TRIANGLES

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1. Triangles

1.1 Angles

A. Terminology and Basics

A triangle is a closed three-sided figure.
It has three sides, three angles and three vertices.

Based on Angles	Type of Triangle	Properties
All the angles are acute	Acute-angled	
One angle is obtuse	Obtuse-angled	You cannot have a second obtuse angle
One angle is a right-angle	Right Angled	Remaining two angles are complementary
Based on Sides		
No two sides are equal	Scalene	
At least two sides are equal	Isosceles	Angles opposite equal sides are equal
All three sides are equal	Equilateral	Angles are also equal - 60 degrees An Equilateral triangle is also isosceles

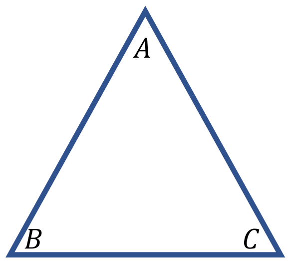
1.1: Sum of Angles of a Triangle
The sum of angles of a triangle is $\mathbf{180°}$

B. Numerical Applications

We can use the property of the sum of angles of a triangle to find missing values.

Example 1.1
See the diagram alongside (which is not drawn to scale), and then answer each question independently.

1. If $\angle A = 65°$, and $\angle B = 70°$, then find the measure of $\angle C$.
2. If $\angle B = 55°$, and $\angle C = 58°$, then find the measure of $\angle A$.
3. If $\angle \boldsymbol{A} = \mathbf{87}°$, and $\angle \boldsymbol{C} = \mathbf{48}°$, then find the measure of $\angle \boldsymbol{B}$.

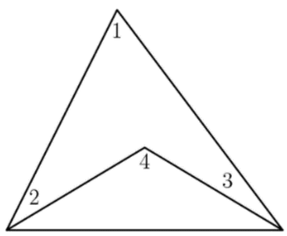


Example 1.2
Find the values of the missing angle in a triangle with angles:

A. $35, 45, b$
B. $25, 75, c$
C. $90, 60, a$

Example 1.3: Finding Angles

A. The measures of the three interior angles of a triangle are 50, 55 and x. What is the degree measure of the largest interior angle of this triangle? (MathCounts 2006 School Sprint)
B. What is the measure of angle 4 if $m\angle 1 = 76$, $m\angle 2 = 27$ and $m\angle 3$=17? (MathCounts 2008 State Countdown)



Part A

$$x = 180 - 50 - 55 = 75$$

Part B

Let's assume that angles

$$\angle 1+\angle 2+\angle 3=x, \qquad \angle 5+\angle 6=y$$

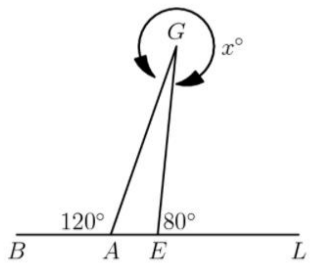
By angles in a triangle:

$$x+y=180 \Rightarrow y=180-x=60$$
$$\angle 4=180-y=180-60=120$$

$$Reflex\ Angle\ of\ \angle 4=360-76-27-17=240$$
$$\angle 4=360-240=120$$

Example 1.4

Four points B, A, E, and L are on a straight line, as shown. The point G is off the line so that $\angle BAG=120$ and $\angle GEL=80$. If the reflex angle at G is x, then what does x equal? **(CEMC 2005 Gauss 8)**



$$\angle A=180-120=60$$
$$\angle E=180-80=100$$
$$\angle G=180-60-100=20$$
$$Reflex\ \angle G=360-20=340$$

Example 1.5

How many triangles exist with two right angles?

Zero

Example 1.6

Find the number of right triangles with angles of integral measure, no two of which are equal.

We can tabulate all the possibilities like this:

First Angle	1°	2°	.	.	.	44°
Second Angle	89°	88°	.	.	.	46°

And hence, the number of triangles is:

$$44$$

C. Conceptual Applications

The sum of angles of a triangle limits values that the angles can take. These are important in ruling out cases which are not possible.

Example 1.7

In a triangle, what is the maximum number of:

A. Acute Angles
B. Right Angles
C. Obtuse Angles

Part A

All three angles can be acute. For example, consider:

$$\Delta ABC: \angle A=\angle B=\angle C=60°$$

Part B

A triangle with one right angle is possible. For example, consider:

$$\Delta ABC: \angle A = 90°, \angle B = \angle C = 45°$$

Consider a triangle with two right angles

$$\Delta ABC: \angle A = \angle B = 90° \Rightarrow \angle A + \angle B = 180° \Rightarrow \angle C = 0° \Rightarrow \textit{Not Valid}$$

Hence, two right angles are not possible.

Part C

A triangle with one obtuse angle is possible. For example, consider:

$$\Delta ABC: \angle A = 100°, \angle B = \angle C = 40°$$

Consider a triangle with two obtuse angles:

$$\Delta ABC: \angle A > 90°, \angle B > 90° \Rightarrow \angle A + \angle B > 180° \Rightarrow \textit{Not Valid}$$

Hence, two obtuse angles are not possible.

Example 1.8

A. If all the angles of a triangle are acute and equal, then find the measure of each angle in the triangle.
B. How many triangles with two obtuse angles exist?
C. If one angle in a triangle is obtuse, then the other two angles must be:
D. How many triangles exist such that one angle is 115°, and the second angle is 89°?
E. How many triangles can be formed such that exactly one of their angles is acute?

Part A

$$3x = 180 \Rightarrow x = 60$$

Part B

$$a + b + c = 180 \Rightarrow c = 180 - (a + b)$$

Since a and b are both obtuse, we must have

$$a > 90, b > 90 \Rightarrow a + b > 180$$

Hence,

$$c = 180 - (a + b) \Rightarrow \textit{Negative} \Rightarrow \textit{Not Possibe}$$

Hence, the number of such triangles is zero.

Part C

Consider a triangle with an obtuse angle:

$$\Delta ABC: \angle A > 90°, \angle B + \angle C < 90° \Rightarrow \angle B, \angle C \textit{ are acute}$$

Hence, the remaining two angles must be acute.

Part D

Consider a triangle with two of the angles given above:

$$\Delta ABC: \angle A = 115°, \angle B = 89° \Rightarrow \angle A + \angle B = 115 + 89 = 204° > 180° \Rightarrow \textit{Not Valid}$$

Part E

Let the measure of the angles be a, x and y, where a is the acute angle. Then:

$$a + x + y = 180$$

Since neither x nor y are acute, the minimum value they can take is 90°. Substitute this in the above equation:

$$a + 90 + 90 = 180 \Rightarrow a + 180 = 180 \Rightarrow a = 0° \Rightarrow \textit{Not Valid}$$

Hence, even if we take the minimum value of 90° for the remaining two angles, we do not get a valid triangle. Hence, such a triangle is not possible.

There are zero such triangles.

Example 1.9: Ratios

A. The angles in a triangle are in the ratio 1:2:3. Find the angles.
B. The measures of the interior angles of a particular triangle are in a 5:6:7 ratio. What is the measure, in degrees, of the smallest interior angle? (MathCounts 2004 School Sprint)
C. If the degree measures of the angles of a triangle are in the ratio $3:3:4$, what is the degree measure of the largest angle of the triangle? (AMC 8 2017/6)
D. The angle measures of the three angles of a triangle are in the ratio 1:3:6. What is the number of degrees in the measure of the largest angle? (MathCounts 2002 Chapter Countdown)
E. The measures of the angles of a triangle are in a ratio of 3:5:7. What is the degree measure of the largest angle? (MathCounts 2009 Chapter Countdown)
F. The measures of the angles of a triangle are in the ratio 5:6:7. What is the number of degrees in the largest of these angles? (MathCounts 1992 Warm-Up 11)
G. The acute angles in a right-angled triangle are in the ratio 1:3. Find the angles.
H. The ratio of the measures of the acute angles of a right triangle is 8:1. In degrees, what is the measure of the largest angle of the triangle? (MathCounts 2010 School Countdown)
I. The sum of two angles of a triangle is $\frac{6}{5}$ of a right angle, and one of these two angles is $30°$ larger than the other. What is the degree measure of the largest angle in the triangle? (AMC 10B 2011/7)

Part A

$$\frac{180}{1+2+3}=\frac{180}{6}=30$$

Angles are

$$30\times 1=30, 30\times 2=60, 30\times 3=90$$

Part B

$$5:6:7\to 5+6+7=18$$
$$\frac{180}{18}=10\to 10\times 5=50$$

Part C

$$3:3:4\to 3+3+4=10$$
$$\frac{180}{10}=18\to 18\times 4=72$$

Part D

1+3+6=10

180/10*18 = 108

Part E

$$3:5:7\to 3+5+7=15$$
$$\frac{180}{15}=12\to 12\times 7=84$$

Part F

$$5+6+7=18$$
$$Largest\ angle=\frac{180}{18}\times 7=70$$

Part G

$$\frac{90}{1+3}=\frac{90}{4}=22.5$$

Angles are $22.5*1=22.5, 22.5*3=67.5$

Part H

90

Part I

Let the angles be

$$a, a+30$$

We know that

$$a+(a+30)=\frac{6}{5}\times 90$$
$$2a+30=108$$
$$a=39\Rightarrow a+30=69$$

Which means that the third angle in the triangle is:

$$180-108=72°\Rightarrow Largest\ Angle=72°$$

Example 1.10: Prime Numbers

The acute angles of a right triangle are $a°$ and $b°$, where $a>b$ and both a and b are prime numbers. What is the least possible value of b? (AMC 10B 2020/4)

In a right triangle, the acute angles must add up to:

$$180-90=90°$$

Hence, we want:

$$a + b = 90, \qquad a, b \text{ both prime}$$

Now, try values of a, b, starting with the smallest. We need only consider prime values for b:

$$b = 2 \Rightarrow a = 88 \Rightarrow Not\ prime$$
$$b = 3 \Rightarrow a = 87 \Rightarrow Not\ prime$$
$$b = 5 \Rightarrow a = 85 \Rightarrow Not\ prime$$
$$b = 7 \Rightarrow a = 83 \Rightarrow Prime \Rightarrow Smallest$$

Example 1.11: Percentage

A. Find the measures of the angles in ΔABC if, $\angle A$ is 50% more than $\angle B$, and $\angle C$ is 50% less than $\angle B$.
B. Find the measure of $\angle B$ in ΔABC if, $\angle A$ is d% more than $\angle B$, and $\angle C$ is d% less than $\angle B$.
C. Find the measures of the angles in ΔABC if, $\angle B$ is 50% of $\angle A$, and $\angle C$ is 50% of $\angle B$.

Part A

Let the measure of

$$\angle B = b \Rightarrow \angle A = 1.5b \Rightarrow \angle C = 0.5b$$
$$b + 1.5b + 0.5b = 180 \Rightarrow 3b = 180 \Rightarrow b = 60$$

Part B

$$\angle B = b$$

$\angle A$ is d% more than $\angle A$

$$\angle A = b + \frac{d}{100} \times b = b + \frac{db}{100}$$

$\angle C$ is d% less than $\angle C$

$$\angle C = b - \frac{d}{100} \times b = b - \frac{db}{100}$$

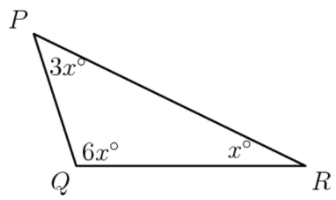
$$\underbrace{b}_{\angle B} + \underbrace{b + \frac{db}{100}}_{\angle A} + \underbrace{b - \frac{db}{100}}_{\angle C} = 180 \Rightarrow 3b = 180 \Rightarrow \angle B = b = 60$$

Part C

Let the measure of

$$\angle C = c \Rightarrow \angle B = 2c \Rightarrow \angle A = 4c$$
$$c + 2c + 4c = 180 \Rightarrow 7c = 180 \Rightarrow \angle C = c = \frac{180}{7} \Rightarrow \angle B = 2c = \frac{360}{7} \Rightarrow \angle A = 4c = \frac{720}{7}$$

Example 1.12: Linear Equations



A. What, in degrees, is the measure of the largest angle in ΔPQR? **(CEMC 2010 Cayley)**
B. In triangle ABC, the measure of angle A is x degrees, the measure of angle B is $2x$ degrees and the measure of angle C is $5x$ degrees. What is the value of x? Express your answer as a decimal to the nearest tenth. **(MathCounts 2008 School Countdown)**
C. In triangle ABC, the measure of $\angle A$ is 86 degrees. The measure of $\angle B$ is 22 degrees more than three times the measure of $\angle C$. What is the measure, in degrees, of $\angle C$? **(MathCounts 2010 State Team)**
D. The angles in a triangle are $2x + 5$, $x - 10$, and $3x + 65$. Find their ratio.
E. Find the measures of the three angles in a triangle if the difference in measure of two of the angles is five degrees, and the sum of these same two angles is three more than twice the third angle.

Part A

$$3x + 6x + x = 180 \Rightarrow x = 18 \Rightarrow 6x = 108$$

Part B

$$x + 2x + 5x = 180 \Rightarrow x = 22.5$$

Part C

$$86 + (3c + 22) + c = 180 \Rightarrow c = 18$$

Part D

$$(2x + 5) + (x - 10) + (3x + 65) = 180 \Rightarrow 6x + 60 = 180 \Rightarrow x = 20 \Rightarrow 45:10:125 \; = \; 9:2:25$$

Part E

Let

$$1st\ Angle \; = x \Rightarrow 2nd\ Angle = x + 5$$

Let

$$3rd\ Angle = y$$
$$x + (x + 5) = 2y + 3 \Rightarrow y = x + 1$$

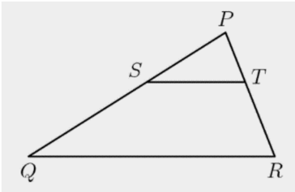
And we also know that the sum of angles in a triangle is 180°:

$$2x + 5 + x + 1 = 180 \Rightarrow x = 58$$

D. Parallel Lines

Example 1.13

In triangle PQR, point T is on PR and point S is on PQ such that $TS \parallel RQ$. The measure of $\angle RPQ$ is 65, and the measure of $\angle TSQ$ is 145. What is the measure of $\angle PRQ$? **(MathCounts 2009 Warm-up 9)**



Draw a diagram.

By co-interior angles:

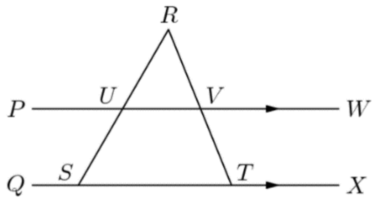
$$\angle SQR = 180 - 145 = 35$$

By sum of angles in a triangle:

$$\angle PRQ = 180 - 65 - 35 = 80$$

Example 1.14

In the diagram, PW is parallel to QX, S and T lie on QX, and U and V are the points of intersection of PW with SR and TR, respectively. If $\angle SUV = 120$ and $\angle VTX = 112$, what is the measure of $\angle URV$? **(CEMC 2010 Pascal)**



By co-interior angles:

$$\angle RST = 180 - 120 = 60$$

By angles in a linear pair:

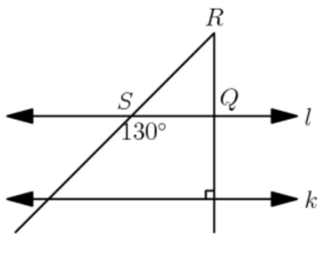
$$\angle RTS = 180 - 112 = 68$$

By angles in a triangle:

$$\angle URV = \angle SRV = 180 - 60 - 68 = 52$$

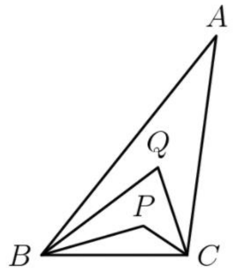
Example 1.15

In the diagram, $l \parallel k$. What is the number of degrees in $\angle SRQ$? **(MathCounts 1997 School Sprint)**



$$\angle RSQ = 180 - 130 = 50$$
$$\angle RQS = 90$$
$$\angle SRQ = 180 - 90 - 50 = 40$$

E. Bisectors and Trisectors



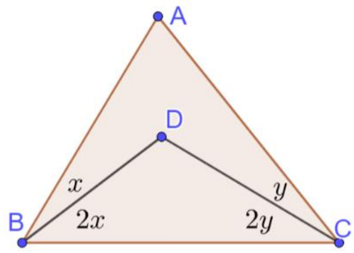
Example 1.16

The trisectors of angles B and C of scalene triangle ABC meet at points P and Q, as shown. Angle A measures 39 degrees and angle QBP measures 14 degrees. What is the measure of angle BPC? **(MathCounts 2008 State Sprint)**

$$\angle ABC = 3 \times 14 = 42$$
$$\angle ACB = 180 - 42 - 39 = 99$$
$$\angle PCB = \frac{99}{3} = 33$$
$$\angle BPC = 180 - 14 - 33 = 133$$

Example 1.17

In triangle ABC,$\angle A = 120$. A point D is inside the triangle such $\angle DBC = 2\angle ABD$ and $\angle DCB = 2\angle ACD$. Find $\angle BDC$.

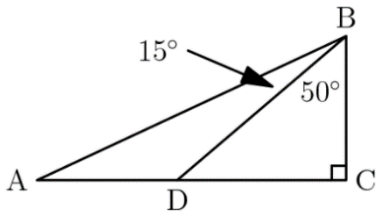


$$\angle ABC + \angle ACB = 180 - 120 = 60°$$
$$\angle ABC + \angle ACB = 3x + 3y$$

$$3x + 3y = 60 \Rightarrow x + y = 20 \Rightarrow \underbrace{2x + 2y}_{\angle DBC + \angle DCB} = 40$$

$$\angle BDC = 180 - (\angle DBC + \angle DCB) = 180 - 40 = 140$$

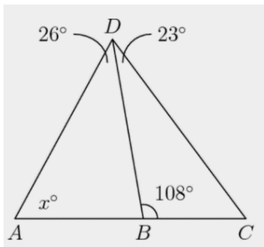
F. Nested Triangles



Example 1.18

Point D is on side AC of triangle ABC, $\angle ABD = 15$ and $\angle DBC = 50$. What is the measure of angle BAD, in degrees? **(MathCounts 2006 Chapter Countdown)**

$$\angle BAD = 180 - 90 - 50 - 15 = 25$$



Example 1.19

Find the number of degrees in the measure of angle x. **(MathCounts 1995 School Sprint)**

By sum of angles in ΔDBC:

$$\angle DCB = 180 - 108 - 23 = 49$$

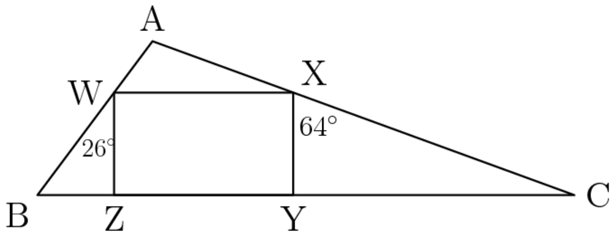
By sum of angles in ΔDAC:

$$x = \angle DAB = 180 - 26 - 23 - 49 = 82$$

G. Quadrilaterals

Example 1.20

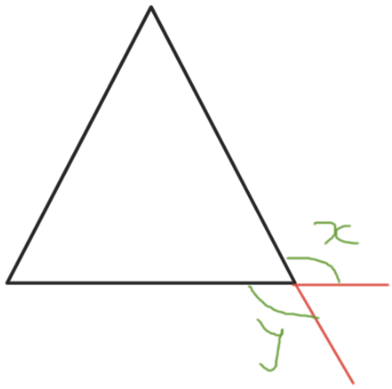
Rectangle $WXYZ$ is drawn on ΔABC, such that point W lies on segment AB, point X lies on segment AC, and points Y and Z lies on segment BC, as shown. If $m\angle BWZ = 26$ and $m\angle CXY = 64$, what is $m\angle BAC$, in degrees? (MathCounts 2010 Chapter Countdown)



$$\angle WBZ = 90 - 26 = 64$$
$$\angle XCY = 90 - 64 = 26$$
$$\angle BAC = 180 - (64 + 26) = 180 - 90 = 90$$

H. Exterior Angles

If the side of a triangle is produced, the angle so formed is called an exterior angle.
Each angle of a triangle has two exterior angles.
In all, there are six exterior angles of a triangle – giving three pairs of vertically opposite angles. Each pair has equal measure.



