Oscillations

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1. Simple Harmonic Motion

1.1 Simple Harmonic Motion

A. Harmonic Motion

1.1 Harmonic Motion
If an object attached to a spring in the equilibrium position is pulled (or pushed), and then released, it will move up and down. This motion is harmonic motion.

An object moving on a spring is *not* the definition of harmonic motion. We will define harmonic motion in terms of Hooke's Law (which you should have seen when you encountered springs).

Further examples of harmonic motion include:
- Motion of a swing
- Oscillation of a pendulum (for small angles)
- Waves: Such as vibrations on a string, or sound waves

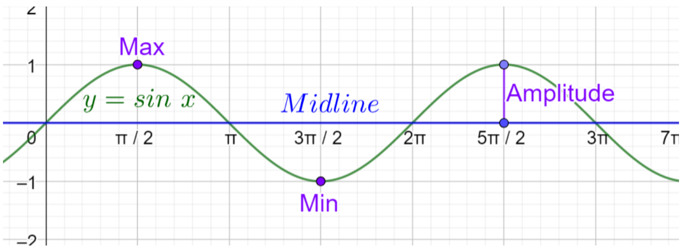
The math and the behavior of these objects is complicated. Hence, we focus on simple harmonic motion first.

B. Simple Harmonic Motion

1.2 Simple Harmonic Motion
Simple harmonic motion is obtained under *ideal conditions* that do not exist, but are useful to understand the mathematical model:
- The spring is an ideal, massless spring.
- There is no friction.
- There is no loss of energy. The spring converts energy from potential to kinetic and vice versa with 100% efficiency.

1.3: Position Equation (SHM)
The position of a wave with $amplitude\ A$, $angular\ frequency\ \omega$, $phase\ shift\ \phi$ and $vertical\ shift\ C$ is a function of time t given by:

$$x = f(t) = A\sin[\omega t + \phi] + C$$



$$Max = Midline + Amplitude = C + A$$
$$Min = Midline - Amplitude = C - A$$

$$Midline = \frac{Max + Min}{2}$$
$$Amplitude = \frac{Max - Min}{2}$$

Example 1.4
In the graph of the position equation for SHM (given above), which variables are on the horizontal and vertical axes?

$$Vertical : Displacement = t$$
$$Horizontal\ axis: time = x$$

Example 1.5
The distance covered by a particle undergoing SHM in one time period is (amplitude=A): **(NEET 2019)**

Assume $y = \sin x$

$$Up: A$$
$$Down: 2A$$
$$Up: A$$

$$Total = A + 2A + A = 4A$$

1.6: *sin* vs *cos* for SHM
The SHM equation can be equally well represented by a *cos* function instead of a sin function since cos is just a phase shift of the sin function:

$$x = A\cos[\omega t + \phi] + C$$

$$x = A\sin[\omega t + \phi] + C$$

Since $y = A\cos\left(x - \frac{\pi}{2}\right) = A\sin x$:

$$x = A\cos\left[\omega t + \phi - \frac{\pi}{2}\right] + C$$

Substitute $\phi - \frac{\pi}{2} = \varphi$:

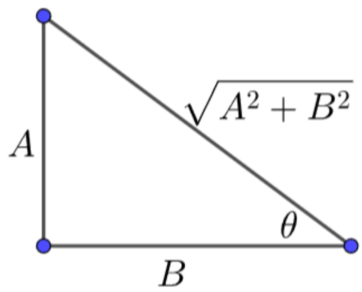
$$x = A\cos[\omega t + \varphi] + C$$

C. Amplitude

1.7: Amplitude
The wave $A\cos\omega t + B\sin\omega t + C$ has

$$Max = \sqrt{A^2 + B^2} + C, \qquad Min = -\sqrt{A^2 + B^2} + C$$

Consider a reference triangle with $Sides = A, B \Rightarrow Hypotenuse = \sqrt{A^2 + B^2}$:



$$\sin\theta = \frac{A}{\sqrt{A^2 + B^2}} \Rightarrow A = \sqrt{A^2 + B^2}\sin\theta$$
$$\cos\theta = \frac{B}{\sqrt{A^2 + B^2}} \Rightarrow B = \sqrt{A^2 + B^2}\cos\theta$$

Hence $A\cos\omega t + B\sin\omega t + C$:

$$= \sqrt{A^2 + B^2}\sin\theta\cos\omega t + \sqrt{A^2 + B^2}\cos\theta\sin\omega t + C$$

Factor $\sqrt{A^2 + B^2}$ and use the sum and difference identity for *sine*:

$$= \sqrt{A^2 + B^2}(\sin(\theta + \omega t)) + C$$

We have reduced the two trigonometric functions to a single trigonometric function.

