Multi Variable Calculus

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Aziz Manva

azizmanva@gmail.com

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1. Differentiation

1.1 Limits

1.1: Limit along all paths
For the limit to exist, it must be the same along all paths.

Example 1.2

$$\lim_{(x,y)\to(0,0)} \frac{y}{x}$$

1.2 Partial Derivatives

A. Functions

In single variable calculus, we usually consider y as a function of x,

$$y = f(x)$$

and plot y and x on the coordinate plane.

1.3: Functions of more than one Variable
If f is a function of the variables x and y we write

$$z = f(x, y)$$

In multivariable calculus, we will consider functions of more than one variable.
To plot this, we will consider the coordinate space in three dimensions (where $z = height$)

$$(x, y, z)$$

B. First Order Derivatives

In multivariable calculus, there is less importance on limits and differentiability (unless it is a proof-based course).

1.4: Partial Derivatives
Given a function $a = f(x, y, z)$

$$Partial\ derivative\ of\ f\ with\ respect\ to\ x = \frac{\partial f}{\partial x} = \frac{\partial a}{\partial x} = f_x$$

$$Partial\ derivative\ of\ f\ with\ respect\ to\ y = \frac{\partial f}{\partial y} = \frac{\partial a}{\partial y} = f_y$$

$$Partial\ derivative\ of\ f\ with\ respect\ to\ z = \frac{\partial f}{\partial z} = \frac{\partial a}{\partial z} = f_z$$

Note that the derivative of $y = f(x)$ with respect to x is written

$$\frac{dy}{dx}$$

Here we use the symbol

$$\partial\ instead\ of\ d$$

To indicate the partial derivative instead of the regular derivative.

1.5: Calculating Partial Derivatives

When calculating a partial derivative with respect to a variable, treat other variables as constants.

Example 1.6

The area of a rectangle as a function of its length(l), width(w) is given by:

$$Area = A = f(l, w) = lw$$

Find

A. f_l
B. f_w

Part A

To find the partial derivative with respective to l, assume that w is a constant:

$$\frac{\partial A}{\partial l} = 1 \cdot w = w$$

Part B

To find the partial derivative with respective to w, assume that l is a constant:

$$\frac{\partial A}{\partial w} = l \cdot 1 = l$$

Example 1.7

The volume of a cuboid as a function of its length(l), width(w) and height(h) is given by

$$Volume = V = f(l, w, h) = lwh$$

Find:

A. f_l
B. f_w
C. f_h

$$\frac{\partial V}{\partial l} = wh$$
$$\frac{\partial V}{\partial w} = lh$$
$$\frac{\partial V}{\partial h} = lw$$

Example 1.8

Given $f(x, y) = x + y + xy$, find:

A. f_x
B. f_y

$$\frac{\partial f}{\partial x} = 1 + 0 + y = 1 + y$$
$$\frac{\partial f}{\partial y} = 0 + 1 + x = 1 + x$$

1.9: Chain Rule Basics

Intuitively, the chain rule from single rule calculus carries over to multivariable calculus.

$$\frac{d}{dt}\sqrt{t} = \frac{1}{2\sqrt{t}}$$

Example 1.10

$$f(x, y) = \sqrt{x^2 + y^3}$$

A. f_x
B. f_y

Use the rule $\frac{\partial}{\partial x}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}} \times f'(x)$:

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x^2 + y^3}} \times 2x = \frac{x}{\sqrt{x^2 + y^3}}$$
$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x^2 + y^3}} \times 3y^2 = \frac{3y^2}{2\sqrt{x^2 + y^3}}$$

Example 1.11

The function f is defined as $f(x, y) = e^{2x+\frac{y}{2}}$. Prove that $\frac{\partial f}{\partial x} = 4\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = \left(e^{2x+\frac{y}{2}}\right)(2) = 2\left(e^{2x+\frac{y}{2}}\right) = 2f$$
$$\frac{\partial f}{\partial y} = \left(e^{2x+\frac{y}{2}}\right)\left(\frac{1}{2}\right) = \frac{1}{2}\left(e^{2x+\frac{y}{2}}\right) = \frac{1}{2}f$$

$$RHS = 4\frac{\partial f}{\partial y} = 4\left[\frac{1}{2}f\right] = 2f = \frac{\partial f}{\partial x} = LHS$$

Example 1.12

Find f_x and f_y

$$f(x, y) = e^{xy}$$

$$f_x = y \cdot e^{xy}$$
$$f_y = x \cdot e^{xy}$$

Example 1.13

Find f_x and f_y:

$$f(x, y) = e^{xy}x^2y^3$$

Apply the product rule:

$$f_x = y^3e^y[ye^xx^2 + e^x(2x)] = e^{xy}xy^3[yx + 2]$$
$$f_y = x^2[xe^{xy}y^3 + 3e^{xy}y^2] = x^2y^2e^{xy}[xy + 3]$$

Example 1.14

Find f_x and f_y if:

$$f(x, y) = \frac{e^x}{e^y}$$

Since we are finding the partial derivative with respective to x, we assume that y is a constant:

$$f_x = \left(\frac{1}{e^y}\right)(e^x) = \frac{e^x}{e^y}$$

$$f_y = e^x e^{-y} = -e^x e^{-y} = -\frac{e^x}{e^y}$$

Example 1.15

Find f_x if:

$$f(x,y) = \ln_{4x-5y}(2x+3y)$$

Use the quotient rule of logarithms to write:

$$f(x,y) = \frac{\ln(2x+3y)}{\ln(4x-5y)}$$

Use the quotient rule of differentiation $\left(\frac{f}{g}\right)' = \frac{gf'-g'f}{g^2}$:

$$f_x = \frac{\ln(4x-5y)\left(\frac{2}{2x+3y}\right) - \left(\frac{4}{4x-5y}\right)\ln(2x+3y)}{[\ln(4x-5y)]^2}$$

