Coordinate Geometry

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Table of contents

0.1 Formula Summary

A. Basics

Distance Formula

The distance between two points (x_1, y_1) and (x_2, y_2) is given by:

$$\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

1. Basics

1.1 Co-ordinate Plane

A. Introduction

Co-ordinate geometry was introduced by Descartes, who was the first mathematician to put the connection between geometry and algebra on firm ground. In his honor, the coordinate plane is called the Cartesian coordinate plane (and points on this plane are called Cartesian coordinates).

He introduced co-ordinate axes. The x-axis is horizontal, and the y-axis is vertical. The axes are perpendicular to each other.

Location of Coordinates as an Ordered Pair

Every point in the co-ordinate plane is uniquely determined by its x coordinate and its y coordinate. The combination of the two coordinates is called an ordered pair written like this

$$(x, y)$$

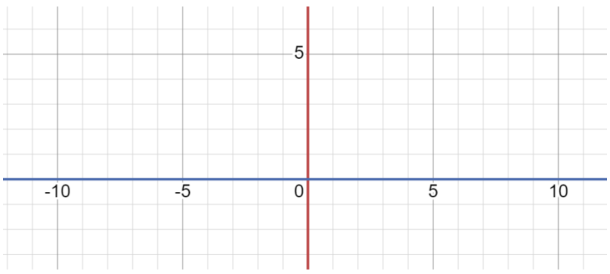
For example, (1,3) is an ordered pair consisting of x co-ordinate 1, and y coordinate 3.
Because (1,3) is an ordered pair, it is not the same as (3,1).

If we are referring to multiple points, we will generally name them using subscripts. For example

$$A = (x_1, y_1), B = (x_2, y_2)$$

Capital letters are generally used to refer to points.

B. Finding the axes



Example 1.1

In the graph alongside:

A. Identify the x-axis and the y-axis.
B. Identify the origin
C. Write the coordinates of the origin.

x-axis and y-axis

The **blue**, horizontal line is the x-axis. The **red**, vertical line is the y-axis.

Origin

The origin is the intersection of the blue and the red lines. In other words, it is the intersection of the x-axis and the y-axis.

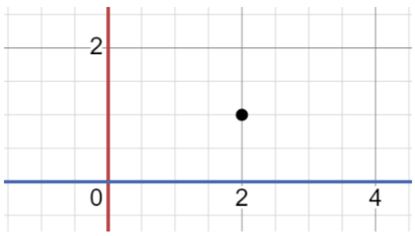
Coordinates of the Origin

At the origin

- the x coordinate is zero.
- the y coordinate is also zero.

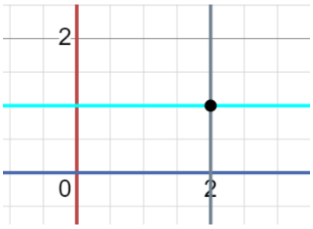
Hence, the coordinates of the origin are

$$(0,0)$$

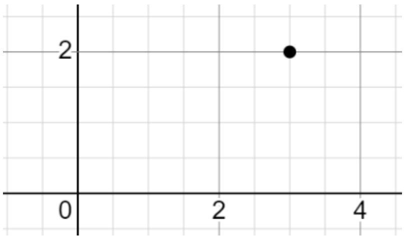


Example 1.2

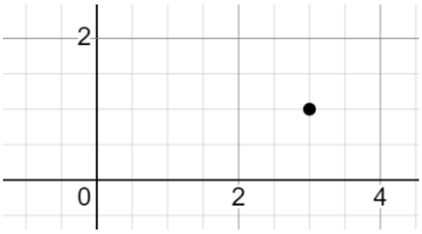
Write the coordinates of the point graphed alongside.



Example 1.3

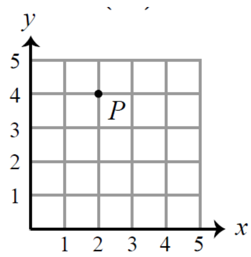


Example 1.4



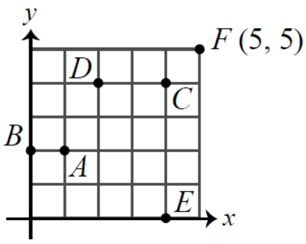
Example 1.5

In the diagram shown, what are the coordinates of point P? (CEMC Gauss 7 2020/2)



Example 1.6

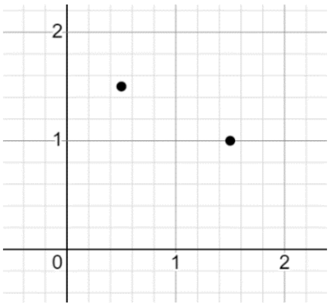
In the diagram, point F has coordinates (5, 5). The point with coordinates (2, 4) is located at (CEMC Gauss 7 2019/2)



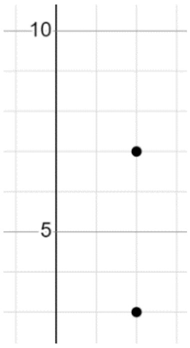
Example 1.7

Micheal has made a diagram showing the location of the library and the school. Michael's house is located at the origin.

A. Find the coordinates of Michael's house.
B. Find the coordinates of the library and the school. The library is to the



left of the school and higher than the school.



C. Distance Between Coordinates

Consider the graph alongside. We cannot see the x coordinate, but it is the same for both the points.

The first point has a y coordinate of 3. The second point has a y coordinate of 7.
The distance between the two point is

$$7 - 3 = 4\ Units$$

Example 1.8

Find the distance between the two points graphed.

D. Stating Coordinates

Example 1.9: Stating Co-ordinates

State the x coordinate and the y coordinate of the following points:

$$(2,3), (-4,3), \left(-\frac{1}{2}, -\frac{8}{9}\right)$$

Example 1.10: Writing Co-ordinates

Write the coordinates of the point that has:

A. x coordinate 5, and y coordinate 7
B. y coordinate $\frac{2}{3}$, and x coordinate $-\frac{5}{4}$

$$(5,7)$$

Example 1.11: Ordered Pairs

State, with reasons, which points represent the same location

$$A = \left(\frac{12}{6}, \frac{27}{9}\right), \quad B = \left(\frac{-8}{-4}, \frac{-81}{-27}\right), \quad C = \left(\frac{27}{9}, \frac{12}{6}\right), \quad D = \left(-\frac{12}{6}, -\frac{27}{9}\right)$$

We simplify the fractions and get:

$$A = (2,3), B = (2,-3), C = (3,2), D = (-2,-3)$$

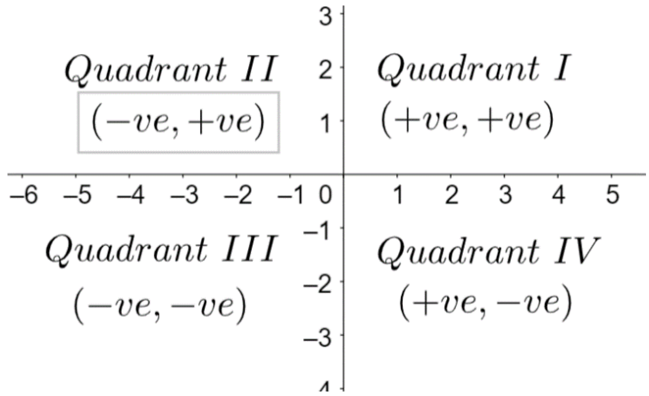
None of the points have both the same x coordinate and the same y coordinate.

Hence, none of the points represent the same location.

E. Quadrants

The coordinate axes divide the coordinate plane into four quadrants.

Note that the numbers of the quadrants are written using Roman Numerals. This is, by convention, and helps ensure that the numbers related to coordinates are not mixed up with number related to Quadrants.



Example 1.12: Finding Quadrants of Points

Find the Quadrants in which the following points lie:

A. $(3,4)$
B. $(-2,-5)$
C. $(-3,2)$
D. $(-4,-6)$
E. $\left(\frac{23}{17},\frac{87}{16}\right)$
F. $\left(\frac{62}{41},-\frac{53}{41}\right)$

Quadrant II $x<0, y>0$	**Quadrant I** $x>0, y>0$
$(\mathbf{-3,2})$	$(\mathbf{3,4})$ $\left(\frac{23}{17},\frac{87}{16}\right)$
Quadrant III $x<0, y<0$	**Quadrant IV** $x>0, y<0$
$(\mathbf{-2,-5})$ $(,-4,-6)$	$\left(\mathbf{\frac{62}{41},-\frac{53}{41}}\right)$

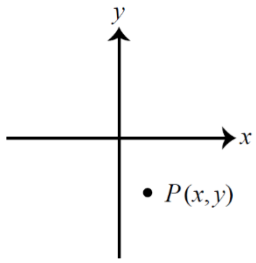
Example 1.13

Give an example of four points, each in a different quadrant. Mention the quadrant that the point lies in.

Example 1.14

In the graph shown, which of the following statements is true about the coordinates of the point P(x, y)?

A. The values of both x and y are positive.
B. The value of x is positive and the value of y is negative.
C. The value of x is negative and the value of y is positive.
D. The values of both x and y are negative.
E. The value of x is 0 and the value of y is negative. (CEMC Gauss 7 2020/7)

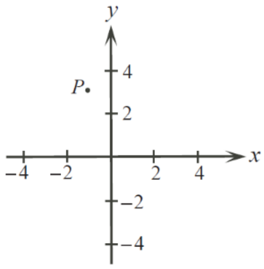


Option B.

Example 1.15

In the diagram, the coordinates of point P could be (CEMC Gauss 7 2009/5)

A. (1,3)
B. (1,-3)
C. (-3,1)
D. (3,-1)
E. (-1,3)



F. Coordinates of the axes

Throughout the x-axis, the y coordinate is zero. Therefore, any point on the x-axis must have coordinates

$$(x, 0)$$

for some value of x.

Throughout the y-axis, the x coordinate is zero. Therefore, similar to the above, any point on the y-axis must have coordinates

$$(0, y)$$

for some value of y.

This fact is very useful when finding the places where a curve will cut the axes. These are called x-intercepts (where a curve cuts the x-axis), and y-intercepts (where a curve cuts the y-axis).

Example 1.16: Identifying axes of coordinates

State which axes do the following points lie on:

A. $(0,4)$
B. $(3,0)$
C. $(0,-2)$
D. $(-7,0)$

$$(0,4) = y - axis$$
$$(3,0) = x - axis$$
$$(0,-2) = y - axis$$
$$(-7,0) = x - axis$$

G. Axes versus Quadrants

A point which lies on an axis does not lie in any quadrant.

Example 1.17: Quadrants and Axes

State the quadrant or the axis that the following points lie on.

A. $(2,3)$
B. $(0,4)$
C. $(-5,7)$
D. $(-2,0)$
E. $(3,-2)$
F. $(-3,-9)$

H. Origin

Suppose a point O lies on both the x-axis and the y-axis.
Since it lies on the x-axis

- it must have y-coordinate zero

Since it lies on the y-axis

- it must have x-coordinate zero

Therefore, both the x coordinate and the y coordinate of the point are zero.

$$O = (0,0)$$

This point, in fact, is called the origin, and is the intersection of the x and the y-axis.

The letter O is commonly used to represent the origin.

I. 3D Coordinate System

The (x, y) coordinate in two dimensions can be extended to three dimensions by adding a z variable. The concepts that we have learnt in two dimensions can usually be extended in a straight forward manner to three dimensions.

The axes divide 3D dimensional coordinate space into eight octants.

Example 1.18

A. Identify the x coordinate, the y coordinate, and the z coordinate in the point $(7, \pi, e)$.
B. Is the point $(7, \pi, e)$ the same as the point $(\pi, e, 7)$?
C. Find the distance between the following pairs of points $(1,2,3)$ and $(1,2,5)$.
D. Find the distance between the following pairs of points (x_1, y_1, z_1) and (x_1, y_2, z_1).

Part A

$$x = 7, y = \pi, z = e$$

Part B

These two points are not the same.
Coordinates are stated as ordered triplets, and the order matters.

Part C

Since the x coordinate and the y coordinate of both the points are the same, we can find the distance by finding the absolute value of the difference between the z coordinates

$$Distance = |3 - 5| = |-2| = 2$$

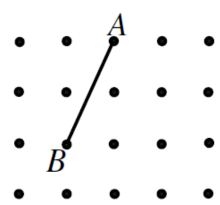
Part D

$$Distance = |y_1 - y_2|$$

J. Counting

Example 1.19

In the diagram, the points are evenly spaced vertically and horizontally. A segment AB is drawn using two of the points, as shown. Point C is chosen to be one of the remaining 18 points. For how many of these 18 possible points is triangle ABC isosceles? (CEMC Gauss 8 2006/23)



Example 1.20

Consider the rectangle with vertices

$$P = (3,6), Q = (9,6), R = (9,2), S = (3,2)$$

A. What is the number of points inside $PQRS$?

B. What is the number of lattice points that lie on or inside $PQRS$?
C. What is the number of lattice points that lie on $PQRS$?
D. What is the number of lattice points that lie inside $PQRS$?

Part A

$$Infinite$$

Part B

The lattice points that lie along the length of $PQRS$ are all of the form:

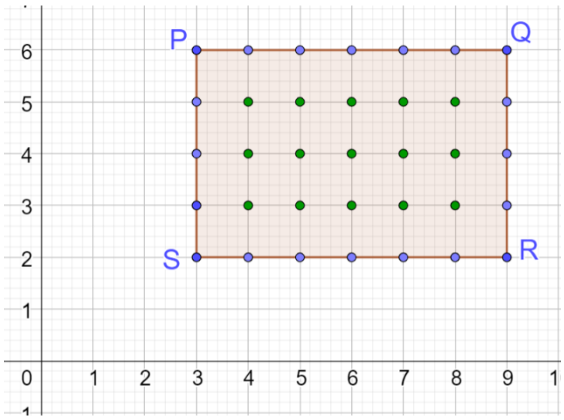
$$(3, y), (4, y), \dots, (9, y)$$

The number of lattice points is the number of elements in the set:

$$\{3,4, \dots ,9\} = 9 - 3 + 1 = 7 \ Points$$

Similarly, the lattice points that lie along the width of $PQRS$ are all of the form:

$$(x, 2), (x, 3), \dots (x, 6) \Rightarrow \{2,3, \dots ,6\} \Rightarrow 6 - 2 + 1 = 5$$



The total number of points

$$\underbrace{7}_{Length} \times \underbrace{5}_{Width} = 35$$

Part C

Number of Lattice Points

$$\underbrace{7}_{PQ} + \underbrace{5}_{QR} + \underbrace{7}_{RS} + \underbrace{5}_{SP} = 24$$

But this method of counting counts the vertices twice. Hence, the vertices need to be subtracted to get the final answer:

$$= 24 - 4 = 20$$

Another way to do it is to recognize that, in this case, the number of lattice points on the polygon is equal to the perimeter of the polygon

$$= 20$$

Part D

We use complementary counting:

$$\underbrace{35}_{\substack{Total \\ Points}} - \underbrace{20}_{\substack{Points \\ on\ Side}} = 15$$

Example 1.21

What is the number of lattice points that lie on the cuboid $ABCDEFGH$ with vertices:

$$A = (1,1,1), B = (1,1,9), C = (7,1,1), D = (7,1,9), E = (1,8,1), F = (1,8,9), G = (7,8,1), H = (7,8,9)$$

The total number of points that lie on or inside the cuboid

$$= \underbrace{7}_{x\ direction} \times \underbrace{8}_{y\ direction} \times \underbrace{9}_{z\ direction} = 504$$

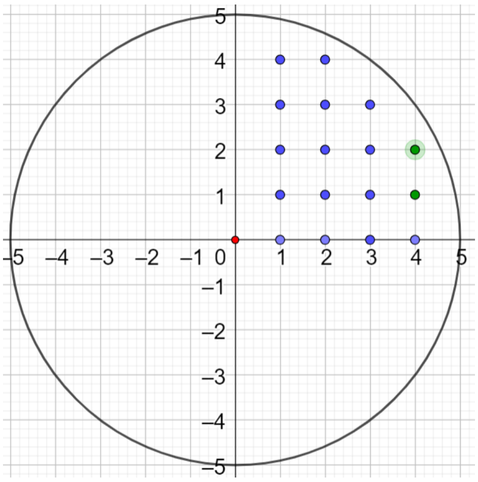
The total number of points that lie inside the cuboid

$$= \underbrace{7 - 2}_{x\ direction} \times \underbrace{8 - 2}_{y\ direction} \times \underbrace{9 - 2}_{z\ direction} = 5 \times 6 \times 7 = 210$$

Using complementary counting

$$504 - 210 = 294$$

Example 1.22
Find the number of lattice points that lie within a circle of area 25π with center at the origin.



$$A = 25\pi \Rightarrow \pi r^2 = 25\pi \Rightarrow r^2 = 25 \Rightarrow r = 5$$

The origin is a lattice point within the circle.

$$1\ Point$$

The number of points on the $x - axis$ and to the right of the origin is

$$4\ Points$$

The distance from $(4,1)$ to the origin is:

$$\sqrt{(4-0)^2 + (1-0)^2} = \sqrt{16+1} = \sqrt{17} < \sqrt{25}$$

The distance from $(4,2)$ to the origin is:

$$\sqrt{4^2 + 2^2} = \sqrt{16+4} = \sqrt{20} < \sqrt{25}$$

The distance from $(4,3)$ to the origin is:

$$\sqrt{4^2 + 3^2} = \sqrt{16+9} = \sqrt{25} \Rightarrow (4,3)\ lies\ on\ the\ circle$$

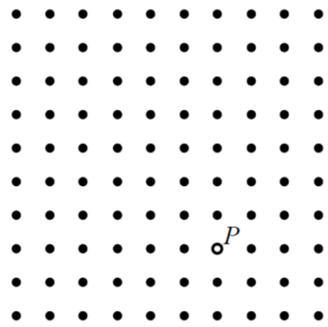
The distance from $(3,4)$ to the origin is:

$$\sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} \Rightarrow (3,4)\ lies\ on\ the\ circle$$

$$4(4+4+4+3+2) + 1 = 4 \times 17 + 1 = 68 + 1 = 69$$

K. Probability

Example 1.23
A 10 by 10 grid is created using 100 points, as shown. Point P is given. One of the other 99 points is randomly chosen to be Q. What is the probability that the line segment PQ is vertical or horizontal? (CEMC Gauss 7 2016/21)

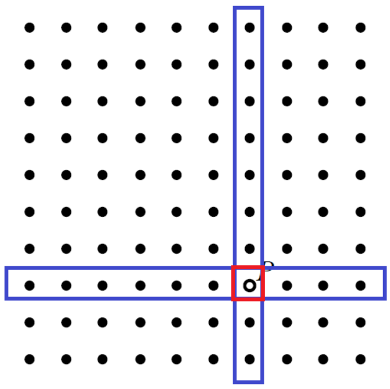


Mark the dots that make PQ vertical with a blue vertical box. There are 10 points in the column, but we can't include P itself, so have

$$10 - 1 = 9\ Points$$

Mark the dots that make PQ horizontal with a blue horizontal box. There are 10 points in the row, but we can't include P itself, so again we have

$$10 - 1 = 9\ Points$$



Then, the required probability is

$$\frac{Sucessful\ Outcomes}{Total\ Outcomes} = \frac{9+9}{99} = \frac{18}{99} = \frac{2}{11}$$

(Continuation) Example 1.24
What is the probability that the line segment PQ is diagonal? (Diagonal in this case means that for every unit

that it goes right, it also goes up or down *exactly* 1 unit).

$$\frac{13}{99}$$

Example 1.25

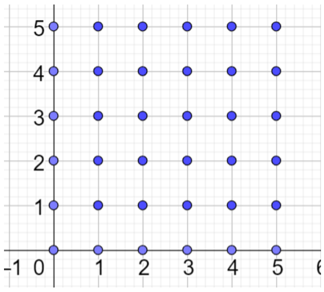
I pick a random lattice point from the points that lie on or inside the square with non-adjacent vertices (0,0), (5,5). Then I pick a second lattice point at random (with the same conditions as before). If the second point is the same as the first, then I repick until the second point is different.

A. What is the number of lattice points in the given region?
B. What is the probability that I need to repick?
C. What is the number of ways to choose two distinct points from the given region?
D. What is the probability that the points lie on a line that is horizontal or vertical?
E. What is the probability that the points lie on a line that is at a $45°$ angle with the x-axis or a $45°$ angle with the y-axis?
F. What is the probability that the points lie on a line that makes an angle of $a°$ with the x-axis or the y-axis, $a \neq 45n, n \in \mathbb{Z}$?

Part A

The number of lattice points will be

$$6 \times 6 = 36$$



Part B

We can pick any point that we want for the first point. We will need to repick only if the second point is the same as the first point, which will happen with probability

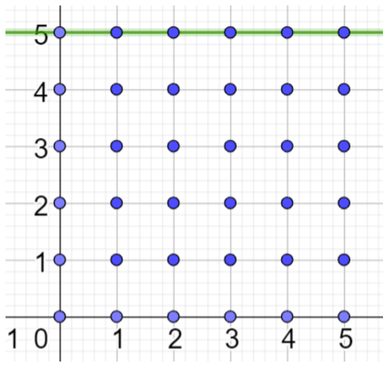
$$P = \frac{1}{36}$$

Part C

The number of ways to choose two points from the 36 points in the region is:

$$\binom{36}{2} = \frac{36 \times 35}{2} = 18 \times 35 = 630$$

Part D



Number of Horizontal Lines

$$= 6$$

Number of Ways for two points to be on a single Horizontal Line

$$= \binom{6}{2} = \frac{6 \times 5}{2} = 15$$

Total Number of Ways

$$= 6 \times 15 = 90$$

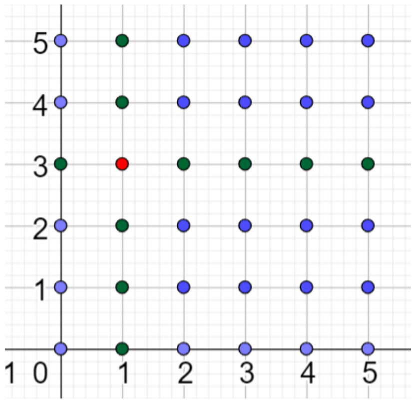
By the same logic, since the width of the region is equal to the height of the region, the number of ways for the two points to lie on a vertical line is also

$$90$$

The probability of the two points lying on a line that is either horizontal or vertical is

$$P = \frac{90+90}{630} = \frac{180}{630} = \frac{18}{63} = \frac{2}{7}$$

Part D: Shortcut Method



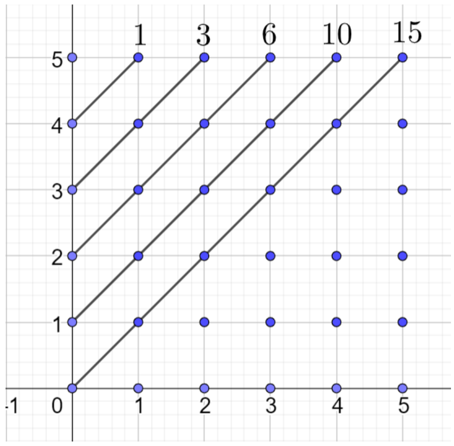
Choose any point as the first point. For instance, pick the red dot in the diagram alongside.

Pick the second point. The points that result in the line being formed as vertical or horizontal are colored green.

$$P = \frac{10}{35} = \frac{2}{7}$$

But, this condition is true irrespective of whichever point you pick in the diagram. Hence, it is also the overall probability.

Part E



The number of ways to get choose a line that goes up from left to right is:

$$2(1+3+6+10)+15 = 2(20)+15 = 40+15 = 55$$

By symmetry, the number of ways to get a line that goes down from left to right is also

$$55$$

Total Number of Ways to choose two points such the points lie on a lie that is 45° to either the x-axis or the y-axis is

$$2 \times 55 = 110$$

Hence, the required probability is:

$$\frac{Successful\ Outcomes}{Total\ Outcomes} = \frac{110}{630} = \frac{11}{63}$$

Part F

We use complementary counting.

$$P = 1 - \frac{4}{21} - \frac{11}{63} = 1 - \left(\frac{12}{63} + \frac{11}{63}\right) = 1 - \frac{23}{63} = \frac{40}{63}$$

1.2 Horizontal Distance and Area

A. Distance between two points with one coordinate same

An important thing to remember is that distance is never negative. Hence, when defining the distance below, we use the absolute value function to remove the negative sign, if it exists.

If one of the coordinates is the same, then the distance is simply the absolute value of the difference of the other coordinate.

