Kinematics

04 February 2024

Revision: 3241

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1. Motion

1.1 1D Motion

A. Displacement

1.1: Vectors and Scalars

Vector quantities have a magnitude and direction.
Scalar quantities only have magnitude.

1.2: Displacement

Displacement is a change in position

$$s = \Delta x = x_f - x_i$$

1.3: Displacement is a Vector

By convention:

- Right is positive
- Left is negative

Example 1.4

I leave for New York from my home on 5th Jan. From New York, I visit Philadelphia, and Denver. I return home on 8th Jan. What is my displacement over the entire time?

$$Zero$$

Example 1.5

Displacement-Time Graph

Example 1.6

At time $t = 0$, a particle is at the origin. Its displacement on the x axis after t seconds is given by te^{-t}. When does the particle returns to the origin

At the origin, the displacement is zero. Hence, we want to solve:

$$te^{-t} = 0$$

Use the zero-product property

$$t = 0 \; OR \; e^{-t} = 0 \Rightarrow \phi$$

$$\lim_{t\to\infty} te^{-t} = \lim_{t\to\infty} \frac{t}{e^t} = \underbrace{\lim_{t\to\infty} \frac{1}{e^t}}_{\substack{Apply \\ L'Hopital}} = 0$$

Hence, when $t \to \infty$, the particle will return to its starting position. But it will not return for any finite value.

B. Distance

1.7: Distance

Distance is the length of path travelled

Example 1.8
Calculate Distance Travelled

Example 1.9
Distance Time Graph

1.10: Distance is a scalar quantity

1.11: Displacement versus Distance

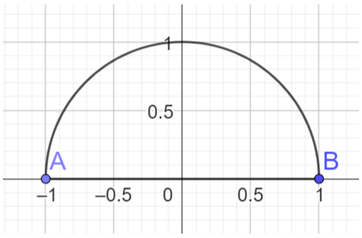
$$s \le d$$

Example 1.12
Graph

$$s(t) = displacement$$
$$d(t) = distance$$

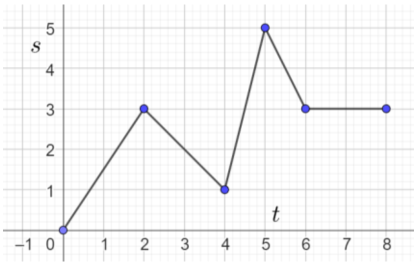
Identify

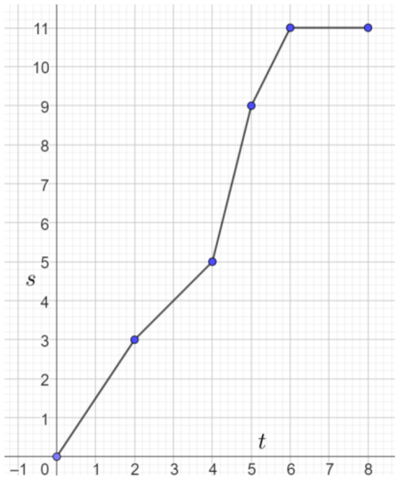
Example 1.13
A monkey travels from point A to point B along the semi-circular arc shown. What is the difference in the distance and the displacement of the monkey?



$$Distance = \frac{1}{2}(2\pi r) = \pi r = \pi$$
$$Displacement = 2$$
$$Difference = \pi - 2$$

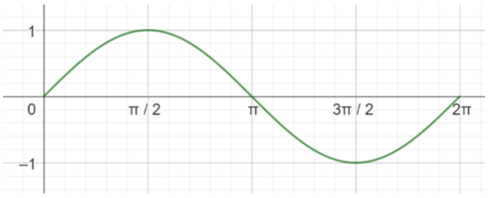
Example 1.14
The adjoining diagram shows a displacement-time graph with displacement on the y axis and time on the x axis. Draw the corresponding distance-time graph with distance travelled on the y axis and time on the x axis.

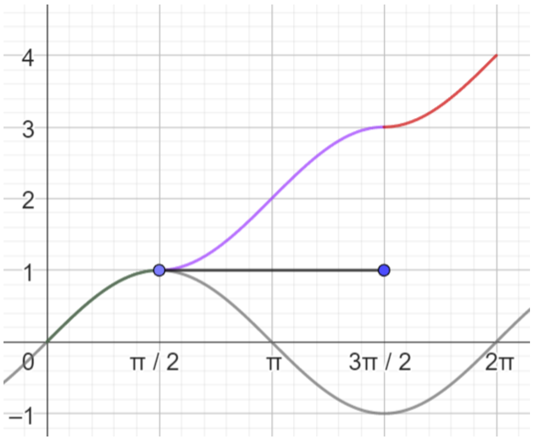




Example 1.15

A particle on the x axis has the displacement $y = \sin x$, graphed alongside. Draw the corresponding distance-time graph with distance travelled on the y axis and time on the x axis.

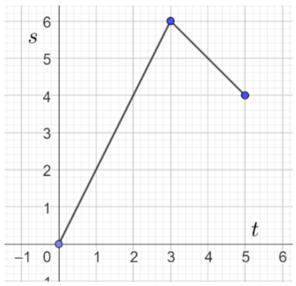




Example 1.16

The adjacent graph shows displacement in meters on the y-axis and time in seconds on the x-axis.

A. What is the distance travelled?
B. What is the maximum displacement, and when?
C. Draw a distance-time graph.



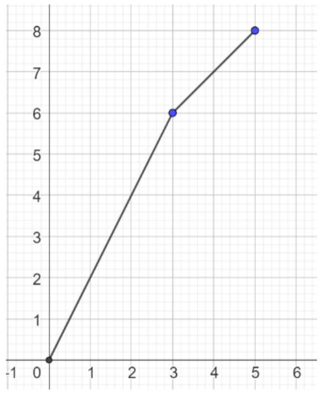
Part A

$$6 + 2 = 8 \; units$$

Part B

$$6 \; units \; at \; t = 3$$

Part C



C. Velocity

1.17: Velocity Vector
Velocity is a vector quantity.

1.18: Interpretation of Velocity
If velocity is positive for an object, it is moving in the positive($right$) direction.
If velocity is negative for an object, it is moving in the negative($left$) direction.

- Positive is right, and negative is usual convention.

Example 1.19
A particle which is at $x = -170$ at $t = 0$ moves to $x = -160$ at $t = 1$. Is the particle moving left, right, or neither?

$$Right$$

Example 1.20
A particle is moving such its velocity is given by $6t^2 - t - 15$. Determine for $t \in \mathbb{R}$ when it is

A. moving left
B. moving right
C. stationary

This in an upward parabola. Determine the critical points

$$6t^2 - t - 15 = 0$$
$$6t^2 - 10t + 9t - 15 = 0$$
$$2t(3t-5) + 3(3t-5) = 0$$
$$(2t+3)(3t-5) = 0$$
$$(2t+3)(3t-5) = 0$$
$$t \in \left\{-\frac{3}{2}, \frac{5}{3}\right\}$$

$$Stationary = \left\{-\frac{3}{2}, \frac{5}{3}\right\}$$
$$Moving\ left = \left(-\frac{3}{2}, \frac{5}{3}\right)$$

$$Moving\ Right = \left(-\infty, -\frac{3}{2}\right) \cup \left(\frac{5}{3}, \infty\right)$$

D. Average Velocity

1.21: Average Velocity

Average velocity is the rate of change of displacement

$$v = \frac{Change\ in\ position}{Time\ Interval} = \frac{s}{t}$$

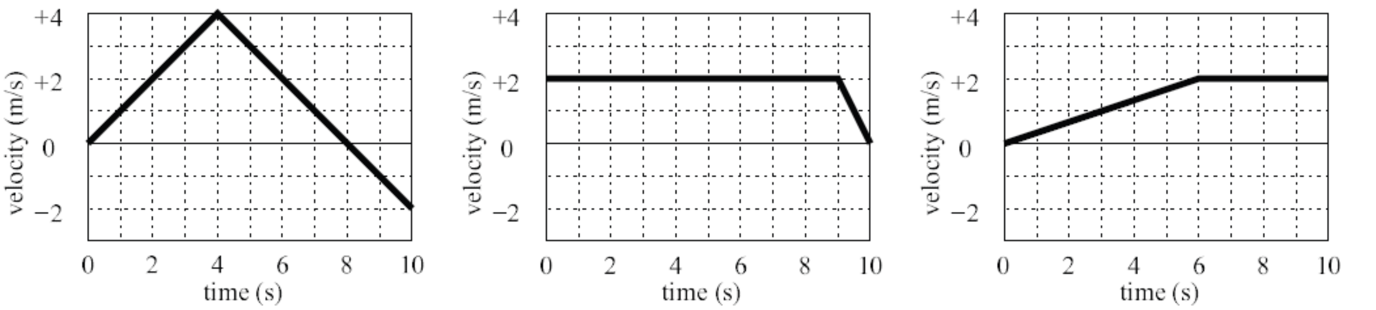
- Velocity in the positive direction is positive, while that in the negative direction is negative.
- Velocity is a vector quantity.

Example 1.22

A. A runner in a 100 meter race runs to the finish line in 20 seconds. He walks back in another 80 seconds. Calculate the velocity when going to the finish line, when back, and the overall velocity.
B. Calculate displacement
C. Calculate Time

Example 1.23

Refer to the three graphs below which show velocity of three objects as a function of time. Each object is moving only in one dimension. Rank the magnitudes of the maximum velocity achieved during the ten second interval. **(F=ma 2011/3)**

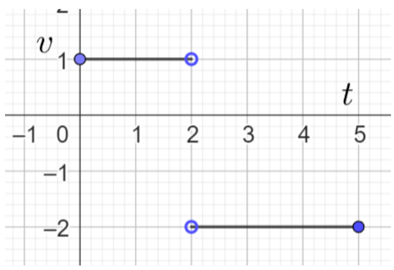


Object I   Object II   Object III

$$V_{Max-I} = 4$$
$$V_{Max-II} = 2$$
$$V_{Max-III} = 2$$

$$I > II = III$$

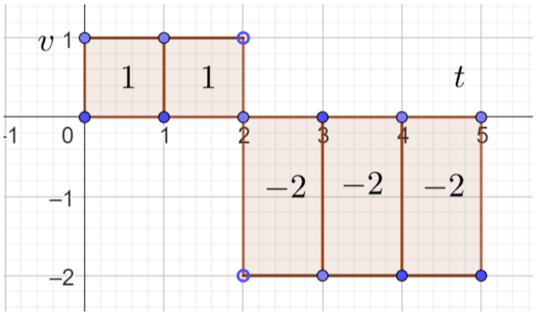
1.24: Graphical Integration



Example 1.25

The velocity-time graph shown alongside is for an object that begins at $x = 3$ and travels on the x −axis.

A. Determine the displacement and the distance travelled.
B. Draw a displacement-time graph

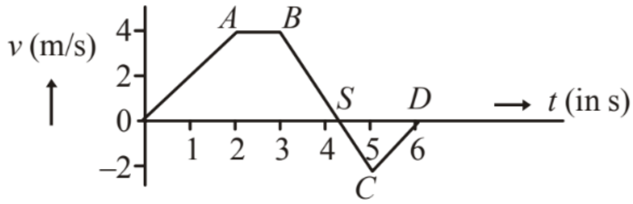


1.26: Distance and displacement

- Distance is the unsigned area between the x −axis and the graph on a velocity-time graph.
- Displacement is the signed area between the x −axis and the graph on a velocity-time graph.

Example 1.27

The velocity (v) and time (t) graph of a body in a straight-line motion is shown in the figure. The point S is at 4.333 seconds.



A. The total distance covered by the body in 6 s is:
(JEE Main, 5 Sep 2020, Shift-II)
B. The displacement of the body in 6 seconds is:

Note: You can approximate 4.333 with a close approximation.

$$OS = 4 + \frac{1}{3} = \frac{13}{3}, \qquad SD = 2 - \frac{1}{3} = \frac{5}{3}$$

We find the unsigned area to calculate the distance:

$$= \underbrace{\frac{1}{2}\left(1 + \frac{13}{3}\right)(4)}_{Trapezoid\ OABS} + \underbrace{\frac{1}{2}\left(\frac{5}{3}\right)(2)}_{Triangle\ SCD} = \frac{32}{3} + \frac{5}{3} = \frac{37}{3}$$

We find the signed area to calculate the displacement:

$$= \frac{32}{3} - \frac{5}{3} = \frac{27}{3} = 9$$

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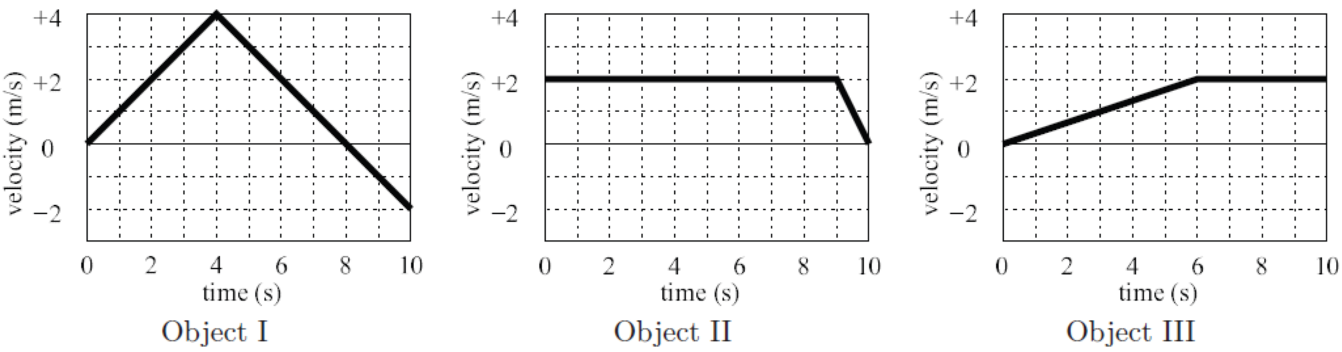
Example 1.28

The velocity-time graphs of a car and a scooter are shown in the figure. (i) the difference between the distance travelled by the car and the scooter in 15 s and (ii) the time at which the car will catch up with the scooter are
(JEE Main April 15, 2018)

Example 1.29

Refer to the three graphs below which show velocity of three objects as a function of time. Each object is moving only in one dimension. Rank the magnitudes of the distance traveled during the ten second interval. **(F=ma 2011/4)**



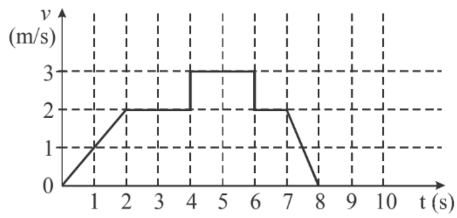
To find the distance travelled, we find the unsigned area between the curve and the time axis:

$$I: \frac{1}{2} \times 8 \times 4 + \frac{1}{2} \times 2 \times 2 = 16 + 2 = 18$$
$$II: 2 \times 9 + \frac{1}{2} \times 2 \times 1 = 18 + 1 = 19$$
$$III: 2 \times 4 + \frac{1}{2} \times 6 \times 2 = 8 + 6 = 14$$

$$s_{II} > s_I > s_{III}$$

Example 1.30

A particle starts from the origin at time $t = 0$ and moves along the positive x-axis. The graph of velocity with respect to time is shown in the figure.



A. What is the position of the particle at time $t = 5s$? **(JEE Main, 10 Jan 2019, Shift-II)**
B. In Part A, will displacement be different from distance? Explain why.

Part A

We find the signed area

$$\underbrace{\frac{1}{2} \times 4}_{\substack{Half\ of\ the\ Square \\ from\ 0\ to\ 2}} + \underbrace{4}_{\substack{Square\ from \\ 2\ to\ 3}} + \underbrace{3}_{\substack{Rectangle \\ from\ 4\ to\ 5}} = 9$$

Part B

Displacement will be same as distance, since the movement is one direction only.

E. Average Acceleration

1.31: Average Acceleration

Average acceleration is the rate of change of velocity.

$$a = \frac{Change\ in\ velocity}{Time\ Interval} = \frac{\Delta v}{\Delta t}$$

- When acceleration has a negative value, it is called deceleration.
- The signed area under an acceleration-time graph gives the velocity.

Example 1.32

At $12:22:23\ pm$, where time is expressed in hours, minutes and seconds my cat is due north of my house, running at a speed of $7\frac{m}{s}$ chasing a rat which is running to save its life away from the house. The cat bumps against a growling dog, and the roles are reversed. At time $12:22:27\ pm$, the cat is running due south at a speed of $21\frac{m}{s}$ to save its life. Calculate the average acceleration of the cat over the given time period. Consider North as positive.

$$a = \frac{\Delta v}{\Delta t} = \frac{-21 - 7}{4} = -\frac{28}{4} = -7\frac{m}{s}$$

Example 1.33

Mark all correct options

A car with constant velocity in one direction has:

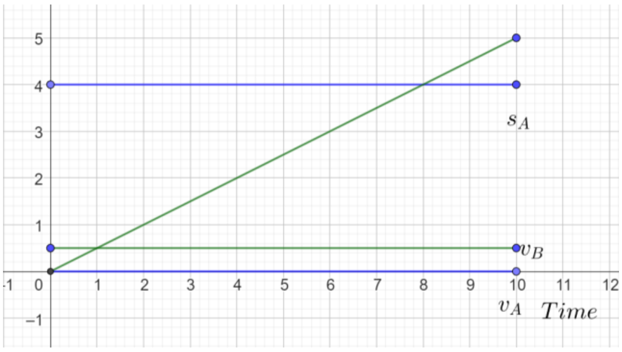
A. constant acceleration
B. constant displacement
C. constant speed
D. constant rate of change of displacement
E. zero acceleration

Consider that $Velocity = 5\frac{m}{s}$

$$Acceleration = 0\frac{m}{s} = Constant$$
$$Displacement\ does\ change \Rightarrow Not\ a\ constant$$

$$Speed = |5| = 5 \Rightarrow Constant$$
$$Velocity = 5 \Rightarrow Constant$$

$$Options\ A, C, D, E$$



Example 1.34

Provide all correct matches.

An object with negative acceleration can experience which of the following. Check the table, and provide all the correct matches.

Change	Parameter
A. Increase	1. Speed
B. Decrease	2. Velocity
	3. Displacement

$$1AB$$

Consider $1A$

$$Speed = 10\frac{m}{s}, Velocity\ is - 10\frac{m}{s}, acceleration\ is - 1\frac{m}{s^2}$$

After some time:

$$Velocity = -11\frac{m}{s} \Rightarrow Speed = 11\frac{m}{s} \Rightarrow Valid$$

Consider $1B$

$$Speed = 10\frac{m}{s}, Velocity = 10\frac{m}{s}, acceleration\ is - 1\frac{m}{s^2}$$

After some time:

$$Velocity = 9\frac{m}{s} \Rightarrow Speed = 9\frac{m}{s} \Rightarrow Valid$$

Consider $2B$
This was already discussed in case 1A and 1B, and it is valid in both.

$$1AB, 2B, 3AB$$

1.35: Increasing and Decreasing Speed

An object will have increasing speed when velocity and acceleration have the same sign.
An object will have decreasing speed when velocity and acceleration have opposite signs.

Example 1.36

Mark all correct options

An object moving in a straight line is speeding up when its acceleration and velocity:

A. are both positive.
B. are both negative.
C. have opposite sign
D. have the same sign
E. None of the above

$$Option\ A, B, D$$

Example 1.37

Mark all correct options

The displacement $(d_1, d_2, \dots, d_n)$ of a particle is measured at equal time intervals. If

$$d_n - d_{n-1} > d_{n-1} - d_{n-2} > \dots > d_2 - d_1$$

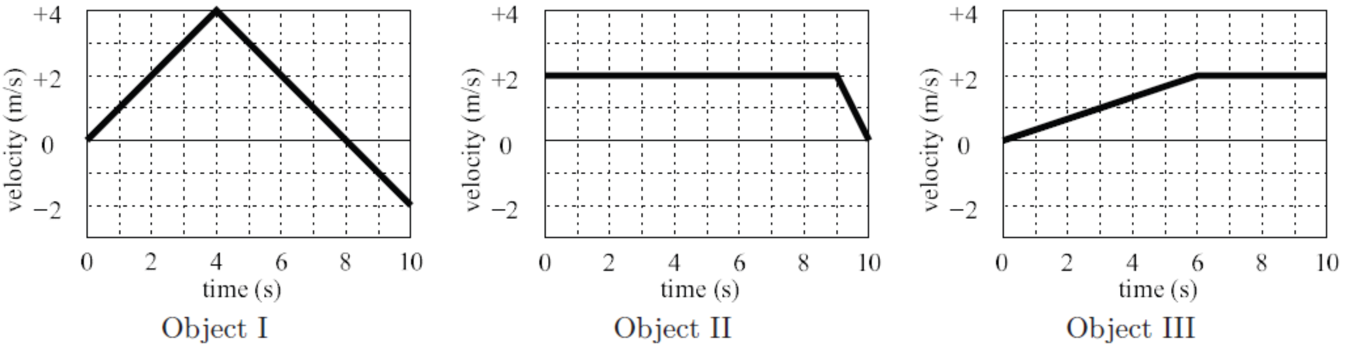
then, we can say that the object is undergoing:

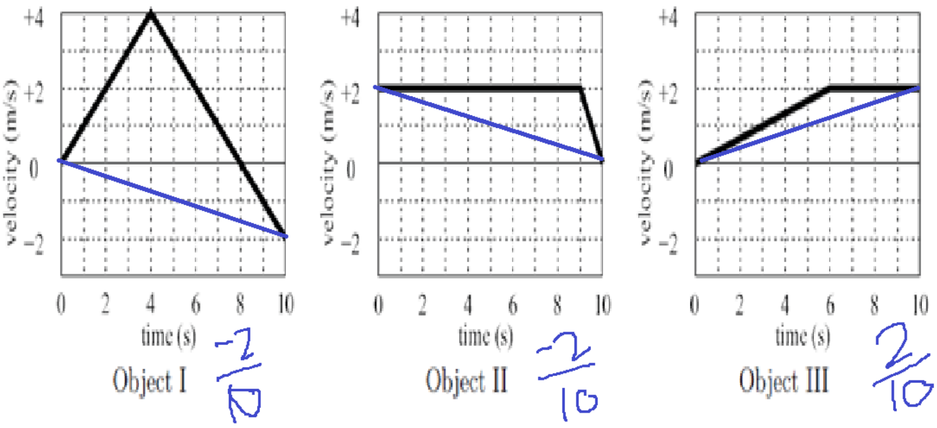
A. Constant Acceleration
B. Positive Acceleration
C. Negative Acceleration
D. Non-Constant Acceleration
E. Constant Velocity

$$Option\ B$$

Example 1.38

Refer to the three graphs below which show velocity of three objects as a function of time. Each object is moving only in one dimension. Rank the magnitudes of the average acceleration during the ten second interval. **(F=ma 2011/2)**





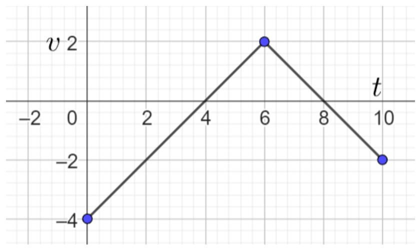
Draw a line connecting the start point and end point for each graph. We get:

$$\underbrace{\left|-\frac{2}{10}\right|}_{\textit{Object I}}=\underbrace{\left|-\frac{2}{10}\right|}_{\textit{Object II}}=\underbrace{\left|\frac{2}{10}\right|}_{\textit{Object III}}$$

Example 1.39

The adjoining velocity-time graph is for an object starting at $x=-5$ on the x-axis, and moving on the x axis. Identify the:

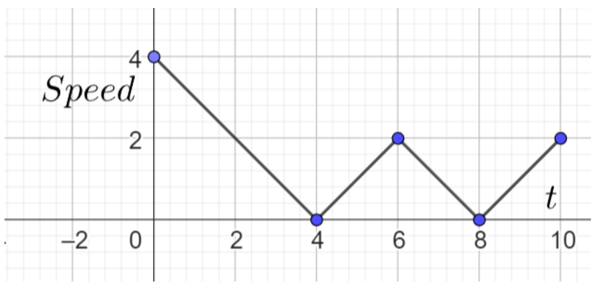
A. intervals where speed is increasing, and decreasing.
B. intervals where acceleration is positive and negative.
C. final position
D. distance travelled



Part A

Speed is a scalar quantity and velocity is a vector quantity. In general, ignoring the direction of velocity gives us speed. That is:

$$Speed=|Velocity|$$



Speed is increasing when

$$t\in(4,6)\cup(8,10)$$

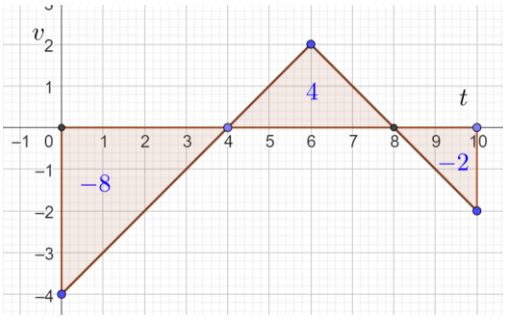
Speed is increasing when

$$t \in (0,4) \cup (6,8)$$

Part B

Acceleration is the rate of change of velocity

$$\text{If velocity is increasing} \Rightarrow \text{Acceleration is} + Ve \Rightarrow t \in (0,6)$$
$$\text{If velocity is decreasing} \Rightarrow \text{Acceleration is} - Ve \Rightarrow t \in (6,10)$$



Part C

Final Position

$$= \underbrace{-5}_{\substack{Intial\\Position}} + \underbrace{-8+4-2}_{\substack{Change\ in\\Position}} = -11$$

Part D

The distance travelled is the sum of the absolute values of the individual displacements:

$$|-8| + |4| + |-2| = 8 + 4 + 2 = 14$$

1.2 Uniform v (TSD)

A. Speed

1.40: Average Speed

Average speed is the rate of change of distance.

$$S = \frac{D}{T}$$

Where

$$S = Average\ Speed$$
$$D = Total\ Distance$$
$$T = Total\ Time$$

- Speed is a scalar.
- Speed is always positive.

