Combinatorics Topics

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1. Counting Topics

1.1 PIE (Principle of Inclusion and Exclusion)

A. Formula

1.1: Number of Elements in Four Sets

$$n(A \cup B \cup C \cup D)$$
$$= \underbrace{n(A) + n(B) + n(C) + n(D)}_{One\ at\ at\ a\ Time}$$
$$- \underbrace{[n(A \cap B) + n(A \cap C) + n(A \cap D) + n(B \cap C) + n(B \cap D) + n(C \cap D)]}_{Two\ at\ a\ Time}$$
$$+ \underbrace{n(A \cap B \cap C) + n(A \cap B \cap D) + n(A \cap C \cap D) + n(B \cap C \cap D)}_{Three\ at\ a\ time}$$
$$- \underbrace{n(A \cap B \cap C \cap D)}_{Four\ at\ a\ Time}$$

A. Summation Notation

1.2: Two at a Time

$$\sum_{1 \le i < j \le n} A_i \cap A_j = \sum_{j=i+1}^{n} \sum_{i=1}^{n-1} A_i \cap A_j$$

$$(A, B, C, D) \Rightarrow AB, AC, AD, BC, BD, CD$$
$$\sum_{1 \le i < j \le n} A_i \cap A_j = \sum_{j=i+1}^{n} \sum_{i=1}^{n-1} A_i \cap A_j$$

$$i = 1 \Rightarrow \sum_{j=2}^{4} A_1 \cap A_j = (A_1 \cap A_2) + (A_1 \cap A_3) + (A_1 \cap A_4)$$
$$i = 2 \Rightarrow \sum_{j=3}^{4} A_2 \cap A_j = (A_2 \cap A_3) + (A_2 \cap A_4)$$
$$i = 3 \Rightarrow \sum_{j=4}^{4} A_3 \cap A_j = A_3 \cap A_3$$

$$\underbrace{A_1A_2 + A_1A_3 + A_1A_4}_{i=1} + \underbrace{A_2A_3 + A_2A_4}_{i=2} + \underbrace{A_3A_4}_{i=4}$$

1.2 Distinguishability

A. Subsets of Indistinguishable Objects

1.3: Distinguishability

- If I can distinguish between two objects, they are distinguishable.
- If two objects are identical, I cannot distinguish between them.

Example 1.4

In how many ways can I do the following, if the order of picking the balls is not important:

A. I have five balls colored red, blue, green, yellow and black. I want to pick two balls from them.
B. I have five balls colored red, and numbered from 1 to 5. I want to pick two balls from them.
C. I have five identical balls. I want to pick two balls from them.
D. I have three balls colored green, yellow and black, respectively, and two red balls. I want to pick two balls from them.

Part A

$$R, B, G, Y, B$$

The color scheme lets me distinguish between the balls.

If order is not important, I can choose the balls in:

$$\binom{5}{2} = \frac{5 \times 4}{2} = 10$$

If order is important, I can pick the balls in:

$$5 \times 4 = 20 \; Ways$$

Part B

$$R_1, R_2, R_3, R_4, R_5$$

The numbering scheme lets me distinguish between the balls.

The answers are the same as Part A

$$Order\ is\ not\ important: 10 \; Ways$$

$$Order\ is\ important: 20 \; Ways$$

Part C

$$R, R, R, R, R$$

Since there is no way for me to distinguish between the balls I pick, whether order is important or not, I can do it in only

$$One\ way$$

Part D

$$G, Y, B, R, R$$

Order is not important:

Zero Red Balls:

$$\binom{3}{2} = 3$$

One Red Balls:

$$\underbrace{\binom{1}{}}_{\substack{Red\\Ball}} \underbrace{\binom{3}{}}_{\substack{Second\\Ball}} = 3$$

Two Red Balls

$$1 \; Way$$

Total

$$= 3 + 3 + 1 = 7$$

Note that, since a single red ball is distinguishable from the others, we can combine the case where we pick a single red ball with the case where we pick one red ball:

$$\binom{4}{2} = 6$$

Order is important:

Zero or One Red Ball

$$4 \times 3 = 12$$

Two Red Balls

$$1 \; Way$$

Example 1.5

I have five identical apples, four identical oranges and six identical pears. In how many ways can I make a non-empty fruit basket with:

A. Only Apples
B. Only Oranges
C. Only Pears
D. Apples and Oranges
E. Apples and Pears
F. Oranges and Pears
G. Apples, Oranges or Pears

Parts A, B, C: Only One Fruit

The minimum number of apples that we need is 1. The maximum number of apples is 5.

$$Apples\ can\ range\ from\ 1\ to\ 5 \Rightarrow \{1,2,3,4,5\} \Rightarrow\ 5\ Choices$$
$$Oranges\ can\ range\ from\ 1\ to\ 4: \Rightarrow \{1,2,3,4\} \Rightarrow 4\ Choices$$
$$Pears\ can\ range\ from\ 1\ to\ 6 \Rightarrow \{1,2,3,4,5,6\} \Rightarrow\ 6\ Choices$$

Apples and Oranges

In this case, count the number of choices if we do not take any apples also:

$$Apples\ can\ range\ from\ 0-5: 6\ Choices$$
$$Oranges\ can\ range\ from\ 0-4: 5\ Choices$$
$$\underbrace{6}_{\substack{Choices\ for\\ Apples}} \times \underbrace{5}_{\substack{Choices\ for\\ Oranges}} = 30$$

But this also includes the case where there are zero apples and zero oranges.
Hence, the final answer is

$$30-1=29$$

Apples and Pears

$$6 \times 7 - 1 = 42 - 1 = 41$$

Oranges and Pears

$$5 \times 7 - 1 = 35 - 1 = 34$$

Apples, Oranges and Pears

$$6 \times 5 \times 7 - 1 = 210 - 1 = 209$$

> Answer each part of the question above, if there are must be at least one fruit of each eligible type placed. For example, in answering Part D, you must place at least one apple and one orange, but you cannot place any pears.

Parts A, B, C: Only One Fruit

Same as above

Apples and Oranges

$$Apples\ can\ range\ from\ 1-5: 5\ Choices$$
$$Oranges\ can\ range\ from\ 1-4: 4\ Choices$$
$$\underbrace{5}_{\substack{Choices\ for\\ Apples}} \times \underbrace{4}_{\substack{Choices\ for\\ Oranges}} = 20$$

Apples and Pears

$$5 \times 6 = 30$$

Oranges and Pears

$$4 \times 6 = 24$$

Apples, Oranges and Pears

$$5 \times 4 \times 6 = 120$$

Core Concept 1.6

In the previous example, a hotel manager comes along and numbers the apples, oranges and pears. Now, answer all parts of the above question again.

First answer with the condition that the fruit basket must be non-empty

Part A

There are five apples.
For the first apple, you have a choice, place it in the fruit basket, or do not place it. This gives you

$$2\ Choices$$

For the second apple also, you have a choice, place it in the fruit basket, or do not place it. This gives you

$$2\ Choices$$

In general, you have:

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$$

But, you cannot the choice where none of the apples have been chosen, so the final answer is:

$$32 - 1 = 31$$

Part B

$$2^4 - 1 = 16 - 1 = 15$$

Part C

$$2^6 - 1 = 64 - 1 = 63$$

Part D

$$2^9 - 1 = 64 - 1 = 63$$

Part E

$$2^{11} - 1 = 2048 - 1 = 2047$$

Part F

$$2^{10} - 1 = 1024 - 1 = 1023$$

Part G

$$2^{15} - 1$$

Now answer with the condition that there must be one fruit of every eligible type.

Parts A, B, C

Same as above: $\{31, 15, 63\}$

Part D

We can do this using complementary counting.

If there are no restrictions, the number of ways to do this is:

$$2^5 \times 2^4 = 2^9 = 512$$

We cannot have the case where no apples are selected, which can be done in:

$$\underbrace{1}_{\substack{Select\ Zero\\ Apples}} \times \underbrace{2^4}_{\substack{Select\ 0-4\ of\\ the\ Oranges}} = 16\ Ways$$

We cannot have the case where no oranges are selected, which can be done in :

$$\underbrace{2^5}_{\substack{Select\ 0-5\ of\\ the\ Apples}} \times \underbrace{1}_{\substack{Select\ Zero\\ Oranges}} = 32\ Ways$$

However, both the cases above count the situation where no apple and no orange is selected, which can be done in exactly

$$1\ Way$$

Hence, the final answer is

$$512 - 16 - 32 + 1 = 465$$

Part E

Practice 1.7

I am going sightseeing in Dubai and have shortlisted four parks and six malls to be visited.

B. Multiplication Principle

Core Concept 1.8: Balls-Distinguishable, Boxes-Distinguishable

Find the number of ways that 5 distinguishable balls can be distributed among 3 distinguishable boxes.

This is a direct application of the multiplication principle:
For the first ball, we have five choices.
For the second ball, we have five choices.
In fact, for each ball, we have five choices, and these choices are independent.

Hence, the total number of choices is:

$$\underbrace{3}_{\substack{First\\ Ball}} \times \underbrace{3}_{\substack{Second\\ Ball}} \times \underbrace{3}_{\substack{Third\\ Ball}} \times \underbrace{3}_{\substack{Fourth\\ Ball}} \times \underbrace{3}_{\substack{Fifth\\ Ball}} = 3^5 = 243$$

C. Stars and Bars/Diophantine Equations

Core Concept 1.9: Balls-Identical, Boxes-Distinguishable

Find the number of ways that 5 identical balls can be distributed among 3 distinguishable boxes.

We have $\underbrace{three\ distinguishable\ boxes}_{\underline{Box\ 1}\ \underline{Box\ 2}\ \underline{Box\ 3}}$, among which we put two dividers:

$$\underline{Box\ 1}\ \ \underline{Divider\ 1}\ \ \underline{Box\ 2}\ \ \underline{Divider\ 2}\ \ \underline{Box\ 3}$$

If we add $\underbrace{two\ dividers}_{\underline{||}}$ among $\underbrace{five\ identical\ balls}_{\underline{0}\,\underline{0}\,\underline{0}\,\underline{0}\,\underline{0}}$, then the dividers divide the balls into three parts. For example

$$\underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{||}$$

corresponds to putting all five balls in the first box:

$$\underbrace{\underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}}_{First\ Box}\ \underline{|}\ \underbrace{\underline{Empty}}_{Second\ Box}\ \underline{|}\ \underbrace{\underline{Empty}}_{Third\ Box}$$

Similarly

$$\underline{0}\ \underline{0}\underline{|}\ \underline{0}\underline{|}\ \underline{0}\ \underline{0}$$

corresponds to two balls in the first box, one ball in the second box, and two balls in the third box:

$$\underbrace{00}_{First\ Box}\ \ \underline{|}\ \ \underbrace{0}_{Second\ Box}\ \ \underline{|}\ \ \underbrace{00}_{Third\ Box}$$

In general, the number of ways we can distribute the balls among the boxes is the way we can select the positions of the two dividers out of the seven objects that we have:

$$^{5\ Balls\ +2\ Dividers}C_{2\ Positions} = {}^{5+2}C_2 = {}^{7}C_2 = \frac{7\times 6}{2} = 21$$

D. Casework

Core Concept 1.10

Find the number of ways that 5 identical balls can be distributed among 3 identical boxes.

Distribution	Meaning
$5-0-0$	Five balls in one box and zero in the other two
$4-1-0$	Four balls in one box, and one ball in one ball
$3-2-0$	Three Balls in one box, and two balls in one box
$3-1-1$	Three balls in one box, and one ball each in the remaining boxes
$2-2-1$	Two balls each in two boxes, and the remaining ball in the remaining box
$5\ Ways$	

E. Casework with Selections

Core Concept 1.11

Find the number of ways that 5 distinguishable balls can be distributed among 3 identical boxes.

$5-0-0$	$^5C_5 = 1$	Choice of box does not matter here. All five balls go in one box.
$4-1-0$	$^5C_4 = {}^5C_1 = 5$	Four balls in one box, and one ball in another box. The choice of ball is going to matter, since the single ball that you pick is placed differently from

		the rest.
$3-2-0$	${}^5C_3 = {}^5C_2 = 10$	Three Balls in one box, and two balls in one box
$3-1-1$	${}^5C_3 = 10$	Three balls in one box, and one ball each in the remaining boxes
$2-2-1$	$\frac{{}^5C_2 \times {}^3C_2}{2} = \frac{10 \times 3}{2} = 15$	Two balls each in two boxes, and the remaining ball in the remaining box
$5\ Ways$	$1+5+10+10+15=41$	

Suppose that the balls are distinguishable on the basis on color: Red(R) , Blue(B), Green(G), Yellow(Y), Magenta(M)

We need to first select two balls from the five available, giving us:

$${}^5C_2 = \frac{5 \times 4}{2} = 10$$

Let us say we get Red and Blue as the balls that go into the first box. We are now left with three balls, from which we must choose two:

$${}^3C_2 = \frac{3 \times 2}{2} = 3$$

Let's say we Green and Yellow. Now, we don't have any choice with respect to the last ball, because it is already selected for us.

Hence, the total number of choices is

$${}^5C_2 \times {}^3C_2 = 10 \times 3 = 30$$

But, this counts as two different arrangements the same situation:

$$\{R, B\} + \{G, Y\}, \qquad \{G, Y\} + \{R, B\}$$

Hence, the number that we found above must be divided by 2, giving us:

$$\frac{30}{2} = 15$$

Example 1.12

Find the number of ways in which the O'Hara quadruplets can go for a wedding in four limousines if

A. Mother can distinguish both quadruplets and limousines
B. Butler can distinguish quadruplets, but not between cars
C. Chauffer can distinguish cars, but not between quadruplets
D. Usher can't distinguish between either

Part A: Multiplication Principle

The first quadruplet can sit in any of the four limousines:

$$4\ Choices$$

The second quadruplet can sit in any of the four limousines:

$$4\ Choices$$

The third quadruplet can sit in any of the four limousines:

$$4\ Choices$$

The fourth quadruplet can sit in any of the four limousines:

$$4\ Choices$$

Each of the choices is independent. Hence, the total number of choices is

$$4 \times 4 \times 4 \times 4 = 256$$

Part B

$4-0-0$	$1\ Choice$
$3-1-0$	$4\ Choices$
$2-2-0$	$\binom{4}{2}$
$2-1-1$	$\binom{4}{2}$

Part C

$$\underbrace{a+b+c+d=4}_{\substack{4\ Stars\\ 3\ Bars}} \Rightarrow \binom{4+3}{3} = \binom{7}{3} = 35$$

Part D

$4-0-0$	
$3-1-0$	
$2-2-0$	
$2-1-1$	

F. Applications

Example 1.13

1.3 Derangements and Rencontres Numbers

A. Definitions

Permutation: For a finite set S, a permutation is a bijective function $f: S \to S$ that maps each element of S with a corresponding element in itself.

1.14: Derangement

A permutation such that no object is returned to its original position is called a derangement.

1.15: Derangement

The numbers of ways to derange n objects is written as:

$$D_n = !n$$

D_n is also called the subfactorial.

Example 1.16

D_1 is the number of ways that one object can be given to one person such that each person gets the wrong object.
Since there is only one person, he must get the right object.

$$D_1 = 0$$

Example 1.17

D_1 is the number of ways that two objects can be given to two people such that each person gets the wrong object.

$$\{A, B\}\{a, b\}$$
$$(A, b)(B, a) \Rightarrow D_2 = 1$$

Example 1.18

A, B, C
Start with A:
A can get any of B, C (2 options)

Suppose, without loss of generality, A get B's paper.
Papers left: A, C; Students left will be B, C
C must get A
B must get C

Right Distribution
A:A, B:B, C:C

Wrong Distribution:
A:B, B:C, C:A
A:C, B:A, C:B

$$D_3 = 3$$

Example 1.19

A, B, C, D
Start with A:
A can get any of B, C, D (3 options)

Suppose, without loss of generality, A get B's paper.
Papers left: A, C, D
B can get any of A, C, D (3 Options)

If B gets A, papers left: C, D
C must get D
D must get C

If B gets C, papers left: A, D
C must get D
D must get A

If B gets D, papers left: A, C
C must get A
D must get C

B. Recursive Formula

1.20: Recursive Formula

$$D_n = (n-1)[D_{n-1} + D_{n-2}]$$

Second Recursive Formula:

$$D_n = n(D_{n-1}) + (-1)^n$$

Writing Assignment 1.1
Prove the first recursive relation above using combinatorial arguments.

Example 1.21

Use the recursive relations for D_n to calculate it from D_1 to D_6

D_1	D_2	D_3	D_4	D_5	D_6
0	1	2	9	44	265

$$D_1 = 0$$
$$D_2 = 1$$

$$D_3 = (3-1)(1+0) = 2(1) = 2$$
$$D_4 = (4-1)(2+1) = 3(3) = 9$$
$$D_5 = (5-1)(9+2) = 4(11) = 44$$

C. Explicit Formula

$$D_n = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!} = n!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{n!}\right]$$

Writing Assignment 1.2
Derive the explicit formula from the first recursive formula above.

Example 1.22

Calculate D_4 using the explicit formula

$$D_n = 4!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right] = 24\left[\frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right] = 24\left[\frac{12-4+1}{24}\right] = 9$$

Example 1.23!

Calculate D_6 using the explicit formula

Example 1.24

If 6 six-sided dice are assigned identifying numbers from 1 to 6, and then rolled, what is the probability that no die has the same number face-up as the number it has been assigned.

$$P(\text{No Die matching its number}) = \frac{D_6}{6!} = \frac{265}{720}$$

Example 1.25

An online MOOC is looking to peer-source grading of tests for the end-term. Each student will grade a test written by someone other than himself. If the number of ways that the students can be assigned to grade tests is 44, then what is the number of students?

$$D_n = 44 \Rightarrow n = 5$$

Example 1.26

In how many ways can we put five different balls with five distinct colors into five different boxes of the same colours as the balls such that every box has one ball in it, if:

A. No box has a ball of the same colour as the box.
B. At least two boxes have balls of the same color

D. Derangements with Repetition

What is the number of ways to derange the letters of the word BOTTLE?

$$P$$

Writing Assignment 1.3

Show that the number of ways to derange n objects with one object repeated twice is given by:

$$\frac{D_n - 2D_{n-1} - D_{n-2}}{2}$$

E. Fixed Points of a Permutation

If in a permutation $f: S \to S$, an element maps to itself, then it is a fixed point. In other words, it does not change position.

Example 1.27

For a finite set S with cardinality n, find the number of bijective functions $f: S \to S$ such that $f(x) = x$ for exactly m points in its domain when $m =$

A. n
B. $(n-1)$
C. $(n-2)$
D. $(n-3)$

$$P$$

A. 1
B. 0
C. ${}^nC_2$
D. $2\,{}^nC_3$

F. Applications

Example 1.28

If 5 letters addressed to 5 people are put randomly into the envelopes, what is the probability that at least one letter goes into the correct envelope?

$$P(E \geq 1) = 1 - P(\mathrm{E} = 0) = 1 - \frac{44}{120} = \frac{76}{120} = \frac{19}{30}$$

Writing Assignment 1.29
(Hat Check Problem) A hat check girl completely loses track of the n hats she has, and hands them back at random. Given the Taylor Series expansion

$$\frac{1}{e} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots$$

show that the probability of every person receiving the wrong hat (as n increases without bound), is given by $\frac{1}{e}$.

1.4 Pigeonhole Principle-I: Basics

A. Definition
The pigeonhole principle is deceptively simple, and very deep. It has applications in solving important mathematical problems, some of which require great subtlety.

1.30: Pigeonhole Principle
If there are n pigeons in m pigeonholes, with $n > m$, then at least some pigeonhole must contain more than one pigeon.

- If your number of pigeons is greater than your number of pigeonholes, then at least one pigeonhole must contain more than one pigeon.
- The pigeonhole which contains more than one pigeon is not specified.

Example 1.31
I have 5 pigeons, which live in 4 pigeonholes.

A. Show, visually that, least one pigeonhole must have two pigeons living in it.
B. What is the maximum possible number of pigeons in a pigeonhole?

Part A
Consider a set of 4 empty pigeonholes, which look like this:

In these empty pigeonholes, we start assigning pigeons. We can assign 4 pigeons without trouble, like the diagram below:

$\underbrace{P}_{Pigeon\ 1}$	$\underbrace{P}_{Pigeon\ 2}$	$\underbrace{P}_{Pigeon\ 3}$	$\underbrace{P}_{Pigeon\ 4}$

However, note that if you want to assign a fifth pigeon, it must go into a pigeon hole that already has a pigeon in in it.
And hence, at least some pigeonhole must have two pigeons.

$\underbrace{P}_{Pigeon\ 1}$	$\underbrace{P}_{Pigeon\ 2}$ $\underbrace{P}_{Pigeon\ 5}$	$\underbrace{P}_{Pigeon\ 3}$	$\underbrace{P}_{Pigeon\ 4}$

Part B
The maximum number of pigeons can be 5 (where all the pigeons go into a single pigeonhole).

B. Birthdays

Example 1.32
What is the maximum number of students in a class with birthdays in distinct months of the year?

There are twelve months in a year.

Jan	Feb	.	.	Dec

Assign a different month of the year to the first twelve students.

Jan	Feb	.	.	Dec
Student 1	Student 2	.	.	Student 12

The thirteenth student must be assigned a month already assigned to one of the previous twelve.
(Though, it could any month, we are not concerned with that at the moment.)

Hence, the maximum number of students is twelve.

Example 1.33
The maximum number of students in a class with birthdays on distinct days of the week is:

Since there are seven days in the week, the maximum number is seven.

Example 1.34
The maximum number of students in a class with birthdays on distinct days of the same month is:

We get different answers depending upon the month that we choose:

$$Feb = 28 \text{ } or \text{ } 29$$
$$Jan, Mar, May, July, Aug, Oct, Dec = 31$$
$$Apr, June, Sep, Nov = 30$$

Out of this, we are looking for the maximum, hence, the answer is:

$$31$$

C. Proof by Contradiction
In proof by contradiction, we start by assuming the exact opposite of what we wish to prove.
We then show that this leads to a logical impossibility (or, in short, contradiction).

Finally, we conclude that our original assumption was wrong.
This lets us prove what we originally started to prove.

Example 1.35
There are 13 students in a class. Prove that at least two students must have their birthday falling in the same month.

$Assume, if\ possible, to\ the\ contrary,$ that no two students have their birthday in the same month.
We can then assign a different month to each student for their birthday.

Assign to the first 12 students, birthdays in the 12 months of the year.

Jan	Feb	.	.	Dec
Student 1	Student 2	.	.	Student 12

The 13^{th} student must be assigned a birthday in a month that already has a student having a birthday.
This is a contradiction.
Hence, our original assumption was wrong.

Hence, at least two students must have their birthday in the same month.

Example 1.36

There are 8 students in a class. Prove that at least two students must have their birthday falling on the same day of the week.

Assume, if possible, to the contrary, that no two students have their birthday falling on the same day of the week.
Then, we can assign every student a different day for his birthday. Suppose, we assign the first 7 students to the 7 days of the week.

Now, for the 8th student we have a problem. He must share the day of the week of his birthday with another student because there are no more days left.

Hence the original assumption was false.
Hence, at least two students must have their birthday falling on the same day of the week.

Practice 1.37

There are 32 students in a class, all of whom have birthdays falling in the same month of the year. Prove that at least two students must have their birthday falling on the same day of the month.

D. Finding Minimum Overlaps

The important thing about the Pigeonhole Principle is that it can be applied to questions that do not immediately look like good candidates for the principle.
We start by looking at a simple variation of the questions done so far.

Example 1.38

A class has fifteen students. What is the minimum number of students that have a birthday falling in the same month as someone else in the class? The maximum?

Part A
Maximum
If we assign all students to have their birthday in the same month, then:

Jan	Feb	.	.	Dec
Student 1				
Student 2				
.				
.				
.				

Student 15				

All 15 students have their birthday in the same month.
And this is the maximum, since there are only 15 students.

Minimum

If we assign a single student to a single month, we get the familiar diagram below:

Jan	Feb	.	.	Dec
Student 1	Student 2	.	.	Student 12

Now, we need to assign three more students. One logical way of doing it is this:

Jan	Feb	Mar	.	Dec
Student 1	Student 2	Student 3	.	Student 12
Student 13	Student 14	Student 15		

Which results in the following students sharing a birthday with other students:

$$(1,13), (2,14), (3,15) \Rightarrow 6 \; Students$$

However, 6 is not the minimum.

Instead of assigning different months to the "extra" students, assign all of them to the same month, in this case, January:

Jan	Feb	Mar	.	Dec
Student 1	Student 2	Student 3	.	Student 12
Student 13				
Student 14				
Student 15				

And now the number of students that share a month is

$$(1,13,14,15) \Rightarrow 4 \; Students$$

Summary

12 students can have 12 distinct months.
If the remaining three students are adjusted

- in three different months, then six students share a birthday with someone else (max)
- in a single month, then four students share a birthday with someone else (min)

Example 1.39

A class has fifteen students. What is the minimum number of

A. months that have more than one birthday? The maximum?

Part A

Jan	Feb	Mar	.	Dec
Student 1	Student 2	Student 3	.	Student 12
Student 13				
Student 14				

Student 15				

Part B

Jan	Feb	Mar	.	Dec
Student 1	Student 2	Student 3	.	Student 12
Student 13	Student 14	Student 15		

E. Finding Minimum Number of Pigeons

In some examples, we are interested in finding the minimum number of pigeons such that they must share a pigeonhole.
In such cases, we can:

- Compare one quantity to pigeons
- Compare another quantity to pigeonholes
- Require that the number of pigeons be greater than the number of pigeonholes

This means that we can then apply the pigeonhole principle.