1.2: Two Angles exterior to same angle are congruent

The two exterior angles formed at a vertex of a triangle are congruent.

In the diagram, because of vertically opposite angles, we must have:

$$x = y$$

1.3: Exterior Angle is supplementary to adjacent interior angle

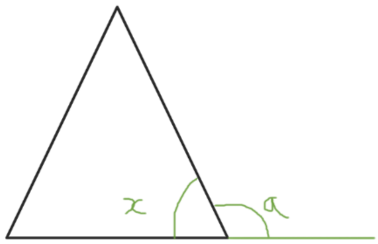
The exterior angle of a triangle is supplementary to its adjacent interior angle.

The exterior angle forms a linear pair with the adjacent interior angle
Hence, the two angles are supplementary.

Example 1.21

In the diagram, which is not drawn to scale:

A. if $\angle x = 74$, find $\angle a$.
B. if $\angle a = 137$, find $\angle x$.
C. if $\angle a = 90$, what kind of triangle is it?
D. If $\angle a$ is acute, what kind of triangle is it?
E. If $\angle a$ is obtuse, can we say anything about what kind of triangle it is?



$\angle x$ and $\angle a$ form a linear pair

Part A

$$x + a = 180 \Rightarrow 74 + a = 180 \Rightarrow a = 106$$

Part B

$$x + a = 180 \Rightarrow x + 137 = 180 \Rightarrow x = 43$$

Part C

Right-Angled Triangle

Part D

Obtuse-Angled Triangle

Part E

We can't conclude that it is an acute angled triangle because one of the other interior angles might be right-angled, or obtuse.

Example 1.22

A. What is the measure of the exterior angle of an equilateral triangle?
B. A triangle has an angle with measure 63. What is the measure of the exterior angle to that angle?

$$180 - 60 = 120$$
$$180 - 63 = 117$$

Example 1.23

A triangle has exterior angle twice its interior angle for each of its angles. What kind of triangle is it?

Method I

$$Interior\ Angle = x \Rightarrow Exterior\ Angle = 2x$$
$$x + 2x = 180 \Rightarrow 3x = 180 \Rightarrow x = 60$$

This is true for each angle in the triangle, and hence the triangle is equilateral.

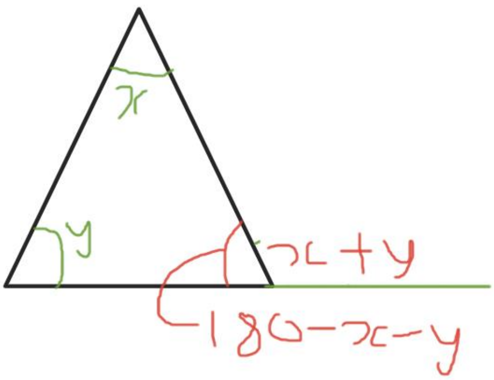
Method II

$$Interior\ Angle = x \Rightarrow Exterior\ Angle = 180 - x$$
$$180 - x = 2x \Rightarrow 3x = 180 \Rightarrow x = 60$$

This is true for each angle in the triangle, and hence the triangle is equilateral.

1.4: Exterior Angle is Sum of Two Interior Angles

Exterior angle of a triangle is the sum of two non-adjacent interior angles.



Let the two non-adjacent interior angles be

$$x\ and\ y$$

Since the sum of angles of a triange is 180°, the third angle must be:

$$180 - x - y$$

Since the exterior angle forms a linear pair with the adjacent interior angle, we must have:

$$Exterior\ Angle = 180 - (180 - x - y) = x + y$$

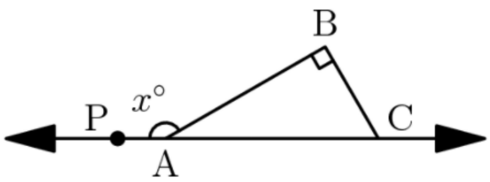
Example 1.24

The measure of an exterior angle of a triangle is 75 degrees. If one of the non-adjacent interior angles measures 28 degrees, what is the number of degrees in the other non-adjacent interior angle? **(MathCounts 2011 State Countdown)**

$$x + 28 = 75 \Rightarrow x = 75 - 28 = 47$$

Example 1.25

Triangle ABC is a right triangle. If the measure of angle PAB is x and the measure of angle ACB is expressed in the form $Mx + N$ with $M = 1$, what is the value of $M + N$? **(MathCounts 2005 State Countdown)**



By exterior angle property:

$$\angle PAB = \angle ABC + \angle ACB$$

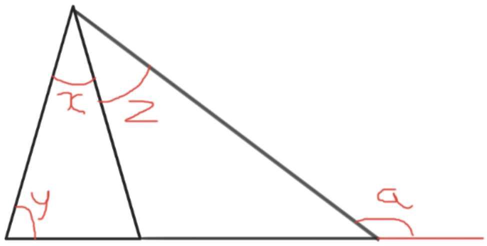
Substitute $\angle PAB = x, \angle ABC = 90$:

$$x = 90 + \angle ACB \Rightarrow \angle ACB = x - 90 \Rightarrow M = 1, N = -90 \Rightarrow M + N = 1 - 90 = -89$$

Example 1.26

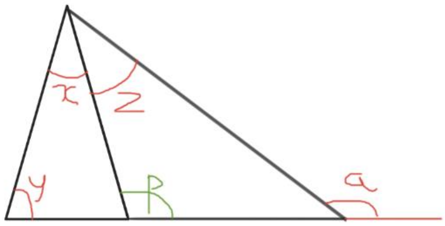
In the diagram, which is not drawn to scale, if

A. $\angle x = 65, \angle y = 75, \angle z = 20$, then find $\angle a$.
B. $\angle x = 55, \angle y = 80, \angle z = 10$, then find $\angle a$.
C. $\angle y = 2.5\angle x, \angle z = 0.5\angle x$ and $\angle a = \angle x + 90$, then find $\angle x$.



Part A

$$\angle p = \angle x + \angle y = 65 + 75 = 140$$
$$\angle a = \angle p + \angle z = 140 + 20 = 160$$



Part B

$$\angle p = \angle x + \angle y = 55 + 80 = 135$$
$$\angle a = p + \angle z = 135 + 10 = 145$$

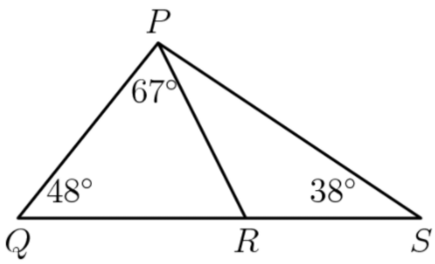
Part C

$$\angle p = \angle x + \angle y = \angle x + 2.5\angle x = 3.5\angle x$$
$$\angle a = \angle p + \angle z = 3.5\angle x + 0.5\angle x = 4\angle x$$

$$4\angle x = \angle x + 90 \Rightarrow \angle x = 30$$

Example 1.27

In the diagram, QRS is a straight line. What is the measure of $\angle RPS$, in degrees? (CEMC 2009 Pascal)



By the exterior angle property:

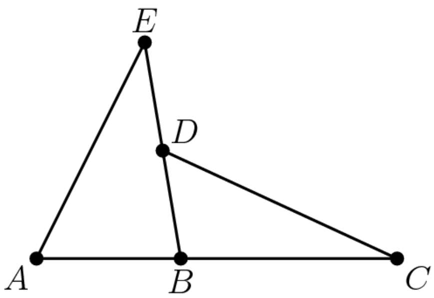
$$\angle PRS = 48 + 67 = 115$$

By the sum of angles in a triangle property:

$$\angle RPS = 180 - 115 - 38 = 27$$

Example 1.28

If $\angle A = 60°, \angle E = 40°$ and $\angle C = 30°$, then $\angle BDC =$(AMC 8 1994/7)



By the exterior angle property:

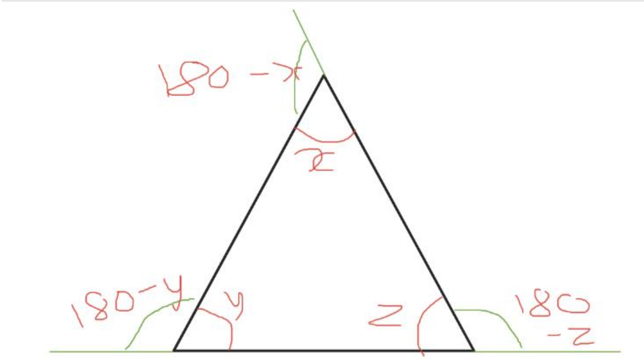
$$\angle B = \angle A + \angle E = 60 + 40 = 100$$

By the sum of angles in a triangle:

$$\angle BDC = 180 - \angle C - \angle B = 180 - 30 - 100 = 50$$

1.29: Sum of Exterior Angles of a Triangle

Sum of exterior angles of a triangle is **360°**



Consider the triangle drawn, with interior angles

$$x, y, z$$

Therefore, by angles in a straight line, the exterior angles are:

$$180 - x, 180 - y, 180 - z$$

And they have sum

$$180 - x + 180 - y + 180 - z = 540 - (x + y + z)$$

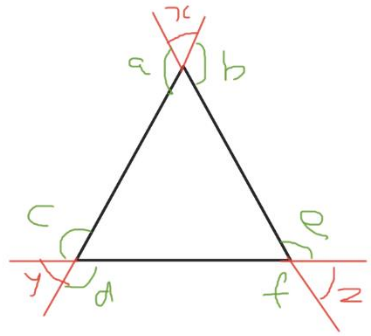
Substitute $x + y + z = 180$ (since these are the angles of a triangle:

$$540 - 180 = 360°$$

Example 1.30

In the diagram alongside (which is not drawn to scale), find

A. $a + b + c + d + e + f$
B. $x + y + z$



Part A

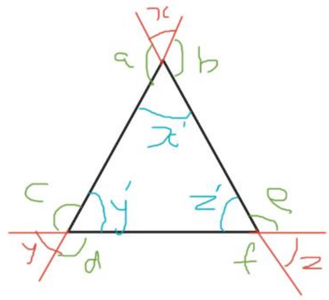
Since the sum of exterior angles of a triangle is 360, we must have

$$a + d + e = 360$$

$$b + c + f = 360$$

Adding the two equations above:

$$a + b + c + d + e + f = 360 + 360 = 720$$



Part B

Name angles x', y', z' as in the diagram. Note that, by vertically opposite angles:

$$x = x', \qquad y = y', \qquad z = z'$$

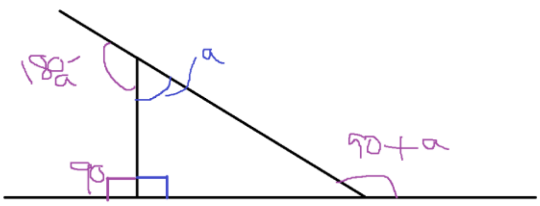
Hence:

$$x + y + z = x' + y + z' = 180$$

Where we get last equality since x', y', z' are the angles of a triangle.

Example 1.31

Consider a right angled triangle with measure a for an angle not opposite its hypotenuse. Find, in terms of a, the measure of the exterior angles of that triangle.



One angle is 90°. Hence, the angle exterior to it is

$$180 - 90 = 90$$

We know one angle which is not the right angle is a°. Hence, the angle exterior to it is:

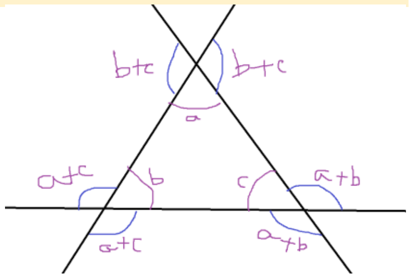
$$(180 - a)°$$

And the third exterior angle is the sum of the two non-adjacent interior angles, which is

$$(90 + a)°$$

Example 1.32

ΔABC is equilateral with side length six units, and ΔXYZ is right-angled with hypotenuse 5 units. What is the positive difference between the sum of the six exterior angles of △ABC and the sum of the six exterior angles of △XYZ?



If the angles of any triangle are a, b, and c, then the exterior angles are:

$$= a + b, b + c, and\ c + a$$

Adding the three:

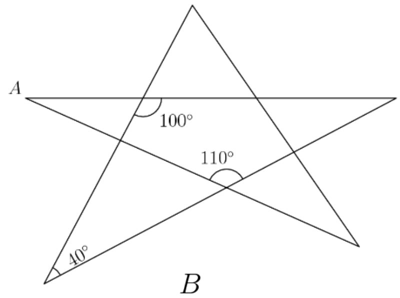
$$2(a + b + c) = 2(180) = 360$$

This is true for any triangle. Hence, difference is zero.

I. Five-Pointed Star

Example 1.33

The degree measure of angle A is (AMC 8 1999/21)



Consider the angle with 110° as an exterior angle:

$$2nd\ Interior\ Angle = 70$$

Angle vertically opposite

$$70$$

Angle forming linear pair with 100

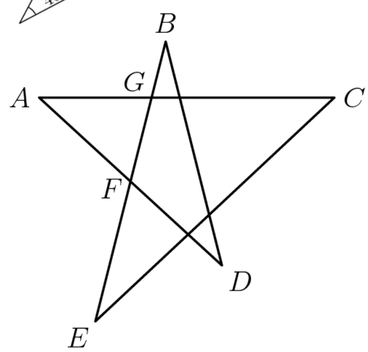
$$= 180 - 100 = 80$$

By sum of angles in a triangle

$$\angle A = 180 - 80 - 70 = 30$$

Example 1.34

If $\angle A = 20°$ and $\angle AFG = \angle AGF$, then $\angle B + \angle D =$ (AMC 8 2000/24)



In Isosceles ΔAFG:

$$\angle AFG = \frac{180 - 20}{2} = \frac{160}{2} = 80$$

By exterior angle property:

$$\angle B + \angle D = \angle AFG = 80$$

Example 1.35

Show that the sum of the measures of the angles of the vertices of a five-pointed star is equal to the sum of the measures of the angles of a triangle. (NMTC Primary/Final 2004/16)

Method I: Using Triangles

$$In\ \Delta ACF: \angle A + \angle C + \angle CFA = \angle 180 \Rightarrow \underbrace{\angle A + \angle C = \angle 180 - \angle CFA}_{Equation\ I}$$

$$In\ \Delta DFE: \angle CED + \angle EDA + \angle DFE = 180 \Rightarrow \underbrace{\angle CED + \angle EDA = 180 - \angle DFE}_{Take\ to\ the\ other\ side} = \underbrace{180 - \angle CFA}_{Vertically\ Opposite\ Angles} = \underbrace{\angle A + \angle C}_{Substitute\ I}$$

$$In\ \Delta BED: 180 = \angle EBD + \angle BDE + \angle BED = \angle B + \angle D + \angle E + \angle CED + \angle EDA = \angle B + \angle D + \angle E + \angle A + \angle C$$

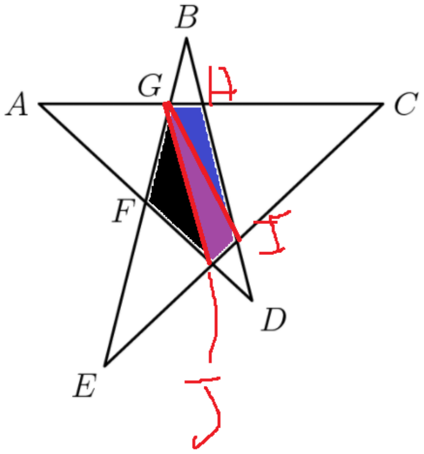
Method I: Using Exterior Angle Property of Triangles:

Method II: Using Triangles and Pentagons

Pentagon

The points of intersection of the lines of the five-pointed star form a pentagon. Note that the angles of the pentagon form three triangles, and hence the sum of the these angles must be

$$180 \times 3$$



Hence, we must have:

$$\angle AJC + \angle BFD + \angle CGE + \angle AHD + \angle BJE = 180 \times 3$$

Consider the five triangles below, and note that the sum of the angles of each triangle must add up to 180°.

$$\text{In } \Delta ACJ: \angle A + \angle C + \angle AJC = 180$$
$$\text{In } \Delta \text{BDF}: \angle B + \angle D + \angle BFD = 180$$
$$\text{In } \Delta \text{CEG}: \angle C + E + \angle CGE = 180$$
$$\text{In } \Delta \text{ADH}: \angle A + \angle D + \angle AHD = 180$$
$$\text{In } \Delta \text{BEJ}: \angle B + \angle E + \angle BJE = 180$$

Add the above five equations:

$$2(\angle A + \angle B + \angle C + \angle D + \angle E) + \angle AJC + \angle BFD + \angle CGE + \angle AHD + \angle BJE = 180 * 5$$

Substitute, as calculated above that sum of angles of a pentagon is 180×3:

$$2(\angle A + \angle B + \angle C + \angle D + \angle E) + 180 \times 3 = 180 \times 5$$
$$2(\angle A + \angle B + \angle C + \angle D + \angle E) = 180 \times 2$$
$$\angle A + \angle B + \angle C + \angle D + \angle E = 180$$

J. Equilateral Triangles

Example 1.36

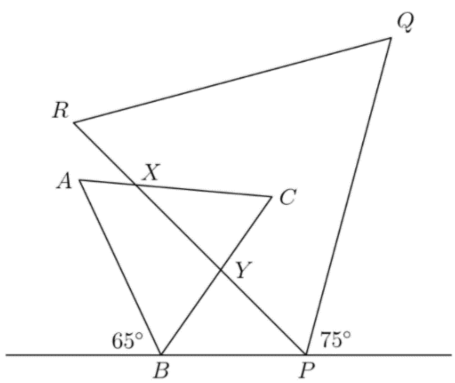
An equilateral triangle has sides 3, $2x + 3$ and $3 \sin y$. What is xy?

Since the triangle is equilateral, all three sides must be equal, and hence, we must have:

$$2x + 3 = 3 \Rightarrow x = 0 \Rightarrow xy = 0$$

Example 1.37

In the diagram, if ΔABC and ΔPQR are equilateral, then what is the measure of $\angle CXY$ in degrees? **(CEMC 2007 Cayley)**[1]



By angles in a linear pair:

$$\angle YBP = 180 - 65 - 60 = 55$$
$$\angle YPB = 180 - 75 - 60 = 45$$

By angles in a triangle:

$$\angle XYC = \angle BYP = 180 - 55 - 45 = 80$$

By vertically opposite angles:

$$\angle XYC = \angle BYP = 80$$

[1] This question has appeared in multiple exams. Solving it requires combining different properties, and some level of creativity.

By angles in a triangle

$$\angle CXY = 180 - 60 - 80 = 40$$

Example 1.38

In an equilateral triangle, the angles have measures of $3p + 10$, $4q - 10$ and $5r$. An exterior angle of the triangle has measure $8s$. What is $p:q:r:s$?

The interior angles are equal to 60.

$3p + 10 = 60 \quad \Rightarrow \quad p = 50/3$
$4q - 10 = 60 \quad \Rightarrow \quad q = 70/4 = 35/2$
$5r = 60 \quad \Rightarrow \quad r = 12$

Exterior angle is equal to 120

$8s = 120 \quad \Rightarrow \quad s = 3/2$

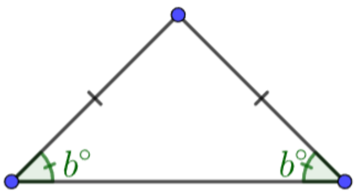
$$p:q:r:s = \frac{50}{3}:\frac{35}{2}:12:\frac{3}{2} = 100:105:72:9$$

1.2 Isosceles Triangles

A. Basics

1.39: Isosceles Triangles

- A triangle with at least two angles equal is called an isosceles triangle.
- The equal angles are called the base angles, and the third angle is called the vertex angle.
- The side between the two base angles is called the base, and the point opposite the base is called the vertex.

