Binomial Theorem

21 June 2025

Revision: 2016

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1. Binomial Theorem (BT)

1.1 Binomial Expansions

A. Basics

1.1: Binomial Expansion
An expansion of the form $(a+b)^n, n \in \mathbb{N}$ is called a binomial expansion

$$(a+b)^0 = 1, \qquad a, b \text{ not both zero}$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

- The power of a keeps decreasing with each term.
- The power of b keeps increasing with each term.

This can be made explicit by introducing a term with $a^0 = 1$, and a term with $b^0 = 1$.
$$(a+b)^4 = a^4b^0 + 4a^3b + 6a^2b^2 + 4ab^3 + a^0b^4$$

1.2: Number of Terms
The number of terms in a binomial expansion to a natural number power is one more than the power:

Example 1.3
Find the number of terms in the following expansions:
1. $(a+b)^3$
2. $(a+b)^4$
3. $(x+y)^7$
4. $(x-y)^9$
5. $\left(\frac{1}{2}x + \frac{3}{4}y\right)^{97}$
6. $\left(\frac{3}{4}x - \frac{5}{7}y\right)^{1001}$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \Rightarrow 4\ Terms$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \Rightarrow 5\ Terms$$
$$8\ Terms$$
$$10\ Terms$$
$$98\ Terms$$
$$1002\ Terms$$

Example 1.4
The number of terms in the expansion of $[(a+3b)^2(a-3b)^2]^2$ when simplified is: (AHSME 1950/16)

$$= [[(a+3b)(a-3b)]^2]^2 = (a^2-9b^2)^4 \Rightarrow 5\ Terms$$

B. Pascal's Triangle

Rule 1: The first and last number of any row are always one.
Rule 2: Every succeeding row has one number more than the previous row.
Rule 3: Any number in Pascals Triangle is the sum of the numbers that are above it.

Row	Numbers
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	**1 4 6 4 1**
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1

Row Zero

Row zero has only one number. So, write a 1 at the top.

Row One

Row One has two numbers. The first number is 1. The last number is also 1.

Row Two

Row Two has three numbers. Put a 1 at the beginning, and a 1 at the end. The middle number is the total of the two numbers above it:

$$1 + 1 = 2$$

1.5: Pascal's Triangle

Consider a binomial expansion:

$$(a+b)^{4} = 1a^4b^0 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4a^0$$

The coefficients of the RHS are also the numbers in the corresponding row of Pascal's Triangle. This is very useful in generating the coefficients of a binomial expansion.

(Pascal's Triangle also has many other applications and connections in Maths).

Example 1.6

Expand the following using Pascal's Triangle:

$$(x+y)^5$$

Take the coefficients from Row 5 of Pascal's Triangle. Write the variables in descending powers of x, and ascending powers of y.

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

1.7: Patterns in Pascal's Triangle

(See this link for an interactive visualization of the patterns in Pascal's Triangle mentioned here, along with a few additional ones).

Ones
Natural Numbers
Triangular Numbers
Tetrahedral Numbers
Serpinski Triangle

Row	Numbers
0	$\binom{0}{0}$
1	$\binom{1}{0}$ $\binom{1}{1}$
2	$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$
3	$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

1.8: Pascal's Triangle as Binomial Coefficients

The numbers in Pascals Triangle can also be represented as Binomial Coefficients.
This way of looking at them gives insight into some of the patterns that we find there.

4	$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$
5	$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$
n	$\binom{n}{0}\ \binom{n}{1} \dots \binom{n}{n-1}\ \binom{n}{n}$

$$\binom{3}{0} = \frac{3!}{0!\,3!} = \frac{3!}{1!\,3!} = 1$$
$$\binom{3}{1} = \frac{3!}{1!\,2!} = 3$$
$$\binom{3}{2} = \frac{3!}{2!\,1!} = 3$$
$$\binom{3}{3} = \frac{3!}{3!\,0!} = 1$$

First Number in n^{th} Row

$$\binom{n}{0} = 1$$

Because it represents the number of ways of choosing zero objects from n objects, which can be done in exactly one way.
From Pascal's Triangle point of view, the first number in any row is always one.

Second Number in the Row

$$\binom{n}{1} = n$$

Because it represents the number of ways of choosing one object from n objects, which can be done in exactly n ways.
From Pascal's Triangle point of view, the second number is any row is always the row number.
The numbers in the second row give the sequence of natural numbers.

C. Binomial Formula

1.9: Binomial Formula

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \dots + \binom{n}{n} x^0 y^n$$

The power of x keeps decreasing with each term. The power of y keeps increasing with each term.

How is Counting related to the Binomial Formula:

$$(a+b)^2 = (a+b)(a+b) = a(a+b) + b(a+b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$$

We don't need a separate formula if the expression has a negative sign. Remember that:

- Odd powers of -1 are -1
- Even powers of -1 are 1

And hence the answer will alternate between positive and negative terms, but otherwise the same formula applies.
(In fact, if you include the minus sign as part of a term, then the formula applies without any change).

Example 1.10

Expand the following using Binomial Theorem:

A. $(x+y)^5$
B. $(x+y)^6$
C. $(2x-3y)^4$
D. $\left(2+\frac{x}{7}\right)^4$
E. $\left(\frac{2x}{3}-\frac{3y}{4}\right)^4$
F. $\left(x^2-\frac{2}{x^3}\right)^5$

Part A

$$(x+y)^5=\binom{5}{0}x^5y^0+\binom{5}{1}x^4y^1+\binom{5}{2}x^3y^2+\binom{5}{3}x^2y^3+\binom{5}{4}x^1y^4+\binom{5}{5}x^0y^5$$
$$=x^5y^0+5x^4y^1+10x^3y^2+10x^2y^3+5x^1y^4+x^0y^5$$

Part B

$$(x+y)^6=\binom{6}{0}x^6y^0+\binom{6}{0}x^5y^1+\binom{6}{0}x^4y^2+\binom{6}{0}x^3y^3+\binom{6}{0}x^2y^4+\binom{6}{0}x^6y^0+\binom{6}{0}x^6y^0$$

Part C

Remember to alternate negative signs:

$$=1(2x)^4(-3y)^0+4(2x)^3(-3y)^1+6(2x)^2(-3y)^2+4(2x)^1(-3y)^2+1(2x)^0(-3y)^4$$
$$=16x^4-96x^3y+216x^2y^2-216xy^3+81y^4$$

Part D

Fractions in the expression can be handled in the usual way by taking powers of fractions. Be careful with cancellation.

$$(x+y)^n=\binom{n}{0}x^ny^0+\binom{n}{1}x^{n-1}y^1+\binom{n}{2}x^{n-2}y^2+\cdots+\binom{n}{n}x^0y^n$$

In the above formula, substitute $n=4, x=2, y=\frac{x}{7}$:

$$First\ Term=\binom{4}{0}2^4\left(\frac{x}{7}\right)^0=\frac{4!}{0!\,4!}\times16\times1=1\times16=16$$
$$Second\ Term=\binom{4}{1}2^3\left(\frac{x}{7}\right)^1=\frac{4!}{1!\,3!}\times8\times\frac{x}{7}=\frac{32x}{7}$$
$$Third\ Term=\binom{4}{2}2^2\left(\frac{x}{7}\right)^2=\frac{4!}{2!\,2!}\times4\times\frac{x^2}{49}=\frac{24x^2}{49}$$
$$Fourth\ Term=\binom{4}{3}2^1\left(\frac{x}{7}\right)^3=\frac{4!}{1!\,3!}\times2\times\frac{x^3}{343}=\frac{8x^3}{343}$$
$$Fifth\ Term=\binom{4}{4}2^0\left(\frac{x}{7}\right)^4=1\times1\times\frac{x^4}{2401}$$

We collate the terms to get the final answer:

$$16+\frac{32x}{7}+\frac{24x^2}{49}+\frac{8x^3}{343}+\frac{x^4}{2401}$$

Part E

$$First\ Term=\binom{4}{0}\left(\frac{2x}{3}\right)^4\left(-\frac{3y}{4}\right)^0=1\times\frac{16x^4}{81}=\frac{16x^4}{81}$$
$$Second\ Term=\binom{4}{1}\left(\frac{2x}{3}\right)^3\left(-\frac{3y}{4}\right)^1=4\times\frac{8x^3}{27}\times\left(-\frac{3y}{4}\right)=-\frac{8x^3y}{9}$$
$$Third\ Term=\binom{4}{2}\left(\frac{2x}{3}\right)^2\left(-\frac{3y}{4}\right)^2=6\times\frac{4x^2}{9}\times\frac{9y^2}{16}=\frac{3x^2y^2}{2}$$

$$Fourth\ Term = \binom{4}{3}\left(\frac{2x}{3}\right)^1\left(-\frac{3y}{4}\right)^3 = 4 \times \frac{2x}{3} \times \left(-\frac{27y^3}{64}\right) = -\frac{9xy^3}{8}$$

$$Fifth\ Term = \binom{4}{4}\left(\frac{2x}{3}\right)^0\left(-\frac{3y}{4}\right)^4 = 1 \times 1 \times \frac{81y^4}{256}$$

Put all the terms together:

$$\left(\frac{2x}{3} - \frac{3y}{4}\right)^4 = \frac{16x^4}{81} - \frac{8x^3y}{9} + \frac{3x^2y^2}{2} - \frac{9xy^3}{8} + \frac{81y^4}{256}$$

Part F

$$(x+y)^n = \binom{n}{0}x^ny^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0y^n$$

Substitute $n = 5, x = x^2, y = -\frac{2}{x^3}$:

$$\binom{5}{0}(x^2)^5\left(-\frac{2}{x^3}\right)^0$$
$$\binom{5}{1}(x^2)^4\left(-\frac{2}{x^3}\right)^1$$
$$\binom{5}{2}(x^2)^3\left(-\frac{2}{x^3}\right)^2$$
$$\binom{5}{3}(x^2)^2\left(-\frac{2}{x^3}\right)^3$$
$$\binom{5}{4}(x^2)^1\left(-\frac{2}{x^3}\right)^4$$
$$\binom{5}{5}(x^2)^0\left(-\frac{2}{x^3}\right)^5$$

$$x^{10} - 10x^5 + 40 - \frac{80}{x^5} + \frac{80}{x^{10}} - \frac{32}{x^{15}}$$

Note that from left to right, the power of x decreases by 5.

Example 1.11

Expand $\left(3x + \frac{y}{2}\right)^4$

$$1\,(3x)^4\left(\frac{y}{2}\right)^0 = 81x^4$$

$$4\,(3x)^3\left(\frac{y}{2}\right)^1 = 54x^3y$$

$$6(3x)^2\left(\frac{y}{2}\right)^2 = 6 \cdot 9x^2 \cdot \frac{y^2}{4} = \frac{27}{2}x^2y^2$$

$$4(3x)^1\left(\frac{y}{2}\right)^3 = 4 \cdot 3x \cdot \frac{y^3}{8} = \frac{3}{2}xy^3$$

$$1(3x)^0\left(\frac{y}{2}\right)^4 = \frac{y^4}{16}$$

$$81x^4 + 54x^3y + \frac{27}{2}x^2y^2 + \frac{3}{2}xy^3 + \frac{y^4}{16}$$

Example 1.12: Expanding Radicals

Expand

A. $\left(3+\sqrt{7}\right)^3$

B. $\left(2+\sqrt{5}\right)^4$

Part A

Substitute $a=3, b=\sqrt{7}$ in $(a+b)^3=a^3+3a^2b+3ab^2+b^3$

$$\left(3+\sqrt{7}\right)^3=3^3+(3)(3^2)\left(\sqrt{7}\right)+(3)(3)\left(\sqrt{7}\right)^2+\left(\sqrt{7}\right)^3$$

$$=27+27\sqrt{7}+63+7\sqrt{7}$$

$$=90+34\sqrt{7}$$

Part B

$$\left(2+\sqrt{5}\right)^4=16+(4)(8)\left(\sqrt{5}\right)+(6)(4)(5)+(4)(2)\left(5\sqrt{5}\right)+25=161+72\sqrt{5}$$

1.2 Specific Terms

A. General Term

1.13: N^{th} Term

$(r+1)^{st}$ term of $(x+y)^n$ is

$$\binom{n}{r}x^{n-r}y^r$$

In the first term of $(x+y)^7$

- r is one less than 1: $r=0$
- n is the power: $n=7$
-

$$(x+y)^n=\underbrace{\binom{n}{0}x^ny^0}_{First\ Term}+\underbrace{\binom{n}{1}x^{n-1}y^1}_{Second\ Term}+\binom{n}{2}x^{n-2}y^2+\cdots+\binom{n}{n}x^0y^n$$

Example 1.14

Find the y^4 term of $(4y-1)^5$.

The first term will have power

$$y^5$$

The second term will have power

$$y^4$$

Hence, we want to find the second term:

$$r+1=2\Rightarrow r=1$$
$$x=4y$$
$$y=-1$$
$$n=5$$

$$\binom{n}{r}x^{n-r}y^r=\binom{5}{1}(4y)^{5-1}(-1)^1=-256$$

Example 1.15

In this example, we look at finding terms from the beginning of an expansion. Find the:

A. 7^{th} term of $(a+b)^{10}$
B. 4^{th} term of $\left(\frac{r}{2}+2s\right)^6$
C. 3^{rd} term of $(m^2+n^3)^7$
D. 3^{rd} term of $\left(q+\frac{2}{q^3}\right)^5$
E. 5^{th}term of $\left(\frac{2a}{9}+\frac{3}{4q^3}\right)^6$

Part A

$$n=10, r=6 \Rightarrow \binom{n}{r}x^{n-r}y^r=\binom{10}{6}a^{10-6}b^6=210a^4b^6$$

Part B

$$n=6, r=3 \Rightarrow \binom{n}{r}x^{n-r}y^r=\binom{6}{3}\left(\frac{r}{2}\right)^{6-3}(2s)^3=20r^3s^3$$

Part C

$$n=7, r=2 \Rightarrow \binom{n}{r}x^{n-r}y^r=\binom{7}{2}x^{n-r}y^r=21(m^2)^{7-2}(n^3)^2=$$

Part D

$$n=5, r=2 \Rightarrow \binom{n}{r}x^{n-r}y^r=\binom{5}{2}q^{5-2}\left(\frac{2}{q^3}\right)^2=10\times q^3\times\frac{4}{q^6}=\frac{40q^3}{q^3\times q^3}=\frac{40}{q^3}$$

Part E

Substitute $n=6, r=4$ in $\binom{n}{r}x^{n-r}y^r$:

$$\binom{6}{4}\left(\frac{2a}{9}\right)^{6-4}\left(\frac{3}{4q^3}\right)^4=15\left(\frac{2a}{9}\right)^2\left(\frac{3}{4q^3}\right)^4=15\left(\frac{4a^2}{81}\right)\left(\frac{81}{256q^{12}}\right)=\frac{15a^2}{64q^{12}}$$

B. Term from End

Example 1.16: From End

A. 3^{rd} term from the end of $\left(q+\frac{2}{q^3}\right)^5$
B. 5^{th} term from the end of $\left(\frac{2a}{9}+\frac{3}{4q^3}\right)^6$
C. 2^{nd} Term from the end of $(a+b)^{100}$

$$\underbrace{1}_{Term}\ \underbrace{2}_{Term}\ \underbrace{3}_{Term}\ \underbrace{4}_{Term}\ \underbrace{5}_{Term}\ \underbrace{6}_{Term}$$

Part A

$3rd\ Term$ from the end

$$=6-(3-1)=6-2=4^{th}\ Term\ from\ the\ Left$$

Part B

$5th\ Term$ from the right

$$=7-(5-1)=7-4=3^{rd}\ Term\ from\ the\ Left$$

Part C

$$n = No.\,of\ Terms = 100 + 1 = 101$$
$$2nd\ Term\ from\ the\ right = 101 - (2 - 1) = 101 - 1 = 100^{th}\ term\ from\ the\ left$$

$$\binom{100}{99} ab^{99}$$

C. Middle Term

1.17: Middle Term

The middle term will depend on whether n is odd or even.

- If n is even:
  - ✓ The number of terms will be $n + 1 \Rightarrow odd$
  - ✓ there will be a single middle term
- If n is odd:
  - ✓ The number of terms will be $n + 1 \Rightarrow$ Even
  - ✓ And hence there will be two middle terms
  - ✓ The binomial coefficient will be the same in both terms, but the values coming from the variables will not be the same.

The middle term or terms has the greatest binomial coefficient across all the terms.

Example 1.18: Identifying Middle Terms

Identify the location of the middle term(s) of the following binomials, and their binomial coefficients. (You don't need to find the terms, just the term number. Similarly, do not calculate the binomial coefficient, only give the expression).

A. $\left(\sqrt{3} + \sqrt[3]{5}\right)^{18}$
B. $\left(\sqrt{5} + \sqrt[3]{3}\right)^{79}$

Recall that the number of terms of a binomial expansion is one more than the power of the expansion.

Part A

$$1,2,3, \dots 9,10,11, \dots 17,18,19$$

Term 10 has nine terms before it, and nine terms after it.
So, it is the middle term.

Using the formula:

$$\frac{19+1}{2} = \frac{20}{2} = 10 \Rightarrow Binomial\ coefficient\ is\ \binom{18}{9}$$

Part B

$$1,2, \dots ,80$$

Now we will do:

$$\frac{80+1}{2} = 40.5 \Rightarrow 40th\ and\ 41st\ Term$$
$$Binomial\ coefficients\ are\ \binom{79}{39}\ and\ \binom{79}{40}$$

D. Specific Power

Example 1.19

$$(H+T)^n$$

A. Given that $n=5$, find the term with H^3
B. Given that $n=8$, find the term with H^2
C. Given that $n=10$, find the term with H^3

Part A

$$n=5 \Rightarrow n-r=3 \Rightarrow 5-r=3 \Rightarrow r=5-3=2$$
$$(H+T)^5=\cdots+\binom{5}{2}H^3T^2+\cdots$$

Part B

$$(H+T)^8=\cdots+\binom{8}{6}H^2T^6+\cdots$$

Part B

$$(H+T)^{10}=\cdots+\binom{10}{7}H^3T^7+\cdots$$

Example 1.20

The coefficient of x^5 in the expansion of $\left(2x^3-\frac{1}{3x^2}\right)^5$ is: **(JEE-M 2023)**

$$\binom{n}{r}x^{n-r}y^r$$

Substitute $n=5, x=x^3, y=x^{-2}$ (ignore the coefficients and $\binom{n}{r}$):

$$(x^3)^{5-r}(x^{-2})^r=x^5$$
$$x^{15-3r}\cdot x^{-2r}=x^5$$
$$15-5r=5$$
$$r=2$$

Substitute $r=2$ in $\binom{n}{r}x^{n-r}y^r$:

$$\binom{5}{2}(2x^3)^{5-2}\left(-\frac{1}{3x^2}\right)^2$$

Drop the variables:

$$(10)(8)\left(\frac{1}{9}\right)=\frac{80}{9}$$

Example 1.21

$$\left(x^4-\frac{1}{x^3}\right)^{15}$$

The coefficient of x^{18} in the expansion above is: **(JEE-M 2023)**

Substitute $n=15, x=x^4, y=x^{-3}$:

$$(x^4)^{15-r}(x^{-3})^r=x^{60-4r}x^{-3r}=x^{18}$$
$$60-7r=18 \Rightarrow r=6$$

$$\binom{15}{6}=\frac{15\cdot14\cdot13\cdot12\cdot11\cdot10}{6!}=5\cdot7\cdot11\cdot13=5(1001)=5005$$

Example 1.22

$$\left(\frac{4x}{5}+\frac{5}{2x^2}\right)^9$$
The coefficient of x^{-6} in the expansion above is: (JEE-M 2023)

The power of the general term is:
$$(x^{9-r})(x^{-2})^r = x^{9-3r} \Rightarrow 9-3r=-6 \Rightarrow r=5$$

The coefficient is:
$$\binom{9}{5}\left(\frac{4}{5}\right)^4\left(\frac{5}{2}\right)^5 = \frac{9\cdot 8\cdot 7\cdot 6}{24}\times\frac{4^4}{5^4}\times\frac{5^5}{2^5} = 9\cdot 7\cdot 2\times 2^3\times 5 = 5040$$

Example 1.23

The absolute difference of the coefficients of x^{10} and x^7 in the expansion of $\left(2x^2+\frac{1}{2x}\right)^{11}$ can be written in the form a^3-a. Find a: (JEE-M 2023)

$$(x^2)^{11-r}(x^{-1})^r = x^{22-3r}$$

$$22-3r=10 \Rightarrow r=4$$
$$22-3r=7 \Rightarrow r=5$$

$$r=4: \binom{11}{4}2^{11-4}\left(\frac{1}{2}\right)^4 = \frac{11\cdot 10\cdot 9\cdot 8}{24}\times 8 = 11\cdot 10\cdot 3\cdot 8$$
$$r=5: \binom{11}{5}2^{11-5}\left(\frac{1}{2}\right)^5 = \frac{11\cdot 10\cdot 9\cdot 8\cdot 7}{120}\times 2 = 11\cdot 3\cdot 4\cdot 7$$

$$\begin{aligned}&11\cdot 10\cdot 3\cdot 8 - 11\cdot 3\cdot 4\cdot 7\\ &= 11\cdot 3\cdot 4(20-7)\\ &= 11\cdot 12\cdot 13\\ &= (12-1)(12)(12+1)\\ &= (12^2-1)(12)\\ &= 12^3-12\end{aligned}$$

$$a=12$$

Example 1.24

$$\left(\frac{\sqrt{x}}{5^{\frac{1}{4}}}+\frac{\sqrt{5}}{x^{\frac{1}{3}}}\right)^{60}$$
If the coefficient of x^{10} in the binomial expansion above is $5^k l$ where $l,k\in\mathbb{N}$ and l is coprime to 5, then k is equal to: (JEE-M 2023)

$$(a+b)^n \textit{ has general term } \binom{n}{r}a^{n-r}b^r$$
Substitute $n=60, a=x^{\frac{1}{2}}, b=x^{-\frac{1}{3}}$:
$$\binom{60}{r}\left(5^{-\frac{1}{4}}x^{\frac{1}{2}}\right)^{60-r}\left(5^{\frac{1}{2}}x^{-\frac{1}{3}}\right)^r = \binom{60}{r}\left(5^{\frac{r-60}{4}}x^{30-\frac{r}{2}}\right)\left(5^{\frac{r}{2}}x^{-\frac{r}{3}}\right) = \binom{60}{r}\left(5^{\frac{3r-60}{4}}x^{\frac{180-5r}{6}}\right)$$

$$\frac{180-5r}{6}=10 \Rightarrow 180-5r=60 \Rightarrow r=24$$

Coefficient of x^{10} in the $\left(\frac{\sqrt{x}}{5^{\frac{1}{4}}}+\frac{\sqrt{5}}{x^{\frac{1}{3}}}\right)^{60}$:

$$\binom{60}{24}5^{\frac{3(24)-60}{4}}=\frac{60!}{24!\,(36!)}\cdot 5^3$$

$$24!\Rightarrow\{5,10,15,20\}\Rightarrow 5^4$$
$$36!\Rightarrow\{5,10,15,20,25,30,35\}+\{25\}\Rightarrow 5^{7+1}=5^8$$
$$60!\Rightarrow\{5,10,\dots,60\}+\{25,50\}\Rightarrow 5^{12+2}=5^{14}$$

$$\frac{5^{14}}{5^4\times 5^8}\cdot 5^3=5^{14+3-4-8}=5^5\Rightarrow k=5$$

E. Constant Term

Example 1.25: Constant Term

Find the constant term in $(x-2)^9$.

