Circles

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1. Circle Geometry

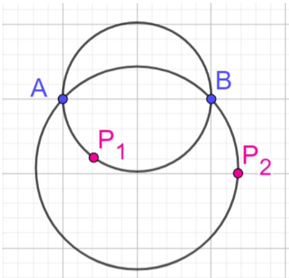
1.1 Basics

A. Basics

1.1: Definition of a circle

A circle is the set of all points that are equidistant from a fixed center.

- The equal distance is called the *radius*.
- The fixed center is called the center of the circle.



1.2: Identifying a Circle: Points on the Circumference

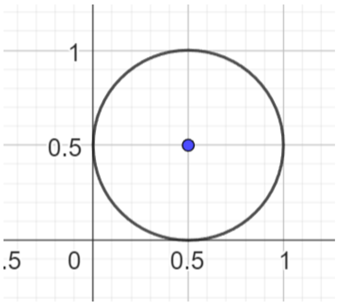
Through two non-collinear points on the circumference, you can draw infinite circles. Through three non-collinear points on the circumference, you can draw exactly one circle.

1.3: Identifying a Circle: Center and Radius

Through a fixed point (*center*), exactly one circle with a given radius can be drawn.

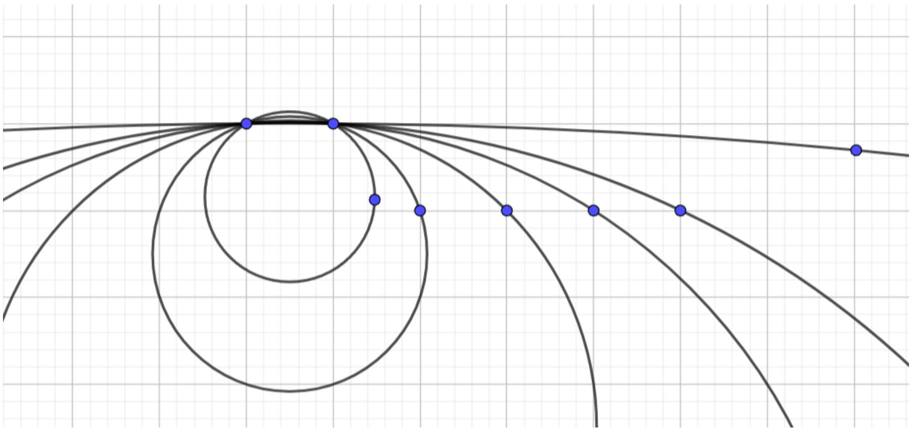
Example 1.4

Draw a circle with radius 0.5, and center (0.5,0.5)



1.5: Circle with Infinite Radius

- As the radius of a circle increases the curvature of the circle decreases, and the circle becomes closer to a straight line.
- A circle with infinite radius is a straight line.

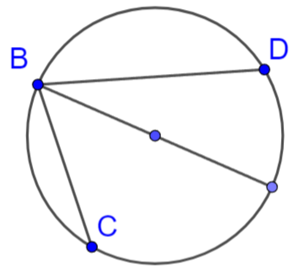


1.2 Chords

A. Definition

1.6: Chord

- A line segment whose endpoints lie on the circumference of circle is a chord of the circle.
- The longest chord in the circle is the diameter. Diameter passes through the center of the circle.

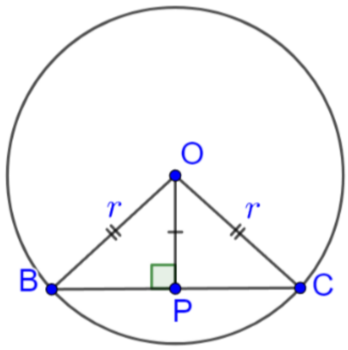


1.7: Perpendicular to Chord

The perpendicular to a chord from the center of the circle bisects the chord.

In a circle with center O, and radius r, draw chord BC.
Construct

$$OP \perp BC$$



$\Delta OPB \cong \Delta OPC$ by RHS:

$$\angle OPB = \angle OPC = 90° \ (Right\ Angle)\ (Definition\ of\ perpendicular)$$
$$OB = OC = r\ (Hyp)(Both\ are\ radii)$$
$$OP = OP\ (Side)\ (Reflexive\ Property)$$

Hence:

$$PB = PC\ (CPCT)$$

Example 1.8

In a circle of radius 5, the perpendicular drawn to a chord has length 3. Find the length of the chord.

We get a Pythagorean Triplet:

$$3, 4, 5$$

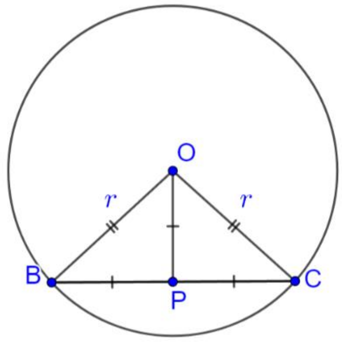
Hence, the length of the chord is:

$$4 \times 2 = 8$$

1.9: Bisector of Chord

The line segment drawn from the center of a circle bisecting the chord is perpendicular to the chord.

In a circle with center O, and radius r, draw chord BC.
Let P be the midpoint of BC. Construct OP.



$\Delta OPB \cong \Delta OPC$ by SSS:

$$OB = OC = r \ (Side)(Both\ are\ radii)$$
$$OP = OP \ (Side) \ (Reflexive\ Property)$$
$$PB = PC \ (Side) \ (P\ is\ the\ midpoint)$$

Hence:

$$\angle OPB = \angle OPC$$

By angles in a linear pair:

$$\angle OPB + \angle OPC = 180°$$
$$\angle OPB + \angle OPB = 180°$$
$$\angle OPB = 90°$$

1.10: Bisector of Chord

The perpendicular bisector of a chord passes through the center of a circle.

B. Distance from Centre

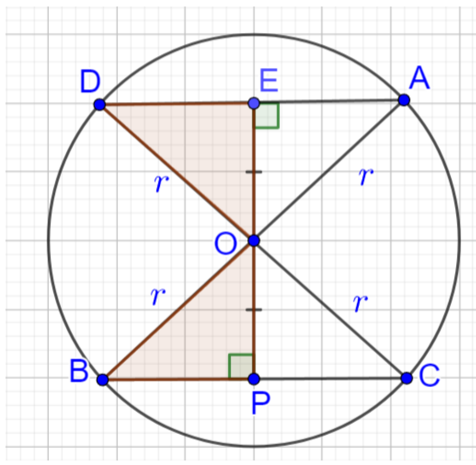
1.11: Distance from Center

The distance of a line, or a chord from the center of a circle is the shortest (perpendicular) distance.

1.12: Congruent Chords & Distance

Chords equidistant from the center of a circle are congruent.

Draw circle O with chords AD and BC equidistant from the center.



$\Delta OED \cong \Delta OPB$ by RHS:

$$\angle OPB = \angle OED = 90° (Definition\ of\ Distance)$$
$$OB = OD = r$$
$$OP = OE \ (Chords\ are\ equidistant)$$

Hence:

$$DE = BP \ (CPCT)$$

Similarly:

$$AE = CP$$

$$DE + AE = BP + CP$$
$$AD = BC$$

1.13: Converse of Above

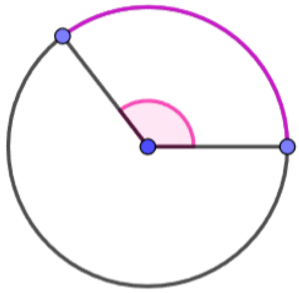
If chords are congruent, they are equidistant from the center.

Example 1.14
Finding Distance from Centre

C. Central Angle, Arcs and Chords

1.15: Arc and Central Angle

- A part of the circumference of the circle is called an arc of the circle.
- The angle made by the radii joining the two endpoints of the arc is called the central angle of the arc.



1.16: Congruent Arcs

In the same or congruent circles, congruent arcs have congruent chords.

1.17: Congruent Chords & Arcs

In the same or congruent circles, congruent chords have congruent arcs

1.18: Congruent Arcs have the same central angle

1.19: Bisector of Arcs

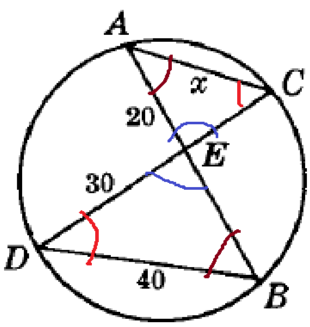
A bisector of a chord also bisects the arcs

D. Power of a Point

1.20: Power of a Point

E. Parallel Chords

1.21: Parallel chords create congruent triangles



(c)

$20:30=2:3$

$\frac{2}{3}=\frac{x}{40}$

$x=\frac{80}{3}$

1.3 Tangents

A. Definition

1.22: Radii and Tangents
A tangent to a circle is perpendicular to the radius at the point of tangency

Proof by contradiction

1.23: Converse of Above
If a line is perpendicular to a radius at the outer end, then it is tangent to the circle.

This property is the converse of the previous property.
Proof by contradiction

1.24: Corollary of Above
If a line is perpendicular to a tangent at its point of contact with a circle, then it passes through the center of the circle.

B. External Points

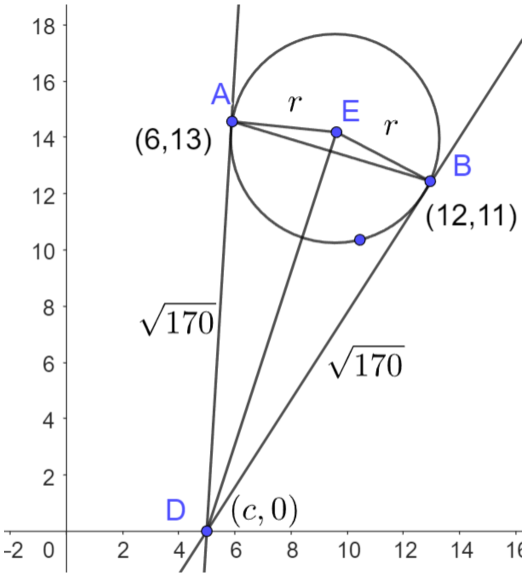
1.25: External Points
- The two tangents drawn from an external point to a circle are congruent.
- The line segment connecting a point outside a circle with the center of a circle bisects the angle formed by the point, and the two tangents to the circle.

We will prove both above properties together.

Example 1.26
Points $A = (6,13)$ and $B = (12,11)$ lie on circle ω in the plane. Suppose that the tangent lines to ω at A and B intersect at a point on the $x - axis$. What is the area of ω? (AMC 10B 2019/23)

Let the point of intersection of the tangents with the $x\ axis$ be

$$(c, 0)$$



Since the length of the tangents to a circle from point is equal:

$$D \text{ from } (c,0) \text{ to } A = D \text{ from } (c,0) \text{ to } B$$

$$\sqrt{(13-0)^2+(6-c)^2} = \sqrt{(11-0)^2+(12-c)^2}$$

Square both sides, and expand

$$169+36-12c+c^2 = 121+144-24c+c^2$$
$$24c-12c = 121+144-169-36$$
$$12c = 60$$
$$c = 5$$

$D \text{ from } (c,0) \text{ to } A$ is

$$\sqrt{(13-0)^2+(6-c)^2} = \sqrt{169+1} = \sqrt{170}$$

Since a radius is perpendicular to its tangent:

$$\angle EAD = \angle EBD = 90°$$

Because:

$$\angle EAD + \angle EBD = 180°$$
$$\Rightarrow \text{Opp. angles are supplementary}$$
$$\Rightarrow \text{Quadrilateral } DAEB \text{ is cyclic}$$

By Pythagoras Theorem, in ΔAED:

$$DE = \sqrt{AE^2+AD^2} = \sqrt{r^2+\left(\sqrt{170}\right)^2} = \sqrt{r^2+170}$$

By the distance formula:

$$AB = \sqrt{(11-13)^2+(12-6)^2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$$

In quadrilateral $DAEB$, by Ptolemy's Theorem:

$$DE \times AB = AE \times DB + EB \times AD$$
$$\sqrt{r^2+170} \times 2\sqrt{10} = 2r\sqrt{170}$$
$$\sqrt{r^2+170} = r\sqrt{17}$$

Square both sides:

$$r^2+170 = 17r^2$$
$$16r^2 = 170$$
$$r^2 = \frac{170}{16} = \frac{85}{8}$$

The area of the circle is then:

$$\pi r^2 = \frac{85}{8}\pi$$

Example 1.27

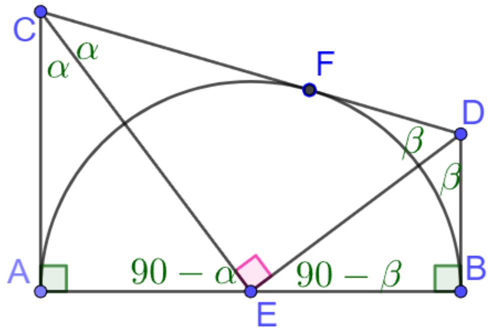
A circle with diameter AB and center E has tangents CA and DB. CD is tangent to the circle at point F. Show that ΔCED is a right triangle.

We draw half the circle. The radius is perpendicular to the tangent:

$$CA \perp AB, DB \perp BE \Rightarrow ABCD \text{ is a trapezoid}$$

The segment from the center (E) to an outside point bisects the angle between the tangents from the point to the circle:

$$\angle ECA = \angle ECF = \alpha, \qquad \angle EDB = \angle EDF = \beta$$



By sum of angles in:

$$\Delta CAE: \angle CEA = 180 - 90 - \alpha = 90 - \alpha$$
$$\Delta CED: \angle CED = 180 - \alpha - \beta$$
$$\Delta DEB: \angle DEB = 180 - 90 - \beta = 90 - \beta$$

By sum of angles in a line:

$$\underbrace{(90-\alpha)}_{\angle CEA} + \underbrace{(180-\alpha-\beta)}_{\angle CED} + \underbrace{(90-\beta)}_{\angle DEB} = 180° \Rightarrow \alpha + \beta = 90$$

Then, the measure of:

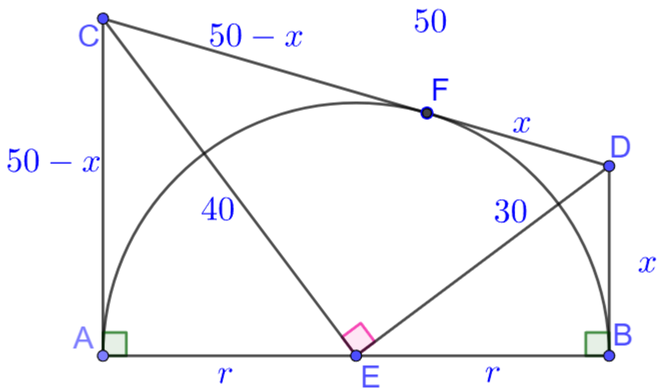
$$\angle CED = 180 - \alpha - \beta = 90 \Rightarrow \Delta CED \text{ is a right } \Delta$$

Example 1.28: Pythagoras Theorem

A circle with diameter AB and center E has tangents CA and DB. CD is tangent to the circle at point F. If $CE = 40$ and $DE = 30$, then determine the perimeter of quadrilateral $ABCD$ by using the property that tangents to a circle from an outside point are congruent.

In right ΔCED, by Pythagorean Triplet (30,40,50):

$$CD = 50$$



Tangents to a circle from an outside point are congruent:

$$DF = DB = x, \qquad CA = CF = CD - DF = 50 - x$$

Using Pythagoras Theorem:

$$\Delta DBE: EA^2 = 900 - x^2$$
$$\Delta CAE: EB^2 = 1600 - (50-x)^2 = 100x - x^2 - 900$$

But since both are radii, we must have $EA^2 = EB^2$:

$$900 - x^2 = 100x - x^2 - 900 \Rightarrow 900 = 10x - 900 \Rightarrow x = 18$$
$$r = \sqrt{900 - x^2} = \sqrt{576} = 24$$

The perimeter is then:

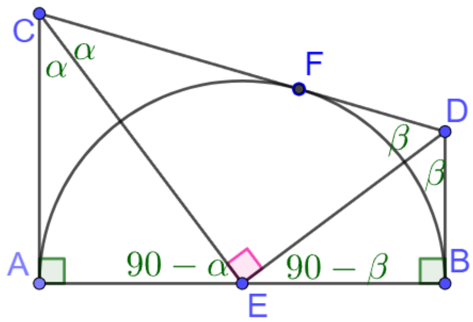
$$CD + CA + DB + AB = 2CD + 2r = 100 + 48 = 148$$

Example 1.29: Similarity

A circle with diameter AB and center E has tangents CA and DB. CD is tangent to the circle at point F. $CE = a$ and $DE = b$. Show that:

A. $\Delta CED \sim \Delta CAE \sim \Delta EBD$

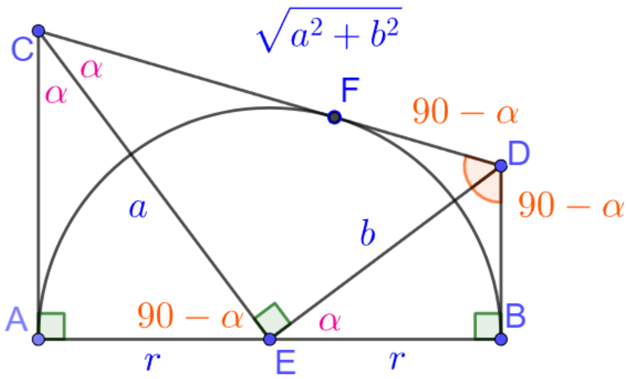
B. the perimeter of quadrilateral $ABCD$ is $2 \cdot \frac{a^2+ab+b^2}{\sqrt{a^2+b^2}}$



Part A
Use the diagram from the previous examples:
$$\alpha+\beta=90 \Rightarrow \alpha=90-\beta, \beta=90-\alpha$$

Then:
$$\angle CAE=\angle CED=\angle DBE=90°$$
$$\angle ACE=\angle DEB=\angle ECD=\alpha$$
By AA Similarity:
$$\Delta CED \sim \Delta CAE \sim \Delta EBD$$

Part B
By similarity in ΔCAE and ΔCED:
$$\frac{r}{a}=\frac{b}{\sqrt{a^2+b^2}} \Rightarrow r=\frac{ab}{\sqrt{a^2+b^2}}$$



$$CD+CA+DB+AB=2CD+2r=2(CD+r)$$

Substitute $CD=\sqrt{a^2+b^2}, r=\frac{ab}{\sqrt{a^2+b^2}}$ in the above:
$$2\left(\sqrt{a^2+b^2}+\frac{ab}{\sqrt{a^2+b^2}}\right)=2\left(\frac{a^2+ab+b^2}{\sqrt{a^2+b^2}}\right)$$

C. Relative Positions of Circles

Overlapping Circles

Circles Away from each other

Example 1.30
Distance between centers of circles in different positions

1.4 Power of a Point

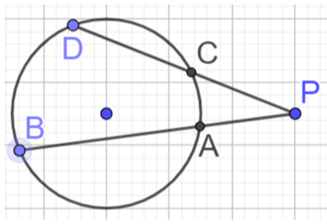
1.31: Power of a Point

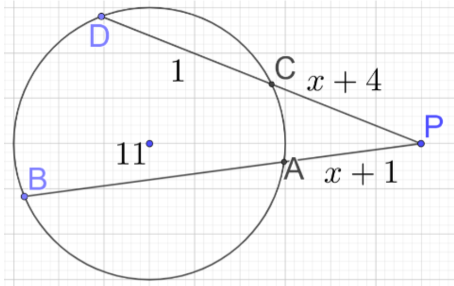
A. Tangent-Secant Theorem

1.32: Tangent-Secant Theorem

B. Two Secants

1.33: Two Secant Theorem
In the diagram alongside, P is a point outside the circle, and CD and AB are secants of the circle.
Then:
$$PA\cdot PB=PC\cdot PD$$





Example 1.34
The adjoining diagram has point P outside circle ω. PAB and PCD are straight lines, and AB and CD are secants. Find the value of x.

Substitute the known values into $PA \cdot PB = PC \cdot PD$:

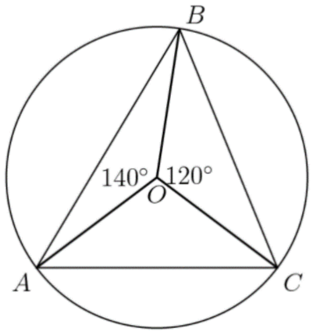
$$(x+1)(x+12) = (x+4)(x+5)$$
$$x^2 + 13x + 12 = x^2 + 9x + 20$$
$$4x = 8$$
$$x = 2$$

1.5 Arcs and Angles

A. Angle Subtended by Arc

A full circle occupies 360.
A semi-circle occupies 180.
A quarter-circle takes up 90.

We generalize this by saying that an arc is measured by its central angle.



B. Inscribed Angle

1.35: Inscribed Angle
An inscribed angle in a circle is one-half of the corresponding central angle.

Example 1.36
The point O is the center of the circle circumscribed about $\triangle ABC$, with $\angle BOC = 120°$ and $\angle AOB = 140°$. What is the degree measure of $\angle ABC$? **(AMC 10B 2007/4)**

$$\angle AOC = 360 - \angle AOB - \angle BOC = 360 - 140 - 120 = 100$$

$$\angle ABC = \angle \frac{AOC}{2} = \frac{100}{2} = 50$$

C. Congruence of Arcs and Angles

1.37: Angles in the same Arc
Two angles inscribed in the same arc are congruent.

1.38: Angles in congruent Arcs
Two angles inscribed in congruent arcs are congruent.

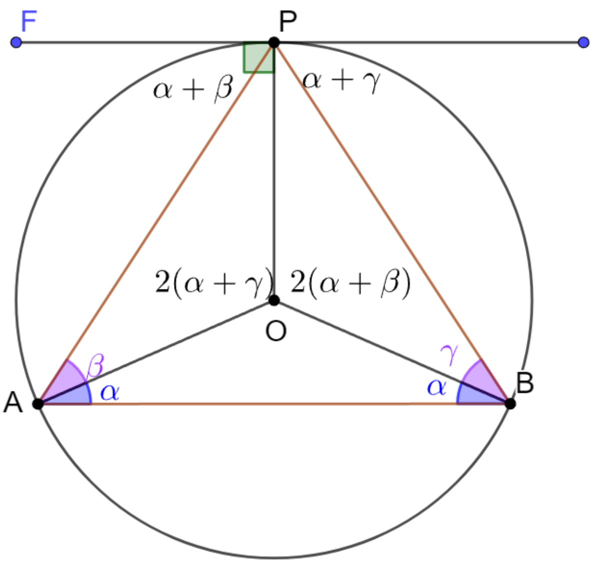
1.39: Congruent Inscribed Angles
If two inscribed angles have the same measure, the arcs they intercept are congruent.

1.40: Inscribed Angles versus Central Angle

An inscribed angle has a measure which is half the measure of the angle subtended by its arc at the center of the circle.

Example 1.41

If the tangent to a circle at P is parallel to chord AB then show that ΔPAB is isosceles.



Let the center of the circle be O. Draw radii

$$OA = OB = OC = r$$

Let $\angle OAB = \alpha, \angle PAO = \beta, \angle PBO = \gamma$. Since:

$$OA = OB \Rightarrow \Delta OAB \text{ is isosceles} \Rightarrow \angle OBA = \alpha$$

Since $FP \parallel AB$ alternate interior angles are congruent:

$$\text{Transversal } PA: \angle PAB = \angle FPA = \alpha + \beta$$
$$\text{Transversal } PB: \angle PBA = \angle GPA = \alpha + \gamma$$

Since radius $\perp$ tangent:

$$FP \perp PO \Rightarrow \angle FPO = \angle GPO = 90°$$

Calculate:

$$\angle APO = \angle FPO - \angle FPA = 90 - \alpha - \beta$$
$$\angle BPO = \angle GPO - \angle FPA = 90 - \alpha - \gamma$$

Since angle subtended by an arc is double of an inscribed angle

$$\angle POA = 2\angle PBA = 2(\alpha + \gamma)$$
$$\angle POB = 2\angle PAB = 2(\alpha + \beta)$$

By sum of angles in

$$\Delta POA: \underbrace{(90 - \alpha - \beta)}_{\angle APO} + \underbrace{\beta}_{\angle PAO} + \underbrace{2(\alpha + \gamma)}_{\angle POA} = 180° \Rightarrow \alpha + 2\gamma = 90°$$
$$\Delta POA: \underbrace{(90 - \alpha - \gamma)}_{\angle BPO} + \underbrace{\gamma}_{\angle PBO} + \underbrace{2(\alpha + \beta)}_{\angle POA} = 180° \Rightarrow \alpha + 2\beta = 90°$$

From the above two statements:

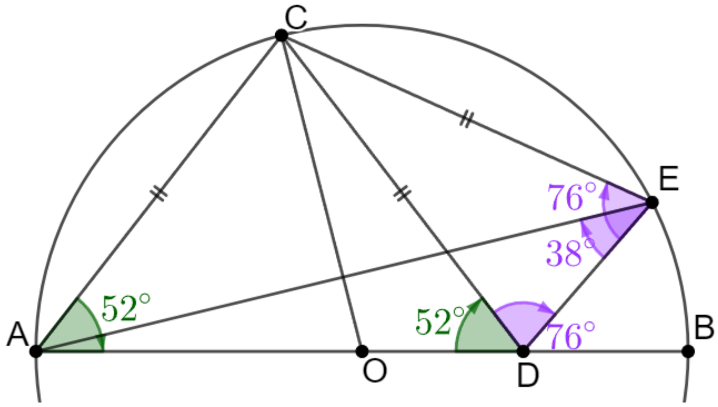
$$\alpha + 2\gamma = \alpha + 2\beta \Rightarrow \alpha + \gamma = \alpha + \beta \Rightarrow \angle PAB = \angle PBA \Rightarrow \Delta PAB \text{ is isosceles}$$

Example 1.42

In a circle with center O, $\angle OAC$ is 52°. D lies on the diameter AB such that $CA = CD$, and E lies on the circumference such that $CA = CE$. Determine the value of $\angle AED$.

In isosceles ΔACD:

$$\angle ACD = 180 - 2(52) = 180 - 104 = 76°$$



Arc AC and arc CEB form a semicircle. Hence, the angles subtended on the circumference are complementary.

$$\angle CEA = 90 - \angle COA = 90 - 52 = 38$$

In isosceles ΔACE

$$\angle ACE = 180 - 2(38) = 104°$$

$$\angle DCE = \angle ACE - \angle ACD = 104 - 76 = 28$$

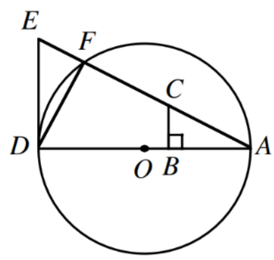
In isosceles ΔCED

$$\angle CED = \frac{180 - 28}{2} = 76$$
$$\angle AED = \angle CED - \angle CEA = 76 - 38 = 38$$

D. Angle Inscribed in a Semi-Circle

1.43: Thales Theorem

An angle inscribed in a semi-circle is a right angle.