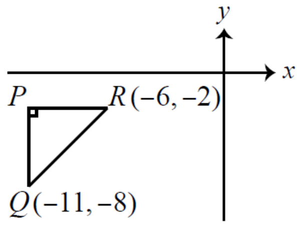
$$Vertical\ distance\ between\ (x_1, y_1)\ and\ (x_1, y_2\ ) = |y_2 - y_1|$$
$$Distance\ between\ (x_1, y_1)\ and\ (x_2, y_1\ ) = |x_2 - x_1|$$

Example 1.26: Finding Distance

Find the distance between the following pairs of points:

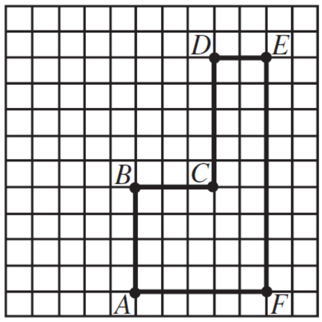
A. $A = (3,4)$ and $B = (3,7)$
B. $P = (2,5)$ and $Q = (8,5)$

$$AB = |7-4| = 3 = |4-7|, \qquad PQ = |2-8| = |8-2| = 6$$



Example 1.27

In \DeltaPQR shown, side PR is horizontal and side PQ is vertical. The coordinates of P are (CEMC Gauss 8 2016/6)



Example 1.28

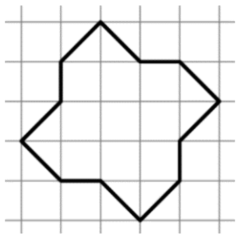
In the diagram, which of the following is the largest? (CEMC Gauss 8 2004/20)

A. AE
B. CD+CF
C. AC+CF
D. FD
E. AC+CE

B. Area Basics

Example 1.29

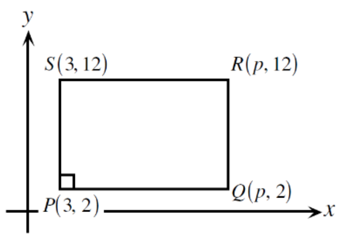
The twelve-sided figure shown has been drawn on 1 cm × 1 cm graph paper. What is the area of the figure in cm^2? (AMC 8 2018/4)

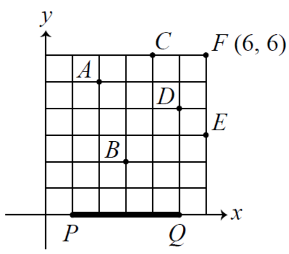


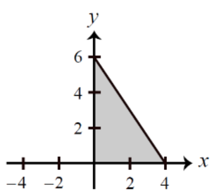
$$\underbrace{9}_{Full\ Squares} + \underbrace{\frac{8}{2}}_{Half\ Squares} = 9 + 4 = 13$$

Example 1.30

A. The coordinates of the vertices of rectangle $PQRS$ are given in the diagram. The area of rectangle $PQRS$ is 120. The value of p is (CEMC Gauss 8 2000/13)
B. In the diagram, point F has coordinates $(6,6)$. Points P and Q are two vertices of a triangle. Which of the following points can be joined to P and Q to create a triangle with an area of 6? (CEMC Gauss 8 2019/9)
C. A line segment is drawn joining the points $(0,6)$ and $(4,0)$, as shown. The area of the shaded triangle is (CEMC Gauss 8 2020/5)







Part A

Since P and S have the same x coordinate, the distance between the two is simply:

$$PS = y_2 - y_1 = 12 - 2 = 10$$

Since area of a rectangle is $length \times width$:

$$A(PQRS) = 10 \times l = 120 \Rightarrow l = 12$$

Since the length is 12, and P and Q have the same y coordinate Q is 12 units to the right of P:

$$p = 3 + 12 = 15$$

Part B

We have to join P and Q.

$$PQ = 4$$

Substitute $A = 6, b = 4$ in the formula for area of a triangle

$$\frac{hb}{2} = A \Rightarrow \frac{h(4)}{2} = 6 \Rightarrow h = 3$$

The point that gives a height of 3 for the triangle:

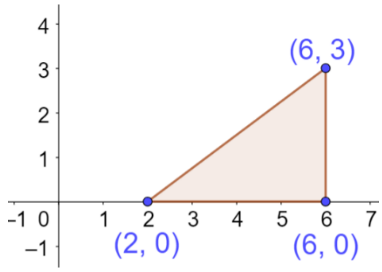
Part C

The area of the triangle is

$$\frac{hb}{2} = \frac{6 \times 4}{2} = 12$$

Example 1.31

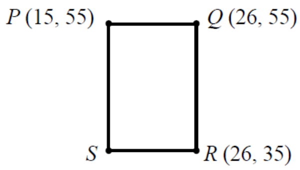
Triangle ABC has its vertices at $A(2, 0)$, $B(6, 0)$ and $C(6, 3)$. The area of the triangle, in square units, is **(CEMC Gauss 7 2003/16)**



$$\frac{hb}{2} = \frac{4 \times 3}{2} = \frac{12}{2} = 6$$

Example 1.32

Points $P(15, 55)$, $Q(26, 55)$ and $R(26, 35)$ are three vertices of rectangle $PQRS$. The area of this rectangle is **(CEMC Gauss 7 2020/13)**



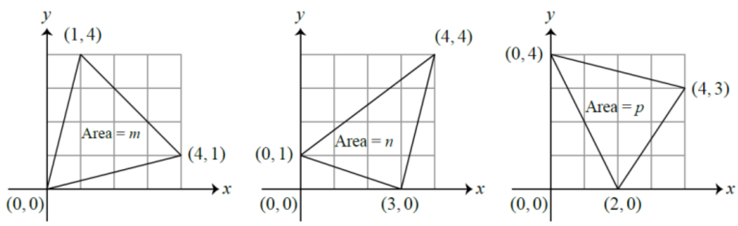
Area of rectangle PQRS

$$= \underbrace{(55 - 35)}_{Length}\underbrace{(26 - 15)}_{Width} = 20 \times 11 = 220$$

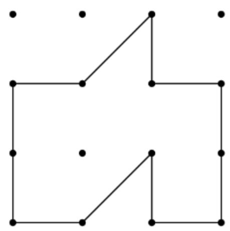
Example 1.33: Complementary Counting

A. Four vertices of a quadrilateral are located at (7,6),(-5,1),(-2,-3), and (10, 2). The area of the quadrilateral in square units is **(CEMC Gauss 7 2017/24)**

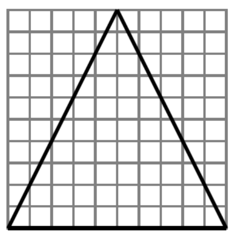
B. Each diagram shows a triangle, labelled with its area. What is the correct ordering of the areas of these triangles? **(CEMC Pascal 9 2015/21)**



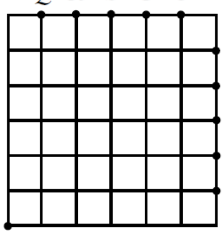
Example 1.34: Rearranging
Dots are spaced one unit apart, horizontally and vertically. The number of square units enclosed by the polygon is (AMC 8 1998/6)



C. More Area



Example 1.35
In the 10 by 10 grid of squares shown, point P can be at any of the 41 points of intersection of pairs of gridlines inside (and not on) 4FGH. For each possible location of P, exactly three triangles are formed: 4FPG, 4GPH, 4HPF. How many of these 123 triangles have an area that is exactly half of the area of 4FGH? (CEMC Gauss 7 2020/24)

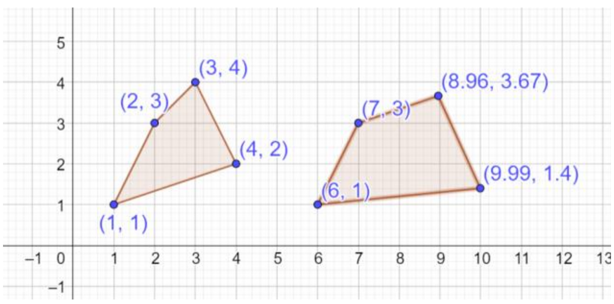


Example 1.36
In the 6 by 6 grid shown, two lines are drawn through point P, dividing the grid into three regions of equal area. These lines will pass through the points (CEMC Gauss 8 2000/20)

D. Pick's Theorem[1]

1.37: Lattice Point
A lattice point is a point in the coordinate plane whose coordinates are both integers.

Example 1.38
The diagram alongside has two quadrilaterals. Decide, for each quadrilateral, whether the coordinates of the vertices are lattice points.



For the quadrilateral on the left, the vertices are:

$$(1,1),(4,2),(3,4),(2,3) \Rightarrow All\ Lattice\ Points$$

For the quadrilateral on the right, the vertices are:

$$(6,1),(9.99,1.4),(8.96,3.67),(7,3) \Rightarrow Not\ all\ Lattice\ Points$$

1.39: Pick's Theorem
If a polygon has lattice points for all of its vertices, then:

$$Area\ of\ Polygon = A = i + \frac{p}{2} - 1$$

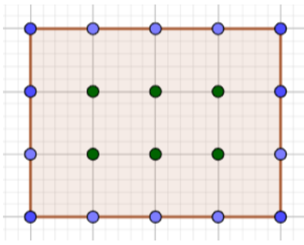
[1] Questions which make use of Pick's Theorem do not occur with high frequency. However, it is very useful in a small set of questions.

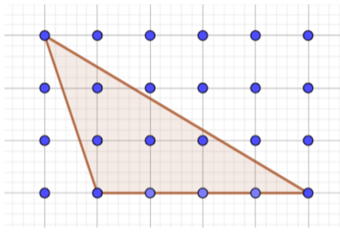
Where the number of lattice points

$$interior\ to\ the\ polygon = i$$
$$on\ the\ perimeter\ of\ the\ polygon = p$$

Example 1.40

The points on the diagram alongside are 1 unit apart (both lengthwise and widthwise). Calculate the area of each figure, first using the standard formula, and then Pick's Theorem





Rectangle

$$Regular\ Formula(Rectangle):\ \underbrace{4}_{Length} \times \underbrace{3}_{Width} = 12$$

$$Pick's\ Theorem: i + \frac{p}{2} - 1 = 6 + \frac{14}{2} - 1 = 6 + 7 - 1 = 12$$

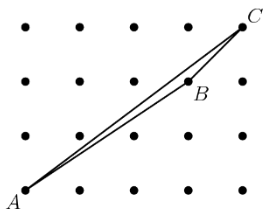
Triangle

$$Regular\ Formula(Triangle): \frac{1}{2} \times \underbrace{3}_{Height} \times \underbrace{4}_{Base} = 6$$

$$Pick's\ Theorem: i + \frac{p}{2} - 1 = 4 + \frac{6}{2} - 1 = 4 + 3 - 1 = 6$$

Example 1.41

The horizontal and vertical distances between adjacent points equal 1 unit. The area of triangle ABC is (AMC 8 1996/22)



The number of black dots inside the triangle

$$= i = 0$$

The number of points on the perimeter of the polygon

$$= p = 3$$

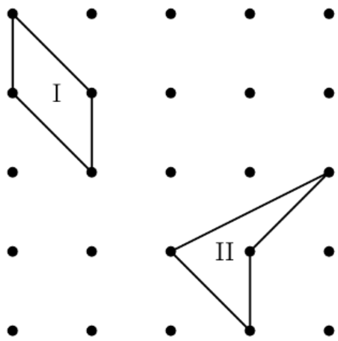
$$Area = i + \frac{p}{2} - 1 = 0 + \frac{3}{2} - 1 = \frac{1}{2}$$

MCQ 1.42

Mark the correct option

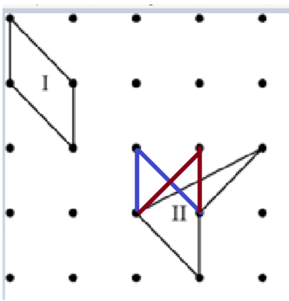
Consider these two geoboard quadrilaterals. Which of the following statements is true? (AMC 8 2000/18)

A. The area of quadrilateral I is more than the area of quadrilateral II.
B. The area of quadrilateral I is less than the area of quadrilateral II.
C. The quadrilaterals have the same area and the same perimeter.
D. The quadrilaterals have the same area, but the perimeter of I is more than the perimeter of II.
E. The quadrilaterals have the same area, but the perimeter of I is less than the perimeter of II.



Area of Quadrilateral One and Two:

$$= i + \frac{p}{2} - 1 = 0 + \frac{4}{2} - 1 = 2 - 1 = 1$$



If you look at the diagram on the right, the quadrilateral on the right has the same bottom as the other quadrilateral.
The blue triangle matches the perimeter of the first quadrilateral.
The red triangle matches the perimeter of the blue triangle.
The actual perimeter of the right quadrilateral is more than the left quadrilateral.

Example 1.43
What is the area enclosed by the geoboard quadrilateral below? **(AMC 8 2004/14)**

$$Area = i + \frac{p}{2} - 1 = 21 + \frac{5}{2} - 1 = 22\frac{1}{2}$$

Example 1.44
Pi using Pick's Theorem
https://www.geogebra.org/m/y2nuDV37

E. Shoelace Theorem

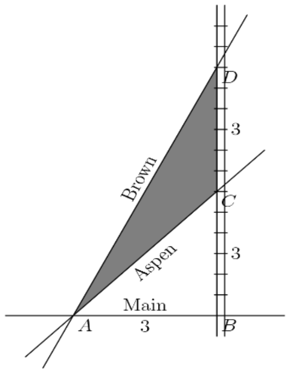
1.45: Shoelace Formula

	x_1	x_2	.	.	.	x_n	x_1
	y_1	y_2	.	.	.	y_n	y_1
		x_1y_2	x_2y_3	.	.	$x_{n-1}y_n$	x_ny_1
		y_1x_2	y_2x_3	.	.	$y_{n-1}x_n$	y_nx_1

$$\frac{1}{2}\left|(x_1y_2 + x_2y_3 + \cdots + x_{n-1}y_n + x_ny_1) - (y_1x_2 + y_2x_3 + \cdots + y_{n-1}x_n + y_nx_1)\right|$$

Example 1.46

The triangular plot of ACD lies between Aspen Road, Brown Road and a railroad. Main Street runs east and west, and the railroad runs north and south. The numbers in the diagram indicate distances in miles. The width of the railroad track can be ignored. How many square miles are in the plot of land ACD? (AMC 8 2009/7)

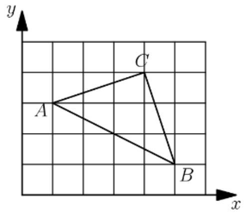


x	-3	0	0	-3	Total
y	0	3	6	0	
		-9	0	0	-9
		0	0	-18	-18

$$\frac{1}{2}|-9-(-18)| = \frac{1}{2}|-9+18| = \frac{1}{2}|9| = \frac{9}{2}$$

Example 1.47

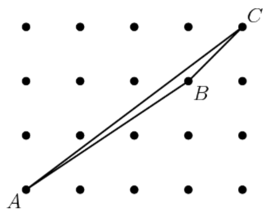
A triangle with vertices as $A=(1,3)$, $B=(5,1)$, and $C=(4,4)$ is plotted on a 6×5 grid. What fraction of the grid is covered by the triangle? (AMC 8 2015/19)



x	1	5	4	1	Total
y	3	1	4	3	
		1	20	12	33
		15	4	4	23

$$\textit{Area of the Triangle} = \frac{1}{2}|33-23| = \frac{1}{2}|10| = 5$$
$$\textit{Area of the Grid} = 5\times 6 = 30$$

$$\textit{Fraction of the Grid} = \frac{5}{30} = \frac{1}{6}$$

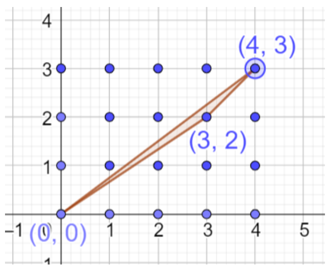


Example 1.48

The horizontal and vertical distances between adjacent points equal 1 unit. The area of triangle ABC is (AMC 8 1996/22)

First, introduce a coordinate system with the origin (0,0) at the bottom left. Calculate the coordinates of the other points, which are:

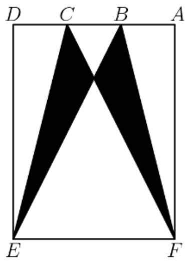
(3,2), (4,3)



x	0	3	4	0	Total
y	0	2	3	0	
		$(0)(2) = 0$	$(3)(3) = 9$	$(4)(0) = 0$	$0 + 9 + 0 = 9$
		$(0)(3) = 0$	$(2)(4) = 8$	$(3)(0) = 0$	$0 + 8 + 0 = 8$

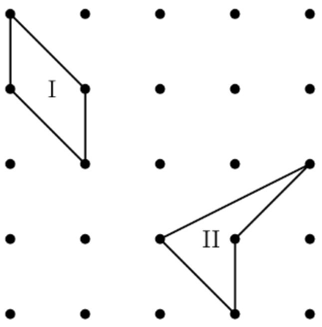
The area of the triangle is:

$$\frac{1}{2}|9-8| = \frac{1}{2}|1| = \frac{1}{2}\times 1 = \frac{1}{2}$$



Example 1.49

Rectangle $DEFA$ below is a 3×4 rectangle with $DC = CB = BA = 1$. The area of the "bat wings" (shaded area) is (AMC 8 2016/22)



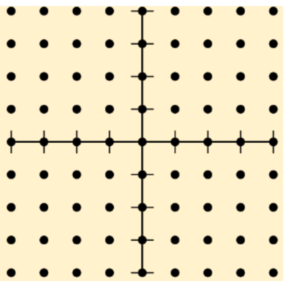
Example 1.50

Consider these two geoboard quadrilaterals. Which of the following statements is true? (AMC 8 2000/18)

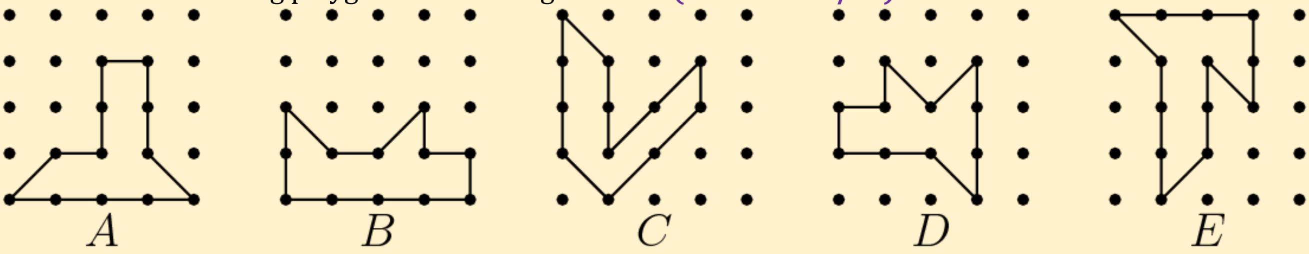
F. The area of quadrilateral I is more than the area of quadrilateral II.
G. The area of quadrilateral I is less than the area of quadrilateral II.
H. The quadrilaterals have the same area and the same perimeter.
I. The quadrilaterals have the same area, but the perimeter of I is more than the perimeter of II.
J. The quadrilaterals have the same area, but the perimeter of I is less than the perimeter of II.

Example 1.51

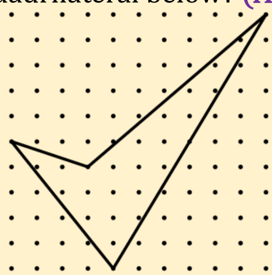
Points A, B, C and D have these coordinates: $A(3,2)$, $B(3,-2)$, $C(-3,-2)$ and $D(-3,0)$. The area of quadrilateral $ABCD$ is (AMC 8 2001/11)



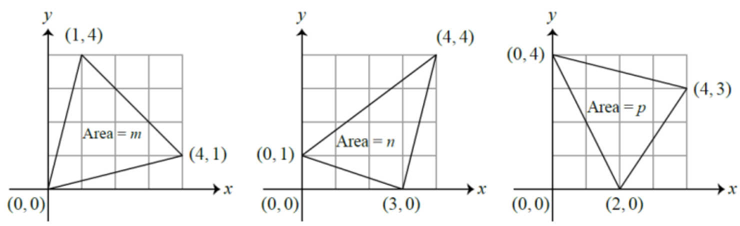
Example 1.52
Which of the following polygons has the largest area? **(AMC 8 2002/15)**



Example 1.53
What is the area enclosed by the geoboard quadrilateral below? **(AMC 8 2004/14)**



Example 1.54
Each diagram shows a triangle, labelled with its area. What is the correct ordering of the areas of these triangles? **(CEMC Pascal 9 2015/21)**



Example 1.55
Two lines with slopes $\frac{1}{2}$ and 2 intersect at $(2,2)$. What is the area of the triangle enclosed by these two lines and the line $x + y = 10$? **(AMC 10A 2019/7)**

2	2			2	2	12
6	4	8		6	4	16
4	6	36		4	6	12
2	2	8		2	2	

The coordinates of the vertices of the triangle are:

$$(2,2),(6,4)(4,6)$$

Use the shoelace formula to calculate the area of the triangle:

$$\frac{1}{2}(8+36+8-12-16-12)=\frac{1}{2}(36-24)=\frac{1}{2}(12)=6$$

1.3 Translations

A. Translations

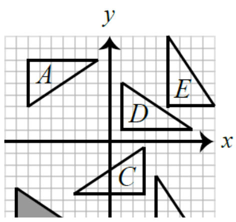
Example 1.56

Zeus starts at the origin (0, 0) and can make repeated moves of one unit either up, down, left or right, but cannot make a move in the same direction twice in a row. For example, he cannot move from (0, 0) to (1,0) to (2,0). What is the smallest number of moves that he can make to get to the point (1056, 1007)? (CEMC Gauss 8 2016/23)

Example 1.57

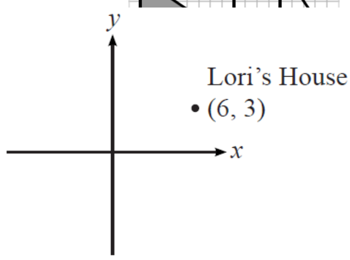
A translation moves point A(-3, 2) to the right 5 units and up 3 units. This translation is done a total of 6 times. After these translations, the point is at (x, y). What is the value of $x + y$? (CEMC Gauss 8 2018/17)

Example 1.58



A. When the shaded triangle shown is translated, which of the following triangles can be obtained? (CEMC Gauss 7 2017/13)
B. Identify the vector used for the translation.

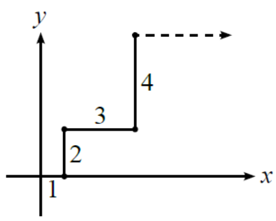
Example 1.59



The diagram shows Lori's house located at (6, 3). If Alex's house is located at (-2,-4), what translation is needed to get from Lori's house to Alex's house? (CEMC Gauss 7 2014/15)

A. 4 units left, 1 unit up
B. 8 units right, 7 units up
C. 4 units left, 1 unit down
D. 8 units left, 7 units down
E. 7 units right, 8 units down

Example 1.60



On a coordinate grid, Paul draws a line segment of length 1 from the origin to the right, stopping at (1, 0). He then draws a line segment of length 2 up from this point, stopping at (1, 2). He continues to draw line segments to the right and up, increasing the length of the line segment he draws by 1 each time. One of his line segments stops at the point (529, 506). What is the endpoint of the next line segment that he draws? (CEMC Gauss 7 2014/25)

List out the endpoints of the first few segments:

$$(1,0),(1,2),(4,2),(4,6),\ldots$$

Note that once the x-coordinate changes, and the next time the y-coordinate changes.
The coordinates of the endpoints are given by:

$$x-coordinate: 1+3+5+\cdots=n^2$$
$$y-coordinate: 2+4+6+\cdots=2(1+2+3+\cdots)=n(n+1)$$

$$529=23^2\Rightarrow n=23$$
$$506=22(23)\Rightarrow n=22$$

Since $n=23$ for the x-coordinate, but only 22 for the $y-coordinate$, the next change must be an increase in y:

$$y=n(n+1)=23(24)=552$$

The next endpoint must be:

$$(529,552)$$

1.61: Translations using Vectors

A point translated using the vector (x_1,y_1) moves:

- x_1 units in the x direction
- y_1 units in the y direction

Example 1.62

Draw a diagram for each part below.