Example 1.40

A school has five different subjects to be taught: Maths, Science, Humanities, Languages, and Arts. Manya is interviewing teachers in the school. What is the minimum number of teachers she must interview such that she is guaranteed an interview with two teachers teaching the same subject?

We can assign 5 teachers to 5 distinct subjects.

Maths	Science	Humanities	Languages	Arts
Teacher 1	Teacher 2	Teacher 3	Teacher 4	Teacher 5

The sixth teacher must be assigned to a subject which has already been assigned before.
Hence, six teachers are sufficient to guarantee repetition.

1.41: Bijection Method

The bijection method says that you set up a $1-1$ correspondence between two situations or two scenarios. And then convert one problem in terms of another problem.

Example 1.42

A school has five different subjects to be taught: Maths, Science, Humanities, Languages, and Arts. Manya is interviewing teachers in the school. What is the minimum number of teachers she must interview such that she is guaranteed an interview with two teachers teaching the same subject?

Solve this using the Bijection Method

Let

$$Subjects \leftrightarrow Pigeonholes = p$$
$$Teachers \leftrightarrow Pigeons = q$$

If

$$q > p \Rightarrow 2\ pigeons\ share\ a\ pigeonhole \Rightarrow 2\ Teachers\ share\ a\ subject$$
$$Minimum\ number\ of\ Teachers = q > p = No.of\ Subjects$$
$$q > 5 \Rightarrow Smallest\ value\ that\ q\ can\ take\ is\ 6$$

Example 1.43

Luke visits a relative whenever they invite him. How many different relatives must he visit before he is guaranteed that he has visited two different relatives on the same day of the week?

Diagram Method

We can assign relatives to distinct days of the week. We can do this for seven relatives.

Mon	Tue	Wed	Thu	Fri	Sat	Sun
Relative 1	Relative 2	Relative 3	Relative 4	Relative 5	Relative 6	Relative 7

The eighth relative must be assigned to a day which has already been assigned before.
Hence, eight relatives are sufficient to guarantee repetition.

Example 1.44

Leena is choosing from among the factors of 6. She has written each factor out on a block, and put the blocks on a table. After choosing a block, she puts it back on the table. How many times must she choose a factor before it is guaranteed that the factor she has chosen is one that she has already chosen before.

The factors of 6 are:

$$\{1,2,3,6\} \Rightarrow 4\ Factors$$

We can assign a distinct factor to a pick.
We can do this four times, maximum.

Then, in the fifth pick, we must pick a factor which has already been picked.

Challenge 1.45

Leena is choosing from among the factors of 144. She has written each factor out on a block, and put the blocks on a table. After choosing a block, she puts it back on the table. How many times must she choose a factor before it is guaranteed that the factor she has chosen is one that she has already chosen before.

Method I

We can count the number of factors of 144 using factor pairs:

$$(1,144), (2,72), (3,48), (4,36), (6,24), (8,18), (9,16), (12,12)$$

$$7 \times 2 + 1 = 14 + 1 = 15$$

Method II

We can use the formula for the number of factors:

$$\tau(144) = \tau(12^2) = \tau((2^2 \times 3)^2) = \tau(2^4 \times 3^2) = (4+1)(2+1) = 5(3) = 15$$

$$\{2^0, 2^1, 2^2, 2^3, 2^4\}\{3^0, 3^1, 3^2\}$$

F. Application: Socks of Two Colours

A standard application of the pigeonhole principle is in drawing socks in order to form a pair. The usual conditions are that:

- The sock is drawn at random.
- There is no light in the room in which the socks are. This has the same effect as saying that the socks are drawn at random.

Example 1.46

I have eight identical black socks and twelve identical white socks in a drawer. The room in which the drawer is, is pitch dark. Hence, I cannot see what is the color of the sock I am taking out of the drawer. What is the minimum number of socks to be drawn so that I am guaranteed a pair of socks of the same color?

Hint: There is no requirement that the socks be of a particular color.

Consider drawing two socks.
There are two cases:

Case I: Both socks are of the same color.
We get a pair. For this case, 2 is sufficient.

Case II: The two socks are of different colors
Then, we can assign two different colors to the first two socks that we draw, giving us:

$1st\ Sock$	$2nd\ Sock$
Black	White

The third sock must be either white or black, and hence a color must repeat, and hence, we must get a pair. Hence, three is the minimum needed to guarantee a pair.

Example 1.47
I have eight identical black socks and twelve identical white socks in a drawer. The room in which the drawer is, is pitch dark. Hence, I cannot see what is the color of the sock I am taking out of the drawer. What is the minimum number of socks to be withdrawn such that I am guaranteed a pair which is white?

Nine Socks does not work because you might end up with:
- Eight black socks
- One White Sock

Ten socks works because:
- The maximum number of black socks is eight.
- Hence, the remaining two socks must be white.

Example 1.48
I have eight identical black socks and twelve identical white socks in a drawer. The room in which the drawer is, is pitch dark. Hence, I cannot see what is the color of the sock I am taking out of the drawer. What is the minimum number of socks to be withdrawn such that I am guaranteed a pair which is black?

13 socks do not work:
- 12 White socks
- 1 Black sock

14 socks do work:
- Maximum number of white socks is 12
- Remaining two socks must be black

1.49: Worst Case Scenario Method
Show that
- k does not work under the worst case scenario.

> ➢ $k+1$ works under any scenario.

G. Socks of Multiple Colors

Example 1.50

I have two pairs of identical green socks, four pairs of identical grey socks, and six identical blue socks in a drawer. What is the minimum number of socks to withdrawn from the drawer such that I am guaranteed a

A. pair of any color?
B. a green pair?
C. a grey pair?
D. a blue pair?

Part A

Three socks can give you socks of different colors

$$Green + Grey + Blue$$

Four

Part B

$$8\,Grey\ +\ 6\,Blue\ +\ 2\,Green\ =\ 16$$

Part C

$$4\,Green\ +\ 6\,Blue\ +\ 2\,Grey\ =\ 12$$

Part D

$$4\,Green\ +\ 8\,Grey\ +\ 2\,Blue\ =\ 14$$

Example 1.51

A parking lot contains cars from different countries: 10 from Brazil, 12 each from Germany and Japan, 25 from the US, and 17 from India. If cars are chosen on a random basis, how many cars must be chosen to guarantee at least 15 cars from a single country.

Consider the worst-case scenario:

$$10+12+12+14+14=62$$

Final Answer:

$$=62+1=63$$

Example 1.52

A box contains 100 balls of different colors: 28 Red, 17 Blue, 21 Green, 10 white, 12 Yellow, and 12 Black. The smallest number of balls drawn from the box so that at least 15 balls are of the same color is: (NMTC Primary-Screening, 2012/A/10)

Since we want a guarantee, we consider the worst-case scenario:

$$\underbrace{10}_{White}+\underbrace{12}_{Yellow}+\underbrace{12}_{Black}+\underbrace{14}_{Green}+\underbrace{14}_{Blue}+\underbrace{14}_{Red}=76\,Balls$$

Now, if we draw one ball, we must have at least one color (Green, or Blue, or Red) for which we have 15 balls.

Hence, the final answer is:

$$76 + 1 = 77$$

H. Generalized Pigeonhole Principle

1.53 Generalized Pigeonhole Principle

If there are N pigeons in m pigeonholes, there is at least one pigeonhole containing:

$$\left\lceil \frac{N}{m} \right\rceil \text{ pigeons}$$

Where $\lceil \quad \rceil$ means that the number inside is rounded upwards.

- The pigeonhole principle guarantees two pigeons in one pigeonhole.
- The generalized pigeonhole principle guarantees $\left\lceil \frac{N}{m} \right\rceil$ pigeons in one pigeonhole.

Example 1.54

Suppose you have 10 pigeons and they roost in three pigeonholes. Then, prove that there is at least one pigeonhole that contains at least four pigeons.

Method I

Let's suppose that no pigeonhole contains four or more pigeons.
Then, assign pigeons to pigeonholes. There is only one way to do it:

- Each pigeonhole gets three pigeons
- There is one pigeon left over

Pigeonhole 1	Pigeonhole 2	Pigeonhole 3
Pigeon 1	Pigeon 2	Pigeon 3
Pigeon 4	Pigeon 5	Pigeon 6
Pigeon 7	Pigeon 8	Pigeon 9

That pigeon must be assigned to some pigeonhole.
And hence, that pigeonhole will have four pigeons.

Method II

$$\left\lceil \frac{10}{3} \right\rceil = \left\lceil 3\frac{1}{3} \right\rceil = 4$$

Example 1.55

Shirley has seven different activities that she likes: horse-riding, mountaineering, swimming, chess, wrestling, skydiving, and cycling. Each day of the week in the year of 2021, starting from Jan 1, she randomly picks an activity to do. How many days must pass before she is guaranteed that she has done at least one activity four times?

Table Method

Do a different activity on each day of the week:

Mon	Tue	Wed	Thu	Fri	Sat	Sun
Horse-riding	Mountaineering	Swimming	Chess	Wrestling	Skydiving	Cycling
Horse-riding	Mountaineering	Swimming	Chess	Wrestling	Skydiving	Cycling
Horse-riding	Mountaineering	Swimming	Chess	Wrestling	Skydiving	Cycling
Horse-riding						

In 21 days, you do each activity 3 times.
On the 22nd day, no matter which activity you choose, that activity will be repeated a fourth time.

Formula

$$\left\lceil \frac{x}{7} \right\rceil = 4$$
$$\left\lceil \frac{21}{7} \right\rceil = \lceil 3 \rceil = 3$$
$$\left\lceil \frac{22}{7} \right\rceil = \left\lceil 3\frac{1}{7} \right\rceil = 4$$
$$Smallest\ value\ of\ x = 22$$

Example 1.56

Shannon is Shirley's brother. He does not like horses, so no horse-riding for him. Also, he is not intellectually inclined, so he does not play chess. Otherwise, he likes the same activities that Shirley does. If he starts from Jan 1, 2021, and randomly picks an activity to do each day, how many days must pass so that he is guaranteed that he has done at least one activity seven times?

Table Method

Do a different activity on each day of the week:

Mountaineering	Swimming	Wrestling	Skydiving	Cycling
Mountaineering	Swimming	Wrestling	Skydiving	Cycling
Mountaineering	Swimming	Wrestling	Skydiving	Cycling
Mountaineering	Swimming	Wrestling	Skydiving	Cycling
Mountaineering	Swimming	Wrestling	Skydiving	Cycling
Mountaineering	Swimming	Wrestling	Skydiving	Cycling
Mountaineering				

In 30 days, you do each activity 5 times.
On the 31nd day, no matter which activity you choose, that activity will be repeated a fourth time.

Formula

$$\left\lceil \frac{x}{5} \right\rceil \geq 7$$
$$\left\lceil \frac{31}{5} \right\rceil = \left\lceil 6\frac{1}{5} \right\rceil = 7$$
$$Smallest\ value\ of\ x = 31$$

Example 1.57

A five-legged Martian has a drawer full of socks, each of which is red, white or blue, and there are at least five socks of each color. The Martian pulls out one sock at a time without looking. How many socks must the Martian remove from the drawer to be certain there will be 5 socks of the same color? **(AMC 8 2005/16)**

Logic

Consider that, in the worst case scenario, you can get
$$Red = White = Blue = 4\ each$$
$$Total = 12$$

Then you can draw one more sock to get five socks of the same color:

$$12 + 1 = 13$$

Formula

$$\left\lceil \frac{x}{3} \right\rceil \geq 5$$

$$\left\lceil \frac{12}{3} \right\rceil = \lceil 4 \rceil = 4$$

$$\left\lceil \frac{13}{3} \right\rceil = \left\lceil 4\frac{1}{3} \right\rceil = 5$$

$$\textit{Smallest value is } 13$$

Table Method

Red	White	Blue
Sock 1	Sock 5	Sock 9
Sock 2	Sock 6	Sock 10
Sock 3	Sock 7	Sock 11
Sock 4	Sock 8	Sock 12
Sock 13		

1.58 Clever Scenarios

It is not necessary that questions on pigeon hole principle be framed in the language of pigeons and pigeonholes.

Difficult questions can be framed in language different from the usual "socks" questions which are often easy to recognize as pigeonhole principle questions.

Example 1.59

A pistol shooting competition has each competitor shoot at six targets. 5 points are given for hitting a target, and 0 points for missing or not attempting a target. What is the minimum number of competitors required to ensure that at least 3 of them get the same number of points?

Method I: Worst Case Scenario

Note that the distinct scores are

$$0,5,10,15,20,25,30 \Rightarrow 7\ different\ values$$

14 competitors can score in such a way that two of them each get a distinct score.
The 15th candidate must repeat one of the scores, and hence 15 is the answer.

Method II: Convert to Pigeons and Pigeonholes

$$Possible\ scores = Pigeonholes$$

$$Actual\ Scores = Pigeons$$

We need to find minimum of pigeons in 7 pigeonholes such that at least pigeonhole has 3 pigeons:

$$\left\lceil \frac{x}{7} \right\rceil \geq 3$$

$$\left\lceil \frac{14}{7} \right\rceil = \lceil 2 \rceil = 2$$

$$\left\lceil \frac{15}{7} \right\rceil = \left\lceil 2\frac{1}{7} \right\rceil = 3$$

Example 1.60

A question paper has six questions. 5 marks are given for a correct answer, 0 marks for not attempting a question, and -1 marks for getting a question wrong. What is the minimum number of students that ensures that at least 3 students have the same marks?

Possible points are:

$$0-\{0,1,2,3,4,5,6\}=\{\mathbf{0},-1,-2,-3,-4,-5,-6\}\Rightarrow 7\ Values$$
$$One\ Correct: 5-\{0,1,2,3,4,5\}=\{5,4,3,2,1,\mathbf{0}\}\Rightarrow 6\ Values$$
$$Two\ Correct: 10-\{0,1,2,3,4\}=\{10,9,8,7,6\}\Rightarrow 5\ Values$$
$$Three\ Correct: 15-\{0,1,2,3\}=\{15,14,13,12\}\Rightarrow 4\ Values$$
$$Four\ Correct: 20-\{0,1,2\}=\{20,19,18\}\Rightarrow 3\ Values$$
$$Five\ Questions\ Correct: 25-\{0,1\}=\{25,24\}\Rightarrow 2\ Values$$
$$Six\ Correct: 30-\{0\}=\{30\}\Rightarrow 1\ Values$$

Total Number of Values:

$$7+6+5+4+3+2+1-1=27\ Values$$

$$Minimum\ No.of\ Competitions=27\times 2+1=55$$

1.5 Pigeonhole Principle-II: Applications

A. Numbers

Example 1.61

There are 226 cards bearing two-digit numbers. What is the minimum number of cards that must be discarded if we keep only the cards that bear unique numbers?

Number of two-digit numbers

$$=99-9=90$$

Minimum number of cards to be discarded

$$= No.of\ cards - No.of\ unique\ two-digit\ numbers$$
$$= 226-90 = 136$$

B. Sums

Example 1.62

In a beach clean-up drive, a set of volunteers went to clean up the beach. At the end of the drive, the waste collected was filled into bins. Someone noticed that each volunteer had filled up a different number of bins.

A. If the number of volunteers was 3, what is the minimum number of bins filled?
B. If the number of volunteers was 5, what is the minimum number of bins filled.
C. If the number of bins filled was 6, what is the minimum number of volunteers?
D. If the number of bins filled was 21, what is the minimum number of volunteers?

Part A

$$0+1+2=3\Rightarrow Min=3$$

Part B

$$0+1+2+3+4=10\ Bins\ Minimum$$

Part C

Assuming that volunteers with the "s" means more than one:

$$4\ Bins + 2\ Bins = 2\ Volunteers$$

Part D

$$10\ Bins + 11\ Bins = 21\ Volunteers$$

1.63 Proof by Contradiction

- Assume the contrary.
- Show that it leads to a contradiction.
- Conclude that the original assumption was not true.
- Hence, what you wish to prove is true.

Example 1.64

Show if you pick 501 numbers from the first 1000 natural numbers, then there must be two numbers that add up to 1001.

Form Pairs

Form pairs of numbers which add up to 1001, as follows:

$$\{(1,1000)(2,999)(3,998)\dots(499,502)(500,501)\} \Rightarrow 500\ unordered\ pairs$$

Assume to the Contrary

Assume, to the contrary, that we can pick 501 numbers out of which no two add up to 501.

Apply PHP

If we pick two numbers which both belong to the same pair, then their total must be 501. Hence, we must pick only a single number from each pair.
However, there are 500 pairs, and we need 501 numbers.
Therefore, there is a contradiction.

Resolve the Contradiction

We got a contradiction because our original assumption was wrong.
Therefore, if you pick 501 numbers from the first 1000 natural numbers, there must be at least 2 numbers that add up to 1001.

Example 1.65

What is the minimum number of numbers that you need to pick from the natural numbers from 1 to 100 to guarantee that you get a sum of 101?

For natural numbers a and b we need $a + b = 101, 1 \le a, b \le 100$ which has solutions:

$$(a, b) = (1,100), (2,99), \dots, (50,51) \Rightarrow 50\ Pairs$$

By pigeonhole principle, we can pick maximum 50 numbers (one from each pair).
So,

$$51\ numbers\ is\ sufficient$$

C. Divisibility

Example 1.66

Consider the numbers 1 to 20. What is the maximum number of numbers that you can pick such that no two numbers have a sum divisible by 4.

Consider the remainders when a number is divided by 4:

Remainder Zero:

We can pick at most one number that remainder zero when divided by 4. Because, if we pick two numbers, then their sum will also be divisible by 4.

Remainders One and Three

Out of numbers which have remainders one and three, we cannot pick both types of numbers. This is because

$$\underbrace{4x+1}_{Remainder\ 1} + \underbrace{4x+3}_{Remainder\ 3} = 8x+4 = \underbrace{4(2x+1)}_{Muliple\ of\ 4}$$

Suppose, we choose to pick numbers with remainder one. We can pick at most three numbers with remainder one because if we pick four numbers, we will get:

$$4x+1+4x+1+4x+1+4x+1 = 20x+4 = 4(5x+1)$$

Suppose, we choose to pick numbers with remainder three. We can pick at most three numbers with remainder one because if we pick four numbers, we will get:

$$4x+3+4x+3+4x+3+4x+3 = 20x+12 = 4(5x+3)$$

Hence, from the numbers with remainders one and three, we can pick:

- three numbers with remainder One OR
- three numbers with remainder three OR

We cannot pick both.

Remainder Two

We can pick a single number with remainder two. Because if we pick numbers with remainder two, they will add up a multiple of 4.

Summarize

Remainder	Maximum Numbers	Example
Zero	One	4
One or Three	Three Numbers	5,9,13
Two	One Number	2
Total	5 Numbers	

Example 1.67

Consider the numbers 1 to 20. What is the maximum number of numbers that you can pick such that no two numbers have a sum divisible by 4.

Remainder		Max Nos. Picked	Final Answer
Zero	4, 8, 12, 16, 20	One Number	1
One	1,5,9,13,17,	Three Numbers	3
Two	2,6,10,14,18	One Number	1
Three	3,7,11,15,19	Three Numbers	0
		Total	5

Also, note that any two numbers, one of which has remainder one, and the other has remainder three, will add up to a multiple of four.
Hence, we can either pick numbers with remainder one, or remainder three, but not both.

$$0+1+2+3+5$$

D. Proof by Contradiction

The main steps of a proof by contradiction are:

- Assume the exact opposite of what we wish to prove
- Arrive at some logical and/or mathematical conclusions based on the assumption.
- Show that the conclusions are
  - ✓ illogical or
  - ✓ contradict either established mathematical facts
  - ✓ or data given in the question itself
- Conclude that the contradiction arose because of the incorrect assumption
- Hence, conclude that the assumption is wrong
  - ✓ Hence the opposite of the assumption is true.

Example 1.68

A car showroom has five different car salesmen. At the end of January 2020, it was found that a total of 9 cars had been sold. Prove that at least two salesmen would have sold the same number of cars.

Assume the Contrary

Assume, to the contrary, that 9 cars have been sold such that every car salesman has a different number of cars sold.

Find minimum number of cars sold

Minimum number of cars sold can only be zero.
Assign zero cars to the first car salesman.
Then, the minimum cars sold by the second salesman must be at least one (since the number of cars sold by each salesman is different).
Similarly, assign cars sold to each of the five salesmen:

$$\underbrace{0}_{\substack{First\\Salesman}}+\underbrace{1}_{\substack{Second\\Salesman}}+\underbrace{2}_{\substack{Third\\Salesman}}+\underbrace{3}_{\substack{Fourth\\Salesman}}+\underbrace{4}_{\substack{Fifth\\Salesman}}=10$$

Find a logical contradiction and, hence, the assumption is wrong

$$\underbrace{10}_{\substack{Minimum\\cars\ sold}}>\underbrace{9}_{\substack{Actual\\cars\ sold}}\Rightarrow Minimum>Actual\Rightarrow Contradiction\Rightarrow Wrong\ Assumption$$

Conclude that the opposite of the original assumption is true

Hence, if 9 cars have been sold, then the same of numbers car must have been sold by two different car salesmen.

E. Number Theory

Example 1.69

Show that if you make pairs of these six numbers, the product of the numbers in at least one pair exceeds eight.

$$\{1,2,3,4,5,6\}$$

First, find the product of all the number in the set:

$$1 \times 2 \times 3 \times 4 \times 5 \times 6 = 10 \times 3 \times 5 \times 6 = 720$$

Assume to the contrary, that you can make pairs such that the product of each pair does not exceed 8. Hence, the maximum value of the product of each pair is 8.

Let each product have the maximum value. Then, we will have

$$x = \underbrace{P_1}_{\substack{\text{Product of}\\ \text{Pair 1}}} \times \underbrace{P_2}_{\substack{\text{Product of}\\ \text{Pair 2}}} \times \underbrace{P_3}_{\substack{\text{Product of}\\ \text{Pair 3}}} = 8 \times 8 \times 8 = 512$$

$$x < 720 \Rightarrow Contradiction \Rightarrow Assumption\ is\ wrong \Rightarrow At\ least\ one\ pair\ must\ exceed\ 8$$

Example 1.70

Given three integers x_1, x_2 and x_3 show that either $\underbrace{one\ of\ them\ is\ divisible\ by\ 3}_{A}$ or $\underbrace{a\ sum\ of\ several\ of\ them\ is\ divisible\ by\ 3}_{b}$.

Assume the Contrary

Assume to the contrary that neither A nor B is true.

List Remainders

The possible remainders when a number is divided by three are

$$0, 1, 2$$

Because of Statement A, we know that none of the numbers have remainder

$$0$$

Hence, the only possible remainders are

$$1,2$$

List the Cases

If the remainders are 1,2, the only possible cases for the remainders of x_1, x_2 and x_3 are:

- All have remainder one
- All have remainder two
- One has remainder one, and the other two have remainder two
- One have remainder two, and the other two have remainder one

Or some arrangement of these remainders, but the arrangement does not concern us.

Case	x_1	x_2	x_3	
1	1	1	1	3
2	2	2	2	6
3	1	2	2	3
4	2	1	1	3

Example 1.71

A. Prove that there exists a natural number that $\underbrace{consists\ of\ only\ one's\ and\ zeros}_{A}$ which is $\underbrace{divisible\ by\ 2020}_{B}$.

> B. Prove that there exists a natural number that $\underbrace{consists\ of\ only\ a's\ and\ zeros}_{A}$ which is $\underbrace{divisible\ by\ 2020}_{B}$ where $1 \le a \le 9$

Part A

Consider 2021 numbers, each made of only ones, such

$$n_1 = 1, n_2 = 11, n_3 = 111, \dots., n_{2021} = \underbrace{111 \dots 1}_{2021\ Ones}$$

Find the remainder when these 2021 numbers are divided by 2020:

$$R\left(\frac{n_1}{2020}\right) = r_1, \qquad R\left(\frac{n_2}{2020}\right) = r_2, \dots, R\left(\frac{n_{2021}}{2020}\right) = r_{2021}$$

Now, the range of remainders that a number has when divided 2020 is:

$$0,1,2,3 \dots 2019 \Rightarrow 2020\ Numbers$$

But, we have 2021 remainders. Hence, by the pigeonhole principle, the remainders must be the same for some two values of r_x. Let the number that have the same remainders be:

$$n_p \text{ and } n_q$$

And let the common remainder of n_p and n_q be R

$$n_q - n_q = (2020m_1 + R) + (2020m_2 + R) = 2020m_1 - 2020m_2 = 2020(m_1 - m_2) \to \underbrace{multiple\ of\ 2020}_{B\ is\ satisfied}$$

$$n_q - n_p = \underbrace{Some\ Number\ made\ of\ only\ zeros\ and\ ones}_{A\ is\ satisfied}$$

Part B

Consider 2021 numbers, each made of only $a's$, such

$$n_1 = a, n_2 = aa, n_3 = aaa, \dots., n_{2021} = \underbrace{aaa \dots a}_{2021\ a's}$$

And then the rest of the proof is exactly the same as Part A.