Simplify:

$$= \frac{\left(\frac{2\ln(4x-5y)}{2x+3y}\right) - \left(\frac{4\ln(2x+3y)}{4x-5y}\right)}{[\ln(4x-5y)]^2}$$

Add the fractions by making the denominators common:

$$= \frac{\left(\frac{2\ln(4x-5y)(4x-5y)}{(2x+3y)(4x-5y)}\right) - \left(\frac{4\ln(2x+3y)(2x+3y)}{(2x+3y)(4x-5y)}\right)}{[\ln(4x-5y)]^2}$$

Denest the fraction:

$$= \frac{2\ln(4x-5y)(4x-5y) - 4\ln(2x+3y)(2x+3y)}{(2x+3y)(4x-5y)[\ln(4x-5y)]^2}$$

Example 1.16

Find f_x if:

$$f(x,y,z) = x^2y - 3z^2y + x^2y^3z^4$$

$$f_x = 2xy - 0 + 2xy^3z^4$$

C. Trigonometry

Example 1.17

Find $f_x - f_y$ if:

$$f(x,y) = \sin x \cos y$$

$$f_x - f_y = \cos x \cos y - (-\sin x \sin y) = \cos x \cos y + \sin x \sin y$$

D. Economics

Example 1.18

Find partial derivatives with respect to each variable of Wilson's Lot Size Formula, given by:

$$A(c,h,k,m,q) = \frac{km}{q} + cm + \frac{hq}{2}$$

$$f_c = 0 + m + 0 = m$$
$$f_h = 0 + 0 + \frac{q}{2} = \frac{q}{2}$$
$$f_k = \frac{m}{q} + 0 + 0 = \frac{m}{q}$$
$$f_m = \frac{k}{q} + 0 + 0$$
$$f_q = -\frac{km}{q^2} + 0 + \frac{h}{2} = -\frac{km}{q^2} + \frac{h}{2}$$

1.19: Cobb Douglas Function

The Cobb Douglas function gives the output from L units of labour and K units of capital. It is defined by:

$$Output = AL^{\alpha}K^{\beta}$$

Where

$$A\ is\ a\ constant$$

It is a common constraint that we want α and β to be positive, and their sum to be 1:

$$a, \beta > 0$$
$$\alpha + \beta = 1 \Rightarrow \alpha = 1 - \beta,$$

Example 1.20

Calculate f_L and f_K of the Cobb Douglas production function given by:

$$Y = f(L,K) = AL^{x}K^{1-x}, \qquad 0 < x < 1$$

Apply the product rule:

$$f_L = AxL^{x-1}K^{1-x} = \frac{xAK^{1-x}}{L^{1-x}} = xA\left(\frac{K}{L}\right)^{1-x}$$
$$f_K = (1-x)\frac{AL^{x}}{K^{x}} = (1-x)A\left(\frac{L}{K}\right)^{x}$$

1.21: Interpretation of Derivatives of the Cobb Douglas Function

Given the Cobb Douglas function $Y = f(L,K) = AL^{x}K^{1-x},\ 0 < x < 1,$

$$f_L = Marginal\ Product\ from\ adding\ one\ more\ unit\ of\ labor$$
$$f_K = Marginal\ Product\ from\ adding\ one\ more\ unit\ of\ capital$$

E. Second Order Derivatives

1.22: Second order Derivatives

Given a function $z = f(x,y)$, we can differentiate twice instead once to get:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

1.23: "Mixed" Partial Derivatives

The mixed partial derivatives of $z = f(x, y)$ are given by differentiating once with respect to x, and once with respect to y:

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$
$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Example 1.24

What is the order of differentiation in f_{xy}?

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

First, you differentiate with respect to x, and then with respect to y.

Example 1.25

Given that $f(x, y) = \ln x \cdot \ln 2y$ find:

A. f_{xx}
B. f_{yy}
C. f_{xy}
D. f_{yx}

Find the first order derivatives:

$$f_x = \frac{\ln 2y}{x}, \qquad f_y = \frac{\ln x}{y}$$

Find the required second order derivatives:

$$f_{xx} = -\frac{\ln 2y}{x^2}$$
$$f_{yy} = -\frac{\ln x}{y^2}$$
$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\ln 2y}{x}\right) = \frac{1}{yx}$$
$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\ln x}{y}\right) = \frac{1}{yx}$$

Example 1.26

In the previous example, compare f_{xy} with f_{yx}.

$$f_{xy} = f_{yx}$$

1.27: Equality of Mixed Partial Derivatives

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b) then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

1.3 Tangent Planes and Linear Approximation

A. Tangent Line and Tangent Plane

1.28: Tangent Line

In single variable calculus, the tangent line to a function gave the direction of the function at a point. One way to define it was a combination of:

- A point on the function $(x, y) = (x, f(x))$
- The slope of the function at (x, y) given by $m = f'(x)$

1.29: Tangent Plane

The analogue to a tangent line for $y = f(x)$ from single variable calculus is a tangent plane for $z = f(x, y)$. One way to define it is a combination of:

- A point on the function $(x, y) = (x, f(x))$
- The slope of the function at (x, y) in the x direction given by $f_x(x, y)$
- The slope of the function at (x, y) in the y direction given by $f_y(x, y)$

In multivariable calculus, we have the function

$$z = f(x, y)$$

We have not just an x direction, but also a y direction.