The constant term is the last $(10th)$ term.

Substitute $r=9, n=9, x=x$:

$$\binom{n}{r}x^{n-r}y^r=\binom{9}{9}x^{9-9}(-2)^9=(1)x^0(-512)=-512$$

Example 1.26

Find the constant term in the expansion of $\left(3x+\frac{1}{x}\right)^8$

Substitute $n=8, x=3x, y=\frac{1}{x}$ in the formula for the general term:

$$\binom{n}{r}x^{n-r}y^r=\binom{8}{r}(3x)^{8-r}\left(\frac{1}{x}\right)^r=\binom{8}{r}(3^{8-r})x^{8-2r}$$

In the constant term, the power is zero. Hence:

$$8-2r=0\Rightarrow 2r=8\Rightarrow r=4$$

Substitute $r=4$ (the fifth term) into the expression for the general term:

$$\binom{8}{r}(3^{8-r})x^{8-2r}=\binom{8}{4}(3^{8-4})x^{8-2(4)}=\frac{8!}{4!\,4!}(3^4)x^0=\frac{8!}{4!\,4!}(3^4)$$

Example 1.27

Find the constant term in the expansion of $\left(6x-\frac{2}{3x^2}\right)^9$

Substitute $n=9, x=6x, y=\frac{2}{3x^2}$ in the formula for the general term:

$$\binom{9}{r}(6\boldsymbol{x})^{9-r}\left(\frac{2}{3\boldsymbol{x^2}}\right)^r$$

Ignore the coefficients, and work only with the power, since only the power matters for the constant term:

$$x^{9-r} \times \frac{1}{x^{2r}} = x^{9-3r}$$

In the constant term, the power is zero. Hence:

$$9 - 3r = 0 \Rightarrow r = 3$$

Substitute $r = 3$ (the fourth term) into the expression for the general term:

$$\binom{9}{3}(6x)^{9-3}\left(\frac{2}{3x^2}\right)^3 = \frac{9!}{3!\,6!}6^6x^6 \times \frac{2^3}{3^3x^6} = \frac{9!}{3!\,6!}2^6 \cdot 3^6 \times \frac{2^3}{3^3} = \frac{9!}{3!\,6!}2^9 \cdot 3^3$$

Example 1.28: Constant Term

Find the value of k given that the value of the constant term in the expansion of $x^2\left(3x^2 + \frac{k}{x}\right)^8$ is 16,128.

We can work out the powers of the first few terms (while ignoring the coefficients):

$$Power\ of\ 1st\ Term = (x^2)(x^2)^8\left(\frac{1}{x}\right)^0 = x^{2+16+0} = x^{18}$$

$$Power\ of\ 2nd\ Term = (x^2)(x^2)^7\left(\frac{1}{x}\right)^1 = x^{2+14-1} = x^{15}$$

$$3rd\ Term: (x^2)(x^2)^6\left(\frac{1}{x}\right)^2 = x^{2+12-2} = x^{12}$$

Now the pattern is clear. The power of x reduces by 3 in each successive term. Hence, without calculating, we keep reducing by 3:

$$4th\ Term: x^9$$
$$5th\ Term: x^6$$
$$6th\ Term: x^3$$
$$7th\ Term: x^0$$

Hence, we need the 7^{th} term. Substitute $n = 8, r = 6, a = 3x^2, b = \frac{k}{x}$:

$$x^2\binom{n}{r}a^{n-r}b^r$$

$$= x^2\binom{8}{6}(3x^2)^{8-6}\left(\frac{k}{x}\right)^6$$

Substitute $x^2 \cdot (3x^2)^{8-6} = x^2 \cdot (3x^2)^2 = x^2 \cdot 9x^4 = 9x^6, \binom{8}{6} = \frac{8\times 7}{2}$:

$$= \frac{8 \times 7}{2} \cdot 9x^6\left(\frac{k^6}{x^6}\right)$$

$$= 252x^6 \cdot \frac{k^6}{x^6}$$

$$= 252k^6$$

$$252k^6 = 16{,}128$$

$$k^6 = 64$$

$$k = \pm\sqrt[6]{64} = \pm 2$$

Example 1.29: Constant Term

Find the constant term in $\left(3x^2 - \frac{1}{x}\right)^9$.

$$1st\ Term: (x^2)^9\left(\frac{1}{x}\right)^0 = x^{18-0} = x^{18}$$
$$2nd\ Term: (x^2)^8\left(\frac{1}{x}\right)^1 = x^{16-1} = x^{15}$$
$$3rd\ Term: (x^2)^7\left(\frac{1}{x}\right)^2 = x^{14-2} = x^{12}$$
$$4th\ Term: x^9$$
$$5th\ Term: x^6$$
$$6th\ Term: x^3$$
$$7th\ Term: x^0$$

We need the $7th$ term.

Substitute $n = 9, r = 6, a = 3x^2, b = -\frac{1}{x}$

$$\binom{n}{r}a^{n-r}b^r = \binom{9}{6}(3x^2)^3\left(-\frac{1}{x}\right)^6 = (84)(27x^2)^3\left(\frac{1}{x}\right)^6 = 2268$$

$$\frac{9!}{6!\,3!} = \frac{9\times 8\times 7}{6} = 84$$

Example 1.30: Constant Term

A. Explain why $\left(2x^2 - \frac{1}{\sqrt{x}}\right)^{12}$ does not have a constant term.

B. $\left(2x^2 - \frac{1}{\sqrt{x}}\right)^{12+n}, n \in \mathbb{N}$ has a constant term. Find the smallest possible value of n

C. Use the value of n from Part B and find that constant term.

Part A

$$1st\ Term: 24$$
$$2nd\ Term: 21.5$$
$$Term\ Number = \frac{24}{2.5} = 9.6$$

Part B

$$\frac{26}{2.5} = 10.4,\qquad \frac{28}{2.5} =,\qquad \frac{30}{2.5} = 12 \Rightarrow n = 3$$

Part C

$$\left(2x^2 - \frac{1}{\sqrt{x}}\right)^{15}$$

1.31: Constant Term

In the binomial expansion

$$(ap + bq)^n, x\ and\ y\ are\ variables, a\ and\ b\ are\ constants$$

the constant term has the variables raised to the power zero:

Hence, we must have

$$p^{n-r}q^r = z^0$$

$(ap + bq)^n$ has general term

$$\binom{n}{r}(ap)^{n-r}(bq)^r = \binom{n}{r}a^{n-r}p^{n-r}b^rq^r = \binom{n}{r}a^{n-r}b^r(p^{n-r}q^r)$$

F. Integral, Rational, Irrational and Radical Free Terms

Example 1.32

A. The number of integral terms in the expansion of $\left(\sqrt{3}+\sqrt[8]{5}\right)^{256}$ is: **(JEE-M 2003)**
B. What is the number of non-integral terms in the above expansion?

Part A

Substitute $n=256, x=\sqrt{3}, y=\sqrt[8]{5}$in the formula for the general term $\binom{n}{r}x^{n-r}y^r$:

$$\binom{256}{r}\left(\left(\sqrt{3}\right)^{256-r}\right)\left(\left(\sqrt[8]{5}\right)^{r}\right)$$

Rewrite the radicals using fractional exponents:

$$\binom{256}{r}\left(\left(3^{\frac{1}{2}}\right)^{256-r}\right)\left(\left(5^{\frac{1}{8}}\right)^{r}\right)$$

Use the power rule $(a^m)^n=a^{mn}$

$$\binom{256}{r}\left(3^{\frac{256-r}{2}}\right)\left(5^{\frac{r}{8}}\right)$$

Any term must be raised to a positive integer power for the final value of the term to be an integer.

For $3^{\frac{256-r}{2}}$ to be an integer, we must have:

$$3^{\frac{256-r}{2}}\in\mathbb{N}\Rightarrow\frac{256-r}{2}\in\mathbb{N}\Rightarrow r\in(0,2,4,\dots,256)$$

For $5^{\frac{r}{8}}$ to be an integer, we must have:

$$5^{\frac{r}{8}}\in\mathbb{N}\Rightarrow\frac{r}{8}\in\mathbb{N}\Rightarrow r\in(0,8,16,\dots,256)$$

We want both the above conditions to apply, and since the second condition is a subset of the first, we want:

$$r\in(0,8,16,\dots,256)=(0,8\times 1,8\times 2,\dots,8\times 32)$$
$$\rightarrow 33\ Values$$

Part B

We use complementary counting:

$$\underbrace{257}_{\substack{Total\\Terms}}-\underbrace{33}_{\substack{Integral\\Terms}}=224$$

Example 1.33

The number of terms in the expansion of $\left(y^{\frac{1}{5}}+x^{\frac{1}{10}}\right)^{55}$ in which powers of x and y are free from radical signs are: **(JEE-M 2012)**

The general term is

$$\binom{55}{r}\left(y^{\frac{1}{5}}\right)^{55-r}\left(x^{\frac{1}{10}}\right)^{r}=\binom{55}{r}\left(y^{\frac{55-r}{5}}\right)\left(x^{\frac{r}{10}}\right)$$

$$\frac{r}{10}\ is\ an\ integer\Rightarrow r\in\{0,10,20,30,40,50\}$$

All the above work for

$$\frac{55-r}{5}$$

$$6\ Terms$$

Example 1.34

The number of integral terms in the expansion of $\left(3^{\frac{1}{2}}+5^{\frac{1}{4}}\right)^{680}$ is equal to: **(JEE-M 2023)**

The general term is

$$\binom{680}{r}\left(3^{\frac{1}{2}}\right)^{680-r}\left(5^{\frac{1}{4}}\right)^{r}=\binom{680}{r}\left(y^{\frac{680-r}{2}}\right)\left(x^{\frac{r}{4}}\right)$$

$$\frac{r}{4}\ is\ an\ integer \Rightarrow r\in\{0,4,8,\dots,680\}=\{0,4\cdot 1,4\cdot 2,\dots,4\cdot 170\}\Rightarrow 171\ Values$$

All the above work for

$$\frac{680-r}{2}=340-\frac{r}{2}$$

$$171\ Values$$

G. Sum of Coefficients

The focus of this section is on the numbers in Pascal's Triangle as coefficients in binomial expansions.

The sum of the numbers in the n^{th} row of the Pascal Triangle is equal to:

$$2^n$$

This is shown in the table for the first few rows.

It can also be proved using:

- Combinatorial Methods
- Mathematical Induction

Row	Numbers	Total
0	1	$1=2^0$
1	1 1	$2=2^1$
2	1 2 1	$4=2^2$
3	1 3 3 1	$8=2^3$
4	1 4 6 4 1	$16=2^4$
5	1 5 10 10 5 1	$32=2^5$

Example 1.35

A. What is the sum of the numbers in the sixth row of Pascal's Triangle?
B. What is the sum of the coefficients of $(x+y)^8$?
C. If the sum of the coefficients in the expansion of $(a+b)^n$ is 4096, then the greatest coefficient in the expansion is: **(JEE-M 2021, 2002)**

$$(x+y)^n=\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r=\binom{n}{0}x^ny^0+\binom{n}{1}x^{n-1}y^1+\binom{n}{2}x^{n-2}y^2+\dots+\binom{n}{n}x^0y^n$$

Part A

Substitute $x=1, y=1$ in $(x+y)^6$

$$(1+1)^6=2^6=64$$

Part B

$$2^8=256$$

Part C

$$2^n=4096\Rightarrow n=12\Rightarrow No.of\ Terms=13$$

To find the greatest coefficient, we want the middle term, which since there are 13 terms is the

$$\frac{13+1}{2}=\frac{14}{2}=7th\ term$$

$$Coefficient\ of\ 7th\ Term=\ Greatest\ coefficient=\binom{12}{6}=924$$

1.36: Substitution

To find the sum of the coefficients of a polynomial $p(x) = ax^n + bx^{n-1} + \cdots + c$ substitute
$$x = 1$$

Example 1.37

A. What is the sum of the coefficients of $(1+x)^{50}$?
B. The sum of the coefficients of the polynomial $(1 + x - 3x^2)^{2163}$ is (JEE-A 1982)

Part A

We can use the binomial expansion:
$$(1+x)^{50} = 1 + \binom{50}{1}x + \binom{50}{2}x^2 + \cdots + \binom{50}{50}x^{50}$$
To find the sum of the coefficients, simply substitute $x = 1$:
$$(1+1)^{50} = 1 + \binom{50}{1} + \binom{50}{2} + \cdots \binom{50}{50}$$
The left-hand side simplifies to:
$$(1+1)^{50} = 2^{50}$$
Hence, the right-hand side must also be the same, and this is precisely the sum of the coefficients we need.
In fact, this is one more method of proving that the sum of the number in the n^{th} row of Pascal's triangle is
$$2^n$$

Part B

Substitute $x = 1$ in $(1 + x - 3x^2)^{2163}$:
$$(1 + 1 - 3)^{2163} = (-1)^{2163} = -1$$

1.38: Symmetry in Coefficients

$$\binom{n}{r} = \binom{n}{n-r}$$

We know from combinations:
$$\binom{n}{r} = \binom{n}{n-r}$$
And this creates a symmetry in Pascal's Triangle.
$$\underbrace{\binom{n}{0}}_{\substack{\text{First Number} \\ \text{in the } n^{th} \text{ Row}}} = \underbrace{\binom{n}{n}}_{\substack{\text{Last Number} \\ \text{in the } n^{th} \text{ row}}}$$
$$\underbrace{\binom{n}{1}}_{\substack{\text{Second Number} \\ \text{in the } n^{th} \text{ Row}}} = \underbrace{\binom{n}{n-1}}_{\substack{\text{Second}-\text{Last Number} \\ \text{in the } n^{th} \text{ row}}}$$
$$\underbrace{\binom{n}{r}}_{\substack{r^{th} \text{ Number} \\ \text{in the } n^{th} \text{ Row}}} = \underbrace{\binom{n}{n-r}}_{\substack{(n-r)^{th} \text{ Number} \\ \text{in the } n^{th} \text{ row}}}$$

Row	Numbers
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 **4** 6 **4** 1
5	1 **5** 10 10 **5** 1

Example 1.39

Mark the correct option

The coefficients of x^p and x^q in the expansion of $(1+x)^{p+q}$ are: (JEE-M 2002)
A. Equal
B. Equal with opposite signs
C. Reciprocals of each other

D. None of these

$$(r+1)^{st} \text{ term of } (x+y)^n : \binom{n}{r} x^{n-r} y^r$$

We are interested only in the coefficients:

$$Coefficient\ of\ x^q = \binom{p+q}{q}$$

Use the property that $\binom{n}{r} = \binom{n}{n-r}$:

$$\binom{p+q}{p+q-q} = \binom{p+q}{p} = Coefficient\ of\ x^p \Rightarrow Option\ A$$

Example 1.40

If r and n are positive integers with $r > 1, n > 2$ and coefficient of $(r+2)^{th}$ and $3r^{th}$ term in the expansion of $(1+x)^{2n}$ are equal then n, in terms of r, equals **(JEE-A 1983, JEE-M 2002)**

The coefficient of the $(r+1)^{st}$ term in $(x+y)^n$ is given by $\binom{n}{r}$:

$$\binom{2n}{r+1} = \binom{2n}{3r-1}$$

If $(r+2)^{th}$ term and $3r^{th}$ term are actually the same term, then there is no restriction on the value on n.

Hence, by symmetry since $\binom{n}{r} = \binom{n}{n-r}$:

$$\binom{2n}{2n-r-1} = \binom{2n}{3r-1}$$

And now we can equate:

$$2n - r - 1 = 3r - 1 \Rightarrow 2n = 4r \Rightarrow n = 2r$$

Example 1.41

The sum of the coefficients of the first 50 terms in the binomial expansion of $(1-x)^{100}$ is equal to $-\binom{n}{r}$. Determine the least value of $n+r$. **(JEE-M 2023)**

Note that the sum of the coefficients is

$$(1-1)^{100} = 0^{100} = 0$$

Write only the coefficients of $(1-x)^{100}$:

$$\binom{100}{0} - \binom{100}{1} + \cdots + \binom{100}{50} + \cdots - \binom{100}{99} + \binom{100}{100} = 0$$

Since $\binom{n}{r} = \binom{n}{n-r}$

$$\binom{100}{0} - \binom{100}{1} + \cdots + \binom{100}{50} + \cdots - \binom{100}{1} + \binom{100}{0} = 0$$

Notice there are 101 terms, of which the 1^{st} is equal to the 101^{st}, the 2^{nd} term is equal to the 100^{th}, and so on:

$$2\left[\binom{100}{0} - \binom{100}{1} + \cdots + \binom{100}{49}\right] + \binom{100}{50} = 0$$

Note that the middle term (51^{st} $term$) does not have a pair. Rearrange:

$$\binom{100}{0} - \binom{100}{1} + \cdots + \binom{100}{49} = -\frac{1}{2}\binom{100}{50}$$

Now the LHS is exactly the sum of the coefficients of the first 50 terms. Hence, we need to write the RHS in the required format:

$$-\frac{1}{2}\binom{100}{50} = -\frac{1}{2} \times \frac{100!}{50!\,50!} = -\frac{1}{2} \times \frac{100 \times 99!}{50!\,50!} = -\frac{50 \times 99!}{50!\,50!} = -\frac{99!}{49!\,50!} = -\binom{99}{49}$$

$$99 + 49 = 100 + 50 - 2 = 148$$

1.3 Back Calculations

A. Expanding to get a specific term

Example 1.42: Coefficients of Specific Terms

A. If the value of the coefficient of the x term in $(1-5x)^n$ is -60, find the value of n.
B. If the value of the coefficient of the x^2 term in $(1-4x)^n$ is 240, find the value of n.

Part A

Expand the given binomial:

$$(1-5x)^n = \underbrace{1}_{\substack{x\text{ to}\\ \text{power zero}}} + \underbrace{\binom{n}{1}(1^{n-1})(-5x)^1}_{\substack{x\text{ to first}\\ \text{power}}} + \underbrace{\binom{n}{2}(1^{n-2})(-5x)^2}_{\substack{x\text{ to second}\\ \text{power}}} + \cdots$$

The term with x is the second term. We only need the coefficient:

$$\binom{n}{1}(-5)^1 = -60 \Rightarrow 5n = 60 \Rightarrow n = 12$$

Part B

The term with x^2 is the third term:

$$\binom{n}{2}(1)^{n-2}(-4x)^2$$

We only need the coefficient:

$$\binom{n}{2}(4)^2 = 240 \Rightarrow \frac{n(n-1)}{2} \times 16 = 240 \Rightarrow n(n-1) = 30 \Rightarrow n = 6$$

Example 1.43

In the expansion of $(4x+1)^n$, the coefficient of the term in x^2 is 1248. Find the value of n.

Expand the given binomial:

$$(1+4x)^n = 1 + \binom{n}{1}1^{n-1}(4x)^1 + \binom{n}{2}1^{n-2}(4x)^2 + \cdots$$

$$\binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)}{2}$$

The x^2 term is the third term, and it simplifies to:

$$\binom{n}{2}1^{n-2}(4x)^2 = \frac{n(n-1)}{2} \times 16x^2 = 8n(n-1)x^2$$

From the given information, the coefficient is 1248:

$$8n(n-1) = 1248$$
$$n(n-1) = 156 = 13(12)$$
$$n = 13$$

Example 1.44

In the expansion of $(5x+1)^n$, the coefficient of the term in x^3 is 7000. Find the value of n.