And the range for $\sin x$ is well known.

$$-1 < \sin(\theta + \omega t) < 1$$
$$-\sqrt{A^2+B^2} < \sqrt{A^2+B^2}\sin(\theta+\omega t) < \sqrt{A^2+B^2}$$
$$-\sqrt{A^2+B^2} + C < \sqrt{A^2+B^2}\sin(\theta+\omega t) + C < \sqrt{A^2+B^2} + C$$

And the result follows.

1.8: Sum of Sinusoidal Functions

For all x and constants a, b, C it is true that:

$$a\sin x + b\cos x = A\sin(x + C)$$

Hence,

$$a\sin x + b\cos x \Rightarrow \textit{Represents Simple Harmonic Motion}$$

Example 1.9

The displacement of a particle executing simple harmonic motion is given by $y = y_0 + A\sin\omega t + B\cos\omega t$. Then the amplitude of its oscillation is given by: **(NEET 2019)**

$$\sqrt{A^2+B^2}$$

D. Definition

1.10: Definition of SHM

SHM is motion that obeys Hooke's Law:

$$F = -kx$$

The restoring force is proportional to the displacement, and in the opposite direction of the displacement.

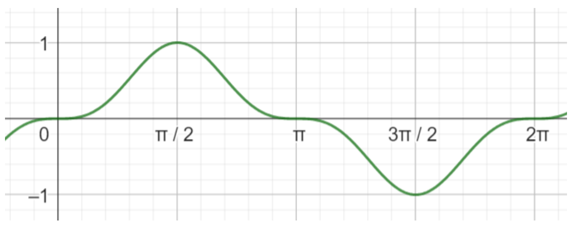
- All harmonic motion is periodic, but all periodic motion is not harmonic.
- Certain quantities are fixed (do not vary): amplitude, frequency, time period

Example 1.11

Mark all correct options

Out of the following functions representing motion of a particle which ones represent SHM?

A. $y = \sin\omega t - \cos\omega t$
B. $y = \sin^3\omega t$
C. $y = 5\cos\left(\frac{3\pi}{4} - 3\omega t\right)$
D. $y = 1 + \omega + \omega^2 t^2$ **(NEET 2011)**



$$x = A\cos[\omega t + \phi] + C$$

Substitute $C = 0, A = 5, \phi = \frac{3\pi}{4}, \omega = -3\omega$

$$A, C\ are\ SHM$$

E. Period, Frequency and Time

1.12: Period

The period of an object that follows simple harmonic motion with $x = f(t) = A\sin[\omega t + \phi] + C$ is

$$Time\ Period = T = \frac{2\pi}{\omega}$$

1.13: Frequency

The frequency with which an object in simple harmonic motion completes one cycle is:

$$f = \frac{1}{T} = \frac{1}{\frac{2\pi}{\omega}} = \frac{\omega}{2\pi}$$

- Frequency is the reciprocal of the time period.

Example 1.14

Mark the correct option

The displacement of a particle along the x axis is given by $x = a\sin^2 \omega t$. The motion of the particle corresponds to: **(NEET 2010)**

A. Simple harmonic motion of frequency $\frac{\omega}{\pi}$
B. Simple harmonic motion of frequency $\frac{3\omega}{2\pi}$
C. Non simple harmonic motion
D. Simple harmonic motion of frequency $\frac{\omega}{2\pi}$

Use the double angle identity:

$$x = a\sin^2 \omega t = a\left(\frac{1 - \cos 2\omega t}{2}\right) = \frac{a}{2} - \frac{\cos 2\omega t}{2}$$

