
POLYGONS

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AZIZ MANVA

AZIZMANVA@GMAIL.COM

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1. SQUARES AND RECTANGLES

1.1 Area and Perimeter Basics

A. Perimeter

1.1: Rectangle

For a rectangle with length l and width w :

$$\begin{aligned} \text{Perimeter} &= 2(l + w) = 2l + 2w \\ \text{Area} &= lw \end{aligned}$$

1.2: Square

For a square with side length s

$$\begin{aligned} \text{Perimeter} &= 4s \\ \text{Area} &= s^2 \end{aligned}$$

Example 1.3

- A. A square has side 3.5 units, while a rectangle has length 2.25 units and breadth 5.25 units. What is the sum of their perimeters?
- B. What is the perimeter of a square with area 64 units?
- C. What is the perimeter of a rectangle with area 56 units, and one side 4 units?
- D. A square and a triangle have equal perimeters. The lengths of the three sides of the triangle are 6.2 cm, 8.3 cm and 9.5 cm. The area of the square is: (AMC 8 1985/12)

Part A

$$3.5 \times 4 + 2(2.25 + 5.25) = 14 + 2 \times 7.5 = 14 + 15 = 29$$

Part B

$$\text{Perimeter} = \text{Side} \times 4 = \sqrt{64} \times 4 = 8 \times 4 = 32$$

Part C

The other side of the rectangle

$$= \frac{56}{4} = 14$$

The perimeter

$$= 2(14 + 4) = 2 \times 18 = 36$$

Part D

Perimeter of triangle

$$= 6.2 + 8.3 + 9.5 = 24$$

Side of square

$$= \frac{24}{4} = 6$$

Area of square

$$= 6^2 = 36$$

Example 1.4

A square with side s is divided into five congruent rectangles stacked on top of each other. If the perimeter of each rectangle is 1 foot, how many inches is the perimeter of the square?

Substitute $l = s, w = \frac{s}{5}$ in $P = 2(l + w)$:

$$2\left(s + \frac{s}{5}\right) = 12 \text{ inches}$$

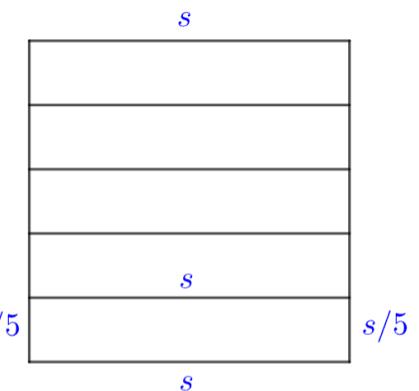
$$\frac{12s}{5} = 12$$

The side length

$$= s = 5$$

The perimeter of the square is:

$$4s = 20$$



Example 1.5

A square playground has a grassy area that is used to play football. The grassy area occupies half of the area of the playground. The perimeter of the grassy area is 100 meters. Find the perimeter and area of the entire playground.

The perimeter of the playground is:

$$s + s + \frac{s}{2} + \frac{s}{2} = 100$$

$$3s = 100$$

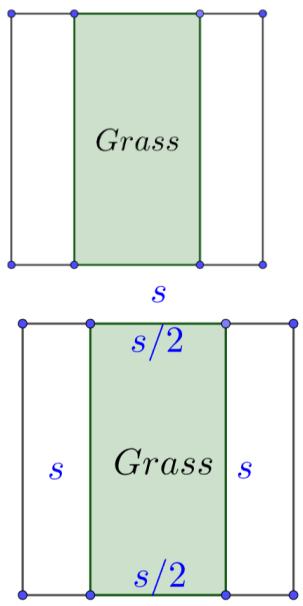
$$s = \frac{100}{3}$$

The perimeter of the entire playground

$$= 4s = \frac{400}{3} \text{ m}$$

The area of the entire playground

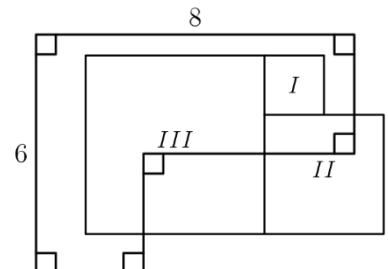
$$= s^2 = \left(\frac{100}{3}\right)^2 = \frac{10,000}{9} \text{ m}^2$$



Example 1.6

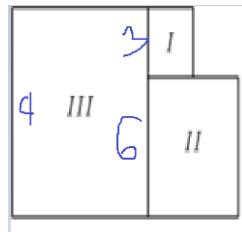
Figures I, II, and III are squares. The perimeter of I is 12 and the perimeter of II is 24. The perimeter of III is (AMC 8 1995/6)

$$P = 4s_{III} = 4(s_I + s_{II}) = 4\left(\frac{12}{4} + \frac{24}{4}\right) = 4\left(\frac{36}{4}\right) = 36$$



Example 1.7: TSD

Betty and Ann are walking around a rectangular park with dimensions 600 m by 400 m, as shown. They both begin at the top left corner of the park and walk at constant but different speeds. Betty



walks in a clockwise direction and Ann walks in a counterclockwise direction. Points P, Q, R, S, T divide the bottom edge of the park into six segments of equal length. When Betty and Ann meet for the first time, they are between Q and R. Which of the following could be the ratio of Betty's speed to Ann's speed? (**Gauss Grade 7 2017/20**)

- A. 5:3
- B. 9:4
- C. 11:6
- D. 12:5
- E. 17:7

Ann's distance:

$$\text{Min: } 600, \quad \text{Max: } 700$$

Betty' distance:

$$\text{Min: } 1300, \quad \text{Max: } 1400$$

Distance Ratios as Fractions:

$$\frac{\text{Min of Betty}}{\text{Max of Ann}} = \frac{13}{7} = 1\frac{6}{7}$$

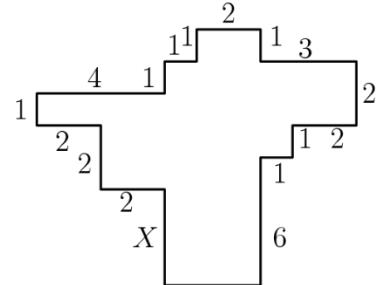
$$\frac{\text{Max of Betty}}{\text{Min of Ann}} = \frac{14}{6} = \frac{7}{3} = 2\frac{1}{3}$$

$$\text{Option A: } \frac{5}{3} = 1\frac{2}{3} < 1\frac{6}{7}$$

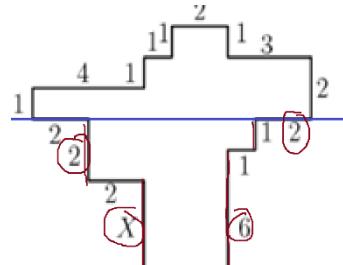
$$\text{Option B: } \frac{9}{4} = 2\frac{1}{4} \Rightarrow 1\frac{6}{7} < 2\frac{1}{4} < 2\frac{1}{3} \Rightarrow \text{Correct}$$

Example 1.8

In the diagram, all angles are right angles and the lengths of the sides are given in centimeters. Note the diagram is not drawn to scale. What is the length in X, in centimeters? (**AMC 8 2012/5**)

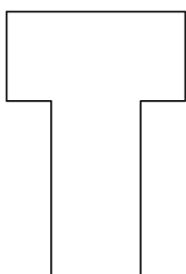


$$2 + X = 6 + 1 \\ X = 5 \text{ cm}$$



Example 1.9

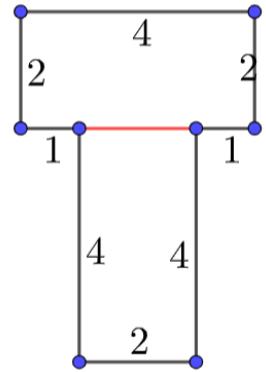
The letter T is formed by placing two 2×4 inch rectangles next to each other, as shown. What is the perimeter of the T, in inches? (**AMC 8 2006/6**)



Method I

Add the sides to get the perimeter

$$= 2 + 4 + 2 + 1 + 4 + 2 + 4 + 1 = 20$$



Method II

Note that the red line is missing from the perimeter of the shape. If the red line had been a part of the perimeter, then the perimeter would have been:

$$\underbrace{2}_{\substack{\text{Double for} \\ \text{Two Rectangles}}} \times \underbrace{2}_{\substack{\text{Double for} \\ \text{the Sides}}} \times \underbrace{(4+2)}_{\substack{\text{Length plus} \\ \text{Width}}} = 4 \times 6 = 24$$

However, the red line must be subtracted to give a final perimeter of:

$$24 - 2 \times 2 = 24 - 4 = 20$$

Example 1.10

Farmer Jane has a square grazing field with area 64 square units. She takes the fence from the field, and uses all of it to fence a rectangular area with length 4 units. What is the percentage change in the grazing area??

$$\text{Side Length(Square)} = \sqrt{64} = 8$$

$$\text{Perimeter(Square)} = \text{Perimeter(Rectangle)} = 32$$

$$2(l+w) = 32$$

$$l+w = 16$$

$$w = 16 - l = 16 - 4 = 12$$

$$\text{Area(Rectangle)} = (12)(4) = 48$$

$$\% \text{ change} = \frac{64 - 48}{64} = \frac{16}{64} = \frac{1}{4} = 25\% \text{ reduction}$$

Example 1.11

A rectangular garden 60 feet long and 20 feet wide is enclosed by a fence. To make the garden larger, while using the same fence, its shape is changed to a square. By how many square feet does this enlarge the garden? (AMC 8 1999/5)

$$\text{Perimeter(Rectangle)} = 2(60 + 20) = 2(80) = 160$$

$$\text{Side(Square)} = \frac{160}{4} = 40$$

$$\text{Area(Square)} - \text{Area(Rectangle)} = 40^2 - (60)(20) = 1600 - 1200 = 400$$

Example 1.12

A rectangular grazing area is to be fenced off on three sides using part of a 100-meter rock wall as the fourth side. Fence posts are to be placed every 12 meters along the fence including the two posts where the fence meets the rock wall. What is the fewest number of posts required to fence an area 36 m by 60 m? (AMC 8 1986/18)

To minimize the number of posts, make the side with length 60 m parallel to the rock wall. Then

$$\text{No. of Fenceposts} = \frac{\text{Length of Fence}}{\text{Interval}} + 1 = \frac{36 + 60 + 36}{6} + 1 = \frac{132}{12} + 1 = 11 + 1 = 12$$

Example 1.13

Carl decided to fence in his rectangular garden. He bought 20 fence posts, placed one on each of the four corners, and spaced out the rest evenly along the edges of the garden, leaving exactly 4 yards between neighboring posts. The longer side of his garden, including the corners, has twice as many posts as the shorter side, including the corners. What is the area, in square yards, of Carl's Garden? (AMC 10B 2016/11)

Out of the 20 fenceposts, 4 are at the corners, leaving 16 for the sides:

$$20 - 4 = 16$$

These 16 fenceposts are equally divided among the opposite sides:

$$\begin{aligned} 2(F_{\text{Length}} + F_{\text{Width}}) &= 16 \\ F_{\text{Length}} + F_{\text{Width}} &= 8 \end{aligned}$$

Method I: Trial and Error

If the width has 1 fencepost not including the corners, then it has the length of 8 posts.

$$8 - 1 = 7 \text{ fenceposts}$$

Including the corners, we have:

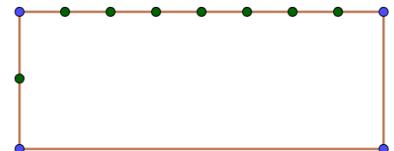
$$\text{Width} = 1 + 2 = 3$$

$$\text{Length} = 7 + 2 = 9 \neq 2(3) = 6 \Rightarrow \text{Not Valid}$$

If we increase the width to have 2 fenceposts, them including the corners, we have

$$\text{Width} = 2 + 2 = 4$$

$$\text{Length} = 6 + 2 = 8 = 2(4) \Rightarrow \text{Valid}$$



Method II: Algebra

$$w, 8 - w$$

The total number of fenceposts on the width is:

$$w + 2$$

The total number of fenceposts on the length is:

$$8 - w + 2 = 10 + w$$

$$10 - w = 2(w + 2)$$

$$w = 2$$

Finding the Area

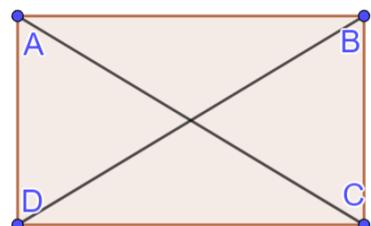
Whichever method we use, the area is:

$$\text{Area} = lw = \underbrace{4(3)}_{\text{Width}} \times \underbrace{4(7)}_{\text{Length}} = (12)(28) = 336 \text{ yards}^2$$

B. Diagonals

1.14: Diagonals of a Rectangle

The diagonals of a rectangle have equal length.



Example 1.15

Milee went from the top left corner of a rectangular field to the bottom right corner, and counted 38 paces. Milee's friend takes paces which are half the size of Milee's.

Milee's. How many paces will she take to go from the bottom left corner to the top right corner?

$$\text{Milee's friend} = 2(38) = 76 \text{ paces}$$

1.16: Proving a Rectangle

If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

Example 1.17

Two sides of a parallelogram are 4 units and 5 units. The diagonals of the parallelogram are each $\sqrt{41}$ units.
 Find the area of the parallelogram.

Since the diagonals are equal, the parallelogram is a rectangle. Hence:

$$A = lw = 4(5) = 20 \text{ units}^2$$

1.18: Diagonal of a Square

The diagonal (d) of a square with side length s is

$$d = \sqrt{2} \times s$$

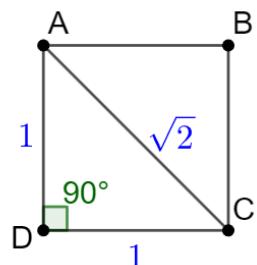
Without loss of generality, let the length of the side of the square be 1.

Then, in right $\triangle ADB$, by Pythagoras Theorem:

$$AC^2 = 1^2 + 1^2 = 1 + 1 = 2$$

Take the square root both sides:

$$AC = \sqrt{2}$$



Example 1.19

- A. What is the length of the diagonal of a square of side 3 units?
- B. A farmer has a square field with a side length of 100m. Determine the length of the diagonal in km.

Part A

Length of the diagonal

$$= \sqrt{2} \times \text{Side} = \sqrt{2} \times 3 = 3\sqrt{2} \text{ units}$$

Part B

$$\text{Side Length} = 100\text{m} = \frac{1}{10}\text{km}$$

$$\text{Diagonal} = \frac{1}{10} \times \sqrt{2} \text{ km} = \frac{\sqrt{2}}{10} \text{ km}$$

Example 1.20

- A. A boy walks three times around his square playground, and takes 120 paces to do so. Determine how many paces the boy will take to go from one corner to a non-adjacent corner. (Your answer need not be an integer).
- B. A square field has area 16 square units. What is the distance from the top left corner of the field to the bottom right corner?

Part A

$$3P = 120 \text{ Paces}$$

$$P = 40 \text{ paces}$$

$$s = \frac{P}{4} = 10 \text{ paces}$$

$$d = \sqrt{2}s = \sqrt{2} \times 10 = 10\sqrt{2}$$

Part B

The area of the square is

$$A = s^2 = 16$$

Then, the side length is:

$$s = \sqrt{16} = 4$$

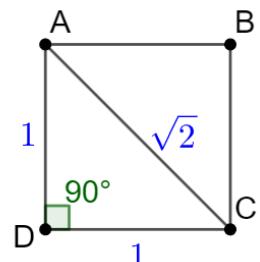
And the distance from top left to bottom right is the diagonal, which is:

$$d = 4 \times \sqrt{2} = 4\sqrt{2}$$

1.21: Side Length of a Square

The diagonal (d) of a square with side length s is

$$d = \sqrt{2} \times s \Rightarrow s = \frac{d}{\sqrt{2}}$$



Example 1.22

- A. Find the perimeter of a square with diagonal $5\sqrt{2}$ m.
- B. What is the length of the side of a square, the sum of whose diagonals is 12 units?
- C. Half the length of the diagonal of a square is 5. Determine the perimeter of the square.

Part A

$$s = \frac{5\sqrt{2}}{\sqrt{2}} = 5$$

$$P = 4s = 20$$

Part B

Since the sum of the diagonals is 12, first find the length of a single diagonal:

$$2 \times \text{diagonal} = 12 \Rightarrow \text{Diagonal} = d = 6$$

The side length is:

$$s = \frac{6}{\sqrt{2}} = \frac{6}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$

Part C

$$\frac{d}{2} = 5 \Rightarrow d = 10$$

$$s = \frac{10}{\sqrt{2}} = \frac{10}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$$

$$P = 4s = 4(5\sqrt{2}) = 20\sqrt{2}$$

Example 1.23

Calculate the cost of fencing a square field which has a distance of 5m between its nonadjacent corners if the cost of fencing 1m is $\sqrt{3}$ dollars.

The side length is:

$$s = \frac{d}{\sqrt{2}} = \frac{5}{\sqrt{2}} = \frac{5}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2} \text{ m}$$

The perimeter

$$P = 4s = 4 \times \frac{5\sqrt{2}}{2} = 10\sqrt{2}$$

$$\text{Total Cost} = P_c = (10\sqrt{2}m) \left(\sqrt{3} \frac{\$}{m} \right) = 10\sqrt{6} \text{ dollars}$$

1.24: Diagonal of a Rectangle

Given a rectangle with length l and breadth b , the length of the diagonal is

$$d = \sqrt{l^2 + b^2}$$

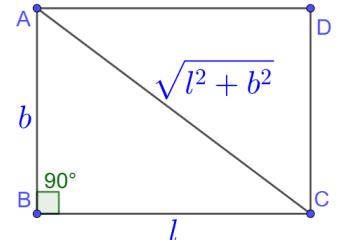
Since $ABCD$ is a rectangle

$$\angle ABC = 90^\circ$$

In right $\triangle ABC$, by Pythagoras Theorem:

$$d^2 = \text{Diagonal}^2 = AC^2 = AB^2 + BC^2 = l^2 + b^2$$

$$d = \sqrt{l^2 + b^2}$$



Example 1.25

- A. What is the length of the diagonal of a rectangle of length 15 units and breadth 8 units?
- B. Find the length of the longest possible straight path in a rectangular park of perimeter 34, and length 5.

Part A

$$\text{Diagonal} = \sqrt{l^2 + b^2} = \sqrt{15^2 + 8^2} = \sqrt{225 + 64} = 17 \text{ units}$$

OR using Pythagorean Triplet (8,15,17) \Rightarrow Diagonal = 17

Part B

$$\text{Breadth} = \frac{34 - 5 \times 2}{2} = \frac{24}{2} = 12$$

$$\text{Diagonal} = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$$

OR using Pythagorean Triplet (5,12,13) \Rightarrow Diagonal = 13

1.26: Area of a Square in terms of its diagonal

The area of a square with diagonal d is

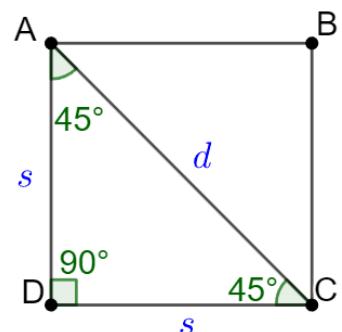
$$\frac{1}{2}d^2$$

By Pythagoras Theorem in $\triangle ADC$

$$s^2 + s^2 = d^2 \Rightarrow s^2 = \frac{d^2}{2}$$

Example 1.27

It takes Paridhi twelve minutes to walk through the shortest path from one vertex to a non-adjacent vertex of a square field while walking at the rate of $6 \frac{\text{km}}{\text{hr}}$. What is the area of the field, in square meters?

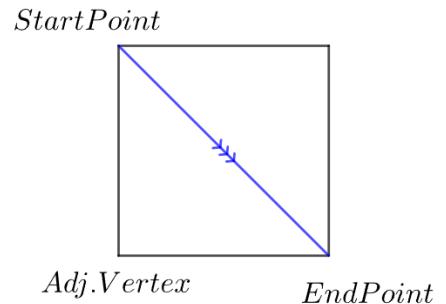


Determine the length of the path using

$$\text{Diagonal} = 6 \underbrace{\frac{\text{km}}{\text{hr}}}_{\text{Speed}} \cdot \underbrace{12 \text{ min}}_{\text{Time}} = 6 \frac{\text{km}}{\text{hr}} \cdot \frac{1}{5} \text{ hr} = \frac{6}{5} \text{ km} = 1.2 \text{ km}$$

The area of the field is:

$$A = \frac{1}{2} d^2 = \frac{1}{2} \cdot 1.2^2 = \frac{1}{2} \cdot 1.2 \cdot 1.2 = 0.6 \cdot 1.2 = 0.72 \text{ km}^2 \\ = 0.72 (\text{km})(\text{km})$$



Convert from km^2 to m^2 by multiplying 1000:

$$= 0.72 (1000\text{m})(1000\text{m}) = 720000 \text{ m}^2$$

Example 1.28

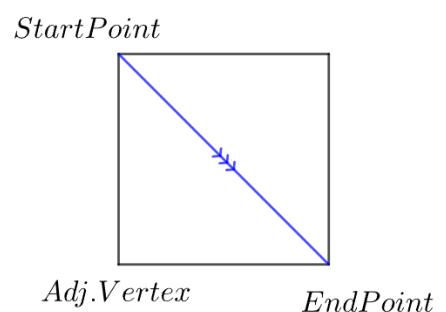
It takes Pari fifteen minutes to walk through the longest straight path from one vertex to a non-adjacent vertex of a square field. She walked at $4 \frac{\text{km}}{\text{hr}}$. Determine the cost of planting the field with grass at \$0.25 per square meter.

Determine the length of the path using

$$\text{Diagonal} = 4 \underbrace{\frac{\text{km}}{\text{hr}}}_{\text{Speed}} \cdot \underbrace{15 \text{ min}}_{\text{Time}} = 4 \frac{\text{km}}{\text{hr}} \cdot \frac{1}{4} \text{ hr} = 1 \text{ km}$$

The area of the field is:

$$A = \frac{1}{2} d^2 = \frac{1}{2} \cdot 1^2 = \frac{1}{2} \text{ km}^2 = \frac{1}{2} (\text{km})(\text{km})$$



Convert from km^2 to m^2 by writing $1 \text{ km} = 1000\text{m}$:

$$= \frac{1}{2} (1000\text{m})(1000\text{m}) = 500,000 \text{ m}^2$$

Calculate the cost:

$$= \frac{1}{4} \times 500,000 = 125,000 \text{ \$}$$

1.29: Area of a Rectangle in terms of its diagonal

A rectangle *cannot* be uniquely identified based on the length of its diagonal.

$$\text{Area} \neq \frac{1}{2} d^2$$

This is proved by way of a counterexample in the next question.

Example 1.30

Consider a rectangle R_1 with dimensions $(b, l) = (15, 20)$. Consider another rectangle R_2 with dimensions $(b, l) = (7, 24)$. Show that the two rectangles have same diagonal, but different areas.

Using Pythagorean Triplets:

$$D_1 = (15, 20) = 5(3, 4, 5) = (15, 20, 25) \\ D_2 = (7, 24, 25)$$

Calculate the area:

$$A_1 = 15 \times 20 = 300$$

$$A_2 = 7 \times 24 = 168$$

The diagonals are the same, but the areas are different.

Example 1.31

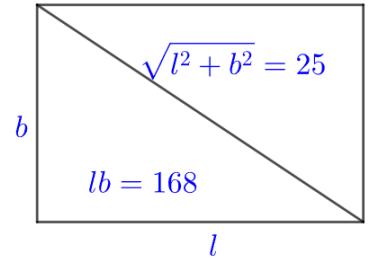
A rectangular parking lot has a diagonal of 25 meters and an area of 168 square meters. In meters, what is the perimeter of the parking lot? (AMC 10B 2011/14)

Square both sides of $\sqrt{\text{---}} = 25$

$$\underbrace{l^2 + b^2 = 625}_{\text{Equation I}}$$

The area is:

$$lb = 168 \Rightarrow 2A = \underbrace{2lb = 336}_{\text{Equation II}}$$



$$\begin{aligned} l^2 + 2lb + b^2 &= 961 \\ (l + b)^2 &= 31^2 \\ l + b &= 31 \\ P &= 2(l + b) = 62 \end{aligned}$$

1.32: Midpoints form a Rectangle

The quadrilateral formed by joining the midpoints of the sides of a square with side length $2s$ in order forms a square with side length

$$\sqrt{2}s$$

Consider square ABCD with side length $2s$

It has midpoints E, F, G and H:

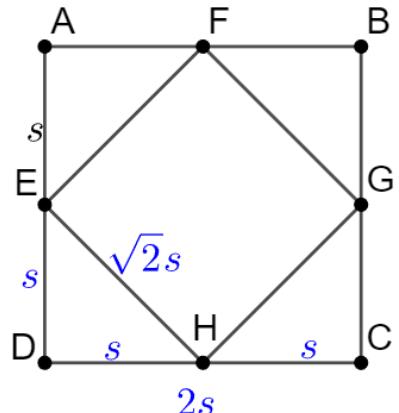
$$ED = DH = s$$

By Pythagoras Theorem in right $\triangle EDH$

$$EH = \sqrt{ED^2 + DH^2} = \sqrt{2s^2} = \sqrt{2}s$$

By similar logic:

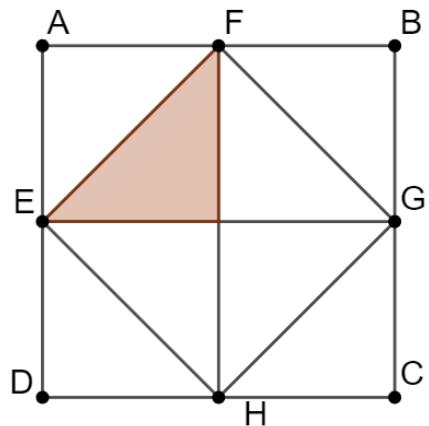
$$EF = FG = HG = \sqrt{2}s$$



Example 1.33

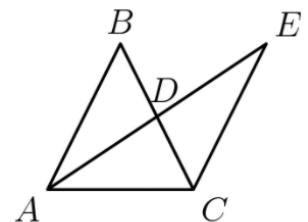
A square $ABCD$ has side length $\sqrt{13}$. P, Q, R and S are the midpoints of sides AB , BC , CD and DA . Determine the ratio of the area of quadrilateral $PQRS$ to the area of quadrilateral $ABCD$.

$$(\sqrt{2}s)^2 : (2s)^2 = 2s^2 : 4s^2 = 2 : 4 = 1 : 2$$



Example 1.34

Triangle ABC is an isosceles triangle with $\overline{AB} = \overline{BC}$. Point D is the midpoint of both \overline{BC} and \overline{AE} , and \overline{CE} is 11 units long. Triangle ABD is congruent to triangle ECD . What is the length of \overline{BD} ? (AMC 8 2006/19)



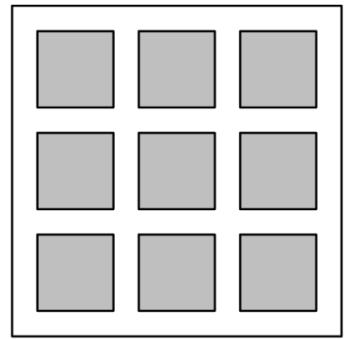
Example 1.35

A large square region is paved with n^2 gray square tiles, each measuring s inches on a side. A border d inches wide surrounds each tile. The figure shows the case for $n = 3$. When $n = 24$, the 576 gray tiles cover 64% of the area of the large square region. What is the ratio $\frac{d}{s}$ for this larger value of n ? (AMC 8 2020/24)

Area of the gray square tiles

The total area of the grey tiles

$$= \underbrace{24^2}_{\substack{\text{No. of} \\ \text{Tiles}}} \cdot \underbrace{s^2}_{\substack{\text{Area of} \\ \text{each tile}}} = 24^2 \cdot s^2$$



Length of Square

For 3 tiles, the length of the square will be:

$$3s + 4d$$

For 24 tiles, the total length of the gaps will be:

$$24s + 25d$$

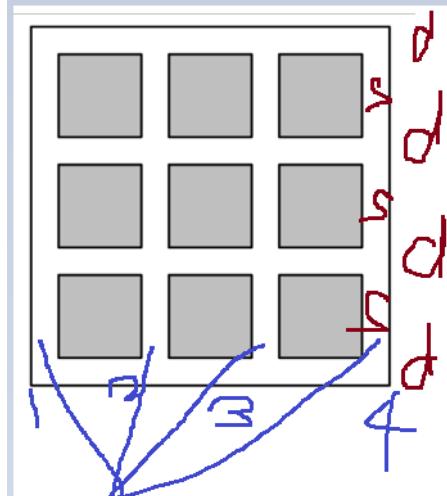
Area of Square

Therefore, the area of the entire square will be:

$$\text{Side}^2 = (24s + 25d)^2$$

The ratio of the grey tiles to the larger square is 64% = $\frac{64}{100}$

$$\frac{24^2 \cdot s^2}{(24s + 25d)^2} = \frac{64}{100}$$



Take the square root both sides:

$$\frac{24s}{24s + 25d} = \frac{8}{10} = \frac{4}{5}$$

Since the numerator and the denominator both have $24s$, take the reciprocal on both sides:

$$\frac{24s}{24s} + \frac{25d}{24s} = \frac{5}{4}$$

Substitute $\frac{24s}{24s} = 1$:

$$1 + \frac{25d}{24s} = \frac{5}{4} \Rightarrow \frac{25d}{24s} = \frac{1}{4} \Rightarrow \frac{25d}{6s} = 1 \Rightarrow \frac{d}{s} = \frac{6}{25}$$

C. Calculations

Example 1.36

Six rectangles each with a common base width of 2 have lengths of 1, 4, 9, 16, 25, and 36. What is the sum of the areas of the six rectangles? (AMC 8 2014/6)

The area that we want is:

$$2 \cdot 1 + 2 \cdot 4 + 2 \cdot 9 + 2 \cdot 16 + 2 \cdot 25 + 2 \cdot 36$$

Factor 2:

$$2(1 + 4 + 9 + 16 + 25 + 36)$$

Note that

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Substituting $n = 6$:

$$2 \cdot \frac{6 \cdot 7 \cdot 13}{6} = 2(91) = 182$$

Example 1.37

Karl's rectangular vegetable garden is 20 feet by 45 feet, and Makenna's is 25 feet by 40 feet. Which of the following statements are true?

- (A) Karl's garden is larger by 100 square feet.
- (B) Karl's garden is larger by 25 square feet.
- (C) The gardens are the same size.
- (D) Makenna's garden is larger by 25 square feet.
- (E) Makenna's garden is larger by 100 square feet. (AMC 8 2011/2)

$$Area_{Karl} = 20(45) = 900$$

$$Area_{Makenna} = 25(40) = 1000$$

Option E

Example 1.38

How many square yards of carpet are required to cover a rectangular floor that is 12 feet long and 9 feet wide?

(There are 3 feet in a yard.) (AMC 8 2015/1)

Convert the dimensions from feet to yards, and then find the area:

$$\frac{12}{3} \times \frac{9}{3} = 4 \times 3 = 12$$

Shortcut

Find the area and then divide by $3^2 = 9$ to convert to square yards:

$$\frac{12 \times 9}{3^2} = 12$$

Example 1.39

At a store, when a length is reported as x inches that means the length is at least $x - 0.5$ inches and at most $x + 0.5$ inches. Suppose the dimensions of a rectangular tile are reported as 2 inches by 3 inches. In square inches, what is the

- A. minimum area for the rectangle? (AMC 10B 2011/3)
- B. maximum area for the rectangle?

Part A

The minimum dimensions are:

$$2 - 0.5 = 1.5 = \frac{3}{2}, \quad 3 - 0.5 = 2.5 = \frac{5}{2}$$

The area

$$= \frac{3}{2} \cdot \frac{5}{2} = \frac{15}{4} = 3.75 \text{ in}^2$$

Part B

The maximum dimensions are:

$$2 + 0.5 = 2.5 = \frac{5}{2}, \quad 3 + 0.5 = 3.5 = \frac{7}{2}$$

The area

$$= \frac{5}{2} \cdot \frac{7}{2} = \frac{35}{4} = 8.75 \text{ in}^2$$

Example 1.40

Tyler is tiling the floor of his 12-foot by 16-foot living room. He plans to place one-foot by one-foot square tiles to form a border along the edges of the room and to fill in the rest of the floor with two-foot by two-foot square tiles. How many tiles will he use? (AMC 8 2018/9)

The area of the border will be:

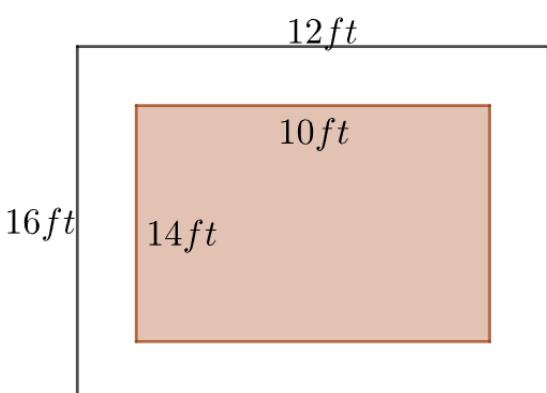
$$\frac{12 \times 16}{\text{Total Area}} - \frac{10 \times 14}{\text{Inner Area}} = 192 - 140 = 52$$

Hence, the number of border tiles

$$= \frac{52}{1} = 52$$

The number of inner tiles

$$\frac{\text{Inner Area}}{\text{Area of each tile}} = \frac{140}{2 \times 2} = 35$$



The total tiles

$$= 52 + 35 = 87$$

Example 1.41

Carrie has a rectangular garden that measures 6 feet by 8 feet. She plants the entire garden with strawberry plants. Carrie is able to plant 4 strawberry plants per square foot, and she harvests an average of 10 strawberries per plant. How many strawberries can she expect to harvest? (AMC 8 2020/3)

The area of the garden is:

$$6 \times 8 = 48 \text{ ft}^2$$

The number of strawberry plants is:

$$48 \times 4 = 192$$

The number of strawberries

$$= 192 \times 10 = 1920$$

Example 1.42

A rectangle with a diagonal of length x is twice as long as it is wide. What is the area of the rectangle? (Answer in terms of x) (AMC 10A 2005/4)

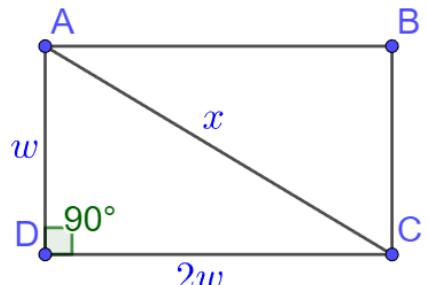
Draw the rectangle with width w and length $2w$. It has area:

$$w \times 2w = 2w^2$$

By Pythagoras Theorem in right $\triangle ADC$:

$$\begin{aligned} w^2 + (2w)^2 &= x^2 \\ w^2 + 4w^2 &= x^2 \\ 5w^2 &= x^2 \\ w^2 &= \frac{x^2}{5} \end{aligned}$$

$$2w^2 = \frac{2}{5}x^2$$



Example 1.43

If four identical squares are stacked in a single row (such that the lowest square touches the bottom, and the highest square touches the top) in a larger square with area 784 units, what is the area of the larger square not covered by the smaller squares?

