Integration

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1. General Resources

1.1 Videos

A. Comprehensive

Example 1.1
Blackpenredpen comprehensive video on Integration

2. Integration (AB)

2.1 Indefinite Integration

A. Antiderivatives

You differentiate a function to find its derivative. If you reverse the process, you are finding an anti-derivative.

2.1: Antiderivative

$F(x)$ is an antiderivative of $f(x)$ if:

$$F'(x) = f(x)$$

Example 2.2

Find an antiderivative for:

$$f(x) = 5$$

$$f(x) = 5 \Rightarrow F(x) = 5x$$

B. Integration

Example 2.3

If $G(x)$ is an antiderivative of $f(x)$, and $F(x)$ is also an antiderivative of $f(x)$ then what is the connection between the two?

$$\int f(x)\,dx = F(x) \Rightarrow F'(x) = f(x)$$
$$\int f(x)\,dx = G(x) \Rightarrow G'(x) = f(x)$$

$$F'(x) = G'(x) = f(x)$$

Since the two functions have the same derivative, they can only differ by a constant.

$$F(x) = G(x) + C, for\ some\ constant\ C$$

2.4: Integration

The collection of all antiderivatives of a function $f(x)$ is called the indefinite integral of $f(x)$ and is written

$$\int f(x)\,dx = F(x) + C$$

Where C is a $constant\ of\ integration$

- The symbol dx is used to indicate that we are finding the antiderivative with respect to x.

Example 2.5

A. Evaluate $\int \frac{2}{7}dx$
B. In the integral that you evaluated above, what are the values that C can take?

Part A

$$\int \frac{2}{7}dx = \frac{2}{7}x + C$$

Part B

$$C \in (-\infty, \infty)\ OR\ C \in \mathbb{R}$$

2.6: Constant Multiple Rule for Integration

For any constant k:

$$\int k \cdot f(x)\,dx = k \int f(x)\,dx$$

- We can generalize the two examples we have been doing so far in the property above. It lets us move constants "out" of the integration symbol.
- This property is directly related to the constant multiple property for derivatives.

Example 2.7

Find the most general indefinite integral for the exercises below. Do not forget the constant of integration.

$$\int -\pi\,dx$$

Constants

Simply putting an x before the given number, and adding a constant of integration works:

$$\int -\pi\,dx = -\pi x + C$$

C. Power Rule (Algebraic Integrals)

2.8: Power Rule for Integration

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$$

We can check this by differentiation:

$$\left(\frac{x^{n+1}}{n+1}\right)' = \frac{n+1}{n+1}x^{n+1-1} = x^n$$

The $n = -1$ case does not work since:

$$\int \frac{1}{x}dx = \ln\ |x|$$

Example 2.9: Integer Exponents

Find the most general indefinite integral for the exercises below. Do not forget the constant of integration.

$$\int x^3\,dx$$

$$\int 5x^2\,dx$$

$$\int -\frac{4}{3}x^7\,dx$$

Here, the formula given for the power rule is useful:

$$\int x^3\,dx = \frac{x^{3+1}}{3+1} = \frac{x^4}{4} + C$$

$$\int 5x^2\,dx = 5 \cdot \frac{x^{2+1}}{2+1} = \left(\frac{5}{3}\right)x^3 + C$$

$$\left(-\frac{4}{3}\right)\frac{x^{7+1}}{7+1} = \left(-\frac{4}{3}\right)\frac{x^8}{8} = -\frac{x^8}{6} + C$$

Example 2.10:
Evaluate, if possible, using the power rule. Explain the answer that you get

$$\int x^{-1}\,dx$$

$$\int x^{-1}\,dx \;= \frac{x^0}{0} \Rightarrow Not\ defined$$

In the power rule,

$$n \neq -1\ for\ precisely\ this\ reason$$

In fact:

$$\int x^{-1}\,dx \;= \int \frac{1}{x}\,dx \;= \ln|x| + C$$

2.11: Sum and Difference Property

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$
$$\int f(x) - g(x)\,dx = \int f(x)\,dx - \int g(x)\,dx$$

Example 2.12: Fractional Exponents

$$\int -x^{\frac{1}{3}} + \frac{2}{3}x^{\frac{1}{2}}\,dx$$

The same power rule formula applies, but since we have fractions, we need to be more careful with the calculations:

$$\int -x^{\frac{1}{3}}\,dx = -\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} = -\frac{x^{\frac{4}{3}}}{\frac{4}{3}} = -\frac{3}{4}x^{\frac{4}{3}}$$
$$\int \frac{2}{3}x^{\frac{1}{2}}\,dx = \left(\frac{2}{3}\right)\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \left(\frac{2}{3}\right)\frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{4}{9}x^{\frac{3}{2}}$$

Hence, the final answer is (remember to add the constant of integration):

$$\int -x^{\frac{1}{3}} + \frac{2}{3}x^{\frac{1}{2}}\,dx = \frac{4}{9}x^{\frac{3}{2}} - \frac{3}{4}x^{\frac{4}{3}} + C$$

Example 2.13: Radicals

$$\int \left(\sqrt{x} - \frac{7}{9}\sqrt[3]{x}\right)dx$$

Split the integral using the sum and difference rule:

$$\int \sqrt{x}\,dx + \int -\frac{7}{9}\sqrt[3]{x}\,dx$$

In the first term, rewrite the radical as an exponent, and then apply the power rule.

$$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x^{\frac{3}{2}} + C_1$$

In the second term, rewrite the radical $\sqrt[3]{x} = x^{\frac{1}{3}}$:

$$\int -\frac{7}{9}\sqrt[3]{x}\,dx = \left(-\frac{7}{9}\right)\left(\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right) = \left(-\frac{7}{9}\right)\left(\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right) = \left(-\frac{7}{9}\right)\left(\frac{3}{4}\right)x^{\frac{4}{3}} = \left(-\frac{7}{12}\right)\left(x^{\frac{4}{3}}\right) + C_2$$

Combine the two to get the final answer:

$$\frac{2}{3}x^{\frac{3}{2}} - \left(\frac{7}{12}\right)\left(x^{\frac{4}{3}}\right) + C$$

Example 2.14: Negative Exponents

$$\int \left(\frac{1}{x^2} + \frac{3}{x^3}\right) dx$$

$$\int \frac{1}{x^2}\,dx + \int \frac{3}{x^3}\,dx$$

If a power of x is in the denominator, move the variable to the numerator by using the property $\frac{1}{a^m} = a^{-m}$.

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x} + C_1$$

$$\int \frac{3}{x^3}\,dx = \int 3x^{-3}\,dx = \left(\frac{3}{-2}\right)x^{-2} + C_2$$

Combine the two to get the final answer:

$$= -\frac{1}{x} - \left(\frac{3}{2x^2}\right) + C$$

Example 2.15: Negative Fractional Exponents

$$\int \frac{1}{x^{\frac{1}{2}}} + \frac{3}{4x^{\frac{2}{3}}}\,dx$$

$$\int \frac{1}{x^{\frac{1}{2}}}\,dx = \int x^{-\frac{1}{2}}\,dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{x^{\frac{1}{2}+1}}{-\frac{1}{2}} = 2\sqrt{x}$$

$$\int \frac{3}{4x^{\frac{2}{3}}}\,dx = \int \left(\frac{3}{4}\right)x^{-\frac{2}{3}}\,dx = \frac{\left(\frac{3}{4}\right)x^{\frac{1}{3}}}{\frac{1}{3}} + C = \frac{9\sqrt[3]{x}}{4} + C$$

$$\int \frac{1}{x^{\frac{1}{2}}} + \frac{3}{4x^{\frac{2}{3}}}\,dx = 2\sqrt{x} + \frac{9\sqrt[3]{x}}{4} + C$$

Example 2.16: Radicals in the denominator

$$\int \frac{1}{\sqrt[3]{x}}\,dx$$

Rewrite the radical as an exponent, and then integrate:

$$\int \frac{1}{\sqrt[3]{x}} dx = \int \frac{1}{x^{\frac{1}{3}}} dx = \int x^{-\frac{1}{3}} dx = \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + C = \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + C = \frac{3}{2} x^{\frac{2}{3}} + C$$

D. Logarithmic and Exponential Integrals

2.17: Integral of $\frac{1}{x}$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$$

- Pay attention to the absolute value sign in the integral, which is important since $\ln x$ is not defined for $x < 0$.

Example 2.18

$$\int -\frac{1}{x} dx$$

$$\int -\frac{1}{x} dx = -\ln|x| + C$$

Use the power rule in logarithms $\ln x^n = n \ln x$:

$$\ln|x^{-1}| + C = \ln\left|\frac{1}{x}\right| + C$$

Example 2.19

$$\int \frac{2}{x} dx$$
$$\int \frac{a}{x} dx$$
$$\int \frac{1}{2x} dx$$

$$\int \frac{2}{x} dx = 2\int \frac{1}{x} dx = 2\ln|x| = \ln|x^2| = \ln x^2 + C$$
$$\int \frac{a}{x} dx = a\int \frac{1}{x} dx = a\ln|x| = \ln|x^a| + C$$
$$\int \frac{1}{2x} dx = \frac{1}{2}\int \frac{1}{x} dx = \frac{1}{2}\ln|x|x = \ln\left|x^{\frac{1}{2}}\right| + C$$

Example 2.20

$$\int \frac{1}{ax} + \frac{5}{3x} + \frac{a}{bx} dx$$

Split the Integral

$$\frac{1}{a}\int \frac{1}{x} dx + \frac{5}{3}\int \frac{1}{x} dx + \frac{a}{b}\int \frac{1}{x} dx$$

Integrate:

$$\frac{1}{a}\ln x + \frac{5}{3}\ln x + \frac{a}{b}\ln x + C$$

Use the power rule for logarithms:

$$\ln x^{\frac{1}{a}} + \ln x^{\frac{5}{3}} + \ln x^{\frac{a}{b}} + C$$

2.21: Integral of $\frac{f'(x)}{f(x)}$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$

$$y = \ln[f(x)] \Rightarrow \frac{dy}{dx} = \frac{1}{f(x)} \times f'(x) = \frac{f'(x)}{f(x)}$$

$$\int \frac{2x}{x^2}dx = \ln x^2 + C$$

Example 2.22

$$\int \frac{3x+1}{6x^2 - 13x - 5}dx$$

Factor the denominator:

$$Product = 6(-5) = -30, Sum = -15 + 2 = -13$$
$$6x^2 - 15x + 2x - 5 \ = 3x(2x-5) + 1(2x-5) = (3x+1)(2x-5)$$

Substitute the factored version in the denominator:

$$\int \frac{3x+1}{(3x+1)(2x-5)}dx = \int \frac{1}{2x-5}dx$$

Use the constant multiple rule:

$$\frac{1}{2}\int \frac{2}{2x-5}dx = \frac{\ln|2x-5|}{2} + C$$

Example 2.23: Simplification

$$\frac{e^3}{-2} - \frac{e^{-1}}{-2} = \frac{e^3}{-2} - \frac{1}{-2e} = \frac{e^4}{-2e} - \frac{1}{-2e} = \frac{e^4 - 1}{-2e}$$
$$\frac{e^3}{-2} - \frac{e^{-1}}{-2} = \frac{e^3 - e^{-1}}{-2} = \frac{e^3 - \frac{1}{e}}{-2} = \frac{\frac{e^4-1}{e}}{-2} = \frac{e^4-1}{e} \times \frac{1}{-2} = \frac{e^4-1}{-2e}$$

E. Some Further Questions

Example 2.24

$$\int |x|\,dx$$

The absolute value function is defined piece-wise to be:

$$|x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

We integrate it using cases.

For $x \geq 0$:

$$\int |x|\, dx = \int x\, dx = \frac{x^2}{2} + C$$

For $x < 0$:

$$\int |x|\, dx = \int -x\, dx = -\frac{x^2}{2} + C$$

F. Initial Value Problems

So far we have not paid any attention to the constant of integration. If we have some further data related to the question, we can get the value of the constant of integration. Such questions are called initial value problems.

Example 2.25

Determine $g(0)$ given that:

$$f(x) = \int x\, dx, \qquad f(2) = 4, \qquad g(x) = \int f(x)\, dx, \qquad g(1) = 1$$

Find $f(x)$

Integrate the first equation to find an expression for $f(x)$:

$$f(x) = \int x\, dx = \frac{x^2}{2} + C_1$$

To find the value of C_1, use the data given that $f(2) = 4$ and substitute it in $\frac{x^2}{2} + C_1$:

$$f(2) = 4 \Rightarrow \frac{2^2}{2} + C_1 = 4 \Rightarrow C_1 = 2$$

Find $g(x)$

Now that we have the explicit definition for $f(x)$, we can integrate it to find $g(x)$:

$$g(x) = \int f(x)\, dx = \int \frac{x^2}{2} + 2\, dx = \frac{1}{2}\frac{x^3}{3} + 2x + C_2 = \frac{x^3}{6} + 2x + C_2$$

To find the value of C_2, use the data given that $g(1) = 1$ and substitute it in $\frac{x^3}{6} + 2x + C_2$:

$$g(1) = 1 \Rightarrow \frac{1}{6} + 2 + C_2 = 1 \Rightarrow C_2 = -\frac{7}{6}$$

$$g(x) = \frac{x^3}{6} + 2x - \frac{7}{6}$$

Find $g(0)$

Substitute $x = 0$ in the above to determine the value asked in the question:

$$g(0) = -\frac{7}{6}$$

Example 2.26

Find $y(2)$ given that $\frac{dy}{dx} = 5x - 2,\ y(3) = 1$.

Integrate both sides of $\frac{dy}{dx} = 5x - 2$ with respect to x:

$$\int \frac{dy}{dx}\, dx = \int 5x - 2\, dx$$

$$\int 1\,dy = \frac{5x^2}{2} - 2x + C$$
$$y = \frac{5x^2}{2} - 2x + C$$

Substitute $y = 3, x = 1$, and solve for C:

$$3 = \frac{5(1)^2}{2} - 2(1) + C \Rightarrow 3 = \frac{1}{2} + C \Rightarrow C = \frac{5}{2}$$

Evaluate $y(2)$:

$$y(2) = \frac{5x^2}{2} - 2x + \frac{4}{2} = \frac{5(2)^2}{2} - 2(2) + \frac{5}{2} =$$

G. Trigonometric Antiderivatives

2.27: Trigonometric Derivatives: Summary

$$\frac{d}{dx}(\sin x) = \cos x\,, \qquad \frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x\,, \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x\,, \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

- The above are the standard forms of the derivatives of the trigonometric functions.

Note that the functions on the right (the co-functions) each have a negative sign in their derivative.

Example 2.28: Basic Formulas

$$\int \sec^2 x + \sec x \tan x + \csc x \cot x \;\, dx$$

$$\tan x + \sec x - \csc x + C$$

Example 2.29: Constant Multiple Rule

$$\int 2\sin\theta + \frac{\cos\theta}{2}\;d\theta$$

Split the integral:

$$\int 2\sin\theta\,d\theta + \int \frac{\cos\theta}{2}\;d\theta$$

Use the constant multiple rule:

$$2\int \sin\theta\,d\theta + \frac{1}{2}\int \cos\theta\;\,d\theta$$

Integrate:

$$-2\cos\theta\; + \frac{1}{2}\sin\theta + C$$

Rewrite:

$$\frac{1}{2}\sin\theta - 2\cos\theta + C$$

Example 2.30: Splitting Fractions

$$\int \frac{1+\sin^2 x}{\sin^2 x}\,dx$$

$$\int \left(\frac{1}{\sin^2 x}+1\right)dx = \int (\csc^2 x + 1)\,dx = -\cot x + x + C$$

Example 2.31: Trig Identities

$$\int \frac{\sin^2 x + \cos^2 x}{\cos^2 x}\,dx$$

Substitute $\sin^2 x + \cos^2 x = 1$:

$$\int \frac{1}{\cos^2 x}\,dx = \int \sec^2 x\,dx = \tan x + C$$

$$\int 1+\tan^2\theta\,d\theta$$

$$\int 1+\tan^2\theta\,d\theta = \int \sec^2\theta\,d\theta = \tan\theta + C$$

Example 2.32: Simplification

$$\int \frac{\csc x}{\csc x - \sin x}\,dx$$

Simplify the integrand:

$$\frac{\frac{1}{\sin x}}{\frac{1}{\sin x}-\sin x} = \frac{\frac{1}{\sin x}}{\frac{1-\sin^2 x}{\sin x}} = \frac{1}{1-\sin^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\int \frac{\csc x}{\csc x - \sin x}\,dx = \int \sec^2 x\,dx = \tan x + C$$

$$\int \sin x\,(\cot x + \csc x + 1)\,dx$$

Multiplying:

$$\int \sin x \cot x + \sin x \csc x + \sin x\,dx$$

$$\int \sin x \frac{\cos x}{\sin x} + \sin x \frac{1}{\sin x} + \sin x\,dx$$

Simplify:

$$\int \cos x \ + 1 + \sin x\,dx$$

Integrate:

$$\sin x + x - \cos x + C$$

$$\int \frac{\tan^2 x + \sin^2 x}{\sin^2 x}\,dx$$

$$\int \frac{\tan^2 x}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x} dx$$
$$\int \frac{\frac{\sin^2 x}{\cos^2 x}}{\sin^2 x} + 1\, dx$$

$$\int \frac{\sin^2 x}{\cos^2 x} \times \frac{1}{\sin^2 x} + 1\, dx$$
$$\int \sec^2 x + 1\, dx$$
$$\tan x + x + C$$

H. Inverse Trigonometric Functions

2.33: Summary of Inverse Trigonometric Derivatives

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}, \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

- All of the cofunction derivatives have a minus sign in front, and the expression is otherwise the same as the corresponding function.

Example 2.34

$$\int \frac{1}{\sqrt{2-x^2}} dx$$

$$\int \frac{1}{\sqrt{2\left(1-\frac{x^2}{2}\right)}} dx = \int \frac{1}{\sqrt{2}\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}} dx = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}} dx$$

$$\frac{d}{dx}\left(\sin^{-1}\frac{x}{\sqrt{2}}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}} \times \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}} = \frac{1}{\sqrt{2\left(1-\frac{x^2}{2}\right)}} = \frac{1}{\sqrt{2-x^2}}$$

I. Preparing for u substitution

In the next section, we will introduce u-substitution, which lets us run the chain rule in reverse. However, we look at the idea by doing some questions with trial and error.

2.35: Constant Multiple Rule

$$\int cf(x)\, dx = c \int f(x)\, dx$$

- We can use the constant multiple to change our guess for an integral if it only differs by a constant multiple.

Example 2.36

$$\int \sin 2x \, dx$$

We know that:

$$\int \sin x \, dx = -\cos x + C$$

Suppose we guess the integral as $-\cos 2x$

$$\frac{d}{dx}(-\cos 2x) = \sin(2x) \times 2$$

This is exactly what we wanted, except that it is double of what we need. Hence, we divide both sides of the above by 2:

$$\frac{d}{dx}\frac{(-\cos 2x)}{2} = \frac{\sin(2x) \times 2}{2} = \sin(2x)$$

Hence:

$$\int \sin 2x \, dx = \frac{(-\cos 2x)}{2} + C$$

Example 2.37

$$\int \sec^2 \frac{x}{2} dx$$

$$\int \sec^2 \frac{x}{2} dx = 2\tan \frac{x}{2} + C$$

$$\int \sec \pi x \tan \pi x \, dx$$

$$\int \sec \pi x \tan \pi x \, dx = \frac{\sec \pi x}{\pi} + C$$

$$\int \frac{2}{3} \cos \frac{ex}{\pi} \, dx$$

$$\int \frac{2}{3} \cos \frac{ex}{\pi} \, dx = \frac{2}{3} \int \cos \frac{ex}{\pi} \, dx = \left(\frac{2}{3}\right)\left(\frac{\pi}{e}\right) \sin \frac{ex}{\pi}$$

Example 2.38

$$\int (2x+3)^2 \, dx$$

Imagine that

$$u = 2x + 3 \Rightarrow u^2 = (2x+3)^2 \Rightarrow \int u^2 \, du = \frac{u^3}{3} + C$$

Based on the above, we can guess that the integral we want is:

$$\frac{u^3}{3} = \frac{(2x+3)^3}{3}$$

But, when we differentiate the above, we get:

$$\frac{d}{dx}\frac{(2x+3)^3}{3} = \frac{3(2x+3)^2}{3} \times 2 = 2(2x+3)^2$$

Divide the above by 2:

$$\frac{\frac{d}{dx}\frac{(2x+3)^3}{3}}{2} = \frac{2(2x+3)^2}{2}$$

Simplify:

$$\frac{d}{dx}\frac{(2x+3)^3}{6} = (2x+3)^2$$

Hence:

$$\int (2x+3)^2\,dx =$$

Example 2.39

$$\int \left(\cos\frac{x}{\pi} + \sin\pi x + \sec^2\frac{x}{2}\right) dx$$

We use a little trial and error

$$Try\,\frac{d}{dx}\left(\sin\frac{x}{\pi}\right) = \cos\frac{x}{\pi} \times \frac{1}{\pi}$$

The answer is off by a multiplicative factor of $\frac{1}{\pi}$. To balance it out, we change our original guess to:

$$\pi\sin\frac{x}{\pi}$$

$$\pi\sin\frac{x}{\pi} - \frac{\cos\pi x}{\pi} + 2\tan\frac{x}{2} + C$$

2.2 *u*-substitution

A. Running the Chain Rule in Reverse

2.40: Integration by Substitution

$$\int f\big(g(x)\big)\,g'(x)\,dx = \int f(u)\,u'dx = \int f(u)\,du$$

Integration by substitution is based on the using the chain rule in reverse. In the above expression, in the leftmost integral, let

$$u = g(x) \Rightarrow \frac{du}{dx} = g'(x) \Rightarrow du = g'(x)\,dx = u'\,dx$$

And then we make the substitution above.

Example 2.41

Use the power rule for integration $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$ to integrate:

$$\int 2(2x+4)^3\,dx$$

Use a change of variable. Decide the substitution:

$$u = 2x+4 \Rightarrow \frac{du}{dx} = 2 \Rightarrow du = 2\,dx$$

Make the substitution:

$$\int \underbrace{(2x+4)^3}_{u^3} \underbrace{2\,dx}_{du} = \int u^3\,du$$

Integrate:

$$\frac{u^4}{4} + C$$

Change back to the original variable:

$$= \frac{(2x+4)^4}{4} + C$$

2.42: Creating the Pattern

If the expression that we have differs from the pattern we need by only a constant, then we can use the property

$$\int f(x)\,dx = \frac{1}{p}\int p\cdot f(x)\,dx$$

Note:

- The chain rule is useful when the derivative of u is available to us in the integrand.
- Hence, the choice of u is critical when using a u-subsitution.

Example 2.43: Creating the pattern

$$\int (5x+2)^9\,dx$$

Use the substitution:

$$u = 5x + 2 \Rightarrow \frac{du}{dx} = 5 \Rightarrow \frac{du}{5} = dx$$

$$\int \underbrace{(5x+2)^9}_{u^9} \underbrace{dx}_{\frac{du}{5}} = \frac{1}{5}\int u^9\,du = \underbrace{\frac{1}{5}\left(\frac{u^{10}}{10}\right) + C}_{\text{Integrate}} = \underbrace{\frac{(5x+2)^{10}}{50} + C}_{\text{Change back to original variable}}$$

Example 2.44: Creating the pattern

A. $\int \frac{\sin 2x}{2}\,dx$
B. $\int \frac{\sec^2 \pi x}{e}\,dx$
C. $\int \cos(\pi x + e)\,dx$
D. $\int (5x^4+4)(x^5+4x)^8\,dx$
E. $\int x(3x^2+4)^3\,dx$

Part A

Use the substitution $u = 2x \Rightarrow \frac{du}{2} = dx$:

$$\int \frac{\sin 2x}{2}\,dx = \frac{1}{2}\int \sin u \frac{du}{2} = \frac{-\cos u}{4} + C = \frac{-\cos 2x}{4} + C$$

Part B

Use the substitution $u = \pi x \Rightarrow du = \pi\,dx$ to get:

$$\int \frac{\sec^2 \pi x}{e}\cdot\frac{1}{\pi}\cdot\pi\,dx = \frac{1}{e\pi}\int \sec^2 u\,du = \frac{\tan u}{e\pi} + C = \frac{\tan \pi x}{e\pi} + C$$

Part C
Substitute $u = \pi x + e \Rightarrow du = \pi\, dx$:
$$\int \cos(\pi x + e)\, dx = \frac{1}{\pi}\int \cos u\, du = \frac{1}{\pi}\sin u + C = \frac{1}{\pi}\sin(\pi x + e) + C$$

Part D
Substitute $u = x^5 + 4x \Rightarrow du = 5x^4 + 4\, dx$:
$$\int \Big(\underbrace{x^5 + 4x}_{u}\Big)^8 \underbrace{(5x^4 + 4)\, dx}_{du} = \int u^8\, du = \frac{u^9}{9} + C = \frac{(x^5 + 4x)^9}{9} + C$$

Part E
Substitute $u = 3x^2 + 4 \Rightarrow du = 6x\, dx$:
$$\frac{1}{6}\int \Big(\underbrace{3x^2 + 4}_{u}\Big)^3 \underbrace{6x\, dx}_{du} = \frac{1}{6}\int u^3\, du = \frac{1}{6}\cdot\frac{u^4}{4} + C = \frac{(3x^2 + 4)^4}{24} + C$$

Example 2.45: Trigonometric Expressions
$$\int (x + 1)\sin(x^2 + 2x)\, dx$$

Let $u = x^2 + 2x \Rightarrow du = 2x + 2\, dx = 2(x + 1)\, dx$
$$\frac{1}{2}\int \underbrace{\sin(x^2 + 2x)}_{\sin u}\underbrace{2(x + 1)\, dx}_{du} = \frac{1}{2}\int \sin u\, du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos(x^2 + 2x) + C$$

2.46: Square Root in the Denominator
$$\int \sqrt{u}\, u'\, dx = \int \sqrt{u}\, du = \frac{2}{3}u^{\frac{3}{2}} + C$$

Example 2.47
A. $\int \cos x \sqrt{\sin x}\, dx$
B. $\int x^2\sqrt{3x^3 + 5}\, dx$

Part A
Let $u = \sin x \Rightarrow du = \cos x\, dx$
$$\int \cos x \sqrt{\sin x}\, dx = \int \sqrt{u}\, du = \int u^{\frac{1}{2}}\, du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}\sin^{\frac{3}{2}} x + C$$

Part B
Let $u = 3x^3 + 5 \Rightarrow du = 9x^2\, dx$
$$\frac{1}{9}\int 9x^2\sqrt{3x^3 + 5}\, dx = \frac{1}{9}\int \sqrt{u}\, du = \frac{1}{9}\times\frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{27}(3x^3 + 5)^{\frac{3}{2}} + C$$

2.48: Square Root in the Denominator
$$\int \frac{u'}{\sqrt{u}}\, dx = \int \frac{1}{\sqrt{u}}\, du = 2\sqrt{u}$$

2.49: Hanging Expressions
Sometimes you may be left over with a "hanging expression" even after the substitution.

This can be sometimes handled by substituting for that expression in terms of u.

Example 2.50

$$\int \frac{x^3}{\sqrt{1-2x^2}}\,dx$$

$$u = 1-2x^2 \Rightarrow du = -4x\,dx$$
$$x^2 = \frac{1-u}{2}$$

$$-\frac{1}{4}\int \frac{x^2}{\sqrt{1-2x^2}}(-4x\,dx)$$

Carry out the substitution:

$$-\frac{1}{4}\int \frac{\frac{1-u}{2}}{\sqrt{u}}du = \frac{1}{8}\int \frac{u-1}{\sqrt{u}}du$$

$$= \frac{1}{8}\int \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)du$$
$$= \frac{1}{8}\left(\frac{2}{3}u^{\frac{3}{2}} - 2\sqrt{u}\right) + C$$
$$= \frac{1}{12}(1-2x^2)^{\frac{3}{2}} - \frac{1}{4}\sqrt{1-2x^2} + C$$

Example 2.51: Trigonometric Expressions

A. $\int \sin^3 x \cos x\,dx$
B. $\int \tan^6 x \cdot \sec^2 x\,dx$
C. $\int \sin^5\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right)dx$ **(CBSE 2020)**

Part A
Let $u = \sin x \Rightarrow du = \cos x\ \ dx$

$$\int \sin^3 x \cos x\,dx = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{\sin^4 x}{4} + C$$

Part C
Let $t = \tan x \Rightarrow dt = \sec^2 x\,dx$

$$\int \tan^6 x \cdot \sec^2 x\,dx = \int t^6 dt = \frac{t^7}{7} + C = \frac{\tan^7 x}{7} + C$$

Part D
Let $\sin\frac{x}{2} = t \Rightarrow dt = \frac{1}{2}\cos\frac{x}{2}dx$:

$$\int \sin^5\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right)dx = 2\int t^5 dt = 2\cdot\frac{t^6}{6} + C = \frac{1}{3}\left(\sin^6\frac{x}{2}\right) + C$$

B. Exponentials and Logarithms

Example 2.52: Exponentials

Use the pattern $\int u' e^u\,dx = \int e^u\,du = e^u + C$

A. $\int \cos x\, e^{\sin x}\,dx$

B. $\int e^{4x}\,dx$
C. $\int \frac{e^{\sqrt{\pi x}}}{\sqrt{ex}}\,dx$
D. $\int \frac{\sin x e^{\sqrt{\cos x}}}{\sqrt{\cos x}}\,dx$
E. $\int \left[\left(e^{2x}+\frac{1}{2}\right)\sec(e^{2x}+x+1)\tan(e^{2x}+x+1)\right]dx$

Part A

Let $u=\sin x \Rightarrow \frac{du}{dx}=\cos x \Rightarrow du=\cos x\;\,dx$:

$$\int \cos x\, e^{\sin x}\,dx=\int e^u\,du=e^u+C=e^{\sin x}+C$$

Part B

Let $u=4x \Rightarrow \frac{du}{dx}=4 \Rightarrow du=4\,dx$

$$\int e^{4x}\,dx=\frac{1}{4}\int \mathbf{4}\times e^{4x}\,\boldsymbol{dx}=\frac{1}{4}\int e^u\,du=\frac{1}{4}e^u+C=\frac{1}{4}e^{4x}+C$$

Part C

Let $u=\sqrt{\pi x}$

$$\Rightarrow du=\frac{\pi}{2}\cdot\frac{1}{\sqrt{\pi x}}dx=\frac{\sqrt{\pi}}{2\sqrt{x}}dx=\left(\frac{\sqrt{\pi}}{2}\right)\left(\frac{1}{\sqrt{x}}\right)dx$$

$$\int \frac{e^{\sqrt{\pi x}}}{\sqrt{ex}}dx=\frac{2}{\sqrt{e\pi}}\int\frac{\sqrt{\pi}e^{\sqrt{\pi x}}}{2\sqrt{x}}dx=\frac{2}{\sqrt{e\pi}}\int e^u\,du=\frac{2}{\sqrt{e\pi}}e^u+C=\frac{2e^{\sqrt{\pi x}}}{\sqrt{e\pi}}+C$$

Part D

Substitute $u=\sqrt{\cos x}\Rightarrow du=-\frac{\sin x}{2\sqrt{\cos x}}dx$:

$$-2\int -\frac{\sin x\, e^{\sqrt{\cos x}}}{2\sqrt{\cos x}}dx=-2\int e^u\,du=-2e^u+C=-2e^{\sqrt{\cos x}}+C$$

Part E

Let $u=e^{2x}+x+1\Rightarrow du=2e^x+1\,dx=2\left(e^{2x}+\frac{1}{2}\right)dx$

$$\frac{1}{2}\int \sec u\tan u\,du=\frac{1}{2}\sec u+C=\frac{1}{2}\sec(e^{2x}+x+1)+C$$

2.53: Integrals leading to Logarithms

If the numerator is the derivative of the denominator, then we get a straight-forward pattern that leads to a logarithm for the integral:

$$\int\frac{u'}{u}dx=\int\frac{1}{u}du=\ln|u|+C$$

Using $\int\frac{1}{x}dx=\ln|x|+C$, we can integrate

$$\int\frac{4}{x}dx=4\int\frac{1}{x}dx=4\ln|x|+C$$

Whenever the numerator is the derivative of the denominator, it will lead to a logarithm in the answer.
Use the pattern

$$\int\frac{u'}{u}dx=\int\frac{1}{u}du=\ln|u|+C$$

Example 2.54

$$\int \frac{e^x}{1-e^x}\,dx$$

Substitute $u = 1 - e^x \Rightarrow dx = \frac{du}{-e^x}$:

$$\int \frac{e^x}{1-e^x} \cdot \frac{du}{-e^x} = -\int \frac{1}{u} du = -\ln|u| = -\ln|1-e^x| + C$$

Example 2.55

$$\int \frac{dx}{\sqrt{x}+x} \quad (CBSE\ 2020, Adapted)$$

$$\int \frac{dx}{\sqrt{x}+x} = \int \frac{dx}{\sqrt{x}(1+\sqrt{x})}$$

Let $t = \sqrt{x} + 1 \Rightarrow= dt = \frac{1}{2\sqrt{x}}dx$:

$$2\int \frac{dx}{2\sqrt{x}(1+\sqrt{x})} = 2\int \frac{dt}{t} = 2\ln|t| + C = 2\ln\left|\sqrt{x}+1\right| + C$$

Example 2.56:

A. $\int \frac{6x^2+3}{2x^3+3x}dx$

B. $\int \frac{1}{3x+2\sqrt{x}}dx$

Part A

Let $u = 2x^3 + 3x \Rightarrow du = \ 6x^2 + 3\,dx$

$$\int \frac{\mathbf{6x^2+3}}{2x^3+3x}\mathbf{dx} = \int \frac{1}{u}du = \ln|u| + C = \ln|2x^3+3x| + C$$

Part B

This doesn't seem to fit the pattern, but since the question asks us to, it should. Factor out $\sqrt{x}$:

$$\int \frac{1}{\sqrt{x}(3\sqrt{x}+2)}dx$$

Let $u = 3\sqrt{x} + 2 \Rightarrow du = \frac{3}{2}\cdot\frac{1}{\sqrt{x}}dx$

Then:

$$\frac{2}{3}\int \frac{1}{\underbrace{(3\sqrt{x}+2)}_{u}} \cdot \underbrace{\frac{3}{2}\cdot\frac{1}{\sqrt{x}}dx}_{du} = \frac{2}{3}\int \frac{1}{u}du = \frac{2}{3}\ln|u| + C = \frac{2}{3}\ln\left|3\sqrt{x}+2\right| + C$$

Example 2.57

True or False

$$\int \frac{u}{u'}dx = \int \frac{1}{u}du = \ln|u| + C$$

$$\frac{dy}{dx}\ln|u| = \frac{1}{u}\times\frac{du}{dx} = \frac{1}{u}\times u' = \frac{u'}{u} \neq \frac{u}{u'} \Rightarrow False$$

C. Trigonometric Integrals

Integrals of trigonometric functions defined as ratios can lead to logarithms since derivatives of the trigonometric functions are also trigonometric functions. We now find the integrals of the four trigonometric functions that we have not yet found.

