solutions to the above equation.
B. Find the range of x that satisfies the above equation.

Part A

$$5 \le \log_2 x < 6$$

Exponentiate throughout:

$$32 \le x < 64$$

Convert the inequality into set notation:

$$x \in \{32,33, \dots ,63\}$$
$$63 - 32 + 1 = 32 \textit{ Solutions}$$

Part B

$$32 \le x < 64 \Rightarrow x \in [32,64)$$

Example 2.59

Find the number of:

A. Even solutions to $\lfloor \log_2 x \rfloor = 5$

B. Solutions which are a perfect power of 2 to $\left\lfloor \frac{\log_2 x}{12} + 7 \right\rfloor = 8$

C. Even solutions to $\left\lfloor \frac{\log_2 x}{12} + 7 \right\rfloor = 8$

Part B

$$x \in \{32, 34, \ldots, 62\}$$

Divide by 2:

$$x \in \{16, 17, \ldots, 31\}$$
$$31 - 16 + 1 = 16 \; Solutions$$

Part C

$$8 \le \frac{\log_2 x}{12} + 7 < 9$$
$$1 \le \frac{\log_2 x}{12} < 2$$
$$12 \le \log_2 x < 24$$
$$2^{12} \le x < 2^{24}$$
$$\{2^{12}, 2^{13}, \ldots, 2^{23}\}$$
$$23 - 12 + 1 = 12 \; Solutions$$

Part D

Smallest number that works is:

$$2^{12}$$

Largest number that works is:

$$2^{24} - 2$$

Hence, the solution set is:

$$\{2^{12}, 2^{12} + 2, \ldots, 2^{24} - 2\}$$

Divide by 2:

$$\left\{\frac{2^{12}}{2}, \frac{2^{12} + 2}{2}, \ldots, \frac{2^{24} - 2}{2}\right\}$$

Simplify

$$\{2^{11}, 2^{11} + 1, \ldots, 2^{23} - 1\}$$

The number of numbers is:

$$2^{23} - 1 - (2^{11}) + 1$$
$$2^{23} - 2^{11}$$

Example 2.60

$$\lfloor \log_C x \rfloor = K, \qquad K \in \mathbb{Z}$$

A. Find the integer solutions, and the number of solutions to the above equation.

B. Find the range of x that satisfies the above equation.

Part A

$$K < \log_C x < K + 1$$

Exponentiate:

$$C^K \le x < C^{K+1}$$
$$x \in \{C^K, C^K + 1, \ldots, C^{K+1} - 1\}$$

Part B

$$C^K \le x < C^{K+1} \Rightarrow x \in [C^K, C^{K+1})$$

Example 2.61

Find the integer solutions to:

$$\lfloor \log_2 x \rfloor = \lfloor \log_3 x \rfloor$$

$$\lfloor \log_2 x \rfloor = \lfloor \log_3 x \rfloor = K, \qquad K \in \mathbb{Z}$$

The smallest value that x can take is 1. If

$$x \ge 1 \Rightarrow K \ge 0 \Rightarrow K \in \mathbb{N}$$

Hence, we do not need to consider negative values for K.

Case I: $K = 0$:

$$\lfloor \log_2 x \rfloor = 0 \Rightarrow x \in \{\mathbf{1}\}$$
$$\lfloor \log_3 x \rfloor = 0 \Rightarrow x \in= \{3^0, 3^0 + 1\} = \{\mathbf{1}, 2\}$$

Case II: $K = 1$:

$$\lfloor \log_2 x \rfloor = 1 \Rightarrow x \in \{2^1, 2^1 + 1\} = \{2, \mathbf{3}\}$$
$$\lfloor \log_3 x \rfloor = 1 \Rightarrow x \in \{3^1, 3^1 + 1, \ldots, 3^2 - 1\} = \{\mathbf{3}, 4, \ldots, 8\}$$

Case III: $K = 2$:

$$\lfloor \log_2 x \rfloor = 2 \Rightarrow$$
$$x \in \{2^2, 2^2+1, \dots, 2^3-1\} = \{4,5,\dots,7\}$$
$$\lfloor \log_3 x \rfloor = 2 \Rightarrow x \in \{3^2, 3^2+1, \dots, 3^3-1\} = \{9,10,\dots,26\}$$
$$Intersection\ is\ \phi$$

We did not get any solutions for the case $K = 2$. Note that

$$\frac{3^x}{2^x} = \left(\frac{3}{2}\right)^x > 1, x > 1$$

Hence, we will not get any solutions for $K \geq 2$ because

$$3^x grows\ faster\ than\ 2^x$$

Hence, the integer solutions are:

$$\{1,3\}$$

Example 2.62

Find the integer solutions to:

$$\lfloor \log_3 x \rfloor = \lfloor \log_4 x \rfloor$$

$$\lfloor \log_3 x \rfloor = \lfloor \log_4 x \rfloor = K$$

Case I: $K = 0$:

$$\lfloor \log_3 x \rfloor = 0 \Rightarrow x \in \{3^0, 3^0+1\} = \{\mathbf{1}, \mathbf{2}\}$$
$$\lfloor \log_4 x \rfloor = 0 \Rightarrow x \in \{\mathbf{1}, \mathbf{2}, 3\}$$
$$x = \{1,2\}$$

Case II: $K = 1$:

$$\lfloor \log_3 x \rfloor = 1 \Rightarrow x \in \{3^1, 3^1+1, \dots, 3^2-1\} = \{\mathbf{3}, 4, \dots, 8\}$$
$$\lfloor \log_4 x \rfloor = 1 \Rightarrow x \in \{4^1, 4^1+1, \dots, 4^2-1\} = \{4,5,\dots,15\}$$
$$x = \{4,5,\dots,8\}$$