Rewrite as

$$(1+5x)^n = 1 + \binom{n}{1}1^{n-1}(5x)^1 + \binom{n}{2}1^{n-2}(5x)^2 + \binom{n}{3}1^{n-3}(5x)^3 + \cdots$$

$$\binom{n}{3} = \frac{n!}{3!\,(n-3)!} = \frac{n(n-1)(n-2)}{6}$$

The x^3 term is the fourth term. Simplify to get:

$$\frac{n(n-1)(n-2)}{6} \times 125x^3$$

Equate coefficients:

$$\frac{125}{6}n(n-1)(n-2) = 7000$$
$$n(n-1)(n-2) = 336 = (8)(7)(6)$$
$$n = 8$$

Example 1.45

If $(1+ax)^n = 1 + 8x + 24x^2 + \cdots$ then $a =$___ and $n =$ ___ (JEE-A 1983)

$$(1+ax)^n = 1 + \binom{n}{1}1^{n-1}(ax) + \binom{n}{2}1^{n-2}(ax)^2 + \cdots = 1 + nax + \frac{n(n-1)}{2}a^2x^2 + \cdots$$

Method of undetermined coefficients

Compare the first terms:

$$nax = 8x \Rightarrow \underbrace{na = 8}_{Equation\ I}$$

Compare the second terms:

$$\frac{n(n-1)}{2}a^2x^2 = 24x^2 \Rightarrow \underbrace{n(n-1)a^2 = 48}_{Equation\ II}$$

Solve the equations

Divide Equation II by Equation I:

$$(n-1)a = 6 \Rightarrow \underbrace{na - a = 6}_{Equation\ III}$$

Subtract Equation III from Equation I:

$$a = 8 - 6 = 2 \Rightarrow n = 4$$

B. Equal Coefficients

Example 1.46

The coefficient of the middle term in the binomial expansion in powers of x of $(1+\alpha x)^4$ and of $(1-\alpha x)^6$ is the same if α equals (JEE-M 2004)

Consider $(1+\alpha x)^4$:

$$No.\ of\ Terms = 4 + 1 = 5$$
$$Middle\ Term = 3rd\ Term$$
$$Value\ of\ r = 3 - 1 = 2$$

Consider$(1-\alpha x)^6$:

$$No.\,of\ Terms = 6+1 = 7$$
$$Middle\ Term = 4th\ \ Term$$
$$Value\ of\ r = 4-1 = 3$$

$$\underbrace{\binom{4}{2}(\alpha)^2}_{\textit{Third Term}} = \underbrace{\binom{6}{3}(-\alpha)^3}_{\textit{Fifth Term}}$$
$$\frac{4!}{2!\,2!}\alpha^2 = -\frac{6!}{3!\,3!}\alpha^3$$
$$\frac{4\times 3}{2!} = -\frac{6\times 5\times 4}{3!}\alpha$$
$$6 = -20\alpha \Rightarrow \alpha = -\frac{3}{10}$$

Example 1.47

If the coefficient of x^7 and x^8 in $\left(2+\frac{x}{3}\right)^n$ are equal, then the value of n is equal to: (JEE-M 2021)

The power of x in the binomial expansion of $\left(2+\frac{x}{3}\right)^n$ is just:

$$x^r \Rightarrow r = \{7,8\}$$

Since the coefficients are equal:

$$\binom{n}{7}(2^{n-7})\left(\frac{1}{3}\right)^7 = \binom{n}{8}(2^{n-8})\left(\frac{1}{3}\right)^8$$

Cancelling, and using $\binom{n}{r} = \frac{n!}{r!(n-r)!}$:

$$\frac{n!}{7!\,(n-7)!}(2) = \frac{n!}{8!\,(n-8)!}\left(\frac{1}{3}\right)$$

$$\frac{1}{n-7}(2) = \frac{1}{8}\left(\frac{1}{3}\right)$$
$$48 = n-7$$
$$n = 55$$

Example 1.48

If for positive integers $r>1, n>2$, the coefficients of the $(3r)^{th}$ and the $(r+2)^{th}$ powers of x in the expansion of $(1+x)^{2n}$ are equal, then find one value of n in terms of r: (JEE-M 2013, 2002)

Since the coefficients of each term in $(1+x)^{2n}$ are both 1, the coefficients in the binomial expansion come only from the binomial coefficients:

$$\binom{2n}{r+2} = \binom{2n}{3r}$$

Expand using $\binom{n}{r} = \frac{n!}{r!(n-r)!}$:

$$\frac{(2n)!}{(r+2)!\,(2n-r-2)!} = \frac{(2n)!}{(3r)!\,(2n-3r)!}$$

Cancel, and eliminate fractions:

$$(3r)!\,(2n-3r)! = (r+2)!\,(2n-r-2)!$$

This will always be equal if one of the terms on the LHS is equal to one of the terms on the RHS, and this also holds for the second term:

Case I:

$$3r = r+2 \Rightarrow r = 1 \Rightarrow \textit{Not Valid}$$

Case II:

$$2n-3r = r+2 \Rightarrow n = 2r+1$$
$$2n-r-2 = 3r \Rightarrow n = 2r+1$$

From both the equations, we get the same answer, and hence the relation is valid.

$$n = 2r+1$$

Example 1.49

If the coefficient of x^9 in $\left(\alpha x^3 + \frac{1}{\beta x}\right)^{11}$ and x^{-9} in $\left(\alpha x - \frac{1}{\beta x^3}\right)^{11}$ are equal then $(\alpha\beta)^2$ is: **(JEE-M 2023)**

Concentrate only on the power of x in the expansion of $\left(\alpha x^3 + \frac{1}{\beta x}\right)^{11}$:

$$(x^3)^{11-r}(x^{-1})^r = x^{33-4r} \Rightarrow 33-4r = 7 \Rightarrow r = 6$$

Concentrate only on the power of x in the expansion of $\left(\alpha x - \frac{1}{\beta x^3}\right)^{11}$:

$$x^{11-r}(x^{-3})^r = x^{11-4r} \Rightarrow 11-4r = -9 \Rightarrow r = 5$$

Since the coefficients are equal:

$$\binom{11}{6}\alpha^5\left(\frac{1}{\beta}\right)^6 = \binom{11}{5}\alpha^6\left(-\frac{1}{\beta}\right)^5$$
$$\alpha\beta = -1$$
$$(\alpha\beta)^2 = 1$$

Example 1.50

If the coefficient of x^7 in $\left(ax^2 + \frac{1}{2bx}\right)^{11}$ and x^{-7} in $\left(ax - \frac{1}{3bx^2}\right)^{11}$ are equal then the value of ab is: **(JEE-M 2023, Adapted)**

Calculate the value of r:

$$\left(ax^2 + \frac{1}{2bx}\right)^{11} : (x^2)^{11-r}(x^{-1})^r = x^{22-3r} \Rightarrow 22-3r = 7 \Rightarrow r = 5$$
$$\left(ax - \frac{1}{3bx^2}\right)^{11} : x^{11-r}(x^{-2})^r = x^{11-3r} \Rightarrow 11-3r = -7 \Rightarrow r = 6$$

$$\binom{11}{5}a^6\left(\frac{1}{2b}\right)^5 = \binom{11}{6}a^5\left(-\frac{1}{3b}\right)^6$$
$$\frac{a}{32b^5} = \frac{1}{729b^6}$$
$$ab = \frac{32}{729}$$

Example 1.51

If the coefficient of x^7 in $\left(ax - \frac{1}{bx^2}\right)^{13}$ and the coefficient of x^{-5} in $\left(ax + \frac{1}{bx^2}\right)^{13}$ are equal, then a^4b^4 is equal to: **(JEE-M 2023)**

The expression for the power is the same in both the binomials:

$$x^{13-r}(x^{-2})^r = x^{13-3r}$$

$$13 - 3r = 7 \Rightarrow r = 2$$
$$13 - 3r = -5 \Rightarrow r = \frac{18}{3} = 6$$

Since the coefficients are equal:

$$\binom{13}{2} a^{11}\left(-\frac{1}{b}\right)^2 = \binom{13}{6} a^7 \left(\frac{1}{b}\right)^6$$
$$\frac{13 \cdot 12}{2} \cdot \frac{a^{11}}{b^2} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot \frac{a^7}{b^6}$$
$$a^4b^4 = 22$$

C. Sum of Two Coefficients

Example 1.52: Sum is Zero

In the binomial expansion of $(a-b)^n, n \geq 5$, the sum of the fifth and the sixth terms is zero. Then $\frac{a}{b} =$ **(JEE-A 2001S, JEE-M 2007)**

$$T_5 + T_6 = 0$$
$$\underbrace{\binom{n}{4} a^{n-4}(-b)^4}_{5th\ Term} + \underbrace{\binom{n}{5} a^{n-5}(-b)^5}_{6th\ Term} = 0$$
$$\binom{n}{4} a^{n-4} b^4 = \binom{n}{5} a^{n-5} b^5$$

Divide both sides by $a^{n-5}b^4$:

$$\frac{n(n-1)(n-2)(n-3)}{4!} a = \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} b$$

Simplify:

$$\frac{a}{b} = \frac{n-4}{5}$$

D. Ratio of Consecutive Coefficients

1.53: Ratio of Consecutive Binomial Coefficients

The ratio of three consecutive binomial coefficients is given by:

$$\binom{N}{r-1} : \binom{N}{r} = \frac{r}{N-r+1}, \qquad \binom{N}{r} : \binom{N}{r+1} = \frac{r+1}{N-r}$$

The first ratio is:

$$\frac{\binom{N}{r-1}}{\binom{N}{r}} = \frac{\frac{N!}{(r-1)!\,(N-r+1)!}}{\frac{N!}{r!\,(N-r)!}} = \frac{\frac{1}{N-r+1}}{\frac{1}{r}} = \frac{r}{N-r+1}$$

$$\frac{\binom{N}{r}}{\binom{N}{r+1}}=\frac{\frac{N!}{r!\,(N-r)!}}{\frac{N!}{(r+1)!\,(N-r-1)!}}=\frac{\frac{1}{N-r}}{\frac{1}{r+1}}=\frac{r+1}{N-r}$$

Example 1.54

If coefficients of the three successive terms in the binomial expansion of $(1+x)^n$ are in the ratio $1:7:42$, then the first of these terms in the expansion has term number: **(JEE-M 2015)**

$$\binom{N}{r-1}:\binom{N}{r}:\binom{N}{r+1}=1:7:42$$

$$\binom{N}{r-1}:\binom{N}{r}=\frac{r}{N-r+1}=\frac{1}{7}\Rightarrow 7r=N-r+1\Rightarrow \underbrace{8r-N=1}_{Equation\ I}$$

$$\binom{N}{r}:\binom{N}{r+1}=\frac{r+1}{N-r}=\frac{7}{42}=\frac{1}{6}\Rightarrow 6r+6=N-r\Rightarrow \underbrace{7r-N=-6}_{Equation\ II}$$

Subtract Equations II from Equation I:

$$r=7\Rightarrow r-1=6\Rightarrow Term\ number\ 7$$

Example 1.55

If some three consecutive coefficients in the binomial expansion of $(x+1)^n$ in powers of x are in the ratio $2:15:70$, then the average of these three coefficients is: **(JEE-M 2019, 2020)**

$$\binom{N}{r-1}:\binom{N}{r}:\binom{N}{r+1}=2:15:70$$

$$\binom{N}{r-1}:\binom{N}{r}=\frac{r}{N-r+1}=\frac{2}{15}\Rightarrow 15r=2N-2r+2\Rightarrow \underbrace{17r=2N+2}_{Equation\ I}$$

$$\binom{N}{r}:\binom{N}{r+1}=\frac{r+1}{N-r}=\frac{15}{70}=\frac{3}{14}\Rightarrow 14r+14=3N-3r\Rightarrow \underbrace{17r=3N-14}_{Equation\ II}$$

From Equations I and II:

$$3N-14=2N+2\Rightarrow N=16$$
$$17r=2(16)+2\Rightarrow r=2$$

$$\frac{\binom{16}{1}+\binom{16}{2}+\binom{16}{3}}{3}=\frac{16+120+560}{3}=\frac{696}{3}=232$$

Example 1.56

The coefficients of three consecutive terms of $(1+x)^{n+5}$ are in the ratio $5:10:14$. Then $n=$ **(JEE-A 2013)**

Use a change of variable. Let $N=n+5$:

$$\binom{N}{r-1}:\binom{N}{r}:\binom{N}{r+1}=5:10:14$$

$$\binom{N}{r-1}:\binom{N}{r}=\frac{r}{N-r+1}=\frac{1}{2}\Rightarrow 2r=N-r+1\Rightarrow 3r-N=1\Rightarrow \underbrace{12r-4N=4}_{Equation\ I}$$

$$\binom{N}{r}:\binom{N}{r+1}=\frac{r+1}{N-r}=\frac{5}{7}\Rightarrow 7r+7=5N-5r\Rightarrow \underbrace{12r-5N=-7}_{Equation\ II}$$

Subtract Equation II from Equation I:

$$N=11\Rightarrow n=6$$

Example 1.57

The sum of the coefficients of three consecutive terms in the binomial expansion of $(1+x)^{n+2}$, which are in the ratio $1:3:5$, is equal to: (JEE-M 2023)

Use a change of variable. Let $N=n+2$.

$$\binom{N}{r-1}:\binom{N}{r}:\binom{N}{r+1}=1:3:5$$

$$\binom{N}{r-1}:\binom{N}{r}=\frac{r}{N-r+1}=\frac{1}{3}\Rightarrow 3r=N-r+1\Rightarrow \underbrace{8r-2N=2}_{Equation\ I}$$

$$\binom{N}{r}:\binom{N}{r+1}=\frac{r+1}{N-r}=\frac{3}{5}\Rightarrow 5r+5=3N-3r\Rightarrow \underbrace{8r-3N=-5}_{Equation\ II}$$

Subtract Equation II from Equation I

$$N=7$$

From Equation I:

$$8r-2(7)=2\Rightarrow 8r=16\Rightarrow r=2$$

$$\binom{7}{1}+\binom{7}{2}+\binom{7}{3}=7+21+35=63$$

Example 1.58

Let the coefficients of three consecutive terms in the binomial expansion of $(1+2x)^n$ be in the ratio $2:5:8$. Then, the coefficient of the term, which is in the middle of these three terms is: (JEE-M 2023)

When $(1+2x)^n$ is expanded, each successive term will have one power of 2 more than the previous term:

$$\binom{N}{r-1}2^{r-1}:\binom{N}{r}2^{r}:\binom{N}{r+1}2^{r+1}=2:5:8$$

Simplify:

$$\binom{N}{r-1}:\binom{N}{r}2:\binom{N}{r+1}4=2:5:8$$

$$\binom{N}{r-1}:\binom{N}{r}2=\frac{r}{N-r+1}\cdot\frac{1}{2}=\frac{2}{5}\Rightarrow 5r=4N-4r+4\Rightarrow \underbrace{9r=4N+4}_{Equation\ I}$$

$$\binom{N}{r}2:\binom{N}{r+1}4=\frac{r+1}{N-r}\cdot\frac{1}{2}=\frac{5}{8}\Rightarrow 4r+4=5N-5r\Rightarrow \underbrace{9r=5N-4}_{Equation\ II}$$

Subtract Equation II from Equation I

$$5N-4=4N+4\Rightarrow N=8$$
$$9r=4(8)+4\Rightarrow r=4$$

Coefficient when $r=4$

$$\binom{8}{4}2^4 = \frac{8\cdot7\cdot6\cdot5}{24}\cdot16 = 1120$$

Example 1.59

The coefficients of three consecutive terms in the binomial expansion of $(1+x)^N$ are in the ratio $a:b:c$. Find the value of the term number of the middle term among the three consecutive terms. Write your answer in terms of a, b, c.[1]

$$\binom{N}{r-1}:\binom{N}{r}:\binom{N}{r+1} = a:b:c$$
$$\binom{N}{r-1}:\binom{N}{r} = \frac{r}{N-r+1} = \frac{a}{b}$$
$$br = aN - ar + a$$
$$(a+b)r - aN = a$$
$$\underbrace{(ab+b^2)r - abN = ab}_{\textit{Equation I}}$$

$$\binom{N}{r}:\binom{N}{r+1} = \frac{r+1}{N-r} = \frac{b}{c}$$
$$cr + c = bN - br$$
$$(b+c)r - bN = -c$$
$$\underbrace{(ab+ac)r - abN = -ac}_{\textit{Equation II}}$$

Subtract Equation II from Equation I:

$$(ab+b^2-ab-ac)r = ab+ac$$
$$(b^2-ac)r = ab+ac$$
$$r = \frac{ab+ac}{b^2-ac}$$

Verify for $a:b:c = 1:3:5$

$$r = \frac{ab+ac}{b^2-ac} = \frac{1(3)+1(5)}{3^2-1(5)} = \frac{3+5}{9-5} = \frac{8}{4} = 2$$

E. Ratio of Two Coefficients

Pending

Example 1.60

75: If the term without x in the expansion of $\left(x^{\frac{2}{3}} + \frac{\alpha}{x^3}\right)^{22}$ is 7315, then the value of $|\alpha|$ is equal to: **(JEE-M 2023)**

$$\alpha = 1$$

Pending

Example 1.61

70: Let $[t]$ denote the greatest integer $\le t$. If the constant term in the expansion of $\left(3x^2 - \frac{1}{2x^5}\right)^7$ is α, then $[\alpha]$ is

[1] Do *not* memorize this result.

equal to: **(JEE-M 2023)**

$$[\alpha] = 1275$$

Example 1.62

If the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}:1$, then the third term from the beginning is: **(JEE-M 2023)**

The ratio of the fifth term from the beginning to the fifth term from the end:

$$\frac{\binom{n}{4}\left(2^{\frac{1}{4}}\right)^{n-4}\left(3^{-\frac{1}{4}}\right)^{4}}{\binom{n}{n-4}\left(2^{\frac{1}{4}}\right)^{4}\left(3^{-\frac{1}{4}}\right)^{n-4}} = \left(2^{\frac{n-4}{4}-\frac{4}{4}}\right)\left(3^{-\frac{4}{4}+\frac{n-4}{4}}\right) = 6^{\frac{n-8}{4}}$$

Equating exponents with $\sqrt{6} = 6^{\frac{1}{2}}$:

$$\frac{n-8}{4} = \frac{1}{2} \Rightarrow n = 10$$

Then, the third term is:

$$\frac{\binom{10}{2}\left(2^{\frac{1}{4}}\right)^{8}}{\left(3^{\frac{1}{4}}\right)^{2}} = \frac{\frac{10\cdot 9}{2}\cdot 4}{\sqrt{3}} = \frac{180}{\sqrt{3}} = 60\sqrt{3}$$

Example 1.63

If the 1011^{th} term from the end in the binomial expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^{2022}$ is 1024 times the 1011^{th} term from the beginning, then $|x|$ is equal to[2]: **(JEE-M 2023)**

The total number of terms is

$$2022 + 1 = 2023$$

The middle term is:

$$\frac{2023+1}{2} = \frac{2024}{2} = 1012$$

Since 1012 is the middle term, it must (by definition) have the same number of terms to its right and left:

$$\underbrace{1,2,3,\dots,1011}_{\textit{From Beginning}},\qquad 1012,\qquad \underbrace{1011,\dots,3,2,1}_{\textit{From End}}$$

Hence, we want:

$$T_{1011-Beg} \Rightarrow r = 1010$$
$$T_{1011-End} = T_{1013-Beg} \Rightarrow r = 1012$$

From the condition given in the question:

$$\binom{2022}{1010}\left(\frac{4x}{5}\right)^{1012}\left(-\frac{5}{2x}\right)^{1010}(1024) = \binom{2022}{1012}\left(\frac{4x}{5}\right)^{1010}\left(-\frac{5}{2x}\right)^{1012}$$

[2] Bonus in the actual exam: none of the given options were correct.

Since $\binom{n}{r} = \binom{n}{n-r}$, we must have $\binom{2022}{1010} = \binom{2022}{1012}$, and it cancels on both sides:

$$\left(\frac{4x}{5}\right)^2 (1024) = \left(-\frac{5}{2x}\right)^2$$

Simplify:

$$\frac{2^4}{5^2}x^2(2^{10}) = \frac{5^2}{2^2 \cdot x^2}$$

Collate x terms on LHS, and numbers on RHS:

$$x^4 = \frac{5^4}{2^{16}} \Rightarrow |x| = \frac{5}{2^4} = \frac{5}{16}$$

1.64: Conjugate

The conjugate of $a+b$ is $a-b$

Example 1.65

When the fifth power of $x = \left(p + \frac{a}{p}\right)$ is multiplied with the fifth power of the conjugate of x, the absolute value of the coefficient of the p^2 term is nine times the absolute value of the coefficient of the $\frac{1}{p^2}$ term. Find the possible value(s) of a.

The conjugate of $\left(p + \frac{a}{p}\right)$ is $\left(p - \frac{a}{p}\right)$. The fifth power of the product of the two conjugates.

$$\left(p + \frac{a}{p}\right)^5 \left(p - \frac{a}{p}\right)^5 = \left[\left(p + \frac{a}{p}\right)\left(p - \frac{a}{p}\right)\right]^5 = \underbrace{\left(p^2 - \frac{a^2}{p^2}\right)^5}_{(a+b)(a-b)=a^2-b^2}$$

The power of the $(r+1)^{st}$ term of $\left(p^2 - \frac{a^2}{p^2}\right)^5$ is given by:

$$(p^2)^{5-r}\left(\frac{1}{p^2}\right)^r = p^{10-2r}\left(\frac{1}{p^{2r}}\right) = p^{10-4r}$$

$$For\ p^2\ Term: 10 - 4r = 2 \Rightarrow r = 2$$

$$For\ \frac{1}{p^2}\ Term: 10 - 4r = -2 \Rightarrow r = 3$$

p^2 term is the third term, and the $\frac{1}{p^2}$ term is the fourth term:

$$\underbrace{\binom{5}{2}(p^2)^{5-2}\left(-\frac{a^2}{p^2}\right)^2}_{Third\ Term\ has\ p^2}, \qquad \underbrace{\binom{5}{3}(p^2)^{5-3}\left(-\frac{a^2}{p^2}\right)^3}_{Fourth\ Term}$$

We only need the absolute value of the coefficients:

$$\binom{5}{2}a^4 = 9\binom{5}{3}(a^6) \Rightarrow a^2 = \frac{1}{9} \Rightarrow a = \pm\frac{1}{3}$$

Example 1.66

If the coefficient of x^7 in $\left[ax^2 + \left(\frac{1}{bx}\right)\right]^{11}$ equals the coefficient of x^{-7} in $\left[ax - \left(\frac{1}{bx^2}\right)\right]^{11}$, then a and b satisfy the relation: (JEE-M 2005)

1.4 Applications: Algebra

A. Terms in Arithmetic Progression

1.67: Arithmetic Progression

If three terms a, b and c are in arithmetic progression, then the middle term is the average of the other two terms:

$$b = \frac{a+b}{2} \Rightarrow 2b = a + c$$

Example 1.68

Let n be a positive integer. If the coefficients of the second, third and fourth terms in the expansion of $(1+x)^n$ are in arithmetic progression, then the value of n is? **(JEE-A 1994)**

$$(1+x)^n = 1 + \boldsymbol{\binom{n}{1}} x + \boldsymbol{\binom{n}{2}} x^2 + \boldsymbol{\binom{n}{3}} x^3 + \cdots$$

The terms in **maroon** above are in arithmetic progression:

$$2\binom{n}{2} = \binom{n}{1} + \binom{n}{3}$$

Use the definition of combinations $\binom{n}{r} = \frac{n!}{r!(n-r)!}$:

$$2 \times \frac{n!}{2!\,(n-2!)} = \frac{n!}{1!\,(n-1)!} + \frac{n!}{3!\,(n-3!)}$$

Simplify by cancelling:

$$\boldsymbol{2}\left(\frac{\boldsymbol{n}(n-1)}{\boldsymbol{2}}\right) = \boldsymbol{n} + \frac{\boldsymbol{n}(n-1)(n-2)}{6}$$

Simplify:

$$n - 1 = \frac{6 + (n-1)(n-2)}{6} \Rightarrow 6n - 6 = 6 + n^2 - 3n + 2$$

Collect all terms on one side to get a quadratic and solve:

$$n^2 - 9n + 14 = 0 \Rightarrow (n-7)(n-2) = 0 \Rightarrow n \in \{\boldsymbol{2}, 7\} \Rightarrow n = 7$$

$$\binom{7}{1} = 7, \qquad \binom{7}{2} = 21, \qquad \binom{7}{3} = 35$$

B. System of Equations

Example 1.69

$$1 + x^4 + x^5 = \sum_{i=0}^{5} a_i (1+x)^i$$

If the above is true for all $x \in \mathbb{R}$, then a_2 is: **(JEE-M 2014)**

Expand the RHS:

$$= a_0 + a_1(1+x) + a_2(1+x)^2 + a_3(1+x)^3 + a_4(1+x)^4 + a_5(1+x)^5$$

$$= a_0 + a_1(1+x) + a_2(1+2x+x^2) + a_3(1+3x+3x^2+x^3) + a_4(1+4x+6x^2+4x^3+x^4)$$
$$+ a_5(1+5x+10x^2+10x^3+5x^4+x^5)$$

$$= (a_0 + a_1 + a_2 + a_3 + a_4 + a_5) + x(a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5) + x^2(a_2 + 3a_3 + 6a_4 + 10a_5) + x^3(a_3 + 4a_4 + 10a_5) + x^4(a_4 + 5a_5) + x^5 a_5$$

Use the method of undetermined coefficients.
Comparing fifth powers:

$$a_5 = 1$$

Comparing fourth powers:

$$a_4 + 5a_5 = a_4 + 5(1) = 1 \Rightarrow a_4 = -4$$

Comparing third powers:

$$a_3 + 4a_4 + 10a_5 = a_3 + 4(-4) + 10(1) = 0 \Rightarrow a_3 = 6$$

Comparing second powers:

$$a_2 + 3a_3 + 6a_4 + 10a_5 = a_2 + 3(6) + 6(-4) + 10(1) = 0 \Rightarrow a_2 = -4$$

Example 1.70

$$\left(a + \frac{x}{a}\right)^n = b^4 + 48b^3x + 1056b^2x^2 + \cdots$$

Given that the binomial on the LHS can be expanded in ascending powers of x as on the RHS, determine the solutions such that n, a, x, and b are nonnegative integers less than or equal to 64.