Example 1.44

In the diagram shown, $\angle ABC = 90°$, $CB \parallel ED$, $AB = DF$, $AD = 24$, $AE = 25$ and O is the centre of the circle. Determine the perimeter of CBDF. **(CEMC Cayley 2000/24)**

In ΔABC and ΔDFE:

$$AB = DF\ (Given)$$
$$\angle DFA = 90\ \ (Angle\ in\ a\ semicircle) \Rightarrow \angle DFE = 90°$$
$$\angle DEF = \angle BCA\ (Corresponding\ angles\ for\ in\ parallel\ lines\ CB\ and\ ED)$$
$$\Delta ABC \cong \Delta DFE\ (SAA\ Congruence)$$
$$EF = CB, DF = BA\ (CPCT)$$

DE is tangent to the circle. Hence, $\angle EDA$ is a right angle. By Pythagoras Theorem in ΔEDA, then by a Pythagorean Triplet, the sides are:

$$(7,24,25) \Rightarrow DE = 7$$
$$AC = DE = 7\ (CPCT\ in\ \Delta ABC\ \text{and}\ \Delta DFE)$$
$$CE = AE - AC = 25 - 7 = 18$$

The perimeter of $CBDF$

$$= BD + DF + FC + CB$$

Substitute $DF = BA, CB = EF$:

$$= BD + BA + FC + EF$$

Substitute $BD + BA = AD = 24, FC + EF = CE = 18$

$$= 24 + 18 = 42$$

E. Parallel lines

1.45: Parallel lines and Arcs

Parallel lines intercept congruent arcs.

F. Angle between tangents and chords

1.46: Angle between tangents and chords
The angle between a tangent and a chord is half of the angle subtended by the intercepted arc.

G. Intersecting Chords

1.47: Angle between Intersecting Chords
The angle between two intersecting chords is half the sum of the angles subtended by the intercepted arcs.

H. Intersecting Secants

1.48: Angle between Intersecting Secants
The angle formed by two secants intersecting outside a circle is half the difference of the angles subtended by the intercepted arcs.

I. Angle between Secant and Tangent

1.49: Angle between Secant and Tangent
The angle formed by a secant and a tangent intersecting outside a circle is half the difference of the angles subtended by the intercepted arcs.

J. Angle between two Tangents

1.50: Angle between Two Tangents
The angle formed by two tangents from a point is half the difference of the angles subtended by the intercepted arcs.

1.6 Cyclic Quadrilaterals

A. Definition

1.51: Concyclic Points
If you can draw a circle through a set of points, the points are concyclic.

1.52: Cyclic Quadrilateral
If the vertices of a quadrilateral are concyclic, then the quadrilateral is cyclic.

B. Opposite Angles

1.53: Opposite Angles
Opposite angles of a cyclic quadrilateral are supplementary

C. Ptolemy's Theorem

1.54: Ptolemy's Theorem

If a cyclic quadrilateral has sides a, b, c and d, and diagonals e and f, then

$$ac + bd = ef$$

Example 1.55

In $\triangle ABC$ we have $AB = 7$, $AC = 8$, and $BC = 9$. Point D is on the circumscribed circle of the triangle so that AD bisects $\angle BAC$. What is the value of $\frac{AD}{CD}$? **(AMC 10B 2004/24)**

Using Ptolemy's Theorem

(Alternate Solution) Example 1.56

In $\triangle ABC$ we have $AB = 7$, $AC = 8$, and $BC = 9$. Point D is on the circumscribed circle of the triangle so that AD bisects $\angle BAC$. What is the value of $\frac{AD}{CD}$? **(AMC 10B 2004/24)**

Using Similarity and Angle Bisector Theorem

D. Area of a Cyclic Quadrilateral

The formula for the area of a cyclic quadrilateral is the generalization of Heron's Formula for the area of a triangle.

1.57: Brahmagupta's Formula

For a cyclic quadrilateral with sides a, b, c and d, the area is:

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Example 1.58

Show that substituting $d = 0$ in Brahmagupta's Formula gives Heron's Formula for the area of a triangle.

2. Circles

2.1 Circles

A. π: An Introduction

For this chapter, give your answer in terms of π, unless mentioned otherwise.

2.1: Definition of π
The ratio of the circumference of a circle to the radius of the circle is called π.

π is an irrational number. Its decimal representation is neither non-recurring, and non-terminating.
- Non-recurring means it does not repeat
- Non-terminating means it does not end.

2.2: Exact Value of π
The exact value of π cannot be determined. Hence, we use approximations to the exact value. Two common approximations are:

$$\pi \approx 3.14$$
$$\pi \approx \frac{22}{7}$$

In particular, π cannot be written as a fraction or a decimal.
The above values are approximations that are used for geometry questions asking for calculation of area and circumference.

Example 2.3
One value of π that is commonly used is 3.14. Another value that is also commonly used is $\frac{22}{7}$.
Are these two values equal?

$$\frac{22}{7} = 3.1428\ldots \neq 3.14$$

Hence, these two values are not equal.

B. Circumference and Area

2.4: Circumference and Area of a Circle
Given a circle with radius r and Diameter $D = 2r$, we have:

$$Circumference = C = 2\pi r = \pi D$$
$$Area = A = \pi r^2$$

$$C = 2\pi r \Rightarrow \frac{C}{2\pi} = r$$

Example 2.5
What are the area and the perimeter of a circle with diameter 6 units?

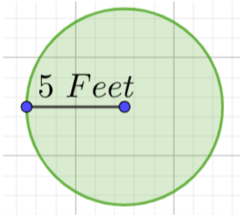
$$A = \pi r^2 = \pi(6^2) = 36\pi$$

$$C = 2\pi r = 12\pi$$

Example 2.6

Harish has a circular garden, in which the distance from the center to the boundary is 5 feet.

A. What are the area and the perimeter of the garden?
B. What is the cost of fencing the garden at \$0.25 per foot of fence?



$$Circumference = C = 2\pi r = 2 \times \pi \times 5 = 10\pi$$
$$Area = A = \pi r^2 = \pi \times 5^2 = 25\pi$$
$$Cost\ of\ fence = 0.25 \times 10\pi = \frac{1}{4} \times 10\pi = \frac{10}{4}\pi \ = 2.5\pi\ dollars$$

Example 2.7

A water wheel has radius seven feet. What are the area and the circumference of the wheel?

$$A = \pi r^2 = \pi(7^2) = 49\pi\ feet^2$$
$$C = 2\pi r = 14\pi\ feet$$

Example 2.8

A circular look-out tower has a radius of twenty-five feet.

A. A guard walks around the tower every three minutes. How many feet must he cover every hour?
B. The Captain of the Guard, and he wants the ground floor covered in carpet. What is the cost of the carpet, in dollars, if one square foot of carpet costs 4 *cents*?
C. The second floor of the tower has a radius of twenty feet. Find the difference in area between the ground floor of the tower, and the second floor of the tower.

Part A

Circumference of Tower

$$= C = 2\pi r = 2\pi \times 25 = 50\pi\ feet$$

The guard will make

$$\frac{60}{3} = 20\ Rounds\ in\ an\ Hour$$

Feet covered in one hour

$$= 20 \times 50\pi = 1000\pi\ feet$$

Part B

Area of the ground floor of tower

$$= A = \pi r^2 = \pi \times 25^2$$

Cost of the carpet

$$= 0.04 \times 25^2 \times \pi = \frac{1}{25} \times 25^2 \times \pi = 25\pi\ dollars$$

Part C

$$A_1 - A_2 = \pi r_1^2 - \pi r_2^2 = 625\pi - 400\pi = 225\pi$$

Example 2.9

Suhani wants to draws a circle with radius 3 cm, but instead draws a circle with radius 4 cm.

A. What is the positive difference in the areas that the two circles occupy?
B. What is the positive difference in the circumference of the two circles?

$$A_1 - A_2 = \pi r_1^2 - \pi r_2^2 = 16\pi - 9\pi = 7\pi$$
$$C_1 - C_2 = 2\pi r_1 - 2\pi r_2 = 2\pi(4) - 2\pi(3) = 8\pi - 6\pi = 2\pi$$

C. Calculating when given area

Till now we have been using the formula to calculate area and circumference of circles. Now, we do questions that are not direct applications of the formula.

Example 2.10

A circle has area 100π. Find its radius and its circumference.

$$A = \pi r^2 \Rightarrow 100\pi = \pi r^2 \Rightarrow 100 = r^2 \Rightarrow r = 10$$
$$C = 2\pi r = 2\pi \times 10 = 20\pi = 20 \times \frac{22}{7} = \frac{440}{7}$$

Example 2.11

A circle has area 81 $units^2$. Find its radius and its circumference.

$$A = \pi r^2 \Rightarrow 81 = \pi r^2 \Rightarrow \frac{81}{\pi} = r^2 \Rightarrow r = \frac{9}{\sqrt{\pi}}$$
$$C = 2\pi r = 2\pi \times \frac{9}{\sqrt{\pi}} = 18\sqrt{\pi}$$

D. Calculating when given Circumference

Example 2.12

A circle has circumference 10π. Find its radius and its area. (Find as a number, not π).

$$C = 10\pi \Rightarrow 2\pi r = 10\pi \Rightarrow r = 5$$
$$A = \pi r^2 = 25\pi = 25 \times \frac{22}{7} = \frac{550}{7}$$

Example 2.13

A circle has circumference 7. Find its radius and its area.

$$C = 7 \Rightarrow 2\pi r = 7 \Rightarrow r = \frac{7}{2\pi}$$
$$A = \pi r^2 = \pi\left(\frac{7}{2\pi}\right)^2 = \frac{49}{4}$$

E. Equating Perimeter and Area

Example 2.14

If the numeric value of the circumference and the area of a circle measured in meters is the same, what is the radius (in centimetres)?

$$A = C \Rightarrow \pi r^2 = 2\pi r \Rightarrow r = 2$$

Example 2.15

A semicircle has radius π meters. What is the numeric value of the sum of its perimeter and area?

$$\text{S1: } \underbrace{(2r+\pi r)}_{Perimeter} + \underbrace{\frac{\pi r^2}{2}}_{Area} = \left(2\pi + \pi(\pi)\right) + \frac{\pi(\pi^2)}{2} = 2\pi + \pi^2 + \frac{\pi^3}{2} = \pi\left(2 + \pi + \frac{\pi^2}{2}\right)$$

Example 2.16

Circle X has a radius of π. Circle Y has a circumference of 8 π. Circle Z has an area of 9 π. List the circles in order from smallest to largest radius. **(AMC 8 2006/7)**

Example 2.17

One circle's area as times of another circle's area

F. Percent Increases and Decreases

Example 2.18

If the radius of a circle is increased by 20%, by what percent does the

A. Circumference increase?
B. Area increase?

20%
44%

Example 2.19

If the circumference of a circle increases by 10%, what is the increase in its area?

$C = 2\pi r$
C ↑ 10%, but 2π remains constant
Hence, r ↑ 10%
Increase in area
$\propto r^2 = (1.1)^2 - 1 = 1.21 - 1 = 0.21 = 21\%$

Example 2.20

The radius of a circle in increased so that its circumference increases by 5%. The area of the circle will increase (in %) by: (NMTC Primary/Screening/2016/17)

$$\frac{C_{New}}{C_{old}} = 1.05$$
$$\frac{2\pi r_{new}}{2\pi r_{old}} = 1.05$$
$$r_{new} = 1.05 r_{old}$$

Hence, the radius has increased by 5%.

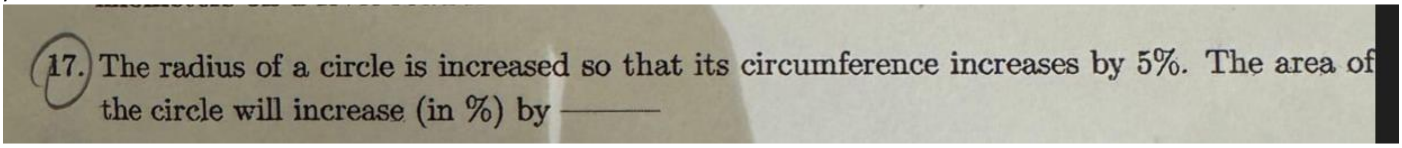
The ratio of areas of the old circle to the new circle will be:

$$\frac{A_{New}}{A_{Old}} = \frac{\pi r_{New}^2}{\pi r_{Old}^2} = \frac{\pi(1.05 r_{old})^2}{\pi r_{Old}^2} = 1.05^2 = 1.1025 = 110.25\%$$

The percent increase is then:

$$110.25\% - 100\% = 10.25\%$$

/



17. The radius of a circle is increased so that its circumference increases by 5%. The area of the circle will increase (in %) by ______

Example 2.21

If the radius of a circle is increased 100%, the area is increased by what percent: (AHSME 1950/8)

$$A = \pi r^2$$
$$New\ Area = \pi(2r)^2 = 4\pi r^2$$
$$Increase = 300\%$$

G. Semi-Circle

2.22: Area of a Semi-Circle

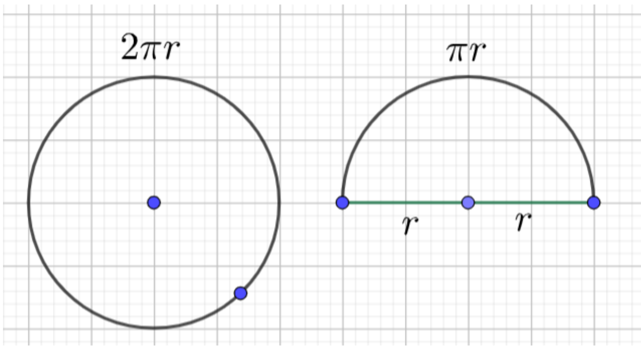
$$\underbrace{A}_{Semi-Circle} = \frac{\pi r^2}{2}$$

A semi-circle is half of a full circle. So, the area of a semi-circle is half of the area of a full-circle.

2.23: Perimeter of a Semi-Circle

The perimeter of a semi-circle is:

$$P = \underbrace{\pi r}_{\substack{Arc\ of \\ the\ Circle}} + \underbrace{2r}_{\substack{Side\ of \\ the\ Circle}}$$



Circumference is applicable only to circles, and refers to the perimeter of the circle.
When we have a half-circle, instead of a full circle, we use perimeter.
When we have a half-circle, instead of a full circle, the circumference of the circle is halved.

$$\underbrace{C}_{Semi-Circle} = \frac{2\pi r}{2} = \pi r$$

But the side of the circle becomes exposed because it is a half-circle. Hence, twice the radius gets added to the perimeter.

$$Side\ of\ the\ Circle = 2r$$

Example 2.24

A semi-circle has radius 6 units. Find the perimeter and the area of the semi-circle.

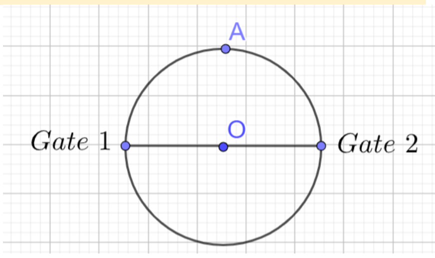
$$P = \pi r + 2r = 6\pi + 12$$
$$A = \frac{\pi r^2}{2} = \frac{\pi}{2} \times 6^2 = 18\pi$$

Example 2.25

A circular park with area 49π has two diametrically opposite gates, and a path connecting them. Asha enters from one gate, walk along the path to the other gate, and walks back along the boundary of the park.

A. What is the distance that she has travelled?
B. What is the area of the region enclosed by the path that she travels?

Note: Diametrically opposite means that they lie on opposite ends of a diameter.



Part A

First, find the radius of the circular park:

$$A = 49\pi \Rightarrow \pi r^2 = 49\pi \Rightarrow r = 7$$

Then, the path taken is:

- Enter at Gate 1
- Go to Gate 2 via the centre of the circle (O)
- Go back to Gate 1 via the boundary of the circle $(via\ A)$

Hence, distance travelled

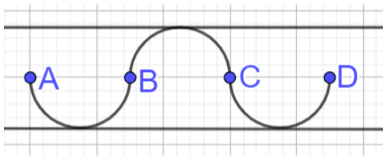
$$= Perimeter\ of\ Semi - circle = \ \pi r + 2r = 7\pi + 14$$

Part B

$$A = \frac{49\pi}{2}$$

Example 2.26

Pedro is in the middle of a wide, empty road at A. He cycles down to D, in the semi-circular pattern shown at a speed of $2\frac{m}{s}$. If the shortest distance from A to D is $100m$, then find the time taken by him.



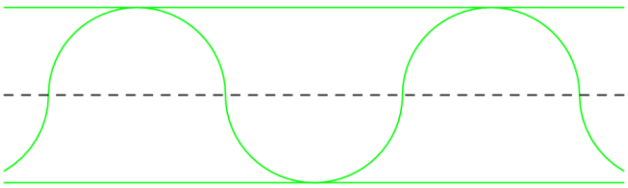
$$AD = 100\ m \Rightarrow Radius = \frac{AD}{6} = \frac{100}{6} = \frac{50}{3}m$$

$$D = \frac{3}{2}C = \frac{3}{2}(2\pi r) = 3\pi\left(\frac{50}{3}\right) = 50\pi\ m$$

$$T = \frac{D}{S} = \frac{50\pi}{2} = 25\pi\ seconds$$

Example 2.27

A straight one-mile stretch of highway, 40 feet wide, is closed. Robert rides his bike on a path composed of semicircles as shown. If he rides at 5 miles per hour, how many hours will it take to cover the one-mile stretch?
Note: 1 mile= 5280 feet (AMC 8 2014/25)



Example 2.28

Diameter ACE is divided at C in the ratio 2:3. The two semicircles, ABC and CDE, divide the circular region into an upper (shaded) region and a lower region. The ratio of the area of the upper region to that of the lower region is: (AMC 8 1997/24)

Assume that the Diameter of the large circle is 10.

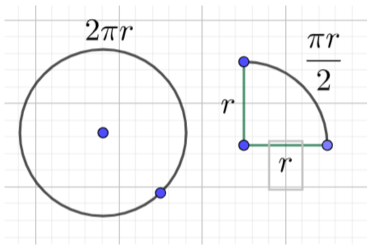
$$A(Shaded\ Region): A(White\ Region)$$
$$\frac{A(Large\ Circle) - A(ABC) + A(CDE)}{2} : \frac{A(Large\ Circle) + A(ABC) - A(CDE)}{2}$$
$$\frac{25\pi - 4\pi + 9\pi}{2} : \frac{25\pi + 4\pi - 9\pi}{2} = 30\pi : 20\pi = 3:2$$

H. Quarter-Circle

2.29: Area of a Quarter-Circle

$$\underbrace{A}_{Quarter-Circle} = \frac{\pi r^2}{4}$$

A semi-circle is one-fourth of a full circle. So, the area of a semi-circle is half of the area of a full-circle.



2.30: Perimeter of a Quarter-Circle

The perimeter of a semi-circle is

$$P = \underbrace{\frac{\pi r}{2}}_{\substack{Arc\ of \\ the\ Circle}} + \underbrace{2r}_{\substack{Side\ of \\ the\ Circle}}$$

When we have a quarter-circle, instead of a full circle, the circumference of the circle is divided by four:

$$\underbrace{C}_{Semi-Circle} = \frac{2\pi r}{4} = \frac{\pi r}{2}$$

But the side of the circle becomes exposed because it is a half-circle. Hence, twice the radius gets added to the perimeter.

$$Side\ of\ the\ Circle = 2r$$

Example 2.31

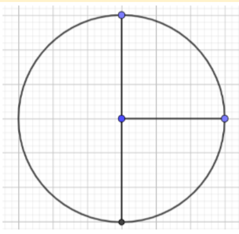
A quarter-circle has radius 6 units. Find the perimeter and the area of the semi-circle. (Give your answer in terms of π).

$$A = \frac{\pi r^2}{4} = \frac{\pi \times 6^2}{4} = 9\pi$$
$$P = \frac{\pi r}{2} + 2r = \frac{\pi \times 6}{2} + 2(6) = 3\pi + 12$$

Example 2.32

A circle with radius 7 cm is cut into two semi-circles, and then each semi-circle into two quarter-circles (giving a total of four quarter circles). Find the ratio:

$$Area(Circle) : Area(Semi - circle) : Area(Quarter - Circle)$$



Using Algebra

$$\pi r^2 : \frac{\pi r^2}{2} : \frac{\pi r^2}{4} = 1 : \frac{1}{2} : \frac{1}{4} = 4:2:1$$

Using Symmetry

By symmetry:

$$Area\ of\ Circle = 4\ Times\ Area\ of\ Quarter\ Circle$$

$$Area\ of\ Circle = 2\ Times\ Area\ of\ Semi - Circle$$

Hence, the required ratio is:

$$4:2:1$$

Repeat the previous question with Perimeter. (Answer in terms of π)

$$2\pi r : \pi r + 2r : \frac{\pi r}{2} + 2r$$

Multiply each of the above by $\frac{2}{r}$:

$$4\pi : 2\pi + 4 : \pi + 4$$

Example 2.33

Reverse Calculations
Semi-Circle
Quarter Circle

Example 2.34: Straight Line versus Circular Path

A circular garden with radius r has two diametrically opposite gates. Path A connects one gate to the other via the shortest way between the two gates, and Path B travels via the edge of the garden. Atul enters via one gate and travels via Path A to the other gate. Ajay enters the same gate, and travels via Path B to the other gate.

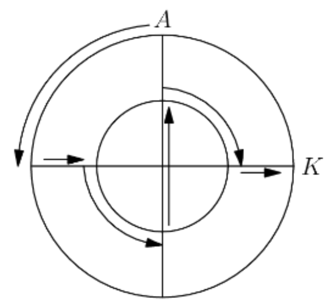
A. Find the difference in the distance travelled by the two in terms of r.
B. If the area of the garden is 16π, then find the value of the expression that you found in Part A.

Part A

$$\pi r - 2r$$

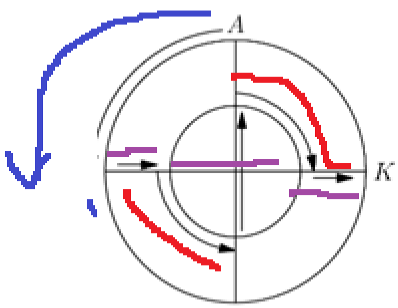
Part B

$$A = 16\pi \Rightarrow \pi r^2 = 16\pi \Rightarrow r = 4 \Rightarrow \pi r - 2r = 4\pi - 2(4) = 4\pi - 8$$



Example 2.35

Two circles that share the same center have radii 10 meters and 20 meters. An aardvark runs along the path shown, starting at A and ending at K. How many meters does the aardvark run? (Answer in terms of π) (AMC 8 2008/18)



Blue Path

$$= Quarter\ of\ Circle = \frac{2\pi r_1}{4} = \frac{2\pi \times 20}{4} = 10\pi$$

Red Path

$$= 2 \times Quarter\ of\ Smaller\ Circle = 2 \times \frac{2\pi r_2}{4} = \pi r_2 = \pi \times 10 = 10\pi$$

Violet Path

$$= Diameter\ of\ Larger\ Circle = 2r = 2 \times 20 = 40$$

Total Length

$$= \underbrace{10\pi}_{Blue\ Path} + \underbrace{10\pi}_{Red\ Path} + \underbrace{40}_{Violet\ Path} = 20\pi + 40$$

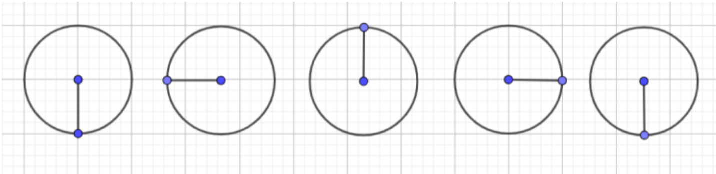
2.2 Rotations

A. Rotations

2.36: Distance covered in a Rotation of a Wheel

The distance covered by the center of the wheel on a single rotation is equal to the circumference.

$$D = C$$



- The diagram shows a circle turns a complete rotation$(360°)$. If you consider a point at the bottom of the circle, it will travel the length of the circumference as the circle makes a complete rotation.
- The circle travels in a straight line. If the path of the circle is not along a straight, this property will not hold.

Example 2.37

Find the exact distance, in inches travelled in one rotation of a wheel for a:

A. Truck with wheels with radius 1.5 feet
B. Car with wheels with a radius of 0.5 feet

The distance travelled in 1 rotation

$$= C = 2\pi r$$

Part A

$$2\pi r = 2\pi(1.5) = 3\pi = 36\pi \ inches$$

Part B

$$2\pi r = 2\pi(0.5) = \pi \ feet = 12\pi \ inches$$

2.38: Distance Travelled given Number of Rotations

Distance covered by a wheel is given by:

$$\underbrace{D}_{Distance} = \underbrace{R}_{\substack{No.of \\ Rotations}} \times \underbrace{C}_{Circumference}$$

Example 2.39

A. A car has wheels, each with radius 7 *inches*. The car breaks down, and is pushed $\frac{1}{22}$ of a rotation of a wheel. What is the approximate distance that the car travels?
B. A girl rolls down an incline on roller-skates. Her wheels, of radius 1 inch, make 50 complete rotations. Find the exact distance that she has travelled?
C. A cylinder, of height 3 feet and radius 1 feet, is rolled down an alleyway. The cylinder makes 20 complete rotations. What is the exact distance that it rolls?

$$D = RC = \frac{1}{22} \times 2\pi r = \frac{1}{22} \times 2 \times \frac{22}{7} \times 7 = 2 \ inches$$

$$D = RC = 50 \times 2\pi r = 100\pi$$

$$D = RC = 20 \times 2\pi r = 20 \times 2\pi(1) = 40\pi$$

Example 2.40

A car travels from one side of the street to another and the wheels rotate 100 times. If the distance travelled by the car is 25 units, find the exact radius of the wheels.

$$D = RC$$

Substitute $D = 25, R = 100, C = 2\pi r$

$$25 = 100 \times 2\pi r$$
$$25 = 200\pi r$$
$$r = \frac{25}{200\pi} = \frac{1}{8\pi}$$

Example 2.41: Calculating Area

A pipe rolls 100 units, and makes 250 rotations. Find the exact area of the cross-section of the pipe.

$$D = RC$$
$$100 = 250 \times 2\pi r$$
$$100 = 500\pi r$$
$$r = \frac{100}{500\pi} = \frac{1}{5\pi} \ units$$
$$A = \pi r^2 = \pi\left(\frac{1}{5\pi}\right)^2 = \pi \times \frac{1}{25\pi^2} = \frac{1}{25\pi} units^2$$

Example 2.42: Multi-Step

The area of a wheel is $64\pi \ unit^2$. The vehicle on which the wheel is mounted travels 5280 units, find the approximate number of complete rotations the wheel makes?

First, find the radius:

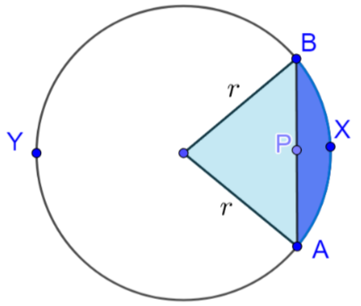
$$A = \pi r^2 = 64\pi \Rightarrow r = 8$$

The number of rotations is:

$$= \frac{D}{C} = \frac{D}{2\pi r} = \frac{5280}{2 \times \frac{22}{7} \times 8} = 5280 \times \frac{1}{2} \times \frac{7}{22} \times \frac{1}{8} = 105$$

2.3 Arc Length

A. Arc and Arc Length



2.43: Arc of a Circle

The length of the curve making an arc in a circle is the arc length. In the diagram alongside:

- BXA is the *minor* arc
- BYA is the *major* arc

Example 2.44

In a circle with radius π, what is the m is a minor arc, and M is its corresponding major arc. Find the sum of the lengths of m and M in terms of π.