Case III: $K = 2$:

$$\lfloor \log_3 x \rfloor = 2 \Rightarrow x \in \{3^2, 3^2+1, \dots, 3^3-1\} = \{9,10,\dots,26\}$$
$$\lfloor \log_4 x \rfloor = 2 \Rightarrow x \in \{4^2, 4^2+1, \dots, 4^3-1\} = \{16,5,\dots,63\}$$
$$x = \{16,17,\dots,26\}$$

Case III: $K = 3$:

$$\lfloor \log_3 x \rfloor = 3 \Rightarrow x \in \{27,28,\dots,80\}$$
$$\lfloor \log_4 x \rfloor = 2 \Rightarrow x \in \{64,65,\dots,255\}$$
$$x = \{64,65,\dots,80\}$$

The final answer is the union of the answers from each case above:

$$\{1,2\} \cup \{4,5,\dots,8\} \cup \{16,17,\dots,26\} \cup \{64,65,\dots,80\}$$

Example 2.63

Find the range of x that satisfies each equation below:

$$\lfloor \log_2 x \rfloor = \lfloor \log_3 x \rfloor = c, c \geq 0$$

$$\lfloor \log_2 x \rfloor = \lfloor \log_3 x \rfloor = K, \qquad K \in \mathbb{Z}$$
$$\lfloor \log_C x \rfloor = K, \qquad K \in \mathbb{Z} \Rightarrow C^K \leq x < C^{K+1} \Rightarrow x \in [C^K, C^{K+1})$$

Case I: $K = 0$:

$$\lfloor \log_2 x \rfloor = 0 \Rightarrow x \in [1,2)$$
$$\lfloor \log_3 x \rfloor = 0 \Rightarrow x \in [3^0, 3^1) = [1,3)$$
$$x \in [1,2)$$

Case II: $K = 1$:

$$\lfloor \log_2 x \rfloor = 1 \Rightarrow x \in [2^1, 2^2) = [2,4)$$
$$\lfloor \log_3 x \rfloor = 1 \Rightarrow x \in [3^1, 3^2) = [3,9)$$

$$x \in [3,4)$$

Hence, the integer solutions are:

$$[1,2) \cup [3,4)$$

Example 2.64

Find the range of x that satisfies each equation below:

$$\lfloor \log_3 x \rfloor = \lfloor \log_4 x \rfloor = c, c \geq 0$$

$$\lfloor \log_3 x \rfloor = \lfloor \log_4 x \rfloor = K$$

Case I: $K = 0$:

$$\lfloor \log_3 x \rfloor = 0 \Rightarrow x \in [3^0, 3^1) = [1,3)$$
$$\lfloor \log_4 x \rfloor = 0 \Rightarrow x \in [4^0, 4^1) = [1,4)$$
$$x = [1,3)$$

Case II: $K = 1$:

$$\lfloor \log_3 x \rfloor = 1 \Rightarrow x \in [3^1, 3^2) = [3,9)$$
$$\lfloor \log_4 x \rfloor = 1 \Rightarrow x \in [4^1, 4^2) = [4,16)$$
$$x = [4,9)$$

Case III: $K = 2$:

$$\lfloor \log_3 x \rfloor = 2 \Rightarrow x \in [3^2, 3^3) = [9,27)$$
$$\lfloor \log_4 x \rfloor = 2 \Rightarrow x \in x \in [4^2, 4^3) = [16,64)$$
$$x = [16,27)$$

Case III: $K = 3$:

$$\lfloor \log_3 x \rfloor = 3 \Rightarrow x \in [3^3, 3^4) = [27,81)$$
$$\lfloor \log_4 x \rfloor = 2 \Rightarrow x \in [4^3, 4^4) = [64,256)$$
$$x = [64,81)$$

The final answer is the union of the answers from each case above:

$$[1,3) \cup [4,9) \cup [16,27) \cup [64,81)$$

Example 2.65[4]

$$\left\lfloor \log_p 1947 \right\rfloor = \left\lfloor \log_q 1947 \right\rfloor, \qquad p, q \in \mathbb{N}, \qquad 1 < p, q < 1950$$

Find the number of

A. unordered pairs (p, q)
B. ordered pairs (p, q).

Part A

Since p and q are not required to be distinct:

$$p = q \in \{2,3,4, \ldots, 1950\} \Rightarrow 1949\ Solutions$$

If $p \neq q$, then we analyze further:

$$10 < \log_2 1947 < 11$$
$$6 < \log_3 1947 < 7$$
$$5 < \log_4 1947 < 6$$
$$4 < \log_5 1947 < 5$$
$$4 < \log_6 1947 < 5$$

$$3 < \log_7 1947 < 4$$
$$3 < \log_8 1947 < 4$$
.
.
.
$$3 < \log_{12} 1947 < 4$$

The number of numbers from 7 to 12 is

$$12 - 7 + 1 = 6$$

Number of unordered pairs should be:

$$\binom{6}{2} = \frac{6 \times 5}{2} = 15$$

[4] 15th Aug 1947 and Jan 26, 1950 are important dates in Indian history.