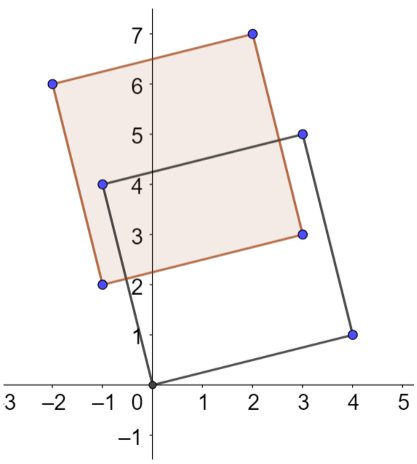
A. Translate the point $(2,1)$ using the vector $(3,-2)$.
B. Translate the square with vertices at $(0,0),(4,1),(3,5),(-1,4)$ using the vector $(-1,+2)$.
C. Consider a square with vertices $(2,1)(5,2)(4,5)(1,4)$. Introduce a coordinate system so that the lower left vertex of the square is at the origin.

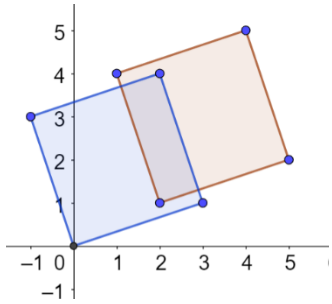
Part A

Part B

We can do this visually. The vector $(-1,2)$ is equivalent to moving left by 1 unit, and moving up by two units. Hence, take each vertex, and move it as per above.
We can also do this algebraically:

$$(0-1,0+2)=(-1,2)$$
$$(4-1,1+2)=(3,3)$$
$$(3-1,5+2)=(2,7)$$
$$(-1-1,4+2)=(-2,6)$$





Part C

If we translate the given square using the vector $(-2,-1)$, then the lower left vertex will be at the origin.

And doing this, we get:

$$(0,0,)(3,1)(2,3)(-1,3)$$

Example 1.63

A. The point $(2,5)$ is translated to the point $(7,-1)$. Identify the vector used for the translation.

B. The triangle $(2,5),(3x,4y),(5a+2,9b-4)$ is translated to $(3,4),(7,2),(2a-6,9b-6)$. Identify the vector used for the translation. Find the values of x,y,a,b.

Part A

$$(2+x,5+y)=(7,-1)$$
$$2+x=7\Rightarrow x=7-2=5$$
$$5+y=-1\Rightarrow y=-6$$
$$Vector=(x,y)=(5,-6)$$

Part B

$$Vector=(1,-1)$$
$$3x+1=7\Rightarrow 3x=6\Rightarrow x=2$$
$$4y-1=2\Rightarrow 4y=1\Rightarrow y=\frac{1}{4}$$
$$5a+2+1=2a-6\Rightarrow 3a=-9\Rightarrow a=-3$$
$$9b-4-1=9b-6\Rightarrow -5=-6\Rightarrow No\ Solutions\ for\ b\Rightarrow b\in\phi$$

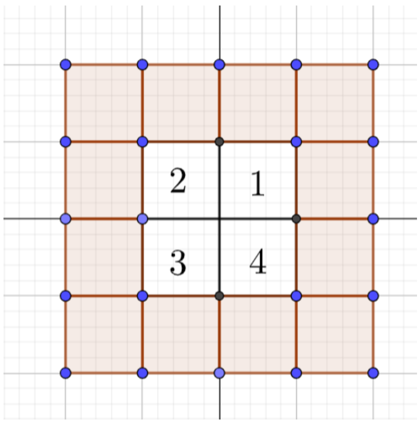
1.64: Lattice Points

A lattice point is a point on the coordinate plane with coordinates

$$(x_1,y_1),\qquad x,y\in\mathbb{Z}$$

Example 1.65

A square of side 1 has one vertex at the origin, and all vertices are lattice points. Let A be the set of possible positions that the square can be in. The square is translated using the vector $(x,y),x,y\in\{\pm1\}$. Let B be the set of possible positions that the square can be in after the translation. Find the number of elements in Set B, but not in Set A.



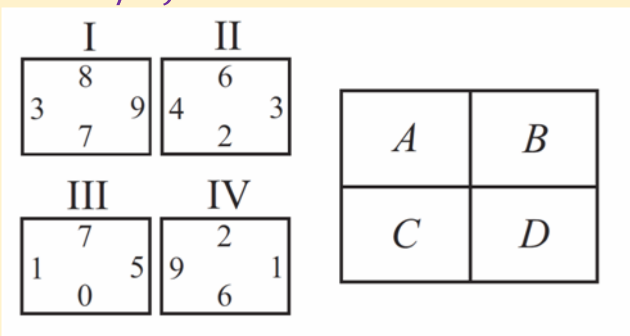
Example 1.66

A dot starts at (20, 19). It can move one unit vertically or horizontally to one of the points (21, 19), (19, 19), (20, 20), or (20, 18). From there it can move two units in either direction that is perpendicular to the first move. All moves thereafter increase in length by one unit (three units, four units, five units, etc.) and must be perpendicular to the direction of the previous move. The dot stops after ten moves. Which of the following final locations is not possible? (CEMC Gauss 7 2019/24)

A. (27,33)
B. (30,40)
C. (21,21)
D. (42,44)
E. (37,37)

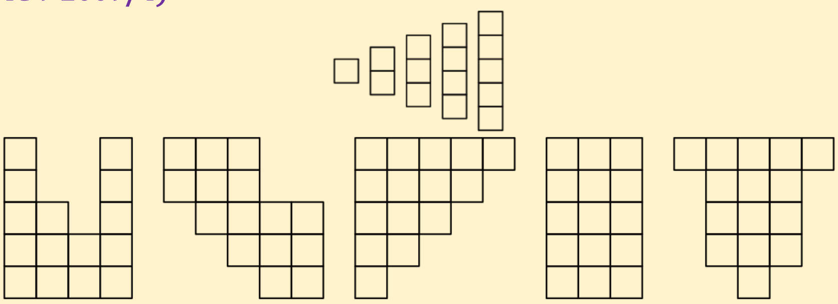
Example 1.67

Tiles I, II, III and IV are translated so one tile coincides with each of the rectangles A, B, C and D. In the final arrangement, the two numbers on any side common to two adjacent tiles must be the same. Which of the tiles is translated to Rectangle C? (AMC 8 2007/11)



Example 1.68

The five pieces shown below can be arranged to form four of the five figures shown in the choices. Which figure cannot be formed? (AMC 9 2009/4)



(A) (B) (C) (D) (E)]

1.4 Reflection

A. Reflection

1.69: Reflecting across the axes

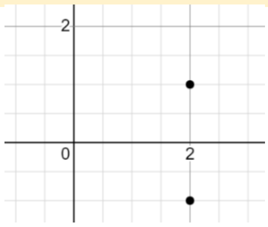
To reflect a point across

- $\boldsymbol{x}$ axis, we negate the $\boldsymbol{y}$ coordinate.
- $\boldsymbol{y}$ axis, we negate the $\boldsymbol{x}$ coordinate.
- origin, we negate both coordinates

Example 1.70

Consider the point $(2,1)$. Reflect the point as directed. In each case, draw the graph, and state the coordinates as well.

A. Across the x axis.
B. Across the y axis
C. Across the origin

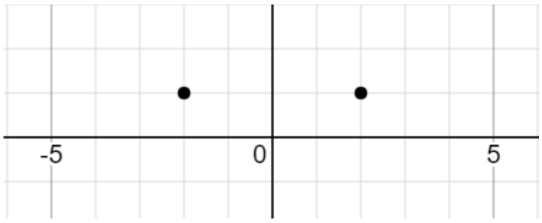


Part A
Algebraically, to reflect a point across the x-axis, we negate the y coordinate

$$(2,1) \rightarrow (2,-1)$$

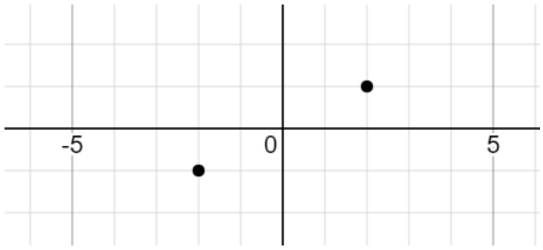
Part B
Algebraically, to reflect a point across the y-axis, we negate the x coordinate

$$(2,1) \rightarrow (-2,1)$$



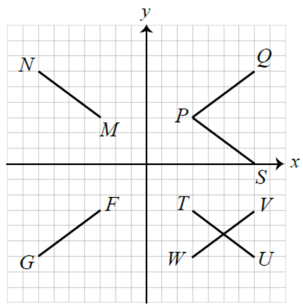
Part C
Algebraically, to reflect a point across the y-axis, we negate both the x and the y coordinate:

$$(2,1) \rightarrow (-2,-1)$$



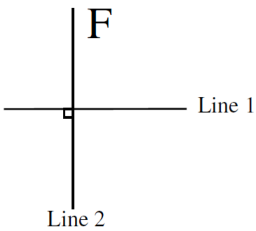
Example 1.71

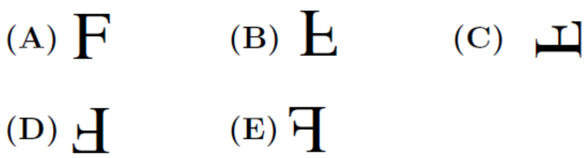
A. If the point (3,4) is reflected in the x-axis, what are the coordinates of its image? (CEMC Gauss 7 2011/9)
B. The point A(1, 2) is reflected in the y-axis. The new coordinates are (CEMC Gauss 8 2014/15)
C. The point (-2,-3) is reflected in the x-axis. What are the coordinates of its image after the reflection? (CEMC Gauss 8 2013/9)
D. In the graph shown, which of the following represents the image of the line segment PQ after a reflection across the x-axis? (CEMC Gauss 7 2015/18, CEMC Gauss 8 2015/13)



Example 1.72

The letter F is reflected in Line 1. The image is then reflected in Line 2. The shape that results is (CEMC Gauss 8 2006/15)



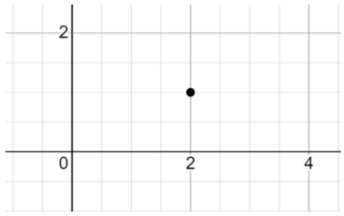


1.73: Reflecting across a vertical/horizontal line

Example 1.74

Reflect the point (2,1)

A. Across the line $x = 4$.
B. Across the line $y = 3$



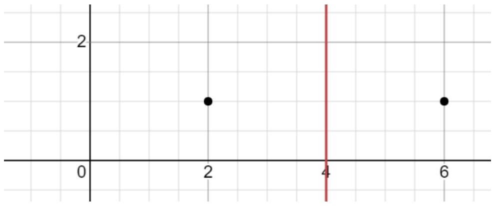
Part A

First, draw the line

$$x = 4$$

Then, the point (2,1) has to be the same distance from $x = 4$ after being reflected as before. But, it has to be on the other side of the line. And, hence it will have coordinates

$$(6,1)$$



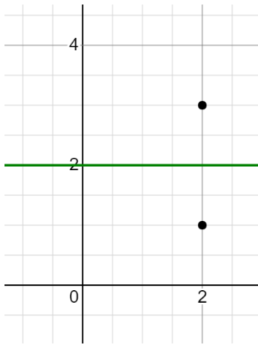
Part B

First, draw the line

$$y = 4$$

Then, the point (2,1) has to be the same distance from $y = 3$ after being reflected as before. But, it has to be on the other side of the line.
And, hence it will have coordinates

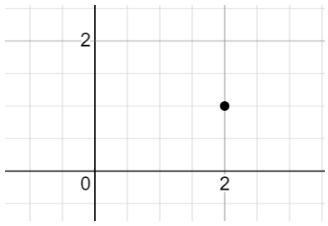
$$(3,1)$$



1.75: Reflecting across $y = x$

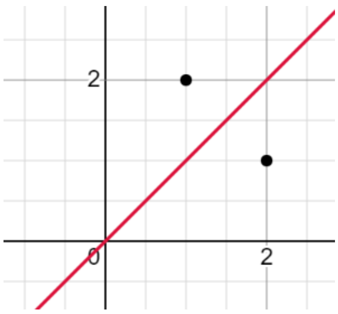
To reflect a point across the line $y = x$

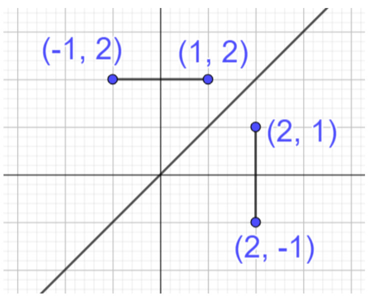
- We interchange the x and the y coordinates



Example 1.76

A. Reflect the point in the diagram alongside across the line $y = x$.
B. Reflect line segment AB, with $A(-1,2)$ and $B(1,2)$ across the line $y = x$.





1.77: Reflecting across any line

Example 1.78

Example 1.79: Reflecting Shapes

ΔABC has vertices $A = (1,1), B = (5,1), C = (5,-4)$.

A. ΔABC is reflected across the x-axis to get ΔXYZ. Draw ΔXYZ.
B. ΔXYZ is reflected across the y-axis to get ΔPQR. Draw ΔPQR.
C. ΔABC is reflected across the line $x = 6$ to get ΔLMN. Draw ΔLMN.
D. ΔABC is reflected across the line $y = -1$ to get ΔDEF. Draw ΔDEF.

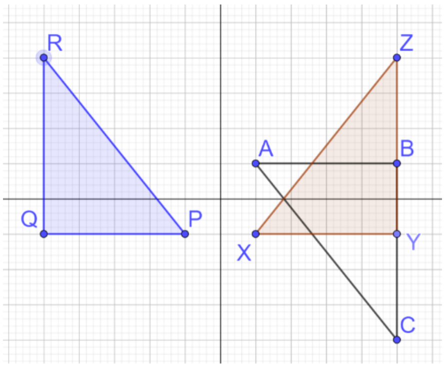
Parts A and B

We can reflect across the x-axis by negating the y coordinate of each point of ΔABC:

$$X = (1,-1), Y = (5,-1), Z = (5,4)$$

We can reflect across the y-axis by negating the x coordinate of each point of ΔXYZ:

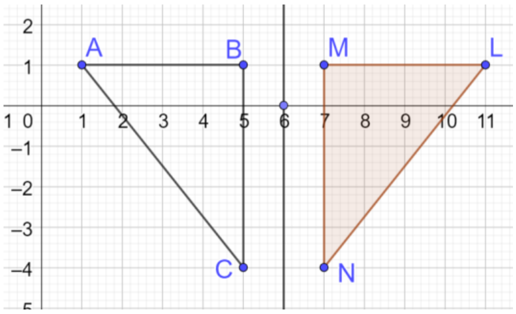
$$P = (-1,-1), Q = (-5,-1), R = (-5,4)$$



Parts C

We can reflect across the line $x = 6$ by calculating the horizontal distance of the point from $x = 6$, and moving it double that many points to the right.

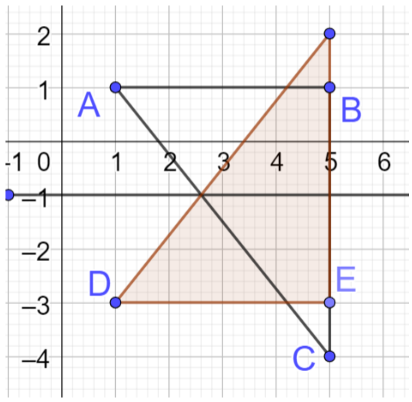
$$A = (1,1), B = (5,1), C = (5,-4)$$
$$L = (1 + 10,1) = (11,1)$$
$$M = (5 + 2,1) = (7,1)$$
$$N = (5 + 2,-4) = (7,-4)$$



Part D

We can reflect across the line $y = -1$ by calculating the vertical distance of the point from $y = -1$, and moving it double that many points down

$$A = (1,1), B = (5,1), C = (5,-4)$$
$$D = (1,1 - 4) = (1,-3)$$
$$E = (5,1 - 4) = (5,-3)$$
$$F = (5,-4 + 6) = (5,2)$$



Example 1.80: Identifying Transformations

A. The line segment $AB = (2,3),(3,9)$ is reflected across an axis to get $A'B' = (-2,3),(-3,9)$. Then, $A'B'$ is reflected across an axis to get $A''B'' = (-2,-3),(-3,-9)$. Identify the transformations.

Example 1.81

The polygon P with n vertices $(p_1, p_2), (p_3, p_4), \dots (p_n, p_{n+1})$ has distinct prime numbers $p_1, p_2, \dots, p_n$ for the coordinates of its vertices, with $p_1 < p_2 < p_3 < \cdots$.

A. State the coordinates of the polygon after it is reflected across the x-axis n times.

B. State the coordinates of the polygon after it is reflected across the y-axis $n+1$ times.

$$n \textit{ is odd} \Rightarrow n+1 \textit{ is even}$$

Part A

By parity, reflecting across the x axis an odd number of times is the same as reflecting it once.
We negate the y coordinates of the points:

$$(p_1, -p_2), (p_3, -p_4), \dots (p_n, -p_{n+1})$$

Part B

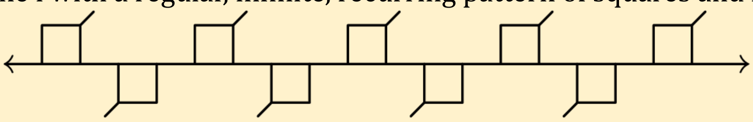
By parity, reflecting across the y axis an even number of times is the same as not reflecting it at all.
There is no change in the coordinates.

Example 1.82

A triangle with vertices $A(0,2)$, $B(-3,2)$, and $C(-3,0)$ is reflected about the x-axis, then the image $\triangle A'B'C'$ is rotated counterclockwise about the origin by $90°$ to produce $\triangle A''B''C''$. Which of the following transformations will return $\triangle A''B''C''$ to $\triangle ABC$? (AMC 10A 2016/16)
(A) counterclockwise rotation about the origin by $90°$.
(B) clockwise rotation about the origin by $90°$.
(C) reflection about the x-axis
(D) reflection about the line $y = x$
(E) reflection about the y-axis.

Example 1.83

The figure below shows line l with a regular, infinite, recurring pattern of squares and line segments.



How many of the following four kinds of rigid motion transformations of the plane in which this figure is drawn, other than the identity transformation, will transform this figure into itself?(AMC 10A 2019/7)

- some rotation around a point of line l
- some translation in the direction parallel to line l
- the reflection across line l
- some reflection across a line perpendicular to line l

Example 1.84

The point in the xy-plane with coordinates $(1000,2012)$ is reflected across the line $y = 2000$. What are the coordinates of the reflected point? (AMC 10B 2012/3)

Example 1.85

Triangle ABC lies in the first quadrant. Points A, B, and C are reflected across the line $y = x$ to points A', B', and C', respectively. Assume that none of the vertices of the triangle lie on the line $y = x$. Which of the following statements is not always true? (AMC 10B 2019/5)
(A) Triangle $A'B'C'$ lies in the first quadrant.
(B) Triangles ABC and $A'B'C'$ have the same area.
(C) The slope of line AA' is -1.
(D) The slopes of lines AA' and CC' are the same.
(E) Lines AB and $A'B'$ are perpendicular to each other.

1.5 Dilations

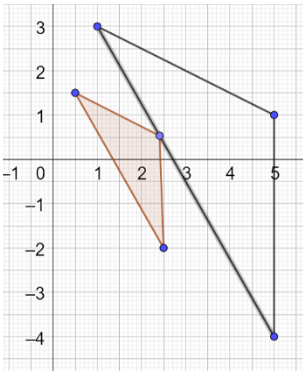
A. Dilations

1.86: Dilation

A dilation will stretch the position of a point.

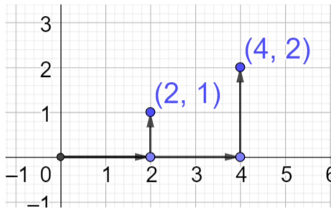
Example 1.87

A. Identify the coordinates of point $(2,1)$ after a dilation by a factor of 2, with respect to the origin.
B. ΔABC has vertices $A=(1,3), B=(5,1), C=(5,-4)$. It is dilated by a factor of $\frac{1}{2}$, with respect to the origin to get $\Delta A'B'C'$. Draw $\Delta A'B'C'$.



Part A

$$(2,1)\to(2\times 2,1\times 2)=(4,2)$$



Part B

$$A'=\left(1\times\frac{1}{2},3\times\frac{1}{2}\right)=\left(\frac{1}{2},\frac{3}{2}\right)$$
$$B'=\left(5\times\frac{1}{2},1\times\frac{1}{2}\right)=\left(\frac{5}{2},\frac{1}{2}\right)$$
$$C'=\left(5\times\frac{1}{2},-4\times\frac{1}{2}\right)=\left(\frac{5}{2},-2\right)$$

Example 1.88

ΔABC has vertices $(2,3),(5,7),(11,13)$. It is dilated with respect to the origin to get ΔXYZ with vertices $\left(1,\frac{3}{2}\right),\left(\frac{5}{2},\frac{7}{2}\right),\left(\frac{11}{2},\frac{13}{2}\right)$. What is the scale factor?

The scale factor will be the same across all points. Check the x coordinate of the first point.

$$2\to 1\Rightarrow Divided\ by\ 2\Rightarrow Dilation\ by\ scale\ factor\ of\ \frac{1}{2}$$

Example 1.89

Line segment AB has endpoints $A=\left(\frac{2}{3},x\right), B=(2p,3a+7)$. It is dilated with respect to the origin to get line segment XY with endpoints $X=\left(\frac{3}{2},y\right), Y=(3q,2a+5)$.

A. What is the scale factor used for the dilation?
B. What is the ratio $x:y$?
C. What is the ratio $p:q$?
D. Find the value of a.