Example 1.72

Prove that there are two powers of 3 that differ by a multiple of 2020.

Consider the first 2021 powers of three:

$$n_1 = 3^1, n_2 = 3^2, n_3 = 3^3, \dots, n_{2021} = 3^{2021}$$

Find the remainder when these powers are divided by 2020;

$$R\left(\frac{n_1}{2020}\right) = r_1, \qquad R\left(\frac{n_2}{2020}\right) = r_2, \dots, R\left(\frac{n_{2021}}{2020}\right) = r_{2021}$$

The maximum number of remainders when a number is divided by 2020 is:

$$\{0,1,2, \dots, 2019\} \Rightarrow 2020\ Remainders$$

However, there are 2021 remainders

$$\{r_1, r_2, \dots, r_{2021}\}$$

And only

$$2020\ distinct\ values\ that\ they\ can\ take$$

Hence, by the Pigeonhole Hole,

$$Two\ numbers\ must\ have\ the\ same\ remainder$$

Hence, their difference

$$must\ be\ a\ multiple\ of\ 2020.$$

Example 1.73

A is a set of 2004 positive integers. Show that there is a pair of elements in A whose difference is divisible by 2003. **(NMTC Sub-Junior/Final 2004/3)**

Let

$$Set\ A = \{x_1, x_2, x_3, \dots, x_{2004}\} \Rightarrow 2004\ elements$$

Divide each number in the set above by 2003, and write the remainder in a new set:

$$Set\ B = \{r_1, r_2, r_3, \dots, r_{2004}\} \Rightarrow 2004\ elements$$

But the possible remainders when dividing a number by 2003 are:

$$R = \{0,1,2, \dots ,2002\} \Rightarrow 2003\ Distinct\ Remainders$$

By the Pigeonhole Principle, since Set B has 2004 elements, and Set R has only 2003 distinct remainders, there must be

$$At\ least\ two\ numbers\ in\ the\ Set\ A\ that\ have\ the\ same\ remainder\ when\ divided\ by\ 2003.$$

Without loss of generality, let the two numbers that have the same remainder be:

$$a_1 = 2003y_1 + r_x, y_1 \in \mathbb{Z}$$
$$a_2 = 2003y_2 + r_x, y_2 \in \mathbb{Z}$$

Then, their difference is:

$$a_1 - a_2 = 2003y_1 + r_x - (2003y_2 + r_x) = 2003y_1 - 2003y_2 = 2003(y_1 - y_2) \Rightarrow Multiple\ of\ 2003$$

Example 1.74

Show that $(abc)(a^3 - b^3)(b^3 - c^3)(c^3 - a^3)$ is divisible by 7. **(NMTC Sub-Junior/Final 2005/7b)**

Part A

Remainders of cubes of numbers when divided by seven:

	Number	Cube	Remainder
$7k + 1$	0	0	0
$7k + 2$	1	1	1
	2	6	6
	3	27	6
	4	64	1
	5	125	6
	6	216	6

The pattern continues. If you want to prove, you can do this

$$(7k + n)^3 = 343k + (3 \times 49)(k^2 n) + (7 \times 9)(kn^2) + n^3$$

Note that all except the last term are divisible by n.

Hence, to check the remainders of the cubes of the numbers when divided by n, it suffices to check the remainders of the numbers upto 6.

Part B: Case I

If any of a, b, c number is divisible by

F. Geometry

Example 1.75

What is the minimum number of smaller equilateral triangles needed to cover a larger equilateral triangle?

We have not been told the triangles do not overlap.
Consider an equilateral triangle which is to be covered by smaller equilateral triangles.

- The first triangle (shown in the first diagram, can cover the first vertex, but no other vertex).
- The second triangle can only cover the second vertex.

Hence, in order to cover the entire triangle, we need a minimum of three triangles.

Example 1.76

Three distinct points are plotted on a circle. Show that at least two points form a reflex angle with the center of the circle as a vertex.

Assume, to the contrary, that:
No two of the three points form a reflex angle with the center of the circle as a vertex
Start adding points to the Circle
Let the points be A, B and C.
Without loss of generality, place the first point anywhere in the circle.
We do not want a reflex angle, so the only place that we can put B is diametrically opposite to A, so that we form a straight angle.
Add the third point
Now, if we try to add the third point, no matter where we place it, it will form a reflex angle with the other two points.
Contradiction
This was because our original assumption was wrong.

Hence, if three distinct points are plotted on a circle, at least two of them must form a reflex angle with the center of the circle as a vertex.

Example 1.77

Five distinct points are plotted on a circle. Show that at least two points form an acute angle with the Centre of the circle as the vertex.

Assume, to the contrary, that:
No two of the five points form an acute angle with the centre of the circle as a vertex
Start adding points to the Circle
Let the points be A, B, C, D and E..
Without loss of generality, place the first point anywhere in the circle.
We do not want an acute angle, so the minimum angular distance that we can put B is 90° away from A.
Similarly, add C and D at a distance of 90° from B and D.
Add the fifth point
Now, if we try to add the fifth point, no matter where we place it, it forms an acute angle with the points adjacent to it

Contradiction
This was because our original assumption was wrong.

Hence, if five distinct points are plotted on a circle, at least two of them must form an acute angle with the center of the circle as a vertex.

Example 1.78
Ten distinct points are plotted on a circle. The angles made by the points with the centers of the circles are measured and found to be integers. Show that at least one angle must be 36 degrees or more.

Assume, to the contrary, that:
No two of the ten points form an angle measuring 36° or more with the centre of the circle as a vertex

Start adding points to the Circle
Let the points be

$$p_1, p_2, p_3, \dots, p_{10}$$

Without loss of generality, place p_1 anywhere in the circle.
We do not want an angle 36°, so the maximum angular distance that we can put p_2 away from p_1 is35° .
Similarly, add points

$$p_3, p_4, \dots . p_9$$

Add the tenth point
Now, if we try to add p_{10}, no matter where we place it
Contradiction
This was because our original assumption was wrong.

Hence, if ten distinct points are plotted on a circle, at least two of them must form an acute angle with the center of the circle as a vertex.

1.6 Combinatorial Identities

A. Introduction
Combinatorial identities are identities that involve combinatorial expressions and hence are encountered while.

1.79: Story Proofs
A "story proof" is a scenario or a situation that can be used to explain why an identity is true.

Example 1.80
Show using a story proof that

$$\binom{n}{k} = \binom{n}{n-k}$$

Story Proof
Choosing a committee of k from n people is the same as rejecting the remaining $n-k$ people.
Hence, they count the same thing.

Algebraic Proof

$$\frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,k!}$$

1.81: Algebra Proofs

Algebra proofs can require a mix of techniques from a variety of topics in Algebra:

- Definition and properties of Combinations
- Sequences and Series
- Binomial Theorem
- Summation Notation and Properties
- Other combinatorial identities

Example 1.82

Give an algebraic proof

$$\sum_{k=0}^{k=n}\binom{n}{k}(-2)^k=(-1)^n$$

$$LHS=\sum_{k=0}^{k=n}\binom{n}{k}(-2)^k(1)^{n-k}=(-2+1)^n=(-1)^n=RHS$$

Example 1.83

Give a story proof

$$\sum_{k=0}^{k=n}2^k\binom{n}{k}=3^n$$

$$LHS=\sum_{k=0}^{k=n}2^k\binom{n}{k}=\sum_{k=0}^{k=n}\binom{n}{k}2^k1^{n-k}=(2+1)^n=3^n=RHS$$

Left Hand Side

There are n numbers. Obtain k chosen numbers and $n-k$ "not chosen" numbers in

$$\binom{n}{k}$$

Classify each of the k chosen numbers to be "good", or "not-good", which can be in

$$2^k\ ways$$

Consider cases on value of k:

$$2^0\binom{n}{0}+2^1\binom{n}{1}+\cdots+2^n\binom{n}{n}=\sum_{k=0}^{k=n}2^k\binom{n}{k}$$

Right Hand Side

Each number can be classified into "good", "not-good", or "not chosen" in three ways. The total number of ways to classify is:

$$3^n$$

Example 1.84

$$\sum_{k=1}^{n}k=\binom{n+1}{2}$$

Algebraic Proof

$$1 + 2 + 3 + \cdots (n-1) + n = \frac{n(n+1)}{2}$$

Story Proof

Count the number of maximum distinct handshakes at a party with $n+1$ attendees.
That is, everyone shakes hands with everyone else.

Right Hand Side

A handshake requires two people – also called a pair.
Since the order is not important, we are choosing (combination), and *not* arranging (permutation).

The number of handshakes at the party is the number of ways of choosing pairs from $n+1$ people:

$$\binom{n+1}{2} \; ways = RHS$$

Left Hand Side

Number the people in the party

$$1,2,3, \dots, n, n+1$$

The person numbered

$$n+1 \; can \; shake \; hands \; with \; \{1,2,3, \dots, n\} = n \; people$$
$$n \; can \; shake \; hands \; with \; \{1,2,3, \dots, n-1\} = n-1 \; people$$
$$\vdots$$
$$2 \; can \; shake \; hands \; with \; \{1\} = 1 \; person$$
$$1 \; has \; already \; shaken \; hands \; with \; everyone$$

Note the number keeps reducing since order is not important, and A shaking hands with B is the same as B shaking hands with A.

The total number of handshakes is then:

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} k = LHS$$

B. Team Captain Identity

1.85: Team Captain Identity: Story Proof

$$n\binom{n-1}{k-1} = k\binom{n}{k}$$

Consider a set of n players, from which a team of k is to be chosen, and a captain is to be chosen from within the team.

Left Hand Side

Choose a captain from among the n players in:

$$n \; ways$$

Choose the remaining $k-1$ team members from the remaining $n-1$ players in

$$\binom{n-1}{k-1} \; ways$$

The total number of ways is, by the multiplication principle:

$$n\binom{n-1}{k-1} \ ways = LHS$$

Right Hand Side

Choose a team of k from the n players in

$$\binom{n}{k} \ ways$$

Choose a captain from among the k teammates in

$$k \ ways$$

The total number of ways is, by the multiplication principle:

$$k\binom{n}{k} = RHS$$

Since we are counting the same scenario

$$LHS = RHS$$

1.86: Team Captain Identity: Algebra Proof

$$n\binom{n-1}{k-1} = k\binom{n}{k}$$

$$LHS = n\binom{n-1}{k-1} = n \cdot \frac{(n-1)!}{(k-1)!\,[n-1-(k-1)]!}$$

Substitute $n(n-1)! = n!$

$$= \frac{k}{k} \cdot \frac{n!}{(k-1)!\,[n-k]!} = k \cdot \frac{n!}{k!\,[n-k]!} = k \cdot \binom{n}{k} = RHS$$

Example 1.87

Use a modified version of the “team captain” method to show that:

$$\binom{2n}{n}\binom{n}{2} = \binom{2n}{2}\binom{2n-2}{n-2}, \qquad n \geq 2$$

Choose a team of n out of $2n$ players, out of which 2 people will be co-captains.

Left Hand Side

Choose the team in $\binom{2n}{n}$ *ways*

Choose the co-captains in $\binom{n}{2}$ *ways*

$$\binom{2n}{n}\binom{n}{2} = LHS$$

Right Hand Side

Choose the captains first in $\binom{2n}{2}$ *ways*

Choose the team in $\binom{2n-2}{n-2}$

$$\binom{2n}{2}\binom{2n-2}{n-2}$$

Example 1.88

$$\binom{2n}{n}\binom{n}{2} = \binom{2n}{2}\binom{2n-2}{n-2}, \qquad n \geq 2$$

An alien creature has n legs and 2 hands. It wants to wear
- n socks of n distinct colors on its legs
- gloves on its hands such that the color of the gloves matches the color of a sock, and each glove has a different color.

The total number of colors to choose from is $2n$.

Left Hand Side

Choose the colors of the socks first:

$$\binom{2n}{n}\ ways$$

From those n colors, choose the colors of the gloves:

$$\binom{n}{2}\ ways$$

By the multiplication principle, the total number of ways

$$= \binom{2n}{n}\binom{n}{2} = LHS$$

Right Hand Side

Choose the colors of the gloves first:

$$\binom{2n}{2}\ ways$$

The number of colors left is $2n-2$. Since the gloves have been chosen, the colors of two socks have also been chosen. So, the number of ways to choose $n-2$ socks from $2n-2$ colors is:

$$\binom{2n-2}{n}\ ways$$

By the multiplication principle, the total number of ways

$$= \binom{2n}{2} \times \binom{2n-2}{n-2} = LHS$$

C. Choosing a committee: Expansion of 2^n

1.89: Expansion of 2^n: Story Proof

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

Consider that we have n people from whom we must choose a committee (of any size).

Left Hand Side

The choice for each person is:

$$Select\ OR\ Do\ not\ select \Rightarrow 2\ choices$$

The total number of choices is:

$$\underbrace{2 \times 2 \times \ldots \times 2}_{n\ times} = 2^n$$

Right Hand Side

$$Choose\ 0\ people\ \binom{n}{0}$$

$$Choose\ 1\ person\ \binom{n}{1}$$

$$\vdots$$

$$Choose\ n\ people\ to\ go\ in\ the\ box: \binom{n}{n}$$

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=RHS$$

1.90: Expansion of 2^n: Story Proof

$$2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}$$

Choose zero or more of n bits to be flipped.
Choices for each bit (flip or do not flip)

$$2^n$$

Decide number of bits to flip and then choose those bits

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}$$

1.91: Expansion of 2^n: Pascal's Triangle Proof

$$2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}$$

Example 1.92

1.93: Expansion of 2^n: Binomial Theorem Proof

$$2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}$$

Rewrite the RHS:

$$\binom{n}{0}1^{n-0}1^0+\binom{n}{1}1^{n-1}1^1+\binom{n}{2}1^{n-2}1^2+\cdots+\binom{n}{n}1^{n-n}1^n$$

Using the Binomial Theorem:

$$=(1+1)^n=2^n=LHS$$

Challenge 1.94

Let $(a_1, a_2, \dots, a_{10})$ be a list of the first 10 positive integers such that for each $2 \le i \le 10$ either a_i+1 or a_i-1 or both appear somewhere before a_i in the list. How many such lists are there? **(AMC 10B 2012/22)**[1]

Consider cases for the first number $=f=1$
$f=1$. There is one possibility:

$$1,2,\dots,10=1=\binom{9}{0}\ Ways$$

$f=2$. The numbers greater than 1 must necessarily be arranged in strictly ascending order. We need to choose a position for 1 out of 9 positions after the 2, which can be done in

[1] Solution by Richard Rusczyk.

$$\binom{9}{1} = 9\ Ways$$

$f = 3$. The freedom now extends to arranging 1 and 2 wherever we want among the 9 available positions, and the rest must be in ascending order, which can be done in

$$\binom{9}{2}\ Ways$$

$f = 10$. The only number that can follow 10 is 9.
In general, the total number of ways to arrange is:

$$\binom{9}{0} + \binom{9}{1} + \cdots + \binom{9}{9}$$

And using the identity $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

$$\binom{9}{0} + \binom{9}{1} + \cdots + \binom{9}{9} = 2^9 = 512$$

D. Choosing a committee-II

Example 1.95

$$\sum_{i=0}^{r} \binom{i+m-1}{m-1} = \binom{m+r}{m}, \qquad m \geq 1, r \geq 0$$

We have $m + r$ players, numbered as below, from which we choose m players.

$$\{1, 2, \dots, m + r\}$$

Right Hand Side

Choosing m people from $m + r$ people:

$$\binom{m+r}{m}\ ways = RHS$$

Left Hand Side

Consider cases on the highest player number chosen. If the highest player chosen has number:

- m, the remaining $m - 1$ team members can be chosen from numbers $1, 2, \dots, m - 1$
$$\binom{m-1}{m-1}\ ways$$
- $m + 1$, the remaining $m - 1$ team members can be chosen from numbers $1, 2, \dots, m$
$$\binom{m}{m-1} = \binom{(m-1)+1}{m-1}\ ways$$

.
.
.

- $m + r$, the remaining $m - 1$ team members can be chosen from numbers $1, 2, \dots, m + r - 1$
$$\binom{m+r-1}{m-1} = \binom{(m-1)+r}{m-1}\ ways$$

Since the cases are disjoint, the total number of ways, by the addition principle is:

$$\binom{m-1}{m-1} + \binom{(m-1)+1}{m-1} + \binom{(m-1)+2}{m-1} + \cdots + \binom{(m-1)+r}{m-1}$$

Which can be written in summation notation as:

$$\sum_{i=0}^{r}\binom{(m-1)+i}{m-1}=\sum_{i=0}^{r}\binom{i+m-1}{m-1}=LHS$$

Since the counting problem is the same

$$LHS=RHS$$

1.96: Vandermonde's Identity

$$\binom{m+n}{k}=\sum_{j=0}^{k}\binom{m}{j}\binom{n}{k-j}$$

A committee of k people is to picked from $m+n$ people. The $m+n$ people consist of m men and n women.

Left Hand Side:
Committee of k people can be chosen from $m+n$ people in

$$\binom{m+n}{k}\ ways=LHS$$

Right Hand Side:
Consider cases.

$$Choose\ 0\ men:\ \underbrace{\binom{m}{0}}_{\substack{Choices\\ for\ Men}}\ \underbrace{\binom{n}{k-0}}_{\substack{Choices\\ for\ Women}}\ ways$$

$$Choose\ 1\ man:\ \binom{m}{1}\binom{n}{k-1}\ ways$$

$$\vdots$$

$$\binom{m}{k}\binom{n}{k-k}\ ways$$

Since the cases are disjoint, the total number by the addition principle is:

$$\binom{m}{0}\binom{n}{k-0}+\binom{m}{1}\binom{n}{k-1}+\cdots\binom{m}{k}\binom{n}{k-k}$$

Which can be written in summation notation as:

$$\sum_{j=0}^{k}\binom{m}{j}\binom{n}{k-j}=RHS$$

Since the counting problem is the same

$$LHS=RHS$$

Example 1.97

$$5+\binom{5}{2}\binom{9}{8}+\binom{5}{3}\binom{9}{7}+\binom{5}{4}\binom{9}{6}=\binom{14}{10}-C$$

$$=\binom{5}{1}\binom{9}{9}+\binom{5}{2}\binom{9}{8}+\binom{5}{3}\binom{9}{7}+\binom{5}{4}\binom{9}{6}+\binom{5}{5}\binom{9}{5}-\binom{5}{5}\binom{9}{5}$$

$$=\binom{5+9}{1+9}-\binom{5}{5}\binom{9}{5}$$

$$= \binom{14}{10} - \binom{9}{5}$$

$$C = \frac{9 \cdot 8 \cdot 7 \cdot 6}{2 \cdot 3 \cdot 4} = 3 \cdot 7 \cdot 6 = 126$$

Example 1.98

Give a story proof for

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Choose a committee of n students from $2n$ students, comprising n boys and n girls

Right Hand Side

n students can be chosen from $2n$ students in

$$\binom{2n}{n} \; ways = RHS$$

Left Hand Side

Consider cases based on the number of boys:

$$0 \; boys \; and \; n \; girls: \binom{n}{0}\binom{n}{n}$$

$$1 \; boy \; and \; (n-1) \; girls: \binom{n}{1}\binom{n}{n-1}$$

$$\vdots$$

$$n \; boys \; and \; 0 \; girls: \binom{n}{n}\binom{n}{0}$$

The total number of ways is:

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2 = LHS$$

Continuation

Give a proof using the Binomial Theorem for

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$[(1+x)^n]^2 = (1+x)^{2n}$$

$$\left[\binom{n}{0}x^0 + \binom{n}{1}x^1 + \cdots + \binom{n}{n}x^n\right]^2 = \binom{2n}{0} + \binom{2n}{1}x^1 + \cdots + \binom{2n}{2n}x^{2n}$$

Consider the coefficient of x^n in the above:

$$RHS \; term \; with \; x^n = \binom{2n}{n}$$

$$LHS \; term \; with \; x^n$$

$$= \left[\binom{n}{0}x^0\right]\left[\binom{n}{n}x^n\right] + \left[\binom{n}{1}x^1\right]\left[\binom{n}{n-1}x^{n-1}\right] + \cdots + \left[\binom{n}{n}x^n\right]\left[\binom{n}{0}x^0\right]$$

Which has coefficient:

$$= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0} = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{k} = \sum_{k=0}^{n}\binom{n}{k}^2 = LHS$$

E. Choosing a committee with a Chair

Example 1.99

Give a story proof for

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$$

Right Hand Side

Choose a committee of n people from $2n$ people (comprising n men and n women), with a female chairperson

$$\underbrace{n}_{\substack{\text{Choose} \\ \text{Chairperson}}} \times \underbrace{\binom{2n-1}{n-1}}_{\substack{\text{Remaining} \\ n-1 \text{ members}}} = RHS$$

Left Hand Side

Select $k, 1 \le k < n$ women for the committee, which can be done in

$$\binom{n}{k} \text{ ways}$$

The remaining $n-k$ committee members must be men, and they can be chosen in

$$\binom{n}{n-k} \text{ ways}$$

The total number of ways of choosing the committee, by the multiplication principle, is:

$$\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}\binom{n}{k} = \binom{n}{k}^2 \text{ ways}$$

Select the female chairperson from the k female committee members in

$$k \text{ ways}$$

The final number of ways is:

$$k\binom{n}{k}^2$$

Sum the above over all possible values of k:

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = LHS$$

F. Casework over the Median

Example 1.100

Question 7: Give a story proof for

$$\sum_{i=1}^{n}(i-1)(n-i) = \binom{n}{3}$$

Choose a committee of 3 people from n people, numbered

$$1,2,\dots,n$$

Right Hand Side

Choose a committee of 3 people from n people in:

$$\binom{n}{3}\ ways = RHS$$

Left Hand Side

We need to choose three people from the numbers:

$$1,2,\dots,i,\dots.,n$$

Let the median of the three number chosen be

$$i$$

There are $i-1$ numbers before i. From which we must choose exactly 1 number, which can be done in

$$i-1\ ways$$

There are $n-i$ numbers after i. From which we must choose exactly 1 number, which can be done in

$$n-i\ ways$$

The total number of ways to choose three numbers (given that i is the median number chosen) is:

$$(i-1)(n-i)$$

Sum over all possible values of i:

$$\sum_{i=1}^{n}(i-1)(n-i) = LHS$$

Specific Cases that we get:

$$i=2: (2-1)(1)(n-2)$$
$$i=3: (3-1)(1)(n-3)$$
$$\vdots$$
$$i=n-1: (n-1-1)(1)\big(n-(n-1)\big) = (n-2)(1)(1)$$

Example 1.101

Give a story proof

$$\sum_{m=k}^{m=n-k}\binom{m}{k}\binom{n-m}{k} = \binom{n+1}{2k+1},\qquad n \ge 2k \ge 0$$

Consider numbers:

$$1,2,3,\dots,n+1$$

Right Hand Side

Pick $2k+1$ numbers from $n+1$ numbers in

$$\binom{n+1}{2k+1}$$

Left Hand Side

Consider the $2k+1$ numbers picked and arrange them in ascending order:

$$1^{st}, 2^{nd}, \dots, k^{th}, \underbrace{(k+1)^{st}}_{Median}, (k+2)^{nd}, \dots, (2k+1)^{st}$$

Consider cases on the value of the median number. Let the median number have value:

$$m+1$$

The numbers $1^{st}, 2^{nd}, \dots, k^{th}$ can be picked from the numbers $1\ to\ m$ in:

$$\binom{m}{k}, \qquad \underbrace{m \geq k}_{Condition\ I}$$

The $(k+2)^{nd}, \dots, (2k+1)^{st}$ numbers to the right of the median can be chosen from the numbers $m+2$ to $n+1$ in:

$$\binom{(n+1)-(m+2)+1}{k} = \binom{n-m}{k}, \qquad n-m \geq k \Rightarrow \underbrace{m \leq n-k}_{Condition\ II}$$

Combining conditions I and II, the possible values that m can take are:

$$m \in \{k, k+1, \dots, n-k\}$$

Summing over all possible values of m gives us

$$\binom{k}{k}\binom{n-k}{k} + \binom{k+1}{k}\binom{n-(k+1)}{k} + \dots + \binom{n-k}{k}\binom{n-(n-k)}{k}$$

Which can be written in summation notation as:

$$\sum_{m=k}^{m=n-k} \binom{m}{k}\binom{n-m}{k} = RHS$$

G. Casework over sets / Venn Diagram Visualization

1 Pending

Example 1.102

Question 8: Give a story proof using Venn Diagrams

$$\sum_{k=0}^{k=n} 2^k \binom{n}{k} = 3^n$$

Consider n students of which

$$M\ are\ good\ at\ Math$$
$$P\ are\ good\ at\ Physics$$
$$M \cap P\ are\ good\ at\ both\ Math\ and\ Physics$$