Step I: Use the Binomial Theorem

Expand the LHS using the Binomial Theorem in ascending powers of x:

$$\left(a + \frac{x}{a}\right)^n = \binom{n}{0}\left(\frac{x}{a}\right)^0 a^{n-0} + \binom{n}{1}\left(\frac{x}{a}\right)^1 a^{n-1} + \binom{n}{2}\left(\frac{x}{a}\right)^2 a^{n-2} + \cdots$$

Simplify:

$$\left(a + \frac{x}{a}\right)^n = a^n + n \cdot \frac{x}{a} \cdot a^{n-1} + \frac{n(n-1)}{2} \cdot \frac{x^2}{a^2} \cdot a^{n-2} + \cdots$$

Further simplify:

$$\left(a + \frac{x}{a}\right)^n = a^n + nxa^{n-2} + \frac{n(n-1)}{2}x^2a^{n-4} + \cdots$$

Using the expansion given in the question, and equating terms:

$$\underbrace{a^n = b^4}_{Equation\ I}, \qquad \underbrace{nxa^{n-2} = 48b^3x}_{Equation\ II}, \qquad \underbrace{\frac{n(n-1)}{2}x^2a^{n-4} = 1056b^2x^2}_{Equation\ III}$$

Step II: Find solutions where one or more of $n, a, x, or\ b$ are zero

Suppose $a = 0$:

$$LHS = \left(a + \frac{x}{a}\right)^n = \left(0 + \frac{x}{0}\right)^n \rightarrow Not\ Defined \Rightarrow a \neq 0$$

Suppose $b = 0$:

$$a^n = b^4 = 0^4 = 0 \Rightarrow a = 0 \Rightarrow No\ Solutions$$

Suppose $n = 0$:

$$b^4 = a^0 = 1 \Rightarrow b = 1$$

$$nxa^{n-2} = 48b^3x$$
$$48b^3x = 0 \cdot xa^{n-2} = 0 \Rightarrow x = 0$$
$$(n, a, x, b) = (0, a, 0, 1), a \in \{1,2,3, \dots, 64\} \Rightarrow 64\ Solutions$$

Suppose $x = 0$:
Equation II and Equation III are satisfied. We need to satisfy Equation I:

$$a^n = b^4$$

We do not need to consider $n = 0$ since that was considered just above.

Suppose $n = 1$:

$$b = 1 \Rightarrow a = 1$$
$$b = 2 \Rightarrow a = 16$$

$$(n, a, x, b) = (1,1,0,1), (1,16,0,2) \Rightarrow 2\ Solutions$$

Suppose $n = 2$:

$$a^2 = b^4 \Rightarrow a = b^2$$
$$b = 1 \Rightarrow a = 1$$
$$b = 2 \Rightarrow a = 4$$
$$b = 3 \Rightarrow a = 9$$
$$\vdots$$
$$b = 8 \Rightarrow a = 64$$

$$(n, a, x, b) = (2,1,0,1), (2,4,0,2), \dots, (2,64,0,8) \Rightarrow 8\ Solutions$$

Suppose $n = 3$:

$$a^3 = b^4 \Rightarrow a = b^{\frac{4}{3}}$$
$$b = 1 \Rightarrow a = 1$$
$$b = 8 \Rightarrow a = 16$$

$$(n, a, x, b) = (3,1,0,1), (3,16,0,8) \Rightarrow 2\ Solutions$$

Suppose $n = 3$:

$$a^3 = b^4 \Rightarrow a = b^{\frac{4}{3}}$$
$$b = 1 \Rightarrow a = 1$$
$$b = 8 \Rightarrow a = 16$$

$$(n, a, x, b) = (3,1,0,1), (3,16,0,8) \Rightarrow 2\ Solutions$$

Suppose $n = 4$:

$$a^4 = b^4 \Rightarrow a = b$$

$$(n, a, x, b) = (4,1,0,1), (4,2,0,2), \dots, (4,64,0,64) \Rightarrow 64\ Solutions$$

Suppose $n = 5$:

$$a^5 = b^4 \Rightarrow a = b^{\frac{4}{5}}$$

$$(n,a,x,b) = (5,1,0,1),(5,16,0,32) \Rightarrow 2\ Solutions$$

Suppose $n = 6$:

$$a^6 = b^4 \Rightarrow a = b^{\frac{4}{6}} = b^{\frac{2}{3}}$$
$$(n,a,x,b) = (6,1,0,1),(6,4,0,8),(6,9,0,27),(6,16,0,64) \Rightarrow 4\ Solutions$$

Suppose $n = 7$

$$a^7 = b^4 \Rightarrow a = b^{\frac{4}{7}}$$
$$(n,a,x,b) = (7,1,0,1)$$
$$(n,a,x,b) = (8,1,0,1)$$
$$(n,a,x,b) = (9,1,0,1)$$

.
.
.

$$(n,a,x,b) = (64,1,0,1)$$

$$64 - 7 + 1 = 65 - 7 = 58$$

Step III: Find solutions in positive integers only

Equations I, II and III are now to be solved in positive integers.
Simplify Equation II:

$$\underbrace{nxa^{n-2} = 48b^3x}_{Equation\ II} \Rightarrow na^{n-2} = 48b^3 \Rightarrow b^3 = \frac{na^{n-2}}{48} \Rightarrow b^{12} = \left(\frac{na^{n-2}}{48}\right)^4$$
$$\underbrace{a^n = b^4}_{Equation\ I} \Rightarrow b^{12} = a^{3n}$$

$$a^{3n} = \left(\frac{na^{n-2}}{48}\right)^4$$
$$48^4a^{3n} = n^4a^{4n-8}$$
$$\left(\frac{48}{n}\right)^4 = a^{4n-8-3n}$$
$$\left(\frac{48}{n}\right)^4 = a^{n-8}$$

Simplify Equation III:

$$\underbrace{\frac{n(n-1)}{2}x^2a^{n-4} = 1056b^2x^2}_{Equation\ III} \Rightarrow \frac{n(n-1)}{2}a^{n-4} = 1056b^2$$

Substitute $\underbrace{a^n = b^4}_{Equation\ I} \Rightarrow a^{\frac{n}{2}} = b^2$:

$$n(n-1)a^{n-4} = 2112a^{\frac{n}{2}}$$
$$n(n-1)a^{n-4-\frac{n}{2}} = 2112$$
$$n(n-1)a^{\frac{n}{2}-4} = 2112$$

Substitute $\left(\frac{48}{n}\right)^4 = a^{n-8} \Rightarrow \left(\frac{48}{n}\right)^2 = a^{\frac{n}{2}-4}$

$$n(n-1)\left(\frac{48}{n}\right)^2 = 2112$$
$$n(n-1)\cdot\frac{48^2}{n^2} = 2112$$
$$(n-1)\cdot\frac{48^2}{n} = 2112$$
$$48^2(n-1) = 2112n$$
$$48(n-1) = 44n$$
$$4n = 48$$
$$n = 12$$

Substitute $n = 12$ in $\left(\frac{48}{n}\right)^2 = a^{\frac{n}{2}-4}$:

$$\left(\frac{48}{12}\right)^2 = a^{\frac{12}{2}-4}$$
$$4^2 = a^2$$
$$a = 4$$

$$a^n =$$
$$b^4 = a^n = 4^{12}$$
$$b = (4^{12})^{\frac{1}{4}} = 4^3 = 64$$

$$(n, a, x, b) = (12, 4, x, 64), x \in \{1, 2, \dots, 64\} \Rightarrow 64\ Solutions$$

$$64 + 2 + 8 + 2 + 2 + 62 + 2 + 4 + 58 + 64 = 270\ Solutions$$

C. Trigonometry

Example 1.71

In the expansion of $\left(\frac{x}{\cos\theta} + \frac{1}{x\sin\theta}\right)^{16}$, if l_1 is the least value of the term independent of x when $\frac{\pi}{8} \le \theta \le \frac{\pi}{4}$ and l_2 is the least value of the term independent of x when $\frac{\pi}{16} \le \theta \le \frac{\pi}{8}$, then the ratio $l_2 : l_1$ is equal to: **(JEE-M 2020)**

The general term is:

$$\binom{16}{r}\left(\frac{x}{\cos\theta}\right)^{16-r}\left(\frac{1}{x\sin\theta}\right)^r = \binom{16}{r}\frac{x^{16-2r}}{\cos^{16-r}\theta\sin^r\theta}$$

$$16 - 2r = 0 \Rightarrow r = 8$$

The term independent of x is:

$$\binom{16}{8}\frac{1}{\cos^8\theta\sin^8\theta} = \binom{16}{8}\frac{2^8}{(\sin 2\theta)^8}$$

Take the ratio:

$$\binom{16}{8}\frac{2^8}{(\sin 2\theta_2)^8} : \binom{16}{8}\frac{2^8}{(\sin 2\theta_1)^8}$$

$$\frac{1}{(\sin 2\theta_2)^8} : \frac{1}{(\sin 2\theta_1)^8}$$

The term is minimum when the quantity in the denominator is maximum

$$\frac{\pi}{16} \le \theta \le \frac{\pi}{8} : \frac{1}{(\sin 2\theta_1)^8} = \frac{1}{\left(\sin \frac{\pi}{4}\right)^8} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^8} = \frac{1}{\frac{1}{16}} = 16$$

$$\frac{\pi}{8} \le \theta \le \frac{\pi}{4} : \frac{1}{(\sin 2\theta_1)^8} = \frac{1}{\left(\sin \frac{\pi}{2}\right)^8} = \frac{1}{1} = 1$$

$$l_2 : l_1 = 16 : 1 = 16$$

1.5 Approximations

A. Applications

Example 1.72: Numerical/Partial Expansions

A. Find the fourth power of 102 using a binomial expansion.
B. Given that $a = 10{,}510{,}100{,}501$ is a perfect fifth power, find $\sqrt[5]{a}$
C. Given that $f(x, y) = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$, evaluate $f(7,13) - 2f(8,12) + 3f(9,11)$

Part A

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Substitute $x = 100, y = 2$ to get:

$$(100 + 2)^4 = 100^4 + (4)(100^3)(2) + (6)(100^2)(2^2) + (4)(100^1)(2^3) + 2^4$$

Simplify:

$$= 100{,}000{,}000 + 8{,}000{,}000 + 240{,}000 + 3200 + 16 = 108{,}243{,}216$$

Part B

$$10{,}510{,}100{,}501 \approx 10^{10} = 100^5$$

Hence, expand:

$$a = 10{,}510{,}100{,}501 = 1 \times 100^5 + 5 \times 100^4 + 10 \times 100^3 + 10 \times 100^2 + 5 \times 100^1 + 100$$

But notice that the above is just the binomial expansion of:

$$= (100 + 1)^5 = (101)^5$$

$$\sqrt[5]{a} = 101$$

Part C

Note that

$$f(x, y) = (x + y)^5$$

And hence $f(7,13) - 2f(8,12) + 3f(9,11)$ becomes:

$$(7 + 13)^5 - 2(8 + 12)^5 + 3(9 + 11)^5$$

Which simplifies to:

$$= (20)^5 - 2(20)^5 + 3(20)^5 = 2(20)^5 = 2 \times 32 \times 10^5 = 64 \times 10^5$$

Example 1.73: Approximation (Calculator-Allowed)

A. Find the first three terms of the binomial expansion of $(1 + x)^4$. Approximate 1.01^4 using the three terms calculated in Part A. In using this approximation, find the Absolute Error, Relative Error and the Percentage Error.
B. Write the first three terms of $(3 + 2x)^6$. Use the three terms written to approximate 3.2^6. What is the

percentage error in the approximation?

Part A

$$(1+x)^4 = 1 + 4x + 6x^2 + \cdots$$

Let $x = 0.01$ in the above to get:

$$(1+0.01)^4 = 1 + 4(0.01) + 6(0.01)^2 + \cdots = 1 + 0.04 + 0.0006 + \cdots \approx 1.0406$$

$$Absolute\ Error = \left|\underbrace{1.04060401}_{Actual} - \underbrace{1.0406}_{Approximation}\right| = 0.00000401$$

$$Relative\ Error = \left|\frac{1.04060401 - 1.0406}{1.04060401}\right| = 3.85 \times 10^{-6}$$

$$\%tage\ error = \left|\frac{1.04060401 - 1.0406}{1.04060401}\right| \times 100 = 3.85 \times 10^{-6} \times 100 = 3.85 \times 10^{-4}\ \%$$

Part B

$$(a+b)^6 = a^6 + \binom{6}{1}a^5 b + \binom{6}{2}a^4 b^2 + \cdots$$

Substitute $a = 3, b = 2x$ in the above:

$$(3+2x)^6 = 3^6 + (6)(3^5)(2)x + (15)(3^4)(2^2)(x^2) + \cdots = 729 + 2916x + 4860x^2 + \cdots$$

Substitute $x = 0.1$ in the above:

$$= 729 + 2916(0.1) + 4860(0.1)^2 + \cdots = 729 + 291.6 + 48.6 + \cdots \approx 1069.2$$

$$\%\ Error = \left|\frac{1073.74 - 1069.2}{1073.74}\right| \times 100 = \frac{4.54}{1073.74} \times 100 = 0.42\%$$

1.6 NT: Remainders

A. Remainders

Example 1.74

The remainder when 7^{103} is divided by 17 is: (JEE-M 2023)

$$7^{103} = 7(49)^{51} = 7(51-2)^{51}$$

Expand the binomial

$$= 7\left[51^{51} + \binom{51}{1}(51)^{50}(-2) + \cdots \binom{51}{50}(51)^1(-2)^{50} + \binom{51}{51}(-2)^{51}\right]$$

If we consider the remainder, we can drop the purple terms since they are all divisible by 17:

$$= 7(-2)^{51}$$
$$= -7(2^{51})$$
$$= -7(2^3 \times 2^{48})$$
$$= -7(8 \times (2^4)^{12})$$
$$= -56(17-1)^{12}$$

We could expand the binomial, but once we did, all but the last term would have a zero remainder, and hence we can ignore those terms. We are left with the last term:

$$= -56(-1)^{12} = -56 = -5 = 12$$

1.75: Remainders

$$(x+a)^n \equiv a^n\ (mod\ x)$$

In the binomial expansion above, all terms except the last term are divisible by x.

$$(x+a)^n = x^n + \binom{n}{1} x^n a + \binom{n}{2} x^{n-1} a^2 + \cdots + \binom{n}{n-1} x a^{n-1} + a^n$$

Example 1.76

The remainder when 7^{103} is divided by 17 is: (JEE-M 2023)

$$7^{103} \equiv 7(49^{51}) \equiv 7(-2)^{51} \equiv -7(2^3)[2^4]^{12} \equiv -7(8)[-1]^{12} \equiv -56 \equiv -5 \equiv 12 \ (mod\ 17)$$

Note: Make sure to compare this method with the one used earlier. We will use this method as far as possible going forward.

Example 1.77

A. The remainder when $7^{2022} + 3^{2022}$ is divided by 5 is: (JEE-M 2022)
B. If 27^{999} is divided by 7, then the remainder is: (JEE-M 2017)
C. The remainder when 2021^{2023} is divided by 7 is: (JEE-M 2022)
D. The remainder when $2021^{2022} + 2022^{2021}$ is divided by 7 is: (JEE-M 2022)
E. The remainder when $11^{1011} + 1011^{11}$ is divided by 9 is: (JEE-M 2022)
F. Determine the remainder when $2023^{2022} - 1999^{2022}$ is divided by 8. (JEE-M 2023, Adapted)
G. If 2021^{3762} is divided by 17, then the remainder is: (JEE-M 2021)

Part A

$$2^{2022} + (-2)^{2022} \equiv 2 \cdot 2^{2022} \equiv 2(4^{1011}) \equiv 2(-1)^{1011} \equiv -2 \equiv 3 \ (mod\ 5)$$

Part B

$$27^{999} \equiv (-1)^{999} \equiv -1 \equiv 6 \ (mod\ 7)$$

Part C

$$-2^{2023} \equiv -2(2^3)^{674} \equiv -2(8^{674}) \equiv -2(1^{674}) \equiv -2 \equiv 5$$

Part D

$$(-2)^{2022} + (-1)^{2021} \equiv (2^3)^{674} - 1 \equiv 8^{674} - 1 \equiv 1^{674} - 1 \equiv 1 - 1 \equiv 0 \ (mod\ 7)$$

Part E

$$2^{1011} + 3^{11} \equiv (2^3)^{337} + 3^2 \cdot 3^9 \equiv (8)^{337} + 0 \cdot 3^9 \equiv (-1)^{337} \equiv -1 \equiv 8$$

Part F

Substitute $2023 \equiv 7 \equiv -1 (mod\ 8), 1999 \equiv 7 \equiv -1 (mod\ 8)$

$$(-1)^{2022} - (-1)^{2022} \equiv 1 - 1 \equiv 0 \ (mod\ 8)$$

Part G

$$(-2)^{3762} \equiv [(-2)^4]^{940} \times (-2)^2 \equiv [16]^{940} \times 4 \equiv [-1]^{940} \times 4 \equiv 1 \times 4 \equiv 4$$

Example 1.78

Let the number $22^{2022} + 2022^{22}$ leave the remainder α when divided by 3, and β when divided by 7. Then $(\alpha^2 + \beta^2) =$ (JEE-M 2023)

$$\alpha \equiv 22^{2022} + 2022^{22} \equiv 1^{2022} + 0^{22} \equiv 1 + 0 \equiv 1 (mod\ 3)$$
$$\beta \equiv 22^{2022} + 2022^{22} \equiv 1^{2022} + (-1)^{22} \equiv 1 + 1 \equiv 2 (mod\ 7)$$

$$\alpha^2 + \beta^2 = 1^2 + 2^2 = 1 + 4 = 5$$

1.79: Fractional Part

Questions that ask for fractional part are effectively asking for the remainder.

Example 1.80

A. If $\{p\}$ denotes the fractional part of the number p, then $\left\{\frac{3^{200}}{8}\right\}$ is equal to: (JEE-M 2020)

B. Fractional part of $\frac{4^{2022}}{15}$ is equal to: (JEE-M 2023)

C. If the fractional part of the number $\frac{2^{400}}{15}$ is $\frac{k}{15}$, then k is equal to: (JEE-M 2019)

$$Part\ A: 3^{200} \equiv 9^{100} \equiv 1^{100} \equiv 1\ (mod\ 8) \Rightarrow \left\{\frac{3^{200}}{8}\right\} = \frac{1}{8}$$
$$Part\ B: 4^{2022} \equiv 16^{1011} \equiv 1^{1011} \equiv 1\ (mod\ 15) \Rightarrow \left\{\frac{4^{2022}}{15}\right\} = \frac{1}{15}$$
$$Part\ C: 2^{400} \equiv 16^{100} \equiv 1^{100} \equiv 1\ (mod\ 15) \Rightarrow k = \frac{1}{15}$$

1.81: Working with Powers

In some questions, the power to use may not be immediately obvious. The next question is one such example. You will then have to work out the remainders for consecutive powers.

Example 1.82

The remainder on dividing 5^{99} by 11 is: (JEE-M 2023)

$$5^2 \equiv 25 \equiv 3$$
$$5^3 \equiv 5 \cdot 5^2 \equiv 5 \cdot 3 \equiv 15 \equiv 4$$
$$5^4 \equiv 5 \cdot 5^3 \equiv 5 \cdot 4 \equiv 20 \equiv 9 \equiv -2$$
$$5^5 \equiv 5 \cdot 5^4 \equiv 5 \cdot (-2) \equiv -10 \equiv 1$$

Hence, we reduce using 5^5:

$$5^4 \cdot (5^5)^{19} \equiv 625 \cdot (3125)^{19} \equiv 9 \cdot 1^{19} \equiv 9$$

Example 1.83

The remainder when 2023^{2023} is divided by 35: (JEE-M 2023)

$$\begin{aligned} 2023^{2023} &\equiv (-7)^{2023} \\ &\equiv (-7)(7^{2022}) \\ &\equiv (-7)(49^{1011}) \\ &\equiv (-7)(50-1)^{1011} \\ &\equiv (-7)(-1)^{1011} \\ &\equiv (-7)(-1) \\ &\equiv 7 \end{aligned}$$

1.84: Product of a Binomial Expansion

$$c(x+a)^n \equiv cx\ (mod\ ca)$$

$$c(x+a)^n = cx^n + \binom{n}{1} cx^n a + \binom{n}{2} cx^{n-1}a^2 + \cdots + \binom{n}{n-1} cxa^{n-1} + ca^n$$

All except the first term are divisible by ca.