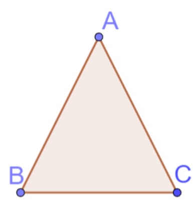


In the diagram alongside, $v°$ is the measure of the vertex angle, and $b°$ is the measure of each base angle.

Example 1.40

In the diagram alongside $\angle B = \angle C$, making ΔABC an Isosceles Triangle. Name the:

A. Base Angles
B. Vertex Angle



$$Base\ Angles: \angle B, \angle C$$
$$Vertex\ Angle: \ \angle A$$

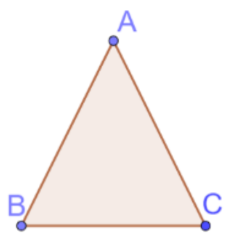
1.41: Base Angle Property

Base Angles of an Isosceles Triangle are equal.

Example 1.42

In the figure alongside, ΔABC is isosceles with base angles $\angle B$ and $\angle C$ and vertex $\angle A$. Find:

A. $\angle A$ if $\angle B$ and $\angle C$ are each equal to $40°$.
B. the base angles if angle $\angle A = 40°$.
C. the remaining angles if base $\angle B = 50°$.
D. $\angle A$ if $\angle B = 3x - 5$ and $\angle C = 2x + 5$.



Part A

$$180 - 40 - 40 = 180 - 80 = 100°$$

Part B

$$\frac{180-40}{2}=\frac{140}{2}=70°$$

Part C

$$\angle C=\angle B=50°$$
$$180-2(50)=180-100=80°$$

Part D

$$\angle B=\angle C\Rightarrow 3x-5=2x+5\Rightarrow x=10\Rightarrow \angle B=\angle C=25$$
$$\angle A=180-25-25=180-50=130$$

Example 1.43: Finding Angles

A. In an isosceles triangle, the vertex angle is 35°. What are the other angles?
B. In an isosceles triangle, the vertex angle is twice a base angle. Find the angles of the triangle.
C. An isosceles, obtuse triangle has one angle with a degree measure that is 50% larger than the measure of a right angle. What is the measure, in degrees, of one of the two smallest angles in the triangle? Express your answer as a decimal to the nearest tenth. (MathCounts 2008 Chapter Countdown)
D. One of the base angles of an isosceles triangle is 35. What is the difference between the other two angles?

Part A

$$\text{Each angle}=\frac{180\ -\ 35}{2}=\frac{145}{2}=72.5°$$

Part B

$$Let\ Base\ Angle=b\Rightarrow Vertex\ Angle=2b$$

And now since the sum of angles of a triangle is 180°, we must have:

$$b+b+2b=180\Rightarrow 4b=180\Rightarrow b=45°$$

And hence the angles must be:

$$45°,45°,90°$$

Part C

$$Larger\ Angle=90\times 1.5=135$$
$$Smaller\ Angle=\frac{180-135}{2}=\frac{45}{2}=22.5$$

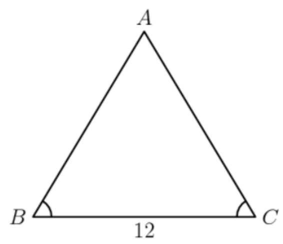
Part D

$$Vertex\ Angle=180-35\times 2=110$$
$$Difference=110-35=75$$

1.44: Sides in an Isosceles Triangle

The sides opposite the base angles in an isosceles triangle are also congruent.

Example 1.45



A. The perimeter of ΔABC is 32. If $\angle ABC=\angle ACB$ and $BC=12$, what is the length of AB? (CEMC 2009 Gauss 8)
B. If the base of an isosceles triangle is 12 units long, and the perimeter is 35 units, what are the lengths of the other two sides?
C. Two sides of an isosceles triangle are 15 cm and 10 cm. What is the greatest possible perimeter of this triangle, in centimeters? (MathCounts 2005 School Countdown)
D. The congruent sides of an isosceles triangle are each 5 cm long, and the perimeter is 17 cm. In centimeters, what is the length of the base? (MathCounts 2004 Chapter Countdown)

Part A

$$AB = \frac{32 - 12}{2} = \frac{20}{2} = 10$$

Part B

$$\text{Each side} = \frac{35 - 12}{2} = \frac{23}{2} = 11.5 \; units$$

Part C

To make the perimeter the largest possible, we need to make the larger side occur twice:

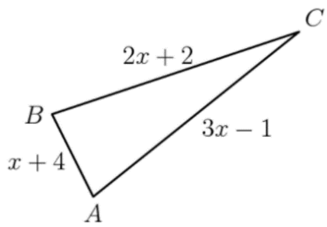
$$15 + 15 + 10 = 40$$

Part D

$$17 - 5 - 5 = 7$$

Example 1.46

In the triangle, $\angle A = \angle B$. What is x? (MathCounts 1996 State Countdown)



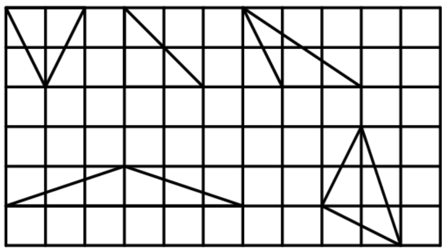
$$2x + 2 = 3x - 1 \Rightarrow x = 3$$

1.47: Isosceles Triangles: Alternate Definition

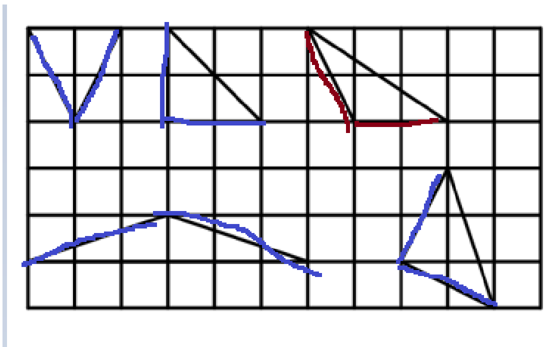
If two sides of a triangle are congruent, it is an isosceles triangle, and the angles opposite the sides are congruent.

Example 1.48

An isosceles triangle is a triangle with two sides of equal length. How many of the five triangles on the square grid below are isosceles? (AMC 8 1988/9)



There are four isosceles triangles with their equal sides marked in blue in the diagram.



1.49: Angles of an Isosceles Right-Angled Triangle

The angles in an isosceles right-angled triangle are

$$\mathbf{45, 45, \textit{and } 90}$$

Since the triangle is right-angled, it must have one angle

$$90°$$

If one angle of the triangle is 90°, then sum of the remaining angles must be:

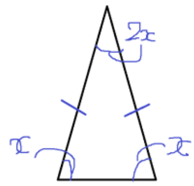
$$\underbrace{180°}_{\substack{Sum\ of\ Angles \\ of\ a\ Triangle}} - \underbrace{90°}_{Right\ Angle} = 90°$$

But, since it is an isosceles triangle, its base angles must be equal. Hence, the value of each base angle must be:

$$b + b = 90 \Rightarrow 2b = 90 \Rightarrow b = 45°$$

Example 1.50

What kind of triangle is formed when, in an isosceles triangle, the sum of the angles opposite the equal sides is equal to the third angle?



Sum of angles property:

$$x + x + 2x = 180 \Rightarrow x = 45$$

And, hence the triangle is:

$$45 - 45 - 90 \text{ (Isosceles Right} - \text{Angled Triangle)}$$

1.51: Casework

If the information given in the question is not specific, then you may need to consider cases.

Example 1.52

One angle of an isosceles triangle is 45. What is the sum of the possible values of the vertex angle?

We need to consider cases based on whether 45° is a vertex angle or a base angle.

Case I: 45 is a base angle

$$Vertex\ Angle = 180 - 45 \times 2 = 180 - 90 = 90°$$

Case II: 45 is a vertex angle

$$Vertex\ Angle = 45$$

Sum of the possible values

$$90 + 45 = 135$$

Example 1.53

Triangle PQR is isosceles and the measure of angle R is 40. The possible measures of angle P are x, y, z. What is the value of the sum $x + y + z$? **(MathCounts 2003 National Countdown)**

We need to consider cases:

Case I: $\angle \boldsymbol{P}$ is a base angle, $\angle \boldsymbol{R}$ is also a base angle

$$\angle P = \angle R = 40°$$

Case II: $\angle \boldsymbol{P}$ is a base angle, $\angle \boldsymbol{R}$ is the vertex

$$\angle P = \frac{180 - 40}{2} = \frac{140}{2} = 70°$$

Case III: $\angle \boldsymbol{P}$ is the vertex angle

$$\angle P = 180 - 40 - 40 = 100°$$

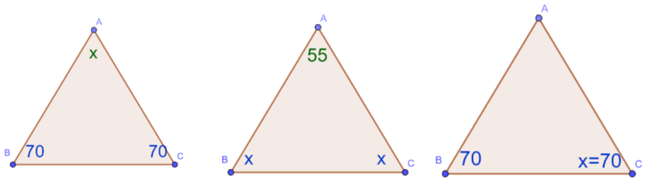
$$x + y + z = 40 + 70 + 100 = 210°$$

Example 1.54

Two angles of an isosceles triangle measure 70° and x°. What is the sum of the three possible values of x? **(AMC 8 2009/19)**

Case I: Base Angles are 70° and 70°
Then, the vertex angle must be

$$x = 180 - 70 - 70 = 40°$$



Case II: Base Angles are $x°$ and $x°$

$$x + x + 70 = 180$$
$$2x = 110°$$

Case III: Base Angles are 70° and $x° = 70°$
In this case, we directly have

$$x = 70°$$

And hence, the final answer is the sum of all possible values, which is:

$$40 + 55 + 70 = 165°$$

Example 1.55

Two isosceles triangles each have at least one angle that measures 70. In the first triangle, the measure in degrees of each of the remaining two angles is even. In the second triangle, the measure in degrees of each of the remaining two angles is odd. In the first triangle, the sum of the equal angles is S. In the second triangle, the sum of the equal angles is T. The value of $S + T$ is **(Gauss 7 2020/18)**

Case I: Both base angles are 70°:

$$One\ Base\ Angle = 70° \Rightarrow Other\ Base\ Angle = 70° \Rightarrow Vertex\ Angle = 40°$$

From the above, we know that we found the first triangle

$$S = 70 + 70 = 140$$

Case II: The vertex angle is 70°

$$Vertex\ Angle = 70° \Rightarrow Each\ base\ Angle = \frac{180 - 70}{2} = \frac{110}{2} = 55$$

From the above, we know that we have found the second triangle.

$$T = 55 + 55 = 110°$$

And we want to find:

$$S + T = 140 + 110 = 250°$$

Example 1.56: Percentages

Find the angles of each triangle below:

A. An isosceles triangle with vertex angle 50% more than a base angle.
B. An isosceles triangle with vertex angle 50% less than a base angle.

Part A

Let the base angle be b:

$$Vertex\ Angle = b + \frac{50}{100} \times b = b + \frac{1}{2}b = \frac{3}{2}b$$

Since sum of angles of a triangle is 180°:

$$b+b+\frac{3}{2}b=180\Rightarrow\frac{7}{2}b=180$$

Solve for b:

$$b=180\cdot\frac{2}{7}=\frac{360}{7}$$

Multiply both sides by $\frac{3}{2}$:

$$\frac{3}{2}b=\frac{360}{7}\cdot\frac{3}{2}=\frac{540}{7}$$

The angles are:

$$\left(\frac{360}{7},\frac{360}{7},\frac{540}{7}\right)$$

Part B

$$Let\ Base\ Angle=b\Rightarrow Vertex\ Angle=b-\frac{50}{100}\times b=b-\frac{1}{2}b=\frac{b}{2}$$

Sum of angles of a triangle is 180°:

$$b+b+\frac{b}{2}=180\Rightarrow\frac{5}{2}b=180\Rightarrow b=72\Rightarrow\frac{b}{2}=36$$

Example 1.57: Ratios and Algebra

A. The vertex angle and the base angle of an isosceles triangle are in the ratio 3: 2. Find the possible values of the vertex angle.
B. One angle in an isosceles triangle is double another angle. Find the sum of the possible values of the vertex angle.
C. Triangle ABC is isosceles with angle A congruent to angle B. The measure of angle C is 30 degrees more than the measure of angle A. What is the number of degrees in the measure of angle C? (MathCounts 2004 Chapter Countdown)

Part A

Case I: Vertex Angle is Smaller

$$2:3:3\rightarrow\frac{180}{8}=22.5\rightarrow 22.5\times 2=45$$

Case II: Vertex Angle is Larger

$$3:2:2\rightarrow\frac{180}{7}\rightarrow\frac{180}{7}\times 3=\frac{540}{7}$$

Part B

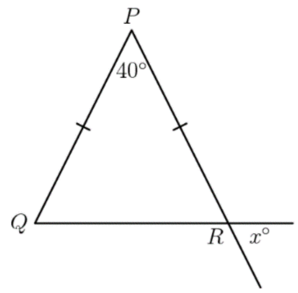
Part C

Let

$$\angle A=\angle B=x\Rightarrow\angle C=x+30$$
$$x+x+x+30=180\Rightarrow 3x=150\Rightarrow x=50\Rightarrow x+30=80$$

Example 1.58: Vertically Opposite Angles

In the diagram, ΔPQR is isosceles. What is the value of x? (CEMC 2008 Gauss 7)



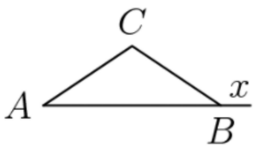
The angle that we want is vertically opposite to $\angle PRQ$, which is given by:

$$x=\frac{180-140}{2}=\frac{140}{2}=70°$$

Example 1.59: Exterior Angles

A. (*Left Diagram*) In triangle ABC, $AC = BC$, and $m\angle BAC = 40$. What is the number of degrees in angle x? **(MathCounts 1998 School Countdown)**

B. If the exterior angle of the vertex angle of an isosceles triangle has measure 40, then what are the measures of the remaining angles of the triangle?



Part A

$$x = 180 - \angle CBA = 180 - \angle BAC = 180 - 40 = 140$$

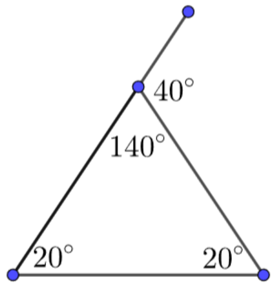
Part B

The vertex angle

$$= 180 - 40 = 140°$$

The remaining angles

$$= \frac{180 - 140}{2} = \frac{40}{2} = 20°$$

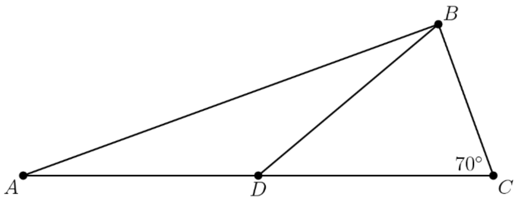


1.60: Exterior Angle Property

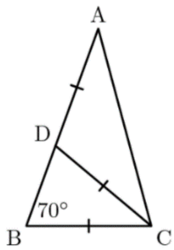
Exterior angle of a triangle is the sum of the two interior angles not adjacent to it.

Example 1.61

In ΔABC, D is a point on side $\overline{AC}$ such that $BD = DC$ and $\angle BCD$ measures 70°. What is the degree measure of $\angle ADB$? **(AMC 8 2014/9)**



$$\angle ADB = \angle DBC + \angle BCD = 70 + 70 = 140°$$



Example 1.62

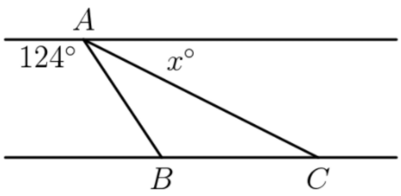
Triangles BDC and ACD are coplanar and isosceles. If we have $m\angle ABC = 70$, what is $m\angle BAC$, in degrees? **(MathCounts 2010 School Countdown)**

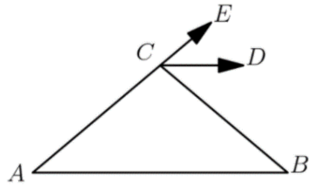
$$\angle BDC = \angle ABC = 70°$$
$$\angle ADC = 180 - 70 = 110°$$
$$\angle BAC = \frac{180 - 110}{2} = \frac{70}{2} = 35$$

Example 1.63: Parallel Lines

A. (*Left Diagram*) $\overline{BC}$ is parallel to the segment through A, and $AB = BC$. What is the number of degrees represented by x? **(MathCounts 2000 State Countdown)**

B. (*Right Diagram*) In ΔABC, $AC = BC$, $m\angle DCB = 40$, and $CD \parallel AB$. What is the number of degrees in $m\angle ECD$? **(MathCounts 1999 School Sprint)**





Part A

By alternate interior angles

$$\angle ABC = 124$$

In Isosceles ΔABC:

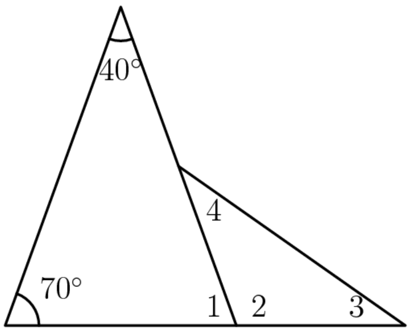
$$\angle BAC = \frac{180 - 124}{2} = \frac{56}{2} = 28$$

By angles on a straight line:

$$x = 180 - 124 - 28 = 28$$

Part B

$$\angle CBA = \angle DCB = 40$$
$$\angle CAB = \angle CBA = 40$$
$$\angle ECD = \angle CAB = 40$$

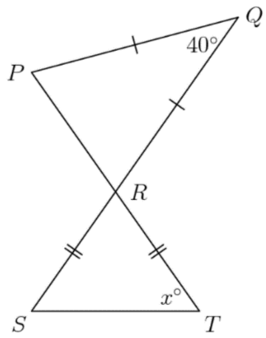


B. Multiple Triangles

Example 1.64

$\angle 1 + \angle 2 = 180°$.
$\angle 3 = \angle 4$
Find $\angle 4$. (AMC 8 1997/12)

$$\angle 2 = 40 + 70 = 110$$
$$\angle 4 = \frac{180 - 110}{2} = \frac{70}{2} = 35$$



Example 1.65

In the diagram, PRT and QRS are straight lines. What is the value of x? (CEMC 2008 Cayley)

In Isosceles ΔPQR:

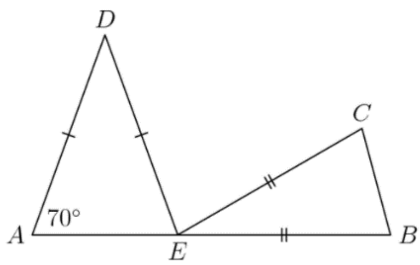
$$\angle PRQ = \frac{180 - 40}{2} = \frac{140}{2} = 70$$

By Vertically Opposite Angles

$$\angle SRT = \angle PRQ = 70$$

In Isosceles ΔRST:

$$x = \angle RTS = \frac{180 - 70}{2} = \frac{110}{2} = 55$$



Example 1.66

In the diagram, point E lies on line segment AB, and triangles AED and BEC are isosceles. Also, $\angle DEC$ is twice $\angle ADE$. What is the measure of $\angle EBC$ in degrees? (CEMC 2006 Pascal)

In Isosceles ΔADE:

$$\angle DEA = \angle DAE = 70°$$
$$\angle DEC = 2 \times \angle ADE = 2 \times (180 - 70 \times 2) = 2 \times 40 = 80°$$

By angles in a straight line:

