Congruence

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1. Congruence

1.1 SSS Theorem

A. Basics

1.1: Summary of Congruence Theorems

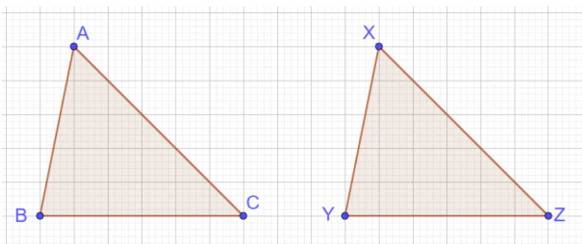
Theorem	Meaning
SSS	Side-Side-Side
SAS	Side-Angle-Side
SAA /ASA	Side-Angle-Angle Angle Side Angle
RHS/Hy Leg	Right Angle, Hypotenuse, Side OR Hypotenuse Leg

1.2: Congruence Definition
Two figures are congruent if they are the same size and shape.

- This definition is very simple, but very powerful. Many problems can be solved using congruence.
- We will focus on congruence of triangles, since it a fundamental skill. It is very useful in the geometry of other shapes, such as quadrilaterals, circles and many other shapes.

1.3: Corresponding Parts of Congruent Triangles are Congruent (CPCTC)
Corresponding parts of congruent triangles are congruent.

- If two triangles are congruent, then their corresponding parts are also congruent.
- The order in which you mention the vertices of two congruent triangles is important.



Example 1.4

$$\Delta ABC \cong \Delta XYZ$$

Identify the corresponding sides and corresponding angles which are congruent.

For angles, we need to pick letters in corresponding positions:

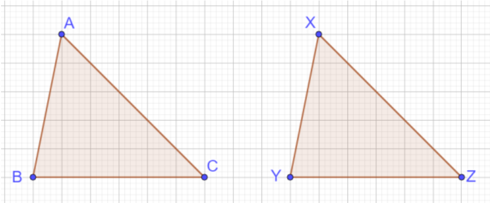
$$\textit{First Position}: \Delta \boldsymbol{A}BC \cong \Delta \boldsymbol{X}YZ \Rightarrow \angle A \cong \angle X$$
$$\text{2nd Position}: \Delta A\boldsymbol{B}C \cong \Delta X\boldsymbol{Y}Z \Rightarrow \angle B \cong \angle Y$$
$$\text{3rd Position}: \Delta AB\boldsymbol{C} \cong \Delta XY\boldsymbol{Z} \Rightarrow \angle C \cong \angle Z$$

We need to do the same for sides:

$$1st\ and\ 2nd\ Position: \Delta \boldsymbol{AB}C \cong \Delta \boldsymbol{XY}Z \Rightarrow AB \cong XY$$
$$2nd\ and\ 3rd\ Position: \Delta A\boldsymbol{BC} \cong \Delta X\boldsymbol{YZ} \Rightarrow BC \cong YZ$$
$$1st\ and\ 3rd\ Position: \Delta \boldsymbol{A}B\boldsymbol{C} \cong \Delta \boldsymbol{X}Y\boldsymbol{Z} \Rightarrow AC \cong XZ$$

Example 1.5

Consider the diagram drawn alongside (which is not drawn to scale). Suppose that $\Delta ABC \cong \Delta YZX$. Identify the corresponding sides and corresponding angles which are congruent.



Sides:

$$AB \cong YZ, BC \cong ZX, AC \cong YX$$

Angles:

$$\angle A \cong \angle Y, \angle B \cong \angle Z, \angle C \cong \angle X$$

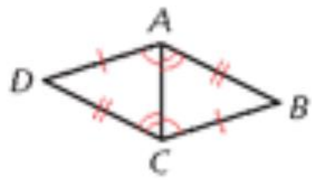
1.6: Showing Congruent Parts

Congruent parts are shown using the same symbols.

$$\textit{Lines are indicated using cuts}$$
$$\textit{Angles are indicated using curves}$$

Example 1.7

Which part are congruent in the figure alongside?



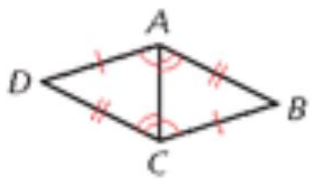
$$AD \text{ has a single cut}, BC \text{ has a single cut} \Rightarrow AD = BC$$
$$DC \text{ has a double cut}, \quad AB \text{ has a double cut} \Rightarrow DC = AB$$

$$\angle DAC \text{ is drawn with a single curve}, \angle ACB \text{ is drawn with a single curve} \Rightarrow \angle DAC = \angle ACB$$
$$\angle CAB \text{ is drawn with a double curve}, \angle ACD \text{ is drawn with a double curve} \Rightarrow \angle CAB = \angle ACD$$

1.8: Reflexive Property

Reflexive property:

$$\textit{Anything is equal to itself}$$



Example 1.9

Use the information in the figure to show that the two triangles are congruent.

Statement	Reason
$AD \cong CB$ $CD \cong AB$ $\angle DAC \cong ACB$ $\angle DCA \cong \angle CAB$	Given
$AC \cong CA$	Reflexive Property
$\angle D \cong \angle B$	Third Angles Theorem

B. SSS Theorem

There are five theorems for proving the congruence of triangles.

1.10: SSS (Side-Side-Side) Congruence Theorem

Two triangles are congruent if:

$$\text{three sides of the first triangle are congruent to three sides of the second triangle}$$

- Order is not mentioned in the property
- Order must be taken into when establishing the congruence

Example 1.11

Show that a triangle is congruent to itself.