Rearrange:

$$x - \frac{a}{2} = -\frac{\cos 2\omega t}{2}$$

Use a change of variable that changes the location of the origin. Let $X = x - \frac{a}{2}$:

$$X = -\frac{\cos 2\omega t}{2}$$

This is SHM with frequency

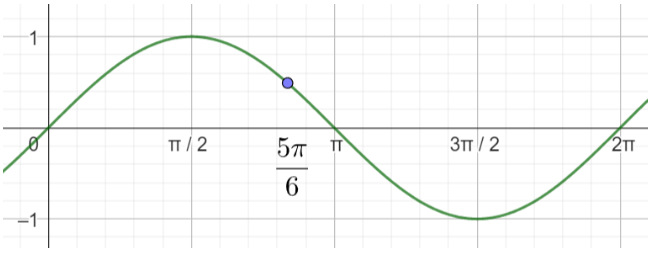
$$= \frac{1}{T} = \frac{2\omega}{2\pi} = \frac{\omega}{\pi}$$

$$Option\ A$$

Example 1.15

A simple harmonic oscillator has an amplitude A and time period T. The time required by it to travel from $x = A$ to $x = \frac{A}{2}$ is: **(NEET 1992)**

$$x = \sin t$$
$$Max = A\ at\ \frac{\pi}{2}$$
$$\frac{A}{2}\ at\ \frac{5\pi}{6}$$



$$Time = \frac{5\pi}{6} - \frac{\pi}{2} = \frac{5\pi}{6} - \frac{3\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$$

Time in terms of $T = 2\pi$

$$= \frac{\frac{\pi}{3}}{2\pi} = \frac{\pi}{3} \times \frac{1}{2\pi} = \frac{1}{6} th\ of\ T$$

Hence, the final answer is

$$\frac{T}{6}$$

Example 1.16

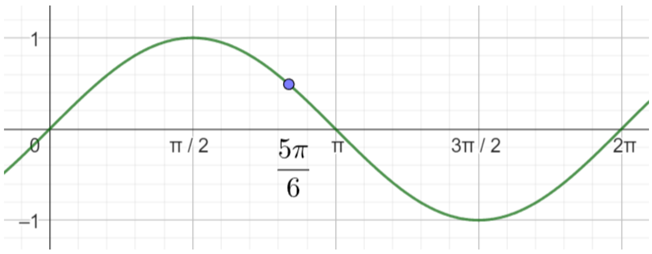
A particle executes simple harmonic oscillation with an amplitude a. The period of oscillation is T. The minimum time taken by the particle to travel half of the amplitude from the equilibrium position is: **(NEET 2007)**

Time in terms of $T = 2\pi$

$$= \frac{\frac{\pi}{6}}{2\pi} = \frac{\pi}{6} \times \frac{1}{2\pi} = \frac{1}{12} th\ of\ T$$

Hence, the final answer is

$$\frac{T}{12}$$



F. 2D Movement

1.17: SHM in 2D Movement

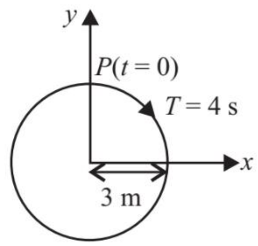
2D movement is movement on the $x - y$ plane. Whereas SHM is movement on a single axis as a function of time. SHM can be recognized by isolating portions of the movement.

Example 1.18

An object undergoes uniform circular motion around a circle at the position (4,3) with a radius of 5. Determine the equation of the projection of the movement

A. on the y axis
B. on the x axis

$$y = \sin(t + \phi), \qquad x = \cos(t + \phi)$$



Example 1.19
The radius of circle, the period of revolution, initial position and sense of revolution are indicated in the figure. y-projection and x projection of the radius vector of rotating particle P are: **(NEET 2019, Adapted)**

$$Amplitude = 3$$
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$y = 3\cos\left(\frac{\pi}{2}t\right), \qquad x = \sin\left(\frac{\pi}{2}t\right)$$

1.20: Periodic Motion
Periodic motion is motion that repeats after an interval.

- All harmonic motion is periodic, but all periodic motion is not harmonic.

Example 1.21
The circular motion of a particle with constant speed is
A. periodic but not simple harmonic
B. simple harmonic but not periodic
C. period and simple harmonic
D. neither periodic nor simple harmonic **(NEET 2005)**

It is a combination of SHM in two directions, hence it is not SHM.
It is periodic.

