Diophantine Equations

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1. Diophantine Equations

1.1 Basics

A. Basics

1.1: Diophantine Equations

- A system of equations where the number of variables is more than the number of equations is called Diophantine[1].
- Diophantine equations are generally solved for integers/whole numbers.

- Integer restrictions can because of real scenarios, or restrictions given in the question.
  - ✓ Number of attendees at a party must be a whole number.
  - ✓ Number of children must be a whole number

Example 1.2

I must distribute five indistinguishable sweets among two children.

A. In how many ways can I do this?
B. In how many ways can I do this if every child must get at least one sweet.

Suppose the children are $Jack$ and Jill.

$Way\ No.$	Jack	Jill
1	0	5
2	1	4
3	2	3
4	3	2
5	4	1
6	5	0

Part A

$$6\ ways$$

Part B

$$4\ ways$$

Example 1.3

Danica wants to arrange her model cars in rows with exactly 6 cars in each row. She now has 23 model cars. What is the smallest number of additional cars she must buy in order to be able to arrange all her cars this way? (AMC 8 2013/1)

If the number of cars in the final arrangement is

$$c = 6k, \qquad k \in \mathbb{N}$$
$$23 + 1 = 24 = 4(6)$$
$$Additional\ Cars\ Required = 1$$

Example 1.4

Soda is sold in packs of 6, 12 and 24 cans. What is the minimum number of packs needed to buy exactly 90 cans of soda? (AMC 8 2005/1)

$$6a + 12b + 24c = 90$$

$$c = 4 \Rightarrow 24c = 96 > 90$$

[1] These kinds of equations are called Diophantine after Diophantus of Greece, a mathematician from antiquity.

$$c = 3 \Rightarrow 24c = 72$$

$$6a + 12b + 72 = 90$$
$$6a + 12b = 18$$
$$c = 3, a = b = 1 \Rightarrow c + a + b = 3 + 1 + 1 = 5$$

Example 1.5

You want to buy dolls and cars. A doll costs $7, and a car costs $11. You pay $93 to the shopkeeper. How many dolls and how many cars did you buy?

7d+11c=93
7d = 93-11c
d = (93-11c)/7

Try different values of c, and see which value is divisible by 7:
93-11=82
93-22=71
93-33=60
93-44=49=7^2

c=4, d=7 works

Example 1.6

A father takes his twins and a younger child out to dinner on the twins' birthday. The restaurant charges 4.95 for the father and 0.45 for each year of a child's age, where age is defined as the age at the most recent birthday. If the bill is 9.45, what is the age of the youngest child? **(AHSME 1996/7, Adapted)**

The bill for the children is:

$$9.45 - 4.95 = 4.5$$

Total of ages of the children

$$= \frac{4.5}{0.45} = 10$$

Let

$$age\ of\ each\ twin = t$$
$$age\ of\ younger\ child = c$$

$$2t + c = 10$$

Apply parity to each side of the equation:

$$\underbrace{2t}_{Even} + \underbrace{c}_{Even} = \underbrace{10}_{Even}$$

Smallest possible value of c is

$$c = 2 \Rightarrow 2t + 2 = 10 \Rightarrow t = 4 \Rightarrow (c, t, t) = (2,4,4) \Rightarrow Works$$

We can also check:

$$c = 4 \Rightarrow 2t + 4 = 10 \Rightarrow t = 3$$

In this case, there is no youngest, since the two twins both have age 3.

$$Age\ of\ youngest\ child = 2$$

Longer Method

You can get the same answer using lengthier algebra:

$$4.95 + 2(0.45)t + (0.45)c = 9.45$$
$$0.9t + 0.45c = 4.5$$
$$90t + 45c = 450$$
$$2t + c = 10$$

Example 1.7

Ike and Mike go into a sandwich shop with a total of \$30.00 to spend. Sandwiches cost \$4.50 each and soft drinks cost \$1.00 each. Ike and Mike plan to buy as many sandwiches as they can, and use any remaining money to buy soft drinks. Counting both sandwiches and soft drinks, how many items will they buy? (AMC 8 2019/1)

$$4.5s + d = 30$$
$$s = \left\lfloor \frac{30}{4.5} \right\rfloor = \left\lfloor \frac{60}{9} \right\rfloor = \left\lfloor 6\frac{2}{3} \right\rfloor = 6 \Rightarrow d = 3$$
$$s + d = 9$$

Example 1.8

On a twenty-question test, each correct answer is worth 5 points, each unanswered question is worth 1 point and each incorrect answer is worth 0 points. Which of the following scores is NOT possible?

A. 90
B. 91
C. 92
D. 95
E. 97(AMC 8 2001/22)

Figure out whether each score is possible:

$$18\ Correct, 2\ Incorrect = 90$$
$$18\ Correct, 1\ Unanswered, 1\ Incorrect = 91$$
$$18\ Correct, 2\ Unanswered = 92$$
$$19\ Correct, 1\ Incorrect = 95$$

$$97\ is\ not\ possible \Rightarrow Option\ E$$

Example 1.9

In a mathematics contest with ten problems, a student gains 5 points for a correct answer and loses 2 points for an incorrect answer. If Olivia answered every problem and her score was 29, how many correct answers did she have? (AMC 8 2002/17)

$$C + I = 10 \Rightarrow I = 10 - C$$
$$5C - 2(10 - C) = 29$$
$$5C - 20 + 2C = 29$$
$$7C = 49$$
$$C = 7$$

Example 1.10

A. Find the ordered pairs (c, t) of whole numbers which are the solutions to $2t + c = 100$. Also, count the number of ordered pairs.
B. In Part B, if $c < t$, then answer again.

Part A

$$(c,t) = (0,50), (2,49), \dots, (100,0) \Rightarrow 51 \ Solutions$$

Part B

$$(c,t) = (0,50), (2,49), \dots, (32,34) \Rightarrow 17 \ Solutions$$

Note that equality will be achieved when $t = c$:

$$c + 2c = 100 \Rightarrow 3c = 100 \Rightarrow c = \frac{100}{3} = 33\frac{1}{3}$$

We can also solve this using an inequality:

$$2t + c = 100 \Rightarrow 2t = 100 - c \Rightarrow t = \frac{100}{2} - \frac{c}{2} = 50 - \frac{c}{2}$$

Substitute $t = 50 - \frac{c}{2}$ in the condition $c < t$:

$$c < 50 - \frac{c}{2} \Rightarrow \frac{3c}{2} < 50 \Rightarrow c < 50 \times \frac{2}{3} = \frac{100}{3} = 33\frac{1}{3}$$

B. General Diophantine Equations

Example 1.11

Find all solutions to the equation below in positive integers a, b, c:

$$a + b + c = abc$$

By observation, we see that $(a,b,c) = (1,2,3)$ works. We now show that it is the only solution that works.
Notice the symmetry in the equation. Interchanging the variables does not change the equation.
Hence, without loss of generality, let

$$a \le b \le c$$

$$abc = a + b + c \le c + c + c \le 3c$$
$$abc \le 3c \Rightarrow ab \le 3$$

Now, we can break this into cases:

Case I: $ab = 1$

$$ab = 1 \Rightarrow (a,b) = (1,1)$$
$$1 + 1 + c = (1)(1)(c) \Rightarrow 2 + c = c \Rightarrow 2 = 0 \Rightarrow \textbf{Contradiction}$$

Case II: $ab = 2$

$$(a,b) = (1,2)$$
$$1 + 2 + c = (1)(2)(c) \Rightarrow 3 + c = 2c \Rightarrow 3 = c$$
$$(a,b,c) = (1,2,3)$$

Case III: $ab = 3$

$$(a,b) = (1,3)$$
$$1 + 3 + c = (1)(3)(c) \Rightarrow 4 + c = 3c \Rightarrow 2 = c \Rightarrow \textbf{Contradiction}$$

Hence the only possible solutions are:

$$(1,2,3) \ and \ its \ permutations$$

$$(a,b,c) = (1,2,3)(1,3,2)(2,1,3)(2,3,1)(3,1,2)(3,2,1)$$

C. Money

Example 1.12
The Amaco Middle School bookstore sells pencils costing a whole number of cents. Some seventh graders each bought a pencil, paying a total of 1.43 dollars. Some of the 30 sixth graders each bought a pencil, and they paid a total of 1.95 dollars. How many more sixth graders than seventh graders bought a pencil? **(AMC 8 2009/11)**

Let the cost of a pencil be p cents.
Let the number of sixth graders and seventh graders be x and y respectively.

$$xp = 195, yp = 143$$
$$xp - yp = 195 - 143 = 52$$
$$p(x - y) = 13(4) = 52(1)$$

Where p must a factor of 195 and 143 both.

$$\underbrace{p}_{13}\underbrace{(x - y)}_{4} = 13(4)$$

$$Ans = 4$$

Example 1.13
Penniless Pete's piggy bank has no pennies in it, but it has 100 coins, all nickels, dimes, and quarters, whose total value is \$8.35. It does not necessarily contain coins of all three types. What is the difference between the largest and smallest number of dimes that could be in the bank? **(AMC 12B 2003/7)**

Since the value of the coins is $\$8.35 = 835\ cents$:

$$5n + 10d + 25q = 835 \Rightarrow \underbrace{n + 2d + 5q = 167}_{Equation\ I}$$

Since the number of coins is 100, we must have:

$$\underbrace{n + d + q = 100}_{Equation\ II}$$

Subtract Equation I from Equation II to get a Diophantine Equation in two variables:

$$d + 4q = 67$$
$$Max(d) = 67, Min(d) = 3$$

The difference is:

$$67 - 3 = 64$$

Example 1.14
Jamar bought some pencils costing more than a penny each at the school bookstore and paid \$1.43. Sharona bought some of the same pencils and paid \$1.87. How many more pencils did Sharona buy than Jamar? **(AMC 8 2012/13)**

$$xp - yp = 187 - 143$$
$$p(x - y) = 11(17 - 13) = 11(4)$$

The difference is:

$$x - y = 4$$

Example 1.15
Paul owes Paula 35 cents and has a pocket full of 5-cent coins, 10-cent coins, and 25-cent coins that he can use to pay her. What is the difference between the largest and the smallest number of coins he can use to pay her? **(AMC 8 2014/2)**

$$5n + 10d + 25q = 35$$
The number of coins is $n + d + q$.
$$Smallest\ value\ is\ when\ d = 1, q = 1 \Rightarrow Total = 2$$
$$Largest\ value\ is\ when\ n = 7 \Rightarrow Total = 7$$

$$Difference = 7 - 2 = 5$$

Example 1.16
You have nine coins: a collection of pennies, nickels, dimes, and quarters having a total value of 1.02, with at least one coin of each type. How many dimes must you have? **(AMC 8 2000/20)**

The total value of the coins is:
$$p + 5n + 10d + 25q = 102$$

We know that there is at least one coin of each type. So, subtract one of each type:
$$P + 5N + 10D + 25Q = 61$$

$P \neq 11 \Rightarrow P = 1$, which we can substitute:
$$5N + 10D + 25Q = 60$$

Try $Q = 0$:
$$N + D = 4, \qquad N, D, Q \in \mathbb{W}$$
$$5N + 10D = 60$$
The number of coins needed is 6, and we have already used 5, for a total of 11.
$$Not\ Valid$$

Try $Q = 1$:
$$5N + 10D + 25 = 60$$
$$5N + 10D = 35$$
$$Min\ No. of\ Coins = 1\ nickel + 3\ Dimes = 4$$
$$Does\ not\ Work$$
Try $Q = 2$:
$$5N + 10D + 50 = 60, D + Q = 2$$
$$5N + 10D = 10, D + Q = 2$$
$$2\ Nickels$$
$$3\ Quarters, 1\ Dime, 3\ Nickels, and\ 2\ Pennies = 75 + 10 + 15 + 2 = 102$$

D. Pythagorean Triplets

Example 1.17
Definition
Format

Application

E. Squares

Example 1.18

The sum of 18 consecutive positive integers is a perfect square. The smallest possible value of this sum is (AMC 12B 2002/13)

Let the numbers be:

$$n, n+1, \dots, n+17$$

The sum is:

$$18n + \frac{17 \cdot 18}{2} = 18n + 17 \cdot 9 = 9(2n+17) = \underbrace{3^2}_{\substack{Perfect \\ Square}}(2n+17)$$

Hence, $2n + 17$ must be a perfect square. The next perfect square after 16 is 25:

$$9 \times 25 = 225$$

And we can also find the value of n as:

$$2n + 17 = 25 \Rightarrow 2n = 8 \Rightarrow n = 4$$

Example 1.19

On the last day of school, Mrs. Wonderful gave jelly beans to her class. She gave each boy as many jelly beans as there were boys in the class. She gave each girl as many jelly beans as there were girls in the class. She brought 400 jelly beans, and when she finished, she had six jelly beans left. There were two more boys than girls in her class. How many students were in her class? (AMC 8 2009/23)

Let the number of girls be g. Then:

$$g^2 + (g+2)^2 + 6 = 400$$
$$g^2 + (g+2)^2 = 394$$

We know that g is an integer. Hence, approximate g and $g + 2$ using $g + 1$:

$$(g+1)^2 \approx 394$$
$$(g+1)^2 \approx 197 \approx 14^2 = 196$$

Try:

$$13^2 + 15^2 = 169 + 225 = 394$$

1.2 Number Theory

A. Divisibility

Example 1.20

Eleven members of the Middle School Math Club each paid the same amount for a guest speaker to talk about problem solving at their math club meeting. They paid their guest speaker $\$1A2$. What is the missing digit A of this 3-digit number? (AMC 8 2014/8)

The amount paid must be divisible by 11. Hence:

$$1 - A + 2 = 0 \Rightarrow A = 3$$

Example 1.21

The 7-digit numbers 74A52B1 and 326AB4C are each multiples of 3. Which could be the possible values of C? **(AMC 8 2014/21)**

Apply the test of divisibility of 3 to 74A52B1

$$7+4+A+5+2+B+1=3x$$
$$A+B+19=3x$$
$$A+B=3x-19$$

$$3+2+6+(3x-19)+4+C=3y$$
$$3x-4+C=3y$$
$$C=3y-3x+4$$
$$C=3y-3x+3+1$$

Substitute $3y-3x+3=3(y-x+1)=3z$

$$C=3z+1$$

Hence, the number is 1 more than a multiple of 3:

$$Multiples\ of\ 3\in\{0,3,6,9\}\Rightarrow C\in\{1,4,7\}$$

B. Fractions

Example 1.22

A. Find the integer solutions to $\frac{48}{n}$.

n must be a factor of 48.

$$Factors\ of\ 48\ are\ \{1,2,3,4,6,8,12,16,24,48\}$$

Example 1.23

Let n be a positive integer such that $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{n}$ is an integer. Find the sum of the possible value(s) of n. **(AMC 10B 2002/7, AMC 12B 2002/4, Adapted)**

Note that $n\in\mathbb{N}\Rightarrow\frac{1}{n}<1$

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{n}=\frac{41}{\underbrace{42}_{\substack{Less\\ than\ One}}}+\frac{1}{\underbrace{n}_{\substack{Less\\ than\ One}}}<2$$

Hence, the only possible value that $\frac{41}{42}$ can take is 1, and we can solve

$$\frac{41}{42}+\frac{1}{n}=1\Rightarrow n=42$$

Example 1.24

For how many integers n is $\frac{n}{20-n}$ the square of an integer? **(AMC 10B 2002/16, AMC 12B 2002/12)**

$$N^2=\frac{n}{20-n}$$
$$20N^2-nN^2=n$$
$$20N^2=n+nN^2$$
$$20N^2=n(1+N^2)$$

$$n = \frac{20N^2}{1+N^2}$$

$HCF(N^2, N^2+1)$ is the HCF of two consecutive numbers and is hence 1.
Hence:

$$1 + N^2 \text{ must be a factor of } 20.$$
$$1 + N^2 \in \{1,2,4,5,10,20\}$$
$$N^2 \in \{0,1,3,4,9,19\}$$

Remove the numbers which are not perfect squares:

$$N^2 \in \{0,1,4,9\} \Rightarrow 4 \text{ Integers}$$

Example 1.25

A. For how many positive integer values of N is the expression $\frac{36}{N+2}$ an integer? **(AMC 8 1994/10)**
B. For how many positive integer values of N is the expression $\frac{5005}{N+13}$ an integer?

Part A

$\frac{36}{N+2}$ will be an integer exactly when $N+2$ is a factor of 36.

$$\text{Factors of } 36 = \{1,2,3,4,6,9,12,18,36\} \Rightarrow 9 \text{ Factors}$$
$$36 = 4 \times 9 = 2^{2} \times 3^{2} \Rightarrow No.\, of \text{ Factors} = (2+1)(2+1) = (3)(3) = 9$$

$$N + 2 \in \{1,2,3,4,6,9,12,18,36\}$$
$$N \in \{\mathbf{-1, 0}, 1,2,4,7,10,16,34\}$$
$$N > 0 \Rightarrow N \in \{1,2,4,7,10,16,34\} \Rightarrow 7 \text{ Values}$$

Part B

$\frac{5005}{N+13}$ will be an integer exactly when $N+13$ is a factor of 5005.

$$5005 = 5^1 \times 7^1 \times 11^1 \times 13^1 = (2)(2)(2)(2) = 16$$

Out of these factors, we cannot have factors which are 13, or less, and those factors are:

$$\{1,5,7,11,13\} \Rightarrow 5 \text{ Factors}$$

The final answer is:

$$16 - 5 = 11 \text{ Numbers}$$

Example 1.26

Each of the letters W, X, Y, and Z represents a different integer in the set {1,2,3,4}, but not necessarily in that order. If $\frac{W}{X} - \frac{Y}{Z} = 1$, then the sum of W and Y is: **(AMC 8 1998/10)**

If you have 3 or 4 in the denominator, your LCM will a multiple of 3, or a multiple or 4, or both. And then, the final answer will not be a whole number.

Hence:

$$\frac{3}{1} - \frac{4}{2} = 3 - 2 = 1$$
$$W + Y = 3 + 4 = 7$$

Example 1.27

Letters A, B, C, and D represent four different digits selected from 0,1,2,...,9. If $\frac{A+B}{C+D}$ is an integer that is as large as possible, what is the value of $A+B$? **(AHSME 1998/2)**

To maximize $\frac{A+B}{C+D}$, we maximize the numerator, and minimize the denominator.

$$\frac{A+B}{C+D}=\frac{9+8}{0+1}=\frac{17}{1}=1\in Integer$$

$$A+B=17$$

Example 1.28

$\frac{p}{9}+\frac{q}{10}=r$ where p, q, r are positive integers, and $p=q$. The biggest value of r less than 100 is **(NMTC Sub-Junior/Screening 2011/11)**

Substitute $p=q$

$$\frac{p}{9}+\frac{p}{10}=r\Rightarrow p=\frac{90r}{19}=\frac{2\times 3^2\times 5\times r}{19}$$

Since the numerator does not have a 19, but the fraction must be an integer, r must be a multiple of 19.

$$r\in\{19,38,57,76,95,114\}$$

Largest multiple of 19 less than 100 is 95. Hence,

$$r=95$$

Example 1.29

How many pairs of positive integers (a, b) are there such that a and b have no common factors greater than 1 and $\frac{a}{b}+\frac{14b}{9a}$ is an integer? **(AMC 10B 2007/25)**

Let:

$$\frac{a}{b}+\frac{14b}{9a}=\frac{9a^2+14b^2}{9ab}=m$$

Divisibility by a

$$\underbrace{9a^2}_{\substack{Multiple\\ of\ a}}+14b^2=\underbrace{9abm}_{\substack{Multiple\\ of\ a}}$$

Since two of the terms are a multiple of a, the third term $(14b^2)$ must be a multiple of a.
Further, since $\gcd(a, b)=1$, a must divide 14 (since it cannot divide b), and hence a is a factor of 14:

$$a\in\{1,2,7,14\}$$

Divisibility by b

$$9a^2+\underbrace{14b^2}_{\substack{Multiple\\ of\ b}}=\underbrace{9abm}_{\substack{Multiple\\ of\ b}}$$

Since two of the terms are a multiple of b, the third term $(9a^2)$ must be a multiple of b.
Further, since $\gcd(a, b)=1$, 9 must be a multiple of b, and hence b must be a factor of 9:

$$b\in\{1,3,9\}$$

Casework for b

Check the above values of b.
If $b = 1$:

$$\underbrace{9a^2}_{\substack{Multiple\\ of\ 3}} + \underbrace{14}_{\substack{Not\ a\\ Multiple\ of\ 3}} = \underbrace{9abm}_{\substack{Multiple\\ of\ 3}} \Rightarrow Not\ Valid$$

If $b = 9$:

$$\frac{a}{b} + \frac{14b}{9a} = \underbrace{\frac{a}{9}}_{\substack{Not\ an\\ Integer}} + \underbrace{\frac{14}{a}}_{Integer} \Rightarrow Not\ Valid$$

If $b = 3$:

$$a = 1 \Rightarrow \frac{a}{b} + \frac{14b}{9a} = \frac{1}{3} + \frac{14 \times 3}{9} = \frac{1}{3} + \frac{14}{3} = \frac{15}{3} = 5$$
$$a = 2 \Rightarrow \frac{a}{b} + \frac{14b}{9a} = \frac{2}{3} + \frac{14 \times 3}{9 \times 2} = \frac{2}{3} + \frac{7}{3} = \frac{9}{3} = 3$$
$$a = 7 \Rightarrow \frac{a}{b} + \frac{14b}{9a} = \frac{7}{3} + \frac{14 \times 3}{9 \times 7} = \frac{7}{3} + \frac{2}{3} = \frac{9}{3} = 3$$
$$a = 14 \Rightarrow \frac{a}{b} + \frac{14b}{9a} = \frac{14}{3} + \frac{14 \times 3}{9 \times 14} = \frac{15}{3} = 5$$

$$4\ Solutions\ in\ all$$

Example 1.30

A charity sells 140 benefit tickets for a total of \$2001. Some tickets sell for full price (a whole dollar amount), and the rest sells for half price. How much money is raised by the full-price tickets? **(AMC 12 2001/7, AMC 10 2001/14)**

Let the price of a full-price ticket be

$$p$$

Let

$$No.of\ full\ price\ tickets = F$$
$$\Rightarrow No.of\ half\ price\ tickets = 140 - F$$

Using the condition given in the question:

$$Fp + (140 - F)\frac{p}{2} = 2001$$

Multiply both sides by 2:

$$2Fp + (140 - F)p = 4002$$

Simplify:

$$Fp + 140p = 4002$$

Factor

$$p(F + 140) = 4002$$

$$\underbrace{p}_{Integer} = \frac{4002}{\underbrace{F + 140}_{Integer}} = \frac{2 \times 3 \times 23 \times 29}{\underbrace{F + 140}_{Integer}}$$

$$F + 140\ must\ be\ a\ factor\ of\ 4002$$

Since:

$$F > 0 \Rightarrow F + 140 > 140$$

$$4002 = \underbrace{23}_{p} \times \underbrace{174}_{F+140}$$

C. Number Systems

1.31: Difference between a Two Digit Number and its reverse

The difference of a two-digit number and its reverse is a multiple of 9.