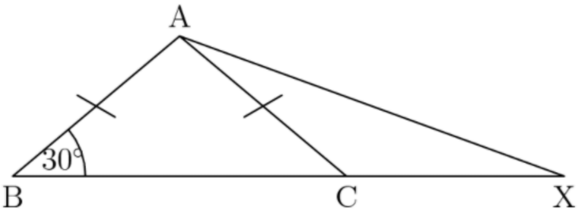
$$\angle CEB = 180 - 70 - 80 = 30°$$

In Isosceles ΔCEB:

$$\angle EBC = \frac{180 - 30}{2} = \frac{150}{2} = 75$$

Example 1.67

In isosceles triangle ABC, if BC is extended to a point X such that $AC = CX$, what is the number of degrees in the measure of angle AXC? (MathCounts 1993 Chapter Countdown)



By angles in a linear pair

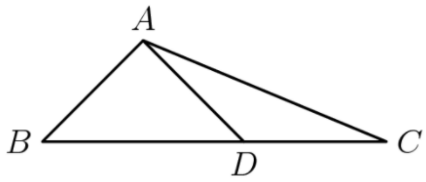
$$Exterior\ Angle\ \angle ACX = 180 - \angle ACB = 180 - \angle ABC = 180 - 30 = 150$$

In Isosceles ΔACX:

$$\angle AXC = \frac{180 - 150}{2} = \frac{30}{2} = 15$$

Example 1.68

In the figure, $BA = AD = DC$ and point D is on segment BC. The measure of angle ACD is 22.5 degrees. What is the measure of angle ABC? (MathCounts 2008 Workout 2)



Since the exterior angle is the sum of the two interior angles

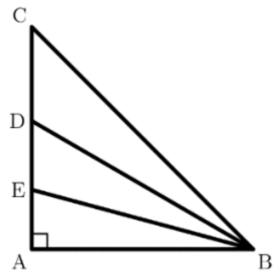
$$\angle ADB = 2 \times 22.5 = 45°$$

In Isosceles ΔABD

$$\angle ABD = \angle ADB = 45°$$

Example 1.69

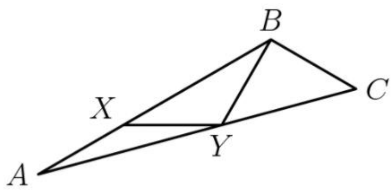
Triangle ABC is an isosceles right triangle with a right angle at A. Segments BD and BE trisect angle ABC. What is the degree measure of angle BDE? (MathCounts 2005 National Sprint)

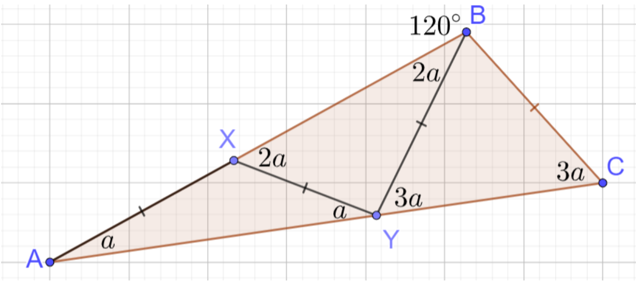


$$\angle DBA = 15 \times 2 = 30$$
$$\angle BDE = 180 - 90 - 30 = 60$$

Example 1.70

In triangle ABC, $AX = XY = YB = BC$ and the measure of angle ABC is 120 degrees. What is the number of degrees in the measure of angle BAC? (MathCounts 1993 National Target)





In Isosceles ΔAXY:

$$\angle XYA = \angle BAC = a$$

In Isosceles ΔBXY, use the exterior angle property:

$$\angle XBY = \angle BXY = a + a = 2a$$

In ΔBXY use the sum of angles property:

$$\angle XYB = 180 - 2a - 2a = 180 - 4a$$

Use the angles in a straight-line property:

$$\angle BCY = \angle BYC = 180 - (180 - 4a) - a = 3a$$

Use the angles in a triangle property:

$$\angle YBC = 180 - 3a - 3a = 180 - 6a$$

Use the adjacent angles property:

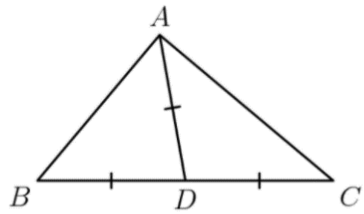
$$\angle ABC = \angle XBY + \angle YBC = 2a + 180 - 6a = 180 - 4a$$

And now that we have an expression for $\angle ABC$, equate it to its value:

$$180 - 4a = 120 \Rightarrow a = 15$$

1.71: Median in a Triangle

If the median to a triangle is equal to the two segments formed at the base, the triangle is right-angled.



In ΔABC, let

$$DB = DA = DC$$

Let

$$\angle BCA = x \Rightarrow \angle DAC = x$$

By the exterior angle property

$$\angle BDA = 2x$$

In Isosceles ΔABD:

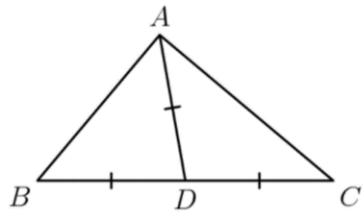
$$\angle BAD = \frac{180 - 2x}{2} = 90 - x$$

And finally, using adjacent angles:

$$\angle BAC = \angle BAD + \angle DAC = (90 - x) + x = 90$$

Example 1.72

In the diagram, $AD = BD = CD$ and $\angle BCA = 40$. What is the measure of $\angle BAC$? (CEMC 2007 Gauss 8)



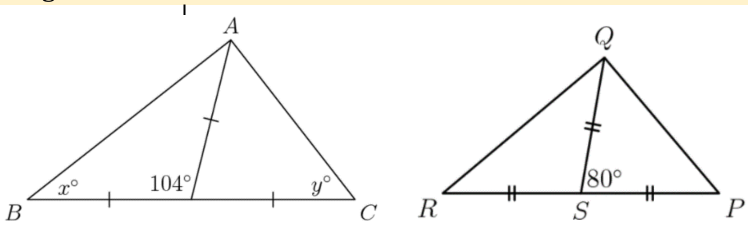
Since the median is equal to the two segments formed at the base:

$$\angle BAC = 90°$$

Example 1.73

A. (*Middle Diagram*) In ΔABC, what is the value of $x + y$? (CEMC 2005 Pascal)
B. (*Right Diagram*) In the diagram, RSP is a straight line and $\angle QSP = 80$. What is the measure of $\angle PQR$, in degrees? (CEMC 2008 Gauss 8)

C. $(Left\ Diagram)$ In the diagram alongside, if $\angle ABC = 35$, what is $\angle ACB$?



Part B

Let the line from A intersect BC at X.

$$\Delta BXA\ is\ isosceles \Rightarrow \angle BAX = x$$
$$\Delta CXA\ is\ isosceles \Rightarrow \angle CAX = y$$

By sum of angles in a triangle:

$$2x + 2y = 180$$
$$x + y = 90$$

Part C

Using the property above,

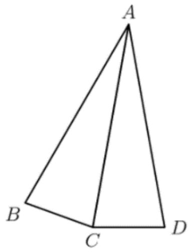
$$\angle PQR = 90$$

Part D

$$\angle ACB = 180 - 90 - 35 = 65$$

Example 1.74: Congruent Triangles

Suppose that there are two congruent triangles ΔABC and ΔACD such that $AB = AC = AD$, as shown in the following diagram. If $\angle BAC = 20$, then what is $\angle BDC$? (AOPS Alcumus, Geometry, Isosceles and Equilateral Triangles)



$$\angle ACB = \frac{180 - 20}{2} = \frac{160}{2} = 80$$

Since ΔABC and ΔACD are congruent:

$$\angle ACD = \angle ACB = 80$$

Then, in isosceles ΔBCD:

$$Vertex\ Angle = \angle BCD = \angle ACB + \angle ACD = 80 + 80 = 160$$
$$\angle BDC = \frac{180 - 160}{2} = \frac{20}{2} = 10$$

Example 1.75: Nested Triangles

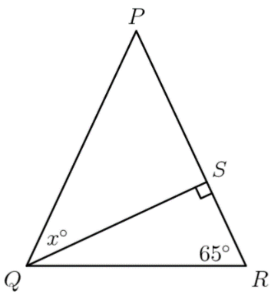
A. $(Left\ Diagram)$ In the diagram, $PQ = PR$ and $\angle QRP = 65$. What is the value of x? (CEMC 2009 Cayley)

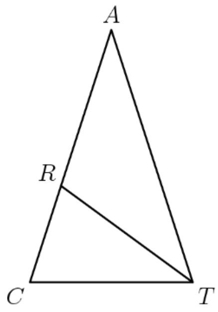
B. $(2nd\ Diagram)$ In triangle CAT, we have $\angle ACT = \angle ATC$ and $\angle CAT = 36°$. If $\overline{TR}$ bisects $\angle ATC$, then $\angle CRT\ =$(AMC 8 2000/13)

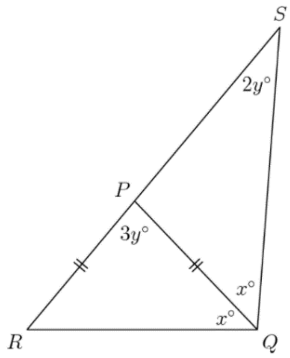
C. Triangles ABC and ADC are isosceles with $AB = BC$ and $AD = DC$. Point D is inside triangle ABC, angle ABC measures 40 degrees, and angle ADC measures 140 degrees. What is the degree measure of angle BAD? (AMC 10A 2007/8)

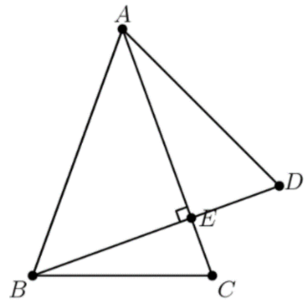
D. $(3rd\ Diagram)$ In the diagram, P is on RS so that QP bisects $\angle SQR$. Also, $PQ = PR$, $\angle RSQ = 2y$, and $\angle RPQ = 3y$. What is the measure, in degrees, of $\angle RPQ$ (CEMC 2008 Cayley)

E. ($Right\ Diagram$) Triangles ABC and ABD are isosceles with $AB = AC = BD$, and BD intersects AC at E. If $BD \perp AC$, then what is the value of $\angle C + \angle D$? (1996 AHSME)









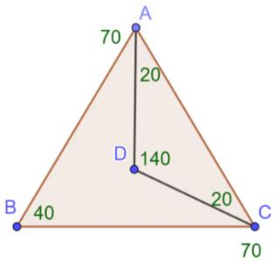
Part A

$$\angle SQR = 180 - 90 - 65 = 25$$
$$x = 65 - 25 = 40$$

Part B

$$\angle ACT = \angle ATC = \frac{180 - 36}{2} = \frac{144}{2} = 72$$
$$\angle RTC = \frac{72}{2} = 36$$
$$\angle CRT = 180 - 72 - 36 = 72$$

Part C



Apply the properties of isosceles triangles in ΔABC:
$$\angle ABC = 40 \Rightarrow \angle BAC = \angle BCA = 70°$$
Apply the properties of isosceles triangles in ΔADC:
$$\angle ACC = 140 \Rightarrow \angle DAC = \angle DCA = 20°$$
And now we can find the needed angle:
$$\angle BAD = \angle BAC - \angle DAC = 50°$$

Part D

By exterior angle of a triangle:
$$\angle RPQ = \angle RSQ + \angle PQR \Rightarrow 3y = 2y + x \Rightarrow x = y$$
In Isosceles ΔRPQ
$$\angle PRQ = \angle PRQ = x$$
By sum of angles property in ΔRPQ:
$$y + 3y + y = 180 \Rightarrow 3y = 108$$

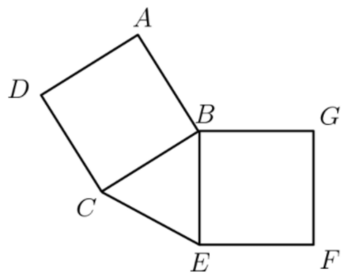
Part E

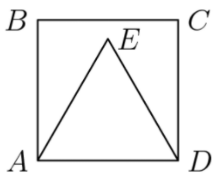
Let:
$$\angle C = c \Rightarrow \angle ABC = c$$
In ΔEBC:
$$\angle BEC = 90 \Rightarrow \angle EBC = 90 - c$$
$$\angle D = d \Rightarrow \angle BAD = d \Rightarrow ABD = 180 - 2d$$
$$\underbrace{c}_{\angle ABC} = \underbrace{180 - 2d}_{\angle ABD} + \underbrace{90 - c}_{\angle EBC}$$
$$2c + 2d = 270 \Rightarrow c + d = 135$$

Example 1.76: Quadrilaterals

A. ($Left\ Diagram$) In the figure, $ABCD$ and $BEFG$ are squares, and BCE is an equilateral triangle. What is the number of degrees in angle GCE? (MathCounts 1991 National Countdown)

B. ($Right\ Diagram$) Square $ABCD$ and equilateral triangle AED are coplanar and share $\overline{AD}$, as shown. What is the measure, in degrees, of angle BAE? (MathCounts 2004 School Sprint)





C. Jane makes a square $WXYZ$. John identifies point P outside the square such that ΔPWX is equilateral. Find $\angle PWZ$, $\angle WPZ$, $\angle ZPY$, $\angle PZY$. Construct equilateral ΔQWZ outside the square. Find $\angle PQW$. Construct equilateral $\Delta Q'WZ$ inside the square. Find $\angle PQ'W$.

Part A

In ΔCBG

$$\angle CBG = 60 + 90 = 150$$

Also ΔCBG is isosceles, so

$$\angle BCG = \frac{180 - 150}{2} = \frac{30}{2} = 15$$

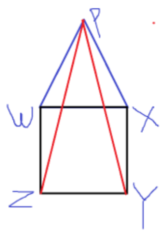
And finally

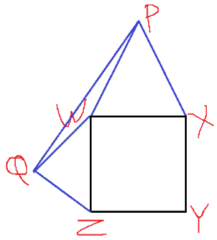
$$\angle GCE = \angle BCE - \angle BCG = 60 - 15 = 45$$

Part B

$\angle BAE = \angle BAD - \angle EAD = 90 - 60 = 30°$

Part C



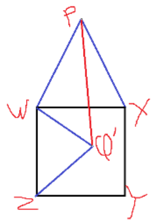


$$\angle PWZ = \angle PWX + \angle XWZ = 60 + 90 = 150$$
$$\angle WPZ = \frac{180 - 150}{2} = \frac{30}{2} = 15$$
$$\angle ZPY = 60 - 15 - 15 = 30$$
$$\angle PZY = 90 - 15 = 75$$

By angles around a point:

$$\angle QWP = 360 - 60 - 90 - 60 = 150$$

In Isosceles ΔPQW:



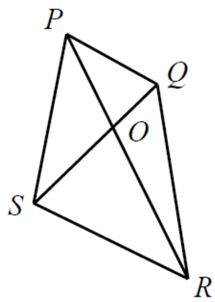
$$\angle PQW = \frac{180 - 150}{2} = \frac{30}{2} = 15$$
$$\angle PWQ' = 60 + 30 = 90$$

In Isosceles $\Delta PWQ'$

$$\angle PQ'W = \frac{180 - 90}{2} = \frac{90}{2} = 45$$

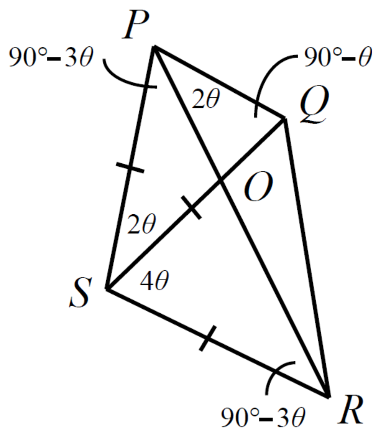
Challenge 1.77: Quadrilaterals

In quadrilateral $PQRS$, diagonals PR and SQ intersect at O inside $PQRS$, $SP = SQ = SR = 1$, and $\angle QSR = 2\angle QSP$. Marc determines the measure of the twelve angles that are the interior angles of $\Delta POS, \Delta POQ, \Delta ROS$, and ΔROQ. The measure of each of these angles, in degrees, is a positive integer, and exactly six of these integers are prime numbers. How many different quadrilaterals have these properties and are not rotations or translations of each other? **(Gauss 8 2019/25)**



Let

$$\angle QSP = 2\theta \Rightarrow \angle QSR = 2 \times 2\theta = 4\theta$$



∴ In Isosceles ΔPSR:

$$\angle SPR = \angle SRP = \frac{180 - \angle PSR}{2} = \frac{180 - 6\theta}{2} = 90 - 3\theta$$

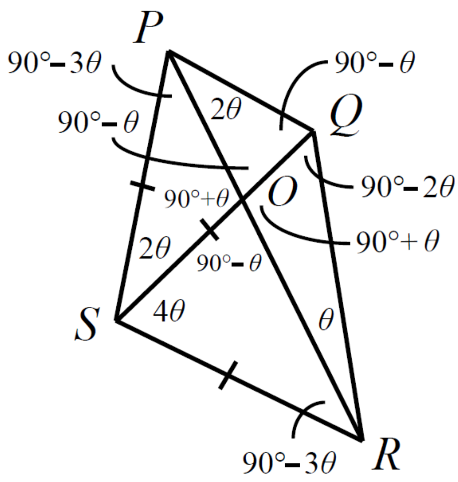
∴ In Isosceles ΔSPQ:

$$\angle SPQ = \angle SQP = \frac{180 - \angle PSQ}{2} = \frac{180 - 2\theta}{2} = 90 - \theta$$

$$\angle RPQ = \angle SPQ - \angle SPR = 90 - \theta - (90 - 3\theta) = 90 - \theta - 90 + 3\theta = 2\theta$$

In ΔPOQ:

$$\angle POQ = 180 - \underbrace{2\theta}_{\angle OPQ} - \underbrace{(90-\theta)}_{\angle PQS} = 180 - 2\theta - 90 + \theta = 90 - \theta$$



By Vertically Opposite Angles:

$$\angle ROS = \angle POQ = 90 - \theta$$

By angles in a linear pair:

$$\angle QOR = 180 - \underbrace{(90-\theta)}_{\angle POQ} = 180 - 90 + \theta = 90 + \theta$$

By Vertically Opposite Angles:

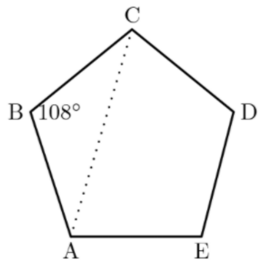
$$\angle POS = \angle QOR = 90 + \theta$$

∴ In Isosceles Triangle SQR:

$$\angle SQR = \angle SRQ = \frac{180 - \angle QSR}{2} = \frac{180 - 4\theta}{2} = 90 - 2\theta$$

In Triangle ROQ:

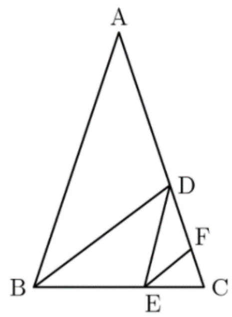
$$\angle ORQ = \underbrace{(90 - 2\theta)}_{\angle SRQ} - \underbrace{(90 - 3\theta)}_{\angle SRP} = \theta$$



Example 1.78: Polygons

In regular pentagon $ABCDE$, diagonal AC is drawn, as shown. Given that each interior angle of a regular pentagon measures 108 degrees, what is the measure of angle CAB? (MathCounts 2010 School Sprint)

$$\angle CAB = \frac{180 - 108}{2} = \frac{72}{2} = 36$$



Challenge 1.79: Counting

In triangle ABC, AB is congruent to AC, the measure of angle ABC is 72 and segment BD bisects angle ABC with point D on side AC. If point E is on side BC such that segment DE is parallel to side AB, and point F is on side AC such that segment EF is parallel to segment BD, how many isosceles triangles are in the figure shown? (MathCounts 2008 State Sprint)

Let's calculate the angles first.

$$AB \cong AC \Rightarrow \angle ABC = \angle ACB = 72$$

Ans=7

Review 1.80:

State true or false:

A. A triangle has six exterior angles.
B. A triangle with one obtuse angle is called obtuse-angled, with one right angle is called right-angled, and with one acute angle is called acute-angled.
C. The sum of the six exterior angles of a triangle is always equal to 360 degrees.
D. The hypotenuse is the shortest side of a right-angled triangle.