$$2 < \log_{13} 1947 < 3$$

Part C

Example 2.66

Let x be chosen at random from the interval $(0,1)$. What is the probability that $\lfloor \log_{10} 4x \rfloor - \lfloor \log_{10} x \rfloor = 0$? Here $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x. **(AMC 12B 2006/20)**

$$\lfloor \log_{10} 4x \rfloor = \lfloor \log_{10} x \rfloor = k$$

$$\lfloor \log_{10} x \rfloor = k$$
$$k \le \log_{10} x < k+1$$
$$10^k \le x < 10^{k+1}$$

$$\lfloor \log_{10} 4x \rfloor = k$$
$$k \le \log_{10} 4x < k+1$$
$$10^k \le 4x < 10^{k+1}$$
$$\frac{10^k}{4} \le x < \frac{10^{k+1}}{4}$$

$$\frac{10^k}{4} < 10^k \le x < \frac{10^{k+1}}{4} < 10^{k+1}$$
$$10^k \le x < \frac{10^{k+1}}{4}$$

But note that since x lies in the interval $(0,1)$:

$$0 < x < 1$$
$$0 < x < 10^0$$

Suppose $k = -1$:

$$10^{-1} \le x < \frac{10^0}{4}$$
$$\frac{1}{10} \le x < \frac{1}{4}$$
$$Length\ of\ Valid\ Interval = \frac{1}{4} - \frac{1}{10} = \frac{6}{40} = \frac{3}{20}$$

Suppose $k = -2$:

$$10^{-2} \le x < \frac{10^{-1}}{4}$$
$$\frac{1}{100} \le x < \frac{1}{40}$$
$$Length\ of\ Valid\ Interval = \frac{1}{40} - \frac{1}{100} = \frac{6}{400} = \frac{3}{200}$$

$$\underbrace{\frac{3}{20}}_{k=-1} + \underbrace{\frac{3}{200}}_{k=-2} + \cdots$$

This is a geometric series with $a = \frac{3}{20}, r = 1/10$ with sum:

$$S = \frac{a}{1-r} = \frac{\frac{3}{20}}{1-\frac{1}{10}} = \frac{3}{20} \cdot \frac{10}{9} = \frac{1}{6}$$

$$Probability = \frac{Length\ of\ Valid\ Interval}{Total\ Interval} = \frac{\frac{1}{6}}{1} = \frac{1}{6}$$

2.67: Conversions

$$\log_e a = 2.303 \cdot \log_{10} a$$

$$\log_{10} a = 0.434 \cdot \log_e a$$

2.3 Inequalities and Modelling

A. Inequalities

Example 2.68: Inequality Proof

If n is a natural number such that $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$, and $p_1, p_2, \ldots p_k$ are distinct primes, then show that $\ln n \ge k \ln 2$. When will equality hold? **(IIT JEE 1984)**

Is α a natural number

Suppose we consider

$$2 = 2^1 \cdot 3^0 \Rightarrow \ln 2 = \ln 2$$

This gives us:

$$p_1 = 2, p_2 = 3, \alpha_1 = 1, \alpha_2 = 0, k = 2$$
$$\ln 2 < k \ln 2 = 2 \ln 2$$

Hence, even though the question does not apply the restriction, we consider $p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}$ to be the factorization of the natural number n, which then enforces $a_n \geq 1$.

Strategy

We have been given an equality, and we need to prove an inequality. We will make the conversion, but first get the equality into a form similar to what we want.

Note that the given equality has n, while the required inequality has $\ln n$. Take the natural log of both sides of the given equality:

$$\ln n = \ln\left(p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k}\right)$$

Use the product rule:

$$= \ln\left(p_1^{\alpha_1}\right) + \ln\left(p_2^{\alpha_2}\right) + \cdots + \ln\left(p_k^{\alpha_k}\right)$$

Use the power rule

$$= \alpha_1 \ln p_1 + \alpha_2 \ln p_2 + \cdots + \alpha_k \ln p_k$$

Since $p_1, p_2, \ldots, p_k$ are primes, $p_1, p_2, \ldots, p_k \geq 2$,

$$\ln n \geq \alpha_1 \ln 2 + \alpha_2 \ln 2 + \cdots + \alpha_k \ln 2$$
$$\ln n \geq \ln 2\,(\alpha_1 + \alpha_2 + \cdots + \alpha_k)$$

Since n is a natural number, $\alpha_n > 1$:

$$\ln n \geq \ln 2\, \underbrace{(1 + 1 + \cdots + 1)}_{k\ times}$$

$$\ln n \geq k \ln 2$$

B. Maximum and Minimum

2.69: AM-GM Inequality

$$\underbrace{\frac{a+b}{2}}_{\substack{Arithmetic\\ Mean}} \geq \underbrace{\sqrt{ab}}_{\substack{Geometric\\ Mean}}$$

Consider the product

$$ab, a > 0, b > 0$$

For a given value of ab:

$$a + b\ will\ be\ minimum\ when\ a = b$$

Suppose $ab = 4$, then:

$$a = 2, b = 2 \Rightarrow a + b = 2 + 2 = 4$$
$$a = 1, b = 4 \Rightarrow a + b = 1 + 4 = 5$$

For a given value of $a + b$:

$$ab\ will\ be\ maximum\ when\ a = b$$

Suppose $a + b = 4$, then:

$$a = 2, b = 2 \Rightarrow ab = 4$$
$$a = 1, b = 3 \Rightarrow ab = 3$$

Example 2.70

If it is known that $\log_2 a + \log_2 b \geq 6$, then the least value that can be taken on by $a + b$ is: (AHSME 1969/25)