Example 1.41: Basics

A. Calculate time
B. Calculate distance

B. Average Speed

1.42: Average Speed for Equal Distances

When averaging for equal distances

- The arithmetic mean is the wrong mean to use.
- The harmonic mean is the right mean to use.

1.43: Harmonic Mean

$$HM(a,b) = \frac{2ab}{a+b}$$

Example 1.44

A car covers the first half of the distance between two places at 40 $\frac{km}{h}$, and the other half at 60 $\frac{km}{h}$. The average speed of the car is: (JEE Main, May 7 2012)

Method I

Let the distance be D.

$$Average\ Speed = \frac{Total\ Distance}{Total\ Time} = \frac{2D}{\frac{D}{40} + \frac{D}{60}} = \frac{2D}{\frac{3D + 2D}{120}} = 2D \times \frac{120}{5D} = 48$$

Method II

Note from the previous method that the distance is not a part of the final calculation. Hence, we can take any distance that we want.

A convenient is $LCM(40,60) = 120$:

$$Average\ Speed = \frac{Total\ Distance}{Total\ Time} = \frac{240}{\frac{120}{40} + \frac{120}{60}} = \frac{240}{3 + 2} = \frac{240}{5} = 48$$

Example 1.45

Car C_1 travels a distance D_1 over time T_1 and car C_2 travels a distance D_2, with $D_2 > D_1$ in a time T_2. If the average speed of both the cars is the same, eliminate D_2 from $D_2 - D_1$.

Since the speeds of the cars are equal, we must have

$$S_1 = S_2$$

Use the definition of speed:

$$\frac{D_1}{T_1} = \frac{D_2}{T_2}$$

Substitute $D_2 - D_1 = \Delta \Rightarrow D_2 = D_1 + \Delta$:

$$\frac{D_1}{T_1} = \frac{D_1 + \Delta}{T_2}$$

Solve for Δ:

$$D_1T_2 = D_1T_1 + \Delta T_1$$
$$D_1T_2 - D_1T_1 = \Delta T_1$$

$$\Delta = \frac{D_1T_2 - D_1T_1}{T_1} = D_1\left(\frac{T_2}{T_1} - \frac{T_1}{T_1}\right) = D_1\left(\frac{T_2}{T_1} - 1\right)$$

(Calculator) Example 1.46

A cyclist travels at a constant speed of 22.0 $\frac{km}{hr}$ except for a 20-minute stop. The cyclist's average speed was 17.5 $\frac{km}{hr}$. How far did the cyclist travel? (F=ma 2011/1)

$$Average\ Speed = \frac{Total\ Distance}{Total\ Time}$$

If you do not consider the stop

$$22 = \frac{D}{T} \Rightarrow \underbrace{D = 22T}_{Equation\ I}$$

If you do consider the stop:

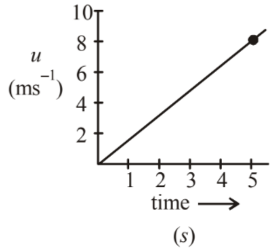
$$17.5 = \frac{D}{T + \frac{1}{3}} \Rightarrow \underbrace{D = 17.5\left(T + \frac{1}{3}\right)}_{Equation\ II}$$

Since the LHS of both Equation I and II is the same, the RHS is also the same:

$$22T = 17.5\left(T + \frac{1}{3}\right)$$
$$22T = 17.5T + \frac{17.5}{3}$$
$$4.5T = \frac{17.5}{3}$$
$$T = \frac{17.5}{3 \times 4.5}$$
$$22T = 22 \times \frac{17.5}{3 \times 4.5} = 28.\overline{518}$$

Example 1.47

The speed versus time graph for a particle is shown in the figure. The distance travelled (in m) by the particle during the time interval $t = 0$ to $t = 5s$ will be (JEE Main 4 Sep 2020, Shift-II)



Note: In the absence of further information, assume that the particle travels in a straight line in one dimension in one direction only.

Then, the distance is the area under the speed-time graph

$$= \frac{1}{2} \times 8 \times 5 = 20m$$

Example 1.48

A person travels x distance with velocity v_1 and then x distance with velocity v_2 in the same direction. The average velocity of the person is v, then the relation between v, v_1 and v_2 will be:

A. $v = v_1 + v_2$
B. $v = \frac{v_1+v_2}{2}$
C. $\frac{2}{v} = \frac{1}{v_1} + \frac{1}{v_2}$
D. $\frac{1}{v} = \frac{1}{v_1} + \frac{1}{v_2}$ (JEE Main Jan 25, Shift-I; JEE Main April 10, Shift-II)

When travelling equal distance with different speed, the average speed is the harmonic mean of the individual speeds:

$$v = \frac{2v_1v_2}{v_1 + v_2}$$
$$\frac{1}{v} = \frac{v_1 + v_2}{2v_1v_2}$$
$$\frac{2}{v} = \frac{v_1}{v_1v_2} + \frac{v_2}{v_1v_2}$$

$$\frac{2}{v}=\frac{1}{v_2}+\frac{1}{v_1}$$

Example 1.49

An object moves with speed v_1, v_2 and v_3 along a line segment AB, BC and CD as shown in figure, where $AB = BC$, and $AD = 3AB$, then average speed of the object will be:

$A.\ \dfrac{v_1v_2v_3}{3(v_1v_2+v_2v_3+v_3v_1)}$

$B.\ \dfrac{3v_1v_2v_3}{v_1v_2+v_2v_3+v_3v_1}$

$C.\ \dfrac{v_1+v_2+v_3}{3}$

$D.\ \dfrac{v_1+v_2+v_3}{3v_1v_2v_3}$ $(\textbf{\textit{JEE Main Feb 1, 2023, Shift} − I})$

$$AB = BC = CD$$

$$\frac{3}{\frac{1}{v_1}+\frac{1}{v_2}+\frac{1}{v_3}}=\frac{3}{\frac{v_2v_3}{v_1v_2v_3}+\frac{v_1v_3}{v_1v_2v_3}+\frac{v_1v_2}{v_1v_2v_3}}=\frac{3v_1v_2v_3}{v_2v_3+v_1v_3+v_1v_2}$$

1.50: Speed versus Velocity

$$Velocity \le Speed$$

Example 1.51

The magnitude of

Example 1.52

Graph

$$s(t) = speed$$
$$v(t) = velocity$$

Identify

1.3 Uniform a (Equations of Motion)

A. Constant Acceleration

1.53: Equations of Motion

The equations of motion are derived under the assumption of constant acceleration a. If the assumption is not correct, then the equation do not hold.

$$Displacement = s \Rightarrow Velocity = v = \frac{ds}{dt} \Rightarrow Acceleration = a = \frac{d^2s}{dt^2}$$

Integrate both sides of $\frac{d^2s}{dt^2} = a$:

$$v = \frac{ds}{dt} = \int a\ dt = at + C$$
$$\underbrace{v = at + C}_{Equation\ I}$$

When

$$t = 0 \Rightarrow Velocity = v(0) = v_i = Inital\ Velocity$$

Substitute $t = 0$ in Equation I:

$$v(0) = a \cdot 0 + C \Rightarrow v(0) = u = C_1$$

The particular solution:

$$Final\ Velocity = v = u + at$$

Substitute $v = \frac{ds}{dt}$ in the above equation:

$$\frac{ds}{dt} = u + at$$

Integrate:

$$s = \int (u + at)\ dt$$
$$\underbrace{s = ut + \frac{1}{2}at^2 + C_2}_{Equation\ II}$$

To find C, use $Initial\ Displacement = s(0) = s_0$:

$$s_0 = v(0) + \frac{1}{2}a(0)^2 + C_2 = C \Rightarrow C_2 = s_0$$

Substitute the constant in Equation II:

$$\underbrace{s = s_0 + ut + \frac{1}{2}at^2}_{Equation\ III}$$

Introduce an origin at the initial position. Then,

$$s_0 = Initial\ Displacement = 0$$

Substituting $s_0 = 0$ into Equation III:

$$s = ut + \frac{1}{2}at^2$$

1.54: Final Velocity

$$v = u + at$$
$$v^2 = u^2 + 2as$$

Where

$$v = final\ velocity$$

$$u = initial\ velocity$$

Final velocity

$$= Initial\ Velocity + \Delta Velocity$$
$$= Initial\ Velocity + \left(\frac{Change\ in\ Velocity}{Time}\right)(Time)$$
$$= v_i + at$$

$$v_f = v_i + at$$
$$v_f^2 = v_i^2 + 2v_i at + a^2t^2$$
$$v_f^2 = v_i^2 + 2a\left(v_i t + \frac{1}{2}at^2\right)$$
$$v_f^2 = v_i^2 + 2as$$

Example 1.55

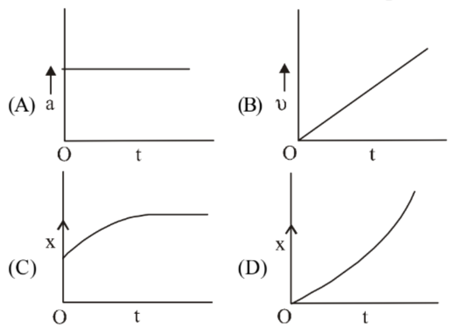
If acceleration is a constant function, then velocity is a ____ function, and displacement is a ________ function

$$Velocity\ is\ linear$$
$$Displacement\ is\ quadratic$$

Example 1.56

Mark all correct options

A particle starts from origin O from rest and moves with a uniform acceleration along the positive x-axis. Identify all figures that correctly represents the motion qualitatively ($a =$ acceleration, $v =$ velocity, $x =$ displacement, $t =$ time) (JEE Main, 8 April 2019, Shift-II, Adapted)



Option A has uniform acceleration. This makes it correct.
Option B has constant slope. This means rate of increase of velocity is constant. Hence, it is correct.

Option D has distance increasing at a increasing rate, which is consistent with increasing velocity, which is consistent with uniform acceleration.
Note that this is *qualitatively* correct (which is asked by the question), not quantitively till we have more information.

$$Options\ A, B, D$$

1.57: Average Velocity

For an object moving with constant acceleration, the average velocity is the average of the final and initial velocity:

$$v_{avg} = \frac{v_i + v_f}{2}$$

1.58: Displacement

For an object moving with constant acceleration, once the average velocity has been calculated we can apply familiar TSD concepts to get displacement:

$$s = (v_{avg})t = \frac{1}{2}(v_i + v_f)t$$

Example 1.59

Distance travelled with constant deceleration

$$s = (v_{avg})t = \frac{1}{2}(v_i + v_f)t =$$

Example 1.60

A cat running at a constant speed of $30\frac{units}{s}$ stumbles over a stationary dog and continues running at the same speed. The dog runs after the cat, accelerating uniformly at $2\frac{units}{s^2}$, till it reaches its maximum speed of $40\frac{units}{s}$. Find the time taken by the dog to catch the cat.

If the dog catches the cat while accelerating, then:

$$s_{Dog} = s_{Cat}$$
$$\left(\frac{0 + 2t}{2}\right)t = 30t$$
$$t^2 = 30t$$
$$t \in \{0,30\}$$
$$30 > 20 \Rightarrow Reject$$

Now, we find the situation at $t = 20$ and work from there:

$$s_{Cat} = 30(20) = 600\ units$$
$$s_{Dog} = 20^2 = 400\ units$$

After $t = 20$, we have uniform velocity:

$$400 + 40T = 600 + 30T$$
$$10T = 200$$
$$T = 20$$

1.61: Displacement

We can write displacement in terms of either the initial velocity and acceleration, or the final velocity and acceleration.

$$s = ut + \frac{1}{2}at^2$$
$$s = vt - \frac{1}{2}at^2$$

- Note the similarity in structure. The second term on the RHS is the same in both (except for the sign).
- The choice of which equation to use from the above depends on the information that you have and the context.

Example 1.62

A body accelerating uniformly from rest travels in a straight line. It travels 2 meters in the 1st second, 6 meters in the 2nd second, 10 meters in 3rd second, and 16 meters in the 4th second. Find the acceleration.

Substitute $Initial\ Velocity = u = 0$ in $s = ut + \frac{1}{2}at^2$

$$s = \frac{1}{2}at^2$$

$$s(1)=\frac{1}{2}a(1)^2 \Rightarrow 2=\frac{1}{2}a \Rightarrow a=4$$

Example 1.63

In a car race on straight road, car A takes a time t less than car B at the finish and passes finishing point with a speed v more than of car B. Both the cars start from rest and travel with constant acceleration a_1 and a_2 respectively. Then v, in terms of a_1, a_2 and t is equal to: **(JEE Main, 9 Jan 2019, Shift-II)**

Substitute $A_{Time}=T, B_{Time}=T+t, v_{i-A}=v_{i-B}=0$ in $v_f=v_i+at$:

$$v_{f-A}=0+a_1T$$
$$v_{f-B}=0+a_2(T+t)$$

v is the difference in final velocities of the cars:

$$v=\underbrace{a_1T}_{v_{f-A}}-\underbrace{a_2(T+t)}_{v_{f-B}}=\underbrace{(a_1-a_2)T-a_2t}_{\textit{Expression I}}$$

To eliminate T from this expression, recognize that displacement is the same for both cars.
Use $s=v_it+\frac{1}{2}at^2$:

$$0+\frac{1}{2}a_1T^2=0+\frac{1}{2}a_2(T+t)^2$$

Simplify:

$$a_1T^2=a_2(T+t)^2$$

Take the square root both sides and rearrange:

$$(\sqrt{a_1}-\sqrt{a_2})T=\sqrt{a_2}t$$

Solve for T to get $\underbrace{T=\frac{\sqrt{a_2}t}{\sqrt{a_1}-\sqrt{a_2}}}_{\textit{Equation II}}$, and substitute this value into Expression I:

$$(a_1-a_2)\left(\frac{\sqrt{a_2}t}{\sqrt{a_1}-\sqrt{a_2}}\right)-a_2t$$

Cancel using a difference of squares:

$$(\sqrt{a_1}+\sqrt{a_2})(\sqrt{a_2}t)-a_2t$$
$$\sqrt{a_1a_2}t+a_2t-a_2t$$
$$\sqrt{a_1a_2}t$$

We can summarize the solution in the table below:

Car	Time	Initial Velocity	Acceleration	Final Velocity	Displacement
Equation				v_i+at	$v_it+\frac{1}{2}at^2$
A	T	0	a_1	a_1T	$\frac{1}{2}a_1T^2$
B	$T+t$	0	a_2	$a_2(T+t)$	$\frac{1}{2}a_2(T+t)^2$

Example 1.64

A scooter accelerates from rest for time t_1 at constant rate a_1, and then retards at constant rate a_2 for time t_2. The correct value of $\frac{t_1}{t_2}$ in terms of a_1 and a_2 will be: **(JEE Main Feb 26, 2021, Shift-II)**

$$v_f = v_i + at$$

The velocity after time t_1

$$0 + a_1 t_1 = a_1 t_1$$

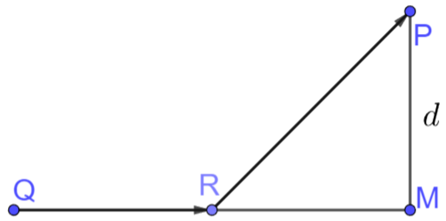
The velocity after time t_2

$$a_1 t_1 - a_2 t_2 = 0$$
$$a_1 t_1 = a_2 t_2$$
$$\frac{t_1}{t_2} = \frac{a_1}{a_1}$$

1.65: Velocity Proportionality

When the final velocity is zero, distance is proportional to the square of initial velocity:

$$s \propto u^2$$



$$v_f^2 = v_i^2 + 2as$$

When $v_f = 0$:

$$0 = v_i^2 + 2as$$
$$s = -\frac{v_i^2}{2a}$$
$$s \propto v_i^2$$

Example 1.66

A. A car is moving with a speed of $150 \frac{km}{hr}$ and after applying the brake it will move $27m$ before it stops. If the same car is moving with a speed of one third the reported speed, then it will stop after travelling _____ m distance. **(JEE Main July 25, 2022, Shift-II)**

B. An automobile, travelling at $40 \frac{km}{h}$, can be stopped at a distance of $40m$ by applying brakes. If the same automobile is travelling at $80 \frac{km}{h}$, the minimum stopping distance, in meters, is (assume no skidding) **(JEE Main 2002; 2003; 2004; April 15 2018)**

C. A car moving with a speed of 40 km/h can be stopped by applying brakes after at least 2 m. If the same car is moving with a speed of 80 km/h, what is the minimum stopping distance? **(NEET 1998)**

$$s \propto v_i^2$$

Part A

When speed is $\frac{1}{3}$, the distance will be $\left(\frac{1}{3}\right)^2 = \frac{1}{9}^{th}$:

$$\frac{27}{9} = 3\ m$$

Part B

When speed is doubled, the distance will be $(2)^2 = 4$ times:

$$40 \times 4 = 160\ m$$

Part C

When speed is doubled, the distance will be $(2)^2 = 4$ times:

$$2 \times 4 = 8\,m$$

Example 1.67

An automobile, travelling at $40\,\frac{km}{h}$, can be stopped at a distance of $40m$ by applying brakes. If the same automobile is travelling at $80\,\frac{km}{h}$, the minimum stopping distance, in meters, is (assume no skidding) **(JEE Main 2002; 2003; 2004; April 15 2018)**

A. Do this question using equations of motion, and not the property.
B. Explain why the property is better.

Part A

$$v^2 = u^2 + 2as$$
$$0 = 40^2 + 2a(0.04)$$
$$0 = 100 \cdot 40^2 + 2a(4)$$
$$0 = 100 \cdot 40^2 + 2a(4)$$
$$a = -\frac{100 \cdot 40^2}{8} = 20{,}000\frac{km}{hr}$$

$$0 = 80^2 + 2(-20{,}000)s$$
$$s = \frac{80 \times 80}{2 \times 20{,}000} = 0.16\,km = 160\,m$$

Part B

This is much lengthier, and has far worse calculations.
Hence, this method is not recommended.

1.68: Displacement proportionality

If an object initially at rest undergoes constant acceleration, then displacement is proportional to the square of the time:

$$s \propto t^2$$

$$s = v_i t + \frac{1}{2}at^2$$

Since the object is initially at rest, substitute $v_i = 0$

$$s = \frac{1}{2}at^2 = \left(\frac{1}{2}a\right)t^2$$

Since $\frac{1}{2}a$ is a constant:

$$s \propto t^2$$

Example 1.69

A small toy starts moving from the position of rest under constant acceleration. If it travels a distance of 10 m in t seconds, the distance travelled by the toy in next t seconds will be: **(JEE Main June 29, 2022, Shift-II)**

$$time = t, s = 10$$

When time is doubled, the displacement will be four times

$$s = 10(2^2) = 40$$

The further distance travelled:

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$$40 - 10 = 30\ m$$

1.4 Free Fall

A. Objects in Free Fall

Free fall is a special case of constant acceleration. Objects in free fall experience gravity.

1.70: Equations of Motion for Gravity

$$s = ut + \frac{1}{2}at^2 \Rightarrow h = ut + \frac{1}{2}gt^2$$
$$s = vt - \frac{1}{2}at^2 \Rightarrow h = vt - \frac{1}{2}gt^2$$
$$v = u + at \Rightarrow v = u + gt$$
$$v^2 = u^2 + 2as \Rightarrow v^2 = u^2 + 2gh$$

By convention:

- Origin is at the object

$$s = displacement$$
$$v = final\ velocity$$
$$u = initial\ velocity$$
$$h = height$$
$$a = g = 9.8\frac{m}{s^2} \approx 10\frac{m}{s^2}$$

B. Objects falling from a height

An object at a given height can be released, and fall due to gravity. Such an object will start from a position of rest.

1.71: Object starting from rest

In the special case of an object falling due to gravity, and starting from rest, the initial velocity is zero.

1.72: Object starting from rest

The height-time path of an object starting from rest is parabolic. Explain why.

$$s = \frac{1}{2}at^2$$

This is a quadratic equation. Hence, the shape is that of a parabola.

Example 1.73

A. A boy standing at the top of a tower of $20m$ height drops a stone. Assuming $g = 10ms^{-2}$, the velocity with which it hits the ground is: **(NEET 2011)**

B. A ball is thrown vertically downward with a velocity of $20\frac{m}{s}$ from the top of a tower. It hits the ground after some time with a velocity of $80\frac{m}{s}$. The height of the tower is: $(g = 10\frac{m}{s^2})$ **(NEET 2020)**

Part A

$$v^2 = 0^2 + 2gh = 2(10)(20) = 400$$
$$v = 20$$

Part B

$$v^2 = u^2 + 2gh$$
$$80^2 = 20^2 + 2(10)h$$
$$6400 = 400 + 20h$$
$$20h = 6000$$
$$h = 300$$

Example 1.74: Two Objects

Two balls A and B are placed at the top of a 180 m tall tower. Ball A is released from the top at $t = 0\ s$. Ball B is thrown vertically downward with an initial velocity u at $t = 2s$. After a certain time, both balls meet 100 m above the ground. Find the value of u in ms^{-1}. (Use $g = 10\ ms^{-2}$) **(JEE Main June 29, 2022, Shift-I)**

For Ball A

$$h = -\frac{1}{2}gt^2$$
$$-80 = -\frac{1}{2}(10)t^2$$
$$16 = t^2$$
$$t = 4$$

For Ball B

Substitute $T = t - 2 = 2, g = 10, h = -80$ in $h = uT - \frac{1}{2}gT^2$:

$$-80 = 2u - \frac{1}{2}(10)(4)$$
$$-80 = 2u - 20$$
$$u = -30$$

1.75: Time to reach the ground

For an object starting from rest and dropped from a height, the time taken to reach the ground is:

$$t = \sqrt{\frac{2h}{g}}$$

Substitute $u = 0$ in $h = ut + \frac{1}{2}gt^2$:

$$h = \frac{1}{2}gt^2 \Rightarrow t^2 = \frac{2h}{g} \Rightarrow t = \sqrt{\frac{2h}{g}}$$

Example 1.76

A NCC parade is going on at uniform speed of $9\frac{km}{h}$ under a mango tree on which a monkey is sitting at a height of $19.6\ m$. At any particular instant, the monkey drops a mango. A cadet will receive the mango whose distance from the tree at time of drop is (given $g = 9.8\frac{m}{s^2}$) **(JEE Main, July 28, 2022, Shift-I)**

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 19.6}{9.8}} = \sqrt{4} = 2$$

$$D = s \times t = \frac{9000\ m}{3600\ s} \times 2s = 5m$$

Example 1.77

Two bodies A (of mass 1 kg) and B (of mass 3 kg) are dropped from heights of 16 m and 25 m, respectively. The ratio of the time taken by A to reach the ground compared to the time taken by B to reach the ground is **(NEET 2006)**

$$t_A : t_B = \sqrt{\frac{2h_A}{g}} : \sqrt{\frac{2h_B}{g}} = \sqrt{h_A} : \sqrt{h_B} = \sqrt{16} : \sqrt{25} = 4 : 5$$

Example 1.78

A ball is released from a height h. If t_1 and t_2 be the time required to complete first half and second half of the distance respectively, then find t_2 in terms of t_1. **(JEE Main, July 29, 2022, Shift-II)**

Since the object is starting from rest $h = ut + \frac{1}{2}gt^2$ simplifies to:

$$h = \frac{1}{2}gt^2$$

Half the distance is travelled in time t_1, and total distance is travelled time $t = t_1 + t_2$:

$$\underbrace{\frac{h}{2} = \frac{1}{2}gt_1^2}_{Equation\ I}, \qquad \underbrace{h = \frac{1}{2}g(t_1 + t_2)^2}_{Equation\ II}$$

The above is a system of equations in t_1, t_2 and h. Eliminate h by dividing Equation I by Equation II:

$$\frac{1}{2} = \frac{t_1^2}{(t_1 + t_2)^2}$$

Take the square root both sides:

$$\frac{1}{\sqrt{2}} = \frac{t_1}{t_1 + t_2} \Rightarrow t_1 + t_2 = \sqrt{2}t_1 \Rightarrow t_2 = (\sqrt{2} - 1)t_1$$

C. Objects moving with an initial velocity

Example 1.79

A balloon was moving upwards with a uniform velocity of $10 \frac{m}{s}$. An object of finite mass is dropped from the balloon when it was at a height of 75m from the ground level. The height of the balloon from the ground when the object strikes the ground was around: (Use $g = 10\ ms^{-2}$) **(JEE Main July 25. 2021, Shift-II)**

$$h = ut + \frac{1}{2}gt^2$$
$$75 = -10t + \frac{1}{2}(10)t^2$$
$$5t^2 - 10t - 75 = 0$$
$$t^2 - 2t - 15 = 0$$
$$(t - 5)(t + 3) = 0$$
$$t \in \{-3, 5\}$$