Let the two-digit number be

$$tu = 10t + u$$

Reversing the digits gives us:

$$ut = 10u + t$$

And the difference between the two is:

$$(10t + u) - (10u + t) = 9t - 9u = 9(t - u) \Rightarrow Multiple\ of\ 9$$

For example:

$$32 - 23 = 9$$
$$93 - 39 = 54$$
$$52 - 25 = 27$$

Example 1.32

When Clara totaled her scores, she inadvertently reversed the units digit and the tens digit of one score. By which of the following might her incorrect sum have differed from the correct one?

A. 45
B. 46
C. 47
D. 48
E. 49 (AMC 8 2013/13)

Must be a multiple of nine

$$Option\ A$$

Example 1.33

The two digits in Jack's age are the same as the digits in Bill's age, but in reverse order. In five years, Jack will be twice as old as Bill will be then. What is the difference in their current ages? (AMC 10B 2004/17, AMC 12B 2004/15)

Method I

Suppose Jack's age is the two-digit number

	Current Age	After 5 Years
Jack	$tu = 10t + u$	$10t + u + 5$
Bill	$ut = 10u + t$	$10u + t + 5$

Then, Bill's age is the two-digit number with the digits reversed:

$$10t + u + 5 = 2(10u + t + 5)$$
$$10t + u + 5 = 20u + 2t + 10$$
$$8t - 19u = 5$$

The minimum value of $19u$ is 19.
Hence,

$$t = 1 \Rightarrow 8t = 8\ does\ not\ work$$
$$t = 2 \Rightarrow 8t = 16\ does\ not\ work$$
$$t = 3 \Rightarrow 8t = 24 \Rightarrow 24 - 19(1) = 5 \Rightarrow Works$$

$$t = 3, u = 1 \Rightarrow tu = 31, ut = 13$$

The difference in their ages

$$= tu - ut = 31 - 13 = 18$$

1.34: Sum of a Two Digit Number and its reverse

The sum of a two-digit number and its reverse is a multiple of 11.

Let the two-digit number be $tu = 10t + u$
Reversing the digits gives us $ut = 10u + t$
Sum is:

$$(10t + u) + (10u + t) = 11t + 11u = 11(t + u) \Rightarrow Multiple\ of\ 11$$

For example:

$$52 + 25 = 77$$

Example 1.35

Determine how many two-digit numbers satisfy the following property: when the number is added to the number obtained by reversing its digits, the sum is 132. **(AMC 8 2016/11)**

Let the two-digit number be tu. The sum of the number and its reverse is:

$$11(t + u) = 132$$
$$t + u = 12$$
$$(t, u) = (3,9), (4,8), \dots, (9,3) \Rightarrow 9 - 3 + 1 = 7\ Numbers$$

1.36: Difference of a Three Digit Number and its Reverse

The difference of a three-digit number and its reverse is a multiple of x.

$$htu - uth = (100h + 10t + u) - (100u + 10t + h) = 99h - 99u = 99(h - u)$$

Example 1.37

The sum of the digits of a two-digit number is subtracted from the number. The units digit of the result is 6. How many two-digit numbers have this property? **(AMC 10A 2005/16)**

Let the number be $ab = 10a + b$.

$$10a + b - a - b = X6$$
$$9a = X6$$
$$a = 4$$

The units digit can be any valid digit:

$$\{40,41,42,43,44,45,46,47,48,49\} \Rightarrow 10\ Numbers$$

Example 1.38

How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits? **(AMC 10A 2005/14, AMC 12A 2005/11)**

Let the number be abc.

$$b = \frac{c+a}{2} \Rightarrow c + a = \underbrace{2b}_{Even}$$

Since the RHS is even, the LHS must also be even. We consider cases:

Case I: *a and c are both odd*

$$\underbrace{5}_{\substack{Choices\\ for\ a}} \times \underbrace{5}_{\substack{Choices\\ for\ b}} = 25$$

Case I: *a and c are both even*

$$\underbrace{4}_{\substack{Choices\\ for\ a}} \times \underbrace{5}_{\substack{Choices\\ for\ b}} = 20$$

All of the above result in valid numbers. Hence, the final answer is:

$$25 + 20 = 45$$

Example 1.39

Let $P(n)$ and $S(n)$ denote the product and the sum, respectively, of the digits of the integer n. For example, $P(23) = 6$ and $S(23) = 5$. Suppose N is a two-digit number such that $N = P(N) + S(N)$. What is the units digit of N? **(AMC 10 2001/6, AMC 12 2001/2)**

Note that

$$P(23) = Product\ of\ the\ Digits = 2 \times 3 = 6$$
$$S(23) = Sum\ of\ the\ Digits = 2 + 3 = 5$$

The condition given in the question is that:

$$N = P(N) + S(N)$$

Since N is a two-digit number, let $N = \overline{tu} = 10t + u$. Then

$$10t + u = \underbrace{t \cdot u}_{\substack{Product\ of\ Digits\\ P(N)}} + \underbrace{t + u}_{\substack{Sum\ of\ Digits\\ S(N)}}$$

Subtract $t + u$ from both sides of the equation:

$$9t = t \cdot u$$

Divide by t both sides:

$$9 = u$$

Example 1.40

How many four-digit numbers N have the property that the three-digit number obtained by removing the leftmost digit is one ninth of N? **(AMC 12B 2002/15)**

Let the four-digit number be

$$\overline{abcd}$$

Removing the left most digit gives us a number that is one ninth of the original number

$$\frac{\overline{abcd}}{9} = \overline{bcd}$$

Multiply both sides by 9:

$$\overline{abcd} = 9\overline{bcd}$$

Just $5342 = 5000 + 342$, we can write:

$$1000a + \overline{bcd} = 9\overline{bcd}$$

Subtract $\overline{bcd}$ from both sides:

$$1000a = 8\overline{bcd}$$

Divide both sides by 8:

$$125a = \overline{bcd}$$

Note that since $100 \le \overline{bcd} < 1000$

$$\overline{bcd} = 125,250,375,500,625,750,875$$

$$a = 1,2, \dots ,7$$

The number of numbers of is equal to the values that a can take.

$$1125$$
$$2250$$
$$3375$$

Example 1.41

For how many positive integer values of n are both $\frac{n}{3}$ and $3n$ three-digit whole numbers? **(AMC 8 2008/22)**

Let

$$x = \frac{n}{3} \Rightarrow n = 3x \Rightarrow 3n = 3(3x) = 9x$$

We want x to be a three-digit whole number

$$100 \le x \le 999$$

We also want $3n = 9x$ to be a three-digit whole number

$$100 \le 9x \le 999$$

$$\frac{100}{9} \le x \le 111$$

Combine the conditions:

$$100 \le x \le 111 \Rightarrow x \in \{100,101, \dots ,111\} \Rightarrow 12\ Numbers$$

D. Triangular Numbers

Example 1.42

Show that k is a factor of $T_n - T_{n-k}$.

Use the definition of triangular numbers:

$$= \frac{n(n+1)}{2} - \frac{(n-k)(n-k+1)}{2} = \frac{n^2+n}{2} - \frac{n^2 - kn + n - kn + k^2 - k}{2}$$

Simplify to get:

$$= -\frac{-2kn + k^2 - k}{2} = \frac{k(2n-k+1)}{2}$$

Hence, k is a factor of $T_n - T_{n-k}$.

Example 1.43

If T_n is the n^{th} triangular number, find all solutions to the equation $T_{14} + T_{n-k} = T_n$.

Step I

Rearrange $T_{14} + T_{n-k} = T_n$ to get:

$$\underbrace{T_n - T_{n-k}}_{Expression\ I} = T_{14} = \frac{14(15)}{2} = 7 \times 15 = 105$$

$$Factors\ of\ 105\ are\ \{1,3,5,7,21,35,105\}$$

Step II: Check values for k

$$\frac{k(2n-k+1)}{2}=105$$

If $k=1$:

$$\frac{k(2n-k+1)}{2}=n=105\Rightarrow n-k=105-1=104$$

If $k=2$:

$$T_n-T_{n-k}=\frac{2(2n-2+1)}{2}=2n-1=105\Rightarrow n=53\Rightarrow n-k=51$$

	(14, 33, 36)	**B1**	
8	Two correct linear equations solved for correct n other than $3n-3=105$	**C2**	**C1** for one of $2n-1=105$ and $n=53$ $4n-6=105$ and $n=27.75$ $5n-10=105$ and $n=23$ $6n-15=105$ and $n=20$ $7n-21=105$ and $n=18$
	(14, 104, 105) (14, 51, 53) [(14, 33, 36)] (14, 18, 23)	**B3**	**B1** for each of (14, 104, 105) (14, 51, 53) (14, 18, 23)

1.3 Multiples and Remainders

Example 1.44

Find the smallest value of $a+b+c$ given that they are positive integers satisfying

$$\frac{c}{a+b}=\frac{1}{3},\qquad \frac{a}{b+c}=\frac{3}{8}$$

Eliminate fractions in the second equation:

$$8a=3b+3c$$

Substitute $\frac{c}{a+b}=\frac{1}{3}\Rightarrow 3c=a+b\Rightarrow a=3c-b$:

$$8(3c-b)=3b+3c\Rightarrow 21c=11b$$

Substitute $c=11, b=21$ in $3c=a+b$:

$$3(11)=a+21\Rightarrow a=33-21=12$$

$$a=12, b=21, c=11$$
$$a+b+c=12+21+11=44$$

Example 1.45

If a, b, c are digits in the decimal number system, find the sum of all values of the product $a \times b \times c$ such that the following equation is satisfied:

$$\frac{abc + bc - a00}{cba + cb - a} = \frac{10}{187}$$

Note: abc represents a three-digit number, not the product $a \times b \times c$. This is true for all the expressions above.

For some natural number K:

$$abc + bc - \ a00 = 10K$$
$$2bc \ = 10K$$
$$bc = 5K$$

$$cba + cb - a = 187K$$
$$10cb + cb = 187K$$
$$cb = 17K$$
$$cb \in \{17,34,51,68,85\}$$

Combining the conditions for bc and cb, we get:

$$cb \ = \ 51, bc \ = 15, bc = 5$$

a can take values from 1 to 9.

$$abc \in \{5,10, \dots ,45\}$$
$$5 + 10 + \cdots + 45 = 5(1 + 2 + \cdots + 9) = 5(45) = 225$$

Example 1.46

Exactly 75% of the members of a club fill exactly 5/6 of the chairs in the room. The smallest possible number of people in the club is:

Let the number of people in the club be P, and the number of chairs be C.

$$\frac{3}{4}P = \frac{5}{6}C$$

Since P must be a multiple of 4, let $P = 4p$. Similarly, C must be a multiple of C. Let $C = 6c$. Then:

$$\frac{3}{4}(4p) = \frac{5}{6}(6c)$$
$$3p = 5c$$

The smallest values that work are:

$$(p, c) = (5,3) \Rightarrow (P, C) = (4p, 6c) = (20,18)$$

Check:

$$People = 75\% \ of \ 20 = 15, \qquad Occupied\ Chairs = \frac{5}{6} of \ 18 = 15$$

Example 1.47

A farmer bought some pigs, sheep and cows, being 20 in all. The cost of each pig was Rs. 5, of each sheep was Rs. 3, and of each cow was Rs. 17. He spent Rs. 198 in all. How many animals of each kind did he buy? **(Pradnya 5, 1987/13)**

The total number of animals is 20:

$$\underbrace{p}_{Pigs} + \underbrace{s}_{Sheep} + \underbrace{c}_{Cows} = 20 \Rightarrow \underbrace{c = 20 - p - s}_{\textbf{Equation I}}$$

The total cost of the animals is 198:

$$\underbrace{5p + 3s + 17c = 198}_{Equation\ II}$$

Substitute the value of c from Equation I in Equation II:

$$5p + 3s + 17(20 - p - s) = 198$$
$$5p + 3s + 340 - 17p - 17s = 198$$
$$-12p - 14s = -142$$

Divide both sides by -2:

$$6p + 7s = 71 \Rightarrow 6p = 71 - 7s \Rightarrow p = \frac{71 - 7s}{6} = \frac{71}{6} - \frac{7s}{6} = 11\frac{5}{6} - \frac{7s}{6}$$
$$If\ s = 1 \Rightarrow p = \frac{64}{6}$$
$$If\ s = 2 \Rightarrow p = \frac{57}{6}$$
$$If\ s = 3 \Rightarrow p = \frac{50}{6}$$
$$If\ s = 4 \Rightarrow p = \frac{43}{6}$$
$$If\ s = 5 \Rightarrow p = \frac{36}{6} = 6 \Rightarrow (s, p) = (5,6)$$

$$c = 20 - p - s = 20 - 6 - 5 = 9$$

And we can check:

$$5p + 3s + 17c = 5(6) + 3(5) + 17(9) = 30 + 15 + 153 = 198$$

Example 1.48

Mr. Likhite bought some books last month. He arranged them on his bookshelf in such a way that every third book was blue, every fourth red, and every fifth green, and all others yellow. One day, he took two green books from the shelf. Finding that he had already read one of them, he kept it in its place on the shelf. He observed that he had read exactly two-thirds of the books that were then on the shelf. How many books were yet to be read? **(Pradnya 5, 1988/6)**

Note: A book is *exactly* one color.

Since there are two green book, the total number of books must be at least

$$2 \times 5 = 10 \Rightarrow b \geq 10$$

Also, since every third book is blue, every fourth book is red, and every fifth book is green, if we arrange the books, we find the $Book\ 12 = LCM(3,4)$ is both blue and red (which is not possible).

1	2	3	4	5	6	7	8	9	10	11	12
		Blue	Red	Green	Blue		Red		Green		Red Blue

Hence,

$$b \leq 11$$

Also, $b - 1$ is divisible by 3. Hence,

$$b = 9 \Rightarrow No. of\ Unread\ books\ on\ shelf = \frac{9}{3} = 3$$

Total number of unread books

$$= \underbrace{3}_{Shelf} + \underbrace{1}_{Hand} = 4$$

Example 1.49

Atul and Vilas play marbles almost every day. It was decided among themselves that Atul being a better player, he should pay Vilas 8 marbles if Vilas wins, but collect only 5 marbles if Vilas loses. They keep a record of their games, and settle their account at the end of the month. One month, at the end of the month, neither owed anything to the other, and they had played G games. Find the sum of the possible two-digit values of G. **(Pradnya 5, 1991/13, Adapted)**

Let the number of games won by Vilas be w, and the number of games lost be l. Then, the number of marbles that Atul gets:

$$\underbrace{8w}_{\substack{Marbles\\Wons}} - \underbrace{5l}_{\substack{Marbles\\Lost}} = 0 \Rightarrow \underbrace{8w}_{\substack{Multiple\\of\ 5}} = \underbrace{5l}_{\substack{Multiple\\of\ 8}}$$

We can solve the equation:

$$(w, l) = (5,8) \Rightarrow 8w = 5l = 40 = LCM(5,8)$$
$$G = k(5+8) = 13k$$

Two digit values of k will be

$$1(13) + 2(13) + \cdots + 7(13) = 13(1 + 2 + \cdots + 7) = 13\left(\frac{7 \times 8}{2}\right) = 13 \times 28 = 364$$

Example 1.50

A four-digit number is exactly divisible by 7, and the quotient obtained is a three-digit number. If the digit in the hundred's place of the four-digit number is dropped, the number formed is the same as the quotient. Find the four-digit number. **(Pradnya 5, 1992/14)**

Let the four-digit number be $\overline{abcd}$.

$$\frac{abcd}{7} = acd \Rightarrow abcd = 7(acd)$$

Expand:

$$1000a + 100b + 10c + d = 700a + 70c + 7d$$
$$300a + 100b - 60c - 6d = 0$$
$$150a + 50b - 30c - 3d = 0$$
$$150a - 30c = 3d - 50b$$
$$\underbrace{30(5a - c)}_{Multiple\ of\ 30} = \underbrace{3d - 50b}_{Multiple\ of\ 30}$$

$$3d - 50b = 30k$$
$$3d = 30k + 50b$$

Since d and b are digits, the only solution to the equation is:

$$b = d = k = 0$$

$$30(5a - c) = 0$$
$$5a - c = 0$$
$$5a = c$$
$$a = 1, c = 5$$

$$1050$$

Example 1.51

Out of the students in a class, one-third are under 12 years of age. Half of the students in the class are under 13 years of age, and the number of students under 11 years of age is 6. If the number of students between the ages of 12 and 13 equals the number of students between the ages of 11 and 12, find the number of students in the class. **(Pradnya 5, 1992/10)**

$$\text{Under } 12 \text{ years of age} = \frac{s}{3}$$
$$\text{Under } 13 \text{ years of age} = \frac{s}{2}$$
$$\text{No. of students under } 11 \text{ years of age} = 6$$

$$\frac{s}{2} - \frac{s}{3} = \frac{s}{3} - 6$$
$$\frac{s}{2} - \frac{2s}{3} = -6$$
$$\frac{3s}{6} - \frac{4s}{6} = -6$$
$$-\frac{s}{6} = -6$$
$$s = 36$$

Example 1.52

A newspaper consists of a certain number of sheets, each sheet having four page numbers. My friend and I distribute sheets of a newspaper among ourselves so that each one of us received one sheet. The sheet which I received bears page numbers 8 and 26. Find the total number of pages of that newspaper, and the number of my friends. **(Pradnya 5, 1993/6)**

In a newspaper with n pages:

$$\text{First Page} = 1, \text{Last Page} = n$$
$$\text{Second Page} = 2, \text{Second Last Page} = n - 1$$

$$\text{Odd Numbered Pages are on Right}$$
$$\text{Even Numbered Pages are on Left}$$

Hence, the sheet had page numbers:

$$7,8,25,26$$

Total Number of Pages

$$= 26 + 6 = 32$$

Total Number of Friends

$$= \frac{32}{4} - 1 = 8 - 1 = 7$$

$$1,2,31,32,$$
$$3,4,29,30$$
$$5,6,27,28$$
$$7,\mathbf{8},25,\mathbf{26}$$

Example 1.53

A. I have a two-digit number in mind such that the sum of 1 and the number is divisible by 2, the sum of 2 and the number is divisible by 3, the sum of 3 and the number is divisible by 4, and the sum of 4 and the number is divisible by 5. Find the number. **(Pradnya 5, 1987/9)**

B. Find all such numbers. Remove the two-digit condition.

Part A

Let the number be x.

$$x+1=2a\Rightarrow x-1=2a-2=\underbrace{2(a-1)}_{\text{Multiple of }2}$$
$$x+2=3b\Rightarrow x-1=3b-3=\underbrace{3(b-3)}_{\text{Multiple of }3}$$
$$x+3=4c\Rightarrow x-1=4c-4=\underbrace{4(c-4)}_{\text{Multiple of }4}$$
$$x+4=5d\Rightarrow x-1=\underbrace{5(d-1)}_{\text{Multiple of }5}$$

$x-1$ is a multiple of 2, 3, 4 and 5. The smallest such positive number is:

$$LCM(2,3,4,5)=60$$
$$x-1=60\Rightarrow x=61$$

Part B

$$x-1=60n\Rightarrow x=60n+1,\qquad n\in\mathbb{Z}$$

Suppose $n=-1$:

$$x=60(-1)+1=-60+1=-59$$
$$x+1=-59+1=-58$$
$$x+2=-59+2=-57$$
$$x+3=-59+3=-56$$
$$x+4=-59+4=-55$$

Example 1.54

Find the smallest positive number which when divided by 3 leaves a remainder of 1, when divided by 4 leaves a remainder of 2 and when divided by 5 leaves a remainder of 3. **(Pradnya 5, 1988/1)**

$$N=3x+1=4y+2=5z+3$$

Add 2 throughout:

$$N+2=3x+3=4y+4=5z+5=LCM(3,4,5)=60$$
$$N=58$$

Example 1.55

A. Find the smallest number which when divided by 3, 7 and 11 leaves remainders 1, 6 and 5 respectively. **(Pradnya 5, 1991/8, Adapted)**

B. What is the range in which the answer for the above must lie (without doing the calculations to find the answer).

Part A

$$\underbrace{3x+1}_{\text{Condition }I}=\underbrace{7y+6}_{\text{Condition }II}=\underbrace{11z+5}_{\text{Condition }III}$$

The numbers which condition III are:

$$\{16,27,38,49,60,71,82,93,104,115,126,137,148,159,170,\mathbf{181},\dots\}$$

The numbers which meet Condition I:

$$\{4,7,10,\mathbf{13},16,19,\dots\}$$

13 is the smallest number that meets Conditions I and II. We want to find the next number that meets these conditions.