G. Lissajous Figures

1.22: 2D Movement / Lissajous Figures
The system of system of parametric equations below describes the movement of an object in the $x - y$ coordinate plane. This kind of movement is called a Lissajous figure.

$$x = f(t) = A\sin(\omega t + \phi)$$
$$y = f(t) = B\sin\omega t$$

Where

$$\omega = Common\ frequency$$
$$\phi = phase\ difference$$

We get a Lissajous Figures if we combine
- an SHM to describe movement in the x direction
- another SHM to describe movement in the y direction

Note that
- x and y are perpendicular to each other

Example 1.23

The first equation in the system below describes an SHM in the x direction, while the second one describes movement in the y direction, with $x \perp y$.

$$x = A\sin(\omega t + \phi), y = A\sin\omega t$$

If $\phi = 0$, then describe the nature of displacement of the object.

$$x = A\sin(\omega t + \phi) = A\sin(\omega t + 0) = A\sin(\omega t) = y$$
$$y = x$$

Straight Line

1.24: Using Perpendicularity

If two SHM's are perpendicular to one another, but not in the x and y direction, we can rotate the axes, and then consider them as usual.

Example 1.25

Two SHM's with same amplitude and time period when acting together in perpendicular direction with a phase difference of $\frac{\pi}{2}$, give rise to what kind of movement? **(NEET 1997)**

$$y = \sin(\omega t)$$
$$x = \sin\left(\omega t + \frac{\pi}{2}\right) = \cos(\omega t)$$

$$x^2 + y^2 = \cos^2\omega t + \sin^2(\omega t) = 1$$

Hence, the Lissajous is a circle with center at the origin.

$$\textit{Circular motion} \Rightarrow \textit{Option C}$$

Example 1.26

The composition of two simple harmonic motions of equal periods at right angles to each other and with a phase difference of π results in the displacement of the particle along what shape? **(NEET 1990)**

$$y = \sin(\omega t)$$
$$x = \sin(\omega t + \pi) = -\sin(\omega t) = -y$$

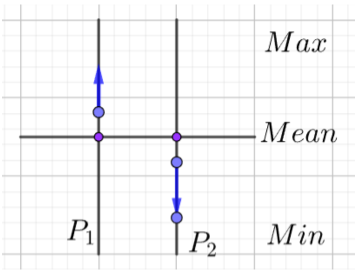
$$-y = x \Rightarrow y = -x$$

$$\textit{Line with Slope} = -1$$

H. Phase Difference (1D Movement)

Example 1.27

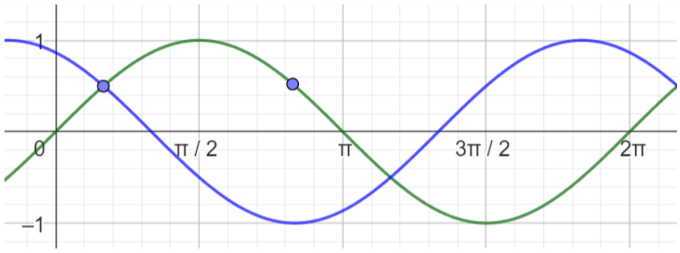
Two particles are oscillating along two parallel lines. The mean positions of the two particles lie on a straight line perpendicular to the paths of the two particles. Draw a diagram.



Example 1.28
Two particles are oscillating along two close parallel straight lines side by side, with the same frequency and amplitudes. They pass each other, moving in opposite directions when their displacement is half of the amplitude. The mean positions of the two particles lie on a straight line perpendicular to the paths of the two particles. The phase difference is **(NEET 2010)**

Diagram from the above question is applicable. We focus on the displacement-time graph.

$$x = \frac{1}{2} \Rightarrow t \in \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$$



Hence the equations are:

$$x_1 = \sin t\,, x_2 = \sin\left(t + \frac{5\pi}{6}\right)$$

Phase difference is:

$$\frac{5\pi}{6} - \frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$$

1.2 Kinematics

A. Basics

Example 1.29
Average velocity of a particle executing SHM in one complete vibration is: **(NEET 2019)**

$$v = \frac{Displacement}{Time} = \frac{0}{Time} = 0$$

1.30: Velocity
Given an SHM equation with displacement function $x = A\sin[\omega t + \phi] + C$

$$v = A\omega\cos[\omega t + \phi]$$

1.31: Acceleration
Given an SHM equation with displacement function $x = A\sin[\omega t + \phi] + C$

$$a = -A\omega^2\sin[\omega t + \phi]$$

Example 1.32

The phase difference between

A. displacement and acceleration of a particle in a simple harmonic motion is: **(NEET 2020)**
B. the instantaneous velocity and acceleration of a particle executing SHM is: **(NEET 2007)**

Consider $x = \sin t$ since it has the same phase difference as $A\sin[\omega t + \phi] + C$.

