# Differential Equations

Revision: 3292

**08 August 2025**

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# 1. Differential Equations

## 1.1 Definition and Basics

### A. Differential Equations

**1.1: Differential Equation**

An equation that has a derivative in it is called a differential equation.

**Example 1.2**

A simple differential equation is $\frac{dy}{dx}=f(x)$. Solve it.

Which can be solved by integrating both sides with respect to $x$:

$$\frac{dy}{dx}=f(x)\Rightarrow\int\frac{dy}{dx}dx=\int f(x)\,dx\Rightarrow y=\int f(x)\,dx$$

**1.3: Order and Degree**

In a differential equation:

- Order is the order of the highest derivative in the equation
- Degree is the highest exponent of the highest order derivative

Notes:

1. Order and degree are always positive integers (if defined).
2. We will see cases where the degree is not defined in the next definition.

**Example 1.4**

Find the order and the degree of the following differential equations.

A. $x^2\frac{d^2y}{dx^2}=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^4$ (CBSE 2019)

B. $x^3\left(\frac{d^2y}{dx^2}\right)^2+x\left(\frac{dy}{dx}\right)^4=0$ (CBSE 2013, 2019)

**Part A**

$$Order=2$$
$$Degree=1$$

**Part B**

$$Order=2$$
$$Degree=2$$

**1.5: "Polynomial" versus Non-polynomial equations**

- The definition of degree assumes that a differential equation can be written as a polynomial equation in terms of its derivatives.
- If a differential equation is not polynomial, then it does not have a degree.

$$\underbrace{y^2+\frac{1}{2}x+x\frac{dy}{dx}=0}_{Polynomial\Rightarrow Degree=1},\quad\underbrace{y-x=\sin\left(\frac{dy}{dx}\right)}_{Not\ Polynomial\Rightarrow No\ Degree}$$

**Example 1.6**

Find the order and the degree (if defined) of the following differential equations

A. $\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = 2x^2 \log\left(\frac{d^2y}{dx^2}\right)$ **(CBSE 2019)**

B. $\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0$ **(CBSE 2015)**

**Part A**

$$Order = 2$$

Since $\log\left(\frac{d^2y}{dx^2}\right)$ is not polynomial:

$$Degree\ is\ not\ defined$$

**Part B**

Differentiate using the chain rule:

$$\left(\frac{dy}{dx}\right)^{3-1}\left(\frac{d^2y}{dx^2}\right) = 0$$
$$Order = 2$$
$$Degree = 1$$

## 1.7: Linear Differential Equation

A differential equation that can be written in the form

## B. Verifying Solutions

Before finding solutions, we learn the useful technique of validating solutions which have already been found. This requires differentiation, which is easier than integration.

### Example 1.8

Verify that $y = \frac{1}{\sqrt{x}}, x > 0$ is a solution to $\frac{dy}{dx} + \frac{y}{2x} = 0$.

Differentiate both sides of $y = \frac{1}{\sqrt{x}}$:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}\left(x^{-\frac{1}{2}}\right) = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x^{\frac{3}{2}}}$$
$$\frac{y}{2x} = y \times \frac{1}{2x} = \frac{1}{\sqrt{x}} \times \frac{1}{2x} = \frac{1}{2x^{\frac{3}{2}}}$$

We can now verify that:

$$LHS = \frac{dy}{dx} + \frac{y}{2x} = -\frac{1}{2x^{\frac{3}{2}}} + \frac{1}{2x^{\frac{3}{2}}} = 0 = RHS$$

### Example 1.9

Verify that $y = \frac{3}{t}$ is a solution of $\frac{dy}{dt} = -\frac{1}{3}y^2$.

Differentiate the given solution:

$$\frac{dy}{dt} = -\frac{3}{t^2}$$

Substitute $y = \frac{3}{t} \Rightarrow t = \frac{3}{y}$

$$\frac{dy}{dt} = -\frac{3}{\left(\frac{3}{y}\right)^2} = -\frac{3}{\frac{9}{y^2}} = -\frac{3}{y^2}$$

## C. Forming Equations

## 1.10: General Solutions

Differential equations are solved by integration, with an arbitrary constant of integration.
Hence, we get, an infinite number of solutions.
Such a solution is a called a general solution.

**1.11: Number of Constants**
An equation of $n^{th}$ order will have $n$ arbitrary constants in its solution.

**Example 1.12: Number of Constants in Solution**

# 1.2 Separable Equations

## A. Basics

**1.13: Separable Equations**
A differential equation that can be written in the form below is called a separable differential equation:

$$y' = f(x) \cdot g(y)$$

- The LHS contains only the derivative of $y$.
- The RHS is a product of a function of $x$, and a function of $y$.

A separable equation can be divided on both sides by the function of $y$ to give only $y$ variables on the LHS and only x variables on the RHS.

$$\underbrace{\frac{y'}{g(y)}}_{y\ variables} = \underbrace{f(x)}_{x\ variables}$$

**Example 1.14: Classifying Equations**
Are the following equations separable?

A. $y' = 2x + 3y$
B. $y' = (2x)(3y)$
C. $y' = f(x) + g(y)$

$$No$$
$$Yes$$
$$No$$

**1.15: Solving Separable Equations**

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

Separate the variables:

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

Integrate both sides with respect to $x$:

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx$$

Note that since the $dx$ on the LHS cancels, you integrate with respect to $y$ on the LHS:

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

Substitute $\int \frac{1}{g(y)}\,dy = G(y), \int f(x)\,dx = F(x)$, and add the constant of integration:

$$G(y) = F(x) + C$$

### Example 1.16

$$\frac{dy}{dt} = -\frac{1}{3}y^2$$

Separate the variables:

$$-\frac{dy}{y^2} = \frac{1}{3}dt$$

Integrate:

$$\int -\frac{dy}{y^2} = \int \frac{1}{3}dt$$

Carry out the integration:

$$\frac{1}{y} = \frac{1}{3}t + C$$

Add the fractions:

$$\frac{1}{y} = \frac{t+3C}{3}$$

Take the reciprocal on both sides:

$$y = \frac{3}{t+3C}$$

Substitute $K = 3C$:

$$y = \frac{3}{t+K}$$

### 1.17: Constant of Integration

When integrating both sides of a differential equation, you will get a constant of integration on the LHS as well as the RHS.

However, the two constants can be replaced by a single constant.

### Example 1.18

Integrate the equation $\frac{dy}{dx} = -\frac{x}{y}$, and determine the curve determined by the equation you get.

Separate the variables:

$$y\,dy = -x\,dx$$

Integrate:

$$\int y\,dy = \int -x\,dx$$

Carry out the integration:

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$y^2 = -x^2 + 2C$$
$$y^2 + x^2 = 2C$$

Substitute $2C = r^2$:

$$y^2 + x^2 = r^2$$

Which is the equation of a circle.