- The remainder when divided by 3 should not change. The only possible numbers that can be added are multiples of 3.
- The remainder when divided by 7 should not change. The only possible numbers that can be added are multiples of 7.
- To meet both the conditions, we need a number which is a multiple of both 3 and 7, which $LCM(3,7)=21$

$$\{13,34,55,21,97,118,139,160,\mathbf{181}\}$$

Part B

$$From\ 1\ to\ LCM(3,7,11)=From\ 1\ to\ 231$$

Example 1.56

Find the greatest number such that the same remainder is obtained when 21, 29, and 33 are divided by it.

$$29 - 21 = 8$$
$$33 - 29 = 4$$
$$33 - 21 = 12$$

$$HCF = (8,4,12) = 4$$

$$21 = 5(4) + 1 = an + r$$
$$29 = 7(4) + 1 = bn + r$$
$$33 = 8(4) + 1 = cn + r$$

$$29 - 21 = 8 = 2(4) = n(b - a)$$
$$33 - 29 = 4 = 1(4) = n(c - b)$$
$$33 - 21 = 12 = 3(4) = n(c - a)$$

Example 1.57

Three numbers are 494, then 726 and finally 1045. Find the greatest number such that the same remainder is obtained when the three numbers are divided by it. **(Pradnya 5, 1986/5)**

Let the number to be found is n. Let the common remainder be r.

$$\underbrace{494 = an + r}_{Equation\ I}, \quad \underbrace{726 = bn + r}_{Equation\ II}, \quad \underbrace{1045 = cn + r}_{Equation\ III}$$

Subtract the equations pairwise from each other:

$$\underbrace{232 = n(b - a)}_{Eq\ I - Eq\ II}, \quad \underbrace{319 = n(c - b)}_{Eq\ II - Eq\ I}, \quad \underbrace{551 = n(c - a)}_{Eq\ III - Eq\ I}$$

The RHS factors into two different numbers. Since both the LHS and RHS are positive integers, the LHS must also factor into the same numbers.
Hence, n must be a factor of 232, 319 and 551:

$$HCF(232,319,551) = 29$$

Example 1.58

Some friends went to a restaurant and decided to eat the same dishes. Each one of them was expected to pay his own bill. But three of them had no money to pay the bill, so each of the others paid Rs. 3 more than his own bill. If the total was Rs. 180, how many friends were there? **(Pradnya 5, 1993/13)**

$$\underbrace{Total\ Cost}_{=180} = \underbrace{No.of\ Friends}_{Natural\ Number} \times Cost\ per\ Friend$$

$$Factor\ Pairs\ of\ 180 = (1,180), (2,90), (3,60), (4,45), (5,36), (6,30), (10,18), (12,15)$$

$$180 = \underbrace{15}_{Friends} \times \underbrace{12}_{\substack{Cost\\ per\ Friend}} = \underbrace{12}_{Friends} \times \underbrace{15}_{\substack{Cost\\ per\ Friend}}$$

$$fc = (f - 3)(c + 3)$$
$$fc = fc + 3f - 3c - 9$$
$$0 = 3f - 3c - 9$$

$$f - c = 3$$
$$c = f - 3$$

$$fc = 180$$
$$f(f-3) = 180$$
$$f^2 - 3f - 180 = 0$$
$$(f-12)(f-15) = 0$$
$$f \in \{12,15\}$$
$$f = 15$$

Example 1.59

Suma said to her father, "A three-digit number is formed by placing my age and the age of two of my friends in a row. If that number is divided by the sum of our ages, the answer is your present age". Her father told her immediately the ages of her two friends. If the age of her father is 32 years, find the ages of Suma and her friends. (Pradnya 5, 1993/11)

$$\frac{abc}{a+b+c} = 32$$

The number is divisible by the sum of its digits.

A number with sum of digits 3 will be divisible by its sum of digits.

$$32 \times 3 = 96 < 100$$

A number with sum of digits 9 will be divisible by its sum of digits.

$$32 \times 9 = 288 \Rightarrow \frac{288}{18} = 16$$
$$32 \times 18 = 576 \Rightarrow \frac{576}{18} = 32$$

Example 1.60

A man encashed a cheque of Rs. 200. He received notes of one rupees, two rupee and five rupees. If the number of two-rupee notes is ten times the number of one-rupee notes, find the number of notes of each denomination received by him. (Pradnya 5, 1993/14)

The total amount received by him:

$$\underbrace{o}_{\substack{One\ Rupee\\ Notes}} + \underbrace{2t}_{\substack{Two\ Rupee\\ Notes}} + \underbrace{5f}_{\substack{Five\ Rupee\\ Notes}} = 200$$

The number of two-rupee notes is ten times the number of one-rupee notes. Substitute $t = 10o$:

$$o + 20o + 5f = 200$$
$$21o + \underbrace{5f}_{\substack{Multiple\\ of\ 5}} = \underbrace{200}_{\substack{Multiple\\ of\ 5}}$$
$$21 \times 5 = 105$$
$$21 \times 10 = 210 > 200 \Rightarrow Not\ Valid$$

$$o = 5$$
$$t = 10o = 50$$
$$f = \frac{200 - 21o}{5} = \frac{200 - 105}{5} = \frac{95}{5} = 19$$

Example 1.61

There are some rectangles which are not squares. The lengths of the sides of the rectangles are integers. The area of each of the rectangles is 144 cm^2.

A. What is the number of rectangles of this type?
B. What are the lengths of the sides of the rectangle whose perimeter is the greatest? Is the least? **(Pradnya 5, 1994/12)**

Part A

We assume here that

$$length > breadth \Rightarrow Rectangles = (b, l)$$

$$Area = 144 = b \times l$$

We need the factor pairs of 144 with $b < l$:

$$(1,144), (2,72), (3,48), (4,36), (6,24), (8,18), (9,16) \Rightarrow 7\ pairs$$

Part B

$$Greatest = 2(1 + 144) = 2(145) = 290\ cm$$
$$Least = 2(9 + 16) = 2(25) = 50\ cm$$

Example 1.62

The telephone number of my friend is seven digits. The number formed by the first and the second digit, the number formed by the fourth and the fifth digit, and the number formed by the sixth and the seventh digit are all prime numbers. The sum of the digits in all these three numbers is the number formed by the second and the third digit. The three prime numbers are in increasing order, and the difference between any two consecutive numbers is 18. Find all such telephone numbers. **(Pradnya 5, 1995/4)**

We do not need to check prime numbers ending with 7 since

$$7 + 18 = 25 \Rightarrow Last\ digit\ 5 \Rightarrow Not\ Prime$$

We do not need to check prime numbers ending with 9 since

$$29 + 36 = 45 \Rightarrow Last\ digit\ 5 \Rightarrow Not\ Prime$$

$1st\ and\ 2nd$	$4th\ and\ 5th$	$6th\ and\ 7th$	$Sum\ of\ Digits$	$2nd\ Digit$	
11	29	47	24	**1**	
13	31	**49**			
23	41	59	24	3	
29	47	65			
31	49				
41	59	77			
43	61	79	30		4306179
53	71	89	33		5337189
61	79	97	39	1	

Example 1.63

Pedhas were bought for Anita's birthday. If three pedhas were given to each of her friends, she would have got only two pedhas. If two pedhas were given to each one of them, including Anita, 14 pedhas would have been left undistributed. How many pedhas were there? **(Pradnya 5, 1995/5)**

Let the number of pedhas be p. Let the number of friends be f.

$$p = 3f + 2 = 2(f + 1) + 14$$
$$3f + 2 = 2f + 2 + 14$$
$$f = 14$$

$$p = 3f + 2 = 42 + 2 = 44$$

Example 1.64

Suresh had some coins in his pocket. Out of them, 3 were not rupee coins, 3 were not 50 paise coins, 3 were not 25 paise coins, and 3 were not 10 paise coins. How much was the least amount Suresh could have? **(Pradnya 5, 1995/8)**

Note: Consider only 10 paise, 25 paise, 50 paise and rupee coins.

$$0.1 + 0.25 + 0.5 + 1 = 1.85$$

A. Multi-Step Linear Equations

Example 1.65

Three people with the help of a monkey took down some coconuts from a tall tree. One of them awoke at midnight, kept one coconut aside for the monkey, divided the remaining into three equal parts, took one part for himself and made a heap of the remaining coconuts. After some time, a second man awoke, kept one coconut aside for the monkey, divided the remaining into three equal parts, took one part for himself, and made a heap of the remaining coconuts. After some time, the third person awoke, and acted in the same way as the other two. In the morning they woke up, gave one coconut to the monkey, and distributed the remaining equally among themselves. If each of them got 7 coconuts, how many coconuts did they take down in all? **(Pradnya 5, 1995/16, Adapted)**

Start with X coconuts. Give one coconut to the monkey:

$$X - 1$$

You keep one-third of the coconuts. Two-thirds are left:

$$(X - 1) \times \frac{2}{3}$$

Let

$$(X - 1) \times \frac{2}{3} = Y \Rightarrow X - 1 = \frac{3}{2}Y \Rightarrow X = \frac{3}{2}Y + 1$$

Before the morning division, there were:

$$7 \times 3 + 1 = 21 + 1 = 22$$

Just before the third person woke up, there were:

$$22 \times \frac{3}{2} + 1 = 33 + 1 = 34$$

Just before the second person woke up, there were:

$$34 \times \frac{3}{2} + 1 = 51 + 1 = 52$$

Before the first person woke up, there were:

$$52 \times \frac{3}{2} + 1 = 78 + 1 = 79$$

B. Multiples

Example 1.66

Multiples of 3 are deleted from the numbers 1 to 25 inclusive. Pairs of remaining numbers are formed such that the sum of the numbers in each pair will be a multiple of 3. How many such unordered pairs can be formed at the most? (Pradnya 5, 1995/15)

If we consider the remainders when a number is divided by three, there are only three possibilities:

$$3n, 3n+1, 3n+2$$

$$\underbrace{3n+1}_{\substack{One\ more\ than \\ a\ multiple\ of\ 3}} + \underbrace{3n+2}_{\substack{Two\ more\ than \\ a\ multiple\ of\ 3}} = 6n+3$$

To make a multiple of three, I need one number which is one more than a multiple, and another number which is two more than a multiple of three.

We divide the numbers into two sets.

$$3n+1 \in \{1,4,7,10,13,16,19,22,25\} \Rightarrow 9\ Numbers$$
$$3n+2 \in \{2,5,8,11,14,17,21,23\} \Rightarrow 8\ Numbers$$

$$\underbrace{9}_{3n+1} \times \underbrace{8}_{3n+2} = 72\ Pairs$$

Example 1.67

The students in Mr. Neatkin's class took a penmanship test. Two-thirds of the boys and 3/4 of the girls passed the test, and an equal number of boys and girls passed the test. What is the minimum possible number of students in the class? (AMC 8 2008/20)

Let

$$b = No.of\ Boys, g = No.of\ girls$$

Then,

$$\frac{2}{3}b = \frac{3}{4}g \Rightarrow 8b = 9g \Rightarrow b = 9, g = 8 \Rightarrow b + g = 9 + 8 = 17$$

Example 1.68

How many different combinations of 5 bills and 2 bills can be used to make a total of 17? Order does not matter in this problem. (AMC 8 2002/2)

$$5a + \underbrace{2b}_{Even} = \underbrace{17}_{Odd} \Rightarrow 5a\ must\ be\ odd \Rightarrow a \in \{1,3\}, a, b \in \{(1,6), (3,1)\} \Rightarrow Two\ combinations$$

Example 1.69

One morning each member of Angela's family drank an 8-ounce mixture of coffee with milk. The amounts of coffee and milk varied from cup to cup, but were never zero. Angela drank a quarter of the total amount of milk and a sixth of the total amount of coffee. How many people are in the family?[2] (AMC 10 2000/22)

Note that the total coffee and the total milk must add up to the total quantity that the entire family drank:

$$\underbrace{c}_{\substack{Total \\ Coffee}} + \underbrace{m}_{\substack{Total \\ Milk}} = 8\underbrace{p}_{People} \Rightarrow \underbrace{2c + 2m = 16p}_{Equation\ I}$$

The total mixture that Angela drinks is 8 ounces. Hence,

[2] You can see a solution for this question based on Inequalities in the Algebra Note on Absolute Value and Inequalities.

$$\frac{c}{6}+\frac{m}{4}=8$$
$$\frac{2c+3m}{12}=8$$
$$2c+3m=96$$
$$\underbrace{2c+2m+m=96}_{Equation\ II}$$

Substitute $2c+2m=16p$ (from Equation I):

$$16p+m=96$$

96 and $16p$ are both multiples of 96. Hence, m is a multiple of 16.
Note that since m is positive:

$$m=16k, \qquad k\in\mathbb{Z}^+$$

Substitute the information that m is a multiple into Equation II:

$$2c+48k=96\Rightarrow c=48-24k$$

Also, the amount of coffee which the family drank is a positive quantity:

$$c>0\Rightarrow 48-24k>0\Rightarrow k<2$$

Hence:

$$k=1\Rightarrow m=16k=16\Rightarrow c=24$$

Finally:

$$p=\frac{c+m}{8}=\frac{16+24}{8}=\frac{40}{8}=5$$

C. Remainders

Example 1.70

Laila took five math tests, each worth a maximum of 100 points. Laila's score on each test was an integer between 0 and 100, inclusive. Laila received the same score on the first four tests, and she received a higher score on the last test. Her average score on the five tests was 82. How many values are possible for Laila's score on the last test? (AMC 8 2018/13)

Suppose the scores that Laila got are

$$t,t,t,t,x \text{ with } x>t$$

$$\frac{t+t+t+t+x}{5}=82$$
$$4t+x=410$$
$$t=\frac{410-x}{4}=\frac{408}{4}+\frac{2-x}{4}$$

x must be 2 more than a multiple of 4. The possible values are:

$$x\in\{98,94,90,86\}\Rightarrow 4\ Values$$

Example 1.71

The list of integers $4,4,x,y,13$ has been arranged from least to greatest. How many different possible ordered pairs (x, y) are there so that the average (mean) of these 5 integers is itself an integer? **(CEMC Gauss, Grade 8/2015/23)**

$$\frac{4+4+x+y+13}{5}=\frac{21+x+y}{5}=\frac{20}{5}+\frac{1+x+y}{5}$$

$$4 \le x, y \le 13$$

$$x+y=9 \Rightarrow (x,y)=(4,5)$$
$$x+y=14 \Rightarrow (x,y)=(4,10)(5,9)(6,8)(7,7)$$
$$x+y=19 \Rightarrow (6,13)(7,12)(8,11)(9,10)$$
$$x+y=24 \Rightarrow (11,13)(12,12)$$

Example 1.72

Find the sum of all prime numbers between 1 and 100 that are simultaneously 1 greater than a multiple of 4 and 1 less than a multiple of 5. **(AHSME 1999/4)**

The numbers which are one less than a multiple of 5 are

$$4,9,14,19,\ldots,94,99$$

Since all primes other than 2 are odd, we only need to consider:

$$9,19,29,39,49,59,69,79,89,99$$

From the above, the primes are:

$$19,29,59,79,89$$

And from the above, the numbers which are one more than a multiple of 4 are:

$$29+89=118$$

Example 1.73

How many positive three-digit integers have a remainder of 2 when divided by 6, a remainder of 5 when divided by 9, and a remainder of 7 when divided by 11? **(AMC 8 2018/21)**

$$N=6x+2=9y+5=11z+7$$

Add 4 throughout:

$$N+4=6x+6=9y+9=11z+11$$

Hence, $N+4$ must be a multiple of:

$$LCM(6,9,11)=198$$

Hence, the number must satisfy:

$$N+4=198k \Rightarrow N=198k-4$$

The values of k that work are:

$$k \in \{1,2,3,4,5\}$$

Example 1.74

A. The smallest number greater than 2 that leaves a remainder of 2 when divided by 3, 4, 5, or 6 is **(AMC 8 2012/15)**
B. The smallest number greater than 2 that leaves a remainder of 2 when divided by 4, 6, 8 and 9, but a remainder of 3 when divided by 5 is:

Part A

$$X = 3a + 2 = 4b + 2 = 5c + 2 = 6c + 2$$

Subtract 2:

$$X - 2 = 3a = 4b = 5c = 6c = LCM(3,4,5,6) = 60$$

Add 2:

$$X = 62$$

Part B

$$X = 4a + 2 = 6b + 2 = 8c + 2 = 9c + 2$$

Subtract 2:

$$X = 4a = 6b = 8c = 9c = LCM(4,6,8,9) = 72$$
$$X + 2 = 74 \Rightarrow 70 + 4$$
$$2X + 2 = 144 + 2 = 146 = 145 + 1$$
$$3X + 2 = 216 + 2 = 218 = 215 + 3 \Rightarrow Works$$

The common multiples are:

$$CM = 72,144,216, \ldots$$

Example 1.75

A. A box contains gold coins. If the coins are equally divided among six people, four coins are left over. If the coins are equally divided among five people, three coins are left over. If the box holds the smallest number of coins that meets these two conditions, how many coins are left when equally divided among seven people? **(AMC 8 2006/23)**

B. What is the smallest number of gold coins that leaves three coins when divided among five people, four coins when divided among six people and five coins when divided among seven people?

Let the number of gold coins be G.

Part A

Since g is 4 more than a multiple of 6, and 3 more than a multiple of 5, we must have

$$g = 6x + 4 = 5y + 3$$

Add 2 throughout:

$$g + 2 = \underbrace{6x + 6}_{\substack{Multiple\\ of\ 6}} = \underbrace{5y + 5}_{\substack{Multiple\\ of\ 5}} = LCM(5,6) = 30$$

Subtract 2:

$$g = 30 - 2 = 28 \Rightarrow \frac{28}{7} has\ remainder\ 0$$

Part B

$$G = 5y + 3 = 6x + 4 = 7z + 5$$

Add 2 throughout:

$$G + 2 = \underbrace{5y + 5}_{\substack{Multiple\\ of\ 5}} = \underbrace{6x + 6}_{\substack{Multiple\\ of\ 6}} = \underbrace{7z + 7}_{\substack{Multiple\\ of\ 7}} = LCM(5,6,7) = 210$$

Subtract 2:

$$G = 210 - 2 = 208$$

Example 1.76

In the United States, coins have the following thicknesses: penny, 1.55 mm; nickel, 1.95 mm; dime, 1.35 mm;

quarter, 1.75 mm. If a stack of these coins is exactly 14 mm high, how many coins are in the stack? **(AMC 10B 2004/13)**

Method I

Consider one coin of each type:

$$155 + 195 + 135 + 175 = 660$$

Two coins of each type then gives us:

$$2(155) + 2(195) + 2(135) + 2(175) = 1320$$

Replace two pennies with two nickels:

$$4(195) + 2(135) + 2(175) = 1400$$

Method II

$$155p + 195n + 135d + 175q = 1400$$

Divide both sides by 5:

$$31p + 39n + 27d + 35q = \underbrace{280}_{\substack{Multiple \\ of\ 4}}$$

$$32p - p + 40n - n + 28d - d + 36q - q = \underbrace{280}_{\substack{Multiple \\ of\ 4}}$$

$$\underbrace{4(8p + 7d + 10n + 9q)}_{\substack{Multiple \\ of\ 4}} - (p + d + n + q) = \underbrace{280}_{\substack{Multiple \\ of\ 4}}$$

Hence, the number of coins must be a multiple of 4.

Consider using 4 coins. If we use the coins with maximum thickness, we get

$$195 \times 4 < 200 \times 4 = 800 < 1400$$

Which is not enough.

Consider using 12 coins. If we use the coins with minimum thickness, we get

$$135 \times 12 = 1620 > 1400$$

Which is too much.

Hence, the number of coins must be:

$$8$$

Example 1.77

Show that $a^2 + b^2 = 4k + 3, a \in \mathbb{Z}, b \in \mathbb{Z}, k \in \mathbb{Z}$ does not have solutions.

Consider the RHS when divided by 4:

$$RHS = 4k + 3 = \underbrace{4k}_{\substack{Multiple \\ of\ 4}} + \underbrace{3}_{Remainder}$$

Consider the RHS when divided by 4.

$$\underbrace{a^2}_{\substack{Square\ of \\ an\ Integer}} + \underbrace{b^2}_{\substack{Square\ of \\ an\ Integer}}$$

Both terms on the RHS are squares of integers. Hence, consider the square of an integer when divided by 4. There are only two possible cases:

$$x\ is\ even \Rightarrow x = 2n, n \in \mathbb{Z} \Rightarrow x^2 = 4n^2 \Rightarrow Remainder = 0\ when\ divided\ by\ 4$$

$$x\ is\ odd \Rightarrow x = 2n + 1, n \in \mathbb{Z} \Rightarrow x^2 = 4n^2 + 4n + 1 \Rightarrow Remainder = 1\ when\ divided\ by\ 4$$

Note that:

$$\underbrace{a^2}_{Remainder:0\ OR\ 1} + \underbrace{b^2}_{Remainder:0\ OR\ 1} = \underbrace{4k}_{Remainder:0} + \underbrace{3}_{Remainder:3}$$

$$Remainder(RHS) = 3$$
$$Maximum\ Value\ of\ Remainder(LHS) = 1 + 1 = 2$$

Since:

$$3 \neq 2 \Rightarrow No\ Solutions$$

Example 1.78

In the land of Binary, the unit of currency is called Ben and currency notes are available in denominations $1, 2, 2^2, 2^3, \ldots$ Bens. The rules of the Government of Binary stipulate that one cannot us more than two notes of any one denomination in any transaction. For example, one can give a change for 2 Bens in two ways: 2 one Ben notes, or 1 two Ben note. For 5 Ben, one can give 1 one Ben note, and 1 four Ben Note or 1 one Ben note and 2 two Ben notes. Using 5 one Ben notes or 3 one Ben notes and 1 two Ben notes for a 5 Ben transaction is prohibited. Find the number of ways in which one can give change for 100 *Bens*, following the rules of the Government. **(IOQM 2023/26)**

Step I: Large Notes

If we allow maximum of one note, then there is exactly one way of giving the change:

$$64 + 32 + \mathbf{4} = 100$$

If we allow two notes, we can convert the 32 *Ben* note as follows:

$$64 + 16(2) + \mathbf{4} = 100$$
$$64 + 16(1) + 8(2) + \mathbf{4} = 100$$
$$64 + 16(1) + 8(1) + \mathbf{12} = 100$$

If we do not use a 64 *Ben* note:

$$2(32) + 2(16) + \mathbf{4} = 100$$
$$2(32) + 1(16) + 2(8) + \mathbf{4} = 100$$
$$2(32) + 1(16) + 1(8) + \mathbf{12} = 100$$

If we do not use a 64 *Ben* note, and also do not use a 32 *Ben* note, the maximum that can be achieved is:

$$1(2) + 2(2) + 4(2) + 8(2) + 16(2) = 2 + 4 + 8 + 16 + 32 = 62$$

Step II: Small Notes

For denominations of 4 and less:

$$4 = 4(1) = 2(2) = 1(2) + 2(1) \Rightarrow 3\ Ways$$
$$12 = 4(2) + 2(2) = 4(2) + 2(1) + 1(2) \Rightarrow 2\ Ways$$

Combine

Combining Step I and II, 4 occurs 5 times, and 12 occurs 2 times. Hence, the final number of ways is:

$$3 \times 5 + 2 \times 2 = 15 + 4 = 19\ Ways$$

D. Squares of Terms

Example 1.79

Olivia runs a soup catering business that uses 3 special ingredients: xylaria mushrooms, yuzu and zucchini. Each of her recipes calls for an integer amount of each ingredient, denoted as x xylaria mushrooms, y yuzu and z zucchini.