Example 1.103

Give a story proof for

$$\binom{n}{a}\binom{a}{k}\binom{n-a}{b-k} = \binom{n}{b}\binom{b}{k}\binom{n-b}{a-k}$$

Consider a universal set of n people:

$$a = No. of\ people\ who\ like\ Almonds$$
$$b = No. of\ people\ who\ like\ books$$
$$k = people\ who\ like\ both$$

Number of people who like either books or almonds or both is:

$$= n(a) + n(b) - n(a \cap b) = a + b - k$$

Number of people like neither books nor almonds

$$= n - ((a + b) - k)) = n - a - b + k$$

Left Hand Side

Choose the people who like almonds from n people in

$$\binom{n}{a}$$

Choose the people who like almonds and books from a people in

$$\binom{a}{k}$$

Choose the $b - k$ people who like only books from $n - a$ people in

$$\binom{n-a}{b-k}$$

Right Hand Side

Choose the people who like books from n people in

$$\binom{n}{b}$$

Choose the people who like almonds and books from b people in

$$\binom{b}{k}$$

Choose the $a - k$ people who like only almonds from $n - b$ people in

$$\binom{n-b}{a-k}$$

$$\frac{n!}{(a-k)!\,(b-k!)(k!)(n-a-b+k)}$$

H. Geometric Interpretation

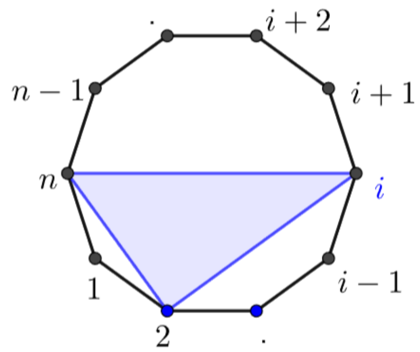
Example 1.104

Question 7: Give a story proof for

$$\sum_{i=1}^{n}(i-1)(n-i) = \binom{n}{3}$$

Consider a polygon with n vertices numbered

$$1,2, \dots, n$$



Right Hand Side
The number of triangles that can be formed is

$$\binom{n}{3}$$

Left Hand Side
Let i be the "median" vertex chosen. We must then choose one vertex from 1 to $i-1$ and one vertex from $i+1$ to n which can be done in:

$$(i-1)(n-1)\ ways$$

Sum over all possible values of i:

$$\sum_{i=1}^{n}(i-1)(n-i) = LHS$$

I. Hockey Stick Identity

1.105: Hockey Stick Identity

$$\sum_{t=k}^{t=n}\binom{t}{k} = \binom{n+1}{k+1}$$

Consider people numbered

$$1,2,\dots,n+1$$

Right Hand Side
Choose $k+1$ people from the $n+1$ people in:

$$\binom{n+1}{k+1} = RHS$$

Left Hand Side
Let the highest number be $t+1$. The remaining k numbers can be chosen from t numbers in

$$\binom{t}{k}$$

Sum over all possible values of t:

$$\binom{k}{k} + \binom{k+1}{k} + \cdots \binom{n}{k} = \sum_{t=k}^{t=n}\binom{t}{k} = LHS$$

Example 1.106
The polynomial $1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$ may be written in the form $a_0 + a_1 y + a_2 y^2 + \cdots + a_{16}y^{16} + a_{17}y^{17}$ where $y = x+1$ and the $a_i's$ are constants. Find the value of a_2. (AIME 1986/11)

Substitute $y = x + 1 \Rightarrow x = y - 1$

$$1 - (y-1) + (y-1)^2 - (y-1)^3 + \cdots + (y-1)^{16} - (y-1)^{17}$$

Incorporate the minus signs into the expansions:

$$1 + (1-y) + (1-y)^2 + (1-y)^3 + \cdots + (1-y)^{16} + (1-y)^{17}$$

Substitute $k = 2, a = 1, b = -y$:

$$(a+b)^n = \sum_{k=0}^{k=n} \binom{n}{k} a^{n-k} b^k$$

$$a = 1, k = 2b = \binom{n}{2} 1^{n-2}(-y)^2 = \binom{n}{2} y^2$$

$$n = 2: \binom{2}{2}$$
$$n = 3: \binom{3}{2}$$
$$.$$
$$.$$
$$.$$
$$n = 17: \binom{17}{2}$$

Sum the above to:

$$0 + 0 + \binom{2}{2} + \binom{3}{2} + \cdots + \binom{16}{2} + \binom{17}{2}$$

Using the hockey stick identity $\sum_{t=k}^{t=n} \binom{t}{k} = \binom{n+1}{k+1}$ with $k = 2, t = 17$

$$0 + 0 + \binom{2}{2} + \binom{3}{2} + \cdots + \binom{16}{2} + \binom{17}{2} = \binom{18}{3}$$

J. Bijective Proof

1.107: Bijective Proof

A bijective proof is when you count the elements of a set in two different ways.
Since the number of elements is the same no matter which way you count, the two different ways of counting are equivalent and lead to the same answer.

- The story proofs that we have been doing above are a specific case of a bijective proof.

2. Probability Topics

2.1 Some Background

A. Creating Frequency Distributions

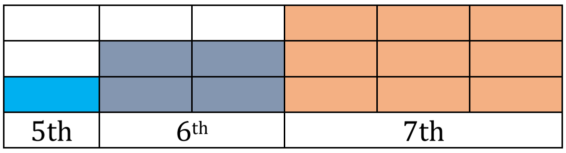
We first look at questions that involve frequency, and then see how symmetry can be applied to frequency.

Example 2.1

A school organizes a beach cleaning drive. It has a single fifth grader, two sixth graders and three seventh graders. Each fifth grader gathers one box of waste, each sixth grader gathers two boxes of waste, and each seventh grader gathers three boxes of waste. After the beach cleaning drive, I select a box at random. Find the probabilities regarding which grade the student belonged to.

Geometric Probability

Imagine the boxes assigned to the grades have colors. After collection, the boxes are stacked, and segregated grade-wise (see table).



The probabilities are then:

$$P(5th) = \frac{1}{14}, \qquad P(6th) = \frac{4}{14} = \frac{2}{7}, \qquad P(7th) = \frac{9}{14}$$

Frequency

We can get the same result as the previous answer by using a tabular method instead of a visual one. We need to count the number of boxes that each grade has collected, given by:

Grade	5^{th}	6^{th}	7^{th}	Total
Students per Grade	1	2	3	
Boxes collected per student per Grade	1	2	3	
No. of Boxes $(Frequency)$	1	4	9	14
Probability= $\frac{Frequency}{Total\ No.of\ Boxes}$	$\frac{1}{14}$	$\frac{4}{14}$	$\frac{9}{14}$	$\frac{14}{14}$

B. Using Frequency Tables

Example 2.2

Jean can't decide her favorite color. It keeps changing. Any time you ask her, she has a $\frac{1}{3}rd$ probability of saying that red is her favorite color, and a $\frac{2}{3}rd$ probability of saying blue.

A. If you ask Jean her favorite color ten times, how many times would you expect to hear red?
B. If you ask Jean her favorite color twenty times, how many times would you expect to hear blue?

Part A

$$P(Red) = \frac{1}{3} \Rightarrow \frac{1}{3} \times 10 = \frac{10}{3}$$

Part B

$$P(Blue) = \frac{2}{3} \Rightarrow \frac{2}{3} \times 20 = \frac{40}{3}$$

C. Symmetry

Example 2.3

AMC 10 – Sum greater than number picked from 1 to 10.

D. Expected Value

Example 2.4

An unfair coin when tossed, shows up heads with probability $\frac{1}{4}$. The coin is tossed twice. Find the expected number of heads.

Example 2.5

Sarn has an urn containing three red and two green balls. He draws a ball from the urn, replaces it, and then draws another ball. Find the expected number of red balls.

Example 2.6

Sarn has an urn containing three red and two green balls. He draws a ball from the urn, replaces it, and then draws another ball. Find the expected number of green balls.

E. Winnings from a Game

Example 2.7

I play a game where I roll an eight-sided die. If the number that comes up is prime, I get a dollar. If the number that comes up is composite, I lose a dollar. If the number is neither prime nor composite, I neither win nor lose anything.

A. Is this game worth playing? How much do I expect to earn if I play this game once?
B. How much do I expect to earn if I play this game 100 times?
C. What is the maximum I should pay to play this game, if I do not want to expect a loss when playing?

Part A

	$Prime$	$Composite$	$Neither$
Numbers	2,3,5,7	4,6,8	1
Frequency	4	3	1
Probability= $p(x)$	$\frac{4}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
Money $=x$	$+1$	-1	0
$x \cdot p(x)$	$\frac{4}{8}$	$-\frac{3}{8}$	0

$$Expected\ Earnings = \underbrace{\frac{4}{8}}_{(+1)\cdot p(Prime)} + \underbrace{-\frac{3}{8}}_{(-1)\cdot p(Composite)} + \underbrace{0}_{0\cdot p(Neither)} = \frac{1}{8}\ Dollars$$

Part B

$$\frac{1}{8} \times 100 = 12.5\ Dollars$$

Part C
Every game you expect to earn

$$\frac{1}{8}\ Dollars$$

Hence, the maximum that you would expect to pay per game is:

$$\frac{1}{8}\ Dollars$$

Example 2.8

A player chooses one of the numbers 1 through 4. After the choice has been made, two regular four-sided (tetrahedral) dice are rolled, with the sides of the dice numbered 1 through 4. If the number chosen appears on the bottom of exactly one die after it has been rolled, then the player wins 1 dollar. If the number chosen appears on the bottom of both of the dice, then the player wins 2 dollars. If the number chosen does not appear on the bottom of either of the dice, the player loses 1 dollar. What is the expected return to the player, in dollars, for one roll of the dice? **(AMC 10B 2007/22)**

	Both	Neither	Exactly One	Total
Probability	$\frac{1}{16}$	$\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$	$1 - \frac{1}{16} - \frac{9}{16} = \frac{6}{16}$	
Money	2	-1	1	
	$\frac{2}{16}$	$-\frac{9}{16}$	$\frac{6}{16}$	$-\frac{1}{16}$

F. Fair Game

Example 2.9

A player pays \$5 to play a game. A die is rolled. If the number on the die is odd, the game is lost. If the number on the die is even, the die is rolled again. In this case the player wins if the second number matches the first and loses otherwise. How much should the player win if the game is fair? (In a fair game the probability of winning times the amount won is what the player should pay.) **(AMC 10A 2006/13)**

$$\frac{1}{2} \times \frac{1}{6} \times m = 5 \Rightarrow m = 60\ dollars$$

G. Geometric Probability

Example 2.10

H. Expected Profit

Example 2.11

A business forecasts three scenarios for the next year. In the optimistic scenario, which has 30% chance of happening, it will make a profit of \$5 million. In the base case scenario, which has a 60% chance of happening, it will make a profit of \$1 million. And in the pessimistic scenario, it will make a loss of \$1 million. What is the expected value of next year's profit?

I. Back Calculations

Example 2.12

A bin has 5 green balls and k purple balls in it, where k is an unknown positive integer. A ball is drawn at random from the bin. If a green ball is drawn, the player wins 2 dollars, but if a purple ball is drawn, the player loses 2 dollars. If the expected amount won for playing the game is 50 cents, then what is k? (AOPS Alcumus, Counting and Probability, Expected Value)

$$\left(\frac{5}{5+k}\right)(2) - \left(\frac{k}{5+k}\right)(2) = \frac{1}{2}$$
$$\frac{10-2k}{5+k} = \frac{1}{2}$$
$$20 - 4k = 5 + k$$
$$k = 3$$

J. Contingency Tables

Example 2.13

2.2 Expected Value

A. Definition

2.14: Expected Value

Expected value is given by:

$$n \cdot P(E)$$

Where

$$n = Number\ of\ times\ the\ event\ will\ happen$$
$$P(E)\ is\ the\ probability\ of\ the\ event\ happening$$

Example 2.15

2.16: Expected Value of Two Events

Expected value is given by:

$$E_1 \cdot P(E_1) + E_2 \cdot P(E_2)$$

Where

$$E_1\ is\ the\ outcome\ if\ Event\ 1\ happens$$
$$E_2\ is\ the\ outcome\ if\ Event\ 2\ happens$$
$$P(E_1)\ is\ the\ probability\ that\ Event\ 1\ happens$$
$$P(E_2)\ is\ the\ probability\ that\ Event\ 2\ happens$$

Example 2.17

Bob rolls a fair six-sided die each morning. If Bob rolls a composite number, he eats sweetened cereal. If he rolls a prime number, he eats unsweetened cereal. If he rolls a 1, then he rolls again. In a non-leap year, what is the expected value of the difference between the number of days Bob eats unsweetened cereal and the number of days he eats sweetened cereal? **(MathCounts 2001 National Team)**

Outcomes
The outcomes when Bob rolls the die are:

$$\{1,2,3,4,5,6\}$$

Out of the above:

$$\{4,6\} \Rightarrow Composite, \qquad \{2,3,5\} \Rightarrow Prime$$

If Bob he rolls a 1, he just rolls again.

$$Hence, ignore\ 1$$

Sweetened Cereal
The expected number of days Bob will eat sweetened cereal is:

$$= 365 \times \underbrace{P(S)}_{Sweetened} = 365 \times \underbrace{P(C)}_{Composite} = 365 \times \frac{2}{5}$$

Unsweetened Cereal
The expected number of days Bob will eat unsweetened cereal is:

$$= 365 \times \underbrace{P(U)}_{Sweetened} = 365 \times \underbrace{P(P)}_{Composite} = 356 \times \frac{3}{5}$$

Difference
The difference between the two is:

$$365\left(\frac{3}{5}\right) - 365\left(\frac{2}{5}\right) = 365\left(\frac{3}{5} - \frac{2}{5}\right) = 365\left(\frac{1}{5}\right) = 73$$

2.18: Expected Value

Expected value is given by:

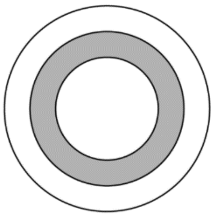
$$E_1 \cdot P(E_1) + E_2 \cdot P(E_2) + \cdots + E_n \cdot P(E_n)$$

Where

$$E_i \text{ is the outcome if Event } i \text{ happens}$$
$$P(E_i) \text{ is the probability that Event } i \text{ happens}$$

B. Geometric Probability

Example 2.19



The dartboard below has a radius of 6 inches. Each of the concentric circles has a radius two inches less than the next larger circle. If nine darts land randomly on the target, how many darts would we expect to land in a non-shaded region? **(MathCounts 2004 State Sprint)**

Single Dart

$$Total\ Area = \pi (r_{outer\ circle})^2 = \pi \times 6^2 = 36\pi$$

To calculate the shaded area, we substitute $r = 2, R = 4$ in

$$\pi R^2 - \pi r^2 = 16\pi - 4\pi = 12\pi$$

$$P(Shaded\ Region) = \frac{Successful\ Area}{Total\ Area} = \frac{12\pi}{36\pi} = \frac{1}{3}$$

Expected Value

The expected number of darts (out of nine) to land in a shaded region is then:

$$= 9 \times \frac{1}{3} = 3$$

And then the number of darts expected to land in a non-shaded region

$$= 9 - 3 = 6$$

Example 2.20

In the example above, suppose that:

- A dart landing in the innermost circle wins \$3
- A dart landing in the shaded region wins \$2
- A dart landing in the outer circle wins \$1

There is a 50% chance that a dart thrown at the dartboard hits the board. If the dart does not hit the board, the player pays \$2.

The areas of the circles are

$$Inner\ Circle = \pi r^2 = 4\pi$$
$$Shaded\ Region\ = \pi R^2 - \pi r^2 = 16\pi - 4\pi = 12\pi$$
$$Outer\ Circle\ \ = 36\pi - 16\pi = 20\pi$$

Hence, the ratio of the areas of these three regions is:

$$4\pi : 12\pi : 20\pi = 1:3:5$$

And the probability of hitting the dartboard is:

$$P(Hit) = 1 - P(Do\ not\ Hit) = 1 - \frac{1}{2} = \frac{1}{2}$$

And we then divide the probability of hitting in the ratio of the areas of the respective regions:

$$\frac{1}{2} \times \frac{1}{9} : \frac{1}{2} \times \frac{3}{9} : \frac{1}{2} \times \frac{5}{9} = \frac{1}{18} : \frac{3}{18} : \frac{5}{18}$$

x	Does not hit	*Inner Circle*	*Shaded Circle*	*Outer Circle*
$p(x)$	$\frac{1}{2}$	$\frac{1}{18}$	$\frac{3}{18}$	$\frac{5}{18}$
Money	-2	3	2	1

Expected Value

$$= \frac{1}{2} \times (-2) + \frac{1}{18} \times 3 + \frac{3}{18} \times 2 + \frac{5}{18} \times 1 = -1 + \frac{3}{18} + \frac{6}{18} + \frac{5}{18} = -1 + \frac{14}{18} = -\frac{4}{18} = -\frac{2}{9}$$

C. Dungeons and Dragons

Example 2.21

In a role-playing game, to determine the outcome of an attack roll, you roll a twenty-sided dice, and add attack modifiers. If the sum of the dice roll and the attack modifiers is greater than the target's Armor Class, the attack succeeds. If the outcome of the die roll is 20, the attack hits, and hits *critically*. What is the expected number of *critical hits* in 10 attacks for an attacker attacking a target with an Armor Class of 7, while using an attack modifier of 5?

There is a lot of extra information in this question: specifically, Armor Class, and attack modifiers.

The question only asks for critical hits, and the only condition required for a critical hit is that the die roll should be 20.
Hence, probability of Critical Hit in a single Attack

$$=\frac{1}{20}$$

Expected number of critical hits in 10 attacks

$$=10\times\frac{1}{20}=\frac{1}{2}$$

D. Pairs

Example 2.22

International checkers is played with twenty light and twenty dark pieces.

A. If the pieces are mixed and arranged in a circle, without regard to their color, what is the expected value of the number of adjacent pairs of pieces that are both light.
B. Answer the previous part, except that this time they are arranged in a line.

Part A: Circle

Imagine that the pieces are mixed and arranged. There are 20 light pieces.

Single Light Piece

Consider a single light piece. Apart from that light piece, there are 19 other light pieces and 20 dark pieces.
We only consider the piece to its right, because when we generalize, all possible pairs will get taken into consideration.

Probability of a Pair

The probability of getting a pair is just the same as the probability that the adjacent piece is light, which is:

$$P(Light)=\frac{19}{39}$$

Expected Value: Pairs

And the expected value for 20 pieces is:

$$20\left(\frac{19}{39}\right)=\frac{20\times 19}{39}=\frac{380}{39}$$

Part B: Line

Start from leftmost.
Here, we have 20 pieces, but the 20^{th} light piece can never have a light piece to its right, because the remaining 19 pieces are already on its left.

$$19\left(\frac{19}{39}\times 1+\frac{20}{39}\times 0\right)=\frac{19\times 19}{39}=\frac{361}{39}$$

E. Maximum

Example 2.23

Aarna picks, at random, a whole number of pizza slices that she is going to eat: from 1 to 3. Aarav picks, at random, a whole number of pizza slices that he is going to eat: from 1 to 4. What is the expected value of the larger of the two numbers? (If two numbers are equal, both are considered "larger").

	Aarav			
Aarna	1	2	3	4
1	1	2	3	4
2	2	2	3	4
3	3	3	3	4

x	1	2	3	4
$p(x)$	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{5}{12}$	$\frac{3}{12}$

$$1\times\frac{1}{12}+2\times\frac{3}{12}+3\times\frac{5}{12}+4\times\frac{3}{12}$$

F. Infinite Series

Example 2.24

Bhairav the Bat lives next to a town where 12.5% of the inhabitants have Type AB blood. When Bhairav the Bat leaves his cave at night to suck of the inhabitant's blood, chooses individuals at random until he bites one with type AB blood, after which he stops. What is the expected value of the number of individuals Bhairav the Bat will bite in any given night? **(2015 CCA Math Bonanza Lightning Round #3.1)**

Example 2.25

Aman has a standard pack of 52 cards with two Jokers added. He selects a card at random from the 54 cards. If he selects a Joker, he replaces it, and selects again at random, till he selects a non-Joker card. What is the expected number of selections that Aman must make?

$$1+\frac{2}{54}+\left(\frac{2}{54}\right)^2+\cdots\Rightarrow Geometric\ Series\ with\ a=1, r=\frac{2}{54}$$

$$Sum=\frac{a}{1-r}=\frac{1}{1-\frac{2}{54}}=\frac{1}{\frac{52}{54}}=\frac{54}{52}=\frac{27}{26}$$

G. Recursion with Tree Diagrams

Let's relook at a couple of the examples that we did just to see a shorter, more elegant solution.

Example 2.26

Bhairav the Bat lives next to a town where 12.5% of the inhabitants have Type AB blood. When Bhairav the Bat leaves his cave at night to suck of the inhabitants blood, chooses individuals at random until he bites one with type AB blood, after which he stops. What is the expected value of the number of individuals Bhairav the Bat will bite in any given night? (2015 CCA Math Bonanza Lightning Round #3.1)

Let the expected number of bites be b.

b = 1/8 + (b+1)(7/8)

b=8

Example 2.27
Aman has a standard pack of 52 cards with two Jokers added. He selects a card at random from the 54 cards. If he selects a Joker, he replaces it, and selects again at random, till he selects a non-Joker card. What is the expected number of selections that Aman must make?

$$E = \frac{52}{54} + \frac{2}{54}(1 + E) \Rightarrow E = \frac{54}{52} = \frac{27}{26}$$

Challenge 2.28
Freddy the frog is jumping around the coordinate plane searching for a river, which lies on the horizontal line y = 24. A fence is located at the horizontal line y = 0. On each jump Freddy randomly chooses a direction parallel to one of the coordinate axes and moves one unit in that direction. When he is at a point where y=0, with equal likelihoods he chooses one of three directions where he either jumps parallel to the fence or jumps away from the fence, but he never chooses the direction that would have him cross over the fence to where y < 0. Freddy starts his search at the point (0, 21) and will stop once he reaches a point on the river. Find the expected number of jumps it will take Freddy to reach the river. **(AIME I 2016/13)**

H. Scenarios

Example 2.29
Suppose an AMC 10 question gives 6 marks for a correct answer, 1.5 marks for an unattempted question, and 0 marks for a incorrect answer. You eliminate two of the five answer choices and mark one of the remaining three answer choices randomly. What is the expected value of the number of marks that you will get?

P(Correct) = 1/3
Marks = 6

Expected Value = (1/3)*6 = 2 Marks

P(Correct) = 1/5
Marks = 6

Expected Value = (1/5)*6 = 6/5 = 1.2 < 1.5 Marks

Example 2.30
The daily demand distribution of a product is given below.

Day	1	2	3	4	5	6	7
Probability	0.2	0.1	0.15	0.25	0.2	0.06	0.04
Demand	1500	1200	1600	1400	1600	1000	1500

The company wants to produce ten percent more than the expected demand, so as to reduce stock-outs. What is the production target? **(JMET 2011/63)**

The expected demand is:

$$= 0.2 \times 1500 + 0.1 + 1200 + 0.15 \times 1600 + 0.25 \times 1400 + 0.2 \times 1600 + 0.06 \times 1000 + 0.04 \times 1500$$
$$= 300 + 120 + 240 + 350 + 320 + 60 + 60$$
$$= 1450$$

Production Target

$$= 110\% \; of \; Demand \; = 1450 + 10\% \; of \; 1450 \; = 1450 + 145 = 1595$$

Example 2.31

In the game of dungeons and dragons, a character playing the fighter class rolls a ten-sided die once per level fo decide his hit points (life). What is the expected value of the number of hit points of a:

A. level one fighter?
B. level five fighter?

Example 2.32

I deal thirteen cards from a standard fifty-two card pack. Consider $Ace = 1, Jack = 11, Queen = 12, King = 13$ and the numbered cards as having their usual values. What is the expected numeric value:

A. Of a single card
B. Of the sum of the numeric values of all thirteen cards

Example 2.33

The contestants in a mixed martial arts fight are aged 22, 24 and 25 years respectively. I pick a contestant at random. What is the expected value of his age?

Example 2.34

I have four 250-gram weights, two 500-gram weights, and one 1-kg weight. I pick a weight at random. What is the expected value of that weight?

Example 2.35

A game of Bingo requires the caller to choose a number from 1 to 90. If the caller chooses a number, what is the expected value of the number of digits in the number.

Example 2.36: Sum and Product of Expected Values

I roll an eight-sided dice, a ten-sided dice, and a twenty-sided dice. What is the expected value of the:

A. sum of the dice rolls
B. product of the dice rolls

$$4.5 + 5.5 + 10.5 = 20.5$$

I. Symmetry

Example 2.37

Two jokers are added to a 52-card deck and the entire stack of 54 cards is shuffled randomly. What is the expected number of cards that will be strictly between the two jokers? **(2009 HMMT General 1)**

Place the Jokers First

$$\underbrace{Before}_{1st\,Joker}, \quad \underbrace{1st\,Joker}, \quad \underbrace{Between}_{the\,Jokers}, \quad \underbrace{2nd\,Joker}, \quad \underbrace{After}_{2nd\,Joker}$$

Each card has $\frac{1}{3}rd$ probability of going to any of the spaces.