### Example 1.19

$$\frac{dy}{dt} = y^2 \sin t$$

Separate the variables:

$$\frac{dy}{y^2} = \sin t \, dt$$

Integrate:

$$\int \frac{dy}{y^2} = \int \sin t \, dt$$

$$-\frac{1}{y} = -\cos t + C$$

Solve for $y$:

$$y = \frac{1}{\cos t - C}$$

Substitute $K = -C$

$$y = \frac{1}{\cos t + K}$$

### Example 1.20

Solve the differential equation $\cos\left(\frac{dy}{dx}\right) = a, (a \in \mathbb{R})$ (CBSE 2018)

Take the cos inverse on both sides:

$$\frac{dy}{dx} = \cos^{-1} a$$

Separate the variables:

$$dy = \cos^{-1} a \, dx$$

Integrate both sides:

$$\int dy = \int \cos^{-1} a \, dx$$

Note that $\cos^{-1} a$ is simply a constant:

$$y = (\cos^{-1} a)(x) + C$$

### Example 1.21

Solve for $y$:

$$\frac{dy}{dt} = -0.6y$$

**General Solution**

$$\frac{dy}{y} = -0.6\,dt$$
$$\int \frac{dy}{y} = \int -0.6\,dt$$

Integrate both sides to obtain the general solution:

$$\ln|y| = -0.6t + C$$

**Solve for $y$**

Exponentiate both sides to $e$:

$$|y| = e^{-0.6t+C}$$

Since the RHS is positive, the LHS is also positive and hence $|y| = y$:

$$y = e^{-0.6t+C}$$

Split the RHS:

$$y = e^{-0.6t}e^{C}$$

Substitute $k = e^C$:

$$y = ke^{-0.6t}$$

## Example 1.22

Write the solution of the differential equation $\frac{dy}{dx} = 2^{-y}$ **(CBSE 2015)**

Separate the variables:

$$2^y dy = dx$$

Integrate both sides:

$$\int 2^y dy = \int 1 \cdot dx$$

Carry out the integration to get:

$$\frac{2^y}{\ln 2} = x + C_1$$

**Explicit Solution for $y$**

Solving explicitly for y is not easy (or required) in all questions. However, in this one, we can do (if we want).

Cross multiply:

$$2^y = x\ln 2 + C_1 \ln 2$$

Substitute $C = C_1 \ln 2$:

$$2^y = x \ln 2 + C$$

Take the natural log of both sides:

$$\ln 2^y = \ln(x\ln 2 + C)$$
$$y = \frac{\ln(x \ln 2 + C)}{\ln 2}$$

## Example 1.23

$$\frac{dy}{dx} = e^{x-y}$$

$$\frac{dy}{dx} = \frac{e^x}{e^y}$$
$$e^y dy = e^x\,dx$$

$$\int e^y dy = \int e^x\, dx$$
$$e^y = e^x + C$$

## 1.24: Verifying Solutions

The solution of a differential equation can be verified by differentiating it.

### Example 1.25

A. Find the solution of the differential equation $\frac{dy}{dx} = x^3 e^{-2y}$ (CBSE 2015)
B. Verify the solution.

**Part A**

Separate the variables:
$$e^{2y} dy = x^3 dx$$
Integrate both sides:
$$\int e^{2y} dy = \int x^3\, dx$$
Carry out the integration:
$$\frac{e^{2y}}{2} = \frac{x^4}{4} + C_1$$
Multiply by 2 both sides to eliminate Fractions:
$$e^{2y} = \frac{x^4}{2} + 2C_1$$
Substitute $C = 2C_1$:
$$2e^{2y} = x^4 + C$$

**Part B**

To verify the solution, differentiate both sides of
$$2e^{2y} = x^4 + C$$
With respect to $x$.
$$\frac{d}{dx}(2e^{2y}) = \frac{d}{dx}(x^4 + C)$$
$$(2)(e^{2y})(2)\left(\frac{dy}{dx}\right) = 4x^3$$
$$e^{2y}\frac{dy}{dx} = 4x^3$$
$$\frac{dy}{dx} = x^3 e^{-2y}$$
And the above is the equation that we solved in Part A.
Hence, the solution has been verified.

### Example 1.26

A. Find the general solution of the differential equation $\ln\left(\frac{dy}{dx}\right) = ax + by$ (CBSE 2022).
B. Verify the solution.

**Part A**

Exponentiate both sides:
$$\frac{dy}{dx} = e^{ax+by} = e^{ax}e^{by}$$
Separate the variables:
$$e^{-by} dy = e^{ax} dx$$
Integrate both sides:
$$\int e^{-by} dy = \int e^{ax}\, dx$$
Carry out the integration:
$$\frac{e^{-by}}{-b} = \frac{e^{ax}}{a} + C_1$$
Eliminate Fractions:
$$ae^{-by} = -be^{ax} - abC_1$$
Substitute $C = -abC_1$:
$$ae^{-by} + be^{ax} = C$$

**Part B**

### Example 1.27

Find $\frac{dy}{dx}$ in terms of $x$ given that $y = \frac{2}{2-e^{-0.8x}}$ and verify that $\frac{dy}{dx} = 0.8y(1-y)$

$$\frac{dy}{dx} = \left(\frac{2}{2-e^{-0.8x}}\right)' = \frac{-1.6e^{-0.8x}}{(2-e^{-0.8x})^2}$$

Substitute $y = \frac{2}{2-e^{-0.8x}}$ in $0.8y(1-y)$:

$$= 0.8\left(\frac{2}{2-e^{-0.8x}}\right)\left(1-\frac{2}{2-e^{-0.8x}}\right) = \left(\frac{1.6}{2-e^{-0.8x}}\right)\left(\frac{-e^{-0.8x}}{2-e^{-0.8x}}\right) = \frac{-1.6e^{-0.8x}}{(2-e^{-0.8x})^2}$$

$$y(0) = 1$$

## B. Initial Value Problems

An initial value problem is a condition that gives you information about the function that is a solution to the differential equation. It lets you find a value of the constant in the differential equation.

### 1.28: Particular Solutions

A solution to a differential equation that has a specific value of the constant is called a particular solution.

### Example 1.29

Find the particular solution to $\frac{dy}{dx} = x(y-2)^2, y(1) = 0$

Separate the variables, and integrate:

$$\int \frac{dy}{(y-2)^2} = \int x\,dx$$

Carry out the integration:

$$-\frac{1}{y-2} = \frac{x^2}{2} + C$$

Take the reciprocal, and move the minus sign to the RHS:

$$y - 2 = -\frac{2}{x^2+2C}$$

Solve for $y$:

$$y = \frac{-2+2x^2+4C}{x^2+2C} = \frac{2x^2-2+4C}{x^2+2C} = \frac{2(x^2-1+2C)}{x^2+2C} = \frac{2(x^2-1+K)}{x^2+K}$$

Substitute $(x,y) = (1,0)$

$$0 = \frac{2(1-1+K)}{1+K} \Rightarrow 0 = 2K \Rightarrow K = 0$$

Substitute $K = 0$:

$$y = \frac{2(x^2-1)}{x^2}$$

### Example 1.30

Find the particular solution to:

$$\frac{dy}{dx} = 2xy^2, \qquad y(5) = 1$$

### General Solution

$$\frac{dy}{y^2} = 2x\,dx$$

Integrate:

$$\int \frac{dy}{y^2} = \int 2x\,dx$$

$$\underbrace{-\frac{1}{y} = x^2 + C}_{\textit{Equation I}}$$

### Particular Solution

Solve it for $C$, and substitute $(x, y) = (5,1)$:

$$C = -\frac{1}{y} - x^2 = -\frac{1}{1} - 5^2 = -26$$

Solve Equation I for $y$, and substitute $C = -26$:

$$y = -\frac{1}{x^2 + C} = -\frac{1}{x^2 - 26} = \frac{1}{26 - x^2}$$

## Example 1.31

Find the particular solution of the differential equation $\frac{dy}{dx} = 1 + x + y + xy$, given that $y = 0$ when $x = 1$. **(CBSE 2014)**

### General Solution

The expression on the RHS is not separated, but we can separate it after factoring:

$$\frac{dy}{dx} = (1 + x)(1 + y)$$

Separate the variables:

$$\frac{dy}{(1 + y)} = (1 + x)\,dx$$

Integrate both sides:

$$\int \frac{dy}{1 + y} = \int (1 + x)\,dx$$

$$\ln|1 + y| = x + \frac{x^2}{2} + C$$

### Particular Solution

Substitute $x = 1 \text{ } and \text{ } y = 0$:

$$\ln|1 + 0| = 1 + \frac{1}{2} + C$$

$$C = -\frac{3}{2}$$

Substitute $C = -\frac{3}{2}$ in the general solution to get the particular solution:

$$\ln|1 + y| = x + \frac{x^2}{2} - \frac{3}{2}$$

## Example 1.32

Find the particular solution of the differential equation $\ln\left(\frac{dy}{dx}\right) = 3x + 4y$, given that $y = 0 \text{ } when \text{ } x = 0$ **(CBSE 2014)**