Hence, the result follows.

Example 1.85
$3 \times 7^{22} + 2 \times 10^{22} - 44$ when divided by 18 leaves remainder: **(JEE-M 2021)**

$$3 \times 7^{22} \equiv 3(1+6)^{22} \equiv (3 + 3 \cdot 6^1 + \cdots) \equiv 3$$
$$2 \times 10^{22} \equiv 2(1+9)^{22} \equiv (2 + 2 \cdot 9^1 + \cdots) \equiv 2$$

$$3 \times 7^{22} + 2 \times 10^{22} - 44 \equiv 3 + 2 - 44 \equiv -39 \equiv 15$$

B. Remainders: Variables in the Power

1.86: Powers of -1
$$(-1)^n = \begin{cases} 1, n \text{ is even} \\ -1, n \text{ is odd} \end{cases}$$

Example 1.87
The remainder left out when $8^{2n} - 62^{2n+1}$ is divided by 9 is: **(JEE-M 2009)**

Substitute $8 \equiv 62 \equiv -1 \ (mod\ 9)$:
$$(-1)^{2n} - (-1)^{2n+1} \equiv 1 - (-1) \equiv 2 \ (mod\ 9)$$

Example 1.88
Mark the correct statements

S1: $2023^{2022} - 1999^{2022}$ is divisible by 8.
S2: $13(13^n) - 11n - 13$ is divisible by 144 for infinitely many $n \in \mathbb{N}$ **(JEE-M 2023)**

Statement 1
$$2023^{2022} - 1999^{2022} = (1999+24)^{2022} - 1999^{2022}$$

$$= \left[1999^{2022} + \binom{2022}{1}(1999)^{2021}(24) + \cdots + 24^{2022}\right] - 1999^{2022}$$
$$= 24\left[\binom{2022}{1}(1999)^{2021} + \cdots + 24^{2021}\right]$$
$$= 8 \cdot 3\left[\binom{2022}{1}(1999)^{2021} + \cdots + 24^{2021}\right]$$

Every term in the above is a multiple of 24, and hence it is divisible by 8.

Statement 2
Rewrite $13 = 1 + 12$:
$$(12+1)^{n+1} - 11n - 13 = \left[1 + (n+1)(12) + \binom{n+1}{2}12^2 + \cdots\right] - 11n - 13$$

If we consider only the remainder of the above:
$$= [1 + (n+1)(12)] - 11n - 13 = [1 + 12n + 12] - 11n - 13 = n$$

If n is a multiple of 144, then $\frac{n}{144}$ has remainder 0. Hence, there are infinitely many values.

Statement is true.

C. Last Few Digits

1.89: Last Digit
Determining the last digit is the same as determining the remainder when divided by 100.

1.90: Last Two Digits
Determining the last two digits is the same as determining the remainder when divided by 100.

Example 1.91
Determine the ten's digit of 17^{1993}. **(MathCounts 1994 Workout 10)**

Write the given number as a binomial expansion:

$$(7+10)^{1993} = 7^{1993} + \binom{1993}{1992} 7^{1992} 10^1 + \cdots$$

All terms after the second term will have 10^2 or higher as a factor. Hence, we only need to check the two terms above.

Note that $7^4 = 2401$.

$$7^{1992} \equiv (7^4)^{498} \equiv (1)^{498} \equiv 1 \ (mod\ 100)$$
$$7^{1993} = 7^{1992} \times 7 = (7^4)^{498} \times 7 = (1)^{498} \times 7 = 1 \times 7 = 7$$

And, again, we are only concerned with last two digits:

$$7 + (1993)(10)(01)$$
$$= 7 + 19930 = 19937$$
$$Ten's\ Digit = 3$$

1.92: Last Three Digits
Determining the last three digits is the same as determining the remainder when divided by 1000.

Example 1.93
What is the hundreds digit of 2011^{2011}? **(AMC 10B 2011/23)**

Ignore the thousands digit and above at all stages of the calculations:

$$(11)^{2011} \equiv (10+1)^{2011} \equiv \cdots + \binom{11}{2}(10^2)(1^{2009}) + \binom{11}{1}(10^1)(1^{2010}) + 1 \ (mod\ 1000)$$

$$\equiv \cdots + \frac{11 \times 10}{2}(100) + 110 + 1 = \cdots + 5500 + 111 = 6611$$
$$Hundred's\ Digit = 6$$

D. Remainders: Further Questions

1.94: Converting from one mod to another
Some questions may require you to convert the remainder in one mod base to a remainder in another mod base.

Example 1.95
If the remainder when x is divided by 4 is 3, then the remainder when $(2020 + x)^{2022}$ is divided by 8 is: **(JEE-M**

2021)

Substitute $x \equiv 3(mod\ 4)$:

$$N \equiv 2020 + 3 \equiv 0 + 3 \equiv 3\ (mod\ 4)$$

Convert from mod arithmetic to algebra and square both sides:

$$N = 4k - 1, \qquad k \in \mathbb{Z}$$
$$N^2 = 16k^2 - 8k + 1 \equiv 1\ (mod\ 8)$$

Hence:

$$(2020 + x)^{2022} \equiv N^{2022} \equiv (N^2)^{1011} \equiv 1^{1011} \equiv 1\ (mod\ 8)$$

Method II: Casework

$3\ (mod\ 4)$ converted to $mod\ 8$ gives us two cases:

$$x \equiv 3 \equiv 7(mod\ 4) \Leftrightarrow x \equiv 3\ (mod\ 8)\ OR\ x \equiv 7\ (mod\ 8)$$

$$Case\ I: 3^{2022} \equiv 9^{1011} \equiv 1^{1011} \equiv 1\ (mod\ 8)$$
$$Case\ II: 7^{2022} \equiv (-1)^{2022} \equiv 1^{2022} \equiv 1\ (mod\ 8)$$

In both cases the answer is 1.

Example 1.96

The total number of two-digit numbers n such that $3^n + 7^n$ is a multiple of 10, is: (JEE-M 2021)

Method I: Pattern

	3^n	7^n	
$n = 1$	3	7	10
$n = 2$	9	9	18
$n = 3$	7	3	10
$n = 4$	1	1	2

Method II: Remainder Theorem

Consider the polynomial

$$P(a) = a^n + b^n, \qquad where\ b\ is\ a\ constant$$

To check if $a + b$ is a root

$$a + b = 0 \Rightarrow a = -b$$
$$P(-b) = (-b)^n + b^n$$

Which equals:

$$2 \cdot b^n\ (n\ is\ even)$$
$$0\ (n\ is\ odd)$$

$a^n + b^n$ is divisible by $a + b$ when n is odd.
$a^n + b^n$ is not divisible by $a + b$ when n is even.

We need to count number of two-digit odd numbers

$$= 45$$

Example 1.97
If $X = \{4^n - 3n - 1 : n \in \mathbb{N}\}$ and $Y = \{9(n-1) : n \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of natural numbers, then $X \cup Y$ is equal to: (JEE-M 2014)

The set Y is the multiples of 9 starting from 0:

$$\{0,9,18,27,\dots\}$$

For the set X we get

$$n = 1: 4^1 - 3(1) - 1 = 0$$
$$n = 2: 4^2 - 3(2) - 1 = 16 - 6 - 1 = 9$$

$$X \cup Y = \{All\ multiples\ of\ 9\ starting\ from\ 0\ and\ increasing\}$$

Method I: Binomial Theorem

$$(1+3)^n - 3n - 1$$
$$= \left[1 + \binom{n}{1}3^1 + \binom{n}{2}3^2 + \cdots\right] - 3n - 1$$
$$= \left[1 + 3n + \binom{n}{2}3^2 + \cdots\right] - 3n - 1$$
$$= \binom{n}{2}3^2 + \binom{n}{3}3^3 + \cdots$$
$$= 9\left[\binom{n}{2} + \binom{n}{3}3 + \cdots\right]$$
$$= 9K, K \in \mathbb{N}$$

Method II: Mod Arithmetic

$$4^n - 3n - 1 \equiv 0\ (mod\ 9)$$
$$4^n \equiv 3n + 1\ (mod\ 9)$$

Consider $n = 3k + m, k \in \mathbb{N}$
Case I: $m = 0$

$$LHS \equiv\ 4^n \equiv 4^{3k} \equiv 64^k \equiv 1^k \equiv 1$$
$$RHS \equiv 3(3) + 1 \equiv 1$$

Case II: $m = 1$

$$LHS \equiv\ 4^n \equiv 4^{3k+1} \equiv 4^{3k} \cdot 4 \equiv 64^k \cdot 4 \equiv 1^k \cdot 4 \equiv 4$$
$$RHS \equiv 3(1) + 1 \equiv 4$$

Case III: $m = 2$

$$LHS \equiv\ 4^n \equiv 4^{3k+2} \equiv 4^{3k} \cdot 16 \equiv 64^k \cdot 7 \equiv 1^k \cdot 7 \equiv 7$$
$$RHS \equiv 3(2) + 1 \equiv 7$$

1.7 NT: Other Topics

A. Number of Factors

Example 1.98
Recall that the number of factors of x with prime factorization $p^a q^b \dots$ is given, using the multiplication principle of counting, by:

$$(a+1)(b+1)\dots$$

Let $N = 69^5 + 5 \times 69^4 + 10 \times 69^3 + 10 \times 69^2 + 5 \times 69 + 1$. How many positive integers are factors of N? (AHSME 1986/23)

Collapse

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

The expression in the question meets the pattern above:

$$\underbrace{69^5 + 5\times 69^4 + 10\times 69^3 + 10\times 69^2 + 5\times 69\ +\ 1}_{a=69,b=1} = (69+1)^5 = 70^5$$

Prime Factorize

We can write the prime factorization of 70 as follows:

$$70^5 = (2\times 5\times 7)^5$$

And then, using the property $(ab)^m = a^m b^m$, we can write:

$$2^5 \times 5^5 \times 7^5$$

Number of Factors

And, then using the formula for the number of factors of a number:

$$(5+1)(5+1)(5+1) = 6^3 = 216\ Factors$$

B. Diophantine Equations

Example 1.99

The natural number m, for which the coefficient of x in the binomial expansion of $\left(x^m + \frac{1}{x^2}\right)^{22}$ is 1540 is: **(JEE-M 2020)**

The general term is

$$\binom{22}{r}(x^m)^{22-r}(x^{-2})^r = \binom{22}{r}x^{22m-mr-2r}$$
$$x^{22m-mr-2r} = x^1$$
$$22m - mr - 2r = 1$$
$$m = \frac{1+2r}{22-r}$$

$$\binom{22}{1} = 22, \binom{22}{2} = 231, \binom{22}{3} = \binom{22}{19} = 1540 \Rightarrow r = 3\ or\ r = 19$$

Try $r = 3$:

$$m = \frac{1+2(3)}{22-3} = \frac{1+2(3)}{22-3} = \frac{7}{19} \Rightarrow Not\ Valid$$

Try $r = 19$:

$$m = \frac{1+2(19)}{22-19} = \frac{39}{3} = \mathbf{13}$$

Example 1.100

A. Determine all ordered pairs of natural numbers (m, r) such that $22m - mr - 2r = 1$.
B. Explain how part A would be useful in the context of the previous example.

Part A

Solve the given equation for r:

$$r = \frac{22m-1}{m+2}$$

Rewrite and divide:

$$r = \frac{22m + 44 - 44 - 1}{m + 2} = 22 - \frac{45}{m + 2}$$

Hence $m + 2$ must be a factor of 45

$$m + 2 \in \{1,3,5,9,15,45\}$$
$$m \in \{-1,1,3,7,13,43\}$$

Reject $m = -1$ since m is a natural number:

$$m \in \{1,3,7,13,43\}$$

Part B

If the condition for r had been more complicated, then we could have determined the solution set for m first, and then worked with the condition for r.

Example 1.101

$$X = \left(\sqrt{x} - \frac{6}{x^{\frac{3}{2}}}\right)^n, n \leq 15$$

Let α be the constant term in the binomial expansion of X. If the sum of the coefficients of the remaining terms is 649, and the coefficient of x^{-n} is $\lambda\alpha$ then $\lambda =$ (JEE-M 2023)

Comparing with $(ax + by)^n$:

$$X = \left(x^{\frac{1}{2}} + (-6)x^{-\frac{3}{2}}\right)^n \Rightarrow p = x^{\frac{1}{2}}, q = x^{-\frac{3}{2}}$$

Substitute in pq:

$$pq = \left(x^{\frac{1}{2}}\right)^{n-r}\left(x^{-\frac{3}{2}}\right)^r = x^{\frac{n-r}{2}} \cdot x^{-\frac{3}{2}r} = x^{\frac{n-r}{2}-\frac{3}{2}r}$$

Since the power of the above expression must be zero in the constant term:

$$\frac{n-r}{2} - \frac{3}{2}r = 0 \Rightarrow \underbrace{n = 4r}_{Equation\ I}$$

$$Sum\ of\ coefficients = \left(\sqrt{1} - \frac{6}{1^{\frac{3}{2}}}\right)^n = (1-6)^n = (-5)^n$$

$$\alpha = Constant\ Term = \binom{n}{0.25n}(-6)^{\frac{n}{4}}$$

$$\underbrace{(-5)^n + \binom{n}{0.25n}(-6)^{\frac{n}{4}} = 649}_{Equation\ II}$$

n and r are positive integers. Since $n \leq 15$, there the only possible cases are:

$$(r, n) \in \{(1,4), (2,8), (3,12)\}$$

Case I: Try $r = 1, n = 4$:

$$(-5)^n + \binom{n}{0.25n}(-6)^{\frac{n}{4}} = (-5)^4 - \binom{4}{1}(-6)^{\frac{4}{4}} = 625625 - (-24) = 649 \Rightarrow Valid$$

To find the term number, where $x^{-n} = x^{-4}$

$$\frac{4-r}{2} - \frac{3}{2}r = -4 \Rightarrow 4 - 4r = -8 \Rightarrow r = 3$$

Substitute $n = 4, r = 3$ to get:

$$\lambda = \frac{\lambda\alpha}{\alpha} = \frac{\binom{n}{r} x^{n-r} y^r}{\alpha} = \frac{\binom{4}{1}(1^1)(-6)^3}{-24} = \frac{-24 \cdot 36}{-24} = 36$$

1.8 Product of Expansions

A. Product of Expansions

Example 1.102: Binomial × Binomial

Consider $(1+x)^5(1-x)^4$, which consists of a binomial multiplied by a binomial. Find the:

A. first three terms
B. number of terms in the expanded expression.

Part A

The given expression consists of two binomials. We can expand each individually:

$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$
$$(1-x)^4 = 1 - 4x + 6x^2 - 4x^3 + x^4$$

$$First\ Term = 1 \times 1 = 1$$
$$Second\ Term = (1)(-4x) + (1)(5x) = -4x + 5x = x$$
$$Third\ Term = (1)(10x^2) + (-4x)(5x) + (1)(6x^2) = 10x^2 - 20x^2 + 6x^2 = -4x^2$$

Part B

$$Lowest\ power\ of\ x = 0$$
$$Highest\ power\ of\ x = 5 + 4 = 9$$

We will have all powers in between. Hence,

$$No. of\ Terms = 9 - 0 + 1 = 10$$

Example 1.103: Binomial × Trinomial

In the expansion of $(1+t+t^2)(1-t)^7$, find the:

A. First three terms
B. term with power t^6
C. number of terms

Part A

Expand:

$$(1-t)^7 = 1 + \binom{7}{1}(-t) + \binom{7}{2}(-t)^2 + \cdots = 1 - 7t + 21t^2 + \cdots$$

Note that we only need the first three terms. Because the further terms will have powers

$$(-t)^3, (-t)^4, \ldots$$

Which will never give a power that we want (such as t or t^2).

$$(1+t+t^2)(1-7t+21t^2+\cdots)$$

Constant Term:

$$1 \times 1 = 1$$

Term with t:

$$[1 \times t] + [(-7t) \times 1] = t - 7t = -6t$$

Term with t^2:

$$[(1)(21t^2)] + [(t)(-7t)] + [(t^2)(1)] = 21t^2 - 7t^2 + t^2 = 15t^2$$

Add the above three terms to get:

$$1 - 6t + 15t^2$$

Part B

Consider:

$$(\mathbf{1} + \boldsymbol{t} + \boldsymbol{t^2})$$

We can get:

$$t^6 = \boldsymbol{t^6} \times \boldsymbol{t^0} = \boldsymbol{t^5} \times \boldsymbol{t^1} = \boldsymbol{t^4} \times \boldsymbol{t^2}$$

Hence, we need the red terms above from $(1-t)^7$:

$$\binom{7}{4}t^4 - \binom{7}{5}t^5 + \binom{7}{6}t^6 = \binom{7}{4}t^4 - \binom{7}{5}t^5 + \binom{7}{6}t^6 = 35t^4 - 21t^5 + 7t^6$$

$$(1 + t + t^2)(\dots + 35t^4 - 21t^5 + 7t^6 + \cdots) = (1)(7t^6) + (t)(-21t^5) + (t^2)(35t^4) = 21t^6$$

Part C

$$(1 + t + t^2)(1-t)^7 = (1 + t + t^2)(1 - 7t + \cdots - t^7)$$

When we multiply the above, we will get terms:

$$t^0, t^1, \dots, t^9 \Rightarrow 10\ Terms$$

Example 1.104

Coefficient of t^{24} in $(1+t^2)^{12}(1+t^{12})(1+t^{24})$ is (JEE-A 2003S)

Multiply the last two terms together:

$$(1+t^2)^{12}(\mathbf{1} + \boldsymbol{t^{12}} + \boldsymbol{t^{24}} + t^{36})$$

Identify the terms that will multiply to give t^{24}:

$$\mathbf{1} \times \boldsymbol{t^{24}} + \boldsymbol{t^{12}} \times \binom{\mathbf{12}}{\mathbf{6}}\boldsymbol{t^{12}} + \boldsymbol{t^{24}} \times \mathbf{1}$$

Factor out t^{24}:

$$= \left[1 + \binom{12}{6} + 1\right]t^{24} = \left[\binom{12}{6} + 2\right]t^{24} = 926t^{24}$$

$$Coefficient\ is\ 924$$

Example 1.105: Back Calculations

If in the expansion of $(1+x)^m(1-x)^n$, the coefficients of x and x^2 are 3 and -6 respectively, then m is ____ (JEE-A 1999)

$$(1+x)^m = 1 + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots$$

$$(1-x)^n = 1 - \binom{n}{1}x + \binom{n}{2}x^2 + \cdots$$

The x term must be:

$$1 \times \binom{m}{1}x + 1 \times \left(-\binom{n}{1}x\right) = \left[\binom{m}{1} - \binom{n}{1}\right]x = (m-n)x \Rightarrow \underbrace{m - n = 3}_{Equation\ I}$$

The x^2 term must be:

$$1 \times \binom{m}{2}x^2 + \left[\binom{m}{1}x\right]\left[-\binom{n}{1}x\right] + 1 \times \binom{n}{2}x^2 = \left(\frac{m(m-1) - 2mn + n(n-1)}{2}\right)x^2$$

$$m^2 - 2mn + n^2 - m - n = (-6)(2) \Rightarrow (m-n)^2 - (m+n) = -12 \Rightarrow \underbrace{m + n = 21}_{Equation\ II}$$

Add Equations I and II:

$$2m = 24 \Rightarrow m = 12$$

Example 1.106: Diophantine Equations

Coefficient of x^{11} in the expansion of $(1+x^2)^4(1+x^3)^7(1+x^4)^{12}$ is **(JEE-A 2014)**

General Term
Find the general term of each term in the above

$$(1+x^2)^4 \ has \ (a+1)^{st} term = \binom{4}{a} x^{2a}$$
$$(1+x^3)^7 \ has \ (b+1)^{st} term = \binom{7}{b} x^{3b}$$
$$(1+x^4)^{12} \ has \ (c+1)^{st} term = \binom{12}{c} x^{4c}$$

Required Terms
To have a power of 11, we must have:

$$2a+3b+4c = 11, a, b, c \in N$$

We do this using casework:

$$c = 0 \Rightarrow 2a+3b = 11 \Rightarrow (2a, 3b, 4c) = \{(8,3,0), (2,9,0)\} \Rightarrow (a, b, c) = (4,1,0)(1,3,0)$$
$$c = 1 \Rightarrow 2a+3b = 7 \Rightarrow (2a, 3b, 4c) = \{(4,3,4)\} \Rightarrow (a, b, c) = (2,1,1)$$
$$c = 2 \Rightarrow 2a+3b = 3 \Rightarrow (2a, 3b, 4c) = \{(0,3,8)\} \Rightarrow (a, b, c) = (0,1,2)$$

Coefficient

$$\binom{4}{4}\binom{7}{1}\binom{12}{0} + \binom{4}{1}\binom{7}{3}\binom{12}{0} + \binom{4}{2}\binom{7}{1}\binom{12}{1} + \binom{4}{0}\binom{7}{1}\binom{12}{2} = 7 + 140 + 504 + 462 = 1113$$

B. Constant Term

The constant term in a binomial expansion is also called the term independent of the variable.