Reject the negative value

$$t = 5$$

Height of balloon
$$= 75 + 5(10) = 125\ m$$

D. Ratio of Times

1.80: Ratio of Distances: Totals

The distance travelled by an object in free fall starting from rest is proportional to the square of the time taken.

$$s \propto t^2 \Leftrightarrow s = \frac{1}{2}gt^2$$

Example 1.81

An object is released by a spaceship from a height h on the planet Mars. The distance travelled by the object in t seconds is given by s_t. Determine the ratio:
$$s_1 : s_2 : s_3 : \dots$$

Let the acceleration due to gravity on Mars be g_{Mars}. Then:
$$-\frac{1}{2}g_{Mars}(1)^2 : -\frac{1}{2}g_{Mars}(2)^2 : -\frac{1}{2}g_{Mars}(3)^2 : \dots$$
$$1 : 4 : 9 : \dots : n^2$$

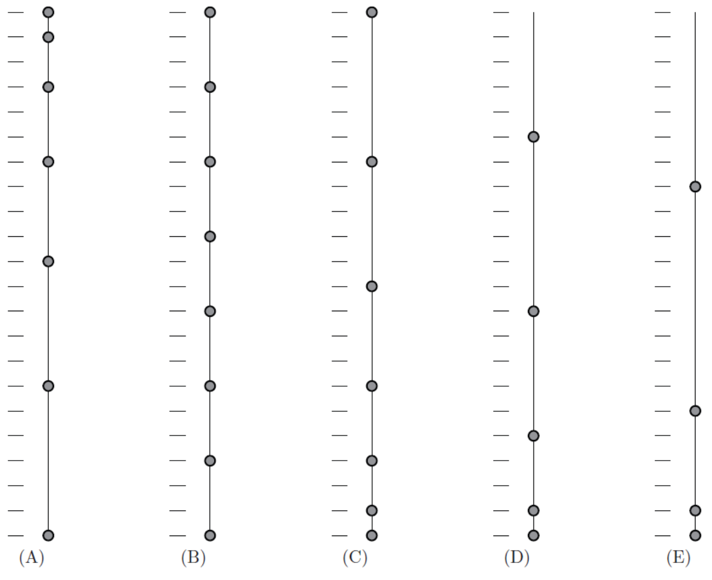
Example 1.82

A particle starts its motion from rest under the action of a constant force. If the distance covered in first 10 seconds is S_1 and that covered in the first 20 seconds is S_2, then S_2 in terms of S_1 is: **(NEET 2009)**

$$h \propto t^2$$
$$S_2 = 4S_1$$

Example 1.83

You have 5 different strings with weights tied at various point, all hanging from the ceiling, and reaching down to the floor. The string is released at the top, allowing the weights to fall. Which one will create a regular, uniform beating sound as the weights hit the floor? **(F=ma 2011/14)**



Using the property from the previous example, we need the distance from the weights to the ground to be in the ratio:

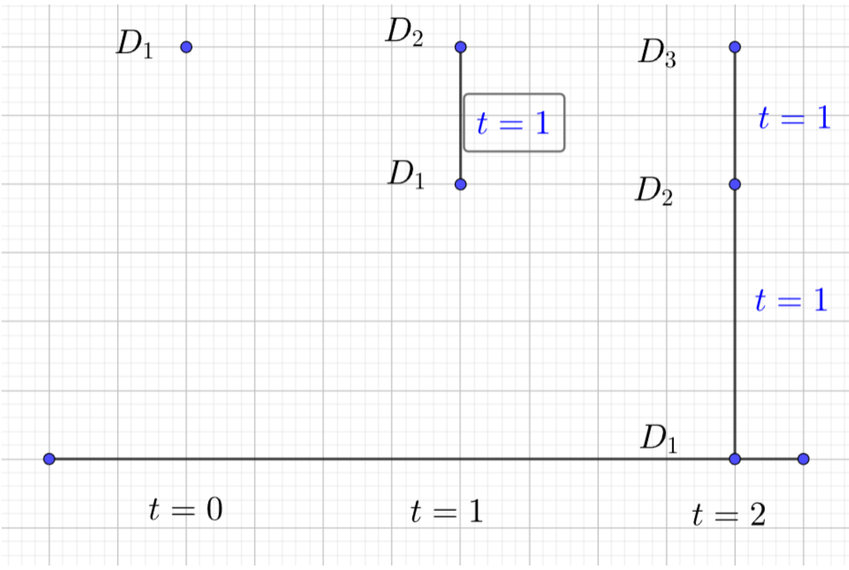
$$1:4:9:16:\ldots.$$

Which is

Option D

Example 1.84

Consider a dripping faucet, where the faucet is 10 cm above the sink. The time between drops is such that when one drop hits the sink, one is in the air and another is about to drop. At what height above the sink will the drop in the air be right as a drop hits the sink? **(F=ma 2012/1)**



The displacement of the first drop:

$$s_1 = \left.\frac{g(2t)^2}{2}\right|_{t=2t} = \underbrace{\frac{4gt^2}{2} = 10}_{Equation\ I}$$

The displacement of the second drop:

$$s_2 = \left.\frac{gt^2}{2}\right|_{t=t} = \frac{gt^2}{2}$$

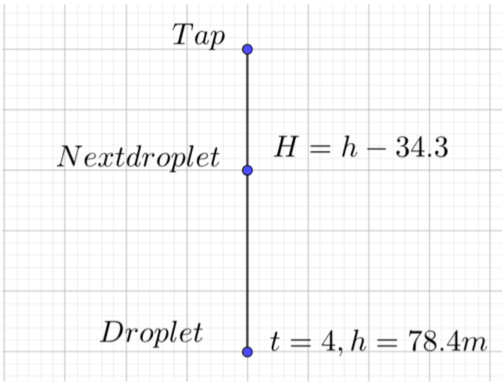
Divide Equation I by 4 to get:

$$\frac{gt^2}{2} = 2.5$$

Example 1.85

Water droplets are coming from an open tap at a particular rate. The spacing between a droplet observed at 4^{th} second after its fall to the next droplet is 34.3 m. At what rate are the droplets coming from the tap? $\left(g = 9.8\frac{m}{s^2}\right)$ **(JEE Main July 25, 2021, Shift-I)**

$$h = \frac{1}{2}g(4^2) = \frac{1}{2} \times 9.8(16) = 78.4\ m$$



The droplet has height

$$H = 78.4 - 34.3 = 44.1$$

Substitute $H = 44.1, g = 9.8$ in $h = \frac{1}{2}gt^2$:

$$44.1 = \frac{1}{2}(9.8)t^2 \Rightarrow t^2 = 9 \Rightarrow t = 3$$

The gap

$$= 4 - 3 = 1\frac{drop}{second}$$

1.86: Ratio of Distances: Differences

The distance travelled by an object in free fall in consecutive intervals has ratio

$$1:3:5:7:\dots\dots$$

$$\frac{1}{2}g(t)^2 : \frac{1}{2}g(2t)^2 : \frac{1}{2}g(3t)^2 : \frac{1}{2}g(4t)^2 : \dots$$
$$t^2 : 4t^2 : 9t^2 : 16t^2 : \dots$$
$$1:4:9:16:\dots$$

Differences are:

$$1 - 0 : 4 - 1 : 9 - 4 : 18 - 9 : \dots$$
$$1:3:5:7:\dots$$

Example 1.87

A stone falls freely under gravity. It covers distances h_1, h_2 and h_3 in the first 5 seconds, the next 5 seconds and the next 5 seconds respectively. The relation between h_1, h_2 and h_3 is

A. $h_2 = 3h_1$ *and* $h_3 = 3h_2$
B. $h_1 = h_2 = h_3$
C. $h_1 = 2h_2 = 3h_3$
D. $h_1 = \frac{h_2}{3} = \frac{h_3}{5}$ **(NEET 2013)**

$$h_1 : h_2 : h_3 = 1:3:5$$

$$\frac{h_1}{h_2} = \frac{1}{3} \Rightarrow h_1 = \frac{h_2}{3}$$
$$\frac{h_1}{h_3} = \frac{1}{5} \Rightarrow h_1 = \frac{h_5}{3}$$
$$h_1 = \frac{h_2}{3} = \frac{h_5}{3} \Rightarrow Option\ D$$

1.5 Objects Thrown Up

A. Objects thrown from the ground

Example 1.88

A football is thrown directly upwards from the ground. Explain why the height-time graph must pass through the origin.

When $t = 0, h = 0$

$$(t, h) = (0,0) = Origin$$

Example 1.89

A football is thrown directly upwards from the ground with initial velocity u. Determine the maximum height that the football will reach in terms of u and g.

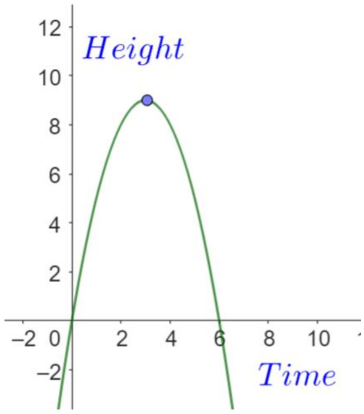
$$v^2 = u^2 + 2as$$

In the case where we have acceleration due to gravity, and displacement is height:

$$v^2 = u^2 - 2gh$$

At the maximum height, the object will have zero velocity, so the final velocity must be zero.

$$0 = u^2 - 2gh$$
$$h = \frac{u^2}{2g}$$



1.90: Maximum height

For an object thrown upwards with initial velocity u, the maximum height is:

$$h = \frac{u^2}{2g}$$

$$v^2 = u^2 + 2as$$
$$0 = u^2 - 2gh$$
$$h = \frac{u^2}{2g}$$

Example 1.91

A football is thrown directly upwards from the ground with initial velocity u. Determine the time taken by the football to reach its maximum height in terms of u and g.

$$v = u + at$$
$$v = u - gt$$

When $v = 0$

$$0 = u - gt$$
$$t = \frac{u}{g}$$

Example 1.92

A. A ball is thrown vertically upward. It has a speed of $10\frac{m}{s}$ when it has reached one half of its maximum height. How high does the ball rise? (Take $g = \frac{10m}{s^2}$) **(NEET 2005)**

B. A particle is thrown vertically upward. Its velocity at half of the height is $10\ \frac{m}{s}$, then the maximum height attained by it (Take $g = \frac{10m}{s^2}$) **(NEET 2001)**

Method I: Short

Rather than considering the entire movement of the ball, focus on the given point in time where the ball has:

$$u = 10\frac{m}{s}$$

From this point, the ball will reach a height of:

$$h = \frac{u^2}{2g} = \frac{10^2}{2(10)} = 5$$

The total height is double of the height from where we considered the initial position, which is:

$$= 2(5) = 10$$

Method II: Long

$$v = 10\frac{m}{s}, s = \frac{h}{2}$$

$$v^2 = u^2 - 2gh$$
$$100 = u^2 - 2g\frac{h}{2}$$
$$u^2 = gh + 100 = 10h + 100$$

$$h = \frac{u^2}{2g} = \frac{10h + 100}{20}$$
$$20h = 10h + 100$$
$$h = 10$$

1.93: Time to Reach maximum height

An object thrown upwards with initial velocity u, reaches its maximum height at

$$t = \frac{u}{g}$$

$$v = u + at$$
$$0 = u - gt$$
$$t = \frac{u}{g}$$

Example 1.94

A ball is thrown vertically with a certain velocity so that it reaches a maximum height h. The two times at which it is at height $\frac{h}{3}$ are t_2 and t_1. If $\frac{t_2}{t_1} > 1$, find $\frac{t_2}{t_1}$ as a number. **(JEE Main July 29 2022, Shift-I, Adapted)**

$$s = ut + \frac{1}{2}at^2$$

Since we are working with gravity, substitute $Acceleration = a = -g, Displacement = s = \frac{h}{3}$:

$$\frac{h}{3} = ut - \frac{1}{2}gt^2$$

Note that this equation is a quadratic in terms of t. Collate all terms on one side:

$$\frac{1}{2}gt^2 - ut + \frac{h}{3} = 0$$

Eliminate fractions by multiplying both sides by 6:

$$\underbrace{3gt^2 - 6ut + 2h = 0}_{Equation\ I}$$

The values of t that are the solutions to the quadratic will be t_2 and t_1 with $t_2 > t_1$:

$$\frac{t_2}{t_1} = \frac{\frac{-b + \sqrt{b^2 - 4ac}}{2a}}{\frac{-b - \sqrt{b^2 - 4ac}}{2a}} = \frac{-b + \sqrt{b^2 - 4ac}}{-b - \sqrt{b^2 - 4ac}}$$

Substitute $a = 3g, b = -6u, c = 2h$ from Equation I into the expression above:

$$\frac{6u + \sqrt{(6u)^2 - (4)(3g)(2h)}}{6u - \sqrt{(6u)^2 - (4)(3g)(2h)}}$$

Simplify:

$$\frac{6u + \sqrt{36u^2 - 24gh}}{6u - \sqrt{36u^2 - 24gh}}$$

Since we want to find a number, we need to reduce the variables.

$$v^2 = u^2 - 2gh$$

When the maximum height is achieved, $v = 0$

$$0 = u^2 - 2gh \Rightarrow 0 = 12u^2 - 24gh$$

$$\frac{6u + \sqrt{24u^2 + 12u^2 - 24gh}}{6u - \sqrt{24u^2 + 12u^2 - 24gh}}$$

$$\frac{6u + \sqrt{24u^2}}{6u - \sqrt{24u^2}}$$

Divide numerator and denominator by $2u$:

$$\frac{3 + \sqrt{6}}{3 - \sqrt{6}}$$

Rationalize:

$$= \frac{3 + \sqrt{6}}{3 - \sqrt{6}} \cdot \frac{3 + \sqrt{6}}{3 + \sqrt{6}} = \frac{9 + 6\sqrt{6} + 6}{9 - 6} \frac{15 + 6\sqrt{6}}{3} = 5 + 2\sqrt{6}$$

1.95: Time to Return

If an object is thrown up with velocity u, the time to return to its original position is:

$$\frac{2u}{g}$$

Take the upward direction to be positive.

$$s = ut + \frac{1}{2}at^2$$

Since the object returns to its original position, the displacement is zero

$$s = 0$$

Substitute $a = -g$:

$$0 = ut - \frac{1}{2}gt^2 \Rightarrow t\left(u - \frac{1}{2}gt\right) = 0 \Rightarrow t \in \left\{0, \frac{2u}{g}\right\}$$

Since the object was thrown up at $t = 0$:

$$t = \frac{2u}{g}$$

Example 1.96

A man throws balls with the same speed vertically upwards one after the other at an interval of 2 seconds. What should be the minimum speed of the throw so that more than two balls are in the sky at any time? $\left(Given\ g = 9.8\frac{m}{s^2}\right)$ **(NEET 2003)**

More than two balls means three balls. Consider the balls being thrown one at a time, beginning at $t = 0$:

$$1st\ ball: t_1 = 0$$
$$2nd\ ball: t_2 = 2$$
$$3rd\ ball: t_3 = 4$$

When the third ball is thrown, we want the 1st and 2nd balls to be in flight. Therefore, the flight time t must Satisfy:

$$t > 4 \Rightarrow \frac{2u}{g} > 4 \Rightarrow u > 2g = 19.6\frac{m}{s}$$

1.97: Time going up

If an object is thrown up with velocity u, the time to reach maximum height is:

$$t = \frac{u}{g}$$

Where

$$t = Time\ to\ reach\ maximum\ height$$
$$t = Time\ taken\ going\ up$$
$$t = Time\ taken\ going\ down$$

Take the upward direction to be positive.
At maximum height, $velocity = 0$:

$$v = u + at$$
$$0 = u - gt$$
$$t = \frac{u}{g}$$

1.98: Time going down

If an object is thrown up with velocity u, the time to reach maximum height is:

$$t = \frac{u}{g}$$

Where

$$t = Time\ taken\ going\ down$$

$$T_{Total} - T_{Up} = \frac{2u}{g} - \frac{u}{g} = \frac{u}{g}$$

1.99: Maximum Height proportional to t^2

For an object which is thrown vertically up, the maximum height is proportional to the square of the time taken.

$$h \propto t^2$$

$$h = vt - \frac{1}{2}gt^2$$

At maximum height $final\ velocity = v = 0$

$$h = -\frac{1}{2}gt^2$$
$$h \propto t^2$$

Example 1.100

An object of mass m reaches a maximum height $10m$ when thrown upwards with velocity. A second object of mass $2m$ (also thrown upwards) takes double the time to return to the ground as the first object. Find the maximum height of the second object.

$$h \propto t^2$$

As time is doubled,

$$H = h(2^2) = 4h = 4(10) = 40m$$

1.101: Time proportional to u

$$t \propto u$$

Time to go up and time to come down are both

$$t = \frac{u}{g}$$
$$t \propto u$$

Example 1.102

An object of mass m is thrown upwards with velocity $5\frac{m}{s}$. A second object of mass $2m$ (also thrown upwards) takes double the time to return to the ground as the first object. Find the velocity of the second object (in terms of u).

$$t \propto u$$

Time is doubles, so is velocity

$$2u$$

1.103: Velocity on Return

If an object is thrown up with a velocity u, it returns to its original position with a velocity $-u$.

Take the upward direction to be positive

$$v = u + at = u - gt = u - g\left(\frac{2u}{g}\right) = u - 2u = -u$$

Example 1.104

I throw a ball vertically up in the air at 30 m/s. When the ball returns to its starting position, what is the velocity? (Take upward direction to be positive).

B. Finding g

1.105: Objects in Free Fall on places other than the Earth

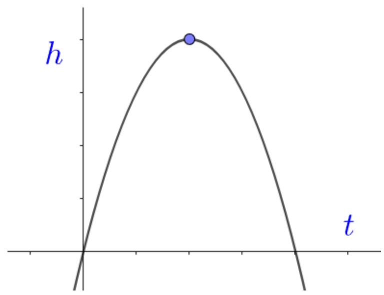
Pending Kinematics

Example 1.106

A ball is dropped from the top of a 100m high tower on a planet. In the last $\frac{1}{2}$ sec before hitting the ground, it covers a distance of $19m$. Acceleration due to gravity in $\frac{m}{s^2}$ near the surface of that planet is: **(JEE Main, 8 Jan 2020, Shift-II)**

C. Distance versus Displacement

1.107: Displacement



Consider an object thrown vertically upward from the horizontal ground with initial velocity u at time $t = 0$.
We want to determine displacement for the object, which we can do from the equation of motion:

$$s = ut - \frac{1}{2}gt^2$$

We might feel this is the final answer, but we need to impose a constraint[1] on t to keep the value of s meaningful.

- If $t < 0$ it does not make sense in this context.
- If $t > \frac{2u}{g}$, the object has returned to its original position and hence $s = 0$

The final equation that we want is:

$$s = ut - \frac{1}{2}gt^2, \qquad 0 \le t \le \frac{2u}{g}$$

1.108: Distance versus displacement

If an object is moving in a single direction, then distance is the absolute value of displacement

$$d = |s|$$

[1] In terms of math, we are saying $s(t)$ has domain $t \in \left[0, \frac{2u}{g}\right]$

Example 1.109

Mark all correct options

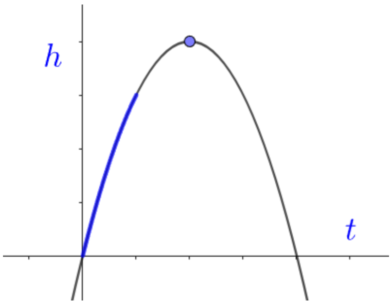
A. Distance is the absolute value of displacement.
B. Distance is the same as displacement.
C. Distance is the same as displacement when movement is in positive direction.
D. Distance is the absolute value of displacement when displacement is in a single direction.

$$C, D$$

Example 1.110

An object is thrown vertically upward from the horizontal ground with initial velocity u at time $t = 0$. Find the distance travelled in t seconds if $0 \le t \le \frac{u}{g}$.

$$s = ut - \frac{1}{2}gt^2, \qquad 0 \le t \le \frac{2u}{g}$$



The total time taken by the object is

$$= \frac{2u}{g}$$

The object reaches its maximum height at half the time

$$= \frac{u}{g}$$

For this case

$$Distance = d = s = ut - \frac{1}{2}gt^2$$

1.111: Calculating Distance with change of direction

If you want to calculate distance for a time interval when there is a change of direction, divide the time interval such that movement happens only in a single direction and then use:

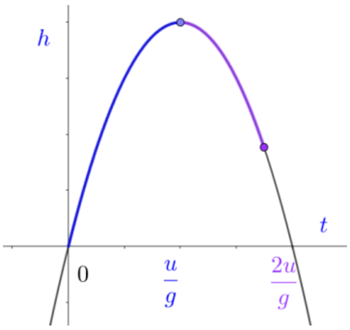
$$d = |s|$$

Example 1.112

A ball is thrown vertically upwards with an initial velocity of $40\frac{m}{s}$. Determine the distance travelled by ball in 6 seconds.

Distance when going up

$$= \frac{u^2}{2g} = \frac{40^2}{2(10)} = \frac{1600}{20} = 80\ m$$



Time to taken to go up

$$= \frac{u}{g} = \frac{40}{10} = 4\ seconds$$

Time taken to go down

$$T = t - \frac{u}{g} = 6 - 4 = 2\ seconds$$

And, distance travelled when going down

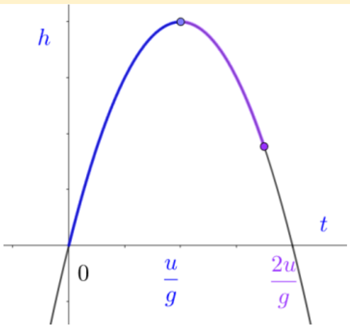
$$= \frac{1}{2}gT^2 = \frac{1}{2}(10)(2^2) = 20\ m$$

Total distance

$$= 80m + 20m = 100m$$

Example 1.113

An object is thrown vertically upward from the horizontal ground with initial velocity u at time $t = 0$. Find the distance travelled in t seconds if $0 \le t \le \frac{u}{g}$.



We need to split this into parts based on when there is a change in direction. If the direction does not change then:

$$Distance = |Displacement| \Rightarrow d = |s|$$

The distance travelled when the object is going up:

$$d = |s| = \left|\frac{u^2}{2g}\right| = \frac{u^2}{2g}$$

The distance travelled when the object is going down:

$$Displacement = s = ut + \frac{1}{2}at^2 = (0)\left(t - \frac{u}{g}\right) - \frac{1}{2}g\left(t - \frac{u}{g}\right)^2 = -\frac{1}{2}g\left(t - \frac{u}{g}\right)^2$$

$$d = |s| = \left|-\frac{1}{2}g\left(t - \frac{u}{g}\right)^2\right| = \frac{1}{2}g\left(t - \frac{u}{g}\right)^2$$

Total distance

$$= d = \frac{u^2}{2g} + \frac{1}{2}g\left(t - \frac{u}{g}\right)^2$$

1.6 Differentiation-Based Problems

A. Velocity as a Derivative

1.114: Instantaneous Velocity

Instantaneous velocity is the rate of change of displacement with respect to time at a point

$$v = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} = \frac{ds}{dt}$$

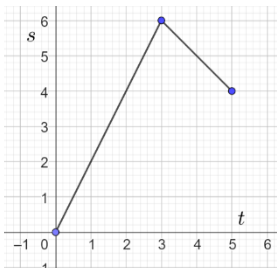
Where

$$s = displacement, \qquad v = velocity, t = time$$

Example 1.115

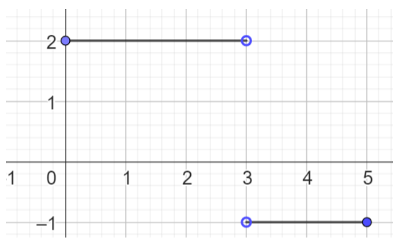
The adjacent graph shows displacement in meters on the y-axis and time in seconds on the x-axis.

A. Draw a velocity graph.
B. Why is the velocity not defined at $t = 3$



Part A

Note that velocity is different over different intervals. But since velocity is constant over a particular interval, we can calculate it using the slope of the lines:



$$t \in (0,3) \Rightarrow Velocity = \frac{Displacement}{Time} = \frac{6}{3} = 2\frac{m}{s}$$

$$t \in (3,5) \Rightarrow Velocity = \frac{Displacement}{Time} = -\frac{2}{2} = -1\frac{m}{s}$$

Part B

$t = 3$ has a sharp turn. Mathematically speaking, velocity is the derivative of displacement. At $t = 3$, the left-hand derivative does not equal the right-hand derivative, and hence the overall derivative does not exist.

In terms of physics, the change in velocity is infinite over an instant. It is not possible for a physical object, but the graph is meant to illustrate a concept.