### General Solution

Exponentiate both sides:

$$\frac{dy}{dx} = e^{3x+4y} = e^{3x} \cdot e^{4y}$$

Separate the variables:

$$\frac{1}{e^{4y}} dy = e^{3x}\, dx$$

Integrate both sides:

$$\int \frac{1}{e^{4y}} dy = \int e^{3x}\, dx \Rightarrow \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C$$

### Particular Solution

Substitute $x = 0$ *and* $y = 0$:

$$\frac{e^0}{-4} = \frac{e^0}{3} + C \Rightarrow C = -\frac{7}{12}$$

Substitute $C = -\frac{7}{12}$ in the general solution:

$$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} - \frac{7}{12} \Rightarrow 4e^{3x} + 3e^{-4y} - 7 = 0$$

## Example 1.33

Find the particular solution of the differential equation $(1 - y^2)(1 + \ln|x|)dx + 2xy\, dy = 0$, given that $y = 0$ when $x = 1$. (CBSE 2016)

Separate the variables:

$$\frac{(1 + \ln|x|)}{x} dx + \frac{2y}{1 - y^2} dy = 0$$

Integrate both sides:

$$\int \left(\frac{1}{x} + \frac{\ln|x|}{x}\right) dx + \int \frac{2y}{1 - y^2} dy = 0$$

$$\ln|x| + \frac{(\ln|x|)^2}{2} - \ln|1 - y^2| = \ln C$$

Substitute $y = 0$ *and* $x = 1$:

$$\ln 1 + \frac{(\ln 1)^2}{2} - \ln|1 - 0| = \ln C$$

$$\ln C = 0$$

Substitute $\ln C = 0$

$$\ln|x| + \frac{(\ln|x|)^2}{2} - \ln|1 - y^2| = 0$$

# C. Tangent Lines

## Example 1.34

Find the tangent line to $\frac{dy}{dx} = x(y - 2)^2$ at $(1,0)$ and use it to approximate $y(0.7)$

Substitute $(x, y) = (1,0)$ in the equation:

$$m = \frac{dy}{dx} = 1(0 - 2)^2 = 4$$

Substitute $(x, y) = (1,0), m = 4$ in the slope-point form of the equation of a line:

$$y - 0 = 4(x - 1)$$

Substitute $x = 0.7$ in the above equation:

$$y = 4x - 4 = 4(0.7) - 4 = 2.8 - 4 = -1.2$$

## D. More Challenging Questions

### Example 1.35

Solve the differential equation $(x+1)\frac{dy}{dx} = 2e^{-y} - 1; y(0) = 0$ (CBSE 2019)

#### Find the General Solution

$$(x+1)\frac{dy}{dx} = 2e^{-y} - 1 = \frac{2}{e^y} - 1 = \frac{2 - e^y}{e^y}$$

Separate the variables:

$$\frac{e^y}{2 - e^y}dy = \frac{1}{x+1}dx$$

Integrate both sides:

$$-\int \frac{e^y}{e^y - 2}dy = \int \frac{1}{x+1}dx$$

Let $u = e^y - 2 \Rightarrow du = e^y\,dy$:

$$-\int \frac{1}{u}du = \ln|x+1| + C_1$$

$$-\ln|u| = \ln|x+1| + C_1$$

$$\ln|x+1| + \ln|e^y - 2| = -C_1$$

Use $\log a + \log b = \log ab$:

$$\ln|(x+1)(e^y - 2)| = -C_1$$

Exponentiate both sides:

$$(x+1)(e^y - 2) = e^{-C_1}$$

Substitute $C = e^{-C_1}$

$$(x+1)(e^y - 2) = C$$

#### Find the particular solution

Substitute $(x, y) = (0,0)$:

$$C = (0+1)(1-2) = -1$$

Substitute $C = -1$:

$$(x+1)(e^y - 2) = -1$$

$$e^y - 2 = -\frac{1}{x+1}$$

$$y = \ln\left(2 - \frac{1}{x+1}\right)$$

### Example 1.36

Find the particular solution of the differential equation $e^x \tan y\,dx + (2 - e^x)\sec^2 y\,dy = 0$, given that $y = \frac{\pi}{4}$ *when* $x = 0$ (CBSE 2018)

#### Find the General Solution

$$e^x \tan y\,dx = (e^x - 2)\sec^2 y\,dy$$

Separate the variables:

$$\frac{\sec^2 y}{\tan y}dy = \frac{e^x}{e^x - 2}dx$$

Integrate both sides:

$$\int \frac{\sec^2 y}{\tan y} dy = \int \frac{e^x}{e^x - 2} dx$$

Note that on both sides the numerator is the derivative of the denominator, and hence the integral is the natural log of the denominator:

$$\ln|\tan y| = \ln|e^x - 2| + C_1$$

Collate the log terms on one side:

$$\ln|\tan y| - \ln|e^x - 2| = C_1$$

Use the log property $\log a - \log b = \log\frac{a}{b}$:

$$\ln\left|\frac{\tan y}{e^x - 2}\right| = C_1$$

Exponentiate both sides:

$$\frac{\tan y}{e^x - 2} = e^{C_1}$$

Eliminate fractions and substitute $e^{C_1} = C$ to find the general solution:

$$\tan y = C(e^x - 2)$$

**Find the value of the constant**

Substitute $y = \frac{\pi}{4}$ *and* $x = 0$ in the general solution to find the value of $C$:

$$\tan\frac{\pi}{4} = e^C(e^0 - 2)$$
$$e^C = -1$$

**Find the particular solution**

Substitute $C = -1$ in the general solution to find the particular solution:

$$\tan y = 2 - e^x$$

# 1.3 Exponential Models

## A. Exponential Growth and Decay

### Example 1.37

Exponential Growth and Decay

## B. Further Examples

### Example 1.38

$$\frac{dP}{dt} = 6.4 - 1.6P, \qquad P(0) = 6$$

Factor $-1.6$ on the RHS, and move the $P$ *term* to the LHS. Move the $dt$ to the RHS. Integrate:

$$\int \frac{1}{P - 4} dP = -1.6dt$$
$$\ln|P - 4| = -1.6t + C$$
$$P - 4 = Ce^{-1.6t}$$

Use the initial condition $P(0) = 6$:

$$6 - 4 = Ce^0 \Rightarrow C = 2$$

The equation is then:

$$P - 4 = 2e^{-1.6t}$$

$$P = 2e^{-1.6t} + 4$$

### C. Newton's Law of Cooling

**Example 1.39**

Newton's law of cooling gives the temperature for an object which is warmer than its surroundings.

$$\frac{dT_t}{dt} = -k(T_t - T_s)$$
$$T_t = Temperature\ at\ time\ t$$
$$T_s = Temperature\ of\ Surroundings = Constant$$

$$\frac{dT_t}{dt} = -k(T_t - T_s)$$

Separate the variables:

$$\frac{dT_t}{T_t - T_s} = -k\,dt$$

Integrate:

$$\int \frac{dT_t}{T_t - T_s} = \int -k\,dt$$

Carry out the integration:

$$\ln|T_t - T_s| = -kt + C_1$$

Exponentiate both sides:

$$T_t - T_s = e^{-kt+C} = e^{-kt}e^{C_1} = e^{-kt}C$$
$$T_t - T_s = e^{-kt}C$$

Use initial value conditions to find the value of $C$. Suppose the temperature at time $t = 0$ is $T_0$.
Substitute $t = 0 \Rightarrow Temperature = T_0$

$$T_0 - T_s = e^{-k\times 0}C$$
$$T_0 - T_s = e^0 C$$
$$T_0 - T_s = C$$

Finally, my equation is:

$$T_t - T_s = (T_0 - T_s)e^{-kt}$$

## 1.4 Slope Fields

### A. Basics

We have seen how to solve simple differential equations that using direct integration, or separation of variables. However, certain differential equations may not have a closed-form solution[1] that is easy to find. In such cases, numerical integration is a way to understand the features of the differential equation.