Consider ΔABC:

$$AB \cong AB\ (Reflexive\ Property)$$
$$BC \cong BC\ (Reflexive\ Property)$$
$$CA \cong CA\ (Reflexive\ Property)$$

Hence, by SSS,

$$\Delta ABC \cong \Delta ABC$$

This is an example of the "reflexive" property: Anything is congruent to itself.

Example 1.12

A right triangle has legs 3 and 4. Another right triangle has hypotenuse 5 and a leg 4. Are the two triangles congruent?

Note: Use SSS Theorem for congruence.

In the first triangle, the sides are:

$$3,4,5$$

In the second triangle, the sides are:

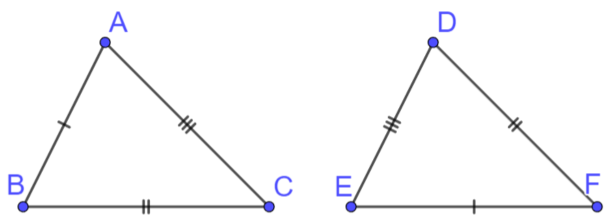
$$3,4,5$$

By SSS:

$$Triangles\ are\ congruent$$

1.13: Order in a Diagram

In a diagram, two congruent triangles need to be oriented the same way. However, if you are able to establish congruence using any order, then the two triangles are congruent.



Example 1.14

The adjoining diagram (which is not drawn to scale) has two triangles.

Prove or disprove the two triangles to be congruent.

$\Delta ABC \cong \Delta EFD$ by SSS Theorem:

$$AB = EF\ (Given)$$
$$BC = DF\ (Given)$$
$$AC = DE\ (Given)$$

C. Insufficient Information

1.15: Insufficient Information

In certain cases, we may not have enough information to decide whether two triangles are congruent.

Example 1.16

A right triangle has sides 3 and 4. Another right triangle has sides 4 and 5. Are the two triangles congruent?

Case I

Suppose the first triangle has

$$Legs\ 3\ and\ 4 \Rightarrow Hypotenuse = 5 \Rightarrow Sides = \{3,4,5\}$$

Suppose the second triangle has

$$Leg\ 4\ and\ Hypotenuse\ 5 \Rightarrow Second\ Leg = 4 \Rightarrow Sides = \{3,4,5\}$$

By SSS:

$$Two\ Triangles\ are\ congruent$$

Case II

Suppose the first triangle has

$$Legs\ 3\ and\ 4 \Rightarrow Hypotenuse = 5 \Rightarrow Sides = \{3,4,5\}$$

Suppose the second triangle has

$$Legs\ 4\ and\ 5 \Rightarrow Hypotenuse = \sqrt{4^2 + 5^2} = \sqrt{16 + 25} = \sqrt{41} \Rightarrow Sides = \{4,5,\sqrt{41}\}$$

Then:

$$The\ Two\ Triangles\ may\ or\ may\ not\ be\ congruent$$

D. Disproving Congruence

1.17: Proving Not Congruent using SSS

SSS requires three sides to be congruent. If even one side does not match, then the two triangles will not be congruent.

1.18: Triangle Inequality

In a triangle:

- Sum of any two sides is greater than the third side
- Third side is than sum of the other two sides, and greater than difference of other two sides

Example 1.19

Consider non-congruent triangles ABC and XYZ with integer side lengths. ΔABC has sides 3,4 and 5. ΔXYZ has sides 3 and 4. The two triangles are not congruent. Determine the sum of the possible values of the third side of ΔXYZ.

Using triangle inequality for ΔXYZ:

$$Third\ Side < 3 + 4 = 7$$
$$Third\ Side > 4 - 3 = 1$$

Possible values using the triangle inequality are:

$$\{2,3,4,5,6\}$$

But ΔABC is not congruent with ΔXYZ. Hence,

$$Third\ Side \neq 5$$

Final answer is:

$$Sum\ of\ possible\ values = 2 + 3 + 4 + 6 = 15$$

1.2 RHS Theorem

A. SSA: Not a Congruence Theorem

1.20: SSA: Not a Congruence Theorem

If two sides and the *angle which is not included* of one triangle are congruent to two sides and the angle which is not included of a second triangle:

$$we\ cannot\ conclude\ anything\ without\ further\ information$$

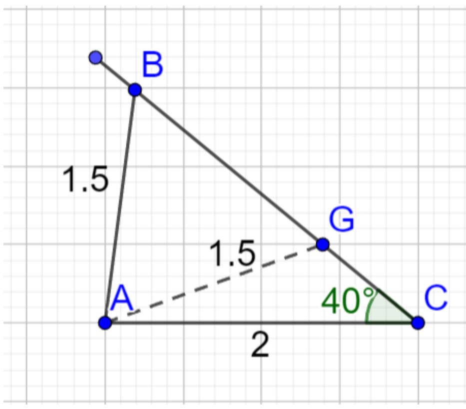
In general:

- Information with SSA will give two triangles that meet the required conditions.
- Hence, this case is called the ambiguous case.

Example 1.21

ΔABC has $AB = 1.5$, $AC = 2$ and $\angle BCA = 40°$. ΔAGC has $AC = 2$, $AG = 1.5$, and $\angle GCA = 40°$. Are the two triangles congruent?

Without further information, we cannot tell.
For example, the diagram on the right shows the case where the two triangles are not congruent.