Use the product rule:

$$\log_2 ab \geq 6$$

Convert to exponential form:

$$ab \geq 2^6$$

If you substitute $a = b = 2^3$, then you get the minimum value as

$$ab \geq 64$$

Take the square root both sides:

$$\sqrt{ab} \geq 8$$

If we combine the AM-GM inequality $\left(\frac{a+b}{2} \geq \sqrt{ab}\right)$, with the above inequality, we get:

$$\frac{a+b}{2} \geq \sqrt{ab} \geq 8$$

Multiply throughout by 2:

$$a + b \geq 2\sqrt{ab} \geq 16$$

And equality is achieved when

$$a = b = 8$$

C. Trigonometric Equations

Example 2.71

D. Exponential and Logarithmic Modelling

2.72: Models

Exponential Growth:

$$y = ab^x, \qquad b = 1 + r$$

Exponential Decay:

$$y = ab^x, \qquad b = 1 - r$$

Example 2.73

$$50 = 100e^{-5730k}$$

$$\frac{1}{2} = e^{-5730k}$$

$$\ln\left(\frac{1}{2}\right) = -5730k$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{-5730} = \frac{-\ln\left(\frac{1}{2}\right)}{5730} = \frac{\ln\left(\frac{1}{2}\right)^{-1}}{5730} = \frac{\ln 2}{5730}$$

Example 2.74

$$V = ab^t, b = 1 + r$$

$$b = 1 + r = 1 + 50\% = 1 + 0.5 = 1.5$$

Substitute $b = 1.5$ in $V = ab^t$

$$V = a(1.5)^t$$

At the start point, the price of the stock is \$0.59
Substitute $(t, V) = (0,0.59)$:

$$0.59 = a(1.5)^0 \Rightarrow 0.59 = a$$

Hence, the equation is:

$$V = 0.59(1.5)^t$$

If the value is 6 dollars, then $V = 6$:

$$6 = 0.59(1.5)^t$$
$$\frac{6}{0.59} = (1.5)^t$$

Take log to the base 10 on both sides:

$$\log \frac{6}{0.59} = \log(1.5)^t$$

Example 2.75

Current value of lamp is 1000 dollars. Lamp appreciates 15% each year.

$$V = ab^t, b = 1 + r$$

Current price is

$$V = 1000\ Dollars$$

Let the current time be

$$t = 0$$

Substitute $V = 1000, t = 0$:

$$1000 = ab^0 \Rightarrow 1000 = a$$

Example 2.76

$$N(t) = N_0 e^{-0.1155t}$$

$$N(t) = N_0 (e^{-0.1155})^t$$
$$b = e^{-0.1155}$$
$$1 - r = e^{-0.1155}$$
$$1 - e^{-0.1155} = r$$

Example 2.77

Richter Scale

$$M = \frac{2}{3} \log\left(\frac{E}{10^{11.8}}\right)$$

$$9.2 = \frac{2}{3} \log\left(\frac{E}{10^{11.8}}\right)$$
$$9.2 \times \frac{3}{2} = \log\left(\frac{E}{10^{11.8}}\right)$$
$$10^{13.8} = \frac{E}{10^{11.8}}$$
$$10^{25.6} = E$$
$$3.98 \times 10^{25} = E$$

2.78: Newton's Law of Cooling

$$T_t - T_S = (T_0 - T_s)b^t$$

Where

$$T_t = Temperature\ at\ Time\ t$$
$$T_s = Surrounding\ Temperature$$
$$T_0 = Initial\ Temperature$$
$$t = Elapsed\ Time$$
$$b\ is\ a\ constant$$

Example 2.79

A potato is taken from the oven, its temperature having reached 350°. After sitting on a plate in a 70° room for twelve minutes, its temperature has dropped to 250°. In how many more minutes will the potato's temperature reach 120°? Assume Newton's Law of Cooling, which says that the difference between an object's temperature and the ambient temperature is an exponential function of time. **(Phillips Exeter Math 4, 2022/5)**

$$\underbrace{T_t}_{\substack{Measured\\Temperature}} - \underbrace{T_S}_{\substack{Room\\Temperature}} = \left[\underbrace{T_0}_{\substack{Initial\\Temperature}} - T_s\right]b^t$$

$$250 - 70 = (350 - 70)b^{12}$$
$$180 = 280b^{12}$$
$$b = \left(\frac{180}{280}\right)^{\frac{1}{12}} = \left(\frac{18}{28}\right)^{\frac{1}{12}} = \left(\frac{9}{14}\right)^{\frac{1}{12}}$$

Substitute $T_0 = 350, T_s = 70, T_t = 120, b = \left(\frac{9}{14}\right)^{\frac{1}{12}}$:

$$120 - 70 = (350 - 70)\left[\left(\frac{9}{14}\right)^{\frac{1}{12}}\right]^t$$

$$50 = 280\left(\frac{9}{14}\right)^{\frac{t}{12}}$$
$$\frac{5}{28} = \left(\frac{9}{14}\right)^{\frac{t}{12}}$$
$$\ln\frac{5}{28} = \ln\left(\frac{9}{14}\right)^{\frac{t}{12}}$$
$$\frac{t}{12} = \frac{\ln\frac{5}{28}}{\ln\frac{9}{14}}$$
$$t = 12\frac{\ln\frac{5}{28}}{\ln\frac{9}{14}} \approx 46.789$$
$$More\ Minutes = 46.789 - 12 = 34.789$$

2.4 Number Theory

A. Diophantine Equations

Example 2.80: Diophantine Equations

$\log_2 x$ and $\log_4 y$ are natural numbers. The pairs (x, y) that are a solution to $\log_2 x + \log_4 y = 8$ can be written

$$(x_1, y_1)(x_2, y_2) \dots (x_n, y_n)$$

Find a given that the product:

$$x_1 x_2 \dots x_n y_1 y_2 \dots y_n = 2^a$$

Type equation here.