Part A

Let the scale factor for the dilation be s.

$$\therefore\frac{2}{3}s=\frac{3}{2}\Rightarrow s=\frac{9}{4}$$

Part B

$$xs=y\Rightarrow\frac{x}{y}=\frac{1}{s}=\frac{1}{\frac{9}{4}}=\frac{4}{9}\Rightarrow x:y=4:9$$

Part C

$$2ps = 3q \Rightarrow \frac{p}{q} = \frac{3}{2s} = \frac{3}{2\left(\frac{9}{4}\right)} = \frac{3}{\left(\frac{9}{2}\right)} = 3 \times \frac{2}{9} = \frac{2}{3} \Rightarrow p : q = 2 : 3$$

Part D

$$(3a+7)(s) = 2a+5$$
$$(3a+7)\left(\frac{9}{4}\right) = 2a+5$$
$$27a+63 = 8a+20$$
$$19a = -43$$
$$a = -\frac{43}{19}$$

Example 1.90

A dilation of the plane—that is, a size transformation with a positive scale factor—sends the circle of radius 2 centered at $A(2,2)$ to the circle of radius 3 centered at $A'(5,6)$. What distance does the origin $O(0,0)$, move under this transformation? **(AMC 10B 2016/20)**

1.6 Rotations

A. Rotations

1.91: Degree Measures of Rotations

Example 1.92

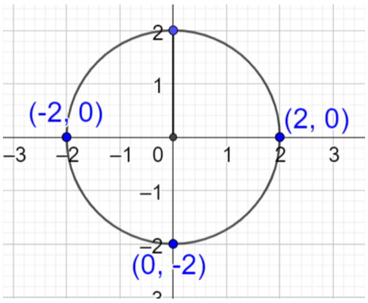
A. Full Circle
B. Half a Circle
C. One-fourth of a Circle
D. One-Fifth of a Circle

- Full Circle $= 360°$
- Half a Circle $= \frac{1}{2} \times 360° = 180°$
- One fourth of a Circle $= \frac{1}{4} \times 360° = 90°$
- One fifth of a Circle $= \frac{1}{4} \times 360° = 72°$

Example 1.93

Consider the point $(0,2)$. Find the coordinates of the point after it has been rotated clockwise with respect to the origin:

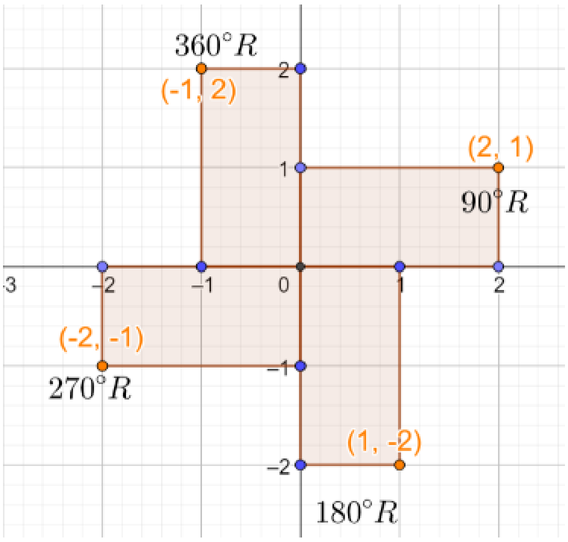
A. $90°$
B. $180°$
C. $270°$
D. $360°$



Example 1.94

Consider the point $(-1,2)$. Find the coordinates of the point after it has been rotated clockwise with respect to the origin:

A. $90°$
B. $180°$
C. $270°$
D. $360°$



After a $90°$ rotation, the point becomes

$$(2,1)$$

After a $180°$ rotation, the point becomes

$$(1,-2)$$

After a $270°$ rotation, the point becomes

$$(-2,-1)$$

1.95: Rotations

$$90°\ Counterclockwise: (x,y) \to (-y,x)$$
$$90°\ Clockwise: (x,y) \to (y,-x)$$
$$180°: (x,y) \to (-x,-y)$$

Example 1.96

Rotate the following points

$90°$ counter-clockwise:

A. $(1,2)$
B. $(3,7)$
C. (x,y)

$90°$ clockwise:

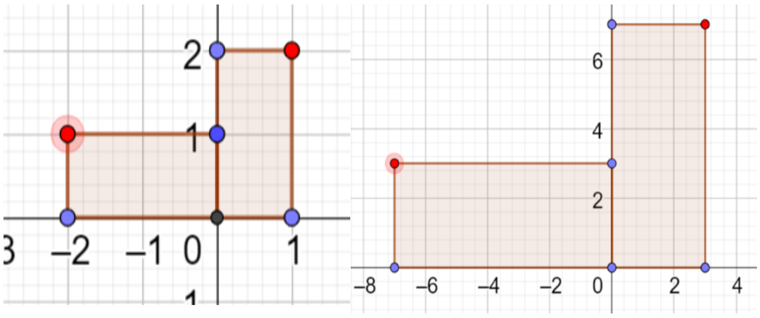
D. $(1,2)$
E. $(3,7)$
F. (x,y)

$180°$ clockwise:

G. $(1,2)$
H. $(3,7)$
I. (x,y)

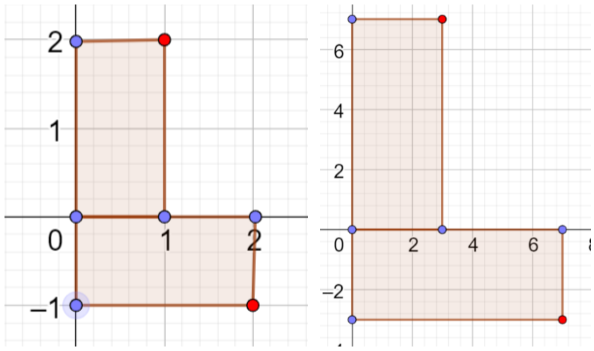
90° counter-clockwise:
Use $(x, y) \to (-y, x)$

$$(1,2) \to (-2,1)$$
$$(3,7) \to (-7,3)$$



90° clockwise:
Use $(x, y) \to (y, -x)$

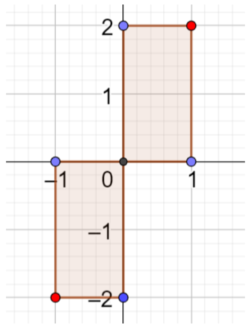
$$(1,2) \to (2,-1)$$
$$(3,7) \to (7,-3)$$



➢

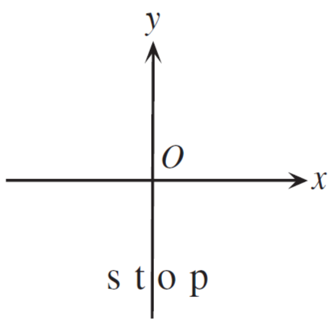
180° clockwise:
Use $(x, y) \to (-x, -y)$

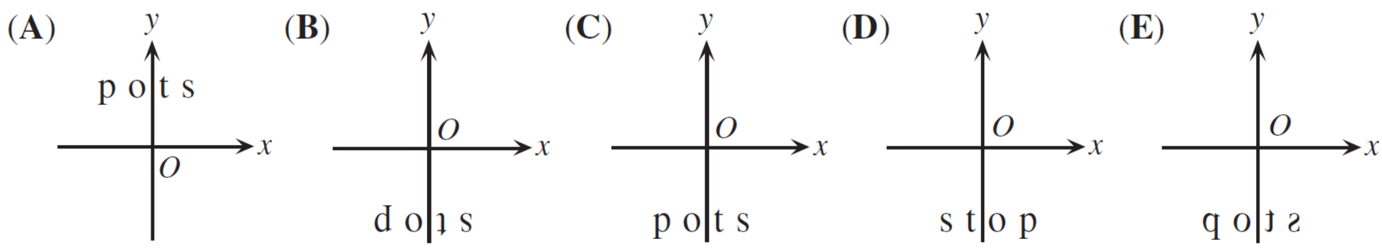
$$(1,2) \to (-1,-2)$$
$$(3,7) \to (-3,-7)$$



Example 1.97

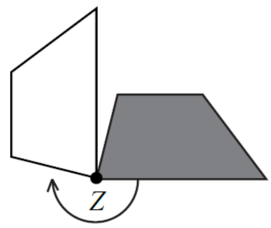
The word "stop" starts in the position shown in the diagram to the right. It is then rotated 180° clockwise about the origin, O, and this result is then reflected in the x-axis. Which of the following represents the final image? (CEMC Gauss 7 2002/20, CEMC Gauss 8 2002/19)





Example 1.98

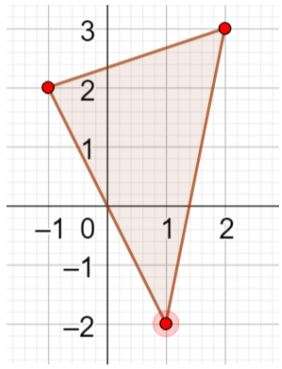
A clockwise rotation around point Z (that is, a rotation in the direction of the arrow) transforms the shaded quadrilateral to the unshaded quadrilateral. The angle of rotation is approximately **(CEMC Gauss 7 2014/9)**



Example 1.99

Triangle ABC has vertices $(2,3)$, $(-1,2)$ and $(1,-2)$. Find the coordinates of the vertices of the triangle after it has been rotated $90°$:

A. Counter-clockwise
B. Clockwise



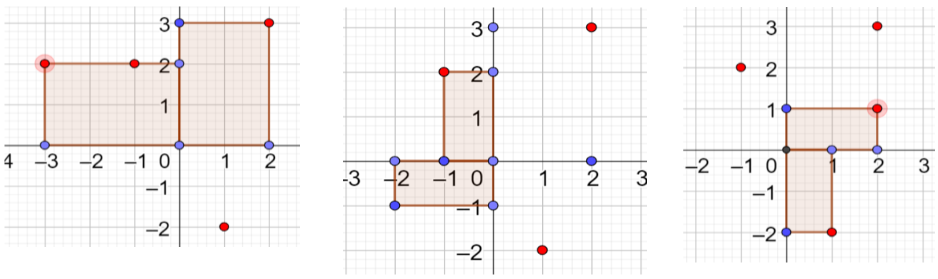
Part A

$$(2,3) \to (-3,2)$$
$$(-1,2) \to (-2,-1)$$
$$(1,-2) \to (2,1)$$

Part B

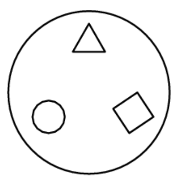
We can get the answer to Part B by simply reflecting the answer to Part A across the origin (which has the effect of rotating it a further $180°$.

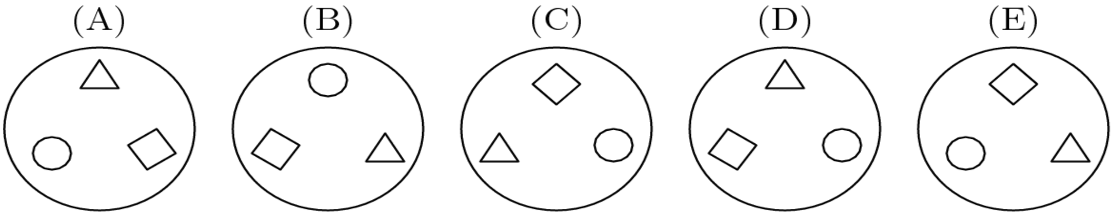
$$(2,3) \to (3,-2)$$
$$(-1,2) \to (2,1)$$
$$(1,-2) \to (-2,-1)$$



Example 1.100

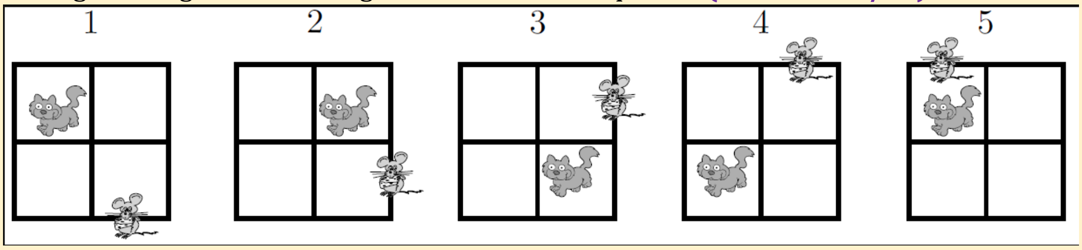
Which of the following represents the result when the figure shown below is rotated clockwise $120°$ around its center? **(AMC 8 1994/4)**



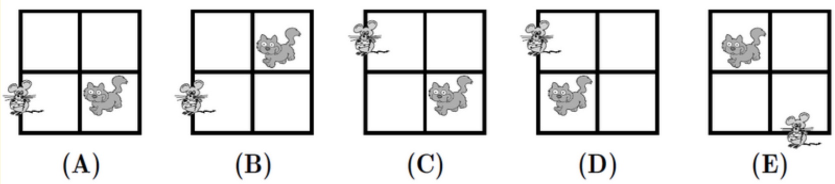


Example 1.101

In the pattern below, the cat moves clockwise through the four squares and the mouse moves counterclockwise through the eight exterior segments of the four squares. (AMC 8 2003/23)



If the pattern is continued, where would the cat and mouse be after the 247th move?



Example 1.102

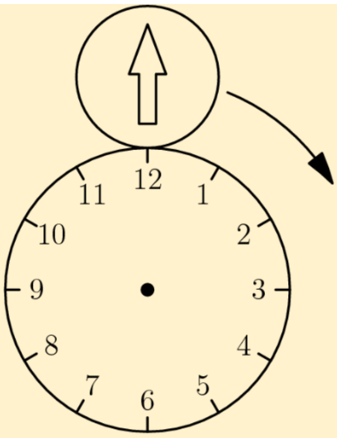
Rectangle $PQRS$ lies in a plane with $PQ = RS = 2$ and $QR = SP = 6$. The rectangle is rotated 90° clockwise about R, then rotated 90° clockwise about the point S moved to after the first rotation. What is the length of the path traveled by point P? (AMC 10A 2008/19)

Example 1.103

A unit square is rotated 45° about its center. What is the area of the region swept out by the interior of the square? (AMC 10A 2013/20)

Example 1.104

The diagram below shows the circular face of a clock with radius 20 cm and a circular disk with radius 10 cm externally tangent to the clock face at 12 o'clock. The disk has an arrow painted on it, initially pointing in the upward vertical direction. Let the disk roll clockwise around the clock face. At what point on the clock face will the disk be tangent when the arrow is next pointing in the upward vertical direction? (AMC 10A 2014/14)

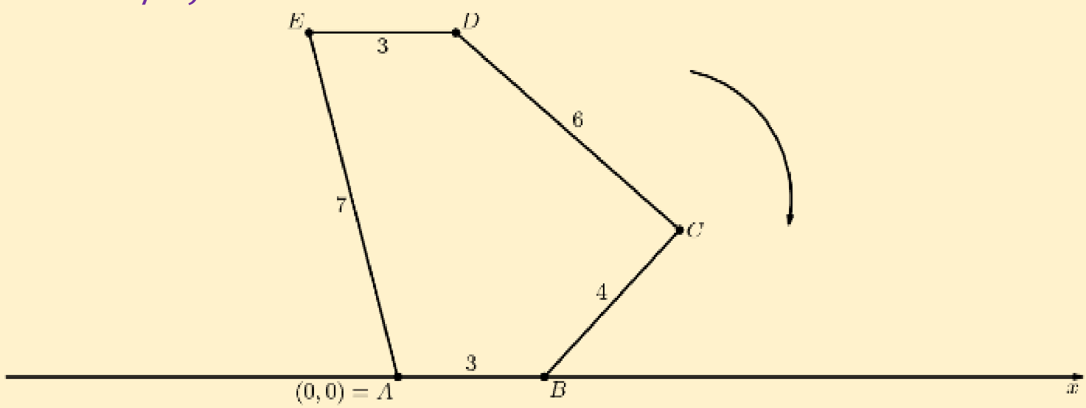


Example 1.105

Triangle OAB has $O = (0,0)$, $B = (5,0)$, and A in the first quadrant. In addition, $\angle ABO = 90^\circ$ and $\angle AOB = 30^\circ$. Suppose that OA is rotated 90° counterclockwise about O. What are the coordinates of the image of A? (AMC 10B 2008/14)

Example 1.106

As shown below, convex pentagon $ABCDE$ has sides $AB = 3$, $BC = 4$, $CD = 6$, $DE = 3$, and $EA = 7$. The pentagon is originally positioned in the plane with vertex A at the origin and vertex B on the positive x-axis. The pentagon is then rolled clockwise to the right along the x-axis. Which side will touch the point $x = 2009$ on the x-axis? (AMC 10B 2009/13)



1.7 Symmetry and Combining Transformations

A. Line Symmetry

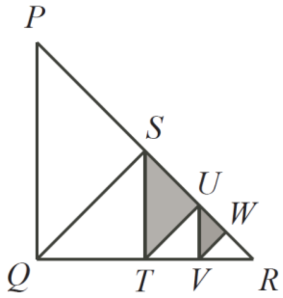
Example 1.107

The letter A has a vertical line of symmetry and the letter B does not. How many of the letters H L O R X D P E have a vertical line of symmetry? (CEMC Gauss 7 2020/12)

$$H, O, X \Rightarrow 3\ Letters$$

Example 1.108

In the right-angled triangle PQR, $PQ = QR$. The segments QS, TU and VW are perpendicular to PR, and the segments ST and UV are perpendicular to QR, as shown. What fraction of PQR is shaded? **(CEMC Gauss 7, 2013/23)**



Since QS is a line of symmetry

$$\Delta SQR = \frac{1}{2}\ of\ \Delta PQR$$

Since ST is a line of symmetry,

$$\Delta STR = 1/2\ of\ \Delta SQR = \frac{1}{4}\ of\ \Delta PQR$$

Since UT is a line of symmetry,

$$\Delta STU\ is\ \frac{1}{2}\ of\ Triangle\ STR = \frac{1}{4}\ of\ \Delta SQR = \frac{1}{8}\ of\ \Delta PQR$$

Similarly,

$$\Delta UTR = \frac{1}{16}\ of\ \Delta PQR$$

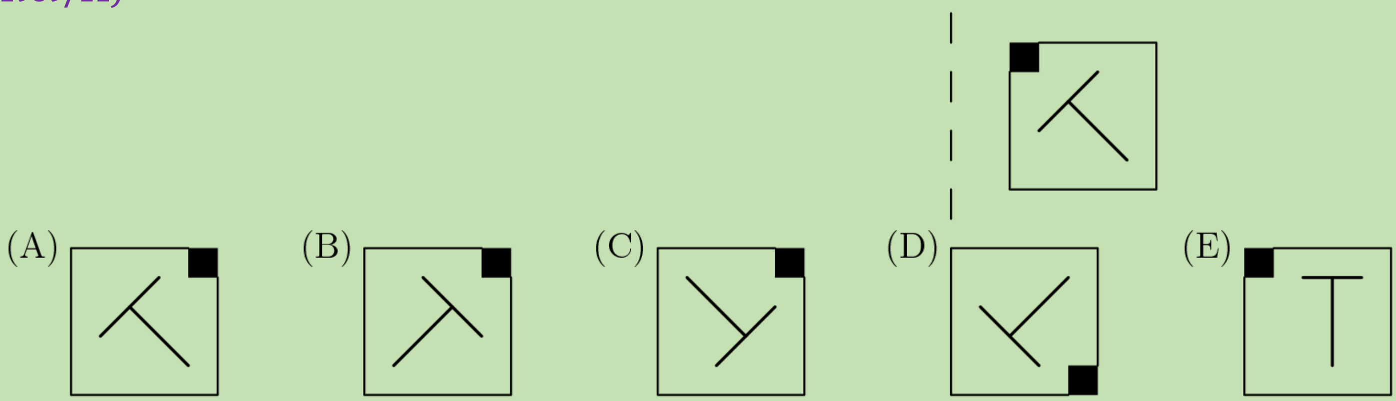
$$\Delta UVW = \frac{1}{32}\ of\ \Delta PQR$$

Hence, the shading that we are looking for is

$$\frac{1}{8} + \frac{1}{32} = \frac{4}{32} + \frac{1}{32} = \frac{5}{32}$$

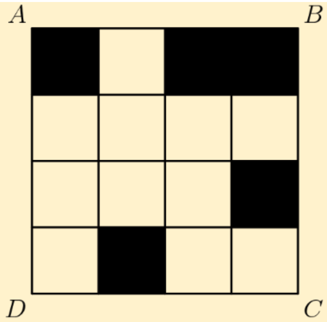
Example 1.109

Which of the five "T-like shapes" would be symmetric to the one shown with respect to the dashed line? **(AMC 8 1989/11)**



Example 1.110

What is the minimum number of small squares that must be colored black so that a line of symmetry lies on the diagonal $\overline{BD}$ of square $ABCD$? (AMC 8 2005/3)



Example 1.111

Which of the following figures has the greatest number of lines of symmetry? (AMC 8 2010/6)

(A) equilateral triangle (B) non-square rhombus (C) non-square rectangle (D) isosceles trapezoid (E) square

How many of the twelve pentominoes pictured below have at least one line of symmetry? **(AMC 10 2001/5)**

Six

B. Rotational Symmetry

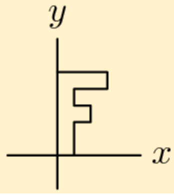
C. Combining Transformations

Example 1.112

Making Transformations
Determining Transformations

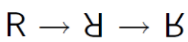
Example 1.113

The letter F shown below is rotated 90° clockwise around the origin, then reflected in the y-axis, and then rotated a half turn around the origin. What is the final image? **(AMC 10B 2015/8)**

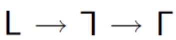


Example 1.114

Using two transformations, the letter R is changed as shown:



Using the same two transformations, the letter L is changed as shown:



Using the same two transformations, the letter G is changed to (CEMC Gauss 2008/15)

(A)  **(B)**  **(C)**  **(D)**  **(E)** 

1.8 Distance Formula

A. Revision: Radicals

To simplify $\sqrt{8}$ use the following procedure:

- Find the prime factorization of the number inside the radical sign.
$$\sqrt{8} = \sqrt{2 \times 2 \times 2}$$
- Separate out perfect squares
$$\sqrt{8} = \sqrt{4 \times 2}$$
- Break the square root into two separate roots
$$\sqrt{8} = \sqrt{4} \times \sqrt{2}$$
- Take the square root of the perfect squares
$$\sqrt{8} = 2 \times \sqrt{2} = 2\sqrt{2}$$

Example 1.115: Simplifying Square Roots

Simplify the following square roots.

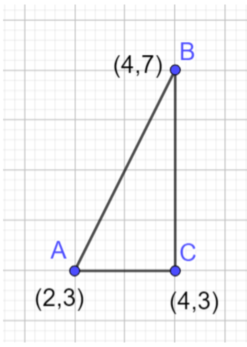
A. $\sqrt{50}$
B. $\sqrt{32}$
C. $\sqrt{72}$
D. $\sqrt{125}$
E. $\sqrt{27}$

$$\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5 \times \sqrt{2} = 5\sqrt{2}$$
$$\sqrt{32} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$$
$$\sqrt{72} = \sqrt{36} \times \sqrt{2} = 6\sqrt{2}$$
$$\sqrt{125} = \sqrt{25} \times \sqrt{5} = 5\sqrt{5}$$
$$\sqrt{27} = \sqrt{9} \times \sqrt{3} = 3\sqrt{3}$$

B. Distance Formula

Example 1.116: Pythagorean Theorem to Find Distance

A. Find the distance between (2,3) and (4,7).
B. Find the distance between (2,1) and (7,5).
C. Find the distance between (3,5) and (7,1).



Part A

The line that goes from A to B is slanted. It's easier to find the straight-line distance. Plot point $C(4,3)$, which has the same vertical value as A, and the same horizontal value as B.

The horizontal distance between (2,3) and (4,7) is given by the difference in their x-coordinates, which is:
$$AC = x_2 - x_1 = 4 - 2 = 2$$

The vertical distance between (2,3) and (4,7) is given by the difference in the y-coordinates, which is:

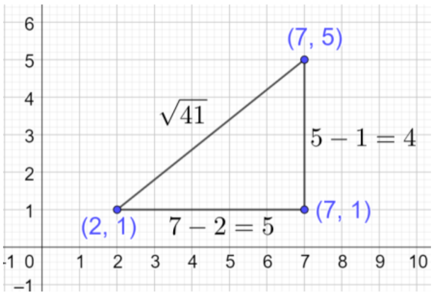
$$BC = y_2 - y_1 = 7 - 3 = 4$$

Apply the Pythagorean Theorem:

$$AB^2 = AC^2 + BC^2 = 2^2 + 4^2 = 4 + 16 = 20$$

Take square roots:

$$AB = \sqrt{20} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$$



Part B

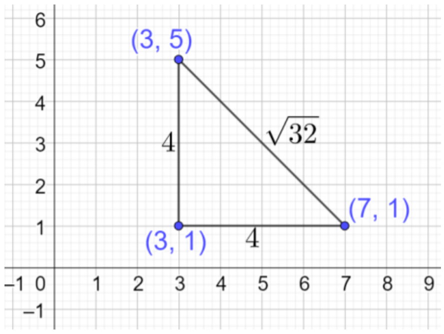
By the Pythagorean Theorem:

$$D = \sqrt{(7-2)^2 + (5-1)^2} = \sqrt{5^2 + 4^2} = \sqrt{25 + 16} = \sqrt{41}$$

Part C

By the Pythagorean Theorem:

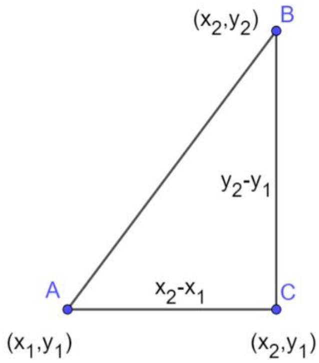
$$D = \sqrt{(7-3)^2 + (5-1)^2} = \sqrt{4^2 + 4^2} = \sqrt{32}$$



1.117: Distance Formula in the 2D Plane

The distance between two points (x_1, y_1) and (x_2, y_2) on the coordinate plane is given by the square root of the sum of the squares of the difference between the respective $\boldsymbol{x}$ coordinates, and the respective $\boldsymbol{y}$ coordinates.

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Proof: Introduce Point $C(x_2, y_1)$, such that ΔABC is right-angled

We want the distance AB. Hence, apply Pythagoras Theorem in right ΔABC:

$$AB^2 = AC^2 + CB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Take the square root of both sides:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Property

The order of the points does not matter, since the square of a number is the same as the square of its negation:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[-(x_2 - x_1)]^2 + [-(y_2 - y_1)]^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

1.118: Distance Formula in 3D

The distance between two points $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ in three-dimensional space is given by:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- Let A and B be at opposite corners of a cuboid in 3D coordinate space.
- Find the length of the base diagonal
- Find the length of the long diagonal using two consecutive applications of Pythagoras Theorem.