Example 1.107: Basics

$$\left(2x^2 - \frac{3}{x}\right)^6$$

$$\binom{n}{r} x^{n-r} y^r$$

$$(x^2)^{6-r}\left(\frac{1}{x}\right)^r = x^0$$

Use the property: $(a^m)^n = a^{mn}$

$$x^{12-2r}x^{-r} = x^0$$
$$x^{12-2r-r} = x^0$$
$$x^{12-3r} = x^0$$
$$12 - 3r = 0$$
$$r = 4$$

$$\binom{6}{4}(x^2)^{6-4}\left(\frac{3}{x}\right)^4 = 15x^4\left(\frac{81}{x^4}\right) = 1{,}215$$

Example 1.108: Reducible to Binomial

$$\left(2y - \frac{1}{y}\right)^6 \left(\frac{1}{2y} + y\right)^6$$

Find the

A. term independent of y in the above expression.
B. number of terms in the expanded expression.

Part A

We can multiply the two expressions to get a binomial expression:

$$\left[\left(2y-\frac{1}{y}\right)\left(\frac{1}{2y}+y\right)\right]^6=\left[1+2y^2-\frac{1}{2y^2}-1\right]^6=\left[2y^2-\frac{1}{2y^2}\right]^6$$

We now want the term independent of y, which will meet the condition that

$$(y^2)^{6-r}\left(\frac{1}{y^2}\right)^r=y^0\Rightarrow(y^{12-2r})(y^{-2r})=y^0\Rightarrow y^{12-4r}=y^0\Rightarrow 12-4r=0\Rightarrow r=3$$

Find the term with $r=3$, which is the fourth term:

$$\binom{6}{3}(2y)^3\left(-\frac{1}{2y}\right)^3=\left(\frac{\mathbf{6}\times 5\times 4}{\mathbf{6}}\right)\left(\frac{-\mathbf{2^3y^3}}{\mathbf{2^3y^3}}\right)=5\times 4\times(-1)=-20$$

Part B

Every term in the above expression has a unique power of y. Hence, the number of terms in the above expression is:

$$6+1=7$$

Example 1.109: Binomial × Binomial

Find the term independent of x in $(2-x)^3\left(\frac{1}{3x}-x\right)^6$. Also, find the number of terms.

Expand each part of the expression

The first part gives us:

$$(2-x)^3=\mathbf{8}-12x+\mathbf{6x^2}+x^3$$

The second part gives us:

$$\binom{6}{0}\left(-\frac{1}{3x}\right)^6x^0\Rightarrow x\ term\ is\ \frac{1}{x^6}$$

$$\binom{6}{1}\left(-\frac{1}{3x}\right)^5x^1\Rightarrow x\ term\ is\ \frac{1}{x^4}$$

$$\boldsymbol{\binom{6}{2}\left(-\frac{1}{3x}\right)^4x^2\Rightarrow x\ term\ is\ \frac{1}{x^2}}$$

$$\boldsymbol{\binom{6}{3}\left(-\frac{1}{3x}\right)^3x^3\Rightarrow x\ term\ is\ \frac{1}{x^0}=1}$$

$$\binom{6}{4}\left(-\frac{1}{3x}\right)^2x^4\Rightarrow x\ term\ is\ x^2$$

$$\binom{6}{5}\left(-\frac{1}{3x}\right)^1x^5\Rightarrow x\ term\ is\ x^4$$

$$\binom{6}{6}\left(-\frac{1}{3x}\right)^0x^6\Rightarrow x\ term\ is\ x^6$$

Determine the terms to multiply

We can get x^0 in two ways

$$\underbrace{8}_{\substack{Constant\\Term}}\times\underbrace{\binom{6}{3}\left(-\frac{1}{3x}\right)^3x^3}_{Constant\ Term}=8\left(\frac{\mathbf{6}\times 5\times 4}{\mathbf{6}}\right)\left(-\frac{1}{27}\right)=\frac{-160}{27}$$

$$\underbrace{6x^2}_{\substack{x^2\\Term}}\times\underbrace{\binom{6}{2}\left(-\frac{1}{3x}\right)^4x^2}_{\frac{1}{x^2}\ Term}=6\boldsymbol{x^2}\left(\frac{\mathbf{6}\times 5}{\mathbf{2}}\right)\left(\frac{1}{81\boldsymbol{x^2}}\right)=\frac{30}{27}$$

Add the two to get:

$$\frac{-160}{27}+\frac{30}{27}=-\frac{130}{27}$$

C. Further Applications

1.110: Trinomial as a Binomial

A trinomial expansion can be "forced" into a binomial expansion by clubbing terms.

Example 1.111

Find the term independent of x in

$$\left(2x+1-\frac{1}{2x^2}\right)^6$$

Consider the above trinomial as a binomial by writing it as:

$$\left((2x+1)-\frac{1}{2x^2}\right)^6$$

Consider the expansion of the above binomial in descending powers of $(2x+1)$:

The first term

$$\binom{6}{0}\frac{(2x+1)^6}{(-2x^2)^0}=(2x+1)^6\Rightarrow Independent\ Term=\mathbf{1}$$

The independent term in the second term is:

$$\binom{6}{1}\frac{(2x+1)^5}{(-2x^2)^1}=6\left[\frac{\dots+\binom{5}{2}(2x)^2+\cdots}{-2x^2}\right]=3\left[\frac{\dots+10\cdot 4x^2+\cdots}{-x^2}\right]=\cdots+(\mathbf{-120})+\cdots$$

The independent term in the third term is:

$$\binom{6}{2}\frac{(2x+1)^4}{(-2x^2)^2}=15\left[\frac{(2x)^4+\cdots}{4x^4}\right]=15\left[\frac{16x^4+\cdots}{4x^2}\right]=\mathbf{60}+\cdots$$

In the fourth term (see below), the power of the denominator is more than the power of the numerator, and hence there will be term independent of x:

$$\binom{6}{3}\frac{(2x+1)^3}{(-2x^2)^3}$$

This is also true of the fifth, sixth and seventh terms since the power of the denominator will keep increasing.

Hence, the final answer is the sum from the first, second, and third terms:

$$1-120+60=-59$$

1.9 Sum and Difference

A. Sum

Till now, we have been looking at using the Binomial Theorem to expand binomials. Now, we look at applications where we have the sum or the difference of the expansion of two expressions, which are conjugates of each other.

Example 1.112

$$a = (x+y)^n, \qquad b = (x-y)^n$$

Find $\underbrace{a+b}_{Sum}, \ \underbrace{a-b}_{Difference}$ for $n = 1,2,3$, and conjecture what is the general pattern that applies

$\boldsymbol{n = 1}$

$$a + b = (x + y) + (x - y) = \underbrace{2x}_{1st \times 2}$$
$$a - b = (x + y) - (x - y) = \underbrace{2y}_{2nd \times 2}$$

$\boldsymbol{n = 2}$

$$a + b = (x+y)^2 + (x-y)^2 = (x^2 + \mathbf{2xy} + y^2) + (x^2 - \mathbf{2xy} + y^2) = \underbrace{2x^2}_{1st\times 2} + \underbrace{2y^2}_{3rd\times 2}$$
$$a - b = (x+y)^2 - (x-y)^2 = (\boldsymbol{x^2} + 2xy + \boldsymbol{y^2}) - (\boldsymbol{x^2} - 2xy + \boldsymbol{y^2}) = \underbrace{4xy}_{2nd\times 2}$$

$\boldsymbol{n = 3}$

$$(x+y)^3 + (x-y)^3 = (x^3 + 3x^2y + 3xy^2 + y^3) + (x^3 - 3x^2y + 3xy^2 - y^3)$$
$$= 2x^3 + 6x^2y$$

We don't know if the pattern applies further, but we can conjecture that:

- Sum: We get double of the odd terms
- Difference: We get double of the even terms

1.113: Sum of Binomial Conjugate Expansions

The sum of binomial conjugate expansion is twice each of the odd numbered terms.

$$(x+y)^n = \binom{n}{0}x^ny^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots$$
$$(x-y)^n = \binom{n}{0}x^ny^0 - \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 - \binom{n}{3}x^{n-3}y^3 + \cdots$$

Add the two equations above and note that the violet terms (even terms) all cancel:

$$(x+y)^n + (x-y)^n = 2\left[\underbrace{\binom{n}{0}x^ny^0}_{1st\ Term} + \underbrace{\binom{n}{2}x^{n-2}y^2}_{3rd\ Term} + \underbrace{\binom{n}{4}x^{n-4}y^4}_{5th\ Term} + \underbrace{\binom{n}{6}x^{n-6}y^6}_{7th\ Term} + \cdots\right]$$
$$(x+y)^n + (x-y)^n = 2\sum_{r \in Even}^{n}\binom{n}{r}x^{n-r}y^r$$

Example 1.114

If α and β be the coefficients of x^4 and x^2 respectively in the expansion of $\left(x + \sqrt{x^2-1}\right)^6 + \left(x - \sqrt{x^2-1}\right)^6$, then $\alpha - \beta$: (JEE-M 2018, 2019, 2020)

We want odd numbered terms:

$$2(T_1 + T_3 + T_5 + T_7)$$
$$= 2\left[\binom{6}{0}x^6 + \binom{6}{2}x^4\left(\sqrt{x^2-1}\right)^2 + \binom{6}{4}x^2\left(\sqrt{x^2-1}\right)^4 + \binom{6}{6}\left(\sqrt{x^2-1}\right)^6\right]$$
$$= 2[x^6 + 15x^4(x^2-1) + 15x^2(x^2-1)^2 + (x^2-1)^3]$$

$$= 2\left[x^6 + 15\left(x^6 - x^4\right) + 15x^2\left(x^4 - 2x^2 + 1\right) + \left(x^6 - 3x^4 + 3x^2 - 1\right)\right]$$
$$= 2[15(-x^4) + 15x^2(-2x^2 + 1) + (-3x^4 + 3x^2)]$$
$$= 2[-15x^4 - 30x^4 + 15x^2 + -3x^4 + 3x^2]$$
$$= 2[-48x^4 + 18x^2]$$
$$= -96x^4 + 36x^2$$

$$\alpha - \beta = -96 - 36 = -132$$

B. Difference

1.115: Difference of Binomial Conjugate Expansions

The difference of binomial conjugate expansions is twice each of the even numbered terms.

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots$$
$$(x-y)^n = \binom{n}{0}x^n y^0 - \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 - \binom{n}{3}x^{n-3}y^3 + \cdots$$

Subtract the second equation from the first, and note the violet terms (odd numbered terms) vanish:

$$(x+y)^n - (x-y)^n = 2\left[\underbrace{\binom{n}{1}x^{n-1}y^1}_{2nd\ Term} + \underbrace{\binom{n}{3}x^{n-3}y^3}_{4th\ Term} + \underbrace{\binom{n}{5}x^{n-5}y^5}_{6th\ Term} + \underbrace{\binom{n}{7}x^{n-7}y^7}_{8th\ Term} + \cdots\right]$$

$$(x+a)^n - (x-a)^n = 2\sum_{r\in Odd}^{n}\binom{n}{r}x^{n-r}y^r$$

Example 1.116

$$(a+b)^2 + (a-b)^2$$

$$(a+b)^2 + (a-b)^2 = a^2 + 2ab + b^2 + a^2 - 2ab + b^2 = 2(a^2 + b^2)$$

$$(a+b)^2 - (a-b)^2 = a^2 + 2ab + b^2 - [a^2 - 2ab + b^2] = 4ab$$

Example 1.117

Find the integer closest to $\left(2+\sqrt{5}\right)^4$ using the Binomial Theorem. Also, find the error in your approximation using a calculator.

We could do a brute force expansion, but that is not very useful.

$$\left(2+\sqrt{5}\right)^4 = 16 + \underbrace{(4)(8)\left(\sqrt{5}\right)}_{2nd\ Term} + (6)(4)(5) + \underbrace{(4)(2)\left(5\sqrt{5}\right)}_{4th\ Term} + 25$$

We want to be rid of the radical terms, which are all the even terms.

$$\left(2-\sqrt{5}\right)^4 \approx (2-2.23)^4 \approx \left(-\frac{1}{4}\right)^4 = \frac{1}{256} \approx Very\ Small\ Number$$

Hence, we are going to find:

$$\left(2+\sqrt{5}\right)^4 + \left(2-\sqrt{5}\right)^4 = 2\left[\underbrace{\binom{4}{0}(2^4)\left(\sqrt{5}\right)^0}_{1st\ Term} + \underbrace{\binom{4}{2}(2^2)\left(\sqrt{5}\right)^2}_{3rd\ Term} + \underbrace{\binom{4}{4}(2^0)\left(\sqrt{5}\right)^4}_{5th\ Term}\right]$$

Simplify:

$$2[16 + (6)(4)(5) + 25] = 2[161] = 322$$

$$Error = 322 - \left(2+\sqrt{5}\right)^4 \approx 0.0031$$

Example 1.118

The larger of $99^{50} + 100^{50}$ and 101^{50} is: (JEE-A 1982)

Let

$$a = 101^{50} = (100+1)^{50}$$
$$b = 99^{50} + 100^{50} = (100-1)^{50} + 100^{50}$$

From the above, we know that we can use the formula for difference of binomial conjugates:

$$a - b$$
$$= (100+1)^{50} - (100-1)^{50} - 100^{50}$$
$$= 2\left[\binom{50}{1}100^{49} + \binom{50}{3}100^{47} + \cdots\right] - 100^{50}$$
$$= \left[2 \times 50 \times 100^{49} + 2 \times \binom{50}{3}100^{47} + \cdots\right] - 100^{50}$$
$$= \left[\mathbf{100^{50}} + 2 \times \binom{50}{3}100^{47} + \cdots\right] \mathbf{- 100^{50}}$$
$$= \left[2 \times \binom{50}{3}100^{47} + \cdots\right]$$
$$= some + ve\ quantity$$

Since

$$a - b > 0 \Rightarrow a > b$$

Example 1.119

If n is a positive integer, then $\left(\sqrt{3}+1\right)^{2n} - \left(\sqrt{3}-1\right)^{2n}$ is: (JEE-M 2012)

a) An irrational number
b) an odd positive integer
c) an even positive integer
d) a rational number other than positive integers

$$\left(\sqrt{3}+1\right)^{2n} - \left(\sqrt{3}-1\right)^{2n} = 2\left[\binom{n}{1}\left(\sqrt{3}\right)^{2n-1} + \cdots\right] = 2\left[\binom{n}{1}3^{\frac{2n-1}{2}} + \cdots\right]$$

Since $\frac{2n-1}{2} = n - \frac{1}{2}$ is not an integer, the final answer will be irrational.
Option A.

Example 1.120[3]

The number of elements in the set (JEE-M 2021)

$$\{n \in \{1,2,3,\dots,100\}, 11^n > 10^n + 9^n\}$$

$$11^n - 9^n > 10^n$$
$$(10+1)^n - (10-1)^n > 10^n$$

Use the formula for the difference of two binomial conjugates:

$$2T_2 + 2T_4 + \cdots > 10^n$$

Since $T_4, T_6, T_8, \dots > 0$

$$2T_2 \ge 10^n \Rightarrow 2n10^{n-1} \ge 10^n \Rightarrow n \ge 5$$

[3] This is solved using Inequality concepts in the Note on that topic.

Check for values of $n < 5$ using $2[T_2 + T_4 + \cdots] = 2\left[\binom{n}{1}10^{n-1} + \binom{n}{3}10^{n-3} + \cdots\right]$:

$$n = 1: T_2 = 2\left[\binom{1}{1}10^{1-1}\right] = 2 < 10$$

$$n = 2: T_2 = 2\left[\binom{2}{1}10^{2-1}\right] = 40 < 100$$

$$n = 3: T_2 + T_4 = 2\left[\binom{3}{1}10^{3-1} + \binom{3}{3}10^{3-3}\right] = 602 < 1000$$

$$n = 4: T_2 + T_4 = 2\left[\binom{4}{1}10^{4-1} + \binom{4}{3}10^{4-3}\right] = 2[4000 + 40] = 8080 < 10{,}000$$

$$\{5,6,7,\ldots,100\} \Rightarrow 100 - 5 + 1 = 96\ Values$$

Example 1.121: Number of Terms

C. Informal

Example 1.122

Find the integer part of $\left(\sqrt{3}+1\right)^6$.

Add $y = \left(\sqrt{3}-1\right)^6$ to both sides of $x = \left(\sqrt{3}+1\right)^6$

$$\underbrace{x + y = \left(\sqrt{3}+1\right)^6 + \left(\sqrt{3}-1\right)^6}_{\textit{Equation I}}$$

Using the sum of binomial conjugates, the RHS is:

$$\underbrace{\left(\sqrt{3}+1\right)^6 + \left(\sqrt{3}-1\right)^6 = 2\left[\left(\sqrt{3}\right)^6 + \binom{6}{2}\left(\sqrt{3}\right)^4 + \binom{6}{4}\left(\sqrt{3}\right)^2 + 1\right] = 2[27 + 15(9) + 15(3) + 1] = 416}_{\textit{Result 1}}$$

$$x + y = 416 \Rightarrow x = 416 - y$$

Apply the integer function both sides:

$$\lfloor x \rfloor = \lfloor 416 - y \rfloor$$

Since $y = \left(\sqrt{3}-1\right)^6 \approx (1.73-1)^6 = (0.73)^6 \rightarrow \textit{very small positive number}$.

$$\lfloor x \rfloor = 415$$

Example 1.123

D. A More Formal Version

We can use the sum of two binomial conjugates to find the integer part of a binomial expansion.[4]

1.124: Integer and Fractional Part

Any number can be written as the sum of its integer and fractional part

$$x = \lfloor x \rfloor + \{x\}$$

[4] Under certain conditions, of course. We will see the conditions after we see the technique.

Where

$$\lfloor x \rfloor = Integer\ Part\ OR\ Greatest\ Integer\ Function$$
$$\{x\} = Fractional\ Part$$

$$0 < \{x\} < 1$$

$$4.75 = \underbrace{4}_{Integer} + \underbrace{0.75}_{Fractional}$$

1.125: A useful result

Given

$$0 < y < 1, \qquad \{x\} + y \in \mathbb{Z}$$

We can conclude:

$$\{x\} + y = 1$$

$$\underbrace{0 < y < 1}_{Inequality\ I}$$

The fractional part of a number is always between 0 and 1:

$$\underbrace{0 < \{x\} < 1}_{Inequality\ II}$$

Add the two inequalities:

$$0 < \{x\} + y < 2$$

If $\{x\} + y$ is an integer, then the only possible value it can take is:

$$\{x\} + y = 1$$

Example 1.126

A. Find the integer part of $\left(\sqrt{3}+1\right)^6$.
B. Verify your answer using a calculator.

Using the sum of binomial conjugates, we get the odd numbered terms:

$$\underbrace{\left(\sqrt{3}+1\right)^6 + \left(\sqrt{3}-1\right)^6 = 2\left[\left(\sqrt{3}\right)^6 + \binom{6}{2}\left(\sqrt{3}\right)^4 + \binom{6}{4}\left(\sqrt{3}\right)^2 + 1\right] = 2[27 + 15(9) + 15(3) + 1] = 416}_{Result\ 1}$$

Note that we have only even powers, which is very nice since we no longer have any radicals.

Let:

$$x = \lfloor x \rfloor + \{x\} = \left(\sqrt{3}+1\right)^6$$

Add $y = \left(\sqrt{3}-1\right)^6$ to both sides,

$$\underbrace{\lfloor x \rfloor + \{x\} + y = \left(\sqrt{3}+1\right)^6 + \left(\sqrt{3}-1\right)^6}_{Equation\ I}$$

Substitute Result 1:

$$\underbrace{\lfloor x \rfloor}_{Integer} + \{x\} + y = \underbrace{416}_{Integer}$$

Since the RHS is an integer, the LHS must also be an integer. On the LHS, $\lfloor x \rfloor$ is an integer. Hence,

$$\{x\} + y \in \mathbb{Z}$$

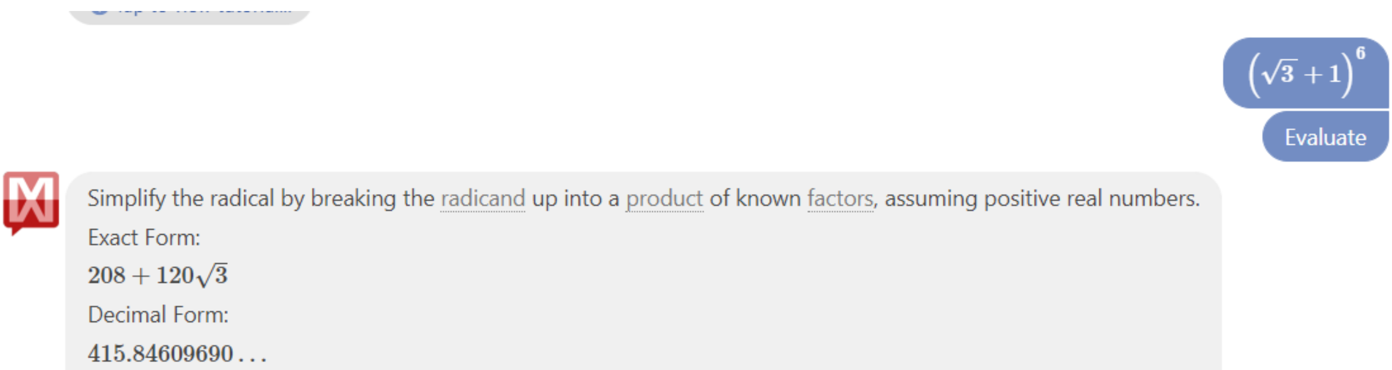
Since

$$y = \left(\sqrt{3} - 1\right)^6 \approx (1.73 - 1)^6 = (0.73)^6 \Rightarrow 0 < y < 1$$

Substitute the result $0 < \{x\} < 1,\ \ 0 < y < 1,\ \ \{x\} + y \in \mathbb{Z} \Rightarrow \{x\} + y = 1$ in the above:

$$\lfloor x \rfloor + 1 = 416$$
$$\lfloor x \rfloor = 415$$

We can verify our work using a calculator:



Example 1.127

$$\sqrt{a} + 1$$

Consider the method from the previous example, which was used to find the integer part of the expression with $a = 3$.