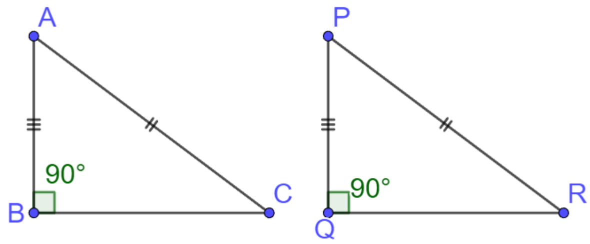


B. RHS

If the angle which is not included is a right angle, then the two triangles that are created are both the same.
In other words, you get only one triangle.
Hence, you can create a congruence condition based on SSA with a right angle.

1.22: RHS (Right Angle-Hyp-Side) / Hyp-Leg

If the hypotenuse and one leg of a right-angled triangle are congruent to the hypotenuse and a leg of another right-angled triangle, then the two triangles are congruent.



Example 1.23

On checking congruence for $\Delta ABC: \angle ABC = 90°, AB = 3, BC = 4$ and $\Delta PQR: \angle PQR = 90°, PQ = 3, QR = 4$, a student says

$$\angle ABC = 90°, AB = 3, BC = 4$$
$$\angle PQR = 90°, PQ = 3, QR = 4$$
$$\therefore \Delta ABC \cong \Delta PQR\ (RHS\ congruence)$$

Is the above solution correct or incorrect? Justify.

In the given solution, the hypotenuse has not been shown to be congruent, and hence RHS congruence is incorrectly used.

To use RHS congruence, we can first calculate, using the Pythagorean Theorem:

$$AC = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$
$$PR = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

And then establish:

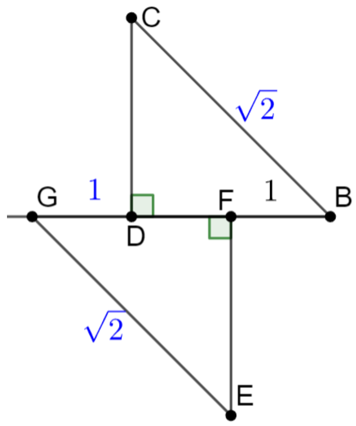
$$\angle ABC = 90°, AB = 3, AC = 5$$
$$\angle PQR = 90°, PQ = 3, PR = 5$$
$$\therefore \Delta ABC \cong \Delta PQR\ (RHS\ congruence)$$

Example 1.24

In the diagram alongside, triangles CDB and GFE are right angled at D and F respectively, and

$$FB = GD = 1$$
$$CB = GE = \sqrt{2}$$

Show that ΔCDB is congruent to ΔGFE.



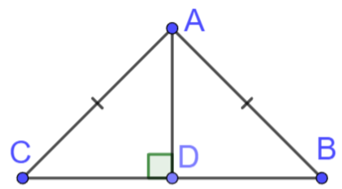
$$GF = GD + DF = 1 + DF$$
$$BD = BF + DF = 1 + DF = GF$$

$\Delta APC \cong \Delta BPC$ by RHS:

$$Right\ Angle: \angle CDB = \angle EFG = 90°\ (Given)$$
$$Hyp: GE = CB\ (Given)$$
$$Leg: GF = BD\ (Shown\ above)$$

Example 1.25

Show that CD is equal to DB in the diagram alongside.

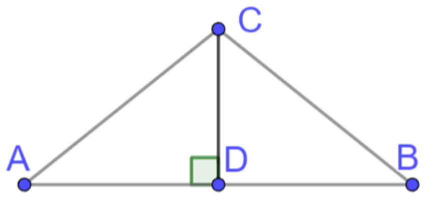


$\Delta ADC \cong ADB$ by RHS because

$$\angle ADC = \angle ADB = 90° \, (Right\ Angle)(Given)$$
$$AC = AB(Hyp)$$
$$AD = AD \, (Leg) \, (Reflexive\ Property)$$

Hence,

$$CD = DB \, (CPCT)$$



Example 1.26

In the diagram, the lengths of AD and DB are equal. Prove that CA is congruent to CB.

$\Delta CDA \cong \Delta CDB$ by SAS

$$\angle CDA = \angle CDB = 90° \, (Right\ Angle)(Given)$$
$$AD = DB(Given)$$
$$CD = CD \, (Reflexive\ Property)$$

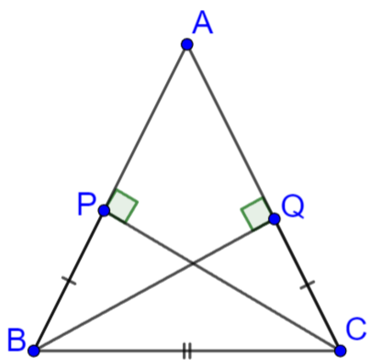
Hence:

$$CA = CB \, (CPCT)$$

C. Altitudes

Example 1.27

In ΔABC, the altitude from C intersects AB at P, and the altitude from B intersects AC at Q. If PB and QC are congruent, then prove that ΔABC is isosceles.



$BPC \cong \Delta CQB$ by RHS because:

$$\angle BPC = \angle BQC = 90° \, (Right\ Angle) \, (Altitudes\ are \perp to\ sides)$$
$$BC = BC \, (Hyp) \, (Reflexive\ Property)$$
$$PB = QC \, (Leg) \, (Given)$$

Hence:

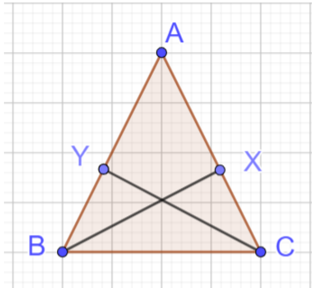
$$\angle PBC = \angle QCB \, (CPCT)$$

Hence, ΔABC is isosceles.

$$SAA, \quad SAS, \quad RHS, \quad SSS$$

Example 1.28

If two altitudes in a triangle are equal, they are drawn from equal angles.