The side length of the larger square is

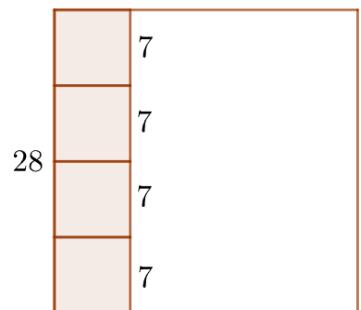
$$\sqrt{784} = 28$$

The side length of the smaller square

$$= \frac{28}{4} = 7$$

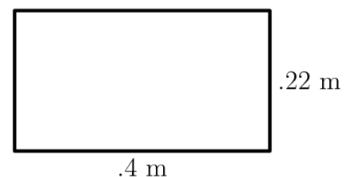
The area of each smaller square

$$= 7^2 = 49$$



The area that we want is:

$$784 - 49 \times 4 = 784 - 200 + 4 = 588$$



Example 1.44

The area of the rectangular region is (AMC 8 1987/5)

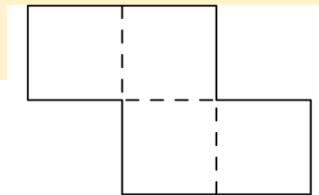
$$0.22 \times 0.4 = 22 \times 4 \times 0.001 = 0.088 \text{ m}^2$$

Example 1.45

The area of this figure is 100 cm^2 . Its perimeter is (AMC 8 1990/15)
 [figure consists of four identical squares]

The area of each square

$$= \frac{100}{4} = 25 \text{ cm}^2$$

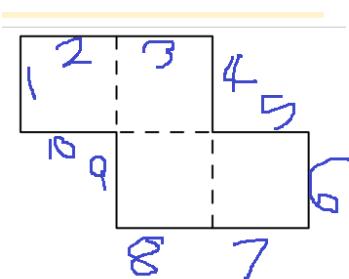


The side length of each square

$$= \sqrt{25} = 5 \text{ cm}$$

Method I

Since there are 10 sides (see diagram to the right), the perimeter
 $= 5 \times 10 = 50 \text{ cm}$



Method II: Complementary Counting

If we had four complete squares, then they would have perimeter:

$$= \underbrace{5 \times 4}_{\text{Perimeter of 1 square}} \times 4 = 80 \text{ cm}$$

But the dotted lines are not part of the perimeter. We have 3 dotted lines. But each dotted is a part of two squares. Hence, we must count it twice:

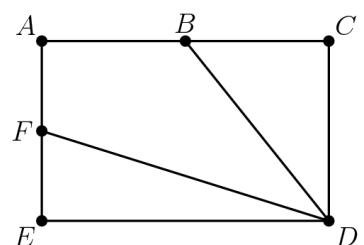
$$\text{Dotted Lines} = 2 \times 3 = 6$$

Hence, the length missing is:

$$= 5 \times 6 = 30 \text{ cm}$$

The final answer is:

$$80 - 30 = 50 \text{ cm}$$

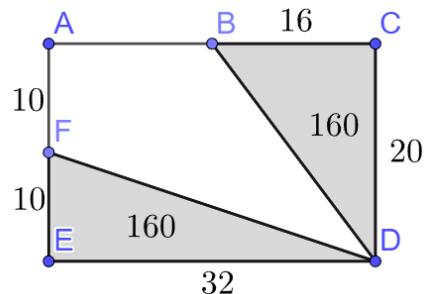


Example 1.46

The rectangle shown has length $AC = 32$, width $AE = 20$, and B and F are midpoints of \overline{AC} and \overline{AE} , respectively. The area of quadrilateral $ABDF$ is: (AMC 8 1993/18)

The area that we want is:

$$\begin{aligned}
 & [ABCD] - [BCD] - [FED] \\
 &= 32 \times 20 - \frac{1}{2} \times 16 \times 20 - \frac{1}{2} \times 10 \times 32 \\
 &= 32 \times 20 - 16 \times 10 - 10 \times 16 \\
 &= 640 - 160 - 160 \\
 &= 320
 \end{aligned}$$

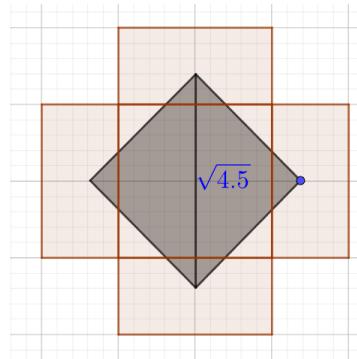


Example 1.47

A checkerboard consists of one-inch squares. A square card, 1.5 inches on a side, is placed on the board so that it covers part or all of the area of each of n squares. The maximum possible value of n is (AMC 8 1993/25)

Calculate the diagonal of the square card:

$$\sqrt{(1.5)^2 + (1.5)^2} = \sqrt{2.25 + 2.25} = \sqrt{4.5} > \sqrt{4} = 2$$



D. Triangles

Example 1.48: Diagonals

$PQRS$ is a square. A is the midpoint of side PQ . B is a point on RS such that RB is $\frac{1}{3}$ rd of RS . If the area of $\triangle AQB$ is t units, find the length of the diagonal of the square in terms of t .

Let the side length of the square be

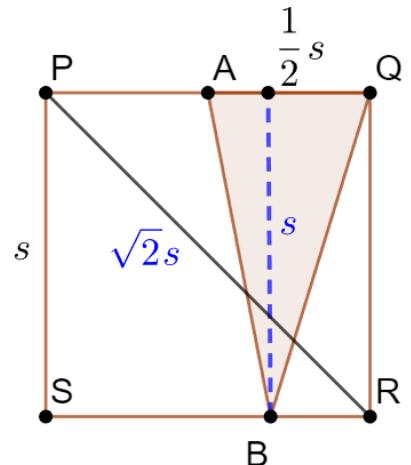
$$s \Rightarrow \text{Diagonal} = \sqrt{2}s$$

The area of $\triangle AQB$

$$= \frac{1}{2}hb = \frac{1}{2}(PS)(AQ) = \frac{1}{2}s\left(\frac{1}{2}s\right) = \frac{1}{4}s^2$$

But the area of the triangle is t

$$\begin{aligned}\frac{1}{4}s^2 &= t \\ s^2 &= 4t \\ s &= 2\sqrt{t} \\ \sqrt{2}s &= 2\sqrt{2t}\end{aligned}$$



Example 1.49: Diagonals

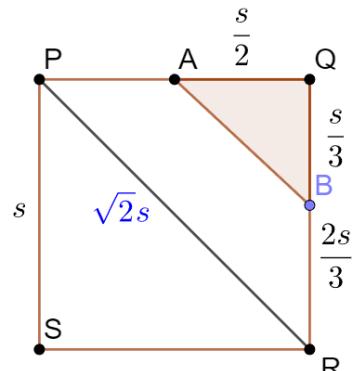
$PQRS$ is a square. A is the midpoint of side PQ . B is a point on RQ such that RB is $\frac{2}{3}$ rd of RQ . If the area of $\triangle AQB$ is t units, find the length of the diagonal of the square in terms of t .

Let the length of side of the square

$$= s$$

$$\begin{aligned}\Delta AQB &= \frac{1}{2}hb = \frac{1}{2}(AQ)(QB) = \frac{1}{2} \times \frac{s}{2}s \times \frac{s}{3}s = \frac{s^2}{12} \\ \frac{s^2}{12} &= t \Rightarrow s^2 = 12t \Rightarrow s = \sqrt{12t}\end{aligned}$$

$$\text{Diagonal} = \sqrt{2} \times \sqrt{12t} = \sqrt{24t} = \sqrt{4} \times \sqrt{6t} = 2\sqrt{6t}$$



Example 1.50

Square $ABCD$ has sides of length 3. Segments CM and CN divide the square's area into three equal parts. How long is segment CM ? (AMC 8 1999/23)

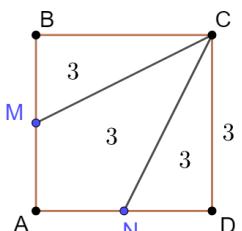
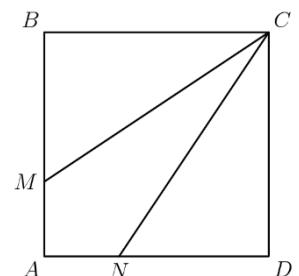
The area of square $ABCD$

$$= 3^2 = 9$$

The area of each triangle

$$= \frac{9}{3} = 3$$

The area of $\triangle BCM$



$$= \frac{1}{2}(BC)(BM) = \frac{1}{2}(3)(BM) = 3 \Rightarrow BM = 2$$

By Pythagoras Theorem in ΔBCM :

$$CM = \sqrt{BC^2 + BM^2} = \sqrt{3^2 + 2^2} = \sqrt{9 + 4} = \sqrt{13}$$

E. Percentage Change

Example 1.51

Each side of a square is increased by 30%. The percentage increase in area is:

Consider a square of side length s . The side is increased by:

$$30\% \text{ of } s = 0.3s$$

New Area will be

$$(s + 0.3s)^2 = (1.3s)^2 = 1.69s^2$$

The percentage increase is:

$$\frac{\text{Increase in Area}}{\text{Old Area}} = \frac{\text{New Area} - \text{Old Area}}{\text{Old Area}} = \frac{1.69s^2 - s^2}{s^2} = \frac{0.69s^2}{s^2} = 0.69 = 69\%$$

1.52: Percentage change in side length

If the side length of a square increases by $p\%$, the area of the square does not depend on the side length of the square.

Example 1.53

Each side of a square with length 23.75 meters is increased by 30%. The percentage increase in area is:

Since the change does not depend on the side length, take a square of side length 1.

	Original	New	Increase
Side	1	1.3	
Area	1	1.69	0.69

$$\% \text{ increase} = \frac{0.69}{1} = 0.69 = 69\%$$

Example 1.54

The dimensions of a square are changed so that the length is squared, and the breadth is halved. The old area as a percentage of the new area is:

$$\frac{2lb}{l^2b} = \frac{2}{l} = \frac{200}{l}\%$$

Example 1.55

If the length and width of a rectangle are each increased by 10%, then the perimeter of the rectangle is increased by what percentage? (AMC 8 1985/19)

Algebraic Method

Let the length and breadth be l and b respectively

$$\text{Old Perimeter} = 2(l + b)$$

$$\text{New Length} = l + 10\% \text{ of } l = l + 0.1l = 1.1l$$

$$\text{New Breadth} = b + 10\% \text{ of } b = b + 0.1b = 1.1b$$

$$\text{New Perimeter} = 2(1.1l + 1.1b) = 2.2(l + b)$$

$$\% \text{ increase} = \frac{\text{New P} - \text{Old P}}{\text{Old P}} = \frac{2.2(l + b) - 2(l + b)}{2(l + b)} = \frac{0.2(\cancel{l + b})}{2(\cancel{l + b})} = \frac{0.2}{2} = 10\%$$

Shortcut Method

Note that in the final calculation, neither the length nor the width appeared. The **violet terms** above cancelled. Hence, the answer does not depend on the value of the length, or the value of the width.

Hence, we can take any length and width that we want. For ease of calculations, take

$$\text{Length} = \text{Width} = 1$$

$$\text{Old P} = 2(1 + 1) = 2(2) = 4$$

$$\text{New Length} = 1 + 10\% \text{ of } 1 = 1 + 0.1 = 1.1$$

$$\text{New Breadth} = 1 + 10\% \text{ of } 1 = 1 + 0.1 = 1.1$$

$$\text{New Perimeter} = 2(1.1 + 1.1) = 2(2.2) = 4.4$$

$$\% \text{ Increase} = \frac{4.4 - 4}{4} = \frac{0.4}{4} = 0.1 = \frac{1}{10} = 10\%$$

Example 1.56

If the length of a rectangle is increased by 20% and its width is increased by 50%, then the area is increased by what percent (**AMC 8 1993/21**)

Let

$$\text{Length} = \text{Width} = 100$$

$$\text{Old Area} = 100^2 = 10,000$$

$$\text{New Length} = 100 + 20\% \text{ of } 100 = 100 + 20 = 120$$

$$\text{New Breadth} = 100 + 50\% \text{ of } 100 = 100 + 5 = 150$$

$$\% \text{ Increase} = \frac{\text{New Area} - \text{Old Area}}{\text{Old Area}} = \frac{(120)(150) - 10,000}{10,000} = \frac{8,000}{10,000} = \frac{80}{100} = 80\%$$

Example 1.57

The length of a rectangle is increased by 10% and the width is decreased by 10%. What percent of the old area is the new area? (**AMC 9 2009/8**)

Let $\text{Length} = \text{Width} = 100$

$$\text{Old Area} = 100^2 = 10,000$$

$$\text{New Length} = 100 + 10\% \text{ of } 100 = 100 + 10 = 110$$

$$\text{New Breadth} = 100 - 10\% \text{ of } 100 = 100 - 10 = 90$$

$$\frac{\text{New Area}}{\text{Old Area}} = \frac{(110)(90)}{10,000} = \frac{9900}{10,000} = \frac{99}{100} = 99\%$$

F. Maximum and Minimum Area

A rectangle will have maximum area when

$$\text{Length} = \text{Width} \Leftrightarrow \text{When it is a Square}$$

If we allow the rectangle to only have integer values for its length and width, then the rectangle with

- minimum area will have maximum difference between the length and the width
- maximum area will have minimum difference between the length and the width:

Example 1.58

The perimeter of a rectangle with integer sides is 52 units. Find the difference between the maximum and the minimum possible area of the rectangle.

$$P = 2(l + w) = 52 \Rightarrow l + w = 26$$

Rectangle with maximum area will have minimum difference between length and width:

$$l = w = 13 \Rightarrow A = lw = 13^2 = 169$$

Rectangle with minimum area will have maximum difference between length and width:

$$l = 25, w = 1 \Rightarrow A = lw = 25 \times 1 = 25$$

Difference

$$169 - 25 = 144$$

Example 1.59

Rishi is making a rectangle using 50 matchsticks of equal length. He first makes a rectangle with minimum area. Then, he takes the same matchsticks, and a rectangle with maximum area. Find the difference in the areas of the two rectangles.

Suppose that each matchstick has length:

$$1 \text{ unit}$$

If you are making a rectangle, the perimeter of the rectangle is

$$P = 2(l + w)$$

From the condition given in the question, we know that:

$$2(l + w) = 50 \Rightarrow l + w = 25$$

The rectangle with minimum area will have maximum difference between the length and the width:

$$l = 24, w = 1 \Rightarrow A = lw = 24 \times 1 = 24$$

The rectangle with maximum area will have minimum difference between the length and the width:

$$l = 13, w = 12 \Rightarrow A = lw = 13 \times 12 = 156$$

And then, the difference between the two areas is:

$$156 - 24 = 132$$

Example 1.60

Ms. Osborne asks each student in her class to draw a rectangle with integer side lengths and a perimeter of 50 units. All of her students calculate the area of the rectangle they draw. What is the difference between the largest and smallest possible areas of the rectangles? (AMC 8 2008/17)

G. Paths and Borders

Identify a smaller figure inside a larger figure. Then:

$$A(\text{Path}) = A(\text{Outside Figure}) - A(\text{Inside Figure})$$

Example 1.61: Path Inside

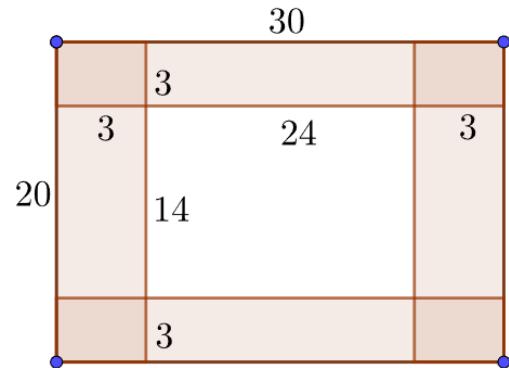
A square garden measuring 20 feet by 30 feet has a three feet marble walkway at its inside border, while the remaining area is covered by grass. What is the area of the walkway?

$$\text{Outer Area} = 20(30) = 600$$

$$\text{Inner Area} = (20 - 6)(30 - 6) = 14 \times 24 = 336$$

Area of the path

$$= [\text{Outer}] - [\text{Inner}] = 600 - 336 = 264 \text{ sq. ft}$$



Example 1.62: Path Outside

A rectangular fort(F) measuring 400 meters by 700 meters has a 3-meter moat(M) around it. What is the area of the moat?

$$A(F + M) - A(C) = 403 * 703 - 400 * 700 = 28000 + 2100 + 1200 + 9 - 28000 = 3309 \text{ sq. m.}$$

Example 1.63: Costs

The page of a book which is 30 cm * 40 cm has a margin of 2 cm. The margin is included in the dimensions given. What percentage of the page is occupied by the margin?

$$(30*40)-(26*36)=1200-936=264$$

$$\% = 264/1200 = 22\%$$

Example 1.64: Costs

Find the cost of adding a zari border of 1 inch on each side to a piece of cloth(C) measuring 18 inches by 36 inches, if zari(Z) is available at Rs. 288 per square foot.

$$A(C + Z) - A(C) = 19 * 37 - 18 * 36 = 703 - 648 = 55 \text{ sq. inches}$$

$$\text{Cost of Zari} = 288 * 55/144 = 110 \text{ Rs.}$$

If more than one path is drawn through the centre, then the paths will overlap.

$$A(\text{Path}) = A(\text{Path} - I) + A(\text{Path} - II) - A(\text{Overlap})$$

Example 1.65

The shaded region formed by the two intersecting perpendicular rectangles, in square units, is (AMC 8 1988/17)

Method I

The area of the longer rectangle is:

$$2(10) = 20$$

The area of the shorter rectangle is:

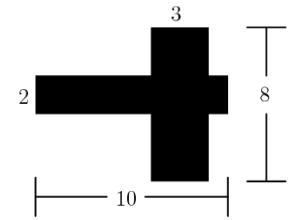
$$3(8) = 24$$

The total area

$$= 20 + 24 - 2(3) = 38$$

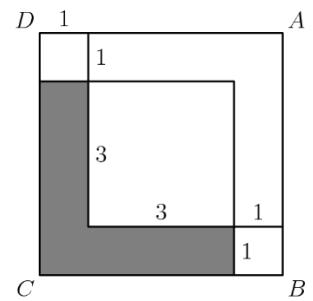
Method II

$$2(10) + 3(8 - 2) = 20 + 3(6) = 20 + 18 = 38$$



Example 1.66

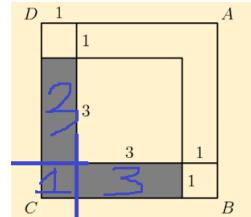
Figure ABCD is a square. Inside this square three smaller squares are drawn with the side lengths as labeled. The area of the shaded L-shaped region is (AMC 8 2000/6)



We split the *L-shaped* region into three rectangles as shown in the diagram.

The total area is:

$$1(3) + 1(3) + 1(1) = 3 + 3 + 1 = 7$$



H. Applications

Example 1.67

Charlyn walks completely around the boundary of a square whose sides are each 5 km long. From any point on her path she can see exactly 1 km horizontally in all directions. What is the area of the region consisting of all points Charlyn can see during her walk, expressed in square kilometers, and rounded to the nearest whole number? (AMC 10 2000/18)

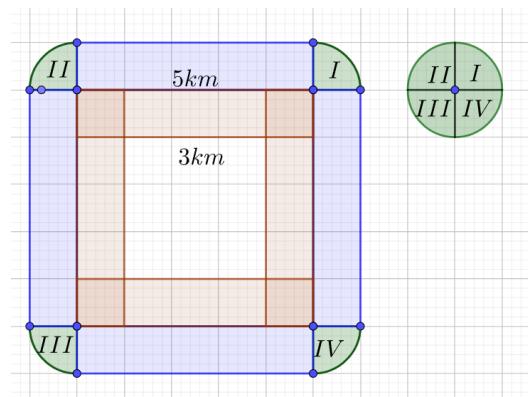
Draw the square with sides 5 km each. Note that Charlyn sees 1 km around her in all directions, which means she can see in the shape of a circle.

Area of the brown region:

$$\begin{aligned} &= \text{Outer Square} - \text{Inner Square} \\ &= 5^2 - 3^2 = 25 - 9 = 16 \end{aligned}$$

Area of the blue region is four rectangles

$$= 4(5 \times 1) = 20$$



Area of the green region is four quarter circles that combine to make a complete circle which has area:

$$= \pi r^2 = \pi(1^2) = \pi \approx 3.14$$

Finally, the total area is

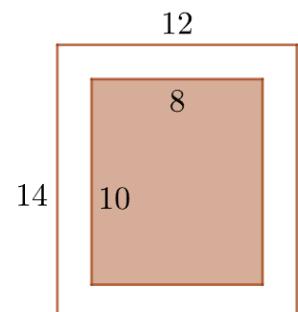
$$16 + 20 + 3.14 = 39.14 \approx 39$$

Example 1.68

A rectangular photograph is placed in a frame that forms a border two inches wide on all sides of the photograph. The photograph measures 8 inches high and 10 inches wide. What is the area of the border, in square inches? (AMC 8 2012/6)

The area of the border

$$\begin{aligned} &= \text{Outer Area} - \text{Inner Area} \\ &= 14 \cdot 12 - 10 \cdot 8 \\ &= 168 - 80 \\ &= 88 \text{ in}^2 \end{aligned}$$



1.69: Arithmetic Progression

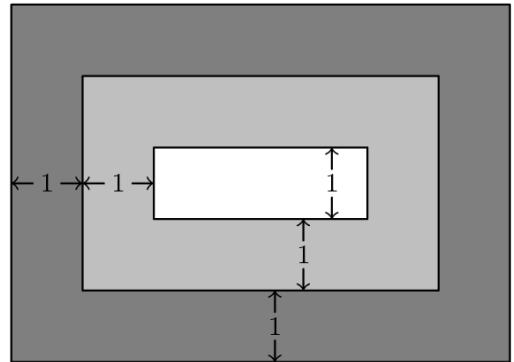
An arithmetic progression is a sequence where each successive term increases by the same amount

Examples of arithmetic progressions are:

$$\begin{aligned} 5, 10, 15 &\Rightarrow \text{Increases by } 5 \\ 3, 10, 17 &\Rightarrow \text{Increases by } 7 \end{aligned}$$

Example 1.70

A rug is made with three different colors as shown. The areas of the three differently colored regions form an arithmetic progression. The inner rectangle is one foot wide, and each of the two shaded regions is 1 foot wide on all four sides. What is the length in feet of the inner rectangle? (AMC 10A 2016/10)



	Inner (white)	Middle (light gray)	Outer (dark gray)
Width	1	3	5
Length	1	3	5
Area of Rectangle	1	9	25
Area of Colored Region	1	$= 9 - 1 = 8$	$= 25 - 9 = 16$

1,8,16 does not form an arithmetic progression

	Inner (white)	Middle (light gray)	Outer (dark gray)
Width	1	3	5
Length	2	4	6
Area of Rectangle	2	12	30
Area of Colored Region		$= 12 - 2 = 10$	$= 30 - 12 = 18$

2,10,18 \Rightarrow Arithmetic Progression

	Inner (white)	Middle (light gray)	Outer (dark gray)
Width	1	3	5
Length	x	$x + 2$	$x + 4$
Area of Rectangle	x	$3(x + 2) = 3x + 6$	$5(x + 4) = 5x + 20$
Area of Colored Region	x	$= 2x + 6$	$= 2x + 14$

$$x, \quad 2x + 6, \quad 2x + 14$$

The difference between the second and third terms is:

$$(2x + 14) - (2x + 6) = 8$$

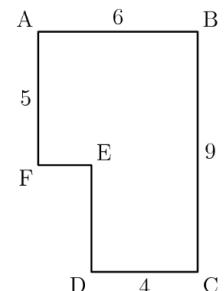
The difference between the first and second terms is:

$$(2x + 6) - x = x + 6 = 8 \Rightarrow x = 2$$

I. Composite Figures

Example 1.71

The area of polygon ABCDEF, in square units, is (AMC 8 1985/4)



$$5 \cdot 6 + 4 \cdot 5 = 30 + 16 = 46 \text{ units}^2$$

Example 1.72

Three identical rectangles are put together to form rectangle ABCD as shown in the figure below. Given that the length of the shorter side of each of the smaller rectangles is 5 feet, what is the area in square feet of rectangle ABCD? (AMC 8 2019/2)

The length of each rectangle is

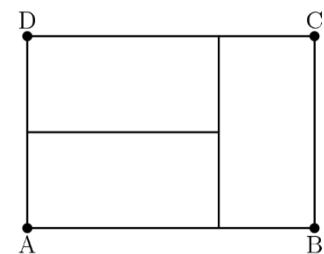
$$5 \times 2 = 10$$

The area of each rectangle is

$$5 \times 10 = 50$$

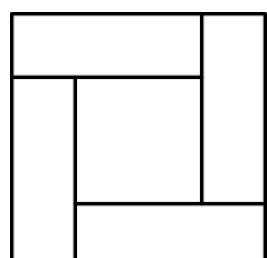
The total area is

$$50 \times 3 = 150 \text{ ft}^2$$



Example 1.73

Four congruent rectangles are placed as shown. The area of the outer square is 4 times that of the inner square. What is the ratio of the length of the longer side of each rectangle to the length of its shorter side? (AMC 10A 2009/14)



Since the question does not give lengths, we assume simple numbers.

Let the inner square have

$$\text{Side} = 1 \Rightarrow \text{Area} = 1$$

Then, the outer square has area four times that of the inner square

$$\text{Area(Outer)} = 4 \times 1 = 4 \Rightarrow \text{Side(Outer)} = 2$$

The width of each rectangle

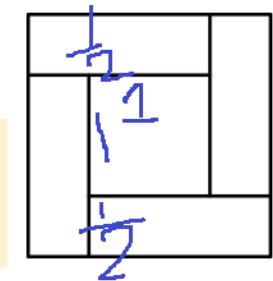
$$= \frac{2-1}{2} = \frac{1}{2}$$

The length of each rectangle

$$= 2 - \frac{1}{2} = \frac{3}{2}$$

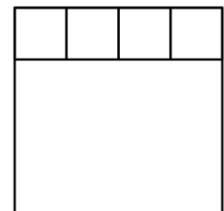
The ratio is:

$$\frac{3}{2} : \frac{1}{2} = 3 : 1$$



Example 1.74

Four identical squares and one rectangle are placed together to form one large square as shown. The length of the rectangle is how many times as large as its width? (AMC 10A 2010/2)



Let the side of each smaller square

$$= 1$$

The side of the larger square

$$= 4$$

The length of the rectangle

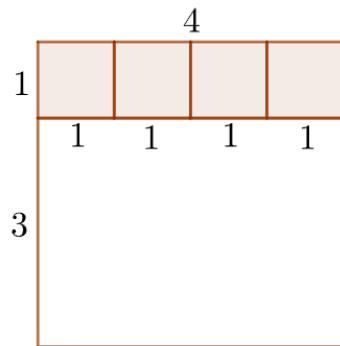
$$= 4$$

The width of the rectangle

$$= 4 - 1 = 3$$

The length is:

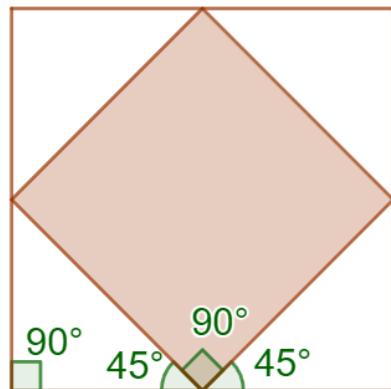
$$\frac{4}{3} \text{ times of the width}$$



J. Inscribed Figures

1.75: Joining the midpoints of a square

The quadrilateral formed by joining the midpoints of a square is itself a square.



1.76: Area of square formed by joining the midpoints of a square

The area of the square formed by joining the midpoints of the sides of a square is half of the larger square.

Method I

Area of larger square

$$= 2 \cdot 2 = 4$$

Area of smaller square

$$= \sqrt{2} \cdot \sqrt{2} = 2$$

Hence, area of smaller square is half of area of larger square.

Method II

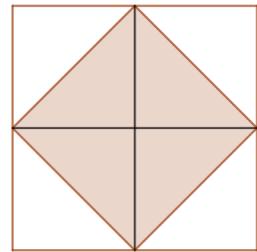
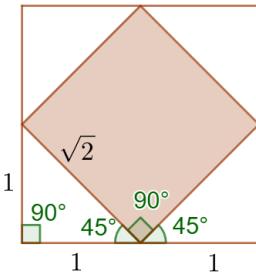
Draw a square and connect its midpoints to form another square. Connect the opposite midpoints to form 4 brown triangles.

There are 4 brown triangles and 4 white triangles.

Each triangle has equal area.

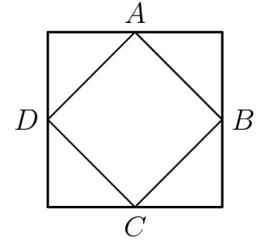
Hence, the area of the brown square

$$= \frac{4}{8} = \frac{1}{2} \text{ of the larger square}$$



Example 1.77

- A. Points A, B, C and D are midpoints of the sides of the larger square. If the larger square has area 60, what is the area of the smaller square? (AMC 8 2006/5)
- B. Each of the sides of a square S_1 with area 16 is bisected, and a smaller square S_2 is constructed using the bisection points as vertices. The same process is carried out on S_2 to construct an even smaller square S_3 . What is the area of S_3 ? (AMC 10A 2008/10)



Part A

$$\text{Area} = \frac{60}{2} = 30$$

Part B

$$S_2 = \frac{S_1}{2} = 8 \Rightarrow S_3 = \frac{S_2}{2} = \frac{8}{2} = 4$$

Example 1.78: Geometric Sequence

Square S has side length s . Square S_1 is drawn by connecting the midpoints of S . Square S_2 is drawn by connecting the midpoint of S_1 . The process continues. What is the area of S_{10} ? S_n

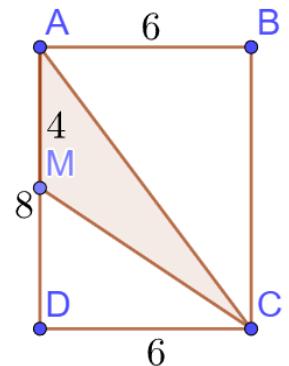
$$\begin{aligned} \text{Area}(S) &= s^2 \\ S_1 &= \frac{s^2}{2} \\ S_2 &= \frac{S_1}{2} = \frac{\frac{s^2}{2}}{2} = \frac{s^2}{2} \times \frac{1}{2} = \frac{s^2}{4} = \frac{s^2}{2^2} \\ S_3 &= \frac{S_2}{2} = \frac{\frac{s^2}{2^2}}{2} = \frac{s^2}{2^3} \\ S_{10} &= \frac{s^2}{2^{10}} \\ S_n &= \frac{s^2}{2^n} \end{aligned}$$

Example 1.79

In rectangle $ABCD$, $AB = 6$ and $AD = 8$. Point M is the midpoint of \overline{AD} . What is the area of $\triangle AMC$? (AMC 8 2016/2)

The area of the triangle is

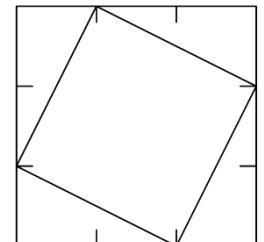
$$\frac{1}{2}hb = \frac{1}{2}(AM)(DC) = \frac{1}{2}(4)(6) = 12$$



K. Trisection

Example 1.80

Each side of the large square in the figure is trisected (divided into three equal parts). The corners of an inscribed square are at these trisection points, as shown. The ratio of the area of the inscribed square to the area of the large square is (AMC 8 1997/15)



Method I

Area of each triangle

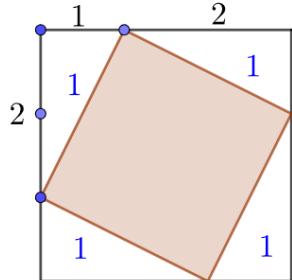
$$= \frac{1}{2}hb = \frac{1}{2}(1)(2) = 1$$

Using complementary areas, area of the smaller square is

$$\begin{aligned} A(\text{Smaller Square}) &= A(\text{Larger Square}) - A(\text{Triangles}) \\ &= 3^2 - 1(4) = 9 - 4 = 5 \end{aligned}$$

The ratio is

$$\frac{5}{9}$$



Method II

By Pythagoras Theorem, the side length of the smaller square is:

$$= \sqrt{1^2 + 2^2} = \sqrt{5}$$

Area of the larger square is

$$(\sqrt{5})^2 = 5$$

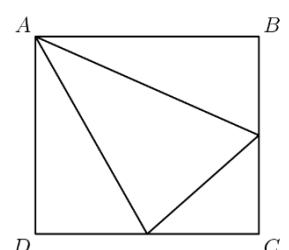
The ratio is

$$\frac{5}{3^2} = \frac{5}{9}$$

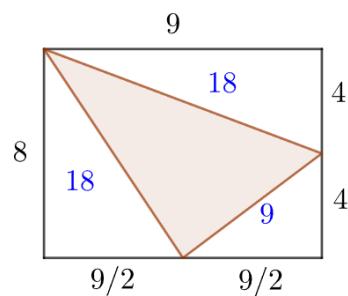
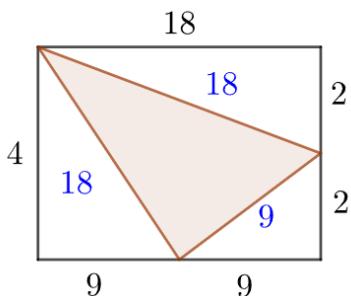
L. Complementary Areas

Example 1.81

The area of rectangle $ABCD$ is 72. If point A and the midpoints of \overline{BC} and \overline{CD} are joined to form a triangle, the area of that triangle is (AMC 8 2000/25)



We can take numbers and do this. The area of the triangles is the same irrespective of the numbers:



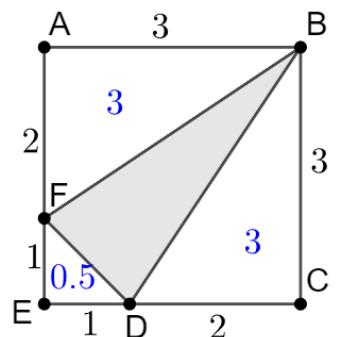
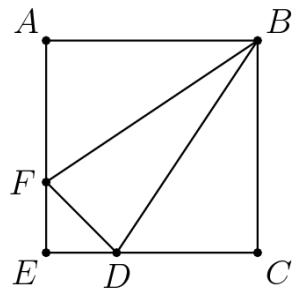
The area is

$$72 - (18 + 18 + 9) = 72 - 45 = 27$$

Example 1.82

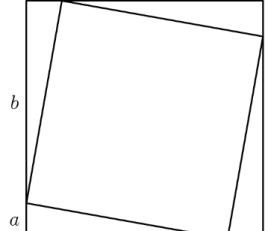
In square $ABCE$, $AF = 2FE$ and $CD = 2DE$. What is the ratio of the area of $\triangle BFD$ to the area of square $ABCE$? (AMC 8 2008/23)

$$\frac{9 - 6.5}{9} = \frac{2.5}{9} = \frac{5}{18} = 5:18$$



Example 1.83

A square with area 4 is inscribed in a square with area 5, with one vertex of the smaller square on each side of the larger square. A vertex of the smaller square divides a side of the larger square into two segments, one of length a , and the other of length b . What is the value of ab ? (AMC 8 2012/25)



The area of the four white triangles is:

$$5 - 4 = 1$$

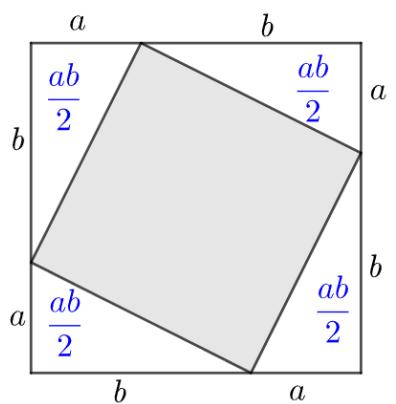
Area of each triangle

$$= \frac{ab}{2}$$

Area of four triangles

$$= 4 \left(\frac{ab}{2} \right) = 2ab$$

$$2ab = 1 \Rightarrow ab = \frac{1}{2}$$

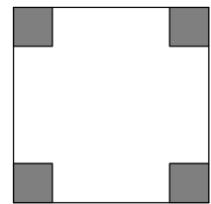


Example 1.84

One-inch squares are cut from the corners of this 5 inch square. What is the area in square inches of the largest square that can fit into the remaining space? (AMC 8 2015/25)

The area of the inner square

$$= 3^2 = 9$$

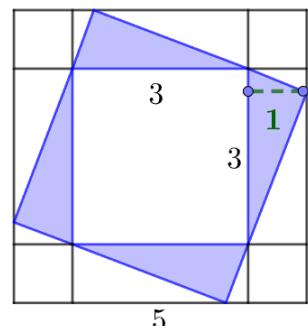


The areas of the four triangles

$$= 4 \left(\frac{1}{2} hb \right) = 4 \left(\frac{1}{2} \cdot 1 \cdot 3 \right) = 6$$

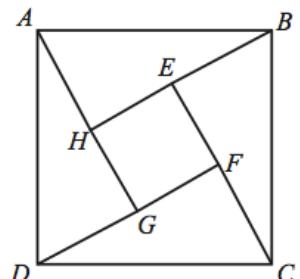
The total area is

$$= 9 + 6 = 15$$



Example 1.85

In the figure, the length of side AB of square $ABCD$ is $\sqrt{50}$ and $BE = 1$. What is the area of the inner square $EFGH$? (AMC 10A 2005/8)



Step I: Prove triangles congruent

$$\angle AHB = \angle AGD = 90^\circ$$

Let

$$\angle ABH = \alpha \Rightarrow \angle BAH = 90 - \alpha$$

Since $\angle BAD = 90^\circ$:

$$\begin{aligned} \angle BAH + \angle GAD &= 90^\circ \\ \angle GAD &= 90 - \angle BAH = 90 - (90 - \alpha) = \alpha \end{aligned}$$

By complementary angles in a right triangle:

$$\angle GDA = 90 - \angle GAD = 90 - \alpha$$

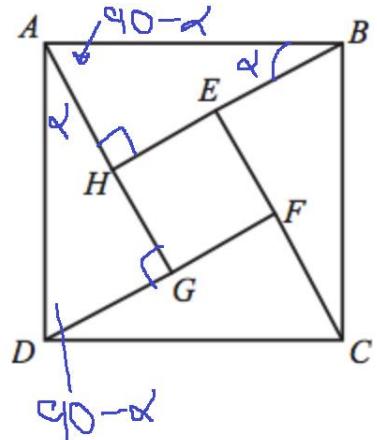
From the diagram, $\Delta BAH \cong \Delta DAG$ by ASA Congruence:

$$BA = DA = \sqrt{50}$$

$$\begin{aligned} \angle BAH &= \angle GDA = 90 - \alpha \\ \angle ABH &= \angle GAD = \alpha \end{aligned}$$

By CPCTC:

$$\begin{aligned} AG &= BH \\ GH + HA &= HE + EB \\ HA &= EB = 1 \end{aligned}$$



Step II: Use the Congruence

$$\angle AHB = 180 - 90 = 90^\circ$$

By Pythagoras Theorem in ΔAHB

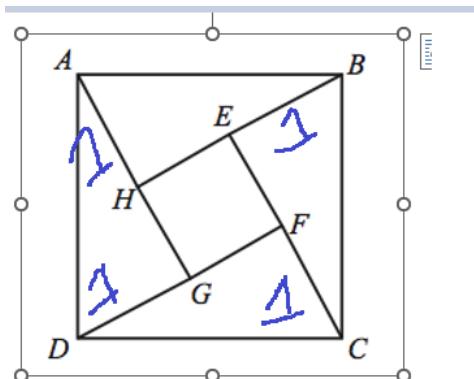
$$AH^2 + HB^2 = AB^2$$

Substitute $AH = 1, HB = HE + EB = HE + 1, AB = \sqrt{50}$

$$1^2 + (HE + 1)^2 = (\sqrt{50})^2$$

$$1 + (HE + 1)^2 = 50$$

$$(HE + 1)^2 = 49$$



Take the square root both sides:

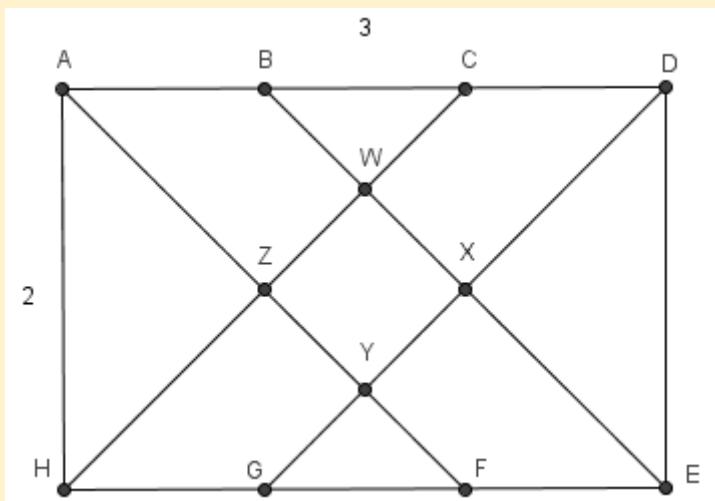
$$HE + 1 = 7$$

$$HE = 6$$

$$HE^2 = 36$$

Example 1.86

In rectangle $ADEH$, points B and C trisect \overline{AD} , and points G and F trisect \overline{HE} . In addition, $AH = AC = 2$, and $AD = 3$. What is the area of quadrilateral $WXYZ$ shown in the figure? (AMC 10A 2006/17)



M. Ratios

Example 1.87

A square is drawn inside a rectangle. The ratio of the width of the rectangle to a side of the square is 2:1. The ratio of the rectangle's length to its width is 2:1. What percent of the rectangle's area is in the square?