2.58: Integrals of the Remaining Trigonometric functions[1]

$$\int \tan x\, dx = \ln|\sec x| + C$$

$$\int \cot x\, dx = \ln|\sin x| + C$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C$$

Part A

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$$

Let $u = \cos x \Rightarrow du = -\sin x\, dx$:

$$= -\int \frac{1}{\underbrace{\cos x}_{u}} \cdot \underbrace{(-\sin x) dx}_{du} = -\int \frac{1}{u} du = -\ln|u| + C = -\ln|\cos x| + C$$

Using the power rule of logarithms: $a \ln x = \ln x^a$

$$= \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$$

Part B

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx$$

Let $u = \sin x \Rightarrow du = \cos x\, dx$:

$$\int \frac{1}{\underbrace{\sin x}_{u}} \times \underbrace{\cos x\, dx}_{du} = \int \frac{1}{u} du = \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C$$

Part C

$$\int \sec x\, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

Let $u = \sec x + \tan x \Rightarrow du = \sec x \tan x + \sec^2 x\ \ dx$:

$$= \int \frac{1}{u}\, du = \ln|u| + C = \ln|\sec x + \tan x| + C$$

Example 2.59: Trigonometric Expressions

A. $\int \frac{1}{x^2} \cot\left(\frac{1}{x}\right) dx$

B. $\int \frac{x + \cos 6x}{3x^2 + \sin 6x} dx$ (CBSE 2012)

Part A

Let $u = \frac{1}{x} \Rightarrow du = -\frac{1}{x^2} dx$

[1] While the integrals are mentioned as a property, the logic behind them is important. Make sure you can derive these whenever needed.

$$-\int \underbrace{\cot\left(\frac{1}{x}\right)}_{u}\underbrace{\left(-\frac{1}{x^2}\right)dx}_{du} = -\int \cot u\, du = -\ln|\sin u| + C = \ln\left|\csc\frac{1}{x}\right| + C$$

Part B

Let $t = 3x^2 + \sin 6x = t \Rightarrow dt = 6(x + \cos 6x)dx$

$$\frac{1}{6}\int \frac{6(x+\cos 6x)}{3x^2+\sin 6x}dx = \frac{1}{6}\int \frac{1}{t}dt = \frac{1}{6}\ln|t| + C = \frac{1}{6}\ln|3x^2+\sin 6x| + C$$

Example 2.60

Find $\int \frac{dx}{x(1+x^2)}$ **(CBSE 2020)**

Let $t = \frac{1}{x^2} + 1 \Rightarrow dt = -\frac{2}{x^3}dx$

$$\int \frac{dx}{x(1+x^2)} = \int \frac{dx}{x^3\left(\frac{1}{x^2}+1\right)} = -\frac{1}{2}\int -\frac{2\,dx}{x^3\left(\frac{1}{x^2}+1\right)} = -\frac{1}{2}\int \frac{1}{t}dt = -\frac{1}{2}\ln|t| + C$$

Change back to the original variable:

$$-\frac{1}{2}\ln\left|\frac{1}{x^2}+1\right| + C = -\frac{1}{2}\ln\left|\frac{x^2+1}{x^2}\right| + C$$

Use the power rule for logarithms:

$$= \ln\sqrt{\frac{x^2}{x^2+1}} + C$$

Example 2.61

D. Long Division

Example 2.62:Basics

$$\int \frac{x}{3x+5}dx$$

Divide

Multiply and divide the given expression by 3:

$$\frac{1}{3}\left(\frac{3x}{3x+5}\right)$$

Add and subtract 5 in the numerator:

$$= \frac{1}{3}\left(\frac{3x+5-5}{3x+5}\right)$$

Split the fraction:

$$= \frac{1}{3}\left(1 - \frac{5}{3x+5}\right)$$

Integrate:

Substitute $\frac{x}{3x+5} = \frac{1}{3}\left(1 - \frac{5}{3x+5}\right)$ in the given integral:

$$\int \frac{1}{3}\left(1 - \frac{5}{3x+5}\right)dx$$

Split the integral:

$$\frac{1}{3}\int 1\,dx - \frac{5}{3}\int \frac{1}{3x+5}dx$$

Evaluate each integral:

$$\frac{1}{3}x - \frac{5}{3}\cdot\frac{1}{3}\ln|3x+5| + C$$

Simplify:

$$\frac{1}{3}x - \frac{5}{9}\ln|3x+5| + C$$

Example 2.63: Polynomial Long Division

$$\int \frac{x^2+3x+2}{2x-1}\,dx$$

Rewrite the expression using polynomial long division:

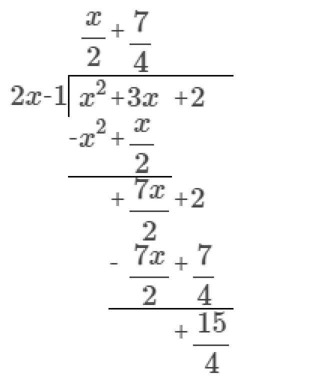
$$\int \left(\frac{x}{2}+\frac{7}{4}\right)dx + \frac{15}{4}\int \frac{1}{2x-1}\,dx$$

Integrate:

$$= \frac{x^2}{4} + \frac{7x}{4} + \frac{15}{4}\cdot\frac{1}{2}\ln|2x-1| + C$$

Simplify:

$$= \frac{x^2}{4} + \frac{7x}{4} + \frac{15}{8}\ln|2x-1| + C$$



E. Integration by Completing the Square

2.64: Summary of Inverse Trigonometric Derivatives

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}, \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}, \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

2.65: Integrals leading to $\tan^{-1}x$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Factor out a^2 from the denominator to get a 1:

$$\int \frac{1}{a^2\left[1+\left(\frac{x}{a}\right)^2\right]}\,dx$$

Substitute $u = \frac{x}{a} \Rightarrow du = \frac{1}{a}dx \Rightarrow dx = a\,du$:

$$\int \frac{a\,du}{a^2(1+u^2)} = \int \frac{du}{a(1+u^2)} = \frac{1}{a}\tan^{-1}u + C$$

Substitute $u = \frac{x}{a}$:

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Example 2.66

$$\int \frac{1}{x^2 + 4x + 8} dx$$

Rewrite the expression in the denominator b by completing the square:

$$\int \frac{1}{x^2 + 4x + 4 + 4} dx = \int \frac{1}{(x+2)^2 + 2^2} dx$$

Use the formula $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$ with $a = 2$:

$$= \frac{1}{2} \tan^{-1}\left(\frac{x+2}{2}\right) + C$$

2.67: Integrals leading to $\boldsymbol{sin^{-1} x}$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{1}{\sqrt{a^2\left(1 - \frac{x^2}{a^2}\right)}} dx = \int \frac{1}{a\sqrt{1 - \left(\frac{x}{a}\right)^2}} dx$$

Substitute $u = \frac{x}{a} \Rightarrow du = \frac{1}{a}\ dx \Rightarrow dx = a\ du$:

$$\int \frac{a\ du}{a\sqrt{1 - u^2}} = \int \frac{du}{\sqrt{1 - u^2}}$$

Integrate and change back to the original variable:

$$= \sin^{-1} u + C = \sin^{-1}\frac{x}{a} + C$$

Example 2.68

$$\int \frac{1}{\sqrt{-x^2 - 6x - 4}} dx$$

Complete the square for the quantity inside the square root:

$$-(x^2 + 6x + 4) = -(x^2 + 6x + 9 - 5) = -[(x+3)^2 - 5] = \left(\sqrt{5}\right)^2 - (x+3)^2$$

Substitute:

$$\int \frac{1}{\sqrt{\left(\sqrt{5}\right)^2 - (x+3)^2}} dx$$

Use the formula $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + C$ with $a = \sqrt{5}$

$$= \sin^{-1}\frac{x+3}{\sqrt{5}} + C$$

F. Initial Value Problems

Example 2.69

Find $f(x)$ given that $f'(x) = (\sin^3 2x)(\cos 2x)$ and $f\left(\frac{\pi}{4}\right) = 1$.

$$f(x) = \int (\sin^3 2x)(\cos 2x)\ dx$$

Substitute $u = \sin 2x \Rightarrow du = 2 \cdot \cos 2x\ dx$:

$$f(x) = \frac{1}{2}\int u^3\, du = \frac{1}{2}\cdot\frac{u^4}{4} + C = \frac{u^4}{8} + C = \frac{\sin^4 2x}{8} + C$$

To find the constant of integration:

$$f\left(\frac{\pi}{4}\right) = \frac{\sin^4\left(2\cdot\frac{\pi}{4}\right)}{8} + C = \frac{\sin^4\left(\frac{\pi}{2}\right)}{8} + C = \frac{1}{8} + C = 1 \Rightarrow C = 1 - \frac{1}{8} = \frac{7}{8}$$

Then:

$$f(x) = \frac{\sin^4 2x}{8} + \frac{7}{8}$$

Example 2.70

Find the particular solution for $\frac{dy}{dx} = \sin 2x$ given the initial value condition $y(0) = \frac{1}{2}$.

$$y = \int \sin 2x\, dx$$

Let $u = 2x \Rightarrow du = 2\, dx$:

$$y = \frac{1}{2}\int \sin u\, du = \frac{1}{2}(-\cos u) + C = \frac{1}{2}(-\cos 2x) + C$$

Substitute $y = 0, x = \frac{1}{2}$ to find the value of C:

$$0 = \frac{1}{2}(-\cos 0) + C \Rightarrow C = \frac{1}{2}$$

The particular solution:

$$y = \frac{1}{2}(-\cos 2x) + \frac{1}{2}$$

G. Challenging Problems

Example 2.71

92.3 Area under a Curve and Reimann Sums

A. Area from Rectangles: Partitions and Heights

Numerical analysis is a subject that uses algorithms to calculate approximate values of interest. We look at some basics related to the subject that lead us to the definition of the integral. These ideas can be extended into the topic of numerical integration (which we will see later).

For many applications, we want to calculate areas. Euclidean geometry gives us the area for a number of geometrical shapes:

- Square $= Side^2$
- Rectangle $= Length \times Width$
- Trapezoid $= h\cdot\frac{b_1+b_2}{2}$
- Circle $= \pi r^2$

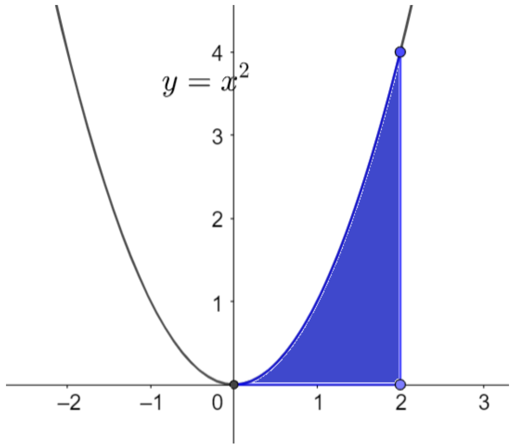
We now look at calculating more general areas than the specific shapes listed above.

Get all the files at: https://bit.ly/azizhandouts

2.72: "Area under the Curve"
The "area under the curve" of a function is taken to mean the area between the curve and the x-axis.

The diagram on the right shows the blue area as the area between the function $y = x^2$ and the x axis

$$from\ x = 0\ \ to\ x = 2$$



2.73: Partition
To approximate area under the curve, we can divide the length under consideration into different parts. Each part is called a partition.

- Partitions can have different shapes.
- We will look at rectangular and trapezoidal partitions, but these are not the only types of partitions.

2.74: Approximation using Rectangles
Since the area of a rectangle is easy to calculate, we can approximate the area under the curve using Rectangles. The number of rectangles is given by the number of partitions that we divide the length into.

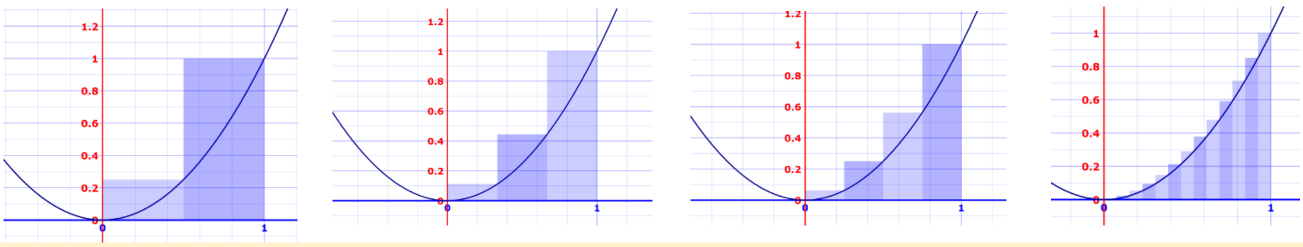
2.75: Right Endpoints
We use right endpoints when we approximate the area under the curve using rectangles with height defined at the right endpoint of the interval.

2.76: Upper Sum
An upper sum is an approximation that is larger than the true area.

2.77: Increasing Function
A function f is called an increasing function for all numbers x_1, x_2 in the domain of f, we have:

$$x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$$

Example 2.78
The area under the curve of $f(x) = x^2$ from $x = 0$ to $x = 1$ has been approximated with n rectangles in the given diagrams using right endpoints.



A. Explain why the diagram shows that right endpoints have been used.
B. Explain why the estimation is an upper sum.
C. Explain the connection between the answers to Parts A and B using the definition of increasing function to help you.
D. Calculate the area when $n = 2, n = 3, n = 4$
E. What is the trend in the areas?
F. Without calculating, what will be the area from $n > 4$. Will it less or more than the areas that you

calculated in Part D.

G. As $n \to \infty$, what does the width of the rectangles tend to. What does the total area that you calculate tend to?

n	$Width$	(x, y)	Area
2	$\frac{1}{2}$	$\left(\frac{1}{2}, \frac{1}{4}\right), (1,1)$	$\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)(1) = \frac{1}{8} + \frac{1}{2} = \frac{5}{8} = 0.625$
3	$\frac{1}{3}$	$\left(\frac{1}{3}, \frac{1}{9}\right), \left(\frac{2}{3}, \frac{4}{9}\right), (1,1)$	$\left(\frac{1}{3}\right)\left(\frac{1}{9}\right) + \left(\frac{1}{3}\right)\left(\frac{4}{9}\right) + \left(\frac{1}{3}\right)(1) = \left(\frac{1}{3}\right)\left(\frac{14}{9}\right) = \frac{14}{27}$
4	$\frac{1}{4}$	$\left(\frac{1}{4}, \frac{1}{16}\right), \left(\frac{1}{2}, \frac{1}{4}\right), \left(\frac{3}{4}, \frac{9}{16}\right), (1,1)$	$\left(\frac{1}{4}\right)\left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right) = \frac{15}{32}$

Write in summation notation

$$\sum_{t=1}^{2} \frac{1}{2} \cdot f\left(\frac{t}{2}\right)$$

$$\sum_{t=1}^{3} \frac{1}{3} \cdot f\left(\frac{t}{3}\right)$$

$$\sum_{t=1}^{4} \frac{1}{4} \cdot f\left(\frac{t}{4}\right) = \frac{1}{4} f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{2}{4}\right) + \frac{1}{4} f\left(\frac{3}{4}\right) + \frac{1}{4} f\left(\frac{4}{4}\right)$$

Example 2.79

Approximate the area under the curve $y = x^3$ from $x = 1$ to $x = 3$ using right endpoints with n rectangles.

A. Calculate the approximation for $n = 2, n = 3, n = 4$.

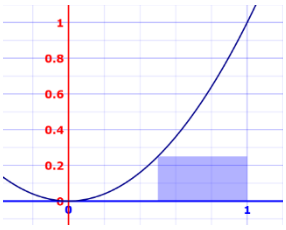
n	$Width$	(x, y)	Area
2	1	$(2,8), (3,27)$	$1(8 + 27) = 35$
3	$\frac{2}{3}$	$\left(\frac{5}{3}, \frac{125}{27}\right), \left(\frac{7}{3}, \frac{343}{27}\right), (3,27)$	$\frac{2}{3}\left(\frac{125}{27} + \frac{343}{27} + 27\right) = \frac{266}{9}$
4	$\frac{1}{2}$	$\left(\frac{3}{2}, \frac{27}{8}\right), (2,8), \left(\frac{5}{2}, \frac{125}{8}\right), (3,27)$	$\frac{1}{2}\left(\frac{27}{8} + 8 + \frac{125}{8} + 27\right)$

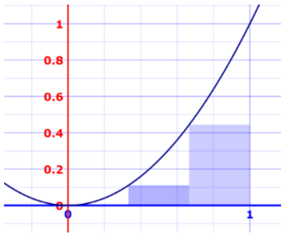
2.80: Left Endpoints

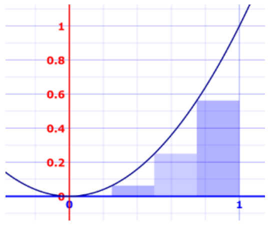
We use left endpoints when we approximate the area under the curve using rectangles with height defined at the left endpoint of the interval.

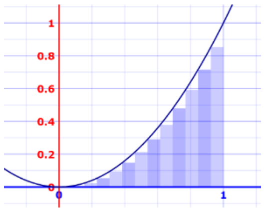
Example 2.81

Approximate the area under the curve $y = x^2$ from $x = 0$ to $x = 1$ using left endpoints with 2, 3 and 4 rectangles.









n	$Width$	(x,y)	Area
2	$\frac{1}{2}$	$(0,0), \left(\frac{1}{2}, \frac{1}{4}\right)$	$\left(\frac{1}{2}\right)(0) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{8} = 0.125$
3	$\frac{1}{3}$	$(0,0), \left(\frac{1}{3}, \frac{1}{9}\right), \left(\frac{2}{3}, \frac{4}{9}\right)$	$\left(\frac{1}{3}\right)(0) + \left(\frac{1}{3}\right)\left(\frac{1}{9}\right) + \left(\frac{1}{3}\right)\left(\frac{4}{9}\right) = \left(\frac{1}{3}\right)\left(\frac{5}{9}\right) = \frac{5}{27}$
4	$\frac{1}{4}$	$(0,0), \left(\frac{1}{4}, \frac{1}{16}\right), \left(\frac{1}{2}, \frac{1}{4}\right), \left(\frac{3}{4}, \frac{9}{16}\right)$	$\left(\frac{1}{4}\right)\left(0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right) = \frac{7}{32}$

Write in summation notation

$$\sum_{t=0}^{1} \frac{1}{2} \cdot f\left(\frac{t}{2}\right)$$
$$\sum_{t=0}^{2} \frac{1}{3} \cdot f\left(\frac{t}{3}\right)$$
$$\sum_{t=0}^{3} \frac{1}{4} \cdot f\left(\frac{t}{4}\right)$$

Example 2.82

Approximate the area under the curve $y = x^3$ from $x = 1$ to $x = 2$ using left endpoints with n rectangles.

A. Calculate the approximation for $n = 2, n = 3, n = 4$.

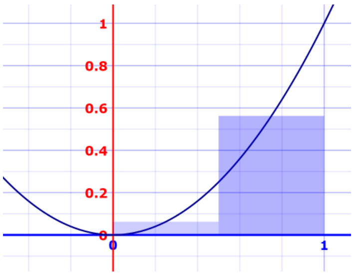
n	$Width$	(x,y)	Area
2	$\frac{1}{2}$	$(1,1), \left(\frac{3}{2}, \frac{27}{8}\right)$	$\frac{1}{2}\left(1 + \frac{27}{8}\right) =$
3	$\frac{1}{3}$	$(1,1), \left(\frac{4}{3}, \frac{64}{27}\right), \left(\frac{5}{3}, \frac{125}{27}\right)$	$\frac{1}{3}\left(1 + \frac{64}{27} + \frac{125}{27}\right) =$
4	$\frac{1}{4}$	$(1,1), \left(\frac{5}{4}, \frac{125}{64}\right), \left(\frac{3}{2}, \frac{27}{8}\right), \left(\frac{7}{4}, \frac{343}{64}\right)$	$\frac{1}{4}\left[1 + \frac{125}{64} + \frac{27}{8} + \frac{343}{64}\right] = \frac{187}{64} = 2.921875$

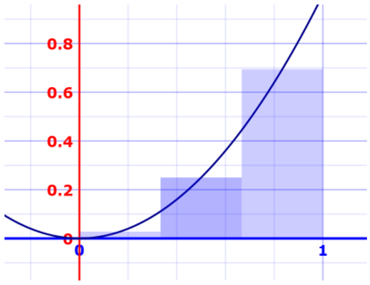
2.83: Midpoints

We use midpoints when we approximate the area under the curve using rectangles with height defined at the midpoint of the interval.

Example 2.84

Approximate the area under the curve $y = x^2$ from $x = 0$ to $x = 1$ using midpoints with 2, 3 and 4 rectangles.





n	$Width$	(x, y)	Area
2	$\frac{1}{2}$	$\left(\frac{1}{4}, \frac{1}{16}\right), \left(\frac{3}{4}, \frac{9}{16}\right)$	$\left(\frac{1}{2}\right)\left(\frac{1}{16}\right) + \left(\frac{1}{2}\right)\left(\frac{9}{16}\right) = \frac{1}{32} + \frac{9}{32} = \frac{10}{32} = \frac{5}{16}$
3	$\frac{1}{3}$	$\left(\frac{1}{6}, \frac{1}{36}\right), \left(\frac{1}{2}, \frac{1}{4}\right), \left(\frac{5}{6}, \frac{25}{36}\right)$	$\left(\frac{1}{3}\right)\left(\frac{1}{36} + \frac{1}{4} + \frac{25}{36}\right) = \frac{35}{108}$
4	$\frac{1}{4}$		

Example 2.85

Approximate the area under the curve $y = x^3$ from $x = 1$ to $x = 2$ using midpoint with n rectangles.

A. Calculate the approximation for $n = 2, n = 3, n = 4$.

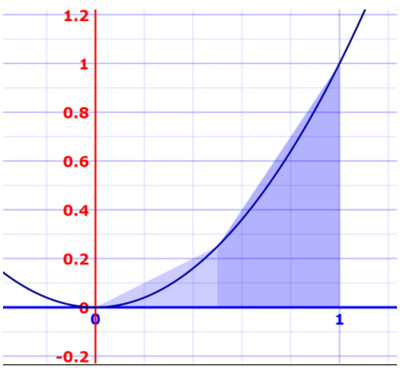
n	$Width$	(x, y)	Area
2	$\frac{1}{2}$	$\left(\frac{5}{4}, \frac{125}{64}\right), \left(\frac{7}{4}, \frac{343}{64}\right)$	
3	$\frac{1}{3}$	$\left(\frac{7}{6}, \frac{343}{216}\right), \left(\frac{3}{2}, \frac{27}{8}\right), \left(\frac{11}{6}, \frac{1331}{216}\right)$	
4	$\frac{1}{4}$	$\left(\frac{9}{8}, \frac{729}{512}\right), \left(\frac{11}{8}, \frac{1331}{512}\right), \left(\frac{13}{8}, \right), \left(\frac{15}{8}, \right)$	

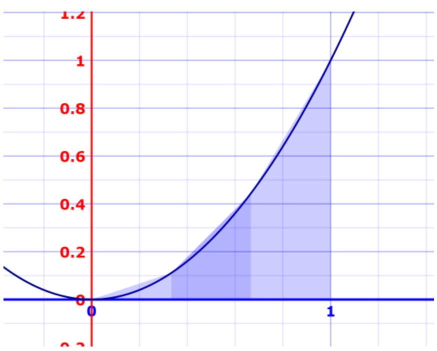
2.86 : Trapezoids

We use the trapezoidal rule when we approximate the area under the curve using trapezoids with height at left and right endpoints of the given interval.

Example 2.87

Approximate the area under the curve $y = x^2$ from $x = 0$ to $x = 1$ using trapezoids with 2, 3 and 4 trapezoids.





n	w	$b_1 + b_2$	$Area = \frac{1}{2} \cdot w \cdot (b_1 + b_2)$
2	$\frac{1}{2}$	$0 + \frac{1}{4} = \frac{1}{4}, \frac{1}{4} + 1 = \frac{5}{4}$	$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4} + \frac{5}{4}\right) = \frac{6}{16} = \frac{3}{8}$
3	$\frac{1}{3}$		
4	$\frac{1}{4}$		

Example 2.88

Find the area under the curve $\frac{1}{x}$ over the interval $x = 1$ to $x = 3$ using 4 rectangles.

$$Width\ of\ Rectangle = \frac{3-1}{4} = \frac{2}{4} = 0.5$$

	1	1.5	2	2.5	3
	1	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$

$$0.5 \cdot \frac{1 + \frac{2}{3}}{2} + 0.5 \cdot \frac{\frac{2}{3} + \frac{1}{2}}{2} + 0.5 \cdot \frac{\frac{1}{2} + \frac{2}{5}}{2} + 0.5 \cdot \frac{\frac{2}{5} + \frac{1}{3}}{2}$$

B. Some Online Tools

This integration calculator will let you decide the function, the interval, the number of rectangles, and the endpoint method.

It is a good way of practicing area under the curve approximation questions.

C. Summation Notation

2.89: Summation independent of the variable

$$\sum_{n=1}^{k} x = kx$$

$$\sum_{n=1}^{k} x = \underbrace{x + x + \cdots + x}_{k\ times} = kx$$

2.90: Property I: Constant Property

A constant can be moved out of, or into the summation sign without changing its value.

$$\sum_{x=1}^{n} cx = c \sum_{x=1}^{n} x$$

For example:

$$\sum_{x=1}^{3} 2x = 2(1) + 2(2) + 2(3) = 2(1+2+3) = 2\sum_{x=1}^{3} x$$

2.91: Property II: Distribution of Summation over an Expression

You can distribute the summation operator over individual terms (for addition or subtraction)

$$\sum_{i=1}^{n} x + y = \sum_{i=1}^{n} x + \sum_{i=1}^{n} y$$

$$\sum_{i=1}^{n} x - y = \sum_{i=1}^{n} x - \sum_{i=1}^{n} y$$

2.92: Sums of n^{th} powers

$$\sum_{x=1}^{n} x = 1 + 2 + .. + n = \frac{n(n+1)}{2}$$

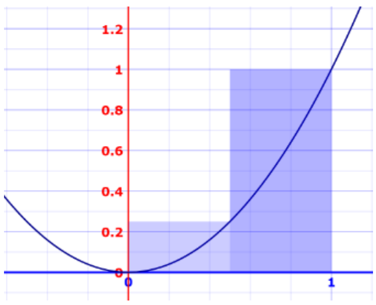
$$\sum_{x=1}^{n} x^2 = 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{x=1}^{n} x^3 = 1^3 + 2^3 + \dots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example 2.93

Suppose the area under the curve $f(x) = x^2$ from $x = 0$ to $x = 1$ is divided into n rectangles of equal width.

A. Show that the area of the i^{th} rectangle is $A(R_i) = \frac{i^2}{n^3}$
B. using right endpoint rectangles that the area under the curve with n rectangles is $\frac{(n+1)(2n+1)}{6n^2}$
C. Show that the limit of the expression as $n \to \infty$ is $\frac{1}{3}$.



Part A

Divide the area into n rectangles of equal width, starting from left to right:

$$R_1, R_2, \dots, R_n$$

Width of each rectangle is:

$$\frac{Length\ of\ Interval}{No.of\ Rectangles} = \frac{1}{n}$$

The right endpoint of the rectangles is given by:

$$\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n}{n}$$

Height of the i^{th} rectangle is given by the function evaluated at the right endpoint of the interval:

$$h(R_i) = f\left(\frac{i}{n}\right) = \left(\frac{i}{n}\right)^2$$

The area of the i^{th} rectangle is:

$$\underbrace{A(R_i) = wh = \frac{1}{n}\left(\frac{i}{n}\right)^2 = \frac{i^2}{n^3}}_{Equation\ I}$$

Part B

The total area under the curve can be approximated by:

$$Area \approx \sum_{i=1}^{n} A(R_i)$$

Substitute the result from Part A in the expression above:

$$= \sum_{i=1}^{n} \frac{i^2}{n^3}$$

Use the property $\sum_{x=1}^{n} cx = c\sum_{x=1}^{n} x$ to move the constant $\frac{1}{n^3}$ out of the summation:

$$= \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

Use the formula for the sum of the squares of the first n natural numbers:

$$= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{(n+1)(2n+1)}{6n^2}$$

Part C

$$\lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} = \frac{2}{6} = \frac{1}{3}$$

2.94: General Approach

$$Width\ of\ each\ Rectangle\ = \frac{Width\ of\ Interval}{No.of\ Rectangles}$$

where x is given by the endpoint rule that you choose. Common endpoint rules include:
Left Endpoint
Right Endpoint
Midpoint

Example 2.95

Approximate under the curve for $f(x) = \sin x$ from $x = 0$ to $x = \pi$ using summation notation

$$Width\ of\ Interval = \pi$$
$$No.Of\ Rectangles = n$$
$$Width\ of\ each\ Rectangle = \frac{Width\ of Interval}{No.of\ Rectangles} = \frac{\pi}{n}$$

Suppose we choose the right endpoint rule.

$$\underbrace{\frac{\pi}{n}}_{Width}\underbrace{\left(\sin\frac{\pi}{n}\right)}_{Height}+\frac{\pi}{n}\left(\sin\frac{2\pi}{n}\right)+\cdots+\frac{\pi}{n}\left(\sin\frac{n\pi}{n}\right)$$

Write the above in summation notation:

$$\sum_{k=1}^{n}\underbrace{\frac{\pi}{n}}_{Width}\underbrace{\left(\sin\frac{k\pi}{n}\right)}_{Height}$$

Suppose we choose the left endpoint rule.

$$\underbrace{\frac{\pi}{n}}_{Width}\underbrace{(\sin 0)}_{Height}+\frac{\pi}{n}\left(\sin\frac{\pi}{n}\right)+\cdots+\frac{\pi}{n}\left(\sin\frac{(n-1)\pi}{n}\right)$$

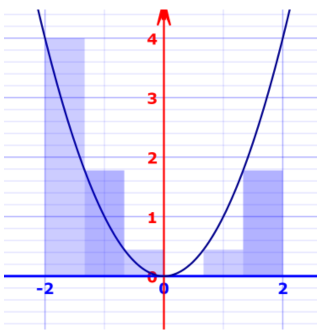
Write the above in summation notation:

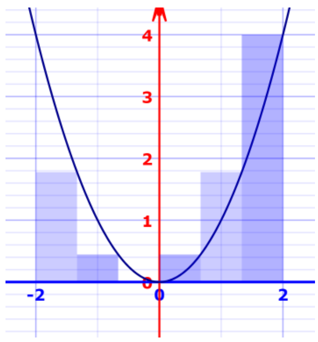
$$\sum_{k=0}^{n-1}\underbrace{\frac{\pi}{n}}_{Width}\underbrace{\left(\sin\frac{k\pi}{n}\right)}_{Height}$$

D. Over and Under Approximation

2.96: Interval

- ➢ Right endpoints will overestimate for an increasing function, and underestimate for a decreasing function.
- ➢ Left endpoints will underestimate for an increasing function, and overestimate for a decreasing function.





2.4 Reimann Sums

A. Reimann sums

We have taken earlier taken area approximations with a small number of rectangles (say 2, or 3, or 4 rectangles) and:

- ➢ rectangles of equal width
- ➢ Some method of choosing the height of the rectangle (left endpoint, right endpoint, midpoint)

We can generalize to find the area for n rectangles. And the sum of these areas is called a Reimann Sum.

2.97: Interval

An interval is a set of real numbers that contains all real numbers lying between any two numbers of the set.

These are the intervals you have seen before, say, when solving inequalities.

$$\underbrace{x\in(2,4)}_{Open\ Interval}\ \Rightarrow All\ numbers\ from\ 2\ to\ 4, but\ not\ including\ 2\ and\ 4$$

$$\underbrace{x\in[2,4]}_{Closed\ Interval}\ \Rightarrow All\ numbers\ from\ 2\ to\ 4, including\ 2\ and\ 4$$

2.98: Partition
A partition $\{x_0, x_1, x_2, \ldots, x_n\}$ such that:
$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$
divides an interval $[a, b]$ into n closed subintervals
$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

- It is not necessary that the width of each subinterval is equal.

2.99: Rectangles from Partitions
The partition can be used to generate rectangles.
The width of the k^{th} rectangle is given by:
$$x_k - x_{k-1} = \Delta x_k$$

Width
The width of each rectangle in the expression above is determined by the partition. Any choice of widths results in a valid partition. Width can be equal but are not required to be equal.

Height
Height is given by evaluating the function f at c_k.
c_k is any point in the interval $[x_{k-1}, x_k]$

2.100: Reimann Sum
The Reimann sum for the function $f(x)$ from $x = a$ to $x = b$ is given by the sum of the area of the rectangles under the curve:
$$\sum_{k=1}^{n} A(R_k) = \sum_{i=1}^{n} \underbrace{f(c_k)}_{Height} \underbrace{\Delta x_k}_{Width}$$

2.101: Norm of a Partition
The norm of a partition is the subinterval in the partition with the largest width.

Notation: The norm of P is written:
$$\|P\|$$

Example 2.102
Determine the norm of the partition in each base.

A. A partition over $[0,1]$ is given by $P = \{0, 0.2, 0.6, 0.7, 0.8, 0.9, 1\}$.

Part A
$$1 - 0.9 = 0.1$$
$$0.9 - 0.8 = 0.1$$
$$0.8 - 0.7 = 0.1$$
$$0.7 - 0.6 = 0.1$$
$$0.6 - 0.2 = 0.4$$
$$0.2 - 0 = 0.2$$
$$Max\{0.1, 0.2, 0.4\} = 0.4$$

$$\|P\| = 0.4$$

2.103: Equal Width Intervals

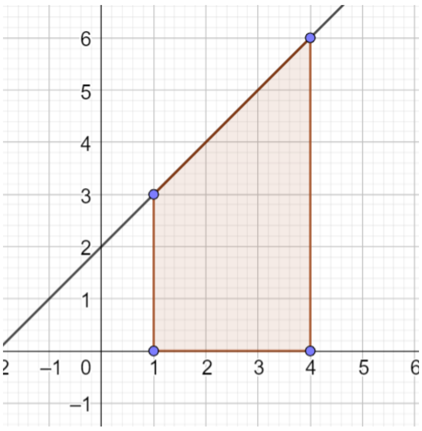
If the intervals have equal width, then the width of each interval

$$= \frac{Length\ of\ Interval}{No.of\ Rectangles}$$

Example 2.104

Find and simplify an expression for the Reimann sum of the area under the curve, using right endpoints, for the function $f(x) = x + 2$ from $x = 1$ to $x = 4$ and find the value of that expression as the number of rectangles approaches infinity.

Area using Geometry



The shape of the area under the curve is a trapezoid:

$$= 3 \cdot \frac{3+6}{2} = 3 \cdot \frac{9}{2} = \frac{27}{2} = 13.5$$

Rectangle Width and Height

$$Width\ of\ each\ Rectangle = \frac{Length\ of\ Interval}{No.of\ Rectangles} = \frac{4-1}{n} = \frac{3}{n}$$

The right endpoint of the rectangles is:

$$1 + \frac{3}{n}, 1 + \frac{6}{n}, 1 + \frac{9}{n}, \dots, 1 + \frac{3n}{n}$$

The right endpoint of the i^{th} rectangle:

$$c_i = 1 + \frac{3i}{n}$$

$$Height\ of\ the\ i^{th} Rectangle = f(c_i) = 1 + \frac{3i}{n} + 2 = 3 + \frac{3i}{n}$$

$$Area\ of\ the\ i^{th}\ rectangle = A(R_i) = \underbrace{\left(3 + \frac{3i}{n}\right)}_{Height} \underbrace{\left(\frac{3}{n}\right)}_{Width} = \frac{9}{n} + \frac{9i}{n^2} = 9\left(\frac{1}{n} + \frac{i}{n^2}\right)$$

Sum of the areas of all Rectangles

$$\sum_{i=1}^{n} A(R_i) = \sum_{i=1}^{n} 9\left(\frac{1}{n} + \frac{i}{n^2}\right) = \underbrace{9\sum_{i=1}^{n} \frac{1}{n} + \frac{i}{n^2}}_{\substack{Constant \\ Multiple\ Rule}}$$

Split the summation:

$$9\left(\sum_{i=1}^{n}\frac{1}{n} + \sum_{i=1}^{n}\frac{i}{n^2}\right) = 9\left(\frac{n}{n} + \frac{1}{n^2}\sum_{i=1}^{n} i\right) = 9\left(1 + \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right)\right) = 9\left(1 + \frac{n^2+n}{2n^2}\right)$$

Find the limit of the expression as the number of rectangles approaches infinity

Move 9 out of $\lim_{n\to\infty} 9\left(1 + \frac{n^2+n}{2n^2}\right)$ using the constant multiple rule:

$$= 9 \lim_{n\to\infty} 1 + \frac{n^2+n}{2n^2}$$

Divide numerator and denominator by n^2:

$$= 9\left(\lim_{n\to\infty} 1 + \frac{\frac{n^2}{n^2} + \frac{n}{n^2}}{\frac{2n^2}{n^2}}\right) = 9\left(\lim_{n\to\infty} 1 + \frac{1 + \frac{1}{n}}{2}\right)$$

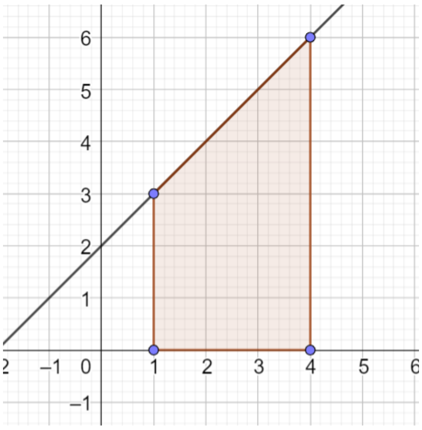
Note that as $n \to \infty, \frac{1}{n} \to 0$:

$$= 9\left(\lim_{n\to\infty} 1 + \frac{1+0}{2}\right) = 9\left(\lim_{n\to\infty} \frac{3}{2}\right) = 9 \cdot \frac{3}{2} = \frac{27}{2}$$

Example 2.105

Evaluate the Reimann sum using left endpoints for $f(x) = x + 2$ from $x = 1$ to $x = 4$ as the number of rectangles approaches infinity.