B. Formula

1.30: Tangent Line Formula

$$y = mx + c$$

Where

$$m = slope =$$

The slope is:

$$m = \frac{x - x_1}{y - y_1}$$

We can rearrange to get the point slope form of the equation of a line:

$$y - y_1 = m(x - x_1)$$

Rearrange to get:

$$y = m(x - x_1) + f(x_1)$$

Substitute $m = slope = f'(x_1) = f_x(x_1)$

$$y = f_x(x_1)(x - x_1) + f(x_1)$$

Let $x_1 = a$:

$$y = \underbrace{f_x(a)}_{\substack{Slope\ in \\ x\ direction}} (x-a) + \underbrace{f(a)}_{Point}$$

1.31: Tangent Line Plane

$$z = \underbrace{f_x(a,b)}_{\substack{Slope\ in \\ x\ direction}} (x-a) + \underbrace{f_y(a,b)}_{\substack{Slope\ in \\ y\ direction}} (y-b) + \underbrace{f(a,b)}_{Point}$$

Since we are working in 3D instead of 2D:

$$Point\ f(a)\ is\ replaced\ by\ f(a,b)$$
$$f_x(a)\ is\ replaced\ by\ f_x(a,b)$$
$$f_y(a,b)\ is\ added\ to\ take\ care\ of\ the\ slope\ in\ the\ y\ direction$$

1.4 Gradient Vectors

A. Basics

Example 1.32

Which of the following are vectors:

A. (2,3)
B. $(2\ apples, 3\ burgers)$
C. $(3\%\ inflation\ rate, 5\%\ deficit\ rate)$
D. $(122\ Systolic\ blood\ pressure, 78\ diastolic\ blood\ pressure, 68\ heart\ rate\ in\ bpm)$

Yes

1.33: Gradient Vector

The gradient for a multivariable function $f(x,y,z)$ is the vector of partial derivatives of the function:

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

Where

$$f_x = \frac{\partial f}{\partial x}, \qquad f_y = \frac{\partial f}{\partial y}, \qquad f_z = \frac{\partial f}{\partial z}$$

Example 1.34

Find the gradient vector for the function $f(x,y) = x + y + xy$.

$$f_x = \frac{\partial f}{\partial x} = 1 + 0 + y = 1 + y$$
$$f_y = \frac{\partial f}{\partial y} = 0 + 1 + x = 1 + x$$

$$\nabla f = \langle f_x, f_y \rangle = \langle 1+y, 1+x \rangle$$

1.35: Evaluating the Gradient Vector

The gradient of a function can be evaluated at a point (x, y, z) by substituting x, y and z in the respective partial derivatives.

Example 1.36
Evaluate ∇f at the point$(1,-1)$ given that $f(x,y) = 2x^3 - 4x^2y^2$

Calculate the partial derivative:

$$\nabla f|_{(1,-1)} = f_x, f_y\big|_{(1,-1)} = 6x^2 - 8xy^2, -8x^2y|_{(1,-1)} = \nabla f(1,-1) = 6 - 8, -8|_{(1,-1)} = \langle -2, -8\rangle$$

1.37: Direction of steepest ascent
The direction of steepest ascent at a point P is given by the gradient vector.

Example 1.38

$$f(x,y) = e^{x+y+xy}$$

A. Find the direction of steepest ascent for f.
B. Evaluate ∇f when at $(-1,1)$

Part A

The direction of steepest ascent is given by the gradient vector:

$$\nabla f = \langle f_x, f_y\rangle = \langle (1+y)e^{x+y+xy}, (1+x)e^{x+y+xy}\rangle$$

Part B

$$\nabla f(-1,1) = \langle (1+1)e^{-1+1+(1)(-1)}, (1-1)e^{1+1+1}\rangle = \langle 2e^{-1}, 0\rangle$$

Example 1.39
Find the gradient of the function below at $(-2,2)$

$$f(x,y) = \frac{64}{x^2+y^2}$$

$$f(x,y) = 64(x^2+y^2)^{-1}$$

$$f_x = \frac{64(-1)}{(x^2+y^2)^2}\cdot 2x = \frac{-128x}{(x^2+y^2)^2} \Rightarrow f_x|_{x=-2} = \frac{-128(-2)}{(4+4)^2} = \frac{2^8}{2^6} = 4$$

$$f_x = \frac{64(-1)}{(x^2+y^2)^2}\cdot 2y = \frac{-128y}{(x^2+y^2)^2} \Rightarrow f_x|_{y=2} = \frac{-128(2)}{(4+4)^2} = -\frac{2^8}{2^6} = -4$$

B. Gradient on Contour Diagrams

1.40: Average Rate of Change

$$= \frac{Change}{Distance} = \frac{z_2 - z_1}{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}}$$

1.5 Optimization

A. First Derivative Test

1.41: Critical points

Critical points are points where partial derivatives of the function are all zero

$$f_x = f_y = f_z = 0$$

OR either f_x or f_y does not exist

1.42: Local Maximum: Informal

If $f(x_0, y_0)$ is greater then or equal to all the points in a disc centered at (x_0, y_0), then it is a local maximum. (You will get a peak/mountain in a graph.)

1.43: Global Maximum

If $f(x_0, y_0)$ is greater then or equal to all points in the domain of f, then it is a global maximum.