(a) Olivia's recipe for Pythagorean Potage calls for her to combine the 3 ingredients such that $x^2 = y^2 + z^2$. This recipe makes xyz servings of soup. Show that she always makes an even number of servings.

Proof by contradiction. Assume, to the contrary, that

$$xyz \text{ is odd} \Rightarrow x, y, z \text{ are each odd} \Rightarrow x^2, y^2, z^2 \text{ are each odd}$$

Where in the last step, we used the property that If n is odd, then n^2 is also odd, proved below:

$$n = 2k + 1 \Rightarrow n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

$$x^2 = y^2 + z^2$$
$$LHS = x^2 = Odd$$
$$RHS = y^2 + z^2 = Odd + Odd = Even$$
$$Contradiction$$

Hence, the original assumption is incorrect. Hence:

$$xyz \text{ is not odd} \Rightarrow xyz \text{ is even}$$

$$n \equiv 1 \Rightarrow n^2 \equiv 1$$

Example 1.80

1. [8 pts] Olivia's Outstanding Omnibus of Organic Offerings

Olivia runs a soup catering business that uses 3 special ingredients: xylaria mushrooms, yuzu and zucchini. Each of her recipes calls for an integer amount of each ingredient, denoted as x xylaria mushrooms, y yuzu and z zucchini.

(b) Olivia's recipe for Consecutive Curry calls for her to use her ingredients such that x, y, z are consecutive integers. Show that a batch of soup which contains $x^2 + y^2 + z^2$ bowls of soup cannot be evenly divided amongst 3 customers.

Assume without loss of generality that

$$x < y < z$$

Then

$$(x, y, z) = (y - 1, y, y + 1)$$

$$x^2 + y^2 + z^2 = (y - 1)^2 + y^2 + (y + 1)^2$$
$$= (y^2 - 2y + 1) + y^2 + (y^2 + 2y + 1)$$
$$= 3y^2 + 2$$

$$2 \text{ more than a multiple of } 3$$

1.4 Primes and Factoring

A. Basics

Example 1.81

A. In how many ways can 47 be written as the sum of two primes? **(AMC 8 1989/16)**
B. In how many ways can 10001 be written as the sum of two primes? **(AMC 8 2011/24)**
C. The sum of two prime numbers is 85. What is the product of these two prime numbers? **(AMC 8 2014/4)**

Based on parity, the only ways to obtain an odd number from the sum of two numbers are when one number is odd and the other number is even:

$$\underbrace{Odd}_{Prime} + \underbrace{Even}_{Prime} = Odd$$

Part A

$$45 + 2 = 47 \Rightarrow No\ Solutions \Rightarrow 0\ Ways$$

Part B

$$9{,}999 + 2 = 10{,}001 \Rightarrow No\ Solutions \Rightarrow 0\ Ways$$

Part C

$$85 = 83 + 2 \Rightarrow (2)(83) = 166$$

Example 1.82

For how many positive integers n is $n^2 - 3n + 2$ a prime number? **(AMC 10B 2002/6, AMC 12B 2002/3)**

The only possible factorization of a prime number:

$$= 1 \times p$$

Then:

$$n^2 - 3n + 2 = \underbrace{(n-2)}_{Smaller}\underbrace{(n-1)}_{Larger}$$

Hence:

$$n - 2 = 1 \Rightarrow n = 3$$

$$(n-2)(n-1) = (1)(2) = 2 \Rightarrow One\ Solution$$

Example 1.83

The positive integers $A, B, A - B$, and $A + B$ are all prime numbers. The sum of these four primes is: **(AMC 10B 2002/15, AMC 12B 2002/11)**

Note that since $A + B$ is prime, and primes are odd, at least one of A and B must be 2.

Try

$$A = 3, B = 2 \Rightarrow A - B = 1 \Rightarrow Not\ Valid$$
$$A = 5, B = 2 \Rightarrow A - B = 3, A + B = 7$$

$$\mathbf{2 + 3 + 5 + 7 = 17}$$

B. Factoring

Example 1.84: Natural Numbers

Find all pairs of positive integers (a, b) with both numbers less than 10, such that that the difference of their squares is a prime.

Since the difference of squares is a prime, we must have:

$$a^2 - b^2 = p$$

Recall that a prime number p has exactly two factors: 1 and p. Factor the LHS and the RHS:

$$\underbrace{(a+b)}_{\substack{Greater\\p}}\underbrace{(a-b)}_{\substack{Smaller\\1}} = \underbrace{p}_{Greater} \times \underbrace{1}_{Smaller}$$

Hence, the smaller term on the LHS must be:

$$a - b = 1$$

Hence, the difference of the two numbers must be 1.

a	b	$a^2 - b^2$
9	8	17
8	7	15
7	6	13
6	5	11
5	4	9
4	3	7
3	2	5
2	1	3
Total Pairs		**6**

Example 1.85

If a and b are natural numbers less than 10, find the ordered pairs of natural numbers (a, b) such that $a^2 - b^2$ is a prime number.

If you pick all values of a and b, the number of values to check is:

$$\underbrace{9}_{a} \times \underbrace{9}_{b} = 81\ possibilities$$

That is a lot of possibilities. We look to narrow down the possibilities. We still have not made use of the fact that we only need prime numbers.

Factor $a^2 - b^2$:

$$a^2 - b^2 = \underbrace{(a+b)}_{Greater}\underbrace{(a-b)}_{Smaller} = A\ prime\ number$$

Any prime number will always factor as:

$$p = \underbrace{p}_{Greater} \times \underbrace{1}_{Smaller}$$

Since the prime factorization is unique, the smaller factor must be 1:

$$a - b = 1 \Rightarrow a = b + 1$$

We only need to check pairs with a difference of 1:

$$(9,8) \Rightarrow 81 - 64 = \mathbf{17}$$
$$(8,7) \Rightarrow 64 - 49 = 15$$

$$(7,6) \Rightarrow 49 - 36 = \mathbf{13}$$
$$(6,5) \Rightarrow 36 - 25 = \mathbf{11}$$
$$(5,4) \Rightarrow 25 - 16 = 9$$
$$(4,3) \Rightarrow 16 - 9 = \mathbf{7}$$
$$(3,2) \Rightarrow 9 - 4 = \mathbf{5}$$
$$(2,1) \Rightarrow 4 - 1 = \mathbf{3}$$

Hence, the pairs that work are:

$$(9,8), (7,6), (6,5), (4,3), (3,2)(2,1)$$

Example 1.86

If a and b are natural numbers less than 50, find the number of ordered pairs of natural numbers (a, b) such that $a^2 - b^2$ is a prime number.

$$(50,49) \Rightarrow 2500 - 2401 = 99$$
$$(49,48) \Rightarrow 2500 - 2401 = 97$$
$$\vdots$$
$$(2,1) \Rightarrow 4 - 1 = 3$$

Hence, we need to find the number of prime numbers in

$$\{3,5,7, \dots, 99\} = \{2,3,5,7, \dots, 99\} - 1 = 25 - 1 = 24$$

Example 1.87: Integer Solutions

Find all integers that satisfy $a^2 - b^2 = 17$

Note that the RHS is a prime. Hence, it can only be factorized as

$$(1)(17)$$
$$(-1)(-17)$$

The terms on the LHS can be factored, and each term must be either 1 or 17.

$$\underbrace{(a+b)}_{\substack{17\\-17}}\underbrace{(a-b)}_{\substack{1\\-1}} = 17$$
$$a + b = 17$$
$$a - b = 1$$

Adding the two equations above:

$$2a = 18 \Rightarrow a = 9 \Rightarrow b = 8 \Rightarrow (9,8) \text{ is a solution}$$

Using the property that $(-a)^2 = a^2$, we get a total of four solutions:

$$\{(9,8), (-9,-8), (-9,8), (9,-8)\}$$

Example 1.88

Choose an integer randomly from the set {1, 2, 3,...100}. Square it and then subtract one. What is the probability that the answer is a prime number? **(Mathcounts 2001 State Countdown)**

Let the number chosen be n. The square of the square is n^2. The final expression is:

$$n^2 - 1 = \underbrace{(n+1)}_{p}\underbrace{(n-1)}_{1} = p \times 1$$

Hence:

$$n - 1 = 1 \Rightarrow n = 2$$

$$P(Prime) = \frac{Favourable\ Outcomes}{Total\ Outcomes} = \frac{1}{100}$$

Example 1.89

Find all integers that satisfy $a^2 - b^2 = 12$

$$(a-b)(a+b) = 12$$

Since we are looking for integer solutions, factor the RHS

Case I	Case II	Case III
12	6	4
1	2	3
$a+b=12$ $a-b=1$	$a+b=6$ $a-b=2$	$a+b=4$ $a-b=3$
Add the two equations: $2a \Rightarrow 13 \Rightarrow a = \frac{13}{2}$	Add the two equations: $2a \Rightarrow 8 \Rightarrow a = 4 \Rightarrow b = 2$	Add the two equations: $2a \Rightarrow 7 \Rightarrow a = \frac{7}{2}$
$Not\ Valid$	$Valid$	$Not\ Valid$

$$\{(4,2), (-4,-2), (-4,2), (-4,-2)\}$$

Example 1.90

Find all integers that satisfy $a^2 - b^2 = 8$

$$(a+b)(a-b) = 8$$

Factor pairs of 8 are:

$$(1,8)(2,4)$$

Case I	Case II
$a+b=8$ $a-b=1$	$a+b=4$ $a-b=2$
Add the two equations: $2a \Rightarrow 9 \Rightarrow a = \frac{9}{2}$	Add the two equations: $2a \Rightarrow 6 \Rightarrow a = 3 \Rightarrow b = 1$
$Not\ Valid$	$Valid$

$$\{(3,1), (-3,-1), (-3,1), (-3,-1)\}$$

Example 1.91

Find all integers that satisfy $a^2 - b^2 = 24$

$$(a+b)(a-b) = 24$$

Factor pairs of 8 are:

$$(1,24)(2,12)(3,8)(4,6)$$

$$1 + 24 = 25 \Rightarrow Odd \Rightarrow Not\ Valid$$
$$3 + 8 = 11 \Rightarrow Odd \Rightarrow Not\ Valid$$

$$a+b=12, a-b=2 \Rightarrow 2a = 14 \Rightarrow a = 7, b = 5$$
$$a+b=6, a-b=4 \Rightarrow 2a = 10 \Rightarrow a = 5, b = 1$$

$$\{(7,6), (-7,-5), (-7,5), (7,-5)\}$$
$$\{(5,1), (-5,-1), (-5,1), (5,-1)\}$$

Example 1.92

Find all pairs $(n, m), n \in \mathbb{N}, m \in \mathbb{N}$ that are solutions to the equation given that $k = m - 1$:

$$2n(n - m + 1) + 2(n - m + 1)^2 = 72$$

Use a change of variable $k = m - 1 \Rightarrow m = k + 1$:

$$2n(n - k) + 2(n - k)^2 = 72$$

Factor $2(n - k)$ on the LHS:

$$2(n - k)[n + n - k] = 72$$

Divide by 2 both sides:

$$\underbrace{(n - k)}_{Smaller}\underbrace{(2n - k)}_{Larger} = 36$$

Note that the RHS is a natural number. And the LHS is the product of two natural numbers. Hence, find the factor pairs of 36:

$$(1,36)(2,18)(3,12)(4,9)(6,6)$$

We could equate each term on the LHS to the factor pairs, but we have a faster method. Note that the second term on the LHS is greater than the first term by exactly n:

$$2n - k - (n - k) = n$$

Hence, the difference of the two factors in each factor pair gives us the value of n. The possible values of n are:

$$\{36 - 1 = 35, 18 - 2 = 16, 12 - 3 = 9, 9 - 4 = 5\}$$
$$6 - 6 = 0 \Rightarrow \textbf{\textit{Not Valid}}$$

The corresponding values of k and m are:

$$35 - k = 1 \Rightarrow k = 34 \Rightarrow m = 35$$
$$16 - k = 2 \Rightarrow k = 14 \Rightarrow m = 15$$
$$9 - k = 3 \Rightarrow k = 6 \Rightarrow m = 7$$
$$5 - k = 4 \Rightarrow k = 1 \Rightarrow m = 2$$

$$(n.m) = (35,35), (16,15), (9,7), (5,2)$$

Example 1.93

Find the ordered pairs that satisfy:

$$a^4 + b^2 = 4b$$

$$a^4 = 4b - b^2 = b(4 - b)$$

Suppose $a = 0$:

$$0 = b(4 - b) \Rightarrow b \in \{0,4\}$$
$$(0,0) \text{ and } (0,4)$$

Suppose $a \neq 0$:

RHS and LHS are both integers, and hence a^4 must factor into $b(4 - b)$:

Case I: $1 \cdot a^4 = b(4 - b)$

$$b = 1 \Rightarrow 4 - b = 4 - 1 = 3 = a^4 \Rightarrow a = \sqrt[4]{3} \Rightarrow Not\ Valid$$

Case II: $a \cdot a^3 = b(4 - b)$

$$b = a \Rightarrow 4 - a = a^3 \Rightarrow a^3 + a = 4 \Rightarrow No\ integer\ Solution$$

Case III: $a^2 \cdot a^2 = b(4 - b)$

$$b = a^2 \Rightarrow 4 - a^2 = a^2 \Rightarrow 2a^2 = 4 \Rightarrow a^2 = 2 \Rightarrow a = \pm\sqrt{2} \Rightarrow Not\ valid$$

Case IV: $a^3 \cdot a = b(4-b)$

$$b = a^3 \Rightarrow 4 - a^3 = a \Rightarrow a^3 + a = 4 \Rightarrow No\ integer\ Solution$$

Case IV: $a^4 \cdot a = b(4-b)$

$$b = a^4 \Rightarrow 4 - a^4 = a \Rightarrow a^4 + a = 4 \Rightarrow No\ integer\ Solution$$

Hence, the only answers that we get are:

$$(0,0)\ and\ (0,4)$$

1.5 Simon's Favorite Factoring Trick

A. Basics

If you have an expression of the form

$$(a+c)(b+c)$$

And you expand the expression using binomial multiplication, you will get:

$$ab + cb + ca + c^2$$

Example 1.94

Expand the following:

A. $(x+11)(y+11)$
B. $(2x+9)(2y+9)$
C. $(3x+11)(3y+11)$

$$(x+11)(y+11) = \mathbf{1}xy + 11x + 11y + 121$$
$$(2x+11)(2y+11) = \mathbf{4}xy + 18x + 18y + \mathbf{81}$$
$$(3x+11)(3y+11) = \mathbf{9}xy + 33x + 33y + 121$$

Note the following:

- Coefficient of the xy term is a **perfect square.**
- The last term is a **perfect square**
- Both the x term and the y term have the same coefficient.

Example 1.95

Factor

A. $ab + 7a + 7b + 49$
B. $mn + 5m + 5n + 25$

$$ab + 7a + 7b + 49 = (a+7)(b+7)$$
$$mn + 5m + 5n + 25 = (m+5)(n+5)$$

Example 1.96

Factor

A. $ab + 7a + 7b$
B. $pq + 8p + 8q$
C. $lm + 4l + 4m$
D. $gh + 9g + 10h$

Part A

Add and subtract 49:

$$ab + 7a + 7b + 49 - 49$$

Now the first four terms of the expression can be factored (which we know from before):

$$= (a+7)(b+7) - 49$$

Part B

Add and subtract 64:

$$pq + 8p + 8q + 64 - 64$$

Now the first four terms of the expression can be factored:

$$(p+8)(q+8) - 64$$

Part C

$$lm + 4l + 4m + 16 - 16$$
$$= (l+4)(m+4) - 16$$

Part D

$$gh + 9g + 10h + 90 - 90$$
$$= (g+10)(h+9) - 90$$

Example 1.97

Two different prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product, which of the following numbers could be obtained? **(AMC 12 2000/6)**

A. 22
B. 60
C. 119
D. 180
E. 231

Let the numbers be p and q. Then, the sum subtracted from the product gives:

$$pq - p - q$$

Add and Subtract 1:

$$pq - p - q + 1 - 1$$

Factor:

$$(p-1)(q-1) - 1$$

Since $p, q > 2$, both p and q are odd:

$$\underbrace{\underbrace{(p-1)}_{Even}\underbrace{(q-1)}_{Even} - 1}_{Odd}$$

Since the answer must be odd, eliminate

$$Options\ A, B, D$$

The largest possible value is:

$$(13-1) \times (17-1) - 1 = 12 \times 16 - 1 = 192 - 1 = 191 \Rightarrow Too\ High\ for\ Option\ E$$

The correct answer is:

$$Option\ C = (13-1)(11-1) - 1 = (12)(10) - 1 = 119$$

Example 1.98

Find all ordered pairs of integers (a, b) that satisfy $ab + 7a + 7b = 0$

Add 49 to both sides:

$$ab + 7a + 7b + 49 = 49$$

Factor the LHS.

$$(a+7)(b+7) = 49$$

Now, factor 49. See table.

Case	$(a+7)$	$(b+7)$
I	1	49
II	7	7
III	49	1
IV	-1	-49
V	-7	-7
VI	-49	-1

Case I	Case II	Case III
$a+7=1 \Rightarrow a=-6$ $b+7=49 \Rightarrow b=42$	$a+7=7 \Rightarrow a=0$ $b+7=7 \Rightarrow b=0$	$a+7=49 \Rightarrow a=42$ $b+7=1 \Rightarrow b=-6$

Case IV	Case V	Case VI
$a+7=-1 \Rightarrow a=-8$ $b+7=-49 \Rightarrow b=-56$	$a+7=-7 \Rightarrow a=-14$ $b+7=-7 \Rightarrow b=-14$	$a+7=-49 \Rightarrow a=-56$ $b+7=-1 \Rightarrow b=-8$

Example 1.99

Find all ordered pairs of positive integers (a,b) that satisfy the equation below:

$$ab+7a+7b=1$$

Factor the LHS

$$ab+7a+7b=1$$
$$ab+7a+7b+49=50$$
$$(a+7)(b+7)=50$$

Case	$(a+7)$	$(b+7)$
I	1	50
II	2	25
III	5	10
IV	10	5
V	25	2
VI	50	1

Casework

Case I	Case II	Case III
$a+7=1 \Rightarrow a=-6$ $b+7=50 \Rightarrow b=43$	$a+7=2 \Rightarrow a=-5$ $b+7=25 \Rightarrow b=18$	$a+7=5 \Rightarrow a=-5$ $b+7=10 \Rightarrow b=3$

Example 1.100

Find all ordered pairs of integers (x,y) that satisfy:

$$xy-3x-6y+8=18$$
$$(x-3)(y-6)\ =\ 18$$

From the table we get:

Case	$(x-3)$	$(y-6)$	x	y
I	1	18	4	24
II	2	9	5	15
III	3	6	6	12
IV	-1	-18	2	-12
V	-2	-9	-1	-3
VI	-3	-6	0	0

$$(x,y)=(4,24),(5,15),(6,12),(2,-12),(-1,-3),(0,0)$$

And interchanging x and y will give another 6 solutions.

B. Reciprocals

Example 1.101

How many distinct ordered pairs of positive integers (m,n) are there so that the sum of the reciprocals of m and n is $\frac{1}{4}$? (MathCounts 1996 State Team)

$$\frac{1}{m}+\frac{1}{n}=\frac{1}{4}$$

Eliminate fractions by multiplying both sides by $4mn$:

$$4m+4n=mn$$
$$mn-4m-4n+16=16$$
$$(m-4)(n-4)=16$$

Factorize 16:

1*16
2*8
4*4
8*2
16*1

Example 1.102

Find all pairs of integers such that the sum of their reciprocals is $\frac{1}{5}$.

Let the integers be a and b. By the condition given in the question:

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{5}$$

Eliminating fractions gives us:

$$\frac{a+b}{ab}=\frac{1}{5}\Rightarrow 5a+5b=ab\Rightarrow ab-5a-5b=0$$

We want this in the form $ab+ay+bx+xy$, so we compare coefficients and add 25 to both sides:

$$\underbrace{ab}_{ab}-\underbrace{5a}_{x=-5}-\underbrace{5b}_{y=-5}+\underbrace{25}_{xy}=25\Rightarrow(a-5)(b-5)=25$$

This is a Diophantine equation: It has two variables, and we only have one equation. In general, we would not be able to solve it, but we know that both a and b are integers.
This means that the LHS must be the product of two integers.
Hence, we factor the RHS.

$(a-5)$	$(b-5)$	a	b
1	25	6	30
5	5	10	10
25	1	30	6
-1	-25	4	-20
-5	-5	-10	-10
-25	-1	-20	4

Example 1.103

Find all ordered pairs of natural numbers such that the sum of their reciprocals is $\frac{2}{5}$

Let the integers be a and b. By the condition given in the question:

$$\frac{1}{a}+\frac{1}{b}=\frac{2}{5}$$

Adding the two fractions:

$$\frac{a+b}{ab}=\frac{2}{5}$$

Cross-multiplying:

$$5a+5b=2ab$$

Collecting all terms on the LHS:

$$2ab-5a-5b=0$$

This is symmetric in a and b, but the coefficient of ab is not a perfect square. So, we multiply by 2:

$$4ab-10a-10b=0$$

We want this in the form $ab+ay+bx+xy$, so we compare coefficients and add 25 to both sides:

$$\underbrace{4ab}_{ab} - \underbrace{10a}_{x=-5} - \underbrace{10b}_{y=-5} + \underbrace{25}_{xy} = 25 \Rightarrow (2a-5)(2b-5) = 25$$

We need to check the values of a and b using the factor pairs of 25 {(1,25)(5,5)}:

$$\underbrace{(2a-5)}_{\substack{1\\5\\25}}\underbrace{(2b-5)}_{\substack{25\\5\\1}} = 25 \Rightarrow (a,b) = \{(3,15)(5,5)(15,3)\}$$

Example 1.104

Find all ordered pairs of natural numbers such that the sum of the reciprocal of the first number and half the reciprocal of the second number is $\frac{1}{7}$.