Area using Geometry



The shape of the area under the curve is a trapezoid:

$$= 3 \cdot \frac{3+6}{2} = 3 \cdot \frac{9}{2} = \frac{27}{2} = 13.5$$

Rectangle Width and Height

$$\textit{Width of each Rectangle} = \frac{\textit{Length of Interval}}{\textit{No. of Rectangles}} = \frac{4-1}{n} = \frac{3}{n}$$

The left endpoint of the rectangles is:

$$1, 1 + \frac{3}{n}, 1 + \frac{6}{n}, \dots, 1 + \frac{3(n-1)}{n}$$

The left endpoint of the i^{th} rectangle:

$$c_i = 1 + \frac{3(i-1)}{n}$$

$$\textit{Height of the } i^{th} \textit{Rectangle} = f(c_i) = 1 + \frac{3(i-1)}{n} + 2 = 3 + \frac{3(i-1)}{n} = 3 + \frac{3i-3}{n}$$

$$\textit{Area of the } i^{th} \textit{ rectangle} = A(R_i) = \underbrace{\left(3 + \frac{3i-3}{n}\right)}_{Height}\underbrace{\left(\frac{3}{n}\right)}_{Width} = \frac{9}{n} + \frac{9i-9}{n^2} = 9\left(\frac{1}{n} + \frac{i-1}{n^2}\right)$$

Sum of the areas of all Rectangles

$$\sum_{i=1}^{n} A(R_i) = \sum_{i=1}^{n} 9\left(\frac{1}{n} + \frac{i-1}{n^2}\right) = 9\underbrace{\sum_{i=1}^{n} \frac{1}{n} + \frac{i-1}{n^2}}_{\substack{\textit{Constant} \\ \textit{Multiple Rule}}}$$

Split the summation:

$$9\left(\sum_{i=1}^{n} \frac{1}{n} + \sum_{i=1}^{n} \frac{i-1}{n^2}\right) = 9\left(\frac{n}{n} + \frac{1}{n^2}\sum_{i=1}^{n} i - 1\right) = 9\left(1 + \frac{1}{n^2}\left(\frac{n(n-1)}{2}\right)\right) = 9\left(1 + \frac{n^2 - n}{2n^2}\right)$$

Find the limit of the expression as the number of rectangles approaches infinity

Move 9 out of $\lim_{n\to\infty} 9\left(1 + \frac{n^2-n}{2n^2}\right)$ using the constant multiple rule:

$$= 9 \lim_{n\to\infty} 1 + \frac{n^2 - n}{2n^2}$$

Divide numerator and denominator by n^2:

$$= 9\left(\lim_{n\to\infty} 1 + \frac{\frac{n^2}{n^2} - \frac{n}{n^2}}{\frac{2n^2}{n^2}}\right) = 9\left(\lim_{n\to\infty} 1 + \frac{1 - \frac{1}{n}}{2}\right)$$

Note that as $n \to \infty, \frac{1}{n} \to 0$:

$$= 9\left(\lim_{n\to\infty} 1 + \frac{1-0}{2}\right) = 9\left(\lim_{n\to\infty} \frac{3}{2}\right) = 9 \cdot \frac{3}{2} = \frac{27}{2}$$

Example 2.106

Find and simplify an expression for the Reimann sum of the area under the curve, using right endpoints, for the function $f(x) = 1 - x^2$ from $x = 0$ to $x = 1$ and find the value of that expression as the number of rectangles approaches infinity.

Width of each rectangle

$$= \frac{Length\ of\ Interval}{No.of\ Rectangles} = \frac{1-0}{n} = \frac{1}{n}$$

Find the Right Endpoint of each rectangle

The right endpoint of the rectangles is given by:

$$\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n}{n}$$

The right endpoint of the i^{th} rectangle:

$$c_i = \frac{i}{n}$$

Find the Height of each rectangle

The height of the i^{th} rectangle:

$$f(c_i) = 1 - c_i^2$$

Find the Area of each rectangle

$$A(R_i) = \underbrace{(1 - c_i^2)}_{Height} \underbrace{\left(\frac{i}{n}\right)}_{Width} = \frac{1}{n} - \frac{c_i^2}{n}$$

Find the sum of the areas of all Rectangles

$$= \sum_{i=1}^{n} A(R_i) = \sum_{i=1}^{n} \frac{1}{n} - \frac{c_i^2}{n}$$

Use the difference property of summation ($\sum_{i=1}^{n} x - y = \sum_{i=1}^{n} x - \sum_{i=1}^{n} y$) to split the expression:

$$= \left(\sum_{i=1}^{n} \frac{1}{n}\right) - \left(\sum_{i=1}^{n} \frac{c_i^2}{n}\right)$$

Use the constant multiple property $\sum_{x=1}^{n} cx = c\sum_{x=1}^{n} x$ to move $\frac{1}{n}$ out of the summation sign in the first term:

$$= \left(\frac{1}{n}\sum_{i=1}^{n} 1\right) - \left(\sum_{i=1}^{n} \frac{c_i^2}{n}\right)$$

Use the property $\sum_{n=1}^{k} x = kx$ to sum the first term:

$$= \left(\frac{1}{n}(n)\right) - \left(\sum_{i=1}^{n} \frac{c_i^2}{n}\right) = 1 - \left(\sum_{i=1}^{n} \frac{c_i^2}{n}\right)$$

Substitute $c_i = \frac{i}{n}$:

$$= 1 - \sum_{i=1}^{n} \frac{\left(\frac{i}{n}\right)^2}{n} = 1 - \sum_{i=1}^{n} \frac{\frac{i^2}{n^2}}{n} = 1 - \sum_{i=1}^{n} \frac{i^2}{n^3}$$

Use the constant multiple property $\sum_{x=1}^{n} cx = c\sum_{x=1}^{n} x$ to move $\frac{1}{n^3}$ out of the summation sign:

$$= 1 - \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

Use the formula for the sum of the squares of the first n natural numbers $\sum_{x=1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}$:

$$= 1 - \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = 1 - \frac{(n+1)(2n+1)}{6n^2} = 1 - \frac{2n^2 + 3n + 1}{6n^2}$$

Find the limit of the expression as the number of rectangles approaches infinity

$$\lim_{n\to\infty} 1 - \frac{2n^2 + 3n + 1}{6n^2}$$

Divide numerator and denominator in the fraction by the highest power of n in the denominator, which is n^2:

$$\lim_{n\to\infty} 1 - \frac{\frac{2n^2}{n^2} + \frac{3n}{n^2} + \frac{1}{n^2}}{\frac{6n^2}{n^2}} = \lim_{n\to\infty} 1 - \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{6} = 1 - \frac{2+0+0}{6} = 1 - \frac{2}{6} = 1 - \frac{1}{3} = \frac{2}{3}$$

2.5 Definite Integral: Reimann Sum Definition

A. Definition

The definite integral is one of the most powerful tools of Calculus. It lets us calculate the area under the curve of a function. However, in this section we are not going to focus on calculations, but on the concept, which is the definition of the definite integral.

2.107: Integral as a Limit of a Reimann Sum

Let f be a function defined on a closed interval $[a, b]$. I is the definite integral over $[a, b]$ if the limit of the Reimann sum below exists:

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

Notation for Definite Integral

$$a = Left\ endpoint\ of\ Interval = Lower\ limit\ of\ Integration$$
$$b = Right\ endpoint\ of\ Interval = Upper\ limit\ of\ Integration$$
$$f(x) = Function\ to\ be\ Integrated = Integrand$$
$$dx = Variable\ with\ respect\ to\ which\ integration\ is\ being\ carried\ out$$

Integral as a limit of a Reimann Sum

Since it is a Reimann sum, we can choose:

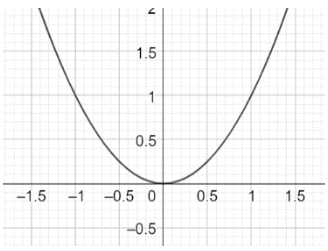
- the partition. That is, we can choose the widths of the rectangles.
- The point within the partition (c_k). This determines $f(c_k)$

The limit should exist (and be same) independent of the choices made above.

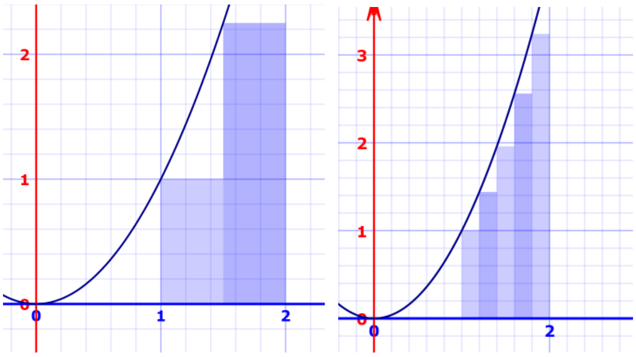
To understand how a Reimann sum works, we take an example and work through it. The parts of the definition above are reproduced below.

Let f be a function defined on a closed interval $[a, b]$.

For example, consider the function $f(x) = x^2$. It is defined over the interval $[1,2]$. The diagram on the right shows the graph of the function.



Consider the partition formed by dividing the interval from $[a, b]$ into n parts. Then, find $f(x)$ for any x within each sub-interval, and that creates n rectangles.
For example, the diagrams below show the interval divided into 2 and 5 rectangles respectively.



The Reimann sum gives an approximation of the area under the curve for this function over the interval $[a, b]$.

$$Reimann\ Sum = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

As the number of rectangles increases the approximation becomes better. If the limit of the Reimann sum as the norm of the partition goes to zero exists, then that is the definition of the definite integral

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \underbrace{f(c_k)}_{Height} \underbrace{\Delta x_k}_{Width}$$

Example 2.108: Converting from Reimann Sums to Integrals

Find the definite integral associated with the Reimann sum

A. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k^2 \Delta x_k$ over a partition P from $x = 0$ to $x = 2$

B. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (c_k + 1)\Delta x_k$ over a partition P from $x = 3$ to $x = 7$.

Part A

We need to write $\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k^2)\Delta x_k$ in the form $\int_a^b f(x)\,dx$. The lower and upper limits of integration are the endpoints of the Partition:

$$\underbrace{a = 0}_{\substack{Left\ endpoint \\ Lower\ limit}}, \qquad \underbrace{b = 2}_{\substack{Right\ Endpoint \\ Upper\ Limit}}$$

Identify the height and the width of the rectangles:

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \underbrace{c_k^2}_{Height} \underbrace{\Delta x_k}_{Width} \Rightarrow f(c_k) = c_k^2 \Rightarrow f(x) = x^2$$

Note that we are integrating with respect to x, and hence we need a dx on the right side of the definite integral. Combine everything:

$$\int_0^2 x^2\,dx$$

Part B

$$\int_3^7 (x + 1)\,dx$$

Example 2.109: Checking for existence of Definite Integrals

For each expression, state whether it corresponds to a definite integral. If it does, state it. If it doesn't, explain why not

A. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\csc c_k)\Delta x_k$ over a partition P from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$

B. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\sin c_k)\Delta x_k$ over a partition P from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$

Part A

The function in the limit expression is:

$$f(x) = \csc x = \frac{1}{\sin x} \Rightarrow 0 \notin D_f$$

And since $\sin 0 = 0$, the number 0 is not in the domain of f.

Hence, the expression does not meet the condition that the function is defined over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Hence, the definite integral does not exist.

Part B

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \ dx$$

Example 2.110: Converting from Integrals to Reimann Sums

Write the following definite integrals as limits of Reimann sums:

A. $\int_{-5}^{7} (x^2 + 5x + 6)dx$

B. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x^2 \, dx$

Part A

$$\int_{-5}^{7} (x^2 + 5x + 6)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \underbrace{(c_k^2 + 5c_k + 6)}_{Height} \underbrace{\Delta x_k}_{Width} \ over\ a\ partition\ of\ the\ interval\ [-5,7]$$

Part B

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x^2 \ dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \underbrace{\sin^2 c_k^2}_{Height} \underbrace{\Delta x_k}_{Width} \ over\ a\ partition\ of\ interval\ \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$$

2.111: Integrals as Area under the Curve

Since a definite integral is the limit of a Reimann sum as the norm of the partition tends to zero, and the Reimann sum gives you the area under the curve, evaluating the definite integral will give you the area under the curve.

$$\underbrace{\int_0^2 x^2 \, dx}_{Definite\ Integral} = \underbrace{\lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k^2 \Delta x_k \text{ over a partition } P[0,2]}_{Reimann\ Sum} = Area\ under\ the\ Curve\ of\ x^2\ for\ [0,2]$$

Example 2.112

Find the area between the line $y = x$, $x = 0$, $x = 3$ and the x-axis:

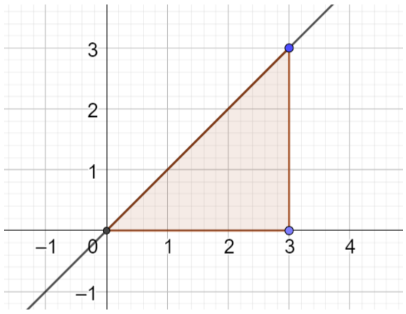
A. By using geometry

B. By setting up a definite integral, and applying the result from Part A.

Part A

The shape that we want the area of is a triangle with:

$$Base = Height = 3 \Rightarrow Area = \frac{1}{2}bh = \frac{1}{2}(3)(3) = \frac{9}{2}$$



Part B

$$Function = f(x) = x$$
$$Left\ endpoint = Lower\ Limit\ of\ Integration = a = 0$$
$$Right\ endpoint = Upper\ Limit\ of\ Integration = b = 0$$

Combine the above, and substitute:

$$\int_a^b f(x)\,dx = \int_0^3 x\,dx = \frac{9}{2}$$

B. Integrability vs. Continuity

The continuity of a function is key in determining whether it is integrable.

2.113: Continuous Functions

If a function is continuous over the interval $[a, b]$ then it is integrable over that interval.

MCMC 2.114

Mark all correct options

A $f(x)$ has domain $\mathbb{R} - \{3\}$. Then it must not be integrable over:

A. the open interval (2,3)
B. the closed interval[2,3]
C. the half open interval [2,3)
D. the half open interval (2,3]

$$Option\ A, B, C\ and\ D$$

Example 2.115

A. Is $f(x) = \frac{1}{x}$ integrable over -3 to 0?
B. Is $f(x) = \frac{1}{x}$ integrable over -3 to -1?

The questions asks whether $\int_0^3 \frac{1}{x}\,dx$ is defined.

For $\frac{1}{x}$ to be integrable over -3 to 0, it must be continuous over $[-3,0]$.

$\frac{1}{x}$ is continuous over the open interval $(-\infty, 0)$, but not over the closed interval $[-\infty, 0]$.

Hence, the function $\frac{1}{x}$ is not integrable over -3 to 0.

2.116: Infinite Discontinuity

If a function has an infinite discontinuity in an interval, then the integral does not exist over that interval.[2]

If $f(x)$ has an infinite discontinuity at $c \in [a, b]$ then

[2] While the integral does not exist if c is within the interval, we can consider c to the right of the interval and calculate the limit as b approaches c from the left. This technique which gives an expression for the area combines the concept of limits and integrals. Such integrals are called improper integrals.

$$\int_a^b f(x)\,dx \text{ is not defined}$$

Example 2.117

The value of the integral $\int\limits_{-1}^{1} \frac{dx}{x^2}$ is $(JMET\ 2011/79)$

When $x = 0$:

$$\frac{1}{x^2} = \frac{1}{0} \Rightarrow Not\ Defined$$

Hence, the function $\frac{1}{x^2}$ has an infinite discontinuity at $x = 0$. Hence,

$$\int\limits_{-1}^{1} \frac{dx}{x^2}\ does\ not\ exist$$

MCQ 2.118

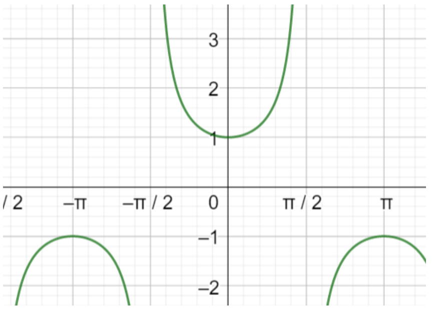
Mark the correct option

Statement: $f(x) = \sec x$ is not integrable over the interval $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Reason: $\cos x$ has a zero at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

A. Statement is true, and the reason explains why it is true.
B. Statement is true, but the reason does not explain why it is true.
C. Statement is true, but reason is false.
D. Statement is false, reason is a true statement by itself.
E. Statement and reason are both false.

Option A



Example 2.119

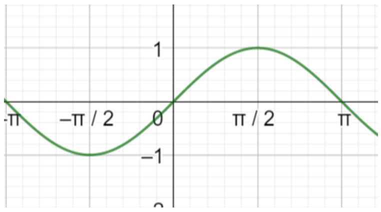
Mark the correct option

Statement: $f(x) = \sin x$ is not integrable over $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

Reason: $\sin x$ has a zero at $x = 0$.

A. Statement is true, and the reason explains why it is true.
B. Statement is true, but the reason does not explain why it is true.
C. Statement is true, but reason is false.
D. Statement is false, reason is a true statement by itself.
E. Statement and reason are both false.

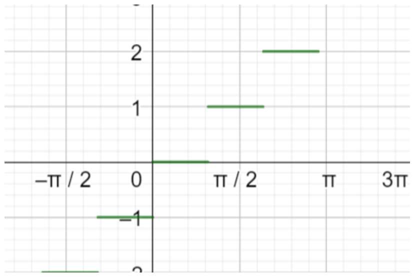
Option D



2.120: Jump Discontinuity
If a function $f(x)$ has a finite number of jump discontinuities over an interval $[a, b]$, then it is integrable over that interval.

Example 2.121
Is the function $y = \lfloor x \rfloor$, $\lfloor x \rfloor$ represents the floor function integrable over the $[a, b]$?

Over a finite interval, the floor function has a finite number of jump discontinuities.
Hence, it is integrable.



Example 2.122
Is the function $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$ integrable over the interval $[0,1]$.

The function oscillates between 0 and 1.
Every time it oscillates there is a jump discontinuity.

There are an infinite number of rational numbers in the interval $[0,1]$, leading to an infinite number of jump discontinuities.

Hence, the function is not integrable.

C. Back Calculations

Example 2.123

Example 2.124

Example 2.125
Challenging JEE Question

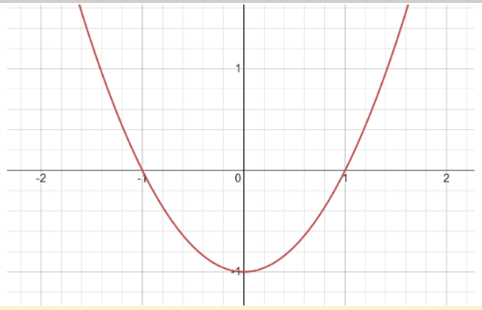
2.6 Definite Integrals (2nd FTC) and Areas

A. Signed Area

2.126: Signed Area

By convention, areas between a function and the x-axis

- above the x-axis (from left to right) will be positive
- above the x-axis (from right to left) will be negative
- below the x-axis (from left to right) will be negative
- below the x-axis (from right to left) will be positive



Example 2.127

State whether each expression below is positive, negative or zero for the function $f(x)$ graphed alongside.

A. $\int_1^2 f(x)\,dx$

B. $\int_2^1 f(x)\,dx$

C. $\int_0^1 f(x)\,dx$

D. $\int_0^{-1} f(x)\,dx$

$$\int_1^2 f(x)\,dx \Rightarrow Left\ to\ right, Above\ the\ x-axis \Rightarrow +ve$$
$$\int_2^1 f(x)\,dx \Rightarrow Right\ to\ left, Above\ the\ x-axis \Rightarrow -ve$$
$$\int_0^1 f(x)\,dx \Rightarrow Left\ to\ right, Below\ the\ x-axis \Rightarrow -ve$$
$$\int_0^{-1} f(x)\,dx \Rightarrow Right\ to\ Left, Below\ the\ x-axis \Rightarrow +ve$$

2.128: Unsigned Area

Unsigned area is found by taking the absolute value of the area found by dividing the integral into parts such that the signed area is either completely positive or completely negative.

- Unsigned area is the area that we consider in geometry.

2.129: Order of Integration

Interchanging the limits of integration keeps the magnitude same, but changes the sign of the integral:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

- We consider area from left to right as positive, and area from right to left as negative.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$
$$\int_b^a f(x)\,dx = F(a) - F(b) = -\big(F(b) - F(a)\big) = -\int_a^b f(x)\,dx$$

Example 2.130

A. If $\int_a^b f(x)\,dx = 4$, then find $\int_b^a f(x)\,dx$.

B. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\,dx + \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \cos x\,dx$

Part A

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx = -4$$

Part B

Using the property $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\,dx + \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \cos x\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\,dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\,dx = 0$$

This can also be solved without the property, with much more work.[3]

B. Fundamental Theorem of Calculus

2.131: 2nd Fundamental Theorem of Calculus (FTC)

$$\int_a^b f(x)\,dx = [F(x)]_{x=a}^{x=b} = F(b) - F(a)$$

- a and b are the limits of integration
  - ✓ a is the lower limit of integration
  - ✓ b is the upper limit of integration
- $F(x)$ is an antiderivative of $f(x)$

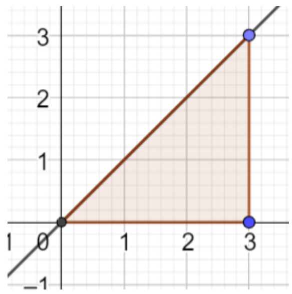
Example 2.132

A. $\int_0^3 x\,dx$

B. $\int_3^7 x\,dx$

$$\int_0^3 x\,dx$$

$$f(x) = x \Rightarrow F(x) = \frac{x^2}{2} + C$$



Part A

$$Lower\ limit\ of\ integration = a = 0$$
$$Upper\ limit\ of\ integration = b = 3$$

Put this all together to get:

$$\int_0^3 x\,dx = F(3) - F(0) = \left(\frac{3^2}{2} + C\right) - \left(\frac{0^2}{2} + C\right) = \frac{9}{2}$$

Part B

Since the constant of integration cancels, we do not need to write the constant when evaluating the definite integral:

---

[3] $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\,dx = -[\sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \underbrace{-\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right)}_{=X} = -\left(1 - \frac{\sqrt{2}}{2}\right) = -\left(\frac{2-\sqrt{2}}{2}\right) = \frac{\sqrt{2}-2}{2}$

$$\int_3^7 x\,dx = F(7) - F(3) = \frac{7^2}{2} - \frac{3^2}{2} = \frac{49}{2} - \frac{9}{2} = \frac{40}{2} = 20$$

Example 2.133

Evaluate the integral below, and interpret it as an area.

$$\int_{-1}^{4} x^2\,dx$$

Calculate the associated indefinite integral:

$$\int x^2\,dx = \frac{x^3}{3} + C$$

Calculate the given definite integral:

$$\int_{-1}^{4} x^2\,dx = \left[\frac{x^3}{3}\right]_{x=-1}^{x=4} = \frac{64}{3} - \left(-\frac{1}{3}\right) = \frac{65}{3}$$

The integral is the area between $y = x^2$ and the x axis for $x = -1$ to $x = 4$.

Example 2.134: Exponential and Logarithmic Integrals

$$\int_e^{e^2} \frac{1}{x}\,dx$$

$$\int_e^{e^2} \frac{1}{x}\,dx = [\ln|x|]_e^{e^2} = \ln|e^2| - \ln|e| = 2 - 1 = 1$$

Suppose we use the power rule:

$$\int_e^{e^2} x^{-1}\,dx = \int_e^{e^2} \frac{x^{-1+1}}{-1+1}\,dx = \int_e^{e^2} \frac{x^0}{0}\,dx \Rightarrow Undefined$$

Example 2.135: Trigonometric Integrals

A. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\ \,dx$

B. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x\,dx$

C. $\int_0^{\pi} \sin x\,dx$

D. $\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$

E. $\int_{\frac{1}{3}}^{\frac{1}{2}} \cos \pi x\,dx$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\,dx = -[\cos x]_{x=\frac{\pi}{4}}^{x=\frac{\pi}{2}} = -\left(0 - \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x\ \,dx = [\tan x]_{x=\frac{\pi}{4}}^{x=\frac{\pi}{3}} = \sqrt{3} - 1$$

$$\int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = -[\cos x]_0^{\pi} = -(-1-1) = 2$$

$$\int_0^{\frac{\pi}{4}} \sec^2 x\,dx = [\tan x]_0^{\frac{\pi}{4}} = \tan\left(\frac{\pi}{4}\right) - \tan 0 = 1 - 0 = 1$$

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos \pi x\,dx = \left[\frac{\sin \pi x}{\pi}\right]_{\frac{1}{3}}^{\frac{1}{2}} = \left(\frac{1}{\pi}\right)\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{3}\right) = \left(\frac{1}{\pi}\right)\left(1 - \frac{\sqrt{3}}{2}\right) = \frac{2-\sqrt{3}}{2\pi}$$

2.136: Integral of Zero Width

$$\int_a^a f(x)\,dx = 0$$

- If you wish to find an integral where the upper and lower limit of integration are the same, then the width of each rectangle is zero.
- The area of a rectangle of zero width is zero. Hence, the property above applies.

Example 2.137

$$\int_{3\pi}^{3\pi} \sin^{1947} x\ dx$$

$$\int_{3\pi}^{3\pi} \sin^{1947} x\ dx = 0$$

C. Sum and Difference Property

2.138: Sum and Difference Property

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$
$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Example 2.139

Split the integral:

$$\int_1^2 (x+1)\,dx$$

$$\int_1^2 (x+1)\,dx = \int_1^2 x\,dx + \int_1^2 1\,dx$$

Example 2.140

If $\int_1^2 [f(x) + x]\,dx = 4$, then find $\int_1^2 f(x)\,dx$

Split the integral $\int_1^2 f(x) + x\,dx = 4$

$$\therefore \int_1^2 f(x)\,dx + \int_1^2 x\,dx = 4$$
$$\int_1^2 f(x)\,dx = 4 - \int_1^2 x\,dx$$

Substitute $\int_1^2 x\,dx = Area\ under\ the\ Curve\ for\ y = x\ over\ the\ interval\ [1,2] = 1.5$

$$\therefore \int_1^2 f(x)\,dx = 4 - 1.5 = 2.5$$

Example 2.141

$$\int_4^{16} \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$$

Rewrite in terms of exponents, and integrate:

$$\int_4^{16} \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) dx = \left[\frac{2x^{\frac{3}{2}}}{3} + 2\sqrt{x}\right]_4^{16}$$

Substitute the limits of integration:

$$\left(\frac{2(16)^{\frac{3}{2}}}{3} + 2\sqrt{16}\right) - \left(\frac{2(4)^{\frac{3}{2}}}{3} + 2\sqrt{4}\right) = \left(\frac{128}{3} + 8\right) - \left(\frac{16}{3} + 4\right) = \left(\frac{112}{3} + 4\right) = \frac{124}{3}$$

D. Constant Multiple Property

2.142: Constant Multiple Property

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

- If you apply a vertical scale to the function, the area under the curve scales proportional to the scale.
- This property is used many times when integrating. Hence, it is important to be able to do this fluently.

Example 2.143

If $\int 3f(x)\,dx = 5$, find $\int f(x)\,dx$

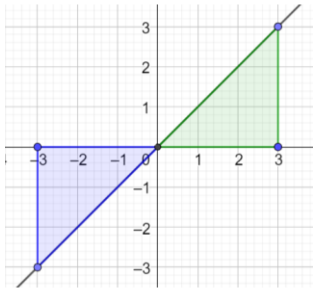
$$\int f(x)\,dx = \frac{1}{3}\int 3f(x)\,dx = \frac{1}{3} \times 5 = \frac{5}{3}$$

2.144: Symmetry

If a function is symmetrical, about the origin, or about a point, the symmetry can be exploited to calculate a definite integral without getting into calculations.

Example 2.145

$$\int_{-3}^{3} x\,dx$$



The integral $\int_{-3}^{3} x\,dx$ represents the area between the x −axis and the function $f(x) = x$ from $x = -3$ till $x = 3$.

The function $f(x) = x$ is symmetric about the origin. Hence, the green area is exactly equal to the blue area.

$$\int_{-3}^{0} x\,dx = \int_{0}^{3} x\,dx = X$$

Hence:

$$\int_{-3}^{3} x\,dx = \int_{-3}^{0} x\,dx + \int_{0}^{3} x\,dx = X - X = 0$$

2.146: Odd Function

If $f(x)$ is an odd function, then

$$\int_{-a}^{a} f(x)\,dx = 0$$

A function which is symmetric about the origin is an odd function. That is:

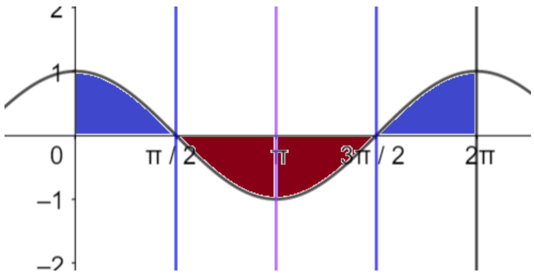
$$f(-x) = -f(x)$$

Example 2.147

$$\int_{0}^{2\pi} \sin^{2021} x\ dx$$

The function $\sin^{2021} x$ is symmetric about $x = \frac{\pi}{2}$

$$\underbrace{\int_{0}^{\frac{\pi}{2}} \sin^{2021} x}_{X} + \underbrace{\int_{\frac{\pi}{2}}^{\pi} \sin^{2021} x}_{-X} + \underbrace{\int_{\pi}^{\frac{3\pi}{2}} \sin^{2021} x}_{-X} + \underbrace{\int_{\frac{3\pi}{2}}^{\pi} \sin^{2021} x}_{X}$$
$$= X - X - X + X = 0$$



2.148: Even Function

If $f(x)$ is an even function, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

A function which is symmetric about the $y - axis$ is an even function. That is:

$$f(-x) = f(x)$$

E. Areas

Example 2.149

Find the area under the curve for the function $f(x) = 1 - x^2$ from $x = 0$ to $x = 1$ using definite integration.

$$f(x) = 1 - x^2 \Rightarrow F(x) = \int (1 - x^2)\,dx = x - \frac{x^3}{3} + C$$

Substitute $f(x) = 1 - x^2$, $a = 0, b = 1$ in

$$\int_a^b f(x)\,dx = \int_0^1 (1-x^2)\,dx = \underbrace{1-\frac{1^3}{3}+C}_{F(1)} - \underbrace{0-\frac{0^3}{3}+C}_{F(0)} = \underbrace{\frac{2}{3}+C}_{F(1)} - \underbrace{C}_{F(0)} = \frac{2}{3}$$

Note that the constant of integration got cancelled.
Hence, we will generally not write the constant of integration when doing definite integration.

Example 2.150

Find the area of the region:

A. Between the graph of $f(x) = \pi x^2$, the x-axis, the line $x = 2$ and the line $x = 4$.
B. Between the graph of $y = \frac{1}{x}$, the x-axis, the line $x = e$ and the line $x = e^2$.

Part A

$$\int_2^4 \pi x^2\,dx = \pi \int_2^4 x^2\,dx = \pi\left[\frac{x^3}{3}\right]_2^4 = \pi\left(\frac{4^3}{3}-\frac{2^3}{3}\right) = \pi\left(\frac{64}{3}-\frac{8}{3}\right) = \frac{56\pi}{3}$$

Part B

$$\int_e^{e^2} \frac{1}{x}\,dx = [\ln x]_e^{e^2} = \ln e^2 - \ln e = 2 - 1 = 1$$

Example 2.151

Determine $\int_0^3 2t + 4\,dt$ and interpret it as an area.

$$\int_0^3 2t + 4\,dt = [t^2 + 4t]_{t=0}^{t=3} = 21 - 0 = 21$$

This represents the area in the region between the lines:

$$x = 0, x = 3, y = 0, y = 2x + 4$$

F. Unsigned Area

2.152: Splitting Limits of Integration

If $c \in [a, b]$, then

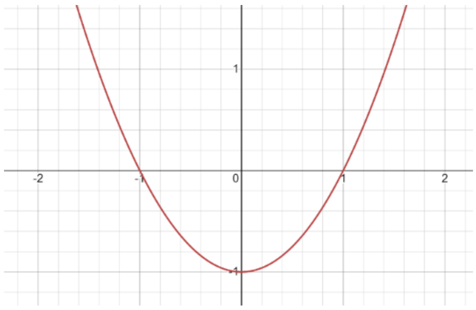
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

- This property lets you "split" an integral into multiple integrals.
- This is useful when part of a function is above the x axis, and part of the function is below the x axis.

Example 2.153

Let $f(x)$ be an even function, graphed alongside. Split the integral:

$$\int_{-1}^2 f(x)\,dx$$



Split the limits of integration:

$$\int_{-1}^2 f(x)\,dx = \int_{-1}^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

Use the property $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ in the first term:

$$= 2\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

2.154: Unsigned Area

Unsigned area is found by taking the absolute value of the area found by dividing the integral into parts such that the signed area is either completely positive or completely negative.

Example 2.155

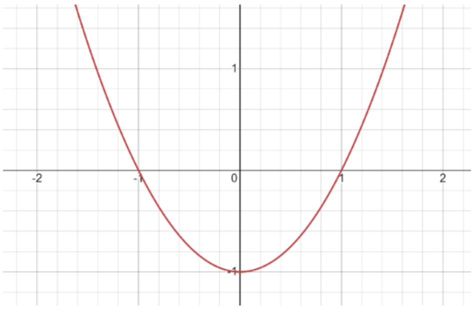
Find an expression for the:

A. signed area for the function $f(x) = x^2 - 1$ over the interval $[-2,2]$.
B. unsigned area for the function $f(x) = x^2 - 1$ over the interval $[-2,2]$.

Part A

The integral for signed area is:

$$\int_{-2}^{2} x^2 - 1\ dx$$



Part B

We need to know when:

$$x^2 - 1 > 0 \Rightarrow (x+1)(x-1) > 0$$

This is a quadratic with leading coefficient >1, which means the graph will be an upward parabola. The solution of the inequality is:

$$x \in (-\infty, -1) \cup (1, \infty) \Rightarrow x > 0$$

Split the integral into three different parts:

$$\left|\int_{-2}^{-1} x^2 - 1\ dx\right| + \left|\int_{-1}^{1} x^2 - 1\ dx\right| + \left|\int_{1}^{2} x^2 - 1\ dx\right|$$

Since the first term and the last term are positive, and the middle term is negative, we get:

$$= \int_{-2}^{-1} (x^2 - 1)\,dx - \int_{-1}^{1} (x^2 - 1)\,dx + \int_{1}^{2} (x^2 - 1)\,dx$$

Example 2.156

Find an expression for the unsigned area for the function $f(x) = x^2 + 5x + 6$ over the interval $[-5,5]$.

$$x^2 + 5x + 6 > 0 \Rightarrow (x+2)(x+3) > 0 \Rightarrow x \in (-\infty, -3) \cup (-2, \infty)$$

The indefinite integral for $f(x)$ is:

$$\int (x^2 + 5x + 6)\,dx = \frac{x^3}{3} + \frac{5x^2}{2} + 6x + C$$

The unsigned area is:

$$\int_{-5}^{-3} f(x)\,dx - \int_{-3}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$$

Example 2.157

Find the unsigned area for the function $f(x) = x^2 + x - 12$ over the interval $[-1,5]$.

$$x^2 + x - 12 > 0$$
$$(x+4)(x-3) > 0$$
$$x \in (-\infty, -4) \cup (3, \infty)$$

$$f(x) = x^2 + x - 12 \Rightarrow F(x) = \frac{x^3}{3} + \frac{x^2}{2} - 12x + C$$
$$-\int_{-1}^{3} (x^2 + x - 12)\,dx + \int_{3}^{5} (x^2 + x - 12)\,dx$$
$$= -\left[\left(\frac{27}{3} + \frac{9}{2} - 36\right) - \left(\frac{-1}{3} + \frac{1}{2} + 12\right)\right] + \left[\left(\frac{125}{3} + \frac{25}{2} - 60\right) - \left(\left(\frac{27}{3} + \frac{9}{2} - 36\right)\right)\right]$$

Example 2.158
Find the unsigned area between the x-axis and the parabola $f(x) = x^2 - 1$ from $x = 0$ to $x = 2$.