1.44: Critical Point

If $f(x_0, y_0)$ is a local maximum, or a local minimum, it is a critical point.
Converse is not true: However, if a point is a critical point, it need not be a local max, or a local min.
Specifically, it can be a saddle point.

B. Second Derivative Test

1.45: Discriminant

The discriminant is:

$$D = f_{xx} f_{yy} - \left(f_{xy}\right)^2$$

1.46: Second Derivative Test

If (a, b) is a critical point of $f(x, y)$, and
$D > 0$ and $f_{xx} > 0$, then (a, b) is a local min
$D > 0$ and $f_{xx} < 0$, then (a, b) is a local max
$D < 0$ then (a, b) is a saddle point
Otherwise, test is inconclusive

Example 1.47

C. Regression

Example 1.48

Find the minimum value of:

$$f(m, b) = \sum_{i=1}^{n} [y_i - (mx_i + b)]^2$$

Using the chain rule, calculate the first partial derivatives:

$$f_m = -2\sum_{i=1}^{n}[y_i - (mx_i + b)(x_i)], \qquad f_b = -2\sum_{i=1}^{n}[y_i - (mx_i + b)]$$

Calculate the second partial derivatives:

$$f_{mm} = 2\sum_{i=1}^{n} x_i^2, \qquad f_{bb} = -2\sum_{i=1}^{n}(-1) = 2n, \qquad f_{mb} = 2\sum_{i=1}^{n} x_i$$

Use the second derivative test:

$$f_{mm}f_{bb} - (f_{mb})^2 = \left(2\sum_{i=1}^{n} x_i^2\right)(2n) - \left(2\sum_{i=1}^{n} x_i\right)^2 = 4n\left[\sum_{i=1}^{n} x_i^2 - 4\left(\sum_{i=1}^{n} x_i\right)^2\right] = 4nS_{xx} > 4(S_x)^2 \geq 0$$

$$4nS_{xx} > 0$$
$$f_{bb} = 2n > 0$$

Hence,

$$\textit{The point is a minimum}$$

1.6 Constrained Optimization (Lagrange)

A. Basics

Lagrange multipliers are used to optimize functions subject to certain constraints.

1.49: Constrained Optimization

To optimize $f(x,y,z)$ subject to the constraint $g(x,y,z) = 0$, solve

$$\nabla f = \lambda \nabla g$$

Where

$$\nabla f = (f_x, f_y, f_z)$$
$$\nabla g = (g_x, g_y, g_z)$$
$$\lambda \text{ is a parameter to be found}$$

Example 1.50

Find the minimum value of $f(x,y,z) = x^2 + y^2 + z^2$ subject to $x + y + z = 27$.

We need to solve the equation:

$$\nabla f = \lambda \nabla g$$

Calculate the partial derivatives and substitute:

$$\langle 2x, 2y, 2z\rangle = \lambda\langle 1,1,1\rangle$$

This is a vector equation. Two vectors are equal *if and only if* their individual components are equal. Hence, equate the components:

$$2x = \lambda \Rightarrow x = \frac{\lambda}{2}$$
$$2y = \lambda \Rightarrow y = \frac{\lambda}{2}$$
$$2z = \lambda \Rightarrow z = \frac{\lambda}{2}$$

Substitute the values of x, y and z into the constraint $x + y + z = 27$:

$$\frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{2} = 27 \Rightarrow \frac{3\lambda}{2} = 27 \Rightarrow \lambda = 18$$

Substitute the value of λ to find the values of the variables:

$$x = y = z = \frac{\lambda}{2} = \frac{18}{2} = 9$$

And the minimum value is:

$$f(9,9,9) = 9^2 + 9^2 + 9^2 = 81 + 81 + 81 = 243$$

Example 1.51

Minimize $f(x, y, z) = 3x^2 + 2y^2 + z^2$ subject to $x + 2y + 3z = 34$. Find the minimum value.

$$\nabla f = \lambda \nabla g$$
$$\langle 6x, 4y, 2z \rangle = \lambda \langle 1,2,3 \rangle$$

Equate components:

$$6x = \lambda \Rightarrow x = \frac{\lambda}{6}, \qquad 4y = 2\lambda \Rightarrow y = \frac{\lambda}{2}, \qquad 2z = 3\lambda \Rightarrow z = \frac{3\lambda}{2}$$

Substitute the above into the constraint to get:

$$x + 2y + 3z = \frac{\lambda}{6} + 2 \cdot \frac{\lambda}{2} + 3 \cdot \frac{3\lambda}{2} = \frac{\lambda + 6\lambda + 27\lambda}{6} = \frac{34\lambda}{6} = \frac{17\lambda}{3} = 34$$
$$\lambda = 6$$

Find the values of the variables:

$$x = \frac{\lambda}{6} = 1, y = 3, z = 9$$

And the minimum value is:

$$f(1,3,9) = 3(1^2) + 2(3^2) + 9^2 = 3 + 18 + 81 = 102$$

B. Applications

Example 1.52

Alan has a budget of \$100 for brownies (\$3 each) and ice-cream (\$2 for a cup). The utility function is the product of the number of brownies and the number of ice cream cups. Find the maximum utility achievable if fractions of brownies and ice cream cups are purchasable.