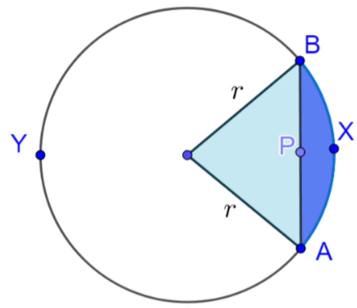
Note that

$$m + M = C = 2\pi r = 2\pi\pi = 2\pi^2$$

2.45: Arc Length of a Circle

The length of arc of a circle with radius r is

$$2\pi r \cdot \frac{\theta}{360°}$$



➢ The arc length of a circle is proportional to the central angle of the circle.

Example 2.46

In a circle, an arc has arc length equal to the radius. Determine the exact value (in degrees) of the central angle.

$$2\pi r \cdot \frac{\theta}{360} = r$$
$$\pi \cdot \frac{\theta}{180} = 1$$
$$\theta = \frac{180}{\pi}[1]$$

Example 2.47

A minor arc of a circle with radius $\sqrt{30}$ subtends a central angle of 100°. Find the exact difference in arc length between the major and the minor arc.

The difference in arc lengths:

$$2\pi r \cdot \frac{\theta_1}{360} - 2\pi r \cdot \frac{\theta_2}{360} = \frac{\pi r}{180}(\theta_1 - \theta_2) = \frac{\pi\sqrt{30}}{180}(260 - 100) = \frac{8}{9}\sqrt{30}\pi$$

Example 2.48

A 45° arc of circle A is equal in length to a 30° arc of circle B. What is the ratio of circle A's area and circle B's area? **(AMC 10A 2002/7)**

As per the given information:

$$45° \text{ arc of A} = 30° \text{ arc of B}$$

Use the formula for arc length:

$$C_A \times \frac{45}{360} = C_B \times \frac{30}{360}$$

Simplify:

$$C_A \times 45 = C_B \times 30$$
$$C_A \times 3 = C_B \times 2$$

Substitute $C = 2\pi r$"

$$\pi r_A \times 3 = \pi r_B \times 2$$
$$r_A \times 3 = r_B \times 2$$

---

[1] This is the definition of a radian.

Determine the ratio of radii and square to get the ratio of areas:

$$\frac{r_A}{r_B}=\frac{2}{3}\Rightarrow\left(\frac{r_A}{r_B}\right)^2=\frac{\pi r_A^2}{\pi r_B^2}=\frac{4}{9}$$

B. Sequences

Example 2.49

A circle with area 64 circle is divided into 12 arcs. Determine the exact value of arc length of the smallest arc given that the ratio of the respective arc lengths is:

A. $1:2:3:\ldots:12$
B. $1:2:4:\ldots:1024$

The area of the circle is:

$$\pi r^2=64\Rightarrow r^2=\frac{64}{\pi}\Rightarrow r=\frac{8}{\sqrt{\pi}}$$

The circumference of the circle is:

$$2\pi r=2\pi\cdot\frac{8}{\sqrt{\pi}}=16\sqrt{\pi}$$

Part A

The ratio of the arc length of the smallest arc to the circumference is:

$$=\frac{1}{1+2+\cdots+12}=\frac{1}{\frac{12(13)}{2}}=\frac{1}{6(13)}=\frac{1}{78}$$

The arc length of the smallest arc is:

$$16\sqrt{\pi}\times\frac{1}{78}=8\sqrt{\pi}\times\frac{1}{39}=\frac{8}{39}\sqrt{\pi}$$

Part B

$$1+2=3=4-1=2^2-1$$
$$1+2+4=1+2+2^2=7=8-1=2^3-1$$
$$\vdots$$
$$1+2+4+\cdots+1024=2(1024)-1=2048-1=2047$$

$$16\sqrt{\pi}\times\frac{1}{2047}=\frac{16}{2047}\sqrt{\pi}$$

C. Extensions

Example 2.50

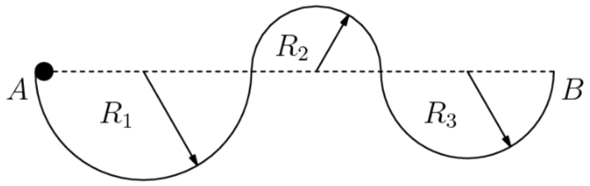
The *Flash* is circumnavigating (going around) the Earth exactly 1 meter above the ground. What is the length of the path for one round? (Answer in terms of R).

Note: For the question, assume that the Earth is a perfect sphere with a radius of $R=6371\ km$.

$$C=2\pi r=2\pi(R+0.001)\ km$$

Example 2.51

A ball with diameter 4 inches starts at point A to roll along the track shown. The track is comprised of 3 semicircular arcs whose radii are $R_1 = 100$ inches, $R_2 = 60$ inches, and $R_3 = 80$ inches, respectively. The ball always remains in contact with the track and does not slip. What is the distance the center of the ball travels over the course from A to B? **(AMC 8 2013/25)**



Perimeter of a semicircle

$$= \frac{C}{2} = \frac{2\pi r}{2} = \pi r$$

The distance travelled by the center of the ball is:

$$\pi r_1 + \pi r_2 + \pi r_3 = \pi \cdot 98 + \pi \cdot 62 + \pi \cdot 78 = \pi(98 + 62 + 78) = 238\pi$$

2.4 Sectors and Segments

A. Angle in a Sector

2.52: Central Angle of a Side of a Polygon

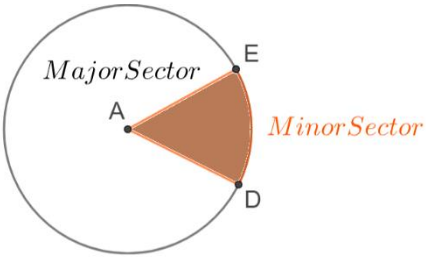
$$\theta = \frac{360}{n}$$

Example 2.53

B. Sectors

2.54: Sector of a Circle

The sector of a circle is formed by two radii, and the corresponding arc. The angle between the two radii is the *angle* or *central angle*.



Example 2.55

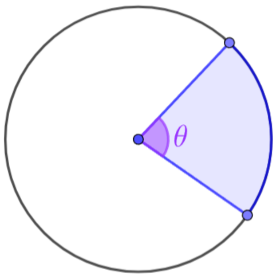
In the diagram alongside, which angle is the angle of the major sector?

$$360° - \angle EAD$$

2.56: Area of Sector

For a circle with radius r, the area of a sector with angle θ is:

$$Area\ of\ Sector = \pi r^2 \cdot \frac{\theta}{360}$$



The area of a sector of a circle is proportional to its central angle.

Example 2.57

Determine the exact area of the sector in each case.

A. The angle subtended is 100° and the radius is 3 $inches$
B. The angle subtended is 20° and the radius is 4 $feet$

Part A

$$Area \ = \pi r^2 \cdot \frac{\theta}{360} = \pi \cdot 3^2 \cdot \frac{100}{360} = 2.5\pi \ in^2$$

Part B

$$Area \ = \pi r^2 \cdot \frac{\theta}{360} = \pi \cdot 4^2 \cdot \frac{20}{360} = \frac{40}{45}\pi \ ft^2 = \frac{8\pi}{9} \ ft^2$$

Example 2.58

A. Find the exact area of a sector of a circle with radius 2, and central angle 210°.
B. A sector of a circle has an area $\frac{1}{\pi}$ times the area of the circle. Find the exact value of the central angle.

Part A

$$Area = \pi r^2 \cdot \frac{\theta}{360} = \pi \cdot 2^2 \cdot \frac{210}{360} = \pi \cdot 4 \cdot \frac{7}{12} = \frac{7\pi}{3}$$

Part B

$$\pi r^2 \cdot \frac{1}{\pi} = \pi r^2 \cdot \frac{\theta}{360}$$
$$1 = \pi \cdot \frac{\theta}{360}$$
$$\theta = \frac{360}{\pi}$$

2.59: Perimeter of a Sector

The perimeter of a sector with radius r and central angle θ will be

$$2r + 2\pi r \cdot \frac{\theta}{360°}$$

Example 2.60

The perimeter of a sector of a circle is equal to the circumference of the circle.

A. If the value of the central angle is less than 360, find its value as a fraction in terms of π.
B. Is it a major sector or a minor sector?

Part A

$$2r + 2\pi r \cdot \frac{\theta}{360} = 2\pi r$$

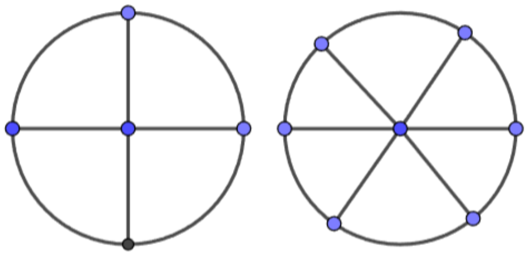
Divide throughout by $2r$:

$$1 + \frac{\theta\pi}{360} = \pi$$
$$\frac{360 + \theta\pi}{360} = \pi$$
$$360 + \theta\pi = 360\pi$$
$$\theta = \frac{360\pi - 360}{\pi} =$$

Part B

$$\frac{360\pi - 360}{\pi} \approx \frac{360(2)}{3} = 240 \Rightarrow Major\ Sector$$

Example 2.61
Aayan cuts a circular pizza into four equal slices, and separates the slices. Aman cuts a circular pizza of same shape as Aayan's into six equal slices and separates them. Determine the exact value of ratio of the total perimeter of the slices for Aayan to those for Aman.



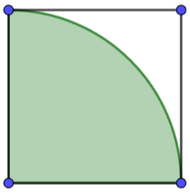
For both Aayan and Aman:

$$\text{Outer perimeter} = C = 2\pi r$$
$$Side\ perimeter\ of\ one\ slice = 2r$$

The ratio will be:

$$\frac{Aayan}{Aman} = \frac{4(2r) + 2\pi r}{6(2r) + 2\pi r} = \frac{8r + 2\pi r}{12r + 2\pi r} = \frac{4 + \pi}{6 + \pi}$$

C. Inscribed Figures



Example 2.62
A square of length 5 cm has a quarter circle inscribed in it, as shown. Find the exact area outside the circle, but inside the square.

The area of the square

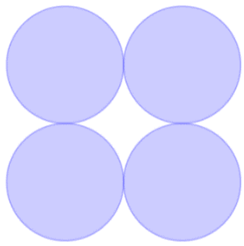
$$= s^2 = 5^2 = 25\ cm^2$$

The area of the quarter circle

$$= \frac{1}{4}\pi r^2 = \frac{1}{4}\pi \cdot 5^2 = 25\left(\frac{\pi}{4}\right)$$

The area which is outside the circle but inside the square is the difference of the two:

$$= 25 - 25\left(\frac{\pi}{4}\right) = 25\left[1 - \frac{\pi}{4}\right]$$



Example 2.63
Four circles of radius π are put next to each other so that each circle touches two other circles in symmetrical arrangement. Determine the area of the shape enclosed by the four circles.

Area of the square

$$= s^2 = (2\pi)^2 = 4\pi^2$$

Area of the circles inside the square

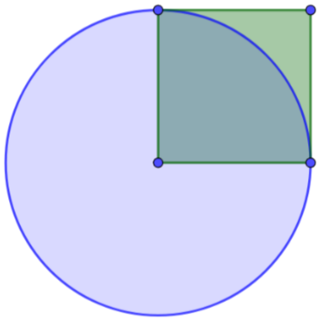
$$= 4\ Quarter\ Circles = 1\ Complete\ Circle = \pi r^2 = \pi(\pi^2) = \pi^3$$

Enclosed area

$$= 4\pi^2 - \pi^3 = \pi^2(4 - \pi)$$

Example 2.64
A square has sides of length 10, and a circle centered at one of its vertices has radius 10. What is the area of the

union of the regions enclosed by the square and the circle? **(AMC 10B 2004/9)**



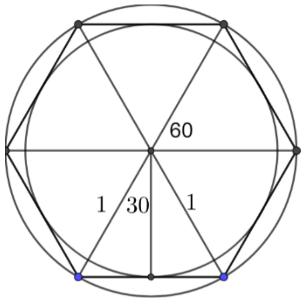
$$Area\ of\ Square = 100$$

$$Area\ of\ Circle\ Outside\ Square = \frac{3}{4} \cdot \pi r^2 = \frac{3}{4} \cdot \pi \cdot 10^2 = 75\pi$$

$$Total = 100 + 75\pi$$

Example 2.65

A circle is inscribed in the polygon formed by the line segments that are part of six congruent sectors in a unit circle. Determine the area between the smaller and the larger circle.



Let the radius of the larger circle be

$$1$$

The six congruent sectors each subtend

$$60°$$

Draw a perpendicular, creating a $30 - 60 - 90$ triangle. Radius of smaller circle

$$= \frac{\sqrt{3}}{2}$$

Area between the two circles

$$= \pi R^2 - \pi r^2 = \pi\left(1 - \frac{3}{4}\right) = \frac{\pi}{4}$$

Example 2.66

A square pasture of side length one unit has a goat tied to one corner, and another tied to the diagonally opposite corner. The length of the rope for each goat is one unit. Determine the area where the grass will be eaten by both the goats.

The area of a quarter circle is

$$x + y$$

The area of two quarter circles

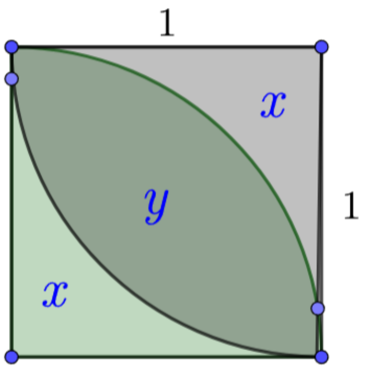
$$= 2x + 2y$$

The area of the square is

$$2x + y$$

The area that we want is:

$$2\underbrace{(x+y)}_{\substack{Area\ of\ the\\ Quarter\ Circle}} - \underbrace{(2x+y)}_{\substack{Area\ of\\ the\ Square}} = 2 \times \left(\frac{1}{4}\pi r^2\right) - 1 = \frac{\pi}{2} - 1$$

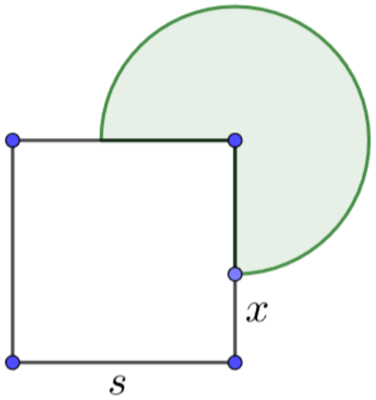


D. Grazing Goats and Dogs

Example 2.67

A goat is tethered to a corner of a square house with side length s. When the goat extends its rope to the maximum length, it is still x feet away from the corner nearest to the one that it is tethered to. Find the area that the goat can graze in terms of s and x if:

A. The goat is outside the house
B. The goat is inside the house



The radius of the circle is

$$s - x$$

The area that the goat can graze

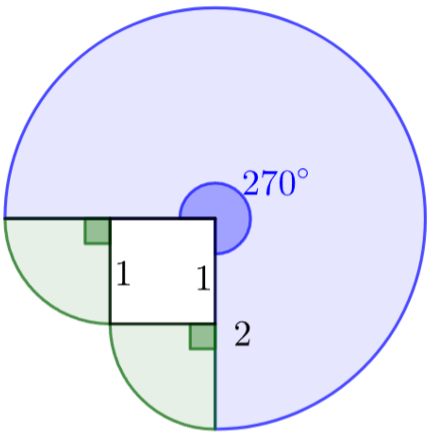
$$= \frac{3}{4}\pi r^2 = \frac{3}{4}\pi(s-x)^2$$

Inside of the house is:

$$= \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(s-x)^2$$

Example 2.68

Spot's doghouse has a regular square base that measures one yard on each side. He is tethered to a vertex with a two-yard rope. What is the exact area, in square yards, of the region outside of the doghouse that Spot can reach?



Sector of 270° in circle of radius 2

$$= \pi r^2 \cdot \frac{3}{4} = \pi \cdot 4 \cdot \frac{3}{4} = 3\pi$$

Two sectors of 90° each in circle of radius 1

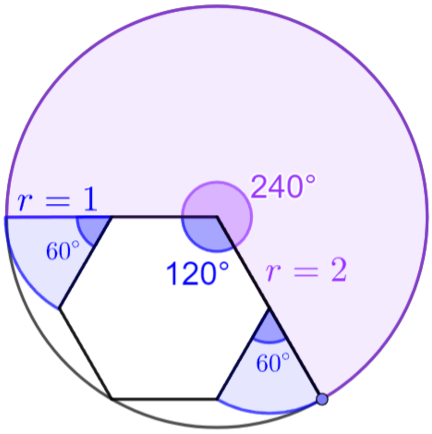
$$= 2\left(\pi r^2 \cdot \frac{1}{4}\right) = 2\left(\pi \cdot 1 \cdot \frac{1}{4}\right) = \frac{\pi}{2} = 0.5\pi$$

The final answer is:

$$3\pi + 0.5\pi = 3.5\pi$$

Example 2.69

Spot's doghouse has a regular hexagonal base that measures one yard on each side. He is tethered to a vertex with a two-yard rope. What is the exact area, in square yards, of the region outside of the doghouse that Spot can reach? **(AMC 10A 2002/19)**



The interior angle of a regular hexagon is 120°. Hence, the exterior angle is

$$360 - 120 = 240°$$

Sector of 240° in circle of radius 2

$$= \pi r^2 \cdot \frac{240}{360} = \pi \cdot 2^2 \cdot \frac{2}{3} = \frac{8\pi}{3}$$

The blue sectors have central angle

$$= 360 - 180° - 120° = 60°$$

Two sectors of 60° in circle of radius 1

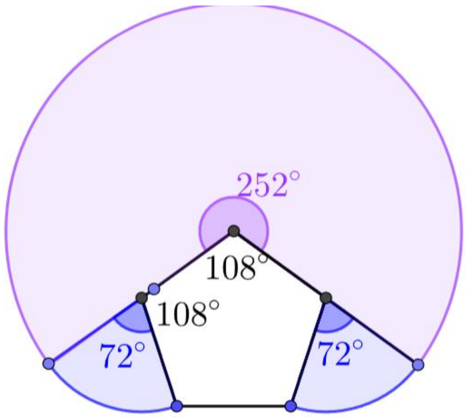
$$= 2\left(\pi r^2 \cdot \frac{60}{360}\right) = \pi \cdot 1 \cdot \frac{1}{3} = \frac{\pi}{3}$$

Total

$$= \frac{8\pi}{3} + \frac{\pi}{3} = \frac{9\pi}{3} = 3\pi$$

Example 2.70

Spot's doghouse has a regular pentagonal base that measures one yard on each side. He is tethered to a vertex with a two-yard rope. What is the exact area, in square yards, of the region outside of the doghouse that Spot can reach?



The interior angle of a regular hexagon is 108°. Hence, the exterior angle is

$$360 - 108 = 252°$$

Sector of 252° in circle of radius 2

$$= \pi r^2 \cdot \frac{252}{360} = \pi \cdot 2^2 \cdot \frac{7}{10} = \frac{14\pi}{5}$$

The blue sectors have central angle

$$= 360 - 180° - 108° = 72°$$

Two sectors of 60° in circle of radius 1

$$= 2\left(\pi r^2 \cdot \frac{72}{360}\right) = 2\left(\pi \cdot 1 \cdot \frac{1}{5}\right) = \frac{2\pi}{5}$$

Total

$$= \frac{14\pi}{5} + \frac{2\pi}{5} = \frac{16\pi}{5} = 3.2\pi$$

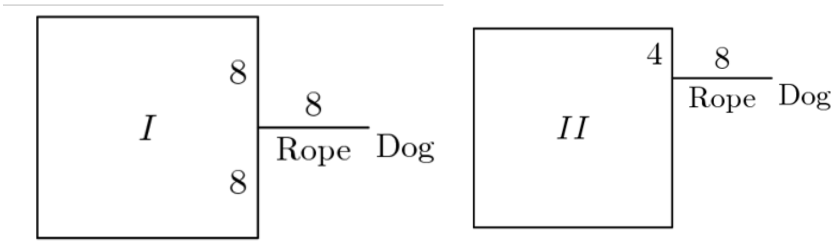
Example 2.71

Spot's doghouse has a regular polygonal base that measures one yard on each side. He is tethered to a vertex with a two-yard rope. The number of sides of the polygon is from 4 to 6 (inclusive of 4 and 6). Based on the number of the sides of the polygon, determine the difference between the maximum exact area, in square yards, of the region outside of the doghouse that Spot can reach, and the minimum exact area that he can reach.

$$3.5\pi - 3\pi = 0.5\pi$$

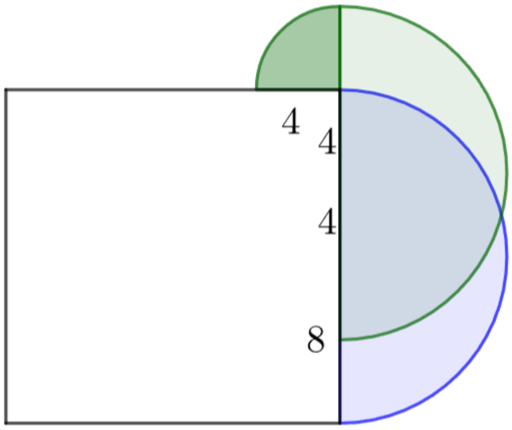
Example 2.72

Rolly wishes to secure his dog with an 8-foot rope to a square shed that is 16 feet on each side. His preliminary drawings are shown. Which of these arrangements give the dog the greater area to roam, and by how many square feet? (AMC 10A 2006/12)



The extra area is:

$$\pi r^2 \cdot \frac{\theta}{360} = \pi \cdot 4^2 \cdot \frac{90}{360} = 4\pi$$

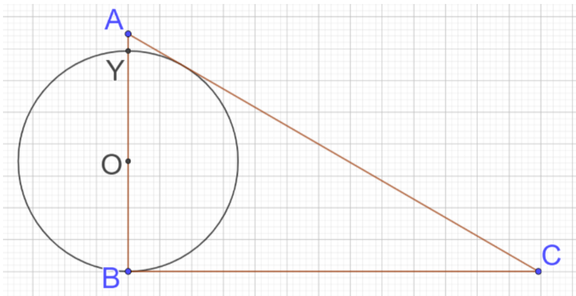


Challenge 2.73

Center O of the circle lies on AB, and the circle is tangent to AC. In right ΔABC, the legs have length

$$AB = 2 + \sqrt{3}, \qquad BC = 3 + 2\sqrt{3}$$

As can be seen in the diagram, there are two regions outside the circle, but inside ΔABC. Find the area of the smaller such region. (Micheal Penn)[2]



Strategy

Let the point of tangency of the circle with the triangle be X. We then need to find

$$[AXO] - [Sector\ OYX]$$

Finding AC

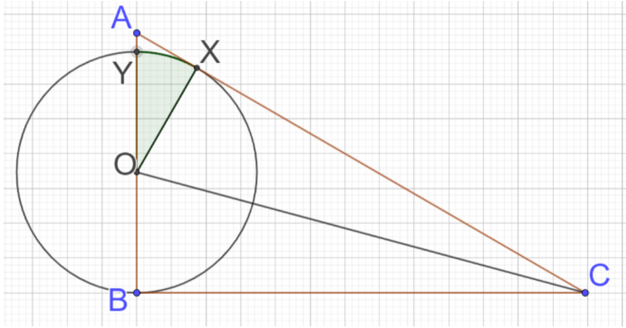
In ΔABC, by the Pythagorean Theorem:

$$AC^2 = AB^2 + BC^2 = \left(2 + \sqrt{3}\right)^2 + \left(3 + 2\sqrt{3}\right)^2$$

[2] This video has a beautiful solution. It does use trigonometry.

Expanding:

$$AC^2 = (4 + 3 + 4\sqrt{3}) + (9 + 12 + 12\sqrt{3})$$
$$= (7 + 4\sqrt{3}) + (21 + 12\sqrt{3}) = 28 + 16\sqrt{3}$$



Taking square roots:

$$AC = \sqrt{28 + 16\sqrt{3}} = \sqrt{4(7 + 4\sqrt{3})} = 2\sqrt{7 + 4\sqrt{3}}$$
$$= 2(2 + \sqrt{3})$$

(where we got the last square by observing that $7 + 4\sqrt{3} = AB^2$)

Showing ΔAOX is $30 - 60 - 90$

Since the radius is perpendicular to the tangent at the point of tangency, we must have:

$$\angle OXC = \angle OBC = 90°$$

Also, in right ΔABC

$$AC = 2 \times AB \Rightarrow \Delta ABC \text{ is } 30 - 60 - 90 \Rightarrow \angle ACB = 30°$$

Hence, in Quadrilateral $XOBC$:

$$\angle XOB = 360 - \angle OXC - \angle OBC - \angle XCB = 360 - 90 - 90 - 30 = 150$$
$$\angle AOX = 180 - 150 = 30°$$

In ΔAOX,

$$\angle AXO = 90°, \angle AOX = 30° \Rightarrow \Delta AOX \text{ is } 30 - 60 - 90$$

Calculating $[AOX]$

Since tangents drawn from a point to a circle are congruent, we must have:

$$CX = CB = 3 + 2\sqrt{3}$$

Hence:

$$AX = AC - XC = 4 + 2\sqrt{3} - (3 + 2\sqrt{3}) = 1$$

In $30 - 60 - 90$ ΔAOX:

$$OX = \sqrt{3} \times AX = \sqrt{3} \times 1 = \sqrt{3}$$
$$[AOX] = \frac{1}{2}hb = \frac{1}{2} \times 1 \times \sqrt{3} = \frac{\sqrt{3}}{2}$$

Calculating $[Sector\ YXO]$

In the circle

$$r = OX = \sqrt{3}$$

Area of Sector YXO

$$= \pi r^2 \times \frac{\theta}{360} = \pi(\sqrt{3})^2 \times \frac{30}{360} = 3\pi \times \frac{1}{12} = \frac{\pi}{4}$$

Area of Shaded Region

$$= [AOX] - [Sector\ YXO] = \frac{\sqrt{3}}{2} - \frac{\pi}{4} = \frac{2\sqrt{3} - \pi}{4}$$

E. Applications of Sectors

Example 2.74: Equations

A sector of a circle has area $\frac{\pi}{2}$ and arc length 3. Find the radius.

$$\underbrace{\pi r^2 \cdot \frac{\theta}{360} = \frac{\pi}{2}}_{\textit{Equation I}}, \qquad \underbrace{2\pi r \cdot \frac{\theta}{360} = 3}_{\textit{Equation II}}$$

Divide Equation I by Equation II:

$$\frac{\pi r^2 \cdot \frac{\theta}{360}}{2\pi r \cdot \frac{\theta}{360}} = \frac{\frac{\pi}{2}}{3}$$
$$\frac{r}{2} = \frac{\pi}{6}$$
$$r = \frac{\pi}{3}$$

2.75: Sum of angles around a point

The sum of angles around a point is:

$$360°$$

Example 2.76

A circle is divided into n sectors with central angle $a°, 2a°, \ldots, na°$ where a is a positive integer, and $n > 1$. Define $Max(x)$ and $Min(x)$ to be the maximum value and minimum value that x can take. Evaluate:

$$\frac{Max(a) - Min(a)}{Max(n) - Min(n)}$$

The sum of the central angles must add up to 360°:

$$a + 2a + \cdots + na = 360°$$

Factor out a:

$$a(1 + 2 + \cdots + n) = 360$$

Use the formula for the sum of the first n natural numbers:

$$a\left[\frac{n(n+1)}{2}\right] = 360$$

The above is a Diophantine equation since a and n are both natural numbers.