True
False. All three angles must be acute for it to be called an acute-angled triangle.

True
False. It is the longest side.

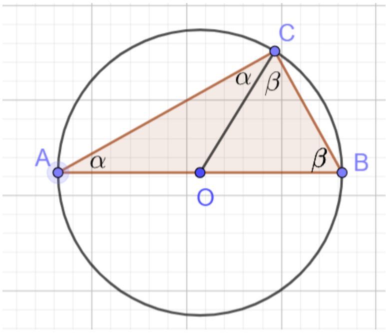
Review 1.81

Point O lies between parallel lines AB and XY, to the left of A and X. What is ∠BAO + ∠ABO + ∠YXO?
Connect points A and X.
Sum of angles of △AXO is 180.
∠BAX + YXA = 180 (Co-Interior Angles)

C. Circle Applications (Optional)

1.82: Angle subtended by diameter of circle is right angle

Consider two points AB on the circumference of a circle. If AB is a diameter of the circle, and C is any point other than A and B on the circumference on the circle, then show ΔABC is a right triangle.



Note that O is center of circle the circle and hence

$$OA = OC = OB = r$$

In Isosceles ΔAOC, let

$$\angle CAO = \angle ACO = \alpha$$

In Isosceles ΔCOB, let

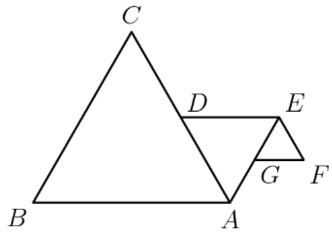
$$\angle OCB = \angle OBC = \beta$$

In ΔACB, by sum of angles of a triangle:

$$\angle CAB + \angle ACB + \angle CBA = 180°$$
$$\alpha + (\alpha + \beta) + \beta = 180°$$
$$2\alpha + 2\beta = 180°$$
$$\alpha + \beta = 90° \Rightarrow \angle ACB = 90° \Rightarrow \Delta ABC\ is\ \ a\ right\ triangle$$

1.3 Perimeter and Area

A. Perimeter

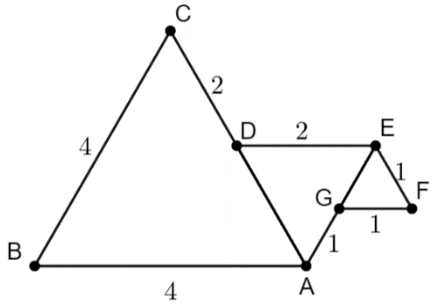


Example 1.83

Triangles ABC, ADE, and EFG are all equilateral. Points D and G are midpoints of $\overline{AC}$ and $\overline{AE}$, respectively. If $AB = 4$, what is the perimeter of figure $ABCDEFG$? **(AMC 8 2000/15)**

Draw a diagram and add up the side lengths:

$$\underbrace{4}_{AB} + \underbrace{4}_{BC} + \underbrace{2}_{CD} + \underbrace{2}_{DE} + \underbrace{1}_{EF} + \underbrace{1}_{FG} + \underbrace{1}_{GA} = 15$$



Example 1.84

The lengths of the sides of a triangle in inches are three consecutive integers. The length of the shortest side is 30% of the perimeter. What is the length of the longest side? **(AMC 8 2010/13)**

Let the sides be

$$p-1, p, p+1$$

The perimeter is then:

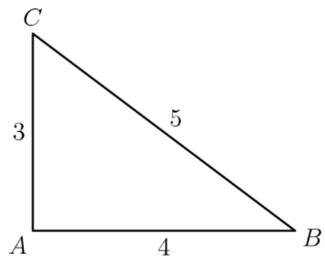
$$(p-1) + p + (p+1) = 3p$$

And the shortest side is 30% of the perimeter:

$$p - 1 = 0.3(3p)$$
$$p - 1 = 0.9p$$
$$0.1p = 1$$
$$p = 10$$

The length of the longest side is:

$$p + 1 = 11$$



Example 1.85

In the figure below, choose point D on $\overline{BC}$ so that ΔACD and ΔABD have equal perimeters. What is the area of ΔABD? **(AMC 8 2017 /16)**

Let the length of side

$$CD = x \Rightarrow DB = 5 - x$$

Perimeter of ΔABD

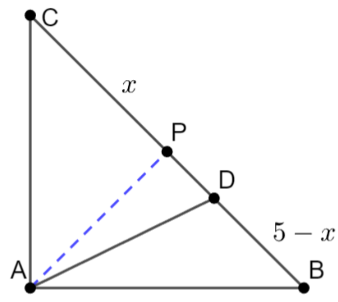
$$= AD + 4 + (5 - x)$$

Perimeter of ΔACD

$$= AD + 3 + x$$

Since the two perimeters are equal, we must have:

$$AD + 4 + (5 - x) = AD + 3 + x$$
$$x = 3$$
$$5 - x = 2$$



ΔABD and ΔABC have the same height. Hence, their areas will be in the ratio of their bases.

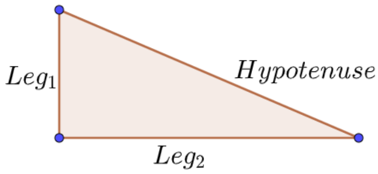
$$\frac{[ADB]}{[ABC]} = \frac{\frac{1}{2}(DB)(AP)}{\frac{1}{2}(BC)(AP)} = \frac{DB}{BC} = \frac{2}{5}$$

Hence, we need to divide $[ABC]$ in the ratio $2:3$ to get

$$= \frac{2}{5} \times [ABC] = \frac{2}{5} \times \left(\frac{1}{2} \times 3 \times 4\right) = \frac{12}{5}$$

1.86: Area of a Right Triangle

Area of a right triangle is half the product of the legs.



Area of a right triangle

$$= \frac{1}{2} \times Leg_1 \times Leg_2$$

Example 1.87: Basics

A. What is the area of a right triangle with legs 5 units and 6 units?
B. What is the number of square inches in one square foot?

$$Area = \frac{1}{2}hb = \frac{1}{2} \times 5 \times 6 = 15 \text{ units}^2$$
$$1\ ft^2 = (12\ in)^2 = 144\ in^2$$

Example 1.88: Basics

A. What is the in square feet, of a right triangle with legs 5 *inches* and 6 *inches*?

Method I

Convert each the length and the width individually to feet and then find the area:

$$Area = \frac{1}{2}hb = \frac{1}{2} \times \frac{5}{12} ft \times \frac{6}{12} \text{ft} = \frac{5}{48}\ ft^2$$

Method II

Find the area first:

$$Area = \frac{1}{2}hb = \frac{1}{2} \times 5 \times 6 = 15\ inches^2$$

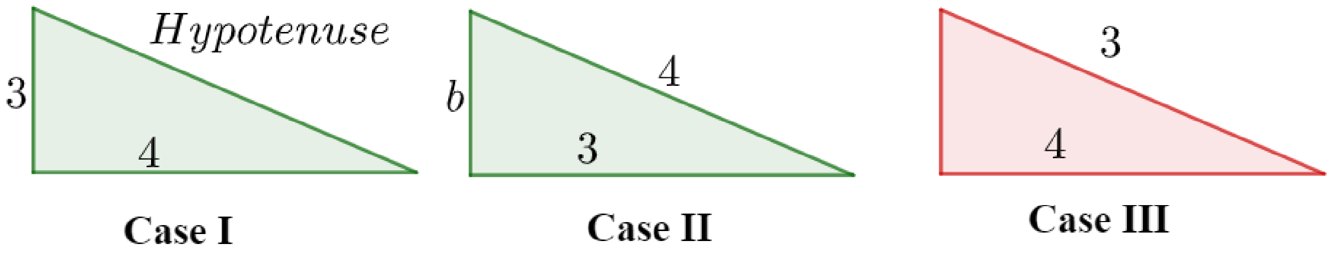
Divide by 12 twice, which is $12 \times 12 = 144$ to convert into:

$$= \frac{15}{144}\ ft^2 = \frac{5}{48}\ ft^2$$

Example 1.89: Casework

Two sides of a right triangle have length 3 units and 4 units. What is the sum of the possible areas of the triangle?

As the question points out, we need the sum of the possible areas. Hence, there is more than one possibility. The diagram below shows the three possible cases:



Case I: Given Sides are the Legs
This is the easiest case to think of:

$$Area = \frac{1}{2}hb = \frac{1}{2} \times 3 \times 4 = 3 \times 2 = 6$$

Case II: 4 is the hypotenuse
Make sure you do not miss out on this case.

$$3^2 + b^2 = 4^2 \Rightarrow b^2 = 16 - 9 = 7 \Rightarrow b = \sqrt{7}$$

$$Area = \frac{1}{2}hb = \frac{1}{2} \times 3 \times \sqrt{7} = \frac{3\sqrt{7}}{2}$$

Case III: 3 is the hypotenuse
The hypotenuse is the longest side. Hence, since

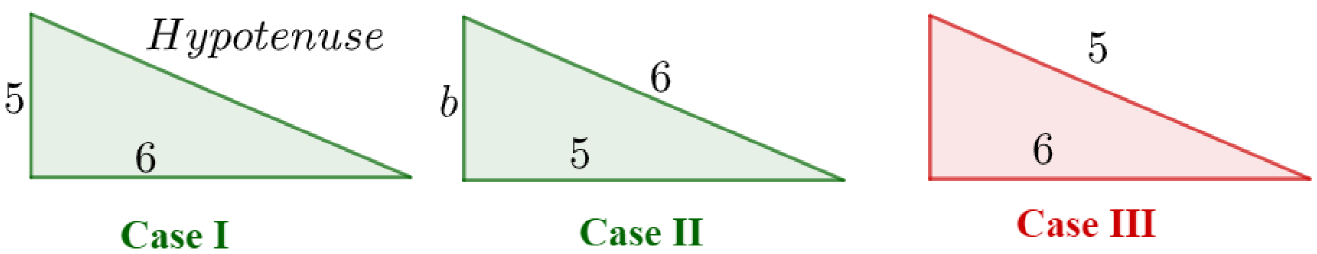
$$4 > 3$$

3 cannot be the hypotenuse.
The final sum:

$$= 6 + \frac{3\sqrt{7}}{2} = \frac{12 + 3\sqrt{7}}{2}\ units^2$$

Example 1.90: Casework

Two sides of a right triangle have length 5 inches and 6 inches. What is the sum of the possible areas of the triangle in square feet?



Case I: Given Sides are the Legs
This is the easiest case to think of:

$$Area = \frac{1}{2}hb = \frac{1}{2} \times 5 \times 6 = 15$$

Case II: 4 is the hypotenuse

Make sure you do not miss out on this case.

$$5^2 + b^2 = 6^2 \Rightarrow b^2 = 36 - 25 = 11 \Rightarrow b = \sqrt{11}$$

$$Area = \frac{1}{2}hb = \frac{1}{2} \times 5 \times \sqrt{11} = \frac{5\sqrt{11}}{2}$$

Case III: 3 is the hypotenuse

The hypotenuse is the longest side. Hence, since

$$5 > 6$$

5 cannot be the hypotenuse.

The final sum:

$$= 15 + \frac{5\sqrt{11}}{2} = \frac{30 + 5\sqrt{11}}{2}\ inches^2$$

Convert to square feet by dividing by 144:

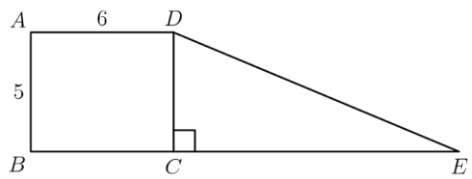
$$= \underbrace{\frac{\frac{30+5\sqrt{11}}{2}}{144}}_{\substack{Nested \\ Fraction}}$$

Write the fraction as a division:

$$= \underbrace{\frac{30 + 5\sqrt{11}}{2} \div 144}_{Fractions\ as\ Division}$$

Convert the division into multiplication by taking the reciprocal:

$$= \frac{30 + 5\sqrt{11}}{2} \times \frac{1}{144} = \frac{30 + 5\sqrt{11}}{288} ft^2$$



Example 1.91

Rectangle $ABCD$ and right triangle DCE have the same area. They are joined to form a trapezoid, as shown. What is DE? (AMC 8 2014/14)

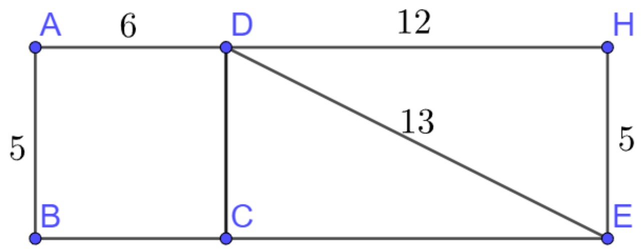
Method I

Flip ΔDCE over DE to get rectangle $DCEH$.

$$[DCEH] = 2[ABCD]$$

But DC is common side:

$$CE = 2DC$$



Method II

The area of the triangle is the same as the area of the square.

$$s^2 = \frac{1}{2}hb$$

In this case, $s = h = DC, b = CE$:

$$DC^2 = \frac{1}{2}(DC)(CE) \Rightarrow CE = DC = 12$$

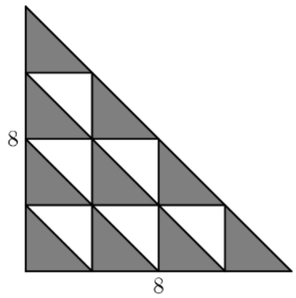
Method III

$$[ABCD] = 5 \times 6 = 30$$
$$DC = AB = 5$$
$$[DCE] = \frac{1}{2}(DC)(CE) = \frac{5}{2}CE = 30 \Rightarrow CE = 12$$

From both the method, we can continue with a Pythagorean Triplet (5,12,13):
$$CE = 13$$

Example 1.92

An isosceles right triangle with legs of length 8 is partitioned into 16 congruent triangles as shown. The shaded area is **(AMC 8 1992/10)**



Method I

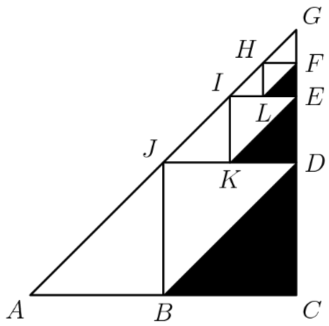
The area of the larger triangle is
$$\frac{1}{2}hb = \frac{1}{2} \times 8 \times 8 = 32$$
Since 10 out of the 16 triangles are shaded, the shaded area is:
$$= \frac{10}{16} \times 32 = \frac{5}{8} \times 32 = 20$$

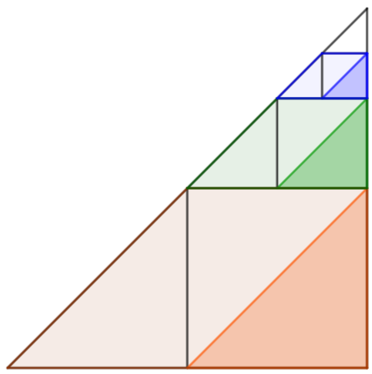
Method II

The area of each shaded triangle
$$\frac{1}{2}hb = \frac{1}{2} \times 2 \times 2 = 2$$
And the area of the shaded region
$$= 2 \times 10 = 20$$

Example 1.93

Points B, D, and J are midpoints of the sides of right triangle ACG. Points K, E, I are midpoints of the sides of triangle JDG, etc. If the dividing and shading process is done 100 times (the first three are shown) and $AC = CG = 6$, then the total area of the shaded triangles is nearest which integer? **(AMC 8 1999/25)**





$$AJDCB: Brown\ Shading = \frac{1}{3}\ Brown\ Part$$
$$Green\ Shading = \frac{1}{3}\ Green\ Part$$
$$Blue\ \ Shading = \frac{1}{3}\ Blue\ Part$$

At each stage the dark shaded part is $\frac{1}{3}$ of the overall area of that color.

Hence, we need to calculate $\frac{1}{3}$ of the area of the triangle:
$$\frac{1}{3}[GCA] = \frac{1}{3} \times \left(\frac{1}{2} \times 6 \times 6\right) = 6\ units$$

We can confirm the $\frac{1}{3}$ numerically as well:
$$[BCD] = [JDG] = \frac{1}{2}hb = \frac{1}{2} \times 3 \times 3 = \frac{9}{2}\ units^2$$

$$[AJDC] = [GCA] - [GDJ] = \frac{36}{2} - \frac{9}{2} = \frac{27}{2}$$

$$\frac{[BCD]}{[AJDC]} = \frac{\frac{9}{2}}{\frac{27}{2}} = \frac{9}{27} = \frac{1}{3}$$

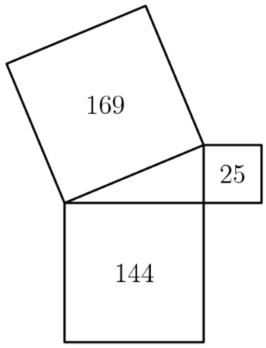
Example 1.94

The two legs of a right triangle, which are altitudes, have lengths $2\sqrt{3}$ and 6. How long is the third altitude of the triangle? **(AMC 10A 2014/9)**

B. Proving that a Triangle is Right

Example 1.95

Given the areas of the three squares in the figure, what is the area of the interior triangle? **(AMC 8 2003/6)**



Example 1.96

The sides x and y of a scalene triangle satisfy $x + \frac{2\Delta}{x} = y + \frac{2\Delta}{y}$, where Δ is the area of the triangle. If $x = 60, y = 63$, what is the length of the largest side of the triangle? **(IOQM 2021/16)**

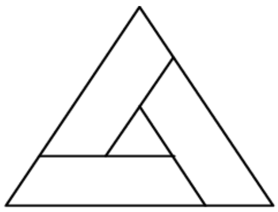
$$\frac{2\Delta}{x} - \frac{2\Delta}{y} = y - x \Rightarrow \frac{2y\Delta - 2x\Delta}{xy} = y - x \Rightarrow \frac{2\Delta(y-x)}{y-x} = xy \Rightarrow \Delta = \frac{xy}{2}$$

$$(x, y, z) = (60, 63, 89)$$

C. Inner and Outer

Example 1.97

In the figure, the outer equilateral triangle has area 16, the inner equilateral triangle has area 1, and the three trapezoids are congruent. What is the area of one of the trapezoids? **(AMC 8 2008/4)**



$$\frac{16-1}{3} = \frac{15}{3} = 5$$

D. Working with Heights and Bases

1.98: Height

Any triangle has three sides. Height is the length of altitude drawn to each side.
Hence, each triangle has three heights.

1.99: Area of a Triangle

The area of a triangle with height h and base b is:

$$A = \frac{1}{2}hb, h = height, b = base$$

In each triangle:

- The shortest side has the longest height, and the longest side has the shortest height.

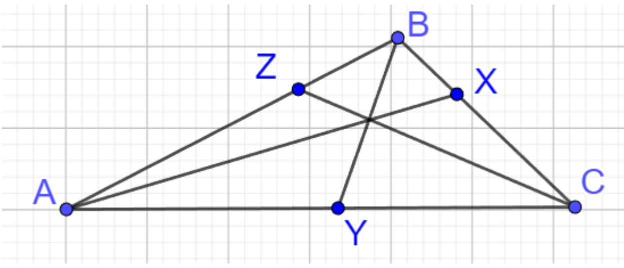
Example 1.100

In ΔABC, the heights to AB, BC and AC are CZ, AX and BY respectively. Also, AC is greater than AB, and AB is greater than BC.