(AMC 10A 2008/2)

Since the question uses ratios, we can take any side lengths that we want.

Take a square with side length 1. The rectangle will have width
 $1 \times 2 = 2$

And it will have length

$$2 \times 2 = 4$$

The area of the rectangle will be

$$2 \times 4 = 8$$

And the percentage will be

$$\frac{1}{8} = 12.5\%$$

Example 1.88

Older television screens have an aspect ratio of 4:3. That is, the ratio of the width to the height is 4:3. The aspect ratio of many movies is not 4:3, so they are sometimes shown on a television screen by "letterboxing" - darkening strips of equal height at the top and bottom of the screen, as shown. Suppose a movie has an aspect ratio of 2:1 and is shown on an older television screen with a 27-inch diagonal. What is the height, in inches, of each darkened strip? (AMC 10A 2008/14)



The TV has height and width in the ratio 4:3. The movie has height which is half of its width.

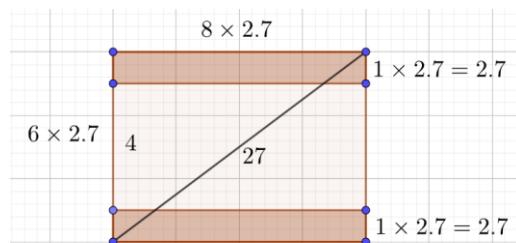
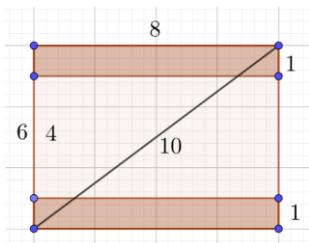
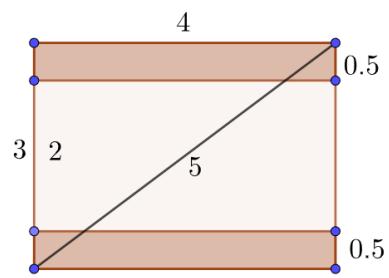
$$\text{Height of movie} = \frac{4}{2} = 2$$

The height of the strip remaining has:

$$\text{Height} = \frac{4 - 2}{2} = \frac{2}{2} = 1$$

This gives us the diagram to the right, which has diagonal

$$(3,4,5) \Rightarrow \text{Diagonal} = 5$$



Double the dimensions to get a diagonal of 10, and then multiply by 2.7 to get a diagonal of 27.

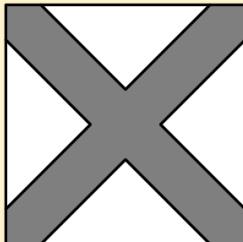
Method II

Scale the height of the side length by $\frac{27}{5}$ to get the height in the TV screen in the question:

$$\frac{1}{2} \times \frac{1}{5} \times 27 = \frac{27}{10} = 2.7 \text{ inches}$$

Example 1.89

A paint brush is swept along both diagonals of a square to produce the symmetric painted area, as shown. Half the area of the square is painted. What is the ratio of the side length of the square to the brush width? (AMC 10A 2007/19)



Example 1.90

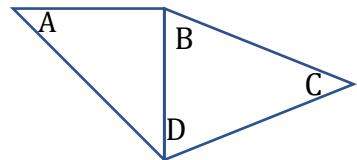
In the question below, mark all options that are correct.

If two adjacent sides and a diagonal of a quadrilateral form a right-angled triangle, then that quadrilateral is definitely a:

- A. Rhombus
- B. Square
- C. Rectangle
- D. Parallelogram
- E. Trapezium
- F. None of these

The quadrilateral in the diagram does not fall under Options A to E, though it meets the conditions in the question.

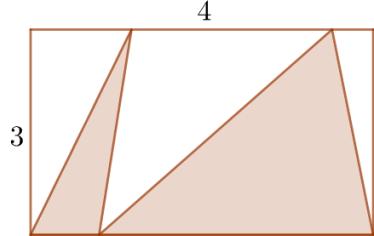
Hence, Option F.



N. Triangles in a Rectangle

Example 1.91

The diagram alongside shows a rectangle with dimension 3×4 . If the triangle on the left has area which is one third of the triangle on the right, then determine the area of the smaller triangle.



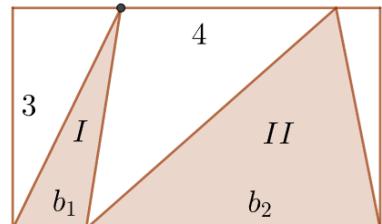
Area of the Triangles

$$\text{Area of } I = \frac{1}{2}hb = \frac{1}{2}(3)b_1 = \frac{3}{2}b_1$$

$$\text{Area of } II = \frac{1}{2}hb = \frac{1}{2}(3)b_2 = \frac{3}{2}b_2$$

The total area is

$$\frac{3}{2}b_1 + \frac{3}{2}b_2 = \frac{3}{2}(b_1 + b_2) = \frac{3}{2}(4) = 6$$



Dividing the area

The ratio of the area of the triangle is:

$$1:3 \Rightarrow 1+3=4$$

The smaller triangle has area

$$\text{Total area} = \frac{1}{4} \text{ of } 6 = \frac{1}{4} \times 6 = 1.5$$

1.2 Visual Techniques

A. Visual Techniques

1.92: Rearrangement

In certain cases, a polygon can be rearranged without changing its perimeter.

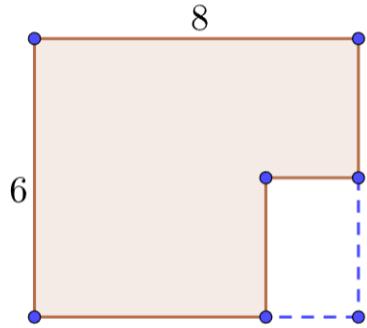
Example 1.93

The perimeter of the polygon shown (in the given units) is (AMC 8 1986/13)

Consider a rectangle that has the same perimeter as the given polygon.
 (Make sure you understand why the perimeters are the same.)

Hence, the perimeter of the polygon

$$= P(\text{Rectangle}) = 2(6 + 8) = 2 \times 14 = 28$$



1.94: Adding a Tile

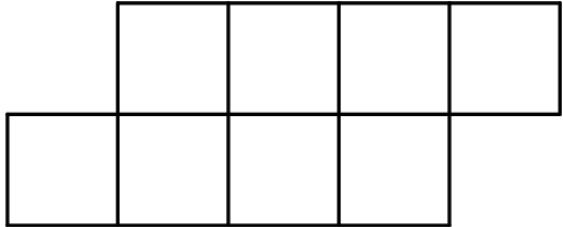
If you have a tiled shape made of squares with side length s , and you add a square to the shape, you will

- NOT change the perimeter if the new square shares two sides with the old shape
- Increase the perimeter by $2s$ if the new square shares a single side with the old shape

Example 1.95

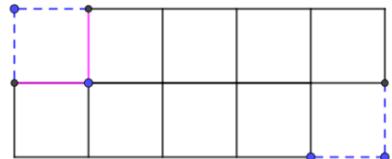
Mark all correct options

Eight 1×1 square tiles are arranged as shown so their outside edges form a polygon with a perimeter of 14 units. Two additional tiles of the same size are added to the figure so that at least one side of each tile is shared with a side of one of the squares in the original figure. Find all possible values of the perimeter of the new figure? (AMC 8 1992/22, Adapted)



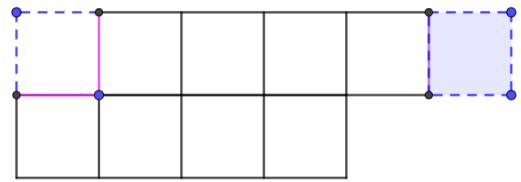
Case I: Perimeter 14

If you add a tile at the top left and the bottom right, the perimeter of the new figure remains the same, since the blue dotted lines get added, and the purple lines get removed.



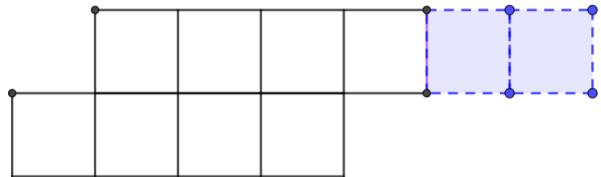
Case II: Perimeter 16

If you add a tile at the top left and top right, the perimeter increases by 2.



Case III: Perimeter 18

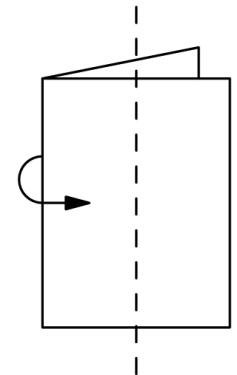
If you add two tiles at the right, the perimeter increases by 4.



Example 1.96

Suppose a square piece of paper is folded in half vertically. The folded paper is then cut in half along the dashed line. Three rectangles are formed—a large one and two small ones.

What is the ratio of the perimeter of one of the small rectangles to the perimeter of the large rectangle? (AMC 8 1989/24)



Perimeter of small rectangle

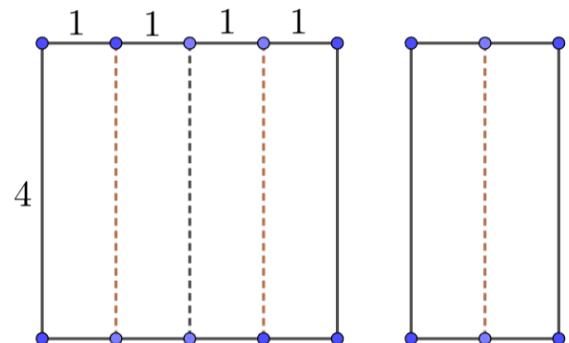
$$= 2(1 + 4) = 2(5)$$

Perimeter of large rectangle

$$= 2(2 + 4) = 2(6)$$

Ratio of perimeter of small rectangle to large rectangle

$$= 2(5):2(6) = 5:6$$



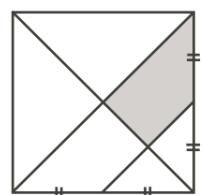
Example 1.97

A square is divided, as shown. What fraction of the area of the square is shaded? (Gauss Grade 7 2007/23)

Method I

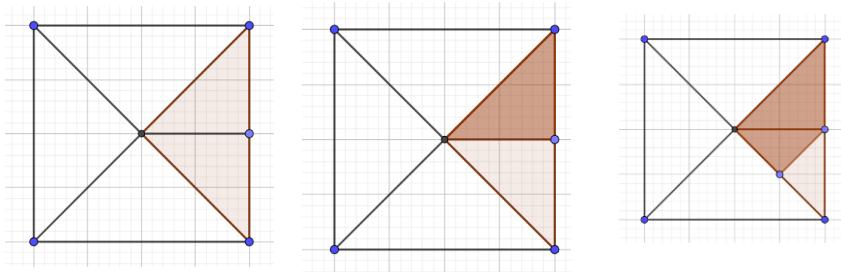
Divide the square into four equal parts using its diagonals. See 1st diagram on the left. The light brown triangle has area

$$\frac{1}{4} \text{ of the square}$$



Divide the light brown triangle into two equal parts, getting a dark brown triangle, which has area

$$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$



Add a further dark brown triangle to get the shape that we want. The smaller dark brown triangle has area

$$\frac{1}{2} \times \frac{1}{8} = \frac{1}{16}$$

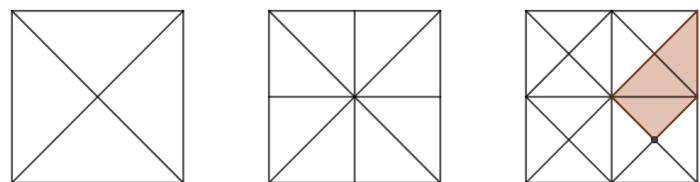
The total area is then:

$$\frac{1}{8} + \frac{1}{16} = \frac{3}{16}$$

Method II:

Divide the square into 16 triangles of equal area as shown. The shaded area occupies 3 of the 16 triangles, and hence the required ratio is

$$\frac{3}{16}$$



Example 1.98

In the diagram shown:

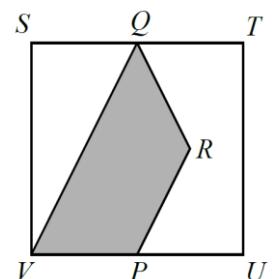
$STUV$ is a square

Q and P are the midpoints of ST and UV

$PR = QR$, and

VQ is parallel to PR

What is the ratio of the shaded area to the unshaded area? (Gauss Grade 7 2014/24)



Let the total area of $STUV$ be 1.

$$A(\Delta QVP) = \frac{1}{4}$$

Also,

$PR \parallel VQ \Rightarrow PRT$ is a diagonal of $QTUP$

$QR = PR \Rightarrow \angle PQU = \angle QPT \Rightarrow QRU$ is a diagonal of $QTUP$

Since diagonals of a rectangle divide it into four triangles with equal area:

$$A(QRP) = \frac{1}{4}A(QTUP) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

Total Shaded Area

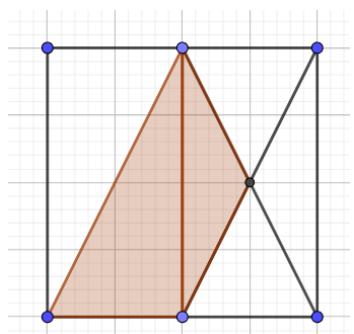
$$= \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

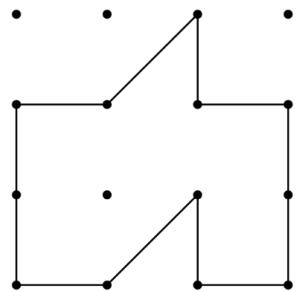
Total Unshaded Area

$$= 1 - \frac{3}{8} = \frac{5}{8}$$

Ratio of Unshaded to Shaded Area

$$= \frac{3}{8} : \frac{5}{8} = 3:5$$





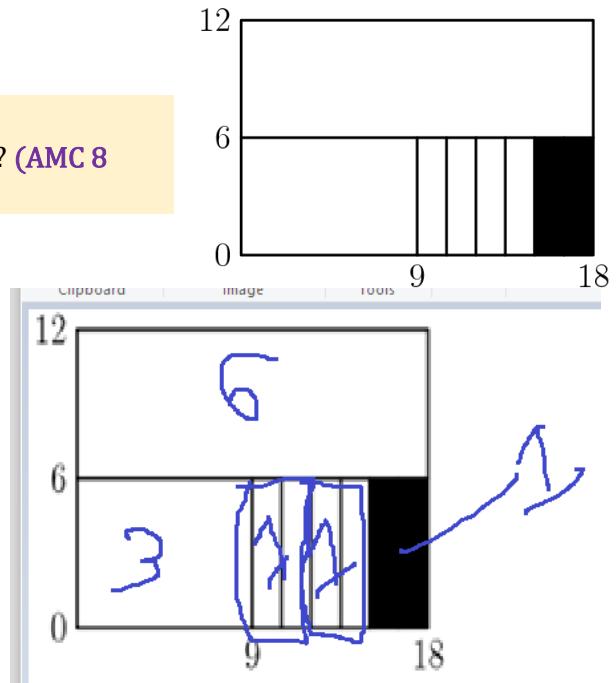
Example 1.99

Dots are spaced one unit apart, horizontally and vertically. The number of square units enclosed by the polygon is (AMC 8 1998/6)

Move the top triangle and fit in the bottom space, giving a rectangle with area
 $2 \times 3 = 6$

Example 1.100

What fraction of the large 12 by 18 rectangular region is shaded? (AMC 8 1987/12)



We can calculate the area of the region.

Let the given shaded region have area

1

The bottom right rectangle will have area

3

The bottom left rectangle will also have area

3

And the top rectangle will have area

6

Total area

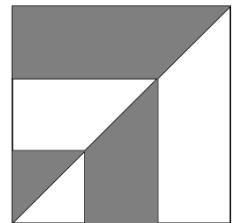
$$= 3 + 3 + 6 = 12$$

And the required fraction will be:

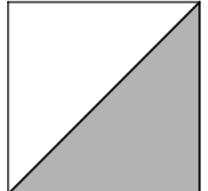
1
12

Example 1.101

What fraction of the square is shaded? (AMC 8 1990/3)

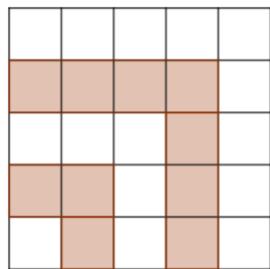


Reflect two of the grey shaded parts across the diagonal of the square. Note the entire area to the right of the diagonal is now shaded.



By symmetry, the two areas are equal. The fraction is

1
—
2



Example 1.102

Determine the number of unshaded squares in the diagram alongside. All of the smallest squares have side length 1.

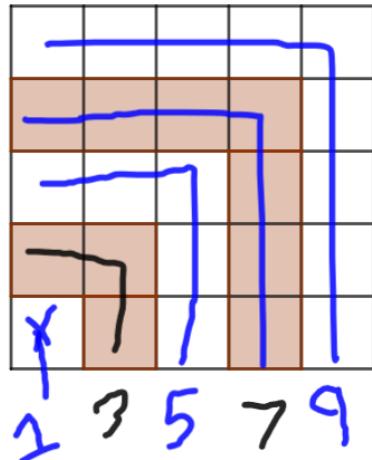
Count the squares as shown to the right. We get:

$$1, 3, 5, 7, 9$$

As the number of squares in each line.

And the number of unshaded squares is:

$$1 + 5 + 9 = 15$$



Example 1.103

What fraction of this square region is shaded? Stripes are equal in width, and the figure is drawn to scale. (AMC 8 1997/10)



Use the square at the bottom left as a unit square, and see the pattern:

$$1, 3, 5, 7, 9, 11$$

Unshaded area:

$$= \frac{1 + 5 + 9}{36} = \frac{15}{36} = \frac{5}{12}$$

Shaded Area

$$= 1 - \frac{5}{12} = \frac{7}{12}$$



Example 1.104

Suppose the figure at right is extended using the same pattern, but so that it is a square with a side length of 20. Find the shaded area.

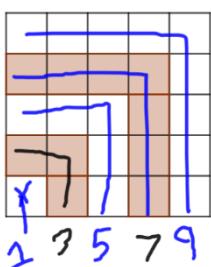
$$1, 3, 5, \dots, 39$$

Where

$$1 \text{ is the } 1\text{st odd number} = 2(1) - 1 = 1$$

$$3 \text{ is the } 2\text{nd odd number} = 2(2) - 1 = 4 - 1 = 3$$

$$39 \text{ is the } 20\text{th odd number} = 2(20) - 1 = 40 - 1 = 39$$

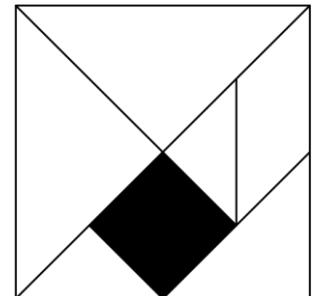


Out of these, we want the sum of alternate numbers:

$$3 + 7 + 11 + 15 + 19 + 23 + 27 + 31 + 35 + 39$$

Arrange the numbers in pairs:

$$\begin{aligned} &= (3 + 39) + (7 + 35) + (11 + 31) + (15 + 27) + (19 + 23) \\ &\quad = 42 + 42 + 42 + 42 + 42 \\ &\quad = 5 \times 42 \\ &\quad = 10 \times 21 \\ &\quad = 210 \end{aligned}$$



Example 1.105

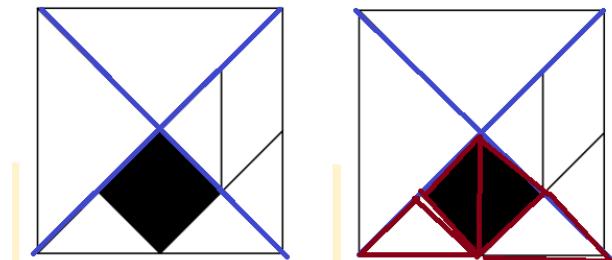
What is the ratio of the area of the shaded square to the area of the large square?
 (The figure is drawn to scale) (AMC 8 1998/13)

Divide the square into four equal parts by drawing the blue diagonals.

Further divide the bottom diagonal into four congruent brown triangles.

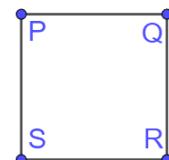
The required area is then:

$$\frac{2}{16} = \frac{1}{8}$$



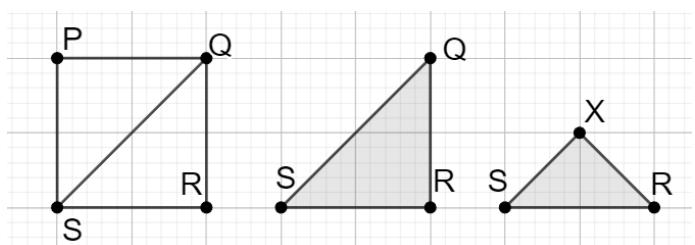
Example 1.106

Let $PQRS$ be a square piece of paper. P is folded onto R and then Q is folded onto S . The area of the resulting figure is 9 square inches. Find the perimeter of square $PQRS$. (AMC 8 1998/20)



Fold the square in half to get ΔQSR . This makes the area half of what it was.

Fold ΔQSR in half to get a smaller triangle. This makes the area $\frac{1}{4}$ of the original square.



Area of square $PQRS$

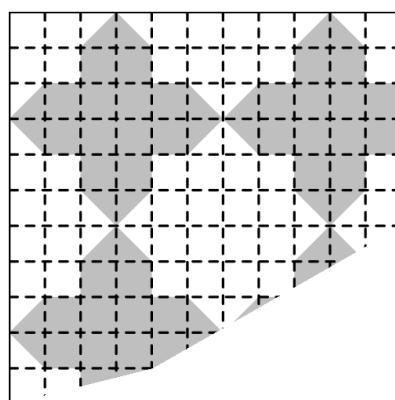
$$= 4(\text{Area of resulting figure}) = 4(9) = 36$$

Side length of $PQRS$

$$= \sqrt{36} = 6$$

Perimeter of $PQRS$

$$= 4 \cdot 6 = 24$$



Example 1.107

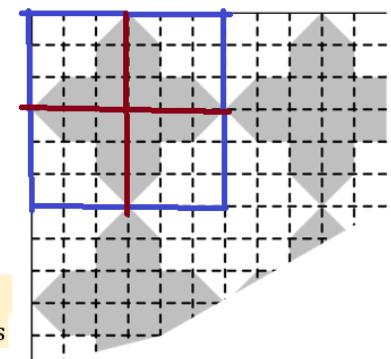
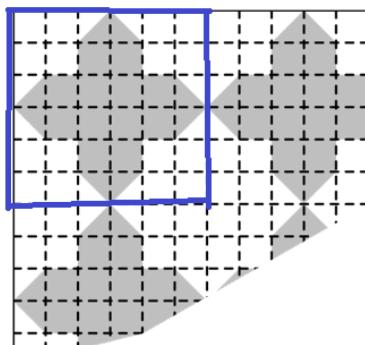
A corner of a tiled floor is shown. If the entire floor is tiled in this

way and

each of the four corners looks like this one, then what fraction of the tiled floor is made of darker tiles? (AMC 8 2002/23)

The pattern repeats every six tiles. See the blue square created in the left diagram. Further, the blue square can be divided into four brown squares, each of which has equal shading.

Then the shading in the brown square

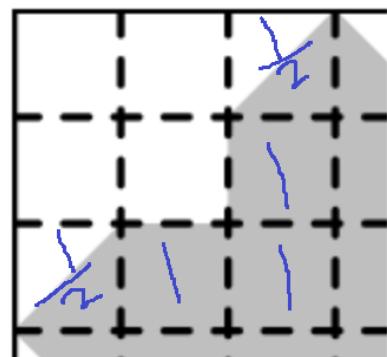


The required shading is

$$\frac{1}{2} + 1 + 1 + 1 + \frac{1}{2} = 4$$

The total number of squares

$$3 \times 3 = 9$$



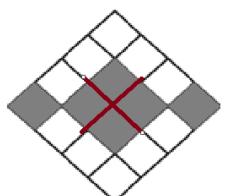
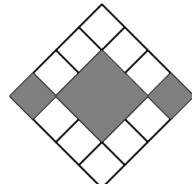
The fraction is

$$\frac{4}{9}$$

Example 1.108

In the figure, what is the ratio of the area of the gray squares to the area of the white squares? (AMC 8 2008/6)

$$6:10 = 3:5$$



Example 1.109

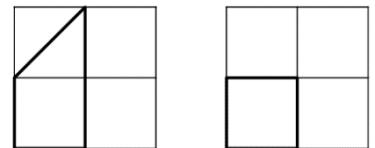
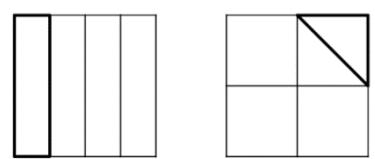
Each of the following four large congruent squares is subdivided into combinations of congruent triangles or rectangles and is partially bolded. What percent of the total area is partially bolded? (AMC 8 2011/7)

Visually rearrange the shapes.

Bottom right occupies a rectangle.

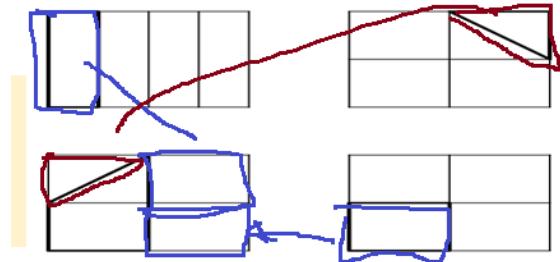
Top left occupies a rectangle.

Top right occupies a triangle at the top left.



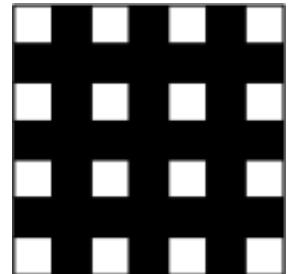
Overall, we get

$$\frac{1}{4} = 25\%$$



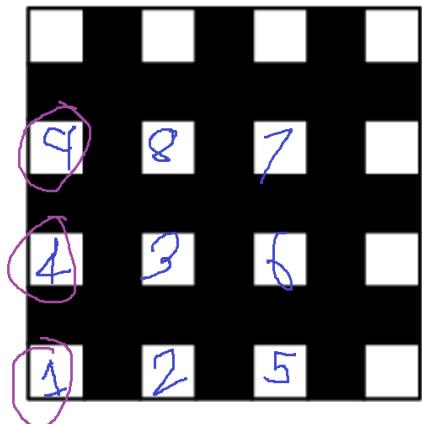
Example 1.110

The diagram represents a 7-foot-by-7-foot floor that is tiled with 1-square-foot black tiles and white tiles. Notice that the corners have white tiles. If a 15-foot-by-15-foot floor is to be tiled in the same manner, how many white tiles will be needed? (AMC 9 2009/18)



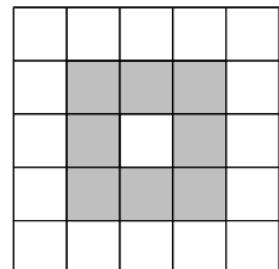
Size of floor

- 1 Foot: 1 White Tile
- 3 Foot: $4 = 2^2$ White Tiles
- 5 Foot: $9 = 3^2$ White Tiles
- 7 Foot: 4^2 White Tiles
- 9 Foot: 5^2 White Tiles
- 11 Foot: 6^2 White Tiles
- 13 Foot: 7^2 White Tiles
- 15 Foot: 8^2 White Tiles



Example 1.111

Extend the square pattern of 8 black and 17 white square tiles by attaching a border of black tiles around the square. What is the ratio of black tiles to white tiles in the extended pattern? (AMC 8 2011/3)



The number of black tiles which get added is:

$$5 \times 4 + 4 = 20 + 4 = 24$$

The total black tiles are now:

$$8 + 24 = 32$$

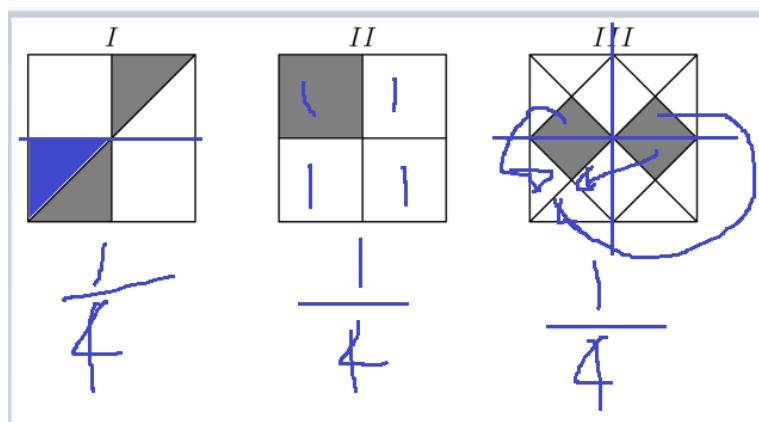
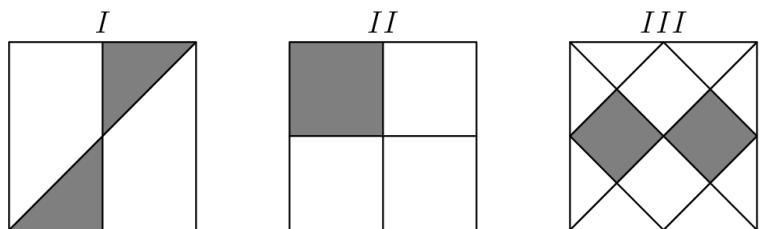
The ratio is:

$$32:17$$

Example 1.112

Each of the three large squares shown below is the same size. Segments that intersect the sides of the squares intersect at the midpoints of the sides. How do the shaded areas of these squares compare? (AMC 8 1994/12)

- A. The shaded areas in all three are equal.
- B. Only the shaded areas of I and II are equal.
- C. Only the shaded areas of I and III are equal.
- D. Only the shaded areas of II and III are equal.
- E. The shaded areas of I, II and III are all different.



Example 1.113

The perimeter of one square is 3 times the perimeter of another square. The area of the larger square is how many times the area of the smaller square? (AMC 8 1994/16)

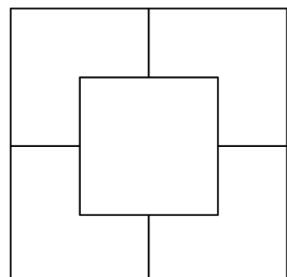
Let

$$\begin{aligned} p_s &= 4 \Rightarrow \text{Side}_s = 1 \Rightarrow \text{Area} = 1 \\ p_L &= 12 \Rightarrow \text{Side}_L = \frac{12}{4} = 3 \Rightarrow \text{Area} = 9 \end{aligned}$$

$$\text{Ratio of areas} = \frac{9}{1} \Rightarrow 9 \text{ times}$$

Example 1.114

The area of each of the four congruent L-shaped regions of this 100-inch by 100-inch square is $\frac{3}{16}$ of the total area. How many inches long is the side of the center square? (AMC 8 1995/18)



The total area of the *L-shaped* regions is:

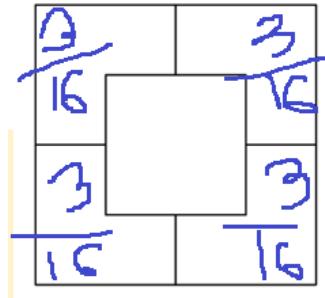
$$\frac{3}{16} \times 4 = \frac{3}{4}$$

The area of the center square

$$= 1 - \frac{3}{4} = \frac{1}{4} \text{ of the entire square}$$

The actual area

$$= \frac{1}{4}(100^2) = \frac{10,000}{4} = 2500$$

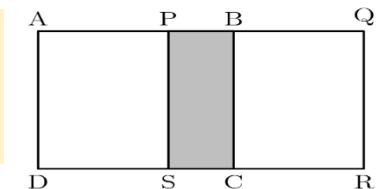


The side length is:

$$\sqrt{2500} = \sqrt{25} \times \sqrt{100} = 5 \times 10 = 50$$

Example 1.115

Two congruent squares, $ABCD$ and $PQRS$, have side length 15. They overlap to form the 15 by 25 rectangle $AQRD$ shown. What percent of the area of rectangle $AQRD$ is shaded? (AMC 8 2011/13)



The length of the shaded portion

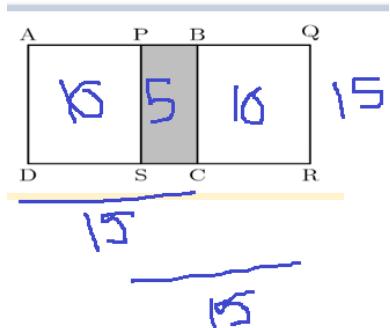
$$= 15 + 15 - 25 = 30 - 25 = 5$$

The percent of area that is shaded is in proportion to the length that is shaded because the height is the same throughout:

$$= \frac{5}{25} = 20\%$$

Method II: Algebra

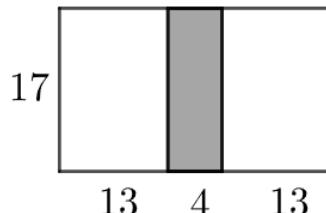
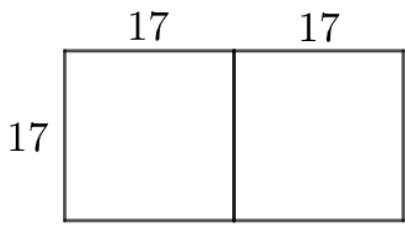
$$\begin{aligned} x + 2(15 - x) &= 25 \\ x + 30 - 2x &= 25 \\ x &= 5 \end{aligned}$$



Example 1.116

Two squares $17 \text{ cm} \times 17 \text{ cm}$ overlap to form a rectangle $17 \text{ cm} \times 30 \text{ cm}$. The area of the overlapping region is: (NMTC Sub-Junior/Screening, 2011/3)

If you do not overlap the squares, and put them next to each other you get:



If you overlap and move the right square to the left, the length will reduce.