To make a as large as possible, we make n as small a possible. Since $n > 1$, we can take $n = 2$:

$$n = 2 \Rightarrow a\left[\frac{2(3)}{2}\right] = 360 \Rightarrow a = 120$$
$$Smallest\ n = 2, Largest\ a = 120$$

To make a as small as possible, we make n as large as possible:

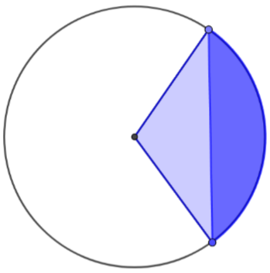
$$Try\ a = 1 \Rightarrow \frac{n(n+1)}{2} = 360 \Rightarrow n(n+1) = 720 \Rightarrow No\ Integer\ Solutions$$
$$Try\ a = 2 \Rightarrow (2)\frac{n(n+1)}{2} = 360 \Rightarrow n(n+1) = 360 \Rightarrow No\ Integer\ Solutions$$

$$a = 3 \Rightarrow (3)\frac{n(n+1)}{2} = 360 \Rightarrow n(n+1) = 240 \Rightarrow n = 15$$

Substitute $Max(a) = 120, Min(a) = 3, Max(n) = 15, Min(n) = 2$ in the required expression:

$$\frac{Max(a) - Min(a)}{Max(n) - Min(n)} = \frac{120-3}{15-2} = \frac{117}{13} = 9$$

F. Segments



2.77: Segment of a Circle

A sector can be divided into a triangle and a segment (see diagram)

- The blue shaded region is the sector.
- The dark blue region is the *minor* segment.
- The rest of the circle other than the dark blue region is the *major* segment.
  - ✓ It consists of the light blue triangle
  - ✓ And the white unshaded sector

- Sum of areas of major and minor segments give the area of the circle.

Example 2.78

A circle with area π has a segment with area $\sqrt{3}$.

A. Is the segment minor or major?
B. Find the area of the other segment of the circle.

Part A

$$Area\ of\ circle = \pi \approx 3.14$$

$$Area\ of\ half\ of\ the\ circle = \frac{\pi}{2} \approx 1.57$$

The area of the segment is

$$\sqrt{3} \approx 1.71$$

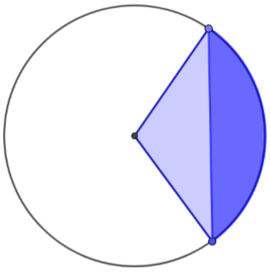
$$1.71 > 1.57 \Rightarrow Major\ Segment$$

Part B

$$\pi - \sqrt{3}$$

2.79: Major vs. Minor

If a segment covers more than half of the area of the circle, it is major. If it covers less than half, it is minor.



2.80: Area of a Segment of a Circle

Area of a segment of a circle is:

$$Area_{Segment} = Area_{Sector} - Area_{Triangle}$$

Example 2.81

In the adjoining diagram (not drawn to scale), the sector has central angle 60°, and the circumference of the circle is 1. Find the area of the minor segment in terms of π, and write your answer as a single fraction.

$$C = 2\pi r = 1 \Rightarrow r = \frac{1}{2\pi}$$

Area of the sector

$$= \pi r^2 \cdot \frac{\theta}{360} = \pi\left(\frac{1}{2\pi}\right)^2 \cdot \frac{60}{360} = \frac{1}{24\pi}$$

The area of the triangle will be

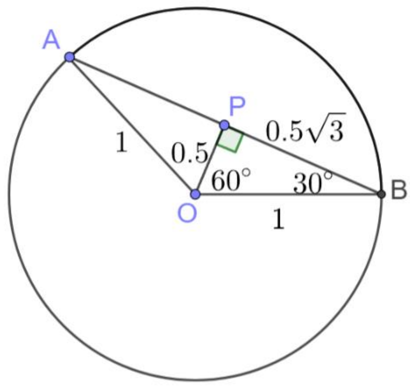
$$\frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}\left(\frac{1}{2\pi}\right)^2 = \frac{\sqrt{3}}{8\pi^2}$$

Area of the segment

$$= \frac{1}{24\pi} - \frac{\sqrt{3}}{8\pi^2} = \frac{\pi - 3\sqrt{3}}{48\pi^2}$$

Example 2.82

In a unit circle, the numerical value of the arc length of a minor sector equals the numerical value of the area of the corresponding major sector. Determine the exact area of the minor segment.



Let $NV = Numerical\ Value$

$$NV(Arc\ Length) = NV(Area\ of\ Sector)$$

$$2\pi r \cdot \frac{\theta}{360°} = \pi r^2 \cdot \frac{360 - \theta}{360°}$$

$$2\theta = r(360 - \theta)$$

Substitute $r = 1$:

$$2\theta = 360 - \theta \Rightarrow \theta = 120°$$

In ΔAOB, draw $OP \perp AB$

$$\angle POB = \frac{\angle AOB}{2} = \frac{120°}{2} = 60°$$

Substitute $PO = \frac{1}{2}OB = \frac{1}{2}, PB = \frac{\sqrt{3}}{2}OB = \frac{\sqrt{3}}{2}$ from $30 - 60 - 90$ ΔOPB in:

$$[OPB] = \frac{1}{2}hb = \frac{1}{2}(OP)(PB) = \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{8}$$

$$[AOB] = 2[OPB] = 2 \cdot \frac{\sqrt{3}}{8} = \frac{\sqrt{3}}{4}$$

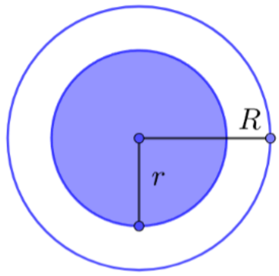
Area of segment

$$[Sector] - [Triangle] = \pi r^2 \cdot \frac{120}{360} - \frac{\sqrt{3}}{4} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}\ units^2$$

G. Nested Sectors

2.83: Concentric Circles

Concentric circles are circles that have the same center.



2.84: Area between concentric circles

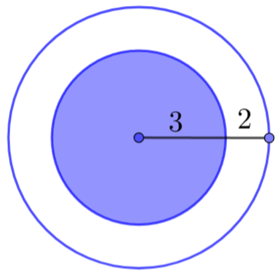
Given two concentric circles with radii R and r, where $R > r$, the area between the circles

$$= \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$$

Example 2.85

A circle of radius 3 is drawn with the same center as a radius of circle 5. What is the area between the circles?

$$\pi(R^2 - r^2) = \pi(5^2 - 3^2) = \pi(25 - 9) = 16\pi$$



2.86: Washers: Volume between shapes

The two-dimensional idea of area between circles can be extended to three dimensions.

Example 2.87

The lines $x = 3$ and $x = 5$ on the coordinate plane are rotated around the y −axis. What is the volume of the shape generated between the two rotated lines from $y = 0$ to $y = 4$.

$$\pi R^2 h - \pi r^2 h = \pi h(R^2 - r^2) = 4\pi(5^2 - 3^2) = 64\pi$$

2.88: Nested Sectors

Given two sectors with common center and radii R and r, where $R > r$, the area between the sectors

$$= \pi \cdot \frac{\theta}{360}(R^2 - r^2)$$

Area between sectors

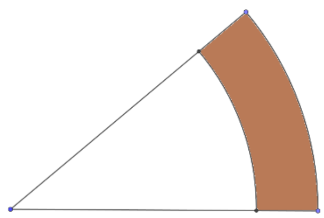
$$= \underbrace{\pi R^2 \cdot \frac{\theta}{360}}_{Area\ (Outer\ Sector)} - \underbrace{\pi r^2 \cdot \frac{\theta}{360}}_{Area\ (Inner\ Sector)}$$

Factor $\pi \frac{\theta}{360}$

$$= \pi \frac{\theta}{360}(R^2 - r^2)$$

Example 2.89

A restaurant makes circular pizzas with radius 12 inches cut into 11 equal slices (*see diagram*). The outermost two inches of one of the slices is burnt, and must be discarded. If a pizza costs 9 dollars, find the cost, in cents, of the discarded part.



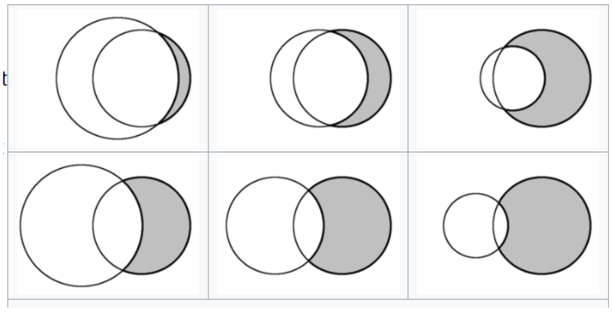
Substitute $R = 12, r = 10, \theta = \frac{360}{11} \Rightarrow \frac{\theta}{360} = \frac{1}{11}$ to find the ratio of the burnt area to the total area:

$$\frac{\pi \cdot \frac{\theta}{360}(R^2 - r^2)}{\pi R^2} = \frac{\pi\left(\frac{1}{11}\right)(12^2 - 10^2)}{\pi \cdot 12^2} = \frac{44}{11 \times 144} = \frac{1}{36}$$

The cost of the burnt area is:

$$9 \times \frac{1}{36} = \frac{1}{4} \; dollars = 25 \; cents$$

H. Lunes

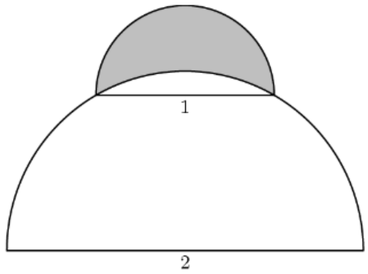


2.90: Lunes

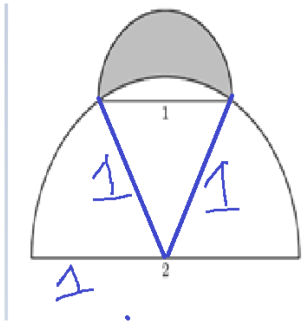
- A lune is the concave-convex region bounded by two circular arcs.
- There are different kinds of lunes.
- The general formula for the area of a lune is known, but mathematically quite complicated.

Example 2.91

A semicircle of diameter 1 sits at the top of a semicircle of diameter 2, as shown. The shaded area inside the smaller semicircle and outside the larger semicircle is called a lune. Determine the exact area of this lune. **(AMC 10A 2003/19, 12A 2003/15)**



Connect the center of the larger circle with the chord to get an equilateral triangle.



In the larger circle:

$$Area_{Sector} = \pi r^2 \times \frac{\theta}{360} = \pi(1)^2 \times \frac{60}{360} = \frac{\pi}{6}$$

$$Area_{Triangle} = \frac{\sqrt{3}}{4} s^2 = \frac{\sqrt{3}}{4}(1)^2 = \frac{\sqrt{3}}{4}$$

$$Area_{Segment} = \frac{\pi}{6} - \frac{\sqrt{3}}{4}$$

In the smaller circle:

$$Area_{Semicircle} = \frac{\pi r^2}{2} = \pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{8}$$

The area of the lune

$$= \underbrace{\frac{\pi}{8}}_{\text{Smaller Semi Circle}} - \underbrace{\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)}_{\text{Segment}} = \frac{\pi}{8} - \frac{\pi}{6} + \frac{\sqrt{3}}{4} = \frac{3\pi}{24} - \frac{4\pi}{24} + \frac{6\sqrt{3}}{24} = \frac{6\sqrt{3} - \pi}{24}$$

I. Complex Figures

Example 2.92

The shortest distance from point P to a circle with center O is 4. The length of the tangent from P to the circle is $4\sqrt{3}$. T is the intersection of the tangent from P with the circle. Determine the area of ΔPOT, that lies outside the circle.

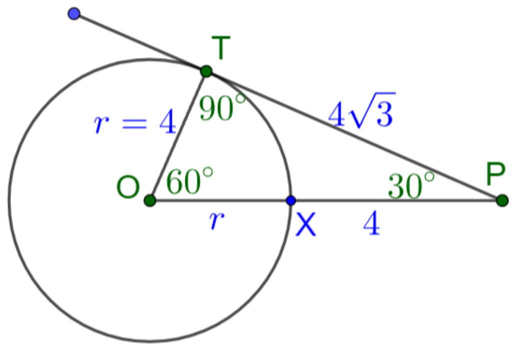
Method I: Tangent Perpendicularity Theorem and Pythagoras Theorem

Since the radius is perpendicular to the tangent at the point of tangency:

$$\angle OTP = 90°$$

Let the radius of the circle

$$= r$$

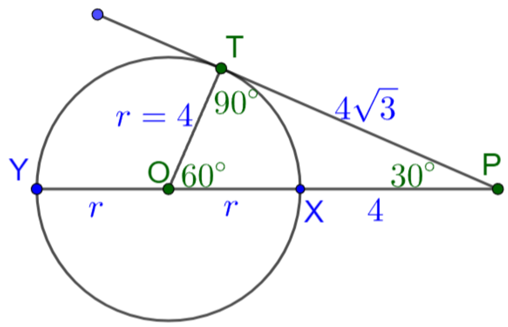


In ΔPOT, by Pythagoras Theorem:

$$(r+4)^2 = r^2 + \left(4\sqrt{3}\right)^2$$
$$r^2 + 8r + 16 = r^2 + 48$$
$$8r = 32$$
$$r = 4$$

$$OP = r + XP = 4 + 4 = 8$$

Method II: Power of a Point



$$PX \cdot PY = PT^2$$
$$4(8+2r) = \left(4\sqrt{3}\right)^2 = 48$$
$$4 + 2r = 12$$
$$2r = 8$$
$$r = 4$$

From either method, once we find $r = 4$:

$$OT = 4 = \frac{1}{2} \times OP \Rightarrow \Delta POT\ is\ a\ 30-60-90\ triangle$$

The area of the sector of the circle

$$= \pi r^2 \times \frac{60}{360} = \pi \cdot 4^2 \times \frac{1}{6} = \frac{8}{3}\pi$$

The area of ΔPOT is:

$$[POT] = \frac{1}{2}hb = \frac{1}{2} \cdot 4 \cdot 4\sqrt{3} = 8\sqrt{3}$$

The area of the triangle that lies outside the circle is:

$$8\sqrt{3} - \frac{8}{3}\pi$$

Example 2.93

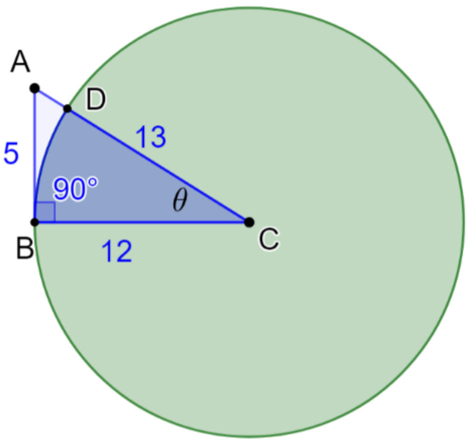
In right ΔABC, the legs are $AB = 5\ units$ and $BC = 12\ units$. A circle is drawn with radius BC and center C. Let X be the parts that lie within exactly one geometrical shape. Find, in terms of π and $\angle ACB = \theta$, the exact:

A. Area of X.
B. Perimeter of X.

Part A

X comprises

$$White - Shaded\ Area + Green - Shaded\ Area$$



Area in only the triangle $(White - Shaded)$

$$= [\Delta ABC] - [Sector\ DCB] = \frac{1}{2} \cdot 5 \cdot 12 - \pi \cdot 12^2 \cdot \frac{\theta}{360} = \underbrace{30 - \frac{2}{5}\pi\theta}_{Expression\ I}$$

Area in only the circle $(Green - Shaded)$

$$= [Circle] - [Sector\ DCB] = \pi \cdot 12^2 - \frac{2}{5}\pi\theta = \underbrace{144\pi - \frac{2}{5}\pi\theta}_{Expression\ II}$$

Add Expressions I and II to get the area of X

$$= 30 + 144\pi - \frac{4}{5}\pi\theta$$

Part B

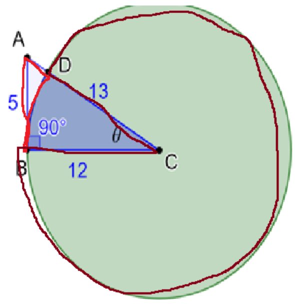
X consists of

$$White\ Region + Green\ Region$$

If you add the perimeters, these add up to the perimeters of the circle and the triangle.

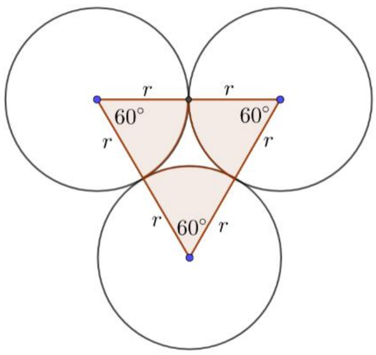
Hence, the perimeter of X

$$= 5 + 12 + 13 + 2\pi \times 12 = 30 + 24\pi$$



Example 2.94

Three circles of radius r are arranged in such a way that each circle is tangent to the other two circles. Find the area enclosed by the three circles.



Area of triangle

$$= \frac{\sqrt{3}}{4}(2r)^2 = \sqrt{3}r^2$$

Area of the three sectors

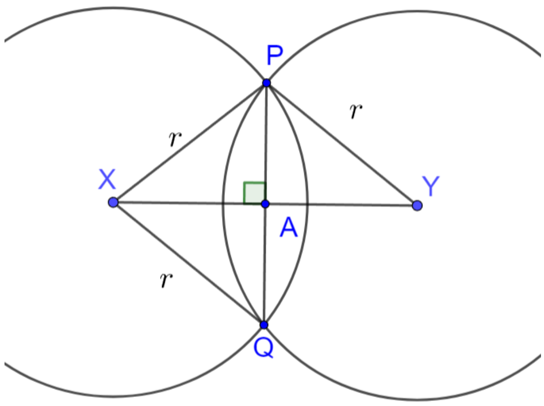
$$= Semicircle = \frac{\pi r^2}{2}$$

$$\sqrt{3}r^2 - \frac{\pi r^2}{2} = r^2\left(\sqrt{3} - \frac{\pi}{2}\right)$$

J. Complex Figures: Two Circles

Example 2.95

Two circles with equal radius r and centers X and Y intersect at P and Q. Given that $\angle XPY$ is $120°$, determine the exact area of intersection of the two circles in terms of r.



$$\angle XPQ = \frac{1}{2} \cdot 120° = 60°$$
$$\angle PAX = 90°$$

Hence:

$$\angle PXA = 180 - 90 - 60 = 30°$$

Hence:

$$\Delta PAX \text{ is a } 30 - 60 - 90 \text{ triangle}$$

Flipping over ΔPAX gives ΔQAX, and hence:

$$\Delta PQX \text{ is equilateral}$$

$$[PQX] = \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}r^2$$

The area of the sector

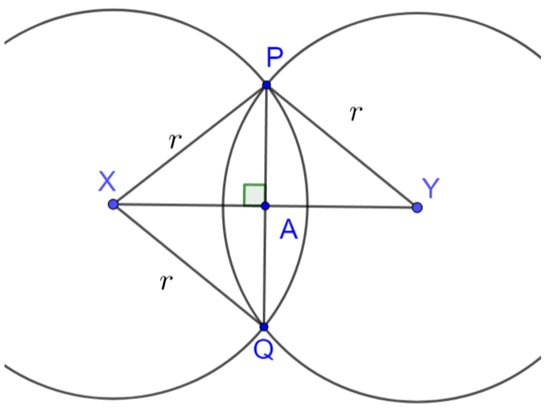
$$= \pi r^2 \times \frac{60}{360} = \frac{\pi r^2}{6}$$

The area of the overlap

$$= 2([Sector] - [PQX]) = 2\left(\frac{\pi r^2}{6} - \frac{\sqrt{3}}{4}r^2\right) = r^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

Example 2.96

Two circles with equal radius r and centers X and Y intersect at P and Q. Given that $\angle XPY$ is $90°$, determine the exact area of intersection of the two circles in terms of r.



$$\angle XPQ = \frac{1}{2} \cdot 90° = 45°$$
$$\angle PAX = 90°$$

Hence:

$$\angle PXA = 180 - 90 - 45 = 45°$$

Hence:

$$\Delta PAX \text{ is a } 45 - 45 - 90 \text{ triangle}$$

Flipping over ΔPAX gives ΔQAX, and hence:

$$\Delta PQX \text{ is } 45 - 45 - 90 \text{ triangle}$$

$$[PQX] = \frac{1}{2}hb = \frac{1}{2} \cdot r \cdot r = \frac{1}{2}r^2$$

The area of the sector

$$= \pi r^2 \times \frac{90}{360} = \frac{\pi r^2}{4}$$

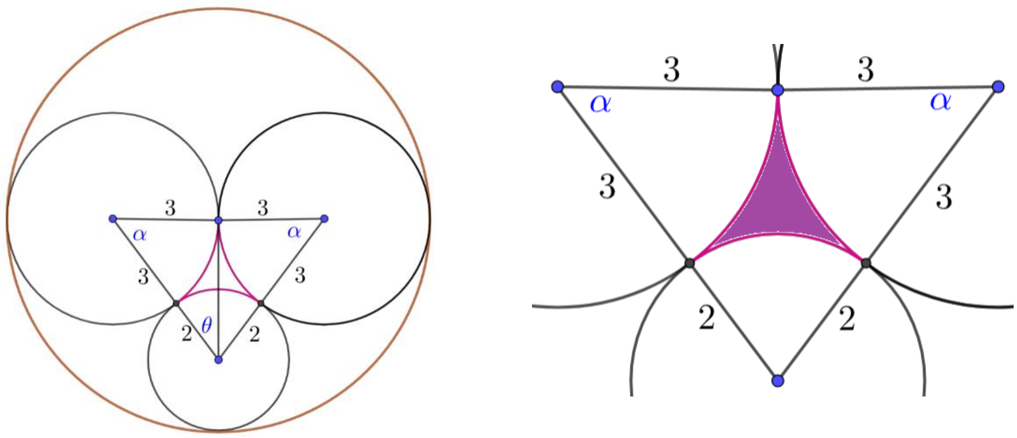
The area of the overlap

$$= 2([Sector] - [PQX]) = 2\left(\frac{\pi r^2}{4} - \frac{1}{2} r^2\right) = r^2\left(\frac{\pi}{2} - 1\right)$$

K. Trigonometry Based Questions

Example 2.97

Find a trigonometric expression in terms of radians for the area enclosed by three circles of radii 3, 3 and 2 that are each tangent to the other two circles externally.



Area of ΔABC

$$= \frac{1}{2} \times 6 \times 4 = 12$$

Area of the shaded region

$$= 12 - 2(3^2 \times \pi)\left(\frac{\alpha}{2\pi}\right) - (2^2 \times \pi)\left(\frac{2\theta}{2\pi}\right)$$
$$= 12 - 9\alpha - 4\theta$$

Where:

$$\sin\alpha = \frac{4}{5} \Rightarrow \alpha = \sin^{-1}\left(\frac{4}{5}\right), \qquad \sin\theta = \frac{3}{5} \Rightarrow \theta = \sin^{-1}\left(\frac{3}{5}\right)$$

Challenge 2.98[3]

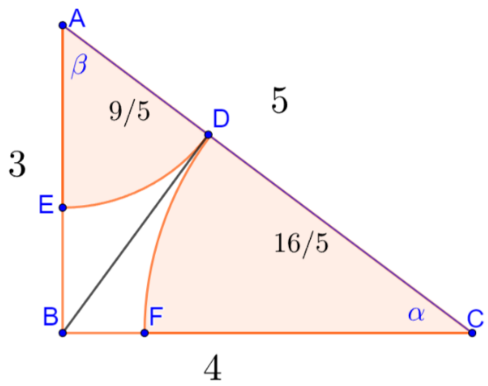
In right ΔABC, right-angled at B, the side lengths are all single digit integers. Point D lies on AC such that $\angle BDC$ is also a right angle. Sectors are drawn with centers at A and C, and radii AD and CD respectively. E lies between A and B such that $AD = AE$, and F lies between C and B such that $CF = CD$. The area of the shape formed by segment EB, segment BF, arc FD, and arc DE can be written as $a - \frac{\pi}{b}\left(c \cot^{-1}\frac{d}{e} + f \cot^{-1}\frac{g}{h}\right)$, where $HCF(a, c, f) = HCF(d, e) = HCF(g, h) = 1$. Find $a + b + c + d + e + f + g$.

---

[3] This "exam-style" question goes out of its way to confuse you. Focus on basics, and it is not so difficult.

Since the side lengths are all integers, we must have a Pythagorean Triplet, and the only one that fits is:

$$(3,4,5)$$



Since ∠BDC is a right angle:

$$BD \perp AC$$

Recall that the two triangles formed by a dropping a perpendicular to the hypotenuse of a right triangle are similar to each other and to the original triangle. Hence:

$$\Delta ABC \sim \Delta ADB \sim \Delta BDC$$

Using the similarity conditions gives us:

$$\frac{Longer\ Leg}{Hypotenuse} = \frac{DC}{BC} = \frac{BC}{AC} \Rightarrow DC = \frac{BC^2}{AC} = \frac{16}{5}$$

$$AD = AC - DC = 5 - \frac{16}{5} = \frac{9}{5}$$

In right ΔABC:

$$\cot\alpha = \frac{4}{3} \Rightarrow \alpha = \cot^{-1}\frac{4}{3}$$

$$\cot\beta = \frac{3}{4} \Rightarrow \beta = \cot^{-1}\frac{3}{4}$$

Area of the required shape

$$= \underbrace{\frac{1}{2}\cdot 3\cdot 4}_{Area\ of\ \Delta ABC} - \underbrace{\pi\cdot\left(\frac{16}{5}\right)^2\cdot\frac{\cot^{-1}\frac{4}{3}}{360}}_{Sector\ DCF} - \underbrace{\pi\cdot\left(\frac{9}{5}\right)^2\cdot\frac{\cot^{-1}\frac{3}{4}}{360}}_{Sector\ DAE}$$

$$= 6 - \pi\cdot\frac{256}{25}\cdot\frac{\cot^{-1}\frac{4}{3}}{360} - \pi\cdot\frac{81}{25}\cdot\frac{\cot^{-1}\frac{3}{4}}{360}$$

$$= 6 - \frac{\pi}{9000}\left(256\cot^{-1}\frac{4}{3} + 81\cot^{-1}\frac{3}{4}\right)$$

$$a + b + c + d + e + f + g$$
$$= 6 + 9000 + 256 + 4 + 3 + 81 + 3 + 4$$
$$= 9357$$

Example 2.99

A. Find an expression (in terms of r) for the exact value of the area enclosed by a set of four circles of radius r, if each of the circles is tangent to two other circles from the set externally.