Example 2.81: Zero and One as Solutions

Find the number of distinct ordered pair solutions (x, y) for:

$$\log_x y = 0, \qquad \log_x y = 1, \qquad \log_y x = -1, x \in \left\{0,1,2,\frac{1}{3}\right\}, y \in \left\{0,1,\frac{1}{2},3\right\}$$

$$\log_x y = 0 \Rightarrow (x, y) = (2,1), \left(\frac{1}{3}, 1\right)$$
$$\log_x y = 1 \Rightarrow (x, y) = No\ Solutions$$
$$\log_x y = -1 \Rightarrow (x, y) = \left(\frac{1}{3}, 3\right)\left(\frac{1}{2}, 2\right)$$

Example 2.82: Integer Solutions

Find integer values of x and y for which
$$x - y\log_3 2 = 10\log_9 6$$

Method I

$$x = \log_3 6^5 + \log_3 2^y = \log_3 3^5 \cdot 2^5 \cdot 2^y$$
$$2^5 \cdot 2^y = 2^0 \Rightarrow 5 + y = 0 \Rightarrow y = -5$$
$$x = 5$$

Method II

$$RHS = 10\log_9 6 = \log_3 3^5 \cdot 2^5 = \log_3 3^5 + \log_3 2^5 = 5 + 5\log_3 2$$
$$x - y\log_3 2 = 5 + 5\log_3 2$$
$$x = 5$$
$$-y = 5 \Rightarrow y = -5$$

Example 2.83

(AMC 12A 2005/21)

How many ordered triples of integers (a, b, c), with $a \geq 2$, $b \geq 1$, and $c \geq 0$, satisfy both $\log_a b = c^{2005}$ and $a + b + c = 2005$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

B. Classification of Numbers

Example 2.84

A. Determine, with proof, whether $\log_2 7$ is rational or irrational. (IIT JEE 1990)
B. Is it possible to generalize your result?

Part A

Assume, if possible, to the contrary that:
$$\log_2 7 \in \mathbb{Q}$$
Then for some integers p and q:
$$\log_2 7 = \frac{p}{q}$$
Exponentiate both sides:
$$7 = 2^{\frac{p}{q}}$$
Raise both sides to the power q:
$$2^p = 7^q$$
However, no power of 2 is ever equal to a power of 7.
Hence:
$$\log_2 7 \notin \mathbb{Q}$$
$$\log_2 7 \ \ is\ an\ irrational\ number$$

Part B

$\log_a b$ for non-zero values of the expression will be rational only if the prime factorization of b and the prime factorization of a both have exactly the same prime numbers.

$$\log_{18} 12 = \log_{2\times 3^2} 2^2 \times 3$$

$$\log_a b$$
$$b = (p_1 \times p_2)^m$$
$$a = (p_1 \times p_2)^n$$

$$\log_{2\times 3^2} 2^2 \times 3^4 = \log_{2\times 3^2}(2\times 3^2)^2 = 2\log_{2\times 3^2}(2\times 3^2) = 2$$

Example 2.85

How many positive integers $b < 1000$ have the property that $\log_b 729$ is a rational number?

$$\log_b 3^6 = 6\log_b 3 = \frac{6}{\log_3 b}$$

$\log_3 b$ should be an integer.

b cannot be an integer which is not a power of 3 since the following do not have rational solutions for x:

$$\log_3 5 = x \Rightarrow 5 = 3^x$$
$$\log_3 b = \frac{1}{2} \Rightarrow b = 3^{\frac{1}{2}} = \sqrt{3}$$

b must be of the form $3^n, n \in \mathbb{N}$

$$n = 3^1, 3^2, \ldots, 3^6$$

C. Number of Factors & Sum of Factors

Example 2.86

Find the sum of the positive integers b that have the property that $\log_b 729$ is a positive integer? **(AMC 12 2007/7, Adapted)**

We rewrite the given expression as:

$$\log_b 3^6 = \frac{6\log 3}{\log b} = \frac{6}{\log_3 b}$$

Hence, $\log_3 b$ must be factors of 6.

$$\log_3 b = x \Rightarrow b = 3^x$$

$$x \in \{1,2,3,6\} \Rightarrow b \in \{3^1, 3^2, 3^3, 3^6\} = \{3,9,27,729\}$$

2.87: Number of Divisors

The number of divisors $\boldsymbol{\tau(x)}$ of a number $\boldsymbol{x}$ with prime factorisation $\boldsymbol{a^p b^q c^r}$ is given by:

$$x = a^p b^q c^r \Rightarrow \tau(x) = (p+1)(q+1)(r+1)$$

(This same result can be generalized to a number with any number of prime factors).

Any nonnegative integer power of a, b, c can be multiplied in any combination to give a factor of x (with the restriction that the number used in the factor must be less than or equal to its value in the number).

	a^p	b^q	c^r	
	a^0	b^0	c^0	
	a^1	b^1	c^1	
	.	.	.	
	.	.	.	
	.	.	.	
	a^p	b^q	c^r	
No. of Choices	$p+1$	$q+1$	$r+1$	$\underbrace{(p+1)(q+1)(r+1)}_{\textit{Total Number of Choices}}$

Example 2.88

Find the number of positive integers b that have the property that $\log_b 25^{36}$ is a positive integer? (AMC 12 2007, Adapted)

$$\log_b 25^{36} = \log_b 5^{72} = 72\log_b 5 = \frac{72}{\log_5 b}$$

$\log_5 b$ should be a factor of 72. The number of factors of 72 is given by:

$$\tau(72) = \tau(8 \times 9) = \tau(2^3 \times 3^2) = (3+1)(2+1) = 4(3) = 12$$

2.89: Sum of Divisors

$$s(n) = (1 + p + p^2 + \cdots + p^a)(1 + q + q^2 + \cdots + q^b)(1 + r + r^2 + \cdots + r^c)$$
$$s(n) = \frac{p^{a+1}-1}{p-1} \times \frac{q^{b+1}-1}{q-1} \times \frac{r^{c+1}-1}{r-1}$$