To reduce the length by $34 - 30 = 4 \text{ cm}$, you move the right square 4cm to the left.

The area of this overlap is

$$4 \times 17 = 68 \text{ cm}^2$$

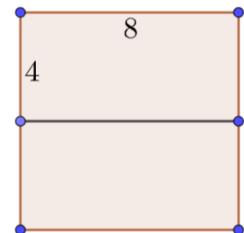
B. Simple Cutting

Example 1.117

A square with side length 8 is cut in half, creating two congruent rectangles. What are the dimensions of one of these rectangles? (AMC 10A 2012/2)

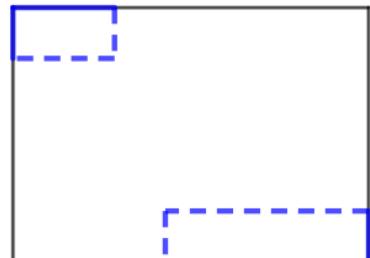
The dimensions are:

$$\begin{matrix} 4 \\ \swarrow \quad \searrow \\ \text{Breadth} \quad \text{Length} \end{matrix}$$



1.118: Types of Angles

If you cut rectangular, non-overlapping regions from a rectangle, the perimeter does not change.



- Does not “change” can be written as does not “vary”, and a single word for this is *invariant*.
- If the cuts overlap, then the perimeter does not remain the same.

Example 1.119

- A. Tanmay bought a rectangular carpet. He cut a rectangle from the top left corner of the carpet. He cut another rectangle from the top right corner of the carpet. The cuts did not overlap. The carpet originally had dimensions 3×4 . What was the perimeter of the carpet after the cuts were made.
- B. If the cuts that Tanmay made had been overlapping, what would have been your answer to Part A.

Part A

The original carpet had

$$\text{Perimeter} = 2(3 + 4) = 2(7) = 14$$

The new carpet (after the cuts) will have the same perimeter

$$= 14$$

Part B

Cannot be determined without further information since the cuts are overlapping.

Example 1.120

Tulsi bought a square-shaped carpet. She removed two unequal, nonoverlapping rectangular regions from two of its corners. Now she could fit the carpet exactly into her study room. If the perimeter of the study room is 16 m, what is the area of the original carpet? (NMTC Primary-Screening, 2008/16)

The perimeter of the square

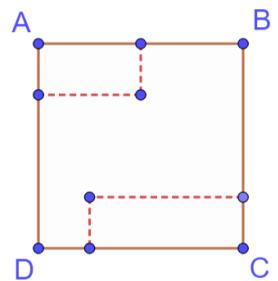
$$= \text{Perimeter of Study Room} = 16m$$

The side length of the square:

$$= \frac{\text{Perimeter}}{4} = \frac{16}{4} = 4m$$

The area of the square

$$= s^2 = 4^2 = 16m^2$$



Example 1.121

A square with integer side length is cut into 10 squares, all of which have integer side length and at least 8 of which have area 1. What is the smallest possible value of the length of the side of the original square? (AMC 8 2012/17)

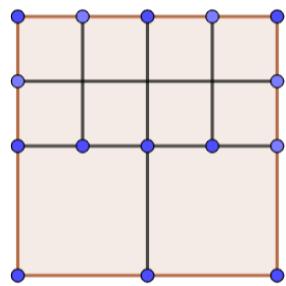
There are 10 squares. Each square has integer side length. Hence, area of these 10 squares must be minimum:

$$10 \times 1 = 10$$

A square of side length will have area

$$3 \times 3 = 9 < 10 \Rightarrow \text{Does not work}$$

Let us try a square of side length 4. We make 8 squares of area 1, and 2 squares of area 4, and we are able to cover the whole square.



Hence,

$$\text{Smallest possible value} = 4$$

2. QUADRILATERALS

2.1 Sum of Angles

Example 2.1

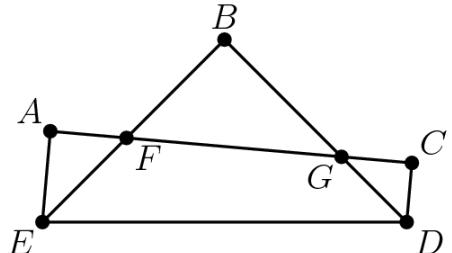
In the figure, $\angle A$, $\angle B$, and $\angle C$ are right angles. If $\angle AEB = 40^\circ$ and $\angle BED = \angle BDE$, then $\angle CDE =$ (AMC 8 1995/13)

In Isosceles Right ΔBED

$$\angle BED = \angle BDE = 45^\circ$$

Sum of angles of a Quadrilateral is 360° . Hence:

$$\angle CDE = 360 - \underbrace{90^\circ}_{\angle A} - \underbrace{90^\circ}_{\angle C} - \underbrace{40^\circ}_{\angle AEB} - \underbrace{40^\circ}_{\angle BED}$$



Example 2.2

The angles of quadrilateral $ABCD$ satisfy $\angle A = 2 \angle B = 3 \angle C = 4 \angle D$. What is the degree measure of $\angle A$, rounded to the nearest whole number? (AMC 10B 2007/15)

$$\begin{aligned} a &= 2b = 3c = 4d = k \\ k + \frac{k}{2} + \frac{k}{3} + \frac{k}{4} &= 360 \Rightarrow k = \frac{360 \times 12}{25} = 172.8 \approx 173 \end{aligned}$$

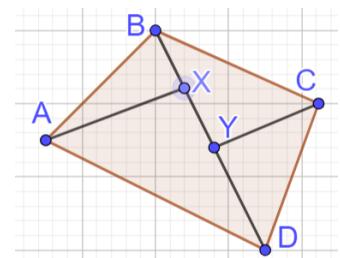
2.3: Area Formula

Consider quadrilateral $ABCD$ with diagonal BD . Also

$$AX \perp BD, \quad CY \perp BD$$

Then, the area of quadrilateral $ABCD$ is:

$$\frac{1}{2}(BD)(AX + CY)$$



$$\begin{aligned} A(\Delta ABD) &= \frac{1}{2}hb = \frac{1}{2}(BD)(AX) \\ A(\Delta CBD) &= \frac{1}{2}hb = \frac{1}{2}(BD)(CY) \end{aligned}$$

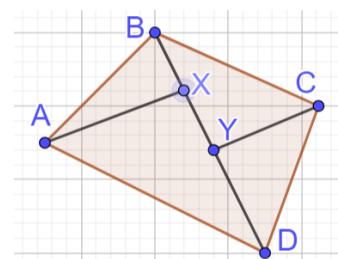
$$A(ABCD) = [ABD] + [CBD] = \frac{1}{2}(BD)(AX) + \frac{1}{2}(BD)(CY) = \frac{1}{2}(BD)(AX + CY)$$

Example 2.4

In quadrilateral $ABCD$, with diagonal BD , AX and CY are perpendicular to BD .

Find the area of the quadrilateral if:

- A. $BD = 13, AX = 5, CY = 4$
- B. $BD = 12, AX = 4, CY = 7$



Part A

$$A(ABCD) = \frac{1}{2}(BD)(AX + CY) = \frac{1}{2}(12)(11) = 66$$

Part B

$$A(ABCD) = \frac{1}{2}(BD)(AX + CY) = \frac{1}{2}(13)(5 + 4) = \frac{117}{2}$$

Example 2.5

The perimeter of a parallelogram is 30 cm. If one of the sides measures 5 cm, find the length of the other three sides.

Example 2.6

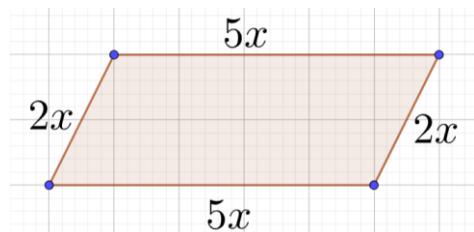
The adjacent sides of a parallelogram are in the ratio 2: 5. If the perimeter of the parallelogram is 10, find the sides of the parallelogram.

$$2x + 5x + 2x + 5x = 10$$

$$14x = 10$$

$$x = \frac{10}{14} = \frac{5}{7}$$

$$2x = \frac{10}{7}, 5x = \frac{25}{7}$$



Example 2.7

A piece of wire 1 m long is used to make four squares of equal size. Each of the squares is then opened up and cut into four smaller, equal, squares. Find the side length of the smaller squares in m.

In all, we are making

$$4 \times = 16 \text{ squares}$$

Total length of wire

$$1 \text{ m}$$

Each square has

$$\text{Perimeter} = 1 \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{16} \text{ m} \Rightarrow \text{Side} = \frac{1}{16} \times \frac{1}{4} = \frac{1}{64} \text{ m}$$

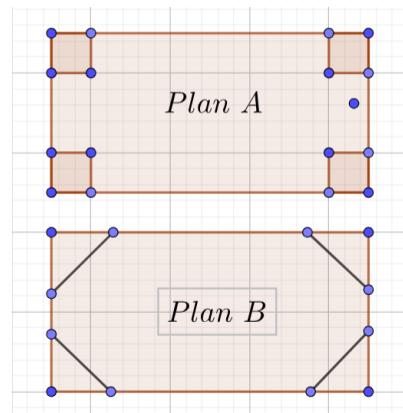
Example 2.8

A rectangular garden of dimensions 15m × 25m has to have four flowerbeds added to its corners. The garden is currently all grass. Under plan

- A: The flowerbeds will be in the shape of rectangles of dimension 2m × 2m, planted with *germaniums*.
- B: The flowerbeds will be in the shape of isosceles right-angled triangles with length of leg 3m, planted with *bougenville*.

Find under each plan:

- the area of the remaining grass.
- the cost of the flowers



Area of garden

$$= 15 \times 25 = 375 \text{ m}^2$$

Area of rectangular flowerbeds

$$= 4 \times 2 \times 2 = 16 \text{ m}^2$$

Area of triangular flowerbeds

$$= 4 \times \frac{1}{2}hb = 4 \times \frac{1}{2} \times 3 \times 3 = 18 \text{ m}^2$$

Part A

$$\text{Plan } A = 375 - 16 = 359 \text{ m}^2$$

$$\text{Plan } B = 375 - 18 = 357 \text{ m}^2$$

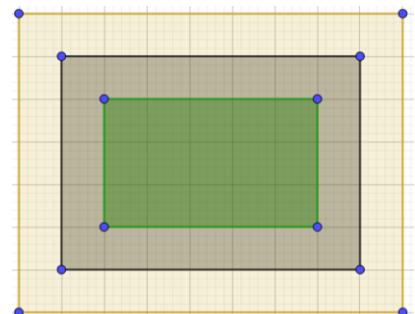
Part B

$$\text{Plan } A =$$

Example 2.9

In a rectangular park, the grass (green area) is surrounded by a walking track (2 m width, dark shaded area), which is surrounded by a jogging track (3 m width, light shaded area). The dimensions of the park (including both tracks) are 15m \times 20m. Find the area of:

- A. The jogging track
- B. The walking track
- C. The Grass



Area of jogging track

$$= (15 \times 20) - (9 \times 14) = 300 - 126 = 174$$

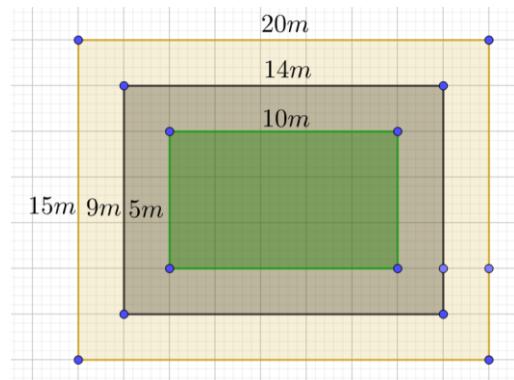
Area of walking track

$$14 \times 9 - 10 \times 5 = 126 - 50 = 76$$

Example 2.10

Rectangle ABCD has sides of length 3 and 4.

- A. Find the length of the diagonal AC.
- B. Find the length of the perpendicular from vertex B to the diagonal.



Similar Triangles

2.2 Trapezoids

A. Trapezoid

2.11: Definition

A quadrilateral with a pair of parallel sides is a trapezoid.

Example 2.12

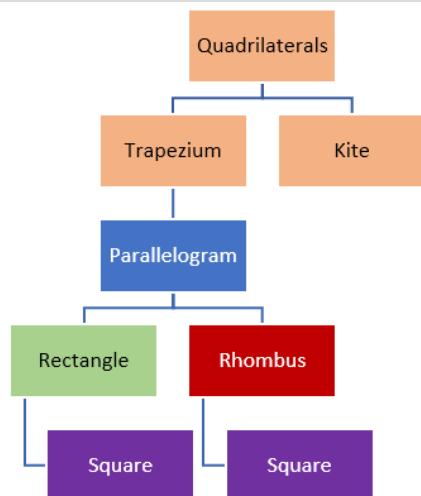
Mark all correct options

Which of the following *must* be trapezoids:

- A. Square
- B. Rectangle
- C. Rhombus
- D. Kite
- E. Parallelogram

A parallelogram is a quadrilateral with two pairs of parallel sides.

Hence, it does have a pair of parallel sides.



Hence, a parallelogram is a special case of a trapezoid.

Squares, rectangles, and rhombuses are all special cases of parallelograms.
 Hence, they are all trapezoids.

However, a kite is a quadrilateral that has two pairs of adjacent sides which are congruent. It may or may not have a pair of parallel sides.

Hence, it is not necessary that a kite is a trapezoid.

Hence, the correct answer is

Options A, B, C, E

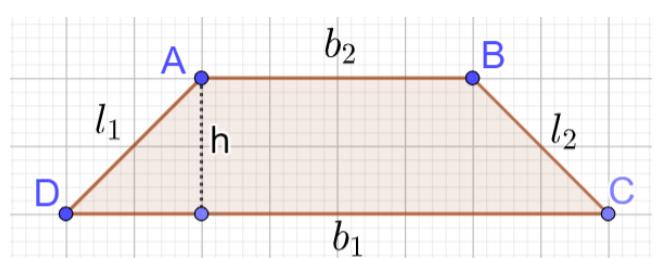
Example 2.13

What is the difference between a trapezoid and a trapezium?

They are the same.

2.14: Terminology

- The two parallel sides are called the bases.
- The two other sides are called the legs.
- The distance between the bases is called the height.

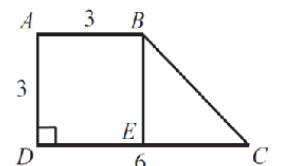


Example 2.15

The quadrilateral drawn alongside is a trapezoid.

- Identify the lines which are parallel to each other.
- Identify the bases, the legs and the height.

*Line AB || Line CD
 Bases are AB and CD
 Legs are AD and BC
 Height = h*



Example 2.167

In trapezoid ABCD, AD is perpendicular to DC, $AD = AB = 3$, and $DC = 6$. In addition, E is on DC, and BE is parallel to AD. Find the area of BEC. (AMC 8 2007/8)

$$\begin{aligned} BE &= AD = 3 \\ EC &= DC - DE = DC - AB = 6 - 3 = 3 \end{aligned}$$

$$[BEC] = \frac{1}{2}hb = \frac{1}{2}(BE)(EC) = \frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2} = 4.5 \text{ units}^2$$

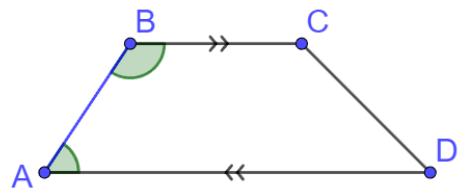
2.17: Angles on the same leg are supplementary

The two angles on the same leg of a trapezoid are supplementary

In trapezoid ABCD, let $BC \parallel AD$. Consider AB as the transversal of BC and AD.

By co-interior angles (angles on the same side of the transversal)

$$\angle CBA + \angle BAD = 180$$



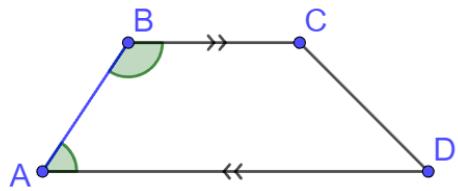
Hence, the angles are supplementary.

Example 2.18

In trapezoid $ABCD$, $\angle A = 50^\circ$. $\angle C = 110^\circ$. Determine the measures of $\angle B$, and $\angle D$.

$$\angle B = 180 - \angle A = 180 - 50^\circ = 130^\circ$$

$$\angle D = 180 - \angle C = 180 - 110^\circ = 70^\circ$$



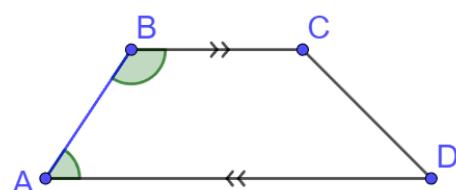
Example 2.19

In trapezoid $ABCD$, $\angle A = 40^\circ$. $\angle C = \angle B + 10$. Determine the measure of $\angle D$.

$$\angle B = 180 - \angle A = 180 - 40 = 140^\circ$$

$$\angle C = \angle B + 10 = 140 + 10 = 150^\circ$$

$$\angle D = 180 - \angle C = 180 - 150 = 30^\circ$$



Example 2.20

In the quadrilateral drawn alongside, side $BC \parallel$ side AD .

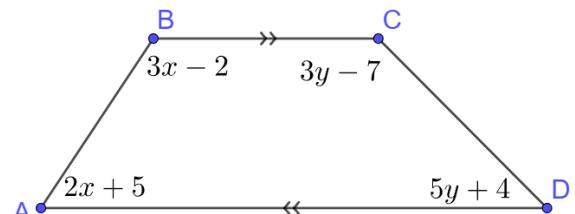
Determine the value of $x + y$ given that:

$$\angle CBA = 3x - 2$$

$$\angle BCD = 3y - 7$$

$$\angle BAD = 2x + 5$$

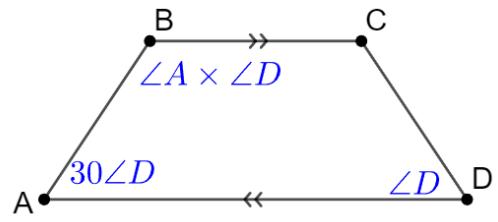
$$\angle CDA = 5y + 4$$



$$3x - 2 + 2x + 5 = 180 \Rightarrow 5x + 3 = 180 \Rightarrow x = \frac{177}{5}$$

$$3y - 7 + 5y + 4 = 180 \Rightarrow 8y - 3 = 180 \Rightarrow y = \frac{183}{8}$$

$$x + y = \frac{177}{5} + \frac{183}{8}$$



Example 2.21

Trapezoid $ABCD$ is shown in the diagram. Determine the ratio of $\angle C$ to $\angle D$ given that $\angle A$ is thirty times $\angle D$, and that $\angle B$ is $\angle A$ times $\angle D$.

Let $\angle D = x$

$$30x^2 + 30x = 180$$

$$x^2 + x = 6$$

$$x^2 + x - 6 = 0$$

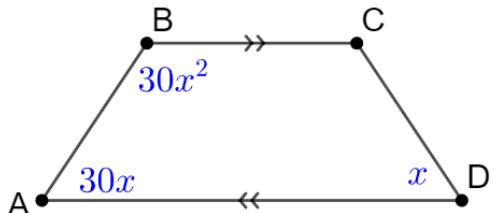
$$(x+3)(x-2) = 0$$

$$x \in \{2, -3\}$$

$$\angle D = 2$$

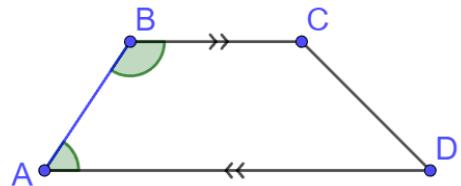
$$\angle C = 180 - 2 = 178$$

$$\angle C : \angle D = 178 : 2 = 89 : 1$$



2.22: Converse: Proving a Trapezoid

In quadrilateral $ABCD$, if $\angle A$ and $\angle B$ are supplementary, then the quadrilateral is a trapezoid.



Since the angles are supplementary:

$$\angle A + \angle B = 180$$

But, $\angle A$ and $\angle B$ are co-interior angles considering AB as the transversal of lines BC and AD.

$$\therefore AB \parallel CD$$

Hence,

Quadrilateral ABCD is a trapezoid

B. Perimeter

2.23: Perimeter

The sum of the lengths of the sides of a trapezoid is the perimeter.

Example 2.24

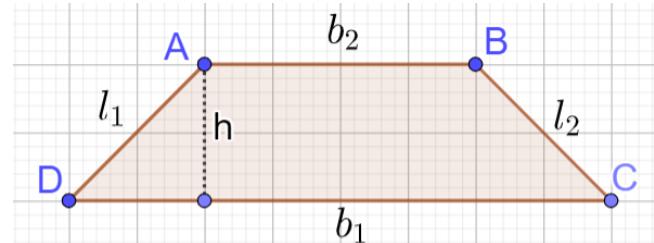
What is the perimeter of a trapezoid with bases 4 cm and 6 cm, and legs each 2 cm.

$$4 + 6 + 2 + 2 = 14$$

Example 2.25

Find the perimeter of the isosceles trapezoid drawn alongside (not to scale)

$$AB = \frac{1}{3}, \quad CD = \frac{1}{2}, \quad AD = \frac{1}{6}$$



$$AB + BC + CD + AD = \frac{1}{3} + \frac{1}{6} + \frac{1}{2} + \frac{1}{6} = \frac{2}{6} + \frac{1}{6} + \frac{3}{6} + \frac{1}{6} = \frac{7}{6} = 1\frac{1}{6}$$

C. Area

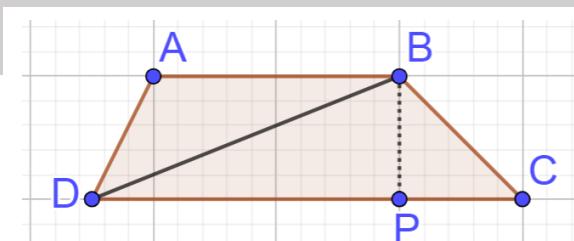
2.26: Area

The area of a trapezoid is given by:

$$\text{Area} = \text{Height} \times \frac{\text{Base}_1 + \text{Base}_2}{2}$$

Consider Trapezoid ABCD (drawn alongside). Draw

- Height BP
- Diagonal DB



The trapezoid is made of $\triangle ABD$ and $\triangle BDC$:

$$[ABCD] = [BDC] + [ABD] = \frac{1}{2}(BP)(CD) + \frac{1}{2}(BP)(AB) = \frac{1}{2} \underbrace{(BP)}_{\text{Height}} (CD + AB)$$

Example 2.27

Find the area of a trapezoid with height 3 cm, and bases 5 cm and 6 cm?

$$\frac{b_1 + b_2}{2} \times h = \frac{5 + 6}{2} \times 3 = \frac{11}{2} \times 3 = \frac{33}{2} = 16.5 \text{ cm}^2$$

Example 2.28

Find the area of a trapezoid with height 0.12 m, and bases 0.3 cm and 0.16 cm

$$\begin{aligned} b_1 &= 0.3 = 0.333333 \dots = \frac{1}{3} \text{ cm} \\ b_2 &= 0.16 = 0.16666 \dots = \frac{1}{6} \text{ cm} \\ h &= 0.12 \text{ m} = 12 \text{ cm} \\ \text{Area} &= \frac{b_1 + b_2}{2} \times h = \frac{\frac{1}{3} + \frac{1}{6}}{2} \times 12 = \frac{3}{6} \times \frac{1}{2} \times 12 = 3 \text{ cm}^2 \end{aligned}$$

Example 2.29

Find the area of a trapezoid with height one-third of a foot, one base five-sixth of a foot, and the other base having $33\frac{1}{3}\%$ greater length than the first base.

$$\begin{aligned} h &= \frac{1}{3} \text{ feet} = \frac{1}{3} \times 12 = 4 \text{ inches} \\ b_1 &= \frac{5}{6} \text{ feet} = \frac{5}{6} \times 12 = 10 \text{ inches} \end{aligned}$$

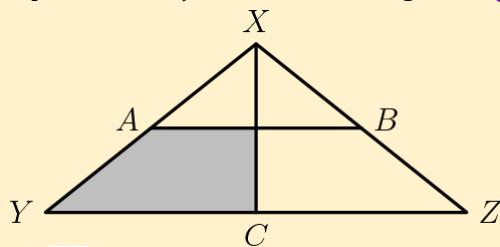
$$33\frac{1}{3}\% = \frac{100}{3}\% = \frac{100}{300} = \frac{1}{3} \text{ greater} \Rightarrow b_2 = b_1 + \frac{1}{3} \times b_1 = b_1 \left(1 + \frac{1}{3}\right) = b_1 \times \frac{4}{3} = 10 \times \frac{4}{3} = \frac{40}{3}$$

$$\text{Area} = \frac{b_1 + b_2}{2} \times h = \frac{10 + \frac{40}{3}}{2} \times 4 = \frac{70}{3} \times 2 = \frac{140}{3} \text{ inch}^2$$

D. Ratios

Example 2.30

The area of triangle XYZ is 8 square inches. Points A and B are midpoints of congruent segments \overline{XY} and \overline{XZ} . Altitude \overline{XC} bisects \overline{YZ} . The area (in square inches) of the shaded region is (AMC 8 2002/20)



Example 2.31

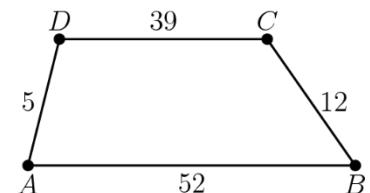
Trapezium ABCD has $AB \parallel CD$ and area 18.

- A. AB is half the length of CD. What is the ratio of the area of $\triangle ABD$ to that of the trapezium? Is it possible to answer this question without calculating the area of $\triangle ABD$?
- B. AB has length 6, and CD has length 8. What is $\text{Area}(\triangle ABD) : \text{Area}(\triangle BCD)$?
- C. If AB has length p , and CD has length q , can you find $\text{Area}(\triangle ABD) : \text{Area}(\triangle BCD)$? Does your final answer increase or decrease if the area of the trapezium changes.

E. Rearrangement

Example 2.32

In trapezoid ABCD with bases AB and CD , we have $AB = 52$, $BC = 12$, $CD = 39$, and $DA = 5$ (diagram not to scale). The area of ABCD is (AMC 10A 2002/25)



Example 2.33

In rectangle ABCD, $AB = 6$, $AD = 30$, and G is the midpoint of \overline{AD} . Segment AB is extended 2 units beyond B to point E, and F is the intersection of \overline{ED} and \overline{BC} . What is the area of $BFDG$? (AMC 10B 2012/19)

$$GD \parallel BF \Rightarrow BGDF \text{ is a trapezoid} \Rightarrow b_1 = GD = 15, h = 6$$

We need to find the other base, which is BF .

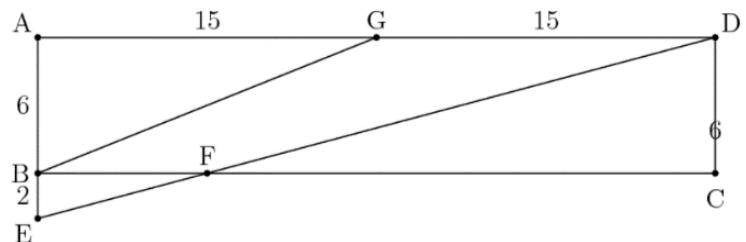
We can prove two triangles similar by AA Similarity:

$$\angle BAD = \angle EBF = 90^\circ, \angle AED = \angle BEF \Rightarrow \Delta BEF \sim \Delta AED$$

W

$$\frac{BE}{AE} = \frac{2}{8} = \frac{1}{4} \Rightarrow \frac{BF}{AD} = \frac{1}{4} \Rightarrow BF = \frac{30}{4} = \frac{15}{2}$$

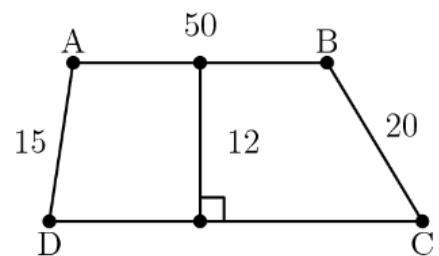
$$[BDFG] = \frac{15 + \frac{15}{2}}{2} \times 6 = \frac{45}{2} \times 3 = \frac{135}{2}$$



F. Pythagorean Theorem

Example 2.34

Quadrilateral ABCD is a trapezoid, $AD = 15$, $AB = 50$, $BC = 20$, and the altitude is 12. What is the area of the trapezoid? (AMC 8 2011/20)



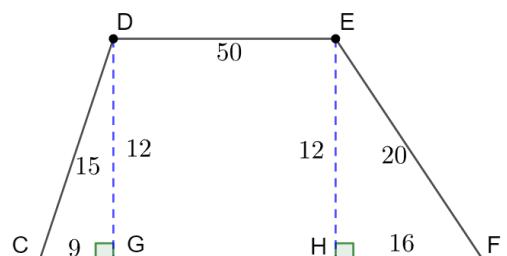
Draw $DG \perp CF$ and $EH \perp CF$. Since the distance between parallel lines is the same,

$$DG = EH = 12$$

By Pythagorean Triplets:

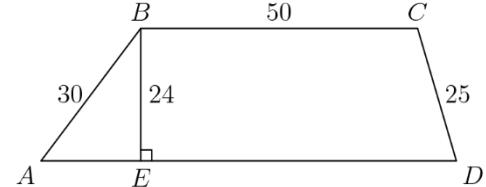
$$3(3,4,5) = (9,12,15) = (CG, DG, DC)$$

$$3(3,4,5) = (12,16,20) = (EH, HF, EF)$$



Then, the area of the trapezoid is:

$$\begin{aligned}
 & [DCG] + [EHF] + [DEHG] \\
 &= \frac{1}{2}(12)(9) + \frac{1}{2}(12)(16) + (50)(12) \\
 &= 6(9) + 6(16) + (100)(6) \\
 &= 6(25) + 100 \\
 &= 750
 \end{aligned}$$



Example 2.35

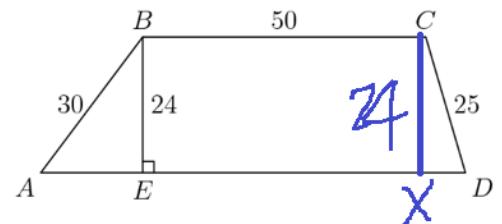
What is the perimeter of trapezoid ABCD? (AMC 8 2005/19)

By Pythagorean Triplet:

$$6(3,4,5) = (18,24,30) = (AE, BE, AB) \Rightarrow AE = 18$$

By Pythagorean Triplet:

$$(7,24,25) = (XD, CX, CD) \Rightarrow XD = 7$$



The perimeter

$$= AB + BC + CD + DA = 30 + 50 + 25 + (7 + 50 + 18) = 180$$

Example 2.36

In trapezoid ABCD, \overline{AB} and \overline{CD} are perpendicular to \overline{AD} , with $AB + CD = BC$, $AB < CD$, and $AD = 7$. What is $AB \cdot CD$? (AMC 10 2001/24)

Let

$$AB = x, CD = y$$

Then

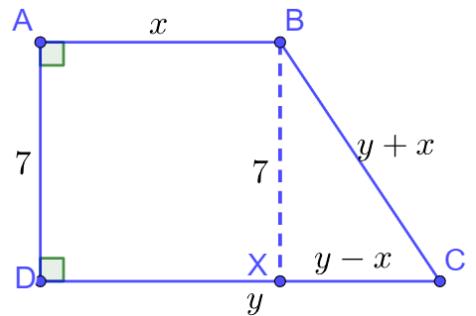
$$BC = AB + CD = x + y, \quad AB \cdot CD = xy$$

Construct

$$BX \perp CD \Rightarrow CX = y - x$$

By the Pythagorean Theorem in $\triangle BXC$:

$$\begin{aligned}
 CX^2 + BX^2 &= BC^2 \\
 (y - x)^2 + 7^2 &= (x + y)^2
 \end{aligned}$$



Now, we have an equation, which we can simplify:

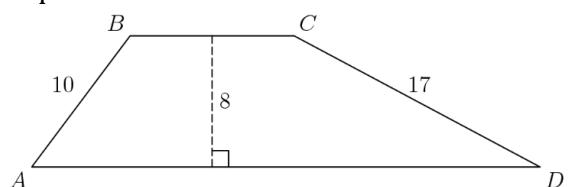
$$y^2 - 2xy + x^2 + 49 = x^2 + 2xy + y^2$$

Simplify and collate like terms on each side:

$$4xy = 49 \Rightarrow xy = \frac{49}{4} = 12\frac{1}{4}$$

Example 2.37

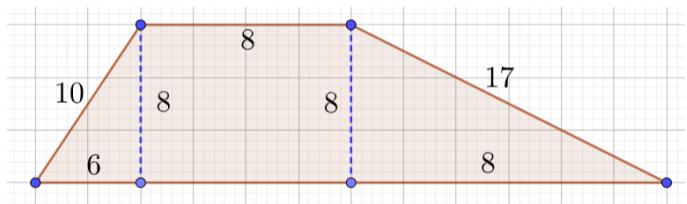
The area of trapezoid ABCD is 164 cm^2 . The altitude is 8 cm, AB is 10 cm, and CD is 17 cm. What is BC, in centimeters? (AMC 8 2003/21)



$$8 \left(\frac{BC + AD}{2} \right) = 164 \Rightarrow BC + AD = 41$$

$$BC + BC + 6 + 15 = 41$$

$$BC = 10 \text{ cm}$$

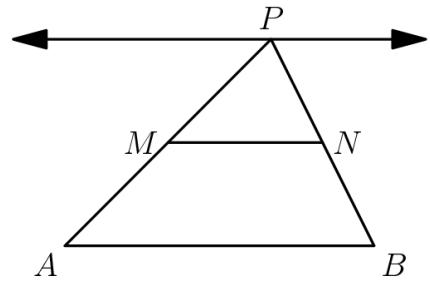


G. General

Example 2.38

Points M and N are the midpoints of sides PA and PB of $\triangle PAB$. As P moves along a line that is parallel to side AB , how many of the four quantities listed below change? (AMC 10 2000/5)

- (a) the length of the segment MN
- (b) the perimeter of $\triangle PAB$
- (c) the area of $\triangle PAB$
- (d) the area of trapezoid $ABNM$



Analysis

Part A

Obviously, AB does not change in length.

Therefore, by the Midpoint Theorem, MN also does not change in length.

Part B

Consider an extreme scenario. P moves far to the left (or far to the right). The perimeter will change.

Part C

$$A(\triangle PAB) = \frac{1}{2}hb$$

Neither the height, nor the base changes, so the area does not change.

Part D

$$A(\text{Trapezoid } ABNM) = \frac{AB + MN}{2} \times h$$

None of the above quantities change, so the area does not change.

Final Answer

Out of the four given quantities, only one changes.

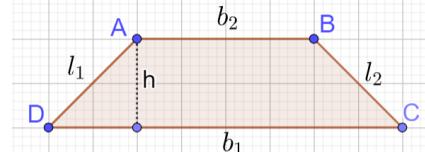
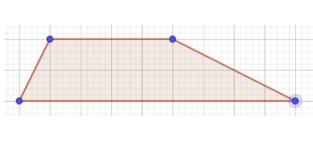
H. Isosceles Trapezoid

2.39: Isosceles Trapezoid

When the legs are equal in length, it is an isosceles trapezoid.

Example 2.40

Decide whether each trapezoid drawn alongside is isosceles or not.



The one on the left is not Isosceles.

The one on the right is Isosceles.

Example 2.41

A rectangular yard contains two flower beds in the shape of congruent isosceles right triangles. The remainder of the yard has a trapezoidal shape, as shown. The parallel sides of the trapezoid have lengths 15 and 25 meters. What fraction of the yard is occupied by the flower beds? (AMC 10B 2009/4)



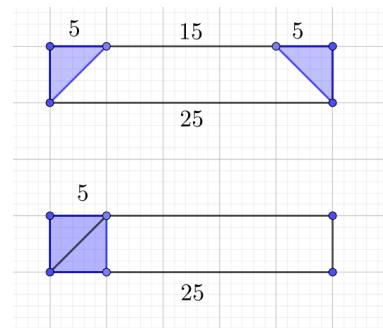
Draw a diagram of the rectangle.

The sides of the triangle are

$$\frac{25 - 15}{2} = \frac{10}{2} = 5$$

The right triangle can be moved to go to the left. The fraction of the yard occupied by the flower beds is

$$\frac{5}{25} = \frac{1}{5}$$



2.42: Isosceles Trapezoid

- The base angles of an isosceles trapezium are equal
- The “top angles” are also equal.