$\Delta BX \cong \Delta CYB$ by RHS:

$$\angle BYC = \angle BXC = 90° \ (Definition\ of\ Altitude)$$
$$BX \cong CY \ (Leg)(Given)$$
$$BC = BC \ (Hyp)(Reflexive\ Property)$$

By CPCTC:

$$\angle YBC = \angle XCB$$

D. Perpendicular Bisector

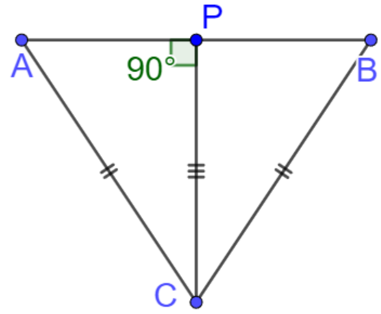
1.29: Perpendicular Bisector of a Line Segment

Line Segment CP is the perpendicular bisector of line segment AB if and only if:

- P bisects AB
- $CP \perp AB$

Example 1.30

C is a point equidistant from the endpoints of line segment AB. Show that it lies on the perpendicular bisector of AB.



Construct line segment CP perpendicular to line segment AB.

$\Delta APC \cong \Delta BPC$ by RHS:

$$Right\ Angle: \angle APC = \angle BPC = 90° \ (Definition\ of\ Perpendicular)$$
$$Hyp: CA = CB \ (Given)$$
$$Leg: PC = PC \ (Reflexive\ Property)$$

By $CPCTC$:

$$PA = PB \Rightarrow P\ bisects\ AB$$

Hence, C lies on the perpendicular bisector of AB because

$$CP \perp AB, P\ bisects\ AB$$

1.3 SAS Theorem

A. Basics

1.31: SAS (Side-Angle-Side)

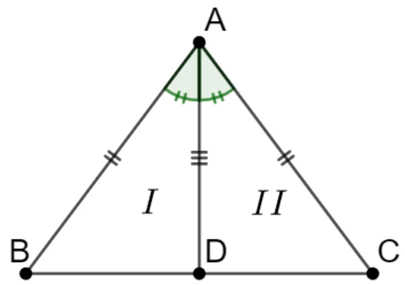
If two sides and the $included\ angle$ of one triangle are congruent to two sides and the included angle of another triangle,
$then$ the two triangles are congruent.

- The angle must be included. Included means it is between the two sides.
- If the angle is not included then, we get ASS, which is not a congruence theorem.

Example 1.32

In ΔABC, $AB = AC$. Line segment AD is the angle bisector of $\angle BAC$. Prove that AD is the median to BC, and the altitude to BC.

(Do not assume any properties of isosceles triangles other than: an Isosceles triangle is a triangle with at least two congruent sides).



$\Delta ADB \cong \Delta ADC$ by SAS:

$$AD = AD\ (Side)\ (Reflexive\ Property)$$
$$\angle BAD = \angle CAD\ (Angle)\ (Definition\ of\ Angle\ Bisector)$$
$$AB = AC\ (Side)\ (Given)$$

Show that AD is the median:

$$BD = DC\ (CPCT)$$
$$AD\ is\ Median\ (BD = DC)$$

Show that that AD is the altitude:

$$\angle ADB = \angle ADC\ (CPCT)$$

By angles in a linear pair:

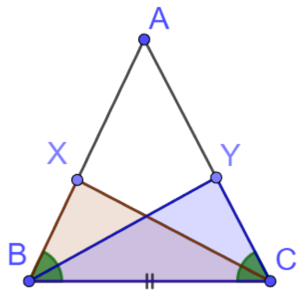
$$\angle ADB + \angle ADC = 180$$
$$\angle ADB + \angle ADB = 180$$
$$\angle ADB = \angle ADC = 90 \Rightarrow AD\ is\ altitude$$

1.4 ASA/SAA Theorem

A. Basics

1.33: ASA / SAA (Angle-Side-Angle / Side-Angle-Angle)
A side and two angles of a triangle are congruent
Note: Both conditions on the left are the same, since if two angles of a triangle are congruent, the third angle is also congruent.

Example 1.34
If a triangle is isosceles, then the altitudes drawn to the congruent sides are also congruent.



$\Delta BYC \cong \Delta BXC$ by ASA:

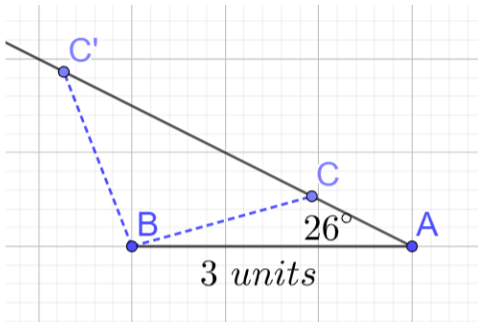
$$\angle BYC = \angle BXC = 90°\ (Angle)\ (Given)$$
$$BC = BC\ (Side)(Reflexive\ Property)$$
$$\angle XBC = \angle YCB\ (Angle)(Given)$$

Hence,

$$CX \cong BY\ (CPCT)$$

1.35: ASS

- Not a congruence theorem
- Ambiguous Case – Be careful



Construct ΔABC with

$$\angle A = 26°, AB = 3\ units, BC = 2\ units$$

$$\angle P = 26°, PQ = 3\ units, QR = 2\ units$$

B. Isosceles Triangles

1.36: Special Lines are Identical

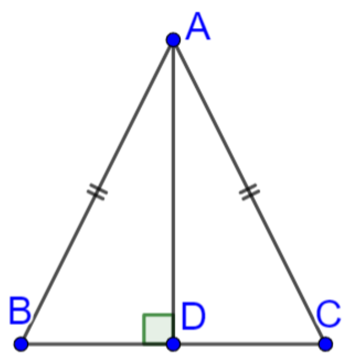
In an Isosceles Triangle, the median, the angle bisector, and the altitude drawn from a vertex are all the same.

