Similarity

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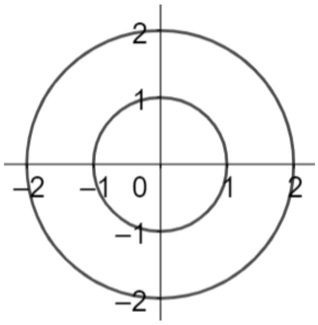
1. Scale Factor; Area; Volume

1.1 Scale Factor

1.1: Similarity
Two objects are similar if one object can be obtained from the other via:

- scaling (increase or decrease in size) without change in shape
- translation (moving)
- reflection (mirror image)
- rotation (move the object around a point)
- Or some combination of the above

For example, apart from size, the two circles in the diagram, have the same kind of shape.

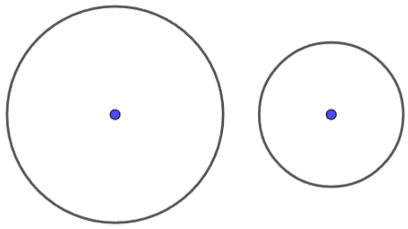


A. Scaling

1.2: Similarity as Scaling
Two objects are similar if they have the same shape, but one object is a larger (or smaller) copy of the other.
That it, it involves a change in size, but no change in shape.

Example 1.3
Decide whether the following statements are true or false. If the statements are false, give a counterexample which shows that they are false.

A. All circles are similar.
B. All rectangles are similar.
C. All squares are similar.
D. All parallelograms are similar.
E. All rhombi are similar.



Part A
All circles have the same kind of shape. Hence, all circles are similar.

$$True$$

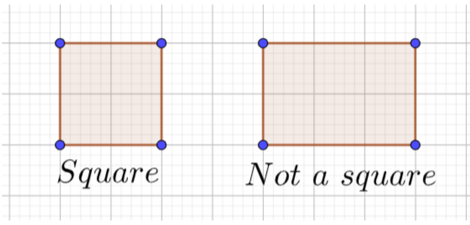
Part B

This is not true. As a counterexample, consider that:

- Some rectangles are squares
- Some rectangles are not squares

The two types of rectangles do not have the same kind of shape.

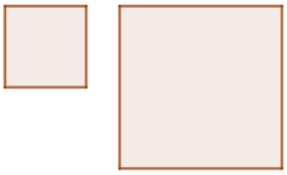
$$False$$



Part C

Squares have the same shape irrespective of their side length, and hence all squares are similar.

$$True$$



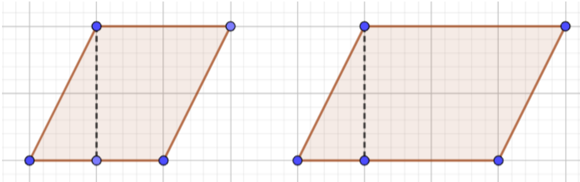
Part D

Two parallelograms need not be similar. For example, in the diagram:

- Both parallelograms have the same height.
- The two parallelograms have different base lengths.

Hence, one parallelogram cannot be obtained from the other by scaling.

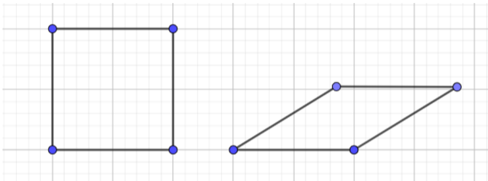
$$False$$



Part E

A rhombus is a quadrilateral with all sides equal.
A rhombus can be a square (or may not be).
Hence, all rhombi are not similar.

$$False$$



Example 1.4

Decide whether the following statements are true or false. If the statements are false, give a counterexample which shows that they are false.

A. All spheres are similar.
B. All cylinders are similar.
C. All cones are similar.
D. All cuboids are similar.
E. All cubes are similar.
F. All pyramids are similar.
G. All square pyramids are similar.

Part A

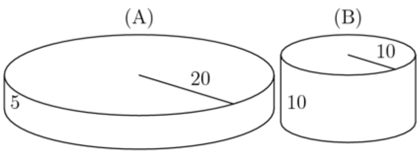
A sphere is essential a ball (think a cricket ball, or a soccer ball). All spheres have the same kind of shape. Hence, all spheres are similar.

$$True$$

Part B

Consider the two objects shown below. They are both cylinders, but they have different kinds of shapes. Hence, all cylinders are not similar.

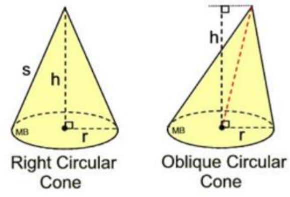
$$False$$



Part C

When we say cone, we often mean right circular cones (such as the one on the left).
However, cones can also be oblique.
And, right circular cones are not similar to oblique circular cones.

False



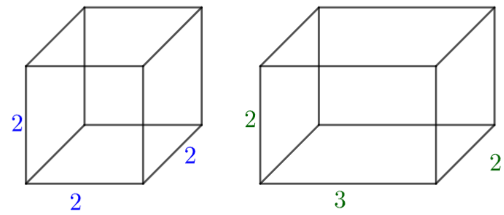
Part D

Consider two cuboids:

- Cuboid A is a cube
- Cuboid B is not a cube

The two shapes are not similar

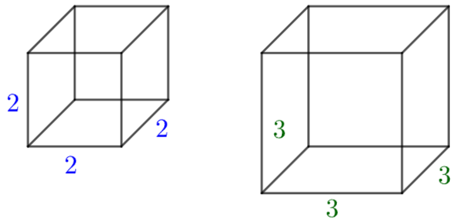
False



Part E

All cubes have the same shape.
Hence, all cubes are similar.

True



Part F

Consider two pyramids:

- The first is a triangular pyramid
- The second is a square pyramid

The two do not have the same shape.

False

Part G

Consider two square pyramids with the same square of side length 1 as the base:

- First pyramid has a height of 1
- Second pyramid has a height of 3

The two do not have the same shape.

False

1.5: Similarity as Scaling

Similar Shapes	Need not be Similar
Squares	Rectangles
Circles	Rhombuses
Equilateral Triangles	Parallelograms
Regular Polygons	
Cubes	Cuboids
Spheres	Cylinders
	Cones

1.6: Self-Similarity

Any object is similar to itself.

1.7: Notation

Similarity is indicated using the symbol:

$$\sim$$

Example 1.8

Write the following using similarity symbols

A. Triangle ABC is similar to Triangle XYZ.
B. Triangle PRQ is similar to itself.

$$\Delta ABC \sim \Delta XYZ$$
$$\Delta PRQ \sim \Delta PRQ$$

B. Using Scale Factor in Geometry

Example 1.9

Find the original and the new area in each case below:

A. A square with a side length of 2 is scaled by a factor of 3.
B. A circle with a radius of $\frac{1}{2}$ is scaled by a factor of $\frac{2}{3}$. Write your answers in terms of π.

Part A

	Original	New	Ratio
Side Length	2	6	$\frac{6}{2} = 3$
Area	4	36	$\frac{36}{4} = 9 = 3^2$

Part B

	Original	New	Ratio
Radius	$\frac{1}{2}$	$\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$	$\frac{1}{3} \div \frac{1}{2} = \frac{2}{3}$
Area	$\pi r^2 = \pi \left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$	$\pi r^2 = \pi \left(\frac{1}{3}\right)^2 = \frac{\pi}{9}$	$\frac{\pi}{9} \div \frac{\pi}{4} = \frac{4}{6} = \left(\frac{2}{3}\right)^2$

C. Using Scale Factor in Maps and Models

Example 1.10: Finding Actual Lengths

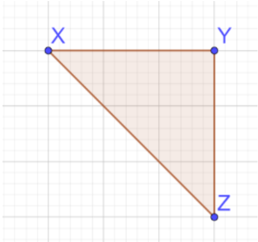
You may use information from one part in another part.

A map of a locality has a scale factor of 1: 100,000.

A. Points X and Y are 3 cm apart on the map. What is the actual length of the (straight) road between them?
B. Points Y and Z are 4 cm apart on the map, what is the actual distance between them?
C. XY and YZ are perpendicular to each other. A road goes directly from X to Z. If a person travels at a speed of $60 \frac{km}{hr}$, what is the time taken to travel from X to Z directly

Parts A and B

$$Actual\ Length_{XY} = 3 \times 100{,}000 = 300{,}000\ cm = 3000\ m = 3\ km$$
$$Actual\ Length_{YZ} = 4 \times 100{,}000 = 400{,}000\ cm = 4000\ m = 4\ km$$



Part C

Draw a diagram. Note that $XY \perp YZ$. In right ΔXYZ, we recognize the Pythagorean Triplet (3,4,5):

$$XZ = 5\ km$$
$$T = \frac{D}{S} = \frac{5\ km}{60\frac{km}{hr}} = \frac{1}{12}hr = 5\ minutes$$

Example 1.11: Finding Lengths in Maps

A. Aarna uses a scale factor of 1: 20 to make a map of her house. Her garden has a width of 12 yards. How many inches should the garden be wide in her actual map. (Note: 1 yard is 3 feet).
B. Akshay walked the width of the town square in 75 paces. His scale map has 15 inches between his house and the town square. It takes Akshay 135 paces to reach the town square from his house. What should be the width of the town square on his scale map?

Part A

$$12\ yards = 36\ feet = 36 \times 12\ inches$$
$$Map\ Distance = \frac{36 \times 12}{20} = 21.6\ inches$$

Part B

Distance from house to town square:

$$15\ inches: 135\ paces$$

Use the unitary method. Divide both sides by 15:

$$1\ inches: 9\ paces$$

Multiply both sides by $\frac{75}{9}$:

$$1 \times \frac{75}{9}\ inches: 9 \times \frac{75}{9} paces$$
$$\underbrace{\frac{25}{3}\ inches}_{\substack{Distance\ on\\ graph\ paper}} : \underbrace{75\ paces}_{\substack{Width\ of\\ town\ square}}$$

D. Finding Scale Factor

Scale factor refers to how much each dimension of a figure is increased or decreased by. Scale factor is a dimensionless quantity.

1.12: Scale Factor

Scale factor is the $\underbrace{ratio\ of\ side\ lengths}_{not\ area}$ of one similar shape to another, with the added condition that the first number in the ratio is one.

$$Ratio\ of\ Side\ Lengths = 1: a \Rightarrow Scale\ Factor\ is\ a$$

Example 1.13

Find the scale factor in each case:

A. An architect's drawing of a square garden has side length half a meter. If the actual garden has a length of $100m$, find the scale factor.

B. A scale model of a drama stage is 2 feet by 3 feet. The actual stage is 40 feet by 60 feet.

Part A

$$0.5m: 100m = 0.5: 100 = 1: 200$$

Part B

We get the same scale factor whether we compare the length or the width:

$$Length: 2\ feet: 40\ feet = 2: 40 = 1: 20$$
$$Width: 3\ feet: 60\ feet = 3: 60 = 1: 20$$

Example 1.14

A scale model of a castle shows the distance between the stables and the greenhouse to be 4 *inches* while the actual distance between the two is 64 feet.

$$4\ inches: 64\ feet$$

Convert the inches to feet by dividing 12:

$$\frac{1}{3} feet: 64\ feet = \frac{1}{3}: 64$$

Eliminate fractions by multiplying both sides by 3:

$$1: 192$$

Example 1.15

Ron is attempting to draw the places of interest on his town on graph paper. He notes that the town market is north-west of his house, and 100 feet away. His graph paper is demarcated using *cm*. He uses 3 cm to represent the distance. ($Find\ an\ expression, not\ the\ final\ answer\ using\ 1\ inch \approx 2.54\ cm$).

$$3\ cm : 100\ feet$$
$$3\ cm : 1200\ inches$$
$$1\ cm : 400\ inches$$

Use an approximate conversion factor:

$$1 : 400 \times 2.54\ cm$$
$$1 : 400 \times 2.54$$

Example 1.16

A map shows the distance between two cities as 5.5 cm. The actual distance between the cities is 275 km.

$$\underbrace{5.5\ cm}_{Map} : \underbrace{275\ km}_{Reality}$$

Convert the km to cm by multiplying by 100,000:

$$5.5\ cm: 27{,}500{,}000\ cm$$

Now that the units are the same, we no longer need to mention them.

Multiply both sides by 10, and remove units since they are the same on both sides:

$$55: 275{,}000{,}000$$

Divide both sides by 55:

$$1: 5{,}000{,}000$$

1.2 Using Scale Factor: Areas & Volumes

A. Area

1.17: Area of Similar Figures

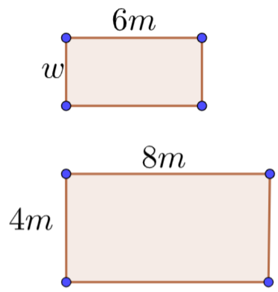
If two figures are similar and their side lengths are s and S respectively, then:

$$Ratio\ of\ Side\ Lengths = s : S$$
$$Ratio\ of\ Areas\ = s^2 : S^2$$

- This is easily observed for figures like squares and circles, but it is true for all similar shapes.

Example 1.18: Two Dimensional Figures

A. A rectangle has length 6 m. Another rectangle similar to the first rectangle has length $8m$, and width $4m$. Find the width of the first rectangle.
B. A triangle has perimeter 6 $feet$. A second triangle, similar to the first, has perimeter 12 feet. If the area of the first triangle is 1 ft^2, find the area of the second triangle.



Draw a diagram.
In the second rectangle

$$\frac{w}{l} = \frac{1}{2}$$

In the first rectangle

$$\frac{w}{l} = \frac{1}{2} \Rightarrow \frac{w}{6} = \frac{1}{2} \Rightarrow w = 3$$

Example 1.19

A. A square has side length 1. Another square has side length 3. What is the ratio of the area of the first square to the area of the second square?
B. Two squares have side lengths in the ratio 3: 4. What is the ratio of their areas?
C. Square X has side length $\frac{3}{4}p$. Square Y has side length $\frac{4}{3}q$. What is the ratio of the areas?

Part A

$$1^2 : 3^2 = 1 : 9$$

Part B

$$3^2 : 4^2 = 9 : 16$$

Part C

$$\left(\frac{3}{4}p\right)^2 : \left(\frac{4}{3}q\right)^2 = \frac{9}{16}p^2 : \frac{16}{9}q^2$$

Multiply both sides by 16×9:

$$81p^2 : 256q^2$$

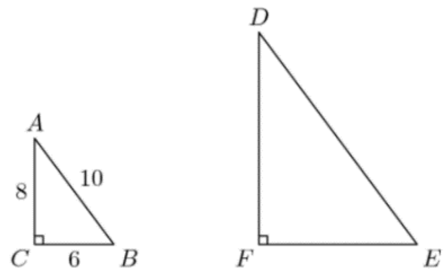
Example 20

A square has side length x. Another square has side length double of the previous square. What is the ratio of the area of the first square to the area of the second square?

$$x^2 : (2x)^2 = x^2 : 4x^2 = 1 : 4 = 1 : 2^2$$

Example 1.21

ΔABC has side lengths 6, 8 and 10, as shown. Each of the side lengths of ΔABC is increased by 50%, forming a new triangle, ΔDEF. What is the area of ΔDEF? **(CEMC 2006 Cayley)**



We can do this by calculating the side lengths of DEF and then using the formula for the area of a triangle:

$$[DEF] = \frac{1}{2}hb = \frac{1}{2}\left(6 \times \frac{3}{2}\right)\left(8 \times \frac{3}{2}\right) = 54$$

Or, we can calculate the required area by determining the relation between the area of the two triangles:

$$Side\ Lengths\ have\ ratio\ 1{:}\frac{3}{2}$$
$$Areas\ have\ ratio\ 1^2{:}\left(\frac{3}{2}\right)^2 = 1{:}\frac{9}{4}$$
$$[ABC] = \left(\frac{1}{2} \times 6 \times 8\right) = 24$$

$$[DEF] = 24 \cdot \frac{9}{4} = 54$$

Example 1.22

A garden has its length and width increased by 10%. Determine the percentage increase in the area?

$$Ratio\ of\ side\ lengths = 1{:}1.1$$
$$Ratios\ of\ Areas = 1^2{:}1.1^2 = 1{:}1.21$$

Percentage increase

$$= \frac{1.21 - 1}{1} = 0.21 = 21\%$$

1.23: Finding Side Length Ratios

Example 1.24

Square A has sixteen times the area of Square B. Find the ratio of the side length of Square A to Square B.

$$Ratio\ of\ Areas = 16{:}1$$
$$Ratio\ of\ side\ lengths = \sqrt{16}{:}\sqrt{1} = 4{:}1$$

Example 1.25

The ratio of side lengths of two squares is $\frac{a}{2}{:}\frac{4}{a}$. If the first square has area 4 times the area of the second square, then find the value of a.

$$Ratio\ of\ Areas = 4{:}1 \Rightarrow Ratio\ of\ Sides = \sqrt{4}{:}\sqrt{1} = 2{:}1 = 16{:}8$$

$$\frac{a}{2}{:}\frac{4}{a} = a^2{:}8 = 16{:}8$$

$$a^2 = 16 \Rightarrow a = 4$$

1.26: Scale Factor
Scale factor is the ratio of side lengths when one of the values is 1.

Example 1.27
Square B (area: 100) is a scaled copy of Square A (area: 25). Find the scale factor.

$$Ratio\ of\ Areas = 100:25 = 4:1$$
$$Ratio\ of\ Side\ Lengths = \sqrt{4}:1 = 2:1$$
$$Scale\ Factor = 2$$

Example 1.28
A. A square and an equilateral triangle have equal perimeters. The area of the triangle is $2\sqrt{3}$ square inches. What is the number of inches in the length of the diagonal of the square? **(MathCounts 2004 State Team)**

B. A square and an equilateral triangle have equal perimeters. The area of the triangle is $16\sqrt{3}$ square centimeters. How long, in centimeters, is a diagonal of the square? Express your answer in simplest radical form. **(MathCounts 2010 School Sprint)**[1]

Part A

The side length of the triangle:

$$A = \frac{\sqrt{3}}{4}s^2 = 2\sqrt{3} \Rightarrow s = \sqrt{8} = 2\sqrt{2}$$

Find the perimeter of the triangle:

$$P = 3s = 6\sqrt{2}$$

Find the side length of the square:

$$S = \frac{P}{4} = \frac{6\sqrt{2}}{4} = \frac{3\sqrt{2}}{2}$$

Find the length of the diagonal:

$$D = \frac{3\sqrt{2}}{2} \times \sqrt{2} = \frac{3 \times 2}{2} = 3$$

Part B

You could apply the same process as in Part A (and you can try doing that). However, we can also use similarity, and the answer from the previous question. Note that:
- All squares are similar
- Are equilateral triangles are similar

$$Area = 2\sqrt{3} \Rightarrow Diagonal = 3$$
$$Area = 16\sqrt{3} \Rightarrow Diagonal = x$$

Ratio of areas:

$$16\sqrt{3}:2\sqrt{3} = 8:1$$

Ratio of lengths is the square root of the ratio of areas:

$$\sqrt{8}:\sqrt{1} = 2\sqrt{2}:1$$

Hence, the diagonal of the square in the current question:

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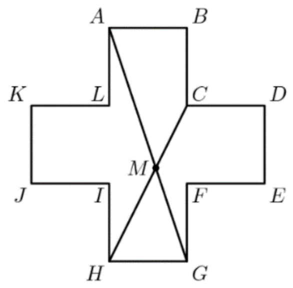
[1] Interesting point: The same question asked for a State level round in 2004 (3rd Round) was considered easy enough in 2010 to be asked in School Round (1st Round)

$$3 \times 2\sqrt{2} = 6\sqrt{2}$$

Example 1.29

Consider the 12-sided polygon $ABCDEFGHIJKL$, as shown. Each of its sides has length 4, and each two consecutive sides form a right angle. Suppose that $\overline{AG}$ and $\overline{CH}$ meet at M. What is the area of quadrilateral $ABCM$? **(AMC 10A 2007/18)**



By AA Similarity:

$$line\ AH \parallel line\ BG \Rightarrow \angle HAG \cong \angle BGA, \angle CHA \cong \angle HCG \Rightarrow \Delta AMH \sim \Delta CMG$$

$$\frac{CG}{AH} = \frac{8}{12} = \frac{2}{3} \Rightarrow \frac{Height\ of\ \Delta CMG}{Height\ of\ \Delta AMH} = \frac{2}{3}$$

Now, we find the height of ΔCMG:

$$h(\Delta CMG) = \frac{h(\Delta CMG) + h(\Delta \mathrm{AMH})}{2+3} \times 2 = \frac{HG}{5} \times 2 = \frac{4}{5} \times 2 = \frac{8}{5}$$

$$A(ABCM) = \underbrace{\frac{1}{2}(4)(12)}_{A(\Delta ABG)} - \underbrace{\frac{1}{2}\left(\frac{8}{5}\right)(8)}_{A(\Delta CMG)} = 24 - \frac{32}{5} = \frac{88}{5}$$

1.30: Ratio of Areas

If two figures are similar, and they have side lengths in the ratio $a: b$, then their areas are in the ratio $a^2: b^2$.

If a square has side length s, then the area of the square is

$$s^2$$

To determine the area from the side length, we square it.
Similarly, to find the ratio of areas, we square the side lengths.

Example 1.31

Square $ABCD$ has side length 1, and square $PQRS$ has side length 3. How many times is the area of square $PQRS$ as compared to square $ABCD$?

$$Ratio\ of\ Side\ Lengths = \underbrace{1}_{ABCD} : \underbrace{3}_{PQRS}$$

In order to find the ratio of areas, we square the ratio of side lengths:

$$Ratio\ of\ Areas = \underbrace{1^2}_{ABCD} : \underbrace{3^2}_{PQRS} = \underbrace{1}_{ABCD} : \underbrace{9}_{PQRS}$$

PQRS has nine times the area.

Example 1.32

Square $ABCD$ has side length 2, and square $PQRS$ has side length 5. How many times is the area of square PQRS as compared to square $ABCD$?

$$Ratio\ of\ Side\ Lengths = \underbrace{2}_{ABCD} : \underbrace{5}_{PQRS} = 1 : \frac{5}{2}$$

In order to find the ratio of areas, we square the ratio of side lengths:

$$Ratio\ of\ Areas = \underbrace{1^2}_{ABCD} : \underbrace{\left(\frac{5}{2}\right)^2}_{PQRS} = \underbrace{1}_{ABCD} : \underbrace{\frac{25}{4}}_{PQRS}$$

PQRS is $\frac{25}{4} = 6\frac{1}{4}$ times the area of square ABCD.

Example 1.33

Square $ABCD$ has side length 3, and square $PQRS$ has side length 6. How many times is the area of square $PQRS$ as compared to square ABCD?

$$Ratio\ of\ Side\ Lengths = \underbrace{3}_{ABCD} : \underbrace{6}_{PQRS} = 1:2$$

In order to find the ratio of areas, we square the ratio of side lengths:

$$Ratio\ of\ Areas = \underbrace{1}_{ABCD} : \underbrace{4}_{PQRS}$$

Hence, the area of square PQRS is 4 times the area of square ABCD.

Example 1.34

Square $ABCD$ has side length 4, and square $PQRS$ has side length 16. How many times is the area of square $ABCD$ as compared to square $PQRS$?

$$Ratio\ of\ Side\ Lengths = \underbrace{4}_{ABCD} : \underbrace{16}_{PQRS} = 1:4$$

In order to find the ratio of areas, we square the ratio of side lengths:

$$Ratio\ of\ Areas = \underbrace{1}_{ABCD} : \underbrace{16}_{PQRS}$$

Hence, the area of square ABCD is $\frac{1}{16}th$ the area of square PQRS.

Example 1.35

Rectangle LMNO has width of 4 cm, and length of 6 cm. Rectangle QWRT has width of 24 cm, and length of 36 cm.

A. What is the ratio of the side lengths of the rectangle $LMNO$ to the rectangle $QWRT$?
B. What is the scale factor?
C. What is the ratio of areas of rectangle $LMNO$ to rectangle $QWRT$?
D. The rectangle $QWRT$ is how many times the area of rectangle $LMNO$?

Part A

$$Ratio\ of\ Width = 4:24 = 1:\mathbf{6}$$
$$Ratio\ of\ Lengths = 6:36 = 1:\mathbf{6}$$

Part B

$$Scale\ Factor = 6$$

Part C

$$Ratio\ of\ Areas = 1:36$$

Part D

$$36\ Times$$

Example 1.36

Polygon A has an area of 5 cm. Polygon B is similar to Polygon A, and has a scale factor of 2. What is the area of Polygon B?

Method I

$$Scale\ Factor = 2 \Rightarrow Ratio\ of\ Side\ Lengths = 1:2$$
$$Ratios\ of\ Areas = 1:4$$

$$Area\ of\ Polgyon\ B = 5 \times 4 = 20$$

Method II

$$Area(B) = Area(A) \times (Scale\ Factor)^2 = 5 \times 2^2 = 5 \times 4 = 20$$

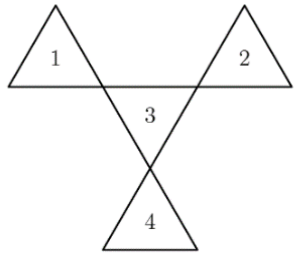
Example 1.37

Polygon A has an area of 7 cm. Polygon B is similar to Polygon A, and has an area of 28 cm. What is the scale factor?

$$Ratio\ of\ Areas = \frac{28}{7} = 4 \Rightarrow Ratio\ of\ Side\ Lengths = \sqrt{4} = \frac{2}{1} \Rightarrow Scale\ Factor = 2$$

Example 1.38

The exact amount of fencing that enclosed the four congruent equilateral triangular corrals shown here is reused to form one large equilateral triangular corral. What is the ratio of the total area of the four small corrals to the area of the new large corral? Express your answer as a common fraction. **(MathCounts 2004 National Team)**



Suppose that for the smaller corrals:

$$Side\ Length = 1 \Rightarrow Perimeter = 3 \Rightarrow Total\ Fence = 12$$

Then, for the larger corral:

$$Perimeter = 12 \Rightarrow Side\ Length = 4$$

Ratio of areas is the square of ratio of side lengths. Hence, the ratio of the large corral to one small corral is:

$$1:4 \Rightarrow 1:16$$

Ratio of area of large corral to four small corrals:

$$4:16 = 1:4$$

Example 1.39

What is the minimum number of equilateral triangles, of side length 1 unit, needed to cover an equilateral triangle of side length 10 units? **(MathCounts 2005 National Countdown)**

$$\underbrace{1:10}_{\substack{Ratio\ of \\ Side\ Lengths}} \rightarrow \underbrace{1^2:10^2}_{\substack{Ratio\ of \\ Areas}} = 1:100$$

Example 1.40

The dimensions of a triangle are tripled to form a new triangle. If the area of the new triangle is 54 square feet, how many square feet were in the area of the original triangle? **(MathCounts 1999 Chapter Countdown)**

$$\underbrace{1:3}_{\substack{Ratio\ of \\ Side\ Lengths}} \rightarrow \underbrace{1^2:3^2}_{\substack{Ratio\ of \\ Areas}} = 1:9$$

Area of Original Triangle

$$= \frac{54}{9} = 6$$

Example 1.41

A thin piece of wood of uniform density in the shape of an equilateral triangle with side length 3 inches weighs 12 ounces. A second piece of the same type of wood, with the same thickness, also in the shape of an equilateral triangle, has side length of 5 inches. The weight, in ounces, of the second piece is: **(AMC 10B 2016/10)**

For objects of uniform density:

$$Density = \frac{Mass}{Volume}$$
$$Mass = Density \times Volume$$

Since the heights of the triangles are the same, the volume is proportional to area.

$$Mass = Density \times Area$$

Also, the question for weight, which is proportional to mass.

$$Wieght \propto Density \times Area$$

The ratio of the areas is the square of the ratio of the side lengths

$$Ratio\ of\ sides = \frac{5}{3} \Rightarrow Ratio\ of\ areas = \frac{25}{9}$$

The weight

$$= 12 \times \frac{25}{9} = \frac{100}{3} = 33\frac{1}{3}$$

B. Volumes

1.42: Similar Figures

Figures with the shape but different sizes are similar. All spheres, cubes, etc are similar.

1.43: Volumes of Similar Figures

If two figures are similar and they have lengths l and L then their volumes are in the ratio:

$$l^3 : L^3$$

Example 1.44

Determine the ratio of the new volume to the old volume of a cube whose edge length is:

A. doubled.
B. Tripled
C. Halved
D. Made two-third of what it was.

Part A

$$\frac{V}{v}=\frac{S^3}{s^3}=\frac{2^3}{1^3}=\frac{8}{1}$$

Part B

$$\frac{V}{v}=\frac{S^3}{s^3}=\frac{3^3}{1^3}=\frac{27}{1}$$

Part C

$$\frac{V}{v}=\frac{S^3}{s^3}=\frac{\left(\frac{1}{2}\right)^3}{1^3}=\frac{1}{8}:1=1:8$$

Part D

$$\boldsymbol{\frac{V}{v}=\frac{S^3}{s^3}=\frac{\left(\frac{2}{3}\right)^3}{1^3}=\frac{\frac{8}{27}}{1}=\frac{8}{27}:1=8:27}$$

Example 1.45

Find the radius of a sphere that has volume half that of a sphere with radius 4 cm.

Method I

With respect to sphere, the ratio of volumes will be the cube of ratio of radii:

$$v:V=r^3:R^3=1:2$$

Take the cube root:

$$r:R=1:\sqrt[3]{2}=1:2^{\frac{1}{3}}$$

Multiply both sides by $2^{\frac{5}{3}}$:

$$1\cdot 2^{\frac{5}{3}}:2^{\frac{1}{3}}\cdot 2^{\frac{5}{3}}=2^{\frac{5}{3}}:2^{\frac{1}{3}+\frac{5}{3}}=2^{\frac{5}{3}}:2^{\frac{6}{3}}=2^{\frac{5}{3}}:2^2=2^{\frac{5}{3}}:4$$

Method II

The volume of the smaller sphere

$$=\frac{V}{2}=\frac{\frac{4}{3}\pi(4^3)}{2}=\frac{4}{3}\pi(32)=\frac{4}{3}\pi(r^3)$$

$$r^3=32\Rightarrow r=\sqrt[3]{32}=\sqrt[3]{8\times 4}=2\sqrt[3]{4}$$

C. Cones

Example 1.46

A conical cup is filled so that the liquid takes up 90% of the depth of the cup. Determine what percent of the volume of the cup is full.

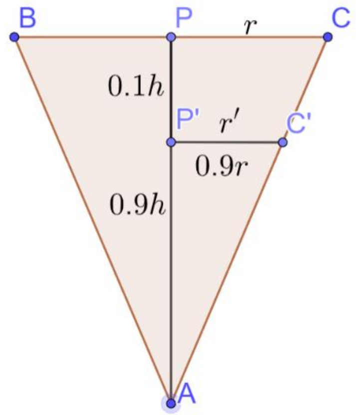
Using Ratio of Volumes

Since the cones are similar, the ratio of the volumes is the cube of the ratio of the height.

$$Ratio\ of\ heights=\ 0.9:1$$

$$Ratio\ of\ volumes=\ 0.9^3:1^3=0.729:1=72.9\%:100\%$$

Therefore, 72.9% of the cup is filled.