Base Angles

Draw the altitudes from B and C .

$$\angle BXA = \angle CYD = 90^\circ \text{ (Right Angle)}$$

The distance between parallel lines is the same:

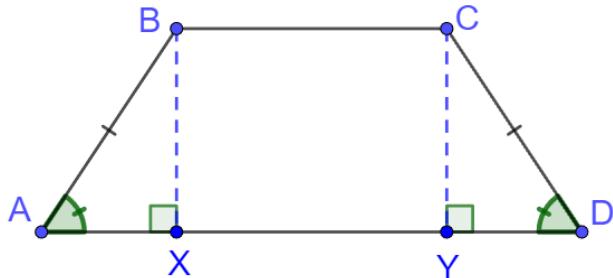
$$BX = CY \text{ (Side)}$$

Since the trapezium is isosceles:

$$BA = CD \text{ (Hyp)}$$

Combining the above three statements:

$$\begin{aligned} \Delta BXA &= \Delta CYD \text{ (RHS)} \\ \angle BAD &= \angle CDA \text{ (CPCTC)} \end{aligned}$$



Top Angles

$$\begin{aligned} \angle ABX &= \angle DCY \text{ (CPCTC)} \\ \angle ABX + 90^\circ &= \angle DCY + 90^\circ \\ \angle ABC &= \angle DCB \end{aligned}$$

2.43: Converse: Proving an Isosceles Trapezium

If the base angles of a trapezoid are congruent, the trapezoid is isosceles.

In trapezoid ABCD draw height BX and CY.

In ΔBAX and ΔCDY

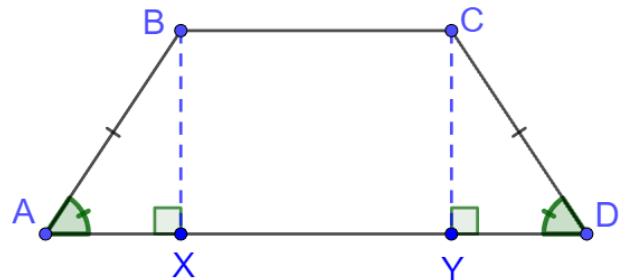
$$\angle BAX = \angle CDY \text{ (Given)}$$

$$BX = CY \text{ (Height is equal)}$$

$$\angle BXA = \angle CYD = 90^\circ \text{ (Def of Height)}$$

$$\Delta BAX \cong \Delta CDY \text{ (SAA)}$$

$$BA = CD \text{ (CPCTC)}$$



2.44: Diagonals of an Isosceles Trapezium

The diagonals of an isosceles trapezium are congruent.

In ΔABC and ΔDCB :

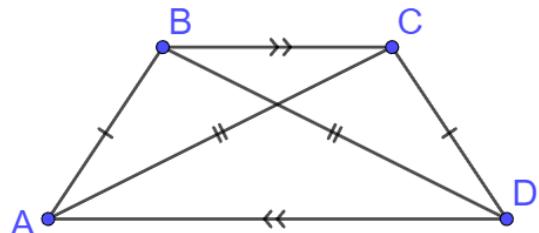
$$AB = CD \text{ (Legs of isosceles trapezium)}$$

$$BC = BC \text{ (Reflexive Property)}$$

$$\angle ABC = \angle BCD \text{ (Top angles of isosceles trapezium)}$$

$$\Delta ABC \cong \Delta DCB$$

$$BD \cong AC \text{ (CPCTC)}$$



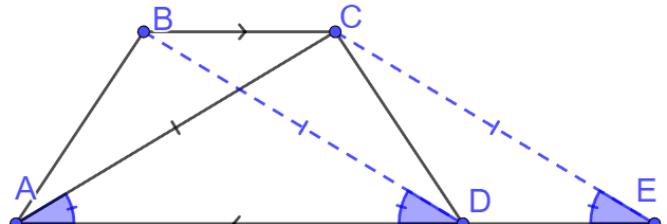
2.45: Converse

If the diagonals of a trapezoid are congruent, it is an isosceles trapezoid.

Consider trapezoid ABCD with $BC \parallel AD$. Construct $CE \parallel BD$, intersecting AD at E .

Consider BD and CE as transversals of BC and AD . Since they are corresponding angles:

$$\angle BDA = \angle CED$$



BD and CE are transversals to parallel lines BC and AD .

Hence, $BD = CE$. Since the diagonals are congruent, $AC = BD$. Combining the two statements:

$$AC = CE \Rightarrow \Delta ACE \text{ is isosceles} \Rightarrow \angle CAD = \angle CEA$$

In ΔCAD and ΔBAD :

$$\begin{aligned} AC &= AC \text{ (Side)} \\ \angle CAD &= \angle CEA \text{ (Angle)} \\ CA &= BD \text{ (Angle) (Given)} \\ \Delta CAD &\cong \Delta BAD \text{ (SAS Congruence)} \\ AB &\cong CD \text{ (CPCTC)} \end{aligned}$$

Example 2.46

What is the perimeter of an isosceles trapezium with height 12, shorter base 23, and longer base 32?

Longer side will have its extra length equally divided on both sides (isosceles trapezium)

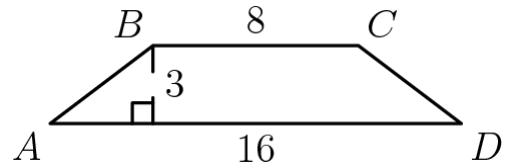
Left-hand side triangle is right-angled. By Pythagoras Theorem:

$$\text{Left Leg} = \text{Right Leg} = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$$

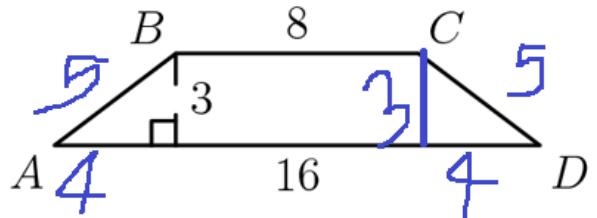
$$\text{Perimeter} = 23 + 32 + 13 \cdot 2 = 81$$

Example 2.47

In trapezoid $ABCD$, the sides AB and CD are equal. The perimeter of $ABCD$ is (AMC 8 1999/14)



$$8 + 2(5) + 16 = 34 \text{ units}$$



I. Trapezoid Midsegment Theorem

2.48: Trapezoid Midsegment Theorem

The line segment joining the midpoints of two legs of a trapezoid is the average of the bases, and parallel to the two bases.

In ΔABY and ΔZCY :

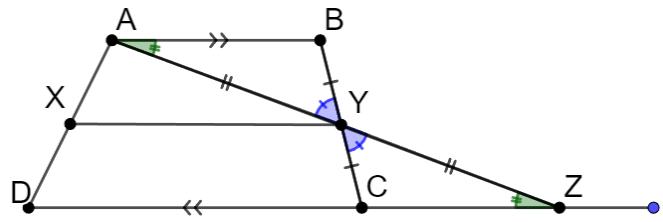
$$BY \cong CY \text{ (midpoint)}$$

$$\angle BAC = \angle YZV \text{ (alternate interior angles)}$$

$$\angle BYC = \angle CYZ \text{ (vertically opposite angles)}$$

$$\Delta ABY \cong \Delta ZCY \text{ (SAA Congruence)}$$

$$AY \cong ZY \text{ (CPCTC)}$$



Since Y is the midpoint of AZ and X is the midpoint of AD , by triangle midpoint theorem:

$$XY = \frac{1}{2}(DZ) = \frac{1}{2}(DC + CZ)$$

Since $AB \cong ZC$ (CPCTC)

$$= \frac{1}{2}(DC + AB)$$

Hence, XY is the average of the two bases.

Since Y is the midpoint of AZ and X is the midpoint of AD , by triangle midpoint theorem:

$$XY \parallel DZ \Rightarrow XY \parallel DC$$

Example 2.49

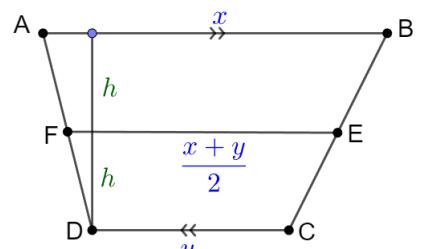
In trapezoid $ABCD$ we have \overline{AB} parallel to \overline{DC} , E as the midpoint of \overline{BC} , and F as the midpoint of \overline{DA} . The area of $ABEF$ is twice the area of $FECD$. What is $\frac{AB}{DC}$? (AMC 10B 2005/23)

$$[ABEF] = 2[FECD]$$

Using Area of parallelogram = $\frac{1}{2}h(b_1 + b_2)$

$$\frac{1}{2}h \left[x + \left(\frac{x+y}{2} \right) \right] = 2 \left[\frac{1}{2}h \left[\left(\frac{x+y}{2} \right) + y \right] \right]$$

$$x + \left(\frac{x+y}{2} \right) = 2 \left[\left(\frac{x+y}{2} \right) + y \right]$$



$$\begin{aligned} \frac{3x + y}{2} &= 2 \left[\frac{x + 3y}{2} \right] \\ 3x + y &= 2x + 6y \\ x &= 5y \\ AB &= 5DC \\ \frac{AB}{DC} &= 5 \end{aligned}$$

2.3 Parallelograms

A. Basics

2.50: Definition

A parallelogram is a quadrilateral with opposite sides parallel.

Any quadrilateral with opposite sides parallel is, by definition, a parallelogram.

2.51: Triangles formed by Diagonals

A diagonal of a parallelogram divides it into two congruent triangles.

In parallelogram $ABCD$

$AD \parallel BC, AB \parallel CD$ (Definition of Parallelogram)

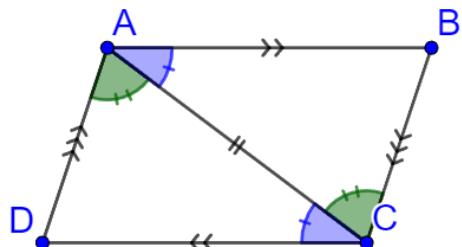
In Triangle ABC and Triangle ADC :

$AD \parallel BC \Rightarrow AC$ is transversal \Rightarrow By Alternate Interior Angles: $\angle DAC = \angle ACB$

$AB \parallel CD \Rightarrow AC$ is transversal \Rightarrow By Alternate Interior Angles: $\angle ACD = \angle BAC$

$AC = AC$ (Reflexive Property)

$\Delta ABC \cong \Delta ADC$ (ASA Congruence)



2.52: Opposite Sides

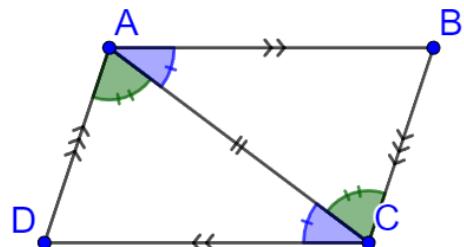
Opposite sides of a parallelogram are congruent.

As above,

$$\Delta ABC \cong \Delta ADC$$

By CPCTC:

$$AB \cong CD$$



2.53: Converse: Opposite Sides

If the opposite sides of a quadrilateral are congruent, the quadrilateral is a parallelogram.

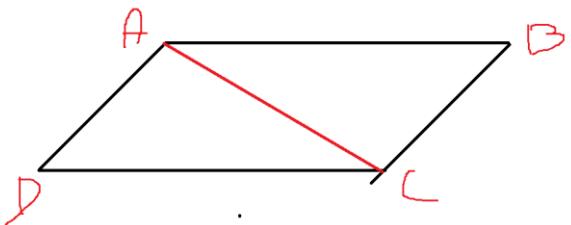
Draw parallelogram $ABCD$.
 Construct Diagonal AC .

In ΔADC and ΔABC :

$$AD = BC \text{ (Given)}$$

$$AB = CD \text{ (Given)}$$

$$AC = AC \text{ (Reflexive Property)}$$



By SSS:

$$\Delta ADC \cong \Delta ABC$$

$$\angle DCA = \angle BAC \quad (\text{---})$$

Consider AC as transversal of AD and CB :

$\angle DAC$ and $\angle ABC$ are co-interior

$\angle DAC = \angle ABC$ (CPCT in $\Delta ADC \cong \Delta ABC$)

$$\therefore AD \parallel CB$$

Consider AC as transversal of AB and CD :

$\angle DCA$ and $\angle BAC$ are co-interior

$\angle DCA = \angle BAC$ (CPCT in $\Delta ADC \cong \Delta ABC$)

$$\therefore AB \parallel CD$$

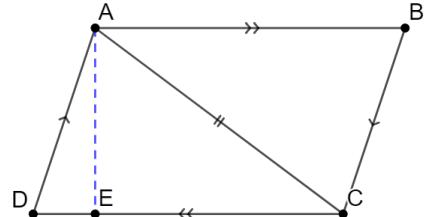
In Quadrilateral ABCD:

$$AD \parallel CB, AB \parallel CD \Rightarrow \text{Opp. Sides are parallel} \Rightarrow \text{Parallelogram}$$

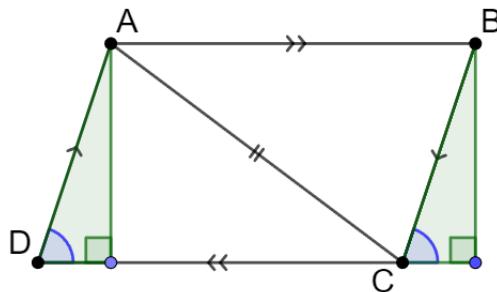
2.54: Area of a Parallelogram

Area of parallelogram

$$= 2(A\Delta ADC) = 2 \left(\frac{1}{2}hb \right) = (AE)(DC)$$



2.55: Area of a Parallelogram



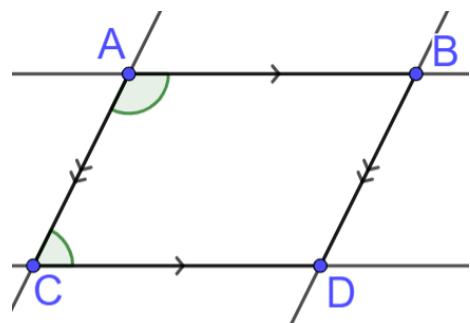
2.56: Adjacent Angles

Adjacent Angles of a parallelogram are supplementary.

In the parallelogram, $AB \parallel CD$:

$$\begin{aligned} &\Rightarrow \angle CD \text{ and } \angle A \text{ are co-interior} \\ &\Rightarrow \angle D \text{ and } \angle A \text{ are supplementary} \end{aligned}$$

$$\begin{aligned} AD \parallel BC \Rightarrow \angle B \text{ and } \angle A \text{ are co-interior} \\ \Rightarrow \angle B \text{ and } \angle A \text{ are supplementary} \end{aligned}$$



2.57: Opposite Angles

Opposite Angles of a parallelogram are congruent

Draw parallelogram $ABCD$.

Since $\angle D$ and $\angle A$ are supplementary

$$\angle A = 180 - \angle D \Rightarrow$$

$\angle B$ and $\angle A$ are supplementary:

$$\angle B = 180 - (180 - \angle D) = \angle D$$

2.58: Diagonals

The diagonals of a parallelogram bisect each other.

Draw parallelogram $ABCD$, and construct diagonals AC and BD , intersecting at X .

To Prove that:

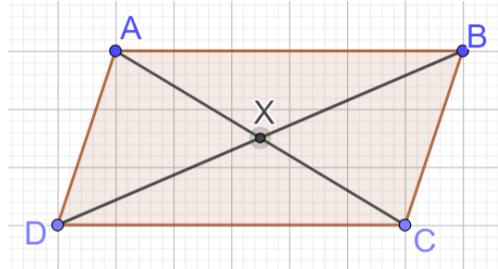
$$AX = XC, DX = XB$$

In $\triangle AXD$ and $\triangle BXC$:

$$AD = BC \text{ (Opp sides of a parallelogram are congruent)}$$

$$\angle DAC = \angle BCA$$

$$\angle ADB = \angle DBC$$



By ASA:

$$\triangle AXD \cong \triangle BXC$$

By CPCTC:

$$XD = XB, XA = XC$$

If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.

B. Perimeter

Example 2.59

Find the perimeter of a parallelogram whose adjacent sides have a total length of 12 cm, and one pair of whose opposite sides have a length of 10 cm. Also, what is the length of each side?

$$\text{Perimeter} = 2 * \text{Adjacent Sides} = 2 * 12 = 24 \text{ cm}$$

$$\text{Pair of opposite sides} = 10 \text{ cm}$$

One of them = 5 cm

Adjacent Sides are 5 cm and $12 - 5 = 7$ cm

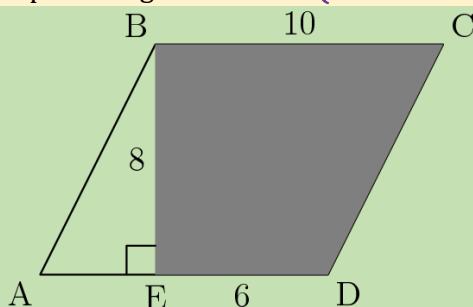
Sides are:

5 cm, 7cm, 5 cm, and 7 cm

C. Area

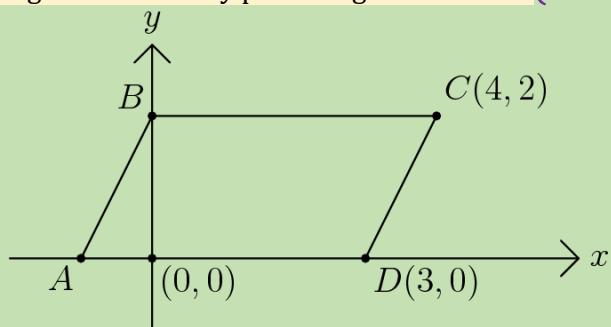
Example 2.60

The area of the shaded region BEDC in parallelogram ABCD is (AMC 8 1989/15)



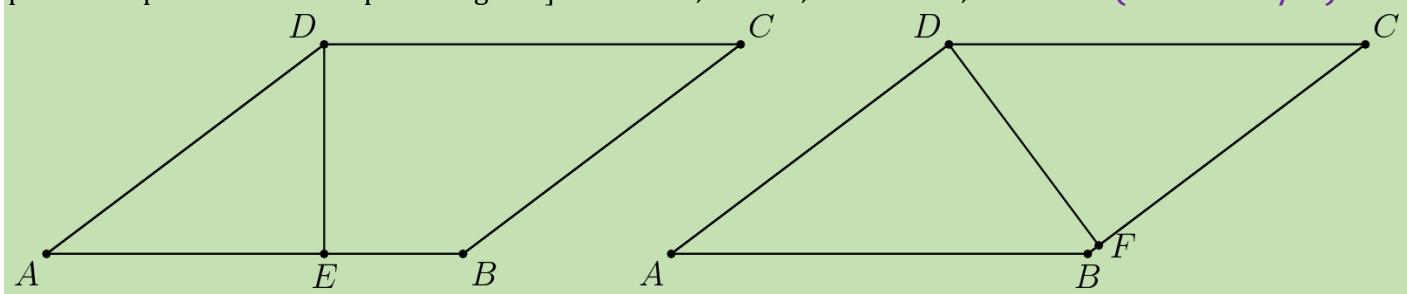
Example 2.61

The area in square units of the region enclosed by parallelogram ABCD is (AMC 8 1991/10)



Example 2.62

In parallelogram $ABCD$, \overline{DE} is the altitude to the base \overline{AB} and \overline{DF} is the altitude to the base \overline{BC} . [Note: Both pictures represent the same parallelogram.] If $DC = 12$, $EB = 4$, and $DE = 6$, then $DF =$ (AMC 8 1995/24)



Example 2.63

A street has parallel curbs 40 feet apart. A crosswalk bounded by two parallel stripes crosses the street at an angle. The length of the curb between the stripes is 15 feet and each stripe is 50 feet long. Find the distance, in feet, between the stripes? (AMC 10 2001/15)

Example 2.64

Quadrilateral $ABCD$ has $AB = BC = CD$, angle $ABC = 70$ and angle $BCD = 170$. What is the measure of angle BAD ? (AMC 10B 2008/24)

Draw

- Diagonal DB
- Line DE congruent to DC and parallel to AB to create parallelogram $ABED$

In Isosceles ΔDCB

$$\angle CDB = \angle CBD = \frac{180 - \angle DCB}{2} = \frac{180 - 170}{2} = 5$$

Then:

$$\angle ABD = \angle ABC - \angle CBD = 70 - 5 = 65$$

In parallelogram $ABED$:

$$\begin{aligned}\angle BDE &= 65 \text{ (Alternate Interior Angles)} \\ \angle EDC &= \angle BDE - \angle BDC = 65 - 5 = 60\end{aligned}$$

In Isosceles ΔDCB

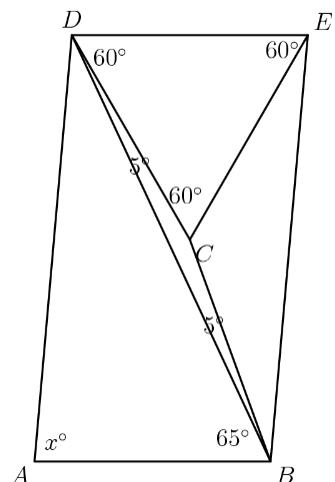
$$\angle EDC = 60 \Rightarrow \angle DEC = \angle DCE = 60^\circ \Rightarrow \Delta ECD \text{ is equilateral.}$$

In parallelogram $ABED$, using parallelogram properties:

$$\begin{aligned}\angle BAD &= \angle DEB = x \\ \angle CEB &= x - 60 \\ \angle ABE &= 180 - x \\ \angle CBE &= \angle ABE - \angle ABC = 110 - x\end{aligned}$$

Since ΔECB is isosceles:

$$x - 60 = 110 - x \Rightarrow x = 85$$



2.4 Rhombi

A. Diagonals

2.65: Diagonals of a Rhombus

The diagonals of a rhombus are perpendicular to each other.

2.66: Diagonals of a Rhombus

The diagonals of a rhombus divide it into four congruent triangles.

2.67: Diagonals of a Rhombus

The diagonals of a rhombus bisect the vertex angles.

B. Proving a Rhombus

2.68: Proving a Rhombus

If the diagonals of a parallelogram are perpendicular to each other, the parallelogram is a rhombus.

If the diagonal of a parallelogram bisects a vertex angle, the parallelogram is a rhombus.

C. Midpoints

2.69: Midpoints form a Rectangle

The quadrilateral formed by joining the midpoints of the sides of a rhombus in order form a rectangle. And this quadrilateral has area half of the rhombus.

D. Perimeter

Example 2.70

What is the perimeter of a rhombus with

- A. one side one foot, a second side 12 inches, and a third side 30.48 cm.
- B. one of whose sides is $\frac{1}{2 + \frac{3}{2 + \frac{3}{4}}}$ units long?

Part A

Rhombuses have all sides equal

$$1 \text{ foot} = 12 \text{ inches} = (12 * 2.54) \text{ cm} = 30.48 \text{ cm}$$

$$P = 4 * 1 \text{ foot} = 4 \text{ feet}$$

Part B

$$4 \times \frac{1}{2 + \frac{2}{3 + \frac{3}{4}}} = \frac{4}{2 + 2 \times \frac{4}{15}} = \frac{4}{\frac{38}{15}} = \frac{30}{19}$$

E. Area

Example 2.71

What is the area of a rhombus with diagonals 4 cm?

Example 2.72

A rhombus has an area of 108 square units. The lengths of its diagonals have a ratio of 3 to 2. What is the length of the longest diagonal, in units? (MathCounts 2007 State Countdown)

Write the given ratio, and use it to form a relation between the diagonals:

$$\begin{array}{c} 3 \\ \text{Longer} \\ \text{Diagonal} \\ =l \end{array} : \begin{array}{c} 2 \\ \text{Shorter} \\ \text{Diagonal} \\ =s \end{array} \Rightarrow 2l = 3s \Rightarrow l = \frac{3s}{2}$$

Create and simplify an expression for the area of the rhombus using the property that the area of a rhombus is half the product of its diagonals:

$$A = \frac{l \times s}{2} = \frac{\frac{3s}{2} \times s}{2} = \frac{3s^2}{4}$$

Equate this expression to the given value of the area, and solve the resulting equation:

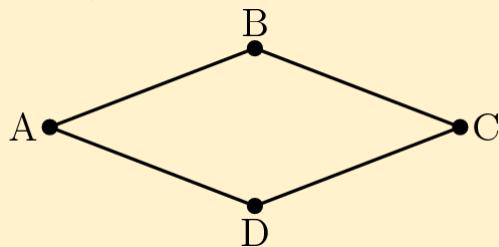
$$\frac{3s^2}{4} = 108 \Rightarrow s^2 = 108 \times \frac{4}{3} = 144 \Rightarrow s = 12$$

Substitute the value of s to find the value of l :

$$l = \frac{3s}{2} = 18$$

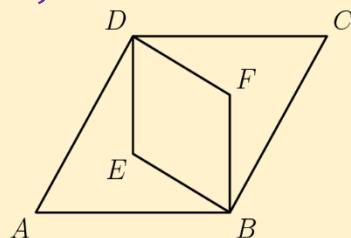
Example 2.73

Quadrilateral $ABCD$ is a rhombus with perimeter 52 meters. The length of diagonal \overline{AC} is 24 meters. What is the area in square meters of rhombus $ABCD$? (AMC 8 2019/4)



Example 2.74

Rhombus $ABCD$ is similar to rhombus $BFDE$. The area of rhombus $ABCD$ is 24 and $\angle BAD = 60^\circ$. What is the area of rhombus $BFDE$? (AMC 10B 2006/15)



2.5 Kites

Example 2.75

Find the perimeter of a kite:

1. with shorter side 0.25 m and longer side 20% longer than the shorter side
2. where the longer side is 20% more than the shorter side, and the shorter side is $\frac{3}{2 - \frac{4}{1 + \frac{3}{4}}}$

Part A

$$P = 2 \times \underbrace{\frac{1}{4}}_{\text{Shorter Side}} + 2 \times \underbrace{\frac{1}{4} \times \frac{5}{4}}_{\text{Longer Side}} = 2 \times \frac{1}{4} \left(1 + \frac{5}{4}\right) = \frac{1}{2} \times \frac{9}{4} = \frac{9}{8}$$

Part B

$$\text{Shorter Side} = s = \frac{3}{2 - \frac{4}{1 + \frac{3}{4}}} = \frac{3}{2 - 4 \times \frac{3}{7}} = 3 \times \frac{7}{2} = \frac{21}{2}$$

$$P = 2(s + l) = 2 \left(s + \frac{6s}{5} \right) = \frac{22s}{5} = \frac{22}{5} \times \frac{21}{2} = 46.2$$

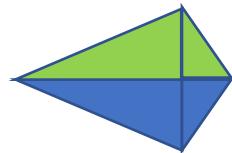
20% more

2.1: Area of a Kite

Consider kite ABCD with $\ell(AC) = p$, and $\ell(BD) = q$

The diagonals of a kite intersect at right angles, with one diagonal bisecting the other:

$$\text{Area} = A(\text{Upper } \triangle) + A(\text{Lower } \triangle) = \left(\frac{1}{2}\right)(p)\left(\frac{q}{2}\right) + \left(\frac{1}{2}\right)(p)\left(\frac{q}{2}\right) = \frac{pq}{2}$$

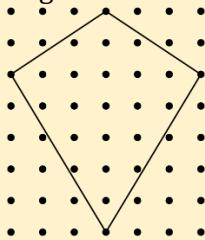


Area of Kite = $\frac{1}{2} * \text{Product of Diagonals}$

Kites on Parade

Problems 7, 8 and 9 are about these kites.

To promote her school's annual Kite Olympics, Genevieve makes a small kite and a large kite for a bulletin board display. The kites look like the one in the diagram below. For her small kite Genevieve draws the kite on a one-inch grid. For the large kite she triples both the height and width of the entire grid. [8]



Example 2.76

What is the number of square inches in the area of the small kite? (AMC 8 2001/7)

Example 2.77

Genevieve puts bracing on her large kite in the form of a cross connecting opposite corners of the kite. How many inches of bracing material does she need? (AMC 8 2001/8)

Example 2.78

The large kite is covered with gold foil. The foil is cut from a rectangular piece that just covers the entire grid. How many square inches of waste material are cut off from the four corners? (AMC 8 2001/9)

2.6 General Quadrilaterals

A. Angles in a Quadrilateral

2.79: Sum of Angles

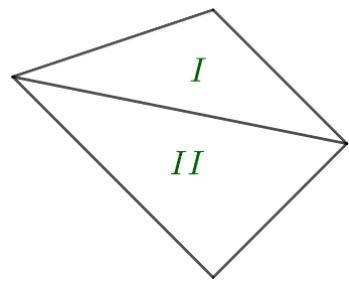
The sum of angles of a quadrilateral is

$$360^\circ$$

A quadrilateral can be divided into two triangles which each have sum 180° .

Hence, the total of the angles of a quadrilateral is

$$180 \times 2 = 360^\circ$$



Example 2.80

The angles of a quadrilateral are in the ratio 1:2:3:4. Determine the values of the angles.

$$\begin{aligned}1 + 2 + 3 + 4 &= 10 \\ \frac{360}{10} &= 36 \\ 36 \times 1 &= 36 \\ 36 \times 2 &= 72 \\ 36 \times 3 &= 108 \\ 36 \times 4 &= 144\end{aligned}$$

Example 2.81

The largest angle in a quadrilateral is 30° more than the smallest angle. The other angles each are right angles. Determine the values of the angles.

Suppose the smallest angle is x . Then the largest is $x + 30$

$$\begin{aligned}x + 90 + 90 + x + 30 &= 360 \\ 2x &= 360 - 210 \\ 2x &= 150 \\ x &= \frac{150}{2} = 75 \\ x + 30 &= 105\end{aligned}$$

$$\{75, 90, 90, 105\}$$

B. Perimeter

Example 2.82

- A. What is the perimeter (in feet) of a regular pentagon with side 6 inches?
- B. What is the perimeter (in inches) of a regular pentagon with side 6 feet?
- C. A pentagonal field with each side five meters is to be fenced using fencing that costs five dollars per meter. Fenceposts that cost five dollars each are to be placed at a distance one meter apart. What is the total cost?

$$\text{Side} * 6 = 6 * 6 = 36 \text{ inches} = 3 \text{ feet}$$

$$\text{Side} * 5 = 6 * 5 = 30 \text{ feet} = 360 \text{ inches}$$

$$\text{Cost} = \text{Fencing} + \text{Fenceposts} = 5 * 5 * 5 + 5 * 5 * 5 = 125 + 125 = 250$$

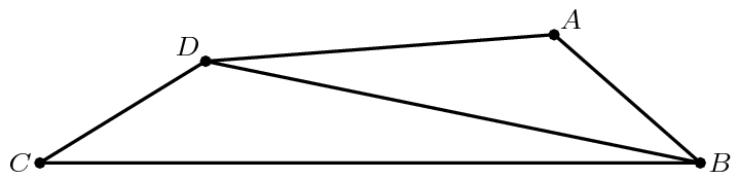
C. Triangle Inequality

2.83: Triangle Inequality

Sum of lengths of any two sides of a triangle is greater than the third side.

Example 2.84

In quadrilateral $ABCD$, $AB = 5$, $BC = 17$, $CD = 5$, $DA = 9$, and BD is an integer. What is BD ? (AMC 10A 2009/12; AMC 12A 2009/10)



In $\triangle ADB$, by the triangle inequality:

$$BD < DA + AB = 9 + 5 = 14$$

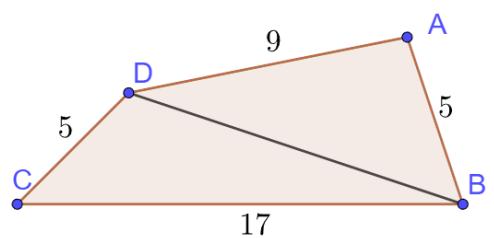
$$\underbrace{BD < 14}_{\text{Inequality I}}$$

In $\triangle CDB$, by the triangle inequality:

$$BD + CD > BC$$

$$BD + 5 > 17$$

$$\underbrace{BD > 12}_{\text{Inequality II}}$$



Combining Inequalities I and II:

$$12 < BD < 14$$

Since BD is an integer:

$$BD = 13$$

D. Quadrilateral Inequality

2.85: Quadrilateral Inequality

Sum of lengths of any three sides of a quadrilateral is greater than the fourth side

Example 2.86

Joy has 30 thin rods, one each of every integer length from 1 cm through 30 cm. She places the rods with lengths 3 cm, 7 cm, and 15 cm on a table. She then wants to choose a fourth rod that she can put with these three to form a quadrilateral with positive area. How many of the remaining rods can she choose as the fourth rod? (AMC 10A 2017/10)

Since the quadrilateral must have positive area, it cannot be a degenerate quadrilateral. That is, the rods should form a "proper quadrilateral."

Let the length of the fourth rod be r . By the quadrilateral inequality:

$$\begin{aligned} r &< 3 + 7 + 15 \\ r &< 25 \end{aligned}$$

The longest(given) side must be less than the sum of the other three sides:

$$\begin{aligned} 15 &< r + 3 + 7 \\ 15 &< r + 10 \\ r &> 5 \end{aligned}$$

$$5 < r < 25$$

$$\{6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24\} \Rightarrow 19 \text{ rods}$$

Get all the files at: <https://bit.ly/azizhandouts>
Aziz Manva (azizmanva@gmail.com)

Remove 7 and 15 since those rods have already been used:

$$19 - 2 = 17 \text{ rods}$$

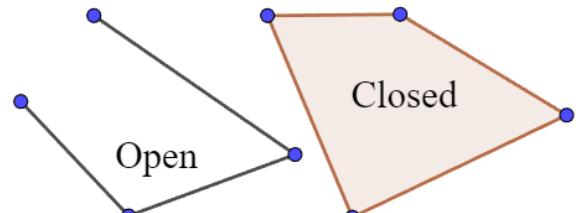
3. POLYGONS

3.1 Angles in a Polygon

A. Basics

3.1: Polygon

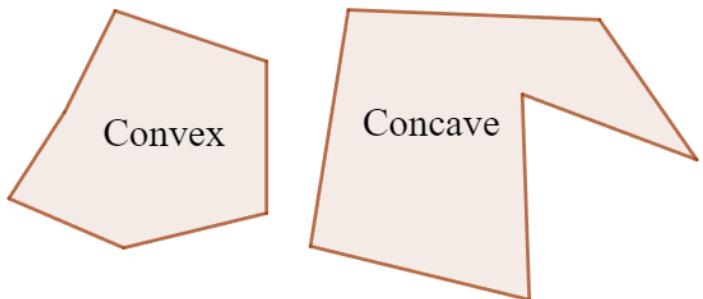
A polygon is a two-dimensional closed shape made with straight lines.



- A polygon cannot have curved lines.
- A polygon must be closed, not open.

3.2: Convex Polygon

If all vertices of a polygon point “outside” of the polygon, it is convex.



- There is great focus in geometry on studying properties of convex polygons.
- Common quadrilaterals that you know are all convex: Square, Rectangle, Parallelogram, Trapezoid, Kite, Rhombus
- The measure of any angle of a convex polygon is less than 180°

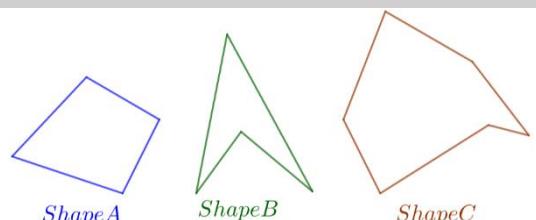
3.3: Concave Polygon

There are multiple, equivalent, definitions of a concave polygon. A polygon is concave if any one of the following is true:

- If at least one vertex of a polygon points “inside”, it is concave.
- At least one angle is greater than 180°
- At least one line connecting two points on the perimeter of the polygon that lies outside the polygon can be drawn.