Let the integers be a and b. By the condition given in the question:

$$\frac{1}{a} + \frac{1}{2b} = \frac{1}{7}$$

Eliminating fractions gives us:

$$\frac{a+2b}{2ab} = \frac{1}{7}$$
$$7a + 14b = 2ab$$
$$2ab - 7a - 14b = 0$$

Compare coefficients with $(a+x)(b+y) = ab + ay + bx + xy$, and add 49 to both sides:

$$\underbrace{2ab}_{ab} - \underbrace{7a}_{x=-7} - \underbrace{14b}_{y=-7} + \underbrace{49}_{xy} = 35$$
$$(a-7)(2b-7) = 49$$

We need to check the values of a and b using the factor pairs of 49 {(1,49)(7,7)(49,1)}:

$$a-7 = 1 \Rightarrow a = 8, 2b-7 = 49 \Rightarrow b = 28$$
$$a-7 = 7 \Rightarrow a = 14, 2b-7 = 7 \Rightarrow b = 7$$
$$a-7 = 49 \Rightarrow a = 56, 2b-7 = 1 \Rightarrow b = 4$$

$$(8.28)(14,7)(56,4)$$

C. Harmonic Mean

1.105: Harmonic Mean of 2 Numbers

The harmonic mean of 2 numbers is twice the reciprocal of sum of the reciprocals of the numbers.

Example 1.106

Find the harmonic mean of x and y.

The reciprocals of the numbers are:

$$\frac{1}{x} \text{ and } \frac{1}{y}$$

The sum of the reciprocals is:

$$\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy}$$

The reciprocal of the sum of the reciprocals is:

$$\frac{xy}{x+y}$$

Twice the reciprocal of the sum of the reciprocals:

$$\frac{2xy}{x+y}$$

Harmonic mean of x and y is:

$$HM(x,y) = 2 \times \frac{1}{\frac{1}{x}+\frac{1}{y}} = \frac{2}{\frac{x+y}{xy}} = \frac{2xy}{x+y}$$

1.107: Harmonic Mean

The harmonic mean of n numbers is n times the sum of the reciprocals of the numbers.

Example 1.108

Find the harmonic mean of

A. $2 \text{ and } 3$

Example 1.109

The harmonic mean of two integers is 10. Find all possible ordered pairs for the integers.

Let the integers be x and y.

$$\frac{2xy}{x+y} = 10$$
$$2xy = 10x + 10y$$
$$xy - 5x - 5y + 25 = 25$$
$$(x-5)(y-5) = 25$$

Factorize 25
1*25
5*5

1.110: Weighted Harmonic Mean

Example 1.111

Find the harmonic mean of

A. $x, y \text{ and } z$

$$HM(x,y,z) = \frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} = \frac{3}{\frac{xy+yz+xz}{xyz}} = \frac{3xyz}{xy+yz+xz}$$

D. Unit Fractions

Example 1.112

$$\frac{1}{m}+\frac{1}{n} = \frac{1}{17} \Rightarrow \frac{m+n}{mn} = \frac{1}{17} \Rightarrow 17m + 17n = mn \Rightarrow mn - 17m - 17m = 0$$
$$mn - 17m - 17n + 289 = 289 \Rightarrow (m-17)(m-17) = 289$$

$$289 = 1 * 289 = 17 * 17 = 289 * 1$$

$$m - 17 = 1 \Rightarrow m = 18, \qquad n - 17 = 289 \Rightarrow n = 306$$

Example 1.113

$$\frac{1}{m} + \frac{1}{n} = \frac{2}{17} \Rightarrow \frac{m+n}{mn} = \frac{2}{17} \Rightarrow 17m + 17n = 2mn \Rightarrow 2mn - 17m - 17m = 0$$
$$4mn - 34m - 34n + 289 = 289 \Rightarrow (2m - 17)(2m - 17) = 289$$
$$289 = 1 * 289 = 17 * 17 = 289 * 1$$
$$2m - 17 = 1 \Rightarrow m = 9, \qquad 2n - 17 = 289 \Rightarrow n = 153$$

Example 1.114

A. Find the positive integers m, n such that $\frac{1}{m} + \frac{1}{n} = \frac{3}{17}$

B. Find the positive integers m, n, p such that $\frac{1}{m} + \frac{1}{n} + \frac{1}{p} = \frac{3}{17}$

C. Using this idea, prove that that we can find for any positive integer k, k distinct integers $n_1, n_2, \dots n_k$ such that $\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k} = \frac{3}{17}$ **(NMTC Sub-Junior/Final/2)**

Part A

$$\frac{m+n}{mn} = \frac{3}{17} \Rightarrow 17m + 17n = 3mn \Rightarrow 3mn - 17m - 17n = 0$$
$$9mn - 51m - 51n + 289 = 289 \Rightarrow (3m - 17)(3n - 17) = 289$$

$$289 = 1 * 289 = 17 * 17 = 289 * 1$$

$$3m - 17 = 1 \Rightarrow m = 6, \qquad 3n - 17 = 289 \Rightarrow n = 102$$

E. Real Life Scenarios

Example 1.115

A small car company operates machinery for t hours per day that produces c cars per hour, where c and t are positive integers. After upgrading the machinery, the company can now produce $2c + 5$ cars per hour. If the machinery now operates for $t - 2$ hours per day and yet produces the same number of cars per day, how many cars per day does it produce? **(AOPS Alcumus)**

Total Cars = No. of Hours * Cars per Hour

$$\underbrace{tc}_{Original} = \underbrace{(t-2)(2c+5)}_{New}$$
$$tc = 2ct - 4c + 5t - 10$$
$$ct - 4c + 5t - 20 = 10 - 20$$
$$(c+5)(t-4) = -10$$

Find the factor pairs of the RHS:

$$-10 = (5)(-2) = (10)(-1)$$

Note that since:

$$c > 0, c + 5 > 5$$
$$c + 5 = 10 \Rightarrow c = 5, \qquad t - 4 = -1 \Rightarrow t = 3$$

F. Three Variables

Example 1.116

Find the integers x, y and z that satisfy

$$6xyz + 30xy + 21xz + 2yz + 105x + 10y + 7z = 812$$

(AOPS Alcumus)

$$z(6xy + 21x + 2y + 7) + 30xy + 105x + 10y + 35 = 812 + 35$$
$$z(6xy + 21x + 2y + 7) + 5(6xy + 21x + 2y + 7) = 847$$
$$(z + 5)(6xy + 21x + 2y + 7) = 847$$
$$(z + 5)(3x + 1)(2y + 7) = 7 \times 11 \times 11$$

$$z + 5 = 7 \Rightarrow z = 2, \qquad 3x + 1 = 11 \Rightarrow x = \frac{10}{3}$$

$$z + 5 = 11 \Rightarrow z = 6, \qquad 3x + 1 = 7 \Rightarrow x = 2, \qquad 2y + 7 = 11 \Rightarrow y = 2$$

Simon's Favorite Factoring Trick

Example 1.117

For how many positive integers m does there exist at least one positive integer n such that mn <=m + n? (AMC 10A 2002/4, AMC 12A 2002/6)

Example 1.118

For how many ordered pairs of positive integers (x,y) is x+2y=100? (AMC 12A 2004/3)

Example 1.119

Two farmers agree that pigs are worth 300 dollars and that goats are worth 210 dollars. When one farmer owes the other money, he pays the debt in pigs or goats, with "change" received in the form of goats or pigs as necessary. (For example, a 390 dollar debt could be paid with two pigs, with one goat received in change.) What is the amount of the smallest positive debt that can be resolved in this way? (AMC 10A 2006/22,AMC 12A 2006/14)

Example 1.120

Oscar buys 13 pencils and 3 erasers for 1.00. A pencil costs more than an eraser, and both items cost a whole number of cents. What is the total cost, in cents, of one pencil and one eraser? (AMC 12A 2006/9)

Example 1.121

Elmo makes N sandwiches for a fundraiser. For each sandwich he uses B globs of peanut butter at 4 cents per glob and J blobs of jam at 5 cents per blob. The cost of the peanut butter and jam to make all the sandwiches is $2.53. Assume that B, J, and N are positive integers with N>1. What is the cost of the jam Elmo uses to make the sandwiches? (AMC 10B 2006/22, AMC 12B 2006/14)

Example 1.122
For how many positive integers n does 1+2+...+n evenly divide from 6n? (AMC 10A 2005/21)

Example 1.123
Let A,M, and C be digits with
(100A+10M+C)(A+M+C) = 2005
What is A? (AMC 12A 2005/8)

Example 1.124
How many ordered pairs (m,n) of positive integers, with m>n, have the property that their squares differ by 96? (AMC 10A 2007/23)

1.6 Other Topics

A. Geometry

Example 1.125
A carpenter is designing a table. The table will be in the form of a rectangle whose length is 4 feet more than its width. How long should the table be if the carpenter wants the table to be 45 square feet. **(JMET 2011/72)**

$$l(l-4)=45$$
$$Factor\ pairs\ of\ 45=(1,45)(3,15)(5,9)\Rightarrow 5\ and\ 9\ have\ difference\ of\ 4\Rightarrow l=9$$

Example 1.126
A cube of edge 3 cm is cut into N smaller cubes, not all the same size. If the edge of each of the smaller cubes is a whole number of centimeters, then $N=$ **(AMC 8 1991/24)**

Since the N cubes are smaller, the only possible edge lengths are:
$$Edge=2\ cm, Edge=1\ cm$$
Let
$$a=No.of\ 2\ cm\ cubes, b=No.of\ 1\ cm\ Cubes$$
Then, the volume of the smaller cubes must add up to the volume of the original cube:
$$8a+b=27$$
If $a=0$
$$b=27\Rightarrow All\ cubes\ are\ same\ size\Rightarrow Not\ Valid$$
If $a=1$:
$$8+b=27\Rightarrow b=19$$

If $a=2$:
$$16+b=27\Rightarrow b=11$$
But this does not fit the geometry of the question, since you cannot cut 2 cubes of edge length 2 from a cube of edge length 3.

Hence, total number of cubes
$$=19+1=20$$

Example 1.127
A square with integer side length is cut into 10 squares, all of which have integer side length and at least 8 of

which have area 1. What is the smallest possible value of the length of the side of the original square? **(AMC 8 2012/17)**

$$a + b = 10 \Rightarrow b \in \{0,1,2\}$$

Minimum area of squares is when

$$b = 0 \Rightarrow a = 10 \Rightarrow Area = 10$$

10 is not a perfect square, so the smallest value we can try is

$$Area = 16, Side\ length = 4$$

This gives us:

$$a + 4b = 16 \Rightarrow a = 8, b = 2$$

Hence, 4 is the smallest that can work, and it is possible.

Example 1.128

An $8 \times 8 \times n$ rectangular prism is made up from $1 \times 1 \times 1$ cubes. Suppose that A is the surface area of the prism and B is the combined surface area of the $1 \times 1 \times 1$ cubes that make up the prism. What is the sum of the values of n for which $\frac{B}{A}$ is an integer? **(Gauss Grade 7, 2019/25)**

Dimensions of the prisms are:

$$\underbrace{8}_{Length}, \underbrace{8}_{Width}, \underbrace{n}_{Height}$$

Surface Area of the rectangular prism:

$$A = 2(lw + wh + lh) = 2(64 + 8n + 8n) = 2(64 + 16n) = 32(4 + n)$$

Surface area of a single $1 \times 1 \times 1$ cube:

$$= 6s^2 = 6$$

No. of Cubes:

$$= \frac{Vol\ of\ prism}{Vol\ of\ 1 \times 1 \times 1\text{ cube}} = \frac{8 \times 8 \times n}{1} = 64n$$

Surface Area of all the cubes:

$$B = 6(64n) = 32(12n)$$

We want to look at the fraction:

$$\frac{B}{A} = \frac{32(12n)}{32(4 + n)} = \frac{12n}{4 + n}$$

We need the above to be an integer. $12n$ is not divisible by $4 + n$. However, $12n + 48$ is divisible by $4 + n$. Hence, we rewrite the above fraction:

$$\frac{12n + 48 - 48}{4 + n} = \frac{12n + 48}{4 + n} - \frac{48}{4 + n} = 12 - \frac{48}{4 + n}$$

12 is an integer. However, $\frac{48}{4+n}$ will be an integer if 48 is divisible by $4 + n$.

Factors of 48 are:

$$\{1,2,3,4,6,8,12,16,24,48\}$$

Since the rectangular prism is made of $1 \times 1 \times 1$ cubes:

$$n \in \mathbb{N}, n \geq 1 \Rightarrow 4 + n \geq 5$$

$$4+n \in \{6,8,12,16,24,48\} \Rightarrow n=\{2,4,8,12,20,44\}$$

The sum of the values of n is:

$$2+4+8+12+20+44=90$$

Example 1.129

Cubical tank A of side 6 units is to be replaced by 112 cubical tanks with integral side lengths. Find the ratio of surface area of Tank A to the surface area of the tanks that will replace it.

Tanks do not have to be same size. We can use tanks of:
Side 1 (with Volume 1)
Side 2 (with Volume $2^3=8$)
Side 3 (with Volume $3^3=27$)
Side 4 (with Volume $4^3=64$)

$$x+8y+27z=216, \qquad x+y+z=112$$

Method I: Solve for y

$$y=\frac{216}{8}-\frac{27z}{8}-\frac{x}{8} \Rightarrow \frac{27z}{8} \in \mathbb{N} \Rightarrow \text{Min}(z)=8 \Rightarrow 27z=216 \Rightarrow x+y+z=8 \Rightarrow \textit{Contradiction}$$

$$\therefore y=0 \Rightarrow x+27z=216 \Rightarrow \underbrace{112-z+27z=216}_{x=112-z} \Rightarrow z=4$$

Method II: Solve for z

Solve the equation for z.

$$z=\frac{216}{27}-\frac{8y}{27}-\frac{x}{27} \Rightarrow z=8-\frac{8y}{27}-\frac{x}{27} \Rightarrow$$

$y \in \mathbb{N} \Rightarrow \frac{8y}{27} \in \mathbb{N} \Rightarrow \text{Min}(y)=27$, which gives us:

$$z=8-8-\frac{x}{27}=-\frac{x}{27}=-ve \Rightarrow \textit{Contradiction}$$

Hence,

$$y=0$$

We are left with:

$$z=8-\frac{x}{27}$$

If we recognize that

$$112-4=108$$

is a multiple of 27, then $z=4$ follows easily.

Method III: Casework

(Very long method. Not recommended).

Case I:	Volume of Tank A $=6^3=216$	

Tanks of Side 1	Tanks of side 1 will not suffice (Volume = 112, against the required 216).	
Case II: x Tanks of Side 1, y Tanks of Side 2	$x + 8y = 216$ Substitute $x = 112 - y$ in the equation above: $112 - y + 8y = 216$ $y = 14\frac{6}{7}$ (Not an Integer)	Solve $x + y = 112$ for x to get $x = 112 - y$

Case III: x Tanks of Side 1, y Tanks of Side 2, z tanks of Side 3	**Brute Force Method:** We need to break this in further cases depending on the value of z. However, we follow the same method as above.	
$z = 1$	$x + 8y + 27 = 216$ $111 - y + 8y + 27 = 216$ $y = \frac{78}{7}$ (Not an Integer)	$x + y = 111$ $x = 111 - y$
$z = 2$	$x + 8y + 54 = 216$ $110 - y + 8y + 27 = 216$ $y = \frac{52}{7}$ (Not an Integer)	$x + y = 110$ $x = 110 - y$
$z = 3$	$x + 8y + 81 = 216$ $109 - y + 8y + 27 = 216$ $y = \frac{26}{7}$ (Not an Integer)	$x + y = 109$ $x = 109 - y$
$z = 4$	$x + 8y + 108 = 216$ $108 - y + 8y + 27 = 216$ $7y = 81$ $y = \frac{81}{7} = 11\frac{4}{7}(Not\ an\ integer)$	$x + y = 108$ $x = 108 - y$

From all of the four cases above we were not able to get a valid value of y as an integer. Hence, the only integer multiple of 8 that will be valid for y will be

$$y = 0\ (\textbf{Integer})$$

Hence,

$$x = 108 - y = 108 - 0 = 108$$

B. Counting Arguments

Example 1.130

There are three cities A, B and C, each of these cities is connected with the other two cities by at least one direct road. If a traveler wants to go from one city(origin) to another city(destination), she can do so either by traversing a road connecting the two cities directly, or by traversing two roads, the first connecting the origin to the third city and the second connecting the third city to the destination. In all there are 33 routes from A to B (including those via C). Similarly, there are 23 routes from B to C (including those via A). How many roads are there from A to C directly? **(CAT 2000/104)**

A. 6
B. 3

C. 5
D. 10

As an example of the multiplication principle, recall that if

$$A \text{ to } B: 3 \text{ ways}, B \text{ to } C: 4 \text{ ways}, A \text{ to } C: 5$$
$$A \text{ to } C \text{ via } B: 3(4) \text{ ways}$$
$$\boldsymbol{A} \text{ to } C \text{ directly}: 5 \text{ ways}$$

$$Total = 3(4) + 5 = 17$$

Set Up the Equation

Assume variables as below. Use the multiplication principle[3] to calculate the number of routes given in the question:

A to C	C to B	A to B	All Routes from A to B	All Routes from B to C
x	y	z	$\underbrace{\underbrace{x}_{A\text{ to }C} \times \underbrace{y}_{C\text{ to }B}}_{\text{With a stopover}} + \underbrace{\underbrace{z}_{A\text{ to }B}}_{\text{Direct}} = 33$	$\underbrace{\underbrace{z}_{B\text{ to }A} \times \underbrace{x}_{A\text{ to }C}}_{\text{With a stopover}} + \underbrace{\underbrace{y}_{B\text{ to }C}}_{\text{Direct}} = 23$

Hence, the equations that we get are:

$$\underbrace{xy + z = 33}_{\text{Equation I}}, \qquad \underbrace{xz + y = 23}_{\text{Equation II}}$$

Add the two equations:

$$xy + xz + y + z = 56 \Rightarrow (y + z)(x + 1) = 56$$

Now, the LHS must be the product of two natural numbers, hence, factor the RHS:

$(y+z)$	$(x+1)$	x	$xy + z = 33$
1	56	55	
2	28	27	
4	14	13	
7	8	7	
8	7	$\underbrace{6}_{\text{Option 1}}$	$\underbrace{3y + z = 33}_{\text{Equation III}}$
14	4	$\underbrace{3}_{\text{Option 2}}$	$\underbrace{6y + z = 33}_{\text{Equation IV}}$
28	2	1	
56	1	0	

Of the options given in the question, only two match the value of x, so we only need to check these two. This we do below using casework.

Casework

$x = 3$ gives a contradiction:

$$y + z = 14 \Rightarrow \underbrace{3y + 3z = 42}_{\text{Equation V}} \Rightarrow \underbrace{2z = 42 - 33 \Rightarrow 2z = 9 \Rightarrow z = \frac{9}{2} \notin \mathbb{N} \Rightarrow \textbf{Not Valid}}_{\text{Equation V}-\text{Equation III}}$$

[3] To review the multiplication principle to calculate routes between cities, refer the document on Counting Rules.

Since the first case does not work, the second must be valid. We can confirm this:

$$x = 6 \Rightarrow y + z = 8 \Rightarrow \underbrace{6y + 6z = 48}_{\text{Equation VI}} \Rightarrow \underbrace{5z = 48 - 33 \Rightarrow z = 3}_{\text{Equation V}-\text{Equation IV}} \Rightarrow \underbrace{6y + 3 = 33 \Rightarrow 6y = 30 \Rightarrow y = 5}_{\text{Substitute } z=3 \text{ in Equation IV}}$$

Final answer:

$$x = 6, z = 3, y = 5$$

1.7 Enumerating Solutions

A. Background

1.131: Diophantine Equations[4]

- Diophantine Equations are a class of equations where the number of variables is more than the number of equations.
- The simplest case is two variables and one equation. This leads to an infinite number of solutions.

is a Diophantine Equation, named after the Greek mathematician of the 3rd Century, who studied equations.

Substituting any value of x in the equation gives a value of y. If $x = 4.1$, then

$$4.1 + y = 5 \quad \Rightarrow \quad y = 0.9$$

Substituting any value of y in the equation gives a value of y. If $x = 4.1$, then

$$4.1 + y = 5 \quad \Rightarrow \quad y = 0.9$$

However, if you introduce conditions, or constraints, on the equations, then the number of solutions is also restricted.

1.132: Ordered Pairs

Solutions to Diophantine Equations are written as ordered pairs.

Example 1.133

Consider the equation

$$a + b = 3$$

Is the solution $a = 2, b = 1$ the same as $a = 1, b = 2$.

$$(a, b) = (2,1)$$
$$(a, b) = (1,2)$$
$$(2,1) \neq (1,2)$$

No.

Example 1.134

$$x + y = 5$$

In the above equation:

A. Find x if y is 4.1
B. Find y is x is 4.1
C. Are the two solutions the same or different?

[4] Diophantine Equations are named the mathematician Diophantus of Greece.

1.135: Integer Solutions

Traditionally, Diophantine Equations are solved for Integer Values only. If there are two variables, their solution can be given as a pair.

$$(4,7) \neq (7,4)$$

B. Conditions on Solutions

1.136: Non-negative Integers

Non-negative integers are integers greater than or equal to zero.

$$x \text{ is non} - \text{negative} \Leftrightarrow x \geq 0$$

1.137: Positive Numbers

Positive numbers are integers greater than zero.

$$x \text{ is positive} \Leftrightarrow x > 0$$

Example 1.138

For the equation $a + b = 15$, find the number of:

A. *nonnegative* integral solutions
B. *positive* integral solutions

Part A

We obtain valid pairs (a, b) by substituting $a = 0, 1, 2, \ldots 15$:

$$(0, 15), (1, 14), \ldots, (15, 0) \Rightarrow 16 \text{ Values}$$

Part B

The smallest number a that satisfies the condition is $a = 1$.
We obtain valid pairs (a, b) by substituting $a = 1, 2, \ldots 14$:

$$(1, 14), (2, 13), \ldots (14, 1) \Rightarrow 14 \text{ Values}$$

Find the reason for the difference in the solutions to the above two parts.

Part A allows 0 as a solution. This allows the minimum solution to be 0, and the maximum solution to be 15.
Part B does not allow 0 as a solution.
Hence, number of solutions are reduced by 2.