Split the integral at the roots, which are: $x^2 - 1 = 0 \Rightarrow x = \pm 1$:
$$\int_0^1 x^2 - 1 = \left[\frac{x^3}{3} - x\right]_0^1 = \left(\frac{1^3}{3} - 1\right) = -\frac{2}{3}$$
$$\int_1^2 x^2 - 1 = \left[\frac{x^3}{3} - x\right]_1^2 = \left(\frac{2^3}{3} - 2\right) - \left(\frac{1^3}{3} - 1\right) = \frac{2}{3} - \left(-\frac{2}{3}\right) = \frac{4}{3}$$
Hence, the area from 0 to 2 (taking the absolute value of the negative area) is:
$$\frac{2}{3} + \frac{4}{3} = \frac{6}{3} = 2$$

Example 2.159
Find the unsigned area for the function $f(x) = x^3 - x$ over the interval $[-4,2]$.

To get unsigned area, we need to split the integral at the places where $f(x) < 0$. Hence, we need to determine the interval(s) over which the function is negative.
$$x^3 - x > 0 \Rightarrow x(x+1)(x-1) > 0$$
The roots of the expression on the LHS above are
$$\{0, \pm 1\}$$

$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
$-ve$	$+ve$	$-ve$	$+ve$

Unsigned area from -4 to 2 is:
$$-\int_{-4}^{-1} (x^3 - x)\,dx + \int_{-1}^{0} (x^3 - x)\,dx - \int_{0}^{1} (x^3 - x)\,dx + \int_{1}^{2} (x^3 - x)\,dx$$

$$\int (x^3 - x)\,dx = \frac{x^4}{4} - \frac{x^2}{2} + C$$

Example 2.160
Find the unsigned area between $f(x) = x^3 + 7x^2 + 16x + 12$ and the $x - axis$ over the interval $[-4,4]$

$$x^3 + 7x^2 + 16x + 12 > 0$$
To find the factors of the cubic on the LHS, we use the Remainder Theorem. We want to see if $x - a$ divides the expression. The values of a come from the factors of 12:
$$\pm\{1,2,3,4,6,12\}$$

We check with only positive values since $16x - 12 > 0, x \geq 1$.
$$f(-1) = -1 + 7 - 16 + 12 = 2 \neq 0$$

$$f(-2) = -8 + 28 - 32 + 12 = 0 \Rightarrow x + 2\ is\ a\ factor$$

$$\frac{x^3 + 7x^2 + 16x + 12}{x + 2} = x^2 + 5x + 6 = (x + 2)(x + 3)$$

Original Polynomial	x^3	x^2	x	$Constant$
	1	7	16	12
-2		-2	-10	-12
	1	5	6	0
Answer	x^2	x	$Constant$	Remainder

$$(x + 2)^2(x + 3) > 0$$

The function will be $+ve$ at $+\infty$.

Since the root -2 is of even multiplicity, the function will not change sign over the interval $(-3, -2)$. Since the root -3 is of odd multiplicity, it will change sign.

$(-\infty, -3)$	$(-3, -2)$	$(-2, \infty)$
$-ve$	$+ve$	$+ve$

Unsigned area from -4 to 4:

$$-\int_{-4}^{-3} f(x)\, dx + \int_{-3}^{4} f(x)\, dx$$

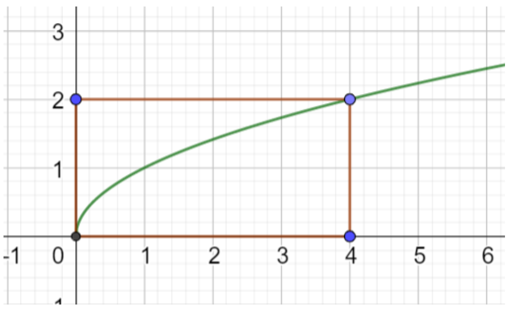
Example 2.161

Find the unsigned area between the curve $f(x) = \sin x$ and the $x - axis$ over one period.

G. Integration with respect to y

Example 2.162

Find the area of the region inside the rectangle formed by the points $(0,0)(4,0), (0,2), (4,2)$ and above the curve $y = \sqrt{x}$.



Integrate with respect to x:

The area of the rectangle

$$= (2)(4) = 8$$

The area between the curve and the $x - axis$ is:

$$\int_0^4 \sqrt{x}\, dx = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4 = \frac{2}{3}\left(4^{\frac{3}{2}}\right) = \frac{16}{3} = 5\frac{1}{3}$$

The area that we want is:

$$\underbrace{8}_{\substack{Area\ of \\ Rectangle}} - \underbrace{5\frac{1}{3}}_{\substack{Area\ below\ the \\ Green\ Curve}} = 2\frac{2}{3}$$

Integrate with respect to y:

$$y = \sqrt{x} \Rightarrow x = y^2$$

$$\int_0^2 y^2\, dy = \left[\frac{y^3}{3}\right]_0^2 = \frac{2^3}{3} = \frac{8}{3} = 2\frac{2}{3}$$

Example 2.163

Find the area of the region inside the rectangle formed by the points $(0,0)(\pi, 0), (0,1), (\pi, 1)$ and above the curve $y = \sin x$.

Integrate with respect to x:
The area between the curve and x −axis is

$$\int_0^\pi (\sin x)\,dx = -[\cos x]_0^\pi = -(\cos\pi - \cos 0) = -\big(-1-(-1)\big) = -(-2) = 2$$

$$\underbrace{\pi}_{\substack{Area\ of\\ Rectangle}} - \underbrace{2}_{\substack{Area\ below\ the\\ Green\ Curve}}$$

Integrate with respect to y:

$$y = \sin^{-1} x \Rightarrow Domain = [-1,1], Range = \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$$2\int_0^1 (\sin^{-1} y)\,dy$$

Example 2.164

$$\int_0^1 (\sin^{-1} y)\,dy$$

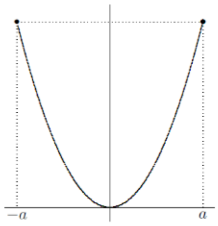
$$x = \sin^{-1} y \Rightarrow y = \sin x$$

$$\int_0^\pi (\sin x)\,dx = -[\cos x]_0^\pi = -(\cos\pi - \cos 0) = -\big(-1-(-1)\big) = -(-2) = 2$$

$$\int_0^1 (\sin^{-1} y)\,dy = \underbrace{\pi}_{\substack{Area\ of\\ Rectangle}} - \underbrace{2}_{\substack{Area\ below\ the\\ Green\ Curve}} = \pi - 2$$

Example 2.165

Phillips Exeter Math 4



717. The diagram shows the parabolic arc $y = x^2$ inscribed in the rectangle $-a \le x \le a$, $0 \le y \le a^2$. This curve separates the rectangle into two regions. Find the ratio of their areas, and show that it does not depend on the value of a.

Area under the curve

$$= \int_{-a}^{a} x^2\,dx = \frac{2a^3}{3}$$

Area of the rectangle

$$= (2a)(a^2) = 2a^3$$

Area above the curve

$$2a^3 - \frac{2a^3}{3} = \frac{4a^3}{3}$$

Ratio

$$= \frac{2a^3}{3} : \frac{4a^3}{3} = 2:4 = 1:2$$

H. Rationalization

Example 2.166
JEE Main 2024

2.7 Derivatives of Integrals (1st FTC)

A. Basics

The Fundamental Theorem of Calculus can be used to find the derivative of an integral.

2.167: Derivative of an Integral

$$y = \int_a^b f(x)\,dx \Rightarrow \frac{dy}{dx} = \frac{d}{dx}[F(b)] - \frac{d}{dx}[F(a)]$$

From the fundamental theorem of Calculus, we know that given a function $f(x)$ with integral $F(x)$, the definite integral over the interval $[a, b]$ is:

$$y = \int_a^b f(x)\,dx = F(b) - F(a)$$

Differentiate throughout with respect to x:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_a^b f(x)\,dx\right) = \frac{d}{dx}[F(b)] - \frac{d}{dx}[F(a)]$$

Example 2.168

A. Find $\frac{dy}{dx}$ given that $y = \int_a^x t^2\,dt$
B. Find $\frac{dy}{dx}$ given that $y = \int_a^x \sin t\,dt$

Part A

Differentiate both sides of the given equation:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_a^x t^2\,dt\right)$$

Use the fundamental theorem of calculus to calculate the integral:

$$= \frac{d}{dx}\left[\frac{t^3}{3} - \frac{a^3}{3}\right]$$

$$= \frac{d}{dx}\left(\frac{x^3}{3}\right) - \frac{d}{dx}\left(\frac{a^3}{3}\right)$$

Since the second term is a constant, its derivative is zero:

$$= x^2$$

Note that in $y = \int_a^x t^2\,dt$:

$$f(t) = t^2 \Rightarrow f(x) = x^2$$

Part B

Differentiate both sides of the given equation:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_a^x \sin t\,dt\right)$$

Use the fundamental theorem of calculus to calculate the integral:

$$= \frac{d}{dx}[-\cos x - (-\cos a)]$$
$$= \frac{d}{dx}(-\cos x) + \frac{d}{dx}(\cos a)$$

Since the second term is a constant, its derivative is zero:

$$= \sin x$$

2.169: Derivative of an Integral with one variable and one constant limit

$$y = \int_a^x f(t)\,dt \Rightarrow \frac{dy}{dx} = f(x)$$
$$F(x) = \int_a^x f(t)\,dt \Rightarrow F'(x) = f(x)$$

Differentiate both sides of the given equation:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right)$$

Use the fundamental theorem of calculus to calculate the integral (where $F(x)$ is an antiderivative of $f(x)$):

$$= \frac{d}{dx}[F(x) - F(a)]$$

Use the difference rule:

$$= \frac{d}{dx}\big(F(x)\big) - \frac{d}{dx}F(a)$$

Since the second term is a constant, its derivative is zero:

$$= f(x)$$

Example 2.170

A. Evaluate $\frac{dy}{dx}$ if $y = \int_a^x \sin(\cos t)\,dt$

B. Evaluate $\frac{dy}{dx}$ if $y = \int_a^x e^{t^2}\,dt$

C. Evaluate $\frac{dy}{dx}$ when $x = 3$ given that $y = \int_a^x \sqrt{t^2 + 5t - 12}\,dt$

Part A

Let $f(t) = \sin(\cos t)\,, F(t) = \int f(t)\,dt$:

$$y = \int_a^x f(t)\,dt$$

Using FTC:

$$= F(x) - F(a)$$

Differentiate the above:

$$\frac{dy}{dx} = \frac{d}{dx}[F(x)] - \frac{d}{dx}[F(a)] = f(x) + 0 = \sin(\cos x)$$

Part B

Let $f(t) = e^{t^2} \Rightarrow F(t) = \int_a^x f(t)dt$:

$$y = \int_a^x f(t)\,dt = F(x) - F(a)$$

Differentiate the above:

$$\frac{dy}{dx} = \frac{d}{dx}[F(x)] - \frac{d}{dx}[F(a)] = f(x) = e^{x^2}$$

Part C

$$\left.\frac{dy}{dx}\right|_{x=3} = \sqrt{x^2 + 5x - 12} = \sqrt{9 + 15 - 12} = \sqrt{12} = 2\sqrt{3}$$

B. Using the Chain Rule

Example 2.171

Find $\frac{dy}{dx}$ given that

$$y = \int_a^{x^2} \sin t \, dt$$

Method I: Finding the Integral

Differentiate both sides of the given equation:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_a^{x^2} \sin t \, dt\right)$$

Use the fundamental theorem of calculus to calculate the integral:

$$= \frac{d}{dx}[-\cos x^2 - (-\cos a)]$$

Use the sum and difference to split the derivative:

$$= \frac{d}{dx}(-\cos x^2) + \frac{d}{dx}(\cos a)$$

Since the second term is a constant, its derivative is zero:

$$= \sin x^2 \times \frac{d}{dx}(x^2) = 2x \sin x^2$$

Method II: Using FTC

Let $f(t) = \sin t$, $\int f(t)\, dt = F(t) + C$. Then by FTC:

$$y = \int_a^{x^2} f(t)\, dt = F(x^2) - F(a)$$

Differentiate both sides of the above:

$$\frac{dy}{dx} = \frac{d}{dx}[F(x^2)] - \frac{d}{dx}[F(a)] = \left[\frac{d}{dx}F(x^2)\right]\frac{d}{dx}x^2 = f(x^2) \cdot 2x = 2x \sin x^2$$

We did not need to find $F(x)$ at any stage in this calculation.

C. Upper and Lower Limits

Example 2.172

Without finding the integral, find $\frac{dy}{dx}$ given that:

$$y = \int_{\sqrt[3]{x}}^{\sqrt{x}} e^{\sin t} \, dt$$

Let $f(t) = e^{\sin t} \Rightarrow \int e^{\sin t} dt = F(t) + C$

$$y = \int_{\sqrt[3]{x}}^{\sqrt{x}} f(t)\, dt = F(\sqrt{x}) - F(\sqrt[3]{x})$$

Differentiate the above:

$$\frac{dy}{dx} = F'(\sqrt{x})\frac{d}{dx}(\sqrt{x}) - F'(\sqrt[3]{x})\frac{d}{dx}(\sqrt[3]{x}) = \frac{f(\sqrt{x})}{2\sqrt{x}} - \frac{f(\sqrt[3]{x})}{3x^{\frac{2}{3}}} = \frac{e^{\sin\sqrt{x}}}{2\sqrt{x}} - \frac{e^{\sin\sqrt[3]{x}}}{3x^{\frac{2}{3}}}$$

Example 2.173

Without finding the integral, find $\frac{dy}{dx}$ given that

$$y = \int_{\sin x}^{\cos x} \frac{1}{t}\, dt$$

Let $f(t) = \frac{1}{t} \Rightarrow \int f(t) = F(t) + C$

$$y = \int_{\sin x}^{\cos x} \frac{1}{t}\, dt = F(\cos x) - F(\sin x)$$

Differentiate the above:

$$\frac{dy}{dx} = f(\cos x)(\cos x)' - f(\sin x)(\sin x)' = -\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x} = -\tan x - \cot x$$

Example 2.174

Without calculating the integral, find $\frac{dy}{dx}$ if:

$$y = \int_{x^2}^{x^3} (t^4 + 1) dt$$

Let $f(t) = t^4 + 1 \Rightarrow \int f(t) = F(t) + C$:

$$y = \int_{x^2}^{x^3} (t^4 + 1)\, dt$$
$$y = F(x^3) - F(x^2)$$

Take the derivative of the first and the last term with respect to x:

$$\frac{dy}{dx} = \frac{d}{dx}[F(x^3)] - \frac{d}{dx}[F(x^2)]$$

Use the chain rule:

$$= F'(x^3)\frac{d}{dx}x^3 - F'(x^2)\frac{d}{dx}x^2 = \underbrace{f(x^3)3x^2 - f(x^2)2x}_{F'(x)=f(x)}$$

Apply the definition of $f(x)$:

$$= [(x^3)^4 + 1]3x^2 - [(x^2)^4 + 1]2x = 3x^{14} + 3x^2 - 2x^9 - 2x = 3x^{14} - 2x^9 + 3x^2 - 2x$$

Example 2.175

JAM MA 2019/47

Example 2.176

JAM MA 2019/49

D. Challenging Questions

Challenge 2.177

This question from JEE Advanced makes use of:
1. Definition of definite integral (Reimann sums) as a limit
2. Logarithms
3. Summation and Product Notation
4. Increasing and Decreasing Functions
5. Inequalities
6. Taking limits by taking the logarithm

2.8 u –Substitution with Definite Integrals

A. Basics

We have already done u substitution with indefinite integrals. We can use u substitution with definite integrals as well.

2.178: Method I: Changing the Limits

When doing definite integration with a change of variable, the limits of integration will also change. This needs to be done carefully.

Example 2.179

Evaluate

A. $\int_0^1 \sqrt{1+8x}\,dx$
B. $\int_1^2 2x\sqrt{x^2+2}\,dx$
C. $\int_0^1 x^2\sqrt{3x^3+1}\,dx$

Part A

Let $u = 1+8x \Rightarrow \frac{du}{8} = dx$. Then:

$$x = 0 \Rightarrow u = 1+8x = 1+8(0) = 1$$
$$x = 1 \Rightarrow u = 1+8x = 1+8(1) = 9$$

Making the above substitutions:

$$\int_0^1 \sqrt{1+8x}\,dx = \int_1^9 \frac{\sqrt{u}}{8}\,du = \frac{1}{8}\times\frac{2}{3}u^{\frac{3}{2}}\Big|_1^9 = \frac{1}{8}\times\frac{2}{3}(27-1) = \frac{1}{8}\times\frac{2}{3}(26) = \frac{13}{6}$$

Part B

Let $u = x^2+2 \Rightarrow du = 2x\,dx$. Then:

$$x = 1 \Rightarrow u = 3, \qquad x = 2 \Rightarrow u = 6$$

Make the above substitutions:

$$\int_{u=3}^{u=6} \sqrt{u}\,du = \frac{2}{3}6^{\frac{3}{2}} - \frac{2}{3}3^{\frac{3}{2}} = \frac{2}{3}\left(6\sqrt{6}-3\sqrt{3}\right) = 2\left(2\sqrt{6}-\sqrt{3}\right) = 4\sqrt{6}-2\sqrt{3}$$

Part C

Let $u = 3x^3+5 \Rightarrow du = 9x^2\,dx$. Then:

$$x = 0 \Rightarrow u = 3x^3+5 = 3(0)^3+1 = 1$$

$$x = 1 \Rightarrow u = 3(1)^3 + 1 = 3 + 1 = 4$$

Make the substitution:

$$\int_0^1 x^2\sqrt{3x^3+1}\,dx = \frac{1}{9}\int_1^4 \sqrt{u}\,du = \frac{1}{9}\times\frac{2}{3}u^{\frac{3}{2}}\Big|_{u=1}^{u=4} = \frac{2}{27}\times u^{\frac{3}{2}}\Big|_{u=1}^{u=4} = \frac{2}{27}(8-1) = \frac{14}{27}$$

2.180: Method II: Find the associated indefinite integral

A definite integral can also be found by finding the associated indefinite integral (with or without u-substitution). In this case, you do not need to change the limits of integration.

Example 2.181

$$\int_1^2 \frac{1}{(x+1)^6}\,dx$$

Method I

Let $u = x + 1 \Rightarrow du = dx$

$$x = 2 \Rightarrow u = 3, \qquad x = 1 \Rightarrow u = 2$$

Make the above substitutions:

$$\int_{x=1}^{x=2} \frac{1}{(x+1)^6}x\,dx = \int_{u=2}^{u=3} \frac{1}{u^6}du = \int_{u=2}^{u=3} -\frac{1}{5u^5}du = \left[\frac{1}{5(3^5)} - \frac{1}{5(2^5)}\right] = \frac{1}{1215} - \frac{1}{160}$$

Method II

Let $u = x + 1 \Rightarrow du = dx$

$$\int \frac{1}{(x+1)^6}dx = \int \frac{1}{u^6}du \Rightarrow \int u^{-6}du = \frac{u^{-5}}{-5} + C = \frac{(x+1)^{-5}}{-5} + C = -\frac{1}{(x+1)^5} + C$$

$$\int_1^2 \frac{1}{(x+1)^6}\,dx = -\left[\frac{1}{5(x+1)^5}\right]_1^2 = -\left[\frac{1}{5(2+1)^5} - \frac{1}{5(1+1)^5}\right] = \frac{1}{1215} - \frac{1}{160}$$

B. Trigonometric Integrals

Example 2.182

$$\int_0^{\frac{\pi}{4}} \cos 2x\,dx$$

Method I

Let $u = 2x \Rightarrow du = 2\,dx$

$$When\ x = 0 \Rightarrow u = 0, \qquad When\ x = \frac{\pi}{4} \Rightarrow u = 2\times\frac{\pi}{4} = \frac{\pi}{2}$$

Make the above substitutions:

$$\int_{x=0}^{x=\frac{\pi}{4}} \cos 2x\,dx = \frac{1}{2}\int_{u=0}^{u=\frac{\pi}{2}} \cos u\,du = \frac{1}{2}\left[\sin\frac{\pi}{2} - \sin 0\right] = \frac{1}{2}(1) = \frac{1}{2}$$

Method II

Let $u = 2x \Rightarrow du = 2\,dx$

$$\int \cos 2x\,dx = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2x + C$$

$$\int_0^{\frac{\pi}{4}} \cos 2x \, dx = \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} = [\quad]$$

C. Inverse Trigonometric Integrals

Example 2.183
If

$$\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$$

where $\tan^{-1}x$ takes only principal values, then the value of $\ln(|1+\alpha|) - \frac{3\pi}{4}$ is: (JEE-A 2015)

Make the substitution:

$$u = 9x + 3\tan^{-1}x \Rightarrow du = 9 + \frac{3}{1+x^2}dx = \frac{12+3x^2}{1+x^2}dx$$

Then the limits of integration change to:

$$x = 0 \Rightarrow u = 0, \qquad x = 1 \Rightarrow u = 9 + \frac{3\pi}{4}$$

Substitute the above in the given integral:

$$\alpha = \int_0^{9+\frac{3\pi}{4}} e^u du = [e^u]_0^{9+\frac{3\pi}{4}} = e^{9+\frac{3\pi}{4}} - 1$$

Add 1 to both sides, and substitute $\alpha + 1 = e^{9+\frac{3\pi}{4}}$ to evaluate:

$$\ln(|1+\alpha|) - \frac{3\pi}{4} = \ln\left(\left|e^{9+\frac{3\pi}{4}}\right|\right) - \frac{3\pi}{4} = 9 + \frac{3\pi}{4} - \frac{3\pi}{4} = 9$$

2.9 Average Value, Area between Curves

A. Mean of a Function

2.184: Mean (Average) of a function
If a function is integrable on a closed interval $[a,b]$, then:

$$Average\ Value\ of\ f(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Since we want the average value, we distribute the total area so that it forms a rectangle. For a rectangle:

$$Area = Height \cdot Width$$

Solve the above expression for $Height$:

$$Height = \frac{Area}{Width} = \frac{\int_a^b f(x)\,dx}{b-a} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example 2.185
Find the average value of:

A. $f(x) = x^2$ over the range $[2,4]$.
B. $f(x) = \sin x$ over the range $\left[0, \frac{\pi}{2}\right]$.

C. $f(x) = e^x$ over the range $[2,4]$

Part A

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{4-2}\int_2^4 x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_2^4 = \frac{1}{2}\left(\frac{64}{3}-\frac{8}{3}\right) = \frac{1}{2}\left(\frac{56}{3}\right) = \frac{28}{3}$$

Part B

$$\frac{1}{\frac{\pi}{2}-0}\int_0^{\frac{\pi}{2}} \sin x\ dx = -\frac{2}{\pi}[\cos x]_0^{\frac{\pi}{2}} = \left(-\frac{2}{\pi}\right)(0-1) = \frac{2}{\pi}$$

Part C

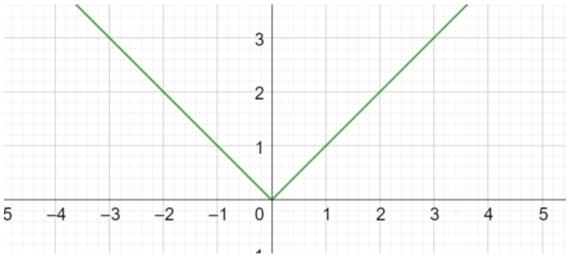
$$\frac{1}{4-2}\int_2^4 e^x\,dx = \frac{1}{2}[e^x]_2^4 = \frac{1}{2}(e^4-e^2)$$

Example 2.186

Find the average value of the function $f(x) = |x|$ over the interval $(-2,3)$.

Use the definition:

$$\frac{1}{3-(-2)}\int_{-2}^3 |x|\,dx = \frac{1}{5}\int_{-2}^3 |x|\,dx$$



Since it is the absolute value function, we need to consider it piece-wise and split the integral:

$$= \frac{1}{5}\left[\int_{-2}^0 |x|\,dx + \int_0^3 |x|\,dx\right]$$

$$= \frac{1}{5}\left[\int_{-2}^0 -x\,dx + \int_0^3 x\,dx\right]$$

$$= \frac{1}{5}\left[\left[-\frac{x^2}{2}\right]_{-2}^0 + \left[\frac{x^2}{2}\right]_0^3\right] = \frac{1}{5}\left[\left(0-\left(-\frac{4}{2}\right)\right)+\left(\frac{9}{2}\right)\right] = \frac{1}{5}\left(\frac{13}{2}\right) = \frac{13}{10} = 1.3$$

B. Area between Curves

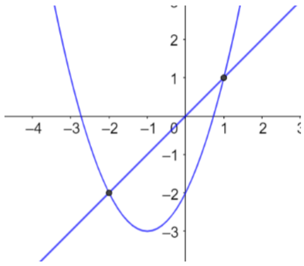
Area between curves is given by the integral of difference of two curves.

Example 2.187

The graph alongside shows:

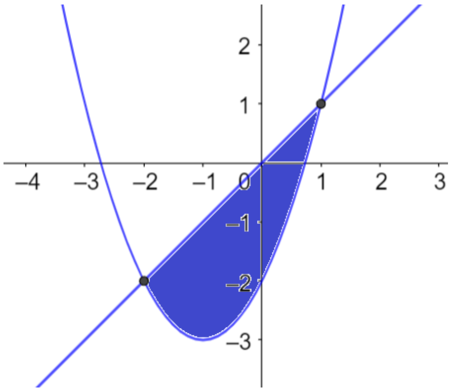
$$f(x) = x^2 + 2x - 2$$
$$g(x) = x$$

Find the area between $f(x)$ and $g(x)$.



From the graph the points of intersection of $f(x)$ and $g(x)$ are -2 and 1.
Hence, we want to find area over the interval:

$$[-2,1]$$



If we divide this area into rectangles, the height of a rectangle at a point x will be given by:

$$g(x) - f(x)$$

Combining the above two, and writing the required area as an integral gives us:

$$\int_{-2}^{1} [g(x) - f(x)]\, dx$$

Substituting the definition of $f(x)$ and $g(x)$ gives us:

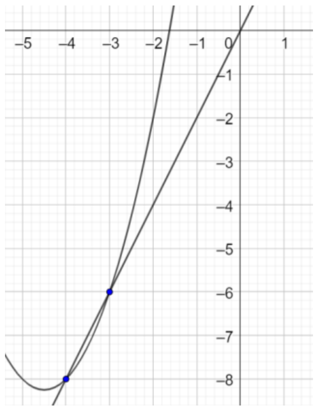
$$\int_{-2}^{1} [x - (x^2 + 2x - 2)]\, dx = \int_{-2}^{1} (-x^2 - x + 2)\, dx$$

Integrate, and evaluate:

$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1} = \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - \frac{4}{2} - 4\right) = -\frac{9}{3} + \frac{3}{2} + 6 = -3 + 1.5 + 6 = 4.5$$

Example 2.188

Find the area between the curves $f(x) = x^2 + 9x + 12$ and $g(x) = 2x$.



Equate the two functions to find the points of intersection:

$$x^2 + 9x + 12 = 2x$$
$$x^2 + 7x + 12 = 0$$
$$(x + 3)(x + 4) = 0$$
$$x \in \{-3, -4\}$$

These points of intersection are also the bounds of integration. Since we do not know whether the area is positive or negative, we take the absolute value:

$$\left|\int_{-4}^{-3} (x^2 + 7x + 12) dx\right|$$

Evaluate the integral:

$$= \left|\left[\frac{x^3}{3} + \frac{7x^2}{2} + 12x\right]_{-4}^{-3}\right|$$

Substitute the limits of integration:

$$= \left|\left(-9 + \frac{63}{2} - 36\right) - \left(-\frac{64}{3} + 56 - 48\right)\right|$$

Simplify:

$$= \left|-45 + \frac{63}{2} + \frac{64}{3} - 8\right| = \left|\frac{-318 + 189 + 128}{6}\right| = \left|\frac{-1}{6}\right| = \frac{1}{6}$$

Example 2.189

Find the area enclosed between the parabola $4y = 3x^2$ and the straight line $3x - 2y + 12 = 0$. (CBSE 2017)

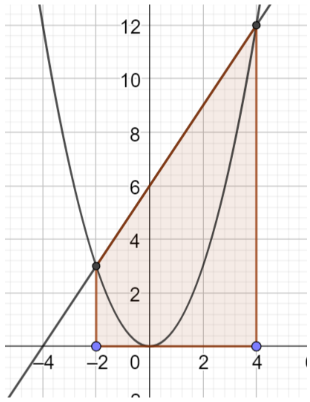
Points of intersection

Solve the equation of the line for y:

$$3x - 2y + 12 = 0 \Rightarrow y = \frac{3x + 12}{2}$$

Substitute the above in $2(2y)y = 3x^2$

$$2(3x + 12) = 3x^2$$
$$3x^2 - 6x - 24 = 0$$
$$x^2 - 2x - 8 = 0$$
$$x \in \{-2, 4\}$$



Integrate using the points of intersection as the limits

Substitute $f(x) = \frac{3x^2}{4}, g(x) = \frac{3x+12}{2}, a = -2, b = 4$ in $\int_a^b [f(x) - g(x)]\ dx$

$$\int_{-2}^{4} \left(\frac{3x + 12}{2} - \frac{3x^2}{4} \right) dx$$

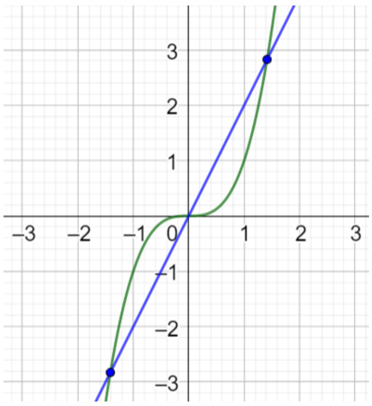
Integrate:

$$= \left[\frac{3x^2}{4} + 6x - \frac{x^3}{4} \right]_{-2}^{4}$$

Evaluate using the limits of integration:

$$= (12 + 24 - 16) - (3 - 12 + 2) = 20 + 7 = 27$$

C. Multiple intersections



Example 2.190

The graph alongside shows:

$$f(x) = x^3$$
$$g(x) = 2x$$

Find the area between $f(x)$ and $g(x)$.

To find the points of intersection, equate the two functions:

$$x^3 = 2x \Rightarrow x^3 - 2x = 0 \Rightarrow x(x^2 - 2) = 0 \Rightarrow x \in \{0, \pm\sqrt{2}\}$$

Method I

To find the bounds of integration:

$$x^3 > 2x \Rightarrow x^3 - 2x > 0$$

	$(-\infty, -\sqrt{2})$	$(-\sqrt{2}, 0)$	$(0, \sqrt{2})$	$(\sqrt{2}, \infty)$
	$-ve$	$+ve$	$-ve$	$+ve$

$$\int_0^{\sqrt{2}} [g(x) - f(x)]\, dx + \int_{-\sqrt{2}}^{0} [f(x) - g(x)]\, dx$$

Method II: Symmetry

$$h(x) = f(x) - g(x) = x^3 - 2x$$
$$h(-x) = (-x)^3 - 2(-x) = -x^3 + 2x = -h(x) \Rightarrow Odd\ Function$$

From symmetry, the second integral is the same as the first integral.

$$= 2\int_0^{\sqrt{2}} [g(x) - f(x)]\, dx = 2\int_0^{\sqrt{2}} (2x - x^3)\, dx = 2\left[x^2 - \frac{x^4}{4}\right]_0^{\sqrt{2}}$$

Substitute the limits of integration and simplify:

$$= 2\left[\left(\sqrt{2}\right)^2 - \frac{\left(\sqrt{2}\right)^4}{4}\right] = 2\left[2 - \frac{4}{4}\right] = 2[1] = 2$$

Example 2.191

Find the area between $f(x) = x^3 - x$ and $g(x) = x$.

To find the points of intersection, equate the two functions:

$$x^3 - x = x \Rightarrow x^3 - 2x = 0 \Rightarrow x(x^2 - 2) = 0 \Rightarrow x \in \{0, \pm\sqrt{2}\}$$

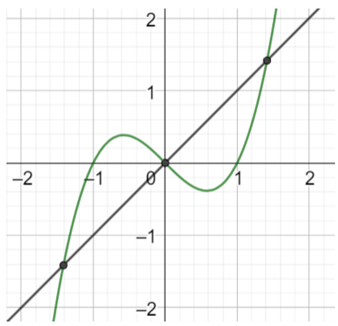
To find the bounds of integration, solve the inequality:

$$h(x) = f(x) - g(x) > 0 \Rightarrow x^3 - 2x > 0$$

We already know the critical points from the equation we solved earlier. Make a sign diagram:

$\left(-\infty, -\sqrt{2}\right)$	$\left(-\sqrt{2}, 0\right)$	$\left(0, \sqrt{2}\right)$	$\left(\sqrt{2}, \infty\right)$
$-ve$	$+ve$	$-ve$	$+ve$
	$\int_{-\sqrt{2}}^{0} h(x)\, dx$	$-\int_0^{\sqrt{2}} h(x)\ dx$	

$$\int_{-\sqrt{2}}^{0} h(x)\, dx = \int_{-\sqrt{2}}^{0} (x^3 - 2x)\, dx = \left[\frac{x^4}{4} - x^2\right]_{-\sqrt{2}}^{0} = -(1 - 2) = 1$$



Note that $h(x) = x^3 - 2x$ is symmetric about the origin:

$$h(-x) = (-x)^3 - 2(-x) = -x^3 + 2x = -h(x) \Rightarrow Odd\ Function$$

$$-\int_0^{\sqrt{2}} h(x)\ dx = \int_{-\sqrt{2}}^{0} h(x)\, dx = 1$$

Hence, the final answer is:

$$1 + 1 = 2$$

Example 2.192

Find the area between $f(x) = \sin x$ and $g(x) = \sin 2x$ over the interval $[0, 2\pi]$.

To find the points of intersection, equate the two functions:

$$\sin 2x = \sin x$$

Use the identity $\sin 2x = 2\sin x \cos x$

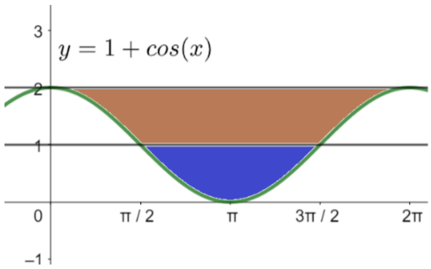
$$2\sin x \cos x = \sin x$$

Case I: If $\sin x = 0$, the equation is satisfied:

$$\sin x = 0 \Rightarrow x \in \{0, \pi, 2\pi\}$$

Case II: If $\sin x \neq 0$, divide both sides by $\sin x$:

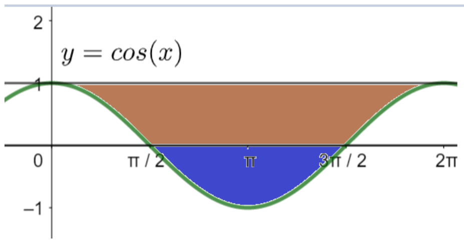
$$2\cos x = 1 \Rightarrow \cos x = \frac{1}{2} \Rightarrow x \in \left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$$



Example 2.193

The blue region is the area between $y = 1$, and $y = 1 + \cos x$. The brown region is the area between $y = 2$, $y = 1$ and $y = 1 + \cos x$. Find the ratio of the blue area to that of the brown area.

Shift the curves down by 1 unit each to get the diagram alongside.