Set Up

If b is the number of brownies, and c is the number of ice cream cups, then:

$$Budget\ Constraint = g(b, c) = 3b + 2c - 100 = 0$$
$$Utility\ function = f(b, c) = bc$$

Solve

The condition to be met is:

$$\nabla f = \lambda \nabla g$$
$$\langle c, b \rangle = \lambda \langle 3,2 \rangle$$

Equating partial derivatives on each side, we get $\underbrace{c = 3\lambda}_{Equation\ I}$, $\underbrace{b = 2\lambda}_{Equation\ II}$, which we substitute into

$$3b + 2c = 100$$
$$3(2\lambda) + 2(3\lambda) = 100$$
$$12\lambda = 100$$
$$\lambda = \frac{100}{12} = \frac{25}{3}$$

Substitute the value of λ in Equations I and II to get the value of b and c:

$$c = 3\left(\frac{25}{3}\right) = 25, \qquad b = 2\left(\frac{25}{3}\right) = \frac{50}{3}$$

$$bc = (25)\left(\frac{50}{3}\right) = \frac{1250}{3}$$

C. Cobb-Douglas Function

Example 1.53

Optimize $f(x, y) = 16x^{0.25}y^{0.75}$ subject to $g(x, y) = 50x + 100y - 500{,}000$

We need to solve:

$$\nabla f = \lambda \nabla g$$

Substitute $f_x = 16 \cdot \frac{1}{4} \cdot x^{\frac{1}{4}-1}y^{\frac{3}{4}} = 4x^{-\frac{3}{4}}y^{\frac{3}{4}}$, $f_y = 16 \cdot \frac{3}{4} \cdot x^{\frac{1}{4}}y^{\frac{3}{4}-1} = 12x^{\frac{1}{4}}y^{-\frac{1}{4}}$:

$$\langle 4x^{-\frac{3}{4}}y^{\frac{3}{4}}, 12x^{\frac{1}{4}}y^{-\frac{1}{4}} \rangle = \lambda\langle 50,100\rangle$$

Equate components:

$$\underbrace{4x^{-\frac{3}{4}}y^{\frac{3}{4}} = 50\lambda}_{\textit{Equation I}}$$

$$12x^{\frac{1}{4}}y^{-\frac{1}{4}} = 100\lambda \Rightarrow \underbrace{6x^{\frac{1}{4}}y^{-\frac{1}{4}} = 50\lambda}_{\textit{Equation II}}$$

Since the LHS of Equations I and II is the same, equate the RHS:

$$4x^{-\frac{3}{4}}y^{\frac{3}{4}} = 6x^{\frac{1}{4}}y^{-\frac{1}{4}}$$
$$y = \frac{3}{2}x$$

Substitute the above into the constraint:

$$50x + 100y = 50x + 100\left(\frac{3}{2}x\right) = 50x + 150x = 200x$$
$$200x = 500{,}000$$
$$x = 2{,}500$$
$$y = \frac{3}{2} \cdot 2500 = 3{,}750$$

Example 1.54

I have a budget of B for purchasing two goods. The price of $Good_1 = p_1$, and the price of $Good_2 = p_2$. On purchasing x quantity of $Good_1$ and y quantity of $Good_2$, the utility is $f(x, y) = x^{\frac{1}{2}}y^{\frac{1}{2}}$

A. Find the budget constraint. If you make maximum use of your budget, what does the constraint become? State the constraint as $g(x, y) = 0$.
B. Find the condition that must be met for the utility function to maximized.

Budget Constraint

$$p_1 x + p_2 y \le B$$

On using the entire budget, we get:

$$p_1 x + p_2 y = B$$

The constraint function is:

$$g(x, y) = p_1 x + p_2 y - B = 0$$

Part C

$$\nabla f = \lambda \nabla g$$

Substitute

$$\langle \frac{1}{2}\left(\frac{y}{x}\right)^{\frac{1}{2}}, \frac{1}{2}\left(\frac{x}{y}\right)^{\frac{1}{2}} \rangle = \langle \lambda p_1, \lambda p_2 \rangle$$

Equate the components:

$$\frac{1}{2}\left(\frac{y}{x}\right)^{\frac{1}{2}} = \lambda p_1, \qquad \frac{1}{2}\left(\frac{x}{y}\right)^{\frac{1}{2}} = \lambda p_2$$

Divide Equation I by Equation II:

$$\frac{\frac{1}{2}\left(\frac{y}{x}\right)^{\frac{1}{2}}}{\frac{1}{2}\left(\frac{x}{y}\right)^{\frac{1}{2}}} = \frac{\lambda p_1}{\lambda p_2} \Rightarrow \left(\frac{y}{x}\right)^{\frac{1}{2}} \times \left(\frac{y}{x}\right)^{\frac{1}{2}} = \frac{p_1}{p_2} \Rightarrow \frac{y}{x} = \frac{p_1}{p_2}$$

Example 1.55

Good 1 = Chocolates
Good 2 = Icecream

Your budget is 100 Rs.
p_1 = Price of a chocolate = Rs. 5
p_2 = Price of an icecreams = Rs. 10

You can buy
10 Icecreams and 0 Chocolates=10I
9 Icecreams and 2 Chocolates

Calculate the relative cost:
p_1/p_2 = 5/10 = 1/2

MRS:
MRS (x,y) is the amount of Icecream that you are willing to give up for an amount of Chocolate.
Suppose that I am willing to give up 0.25 Icecream for 1 Chocolates
MRS=1/4