Part A

$$a = -\sin t = \sin[t + \pi] \Rightarrow Phase\ difference = \pi$$

Part B

$$v = \cos t \Rightarrow a = -\sin t = \sin[t + \pi] = \cos\left(t + \frac{\pi}{2}\right) \Rightarrow Phase\ difference = \frac{\pi}{2}$$

Example 1.33

$$\sin x, \cos x \Rightarrow Phase\ difference = \frac{\pi}{2}$$

$$\sin 2y, \cos 2y$$

Substitute $x = 2y$

$$\sin x, \cos x \Rightarrow Phase\ difference = \frac{\pi}{2}$$

Example 1.34

A particle is executing a simple harmonic motion. Its maximum acceleration is α and maximum velocity is β. Then, its time period of vibration will be: **(NEET 2017)**

A. $\frac{\beta^2}{\alpha}$
B. $\frac{2\pi\beta}{\alpha^2}$
C. $\frac{\beta^2}{\alpha^2}$
D. $\frac{\alpha}{\beta}$

In the interest of simplicity, consider:

$$x = \sin\omega t \Rightarrow v = \omega\cos\omega t \Rightarrow a = -\omega^2\sin\omega t$$

$$\alpha = \omega^2, \beta = \omega$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega}\cdot\frac{\omega}{\omega} = \frac{2\pi\omega}{\omega^2} = \frac{2\pi\beta}{\alpha^2} \Rightarrow Option\ B$$

Example 1.35

Why can we take $x = \sin t$ when comparing phase difference but need to take $x = \sin\omega t$ when calculating time period.

Time period depends on ω which is different in velocity and acceleration:

$$v = A\omega\cos[\omega t + \phi]$$
$$a = -A\omega^2\sin[\omega t + \phi]$$

On the other hand, $\omega t + \phi$ is the same in both and hence the phase difference can be calculated using a change of variable:

$$T = \omega t + \phi$$

And then another change of variable:

$$t = T$$

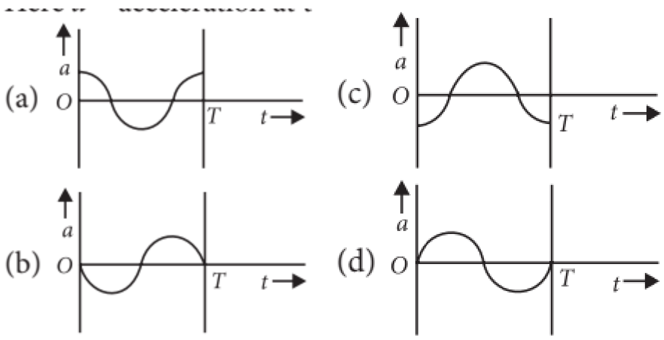
Example 1.36

The oscillation of a body on a smooth horizontal surface is represented by the equation, $X = A\, cos(\omega t)$ where:

$$X = displacement\ at\ time\ t$$
$$\omega = frequency\ of\ oscillation$$

Which one of the graphs shows correctly the variation of a with t? Here

$$a = acceleration\ at\ time\ t,$$
$$T = time\ period$$



(NEET 2014)

$$Displacement = X = A\cos(\omega t)$$
$$Acceleration = -A\omega^2\cos(\omega t)$$

Which has the graph:

$$Option\ C$$

B. Completing the Cycle

Example 1.37

$$x = \sin t$$

When t=pi/2:
Displacement=1=Max
Velocity=0
Acceleration =-1=Min

When t=3pi/2:
Displacement=-1=Max
Velocity=0
Acceleration =1=Max

When t=0:
Displacement=0
Velocity=1
Acceleration =0

C. Relating Quantities

1.38: Velocity in terms of Displacement

For an object in SHM with mean position at the origin and displacement $x = A\sin[\omega t + \phi]$

$$v^2 = \omega^2(A^2 - x^2)$$

$$RHS = \omega^2(A^2 - x^2)$$

Substitute $x = A\sin[\omega t + \phi]$, and factor out A:

$$= \omega^2[A^2 - A^2\sin^2(\omega t + \phi)] = A^2\omega^2(1 - \sin^2(\omega t + \phi))$$