**1.40: $\frac{dy}{dx}$ as slope**

Given a function $y = f(x)$, the quantity $\frac{dy}{dx}$ represents the slope of the function $f$ at the point $x$.

**1.41: Slope Field**

A slope field gives the slope at any point in the coordinate plane due to a differential equation.

- A slope field is a way of visualizing the behavior of a differential equation qualitatively, rather than

[1] A closed form solution is an explicit solution that connects the variables.

quantitatively.

- Every point in the coordinate plane is represented by a pair of coordinates $(x, y)$, and the slope field associates with that pair $(x, y)$, the slope of the differential equation at that point.
- Slope can be represented using a vector. Hence, the input is a point on the coordinate plane, and the output is a vector.

### 1.42: Single Variable Function

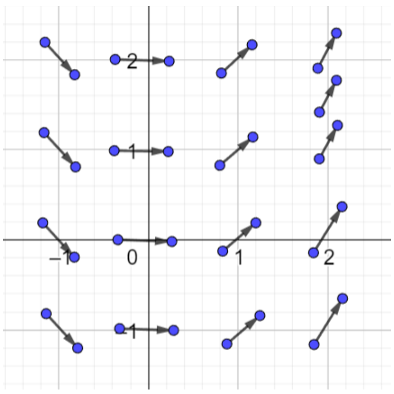
$$y = f(x)$$

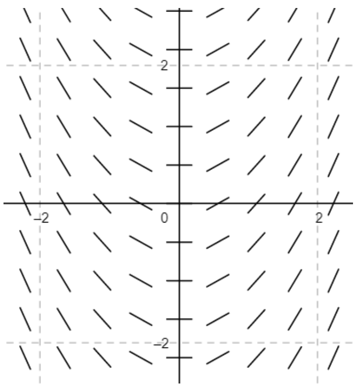
### 1.43: Vector Function

$$\vec{v} = f(x, y)$$

### Example 1.44

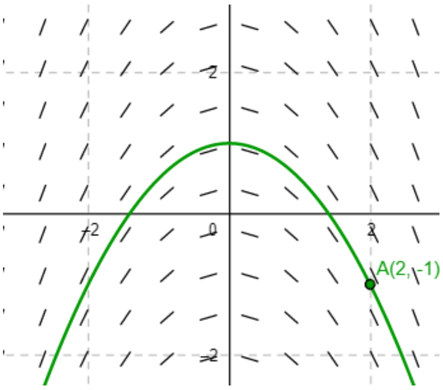
Sketch a slope field for the differential equation $\frac{dy}{dx} = x$ over $-1 \le x \le 2, -1 \le y \le 2$[2].





### Example 1.45

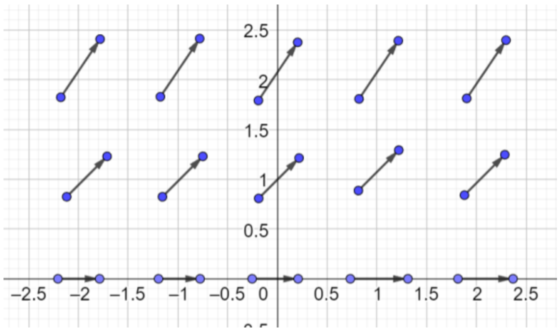
Sketch a slope field for the differential equation $\frac{dy}{dx} = -x$ over $-2 \le x \le 2, -2 \le y \le 2$.

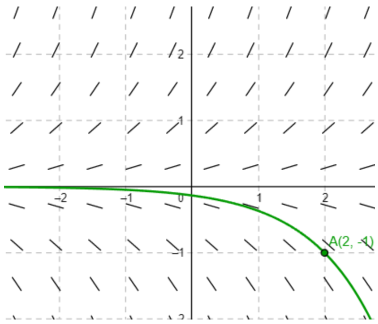



[2] https://www.geogebra.org/m/W7dAdgqc lets you plot slope fields.

### Example 1.46

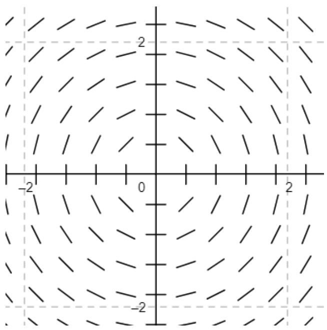
Sketch a slope field for the differential equation $\frac{dy}{dx} = y$ over $-2 \le x \le 2, -2 \le y \le 2$.




### Example 1.47

Sketch a slope field for the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ over $-3 \le x \le 3, -3 \le y \le 3$.



Rather than attempting this numerically, we integrate the differential equation using separation of variables:

$$\int y\,dy = -\int x\,dx$$

Integrate the above to get:

$$y^2 = -x^2 + C$$

Substitute $C = r^2$, and collate all variables on the left to get the equation of a circle:

$$y^2 + x^2 = r^2$$

Hence, the slope field must have solution curve that are circles, and now drawing the slopes becomes much easier.

# 1.5 Euler's Method

## A. Basics

### 1.48: Euler's Method

$$L(x) = y_0 + f(x_0, y_0)dx$$

### Example 1.49

$$Differential\ Equation: \frac{df}{dx} = f - 2$$
$$Initial\ Value\ or\ Starting\ point:\ f(0) = 3$$
$$x_0 = 0$$
$$Increment: dx = 0.1$$

$$f(0.1) = y_1 = y_0 + f(x_0, y_0)dx = 3 + (3 - 2)(0.1) = 3.1$$

$$f(0.2) = y_2 = y_1 + f(x_1, y_1)dx = 3.1 + (3.1 - 2)(0.1) = 3.21$$

$$f(0.2) = y$$

### Example 1.50

Solve the differential equation

$$\frac{df}{dx} = f - 2, \qquad f(0) = 3$$

Separate variables and integrate both sides:

$$\int \frac{df}{f-2} = \int dx$$
$$\ln|f - 2| = x + C$$

Determine the value of C using the initial value condition $f(0) = 3$:

$$\ln|3 - 2| = 0 + C \Rightarrow C = 0$$

The particular solution to the differential equation is:

$$\ln|f - 2| = x$$

### Example 1.51

Compare the value found using Euler's Method for $x = 0.2$ and the actual value using the particular solution to determine the difference between the two.

$$\ln|f - 2| = 0.2$$
$$f - 2 = e^{0.2}$$
$$f = e^{0.2} + 2 = 3.22140275$$

Difference

$$= 3.22140275 - 3.21$$

## 1.6 Logistic Growth

Exponential Model

Assume a population P. The death rate in the population
=dP

The birth rate in the population
=bP

The net change in the population is
=bP-dP=kP

Since the above is the rate of change of population, we can write
dP/dt = kP
The above model results in constant growth or constant decay (exponential model).

Logistic Growth Model

Assume an existing population P, and a carrying capacity or limiting population M, where M is the long term value of the population.

k = r(M-P), r>0
Growth rate increases if the population is far below the carrying capacity, and decreases closer to (but still below) the carrying capacity.

If, for some reason, the population increases above the carrying capacity, then P>M, and M-P<0.
Hence, the rate is negative.