A. Explain why the method will work with $a = 2$
B. Explain why the method will work with $a = 5$
C. Generalize your answer to Parts A and B

Part A

It will work with $a = 2$

Part B

$$\sqrt{5} - 1 \approx 2.23 - 1 = 1.23 > 1$$

Hence, this method will not work to find the integer part of:

$$\sqrt{5} + 1$$

$$y = \left(\sqrt{5} - 1\right)^n \approx (2.23 - 1)^n = (1.23)^n = \textit{some large number}$$

1.128: A useful result

Given

$$0 < y < 1, \qquad \{x\} + y \in \mathbb{Z}$$

We can conclude:

$$\{x\} - y = 0$$

$$\underbrace{0 < y < 1}_{Inequality\ I}$$

The fractional part of a number is always between 0 and 1:

$$\underbrace{0 < \{x\} < 1}_{Inequality\ II}$$

$$Max(\{x\} - y) = Max(\{x\}) - Min(y) = 1 - 0 = 1$$
$$Min(\{x\} - y) = Min(\{x\}) - Max(y) = 0 - 1 = -1$$

Note that since the endpoints are not included in the inequalities, 1 and -1 cannot be achieved, but they are the endpoints of what can be achieved.

$$-1 < \{x\} - y < 1$$

If $\{x\} + y$ is an integer, then the only possible value it can take is:

$$\{x\} - y = 0$$

Example 1.129

Explain why the fractional part of $\left(8\sqrt{3} + 13\right)^{13}$ is given by $\left(8\sqrt{3} - 13\right)^{13}$. In other words, explain why

$$\left\{\left(8\sqrt{3} + 13\right)^{13}\right\} = \left(8\sqrt{3} - 13\right)^{13}$$

Where

$$\{x\} = Fractional\ part\ of\ x$$

Let

$$x = \lfloor x \rfloor + \{x\} = \left(8\sqrt{3} + 13\right)^{13}$$

Subtract $y = \left(8\sqrt{3} - 13\right)^{13}$ from both sides:

$$\lfloor x \rfloor + \{x\} - Y = \left(8\sqrt{3} + 13\right)^{13} - \left(8\sqrt{3} - 13\right)^{13}$$

The RHS is a difference of binomial conjugates. You can simplify keeping double of only the even numbered terms:

$$\lfloor x \rfloor + \{x\} - Y = 2\left[\binom{13}{1}\left(8\sqrt{3}\right)^{12}(13) + \binom{13}{3}\left(8\sqrt{3}\right)^{10}(13)^3 + \cdots + \binom{13}{13}\left(8\sqrt{3}\right)^{0}(13)^{13}\right]$$

Note that the RHS has radicals to an even power only. Hence, the RHS is an even integer.

$$\lfloor x \rfloor + \{x\} - Y \in \mathbb{Z}$$

Since $\lfloor x \rfloor \in \mathbb{Z}$:

$$\{x\} - Yy \in \mathbb{Z}$$
$$Y = \left(8\sqrt{3} - 13\right)^{13} \approx (8 \cdot 1.73 - 13)^{13} = (0.84)^{13} < 1 \Rightarrow 0 < Y < 1$$
$$0 < \{x\} < 1$$

Combine the above three to get:

$$\{x\} - Y = 0 \Rightarrow \{x\} = Y$$

Example 1.130

Explain why the fractional part of $\left(7\sqrt{2}+9\right)^9$ is given by $\left(7\sqrt{2}-9\right)^9$. In other words, explain why

$$\left\{\left(7\sqrt{2}+9\right)^9\right\} = \left(7\sqrt{2}-9\right)^9$$

Where

$$\{t\} = Fractional\ part\ of\ x$$

Let

$$y = \lfloor y \rfloor + \{y\} = \left(7\sqrt{2}+9\right)^9$$

Subtract $Y = \left(7\sqrt{2}-9\right)^9$ from both sides:

$$\lfloor y \rfloor + \{y\} - Y = \left(7\sqrt{2}+9\right)^9 - \left(7\sqrt{2}-9\right)^9$$

The RHS is a difference of binomial conjugates. You can simplify keeping double of only the even numbered terms:

$$\lfloor y \rfloor + \{y\} - Y = 2\left[\binom{9}{1}\left(7\sqrt{2}\right)^8(9) + \cdots + \binom{9}{9}\left(7\sqrt{2}\right)^0(9)^9\right]$$

Note that the RHS has radicals to an even power only. Hence, the RHS is an even integer.

$$\lfloor y \rfloor + \{y\} - Y \in \mathbb{Z}$$

Since $\lfloor y \rfloor \in \mathbb{Z}$:

$$\{y\} - Y \in \mathbb{Z}$$

$$Y = \left(7\sqrt{2}-9\right)^9 \approx (7 \cdot 1.41 - 9)^9 = (0.87)^9 < 1 \Rightarrow 0 < Y < 1$$

$$0 < \{y\} < 1$$

Combine the above three to get:

$$\{y\} - Y = 0 \Rightarrow \{y\} = Y$$

Example 1.131

Let $x = \left(8\sqrt{3}+13\right)^{13}$ and $y = \left(7\sqrt{2}+9\right)^9$. If $\lfloor t \rfloor$ denotes the greatest integer $\le t$, then:

A. $\lfloor x \rfloor + \lfloor y \rfloor$ is even
B. $\lfloor x \rfloor$ is odd but $\lfloor y \rfloor$ is even
C. $\lfloor x \rfloor$ is even but $\lfloor y \rfloor$ is odd
D. $\lfloor x \rfloor$ and $\lfloor y \rfloor$ are both odd **(JEE-M 2023)**

$$\lfloor x \rfloor + \{x\} - Y = \left(8\sqrt{3}+13\right)^{13} - \left(8\sqrt{3}-13\right)^{13}$$

And we also that know that $\{x\} = Y$:

$$\lfloor x \rfloor = \left(8\sqrt{3}+13\right)^{13} - \left(8\sqrt{3}-13\right)^{13}$$

And we also know that the RHS is an even integer:

$$\lfloor x \rfloor \in Even\ Integer$$

$$\lfloor y \rfloor + \{y\} - Y = \left(7\sqrt{2}+9\right)^9 - \left(7\sqrt{2}-9\right)^9$$

And we also that know that $\{y\} = Y$:

$$\lfloor y \rfloor = \left(7\sqrt{2}+9\right)^9 - \left(7\sqrt{2}-9\right)^9$$

And we also know that the RHS is an even integer

$$\lfloor y \rfloor \in Even\ Integer$$

$$Option\ A$$

2. Extending the BT

2.1 Multinomial Theorem

A. Square of an Expression

2.1: Trinomial as a Binomial

A trinomial expansion can be "forced" into a binomial expansion by clubbing terms.

Example 2.2

Expand $(a+b+c)^2$

Use a change of variable. Let $b+c=x$, which gives us:

$$(a+x)^2=a^2+2ax+x^2$$

Substitute $x=b+c$:

$$a^2+2a(b+c)+(b+c)^2$$
$$=a^2+2ab+2ac+b^2+2bc+c^2$$
$$=a^2+b^2+c^2+2ab+2ac+2bc$$

Example 2.3

Expand $(a+b+c+d)^2$

$$(a+b+c+d)(a+b+c+d)$$
$$=a^2+ab+ac+ad+ab+b^2+bc+bd+ca+cb+c^2+cd+da+db+dc+d^2$$
$$=a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)$$

2.4: Square of an expression

$$(x_1+x_2+\cdots+x_n)^2=\sum_{i=1}^{n}x_i^2+2\sum_{1\le i<j\le n}x_ix_j$$

We can recognize a pattern from the previous two examples.

- The square of every term occurs once.
- The product of two distinct terms occurs twice: once from the first term getting multiplied with the second, and the second when the order is reversed.

Example 2.5

Expand $(x_1+x_2+x_3+x_4)^2$

We solved this with different variables when we solved $(a+b+c+d)^2$. This question we can solve using the formula rather than expanding it out:

$$(x_1+x_2+x_3+x_4)^2=\sum_{i=1}^{4}x_i^2+2\sum_{1\le i<j\le 4}x_ix_j$$
$$=x_1^2+x_2^2+x_3^3+x_4^4+2(x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4)$$

Example 2.6

$$(x_1+x_2+x_3+x_4)^2$$

When the above expression is expanded and simplified, determine the number of:

A. Perfect Square terms
B. Non perfect square terms
C. Total Terms

The expression $(x_1 + x_2 + x_3 + x_4)^2$ has four variables $\{x_1, x_2, x_3, x_4\}$, and each of those four variables will be squared and present in the final answer:

$$\{x_1^2, x_2^2, x_3^3, x_4^4\} \Rightarrow 4\ terms$$

The number of variables with two different variables has all possible choices of the variable taken two at a time (where order is not important):

$$\binom{4}{2} = 6$$

2.7: Square of an expression

$$(x_1 + x_2 + \cdots + x_n)^2 = \sum_{i=1}^{n} x_i^2 + 2 \sum_{1 \le i < j \le n} x_i x_j$$

$$Perfect\ Square: n\ terms$$

$$Non\ perfect\ Square: \frac{n(n-1)}{2}\ terms$$

$$Perfect\ Square\ Terms: \{x_1^2, x_2^2, \dots, x_n^2\} \Rightarrow n\ terms$$

The non perfect square terms consist of all possible pairs of the terms, taken two at a time:

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

Example 2.8

$$(a + b + \cdots z)^2$$

The expression above has the letters of the English Alphabet, and each is a distinct real number.

When the above expression is expanded and simplified, determine the number of:

A. Perfect Square terms
B. Non perfect square terms
C. Total Terms

$$Perfect\ Square: 26\ Terms$$

$$Non\ perfect\ Square: \binom{26}{2} = 325\ Terms$$

B. Cube of an Expression

Example 2.9

Expand $(a + b + c)^3$

Let $b + c = x$, which gives us:

$$(a + x)^3 = a^3 + 3a^2 x + 3ax^2 + x^3$$

Substitute $x = b + c$:

$$= a^3 + 3a^2(b + c) + 3a(b + c)^2 + (b + c)^3$$

$$= a^3 + 3a^2b + 3a^2c + 3a(b^2 + 2bc + c^2) + b^3 + 3b^2c + 3bc^2 + c^3$$

Example 2.10

Expand $(a + 2b - 3c)^3$

Use a change of variable. Let $y = 2b - 3c$. Then:
$(a + 2b - 3c)^3 = (a + y)^3$
$= a^3 + 3a^2y + 3ay^2 + y^3$
$= a^3 + 6a^2b - 9a^2c + 12ab^2 - 36abc + 27ac^2 + 8b^3 - 36b^2c + 54bc^2 - 27c^3$

2.11: Number of Terms in a Trinomial

The number of terms in the trinomial expansion $(a + b + c)^n$ is

$$\frac{(n+1)(n+2)}{2}$$

$$(a + b + c)^n = [a + (b + c)]^n$$

Consider the above trinomial as a binomial and use the binomial expansion:

$$\underbrace{a^n(b+c)^0}_{First\ Term = T_1} + \underbrace{\binom{n}{1} a^{n-1}(b+c)}_{Second\ Term = T_2} + \underbrace{\binom{n}{2} a^{n-2}(b+c)^2}_{Third\ Term = T_3} + \cdots + \underbrace{\binom{n}{n} a^0(b+c)^n}_{Last\ terms = T_n}$$

Since our "second term" was itself a binomial, the expression above has some binomial expansions:

$$\underbrace{a^n}_{T_1 = 1\ Term} + \underbrace{na^{n-1}b + na^{n-1}bc}_{T_2 = 2\ Terms} + \underbrace{\frac{n(n-1)}{2} a^{n-2}(b^2 + 2bc + c^2)}_{T_3 = 3\ Terms} + \cdots$$

Because of the binomial expansion, and the result when we multiply it out completely, we have the very useful observation that:

$$T_n \to n\ terms$$

Hence, the total number of terms

$$\underbrace{\underbrace{1}_{T_1}}_{1\ Term} + \underbrace{\underbrace{2}_{T_2}}_{2\ Terms} + \cdots + \underbrace{\underbrace{(n+1)}_{T_{n+1}}}_{n+1\ terms}$$

Using the formula for the sum to n terms, the final answer is:

$$= \frac{(n+1)(n+2)}{2}$$

Example 2.12

If the number of terms in the expansion of $\left(1 - \frac{2}{x} + \frac{4}{x^2}\right)^n, x \neq 0$ is 28, then the sum of the coefficients of all the terms in the terms in this expansion is (JEE-M 2016)

We can determine the number of terms by using the formula above, and solving for n:

$$\frac{(n+1)(n+2)}{2} = 28 \Rightarrow (n+1)(n+2) = 56 \Rightarrow n = 6$$

Substitute $x = 1$ to determine the sum of the coefficients:

$$\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^6=(1-2+4)^6=3^6=729$$

C. Multinomial Coefficients

We begin by considering patterns in expansions which we already know the answer to.

Example 2.13

Determine the values of the coefficients in without expanding it out algebraically:

$$(a+b)^2$$

Note that each term in the expansion has two variables, and the sum of the powers of the two variables is always 2.

$$(a+b)(a+b)=a(a+b)+b(a+b)=aa+ab+ba+b^2$$

Coefficient of a:

The number of ways to choose both a will give the coefficient of a^2:

$$\binom{2}{2}=1$$

Coefficient of ab:

The number of ways to choose exactly one a will give the coefficient of ab, which is:

$$\binom{2}{1}=2$$

(Once we choose one a, and we know that the sum of the powers is 2, then automatically, we are going to choose one b.)

2.14: Power of a Term

$$(x_1+x_2+\cdots+x_n)^k$$

Each term in the above expression has a sum of powers, which is k.

$$x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}\Rightarrow a_1+a_2+\cdots a_n=k$$

We know that the binomial expansion is:

$$(x+y)^n=\binom{n}{r}x^ry^{n-r}$$

Sum of powers is:

$$Sum\ of\ powers\ is\ r+(n-r)=n$$

Example 2.15: Using Combinations

Find the coefficient of $x^3y^3z^2$ in the expansion of $(x+y+z)^8$.

We can think of the given expression as $x+y+z$ multiplied by itself 8 times:

$$(x+y+z)^8=(x+y+z)(x+y+z)\ldots(x+y+z)$$

The sum of the powers of $(x+y+z)^8$ is 8. Out of the eight variables available to us, we want exactly 3 of them variables to be x. Hence, we need to choose the positions of x among 8 variables (see diagram below)

$$\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_$$

which can be done in:

$$\binom{8}{3}\ ways$$

Now that the $x's$ are in position, we can consider the remaining variables only.
Out of the remaining 5 variables, we want 3 of the variables to be y, which can be done in:

$$\binom{5}{3}\ Ways$$

And once we have chosen the positions of the x variables, and the positions of the y variables, the positions of the z variables are automatically chosen. Hence, the final answer is:

$$\binom{8}{3}\binom{5}{3} = \frac{8!}{5!\,3!} \times \frac{5!}{3!\,2!} = \frac{8!}{3!\,3!\,2!} = 560$$

Example 2.16

Find the coefficient of $x^3y^3z^2$ in the expansion of $(x+y+z)^8$.

$$(x+y+z)^8 = (x+y+z)(x+y+z)\dots(x+y+z)$$

Permutations of Repeated Objects

We need to pick 8 variables, out of which 3 are x, 3 are y, and 2 are z. This is the same as the number of ways to arrange 8 objects, out of which:

- There are three identical x's
- There are three identical y's
- There are two identical z's

Which is given by:

$$\frac{8!}{3!\,3!\,2!} = 560$$

2.17: Multinomial Coefficient (Two Variables)

The binomial coefficient can be written in a different way with redundant information:

$$\binom{n}{r,s} = \frac{n!}{r!\,s!}, \qquad r+s=n$$

The usual binomial coefficient is:

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Use a change of variable. Let $n-r=s$:

$$\binom{n}{r} = \frac{n!}{r!\,s!}, \qquad r+s=n$$

And if we want to include the information not just about r,but also s, we can write the above without changing the meaning as:

$$\binom{n}{r,s} = \frac{n!}{r!\,s!}, \qquad r+s=n$$

2.18: Multinomial Coefficient

The coefficient of $a^pb^qc^r$ in $(a+b+c)^n$ is given by

$$\binom{n}{p,q,r} = \frac{n!}{p!\,q!\,r!}, \qquad p+q+r=n$$

- A binomial coefficient is a special case of the multinomial coefficient where there are only two variables (a, b) instead of 3.

Example 2.19
Find the coefficient of $x^3y^3z^2$ in the expansion of $(x+y+z)^8$.

Substitute $n = 8, p = 3, q = 3, r = 2$:

$$\frac{n!}{p!\,q!\,r!} = \frac{8!}{3!\,3!\,2!} = 560$$

D. Independent Term

Example 2.20
Find the term independent of x in

$$\left(x + x^3 - \frac{1}{x}\right)^8$$

The general term of the expansion has the form:

$$(x)^a(x^3)^b\left(-\frac{1}{x}\right)^c = -x^a x^{3b} x^{-c} = -x^{a+3b-c}$$

For the term to be independent of x, the power must be zero:

$$a + 3b - c = 0$$

Since the sum of the powers must be 8, we must have the restriction $a + b + c = 8 \Rightarrow c = 8 - a - b$, which we can we substitute in the above:

$$a + 3b - (8 - a - b) = 0$$
$$a + 3b - 8 + a + b = 0$$
$$2a + 4b = 8$$

Divide both sides by 2:

$$a + 2b = 4$$

The above is a Diophantine to be solved in positive integers. Considering cases for b, and adding the corresponding values of c gives:

$$b = 1, a = 2, c = 5$$
$$b = 2, a = 0, c = 6$$

And now we know the terms that are independent of x, and hence we can calculate their coefficients (using the multinomial coefficient formula) as:

$$(a, b, c) = (2,1,5) \Rightarrow \binom{8}{2,1,5} = \frac{8!}{2!\,1!\,5!} =$$
$$(a, b, c) = (0,2,6) \Rightarrow \binom{8}{0,2,6} = \frac{8!}{0!\,2!\,6!} =$$

Example 2.21
Find the term independent of x in

$$\left(2x+1-\frac{1}{2x^2}\right)^6$$

$$(2x)^a(1)^b\left(-\frac{1}{2x^2}\right)^c$$

Observe that for the power to be zero, we must have:

$$a=2c$$

$$c=1, a=2, b=3$$

$$c=0, a=0, b=6$$

Consider $c=1, a=2, b=3$:

$$\frac{6!}{2!\,3!\,1!}(2x)^2(1)^3\left(-\frac{1}{2x^2}\right)=-120$$

Consider $c=2, a=4, b=0$:

$$\frac{6!}{4!\,0!\,2!}(2x)^4(1)^0\left(-\frac{1}{2x^2}\right)^2=60$$

Consider $c=0, a=0, b=6$:

$$\frac{6!}{0!\,0!\,c!}(2x)^0(1)^6\left(-\frac{1}{2x^2}\right)^0=1$$

The final answer is:

$$-120+60+1=-59$$

Follow-up

Find the term independent of x in

$$\left(2x+1-\frac{1}{2x^2}\right)^6$$

$$(2x)^a(1)^b\left(-\frac{1}{2x^2}\right)^c$$

Drop the numerical values, and the sign of the numbers (not the exponent) since we are interested only in the power:

$$(x)^a\left(-\frac{1}{x^2}\right)^c=x^a x^{-2c}=x^{a-2c}$$

$$a-2c=0$$

E. Further Resources

Video 2.22

Acadza – Anshul Sir

2.2 Pascal's Triangle

A. Pascal's Triangle

2.23: Pascal's Triangle: As Numbers

2.24: Pascal's Rule: Combinatorial

$$\binom{\boldsymbol{n-1}}{\boldsymbol{k-1}} + \binom{\boldsymbol{n-1}}{\boldsymbol{k}} = \binom{\boldsymbol{n}}{\boldsymbol{k}}$$

We use the double counting strategy.

Choose k out of n objects

Choosing k out of n objects can be done in:

$$\binom{n}{k}\ ways$$

Cases for a particular object x

Consider there is a particular object x among the n objects, and we wish to choose k out of n objects. We can consider two cases:

Case I: Object x is included among the k

This leaves us $k-1$ objects to be chosen from the remaining $n-1$ objects, which can be done in

$$\binom{n-1}{k-1}\ ways$$

Case II: Object x is not included among the k

This leaves us k objects to be chosen from the remaining $n-1$ objects, which can be done in

$$\binom{n-1}{k}\ ways$$

Total Ways

The total ways are then the sum of Case I, and Case II:

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

Equate the Two Methods

But, both methods of counting must lead to the same answer, and hence

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

2.25: Pascal's Rule: Algebraic

$$\binom{\boldsymbol{n-1}}{\boldsymbol{k-1}} + \binom{\boldsymbol{n-1}}{\boldsymbol{k}} = \binom{\boldsymbol{n}}{\boldsymbol{k}}$$

Use the formula for combinations on the LHS:

$$LHS = \binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-k-1)!}$$

Add the two fractions by taking the LCM:

$$\frac{\boldsymbol{k}(n-1)!}{\boldsymbol{k}(k-1)!\,(n-k)!} + \frac{(n-1)!\,(\boldsymbol{n-k})}{k!\,(n-k-1)!\,(\boldsymbol{n-k})!}$$

Combine the factorials in the denominator using the property $n! = n(n-1)!$:

$$\frac{\boldsymbol{k}(n-1)! +}{k!\,(n-k)!} + \frac{(n-1)!\,(\boldsymbol{n-k})}{k!\,(n-k)!}$$

Take $(n-1)!$ common in the numerator, and note that we get precisely the expression for the combinatorial formula for the RHS:

$$\frac{(n-1)!\,(\boldsymbol{k+n-k})}{k!\,(n-k)!} = \frac{(n-1)!\,(\boldsymbol{n})}{k!\,(n-k)!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k} = RHS$$

2.26: Using Pascal's Rule

2.3 BT: Negative and Fractional Exponents

A. Binomial Coefficients for Real Numbers

2.27: Binomial Coefficients

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

- For the binomial theorem with positive integer exponents, we made use of binomial coefficients.
- The binomial coefficient $\binom{n}{r}$ was interpreted as the number of ways of choosing r distinct objects out of n distinct objects.

2.28: Binomial Coefficients: Alternate Version

$$\binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

In order to extend the binomial theorem to negative and fractional exponents, we first extend the binomial coefficient to handle negative and fractional values.

We begin with the definition of binomial coefficients that we are already aware of.
For positive integer n and nonnegative integer r such that $0 \le r \le n$ we have

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Expand the RHS using the recursive definition of the factorial:

$$= \frac{n(n-1)(n-2)\dots(n-r+1)(n-r)!}{r!\,(n-r)!}$$

Cancel:

$$= \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

- This is our new definition of the binomial coefficient.
- When n is a positive integer, and r is between 0 to n, this new definition behaves the same as the old definition.
- But it is useful even when n is not a positive integer.