So, expected value

$$\frac{1}{3} \times 52 = \frac{52}{3}$$

Example 2.38

I turn over 13 cards from a well shuffled deck of cards. What is the expected number of aces?

The probability of each card being an ace is:

$$\frac{4}{52} = \frac{1}{13}$$

The expected number of aces is:

$$13 \cdot \frac{1}{13} = 1$$

Example 2.39

I turn over cards from a well shuffled deck of cards. What is the expected number of cards before the first ace is drawn? (Including the first ace in the number of cards drawn)

View the four aces as dividing the cards into five groups

$$- A_1 - A_2 - A_3 - A_4 -$$

Each of the remaining 48 cards can be in any of the five groups. So the number of cards (upto and including the first ace) is:

$$\frac{48}{5} + 1 = 10.6$$

Example 2.40

Forty two cards are labeled with the natural numbers 1 through 42 and randomly shuffled into a stack. One by one, cards are taken off the top of the stack until a card labeled with a prime number is removed. How many cards are removed on average? **(HMMT 2007, Individual, Combinatorics)**

We can view the 13 prime numbers from 1 to 42 as dividing the cards into 14 groups.
Arrange the 13 primes first. The remaining 29 primes can be in any of the 14 groups, by symmetry.

The number of cards removed is the expected number of cards in the first group, which is:

$$\frac{29}{14} + 1 = \frac{43}{14}$$

J. Economics/Finance (Optional)

Example 2.41: Value of a Bond

The expected value of the income from a bond is given by

$$P(D) \times CF(D) + \big(1 - P(D)\big)\big(CF(P)\big)$$

Where:

$$P(D) = Probability\ of\ Default$$
$$CF(D) = Cash\ Flow\ given\ default$$
$$CF(P) = Promised\ Cash\ Flow$$

A junk bond has a promised cash flow of $1000 at maturity. The probability of default is 10%, and the cash flow in case of default is estimated to 40% of the promised cash flow. Calculate the expected value of the cash flow from the bond at the time of maturity.

$$= 0.1 \times 0.4 \times 1000 + 0.9 \times 1000 = 40 + 900 = 940$$

2.3 Recursion

A. Recursion with Induction

Example 2.42

Flora the frog starts at 0 on the number line and makes a sequence of jumps to the right. In any one jump, independent of previous jumps, Flora leaps a positive integer distance m with probability $\frac{1}{2^m}$. What is the probability that Flora will eventually land at 10? **(AMC 12A 2023/17)**

Let $p(n)$ be the probability that Flora ever reaches or has reached n.

The starting point for Flora is 0. Hence:

$$p(0) = 1$$

From, 0 there is only one way for Flora to reach 1 and hence:

$$p(1) = \frac{1}{2}p(0) = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

To reach 2, Flora can jump from either 1 or 0. We add the probabilities that Flora reaches 2 directly from 1 and directly from 0.
(The probability that Flora go from 0 to 1 can be bypassed because we multiply the probability of reaching 2 from 1 with the probability of reaching 1 itself.).

$$p(2) = \frac{1}{2}p(1) + \frac{1}{4}p(0) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot 1 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Similarly, we can calculate the probability of reaching 3, as being reached from any one of 0,1 or 2:

$$p(3) = \frac{1}{2}p(2) + \frac{1}{4}p(1) + \frac{1}{8}p(0) = \frac{1}{2}\left[p(2) + \frac{1}{2}p(1) + \frac{1}{4}p(0)\right] = \frac{1}{2}[p(2) + p(2)] = p(2) = \frac{1}{2}$$

We can generalize this using induction.
Base Case: $n \geq 2$

$$p(2) = \frac{1}{2}\ (As\ shown\ above)$$

Inductive Case:

$$p(n) = \frac{1}{2}p(n-1) + \frac{1}{4}p(n-2) + \cdots + \frac{1}{2^n}p(0)$$

$$p(n) = \frac{1}{2}\left[p(n-1) + \frac{1}{2}p(n-2) + \cdots + \frac{1}{2^{n-1}}p(0)\right]$$
$$p(n) = \frac{1}{2}[p(n-1) + p(n-1)] = p(n-1)$$

$$p(n) = \frac{1}{2}, n \geq 2$$

B. Recursion: Self-Similarity

We begin by looking at symmetric recursion, which leads to a summation pattern involving geometric series. The geometric series exhibit self-similarity, which can we exploit to arrive at an answer.

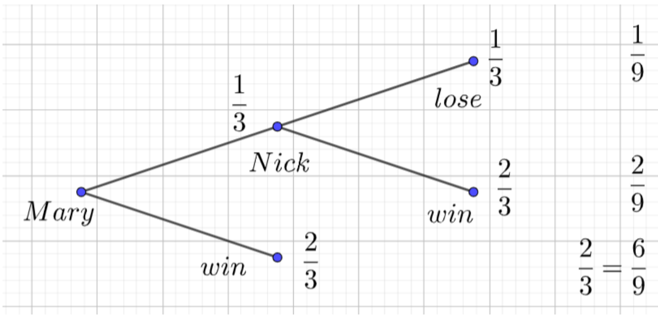
Example 2.43

Mary and Nick are playing a game. In round 1, Mary rolls a standard six-sided die. If she rolls a number that is not composite, she wins. If she doesn't win, then Nick gets his turn. He too, wins, if he rolls a number that is not composite. They alternate rolling the die till a winner is decided. Calculate Mary and Nick's chances of winning the game.[2]

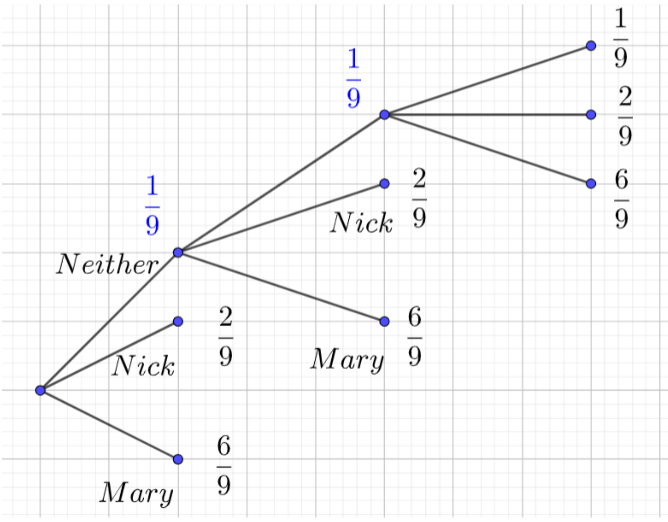
Method I

Use a tree diagram to calculate the probabilities in the first round. Note that:

$$P(Mary\ Winning) = \frac{6}{9}$$
$$P(Nick\ Winning) = \frac{2}{9}$$
$$P(Neither\ Winning) = \frac{1}{9}$$



Use these probabilities to create a tree diagram with the 1st, 2nd and 3rd rounds. From the tree diagram, we can see that the probability of Mary winning is given by:



$$\underbrace{\frac{2}{3}}_{\substack{1st\\Round}} + \underbrace{\left(\frac{2}{3}\right)\left(\frac{1}{9}\right)}_{\substack{2nd\\Round}} + \underbrace{\left(\frac{2}{3}\right)\left(\frac{1}{9^2}\right)}_{\substack{3rd\\Round}} + \cdots$$

While the probabilities have been calculated only till the third round, we can see that the process continues, and we recognize an infinite geometric series with $a = \frac{2}{3}, r = \frac{1}{9}$.

Substitute the above values in the formula for the sum of an infinite geometric series:

$$S = \frac{a}{1-r} = \frac{\frac{2}{3}}{1 - \frac{1}{9}} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{2}{3} \times \frac{9}{8} = \frac{3}{4}$$

Method II

[2] Using the shortcut formula developed later, we get $P = \frac{1}{2-p} = \frac{1}{2-\frac{2}{3}} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$

1st Round	Mary	Nick
Win	$\frac{2}{3}$	$\left(1-\frac{2}{3}\right)\left(\frac{2}{3}\right)=\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$

$$P(Mary)=p$$

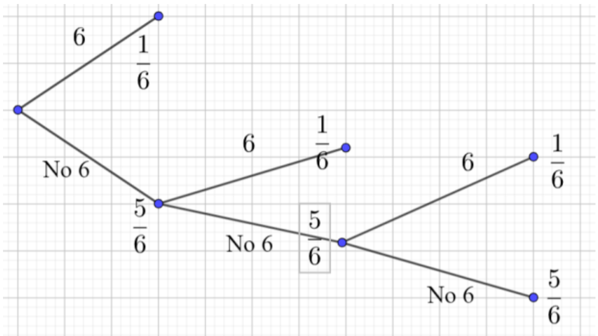
At each round, the probability that Nick will win game is $\frac{1}{3}rd$ of the probability that Mary will the game:

$$\therefore P(Nick)=\frac{p}{3}$$

$$p+\frac{p}{3}=1\Rightarrow\frac{4p}{3}=1\Rightarrow p=\frac{3}{4}$$

Example 2.44

Henrik and Zhao play a game in which they take turns successively rolling dice. The first person who rolls a 6 wins. If Zhao goes first, what is the probability that he wins the game? [3] **(*MAΘ*, Combinations and Probability 2021/14)**



Method I: Geometric Series

The probability that Zhao wins in Round 1:

$$Round\ 1=\frac{1}{6}$$

The probability that Zhao wins in Round 2:

$$=\underbrace{\frac{1}{6}}_{\substack{Winning\\2nd\ Round}}\times\underbrace{\left(\frac{5}{6}\right)^2}_{\substack{Both\ losing\\1st\ Round}}$$

The probability that Zhao wins in Round 3:

$$=\underbrace{\frac{1}{6}}_{\substack{Winning\\3rd\ Round}}\times\underbrace{\left(\frac{5}{6}\right)^2}_{\substack{Both\ losing\\1st\ Round}}\times\underbrace{\left(\frac{5}{6}\right)^2}_{\substack{Both\ losing\\2nd\ Round}}=\frac{1}{6}\times\left(\frac{5}{6}\right)^4$$

The final answer is given by a geometric series with $a=\frac{1}{6}, r=\left(\frac{5}{6}\right)^2=\frac{25}{36}$.

$$\frac{1}{6}+\left(\frac{1}{6}\right)\left(\frac{25}{36}\right)+\left(\frac{1}{6}\right)\left(\frac{25}{36}\right)^2+\cdots$$

Substitute the above values in the formula for the sum of an infinite geometric series:

$$S=\frac{a}{1-r}=\frac{\frac{1}{6}}{1-\frac{25}{36}}=\frac{1}{6}\times\frac{36}{11}=\frac{6}{11}$$

Method II: Equation

Notice that at every stage, the probability of Zhao winning is greater than Henrik winning. In fact:

$$\underbrace{P(H)=\frac{5}{6}P(Z)}_{Equation\ I}$$

Since one of them must win, we have:

$$\underbrace{P(H)+P(Z)=1}_{Equation\ II}$$

Substitute the value from Equation I into Equation II:

$$P(Z)+\frac{5}{6}P(Z)=1$$

$$\frac{11}{6}P(Z)=1$$

$$P(Z)=\frac{6}{11}$$

Method III: Recursion

We have calculated that the probability of winning is:

$$p=\frac{1}{6}+\left(\frac{1}{6}\right)\left(\frac{25}{36}\right)+\left(\frac{1}{6}\right)\left(\frac{25}{36}\right)^2+\cdots$$

Factor $\left(\frac{25}{36}\right)$ from the second and further terms:

[3] Using the shortcut formula developed later, we get $P=\frac{1}{2-p}=\frac{1}{2-\frac{1}{6}}=\frac{1}{\frac{11}{6}}=\frac{6}{11}$

$$p=\frac{1}{6}+\left(\frac{25}{36}\right)\underbrace{\left[\left(\frac{1}{6}\right)+\left(\frac{1}{6}\right)\left(\frac{25}{36}\right)+\cdots\right]}_{P(A)}$$

But note that after factoring, the expression that we get is itself p:

$$p=\frac{1}{6}+\left(\frac{25}{36}\right)p$$
$$p-\left(\frac{25}{36}\right)p=\frac{1}{6}$$

$$p\left[1-\frac{25}{36}\right]=\frac{1}{6}$$
$$p\left[\frac{11}{36}\right]=\frac{1}{6}$$
$$p=\frac{6}{11}$$

Example 2.45

Jack and Jill take turns tossing a fair coin. Whoever gets the first *tail* goes to get the water. Jack has the first turn. Find the probability that he is one to fetch the water. [4]

The probability that Jack wins is given by:

$$\underbrace{\frac{1}{2}}_{\substack{Jack\ Wins\\ 1st\ Round}}+\underbrace{\left(\frac{1}{4}\right)}_{\substack{Both\ lose\\ 1st\ Round}}\left(\frac{1}{2}\right)+\underbrace{\left(\frac{1}{16}\right)}_{\substack{Both\ lose\\ first\ 2\ Rounds}}\left(\frac{1}{2}\right)+\cdots=\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\cdots$$

The probability that Jill wins is given by:

$$\underbrace{\frac{1}{2}}_{\substack{Jack\\ Loses}}\underbrace{\frac{1}{2}}_{\substack{Jill\ Wins\\ 1st\ Round}}+\underbrace{\left(\frac{1}{4}\right)}_{\substack{Both\ lose\\ 1st\ Round}}\underbrace{\frac{1}{2}}_{\substack{Jack\ Loses\\ 2nd\ Round}}\left(\frac{1}{2}\right)+\underbrace{\left(\frac{1}{16}\right)}_{\substack{Both\ lose\\ first\ 2\ Rounds}}\underbrace{\frac{1}{2}}_{\substack{Jack\ Loses\\ 3rd\ Round}}\left(\frac{1}{2}\right)+\cdots=\frac{1}{4}+\frac{1}{16}+\frac{1}{64}$$

We could add the geometric series and get the answers. But note that the geometric series for Jack is exactly double at each stage compared to Jill.
Hence:

$$P(Jack)=p\Rightarrow P(Jill)=\frac{p}{2}$$
$$p+\frac{p}{2}=1\Rightarrow\frac{3p}{2}=1\Rightarrow p=\frac{2}{3}$$

Example 2.46

Sam is playing a game of cards with his brother Dan. Sam goes first. If he gets a King, he wins. If he does not get a King, he puts the card back in the deck. Then it's Dan's turn. Same rules apply. Find the probability that:

A. Sam wins two games in a row.
B. Dan wins two games in a row.
C. Sam and Dan each win one game (given that they play two games)

$$P(D)=\frac{12}{13}P(S)$$
$$P(S)+P(D)=1$$
$$P(S)+\frac{12}{13}P(S)=1$$
$$P(S)=\frac{13}{25}$$
$$P(D)=1-P(S)=1-\frac{13}{25}=\frac{12}{25}$$

Part A

$$P(SS)=\left(\frac{13}{25}\right)^2=\frac{169}{625}$$

Part B

$$P(DD)=\left(\frac{12}{25}\right)^2=\frac{144}{625}$$

Part C

[4] Using the shortcut formula developed later, we get $P=\frac{1}{2-\frac{1}{2}}=\frac{1}{2-\frac{1}{2}}=\frac{1}{\frac{3}{2}}=\frac{2}{3}$

$$P(DS)+P(SD)=2\left(\frac{12}{25}\right)\left(\frac{13}{25}\right)=\frac{312}{625}$$

Example 2.47

A and B are throwing darts at a dart-board, in turns, with A going first. The first person whose dart lands in the center ring wins. On their turn, A and B have equal probability of success, p. Find the probability that A wins.

$$P(A)=\underbrace{p}_{\substack{\text{Wins First}\\ \text{Round}}}+\underbrace{(1-p)^2}_{\substack{A\text{ and }B\\ \text{lose 1st Round}}}\underbrace{p}_{\substack{\text{Wins Second}\\ \text{Round}}}+(1-p)^4p+\cdots$$

Geometric Series

Substitute $a=p, r=(1-p)^2$ in $S=\frac{a}{1-r}$:

$$P(A)=\frac{p}{1-(1-p)^2}$$

Recursion Method

$$P(A)=p+(1-p)^2P(A)$$
$$P(A)[1-(1-p)^2]=p$$
$$P(A)=\frac{p}{1-(1-p)^2}=\frac{p}{1-(1-2p+p^2)}=\frac{p}{2p-p^2}=\frac{1}{2-p}$$

Example 2.48

Juan, Carlos and Manu take turns flipping a coin in their respective order. The first one to flip heads wins. What is the probability that Manu will win? Express your answer as a common fraction. (MathCounts 2002 National Sprint)

TTH: (1/2)^3
TTTTTH: (1/2)^6

1/2^3 + 1/2^6 +... = 1/7

Logical Method

In every round, Carlos has half the probability that Juan has. Manu has half the probability that Carlos has.
Let the probability that Manu wins be:

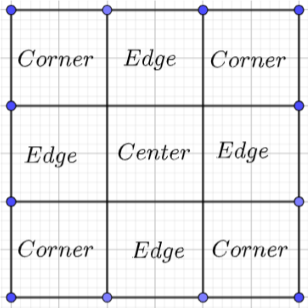
$$P(Manu)=m$$
$$\underbrace{4m}_{Juan}+\underbrace{2m}_{Carlos}+\underbrace{m}_{Manu}=1$$
$$m=\frac{1}{7}$$

C. Asymmetrical Recursion

Example 2.49

Frieda the frog begins a sequence of hops on a 3×3 grid of squares, moving one square on each hop and choosing at random the direction of each hop-up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she "wraps around" and jumps to the opposite edge. For example, if Frieda begins in the center square and makes two hops "up", the first hop would place her in the top row middle square, and the second hop would cause Frieda to jump to the opposite edge, landing in the bottom row middle square. Suppose Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops? (AMC 10A 2021 Spring /23, AMC 12A 2021 Spring/23)

Strategy

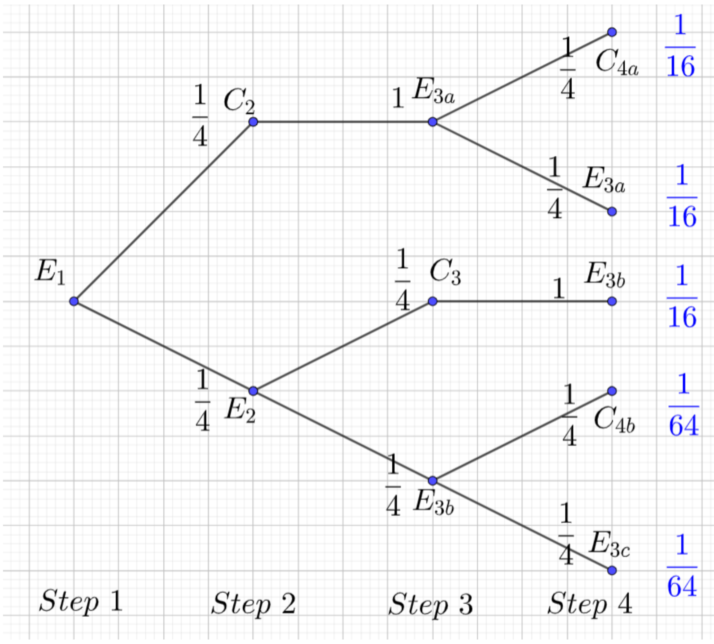


Frieda begins at the center. We use complementary counting, and find the probability that Frieda visits only Center (C) or Edge (E) squares.
In Step I, she has four choices, all valid. The four choices also lead to symmetrical outcomes. Hence, we begin our analysis at Step 2.

Tree Diagram

We want the sum of the probabilities:

$$3 \times \frac{1}{16} + 2 \times \frac{1}{64} = \frac{3}{16} + \frac{1}{32} = \frac{7}{32}$$



And, since we want the complementary probability, we calculate:

$$P(C) = 1 - P(C') = 1 - \frac{7}{32} = \frac{25}{32}$$

Establish a recurrence relation,[5] and verify the probability calculated in the example above.

Base Cases

We know that Frieda begins in the center, and moves to an edge on the first step. Hence:

$$C_0 = E_1 = 1$$
$$E_0 = C_1 = 0$$

Where

$$C_0 = Probability\ of\ being\ at\ Center\ at\ step\ 0$$
$$E_0 = Probability\ of\ being\ at\ Edge\ at\ step\ 0$$

Recurrence Relation for C_n

The probability of reaching the center at step n is $\frac{1}{4}^{th}$ the probability of reaching an edge at step $n-1$:

$$\underbrace{C_n = \frac{1}{4}E_{n-1}, n \geq 2}_{Recurrence\ I}$$

Recurrence Relation for E_n

The probability of reaching an edge on step n is:

- Same as the probability of reaching the center on step $n-1$
- $\frac{1}{4}^{th}$ the probability of reaching an edge on step $n-1$

Since both the above are valid ways, combine the two to get (for $n \geq 2$):

$$E_n = C_{n-1} + \frac{1}{4}E_{n-1} = \underbrace{\frac{1}{4}E_{n-2} + \frac{1}{4}E_{n-1}}_{Using\ Recurrence\ I} = \frac{1}{4}(E_{n-2} + E_{n-1})$$

Using the base cases, and the recurrence relation above, calculate

$$E_2 = \frac{1}{4}(E_1 + E_0) = \frac{1}{4}(1 + 0) = \frac{1}{4}$$
$$E_3 = \frac{1}{4}(E_2 + E_1) = \frac{1}{4}\left(\frac{1}{4} + 1\right) = \frac{1}{4}\left(\frac{5}{4}\right) = \frac{5}{16}$$

Recurrence Relation for R_n

Let the probability that Frieda does not visit a corner square upto (and including) step n be

[5] This is not an exam solution. It is meant to understand how recurrence relations work.

$$R_n$$

In step $n-1$, Frieda can be at

- the center, and must then visit an edge square with probability 1.
- An edge square, and must then visit an edge square with probability $\frac{1}{4}$, or the center square with probability $\frac{1}{4}$.

Hence,

$$R_n = C_{n-1} + \frac{1}{2}E_{n-1} = \underbrace{\frac{1}{4}E_{n-2} + \frac{1}{2}E_{n-1}}_{\text{Using Recurrence I}}$$

We can calculate the probability of not visiting a corner square as:

$$R_4 = \frac{1}{4}E_2 + \frac{1}{2}E_3 = \frac{1}{4}\left(\frac{1}{4}\right) + \frac{1}{2}\left(\frac{5}{16}\right) = \frac{1}{16} + \frac{5}{32} = \frac{7}{32}$$

And, finally, the probability of visiting a corner square:

$$= 1 - R_4 = 1 - \frac{7}{32} = \frac{25}{32}$$

Calculate R_6. Would you prefer a tree diagram or the recurrence relation to do it?

$$E_2 = \frac{1}{4}, \qquad E_3 = \frac{5}{16}$$

$$E_4 = \frac{1}{4}(E_2 + E_3)\frac{1}{4}\left(\frac{1}{4} + \frac{5}{16}\right) = \frac{1}{4}\left(\frac{9}{16}\right) = \frac{9}{64}$$

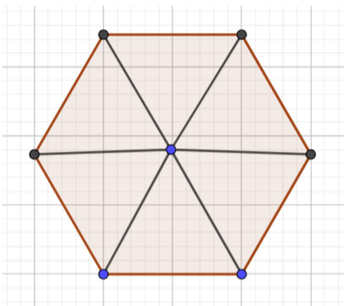
$$E_5 = \frac{1}{4}(E_3 + E_4) = \frac{1}{4}\left(\frac{5}{16} + \frac{9}{64}\right) = \frac{1}{4}\left(\frac{29}{64}\right) = \frac{29}{256}$$

$$R_6 = \frac{1}{4}E_4 + \frac{1}{2}E_5 = \frac{1}{4}\left(\frac{9}{64}\right) + \frac{1}{2}\left(\frac{29}{256}\right) = \frac{18}{512} + \frac{29}{512} = \frac{47}{512}$$

For small values of n, the tree diagram is faster. As n increases, the tree diagram will be difficult to draw, and the recurrence relations will be better.