Example 1.116

At time $t = 0$, a particle is at the origin. Its displacement on the x axis after t seconds is given by te^{-t}. Considering $t > 0$, classify the particle based on when it

A. remains stationary
B. moves right
C. moves left

The velocity is the derivative of the displacement function:

$$v = \frac{dx}{dt} = \frac{d}{dt}(te^{-t}) = e^{-t} - te^{-t} = e^{-t}(1-t)$$

The particle will

- be stationary when the velocity is zero
- move to the right when the velocity is positive
- move to the left when the velocity is negative

$$e^{-t}(1-t) > 0$$

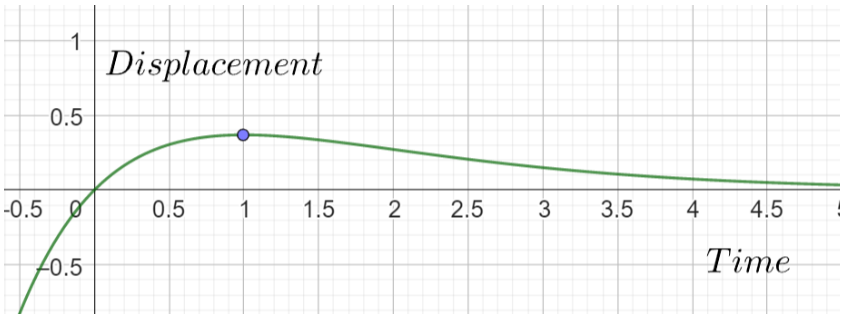
Divide both sides by e^{-t} since $e^{-t} > 0$:

$$1 - t > 0$$
$$t < 1$$

Note that

$$t = 1 \Rightarrow v = 0 \Rightarrow Particle\ is\ stationary$$

$$\underbrace{(0,1)}_{Moving\ Right}, \underbrace{\{1\}}_{Stationary}, \underbrace{(1,\infty)}_{Moving\ Left}$$



Example 1.117

If $t = \sqrt{x} + 4$, then $\left(\frac{dx}{dt}\right)_{t=4}$ is: (JEE Main July 2022, Shift-I)

Method I

$$t-4=\sqrt{x}$$
$$x=(t-4)^2=t^2-8t+16$$
$$\frac{dx}{dt}=2t-8$$
$$\left(\frac{dx}{dt}\right)_{t=4}=2(4)-8=0$$

Method II: Implicit Differentiation

Differentiate both sides of $t=\sqrt{x}+4$ with respect to t:

$$1=\frac{1}{2\sqrt{x}}\cdot\frac{dx}{dt}\Rightarrow\frac{dx}{dt}=2\sqrt{x}$$

When $t=4\Rightarrow 4=\sqrt{x}+4\Rightarrow x=0$

$$\frac{dx}{dt}=0$$

Example 1.118

Two buses P and Q start from a point at the same time and move in a straight line and their positions are represented by $X_p(t)=\alpha t+\beta t^2$ and $X_Q(t)=ft-t^2$. At what time will both the buses have the same velocity. **(JEE Main June 25, 2022, Shift-II)**

Velocity for bus P:

$$\frac{d}{dt}X_p(t)=\frac{d}{dt}(\alpha t+\beta t^2)=\alpha+2\beta t$$

Velocity for bus Q:

$$\frac{d}{dt}X_Q(t)=\frac{d}{dt}(ft-t^2)=f-2t$$

When velocities are same, we must have:

$$\alpha+2\beta t=f-2t$$
$$t(2\beta+2)=f-\alpha$$
$$t=\frac{f-\alpha}{2\beta+2}$$

B. Acceleration

Recall that acceleration is the rate of change of velocity with respect to time.

1.119: Instantaneous Acceleration

$$a=\lim_{h\to 0}\frac{v(t+h)-v(t)}{h}=\frac{dv}{dt}=\frac{d^2s}{dt^2}$$

Example 1.120

The x and y coordinates of the particle at any time are $x=5t–2t^2$ and $y=10t$ respectively, where x and y are in meters and t in seconds. The acceleration of the particle at $t=2s$ is **(NEET 2017)**

$$\vec{\boldsymbol{s}}(t)=(5t-2t^2,10t)$$
$$\vec{\boldsymbol{v}}(t)=(5-4t,10)$$
$$\vec{\boldsymbol{a}}(t)=(-4,0)$$
$$\vec{\boldsymbol{a}}(2)=-4\hat{\boldsymbol{\imath}}$$

Example 1.121

The position of a particle as a function of time t, is given by $x(t) = at + bt^2 - ct^3$ where a, b and c are constants. When the particle attains zero acceleration, then its velocity will be: **(JEE Main, 9 April 2019, Shift-II)**

Differentiate the given relation to get velocity:

$$\dot{x} = a + 2bt - 3ct^2$$

Differentiate a second time to get acceleration and equate it to zero to find the time:

$$\ddot{x} = 2bt - 3ct = 0 \Rightarrow t = \frac{b}{3c}$$

Substitute $t = \frac{b}{3c}$ in the expression for velocity:

$$\dot{x}\Big|_{t=\frac{b}{3c}} = a + 2b\left(\frac{b}{3c}\right) - 3c\left(\frac{b}{3c}\right)^2 = a + \frac{2b^2}{3c} - \frac{b^2}{3c} = a + \frac{b^2}{3c}$$

1.122: Interpretation of Acceleration

If acceleration is positive for an object, its velocity is increasing.
If acceleration is negative for an object, its velocity is decreasing.

Example 1.123

At time $t = 0$, a particle is at the origin. Its displacement on the x axis after t seconds is given by te^{-t}. Considering $t > 0$, classify the particle based on when it:

A. has constant velocity, increasing velocity, or decreasing velocity
B. has constant speed, increasing speed, or decreasing speed

Part A

Acceleration is the derivative of the velocity function.

$$a = \frac{dv}{dt} = \frac{d}{dt}e^{-t}(1-t) = e^{-t}(-1) - e^{-t}(1-t) = -e^{-t} - e^{-t} + te^{-t} = te^{-t} - 2e^{-t} = e^{-t}(t-2)$$

Check where the acceleration is positive and zero:

$$e^{-t}(t-2) > 0 \Rightarrow t - 2 > 0 \Rightarrow t > 2$$
$$e^{-t}(t-2) = 0 \Rightarrow t = 2$$

Write the intervals to get:

$$\underbrace{(0,2)}_{\substack{\textit{Decreasing}\\ \textit{Velocity}}}, \quad \underbrace{\{2\}}_{\substack{\textit{Constant}\\ \textit{Velocity}}}, \quad \underbrace{(2,\infty)}_{\substack{\textit{Increasing}\\ \textit{velocity}}}$$

Part B

	$(0,1)$	$\{1\}$	$(1,2)$	$\{2\}$	$(2,\infty)$
Displacement	$+ve$	$+ve$	$+ve$	$+ve$	$+ve$
Velocity	$+ve$	0	$-ve$	$-ve$	$-ve$
Acceleration	$-ve$	$-ve$	$-ve$	0	$+ve$
Signs of v and a	Opp	Type equation here.	$Same$		Opp
Speed	$Decreasing$	$Decreasing$	$Increasing$	$Constant$	$Decreasing$

Example 1.124

The distance x covered by a particle in one dimensional motion varies with time t as $x^2 = at^2 + 2bt + c$. If the acceleration of the particle depends on x as x^n, where n is an integer, the value of n is (JEE Main 9 Jan 2020, Shift-I)

Differentiate both side of the above with respect to time:

$$2x\left(\frac{dx}{dt}\right) = 2at + 2b$$

Divide both sides by 2:

$$\underbrace{x(\dot{x}) = at + b}_{\textit{Equation I}}$$

Solve for $\dot{x}$:

$$\underbrace{\dot{x} = \frac{at+b}{x}}_{\textit{Equation II}}$$

Differentiate Equation I with respect to t:

$$(\dot{x})^2 + x\ddot{x} = a$$

$$\ddot{x} = \frac{a - (\dot{x})^2}{x} = \frac{a - \left(\frac{at+b}{x}\right)^2}{x} = \frac{ax^2 - (at+b)^2}{x^3} = \frac{a^2t^2 + 2abt + ac - (a^2t^2 + 2abt + b^2)}{x^3}$$

$$Acceleration = \ddot{x} = \frac{ac - b^2}{x^3}$$
$$Acceleration \propto \frac{1}{x^3} = x^{-3} \Rightarrow n = -3$$

Example 1.125

An object starts at $x = 0$ on the x axis at time $t = 0$, and has position function given by $s = \sin t$. Determine the:

A. Velocity Function and Acceleration Function
B. Maximum Displacement, and when it happens
C. Maximum Velocity
D. Maximum Acceleration

Part A

$$v = \frac{ds}{dt} = \cos t$$
$$a = \frac{dv}{dt} = -\sin t$$

Part B

$$Maximum\ Displacement = 1\ at\ t = \left\{\frac{\pi}{2} + 2n\pi, n \in \mathbb{W}\right\}$$
$$Maximum\ Velocity = 1\ at\ t = \{0 + 2n\pi, n \in \mathbb{W}\}$$
$$Maximum\ Acceleration = 1\ at\ t = \left\{\frac{3\pi}{2} + 2n\pi, n \in \mathbb{W}\right\}$$

Challenge 1.126

A particle moves along a straight line. Its displacement S varies with time t according to the law $S^2 = at^2 + 2t + c$ (a, b, and c are constants.). The acceleration of this particle varies as:

A. S^0
B. S^{-1}
C. S^{-2}
D. S^{-3} (NSEP 2022)

Since we have S^2, differentiating directly will require implicit differentiation. To avoid this, take the square root both sides:

$$S = \pm\sqrt{at^2 + 2bt + c}$$

Differentiate both sides to get velocity:

$$v = \frac{dS}{dt} = \frac{1}{2S}(2at + 2b) = \frac{at + b}{S}$$

Differentiate one more time using the quotient rule to get acceleration and simplify:

$$\frac{dv}{dt} = \frac{S(at+b)' - (at+b)(S')}{S^2} = \frac{aS - (at+b)\frac{at+b}{S}}{S^2} = \frac{aS^2 - (at+b)^2}{S^3}$$

Substitute $S^2 = at^2 + 2t + c$ and simplify:

$$\frac{dv}{dt} = \frac{a^2t^2 + 2abt + ac - a^2t^2 - 2atb - b^2}{S^3} = \frac{ac - b^2}{S^3} = (ac - b^2)S^{-3}$$

Since a, c and b are given to be constants:

$$\frac{dv}{dt} \propto S^{-3}$$

Option D

C. Acceleration: Alternate Formula

1.127: Acceleration

$$a = v \cdot \frac{dv}{ds}$$

$$a = \frac{dv}{dt}$$

Using chain rule:

$$a = \frac{dv}{ds} \cdot \frac{ds}{dt}$$

Substitute $\frac{ds}{dt} = v$

$$a = \frac{dv}{ds} \cdot v$$

Rearrange:

$$= v \cdot \frac{dv}{ds}$$

Example 1.128

Show that $a = v \cdot \frac{dv}{ds}$ follows the rules of dimensional analysis.

LHS

$$LHS = a\ has\ units\ \frac{m^2}{s}$$

RHS

Unit of

$$v \cdot \frac{dv}{ds} = \frac{m}{s} \cdot \frac{1}{s} = \frac{m^2}{s} = unit\ of\ acceleration$$

Example 1.129

A particle is moving in a straight line such that its velocity is increasing at $5\frac{m}{s}$ per meter. The acceleration of the particle is _____ $\frac{m}{s^2}$ at a point where the velocity is $20\frac{m}{s}$. (JEE Main July 25, 2022, Shift-II)

$$a = v \cdot \frac{dv}{ds} = 20 \times 5 = 100$$

Example 1.130

If the velocity of a body related to displacement is given by $v = \sqrt{5000 + 24x}\frac{m}{s}$, then the acceleration of the body is _______ $\frac{m}{s^2}$. (JEE Main 2021, Aug 27, Shift-I)

$$a = v \cdot \frac{dv}{ds} = \sqrt{5000 + 24x} \times \frac{12}{\sqrt{5000 + 24x}} = 12\frac{m}{s}$$

D. Optimization

Example 1.131

A man in a car at location Q on a straight highway is moving with speed v. He decides to reach a point P in a field at a distance d from highway (point M) as shown in the figure. Speed of the car in the field is half that on the highway. What should be the distance RM so that the time taken to reach P is minimum? (JEE Main April 15, 2018)

Let

$$v = 2V$$

Total time

$$t = Time_{QR} + Time_{RP} = \frac{QM - x}{2V} + \frac{\sqrt{d^2 + x^2}}{V}$$

To minimize, find and equate the first derivative to zero:

$$\frac{dt}{dx} = \frac{-1}{2V} + \frac{2x}{2\sqrt{d^2 + x^2}V} = 0$$

$$\frac{1}{V}\left[-\frac{1}{2} + \frac{x}{\sqrt{d^2 + x^2}}\right] = 0$$

$$\frac{x}{\sqrt{d^2 + x^2}} = \frac{1}{2}$$

$$2x = \sqrt{d^2 + x^2}$$

$$4x^2 = d^2 + x^2$$

$$3x^2 = d^2$$

$$x = \frac{d}{\sqrt{3}}$$

And this is a minimum.

1.7 Integration-Based Problems

A. Displacement as an Integral

1.132: Displacement as Integral of Velocity

Displacement is the integral of velocity.

$$\int v\,dt = s$$

Integrate both sides of $v = \frac{ds}{dt}$ with respect to time:

$$\int v\,dt = \int \frac{ds}{dt}\,dt$$

Substitute $\frac{dt}{dt} = 1$:

$$\int v\,dt = \int ds$$

$$\int v\,dt = \int 1 \cdot ds$$

$$\int v\,dt = s + C$$

If you take the initial displacement as zero, this simplifies to:

$$\int v\,dt = s$$

Example 1.133

The velocity of a particle is $v_0 + gt + ft^2$. If its position is $x = 0$ at $t = 0$, then its displacement after unit time

$(t = 1)$ is: (JEE Main 2007)

$$\int_0^1 (v_0 + gt + ft^2)\,dt = \left[v_0 t + gt^2 + \frac{ft^3}{3}\right]_{t=0}^{t=1} = v_0 + \frac{g}{2} + \frac{f}{3}$$

Example 1.134

The instantaneous velocity of a particle moving in a straight line is given as $v = \alpha t + \beta t^2$, where α and β are constants. The distance travelled by the particle between 1s and 2s is: (JEE Main July 25, 2021, Shift-II)

$$s = \int_1^2 v\,dt = \int_1^2 (\alpha t + \beta t^2)\,dt$$
$$= \left[\frac{\alpha t^2}{2} + \frac{\beta t^3}{3}\right]_1^2$$
$$= \left[\frac{\alpha(4-1)}{2}\right]_1^2 + \left[\frac{\beta(8-1)}{3}\right]_1^2$$
$$= \frac{3\alpha}{2} + \frac{7\beta}{3}$$

Example 1.135

An object initially at $x = 1$ moves along the x axis with velocity $1 - \sin(2t)$, where t is time.

A. What is the object's position when $t = \frac{\pi}{2}$.
B. Is the object moving to the right or the left when $t = \pi$.

Part A

$$\frac{dx}{dt} = 1 - \sin(2t)$$
$$x = \int [1 - \sin(2t)]\,dt = t + \frac{1}{2}\cdot\cos(2t) + C$$

When $t = 0, x = 1$

$$1 = 0 + \frac{1}{2}\cdot\cos(2\cdot 0) + C$$
$$1 = \frac{1}{2} + C$$
$$\frac{1}{2} = C$$

$$x = t + \frac{1}{2}\cdot\cos(2t) + \frac{1}{2} = \frac{\pi}{2} + \frac{1}{2}\cdot\cos(\pi) + \frac{1}{2} = \frac{\pi}{2} + \frac{1}{2}(-1) + \frac{1}{2} = \frac{\pi}{2}$$

Part B

$$\left.\frac{dx}{dt}\right|_{t=\pi} = 1 - \sin(2\pi) = 1 > 0 \Rightarrow \textit{Moving to the right}$$

Example 1.136

A particle located at $x = 0$ at time $t = 0$ starts moving along the positive x-direction with a velocity v that varies as $v = \alpha\sqrt{x}$. The displacement of the particle varies with time as: (JEE Main 2006)

Substitute $s = x \Rightarrow v = \frac{ds}{dt} = \frac{dx}{dt}$:

$$\frac{dx}{dt} = \alpha\sqrt{x}$$

Separate the variables:

$$\frac{dx}{\sqrt{x}} = \alpha\, dt$$

Integrate:

$$\int \frac{dx}{\sqrt{x}} = \int \alpha\, dt \Rightarrow 2\sqrt{x} = \alpha t + x_0$$

Substitute $x_0 = 0$

$$2\sqrt{x} = \alpha t \Rightarrow x = \frac{\alpha^2}{4}t^2$$

Remove the constant and replace the equality sign with a proportionality sign:

$$s = x \propto t^2$$

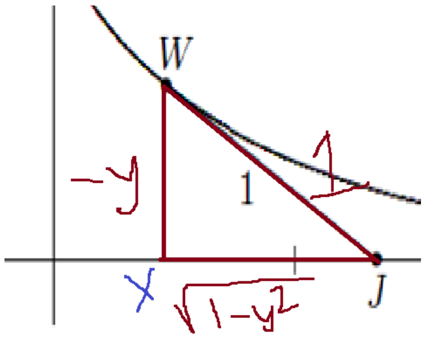
Example 1.137

Walking along the x axis, Jamie uses a rope of unit length to draw a wagon W that is initially at (0,1). Thus $W = (x, y)$ rolls along a tractrix. Suppose that Jamie walks in such a way that the speed of W is 1 unit per second.

A. Explain why $\frac{dy}{dt} = -y$
B. Show that $y = e^{-t}$ **(Phillips Exeter, Math 4)**

Consider the velocity vector $\overrightarrow{\boldsymbol{JW}}$

$$Speed = |Velocity| = \left|\overrightarrow{\boldsymbol{JW}}\right| = 1$$



As can be seen from the diagram:

$$y\ component\ of\ \overrightarrow{\boldsymbol{JW}} = \underbrace{\frac{dy}{dt} = -y}_{Equation\ I}$$

- In other words, the rate at which the wagon approaches the x axis is proportional to the distance between the x axis and the wagon. This is precisely the negative of the y coordinate of the wagon.
- Since the derivative of the function is proportional to the function itself, it is similar to the form where the derivative of a function is equal to the function itself. This happens when the function is exponential.

In right ΔWXJ, by Pythagorean Theorem

$$\frac{dx}{dt} = \sqrt{WX^2 - XJ^2} = \sqrt{1 - y^2}$$

Separate the variables in Equation I and

$$\frac{dy}{-y} = -dt$$

Integrate both sides:

$$\int \frac{dy}{y} = \int -dt \Rightarrow \ln|y| = -t$$

Exponentiate both sides:

$$y = e^{-t} + C$$

B. Velocity as an Integral

1.138: Velocity

Velocity is the integral of acceleration with respect to time

$$v = \int a\, dt$$

Example 1.139

Kelly completed a 250-mile drive in exactly 5 hours – an average speed of 50 mph. The trip was not actually made at a constant speed of 50 mph, of course, for there were traffic lights, slow moving trucks in the way, etc. Nevertheless there must have been at least one instant during the trip when Kelly's speedometer showed exactly 50 mph. Give two explanations – one using a distance-versus-time graph, and the other using a speed-versus-time graph. Make your graphs consistent with each other. **(Phillips Exeter, Math 4)**

Assume, for simplicity, that the trip was made in one direction. In other words:

$$Distance = Displacement$$

$$Velocity = Speed$$

The trip was made at an average speed of 50 mph, while not being a constant speed of 50 mph.

The trip must have started at a speed increasing from zero. And ending at zero.

Consider contrary to fact that, the speed was never above 50 mph. In this case, the average speed cannot be 50 mp.

Hence, we know that:

- The starting speed was zero
- At some point the speed was above 50.

$$Speed = f(t), Domain = [0,5]$$

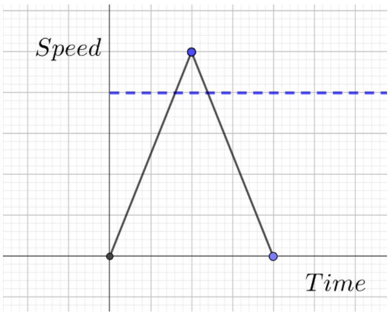
$$f(0) = 0$$

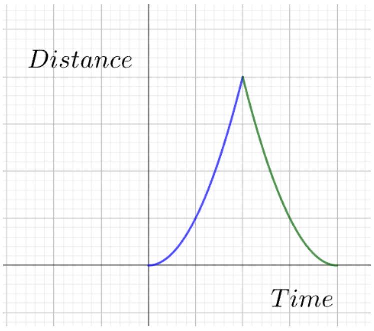
$$f(t) > 50, for\ some\ t$$

Since speed is a continuous function with respect to time, by the Intermediate Value Theorem, for some in [0,5], we must have

$$f(t) = 50$$

If $s = ax + b$, then displacement is a linear function of time, and its integral (velocity) will be a quadratic function.





(Diagram not drawn to scale)

1.8 2D/3D Motion

A. 2D and 3D Motion

Multi-dimensional motion is best handled with *vectors*.

Example 1.140

Classify into one-dimensional and two-dimensional motion:

A. Stone dropped from a tower
B. Stone thrown in the horizontal direction from a tower
C. Stone thrown in the vertical direction from a tower
D. Object released from a stationary balloon
E. Object released from a moving plane

$$1D: A, C, D$$
$$2D: B, E$$

1.141: Vectors in Component Form

A vector is written in component form as

$$\vec{\boldsymbol{r}} = (r_x, r_y, r_z)$$

Where

$$r_x = x \text{ component of } \vec{\boldsymbol{r}}$$
$$r_y = y \text{ component of } \vec{\boldsymbol{r}}$$
$$r_z = z \text{ component of } \vec{\boldsymbol{r}}$$

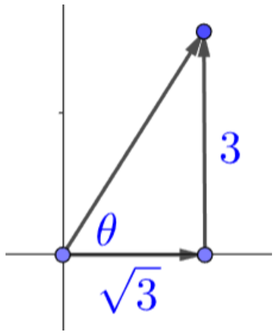
The advantage of vectors in component form is that forces in one component have $no\ effect$ on forces in another component. Hence, we write a vector in component and work with each component individually.

Example 1.142

A particle starting from the origin $(0, 0)$ moves in a straight line in the (x, y) plane. Its coordinates at a later time are $(\sqrt{3}, 3)$. The path of the particle makes with the x-axis an angle of _______ in radians. **(NEET 2007)**

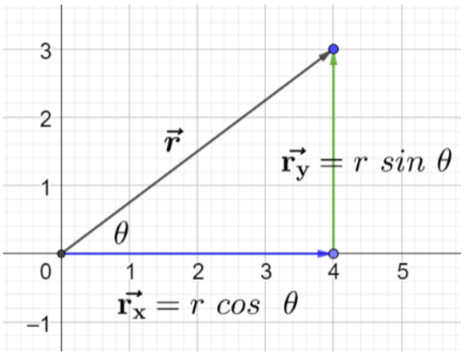
Since

$$\tan\theta = \frac{3}{\sqrt{3}} = \sqrt{3} \Rightarrow \theta = 60° = \frac{\pi}{3}$$



1.143: Converting into component form

$$\vec{r} = \underbrace{(r,\theta)}_{Polar\ Form} = \underbrace{(r\cos\theta, r\sin\theta)}_{Component\ Form}$$



We can determine the magnitude of $\vec{r_y}$:

$$\sin\theta = \frac{r_y}{r} \Rightarrow r_y = r\sin\theta$$

$$\cos\theta = \frac{r_x}{r} \Rightarrow r_x = r\cos\theta$$

Example 1.144

A man is slipping on a frictionless inclined plane and a bag falls down from the same height. Ignoring air resistance, the speed of both is related as

A. Speed of bag is greater
B. Speed of man is greater
C. Speeds are equal
D. Cannot be determined **(NEET 2000, Adapted)**

Option C

1.145: Position Vector

The position of an object is given by a position vector:

$$\vec{r} = (r_x, r_y) = Position\ Vector$$

1.146: Displacement Vector

The displacement of an object is given by change in position:

$$\Delta\vec{r} = \vec{r_2} - \vec{r_1}$$

1.147: Calculating Displacement

If $A = (x_A, y_A)$, $B = (x_B, y_B)$

Displacement from A to B is:

$$\overrightarrow{AB} = (x_B - x_A, y_B - y_A)$$

Displacement from B to A is:

$$-\overrightarrow{AB} = \overrightarrow{BA} = (x_A - x_B, y_A - y_B)$$

1.148: Velocity Vector

Velocity is rate of change of position.