9 Icecreams and 2 Chocolates= 9 Icecream + 0.5 Icecreams = 9.5 Icecreams

Suppose that I am willing to give up 1 Icecream for 1 Chocolate
MRS=1

9 Icecreams and 2 Chocolates= 9 Icecreams + 2 Icecreams=11 Icecreams

M=72, p_1=1,p_2=3
p_1/p_2 = 1/3
x_max=M/p_1=72/1=72
y_max=M/p_2=72/3=24

MRS at the left corner point
=MRS(0,y_max)
=y/x (x=0, y=24)
=24/0
=Infinity

MRS at the right corner point
=MRS(x_max,0)
=y/x (y=0,x=72)
=0/72
=0

The optimality condition is that
MRS = p_1/p_2

y/x = 1/3
3y=x

Budget Constraint:
x+3y=72
3y+3y=72
6y=72
y=72/6=12
x=3y=3(12)=36

D. Geometry

Example 1.56

$$f(r,h) = 2\pi r^2 + 2\pi rh$$
$$g(r,h) = \pi r^2 h - V$$

$$\nabla f = \lambda \nabla g$$
$$\langle 4\pi r + 2\pi h, 2\pi r\rangle = \lambda\langle 2\pi rh, \pi r^2\rangle$$

$$2\pi r = \lambda\pi r^2 \Rightarrow r = \frac{2}{\lambda}$$

$$4\pi r + 2\pi h = 2\lambda\pi r h$$
$$2r + h = \lambda r h$$
$$2\left(\frac{2}{\lambda}\right) + h = \lambda\left(\frac{2}{\lambda}\right)h$$
$$\frac{4}{\lambda} + h = 2h$$
$$\frac{4}{\lambda} = h$$
$$\lambda = \frac{4}{h}$$
$$r = 2 \times \frac{h}{4} = \frac{h}{2}$$

2. 3D Geometry

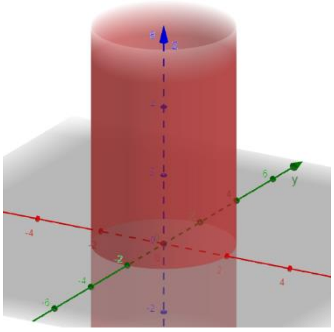
2.1 Cylinders

A. Circular Cylinder

2.1: Cylinder
The base of a cylinder is a circle.

The equation of a circular cylinder with center at the origin and no restriction on height is:

$$x^2 + y^2 = r^2$$



Example 2.2
Explain why $x^2 + y^2 = 4$ has the shape that it does.

This is a cylinder with:

$$Base = circle\ with\ radius\ 2, and\ center\ at\ the\ origin$$
$$z\ has\ no\ restriction$$

Example 2.3
What shape does the below have:

$$x^2 + y^2 = 5, \qquad 2 < z < 7$$

This is a cylinder with:

$$Base = circle\ with\ radius\ \sqrt{5}, and\ center\ at\ the\ origin$$
$$2 < z < 7 \Rightarrow Cylinder\ ranges\ from\ 2\ to\ 7$$

2.4: Projection
If you shine a light on a 3D shape from above ($z = height$), the shadow that will be formed on the xy plane is the projection of the 3D shape.

B. Elliptic Cylinder

2.5: Elliptic Cylinder
The base of an elliptic cylinder is an ellipse.

The equation of an elliptical cylinder with center at the origin and no restriction on height is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

3. Integration

3.1 Double Integration

A. Double Integration

3.1: Logic behind double integration

3.2: Double Integral

$$\iint_R f(x,y)\,dA$$

Example 3.3

Write a double integral of $f(x,y) = 4x^2y$ over the region R.

$$\iint_R 4x^2y\,dA$$

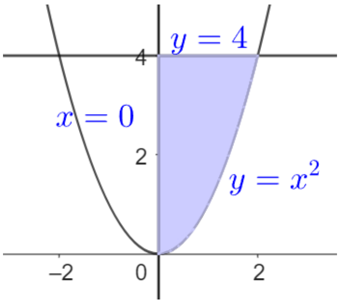
B. Iterated Integral

3.4: Iterated Integral

$$\int_a^b \int_c^d f(x,y)\,dy\,dx$$

Example 3.5

Evaluate the double integral of $f(x,y) = 4x^2y$ over the region R in the first quadrant of the coordinate plane bounded by $y = x^2$, $y = 4$ and $x = 0$.



Consider a strip parallel to the y axis.

$$\int_{x=0}^{x=2} \int_{y=x^2}^{y=4} 4x^2y\,dy\,dx$$

Evaluate the "inner" integral:

$$\int_{x^2}^{4} 4x^2y\,dy = 2x^2[y^2]_{x^2}^4 = 2x^2(16 - x^4) = 2(16x^2 - x^6)$$

Substitute in the double integral:

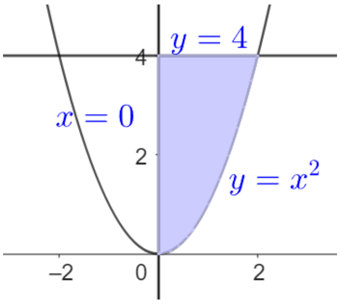
$$\int_0^2 2(16x^2 - x^6)dx = 2\left[\frac{16x^3}{3} - \frac{x^7}{7}\right]_0^2 = 2\left(\frac{128}{3} - \frac{128}{7}\right) = 256\left(\frac{1}{3} - \frac{1}{7}\right) = 256\left(\frac{4}{21}\right) = \frac{1024}{21}$$

C. Order of Integration

Example 3.6

Write the integral below with the order of integration reversed:

$$\int_0^2 \int_{x^2}^4 4x^2 y \, dy \, dx$$



D. Area and Average Value

3.7: Area of the Region *R*

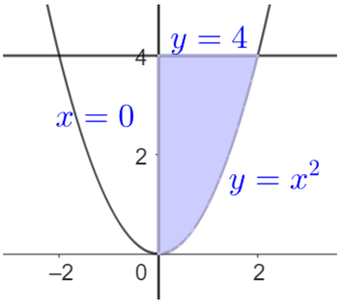
To find the area of the region R, integrate $f(x, y) = 1$. That is, find:

$$\iint_R 1 \, dA$$

- This is the same concept as the "area between curves" from single variable calculus.
- We can integrate more general regions compared to single variable calculus.