Example 1.139

I did twelve homework problems over the weekend, and I did at least one problem each day. Find the number of ordered pairs that represent the number of problems I could have solved on Saturday and Sunday.

$$(Sat, Sun) = (1,11), (2,10), (3,9), (4,8), (5,7), (6,6), (7,5), (8,4), (9,3), (10,2), (11,1)$$
$$11 \text{ ordered pairs}$$

1.140: Number of Solutions

A two-variable Diophantine equation will have two fewer *positive* integral solutions as compared to *nonnegative* integral solutions.

The number of solutions to

$$x + y = c, \qquad c \in \mathbb{N}$$

x can take:

$$Non - negative\ Values: \{0, 1, 2 \dots, c - 1, c\} \Rightarrow c + 1\ values$$

For positive values, it cannot take the value zero, and the value c:

$$Positive\ Values: \{\mathbf{0}, 1, 2 \dots, c - 1, \boldsymbol{c}\} \Rightarrow \{1, 2 \dots, c - 1\} \Rightarrow c - 1\ values$$

1.141: Zero Solutions

If the conditions of a problem cannot be met, then the number of solutions will be zero.

1.142: Repeating Values in Ordered Pairs

If the values in an ordered pair are not repeated, then interchanging the elements results in a different ordered pair.

$$(a, b) \neq (b, a)$$

If the values in an ordered pair are the same, then interchanging the elements does not result in a different ordered pair:

$$(a, a) = (a, a)$$

1.143: Distinct Solutions

If we want distinct solutions, the value of both the variables cannot be the same. They must be different.

Example 1.144

A. Find the number of *nonnegative* integral solutions of $p + q = 7$
B. How many *positive* integral solutions of $x + y = 100$ exist?
C. What is the number of *positive* integral solutions of $r + s = 0$?
D. What is the number of solutions to $r + s = 0$ such that r and s are *nonnegative* integers?
E. If m and n are distinct positive integers, then what is the number of solutions of $m + n = 8$?
F. If m and n are distinct positive integers, then what is the number of solutions of $m + n = -8$?

$$p + q = 7 \Rightarrow (0,7)(1,6), \dots, (7,0) \Rightarrow 8\ Values$$
$$x + y = 100 \Rightarrow (1,99)(2,98), \dots, (99,1) \Rightarrow 99\ Values$$
$$r + s = 0 \Rightarrow No\ Solutions \Rightarrow Zero\ Solutions$$
$$r + s = 0 \Rightarrow (0,0) \Rightarrow 1\ Solution$$
$$m + n = 8 \Rightarrow (1,7)(2,6), \dots, (4, 4), \dots, (7,1) \Rightarrow 6\ Solutions$$
$$m + n = -8 \Rightarrow Zero\ Solutions$$

C. Range Restrictions

If there are range restrictions, as before, you can find the minimum and the maximum, and use the concepts of counting lists in order to find the number of solutions that satisfy the constraints.

Example 1.145

Find the number of positive integral solutions to $y + z = 16$, such that

A. y and z are both less than 12.
B. y and z are both more than 5
C. y and z have a difference of more than two.

D. y and z have a difference of not more than five.
E. y is less than 12.
F. y is more than 5.

$$(5,11),(6,10),\ldots,(11,5) \Rightarrow y \in \{5,6,\ldots,11\} \Rightarrow 11-5+1=7\ pairs$$
$$(6,10),(7,9),\ldots,(10,6) \Rightarrow y \in \{6,\ldots,10\} \Rightarrow 10-6+1=5\ pairs$$
$$(1,15),(2,14),\ldots,(15,1)-(7,9)(8,8),(9,7) \Rightarrow 15-3=12$$
$$(6,10)(7,9)(8,8),(9,7)(10,6) \Rightarrow 5\ Solutions$$
$$(y,z)=(5,11),(6,10),\ldots,(15,1) \Rightarrow 15-5+1=11\ pairs$$
$$(y,z)=(6,10),\ldots,(15,1) \Rightarrow 15-6+1=10\ pairs$$

D. Unordered Pairs

So we have been considering pairs, which can be ordered or unordered. However, if consider sets, order is not important in sets, and hence, we have to work accordingly.

Example 1.146

There are several sets of three different numbers whose sum is 15 which can be chosen from $\{1,2,3,4,5,6,7,8,9\}$. How many of these sets contain a 5? **(AMC 8 1991/11)**

Let the set of three numbers be:

$$\{a,b,5\}$$

We know that

$$a+b+5=15 \Rightarrow a+b=10 \Rightarrow \{a,b\} \in \big\{\{1,9\}\{2,8\}\{3,7\}\{4,6\}\big\}$$

Note that

$$\{9,1\}=\{1,9\}$$

Since we are using sets and not ordered pairs.

E. Negative Integers

So far, the solutions that we have considered have been either positive, or non-negative. However, if we introduce restrictions, then negative numbers will also give a finite number of solutions.

Example 1.147

A. Find the number of integral solutions to $m+n=9$, such that both variables are more than -4.
B. Find the number of integral solutions to $a+b=24$, such that both variables are greater than -5.
C. Find the number of integral solutions to $s+t=13$, such that both variables are between -2 and 12.
➢ Find the number of integral solutions to $u+v=23$, such that u is greater than -12, and v is greater than -8.

Part A

$$\underbrace{(-3,12)}_{\substack{Smallest\\ Value\ of\ m}},(-2,11),\ldots,(11,-2)\ \underbrace{(12,-3)}_{\substack{Largest\\ Value\ of\ m}}$$

Use the method of counting lists:

$$\underbrace{12}_{\substack{Largest\\ Value}}-\underbrace{(-3)}_{\substack{Smallest\\ Value}}+\underbrace{1}_{Add\ 1}=12+3+1=16\ solutions$$

Part B

$$(-4,28)(-3,27),\ldots,(28,-4) \Rightarrow a \in \{-4,-3,\ldots,28\} \Rightarrow 28-(-4)+1=33\ Pairs$$

Part C

$$(2,11)(3,10)\dots(11,2) \Rightarrow 11-2+1 = 10\ Pairs$$

Part D

If we had the restriction that both variables were greater than -12, we would have this solution:

$$(u,v) = (-11,34)(-10,33),\dots,(30,-7),(31,-8),\dots,(34,-11)$$

But note that the last few values are valid because of the additional restriction on the values of v:

$$(u,v) = (-11,34)(-10,33),\dots,(30,-7),\mathbf{(31,-8),\dots,(34,-11)}$$

And hence, we only consider the valid values for u:

$$u \in \{-11,-10,\dots,30\} \Rightarrow 30-(-11)+1 = 42\ Pairs$$

F. Restrictions on Solutions Set

Example 1.148

What is the number of solutions to $m+n=12$ for positive integers such that m is even?

Listing Method

We list out the valid pairs:

$$(2,10)(4,8)(6,6)(8,4)(10,2) \Rightarrow m \in \{2,4,6,8,10\} \Rightarrow \{1,2,3,4,5\} \Rightarrow 5\ Solutions$$

Change of Variable

Let

$$m\ be\ even \Rightarrow m = 2x$$
$$n\ is\ also\ even \Rightarrow n = 2y$$

Hence, our equation becomes:

$$2x+2y = 12 \Rightarrow x+y = 6 \Rightarrow x \in \{1,2,3,4,5\} \Rightarrow 5\ Solutions$$

Example 1.149

What is the number of solutions to $m+n=12$ for positive integers such that m is odd?

Again, we list out the valid pairs:

$$(1,11)(3,9)(5,7)(7,5)(9,3)(11,1) \Rightarrow m \in \{1,3,5,7,9,11\}$$

Add 1 to each element in the list:

$$\{2,4,6,8,10,12\}$$

Divide each element in the list by 2:

$$\{1,2,3,4,5,6\} \Rightarrow 6\ Solutions$$

Example 1.150

I have ten cars, which are either blue or red. The number of red cars I have is even and positive. The number of blue cars I have is also positive. Find the number of cars of each type I can have.

(Red, Blue) = (2,8),(4,6),(6,4),(8,2)

Example 1.151

Solve $s+t=20$, if both s and t are integers greater than -5 and (answer each separately):

A. s is even
B. s is odd

Part A

$$(-4,24),(-2,22),\dots,(24,-4)$$

This means that:

$$s \in \{-4,-2,\dots,24\}$$

Divide each number by 2:

$$\frac{s}{2} \in \{-2,-1,\dots,12\} \Rightarrow 12-(-2)+1=15$$

Part B

$$(-3,23),(-1,21),\dots,(23,-3)$$

This means that:

$$s \in \{-3,-1,\dots,23\}$$

Add 1 to each number:

$$s+1 \in \{-2,0,\dots,24\}$$

Divide each number by 2:

$$\frac{s+1}{2} \in \{-1,0,\dots,12\} \Rightarrow 12-(-1)+1=14$$

G. Number Theory

Example 1.152

How many two-digit numbers have digits whose sum is a perfect square? (AMC 8 2006/11)

Strategy

Two-digit numbers range from:

$$10 \to Sum = 1, 99 \to Sum = 18$$

And hence the range of the sum of digits is:

$$Range = \{1,2,3,4,\dots 16,17,18\}$$

Out of the above, the following are perfect squares:

$$Perfect\ Squares = \{1,4,9,16\}$$

Let the digits of the number be

$$t,u, \qquad 1 \le t \le 9, \qquad 0 \le u \le 9$$

Casework

We do this using casework

$$Sum = 1: 10 \Rightarrow 1\ Numbers$$
$$Sum = 4 \Rightarrow t+u=4 \Rightarrow \{13,22,31,41\} \Rightarrow 4\ Numbers$$
$$Sum = 9 \Rightarrow t+u=9 \Rightarrow \{18,27,\dots,90\} \Rightarrow 9\ Numbers$$
$$Sum = 16 \Rightarrow t+u=16 \Rightarrow \{79,88,97\} \Rightarrow 3\ Numbers$$

H. Average

Example 1.153

The list of integers 4, 4, x, y, 13 has been arranged from least to greatest. How many different possible ordered pairs (x; y) are there so that the average (mean) of these 5 integers is itself an integer? (Gauss Grade 8 2015/24)

For some integer n, we must have:

$$\frac{4+4+x+y+13}{5}=\frac{21+x+y}{5}=n\Rightarrow 21+x+y=5n$$

$$n=6\Rightarrow (x,y)=(4,5)\Rightarrow 1\text{ Pair}$$
$$n=7\Rightarrow x+y+21=35\Rightarrow x+y=14\Rightarrow (x,y)=(4,10)(5,9)(6,8)(7,7)\Rightarrow 4\text{ Pairs}$$
$$n=8\Rightarrow x+y+21=40\Rightarrow x+y=19\Rightarrow (x,y)=(6,13),(7,12),(8,11),(9,10)\Rightarrow 4\text{ Pairs}$$
$$n=9\Rightarrow x+y+21=45\Rightarrow x+y=24\Rightarrow (x,y)=(11,13),(12,12)\Rightarrow 2\text{ Pairs}$$

Final Answer:

$$1+4+4+2=11$$

I. Triangle Inequality

Example 1.154

In any triangle, the length of the longest side is less than half of the perimeter. All triangles with perimeter 57 and integer side lengths x, y, z, such that $x<y<z$ are constructed. How many such triangles are there? (Guass 2013/24)

Triangles cannot be Isosceles

$$P=x+y+z=57$$

Maximum value of z is 57/2 =28.5
Since z is integer, maximum value is 28

z	$x+y$	$Pairs$	Number of Pairs
28	29	$(27,2)(26,3),\dots,(15,14)$	$27-15+1=13$
27	30	$(26,4)(25,5),\dots,(16,14)$	$26-14+1=11$
26	31	$(25,6)(25,5),\dots,(16,15)$	$25-16+1=10$
25	32	$(24,8)(23,9),\dots,(17,15)$	$24-17+1=8$
24	33	$(23,10)(22,11),\dots,(17,16)$	$23-17+1=7$
23	34	$(22,12)(21,13),\dots,(18,16)$	$22-18+1=5$

J. Pigeonhole Principle

Example 1.155

A subset B of the set of integers from 1 to 100, inclusive, has the property that no two elements of B sum to 125. What is the maximum possible number of elements in B? **(AMC 10B 2005/25)**

$$a+b=125\Rightarrow (a,b)=\{(1,124),(2,123),\dots,(62,63)\}$$

By the Pigeonhole Principle, from each of the above unordered pairs, we can take maximum one number. Hence, we can take 62 numbers.

(Continuation) Example 1.156

What is the number of ways to make the above selection such there are 62 elements in the subset?

We can only take the first number from the 24 pairs below, since the second number of each pair can never be a part of the subset:

$$(1,124),(2,123),\dots,(24,101) \Rightarrow 24\ Numbers \Rightarrow 1\ Choice\ Only$$

From the 38 pairs remaining, we can take exactly one number from each pair:

$$(25,100),(26,99),\dots(100,25)$$

For each pair, we have 2 choices. Total number of choices is:

$$2^{38}$$

K. Distinguishability

Example 1.157

7 indistinguishable prizes have to be distributed among 2 boys based on their performance in a sport competition. What is the number of ways in which the prizes can be distributed?

Let the number of awards given to the boys be

$$b_1\ and\ b_2$$

Now, by the condition given in the question:

$$b_1 + b_2 = 7$$

- Condition I: The minimum number of prizes that can be given is zero.
- Condition II: Further, the number of prizes given must be an integer.

Combining the two conditions tells us that we are looking for nonnegative integral solution.
Hence, the number of solutions that we get is:

$$(0,7)(1,6),\dots,(7,0) \Rightarrow 7 - 0 + 1 = 8\ Solutions$$

Suppose that, in the question on distribution of prizes to boys above, the prizes were distinguishable. What would be your answer in this case? (This will need the Multiplication Principle from Counting)

Let's call the prizes

$$A, B, \dots, G$$

Prize A can be given either to the first boy, or the second boy. This gives us

$$2\ choices$$

Similarly, Prize B can be given either to the first boy, or the second boy. This also gives us:

$$2\ choices$$

In fact, we have two choices for each prize, giving us:

$$\underbrace{2}_{\substack{First\\Prize}} \times \underbrace{2}_{\substack{Second\\Prize}} \times \dots \times \underbrace{2}_{\substack{Seventh\\Prize}} = 2^7 = 128\ Choices$$

Example 1.158

Two leading scientists have been nominated for 9 indistinguishable awards at a scientific symposium. In order to not upset the competitive scientists, each scientist must receive at least one award. If they are the only ones who have been nominated for these awards, then in how many ways can the awards be distributed?

Let the number of awards given to the scientists be

$$s_1\ and\ s_2$$

Now, by the condition given in the question:

$$s_1 + s_2 = 9$$

And we know that each scientist must get at least one award. Further, the number of awards given must be an

integer.
Hence, we are looking for positive integral solutions to the equation, which are:

$$(1,8)(2,7),\dots,(8,1) \Rightarrow 8-1+1=8\ Solutions$$

Example 1.159

Suppose that, in the question on distribution of awards to scientists above, the awards were distinguishable. What would be your answer in this case?

Method I: Complementary Counting

If there are no restrictions, then, as in the previous example, the number of ways to distribute the awards is:

$$\underbrace{2}_{\substack{First\\Award}} \times \underbrace{2}_{\substack{Second\\Award}} \times \dots \times \underbrace{2}_{\substack{Ninth\\Award}} = 2^9\ Choices$$

But, we have to give at least one award to each scientist. Hence, there are two ways in which we cannot distribute the awards

$$Way\ 1: All\ awards\ to\ first\ scientist$$
$$Way\ 2: All\ awards\ to\ second\ scientist$$

Hence, the final answer is:

$$2^9-2=512-2=510$$

Method II

We can do this using direct counting by splitting into cases. From the above, we know that the solutions for the distribution equation

$$s_1+s_2=9$$

Are the following:

$$(1,8)(2,7),\dots,(8,1) \Rightarrow 8-1+1=8\ Solutions$$

Case I: $(1,8),(8,1)$
We need to choose one award to give to one of the scientists and the remaining awards must be given to the other scientists:

$$2\times\binom{9}{1}=2\times 9\ Ways=18\ Ways$$

We can continue with the rest of the cases and get the same answer.

Example 1.160

Two tour guides are leading six tourists. The guides decide to split up. Each tourist must choose one of the guides, but with the stipulation that each guide must take at least one tourist. How many different groupings of guides and tourists are possible? **(AMC 10A 2004/12)**

Consider the number of choices for each tourist:

$$\underbrace{2}_{\substack{First\\Tourist:\\Two\ Choices}} \times \underbrace{2}_{\substack{Second\\Tourist:\\Two\ Choices}} \times \dots \times \underbrace{2}_{\substack{Sixth\\Tourist:\\Two\ Choices}} = 2^6=64$$

$$64-2=62$$

Do the above question using combinations:

$$\binom{6}{1}+\binom{6}{2}+\binom{6}{3}+\binom{6}{4}+\binom{6}{5}=6+15+20+15+6=62$$

1.161: Identity
Prove the identity below by counting in two different ways:
$$\mathbf{2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}}$$

I have n distinguishable objects. I want to select zero or more of these objects.

One way to think about this is that for each object, I have a choice:
$$\text{To pick or not to pick} \Rightarrow 2 \text{ Choices}$$

$$\underbrace{2}_{\substack{First\\Object}} \times \underbrace{2}_{\substack{Second\\Object}} \times \ldots \times \underbrace{2}_{\substack{Nth\\Object}} \Rightarrow 2^n = LHS$$

Another way to think about this, I can pick
$$0\ Objects, 1\ Object, \ldots, n\ objects$$
And the number of ways to do this is:
$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = RHS$$

But the number of ways to choose the objects has to be the same, irrespective of which method we use to count it.
Hence,
$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

L. Casework

Example 1.162
Three friends have a total of 6 identical pencils, and each one has at least one pencil. In how many ways can this happen? (AMC 8 2004/17)

$$a + b + c = 6, \qquad a, b, c \ge 1$$
Since every friend has at least one pencil, give them one pencil:
$$A + B + C = 3, \qquad A, B, C \ge 0$$

$$A = 3 \Rightarrow B + C = 0 \Rightarrow (B, C) = (0,0) \Rightarrow 1\ Solution$$
$$A = 2 \Rightarrow B + C = 1 \Rightarrow (B, C) = (1,0)(0,1) \Rightarrow 2\ Solutions$$
$$A = 1 \Rightarrow B + C = 2 \Rightarrow (B, C) = (0,2)(1,1)(2,0) \Rightarrow 3\ Solutions$$
$$A = 0 \Rightarrow B + C = 3 \Rightarrow (B, C) = (0,3)(1,2)(2,1)(3,0) \Rightarrow 4\ Solutions$$

Total number of solutions
$$= 1 + 2 + 3 + 4 = 10\ Solutions$$

Example 1.163
Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible? (AMC 10 2001/19)

$$a + b + c = 4$$

Total number of solutions

$$= 1 + 2 + 3 + 4 + 5 = 15\ Solutions$$

Example 1.164

Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected? **(AMC 10A 2003/21)**

$$a + b + c = 6$$

Total number of solutions

$$= 1 + 2 + 3 + 4 + 5 + 6 + 7 = \frac{7 \times 8}{2} = 28$$

Example 1.165

Alice has 24 apples. In how many ways can she share them with Becky and Chris so that all of them have at least 2 apples? **(AMC 8 2019/25)**

$$\underbrace{a}_{Alice} + \underbrace{b}_{Becky} + \underbrace{c}_{Chris} = 24$$

Each one needs to have at least two apples. So, give them the required two apples, and define new variables:

$$a' = a - 2, \qquad b' = b - 2, \qquad c' = c - 2 \Rightarrow a' + b' + c' = 18$$

In order to solve this equation, fix the value of a':

$Value\ of\ a'$		Pairs for $(b' + c')$	Number of Pairs
18	$b' + c' = 0$	(0,0)	1
17	$b' + c' = 1$	(0,1)(1,0)	2
16	$b' + c' = 2$	(0,2)(1,1)(2,0)	3
.		.	.
.		.	.
.		.	.
0			19

$$1 + 2 + 3 + \cdots + 19 = \frac{19 \times 20}{2} = 190$$

Example 1.166

Find the number of solutions to:

$$x + y + z = 20, \qquad x\ is\ an\ even\ number$$

We do casework based on x, which can only take values from 0 to 10.

x	$y + z$	$No.of\ Solutions$
0	20	21
1	18	19
2	16	17
.	.	.
.	.	.
.	.	.

10	0	1

The tota number of solutions is:

$$1+3+5+\cdots+21=11^2=121$$

Note:
In the above, (where we used the result for the sum of odd numbers

$$1+3+5+\cdots+(2n-1)=n^2$$

Example 1.167

Find the number of solutions to:

$$x+y+z=20, x, y \text{ are even numbers}$$

We do casework based on z, which can take values from 0 to 20.

z	$x+y$	$\frac{x+y}{2}$	$No. of\ Solutions$
0	20	10	11
2	18	9	10
4	16	8	9
.	.	.	.
.	.	.	.
.	.	.	.
20	0	0	1

The tota number of solutions is:

$$1+2+3+\cdots+11=\frac{11\times 12}{2}=66$$

M. Basics

Example 1.168

Brady is stacking 600 plates in a single stack. Each plate is colored black, gold, or red. Any black plates are always stacked below any gold plates, which are always stacked below any red plates. The total number of black plates is always a multiple of two, the total number of gold plates is always a multiple of three, and the total number of red plates is always a multiple of six. For example, the plates could be stacked with:

- 180 black plates below 300 gold plates below 120 red plates, or
- 450 black plates below 150 red plates, or
- 600 gold plates

In how many different ways could Brady stack the plates **(Gauss 8/2017/25)**

Let the number of:

$$black\ plates=2b, \qquad gold\ plates=3g, \qquad red\ plates=6r$$

We want to determine the number of nonnegative integer solutions to

$$2b+3g+6r=600$$

Rearrange and note that since the RHS is even, the LHS must also be even:

$$\underbrace{2b}_{Even} + 3g = \underbrace{600 - 6r}_{Even}$$

Since $2b$ is even $3g$ must also be even:

$$3g \text{ is even} \Rightarrow g \text{ is a multiple of } 2 \Rightarrow g = 2G \Rightarrow 3g = 6G$$

Again, look at the equation:

$$2b + 6G = 600 - 6r$$
$$b = 3(100 - r - G)$$

Since the RHS is a multiple of 3, the LHS must also be a multiple of 3.

$$b \text{ is a multiple of } 3 \Rightarrow b = 3B \Rightarrow 2b = 6B$$

Hence, with a further substitution $r = R$ our equation becomes:

$$6B + 6G + 6R = 600$$
$$B + G + R = 100$$

We need non-negative integer solutions for the above:

$$B = 100 \Rightarrow G + R = 0 \Rightarrow 1 \text{ Solution}$$
$$B = 99 \Rightarrow G + R = 1 \Rightarrow 2 \text{ Solutions}$$
$$B = 98 \Rightarrow G + R = 2 \Rightarrow 3 \text{ Solutions}$$
$$\vdots$$
$$B = 0 \Rightarrow G + R = 100 \Rightarrow 101 \text{ Solutions}$$

Total number of solutions is:

$$1 + 2 + 3 + \cdots + 101 = \frac{101(102)}{2} = 5151$$

1.8 Divisibility

A. Basics

Example 1.169

How many five digit numbers of the form $347AB$, where A and B are digits, are divisible by 3.