Example 3.4

Decide whether the polygons in the diagram are convex or concave.



$$\begin{aligned} A &= \text{Convex} \\ B, C &\Rightarrow \text{Concave} \end{aligned}$$

3.5: Names of Polygons

Quadrilateral	4	Undecagon	11
Pentagon	5	Dodecagon	12
Hexagon	6		
Heptagon	7		
Octagon	8		
Nonagon	9		

B. Sum of Angles

3.6: Sum of Angles of a Polygon

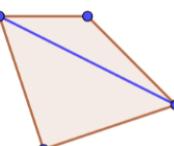
The sum of angles of a polygon with n sides is

$$(n - 2) \times 180$$

Quadrilateral: Draw a diagonal, creating two triangles.

$$\text{Sum of angles} = 180 \times 2 = 360$$

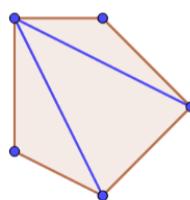
Quadrilateral



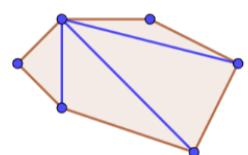
Pentagon: Draw two diagonals from any vertex, creating three triangles.

$$\text{Sum of angles} = 180 \times 3 = 540$$

Pentagon



Hexagon



General Polygon: Draw the diagonals from any vertex, creating $(n - 2)$ triangles. Sum of all angles

$$= \underbrace{(n - 2)}_{\substack{\text{No.of} \\ \text{Triangles}}} \times \underbrace{180}_{\substack{\text{Sum of Angles} \\ \text{of A Triangle}}}$$

Example 3.7

Find the sum of angles of the following polygons

- A. Hexagon
- B. Nonagon
- C. Decagon
- D. Dodecagon (12 Sides)
- E. Polygon with 20 sides

$$\text{Hexagon: } 4 \times 180 = 720$$

$$\text{Nonagon: } 7 \times 180 = 1,260$$

$$\text{Decagon: } 8 \times 180 = 1,440$$

$$\text{Dodecagon: } 10 \times 180 = 1800$$

$$20-\text{gon: } 18 \times 180 = 3,240$$

Example 3.8: Missing Angle

Calculate the value of the missing angle in each part below:

- A. Triangle: $40^\circ, 70^\circ, y^\circ$
- B. Quadrilateral: $30^\circ, 120^\circ, 70^\circ, x^\circ$
- C. Pentagon: $80^\circ, 120^\circ, q^\circ, 100^\circ, 120^\circ$
- D. Pentagon: $50^\circ, 80^\circ, p^\circ, 2p^\circ, 140^\circ$

$$y = 180 - 40 - 70 = 180 - 110 = 70$$

$$x = 360 - (30 + 120 + 70) = 360 - 220 = 140$$

$$q = (5 - 2)(180) - (80 + 120 + 100 + 120) = 540 - 420 = 120$$

We form an equation:

$$50 + 80 + p + 2p + 140 = (5 - 2)(180)$$

$$270 + 3p = 540$$

$$\begin{aligned}3p &= 270 \\p &= 90\end{aligned}$$

3.9: Number of Sides

If information on the angles is given, we can use it to calculate the number of sides of the polygon.

Example 3.10: Number of Sides

- A. The sum of the interior angles of a polygon is 1260° . Find the number of sides of the polygon.
- B. A polygon has one angle 179° and a second angle 178° . What is the smallest number of sides it can have?

Part A

$$\begin{aligned}(n - 2)180 &= 1260 \\n - 2 &= \frac{1260}{180} = 7 \\n &= 9\end{aligned}$$

Part B

$$179 + 178 = 357 \Rightarrow \text{Smallest number of sides} = 4$$

Example 3.11: Ratios

- A. The angles in a pentagon are in the ratio $1: 2: 3: 4: 5$. Find the difference between the smallest and the largest angle.
- B. If a convex pentagon has two congruent angles, and the other interior angles each have measure equal to the sum of the measures of the two congruent angles, then what is the sum of the measures of the large angles?

Part A

$$x: 2x: 3x: 4x: 5x \Rightarrow x + 2x + 3x + 4x + 5x = 180(5 - 2) \Rightarrow 15x = 540 \Rightarrow x = 36 \Rightarrow 5x - x = 4x = 144$$

Part B

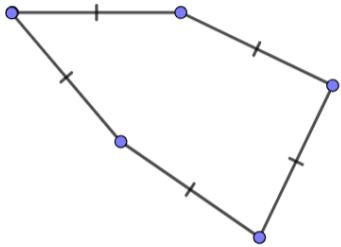
$$x + x + 2x + 2x + 2x = 540 \Rightarrow 8x = 540 \Rightarrow 6x = \frac{540}{8} \times 6 = 405$$

C. Properties

3.12: Equilateral, Equiangular, and Regular Polygons

Example 3.13

- A. Name a quadrilateral that is equilateral, but not equiangular.
- B. Draw a pentagon that is equilateral, but not equiangular.



3.14: Maximum Measure of Angle of Convex Polygon

Is 360

This property is very commonly used.

Example 3.15

How many sides would there be in a convex polygon if the sum of all but one of its interior angles is 1070? (MathCounts 1993 National Countdown)

$$\text{No. of right angles} = n \Rightarrow \text{Sum of remaining } (8 - n) \text{ angles} = \frac{1080}{(8-2) \times 180} - 90n$$

D. Inequalities

Example 3.16

What is the greatest number of interior right angles a convex octagon can have? (MathCounts 2008 State Sprint)

$$\text{No. of right angles} = n \Rightarrow \text{Sum of remaining } (8 - n) \text{ angles} = \frac{1080}{(8-2) \times 180} - 90n$$

To maximise the right angles, we make the remaining angles equal, since that lets them be greater:

$$\text{Average of } (8 - n) \text{ angles} = \frac{1080 - 90n}{8 - n}$$

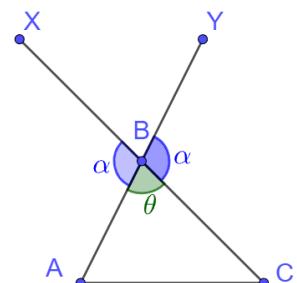
Each angle must be less than 180° in a convex polygon+:

$$\frac{1080 - 90n}{8 - n} < 180 \Rightarrow \underbrace{1080 - 90n < 1440 - 180n}_{\substack{\text{Multiply both sides} \\ \text{by } (8-n)}} \Rightarrow 90n < 360 \Rightarrow n < 4 \Rightarrow \text{Max}(n) = 3$$

E. Exterior Angles

3.17: Exterior Angle

The angle formed by extending the side of a polygon is called the exterior angle of a polygon.



Example 3.18

Identify the exterior angles

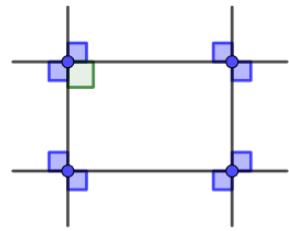
$\angle ABC$ is an angle of the triangle.

Exterior angles are

$\angle XBA$ and $\angle YBC$

3.19: Number of Exterior Angles

The number of exterior angles of a polygon with n sides is $2n$.



Example 3.20

Count the exterior angles of the rectangle alongside.

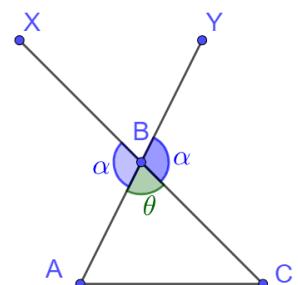
8

3.21: Linear Pair

An exterior angle and its *corresponding* interior angle form a linear pair.

∴ An exterior and its corresponding interior angle are supplementary.

∴ The two angles add up to 180° .



If I draw the exterior angles at vertex C , they may not add up to 180° , if added to $\angle ABC$.

Example 3.22

- If the exterior angle of a polygon measures 12° , what is the measure of the corresponding interior angle?
- If the exterior angle of a polygon has degree measure as the largest two digit prime, what is the measure of the corresponding interior angle?

$$IA = 180 - EA = 180 - 12 = 168^\circ$$

$$IA = 180 - EA = 180 - 97 = 83^\circ$$

Example 3.23

- If the exterior angle of a polygon has degree measure $3x - 4$, what is the measure of the corresponding interior angle?
-

$$IA = 180 - (3x - 4) = 180 - 3x + 4 = 184 - 3x$$

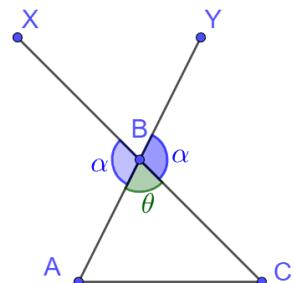
Example 3.24

If the exterior angle of a polygon and the corresponding interior angle have measures $3x - 4$ and $2x - 3$ then determine the value of x .

$$\begin{aligned} (3x - 4) + (2x - 3) &= 180 \\ 5x - 7 &= 180 \\ 5x &= 187 \\ x &= \frac{187}{5} \end{aligned}$$

3.25: Sum of Exterior Angles

Sum of exterior angles of a polygon = 360°



- When we form exterior angles, for each interior angle, two exterior angles are created.
- For this property, we only take one exterior into consideration.

Example 3.26

As the number of sides of a polygon increases from 3 to n , the sum of the exterior angles formed by extending each side in succession:

- A. Increases
- B. Decreases
- C. Remains constant
- D. Cannot be predicted
- E. Becomes $(n - 3)$ straight angles (AHSME 1950/12)

Remains Constant.

Option C

3.2 Regular Polygons

A. Regular Polygons

3.27: Regular Polygon

If a polygon has equal angles, then it is equiangular.

A polygon with all sides equal is an equal sided polygon.

A regular polygon is a polygon with equal sides and equal angles.

Example 3.28

Classify the following as regular polygons, or not.

- A. Square
- B. Rectangle
- C. Isosceles Triangle
- D. Right Triangle
- E. Equilateral Triangle

	Equal Sides	Equal Angles	Regular
Square	Yes	Yes	Yes
Rectangle	No	Yes	No
Isosceles Triangle	No	No	No
Right Triangle	No	No	No
Equilateral Triangle	Yes	Yes	Yes

3.29: Regular Polygon

Regular Triangle = Equilateral Triangle

Regular Quadrilateral = Square

Regular Pentagon = Regular Pentagon

3.30: Interior Angle of a Regular Polygon

Interior angles of a regular polygon are equal.

$$IA = \frac{(n - 2)(180)^\circ}{n} = \frac{180n - 360}{n}$$

We can use this to find the measure of each angle of a regular polygon

$$\text{Value of each angle} = \frac{\text{Sum of Angles}}{\text{No. of Angles}} = \frac{(n - 2)(180)^\circ}{n}$$

Example 3.31

Find the value of the measure of the interior angle of the following regular polygons:

- A. Triangle
- B. Pentagon
- C. Hexagon
- D. Octagon

$$\text{Triangle} = \frac{(3 - 2)(180)}{3} = \frac{180}{3} = 60$$

$$\text{Pentagon} = \frac{(5 - 2)(180)}{5} = \frac{540}{5} = 108^\circ$$

$$\text{Hexagon} = \frac{(6 - 2)(180)}{6} = \frac{4(180)}{6} = 4 \times 30 = 120^\circ$$

$$\text{Octagon} = \frac{(8 - 2)(180)}{8} = \frac{6(180)}{8} = 135^\circ$$

3.32: Interior Angle based on number of side

As the number of sides of a regular polygon increase, its interior angle
increasing

Example 3.33: Back-Calculations

Each interior angle of a regular polygon measures 140° . How many sides does the polygon have?

Let n be the number of sides of the regular polygon. The measure of its interior angle will be

$$\frac{180n - 360}{n} = 140$$

Multiply both sides by n :

$$180n - 360 = 140n$$

$$40n = 360$$

$$n = 9$$

B. Shapes inside Regular Polygons

Example 3.34

Point X is inside regular quadrilateral $ABCD$ such that $\triangle XCD$ is equilateral. Find $\angle XCB$.

$\angle CBA$ is an angle of a square:

$$\angle CBA = 90^\circ$$

In the square, the interior angle is made of the sum of two angles:

$$\angle CBA = \angle CBX + \angle XBC$$

$$\angle CBX = \angle CBA - \angle XBA = 90 - 60 = 30^\circ$$

In square $ABCD$

$$\overbrace{AB = BC}^{\text{Statement I}}$$

In equilateral triangle XBA

$$\overbrace{AB = BX}^{\text{Statement II}}$$

Combine Statements I and II

$$AB = BC = BX \Rightarrow \Delta BXC \text{ is isosceles}$$

In isosceles ΔBXC

$$\text{Base } \angle BCX = \frac{180 - 30}{2} = \frac{150}{2} = 75^\circ$$

Example 3.35

Point F is inside regular pentagon $ABCDE$ such that ΔFAB is equilateral. Find $\angle BCF$.

$\angle CBA$ is an angle of a regular pentagon:

$$\angle CBA = \frac{180(n-2)}{n} = \frac{180(3)}{5} = 108^\circ$$

In the pentagon, the interior angle is made of the sum of two angles:

$$\angle CBA = \angle CBF + \angle FBC$$

$$\angle FBC = \angle CBA - \angle FBA = 108 - 60 = 48^\circ$$

In regular pentagon $ABCDE$

$$\overbrace{AB = BC}^{\text{Statement I}}$$

In equilateral triangle FBA

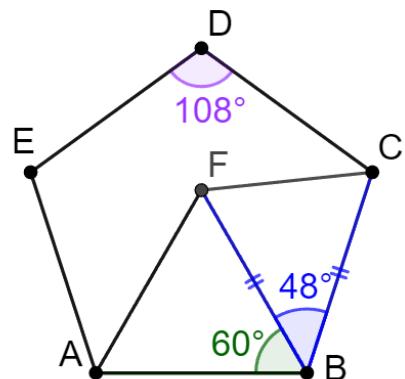
$$\overbrace{AB = BF}^{\text{Statement II}}$$

Combine Statements I and II

$$AB = BC = BF \Rightarrow \Delta BFC \text{ is isosceles}$$

In isosceles ΔBFC

$$\text{Base } \angle BCF = \frac{180 - 48}{2} = 66^\circ$$



Example 3.36

Square $XYBA$ is constructed such that it shares a side with regular pentagon $ABCDE$. Determine the possible values of $\angle BCY$.

$\angle CBA$ is an angle of a regular pentagon:

$$\angle CBA = \frac{180(n-2)}{n} = \frac{180(3)}{5} = 108^\circ$$

Since they are the sides of a square and a regular pentagon (with a common side):

$$AB = BY = BC \Rightarrow \Delta BCY \text{ is isosceles}$$

Case I: Square is drawn inside the pentagon

$$\angle CBY = \angle CBA - \angle YBA = 108 - 90 = 18^\circ$$

In Isosceles ΔBCY

$$\angle BCY = \frac{180 - 18}{2} = \frac{162}{2} = 81^\circ$$

Case II: Square is drawn outside the pentagon

$$\angle CBY = 360 - \angle CBA - \angle YBA = 360 - 108 - 90 = 162^\circ$$

In Isosceles ΔBCY

$$\angle BCY = \frac{180 - 162}{2} = \frac{18}{2} = 9^\circ$$

The possible values are:

$$\angle BCY \in \{9^\circ, 81^\circ\}$$

3.37: Interior Angle of a Regular Polygon (Alternate Version)

The interior angle of a regular polygon has degree measure:

$$180 - \frac{360}{n}$$

$$\frac{180(n-2)}{n}$$

Open the brackets:

$$= \frac{180n - 360}{n}$$

Split the fraction:

$$\begin{aligned} &= \frac{180n}{n} - \frac{360}{n} \\ &= 180 - \frac{360}{n} \end{aligned}$$

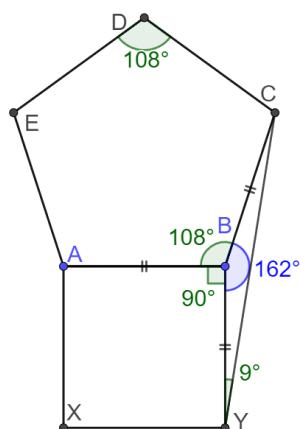
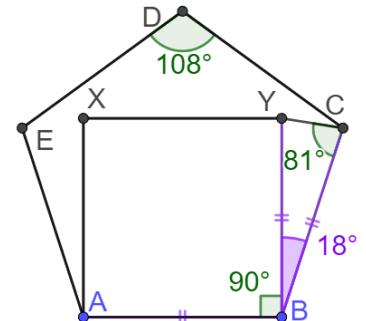
Example 3.38

Point F is inside a regular polygon AB ... with n sides such that ΔFAB is equilateral. Find $\angle BCF$ in terms of n.

$$\angle CBF = \angle CBA - \angle FBA$$

Substitute $\angle CBA = 180 - \frac{360}{n}$, $\angle FBA = 60^\circ$:

$$\angle CBF = \left(180 - \frac{360}{n}\right) - 60 = 120 - \frac{360}{n}$$



Since

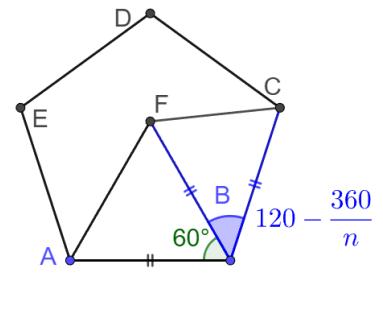
$$AB = BF = BC \Rightarrow \Delta BCF \text{ is isosceles}$$

In Isosceles ΔBCF :

$$\angle BCF = \frac{1}{2} \left[180 - \left(120 - \frac{360}{n} \right) \right] = \frac{1}{2} \left[60 + \frac{360}{n} \right] = 30 + \frac{180}{n}$$

$$n = 4: 30 + \frac{180}{4} = 30 + 45 = 75$$

$$n = 5: 30 + \frac{180}{5} = 30 + 36 = 66$$



Example

Angles of an isosceles trapezoid in a polygon

C. Exterior Angle of a Regular Polygon

3.39: Exterior Angle of a Regular Polygon

The measure of an exterior angle of a regular polygon with n sides is:

$$\frac{360^\circ}{n}$$

Sum of exterior angles of a polygon adds up to 360° . But each exterior angle is equal. Hence, the measure of each angle is

$$\frac{360^\circ}{n}$$

Exterior angles (EA) of a regular polygon are equal.

Example 3.40

Find the measure of an exterior angle of the following regular polygons

- A. Pentagon
- B. Hexagon
- C. Nonagon

$$\text{Pentagon} = \frac{360}{5} = 72^\circ$$

$$\text{Hexagon} = \frac{360}{6} = 60^\circ$$

$$\text{Nonagon} = \frac{360}{9} = 40^\circ$$

Example 3.41: Back-Calculations

Each exterior angle of a polygon measures 45° degrees. What is the number of sides?

$$\frac{360}{n} = 45$$

$$n = \frac{360}{45} = 8$$

Example 3.42

A regular polygon with n sides and side length x is such that the measure of its exterior angle is equal to the measure of its interior angle. Find the area of that regular polygon.

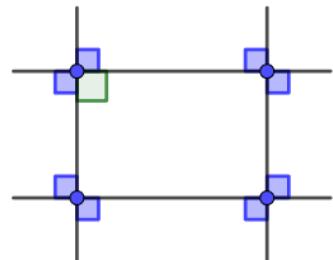
By observation, the interior angle of a square is equal to the exterior angle.

Hence,

$$n = 4$$

Alternatively, we can use the formula for an exterior angle and the formula for an interior angle (since the two are equal):

$$IA = EA$$



$$\begin{aligned} \frac{(n-2)180}{n} &= \frac{360^\circ}{n} \\ (n-2)180 &= 360^\circ \\ n-2 &= 2 \\ n &= 4 \end{aligned}$$

Hence, the polygon is a square, and it has area

$$Side^2 = x^2$$

Example 3.43

A regular polygon with n sides has exterior angle x° . A regular polygon with $n+3$ sides has exterior angle y° . If the difference between x and y is 60° , determine $x+y$ as an integer.

We can do this with trial and error. The smallest possible value of n is

$$n = 3: \frac{360}{3} - \frac{360}{6} = 120 - 60 = 60^\circ$$

You can do this with algebra, but the process is much longer, and not recommended:

$$\begin{aligned} \frac{360}{n} - \frac{360}{n+3} &= 60 \\ \frac{360(n+3) - 360n}{n(n+3)} &= 60 \\ 1080 &= 60(n)(n+3) \\ 18 &= (n)(n+3) \\ n &= 3 \end{aligned}$$

3.44: Connecting Interior and Exterior Angle of a Regular Polygon

$$IA = 180 - EA = 180 - \frac{360}{n}$$

This property is true because the interior and the exterior angle are supplementary.

Example 3.45: Back-Calculations

An interior angle of a regular polygon measures 140° . Determine the number of sides of the polygon.

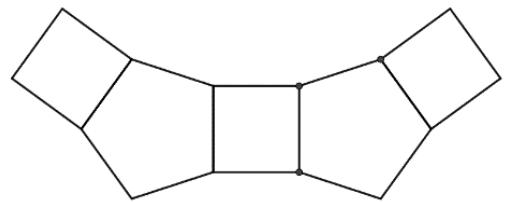
Rather than using the formula for an interior angle (which is complicated), it is easier to find the exterior angle first.

$$\begin{aligned} EA &= 180 - 140 = 40 \\ 40 &= \frac{360}{n} \\ n &= \frac{360}{40} = 9 \end{aligned}$$

Example 3.46

The diagram alongside has a pattern of alternating squares and regular pentagons. If the pattern is continued what is the number of sides of

- A. The inside of the shape
- B. The outside of the shape



Part A

The interior angle of the regular pentagon
 $= 108^\circ$

The interior angle of the square is
 90°

By sum of angles around a point, the interior angle of the polygon:

$$= 360 - 90 - 108 = 162$$

Extend the top side of the middle square to the right. By angles in a linear pair, the exterior angle of the polygon:
 $= 180 - 162 = 18$

Number of sides

$$= \frac{360}{EA} = \frac{360}{18} = 20 \text{ sides}$$

Part B

Number of sides on the outside

$$\begin{aligned} &= 1 \times \text{No. of Squares} + 2 \times \text{No. of Pentagon} \\ &= 10 + 20 = 30 \end{aligned}$$

Note: The number of sides can also be calculated directly from the value of the interior angle by using the formula for the interior angle of a regular polygon:

$$\begin{aligned} \frac{(n-2)(180)}{n} &= 162 \\ 180n - 360 &= 162n \\ 18n &= 360 \\ n &= \frac{360}{18} = 20 \end{aligned}$$

D. Circle of a Regular Polygon

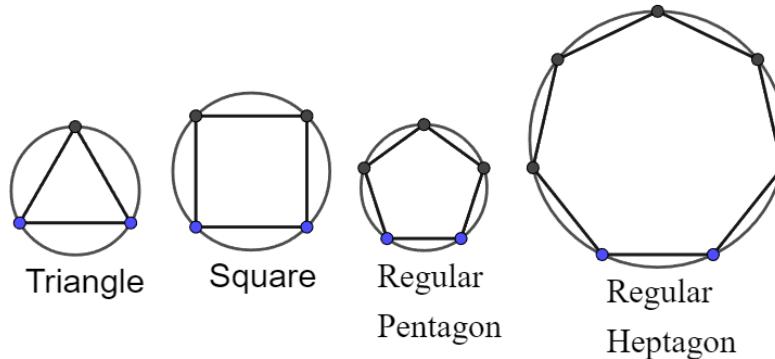
3.47: Number of Points to define a circle

3 distinct points define a circle

If the three points lie on the same line, then the circle has infinite radius.

3.48: Some Terminology

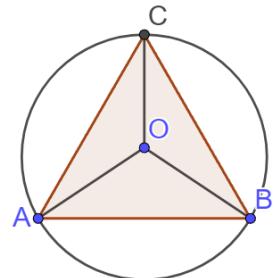
Circle that passes through the three vertices of a triangle is called the circumcircle of the triangle.
 If a quadrilateral has a circle passing through its vertices, that quadrilateral is called concyclic.



3.49: Regular polygon subtends equal angles on a circle

By sum of angles around a point:

$$\angle AOC + \angle AOB + \angle BOC = 360^\circ$$



Since it is a regular polygon, the sides have equal lengths.

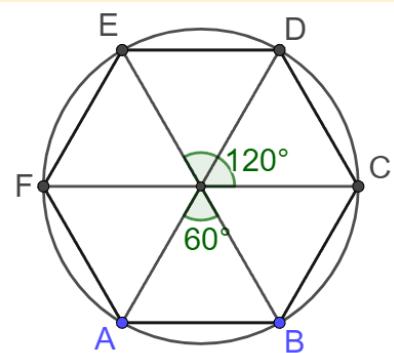
The sides are chords of the circle, and since they have equal lengths, they subtend equal angles.

$$\begin{aligned} 3 \times \angle AOC &= 360^\circ \\ \angle AOC &= 120^\circ \end{aligned}$$

Example 3.50

A circle with center O is drawn through the vertices of regular hexagon ABCDEF. Calculate $\angle AOB + \angle COE$

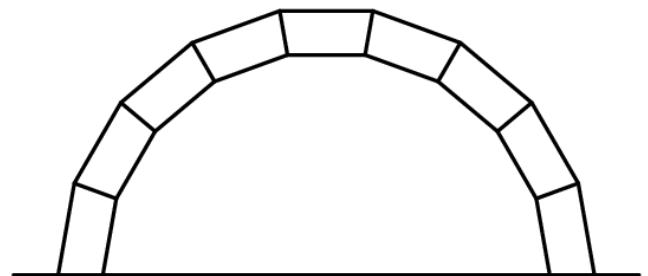
$$\frac{360}{n} = \frac{360}{6} = 60^\circ$$



Example 3.51

The keystone arch is an ancient architectural feature. It is composed of congruent isosceles trapezoids fitted together along the non-parallel sides, as shown. The bottom sides of the two end trapezoids are horizontal. In an arch made with 9 trapezoids, let x be the angle measure in degrees of the larger interior angle of the trapezoid.

What is x ? (AMC 10B 2009/24)



Method I: Regular Polygon on a Circle

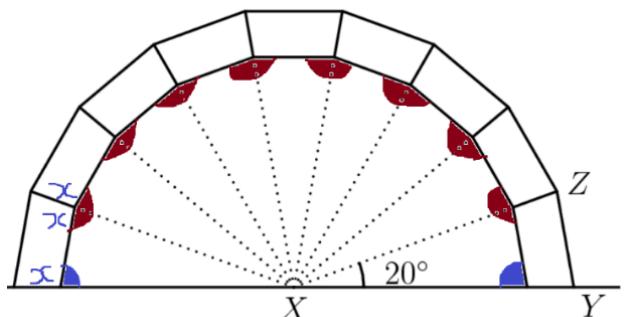
Recognize the inner sides of the trapezoids form one-half of a 18-sided regular polygon, and that the vertices of the polygon lie on a circle.

Connect the vertices with the center of the circle. The central angle subtended by each sector is then

$$\frac{360}{n} = \frac{360}{18} = 20^\circ$$

Let the base angle of each isosceles triangle be y

$$y + y + 20 = 180 \Rightarrow 2y = 160 \Rightarrow y = 80$$



The interior angle of the trapezoid is then

$$= 180 - 80 = 100^\circ$$

Method II: Angle Chasing

The angle we want is x . Each blue angle (two of them) is:

$$180 - x$$

Each maroon angle (eight of them) is:

$$360 - 2x$$

Since the blue and maroon angles together form a decagon with sum of interior angles = $180(10 - 2) = 180 \times 8$:

$$\begin{aligned} 2(180 - x) + 8(360 - 2x) &= 180 \times 8 \\ 2 \times 180 - 2x + 16 \times 180 - 16x &= 180 \times 8 \\ 10 \times 180 &= 18x \\ x &= 100^\circ \end{aligned}$$

Method III: Polygon Angle Chasing

Reflect the inner side of the arch to get an 18-sided regular polygon. It has

$$\text{Interior Angle} = 180 - \frac{\frac{360}{18}}{\text{Exterior Angle}} = 180 - 20 = 160$$

The angle that we want is:

$$\frac{360 - 160}{2} = \frac{200}{2} = 100$$

E. Applications: Ratios

Example 3.52

The exterior angles of a triangle are in the ratio 1:2:2. What kind of triangle is it?

$$x:2x:2x \Rightarrow x + 2x + 2x = 360 \Rightarrow 5x = 360 \Rightarrow x = 72 \Rightarrow 2x = 144$$

Angles of the triangle = $\{180 - 72, 180 - 144, 180 - 144\} = \{108, 36, 36\}$
 It is an obtuse-angled, isosceles triangle.

3.3 Regular Hexagons

A. Diagonals

Example 3.53

$ABCDEF$ is a regular hexagon with side length 1. What is the length of diagonal AE ?

Since $\angle AFE$ is an angle of a regular hexagon, it is:

$$\frac{(n-2)(180)}{n} = \frac{(4)(180)}{6} = 120^\circ$$

In Isosceles ΔAFE :

$$\angle FAE = \angle AEF = \frac{180 - 120}{2} = \frac{60}{2} = 30^\circ$$

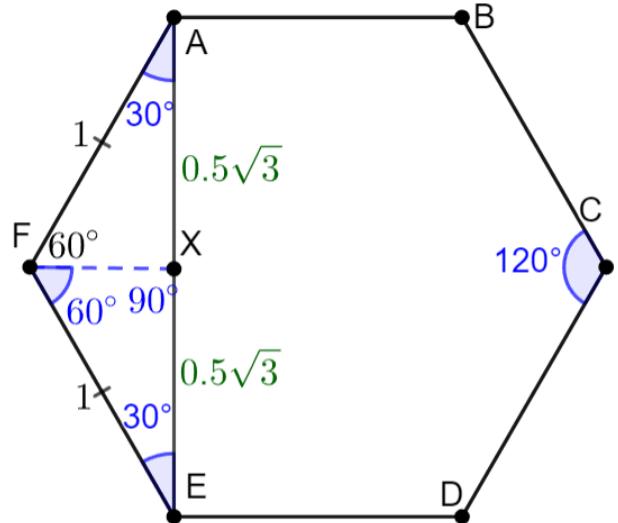
Draw $FX \perp AE$. In ΔFXE and ΔFXA :

$$\angle XFE = \angle XFA = 180 - 30 - 90 = 60^\circ$$

Hence, ΔFXE and ΔFXA are both $30 - 60 - 90$ triangles.

Using the properties of $30 - 60 - 90$ triangles

$$\begin{aligned} XA &= XE = \frac{\sqrt{3}}{2} \\ XA + XE &= 0.5\sqrt{3} + 0.5\sqrt{3} = \sqrt{3} \end{aligned}$$



Example 3.54

$ABCDEF$ is a regular hexagon with side length 1. What is the length of diagonal AD ?

$$\angle AED = \angle FED - \angle AEF = 120 - 30 = 90^\circ$$

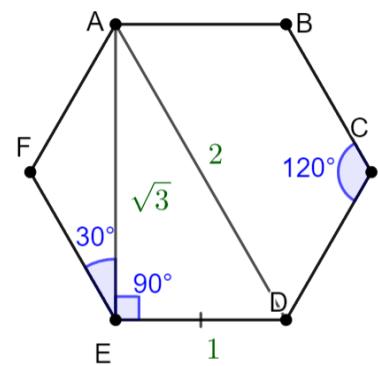
By Pythagoras Theorem in ΔAED :

$$AD = \sqrt{AE^2 + ED^2} = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$$

3.55: Diagonals of a Regular Hexagon

In regular hexagon, $ABCDEF$ with side length s :

$$\begin{aligned} \text{Diagonal } AE &= \sqrt{3}s \\ \text{Diagonal } AD &= 2s \end{aligned}$$



These results follow from the previous examples, since any two regular hexagons have the same shape (are similar).

Hence, their corresponding diagonals are in the same ratio.

Example 3.56

$ABCDEF$ is a regular hexagon with side length 2 feet 5 inches. Determine the length of:

- A. Diagonal AE

B. Diagonal AD

$$\text{Side Length} = 2 \text{ feet } 5 \text{ inches} = 2 \frac{5}{12} \text{ feet} = \frac{29}{12} \text{ feet}$$

Since AE is $\sqrt{3}$ times the side length:

$$AE = \frac{29}{12} \cdot \sqrt{3} = \frac{29\sqrt{3}}{12}$$

Since AD is 2 times the side length:

$$AE = \frac{29}{12} \cdot 2 = \frac{29}{6}$$

Example 3.57

Use the result above to calculate the area of a regular hexagon by splitting it into two trapezoids.

Draw diagonal FC

$$\angle FAB + \angle AFC = 120^\circ + 60^\circ = 180^\circ \Rightarrow FC \parallel AB$$

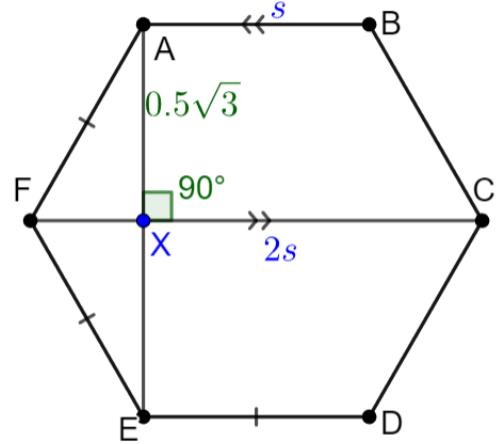
Similarly:

$$FC \parallel ED$$

Combine:

$$FC \parallel AB \parallel ED$$

Hence, diagonal FC splits hexagon $ABCDEF$ into two trapezoids $FABC$ and $FECD$.



The area of trapezoid $FABC$

$$= \frac{(b_1 + b_2)}{2} h = \frac{AB + FC}{2} \cdot AX = \frac{(s + 2s)}{2} \cdot \frac{\sqrt{3}}{2} s = \frac{3\sqrt{3}}{4} s^2$$

The area of hexagon $ABCDEF$

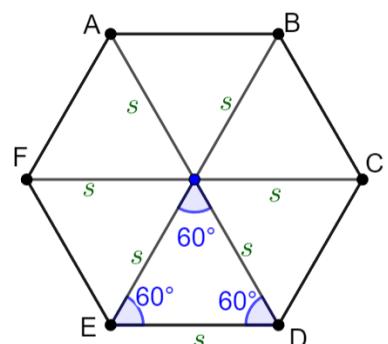
$$2 \cdot [FABC] = 2 \cdot \frac{3\sqrt{3}}{4} s = \frac{3\sqrt{3}}{2} s^2$$

3.58: Six Equilateral Triangles

The diagonals of a regular hexagon connecting opposite vertices are twice the side length of the hexagon.

They intersect at the center of the hexagon, dividing it into

Six equilateral triangles



B. Area

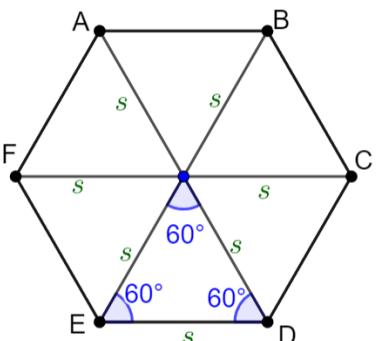
3.59: Area of a Regular Hexagon

The area of a regular hexagon is $\frac{3\sqrt{3}}{2}$ times the square of the side length.

$$A(\text{Regular Hexagon}) = \frac{3\sqrt{3}}{2} s^2$$

A regular hexagon can be split into six equilateral triangles by drawing lines connecting opposite vertices. These lines will all pass through a common point at the center of the hexagon.

$$\frac{\sqrt{3}}{4} s^2 \cdot 6 = \frac{3\sqrt{3}}{2} s^2$$



Example 3.60

ABCDEF is a regular hexagon with side length 1 foot 6 inches. Determine the area of the hexagon in square yards. Note: 1 yard is 3 feet.

$$\text{Side Length} = 1 \text{ foot } 6 \text{ inches} = 1 \frac{6}{12} \text{ inches} = \frac{3}{2} \text{ ft}$$

The area is:

$$\frac{3\sqrt{3}}{2} s^2 = \frac{3\sqrt{3}}{2} \left(\frac{3}{2}\right)^2 = \frac{27\sqrt{3}}{8} \text{ ft}^2$$

To convert from square feet to square yards, divide by $3^2 = 9$

$$\frac{27\sqrt{3}}{8} \times \frac{1}{9} = \frac{3\sqrt{3}}{8} \text{ yards}^2$$

Example 3.61

ABCDEF is a regular hexagon with side length $\frac{17}{12}$ yards. Determine the area of the hexagon in square inches.
 Note: 1 yard is 3 feet.

Convert the side length to inches by multiplying by $12 \times 3 = 36$

$$\frac{17}{12} \times 36 = 51 \text{ inches}$$

The area is:

$$\frac{3\sqrt{3}}{2} \cdot 51^2 \text{ in}^2$$

Example 3.62

ABCDEF is a regular hexagon with side length 1 foot 5 inches.

- A. Determine the area of the hexagon in square inches
- B. Convert your answer from part A to square yards. Note: 1 yard is 3 feet.

$$\text{Side Length} = 1 \text{ foot } 5 \text{ inches} = 12 + 6 \text{ inches} = 18 \text{ inches}$$

The area is:

$$\frac{3\sqrt{3}}{2}s^2 = \frac{3\sqrt{3}}{2}(17)^2 = \frac{3\sqrt{3}}{2} \cdot 289 \text{ in}^2$$

To convert from square inches to square feet multiply by $\frac{1}{12^2} \times \frac{1}{3^2} = \frac{1}{144} \times \frac{1}{9}$

$$\frac{3\sqrt{3}}{2} \cdot 289 \times \frac{1}{144} \times \frac{1}{9} = \frac{289\sqrt{3}}{864} \text{ yards}^2$$

Example 3.63

A square and a regular hexagon have equal perimeters. If the area of the hexagon is 4, what is the area of the square?