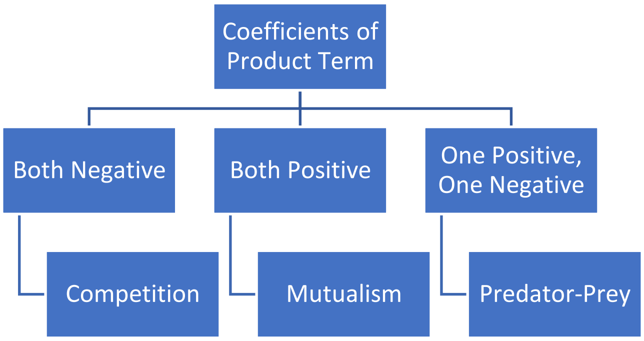
Second Derivative and Interpretations
For M/2 < P<M:
First Derivative is positive and second derivative is negative.
The population is increasing, but the rate of increase is decreasing.

For P<M/2:
The first and second derivative are both positive. The population is increasing, and the rate of increase is also increasing.

For P=M/2:
The first is positive, but the second derivative is zero. The population is increasing, and the rate of increase is constant.

# 1.7 Systems of Differential Equations

## A. Biology Models




**1.52: Biology Models**
A system of two differential equations in two variables of a specific type can be interpreted to have different relationships depending upon the coefficients of the $xy$ term.

**1.53: Competition**
- The $xy$ coefficients in both equations are negative.
- Therefore, the presence of each negatively affects the other.

$$\text{(b)} \quad \begin{cases} \dfrac{dx}{dt} = 0.02x\left(1-\dfrac{x}{300}\right) - 0.1xy \\ \dfrac{dy}{dt} = 0.03y\left(1-\dfrac{y}{60}\right) - 0.3xy \end{cases}$$

**1.54: Mutualism**
- The $xy$ coefficients in both equations are positive.
- Therefore, the presence of each positively affects the other.

$$\text{(c)} \quad \begin{cases} \dfrac{dx}{dt} = 0.02x + 0.1xy \\ \dfrac{dy}{dt} = 0.03y + 0.3xy \end{cases}$$

(d) $\begin{cases} \dfrac{dx}{dt} = -0.02x + 0.1xy \\ \dfrac{dy}{dt} = 0.03y\left(1 - \dfrac{y}{5000}\right) - 7xy \end{cases}$

**1.55: Predator-Prey Relations**

- The $xy$ coefficients in one equation is positive and the other equation is negative.
- The presence of $x$ negatively affects $y$
  - ✓ $x$ is predator
- The presence of $y$ positively affects $x$
  - ✓ $y$ is prey

## B. Infection Model

**Example 1.56**

$$\frac{dS}{dt} = -rSI, \qquad \frac{dI}{dt} = rSI - \alpha I, \qquad \frac{dR}{dt} = \alpha I$$

$$\frac{dS}{dt} = -rSI = 0 \Rightarrow SI = 0 \Rightarrow S = 0\ OR\ I = 0$$

$$\frac{dI}{dt} = rSI - \alpha I = 0 \Rightarrow I(rS - \alpha) = 0 \Rightarrow I = 0\ OR\ S = \frac{\alpha}{r}$$

For the system to be at equation, we need both $\frac{dS}{dt} = 0$ and $\frac{dI}{dt} = 0$:

$$I = 0 \Rightarrow S\ can\ be\ any\ value$$

Every pair $(S, 0)$ is an equilibrium solution.

# 2. Further Topics

## 2.1 Homogenous Functions and DE

### A. Further Resources

**Example 2.1**
Nishant Vora on Homogenous DE

### B. Homogenous Functions

**2.2: Homogenous Function**
A multivariable function $f(x, y)$ is said to be homogenous if for any real number $t \neq 0$:
$$f(tx, ty) = t^n f(x, y)$$

**Example 2.3**
Show that the function below is homogenous, and determine the value of $n$ in $f(tx, ty) = t^n f(x, y)$:
$$f(x, y) = x^{\frac{2}{5}} + x^{\frac{1}{5}} y^{\frac{1}{5}} + y^{\frac{2}{5}}$$

$$\begin{aligned} f(tx, ty) &= (tx)^{\frac{2}{5}} + (tx)^{\frac{1}{5}}(ty)^{\frac{1}{5}} + (ty)^{\frac{2}{5}} \\ &= t^{\frac{2}{5}} x^{\frac{2}{5}} + t^{\frac{1}{5}} x^{\frac{1}{5}} t^{\frac{1}{5}} y^{\frac{1}{5}} + t^{\frac{2}{5}} y^{\frac{2}{5}} \\ &= t^{\frac{2}{5}} \left( x^{\frac{2}{5}} + x^{\frac{1}{5}} y^{\frac{1}{5}} + y^{\frac{2}{5}} \right) \end{aligned}$$

Substitute $f(x, y) = x^{\frac{2}{5}} + x^{\frac{1}{5}} y^{\frac{1}{5}} + y^{\frac{2}{5}}$:
$$= t^{\frac{2}{5}} f(x, y)$$

Hence:
$$f(tx, ty) = t^{\frac{2}{5}} f(x, y) \Rightarrow n = \frac{2}{5}$$

**2.4: "Same Power" Shortcut**
An algebraic expression where the sum of powers of each term is the same is homogenous.

**Example 2.5**
Show that the function below is homogenous by the "same power" shortcut:
$$f(x, y) = x^{\frac{2}{5}} + x^{\frac{1}{5}} y^{\frac{1}{5}} + y^{\frac{2}{5}}$$

$$\begin{aligned} x^{\frac{2}{5}} &\Rightarrow Sum\ of\ Powers = \frac{2}{5} \\ x^{\frac{1}{5}} y^{\frac{1}{5}} &\Rightarrow Sum\ of\ Powers = \frac{1}{5} + \frac{1}{5} = \frac{2}{5} \\ y^{\frac{2}{5}} &\Rightarrow Sum\ of\ Powers = \frac{2}{5} \end{aligned}$$

Since the powers are the same, the function is homogenous.

**2.6: Rational Functions**
A rational function whose numerator and denominator are homogenous is also homogenous.

$$f(x) = \frac{Numerator\ is\ homogenous}{Denominator\ is\ homogenous} \Rightarrow f(x)\ is\ homogenous$$

**Example 2.7**
Check the homogeneity of the function:

$$f(x, y) = -1 + \frac{y}{-x}$$

$$f(tx, ty) = -1 + \frac{ty}{tx} = -1 + \frac{y}{x} = f(x, y) = t^0 f(x, y)$$

## C. Non-Homogenous Functions

**Example 2.8**
Are the following functions homogenous:

A. $f(x, y) = e^{x+y}$
B. $g(x, y) = \ln(x + y)$

$$f(tx, ty) = e^{tx+ty} = e^{t(x+y)} \neq t^n e^{x+y} \Rightarrow Not\ homogenous$$

$$g(tx, ty) = \ln(tx + ty) = \ln\big(t(x + y)\big) = \ln t + \ln(x + y) \neq t \ln(x + y) \Rightarrow Not\ homogenous$$

**Example 2.9**
Given that $f(x, y) = e^{x+y}$, check the homogeneity of:

A. $\ln\big(f(x, y)\big)$
B. $\ln\Big(\ln\big(f(x, y)\big)\Big)$

$$\ln\big(f(x, y)\big) = \ln(e^{x+y}) = x + y \Rightarrow Homogenous$$
$$\ln\Big(\ln\big(f(x, y)\big)\Big) = \ln(x + y) \Rightarrow Not\ Homogenous$$

## D. The substitution method

**2.10: Homogenous Differential Equation**
A differential equation of the form

$$\frac{dy}{dx} = f(x, y)$$

Where $f(x, y)$ is homogenous is a homogenous differential equation.

**2.11: Substitution Method**
To solve a homogenous differential equation

- Substitute $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$
- The equation will transform to a separable variable equation. Separate the variables and integrate.
- Change back to the original variable.