We already showed that

$$s(n) = \underbrace{(1 + p + p^2 + \cdots + p^a)}_{\substack{\textit{First term}=1 \\ \textit{Common Ratio}=p}}(1 + q + q^2 + \cdots + q^b)(1 + r + r^2 + \cdots + r^c)$$

Each term above is a geometric series. For example, $1 + p + p^2 + \cdots + p^a$ is a geometric series with $first\ term = 1, common\ ratio = p, number\ of\ terms = a + 1$

The geometric series has sum

$$= \frac{(1)(p^{a+1}-1)}{p-1}$$

letting us rewrite the above as:

$$s(n) = \frac{p^{a+1}-1}{p-1} \times \frac{q^{b+1}-1}{q-1} \times \frac{r^{c+1}-1}{r-1}$$

Example 2.90

Find the product of all positive integers b that have the property that $\log_b 25^{36}$ is a positive integer? (AMC 12 2007, Adapted)

$$\log_b 25^{36} = \log_b 5^{72} = 72\log_b 5 = \frac{72}{\log_5 b}$$

$$\tau(72) = \tau(8 \times 9) = \tau(2^3 \times 3^2) = (3+1)(2+1) = 4(3) = 12$$

$\log_5 b$ should be a factor of 72. The values of b that will work are:

$$5^{n_1}, 5^{n_2}, \dots, 5^{n_{12}}$$

The product of the numbers will be:

$$5^{n_1+n_2+\dots+n_{12}}$$

Hence, to find the exponent, we need to find:

$$n_1 + n_2 + \dots + n_{12} = Sum\ of\ Factors\ of\ 72$$

Hence, we want the sum of the factors of 72:

$$s(72) = s(2^3 \cdot 3^2) = (1+2+4+8)(1+3+9) = 15(13) = 195$$

$$s(72) = s(2^3 \cdot 3^2) = \frac{2^4-1}{2-1} \cdot \frac{3^3-1}{3-1} = \frac{16-1}{1} \cdot \frac{27-1}{2} = 15 \cdot 13 = 195$$

Hence, the final answer is:

$$5^{195}$$

D. Number of Digits

Example 2.91 (Calculator)

Number of digits in 15^{6324}
Find first five digits of the number

Part A

$$y = 15^{6324}$$

Take logs to the base-10 both sides:

$$\log y = 6324 \log 15 \approx 7437.601122$$
$$y = 10^{7437.601122} \Rightarrow 7438\ Digits$$

Part B

$$y = 10^{7437.601122} = 10^{0.601122} \times 10^{7437}$$

$$10^{0.601122} = 3.9913 \dots\ (not\ rounded)$$
$$First\ five\ digits = 39913$$

Example 2.92 (Calculator)

Number of digits in 12^{5764}
Find first five digits of the number

Part A

$$y = 12^{5764}$$

Take logs to the base-10 both sides:

$$\log y = 5764 \log 12 \approx 6220.40070$$
$$y = 10^{6220.40070} \Rightarrow 6221\ Digits$$

Part B

$$y = 10^{6220.40070} = 10^{0.40070} \times 10^{6220}$$

$$10^{0.40070} = 2.5159 \ldots \ (not\ rounded)$$
$$First\ five\ digits = 25159$$

E. $y = \log x$ as a Function

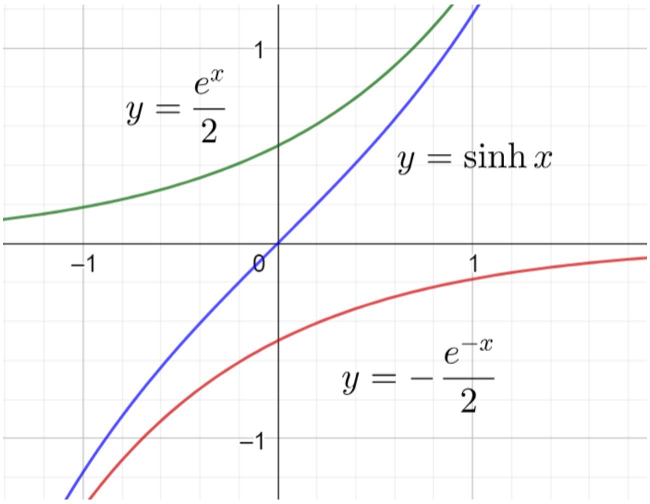
2.5 Hyperbolic Functions

A. Hyperbolic Sine

The hyperbolic sine is called such because of its similarity to the trigonometric sine function.
It is defined in terms of exponentials, so its similarity is not immediately apparent, but we will see it as go along.

2.93: sinh x

$$Hyperbolic\ sine = \sinh x = \frac{e^x - e^{-x}}{2}$$



There are no restrictions on the inputs for the function.
Hence,

$$Domain\ of \sinh x = \mathbb{R}$$

Also,

$$Range\ of \sinh x = \mathbb{R}$$

$\sinh x$ is a continuous function.