Example 1.119: Using the Distance Formula

Find the distance between:

A. $\left(\underbrace{-5}_{x_1},\underbrace{5}_{y_1}\right)$ and $\left(\underbrace{5}_{x_2},\underbrace{-5}_{y_2}\right)$
B. $(-1,4)$ and $(2,-7)$
C. $(3,-1)$ and $(0,4)$
D. $\left(\frac{1}{2},\frac{1}{3}\right)$ and $\left(\frac{2}{3},\frac{3}{4}\right)$
E. $(2,3,7)$ and $(1,-2,5)$

Part A

$$D=\sqrt{\left(5-(-5)\right)^2+(-5-5)^2}=\sqrt{100+100}=\sqrt{200}=\sqrt{100}\times\sqrt{2}=10\sqrt{2}$$

Part B

$$D=\sqrt{\left(2-(-1)\right)^2+(-7-4)^2}=\sqrt{3^2+(-11)^2}=\sqrt{9+121}=\sqrt{130}$$

Part C

$$D=\sqrt{(0-3)^2+\left(4-(-1)\right)^2}=\sqrt{(-3)^2+5^2}=\sqrt{9+25}=\sqrt{34}$$

Part D

$$D=\sqrt{\left(\frac{2}{3}-\frac{1}{2}\right)^2+\left(\frac{3}{4}-\frac{1}{3}\right)^2}=\sqrt{\left(\frac{1}{6}\right)^2+\left(\frac{5}{12}\right)^2}=\sqrt{\frac{1}{36}+\frac{25}{144}}=\sqrt{\frac{29}{144}}=\frac{\sqrt{29}}{12}$$

Part E

$$D=\sqrt{(2-1)^2+\left(3-(-2)\right)^2+(7-5)^2}=\sqrt{1^2+5^2+2^2}=\sqrt{1+25+4}=\sqrt{30}$$

Example 1.120: Basic Word Problems

A. Find the distance between the points $(2,2)$ and $(-1,-1)$. (MathCounts 1992 Warm-Up 6)
B. What is the number of units in the distance between $(2,5)$ and $(-6,-1)$? (MathCounts 2000 Warm-Up 17)

1.121: Distance From the Origin

$$2D\ Geometry: \sqrt{x^2+y^2}$$
$$3D\ Geometry: \sqrt{x^2+y^2+z^2}$$

When we want to find the distance of a point from the origin, the origin has coordinates $O=(0,0)$. Therefore, the distance of $A=(x_2,y_2)$ from the origin becomes

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(x_2-0)^2+(y_2-0)^2}=\sqrt{(x_2)^2+(y_2)^2}=\sqrt{x^2+y^2}$$

Don't bother memorizing the formula. Keep the concept in mind, and use the main formula only.

Example 1.122: Pythagorean Triplets

Find the distance that each point lies from the origin.

A. $(3,4)$
B. $(5,12)$
C. $(7,24)$
D. $(8,15)$
E. $(9,40)$

$$A.\sqrt{3^2+4^2}=\sqrt{9+16}=\sqrt{25}=5$$

$$B.\sqrt{5^2+12^2}=\sqrt{25+144}=\sqrt{169}=13$$
$$C.\sqrt{7^2+24^2}=\sqrt{49+576}=\sqrt{625}=25$$
$$D.\sqrt{8^2+15^2}=\sqrt{64+225}=\sqrt{289}=17$$
$$E.\sqrt{9^2+40^2}=\sqrt{81+1600}=\sqrt{1681}=41$$

Example 1.123

A. The point P has the same x coordinate and the same y coordinate. Find the coordinates of point P if it is at a distance of $\sqrt{8}$ units from the origin on the coordinate plane.

B. A mouse at point O moves n units in the x direction, and n units in the y direction to reach point P. Then, he comes back in a straight line to O, and travels $\sqrt{5}$ units. Find n.

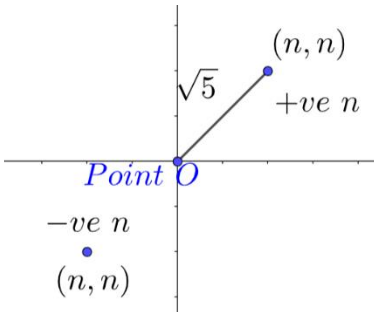
C. A mouse at point O moves n units in the x direction, and n units in the y direction to reach point P. Then, he comes back in a straight line to O. Find the difference between the distance travelled when going, and the distance travelled when coming back.

Part A

Let $P=(a,a)$. By the distance formula, the distance from the origin (0,0):

$$\sqrt{(a-0)^2+(a-0)^2}=\sqrt{8}$$
$$\sqrt{2a^2}=\sqrt{8}\Rightarrow 2a^2=8\Rightarrow a^2=4\Rightarrow a=\pm 2$$

Part B



Introduce an origin at point O. Then the coordinates of point P are:

$$(n,n)$$

By the distance formula:

$$\sqrt{(n-0)^2+(n-0)^2}=\sqrt{5}$$
$$\sqrt{2n^2}=\sqrt{5}\Rightarrow 2n^2=5\Rightarrow n^2=\frac{5}{2}\Rightarrow n=\pm\sqrt{\frac{5}{2}}$$

Part C

The distance when going is:

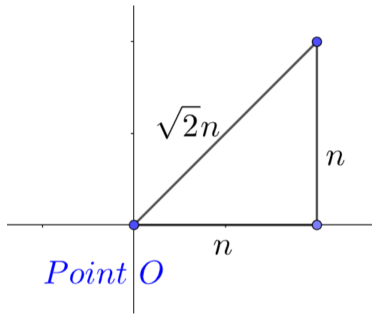
$$n+n=2n$$

The distance when coming back is:

$$\sqrt{(n-0)^2+(n-0)^2}=\sqrt{2n^2}=\sqrt{2}n$$

Difference

$$=2n-\sqrt{2}n=n\left(2-\sqrt{2}\right)$$



Example 1.124

$$a_n=2^n, n=1,2,3,\dots$$

A. Find the distance of the point $\left(a_1,a_2,a_3\right)$ from the origin.

B. $P_n=\left(a_n,a_{n+1},a_{n+2}\right)$. The distance between P_n and the origin is given by D_n. Find D_3-D_2.

Part A

$$\left(a_1,a_2,a_3\right)=\left(2^1,2^2,2^3\right)=(2,4,8)$$
$$Distance=\sqrt{2^2+4^2+8^2}=\sqrt{4+16+64}=\sqrt{84}=2\sqrt{21}$$

Part B

$$D_2=\sqrt{(2^2)^2+(2^3)^2+(2^4)^2}=\sqrt{2^4+2^6+2^8}=\sqrt{16+64+256}=\sqrt{336}=4\sqrt{21}$$
$$D_3=\sqrt{(2^3)^2+(2^4)^2+(2^5)^2}=\sqrt{2^6+2^8+2^{10}}=\sqrt{64+256+1024}=\sqrt{1344}=8\sqrt{21}$$
$$D_3-D_2=8\sqrt{21}-4\sqrt{21}=4\sqrt{21}$$

Example 1.125

$$f(x) = an^3 + bn^2 + cn + d = 0, \qquad a \neq 0$$

Point P is a point in $3D$ space. Find the distance from the origin of $P = (x, y, z)$ if

- x is equal to the sum of the roots of the above equation.
- y is equal to the product of the zeroes of f.
- z is equal to the sum of the pair-wise products of the roots of the above equation.

$$x = -\frac{b}{a}, \qquad y = -\frac{d}{a}, \qquad z = \frac{c}{a}$$

$$Distance = \sqrt{\left(-\frac{b}{a}\right)^2 + \left(-\frac{d}{a}\right)^2 + \left(\frac{c}{a}\right)^2} = \sqrt{\frac{b^2 + c^2 + d^2}{a^2}} = \frac{\sqrt{b^2 + c^2 + d^2}}{a}$$

Example 1.126

$$f(x) = \frac{x - 3}{(x^2 - x - 6)(x^2 + 7x + 10)}$$

The domain of $f(x)$ is $D_f = \mathbb{R} - \{x, y, z\}$. Find the distance of the point $P = \{x, y, z\}$ from the origin.

$$f(x) = \frac{x - 3}{(x - 3)(x + 2)^2(x + 5)}$$

$$D_f = \mathbb{R} - \{-5, -2, 3\}$$

$$Distance = \sqrt{(-5)^2 + (-2)^2 + 3^2} = \sqrt{25 + 4 + 9} = \sqrt{38}$$

Example 1.127

The function used for the Collatz conjecture is defined piece-wise:

$$f(n) = \begin{cases} \frac{n}{2}, & if\ n\ is\ even \\ 3n + 1, & if\ n\ is\ odd \end{cases}$$

Recall that composition means that the function is applied twice.

$$f^2(n) \rightarrow f\ applied\ twice$$
$$f^m(n) \rightarrow f\ applied\ m\ times$$

For some y:

$$f^x(19) = f^{x+3}(19), \qquad f^{x+1}(19) = f^{x+4}(19), \qquad f^{x+2}(19) = f^{x+5}(19)$$

Find the distance from the origin of the point:

$$\left(f^x(19), f^{x+1}(19), f^{x+2}(19)\right)$$

$$19,58,29,88,44,22,11,34,17,52,26,13,40,20,10,5,16,8, \underbrace{4,2,1}_{Cycle}, \underbrace{4,2,1}_{Cycle}$$

$$Distance = \sqrt{4^2 + 2^2 + 1^2} = \sqrt{16 + 4 + 1} = \sqrt{21}$$

Example 1.128

How many lattice points are exactly 1 unit away from the origin:

A. on the coordinate plane.

B. in 3D coordinate space.
C. in n −dimensional coordinate space.

Part A

$$(\pm 1,0),(0,\pm 1) \Rightarrow 4\ Points$$

Part B

$$(\pm 1,0,0),(0,\pm 1,0)(0,0,\pm 1) \Rightarrow 6\ Points$$

Part C

$$(a_1,a_2,\dots,a_n)$$

Where exactly one out of $a_1,a_2,\dots,a_n$ is non-zero, others are zero.
And the non-zero variable can ± 1.

$$\underbrace{n}_{\substack{Choice\\ of\ Direction}} \times \underbrace{2}_{\pm 1} = 2n$$

Example 1.129

How many lattice points are exactly 3 units away from the origin:

A. on the coordinate plane.
B. in 3D coordinate space.
C. in 4D dimensional coordinate space.
D. in 5D dimensional coordinate space.
E. in n-dimensional coordinate space.

Part A

$$(0,\pm 3),(\pm 3,0) \Rightarrow 4\ Points$$

Part B

Case I

$$(\pm 3,0,0),(0,\pm 3,0)(0,0,\pm 3) \Rightarrow 6\ Points$$

Case II

Let the coordinates of a point satisfying this condition be $P=(x,y,z), x,y,z \in \mathbb{Z}$

$$\sqrt{x^2+y^2+z^2}=3 \Rightarrow x^2+y^2+z^2=9$$

If we take

$$x=3 \Rightarrow x^2=9 \Rightarrow y^2=z^2=0 \Rightarrow Not\ Valid$$
$$x=1 \Rightarrow y^2+z^2=8 \Rightarrow y=z=2\ Works$$
$$(1,2,2,3)\ is\ a\ Pythagorean\ Quadruple$$

The points which will work are:

$$(1,2,2),(2,1,2),(2,2,1)$$

And each of the above can be positive or negative.

$$\underbrace{2}_{x\ coordinate} \times \underbrace{2}_{y\ coordinate} \times \underbrace{2}_{z\ coordinate} = 8$$

Total from this case

$$=3\times 8=24$$

Total for this Part

$$=6+24=30$$

Part D

Case I

$$(\pm 3,0,0,0),(0,\pm 3,0,0)(0,0,\pm 3,0),(0,0,0,\pm 3) \Rightarrow 8\ Points$$

Case II
Let the coordinates of a point satisfying this condition be $P=(x,y,z), x,y,z \in \mathbb{Z}$
$$\sqrt{x^2+y^2+z^2+a^2}=3 \Rightarrow x^2+y^2+z^2+a^2=9$$
If we take
$$x=3 \Rightarrow x^2=9 \Rightarrow y^2=z^2=0 \Rightarrow Not\ Valid$$
$$x=1 \Rightarrow y^2+z^2=8 \Rightarrow y=z=2\ Works$$

The points which will work are a distinct rearrangement of:
$$(0,1,2,2)$$
There are
$$\underbrace{4}_{\substack{Choices\\ for\ Zero}} \times \underbrace{3}_{\substack{Way\ to\\ Arrange\ (1,2,2)}} \times \underbrace{8}_{\substack{Positive\\ and\ Negative\ Values}} = 96$$

Total for this Part
$$=6+96=102$$

Part E
Points on the coordinate axes:
$$2n\ Points$$

Case (1,2,2)
$$\underbrace{n}_{Position\ of\ 1} \times \underbrace{\binom{n-1}{2}}_{Position\ of\ 2} \times \underbrace{2^3}_{\substack{Positive\\ and\ Negative\ Values}} = n \cdot \frac{(n-1)(n-2)}{2} \cdot 8 = 4n(n-1)(n-2)$$
Case (2,1,1,1,1,1)
$$\underbrace{n}_{Position\ of\ 2} \times \underbrace{\binom{n-1}{5}}_{Position\ of\ 1's} \times \underbrace{2^6}_{\substack{Positive\\ and\ Negative\ Values}} = 2^6 n\binom{n-1}{5}$$
Case (1,1,1,1,1,1,1,1,1)
$$\underbrace{\binom{n}{9}}_{\substack{Choose\ the\\ 9\ 1's}} \times \underbrace{2^9}_{\substack{Positive\\ and\ Negative\ Values}} = 2^9\binom{n}{9}$$

Challenge 1.130
$A=(a,b)$ and $B=(a,b,c), abc \neq 0$ are lattice points in the xy coordinate plane, and 3D coordinate space, respectively. A is an integral number of units from the origin. B is 13 units from the origin. The n distinct ordered triples satisfying these conditions can be written $(a_1,b_1,c_1),(a_2,b_2,c_2),\dots,(a_n,b_n,c_n)$. Find $x+y$ given that:
$$\prod_{i=1}^{n} a_i b_i c_i = (a_1b_1c_1)\times(a_2b_2c_2)\times\dots\times(a_nb_nc_n)=2^x3^y$$

A is a lattice point, and also it is an integer number of units from the origin:
$$\therefore \left(a,b,\sqrt{a^2+b^2}\right)\ is\ a\ Pythagorean\ Triplet$$

Introduce a cuboid with $O=(0,0)$ at the origin, $A=(a,b)$ on the base, diagonally opposite to O.
Point B is on the longest diagonal of the cuboid.
$$Base\ diagonal\ of\ the\ cuboid=\sqrt{a^2+b^2} \Rightarrow Integer$$

$$\text{Height of the cuboid} = c \Rightarrow \text{Integer}$$
$$\left(\sqrt{a^2+b^2}, c, 13\right) \text{ is a Pythagorean Triplet}$$

Hence, we must have

$$(5,12,13) \text{ or } (12,5,13)$$

12 does not work, since 12 is not the hypotenuse of any Pythagorean Triplet.
However, 5 is the hypotenuse of

$$(3,4,5)$$

$$(a,b,c) = (3,4,12)$$

We have a total of

$$\underbrace{2\ Choices}_{\pm 3} \times \underbrace{2\ Choices}_{\pm 4} \times \underbrace{2\ Choices}_{\pm 12}$$

$$(3,4,12), (-3,4,12), (3,-4,12), (3,4,-12), (-3,-4,12), (-3,4,-12), (3,-4,-12), (-3,-4,-12)$$

Now we need to take the product of the above:

$$(3 \times 4 \times 12)^8 = (3^1 \times 2^2 \times 3 \times 2^2)^8 = (3^2 \times 2^4)^8 = 3^{16} \times 2^{32}$$
$$a + b = 16 + 32 = 48$$

Example 1.131: Radicals in coordinate points

Recall the following property from exponents and radicals"

$$\left(\sqrt{x}\right)^2 = \left(x^{\frac{1}{2}}\right)^2 = x^{\frac{1}{2}\times 2} = x^1 = x \Rightarrow \text{The square and the square root cancel}$$

Find the distance that each point lies from the origin.

A. $\left(\sqrt{40}, \sqrt{41}\right)$
B. $\left(\sqrt{3}, \sqrt{22}\right)$
C. $\left(-\sqrt{3}, \sqrt{5}\right)$
D. $\left(\sqrt{7}, -\sqrt{11}\right)$

$$A. \sqrt{\left(\sqrt{40}\right)^2 + \left(\sqrt{41}\right)^2} = \sqrt{40+41} = \sqrt{81} = 9$$
$$B. \sqrt{\left(\sqrt{3}\right)^2 + \left(\sqrt{22}\right)^2} = \sqrt{3+22} = \sqrt{25} = 5$$
$$C. \sqrt{\left(-\sqrt{3}\right)^2 + \left(\sqrt{5}\right)^2} = \sqrt{3+5} = \sqrt{8} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$$
$$D. \sqrt{\left(\sqrt{7}\right)^2 + \left(-\sqrt{11}\right)^2} = \sqrt{7+11} = \sqrt{18} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$$

Example 1.132

What is the minimum distance between two points in 3D coordinate space?

A. Without any further condition
B. If they are both lattice points
C. If they are both distinct lattice points

Parts A and B

$$Minimum\ Distance = 0$$

Part C

$$Minimum\ Distance = 1$$

C. Algebraic Applications

In the prior section, we looked at direct applications of the distance formula, which let us calculate the distance from

- One point to another
  - ✓ Remember that distance is never negative. Hence, the smallest value for distance is zero.
- A point to the origin
  - ✓ This is just a special case of one point to another, but it is quite useful.

In this section, we look at calculations using the distance formula, where:

- some values are missing
- OR special conditions are given

These missing values or special conditions lead to the formation of equations. Solving these equations gives us the values that we need to find the answer to the question.

Example 1.133: Distance from the origin

Sherlock finds that the distance of a well from a tree on an alfalfa farm is 23 units. He plots a coordinate system, and puts the well at the origin. What is the sum of the squares of the coordinates of the tree?

$$D\left(\underbrace{Tree}_{(x_2,y_2)}, \underbrace{Well}_{(0,0)}\right) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(x_2-0)^2+(y_2-0)^2} = \underbrace{\sqrt{(x_2)^2+(y_2)^2} = 23}_{Equation\ I}$$

Square both sides of Equation I:

$$(x_2)^2+(y_2)^2 = \underbrace{529}_{\substack{Sum\ of\ the\ squares\ of\ the\\ coordinates\ of\ the\ tree}}$$

Example 1.134: Equidistant points

A. Find the point A on the y-axis that is equidistant from the points $X(3,-3)$ and $Y(-2,2)$.
B. Find the point A on the y-axis that is equidistant from the points $M(-1,5)$ and $N(4,-7)$.
C. Point A lies on the x-axis at the same distance from $P(-2,0)$ and $Q(0,4)$. Find the coordinates of A.
D. Point P is the point on the x-axis equidistant from $A(4,-2)$ and $B(1,5)$. Find the coordinates of P.
E. Point $P(a-3, a+2)$ is equidistant from the points $(4,1)$ and $(5,2)$. Find the coordinates of P.

Part A

Any point on the y −axis must have coordinates $(0,a)$.

Since point A is equidistant from X and Y:

$$\underbrace{AX}_{\substack{Distance\\ from\ A\ to\ X}} = \underbrace{AY}_{\substack{Distance\\ from\ A\ to\ Y}}$$

Use the distance formula:

$$\sqrt{(0-3)^2+(a+3)^2} = \sqrt{(0+2)^2+(a-2)^2}$$

Square both sides and simplify:

$$9+a^2+6a+9 = 4+a^2-2a+4$$

Solve for a:

$$8a+18 = 8 \Rightarrow 8a = -10 \Rightarrow a = -\frac{10}{8} = -\frac{5}{4}$$

Part B

Any point on the y −axis must have coordinates $(0,a)$.

Since point A is equidistant from $M(-1,5)$ and $N(4,-3)$:

$$\underbrace{AM}_{\substack{Distance\\ from\ A\ to\ M}} = \underbrace{AN}_{\substack{Distance\\ from\ A\ to\ N}}$$

Use the distance formula:

$$\sqrt{(0+1)^2+(a-5)^2} = \sqrt{(0-4)^2+(a+7)^2}$$

Square both sides and simplify:

$$1 + a^2 - 10a + 25 = 16 + a^2 + 14a + 49$$

Solve for a:

$$-10a + 26 = 14a + 65 \Rightarrow 4a = 39 \Rightarrow a = \frac{39}{4}$$

Part C

Since the point is on the x-axis, it must be of the form $(x, 0)$:

$$\underbrace{PA}_{\substack{\text{Distance}\\ \text{from } P \text{ to } A}} = \underbrace{QA}_{\substack{\text{Distance}\\ \text{from } Q \text{ to } A}}$$

Substitute $P(-2, 0)$ in the LHS, and $Q(0,4)$ in the RHS and use the distance formula:

$$\sqrt{(x+2)^2 + (0-0)^2} = \sqrt{(x-0)^2 + (0-4)^2}$$

Simplify, and square both sides:

$$\boldsymbol{x^2} + 4x + 4 = \boldsymbol{x^2} + 16$$

Solve for x:

$$4x = 12 \Rightarrow x = 3 \Rightarrow A = (3,0)$$

Part D

Since the point is on the x-axis, it must be of the form $(x, 0)$:

Part E

$$\begin{aligned} PQ^2 &= (a-3-4)^2 + (a+2-1)^2 \\ &= (a-7)^2 + (a+1)^2 \\ &= a^2 - 14a + 49 + a^2 + 2a + 1 \\ &= 2a^2 - 12a + 50 \end{aligned}$$

$$\begin{aligned} PR^2 &= (a-3-5)^2 + (a+2-2)^2 \\ &= (a-8)^2 + a^2 \\ &= 2a^2 - 16a + 64 \end{aligned}$$

$$\underbrace{2a^2 - 12a + 50}_{PQ^2} = \underbrace{2a^2 - 16a + 64}_{PR^2}$$

$$4a = 14 \Rightarrow a = \frac{14}{4} = \frac{7}{2}$$

Example 1.135: Finding the coordinates of a point

The distance between $(x, 4)$ and $\left(7, \frac{1}{2}\right)$ is 6. Find the sum of the possible values of x.

Substitute the known values in the distance formula $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$:

$$\sqrt{(x-7)^2 + \left(4 - \frac{1}{2}\right)^2} = 6$$

Square both sides:

$$(x-7)^2 + \frac{49}{4} = 36$$

Isolate $(x-7)^2$ on the left-hand side:

$$(x-7)^2 = 36 - \frac{49}{4} \Rightarrow (x-7)^2 = \frac{95}{4}$$

Take the square root both sides, and solve for x:

$$x - 7 = \pm\sqrt{\frac{95}{4}} \Rightarrow x = 7 \pm \frac{\sqrt{95}}{2}$$

Find the sum:

$$\left(7 + \frac{\sqrt{95}}{2}\right) + \left(7 - \frac{\sqrt{95}}{2}\right) = 14$$

Example 1.136: Finding the coordinates of a point

The distance between $(4,5)$ and $(6, y)$ is 12. Find the product of the possible values of y.

Substitute the known values in the distance formula $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$:

$$\sqrt{(6-4)^2 + (y-5)^2} = 12$$

Square both sides, and simplify within the square root:

$$(2)^2 + (y-5)^2 = 144$$

Isolate the $(y-5)^2$ term, and take square roots both sides:

$$(y-5)^2 = 140 \Rightarrow y - 5 = \pm\sqrt{140}$$

Simplify the right-hand side, and then solve for y:

$$y - 5 = \pm\left(\sqrt{4} \times \sqrt{35}\right) \Rightarrow y - 5 = \pm 2\sqrt{35} \Rightarrow y = 5 \pm 2\sqrt{35}$$

$$Product = \left(5 + 2\sqrt{35}\right)\left(5 - 2\sqrt{35}\right)$$

This is of the form $(a+b)(a-b) = a^2 - b^2$:

$$5^2 - \left(2\sqrt{35}\right)^2 = 25 - 4 \times 35 = 25 - 140 = -115$$

Example 1.137: Finding the coordinates of a point

The points $(a, 2,3)$ and $(2,4,5)$ have a distance of 5 between them. Find the possible values of a.

$$\sqrt{(a-2)^2 + (2-4)^2 + (3-5)^2} = 5$$
$$(a-2)^2 + 4 + 4 = 25$$
$$(a-2)^2 = 17$$
$$a - 2 = \pm\sqrt{17}$$
$$a = 2 \pm \sqrt{17}$$

Example 1.138: Finding the coordinates of a point

A. Find the coordinates of the point X on the z axis equidistant from $P(3,1,4)$ and $Q(2,7,1)$.
B. Find the coordinates of the point on the x axis is twice as far from $(2,4,-3)$ as it is from $(1,6,3)$.