Blue Region

Integrating gives us:

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\, dx = [\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) = -1 - 1 = -2$$

And the area of the blue region is the absolute of the above:

$$Blue = |-2| = 2$$

Brown Region

The area between $y = 1$ and $y = \cos x$ is $Blue + Brown$:

$$Blue + Brown = \int_0^{2\pi} (1 - \cos x) dx = [x - \sin x]_0^{2\pi} = [2\pi - \sin(2\pi)] - [0 - \sin(0)] = 2\pi$$

Subtract the area of the blue region from the above:

$$Brown = 2\pi - 2$$

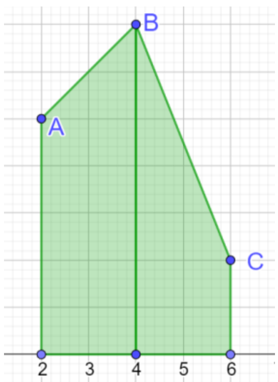
Finally, the required ratio is:

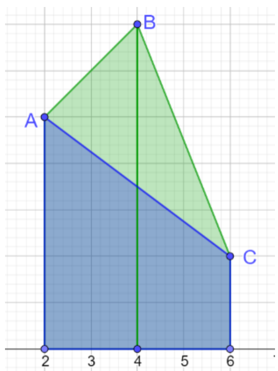
$$2: 2\pi - 2 = 1: \pi - 1$$

D. Area of a Triangle

Example 2.194

Using integration, find the area of ΔABC, the coordinates of whose vertices are $A(2,5)$, $B(4,7)$ and $C(6,2)$. **(CBSE 2010, 2011, 2019)**





Substitute $m = 1, (x_1, y_1) = (2,5)$ in $y - y_1 = m(x - x_1)$ to find the equation of the line AB:

$$y - 5 = 1(x - 2) \Rightarrow y = x + 3$$

$m = \frac{5}{2}, (x_1, y_1) = (4,7)$ in $y - y_1 = m(x - x_1)$ to find the equation of the line BC:

$$y - 7 = -\frac{5}{2}(x - 4) \Rightarrow y = -\frac{5}{2}x + 17$$

$m = -\frac{3}{4}, (x_1, y_1) = (2,5)$ in $y - y_1 = m(x - x_1)$ to find the equation of the line AC:

$$y-5=-\frac{3}{4}(x-2)\Rightarrow y=-\frac{3}{4}x+\frac{13}{2}$$

$$\int_2^4 Line\ AB+\int_4^6 Line\ BC-\int_2^6 Line\ AC$$

$$\int_2^4 (x+3)\,dx+\int_4^6\left(-\frac{5}{2}x+17\right)dx-\int_2^6\left(-\frac{3}{4}x+\frac{13}{2}\right)dx$$

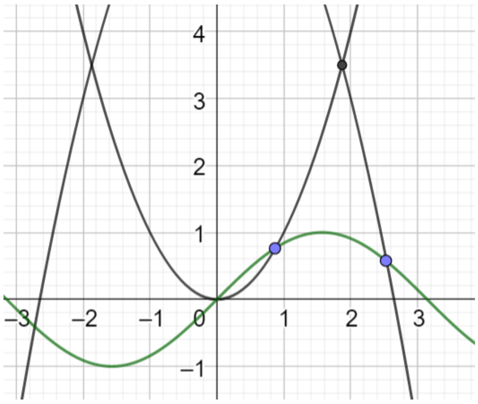
The final answer is:

$$Area=7$$

$$\int_2^4 (Line\ AB-Line\ AC)+\int_4^6 (Line\ BC-Line\ AC)$$

Example 2.195

$$y=x^2$$
$$y=-x^2+7$$
$$y=\sin x$$



E. Integrating with respect to y

So far, we have been integrating with respect to x, drawing vertical rectangles (in terms of the Reimann sum), and finding the area. Instead of drawing vertical rectangles, we can draw horizontal rectangles, create a Reimann sum, write the Reimann sum as an integral, and evaluate. Most importantly, note that we will now be integrating with respect to y.

Example 2.196

Find the area between the curves $y=\sqrt[3]{x}$ and $y=\sqrt{x}$ by

A. Integrating with respect to x
B. Integrating with respect to y

Integrate with respect to x:

Find the points of intersection:

$$\sqrt[3]{x} = \sqrt{x} \Rightarrow x \in \{0,1\}$$

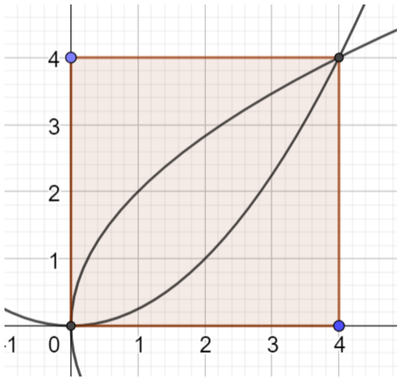
Find the integral:

$$\int_0^1 \left(\sqrt[3]{x} - \sqrt{x}\right) dx = \frac{1}{12}$$

Integrate with respect to y:

$$y = \sqrt[3]{x} \Rightarrow x = y^3, \qquad y = \sqrt{x} \Rightarrow x = y^2$$
$$y^3 = y^2 \Rightarrow y \in \{0,1\}$$
$$\int_0^1 (y^3 - y^2)\, dx = -\frac{1}{12} \Rightarrow Area = \left|-\frac{1}{12}\right| = \frac{1}{12}$$

F. Bounded Regions



Example 2.197

Using integration, prove that curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0, x = 4, y = 4$ and $y = 0$ into three equal parts. (CBSE 2009, 2015, 2019)

Points of Intersection

G. Back Calculations

2.10 Volumes with Cross Sections

A. Volumes with Cross Sections

2.198: Volume using Cross Sections

A solid with cross sectional area $A(x)$ from $x = a$ to $x = b$ has volume given by the definite integral of $A(x)$ from a to b:

$$Volume = \int_a^b A(x)\, dx$$

Example 2.199: Warmup

Show that:

A. A cuboid with length l, width w, and height h has volume lwh.
B. A cylinder with base radius r, and height h has volume $\pi r^2 h$.
C. (*Non Constant Cross Sectional Area*) A rectangular pyramid with length l, width w, and height h has volume $\frac{1}{3}lwh$.
D. The volume of a circular cone with radius r is $\frac{1}{3}\pi r^2 h$.
E. A pyramid with area of base B has volume $\frac{1}{3}Bh$.

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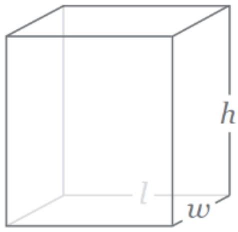
Part A



Introduce a coordinate system with origin at bottom left of the cuboid. The bottom left of the cuboid has $y = 0$ and the top left of the cuboid has $y = h$, where h is the height of the cuboid.

The limits of integration are:

$$y = 0, y = h$$

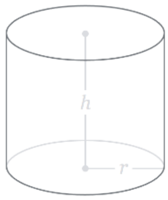
Any horizontal cross section of the cuboid at height y has area:

$$A(y) = lw = Constant$$

Hence, the volume is:

$$\int_0^h lw\, dy = lw \int_0^h 1\, dy = lw[y]_0^h = lwh$$

Part B



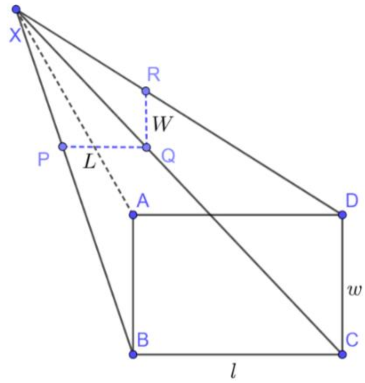
Any horizontal cross section of the cylinder at height y has area:

$$A(y) = \pi r^2 = Constant$$

Hence, the volume is:

$$\int_0^h \pi r^2\, dy = \pi r^2 \int_0^h 1\, dy = \pi r^2 [y]_0^h = \pi r^2 h$$

Part C



Note that for a general rectangular pyramid, it is not necessary that the apex of the pyramid is above the center of the base.

Introduce a coordinate system with origin at the apex of the pyramid, and positive in the downward direction.

Note that any horizontal cross section of the pyramid will be similar to the base. The scaling factor due to the similarity will be:

$$\frac{y}{h}$$

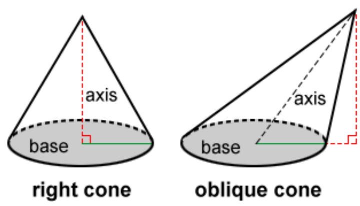
In fact, if y is the distance from the top of the pyramid, then the area of a horizontal cross section of the pyramid is:

$$A(y) = \underbrace{\left(l \cdot \frac{y}{h}\right)}_{Length} \underbrace{\left(w \cdot \frac{y}{h}\right)}_{Width} = \frac{lw}{h^2} \cdot y^2$$

The volume is:

$$\int_0^h \frac{lw}{h^2} \cdot y^2\, dy = \frac{lw}{h^2} \int_0^h y^2\, dy = \frac{lw}{h^2} \left[\frac{y^3}{3}\right]_0^h = \frac{lw}{h^2} \cdot \frac{h^3}{3} = \frac{1}{3} lwh$$

Part D



Using similarity, if y is the distance from the top of the pyramid, then the area of a horizontal cross section of the pyramid is:

$$A(y) = \pi r^2 \cdot \frac{y^2}{h^2}$$

The volume is:

$$\int_0^h \pi r^2 \cdot \frac{y^2}{h^2} dy = \frac{\pi r^2}{h^2} \int_0^h y^2\, dy = \frac{\pi r^2}{h^2} \left[\frac{y^3}{3}\right]_0^h = \frac{\pi r^2}{h^2} \cdot \frac{h^3}{3} = \frac{1}{3} \pi r^2 h$$

Part E

Using similarity, if y is the distance from the top of the pyramid, then the area of a horizontal cross section of the pyramid is:

$$A(y) = B \cdot \frac{y^2}{h^2}$$

The volume is:

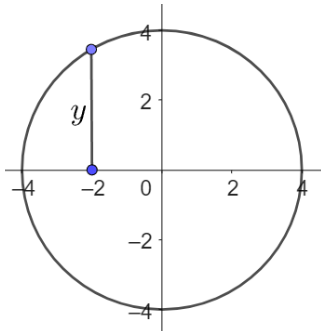
$$\int_0^h B \cdot \frac{y^2}{h^2} dy = \frac{B}{h^2} \int_0^h y^2 \, dy = \frac{B}{h^2} \left[\frac{y^3}{3}\right]_0^h = \frac{B}{h^2} \cdot \frac{h^3}{3} = \frac{1}{3} Bh$$

Example 2.200: Semi-Circular Cross Sections

A solid region has a circular base with center at the origin and a radius of 4. Cross sections perpendicular to the x-axis are semi-circles. Find the volume of the solid using integration.

Recognize that the shape is a hemisphere with volume:

$$= \frac{1}{2} \cdot \frac{4}{3} \pi r^3 = \frac{1}{2} \cdot \frac{4}{3} \pi (4)^3 = \frac{128}{3} \pi$$



Verify the calculation above using Calculus. As shown in the diagram, the radius of the semi-circle is y. Area of the semicircle

$$= A(y) = \frac{1}{2} \cdot \pi r^2 = \frac{1}{2} \cdot \pi y^2 = \frac{\pi}{2} y^2$$

Substitute $y^2 + x^2 = r^2 \Rightarrow y^2 + x^2 = 4^2 \Rightarrow y^2 = 16 - x^2$

$$A(x) = \frac{\pi}{2}(16 - x^2)$$

The limits of integration are:

$$a = -4, b = 4$$

$$Volume = \int_a^b A(x) \, dx = \int_{-4}^{4} \left[\frac{\pi}{2}(16 - x^2)\right] dx = \frac{\pi}{2} \int_{-4}^{4} [16 - x^2] dx$$

Since $A(x)$ is an even function, we can find use the property of a symmetric function:

$$= \pi \int_0^4 [16 - x^2] \, dx = \pi \left[16x - \frac{x^3}{3}\right]_0^4 = \pi \left[\left(64 - \frac{64}{3}\right) - (0 - 0)\right] = \frac{128}{3} \pi$$

Example 2.201

The region R is the circle with radius 3 centered at the origin. Cross sections perpendicular to the x axis that lie in the region R are squares. Find the volume.

Solve the equation of a circle for y^2:

$$x^2 + y^2 = 9 \Rightarrow y^2 = 9 - x^2$$

Let the side of the square be:

$$s(x) = 2y$$

The area function is then:

$$A(x) = [s(x)]^2 = (2y)^2 = 4y^2 = 4(9 - x^2)$$

The integral that gives us the volume is:

$$\int_a^b A(x)\,dx = 4\int_{-3}^{3}(9-x^2)\,dx = 8\int_0^3 (9-x^2)\,dx = 8(18) = 144$$

B. Disc Method

2.202: Disc Method: Revolving around the $x - axis$

A solid generated by revolving the curve $y = f(x)$ around the $x - axis$ from $x = a$ to $x = b$ has volume given by:

$$Volume = \pi \int_a^b [f(x)]^2\,dx$$

- The cross section of the solid is a circle with $radius = R(x) = f(x)$
- Hence, the area is found by using the formula for the area of a circle:

$$= A(x) = \pi[R(x)]^2 = \pi[f(x)]^2$$

Example 2.203

Setup an integral to find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x-3}$ and $y = \sqrt{6-x}$ around the x axis

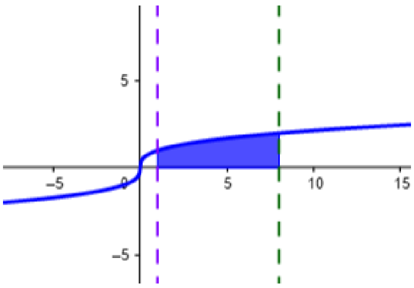
Since the graph is symmetrical about the line x=4.5, we can double the result of the integration from 3 to 4.5

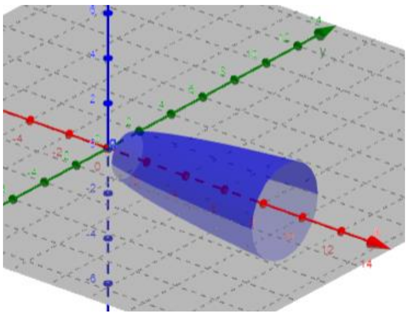
Substitute $[R(x)]^2 = \left(\sqrt{x-3}\right)^2 = x - 3$ in $2\pi\int_3^{4.5}[R(x)]^2\,dx$

$$2\pi\int_3^{4.5} x - 3\,dx$$

Example 2.204

Find, in terms of π, the volume of the solid generated by revolving the curve $y = \sqrt[3]{x}, 1 \le x \le 8$ around the x −axis.





Substitute $a = 1, b = 8, f(x) = \sqrt[3]{x}$:

$$Volume = \pi\int_a^b [f(x)]^2\,dx = \pi\int_1^8 \left[\sqrt[3]{x}\right]^2 dx = \pi\int_1^8 x^{\frac{2}{3}}\,dx$$

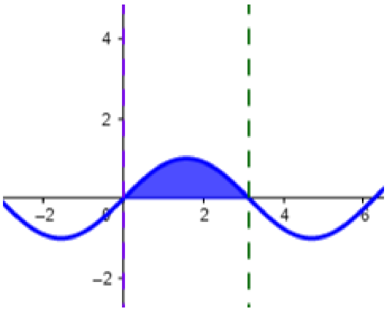
Integrate and simplify:

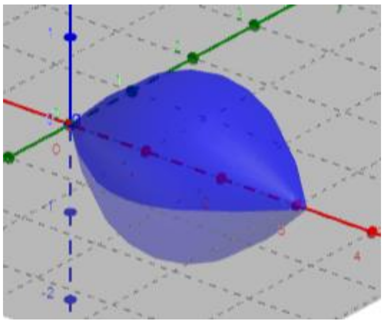
$$= \pi \cdot \frac{3}{5}\left[x^{\frac{5}{3}}\right]_1^8 = \frac{3\pi}{5}\left(8^{\frac{5}{3}} - 1^{\frac{5}{3}}\right) = \frac{3\pi}{5}(32-1) = \frac{93\pi}{5}$$

Example 2.205

Find, in terms of π, the volume of the solid generated by revolving the curve $y = \sin x\,, 0 \le x \le \pi$ around the

x –axis.





Substitute $a = 0, b = \pi, f(x) = \sin x$:

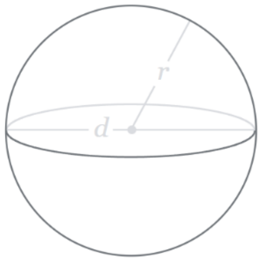
$$Volume = \pi \int_a^b [f(x)]^2\, dx = \pi \int_0^\pi [\sin x]^2\, dx = \pi \int_0^\pi \left(\frac{1 - \cos 2x}{2}\right) dx$$

Integrate and simplify:

$$= \pi \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^\pi = \pi\left(\frac{\pi}{2} - \frac{\sin 2\pi}{4}\right) = \pi\left(\frac{\pi}{2} - \frac{0}{4}\right) = \frac{\pi^2}{2}$$

Example 2.206

Find the volume of a sphere with radius r.



The equation of a circle:

$$y^2 + x^2 = r^2 \Rightarrow y = \pm\sqrt{r^2 - x^2}$$

Because of the $\pm$, this is not a function. However, we take only the positive value, we get a semicircle with radius r centered at the origin:

$$R(x) = y = \sqrt{r^2 - x^2} \Rightarrow [R(x)]^2 = y^2 = r^2 - x^2$$

Evaluate the integral below:

$$\int_{-r}^{r} \pi\big(R(x)\big)^2\, dx$$

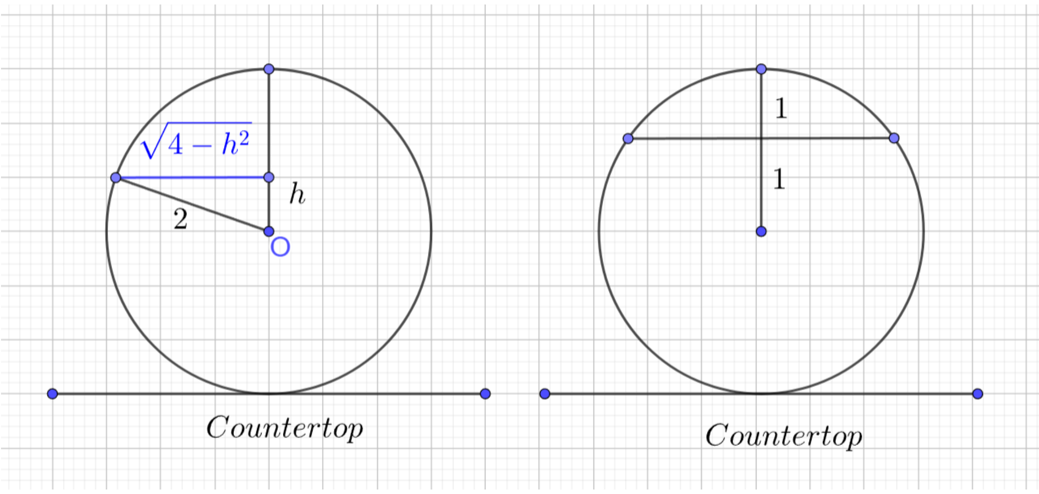
Since this is an even function, the integral from $-r$ to r is twice the integral from 0 to r:

$$2\pi \int_0^r (r^2 - x^2)\, dx = 2\pi\left[r^2 x - \frac{x^3}{3}\right]_0^r = 2\pi\left(r^3 - \frac{r^3}{3}\right) = \frac{4}{3}\pi r^3$$

Example 2.207

A chef has cheese in a spherical shape with a radius of 2 inches. He asks his assistant to get him one-fourth of the cheese. The assistant places the cheese on the countertop, slices it into two parts using a cut parallel to the countertop and 3 inches above the countertop. He brings the smaller part to the chef. What is the ratio of the cheese that the chef got to the cheese that he expected.

Consider a vertical cross section of the cheese ball (spherical ball), and introduce a coordinate system with origin at the center of the sphere.



Then:

$$Radius\ of\ Sphere = 2$$
$$Height = h$$

Using Pythagoras,

$$Radius\ of\ Horizontal\ CrossSection = \sqrt{4-h^2}$$

The formula for volume using a disc is:

$$Volume = \pi \int_a^b [f(x)]^2\, dx$$

We will integrate with respect to h:

$$Volume = \pi \int_1^2 \left(\sqrt{4-h^2}\right)^2\, dh = \pi \int_1^2 (4-h^2)\, dh = \pi \left[4h - \frac{h^3}{3}\right]_1^2$$
$$= \pi \left[\left(8-\frac{8}{3}\right) - \left(4-\frac{1}{3}\right)\right] = \pi\left(4-\frac{7}{3}\right) = \frac{5}{3}\pi$$

$$\frac{\frac{5}{3}\pi}{\frac{1}{4}\cdot\frac{4}{3}\pi(2^3)} = \frac{\frac{5}{3}}{\frac{8}{3}} = \frac{5}{8}$$

Example 2.208

A hemispherical bowl is filled with water to one half of its depth. Find the ratio of water to air in the bowl.

Since all bowls are similar, consider a bowl with radius 10. Consider a vertical cross section of the bowl, and introduce a coordinate system with origin at the center of the sphere.

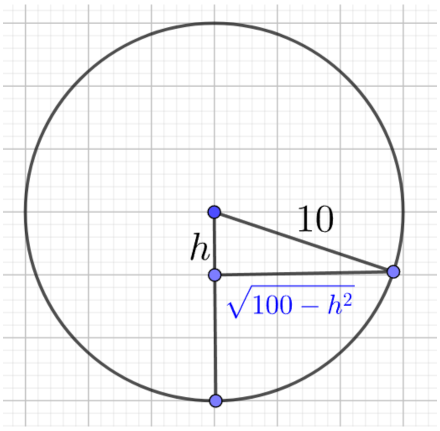
$$Radius\ of\ Sphere = 10, \qquad Height = h$$

Using Pythagoras,

$$Radius\ of\ Horizontal\ CrossSection = \sqrt{100-h^2}$$

Substitute $\sqrt{100-h^2}$ in

$$Volume = \pi \int_a^b [R(x)]^2\, dx$$



To get:

$$Volume = \pi \int_{-10}^{-5} \left(\sqrt{100-h^2}\right)^2 dh = \pi \int_{-10}^{-5} (100-h^2)\, dh$$

Integrate:

$$= \pi \left[100h - \frac{h^3}{3}\right]_{-10}^{-5}$$

Simplify:

$$= \pi \left[\left(-500 - \frac{-125}{3}\right) - \left(-1000 - \frac{-1000}{3}\right)\right]$$

$$= \pi \left[-500 + \frac{125}{3} + 1000 - \frac{1000}{3}\right] = \pi \left[500 - \frac{875}{3}\right] = \pi \left[\frac{1500}{3} - \frac{875}{3}\right] = \pi \left[\frac{625}{3}\right]$$

The ratio of the volume of the filled bowl to that of the entire bowl is:

$$\frac{\pi \left[\frac{625}{3}\right]}{\frac{1}{2}\times\frac{4}{3}\pi r^3} = \frac{\frac{625}{3}}{\frac{2}{3}(10)^3} = \frac{\frac{625}{3}}{\frac{2000}{3}} = \frac{625}{3}\times\frac{3}{2000} = \frac{5}{16}$$

$$\frac{\pi \left[\frac{625}{3}\right]}{\frac{1}{2}\times\frac{4}{3}\pi r^3} = \frac{5}{16}$$

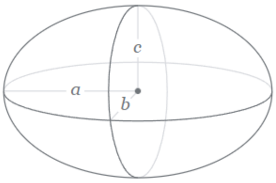
$$\pi \left[\frac{625}{3}\right] = \frac{5}{16}\times\frac{1}{2}\times\frac{4}{3}\pi r^3$$

$$\pi \left[\frac{625}{3}\right] = \frac{5}{16}\times\frac{2000}{3}\pi$$

Example 2.209

Find the volume of the solid generated by rotating the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ around the x −axis.

The limits of integration are from $-a$ to a. Since the function is symmetric, we instead integrate from 0 to a, and double the answer.



Substitute $[f(x)]^2 = y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$ in the formula for

$$Volume = \pi \int_a^b [f(x)]^2\, dx = 2\pi \int_0^a b^2\left(1-\frac{x^2}{a^2}\right) dx$$

Factor out b^2 and carry out the integration:

$$= 2\pi b^2 \left[x - \frac{x^3}{3a^2}\right]_0^a$$

Substitute the limits of integration and simplify:

$$= 2\pi b^2\left(a - \frac{a^3}{3a^2}\right) = 2\pi b^2\left(a-\frac{a}{3}\right) = 2\pi b^2\left(\frac{2}{3}a\right) = \frac{4}{3}\pi a b^2$$

C. Washer Method

2.210: Washer Method

The volume of the solid between $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is:

$$Volume = \pi \int_a^b [R(x)]^2 - [r(x)]^2 \, dx$$

Example 2.211

Setup an integral to find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x-3}$ and $y = \sqrt{6-x}$ around the line $y = -2$.

Since the graph is symmetrical about the line x=4.5, we can double the result of the integration from 3 to 4.5

$$Outer = R^2 = \left[\left(\sqrt{x-3}\right) + 2\right]^2$$
$$Inner = r^2 = 2^2 = 4$$

$$2\pi \int_3^{4.5} \left[\left(\sqrt{x-3}\right) + 2\right]^2 - 4 \, dx$$

Example 2.212

Find, but do not evaluate, an integral expression for the volume of the solid between $y = 1$ and $y = \sqrt[3]{x}$ for $1 \le x \le 8$.

The limits of integration are:

$$x = 1, x = 8$$

$$Volume = \pi \int_1^8 \left[\sqrt[3]{x}\right]^2 - [1]^2 \, dx = \pi \left(\int_1^8 \left[\sqrt[3]{x}\right]^2 dx - \int_1^8 1 \, dx \right)$$

Example 2.213

Find, in terms of π, the volume of the solid obtained by rotating, around the x-axis, the area between $f(x) = \sin(x) + 2$ and $g(x) = 1$ for $0 \le x \le \pi$.

$$Volume = \pi \int_a^b [R(x)]^2 - [r(x)]^2 \, dx$$

Substitute $R(x) = \sin(x) + 2, r(x) = 1$:

$$\pi \int_a^b [\sin(x) + 2]^2 - [1]^2 \, dx$$

2.214: Revolving around a general function

A solid generated by revolving the curve $y = f(x)$ around the function $y = g(x)$ from $x = a$ to $x = b$ has volume given by:

$$Volume = \int_a^b \pi [R(x)]^2 \, dx = \pi \int_a^b [f(x) - g(x)]^2 \, dx$$

Example 2.215
Setup an integral to find the volume of the solid generated by revolving the curve $y = \sqrt[3]{x}, 1 \le x \le 8$ around the line $y = 1$.

The limits of integration are:

$$x = 1, x = 8$$

$$Volume = \pi \int_a^b [f(x) - g(x)]^2\, dx = \pi \int_1^8 \left[\sqrt[3]{x} - 1\right]^2 dx$$

Example 2.216
The curve $f(x) = \sin x$ is rotated around $g(x) = \frac{1}{2}$. Setup an integral to find the volume of the solid between $f(x)$ and $g(x)$ for $0 \le x \le \pi$.

$$\pi \int_a^b [f(x) - g(x)]^2\, dx = \pi \int_0^\pi \left[\sin(x) - \frac{1}{2}\right]^2 dx$$

D. Rotations around the y-axis

2.217: Disc Method: Revolving around the $y - axis$
A solid generated by revolving the curve $x = f(y)$ around the $y - axis$ from $y = a$ to $y = b$ has volume given by:

$$Volume = \pi \int_a^b [f(y)]^2\, dy$$

Example 2.218
Setup an integral to find the volume of the solid generated by revolving the curve $y = x^2, 1 \le y \le 4$ around the y −axis.

Take circular cross sections perpendicular to the y-axis.
Substitute $a = 1, b = 4, x^2 = [f(y)]^2 = y$ in the formula for the volume of a solid rotated about the y-axis:

$$Volume = \pi \int_a^b [f(y)]^2\, dy = \pi \int_1^4 y^2\, dy =$$

E. Rotating around a Horizontal Line

Example 2.219
Setup an integral to find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x - 3}$ and $y = \sqrt{6 - x}$ around the line $x = \ln\left(\frac{1}{2}\right)$.

Determine the point of intersection of the curves:

$$\sqrt{x - 3} = \sqrt{-x + 6} \Rightarrow x - 3 = -x + 6 \Rightarrow x = \frac{9}{2}$$

$$y = \sqrt{x-3} = \sqrt{\frac{9}{2} - 3} = \sqrt{\frac{3}{2}}$$

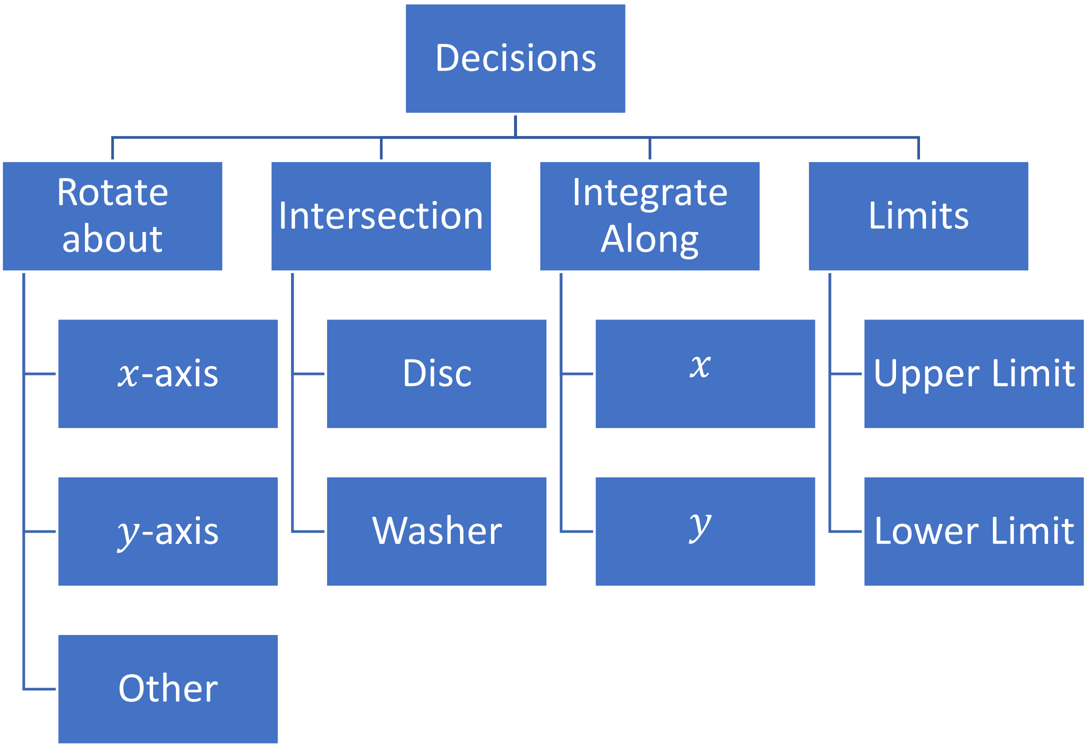
Solve the given equations for y:

$$y = \sqrt{x-3} \Rightarrow y^2 = x - 3 \Rightarrow x = y^2 + 3$$
$$y = \sqrt{-x+6} \Rightarrow y^2 = -x + 6 \Rightarrow x = 6 - y^2$$

Using the washer method, the integral is $\pi \int_0^{\sqrt{\frac{3}{2}}} R^2 - r^2\,dx$ where:

$$Outer = R^2 = \left[(6-y^2) - \ln\left(\frac{1}{2}\right)\right]^2 = [(6-y^2) + \ln 2]^2$$
$$Inner = r^2 = \left[(y^2+3) - \ln\left(\frac{1}{2}\right)\right]^2 = [(y^2+3) + \ln 2]^2$$

F. Summary



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2.11 Volumes with Cylindrical Shells

A. Cylindrical Shells

Example 2.220
Setup an integral to find the volume of the solid of revolution formed by revolving the region bounded by $x = 3y + 5$ and $x = y^2 + 1$ about $y = 6$

Use a change of variable. Let:

$$X = y, Y = x \Rightarrow R: Y = 3X + 5, Y = X^2 + 1$$

The limits of integration are:

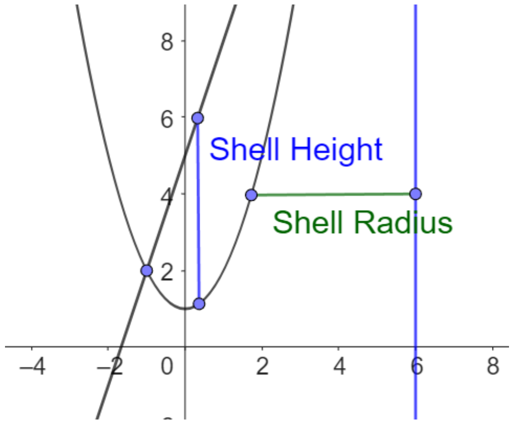
$$3X + 5 = X^2 + 1 \Rightarrow X \in \{-1,4\}$$

Hence, the integral is:

$$\int_{-1}^{4} \underbrace{(6 - X)}_{\substack{Shell\\Radius}} \underbrace{[(3X + 5) - (X^2 + 1)]}_{\substack{Shell\\Height}} \, dX$$

Change back to the original variable:

$$\int_{-1}^{4} \underbrace{(6 - y)}_{\substack{Shell\\Radius}} \underbrace{[(3y + 5) - (y^2 + 1)]}_{\substack{Shell\\Height}} \, dy$$



Example 2.221

Determine the volume of the solid generated by revolving the triangle with vertices at (0,0), (2,2) and (3,0) around the x-axis using the:

A. Disc/washer method
B. Shell Method

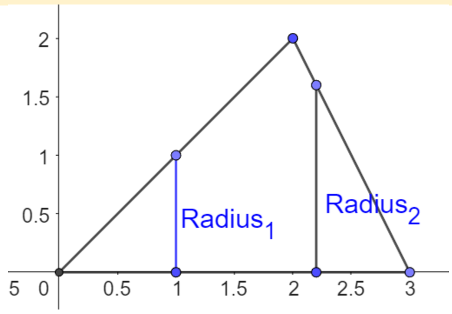
Part A: Disc/Washer Method

The volume is:

$$\pi \int_0^2 R_1^2 \, dx + \pi \int_2^3 R_2^2 \, dx$$

Substitute $R_1 = y = x, R_2 = y = 6 - 2x$:

$$\pi \int_0^2 x^2 \, dx + \pi \int_2^3 (6 - 2x)^2 \, dx$$



Part B: Shell Method

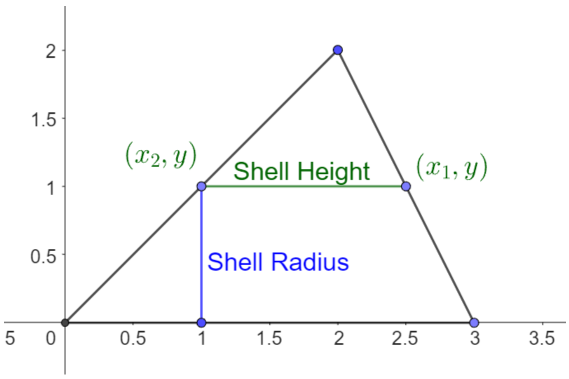
$$y = 6 - 2x_1 \Rightarrow x_1 = 3 - \frac{y}{2}$$

$$x_2 = y$$

$$x_1 - x_2 = 3 - \frac{y}{2} - y = 3 - \frac{3y}{2}$$

The volume is then:

$$2\pi \int_0^2 \underbrace{y}_{\substack{Shell\\Radius}} \underbrace{3 - \frac{3y}{2}}_{\substack{Shell\\Height}} \, dy = 2\pi \int_0^2 6y - 3y^2 \, dy$$



2.12 Kinematics & Biology

A. Kinematics

2.222: Acceleration, Velocity and Displacement

Displacement is the integral of velocity with respect to time

$$s = \int \frac{ds}{dt}\, dt = \int v\, dt + C$$

Velocity is the integral of acceleration with respect to time

$$\frac{ds}{dt} = v = \int \frac{dv}{dt}\, dt = \int a\, dt + C$$

Example 2.223

A. A car starting from rest goes in the positive direction in a straight line with a constant acceleration of $3\frac{m^2}{s}$. Find the distance travelled by the car in 30 seconds.

B. An object with an initial velocity v_0 goes in a straight line with a constant acceleration of $a\frac{m^2}{s}$. If the initial displacement is s_0, find the displacement function of the car.

Part A

Since the car is starting from rest:

$$Initial\ Velocity = v_0 = 0$$

$$v(t) = \int 3\, dt = 3t + v_0 = 3t$$

Let the position of the car at time $t = 0$ be $x = 0$.

$$Initial\ displacement = s_0 = 0$$

$$s(t) = \int 3t\, dt = \frac{3t^2}{2} + s_0 = \frac{3t^2}{2}$$

To find the distance travelled by the car in 30 seconds, evaluate:

$$s(30) = \frac{3(30)^2}{2} = 1350\ m$$

Part B

$$Initial\ Velocity = v_0$$

$$v(t) = \int a\, dt = at + v_0$$

$$Initial\ displacement = s_0$$

$$s(t) = \int (at + v_0)\, dt = \frac{at^2}{2} + v_0 t + s_0$$

Example 2.224

A particle moves on the x-axis with velocity at time $t \geq 0$ given by $v = t^2 - 9t + 20$. Find the maximum possible difference between the displacement and the distance travelled by the particle.