Example 2.94

Evaluate:

A. $\sinh 0$
B. $\sinh 1$
C. $\sinh(\ln 3)$

Example 2.95

Show that the $Domain\ of \sinh x = \mathbb{R}$

There are no restrictions on the inputs to the function because of denominators, roots, etc.
Hence:

$$Domain\ of \sinh x = \mathbb{R}$$

Example 2.96

Show that the range of $\sinh x$ is $(-\infty, \infty)$

Calculate the limit as it goes to positive infinity:

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{2} = \lim_{x\to\infty} \frac{e^x}{2} - \lim_{x\to\infty} -\frac{1}{2e^x} = \infty - 0 = \infty$$

Similarly, calculate the limit as it goes to negative infinity:

$$\lim_{x\to-\infty}\frac{e^x-e^{-x}}{2}=\lim_{x\to-\infty}\frac{e^x}{2}-\lim_{x\to-\infty}-\frac{1}{2e^x}=0-\infty=-\infty$$

Also, note that it is a continuous function with

$$Domain=\mathbb{R}$$

And hence, it will achieve all values between the above two limits. Hence,

$$Range\ of\ \sinh x=\mathbb{R}$$

2.97: sinh x is an odd function

- $\sinh(-x)=-\sinh x$
- $\sinh x\ \ is\ symmetric\ about\ the\ origin$

Recall that a function is odd if and only if:

$$f(-x)=-f(x)$$

And also that

$$Odd\ functions\ are\ symmetric\ about\ the\ origin$$

We can now prove that $\sinh x$ is odd:

$$\sinh(-x)=\frac{e^{-x}-e^{-(-x)}}{2}=\frac{e^{-x}-e^{x}}{2}=-\frac{e^x-e^{-x}}{2}=-\sinh x$$

And the symmetry directly follows from the fact that it is odd.

B. Hyperbolic Cosine

Just like define the hyperbolic sine, we define the hyperbolic cosine in terms of exponentials.

2.98: cosh x

$$Hyperbolic\ cosine=\cosh x=\frac{e^x+e^{-x}}{2}$$

Note the similarity between the definition of the hyperbolic sine and the hyperbolic cosine: the only difference between sinh x and cosh x is that the minus sign is replaced with a plus sign.

Like the hyperbolic sine, the hyperbolic cosine also has:

$$Domain\ of\ \cosh x=\mathbb{R}$$

However, the hyperbolic cosine is the sum of the two functions, making its minimum 1. Hence, it has

$$Range\ of\ \cosh x=[1,\infty)$$

2.99: cosh x is an even function

- $\cosh(-x)=\cosh x$
- $\cosh x\ \ is\ symmetric\ about\ the\ y-axis$

$$\cosh(-x)=\frac{e^{-x}+e^{-(-x)}}{2}=\frac{e^{-x}+e^{x}}{2}=\frac{e^x+e^{-x}}{2}=\cosh x$$

Example 2.100

Show that the $Domain\ of\ \cosh x = \mathbb{R}$

There are no restrictions on the inputs to the function because of denominators, roots, etc.
Hence:

$$Domain\ of\ \cosh x = \mathbb{R}$$

Example 2.101[5]

Show that the range of $\cosh x$ is $[1, \infty)$

$$y = \cosh x = \frac{e^x + e^{-x}}{2} = \underbrace{\frac{e^x}{2}}_{Positive} + \underbrace{\frac{1}{2e^x}}_{Positive}$$

Since both the terms of the above are positive, we know that

$$y > 0$$

However, we do not know the minimum value.
Recall that we can find the range of y by solving for x, and then finding the domain of y.
To find the minimum value, find the inverse function of

$$y = \cosh x$$

And then find the domain of y.

Multiply $y = \frac{e^x+e^{-x}}{2}$ by 2 both sides:

$$2y = e^x + \frac{1}{e^x}$$

Multiply by e^x to eliminate fractions:

$$2ye^x = e^{2x} + 1$$

Collate all terms on one side:

$$e^{2x} - 2ye^x + 1 = 0$$

Substitute $z = e^x \Rightarrow z^2 = e^{2x}$:

$$z^2 - 2yz + 1 = 0$$

This is a quadratic in z, with $a = 1, b = -2y, c = 1$:

$$z = e^x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2y \pm \sqrt{4y^2 - 4(1)(1)}}{2} = \frac{2y \pm 2\sqrt{y^2 - 1}}{2} = y \pm 2\sqrt{y^2 - 1}$$

Since y is real, we must have:

$$y^2 - 1 \geq 0 \Rightarrow y^2 \geq 1 \Rightarrow y \geq 1\ OR\ y \leq -1$$

But we already know that $y > 0$. Hence:

$$y \geq 1$$

Example 2.102

$$\cosh(\ln a)$$

[5] Copy to Disguised Quadratics, Range of Exponential Functions

$$\cosh(\ln a) = \frac{e^{\ln a} + e^{-\ln a}}{2} = \frac{e^{\ln a} + e^{\ln\frac{1}{a}}}{2} = \frac{a + \frac{1}{a}}{2} = \frac{a^2+1}{2a}$$

C. Decomposing e^x to get the hyperbolic functions (Optional)

We now look at one way to think about how the hyperbolic functions are obtained. We already know that

- sinh x is odd
- cosh x is even

We first see that these two functions add up to e^x. That is:

$$\sinh x + \cosh h = e^x$$

But, if we had e^x, and we wanted to get $\sinh x$ and $\cosh h$, how would we do that?
That is a based on a property that we prove (and then use).

Example 2.103

Show that $\sinh x + \cosh h = e^x$

$$\sinh x + \cosh h = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} = \frac{2e^x}{2} = e^x$$

Example 2.104

Any function $f(x)$ can be decomposed into an even function and an odd function by using the following definitions:

$$Even\ Function = f_e = \frac{f(x) + f(-x)}{2}$$
$$Odd\ Function = f_o = \frac{f(x) - f(-x)}{2}$$

Show that

A. $f(x) = f_e + f_o$
B. f_e is even
C. f_o is odd

Example 2.105

Use the property stated in the previous example to decompose $f(x) = e^x$ into an even function and odd function. Name the functions so obtained.

$$f_e = \frac{f(x) + f(-x)}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$
$$f_o = \frac{f(x) - f(-x)}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$$

D. Hyperbolic tangent

We can now start exploiting the similarity between the trigonometric functions and the hyperbolic functions.
tanh x is defined just as tan x would be, making it easier to remember.