A. Write an expression for the area of the triangle in terms of the sides and the heights.
B. Rank the heights in descending order of their value.

Part A

$$A = \frac{1}{2}hb = \frac{(AX)(BC)}{2} = \frac{(BY)(AC)}{2} = \frac{(CZ)(AB)}{2}$$



Part B

$$AC > AB > BC$$
$$BY < CZ < AX$$

1.101: Product of Height and Base

The product of the height and the base of a triangle is twice the area.

$$2A = (AX)(BC) = (BY)(AC) = (CZ)(AB)$$

Multiply the result from Part A of the example above:

$$2A = (AX)(BC) = (BY)(AC) = (CZ)(AB)$$

1.102: Considering an Area Twice

Certain quantities can be found by calculating the area of a triangle (or other geometrical figure) in two different ways.

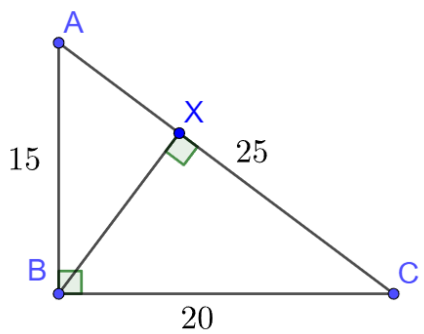
In terms of approach, this is like the idea of setting up an equation that calculates a quantity in two different ways.

Example 1.103

Given a triangle with side lengths 15, 20, and 25, find the triangle's smallest height. (AMC 10A 2002/13)

The sides of the triangle form a Pythagorean Triplet:

$$(15,20,25) = 5 \times (3,4,5) \Rightarrow Right - angled\ Triangle$$



Consider the area of ΔABC calculated in two different ways:

$$\frac{1}{2}(AC)(BX) = \frac{1}{2}(AB)(BC)$$
$$(AC)(BX) = (AB)(BC)$$

Substitute $AC = 25, AB = 15, BC = 20$:

$$25BX = (15)(20)$$
$$BX = 12$$

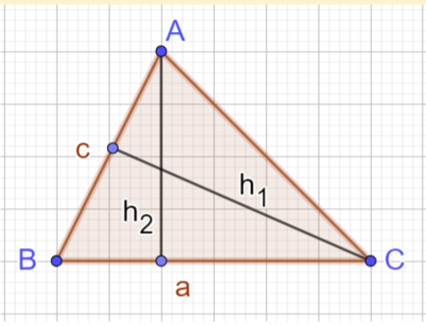
Note: The heights of the triangle are:

$$Side\ AB \Rightarrow Height = BC = 20$$
$$Side\ BC \Rightarrow Height = AB = 15$$
$$Side\ AC \Rightarrow Height = BX = 12$$

$$Smallest\ height = 12$$

Example 1.104[2]

Sarah and Bob are trying to find the area of ΔABC. Sarah mistakenly uses AB and the height from A (instead of the height from C) and Bob mistakenly uses BC and the height from C (instead of the height from A). Sarah finds an area of 12 and Bob finds an area of 27. What is the area of ΔABC?



$$Sarah: \frac{1}{2}(AB)(Height\ from\ A) = \underbrace{\frac{1}{2}ch_2 = 12}_{Equation\ I}$$

$$Bob: \frac{1}{2}(BC)(Height\ from\ C) = \underbrace{\frac{1}{2}ah_1 = 27}_{Equation\ II}$$

Multiply Equation I and II:

$$\left(\frac{1}{2}ch_2\right)\left(\frac{1}{2}ah_1\right) = (12)(27)$$

Rearrange, and substitute $12 \times 27 = (2^2 \times 3) \times (3^3) = 2^2 \times 3^4 = 2^2 \times 9^2$

$$\left(\frac{1}{2}ch_1\right)\left(\frac{1}{2}ah_2\right) = 2^2 \times 9^2$$

Substitute $[ABC] = \frac{1}{2}ch_1 = \frac{1}{2}ah_2$

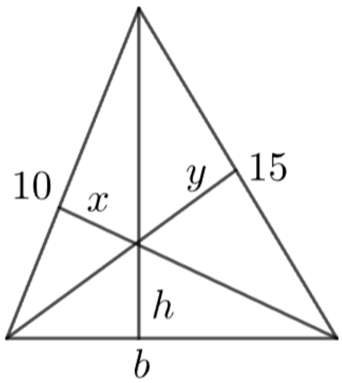
$$[ABC]^2 = 2^2 \times 9^2$$

Take the square root both sides:

$$A(\Delta ABC) = 2 \times 9 = 18$$

Example 1.105

Two sides of a triangle have lengths 10 and 15. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side? **(AMC 10A 2013/15)**



Let the

$$Height\ to\ side\ 10 = x$$
$$Height\ to\ side\ 15 = y$$

For the third side, let

$$Height = h, Base = b$$

We can calculate the area of the triangle in three different ways:

$$\Delta = \frac{1}{2}\cdot 10 \cdot x = \frac{1}{2}\cdot 15 \cdot y = \frac{1}{2}\cdot b \cdot h$$

Multiplying throughout by 2 and equating twice the area to k:

[2] https://artofproblemsolving.com/community/c3t48f3h2559714_mathematics

$$2\Delta = 10x = 15y = hb = k$$

Solve for the base b of the third side in terms of k:

$$hb = k \Rightarrow b = \frac{k}{h}$$

But, since the third side has height which is the average of the heights to the other two sides, substitute $h = \frac{x+y}{2}$:

$$\frac{k}{\frac{x+y}{2}} = \frac{2k}{x+y}$$

Substitute $10x = k \Rightarrow x = \frac{k}{10}$, $15y = k \Rightarrow y = \frac{k}{15}$:

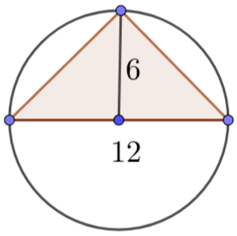
$$\frac{2k}{\frac{k}{10}+\frac{k}{15}} = \frac{2k}{\frac{5k}{30}} = 2k \cdot \frac{30}{5k} = 12$$

E. Inscribed Triangle

Example 1.106

Circle C has radius 6 cm. How many square centimeters are in the area of the largest possible inscribed triangle having one side as a diameter of circle C? **(MathCounts 2000 State Countdown)**

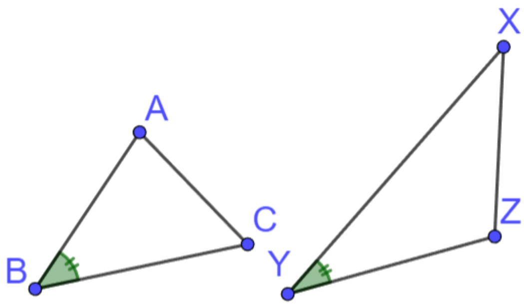
$$\frac{1}{2} \times \underbrace{6}_{\substack{Height\\=Radius}} \times \underbrace{12}_{\substack{Base\\=Diameter}} = 36$$



F. Common Angle

1.107: Ratio of Area of Two Triangles with Angle of Same Measure

If two triangles each have an angle with the same measure, then the ratio of the areas of the two triangles is given by the ratio of the product of the included sides.



- We only need one angle to be the same for the property to hold.

For example, in ΔABC and ΔXYZ, from the diagram, $\angle B = \angle Y$, then we can conclude:

$$\frac{[ABC]}{[XYZ]} = \frac{AB \cdot BC}{XY \cdot YZ}$$

Example 1.108

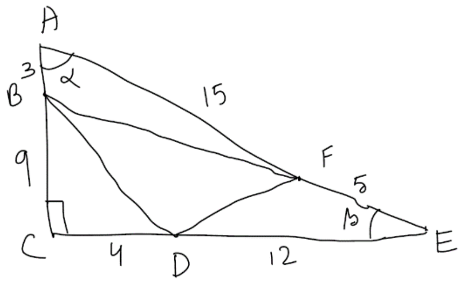
In ΔABC and ΔXYZ, the measure of $\angle B = \angle Y = \alpha$. Given that $AB = 3\ feet$ and $BC = 4\ cm$ and $XY = 3\ cm$ and $YZ = 4\ feet$, determine the ratio of the areas of the two triangles.

$$\frac{[ABC]}{[XYZ]} = \frac{AB \cdot BC}{XY \cdot YZ} = \frac{3\ ft \cdot 4\ cm}{3\ cm \cdot 4\ ft} = \frac{3 \cdot 4 \cdot ft \cdot cm}{3 \cdot 4 \cdot ft \cdot cm} = 1$$

Example 1.109

In the right triangle ΔACE, we have $AC = 12$, $CE = 16$, and $EA = 20$. Points $B, D,$ and F are located on AC, CE, and EA, respectively, so that $AB = 3$, $CD = 4$, and $EF = 5$. What is the ratio of the area of ΔDBF to that of ΔACE? **(AMC 10B 2004/18)**

$$\frac{[ABF]}{[ACE]} = \frac{AB \times AF}{AC \times AE} = \frac{3 \times 15}{12 \times 20} = \frac{3}{16}$$
$$\frac{[BCD]}{[ACE]} = \frac{BC \times CD}{AC \times CE} = \frac{9 \times 4}{12 \times 16} = \frac{3}{16}$$
$$\frac{[FED]}{[ACE]} = \frac{EF \times ED}{AE \times EC} = \frac{5 \times 12}{20 \times 16} = \frac{3}{16}$$



Using Complementary Areas:

$$\frac{[DBF]}{[ACE]} = 1 - \left(\frac{[ABF]}{[ACE]} + \frac{[BCD]}{[ACE]} + \frac{[FED]}{[ACE]}\right) = 1 - \left(\frac{3}{16} + \frac{3}{16} + \frac{3}{16}\right) = 1 - \frac{9}{16} = \frac{7}{16}$$

G. Mixing Concepts

Example 1.110

What is the area of a right-angled triangle (with hypotenuse 3 units) that has one angle equivalent to that of an equilateral triangle?

Double the triangle to get an equilateral triangle.
Area of Right-angled triangle
= 0.5 * Area of Equilateral Triangle
= 0.5 * Root (3)/4 * 3^2
= 9 * Root (3)/8

Example 1.111

What is the maximum possible difference in areas of two triangles with perimeter 21 and integer side lengths (if the one with less area is isosceles)?

A. [21Root (3) – 10]/3
B. [49Root (3) – 20]/4
C. [441Root (3)]/12
D. [64Root (5) – 10]/7

Dimensions of smaller triangle are 1, 10, 10

$h(\text{Smaller } \Delta) = \sqrt{10^2 - 0.5^2} = \sqrt{99.75} \approx 10$

$A(\text{Smaller } \Delta) = \frac{bh}{2} = \frac{1 \times 10}{2} = 5$

$\text{A[Greater } \triangle \text{ (equilateral)]} = \frac{\sqrt{3}}{4} \times 7^2 = \frac{49\sqrt{3}}{4}$

$\text{Difference} = \frac{49\sqrt{3}}{4} - 5 = \frac{49\sqrt{3} - 20}{4}$

Example 1.112

An isosceles right-angled triangle has 2/3 the area of an equilateral triangle with length 3. What is the difference between the lengths of the longest sides of the two triangles?

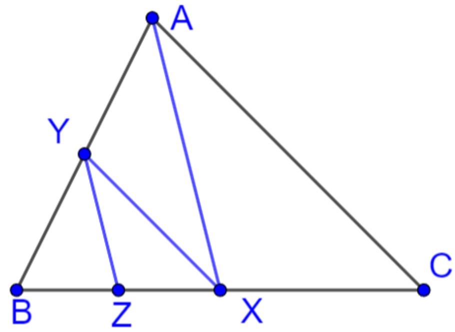
H. Equal Bases and Equal Heights

1.113: Triangles with Equal Heights and Equal Bases

Triangles with equal heights and equal bases have equal areas.

Example 1.114

In ΔABC, draw the median from A to point X lying on BC, the line segment from X to the midpoint Y of AB, the line segment from Y to the midpoint Z of BD. Find the ratio of the area of ΔABC to the area of ΔBYZ.



In ΔAXB and ΔAXC:

$$Base = XB = XC$$
$$Heights\ are\ equal$$
$$[AXB] = [AXC] = \frac{1}{2}[ABC]$$

In ΔXYA and ΔXYB:

$$Base = YA = YB$$
$$Heights\ are\ equal$$
$$[XYB] = [XYC] = \frac{1}{2}[AXB] = \frac{1}{4}[ABC]$$

In ΔYZB and ΔYZX:

$$Base = BZ = ZX$$
$$Heights\ are\ equal$$
$$[YZB] = [YZX] = \frac{1}{2}[XYB] = \frac{1}{4}[AXB] = \frac{1}{8}[ABC]$$

$$[YZB]:[ABC] = 1:8$$

1.115: Triangles with Equal Bases

Triangles with equal bases have areas in the ratio of their heights.

1.116: Triangles with Equal Heights

Triangles with equal heights have areas in the ratio of their heights.

I. Median

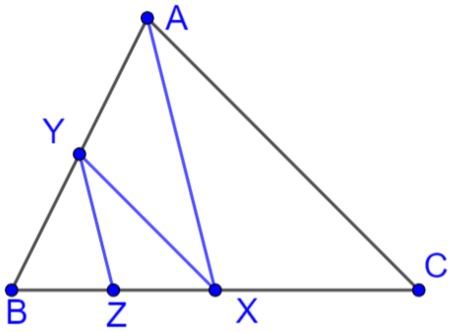
1.117: Median divides triangle into two triangles with equal areas

The two triangles formed by drawing the median of a triangle have equal areas

The two triangles formed have equal heights and equal bases. Hence, their areas are equal.

Example 1.118

In ΔABC, draw the median from A to point X lying on BC, the line segment from X to the midpoint Y of AB, the line segment from Y to the midpoint Z of BD. Find the ratio of the area of ΔABC to the area of ΔBYZ.



AX is the median in ΔABC:

$$[AXB] = [AXC] = \frac{1}{2}[ABC]$$

XY is the median in ΔAXB:

$$[XYB] = [XYC] = \frac{1}{2}[AXB] = \frac{1}{4}[ABC]$$

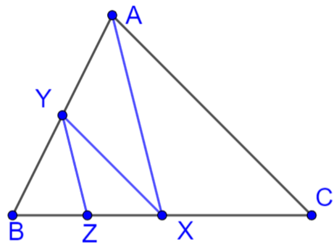
YZ is the median in ΔYBX:

$$[YZB] = [YZX] = \frac{1}{2}[XYB] = \frac{1}{4}[AXB] = \frac{1}{8}[ABC]$$

$$[YZB]:[ABC] = 1:8$$

1.119: Successive medians in a triangle

When drawing successive medians in a triangle, at each stage the area becomes half.



Example 1.120

In ΔABC, draw the median from A to point X lying on BC, the line segment from X to the midpoint Y of AB, the line segment from Y to the midpoint Z of BD. Find the ratio of the area of ΔABC to the area of ΔBYZ.

We have drawn medians three times successively.

Hence, ratio of area of original triangle to smallest triangle is

$$1:2^3 = 1:8$$

1.121: Area of Quadrilateral formed by Midpoints

Show that the area of the quadrilateral formed by joining the midpoints of the sides of a quadrilateral in order has area half that of the original quadrilateral.

J. Ratio of Areas

1.122: Ratio of Areas

If two similar shapes have side lengths in the ratio $a:b$, then their areas are in the ratio

$$a^2:b^2$$

Example 1.123

Two similar triangles have sides in the ratio 1:3. What is the ratio of their areas?

$$Ratio\ of\ areas\ =\ 12:32:\ =\ 1:9$$

Example 1.124

What is the ratio of the area of the smaller triangle formed by joining the midpoints of a triangle to the original triangle?

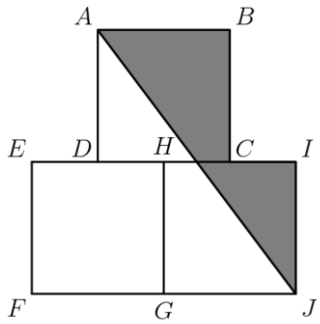
Example 1.125

An equilateral triangle of side length 10 is completely filled in by non-overlapping equilateral triangles of side length 1. How many small triangles are required? **(AMC 10B 2008/7)**

Example 1.126

Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$? **(AMC 10B 2017/19)**

K. Complementary Areas



Example 1.127

Squares $ABCD$, $EFGH$, and $GHIJ$ are equal in area. Points C and D are the midpoints of sides IH and HE, respectively. What is the ratio of the area of the shaded pentagon $AJICB$ to the sum of the areas of the three squares? **(AMC 8 2013/24)**

Add Point D to make Rectangle $BDIC$.

$$[AJICB] = [AJD] - [BDIC] = \frac{1}{2} \times 2 \times 1.5 - 1 \times \frac{1}{2} = 1.5 - 0.5 = 1$$

Ratio

$$= 1:3$$

L. Relationships via Equations

1.128: Sum of Parts

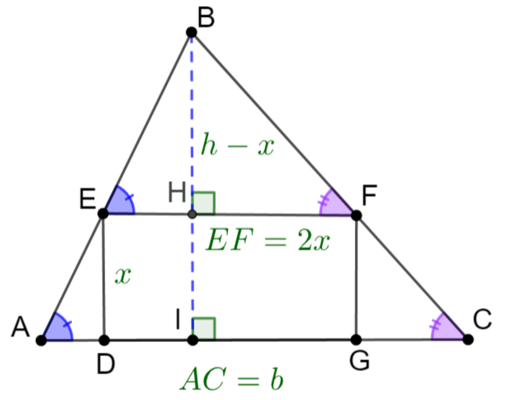
The area of parts of a triangle adds up to the area of the entire triangle.

1 Pending

Example 1.129

A rectangle inscribed in a triangle has its base coinciding with the base b of the triangle. If the altitude of the triangle is h, and the altitude x of the rectangle is half the base of the rectangle, then, find x in terms of h and b: **(AHSME 1950/47)**

Draw ΔABC with rectangle $DEFG$. Draw altitude BI[3].



$$[ABC] = [BEF] + [AED] + [GFC] + [DEFG]$$

Calculate the areas:

$$[BEF] = \frac{1}{2} \cdot EF \cdot BH = \frac{1}{2} \cdot 2x \cdot (h-x) = xh - x^2$$

$$[AED] + [GFC] = \frac{1}{2} \cdot (AD + GC) \cdot ED = \frac{1}{2} \cdot (b - 2x) \cdot x = \frac{bx}{2} - x^2$$

$$[DEFG] = x \cdot 2x = 2x^2$$

Substitute into Equation I:

$$\frac{1}{2}hb = (xh - x^2) + \left(\frac{bx}{2} - x^2\right) + 2x^2$$

$$\frac{1}{2}hb = xh + \frac{bx}{2}$$

$$hb = 2xh + bx$$

$$hb = x(2h + b)$$

$$x = \frac{hb}{2h + b}$$

M. Other Concepts

Area of a triangle (or any polygon) will be maximum when the side lengths are equal.
Area of a triangle (or any polygon) will be minimum when the difference in side lengths is maximum.

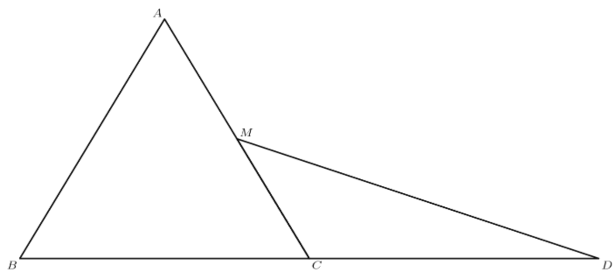
Consider a rectangle with perimeter 18, and integer side lengths.
Hence, the sum of the length and the breadth is 9.