B. Can you generalize the previous example? That is, the circles have different radii, but are still externally tangent in a way as to enclose an area.

2.5 Descartes' Theorem

A. Curvature

2.100: Curvature

The curvature of a circle is the reciprocal of its radius:

$$Curvature = k = \pm\frac{1}{r}$$

- The concept of curvature has extension beyond circles. It is relevant in analyzing curves, and in physics.

Example 2.101

A. As the radius of a circle decreases, does its curvature increase or decrease?
B. As the radius of a circle increases, does its curvature increase or decrease?
C. As the radius of a circle becomes very large, what happens to its curvature?
D. Using the answers to the answers to the above questions, explain what is the physical interpretation of curvature.

Part A

$$\frac{1}{0.1} = 10, \frac{1}{0.01} = 100, \dots$$

As the radius decreases, the curvature increases.

Part B

$$\frac{1}{10} = 0.1, \frac{1}{0.01} = 0.01, \dots$$

As the radius decreases, the curvature increases.

Part C

As

$$r \to \infty, k = \frac{1}{r} \to 0$$

Part D

A very large circle looks like a straight line.
It has very low curvature.

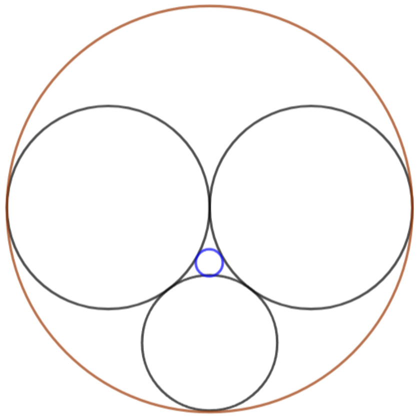
A very small circle is highly curved. It has high curvature.

Example 2.102

A circle has circumference 1. Find its curvature in terms of π.

$$C = 1$$
$$2\pi r = 1$$
$$r = \frac{1}{2\pi}$$
$$k = \pm\frac{1}{r} = \pm 2\pi$$

B. Descartes' Theorem



2.103: Descartes' Theorem (Curvature)

The curvature of any four kissing, or mutually tangent circles with curvature k_1, k_2, k_3 and k_4 satisfy the equation:

$$(k_1 + k_2 + k_3 + k_4)^2 = 2(k_1^2 + k_2^2 + k_3^2 + k_4^2)$$

- This can be proved, but we will not do so here.
- Note that, as shown in the diagram, there are two circles that are mutually tangent to the three black circles.

2.104: Descartes' Theorem (Finding Curvature)

We can solve the equation from Descartes' Theorem to find the radius of a fourth circle that is tangent to three given circles.

$$k_4 = k_1 + k_2 + k_3 \pm 2\sqrt{k_1k_2 + k_2k_3 + k_1k_3}$$

- The equation above can be derived from the statement of Descartes' Theorem, but we will do not do so here.
- The above equation has two solutions (from the $\pm$ sign).
- The negative solution is for the large circle. The other three circles are internally tangent.
- The positive solution is for the small circle. The other three circles are externally tangent to it.

Example 2.105

Fill in the blanks

The blue circle has _____ curvature. The brown circle has _____ curvature.

A. Positive
B. Negative

$$\begin{gathered} Blue: positive \\ Brown: Negative \end{gathered}$$

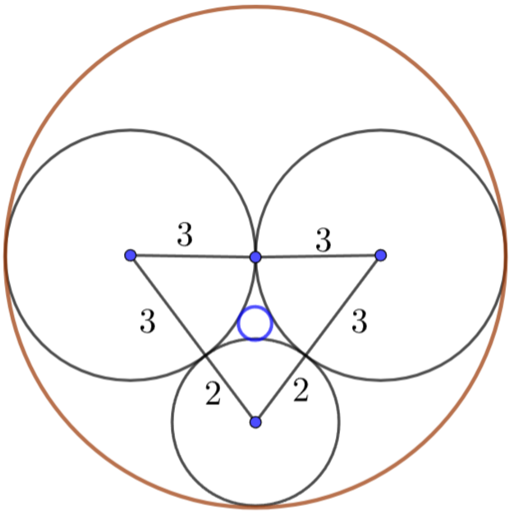
Example 2.106

Find the radius of all circles tangent to three mutually tangent circles with radii 3,3 and 2 respectively.

$$k_1 + k_2 + k_3 = \frac{1}{3} + \frac{1}{3} + \frac{1}{2} = \frac{2}{3} + \frac{1}{2} = \frac{4}{6} + \frac{3}{6} = \frac{7}{6}$$

And $2\sqrt{k_1k_2 + k_2k_3 + k_1k_3}$

$$= 2\sqrt{\frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3}} = 2\sqrt{\frac{1}{6} + \frac{1}{6} + \frac{1}{9}} = 2\sqrt{\frac{1}{3} + \frac{1}{9}} = 2\sqrt{\frac{4}{9}} = \frac{4}{3}$$



The larger circle has curvature

$$= k_4 = \frac{7}{6} - \frac{4}{3} = \frac{7}{6} - \frac{8}{6} = -\frac{1}{6} \Rightarrow Radius = 6$$

The smaller circle has curvature

$$= k_4 = \frac{7}{6} + \frac{4}{3} = \frac{7}{6} + \frac{8}{6} = \frac{15}{6} = \frac{5}{2} \Rightarrow Radius = \frac{2}{5}$$

2.107: Descartes' Theorem (Radius)

We can also state Descartes' Theorem in terms of three mutually tangent circles with radius r_1, r_2, r_3 with radius of the fourth circle r_4 given by:

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \pm 2\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_1 r_3}}$$

Example 2.108

Three circles with radii 3, 3 and 2 are mutually tangent to the other two circles externally. Find the area between the circles, but outside the fourth circle that is externally tangent to the three circles.

Combine the results from above to get:

$$= 12 - 9\alpha - 4\theta - \frac{4}{25}\pi$$

Where:

$$\sin\alpha = \frac{4}{5} \Rightarrow \alpha = \sin^{-1}\left(\frac{4}{5}\right), \qquad \sin\theta = \frac{3}{5} \Rightarrow \theta = \sin^{-1}\left(\frac{3}{5}\right)$$

Example 2.109

Area not inside four circles and tangent circle (internal tangency)

Example 2.110

Circles of radii 5, 5, 8 and $\frac{m}{n}$ are mutually externally tangent, where m and n are relatively prime positive integers. Find $m + n$. (AIME 1997/4)

$$k_1 = \frac{1}{5}, k_2 = \frac{1}{5}, k_3 = \frac{1}{8}$$
$$k_1 + k_2 + k_3 = \frac{1}{5} + \frac{1}{5} + \frac{1}{8} = \frac{2}{5} + \frac{1}{8} = \frac{16}{40} + \frac{5}{40} = \frac{21}{40}$$

Then $2\sqrt{k_1k_2 + k_2k_3 + k_1k_3}$

$$= 2\sqrt{\frac{1}{5}\cdot\frac{1}{5} + \frac{1}{5}\cdot\frac{1}{8} + \frac{1}{5}\cdot\frac{1}{8}} = 2\sqrt{\frac{1}{25} + \frac{1}{40} + \frac{1}{40}} = 2\sqrt{\frac{1}{25} + \frac{1}{20}} = 2\sqrt{\frac{45}{500}} = 2\sqrt{\frac{45}{500}} = 2\sqrt{\frac{9}{100}} = \frac{6}{10} = \frac{24}{40}$$

The smaller circle has curvature:

$$= k_4 = \frac{21}{40} + \frac{24}{40} = \frac{45}{40} = \frac{9}{8} \Rightarrow Radius = \frac{8}{9}$$
$$m + n = 8 + 9 = 17$$

Example 2.111

Circle C with radius 2 has diameter AB. Circle D is internally tangent to circle C at A. Circle E is internally tangent to circle C, externally tangent to circle D, and tangent to AB. The radius of circle D is three times the radius of circle E, and can be written in the form $\sqrt{m} - n$, where m and n are positive integers. Find $m + n$. (AIME II 2014/8)[4]

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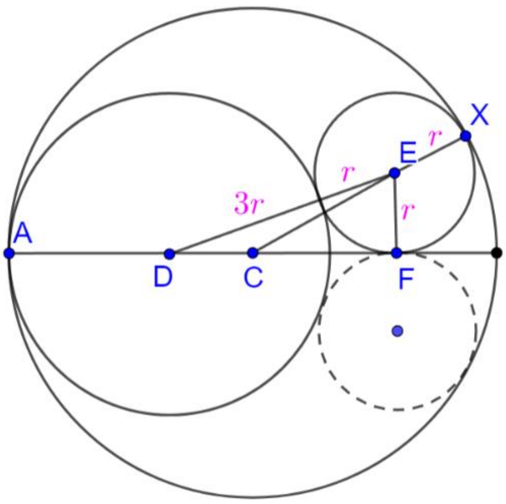
[4] A solution to this question using Law of Cosines is in that section. And a solution using Pythagorean Theorem is in the Note on Triangles.



Reflect circle E over diameter AB to get circle E'. The four circles that we now have meet the conditions of Descartes' Theorem.

Let

$$Radius\ of\ E = r \Rightarrow Radius\ of\ D = 3r$$

Descartes' Theorem is:

$$(k_E + k_{E'} + k_D + k_C)^2 = 2\left(k_E^2 + k_{E'}^2 + k_D^2 + k_C^2\right)$$

Substituting $k_D = \frac{1}{3r} = k \Rightarrow k_E = K_{E'} = \frac{1}{r} = 3k,\ \ k_C = -\frac{1}{r} = -\frac{1}{2}$:

$$\left(3k + 3k + k - \frac{1}{2}\right)^2 = 2\left[(3k)^2 + (3k)^2 + k^2 + \left(-\frac{1}{2}\right)^2\right]$$

Simplify:

$$\left(7k - \frac{1}{2}\right)^2 = 2\left[19k^2 + \frac{1}{4}\right]$$

Expand:

$$49k^2 - 7k + \frac{1}{4} = 38k^2 + \frac{1}{2}$$

Collate all terms on one side:

$$11k^2 - 7k - \frac{1}{4} = 0$$

Use the quadratic formula:

$$k = \frac{7 \pm \sqrt{49 - (4)(11)\left(\frac{1}{4}\right)}}{(2)(11)} = \frac{7 \pm \sqrt{60}}{22}$$

Take the reciprocal and rationalize:

$$3r = \frac{1}{k} = \frac{1}{\frac{7 \pm \sqrt{60}}{22}} = \frac{22}{7 \pm \sqrt{60}} \cdot \frac{7 \pm \sqrt{60}}{7 \pm \sqrt{60}} = \frac{22\left(7 \pm \sqrt{60}\right)}{-11} = -14 \pm 2\sqrt{60} = \pm\sqrt{240} - 14$$

Then:

$$\sqrt{m} - n = \sqrt{240} - 14 = 240 + 14 = 254$$

C. Special Cases

2.112: Circles with infinite radius

A line is a circle with infinite radius.

Consider a circle with infinite radius.

$$r_3 \to \infty, k_3 = \frac{1}{r_3} \to 0$$

2.113: Three Circles

$$k_4 = k_1 + k_2 \pm 2\sqrt{k_1 k_2}$$

A line is a circle with infinite radius. Substitute $k_3 = 0$ in the above

$$k_4 = k_1 + k_2 \pm 2\sqrt{k_1 k_2}$$

Example 2.114

Given that $k_4 = k_1 + k_2 \pm 2\sqrt{k_1 k_2}$, find $\sqrt{k_4}$

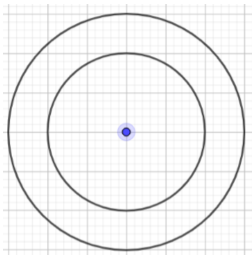
$$k_4 = k_1 + k_2 \pm 2\sqrt{k_1 k_2}$$
$$k_4 = \left(\sqrt{k_1} + \sqrt{k_2}\right)^2$$
$$\sqrt{k_4} = \sqrt{k_1} + \sqrt{k_2}$$

Example 2.115

A. More questions on Descartes' Theorem

2.6 Multiple Circles

A. Concentric Circles



2.116: Concentric Circles

Concentric circles are circles with the same center, but different radii.

Example 2.117

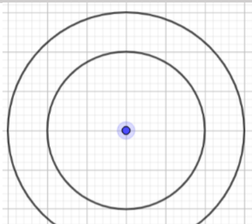
In the diagram, which is drawn to scale, find the distance between the centers of the two circles.

$$D = 0$$

2.118: Distance between Centers

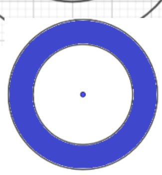
The distance between the centers of two concentric circles is

$$0$$



Example 2.119

In the diagram, which is drawn to scale, find the area inside the outer circle, but outside the inner circle.



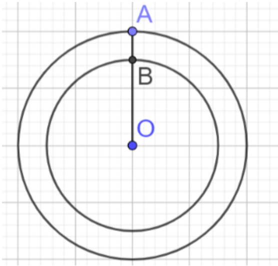
In the diagram, we want the blue area. This is given by:

$$Area(Outer\ Circle) - Area(Inner\ Circle)$$
$$= \pi(r_1)^2 - \pi(r_2)^2 = \pi \times 3^2 - \pi \times 2^2 = 9\pi - 4\pi = 5\pi$$

Example 2.120

A circular park of radius $10m$ has a two-meter-wide walking path along its inside, at the edge.

A. If you walk along the path at its outside, what is the distance walked in one round.
B. If you walk along the path at its inside, what is the distance walked in one round.
C. What is the area of the walking path?



Part A

$$Distance = C = 2\pi r = 2\pi \times 10 = 200\pi\ m$$

Part B

The radius will reduce to become

$$= 10 - 2 = 8\ m$$
$$Distance = C = 2\pi r = 2\pi \times 8 = 16\pi\ m$$

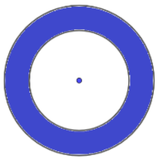
Part C

Area of the walking path

$$= A(Park) - A(Smaller\ Circle) = \pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi(10^2 - 8^2) = 36\pi$$

Example 2.121

A circular park of radius 10 m has a walkway of breadth 2 m along its inside. Find the area of the walkway.



In the diagram, we want the blue area. This is given by:

$$Area(Outer\ Circle) - Area(Inner\ Circle)$$
$$= \pi(r_1)^2 - \pi(r_2)^2 = \pi \times 10^2 - \pi \times 8^2 = 100\pi - 64\pi = 36\pi$$

The walkway above has to be paved at a cost of $\frac{5}{\pi}$ dollars per square meter. Find the cost of paving the walkway.

$$Cost = Area \times \frac{Cost}{Sq.Meter} = 36\pi \times \frac{5}{\pi} = 180\ Dollars$$

Challenge 2.122: Telescoping

Suppose circles $C_n(n = 1,2,\dots)$ concentric are point O are drawn in such a way that the line P_nP_{n+1} is of unit length and is perpendicular to OP_n for every $n = 1,2,\dots$. If the point P_n is on the circle C_n, and S_n is the area of the region between circles C_n and C_{n+1} for $n = 1,2,\dots$ then the value of **(JMET 2011/84)**

$$\sum_{m=1}^{100} S_{2m} - \sum_{m=1}^{100} S_{2m-1} \text{ is}$$

Expanding the Summation Notation

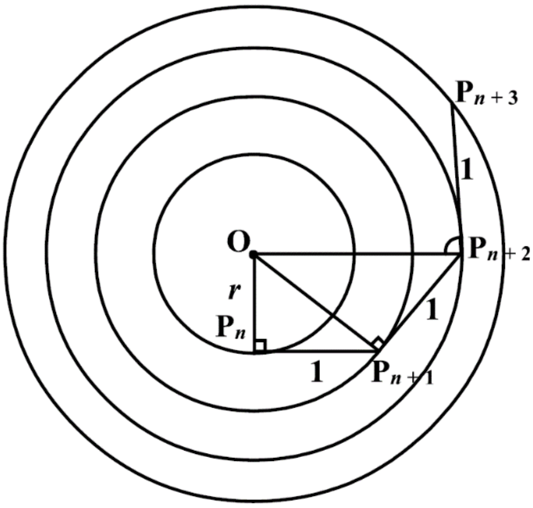
$$\sum_{m=1}^{100} S_{2m} = S_2 + S_4 + \dots + S_{100} \Rightarrow 50\ Terms$$
$$\sum_{m=1}^{100} S_{2m-1} = S_1 + S_2 + \dots + S_{99} \Rightarrow 50\ Terms$$

Finding the Area

By Pythagorean Theorem

$$OP_1 = \sqrt{(OP_1)^2 + (P_1P_2)^2} = \sqrt{r^2 + 1}$$

$$OP_2 = \sqrt{(OP_{+1})^2 + (P_2P_3)^2} = \sqrt{(r^2 + 1) + 1} = \sqrt{r^2 + 2}$$



The radius of the k^{th} circle is:

$$r_k = OP_k = \sqrt{r^2 + k}$$

Calculate:

$$S_n = C_{n+1} - C_n = \pi(r_{n+1})^2 - \pi(r_n)^2$$
$$= \pi(r^2 + k + 1) - \pi(r^2 + k) = \pi$$

The difference only depends on the number of terms:

$$S_2 + S_4 + \cdots + S_{100} = 50\pi$$
$$S_1 + S_2 + \cdots + S_{99} = 50\pi$$

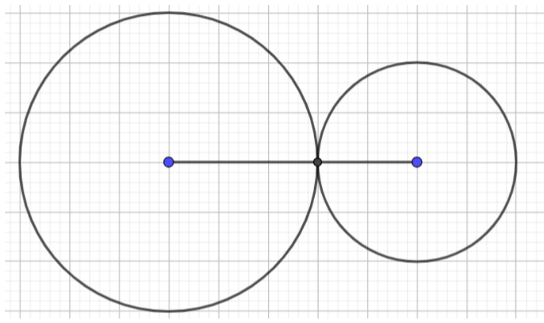
And hence their difference is

$$50\pi - 50\pi = 0$$

B. Tangent Circles

2.123: Circles Externally Tangent

If a circle is externally tangent to another circle, it touches the circle from the outside.



Note: Touches means exactly one place.
It does not intersect or cross.

Example 2.124

In the diagram, which is drawn to scale, find the distance between the centers of the two circles.

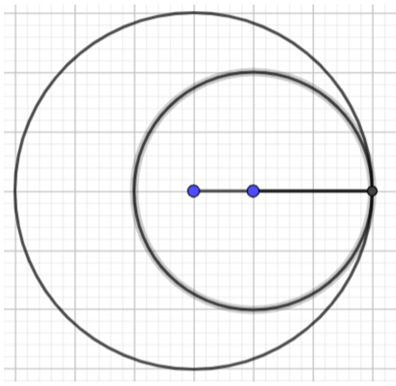
$$Distance = 3 + 2 = 5$$

2.125: Distance between Centers

The distance between the centers of two externally tangent circles with radius r_1 and r_2 is given by:

$$D = r_1 + r_2$$

C. Circles Tangent Internally



2.126: Circles Internally Tangent

If a circle is internally tangent to another circle, it touches the circle from the inside.

Note: Touches means exactly one place.
It does not intersect or cross.

Example 2.127

In the diagram, which is drawn to scale, find the distance between the centers of the two circles.

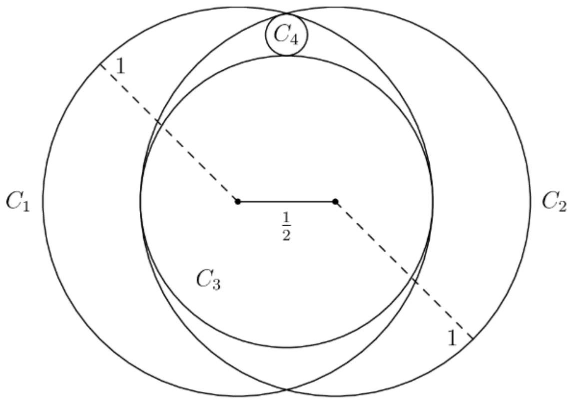
$$Distance = 3 - 2 = 1$$

2.128: Distance between Centers

The distance between the centers of two internally tangent circles with radius r_1 and r_2 is given by:

$$D = r_1 - r_2$$

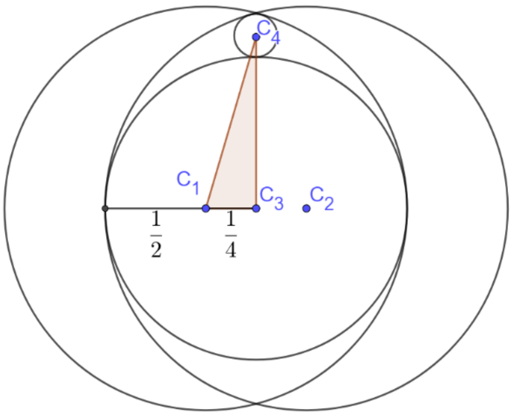
D. Multiple Tangent Circles

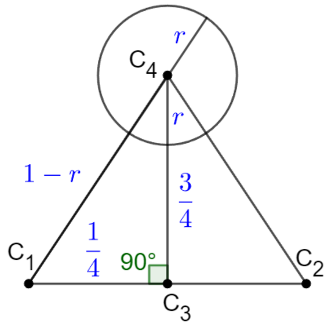


Example 2.129

Circle C_1 and C_2 each have radius 1, and the distance between their centers is $\frac{1}{2}$. Circle C_3 is the largest circle internally tangent to both C_1 and C_2. Circle C_4 is internally tangent to both C_1 and C_2 and externally tangent to C_3. What is the radius of C_4? **(AMC 10A 2023/22, 12A 2023/18)**

Since C_3 is symmetric with respect C_1 and C_2, the center of C_3 is between C_1 and C_2. Radius of circle C_3





$$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

Connect the centers of the circles C_1, C_3 and C_4 to get a right-angled triangle. By Pythagoras Theorem:

$$(1-r)^2 = \left(\frac{1}{4}\right)^2 + \left(r + \frac{3}{4}\right)^2$$

$$r^2 - 2r + 1 = \frac{1}{16} + \left(r^2 + \frac{3}{2}r + \frac{9}{16}\right)$$

$$r = \frac{3}{28}$$

E. Packing Problems

2.130: Packing Problems

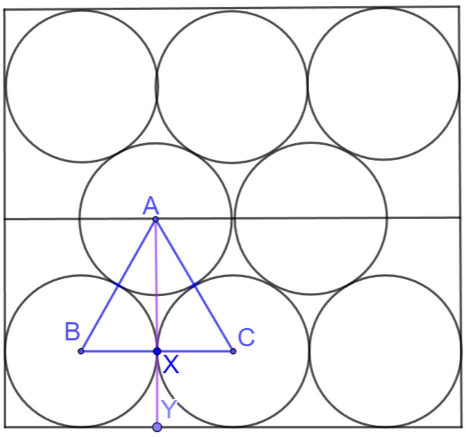
Packing problems require you to "pack" a set of objects in minimum space.
They are an important class of mathematical problems.

Example 2.131

A. What is the minimum area of the rectangle that encloses eight circles of diameter D.
B. $(Calc)$ Compare the area with the arrangement where they are stacked three in a row.
C. Find the area if $D = 4$

Part A

The arrangement with minimum area has two rows of three circles with the middle row having two circles. Join the centers of three circles to get an equilateral triangle with:



$$Side = BC = D \Rightarrow Height = AX = \frac{\sqrt{3}}{2}D$$

For the rectangle:

$$Width = 2(AX + XY) = 2\left(\frac{\sqrt{3}}{2}D + \frac{D}{2}\right) = D\left(1 + \sqrt{3}\right)$$

$$Length = 3D$$

Then, the area

$$= D\left(1 + \sqrt{3}\right)(3D) = D^2\left(3 + 3\sqrt{3}\right)$$

Part B

$$9D^2 : D^2\left(3 + 3\sqrt{3}\right) = 9 : \left(3 + 3\sqrt{3}\right) \approx 9 : 8.20$$

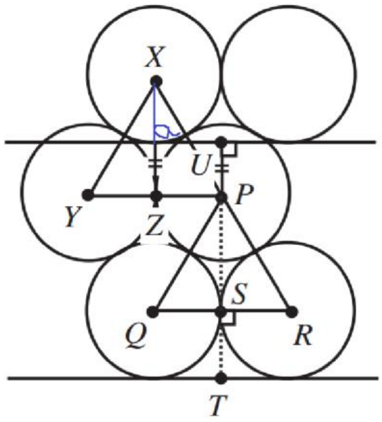
Part C

$$D^2\left(3 + 3\sqrt{3}\right) = 16\left(3 + 3\sqrt{3}\right) = 48 + 48\sqrt{3}$$

(Calculator Allowed) Example 2.132

One way to pack a 100 by 100 square with 10,000 circles, each of diameter 1, is to put them in 100 rows with 100 circles in each row. If the circles are repacked so that the centers of any three tangent circles form an equilateral triangle, what is the maximum number of additional circles that can be packed? (CEMC Cayley

1998/25)



Number Of Circles

The bottom row of circles (with centers Q and R) will contain 100 circles. The second row of circles (with centers Y and P) will contain 99 circles.
A double row then has:

$$100 + 99 = 199\ Circles$$

Height of a Double Row

ΔPQR is equilateral with side length 1.

$$\therefore Height(\Delta PRQ) = \frac{\sqrt{3}}{2}h = \frac{\sqrt{3}}{2} \cdot 1 = \frac{\sqrt{3}}{2}$$

The vertical distance between T and U is:

$$TU = TS + SP + PU = aX + SP + PU = PS + ZX = 2PS = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

Number of Double Rows

The number of double rows that we can fit is:

$$\frac{100}{\sqrt{3}} \approx 57.73$$

We can fit 57 rows at a minimum. Note that:

$$100 - 57\sqrt{3} \approx 1.27 > 1$$

We can fit in a last row.

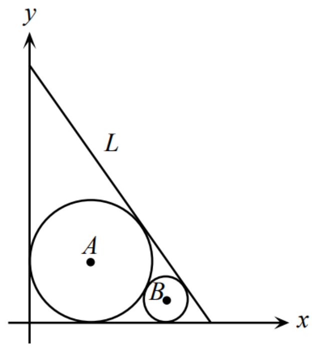
$$57(199) + 100 = 11{,}443$$

The number of additional circles is:

$$11{,}443 - 10{,}000 = 1{,}443$$

Example 2.133

The circle with center A has radius 3 and is tangent to both the positive x-axis and positive y-axis, as shown. Also, the circle with center B has radius 1 and is tangent to both the positive x-axis and the circle with center A. The line L is tangent to both circles. The y-intercept of line L is (CEMC Cayley 2000/25)

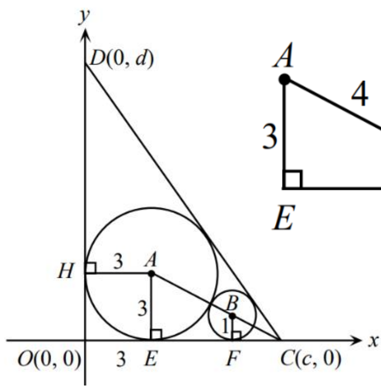


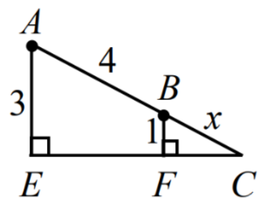
Join the centers of the circles, and note that

$$AB = 3 + 1 = 4$$

Let the line connecting the circles intersect the x-axis at C.
Drop a perpendicular from B and A to the x-axis, and another from A to the y-axis.