2.106: tanh x

$$\tanh x = \frac{\sinh x}{\cosh x}$$

Hyperbolic tan has

$$Domain\ of \tanh x = \mathbb{R}$$

And it has

$$Range\ of \tanh x = (1,1)$$

Example 2.107

Show that the $Domain\ of \tanh x = \mathbb{R}$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

Both functions have

$$Domain = \mathbb{R}$$

The numerator is not an issue.

$$Denominator = \cosh x\ \ has\ range\ [1, \infty)$$

And hence is never zero.

Hence,

$$Domain\ of \tanh x = \mathbb{R}$$

Example 2.108

Show that range of tanh x is $(-1,1)$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Property I: tanh x passes through the origin

$$\tanh 0 = \frac{e^0 - e^{-0}}{e^0 + e^{-0}} = \frac{0}{2} = 0$$

Property II: tanh x is always less than 1:

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} < \frac{e^x}{e^x + e^{-x}} < 1$$

Property III: As x increases without bound, the limit of tanh x is 1.

$$\lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \to \infty} \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = 1$$

If we combine the above three properties, then by the Intermediate Value Theorem

$$Range\ of \tanh x\ \ over\ x \in [0, \infty)\ is\ y \in [0,1)$$

Also, tanh x is an odd function, and hence:

$$Range\ of\ \tanh x\ \ over\ x \in (-\infty, 0]\ is\ y \in [0, -1)$$

If we had not used the fact that tanh x is odd, we would have had to repeat the entire process for $x \in (-\infty, 0]$

Consider $x < 0$, and substitute $y = -x$:

$$\lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{y\to\infty} \frac{e^{-y} - e^y}{e^{-y} + e^y} = \lim_{y\to\infty} -\frac{e^y - e^{-y}}{e^{-y} + e^y} = -1$$

E. Reciprocal Hyperbolic Functions

Like the trigonometric functions, the hyperbolic functions also have reciprocal counterparts. We define each of these below.

2.109: Reciprocal Functions: csch x

$$\operatorname{csch} x = \frac{1}{\sinh x}$$

2.110: Reciprocal Functions: sech h

$$\operatorname{sech}\ x = \frac{1}{\cosh x}$$

2.111: Reciprocal Functions: coth x

$$\coth x = \frac{\cosh x}{\sinh x}$$

F. Identities

2.112: "Pythagorean Identity"

The Pythagorean identity holds in a modified form, so be careful.

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

This is possible to square, but would be messy, increasing the chances of making a mistake.
Instead exploit the symmetry by using a change of variables.

Use a change of variables. Let $a = e^x, b = e^{-x}$. Then:

$$LHS = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = \frac{a^2 + 2ab + b^2}{4} - \frac{a^2 - 2ab + b^2}{4} = \frac{4ab}{4} = ab = e^x e^{-x} = 1$$

2.113: Alternate Version "Pythagorean Identity"

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

We have just proved:

$$\cosh^2 x - \sinh^2 x = 1$$

Dividing the above by $\cosh^2 x$ gives us:

$$\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$$1-\tanh^2 x=\operatorname{sech}^2 x$$

G. Double Identity

2.114: Double Angle Identity

$$\sinh^2 x=\frac{\cosh 2x-1}{2}$$

$$RHS=\frac{\left(\frac{e^{2x}+e^{-2x}}{2}\right)-1}{2}=\frac{\left(\frac{a^2+b^2}{2}\right)-1}{2}=\frac{\frac{a^2+b^2-2}{2}}{2}$$

Substitute $ab=e^xe^{-x}=1$

$$=\frac{a^2-2ab+b^2}{4}=\left(\frac{a-b}{2}\right)^2=\left(\frac{e^x-e^{-x}}{2}\right)^2=LHS$$

2.115: Double Angle Identity

$$\cosh^2 x=\frac{\cosh 2x+1}{2}$$

$$RHS=\frac{\left(\frac{e^{2x}+e^{-2x}}{2}\right)+1}{2}=\frac{\left(\frac{a^2+b^2}{2}\right)+1}{2}=\frac{\frac{a^2+b^2+2}{2}}{2}$$

Substitute $ab=e^xe^{-x}=1$

$$=\frac{a^2+2ab+b^2}{4}=\left(\frac{a+b}{2}\right)^2=\left(\frac{e^x+e^{-x}}{2}\right)^2=LHS$$

H. Sum to Product Rules

2.116: Sum to Product for sinh x

$$\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y$$

2.117: Sum to Product for cosh x

$$\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y$$

$$RHS=\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^y+e^{-y}}{2}\right)+\left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^y-e^{-y}}{2}\right)$$

Use a change of variable. Let $a=e^x, b=e^{-x}, c=e^y, d=e^{-y}$

$$=\left(\frac{a+b}{2}\right)\left(\frac{c+d}{2}\right)+\left(\frac{a-b}{2}\right)\left(\frac{c-d}{2}\right)$$

$$=\frac{ac+ad+bc+bd}{4}+\frac{ac-ad-bc+bd}{4}$$

$$=\frac{2ac+2bd}{4}=\frac{ac+bd}{2}=\frac{ac+bd}{2}$$

$$=\frac{e^{x+y}+e^{-(x+y)}}{2}=LHS$$

2.6 Further Topics

118 Examples