Example 2.50

A bug starts at a vertex of a grid made of equilateral triangles of side length 1. At each step the bug moves in one of the 6 possible directions along the grid lines randomly and independently with equal probability. What is the probability that after 5 moves the bug never will have been more than 1 unit away from the starting position? (AMC 12B 2021 Fall/17)



The bug begins at the center. There are six paths for the bug, and all are valid and symmetrical. So, we ignore step I. The paths from Step II to Step V, using the multiplication principle (with repetition) is:

$$6 \times 6 \times 6 \times 6 = 6^4$$

Draw a tree diagram[6]. Let

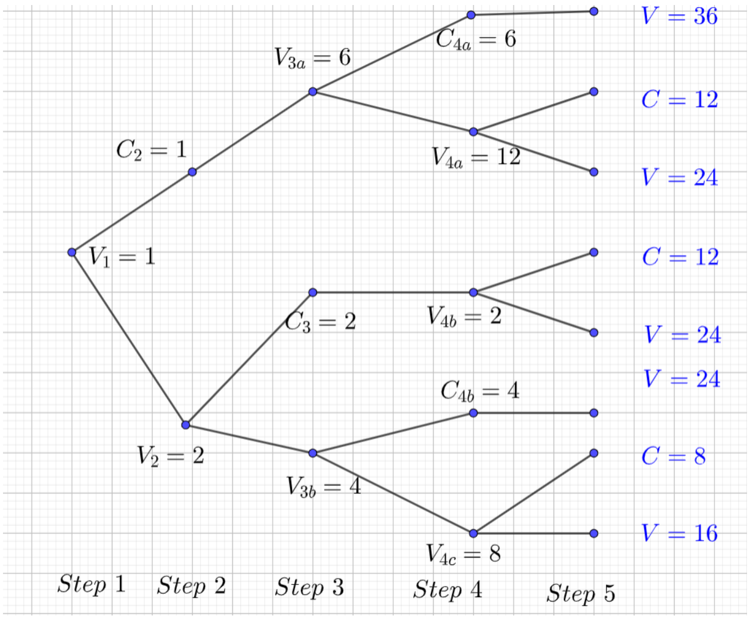
- C represent the center of the grid
- V represent any other vertex of the grid,

Valid paths

$= 36 + 12 + 24 + 12 + 24 + 24 + 8 + 16 = 156$

Hence, the required probability is:

$$\frac{156}{6^4} = \frac{13}{108}$$



Calculate the probability that after 6 moves the bug never will have been more than 1 unit away from the starting position? Use a recurrence relation.

Recurrence Relation for C_n

The same tree diagram that we drew earlier holds.

- C_2 is the paths that reach the center at Step 2.

$$C_2 = V_1 = 1$$
$$C_3 = V_2 = 2$$

- In some places, we have V_{3a} and V_{3b} since you can reach a vertex in multiple ways

$$C_4 = C_{4a} + C_{4b} = V_{3a} + V_{3b} = V_3$$

The paths to the center at step n are the same as the paths to reach a vertex at step $n-1$:

$$\underbrace{C_n = V_{n-1}}_{Recurrence\ I}$$

Recurrence Relation for V_n

We can reach a Vertex in two ways:

- From another Vertex in two ways: $V_{n-Vertex} = 2V_{n-1}$
- From the center in six ways: $V_{n-Centre} = 6C_{n-1}$

Since both the above ways are valid, the total number of ways to reach a vertex on the n^{th} step is given by:

$$\underbrace{V_n = 2V_{n-1} + 6C_{n-1}}_{Recurrence\ II}$$

We first need to calculate the base cases. There are actually six ways to reach a vertex on Step 1. All six are valid. Hence, we can ignore the first step and set:

$$V_1 = 1, C_1 = 0$$
$$V_2 = 2V_1 + 6C_1 = 2 + 6(0) = 2$$

Substitute Recurrence I in Recurrence II to get a relation only in terms of V_n:

$$V_n = 2V_{n-1} + 6V_{n-2}, n \ge 2$$

Now, we use the recurrence:

$$V_3 = 2V_2 + 6V_1 = (2)(2) + 6(1) = 4 + 6 = 10$$
$$V_4 = 2V_3 + 6V_2 = (2)(10) + (6)(2) = 20 + 12 = 32$$
$$V_5 = 2V_4 + 6V_3 = (2)(32) + (6)(10) = 64 + 60 = 126$$
$$V_6 = 2V_5 + 6V_4 = (2)(126) + (6)(32) = 252 + 192 = 444$$

Recurrence Relation for S_n

The number of valid paths S_n for step n is given by:

- Valid ways to reach a vertex at step n
- Valid ways to reach the center at step $n-1$

$$S_n = V_n + C_{n-1} = V_n + V_{n-1}$$

[6] Compare the tree diagram here, which is in terms of probabilities with the one in the previous example (about the frog), which is terms of paths. The calculations are similar, but the method is slightly different.

Hence:

$$S_6 = V_6 + V_5 = 444 + 126 = 570$$

Calculating the Probability

Since we ignored the first step when calculating successful outcomes, we do the same when calculating total outcomes:

$$Total\ Outcomes = 6^5$$

$$Probability = \frac{Successful\ Outcomes}{Total\ Outcomes} = \frac{570}{6^5} = \frac{95}{1296}$$

Example 2.51

Let A, B, C and D be the vertices of a regular tetrahedron, each of whose edges measures 1 meter. A bug, starting from vertex A, observes the following rule: at each vertex it chooses one of the three edges meeting at that vertex, each edge being equally likely to be chosen, and crawls along that edge to the vertex at its opposite end. Let $p = \frac{n}{729}$ be the probability that the bug is at vertex A when it has crawled exactly 7 meters. Find the value of n. **(AIME 1985/12)**

Example 2.52

A bug starts at a vertex of an equilateral triangle. On each move, it randomly selects one of the two vertices where it is not currently located, and crawls along a side of the triangle to that vertex. Given that the probability that the bug moves to its starting vertex on its tenth move is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$. **(AIME II 2003/13)**

The bug can be at the starting vertex on step n *only if* it is not at the starting vertex on step $n - 1$:

$$P_n = \frac{1}{2}(1 - P_{n-1})$$

$$P_0 = 1$$
$$P_1 = 0$$
$$P_2 = \frac{1}{2}(1 - P_1) = \left(\frac{1}{2}\right)(1) = \frac{1}{2}$$
$$P_3 = \frac{1}{2}(1 - P_2) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$
$$P_4 = \frac{1}{2}(1 - P_3) = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) = \frac{3}{8}$$

$$P_5 = \frac{1}{2}(1 - P_4) = \left(\frac{1}{2}\right)\left(\frac{5}{8}\right) = \frac{5}{16}$$
$$P_6 = \frac{1}{2}(1 - P_5) = \left(\frac{1}{2}\right)\left(\frac{11}{16}\right) = \frac{11}{32}$$
$$P_7 = \frac{1}{2}(1 - P_6) = \left(\frac{1}{2}\right)\left(\frac{21}{32}\right) = \frac{21}{64}$$
$$P_8 = \frac{1}{2}(1 - P_7) = \left(\frac{1}{2}\right)\left(\frac{43}{64}\right) = \frac{43}{128}$$
$$P_9 = \frac{1}{2}(1 - P_8) = \left(\frac{1}{2}\right)\left(\frac{85}{128}\right) = \frac{85}{256}$$
$$P_{10} = \frac{1}{2}(1 - P_9) = \left(\frac{1}{2}\right)\left(\frac{85}{128}\right) = \frac{171}{512}$$

Example 2.53

A moving particle starts at the point (4,4) and moves until it hits one of the coordinate axes for the first time. When the particle is at the point (a, b), it moves at random to one of the points (a-1,b), (a,b-1), or (a-1,b-1), each with probability $\frac{1}{3}$, independently of its previous moves. The probability that it will hit the coordinate axes at (0,0) is $\frac{m}{3^n}$, where m and n are positive integers. Find m + n. **(AIME I 2019/5)**

2.4 State Spaces

A. Basics

Example 2.54

Two players, P_1 and P_2 play a game against each other. In every round of the game, each player rolls a fair die once, where the six faces of the die have six distinct numbers. Let x and y denote the readings on the die rolled by P_1 and P_2, respectively. If

- $x > y$, then P_1 scores 5 points, and P_2 scores 0 points.
- $x = y$, then each player scores 2 points.
- $x < y$, then P_1 scores 0 points and P_2 scores 5 points.

Let X_i and Y_i be the total scores of P_1 and P_2 respectively, after playing the i^{th} round. **(JEE Advanced, 2022/Paper-I/16, Adapted)**

Find the probability that:

A. $X_2 \geq Y_2$
B. $X_2 = Y_2$
C. $X_3 = Y_3$
D. $X_3 > Y_3$

Note that the six distinct numbers have not been specified. But, for the winning condition, it is only important that the numbers be distinct. Hence, without loss of generality, let the numbers be:

$$\{1,2,3,4,5,6\}$$

The die is fair. Hence, the sample space is equiprobable.

	Total					
	1	**2**	**3**	**4**	**5**	**6**
1						
2						
3						
4						
5						
6						

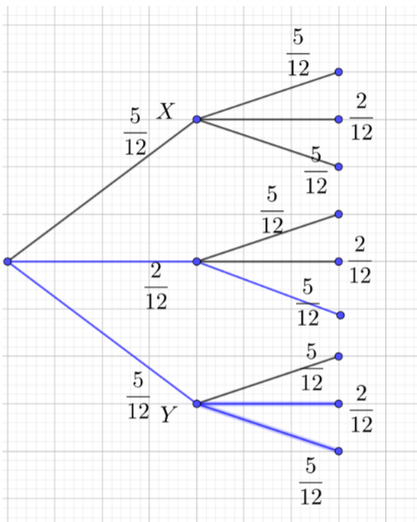
$$P(X\ Wins) = P(Y\ Wins) = \frac{15}{36} = \frac{5}{12}$$

$$P(Draw) = \frac{6}{36} = \frac{2}{12}$$

Part A

$$P(Y_2 > X_2) = \left(\frac{2}{12}\right)\left(\frac{5}{12}\right) + \left(\frac{5}{12}\right)\left(\frac{2}{12}\right) + \left(\frac{5}{12}\right)\left(\frac{5}{12}\right) = \frac{5}{12}\left(\frac{2}{12} + \frac{2}{12} + \frac{5}{12}\right)$$
$$= \frac{5}{12}\left(\frac{9}{12}\right) = \frac{5}{16}$$

$$P(X_2 \geq Y_2) = 1 - \frac{5}{16} = \frac{11}{16}$$



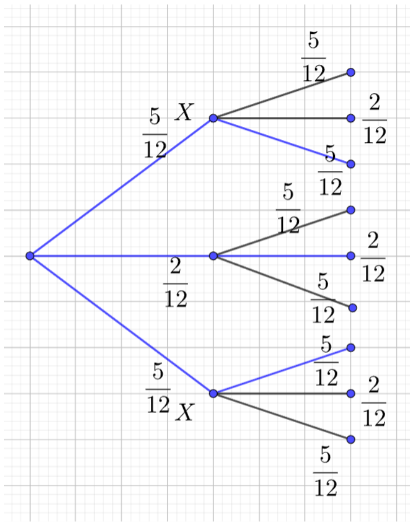
Part B

By symmetry:

$$P(X_2 > Y_2) = P(Y_2 > X_2) = \frac{5}{16}$$

Part C and D

The tree diagram with 2 rounds has 9 ending nodes. If you extend it to 3 rounds, it will have 27 ending nodes.



$$P(X_2 = Y_2) = \left(\frac{5}{12}\right)\left(\frac{5}{12}\right) + \left(\frac{2}{12}\right)\left(\frac{2}{12}\right) + \left(\frac{5}{12}\right)\left(\frac{5}{12}\right) = \frac{54}{144} = \frac{3}{8}$$

2.55: State Space

A state space uses variables to track the behavior of a system through time.

Example 2.56

Two players, P_1 and P_2 play a game against each other. In every round of the game, each player rolls a fair die once, where the six faces of the die have six distinct numbers. Let x and y denote the readings on the die rolled by P_1 and P_2, respectively. If

- $x > y$, then P_1 scores 5 points, and P_2 scores 0 points.
- $x = y$, then each player scores 2 points.
- $x < y$, then P_1 scores 0 points and P_2 scores 5 points.

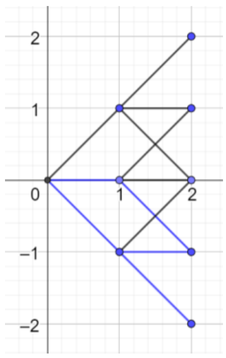
Let X_i and Y_i be the total scores of P_1 and P_2 respectively, after playing the i^{th} round. (JEE Advanced, 2022/Paper-I/16, Adapted)

Find the probability that:

A. $X_2 \geq Y_2$
B. $X_2 = Y_2$
C. $X_3 = Y_3$
D. $X_3 > Y_3$

Note that instead of tracking the probabilities on a tree diagram, we can simply track

$$g = game\ number$$
$$W = No.of\ Wins\ by\ X - No.of\ Wins\ by\ Y$$



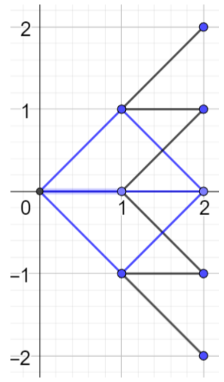
Part A

$$P(Y_2 > X_2) = \left(\frac{2}{12}\right)\left(\frac{5}{12}\right) + \left(\frac{5}{12}\right)\left(\frac{2}{12}\right) + \left(\frac{5}{12}\right)\left(\frac{5}{12}\right) = \frac{5}{12}\left(\frac{2}{12} + \frac{2}{12} + \frac{5}{12}\right) = \frac{5}{12}\left(\frac{9}{12}\right) = \frac{5}{16}$$

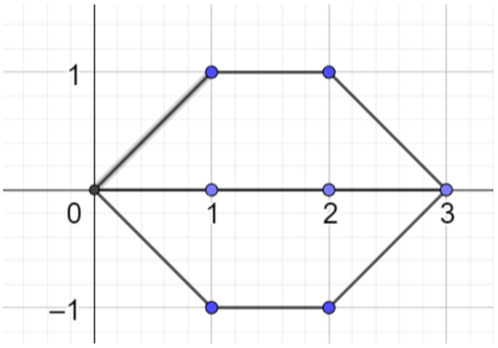
$$P(X_2 \geq Y_2) = 1 - \frac{5}{16} = \frac{11}{16}$$

Part B

$$P(X_2 = Y_2) = \left(\frac{5}{12}\right)\left(\frac{5}{12}\right) + \left(\frac{2}{12}\right)\left(\frac{2}{12}\right) + \left(\frac{5}{12}\right)\left(\frac{5}{12}\right) = \frac{54}{144} = \frac{3}{8}$$

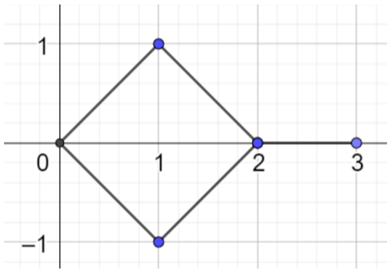


Part C

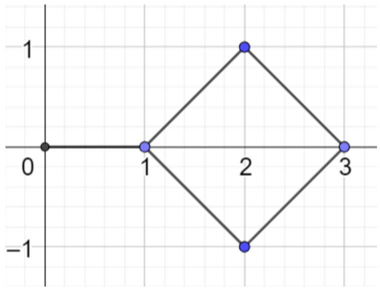


$$P(D,D,D) = \left(\frac{2}{12}\right)^3$$

$$P(X,D,Y) = P(Y,D,X) = \left(\frac{5}{12}\right)^2\left(\frac{2}{12}\right)$$



$$P(X,Y,D) = P(Y,X,D) = \left(\frac{5}{12}\right)^2\left(\frac{2}{12}\right)$$



$$P(D,X,Y) = P(D,Y,X) = \left(\frac{5}{12}\right)^2\left(\frac{2}{12}\right)$$

If you add all the cases, you get:

$$= \frac{8}{1728} + \frac{300}{1728} = \frac{308}{1728} = \frac{77}{432}$$

Part D

Case I: Y has 2 or 3 wins:

$$Not\ possible$$

Case I: Y has 1 win:

$$(Y,X,X),(X,Y,X),(X,X,Y) = \left(\frac{5}{12}\right)^3(3)$$

Case II: Y has 0 wins, 0 Draws

$$(X,X,X) = \left(\frac{5}{12}\right)^3(1)$$

Case III: Y has 0 wins, 1 Draws

$$(X,X,D),(X,D,X),(D,X,X)=\left(\frac{5}{12}\right)^2\left(\frac{2}{12}\right)(3)$$

Case IV: Y has 0 wins, 2 Draws

$$(X,D,D),(D,X,D),(D,D,X)=\left(\frac{2}{12}\right)^2\left(\frac{5}{12}\right)(3)$$

$$\left(\frac{5}{12}\right)^3(4)+\left(\frac{5}{12}\right)^2\left(\frac{2}{12}\right)(3)+\left(\frac{2}{12}\right)^2\left(\frac{5}{12}\right)(3)$$
$$=\frac{500}{1728}+\frac{150}{1728}+\frac{60}{1728}=\frac{710}{1728}=\frac{355}{864}$$

2.5 Geometric Probability

A. Area as Probability

Example 2.57

Rachel and Robert run on a circular track. Rachel runs counterclockwise and completes a lap every 90 seconds, and Robert runs clockwise and completes a lap every 80 seconds. Both start from the same line at the same time. At some random time between 10 minutes and 11 minutes after they begin to run, a photographer standing inside the track takes a picture that shows one-fourth of the track, centered on the starting line. What is the probability that both Rachel and Robert are in the picture? **(AMC 10B 2009/23, AMC 12B 2009/18)**

After 10 minutes:

$$\frac{600}{90}=\frac{60}{9}=\frac{20}{3}=6\frac{2}{3}$$

$$18.75<x<41.25\ AND\ 30<x<50\Rightarrow 30<x<41.25$$

$$P=\frac{41.25-30}{60}=\frac{11.25}{60}=\frac{45}{240}=\frac{3}{16}$$

Example 2.58

On a checkerboard composed of 64 unit squares, what is the probability that a randomly chosen unit square does not touch the outer edge of the board? **(AMC 8 2009/10)**

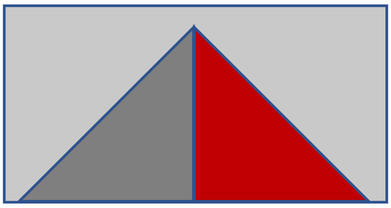
Ans $=9/16$

Warmup 2.59

Kunal throws a dart at the dartboard in the diagram. If the dart has equal probability of any part of the dartboard, find the probability of hitting each colored region. (Note: Areas that look equal are actually equal).

$$p+p=1\Rightarrow 2p=1\Rightarrow p=\frac{1}{2}$$

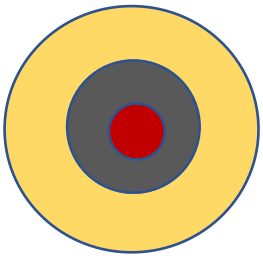
Example 2.60



A surveyor releases a survey probe from a balloon at a mountain (side view shown in the diagram). If it is known that the probe hits the mountain, and it has equal probability of reaching any part of the diagram, find the probability of hitting each colored region.

$$p + p = 1 \Rightarrow 2p = 1 \Rightarrow p = \frac{1}{2}$$

Example 2.61



A dart is thrown at the circular dartboard shown alongside. The circles are concentric, with the innermost circle having a radius of two units, the middle circle a radius of two units, and the largest circle a radius of four units. Find the probability of hitting each color.

Total Area

$$A = \pi r^2 = 16\pi$$

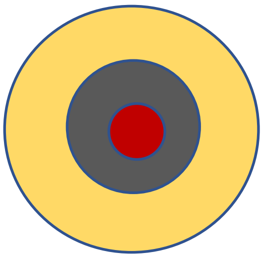
Coloured Areas

$$Red\ Area = \pi r^2 = \pi$$
$$Gray\ Area = 4\pi - \pi = 3\pi$$
$$Yellow\ Area = 16\pi - 4\pi = 12\pi$$

Probability

$$P(Red) = \frac{\pi}{16\pi} = \frac{1}{16}$$
$$P(Gray) = \frac{3\pi}{16\pi} = \frac{3}{16}$$
$$P(Yellow) = \frac{12\pi}{16\pi} = \frac{12}{16} = \frac{3}{4}$$

Example 2.62



A dart is thrown at the circular dartboard shown alongside. The circles are concentric. The probability of hitting the gray region is half, and the probability of hitting the yellow region is seven-sixteenth. Find

A. The probability of hitting the red region.
B. The ratio of the radii of the respective circular regions.

Part A

We use complementary probability

$$P(Red) = 1 - P(Yellow) - P(Gray) = 1 - \frac{7}{16} - \frac{1}{2} = \frac{1}{16}$$

Part B

$$P(Red) = \frac{1}{16} \Rightarrow \frac{Area(Red)}{Total\ Area} = \frac{1}{16}$$

Recall that radius is a linear measure, while area is squared. Hence, the radii of the circles will be proportional to the square root of the areas of the circles.

Hence, take square roots to find the ratio of the radii:

$$\frac{Radius(Red)}{Radius(Largest\ Circle)} = \sqrt{\frac{1}{16}} = \frac{1}{4}$$

Example 2.63

A point is chosen at random from within a circular region. What is the probability that the point is closer to the center of the region than it is to the boundary of the region? (AMC 8 1996/25)
Ans = ¼

Example 2.64

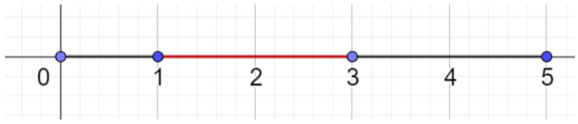
On the dart board shown in the figure below, the outer circle has radius 6 and the inner circle has radius 3. Three radii divide each circle into three congruent regions, with point values shown. The probability that a dart will hit a given region is proportional to the area of the region. When two darts hit this board, the score is the sum of the point values in the regions. What is the probability that the score is odd? (AMC 8 2007/25)
Ans = 35/72

B. Real Number Line

Example 2.65

Nick picks a real number at random from the interval (0,5). What is the probability that the number lies in the interval (1,3)?

$$Probability = \frac{Length\ of\ Valid\ Interval}{Total\ Interval} = \frac{2}{5}$$



Example 2.66

Britney picks a random number on the real number line between 0 and 1 and then rounds it off to the nearest integer. Find the probability of each possible integer that she can get.

Suppose the number Britney picks is x.
Apply the rounding off rules:

$$Round(x) = 1 \Rightarrow 0.5 \le x \le 1$$
$$Round(x) = 0 \Rightarrow 0 \le x \le 0.5$$

$$Probability = \frac{Length\ of\ Valid\ Interval}{Total\ Interval} = \frac{0.5}{1} = \frac{1}{2}$$

Example 2.67

Lee picks a random number on the real number line between 0 and 2 and then rounds it off to the nearest integer. Find the probability:

A. Of each possible integer that Lee can get.
B. That, before rounding, the number that Lee picks is exactly one.

C. Coordinate Plane

Example 2.68

Sean picks a random non-zero number. Sally picks another random non-zero number distinct from the one that Sean picks. Stephen makes an ordered pair from the numbers that Sean and Sally picked, writing them in the form

$$(x, y)$$

D. Triangles

Example 2.69

A point is selected at random from ΔABC. What is the probability that it lies inside the region formed by joining the midpoints of AB, BC and CA.

Connect the midpoints of AB, BC and CA to form a triangle, and let them be

$$X, Y \ and \ Z \ respectively$$

By the Midpoint Theorem (which is a special case of the Basic Proportionality Theorem),

$$XY = \frac{1}{2}BC, \qquad XZ = \frac{1}{2}CA, \qquad YZ = \frac{1}{2}AB$$

Hence,

$$\Delta XYZ \sim \Delta ABC$$

By similarity, the ratio of the areas are proportional to the square of the ratios of the sides, and this is the same as the probability of the point being inside the region:

$$\frac{P(\Delta XYZ)}{P(\Delta ABC)} = \frac{A(\Delta XYZ)}{A(\Delta ABC)} = \frac{\left(\frac{1}{2}\right)^2}{1} = \frac{1}{4}$$

E. Triangle Inequality

Example 2.70

Two sides of a nondegenerate triangle measure 2" and 4" and the third side measures some whole number of inches. If a cube with faces numbered 1 through 6 is rolled, what is the probability, expressed as a common fraction, that the number showing on top could be the number of inches in the length of the third side of the triangle? (MathCounts 1995 Chapter Target)

By the triangle inequality:

$$4 - 2 < x < 4 + 2 \Rightarrow 2 < x < 6 \Rightarrow x \in (3,4,5) \Rightarrow P = \frac{3}{6} = \frac{1}{2}$$

F. Coordinate Geometry

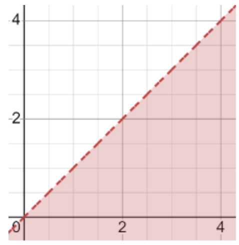
Example 2.71

x and y are each random values between 0 and 2 (exclusive).

A. What is the probability that x is greater than y?
B. If x and y are changed to be random values between 0 and 2 (inclusive), will your answer change? Your

calculations?

Part A
By Symmetry



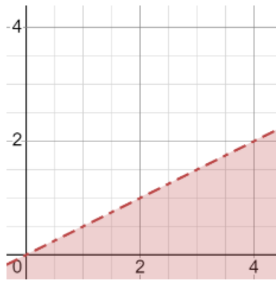
$$p + p = 1 \Rightarrow 2p = 1 \Rightarrow p = \frac{1}{2}$$

By Coordinate Geometry
Shade the region on the coordinate where x is greater than y. By symmetry, the probability is

$$\frac{1}{2}$$

Part B
Neither answers nor calculations will change.