Extract ΔAEC from the larger diagram, and draw it separately.





In ΔAEC and ΔBFC:

$$\angle AEC = \angle BFC = 90^\circ\ (Radius \perp Tangent)$$
$$\angle CBF = \angle CAE\ (Corr.\ angles\ in \parallel\ lines\ AE\ and\ BF)$$
$$\Delta AEC \sim \Delta BFC\ (AA\ Similarity)$$

Let $CB = x$. Then, in similar triangles ΔAEC and ΔBFC:

$$\frac{CB}{BF} = \frac{CA}{AE} \Rightarrow \frac{x}{1} = \frac{x+4}{3} \Rightarrow 3x = x + 4 \Rightarrow x = 2 \Rightarrow CB = 2 \Rightarrow AC = 4 + 2 = 6$$

By Pythagoras Theorem in ΔBFC

$$FC = \sqrt{CB^2 - BF^2} = \sqrt{2^2 - 1^2} = \sqrt{4-1} = \sqrt{3}$$

Note that ΔBFC is a $30-60-90$ triangle since its sides are:

$$CB: BF: FC = 2: 1: \sqrt{3} = 1: \frac{1}{2}: \frac{\sqrt{3}}{2}$$

Since ΔAEC is similar to ΔBFC, it is also a $30-60-90$ triangle

$$EC = AC \times \frac{\sqrt{3}}{2} = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$$

Note that

$$\angle DCO = 2\angle BCF = 2 \times 30° = 60°$$

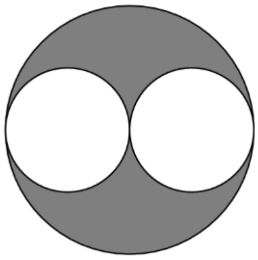
In $30-60-90$ ΔDCO:

$$OD = \sqrt{3}OC = \sqrt{3}(OE + EC) = \sqrt{3}\left(3 + 3\sqrt{3}\right) = 3\sqrt{3} + 9$$

F. Tangent Circles: Applications

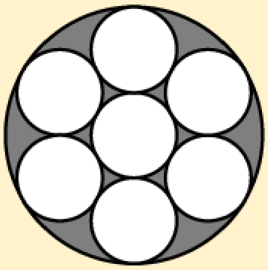
Example 2.134

In the diagram below, a diameter of each of the two smaller circles is a radius of the larger circle. If the two smaller circles have a combined area of 1 square unit, then what is the area of the shaded region, in square units? (AMC 8 2018/15)



Example 2.135

Each of the small circles in the figure has radius one. The innermost circle is tangent to the six circles that surround it, and each of those circles is tangent to the large circle and to its small-circle neighbors. Find the area of the shaded region. Write your answer in terms of π (AMC 10A 2002/5)



$$\pi \times 3^2 - 7 \times \pi \times 1^2 = 9\pi - 7\pi = 2\pi$$

Example 2.136

Circles ω and γ are drawn such that ω is internally tangent to γ, the distance between their centers are 5, and the area inside of γ but outside of ω is 100π. What is the sum of the radii of the circles? (CCA Math Bonanza, Individual Round, 2020/2)

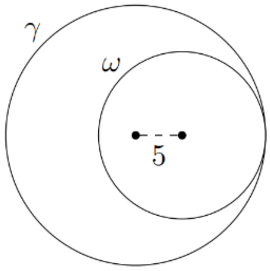


Figure not drawn to scale

Let

$$Radisu\ of\ \gamma = r \Rightarrow Radius\ of\ \omega = r + 5$$

Then, by the given condition:

$$\pi(r+5)^2 - \pi r^2 = 100\pi$$

Divide by π both sides:

$$(r+5)^2 - r^2 = 100$$

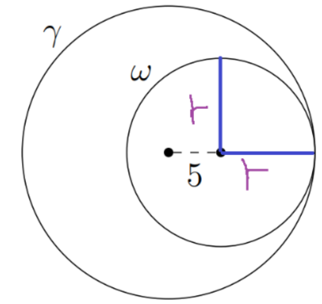


Figure not drawn to scale

Use $(r+5)^2 = (r+5)(r+5) = r(r+5) + 5(r+5) = r^2 + 5r + 5r + 25$:

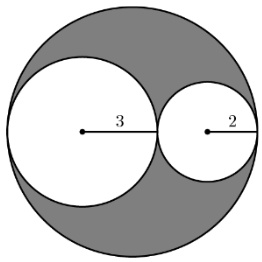
$$r^2 + 10r + 25 - r^2 = 100$$
$$10r + 25 = 100$$
$$r = 7.5$$
$$r + 5 = 12.5$$

Sum of the radii

$$= 7.5 + 12.5 = 20$$

Example 2.137

Circles of radius 2 and 3 are externally tangent and are circumscribed by a third circle. Find the area of the region outside the two circles, but inside the third circle. **(AMC 10B 2002/5)**



The area of the outer circle

$$\pi r^2 = \pi(2+3)^2 = 25\pi$$

The area of the inner circles is

$$\pi r_1^2 + \pi r_2^2 = \pi \cdot 2^2 + \pi \cdot 3^2 = 13\pi$$

The area outside the two circles, but inside the third circle is:

$$25\pi - 13\pi = 12\pi$$

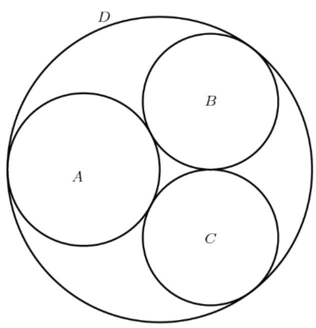
Example 2.138

Margie's winning art design is shown. The smallest circle has radius 2 inches, with each successive circle's radius increasing by 2 inches. What is the percent of the design that is black? **(AMC 8 2008/25)**

Radius	2	4	6	8	10	12
Area	4π	16π	36π	64π	100π	144π

$$\text{Outermost black area} = 100\pi - 64\pi = 36\pi$$
$$\text{Middle black area} = 36\pi - 16\pi = 20\pi$$
$$\text{Innermost black area} = 4\pi$$

$$\frac{36\pi + 20\pi + 4\pi}{144\pi} = \frac{60\pi}{144\pi} = \frac{5}{12} = \frac{500}{12}\% = \frac{125}{3}\%$$



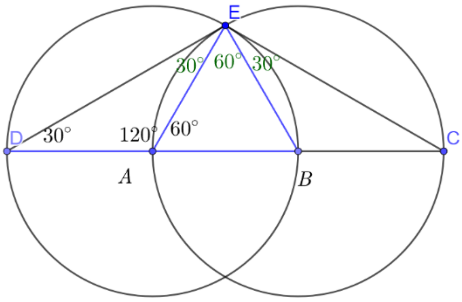
Example 2.139

Circles $A, B,$ and C are externally tangent to each other and internally tangent to circle D. Circles B and C are congruent. Circle A has radius 1 and passes through the center of D. What is the radius of circle B? **(AMC 10A 2004/23)**

G. Overlapping Circles

Example 2.140

Two congruent circles centered at points A and B each pass through the other circle's center. The line containing both A and B is extended to intersect the circles at points C and D. The circles intersect at two points, one of which is E. What is the degree measure of $\angle CED$? **(AMC 8 2016/23)**



In ΔAEB, all three of the sides are radii, and hence

$$\Delta AEB \text{ is equilateral} \Rightarrow \angle EAB = \angle AEB = 60°$$

$$\angle EAD = 180 - \angle EAB = 180 - 60 = 120°$$

$\Delta\ EAD$ is isosceles because $AE = AD = 1$

$$\angle DEA = \frac{180 - 120}{2} = 30°$$

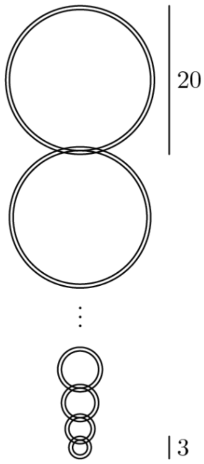
Similarly,

$$\angle CEB = 30°$$

$$\angle CED = 30 + 60 + 30 = 120°$$

Example 2.141

A number of linked rings, each 1 cm thick, are hanging on a peg. The top ring has an outside diameter of 20 cm. The outside diameter of each of the other rings is 1 cm less than that of the ring above it. The bottom ring has an outside diameter of 3 cm. What is the distance, in cm, from the top of the top ring to the bottom of the bottom ring? **(AMC 10A 2006/14)**



Method I

If the rings were *not* linked, then the total distance would be:

$$3 + 4 + \cdots + 20 = 1 + 2 + \cdots + 20 - 3 = \frac{20 \times 21}{2} - 3 = 210 - 3 = 207$$

Since the rings are linked, we must subtract the overlap. There are

$$20 - 2 = 18\ Rings \Rightarrow 18 - 1 = 17\ Overlaps \Rightarrow Overlap\ Distance = 2 \times 17 = 34$$

And hence the final answer is:

$$207 - 34 = 173$$

Method II

The solution above focused on finding the total distance, and then subtracting the overlap. We can also find the distance ring by ring.

The distance from the top of the largest, to its bottom, is simply:

$$20$$

The bottom of the top ring is 2 cm below the top of the 2nd ring. Hence, its distance from the bottom of the 2nd ring is

$$19 - 2 = 17$$

Similarly, the bottom of the 2nd ring is 2 cm below the top of the 3rd ring, and hence its distance from the bottom of the 3rd ring is:

$$18 - 2 = 16$$

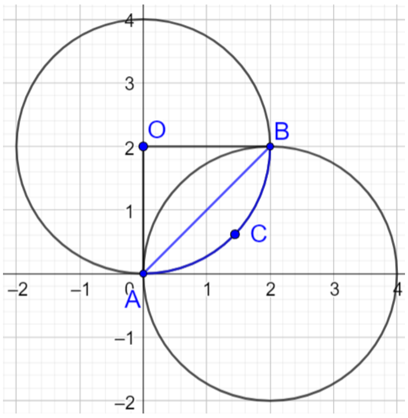
We continue this pattern till we reach the last ring, getting a total of:

$$20 + 17 + 16 + \cdots + 1 = 20 + \frac{17 \times 18}{2} = 20 + 153 = 173$$

H. Coordinate Geometry

Example 2.142

Two circles of radius 2 are centered at (2,0) and at (0,2). What is the area of the intersection of the interiors of the two circles? **(AMC 10B 2007/13)**



The circles intersect at

$$(0,0) \text{ and } (2,2)$$

The area of $Segment\ ABC$

$$= \underbrace{\frac{1}{4}\pi r^2}_{Sector\ ABC} - \underbrace{\frac{1}{2}hb}_{\Delta OAB} = \frac{1}{4}\pi(2^2) - \frac{1}{2}(2)(2) = \pi - 2$$

By symmetry, the area that we want is:

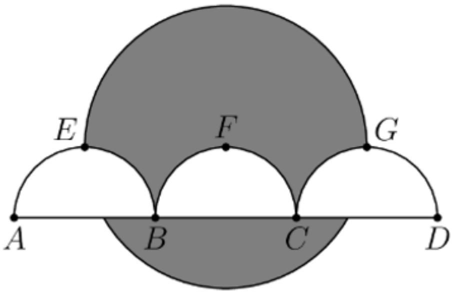
$$2 \times A(Segment\ ABC) = 2(\pi - 2)$$

Example 2.143

As shown in the figure, line segment AD is trisected by points B and C so that $AB = BC = CD = 2$. Three semicircles of radius 1, AEB, BFC, and CGD have their diameters on AD, and are tangent to line EG at E, F and G respectively. A circle of radius 2 has its center on F. The area of the region inside the circle but outside the three semicircles, shaded in the figure, can be expressed in the form

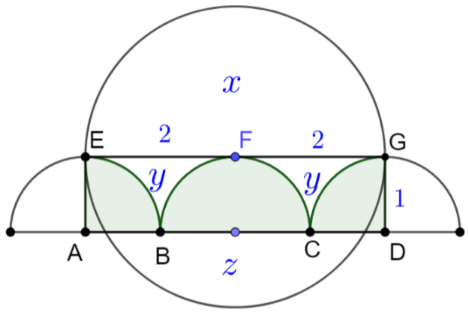
$$\frac{a}{b} \cdot \pi - \sqrt{c} + d$$

Where a, b, c and d are positive integers, and a and b are relatively prime. What is $a + b + c + d$? **(AMC 10B 2019/20)**



Let

$$x \text{ be the area above the diameter}$$
$$y \text{ be the white part shown}$$
$$z \text{ be the area below } AD$$



Step I

x is half the area of the large circle

$$= \frac{\pi r^2}{2} = \frac{\pi \cdot 2^2}{2} = 2\pi$$

Step II

$2y$ is the area of the rectangle minus the area of the shaded circular regions

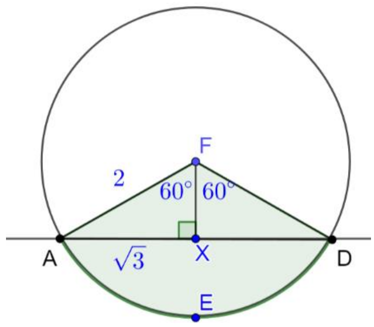
$$[EGDA] - 2(Quartercircles) - 1(Semicircle)$$
$$= 4(1) - 1\ Circle$$
$$= 4 - \pi r^2 = 4 - \pi$$

Step III

Drop a perpendicular from F to AD, intersecting AD at X.

By Pythagoras Theorem

$$AX = \sqrt{AF^2 - FX^2} = \sqrt{2^2 - 1^2} = \sqrt{4-1} = \sqrt{3}$$
$$AD = 2 \cdot AX = 2\sqrt{3}$$



The area of ΔFAD

$$=\frac{1}{2}hb=\frac{1}{2}(FX)(AD)=\frac{1}{2}(1)(2\sqrt{3})=\sqrt{3}$$

The area of sector $FAED$

$$=\pi r^2\cdot\frac{\theta}{360°}=\pi\cdot 2^2\cdot\frac{120°}{360°}=\frac{4}{3}\pi$$

The area of segment $AEDX$

$$=z=\frac{4}{3}\pi-\sqrt{3}$$

Shaded Area

$$x+2y+z=2\pi+(4-\pi)+\left(\frac{4}{3}\pi-\sqrt{3}\right)=\pi\left(2-1+\frac{4}{3}\right)-\sqrt{3}+4=\frac{7}{3}\pi-\sqrt{3}+4$$

Compare with $\frac{a}{b}\cdot\pi-\sqrt{c}+d$:

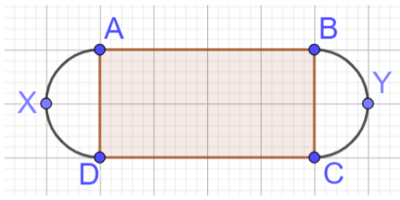
$$a=7, b=3, c=3, d=4\Rightarrow a+b+c+d=17$$

2.7 Applications

A. Composite Shapes

Example 2.144

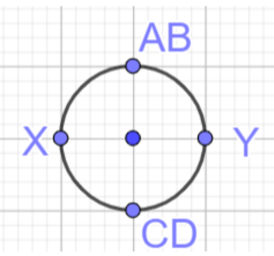
A cycling track consists of rectangle $ABCD$ with length 100 m, and width 50 m. AXD and BYC are semi-circles. A cyclist goes around the track at a speed of $5\frac{m}{s}$, and comes back to his starting point.



A. What is the distance he travels on path $ABCD$, and how much time does he take?
B. What is the distance he travels on path $ABYCDX$, and how much time does he take?
C. What is the area enclosed by the entire cycling track, $ABYCDXA$?

Part A

$$Perimeter=2(l+w)=2(100+50)=2(150)=300m$$
$$T=\frac{D}{S}=\frac{300}{5}=60\ seconds$$



Part B

This consists of two parts

$$AB+CD=100+100=200\ m$$

The other two parts can be "cut and combined" into a single circle, which has

$$Diameter=AD=50\Rightarrow Radius=\frac{AD}{2}=25$$

And the length of the circular path

$$=2\pi r=2\times\pi\times 25=50\pi$$

Hence, total length

$$=200+50\pi\ m$$

$$T=\frac{D}{S}=\frac{200+50\pi}{5}=\frac{200}{5}+\frac{50\pi}{5}=40+10\pi\ seconds$$

Part C

$$[ABCD]=lw=100\times 50=5000\ m^2$$

As before, cut and combine the two semi-circles into a single circle with $25m$:

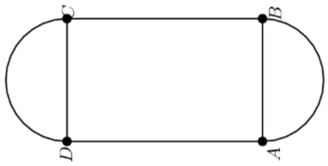
$$= \pi r^2 = \pi \times 25^2 = 625\pi$$

Hence, the total area is:

$$5000 + 625\pi$$

Example 2.145

A decorative window is made up of a rectangle with semicircles on either end. The ratio of AD to AB is $3:2$, and AB is 30 inches. What is the ratio of the area of the rectangle to the combined areas of the semicircles? **(AMC 8 2010/18)**



Example 2.146

Four figures are drawn (with none of them overlapping) on four sides of a square with length one unit as follows:An equilateral triangle with side length same as the side of the square

A. An isosceles triangle with base length same as the side of the square, and legs twice the side of the square.
B. A semi-circle with diameter same as the side of the square.
C. A scalene triangle with sides in the ratio 1: 2:2.5 and smallest side same as the side of the square.

Q1: What is the perimeter of the figure?
Q2: What is the area of the figure?

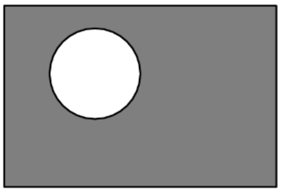
P (Eq. Δ) = 3 * 1 - 1= 2
P (Iso. Δ) = 2 + 2 = 4
P(Semi-circle) = πr = π (1/2) = 11/7
P (Scalene Δ) = 2 + 2.5 = 4.5

$$\text{Total} = 2 + 4 + \frac{11}{7} + 4.5 = \frac{11}{7} + 10.5 = 79/7$$

S2: A (Eq. Δ)

Example 2.147

A circle of diameter 1 is removed from a 2×3 rectangle, as shown. Which whole number is closest to the area of the shaded region? **(AMC 8 1992/5)**



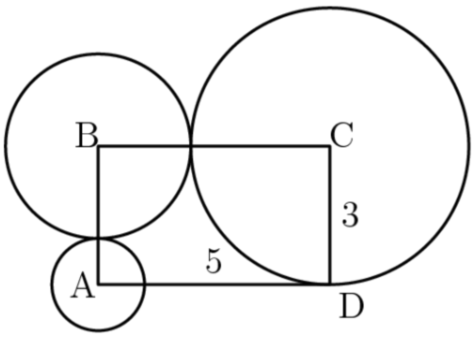
$$D = 1 \Rightarrow r = \frac{1}{2}$$

$$A(Rectangle) - A(Circle) = lw - \pi r^2 = 2 \times 3 - \pi\left(\frac{1}{2}\right)^2 = 6 - \frac{\pi}{4} = 6 - \frac{3.14}{4} \approx 5$$

Example 2.148

Rectangle $ABCD$ has sides $CD = 3$ and $DA = 5$. A circle with a radius of 1 is centered at A, a circle with a radius of 2 is centered at B, and a circle with a radius of 3 is centered at C. Which of the following is closest to the area of the region inside the rectangle but outside all three circles? **(AMC 8 2014/20)**

A. 3.5
B. 4.0
C. 4.5
D. 5.0

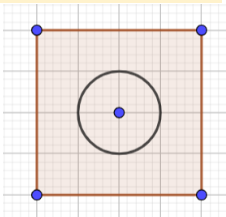


E. 5.5

B. Grazing Animals

Example 2.149

A goat is tied to a pole in the center of a square field, with a rope of length 7 m. If the side of the field is 20 m, find the area of the field that cannot be grazed by the goat.

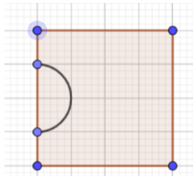


The area that can be grazed by goat will be a circle of radius 7 m.
We want the area outside the circle but inside the square, which will be given by:

$$A(Square) - A(Circle) = s^2 - \pi r^2 = 20^2 - \pi(14^2) = 400 - 196\pi$$

Example 2.150

A goat is tied to a pole at the midpoint of the side of a square field, with a rope of length 4 m. If the side of the field is 10 m, find the area of the field that cannot be grazed by the goat.

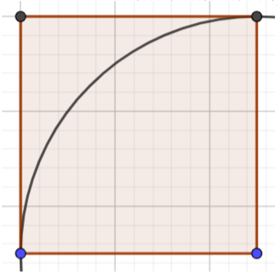


The area that can be grazed by goat will be a semi-circle of radius 4 m.
We want the area outside the semi-circle but inside the square, which will be given by:

$$A(Square) - A(Semi - Circle) = s^2 - \pi r^2 = 10^2 - \pi(4^2) = 100 - 16\pi$$

Example 2.151

A goat is tied to a pole at a corner of a square field, with a rope of length 4 m. If the side of the field is also 4 m, find the area of the field that cannot be grazed by the goat.



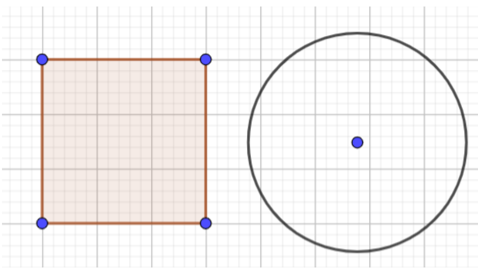
$$A(Square) - A(Quarter - Circle) = s^2 - \frac{\pi r^2}{4} = 4^2 - \frac{\pi(4^2)}{4} = 16 - 4\pi$$

C. Comparing Figures

Example 2.152

A wire is bent to form a square with side length 10 cm. The wire is cut, and formed into a circle. What is the

A. Circumference of the circle?
B. Area of the Circle?



Part A

The perimeter of the square is the same as the circumference of the circle.

$$Perimeter\ of\ Square = 10 \times 4 = 40\ cm$$
$$Circumference = Perimeter\ of\ Square = 40\ cm$$

Part B

$$C = 2\pi r \Rightarrow 40 = 2\pi r \Rightarrow r = \frac{20}{\pi} \Rightarrow A = \pi r^2 = \pi\left(\frac{20}{\pi}\right)^2 = \frac{400}{\pi}$$

Example 2.153
A square and a circle have the same area. What is the ratio of the side length of the square to the radius of the circle? (AMC 8 2010/16)

D. Intervals

Example 2.154
If rose bushes are spaced about 1 foot apart, approximately how many bushes are needed to surround a circular patio whose radius is 12 feet? (AMC 8 1988/13)

E. Similarity

Example 2.155
The number of circular pipes with an inside diameter of 1 inch which will carry the same amount of water as a pipe with an inside diameter of 6 inches is: (AHSME 1950/33)

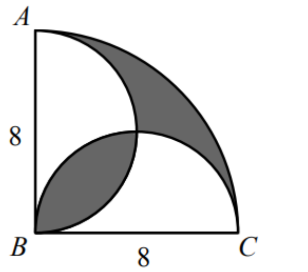
Using Ratios:

$$\pi r_1^2 : \pi r_2^2 = 1^2 : 6^2 = 1:36$$

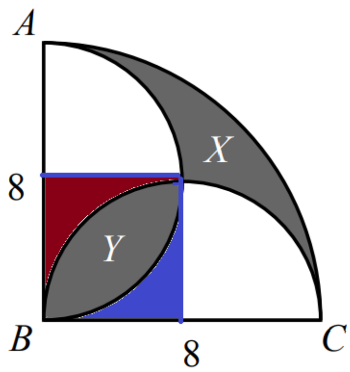
By Similarity, the area of the larger pipe is:

$$\frac{r_{larger\ pipe}}{r_{smaller\ pipe}} = \left(\frac{6}{1}\right)^2 = 36$$

2.8 Inscribed Figures



Example 2.156
In the diagram, ABC is a quarter of a circle with radius 8. A semi-circle with diameter AB is drawn, as shown. A second semi-circle with diameter BC is also drawn. The area of the shaded region is closest to (Gauss Grade 8 2017/24)



Split the quarter into multiple parts as shown. First, complete the square.
The area of the square

$$= 4^2 = 16$$

$$Blue\ Area = 16 - \left(\frac{1}{4}\pi r^2\right) = 16 - 4\pi$$

By complementary areas:

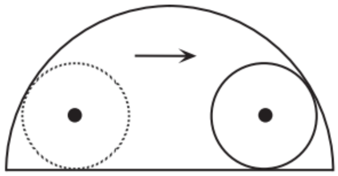
$$Y = 16 - 2(Blue\ Area) = 16 - 2(16 - 4\pi) = 8\pi - 16$$

$$X = \underbrace{\frac{1}{4}\pi r^2}_{\substack{Large\ Quarter \\ Circle}} - \underbrace{2(4\pi)}_{\substack{Small\ Quarter \\ Circle}} - 16 = 16\pi - 8\pi - 16 = 8\pi - 16$$

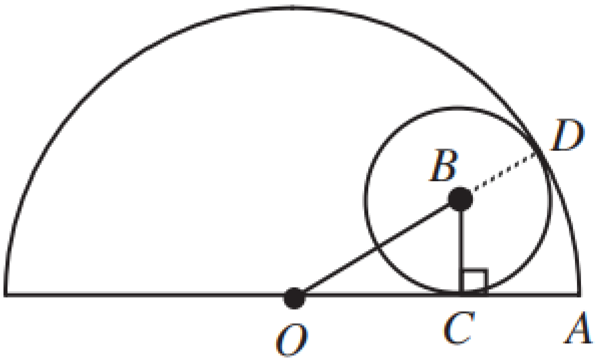
$$X + Y = 2Y = 2(8\pi - 16) = 16\pi - 32$$

Example 2.157

A wheel of radius 8 rolls along the diameter of a semicircle of radius 25 until it bumps into this semicircle. What is the length of the portion of the diameter that cannot be touched by the wheel? **(CEMC Cayley 1998/22)**



$$Radius\ of\ small\ circle = BC = BD = 8$$
$$Radius\ of\ semicircle = OA = OD = 25$$
$$OB = 25 - BD = 25 - 8 = 17$$



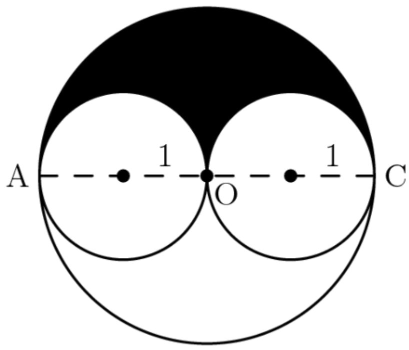
By Pythagorean Triplet (17,15,8) in right ΔOCB:
$$OC = 15$$