Using Similarity of Triangles

By AA Similarity of Triangles, we know that $\Delta APC \sim \Delta AP'C'$:

$$\frac{r'}{r} = \frac{0.9h}{h} \Rightarrow r' = 0.9r$$

Substitute $h' = 0.9h, r' = 0.9r$:

$$V' = \frac{1}{3}\pi(r')^2 h' = \frac{1}{3}\pi(0.9r)^2(0.9h)^2$$

Find the percentage of the new cup

$$\frac{V'}{V} = \frac{\frac{1}{3}\pi(0.9r)^2(0.9h)^2}{\frac{1}{3}\pi r^2 h} = \frac{0.9^3}{1} = 0.729 = 72.9\%$$

D. Pyramids

Example 1.47

In triangular pyramid $ABCD$, points E,F,G are located on AB, AC and AD respectively so that $\frac{AE}{AB} = \frac{AF}{AC} = \frac{AG}{AD} = \frac{17}{20}$. H is the intersection of altitudes of triangle BCD. If the ratio of the volumes of $EFGH$ and $ABCD$ is $\frac{a}{b}$, find $b - a$. **(Mock AMC 8 2017/7, Adapted)**[2]

$$\frac{AE}{AB} = \frac{AF}{AC} = \frac{AG}{AD} = \frac{17}{20} \Rightarrow \Delta EFG \sim \Delta BCD$$

Hence, the sides of ΔEFG and ΔBCD are

$$17:20$$

Ratio of area of bases is square of ratio of sides:

$$\frac{[EFG]}{[BCD]} = \frac{17^2}{20^2} = \frac{289}{400}$$

Since pyramid $AEFG \sim$ pyramid $ABCD$, their heights are in the ratio of their sides:

$$\frac{Height(AEFG)}{Height(ABCD)} = \frac{17}{20}$$

Since $Height(AEFG) + Height(EFGH) = Height(ABCD)$:

$$\frac{Height(EFGH)}{Height(ABCD)} = 1 - \frac{17}{20} = \frac{3}{20}$$

Finally, the ratio of volumes of $\frac{V(EFGH)}{V(ABCD)}$:

$$= \frac{\frac{1}{3}}{\frac{1}{3}} \times \frac{Height_{EFGH}}{Height_{ABCD}} \times \frac{Base_{EFGH}}{Base_{ABCD}}$$

Substitute the values from above:

$$= \frac{3}{20} \times \frac{289}{400} = \frac{867}{8000} = \frac{b}{a}$$

The value of the expression we need is then:

$$b - a = 8000 - 867 = 7133$$

[2] By eisirrational, pretzel, AOPS12142015, Th3Numb3rThr33, e_power_pi_times_i. Details here

E. Surface Area

1.48: Surface Area
If two figures are similar, their surface areas are in the ratio of the square of their lengths.

Example 1.49
Cube X has surface area 64 square units. Determine the surface area for a cube that has edge length half of the edge length of X.

The surface area will be in the ratio of the square of edge lengths

$$s_1^2 : s_2^2 = 1^2 : 2^2 = 1 : 4 = 16 : 64$$

The smaller cube has surface area

$$64$$

1.3 Area Theorems

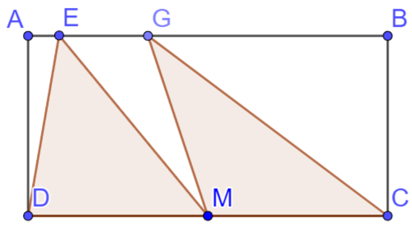
A. Equal Areas

Area theorems let us compare triangles that have something in terms of their base, their height, or both.

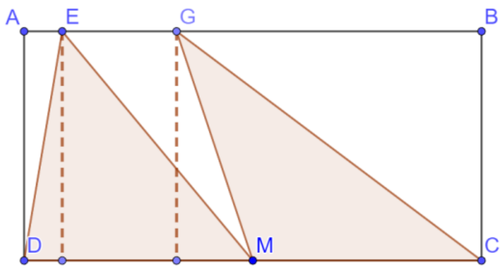
1.50: Equal Bases and Heights
If two triangles have the same base and the same height, then their areas are equal.

$$\textit{Triangle } 1 \textit{ has height } h \textit{ and base } b$$
$$\textit{Triangle } 2 \textit{ has height } h \textit{ and base } b$$

$$\textit{Area of each triangle} = \frac{1}{2}$$



Example 1.51
In rectangle $ABCD$, M is the midpoint of CD. Determine the ratio of the areas of the two triangles.



$$Area = \frac{1}{2}hb$$

$$\frac{[EDM]}{[GMC]} = \frac{\frac{1}{2}(AD)(DM)}{\frac{1}{2}(AD)(MC)}$$

Since $DM = MC$, the ratio is simply:

$$1:1$$

1.52: Equal Areas
If two triangles have equal areas:

- the ratio of the bases is the reciprocal of the ratio of the heights
- Equivalently, the ratio of the heights is the reciprocal of the ratio of the bases

$$\frac{h_1}{h_2} = \frac{b_2}{b_1}$$

$$A_1 = A_2$$
$$\frac{1}{2}h_1b_1 = \frac{1}{2}h_2b_2$$
$$\frac{h_1}{h_2} = \frac{b_2}{b_1}$$

Example 1.53
Two triangles with equal area have heights in the ratio $1:5$. Determine the ratio of their bases.

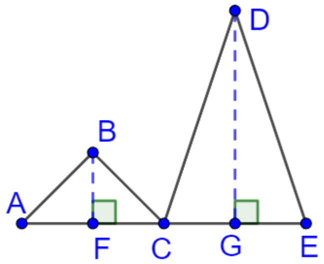
$$\frac{1}{5} = \frac{h_1}{h_2} = \frac{b_2}{b_1} \Rightarrow \frac{b_1}{b_2} = \frac{5}{1}$$

B. Equal Bases

1.54: Triangles with Equal Bases
If two triangles have equal bases, then the ratio of their areas is in the ratio of their heights.

Consider triangles T_1 and T_2 with equal bases b and heights h_1 and h_2 respectively. The ratio of the areas is:

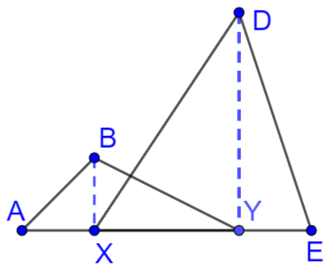
$$\frac{[T_1]}{[T_2]} = \frac{\frac{1}{2}h_1b}{\frac{1}{2}h_2b} = \frac{h_1}{h_2}$$



Example 1.55
In the diagram alongside, diagram alongside, $AC = CE$. $BF = x$, and $DG = y$. Determine the ratio of ΔBAC to ΔDCE.

Since the bases are equal, the areas will be in the ratio of their heights:

$$\frac{[BAC]}{[DCE]} = \frac{x}{y}$$



Example 1.56
In the given diagram, $BX \perp AY$, $DY \perp XE$ and

$$\frac{[BAY]}{[DXE]} = \frac{BX}{DY}$$

Determine the ratio of AX to YE.

Method I
- Since the areas are in the ratio of the heights, the bases are equal:

$$AY = XE$$
$$AY - XY = XE - XY$$
$$AX = YE$$
$$\frac{AX}{YE} = \frac{1}{1}$$

Method II
Use the formula for the area of a triangle in the given relationship:

$$\frac{\frac{1}{2}(BX)(AY)}{\frac{1}{2}(DY)(XE)} = \frac{BX}{DY}$$
$$\frac{AY}{XE} = \frac{1}{1}$$
$$AY = XE$$

C. Equal Heights

1.57: Triangles with Equal Heights

If two triangles have equal heights, then the ratio of their areas is in the ratio of their bases.

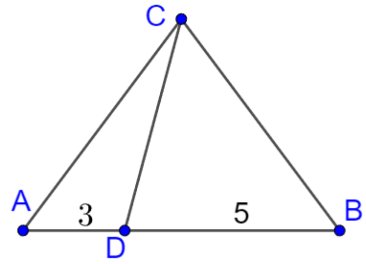
Consider triangles T_1 and T_2 with equal heights h and bases b_1 and b_2 respectively. The ratio of the areas is:

$$\frac{[T_1]}{[T_2]} = \frac{\frac{1}{2}hb_1}{\frac{1}{2}hb_2} = \frac{b_1}{b_2}$$

Example 1.58

ΔACB has area 100 $units^2$. As shown in the diagram, $AD = 3, DB = 5$.
Determine:

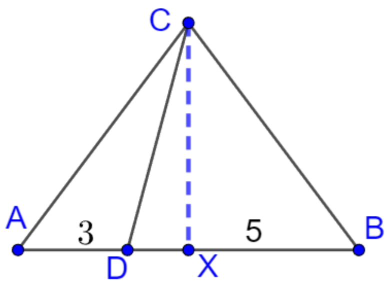
A. the ratio of the areas of the two smaller triangles created by line segment CD.
B. the area of the two smaller triangles



Part A

Draw $CX \perp AB$. Since the two each have height CX, their areas are in the ratio of their bases:

$$\frac{[ACD]}{[CDB]} = \frac{3}{5}$$



Part B

The total area must get divided in the ratio

$$3:5 \to 3+5 = 8$$

Hence:

$$[ACD] = \frac{3(100)}{8} = 37.5, \qquad [CDB] = \frac{5(100)}{8} = 62.5$$

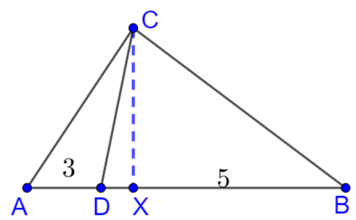
Example 1.59

In the previous example, can you conclude (from the diagram or otherwise) that

$$AC = CB$$

You cannot reach this conclusion.
Even if you draw the diagram to be asymmetrical, you can still do the same algebra/calculations as before and the answer remains the same.

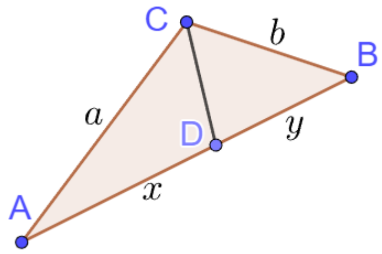
In general, diagrams are not drawn to scale in exam questions unless indicated otherwise.



Example 1.60

ΔACB has area 54 $units^2$. As shown in the diagram, $AC = a, CB = b, AD = x, DB = y$. Determine, in terms of a, b, x, y:

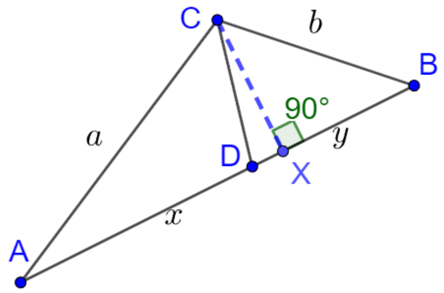
C. the ratio of the areas of the two smaller triangles created by line segment CD.
D. the area of the two smaller triangles



Part A

Draw $CX \perp AB$. Since the two triangles each have height CX, their areas are in the ratio of their bases:

$$\frac{[ACD]}{[CDB]} = \frac{x}{y}$$



Part B

The total area must get divided in the ratio

$$x : y \to Total = x + y$$

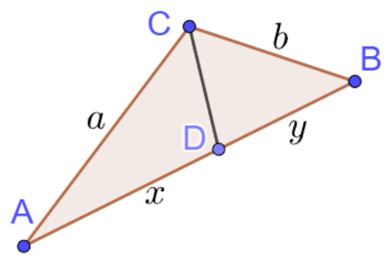
Hence:

$$[ACD] = 54\left(\frac{x}{x+y}\right), \qquad [CDB] = 54\left(\frac{y}{x+y}\right)$$

Example 1.61

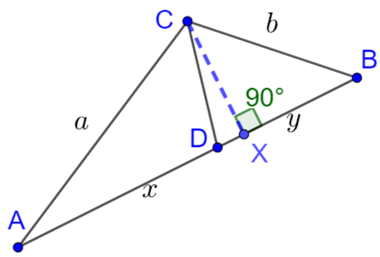
Refer the diagram alongside. Determine the value of z given that

$$\frac{[ACD]}{[CDB]} = \frac{3}{5}, \qquad x = 2z - 6, \qquad y = 3z + 4$$



Since the heights are equal, the areas are in the ratio of their bases:

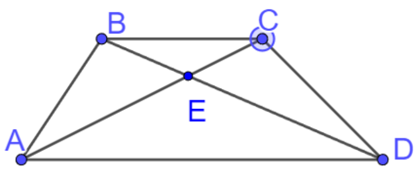
$$\frac{[ACD]}{[CDB]} = \frac{x}{y} = \frac{2z-6}{3z+4} = \frac{3}{5}$$
$$10z - 30 = 9z + 12$$
$$z = 42$$



D. Quadrilaterals

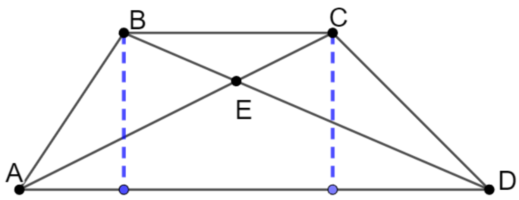
Example 1.62

$ABCD$ is a trapezoid with $BC \parallel AD$. List all pairs of triangles with equal heights in the diagram.



Pair 1

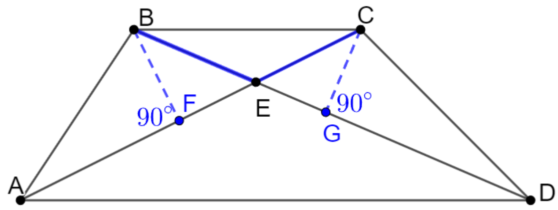
ABD and ACD are between two parallel lines and hence they have the same height



Pair 2

ABC and BCD are between two parallel lines and hence they have the same height

Pair 3

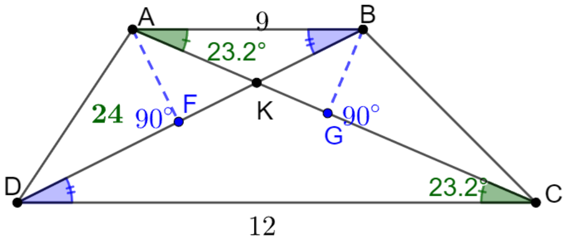


In ABE, consider AE as the base and B as the vertex. The height is BF.
In BEC, consider EC as the base and B as the vertex. The height is BF.
Hence, ABE and BEC have the same height.

Pair 4
CED and BEC have height CG.

Example 1.63

Trapezoid $ABCD$ has bases $\overline{AB}$ and $\overline{CD}$ and diagonals intersecting at K. Suppose that $AB = 9$, $DC = 12$, and the area of $\triangle AKD$ is 24. What is the area of trapezoid $ABCD$? **(AMC 10A 2008/20)**



In ΔAKB and ΔCKD

$$\angle BAC = \angle ACD \ (Alternate\ interior\ angles)$$
$$\angle ABD = \angle BDC \ (Alternate\ interior\ angles)$$
$$\Delta AKB \sim \Delta CKD \ (AA\ Similarity)$$

Hence,

$$\frac{AB}{CD} = \frac{KA}{KC} = \frac{KB}{KD} = \frac{9}{12} = \frac{3}{4}$$

Since ΔAKB and ΔAKD have same height AF, their areas are in the ratio of their bases:

$$\frac{[AKB]}{[AKD]} = \frac{KB}{KD} = \frac{3}{4} \Rightarrow \frac{[AKB]}{24} = \frac{3}{4} \Rightarrow [AKB] = 18$$

Since ΔKDA and ΔCKD have same height, their areas are in the ratio of their bases:

$$\frac{[CKD]}{[AKD]} = \frac{KC}{KA} = \frac{4}{3} \Rightarrow [CKD] = \frac{4}{3}(24) = 32$$

Since ΔBKA and ΔBKC have same height BG, their areas are in the ratio of their bases:

$$\frac{[BKC]}{[BKA]} = \frac{KC}{KA} = \frac{4}{3} \Rightarrow [BKC] = \frac{4}{3}(18) = 24$$

$$[AKD] + [AKB] + [CKD] + [BKC] = 24 + 18 + 32 + 24 = 98$$

E. Median Properties

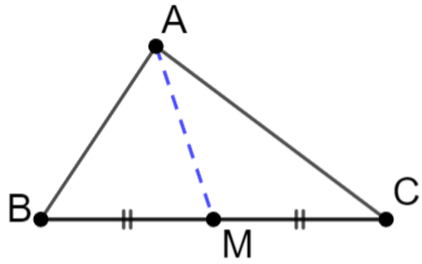
1.64: Median: Revision

- The median of a triangle is the line segment drawn from the vertex of a triangle to the midpoint of the opposite side.
- Every triangle has three medians, corresponding to its three vertices.
- The three medians of a triangle are concurrent at the centroid.

1.65: Single Median

The median of a triangle divides it into two triangles with equal area.

Note: This property is not difficult to prove, or remember. Its usefulness lies in speeding up calculations.



In ΔABC, let AM be the median. Then:

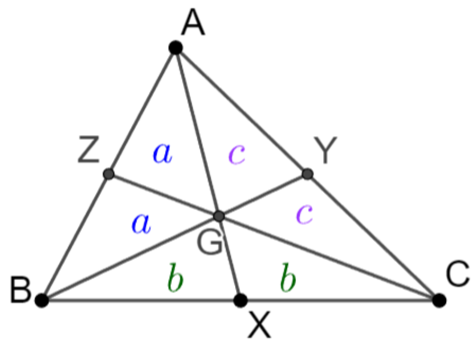
$$\frac{[ABM]}{[AMC]}=\frac{AH\cdot BM}{AH\cdot MC}=1$$

(where $BM=MC$ since AM is the median)

1.66: Six Triangles from Three Medians

The three medians of a triangle divide it into six triangles of equal area, each with a vertex at the centroid.

Draw ΔABC with medians AX, BY, and CZ.
The three medians will be concurrent at the centroid, say
$$G$$



The median divides the triangle into two equal triangles:

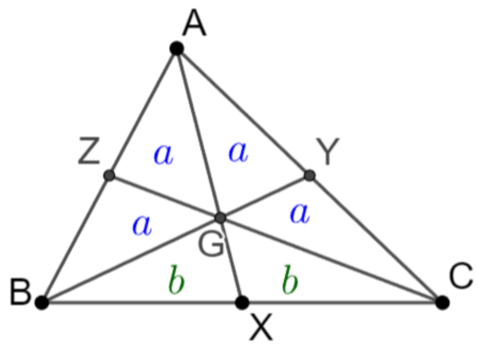
$$\Delta AGB: [AGZ]=[ZGB]=a$$
$$\Delta \mathrm{BGC}: [BGX]=[XGC]=b$$
$$\Delta \mathrm{AGC}: [AGY]=[YGC]=c$$

Median AX divides ΔABC into two triangles of equal area:

$$[AXB]=[AXC]$$
$$a+a+b=c+c+b$$
$$\underbrace{a=c}_{Result\ I}$$

Median BY divides ΔABC into two triangles of equal area:

$$[BYC]=[BYA]$$
$$b+b+a=a+a+a$$
$$\underbrace{b=a}_{Result\ II}$$



From I and II:

$$a=b=c$$

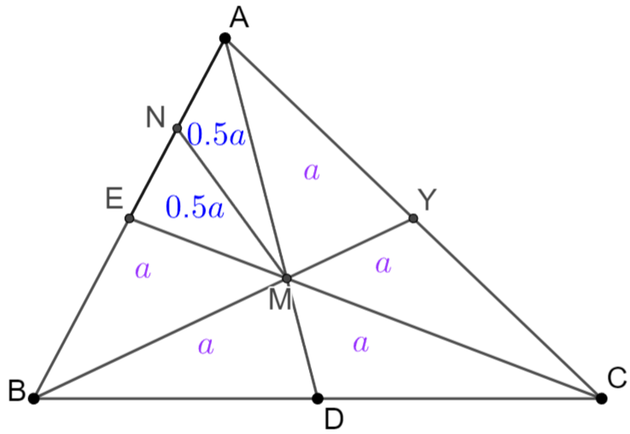
Hence, the areas of all six of the triangles are equal.

Example 1.67

Medians AD and CE of ΔABC intersect in M. The midpoint of AE is N. Let the area of ΔMNE be k times the area of ΔABC. Then k equals: (AHSME 1961/21)

The three medians of a triangle divide it into six triangles of equal area. Hence:

$$[AME] = \frac{[ABC]}{6}$$



MN is the median of ΔAME. Hence:

$$[MNE] = \frac{[AME]}{2} = \frac{\frac{[ABC]}{6}}{2} = \frac{[ABC]}{12}$$

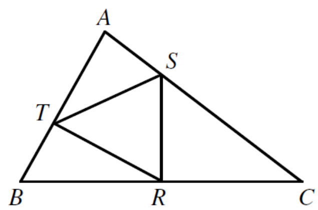
Hence:

$$k = \frac{1}{12}$$

F. Challenging Problems

Example 1.68

In triangle ABC, $BR = RC$, $CS = 3SA$, and $\frac{AT}{TB} = \frac{p}{q}$. If the area of ΔRST is twice the area of ΔTBR, then $\frac{p}{q}$ is equal to **(CEMC Cayley 1997/25)**



$$[RST] = 2[TBR]$$

Using complementary areas, ΔRST is the area of the larger triangle minus the three smaller triangles:

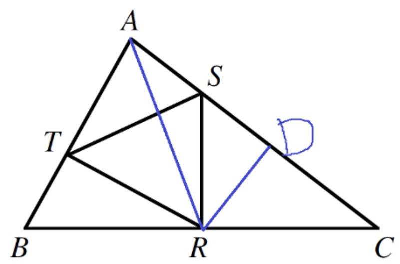
$$[ABC] - [CRS] - [TBR] - [ATS] = 2[TBR]$$

Rearrange to get:

$$\underbrace{[ABC] - [CRS] - [ATS] = 3[TBR]}_{\textit{Equation I}}$$

Since we are interested in the ratio, without loss of generality, let $[ABC] = 1$. Since ΔABR and ΔCRA have the same height and the same base:

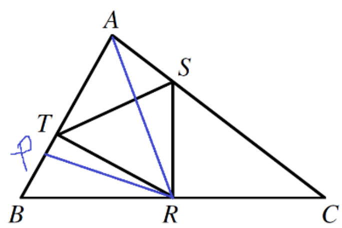
$$[ABR] = [CRA] = \frac{1}{2}$$



Since ΔCRS and ΔCRA have the same height, their areas are in the ratio of their bases:

$$\frac{[CRS]}{[CRA]} = \frac{CS}{CA} = \frac{CS}{CS + SA} = \frac{3}{3+1} = \frac{3}{4}$$

$$[CRS] = \frac{3}{4}[CRA] = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$$



Since ΔTBR and ΔABR have the same height, their areas are in the ratio of their bases:

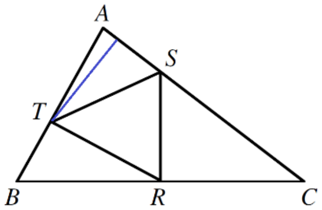
$$\frac{[TBR]}{[ABR]} = \frac{BT}{BA} = \frac{BT}{BT + TA} = \frac{q}{q+p}$$

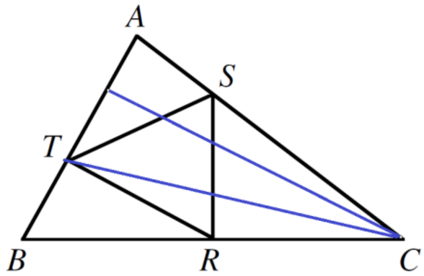
$$[TBR] = \frac{q}{q+p}[ABR] = \frac{q}{2(q+p)}$$

ΔATS has the same height as ΔATC but a base that is $\frac{1}{4}^{th}$ of ΔATC:

$$[ATS] = \frac{1}{4}[ATC] = \left(\frac{1}{4}\right)\left(\frac{p}{q+p}\right)[ABC] = \frac{p}{4(q+p)}$$

(Where $[ATS]$ can be calculated in terms of $[ABC]$ since they have the same height but different bases).





Make the substitutions of the quantities calculated above into Equation I:

$$\underbrace{1}_{[ABC]} - \underbrace{\frac{3}{8}}_{[CRS]} - \underbrace{\frac{p}{4(q+p)}}_{[ATS]} = \underbrace{\frac{3q}{2(q+p)}}_{[TBR]}$$

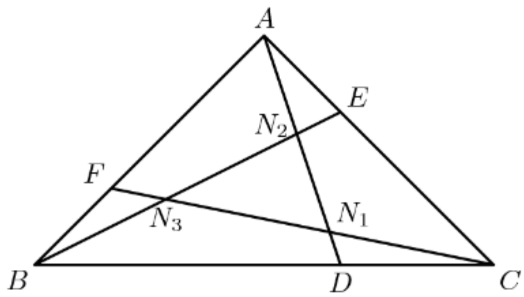
Multiply both sides by $8(q+p)$ to clear fractions, and solve for the ratio we want:

$$5q + 5p - 2p = 12q$$
$$3p = 7q$$
$$\frac{p}{q} = \frac{7}{3}$$

1.69: Ratios of Areas of Two Triangles

The ratio of areas of two triangles is the ratio of the product of their respective heights and bases:

$$\frac{\Delta_1}{\Delta_2} = \frac{\frac{1}{2}h_1b_1}{\frac{1}{2}h_2b_2} = \frac{h_1b_1}{h_2b_2}$$



Example 1.70

In the figure, CD, AE and BF are one-third of their respective sides. It follows that $AN_2 : N_2N_1 : N_1D = 3:3:1$, and similarly for lines BE and CF. Then the area of triangle $N_1N_2N_3$ (in terms of the area of ΔABC) is: **(AHSME 1952/49)**

Draw the altitudes of ΔABC

$$AX \perp BC, \qquad BY \perp AC, \qquad CZ \perp AB$$

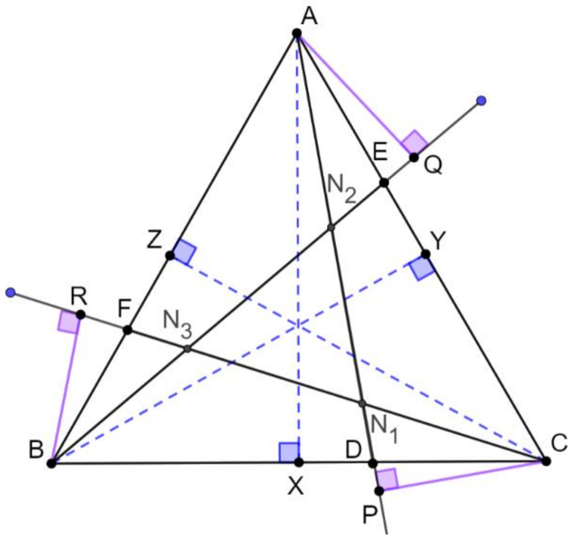
Step I: Area of Triangles

ΔADC, ΔBEA, and ΔCFB each have the same height as ΔABC, but base which is one-third of ΔABC and hence:

$$\frac{[ADC]}{[ABC]} = \frac{AX \cdot DC}{AX \cdot BC} = \frac{1}{3}, \qquad \frac{[BEA]}{[ABC]} = \frac{BY \cdot EA}{BY \cdot CA} = \frac{1}{3}, \qquad \frac{[CFB]}{[ABC]} = \frac{CZ \cdot FB}{CZ \cdot AB} = \frac{1}{3}$$

From the above three, we can conclude, for some constant $[ABC] = K$:

$$[ADC] = [BEA] = [CFB] = \frac{1}{3}[ABC] = \frac{K}{3}$$



Step II: Area of Smaller Triangles

Draw the altitudes in three smaller triangles. In

$\Delta ADC: CP \perp AD$, $\quad \Delta\text{BEA}: AQ \perp BE$, $\quad \Delta CFB: BR \perp CR$

The question gives us the ratios:

$$AN_2 : N_2N_1 : N_1D \; = \; 3:3:1 \Rightarrow N_1D = \frac{1}{7}AD$$

And the above result comes from converting to totals.
Similarly:

$$N_2E = \frac{1}{7}BE, \qquad N_3F = \frac{1}{7}CF$$

Then, since they have the same heights, the ratio of areas is given by the ratio of their bases for:

$$\frac{[N_1DC]}{[ADC]} = \frac{CP \cdot N_1D}{CP \cdot AD} = \frac{1}{7}, \qquad \frac{[N_2EA]}{[BEA]} = \frac{AQ \cdot N_2E}{AQ \cdot BE} = \frac{1}{7}, \qquad \frac{[N_3FB]}{[CFB]} = \frac{BR \cdot N_3F}{BR \cdot CF} = \frac{1}{7}$$

From the above three, and since $[ADC] = [BEA] = [CFB] = \frac{K}{3}$, we can conclude:

$$[N_1DC] = [N_2EA] = [N_3FB] = \frac{1}{7}[ADC] = \frac{K}{21}$$

Step III: Calculation of Areas

$$[CN_1N_2E] = [ADC] - [N_1DC] - [N_2EA] = \frac{K}{3} - \frac{K}{21} - \frac{K}{21} = \frac{7K}{21} - \frac{2K}{21} = \frac{5K}{21}$$

$$[AN_2N_3F] = [BEA] - [N_1DC] - [N_3FB] = \frac{5K}{21}$$

$$[DN_1N_3B] = [CFB] - [N_1DC] - [N_1DC] = \frac{5K}{21}$$

By complementary areas:

$$[N_1N_2N_3]' = [N_1DC] + [N_2EA] + [N_2FB] + [CN_1N_2Q] + [AN_2N_3F] + [DN_1N_3B]$$

$$= 3[N_1DC] + 3[CN_1N_2Q] = 3\left[\frac{K}{21}\right] + 3\left[\frac{5K}{21}\right] = \frac{3K}{21} + \frac{15K}{21} = \frac{18K}{21} = \frac{6K}{7}$$

The final answer is:

$$[N_1N_2N_3] = [ABC] - [N_1N_2N_3]' = K - \frac{6K}{7} = \frac{K}{7} = \frac{[ABC]}{7}$$

Hence, the area of the "inner" triangle is

$$\frac{1}{7}$$

Of the area of the larger triangle.

2. Similarity with Angles

2.1 AA Similarity

A. Angles in Similar Triangles

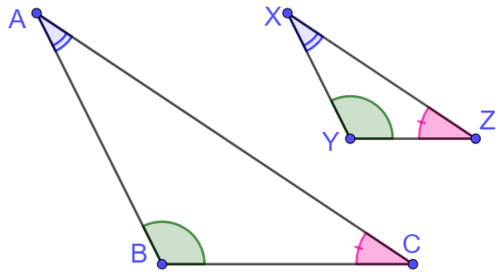
2.1: Similar Triangles

If two triangles are similar, they have

- the same shape
- the same angles.

Notation for similarity. Similarity is indicated using the symbol

$$\sim$$



Example 2.2

Use the diagram alongside to answer the questions. The two triangles are similar. Information from one part can be used in subsequent parts.

A. Identify which angles are equal.
B. $\angle B = 120°$, what is $\angle Y$?
C. $\angle X = 32°$, what is $\angle C$?

Part A

The angles which are equal are:

$$\angle A = \angle X$$
$$\angle B = \angle Y$$
$$\angle C = \angle Z$$

Part B

$$\angle B = \angle Y = 120°$$

Part C

$$\angle A = \angle X = 32°$$
$$\angle C = 180 - 120 - 32 = 28°$$

2.3: Angles in Similar Triangles

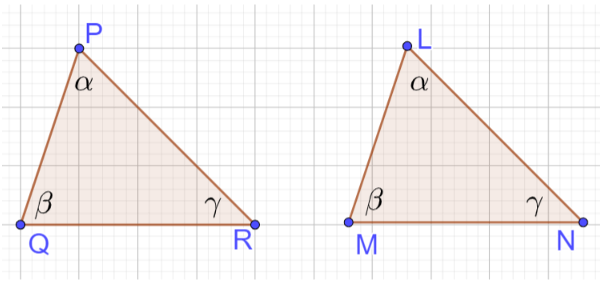
If two triangles are similar, their corresponding angles are equal.

$$\underbrace{\Delta ABC \sim \Delta XYZ}_{Triangles} \Leftrightarrow \underbrace{\angle A = \angle X, \angle B = \angle Y, \angle C = \angle Z}_{Angles}$$

- $\Delta ABC \sim \Delta XYZ$ (Read triangle ABC similar to triangle XYZ) is called a similarity statement.