Example 2.72
x is a random value between 0 and 4. y is also a random value between 0 and 4. What is the probability that x is greater than $2y$?



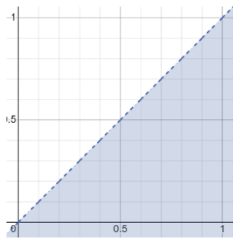
Shade the region on the coordinate plane where x is greater than y. We want to find the area of the triangle divided by the area of the square, which is given by:

$$\frac{A(Triangle)}{A(Square)} = \frac{4 \times 2 \times \frac{1}{2}}{4 \times 4} = \frac{1}{4}$$

G. Triangle Inequality

Challenge 2.73
A one-meter long stick is cut at two random places. What is the probability that the three sticks so formed can be used to form a triangle?

Total Outcomes / Area
Place a real number line along the one-meter stick (see diagram below). Let the cuts be made at the number x and the number y, with



As shown in the number line, we assume that

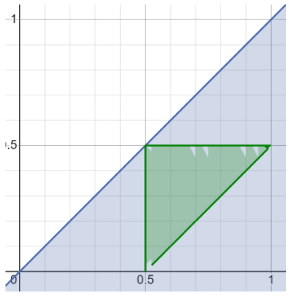
$$x > y$$

Hence, the lengths of the three sticks obtained are:

$$y, \qquad x - y, \qquad 1 - x$$

Also, we can calculate the total area that satisfies $x > y$ as the blue region shown to the right.

Successful Outcomes / Area



The three sides of the triangle must satisfy the triangle inequality:

$$(1-x)+(x-y)>y \Rightarrow 1-y>y \Rightarrow y<\frac{1}{2}$$

$$y+(x-y)>1-x \Rightarrow x>1-x \Rightarrow x>\frac{1}{2}$$

$$(1-x)+y>x-y \Rightarrow y>x-\frac{1}{2}$$

The region that satisfies all three inequalities is shown in the green region to the right.

Probability

$$Probability=\frac{Green\ Area}{Blue\ Area}=\frac{\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}}{1\times 1\times\frac{1}{2}}=\frac{1}{4}$$

H. Time Arrivals

Example 2.74

Romeo and Juliet have a date at a given time, and each will arrive at the meeting place with a delay between 0 and 1 hour, with all pairs of delays being equally likely. The first to arrive will wait for 15 minutes and will leave if the other has not yet arrived. What is the probability that they will meet? **(MIT OpenCourses)**

Solution

2.6 Reducing Counting and Symmetry

A. Basics

Example 2.75: Basics

A five-digit integer will be chosen at random from all possible positive five-digit integers. What is the probability that the number's units digit will be less than 5? Express your answer as a common fraction. **(MathCounts 2005 School Countdown)**

If we have a one-digit integer:

$$Probability=\frac{Successful\ Outcomes}{Total\ Outcomes}=\frac{5}{10}=\frac{1}{2}$$

Even if we have a five-digit integer, the question is only concerned about the last digit, and hence, the probability is still

$$\frac{Successful\ Outcomes}{Total\ Outcomes}=\frac{5}{10}=\frac{1}{2}$$

Example 2.76: Tossing Coins

A fair coin is tossed twice. What is the probability of getting:

A. Different outcomes in the two tosses
B. Same outcomes in the two tosses

Part A

There are no restrictions on the first roll. Hence, we accept the first roll with probability

$$1$$

We want the second roll to be different from the first roll, which will happen with probability

$$\frac{1}{2}$$

Overall probability is

$$1 \times \frac{1}{2} = \frac{1}{2}$$

Part B

There are no restrictions on the first roll. Hence, we accept the first roll with probability

$$1$$

We want the second roll to be the same as the first roll, which will happen with probability

$$\frac{1}{2}$$

Overall probability is

$$1 \times \frac{1}{2} = \frac{1}{2}$$

Example 2.77: Drawing Cards

Cybil and Ronda are sisters. The 10 letters from their names are placed on identical cards so that each of 10 cards contains one letter. Without replacement, two cards are selected at random from the 10 cards. What is the probability that one letter is from each sister's name? Express your answer as a common fraction. **(MathCounts 2008 State Team)**

Method I

Let the first pick be any letter. It will be from some sister's name. The second pick should be from a different sister's name, which has probability:

$$= \frac{5}{9}$$

Method II

The first letter could be from Cybil, and the second letter from Rhonda, which has probability:

$$\frac{5}{10} \times \frac{5}{9}$$

Or, the first letter could be from Rhonda, and the second letter from Cybil, which also has probability:

$$\frac{5}{10} \times \frac{5}{9}$$

Hence, the required probability:

$$= \left(\frac{5}{10} \times \frac{5}{9}\right) + \left(\frac{5}{10} \times \frac{5}{9}\right) = 2\left(\frac{1}{2} \times \frac{5}{9}\right) = \frac{5}{9}$$

Example 2.78: Rolling Dice

A. Beatrice is playing Dungeons and Dragons with her sister Rose. First, she rolls an attack roll using an eight-sided dice, and then Rose rolls a defense roll using the same eight-sided dice. If the values on both the rolls are the same, then it's a tie, and they roll again. What is the probability that must roll more than once?
B. Two cubical dice each have removable numbers 1 through 6. The twelve numbers on the two dice are removed, put into a bag, then drawn one at a time and randomly reattached to the faces of the cubes, one number to each face. The dice are then rolled and the numbers on the two top faces are added. What is the probability that the sum is 7? **(AMC 10A 2009/22)**

Part A
There is no restriction on the first roll. Hence, we simply ignore the first roll.
However, the second roll must match the first roll, and this will happen with probability

$$\frac{1}{8}$$

Part B
We are drawing two numbers from the twelve numbers:

$$\{1,1,2,2,3,3,4,4,5,5,6,6\}$$

We can get a sum of 7 in the following ways

$$1+6, 2+5, 3+4 \Rightarrow \textit{The two numbers are never equal}$$

Suppose we draw the first number and it is

$$n$$

For the second draw, the number of numbers remaining

$$= \textit{Total Outcomes} = 11$$

For the sum to be 7, the second number must be

$$7-n \Rightarrow 2\ \textit{Ways}$$

And hence the probability is

$$\frac{2}{11}$$

Example 2.79
A standard six-sided die is rolled five times. What is the probability that the five rolls are all the same or all different?

All the same
The first roll can be any number. The second and further rolls must then match the first number, which will happen with probability

$$\underbrace{\left(\frac{1}{6}\right)}_{2nd\ Roll} \times \underbrace{\left(\frac{1}{6}\right)}_{3^{rd}\ Roll} \times \underbrace{\left(\frac{1}{6}\right)}_{4^{th}\ Roll} \times \underbrace{\left(\frac{1}{6}\right)}_{5^{th}\ Roll} = \left(\frac{1}{6}\right)^4 = \frac{1}{1296}$$

All different
Again, the first roll can be any number. The second roll cannot repeat the first, and has only five numbers that work for us with probability:

$$\frac{5}{6}$$

Similarly, the third roll cannot repeat the first, or the second, and will be successful with probability

$$\frac{4}{6}$$

Continuing the pattern, we get:

$$\frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} \times \frac{2}{6} = \frac{120}{1296}$$

Final Probability
The final probability is then:

$$\frac{120}{1296}+\frac{1}{1296}=\frac{121}{1296}$$

Example 2. 80: Seating Arrangements

Angie, Bridget, Carlos, and Diego are seated at random around a square table, one person to a side. What is the probability that Angie and Carlos are seated opposite each other? **(AMC 8 2011/12)**

Method I: Symmetry

We are not bothered about opposite till we seat both of them. Only seat Angie. He can sit where ever he wants.

	Seat 2	
Seat 1		Seat 3
	Angie	

Now let Carlos sit. He has to select one of three seats, out of which one is opposite Angie. Hence, the probability is:

$$\frac{1}{3}$$

Combinations

Angie and Carlos are selecting two seats out of four seats, which can be done in

$$\binom{4}{2}=6\ ways$$

Out of the 6 ways of being seated in a four-seater square table, the number of ways where Angie and Carlos are opposite each are

$$2\ ways$$

Hence, required probability is

$$\frac{2}{6}=\frac{1}{3}$$

B. Dealing Cards

Example 2.81

A dealer deals a well-shuffled pack of cards to four people. Consider Ace to be worth 1 point, Jack to be worth 11, Queen to worth 12, and King to be worth 13. What is the probability that:

A. The Ace of Hearts is dealt before the Seven of Spades?
B. The Ace of Spades is the first of the Aces to be dealt?
C. A king is the first among all the face cards which are dealt?
D. The first card dealt has a $different$ "value" from the second card.
E. The first card dealt has the *same* "value" from the second card. (Here, consider Ace to be worth 1 point, Jack to be worth 11, Queen to worth 12, and King to be worth 13).

Strategy: Narrow the Problem Down

Here, there is a lot of information being thrown at us. 52 cards are being dealt. Taking all of them into consideration makes the problem more difficult.

Instead, we only focus on the part of the pack of cards that we need to answer our specific question.

Part A

We only consider two cards out of 52, since those are the only ones we are concerned with. This gives us two arrangements:

$$(Ace, Seven), (Seven, Ace) \Rightarrow P = \frac{1}{2}$$

Part B
We only consider the four Aces. They have to be in the deck in some sequence:

$$\underbrace{}_{Ace\ 1}\underbrace{}_{Ace\ 2}\underbrace{}_{Ace\ 3}\underbrace{}_{Ace\ 4}$$

We are interested in placing the Ace of Spades in this sequence. We have four choices, of which, only in the first choice will the Ace of Spades be dealt first.

$$P = \frac{1}{4}$$

Part C
We only consider the twelve face cards, which are arranged in the deck in some sequence

$$\underbrace{}_{\substack{Face\\Card\ 1}}\underbrace{}_{\substack{Face\\Card\ 2}}\ \dots\ \underbrace{}_{\substack{Face\\Card\ 12}}$$

For the first place, we want a King, so out of the twelve face cards, we will pick a King with probability:

$$\frac{4}{12} = \frac{1}{3}$$

(Compare the logic used in Part C with the logic used in Part B.)

Part D
There are no restrictions on the value of the first card.
Having dealt the first card, we no longer have a 52-card deck. Instead, we have a 51-card deck.

Whatever be the value of the card that we dealt, there will be

- 3 Cards with the same value
- 48 Cards with a different value

Hence, the probability of getting a card with a different value is

$$\frac{48}{51} = \frac{16}{17}$$

Part E

$$1 - \frac{16}{17} = \frac{1}{17}$$

Example 2.82
A card sharp is playing cards. He deals the cards for bridge to four people, including himself. The cards are shuffled, and he deals them randomly. But from long years of experience, and secret raised markings on the cards, he knows which cards are being dealt. What is the probability that, out of the exactly

A. thirteen cards that comprise the suit of hearts, the two of hearts is dealt second.
B. twelve face cards, the Queen of Clubs is dealt third.

Example 2.83
Two cards are dealt from a deck of four red cards labeled A, B, C, D and four green cards labeled A, B, C, D. A winning pair is two of the same color or two of the same letter. What is the probability of drawing a winning

pair? **(AMC 8 2007/21)**

Logic

A	B	C	D
A	B	C	D

Choose the first card (on which there are no restrictions), leaving seven cards.

A	B	C	D
A	B	C	D

Of these, three have the same color, and one has the same letter. Hence, the probability is:

$$\frac{4}{7}$$

Permutations

By multiplication principle, total outcomes for drawing two cards is:

$$Total\ Outcomes(TO) = 8 \times 7$$

To get a winning pair, we don't have any restrictions on the first card, giving us 8 choices. Once we do choose the first card, the second card must either be same letter (1 Choice) or same color (3 choices):

$$Successful\ Outcomes(SO) = 8 \times (1 + 3) = 8 \times 4$$

The required probability is

$$P = \frac{SO}{TO} = \frac{8 \times 4}{8 \times 7} = \frac{4}{7}$$

Combinations

The number of ways of choosing two cards from eight is:

$$\binom{8}{2} = \frac{8 \times 7}{2} = 28$$

Out of these, the ways that we want are:

$$Both\ Letters\ Same = 4\binom{2}{2} = 4\ Ways$$

$$Both\ Colours\ Same = 2\binom{4}{2} = 2 \times \frac{4 \times 3}{2} = 12$$

Probability of Success

$$\frac{4 + 12}{28} = \frac{16}{28} = \frac{4}{7}$$

2.7 Symmetry

A. Basics

Read the information below and answer the questions that follow

Example 2.84

Ashani is tossing coins.

A. He tosses the coin once
B. He tosses the coin thrice
C. He tosses the thirty-three times.

For each part, what is the probability that the number of heads is equal to the number of tails.

Since the number of coins tosses is always odd, the number of heads can never be equal to the number of tails. Hence:

$$P = 0$$

For each part above, what is the probability that the number of heads is more than the number of tails?

Parts A and B

$$H, T \Rightarrow \frac{1}{2}$$

$$(HHH)(HHT)(HTH)(HTT)(THH)(THT)(TTH)(TTT) \Rightarrow \frac{4}{8} = \frac{1}{2}$$

Part C

Parts A and B can be done easily using enumeration. However, Part C involves really large numbers, and enumeration is not recommended.

Consider the probability of getting more heads than tails, and let this probability be

$$p$$

By symmetry, the probability of getting more tails than heads is also:

$$p$$

Since the sum of mutually exclusive and exhaustive outcomes is 1, we must have:

$$p + p = 1 \Rightarrow 2p = 1 \Rightarrow p = \frac{1}{2}$$

Example 2.85: Even Parity

A. Two standard six-sided dice are tossed. One die is red and the other die is blue. What is the probability that the number appearing on the red die is greater than the number appearing on the blue die? (Gauss 7, 2014/23)

B. Diana and Apollo each roll a standard die obtaining a number at random from 1 to 6. What is the probability that Diana's number is larger than Apollo's number? (AMC 8 1995/20)

C. A dice is rolled twice. What is the probability that the number in the second roll will be higher than that in the first? (XAT 2007/52)

D. A fair 6-sided die is rolled twice. What is the probability that the first number that comes up is greater than or equal to the second number? (AMC 8 2011/18)

Part A

Enumeration

Write the outcomes in the table alongside as follows:

$$\left(\underbrace{1}_{Red\ Die}, \underbrace{1}_{Blue\ Die}\right)$$

Shade the region which is the valid region for us, and count the valid outcomes. Hence, the final probability is:

$$\frac{15}{36} = \frac{5}{12}$$

	1	2	3	4	5	6	
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)	5
2	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)	4
3	(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)	3
4	(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)	2
5	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)	1
6	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)	0
							15

Symmetry

When we roll a red die and a blue die, there are three possible cases:

- $Red > Blue$

- $Red = Blue$
- $Red < Blue$

There is nothing to distinguish between the red and the blue dice. Hence, by symmetry

$$P(Red > Blue) = P(Red < Blue) = p(say)$$

Also, we can calculate that there are exactly six outcomes where red is equal to blue, and hence,

$$P(Red = Blue) = \frac{6}{36} = \frac{1}{6}$$

And we know that:

$$P(Red > Blue) + P(Red < Blue) + P(Red = Blue) = 1$$

Substitute $P(Red > Blue) = P(Red < Blue) = p$, and $P(Red = Blue) = \frac{1}{6}$:

$$p + p + \frac{1}{6} = 1 \Rightarrow 2p = \frac{5}{6} \Rightarrow p = \frac{5}{12}$$

Parts B and C

For both the parts, by the same logic as above, the answer is

$$\frac{5}{12}$$

Part D

From the above, we know that

$$P(1st\ Number > 2nd\ Number) = \frac{5}{12}$$
$$P(1st\ Number = 2nd\ Number) = \frac{1}{6}$$

Hence, the probability that we are looking for

$$= \frac{5}{12} + \frac{1}{6} = \frac{5}{12} + \frac{2}{12} = \frac{7}{12}$$

Example 2.86

Harold tosses a coin four times. The probability that he gets at least as many heads as tails is: (AMC 8 2002/21)

Example 2.87

Alan, Jason, and Shervin are playing a game with MafsCounts questions. They each start with 2 tokens. In each round, they are given the same MafsCounts question. The first person to solve the MafsCounts question wins the round and steals one token from each of the other players in the game. They all have the same probability of winning any given round. If a player runs out of tokens, they are removed from the game. The last player remaining wins the game. If Alan wins the first round but does not win the second round, what is the probability that he wins the game? (CCA Math Bonanza, Individual Round, 2020/4)

	Alan	Jason	Shervin
Start	2	2	2
After 1st Round	4	1	1
After 2nd Round	3	3	0

Note that Shervin could have won the 2nd round either. It will not affect the answer.

By symmetry, both players have equal chance of winning, and hence

$$P(Alan\ wins) = \frac{1}{2}$$

Example 2.88

A box contains 3 red chips and 2 green chips. Chips are drawn randomly, one at a time without replacement, until all 3 of the reds are drawn or until both green chips are drawn. What is the probability that the 3 reds are drawn? **(AMC 8 2016/21)**

The question asks for the probability that the three reds are drawn first. Imagine continuing to draw even after the three reds are drawn. In such a case, if the three reds are drawn first, the last chip drawn must be green, which gives us the arrangement below:

$$\underbrace{C}_{Chip\ 1}\underbrace{C}_{Chip\ 2}\underbrace{C}_{Chip\ 3}\underbrace{C}_{Chip\ 4}\underbrace{\boldsymbol{C}}_{\boldsymbol{Chip\ 5}}$$

An arrangement like the one below will not work since the three reds will not be drawn first:

$$\underbrace{C}_{Chip\ 1}\underbrace{C}_{Chip\ 2}\underbrace{C}_{Chip\ 3}\underbrace{C}_{Chip\ 4}\underbrace{\boldsymbol{C}}_{\boldsymbol{Chip\ 5}}$$

Hence, what we really need to find is the probability that:

$$P(Last\ chip\ is\ Green)$$

Symmetry

Think of the drawing the chips and arranging them in a line:

$$\underbrace{C}_{Chip\ 1}\underbrace{C}_{Chip\ 2}\underbrace{C}_{Chip\ 3}\underbrace{C}_{Chip\ 4}\underbrace{\boldsymbol{C}}_{\boldsymbol{Chip\ 5}}$$

Now, remove the numbers from the chips:

$$CCCC\boldsymbol{C}$$

The green chip is last from the left, but first from the right.

Consider renumbering the line so that the green chip is now numbered first, the other four chips are numbered two to five:

$$\underbrace{C}_{Chip\ 5}\underbrace{C}_{Chip\ 4}\underbrace{C}_{Chip\ 3}\underbrace{C}_{Chip\ 2}\underbrace{\boldsymbol{C}}_{\boldsymbol{Chip\ 1}}$$

Then, by symmetry:

$$\therefore \boldsymbol{P(Last\ chip\ is\ Green) = P(First\ chip\ is\ Green) = \frac{2}{5}}$$

From an arrangements point of view:
Successful Outcomes: We want to put the green chip last, and we can arrange the remaining green chip and the three red chips in any way we want.
Total Outcomes: We want to arrange three red and two green chips in any way we want.

By symmetry, we can make the green chip first (instead of last) since the number of arrangements of the remaining chips do not change.

$$P = \frac{Sucessful\ Outcomes}{Total\ Outcomes} = \frac{CCCC\boldsymbol{C}}{CCCCC} = \frac{\boldsymbol{C}CCCC}{CCCCC}$$

Permutations

Successful Outcomes

The last chip is green. Hence, we are left with three red and one green chip

The only choice is the position of the green chip, for which there are

$$\{CCCC, CCCC, CCCC, CCCC\} \Rightarrow 4\ Ways$$

Total Outcomes

There are five chips (3 red, 2 green), which can be arranged (using identical objects with repetition) in

$$\frac{5!}{2!\,3!} = \frac{5 \times 4}{2} = 10$$

Probability

$$Probability = \frac{Sucessful\ Outcomes}{Total\ Outcomes} = \frac{4}{10} = \frac{2}{5}$$

Combinations

Successful Outcomes

Since the last chip is green, we need to arrange three red and one green chip, which can be done in the following ways

$$CCCC$$

From the combination point of view, we can choose the position of the green chip out of four places, which can be done in:

$$4\ Choose\ 1 = \binom{4}{1} = \frac{4!}{1!\,3!} = 4\ Ways$$

Total Outcomes

We choose the location of the two green chips (or the three red chips) in

$$\binom{5}{2} = \frac{5!}{2!\,3!} = 10\ Ways$$

$$P = \frac{Sucessful\ Outcomes}{Total\ Outcomes} = \frac{4}{10} = \frac{2}{5}$$

Example 2.89

A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white? **(AMC 10 2001/23, AMC 12 2001/11)**

Make sure you read and see the similarity between this example and the previous one. Also, a small but important difference – this question is asking for the complement of the probability that was asked in the previous example:[7]

$$P(White) = 1 - P(Red) = 1 - \frac{2}{5} = \frac{3}{5}$$

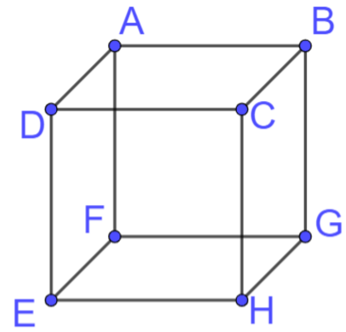
B. 3D Paths

Example 2.90

I pick two vertices of a cube, at random. What is the probability that they form:

A. An edge
B. A face diagonal (a diagonal that lies completely on a face of the cube)
C. A space diagonal (a diagonal that lies completely in space)

[7] You can also refer a detailed solution online.



Note that a cube has

$$8\ Vertices$$

Because of the symmetry of the cube, it does not matter which is the first point to get picked. For all parts below, suppose, without loss of generality, that the first point to be picked is

$$A$$

We can get edges, face diagonals and space diagonals from:

$$\underbrace{AB, AD, AF}_{Edges}, \underbrace{AC, AG, AE}_{\substack{Face\\Diagonals}}, \underbrace{AH}_{Space\ Diagonals}$$

Hence, the required probabilities are:

$$\underbrace{\frac{3}{7}}_{P(Edge)}, \underbrace{\frac{3}{7}}_{P\left(\substack{Face\\Diagonal}\right)}, \underbrace{\frac{1}{7}}_{P\left(\substack{Space\\Diagonal}\right)}$$

C. Games

Example 2.91

Two players, P_1 and P_2 play a game against each other. In every round of the game, each player rolls a fair die once, where the six faces of the die have six distinct numbers. Let x and y denote the readings on the die rolled by P_1 and P_2, respectively. If

- $x > y$, then P_1 scores 5 points, and P_2 scores 0 points.
- $x = y$, then each player scores 2 points.
- $x < y$, then P_1 scores 0 points and P_2 scores 5 points.

Let X_i and Y_i be the total scores of P_1 and P_2 respectively, after playing the i^{th} round. (JEE Advanced, 2022/Paper-I/16)

I.	Probability of $(X_2 \geq Y_2)$ is	(P) $\frac{3}{8}$
II.	Probability of $(X_2 > Y_2)$ is	(Q) $\frac{11}{16}$
III.	Probability of $(X_3 = Y_3)$ is	(R) $\frac{5}{16}$
IV.	Probability of $(X_3 > Y_3)$ is	(S) $\frac{355}{864}$
		(T) $\frac{77}{432}$

The correct option is:
(A) (I) → (Q); (II) → (R); (III) → (T); (IV) → (S)
(B) (I) → (Q); (II) → (R); (III) → (T); (IV) → (T)
(C) (I) → (P); (II) → (R); (III) → (Q); (IV) → (S)
(D) (I) → (P); (II) → (R); (III) → (Q); (IV) → (T)

Since all options have $II \to R$

$$P(X_2 > Y_2) = \frac{5}{16}$$

By symmetry rules are same for X, and Y.

$$P(X_2 \geq Y_2) = 1 - P(Y_2 > X_2) = 1 - P(X_2 > Y_2)$$
$$= 1 - \frac{5}{16} = \frac{11}{16} \Rightarrow \boldsymbol{I \to Q}$$

Since $I \to Q$, options C and D are ruled out. Options A and are the only ones left.

Hence, $III \to T$

$$P(X_3 > Y_3) = P(Y_3 > X_3) = p$$
$$p + p + \frac{77}{432} = 1$$
$$2p = 1 - \frac{77}{432} = \frac{355}{432}$$
$$p = \frac{355}{864}$$

D. Symmetry with Recursion

Example 2.92

Flora the frog starts at 0 on the number line and makes a sequence of jumps to the right. In any one jump, independent of previous jumps, Flora leaps a positive integer distance m with probability $\frac{1}{2^m}$. What is the probability that Flora will eventually land at 10? **(AMC 12A 2023/17)**

Recursion
Let Flora be at any point to the left of 10 on the number line. Let her jump once.
Suppose that Flora lands to the left of 10. Then, let her jump one more time.