- We prove this in three different ways below.
- This property does not hold true in general for all triangles. In fact, it is not true for scalene triangles.

Example 1.37

Prove, using *RHS Congruence*, that, in an Isosceles Triangle, the median to the base, the angle bisector to the base, and the altitude to the base are the same.

In Isosceles ΔABC, with $AB = AC$, draw the altitude from point A, intersecting base BC at D. $(Construction)$



$\Delta ADB \cong \Delta ADC$ by RHS

$$AD = AD\ (Reflexive\ Property)$$
$$AB = AC\ (Definition\ of\ Isosceles\ Triangle)$$
$$\angle ADB = \angle ADC = 90°\ (Definition\ of\ Altitude)$$

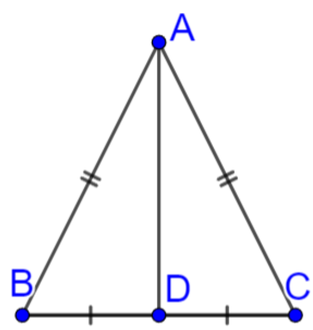
Hence, by CPCT:

$$BD = DC \Rightarrow AD\ is\ Median$$
$$\angle BAD = \angle CAD \Rightarrow AD\ is\ angle\ bisector$$

Example 1.38

Prove, using *SSS Congruence*, that, in an Isosceles Triangle, the median to the base, the angle bisector to the base, and the altitude to the base are the same.
(Don't assume any properties of an isosceles triangle other than the definition, which for this question is:
An Isosceles triangle is a triangle with at least two congruent sides).

Construction:
In Isosceles ΔABC, with $AB = AC$, draw the median from point A, intersecting base BC at D.



$\Delta ADB \cong \Delta ADC$ by SSS:

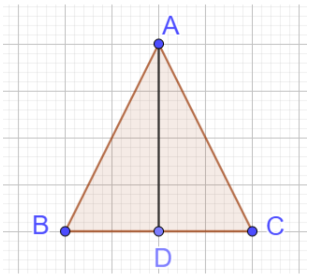
$$AD = AD\ (Reflexive\ Property)$$
$$AB = AC\ (Definition\ of\ Isosceles\ Triangle)$$
$$BD = DC\ (Definition\ of\ Median)$$

In ΔADB and ΔADC, by CPCT

$$\angle BAD = \angle CAD \Rightarrow AD \text{ is angle bisector}$$
$$\angle ADB = \angle ADC \Rightarrow \angle ADB + \angle ADC = 180 \Rightarrow \angle ADB = \angle ADC = 90 \Rightarrow AD \text{ is altitude}$$

Example 1.39

Prove that the base angles of an isosceles triangle are congruent.
(Don't assume any properties of an isosceles triangle other than the definition, which for this question is:
An Isosceles triangle is a triangle with at least two congruent sides).



Take any of the three proofs above, and show that

$$\Delta ADB \cong \Delta ADC$$

Then, since corresponding parts of congruent triangles are equal:

$$\angle B = \angle C \Rightarrow \text{Base angles are equal}$$

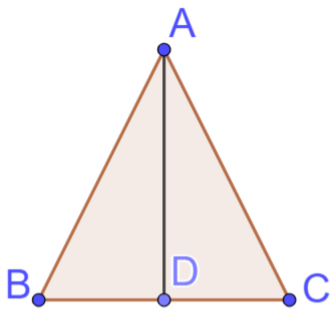
1.40: Converse: Special Lines are Identical

If, in a triangle:

- The altitude is the same as the angle bisector, then the triangle is Isosceles
- The altitude is the same as the median, then the triangle is Isosceles
- The median is the same as the angle bisector, then the triangle may or may not be Isosceles

Example 1.41

Prove that a triangle where the altitude is the same as the angle bisector is an Isosceles Triangle.



In ΔABC let AD be the common altitude and angle bisector.

$\Delta ABD \cong \Delta ACD$ by ASA:

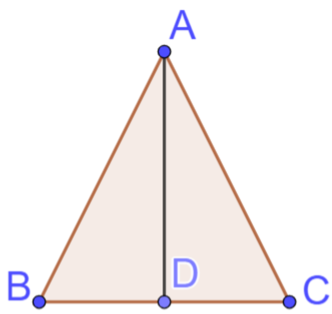
$$\angle BDA = \angle CDA = 90° \ (Angle) \ (Definition\ of\ Altitude)$$
$$AD \cong AD \ (Side) \ (Reflexive\ Property)$$
$$\angle BAD \cong \angle CAD \ (Angle) \ (Definition\ of\ Angle\ Bisector)$$

By CPCT:

$$AB \cong AC$$

Example 1.42

Prove that a triangle where the altitude is the same as the median is an Isosceles Triangle.