B. AAA Test

2.4: AAA Test of Similarity
If the angles of two triangles are equal, the triangle are similar.



Example 2.5
Explain why $\Delta PQR \sim \Delta LMN$.

The three angles of the first triangle are equal in measure to the three angles of the second triangle. Specifically:

$$\angle P = \angle L = \alpha$$
$$\angle Q = \angle M = \beta$$
$$\angle R = \angle N = \gamma$$

$$\therefore \Delta PQR \sim \Delta LMN \ (By\ definition\ of\ similarity)$$

2.6: Order of Vertices
You need to match vertex for vertex when writing two triangles as similar.

$$\Delta ABC \sim \Delta XYZ$$

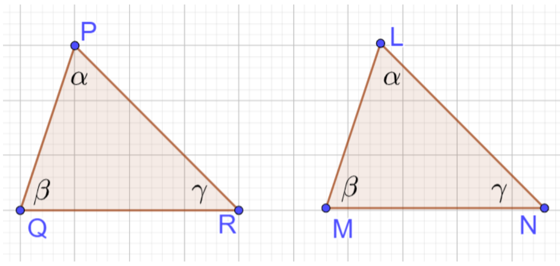
Note that
- A and X are both the first letter
- B and Y are both the second letter
- C and Z are both the third letter

Hence:

$$\angle A = \angle X, \angle B = \angle Y, \angle C = \angle Z$$

Example 2.7
Mark all correct options
From the diagram, we can see that $\Delta PQR \sim \Delta LMN$. Which of the following similarity statements must also be true:

A. $\Delta PQR \sim \Delta MNL$
B. $\Delta QRP \sim \Delta MNL$



Option Part A

$$\angle P = \alpha, \angle M = \beta$$

We do not know whether

$$\alpha = \beta$$

Hence, we cannot mark Option A.

Part B

$$\angle Q = \angle M = \beta$$
$$\angle R = \angle N = \gamma$$
$$\angle P = \angle L = \alpha$$

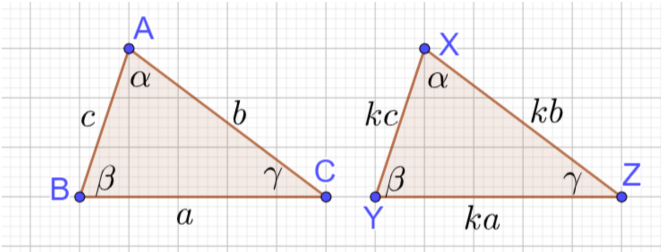
$$\therefore \Delta QRP \sim \Delta MNL (By\ definition\ of\ similarity)$$

Option B is correct

C. Scale Factor

2.8: Scale Factor in Similar Triangles

If ΔABC is similar to ΔXYZ, then their sides have the same ratio.



Since $\Delta ABC \sim \Delta XYZ$, we can conclude:

$$\frac{AB}{XY} = \frac{AC}{XZ} = \frac{BC}{YZ} = k$$

Where k is the scale factor.

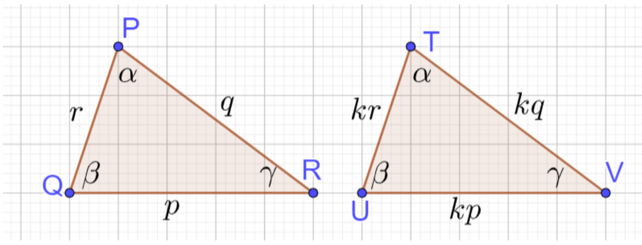
Example 2.9: Similarity Ratios, Scale Factor

ΔPQR is similar to ΔTUV.

A. Draw a diagram, state the similarity ratios, and show how the angles in the two triangles are related.
B. Given that $PQ = 3, QR = 4, PR = 5, TU = 6, UV = 8, TV = 10$, verify that the scale factor is 2.

Part A

$$\frac{PQ}{TU} = \frac{QR}{UV} = \frac{PR}{TV} = k$$
$$\angle P = T$$
$$\angle Q = \angle U$$
$$\angle R = \angle V$$



Part B

$$\frac{PQ}{TU} = \frac{3}{6} = \frac{1}{2}$$
$$\frac{QR}{UV} = \frac{4}{8} = \frac{1}{2}$$
$$\frac{PR}{TV} = \frac{5}{10} = \frac{1}{2}$$

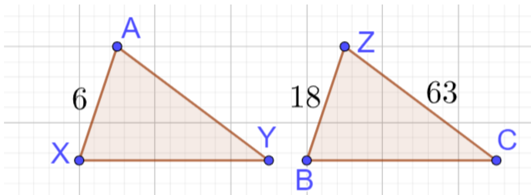
Example 2.10

Triangle AXY is similar to triangle ZBC. If $AX = 6cm$, $ZB = 18cm$ and $ZC = 63cm$, what is the length of segment AY, in centimeters? **(MathCounts 2005 State Countdown)**

$$\frac{AX}{ZB} = \frac{AY}{ZC} = \frac{XY}{BC} = k$$

Calculate the value of the scale factor, k:

$$k = \frac{AX}{ZB} = \frac{6}{18} = \frac{1}{3}$$



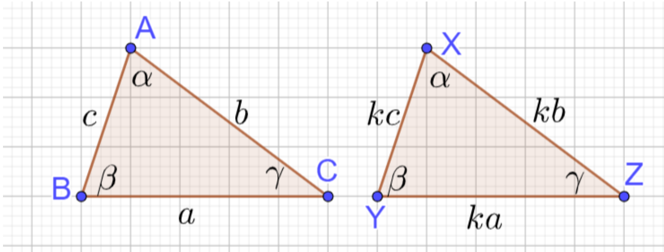
This means ΔAXY has sides $\frac{1}{3}rd$ those of ΔZBC. Use the value of k to calculate AY:

$$\frac{AY}{ZC} = \frac{1}{3} \Rightarrow AY = \frac{1}{3} \cdot ZC = \frac{1}{3} \cdot 63 = 21\ cm$$

2.11: Similarity ratios with four sides

Similarity ratios that involve only four of the six sides of two similar triangles are very useful in solving problems. For example, for the diagram alongside, we can conclude the following three equations:

$$\frac{AB}{XY} = \frac{AC}{XZ}, \qquad \frac{AC}{XZ} = \frac{BC}{YZ}, \qquad \frac{AB}{XY} = \frac{BC}{YZ}$$



Since $\Delta ABC \sim \Delta XYZ$, we can conclude:

$$\frac{AB}{XY} = \frac{AC}{XZ} = \frac{BC}{YZ} = k$$

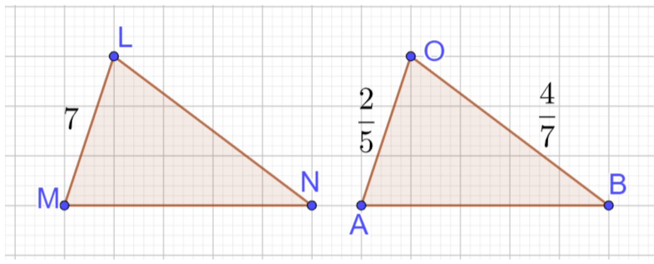
And the equations given above follow from dropping one part each time.

Example 2.12

ΔLMN is similar to triangle ΔOAB. If $LM = 7\ cm$, $OB = \frac{4}{7}\ cm$ and $OA = \frac{2}{5}\ cm$, what is the length of segment LN, in centimeters?

Write the similarity ratios:

$$\frac{LM}{OA} = \frac{LN}{OB}$$



Rearrange to solve for LN, and substitute the given values:

$$LN = \frac{LM}{OA} \times OB = \frac{7}{\frac{2}{5}} \times \frac{4}{7} = \frac{4 \times 5}{2} = 10$$

D. Parts Other than Sides

2.13: Parts other than Sides

If two triangles are similar, the scale factor is applicable to all "lengths" in the triangle. This includes:

- Perimeter
- Length of altitude
- Length of median
- Length of angle bisector

Example 2.14

An isosceles triangle has side lengths 8 cm, 8 cm and 10 cm. The longest side of a similar triangle is 25 cm. What is the perimeter of the larger triangle, in centimeters? (MathCounts 2011 Chapter Countdown)

Perimeter of the smaller triangle:

$$= 8 + 8 + 10 = 26$$

The scale factor (ratio of corresponding sides) is:

$$\frac{25}{10} = \frac{5}{2}$$

Perimeter of the larger triangle

$$= 26 \times \frac{5}{2} = 65$$

E. AA Test

- Since the sum of angles of a triangle is 180°, two angles (rather than three) are enough to uniquely identify a triangle. This is the basis of the AA test of similarity.
- The AA test is more powerful than the AAA Test that we learnt earlier. Hence, we focus on the AA test henceforth.

2.15: AA Test of Similarity

If two angles of a triangle are equal in measure to two angles of another triangle, then the two triangles are similar.

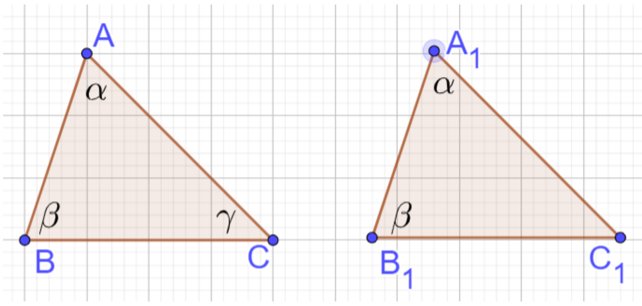
Consider ΔABC and $\Delta A_1B_1C_1$, with α and β equal in measure.

In $\Delta A_1B_1C_1$:

$$\angle A_1C_1B_1 = 180 - \alpha - \beta$$

In ΔABC:

$$\angle ACB = 180 - \alpha - \beta$$



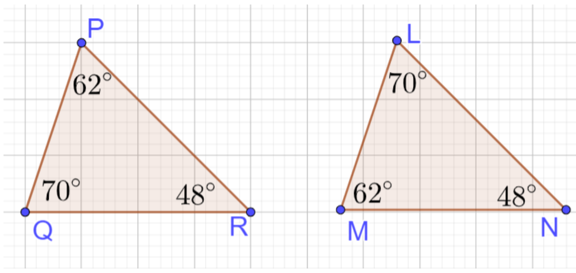
Hence, all three angles of the two triangles are the same. Hence, the two triangles are similar.

Example 2.16

ΔPQR has angles $\angle QPR = 62°$ and $\angle QRP = 48°$. ΔLMN has angles $\angle LMN = 62°$ and $\angle MLN = 70°$. Are the two triangles similar? If so, write a similarity statement and ratios.

Draw a diagram, and fill in the missing angles.

$$\angle P = \angle M = 62°$$
$$\angle Q = \angle L = 70°$$



Hence:

$$\Delta PQR \sim \Delta MLN$$

And from the above similarity, we can write the ratios:

$$\frac{PQ}{ML} = \frac{QR}{LN} = \frac{PR}{MN}$$

Example 2.17

ΔPQR has angles 35° and 55°. ΔJKL has one angle 35°. What value(s) of $\angle J$ will let us prove the two triangles similar?

Case I

If $\angle J = 55$, we know that a second angle is 35°, then by AA Test

$$\Delta PQR \sim \Delta JKL$$

Case II

If $\angle J = 180 - 35 - 55 = 90°$, and we know that a second angle is 35°, then the third angle must be:

$$180 - 35 - 90 = 55°$$

Hence, by AA Test:

$$\Delta PQR \sim \Delta JKL$$

The values of $\angle J$ that work are:

$$55°, 90°$$

2.18: Order of Similarity

When writing similarity statements, the order of the triangles does not matter. If $\Delta I \sim \Delta II$, then

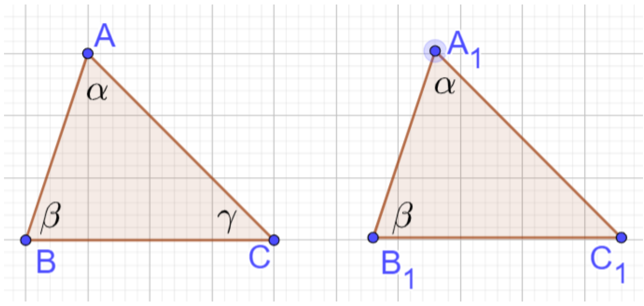
$$\Delta II \sim \Delta I$$

Example 2.19

Mark all correct options

The measures of angles α, β, γ are distinct. State which of the following options must be true:

A. $\Delta ABC \sim \Delta A_1B_1C_1$
B. $\Delta ACB \sim \Delta A_1B_1C_1$
C. $\Delta A_1B_1C_1 \sim \Delta ABC$
D. $\Delta B_1A_1C_1 \sim \Delta CAB$
E. $\Delta BCA \sim \Delta B_1C_1A_1$



$$\angle A = \angle A_1 = \alpha$$
$$\angle B = \angle B_1 = \beta$$
$$\therefore \Delta ABC \sim \Delta A_1B_1C_1 \ (AA\ Test) \Rightarrow Option\ A: True$$

Compare:

$$\angle C = \gamma, \angle B_1 = \beta, \gamma \neq \beta, \angle C \neq \angle B_1 \Rightarrow Option\ B\ is\ not\ correct$$

Option C interchanges the RHS and the LHS of Option C. Hence:

$$Option\ C: True$$

2.2 Scale Factor

A. Using Similarity Ratios

Example 2.20

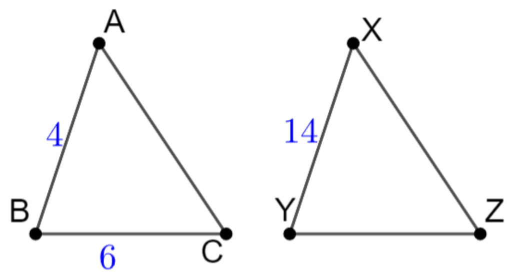
Triangle ABC is similar to triangle XYZ with side AB measuring 4 units, side BC measuring 6 units and side XY measuring 14 units. What is the measure of side YZ? **(MathCounts 2011 School Sprint)**

The similarity ratios give us:

$$\frac{YZ}{BC} = \frac{XY}{AB}$$

Rearrange and substitute:

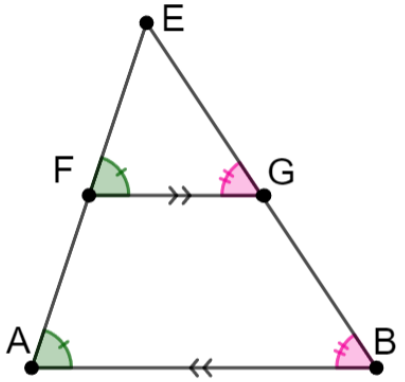
$$YZ = \frac{XY}{AB} \cdot BC = \frac{14}{4} \cdot 6 = 21$$



B. Nested Triangles

2.21: Parallel Lines

Parallel lines in a triangle create similar triangles.
This pattern is very useful, and is part of more complex questions.



In the diagram alongside,

$$FG \parallel AB$$

Considering AE and BE as transversals, by corresponding angles:

$$\angle EFG = \angle EAB$$
$$\angle EGF = \angle EBA$$

And hence, by AA similarity:

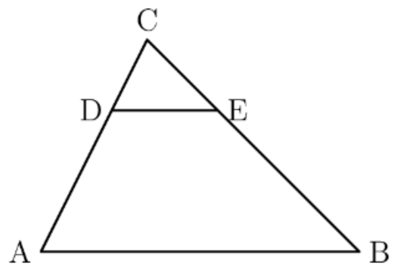
$$\Delta EFG \sim \Delta EAB$$

The ratios of the sides are:

$$\frac{EF}{EA} = \frac{EG}{EB} = \frac{FG}{AB} = k$$

Example 2.22

In ΔABC, $DE \parallel AB$, $CD = 4\ cm$, $DA = 10\ cm$, and $CE = 6\ cm$. What is the number of centimeters in the length of CB? **(MathCounts 1999 Chapter Countdown)**

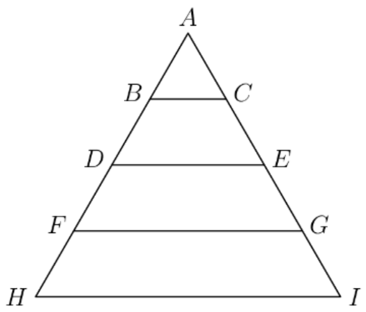


$\Delta CDE \sim \Delta CAB$ by Parallel Line Symmetry. Hence:

$$\frac{CB}{CE} = \frac{CA}{CD}$$
$$\frac{CB}{6} = \frac{4+10}{4}$$
$$CB = \frac{7}{2} \cdot 6 = 21\ cm$$

Example 2.23

Triangle AHI is equilateral. Points A, B, D, F, and H are in a straight line, in that order. Points A, C, E, G, and I are also in a straight line, in that order. We know BC, DE and FG are all parallel to HI and $AB = BD = DF = FH$. What is the ratio of the area of trapezoid $FGIH$ to the area of triangle AHI? Express your answer as a common fraction. **(MathCounts 2004 State Sprint)**



$\Delta AFG \sim AHI$ by Parallel Line Similarity. Hence:

$$\frac{AF}{AH} = \frac{AG}{AI} = \frac{3}{4}$$

The ratio of areas is the square of the ratio of sides. Hence:

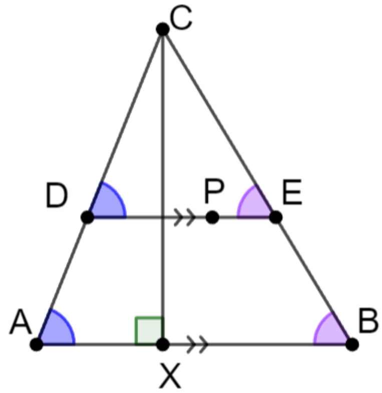
$$\frac{[AFG]}{[AHI]} = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

The ratio that we want is:

$$\frac{[FGIH]}{[AHI]} = \frac{[AHI]-[AFG]}{[AHI]} = 1 - \frac{[AFG]}{[AHI]} = 1 - \frac{9}{16} = \frac{7}{16}$$

Example 2.24

Through a point P inside ΔABC a line is drawn parallel to the base AB, dividing the triangle into two equal areas. If the altitude to AB has a length of 1, then the distance from P to AB is: (AHSME 1959/2)



$\Delta CDE \sim \Delta CAB$ by parallel line similarity.
The ratio of sides

$$= Ratio\ of\ Altitudes = \sqrt{Ratio\ of\ Areas} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

The length of CP

$$= \frac{\sqrt{2}}{2} \cdot CX = \frac{\sqrt{2}}{2} \cdot 1 = \frac{\sqrt{2}}{2}$$

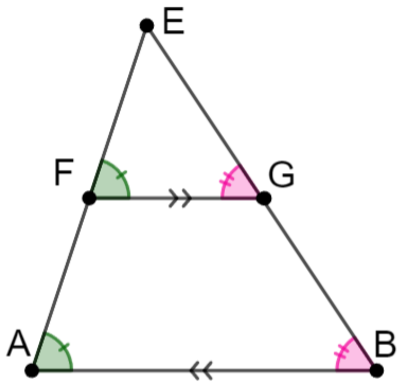
The distance from P to AB

$$= PX = CX - CP = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}$$

2.25: Ratio of Parts[3]

A line in a triangle drawn parallel to the base divides it in proportion:

$$\frac{FA}{EF} = \frac{GB}{EG}$$



$\Delta EFG \sim \Delta EAB$ by Parallel Line Similarity:

$$\frac{EA}{EF} = \frac{EB}{EG}$$

Subtract 1 from both sides:

$$\frac{EA}{EF} - 1 = \frac{EB}{EG} - 1$$

Simplify to get:

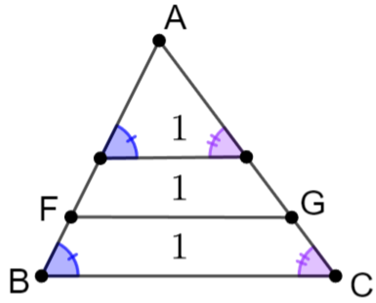
$$\frac{EA-EF}{EF} = \frac{EB-EG}{EG}$$

$$\frac{FA}{EF} = \frac{GB}{EG}$$

Example 2.26

The base of a triangle is 15 inches. Two lines are drawn parallel to the base, terminating in the other two sides, and dividing the triangle into three equal areas. The length of the parallel closer to the base is: (AHSME 1953/26)

[3] This is essentially the Basic Proportionality Theorem under a different name.



$\Delta AFG \sim \Delta ABC$ by Parallel Line Similarity.
The ratio of sides is the square root of the ratio of areas:

$$\frac{FG}{BC} = \sqrt{\frac{[AFG]}{[ABC]}} = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$$

The length of the parallel closer to the base:

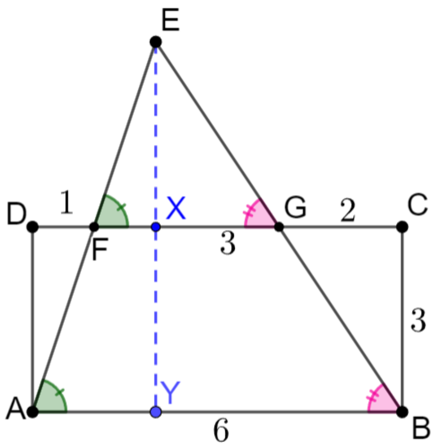
$$FG = BC \cdot \frac{\sqrt{6}}{3} = 15 \cdot \frac{\sqrt{6}}{3} = 5\sqrt{6}$$

Note: The parallel away from the base is not really required for the solution.

C. Nested Triangles in Quadrilaterals

Example 2.27

In rectangle $ABCD$, side AB measures 6 units and side BC measures 3 units. Points F and G are on side CD with segment DF measuring 1 unit and segment GC measuring 2 units, and lines AF and BG intersect at E. What is the area of triangle AEB? **(MathCounts 2010 Chapter Sprint)**



Since $DC \parallel AB$, by corresponding angles (as shown in the diagram)

$$\Delta EFG \sim \Delta EAB$$

The ratio of the bases

$$= \frac{FG}{AB} = \frac{6-2-1}{6} = \frac{3}{6} = \frac{1}{2}$$

Hence, the ratio of altitudes is also:

$$\frac{EX}{EY} = \frac{1}{2}$$
$$\frac{EX}{EX+3} = \frac{1}{2}$$
$$2EX = EX + 3$$
$$EX = 3$$
$$EY = EX + XY = 3 + 3 = 6$$

The area of ΔAEB is then:

$$= \frac{1}{2}hb = \frac{1}{2}(AB)(EY) = \frac{1}{2} \cdot 6 \cdot 6 = 18$$

Example 2.28

In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on CD so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. Find the area of ΔAEB. Express your answer as a common fraction. **(AMC 10B 2003/20, AMC 12B 2003/14)**

Since $DC \parallel AB$, by corresponding angles (as shown in the diagram)

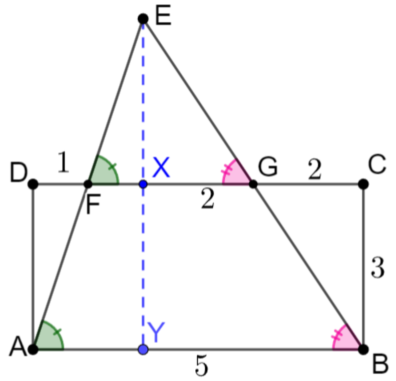
$$\Delta EFG \sim \Delta EAB$$

The ratio of the bases

$$= \frac{FG}{AB} = \frac{5-2-1}{5} = \frac{2}{5} = \frac{2}{5}$$

Hence, the ratio of altitudes is also:

$$\frac{EX}{EY}=\frac{2}{5}$$
$$\frac{EX}{EX+3}=\frac{2}{5}$$
$$5EX=2EX+6$$
$$3EX=6$$
$$EX=2$$
$$EY=EX+XY=2+3=5$$



The area of ΔAEB is then:

$$=\frac{1}{2}hb=\frac{1}{2}(AB)(EY)=\frac{1}{2}\cdot 5\cdot 5=\frac{25}{2}$$

Example 2.29

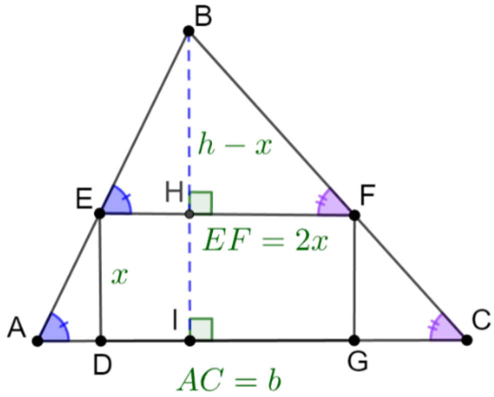
A. A rectangle inscribed in a triangle has its base coinciding with the base b of the triangle. If the altitude of the triangle is h, and the altitude x of the rectangle is half the base of the rectangle, then, find x in terms of h and b: **(AHSME 1950/47)**

B. The base of a triangle is of length b, and the altitude is of length h. A rectangle of height x is inscribed in the triangle with the base of the rectangle in the base of the triangle. The area of the rectangle is: **(AHSME 1960/37)**

Part A

Draw ΔABC with rectangle $DEFG$[4].
Draw altitude BI. Since $EF \parallel DG$

$$\angle BHF=\angle BIC=90^\circ\ (Corresponding\ Angles)$$



$\Delta BEF \sim \Delta BAC$ since $EF \parallel DG$:

$$\angle BEF=\angle BAC\ (Corresponding\ Angles)$$
$$\angle BFE=\angle BCA\ (Corresponding\ Angles)$$

The key part of this question is deciding which parts to compare. Here we compare the base of each triangle with the altitude of each triangle. Hence:

$$\underbrace{\frac{EF}{AC}=\frac{BH}{BI}}_{Equation\ I}$$

Substitute $EF=2x, AC=b, BH=h-x, BI=h$:

$$\frac{2x}{b}=\frac{h-x}{h}\Rightarrow 2xh=bh-bx$$

Collate all x terms on one side, and factor:

$$2xh+bx=bh\Rightarrow x(2h+b)=bh$$

Solve for x:

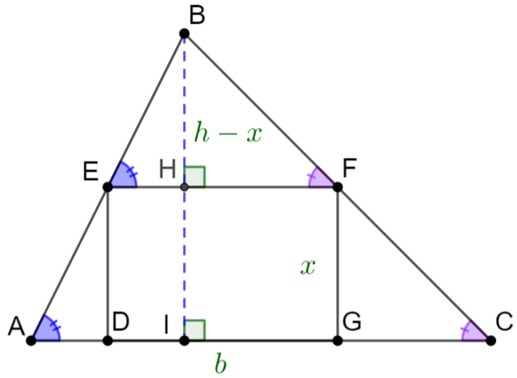
$$x=\frac{bh}{2h+b}$$

[4] This is solved using areas in the Note on Triangles.

Part B

As in Part A, $\Delta BEF \sim \Delta BAC$ (the diagrams are similar). Then, rearranging Equation I from Part A, the base of the rectangle,

$$EF = \frac{BH}{BI} \cdot AC = \frac{h-x}{h} \cdot b$$



The area of the rectangle is:

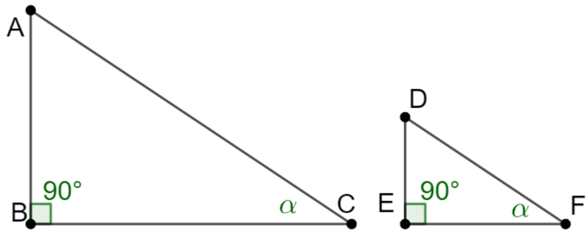
$$\underbrace{FG}_{Height} \cdot \underbrace{EF}_{Base} = x \cdot \frac{h-x}{h} \cdot b = \frac{bx}{h}(h-x)$$

D. Right Triangles

Right angled triangles are special because one angle is a right angle. Hence, we need only one more angle equality to prove similarity.

2.30: Right Triangle Similarity

Two right angled triangles are similar if one acute angle of the first triangle has the same measure as an acute angle of the other triangle.



A right-angled triangle has a right angle. Hence, if you have two right-angled triangles, one angle is already equal.

We will refer to this as

$$Right\ Triangle\ Similarity$$

Example 2.31

In each part below, two triangles are mentioned. Determine if they are similar? If they are, write a ratio comparing the sides.

A. ΔABC is a right-angled triangle, right-angled at B, with $\angle BAC = 34°$. ΔXYZ is a right-angled triangle, right-angled at X, with $\angle XYZ = 56°$.
B. ΔPQR has its angles in the ratio $\angle P : \angle Q : \angle R = 1:2:3$. ΔLMN is a right-angled triangle where the smallest angle, $\angle L$ is one-third of the largest angle, $\angle N$.

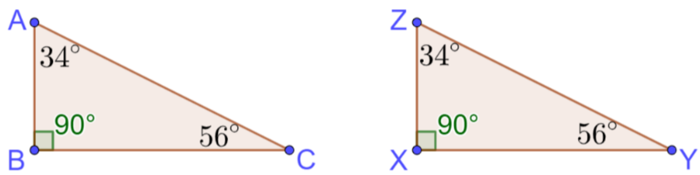
Part A

Draw a diagram:

$$\angle Z = 90 - \angle Y = 34°$$

In ΔABC and ΔZXY:

$$\angle B = \angle X = 90°, \angle A = \angle Z = 34°$$



By the AA Test of Similarity:

$$\Delta ABC \sim \Delta ZXY \Rightarrow \frac{AB}{ZX} = \frac{BC}{XY} = \frac{AC}{ZY}$$

Part B

For ΔPQR:

$$\angle P : \angle Q : \angle R = 1:2:3 = 30:60:90$$

For ΔLMN:

Since it is not possible to have a right-angled obtuse triangle, the triangle must have remaining two angles

acute. This means that the largest angle is $\angle N$.
Hence, the smallest angle

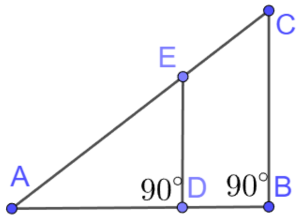
$$= \angle L = \frac{1}{3} \times \angle N = \frac{1}{3} \times 90 = 30°$$

In ΔPQR and ΔLMN:

$$\angle R = \angle N = 90°$$
$$\angle P = \angle L = 30°$$

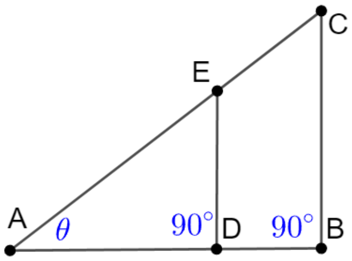
By the AA Test of Similarity:

$$\Delta PQR \sim \Delta LMN \Rightarrow \frac{PQ}{LM} = \frac{QR}{MN} = \frac{PR}{LN}$$



Example 2.32

In the diagram alongside, $AB = 4, AC = 5$. If $AD = 3$, find the length of ED.