1.149: Change in Velocity

The change in velocity

$$\Delta\vec{v} = Old\ Velocity - New\ Velocity = \vec{v_2} - \vec{v_1}$$

Example 1.150

A bus is moving on a straight road towards north with a uniform speed of $50\frac{km}{hour}$ then it turns left through 90°. If the speed remains unchanged after turning, the increase in the velocity of bus in the turning process is

A. $70.7\frac{km}{hr}$ along south-west direction
B. zero
C. $50\frac{km}{hr}$ along west
D. $70.7\frac{km}{hr}$ along north-west direction **(NEET 1989)**

$$\Delta\vec{v} = \vec{v_2} - \vec{v_1} = (-50,0) - (0,50) = (-50,-50)$$

$$|\vec{v}| = \sqrt{50^2 + 50^2} = 50\sqrt{2} \approx 70\frac{km}{hr}$$

$\Delta\vec{v} = (-50,-50)$ is in quadrant III, making a $45 - 45 - 90$ triangle going in the south-west direction.

$$Option\ A$$

B. Average Velocity and Acceleration

1.151: Average Velocity and Acceleration

$$Average\ Velocity = \vec{v} = (v_x, v_y) = \frac{Displacement}{Time} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r_2} - \vec{r_1}}{\Delta t}$$

Example 1.152

A particle is moving such that its position coordinates (x,y) are $(2m, 3m)$ at time $t = 0$, $(6m, 7m)$ at time $t = 2s$ and $(13m, 14m)$ at time $t = 5s$. Average velocity vector from $t = 0$ to $t = 5s$, written in $x\hat{\imath} + y\hat{\jmath}$ notation is **(NEET 2014)**

$$\vec{v} = \frac{\Delta\vec{r}}{\Delta t} = \frac{(13,14) - (2,3)}{5} = \frac{(11,11)}{5} = \left(\frac{11}{5}, \frac{11}{5}\right) = \frac{11}{5}\hat{\imath} + \frac{11}{5}\hat{\jmath}$$

1.153: Average Acceleration

$$Average\ Acceleration\ \vec{a} = (a_x, a_y) = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v_2} - \vec{v_1}}{\Delta t}$$

Example 1.154

A body is moving with velocity $30\frac{m}{s}$ towards east. After 10 seconds its velocity becomes $40\frac{m}{s}$ towards north. The magnitude of average acceleration of the body is: **(NEET 2011)**

$$\vec{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{(0,40) - (30,0)}{10} = \frac{(-30,40)}{10} = (-3,4)$$
$$|\vec{a}| = \sqrt{(-3)^2 + 4^2} = 5$$

C. Equations of Motion

Recall that equations of motion assume *constant* acceleration. In one dimensional motion, the variables in the equations were scalars. There are equivalent versions of the equations of motion with *vectors* that work in multiple dimensions.

1.155: Final Velocity

$$\vec{v} = \vec{u} + \vec{a}t$$

Where

$$\vec{v} = Final\ Velocity\ vector$$
$$\vec{u} = Initial\ Velocity\ vector$$
$$\vec{a} = acceleration\ vector$$

The scalar version of this equation of motion is:

$$v = u + at$$

Example 1.156

A. A particle has initial velocity $(2\hat{\imath} + 3\hat{\jmath})$ and acceleration $(0.3\hat{\imath} + 0.2\hat{\jmath})$. The magnitude of velocity after 10 seconds will be **(NEET 2012)**
B. A particle has an initial velocity of $3\hat{\imath} + 4\hat{\jmath}$ and an acceleration of $0.4\hat{\imath} + 0.3\hat{\jmath}$. Its speed after $10s$ is: **(JEE Main 2009, NEET 2010)**

Part A

$$\vec{v} = \vec{u} + \vec{a}t = (2,3) + (0.3,0.2)10 = (2,3) + (3,2) = (5,5)$$
$$|\vec{v}| = \sqrt{5^2 + 5^2} = 5\sqrt{2}$$

Part B

$$\vec{v} = \vec{u} + \vec{a}t = (3,4) + (0.4,0.3)10 = (7,7)$$
$$|\vec{v}| = \sqrt{7^2 + 7^2} = 7\sqrt{2}$$

1.157: Equations of Motion

$$\vec{s} = \vec{r} + \vec{u}t + \frac{1}{2}\vec{a}t^2$$

Where

$$\vec{r} = Initial\ position\ vector$$
$$\vec{u} = Initial\ position\ vector$$
$$\vec{a} = Acceleration\ vector$$

Example 1.158

A particle moves from the point $(2.0\hat{\imath} + 4.0\hat{\jmath})m$ at $t = 0$ with an initial velocity of $(5.0\hat{\imath} + 4.0\hat{\jmath})\frac{m}{s}$. It is acted upon by a constant force which produces a constant acceleration $(4.0\hat{\imath} + 4.0\hat{\jmath})\frac{m}{s^2}$. What is the distance of the particle from the origin at time $2s$? **(JEE Main, 11 Jan 2019, Shift-II)**

$$\vec{s} = \vec{r} + \vec{u}t + \frac{1}{2}\vec{a}t^2 = (2,4) + (5,4)2 + \frac{1}{2}(4,4)(2^2) = (2,4) + (10,8) + (8,8) = (20,20)$$

$$|\vec{s}| = \sqrt{20^2 + 20^2} = 20\sqrt{2}$$

Example 1.159

Starting from the origin at time $t = 0$, with initial velocity $5\hat{j}\frac{m}{s}$, a particle moves in the $x - y$ plane with a constant acceleration of $(10\hat{\imath} + 4\hat{\jmath})\frac{m}{s^2}$. At time t, its coordinates are $(20\ m, y_0\ m)$. The values of t and y_0 respectively are: **(JEE Main 9 Jan 2020, Shift-II; JEE Main 4 Sep 2020, Shift-I)**

$$\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$$
$$(s_x, s_y) = (0,5)t + \frac{1}{2}(10,4)t^2$$
$$(s_x, s_y) = (0,5t) + (5t^2, 2t^2)$$
$$(s_x, s_y) = (5t^2, 5t + 2t^2)$$

Two vectors are equal if and only if their components are equal.
Equating components:

$$s_x = 5t^2 = 20 \Rightarrow t^2 = 4 \Rightarrow t = 2$$
$$y_0 = 5t + 2t^2 = 5(2) + 2(2^2) = 10 + 8 = 18$$

D. Instantaneous Velocity and Acceleration

1.160: Instantaneous Velocity

The *velocity vector* is the derivative of the position vector:

$$\vec{v} = (v_x, v_y) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r_x, r_y) = \left(\frac{d}{dt}r_x, \frac{d}{dt}r_y\right)$$

Example 1.161

The coordinates of a moving particle at any time t are given by $x = \alpha t^3$and $y = \beta t^3$. The speed of the particle at time t is given by: **(JEE Main 2003)**

$$\vec{r} = (\alpha t^3, \beta t^3)$$
$$\vec{v} = \frac{d\vec{r}}{dt} = (3\alpha t^2, 3\beta t^2)$$

$$Speed = |\vec{v}| = \sqrt{(3\alpha t^2)^2 + (3\beta t^2)} = \sqrt{(3t^2)^2(\alpha^2 + \beta^2)} = 3t^2\sqrt{\alpha^2 + \beta^2}$$

1.162: Instantaneous Acceleration

The *acceleration vector* is the derivative of the velocity vector:

$$Acceleration\ Vector = \vec{a} = (a_x, a_y) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Example 1.163

At time $t = 0$, a particle starts travelling from a height $7\hat{z}\ cm$ in a plane keeping z coordinate constant. At any instant of time, its position along the $\hat{x}$ and $\hat{y}$ directions is defined as $3t$ and $5t^3$ respectively. At $t = 1s$, acceleration of the particle will be: **(JEE Main, 28 July 2022, Shift-II)**

$$\vec{r} = (3t, 5t^3, 7)$$

$$\vec{\boldsymbol{v}} = \frac{d\vec{\boldsymbol{r}}}{dt} = (3,15t^2,0)$$
$$\vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{v}}}{dt} = (0,30t,0)$$

Example 1.164

The position vector of a particle changes with time according to the relation $\vec{\boldsymbol{r}} = 15t^2\hat{\imath} + (4 - 20t^2)\hat{\jmath}$. What is the magnitude of the acceleration at $t = 1$? **(JEE Main, 9 April 2019, Shift-II)**

$$\vec{\boldsymbol{r}} = (15t^2, 4 - 20t^2)$$
$$\vec{\boldsymbol{v}} = \frac{d\vec{\boldsymbol{r}}}{dt} = (30t, -40t)$$
$$\vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{v}}}{dt} = (30, -40)$$

$$|\vec{\boldsymbol{a}}| = \sqrt{30^2 + (-40)^2} = 50$$

2. Projectile Motion

2.1 Projectiles: Horizontal and Ground

A. Ideal Projectiles

2.1: Projectile Motion
A particle is in projectile motion if the only acceleration on it is due to gravity.

Example 2.2
Is a flying plane in projectile motion?

No. It has acceleration due to its engine. It is not in projectile motion.
To take off the plane must accelerate. Hence, the plane does not undergo constant velocity.

2.3: Ideal Projectile Motion
To begin the analysis of projectile motion we introduce simplifying assumptions:
- No forces other than gravity act on the projectile.
- Specifically, no air resistance or drag.
- Turning of the Earth when the projectile is in motion does not matter. This wi

Example 2.4
A ball is thrown horizontally from a building. Neglecting air resistance:
A. What are the forces causing acceleration on the ball?
B. Which direction does gravity act in?
C. What are the forces acting in the x direction?
D. What is the acceleration in the x direction?
E. What is the change in velocity in the x direction?

Part A

$$\text{Only Gravity}$$

Part B

$$y\ \text{direction}$$

Part C

$$\text{No Forces}$$

Part D

$$a = 0$$

Part E

$$\Delta v = 0$$

2.5: Gravity is the only force
This assumption is not valid for objects that have their power. For example, planes and missiles will use their engines.

2.6: Air Resistance
Air resistance is an important force other than gravity which has a noticeable effect when objects are travelling very fast.
Air resistance will slow down an object

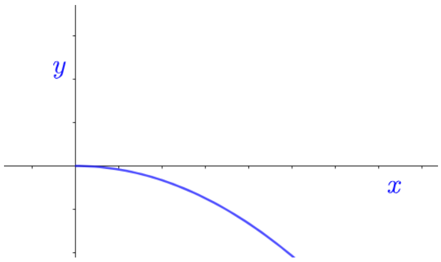
- Planes, missiles and the like. The objective is minimize the air resistance to travel at maximum speed.
- Trains: The objective is to minimize the air resistance.
- Parachute: The objective is to minimize the air resistance so as to travel at minimum speed for a given size of parachute.

2.7: Turning of the Earth
Turning of the Earth will matter when the projectile is in motion for a very long time.
For example, missiles with long range.

B. Objects Thrown Horizontally

2.8: Objects thrown horizontally

The object may initially be at a height as well:
- Stone thrown from a cliff
- Object released from a plane



2.9: Equations of Motion

$$\vec{s} = \vec{r} + \vec{u}t + \frac{1}{2}\vec{a}t^2$$
$$\vec{s} = \vec{r} + \vec{v}t - \frac{1}{2}\vec{a}t^2$$
$$\vec{v} = \vec{u} + \vec{a}t$$
$$|\vec{v}|^2 = |\vec{u}|^2 + 2\vec{a}\cdot\vec{s}$$

Where

$$\vec{r} = Initial\ position\ vector$$
$$\vec{u} = Initial\ position\ vector$$
$$\vec{a} = Acceleration\ vector$$
$$\vec{v} = Final\ velocity\ vector$$

If we introduce an origin at the initial position, then the equations simplify to:

$$\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$$
$$\vec{s} = \vec{v}t - \frac{1}{2}\vec{a}t^2$$

2.10: Components are independent
x component and y component of velocity and acceleration have no effect on each other.

Example 2.11
Mark all correct options

A particle A is dropped from a height and another particle B is projected in horizontal direction with speed of 5 m/s from the same height then correct statement is

A. particle A will reach the ground first with respect to particle B
B. particle B will reach the ground first with respect to particle A
C. both particles will reach the ground simultaneously
D. both particles will reach at ground with same speed
E. both particles will reach at ground with same velocity (NEET 2002, Adapted)

Option C

2.12: Projectile Motion

Consider the quantity of interest component wise.
Combine for the final answer.

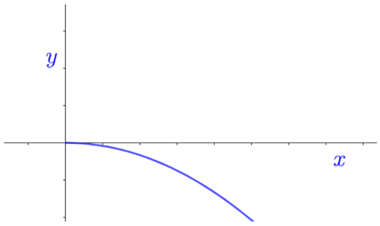
Example 2.13

A child stands on the edge of a cliff $10m$ above the ground and throws a stone horizontally with an initial speed of $5\frac{m}{s}$. Neglecting the air resistance, the speed with which the stone hits the ground will be: (given, $g = 10\text{m/s}$)
(JEE Main, Feb 1, 2023, Shift-I)

Method I

There is no acceleration in the x direction. The velocity remains constant until the stone hits the ground. The velocity in the x direction

$$= v_x = 5$$



The velocity in the y direction[2]

$$v_y^2 = u_y^2 + 2gh = 0 + 2gh = 2(10)(10) = 200$$

Speed is the magnitude of velocity:

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{5^2 + 200} = \sqrt{225} = 15\frac{m}{s}$$

Method II: Using Equations of Motion

$$\begin{gathered} Initial\ Velocity = \vec{u} = (v_x, v_y) = (5,0) \\ Acceleration = \vec{a} = (a_x, a_y) = (0,10) \\ Displacement = \vec{s} = (s_x, s_y) = (s_x, 10) \end{gathered}$$

Make the substitutions above into $|\vec{v}|^2 = |\vec{u}|^2 + 2\vec{a} \cdot \vec{s}$:

$$|\vec{v}|^2 = \underbrace{|(5,0)|^2}_{|\vec{u}|^2} + 2\underbrace{(0,10)}_{\vec{a}} \cdot \underbrace{(s_x, 10)}_{\vec{s}}$$

Simplify:

$$|\vec{v}|^2 = 5^2 + 2(0 \cdot s_x + 10 \cdot 10) = 25 + 200 = 225$$

Take the square root both sides:

$$|\vec{v}| = 15$$

Example 2.14

At $t = 0$, a truck starting from rest, moves in the positive x direction at uniform acceleration of $5\frac{m}{s^2}$. At $t = 20s$, a ball is released from the top of the truck. The ball strikes the ground in $1s$ after the release. The velocity of the ball when it strikes the ground, written in $\hat{\imath}, \hat{\jmath}$ notation and considering upward direction as positive will be (given, $g = 10\text{m/s}$) **(JEE Main, June 30, 2023 Shift-I)**

The velocity of the truck when the ball is released is:

$$v_x = u + at = 0 + (5)(20) = 100$$

---

[2] Take the downward direction as positive.

The velocity in the y direction is:

$$v_y = u + at = 0 - gt = 0 - (10)(1) = -10$$

The overall velocity:

$$= \vec{\boldsymbol{v}} = (100, -10) = 100\hat{\boldsymbol{\imath}} - 10\hat{\boldsymbol{\jmath}}$$

Example 2.15

A helicopter flying horizontally with a speed v at an altitude h has to drop a food packet for a man on the ground. What is the distance of the helicopter from the man when the food packet is dropped in terms of v, h and g. (JEE Main, Aug 31, 2021, Shift-I)

In the y direction:

Since the helicopter is flying horizontally:

$$u_y = initial\ velocity = 0, a_y = g$$

Substitute the above in $s_y = u_y t + \frac{1}{2} a_y t^2$

$$h = 0 + \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2h}{g}}$$

In the x direction:

Substitute $u_x = v, a = 0, t = \sqrt{\frac{2h}{g}}$:

$$s_x = u_x t + \frac{1}{2}a_x t^2 = vt + \frac{1}{2}(0)t^2 = vt = v\sqrt{\frac{2h}{g}}$$

Calculate distance as magnitude of displacement vector of food packet:

$$s = \sqrt{s_x^2 + s_y^2} = \sqrt{\left(v\sqrt{\frac{2h}{g}}\right)^2 + h^2} = \sqrt{\frac{2v^2h}{g} + h^2}$$

Example 2.16: Important

A bomb is dropped by fighter plane flying horizontally. To an observer on the ground directly below the plane at the time of release of the bomb, the trajectory of the bomb is a:

A. Hyperbola
B. Parabola in the direction of motion of the plane
C. Straight line vertically down the plane
D. Parabola in a direction of motion opposite the plane (JEE Main, Aug 26, 2021-Shift-I, Adapted)

Take the downward direction as negative.

$$s_x = x = ut \Rightarrow t = \frac{x}{u}$$

$$s_y = y = 0 - \frac{1}{2}gt^2 \Rightarrow t = \sqrt{-\frac{2y}{g}}$$

$$\sqrt{-\frac{2y}{g}} = \frac{x}{u}$$

Square both sides:

$$\frac{2y}{g} = -\frac{x^2}{u^2}$$

Solve for y

$$y = -\frac{gx^2}{2u^2} = \left(-\frac{g}{2u}\right)x^2$$

And the above is the equation of a parabola

$$y = ax^2, a = -\frac{g}{2u}$$

Option B

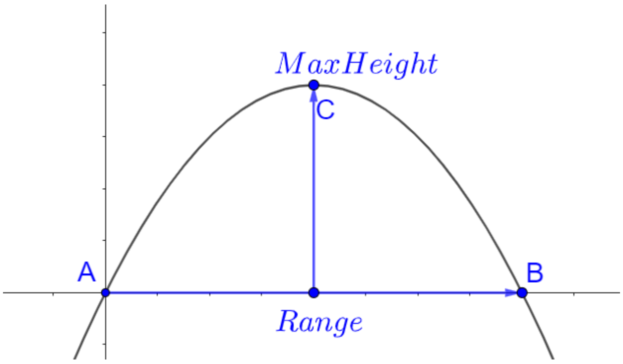
C. Ground Projectiles: Basics

2.17: Object initially on the ground

The object may initially also be on the ground:
- Football kicked into the air
- Cannonball shot from a cannon

2.18: Notation

A is the initial position of the object
B is the final position of the object
C is the maximum height of the object. At maximum height, the object has zero velocity in vertical direction.

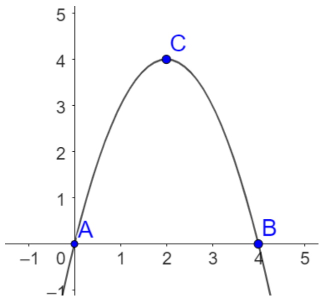


Example 2.19

True or False

At maximum height, an object in projectile motion has zero velocity.

At maximum height:
- the velocity in the y direction is zero
- the velocity in the x direction is equal to the initial velocity in the x direction, and this velocity takes the object from 2 to 4 (in the x direction).



False

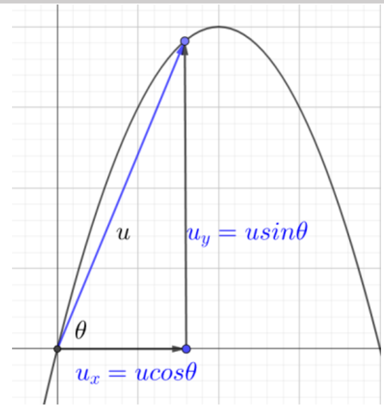
2.20: Launch Parameters

The launch parameters for an object in projectile motion will completely determine the path of the projectile. The launch parameters are:
- Launch Velocity
- Initial Height

- Once the projectile is in motion, the only acceleration is due to gravity, and hence the path is determined.

- We continue to assume that there is no air resistance.

2.21: Launch Speed and Launch Angle
Launch speed is the magnitude of the launch velocity vector.
Launch angle is the direction the launch velocity vector.



$$\textit{Launch speed} = |\boldsymbol{u}| = u$$
$$\textit{Launch angle} = \theta$$

$$0° \le \theta \le 90°$$

2.22: Components of Launch Velocity
Launch velocity can be converted into its components using the launch speed (u) and launch angle(θ).

$$u_y = u\sin\theta$$
$$u_x = u\cos\theta$$

2.23: Components are independent
One component has no effect on the other components.

Example 2.24
The speed of a projectile at its maximum height is half of its initial speed. The angle of projection is **(NEET 2010)**

At max height, vertical component is zero, and hence only horizontal component is left:

$$u_x = u\cos\theta = \frac{u}{2}$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = 60°$$

Example 2.25
A projectile is projected from 30° to the ground with initial velocity $40\frac{m}{s}$. The velocity of the projectile at $t = 2s$ from the start will be: (given, $g = 10\mathrm{m/s}$) **(JEE Main, June 30, 2023, Shift-I)**

Take downward direction as negative. The components of the initial velocity are:

$$u_x = 40\cos 30° = 40\left(\frac{\sqrt{3}}{2}\right) = 20\sqrt{3}$$
$$u_y = 40\sin 30° = 40\left(\frac{1}{2}\right) = 20$$

At $t = 2$:

$$v_y = u_y + a_y t = u_y - gt = 20 - (10)(2) = 0$$
$$v_x = u_x + a_x t = u_x + 0 = 20\sqrt{3}$$

$$\vec{\boldsymbol{v}} = (20\sqrt{3}, 0) = 20\sqrt{3}\hat{\boldsymbol{\imath}}$$

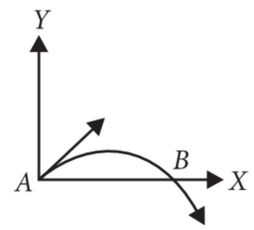
2.26: Landing Velocity
If the launch height is equal to the landing height, the landing velocity of a projectile is the negative of the launch velocity.

Take the upward direction to be positive.
The x component has no acceleration, so the velocity in the x direction remains unchanged.

$$So\ v_x = u_x = u\cos\theta$$

The y component is independent of the x component. The velocity is negative of the original y velocity

$$v_y = -u_y = -u\sin\theta$$



Example 2.27
The velocity of a projectile at the initial point A is $2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$. Its velocity (in $\frac{m}{s}$) at point B is **(NEET 2013)**

$$2\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}$$

2.2 Projectiles: Flight Time

A. Summary

	$Variable$	$1D$	$2D$
$Total\ Flight\ time$	t	$\frac{2u}{g}$	$\frac{2u\sin\theta}{g}$
$Flight\ time$ $for\max height$	t	$\frac{u}{g}$	$\frac{u\sin\theta}{g}$
$Range$	R	0	$\frac{u^2\sin 2\theta}{g}$
$Max\ Height$	h_{max}	$\frac{u^2}{2g}$	$\frac{u^2\sin^2\theta}{2g}$

Notice that the $2D$ equations reduce to the $1D$ equations if we substitute $\theta = 90°$.

B. Flight Time

As we discuss flight time, we learn

- $Flight\ Time = t = \frac{2u\sin\theta}{g}$
- That the path of a projectile is symmetric, and we can exploit that symmetry

2.28: Flight Time
Assuming that the ground is horizontal, the flight time of a projectile with launch speed u and launch angle θ is:

$$Flight\ Time = t = \frac{2u\sin\theta}{g}$$

Take the upward direction to be positive.
The displacement in the x direction does not affect the height of the projectile so we can ignore it completely.
Focus on the displacement in the y direction. The equation of motion in the y direction is:

$$v_y = u_y + a_y t$$

If an object is thrown up with a velocity u_y, it returns to its original position with a velocity $-u_y$.

$$-u_y = u_y - gt \Rightarrow gt = 2u_y \Rightarrow t = \frac{2u_y}{g}$$

Substitute $u_y = u\sin\theta$:

$$t = \frac{2u\sin\theta}{g}$$

Example 2.29

The initial speed of a projectile fired from ground is u. At the highest point during its motion, the speed of projectile is $\frac{\sqrt{3}}{2}u$. The time of flight of the projectile is, in terms of u and g is: **(JEE Main, Jan 23, 2023, Shift-I)**

At the highest point, the only component of velocity is the horizontal component.

$$u_x = \frac{\sqrt{3}}{2}u$$
$$u\cos\theta = \frac{\sqrt{3}}{2}u$$
$$\cos\theta = \frac{\sqrt{3}}{2}$$
$$\theta = 30^\circ$$

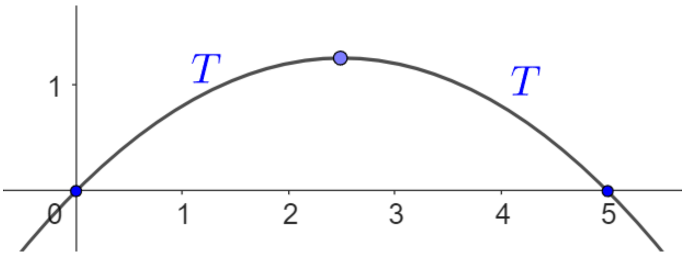
$$t = \frac{2u\sin\theta}{g} = \frac{2u\sin 30^\circ}{g} = \frac{2u\left(\frac{1}{2}\right)}{g} = \frac{u}{g}$$

2.30: Symmetry

$$Total\ Time = t = 2T$$

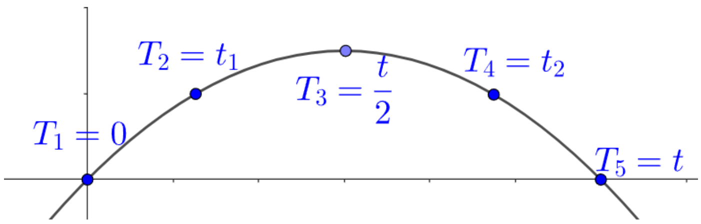
Where

$$T = Time\ going\ up = Time\ going\ down$$



The path of a projectile is parabolic. A parabola has vertical symmetry: reflecting it across its vertex does not change the parabola.
This can be exploited to answer some questions very easily.