Substitute using the Pythagorean Identity:

$$= A^2\omega^2\cos^2(\omega t + \phi)$$
$$= [A\omega\cos(\omega t + \phi)]^2$$
$$= v^2 = LHS$$

1 Pending

Example 1.39

A particle executes linear simple harmonic motion with an amplitude of 3 cm. When the particle is at 2 cm from the mean position, the magnitude of its velocity is equal to that of its acceleration. Determine its time period. **(NEET 2017)**

$$|v| = |a| \Rightarrow v^2 = a^2$$

$$\omega^2(A^2 - x^2) = \omega^4 x^2$$
$$A^2 - x^2 = \omega^2 x^2$$
$$\omega^2 = \frac{A^2 - x^2}{x^2} = \frac{9-4}{4} = \frac{5}{4}$$
$$\omega = \pm\frac{\sqrt{5}}{2}$$

Considering the positive value of ω gives us:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\sqrt{5}}{2}} = 2\pi \cdot \frac{2}{\sqrt{5}} = \frac{4\pi}{\sqrt{5}} = \frac{4\sqrt{5}\pi}{5}$$

2 Pending

Example 1.40

A particle is executing SHM along a straight line. Its velocities at distances x_1 and x_2 from the mean position are V_1 and V_2, respectively. Its time period is: (NEET 2015)

A. $2\pi\sqrt{\frac{V_1^2+V_2^2}{x_1^2+x_2^2}}$

B. $2\pi\sqrt{\frac{V_1^2-V_2^2}{x_1^2-x_2^2}}$

C. $2\pi\sqrt{\frac{x_1^2+x_2^2}{V_1^2+V_2^2}}$

D. $2\pi\sqrt{\frac{x_2^2-x_1^2}{V_1^2-V_2^2}}$

$$\underbrace{V_1^2 = \omega^2(A^2 - x_1^2)}_{Equation\ I}, \qquad \underbrace{V_2^2 = \omega^2(A^2 - x_2^2)}_{Equation\ II}$$

Subtracting Equation II from Equation I, and solving for ω:

$$V_1^2 - V_2^2 = \omega^2(x_2^2 - x_1^2) \Rightarrow \omega = \sqrt{\frac{V_1^2 - V_2^2}{x_2^2 - x_1^2}}$$

The time period is:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{V_1^2 - V_2^2}} \Rightarrow Option\ D$$

D. Angular Frequency

1.41: Angular Frequency

For an object in simple harmonic motion, the angular frequency ω is given by:

$$\omega = \sqrt{\frac{k}{m}}$$

Where

$$k = spring\ constant$$
$$m = mass\ of\ object$$

Substitute the acceleration from Hooke's Law into Newton's Second Law:

$$F = ma = m(-\omega^2 x)$$

Compare the above the displacement from Hooke's Law:

$$F = -kx$$

Equating the RHS on both sides:

$$k = m\omega^2 \Rightarrow \omega = \sqrt{\frac{k}{m}}$$

Example 1.42

An object of mass m is attached to a massless spring of spring constant k, and moves in simple harmonic motion with angular frequency ω. Determine what happens if, other things remaining constant:

A. The spring constant is increased
B. The mass of the object is increased

$$k \uparrow \Rightarrow \omega \uparrow \Rightarrow T \downarrow \Rightarrow Faster\ Oscillations$$
$$m \uparrow \Rightarrow \omega \downarrow \Rightarrow T \uparrow \Rightarrow Slower\ Oscillations$$

1.43: Time Period

We can use the expression for angular frequency to write the time period in terms of the mass of the object m and the spring constant k:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}}$$

1.3 Energy

A. Conservation of Energy

1.44: Conservation of Mechanical Energy

An object in simple harmonic motion (under the assumptions of no friction and 100% efficiency in energy conversion) has

$$No\ Loss\ of\ Energy = Constant\ Energy$$

The assumptions that we have made for simple harmonic motion are very strong.

- No friction
- 100% efficiency in energy conversion, which means there is no energy loss

Hence, energy can only be converted from potential to kinetic and vice versa.