Let $y = vx$ where $v$ is a variable. Since $v$ is a variable, we need to apply the product rule to differentiate:

$$y = vx \Rightarrow \frac{dy}{dx} = v \cdot \frac{d}{dx}x + \frac{dv}{dx} \cdot x = v + x\frac{dv}{dx}$$

### Example 2.12

$$\frac{dy}{dx} = \frac{x - y}{-x}$$

$$\frac{dy}{dx} = \frac{y}{x} - 1$$

**Step I: Substitute $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$:**

$$v + x\frac{dv}{dx} = \frac{vx}{x} - 1$$

Simplify:

$$x\frac{dv}{dx} = -1$$

**Step II: Separate the variables and Integrate**

$$\int dv = -\int \frac{1}{x}dx$$
$$v = -(\ln|x|) + C$$

**Step III: Change back to the original variables**

$$\frac{y}{x} = -(\ln|x|) + C$$
$$y = -x(\ln|x|) + xC$$

### Example 2.13

Solve

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$$

**Step I: Substitute $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$:**

$$v + x\frac{dv}{dx} = \frac{vx}{x} + \tan\frac{vx}{x}$$
$$v + x\frac{dv}{dx} = v + \tan v$$
$$x\frac{dv}{dx} = \tan v$$

**Step II: Separate the variables and Integrate**

$$x\frac{dv}{dx} = \frac{\sin v}{\cos v}$$

Integrating:

$$\int \frac{\cos v}{\sin v}dv = \int \frac{dx}{x}$$
$$\ln|\sin v| = \ln|x| + \ln C$$
$$\ln|\sin v| = \ln|Cx|$$

$$\sin v = Cx$$

**Step III: Change back to the original variables**

$$\sin\frac{y}{x} = Cx$$
$$\frac{y}{x} = \sin^{-1} Cx$$
$$y = x\sin^{-1} Cx$$

## Example 2.14

Solve:

$$\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}$$

(Gorakhpur 2007, Agra 2008)

**Step I: Substitute $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$:**

$$v + x\frac{dv}{dx} = \frac{x^2(vx)}{x^3 + v^3x^3} = \frac{v}{1+v^3}$$

Solve for $x\frac{dv}{dx}$:

$$x\frac{dv}{dx} = \frac{v}{1+v^3} - v = \frac{v - v(1+v^3)}{1+v^3} = -\frac{v^4}{1+v^3}$$

**Step II: Separate the variables and Integrate**

Separate the variables:

$$\int -\frac{1+v^3}{v^4}dv = \int \frac{dx}{x}$$

Split the LHS:

$$\int -\frac{1}{v^4}dv - \int \frac{v^3}{v^4}dv = \ln|x| + \ln C$$
$$\frac{1}{3v^3} - \ln|v| = \ln|Cx|$$
$$\frac{1}{3v^3} = \ln|Cvx|$$

Exponentiate both sides:

$$e^{\frac{1}{3v^3}} = Cvx$$

**Step III: Change back to the original variables**

$$e^{\frac{1}{3\left(\frac{y}{x}\right)^3}} = Cy$$
$$y = \frac{e^{\frac{x^3}{3y^3}}}{C} = Ke^{\frac{x^3}{3y^3}}$$

## Example 2.15

Solve

$$\frac{dy}{dx} = \frac{y-x}{y+x}$$

(Gorakhpur 2010)

**Step I: Substitute $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$:**

$$v + x\frac{dv}{dx} = \frac{vx - x}{vx + x} = \frac{v - 1}{v + 1}$$

$$x\frac{dv}{dx} = \frac{v - 1}{v + 1} - v = \frac{v - 1 - v^2 - v}{v + 1} = -\frac{v^2 + 1}{v + 1}$$

**Step II: Separate the variables and Integrate**

Separate the variables and integrate:

$$\int \frac{v + 1}{v^2 + 1} dv = -\int \frac{dx}{x}$$

Split the integral in the LHS:

$$\int \frac{v}{v^2 + 1} dv + \int \frac{1}{v^2 + 1} dv = -\ln|x| + \ln|c|$$

$$\ln\sqrt{v^2 + 1} + \tan^{-1} v + \ln|x| = \ln|c|$$

**Step III: Change back to the original variables**

$$\ln\sqrt{\left(\frac{y}{x}\right)^2 + 1} + \tan^{-1}\left(\frac{y}{x}\right) + \ln|x| = \ln|c|$$

$$\ln\sqrt{\frac{y^2 + x^2}{x^2}} + \tan^{-1}\left(\frac{y}{x}\right) + \ln|x| = \ln|c|$$

$$\ln\sqrt{\frac{y^2 + x^2}{x^2} \cdot x^2} + \tan^{-1}\left(\frac{y}{x}\right) = \ln|c|$$

$$\ln\sqrt{y^2 + x^2} + \tan^{-1}\left(\frac{y}{x}\right) = \ln|c|$$

## E. Partial Fractions

### Example 2.16

Solve the differential equation:

$$\frac{dy}{dx} = \frac{2y - x}{y}$$

(Gorakhpur 2005)

**Step I: Substitute $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$:**

$$v + x\frac{dv}{dx} = \frac{2vx - x}{vx} = \frac{2v - 1}{v}$$

$$x\frac{dv}{dx} = \frac{2v - 1}{v} - v = \frac{2v - 1 - v^2}{v} = -\frac{(v - 1)^2}{v}$$

**Step II: Separate the variables and Integrate**

$$\int \frac{v}{(v-1)^2} dv = -\int \frac{dx}{x}$$

Add and subtract 1 in the LHS integral:

$$\int \frac{v-1}{(v-1)^2} dv + \int \frac{1}{(v-1)^2} dv = -\int \frac{dx}{x}$$

Substitute $\int \frac{v-1}{(v-1)^2} dv = \int \frac{1}{v-1} dv = \ln|v-1|$:

$$\ln|v-1| - \frac{1}{v-1} = -\ln x + \ln|C|$$

Combine logarithms to the LHS

$$\ln\left|\frac{x(v-1)}{C}\right| = \frac{1}{v-1}$$

Exponentiate both sides:

$$\frac{xv-x}{C} = e^{\frac{1}{v-1}}$$

### Step III: Change back to the original variables

$$\frac{y-x}{C} = e^{\frac{1}{\frac{y}{x}-1}}$$

$$y-x = Ce^{\frac{x}{y-x}}$$

## Example 2.17

$$\frac{dy}{dx} = -\frac{y^2}{xy+x^2}$$

(Avadh 2010)

### Step I: Substitute $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$:

$$v + x\frac{dv}{dx} = -\frac{v^2x^2}{x(vx)+x^2} = -\frac{v^2}{v+1}$$

$$x\frac{dv}{dx} = -\frac{v^2}{v+1} - v = -\left(\frac{v^2}{v+1} + v\right) = -\left(\frac{v^2+v^2+v}{v+1}\right) = -\frac{v(2v+1)}{v+1}$$

### Step II: Separate the variables and Integrate

$$-\int \frac{dx}{x} = \int \frac{v+1}{v(2v+1)} dv$$

The LHS is:

$$-\ln|x| + \ln C$$

The RHS side can be solved using partial fractions:

$$\frac{v+1}{v(2v+1)} = \frac{A}{v} + \frac{B}{2v+1}$$

$$v+1 = (2v+1)A + vB$$

$$v = 0 \Rightarrow A = 1, v = -\frac{1}{2} \Rightarrow B = -1$$

$$\int \frac{v+1}{v(2v+1)} dv = \int \frac{1}{v} dv - \int \frac{1}{2v+1} dv = \ln|v| - \frac{1}{2}\ln|2v+1|$$

$$\ln|v| - \frac{1}{2}\ln|2v+1| = -\ln|x| + \ln C$$
$$\ln|v| + \ln|x| - \ln C = \frac{1}{2}\ln|2v+1|$$
$$\ln\left|\frac{vx}{C}\right| = \ln\sqrt{2v+1}$$
$$\frac{vx}{C} = \sqrt{2v+1}$$

### Step III: Change back to the original variables

$$\frac{y}{C} = \sqrt{2\left(\frac{y}{x}\right) + 1}$$
$$\frac{y^2}{C^2} = \frac{2y+x}{x}$$
$$xy^2 = C^2(2y+x)$$