$\Delta ADE \sim \Delta ABC$ by Right Triangle Similarity since:

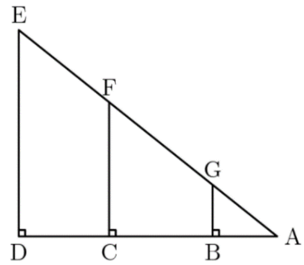
$$\angle EAD = \angle CAB = \theta$$

Hence:

$$\frac{\text{ED}}{\text{AD}} = \frac{CB}{AB} \Rightarrow ED = \frac{CB}{AB} \cdot AD$$

Using Pythagorean Triplet (3,4,5), $CB = 3$:

$$ED = \frac{3}{4} \cdot 3 = \frac{9}{4}$$



Example 2.33

Given $DC = 7, CB = 8, AB = \frac{1}{4}AD$, and $ED = \frac{4}{5}AD$, find FC. Express your answer as a decimal. (MathCounts 1994 State Team)

Form an equation using the lengths of the base:

$$AD = DC + CB + BA = 7 + 8 + \frac{1}{4}AD$$
$$\frac{3}{4}AD = 15$$
$$AD = 20$$
$$AB = \frac{1}{4}AD = \frac{1}{4} \cdot 20 = 5$$

$\Delta ACF \sim \Delta ADE$ by Right Triangle Similarity since:

$$\angle FAC = \angle EAD \; (Same\ angle)$$

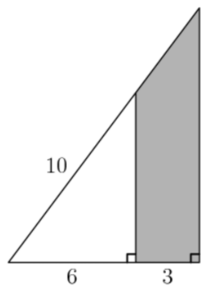
Hence:

$$\frac{FC}{AC}=\frac{ED}{AD}$$

$$FC=\frac{ED}{AD}\cdot AC=\frac{\frac{4}{5}AD}{AD}\cdot(8+5)=\frac{4}{5}\cdot(13)=10.4$$

Example 2.34

What is the number of square centimeters in the shaded area? (The 10 represents the hypotenuse of the white triangle only.) (MathCounts 2000 Warm-Up 10)



The white triangle has sides

$$2\times(3,4,5)=(10,8,6)$$

By AA similarity, the smaller right triangle is similar to the larger right triangle, and it will have sides

$$3\times(3,4,5)=(9,12,15)$$

Shaded Area

$$\frac{1}{2}\times 9\times 12-\frac{1}{2}\times 6\times 8=54-24=30$$

Example 2.35

A telephone pole is supported by a steel cable which extends from the top of the pole to a point on the ground 3 meters from its base. When Leah walks 2.5 meters from the base of the pole toward the point where the cable is attached to the ground, her head just touches the cable. Leah is 1.5 meters tall. How many meters tall is the pole? (MathCounts 2000 Warm-Up 2)

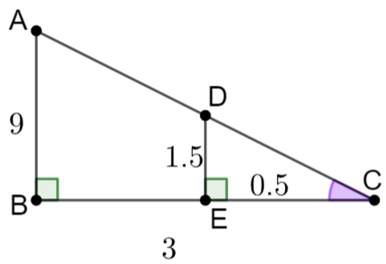
$\Delta ABC\sim\Delta DEC$ by Right Triangle Similarity since:

$$\angle DCE=\angle ACB\ (Same\ Angle)$$

$$\frac{AB}{BC}=\frac{DE}{EC}$$

$$\frac{AB}{3}=\frac{1.5}{0.5}=3$$

$$AB=3\times 3=9\ meters$$



Example 2.36

In the figure below, isosceles ΔABC with base AB has altitude $CH=24$ cm, $DE=GF$, $HF=12\ cm$, and $FB=6\ cm$. What is the number of square centimeters in the area of pentagon CDEFG? (MathCounts 2001 National Target)

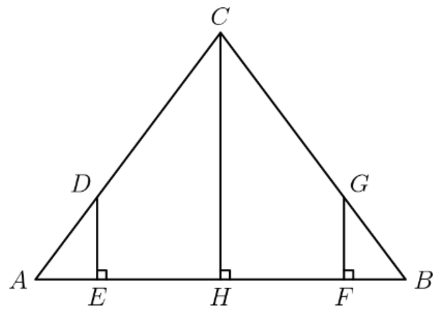
$\Delta CHB\sim\Delta GFB$ by Right Triangle Similarity since:

$$\angle CHB=\angle GFB=90°$$

The ratio of sides

$$=\frac{FB}{HB}=\frac{FB}{HF+FB}=\frac{6}{12+6}=\frac{6}{18}=\frac{1}{3}$$

The ratio of areas is the square of the ratio of sides:



$$= \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

By complementary areas, the area of pentagon CDEFG:

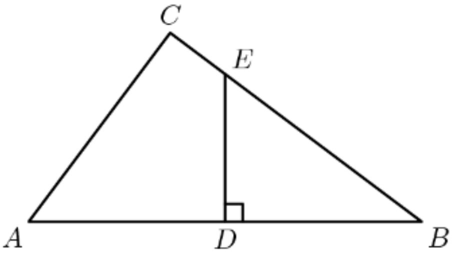
$$= [CAB]\left(1 - \frac{1}{9}\right) = [CAB]\left(\frac{8}{9}\right)$$

Substitute $h = CH = 24, b = AB = 2(12 + 6) = 2(18) = 36$

$$= \frac{1}{2}hb\left(\frac{3}{4}\right) = \frac{1}{2}(24)(36)\left(\frac{8}{9}\right) = \frac{1}{2}(24)(36)\left(\frac{8}{9}\right) = 384$$

Example 2.37

In the figure, it is given that angle $C = 90°, AD = DB, DE \perp AB, AB = 20,$ and $AC = 12$. The area of quadrilateral $ADEC$ is: **(AHSME 1952/24)**



The sides of ΔACB are:

$$(12, CB, 20) = 4(3,4,5) = (12,16,20) \Rightarrow CB = 16$$

$\Delta EDB \sim \Delta ACB$ by Right Triangle Similarity since

$$\angle EBD = \angle CBA\ (Same\ Angle)$$

The ratio of sides

$$= \frac{DB}{CB} = \frac{10}{16} = \frac{5}{8}$$

The ratio of areas is

$$= \left(\frac{5}{8}\right)^2 = \frac{25}{64}$$

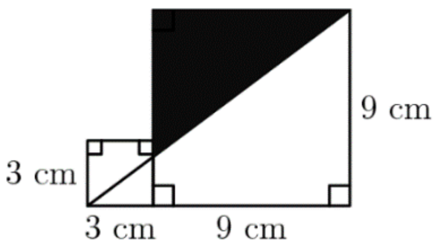
By complementary areas:

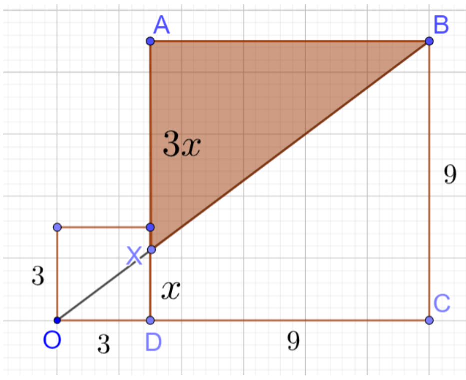
$$[ADEC] = [ABC] - [EDB]$$
$$= [ABC] - \frac{25}{64}[ABC]$$
$$= [ABC]\left(1 - \frac{25}{64}\right)$$
$$= \left[\frac{1}{2} \cdot 12 \cdot 16\right]\left(\frac{39}{64}\right)$$
$$= \frac{137}{2}$$

E. Right Triangles in Quadrilaterals

Example 2.38

What is the area of the shaded region in the figure below? Round your answer to the nearest square centimeter. **(MathCounts 1991 National Team)**





$\Delta XOD \sim \Delta BOC$ by Right Triangle Similarity since

$$\angle XOD = \angle BOC \ (Same\ Angle)$$

The ratio of sides in the larger triangle (ΔBOC) is:

$$\frac{BC}{OC} = \frac{9}{9+3} = \frac{9}{12} = \frac{3}{4}$$

The ratio of corresponding sides in the smaller triangle (ΔXOD) will be the same:

$$\frac{XD}{OD} = \frac{3}{4}$$

$$XD = \frac{3}{4}OD = \frac{3}{4} \cdot 3 = \frac{9}{4}$$

By collinear points:

$$AX = AD - XD = 9 - \frac{9}{4} = \frac{27}{4}$$

The area of ΔBAX

$$= \frac{1}{2}hb = \frac{1}{2}(9)\left(\frac{27}{4}\right) = \frac{243}{8} = 30\frac{3}{8} \approx \underbrace{30\ cm^2}_{Rounded}$$

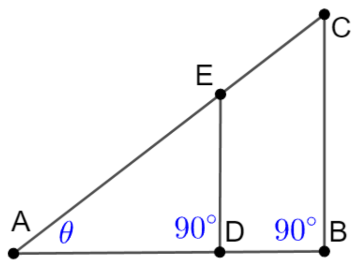
2.3 Working with Similarity

A. Right Triangles to Infer Heights

2.39: Heights from Line Segments

Line segment ratios in a triangle are proportional to heights. In the diagram:

$$\frac{Height\ of\ \Delta EAD}{Height\ of\ \Delta CAB} = \frac{ED}{CB} = \frac{AE}{AC}$$



$\Delta ADE \sim \Delta ABC$ by Right Triangle Similarity since:

$$\angle EAD = \angle CAB = \theta$$

Hence:

$$\frac{Height\ of\ \Delta EAD}{Height\ of\ \Delta CAB} = \frac{ED}{CB} = \frac{EA}{CA}$$

Example 2.40

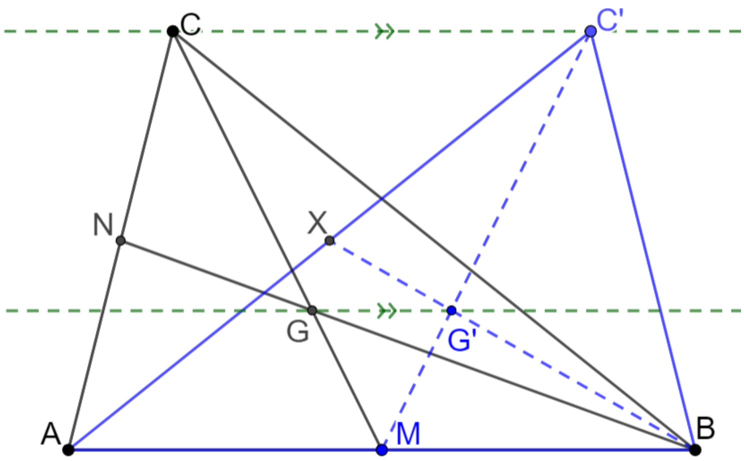
Given ΔABC with base AB fixed in length and position. As the vertex C moves on a straight line parallel to the base, what shape does the intersection point of the three medians move on? **(AHSME 1962/15, Adapted)**

Step I: Draw ΔABC with

$$CM\ as\ median\ to\ AB, \qquad BN\ as\ median\ to\ AC$$

The intersection of the three medians is the centroid of the triangle, which is

$$G$$



The centroid G divides the medians in the ratio $2:1$

$$\frac{GM}{CM}=\frac{1}{3}$$

The ratio of heights is equal to the ratio of line segments

$$\frac{h_{G\ to\ AB}}{h_{C\ to\ AB}}=\frac{GM}{CM}=\frac{1}{3}$$

$$h_{G\ to\ AB}=\frac{h_{C\ to\ AB}}{3}$$

Step II: Suppose C moves to C' on a line parallel to the base.

Then, let the medians be

$$C'M\ and\ BX$$

Let then the intersection of $C'M$ and BX' is the new centroid. Call it:

$$G'$$

Again, since the centroid G divides the medians in the ratio $2:1$, we have:

$$\frac{G'M}{C'M}=\frac{1}{3}$$

And since the ratio of heights is equal to the ratio of line segments:

$$\frac{h_{G'\ to\ AB}}{h_{C'\ to\ AB}}=\frac{G'M}{C'M}=\frac{1}{3}$$

$$h_{G'to\ AB}=\frac{h_{C'to\ AB}}{3}$$

But C' is moving on a line parallel to AB. Hence, $h_{C'to\ AB}=h_{C\ to\ AB}$:

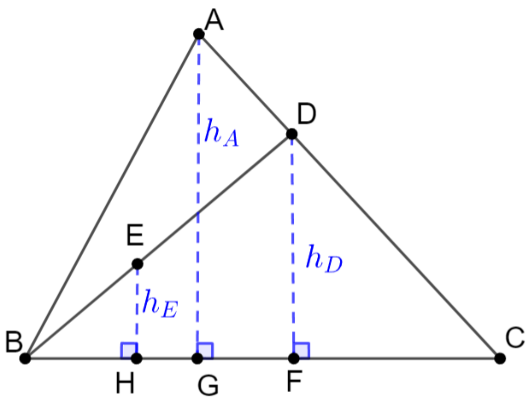
$$h_{G'to\ AB}=\frac{h_{C\ to\ AB}}{3}=h_{G\ to\ AB}$$

Hence, the distance of the centroid from the base remains the same even the vertex moves to a new point. Hence, the shape formed will be

$$A\ line\ parallel\ to\ AB$$

Example 2.41

In ΔABC, point D is on AC such that $\frac{AD}{DC}=\frac{2}{3}$, and point E is on BD such that $\frac{BE}{ED}=\frac{3}{4}$. Determine the ratio of the altitude from E to BC with the altitude from A to BC.



Draw ΔABC, and draw

$$DF \perp BC, \qquad AG \perp BC, \qquad EF \perp BC$$

Let

$$DF = h_D, \qquad AG = h_A, \qquad EH = h_E$$

Using the property on heights from line segments, in ΔDBF:

$$\underbrace{\frac{h_E}{h_D} = \frac{BE}{BD} = \frac{3}{7}}_{\text{Equation I}}$$

Using the property again, in ΔAGC gives

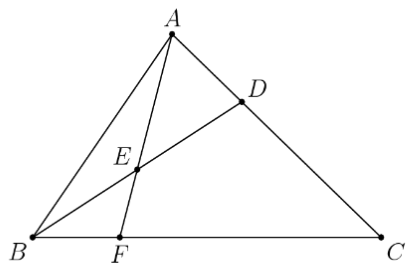
$$\frac{h_D}{h_A} = \frac{DC}{AC} = \frac{3}{5} \Rightarrow h_D = \frac{3}{5}h_A$$

which we can substitute in Equation I to get:

$$\frac{h_E}{\frac{3}{5}h_A} = \frac{3}{7} \Rightarrow \frac{h_E}{h_A} = \frac{3}{7}\cdot\frac{3}{5} = \frac{9}{35}$$

Example 2.42

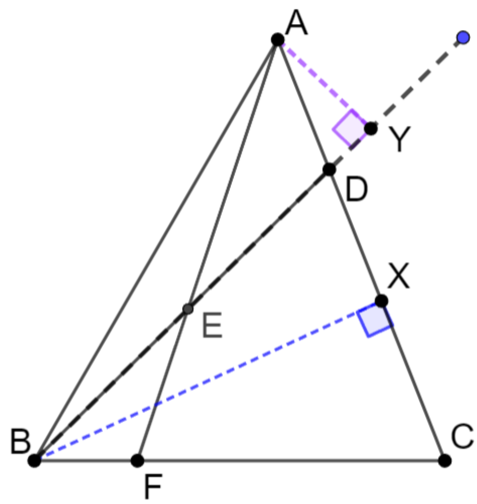
In triangle ABC, point D divides side $\overline{AC}$ so that $AD: DC = 1: 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of ΔABC is 360, what is the area of ΔEBF? (AMC 8 2019/24)



Step I: Area of ABE

Calculate the ratio of areas:

$$\frac{[ABD]}{[ABC]} = \frac{BX\cdot AD}{BX\cdot AC} = \frac{1}{3}, \qquad \frac{[ABE]}{[ABD]} = \frac{AY\cdot BE}{AY\cdot BD} = \frac{1}{2}$$



Then:

$$\frac{[ABD]}{[ABC]}\cdot\frac{[ABE]}{[ABD]} = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

$$[ABE] = \frac{[ABC]}{6} = \frac{360}{6} = 60$$

Step II: Ratio of Heights

Calculate the ratio of the heights

$$\frac{h_{D\ to\ BC}}{h_{A\ to\ BC}} = \frac{h_D}{h_A} = \frac{DC}{AC} = \frac{2}{3}, \qquad \frac{h_{E\ to\ BC}}{h_{D\ to\ BC}} = \frac{h_E}{h_D} = \frac{1}{2}$$

Multiply the results to get:

$$\frac{h_E}{h_D}\cdot\frac{h_D}{h_A} = \frac{2}{3}\cdot\frac{1}{2} \Rightarrow \frac{h_E}{h_A} = \frac{1}{3} \Rightarrow \frac{h_A}{h_E} = 3$$

Step II: Final Answer

Calculate the ratio of areas:

$$\frac{[ABF]}{[EBF]} = \frac{h_A\cdot BF}{h_E\cdot BF}$$

Substitute $[ABF] = [EBF] + [ABE]$

$$\frac{[EBF] + [ABE]}{[EBF]} = \frac{h_A}{h_E}$$

Split the fraction:

$$1 + \frac{[ABE]}{[EBF]} = 3$$
$$\frac{[ABE]}{[EBF]} = 2$$
$$[EBF] = \frac{[ABE]}{2} = \frac{60}{2} = 30$$

B. Construction

2.43: Constructions

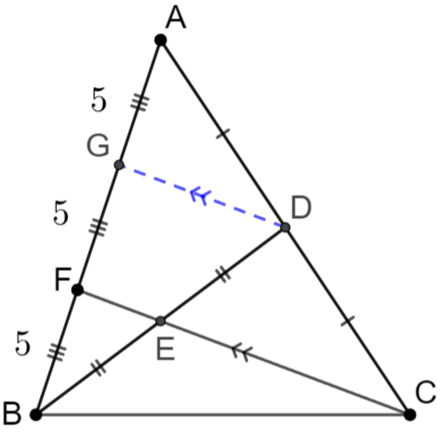
- Some questions require constructions to proceed with the solution.
- If the diagram looks "incomplete", then it is easier to observe the need for a construction.
- However, many constructions can feel "out of the blue", making them among the more difficult questions in geometry.
- Revisiting such questions a second time can help understand the motivation for the construction.

Example 2.44

In ΔABC, BD is a median. CF intersects BD at E so that $BE = ED$. Point F is on AB. Then, if $BF = 5$, BA equals: (AHSME 1959/40)

The key step here is the construction

$$Draw\ DG \parallel CF$$



$\Delta BEF \sim \Delta BDG$ by Parallel Line Symmetry. By ratio of parts:

$$\frac{FG}{BF} = \frac{ED}{BE} = 1 \Rightarrow FG = 1 \cdot BG = 1 \cdot 5 = 5$$

$\Delta AGD \sim \Delta$AFC by Parallel Line Symmetry. By ratio of parts:

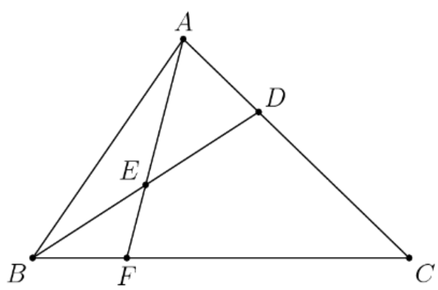
$$\frac{AG}{GF} = \frac{AD}{DC} = 1 \Rightarrow AG = 1 \cdot GF = 1 \cdot 5 = 5$$

The required length is:

$$BA = AG + FG + BF = 5 + 5 + 5 = 15$$

Example 2.45

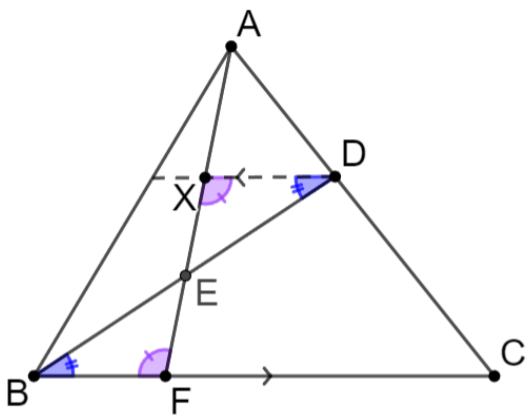
In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of ΔABC is 360, what is the area of ΔEBF? (AMC 8 2019/24)



Note:

1. This was solved above by considering heights and ratios of areas.

Now, we will use a similarity-focused construction that will also give us the ratio of BF to BC.
2. The solution is $illustrative$ (meant to focus on a concept), rather than exam-oriented.



Construct $XD \parallel BC$ with X lying on AF.

Step I: $\Delta DEX \sim \Delta BEF$ by AA Similarity[5]**:**

$$\angle DXE = \angle BFE \ (Corresponding\ Angles)$$
$$\angle XDE = \angle EBF \ (Corresponding\ Angles)$$

The ratio of sides is:

$$\frac{DE}{BE} = 1 \Rightarrow Scale\ Factor = 1 \Rightarrow \Delta's\ are\ congruent$$

Hence:

$$\underbrace{DX = BF}_{I}, \qquad \underbrace{FE = EX}_{II}$$

Step II: ΔAXD~ΔAFC by Parallel Line Similarity:

The ratio of parts

$$= \frac{AD}{DC} = \frac{1}{2} \Rightarrow \frac{AX}{FX} = \frac{1}{2} \Rightarrow \underbrace{AX = \frac{FX}{2} = \frac{FE + EX}{2} = \frac{FE + FE}{2} = FE}_{III}$$

From II and III, AF is divided into three equal parts with $AX = FE = EX$:

$$\frac{EF}{AF} = \frac{1}{3}$$

Since the ratio of heights is proportional to the line segments:

$$\underbrace{\frac{Height_{BEF}}{Height_{ABC}} = \frac{1}{3}}_{Ratio\ of\ Heights = IV}$$

Further, the ratio of sides $= \frac{AD}{AC} = \frac{1}{3}$. Hence:

$$\frac{XD}{FC} = \frac{1}{3}$$

From I, $DX = BF$

$$\frac{BF}{FC} = \frac{1}{3} \Rightarrow FC = 3BF$$

$$\underbrace{\frac{BF}{BC} = \frac{BF}{BF + FC} = \frac{BF}{BF + 3BF} = \frac{1}{4}}_{Ratio\ of\ Bases = V}$$

Step III: Final Answer

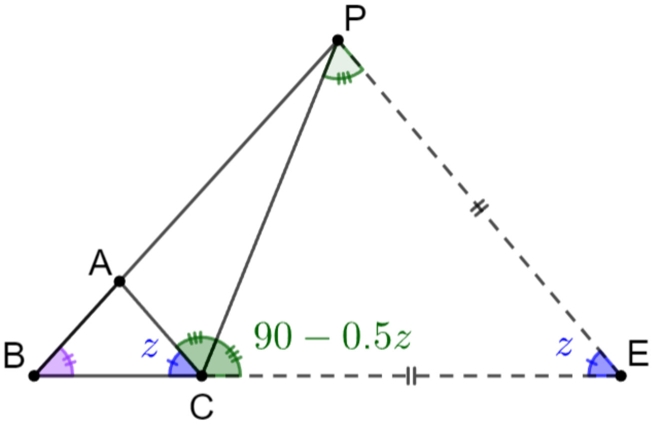
The ratio of areas of the required triangle to the complete triangle (using IV and V)

$$\frac{[EBF]}{[ABC]} = \frac{h_{EBF}b_{EBF}}{h_{ABC}b_{ABC}} = \underbrace{\frac{1}{3}}_{\substack{Ratio\ of \\ Heights}} \times \underbrace{\frac{1}{4}}_{\substack{Ratio\ of \\ Bases}} = \frac{1}{12} \Rightarrow [EBF] = \frac{[ABC]}{12} = \frac{360}{12} = 30$$

Example 2.46

[5] This is an example of Bow Tie Similarity, which often occurs in questions.

In ΔABC the ratio $AC:CB$ is 3:4. The bisector of the exterior angle at C intersects BA extended at P (A is between P and B). The ratio $PA:AB$ is: **(AHSME 1961/31)**



Draw $PE \parallel AC$ and so that it intersects BC at E.

$$\therefore \angle PEC = \angle ACB = z\ (Corresponding\ Angles)$$

Since CP is the bisector of $\angle ACE$:

$$\angle PCE = 0.5(\angle ACE) = 0.5(180 - z) = 90 - 0.5z$$

ΔPCE is isosceles because:

$$\angle CPE = 180 - \underbrace{z}_{\angle PEC} - \underbrace{(90 - 0.5z)}_{\angle PCE}$$
$$= 90 - 0.5z = \angle PCE$$
$$\therefore \underbrace{CE = PE}_{Result\ I}$$

$\Delta BCA \sim \Delta BEP$ by Parallel Line Symmetry since $PE \parallel AC$. Hence, the ratio of sides:

$$= \frac{AC}{BC} = \frac{PE}{BE} = \frac{PE}{BC + CE} = \frac{3}{4}$$

Substitute Result I ($CE = PE$):

$$\frac{CE}{BC + CE} = \frac{3}{4} \Rightarrow 4CE = 3BC + 3CE \Rightarrow CE = 3BC$$

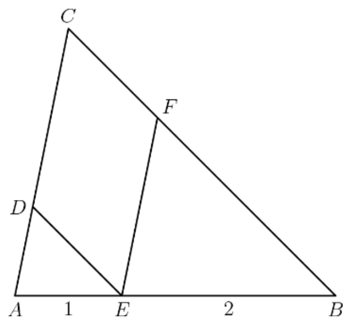
By ratio of parts, the required ratio is also:

$$\frac{PA}{AB} = \frac{3}{1}$$

C. Nested Triangles-II

Example 2.47

In ΔABC, a point E is on $\overline{AB}$ with $AE = 1$ and $EB = 2$. Point D is on $\overline{AC}$ so that $\overline{DE} \parallel \overline{BC}$ and point F is on $\overline{BC}$ so that $\overline{EF} \parallel \overline{AC}$. What is the ratio of the area of $CDEF$ to the area of ΔABC? **(AMC 8 2018/20)**

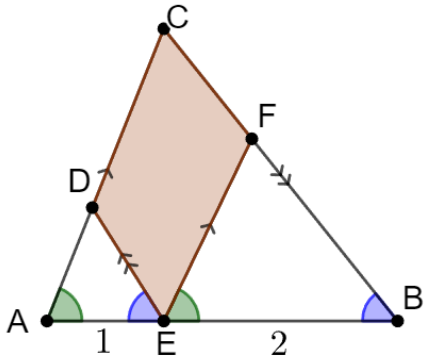


Line $\overline{DE} \parallel \overline{BC}$. Consider *line AB* as a transversal:

$$\angle DEA = \angle CBA\ (Corresponding\ angles)$$

Line $\overline{EF} \parallel \overline{AC}$. Consider *line AB* as a transversal:

$$\angle CEA = FEB\ (Corresponding\ angles)$$



By AA Similarity

$$\Delta AED \sim \Delta EBF \sim \Delta ABC$$

In similar triangles, the ratio of areas is the square of the ratio of sides

$$[AED]:[EBF]:[ABC] = 1^2:2^2:(1+2)^2 = 1:4:9$$

Using complementary areas, we want to find $\frac{[CDEF]}{[ABC]}$, where the numerator is the larger triangle minus the smaller triangles:'

$$\frac{[CDEF]}{[ABC]}=\frac{[ABC]-[ADE]-[EFB]}{[ABC]}$$

Split the fraction to get:

$$=1-\frac{A(\Delta ADE)+A(\Delta \text{FEB})}{(\Delta ABC)}=1-\frac{1+4}{9}=\frac{4}{9}=4{:}9$$

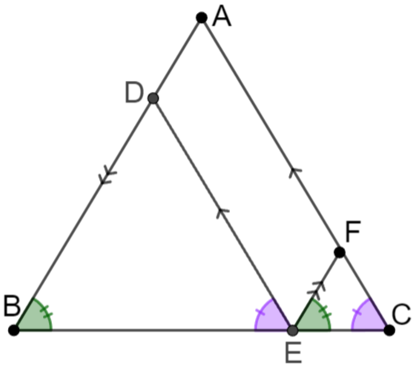
2.48: Variables that "cancel"

Introducing variables that "cancel" later on in the calculations can be a useful technique.

Example 2.49

In ΔABC, $AB=AC=28$ and $BC=20$. Points D,E, and F are on sides AB, BC, and AC, respectively, such that DE and EF are parallel to AC and AB, respectively. What is the perimeter of parallelogram $ADEF$? **(AMC 10A 2013/12, AMC 12A 2013/9)**

Method I: Adding up to 1



$\Delta DBE \sim \Delta ABC$ by AA Similarity:

$$\angle DBE=\angle ABC\ (Same\ Angle)$$
$$\angle DEB=\angle ACE\ (Corresponding\ Angles)$$

Hence:

$$\frac{DE}{AC}=\frac{BE}{BC}\Rightarrow \underbrace{DE=\frac{BE}{BC}\cdot AC}_{Equation\ I}$$

ΔFEC$\sim\Delta ABC$ by AA Similarity:

$$\angle ACE=\angle FCE\ (Same\ Angle)$$
$$\angle ABE=\angle FEC\ (Corresponding\ Angles)$$

Hence:

$$\frac{FE}{AB}=\frac{EC}{BC}\Rightarrow\frac{FE}{AC}=\frac{EC}{BC}\Rightarrow \underbrace{FE=\frac{EC}{BC}\cdot AC}_{Equation\ II}$$

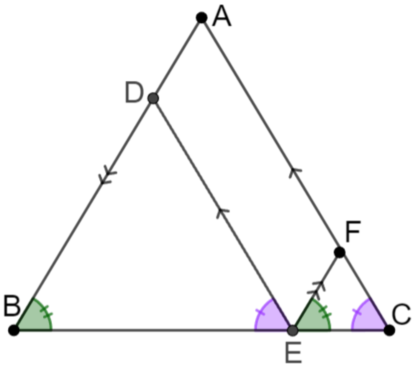
Add Equation I and II:

$$DE+FE=AC\left(\frac{BE}{BC}+\frac{EC}{BC}\right)=AC\left(\frac{BC}{BC}\right)=AC$$

Since the opposite sides of a parallelogram are equal, the perimeter is:

$$2(DE+FE)=2(AC)=2(28)=56$$

Method II: Rearrangement and Isosceles Triangle Properties



As above:

$$\Delta DBE\sim\Delta ABC,\qquad \Delta\text{FEC}\sim\Delta ABC$$

Since ΔABC is isosceles, so are the above two triangles. Hence:

$$DE+FE=DB+AD=AB$$

Since the opposite sides of a parallelogram are equal, the perimeter is:

$$2(DE+FE)=2(AB)=2(28)=56$$

Method III: Limiting Value as $EC\to 0$

Note that the question does not mention the location of E in the base.

Hence, we can assume that the perimeter does not depend on the location of E.

Suppose[6]

$$\frac{BE}{BC} \approx 1 \Rightarrow Perimeter(ADEF) = 2AC = 2(28) = 56$$

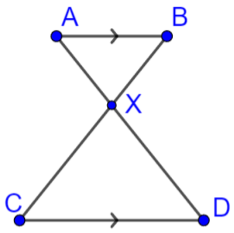
2.4 Bow Tie Similarity

A. Triangles

Example 2.50

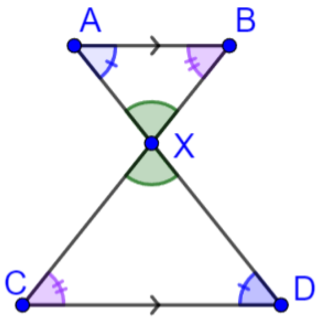
Use the adjoining diagram, which is not drawn to scale, to answer the following questions. Answer each part separately.

A. Explain why $\Delta ABX \sim \Delta DXC$
B. Write the similarity ratios of the two triangles.
C. AX is one-third of XD. BX is one less than XC. Determine XC.