In ΔABC let AD be the common altitude and median.

$\Delta ABD \cong \Delta ACD$ by SAS:

$$AD \cong AD \ (Side) \ (Reflexive\ Property)$$
$$\angle BDA = \angle CDA = 90° \ (Angle) \ (Definition\ of\ Altitude)$$
$$BD = DC \ (Side) \ (Definition\ of\ Median)$$

By CPCT:

$$AB \cong AC$$

Example 1.43

A. Draw a counterexample where the median is also the angle bisector in a triangle, but the triangle is not isosceles.
B. If you attempt to set up a congruence, which "congruence theorem" do you get? What is the pitfall here?

In ΔABC let AD be the common median and angle bisector.

$$BD = DC\ (Side)\ (Definition\ of\ Median)$$
$$AD \cong AD\ (Side)\ (Reflexive\ Property)$$
$$\angle BAD = \angle CAD = 90°\ (Angle)\ (Definition\ of\ Angle\ Bisectore)$$

What we get is

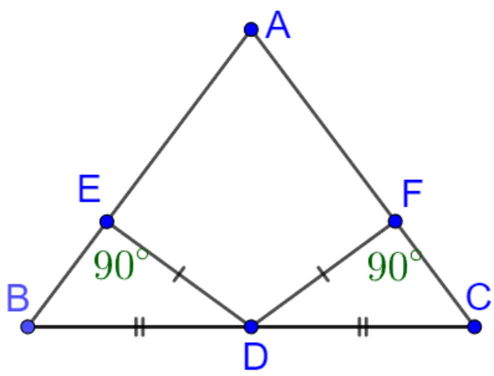
$$SSA \Rightarrow Not\ a\ congruence\ theorem$$

1.44: Altitudes in an isosceles triangle

In an isosceles triangle, the altitudes drawn to equal sides are equal.
The converse is also true: if the altitudes drawn to two sides are equal, then the triangle is isosceles.

Example 1.45

Prove that if the perpendiculars to two sides of a triangle from the midpoint of the third side are congruent, then the triangle is isosceles.



Draw ΔABC. Let D be the midpoint of side BC. Then:

$$DB = DC\ (Hyp)$$

Draw $DE \perp AB$ and $DF \perp AC$.

$$\angle DEB = \angle DFC = 90°\ (Right\ Angle)$$

From the question, we are given that:

$$DE = DF\ (Side)$$

From the above three statements:

$$\Delta DEB \cong \Delta DFC\ (RHS\ Congruence)$$
$$\angle EBD = \angle FCD\ (CPCT)$$

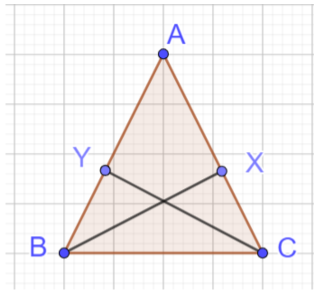
Hence, since its base angles are congruent, ΔABC is isosceles.

1.46: Medians in an isosceles triangle

In an isosceles triangle, the medians drawn to equal sides are equal.
The converse is also true: if the medians drawn to two sides are equal, then the triangle is isosceles.

Example 1.47

Show that the medians drawn to equal sides in a triangle are equal.



Prove the converse of the above statement: If two medians in a triangle are equal, they are drawn from equal angles

1.48: Angle Bisectors in an isosceles triangle
In an isosceles triangle, the angle bisectors drawn to equal sides are equal.
The converse is also true: if the angle bisectors drawn to two sides are equal, then the triangle is isosceles.

Example 1.49
Show that the angle bisectors drawn to equal sides in a triangle are equal.

Example 1.50
Show that if the angle bisectors drawn to two sides in a triangle are equal, then the triangle is isosceles.

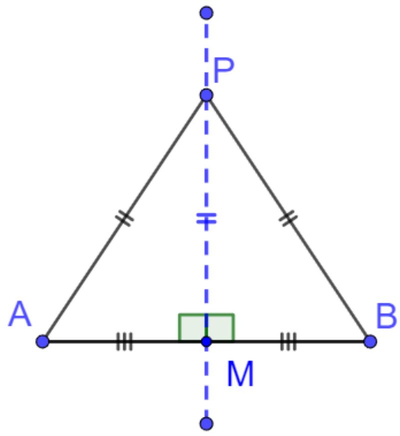
Note: the proof here is based on inequalities, not on congruence.

C. Perpendicular Bisectors

Example 1.51
Prove

A. If a point is on the perpendicular bisector of a line segment, then it is equidistant from the ends of the line segment.
B. If a point is equidistant from the ends of a line segment, then it is on the perpendicular bisector of the line segment.

Part A
On line segment AB let M be the midpoint.
Construct the perpendicular bisector to line segment AB. Note that the bisector passes through M.

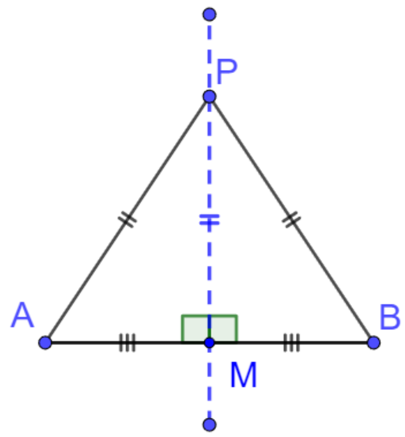


Let P be a point on the perpendicular bisector.