Method I: Enumerating Solutions

Apply the test of divisibility by 3:

$$3 + 4 + 7 + A + B = 3k, k \in \mathbb{N}$$
$$A + B + 14 = 3k, k \in \mathbb{N}$$

Subtract 12 from the left hand side:

$$A + B + 14 - 12 = 3k - 12, k \in \mathbb{N}$$
$$A + B + 12 = 3(k - 4), k \in \mathbb{N}$$

Substitute $K = k - 4$:

$$A + B + 2 = 3K, \qquad K \in \mathbb{N}$$

$$A + B = 1 \Rightarrow (A, B) = (0,1), (1,0) \Rightarrow 2 \text{ Solutions}$$

$$A+B=4 \Rightarrow (A,B)=(0,4),(1,3),\dots,(4,0) \Rightarrow 5\ Solutions$$
$$A+B=7 \Rightarrow (A,B)=(0,7),(1,6),\dots,(7,0) \Rightarrow 8\ Solutions$$
$$A+B=10 \Rightarrow (A,B)=(1,9),(2,8),\dots,(9,1) \Rightarrow 9\ Solutions$$
$$A+B=13 \Rightarrow (A,B)=(4,9),(5,8),\dots,(9,4) \Rightarrow 6\ Solutions$$
$$A+B=16 \Rightarrow (A,B)=(7,9),(8,8),\dots,(9,7) \Rightarrow 3\ Solutions$$

$$2+5+8+9+6+3=33$$

Method II: Counting Lists

Note that 34701 *is divisible by* 3. Hence, the numbers that are divisible by 3 are:
$$34701, 34704, \dots, 34797$$

The first three digits in each number above are the same. They can be ignored (for the purposes of counting the number of elements in the list)
$$AB \in \{01, 04, \dots, 97\}$$

We want to count the number of elements in the list above. Add 2 to each number
$$\{03, 06, \dots, 99\}$$

Divide each number by 3:
$$\{1, 2, \dots, 33\} \to 33\ Values$$

Example 1.170

How many five digit numbers of the form $XY614$, where X and Y are digits, are divisible by 3.

Apply the test of divisibility by 3:
$$X+Y+6+1+4=3k, k \in \mathbb{N}$$
$$X+Y+11=3k, k \in \mathbb{N}$$

Subtract 9 from the left hand side:
$$X+Y+11-9=3k-9, k \in \mathbb{N}$$
$$X+Y+2=3(k-3), k \in \mathbb{N}$$

Substitute $K=k-4$:
$$X+Y+2=3K, \qquad K \in \mathbb{N}$$

This is the same as the previous example$347AB$, except that $X \neq 0$. Since there are 33 solutions, of which 3 have the first digit as zero, subtract those solutions:
$$33-3=30$$

$$A+B=1 \Rightarrow (A,B)=(0,1),(1,0) \Rightarrow 2\ Solutions$$
$$A+B=4 \Rightarrow (A,B)=(0,4),(1,3),\dots,(4,0) \Rightarrow 5\ Solutions$$
$$A+B=7 \Rightarrow (A,B)=(0,7),(1,6),\dots,(7,0) \Rightarrow 8\ Solutions$$
$$A+B=10 \Rightarrow (A,B)=(1,9),(2,8),\dots,(9,1) \Rightarrow 9\ Solutions$$
$$A+B=13 \Rightarrow (A,B)=(4,9),(5,8),\dots,(9,4) \Rightarrow 6\ Solutions$$
$$A+B=16 \Rightarrow (A,B)=(7,9),(8,8),\dots,(9,7) \Rightarrow 3\ Solutions$$

Example 1.171

How many five digit numbers of the form $3A41B$, where A and B are digits, are divisible by 11, but not divisible by 110.

Example 1.172

Let $x = 352, ABC, 052$ where A, B and C represent digits that are not necessarily distinct. What is the probability that x is not divisible by 11? **(MathCounts State Sprint, 2023, Adapted)**

Total Outcomes:

The number of total outcomes for A, B, C is

$$10^3 = 1{,}000$$

Successful Outcomes

Use complementary counting:

$$352, ABC, 052 \text{ is divisible by } 11 \text{ if and only if } 352{,}052, ABC \text{ is divisible by } 11$$

$$352, ABC, 052: (5 + A + C + 5) - (3 + 2 + B + 0 + 2) = (A + C + 10) - (7 + B) = 3 + A + C - B$$
$$352{,}052, ABC: 3 + 2 + 5 + A + C - (5 + 0 + 2 + B) = 10 + A + C - (7 + B) = 3 + A + C - B$$

$$ABC \in \{008, 019, \dots, 998\}$$

Add 3

$$ABC \in \{011, 022, \dots, 1001\} \to 91 \text{ values}$$

The number of values which are not divisible by 11 is:

$$1000 - 91 = 909$$

Probability

$$\frac{\text{Successful Outcomes}}{\text{Total Outcomes}} = \frac{909}{1000}$$

Apply complementary counting. Count the number of values that are divisible by 11:

Apply the test of divisiblity by 11:

$$(5 + A + C + 5) - (3 + 2 + B + 0 + 2) = (A + C + 10) - (7 + B) = 3 + A + C - B$$

The above expression has

Maximum value when: $A = 9, C = 9, B = 0$

$$3 + A + C - B = 3 + 9 + 9 - 0 = 21$$

Minimum value when $A = 0, C = 0, B = 9$

$$3 + A + C - B = 3 + 0 + 0 - 9 = -6$$

So, the cases that we need to consider are:

$$-6 < 3 + A + C - B < 21 \Rightarrow \text{Possible values are } 0 \text{ and } 11$$

Case I: 3+A+C-B=0

$$A + C + 3 = B$$

Consider cases for B:

$$B=0,1,2: \textit{Not possible}$$
$$B=3: A+C=0: (A,C)=(0,0): 1\ \textit{Solution}$$
$$B=4: A+C=1: (A,C)=(0,1),(1,0): 2\ \textit{Solutions}$$
$$B=5: A+C=2: (A,C)=(0,2),(1,1),(2,0): 3\ \textit{Solutions}$$
$$B=6: A+C=3: 4\ \textit{Solutions}$$
$$B=7: A+C=4: 5\ \textit{Solutions}$$
$$B=8: A+C=7: 6\ \textit{Solutions}$$
$$B=9: A+C=8: 7\ \textit{Solutions}$$

$$1+2+\ldots+7=\frac{7(8)}{2}=28$$

Case II: 3+A+C-B=11

$$A+C=B+8$$

Consider cases for B:

$$B=0: A+C=8: 9\ \textit{Solutions}$$
$$B=1: A+C=9: 10\ \textit{Solutions}$$
$$B=2: A+C=10: (1,9)(2,8),\ldots(9,1): 9\ \textit{Solutions}$$
$$B=3: A+C=11: (A,C)=(2,9),(3,8),\ldots(9,2): 8\ \textit{Solutions}$$
$$B=4: A+C=12: 7\ \textit{Solutions}$$
$$B=5: A+C=13: 6\ \textit{Solutions}$$
$$B=6: A+C=14: 5\ \textit{Solutions}$$
$$B=7: A+C=15: 4\ \textit{Solutions}$$
$$B=8: A+C=16: 3\ \textit{Solutions}$$
$$B=9: A+C=17: 2\ \textit{Solutions}$$

$$9+10+(9+8+\ldots+1)-1=19+45-1=63$$

The final answer, obtained by adding Cases I and II is:

$$28+63=91$$

1.9 Stars and Bars

A. Basics

Example 1.173

Find the number of ways that 9 identical chocolates can be distributed among two children, such that each child gets at least one chocolate.

The above question is equivalent to solving:

$$\underbrace{a}_{\substack{\text{Chocolates for}\\ \text{first child}}}+\underbrace{b}_{\substack{\text{Chocolates for}\\ \text{second child}}}=9,\qquad a,b\in\mathbb{N}$$

The minimum number of chocolates that each child gets must be 1. So, give them those chocolates. Mathematically, this is a change of variables.

$$A=a-1,\qquad B=b-1\Rightarrow A+B=7$$

We have seven chocolates to be distributed among two children.

$$CCCCCCC$$

Let us introduce a divider to help us decide how to distribute the chocolates.

$$\underbrace{|}_{Divider}$$

So, in all, we have seven chocolates and one divider:

$$CCCCCCC|$$

The position of the divider tells us which child gets how many chocolates. Some sample positions of the divider are given below:

$$\underbrace{\underbrace{\underbrace{CC}_{Child\ 1}}_{Two\ Chocolates}\quad | \quad \underbrace{\underbrace{CCCCC}_{Child\ 2}}_{Five\ Chocolates}}_{Position\ 1}, \qquad \underbrace{\underbrace{\underbrace{CCCC}_{Child\ 1}}_{Four\ Chocolates}\quad | \quad \underbrace{\underbrace{CCC}_{Child\ 2}}_{Three\ Chocolates}}_{Position\ 2}$$

Note that once we place the divider, we have only one way to place the chocolates.
Hence, the number of ways of distributing the chocolates is the same as the number of ways of placing the divider.
Number of Chocolates $= 7$
Number of Dividers $= 1$
Total Objects $= 8$

Number of ways of placing one divider in eight places:

$$8\ Choose\ 1 = \binom{8}{1} = \frac{8!}{1!\,7!} = 8$$

Number of bars needed to divide the stars is the same as the number of plus signs in the equation.

Example 1.174

Find the numbers of ways that 9 distinguishable chocolates can be distributed among two children, such that each child gets at least one chocolate.

$$\underbrace{2}_{\substack{First\\Chocolate:\\Two\ Choices}} \times \underbrace{2}_{\substack{Second\\Chocolate:\\Two\ Choices}} \times \ldots \times \underbrace{2}_{\substack{Ninth\\Chocolate:\\Two\ Choices}} = 2^9 = 512$$

$$512 - 2 = 510$$

Example 1.175

How many ways are there to distribute 7 indistinguishable balls to two distinguishable boxes?

$$a + b = 7, \qquad a, b \in \mathbb{W}$$

We have seven balls:

$$BBBBBBB$$

Let us introduce a divider to help us decide how to distribute the chocolates.

$$\underbrace{|}_{Divider}$$

This gives us eight objects in all. By placing the divider, we can decide which box gets how many balls.

Total Number of Objects

$$= 7\ Balls + 1\ Divider = 8\ Objects$$

And the number of ways of placing the divider uniquely determines the number of ways to distribute the balls:

$$\binom{No.of\ Balls + No.of\ Dividers}{No.of\ Dividers} = \binom{7+1}{1} = \binom{8}{1} = 8$$

Example 1.176

How many ways are there to distribute 7 distinguishable balls to two distinguishable boxes?

$$\underbrace{2}_{\substack{First\\Ball:\\Two\ Choices}} \times \underbrace{2}_{\substack{Second\\Ball:\\Two\ Choices}} \times \ldots \times \underbrace{2}_{\substack{Seventh\\Ball:\\Two\ Choices}} = 2^7 = 128$$

Example 1.177

Solve

$$a + b + c = 10, \qquad a, b, c \in \mathbb{N}$$

Since each of the above variables must be at least one, define new variables such that

$$A = a - 1, \qquad B = b - 1, \qquad C = c - 1$$

Which gives us:

$$A + B + C = 7$$

Objects and Dividers

We have seven stars

$$\boldsymbol{C\ C\ C\ C\ C\ C\ C}$$

We will need two dividers:

$$|\ |$$

This gives nine objects in all:

$$\boldsymbol{C\ C\ C\ C\ C\ C\ C}\,|\,|$$

A sample position of the dividers is given below:

$$\underbrace{\underbrace{\boldsymbol{C}}_{\substack{\boldsymbol{Child\ 1}\\\boldsymbol{One\ Chocolates}}} \quad | \quad \underbrace{\boldsymbol{C\ C\ C\ C}}_{\substack{\boldsymbol{Child\ 2}\\\boldsymbol{Four\ Chocolates}}} \quad | \quad \underbrace{\boldsymbol{C\ C}}_{\substack{\boldsymbol{Child\ 3}\\\boldsymbol{Two\ Chocolates}}}}_{\boldsymbol{Position\ 1}}$$

Placing the Dividers

Out of these objects, we need to choose the position of the two dividers, giving us:

$$\binom{No.of\ Chocolates + No.of\ Dividers}{No.of\ Dividers} = \binom{7+2}{2} = \binom{9}{2} = \frac{9!}{7!\,2!} = \frac{9 \times 8}{2} = 36\ Solutions$$

1.178: Number of Dividers

The number of dividers is always one less than the number of variables.
A quick way to remember this is:

$$No.of\ Dividers = No.of\ Plus\ Signs$$

B. Applications

Example 1.179
Three friends have a total of 6 identical pencils, and each one has at least one pencil. In how many ways can this happen? **(AMC 8 2004/17)**

$$a + b + c = 6$$
$$a' + b' + c' = 3$$

Use Stars and Bars with three objects and two dividers to get:

$$\binom{No. of\ Pencils + No. of\ Dividers}{No. of\ Dividers} = \binom{3+2}{2} = \binom{5}{2} = 10\ Ways$$

Example 1.180 (Re-visited)
Alice has 24 apples. In how many ways can she share them with Becky and Chris so that all of them have at least 2 apples? **(AMC 8 2019/25)**

$$a + b + c = 24$$
$$A + B + C = 18$$

$$\underbrace{Apples}_{Alice},\quad \underbrace{Divider}_{\square},\quad \underbrace{Apples}_{Becky},\quad \underbrace{Divider}_{\square},\quad \underbrace{Apples}_{Chris}$$

We have $\underbrace{18\ apples}_{Stars}$ and $\underbrace{two\ dividers}_{Bars}$. We use the 2 bars as dividers to decide which variable the 18 stars are allocated to.
So, the final number of possible arrangements is:

$$\binom{No. of\ Apples + No. of\ Dividers}{No. of\ Dividers} = \binom{18+2}{2} = \binom{20}{2} = \frac{20!}{2!\,18!} = \frac{20 \times 19}{2!} = 190$$

Example 1.181
Seven people are to be divided into three teams called A, B and C. Such that each team has at least one person. (Teams can have different number of people).

$$\underbrace{3}_{\substack{First\\Person:\\Three\ Choices}} \times \underbrace{3}_{\substack{Second\\Person:\\Three\ Choices}} \times \ldots \times \underbrace{3}_{\substack{Seventh\\Person:\\Three\ Choices}} = 3^7$$

Example 1.182
You roll three, standard, six-sided dice. How many outcomes are possible, if order is important?

$$6 \times 6 \times 6 = 6^3 = 216$$

What if order of outcomes is not important?

$$a = No. of\ 1's\ Rolled$$
$$b = No. of\ 2's\ Rolled$$
$$And\ so\ on$$

$$a+b+c+d+e+f=3$$

$$\binom{No.of\ Dice\ Outcomes\ +No.of\ Dividers}{No.of\ Dividers}=\binom{3+5}{5}=\binom{8}{5}=\frac{8!}{3!\,5!}=\frac{8\times 7\times 6}{6}=56$$

Example 1.183

You draw a card from a standard pack of playing cards, and then put it back. You repeat this three times, to draw three cards. What is the number of possible outcomes?

$$52\times 52\times 52=52^3$$

What if order of outcomes is not important?

$$c_1+c_2+\cdots+c_{52}=3$$

$$\binom{No.of\ Cards\ Chosen\ +No.of\ Dividers}{No.of\ Dividers}=\binom{3+51}{51}=\binom{54}{51}=\binom{54}{3}$$

Example 1.184

PizzaExpress, your favorite pizza chain has a choice of six toppings: cherry tomatoes, chargrilled vegetables, mushrooms, mixed jalapenos, cheese, finely chopped green chilies. You have the choice of picking one or more toppings. However, toppings cannot be repeated. In how many ways can you order a pizza? (The order of picking the toppings does not matter).

$$Cherry\ Tomato: Pick\ or\ Don't\ Pick: 2\ Choices$$
$$Vegetables: Pick\ or\ Don't\ Pick: 2\ Choices$$
$$And\ So\ On$$

$$2^6=64$$

$$64-1=63$$

You ask for six toppings(*one of each type*), and the order of placing the toppings matters. How many ways can the chef do that?

$$6\times 5\times 4\times 3\times 2\times 1=6!=720$$

Your friend orders six toppings, but all six toppings should not all be the same. How many different pizzas fit this condition? (The order of toppings does *not* matter.)

$$a+b+c+d+e+f=6$$

$$\binom{No.of\ Toppings\ Chosen\ +No.of\ Dividers}{No.of\ Dividers}=\binom{6+5}{5}=\binom{11}{5}$$

C. More Variables

Example 1.185

Bill is sent to a donut shop to purchase exactly six donuts. If the shop has four kinds of donuts and Bill is to get at least one of each kind, how many combinations will satisfy Bill's order requirements? **(MathCounts 2008 State Target)**

$$a+b+c+d=6$$

We have to get one of each kind of donut.

$$a'+b'+c'+d'=2$$

$$\binom{No.of\ Donuts\ +No.of\ Dividers}{No.of\ Dividers}=\binom{2+3}{3}=\binom{5}{3}=\frac{5!}{2!\,3!}=10$$

Example 1.186

How many seven letter words can you make using the vowels from the English language, if the order of the letters is not important.

If order were important:

$$AAAAEEE\neq EEEAAAA$$

However, order is not important, and hence:

$$AAAAEEE=EEEAAAA$$

So, we are only concerned with how many times we are taking each vowel.

$$a+b+c+d+e=7$$

$$\binom{No.of\ Letters\ +No.of\ Dividers}{No.of\ Dividers}=\binom{7+4}{4}=\binom{11}{4}=\frac{11\times 10\times 9\times 8}{2\times 3\times 4}=330$$

D. Non-Increasing Numbers

Example 1.187

What is the number of five digit numbers with non-increasing digits?

$$5\ Digits\Rightarrow 5\ Stars$$
$$10\ Digits\Rightarrow 9\ Dividers$$

Apply Stars and Bars with 5 Stars and 9 Dividers

$$\binom{No.of\ Stars\ +No.of\ Dividers}{No.of\ Dividers}=\binom{5+9}{2}=\binom{14}{9}=\frac{14\times 13\times 12\times 11\times 10}{2\times 3\times 4\times 5}=2002$$

Example 1.188

AIME Reference.
More Questions

E. Probability

1.189: Probability

$$Probability = \frac{Successful\ Outcomes}{Total\ Outcomes}$$

We will use the formula above, and count both types of outcomes using counting rules.
If you need to understand probability, you can look at the Notes on Probability.

F. Sum of Numbers on Dice

Example 1.190

Rolling Two Dice to get a sum of 5

Example 1.191

If three standard, six-faced dice are rolled, what is the probability that the sum of the three numbers rolled is 9? Express your answer as a common fraction. **(MathCounts 2009 State Sprint)**

Sucessful Outcomes

$$a + b + c = 9, \qquad a, b, c \le 6$$
$$a' + b' + c' = 6, \qquad a', b'c' \le 5$$

Use Stars and Bars with 6 stars and 2 bars

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{6+2}{2} = \binom{8}{2} = \frac{8 \times 7}{2} = 28$$

We do not want:

$$6 + 0 + 0, 0 + 6 + 0, 0 + 0 + 6 \Rightarrow 3\ Cases$$

Hence, subtract these to get:

$$28 - 3 = 25$$

Total Outcomes

By the multiplication Rule, the number of outcomes when you roll three dice is

$$6 \times 6 \times 6 = 216$$

Probability

$$P = \frac{Successful\ Outcomes}{Total\ Outcomes} = \frac{25}{216}$$

Example 1.192

Rolling Four Dice

G. Missing Numbers

Example 1.193

A fair, twenty-faced die has 19 of its faces numbered from 1 through 19 and has one blank face. Another fair, twenty-faced die has 19 of its faces numbered from 1 through 8 and 10 through 20 and has one blank face. When the two dice are rolled, what is the probability that the sum of the two numbers facing up will be 24? Express your answer as a common fraction. **(MathCounts 2008 Chapter Sprint)**

Successful Outcomes

$$a + b = 24, a \le 19, b \le 20, b \ne 9$$

$$(a, b) = (4,20), (6,18), \dots, (19,5) \Rightarrow 19 - 4 + 1 = 16$$

Also, we cannot have

$$(15,9)$$

Hence, final number of succesful outcomes is:

$$16 - 1 = 15$$

Total Outcomes

$$20 \times 20 = 400$$

Probability

$$P = \frac{Successful\ Outcomes}{Total\ Outcomes} = \frac{15}{400} = \frac{3}{80}$$

H. Multiples

Example 1.194

Brady is stacking 600 plates in a single stack. Each plate is colored black, gold, or red. Any black plates are always stacked below any gold plates, which are always stacked below any red plates. The total number of black plates is always a multiple of two, the total number of gold plates is always a multiple of three, and the total number of red plates is always a multiple of six. For example, the plates could be stacked with:

- 180 black plates below 300 gold plates below 120 red plates, or
- 450 black plates below 150 red plates, or
- 600 gold plates

In how many different ways could Brady stack the plates **(Gauss 8/2017/25)**

Let the number of:

$$black\ plates = 2b, \qquad gold\ plates = 3g, \qquad red\ plates = 6r$$

We want to determine the number of nonnegative integer solutions to

$$2b + 3g + 6r = 600$$

Rearrange and note that since the RHS is even, the LHS must also be even:

$$\underbrace{2b}_{Even} + 3g = \underbrace{600 - 6r}_{Even}$$

Since $2b$ is even $3g$ must also be even:

$$3g\ is\ even \Rightarrow g\ is\ a\ multiple\ of\ 2 \Rightarrow g = 2G \Rightarrow 3g = 6G$$

Again, look at the equation:

$$2b + 6G = 600 - 6r$$
$$b = 3(100 - r - G)$$

Since the RHS is a multiple of 3, the LHS must also be a multiple of 3.

$$b\ is\ a\ multiple\ of\ 3 \Rightarrow b = 3B \Rightarrow 2b = 6B$$

Hence, with a further substitution $r = R$ our equation becomes:

$$6B + 6G + 6R = 600$$
$$B + G + R = 100$$

We need non-negative integer solutions for the above. Use stars and bars with:

$$100\ Stars, 2\ Bars$$

We have to choose the position of the 2 bars among the 100 stars, which is:

$$\binom{102}{2} = \frac{102!}{2!\,100!} = \frac{101(102)}{2} = 5151$$

I. Gap Method Questions

Example 1.195

What is the number of ways to pick three integers out of the numbers 1 to 10 such that no two are consecutive.