Total portion of the diameter that cannot be touched:
$$= 2(CA) = 2(OA - OC) = 2(25 - 15) = 2(10) = 20$$

A. Similarity

Example 2.158

The large circle has diameter AC. The two small circles have their centers on AC and just touch at O, the center of the large circle. If each small circle has radius 1, what is the value of the ratio of the area of the shaded region to the area of one of the small circles? **(AMC 8 1986/23)**



The area of the large circle will be
$$\pi r^2 = \pi(2^2) = 4\pi$$

The area of the small circle will be
$$\pi r^2 = \pi(1^2) = \pi$$

The area of the shaded region
$$= \frac{4\pi}{2} - 2\left(\frac{\pi}{2}\right) = 2\pi - \pi = \pi$$

The ratio is:
$$\frac{\pi}{\pi} = 1$$

Example 2.159

Six pepperoni circles will exactly fit across the diameter of a 12-inch pizza when placed. If a total of 24 circles of pepperoni are placed on this pizza without overlap, what fraction of the pizza is covered by pepperoni? **(AMC 8 2010/10)**

B. Triangles

Example 2.160

The area of the largest triangle that can be inscribed in a semi-circle whose radius is r is: (AHSME 1950/9)

$$A = \frac{1}{2}hb = \frac{1}{2} \times r \times 2r = r^2$$

C. Squares

Example 2.161

The area of the smallest square that will contain a circle of radius 4 is **(AMC 8 1997/7)**

Example 2.162

A square with area 196 $inches^2$ is inscribed in a circle. What is the area of the circle (in terms of π)?
Find the side length of the square:

$$\mathrm{A} = 196 \Rightarrow s^2 = 196 \Rightarrow s = 14$$

Find the diagonal using the property that the hypotenuse of an isosceles right angled triangle is $\sqrt{2}$ times its side:

$$Diagonal = 14\sqrt{2}$$

But the diagonal of the square is the diameter of the circle

$$Diameter = Diagonal = 14\sqrt{2} \Rightarrow Radius = \frac{Diameter}{2} = \frac{14\sqrt{2}}{2} = 7\sqrt{2}$$

Area of the Circle

$$= \pi r^2 = \pi\left(7\sqrt{2}\right)^2 = \pi \times 49 \times 2 = 98\pi$$

Example 2.163

A circle is inscribed in a square of area 196 square units. What are its circumference and area?
Diameter of the Circle is the same as the side length of the square

$$Diameter = Side\ Length = s = \sqrt{196} = 14 \Rightarrow r = \frac{s}{2} = \frac{14}{2} = 7$$

Circumference

$$C = 2\pi r = 2\pi \times 7 = 14\pi$$
$$A = \pi r^2 = \pi \times 7^2 = 49\pi$$

Example 2.164

What is the perimeter of a square inscribed in a circle of area 44 units?

$$r = \sqrt{\frac{A}{\pi}} = \sqrt{44 \times \frac{7}{22}} = \sqrt{14} \Rightarrow P = 4s = 4\left(\frac{2\sqrt{14}}{\sqrt{2}}\right) = 8\sqrt{7}$$

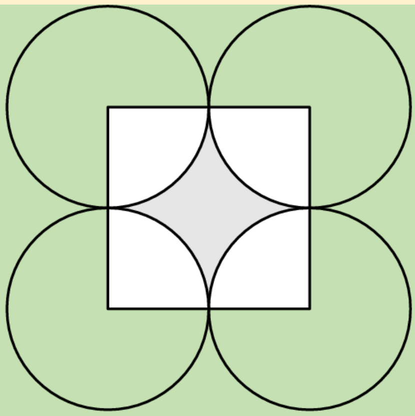
Example 2.165

Two equal half-circles are inscribed in a square (of side 4 cm) with the side of the square as their diameter. Let X be the part of the square that is not inside the circles. What is the ratio of the area of X to the perimeter of X?

$$Ratio = \frac{A(X)}{P(X)} = \frac{A(Square) - A(Circle)}{C(Circle) + 2s} = \frac{16 - (2^2)\pi}{2\pi r + 4(2)} = \frac{16 - 4\pi}{4\pi + 8} = \frac{4 - \pi}{2 + \pi}$$

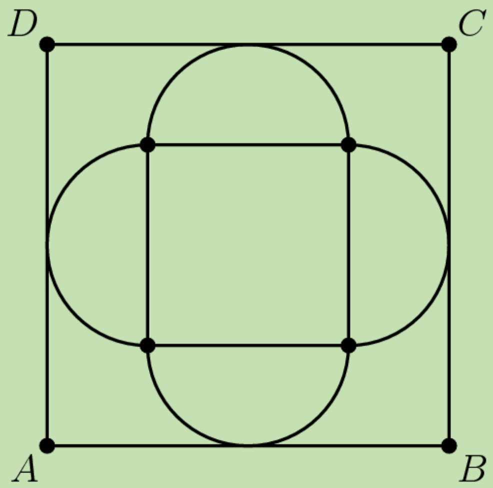
Example 2.166

Four circles of radius 3 are arranged as shown. Their centers are the vertices of a square. The area of the shaded region is closest to **(AMC 8 1992/24)**



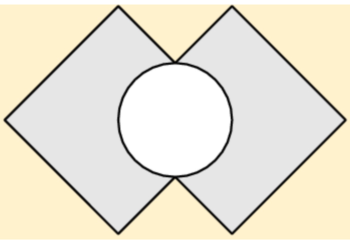
Example 2.167

Around the outside of a 4 by 4 square, construct four semicircles (as shown in the figure) with the four sides of the square as their diameters. Another square, $ABCD$, has its sides parallel to the corresponding sides of the original square, and each side of $ABCD$ is tangent to one of the semicircles. The area of the square $ABCD$ is **(AMC 8 1994/19)**

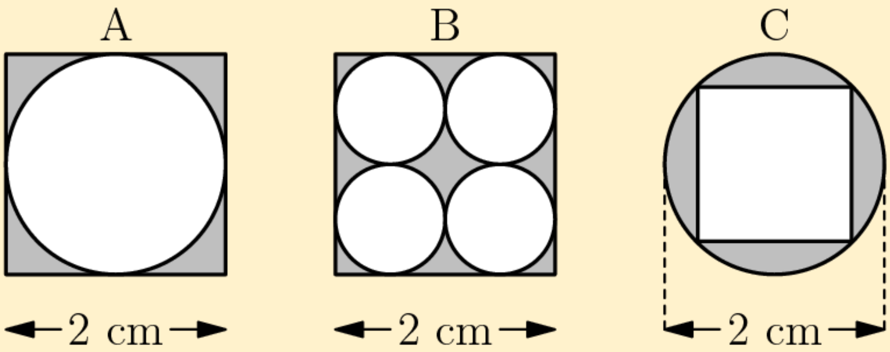


Example 2.168

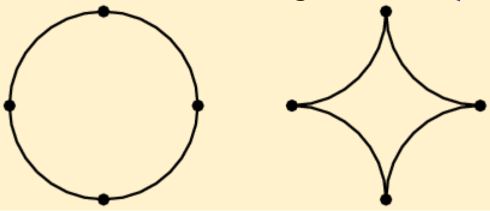
Two 4 $\times$ 4 squares intersect at right angles, bisecting their intersecting sides, as shown. The circle's diameter is the segment between the two points of intersection. What is the area of the shaded region created by removing the circle from the squares? **(AMC 8 2004/25)**



Example 2.169
The following figures are composed of squares and circles. Which figure has a shaded region with largest area? (AMC 8 2003/22)

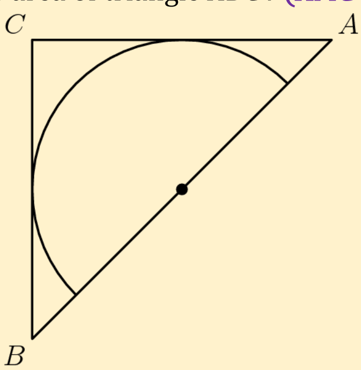


Example 2.170
A circle of radius 2 is cut into four congruent arcs. The four arcs are joined to form the star figure shown. What is the ratio of the area of the star figure to the area of the original circle? (AMC 8 2012/24)



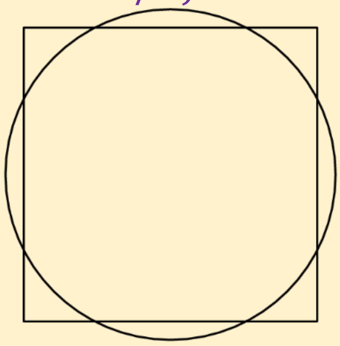
Example 2.171
A 1×2 rectangle is inscribed in a semicircle with longer side on the diameter. What is the area of the semicircle? (AMC 8 2013/20)

Example 2.172
Isosceles right triangle ABC encloses a semicircle of area 2π. The circle has its center O on hypotenuse $\overline{AB}$ and is tangent to sides $\overline{AC}$ and $\overline{BC}$. What is the area of triangle ABC? (AMC 8 2005/23)



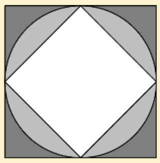
Example 2.173

A square with side length 2 and a circle share the same center. The total area of the regions that are inside the circle and outside the square is equal to the total area of the regions that are outside the circle and inside the square. What is the radius of the circle? **(AMC 8 2005/25)**



Example 2.174

A circle with radius 1 is inscribed in a square and circumscribed about another square as shown. Which fraction is closest to the ratio of the circle's shaded area to the area between the two squares?**(AMC 8 2011/25)**



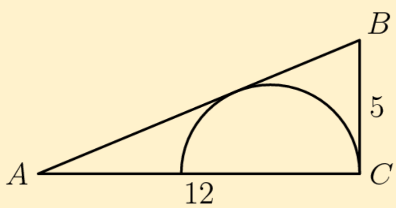
$(A)\ \frac{1}{2}$ $\quad (B)\ 1$ $\quad (C)\ \frac{3}{2}$ $\quad (D)\ 2$ $\quad (E)\ \frac{5}{2}$

Example 2.175

A semicircle is inscribed in an isosceles triangle with base 16 and height 15 so that the diameter of the semicircle is contained in the base of the triangle as shown. What is the radius of the semicircle? **(AMC 8 2016/25)**

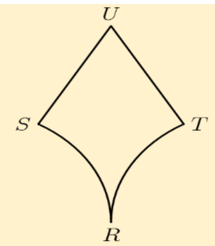
Example 2.176

In the right triangle ABC, $AC = 12$, $BC = 5$, and angle C is a right angle. A semicircle is inscribed in the triangle as shown. What is the radius of the semicircle?**(AMC 8 2017 /22)**



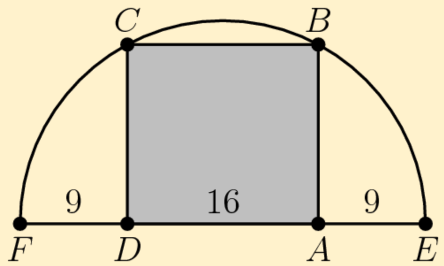
Example 2.177

In the figure shown, $\overline{US}$ and $\overline{UT}$ are line segments each of length 2, and $m\angle TUS = 60°$. Arcs TR and SR are each one-sixth of a circle with radius 2. What is the area of the region shown? **(AMC 8 2017 /25)**



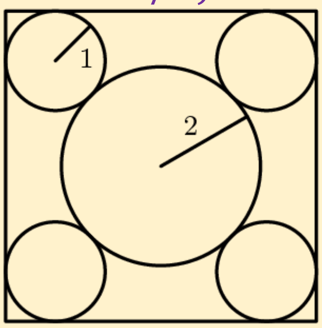
Example 2.178

Rectangle $ABCD$ is inscribed in a semicircle with diameter $\overline{FE}$, as shown in the figure. Let $DA = 16$, and let $FD = AE = 9$. What is the area of $ABCD$? **(AMC 8 2020/18)**



Example 2.179

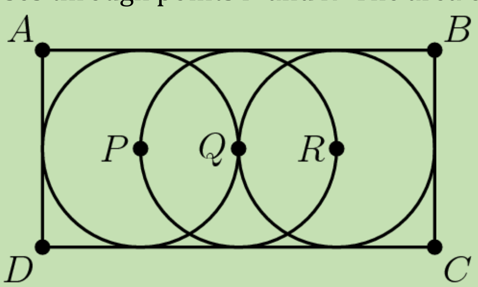
Four circles of radius 1 are each tangent to two sides of a square and externally tangent to a circle of radius 2, as shown. What is the area of the square? **(AMC 10A 2007/15)**



D. Rectangles

Example 2.180

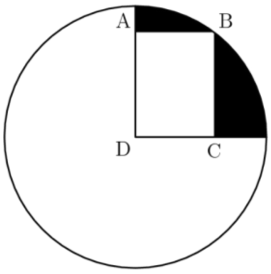
Three congruent circles with centers P, Q, and R are tangent to the sides of rectangle $ABCD$ as shown. The circle centered at Q has diameter 4 and passes through points P and R. The area of the rectangle is **(AMC 8 1995/9)**



Example 2.181

ABCD is a rectangle, D is the center of the circle, and B is on the circle. If AD = 4 and CD = 3, then the area of the shaded region is between **(AMC 8 1987/22)**

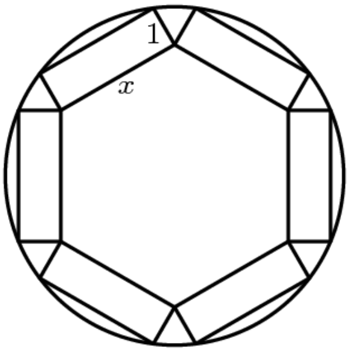
(A) 4 and 5 (B) 5 and 6 (C) 6 and 7 (D) 7 and 8 (E) 8 and 9



E. Regular Polygons

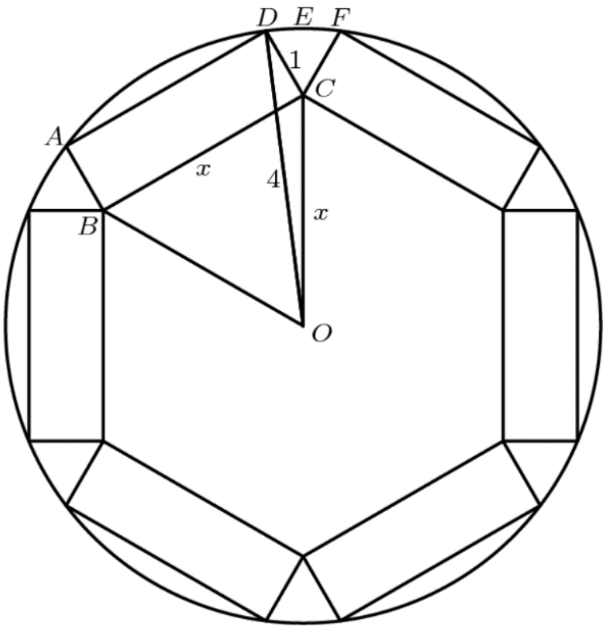
Example 2.182

A round table has radius 4. Six rectangular place mats are placed on the table. Each place mat has width 1 and length x as shown. They are positioned so that each mat has two corners on the edge of the table, these two corners being end points of the same side of length x. Further, the mats are positioned so that the inner corners each touch an inner corner of an adjacent mat. What is x? **(AMC 10A 2008/25)**[5]



Let the circle have center O.

$$OD = Radius = 4$$



BC forms one side of a regular hexagon.

$$Each\ angle\ of\ Regular\ Hexagon = 120°$$
$$\Delta BCO\ is\ equilateral \Rightarrow BC = OC = x$$

By Angles around a point:

$$\angle DCF = 360 - 90 - 90 - 120 = 60°$$

Also, $CD = CF = 1$

$$\Rightarrow \Delta CFD\ is\ Isosceles$$

Since the base angles of an Isosceles triangle are equal:

$$\angle CDF = \angle CFD = \frac{180 - 60}{2} = 60°$$

Hence, $\Delta CFD\ is\ equilateral$

$$\Rightarrow DF = 1, CE = Height = \frac{\sqrt{3}}{2} \times 1 = \frac{\sqrt{3}}{2}$$

In ΔODF, draw $\perp$ from O intersecting DF at E (and passing through C).

$$DE = \frac{1}{2}DF = \frac{1}{2} \times 1 = \frac{1}{2}$$

In right ΔOED, by Pythagoras Theorem,

$$OE = \sqrt{OD^2 - DE^2} = \sqrt{4^2 - \left(\frac{1}{2}\right)^2} = \sqrt{16 - \frac{1}{4}} = \sqrt{\frac{63}{4}} = \frac{3\sqrt{7}}{2}$$

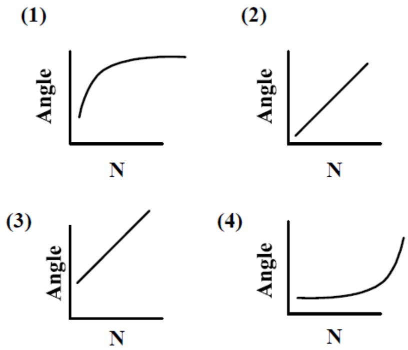
---

[5] This can be solved faster if you know Trigonometry. Look for this question in the Section on Law of Cosines in the Trigonometry Notes.

$$x = OC = \underbrace{\frac{3\sqrt{7}}{2}}_{OE} - \underbrace{\frac{\sqrt{3}}{2}}_{CE} = \frac{3\sqrt{7} - \sqrt{3}}{2}$$

Example 2.183

Consider the internal angle between any two contiguous sides of the largest regular polygon of N sides drawn inside a circle. Which of the following graphs represents the internal angle between two contiguous sides as a function of N? **(JMET 2010/64)**



Consider the first few regular polygons:

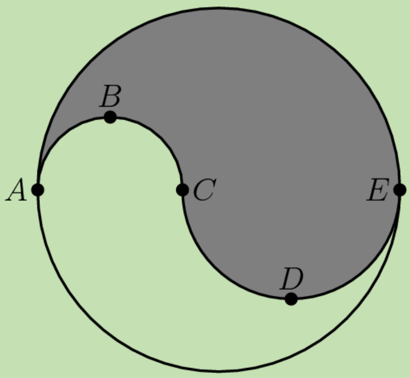
		Increase
Triangle	60°	
Quadrilateral	90°	30
Pentagon	108°	18
Hexagon	120°	12

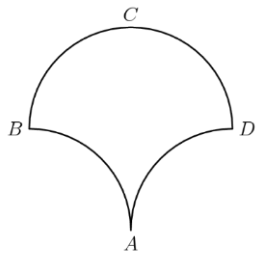
The values are increasing at a decreasing rate, which is represented in Graph (1).
Also, the values will never cross 360°.

F. Circles

Example 2.184

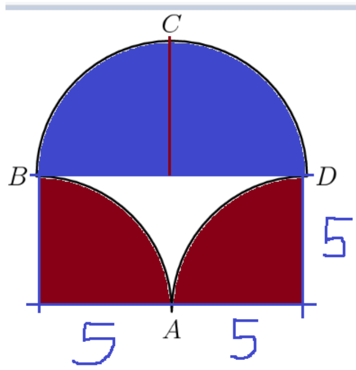
Diameter ACE is divided at C in the ratio $2:3$. The two semicircles, ABC and CDE, divide the circular region into an upper (shaded) region and a lower region. The ratio of the area of the upper region to that of the lower region is **(AMC 8 1997/24)**





Example 2.185

Three circular arcs of radius 5 units bound the region shown. Arcs AB and AD are quarter-circles, and arc BCD is a semicircle. What is the area, in square units, of the region? **(AMC 8 2000/19)**



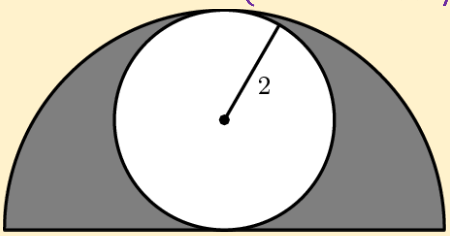
Note that the semicircle can be cut into two quarter circles and fill exactly the remaining gap in the rectangle.

Hence,

$$Area\ of\ Region = Area\ of\ Rectangle = 5(10) = 50$$

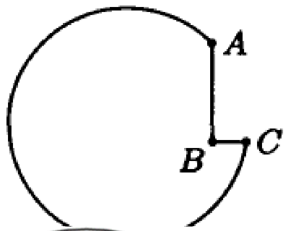
Example 2.186

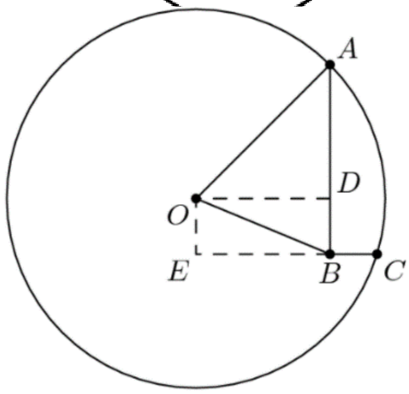
A circle of radius 2 is inscribed in a semicircle, as shown. The area inside the semicircle but outside the circle is shaded. What fraction of the semicircle's area is shaded? **(AMC 10A 2009/6)**



Example 2.187

A machine-shop cutting tool has the shape of a notched circle, as shown. The radius of the circle is $\sqrt{50}$ cm, the length of AB is 6 cm and that of BC is 2 cm. The angle ABC is a right angle. Find the square of the distance (in centimeters) from B to the center of the circle. **(AIME 1983/4)**





Shortcut

We want to find $OB^2 = OE^2 + EB^2$. By observation:

$$x = OE = 1, \qquad y = EB = 5$$

Algebraic Method

Set up equations using the Pythagorean Theorem:

$$In\ \Delta AOD: OA^2 = OD^2 + AD^2 \Rightarrow \underbrace{\left(\sqrt{50}^2\right) = y^2 + (6-x)^2}_{Equation\ I}$$

$$In\ \Delta \text{OEC}: OC^2 = OE^2 + EC^2 \Rightarrow \underbrace{\left(\sqrt{50}^2\right) = x^2 + (y+2)^2}_{Equation\ II}$$

Since the LHS of both Eq. I and II is equal, equate the RHS of both equations:

$$y^2 + 36 - 12x + x^2 = x^2 + y^2 + 4y + 4 \Rightarrow 32 - 12x = 4y \Rightarrow \underbrace{8 - 3x = y}_{Equation\ III}$$

Substitute the value of y from Equation III in Equation I:

$$\left(\sqrt{50}^2\right) = (8-3x)^2 + (6-x)^2$$
$$50 = 64 - 48x + 9x^2 + 36 - 12x + x^2$$
$$0 = 10x^2 - 60x + 50$$
$$0 = x^2 - 6x + 5$$
$$0 = (x-6)(x-1)$$
$$x \in \{1,5\}$$

If $x = 5$:

$$y = 7\ from\ Eq.I, y = 3\ from\ Eq.II \Rightarrow \boldsymbol{Contradiction}$$

If $x = 1$:

$$y = 5\ from\ both\ Eq.I\ and\ Eq.II \Rightarrow \boldsymbol{Works}$$

Hence, the final answer is:

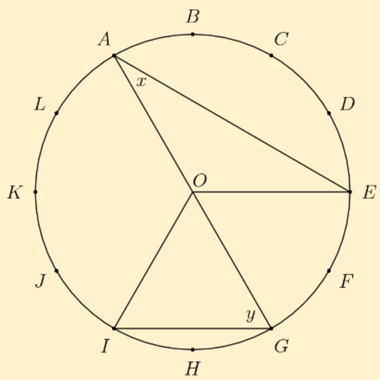
$$x^2 + y^2 = 1^2 + 5^2 = 1 + 25 = 26$$

2.9 Exam Questions

A. Angles

Example 2.188

The circumference of the circle with center O is divided into 12 equal arcs, marked the letters A through L as seen below. What is the number of degrees in the sum of the angles x and y? **(AMC 8 2014/15)**



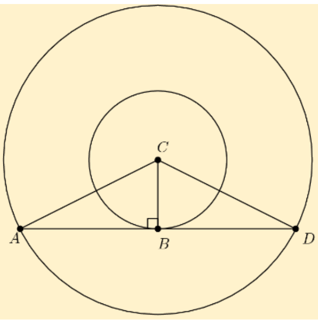
Example 2.189

Riders on a Ferris wheel travel in a circle in a vertical plane. A particular wheel has radius 20 feet and revolves at the constant rate of one revolution per minute. How many seconds does it take a rider to travel from the bottom of the wheel to a point 10 vertical feet above the bottom? **(AMC 10B 2002/24)**

B. Area

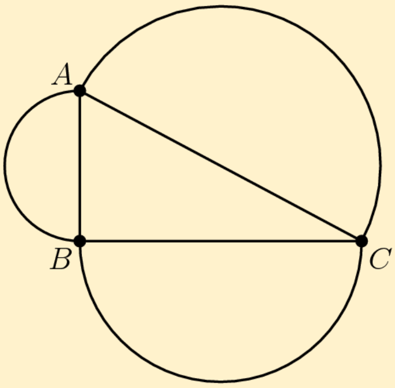
Example 2.190

The two circles pictured have the same center C. Chord $\overline{AD}$ is tangent to the inner circle at B, AC is 10, and chord $\overline{AD}$ has length 16. What is the area between the two circles? **(AMC 8 2010/19)**



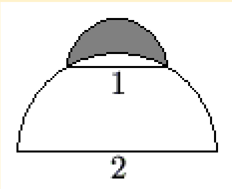
Example 2.191

Angle ABC of $\triangle ABC$ is a right angle. The sides of $\triangle ABC$ are the diameters of semicircles as shown. The area of the semicircle on $\overline{AB}$ equals 8π, and the arc of the semicircle on $\overline{AC}$ has length 8.5π. What is the radius of the semicircle on $\overline{BC}$? **(AMC 8 2013/23)**



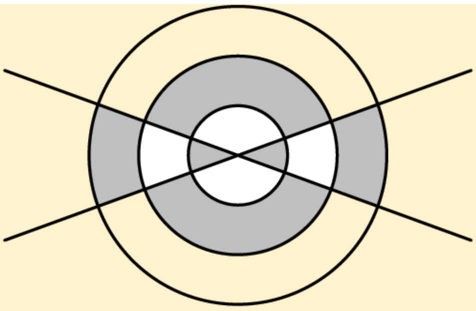
Example 2.192

A semicircle of diameter 1 sits at the top of a semicircle of diameter 2, as shown. The shaded area inside the smaller semicircle and outside the larger semicircle is called a lune. Determine the area of this lune. **(AMC 10A 2003/19)**



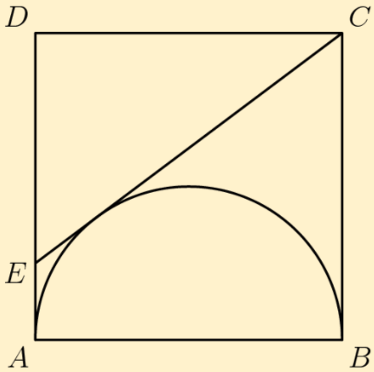
Example 2.193

Two distinct lines pass through the center of three concentric circles of radii 3, 2, and 1. The area of the shaded region in the diagram is $\frac{8}{13}$ of the area of the unshaded region. What is the radian measure of the acute angle formed by the two lines? (Note: π radians is 180 degrees.) **(AMC 10A 2004/21)**