Then, in ΔPAM and ΔPBM

$$AM = BM\ (M\ is\ midpoint)$$
$$\angle PMA = \angle PMB = 90°(Def\ of\ \perp)$$
$$PM = PM\ (Reflexive\ property)$$
$$\Delta PAM \cong \Delta PBM\ (SAS\ Theorem)$$
$$PA = PB(CPCTC)$$
$$P\ is\ equidistant\ from\ A\ and\ B.\ (Def\ of\ distance\ and\ PA = PB)$$

Part B
Let P be equidistant from A and B.
Draw the perpendicular from P to AB, intersecting AB at M.



Then, in ΔPAM and ΔPBM

$$PA = PB \; (Def\ of\ Distance)$$
$$PM = PM \; (Reflexive\ Property)$$
$$\angle PMA = \angle PMB = 90° \; (Def\ of\ \perp)$$
$$\Delta PMBA \cong \Delta PMB \; (RHS\ Theorem)$$
$$AM = BM \; (CPCTC)$$
$$P\ lies\ on\ perpendicular\ bisector$$

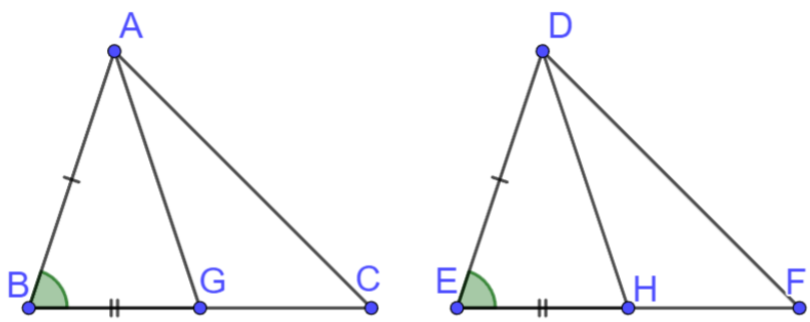
1.5 Applications

A. Parts of Congruent Triangles

1.52: More corresponding parts
The heights, angle bisectors, medians, and other parts of two congruent triangles are also congruent.

Example 1.53
Prove that if two triangles are congruent, their medians are also congruent.

Let $\Delta ABC \cong \Delta DEF$.
Draw medians AG and DH



$\Delta ABG \cong \Delta DEH$ by SAS:

$$AB = DE \; (CPCT)$$
$$\angle ABG = \angle DEH \; (CPCT)$$
$$BC = EF \; (CPCT)$$
$$\frac{BC}{2} = BG = EH = \frac{EF}{2} \; (Def.of\ Median)$$

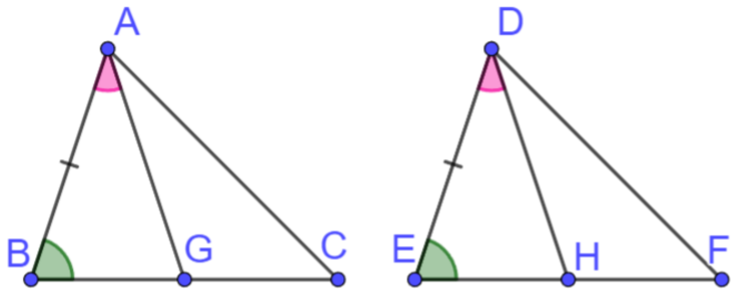
Hence:

$$Angle\ Bisectors\ AG\ and\ DH\ are\ congruent\ (CPCT)$$

Similarly, the other two medians can also be proved congruent.

Example 1.54
Prove that if two triangles are congruent, their angle bisectors are also congruent.

Let $\Delta ABC \cong \Delta DEF$.
Draw angle bisectors AG and DH



$\Delta ABG \cong \Delta DEH$ by SAA:

$$\angle ABG = \angle DEH\ (CPCT)$$
$$AB = DE\ (CPCT)$$
$$\angle BAC = \angle EDF\ (CPCT)$$
$$\frac{\angle BAC}{2} = \angle BAG = \angle EDH = \frac{\angle EDF}{2}$$

Hence:

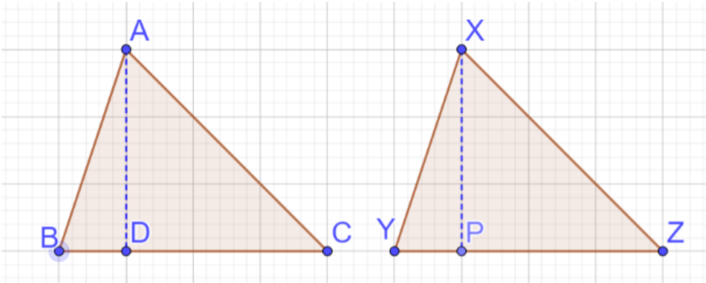
$$Angle\ Bisectors\ AG\ and\ DH\ are\ congruent\ (CPCT)$$

Similarly, the other two angle bisectors can also be proved congruent.

Example 1.55

Prove that if two triangles are congruent, their heights are also the same.

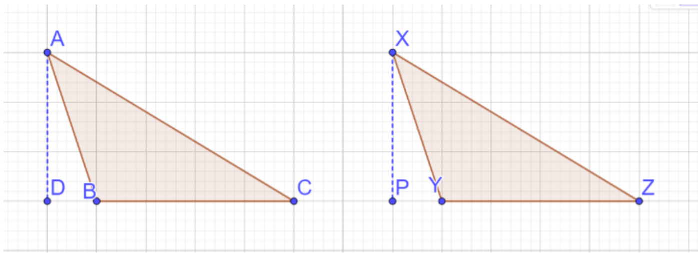
Let $\Delta ABC \cong \Delta XYZ$.
Draw altitudes AD and XP.