We need to pick three numbers from 1 to 10. Lets these numbers be

$$N_1, N_2, N_3$$

We must have numbers between N_1 and N_2, and N_2 and N_3

$$N_1\ g_1\ N_2\ g_2 N_3, \qquad g_1, g_2 > 0$$

And we can (optionally) have numbers before N_1 and after N_3

$$G_3\ N_1\ g_1\ N_2\ g_2 N_3\ G_4, \qquad g_1, g_2 > 0, \qquad G_3, G_4 \geq 0$$

Then, we must have

$$G_3 + g_1 + g_2 + G_4 = 7$$

Since $g_1, g_2 > 0$, substitute $G_1 = g_1 + 1, G_2 = g_2 + 1$

$$G_3 + (G_1 + 1) + (G_2 + 1) + G_4 = 7$$
$$G_3 + G_1 + G_2 + G_4 = 5$$

Now, apply stars and bars with

$$\binom{5\ bars + 3\ stars}{3\ stars} = \binom{8}{3} = 56$$

1.10 Diophantine Inequalities

A. Basics

1.196: Positive Integers

Positive integers are integers which are greater than zero, that is

$$\mathbb{N} = \{1,2,3, \dots\}$$

They are also called natural numbers.

Example 1.197

$$a + b < 7$$

A. Find the number of ordered pairs $(a, b), a, b \in \mathbb{N}$ which are solutions for the above inequality.
B. Find the number of unordered pairs $(a, b), a, b \in \mathbb{N}$ which are solutions for the above inequality.

Part A

Minimum value of

$$a = 1, b = 1 \Rightarrow a + b = 2$$

So, we need to find solutions for $2 \le a + b < 7$:

						Total
$a+b$	2	3	4	5	6	
Solutions	1	2	3	4	5	$= \frac{5 \times 6}{2} = 15$
	(1,1)	(1,2) (2,1)	(3,1) (2,2) (1,3)	(4,1) (2,3) (3,2) (1,4)	(5,1) (4,2) (3,3) (2,4) (1,5)	

Part B

						Total
$a+b$	2	3	4	5	6	
Solutions	1	1	2	2	3	$= 9$
	$\mathbf{(1,1)}$	(1,2) (2,1)	(3,1) $\mathbf{(2,2)}$ (1,3)	(4,1) (2,3) (3,2) (1,4)	(5,1) (4,2) $\mathbf{(3,3)}$ (2,4) (1,5)	

1.198: Slack Variable

A slack variable is used to convert an inequality into an equality

$$a < 7$$
$$a + Slack = 7$$

Where

$$Slack = 7 - a$$

Example 1.199

Find the number of positive integer solutions for:

$$a + b < 7$$

Use a change of variable. $a = A + 1, b = B + 1$

$$A + B < 5$$

Convert the inequality to also include an equality condition:

$$A + B \le 4$$

If $A + B$ is less than 4, then we can asssign the leftover
Introduce a new variable $C = 4 - A - B$, and convert the inequality into an equation.

$$A + B + C = 4$$

The equation on the right corresponds to the inequality on the left because they both have the same number of solutions.

Consider cases for C:

C	4	3	2	1	0	Total
$A + B$	0	1	2	3	4	
Solutions	1	2	3	4	5	15

Example 1.200 (Stars and Bars)

Find the number of positive integer solutions for:

$$a + b < 7$$

As before

$$A + B + C = 4$$

$$\mathbf{CCCC}||$$

Now, we have 4 stars and two bars, giving us:

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{4+2}{2} = \binom{6}{2} = \frac{6 \times 5}{2} = 15$$

Example 1.201

Find the number of positive integer solutions for:

$$a + b < 9$$

$$a' = a - 1 \Rightarrow a = a' + 1$$
$$b' = b - 1 \Rightarrow b' + 1$$

$$a' + b' < 7$$
$$a' + b' \le 7$$
$$a' + b' + c' = 6$$

Now, we have 6 Stars and 2 Dividers, giving us:

$$\binom{No.of\ Stars\ + No.of\ Dividers}{No.of\ Dividers} = \binom{6+2}{2} = \binom{8}{2} = \frac{8 \times 7}{2} = 28$$

1.202: Positive Integer Solutions

$$a + b < n \text{ has } \frac{(n-1)(n-2)}{2} \textit{ positive integer solutions}$$

Note: Don't memorize this result. Instead, follow the process every time.

$$a + b < n, \qquad n \in \mathbb{N}, n \geq 2$$

$$a' + b' < n - 2$$
$$a' + b' + c' = n - 3$$

Now, we have $n - 3$ stars and 2 dividers, giving us:

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{(n-3)+2}{2} = \binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$$

B. Questions with $\leq$

Example 1.203

Find the number of positive integer solutions:

$$a + b \leq 9$$

$$a' + b' \leq 7$$
$$a' + b' + c = 7$$

Use Stars and Bars with 7 Stars and 2 Dividers:

$$\binom{7+2}{2} = \binom{9}{2} = \frac{9 \times 8}{2} = 36$$

Example 1.204

I need to distribute upto ten pencils to my two friends, such that each friend gets at least one pencil. Find the number of ways in which I can do it.

$$a + b \leq 10$$
$$a' + b' \leq 8$$
$$a' + b' + c' = 8$$

We now have 8 Stars, and 2 Bars, giving us:

$$\binom{No.of\ Pencils\ + No.of\ Dividers}{No.of\ Dividers} = \binom{8+2}{2} = \binom{10}{2} = \frac{10 \times 9}{2} = 45$$

1.205: Positive Integer Solutions with $\leq$

$$a + b \leq n \text{ has } \frac{(n)(n-1)}{2} \textit{ positive integer solutions}$$

Note: Don't memorize this result. Instead, follow the process every time.

$$a + b \leq n, \qquad n \in \mathbb{N}, n \geq 2$$

$$a' + b' \le n - 2$$
$$a' + b' + c' = n - 2$$

Now, we have $n - 2$ stars and 2 dividers, giving us:

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{(n-2)+2}{2} = \binom{n}{2} = \frac{(n)(n-1)}{2}$$

Example 1.206

A charity worker has twenty food packets to be distributed in two flood hit areas such that each area receives at least two packets, and at least ten packets are distributed overall. Find the number of ways in which the distribution can take place.

Let the number of packets

$$Distributed\ in\ the\ first\ area = a, \qquad a \in \mathbb{W}$$
$$Distributed\ in\ the\ second\ area = b, \qquad b \in \mathbb{W}$$

Now, the conditions the question imposes are:

$$10 \le a + b \le 20, \qquad a \ge 2, \qquad b \ge 2$$

$$a' = a - 2 \Rightarrow a = a' + 2$$
$$b' = b - 2 \Rightarrow b = b' + 2$$

$$10 \le a' + 2 + b' + 2 \le 20$$
$$6 \le a' + b' \le 16$$

$$a' + b' \le 16$$
$$a' + b' + c' = 16$$

Use Stars and Bars with 16 Stars and 2 Bars

$$\binom{16+2}{2} = \binom{18}{2} = \frac{18 \times 17}{2} = 153$$

$$a' + b' \le 6$$
$$a' + b' + c' = 6$$

Use Stars and Bars with 6 Stars and 2 Bars

$$\binom{6+2}{2} = \binom{8}{2} = \frac{8 \times 7}{2} = 28$$

$$153 - 28 = 125$$

C. Back Calculations

Example 1.207

The number of solutions to the inequality below for positive integer values is 6. Find the value of n.

$$a + b < n$$

$$\frac{(n-1)(n-2)}{2} = 6 \Rightarrow (n-1)(n-2) = 12 = 3 \times 4 \Rightarrow n = 5$$

D. Non-Negative Integer Solutions

1.208: Non-Negative Integers

Non-negative integers are integers which are not negative, that is

$$\mathbb{W} = \{0,1,2,3,\ldots\}$$

They are also called whole numbers.

Example 1.209

Solve for non-negative integer values:

$$a + b < 6$$

$$a + b + c = 5$$

Use Stars and Bars with 5 Stars and 2 Dividers

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{5+2}{2} = \binom{7}{2} = \frac{7 \times 6}{2} = 21$$

Example 1.210

What is the number of ways to distribute a maximum of eleven chocolates to two children?

$$a + b < 11$$
$$a + b \le 10$$
$$a + b + c = 10$$

Use Stars and Bars with 10 Stars and 2 Dividers:

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{10+2}{2} = \binom{12}{2} = \frac{12 \times 11}{2} = 66$$

1.211: Non-Negative Integer Solutions

$$a + b < n \text{ has } \frac{(n+1)(n)}{2}\ non - negative\ integer\ solutions$$

Note: Don't memorize this result. Instead, follow the process every time.

$$a + b < n, \qquad n \in \mathbb{N}, n \ge 2$$
$$a + b \le n - 1$$
$$a + b + c = n - 1$$

Now, we have n stars and 2 dividers, giving us:

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{n-1+2}{2} = \frac{(n+1)(n)}{2}$$

Example 1.212

Solve for non-negative integer values:

$$a + b \le 9$$

$$a + b + c = 9$$

Use Stars and Bars with 9 Stars and 2 Dividers:

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{9+2}{2} = \binom{11}{2} = \frac{11 \times 10}{2} = 55$$

1.213: Non-Negative Integer Solutions

$$a + b \leq n \text{ has } \frac{(n+2)(n+1)}{2}\ non - negative\ integer\ solutions$$

Note: Don't memorize this result. Instead, follow the process every time.

$$a + b \leq n, \qquad n \in \mathbb{N}, n \geq 2$$

$$a + b + c = n$$

Now, we have n stars and 2 dividers, giving us:

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{n+2}{2} = \frac{(n+2)(n+1)}{2}$$

E. "More than One" Minimum

Example 1.214

In how many ways can Alice give upto 24 apples to Becky and Chris so that each has at least 2 apples?

$$b + c \leq 24$$

Now, each one of them needs at least two apples. So, give them the two apples:

$$b' + c' \leq 20$$
$$a' + b' + c' = 20$$

Use Stars and Bars with 20 Stars and 2 Dividers

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{20+2}{2} = \binom{22}{2} = \frac{22 \times 21}{2} = 231$$

F. Negative Integer Ranges

Example 1.215

Solve for integer values:

$$a + b < 5, \qquad a, b > -4$$

$$a + b < 5$$

$$a' = a + 4 \Rightarrow a = a' - 4$$
$$b' = b + 4 \Rightarrow b = b' - 4$$

$$a' - 4 + b' - 4 < 5$$
$$a' + b' < 13$$
$$a' + b' + c' = 12$$

Use Stars and Bars with 12 Stars and 2 Dividers

$$\binom{No.of\ Objects\ +No.of\ Dividers}{No.of\ Dividers} = \binom{12+2}{2} = \binom{14}{2} = \frac{14\times 13}{2} = 91$$

G. Three Variables

Example 1.216

A total of n ordered quadruples (a, b, c, d) satisfy $a+b+c+d<24$, where a, b, c, and d are positive integers. What are the last two digits of n? **(Stormersyle Mock AMC 8, 2018)**

$$a+b+c+d<24$$

Convert the constraint that each variable must be positive by subtracting one from each variable (and hence 4 from the RHS):

$$a'+b'+c'+d'<20$$
$$a'+b'+c'+d'\le 19$$

Convert the inequality into an equation by introducing a new variable:

$$a'+b'+c'+d'+e'=19$$

Now, we have 19 objects and we need 4 dividers.

Number of ways to solve the equation above is the same as the number of ways of placing 4 dividers among $19+4=23$ places, which is given by:

$$\binom{19+4}{4} = \binom{23}{4} = \frac{23\times 22\times 21\times 20}{2\times 3\times 4} = 23\times 11\times 7\times 5 = 8{,}855 \Rightarrow Last\ two\ digits = 55$$

H. Multiples

Example 1.217

Find the number of positive integer solutions for a and b:

$$2a+3b<30$$

Consider the values that b can take:

$$b = \{1,2,3,4,5,6,7,8,9\} \Rightarrow 3b = \{3,6,9,12,15,18,21,24,27\}$$

We tabulate:

b	$3b$	$30-3b$	$2A$	A
1	3	27	$\{2,4,\dots,26\}$	13
2	6	24	$\{2,4,\dots,24\}$	12
3	9	21	$\{2,4,\dots,20\}$	10
4	12	18	$\{2,4,\dots,18\}$	9
5	15	15	$\{2,4,\dots,14\}$	7
6	18	12	$\{2,4,\dots,12\}$	6
7	21	9	$\{2,4,\dots,8\}$	4
8	24	6	$\{2,4,6\}$	3
9	27	3	$\{2\}$	1
			Total	65

Example 1.218

Find the number of positive integer solutions for a and b:

$$5a + 7b < 100$$

Example 1.219

Find the number of ordered triples (a, b, c) of positive integers such that: $30a + 50b + 70c \le 343$ (IOQM 2019/17)

Divide both sides by 10:

$$3a + 5b + 7c \le 34.3$$

Since a, b, c are integers, we must have:

$$3a + 5b + 7c \le 34$$
$$3(x + 1) + 5(y + 1) + 7(z + 1) \le 34$$

Use a change of variable. Let $a = x + 1, b = y + 1, c = z + 1$:

$$3x + 5y + 7z + 3 + 5 + 7 \le 34$$
$$3x + 5y + 7z \le 19$$

Split into cases on the basis of the possible values of z, starting with the largest possible value of z is $z = 2$.

Case I: $z = 2$

$$3x + 5y + 14 \le 19$$
$$3x + 5y \le 5$$

$$(3x, 3y) = (0,0), (0,5), (3,0) \Rightarrow 3\ values$$

Case II: $z = 1$

$$3x + 5y + 7 \le 19$$
$$3x + 5y \le 12$$

$$5y = 0 \Rightarrow 3x \in \{0,3,6,9,12\} \Rightarrow 5\ values$$
$$5y = 5 \Rightarrow 3x \in \{0,3,6\} \Rightarrow 3\ values$$
$$5y = 10 \Rightarrow 3x \in \{0\} \Rightarrow 1\ values$$
$$Total = 5 + 3 + 1 = 9$$

Case III: $z = 0$

$$3x + 5y \le 19$$

$$5y = 0 \Rightarrow 3x \in \{0,3, \dots ,18\} \Rightarrow 7\ values$$
$$5y = 5 \Rightarrow 3x \in \{0,3,6,9,12\} \Rightarrow 5\ values$$
$$5y = 10 \Rightarrow 3x \in \{0,3,6,9\} \Rightarrow 4\ values$$
$$5y = 15 \Rightarrow 3x \in \{0,3\} \Rightarrow 2\ values$$
$$Total = 7 + 5 + 4 + 2 = 18$$

$$18 + 9 + 3 = 30$$

2. Further Topics

2.1 Logarithms

A. Warmup

Example 2.1
If each term is a positive integer, determine the number of solutions to:

$$\log_2 a + \log_2 b + \log_2 c \le 10$$

Each term is a positive integer. Use a change of variable. Let $x = \log_2 a$, $y = \log_2 b$, $z = \log_2 c$

$$x + y + z \le 10$$

Convert to a strict inequality (which we can because the LHS is only integers):

$$x + y + z < 11$$

Convert it into an equation by adding an extra variable:

$$x + y + z + n = 11$$

Apply stars and bars with 11 stars, and 3 bars:

$$\binom{11+3}{3} = \binom{14}{3} = \frac{14 \times 13 \times 12}{6} = 364$$

Example 2.2
Determine the number of ordered triples (a, b, c) that are solutions to:

$$(\log_2 a)(\log_2 b)(\log_2 c) \le 10$$

$\log_2 a$ is a positive integer, and so are each of $\log_2 b$ and $\log_2 c$.

Each term is a positive integer. Use a change of variable. Let $x = \log_2 a$, $y = \log_2 b$, $z = \log_2 c$:

$$xyz \le 10$$

Consider cases for xyz.
Case I: All three are equal:

$$(1,1,1), (2,2,2) \Rightarrow 2\ Solutions$$

Case II: Exactly two of them are the same:

$$(1,1,2), (1,1,3), \dots, (1,1,10) \Rightarrow 9\ Solutions$$
$$(1,2,2), (1,3,3) \Rightarrow 2\ Solutions$$

Each of the above solutions has three different arrangements. For example: (1,1,2), (1,2,1) and (2,1,1). So, the number of solutions from this case is:

$$(9 + 2)(3) = (11)(3) = 33$$

Case III: All three are different:

$$(1,2,3), (1,2,4), (1,2,5)$$

Each of the above solutions can be arranged in $3! = 6\ ways$.

$$Total = 3(6) = 18$$

Final Answer is:

$$2 + 33 + 18 = 33 + 20 = 53$$

B. Floor Function

Example 2.3

$$\lfloor \log_2 a \rfloor + \lfloor \log_2 b \rfloor = 10$$

Given that a, b are distinct integers, determine the number of ordered pairs (a, b) that are solutions of the equation above.

Hint: Logs are defined only for positive integers.

Use a change of variable. Let $x = \lfloor \log_2 a \rfloor, y = \lfloor \log_2 b \rfloor$:

$$x + y = 10$$
$$(x, y) = (0,10), (1,9), (2,8), (3,7), (4,6), (5,5)$$

$$x = 0 \Rightarrow \lfloor \log_2 a \rfloor = 0 \Rightarrow a = 1 \Rightarrow No. of\ Solutions = 1$$
$$x = 1 \Rightarrow \lfloor \log_2 a \rfloor = 1 \Rightarrow a \in \{2,3\} \Rightarrow No. of\ Solutions = 2$$
$$x = 2 \Rightarrow \lfloor \log_2 a \rfloor = 2 \Rightarrow a \in \{4,5,6,7\} \Rightarrow No. of\ Solutions = 4$$
$$x = 3 \Rightarrow \lfloor \log_2 a \rfloor = 3 \Rightarrow a \in \{8,9, \dots ,15\} \Rightarrow No. of\ Solutions = 8$$

$$x = 3 \Rightarrow \lfloor \log_2 a \rfloor = 3 \Rightarrow a \in \{2^3, 2^3, \dots , 2^4 - 1\} \Rightarrow No. of\ Solutions = 8$$
$$x = m \Rightarrow \lfloor \log_2 a \rfloor = m \Rightarrow a \in \{2^m, 2^m + 1, \dots , 2^{m+1} - 1\} \Rightarrow No. of\ Solutions = 8$$
$$x = m \Rightarrow No.\ of\ Solutions = 2^m$$

Consider (0,10):

$$a\ has\ 1\ solution, b\ has\ 2^{10}\ solutions \Rightarrow Total\ Solutions = 2^{10}$$

Since (10,0) has same number of solutions:

$$(0,10) + (10,0) \rightarrow 2 \cdot 2^{10}$$

Consider (1,9) and (9,1)

$$a\ has\ 2\ solution, b\ has\ 2^9\ solutions \Rightarrow Total\ Solutions = 2 \cdot 2^9 \cdot 2 = 2 \cdot 2^{10}$$

Similarly:

$$(2,8)\ \&\ (8,2) \Rightarrow 2 \cdot 2^{10}$$
$$(3,7)\ \&\ (7,3) \Rightarrow 2 \cdot 2^{10}$$
$$(4,6)\ \&\ (6,4) \Rightarrow 2 \cdot 2^{10}$$

Consider (5,5)

$$a\ has\ 2^5\ solution, b\ has\ 2^5\ solutions \Rightarrow 2^{10}$$

We need to subtract

$$(a, b) = (32,32), (33,33), \dots , (63,63) \Rightarrow 32\ Solutions$$
$$Answers\ for\ (5,5): 2^{10} - 32$$

Final answer is:

$$11(2^{10}) - 32$$

Example 2.4

$$\lfloor \log_2 a \rfloor + \lfloor \log_2 b \rfloor = 10$$

Given that a, b are distinct integers, the number of ordered pairs (a, b) that are solutions of the equation above can be written

$$5 \cdot 2^{11} - n$$

Determine the value of n.

Use a change of variable. Let $x = \lfloor \log_2 a \rfloor, y = \lfloor \log_2 b \rfloor$:

$$x + y = 10$$

Apply stars and bars with 10 *stars* and 1 *bar*:

$$\binom{10+1}{1} = \binom{11}{1} = 11 \text{ } cases$$

For each case:

$$2^{10} \text{ } solutions$$

Total Cases

$$= 11 \times 2^{10}$$

Subtract cases from (5,5) where $a = b$:

$$= 11 \times 2^{10} - 32$$

Example 2.5

$$\lfloor \log_2 a \rfloor + \lfloor \log_2 b \rfloor + \lfloor \log_2 c \rfloor = 5$$

Given that a, b, c are integers, the number of ordered pairs (a, b, c) that are solutions of the equation above.

Use a change of variable. Let $x = \lfloor \log_2 a \rfloor, y = \lfloor \log_2 b \rfloor, z = \lfloor \log_2 c \rfloor$:

$$x + y + z = 5$$

Apply stars and bars with 5 *stars* and 2 *bars*:

$$\binom{5+2}{2} = \binom{7}{2} = 21 \text{ } cases$$

Each case above has:

$$2^5 \text{ } solutions$$

Final Answer

$$21(2^5) = 21(32)$$

2.2 Further Topics

6 Examples