The area of the hexagon

$$\frac{3\sqrt{3}}{2}s^2 = 4 \Rightarrow s^2 = \frac{8}{3\sqrt{3}} \Rightarrow s = \sqrt{\frac{8}{3\sqrt{3}}}$$

The perimeter of the hexagon (and the square):

$$= P = 6s = 6\sqrt{\frac{8}{3\sqrt{3}}}$$

The side length of the square is:

$$\frac{P}{4} = \frac{6}{4}\sqrt{\frac{8}{3\sqrt{3}}} = \frac{3}{2}\sqrt{\frac{8}{3\sqrt{3}}}$$

The area of the square

$$= \frac{9}{4} \cdot \frac{8}{3\sqrt{3}} = \frac{3 \cdot 2}{\sqrt{3}} = 2\sqrt{3}$$

Example 3.64

An equilateral triangle and a regular hexagon have equal perimeters. If the area of the triangle is 4, what is the area of the hexagon? (AMC 8 2012/23)

Let the length the side of the equilateral triangle be s .

$$\frac{\sqrt{3}}{4}s^2 = 4 \Rightarrow s^2 = \frac{16}{\frac{1}{3^2}} \Rightarrow s = \frac{4}{3^{\frac{1}{2}}}$$

Length of side of the hexagon will be half of the side of the equilateral triangle

$$\frac{s}{2} = \frac{2}{3^{\frac{1}{2}}} \Rightarrow \text{Area} = \frac{3\sqrt{3}}{2}(\text{Side})^2 = \frac{3\sqrt{3}}{2} \times \left(\frac{2}{3^{\frac{1}{2}}}\right)^2 = \frac{3\sqrt{3}}{2} \times \frac{4}{3^2} = 6$$

Example 3.65

In regular hexagon $ABCDEF$ with side length 8, determine the area

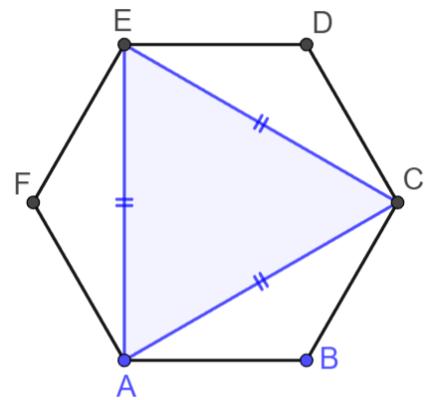
- A. of $\triangle ACE$
- B. of $\triangle EFA$

Part A

$$EA = AC = CE = 8\sqrt{3}$$

Area of ΔACE

$$= \frac{\sqrt{3}}{4} s^2 = \frac{\sqrt{3}}{4} (8\sqrt{3})^2 = 48\sqrt{3}$$



Part B

Area of ΔEFA

$$= \frac{1}{2} hb = \frac{1}{2} (EA)(FX) = \frac{1}{2} (8\sqrt{3})(4) = 16\sqrt{3}$$

C. Regular Octagons

3.66: Area of a Regular Octagon

The area of a regular octagon with side length 1 is

$$2(1 + \sqrt{2})s^2$$

Angle of a regular octagon

$$= \frac{(n - 2)(180)}{n} = \frac{6(180)}{8} = \frac{3(180)}{4} = 135^\circ$$

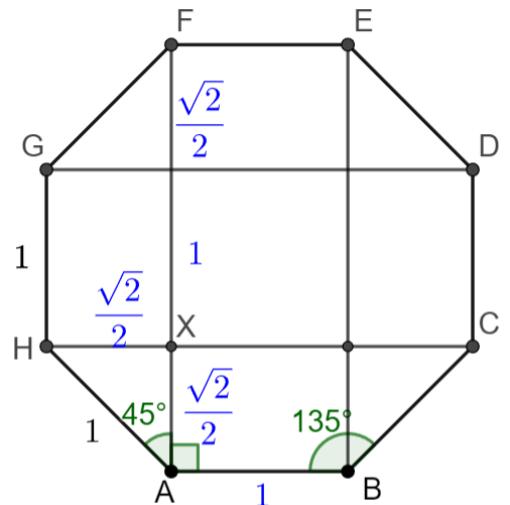
$$FA = 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 1 + \frac{2\sqrt{2}}{2} = 1 + \sqrt{2}$$

Area of rectangle $ABEF$

$$= lw = (AB)(FA) = 1(1 + \sqrt{2}) = 1 + \sqrt{2}$$

Area of trapezoid $GFAH$

$$\begin{aligned} &= \frac{GH + FA}{2} \cdot HX = \frac{1 + 1 + \sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{2 + \sqrt{2}}{4} \cdot \sqrt{2} = \frac{2\sqrt{2} + 2}{4} \\ &= \frac{1 + \sqrt{2}}{2} \end{aligned}$$



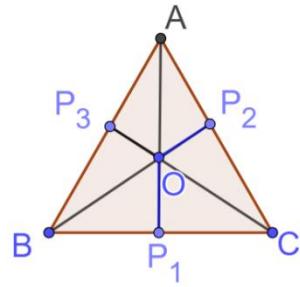
The final area is:

$$\begin{aligned} &[ABEF] + [GFAH] + [EDBC] \\ &1 + \sqrt{2} + 2\left(\frac{1 + \sqrt{2}}{2}\right) = 1 + \sqrt{2} + (1 + \sqrt{2}) = 2(1 + \sqrt{2}) \end{aligned}$$

D. Regular Polygons: Area

3.67: Area of a Regular Polygon

$$\begin{aligned}
 A(\Delta ABC) &= A(\Delta OAB) + A(\Delta OBC) + A(\Delta OCA) \\
 &= \frac{1}{2}(OP_1)(BC) + \frac{1}{2}(OP_2)(AC) + \frac{1}{2}(OP_3)(AB) \\
 &= \frac{1}{2}as + \frac{1}{2}as + \frac{1}{2}as \\
 &= \frac{1}{2}a(s+s+s) = \frac{1}{2}a(3s) = \frac{1}{2}ap
 \end{aligned}$$



(1/2)(Perimeter* Apothem)

Example 3.68

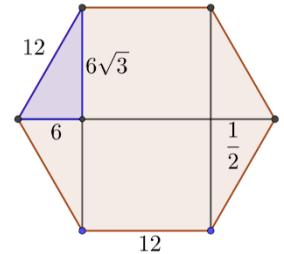
- A. What is the area of a regular hexagon with side 12 units?
- B. What is the area of a regular pentagon with side 3 units?
- C. What is the maximum area of a pentagon which has the same perimeter as a regular hexagon with area X?
- D. perimeter of the hexagon in terms of x , and the area of the hexagon in terms of y .

$$\text{Hexagon} = 2 \times \frac{(b_1 + b_2)h}{2} = 2 \frac{(12 + 24)6\sqrt{3}}{2} = 216\sqrt{3}$$

$$12^2 * 3 * \text{Root}(3)/2 = 216 \text{ Root}(3)$$

$$11.72 * 3^2 = 1.72 * 9$$

1



Example 3.69

A regular hexagon with side 8 cm has the same perimeter as an equilateral triangle. What is the difference in the areas of the two shapes?

Area of Hexagon

$$= 6 \times \frac{\sqrt{3}}{4}s^2 = 6 \times \frac{\sqrt{3}}{4} \times 8^2 = 96\sqrt{3}$$

Side of Triangle

$$= \frac{P(\text{Hexagon})}{3} = \frac{6 \times 8}{3} = 16$$

Area of Triangle

$$= \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4} \times 16^2 = 64\sqrt{3}$$

$$\text{Difference} = 96\sqrt{3} - 64\sqrt{3} = 32\sqrt{3}$$

Example 3.70

A wire is cut into two pieces, one of length a and the other of length b . The piece of length a is bent to form an equilateral triangle, and the piece of length b is bent to form a regular hexagon. The triangle and the hexagon have equal area. What is $\frac{a}{b}$? (AMC 10B 2013/15)

Calculation of Areas

$$\left(\frac{a}{3}\right)^2 \times \frac{\sqrt{3}}{4} = 6 \times \left(\frac{b}{6}\right)^2 \times \frac{\sqrt{3}}{4} \Rightarrow \frac{a}{b} = \frac{\sqrt{6}}{2}$$

Similarity

The hexagon can be split into six congruent equilateral triangles.

$$\begin{array}{c} \text{Ratio of} \\ \text{Areas} \end{array} \Leftrightarrow \begin{array}{c} \text{Ratio of} \\ \text{Side Lengths} \end{array}$$

Hence, we must have

$$\frac{a}{3} = \sqrt{6} \times \frac{b}{6} \Rightarrow \frac{a}{b} = \frac{\sqrt{6}}{2}$$

Example 3.71

A regular hexagon is divided by its diagonals into six congruent triangles, each with area x and perimeter y . Find:

- A. The perimeter of the hexagon in terms of x
- B. The area of the hexagon in terms of y

The congruent triangles are equilateral. Let the side of each equilateral triangle = s

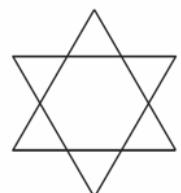
$$\frac{\sqrt{3}}{4}s^2 = x \Rightarrow s = \frac{2x}{\sqrt[4]{3}} \Rightarrow P = 6s = \frac{12x}{\sqrt[4]{3}}$$

$$\text{Area of regular hexagon} = \frac{3\sqrt{3}}{2} \text{ Side}^2 = \frac{3\sqrt{3}}{2} \left(\frac{y}{6}\right)^2 = \frac{y\sqrt{3}}{24}$$

E. Applications

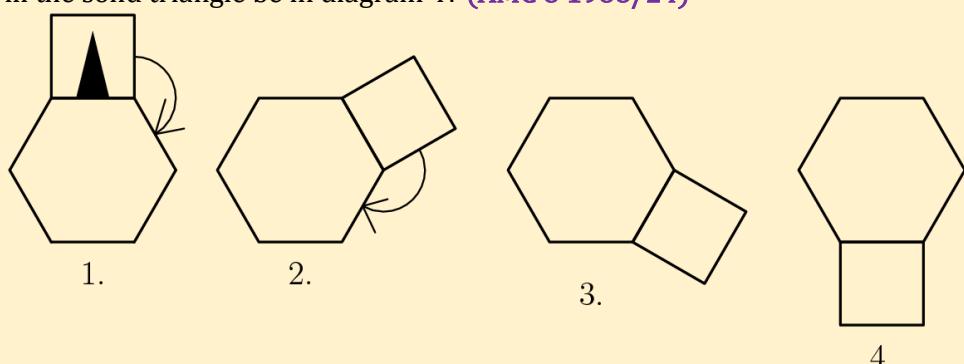
Example 3.72

A unit hexagram is composed of a regular hexagon of side length 1 and its 6 equilateral triangular extensions, as shown in the diagram. What is the ratio of the area of the extensions to the area of the original hexagon? (AMC 8 2007/12)



Example 3.73

The square in the first diagram "rolls" clockwise around the fixed regular hexagon until it reaches the bottom. In which position will the solid triangle be in diagram 4? (AMC 8 1988/24)



(A)



(B)



(C)



(D)

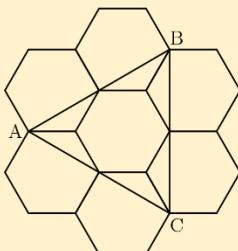


(E)



Example 3.74

Six regular hexagons surround a regular hexagon of side length 1 as shown. What is the area of $\triangle ABC$? (AMC 10B 2014/13)



Example 3.75

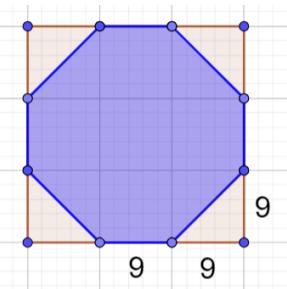
Let $ABCDEF$ be a regular hexagon with side length 1. Denote by X , Y , and Z the midpoints of sides \overline{AB} , \overline{CD} , and \overline{EF} , respectively. What is the area of the convex hexagon whose interior is the intersection of the interiors of $\triangle ACE$ and $\triangle XYZ$? (AMC 10B 2018/24)

F. Inscribed Figures

Example 3.76

An octagon is inscribed in a square so that the vertices of the octagon trisect the sides of the square. The perimeter of the square is 108 centimeters. What is the number of square centimeters in the area of the octagon? (MathCounts 1999 Warm-Up 15)

Draw a diagram. Note that the octagon is not regular.



Find the side length:

$$s = \frac{p}{4} = \frac{108}{4} = 27$$

Find the area of the square:

$$s^2 = 27^2 = 729$$

Area of four corner triangles

$$= 9 \times 9 \times \frac{1}{2} \times 4 = 162$$

Using complementary area:

$$\text{Area of Octagon} = \underbrace{729}_{\text{Area of Square}} - \underbrace{\frac{162}{4 \text{ Corner Triangles}}}_{\text{4 Corner Triangles}} = 567$$

Example 3.77

Regular Hexagon inscribed in a circle
 Circle inscribed in a Regular Hexagon

Example 3.78: Nested

Find the angle of sector of larger circle that is equivalent of area of circle inscribed inside hexagon inside the larger circle

Larger circle has radius 1

Hexagon has side 1

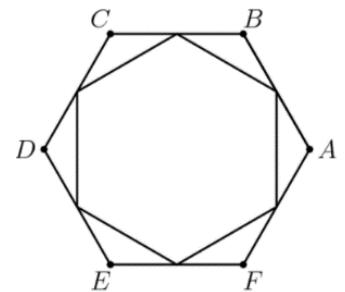
Inner circle has radius $\frac{\sqrt{3}}{2}$

Ratio of areas = $\frac{3}{4}$

Angle of Sector = $360 \times \frac{3}{4} = 270^\circ$

Example 3.79

The midpoints of the sides of a regular hexagon ABCDEF are joined to form a smaller hexagon. What fraction of the area of ABCDEF is enclosed by the smaller hexagon? (AHSME 1996)



Let X and Y be the midpoints of CB and BA respectively.

$$\text{Let } CB = 1 \Rightarrow XB = \frac{1}{2}$$

Drop a perpendicular from B intersecting XY at P .

ΔXBA is an isosceles \triangle with $\angle XBA = 120^\circ \Rightarrow \Delta BPA$ is a 30-60-90 triangle.

Apply the properties of 30-60-90 triangle:

$$\therefore PY = \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} \Rightarrow XY = 2 \times PY = \frac{\sqrt{3}}{2}$$

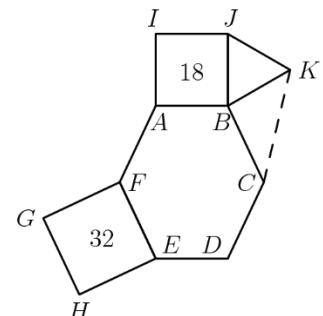
We can now find:

$$\underbrace{1:\frac{\sqrt{3}}{2}}_{\text{Ratio of Side Lengths}} \Rightarrow \underbrace{1:\frac{3}{4}}_{\text{Ratio of Areas}}$$

G. Equiangular Hexagons

Example 3.80

In the given figure hexagon $ABCDEF$ is equiangular, $ABJI$ and $FEHG$ are squares with areas 18 and 32 respectively, $\triangle JBK$ is equilateral and $FE = BC$. What is the area of $\triangle KBC$? (AMC 8 2015/21)



Example 3.81

Equiangular hexagon $ABCDEF$ has side lengths $AB = CD = EF = 1$ and $BC = DE = FA = r$. The area of $\triangle ACE$ is 70% of the area of the hexagon. What is the sum of all possible values of r ? (AMC 10A 2010/19)¹

¹ This can be solved faster using Law of Cosines (See the Trigonometry Notes)

$$\begin{aligned} A(\Delta CED) + A(\Delta AFE) + A(\Delta ABC) \\ = \frac{100\%}{A(ABCDEF)} - \frac{70\%}{A(\Delta ACE)} = 30\% \text{ of } A(ABCDEF) \end{aligned}$$

Each exterior angle of $ABCDEF$:

$$= \frac{360}{6} = 60$$

Hence, $\Delta CED \cong \Delta AFE \cong \Delta ABC$ by SAS Theorem since each triangle has $s_1 = 1, s_2 = r, \text{ Included angle} = 180 - 60^\circ = 120^\circ$

Therefore,

$$3A(\Delta CED) = 30\% \Rightarrow A(CED) = 10\% A(ABCDEF)$$

We now need to find this area in terms of r . Extend AF, BC , and ED on both sides to form ΔGHI . Since they are exterior angles:

$$\begin{aligned} \angle CDH = \angle HDC = 60^\circ \Rightarrow \angle CHD = 60^\circ \\ \Rightarrow \Delta CDH \text{ is equilateral} \end{aligned}$$

By similar logic:

$$\Delta IFE \text{ and } \Delta GAB \text{ are equilateral}$$

Since it is equilateral, we know the side length, and the area:

$$A(\Delta CDH) = 1^2 \times \frac{\sqrt{3}}{4}$$

ΔCED has same height as ΔCDH , but a different base. Hence, the ratio of the areas of the triangles depends only on their bases.

$$A(\Delta CED) = rA(\Delta CDH) = r \frac{\sqrt{3}}{4}$$

$$A(\Delta GHI) = \underbrace{3\left(\frac{\sqrt{3}}{4}\right)}_{\Delta CDH, \Delta IFE, \Delta GAB} + 10 \times r \underbrace{\frac{\sqrt{3}}{4}}_{\Delta CED} = \frac{\sqrt{3}}{4}(3 + 10r)$$

But, ΔGHI is equilateral with side length $r + 2$, and hence we can calculate its area as:

$$\frac{\sqrt{3}}{4}(r+2)^2 = \frac{\sqrt{3}}{4}(r^2 + 4r + 4)$$

Since the area calculated both ways must be equal:

$$\begin{aligned} \frac{\sqrt{3}}{4}(r^2 + 4r + 4) &= \frac{\sqrt{3}}{4}(3 + 10r) \\ r^2 + 4r + 4 &= 3 + 10r \\ r^2 - 6r + 1 &= 0 \end{aligned}$$

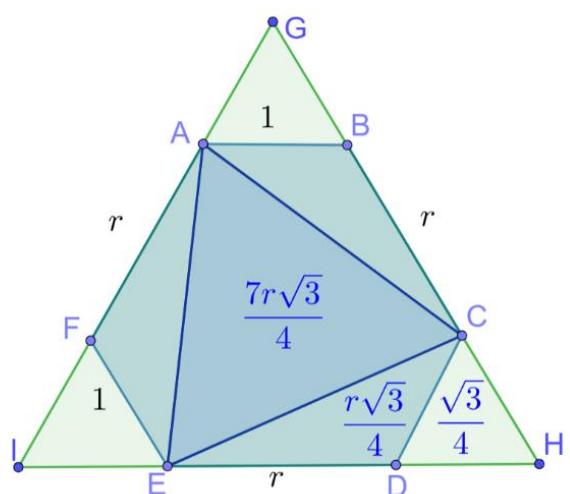
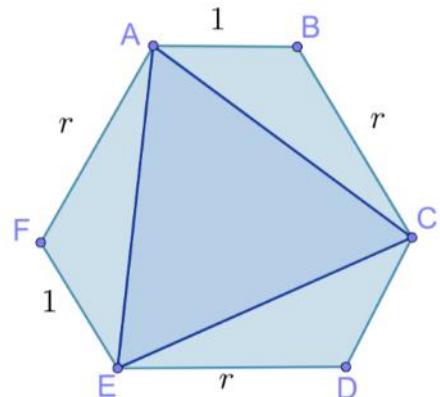
By Vieta's Formulas, the sum of the roots of the above quadratic is:

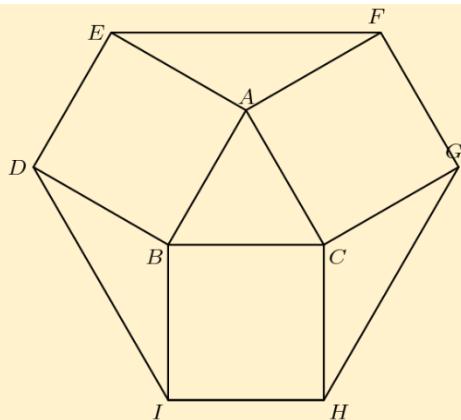
$$-\frac{b}{a} = -\frac{-6}{1} = 6$$

H. General Hexagons

Example 3.82

Equilateral $\triangle ABC$ has side length 1, and squares $ABDE, BCHI, CAFG$ lie outside the triangle. What is the area of hexagon $DEFGHI$? (AMC 10A 2014/13)





I. Rotating a Square to Form an Octagon

Example 3.83

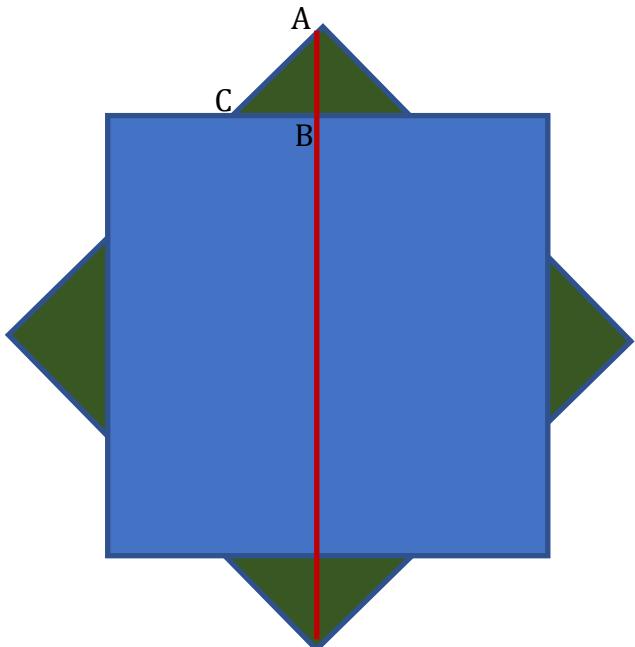
Two squares of side 1, with the same centre, are colored green and blue respectively. If the green square is rotated, what is the minimum area of intersection of the blue and the green square.

Finding Angle of Rotation

- Start with the green square perfectly aligned with the blue square, giving an angle of rotation of 0° . Here, the intersected area is 100%.
- Increase the angle of rotation. As the angle of rotation increases, the intersected area decreases.
- The minimum area will occur when the angle of rotation is 90° .
- If you rotate the square more than 90° , the area of intersection will increase.

Hence, minimum area of intersection is when \square is rotated

90 degrees to the orientation of the \square .



Strategy

Before we move to calculating the area, here are a couple of important observations:

- Each of the green triangles is congruent to a blue triangle that is “outside” the area of intersection of the two squares.
- Due to this congruence, the area of the “blue triangles” is the same as the area of the “green triangles”.

Green Triangle

A single green triangle is a right-angled triangle, since it is also the vertex of a square.

By symmetry, since the green square has been rotated 90° , it is also an isosceles triangle.

Hence, each green triangle is an isosceles right-angled triangle.

Drop a perpendicular at a green triangle so that it touches the base. The two smaller green triangles (and specifically $\triangle ABC$) are also isosceles right-angled triangles.

Green Area

ΔABC is 45-45-90 triangle. Hence,

$$AB = \frac{\text{Length of Diagonal} - \text{Length of Side}}{2} = \frac{\sqrt{2} - 1}{2}$$

Area of isosceles right-angled triangle ΔABC

$$= \frac{1}{2} \times (\text{Leg})^2 = \frac{1}{2} \left(\frac{\sqrt{2} - 1}{2} \right)^2 = \frac{3 - 2\sqrt{2}}{8}$$

The green area consists of eight congruent triangles, of which ΔABC is one.

Hence, the entire

$$\text{Green Area} = 8A(\Delta ABC) = 8 \left(\frac{3 - 2\sqrt{2}}{8} \right) = 3 - 2\sqrt{2}$$

Intersection

To find the area of intersection, we use the fact that the blue triangles have same area as the green triangles.

$$\text{Intersection} = A(\text{Blue } \square) - A(\text{Green } \Delta's) = 1 - (3 - 2\sqrt{2}) = 2\sqrt{2} - 2$$

Example 3.84

In the previous example, the blue triangles and the green triangles are cut from the square

- A. Find the remaining shape
- B. Show that the remaining shape is regular.
- C. Find the length of each side of the regular octagon

Part A

If you cut the blue and the green triangles, the remaining shape has eight sides and eight vertices. Hence, it is a Octagon.

Part B

The octagon consists of:

- Four sides which are the hypotenuse of blue triangles
- Four sides which are the hypotenuse of green triangles

But, the blue triangles are congruent to the green triangles.

Therefore, the eight sides are all equal.

Therefore, the octagon is a regular octagon.

Part C

The side of the octagon

$$= \text{Hypotenuse of Green Triangle} = 2 \times \frac{\sqrt{2} - 1}{2} = \sqrt{2} - 1$$

Length of each side of a regular octagon formed by cutting the corners of a square = $(\sqrt{2} - 1) \times \text{Side}$

Example 3.85

A regular octagon is formed by cutting an isosceles right triangle from each of the corners of a square with sides of length 2000. What is the length of each side of the octagon? (AMC 10 2001/20)

Find a General Formula

Let

$$\text{Length of Square} = 1$$

Length of each side of the regular octagon be x times the sides of the square.

This length (for the triangles, which are being cut-off) is also
 Length of Hypotenuse of the isoceles right – angled Δ's

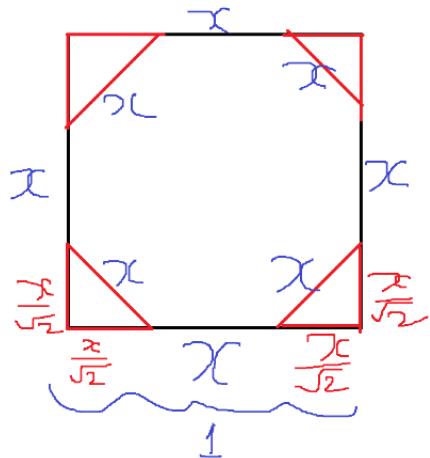
Since the triangles are $45 - 45 - 90$, the length of:

$$\text{Each leg of the triangle} = \frac{x}{\sqrt{2}} = \frac{x\sqrt{2}}{2}$$

However, the hypotenuse and the legs add up to the side of the square, so:

$$\begin{aligned} & 2 \times \frac{x\sqrt{2}}{2} + \underbrace{x}_{\text{Hypotenuse}} = 1 \\ & x\sqrt{2} + x = 1 \\ & x(\sqrt{2} + 1) = 1 \end{aligned}$$

$$x = \frac{1}{\sqrt{2} + 1} = \sqrt{2} - 1$$



Use the formula

$$2000(\sqrt{2} - 1)$$

Example 3.86

From a square with sides of length 5, triangular pieces from the four corners are removed to form a regular octagon. Find the area removed to the nearest integer? (PRMO, 2019/1)

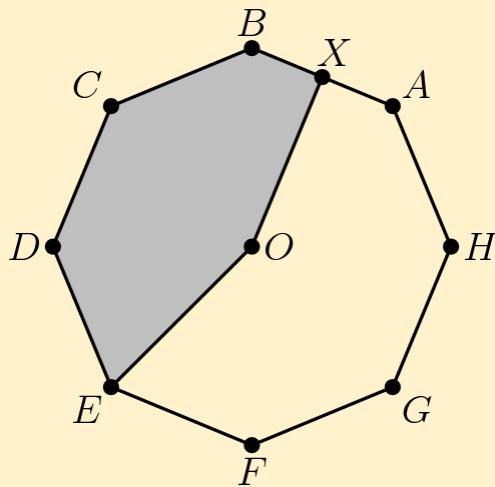
In similar figures, areas are in proportion to squares of sides. So, area removed

$$= 5^2 (\text{Green Area}) = 25(3 - 2\sqrt{2}) = 75 - 50\sqrt{2} \approx 75 - 50 \times 1.41 = 4.5$$

But note that we only took two decimal places for $\sqrt{2}$, so the answer will be less than 4.5, and when we round it, it will become 4.

Example 3.87

Point O is the center of the regular octagon $ABCDEFGH$, and X is the midpoint of the side \overline{AB} . What fraction of the area of the octagon is shaded? (AMC 8 2015/2)



Example 3.88

An equiangular octagon has four sides of length 1 and four sides of length $\frac{\sqrt{2}}{2}$, arranged so that no two

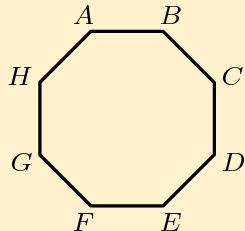
consecutive sides have the same length. What is the area of the octagon? (AMC 10A 2005/20)

Example 3.89

A regular octagon $ABCDEFGH$ has sides of length two. Find the area of $\triangle ADG$. (AMC 10B 2002/17)

Example 3.90

A regular octagon $ABCDEFGH$ has an area of one square unit. What is the area of the rectangle $ABEF$? (AMC 10B 2003/23)



J. General Octagons

Example 3.91

In rectangle $PQRS$, $PQ = 8$ and $QR = 6$. Points A and B lie on \overline{PQ} , points C and D lie on \overline{QR} , points E and F lie on \overline{RS} , and points G and H lie on \overline{SP} so that $AP = BQ < 4$ and the convex octagon $ABCDEFGH$ is equilateral. The length of a side of this octagon can be expressed in the form $k + m\sqrt{n}$, where k , m , and n are integers and n is not divisible by the square of any prime. What is $k + m + n$? (AMC 10B 2018/17)

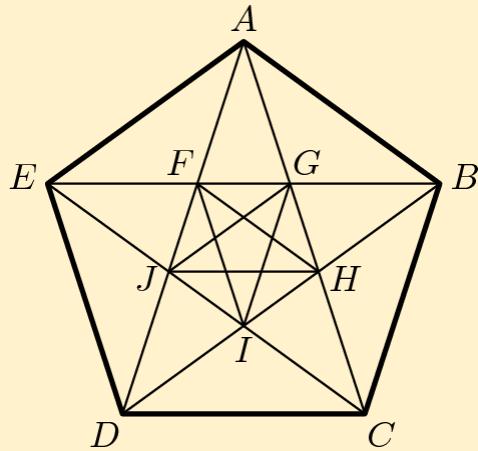
K. Pentagons

Example 3.92

All sides of the convex pentagon $ABCDE$ are of equal length, and $\angle A = \angle B = 90^\circ$. What is the degree measure of $\angle E$? (AMC 10B 2007/7)

Example 3.93

In the figure shown below, $ABCDE$ is a regular pentagon and $AG = 1$. What is $FG + JH + CD$? (AMC 10B 2015/22)



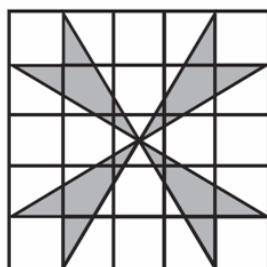
Example 3.94

The plane is tiled by congruent squares and congruent pentagons as indicated. The percent of the plane that is enclosed by the pentagons is closest to (AMC 10 2001/18)

L. Others

Example 3.95

What is the area of the shaded part shown in the 5×5 grid? (AMC 8 2007/23)

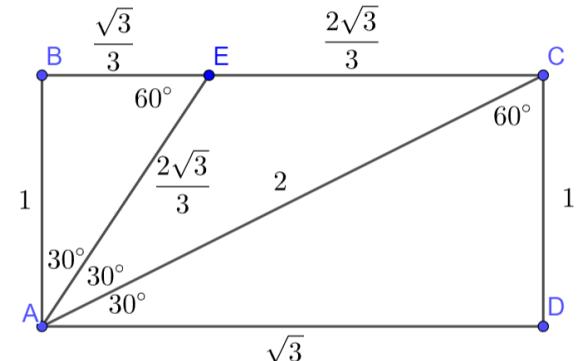
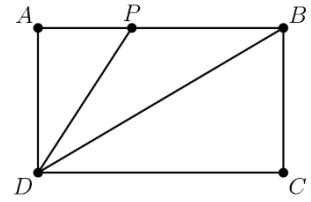


4. AMC QUESTIONS

4.1 AMC Questions-I

Example 4.1

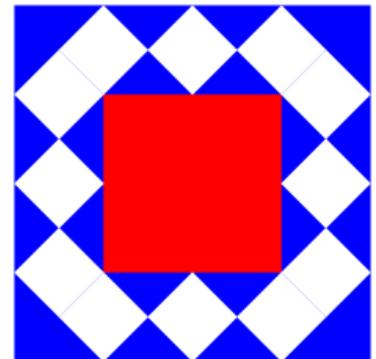
In rectangle $ABCD$, $AD = 1$, P is on \overline{AB} , and \overline{DB} and \overline{DP} trisect $\angle ADC$. What is the perimeter of $\triangle BDP$? (AMC 10 2000/7)



Example 4.2

Betsy designed a flag using blue triangles, small white squares, and a red center square, as shown. Let B be the total area of the blue triangles, W the total area of the white squares, and R the area of the red square. Which of the following is correct? (AMC 10A 2002/8)

- A. $B = W$
- B. $W = P$
- C. $B = P$
- D. $3B = 2P$
- E. $2P = W$

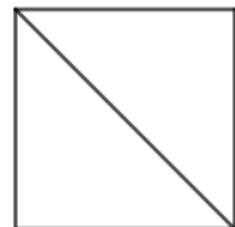


Notice that the diagonal of a square divides it into two congruent triangles.

$$1 \text{ Blue Triangle} = \frac{1}{2} \text{ White Square}$$

Counting gives us:

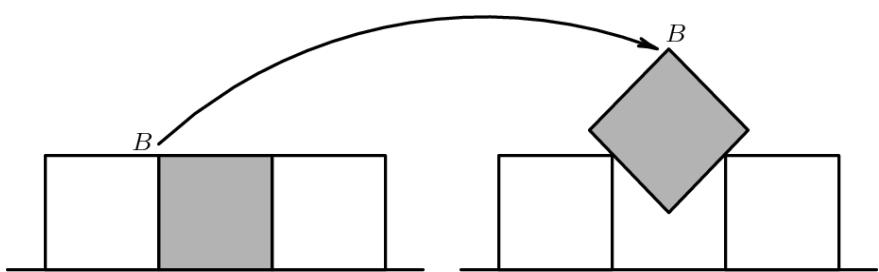
$$\begin{aligned} 24 \text{ Blue Triangles} &= 24 \Rightarrow 12 \text{ Blue Squares} \\ &12 \text{ White Squares} \end{aligned}$$



Option A

Example 4.3

Three one-inch squares are placed with their bases on a line. The center square is lifted out and rotated 45 degrees, as shown. Then it is centered and lowered into its original location until it touches both of the adjoining squares. How many inches is the point B from the line



on which the bases of the original squares were placed? (AMC 10A 2005/19)

Since the figure is symmetrical:

$$OB = OE$$

Hence, $\triangle OBE$ is isosceles:

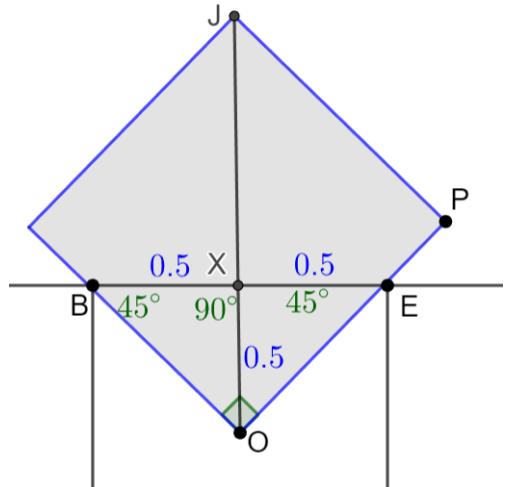
$$\angle OBX = \angle OEX = 45^\circ$$

Drop a perpendicular from O to BE , and note that OX lies on diagonal OJ :

$$XB = XE = XO = \frac{1}{2}$$

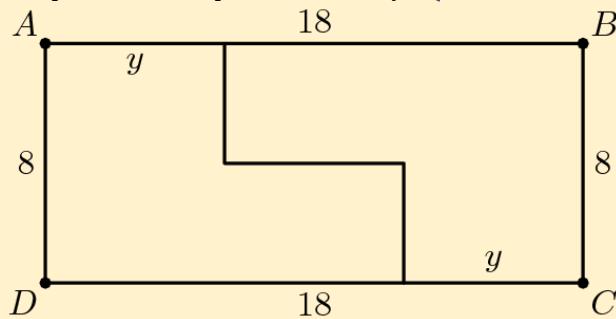
$$JX = JO - XO = \sqrt{2} - \frac{1}{2}$$

$$BC + JX = 1 + \left(\sqrt{2} - \frac{1}{2}\right) = \sqrt{2} + \frac{1}{2}$$



Example 4.4

The 8×18 rectangle $ABCD$ is cut into two congruent hexagons, as shown, in such a way that the two hexagons can be repositioned without overlap to form a square. What is y ? (AMC 10A 2006/7)



Example 4.5

Square $EFGH$ has one vertex on each side of square $ABCD$. Point E is on \overline{AB} with $AE = 7 \cdot EB$. What is the ratio of the area of $EFGH$ to the area of $ABCD$? (AMC 10A 2011/11)

Assume numbers instead of variables (since only want a ratio). Let

$$AE = 7 \Rightarrow EB = 1$$

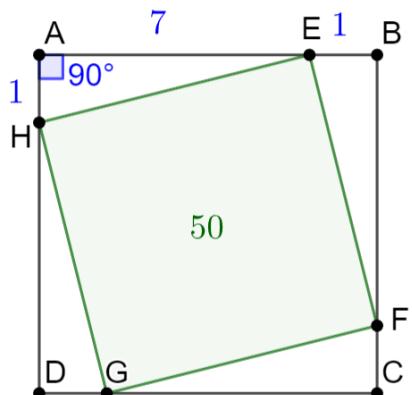
Note that because $EFGH$ is a square, the diagram is symmetrical, and hence the area outside $EFGH$ can be divided into four congruent triangles.