Possible values of the sides are:

Area $(1, 8) = 8$	8	7
Area $(2, 7) = 14$	14	5
Area $(3, 6) = 18$	18	3
Area $(4, 5) = 20$	20	1

Example 1.130

Equilateral $\triangle ABC$ has side length 2, M is the midpoint of $\overline{AC}$, and C is the midpoint of $\overline{BD}$. What is the area of $\triangle CDM$? (AMC

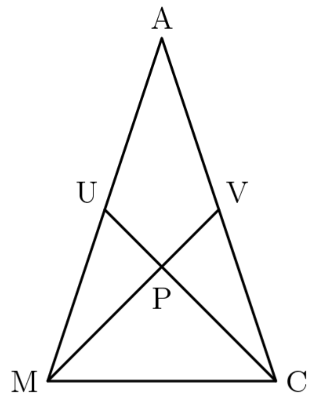


[3] This is solved using Similarity using in the Note on that topic.

10B 2005/14)

Example 1.131

Triangle AMC is isosceles with $AM = AC$. Medians $\overline{MV}$ and $\overline{CU}$ are perpendicular to each other, and $MV = CU = 12$. What is the area of $\triangle AMC$? **(AMC 10A 2020/12)**



1.4 Area: Hero's Formula

A. Area in Terms of Height and Base

We already know one formula for the area of a triangle.

1.132: Area of a Triangle

$$Area = A = \frac{1}{2}hb$$

- If we do not know the height, we can still calculate the area using Heron's Formula.
- Heron's Formula is computationally involved, so we usually prefer other formulas before it.

B. Heron's Formula / Hero's Formula[4]

1.133: Perimeter

If $a, b \text{ and } c \text{ are the sides of the triangle,}$

$$Perimeter = p = a + b + c$$

$$Semiperimeter = s = \frac{p}{2} = \frac{a+b+c}{2}$$

Where

$$a, b \text{ and } c \text{ are the sides of the triangle}$$

The sum of the lengths of the sides of a triangle is the perimeter
Half of the sum of the lengths of the sides of a triangle is the perimeter

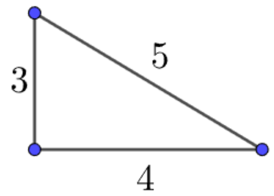
Example 1.134

A triangle has side lengths 3,4 and 5. Find its

A. Perimeter
B. Semiperimeter

$$Perimeter = p = 3 + 4 + 5 = 12$$

$$Semiperimeter = s = \frac{p}{2} = \frac{12}{2} = 6$$



1.135: Heron's Formula

If a, b and c are the sides of a triangle and s is its semiperimeter, then the area of the triangle is:

$$\sqrt{s\,(s-a)(s-b)(s-c)}$$

Example 1.136

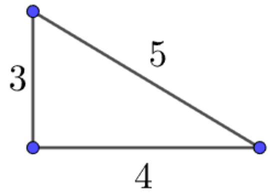
Find the area of a triangle with side length 3,4 and 5 using

A. $Area = \frac{1}{2}hb$
B. $Area = \sqrt{s\,(s-a)(s-b)(s-c)}$

[4] Hero of Alexandria (also called of Alexandria) was a Greek mathematician. This formula for the area of a triangle is generally called by his name.

Part A

$$A = \frac{1}{2}hb = \frac{1}{2}(3)(4) = 6$$



Part B

$$s = \frac{p}{2} = \frac{3+4+5}{2} = \frac{12}{2} = 6$$

$$A = \sqrt{s\,(s-a)(s-b)(s-c)}$$

Substitute $s = 6, a = 3, b = 4, c = 5$ in the above:

$$= \sqrt{6\,(6-3)(6-4)(6-5)}$$

Simplify:

$$= \sqrt{6\,(3)(2)(1)} = \sqrt{6(6)} = \sqrt{36} = 6$$

Example 1.137

A triangle has side lengths 20 units, 15 units and 7 units?

A. What is its perimeter?
B. What is its area?
C. Compare perimeter with area.

Part A

$$Perimeter = 20 + 15 + 7 = 42$$

Part B

The semiperimeter is:

$$s = \frac{p}{2} = \frac{42}{2} = 21$$

Substitute $s = 21, a = 20, b = 15, c = 7$ in $\sqrt{s\,(s-a)(s-b)(s-c)}$

$$Area = \sqrt{21\,(21-20)(21-15)(21-7)} = \sqrt{21\,(1)(6)(14)}$$

Rather than multiplying and taking the square root, we prime factor:

$$= \sqrt{7 \times 3 \times 3 \times 2 \times 2 \times 7}$$

Collate like numbers together:

$$= \sqrt{7^2 \times 3^2 \times 2^2}$$

Take the square roots:

$$= 7 \times 3 \times 2 = 42$$

Part C

For this triangle, the numerical value of the perimeter is the same as the numerical value of the area.

Note: We cannot say that the perimeter is the same as the area because their units are different.

C. Heronian Triangles

Because of the square root in the formula, we are not guaranteed that answers for area will be integers. Hence, triangles with integers have a name.

Three integers that are the sides of a right triangle are a Pythagorean Triplet. Similarly, three integers that are the sides of a triangle with integer area are a Heronian Triangle.

1.138: Heronian Triangle

A triangle with integer side lengths and integer area is called a Heronian Triangle

Example 1.139

Find the area of the following triangles with side lengths:

A. 15, 13 and 4
B. 6, 5 and 5
C. 8, 5 and 5
D. 13, 12 and 5

$$\sqrt{s\,(s-a)(s-b)(s-c)}$$

Part A

Substitute $s = \frac{p}{2} = \frac{15+13+4}{2} = \frac{32}{2} = 16, a = 15, b = 13, c = 4$:

$$\sqrt{16\,(1)(3)(12)} = \sqrt{16\,(36)} = 4 \times 6 = 24$$

Part B

Substitute $s = \frac{p}{2} = \frac{6+5+5}{2} = \frac{16}{2} = 8, a = 6, b = 5, c = 5$

$$\sqrt{8\,(2)(3)(3)} = \sqrt{16\,(9)} = 4 \times 3 = 12$$

Part C

Substitute $s = \frac{p}{2} = \frac{8+5+5}{2} = \frac{18}{2} = 9, a = 8, b = 5, c = 5$:

$$\sqrt{9\,(1)(4)(4)} = \sqrt{16\,(9)} = 4 \times 3 = 12$$

Part D

Substitute $s = \frac{p}{2} = \frac{13+12+5}{2} = \frac{30}{2} = 15, a = 13, b = 12, c = 5$:

$$\sqrt{15\,(2)(3)(10)} = \sqrt{900} = 30$$

Example 1.140

A garden is in the shape of a triangle. The side lengths of the garden are 17 meters, 10 meters and 9 meters. What is the area of the garden?

The semiperimeter is:

$$s = \frac{17+10+9}{2} = \frac{36}{2} = 18$$

The area is:

$$A = \sqrt{(18)(1)(8)(9)} = \sqrt{(9)(1)(16)(9)} = 9 \times 4 = 36$$

D. Nested Triangles

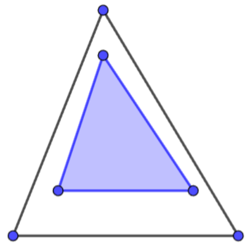
Example 1.141

A triangle with side lengths of 26 units, 25 units and 3 units is drawn inside a triangle with side lengths of 30 units, 29 units and 5 units. Find the area of the region between the two triangles.

Inner Triangle

$$s = \frac{26+25+3}{2} = \frac{54}{2} = 27$$

$$A = \sqrt{(27)(1)(2)(24)} = \sqrt{(9)(1)(12)(12)} = 3 \times 12 = 36$$



Outer Triangle

$$s = \frac{30 + 29 + 5}{2} = \frac{64}{2} = 32$$
$$A = \sqrt{(32)(2)(3)(27)} = \sqrt{(64)(81)} = 8 \times 9 = 72$$

Shaded Area

Area between the two triangles

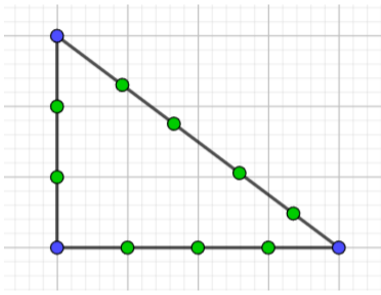
$$= 72 - 36 = 36\, unit^2$$

E. Costs

Challenge 1.142

A triangular garden with sides $29m$, $25m$ and $6m$ is to be fenced, and planted with grass. Fence poles are $1m$ apart, but vertices of the triangle need two poles each. Fence costs $\frac{\$4}{m}$. Each pole costs \$5. Grass costs $\frac{\$7}{m^2}$. What is the total cost?

Number of Poles Needed - Concept



We need two poles each at the vertices, which are represented using blue points on the diagram.
If the side length is 2, then we need a single pole between the vertices, colored green.
If the side length is 3, then we need two poles between the vertices, colored green.

In general, the number of green poles needed for side length n is

$$n - 1$$

Cost of Poles

Number of poles needed at the vertices

$$= 3 \times 2 = 6$$

Number of poles needed at the sides

$$= (29 - 1) + (25 - 1) + (6 - 1) = 57$$

Total number of poles

$$= 57 + 6 = 63$$

Total cost of the poles

$$= 5 \times 63 = \$315$$

Cost of Fence

Cost of the fence

$$= 4p = 4(29 + 25 + 6) = 4(60) = 240$$

Cost of Grass

$$Semiperimeter = s = \frac{29 + 25 + 6}{2} = \frac{60}{2} = 30$$
$$Area = A = \sqrt{(30)(1)(5)(24)} = \sqrt{(12)(1)(25)(12)} = 12 \times 5 = 60$$
$$Cost = 7A = 7 \times 60 = 420$$

Total Cost

$$= 315 + 240 + 420 = 975$$

F. Back Calculations / Number Theory

Since Heronian Triangles have integer side lengths, we use this while solving for the values of the sides of the triangle. This takes us in the direction of Number Theory, and Diophantine Equations.

Example 1.143

A triangle with area 24 units and perimeter 32 units has a side length of 15. If all sides of the triangle are integers, find the length of the other two sides.

$$Perimeter = 15 + b + c = 32$$
$$b + c = 17$$

Case I:

$$b = 1 \Rightarrow c = 16$$
$$15,1,16$$
$$15 + 1 = 16 \Rightarrow Not\ Valid$$

Case II:

$$b = 2 \Rightarrow c = 15$$
$$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{16(1)(14)(1)} = \sqrt{16(1)(14)(1)} = 4\sqrt{14}$$

Case III:

$$b = 3 \Rightarrow c = 14$$
$$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{16(1)(13)(2)}$$

Case IV:

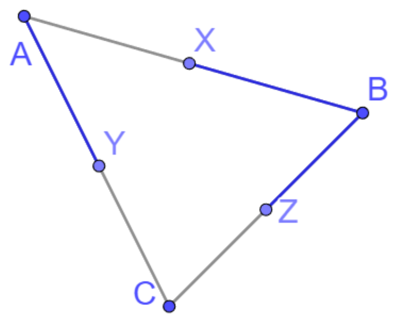
$$b = 4 \Rightarrow c = 13$$
$$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{16(1)(12)(3)} = \sqrt{16(36)} = 4 \times 6 = 24$$

1.144: Semiperimeter is greater than any side

For a triangle with sides a, b and c its semiperimeter is greater than the length of any side.

$$s - a > 0$$
$$s - b > 0$$
$$s - c > 0$$

Consider a general ΔABC with sides a, b, c.



$$s > c$$
$$\frac{a+b+c}{2} > c$$

Split the fraction on the LHS:

$$\frac{a+b}{2} + \frac{c}{2} > c$$

Subtract $\frac{c}{2}$ from both sides:

$$\frac{a+b}{2} > \frac{c}{2}$$

Multiply both sides by 2:

$$a + b > c$$ [5]

[5] We cannot begin a proof with the property to be proved, but in this case the steps are reversible, and hence it is valid.

Example 1.145

A triangle with area 24 units and perimeter 32 units has a side length of 15. If all sides of the triangle are integers, find the length of the other two sides.

$$Semiperimeter = s = \frac{32}{2} = 16$$

Let the sides of the triangle be

$$15, b \text{ and } c$$

Substitute $a = 15, s = 16, A = 24$ in $A = \sqrt{s(s-a)(s-b)(s-c)}$:

$$24 = \sqrt{(16)(1)(16-b)(16-c)}$$

Prime Factorize 24 and 16:

$$2^3 \times 3 = \sqrt{(2^4)(1)(16-b)(16-c)}$$

Square both sides. Note that on the RHS, the square and the square root cancel:

$$2^6 \times 3^2 = 2^4(16-b)(16-c)$$

Divide both sides by 2^4:

$$36 = (16-b)(16-c)$$

Because the sides of the triangle are integers, each factor on the right must be an integer.
Hence, we find the factor pairs of 36:

$$(1,36)(2,18)(3,12)(4,9)$$

Any of the individual sides cannot be greater than the semiperimeter.

$$16 - b > 0 \Rightarrow 16 > b \Rightarrow b < 16$$
$$16 - c > 0 \Rightarrow 16 > c \Rightarrow c < 16$$

$$Valid\ Options\ are\ (3,12)(4,9)$$

Try the factor pair (3,12):

$$16 - b = 3 \Rightarrow b = 13$$
$$16 - c = 12 \Rightarrow c = 4$$

$$Perimeter = a + b + c = 15 + 13 + 4 = 32 \Rightarrow Works$$

(Alternate Solution) Example 1.146

A triangle with area 24 units and perimeter 32 units has a side length of 15. Find the length of the other two sides.

This time we do not know that the sides are integers, so we change our approach:

$$15 + b + c = 32 \Rightarrow c = 17 - b$$

$$24 = \sqrt{(16)(1)(16-b)\big(16-(17-b)\big)}$$
$$36 = (16-b)(-1+b)$$
$$36 = -16 + 16b + b - b^2$$
$$b^2 - 17b + 52 = 0$$

$$(b-13)(b-4)=0$$
$$b \in \{4,13\}$$

Example 1.147

A triangle with integer side lengths has area 84, perimeter 48 units and a side length of 21. Find the sides of the triangle.

$$s=\frac{48}{2}=24$$
$$84=\sqrt{(24)(3)(24-b)(24-c)}$$
$$2^4 \times 3^2 \times 7^2 = 2^3 \times 3^2(24-b)(24-c)$$
$$2 \times 7^2 = (24-b)(24-c)$$
$$98=(24-b)(24-c)$$

$$Factor\ Pairs\ of\ 98 = (1,98)(2,49)(7,14)$$

$$24-b=7 \Rightarrow b=24-7=17$$
$$24-c=14 \Rightarrow c=24-14=10$$

$$21+17+10=48 \Rightarrow Works$$

Example 1.148

A triangle with integer side lengths has area 84, perimeter 72 units and a side length of 35. Find the sides of the triangle.

$$s=\frac{72}{2}=36$$
$$84=\sqrt{(36)(1)(36-b)(36-c)}$$
$$2^4 \times 3^2 \times 7^2 = 2^2 \times 3^2(36-b)(36-c)$$
$$2^2 \times 7^2 = (36-b)(36-c)$$
$$196=(36-b)(36-c)$$

$$Factor\ Pairs\ of\ 196 = (1,196)(2,98)(4,49)(7,28)$$

$$36-b=7 \Rightarrow b=36-7=29$$
$$36-c=28 \Rightarrow c=36-28=8$$

$$35+29+8=72 \Rightarrow Works$$

G. Primitive Heronian Triangles

When we learn Pythagorean Triplets, which are the side lengths of a right triangle, which are integers, we are most interested in Primitive Pythagorean Triplets, since all other triplets are multiples of these.
Similarly, for Heronian Triangles, we are interested in primitive Heronian Triangles.

1.149: Primitive Heronian Triangles

A Heronian triangle with sides a, b, and c is primitive if
$$HCF(a,b,c)=1$$

Example 1.150

The following triangles are Heronian Triangles. Are they primitive Heronian Triangles?

A. 3,4,5
B. 6,5,5
C. 26,25,3
D. 20,15,7
E. 30,26,8
F. 65,55,12

$$HCF(3,4,5) = 1 \Rightarrow Primitive$$
$$HCF(6,5,5) = 1 \Rightarrow Primitive$$
$$HCF(26,25,3) = 1 \Rightarrow Primitive$$
$$HCF(20,15,7) = 1 \Rightarrow Primitive$$
$$HCF(30,26,8) = 2 \Rightarrow Non-Primitive$$
$$HCF(65,55,12) = 1 \Rightarrow Primitive$$

1.151: List of Primitive Heronian Triangles

The table below gives a list of Heronian triangles, arranged in order of increasing area. These are all primitive.

Side Lengths	Perimeter	Semi-Perimeter	Area
15, 13, 4			24
17, 10, 9			36
26, 25, 3			36
20, 15, 7			42
29,25,6			60
20,13,11			66
30,29,5			72
15,14,13			84
21,17,10			84
35,29,8			84

1.152: Pythagorean Triplet Triangles

All triangles with side lengths which are Pythagorean Triplets are Heronian Triangles.

All Pythagorean Triplets have integer side lengths. The area of the triangle is given by:

$$\frac{1}{2}hb$$

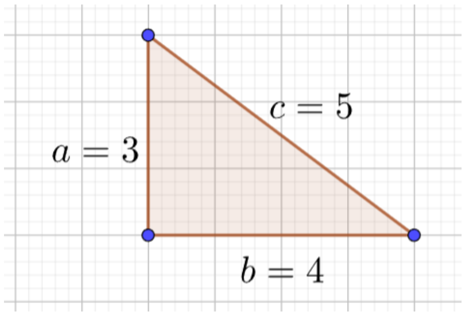
In the adjoining table, note that exactly one of the legs is even in every Pythagorean Triplet.

Primitive Pythagorean Triplets			
3, 4, 5	20, 21, 29	11, 60, 61	36, 77, 85
5, 12, 13	12, 35, 37	33, 56, 65	13, 84, 85
8, 15, 17	9, 40, 41	16, 63, 65	39, 80, 89
7, 24, 25	28, 45, 53	48, 55, 73	65, 72, 97

Hence:

$$\frac{1}{2}hb \text{ is always an integer} \Rightarrow \text{All Pythagorean Triplets are Heronian Triangles}$$

(Note: This is not a proof of the property).



1.153: Pythagorean Triple

For a right-angled triangle with sides p, q, r, where (p, q, r) is a primitive Pythagorean Triplet, and r is the hypotenuse,

$$Area = \frac{1}{2}hb = \frac{1}{2}pq \in \mathbb{N}$$

We want to show that exactly one of p and q is even. That is, in the table alongside, either Case III is true, or Case IV is true.

We will do this using proof by contradiction.

	I	II	III	IV
p	Even	Odd	Even	Odd
q	Even	Odd	Odd	Even

Suppose Case I is true.

$$p \text{ is even} \Rightarrow p^2 \text{ is even}$$
$$q \text{ is even} \Rightarrow q^2 \text{ is even}$$
$$\underbrace{p^2}_{Even} + \underbrace{q^2}_{Even} = \underbrace{r^2}_{Even} \Rightarrow HCF(p, q, r) \geq 2 \Rightarrow (p, q, r) \text{ is not primitive} \Rightarrow Contradiction$$

Hence, Case I cannot be true.

Suppose Case II is true.

$$p \text{ is odd}, q \text{ is odd}$$

Any odd number must be of the form

$$4n \pm 1$$
$$(4n \pm 1)^2 = 16n^2 \pm 8n + 1$$

$$x \equiv 1(mod\ 4)\ Or\ x \equiv 3(mod\ 4) \Rightarrow x \equiv \pm 1(mod\ 4)$$

$$x^2 \equiv 1(mod\ 4)$$

$1 = 0 + 1$	$3 = 4 - 1$
$5 = 4 + 1$	$7 = 8 - 1$
One More than a Multiple of 4, then it can be written as: $4n + 1$	Three More than a Multiple of 4, then it can be written as: $4m + 3 = 4n - 1$

$$p^2 + q^2 \equiv 1 + 1 \equiv 2(mod\ 4)$$

The above statement means that $p^2 + q^2$ is divisible by 2, but not by 4.