2.31: Symmetry



By symmetry, T_3 is the midpoint of T_2 and T_4.

$$\frac{t}{2} = \frac{1}{2}(t_2 + t_1)$$

$$t = t_1 + t_2$$

By symmetry, the distance from T_2 to T_3 is the same as the distance from T_3 to T_4:

$$\frac{t}{2} = \frac{t_1 + t_2}{2}$$
$$t = t_1 + t_2$$

Example 2.32

A projectile fired at 30° to the ground is observed to be at the same height $3s$ and $5s$ after projection, during its height. The speed of projection of the projectile is ____ (given, $g = 10\frac{m}{s}$)(JEE Main April 11, 2023, Shift-I)

Substitute $t = 3 + 5 = 8$ in $t = \frac{2u\sin\theta}{g}$:

$$8 = \frac{2u\sin 30°}{10} \Rightarrow u = 80\frac{m}{s}$$

2.33: $\boldsymbol{Final\ Height \neq Launch\ Height}$

If the final height is not equal to the launch height, then the flight time and flight velocity will be different.

If the final height is greater than the launch height,

$$|v_y| < |u\sin\theta|$$

If the final height is less than the launch height,

$$|v_y| > |u\sin\theta|$$

Example 2.34

A projectile is projected from 30° to the ground with initial velocity $40\frac{m}{s}$. The projectile travels over horizontal ground. The velocity of the projectile at $t = 6s$ from the start will be (given, $g = 10\text{m/s}$). Neglect friction from the air, but not assume ground friction is very high.

$$v_y = u_y + at = 40\sin 30° - gt = 20 - 60 = -40$$

When the object has height 0, it will have velocity

$$-u_y = -40\sin 30° = -20$$

However,

$$v_y = -40 < -20$$

Which is only possible if the object goes below the original launch height.

In this case, since the ground is horizontal, the object will hit the ground, and it will stop (assuming ground friction).

$$v = 0$$

2.35: Time for Maximum Height

The vertical component is independent of the horizontal component and vice versa.

To find maximum height, we must find

$$Maximum\ value\ of y - component\ of\ \vec{\boldsymbol{r}}(\boldsymbol{t}) = r_y(t)$$

$$\vec{\boldsymbol{v}} = (u\cos\theta)\hat{\boldsymbol{\imath}} + (u\sin\theta - gt)\hat{\boldsymbol{\jmath}}$$

$$\frac{d}{dt}r_y(t) = v_y(t) = u\sin\theta - gt$$

At maximum height, the vertical component must be zero.

$$u \sin\theta = gt$$
$$t = \frac{u\sin\theta}{g}$$

Example 2.36

A ball of mass m is thrown vertically upward. Another ball of mass $2m$ is thrown at an angle θ with the vertical. Both balls stay in the air for the same period of time. The ratio of heights attained by the two balls respectively is $\frac{1}{x}$. The value of x is: **(JEE Main, July 27, 2022, Shift-I, Adapted)**

Since times are same, vertical component of velocity is equal

$$t_1 = t_2 \Rightarrow \frac{u_{1-y}}{g} = \frac{u_{2-y}}{g} \Rightarrow u_{1-y} = u_{2-y}$$

If the vertical component of velocity is equal, heights are equal.

$$h_1 = h_2 \Rightarrow \frac{h_1}{h_2} = 1 = \frac{1}{x} \Rightarrow x = 1$$

Example 2.37

Two projectiles thrown at 30° and 45° with the horizontal respectively reach the maximum height in same time. The ratio of their initial velocities is: **(JEE Main, July 26, 2022, Shift-I)**

$$\frac{u_{1-y}}{g} = \frac{u_{2-y}}{g}$$
$$u_1 \sin\theta_1 = u_2 \sin\theta_2$$
$$\frac{u_1}{u_2} = \frac{\sin\theta_2}{\sin\theta_1} = \frac{\sin 45^\circ}{\sin 30^\circ} = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{2}} = \frac{\sqrt{2}}{1}$$

Example 2.38

Identify the mistake, if any

Problem: A projectile is fired from the surface of the earth with a velocity of $5\frac{m}{s}$ and angle θ with the horizontal. Another projectile fired from another planet with a velocity of $3\frac{m}{s}$ at the same angle follows a trajectory which is identical with the trajectory of the projectile fired from the earth. The value of the acceleration due to gravity ($\mathcal{g}$) on the planet is (in $\frac{m}{s^2}$ in terms of g) is **(NEET 2014, Adapted)**

Solution: Since trajectory is same, flight time must be same:

$$t_1 = t_2 \Rightarrow \frac{u_1 \sin\theta}{g} = \frac{u_2\sin\theta}{\mathcal{g}} \Rightarrow \mathcal{g} = \frac{u_2}{u_1} g$$

The trajectory of the two projectiles is the same.
Trajectory plots x coordinate versus y coordinate. It does not plot time. Hence, time being equal is not guaranteed.

Hence, the answer uses incorrect logic, and is wrong.

2.3 Projectiles: Range

A. Range

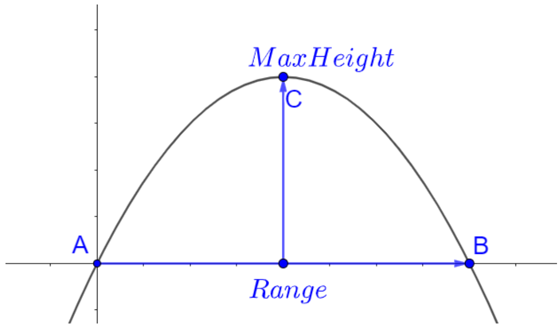
As we discuss range, we learn

- $Range = R = \frac{u^2}{g} \sin 2\theta$
- That complementary launch angles result in the same range.
- Range increases over $0 < \theta < 45°$, decreases over $45° < \theta < 90°$, and is maximum at $\theta = 45°$.

2.39: Range

Range is the displacement from the launch position to the landing position. If launch speed is u, and launch angle is θ, then:

$$Range = R = \frac{u^2}{g} \sin 2\theta$$



Note that:

$$Displacement\ in\ x\ direction = u_x = u\cos\theta$$

$$Flight\ time = t = \frac{2u\sin\theta}{g}$$

$$a_x = 0$$

Substitute the above in the equation of motion in the x direction:

$$s_x = u_x t + \frac{1}{2} a_x t^2 = (u\cos\theta)\left(\frac{2u\sin\theta}{g}\right) + \frac{1}{2}(0)t^2$$

The second term will become zero, and carry out the multiplication in the first term:

$$= \frac{u^2(2\sin\theta\cos\theta)}{g}$$

Substitute $2\sin\theta\cos\theta = \sin 2\theta$ using the double angle formula:

$$= \frac{u^2 \sin 2\theta}{g}$$

Example 2.40

Two projectiles are thrown with same initial velocity making an angle of 45° and 30° with the horizontal respectively. The ratio of their respective ranges will be: **(JEE Main July 26, 2022, Shift-II)**

$$R_1 : R_2 = \frac{u^2 \sin 90°}{g} : \frac{u^2 \sin 60°}{g} = \sin 90° : \sin 60° = 1 : \frac{\sqrt{3}}{2} = 2 : \sqrt{3}$$

2.41: Maximum Range

$$R_{max} = \frac{u^2}{g}\ when\ \theta = 45°$$

Example 2.42

A. A missile is fired for maximum range with an initial velocity of $20\frac{m}{s}$. If $g=\frac{10m}{s^2}$, the range of the missile is **(NEET 2011)**

B. The maximum range of a gun of horizontal terrain is 16 km. If $g=10\frac{m}{s^{-2}}$, then muzzle velocity of a shell $(in\frac{m}{s})$ must be **(NEET 1990)**

Part A

$$R_{max}=\frac{u^2}{g}=\frac{20^2}{10}=\frac{400}{10}=40$$

Part B

$$R_{max}=\frac{u^2}{g}\Rightarrow u=\sqrt{R_{max}g}=\sqrt{(16\cdot 1000)(10)}=400\frac{m}{s}$$

Example 2.43

Mark the correct option

Assertion A: When a body is projected at an angle 45°, its range is maximum.
Reason R: For maximum range, the value of $\sin 2\theta$ should be equal to one.

A. Both Assertion and Reason are correct, but Reason is not the correct explanation of Assertion.
B. Both Assertion and are correct. Reason is the correct explanation of Assertion.
C. Assertion is true but Reason is false.
D. Assertion is false but Reason is true. **(JEE Main, April 6, 2022, Shift-I)**

$$R=\frac{u^2}{g}\sin 2\theta$$

The maximum value that $\sin 2\theta$ can achieve is 1:

$$\sin 2\theta=1\Rightarrow 2\theta=90\Rightarrow\theta=45°$$

Option B

2.44:

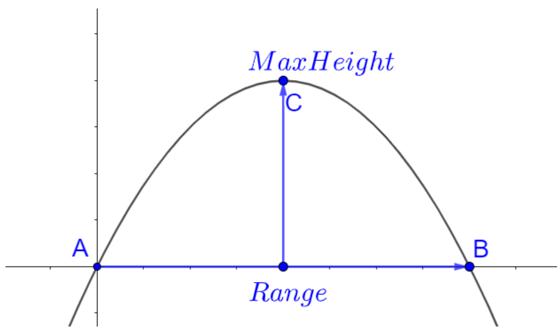
$$\sin x=\sin y$$

$$x=y$$
$$x=180°-y$$

2.45: Complementary Launch Angles

If two projectiles have complementary launch angles θ_1 and θ_2, then the ratios of the ranges are in the ratio of their launch speeds.

$$R_1:R_2=u_1^2:u_2^2$$



The ratio of the ranges is:

$$R_1:R_2=\frac{u_1^2\sin 2\theta_1}{g}:\frac{u_2^2\sin 2\theta_2}{g}=u_1^2\sin 2\theta_1:u_2^2\sin 2\theta_2$$

$$=u_1^2\sin 2\theta_1:u_2^2\sin(180-2\theta_1)$$

Substitute $\sin(180-2\theta_1)=\sin 2\theta_1$

$$= u_1^2 \sin 2\theta_1 : u_2^2 \sin 2\theta_1$$
$$= u_1^2 : u_2^2$$

Example 2.46

A. Two projectiles A and B are thrown with initial velocities of $40\frac{m}{s}$ and $60\frac{m}{s}$ at angles 30° and 60° with the horizontal respectively. The ratio of their ranges respectively is: $\left(g = \frac{10m}{s}\right)$ **(JEE Main, April 8, 2023, Shift-II)**

B.

Since the launch angles are complementary, the ranges are in the ratio of launch speeds:

$$R_1 : R_2 = u_1^2 : u_2^2 = 40^2 : 60^2 = 1600 : 3600 = 16 : 36 = 4 : 9$$

2.47: Complementary Launch Angles with same Speed

If two projectiles have complementary launch angles θ_1 and θ_2, and same launch speed then they have equal range.

The ratio of the ranges is:

$$u_1^2 : u_2^2 = 1 : 1 = Same$$

Example 2.48

A. Two objects are projected with same velocity u however at different angles α and β with the horizontal. If $\alpha + \beta = 90°$, the ratio of horizontal range of the first object to the second object is: **(JEE Main, Jan 25, 2023, Shift-II)**

B. For angles of projection of a projectile at angle $(45° - \theta)$ and $(45° + \theta)$, the horizontal range described by the projectile are in the ratio of **(NEET 2006)**

C. If a body A of mass M is thrown with velocity v at an angle of 30° to the horizontal and another body B of the same mass is thrown with the same speed at an angle of 60° to the horizontal, the ratio of horizontal range of A to B will be **(NEET 1990, 1992)**

Part A

Angles are complementary. Magnitude of velocity is same. Range is equal.

$$Ratio = 1 : 1$$

Part B

$$45° - \theta + 45° + \theta = 90°$$

Angles are complementary. Range is equal.

$$1 : 1$$

Part C

Angles are complementary. Speed is same. Range is equal.

$$Ratio = 1 : 1$$

Example 2.49

Mark all correct options

Two objects of same mass are projected with the same speed, but at angles 60° and 30° with the horizontal. Which of the following quantities are equal:

A. time of flight
B. range
C. maximum height
D. launch velocity

E. landing velocity
F. landing speed

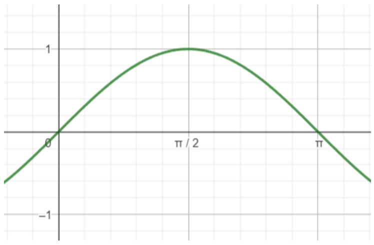
Time of flight depends on vertical components, which are different, so time is different.
Range is same because launch angles are complementary.
Because the launch angle is different, the launch and landing velocities are different.
The landing speed will match the launch speed, which is same.

Options B and F

2.50: Range as a Function of θ

Consider a projectile with fixed launch velocity u, and launch angle in the first quadrant. That is

$$0° < \theta < 90°$$



$$R = \frac{u^2}{g}\sin 2\theta$$

Since $g = constant, u = constant$

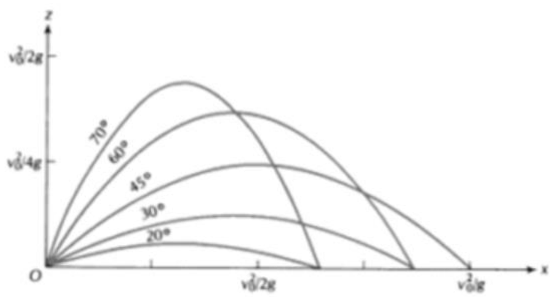
$$R \propto \sin 2\theta$$

$$0° < \theta < 90°$$
$$0° < 2\theta < 180°$$

2.51: Range/Height over different launch angles

As launch angle increases from 0° to 45°
- Height increases
- Range also increases

As launch angle increases from 45° to 90°
- Height keeps on increases
- Range decreases, going to zero



2.52: Half of Maximum Range

For a given initial velocity u, half of the maximum range is achieved when the launch angle is

$$\theta_2 \in \{15°, 75°\}$$

The maximum range is achieved when $\theta = 45°$.

$$R_1 = \frac{u^2}{g}\sin 2\theta_1 = \frac{u^2}{g}\sin(2 \times 45°) = \frac{u^2}{g}$$

To determine the angles when the range achieved is half of the maximum range, let:

$$R_2 = \frac{1}{2}R_1$$

$$\frac{u^2}{g}\sin 2\theta_2 = \frac{1}{2}\frac{u^2}{g} \Rightarrow \sin 2\theta_2 = \frac{1}{2} \Rightarrow 2\theta_2 \in \{30°, 150°\} \Rightarrow \theta_2 \in \{15°, 75°\}$$

Example 2.53

A. The range of the projectile projected at an angle of 15° with horizontal is $50m$. If the projectile is projected with same velocity, at an angle of 45° with horizontal, then its range will be: **(JEE Main, April 10, 2023, Shift-II)**

B. An object is projected in the air with initial velocity u at an angle θ. The projectile motion is such that the horizontal range R is maximum. Another object is projected in the air with a horizontal range half of the range of the first object. The initial velocity remains the same in both cases. The value(s) of the angle of projection at which the second object is projected will be ______ degree. **(JEE Main, July 29, 2022, Shift-I)**

Part A

By the property above, range at 45° will be double of range at 15°:Using t

$$R_2 = 2(50) = 100\ m$$

Part B

Since range is maximum:

$$\theta = 45°$$

Using the property for half of maximum range:

$$\theta_2 \in \{15°, 75°\}$$

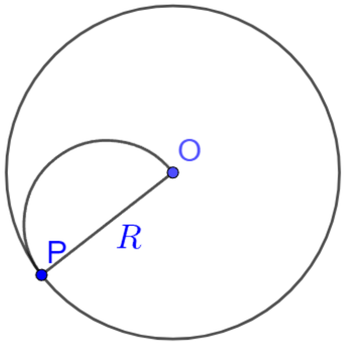
2.54: Maximum Area covered by a Projectile

Maximum area covered by a projectile with maximum range R_{max} is

$$\pi R_{max}^2$$

Where

$$R_{max} = \frac{u^2}{g}$$



$$R_{max} = \frac{u^2 \sin 2\theta}{g} = \frac{u^2 \sin 90°}{g} = \frac{u^2}{g}$$

Example 2.55

A. A water fountain on the ground sprinkles water all around it. If the speed of water coming out of the fountain is v, the total area around the fountain that gets wet is: (in terms of g and u) **(JEE Main 2011).** Note: Question should state "maximum speed".

B. Two guns A and B can fire bullets at speeds $1\frac{km}{s}$ and $2\frac{km}{s}$ respectively. From a point on the horizontal ground, they are fired in all possible directions. The ratio of maximum area covered by the bullets fired on the ground, is: **(JEE Main, 10 Jan 2019, Shift-I)**

Part A

$$A = \pi R_{max}^2 = \pi\left(\frac{v^2}{g}\right)^2 = \pi\frac{v^4}{g^2}$$

Part B

$$Ratio = \pi {R_1}_{max}^2 : \pi {R_2}_{max}^2 = {R_1}_{max}^2 : {R_2}_{max}^2$$
$$= \left(\frac{1^2}{g}\right)^2 : \left(\frac{2^2}{g}\right)^2 = 1^4 : 2^4 = 1:16$$

Example 2.56

A projectile is fired from the surface of the earth with a velocity of $5\frac{m}{s}$ and angle θ with the horizontal. Another projectile fired from another planet with a velocity of $3\frac{m}{s}$ at the same angle follows a trajectory which is identical with the trajectory of the projectile fired from the earth. The value of the acceleration due to gravity ($\mathcal{g}$) on the planet is (in $\frac{m}{s^2}$ in terms of g) is **(NEET 2014, Adapted)**

Since trajectory is same, range must be same:

$$R_1 = R_2$$
$$\frac{u_1^2 \sin 2\theta}{g} = \frac{u_2^2 \sin 2\theta}{\mathcal{g}}$$
$$\mathcal{g} = \frac{u_2^2}{u_1^2} g = \frac{3^2}{5^2} g = \frac{9}{25} g$$

2.4 Projectiles: Max Height

A. Maximum Height

As we discuss maximum height, we learn:

- Since the path of a projectile is a
  - ✓ parabola, the maximum height is its vertex (using quadratic functions)
  - ✓ parabolic function, we can apply optimization techniques (Calculus) to find the maximum
- $h_{max} = \frac{u^2 \sin^2\theta}{2g}$, for cases where we know the launch angle and launch velocity
- By symmetry, the time at which maximum height is achieved is $t = \frac{u \sin\theta}{g}$

2.57: Maximum of a Parabola

The x value of the maximum of the parabola $f(x) = y = ax^2 + bx + c$ is

$$-\frac{b}{2a}$$

The y value of the maximum of the parabola $f(x) = y = ax^2 + bx + c$ is

$$f\left(-\frac{b}{2a}\right)$$

Example 2.58

The trajectory of projectile, projected from the ground is given by $y = x - \frac{x^2}{20}$, where x and y are measured in meter. The maximum height attained by the projectile will be: (answer as a number) **(JEE Main, April 8, 2023, Shift-II)**

$$f(x) = y = x - \frac{x^2}{20}$$

The maximum of the parabola $y = ax^2 + bx + c$ is

$$f\left(-\frac{b}{2a}\right) = f\left(-\frac{1}{2\left(-\frac{1}{20}\right)}\right) = f\left(\frac{1}{\frac{1}{10}}\right) = f(10) = 10 - \frac{10^2}{20} = 10 - 5 = 5\,m$$

2.59: Maximum Height

$$Max\ Height = h_{max} = \frac{u^2 \sin^2 \theta}{2g}$$

$$s_y = u_y t + \frac{1}{2} a_y t^2$$
$$= (u \sin \theta)\left(\frac{u \sin \theta}{g}\right) + \frac{1}{2}(-g)\left(\frac{u \sin \theta}{g}\right)^2$$
$$= \frac{(u \sin \theta)^2}{2g}$$

2.60: Maximum Height of a Projectile

$$y = \underbrace{-\left(\frac{g}{2v_0^2 \cos^2 \theta}\right)x^2}_{ax^2} + \underbrace{(\tan \theta)x}_{bx}$$

$$y\left(-\frac{b}{2a}\right) = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) = a \cdot \frac{b^2}{4a^2} - \frac{b^2}{2a} = \frac{b^2}{4a} - \frac{2b^2}{4a} = -\frac{b^2}{4a}$$

Substitute

$$a = -\left(\frac{g}{2v_0^2 \cos^2 \theta}\right), b^2 = \tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta}$$
$$\frac{-\left(\frac{\sin^2 \theta}{\cos^2 \theta}\right)}{4\left[-\left(\frac{g}{2v_0^2 \cos^2 \theta}\right)\right]} = \frac{v_0^2 \sin^2 \theta}{2g}$$

Example 2.61

Two projectiles are projected at 30° and 60° with the horizontal with the same speed. The ratio of the maximum height attained by the two projectiles respectively is: **(JEE Main, April 10, 2023, Shift-II)**

Use the formula for the maximum height:

$$h_1 : h_2 = \frac{(u \sin \theta_1)^2}{2g} : \frac{(u \sin \theta_2)^2}{2g}$$

Cancel $\frac{u^2}{2g}$, and substitute $\theta_1 = 30°, \theta_2 = 60°$:

$$= \sin^2 30° : \sin^2 60° = \left(\frac{1}{2}\right)^2 : \left(\frac{\sqrt{3}}{2}\right)^2 = 1:3$$

Example 2.62

Two projectiles P_1 and P_2 thrown with speed in the ratio $\sqrt{3}:\sqrt{2}$ attain the same height during their motion. If P_2 is thrown at an angle of 60° with the horizontal, the angle of projection of P_1 with the horizontal will be: **(JEE Main, June 30, 2022, Shift-I)**

Example 2.63

Two bodies are projected from ground with same speed $40\frac{m}{s}$ at two different angles with respect to horizontal.

The bodies were found to have same range. If one of the bodies was projected at an angle 60° with horizontal, then sum of the maximum heights attained by the two projectiles is: $\left(g = \frac{10m}{s}\right)$ **(JEE Main, Jan 31, 2023, Shift-I)**

Speeds are equal, the range is same, but the launch angles are different, the launch angles must be complementary.

$$\theta_2 = 90 - \theta_1 = 90 - 60 = 30°$$

$$h_1 + h_2 = \frac{u^2 \sin^2 \theta_1}{2g} + \frac{u^2 \sin^2 \theta_2}{2g} = \frac{u^2}{2g}(\sin^2 \theta_1 + \sin^2 \theta_2)$$

Substitute $u = 40, g = 10, \theta_1 = 60°, \theta_2 = 30°$:

$$= \frac{40^2}{20}\left[\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right] = 80\left[\frac{3}{4} + \frac{1}{4}\right] = 80\ m$$

B. Height versus Range

Example 2.64

A. $(Calc)$ A cannonball is launched with initial velocity of magnitude v_0 over a horizontal surface. At what minimum angle θ_{min} above the horizontal should the cannonball be launched so that it rises to a height H which is larger than the horizontal distance R that it will travel when it returns to the ground? **(F=ma 2012/2)**

B. A ball is projected from the ground with a speed $15\frac{m}{s}$ at an angle θ with horizontal so that its range and maximum height are equal. Then $\tan\theta$ will be equal to: **(JEE Main, July 25, 2022, Shift-I)**

C. The horizontal range and the maximum height of a projectile are equal. The angle of projection of the projectile is **(NEET 2012)**

Part A

Substitute the formulas for height and range in $Height > Range$

$$\frac{u^2 \sin^2 \theta}{2g} > \frac{u^2 \sin 2\theta}{g}$$

Simplify, use the identity for $\sin 2\theta$, and simplify further:

$$\begin{aligned} \sin^2\theta &> 2\sin 2\theta \\ \sin^2\theta &> 2(2\sin\theta\cos\theta) \\ \sin\theta &> 4\cos\theta \end{aligned}$$

Collate trigonometric terms on one side, and take the tan $inverse$ on both sides:

$$\tan\theta > 4$$
$$\theta > \tan^{-1} 4 \approx 75.96°$$

Parts B

Solve as per Part A (except that it is an equality)

$$H = R \Rightarrow \tan\theta = 4$$

Parts B and C

Solve as per Part A (except that it is an equality)

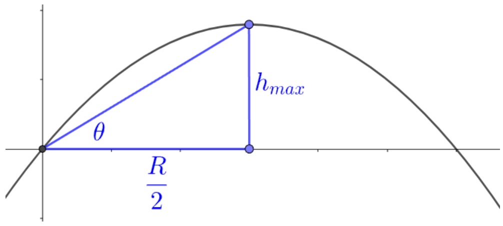
$$\theta = \tan^{-1} 4$$

Example 2.65

A projectile is fired at an angle of 45° with the horizontal. Elevation angle of the projectile at its highest point as seen from the point of projection can be written in the form $\tan^{-1} x$, where x is a real number. Find x. **(NEET 2011)**

$$\tan\theta = \frac{h_{max}}{\frac{R}{2}} = \frac{\frac{u^2 \sin^2 45°}{2g}}{\frac{u^2 \sin 2 \times 45°}{2g}} = \frac{\left(\frac{1}{\sqrt{2}}\right)^2}{1} = \frac{1}{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{2}\right)$$



2.66: Vertical versus Horizontal

A projectile with initial velocity u, and maximum vertical height h, has

$$Maximum\ Range = R = 2h$$

The maximum height is achieved when we launch the projectile vertically up, with launch angle $\theta = 90°$.
Since the projectile is launched vertically, we can use the $1D$ formula:

$$h = \frac{u^2}{2g} \Rightarrow u^2 = 2gh$$

Now to maximize the range, we change the launch angle.
Maximum range is achieved when launch angle $= 45°$.