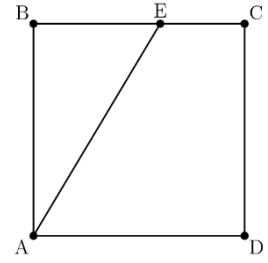
The ratio that we want is:

$$\frac{[EFGH]}{[ABCD]} = \frac{EH^2}{AB^2}$$

Using Pythagoras Theorem in right $\triangle EAH$, $EH^2 = AE^2 + AH^2$:

$$= \frac{AE^2 + AH^2}{AB^2} = \frac{7^2 + 1^2}{8^2} = \frac{50}{64} = \frac{25}{32}$$





Example 4.6

Square $ABCD$ has side length 10. Point E is on \overline{BC} , and the area of $\triangle ABE$ is 40. What is BE ? (AMC 10A 2013/3)

$$[ABE] = \frac{1}{2}hb = \frac{1}{2}(AB)(BE) = \frac{1}{2}(10)(BE) = 5BE = 40 \Rightarrow BE = 8$$

Example 4.7

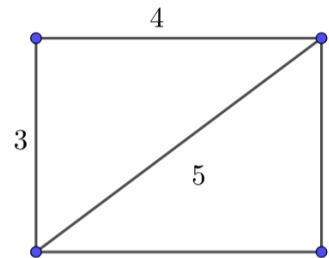
The ratio of the length to the width of a rectangle is 4:3. If the rectangle has diagonal of length d , then the area may be expressed as kd^2 for some constant k . What is k ? (AMC 10A 2015/11)

Using a Pythagorean Triplet the ratio *Width:Length:Diagonal* is:

$$3:4:5 = \frac{3}{5}:\frac{4}{5}:1 = \frac{3}{5}d:\frac{4}{5}d:d$$

Then the area is:

$$\left(\frac{3}{5}d\right)\left(\frac{4}{5}d\right) = \frac{12}{25}d^2 \Rightarrow k = \frac{12}{25}$$



Example 4.8

In rectangle $ABCD$, $AB = 20$ and $BC = 10$. Let E be a point on \overline{CD} such that $\angle CBE = 15^\circ$. What is AE ? (AMC 10A 2014/22)

Reflect rectangle $ABCD$ across CD to get rectangle $DCGF$.

EB gets reflected to EG

AE gets reflected to FE

Construct equilateral triangle HGE such that H lies inside rectangle $DCGF$.

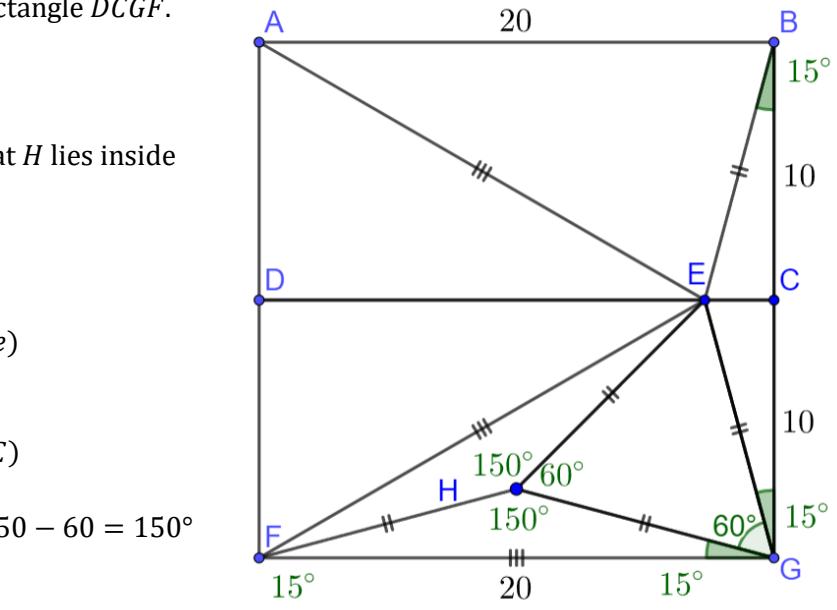
In $\triangle GHF$ and $\triangle GEB$:

- $GF = GB = 20$
- $\angle FGB = \angle BGE = 15^\circ$ (Angle)
- $GH = GE$
- $\triangle GHF \cong \triangle GEB$ (SAS)
- $\angle HFG = \angle EBG = 15^\circ$ (CPCTC)

$$\angle FHE = 360 - \angle FHG - \angle EHG = 360 - 150 - 60 = 150^\circ$$

In $\triangle FHE$ and $\triangle FHG$:

$$\angle FHE = \angle FHG = 150^\circ$$

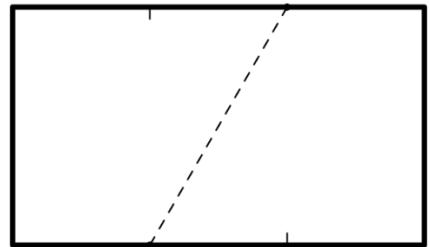


- $HF = HG$
- $HE = HG$
- $\triangle FHE \cong \triangle FHG$ (SAS)
- $FE = FG = 20$ (CPCTC)

$$AE = FE = 20(\text{Reflection})$$

Example 4.9

A rectangular piece of paper whose length is $\sqrt{3}$ times the width has area A . The paper is divided into three equal sections along the opposite lengths, and then a dotted line is drawn from the first divider to the second divider on the opposite side as shown. The paper is then folded flat along this dotted line to create a new shape with area B . What is the ratio $B:A$? (AMC 10A 2014/23)



Example 4.10

A rectangle with positive integer side lengths in cm has area A cm^2 and perimeter P cm. Which of the following numbers cannot equal $A + P$? (AMC 10A 2015/20)

- A. 100
- B. 102
- C. 104
- D. 106
- E. 108

Let $w = \text{width}, l = \text{length}$:

$$\begin{aligned} A + P &= wl + 2(w + l) = wl + 2w + 2l \\ &= wl + 2w + 2l + 4 - 4 \\ &= w(l + 2) + 2(l + 2) - 4 \\ &= (l + 2)(w + 2) - 4 \end{aligned}$$

$$\text{Option A: } (l + 2)(w + 2) = 100 + 4 = 104 = (4)(26) = (2 + 2)(24 + 2) \Rightarrow l = 24, w = 2$$

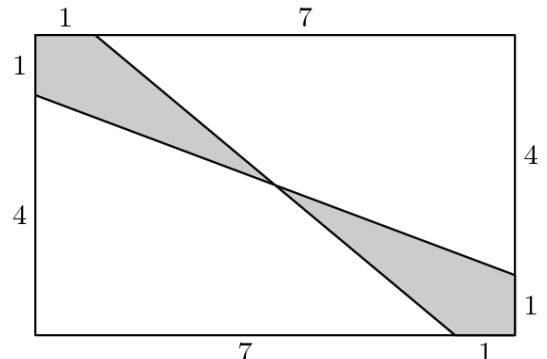
$$\text{Option B: } (l + 2)(w + 2) = 102 + 4 = 106 = (1)(106) = 2(53) \Rightarrow \text{Not Valid} \Rightarrow \text{Correct Option}$$

We can check that the remaining options create valid rectangles:

$$(l + 2)(w + 2) = 104 + 4 = 108 = (3)(36) = (1 + 2)(34 + 2) \Rightarrow l = 34, w = 1$$

$$(l + 2)(w + 2) = 106 + 4 = 110 = (5)(22) = (3 + 2)(19 + 2) \Rightarrow l = 19, w = 3$$

$$(l + 2)(w + 2) = 108 + 4 = 112 = (4)(28) = (2 + 2)(26 + 2) \Rightarrow l = 2, w = 26$$



Example 4.11

Find the area of the shaded region. (AMC 10A 2016/11)

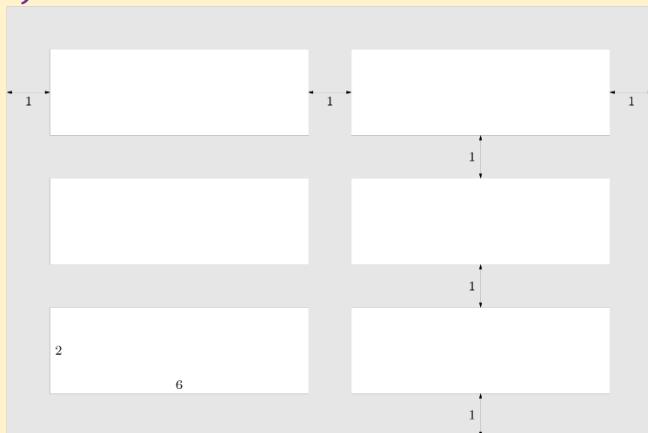
Example 4.12

In rectangle $ABCD$, $AB = 6$ and $BC = 3$. Point E between B and C , and point F between E and C are such that $BE = EF = FC$. Segments \overline{AE} and \overline{AF} intersect \overline{BD} at P and Q , respectively. The ratio $BP:PQ:QD$ can be written as $r:s:t$ where the greatest common factor of r, s and t is 1. What is $r + s + t$?

(AMC 10A 2016/19)

Example 4.13

Tamara has three rows of two 6-feet by 2-feet flower beds in her garden. The beds are separated and also surrounded by 1-foot-wide walkways, as shown on the diagram. What is the total area of the walkways, in square feet? (AMC 10A 2017/3)

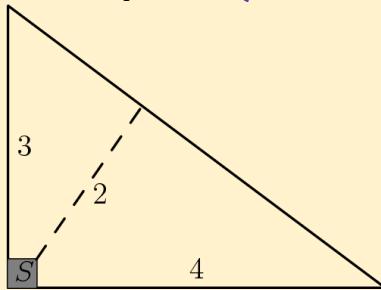


Example 4.14

A square with side length x is inscribed in a right triangle with sides of length 3, 4, and 5 so that one vertex of the square coincides with the right-angle vertex of the triangle. A square with side length y is inscribed in another right triangle with sides of length 3, 4, and 5 so that one side of the square lies on the hypotenuse of the triangle. What is $\frac{x}{y}$? (AMC 10A 2017/21)

Example 4.15

Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted? (AMC 10A 2018/23)



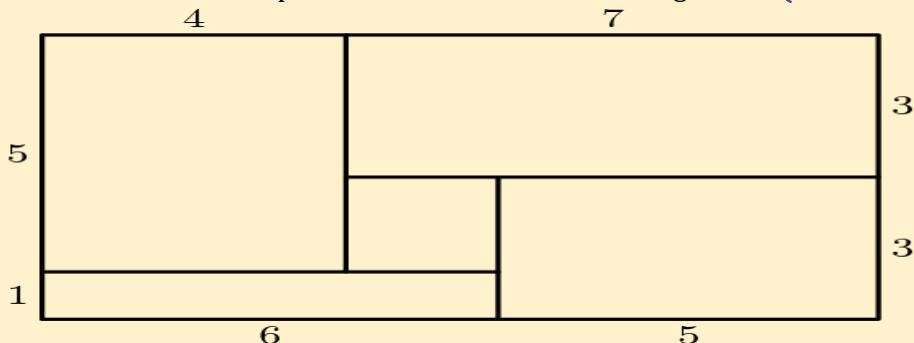
Example 4.16

Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of \overline{AB} , and let E be the midpoint of \overline{AC} . The angle bisector of $\angle BAC$ intersects \overline{DE} and \overline{BC} at F and G , respectively. What is the area of quadrilateral $FDBG$? (AMC 10A 2018/24)

Example 4.17

Rose fills each of the rectangular regions of her rectangular flower bed with a different type of flower. The lengths, in feet, of the rectangular regions in her flower bed are as shown in the figure. She plants one flower per square foot in each region. Aster cost \$1 each, begonias \$1.50 each, canna \$2 each, dahlia \$2.50 each, and

Easter lilies \$3 each. What is the least possible cost, in dollars, for her garden? (AMC 10B 2003/4)



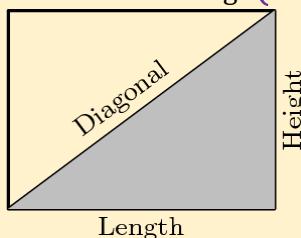
Example 4.18

Moe uses a mower to cut his rectangular 90-foot by 150-foot lawn. The swath he cuts is 28 inches wide, but he overlaps each cut by 4 inches to make sure that no grass is missed. He walks at the rate of 5000 feet per hour while pushing the mower. Which of the following is closest to the number of hours it will take Moe to mow the lawn? (AMC 10B 2003/5)

- (A) 0.75 (B) 0.8 (C) 1.35 (D) 1.5 (E) 3

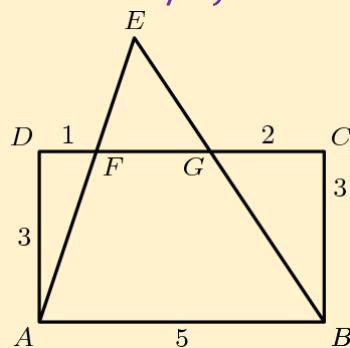
Example 4.19

Many television screens are rectangles that are measured by the length of their diagonals. The ratio of the horizontal length to the height in a standard television screen is 4 : 3. The horizontal length of a "27-inch" television screen is closest, in inches, to which of the following? (AMC 10B 2003/6)



Example 4.20

In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on \overline{CD} so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E . Find the area of $\triangle AEB$. (AMC 10B 2003/20)



Example 4.21

A 2×3 rectangle and a 3×4 rectangle are contained within a square without overlapping at any point, and the sides of the square are parallel to the sides of the two given rectangles. What is the smallest possible area of the square? (AMC 10B 2006/5)

Example 4.22

Right $\triangle ABC$ has $AB = 3$, $BC = 4$, and $AC = 5$. Square $XYZW$ is inscribed in $\triangle ABC$ with X and Y on \overline{AC} , W on \overline{AB} , and Z on \overline{BC} . What is the side length of the square? (AMC 10B 2007/21)

Example 4.23

Rectangle $ABCD$ has $AB = 8$ and $BC = 6$. Point M is the midpoint of diagonal \overline{AC} , and E is on AB with $\overline{ME} \perp \overline{AC}$. What is the area of $\triangle ABE$? (AMC 10B 2009/18)

Example 4.24

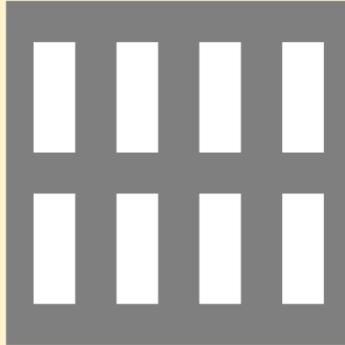
Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on side AB so that $\angle AMD = \angle CMD$. What is the degree measure of $\angle AMD$? (AMC 10B 2011/18)

Example 4.25

Mr. Green measures his rectangular garden by walking two of the sides and finding that it is 15 steps by 20 steps. Each of Mr. Green's steps is 2 feet long. Mr. Green expects a half a pound of potatoes per square foot from his garden. How many pounds of potatoes does Mr. Green expect from his garden? (AMC 10B 2013/2)

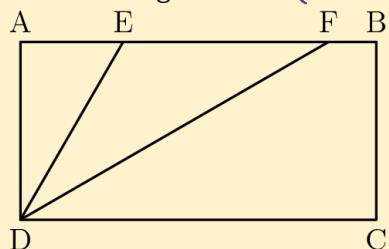
Example 4.26

Camden constructs a square window using 8 equal-size panes of glass, as shown. The ratio of the height to width for each pane is 5 : 2, and the borders around and between the panes are 2 inches wide. In inches, what is the side length of the square window? (AMC 10B 2014/5)



Example 4.27

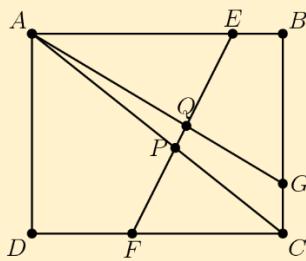
In rectangle $ABCD$, $DC = 2 \cdot CB$ and points E and F lie on \overline{AB} so that \overline{ED} and \overline{FD} trisect $\angle ADC$ as shown. What is the ratio of the area of $\triangle DEF$ to the area of rectangle $ABCD$? (AMC 10B 2014/15)



Example 4.28

Rectangle $ABCD$ has $AB = 5$ and $BC = 4$. Point E lies on \overline{AB} so that $EB = 1$, point G lies on \overline{BC} so that $CG = 1$, and point F lies on \overline{CD} so that $DF = 2$. Segments \overline{AG} and \overline{AC} intersect \overline{EF} at Q and P , respectively. What is the

value of $\frac{PQ}{EF}$? (AMC 10B 2016/19)

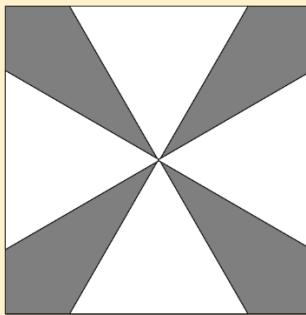


Example 4.29

Rectangle $ABCD$ has $AB = 3$ and $BC = 4$. Point E is the foot of the perpendicular from B to diagonal \overline{AC} . What is the area of $\triangle AED$? (AMC 10B 2017/15)

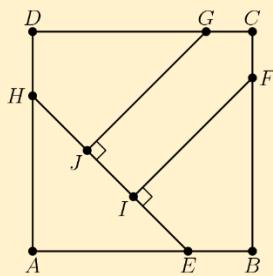
Example 4.30

The figure below shows a square and four equilateral triangles, with each triangle having a side length 2 and the third vertices of the triangles meet at the center of the square. The region inside the square but outside the triangles is shaded. What is the area of the shaded region? (AMC 10B 2019/8)



Example 4.31

In square $ABCD$, points E and H lie on \overline{AB} and \overline{DA} , respectively, so that $AE = AH$. Points F and G lie on \overline{BC} and \overline{CD} , respectively, and points I and J lie on \overline{EH} so that $\overline{FI} \perp \overline{EH}$ and $\overline{GJ} \perp \overline{EH}$. See the figure below. Triangle AEH , quadrilateral $BFIE$, quadrilateral $DHJG$, and pentagon $FCGJI$ each has area 1. What is FI^2 ? (AMC 10B 2020/21)



Example 4.32

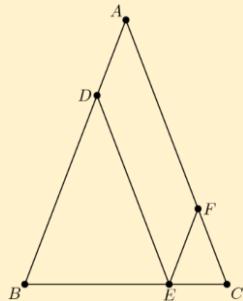
In ABC we have $AB = 25$, $BC = 39$, and $AC = 42$. Points D and E are on AB and AC respectively, with $AD = 19$ and $AE = 14$. What is the ratio of the area of triangle ADE to the area of the quadrilateral $BCED$? (AMC 10A 2005/25)

Example 4.33

Convex quadrilateral $ABCD$ has $AB = 9$ and $CD = 12$. Diagonals AC and BD intersect at E , $AC = 14$, and $\triangle AED$ and $\triangle BEC$ have equal areas. What is AE ? (AMC 10A 2009/23)

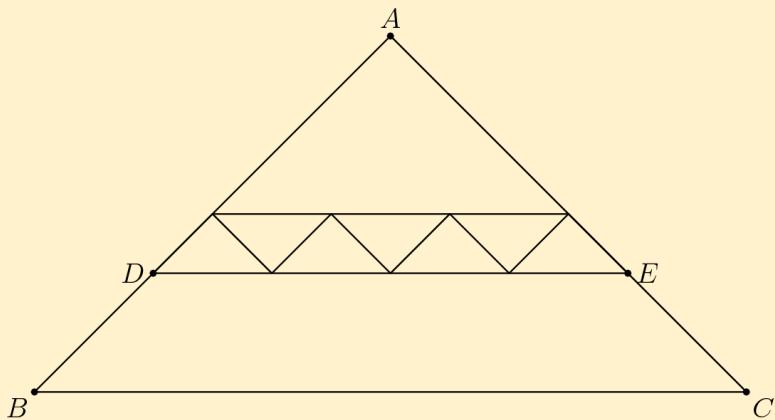
Example 4.34

In $\triangle ABC$, $AB = AC = 28$ and $BC = 20$. Points D , E , and F are on sides \overline{AB} , \overline{BC} , and \overline{AC} , respectively, such that \overline{DE} and \overline{EF} are parallel to \overline{AC} and \overline{AB} , respectively. What is the perimeter of parallelogram $ADEF$? (AMC 10A 2013/12)



Example 4.35

All of the triangles in the diagram below are similar to isosceles triangle ABC , in which $AB = AC$. Each of the 7 smallest triangles has area 1, and $\triangle ABC$ has area 40. What is the area of trapezoid $DBCE$? (AMC 10A 2018/9)



Example 4.36

For how many of the following types of quadrilaterals does there exist a point in the plane of the quadrilateral that is equidistant from all four vertices of the quadrilateral? (AMC 10A 2019/6)

- A square
- A rectangle that is not a square
- A rhombus that is not a square
- A parallelogram that is not a rectangle or a rhombus
- An isosceles trapezoid that is not a parallelogram

Example 4.37

Quadrilateral $ABCD$ satisfies $\angle ABC = \angle ACD = 90^\circ$, $AC = 20$, and $CD = 30$. Diagonals \overline{AC} and \overline{BD} intersect at point E , and $AE = 5$. What is the area of quadrilateral $ABCD$? (AMC 10A 2020/20)

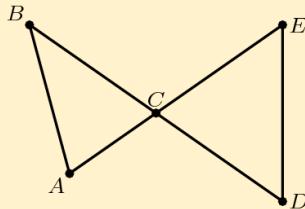
Example 4.38

Quadrilateral $ABCD$ has $AB = BC = CD$, angle $ABC = 70$ and angle $BCD = 170$. What is the measure of angle

BAD?(AMC 10B 2008/24)

Example 4.39

Segment BD and AE intersect at C , as shown, $AB = BC = CD = CE$, and $\angle A = \frac{5}{2}\angle B$. What is the degree measure of $\angle D$? (AMC 10B 2009/9)



Example 4.40

Rhombus $ABCD$ has side length 2 and $\angle B = 120^\circ$. Region R consists of all points inside the rhombus that are closer to vertex B than any of the other three vertices. What is the area of R ? (AMC 10B 2011/20)

Example 4.41

Two equilateral triangles are contained in a square whose side length is $2\sqrt{3}$. The bases of these triangles are the opposite sides of the square, and their intersection is a rhombus. What is the area of the rhombus? (AMC 10B 2012/14)

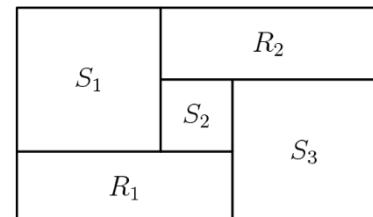
Example 4.42

Trapezoid $ABCD$ has parallel sides \overline{AB} of length 33 and \overline{CD} of length 21. The other two sides are of lengths 10 and 14. The angles at A and B are acute. What is the length of the shorter diagonal of $ABCD$? (AMC 10B 2014/21)

4.2 AMC Questions-II

Example 4.43

Rectangles R_1 and R_2 , and squares S_1 , S_2 , and S_3 , shown below, combine to form a rectangle that is 3322 units wide and 2020 units high. What is the side length of S_2 in units? (AMC 8 2020/25)



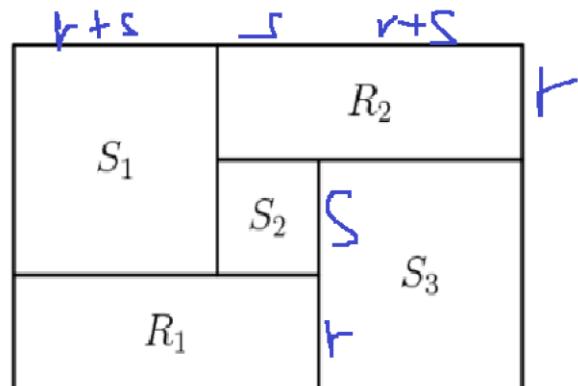
The side length of squares 1 and 2 is not given. Similarly, the dimensions of rectangles 1 and 2 are also not given.

But, if the question is correct, the diagram should work even when the squares have the same dimensions, and the rectangles have the same dimensions.

$$r + s + r = 2020, 2r + s = 2020$$

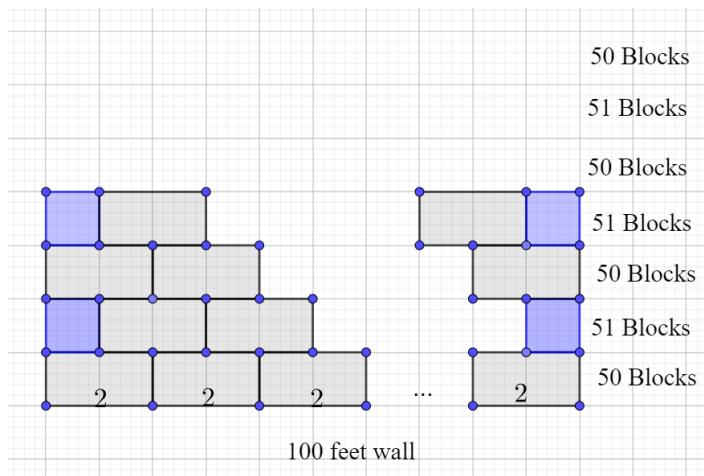
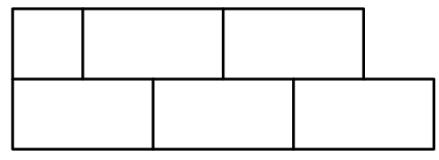
$$(r + s) + s + (r + s) = 3322 \\ 2r + s + 2s = 3322 \\ 2020 + 2s = 3322$$

$$2s = 1302 \\ s = 651$$



Example 4.44

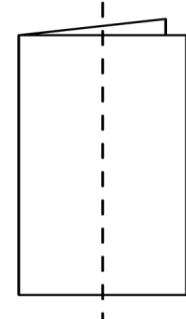
A block wall 100 feet long and 7 feet high will be constructed using blocks that are 1 foot high and either 2 feet long or 1 foot long (no blocks may be cut). The vertical joins in the blocks must be staggered as shown, and the wall must be even on the ends. What is the smallest number of blocks needed to build this wall? (AMC 8 2000/12)



$$50 \times 4 + 51 \times 3 = 200 + 153 = 353 \text{ Blocks}$$

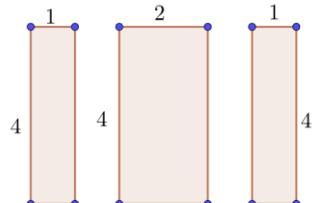
Example 4.45

A square piece of paper, 4 inches on a side, is folded in half vertically. Both layers are then cut in half parallel to the fold. Three new rectangles are formed, a large one and two small ones. What is the ratio of the perimeter of one of the small rectangles to the perimeter of the large rectangle? (AMC 8 2001/16)



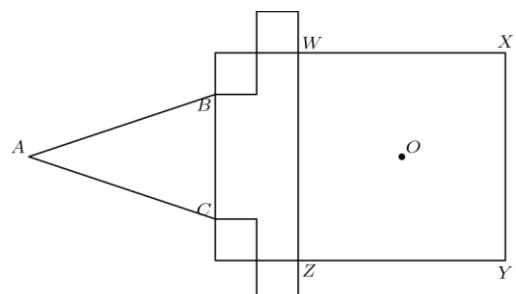
The ratio is

$$\frac{2(4+1)}{2(4+2)} = \frac{5}{6}$$

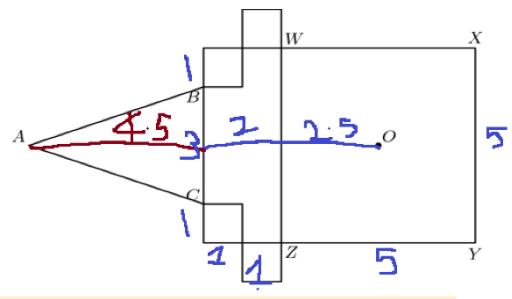


Example 4.46

In the figure, the area of square $WXYZ$ is 25 cm^2 . The four smaller squares have sides 1 cm long, either parallel to or coinciding with the sides of the large square. In $\triangle ABC$, $AB = AC$, and when $\triangle ABC$ is folded over side \overline{BC} , point A coincides with O , the center of square $WXYZ$. What is the area of $\triangle ABC$, in square centimeters? (AMC 8 2003/25)

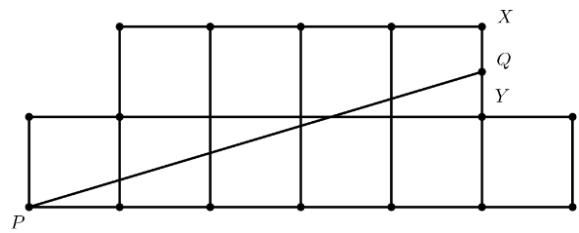


$$\left(\frac{1}{2}\right)\left(\frac{9}{2}\right)(3) = \frac{27}{4} = 6.75$$



Example 4.47

The diagram shows an octagon consisting of 10 unit squares. The portion below \overline{PQ} is a unit square and a triangle with base 5. If \overline{PQ} bisects the area of the octagon, what is the ratio $\frac{XQ}{QY}$? (AMC 8 2010/17)

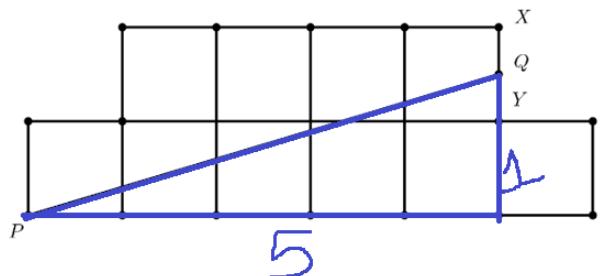


Area of the triangle

$$= 5 - 1 = 4$$

Then, the area of the blue triangle:

$$\begin{aligned} \frac{1}{2}hb &= \left(\frac{1}{2}\right)(1 + QY)(5) = 4 \Rightarrow 1 + QY = \frac{8}{5} \Rightarrow QY = \frac{3}{5} \\ XQ &= 1 - QY = 1 - \frac{3}{5} = \frac{2}{5} \end{aligned}$$



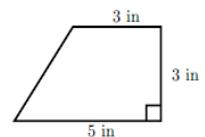
$$\frac{XQ}{QY} = \frac{\frac{2}{5}}{\frac{3}{5}} = \frac{2}{3}$$

Example 4.48

Four friends, Art, Roger, Paul and Trisha, bake cookies, and all cookies have the same thickness. The shapes of the cookies differ, as shown. Each friend uses the same amount of dough, and Art makes exactly 12 cookies.

- A. Who gets the fewest cookies from one batch of cookie dough? (AMC 8 2003/8)
- B. Art's cookies sell for 60 cents each. To earn the same amount from a single batch, how much should one of Roger's cookies cost in cents? (AMC 8 2003/9)
- C. How many cookies will be in one batch of Trisha's cookies? (AMC 8 2003/10)

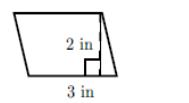
- Art's cookies are trapezoids:



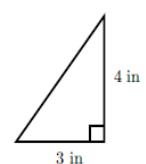
- Roger's cookies are rectangles:



- Paul's cookies are parallelograms:



- Trisha's cookies are triangles:



Part A

$$\begin{aligned} Art: \frac{3+5}{2} \cdot 3 &= \frac{8}{2} \cdot 3 = 12 \text{ in}^2 \\ Roger: 2(4) &= 8 \text{ in}^2 \\ Paul: 2(3) &= 6 \text{ in}^2 \\ Trisha: \frac{1}{2}(3)(2) &= 6 \text{ in}^2 \end{aligned}$$

Part B

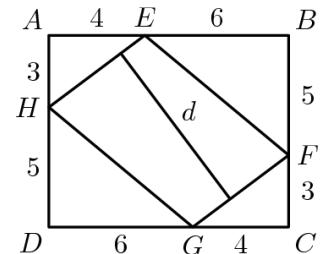
$$60 \cdot \frac{8}{12} = 40 \text{ cents}$$

Part C

$$12 \times \frac{12}{6} = 24 \text{ cookies}$$

Example 4.49

In the figure, $ABCD$ is a rectangle and $EFGH$ is a parallelogram. Using the measurements given in the figure, what is the length d of the segment that is perpendicular to \overline{HE} and \overline{FG} ? (AMC 8 2004/24)



Area of parallelogram $EFGH$

$$\begin{aligned} [ABCD] - [AEH] - [FCG] - [EBF] - [GDH] &= [ABCD] - 2[AEH] - 2[EBF] \\ &= (10)(8) - 2 \cdot \frac{1}{2} \cdot 3 \cdot 4 - 2 \cdot \frac{1}{2} \cdot 5 \cdot 6 = 80 - 12 - 30 = 38 \end{aligned}$$

Area of parallelogram $EFGH$

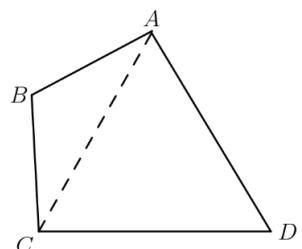
$$= bh = (FG)(d) = 5d = 38 \Rightarrow d = \frac{38}{5}$$

By Pythagorean Triplet (3,4,5)

$$FG = 5$$

Example 4.50

In quadrilateral $ABCD$, sides \overline{AB} and \overline{BC} both have length 10, sides \overline{CD} and \overline{DA} both have length 17, and the measure of angle ADC is 60° . What is the length of diagonal \overline{AC} ? (AMC 8 2005/9)

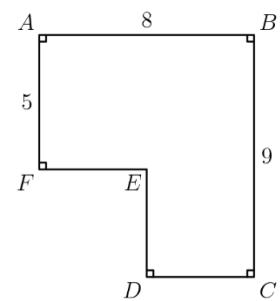


In Isosceles $\triangle DAC$

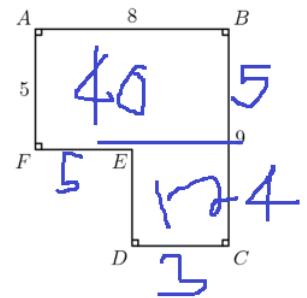
$$\begin{aligned} \angle DAC = \angle DCA &= \frac{180 - 60}{2} = \frac{120}{2} = 60^\circ \\ \triangle DAC \text{ is equilateral} &\Rightarrow AC = 17 \end{aligned}$$

Example 4.51

The area of polygon $ABCDEF$ is 52 with $AB = 8$, $BC = 9$ and $FA = 5$. What is $DE + EF$? (AMC 8 2005/13)

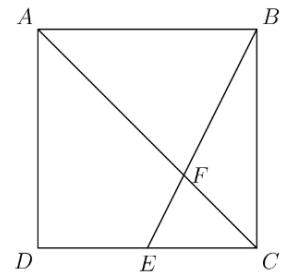


$$DE + EF = 4 + 5 = 9$$



Example 4.52

Point E is the midpoint of side \overline{CD} in square $ABCD$, and \overline{BE} meets diagonal \overline{AC} at F . The area of quadrilateral $AFED$ is 45. What is the area of $ABCD$? (AMC 8 2018/22)



In $\triangle ABF$ and $\triangle CEF$

$$\angle BAC = \angle ACE \text{ (Alternate Interior Angles)}$$

$$\angle BEC = \angle ABE \text{ (Alternate Interior Angles)}$$

$$\triangle ABF \sim \triangle CEF \text{ (AA Similarity)}$$

Draw $FX \perp DC$

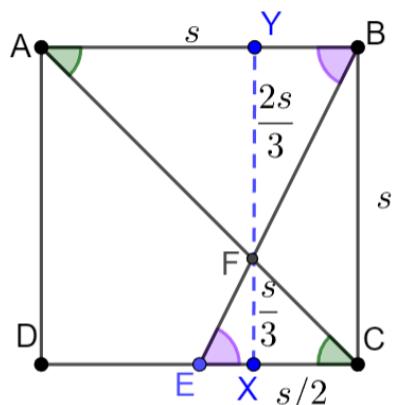
$$\frac{FX}{FY} = \frac{AB}{EC} = \frac{1}{2} \Rightarrow 2FX = FY$$

$$FX + FY = FX + 2FX = s \Rightarrow FX = \frac{s}{3}$$

$$[CEF] = \frac{1}{2}hb = \frac{1}{2}\left(\frac{s}{3}\right)\left(\frac{s}{2}\right) = \frac{s^2}{12}$$

$$[AFED] = [ABCD] - [ABC] - [FEC] = s^2 - \frac{s^2}{2} - \frac{s^2}{12} = \frac{5s^2}{12} = 45$$

$$\frac{5s^2}{12} = 45 \Rightarrow s^2 = 45 \cdot \frac{12}{5} = 108$$



4.3 Further Topics

53 Examples