Any point on the z axis must be of the form $(0,0,z)$:

$$XP = XQ$$
$$\sqrt{(3-0)^2 + (1-0)^2 + (4-z)^2} = \sqrt{(2-0)^2 + (7-0)^2 + (1-z)^2}$$
$$9 + 1 + (z^2 - 8z + 16) = 4 + 49 + (z^2 - 2z + 1)$$
$$-8z + 26 = -2z + 44$$
$$-6z = 28$$
$$z = -\frac{18}{6} = -3$$

Challenge 1.139: Finding the coordinates of a point

The points $(a, -3,2)$ and $(4, b, 5)$ have a distance of 7 between them. If b is twice the absolute value of a, find the sum of all possible values of a.

$$b = 2|a|$$

Case I: $a \geq 0 \Rightarrow b = 2a$

$$\sqrt{(4-a)^2 + (2a+3)^2 + (5-2)^2} = 7$$
$$(a^2 - 8a + 16) + (4a^2 + 12a + 9) + 9 = 49$$
$$5a^2 + 4a - 15 = 0$$

Solve the above quadratic using $a = 5, b = 4, c = -15$:

$$a = \frac{-4 \pm \sqrt{16 - (4)(5)(-15)}}{2(5)} = \frac{-4 \pm \sqrt{316}}{10} = \frac{-4 \pm 2\sqrt{79}}{10} = \frac{-2 \pm \sqrt{79}}{5}$$

$$\frac{-2 + \sqrt{79}}{5} \Rightarrow +ve$$

$$-2 - \sqrt{19} \Rightarrow -ve \Rightarrow Contradiction$$

$$Sum\ of\ Values = -\frac{b}{a} - \frac{4}{5}$$

Case II: $a < 0 \Rightarrow b = -2a$

$$\sqrt{(4-a)^2 + (-2a+3)^2 + (5-2)^2} = 7$$
$$(a^2 - 8a + 16) + (4a^2 - 12a + 9) + 9 = 49$$
$$5a^2 - 20a - 15 = 0$$
$$a^2 - 4a - 3 = 0$$

Solve the above quadratic using $a = 1, b = -4, c = -3$:

$$a = \frac{4 \pm \sqrt{16 - (4)(1)(-3)}}{2(1)} = \frac{4 \pm \sqrt{28}}{2} = \frac{4 \pm 2\sqrt{7}}{2} = 2 \pm \sqrt{7}$$
$$2 + \sqrt{7} > 0 \Rightarrow +ve \Rightarrow Contradiction$$
$$2 - \sqrt{4} = 2 - 2 = 0 \Rightarrow 2 - \sqrt{7} < 0 \Rightarrow -ve$$

The final answer is:

$$\frac{-2 + \sqrt{79}}{5} + 2 - \sqrt{7} = \frac{8 + \sqrt{79} - 5\sqrt{7}}{5}$$

Challenge 1.140: Finding the coordinates of a point

Recall that the floor function $f(x) = \lfloor x \rfloor$ gives the largest value of x that is the left of x on the real number line. Find the value of x given that

A. the points $(\lfloor x \rfloor, 1, 6)$ and $(2, \lfloor x \rfloor, 4)$ have a distance of $\sqrt{21}$ between them.
B. the points $(\lfloor x \rfloor, 1, 6)$ and $(2, \lfloor x \rfloor, 4)$ have a distance of $\sqrt{17}$ between them.

Part A

Use a change of variable. Let $y = \lfloor x \rfloor$:

$$\sqrt{(y-2)^2 + (1-y)^2 + (6-4)^2} = \sqrt{21}$$
$$(y^2 - 4y + 4) + (y^2 - 2y + 1) + 4 = 21$$
$$2y^2 - 6y - 12 = 0$$
$$y^2 - 3y - 6 = 0$$

Solve the above quadratic using $a = 1, b = -3, c = -6$:

$$y = \frac{3 \pm \sqrt{9 - (4)(1)(-6)}}{2(1)} = \frac{3 \pm \sqrt{33}}{2}$$
$$\underbrace{\lfloor x \rfloor}_{Integer} = \underbrace{\frac{3 \pm \sqrt{33}}{2}}_{Not\ an\ Integer} \Rightarrow Contradiction \Rightarrow No\ Solutions$$

Part B

$$\sqrt{(y-2)^2 + (1-y)^2 + (6-4)^2} = \sqrt{17}$$
$$(y^2 - 4y + 4) + (y^2 - 2y + 1) + 4 = 17$$
$$2y^2 - 6y - 8 = 0$$
$$y^2 - 3y - 4 = 0$$
$$(y-4)(y+1) = 0$$

$$y = 4 \Rightarrow \lfloor x \rfloor = 4 \Rightarrow 4 \le x < 5 \Rightarrow x \in [4,5)$$
$$y = -1 \Rightarrow \lfloor x \rfloor = -1 \Rightarrow -1 \le x < 0 \Rightarrow x \in [-1,0)$$

Hence, the final answer is:

$$x \in [-1,0) \cup [4,5)$$

D. Geometrical Applications

Example 1.141

Wanda is trying to locate the Fermat point P of ΔABC, where A is at the origin, B is at $(8,-1)$, and C is at $(5,4)$ (the Fermat point is the point such that the sum of its distances from the vertices of a triangle is minimized). She guesses that the point is at $P=(4,2)$, and computes the sum of the distances from P to the vertices of ΔABC. If she obtains $m+n\sqrt{5}$, where m and n are integers, what is $m+n$? **(AOPS Alcumus, Algebra, Distance in the Plane)**

$$PA=\sqrt{4^2+2^2}=\sqrt{20}=2\sqrt{5}$$
$$PB=\sqrt{(4-8)^2+(2+1)^2}=\sqrt{16+9}=\sqrt{25}=5$$
$$PC=\sqrt{(4-5)^2+(2-4)^2}=\sqrt{1+2^2}=\sqrt{5}$$

$$PA+PB+PC=5+3\sqrt{5}\Rightarrow m+n=5+3=8$$

Example 1.142

Consider

$$\Delta PQR \text{ with } P=(3,-4), Q=(-2,-5), R=(2,1)$$

A. The perimeter of the triangle can be written as $a(\sqrt{b}+\sqrt{c})$, where a is a natural number, and b and c are natural numbers with no perfect square factor. Find $a+b+c$.
B. Classify the triangle. (If more than one classification fits, use them all).

$$PQ=\sqrt{(3+2)^2+(-4+5)^2}=\sqrt{25+1}=\sqrt{26}$$
$$QR=\sqrt{(-2-2)^2+(-5-1)^2}=\sqrt{16+36}=\sqrt{52}=2\sqrt{13}$$
$$PR=\sqrt{(3-2)^2+(-4-1)^2}=\sqrt{1+25}=\sqrt{26}$$

Part A

$$Perimeter=2(\sqrt{13}+\sqrt{26})$$

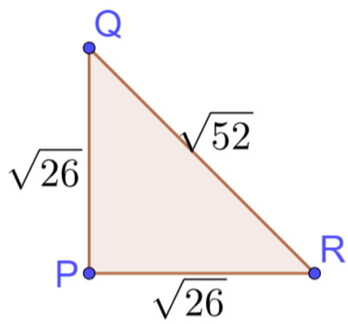
Part B

Note that:

$$PQ^2+PR^2=(\sqrt{26})^2+(\sqrt{26})^2=26+26=52=(\sqrt{52})^2=QR^2$$

Since $PQ^2+PR^2=QR^2$

$$\therefore \Delta PQR \text{ is right}-\text{angled by Converse of Pythagoreas Theorem}$$



Draw a diagram. From the diagram:

$$\angle P \text{ is the right}-\text{angle}$$

Also, the triangle is isosceles.

$$\therefore \Delta PQR \text{ is a } 45-45-90 \text{ triangle}$$

Example 1.143

What is the second smallest number of sides whose length must be changed (and by how much), to make the triangle ΔPQR with $P=(\sqrt{2},0), Q=(-\sqrt{2},0), R=(0,-\sqrt{5})$ equilateral?

$$PQ = \sqrt{\left[\sqrt{2} - (-\sqrt{2})\right]^2 + (0-0)^2} = \sqrt{\left[\sqrt{2} + \sqrt{2}\right]^2} = \sqrt{\left[2\sqrt{2}\right]^2} = 2\sqrt{2}$$

$$QR = \sqrt{\left(-\sqrt{2} - 0\right)^2 + \left[0 - (-\sqrt{5})\right]^2} = \sqrt{\left(-\sqrt{2}\right)^2 + \left(\sqrt{5}\right)^2} = \sqrt{2+5} = \sqrt{7}$$

$$PR = \sqrt{\left(\sqrt{2} - 0\right)^2 + \left[0 - (-\sqrt{5})\right]^2} + \sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{5}\right)^2} = \sqrt{2+5} = \sqrt{7}$$

$$QR = PR \neq PQ \Rightarrow \Delta PQR \text{ is isosceles}$$

The smallest number of sides that must be changed is:

$$1$$

Second smallest number of sides that must be changed is:

$$2$$

Decrease by

$$2\sqrt{2} - \sqrt{7}$$

Example 1.144

If $AC = p$ and $BD = q$, find the area of quadrilateral $ABCD$ with $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3), D(x_4, y_4)$ and

$$AB = BC = CD = DA = d$$

Since all sides of the quadrilateral are equal, it must be a rhombus.

$$Area\ of\ Rhombus = \frac{Product\ of\ Diagonals}{2} = \frac{pq}{2}$$

E. Conceptual Applications

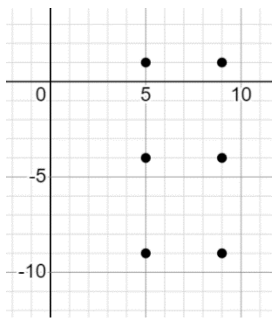
Along with other things, conceptual applications focus on visualization of possible locations of points that meet criteria for geometrical shapes given in the question.

Example 1.145

Points $P(5, -4)$ and $Q(9, -4)$ are two vertices of rectangle $PQRS$ whose area is 20 square units. What are the possible values of the co-ordinates of R and S?

$$PQ = 9 - 4 = 5 \Rightarrow PR = \frac{A(PQRS)}{PQ} = \frac{20}{5} = 4$$

S has to be either 4 units above or below $P \Rightarrow S = \{(5,0), (5,-8)\}$
R has to be either 4 units above or below $P \Rightarrow R = \{(9,0), (9,-8)\}$



Example 1.146

A. The center of a circle has coordinates $P(5,7)$. The point $Q(2,1)$ lies on the boundary of the circle. Find the area of the circle in terms of π.
B. $A(3,6)$ and $B(-2,4)$ are the endpoints of the chord of maximum length of a circle. Find the area of the circle.

Part A

The radius is the distance between the center of the circle, and the boundary of the circle:

$$Radius = PQ = \sqrt{(5-2)^2 + (7-1)^2} = \sqrt{3^2 + 6^2} = \sqrt{9+36} = \sqrt{45}$$

$$A = \pi r^2 = \pi\left(\sqrt{45}\right)^2 = 45\pi$$

Part B

$$\text{Diameter} = AB = \sqrt{(3+2)^2 + (6-4)^2} = \sqrt{25+4} = \sqrt{29}$$

$$\text{Area} = \pi r^2 = \pi\left(\frac{\sqrt{29}}{2}\right)^2 = \frac{29}{4}\pi$$

Example 1.147: Area of a Square

$A(2,3)$ and $B(4,5)$ are vertices of a square. Find the difference in the possible areas of the square.

First, find the distance between the two points using the distance formula:

$$AB = \sqrt{(2-4)^2 + (3-5)^2} = \sqrt{4+4} = \sqrt{8}$$

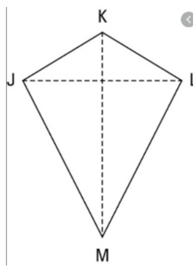
Now, there are two cases to consider:
Case I: AB is the side of the square:

$$Area = AB^2 = \left(\sqrt{8}\right)^2 = 8$$

Case II: AB is the diagonal of the square:

$$Area = \frac{AB^2}{2} = \frac{\left(\sqrt{8}\right)^2}{2} = \frac{8}{2} = 4$$

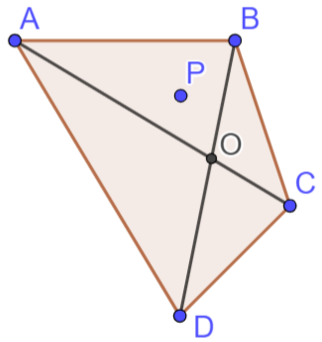
$$\text{Difference} = \underbrace{A(\text{Side})}_{Adjacent\ Vertices} - \underbrace{A(\text{Diagonal})}_{Non-Adjacent\ Vertices} = 8 - 4 = 4$$



Example 1.148: Area of a Kite

1.149: Minimum sum of distances to vertices in a convex quadrilateral

In a convex quadrilateral, the point of intersection of the diagonals minimizes the sum of the distance to its vertices.
Hence, the minimum distance is given by the sum of the length of the diagonals.



In Quadrilateral $ABCD$, let the point of intersection of the diagonals be O, and P be any point not O.

By the triangle inequality:

$$\underbrace{AC < PA + PC}_{Inequality\ I}, \quad \underbrace{BD < PB + PD}_{Inequality\ II}$$

Add Inequality I and II:

$$AC + BD < PA + PB + PC + PD$$
$$OA + OB + OC + OD < PA + PB + PC + PD$$

Hence proved

Example 1.150: Point equidistant from Vertices

Points A, B, C, and D are in a Cartesian plane such that $A = (2048, 2058)$, $B = (2018, 2018)$, $C = (2036, 2046)$, and $D = (2060, 2039)$. Given that point P is in the same plane, the minimum possible value of $PA + PB + PC + PD$ is closest to which integer? (AOPS/Stormersyle Mock AMC 8 2018/20)

The underlying concept is simple, but the question is high quality because:

- The values of the coordinates are large, leading to information overload
- The minimum value is actually an integer, but the question does not give this away.

Change of Origin

We introduce a new pair of coordinate axes at the location (2018,2018) to get "nicer" numbers:

$$A = (2048 - 2018, 2058 - 2018) = (30,40)$$
$$B = (2018, 2018) = (0,0)$$
$$C = (2036, 2046) = (18,28)$$
$$D = (2060, 2039) = (42,21)$$

Form a convex quadrilateral

The points $BCAD$, in that order, form a convex quadrilateral. This can be checked since BA lies on the line

$$y = \frac{4}{3}x$$

And point C lies above the line, whereas point D lies below the line.

Find the minimum distance

$$BA + CD = \sqrt{(30-0)^2 - (40-0)^2} + \sqrt{(42-18)^2 + (21-28)^2} = 50 + 25 = 75$$

Shortcut:

The last step can be done much faster (and possibly more accurately) if you know your Pythagorean Triplets:

$$(30,40,l_1) = 10(3,4,x) = 10(3,4,5) = (30,40,50)$$
$$(42-18,28-21,l_2) = (24,7,l_2) = (24,7,25)$$

And finally:

$$50 + 25 = 75$$

1.9 Midpoint and Section Formula

A. Midpoint on the Number Line

1.151: Midpoint on the Number Line

The midpoint of the line segment connecting two points x_1 and x_2 on the number line is given by the average of the two points:

$$M_{AB} = \frac{x_1 + x_2}{2}$$

Example 1.152: Finding Midpoint

Find the midpoint of the following numbers on the real number line:

A. $3\ and\ 4$
B. $-2\ and\ 2$
C. $-3\ and\ 7$
D. 0.3 and 0.01
E. $\frac{1}{2}\ and\ \frac{1}{3}$

$$\frac{3+4}{2} = \frac{7}{2}$$
$$\frac{-2+2}{2} = \frac{0}{2} = 0$$
$$\frac{-3+7}{2} = \frac{4}{2} = 2$$

Example 1.153: Back Calculations

A. Points P,Q and R are on a number line. Q is halfway between P and R. If P is at -6 and Q is at -1, then R is at **(CEMC Gauss 7 2016/10)**

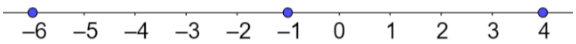
B. Point X on the number line is halfway between Y and Z. If $X = \frac{3}{4}$, and $Z = -\frac{1}{2}$, then find Y.

C. Sally is sixteen and one-third feet due west of a candy floss stand on the beach. Sally's mother just finished buying candy floss for Sally and is five feet due east of the candy floss stand. Sally's dog is due east of Sally. Sally's mother is exactly between her daughter, and her dog. Sally's father is building a sand castle, twelve and one-fourth feet due north of the candy floss stand. He notes that a pizza joint, six feet due south of the candy floss stand is exactly halfway between him and the road. Sally whistles to call her dog back. He runs to the candy floss stand, where Sally joins him, and they go together to the road. Find the total distance travelled by the dog.

D. Suzy's 5m long ribbon has shaded and unshaded sections of equal length, as shown. Points A,B,C,D,E are equally spac ed along the ribbon. If Suzy wants a ribbon that is $\frac{11}{15}$ of the size of this ribbon, at which point could she make a single vertical cut? **(CEMC Gauss 7 2016/19)**

Part A

Substitute $P = -6, Q = -1$ in $\frac{P+R}{2} = Q$:



$$\frac{-6+R}{2} = -1 \Rightarrow -6 + R = -2 \Rightarrow R = 4$$

Part B

Part C

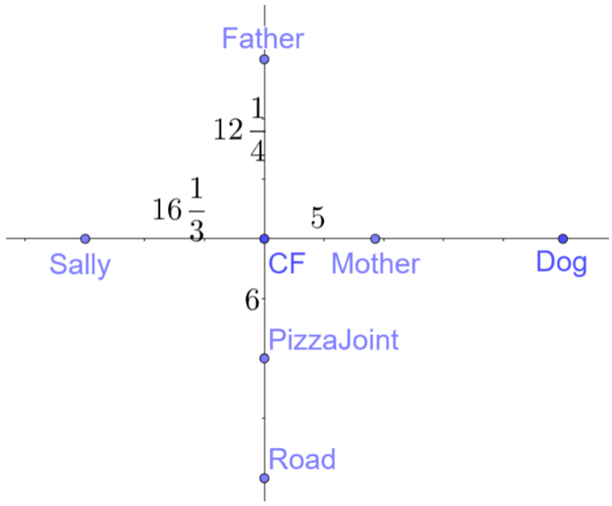
Introduce a coordinate system, and let the candy floss stand be at the origin.

The coordinates from the diagram for the points on the x-axis are:

$$Sally\left(-16\frac{1}{3}, 0\right), Mother(5,0), Dog(x,0)$$

$$\frac{-16\frac{1}{3}+x}{2} = 5 \Rightarrow -16\frac{1}{3} + x = 10 \Rightarrow x = 26\frac{1}{3}$$

Hence, the dog is $26\frac{1}{3}$ feet to the right of the candy floss stand.



The coordinates from the diagram for the points on the y-axis are:

$$Father\left(0, 12\frac{1}{4}\right), PizzaJoint = (0,-6), Road(0,y)$$

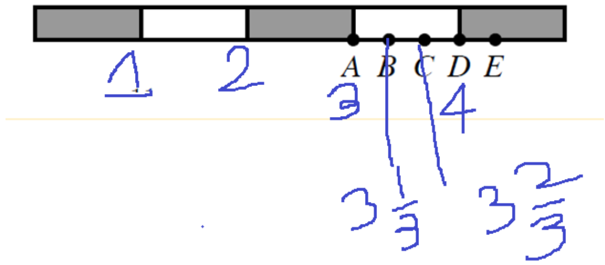
$$\frac{12\frac{1}{4}+y}{2} = -6 \Rightarrow 12\frac{1}{4} + y = -12 \Rightarrow y = 24\frac{1}{4}$$

Total Distance travelled by the dog

$$= x + y = 26\frac{1}{3} + 24\frac{1}{4} = 50\frac{7}{12}$$

Part D

Introduce a number line with zero at the left end of the ribbon.
Then, the right side of the ribbon is at 5 m.



We want to make a cut at:

$$\frac{11}{15}\times 5=\frac{11}{3}=3\frac{2}{3}$$

$$C=3\frac{2}{3}$$

B. Midpoint on the Coordinate Plane

1.154: Midpoint on the Coordinate Plane

The midpoint of the line segment connecting two points $A(x_1,y_1)$ and $B(x_2,y_2)$ is given by:

$$M_{AB}=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

The x coordinate of the midpoint will be given by x_1 added to half the distance of the x coordinates of the two points:

$$M_x=x_1+\frac{x_2-x_1}{2}=\frac{2x_1+x_2-x_1}{2}=\frac{x_1+x_2}{2}$$

Similarly:

$$M_y=\frac{y_1+y_2}{2}$$

Example 1.155

For each part below, find the midpoint of the line segment connecting the points. Also, state the quadrant in which the midpoint lies.

A. Find the midpoint of (2,3) and (6,9).
B. Consider the points $A=(-5,7)$ and $B=(12,-8)$. What is the midpoint of the line segment joining the two points?
C. What is the midpoint of $\left(\frac{1}{3},\frac{2}{5}\right)$ and $\left(\frac{7}{9},\frac{5}{7}\right)$?
D. Find the coordinates of the midpoint of $\left(\frac{3}{4},-\frac{3}{7}\right)$ and $\left(-\frac{3}{7},\frac{2}{11}\right)$.
E. Find the quadrant for each of the answer that you got in Parts $A-D$.
F. Modeltown has its library located at $(-1,2)$ on the coordinate plane, and its central fountain at $(-5,2)$. A straight road runs directly from the library to the central fountain. The mayor's office is exactly between the library and the fountain. Find the coordinates of the mayor's office.
G. X is the sum of the coordinates of the midpoint of the line segment connecting $(3,4)$ and $(7,12)$. Y is the product of the coordinates of the midpoint of the line segment connecting $\left(-\frac{3}{2},\frac{4}{5}\right)$ and $\left(\frac{1}{3},-\frac{2}{7}\right)$. Find $X+Y$.

Part A

$$\left(\frac{2+6}{2},\frac{3+9}{2}\right)=\left(\frac{8}{2},\frac{12}{2}\right)=(4,6)\Rightarrow Lies\ in\ Q-I$$

Part B

$$\left(\frac{-5+12}{2},\frac{7-8}{2}\right)=\left(\frac{7}{2},\frac{-1}{2}\right)\Rightarrow Lies\ in\ Q-IV$$

Part C

$$\left(\frac{\frac{1}{3}+\frac{7}{9}}{2},\frac{\frac{2}{5}+\frac{5}{7}}{2}\right)=\left(\frac{\frac{10}{9}}{2},\frac{\frac{39}{35}}{2}\right)=\left(\frac{10}{9}\times\frac{1}{2},\frac{39}{35}\times\frac{1}{2}\right)=\left(\frac{5}{9},\frac{39}{70}\right)\Rightarrow Lies\ in\ Q-I$$

Part D

$$\left(\frac{\frac{3}{4}-\frac{3}{7}}{2},\frac{-\frac{3}{7}+\frac{2}{11}}{2}\right)=\left(\frac{\frac{9}{28}}{2},\frac{\frac{-19}{77}}{2}\right)=\left(\frac{9}{56},-\frac{19}{154}\right)$$

Part E

Part F

$$\left(\frac{3+7}{2},\frac{4+12}{2}\right)=(5,8)\Rightarrow X=5+8=13$$

$$\left(\frac{-\frac{3}{2}+\frac{1}{3}}{2},\frac{\frac{4}{5}-\frac{2}{7}}{2}\right)=\left(\frac{-\frac{7}{6}}{2},\frac{\frac{18}{35}}{2}\right)=\left(-\frac{7}{12},\frac{9}{35}\right)\Rightarrow Y=-\frac{7}{12}\times\frac{9}{35}=-\frac{3}{20}$$

$$X+Y=13-\frac{3}{20}=12\frac{17}{20}$$

Example 1.156: Back-Calculations

The midpoint of $A(x,y)$ and $B(2,3)$ is $(1,4)$. Find the co-ordinates of Point A.

We can frame an equation for the x coordinate of the midpoint using the midpoint formula:

$$\frac{x+2}{2}=1\Rightarrow x+2=2\Rightarrow x=0$$

And we can frame a similar equation for the y coordinate of the midpoint:

$$\frac{y+3}{2}=4\Rightarrow y+3=8\Rightarrow y=5\Rightarrow A=(0,5)$$

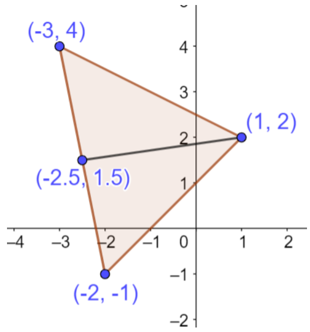
Example 1.157: Geometrical Applications

A. The points $(2,3)$ and $(7,10)$ lie on the endpoints of the diameter of a circle. Find the positive difference of the coordinates of the $\underbrace{centre}_{C}$ of the circle.
B. ΔABC has vertices $A(1,2)$, $B(-2,-1)$ and $C(-3,4)$. Find the co-ordinates of the vertices of the triangle formed by the midpoints of the sides of ΔABC.
C. Find the midpoint of the median from the vertex A in ΔABC from the above part.
D. Find the coordinates of the intersection of the diagonals of the parallelogram with vertices at $(-2,-1)$, $(6,3)$, $(1,3)$, $(3,-1)$.
E. Find the point of intersection of the diagonals of the square $ABCD$, which has vertices $A(\quad), B(\quad), C(\quad)$ and $D(\quad)$.