We first check when the velocity is negative:

$$t^2 - 9t + 20 < 0 \Rightarrow (t-4)(t-5) < 0 \Rightarrow t \in (4,5)$$

If velocity does not change sign, distance and displacement will increase at the same rate.

The distance is the length of path travelled. Hence, we will need to take the absolute value of the integral. Divide the integral into intervals based on $t \in (4,5)$:

$$s = \left|\int_0^4 v(t)\, dt\right| + \left|\int_4^5 v(t)\ \ dt\right| + \left|\int_5^n v(t)\, dt\right|$$

Velocity is positive in the first and the third intervals, and negative in the second interval. Hence, the above expression becomes:

$$= \int_0^4 v(t)\, dt - \int_4^5 v(t)\, dt + \int_5^n v(t)\, dt$$

The displacement is:

$$d = \int_0^4 v(t)\, dt + \int_4^5 v(t)\, dt + \int_5^n v(t)\ \ dt$$

The difference between the two is maximum for $t \geq 5$:

$$s - d = -2\int_4^5 v(t)$$

Example 2.225
A particle moves on the x-axis with displacement function $s(x) = \frac{1}{\sqrt{1+3x}}$. Show that the velocity of the particle is never constant.

B. Biology

Example 2.226: Discrete Version
Rate of increase of population is 10 people per year.
What is the change in population in three years?

$$10 \cdot 3 = 30$$

2.227: Rate of Change Function
$$P(t) = Population\ at\ time\ t \Rightarrow P'(t) = Rate\ of\ change\ of\ population$$

Example 2.228
Suppose that the population of fish in a particular lake t years after 1 Jan 2000 is given by
$$P(t) = t^3 + 3t^2 + 25, \qquad 0 \le t \le 6$$

A. What is the meaning of $P'(t) = \frac{dP}{dt}$?
B. What is the population of the lake at the end of 2003?
C. At the beginning of the year 2005, what is the rate of change of the population of fish?

Part A

$$\frac{dP}{dt}\ represents\ rate\ of\ change\ of\ P\ with\ respect\ to\ change\ in\ t.$$

Part B

$$P(4) = 4^3 + 3(4^2) + 25 = 64 + 48 + 25 = 137$$

Part C

$$P'(t) = 3t^2 + 6t \Rightarrow P(5) = 3(5^2) + 6(5) = 105\frac{fish}{year}$$

Example 2.229
In Orange County, the population at time $t = 0$ is p.
$$Birth\ rate = b(t)$$
$$Death\ rate = d(t)$$
where t represents time in years.

A. What is the interpretation of $c(t) = b(t) - d(t)$
B. What is the interpretation of $p + \int_0^t c(t)\,dt$?

Part A
Part B

$$Starting\ population = p$$

$$Change\ in\ population\ from\ t = 0\ to\ t = t: \int_0^t c(t)$$

$$p + \int_0^t c(t) = Population\ at\ time\ t$$

Example 2.230

In Orange County

$$Population\ at\ time\ t = 0\ is\ 4$$
$$Change\ in\ population\ at\ time\ t = c(t) = 2t + 4$$
$$Population\ at\ time\ t = P(t)$$

A. Find $\int c(t)\,dt$ and interpret it.
B. Find $P(t)$
C. Determine the increase in population from $t = 0$ to $t = 3$ by:
    a. Using $P(t)$
    b. Solving a definite integral of the form $\int_a^b c(t)\,dt$

Part A

$$\int c(t)\,dt = \int 2t + 4\,dt = t^2 + 4t + C$$

Part B

$$P(0) = 4 \Rightarrow C = 4$$
$$P(t) = t^2 + 4t + 4 = (t + 2)^2$$

Part B

$$Ending\ Population - Starting\ Population = P(3) - P(0) = (3 + 2)^2 - (0 + 2)^2 = 25 - 4 = 21$$

Part C

$$\int_a^b c(t)\,dt = \int_0^3 2t + 4\,dt = [t^2 + 4t]_{t=0}^{t=3} = 21 - 0 = 21$$

3. Integration (BC)

3.1 Integration by Parts

A. Background

Running the chain rule in reverse gives u substitution. Correspondingly, integration by parts is based on the product rule for differentiation.

3.1: Integration by Parts

$$\int u \cdot dv = uv - \int v \cdot du$$

Start with the formula for the derivative of a product:

$$(uv)' = uv' + vu'$$

Integrate both sides:

$$\int (uv)' = \int uv' + vu'$$

On the LHS, integration and differentiation cancel.
On the RHS, separate out the two integrals:

$$uv = \int uv' + \int vu'$$

Rearrange:

$$\int uv' = uv - \int vu'$$

Rewrite in differentials:

$$\int u\, dv = uv - \int v\, du$$

3.2: Choice of u and dv

$$\int u \cdot dv = uv - \int v \cdot du$$

Since u gets differentiated to du, and dv gets integrated to v, choose:

$$u \text{ so that its derivative is easier}$$
$$dv \text{ so that its integral is easier}$$

3.3: ILATE

Order of preference for u.

$$I = \text{Inverse}$$
$$L = \text{Logarithmic}$$
$$A = \text{Algebraic}$$
$$T = \text{Trigonometric}$$
$$E = \text{Exponential}$$

B. Logarithmic Integrals

Example 3.4

Evaluate

$$\int \ln x \, dx$$

We want to choose u so that the derivative is easier. Taking $u = \ln x$ converts a logarithmic function to an algebraic function.

$$\frac{d}{dx}\underbrace{(\ln x)}_{\textit{Logarithmic}} = \underbrace{\frac{1}{x}}_{\textit{Algebraic}}$$

There is no v that we can seemingly take. However, we can always multiply by 1. Take $dv = 1 \cdot dx$:

$$u = \ln x\,, \qquad dv = 1 \cdot dx$$
$$du = \frac{1}{x}dx, \qquad v = x$$

Substitute in $\int u\,dv = uv - \int v\,du$:

$$\int \ln x \, dx = x \cdot \ln x - \int x\frac{1}{x}dx$$
$$= x \cdot \ln x - \int 1 \cdot dx$$
$$= x \cdot \ln x - x + C$$

Example 3.5

Evaluate

$$\int x^3 \ln x \;\, dx$$

$$u = \ln x\,, \qquad dv = x^3\,dx$$
$$du = \frac{1}{x}\,dx, \qquad v = \frac{x^4}{4}$$

Substitute $\int u\,dv = uv - \int v\,du$:

$$\ln x\left(\frac{x^4}{4}\right) - \int\left(\frac{x^4}{4}\right)\left(\frac{dx}{x}\right)$$

Simplify:

$$= \frac{x^4 \ln x}{4} - \int\left(\frac{x^3}{4}\right)dx$$

Integrate:

$$\frac{x^4 \ln x}{4} - \frac{x^4}{16} + C$$

Example 3.6

$$\int x^n \ln x \;\, dx$$

Evaluate the above. Ensure you take all special cases into consideration.[4]

$$u = \ln x\,, \qquad dv = x^n dx$$

[4] You can be asked to integrate in terms of unknown constants. This is equivalent to proving a formula. Use this example to improve your manipulation skills. (Don't Memorize).

$$du = \frac{1}{x}dx, \qquad v = \frac{x^{n+1}}{n+1}$$

Apply $\int u\,dv = uv - \int v\,du$:

$$\int x^n \ln x \ dx = \ln x \left(\frac{x^{n+1}}{n+1}\right) - \int \left(\frac{x^{n+1}}{n+1}\right)\left(\frac{1}{x}dx\right)$$

Simplify:

$$= \frac{x^{n+1}\ln x}{n+1} - \int \frac{x^n}{n+1}dx$$

Integrate

$$= \frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C$$

Special Case: $n = -1$

But wait. The above is not applicable when $n = -1$:

$$\int x^{-1}\,dx = \int \frac{1}{x}dx = \ln x \neq \frac{x^{n+1}}{n}$$

Hence, we need to find this separately:

$$\int x^{-1}\ln x \ dx = \int \frac{\ln x}{x}dx = \int u\,du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$$

Where we made a u-substitution using

$$u = \ln x \Rightarrow du = \frac{1}{x}dx$$

Example 3.7

Evaluate

$$\int xe^x\,dx$$

$$\begin{aligned} u &= x, \qquad & dv &= e^x\,dx \\ du &= dx, \qquad & v &= e^x \end{aligned}$$

Apply $\int u\,dv = uv - \int v\,du$:

$$\int xe^x\,dx = x\cdot e^x - \int e^x\,dx = x\cdot e^x - e^x + C$$

Example 3.8

Evaluate

$$\int xe^{-x}\,dx$$

$$\begin{aligned} u &= x, \qquad & dv &= e^{-x}\,dx \\ du &= dx, \qquad & v &= -e^{-x}\,dx \end{aligned}$$

Apply $\int u\,dv = uv - \int v\,du$:

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,du = -xe^{-x} - e^{-x} + C$$

C. Successive Integration

It is sometimes necessary to apply integration by parts multiple times. The key here is that the integral that you are left with should be simpler than the one that you started with. If that is not the case, you should re-consider your strategy.

Example 3.9

Evaluate

$$\int x^2 e^x \, dx$$

Integration by Parts

$$u = x^2, \qquad dv = e^x dx$$
$$du = 2x \, dx, \qquad v = e^x$$

Substitute $\int u \, dv = uv - \int v \, du$:

$$= x^2 e^x - \int 2xe^x dx$$
$$= x^2 e^x - 2\int xe^x dx$$

Integration by Parts One More Time

We now need to find $\int xe^x \, dx$:

$$u = x, \qquad dv = e^x dx$$
$$du = dx, \qquad v = e^x$$

Substitute $\int u \, dv = uv - \int v \, du$:

$$\int xe^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x + C$$

And, hence the final answer is:

$$= x^2 e^x - 2(xe^x - e^x) + C$$
$$= x^2 e^x - 2xe^x + 2e^x + C$$

D. Tabular Integration

Successive integration by parts becomes much easier if you put it in a table.

Example 3.10

Evaluate

$$\int x^2 e^x \, dx$$

	u	dv
		e^x
$+$	x^2	e^x
$-$	$2x$	e^x
$+$	2	e^x

$$x^2 e^x - 2xe^x + 2e^x + C$$

Example 3.11

Evaluate

$$\int x^2 e^{-x}\, dx$$

	u	dv
		e^{-x}
$+$	x^2	$-e^{-x}$
$-$	$2x$	e^{-x}
$+$	2	$-e^{-x}$

$$\int x^2 e^{-x}\, dx = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

Example 3.12

Formula for integration by parts equivalent to tabular integration

$$\int uv = uv_1 - u'v_2 + u''v_3 - \cdots$$

Where

$$u\ gets\ differentiated\ each\ time$$
$$v\ gets\ integrated\ each\ time$$

Stop when one of the terms becomes zero.

E. Reduction Formulas

Reduction formulas are formulas that let you "reduce" the power of an integral. In other words, they let you replace

$$\int f(x) \to \int g(x)$$

Where

$$f(x)\ has\ power\ n$$
$$g(x)\ has\ power\ m < n$$

In the earlier example, we evaluated

- $\int xe^x\, dx$ using integration by parts
- $\int x^2 e^x\, dx$ using integration by parts twice

This makes us conjecture that if we needed to evaluate $\int x^3 e^x\, dx$, we could do so by using integration by parts three times.
But rather than doing this, we prove a reduction formula that lets us evaluate $\int x^n e^x\, dx$ in terms of the simpler integral given by $\int nx^{n-1} e^x\, dx$

3.13: Reduction Formula

$$\int x^n e^x\, dx = x^n e^x - \int nx^{n-1} e^x\, dx, n > 0, n \in \mathbb{N}$$

$$u = x^n, \qquad dv = e^x dx$$
$$du = nx^{n-1}\, dx, \qquad v = e^x$$

Substitute $\int u\, dv = uv - \int v\, du$:

$$\int x^n e^x\, dx = x^n e^x - \int nx^{n-1} e^x\, dx$$

Example 3.14

$$\int x^5 e^x \, dx$$

Apply the reduction formula:

$$\int x^5 e^x \, dx = x^5 e^x - \int 5x^4 e^x dx$$

Apply the reduction formula a second time:

$$= x^5 e^x - 5\left(x^4 e^x - \int 4x^3 e^x dx\right) = x^5 e^x - 5x^4 e^x + 20 \int x^3 e^x dx$$

Apply the reduction formula a third time:

$$= x^5 e^x - 5x^4 e^x + 20\left(x^3 e^x - \int 3x^2 e^x dx\right)$$

$$= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60 \int x^2 e^x dx$$

Apply the reduction formula a fourth time:

$$= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60\left(x^2 e^x - \int 2x e^x dx\right)$$

$$= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120 \int x e^x dx$$

Apply the reduction formula a last time:

$$= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120\left(x e^x - \int e^x dx\right)$$

$$= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120x e^x - 120 \int e^x dx$$

Now we don't need to apply the reduction formula, since we directly integrate (and also it is not applicable):

$$= x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120x e^x - 120 e^x + C$$

	u	dv
		e^x
$+$	x^5	e^x
$-$	$5x^4$	e^x
$+$	$20x^3$	e^x
$-$	$60x^2$	e^x
$+$	$120x$	e^x
$-$	120	e^x

$$x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120x e^x - 120 e^x + C$$
$$e^x(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120) + C$$

F. Trigonometric Integrals

Example 3.15

Evaluate

$$\int x \sin x \, dx$$

$$u = x, \qquad dv = \sin x \, dx$$
$$du = dx, \qquad v = -\cos x$$

Apply $\int u \, dv = uv - \int v \, du$:

$$= x(-\cos x) - \int -\cos x \, dx$$
$$= x(-\cos x) + \int \cos x \, dx$$
$$= -x(\cos x) + \sin x + C$$

Example 3.16

Evaluate

$$\int x^2 \sin x \, dx$$

Integration by Parts

Choose the algebraic function for u:

$$u = x^2, \qquad dv = \sin x \, dx$$
$$du = 2x \, dx, \qquad v = -\cos x$$

Apply $\int u \, dv = uv - \int v \, du$:

$$\int x^2 \sin x \, dx = x^2(-\cos x) - \int -\cos x \, 2x \, dx = -x^2(\cos x) + 2\int x \cos x \, dx$$

Apply Integration by Parts Again

$$u = 2x, \qquad dv = \cos x \, dx$$
$$du = 2 \, dx, \qquad v = \sin x$$

Substitute $\int u \, dv = uv - \int v \, du$:

$$2\int x \cos x \, dx = 2x \sin x - \int (\sin x) \, 2 \, dx = 2x \sin x + 2\cos x + C$$

Bring everything together:

$$\int x^2 \sin x \, dx = -x^2(\cos x) + 2x \sin x + 2\cos x + C$$

Example 3.17

Evaluate

$$\int x^2 \cos 3x \, dx$$

Integration by Parts

Choose the algebraic function for u:

$$u = x^2, \qquad dv = \cos 3x \, dx$$
$$du = 2x \, dx, \qquad v = \frac{1}{3}\sin 3x$$

Apply $\int u \, dv = uv - \int v \, du$:

$$\int x^2 \cos 3x \, dx = \frac{1}{3}x^2 \sin 3x - \frac{1}{3}\int \sin 3x \, 2x \, dx$$

Apply Integration by Parts Again

$$\int 2x \sin 3x \; dx$$

$$u = 2x, \qquad dv = \sin 3x \, dx$$

$$du = 2\, dx, \qquad v = -\frac{1}{3}\cos 3x$$

Substitute $\int u\, dv = uv - \int v\, du$:

$$\int \sin 3x \, 2x \, dx = -\frac{1}{3}(2x)(\cos 3x) + \frac{2}{3}\int \cos 3x \, dx = -\frac{2}{3}x\cos 3x + \frac{2}{9}\sin 3x$$

$$\frac{1}{3}x^2 \sin 3x - \frac{1}{3}\left(-\frac{2}{3}x\cos 3x + \frac{2}{9}\sin 3x\right)$$

$$= \frac{1}{3}x^2 \sin 3x - \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x$$

G. Solving for the Integral

When using integration by parts, the remaining integral can cycle and give you the original integral. This may feel like a dead end, but you can then solve for the integral.

Example 3.18

Evaluate

$$\int e^x \sin x \, dx$$

Choose the algebraic function for u:

$$u = \sin x\,, \qquad dv = e^x \, dx$$
$$du = \cos x \, dx, \qquad v = e^x$$

Apply $\int u\, dv = uv - \int v\, du$:

$$e^x \sin x - \int e^x \cos x \, dx$$

Use integration by parts one more time

$$u = \cos x\,, \qquad dv = e^x \, dx$$
$$du = -\sin x \, dx, \qquad v = e^x$$

$$-\int e^x \cos x \, dx = -\left[e^x \cos x - \int e^x(-\sin x)dx\right] = -e^x \cos x - \int e^x \sin x \, dx$$

Substitute back into the original expression:

$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$2\int e^x \sin x \, dx = e^x \sin x - e^x \cos x$$

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x)$$

Example 3.19

Evaluate

$$\int \sec^3 \theta \; d\theta$$

Here $ILATE$ does not help. We need to go back to basic principles and choose

$$u \text{ is easy to differentiate}$$
$$dv \text{ easy to integrate}$$

$$u = \sec\theta, \qquad dv = \sec^2\theta \; d\theta$$
$$du = \sec\theta \tan\theta \, d\theta, \qquad v = \tan\theta$$

Apply $\int u \, dv = uv - \int v \, du$:

$$= \sec\theta \tan\theta - \int \underbrace{\tan\theta}_{v} \underbrace{\sec\theta \tan\theta \, d\theta}_{du} = \sec\theta\tan\theta - \int \tan^2\theta \sec\theta \, d\theta$$

Substitute $\tan^2\theta = \sec^2\theta - 1$:

$$= \sec\theta \tan\theta - \int (\sec^2\theta - 1) \sec\theta \, d\theta$$

$$\int \sec^3\theta \, d\theta = \sec\theta\tan\theta - \int \sec^3\theta \, d\theta + \int \sec\theta \, d\theta$$

Substitute $\int \sec\theta \, d\theta = \ln|\sec\theta + tan\,\theta|$, and move $\int \sec^3\theta \, d\theta$ from RHS to LHS:

$$2\int \sec^3\theta \; d\theta = \sec\theta\tan\theta + \ln|\sec\theta + tan\,\theta|$$

Solve for $\int \sec^3\theta \; d\theta$:

$$\int \sec^3\theta \; d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + tan\,\theta|)$$

Example 3.20

$$I = \int \csc^3 x \; dx$$

$$u = \csc x, dv = \csc^2 x \; dx$$
$$du = -\csc x \cot x, v = -\cot x$$

Apply $\int u \, dv = uv - \int v \, du$:

$$I = -\cot x \csc x - \int \csc x \cot^2 x \; dx$$

Substitute $\cot^2 x = \csc^2 x - 1$:

$$I = -\cot x \csc x - \int \csc x \, (\csc^2 x - 1) \, dx$$
$$I = -\cot x \csc x - \int \csc^3 x \, dx + \int \csc x \; dx$$

$$2I = -\cot x \csc x + \int \csc x \;\; dx$$
$$I = -\frac{1}{2}\cot x \csc x - \frac{1}{2}\ln|\csc x + \cot x|$$

Example 3.21

$$I = \int \sin 6x \cos 2x \, dx$$

$$u = \sin 6x\,, \qquad dv = \cos 2x \, dx$$
$$du = 6\cos 6x \, dx, \qquad v = \frac{\sin 2x}{2}$$

Apply $\int u \, dv = uv - \int v \, du$:

$$I = (\sin 6x)\left(\frac{\sin 2x}{2}\right) - \int \left(\frac{\sin 2x}{2}\right)(6\cos 6x)dx$$

$$u = 6\cos 6x\,, \qquad dv = \frac{\sin 2x}{2}dx$$
$$du = -36\sin 6x \, dx, \qquad v = -\frac{\cos 2x}{4}$$

Apply $\int u \, dv = uv - \int v \, du$:

$$I = (\sin 6x)\left(\frac{\sin 2x}{2}\right) - \left[(6\cos 6x)\left(-\frac{\cos 2x}{4}\right) - \int \left(-\frac{\cos 2x}{4}\right)(-36\sin 6x \, dx)\right]$$

Substitute $I = \int \sin 6x \cos 2x \, dx$ in the last term, and open the parenthesis:

$$I = (\sin 6x)\left(\frac{\sin 2x}{2}\right) + \frac{3}{2}\cos 6x \cos 2x + 9I$$
$$I = -\frac{1}{16}(\sin 6x \sin 2x) - \frac{3}{16}(\cos 6x \cos 2x)$$

H. *u* substitution

3.22: *u* substitution with Integration by parts

Some integrals may need u substitution prior to using integration by parts.

Example 3.23

$$\int e^{\sqrt{x}} dx$$

Substitute $\sqrt{x} = t \Rightarrow x = t^2 \Rightarrow dx = 2t \, dt$:

$$= 2\int te^t \, dt$$

Use integration by parts:

$$u = t, \qquad dv = e^t dt$$
$$du = dt, \qquad v = e^t$$

Substitute in $\int u \, dv = uv - \int v \, du$:

$$2\left[te^t - \int e^t dt\right] = 2[te^t - e^t] + C$$

Change back to the original variable:

$$= 2\left(\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}}\right) + C$$

Example 3.24

$$\int_0^{\pi^2} \cos\sqrt{x}\, dx$$

Indefinite Integral

Substitute $\sqrt{x} = t \Rightarrow x = t^2 \Rightarrow dx = 2t\, dt$:

$$\int (\cos t)(2t) dt$$

$$u = 2t, \qquad dv = \cos t\, dt$$
$$du = 2\, dt, \qquad v = \sin t$$

Substitute in $\int u\, dv = uv - \int v\, du$:

$$2t \sin t - \int (\sin t)(2\, dt)$$
$$= 2t \sin t + 2\cos t + C$$

Change back to the original variable:

$$= 2\sqrt{x} \sin\sqrt{x} + 2\cos\sqrt{x} + C$$

Definite Integral

$$\int_0^{\pi^2} \cos\sqrt{x}\, dx = \left[2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x}\right]_0^{\pi^2}$$
$$= \left(2\sqrt{\pi^2}\sin\sqrt{\pi^2} + 2\cos\sqrt{\pi^2}\right) - \left(2\sqrt{0}\sin 0 + 2\cos 0\right)$$
$$= (0 - 2) - (2 + 0) = -4$$

Example 3.25

$$\int \cos(\ln x)\, dx$$

Substitute $\ln x = t \Rightarrow x = e^t \Rightarrow dx = e^t\, dt$

$$\underbrace{I = \int e^t \cos t\, dt}_{\textit{Equation I}}$$

Use integration by parts:

$$u = e^t, \qquad dv = \cos t\, dt$$
$$du = e^t\, dt, \qquad v = \sin t$$

Substitute in $\int u\, dv = uv - \int v\, du$:

$$\underbrace{I = e^t \sin t - \int e^t \sin t\, dt}_{\textit{Equation II}}$$

Use integration by parts one more time:

$$u = e^t, \qquad dv = \sin t\, dt$$
$$du = e^t\, dt, \qquad v = -\cos t$$

Substitute in $\int u\, dv = uv - \int v\, du$:

$$\underbrace{\int e^t \sin t\, dt = -e^t \cos t - \int -e^t \cos t\, dt}_{\text{Equation III}}$$

Substitute Equation II in Equation III:

$$I = e^t \sin t - \left[-e^t \cos t - \int -e^t \cos t\, dt\right]$$
$$I = e^t \sin t + e^t \cos t - \int e^t \cos t\, dt$$

Substitute $I = \int \cos(t)\, e^t\, dt$ for the last term:

$$I = e^t \sin t\, dt + e^t \cos t - I$$

Isolate I:

$$2I = e^t \sin t + e^t \cos t$$

Divide both sides by 2:

$$I = \frac{1}{2} e^t (\sin t + \cos t) + C$$

Change back to the original variable ($t = \ln x$):

$$I = \frac{1}{2} e^{\ln x} (\sin(\ln x) + \cos(\ln x)) + C$$
$$I = \frac{1}{2} x (\sin(\ln x) + \cos(\ln x)) + C$$

I. Cancellation

3.26: Cancellation

Some of the integral that you come up with cancels with another part of the integral, making an otherwise much difficult integral easier to solve.

Example 3.27

$$\int e^{2x} \left(\frac{1 - \sin 2x}{1 - \cos 2x}\right) dx \;\; (CBSE\ 2013)$$

Substitute $1 - \sin 2x = 1 - 2\sin x \cos x$, $1 - \cos 2x = 2\sin^2 x$:

$$\int e^{2x} \left(\frac{1 - 2\sin x \cos x}{2\sin^2 x}\right) dx$$

Split the fraction:

$$\int e^{2x} \left(\frac{1}{2\sin^2 x} - \frac{\sin x \cos x}{\sin^2 x}\right) dx$$

Simplify and split the integral:

$$= \frac{1}{2}\int e^{2x}(\csc^2 x)dx - \int e^{2x}(\cot x)dx$$
$$u = e^{2x}, \qquad dv = \csc^2 x\, dx$$
$$du = 2e^{2x}\, dx, \qquad v = -\cot x$$

Substitute in $\int u\, dv = uv - \int v\, du$:

$$= -\frac{1}{2}e^{2x}\cot x - \frac{1}{2}\int (-\cot x)(2e^{2x})\, dx - \int e^{2x}(\cot x)dx$$
$$= -\frac{1}{2}e^{2x}\cot x + C$$

Example 3.28

$$\int \left[\ln(\ln x) + \frac{1}{(\ln x)^2}\right] dx \; (CBSE\ 2010)$$

Split into two integrals using the additive property:

$$\int \ln(\ln x)\, dx + \int \frac{1}{(\ln x)^2} dx$$

First Part: $I_1 = \int \ln(\ln x)\, dx$

Use integration by parts:

$$u = \ln(\ln x), \qquad dv = dx$$
$$du = \frac{1}{x \ln x}, \qquad v = x$$

Apply $\int u\, dv = uv - \int v\, du$:

$$I_1 = x\ln(\ln x) - \int \frac{1}{\ln x} dx$$

Second Part: $I_2 = \int \frac{1}{(\ln x)^2} dx$

Use a substitution. Let $t = \ln x \Rightarrow x = e^t, dt = \frac{1}{x} dx$

$$\int \frac{e^t}{t^2}\, dt$$

Use integration by parts:

$$u = e^t, \;\; dv = \frac{1}{t^2}\, dt$$
$$du = e^t\, dt, \qquad v = -\frac{1}{t}$$

Apply $\int u\, dv = uv - \int v\, du$:

$$= -\frac{e^t}{t} - \int -\frac{e^t}{t}\, dt$$

Change back to the original variable:

$$I_2 = -\frac{x}{\ln x} + \int \frac{1}{\ln x} dx$$

Combine the first and second parts

$$I_1 + I_2 = \underbrace{x\ln(\ln x) - \int \frac{1}{\ln x}dx}_{I_1} + \underbrace{-\frac{x}{\ln x} + \int \frac{1}{\ln x}dx}_{I_2}$$

Simplify, and add the constant of integration to get the final answer:

$$x\ln(\ln x) - \frac{x}{\ln x} + C$$

J. Inverse Trigonometric Integrals

Example 3.29

Evaluate

$$\int \sin^{-1}(\cos x)\ dx$$

$$\int \sin^{-1}\left(\sin\left(\frac{\pi}{2} - x\right)\right) dx = \int \frac{\pi}{2} - x\, dx = \frac{\pi}{2}x - \frac{x^2}{2} + C$$

Example 3.30

$$\int x\tan^{-1}x\ dx$$

$$u = \tan^{-1}x\,, \qquad dv = x\,dx$$
$$du = \frac{1}{1+x^2}\,dx, \qquad v = \frac{x^2}{2}$$

Apply $\int u\,dv = uv - \int v\,du$:

$$\int x\tan^{-1}x\ dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}$$

$$= \frac{1}{2}\left(x^2\tan^{-1}x - \int \frac{x^2+1-1}{1+x^2}dx\right)$$
$$= \frac{1}{2}\left(x^2\tan^{-1}x - \int\left(1 - \frac{1}{1+x^2}\right)dx\right)$$
$$= \frac{1}{2}(x^2\tan^{-1}x - x + \tan^{-1}x) + C$$

Example 3.31

$$\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx\ (CBSE\ 2012, 2016)$$

Let

$$\sin^{-1}x = t \Rightarrow \frac{1}{\sqrt{1-x^2}}dx = dt$$
$$\sin^{-1}x = t \Rightarrow x = \sin t$$

Substitute the above:

$$\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx = \int t\sin t\,dt$$

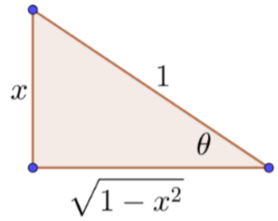
Using $\int x\sin x\,dx = -x(\cos x) + \sin x + C$:

$$\int t \sin t \, dt = -t(\cos t) + \sin t + C$$

Change back to the original variable ($\sin^{-1} x = t$):

$$= -(\sin^{-1} x)(\cos(\sin^{-1} x)) + \sin(\sin^{-1} x) + C$$
$$= -(\sin^{-1} x)(\cos(\sin^{-1} x)) + x + C$$

Draw a reference triangle:



$$\cos(\sin^{-1} x) = \cos t = \sqrt{1 - x^2}$$

The integration becomes:

$$= -(\sin^{-1} x)\sqrt{1 - x^2} + x + C$$

Example 3.32

Evaluate $\int \sin^{-1} x \, dx$ by:

A. Integration by parts, followed by u substitution.
B. u substitution, followed by integration by parts

Part A

$$u = \sin^{-1} x, \qquad dv = dx$$
$$du = \frac{1}{\sqrt{1 - x^2}} dx, \qquad v = x$$

Apply $\int u \, dv = uv - \int v \, du$:

$$\int \sin^{-1} x = x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}} dx$$

To evaluate $-\int \frac{x}{\sqrt{1-x^2}} dx$ let $u = \sqrt{1 - x^2} \Rightarrow du = -2x \, dx$:

$$\int \frac{-2x}{2\sqrt{1 - x^2}} dx = \int \frac{1}{2\sqrt{u}} du = \sqrt{u} = \sqrt{1 - x^2}$$

Hence, the final answer is:

$$x \cdot \sin^{-1} x + \sqrt{1 - x^2} + C$$

Part B

To "cancel" the $\sin^{-1} x$ substitute $t = \sin^{-1} x \Rightarrow x = \sin t \Rightarrow dx = \cos t \, dt$

$$\int \sin^{-1} x \, dx = \int t \cos t \, dt$$

$$u = t, \qquad dv = \cos t \, dt$$
$$du = dt, \qquad v = \sin t$$

Apply $\int u \, dv = uv - \int v \, du$:

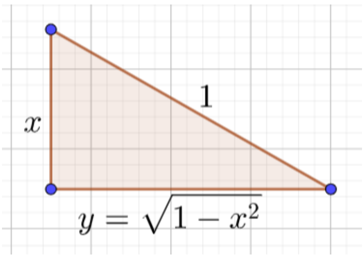
$$\int t \cos t \, dt = t \sin t \, dt - \int \sin t \, dt = t \sin t \, dt + \cos t + C$$

Change back to the original variable:

$$t = \sin^{-1} x, \sin t = x$$

To find $\cos t$, draw a reference triangle using 1 as the hypotenuse to simplify calculations:

$$\sin t = x \Rightarrow \frac{opp}{hyp} = x \Rightarrow opp = x \cdot hyp = x$$



Find y which satisfies the relation:

$$x^2 + y^2 = 1 \Rightarrow y^2 = 1 - x^2 \Rightarrow y = \sqrt{1-x^2}$$

Find $\cos t$ from our reference triangle:

$$\cos t = \frac{adj}{hyp} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

Substitute everything back to get our answer in terms of x:

$$t \sin t \, dt + \cos t + C = x \sin^{-1} x + \sqrt{1-x^2} + C$$

Example 3.33

Show that

$$\int \cos^{-1} x = x \cos^{-1} x - \sqrt{1-x^2} + C$$

Example 3.34

Evaluate $\int \tan^{-1} x \, dx$ by

A. Integration by parts, followed by u substitution.
B. u substitution, followed by integration by parts

Part A

$$u = \tan^{-1} x\,, \qquad dv = dx$$
$$du = \frac{1}{1+x^2} dx\,, \qquad v = x$$

Apply $\int u \, dv = uv - \int v \, du$:

$$\int \tan^{-1} x \, dx = x \cdot \tan^{-1} x - \int \frac{x}{1+x^2} dx$$

Substitute $u = 1 + x^2 \Rightarrow du = 2x \, dx$ in

$$-\int \frac{x}{1+x^2} dx = -\frac{1}{2}\int \frac{2x}{1+x^2} dx = -\frac{1}{2}\int \frac{1}{u} du = -\frac{1}{2}\ln|u| = -\frac{1}{2}\ln(1+x^2)$$

The final answer is:

$$\int \tan^{-1} x \, dx = x \cdot \tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$$

Part B

Let

$$t = \tan^{-1} x \Rightarrow x = \tan t \Rightarrow dx = \sec^2 t$$

$$\int \tan^{-1} x \, dx = \int t \sec^2 dt$$

Use integration by parts:

$$u = t, \qquad dv = \sec^2 dt$$
$$du = dt, \qquad v = \tan t$$

Apply $\int u\, dv = uv - \int v\, du$:

$$\int t \sec^2 dt = t \cdot \tan t - \int \tan t\, dt = t \cdot \tan t - \ln|\sec t| + C$$

Now convert back:

Example 3.35

$$\int \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}\, dx$$

Substitute $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx$

$$\int \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}\, dx = 2\int \tan^{-1} u\, du$$
$$= 2\left(u \cdot \tan^{-1} u - \frac{1}{2}\ln(1 + u^2)\right) + C$$

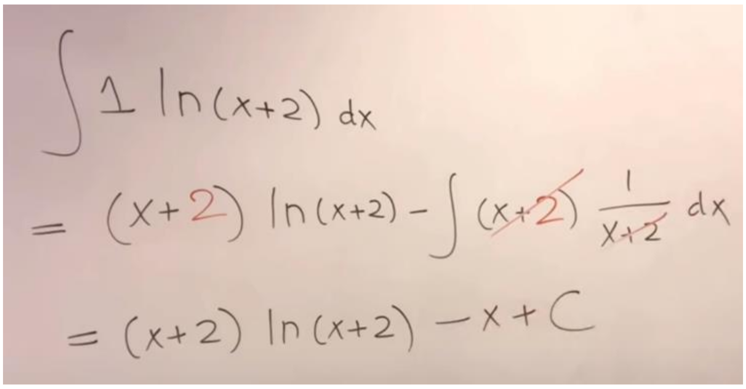
Change back to the original variable. Substitute $u = \sqrt{x}$:

$$= 2\left(\sqrt{x} \cdot \tan^{-1}\sqrt{x} - \frac{1}{2}\ln\left(1 + \left(\sqrt{x}\right)^2\right)\right) + C$$
$$= 2\sqrt{x} \cdot \tan^{-1}\sqrt{x} - \ln(1 + |x|) + C$$

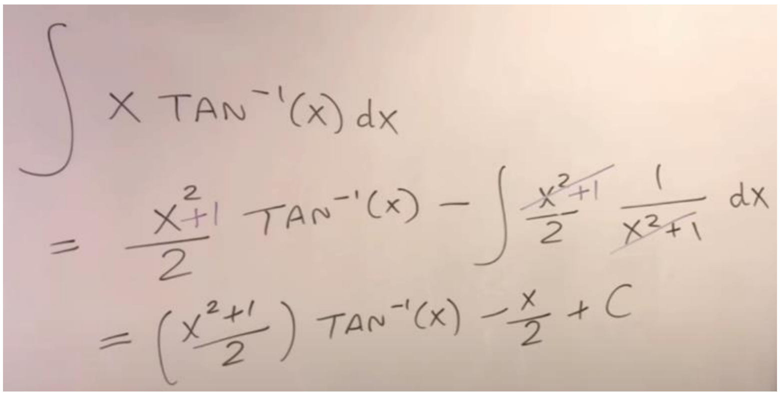
K. Constant of Integration

Using a constant of integration to speed things up
Can be achieved using the standard method also

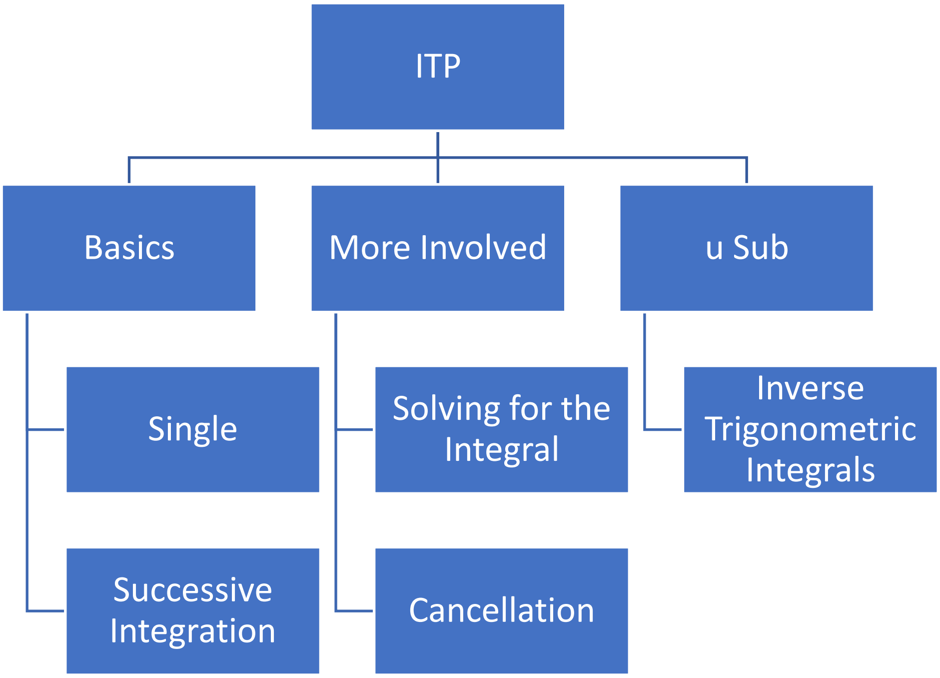
Example 3.36



Example 3.37



L. Summary of Ideas



3.2 Trigonometric Integrals

A. Odd Powers of sine and cosine

- Knowing trig identities is essential to work with the integrals in this section.
- Practice is important for these kinds of questions.