Example 2.29

Show that the result from the two definitions is the same for

$$\binom{10}{4}$$

$$\binom{10}{4} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6!}{4!\,6!} = \frac{10\cdot 9\cdot 8\cdot 7}{4!}$$

2.30: Generalized Binomial Coefficients

We define the generalized binomial coefficient $\binom{n}{r}$ to be:

$$\binom{n}{r} := \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}, \qquad n \in \mathbb{R}, \qquad r\ is\ a\ positive\ integer$$

- $:=$ is used to emphasize that it is a definition
- The formula is valid when $n \in \mathbb{C}$. That is, it works when n is a complex number. However, we will not consider such cases.

Example 2.31

How many terms does the numerator of the formula below have:

$$\binom{n}{r} := \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

$$r\ terms$$

Example 2.32

Evaluate

$$\binom{\frac{1}{2}}{3}$$

$$\binom{\frac{1}{2}}{3} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6} = \frac{\frac{3}{8}}{6} = \frac{3}{8}\cdot\frac{1}{6} = \frac{1}{16}$$

Example 2.33

Evaluate

$$\binom{\frac{1}{3}}{4}$$

$$\binom{\frac{1}{3}}{4} = \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)}{4!} = \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{24} = \frac{-\frac{80}{81}}{24} = -\frac{80}{81}\cdot\frac{1}{24} = -\frac{10}{243}$$

Example 2.34

$$\binom{-5}{3}$$

$$\binom{-5}{3} = \frac{(-5)(-6)(-7)}{6} = -35$$

Example 2.35

Explanation of generalized binomial coefficients using Pascal's Triangle.

B. Generalized Binomial Theorem

We know that the binomial expansion for $(x+y)^n$ where n is a positive integer has one term more than n. For example:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0 y^n$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

- The binomial expansion for $(x+y)^{-n}$ where n is an integer is an infinite series.

2.36: Generalized Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

The binomial theorem for positive exponents is:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}a^{n-n}b^n$$

Substitute $a = 1, b = x$:

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots$$

Substitute the generalized binomial coefficients: $\binom{n}{1} = n, \binom{n}{2} = \frac{n(n-1)}{2!}, \binom{n}{3} = \frac{n(n-1)(n-2)}{3!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

2.37: Conditions for the Generalized Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

The above formula requires:

$$|x| < 1$$

Example 2.38

$$\frac{1}{1-r}$$

A. Expand using the Generalized Binomial Theorem
B. Confirm the expansion using geometric series

Part A: Generalized Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$\frac{1}{1-r} = (1-r)^{-1} \Rightarrow$substitute $x = -r, n = -1$ in the above:

$$(1-r)^{-1} = 1 + (-1)(-r) + \frac{(-1)(-2)}{2!}(-r)^2 + \frac{(-1)(-2)(-3)}{3!}(-r)^3 + \cdots$$
$$(1-r)^{-1} = 1 + r + r^2 + r^3 + \cdots$$

For the general term, consider the term with power k, which is the $(k+1)st$ term:

$$\frac{(-1)(-2)(-3)\dots(-k)}{k!}(-r)^k$$

Split $(-r)^k = [(-1)r]^k = (-1)^k(r^k)$

$$= \frac{(-1)^k(1)(2)(3)\dots(k)}{k!}(-1)^k(r^k)$$

$$= \frac{(-1)^{2k} k!}{k!} r^k$$

Since k is an integer, $2k$ is an even integer, and hence $(-1)^{2k} = 1$

$$= r^k$$

Part B: Infinite Geometric Series

$$(1-r)^{-1} = 1 + r + r^2 + r^3 + \cdots$$

The RHS is an infinite geometric series with $first\ term = a = 1, common\ ratio = r = r$, and sum:

$$\frac{a}{1-r} = \frac{1}{1-r}$$

And convergence condition:

$$|r| < 1$$

Example 2.39

$$\frac{1}{(1+r)^2}$$

Expand using the Generalized Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$\frac{1}{(1+r)^2} = (1+r)^{-2}$ ⇒Substitute $x = r, n = -2$ in the above:

$$(1+r)^{-2} = 1 + \frac{-2}{1!}r + \frac{(-2)(-3)}{2!}r^2 + \frac{(-2)(-3)(-4)}{3!}r^3 + \cdots$$

$$(1+r)^{-2} = 1 - 2r + 3r^2 - 4r^3 + \cdots$$

$$\frac{(-2)(-3)\ldots(-k-1)}{k!}r^k = (-1)^k(k+1)r^k$$

C. Shortcut Formulas for Negative Integers

2.40: Binomial Coefficients for negative integers

Binomial coefficients of the form $\binom{-n}{r}$ where n is a positive integer can be written using positive values in the binomial coefficient:

$$\binom{-n}{r} = (-1)^r\binom{n+r-1}{r}$$

Expand the LHS using the definition:

$$\binom{-n}{r} = \frac{(-n)(-n-1)(-n-2)\ldots(-n-r+1)}{r!}$$

Collate and pull out all the minus signs:

$$\frac{(-1)^r(n)(n+1)(n+2)\ldots(n+r-1)}{r!}$$

Rewrite using factorials:

$$(-1)^r \frac{(n+r-1)!}{r!\,(n-1)!}$$

Substitute $\frac{(n)(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}$:

$$= (-1)^r \binom{n+r-1}{r} = RHS$$

Example 2.41

$$\binom{-6}{10} = \binom{6+10-1}{10}(-1)^{10} = \binom{15}{10}$$

$$\binom{-17}{9} = \binom{17+9-1}{9}\underline{(-1)^{9}} = -\binom{25}{9}$$

2.42: Binomial Theorem: Negative Integer Exponents

$$\frac{1}{(1-x)^n} = (1-x)^{-n} = 1 + \binom{n}{1}x + \binom{n+1}{2}x^2 + \cdots + \binom{n+r-1}{r}x^r + \cdots = \sum_{r=0}^{\infty}\binom{n+r-1}{r}x^r$$

- Why is it always positive?

The $(-1)^r$ from the binomial coefficient and $(-x)^r$ from the binomial formula cancel out.

$$= \sum_{k=0}^{\infty}\binom{r}{k}x^{r-k}y^k$$

$$= x^r + rx^{r-1}y + \frac{r(r-1)}{2!}x^{r-2}y^2 + \frac{r(r-1)(r-2)}{3!}x^{r-3}y^3 + \cdots.$$

D. Extracting Coefficients

2.43: Binomial Theorem: Negative Integer Exponents

By the generalized binomial theorem:

$$(1+x)^{-n} = 1 + \binom{-n}{1}x + \binom{-n}{2}x^2 + \binom{-n}{3}x^3 + \cdots$$

Hence, the coefficient of x^k is

$$\binom{-n}{k}$$

Example 2.44

Write the coefficient of x^6 in the expression below in the form $\binom{n}{k}$ where n and k are positive integers, and then evaluate it:

$$\frac{1}{(1-x)^4}$$

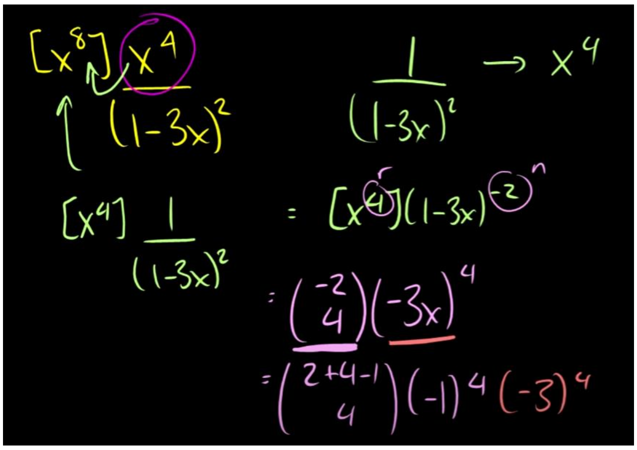
$$\binom{-n}{k}$$

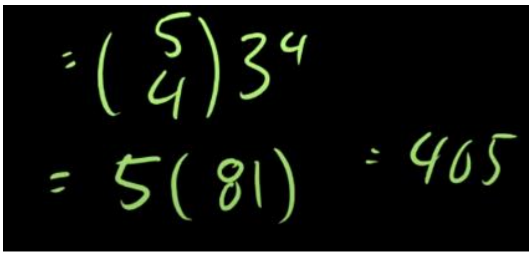
$\frac{1}{(1-x)^4} = (1-x)^{-4}$ ⇒Subsitute $n = 4, k = 6$ in the above:

$$\binom{-4}{6} = \frac{(-4)(-5)(-6)(-7)(-8)(-9)}{6!} = \frac{(-1)^6 9!}{6!\,4!} = \binom{9}{6}$$

$$\binom{9}{6} = \frac{9!}{6!\,4!} =$$

Example 2.45





Example 2.46

Find the coefficient of x^5 in $(1+x+x^2)^8$ **(EAMCET, 18 Sep 2020, Shift-II)**

$$(1+x+x^2)^8 = \left(\frac{1-x^3}{1-x}\right)^8 = (1-x^3)^8(1-x)^{-8} = (1-8x^3+\cdots)(1-x)^{-8}$$

$$0+5: (1)\binom{8+5-1}{5}=\binom{12}{5}=11\cdot 9\cdot 8=8(99)$$
$$3+2: (-8)\binom{8+2-1}{5}=-8\binom{9}{2}=-8(36)$$

$$8(99)-8(36)=8(63)=8(64)-8=512-8=504$$

3. Further Topics

3.1 Generating Functions

A. The Idea

3.1: Generating Function
A counting generating function is a series that we can use to count some quantity of interest.

Example 3.2
Solve in natural numbers:
$$a + b = 5$$

Since $a \geq 1, b \geq 1$ let $a = A + 1, b = B + 1$
$$A + 1 + B + 1 = 5$$
$$A + B = 3$$
$$(3,0), (2,1), (1,2)(0,3) \Rightarrow 4\ Solutions$$

Example 3.3
Solve in natural numbers:
$$a + b = 5$$

Consider x raised to the above as a power:
$$x^{a+b} = x^5$$

The restriction on a is that it is a natural number:
$$1 \leq a \leq 4: P_1 = x + x^2 + x^3 + x^4$$
$$1 \leq b \leq 4: P_2 = x + x^2 + x^3 + x^4$$

The number of solutions to $a + b = 5$ is the same as the coefficient of x^5 in the multiplication:
$$P_1 P_2$$
$$= (x + x^2 + x^3 + x^4)(x + x^2 + x^3 + x^4)$$
$$= x^2(1 + x + x^2 + x^3)(1 + x + x^2 + x^3)$$
$$= x^2[x^3 \cdot 1 + x^2 \cdot x + x \cdot x^2 + 1 \cdot x^3]$$
$$= x^2[4x^3]$$
$$= 4x^5$$

Coefficient of x^5 is
$$4 \Rightarrow 4\ solutions$$

Example 3.4
I have an urn with an infinite number of balls. I wish to pick an odd non-zero number of red balls, an even (possibly zero) number of blue balls, and green balls in multiples of three such that the number of green balls is non-zero. I pick a total of n balls.

A. Write the generating function.
B. In how many ways can we pick a hundred balls. Write your answer as a coefficient of a term in the generating function from Part A. (Numerical calculations not required).

Part A

With $R = Red, B = Blue, G = Green$:

$$R + B + G = n$$

Valid values for the number of red balls are:

$$\{1,3,5,7, \dots, 2k+1, \dots\}$$

We convert these into exponents:

$$Red = x^1 + x^3 + x^5 + \dots + x^{2k+1} + \cdots$$

$$\{0,2,4, \dots, 2k, \dots\}$$
$$Blue = x^0 + x^2 + x^4 + \dots + x^{2k} + \cdots$$

$$\{3,6,9, \dots, 3k, \dots\}$$
$$Green = x^3 + x^6 + \dots + x^{3k} + \cdots$$

The generating function is:

$$G(x) = \underbrace{\left(x^1 + x^3 + x^5 + \dots + x^{2k+1} + \cdots\right)}_{Red}\underbrace{\left(x^0 + x^2 + x^4 + \dots + x^{2k} + \cdots\right)}_{Blue}\underbrace{\left(x^3 + x^6 + \dots + x^{3k} + \cdots\right)}_{Green}$$

Part B

The number of ways is the coefficient of x^{100} in

$$G(x)$$

3.5: Formal Power Series

We will be treating the generating functions that we are using as $formal$ power series.
We can add, subtract, multiply and divide the series without thinking about the validity of the operations.

- This is because we are concerned with coefficients, and not with convergence.

B. Using Coefficients/Probability Generating Function

3.6: Probability Generating Function

A series whose coefficients can be used to determine the probabilities of a quantity of interest is a probability generating function.

Example 3.7

An urn has N balls, of which g are green and r are red. You draw n balls with replacement. Write a generating function to give the probability of drawing a are green and b are red.

.

$$a + b = n$$
$$n \le N$$

For a single ball, let

$$P(Green) = \frac{g}{N} = G, \qquad P(Red) = \frac{r}{N} = R(say)$$

$$(Gx + Ry)^2 = G^2x^2 + GxRy + RyGx + R^2y^2$$
$$Value\ of\ the\ coefficient\ of\ x^a y^b\ in\ (Gx + Ry)^n$$

3.8: Multiplication

C. Geometric Series

3.9: Finite Geometric Series

The sum of a finite geometric series with $first\ term = a$ and $common\ ratio = r$ and n terms is:

$$S = \frac{a(1-r^n)}{1-r}$$

Example 3.10

$$1 + x + x^2 + x^3 + \cdots + x^n$$

A. Write the sum of the above using a finite geometric series.
B. Write the coefficients that it generates.

This is a finite geometric series with $first\ term = a = 1, common\ ratio = r = x$ and $n+1$ terms. It has sum:

$$S = \frac{a(1-r^n)}{1-r} = \frac{1-x^{n+1}}{1-x}$$

$$Coefficents\ are\ \left(\underbrace{1,1,\ldots,1}_{n\ 1's}, 0,0,0,\ldots\right)$$

Example 3.11

A. Write a generating function to determine the number of ways a sum of n can be obtained from rolling two dice: one with six faces numbered from 1 to 6, and the other with four faces numbered from 1 to 4.
B. Treat the generating functions as a geometric series and write their sum.

$$Die\ with\ 4\ faces: x^1 + x^2 + \cdots + x^4 = \frac{x(1-x^4)}{1-x}$$
$$Die\ with\ 6\ faces: x^1 + x^2 + \cdots + x^6 = \frac{x(1-x^6)}{1-x}$$

We get the final answer, we need the multiplication of the two generating functions:

$$\left[\frac{x(1-x^4)}{1-x}\right]\left[\frac{x(1-x^6)}{1-x}\right] = \frac{x^2(1-x^4)(1-x^6)}{(1-x)^2}$$

3.12: Infinite Geometric Series

The sum of an infinite geometric series with $first\ term = a$ and $common\ ratio = r$ is:

$$S = \frac{a}{1-r}, -1 < r < 1$$

- Again, we are concerned with coefficients, and hence we will treat it as a $formal$ power series.

Example 3.13

$$1 + x + x^2 + x^3 + \cdots$$

A. Write the sum of the above using a infinite geometric series.
B. Write the coefficients that it generates.

This is an $\underbrace{infinite\ geometric\ series}_{IGS}$ with $a = 1, r = x$ and it has sum:

$$S = \frac{a}{1-r} = \frac{1}{1-x} = (1-x)^{-1}$$

$$(1,1,1,\dots)$$

Example 3.14

$$1 - x + x^2 - x^3 + \cdots$$

A. Write the sum of the above using a infinite geometric series.
B. Write the coefficients that it generates.

Substitute $a = 1, r = -x$:

$$S = \frac{a}{1-r} = \frac{1}{1-(-x)} = \frac{1}{1+x} = (1+x)^{-1}$$

Example 3.15

$$1 + ax + a^2x^2 + a^3x^3 + \cdots$$

A. Write the sum of the above using a infinite geometric series.
B. Write the coefficients that it generates.

Substitute $first\ term = 1, common\ ratio = ax$

$$S = \frac{a}{1-r} = \frac{1}{1-ax} = (1-ax)^{-1}$$

$$(a^0, a^1, a^2, a^3, \dots)$$

Example 3.16

Write a generating function to determine the number of ways to hand over exactly $n, n \in \mathbb{N}$ rupees if you have many $1 - rupee$ coins, $2 - rupee\ coins$ and $5 - rupee$ coins. You do not need to use each type of coin.

$$G(x) = \underbrace{(1 + x + x^2 + \cdots)}_{1\ Rupee\ Coin}\underbrace{(1 + x^2 + x^4 + \cdots)}_{2\ Rupee\ Coins}\underbrace{(1 + x^5 + x^{10} + \cdots)}_{5\ Rupee\ Coins}$$

$$G(x) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^5}\right) = (1-x)^{-1}(1-x^2)^{-1}(1-x^5)^{-1}$$

D. Extracting Coefficients

Earlier, we looked at writing a generating function as a geometric series.
Now, we turn the problem around, and ask for the coefficient of a generating function that a geometric series represents.

Example 3.17

What is the coefficient of x^7 in:

$$\frac{1}{1-x}$$

$$\frac{1}{1-2x}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^7 + \cdots$$
$$Coefficient\ of\ x^7 = 1$$

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + \cdots + 2^n x^n + \cdots$$
$$Coefficient\ of\ x^7 = 2^7 = 128$$

E. Multiplication

Example 3.18

$$\frac{x^4}{1-3x}$$

A. Expand.
B. Determine coefficient of x^7.
C. The coefficient of x^7 in $\frac{x^4}{1-3x}$ is equal to the coefficient of x^n in $\frac{1}{1-3x}$. What is n?
D. What is the series of coefficents that the above function generates?

$$\frac{x^4}{1-3x} = x^4 \cdot \frac{1}{1-3x} = x^4(1 + 3x + 9x^2 + 27x^3 \ldots) = x^{0+4} + 3x^{1+4} + 9x^{2+4} + 27x^{3+4} + \cdots$$

$$Coefficient\ of\ x^7 = 3^{7-4} = 3^3 = 27$$
$$n = 3$$

Coefficients:

$$(0,0,0,0,1,3,9, \ldots,)$$

3.19: Multiplying Two Functions

Example 3.20

Use multiplication to get the series associated with:

$$\frac{1}{(1-x)^2}$$

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)^2$$

Expand the expression inside the brackets as an infinite series:

$$= (1 + x + x^2 + x^3 + \cdots)^2$$

Write the square as a product:

$$= (1 + x + x^2 + x^3 + \cdots)(1 + x + x^2 + x^3 + \cdots)$$

$$\begin{aligned} = 1 + x + x^2 + x^3 + x^4 + \cdots \\ x + x^2 + x^3 + x^4 + \cdots \\ x^2 + x^3 + x^4 + \cdots \\ x^3 + x^4 + \cdots \end{aligned}$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Example 3.21

The generating function to determine the number of ways a sum of n can be obtained from rolling two dice: one with six faces numbered from 1 to 6, and the other with four faces numbered from 1 to 4 is given by:

$$\frac{x^2(1-x^4)(1-x^6)}{(1-x)^2}$$

Move the denominator to the numerator:

$$= x^2(1-x^4)(1-x^6)(1-x)^{-2}$$

Substitute using the expansion from the previous example:

$$= x^2(1-x^4)(1-x^6)(1 + 2x + 3x^2 + 4x^3 + \cdots)$$

Multiplying:

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10

$$G(x) = (1-x^4)(1-x^6)(x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 7x^8 + \cdots)$$

$Number\ of\ ways\ of\ making\ 2$ =Coefficient of x^2 in $G(x)$

$$= 1 \times 1 \times x^2 = x^2 \Rightarrow Coefficient = 1$$

$Number\ of\ ways\ of\ making\ 3$ =Coefficient of x^3 in $G(x)$

$$= 1 \times 1 \times 2x^3 = 2x^3 \Rightarrow Coefficient = 2$$

$Number\ of\ ways\ of\ making\ 4$ =Coefficient of x^4 in $G(x)$

$$= 1 \times 1 \times 3x^4 = 3x^4 \Rightarrow Coefficient = 3$$

$Number\ of\ ways\ of\ making\ 5$ =Coefficient of x^5 in $G(x)$

$$= 1 \times 1 \times 4x^5 = 4x^5 \Rightarrow Coefficient = 4$$

$Number\ of\ ways\ of\ making\ 6$ =Coefficient of x^6 in $G(2)$

$$= (1 \times 1 \times 5x^6) + (-x^4)(x^2) = 5x^6 - x^6 = 4x^6 \Rightarrow Coefficient = 4$$

F. Finding Generating Functions

Example 3.22

If a fair 6 sided die is rolled 12 times, how many ways can they sum to 30?

$$[x^{30}](x+x^2+x^3+x^4+x^5+x^6)^{12}$$

$$[x^{30}]\,x^{12}(1+x+x^2+x^3+x^4+x^5)^{12}$$

$$[x^{18}](1+x+x^2+\dots+x^5)^{12}$$

$$[x^{18}]\left(\frac{1-x^6}{1-x}\right)^{12}$$

$$[x^{18}](1-x^6)^{12}(1-x)^{-12}$$

$$[x^{18}](1-x^6)^{12}(1-x)^{-12}$$

$$=\binom{12}{0}\binom{-12}{18}+\binom{12}{1}\binom{-12}{12}(-1)+\binom{12}{2}\binom{-12}{6}+\binom{12}{3}(-1)\binom{-12}{0}$$

$$=\binom{-12}{18}-\binom{12}{1}\binom{-12}{12}+\binom{12}{2}\binom{-12}{6}-\binom{12}{3}$$

$$=\binom{29}{18}-12\binom{23}{12}+\binom{12}{2}\binom{17}{6}-\binom{12}{3}$$

A. Using Generating Functions to Count

Example 3.23

3.2 Further Topics/Binomial Identities

A. Binomial Identities

Example 3.24: Binomial Identities

Use the binomial theorem to prove the identities below:

A. $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$
B. $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + \binom{n}{n} = 0$
C. $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{n} = \binom{n}{1} + \binom{n}{3} + \ \dots + \binom{n}{n}$

Part A

Substitute $x = 1, y = 1$ in the Binomial Theorem:

$$LHS = (x + y)^n = (1 + 1)^n = 2^n$$
$$RHS = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

And we know that:

$$LHS = RHS \Rightarrow 2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

Part B

Let $x = 1, y = -1$ in the Binomial Theorem:

$$LHS = (1 - 1)^n = 0^n = 0, n \neq 0$$
$$RHS = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + \binom{n}{n}$$

Part C

Take all the negative terms to the RHS in the identity proved in Part B, we are done.

25 Examples