Part A

$$\angle BAD = \angle CDA \; (Alternate\ Interior\ Angles)$$
$$\angle ABC = \angle ADC \; (Alternate\ Interior\ Angles)$$
$$\angle AXB = \angle CXD \; (Vertically\ opposite\ Angles)$$
$$\Delta ABX \sim \Delta DXC \; (AAA\ Similarity)$$



Part B

$$\frac{AB}{CD} = \frac{XA}{XD} = \frac{XB}{XC}$$

Part C

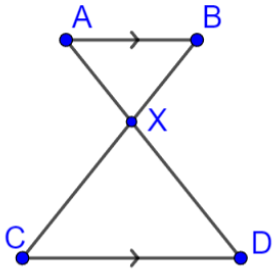
$$XC = BX + 1, \qquad 3BX = XC$$

$$3BX = BX + 1 \Rightarrow BX = \frac{1}{2} \Rightarrow XC = 3(BX) = \frac{3}{2}$$

Example 2.51

Mark all correct options

A. $\Delta ABX \sim \Delta DXC$
B. $\Delta BAX \sim \Delta CXD$
C. $\Delta XAB \sim \Delta XCD$
D. $\Delta AXB \sim \Delta DXC$

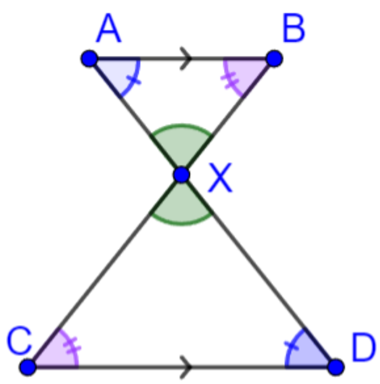


$$Option\ D$$

[6] Technically, we are taking the limit as $\lim_{BE \to BC} \frac{BE}{BC}$

Example 2.52

A. $\angle BAX = 2\angle CXD = 3\angle XCD$, determine the value of $\angle ABC$.
B. Comment on your results.

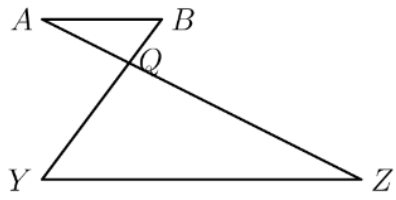


$$\angle BAX = x$$
$$2\angle CXD = x \Rightarrow \angle CXD = \angle AXB = \frac{x}{2}$$
$$3\angle XCD = x \Rightarrow \angle XCD = \angle ABX = \frac{x}{3}$$

By sum of angle of ΔABX

$$x + \frac{x}{2} + \frac{x}{3} = 180$$
$$\frac{6x + 3x + 2x}{6} = 180$$
$$x = \frac{1080}{11} \approx 98.18°$$
$$\angle BAX > 90° \Rightarrow Angle\ is\ obtuse$$

The given information is inconsistent because the triangles are not similar if $\angle BAX$ is obtuse.



Example 2.53

In the figure shown, segment AB is parallel to segment YZ. If $AZ = 42$ units, $BQ = 12$ units, and $QY = 24$ units, what is the length of segment QZ? (MathCounts 2008 Proportional Reasoning Stretch)

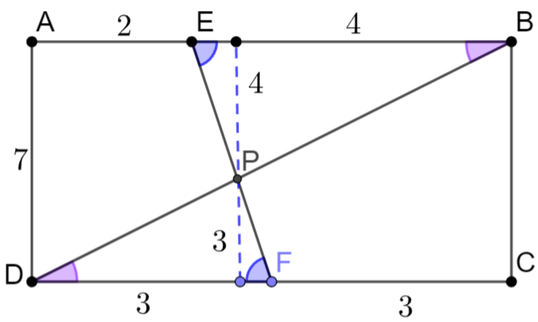
$\Delta AQB \sim \Delta ZQY$ by Bow Tie Symmetry. Hence:

$$\frac{QB}{QY} = \frac{12}{24} = \frac{1}{2}$$
$$\frac{AQ}{QZ} = \frac{1}{2}$$
$$\frac{AQ}{QZ} + 1 = \frac{1}{2} + 1$$
$$\frac{AZ}{QZ} = \frac{3}{2}$$
$$QZ = \frac{2}{3} \cdot AZ = \frac{2}{3} \cdot 42 = 28$$

B. Quadrilaterals

Example 2.54

In rectangle $ABCD$, points E and F lie on segments AB and CD, respectively, such that $AE = \frac{AB}{3}$ and $CF = \frac{CD}{2}$. Segment BD intersects segment EF at P. What fraction of the area of rectangle $ABCD$ lies in triangle EBP? Express your answer as a common fraction. (MathCounts 2010 State Sprint)



Assume without loss of generality that

$$AD = 7, AB = DC = 6$$

$\Delta EPB \sim \Delta FPD$ by Bow Tie Similarity. Hence:

$$\frac{EB}{DF} = \frac{4}{3}$$

The ratio of altitudes is the same as the ratio of heights.

$$\frac{YP}{XP} = \frac{3}{4}$$

$$\frac{YP}{XP} + 1 = \frac{3}{4} + 1$$

$$\frac{XY}{XP} = \frac{7}{4}$$

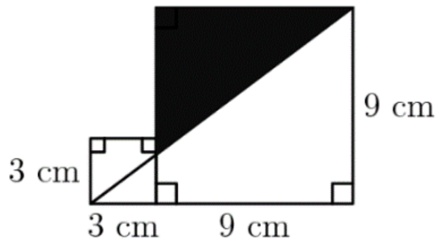
$$\frac{XP}{XY} = \frac{4}{7}$$

The ratio of areas is then:

$$\frac{1}{2} \times \underbrace{\frac{4}{6}}_{\substack{Ratio\\ of\ Bases}} \times \underbrace{\frac{4}{7}}_{\substack{Ratio\\ of\ Heights}} = \frac{4}{21}$$

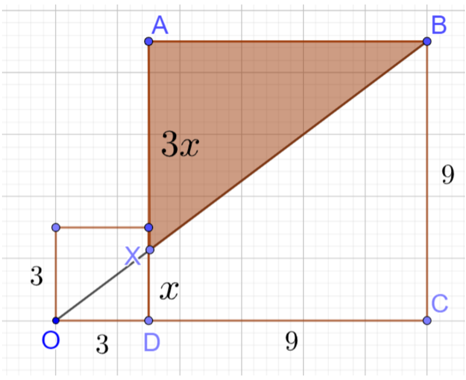
Example 2.55

What is the area of the shaded region in the figure below? Round your answer to the nearest square centimeter. **(MathCounts 1991 National Team)**



Method I

$\Delta DXO \sim \Delta AXB$ by Bow Tie Similarity. Hence, the ratio of sides is:



$$\frac{DO}{AB} = \frac{3}{9} = \frac{1}{3}$$

Hence:

$$\frac{DX}{AX} = \frac{1}{3}$$

Add 1 to both sides:

$$\frac{AD}{AX} = \frac{4}{3}$$

Solving for AX:

$$AX = \frac{3}{4}AD = \frac{3}{4} \cdot 9 = \frac{27}{4}$$

The area of ΔBAX

$$= \frac{1}{2}(9)\left(\frac{27}{4}\right) = \frac{243}{8} = 30\frac{3}{8} \approx \underbrace{30\ cm^2}_{Rounded}$$

Method II

Introduce a coordinate system with the origin$(0,0)$ at O.

$$Slope_{OB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-0}{12-0} = \frac{3}{4}$$

$$y - intercept = 0$$

Equation of the line:

$$y = \frac{3}{4}x$$

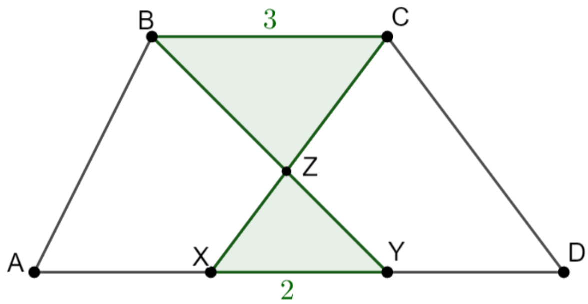
Point X has coordinates $(3, a)$:

$$a = \frac{3}{4}x = \frac{3}{4}(3) = \frac{9}{4}$$
$$AX = AD - DX = 9 - \frac{9}{4} = \frac{27}{4}$$

Example 2.56

In trapezoid $ABCD$, base AD is twice the length of base BC. X and Y trisect AD. BY and CX intersect at Z. The ΔXZY and ΔBZC each have integer areas less than one hundred units.

A. Determine the sum of all possible areas of ΔXZY.
B. Determine the sum of all possible areas of ΔBZC.



Let$BC = 3 \Rightarrow XY = 2$. By bow-tie similarity:

$$\Delta XZY \sim \Delta BZC \Rightarrow \frac{XY}{BC} = 2:3$$

The ratio of the areas is the square of the ratio of the side lengths:

$$\frac{[XZY]}{[BZC]} = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

Part B

The possible values for $[BZC]$ are $\{9,18,27, \dots, 99\}$.

$$Sum = 9(1 + 2 + \cdots + 11) = 9\left(\frac{11 \times 12}{2}\right) = 594$$

Part A

The possible values for $[XZY]$ are $\{4,8,12, \dots, 44\}$.

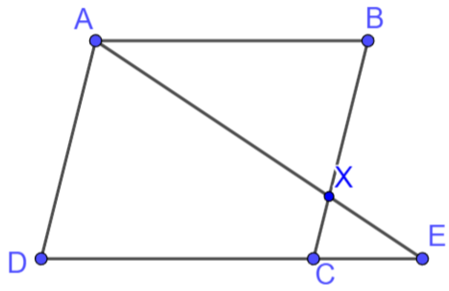
$$Sum = 4(1 + 2 + \cdots + 11) = 4\left(\frac{11 \times 12}{2}\right) = 264$$

Note: $[XYZ]$ cannot exceed 44 since that would mean $[BZC]$ would exceed 100. Hence, it makes sense to solve part B first.

Example 2.57

In parallelogram $ABCD$, $AB \parallel CD$. The line from vertex A intersects segment DC produced to E. XA is greater than XE by 2 units. XC is less than XB by three units.

A. Determine the ratio of XE and XC.
B. If BA is greater than CE by 4 units, determine the ratios of the side lengths of ΔCXE. Comment on your result.



Part A

Let $XE = x \Rightarrow XA = x + 2, XC = y \Rightarrow XB = y + 3$:

$$\frac{x}{x+2} = \frac{y}{y+3} \Rightarrow xy + 3x = xy + 2y \Rightarrow \frac{XE}{XC} = \frac{2}{3}$$

Part B

Let $CE = z \Rightarrow AB = z + 4$

$$\frac{x}{x+2} = \frac{y}{y+3} = \frac{z}{z+4}$$

$$\frac{x}{x+2} = \frac{z}{z+4} \Rightarrow xz + 4x = xz + 2z \Rightarrow x = \frac{z}{2}$$

Then:

$$x = \frac{2}{3}y = \frac{z}{2}$$
$$6x = 4y = 3z = k$$
$$x = \frac{k}{6}, y = \frac{k}{4}, z = \frac{k}{3}$$

Let $k = LCM(6,4,3) = 12$

$$x = \frac{12}{6} = 2, y = \frac{12}{4} = 3, z = \frac{12}{3} = 4$$

$$2 + 3 = 5 > 4$$
$$3 + 4 = 7 > 2$$
$$2 + 4 = 6 > 3$$
$$Results\ are\ consistent$$

C. Harmonic Mean

2.58: Harmonic Mean

The harmonic mean of two numbers is twice the reciprocal of the sum of the reciprocals of the numbers.

The harmonic mean of x and y is:

$$HM(x,y) = \frac{2xy}{x+y}$$

The reciprocals of x and y are

$$\frac{1}{x}\ and\ \frac{1}{y}$$

And they have sum

$$\frac{1}{x} + \frac{1}{y}$$

Twice the reciprocal of the above sum is:

$$\frac{2}{\frac{1}{x} + \frac{1}{y}} = \frac{2}{\frac{x+y}{xy}} = \frac{2xy}{x+y}$$

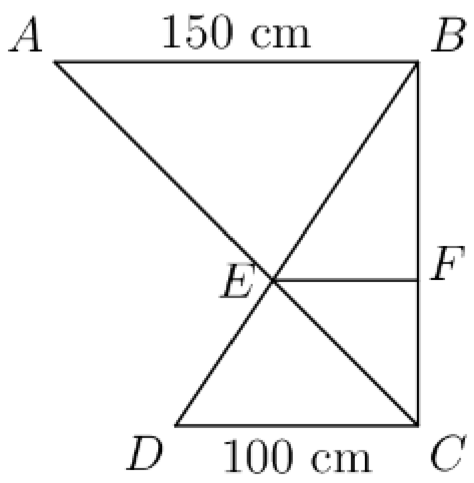
2.59: Harmonic Mean in Bow Tie Similarity

Given that $AB \parallel EF \parallel DC$,

$$EF = \frac{1}{2} HM(AB, CD)$$

Where

$$HM = Harmonic\ Mean$$



Step I: $\Delta CFE \sim \Delta CBA$ by AA Similarity since

Consider AE and BC as transversals of parallel lines AB and EF:

$$\angle BAC = \angle FEC\ (Corresponding\ Angles)$$
$$\angle ABC = \angle EFC\ (Corresponding\ Angles)$$

Hence:

$$\underbrace{\frac{CF}{CB} = \frac{FE}{BA}}_{Equation\ I}$$

Step II: $\Delta BEF \sim \Delta BDC$ by AA Similarity since

Consider DE and CF as transversals of parallel lines CD and EF:

$$\angle BED = \angle BDC\ (Corresponding\ Angles)$$
$$\angle DCB = \angle EFB\ (Corresponding\ Angles)$$

Hence:

$$\underbrace{\frac{BF}{CB} = \frac{FE}{CD}}_{Equation\ II}$$

Add Equations I and II:

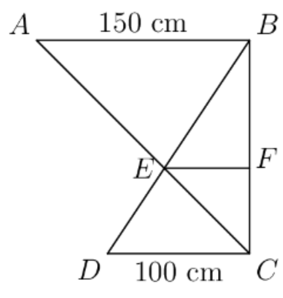
$$LHS = \frac{CF}{CB} + \frac{BF}{CB} = \frac{CB}{CB} = 1$$
$$RHS = \frac{FE}{BA} + \frac{FE}{CD} = FE\left(\frac{1}{BA} + \frac{1}{CD}\right)$$

Hence:

$$FE\left(\frac{1}{BA} + \frac{1}{CD}\right) = 1$$
$$FE = \frac{1}{\frac{1}{BA} + \frac{1}{CD}} = \frac{1}{\frac{CD + BA}{BA \cdot CD}} = \frac{BA \cdot CD}{CD + BA} = \frac{1}{2} HM(AB, CD)$$

Example 2.60

What is the number of centimeters in the length of EF if $AB \parallel CD \parallel EF$? **(MathCounts 1999 National Target)**



$$EF = \frac{1}{2} HM(AB, CD) = \frac{BA \cdot CD}{CD + BA} = \frac{150 \cdot 100}{150 + 100} = \frac{150 \cdot 100}{250} = \frac{150 \cdot 2}{5} = 60\ cm$$

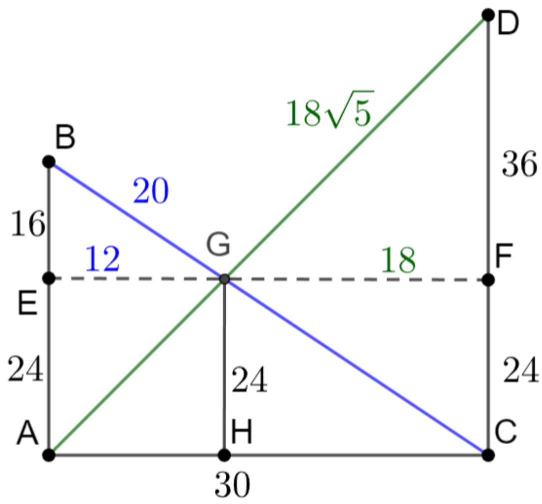
Example 2.61

If two poles $20''$ and $80''$ high are $100''$ apart, then the height of the intersection of the lines joining the top of each pole to the foot of the opposite pole is: **(AHSME 1951/30)**

$$EF = \frac{1}{2}HM = \frac{20 \cdot 80}{20 + 80} = \frac{1600}{100} = 16''$$

Example 2.62

Two vertical poles (height 40 cm and 60 cm) are 30 cm apart on level ground. A blue rope goes from the top of the smaller pole to the bottom of the longer pole. A green rope goes from the top of the longer pole to the bottom of the shorter pole. An insect at the top of the smaller pole crawls down the blue rope, changes rope at the intersection of the blue and green ropes and crawls up to the top of the longer pole. The time on the blue rope is 1 minute, and the time on the green rope is 3 minutes. Due to gravity, the speed of the insect increases (from its default) by $G\,\frac{cm}{min}$ when going down a rope, and decreases (from its default) by $G\,\frac{cm}{min}$ when going up a rope. Given that $\sqrt{5} \approx 2.236$, determine the approximate value of G.



The height of the intersection of the two ropes is half the harmonic mean of the two heights:

$$GH = \frac{1}{2}HM = \frac{40 \cdot 60}{40 + 60} = \frac{2400}{100} = 24\ cm$$

$\Delta BGA \sim \Delta DGC$ by bow tie similarity.
The ratio of sides is:

$$\frac{40}{60} = \frac{4}{6} = \frac{2}{3}$$

The horizontal displacement is the altitudes, which are in the same ratio as the sides:

$$2:3 = 12:18$$

The vertical displacements of the insect are:

$$Blue\ Rope: 40 - 24 = 16$$
$$Green\ Rope: 60 - 24 = 36$$

The distance travelled by the insect is:

$$Blue: (12,16,x) = 4(3,4,5) = (12,16,20)$$
$$Red: \sqrt{18^2 + 36^2} = \sqrt{18^2(1^2 + 2^2)} = 18\sqrt{1+4} = 18\sqrt{5}$$

Let the default speed of the insect be S.
The time taken by the insect is:

$$T_{Blue} = \frac{D}{S+G} = \frac{20}{S+G} = 1 \Rightarrow \underbrace{S + G = 20}_{Equation\ I}$$

$$T_{Green} = \frac{D}{S-G} = \frac{18\sqrt{5}}{S-G} = 3 \Rightarrow \underbrace{S - G = 6\sqrt{5}}_{Equation\ II}$$

Add Equations I and II:

$$2S = 20 + 6\sqrt{5}$$

Substitute $S = 10 + 3\sqrt{5}$ in Equation I

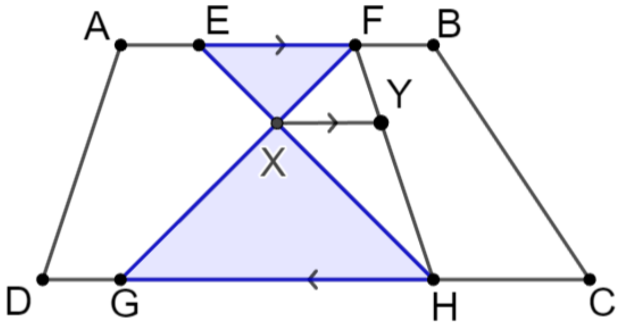
$$10 + 3\sqrt{5} + G = 20$$

Solve for G:

$$G = 10 - 3\sqrt{5} \approx 10 - 3(2.236) = 10 - 6.708 = 3.292$$

Example 2.63

In trapezoid $ABCD$, the parallel sides are AB and CD. Points E and F lie on AB, in the order A, E, F, B from left to right. Points G and H lie on CD, in the order D, G, H, C from left to right. EH and GF intersect at X. XY is parallel to AB and Y lies on FH. Given that $XY = EF \cdot GH$, determine the value of $EF + GH$.



$\Delta EFX \sim \Delta HXG$ by Bow Tie Similarity since

$$EF \parallel GH$$

Using the harmonic mean in bow tie similarity property:

$$XY = \frac{EF \cdot GH}{EF + GH}$$

Substitute $XY = EF \cdot GH$:

$$EF \cdot GH = \frac{EF \cdot GH}{EF + GH}$$

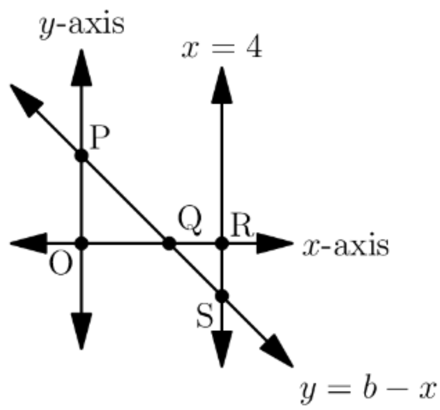
Simplify and solve for $EF + GH$:

$$1 = \frac{1}{EF + GH}$$
$$EF + GH = 1$$

D. Coordinate Geometry

Example 2.64

The line $y = b - x$ with $0 < b < 4$ intersects the y-axis at P, the x-axis at Q, and the line $x = 4$ at S. The line $x = 4$ intersects the x-axis at R. O is the origin. If the ratio of the area of triangle QRS to the area of triangle QOP is $9:25$, what is the value of b? Express the answer as a decimal to the nearest tenth. **(MathCounts 2005 National Target, Adapted)**



When $x = 4$:

$$y = b - 4 < 0 \text{ since } 0 < b < 4$$

Hence,

$$S \text{ is below the } x \text{ axis}$$
$$\therefore Q \text{ is between } O \text{ and } R$$

$\Delta PQO \sim \Delta RQS$ by Bow Tie Similarity since $PO \parallel RS$. Hence:

$$\frac{QR}{QO} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\frac{OR}{QO} = \frac{8}{5}$$

Hence, the x coordinate of Q

$$= QO = OR \cdot \frac{5}{8} = 4 \cdot \frac{5}{8} = \frac{5}{2}$$

If you recognize that ΔPOQ is an isosceles right-angled triangle, then you can directly write

$$b = 2.5$$

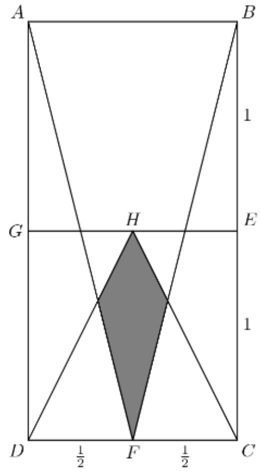
Else substitute $(x, y) = (2.5, 0)$ in $y = b - x$

$$0 = b - 2.5 \Rightarrow b = 2.5$$

E. Challenging Questions

Example 2.65

In rectangle $ABCD$, $AB = 1$, $BC = 2$, and points E, F, and G are midpoints of $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$, respectively. Point H is the midpoint of $\overline{GE}$. What is the area of the shaded region? (AMC 10A 2014/16)

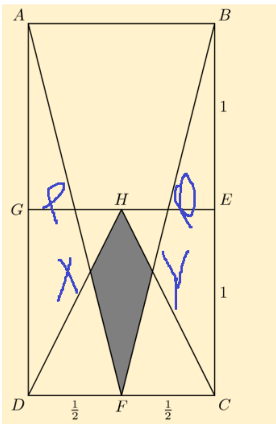


By alternate interior angles:

$$HYQ \sim CYF$$

$$HQ = HE - QE = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\frac{FC}{HQ} = \frac{Altitude\ of\ FYC}{Altitude\ of\ HYQ} = 2$$



Distributing in the ratio $2:1$

$$Altitude\ of\ FYC\ is\ 2/3, and\ Altitude\ of\ HYQ\ is\ 1/3$$

$$Area\ of\ FYC = \frac{1}{2}hb = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

$$Area\ of\ DHC = \frac{1}{2} = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

By symmetry

$$FYC \cong DXF$$

Area of FXHY

$$= [DHC] - 2[FYC] = \frac{1}{2} - 2 \cdot \frac{1}{6} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

1 Pending

Example 2.66

Point F is taken on the extension of side AD of parallelogram ABCD. BF intersects diagonal AC at E and side DC at G. If EF=32 and GF=24, then BE equals: (AHSME 1963/38)

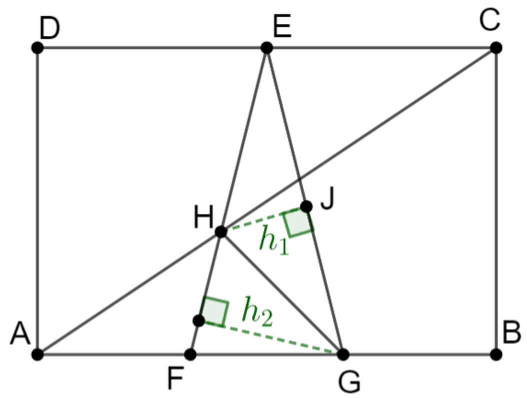
Example 2.67

In rectangle $ABCD$, points F and G lie on AB so that $AF = FG = GB$ and E is the midpoint of DC. Also, AC intersects EF and EG at H and J. The area of the rectangle $ABCD$ is 70. Find the area of triangle EHJ. (AMC 12

2001/22)

Assume that

$$AB = CD = 6 \Rightarrow AF = 2, EC = 3$$



Step I: Bow Tie Similarity in $\Delta AFH \sim \Delta CHE$

Ratio of sides

$$= \frac{AF}{EC} = \frac{HF}{EF} = \frac{2}{3}$$

Add 1 to both sides of the last equality:

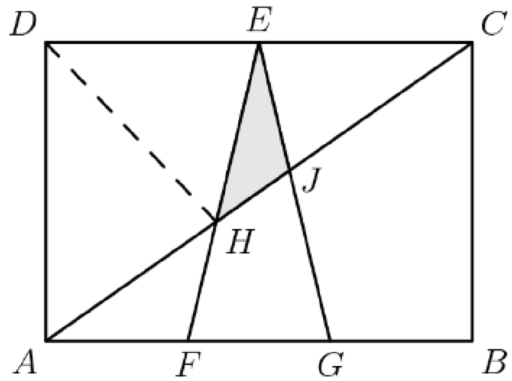
$$\frac{HF}{EH} + 1 = \frac{2}{3} + 1$$

Substitute $HF + EH = EF$:

$$\frac{EF}{EH} = \frac{5}{3}$$

The ratio of areas of triangles

$$\underbrace{\frac{[EHJ]}{[EFG]} = \frac{\frac{1}{2} \cdot h_2 \cdot EH}{\frac{1}{2} \cdot h_2 \cdot EF} = \frac{EH}{EF} = \frac{3}{5}}_{\textit{Result 1}}$$



Step II: Bow Tie Similarity in $\Delta AJG \sim \Delta CJE$

Ratio of sides

$$= \frac{AG}{EC} = \frac{GJ}{EJ} = \frac{4}{3}$$

Add 1 to both sides of the last equality:

$$\frac{GJ}{EJ} + 1 = \frac{4}{3} + 1$$

Substitute $GJ + EJ = EG$:

$$\frac{EG}{EJ} = \frac{7}{3}$$

The ratio of areas of triangles:

$$\underbrace{\frac{[EHJ]}{[EHG]} = \frac{\frac{1}{2} \cdot h_1 \cdot EJ}{\frac{1}{2} \cdot h_1 \cdot EG} = \frac{EJ}{EG} = \frac{3}{7}}_{\textit{Result 2}}$$

Step III: Ratio of ΔEFG to $ABCD$

$$\underbrace{\frac{[EFG]}{[ABCD]} = \frac{\frac{1}{2} \cdot AD \cdot FG}{AD \cdot AB} = \frac{\frac{1}{2} \cdot FG}{AB} = \frac{\frac{1}{2} \cdot 1}{3} = \frac{1}{6}}_{\textit{Result 3}}$$

Step IV: Final Answer

We know that

$$\underbrace{[ABCD] = 70}_{\textit{Result 4}}$$

Multiply Results 1, 2, 3 and 4:

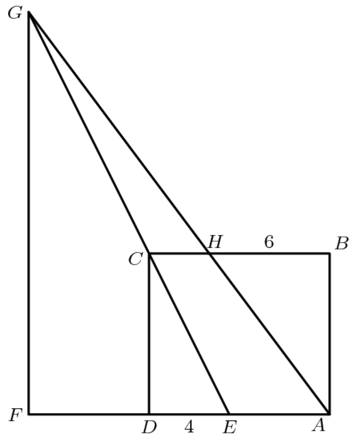
$$[ABCD] \cdot \frac{[EFG]}{[ABCD]} \cdot \frac{[EHG]}{[EFG]} \cdot \frac{[EHJ]}{[EHG]} = 70 \cdot \frac{1}{6} \cdot \frac{3}{7} \cdot \frac{3}{5}$$

$$[EHJ] = 3$$

2.5 Right Triangles

A. Quadrilaterals

We have considered simple questions on right triangles in the section on scale factor. We will look at more complicated questions in this section.