## F. Reciprocal Rule for Derivatives

### Example 2.18

Find the particular solution of the differential equation below that passes through (1,1):

$$2xy\frac{dy}{dx} = y^2 - x^2$$

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

### Step I: Make the substitution

Substitute $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$:

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x(vx)}$$

Simplify the RHS by cancelling the $x^2$:

$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$
$$x\frac{dv}{dx} = -\frac{v^2+1}{2v}$$

### Step II: Separate the variables and Integrate

Separate the variables, and integrate:

$$\int \frac{2v}{v^2+1} dv = -\int \frac{1}{x} dx$$

The LHS has the form $\frac{f'(x)}{f(x)}$, where the derivative of the denominator is present in the numerator, and hence:

$$\ln|v^2+1| = -\ln|x| + \ln C$$

Combine the RHS using log rules:

$$\ln|v^2+1| = \ln\left|\frac{C}{x}\right|$$

Exponentiate both sides:

$$v^2+1 = \frac{C}{x}$$

Change back to the original variable, and eliminate fractions:

$$\frac{y^2}{x^2}+1 = \frac{C}{x} \Rightarrow y^2+x^2 = Cx$$

### Step III: Find the particular solution

Substitute $(x, y) = (1,1)$ in the above equation to find the constant of integration:

$$1^2+1^2 = C(1) \Rightarrow C = 2$$

The particular solution is then:

$$y^2+x^2 = 2x$$
$$y^2+x^2-2x = 0$$

## 2.19: Reciprocal of a derivative

If $\frac{dy}{dx}$ is non-zero, then:

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

## Example 2.20

Find the particular solution of the differential equation below that passes through $(1,1)$:

$$\frac{dy}{dx} = \frac{2xy}{y^2-x^2}$$

Use a change of Variable. Let $Y = x, X = y$:

$$\frac{dX}{dY} = \frac{2YX}{X^2-Y^2}$$

Using $\frac{dx}{dy} = \frac{1}{dy/dx}$:

$$\frac{dY}{dX} = \frac{X^2-Y^2}{2YX}$$

Which we already know from above has solution:

$$Y^2+X^2-2X = 0$$

Change back to the original variables:

$$x^2+y^2-2y = 0$$

## 2.21: Equation of a circle

The equation of a circle in the coordinate plane with center $(h, k)$ and radius $r$ is given by:

$$(x-h)^2+(y-k)^2 = r^2$$

**Example 2.22**

$$A: x^2 + y^2 - 2x = 0$$
$$B: x^2 + y^2 - 2y = 0$$

A. Find the center and radius of the circles above by completing the square.
B. Graph them on the coordinate plane.
C. Find the area that lies in both the circles.

Add 1 to both sides:

$$(x^2 - 2x + 1) + y^2 = 1$$
$$(x-1)^2 + (y-0)^2 = 1$$
$$Center = (1,0), Radius = 1$$

Add 1 to both sides of the second equation:

$$(x-0)^2 + (y-1)^2 = 1$$
$$Center = (0,1), Radius = 1$$

**Example 2.23**

Let $C_1$ be the curve obtained by solution of the differential equation:

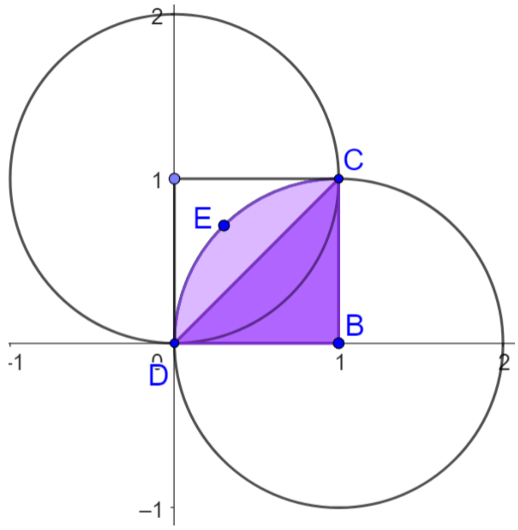
$$2xy\frac{dy}{dx} = y^2 - x^2, \qquad x > 0$$

Let the curve $C_2$ be the solution of

$$\frac{2xy}{x^2 - y^2} = \frac{dy}{dx}$$

If both the curves pass through $(1,1)$, then the area enclosed by the curves $C_1$ and $C_2$ is equal to: **(JEE-M 2021)**

We solved these differential equations, and graphed the curves.
We can now find the area between them.




The area between the two circles is twice the segment in the diagram, which is given by:

$$A(Segment) = 2[A(Sector) - A(Triangle)]$$
$$= 2\left[\pi r^2 \times \frac{\theta}{360} - \frac{1}{2}hb\right]$$
$$= 2\left[\pi \cdot 1^2 \times \frac{90}{360} - \frac{1}{2} \cdot 1 \cdot 1\right]$$
$$= 2\left[\frac{\pi}{4} - \frac{1}{2} \cdot 1 \cdot 1\right]$$
$$= \frac{\pi}{2} - 1$$

# 2.2 Integrating Factors

## A. Resources

**Video 2.24**

Nishant Vora on LDE with Integrating Factors

## B. Linear ODE

### 2.25: Linear Differential Equation (LDE)

A linear differential equation is an equation of the form:

$$a_1(x)\frac{dy}{dx}+a_2(x)\frac{d^2y}{dx^2}+\cdots+a_n(x)\frac{d^ny}{dx^n}+a_0(x)y=b(x)$$

Where:

- $a_1(x)$, ... etc are differentiable functions of $x$, that need not be linear.
- The derivatives are raised to the first power only.
- The right-hand side is in terms of $x$ only.

### Example 2.26

$$y\frac{dy}{dx}+x\frac{d^2y}{dx^2}=7x$$

$$y\frac{dy}{dx}=f(y)\frac{dy}{dx}$$

And since we need $f(x)$ only to be multiplied with the derivative, the equation is not linear.

### Example 2.27

*Mark all correct options*

$$x\left(\frac{dy}{dx}\right)^2+x^2\frac{d^2y}{dx^2}=\sin x$$

is not a linear differential equation because:

A. The derivative is squared, which is not linear.
B. $x^2\frac{d^2y}{dx^2}$ has an $x^2$ in it, which is not linear.
C. $\frac{d^2y}{dx^2}$ is a second derivative, which is not linear.
D. $\sin x$ is not a linear function.

$$x\left(\frac{dy}{dx}\right)^2$$

The derivative is squared, and hence the equation is not linear.

$$a_1(x)\frac{dy}{dx}+a_2(x)\frac{d^2y}{dx^2}+\cdots+a_n(x)\frac{d^ny}{dx^n}+a_0(x)y=b(x)$$

Second derivatives are allowed.
$a_n(x)$ can be any differentiable function of $x$. Hence, $x^2$ is valid.
$b(x)$ can be any differentiable function of $x$. Hence, $\sin x$ is valid.

### 2.28: Revision: Linear Differential Equation

- A linear differential equation is linear in the *derivatives*. The derivative cannot be raised to a power, or multiplied with a function of $y$.
- The functions of $x$ need not be linear.

**2.29: Degree of a DE**

The degree of a differential equation is the power of the highest derivative in the equation.

**Example 2.30**

$$x\left(\frac{dy}{dx}\right)^2 + x^2\frac{d^2y}{dx^2} = \sin x$$

A. What is the degree of the equation above?
B. Is it a linear DE?

$$Degree = Power\ of\ \frac{d^2y}{dx^2} = 1$$

$$Not\ linear, because\ \left(\frac{dy}{dx}\right)^2\ is\ squared$$

## C. LDE of First Order

**2.31: Order of a DE**

The order of a DE is the highest derivative present in the equation.