Example 3.8

A. Evaluate the double integral of $f(x, y) = 1$ over the region R in the first quadrant of the coordinate plane bounded by $y = x^2$, $y = 4$ and $x = 0$.
B. What does it represent?



Consider a strip parallel to the y axis.

$$\int_0^2 \int_{x^2}^4 1 \, dy \, dx$$

Evaluate the "inner" integral:

$$\int_{x^2}^4 1 \, dy = [y]_{x^2}^4 = 4 - x^2$$

Substitute in the double integral:

$$\int_0^2 (4 - x^2) \, dx = Area\ between\ y = 4\ \&\ y = x^2$$

3.9: Average Value of a Single Integral

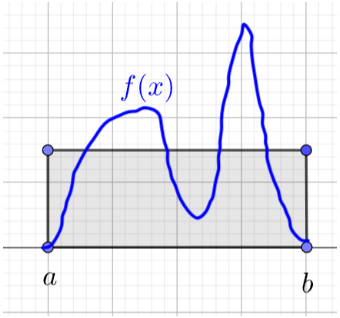
The average value of a single integral is the value of the integral divided by the length of interval over which it is integrated:

$$\frac{1}{b-a} \int_a^b f(x) \, dx$$

Example 3.10
Explain why

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Represents the average value of $f(x)$ using the diagram alongside.



The area between $f(x)$ and the x −axis is given by the integral:

$$\int_a^b f(x)\,dx$$

We want the average which means we want the value if the area were to be distributed equally.

Draw a rectangle with:

$$Width = Width\ of\ interval = b-a$$

$$Height = \frac{Area}{Width} = \frac{\int_a^b f(x)\,dx}{b-a} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

And the last expression is the expression that we started with.

3.11: Average Value of a Double Integral
The average value of a double integral over a region R is given by the value of the integral, divided by the area of the region:

$$\frac{Double\ Integral\ of\ f}{Area\ of\ R} = \frac{1}{A}\left(\iint_R f(x,y)\,dA\right)$$

Where

$$A = Area = \iint_R 1\,dA$$

The concept in double integration is analogous to single integration:
Divide the volume by the area to get the average height

$$Average\ Height = \frac{Volume}{Area} = \frac{\iint_R f(x,y)\,dA}{A} = \frac{1}{A}\left(\iint_R f(x,y)\,dA\right)$$

Example 3.12
The average value of $f(x,y)$ over a region R is positive. Determine the range of the double integral

$$\iint_R f(x,y)\,dA$$

$$\iint_R f(x,y)\,dA = \underbrace{(Average\ Value)}_{+ve}\underbrace{(Area)}_{+ve} > 0$$

E. Separation of Integrals

3.13: Separation of Integrals

If the integrand can be written as a product of two functions (one in x, and the other in y), then an iterated integral can be split into the product of two integrals:

$$\int_{y=a}^{y=b}\int_{x=c}^{x=d} f(x)f(y)\,dx\,dy = \left(\int_c^d e^{-\frac{x^2}{2}}\,dx\right)\left(\int_a^b e^{-\frac{y^2}{2}}\,dy\right)$$

- This property is useful in evaluating iterated integrals of this type since they reduce to a product of two integrals from single variable calculus.
- The property also allows us to convert a product of integrals into an iterated integral.

3.2 Double Integrals: Polar

A. Area and Average Value

3.14: Changing to Polar Coordinates

$$x = r\cos\theta,\ y = r\sin\theta$$
$$x^2 + y^2 = r^2$$
$$dA = dx\,dy = r\,dr\,d\theta$$

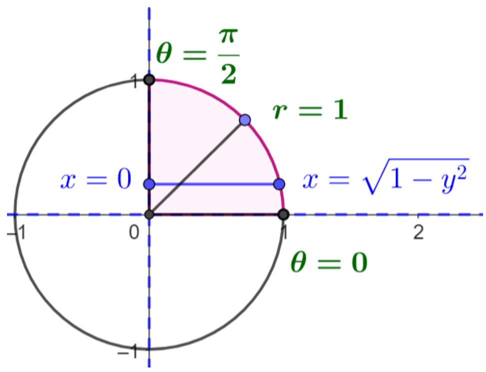
Example 3.15

Convert to polar coordinates and evaluate:

$$\int_{y=0}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}} x^2 + y^2\,dx\,dy$$