Example 2.68

In rectangle $ABCD$, we have $AB = 8$, $BC = 9$, H is on BC with $BH = 6$, E is on AD with $DE = 4$, line EC intersects line AH at G, and F is on line AD with $GF \perp AF$. Find the length of GF. **(AMC 10A 2003/22)**

$\Delta GCH \sim \Delta GEA$ by Parallel Line Similarity. Hence:

$$\frac{GC}{GE} = \frac{CH}{EA} = \frac{BC - BH}{DA - DE} = \frac{9-6}{9-4} = \frac{3}{5}$$

Use this to calculate:

$$\frac{CE}{GE} = \frac{GE - GC}{GE} = 1 - \frac{GC}{EA} = 1 - \frac{3}{5} = \frac{2}{5}$$

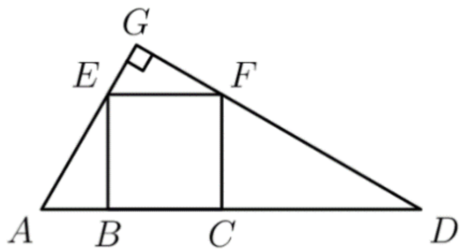
$\Delta CDE \sim \Delta GFE$ by Right Triangle similarity since

$$\angle CDE = \angle GFE \ (Same\ Angle)$$

Hence:

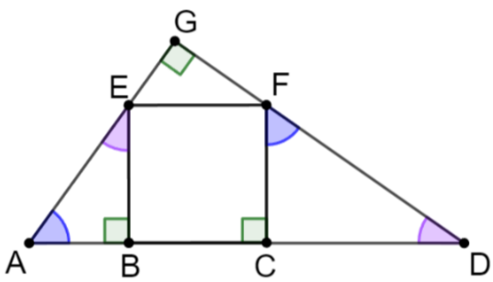
$$\frac{GF}{CD} = \frac{GE}{CE} = \frac{5}{2} \Rightarrow GF = \frac{GE}{CE} \cdot CD = \frac{5}{2} \cdot 8 = 20$$

B. Inscribed Squares



Example 2.69

Square $BCFE$ is inscribed in right triangle AGD, as shown below. If $AB = 28$ units and $CD = 58$ units, what is the area of square $BCFE$? **(MathCounts 2005 State Sprint)**



Let $\angle GAD = \theta$. Then:

$$\angle GDA = 90 - \theta$$
$$\angle CFD = 90 - (90 - \theta) = \theta$$
$$\angle AEB = 90 - \theta$$

$\Delta AEB \sim FDC$ by AA Similarity since:

$$\angle GDA = \angle AEB = 90 - \theta$$
$$\angle EAB = \angle FCD = 90°$$

Hence:

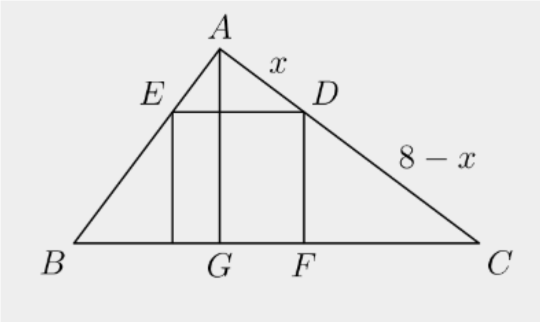
$$\frac{EB}{AB} = \frac{CD}{FC}$$

Substitute $AB = 28, CD = 58, FC = EB$:

$$\frac{EB}{28} = \frac{58}{EB} \Rightarrow EB^2 = [BCFE] = 28 \times 58 = 1624$$

Example 2.70

Right triangle ABC has one leg of length 6 cm, one leg of length 8 cm and a right angle at A. A square has one side on the hypotenuse of triangle ABC and a vertex on each of the two legs of triangle ABC. What is the length of one side of the square, in cm? Express your answer as a common fraction. (MathCounts 2007 State Sprint)



Step I: $\Delta AED \sim \Delta ABC$ by Parallel Line Similarity since

$$ED \parallel BC$$

Hence:

$$\frac{ED}{BC} = \frac{AD}{AC}$$

Substitute $BC = 10$, and rearrange:

$$ED = \frac{AD}{AC} \cdot 10$$

Step II: Calculate the area of ΔABC in two different ways:

$$\frac{1}{2} \cdot AG \cdot BC = \frac{1}{2} \cdot AB \cdot AC$$

$$10AG = \frac{1}{2} \cdot 6 \cdot 8$$

$$AG = \frac{24}{5}$$

Step III: $\Delta AGC \sim \Delta DFC$ by Right Triangle Similarity since

$$\angle ACG = \angle DFC \ (Same\ Angle)$$

Hence:

$$\frac{DF}{AG} = \frac{DC}{AC}$$

Substitute $AG = \frac{24}{5}$, and rearrange:

$$\underbrace{DF = \frac{DC}{AC} \cdot \frac{24}{5}}_{Equation\ I}$$

Step IV: Final Answer

Since $Side\ of\ square = ED = DF = \frac{AD}{AC} \cdot 10$:

$$\frac{AD}{AC} \cdot 10 = \frac{DC}{AC} \cdot \frac{24}{5}$$

$$\frac{AD}{DC} = \frac{12}{25}$$

Add 1 to both sides:

$$\frac{AC}{DC} = \frac{37}{25}$$

Substitute the above into Equation I:

$$DF = \frac{DC}{AC} \cdot \frac{24}{5} = \frac{25}{37} \cdot \frac{24}{5} = \frac{120}{37}$$

Example 2.71

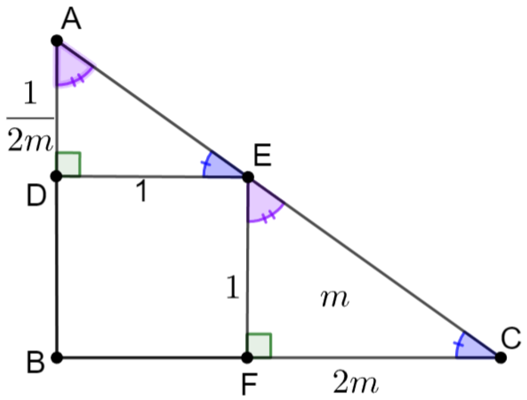
Right ΔABC has $AB = 3$, $BC = 4$, and $AC = 5$. Square $XYZW$ is inscribed in ΔABC with X and Y on AC, W on AB, and Z on BC. What is the side length of the square? **(AMC 10B 2007/21)**

This is the same as the previous example, except that all the lengths are halved. Hence, the final answer is:

$$\frac{120}{37} \times \frac{1}{2} = \frac{60}{37}$$

Example 2.72

Through a point on the hypotenuse of a right triangle, lines are drawn parallel to the legs of the triangle so that the triangle is divided into a square and two smaller right triangles. The area of one of the two small right triangles is m times the area of the square. The ratio of the area of the other small right triangle to the area of the square is ______ **(AMC 10 2000/19)**



Since this we need ratios, without loss of generality, let

$$Side\ of\ square = DE = EF = 1$$

Again, without loss of generality, let the triangle which has area m times the square be

$$\Delta EFC$$

Then it has area

$$= [EFC] = m \times 1 = m$$

We can also calculate the area of ΔEFC as

$$\frac{1}{2}bh = A$$

Substitute $A = m, h = DE = 1, b = FC$:

$$\frac{1}{2} \times FC \times 1 = m \Rightarrow FC = 2m$$

$\Delta ADE \sim \Delta EFC$ by Right Triangle Similarity since

$$\angle AED = \angle ECF\ (Corresponding\ Angles)$$

Hence:

$$\frac{AD}{DE} = \frac{FE}{FC} \Rightarrow \frac{AD}{1} = \frac{1}{2m}$$

The ratio that we want is:

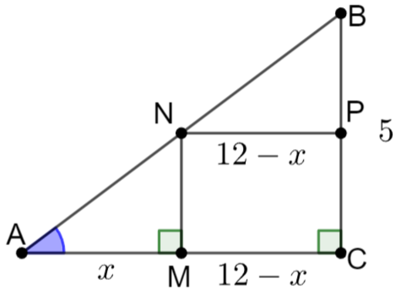
$$\frac{[ADE]}{[DEFG]} = \frac{\frac{1}{2}(DE)(AD)}{1^2} = \frac{1}{2} \times 1 \times \frac{1}{2m} = \frac{1}{4m}$$

C. Inscribed Rectangles

Example 2.73

In right triangle ABC right angled at C, $BC = 5$, $AC = 12$, and $AM = x$; $MN \perp AC$, $NP \perp BC$; N is on AB; M is on AC, P is on BC. If $y = MN + NP$, one-half the perimeter of rectangle $MCPN$, then find the value of y in terms of x. Write your answer as a single fraction. **(AHSME 1957/37)**

We need to calculate MN and NP in terms of x.



Step I: Find NP
Since A, M and C are collinear:

$$NP = MC = AC - AM = 12 - x$$

Step II: Find MN
$\Delta AMN \sim \Delta ACB$ by Right Triangle Similarity:

$$\angle NAM = \angle BAC \text{ (Same Angle)}$$

Hence:

$$\frac{MN}{AM} = \frac{BC}{AC}$$

Rearrange:

$$MN = \frac{BC}{AC} \cdot AM = \frac{5x}{12}$$

Step II: Find $NP + MN$

$$NP + MN = 12 - x + \frac{5x}{12} = \frac{144 - 12x + 5x}{12} = \frac{144 - 7x}{12}$$

Example 2.74

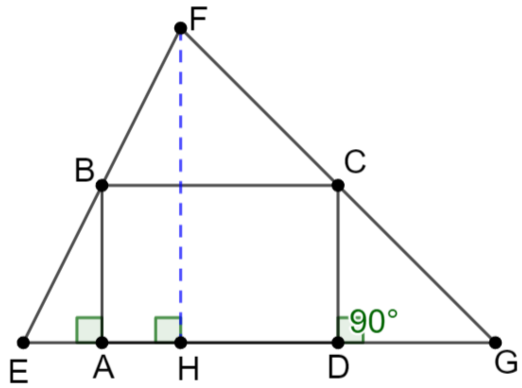
Rectangle $ABCD$ is inscribed in triangle EFG such that side AD of the rectangle is on side EG of the triangle. B lies on EF. C lies on FG. The triangle's altitude from F to side EG is 7 inches, and $EG = 10$. The length of segment AB is equal to half the length of segment AD. What is the area of rectangle $ABCD$? Express your answer as a common fraction. **(MathCounts 2007 National Team, Adapted)**

Step I: Establish Similarity
$\Delta EAB \sim \Delta EHF$ by Right Triangle Similarity since:

$$\angle BEA = \angle FEH \text{ (Same Angle)}$$

Hence:

$$\underbrace{\frac{HE}{HF} = \frac{AE}{AB}}_{\text{Equation I}}$$



$\Delta GDC \sim \Delta GHF$ by Right Triangle Similarity since:

$$\angle CGD = \angle FGH \text{ (Same Angle)}$$

Hence:

$$\underbrace{\frac{HG}{HF} = \frac{DG}{DC}}_{\text{Equation II}}$$

Step II: Solve the system of Equations
Add Equations I and II, and substitute $DC = AB$ in Equation II:

$$\underbrace{\frac{HE + HG}{HF} = \frac{AE + DG}{AB}}_{\text{Equation III}}$$

Note that:

$$HE + HG = EG = 10$$

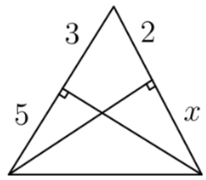
$$HF = 7$$
$$AE + DG = EG - AD = 10 - 2AB$$

Make the above substitutions in Equation III:

$$\frac{10}{7} = \frac{10 - 2AB}{AB}$$
$$10AB = 70 - 14AB$$
$$24AB = 70$$
$$AB = \frac{35}{12}$$

$$[ABCD] = AB \cdot CD = AB \cdot 2AB = \frac{35}{12} \cdot 2 \cdot \frac{35}{12} = \frac{1225}{72}$$

D. Altitudes to Sides

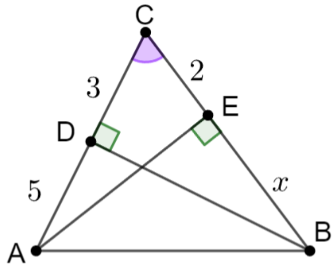


Example 2.75

Two of the altitudes of an acute triangle divide the sides into segments of lengths 5,3,2 and x units, as shown. What is the value of x? **(MathCounts 2010 National Countdown)**

$\Delta ACE \sim \Delta BCD$ by Right Triangle Similarity since[7]:

$$\angle ACE = \angle BCD\ (Same\ Angle)$$



The ratio of sides is:

$$\frac{CB}{CD} = \frac{AC}{CE}$$

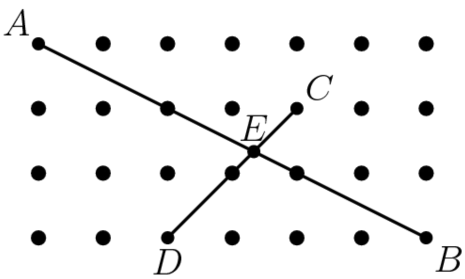
Substituting the values:

$$\frac{x+2}{3} = \frac{8}{2} \Rightarrow x + 2 = 12 \Rightarrow x = 10$$

E. Coordinate Geometry

Example 2.76

The diagram shows 28 lattice points, each one unit from its nearest neighbors. Segment AB meets segment CD at E. Find the length of segment AE[8]. **(AMC 10 2000/16)**



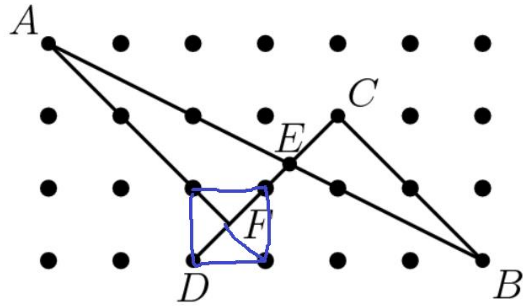
$$BC \perp CD$$

Let the perpendicular from A to CD intersect CD at F. Note that F is the midpoint of the diagonal of the blue square.

[7] This is solved in the Note on Triangles using the Pythagorean Theorem.
[8] An alternate solution using multiple lines is available in the Note on Coordinate Geometry.

$\Delta AFE \sim \Delta BCE$ by Right Triangle Similarity:

$$\angle BCE = \angle AFE = 90°$$
$$\angle AEF = \angle CEB \ (Vertically\ Opposite\ Angles)$$



Hence:

$$\frac{AF}{AE} = \frac{BC}{BE}$$

$$AF = 2.5\sqrt{2}\ (45 - 45 - 90\ Triangle)$$
$$BC = 2\sqrt{2}$$
$$AB = \sqrt{3^2 + 6^2} = \sqrt{9 + 36} = \sqrt{45} = 3\sqrt{5}$$
$$BE = 3\sqrt{5} - AE$$

Substitute the values from above in $\frac{AF}{AE} = \frac{BC}{BE}$:

$$\frac{2.5\sqrt{2}}{AE} = \frac{2\sqrt{2}}{3\sqrt{5} - AE}$$

Cross-multiply:

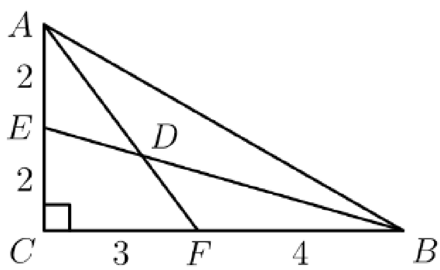
$$7.5\sqrt{10} - 2.5\sqrt{2}AE = 2\sqrt{2}AE$$
$$7.5\sqrt{10}AE = 4.5\sqrt{2}AE$$
$$\frac{15}{2}\sqrt{5}AE = \frac{9}{2}AE$$
$$15\sqrt{5}AE = 9AE$$
$$15\sqrt{5}AE = 9AE$$

$$AE = \frac{5\sqrt{5}}{3}$$

F. Challenging Questions

Example 2.77

In the figure, what is the area of triangle ABD? Express your answer as a common fraction. (MathCounts 2008 National Team)



We will split the area that we do not want by drawing DC, and determining

$$[ADC] + [CDB]$$

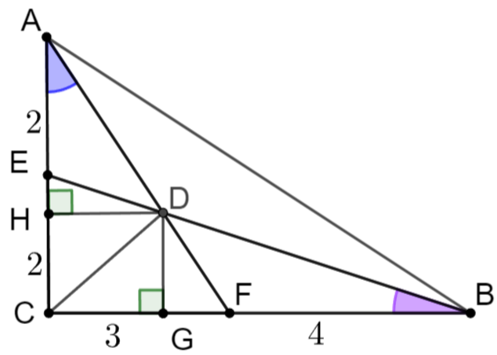
Step I: Draw

We already know the bases. Draw the altitudes:

$$DH \perp CB, \qquad DG \perp AC$$

Let

$$DH = x \Rightarrow GB = 7 - x$$
$$DG = y \Rightarrow AH = 4 - y$$



Step II: Similarity

ΔADH~ΔAFC by Right Triangle Similarity since

$$\angle HAD = \angle CAF \ (Same\ Angle)$$

Hence:

$$\frac{DH}{FC} = \frac{AH}{AC} \Rightarrow \frac{x}{3} = \frac{4-y}{4} \Rightarrow \underbrace{3y + 4x = 12}_{Equation\ I}$$

$\Delta DBG \sim \Delta ECB$ by Right Triangle Similarity since

$$\angle DBG = \angle EBC\ (Same\ Angle)$$

Hence:

$$\frac{DG}{EC} = \frac{GB}{CB} \Rightarrow \frac{y}{2} = \frac{7-x}{7} \Rightarrow 7y = 14 - 2x \Rightarrow \underbrace{14y + 4x = 28}_{Equation\ II}$$

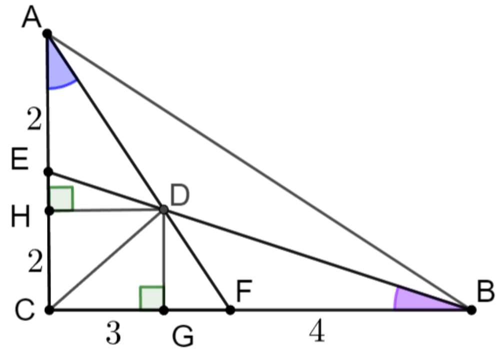
Step III: Solve the System of Equations

Subtract Equation I from Equation II:

$$11y = 16 \Rightarrow y = DG = \frac{16}{11}$$

Substitute $y = \frac{16}{11}$ in Equation I:

$$x = DH = \frac{12-3y}{4} = \frac{12 - 3\left(\frac{16}{11}\right)}{4} = \frac{132-48}{(11)4} = \frac{84}{44} = \frac{21}{11}$$



Step IV: Calculate using complementary areas

$$\underbrace{[ADB] = [ABC] - [ADC] - [CDB]}_{Equation\ III}$$

Calculate the areas of the three triangles on the RHS;

$$[ABC] = \frac{1}{2}hb = \frac{1}{2}(AC)(CB) = \frac{1}{2} \cdot 4 \cdot 7 = 14$$

$$[ADC] = \frac{1}{2}hb = \frac{1}{2} \cdot DH \cdot AC = \frac{1}{2} \cdot \frac{21}{11} \cdot 4 = \frac{42}{11}$$

$$[CDB] = \frac{1}{2}hb = \frac{1}{2} \cdot DG \cdot CB = \frac{1}{2} \cdot \frac{16}{11} \cdot 7 = \frac{56}{11}$$

Substitute the three areas into Equation III:

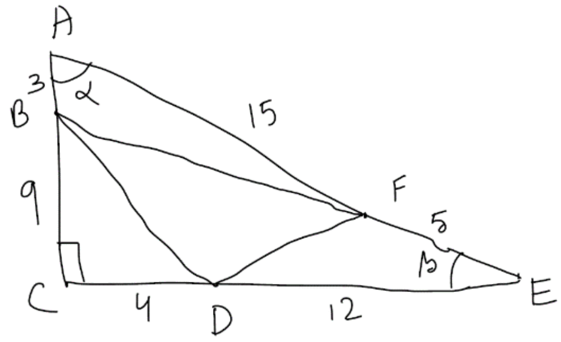
$$[ADB] = 14 - \left(\frac{42}{11} + \frac{56}{11}\right) = 14 - \frac{98}{11} = 14 - 9 + \frac{1}{11} = 5\frac{1}{11} = \frac{56}{11}$$

Example 2.78

In the right triangle ΔACE, we have $AC = 12$, $CE = 16$, and $EA = 20$. Points $B, D,$ and F are located on AC, CE, and EA, respectively, so that $AB = 3$, $CD = 4$, and $EF = 5$. What is the ratio of the area of ΔDBF to that of ΔACE?
(AMC 10B 2004/18)

Solution I[9]

Draw a perpendicular from F intersecting CE at X.



In ΔFXE and ΔACE:

$$\angle FEX = \angle ACE \ (Same\ Angle)$$
$$\angle ACE = \angle FXE = 90°$$

Hence, by AA Similarity:

$$\Delta FXE \sim \Delta ACE \Rightarrow \frac{EF}{EA} = \frac{5}{20} = \frac{1}{4}$$
$$FX = \frac{1}{4} \times AC = \frac{1}{4} \times 12 = 3$$
$$EX = \frac{1}{4} \times EC = \frac{1}{4} \times 16 = 4$$

Area of Trapezoid:

$$[BFXC] = h \times \frac{b_1 + b_2}{2} = CX \times \frac{FX + BC}{2} = (16 - 4) \times \frac{3 + 9}{2} = 72$$

Area of Triangles:

$$[BCD] = \frac{1}{2} \times 9 \times 4 = 18$$
$$[FXD] = \frac{1}{2} \times FX \times XD = \frac{1}{2} \times 3 \times (16 - 4 - 4) = 12$$

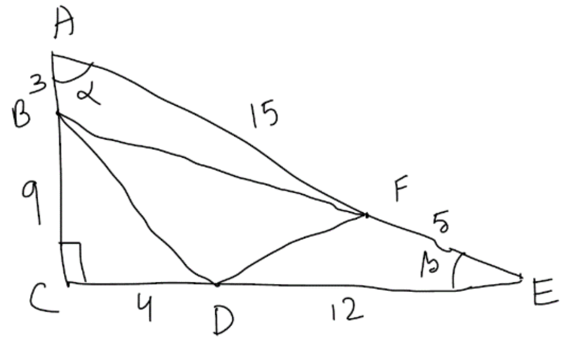
By Complementary Areas:

$$[BDF] = [BFXC] - [BCD] - [FXD] = 72 - 18 - 12 = 42$$

$$\frac{[DBF]}{[ACE]} = \frac{42}{\frac{1}{2} \times 12 \times 16} = \frac{42}{96} = \frac{7}{16}$$

Solution II

Drop a perpendicular from F intersecting AC at X.



In ΔAXF and ΔACE:

$$\angle AXF = \angle ACE = 90°$$
$$\angle CAE = \angle XAF \ (Same\ Angle)$$

By AA Similarity:

$$\Delta AXF \sim \Delta ACE \Rightarrow \frac{AF}{AE} = \frac{15}{20} = \frac{3}{4}$$
$$\frac{AX}{AC} = \frac{3}{4} \Rightarrow AX = \frac{3}{4} \times AC = \frac{3}{4} \times 12 = 9 \Rightarrow XC = 3$$
$$\Rightarrow Height\ of\ \Delta FDE = 3$$
$$\frac{XF}{CE} = \frac{3}{4} \Rightarrow XF = \frac{3}{4} \times CE = \frac{3}{4} \times 16 = 12$$

Area of Triangles:

$$\underbrace{\frac{1}{2} \times 9 \times 4 = 18}_{[BCD]}, \quad \underbrace{\frac{1}{2} \times 3 \times 12 = 18}_{[FDE]}, \quad \underbrace{\frac{1}{2} \times 3 \times 12 = 18}_{[ABF]}, \quad \underbrace{\frac{1}{2} \times 12 \times 16 = 96}_{[ACE]}$$

By Complementary Areas:

$$[BDF] = [ACE] - ([BCD] + [FDE] + [ABF]) = 96 - (18 \times 3) = 96 - 54 = 42$$

And, hence, the required ratio is:

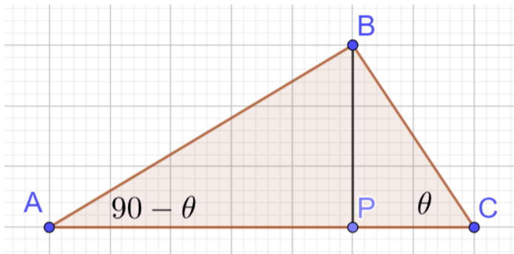
$$\frac{[BDF]}{[ACE]} = \frac{42}{96} = \frac{7}{16}$$

[9] This is solved in the Note on Trigonometry using the formula for the area of a triangle.

2.6 Right Triangles: Altitude to Hypotenuse

A. Self-Similar Right Triangles

2.79: Similarity in Right Triangles with Perpendicular
The two right triangles formed by dropping a perpendicular from the vertex of a right-angled triangle to its hypotenuse are similar to each other and to the original triangle.



Consider right ΔABC, right-angled at B.
Drop a perpendicular from B to AC, intersecting at P.

ΔABC	ΔAPB	ΔBPC
$\angle ABC = 90°$	$\angle APB = 90°$	$\angle CPB = 90°$
$\angle BCP = \theta$		$\angle BCP = \theta$
$\angle BAC = 90-\theta$	$\angle BAP = 90-\theta$	

By AA similarity:

$$\Delta ABC \sim \Delta APB \sim \Delta BPC$$

Using $\Delta ABC \sim \Delta APB$, the ratios that we get are:

$$\frac{AC}{AB} = \frac{BC}{BP} = \frac{AB}{AP}$$

Using $\Delta ABC \sim \Delta BPC$, the ratios that we get are:

$$\frac{AC}{BC} = \frac{BC}{PC} = \frac{AB}{BP}$$

Example 2.80

A perpendicular segment is drawn from B in rectangle $ABCD$ to meet diagonal AC at point X. Side AB is 6 cm and diagonal AC is 10 cm. How many centimeters away is point X from the midpoint M of the diagonal AC? Express your answer as a decimal to the nearest tenth. **(MathCounts 2007 School Sprint, Adapted)**

..

Method I: $\Delta ABX \sim \Delta ACB$ by Right Triangle Similarity since

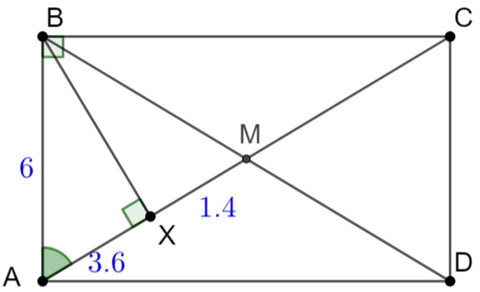
$$\angle BAC = \angle BAX \ (Same\ Angle)$$

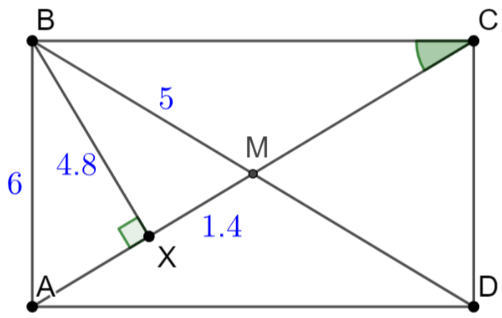
Hence:

$$\frac{AX}{AB} = \frac{AB}{AC} \Rightarrow AX = \frac{AB^2}{AC} = \frac{6^2}{10} = \frac{36}{10} = 3.6$$

$$XM = AM - AX = 5 - 3.6 = 1.4$$

(See Left Diagram)





Method II: $\Delta BXC \sim \Delta ABC$ by Right Triangle Similarity since

$$\angle BCX = \angle ABC \ (Same\ Angle)$$

Hence,

$$\frac{BX}{AB} = \frac{BC}{AC}$$

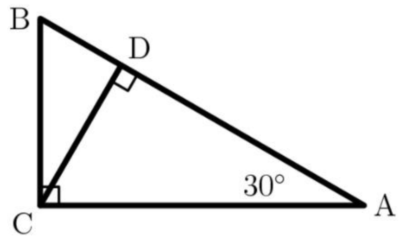
Substitute $AB = 6, AC = 10, BC = \sqrt{10^2 - 6^2} = 8$:

$$\frac{BX}{6} = \frac{8}{10} \Rightarrow BX = 4.8$$

In right triangle BXM

$$(BM, BX, XM) = (5, 4.8, XM) = \frac{1}{10}(50, 48, XM) = \frac{1}{5}(25, 24, XM) = \frac{1}{5}(25, 24, 7)$$

$$XM = \frac{1}{5} \cdot 7 = \frac{14}{10} = 1.4$$



Example 2.81

What is the ratio of the area of triangle BDC to the area of triangle ADC? Express your answer as a common fraction. **(MathCounts 2005 State Countdown)**

Since the question asks for a ratio, without loss of generality, let

$$AB = 1$$

Since ΔBCA is a $30 - 60 - 90$ triangle:

$$BC = \frac{1}{2}, \qquad AC = \frac{\sqrt{3}}{2}$$

$\Delta DBC \sim \Delta DCA$ by Similarity in Right Triangles with Perpendicular.
The ratio of sides is:

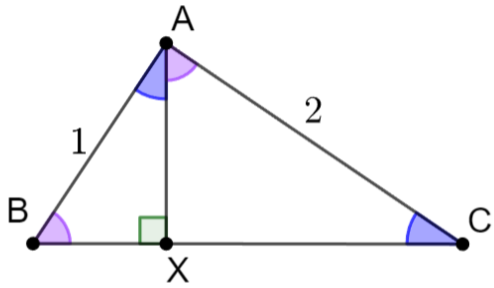
$$\frac{Hyp_{\Delta DBC}}{Hyp_{\Delta DCA}} = \frac{BC}{AC} = \frac{1}{2} : \frac{\sqrt{3}}{2} = 1 : \sqrt{3}$$

The ratio of areas is the square of the ratio of sides

$$= 1 : 3$$

Example 2.82

If the ratio of the legs of a right triangle is 1:2, then the ratio of the corresponding segments of the hypotenuse made by a perpendicular upon it from the vertex is: **(AHSME 1954/29)**



Draw a right triangle. The hypotenuse

$$= BC = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$\Delta ABC \sim \Delta XBA$ by Right Triangle Similarity because

$$\angle ABC = \angle ABX \ (Common\ angle)$$

The ratio of sides is:

$$\frac{Hyp_{XBA}}{Hyp_{ABC}} = \frac{AB}{BC} = \frac{1}{\sqrt{5}}$$

We can then calculate:

$$\frac{(Side\ opp\ blue)_{XBA}}{(Side\ opp\ blue)_{ABC}} = \frac{BX}{AB} = \frac{BX}{1} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

And since B, X and C are collinear:

$$XC = BC - BX = \sqrt{5} - \frac{\sqrt{5}}{5} = \frac{4\sqrt{5}}{5}$$

The ratio of the segments is:

$$BX : XC = \frac{\sqrt{5}}{5} : \frac{4\sqrt{5}}{5} = 1:4$$

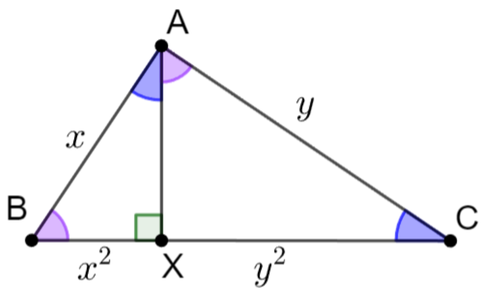
2.83: Ratio of Segments of the Hypotenuse

If the ratio of the legs of a right triangle is $x : y$, then the corresponding segments of the hypotenuse made by a perpendicular upon it from the vertex are:

$$\frac{x^2}{\sqrt{x^2 + y^2}}, \qquad \frac{y^2}{\sqrt{x^2 + y^2}}$$

And they have ratio

$$x^2 : y^2$$



Draw a right triangle. The hypotenuse

$$= BC = \sqrt{x^2 + y^2}$$

$\Delta ABC \sim \Delta XBA$ by Right Triangle Similarity because

$$\angle ABC = \angle ABX \ (Common\ angle)$$

The ratio of sides is:

$$\frac{Hyp_{XBA}}{Hyp_{ABC}} = \frac{AB}{BC} = \frac{x}{\sqrt{x^2 + y^2}}$$

We can then calculate:

$$\frac{(Side\ opp\ blue)_{XBA}}{(Side\ opp\ blue)_{ABC}} = \frac{BX}{AB} = \frac{BX}{x} = \frac{x}{\sqrt{x^2 + y^2}} \Rightarrow BX = \frac{x^2}{\sqrt{x^2 + y^2}}$$

And since B, X and C are collinear:

$$XC = BC - BX = \sqrt{x^2 + y^2} - \frac{x^2}{\sqrt{x^2 + y^2}} = \frac{y^2}{\sqrt{x^2 + y^2}}$$

Hence:

$$BX = \frac{x^2}{\sqrt{x^2 + y^2}}, XC = \frac{y^2}{\sqrt{x^2 + y^2}}$$

The ratio of the segments is:

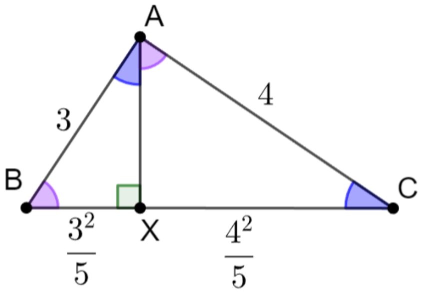
$$BX : XC = \frac{x^2}{\sqrt{x^2 + y^2}} : \frac{y^2}{\sqrt{x^2 + y^2}} = x^2 : y^2$$

Example 2.84

If the legs of a right triangle are 3 and 4, then the lengths of the corresponding segments of the hypotenuse made by a perpendicular upon it from the vertex are:

$$Hyp = 5$$

$$BX = \frac{x^2}{\sqrt{x^2 + y^2}} = \frac{3^2}{5} = \frac{9}{5}, \qquad XC = \frac{x^2}{\sqrt{x^2 + y^2}} = \frac{4^2}{5} = \frac{16}{5}$$



Example 2.85

The sides of a right triangle are a and b and the hypotenuse is c. A perpendicular from the vertex divides c into segments r and s, adjacent respectively to a and b. If $a : b = 1 : 3$, then the ratio of r to s is: **(AHSME 1958/19)**

$$r : s = a^2 : b^2 = 1^2 : 3^2 = 1 : 9$$

B. Altitude of a Right Triangle

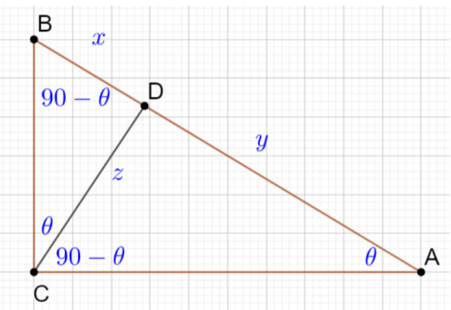
2.86: Altitude of a Right Triangle

The length of the altitude of a right triangle is the geometric mean of the segments that it divides the hypotenuse into.

$$z = Geometric\ Mean(x, y) = \sqrt{xy}$$

ΔBCD is similar to ΔCAD. Hence:

$$\frac{(Side\ opp\ \theta)_{BCD}}{(Side\ opp\ \theta)_{CAD}} = \frac{(side\ opp\ 90 - \theta)_{BCD}}{(side\ opp\ 90 - \theta)_{CAD}}$$



The ratio of sides is:

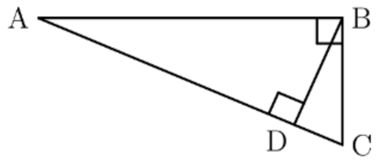
$$\frac{BD}{CD} = \frac{CD}{AD}$$

Rearrange:

$$CD^2 = BD \cdot AD$$

Take the square root on both sides:

$$CD = \sqrt{BD \cdot AD}$$

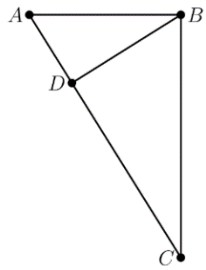


Example 2.87

In the figure shown, $AC = 13$ and $DC = 2$ units. What is the length of the segment BD? Express your answer in simplest radical form. **(MathCounts 2008 School Sprint)**

The altitude is the geometric mean of the segment that it divides the hypotenuse into:

$$BD = \sqrt{DC \cdot AD} = \sqrt{2(13-2)} = \sqrt{2(11)} = \sqrt{22}$$

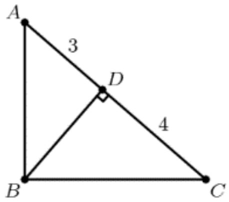


Example 2.88

In the figure, $\angle ABC$ and $\angle ADB$ are each right angles. Additionally, $AC = 17.8$ units and $AD = 5$ units. What is the length of segment DB? **(MathCounts 2009 Warm-up 16)**

The altitude is the geometric mean of the segment that it divides the hypotenuse into:

$$BD = \sqrt{AD \cdot CD} = \sqrt{5(17.8-5)} = \sqrt{5(12.8)} = \sqrt{10(6.4)} = \sqrt{64} = 8$$



Example 2.89

Triangle ABC has a right angle at B. Point D is the foot of the altitude from B, $AD = 3$, and $DC = 4$. What is the area of ΔABC? **(AMC 10A 2009/10)** [10]

The altitude is the geometric mean of the segment that it divides the hypotenuse into:

$$BD = \sqrt{3 \times 4} = 2\sqrt{3}$$

The area is then:

$$= \frac{1}{2}hb = \frac{1}{2}(BD)(AC) = \frac{1}{2}(2\sqrt{3})(3+4) = 7\sqrt{3}$$

2.90: Altitude given legs

If a right triangle has legs x and y then it has altitude

$$\frac{xy}{\sqrt{x^2+y^2}}$$

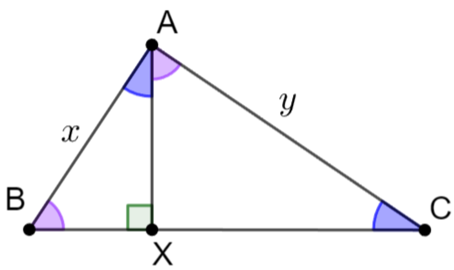
Calculate the area of the triangle in two different ways:

$$[ABC] = \frac{1}{2} \cdot BC \cdot AX = \frac{1}{2} \cdot AB \cdot BC$$

$$BC \cdot AX = AB \cdot BC$$

$$\sqrt{x^2+y^2} \cdot AX = xy$$

$$AX = \frac{xy}{\sqrt{x^2+y^2}}$$



Example 2.91

[10] For an AMC 10 question (that too at number 10), this is quite direct. It does require knowledge of the property. This is an *important* property.