Substitute $u^2 = 2gh, \theta = 45°$ in the formula for range:

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{(2gh)\sin(2 \cdot 45°)}{g} = 2h$$

Example 2.67

A. The maximum vertical height to which a man can throw a ball is 136m. The maximum horizontal distance to which he can throw the same ball is: **(JEE Main, Jan 24, 2023, Shift-I)**
B. A boy can throw a stone up to a maximum height of 10m. The maximum horizontal distance that the boy can throw the same stone will be: **(JEE Main, 2012)**
C. A person can throw a ball up to a maximum range of 100 m. How high above the ground can he throw the same ball. **(JEE Main, June 29, 2022, Shift-II)**

Part A

$$R = 2h = 272m$$

Part B

$$R = 2h = 20m$$

Part C

$$h = \frac{R}{2} = 50$$

2.68: Vertical versus Horizontal

A projectile with fixed initial velocity u, and maximum vertical height h, has maximum range $2h$.
If it is launched on the path for maximum range, the maximum height achieved on that path will be:

$$\textit{Maximum height on path with maximum range} = \frac{h}{2}$$

We know from the previous property that

$$u^2 = 2gh$$

The path for maximum range with have launch angle $\theta = 45°$.

Substitute $u^2 = 2gh, \theta = 45°$ to find height achieved on the path for maximum range:

$$= H = \frac{u^2 \sin^2 \theta}{2g} = \frac{(2gh)\left(\frac{1}{\sqrt{2}}\right)^2}{2g} = \frac{h}{2}$$

Note:

- h is maximum vertical height when there is no restriction on launch angle. It is achieved when $\theta = 90°$.
- H is maximum vertical height when the range is maximum, which means $\theta = 45°$

C. System of Equations

Example 2.69

A projectile is projected with velocity of 25 $\frac{m}{s}$ at an angle θ with the horizontal. After t seconds, its inclination with the horizontal becomes zero. If R represents horizontal range of the projectile, the value of θ will be:

A. $\frac{1}{2}\sin^{-1}\left(\frac{5t^2}{4R}\right)$
B. $\frac{1}{2}\sin^{-1}\left(\frac{4R}{5t^2}\right)$
C. $\tan^{-1}\left(\frac{4t^2}{5R}\right)$
D. $\cot^{-1}\left(\frac{R}{20t^2}\right)$ (JEE Main, June 24, 2022, Shift-I)

$$t = \frac{u \sin \theta}{g} \Rightarrow u = \frac{gt}{\sin \theta} \Rightarrow u^2 = \frac{g^2 t^2}{\sin^2 \theta}$$

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{\left(\frac{g^2t^2}{\sin^2\theta}\right)^2 (2 \sin \theta \cos \theta)}{g} = \frac{\frac{g^2t^2}{\sin^2\theta}(2 \sin \theta \cos \theta)}{g} = gt^2\left(2\frac{\cos\theta}{\sin\theta}\right)$$

$$R = gt^2(2\cot\theta)$$
$$\cot\theta = \frac{R}{2gt^2} = \frac{R}{20t^2}$$
$$\theta = \cot^{-1}\left(\frac{R}{20t^2}\right)$$

Example 2.70

Two particles are projected from the same point with the same speed u such that they have the same range R, but different maximum heights h_1 and h_2. Find the value of R in terms of h_1 and h_2. (JEE Main, 12 April 2019, 2022, Shift-II, Adapted; JEE Main 2004)

$$h_1 = \frac{u^2 \sin^2 \theta}{2g}$$
$$h_2 = \frac{u^2 \sin^2(90-\theta)}{2g} = \frac{u^2 \cos^2(\theta)}{2g}$$

$$h_1 h_2 = \left(\frac{u^2 \sin^2 \theta}{2g}\right)\left(\frac{u^2 \cos^2 \theta}{2g}\right) = \frac{u^4(\sin\theta\cos\theta)^2}{4g^2} = \frac{u^4\left(\frac{\sin 2\theta}{2}\right)^2}{4g^2} = \frac{u^4 \sin^2 2\theta}{16g^2}$$
$$\sqrt{h_1 h_2} = \frac{u^2 \sin 2\theta}{4g} = \frac{1}{4}R$$
$$R = 4\sqrt{h_1 h_2}$$

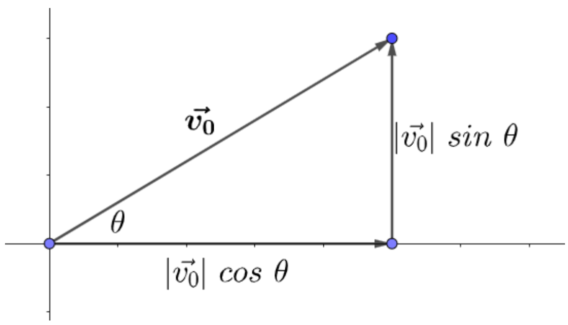
Example 2.71
A shell is fired from a fixed artillery gun with an initial speed u such that it hits the target on the ground at a distance R from it. If t_1 and t_2 are the values of the time taken by it to hit the target in two possible ways, then the value of R in terms of t_1, t_2 and g is: **(JEE Main, 12 April 2019, 2022, Shift-I, Adapted)**

2.5 Projectiles: Parabolic Path

A. Parabolic Path

Example 2.72
Given a projectile with
$$Initial\ Position = \vec{r_0} = 0\hat{\imath} + 0\hat{\jmath}$$
$$Initial\ Velocity = \vec{v_0} = (v_0\cos\theta)\hat{\imath} + (v_0\sin\theta)\hat{\jmath}$$
$$Magnitude\ of\ initial\ velocity = v_0 = |\vec{v_0}|$$
$$Acceleration = \vec{a} = -g\hat{\jmath}$$

A. Determine $Velocity\ vector = \vec{v}$
B. Determine $Position\ vector = \vec{r}$



Part A
Integrate both sides of $\vec{a} = -g\hat{\jmath}$:
$$\vec{v} = \int -g\hat{\jmath}\ dt = (-gt)\hat{\jmath} + \vec{C_1}$$
To find the constant vector $\vec{C}$, substitute $t = 0$:
$$\vec{v_0} = (-g\cdot 0)\hat{\jmath} + \vec{C_1} \Rightarrow \vec{v_0} = \vec{C_1}$$
$$\vec{v} = (-gt)\hat{\jmath} + \vec{v_0} = (v_0\cos\theta)\hat{\imath} + (v_0\sin\theta - gt)\hat{\jmath}$$

Part B
Integrate both sides of the above:
$$\vec{r} = \left(-\frac{1}{2}gt^2\right)\hat{\jmath} + \vec{v_0}t + \vec{C_2}$$

To find the constant vector $\vec{C}$, substitute $t = 0$:

$$\vec{r_0} = \left(-\frac{1}{2}g\cdot 0^2\right)\hat{\jmath} + \vec{v_0}(0) + \vec{C_2} \Rightarrow \vec{r_0} = \vec{C_2}$$

$$\vec{r} = \left(-\frac{1}{2}gt^2\right)\hat{\boldsymbol{j}} + \vec{\boldsymbol{v_0}}\boldsymbol{t} + \vec{\boldsymbol{r_0}}$$

$$\vec{r} = \left(-\frac{1}{2}gt^2\right)\hat{\boldsymbol{j}} + (v_0 \cos\theta\, \hat{\boldsymbol{\imath}} + v_0 \sin\theta\, \hat{\boldsymbol{j}})t + 0$$

Rearrange to write component wise:

$$\vec{r} = (v_0 \cos\theta t)\hat{\boldsymbol{\imath}} + \left(v_0 \sin\theta\, t - \frac{1}{2}gt^2\right)\hat{\boldsymbol{j}}$$

2.73: Parametric Form

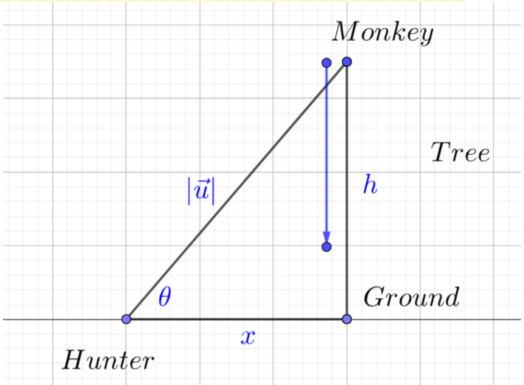
$$x = (u\cos\theta)t$$

$$y = (u\sin\theta)t - \frac{1}{2}gt^2$$

Example 2.74: Hunter and the Monkey

A hypothetical hunter on the ground is aiming at a real monkey hanging on a tree of height h. The hunter fires a bullet, aiming directly at the monkey. At the same time the monkey loses its grip on the tree, and starts falling. With reference to the monkey, at what place on the tree will the bullet hit?

Note: Ignore the width between the monkey and the tree.



Trajectory of the monkey

Introduce an origin at the hunter.

The position of the monkey at time t using free fall concept from 1D motion:

$$y = h - \frac{1}{2}gt^2$$

Trajectory of the bullet

Match horizontal coordinate: We need the bullet at the same x coordinate as the monkey. The time for the bullet to travel the horizontal distance (x) is:

$$x = (u\cos\theta)t \Rightarrow t = \frac{x}{u\cos\theta} = \frac{Distance}{Speed}$$

Determine vertical coordinate corresponding to x:

Substitute $t = \frac{x}{u\cos\theta}$ in the parametric equation for the y coordinate $y = (u\sin\theta)t - \frac{1}{2}gt^2$:

$$y = (u\sin\theta)\frac{x}{u\cos\theta} - \frac{1}{2}gt^2 = (\tan\theta)x - \frac{1}{2}gt^2$$

Substitute $\tan\theta = \frac{h}{x} \Rightarrow h = (\tan\theta)x$:

$$y = h - \frac{1}{2}gt^2$$

The vertical coordinate of the bullet matches the vertical coordinate of the monkey when the bullet has the same horizontal coordinate as the monkey. The bullet will exactly hit the monkey.[3]

2.75: Projectile Path is Parabolic

$$y = (\tan\theta)x - \left(\frac{g}{2u^2\cos^2\theta}\right)x^2$$

$$x = (u\cos\theta)t \Rightarrow t = \frac{x}{v_0\cos\theta}$$

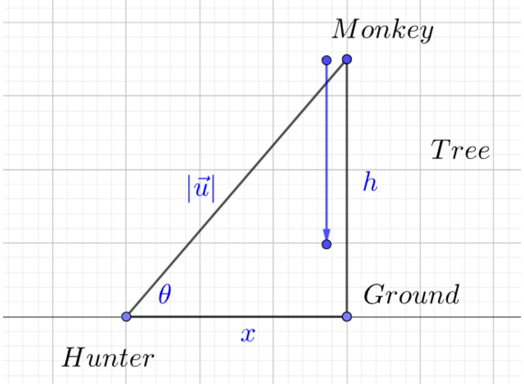
[3] If you have trouble believing this solution, a physical demonstration from MIT might help convince you.

$$y = (\boldsymbol{u}\sin\theta)t - \frac{1}{2}gt^2$$
$$y = (\boldsymbol{u}\sin\theta)\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2 = (\tan\theta)x - \left(\frac{g}{2u^2\cos^2\theta}\right)x^2$$
$$y = \underbrace{-\left(\frac{g}{2u^2\cos^2\theta}\right)x^2}_{ax^2} + \underbrace{(\tan\theta)x}_{bx}$$

$$y = ax^2 + bx$$

Example 2.76: Hunter and the Monkey

A hypothetical hunter on the ground is aiming at a real monkey hanging on a tree of height h. The hunter fires a bullet, aiming directly at the monkey. At the same time the monkey loses its grip on the tree, and starts falling. With reference to the monkey, at what place on the tree will the bullet hit?
Note: Ignore the width between the monkey and the tree.



Trajectory of the monkey
Introduce an origin at the hunter.
The position of the monkey at time t using free fall concept from 1D motion:

$$y = h - \frac{1}{2}gt^2$$

Trajectory of the bullet

$$y = (\tan\theta)x - \left(\frac{g}{2u^2\cos^2\theta}\right)x^2$$

Substitute $\tan\theta = \frac{h}{x} \Rightarrow h = (\tan\theta)x$:

$$y = h - \left(\frac{g}{2u^2\cos^2\theta}\right)x^2 = h - \left(\frac{g}{2}\right)\left(\frac{x}{u\cos\theta}\right)^2$$

Using $Time = \frac{Distance}{Speed}$, we get $t = \frac{x}{u\cos\theta}$:

$$y = h - \frac{1}{2}gt^2$$

This matches. So, the bullet hits the monkey exactly[4].

Example 2.77

A projectile is fired from the surface of the earth with a velocity of $5\frac{m}{s}$ and angle θ with the horizontal. Another projectile fired from another planet with a velocity of $3\frac{m}{s}$ at the same angle follows a trajectory which is identical with the trajectory of the projectile fired from the earth. The value of the acceleration due to gravity (g) on the planet is (in $\frac{m}{s^2}$ in terms of g) is **(NEET 2014, Adapted)**

The equation of the trajectory of the projectile is:

$$y = (\tan\theta)x - \left(\frac{g}{2u^2\cos^2\theta}\right)x^2$$

For both the projectiles, θ *is same* and x *is also same*. Hence, y depends only on the part of the term that has g and u.

[4] Here is one more demo video showing the same experiment. And another.

Hence, we must have

$$\frac{g}{u_1^2} = \frac{\mathscr{g}}{u_2^2} \Rightarrow \mathscr{g} = \frac{u_2^2}{u_1^2} g = \frac{3^2}{5^2} g = \frac{9}{25} g$$

Example 2.78

The initial velocity in horizontal direction of a projectile is unit vector $\hat{\boldsymbol{\imath}}$ and the equation of trajectory is $y = 5x(1 - ax)$.

A. The y component vector of the initial velocity is: **(JEE Main, July 26 2022, Shift-II, Adapted)**
B. The value of a is ____.

Compare the given equation of the trajectory with the standard equation of the trajectory of a parabola:

$$y = 5x - 5ax^2 \Leftrightarrow y = (\tan\theta)x - \left(\frac{g}{2u^2\cos^2\theta}\right)x^2$$

Part A

Comparing coefficients tells us:

$$\tan\theta = 5$$
$$\frac{u\sin\theta}{u\cos\theta} = 5$$

Substitute $u_x = u\cos\theta = 1$:

$$u_y = u\sin\theta = 5$$

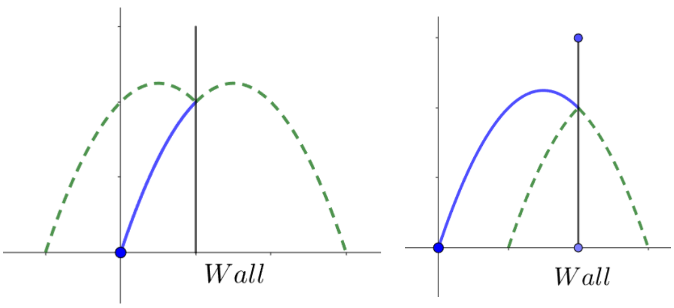
Part B

$$\frac{g}{2u^2\cos^2\theta} = 5a$$
$$\frac{10}{2(1)^2} = 5a$$
$$a = 5$$

B. Collisions and Reflections

2.79: Elastic Collisions[5]

If a projectile collides with a vertical, stationary, wall, the angle at which it bounces back is the angle at which it collides.



We can consider a couple of cases. The projectile bounces against the wall, and comes back. But its trajectory is "preserved" in the sense that the new trajectory is the reflection of the old trajectory if the wall had not been present.

(Calc) Example 2.80

Jordi stands $20m$ from a wall and Diego stands $10m$ from the same wall. Jordi throws a ball at an angle of $30°$ above the horizontal, and it collides elastically with the wall. How fast does Jordi need to throw the ball so that Diego will catch it? Consider Jordi and Diego to be the same height, and both are on the same perpendicular line from the wall. **(F=ma 2013/2)**

$$\theta = 30°$$

[5] Elastic collisions preserve kinetic energy. For non-elastic collisions, see the Note on Work, Power, Energy (which includes a section on collisions).

Since Jordi is to the left of Diego, we want the case on the right.

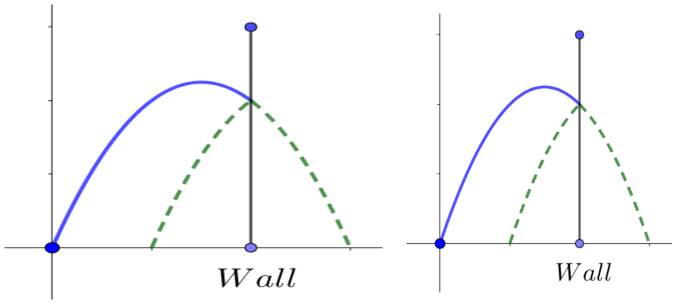
If the ball had not been reflected, it would have continued on the dotted green path, and would have had

$$Range = 20 + 10 = 30\ m$$

$$R = \frac{u^2 \sin 2\theta}{g}$$

$$30 = \frac{u^2 \sin(2 \cdot 30°)}{g} = \frac{u^2 \frac{\sqrt{3}}{2}}{g}$$

$$u = \sqrt{\frac{60g}{\sqrt{3}}} = \sqrt{\frac{600}{\sqrt{3}}} = 18.61$$



3. Relative Motion

3.1 Relative Motion: Two Objects

A. Relative Velocity

3.1: Relative Velocity

Relative Velocity is the difference in velocity. If object I has velocity v_I and object II has velocity v_{II} then

$$\text{Relative velocity of object I with respect to object II} = v_I - v_{II}$$
$$\text{Relative velocity of object II with respect to object I} = v_{II} - v_I$$

Example 3.2

Object I is moving at $5\frac{m}{s}$ to the right. Object II is moving at $7\frac{m}{s}$ to the left. Find the relative velocity of:

A. Object I with respect to Object II
B. Object II with respect to Object I

Part A

$$\underbrace{5}_{Object\ I} - \underbrace{(-7)}_{Object\ II} = 12\frac{m}{s}$$

Imagine that at $t = 0$, both objects are at the origin. At:

$$t = 1, x_I = 5, x_{II} = -7 \Rightarrow Object\ I\ is\ moving\ right\ by\ 12\frac{m}{s}$$

Part B

Relative velocity of Object II with respect to Object I is:

$$\underbrace{-7}_{Object\ II} - \underbrace{5}_{Object\ I} = -12\frac{m}{s}$$

Imagine that at $t = 0$, both objects are at the origin. At:

$$t = 1, x_I = 5, x_{II} = -7 \Rightarrow Object\ II\ is\ moving\ left\ by\ 12\frac{m}{s}$$

3.3: Objects moving in opposite directions

If two objects move in opposite directions with velocity $\boldsymbol{v_1}$ and $-\boldsymbol{v_2}$ their relative speed is

$$v_1 + v_2$$

$$\boldsymbol{v_1} - (-\boldsymbol{v_2}) = v_1 + v_2$$

Example 3.4

A passenger sitting in a train A moving at $90\frac{km}{hr}$ observes another train B moving in the opposite direction for $8s$. If the velocity of the train B is $54\ \frac{km}{hr}$, then length of train B is, in meters: (JEE Main, April 13, 2023, Shift-II)

$$90 + 54 = 144\frac{km}{hr} = 144 \times \frac{5}{18}\frac{m}{s} = 40\frac{m}{s}$$
$$Distance = Speed \times Time = 40 \times 8 = 320\ m$$

Example 3.5

Two trains A and B of length l and $4l$ are travelling into a tunnel of length L in parallel tracks from opposite directions with velocities $108\frac{km}{hr}$ and $72\frac{km}{hr}$, respectively. If train A takes $35s$ less time than train B to cross the tunnel then length of tunnel L, in meters, is (given $L = 60l$): (JEE Main, April 13, 2023 II)

$$\frac{64l}{20} - \frac{61l}{30} = 35$$
$$\frac{192l - 122l}{60} = 35$$
$$70l = 2100$$
$$l = \frac{2100}{70}$$
$$60l = \frac{2100}{70} \times 60 = \frac{2100}{7} \times 6 = 1800\ m$$

Example 3.6

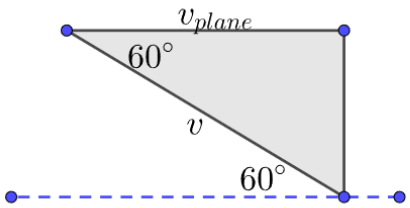
A person standing on open ground hears the sound of a jet aeroplane, coming from north at an angle 60° with ground level. But he finds the aeroplane right vertically above his position. If v is the speed of sound, speed of the plane, in terms of v is: (JEE Main, Jan 12 2019 II)

The time taken by the plane and sound is the same. Hence, the ratio of the speed is the ratio of the distances travelled.

$$\frac{v_{plane}}{v} = \cos 60° = \frac{1}{2} \Rightarrow v_{plane} = \frac{v}{2}$$

We can derive the property used as:

$$t_1 = t_2 \Rightarrow \frac{d_1}{s_1} = \frac{d_2}{s_2} \Rightarrow \frac{s_1}{s_2} = \frac{d_1}{d_2}$$



3.7: Objects moving in same directions

If two objects move in same directions with velocity $\vec{v}_1$ and $\vec{v}_2$ their relative speed is

$$v_1 - v_2, \qquad v_1 > v_2$$
$$v_2 - v_1, \qquad v_1 > v_2$$

Example 3.8

Two cars with speed A and B get $4m$ away from each other when travelling in directions which are exactly opposite to each other. When travelling in the same direction, they get $1m$ closer to each other.

$$A + B = 4$$
$$A - B = 1$$
$$2A = 5 \Rightarrow A = \frac{5}{2} = 2.5m$$
$$B = 4 - 2.5 = 1.5m$$

Example 3.9

A passenger train of length $60m$ travels at a speed of $80\frac{km}{hr}$. Another freight train of length 120m travels at a speed of $30\frac{km}{hr}$. The ratio of times taken by the passenger train to completely cross the freight train when (i) they are moving in same direction, and (ii) in the opposite direction is: (JEE Main, 12 Jan 2019, Shift-II)

Length to completely cross

$$= 120 + 60 = 180$$

$$Ratio = \frac{t_{Same\ Direction}}{t_{Opposite\ Direction}} = \frac{\frac{180\ m}{(80-30)\frac{km}{hr}}}{\frac{180\ m}{(80+30)\frac{km}{hr}}} = \frac{180\ m}{50\ \frac{km}{hr}} \cdot \frac{110\ \frac{km}{hr}}{180\ m} = \frac{11}{5}$$

B. Object inside another Object

3.10: Object moving inside another object

A Person moves with a speed of s_1 *units* inside a train travelling at a speed of s_2 units. For an observer on the ground, the speed is:

- $s_1 + s_2$ if the person moves in the direction of motion of the train
- $s_1 - s_2$ if the person moves against the direction of motion of the train

Example 3.11

Train A and train B are running on parallel tracks in opposite directions with speeds of $36\frac{km}{hr}$ and $72\frac{km}{hr}$, respectively. A person is walking in train A in the direction opposite to its motion with a speed of $1.8\frac{km}{hr}$ with respect to the train A. Speed in $\frac{m}{s}$ of this person as observed from train B will be: (take the distance between the tracks as negligible). **(JEE Main, 2 Sep, 2020, Shift-I)**

Net speed of person on the train:

$$= 36 - 1.8 = 34.2\frac{km}{hr}$$

The

$$72 + 34.2 = 106.2\frac{km}{hr}$$

$$106.2 \times \frac{5}{18} = 29.5\frac{m}{s}$$

Example 3.12

A person climbs up a stalled escalator in $60s$. If standing on the same escalator but escalator running with constant velocity, he takes 40s. How much time is taken by the person to walk up the moving escalator. **(JEE Main, April 12, 2014)**

Let the length (distance to be travelled) of escalator be 1. Speed on stalled escalator

$$= \frac{1}{t} = \frac{1}{60}$$

Speed on running escalator with standing person

$$= \frac{1}{t} = \frac{1}{40}$$

Combined speed

$$= \frac{1}{60} + \frac{1}{40} = \frac{2+3}{120} = \frac{5}{120} = \frac{1}{24}$$

Then:

$$T = \frac{D}{S} = \frac{1}{\frac{1}{24}} = 24s$$

Example 3.13

A boy reaches the airport and finds that the escalator is not working. He walks up the stationary escalator in time t_1. If he remains stationary on a moving escalator, then the escalator takes him up in time t_2. The time taken by him to walk up the moving escalator (in terms of t_1 and t_2) will be: **(JEE Main, July 20, 2021, Shift-II)**

Let the length (distance to be travelled) of escalator be 1. Speed on stalled escalator

$$= \frac{1}{t_1}$$

Speed on running escalator with standing person

$$= \frac{1}{t_2}$$

Combined speed

$$= \frac{1}{t_1} + \frac{1}{t_2} = \frac{t_2 + t_1}{t_1 t_2}$$

Then:

$$T = \frac{D}{S} = \frac{1}{\frac{t_2 + t_1}{t_1 t_2}} = \frac{t_1 t_2}{t_1 + t_2}$$

3.14: Medium

From the previous example, the time taken is half the harmonic mean of t_1 and t_2.

C. Medium

3.15: Medium

A medium is a frame of reference which changes the speed of an object relative to someone outside the frame of reference.

Example 3.16

A. Is a river a medium? Explain why or why not?
B. For a plane or a ship, is air a medium?

Part A

A river is a medium.
With reference to an observer on the river bank, it will increase the speed of a boat travelling downstream, and decrease the speed of a boat travelling upstream.