B. Horizontal SHM

1.45: Potential Energy

An object in simple harmonic motion with equilibrium at the origin, and displacement $x = A\sin(\omega t + \phi)$ has:

$$Potential\ Energy = U = \frac{1}{2}kA^2\sin^2(\omega t + \phi)$$

Since the spring stores the energy used to push or pull it, and we are assuming no loss of energy, the potential energy in the spring is equal to the work done to move it from its equilibrium position:

$$U = \frac{1}{2}kx^2$$

Substitute the displacement for simple harmonic motion $x = A\sin(\omega t + \phi)$

$$U = \frac{1}{2}kA^2\sin^2(\omega t + \phi)$$

1.46: Kinetic Energy

An object in simple harmonic motion with equilibrium at the origin, and displacement $x = A\sin(\omega t + \phi)$ has:

$$Kinetic\ Energy = KE = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$$

$$KE = \frac{1}{2}mv^2$$

Substitute the velocity $v = A\omega\cos(\omega t + \phi)$

$$= \frac{1}{2}mA^2\omega^2\cos^2(\omega t + \phi)$$

Substitute $\omega^2 = \frac{k}{m}$:

$$= \frac{1}{2}kA^2\cos^2(\omega t + \phi)$$

1.47: Total Energy
An object in simple harmonic motion with equilibrium at the origin, and displacement $x = A\sin(\omega t + \phi)$ has total energy

$$E = U + KE = \frac{1}{2}kA^2$$

The sum of potential and kinetic energy equals

$$= \underbrace{\frac{1}{2}kA^2\sin^2(\omega t + \phi)}_{Potential\ Energy} + \underbrace{\frac{1}{2}kA^2\cos^2(\omega t + \phi)}_{Kinetic\ Energy}$$

Factor $\frac{1}{2}kA^2$:

$$= \frac{1}{2}kA^2[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)]$$

Using the Pythagorean Identity, $\sin^2\theta + \cos^2\theta = 1$:

$$= \frac{1}{2}kA^2[1] = \frac{1}{2}kA^2$$

Example 1.48
A particle executing simple harmonic motion has a kinetic energy $K_0\cos^2\omega t$. The maximum values of the potential energy and the total energy are respectively: **(NEET 2007)**

$$-1 \le \cos\omega t \le 1$$
$$0 \le \cos^2\omega t \le 1$$
$$0 \le K_0\cos^2\omega t \le 1$$

When KE is zero, $PE = Max = Total\ Energy$:

$$Max\ KE = Max\ PE = Total\ Energy = K_0$$

1.4 Systems in SHM

A. Springs

B. Simple Pendulum

3 Pending

1.49: Angle of a Simple Pendulum
The angle of a simple pendulum that moves through small angles is given by:

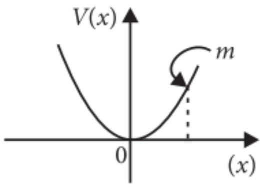
$$\theta(t) = \theta_0\sin\left(\sqrt{\frac{g}{l}}t\right)$$

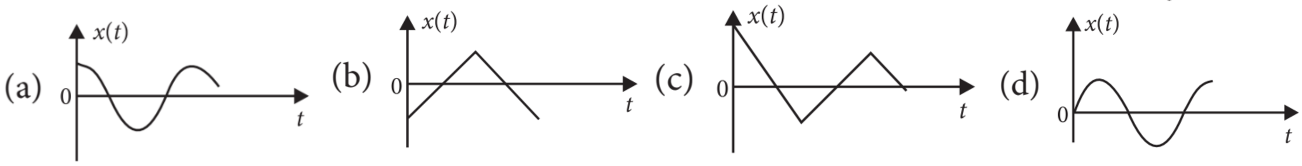
Note that

- the given equation is an equation for simple harmonic motion
- the simple pendulum does follow simple harmonic motion approximately for small angles.

Example 1.50

A particle of mass m is released from rest and follows a parabolic path as shown. Assuming that the displacement of the mass from the origin is small, which graph correctly depicts the position of the particle as a function of time? **(NEET 2011)**





$$Option\ A$$

1.51: Simple Pendulum

The time period of a simple pendulum that moves through small angles is:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Since the pendulum moves through small angles, we can assume:

$$\theta(t) = \theta_0 \sin\left(\sqrt{\frac{g}{l}}t\right)$$

From the above, we can substitute the coefficient of t as $\omega = \sqrt{\frac{g}{l}}$ in the formula for the time period:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{g}{l}}} = 2\pi\sqrt{\frac{l}{g}}$$