Symmetry
We continue the jumping till she lands on 10, or to the right of 10. Now calculate the probabilities:

Let $P(n)$ be the probability that she lands on n. If Flora is k units to the left of 10, then

$$P(10)=\frac{1}{2^k},\qquad P(11)=\frac{1}{2^{k+1}},\qquad P(12)=\frac{1}{2^{k+2}}$$

Summing up the probabilities gives:

$$P(11)+P(12)+\cdots=\frac{1}{2^{k+1}}+\frac{1}{2^{k+2}}+\cdots$$

The above is an infinite geometric series with $a=\frac{1}{2^{k+1}}, r=\frac{1}{2}$ and sum:

$$S=\frac{a}{1-r}=\frac{\frac{1}{2^{k+1}}}{\frac{1}{2}}=\frac{1}{2^k}$$

Hence

$$P(10)=P(11)+P(12)+\cdots$$

Let

$$p=P(10)=P(11)+P(12)+\cdots$$

But Flora must eventually land on either 10 or a point to the right of 10. Hence:

$$p+p=1\Rightarrow p=\frac{1}{2}$$

E. Reframing the Question

Example 2.93
Kamal picks two points on a circle. Harris picks two more points on the circle. What is the probability that the line segments formed by the two points that Kamal picked, and the two points that Harris picked intersect?

Consider the quadrilateral ABCD formed by the four points. Suppose, without loss of generality, that A is the first point picked.
If C is the next point picked, the quadrilateral will be concave, and the line segments will intersect.
If B or D is the next point picked, the quadrilateral will be convex, and the line segments will not intersect.
Hence, the probability is
1/3

Example 2.94
Four distinct points, A, B, C, and D, are to be selected from 1996 points evenly spaced around a circle. All quadruples are equally likely to be chosen. What is the probability that the chord AB intersects the chord CD?

(AHSME 1996)

The same solution as the previous example applies. In fact, this is a specific case of the previous example.

Example 2.95[8]

A. What is the probability that the triangle formed by three random points picked on a circle include the center of the circle?
B. What is the probability that the tetrahedron formed by picking four points on a sphere contains the center of the sphere?

2.8 Binomial Probability Distribution

A. Binomial Probabilities

2.96: Binomial Probabilities

Binomial requires n independent, identically distributed trials.
Each trial has exactly two outcomes ($success$) and $failure$ with:

$$P(success) = p, \qquad P(failure) = 1 - p = q$$

Trials can refer to any events that occurs repeatedly. For example

- Tossing a coin
- Drawing a ball from an urn with replacement
- Raining on a particular day
- Being able to solve a problem

In the case of Binomial probabilities, it is assumed that:

- Trials are independent: Success or failure in one trial does not affect success or failure in other trials.

Trials are identical: Each trial has same probability of success or failure.

Example 2.97

I have a bowl with 10 jellybeans of which 4 are red, and 6 are green. I draw 3 jellybeans without replacement. Consider each drawing of a jellybean as a trial. Are the trials independent?

On the first draw:

$$P(\quad)$$

B. Binomial probability mass function

2.98: Binomial Probability Distribution Function

Given n independent, identically distributed(iid) trials, the probability of r successes in n trials is:

$$P = \binom{n}{r} p^r (1-p)^{n-r}$$

Where

$$p = probability\ of\ success\ in\ a\ single\ trial$$

Example 2.99: Finding a general expression

[8] Refer to this video for an animated solution of these two questions.

I toss a coin four times that has $P(Heads) = \frac{3}{4}$. Derive the probability mass function.

Since I toss a coin four times, we can make each coin toss a trial. Then:

$$No. = n = 4$$

Let X be the number of successes in the four trials:

$$No.\ of\ Successes = No.\ of\ Heads = r \in \{0,1,2,3,4\}$$

Define a head to be a success. Then:

$$P(Success) = p = \frac{3}{4}$$

And since heads and tails are complementary, the probability of tails

$$= P(Failure) = 1 - p = \frac{1}{4}$$

Substitute $n = 4, p = \frac{3}{4}, 1 - p = \frac{1}{4}$ in $\binom{n}{r} p^r (1-p)^{n-r}$:

$$P(X = r) = \binom{4}{r} \left(\frac{3}{4}\right)^r \left(\frac{1}{4}\right)^{n-r}$$

C. Adding Probabilities

2.100: Addition Rule

The probability that a binomial variable

$$P(X =)$$

Example 2.101

An electronic system contains three cooling components that operate independently. The probability of each component's failure is 0.05. The system will overheat if and only if at least two components fail. Calculate the probability that the system will overheat. (Exam P Sample Question, SoA)

Let the number of components that fail be X.

$$X \sim Binomial(3, 0.95)$$

The system will fail if and only if at least two components fail. Hence, we need to consider two possibilities:

Case I: Two Components Fail

Substitute $n = 3, p = 0.95, x = 2$ in $\binom{n}{x} p^x (1-p)^{n-x}$

$$P(X = 2) = \binom{3}{2} \underbrace{(0.95)^1}_{1\ Success} \underbrace{(1 - 0.95)^{3-2}}_{2\ Failures} = 3 \times \left(\frac{19}{20}\right) \times \left(\frac{1}{20}\right)^2 = \frac{57}{8000}$$

Case II: Three Components Fail

Substitute $n = 3, p = 0.95, x = 3$ in $\binom{n}{x} p^x (1-p)^{n-x}$

$$P(X = 3) = \binom{3}{3} \underbrace{(0.95)^0}_{0\ Success} \underbrace{(0.05)^3}_{3\ Failures} = 1 \times \left(\frac{1}{20}\right)^3 = \frac{1}{8000}$$

$$\frac{57}{8000} + \frac{1}{8000} = \frac{58}{8000} = \frac{29}{4000}$$

(Calc) Example 2.102

A company prices its hurricane insurance using the following assumptions. (i) In any calendar year, there can be at most one hurricane. (ii) In any calendar year, the probability of a hurricane is 0.05. (iii) The numbers of hurricanes in different calendar years are mutually independent. Using the company's assumptions, calculate the probability that there are fewer than 3 hurricanes in a 20-year period. (Exam P Sample, SoA)

$$n = 20, p = 0.05, 1 - p = 0.95$$

$$\underbrace{\binom{20}{0}(0.05)^0(0.95)^{20}}_{P(X=0)} + \underbrace{\binom{20}{1}(0.05)^1(0.95)^{19}}_{P(X=1)} + \underbrace{\binom{20}{2}(0.05)^2(0.95)^{18}}_{P(X=2)}$$

Simplify the binomial coefficients:

$$= (0.95)^{20} + 20(0.05)^1(0.95)^{19} + 190(0.05)^2(0.95)^{18}$$

Calculate:

$$0.9245$$

D. Complementary Probability

2.103: Complementary Probability

$$P(X \geq k) = 1 - P(X \leq k)$$

We can expand the above to write:

$$1 - [P(X = 0) + P(X = 1) + \cdots + P(X = k - 1)]$$

Example 2.104

A study is being conducted in which the health of two independent groups of ten policyholders is being monitored over a one-year period of time. Individual participants in the study drop out before the end of the study with probability 0.2 (independently of the other participants). Calculate the probability that at least nine participants complete the study in one of the two groups, but not in both groups? (Exam P Sample Question, Society of Actuaries, USA)

The number of participants who continue follows a binomial distribution with:

$$n = 10, \quad p = 0.8, \quad 1 - p = 0.27$$

In a single group,

$$P(X \geq 9) = \binom{10}{9}(0.8)^9(0.2)^1 + (0.8)^{10} = 0.376$$

$$P(X < 9) = 1 - 0.376 = 0.624$$

Across both groups, we need an XOR probability. That is, exactly one of the groups should meet the condition above, given by:

$$\underbrace{(0.376)}_{\substack{Group\ I \\ X \geq 9}} \underbrace{(0.624)}_{\substack{Group\ II \\ X < 8}} + \underbrace{(0.376)}_{\substack{Group\ II \\ X \geq 9}} \underbrace{(0.624)}_{\substack{Group\ I \\ X < 8}} = 0.469$$

2.105: Converting to Binomial

Example 2.106

I have an urn that has 33 red balls, 33 green balls, and 33 blue balls. I draw a ball (with replacement) from the urn nine times. What is the probability that I get exactly nine red balls? eight or more green balls? two or more blue balls? (Keep numbers with large exponents in exponent form for the answers to this question).

Nine Red Balls
This is not binomial because it has more than two outcomes. But, we can classify the events as:

$$P(Red\ Ball) = \frac{33}{99} = \frac{1}{3} \Rightarrow P(Not\ Red) = 1 - \frac{1}{3} = \frac{2}{3}$$

Now, the probability of getting nine red balls:

$$\binom{9}{9}\left(\frac{1}{3}\right)^9\left(\frac{2}{3}\right)^0 = \frac{1}{3^9}$$

Eight or More Green Balls
As above, we get:

$$P(Green\ Ball) = \frac{33}{99} = \frac{1}{3} \Rightarrow P(Not\ Green) = 1 - \frac{1}{3} = \frac{2}{3}$$

Let the number of green balls drawn be G.

$$P(G = 8) = \binom{9}{8}\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)^1 = 9 \times \frac{1}{3^8} \times \frac{2}{3} = \frac{18}{3^9}$$

$$P(G = 9) = \frac{1}{3^9}$$

$$P(G \geq 8) = P(G = 8) + P(G = 9) = \frac{18}{3^9} + \frac{1}{3^9} = \frac{19}{3^9}$$

Two or More Blue Balls
We need to find:

$$P(B = 2) + P(B = 3) + \cdots + P(B = 9)$$

One option is to stoically calculate the individual probabilities in the above, which is eight different probabilities.

$$P(B = 0) = \binom{9}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^9 = \frac{512}{3^9}$$

$$P(B = 1) = \binom{9}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^8 = 9 \times \frac{1}{3} \times \frac{2^8}{3^8} = \frac{256 \times 9}{3^9}$$

Then the expression we need to find is:

$$1 - P(B = 0) - P(B = 1) = 1 - \frac{256 \times 2}{3^9} - \frac{256 \times 9}{3^9} = \frac{3^9 - 256 \times 11}{3^9}$$

Example 2.107: Ratio of Probabilities

Phillip flips an unfair coin eight times. This coin is twice as likely to come up heads as tails. How many times as likely is Phillip to get exactly three heads than exactly two heads? **(AOPS, Alcumus, Binomial Theorem, Counting and Probability)**

$$\binom{8}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^5 : \binom{8}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^6$$

$$\left(\frac{8 \times 7 \times 6}{6}\right)\left(\frac{2}{3}\right) : \left(\frac{8 \times 7}{2}\right)\left(\frac{1}{3}\right) = 2 : \frac{1}{2} = 4 : 1 \Rightarrow 4\ Times$$

Example 2.108: Back Calculations

($Calculator$)A company establishes a fund of 120 from which it wants to pay an amount, C, to any of its 20 employees who achieve a high-performance level during the coming year. Each employee has a 2% chance of achieving a high-performance level during the coming year. The events of different employees achieving a high-performance level during the coming year are mutually independent. Calculate the maximum value of C for which the probability is less than 1% that the fund will be inadequate to cover all payments for high performance. **(Exam P Sample Question, Society of Actuaries, USA)**

We need to cover 99% of the cases. Consider this a binomial distribution with:

$$n = 20, p = 0.02$$

$$P(X=0) = \binom{20}{0}(0.02)^0(0.98)^{20} = 0.668$$
$$P(X=1) = \binom{20}{1}(0.02)^1(0.98)^{19} = 0.272$$
$$P(X=1) = \binom{20}{2}(0.02)^2(0.98)^{19} = 0.053$$

Note that:

$$P(X=0) + P(X=1) = 0.668 + 0.272 = 0.94 < 0.99$$
$$P(X=0) + P(X=1) + P(X=2) = 0.668 + 0.272 + 0.053 = 0.993 > 0.99$$

Hence, if we consider upto two employees, we cover 99% of the cases.
Hence, the maximum value of C is:

$$\frac{120}{2} = 60$$

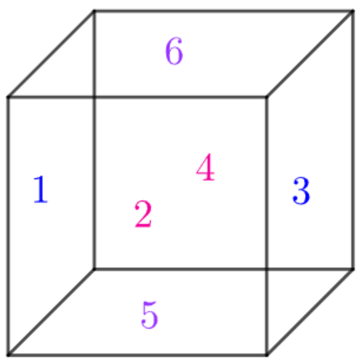
E. Geometrical Counting

Example 2.109[9]
(AMC 10B 2004/23)

Problem

Each face of a cube is painted either red or blue, each with probability 1/2. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

(A) $\frac{1}{4}$ (B) $\frac{5}{16}$ (C) $\frac{3}{8}$ (D) $\frac{7}{16}$ (E) $\frac{1}{2}$



Let X be the number of pairs of opposite faces that have the same color. X can take the values:

$$X \in \{0,1,2,3\}$$

Opposite faces have the same color with probability

$$\frac{1}{2}$$

Case I: $X = 0$ *Or* $X = 1$

If $X = 0$ OR $X = 1$, then four vertical faces will not be the same color.

Case II: $X = 3$

3 opposite pairs have the same color. This does not mean that all faces have the same color. Rather the possible cases are:

$$RRR, BBB, RRB, BBR$$

However, in all of these cases, we get four vertical faces that are the same color.

[9] A simpler solution for this can be found in the Note on Probability.

$$P(X=3)=\left(\frac{1}{2}\right)^3=\frac{1}{8}=\frac{2}{16}$$

Case III: $X = 2$
If 2 opposite pairs have the same color, then the possible cases are:

$$RR, BB, RB, BR$$

Hence, the successful probability (if this case holds) is

$$\frac{2}{4}=\frac{1}{2}$$

Using $n=3, r=2, p=q=\frac{1}{2}$, the probability that:

$$P(X=2)=\binom{n}{r}p^r q^{n-r}=\binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1=\frac{3}{8}$$

And the probability of being successful via this case is:

$$\frac{3}{8}\cdot\frac{1}{2}=\frac{3}{16}$$

The final probability is:

$$\frac{3}{16}+\frac{2}{16}=\frac{5}{16}$$

2.9 Sum of Dice Rolls

A. Basics

Example 2.110

Ava rolls two standard, fair, six-sided dice. The probability of getting a value of $n, 2 \le 2 \le 12$ for the total of the two dice is given by the piecewise function:

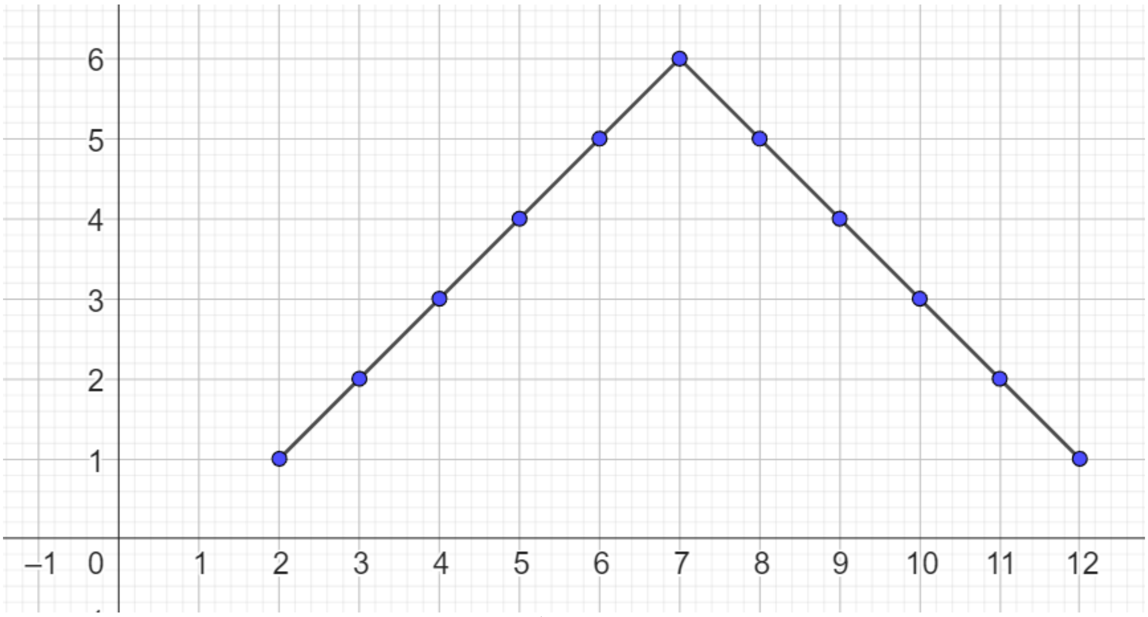
$$P(n)=\begin{cases}\frac{a}{36}, 2 \le n \le 7\\ \frac{b}{36}, 8 \le n \le 12\end{cases}=\begin{cases}\frac{a}{36}, 1 \le n \le 6\\ \frac{b}{36}, 7 \le n \le 12\end{cases}$$

Determine the value of:

$$a+b=k, \qquad k \in \mathbb{N}$$

	Total					
	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$Sum = n$	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$



$$P(n) = \begin{cases} \dfrac{n-1}{36}, 1 \le n \le 7 \\ \dfrac{13-n}{36}, 8 \le n \le 12 \end{cases}$$

$$a + b = (n-1) + (13-n) = 12$$

Example 2.111

Ava rolls two standard, fair dice, one of which has 4 sides and the other has 5 sides. The probability of getting a value of $n, 2 \le n \le 9$ for the total of the two dice is given by the piecewise function:

$$P(n) = \begin{cases} a, 2 \le n \le c \\ b, c \le n \le 9 \end{cases}$$

Determine the value of

$$a + b + c = k, \qquad k \in \mathbb{N}$$

	1	2	3	4	5
1	2	3	4	5	6
2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9

$Sum = n$	2	3	4	5	6	7	8	9
Probability	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{4}{20}$	$\frac{4}{20}$	$\frac{3}{20}$	$\frac{2}{20}$	$\frac{1}{20}$

$$P(n) = \begin{cases} \frac{n-1}{20}, 2 \le n \le 5 \\ \frac{10-n}{20}, 5 \le n \le 9 \end{cases}$$

Example 2.112

Ava rolls two standard, fair dice, one of which has 4 sides and the other has 6 sides. The probability of getting a value of $n, 2 \le n \le 9$ for the total of the two dice is given by the piecewise function:

$$P(n) = \begin{cases} a, 2 \le n \le 4 \\ b, 5 \le n \le 7 \\ c, 8 \le n \le 10 \end{cases}$$

Determine the value of

$$a + b + c = k, \qquad k \in \mathbb{Q}$$

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10

$Sum = n$	2	3	4	5	6	7	8	9	10
Probability	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{4}{20}$	$\frac{4}{20}$	$\frac{4}{20}$	$\frac{3}{20}$	$\frac{2}{20}$	$\frac{1}{20}$

$$P(n) = \begin{cases} \frac{n-1}{24}, 2 \le n \le 4 \\ \frac{4}{24}, 5 \le n \le 7 \\ \frac{11-n}{24}, 8 \le n \le 10 \end{cases}$$

$$a + b + c = \frac{n-1}{24} + \frac{4}{24} + \frac{11-n}{24} = \frac{14}{24} = \frac{7}{12}$$

2.10 AMC Questions

Start from 10A 2005

A. Decimal System

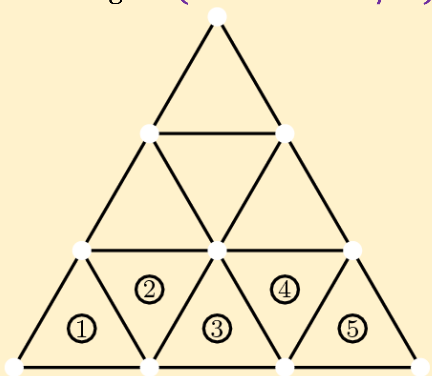
The sum of the digits of a two-digit number is subtracted from the number. The units digit of the result is 6. How many two-digit numbers have this property? **(AMC 10A 2005/16)**

B. Lists

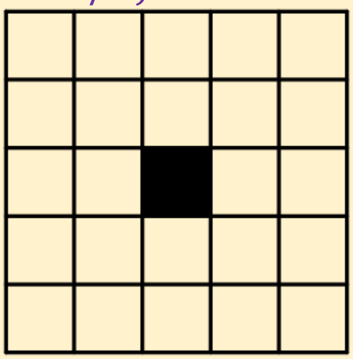
A game is played with tokens according to the following rule. In each round, the player with the most tokens gives one token to each of the other players and also places one token in the discard pile. The game ends when some player runs out of tokens. Players $A, B, and\ C$ start with 15, 14, and 13 tokens, respectively. How many rounds will there be in the game? **(AMC 10A 2004/8)**

C. Geometrical

A large equilateral triangle is constructed by using toothpicks to create rows of small equilateral triangles. For example, in the figure we have 3 rows of small congruent equilateral triangles, with 5 small triangles in the base row. How many toothpicks would be needed to construct a large equilateral triangle if the base row of the triangle consists of 2003 small equilateral triangles? **(AMC 10A 2003/23)**

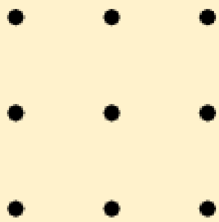


The 5×5 grid shown contains a collection of squares with sizes from 1×1 to 5×5. How many of these squares contain the black center square? **(AMC 10A 2004/16)**



D. Coordinate Geometry

A set of three points is randomly chosen from the grid shown. Each three-point set has the same probability of being chosen. What is the probability that the points lie on the same straight line? **(AMC 10A 2004/5)**



E. Strategies

There are 5 yellow pegs, 4 red pegs, 3 green pegs, 2 blue pegs, and 1 orange peg to be placed on a triangular peg board. In how many ways can the pegs be placed so that no (horizontal) row or (vertical) column contains two pegs of the same color? **(AMC 10 2000/13)**

We have a problem with restrictions. It usually makes sense to apply the most restrictive condition first. There are five rows, and five columns and five yellow pegs, so we need to fit in those first.

Y				
	Y			
		Y		
			Y	
				Y

Y				
G	Y			
	G	Y		
		G	Y	
			G	Y

Y				
R	Y			
	R	Y		
		R	Y	
			R	Y

Y				
R	Y			
G	R	Y		
	G	R	Y	
		G	R	Y

Y				
R	Y			
G	R	Y		
B	G	R	Y	
	B	G	R	Y

Y				
R	Y			
G	R	Y		
B	G	R	Y	
O	B	G	R	Y

What is the maximum number for the possible points of intersection of a circle and a triangle? **(AMC 10**

2001/4)

Six

How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits? (AMC 10A 2005/14)

F. Multiplication Rule

Henry's Hamburger Heaven offers its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties, and any collection of condiments. How many different kinds of hamburgers can be ordered? (AMC 10A 2004/12)

The sum of the digits of a two-digit number is subtracted from the number. The units digit of the result is 6. How many two-digit numbers have this property? (AMC 10A 2005/16)

G. Permutations

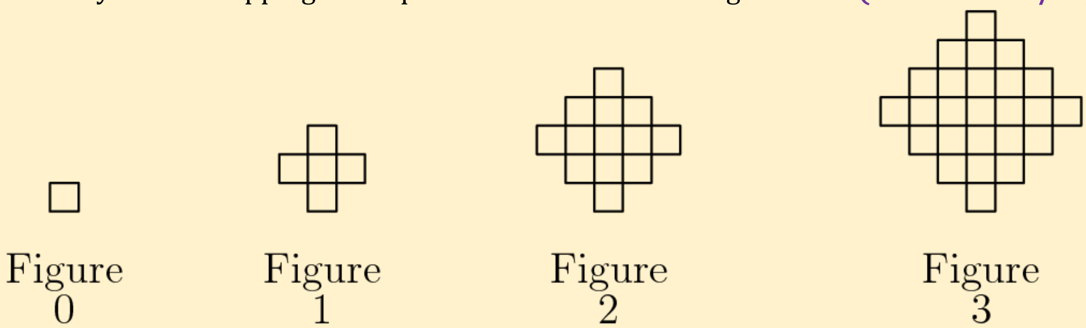
H. Combinations

Three tiles are marked X and two other tiles are marked O. The five tiles are randomly arranged in a row. What is the probability that the arrangement reads $XOXOX$? (AMC 10A 2005/9)

I. Counting: Other Topics

Problem 12

Figures 0, 1, 2, and 3 consist of 1, 5, 13, and 25 nonoverlapping unit squares, respectively. If the pattern were continued, how many nonoverlapping unit squares would there be in figure 100? (AMC 10 2000/12)



J. Diophantine/Distribution

Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible? **(AMC 10 2001/19)**

Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected? **(AMC 10A 2003/21)**

K. Probability: Frequency and Distributions

Tina randomly selects two distinct numbers from the set {1, 2, 3, 4, 5}, and Sergio randomly selects a number from the set {1, 2, ..., 10}. What is the probability that Sergio's number is larger than the sum of the two numbers chosen by Tina? **(AMC 10A 2002/24)**

Coin A is flipped three times and coin B is flipped four times. What is the probability that the number of heads obtained from flipping the two fair coins is the same? **(AMC 10A 2004/10)**

L. Geometric Probability

A point (x, y) is randomly picked from inside the rectangle with vertices (0,0), (4,0), (4,1), and (0,1). What is the probability that $x < y$? **(AMC 10A 2003/12)**

113 Examples