**Example 2.32**

$$x\left(\frac{dy}{dx}\right)^2 + x^2\frac{d^2y}{dx^2} = \sin x$$

What is the order?

$$Order = 2$$

**2.33: LDE of First Order: Standard Form**

$$\frac{dy}{dx} + P(x)y = Q(y)$$

$$a_1(x)\frac{dy}{dx} + a_2(x)\frac{d^2y}{dx^2} + \cdots + a_n(x)\frac{d^ny}{dx^n} + a_0(x)y = b(x)$$

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{b(x)}{a_1(x)}$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

## D. Solving LDE

**2.34: Solution of LDE of First Order**

The solution to $\frac{dy}{dx} + P(x)y = Q(y)$ is:

$$y \cdot IF = \int (Q(x) \cdot IF)\, dx$$

Where

$$IF = Integrating\ Factor = e^{\int P(x)\,dx}$$

### Example 2.35: A trivial example

$$y = y'$$

$$\frac{dy}{dx} - 1 \cdot y = 0$$

$$y \cdot IF = \int (Q(x) \cdot IF)\ dx$$

Substitute $IF = e^{\int P(x)\,dx} = e^{\int -1\,dx} = e^{-x}$:

$$y \cdot e^{-x} = \int (0 \cdot IF)\,dx = \int 0\,dx = C$$

$$y = Ce^{x}$$

### Example 2.36

$$\frac{dy}{dx} + 2xy = e^{-x^2}$$

(Meerut Univ. 2001, 2003)

$$y \cdot IF = \int (Q(x) \cdot IF)\ dx$$

Substitute $IF = e^{\int P(x)\,dx} = e^{\int 2x\,dx} = e^{x^2}$:

$$y \cdot e^{x^2} = \int \left(e^{-x^2} \cdot e^{x^2}\right)\ dx = x + C$$

$$y \cdot e^{x^2} = x + C$$

### 2.37: Converting to Standard Form

### 2.38: Ignoring Absolute Value

### Example 2.39

$$(x^2 - 1)\frac{dy}{dx} + 2xy = 1$$

(Meerut Univ. 2001, 2003, 2010)

Convert the equation to standard form:

$$\frac{dy}{dx} + \frac{2x}{x^2 - 1}y = \frac{1}{x^2 - 1}$$

The integrating factor is;

$$\int P(x)\,dx = \int \frac{2x}{x^2 - 1}\ dx = \ln(x^2 - 1) \Rightarrow IF = e^{\int P(x)\,dx} = e^{\ln(x^2 - 1)} = x^2 - 1$$

Using $y \cdot IF = \int (Q(x) \cdot IF)\ dx$:

$$y(x^2-1)=\int\left(\frac{1}{x^2-1}\cdot x^2-1\right)\,dx$$
$$y(x^2-1)=x+C$$

### Example 2.40

$$(x^2+1)\frac{dy}{dx}+2xy=4x^2$$

(Agra Univ. 2008)

Divide both sides by $x^2+1$:

$$\frac{dy}{dx}+\frac{2x}{x^2+1}y=\frac{4x^2}{x^2+1}$$

The integrating factor is;

$$\int P(x)\,dx=\int\frac{2x}{x^2+1}\,dx=\ln(x^2+1)\Rightarrow IF=e^{\int P(x)\,dx}=e^{\ln(x^2+1)}=x^2+1$$

Using $y\cdot IF=\int(Q(x)\cdot IF)\ dx$:

$$y(x^2+1)=\int\left(\frac{4x^2}{x^2+1}\cdot(x^2+1)\right)\,dx\ =\int 4x^2\,dx$$

$$y(x^2-1)=x+C$$

$$\frac{dy}{dx}+\frac{2x}{x^2+1}\cdot y=\frac{4x^2}{x^2+1}\cdot \qquad \left[\text{Form } \frac{dy}{dx}+Py=Q\right]$$

Here $P=\dfrac{2x}{1+x^2}$ and so $\displaystyle\int P\,dx=\int\frac{2x}{1+x^2}\cdot dx=\log(1+x^2)$.

$\therefore$ Integrating factor $=e^{\int P\,dx}=e^{\log(1+x^2)}=(1+x^2)$.

$\therefore$ The solution is $y\cdot(\text{I.F.})=\displaystyle\int Q\cdot(\text{I.F.})\,dx=c$

*i.e.,* $$y\cdot(1+x^2)=\int\frac{4x^2}{(x^2+1)}\cdot(1+x^2)\,dx+c \qquad \left[\because Q=\frac{4x^2}{x^2+1}\right]$$

or $$y\,(1+x^2)=\int 4x^2\,dx+c=\frac{4}{3}x^3+c.$$

## E. Integration by Parts

### Example 2.41

$$\cos^2 x\frac{dy}{dx}+y=\tan x$$

(Meerut(U) 2003, Avadh(U) 2005)

Convert the equation to standard form:

$$\frac{dy}{dx} + (\sec^2 x)y = \tan x \sec^2 x$$

The integrating factor is:

$$\int P(x)\,dx = \int \sec^2 x\ dx = \tan x \Rightarrow IF = e^{\int P(x)\,dx} = e^{\tan x}$$

Using $y \cdot IF = \int (Q(x) \cdot IF)\ dx$:

$$ye^{\tan x} = \int (\tan x \sec^2 x\, e^{\tan x})\, dx$$

Use $u$ substitution. Let $t = \tan x \Rightarrow dt = \sec^2 x\, dt$ to get the RHS as

$$\int te^t\ du$$

Use integration by parts with

$$\begin{aligned} u &= t, & dv &= e^t\ dt \\ du &= dt, & v &= e^t \end{aligned}$$

Apply $\int u\, dv = uv - \int v\, du$:

$$\int te^t\ dt = t \cdot e^t - \int e^t\, dt = t \cdot e^t - e^t + C = e^t(t-1) + C$$

$$ye^{\tan x} = e^{\tan x}(\tan x - 1) + C$$

## 2.3 Series Solutions

### A. Basics

**Example 2.42**

Determine the first four terms of the Maclaurin series of the particular solution of the curve below that passes through (0,2):

$$\frac{dy}{dx} = \frac{y^2}{2} - x^2$$

Evaluate the first derivative using the initial condition:

$$y' = \frac{y^2}{2} - x^2 \Rightarrow y'(0) = \frac{2^2}{2} - 0^2 = 2$$

Calculate the second derivative using implicit differentiation, and evaluate it at (0,2):

$$y'' = yy' - 2x \Rightarrow y''(0) = 2(2) - 2(0) = 4$$

Calculate the third derivative using implicit differentiation, and evaluate it at (0,2):

$$y''' = (y')^2 + yy'' - 2 \Rightarrow y'''(0) = 2^2 + 2(4) - 2 = 4 + 8 - 2 = 10$$

Calculate the fourth derivative using implicit differentiation, and evaluate it at (0,2):

$$y^{(4)} = 2y'y'' + y'y'' + yy''' = 3y'y'' + y'y''' \Rightarrow y^{(4)}(0) = 3(2)(4) + 2(10) = 24 + 20 = 44$$

$$S(x) = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$= 2 + 2x + \frac{4}{2!}x^2 + \frac{10}{3!}x^3 + \frac{44}{4!}x^4 + \cdots$$
$$= 2 + 2x + 2x^2 + \frac{5}{3}x^3 + \frac{11}{6}x^4 + \cdots$$

# 2.4 State Space

## A. Basics

### Example 2.43

Suppose you have a patient. You want to capture some basic information about the patient.

- Heart Rate
- Blood Pressure

$$State\ Space = S = (Heart\ Rate, Diastolic\ Blood\ Pressure)$$

Patient 1 has a heart rate of 60 beats per minute, and a diastolic blood pressure of 80. Record these in a state space.

$$S = (60,80)$$

### Example 2.44

# 2.5 Further Topics

### 45 Examples