In a right triangle with sides a and b, and hypotenuse c, the altitude drawn on the hypotenuse is x. Then: (AHSME 1956/38)

A. $ab = x^2$
B. $\frac{1}{a} + \frac{1}{b} = \frac{1}{x}$
C. $a^2 + b^2 = 2x^2$
D. $\frac{1}{x^2} = \frac{1}{a^2} + \frac{1}{b^2}$
E. $\frac{1}{x} = \frac{b}{a}$

$$x = \frac{ab}{\sqrt{a^2 + b^2}}$$

Square both sides:

$$x^2 = \frac{a^2b^2}{a^2 + b^2}$$

Take the reciprocal:

$$\frac{1}{x^2} = \frac{a^2 + b^2}{a^2b^2} = \frac{1}{a^2} + \frac{1}{b^2}$$
$$\mathit{Option\ D}$$

Example 2.92

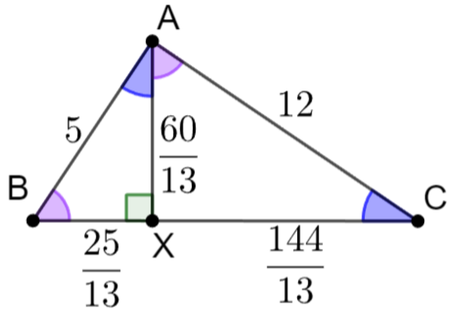
If the legs of a right triangle are 5 and 12, then find:

A. The lengths of the corresponding segments of the hypotenuse made by a perpendicular upon it from the vertex
B. The length of the perpendicular itself

$$(5,12,13) \Rightarrow BC = 13$$

$$BX = \frac{x^2}{\sqrt{x^2 + y^2}} = \frac{5^2}{13} = \frac{25}{13}, \quad XC = \frac{x^2}{\sqrt{x^2 + y^2}} = \frac{12^2}{13} = \frac{144}{5}$$

$$AX = \frac{5(12)}{13} = \frac{60}{13}$$

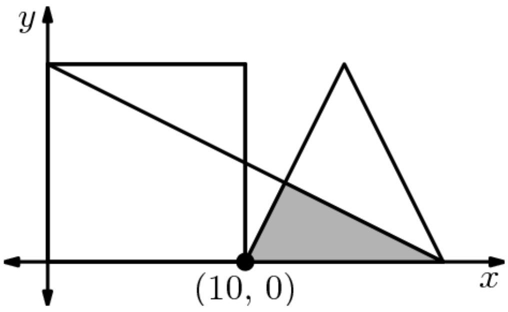


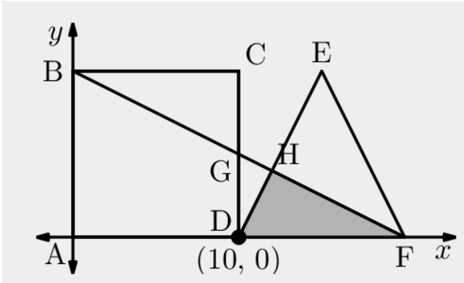
2.7 Further Examples

A. Adjacent Figures

Example 2.93

A square and isosceles triangle of equal height are side-by-side, as shown, with both bases on the x-axis. The lower right vertex of the square and the lower left vertex of the triangle are at $(10, 0)$. The side of the square and the base of the triangle on the x-axis each equal 10 units. A segment is drawn from the top left vertex of the square to the farthest vertex of the triangle, as shown. What is the area of the shaded region? (MathCounts 2010 State Team)





Slope of

$$Line\ BF = m_{BF} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10 - 0}{0 - 20} = -\frac{10}{20} = -\frac{1}{2}$$

$$Line\ DE = m_{DE} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10 - 0}{15 - 10} = \frac{10}{5} = 2$$

$$(m_{BF})(m_{DE}) = -\frac{1}{2}(2) = -1$$

Since the product of the slopes is -1, the lines are perpendicular. Hence:

$$\angle DHF = 90°$$

$\Delta BAF \sim \Delta DHF$ by AA similarity since:

$$\angle BAF = \angle DHF = 90°$$
$$\angle BFA = \angle HFA\ (Common\ Angle)$$

The ratio of side lengths is then:

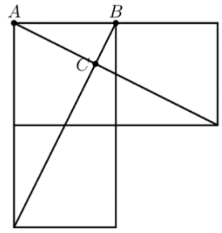
$$\frac{AF}{BF} = \frac{10}{\sqrt{BA^2 + AF^2}} = \frac{10}{\sqrt{10^2 + 20^2}} = \frac{10}{\sqrt{500}} = \frac{10}{10\sqrt{5}} = \frac{1}{\sqrt{5}}$$

Ratio of Areas is the square of ratio of side lengths:

$$= \left(\frac{AF}{BF}\right)^2 = \frac{1}{5}$$

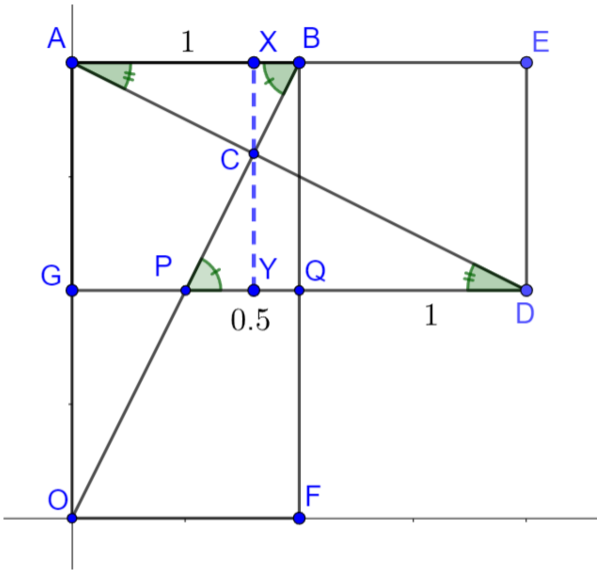
Area of ΔDHF

$$= \frac{1}{5}[\Delta BAF] = \frac{1}{5}\left(\frac{1}{2} \cdot BA \cdot AF\right) = \frac{1}{5}\left(\frac{1}{2} \times 10 \times 20\right) = 20$$



Example 2.94

Three unit squares and two line segments connecting two pairs of vertices are shown. What is the area of ΔABC? **(AMC 10A 2012/15)**



Step I: $\Delta BQP \sim \Delta BFO$ by Parallel Line Similarity since

$$PQ \parallel OF$$

Hence:

$$\frac{PQ}{OF} = \frac{BQ}{BF} \Rightarrow PQ = \frac{BQ}{BF} \cdot OF = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

Step II: $AB \parallel GD$. $\therefore$ $\Delta BCA \sim \Delta PCD$ by Bow Tie Similarity:

Hence:

$$\frac{AB}{PD} = \frac{AB}{PQ + QD} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}$$

The ratio of heights is also:

$$\frac{CX}{CY} = \frac{2}{3} \Rightarrow CY = \frac{3}{2}CX$$

Since we have unit squares:

$$CX + CY = CX + \frac{3}{2}CX = 1 \Rightarrow \frac{5}{2}CX = 1 \Rightarrow CX = \frac{2}{5}$$

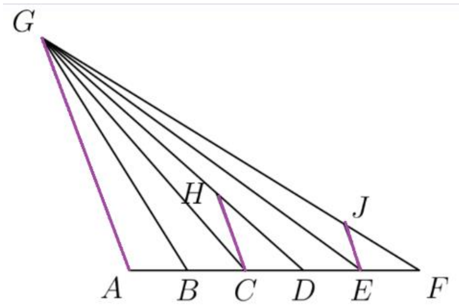
The area of ΔABC:

$$= \frac{1}{2}hb = \frac{1}{2}\left(\frac{2}{5}\right)(1) = \frac{1}{5}$$

B. Nested Triangles

Example 2.95

Points A, B, C, D, E and F lie, in that order, on $\overline{AF}$, dividing it into five segments, each of length 1. Point G is not on line $\overline{AF}$. Point H lies on $\overline{GD}$, and point J lies on $\overline{GF}$. The line segments $\overline{HC}, \overline{JE}$, and $\overline{AG}$ are parallel. Find HC/JE. **(AMC 10A 2002/20)**



$$\angle ADG = \angle CDH \ (Common\ Angle)$$
$$\angle GAD = \angle HCF \ (Corresponding\ Angles)$$
$$\Delta GAD \sim \Delta HCD \ (AA\ Similarity)$$

$$\frac{HC}{GA} = \frac{CD}{AD} = \frac{1}{3}$$

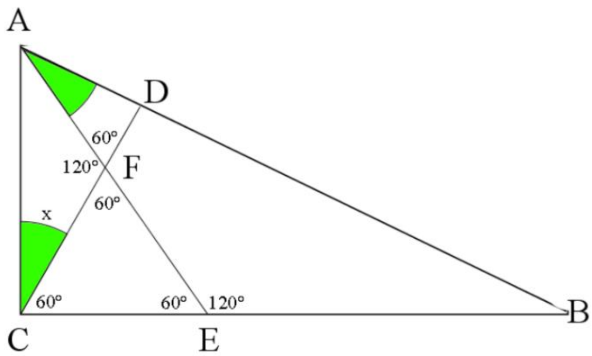
$$\angle F = \angle F \ (Common\ Angle)$$
$$\angle GAF = \angle JEF \ (Corresponding\ Angles)$$
$$\Delta GAF \sim \Delta JEF \ \ (AA\ Similarity)$$

$$\frac{GA}{JE} = \frac{AF}{FE} = \frac{5}{1}$$

$$\frac{HC}{JE} = \frac{HC}{GA} \cdot \frac{GA}{JE} = \frac{1}{3} \cdot \frac{5}{1} = \frac{5}{3}$$

Example 2.96

Triangle ABC has $AB = 2 \cdot AC$. Let D and E be on $\overline{AB}$ and $\overline{BC}$, respectively, such that $\angle BAE = \angle ACD$. Let F be the intersection of segments AE and CD, and suppose that ΔCFE is equilateral. What is $\angle ACB$? **(AMC 10A 2010/14)**



Draw a Diagram

ΔCFE is equilateral, so each angle in the triangle is $60°$.
By Angles in a Linear Pair:

$$\angle AEB = \angle AFC = 180 - 60 = 120°$$
$$\angle AFC = 180° - CFE = 180 - 60 = 120°$$

Prove ΔAFC Similar to ΔAEB

$\Delta AFC \sim \Delta AEB$ by AA Similarity since:

$$\angle ACD = \angle BAE \ (Given)$$
$$\angle AEB = \angle AFC = 120°$$

Prove ΔAFC Isosceles

$$\frac{AE}{AF} = \frac{AB}{AC}$$

But we know from the question that $AB = 2 \cdot AC \Rightarrow \frac{AB}{AC} = 2$

$$\frac{AE}{AF} = 2 \Rightarrow F\ is\ the\ midpoint\ of\ AE$$
$$AF = FE$$
$$FE = FC$$
$$AF = FC$$

$$FC = FE = AF \Rightarrow \Delta AFC\ is\ isosceles$$

Find the values of the angles

In Isosceles ΔAFC

$$\angle ACD = \frac{180 - 120}{2} = \frac{60}{2} = 30$$

$$\angle ACB = 60 + x = 60 + 30 = 90°$$

C. Quadrilaterals

Challenge 2.97

In a rectangle $ABCD$, E is the midpoint of AB. F is a point in AC such that BF is perpendicular to AC, and FE perpendicular to BD. Suppose $BC = 8\sqrt{3}$.

A. Draw $FH \parallel GB$ and show that $\Delta CFH \sim \Delta CAB \sim \Delta CFB$
B. Find AB. **(IOQM 2017/13, Adapted)**

Proving Similarity

The angles of the diagonals of a rectangle with its base side are equal:

$$\angle DBA = \angle CAB = \alpha\ (say)$$
$$\angle NEB = \angle FBA = 90 - \alpha \Rightarrow \Delta FEB \text{ is isosceles}$$
$$In\ \Delta CBF: \angle ACB = 90 - \alpha \Rightarrow \angle CBF = \alpha$$

ΔCFH~ΔCAB~ΔCFB (AA Similarity)

$$\angle CFH = \angle CAB = \angle CBF = \alpha$$
$$\angle CBA = \angle CHF = \angle CFB = 90°$$

Using Similarity

From the similarity condition in ΔCFB and ΔCAB:

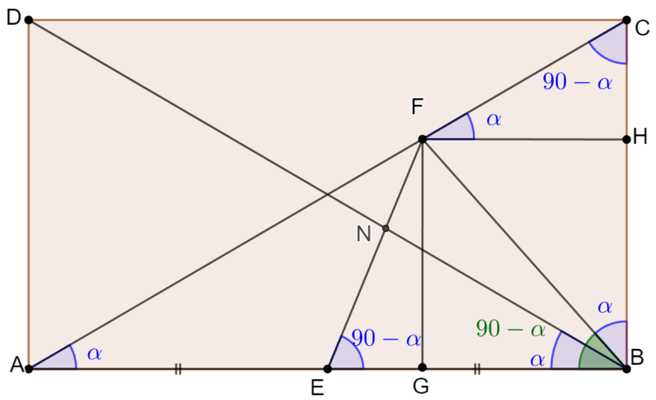
$$\frac{FC}{CB} = \frac{CB}{AC} \Rightarrow FC \cdot AC = CB^2$$

The altitude of isosceles ΔFEB divides its base into equal parts:

$$Altitude\ FG \perp EB \Rightarrow \underbrace{FH = GB = \frac{1}{2}EB = \frac{1}{4}AB}_{Equation\ I}$$

From Equation I and $\Delta CFH \sim \Delta CAB$, sides of ΔCFH are $\frac{1}{4}$ the sides of ΔCAB. Substitute $FC = \frac{1}{4}AC$:

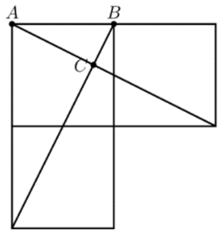
$$\frac{1}{4}AC \cdot AC = CB^2 \Rightarrow AC = 2CB$$



Since the side opposite $\angle\alpha$ is $\frac{1}{2}$ the hypotenuse ΔCAB is a $30-60-90$ *triangle*

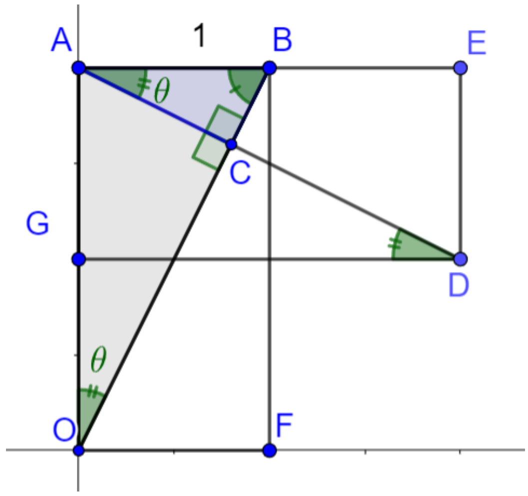
$$AB = \sqrt{3}CB = \sqrt{3}(8\sqrt{3}) = 24$$

D. Rotation



Example 2.98

Three unit squares and two line segments connecting two pairs of vertices are shown. What is the area of ΔABC? **(AMC 10A 2012/15)**



Rotate $ABFO$ (which is a 1 unit by 2 units rectangle) counterclockwise $90°$ about point B, translate it 1 unit to the left to get rectangle $AEDG$.

Diagonal OB gets rotated $90°$ to become diagonal AD. Hence,

$$OB \perp AD$$

Alternatively, translate $ABFO$ 1 unit down and then rotate it $90°$ counterclockwise about A to get rectangle $AEDG$.
$\angle AOB$ gets rotated to become $\angle GDA$, which is equal to $\angle EAD$ by alternate interior angles. Hence:

$$\angle AOB \cong \angle GDA \cong \angle EAD$$

In ΔBCA and ΔBAO

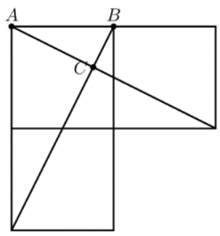
$$\angle AOB \cong \angle EAD$$
$$\angle BAO = \angle BCO = 90°$$
$$\angle ABC \cong \angle ABO \ (Same\ angle)$$
$$\Delta BCA \sim \Delta BAO \ (AAA\ Similarity)$$

Using the similarity of the above two triangles:

$$\frac{AB}{BO} = \frac{BC}{AB} \Rightarrow \frac{1}{\sqrt{5}} = \frac{BC}{1} \Rightarrow BC = \frac{1}{\sqrt{5}}$$
$$\frac{AO}{BO} = \frac{AC}{AB} \Rightarrow \frac{2}{\sqrt{5}} = \frac{AC}{1} \Rightarrow AC = \frac{2}{\sqrt{5}}$$

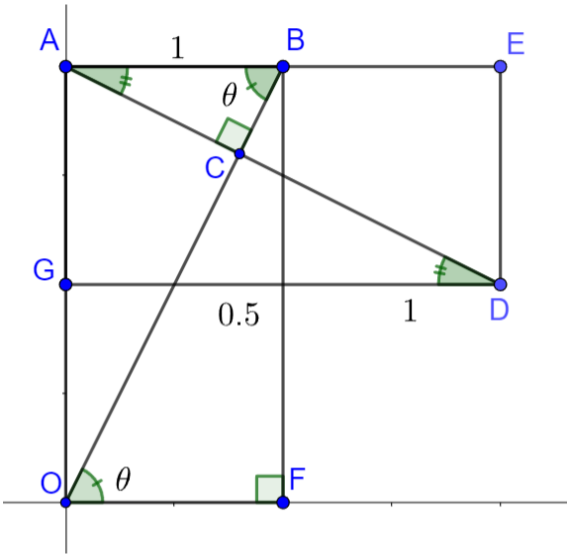
The area of ΔABC:

$$= \frac{1}{2}hb = \frac{1}{2}(BC)(AC) = \frac{1}{2}\left(\frac{1}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}\right) = \frac{1}{5}$$



Example 2.99

Three unit squares and two line segments connecting two pairs of vertices are shown. What is the area of ΔABC? **(AMC 10A 2012/15)**



Rotate $ABFO$ about point B, translate it 1 unit to the left to get rectangle $AEDG$.

Diagonal OB gets rotated 90° to become diagonal AD. Hence,

$$OB \perp AD$$

In ΔBCA and ΔBFO:

$$\angle ABO = \angle BOF \; (Alternate\ Interior\ Angles)$$
$$\angle BCA = \angle BFO = 90°$$
$$\Delta BCA \sim \Delta BFO \; (AA\ Similarity)$$

In right $\Delta\ BCA$, by Pythagoras Theorem:

$$BC^2 + AC^2 = 1$$

Substitute $\frac{BF}{FO} = \frac{2}{1} \Rightarrow BF = 2FO \Rightarrow AC = 2BC$

$$BC^2 + (2BC)^2 = 1 \Rightarrow BC = \frac{1}{\sqrt{5}}, AC = \frac{2}{\sqrt{5}}$$

The area is

$$= \frac{1}{2}hb = \frac{1}{2}(BC)(AC) = \frac{1}{2}\left(\frac{1}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}\right) = \frac{1}{5}$$

3. Similarity with Sides

3.1 SSS & SAS Similarity

A. SSS Test of Similarity

Here we consider test of similarity which can also be used to prove

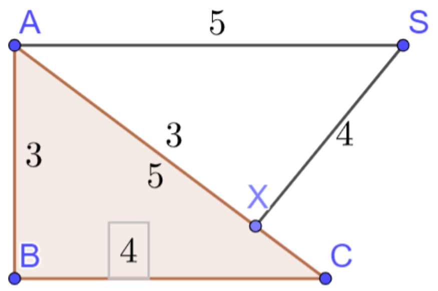
3.1: SSS Test of Similarity
If three sides of a triangle have the same ratio to the three sides of another triangle, then the two triangles are similar.

Example 3.2
Identify the similar triangles and write the similarity.

$$AB = 3, BC = 4, CA = 5$$
$$SA = 5, SX = 4, AX = 3$$

Draw a diagram. From the diagram, by SSS Similarity:

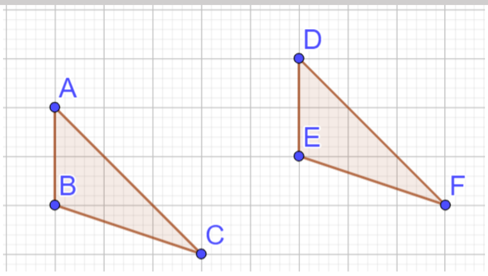
$$\Delta ABC \sim \Delta AXS$$
$$\frac{AC}{SA} = \frac{AB}{AX} = \frac{BC}{SX} = 1$$



Note: In this case, the triangles are not just similar but also congruent.

3.3: SSS: Similarity versus Congruence
If the ratio of sides is 1, then the triangles are not only similar, they are also congruent.

3.4: Translation
Translation is moving an object without changing its shape or its size. On a two-dimensional surface, translation can happen in either the vertical dimension, or the horizontal dimension, or both.



Example 3.5
ΔABC has the same shape that ΔDEF does. Show this by a translation.

If we translate each point of ΔABC by

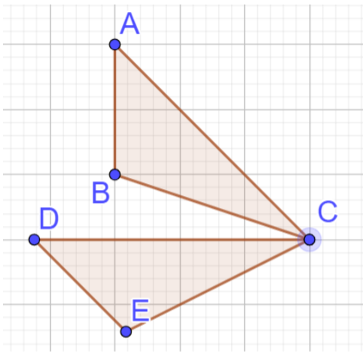
$$(x, y) = (1,5) \Rightarrow Up\ by\ 1\ Unit, Right\ by\ 5\ Units$$

Then we obtain

$$\Delta DEF$$

3.6: Rotation

A rotation of an object preserves the shape of the object while moving the object through an angle around a point.



Example 3.7

Consider the diagram drawn alongside. ΔABC is rotated 45° counter-clockwise to obtain the second triangle shown. Determine for each statement below whether it is true or false:

A. $\Delta ABC \cong \Delta CDE$
B. $\Delta ABC \sim \Delta CDE$
C. $\Delta ABC \cong \Delta CED$
D. $\Delta ABC \sim \Delta CED$

It should actually be:

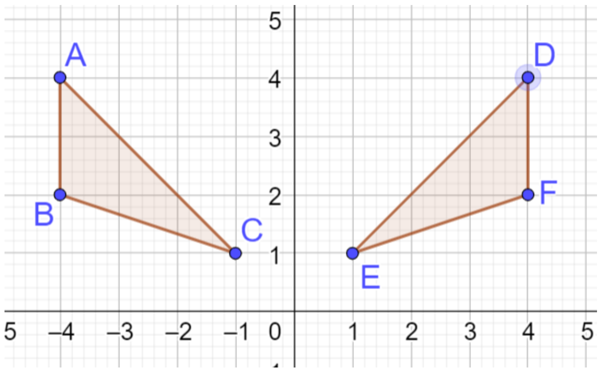
$$\Delta ABC \cong \Delta DEC$$
$$\Delta ABC \sim \Delta DEC$$

Which means none of the options are true.

3.8: Reflection

A reflection is a mirror image of an object across a line.

Example 3.9



Consider the diagram drawn alongside. If A is true, it has value 1, and if is false, it has value zero. If B is true, it has value 2, and if it is false, it has value zero. If C is true, it has value 4, and if it is false, it has value zero. If D is true, it has value 8, and if it is false, it has value zero.
Find

$$A + B + C + D$$

A. The two triangles shown in the figure are congruent.
B. The two triangles shown in the figure are similar.
C. $\Delta ABC \cong \Delta DEF$
D. $\Delta ABC \sim \Delta DEF$

The two triangles are congruent:

$$A \text{ is true}: A = 1$$

If two triangles are congruent, they are also similar:

$$B \text{ is true}: B = 2$$

$$C \text{ is false}: C = 0$$
$$D \text{ is false}: D = 0$$

$$A + B + C + D = 1 + 2 + 0 + 0 = 3$$

$$\Delta ABC \sim \Delta DEF \Leftrightarrow \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

Example 3.10

A. ΔJKL has sides JK, KL and JL of length 3, 4 and 6. ΔQWE has sides QW, WE and QE of length 4,8 and 12.ΔJKL has $\angle J = \alpha, \angle K = \beta, \angle L = \gamma$. Find, with proof, the measures of each angle of ΔQWE.

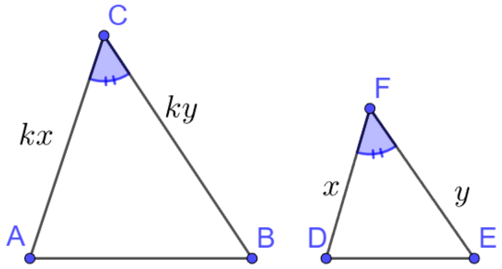
$$\frac{JK}{QW} = \frac{KL}{WE} = \frac{JL}{QE} = \frac{1}{2} \Rightarrow \underbrace{\Delta JKL \sim \Delta QWE}_{SSS\ Test}$$
$$\angle Q = \angle J = \alpha$$
$$\angle W = \angle K = \beta$$
$$\angle E = \angle L = \gamma$$

Example 3.11
SAS
ASA
RHS

B. SAS Test of Similarity

3.12: SAS Test of Similarity $(\boldsymbol{Side - Angle - Side})$
If
two sides of one triangle have the same ratio as two sides of another triangle
the included angle has the same measure

then the two triangles are similar.



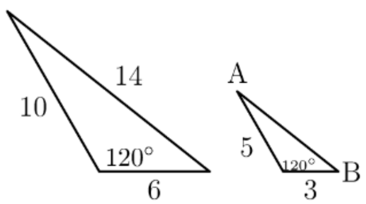
Example 3.13
The diagram alongside (not drawn to scale) has two triangles.

A. Explain why the two triangles are similar.
B. Write the similarity ratios.
C. If $AB = 4, k = 3$, determine the length of DE.

Part A
In the adjoining diagram, two adjacent sides are in the same ratio:
$$\frac{CA}{FD} = \frac{kx}{x} = k, \qquad \frac{CB}{FE} = \frac{ky}{y} = k$$
$$\angle ACB = \angle DFE$$
$$\Delta CAB \sim \Delta FDE\ (SAS\ Test\ of\ Similarity)$$

Part B
$$\frac{CA}{FD} = \frac{CB}{FE} = \frac{AB}{DE} = k$$

Part C
$$\frac{AB}{DE} = k \Rightarrow DE = \frac{AB}{k} = \frac{4}{3}$$



Example 3.14
The side lengths of both triangles to the right are given in centimeters. What is the length of segment AB? (MathCounts 2007 Warm-Up 6)

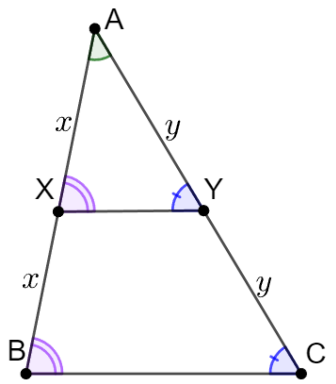
$$\frac{x}{14} = \frac{3}{6} \Rightarrow x = \frac{3}{6} \times 14 = \frac{1}{2} \times 14 = 7$$

3.2 Midpoint Theorem

A. Midpoint Theorem

3.15: Midpoint Theorem[11]

The line joining the midpoints of two sides of a triangle is parallel to the third side and half of the third side.