3.38: Pythagorean Identity

$$\sin^2 x + \cos^2 x = 1$$

We being with the most common trigonometric identity, which is the Pythagorean Identity And see how it can be applied to u −substitution.

Example 3.39

$$\int \cos^5 x \, dx$$

Separate out a single power:

$$\int \cos x \cos^4 x \, dx$$

Rewrite $\cos^4 x = (\cos^2 x)^2$

$$\int \cos x \, (\cos^2 x)^2 \, dx$$

Substitute $\cos^2 x = 1 - \sin^2 x$:

$$= \int \cos x \, (1 - \sin^2 x)^2 dx$$

Now we can make a u −substitution since

$$u = \sin x \Rightarrow du = \cos x \, dx$$

Make the above substitution:

$$\int \underbrace{(1 - \sin^2 x)^2}_{(1-u^2)^2} \underbrace{\cos x \, dx}_{du} = \int (1 - u^2)^2 du$$

Expand using $(a + b)^2 = a^2 + 2ab + b^2$:

$$= \int 1 - 2u^2 + u^4 \, du$$

This is now a polynomial in u which can be integrated using the power rule $\int u^n = \frac{u^{n+1}}{n+1}, n \neq 1$:

$$= u - \frac{2u^3}{3} + \frac{u^5}{5} + C$$

Change back to the original variable and substitute $u = \sin x$:

$$= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + C$$

3.40: Integrating $\int \cos^{2n+1} x$, $\int \sin^{2n+1} x, n \in \mathbb{N}$

To integrate an expression which contains an odd power of sin or cos:

- Separate out a single power.
- Convert the remaining powers from cos to sin or sin to cos using the identity $\sin^2 x + \cos^2 x = 1$
- Do a u-substitution with $u = \sin x \Rightarrow du = \cos x \, dx$

This strategy works since

$$\frac{d}{dx} \sin x = \cos x$$

Example 3.41

$$\int \cos^3 x \, dx$$

Separate out a single power:

$$\int \cos x \cos^2 x \, dx$$

Substitute $\cos^2 x = 1 - \sin^2 x$:

$$= \int \cos x \, (1 - \sin^2 x) \, dx$$

Substitute $u = \sin x \Rightarrow du = \cos x \, dx$

$$= \int (1 - u^2) du = u - \frac{u^3}{3} + C$$

Change back to the original variable and substitute $u = \sin x$:

$$= \sin x - \frac{\sin^3 x}{3} + C$$

Example 3.42

$$\int_0^{\pi} \sin^3 x$$

Substitute $\sin^2 x = 1 - \cos^2 x$, and separate out a power of $\sin x$:

$$\int_0^{\pi} (1 - \cos^2 x \,) \sin x \, dx$$

Substitute

$$u = \cos x \Rightarrow du = -\sin x \, dx$$
$$x = 0 \Rightarrow u = \cos x = \cos 0 = 1$$
$$x = \pi \Rightarrow u = \cos x = \cos \pi = -1$$

$$\int_1^{-1} -(1 - u^2) \, du = \int_1^{-1} u^2 - 1 \, du = \left[\frac{u^3}{3} - u\right]_1^{-1}$$

Substitute the values:

$$= \left(\frac{-1}{3} - (-1)\right) - \left(\frac{1}{3} - 1\right) = \left(\frac{2}{3}\right) - \left(-\frac{2}{3}\right) = \frac{4}{3}$$

Example 3.43

Consider the rectangle formed by the four points $A = (0,0), B = (0,1), C = (\pi, 0), D = (\pi, 1)$.

A. What proportion of the area of this rectangle lies between the curves $y = \sin x$ and $y = \sin^3 x$.
B. Is this proportion rational or irrational?

$$\int \sin x - \sin^3 x \;\, dx$$

$$\int \sin x \, (1 - \sin^2 x) \, dx = \int \sin x \cos^2 x \;\, dx$$

Use the substitution $u = \cos x \Rightarrow du = -\sin x \, dx$:

$$\int -u^2 \, du \;\, = -\frac{1}{3} u^3 + C = -\frac{1}{3} \cos^3 x + C$$

$$\left[-\frac{1}{3} \cos^3 x\right]_0^{\pi} = \left(-\frac{1}{3}(-1)^3\right) - \left(-\frac{1}{3}(1)\right) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Area of the Rectangle

$$= 1 \times \pi = \pi$$

Proportion

$$\frac{2}{3\pi}$$

3.44: Integrating with sin and cos

The strategy that we learnt can be used if both powers of sin and cos are present in the expression. As before, separate out the function that has an odd power.

Example 3.45

$$\int \sin^5 x \cos^4 x \, dx$$

Separate out a single power of $\sin x$:

$$\int \sin x \sin^4 x \cos^4 x \, dx$$

Substitute $\sin^4 x = (1 - \cos^2 x)^2$

$$= \int \sin x \,(1 - \cos^2 x)^2 \cos^4 x \, dx$$

Substitute $u = \cos x \Rightarrow du = -\sin x \, dx$:

$$= \int -(1 - u^2)^2 \, u^4 \, du$$

Expand:

$$-\int (1 - 2u^2 + u^4) u^4 \, du$$

Multiply:

$$= -\int u^4 - 2u^6 + u^8 \, du$$

Integrate:

$$= -\left(\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}\right) + C$$

Change back to the original variable. Substitute $u = \cos x$:

$$= -\left(\frac{\cos^9 x}{9} - \frac{2\cos^7 x}{7} + \frac{\cos^5 x}{5}\right) + C$$

B. Products of Even Powers

3.46: Strategy: Even powers

Reduce the power of the trigonometric function by using the double angle identities for cos:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Example 3.47

A. $\int \sin^2 x \, dx$
B. $\int \cos^2 x \, dx$

Part A

Use the double angle trigonometric identity to rewrite the integral:

$$\int \frac{1}{2}(1 - \cos 2x)dx$$

Split the integral:

$$\int \frac{1}{2}dx - \int \frac{\cos 2x}{2}dx$$

Let $u = 2x \Rightarrow du = 2\,dx$:

$$\int \frac{1}{2}dx - \int \frac{\cos 2x}{2} \cdot \frac{1}{2} \cdot \underbrace{2 \cdot dx}_{du}$$

Make the substitution:

$$\int \frac{1}{2}dx - \int \frac{\cos u}{4}du$$

Integrate:

$$\frac{x}{2} - \frac{\sin u}{4} + C$$

Change back to the original variable:

$$= \frac{x}{2} - \frac{\sin 2x}{4} + C$$

Part B
Similarly:

$$\int \frac{1}{2}(1 + \cos 2x)dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$$

3.48: Using the Double Angle more than once

If $\sin x$ or $\cos x$ has an even power greater than 2, the double identity will need to be used more than once.

Example 3.49

$$\int \sin^4 x\, dx$$

The key steps relate to rewriting the integrand. Once that is done, the "actual" integration is simpler:
Use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$:

$$\sin^4 x = (\sin^2 x)^2 = \left[\frac{1}{2}(1 - \cos 2x)\right]^2 = \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$$

Use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ one more time

$$= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right) = \frac{1}{4}\left(\frac{3}{2} - 2\cos 2x + \frac{\cos 4x}{2}\right)$$

Integrate:

$$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{\sin 4x}{8}\right) + C = \frac{3}{8}x - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$$

Example 3.50

$$\int \cos^4 x\, dx$$

Example 3.51

$$\int \sqrt{1+\cos 2x}\, dx$$

Substitute $1+\cos 2x = 2\cos^2 x$

$$\int \sqrt{2}\cos x\, dx = -\sqrt{2}\sin x + C$$

C. Products of sines and cosines

3.52: Product to Sum Formulas

$$\sin A \cos B = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$$
$$\sin A \sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$$
$$\cos A \cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$$

Example 3.53

$$\int \sin 6x \cos 2x\, dx$$

Use $\sin A \cos B = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$ with $A=6x$ and $B=2x$[5]:

$$= \int \frac{1}{2}(\sin 4x + \sin 8x)\, dx$$

Integrate:

$$= -\frac{1}{2}\left(\frac{\cos 4x}{4}+\frac{\cos 8x}{8}\right)+C$$
$$= -\left(\frac{\cos 4x}{8}+\frac{\cos 8x}{16}\right)+C$$

Example 3.54

$$\int \cos 5t \cos 10t\, dt$$

Using $\cos A \cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$

$$\frac{1}{2}\int \cos 5t + \cos 15t\, dt = \frac{\sin 5t}{10}+\frac{\sin 15t}{30}+C$$

D. Integrals with $sec^2 x$

3.55: Pythagorean Identity: Alternate Version

$$\tan^2 x = \sec^2 x - 1$$

Example 3.56: Observation

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[5] The focus of this question is more on trigonometry than on integration. In fact, that is the focus of the entire section: using trigonometric identities to make integration easier.

You should be able to get the examples below by using derivatives of basic functions, or simple identities.
Evaluate

A. $\int \sec^2 x\,dx$
B. $\int \sec x \tan x\,dx$
C. $\int \tan^2 x\,dx$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

Use the Pythagorean Identity to substitute $\tan^2 x = \sec^2 x - 1$

$$\int (\sec^2 x - 1)\,dx = \tan x - x + C$$

3.57: Integrating $sec^n x$ for even n

- Separate out $\sec^2 x$
- Convert remaining powers of $\sec x$ to $\tan x$ using $\sec^2 x = \tan^2 x + 1$
- Substitute $u = \tan x \Rightarrow du = \sec^2 x\,dx$

Example 3.58

Evaluate

$$\int \sec^4 x\,dx$$

Separate out $\sec^2 x$:

$$\int \sec^2 x \sec^2 x\,dx$$

Convert remaining powers of $\sec x$ to $\tan x$ using $\sec^2 x = \tan^2 x + 1$

$$= \int \sec^2 x\,(\tan^2 x + 1)dx$$

Substitute $u = \tan x \Rightarrow du = \sec^2 x\,dx$:

$$= \int (u^2 + 1)\,du$$

Integrate:

$$= \frac{u^3}{3} + u + C$$

Change back to the original variable:

$$= \frac{\tan^3 x}{3} + \tan x + C$$

Example 3.59

Evaluate

$$\int \tan\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx$$

We will u substitution twice.

Substitute $u = \frac{x}{2} \Rightarrow du = \frac{1}{2}dx$

$$2\int \tan u \sec^2 u \, du$$

Substitute $v = \tan u \Rightarrow dv = \sec^2 u$:

$$= \int 2v \, dv = v^2 + C = \tan^2 v + C = 2\left(\tan\frac{x}{2}\right)^2 + C$$

Example 3.60

Evaluate

$$\int \tan^8 2z \sec^6 2z \, dz$$

Use $x = 2z \Rightarrow dx = 2\,dz$

$$\frac{1}{2}\int \tan^8 x \sec^6 x \, dx$$

Separate out $\sec^2 x$:

$$\frac{1}{2}\int \tan^8 x \sec^4 x \sec^2 x \, dx$$

Rewrite using the identity $\sec^2 x = \tan^2 x + 1$:

$$\frac{1}{2}\int \tan^8 x \, (\tan^2 x + 1)^2 \sec^2 x \, dx$$

Let $u = \tan x \Rightarrow du = \sec^2 x \, dx$:

$$\frac{1}{2}\int u^8 \, (u^2 + 1)^2 du$$

Expand:

$$\frac{1}{2}\int u^8 \, (u^4 + 2u^2 + 1) \, du$$

Multiply:

$$= \frac{1}{2}\int u^{12} + 2u^{10} + u^8 \, du$$

Integrate:

$$\frac{1}{2}\left(\frac{u^{13}}{13} + \frac{2u^{11}}{11} + \frac{u^9}{9}\right) + C$$

Substitute $u = \tan x$:

$$\frac{1}{2}\left(\frac{\tan^{13} x}{13} + \frac{2\tan^{11} x}{11} + \frac{\tan^9 x}{9}\right) + C$$

Substitute $x = 2z$:

$$\frac{\tan^{13} 2z}{26} + \frac{2\tan^{11} 2z}{11} + \frac{\tan^9 2z}{18} + C$$

3.61: Creating $\sec^2 x$

If $\sec^2 x$ is not available, then it can be created using the identity

$$\tan^2 x = \sec^2 x - 1$$

Example 3.62

Evaluate

$$\int \tan^4 x \, dx$$

Expand:

$$\int \tan^2 x \tan^2 x \, dx$$

Substitute $\tan^2 x = \sec^2 x - 1$:

$$\int \tan^2 x \, (\sec^2 x - 1) \, dx$$

Multiply and separate into two integrals:

$$\int \tan^2 x \sec^2 x \ dx - \int \tan^2 x \, dx$$

In the first integral let $u = \tan x \Rightarrow du = \sec^2 x \, dx$:

$$\int \tan^2 x \sec^2 x \ dx = \int u^2 \, du = \frac{u^3}{3} + C_1 = \frac{\tan^3 x}{3} + C_1$$

The second integral is one we solved in the earlier examples:

$$-\int \tan^2 x \, dx = -\int (\sec^2 x - 1) \, dx \ = -\tan x + x + C_2$$

And we combine the two to get:

$$\int \tan^4 x \, dx = \frac{\tan^3 x}{3} - \tan x + x + C$$

E. Separating out $\sec x \tan x$

3.63: Creating $\sec x \tan x$

Separate out $\sec x \tan x$, then we can make the substitution

$$u = \sec x \Rightarrow du = \sec x \tan x \, dx$$

Convert remaining powers of $\tan x$ to $\sec x$ using

$$\tan^2 x = \sec^2 x - 1$$

F. Integrals with $csc^2 x$

3.64: Integrating $csc^n \ x$ with even n

- Separate out $\csc^2 x$
- Convert remaining powers of $\csc x$ to $\cot x$ using $\csc^2 x = \cot^2 x + 1$
- Substitute $u = \cot x \Rightarrow du = -\csc^2 x \, dx$

Example 3.65

$$\int \csc^4 x\ dx$$

Split the product:

$$\int (\csc^2 x)(\csc^2 x)\, dx$$

Rewrite using $csc^2 x = \cot^2 x + 1$:

$$\int (\csc^2 x)(\cot^2 x + 1)\, dx$$

Substitute $u = \cot x \Rightarrow du = -\csc^2 x\, dx$

$$= -\int (u^2 + 1)\, du$$

Integrate:

$$= -\frac{u^3}{3} - u + C$$

Change back to the original variable:

$$= -\frac{\cot^3 x}{3} - \cot x + C$$

3.66: Creating $csc^2\ x$

$\boldsymbol{csc^2\ x}$ can be created using the identity

$$\cot^2 x = \csc^2 x - 1$$

Example 3.67

$$\int \cot^4 x\ dx$$

Split the powers inside the integral:

$$\int (\cot^2 x)(\cot^2 x)\, dx$$

Substitute $\cot^2 x = \csc^2 x - 1$:

$$\int (\cot^2 x)(\csc^2 x - 1)\, dx$$

Multiply and separate into two integrals:

$$\int \cot^2 x \csc^2 x\ dx - \int \cot^2 x\, dx$$

For the first integral $let\ u = \cot x \Rightarrow du = -\csc^2 x\, dx$

$$-\int u^2\, du = -\frac{u^3}{3} = -\frac{\cot^3 x}{3}$$

For the second integral:

$$-\int \cot^2 x\, dx = -\int \csc^2 x - 1\, dx = \int 1 - \csc^2 x\ dx = x + \cot x$$

Combining the first and the second integrals:

$$-\frac{\cot^3 x}{3} + \cot x + x + C$$

G. Integration by Parts

Example 3.68

$$\int \csc^3 x \;\, dx$$

Substitute $\csc^2 x = \cot^2 x + 1$

$$\int \csc x \,(1 + \cot^2 x)\, dx$$

Split the integrals:

$$= \int \csc x \, dx + \int \cot x \,(\csc x \cot x)\, dx$$

The first integral is:

$$\int \csc x \, dx = -\ln|\csc x + \cot x|$$

Use integration by parts for $\int \csc x \cot^2 x \, dx$

$$u = \cot x\,, \qquad dv = \csc x \cot x \;\, dx$$
$$du = -\csc^2 x \;\, dx, \qquad v = -\csc x \, dx$$

Apply $\int u \, dv = uv - \int v \, du$:

$$-\cot x \csc x - \int (-\csc x)\,(-\csc^2 x \;\, dx)$$
$$= -\cot x \csc x - \int \csc^3 x \, dx$$

Combine the two integrals:

$$\int \csc^3 x \;\, dx = -\ln|\csc x + \cot x| - \cot x \csc x - \int \csc^3 x \, dx$$
$$2\int \csc^3 x \;\, dx = -\ln|\csc x + \cot x| - \cot x \csc x$$
$$\int \csc^3 x \;\, dx = -\frac{1}{2}\ln|\csc x + \cot x| - \frac{1}{2}\cot x \csc x$$

Example 3.69

$$\int x \sin^3 x$$

Use integration by parts:

$$u = x, \qquad dv = \sin^3 x$$
$$du = dx, \qquad v = \frac{\cos^3 x}{3} - \cos x$$

Apply $\int u \, dv = uv - \int v \, du$:

$$x\left(\frac{\cos^3 x}{3}-\cos x\right)-\int \frac{\cos^3 x}{3}-\cos x\ dx$$

Using $\int \cos^3 x\, dx = \sin x - \frac{\sin^3 x}{3} + C$

$$= x\left(\frac{\cos^3 x}{3}-\cos x\right)-\frac{1}{3}\left(\sin x-\frac{\sin^3 x}{3}\right)-\sin x + C$$

H. Challenging Examples

Example 3.70

Reduction Formulas

3.71: Generalizing odd power of cos (Optional)[6]

For natural number n, we have:

$$\int \cos^{2n+1} x\ dx = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\sin^{2k+1} x}{2k+1} + C$$

Separate out a single power of $\cos x$:

$$\int (\cos^2 x)^n \cos x\ dx$$

Substitute $\cos^2 x = 1 - \sin^2 x$:

$$\int (1-\sin^2 x)^n\ \cos x\ dx$$

Substitute $u = \sin x \Rightarrow du = \cos x\, dx$:

$$\int (1-u^2)^n du$$

Expand using the binomial theorem:

$$\int \sum_{k=0}^{n} \binom{n}{k} (-1)^k u^{2k}\ du$$

Get the constants out of the integral sign:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \int u^{2k}\ du$$

Integrate using the reverse power rule:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{u^{2k+1}}{2k+1} + C$$

3.3 Trigonometric Substitutions

A. Summary of Substitutions

3.72: Substitutions

[6] This needs the binomial theorem. You can refer the Note on Binomial Theorem if you need a review, or skip it without loss of continuity.

$$x = a\sin\theta \Rightarrow \sqrt{a^2 - x^2} = a\cos\theta$$
$$x = a\tan\theta \Rightarrow \sqrt{a^2 + x^2} = a\sec\theta$$
$$x = a\sec\theta \Rightarrow \sqrt{x^2 - a^2} = a\tan\theta$$

Part A

Substitute $x = a\sin\theta$ in $\sqrt{a^2 - x^2}$:

$$\sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = a\sqrt{1 - \sin^2\theta} = a\sqrt{\cos^2\theta} = a\cos\theta$$ [7]

Part B

Substitute $x = a\tan\theta$ in $\sqrt{a^2 + x^2}$:

$$\sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(1 + \tan^2\theta)} = a\sqrt{(1 + \tan^2\theta)} = a\sqrt{\sec^2\theta} = a\sec\theta$$

Part C

Substitute $x = a\sec\theta$ in $\sqrt{x^2 - a^2}$:

$$\sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = a\sqrt{\sec^2\theta - 1} = a\sqrt{\tan^2\theta} = a\tan\theta$$

3.73: Substitutions

$$x = a\tan\theta \Rightarrow \sqrt{a^2 + x^2} = a\sec\theta$$

Substitute $x = a\tan\theta$ in $\sqrt{a^2 + x^2}$:

$$\sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(1 + \tan^2\theta)} = a\sqrt{(1 + \tan^2\theta)} = a\sqrt{\sec^2\theta} = a\sec\theta$$

B. $\cos^2 t = 1 - \sin^2 t$

The idea behind trigonometric substitutions is to use identities (such as the Pythagorean Identity) to convert expressions that have square roots into ones where the square root is no longer there.
We do this using a change of variable. Hence, this is a specific case of u substitution.

3.74: Change of Variable using $\cos^2 t = 1 - \sin^2 t$

The square root expression $\sqrt{r^2 - x^2}$ can be converted into one that does not have the square root using the change of variable $x = r\sin t$. That is:

$$x = r\sin t \Rightarrow \sqrt{r^2 - x^2} = r\cos t$$

We begin with the expression:

$$\sqrt{r^2 - x^2}$$

Use a change of variable. Substitute $x = r\sin t$:

$$\sqrt{r^2 - (r\sin t)^2} = \sqrt{r^2 - r^2\sin^2 t} = r\sqrt{1 - \sin^2 t}$$

Substituting $1 - \sin^2 t = \cos^2 t$:

$$= r\sqrt{\cos^2 t} = r\cos t$$

3.75: Equation of a Circle

The equation of a circle with center at the origin and radius r is

---

[7] Specifically $\sqrt{x^2} = |x|$. For the time being, let us assume that $\cos\theta$ is positive.

$$y^2 + x^2 = r^2$$

Using the distance formula, distance of any point on the circle will be

$$\sqrt{(y-0)^2 + (x-0)^2} = \sqrt{y^2 + x^2} = r$$

Example 3.76

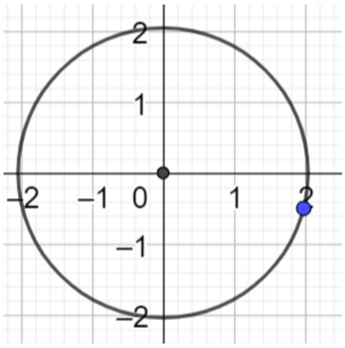
Show that the area of a circle with center at the origin and radius r is

$$4\int_0^r \sqrt{r^2 - x^2}\, dx$$

Evaluate the above integral.

Solve the equation of a circle $y^2 + x^2 = r^2$ for y:

$$y = \pm\sqrt{r^2 - x^2}$$



This is not a function, but we consider the positive square root and double the area (by symmetry).
In fact, since $f(x) = \sqrt{r^2 - x^2}$ is even, we can find four times the area in the first quadrant:

$$4\int_0^r \sqrt{r^2 - x^2}\, dx$$

Use a change of variable. Substitute $x = r\sin\theta$ and change the limits of integration:

$$x = r\sin\theta \Rightarrow dx = r\cos\theta\, d\theta$$

$$Lower\ Limit: x = r\sin\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$$

$$Upper\ Limit: x = r\sin\theta = r \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

Hence, we get:

$$4\int_0^{\frac{\pi}{2}} \sqrt{r^2 - (r\sin\theta)^2}\, r\cos\theta\, d\theta$$

Factor out r, combine with the other r, and move both outside the integral:

$$= 4r^2\int_0^{\frac{\pi}{2}} \sqrt{1^2 - \sin^2\theta}\, \cos\theta\, d\theta$$

Substitute $1 - \sin^2 t = \cos^2 t$:

$$= 4r^2\int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta}\, \cos\theta\, d\theta$$

Substitute $\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$, since θ is in the first quadrant or zero, and hence never negative:

$$= 4r^2\int_0^{\frac{\pi}{2}} \cos\theta\cos\theta\, d\theta = 4r^2\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

Reduce the power of cosine by using the half-angle formula $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$:

$$= 4r^2\int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2\theta) d\theta$$

Move $\frac{1}{2}$ outside the integral, and integrate term by term:

$$= 2r^2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

Substitute the limits of integration:

$$= 2r^2\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right) - \left(0 + \frac{1}{2}\sin 0\right)\right]$$

Evaluate and simplify:

$$= 2r^2\left[\left(\frac{\pi}{2} + 0\right) - (0+0)\right] = 2r^2\left[\frac{\pi}{2}\right] = \pi r^2$$

3.77: Equation of an Ellipse

The equation of an ellipse located at the origin with a, b as the lengths of the axes is

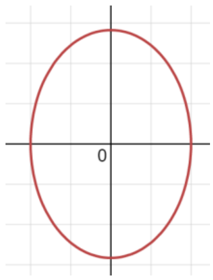
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- A circle is a special case of an ellipse with $a = b = r$.

Example 3.78: Area of an Ellipse

Show that the area of the ellipse below is πab.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Solve the equation for y:

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = \left(\frac{b^2}{a^2}\right)(a^2 - x^2)$$

Take the square root both sides:

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

Since this is symmetrical with respect to both the axes, we can find four times the area in the first quadrant, and hence we want:

$$4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

Let $x = a\sin\theta$:

$$x = 0 \Rightarrow 0 = a\sin\theta \Rightarrow 0 = \sin\theta \Rightarrow \theta = 0$$

$$x = a \Rightarrow a = a\sin\theta \Rightarrow 1 = \sin\theta \Rightarrow \theta = \frac{\pi}{2}$$

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - (a\sin\theta)^2} = a\sqrt{1 - \sin^2\theta} = a\sqrt{\cos^2\theta} = a\cos\theta$$

$$dx = a\cos\theta\,d\theta$$

Make the substitutions above:

$$4\int_0^{\frac{\pi}{2}} \frac{b}{a} a\cos\theta\, a\cos\theta\,d\theta = 4ab\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

By comparing this with the integral solved previously and using the constant multiple rule, we see that is it:

$$\pi ab$$

Example 3.79

Evaluate

$$\int \sqrt{a^2 - x^2}\, dx$$

Use u-substitution

Substitute $x = a \sin\theta$ to get:

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - (a\sin\theta)^2} = a\cos\theta$$
$$x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$$

Making the above substitutions yields:

$$\int \sqrt{a^2 - x^2}\, dx = \int a\cos\theta\, a\cos\theta\, d\theta = a^2 \int \cos^2\theta\, d\theta$$

Substitute using the half-angle identity:

$$= a^2 \int \frac{1}{2}(1 + \cos 2\theta)\, d\theta$$

Integrate term by term:

$$= \frac{a^2}{2}\left[\left(\theta + \frac{\sin 2\theta}{2}\right)\right] + C$$

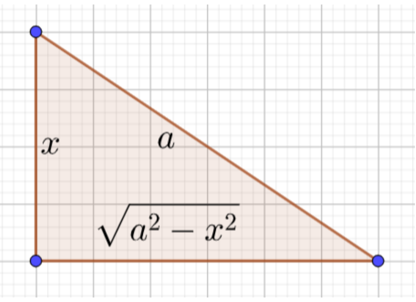
Substitute using the double-angle identity:

$$= \frac{a^2}{2}\left[\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right)\right] + C$$

Convert back to original variable

Get our answer back in terms of the original variable. Draw a reference triangle.

Note that:



$$\sin\theta = \frac{x}{a}$$
$$\theta = \sin^{-1}\left(\frac{x}{a}\right)$$
$$\cos\theta = \frac{\sqrt{a^2 - x^2}}{a}$$

And now we can make the substitutions:

$$= \frac{a^2}{2}\left[\left(\sin^{-1}\left(\frac{x}{a}\right) + \left(\frac{x}{a}\right)\frac{\sqrt{a^2 - x^2}}{a}\right)\right] + C$$
$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{ax}{2}\sqrt{a^2 - x^2} + C$$

Example 3.80

$$\int \frac{x^2}{(9 - x^2)^{\frac{3}{2}}} dx$$

Use the substitution $x = 3\sin\theta \Rightarrow dx = 3\cos\theta\, d\theta$:

$$(9-x^2)^{\frac{3}{2}} = \left[\sqrt{9-x^2}\right]^3 = \left[\sqrt{9-9\sin^2\theta}\right]^3 = \left[3\sqrt{\cos^2\theta}\right]^3 = [3\cos\theta]^3 = 27\cos^3\theta$$

The integral becomes:

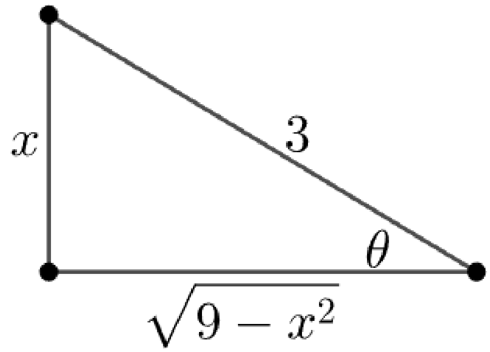
$$\int \frac{9\sin^2\theta}{27\cos^3\theta}\cdot 3\cos\theta\ d\theta = \int \tan^2\theta\, d\theta$$

Substitute $\tan^2\theta = 1-\sec^2\theta$

$$= \int (1-\sec^2\theta)\, d\theta = \theta - \tan\theta + C$$

$$x = 3\sin\theta \Rightarrow \frac{x}{3} = \sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{3}\right)$$

$$\tan\theta = \frac{x}{\sqrt{9-x^2}}$$



Changing back to the original variable:

$$= \sin^{-1}\left(\frac{x}{3}\right) + \frac{x}{\sqrt{9-x^2}} + C$$

C. $\tan^2 t = \sec^2 t - 1$

3.81: Integral of $\int \sec x\, dx$

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C$$

We found $\int \sec x$ in the chapter on u-substitution. Let's redo it. We will use this formula later.

Bring the integral into the form $\int \frac{f\prime(x)}{f(x)} dx = \ln|f(x)|$, but we need to employ a clever trick by multiplying by a $form-of-unity$:

$$\int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

Let $u = \sec x + \tan x \Rightarrow du = \sec x \tan x + \sec^2 x\ \ dx$:

$$= \int \frac{1}{u}\, du = \ln|u| + C = \ln|\sec x + \tan x| + C$$

3.82: Change of Variable using $\tan^2 t = \sec^2 t - 1$

$$x = a\sec t \Rightarrow \sqrt{x^2-a^2} = a\tan t$$

The identity $\tan^2 t = \sec^2 t - 1$ lets us eliminate square roots in expressions of the form $\sqrt{x^2-a^2}$.

Use a change of variable. Substitute $x = a\sec t$:

$$\sqrt{(a\sec t)^2 - a^2} = \sqrt{a^2\sec^2 t - a^2} = a\sqrt{\sec^2 t - 1}$$

Substituting $\sec^2 t - 1 = \tan^2 t$:

$$= a\sqrt{\tan^2 t} = a\tan t$$

Example 3.83

$$\int \frac{1}{\sqrt{x^2-1}}\,dx$$

Substitute $x = \sec t \Rightarrow dx = \sec t \tan t\, dt$

$$\int \frac{1}{\sqrt{\sec^2 t - 1}} \sec t \tan t\, dt$$

Substitute $\tan^2 t = \sec^2 t - 1$:

$$\int \frac{1}{\sqrt{\tan^2 t}} \sec t \tan t\, dt = \int \frac{1}{\tan t} \sec t \tan t\, dt = \int \sec t\, dt$$

Use the formula for $\int \sec t\, dt$ to integrate (see above):

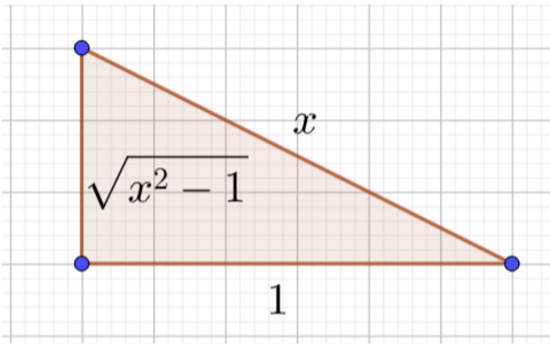
$$\int \sec t\, dt = \ln|\sec t + \tan t| + C$$

We now need to convert back in terms of x.

$$\sec t = x \Rightarrow \frac{hyp}{adj} = \frac{x}{1}$$

Then:

$$\tan t = \frac{opp}{adj} = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$$



Hence, the final answer is:

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln\left|x + \sqrt{x^2-1}\right| + C$$

Example 3.84

Evaluate

$$\int \sin^{-1}\left(\frac{1}{x}\right) dx$$

Integration by parts

$$u = \sin^{-1}\left(\frac{1}{x}\right), \qquad dv = dx$$

$$du = -\frac{1}{\sqrt{1-\frac{1}{x^2}}\cdot x^2}\,dx = -\frac{1}{\frac{\sqrt{x^2-1}}{x}\cdot x^2}\,dx = -\frac{1}{\sqrt{x^2-1}\cdot x}\,dx, \qquad v = x$$

Apply $\int u\, dv = uv - \int v\, du$:

$$\int \sin^{-1}\left(\frac{1}{x}\right) dx = x\cdot \sin^{-1}\frac{1}{x} - \int x \times -\frac{1}{\sqrt{x^2-1}\cdot x}\,dx = x\cdot\sin^{-1}\frac{1}{x} + \int \frac{1}{\sqrt{x^2-1}}\,dx$$

From the examples above, we know that $\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln\left|x+\sqrt{x^2-1}\right| + C$. Hence, the final answer is:

$$\int \sin^{-1}\left(\frac{1}{x}\right) dx = x\cdot\sin^{-1}\frac{1}{x} + \ln\left|x+\sqrt{x^2-1}\right| + C$$

D. $\sec^2 t = 1 + \tan^2 t$

3.85: Substitutions

$$x = a\tan\theta \Rightarrow \sqrt{a^2 + x^2} = a\sec\theta$$

Substitute $x = a\tan\theta$ in $\sqrt{a^2 + x^2}$:

$$\sqrt{a^2 + a^2\tan^2\ \theta} = \sqrt{a^2(1 + \tan^2\ \theta)} = a\sqrt{(1 + \tan^2\ \theta)} = a\sqrt{\sec^2\theta} = a\sec\theta$$

3.86: A trigonometric Integral

$$\int \sec^3\theta\, d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + tan\,\theta|) + C$$

This was solved in the chapter on integration by parts.