1.5 Calculus

A. Basics

1.52: Velocity and Acceleration

Given an SHM equation with displacement function $x = A\sin[\omega t + \phi] + C$, we can differentiate with respect to time to obtain the velocity and acceleration:

$$v = \frac{dx}{dt} = A\omega\cos[\omega t + \phi]$$
$$a = \frac{dx^2}{dt^2} = -A\omega^2\sin[\omega t + \phi]$$

$$a = -A\omega^2 x = -kx, \qquad k = -A\omega^2$$

B. Hooke's Law

1.53: Hooke's Law

$$F = ma = -ks$$
$$m = mass, k = spring\ constant, a = acceleration, s = displacement$$

Has solution:

$$y = C_1 \sin\left(\sqrt{\frac{k}{m}}x\right) + C_2 \cos\left(\sqrt{\frac{k}{m}}x\right)$$

Where

$$k = spring\ constant$$
$$m = mass\ of\ the\ object$$
$$C_1\ and\ C_2\ are\ constants$$

Substitute $F = ma$ in $F = -kx$:

$$ma = -kx$$

Acceleration is the second derivative of position:

$$m\frac{dx^2}{dt^2} = -kx$$

Use a change of variable. Let $y = x$:

$$my'' = -ky \Rightarrow y'' = -\frac{k}{m}y$$

We are looking for a function that is the negative of its second derivative.

$$y = \sin t \Rightarrow y'' = (\sin t)'' = (\cos t)' = -\sin t = -y$$
$$y = \cos t \Rightarrow y'' = (\cos t)'' = (-\sin t)' = -\cos t = -y$$

And we can take care of the $\frac{k}{m}$ also.

If we set $y = C_1 \sin\left(\sqrt{\frac{k}{m}}t\right)$ then

$$y' = C_1\sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}t\right) \Rightarrow y'' = -C_1\frac{k}{m}\sin\left(\sqrt{\frac{k}{m}}t\right) = -y$$

Similarly, if we set $y = C_2 \cos\left(\sqrt{\frac{k}{m}}t\right)$ then

$$y' = -C_2\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}t\right) \Rightarrow y'' = -C_2\frac{k}{m}\cos\left(\sqrt{\frac{k}{m}}t\right) = -y$$

A little more Calculus will show that a linear combination of the above two is also a solution:

$$y = C_1 \sin\left(\sqrt{\frac{k}{m}}t\right) + C_2 \cos\left(\sqrt{\frac{k}{m}}t\right)$$

Example 1.54

Mark all correct options

Out of the following functions representing motion of a particle which ones represent SHM?

A. $y = \sin \omega t - \cos \omega t$
B. $y = \sin^3 \omega t$
C. $y = 5\cos\left(\frac{3\pi}{4} - 3\omega t\right)$
D. $y = 1 + \omega + \omega^2 t^2$ **(NEET 2011)**

Option A

Option B

$$y = \sin^3 \omega t$$
$$y' = 3\omega \sin^2 \omega t \cos \omega t = 3\omega(1 - \cos^2 \omega t)\cos \omega t = 3\omega(\cos \omega t - \cos^3 \omega t)$$
$$y'' = 3\omega(-\sin \omega t + 3\cos^2 \omega t \sin \omega t) \neq -ky$$
$$Not\ SHM$$

Option D
Second degree polynomial in t.
Second derivative will be constant function.
Hence, not SHM.

C. Simple Pendulum

1.55: Small Angle Approximation

If a simple pendulum[1] moves through a *small angle*, then you can approximate its behaviour using the differential equation:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

Which has the solution:

$$\theta(t) = \theta_0 \sin\left(\sqrt{\frac{g}{l}}t\right), \qquad \theta_0 = Initial\ Position$$

A simple pendulum has an angle that satisfies the differential equation:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

Recall that the Maclaurin expansion of $\sin\theta$ is:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

By dropping the third and further terms, we can approximate $\sin\theta \approx \theta$, giving us:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

This is like the differential equation solved above (with different variables), and it has solution:

[1] It is a good exercise to follow through the derivation of the differential equation at the given link, but it requires careful reading and patience.

$$\theta(t) = \theta_0 \sin\left(\sqrt{\frac{g}{l}}\,t\right), \qquad \theta_0 = Initial\ Position$$

1.6 Further Topics

56 Examples