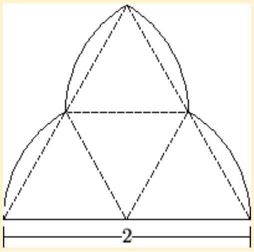
Example 2.194

Square $ABCD$ has side length 2. A semicircle with diameter $\overline{AB}$ is constructed inside the square, and the tangent to the semicircle from C intersects side $\overline{AD}$ at E. What is the length of $\overline{CE}$?. **(AMC 10A 2004/22)**



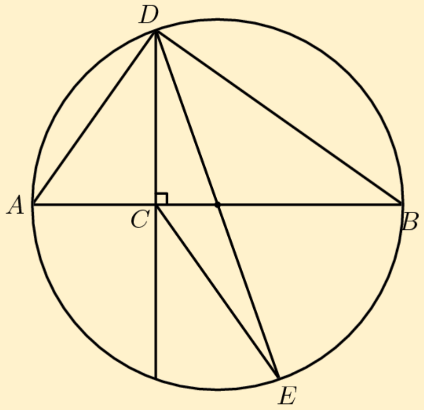
Example 2.195

The figure shown is called a trefoil and is constructed by drawing circular sectors about the sides of the congruent equilateral triangles. What is the area of a trefoil whose horizontal base has length 2?
(AMC 10A 2004/12)



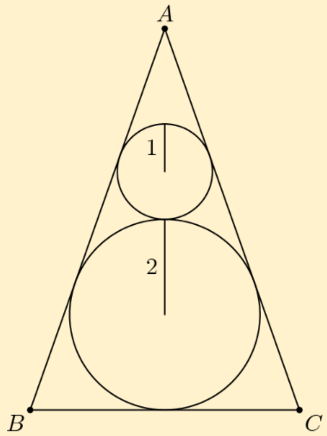
Example 2.196

Let AB be a diameter of a circle and let C be a point on AB with $2 \cdot AC = BC$. Let D and E be points on the circle such that $DC \perp AB$ and DE is a second diameter. What is the ratio of the area of $\triangle DCE$ to the area of $\triangle ABD$?
(AMC 10A 2005/23)



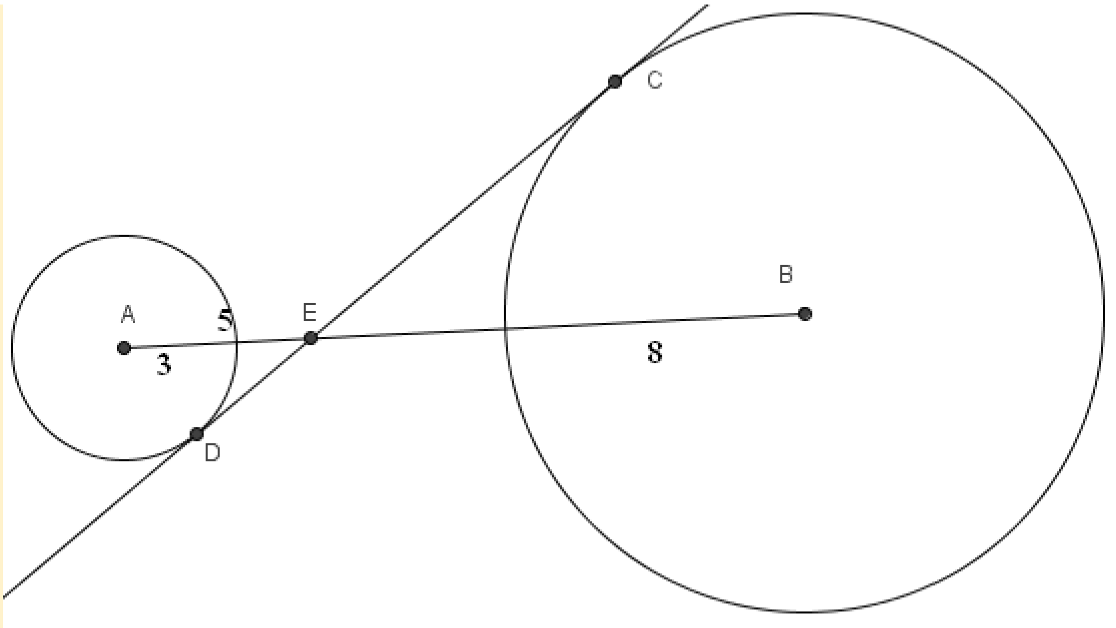
Example 2.197

A circle of radius 1 is tangent to a circle of radius 2. The sides of $\triangle ABC$ are tangent to the circles as shown, and the sides $\overline{AB}$ and $\overline{AC}$ are congruent. What is the area of $\triangle ABC$? **(AMC 10A 2006/16)**



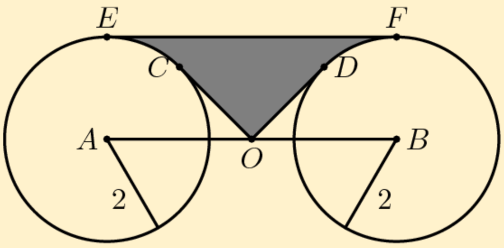
Example 2.198

Circles with centers A and B have radii 3 and 8, respectively. A common internal tangent intersects the circles at C and D, respectively. Lines AB and CD intersect at E, and $AE = 5$. What is CD? **(AMC 10A 2006/23)**



Example 2.199

Circles centered at A and B each have radius 2, as shown. Point O is the midpoint of $\overline{AB}$, and $OA = 2\sqrt{2}$. Segments OC and OD are tangent to the circles centered at A and B, respectively, and EF is a common tangent. What is the area of the shaded region $ECODF$? (AMC 10A 2007/24)



Example 2.200

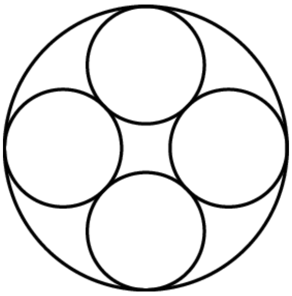
Points A and B lie on a circle centered at O, and $\angle AOB = 60°$. A second circle is internally tangent to the first and tangent to both $\overline{OA}$ and $\overline{OB}$. What is the ratio of the area of the smaller circle to that of the larger circle? (AMC 10A 2008/16)

Example 2.201

An equilateral triangle has side length 6. What is the area of the region containing all points that are outside the triangle but not more than 3 units from a point of the triangle? (AMC 10A 2008/17)

Example 2.202

Circle A has radius 100. Circle B has an integer radius $r < 100$ and remains internally tangent to circle A as it rolls once around the circumference of circle A. The two circles have the same points of tangency at the beginning and end of circle B's trip. How many possible values can r have? **(AMC 10A 2009/19)**



Example 2.203

Many Gothic cathedrals have windows with portions containing a ring of congruent circles that are circumscribed by a larger circle. In the figure shown, the number of smaller circles is four. What is the ratio of the sum of the areas of the four smaller circles to the area of the larger circle? **(AMC 10A 2009/21)**

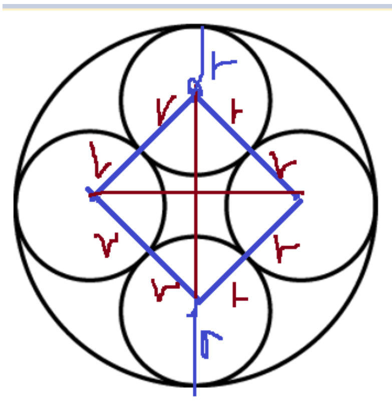
Connect the centers of the smaller circles. The quadrilateral so formed is a square.

The diagonal of the square

$$= \sqrt{2}(Side\ Length) = \sqrt{2}(2r) = 2\sqrt{2}r$$

The radius of the outer circle

$$R = \sqrt{2}r + r = r(\sqrt{2}+1)$$



The ratio that we need is:

$$\frac{4 \times Smaller}{Larger} = \frac{4\pi r^2}{\pi R^2} = \frac{4r^2}{r^2(\sqrt{2}+1)^2} = \frac{4}{3+2\sqrt{2}}$$

$$\frac{4}{3+2\sqrt{2}} \times \frac{3-2\sqrt{2}}{3-2\sqrt{2}} = \frac{4(3-2\sqrt{2})}{9-8} = 4(3-2\sqrt{2})$$

Example 2.204

The area of a circle whose circumference is 24π is $k\pi$. What is the value of k? **(AMC 10A 2010/5)**

Example 2.205

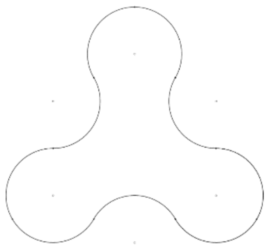
Circles A, B, and C each have radius 1. Circles A and B share one point of tangency. Circle C has a point of tangency with the midpoint of $\overline{AB}$. What is the area inside Circle C but outside circle A and circle B ? **(AMC 10A 2011/18)**

Example 2.206

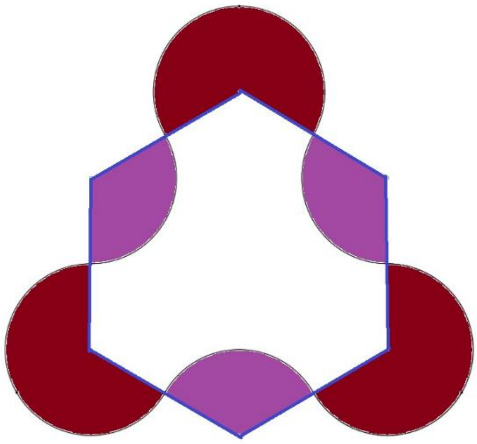
Externally tangent circles with centers at points A and B have radii of lengths 5 and 3, respectively. A line externally tangent to both circles intersects ray AB at point C. What is BC? **(AMC 10A 2012/11)**

Example 2.207

The closed curve in the figure is made up of 9 congruent circular arcs each of length $\frac{2\pi}{3}$, where each of the centers of the corresponding circles is among the vertices of a regular hexagon of side 2. What is the area enclosed by the curve? **(AMC 10A 2012/18)**



Complete the hexagon.



Area of hexagon

$$= (6)\left(\frac{\sqrt{3}}{4}\times 2^2\right) = 6\sqrt{3}$$

Add the areas outside, and subtract the areas inside:

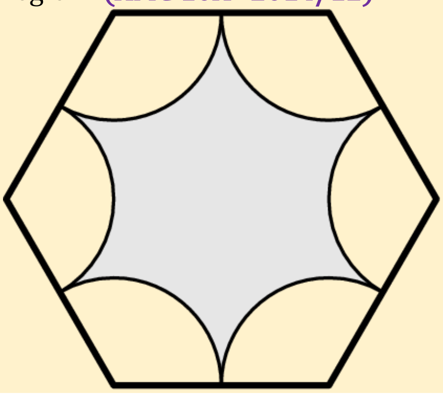
$$= 6\sqrt{3} + 3\left(\frac{2}{3}\times \pi(1^2)\right) - 3\left(\frac{1}{3}\times \pi(1^2)\right) = 6\sqrt{3} + \pi$$

Example 2.208

In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC? **(AMC 10A 2013/23)**

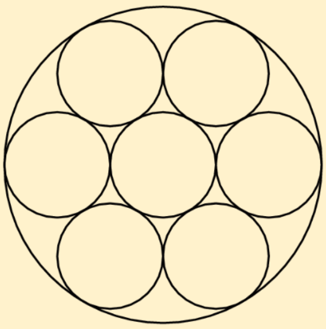
Example 2.209

A regular hexagon has side length 6. Congruent arcs with radius 3 are drawn with the center at each of the vertices, creating circular sectors as shown. The region inside the hexagon but outside the sectors is shaded as shown. What is the area of the shaded region? **(AMC 10A 2014/12)**



Example 2.210

Seven cookies of radius 1 inch are cut from a circle of cookie dough, as shown. Neighboring cookies are tangent, and all except the center cookie are tangent to the edge of the dough. The leftover scrap is reshaped to form another cookie of the same thickness. What is the radius in inches of the scrap cookie? **(AMC 10A 2016/15)**



Example 2.211

Circles with centers P, Q and R, having radii 1, 2 and 3, respectively, lie on the same side of line l and are tangent to l at P', Q' and R', respectively, with Q' between P' and R'. The circle with center Q is externally tangent to each of the other two circles. What is the area of triangle PQR? **(AMC 10A 2016/21)**

Example 2.212

A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of the fourth side? **(AMC 10A 2016/24)**

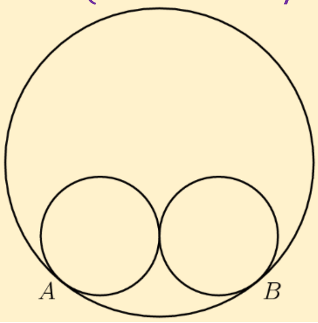
Example 2.213

Sides $\overline{AB}$ and $\overline{AC}$ of equilateral triangle ABC are tangent to a circle at points B and C respectively. What fraction of the area of $\triangle ABC$ lies outside the circle? **(AMC 10A 2017/22)**

$(A)\ \frac{4\sqrt{3}\pi}{27}-\frac{1}{3}$ $\quad (B)\ \frac{\sqrt{3}}{2}-\frac{\pi}{8}$ $\quad (C)\ \frac{1}{2}$ $\quad (D)\ \sqrt{3}-\frac{2\sqrt{3}\pi}{9}$ $\quad (E)\ \frac{4}{3}-\frac{4\sqrt{3}\pi}{27}$

Example 2.214

Two circles of radius 5 are externally tangent to each other and are internally tangent to a circle of radius 13 at points A and B, as shown in the diagram. The distance AB can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$? **(AMC 10A 2018/15)**

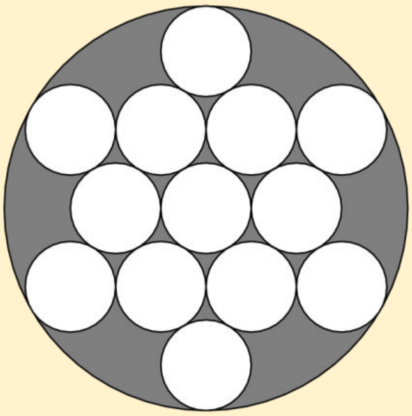


Example 2.215

Let $\triangle ABC$ be an isosceles triangle with $BC = AC$ and $\angle ACB = 40°$. Construct the circle with diameter $\overline{BC}$, and let D and E be the other intersection points of the circle with the sides $\overline{AC}$ and $\overline{AB}$, respectively. Let F be the intersection of the diagonals of the quadrilateral $BCDE$. What is the degree measure of $\angle BFC$? **(AMC 10A 2019/13)**

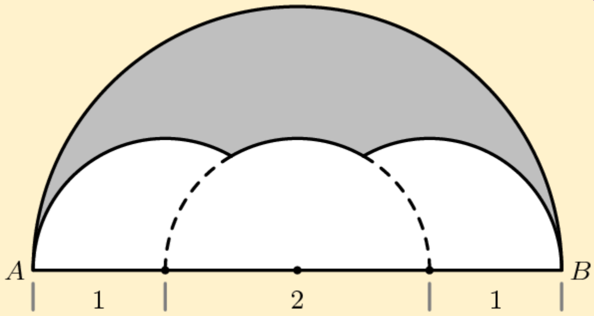
Example 2.216

The figure below shows 13 circles of radius 1 within a larger circle. All the intersections occur at points of tangency. What is the area of the region, shaded in the figure, inside the larger circle but outside all the circles of radius 1 ? **(AMC 10A 2019/16)**



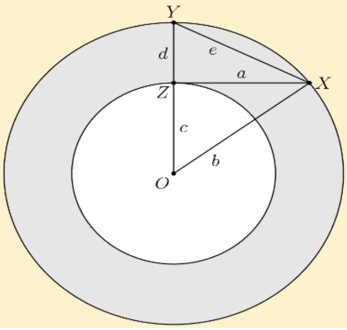
Example 2.217

Three semicircles of radius 1 are constructed on diameter $\overline{AB}$ of a semicircle of radius 2. The centers of the small semicircles divide $\overline{AB}$ into four line segments of equal length, as shown. What is the area of the shaded region that lies within the large semicircle but outside the smaller semicircles? **(AMC 10B 2003/19)**



Example 2.218

An annulus is the region between two concentric circles. The concentric circles in the figure have radii b and c, with $b > c$. Let OX be a radius of the larger circle, let XZ be tangent to the smaller circle at Z, and let OY be the radius of the larger circle that contains Z. Let $a = XZ$, $d = YZ$, and $e = XY$. What is the area of the annulus? **(AMC 10B 2004/12)**

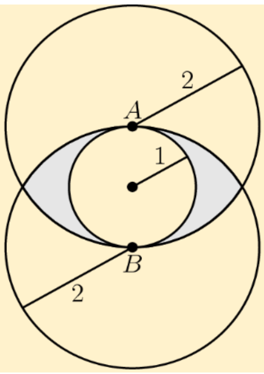


Example 2.219

Three circles of radius 1 are externally tangent to each other and internally tangent to a larger circle. What is the radius of the large circle? **(AMC 10B 2004/16)**

Example 2.220

A circle of radius 1 is internally tangent to two circles of radius 2 at points A and B, where AB is a diameter of the smaller circle. What is the area of the region, shaded in the picture, that is outside the smaller circle and inside each of the two larger circles? **(AMC 10B 2004/25)**

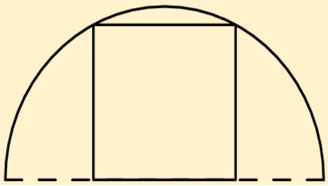


Example 2.221
A circle is inscribed in a square, then a square is inscribed in this circle, and finally, a circle is inscribed in this square. What is the ratio of the area of the smaller circle to the area of the larger square? **(AMC 10B 2005/7)**

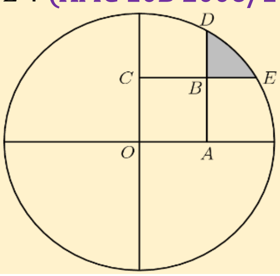
Example 2.222
An 8-foot by 10-foot floor is tiled with square tiles of size 1 foot by 1 foot. Each tile has a pattern consisting of four white quarter circles of radius 1/2 foot centered at each corner of the tile. The remaining portion of the tile is shaded. How many square feet of the floor are shaded? **(AMC 10B 2005/8)**

Example 2.223
Circles of diameter 1 inch and 3 inches have the same center. The smaller circle is painted red, and the portion outside the smaller circle and inside the larger circle is painted blue. What is the ratio of the blue-painted area to the red-painted area? **(AMC 10B 2006/4)**

Example 2.224
A square of area 40 is inscribed in a semicircle as shown. What is the area of the semicircle? **(AMC 10B 2006/8)**

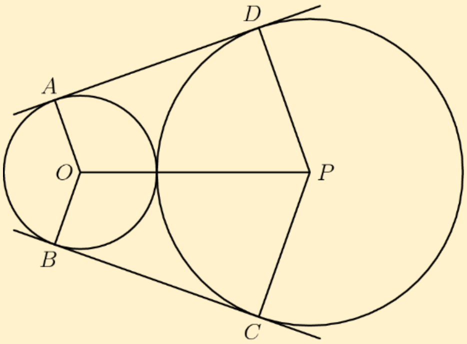


Example 2.225
A circle of radius 2 is centered at O. Square $OABC$ has side length 1. Sides AB and CB are extended past B to meet the circle at D and E, respectively. What is the area of the shaded region in the figure, which is bounded by BD, BE, and the minor arc connecting D and E ? **(AMC 10B 2006/19)**



Example 2.226
Circles with centers O and P have radii 2 and 4, respectively, and are externally tangent. Points A and B on the circle with center O and points C and D on the circle with center P are such that AD and BC are common

external tangents to the circles. What is the area of the concave hexagon $AOBCPD$? (AMC 10B 2006/24)

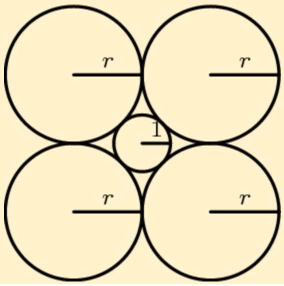


Example 2.227

A circle passes through the three vertices of an isosceles triangle that has two sides of length 3 and a base of length 2. What is the area of this circle? (AMC 10B 2007/11)

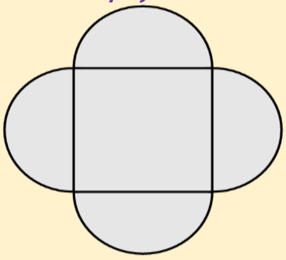
Example 2.228

A circle of radius 1 is surrounded by 4 circles of radius r as shown. What is r? (AMC 10B 2007/18)



Example 2.229

A region is bounded by semicircular arcs constructed on the side of a square whose sides measure $\frac{2}{\pi}$, as shown. What is the perimeter of this region? (AMC 10B 2006/6)



Example 2.230

Points A and B are on a circle of radius 5 and $AB = 6$. Point C is the midpoint of the minor arc AB. What is the length of the line segment AC? (AMC 10B 2008/10)

Example 2.231

Points A and C lie on a circle centered at O, each of $\overline{BA}$ and $\overline{BC}$ are tangent to the circle, and $\triangle ABC$ is equilateral. The circle intersects $\overline{BO}$ at D. What is $\frac{BD}{BO}$? (AMC 10B 2009/16)

Example 2.232

A circle is centered at O, $\overline{AB}$ is a diameter and C is a point on the circle with $\angle COB = 50°$. What is the degree measure of $\angle CAB$? (AMC 10B 2010/6)

Example 2.233

A square of side length 1 and a circle of radius $\frac{\sqrt{3}}{3}$ share the same center. What is the area inside the circle, but outside the square? **(AMC 10B 2010/16)**

Example 2.234

A circle with center O has area 156π. Triangle ABC is equilateral, $\overline{BC}$ is a chord on the circle, $OA = 4\sqrt{3}$, and point O is outside $\triangle ABC$. What is the side length of $\triangle ABC$? **(AMC 10B 2010/19)**

Example 2.235

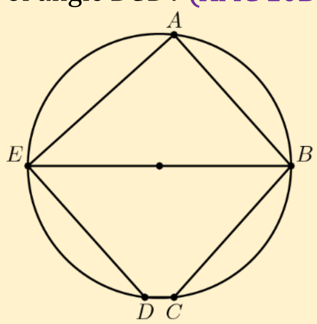
Two circles lie outside regular hexagon $ABCDEF$. The first is tangent to $\overline{AB}$, and the second is tangent to $\overline{DE}$. Both are tangent to lines BC and FA. What is the ratio of the area of the second circle to that of the first circle? **(AMC 10B 2010/20)**

Example 2.236

Keiko walks once around a track at exactly the same constant speed every day. The sides of the track are straight, and the ends are semicircles. The track has a width of 6 meters, and it takes her 36 seconds longer to walk around the outside edge of the track than around the inside edge. What is Keiko's speed in meters per second?**AMC 10B 2011/12)**

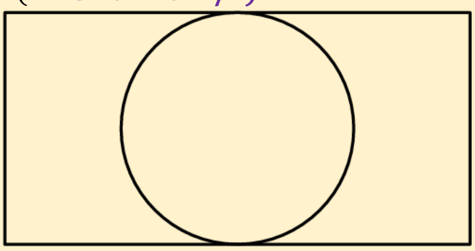
Example 2.237

In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio $4 : 5$. What is the degree measure of angle BCD? **(AMC 10B 2011/17)**



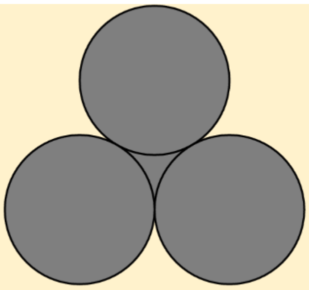
Example 2.238

A circle of radius 5 is inscribed in a rectangle as shown. The ratio of the length of the rectangle to its width is 2:1. What is the area of the rectangle? (**AMC 10B 2012/2)**



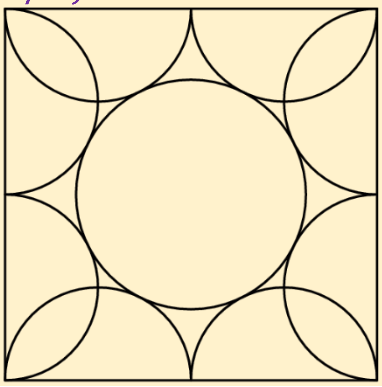
Example 2.239

Three circles with radius 2 are mutually tangent. What is the total area of the circles and the region bounded by them, as shown in the figure? **(AMC 10B 2012/16)**



Example 2.240
Six points are equally spaced around a circle of radius 1. Three of these points are the vertices of a triangle that is neither equilateral nor isosceles. What is the area of this triangle? (AMC 10B 2013/7)

Example 2.241
Eight semicircles line the inside of a square with side length 2 as shown. What is the radius of the circle tangent to all of these semicircles? (AMC 10B 2014/22)

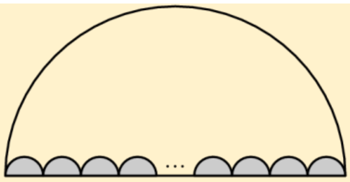


Example 2.242
In $\triangle ABC$, $\angle C = 90°$ and $AB = 12$. Squares $ABXY$ and $ACWZ$ are constructed outside of the triangle. The points X, Y, Z, and W lie on a circle. What is the perimeter of the triangle? (AMC 10B 2015/19)

Example 2.243
In $\triangle ABC$, $AB = 6$, $AC = 8$, $BC = 10$, and D is the midpoint of $\overline{BC}$. What is the sum of the radii of the circles inscribed in $\triangle ADB$ and $\triangle ADC$? (AMC 10B 2017/21)

Example 2.244
The diameter AB of a circle of radius 2 is extended to a point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and line ED is perpendicular to line AD. Segment AE intersects the circle at a point C between A and E. What is the area of $\triangle ABC$? (AMC 10B 2017/22)

Example 2.245
In the figure below, N congruent semicircles are drawn along a diameter of a large semicircle, with their diameters covering the diameter of the large semicircle with no overlap. Let A be the combined area of the small semicircles and B be the area of the region inside the large semicircle but outside the small semicircles. The ratio $A: B$ is $1: 18$. What is N? AMC 10B 2018/7)

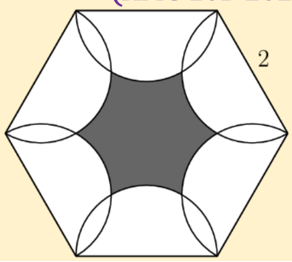


Example 2.246

Line segment $\overline{AB}$ is a diameter of a circle with $AB = 24$. Point C, not equal to A or B, lies on the circle. As point C moves around the circle, the centroid (center of mass) of $\triangle ABC$ traces out a closed curve missing two points. To the nearest positive integer, what is the area of the region bounded by this curve?**(AMC 10B 2018/12)**

Example 2.247

As shown in the figure below, six semicircles lie in the interior of a regular hexagon with side length 2 so that the diameters of the semicircles coincide with the sides of the hexagon. What is the area of the shaded region — inside the hexagon but outside all of the semicircles? **(AMC 10B 2020/14)**



2.10 Further Topics

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