Example 3.87

Evaluate

$$\int \sqrt{a^2 + x^2}\, dx$$

Let $x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$:

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2\theta} = a\sqrt{1 + \tan^2\theta} = a\sec\theta$$

Making the substitutions above:

$$\int a\sec\theta\ \ a\sec^2\theta\, d\theta = a^2\int \sec^3\theta\, d\theta$$

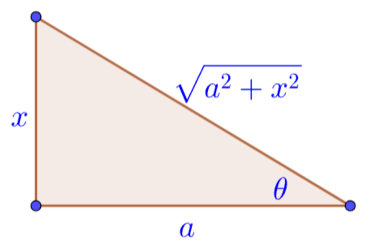
Which as per the property mentioned above is:

$$= \frac{a^2}{2}(\sec\theta\tan\theta + \ln|\sec\theta + tan\,\theta|) + C_1$$

Substitute $\tan\theta = \frac{x}{a} \Rightarrow \sec\theta = \frac{\text{hyp}}{adj} = \frac{\sqrt{a^2+x^2}}{a}$

$$= \frac{a^2}{2}\left(\frac{x}{a}\cdot\frac{\sqrt{a^2 + x^2}}{a} + \ln\left|\frac{\sqrt{a^2 + x^2}}{a} + \frac{x}{a}\right|\right) + C_1$$

$$= \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\left(\ln\left|x + \sqrt{a^2 + x^2}\right|\right) - \frac{a^2}{2}\ln|a| + C_1$$



Substitute $-\frac{a^2}{2}\ln|a| + C_1 = C$:

$$= \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\left(\ln\left|x + \sqrt{a^2 + x^2}\right|\right) + C$$

3.4 Partial Fractions[8]

A. Linear Factors

- It is difficult to integrate expressions that have a quadratic in the denominator.
- It is much easier to integrate expressions which have a linear factor in the denominator.
- Hence, the technique of partial fractions is useful.
- Partial fractions splits fractions which have a quadratic or higher polynomial into fractions which only have linear factors in their denominator.
- Partial fractions itself is an Algebra technique and does not require any Calculus.

Example 3.88

$$\int \frac{1}{x^2 - a^2} dx$$

Since $x^2 - a^2 = (x + a)(x - a)$, we write:

$$\underbrace{\frac{1}{x^2 - a^2} = \frac{A}{x + a} + \frac{B}{x - a}}_{Identity\ I}$$

We wish to determine constants A and B such that the above is an identity. Which means that it is true for all values of x and a.

Eliminate fractions by multiplying both sides by $x^2 - a^2$:

$$1 = A(x - a) + B(x + a)$$

Since Identity I is true for all values of x, it must be true for $x = a$:

$$1 = A(a - a) + B(x + a) \Rightarrow 1 = 2Ba \Rightarrow B = \frac{1}{2a}$$

Since Identity I is true for all values of x, it must be true for $x = -a$:

$$1 = A(-a - a) + B(-a + a) \Rightarrow 1 = -2aA \Rightarrow A = \frac{1}{-2a}$$

Substitute $A = \frac{1}{-2a}$ and $B = \frac{1}{2a}$ in the integral. Using the technique of partial fractions, we no longer have a quadratic in the denominator.

$$\int \frac{1}{x^2 - a^2} dx = -\frac{1}{2a} \int \frac{1}{x + a} dx + \frac{1}{2a} \int \frac{1}{x - a} dx$$

The above integrals can be evaluated using $\int \frac{1}{t} = \ln|t| + C$

$$= -\frac{1}{2a} \ln|x + a| + \frac{1}{2a} \ln|x - a| + C$$

Example 3.89

$$\int \frac{x + 2}{(x - 3)(x + 5)} dx$$

[8] You should be able to write an algebraic fraction in parts using partial fractions. This is covered in the Note on Rational Functions, and you should review it before you continue.

$$\frac{x+2}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5}$$

Eliminate fractions:

$$x + 2 = A(x+5) + B(x-3)$$

Substitute:

$$x = -5 \Rightarrow -5 + 2 = A(-5+5) + B(-5-3) \Rightarrow -3 = -8B \Rightarrow B = \frac{3}{8}$$
$$x = 3 \Rightarrow 3 + 2 = A(3+5) + B(3-3) \Rightarrow 5 = 8A \Rightarrow A = \frac{5}{8}$$

Hence, the integration is:

$$\frac{5}{8}\int \frac{1}{x-3}dx + \frac{3}{8}\int \frac{1}{x+5}dx$$

Which we can integrate as:

$$\frac{5}{8}\ln|x-3| + \frac{3}{8}\ln|x+5| + C$$

Example 3.90

$$\int \frac{1}{2x^2 - 3x - 5}dx$$

$$2x^2 - 3x - 5$$
$$Product = (-5)(2) = -10 =$$
$$Sum = -3 = -5 + 2$$

$$2x^2 - 5x + 2x - 5$$
$$x(2x-5) + 1(2x-5)$$
$$(2x-5)(x+1)$$

Step I: Split the Fraction

$$\frac{1}{(2x-5)(x+1)} = \frac{A}{2x-5} + \frac{B}{x+1}$$

Eliminate fractions:

$$1 = A(x+1) + B(2x-5)$$
$$0x + 1 = (A+2B)x + A - 5B$$

Use the method of undetermined coefficients. Equate coefficients on the left with coefficients on the right:

$$A + 2B = 0, A - 5B = 1 \Rightarrow B = -\frac{1}{7}, A = \frac{2}{7}$$

Hence, the above integral can be written:

$$\int \frac{1}{2x^2 - 3x - 5}dx = \int \frac{\frac{2}{7}}{2x-5} + \frac{-\frac{1}{7}}{x+1}dx$$

Step II: Split the Fraction

Split the integral:

$$\int \frac{\frac{2}{7}}{2x-5}dx + \int \frac{-\frac{1}{7}}{x+1}dx$$

Move the constant outside:

$$\frac{1}{7}\int \frac{2}{2x-5}dx - \frac{1}{7}\int \frac{1}{x+1}dx$$

By observation or by using u substitution:

$$\frac{1}{7}\ln|2x-5| - \frac{1}{7}\ln|x+1| + C$$

3.91: Linear Factors

An expression with a denominator that can be written as a product of linear factors can always be expressed as the sum of fractions that only have linear factors in their denominator[9]:

$$\frac{C}{(a_1x+b_1)(a_2x+b_2)\dots(a_nx+b_n)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \dots + \frac{A_n}{a_nx+b_n}$$

Note: Do not memorize the formula. It is meant to illustrate the concept.

3.92:Integration of Linear Factors

$$\int \frac{A}{ax+b} = \frac{A}{a}\ln|ax+b|$$

Example 3.93

Write the form of the partial form decomposition (do not find the constants) of the following:

(Calculator Allowed) Example 3.94

Use long division to write the fraction as the sum of a polynomial and a fraction where the degree of the numerator is less than the degree of the denominator

$$\frac{x^5+5x^4+6x^3+x^2+x+1}{x^3+6x^2+11x+6}$$

$$\begin{array}{r|llllll}
 & x^2 & -x & +1 & & & \\ \hline
x^3+6x^2+11x+6 & x^5+5x^4 & +6x^3 & +x^2 & +x & +1 & \\
 & -x^5-6x^4 & -11x^3 & -6x^2 & & & \\ \hline
 & -x^4 & -5x^3 & -5x^2 & +x & & \\
 & +x^4 & +6x^3 & +11x^2 & +6x & & \\ \hline
 & & +x^3 & +6x^2 & +7x & +1 & \\
 & & -x^3 & -6x^2 & -11x & -6 & \\ \hline
 & & & & -4x & -5 &
\end{array}$$

Factor $x^3+6x^2+11x+6$ using the rational roots, and factor theorems.

The possible roots are

[9] This can be proved using Algebra, though we will not do so here.

$$\pm\{1,2,3\}$$

Substitute $x = -1$ in $x^3 + 6x^2 + 11x + 6$:

$$(-1)^3 + 6(-1)^2 + 11(-1) + 6 = 0$$

Use Polynomial Long Division:

Use partial fraction decomposition to write the fraction as the sum of fractions that only have a linear factor in the denominator

Carry out the integration.

B. Repeated Linear Factors

Example 3.95

Show that the equality below is true and hence find the exact value of the definite integral:

$$\int_3^5 \frac{2x+3}{(2x-1)^2}dx = \int_3^5 \left(\frac{1}{2x-1} + \frac{4}{(2x-1)^2}\right)dx$$

$$\int_3^5 \frac{2x-1+4}{(2x-1)^2} = \int_3^5 \frac{2x-1+4}{(2x-1)^2} = \int_3^5 \left(\frac{1}{2x-1} + \frac{4}{(2x-1)^2}\right)dx$$

Example 3.96

7 **(a)** Show that $\frac{2}{2x+3} - \frac{1}{x-1} + \frac{1}{(x-1)^2}$ can be written as $\frac{8-3x}{(x-1)^2(2x+3)}$. [2]

(b) Find $\int_2^a \frac{8-3x}{(x-1)^2(2x+3)}\mathrm{d}x$ where $a > 2$. Give your answers in the form $c + \ln d$, where c and d are functions of a. [6]

C. Irreducible Quadratic Factors

Example 3.97

D. Repeated Irreducible Quadratic Factors

Example 3.98

3.5 Arc Length

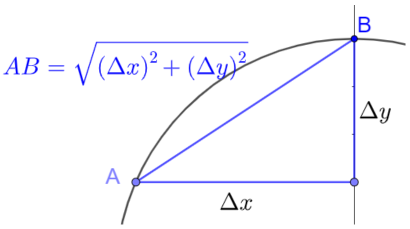
A. Basics

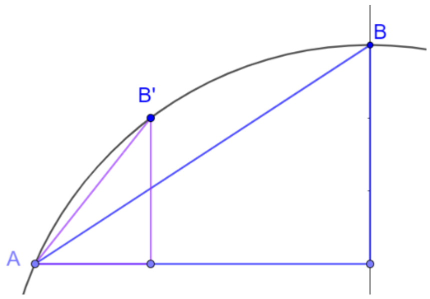
3.99: Arc Length Formula (Informal Derivation)

$$s = \int ds = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$AB = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Notice that line segment AB has a different length from $arc\ AB$.





To improve the approximation, we can reduce the distance between A and B. ($right\ diagram$)
To make the approximation very, very close, take the limit as the distance between the points A and B becomes close to zero:

$$As\ AB \to 0, arc\ length\ AB \to\ AB$$

Let $AB = ds, \Delta x = dx, \Delta y = dy$ in $AB = \sqrt{(\Delta x)^2 + (\Delta y)^2}$:

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

Multiply and divide by dx:

$$ds = \sqrt{(dx)^2 + (dy)^2} \cdot \frac{1}{dx} \cdot dx$$

Move the $\frac{1}{dx}$ inside the square root:

$$ds = \sqrt{\frac{(dx)^2}{(dx)^2} + \frac{(dy)^2}{(dx)^2}} \cdot dx = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

Since we took the limit as $AB = ds \to 0, \Delta x = dx \to 0, \Delta y = dy \to 0$. Integrate both sides:

$$s = \int ds = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

This is the formula for arc length.

Example 3.100

$$y = x^2$$

Show that

$$Second\ derivative = \frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2 = Square\ of\ the\ derivative$$

Note that

$$\frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)^2 = 4x^2$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(2x) = 2$$

3.101: Arc Length of $y = f(x), a \le x \le b$

Arc length over the interval $[a, b]$ for a function $y = f(x)$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

- f' must be continuous over the interval $[a, b]$
- To be continuous, f' must exist over the interval $[a, b]$

When we say the interval $[a, b]$, we mean the points:

$$A = (a, f(a)), B = (b, f(b))$$

Example 3.102

Find the arc length of the curve $f(x) = y = x^{\frac{3}{2}} - 4$ over the interval $4 \le x \le 8$. Write your final answer in exact radical form.

Make the substitutions $a = 4, b = 8, \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{9}{4}x$ in

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_4^8 \sqrt{1 + \frac{9}{4}x\, dx}$$

Use u substitution with:

$$u = 1 + \frac{9}{4}x \Rightarrow du = \frac{9}{4}dx \Rightarrow dx = \frac{4}{9}du$$

The limits of integration change to:

$$x = 4 \Rightarrow u = 1 + 9 = 10$$
$$x = 8 \Rightarrow u = 1 + 18 = 19$$

$$= \int_{10}^{19} \sqrt{u} \cdot \frac{4}{9} du$$

Carry out the integration and move the constant outside:

$$= \frac{4}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{10}^{19} = \frac{8}{27}\left[u^{\frac{3}{2}}\right]_{10}^{19}$$

Substitute the limits of integration and simplify:

$$= \frac{8}{27}\left(19^{\frac{3}{2}} - 10^{\frac{3}{2}}\right) = \frac{8}{27}\left(19\sqrt{19} - 10\sqrt{10}\right)$$

Example 3.103

Concept Questions: Continuity

3.104: Perfect Square

If the expression $1+\left(\frac{dy}{dx}\right)^2$ is a perfect square, then

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Will have the square and the square root cancel, and the integration will become simpler.

Example 3.105: Perfect Square

Find the arc length of $y=\frac{e^x+e^{-x}}{2}$ from $x=1$ to $x=2$.

Find the first derivative:

$$\frac{dy}{dx}=\left(\frac{1}{2}\right)(e^x-e^{-x})$$

Square both sides:

$$\left(\frac{dy}{dx}\right)^2=\left(\frac{1}{4}\right)(e^{2x}-2+e^{-2x})$$

Add 1 to both sides:

$$1+\left(\frac{dy}{dx}\right)^2=\left(\frac{1}{4}\right)(e^{2x}+2+e^{-2x})=\left(\frac{e^x+e^{-x}}{2}\right)^2$$

Substitute $\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{e^x+e^{-x}}{2}, a=1, b=2$ in $L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$L=\int_1^2\left(\frac{e^x+e^{-x}}{2}\right)dx=\frac{1}{2}[e^x-e^{-x}]_1^2$$

$$=\frac{1}{2}(e^2-e^{-2}-e+e^{-1})=\frac{1}{2}\left(e^2+e-\frac{1}{e}-\frac{1}{e^2}\right)$$

B. Integrating in terms of y

If the derivative does not exist, or the derivative is discontinuous, then we cannot use the arc length formula. In such a case, we might still be able to find the arc length by integrating in terms of y rather than x.

3.106: Arc Length of $x=g(y), a\le x\le b$

Arc length over the interval $[a,b]$ for a function $x=g(x)$ is

$$L=\int_c^d\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_c^d\sqrt{1+[g'(x)]^2}\,dy$$

- g' must be continuous over the interval $[a,b]$
- To be continuous, g' must exist over the interval $[a,b]$

When we say the interval $[c,d]$, we mean the points

$$A=(g(c),c), B=(g(d),d)$$

Example 3.107

$$y = x^{\frac{2}{3}}, 0 \le x \le 1$$

A. Explain why the arc length formula that integrates in terms of x will not work
B. Find the arc length by using the formula that integrates in terms of y.

Part A

$$\frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3x^{\frac{1}{3}}}$$

When

$$x = 0 \Rightarrow \frac{dy}{dx} = \frac{2}{0} \Rightarrow Not\ defined$$

Part B

$$x = y^{\frac{3}{2}}$$
$$\frac{dx}{dy} = \frac{3}{2}y^{\frac{1}{2}}$$
$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{9}{4}y$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + \frac{9}{4}y}\, dy = \frac{8}{27}\left(13\sqrt{13} - 1\right)$$

C. Differentials

3.108: Differential Formula

D. Arc Length Function

Example 3.109

Arc length of circle

3.6 Improper Integrals

A. Type I Improper Integrals

Till now, we have learnt how to deal with integrals that have finite limits of integration. However, we may sometimes wish to calculate integrals where one or both limits are infinite.

This cannot be done directly, but we can find the limit of the integral, and one of the limits of integration tends to infinity.

3.110: Integrals: Upper Limit $\to \infty$

If $f(x)$ is continuous on (a, ∞) and the limit below exists and is finite, then:

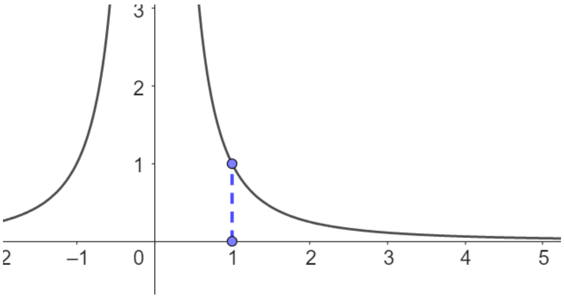
$$\int_a^\infty f(x)\, dx = \lim_{t \to \infty} \int_a^t f(x)\, dx$$

- For the limit to exist, the integral must exist.
- An improper integral of Type I has a limit tending to infinity on the x axis.

Example 3.111

Evaluate $\int_1^a \frac{1}{x^2} dx$ and use it to find the value of:

A. $\int_1^{10} \frac{1}{x^2} dx$
B. $\int_1^{100} \frac{1}{x^2} dx$
C. $\int_1^{1000} \frac{1}{x^2} dx$
D. $\int_1^{\infty} \frac{1}{x^2} dx$



$$\int_1^a \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_1^a = -\frac{1}{a} - \left(-\frac{1}{1}\right) = 1 - \frac{1}{a}$$

Part A

$$1 - \frac{1}{10} = \frac{9}{10} = 0.9$$

Part B

$$1 - \frac{1}{100} = \frac{99}{100} = 0.99$$

Part C

$$1 - \frac{1}{1000} = \frac{999}{1000} = 0.999$$

Part D

Determine the limit as the variable a upper limit tends to infinity:

$$\lim_{a\to\infty}\left(1 - \frac{1}{a}\right) = 1 + 0 = 1$$

3.112: Convergence and Divergence

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

If the limit is:

- a finite number, the integral is convergent.
- infinite, or does not exist, the above integral is divergent.

Improper integrals can be evaluated only if they can be assigned a finite value (that is, are convergent).

For example, a geometric series:

$$a + ar + \cdots \begin{cases} converges\ for -1 < r < 1 \\ diverges\ otherwise \end{cases}$$

Example 3.113

Evaluate the integral below:

$$\int_1^{\infty} \frac{1}{x^4}\,dx$$

Replace the upper limit with a variable, and evaluate the integral:

$$\int_a^t \frac{1}{x^2}\,dx = -\frac{1}{3}\left[\frac{1}{x^3}\right]_1^t = -\frac{1}{3}\left(\frac{1}{t^3} - \frac{1}{1}\right) = \frac{1}{3}\left(1 - \frac{1}{t^3}\right)$$

$$\lim_{t\to\infty}\frac{1}{3}\left(1 - \frac{1}{t^3}\right) = \frac{1}{3}$$

Example 3.114

Evaluate the integral below:

$$\int_1^\infty \frac{1}{x}\,dx$$

Replace the upper limit with a variable, and evaluate the integral:

$$\int_1^t \frac{1}{x}\,dx = [\ln|x|]_1^t = \ln t - \ln 1 = \ln t - 0 = \ln t$$

$$As\ t \to \infty \Rightarrow \ln t \to \infty \Rightarrow Limit\ DNE \Rightarrow Integral\ diverges$$

Example 3.115

Evaluate the integral below:

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$$

Replace the upper limit with a variable, and evaluate the integral:

$$\int_1^t \frac{1}{\sqrt{x}}\,dx = \left[\frac{1}{2}\sqrt{x}\right]_1^t = \frac{1}{2}\sqrt{t} - \frac{1}{2}$$

$$\lim_{t\to\infty}\left(\frac{1}{2}\sqrt{t} - \frac{1}{2}\right)\ does\ not\ exist \Rightarrow Integral\ diverges$$

3.116: Integrals: Lower Limit $\to -\infty$

$$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx$$

There are some important conditions:

- $\int_t^a f(x)\,dx$ exists for every number $t \le a$
- The limit above exists and is finite

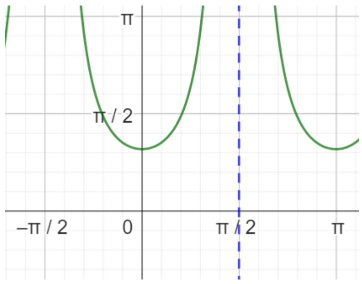
Example 3.117

B. Type II: Functions tending to Infinity on the y-axis

Type II improper integrals have integrands whose y values tend to infinity on the y axis. They have an infinite discontinuity (also called a vertical asymptote) because the denominator is not defined - either in the interval or at an endpoint.

3.118: Types of Discontinuity

- Lower limit has a discontinuity
- Upper limit has a discontinuity
- Discontinuity exists between the limits



Example 3.119

$$\int_0^{\frac{\pi}{2}} \sec^2 x \ dx$$

A. Can you evaluate the integral by integrating and substituting the limits?
B. Evaluate the above integral as an improper integral.

Part A

$\sec^2 x = \frac{1}{\cos^2 x}$ does not have $\frac{\pi}{2}$ in its domain.

Part B

Replace the upper limit with a variable, and evaluate the integral:

$$\int_0^k \sec^2 x \ dx = [\tan x]_0^k = \tan k - \tan 0 = \tan k$$

Determine the limit as the variable k tends to infinity:

$$\lim_{k \to \frac{\pi}{2}} \tan k = \infty$$

C. Area under the Curve

3.120: Improper Integrals as Area

One interpretation of a definite integral is as area under the curve.
Improper integrals can be used to calculate the area under a curve, if it has a finite value.

- This has applications in probability distributions.

Example 3.121

Determine the area to the right of the y axis, above the x axis and under the curve $y = e^{-x}$.

Replace the upper limit of $\int_0^\infty e^{-x} \ dx$with a variable, and evaluate the integral:

$$\int_0^k e^{-x} \, dx = \left[-\frac{1}{e^x}\right]_0^k = -\frac{1}{e^k} - \left(-\frac{1}{e^0}\right) = 1 - \frac{1}{e^k}$$

Determine the limit as the variable k tends to infinity:

$$= 1 - 0 = 1$$

3.122: Integration by Parts

Improper integrals can require the use of integration techniques such as integration by parts.

Example 3.123

Determine the area to the right of the y axis and under the curve[10] $y = xe^{-x}$.

Find the associated indefinite integral of $\int_0^\infty xe^{-x} \ dx$ using integration by parts:

$$u = x, \qquad dv = e^{-x} \, dx$$
$$du = dx, \qquad v = -e^{-x}$$

[10] This is an example of the gamma function, an important function in higher mathematics.

Apply $\int u\,dv = uv - \int v\,du$:

$$\int xe^{-x}\,dx = x(-e^{-x}) - \int -e^{-x}\,dx = -xe^{-x} - e^{-x} + C = -\frac{x+1}{e^x} + C$$

$$\int_0^\infty xe^{-x}\,dx = \left[-\frac{x+1}{e^x}\right]_0^\infty = \left(\lim_{x\to\infty} -\frac{x+1}{e^x}\right) - \left(-\frac{1-0}{e^0}\right) = 0 + 1 = 1$$

Example 3.124

Determine the area to the right of the y axis and under the curve[11] $y = x^2e^{-x}$.

The associated indefinite integral is:

$$\int x^2e^{-x}\,dx = -(x^2e^{-x} + 2xe^{-x} + 2e^{-x}) = -\left(\frac{x^2+2x+2}{e^x}\right) + C$$

The improper integral becomes:

$$\int_0^\infty x^2e^{-x}\,dx = \left[-\left(\frac{x^2+2x+2}{e^x}\right)\right]_0^\infty = 0 - (-2) = 2$$

Where we evaluated the upper and lower limit as:

$$\lim_{x\to\infty}\left(-\frac{x^2+2x+2}{e^x}\right) = 0$$

$$x = 0 \Rightarrow \left(-\frac{0^2+2(0)+2}{e^0}\right) = -2$$

D. Doubly Improper Integrals: Type I

3.125: Doubly Improper Integrals: Limits $\to \pm\infty$

To handle improper integrals which range from $-\infty$ to ∞, we need to divide (cut) the integral into two integrals, each of which is a (singly) improper integral:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

Where

$$c\ is\ any\ real\ number$$

Conditions:

- $f(x)$ must be continuous over $(-\infty, \infty)$
- Each of the integrals must separately converge.

Note:

- The choice of c is usually to make the calculations easier.

Example 3.126

[11] As you might have noticed, $\int_0^\infty e^{-x}\,dx = 0! = 1$, $\int_0^\infty xe^{-x}\,dx = 1! = 1$, $\int_0^\infty x^2e^{-x}\,dx = 2! = 2$. The gamma function is a generalization of the factorial function.

$$\int_{-\infty}^{\infty} \frac{1}{1+9x^2} dx$$

Since $1+9x^2 = 1+(3x)^2$, substitute $u = 3x \Rightarrow du = 3\,dx$

$$\int \frac{1}{1+9x^2} dx = \frac{1}{3}\int \frac{1}{1+u^2} dx = \frac{1}{3}(\arctan u) = \frac{1}{3}\tan^{-1} 3x$$

Split the integral at 0:

$$\int_{-\infty}^{0} \frac{1}{1+9x^2} dx = \frac{1}{3}[\tan^{-1} 3x]_a^0 = \frac{1}{3}\lim_{a\to-\infty}(\tan^{-1} 0 - \tan^{-1} 3a) = \frac{\pi}{6}$$
$$\int_{0}^{\infty} \frac{1}{1+9x^2} dx = \frac{1}{3}[\tan^{-1} 3x]_0^a = \frac{1}{3}\lim_{a\to-\infty}(\tan^{-1} 3a - \tan^{-1} 0) = \frac{\pi}{6}$$

The final answer is:

$$\frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$$

Example 3.127

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+5)^2}\, dx$$

Split the integral at a suitable value ($c = 0$):

$$\int_{-\infty}^{0} \frac{x}{(x^2+5)^2}\, dx + \int_{0}^{\infty} \frac{x}{(x^2+5)^2}\, dx$$

Substitute $u = x^2 + 5 \Rightarrow du = 2x\,dx$. The limits of integration are:

$$x = 0 \Rightarrow u = 0^2 + 5 = 5$$
$$x \to \pm\infty \Rightarrow u \to \pm\infty$$

Substitute the values calculated above:

$$\frac{1}{2}\int_{\infty}^{5} \frac{1}{u^2}\, du + \frac{1}{2}\int_{5}^{\infty} \frac{1}{u^2}\, du = -\frac{1}{10} + \frac{1}{10} = 0$$

Where we evaluated the integrals above using:

$$\frac{1}{2}\int_{\infty}^{5} \frac{1}{u^2}\, du = \lim_{p\to\infty} \frac{1}{2}\left[-\frac{1}{u}\right]_p^5 = -\frac{1}{2}\lim_{p\to\infty}\left(\frac{1}{5} - \frac{1}{p}\right) = -\frac{1}{2}\cdot\frac{1}{5} = -\frac{1}{10}$$
$$\frac{1}{2}\int_{5}^{\infty} \frac{1}{u^2}\, du = \lim_{q\to\infty} \frac{1}{2}\left[-\frac{1}{u}\right]_5^q = -\frac{1}{2}\lim_{q\to\infty}\left(\frac{1}{q} - \frac{1}{5}\right) = \frac{1}{2}\cdot\frac{1}{5} = \frac{1}{10}$$

E. Doubly Improper Integrals: Type II

3.128: Doubly Improper Integrals of Type II

If a function $f(x)$ is such that its integral $F(x)$ is not defined at $x = a$, and $x = b$, we have a doubly improper integral of Type II.

As before, this can be evaluated by splitting the integral at a value c in the open interval (a, b) and evaluating each (singly) improper integral:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Conditions:

- Each of the integrals must separately converge.

Note:
The choice of c is usually to make the calculations easier.

Example 3.129

$$\int_{-1}^{1} \frac{1}{\sqrt[3]{x}}\,dx$$

The expression $\frac{1}{\sqrt[3]{x}}$ is everywhere over $[-1,1]$ except at $x = 0$.
Hence, this is an improper integral of Type II.

Split the integral at $x = 0$:

$$A = \int_{-1}^{t} \frac{1}{\sqrt[3]{x}}\,dx = \frac{3}{2}\left[x^{\frac{2}{3}}\right]_{-1}^{t} = \frac{3}{2}\left(t^{\frac{2}{3}} - (-1)^{\frac{2}{3}}\right) = \frac{3}{2}\left(t^{\frac{2}{3}} - 1\right) \Rightarrow \lim_{t\to 0}\frac{3}{2}\left(t^{\frac{2}{3}} - 1\right) = -\frac{3}{2}$$

$$B = \int_{t}^{1} \frac{1}{\sqrt[3]{x}}\,dx = \left[\frac{3}{2}x^{\frac{2}{3}}\right]_{t}^{1} = \frac{3}{2}\left(1^{\frac{2}{3}} - t^{\frac{2}{3}}\right) \Rightarrow \lim_{t\to 0}\frac{3}{2}\left(1^{\frac{2}{3}} - t^{\frac{2}{3}}\right) = \frac{3}{2}$$

$$\int_{-1}^{1} \frac{1}{\sqrt[3]{x}}\,dx = A + B = -\frac{3}{2} + \frac{3}{2} = 0$$

Example 3.130: Arc Length

Determine the length of the semicircle $y = \sqrt{1 - x^2}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = -\frac{x}{\sqrt{1-x^2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{1-x^2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{1-x^2} = \frac{1 - x^2 + x^2}{1-x^2} = \frac{1}{1-x^2}$$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_{-1}^{1} \sqrt{\frac{1}{1-x^2}}\,dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$$

This integral is doubly improper since $\frac{1}{\sqrt{1-x^2}}$ is not defined at $x \in \{-1,1\}$.
To calculate the integral, we split it in between the limits of integration:

$$= \int_{-1}^{0} \frac{1}{\sqrt{1-x^2}}\,dx + \int_{0}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$$

Replace the -1 with an a, and 1 with a b:

$$= \int_a^0 \frac{1}{\sqrt{1-x^2}} dx + \int_0^b \frac{1}{\sqrt{1-x^2}} dx$$

Since $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$:

$$= [\sin^{-1} x]_a^0 dx + [\sin^{-1} x]_0^b dx$$

Substitute the limits of integration:

$$= (\sin^{-1} 0 - \sin^{-1} a) - (\sin^{-1} b - \sin^{-1} 1)$$

Take the

$$= \lim_{a \to -1} (\sin^{-1} 0 - \sin^{-1} a) - \lim_{b \to 1} (\sin^{-1} b - \sin^{-1} 0)$$

$$= \left[0 - \left(-\frac{\pi}{2}\right)\right] + \left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

F. Volumes with Improper Integrals

Example 3.131: Volumes with Shells

Find the volume between $z = e^{-x^2}$ is revolved around the z axis and the xy plane.

Use a change of variables. Let Y be the z axis, X be the x axis, Z be the y axis.

$Y = e^{-X^2}$ is revolved around the Y axis, find the volume between this and the XZ plane

$$V = 2\pi \int \binom{Shell}{Radius} \binom{Shell}{Height}$$

$$Shell\ Radius = x$$
$$Shell\ Height = e^{-x^2}$$

$$2\pi \int_0^\infty x e^{-x^2}\, dx$$

$$-\pi \left[e^{-x^2}\right]_0^t = -\pi\left(e^{-t^2} - e^0\right) = \pi\left(e^0 - e^{-t^2}\right)$$

$$\lim_{t \to \infty} \pi\left(e^0 - e^{-t^2}\right) = \pi(e^0) = \pi$$

G. Interval of Convergence

Example 3.132

Determine the x values for which the integral below converges:

$$\int_0^x \frac{u}{\sqrt{1-u}} du$$

Calculate the associated indefinite integral first. Substitute $t = 1 - u, dt = -du, u = 1 - t$

$$\int \frac{u}{\sqrt{1-u}}du = \int \frac{t-1}{\sqrt{t}}dt = \frac{2}{3}t^{\frac{3}{2}} - 2\sqrt{t} = \frac{2}{3}(1-u)^{\frac{3}{2}} - 2\sqrt{1-u} + C$$

The given integral is then:

$$\int_0^x \frac{u}{\sqrt{1-u}}du = \left[\frac{2}{3}(1-u)^{\frac{3}{2}} - 2\sqrt{1-u}\right]_0^x = \left(\frac{2}{3}(1-x)^{\frac{3}{2}} - 2\sqrt{1-x}\right) - \left(\frac{2}{3} - 2\right)$$

Simplify:

$$= \left(\frac{2}{3}(1-x)^{\frac{3}{2}} - 2\sqrt{1-x}\right) + \frac{4}{3}$$

For the expression above to be defined:

$$1 - x \ge 0 \Rightarrow x \le 1$$

At $x = 1$, the original function is undefined. Hence, take the limit of the expression as $x \to 1$:

$$\lim_{x\to 1}\left(\frac{2}{3}(1-x)^{\frac{3}{2}} - 2\sqrt{1-x}\right) + \frac{4}{3}$$

Evaluate the limit by substitution:

$$\left(\frac{2}{3}(1-1)^{\frac{3}{2}} - 2\sqrt{1-1}\right) + \frac{4}{3} = \frac{4}{3}$$

Hence, the interval of convergence of the integral is:

$$x \le 1$$

H. Type I and II Combined

Example 3.133

$$\int_0^{\infty} \frac{1}{x^2}\,dx$$

Since the function is not defined at the lower limit of integration $(x = 0)$, it is an improper integral of Type II. Since the function has an upper limit of ∞, it is an improper integral of Type I.

Hence, this is an improper integral of both Type I and Type II

I. Applications

Example 3.134

Inequalities
Laplace Transforms
Escape Velocity

4. Further Topics

4.1 Hyperbolic Integrals

A. Antiderivatives

Example 4.1

$$\int \pi \sinh x + \frac{1}{\pi} \cosh x \, dx$$

Write as the sum of two integrals, and then integrate:

$$\pi \int \sinh x + \frac{1}{\pi} \int \cosh x \, dx = \pi \cosh x + \frac{1}{\pi} \sinh x + C$$

B. u-substitution

Example 4.2

$$\int \frac{\sinh \pi x}{\pi} dx$$

$$\int \frac{\sinh \pi x}{\pi} dx = \frac{1}{\pi} \int \sinh \pi x \, dx$$

Substitute $u = \pi x \Rightarrow du = \pi \, dx$:

$$\frac{1}{\pi^2} \int \sinh u \; du = \frac{1}{\pi^2} \cosh u = \frac{\cosh \pi x}{\pi^2}$$

C. Integration by Parts

Example 4.3

$$\int x \sinh ex \, dx$$

We use integration by parts:

$$u = x, \qquad dv = \sinh ex \, dx$$

$$du = dx, \qquad v = \frac{\cosh ex}{e}$$

Apply $\int u \, dv = uv - \int v \, du$:

$$\int x \sinh ex \, dx = x \cdot \frac{\cosh ex}{e} - \int \frac{\cosh ex}{e} du = x \cdot \frac{\cosh ex}{e} - \frac{1}{e^2} \sinh ex + C$$

D. Inverse Trigonometric Functions

Example 4.4

Evaluate the integral below by using u-substitution, followed by integration by parts.

$$\int \sin^{-1}\left(\frac{1}{x}\right) dx$$

Integration by parts

$$u = \sin^{-1}\left(\frac{1}{x}\right), \qquad dv = dx$$

$$du = -\frac{1}{\sqrt{1-\frac{1}{x^2}}\cdot x^2}dx = -\frac{1}{\frac{\sqrt{x^2-1}}{x}\cdot x^2}dx = -\frac{1}{\sqrt{x^2-1}\cdot x}dx, \qquad v = x$$

Apply $\int u\,dv = uv - \int v\,du$:

$$\int \sin^{-1}\left(\frac{1}{x}\right)dx = x\cdot\sin^{-1}\frac{1}{x} - \int x \times -\frac{1}{\sqrt{x^2-1}\cdot x}dx$$

Focus on the last integral. Start by simplifying:

$$-\int x \times -\frac{1}{\sqrt{x^2-1}\cdot x}dx = \int \frac{1}{\sqrt{x^2-1}}dx$$

u-substitution

You can recognize the above as the derivative of $\cosh^{-1} x$ or understand the method better by substituting:

$$x = \cosh u \Rightarrow dx = \sinh u\, du$$

To get:

$$\int \frac{1}{\sqrt{x^2-1}}dx = \int \frac{1}{\sqrt{\cosh^2 u - 1}}\sinh u\, du$$

Use the identity $\cosh^2 u - \sinh^2 u = 1 \Rightarrow \sinh^2 u = 1 - \cosh^2 u$:

$$= \int \frac{1}{\sqrt{\sinh^2 u}}\sinh u\, du = \int \frac{1}{\sinh u}\sinh u\, du = \int du = u + c$$

And now $x = \cosh u \Rightarrow u = \cosh^{-1} x$:

$$u + c = \cosh^{-1} x + C$$

And, hence, the final answer is:

$$\int \sin^{-1}\left(\frac{1}{x}\right)dx = x\cdot\sin^{-1}\left(\frac{1}{x}\right) + \cosh^{-1} x + C$$

E. Using equations to solve for the Integral

4.2 Leibniz Rule

A. Basics

4.5: Leibniz Rule

Proof

Example 4.6

Explanation

Example 4.7

Practice examples

Example 4.8

JEE 2024 question

4.3 Walli's Theorem

Example 4.9
Statement and Basic Examples

10 Examples