Draw ΔABC.
Let X, Y be the midpoints of sides AB and AC respectively. Hence:

$$AX = XB = x, \qquad AY = YC = y$$

$\Delta ABC \sim \Delta AXY$ by SAS:

$$\frac{AB}{AX} = \frac{x+x}{x} = 2\ (Side)$$
$$\frac{AC}{AY} = \frac{y+y}{y} = 2\ (Side)$$
$$\angle XAY = \angle BAC\ \ (Same\ Angle)$$

Hence:

$$\frac{AX}{AB} = \frac{AY}{AC} = \frac{XY}{BC} = \frac{1}{2}$$

Since

$$\frac{XY}{BC} = \frac{1}{2} \Rightarrow XY = \frac{1}{2}BC \Rightarrow XY\ is\ half\ of\ BC$$

Since the triangles are similar, their corresponding angles are congruent. Therefore

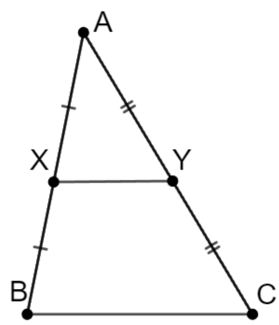
$$\angle AXY = \angle ABC, \angle AYX = \angle ACB$$

Consider line AB as a transversal of lines XY and BC

$$\angle AXY = \angle ABC \Rightarrow Corresponding\ angles\ are\ congruent \Rightarrow XY \parallel BC$$

Example 3.16

In ΔABC, X and Y are the midpoints of the AB and BC respectively. If $AX = 3, AY = 4, BC = 9$ determine the perimeter of quadrilateral $XYCB$.



Draw a diagram, and use the midpoint theorem:

$$XY + YC + CB + BX = 4.5 + 4 + 9 + 3 = 20.5$$

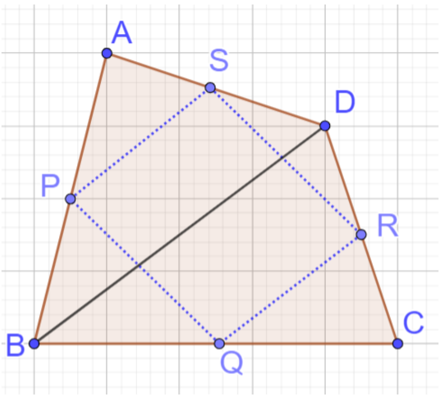
Example 3.17

Show that the quadrilateral obtained by joining the midpoints of the adjacent sides of a quadrilateral is a

[11] The midpoint is proved using congruence properties in the Note on Triangles.

parallelogram.

Draw general quadrilateral $ABCD$.
Draw diagonal BD.
Let the midpoints be P, Q, R, S as shown in the diagram.



By the midpoint theorem:

$$In\ \Delta ABD: PS = \frac{1}{2}BD, \qquad PS \parallel BD$$
$$In\ \Delta CBD: \ RQ = \frac{1}{2}BD, \qquad RQ \parallel BD$$

$$PS = \frac{1}{2}BD = RQ \Rightarrow PS = RQ$$
$$PS \parallel BD \parallel RQ \Rightarrow PS \parallel RQ$$

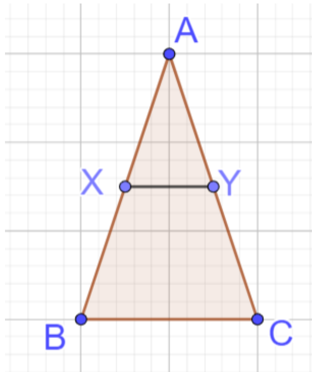
In a quadrilateral, if one pair of opposite sides is equal and parallel, then the quadrilateral is a parallelogram.

Example 3.18

Answer each part separately

In ΔABC, draw XY connecting the midpoints of AB and BC respectively. If

A. $AX = 7$, find the length of AB?
B. $AC = 12$, determine the distance between A and Y
C. $XY = 9$, calculate BC?
D. $AX = 2\frac{2}{3}$, $AC = 1\frac{4}{5}$, and $XY = 1\frac{1}{4}$, find $AB + AY + BC$.



Part A

$$AB = 2(AX) = 2(7) = 14$$

Part B

$$AY = \frac{AC}{2} = \frac{12}{2} = 6$$

Part C

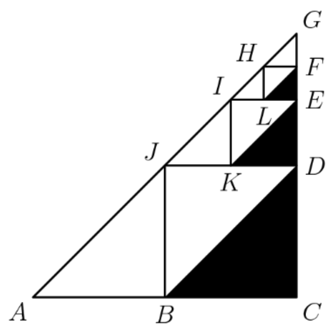
$$BC = 2(XY) = 2(9) = 18$$

Part D

$$AB = 2(AX) = 2\left(2\frac{2}{3}\right) = 2\left(\frac{8}{3}\right) = \frac{16}{3}$$
$$AY = \frac{1}{2}(AC) = \frac{1}{2}\left(1\frac{4}{5}\right) = \frac{1}{2}\left(\frac{9}{5}\right) = \frac{9}{10}$$
$$BC = 2XY = 2\left(1\frac{1}{4}\right) = 2\left(\frac{5}{4}\right) = \frac{5}{2}$$

Example 3.19

Points B, D, and J are midpoints of the sides of right triangle ACG. Points K, E, I are midpoints of the sides of triangle JDG, etc. If the dividing and shading process is done 100 times (the first three are shown) and $AC = CG = 6$, then the total area of the shaded triangles is nearest to which integer? (AMC 8 1999/25)



Midpoint Theorem

By the Midpoint Theorem:

Since J and D are midpoints of AG and CG respectively,

$$JD \parallel AC, JD = \frac{1}{2}ABC$$

Since B and J are midpoints of AC and AG respectively,

$$BJ \parallel CG, BJ = \frac{1}{2}CG$$

Also,

$$\angle BJD = 90°$$

Hence

$$\Delta AJB \cong \Delta DJB \cong \Delta DCB$$

Hence,

$$[DCB] = \frac{1}{3}[AJDC]$$

This same logic is applicable to each of the 100 quadrilaterals which will be created.
And hence

$$Shaded\ Area \approx \frac{1}{3}[ABC] = \frac{1}{3}\left(\frac{1}{2}hb\right) = \frac{1}{3} \times \frac{1}{2} \times 6 \times 6 = 6$$

Geometric Series

$$[BCD] = \frac{1}{4}[ABC]$$

$$[KDE] = \frac{1}{4}[BCD] = \frac{1}{16}[ABC]$$

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots$$

$$a = \frac{1}{4}, r = \frac{1}{4} \Rightarrow \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

3.20: Converse of Midpoint Theorem

The straight line drawn through the midpoint of one side of a triangle parallel to another, bisects the third side.

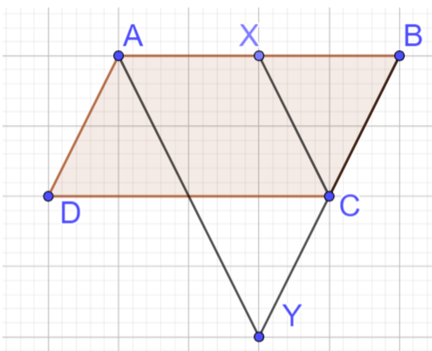
Example 3.21

In parallelogram $ABCD$, X is the midpoint of side AB. $AY \parallel XC$ and intersects BC (produced) at Y. Show that:

A. $AD = \frac{1}{2}BY$
B. $AY = 2XC$

Part A

In ΔABY:



$$X\ is\ midpoint\ of\ AB\ (Given)$$

$$AY \parallel XC$$

$$\therefore C\ is\ midpoint\ of\ BY\ (Converse\ of\ midpoint\ theorem)$$

$$BC = \frac{1}{2}CY$$

$$AD = BC\ (Opp. sides\ of\ a\ parallelogram\ are\ equal)$$

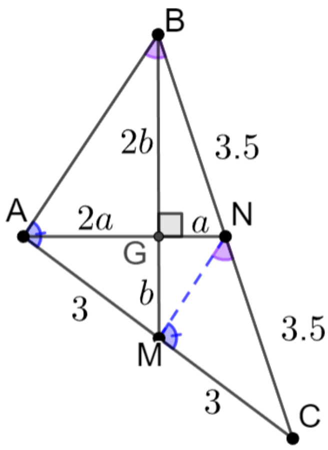
$$AD = \frac{1}{2}CY$$

Part B

B. Challenging Questions

Example 3.22

In ΔABC the median from A is given perpendicular to the median from B. If $BC = 7$ and $AC = 6$, find the length of AB. (AHSME 1961/36)



$\Delta AGB \sim \Delta NGM$ by Bow Tie Similarity since

$$MN \parallel AB \text{ (Midpoint Theorem)}$$

Since G is the centroid of ΔABC, it divides the medians in the ratio $2:1$. Hence:

$$Let\ GN = a \Rightarrow GA = 2a$$
$$Let\ GM = b \Rightarrow GM = 2b$$

By the Pythagorean Theorem, in:

$$\Delta AGM: 4a^2 + b^2 = 9, \qquad \Delta BGN: a^2 + 4b^2 = \frac{49}{4}$$

Adding the two equations above:

$$5a^2 + 5b^2 = \frac{85}{4} \Rightarrow 4a^2 + 4b^2 = 17$$

By the Pythagorean Theorem, in ΔAGB:

$$AB = \sqrt{4a^2 + 4b^2} = \sqrt{17}$$

3.3 Basic Proportionality Theorem

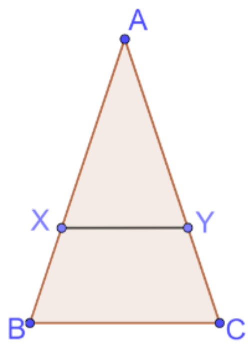
A. Basics

3.23: Basic Proportionality Theorem (BPT)

The line joining two sides of a triangle that divides each of those sides in the same ratio creates a similar triangle.

In the diagram, if $\frac{AX}{AB} = \frac{AY}{YC}$

$$\Delta AXY \sim \Delta ABC$$



- Other names for BPT include Side Splitter Theorem, and the Thales intercept theorem

3.24: Basic Proportionality Theorem

$$\frac{AX}{AB} = \frac{AY}{AC} \Leftrightarrow \frac{AX}{BX} = \frac{AY}{YC}$$

Take the reciprocal both sides:

$$\frac{AB}{AX} = \frac{AC}{AY}$$

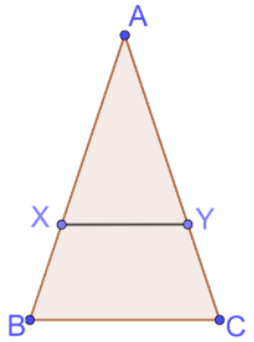
Subtract 1 from both sides:

$$\frac{AB - AX}{AX} = \frac{AC - AY}{AY}$$
$$\frac{BX}{AX} = \frac{YC}{AY}$$
$$\frac{AX}{BX} = \frac{AY}{YC}$$

Example 3.25

In the diagram, $XY \parallel BC$.

A. If $AX = 7, XB = 4, AY = 6$, then find YC.
B. If $XY = \frac{2}{3}, AX = \frac{4}{5}, BC = \frac{7}{4}$, then find BX.



Part A

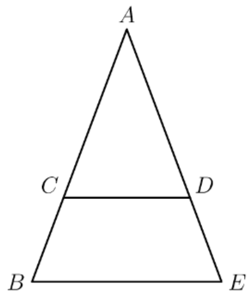
$$\frac{AX}{XB} = \frac{AY}{YC} \Rightarrow \frac{7}{4} = \frac{6}{YC} \Rightarrow YC = 6 \times \frac{4}{7} = \frac{24}{7}$$

Part B

$$\frac{AX}{BX} = \frac{XY}{BC} \Rightarrow \frac{\frac{4}{5}}{BX} = \frac{\frac{2}{3}}{\frac{7}{4}} \Rightarrow BX = \frac{4}{5} \times \frac{7}{4} \times \frac{3}{2} = \frac{21}{10} = 2.1$$

Example 3.26

Isosceles triangle ABE of area 100 square inches is cut by CD into an isosceles trapezoid and a smaller isosceles triangle. The area of the trapezoid is 75 square inches. If the altitude of triangle ABE from A is 20 inches, what is the number of inches in the length of CD? (MathCounts 2004 Chapter Team)



In ΔBAE

$$Area = \frac{1}{2}bh = \frac{1}{2} \cdot BE \cdot h = \frac{1}{2} \cdot BE \cdot 20 = 100 \Rightarrow BE = 10$$

$$[ACD] = [ABE] - [CDBE] = 100 - 75 = 25$$

$\Delta CAD \sim \Delta BAE$ by SAS Similarity

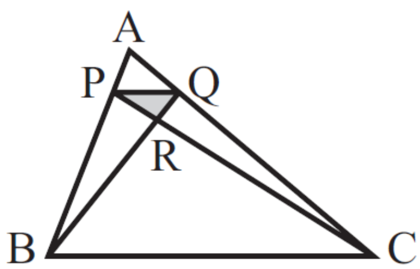
The ratio of sides is the square root of the ratio of areas:

$$\frac{CD}{BE} = \sqrt{\frac{[ACD]}{[ABE]}} = \sqrt{\frac{25}{100}} = \sqrt{\frac{1}{4}} = \frac{1}{2} \Rightarrow CD = \frac{1}{2} \cdot BE = \frac{1}{2} \cdot 10 = 5$$

B. Challenging Questions

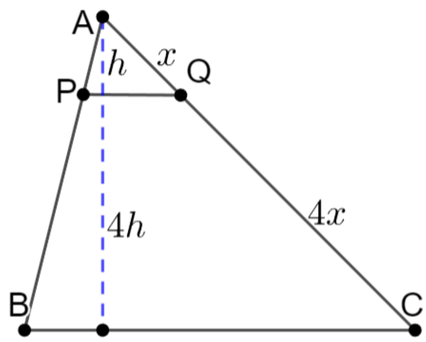
Example 3.27

In triangle ABC, shown here, P and Q lie on sides AB and AC, respectively, so that $\frac{AP}{AB} = \frac{AQ}{AC} = \frac{1}{5}$. Segments PC and QB intersect at R. What is the ratio of the area of triangle PQR to the area of triangle ABC? Express your answer as a common fraction. (MathCounts 2019 Chapter Sprint Round/29)



Step I: By the Basic Proportionality Theorem:

$$\Delta APQ \sim \Delta ABC$$



The ratio of bases

$$=\frac{PQ}{BC}=\frac{1}{5}$$

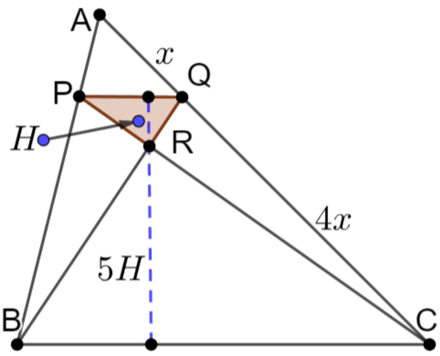
The ratio of heights is equal to the ratio of bases

$$=\frac{1}{5}=\frac{h}{5h}$$

The ratio of height of the trapezoid $PQCB$ to ΔABC is:

$$\frac{Height(PQCB)}{Height(ABC)}=\frac{5h-h}{5h}=\frac{4}{5}$$

Step II: By Bow Tie Similarity



$$\Delta PQR \sim \Delta BRC$$

The ratio of bases is:

$$\underbrace{\frac{PQ}{BC}=\frac{1}{5}}_{\textit{Ratio of Bases}}$$

The ratio of heights

$$=\frac{1}{5}=\frac{H}{5H}$$

The ratio of height of ΔPQR to trapezoid $PQCB$ is:

$$\frac{Height(PQR)}{Height(PQCB)}=\frac{H}{H+5H}=\frac{1}{6}$$

Step III: Combine Step I and Step II

Multiplying the results from Step I and II:

$$\frac{Height(PQCB)}{Height(ABC)}\times\frac{Height(PQR)}{Height(PQCB)}=\frac{4}{5}\times\frac{1}{6}$$

Simplifying:

$$\underbrace{\frac{Height(PQR)}{Height(ABC)}=\frac{2}{15}}_{\textit{Ratio of Heights}}$$

Step IV: Final Answer

Ratio of Areas

$$=\underbrace{\frac{1}{5}}_{\substack{\textit{Ratio}\\ \textit{of Bases}}}\times\underbrace{\frac{2}{15}}_{\substack{\textit{Ratio}\\ \textit{of Heights}}}=\frac{2}{75}$$

Example 3.28

Given ΔABC with base AB fixed in length and position. As the vertex C moves on a straight line, the intersection point of the three medians moves on ______ (AHSME 1962/15)

Step I: Draw ΔABC with

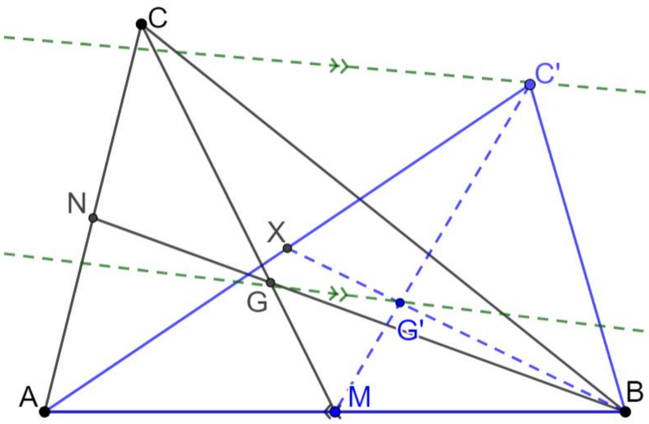
$CM\ as\ median\ to\ AB, \qquad BN\ as\ median\ to\ AC$

The intersection of the three medians is the centroid of the triangle, which is

$$G$$

The centroid G divides the medians in the ratio $2:1$

$$\frac{GM}{CM} = \frac{1}{3}$$



Step II: Suppose C moves to C' on a straight line

Let the medians be

$$CM'\ and\ BX'$$

The intersection of CM' and BX' is the new centroid. Call it:

$$G'$$

Again, since the centroid G' divides the medians in the ratio $2:1$, we have:

$$\frac{G'M}{C'M} = \frac{1}{3}$$

Hence, in $\Delta MCC'$:

$$\frac{GM}{CM} = \frac{G'M}{C'M} = \frac{1}{3}$$

Hence, by the basic proportionality theorem:

$$GG' \parallel CC'$$

Hence, the intersection point of the three medians moves on

$$A\ line\ parallel\ to\ CC'$$

3.4 Congruence via ASA & AAS

A. Basics

3.29: Congruence: Similarity with scale ratio 1

If the scale factor in similarity is 1, then the two shapes are not only similar, but also congruent.

B. Trapezoids

3.30: Median of a Trapezoid

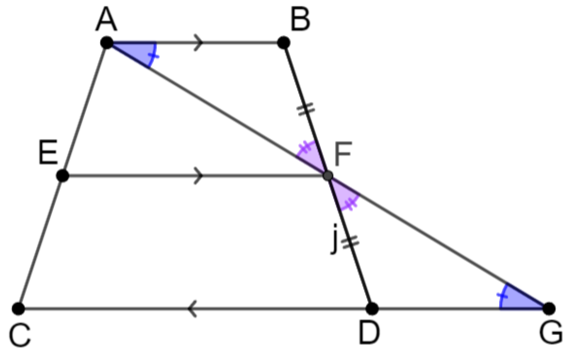
The median of a trapezoid is the line joining the midpoints of its two legs.

3.31: Median: Length and Parallel to Base Properties

The median of a trapezoid is parallel to its bases.

The length of the median of a trapezoid is the average of the two bases of the trapezoid.

$$Median = EF = \frac{AB + CD}{2}$$



Draw trapezoid $ABDC$ with $AB \parallel CD$.
Let E be the midpoint of AC, and F be the midpoint of BD.

Extend AF to meet CD at G.

$\Delta ABF \cong \Delta DFG$ by AAS since:

$$\angle BAF = \angle DGF\ (Alternate\ Interior\ Angles)$$
$$\angle AFB = \angle DFG\ (Vertically\ Opposite\ Angles)$$
$$BF = FD\ (Midpoint\ Definition)$$

By CPCT:

$$AF = FG \Rightarrow F\ is\ midpoint\ of\ AG$$

By Midpoint Theorem:$EF \parallel CD$. Hence:

$$EF = \frac{CG}{2} = \frac{CD + DG}{2}$$

Substitute $DG = AG$ $(CPCT\ in\ \Delta ABF \cong \Delta DFG)$

$$= \frac{CD + AB}{2}$$

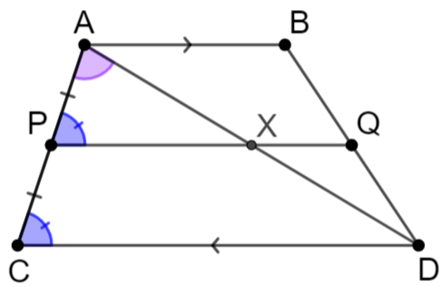
3.32: Median: Midpoint of Diagonals

The median of a trapezoid passes through the midpoints of its diagonals.
In the diagram,

$$PQ\ passes\ through\ Y\ and\ X$$

Where

$$Y\ is\ midpoint\ of\ CB$$
$$X\ is\ midpoint\ of\ AD$$



Draw trapezoid $ABDC$ with median:

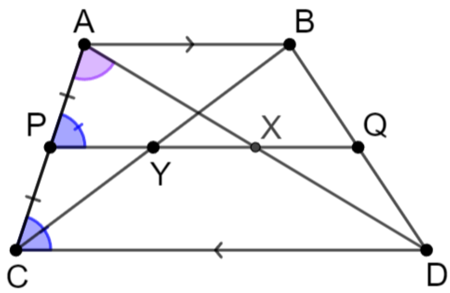
$$PQ \parallel AB \parallel CD$$

Since a parallel line divides the sides of a triangle in the same ratio:

$$\frac{AX}{XD} = \frac{AP}{PC} = 1$$

Hence,

$$X\ is\ the\ midpoint\ of\ Diagonal\ AD$$



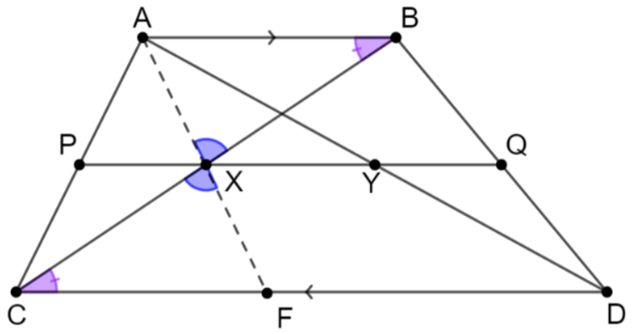
That PQ passes through the midpoint Y of the other diagonal can be proved similarly.

3.33: Midpoint of Diagonals

The line joining the mid points of the diagonals of a trapezoid is equal to half of the difference of the parallel sides.

In the diagram:

$$XY = \frac{1}{2}(CD - AB)$$



Draw trapezoid $ABDC$ with midpoints of diagonals X and Y as shown.
Let P and Q be the midpoints of AC and BD respectively.
From above, we know that the median PQ passes through X and Y.

Construction:
Extend AX till it intersects CD at F.

$\Delta BXA \cong \Delta FXC$ by ASA:

$$\angle ABC = \angle BCD \text{ (Alternate Interior Angles)}$$
$$\angle BXA = \angle FXC \text{ (Vertically Opposite Angles)}$$
$$BX = XC \text{ (X is midpoint of diagonal BC)}$$

Hence, by CPCT:

$$\underbrace{AX = XF}_{\text{Equation I}}, \qquad \underbrace{CF = AB}_{\text{Equation II}}$$

From the first equality above, X and Y are midpoints of AF and AD respectively.
Also $XY \parallel FD$ since $PQ \parallel CD$.
Therefore, by Midpoint Theorem:

$$XY = \frac{1}{2}FD$$

By complementary lengths, $FD = CD - CF$

$$XY = \frac{1}{2}(CD - CF)$$

From Equation II, $CF = AB$

$$XY = \frac{1}{2}(CD - AB)$$

Which is what we wanted to prove.

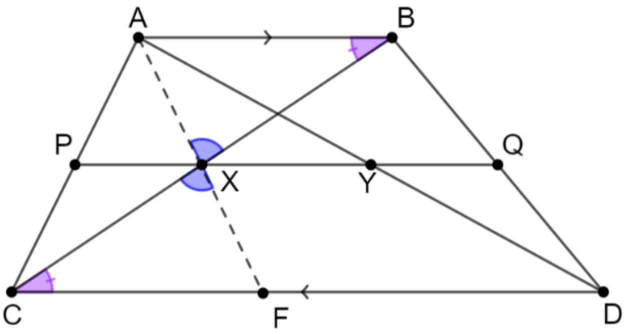
Example 3.34

The line joining the midpoints of the diagonals of a trapezoid has length 3. If the longer base is 97, then the shorter base is: (AHSME 1959/22)

$$XY = \frac{1}{2}(CD - AB)$$

Substitute $XY = 3, CD = 97$

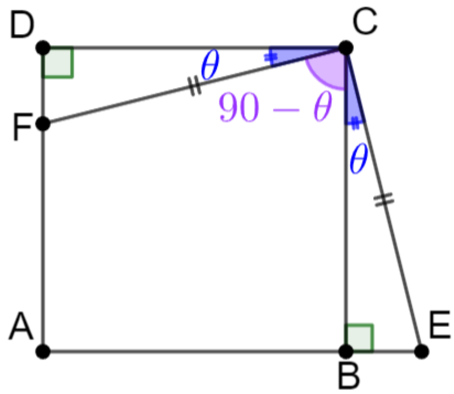
$$3 = \frac{1}{2}(97 - AB)$$
$$6 = 97 - AB$$
$$AB = 97 - 6 = 91$$



C. Squares

Example 3.35

Point F is taken in side AD of square $ABCD$. At C a perpendicular is drawn to CF, meeting AB extended at E. The area of $ABCD$ is 256 square inches and the area of ΔCEF is 200 square inches. Then the number of inches in BE is: (AHSME 1963/25)



Let

$$\angle DCF = \theta \Rightarrow \angle FCB = 90 - \angle DCF = 90 - \theta$$

And then

$$\angle BEC = 90 - (90 - \theta) = \theta$$

$\Delta DCF \cong \Delta BCE$ by ASA congruence since

$$BC = CD = \sqrt{256} = 16$$
$$\angle DCF = \angle BEC = \theta$$
$$\angle CDF = \angle CBE = 90°$$

Hence:

$$CF = CE\ (CPCT)$$

$$[CEF] = \frac{1}{2}hb = \frac{1}{2}(CF)(CE) = \frac{1}{2}CE^2 = 200 \Rightarrow CE = 20$$
$$BC = 16$$

By Pythagorean Triplet:

$$(BE, CB, CE) = (BE, 16, 20) = 4\left(\frac{BE}{4}, 4, 5\right) = 4(3,4,5) = (12,16,20)$$
$$BE = 12$$

D. Rectangles

Example 3.36

$ABCD$ is a rectangle with P any point on AB. $PS \perp BD$ and $PR \perp AC$. $AF \perp BD$ and $PQ \perp AF$. Then $PR + PS$ is equal to: (AHSME 1958/47)

A. PQ
B. AE
C. $PT + AT$
D. AF
E. EF

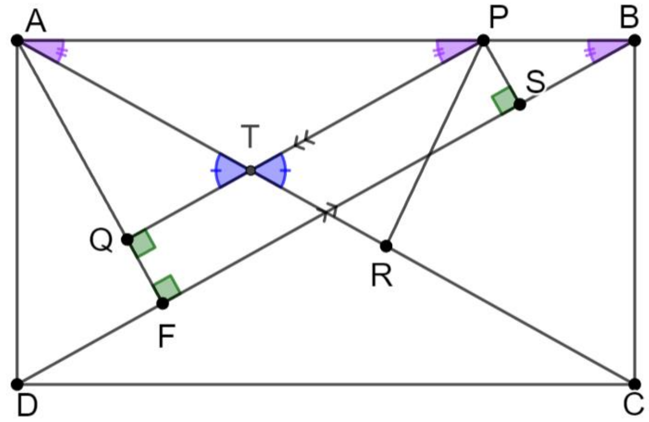
Step 1: Use Parallel Lines

$PQ \parallel BD$ since corresponding angles are equal:

$$PQ \perp AF, \qquad AF \perp BD$$

AB is a transversal of parallel lines PQ and BD:

$$\underbrace{\angle APQ = \angle ABD}_{I} \ (Corresponding\ Angles)$$



Step 2: $\Delta ABC \cong \Delta BAD$ by RHS/Hy-Leg since

$$\angle ABC = \angle BAD = 90° \ (Rectangle)$$
$$AB = AB \ (Common\ side)$$
$$BC = AD \ (Rectangle)$$
$$\therefore \underbrace{\angle BAC = \angle ABD}_{II} \ (CPTC)$$

From I and II:

$$\angle APQ = \angle BAC \Rightarrow \Delta APT\ is\ isosceles \Rightarrow \underbrace{TA = TP}_{III}$$

Step 3: $\Delta ATQ \cong \Delta PTR$ by RHS/Hy-Leg since

$$\angle AQT = \angle PRT = 90° \ (Given)$$
$$\angle ATQ = \angle PTR \ (Vertically\ Opposite\ Angles)$$
$$TA = TP \ (From\ III)$$
$$\therefore \underbrace{AQ = PR}_{IV} \ (CPCT)$$

Step 4: PSFQ is a rectangle since

$$PS \perp BD, BD \perp AF, PQ \perp AF$$
$$\therefore \underbrace{QF = PS}_{V}$$

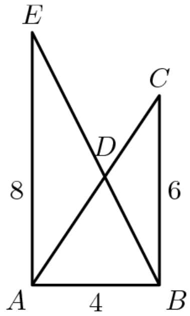
From IV and V:

$$PR + PS = AQ + QF = AF$$
$$Option\ D$$

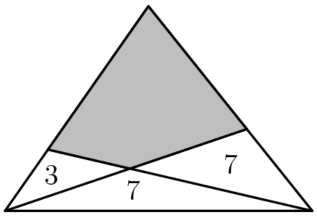
4. 3D Similarity

4.1 Further Topics

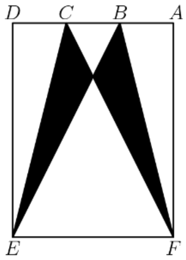
A. Further Exam Questions



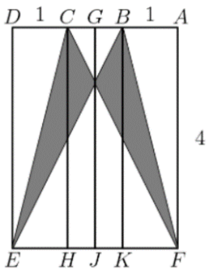
Example 4.1
In the figure, $\angle EAB$ and $\angle ABC$ are right angles. $AB = 4, BC = 6, AE = 8$, and AC and BE intersect at D. What is the difference between the areas of $\triangle ADE$ and $\triangle BDC$? **(AMC 10A 2004/9)**



Example 4.2
A triangle is partitioned into three triangles and a quadrilateral by drawing two lines from vertices to their opposite sides. The areas of the three triangles are 3, 7, and 7 as shown. What is the area of the shaded quadrilateral? **(AMC 10B 2006/23)**



Example 4.3
Rectangle $DEFA$ below is a 3×4 rectangle with $DC = CB = BA = 1$. The area of the "bat wings" (shaded area) is (AMC 8 2016/22)



Method I: Similarity
Trapezoid CDEF

4 Examples