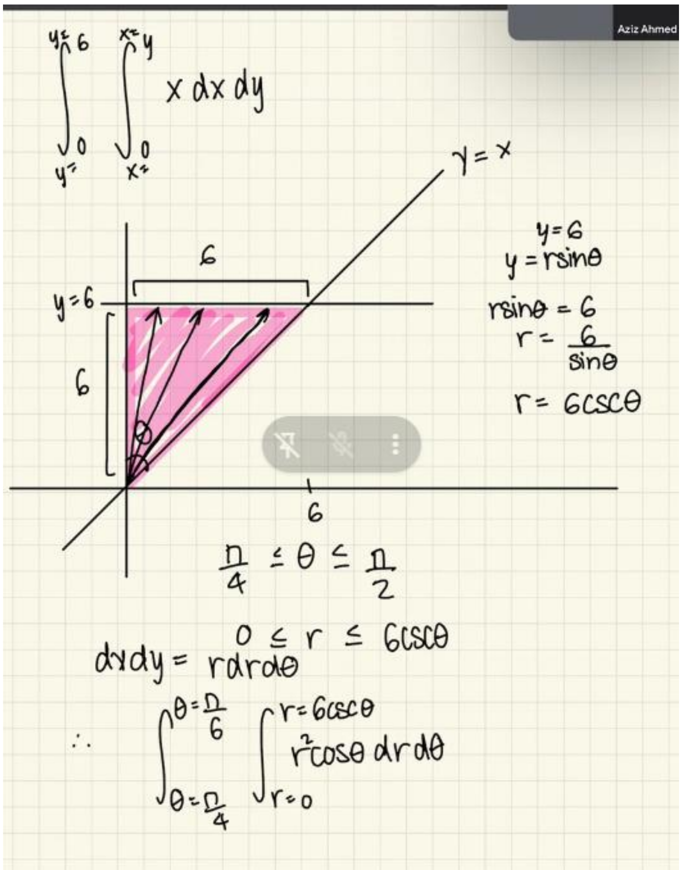
$$x^2 + y^2 = r^2, \qquad dx\,dy = r\,dr\,d\theta$$

$$\int_{y=0}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}} r^2\,r\,dr\,d\theta$$



Example 3.16

$$\int_0^6\int_0^y x\,dx\,dy$$



Example 3.17

$$\int_1^{\sqrt{3}} \int_1^x dy\,dx$$

$$\int_{1}^{\sqrt{3}}\int_{1}^{x} dy\,dx$$

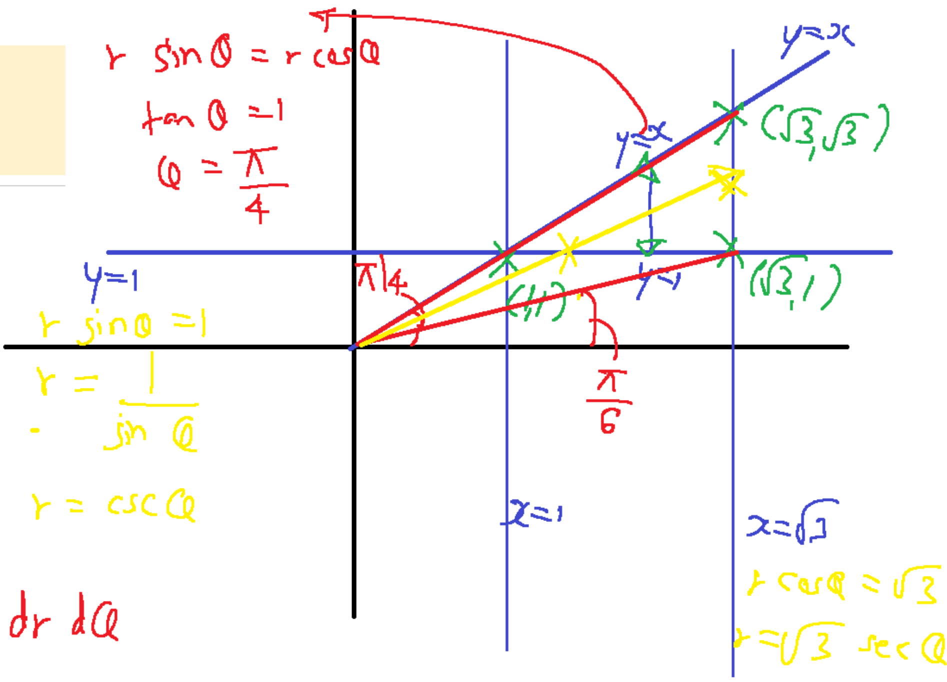
$x = r\cos\theta$

$y = r\sin\theta$

$dy\,dx = r\,dr\,d\theta$

$x^2 + y^2 = r^2$

$$\int_{\theta=\pi/6}^{\theta=\pi/4}\int_{r=\csc\theta}^{r=\sqrt{3}\sec\theta} r\,dr\,d\theta$$



3.3 Triple Integrals

A. Basics

3.18: Integrating 1

$$\int 1\,dx = Length\ of\ Interval$$

$$\iint 1\,dA = \iint 1\,dx\,dy = Area\ of\ Region$$

$$\iiint 1\,dV = \iiint 1\,dx\,dy\,dz = Volume\ of\ Region$$

Example 3.19

Interpret

$$\iiint 3\,dV$$

$$\iiint 3\,dV = 3\iiint 1\,dV = 3(Volume\ of\ the\ Region)$$

Example 3.20

$$\int_a^b f(x)\,dx$$

- **Slice** the interval (a, b) into n parts.
- Determine the height of a rectangle above each part. Use the area of the rectangle to get the **approximate** area under the function for that part.
- Take the **sum** of the areas of all the rectangles
- **Take the limit** of the sum as the width of each slice goes to zero.

Example 3.21

$$\iint f(x, y)\, dA$$

- **Slice** the region (a, b) into n rectangles.
- Determine the height of a box above each part. Use the volume of the box to get the **approximate** volume of the function under the function for that part.
- Take the **sum** of the volumes of all the boxes
- **Take the limit** of the sum as the width and length of each rectangle goes to zero.

3.4 Further Topics

22 Examples