Part B

Air is a medium.
With reference to an observer on the ground, windspeed in the direction of travel will increase the speed, and windspeed against the direction of travel will decrease the speed.

Example 3.17

A boat can go at a speed s_1 upstream in a river and s_2 downstream. Find the speed of the boat and the river in terms of s_1 and s_2.

r

Let speed of

$$boat = b, \qquad river = r$$

When the boat is travelling upstream, it travels against the direction of the current, so its speed is reduced by the speed of the river:

$$\underbrace{b - r = s_1}_{Equation\ I}$$

Conversely, when the boat is travelling downstream, it travels with the current, so its speed is increased by the speed of the river:

$$\underbrace{b + r = s_2}_{Equation\ II}$$

Add Equations I and II:

$$2b = s_1 + s_2 \Rightarrow b = \frac{s_1 + s_2}{2}$$

Subtract Equation II from Equation I

$$2r = s_2 - s_1 \Rightarrow r = \frac{s_2 - s_1}{2}$$

Example 3.18: Queuing Theory

A toll booth is set up at a sea link connecting two parts of a city. Cars can pass through the toll booth at a speed of $v\frac{m}{s}$. Cars on the road travel at a speed of $kv\frac{m}{s}$. Cars have a length of $l\ meters$, and two cars need a buffer of $b\ meters$ between them for safety considerations.

A. Under practical conditions, what is the range of values for k.
B. What is the minimum distance D between the fronts of two cars.
C. Draw a diagram showing a queue of q cars and one car approaching the queue at a distance d.
D. What should be the distance d between two cars on the road so that q neither increases nor decreases.

Part A

Cars on the road should travel faster than cars through the toll booth. Hence:

$$k > 1$$

Part B

$$D = l + b$$

Part C

Cars in the queue are moving at

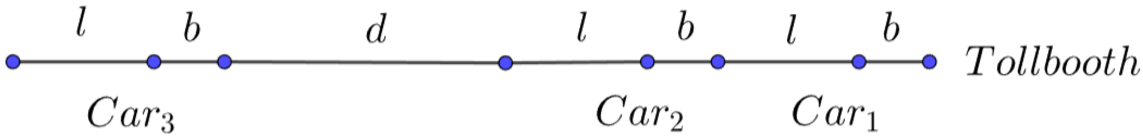
$$v\frac{m}{s}$$

Cars on the road are moving at

$$kv\frac{m}{s}$$

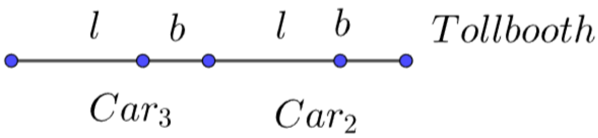
From the perspective of a car in the queue, the cars on the road are moving at a relative velocity of:

$$(kv - v) = (k-1)v\ \frac{m}{s}$$



The time taken for a car in the queue to cover the distance D, and move ahead one spot in the queue is:

$$T = \frac{Distance}{Speed} = \frac{D}{v} = \frac{l+b}{v}s$$



In the same time that a car in queue moves ahead one spot, a car on the road must cover distance d to join the queue, and hence maintain the queue length:

$$d = \underbrace{(k-1)v\,\frac{m}{s}}_{Speed} \cdot \underbrace{\frac{l+b}{v}s}_{Time} = (k-1)(l+b)\,m$$

D. Frame of Reference

3.19: Frame of Reference

A system which is moving at the same speed is a frame of reference.

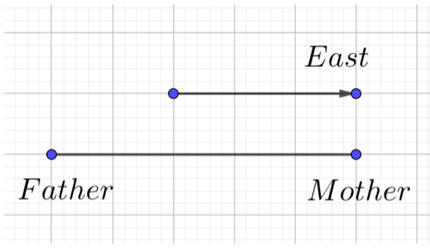
- The earth is a frame of reference.
- A moving train is a frame of reference.
- A river is a frame of reference.

Example 3.20

A girl is travelling in a metro train in the first compartment (immediately after the engine, which is at the front of the train) with her father. She wants to go to her mother, who is in the last compartment. The distance between mother and father is 20 meters. The girl runs at $1\frac{m}{s}$. The train is travelling at a constant speed of $80\frac{m}{s}$ due West.

Find the velocity of the girl with respect to:

A. her mother
B. her father
C. a bystander on the platform



With reference to both the mother and the father, the girl is moving:

$$1\frac{m}{s}East = 1\frac{m}{s}$$

For the bystander, we need to add the velocities:

$$-80\frac{m}{s} + 1\frac{m}{s} = -79\frac{m}{s}$$

Example 3.21

A rocket is moving in a gravity free space with a constant acceleration of $2\frac{m}{s^2}$ along x direction. The length of a chamber inside the rocket is $4m$. A ball is thrown from the left end of the chamber in $+x$ direction with a speed of $0.3\frac{m}{s}$ relative to the rocket. At the same time another ball is thrown in $-x$ direction with a speed of $0.2\frac{m}{s}$ from its right end relative to the rocket. The time in seconds when the two balls hit each other is: **(JEE Adv. 2014)**

Consider the rocket as a frame of reference, and assume that the balls continue to have constant acceleration of $2\frac{m}{s^2}$ even after being thrown.

Relative velocity

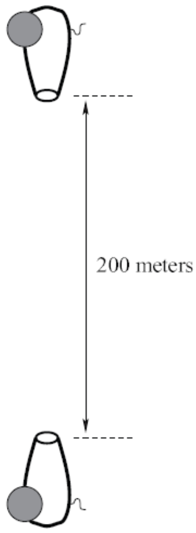
$$= 0.3 - (-0.2) = 0.5$$

$$T = \frac{d}{s} = \frac{4}{0.5} = 8\ seconds$$

(Calc) Example 3.22

Two cannons are arranged vertically, with the lower cannon pointing upward (towards the upper cannon) and the upper cannon pointing downward (towards the lower cannon), $200m$ above the lower cannon. Simultaneously, they both fire. The muzzle velocity of the lower cannon is $25\frac{m}{s}$ and the muzzle velocity of the upper cannon is $55\frac{m}{s}$.

A. How long after the cannons fire do the projectiles collide? **(F=ma 2012/6)**
B. How far beneath the top cannon do the projectiles collide? **(F=ma 2012/7)**



Method I: Frame of Reference

Both cannonballs are in the frame of reference of Earth, and both are influenced by gravity. Hence, we can ignore gravity for the purposes of our calculations.

Part A

Relative velocity is

$$t = \frac{s}{v_{avg}} = \frac{200}{25-(-55)} = \frac{200}{25+55} = \frac{200}{80} = 2.5\ s$$

Part B

$$-55 - \frac{1}{2}gt^2\Big|_{t=2.5,g=9.8} = -168.125\ m$$

The distance beneath the cannon is:

$$168.125\ m$$

Method I

If you do not ignore gravity, you get the same answer, as shown below.

Part A

Relative velocity is

$$\underbrace{25 - \frac{1}{2}gt^2}_{v_{Lower}} - \underbrace{\left(-55 - \frac{1}{2}gt^2\right)}_{v_{Upper}} = 25 - \frac{1}{2}gt^2 + 55 + \frac{1}{2}gt^2 = 25 + 55 = 80\frac{m}{s}$$

$$s = v_{avg}t \Rightarrow t = \frac{s}{v_{avg}} = \frac{200}{80} = 2.5\ s$$

3.2 Relative Motion in 2D

A. General Concepts

Example 3.23

A person aiming to reach exactly opposite point on the bank of a stream is swimming with a speed of 0.5 m/s at an angle of 120° with the direction of flow of water. The speed of water in the stream, is (NEET 1999)

0.25 m/s

Example 3.24

A swimmer wishes to travel across a river with speed r km/hr using the shortest possible path. Determine the minimum possible speed of the swimmer.

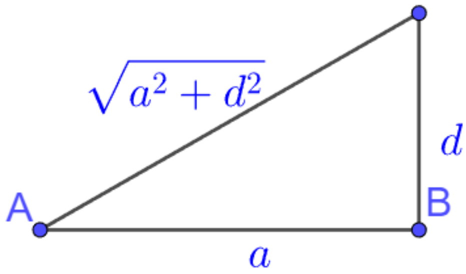
Example 3.25

Two particles A and B are connected by a rigid rod AB. The rod slides along perpendicular rails as shown here. The velocity of A to the left is 10 m/s. What is the velocity of B when angle a = 60°? (NEET 1998)

Example 3.26

Two boys are standing at the ends A and B of a ground where $AB = a$. The boy at B starts running in a direction perpendicular to AB with velocity v_1. The boy at A starts running simultaneously with velocity v and catches the other in a time t. Find t in terms of a, v and v_1 only. **(NEET 2005)**

Method I: Using TSD Concepts



Let the distance travelled by boy at point B be d.

Using $Speed = \frac{Distance}{Time}$:

$$v = \frac{\sqrt{a^2+d^2}}{t} \Rightarrow v^2t^2 = a^2 + d^2 \Rightarrow \underbrace{d^2 = v^2t^2 - a^2}_{Equation\ I}$$

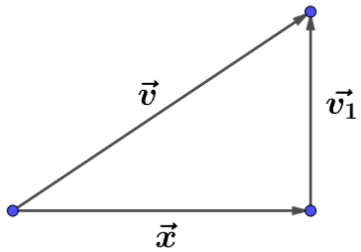
$$v_1 = \frac{d}{t} \Rightarrow d = v_1 t \Rightarrow \underbrace{d^2 = v_1^2 t^2}_{Equation\ II}$$

Equate the RHS of Equations I and II:

$$v^2t^2 - a^2 = v_1^2t^2$$
$$t^2(v^2 - v_1^2) = a^2$$
$$t = \sqrt{\frac{a^2}{v^2 - v_1^2}} = \frac{a}{\sqrt{v^2 - v_1^2}}$$

Method II: Using Vectors

$$\vec{v} = \vec{x} + \vec{v}_1$$



Square both sides:

$$|\vec{v}|^2 = |\vec{x} + \vec{v}_1|^2$$
$$|\vec{v}|^2 = |\vec{x}|^2 + |\vec{v}_1|^2 + 2|\vec{x}| \cdot |\vec{v}_1|$$

Since $\vec{x}$ and $\vec{v}_1$ are perpendicular, their dot product is zero:

$$|\vec{v}|^2 = |\vec{x}|^2 + |\vec{v}_1|^2$$
$$v^2 = x^2 + v_1^2$$
$$x^2 = v^2 - v_1^2$$

Hence, the horizontal component of the vector $\vec{v}$ has magnitude:

$$x = \sqrt{v^2 - v_1^2}$$

Using $Speed = \frac{Distance}{Time}$:

$$v = \frac{a}{x} = \frac{a}{\sqrt{v^2 - v_1^2}}$$

B. River-Swimmer Problems

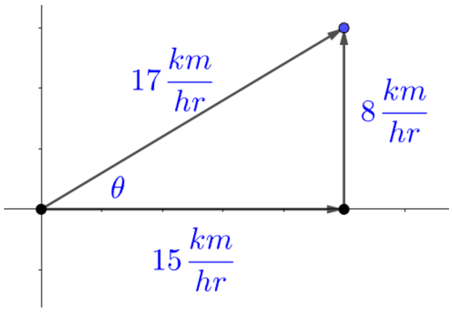
Example 3.27

A river with a width of 1 km flows due east at a speed of $15\frac{km}{hr}$. A swimmer with a speed of $8\frac{km}{hr}$ in still water wishes to cross the river. He jumps, ignoring the current. He expected to find himself at the point opposite where he jumped. Where he actually find himself?

A. Did he find himself upstream or downstream of where he jumped? By how many km?
B. To an observer on the bank of the river what is the swimmer's relative velocity.
C. To an observer on a boat (which is not being rowed), what is the swimmer's velocity.

Part A

$$Distance = Speed \times Time = (15\ km)\left(\frac{1}{8}\right) = \frac{15}{8}\ km$$



Part B

Using Pythagorean Triplet (8,15,17)

$$|\vec{v}| = v = 17\frac{km}{hr}$$
$$\theta = \tan^{-1}\frac{8}{15}$$

Part C

$$|\vec{v}| = v = 8\frac{km}{hr}$$
$$\theta = 90°$$

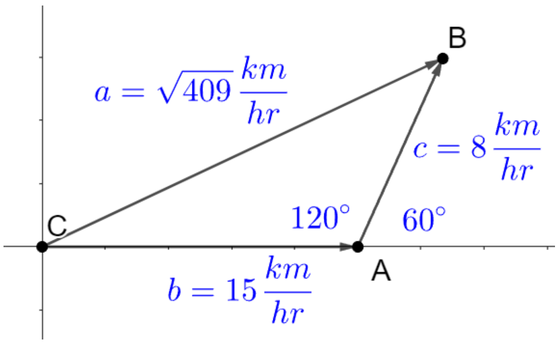
Example 3.28

A river with a width of 1 km flows due east at a speed of $15\frac{km}{hr}$. A swimmer with a speed of $8\frac{km}{hr}$ in still water jumps into the river at an angle of 60° with the current.

A. To an observer on the bank of the river, what is the swimmer's relative speed.
B. On the other bank, did the swimmer find himself upstream or downstream of where he jumped? By how many km?

Part A

The swimmer's relative velocity is the resultant vector from adding the velocity vector for the current and the velocity vector for the swimmer's movement.



Use the law of cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Substitute $b = 15, c = 8, A = 120°$ in the above

$$a^2 = 15^2 + 8^2 - 2(8)(15)\left(-\frac{1}{2}\right) = 40 \Rightarrow a = \sqrt{409}\frac{km}{hr}$$

Part B

Determine the horizontal and vertical components of the swimmer's velocity:

$$Horizontal\ component = c\cos 60° = 8\left(\frac{1}{2}\right) = 4\frac{km}{hr}$$

$$Vertical\ Component = c\sin 60° = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}\frac{km}{hr}$$

Swimmer's speed in the direction of the river flow (using the horizontal component)

$$= 15\frac{km}{hr} + 4\frac{km}{hr} = 19\frac{km}{hr}$$

Time taken to cross the river (using river speed + vertical component)

$$= \frac{Distance}{Speed} = \frac{1}{4\sqrt{3}} = \frac{\sqrt{3}}{12}hr$$

Horizontal distance travelled by the swimmer (using time from previous step):

$$Distance = Speed \times Time = (19\ km)\left(\frac{\sqrt{3}}{12}\right) = \frac{19\sqrt{3}}{12}\ km$$

Example 3.29

A river of width w flows at a speed of $r\frac{m}{s}$. A girl with a speed of $s\frac{m}{s}$ in still water jumps into the river at an angle θ to the river bank. Find an expression for the positive difference between the two possible values of the speed of the girl as seen by an observer on the river bank.

Using the law of cosines, the first speed (*blue triangle*):

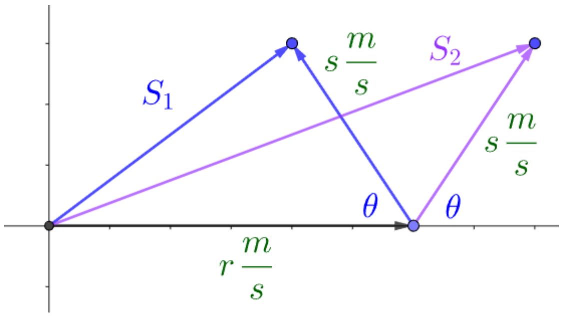
$$S_1 = r^2 + s^2 - 2rs\cos\theta$$

Using the law of cosines, the second speed (*violet triangle*):

$$S_2 = r^2 + s^2 - 2rs\cos(180 - \theta) = r^2 + s^2 + 2rs\cos\theta$$

The difference in the speeds is then:

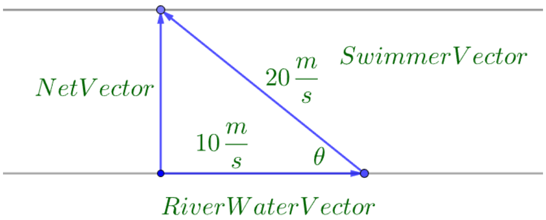
$$S_2 - S_1 = 4rs\cos\theta$$



Example 3.30

The speed of a swimmer in still water is $20\frac{m}{s}$. The speed of river water is $10\frac{m}{s}$ and is flowing due east. If he is standing on the south bank and wishes to cross the river along the shortest path, the angle at which he should make his strokes

A. with respect to north is $x°$ *west*. Find x. **(NEET 2019)**
B. with respect to river water flow.



$$\cos\theta = \frac{10}{20} = \frac{1}{2} \Rightarrow \theta = 60°$$

Part A

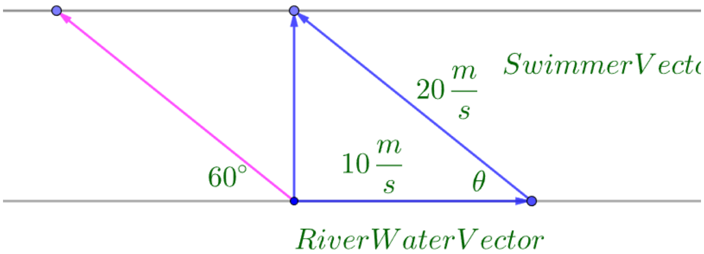
The angle with north is

$$30°\ West \Rightarrow x = 30$$

Part B

The angle with respect to river water flow is the angle between vectors

$$= 180 - 60° = 120°$$



Example 3.31

The width of river is 1 km. The velocity of boat is $5\frac{km}{hr}$. The velocity of river is $4\frac{km}{hr}$.

A. Find the minimum possible time t to cross the river.
B. Find the distance travelled when crossing in time t.

If you want to minimize travel time, the entire focus should be maximizing the vertical component of travel.

$$t = \frac{d}{s} = \frac{1}{5}hr = 12\ min$$

Example 3.32

Find the velocity of the river stream if a boat with velocity 5 km/hr traversed the distance between the banks (1 km) using the shortest possible path in 15 min.

C. Frame of Reference

Example 3.33

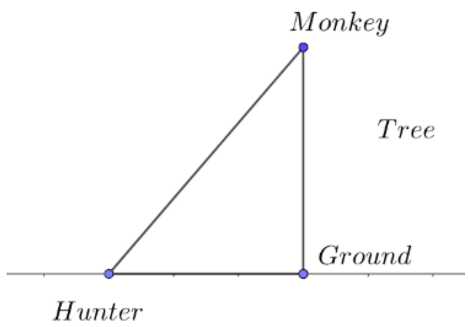
When we throw a ball and wish to calculate data related to its trajectory, why can we ignore the movement of the Earth?

We think of the Earth as a Frame of reference.
Both the ground and the ball have same acceleration and hence we can ignore it.

Example 3.34: Hunter and the Monkey

A hypothetical hunter on the ground is aiming at a real monkey hanging on a tree of height h. The hunter fires a bullet, aiming directly at the monkey. At the same time the monkey loses its grip on the tree, and starts falling. With reference to the monkey, at what place on the tree will the bullet hit?
Note: Ignore the width between the monkey and the tree.

Once the bullet is fired:
- Monkey is accelerating at $-g\frac{m}{s^2}$
- Bullet is accelerating at $-g\frac{m}{s^2}$



Acceleration on bullet is same as acceleration on monkey. We can introduce a frame of reference that includes only monkey and bullet. Since both have same acceleration, we can ignore acceleration on bullet if at the same time we ignore acceleration on monkey.

Hence, we can think of monkey and bullet on a coordinate plane:
- If you ignore acceleration on monkey, it will remain at the same position on the coordinate plane.
- If you ignore acceleration on the bullet, it will travel in straight line motion and hit the monkey.

Example 3.35

F=ma 2015/5

Example 3.36

A bomb is dropped by fighter plane flying horizontally. To an observer sitting in the plane, the trajectory of the bomb is a:

E. Hyperbola
F. Parabola in the direction of motion of the plane
G. Straight line vertically down the plane
H. Parabola is a direction of motion opposite the plane (JEE Main, Aug 26, 2021-Shift-I)

$$Option\ C: Vertically\ Down$$

D. Motion with Vectors

Example 3.37

Ship P is at $50\hat{j}\ km$ and has velocity $(20\hat{\imath} + 10\hat{j})\frac{km}{h}$. Ship Q is $(80\hat{\imath} + 20\hat{j})\ km$ and has velocity $(-10\hat{\imath} + 30\hat{j})\ km$. Will P and Q meet?

Consider Ship P

$$\vec{\boldsymbol{P}} = \underbrace{(0,50)}_{\substack{Starting\\Position}} + \underbrace{t(20,10)}_{\substack{Velocity\\Vector}} = (20t, 50 + 10t)$$

Consider Ship Q

$$\vec{\boldsymbol{Q}} = \underbrace{(80,20)}_{\substack{Starting\\Position}} + t\underbrace{(-10,30)}_{\substack{Velocity\\Vector}} = (80 - 10t, 20 + 30t)$$

For the ships to meet, they must be at the same point in the coordinate plane. Which means they must have the same x and y coordinates:

$$20t = 80 - 10t \Rightarrow 30t = 80 \Rightarrow t = \frac{8}{3}$$
$$50 + 10t = 20 + 30t \Rightarrow 30 = 20t \Rightarrow t = \frac{3}{2} \neq \frac{8}{3}$$

Since the value of t is not equal, the two ships will not have same x and y coordinates at the same time. Which means they will not meet.

(Calculator) Example 3.38

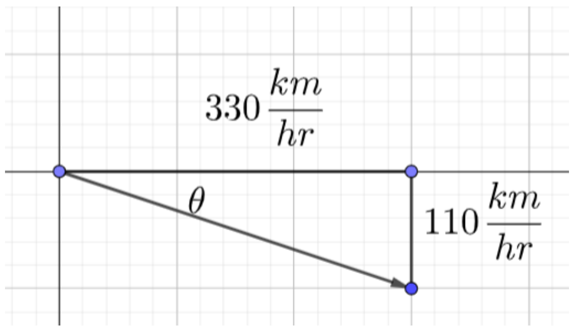
A plane flies from P to Q, which are 273 km apart. The velocity, in still air of the plane is $(280\hat{\imath} - 40\hat{\jmath})\frac{km}{hr}$ and there is a constant wind blowing with velocity $(50\hat{\imath} - 70\hat{\jmath})\frac{km}{hr}$. Find:

A. The bearing of Q from P
B. The time of flight

$$\overrightarrow{\boldsymbol{v_{Net}}} = (280\hat{\boldsymbol{\imath}} - 40\hat{\boldsymbol{\jmath}}) + (50\hat{\boldsymbol{\imath}} - 70\hat{\boldsymbol{\jmath}}) = (330\hat{\boldsymbol{\imath}} - 110\hat{\boldsymbol{\jmath}})\frac{km}{hr}$$

$$\tan\theta = \frac{110}{330} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{3}\right) = 18.43°$$
$$Bearing = 90 + 18.43 = 108.43$$

$$T = \frac{D}{S} = \frac{273\ km}{\sqrt{330^2 + 110^2}\frac{km}{hr}} = \frac{273}{110\sqrt{10}}$$



Which we can approximate to:

$$\approx 0.7848\ hr \approx 47.08\ min \approx 47\ min$$

Example 3.39

A fly starting at $(\hat{\imath} + 12\hat{\jmath})$ has velocity $(3\hat{\imath} + 2\hat{\jmath})\frac{cm}{s}$. A spider at $(85\hat{\imath} + 5\hat{\jmath})\ cm$ starts moving at the same time with velocity $(-5\hat{\imath} + k\hat{\jmath})\frac{cm}{s}$, where k is a constant. Given that the spider catches the fly, find the value of k.

$$\overrightarrow{\boldsymbol{Fly}} = (1,12) + t(3,2) = (1 + 3t, 12 + 2t)$$
$$\overrightarrow{\boldsymbol{Spider}} = (85,5) + t(-5,k) = (85 - 5t, 5 + kt)$$

$$1 + 3t = 85 - 5t \Rightarrow 8t = 84 \Rightarrow t = 10.5$$

Substitute $t = 10.5$ in the equations for the y coordinates:

$$12 + 2(10.5) = 5 + t(10.5) \Rightarrow 28 = 10.5k \Rightarrow k = \frac{8}{3}$$

Example 3.40

A plane flies from P to Q in 4 hours. If $PQ = (960\hat{\imath} + 400\hat{\jmath})\ km$ and a wind is blowing with velocity $(60\hat{\imath} + 60\hat{\jmath})\ km$ find, for the plane:

A. The velocity in still air
B. The bearing

$$T = \frac{D}{S}$$
$$4 = \frac{\sqrt{960^2 + 400^2}}{S}$$
$$S = \frac{1040}{4} = 260$$

$$\boldsymbol{v_{net}} = \frac{Displacement}{Time} = \frac{(960,400)}{4} = (240,100)$$

$$\boldsymbol{v_{still\ air}} + \boldsymbol{v_{wind}} = \boldsymbol{v_{net}}$$
$$(x, y) + (60,60) = (240,100)$$
$$\boldsymbol{v_{still\ air}} = (x, y) = (180,40)$$

$$\tan\theta = \frac{40}{180} \Rightarrow \theta = \tan^{-1}\left(\frac{2}{9}\right)$$
$$Bearing = 90 -$$

3.3 Further Topics

41 Examples