Case I: The altitude lies inside the triangle

$\Delta ABD \cong \Delta XYP$ By SAA

$$AB \cong XY\ (CPCT)$$
$$\angle ABD \cong \angle XYP\ (CPCT)$$
$$\angle ADB = \angle XPY = 90°\ (AD \perp BC, XP \perp YZ)$$



Case II: The altitude lies outside the triangle

$\Delta ACD \cong \Delta XZP$ By SAA

$$AC \cong XZ\ (CPCT)$$
$$\angle ACD \cong \angle XZP\ \ (CPCT)$$
$$\angle ADC = \angle XPZ = 90°\ (AD \perp BC, XP \perp YZ)$$

In both the cases above, by CPCT:

$$AD \cong XP \Rightarrow Altitudes\ are\ congruent$$

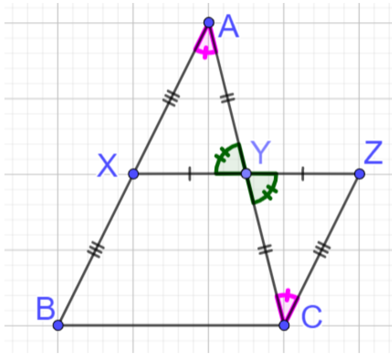
Since there is nothing special about the altitude from A, we can prove similarly that the altitudes from B and C are also congruent.

B. Midpoint Theorem

1.56: Midpoint Theorem

➢ In a triangle, the line joining the midpoints of two sides is parallel to the third side, and half of the third side.

In ΔABC, let X and Y be the midpoints of AB and AC, respectively. Draw XY and produce XY to Z such that $XY = YZ$.



In ΔAYX and ΔCYZ:

$$\angle AYX = \angle CYZ\ (Vertically\ opposite\ angles)$$
$$XY = YZ\ (Construction)$$
$$AY = YC\ (Y\ is\ midpoint\ of\ AC)$$
$$\therefore \Delta AYZ \cong \Delta CYZ\ (SAS\ Theorem)$$

Consider AC as a transversal of BA and CZ:

$$\angle XAY\ and\ ZCY\ are\ alternate\ interior\ angles$$
$$\angle XAY \cong ZCY\ (CPCTC\ in\ \Delta AYX \text{ and } \Delta CYZ)$$
$$\therefore AB \parallel CZ$$
$$\therefore BX \parallel CZ$$

Also:

$$BX \cong AX\ (X\ is\ midpoint\ of\ AB)$$
$$AX \cong CZ (CPCTC\ in\ \Delta AYX \text{ and } \Delta CYZ)$$
$$\therefore BX \cong CZ$$

Hence,

$$BX \parallel CZ, BX \cong CZ \Rightarrow XZCB\ is\ a\ parallelogram$$

Since $XZCB$ is a parallelogram:

$$XZ \parallel BC \Rightarrow Parallel\ condition$$
$$XZ = BC \Rightarrow XY = \frac{1}{2}XZ = \frac{1}{2}BC \Rightarrow Half\ Condition$$

C. Right Triangles

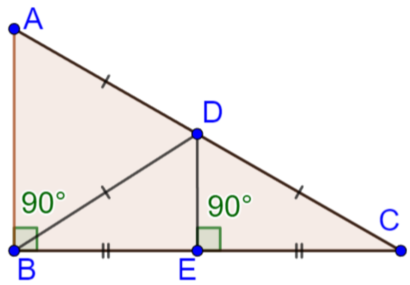
1.57: Median to Hypotenuse

The median to the hypotenuse of a right-angled triangle is half the length of the hypotenuse.

Using Congruence

In right ΔABC, draw median BD, and hence

$$AD = DC$$



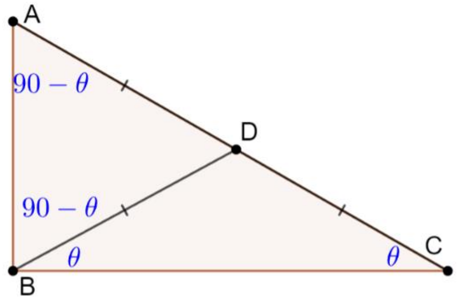
Draw DE such that E is the midpoint of BC.
Then, in ΔBDE and ΔCDE:

$$DE = DE\ (Common\ Side)$$
$$BE = EC\ (Construction)$$
$$\angle DEC = \angle DEB = 90°\ (Midpoint\ Theorem)$$
$$\Delta BDE \cong \Delta CDE\ (SAS)$$
$$BD \cong CD\ (CPCTC)$$

Using Isosceles Triangles

In right ΔABC, draw BD such that

$$\angle DBC = \angle DCB = \theta$$



This means that ΔDBC is isosceles, and hence

$$\underbrace{BD = DC}_{Equation\ I}$$

By sum of angles of a triangle,

$$\angle DAB = 180 - 90 - \theta = 90 - \theta$$

Also, since $\angle ABC = 90°$:

$$\angle DAB = 90 - \theta$$

Hence, ΔADB is isosceles and

$$\underbrace{AD = BD}_{Equation\ II}$$

Combine Equations I and II to get:

$$AD = BD = DC \Rightarrow AD = DC$$

Hence, BD is the median, and since the median is unique, we have also shown that

$$BD = DC = \frac{1}{2}AC$$

1.6 Further Topics

58 Examples