Hints:

1. The center of a circle is the midpoint of its diameter.
2. The median of a triangle is the line segment joining a vertex to the midpoint of the opposite side.
3. The diagonals of a parallelogram bisect each other.
4. A square is a special case of a parallelogram.

Part A



Recall that the center of a circle is the midpoint of the diameter of the circle.

$$C=\left(\frac{2+7}{2},\frac{3+10}{2}\right)=\left(\frac{9}{2},\frac{13}{2}\right)=(4.5,6.5)\Rightarrow \text{Difference}=6.5-4.5=2$$

Part B

$$\underbrace{\left(-\frac{1}{2},\frac{1}{2}\right)}_{M(AB)},\underbrace{\left(\frac{-5}{2},\frac{3}{2}\right)}_{M(BC)},\underbrace{(-1,3)}_{M(CA)}$$

Part C

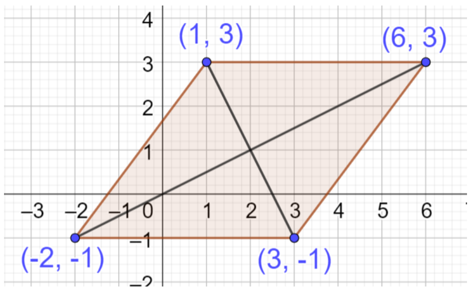
Draw median AD, intersecting BC at D.

$$D=Midpoint\ of\ BC=\left(\frac{-5}{2},\frac{3}{2}\right)$$

Now, we can find the midpoint of the median as:

$$Midpoint(AD)=\left(-\frac{3}{4},\frac{7}{4}\right)$$

Part D



Midpoint of $(1,3)$ and $(3,-1)$:

$$\left(\frac{1+3}{2},\frac{3-1}{2}\right)=\left(\frac{4}{2},\frac{2}{2}\right)=(2,1)$$

Part E

Example 1.158

The midpoints of the sides of a triangle are $(1,2)$, $(-2,5)$ and $(9,1)$. Find the coordinates of the original triangle.

Let the triangle have vertices $A(x_1,y_1),B(x_2,y_2)\ and\ C(x_3,y_3)$.

$$\frac{x_1+x_2}{2}=1\Rightarrow x_1+x_2=2$$
$$\frac{x_2+x_3}{2}=-2\Rightarrow x_2+x_3=-4$$
$$\frac{x_1+x_3}{2}=9\Rightarrow x_1+x_3=18$$

Add the three equations:

$$2x_1+2x_2+2x_3=2-4+18$$
$$\underbrace{x_1+x_2+x_3=8}_{Equation\ IV}$$

Substitute $x_1+x_2=2$ in the last equation:

$$6+x_3=8\Rightarrow x_3=2$$
$$x_1-4=8\Rightarrow x_1=12$$
$$18+x_2=8\Rightarrow x_2=-10$$

Example 1.159

Polygon A has vertices given by points $P_1(x_1,y_1),P_2(x_2,y_2),\dots,P_n(x_n,y_n)$. Polygon B is formed by joining the midpoints

The midpoints are:

$$M_1\left(\frac{x_1+x_2}{2}\right),M_2\left(\frac{x_2+x_3}{2}\right),\dots M_n\left(\frac{x_n+x_1}{2}\right)$$

Sum of the x coordinates is:

$$\frac{x_1+x_2}{2}+\frac{x_2+x_3}{2}+\dots+\frac{x_n+x_1}{2}$$

Note that each term from x_1 to x_{100} appears exactly twice:

$$\frac{2(x_1 + x_2 + \cdots + x_n)}{2}$$
$$= x_1 + x_2 + \cdots + x_n$$

By the same logic, if you repeat the process, the sum does not change.

Example 1.160

Prove that the midpoint of the line segment connecting two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by:

$$\text{Midpoint} = M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

C. Algebraic Applications

D. Section Formula: Internal Division

Background

The midpoint of a line segment divides the line segment into two equal parts. In other words, the midpoint divides a line segment in the ratio $1:1$.

This concept of a midpoint can be generalised to have a point that divides a line segment in the ratio $m:n$.

Definition

Let $\boldsymbol{AB = 6}$. Let point M be $\frac{1}{3}^{\text{rd}}$ of the way between A and B. $\Rightarrow$ $\boldsymbol{AM = 2, MB = 4}$

A 2 units M 4 units B

Point $\boldsymbol{M}$ divides $\boldsymbol{AB}$ internally in the ratio $\boldsymbol{AM:MB = m:n = 2:4 = \underbrace{1}_{m=1} : \underbrace{2}_{n=2}}$.

1.161: Section Formula for Internal Division

The point M dividing $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m:n$ is:

$$M = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

The right-angled triangles formed by dropping perpendiculars from point M, and point B to lines parallel to the x-axis drawn from point A and M, respectively are similar.

Point M is $\frac{m}{m+n}$ times the distance from A to B.

$$\therefore \ x = x_1 + \frac{m}{m+n}(x_2 - x_1) = \frac{(m+n)(x_1) + (m)(x_2 - x_1)}{m+n} = \frac{mx_2 + nx_1}{m+n}$$

Similarly, for y.

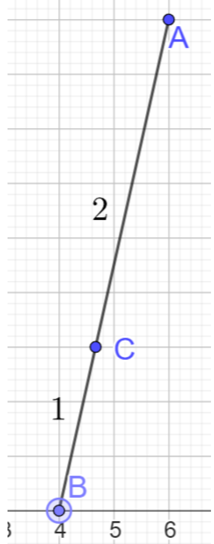
Example 1.162: Internal Division

Point C is $\frac{2}{3}$rd of the way between $A(6,9)$ and $B(4,0)$. Find C.

Method I: Using the Section Formula

Point C divides AB in the ratio $2:1$.

The section formula is:



$$C=\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right)$$

Substitute $m=2, n=1$ in the above:

$$=\left(\frac{2\times 4+1\times 6}{2+1},\frac{2\times 0+1\times 9}{2+1}\right)=\left(\frac{8+6}{3},\frac{0+9}{3}\right)=\left(\frac{14}{3},3\right)$$

Method II: Using Logic

Start from x coordinate 6. To go from 6 to 4, we need to go

$$-2$$

$\frac{2}{3}rd$ of this distance is

$$-2\times\frac{2}{3}=-\frac{4}{3}$$

Hence, we need to add the above distance to the starting point:

$$6+\left(-\frac{4}{3}\right)=\frac{18}{3}-\frac{4}{3}=\frac{14}{3}$$

And we can do the same thing for the y coordinate

$$9+\left(-9\times\left(\frac{2}{3}\right)\right)=9-6=3$$

Example 1.163: Internal Division

Point X divides $P(-3,4)$, and $Q(2,-7)$ in the ratio $3:5$. Find X.

$$X=\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n}\right)=\left(\frac{3\times 2+5\times(-3)}{3+5},\frac{3\times(-7)+5\times 4}{2+1}\right)=\left(\frac{-9}{8},\frac{-1}{8}\right)$$

Example 1.164: Back Calculations

Point $P(2,3)$ divides $M(1,2)$, and Q in the ratio $5:7$. Find Q.

S1: $(2,3)=\left(\frac{5x_2+7\times 1}{5+7},\frac{5y_2+7\times 2}{5+7}\right)$

$$2=\frac{5x_2+7}{12}\Rightarrow 24=5x_2+7\Rightarrow x_2=\frac{17}{5}$$

$$3=\frac{5y_2+14}{12}\Rightarrow 36=5y_2+14\Rightarrow y_2=\frac{22}{5}$$

E. Section Formula: External Division

Writing Assignment 1.1

Prove that the coordinates of the point that divides the points $A(x_1,y_1)$ and $B(x_2,y_2)$ externally in the ratio $m:n$ are given by:

$$M=\left(\frac{mx_1-nx_2}{m+n},\frac{my_1-ny_2}{m+n}\right)$$

Example 1.165: External Division

Perimeter of a Polygon

Q1: F

S2: S

1.10 Triangles

A. Centroid

Revision

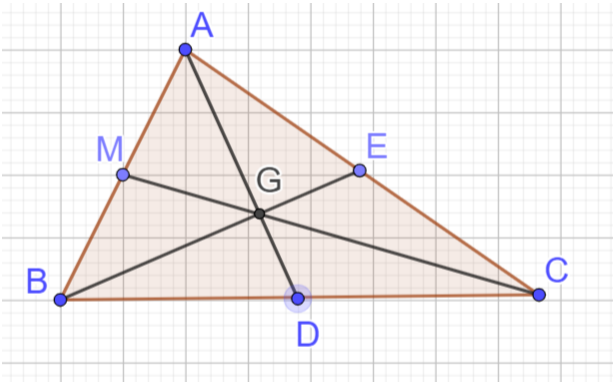
1. The median is the line joining the vertex of a triangle to the midpoint of its opposite side. Every triangle has three midpoints.
2. If three or more lines intersect at a point, the lines are called concurrent.

Definition

The point of concurrency (intersection) of the three medians of a triangle is its centroid.

Property

The centroid of a triangle divides the median in the ratio $2:1$



1.166: Centroid of a Triangle

A triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ has centroid

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

$$\text{Midpoint of } AB = M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Centroid divides CM in the ratio $2:1$. Hence, using the section formula with $m = 2, n = 1$:

$$\text{Centroid} = G = \left(\frac{2\left[\frac{x_1 + x_2}{2}\right] + x_3}{2 + 1}, \frac{2\left[\frac{y_1 + y_2}{2}\right] + y_3}{2 + 1}\right) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Example 1.167

Find the centroid of the following triangles with vertices:

A. $X(2,3)$, $Y(-3,4)$ and $Z(5,-2)$
B. $X\left(\frac{1}{2}, -\frac{2}{3}\right)$, $Y\left(-0.75, \frac{2}{7}\right)$ and $Z\left(\frac{5}{3}, -\frac{5}{6}\right)$

Part A

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right) = \left(\frac{2 - 3 + 5}{3}, \frac{3 + 4 - 2}{3}\right) = \left(\frac{4}{3}, \frac{5}{3}\right)$$

Part B

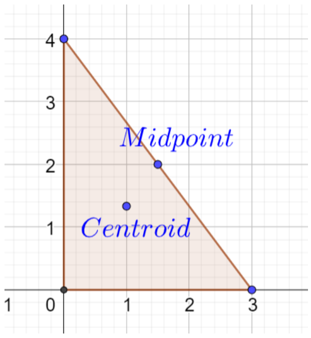
$$\left(\left[\frac{1}{2} - \frac{3}{4} + \frac{5}{3}\right] \times \frac{1}{3}, \left[-\frac{2}{3} + \frac{2}{7} - \frac{5}{6}\right] \times \frac{1}{3}\right)$$

$$= \left(\left[\frac{6 - 9 + 20}{12}\right] \times \frac{1}{3}, \left[\frac{-28 + 12 - 35}{42}\right] \times \frac{1}{3}\right)$$

$$= \left(\frac{17}{36}, \frac{-17}{42}\right)$$

Example 1.168

A right-angled triangle has one vertex at the origin, and the other two vertices on the x axis and the y axis, respectively. If the vertex on the x axis is 3 units away from the origin, and the vertex on the y axis is 4 units away from the origin. Find the distance between the midpoint of the hypotenuse and the centroid.



The first vertex is at the origin.

$$O = (0,0)$$

The second vertex is 3 units away from the origin, on the x axis, giving it coordinates

$$(\pm 3,0)$$

The third vertex is 3 units away from the origin, on the y axis, giving it coordinates

$$(0,\pm 4)$$

This gives us four possible triangles, but due to symmetry, we can find the distance for any triangle.

We take the triangle in the first quadrant with vertices

$$(0,0)(3,0)(0,4)$$

$$Midpoint = \left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right) = \left(\frac{3+0}{2},\frac{0+4}{2}\right) = \left(\frac{3}{2},2\right)$$

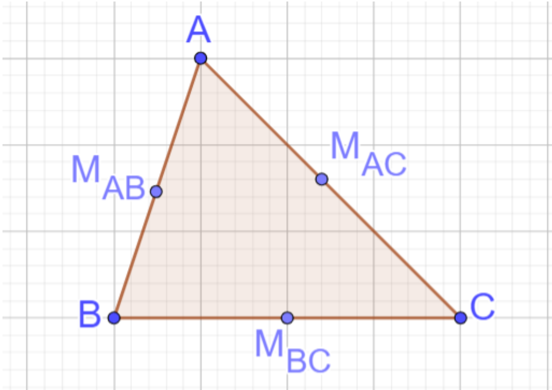
$$\text{Centroid} = \left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right) = \left(\frac{0+3+0}{3},\frac{0+0+4}{3}\right) = \left(1,\frac{4}{3}\right)$$

Find the distance:

$$\sqrt{\left(\frac{3}{2}-1\right)^2+\left(2-\frac{4}{3}\right)^2} = \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{2}{3}\right)^2} = \sqrt{\frac{1}{4}+\frac{4}{9}} = \sqrt{\frac{9+16}{36}} = \sqrt{\frac{25}{36}} = \frac{5}{6}$$

1.169: Centroid of Triangle formed by joining Midpoints

The triangle formed by joining the midpoints of the sides of a triangle has the same centroid as the original triangle.



Consider a triangle with vertices

$$A(x_1,y_1), B(x_2,y_2)\ and\ C(x_3,y_3)$$

$$M_{AB} = \left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

$$M_{BC} = \left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right)$$

$$M_{AC} = \left(\frac{x_1+x_3}{2},\frac{y_1+y_3}{2}\right)$$

The centroid of the triangle formed by joining the midpoints

$$\left(\left[\frac{x_1+x_2}{2}+\frac{x_2+x_3}{2}+\frac{x_1+x_3}{2}\right]\times\frac{1}{3},\left[\frac{y_1+y_2}{2}+\frac{y_2+y_3}{2}+\frac{y_1+y_3}{2}\right]\times\frac{1}{3}\right)$$

$$= \left(\frac{2x_1+2x_2+2x_3}{2}\times\frac{1}{3},\frac{2y_1+2y_2+2y_3}{2}\times\frac{1}{3}\right)$$

$$= \left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$$

B. Incentre

Writing Assignment 1.2 (Formula for Incentre of a Triangle)

Show that the incentre of a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_2, y_2)$ is:

$$G = \left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$$

C. Circumcenter

1.170: Circumcenter of a Triangle

Circumcenter of a triangle is the point of concurrency of the perpendicular bisectors of the sides

Find the midpoint of a side.
Find the line perpendicular to the side.
Same way, find the line perpendicular to second side.
Finally find the point of intersection of the two lines.

Example 1.171: Incentre of a Triangle

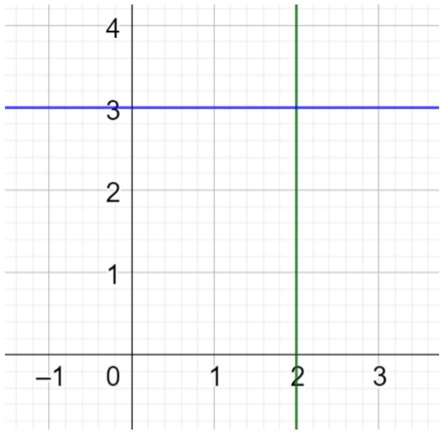
Perimeter of a Polygon
Q2: F
S3: S

2. Lines

2.1 Basics

A. Horizontal and Vertical Lines

2.1: Vertical Line
A vertical line cutting the x-axis at a has equation
$$x = \underbrace{a}_{Any\ Constant}, \quad a = x - intercept$$



This is true because, for a vertical line, any point on the line has the x coordinate a.
The place where a line cuts the x-axis is called its x-intercept.

2.2: Horizontal Line
A horizontal line cutting the y-axis at a has equation
$$y = \underbrace{a}_{Any\ Constant}, \quad a = y - intercept$$

This is true because, for a horizontal line, any point on the line has y coordinate a.
The place where a line cuts the y-axis is called its y-intercept.

Summary

Equation	Nature of Line	Example
$x = \underbrace{a}_{Any\ Constant}$	Vertical Line cutting the x-axis at a	$x = 5$
$y = \underbrace{a}_{Any\ Constant}$	Horizontal Line cutting the y-axis at a	$y = 4$

Example 2.3
Identify the following lines as horizontal or vertical. Also, state their x-intercept, or y-intercept, as the case may be.

Basics
A. $x = 4$
B. $y = 3$

Fractions
C. $x = \frac{4}{3}$
D. $y = -\frac{5}{7}$

Equations
E. $\frac{x}{4} = 2$
F. $\frac{1}{y} = 2$
G. $\frac{1}{x} = \frac{1}{3}$

Decimals
H. $\frac{x}{0.2} = 0.7$
I. $\frac{0.3}{y} = \frac{0.7}{0.03}$

Basics

$$x = 4 \Rightarrow Vertical\ Line, \qquad x - intercept: 4$$
$$y = 3 \Rightarrow Horizontal\ Line, \qquad y - intercept: 3$$

Fractions

$$x = \frac{4}{3} \Rightarrow Vertical\ Line, \qquad x - intercept: \frac{4}{3}$$
$$y = -\frac{5}{7} \Rightarrow Horizontal\ Line, \qquad y - intercept: -\frac{5}{7}$$

Equations

$$\frac{x}{4} = 2 \Rightarrow x = 8 \Rightarrow Vertical\ Line, \qquad x - intercept: 8$$
$$\frac{1}{y} = 2 \Rightarrow y = \frac{1}{2} \Rightarrow Horizontal\ Line, \qquad y - intercept: \frac{1}{2}$$

Decimals

$$\frac{x}{0.2} = 0.7 \Rightarrow x = 0.7 \times 0.2 = \frac{7}{10} \times \frac{2}{10} = \frac{14}{100} = 0.14 \Rightarrow Vertical\ Line, \qquad x - intercept: 0.14$$
$$\frac{0.3}{y} = \frac{0.7}{0.03} \Rightarrow \frac{y}{0.3} = \frac{0.03}{0.7} \Rightarrow y = \frac{0.009}{0.7} \Rightarrow Horizontal\ Line, \qquad y - intercept: \frac{0.009}{0.7} = \frac{9}{700}$$

2.4: Slope

Slope is the steepness of a line.
A horizontal line is flat. Hence, the slope of a horizontal line is zero.
A vertical line is infinitely steep. It is said to have no slope

Example 2.5

State the slope of the lines below:

A. $x = 12$
B. $y = 7$
C. $x = \frac{4}{9}$
D. $y = \frac{6}{11}$

$$x = 12 \Rightarrow Vertical\ Line = No\ Slope$$
$$y = 7 \Rightarrow Horizontal\ Line = Zero\ Slope \Rightarrow Slope = 0$$
$$x = \frac{4}{7} \Rightarrow Vertical\ Line = No\ Slope$$
$$y = \frac{6}{11} \Rightarrow Horizontal\ Line = Zero\ Slope \Rightarrow Slope = 0$$

Example 2.6: Review

For the following lines, state whether they are horizontal or vertical. State their x-intercept, or y-intercept, as the case may be. Also, state their slope.

B. Slope

2.7: Positive and Negative Slope

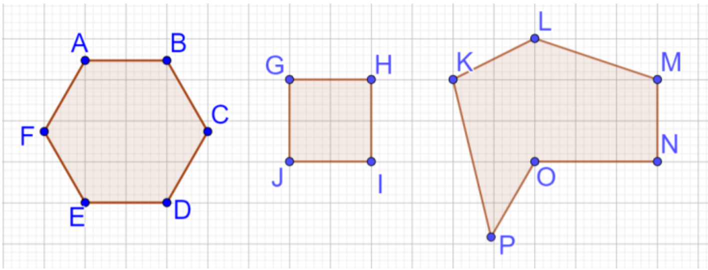
- If a line goes up (from left to right), it has positive slope.
- If a line goes down (from left to right), it has negative slope.

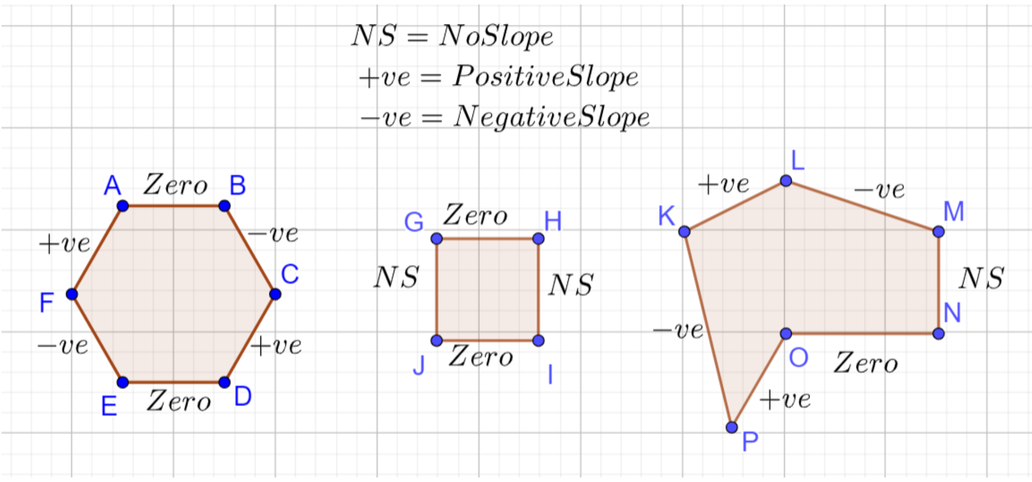
- Slope is a numerical measure of the steepness of a line.
- Any two points on a line will always have the same slope.

Slope can be defined using different (equivalent) conceptualizations. Questions can be solved faster if you know the right choice of definition to use.

Example 2.8

The diagram alongside has three polygons. The first is a regular hexagon. The second is a square. The third is an irregular polygon. Classify the slope of each line segment in each polygon as positive, negative, zero, or no slope.





2.9: Slope: Rise over Run

The slope between (x_1, y_1) and (x_2, y_2) is given by:

$$m = \frac{\text{Rise}}{\underbrace{\text{Run}}_{\textit{Concept}}}$$

- m is the variable which used for slope
- Rise is the change in the y coordinate.
- Run is the change in the x coordinate.

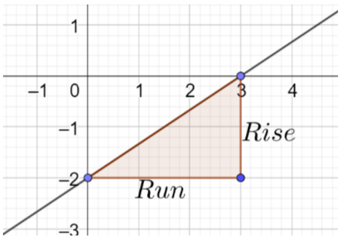
You can pick two points and find the slope between by using the $\frac{Rise}{Run}$ concept.

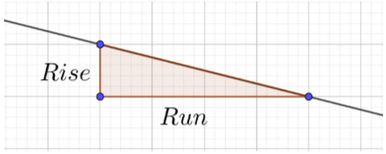
Interpretation of slope is best done using the $\frac{\text{Rise}}{\text{Run}}$ concept.

$$Rise = Change\ in\ y - variable$$
$$Run = Change\ in\ x - variable$$

Example 2.10: Slope from Graph

Identify the slope of the line graphed alongside.





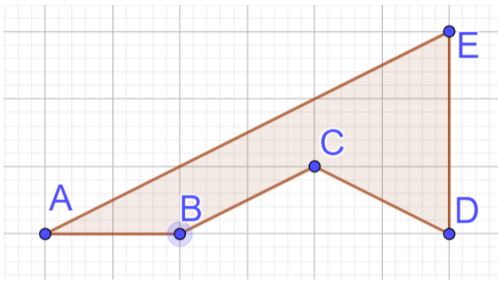
$$Slope = \frac{Rise}{Run} = \frac{2}{3}$$
$$Slope = \frac{Rise}{Run} = \frac{-1}{4}$$

2.11: Slope of Parallel Lines

- Parallel lines have the same slope.
- If two lines are vertical, then they both have no slope, and they are parallel.