However, $p^2 + q^2 = r^2$ and r^2 is a perfect square.
A perfect square that is divisible by 2, must be divisible by 4.

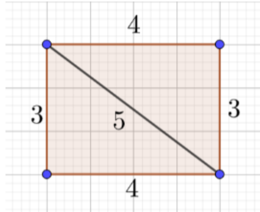
$$Contradiction$$
$$Case\ II\ is\ not\ possible.$$

1.154: Joining Pythagorean Triplet Triangles

A Pythagorean Triplet results in a Heronian Triangle. Putting two triangles which have side lengths from a Pythagorean Triplet also results in a Heronian Triangle.

Example 1.155

Two triangles with sides 3, 4 and 5 are joined together at a side of the same length. Find the possible values of the perimeter of the resulting shape.

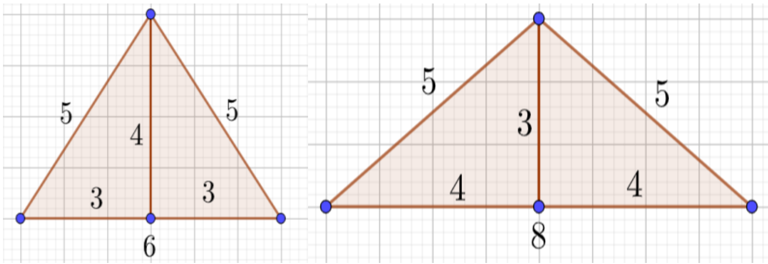


Case I: Joined at the Hypotenuse
This results in a square.

$$Perimeter = 2(4+3) = 2 \times 7 = 14$$

Case II: Joined at a Leg
This has two possible outcomes, both resulting in a Heronian Triangle with

$$Perimeter = 5+5+6 = 16$$
$$Perimeter = 5+5+8 = 18$$



H. Integer Multiples

A triangle with sides which are an integer multiple of a Heronian Triangle will also be Heronian.

1.156: Multiple of a Heronian Triangle

If (a, b, c) is a Heronian Triangle with area A^2, then the triangle (xa, xb, xc) is also a Heronian Triangle with area x^2A.

$$Area(a, b, c) = A \Rightarrow Area(xa, xb, xc) = x^2A$$

Each side of the first triangle is multiplied by the (same) constant multiple to get the second triangle.
Hence, the second

I. Non-Heronian Triangles: Integer Semi-Perimeter

Example 1.157

What is the area of a triangle with sides 5, 6, and 7?

$$Semiperimeter = s = \frac{a+b+c}{2} = \frac{5+6+7}{2} = \frac{18}{2} = 9$$
$$A = \sqrt{9\,(2)(3)(4)} = 6\sqrt{6}$$

J. Non-Heronian Triangles: Non-Integer Semi-Perimeter

If the semiperimeter of a triangle is odd, then each term in the formula is divided by 2. This generally increases the calculations.

Example 1.158

What is the area of a triangle with sides 3, 4, and 6?

$$Semiperimeter = s = \frac{a+b+c}{2} = \frac{3+4+6}{2} = \frac{13}{2} = 6.5$$

Substitute $a = 3, b = 4, c = 6$ into Heron's formula $\left(A = \sqrt{s(s-a)(s-b)(s-c)}\right)$:

$$A = \sqrt{6.5\,(6.5-3)(6.5-4)(6.5-5)} = \sqrt{\left(\frac{13}{2}\right)\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{1}{2}\right)} = \frac{1}{4}\sqrt{13\times 7\times 5} = \frac{\sqrt{455}}{4}$$

Example 1.159

A triangle with sides 4, 5, 6 and another triangle with height 4 have the same area. What is the length of the base of the second triangle?

$$s = \frac{4+5+6}{2} = \frac{15}{2} = 7.5$$

$$A = \sqrt{7.5(7.5-4)(7.5-5)(7.5-6)} = \sqrt{\left(\frac{15}{2}\right)\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)} = \frac{15\sqrt{7}}{4}$$

$$\frac{1}{2}hb = A \Rightarrow \frac{1}{2}\times 4\times b = \frac{15\sqrt{7}}{4} \Rightarrow\ b = \frac{15\sqrt{7}}{8}$$

K. Triangle Inequality

1.160: Triangle Inequality

In a triangle with sides a, b and c:

$$a < b + c$$
$$b < a + c$$
$$c < a + b$$

In a triangle, any side is less than the sum of the other two sides.

We take it as an axiom in geometry that

A straight line is the shortest distance between two points.

And the property above directly follows from the axiom.

Example 1.161

In $\Delta ABC, AB = 4, BC = 6$ and $AC = 10$.

A. Does this triangle satisfy the triangle inequality?
B. What is the area of ΔABC?
C. Draw a diagram of ΔABC? What is the interpretation of area that you get?

Part A

$$AB + BC = 4 + 6 = 10 = AC$$

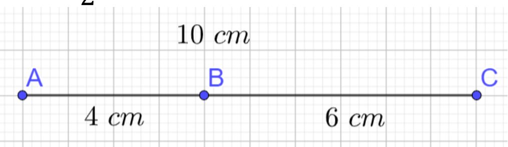
This violates the triangle inequality.

Part B

$$Semiperimeter = s = \frac{a+b+c}{2} = \frac{4+6+10}{2} = \frac{20}{2} = 10$$

$$A = \sqrt{10(0)(6)(4)} = 0$$

Part C



There is only one interpretation of the data above. The points A, B and C are collinear.

1.162: Degenerate Triangles

Triangles with zero area are degenerate triangles.

Example 1.163

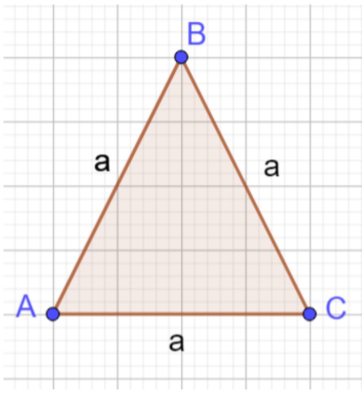
A. What is the area of a triangle with $AB = 4, BC = 5$ and $AC = 10$?
B. Is it possible to draw a diagram of the "triangle" above? Does that let you interpret the area you got?

L. Equilateral Triangles

1.164: Heron's Formula: Equilateral Triangle

If an equilateral triangle has side length a, then show that:

$$\Delta = \frac{\sqrt{3}}{4} \times a^2$$



Heron's Formula for the area of a triangle is:

$$\sqrt{s\,(s-a)(s-b)(s-c)}$$

In Equilateral ΔABC:

$$a = b = c$$

Hence,

$$s = \frac{a+b+c}{2} = \frac{a+a+a}{2} = \frac{3a}{2}$$

Substitute, $s = \frac{3a}{2}, c = a, b = a$ in Heron's Formula:

$$\Delta = \sqrt{\left(\frac{3a}{2}\right)\left(\frac{3a}{2} - a\right)\left(\frac{3a}{2} - a\right)\left(\frac{3a}{2} - a\right)}$$

Simplify:

$$= \sqrt{\left(\frac{3a}{2}\right)\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)} = \frac{a^2}{4}\sqrt{3} = \frac{\sqrt{3}}{4} \times a^2$$

Which is what we wanted to prove.

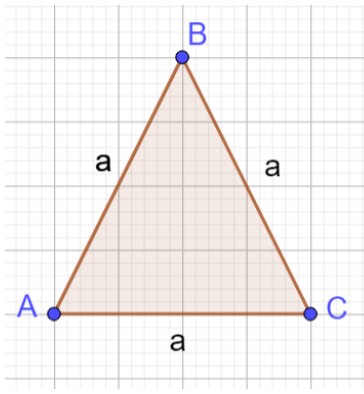
$$Q.E.D$$

Example 1.165

Find the area of an equilateral triangle with side length $\sqrt{3}$ units.

Example 1.166

If an equilateral triangle has area 36, then find the perimeter of the triangle.



1.167: Heron's Formula: Equilateral Triangle

If an equilateral triangle has side length a, then show that the height of the triangle is

$$h = \frac{\sqrt{3}}{2}a$$

We showed earlier that:

$$\Delta = \frac{\sqrt{3}}{4} \times a^2$$

Also, we know that:

$$\Delta = \frac{1}{2} \times hb = \frac{1}{2} \times ha$$

And since both the formulas have to give the same answer:

$$\frac{1}{2} \times ha = \frac{\sqrt{3}}{4} \times a^2 \Rightarrow h = \frac{\sqrt{3}}{2}a$$

Example 1.168

Find the height of an equilateral triangle with side length 4 units.

Example 1.169

Find the height of an equilateral triangle with area $\sqrt{4}$ $units$.

M. Isosceles Triangle

1.170: Heron's Formula: Isosceles Triangle

If an isosceles triangle has two equal sides, each with length a and base b, then show that:

$$Area\ of\ the\ Triangle = \Delta = \frac{b}{4}\sqrt{4a^2 - b^2}$$

Heron's Formula for the area of a triangle is:

$$\sqrt{s\,(s-a)(s-b)(s-c)}$$

In Isosceles ΔABC:

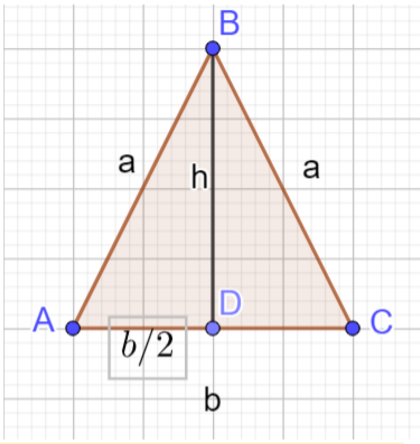
$$c = third\ side = a$$

Hence,

$$s = \frac{a+b+c}{2} = \frac{a+a+b}{2} = \frac{2a+b}{2}$$

Substitute, $s = \frac{2a+b}{2}$, $c = a$ in Heron's Formula:

$$\Delta = \sqrt{\left(\frac{2a+b}{2}\right)\left(\frac{2a+b}{2} - a\right)\left(\frac{2a+b}{2} - b\right)\left(\frac{2a+b}{2} - a\right)}$$



$$= \sqrt{\left(\frac{2a+b}{2}\right)\left(\frac{b}{2}\right)\left(\frac{2a-b}{2}\right)\left(\frac{b}{2}\right)} = \sqrt{\left(\frac{2a+b}{2}\right)\left(\frac{2a-b}{2}\right)\left(\frac{b}{2}\right)^2}$$

Apply $(x+y)(x-y) = x^2 - y^2$ inside the square root, and note that we can move some perfect squares outside the square root:

$$= \frac{b}{4}\sqrt{4a^2 - b^2}$$

Which is what we wanted to prove.

$Q.E.D$

Example 1.171

Show that a triangle with sides 5, 5 and 6 is Heronian.

Example 1.172

If an isosceles triangle with area 24 has base 6, then find the perimeter of the triangle.

N. Right-Angled Triangle

We can exploit the symmetry in right-angled triangles

1.173: Heron's Formula: Right-Angled Triangle

If a right-angled triangle has legs a and b, and hypotenuse c, then show that:

$$Area\ of\ the\ Triangle = s(s-c), \qquad s = \frac{a+b+c}{2}$$

Substitute $s = \frac{a+b+c}{2}$ in $s(s-c)$:

$$\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2} - c\right) = \left(\frac{(a+b)+c}{2}\right)\left(\frac{(a+b)-c}{2}\right)$$

Carry out the multiplication using $(x+y)(x-y) = x^2 - y^2$:

$$\frac{(a+b)^2 - c^2}{4} = \frac{a^2 + 2ab + b^2 - c^2}{4} \to \boldsymbol{I}$$

Since the triangle is right-angled, so, by the Pythagorean Theorem:

$$a^2 + b^2 = c^2 \Rightarrow a^2 + b^2 - c^2 = 0 \to \boldsymbol{II}$$

Substitute **II** in **I** to get:

$$\frac{2ab}{4} = \frac{ab}{2}$$

Note that the last expression is the area of a right-angled triangle.

Hence,

$$s(s-c) = \frac{ab}{2} = Area\ of\ the\ Triangle$$

Where $\frac{ab}{2}$ is the area of a right triangle by substituting $h = a, b = b$ in $\Delta = \frac{1}{2} \times hb$

Example 1.174

The perimeter of a right-angled triangle is 11. The hypotenuse of the triangle is 5. Find the area of the triangle.

$$s = \frac{p}{2} = \frac{11}{2} = 5.5$$
$$A = s(s-c) = 5.5(5.5-5) = 5.5(0.5) = 2.75$$

O. Joining Midpoints

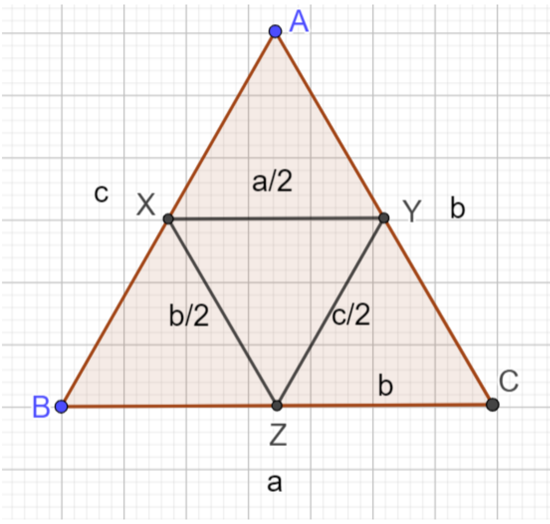
1.175: Area of Triangle formed by joining midpoints of a Triangle

Show that the triangle formed by joining the midpoints of the sides of a triangle has area $\frac{1}{4}th$ the original triangle.

Note: This property can be proved using similar triangles, but we will do it here using Heron's Formula. It is useful to be able to prove a property in multiple ways, showing the connection between different areas of mathematics.

Area of ΔABC

Consider ΔABC with sides lengths $a, b\ and\ c$.



Substituting $s = \frac{a+b+c}{2}$ in $\sqrt{s(s-a)(s-b)(s-c)}$:

$$= \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2}-a\right)\left(\frac{a+b+c}{2}-b\right)\left(\frac{a+b+c}{2}-c\right)}$$
$$= \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{a+b-c}{2}\right)}$$
$$= \frac{1}{4}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}$$

Area of ΔXYZ

Let $X, Y\ and\ Z$ be the midpoints of AB, BC and CA respectively.
Then, by the Midpoint Theorem:

$$XY = \frac{a}{2}, \qquad XZ = \frac{b}{2}, \qquad YZ = \frac{c}{2}$$

For ΔXYZ:

$$s = \frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2} = \frac{a+b+c}{4}$$

Substituting $s = \frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2} = \frac{a+b+c}{4}$ and the sides as $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ in $\sqrt{s(s-a)(s-b)(s-c)}$, we get:

$$= \sqrt{\left(\frac{a+b+c}{4}\right)\left(\frac{a+b+c}{4}-\frac{a}{2}\right)\left(\frac{a+b+c}{4}-\frac{b}{2}\right)\left(\frac{a+b+c}{4}-\frac{c}{2}\right)}$$
$$= \sqrt{\frac{a+b+c}{4}\left(\frac{b+c-a}{4}\right)\left(\frac{a+c-b}{4}\right)\left(\frac{a+b-c}{4}\right)}$$
$$= \frac{1}{16}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}$$
$$= \frac{1}{4}\left(\frac{1}{4}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}\right)$$
$$= \frac{1}{4}([ABC])$$

P. Area of Quadrilaterals

1.176: Area of General Quadrilateral

Example 1.177

In a quadrilateral-shaped field, the corner connecting the sides with length 20 and 11 has a fence put up going to the other corner, which has sides of 15 and 14. The length of the fence is 13. Find the difference in areas of the two parts of the quadrilateral.

Q. Quadrilateral with a right angle

Example 1.178

Quadrilateral with a right angle

R. Rhombus

Example 1.179

Find second diagonal given perimeter and first diagonal of rhombus

S. Arithmetic Sequence

Challenge 1.180

A carpenter makes a triangular table, the lengths of whose sides are integers in Arithmetic Progression. If the area of the table is 6 sq. ft., then the perimeter of the smallest possible such table is: (JMET 2011/90, Adapted)

If $d = 0$, then the triangle is equilateral. If each side is x, then:

$$A = \frac{\sqrt{3}}{4}x^2 = 6 \Rightarrow x \notin \mathbb{Z}$$

Since x is not an integer, d cannot be zero.

Let the sides of the triangle be:

$$x - d, x, x + d$$

Then

$$p = (x - d) + (x) + (x + d) = 3x \Rightarrow s = \frac{3x}{2}$$

Substitute $s = \frac{3x}{2}, a = x - d, b = x, c = x + d$ as the sides in Heron's Formula $A = \sqrt{s\,(s-a)(s-b)(s-c)}$:

$$A = \sqrt{\frac{3x}{2}\left(\frac{3x}{2} - x - d\right)\left(\frac{3x}{2} - x\right)\left(\frac{3x}{2} - x + d\right)}$$

Simplify:

$$A = \sqrt{\frac{3x}{2}\left(\frac{x}{2} - d\right)\left(\frac{x}{2}\right)\left(\frac{x}{2} + d\right)} = \sqrt{\frac{3x^2}{4}\left(\frac{x^2}{4} - d^2\right)} \Rightarrow A^2 = \frac{3x^2}{4}\left(\frac{x^2}{4} - d^2\right)$$

We also know that

$$A = 6 \Rightarrow A^2 = 36$$

Hence:

$$\frac{3x^2}{4}\left(\frac{x^2}{4}-d^2\right)=36 \Rightarrow x^2\left(\frac{x^2}{4}-d^2\right)=48$$

From above, we know that $d \neq 0$. Values of d which will work are:

$$d=1,2,3,\ldots$$

Also,

$$\frac{x^2}{4}\in\mathbb{Z} \Rightarrow x^2 \text{ must be a multiple of } 4 \Rightarrow x \text{ must be a multiple of } 2:$$

Try $d=1, x=2$:

$$2^2\left(\frac{2^2}{4}-1^2\right)=2^2\,(0)=0$$

Try $d=1, x=4$:

$$4^2\left(\frac{4^2}{4}-1^2\right)=16\,(3)=48 \Rightarrow Works$$

Challenge 1.181

A carpenter makes a triangular table, the lengths of whose sides are in Arithmetic Progression. If the area of the table is 6 sq. ft., then the perimeter of one such table is: **(JMET 2011/90)**

A. 6 ft
B. 24 ft
C. 12 ft
D. 36 ft

Note: As given in the exam, the question had multiple correct answers. Hence, add the further condition that the sides of the triangle are integers. This guarantees a single correct answer.

$$A=6 \Rightarrow \frac{1}{2}hb=6 \Rightarrow hb=12$$

If the base is an integer, height also has to be an integer.
Hence, the only possible values for the base are:

$$12=1\times12=2\times6=3\times4$$

If in the above, you recognize that

$$3,4,5 \text{ is a Pythagorean Triplet} \Rightarrow p=3+4+5=12 \Rightarrow \textbf{Option C}$$

T. Area of Cyclic Quadrilaterals (Brahmagupta's Formula)

Heron's formula for the area of a triangle is a special case of a different formula for the area of a cyclic quadrilateral.

1.182: Brahmagupta's Formula: Area of a Cyclic Quadrilateral

If a cyclic quadrilateral has sides a, b, c and d, then the area of the quadrilateral is given by

$$\sqrt{(s-a)(s-b)(s-c)(s-d)},\qquad s=\frac{a+b+c+d}{2}$$

Note that if we substitute $d=0$ in the formula, we get:

$$\sqrt{(s-a)(s-b)(s-c)(s-0)} = \sqrt{s(s-a)(s-b)(s-c)}$$

Which is Heron's formula.

1.5 Further Topics

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