Example 2.12

Polygon $ABCDE$ is graphed alongside. Find the slope of each line segment given that $AE \parallel BC$.



$$AB \text{ is horizontal} \Rightarrow Zero\ Slope$$
$$ED \text{ is vertical} \Rightarrow No\ Slope$$

Note that $AE \parallel BC$. Both of them have the same slope, and the slope is positive since the lines go up from left to right.

$$Slope\ for\ AE\ \&\ BC = \frac{Rise}{Run} = \frac{2}{2} = 1$$
$$Slope\ for\ AE\ \&\ BC = \frac{Rise}{Run} = \frac{2}{2} = 1$$

CD goes down from left to right. So, it has negative slope.

$$Slope = \frac{Rise}{Run} = \frac{-1}{2} = -\frac{1}{2}$$

2.13: Change in y over Change in x

$$m = \underbrace{\frac{\Delta y}{\Delta x}}_{\frac{\text{Change in } y}{\text{Change in } x}}$$

Recall that

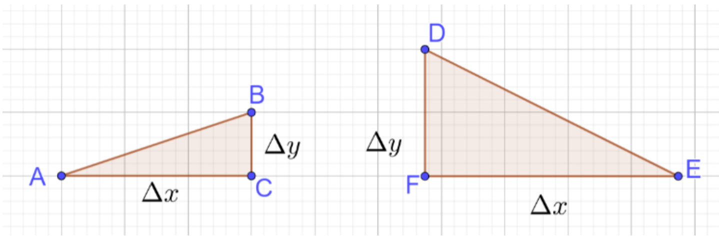
- Rise is the change in the y coordinate $= \Delta y$
- Run is the change in the x coordinate $= \Delta x$

The symbol Δ is the Capital Greek Letter Delta, which is used when a change is to be measured. Using this notation, we can write the formula for slope as:

$$m = \underbrace{\frac{\text{Rise}}{\text{Run}}}_{\text{Concept}} = \underbrace{\frac{\Delta y}{\Delta x}}_{\frac{\text{Change in } y}{\text{Change in } x}}$$

Example 2.14

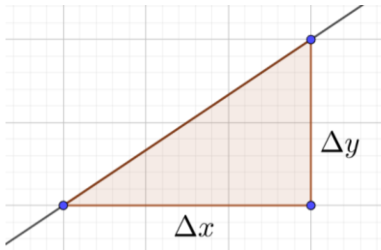
Find the slope of line segment AB, and the slope of line segment DE. The triangles have been used to mark the change in x, and the change in y for each line segment.



$$Slope_{AB} = \frac{\Delta y}{\Delta x} = \frac{1}{3}$$
$$Slope_{DE} = \frac{\Delta y}{\Delta x} = \frac{-2}{4} = -\frac{1}{2}$$

Example 2.15: Interpreting Slope

The slope of a line is given to be $\frac{2}{3}$. Interpret the slope.



$$m = \frac{\Delta y}{\Delta x} = \frac{Change\ in\ y - variable}{Change\ in\ x - variable} = \frac{2}{3}$$

A slope of $\frac{2}{3}$ means that a change of:

- 3 units in the x-direction $\Leftrightarrow$ 2 units in the y-direction.
  - ✓ Since the slope is positive, the line goes up when the x coordinate increases.
- -3 units in the x-direction, there will be a change of -2 units in the y-direction.
  - ✓ Since the slope is positive, the line goes down when the x coordinate decreases.

Example 2.16: Calculating change using given slope

The slope of a line is given to be $\frac{4}{7}$. Find the change in the

A. y coordinate when the x coordinate increases by 7
B. y coordinate when the x coordinate decreases by 7
C. x coordinate when the y coordinate increases by 4
D. x coordinate when the y coordinate decreases by 4

Slope

$$Slope = \frac{4}{7} = \frac{\Delta y}{\Delta x} = \frac{Change\ in\ y - variable}{Change\ in\ x - variable}$$

Part A

The x coordinate has increased by seven. This means that $x_2 - x_1 = \Delta x = 7$.

But, from the formula for slope, we know that $\frac{\Delta y}{\Delta x} = \frac{4}{7}$. Substitute $\Delta x = 7$:

$$\frac{\Delta y}{7} = \frac{4}{7} \Rightarrow \Delta y = 4$$

Part B

Substitute $\Delta x = -7$ in the formula for slope:

$$\frac{\Delta y}{\Delta x} = \frac{4}{7} \Rightarrow \frac{\Delta y}{-7} = \frac{4}{7} \Rightarrow \Delta y = -4$$

Part C

Substitute $\Delta y = 4$ in the formula for slope:

$$\frac{\Delta y}{\Delta x} = \frac{4}{7} \Rightarrow \frac{4}{\Delta x} = \frac{4}{7} \Rightarrow \frac{\Delta x}{4} = \frac{7}{4} \Rightarrow \Delta x = 7$$

Part D

Substitute $\Delta y = -4$ in the formula for slope:

$$\frac{\Delta y}{\Delta x} = \frac{4}{7} \Rightarrow \frac{-4}{\Delta x} = \frac{4}{7} \Rightarrow \frac{-\Delta x}{4} = \frac{7}{4} \Rightarrow \Delta x = -7$$

Example 2.17: Calculating change using given slope

The slope of a line is given to be $-\frac{2}{5}$. Find the change in the

A. y coordinate when the x coordinate increases by 2

B. y coordinate when the x coordinate decreases by $3\frac{1}{3}$
C. x coordinate when the y coordinate increases by 5
D. x coordinate when the y coordinate decreases by $2\frac{1}{5}$

Part A

Substitute $\Delta x = 2$ in $\frac{\Delta y}{\Delta x} = -\frac{2}{5}$:

$$\frac{\Delta y}{2} = -\frac{2}{5} \Rightarrow \Delta y = -\frac{4}{5}$$

Part B

Substitute $\Delta x = -3\frac{1}{3} = -\frac{10}{3}$ in $\frac{\Delta y}{\Delta x} = -\frac{2}{5}$:

$$\frac{\Delta y}{-\frac{10}{3}} = -\frac{2}{5} \Rightarrow \Delta y = -\frac{2}{5} \times -\frac{10}{3} = \frac{4}{3}$$

Part C

Substitute $\Delta y = 5$ in $\frac{\Delta y}{\Delta x} = -\frac{2}{5}$:

$$\frac{5}{\Delta x} = -\frac{2}{5} \Rightarrow 5 \times \left(-\frac{5}{2}\right) = \Delta x \Rightarrow \Delta x = -\frac{25}{2}$$

Part D

Substitute $\Delta y = -2\frac{1}{5} = -\frac{11}{5}$ in $\frac{\Delta y}{\Delta x} = -\frac{2}{5}$:

$$\frac{-\frac{11}{5}}{\Delta x} = -\frac{2}{5} \Rightarrow -\frac{11}{5} \times \left(-\frac{5}{2}\right) = \Delta x \Rightarrow \Delta x = \frac{11}{2}$$

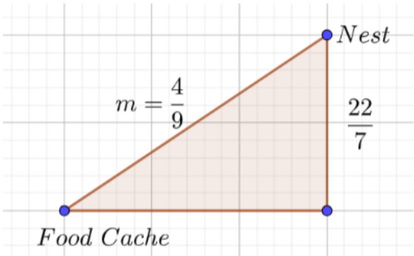
Example 2.18

Robert the raccoon lives on a tilted tree. Robert measures the slope of the tree as $-\frac{4}{9}$ (using an x-axis parallel to the ground, but using the left direction as the positive x-direction since Robert is dyslexic). Assuming that the other parts of Robert's calculations are correct, find the horizontal distance (and direction) from Robert's nest to his food cache if his nest is $3\frac{1}{7}$ feet above his food cache.

Substitute $\Delta y = -\frac{22}{7}$ in $\frac{\Delta y}{\Delta x} = \frac{4}{9}$

$$\frac{-\frac{22}{7}}{\Delta x} = \frac{4}{9} \Rightarrow -\frac{22}{7} \times \frac{9}{4} = \Delta x \Rightarrow \Delta x = -\frac{99}{14}$$

$$\frac{99}{14}\ in\ the\ left\ direction$$



2.19: Slope Formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

We can convert the above expression into a formula that is directly applicable to the coordinates of the points that we get.

$$\Delta y = Change\ in\ y\ coordinate = y_2 - y_1$$
$$\Delta x = Change\ in\ x\ coordinate = x_2 - x_1$$

Using this, we can now write our formula for slope as:

$$Slope = m = \underbrace{\frac{\text{Rise}}{\text{Run}}}_{Concept} = \underbrace{\frac{\Delta y}{\Delta x}}_{\frac{Change\ in\ y}{Change\ in\ x}} = \frac{y_2 - y_1}{x_2 - x_1}$$

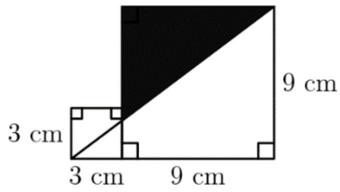
Alternate Formula for slope

The above formula can be used to calculate slope. But we sometimes want to put coordinates of the first point first. Hence, we find an alternate version of the formula above:

$$\frac{y_2-y_1}{x_2-x_1}=\frac{y_2-y_1}{x_2-x_1}\times\frac{-1}{-1}=\frac{-y_2+y_1}{-x_2+x_1}=\frac{y_1-y_2}{x_1-x_2}$$

Example 2.20

What is the area of the shaded region in the figure below? Round your answer to the nearest square centimeter. (MathCounts 1991 National Team)

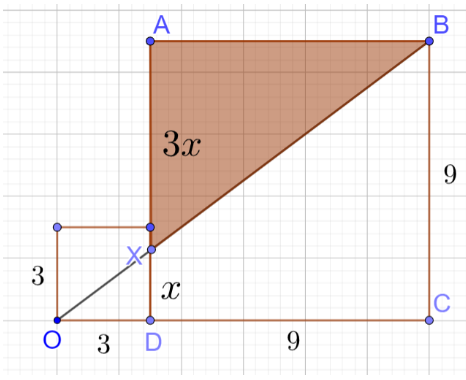


Method I

By AA Similarity:

$$\Delta DXO \sim \Delta AXO$$

$$\frac{DX}{AX}=\frac{DO}{AB}=\frac{3}{9}=\frac{1}{3}\Rightarrow 3DX=AX$$
$$DX+AX=9$$
$$DX+3DX=9$$
$$DX=\frac{9}{4}$$
$$AX=\frac{27}{4}$$



$$Area\ of\ \Delta BAX=\frac{1}{2}(9)\left(\frac{27}{4}\right)=\frac{243}{8}=30\frac{3}{8}=\underbrace{30\ cm^2}_{Rounded}$$

Method II

Introduce a coordinate system with the origin$(0,0)$ at O.

$$Slope_{OB}=\frac{y_2-y_1}{x_2-x_1}=\frac{9}{12}=\frac{3}{4}$$

Point X has coordinates $(3,a)$. It is 3 units in the x direction to the right of the origin. Hence,

$$a=\frac{3}{4}x=\frac{3}{4}(3)=\frac{9}{4}$$
$$AX=AD-DX=9-\frac{9}{4}=\frac{27}{4}$$

2.21: Slope Summary

$$m=\underbrace{\frac{\text{Rise}}{\text{Run}}}_{Concept}=\underbrace{\frac{\Delta y}{\Delta x}}_{\frac{Change\ in\ y}{Change\ in\ x}}=\frac{y_2-y_1}{x_2-x_1}=\underbrace{\frac{y_1-y_2}{x_1-x_2}}_{Multiply\ by\frac{-1}{-1}}$$

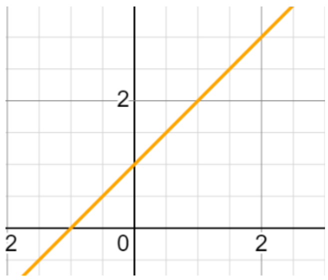
We can summarize the different form of the slope formula that we have done so far in the expression below.

Example 2.22

A. Find the slope of the line graphed alongside.

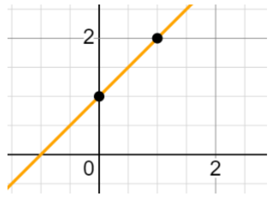
B. Find the slope between $\left(\underbrace{2}_{x_1},\underbrace{4}_{y_1}\right)$ and $\left(\underbrace{1}_{x_2},\underbrace{3}_{y_2}\right)$

C. Find the slope between $\left(\underbrace{2}_{x_1},\underbrace{5}_{y_1}\right)$ and $\left(\underbrace{-3}_{x_2},\underbrace{-1}_{y_2}\right)$



Part A

Identify two points on the line as $(0,1)$ and $(1,2)$



$$m=\frac{y_1-y_2}{x_1-x_2}=\frac{2-1}{1-0}=\frac{1}{1}=1$$

Part B

$$m=\frac{y_1-y_2}{x_1-x_2}=\frac{4-3}{2-1}=\frac{1}{1}=1$$

Part C

$$\text{Slope}=m=\frac{y_1-y_2}{x_1-x_2}=\frac{5-(-1)}{2-(-3)}=\frac{6}{5}$$

We can also calculate the slope using the other version of the formula:

$$m=\frac{y_2-y_1}{x_2-x_1}=\underbrace{\frac{-1-5}{-3-2}=\frac{-6}{-5}}_{\substack{\textit{Minus signs cancel}\\ \textit{Slope is same}}}=\frac{6}{5}$$

Example 2.23: Slope from Table

x					
y					

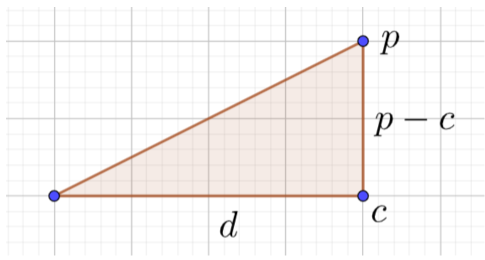
Example 2.24: Application

The grade of a railway is the slope between the two points. Apart from slope, grade can also be stated as a percentage or *per mille* (like percentage, but with a denominator of 1000 instead of 100). Find the grade from the base station to the Pikes Peak as a slope, as a percentage and as *per mille* using the information below:
The Manitou and Pike's Peak Railway takes tourists from its base station in Colorado to Pikes Peak of the Rocky Mountains. Colorado has an elevation of c feet, Pikes Peak has an elevation of p feet, and the horizontal distance between the Colorado base station and the Pike's peak station is d feet.

$$Slope=\frac{p-c}{d}$$

$$Percentage=\frac{100(p-c)}{d}\%$$

$$Per\ Mille\ =\frac{1000(p-c)}{d}\ per\ mille$$



Example 2.25: Back Calculations

Given the slope of two points, it is possible to identify a missing value of one of the points.

A. Find the value of x if the line going through the points $(x,7)$ and $(-1,8)$ has a slope of 1.
B. Find x if the slope of the line connecting $(4,3)$ and $(x,2)$ is 3.
C. A line with slope 7 passes through the points $(2,7)$ and $(d,-7)$
D. A line with slope $-\frac{1}{8}$ passes through $(2,5)$ and $(a,6)$

Part A

Substitute $m=1, y_2=8, y_1=7, x_2=-1, x_1=x$ in $m=\frac{y_2-y_1}{x_2-x_1}$:

$$1=\frac{8-7}{-1-x}\Rightarrow -1-x=1\Rightarrow x=-2$$

Part B

$$m=\frac{y_2-y_1}{x_2-x_1}\Rightarrow \underbrace{3=\frac{3-2}{4-x}}_{Substitute\ Known\ Values}\Rightarrow 12-3x=1\Rightarrow 12=3x\Rightarrow x=\frac{11}{3}$$

Part C

$$m=\frac{y_2-y_1}{x_2-x_1}\Rightarrow 7=\frac{-7-7}{d-2}\Rightarrow 7=\frac{-14}{d-2}\Rightarrow 1=\frac{-2}{d-2}\Rightarrow d-2=-2\Rightarrow d=0$$

Part D

$$\frac{6-5}{a-2}=-\frac{1}{8}$$
$$\frac{1}{a-2}=-\frac{1}{8}$$
$$8=2-a$$
$$a=2-8=-6$$

C. Zero Slope and No Slope

We now consider some edge cases with respect to calculation of slope.

Vertical Lines do not have Slope

If the denominator is zero, recall that division by zero is not defined

- Hence, there is no slope.
- The value of the numerator is not important in this case. It can be anything (including zero).

Horizontal Lines have Zero Slope

If the numerator is zero, then there is no change in the y coordinate and hence the slope is zero.

Summary Table

$Numerator = n$	Denominator	Slope
$n \neq 0$	$n=0$	Not Defined
$n=0$	$n=0$	Not Defined
$n=0$	Any non-zero number	0

Example 2.26: Zero Slope and No Slope

The information gives six pairs of points. For each pair of points:

A. Find the slope between the points.
B. Hence, identify whether the lines connecting the points are horizontal or vertical.

I. $\left(\underbrace{6}_{x_1},\underbrace{9}_{y_1}\right)$ and $\left(\underbrace{4}_{x_2},\underbrace{9}_{y_2}\right)$
II. $(3,4)$ and $(3,7)$
III. $(a,5)$ and $(a,7)$
IV. $(b,8)$ and $(c,8)$ where $b\neq c$
V. (x,y) and (x,z) where $y\neq z$
VI. (p,q) and (r,q) where $p\neq r$

$$m=\frac{y_1-y_2}{x_1-x_2}=\frac{9-9}{6-4}=\frac{0}{2}=0$$
$$m=\frac{y_1-y_2}{x_1-x_2}=\frac{4-7}{3-3}=\frac{-3}{0}\Rightarrow Not\ Defined\Rightarrow No\ Slope$$
$$m=\frac{y_1-y_2}{x_1-x_2}=\frac{5-7}{a-a}=-\frac{2}{0}\Rightarrow Not\ Defined\Rightarrow No\ Slope$$

$$m=\frac{y_1-y_2}{x_1-x_2}=\frac{8-8}{b-c}=\frac{0}{b-c}=0$$
$$m=\frac{y_1-y_2}{x_1-x_2}=\frac{y-z}{x-x}=\frac{y-z}{0}\Rightarrow Not\ Defined\Rightarrow No\ Slope$$
$$m=\frac{y_1-y_2}{x_1-x_2}=\frac{q-q}{p-r}=\frac{0}{p-r}=0$$

2.27: Perpendicular Lines

- Case I: If the product of the slopes of two lines is -1, then the two lines are perpendicular.
- Case II: Additionally, a line which is no slope is perpendicular to a line which has zero slope.

Case I: Both lines have slopes

If two lines are perpendicular, the product of their slopes is -1.

$$m_1m_2=-1\Rightarrow m_2=-\frac{1}{m_1}$$

Example 2.28: Identifying Lines

From the above, we can conclude that if two lines are perpendicular, they will have slopes

$$m\ and\ -\frac{1}{m}\Rightarrow(m)\left(-\frac{1}{m}\right)=-1$$

Example 2.29: Identifying Lines

The information below is on nine lines, named Lines $A-I$. Identify pairs of lines which are

A. Parallel
B. Perpendicular

$$Slopes:\underbrace{No\ Slope}_{Line\ A},\underbrace{0}_{Line\ B},\underbrace{1}_{Line\ C},\underbrace{2}_{Line\ D},\underbrace{\frac{1}{2}}_{Line\ E},\underbrace{-\frac{1}{2}}_{Line\ F},\underbrace{1}_{Line\ G}$$
$$Nature\ of\ Line:\underbrace{Horizontal}_{Line\ H}\ \underbrace{Vertical}_{Line\ I}$$

Perpendicular Lines

First Pair

$\underbrace{Line\ A\ (no\ slope)}_{Vertical\ Line}$ is perpendicular to the line $\underbrace{B\ (zero\ slope)}_{Horizontal\ Line}$

Second Pair

$$2\times-\frac{1}{2}=-1\Rightarrow Lines\ are\ perpendicular$$
$$Line\ D\ and\ Line\ F\ are\ perpendicular$$

Third and Fourth Pair

Line H is horizontal. Line A is vertical. They are perpendicular
Line H is horizontal. Line I is vertical. They are perpendicular

Fifth Pair

Line I is vertical. Line B is horizontal. They are perpendicular

Parallel Lines

First Pair

Both Line C and Line G have slope 1. Line C parallel to Line G.

Second Pair

Line H is horizontal. Line B is also horizontal. They are parallel.

Third Pair

Line I is vertical. Line A is also vertical. They are parallel.

Example 2.30: Finding Slope

$$Slopes: \underbrace{No\ Slope}_{Line\ A}, \underbrace{0}_{Line\ B}, \underbrace{3}_{Line\ C}, \underbrace{7}_{Line\ D}, \underbrace{\frac{1}{4}}_{Line\ E}, \underbrace{-\frac{1}{9}}_{Line\ F}, \underbrace{1}_{Line\ G}$$

Use the information given above on the slopes of various lines to find the slope of:

A. A line parallel to line D
B. A line perpendicular to Line C
C. A line parallel to Line A
D. A line perpendicular to line A
E. A line perpendicular to Line F
F. A line perpendicular to Line B

$$Part\ A: 7$$
$$Part\ B: -\frac{1}{3}$$
$$Part\ C: No\ Slope$$
$$Part\ D: 0$$
$$Part\ E: 9$$
$$Part\ F: No\ Slope$$

D. Calculating Points on the Line

Example 2.31: Line Passing Through the Origin

The slope of a line is given to be $\frac{3}{5}$. It is known that the origin$\left(O = (0,0)\right)$ lies on the line.

A. Find five points to the right of the origin that lie on the line.
B. Find five points to the left of the origin that lie on the line
C. Hence, find a pattern for these points.
D. Also, write a real-world interpretation of the slope.

Parts A, B and C

x	-25	-20	-15	-10	-5	0	5	10	15	20	25
y	-15	-12	-9	-6	-3	0	3	6	9	12	15
	Left of Origin					Origin	Right of Origin				

Pattern

The x coordinate is a multiple of 5.
The y coordinate is a multiple of 3.

Real World Interpretation

Suppose that five chocolates cost $3. Then, the cost of one chocolate is given by:

$$\frac{Cost\ of\ Chocolates}{No.\ of\ Chocolates} = \frac{3\ dollars}{5\ chocolates}$$

Hence, the cost of ten chocolates will be

$$\frac{3\ dollars}{5\ chocolates} \times \frac{2}{2} = \frac{6\ dollars}{10\ chocolates}$$

Example 2.32: General Line

The slope of a line is given to be 2. It is known that the point (0,8) lies on the line.

A. Find five points to the right of (0,8) that lie on the line.
B. Find five points to the left of (0,8) that lie on the line
C. Hence, find a pattern for these points.
D. Also, write a real-world interpretation of the slope.

Parts A, B and C

$$Slope = 2 = \frac{2}{1}$$

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	-2	0	2	4	6	8	10	12	14	16	18
	Left of (0,8)					(0,8)	Right of (0,8)				

Pattern

The y coordinate is given by

$$y = 2x + 8$$

Example 2.33: Real World Interpretation

The cost of x hours of internet time in a subscription plan is $3x$ dollars.

A. Tabulate the relation between internet time and the cost of that time
B. Frame this as a linear equation.
C. Plot the equation on a graph.
D. Calculate the slope of the graph that you plotted in part B.
E. Interpret the slope that you calculated in part C
   I. Algebraically
   II. Graphically
   III. In the real world

A. We can tabulate below

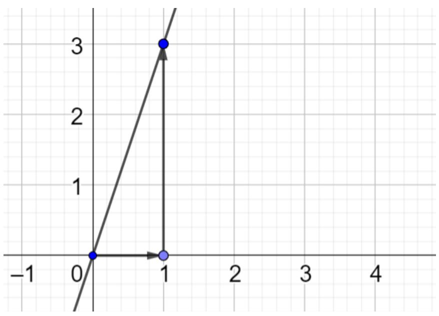
Internet Time (Hours)	Cost (Dollars)
1	3
2	6
5	15

B. $$\underbrace{y}_{Cost\ of\ x\ hours} = \underbrace{3x}_{x=No.of\ Hours\ of\ Internet\ used}$$

C. Since this is a linear equation, we know that its graph is a line.
Two points are enough to define a line.
Substitute simple values of x in the equation:

$$x = 0 \Rightarrow y = 0, x = 1 \Rightarrow y = 3$$

This tells us that $\underbrace{(0,0), (1,3)}_{\substack{Plot\ the\ points \\ Join\ them\ using\ a\ ruler}}$ are two points that lie on the graph of the line.



D. $m = \frac{3-0}{1-0} = \frac{3}{1} = 3, \quad \frac{0-3}{0-1} = \frac{-3}{-1} = \frac{3}{1} = 3 \ also\ works$

E. Interpretation	
Algebraic	For every increase in x by one, there will be a change of 3 units in y.
Graphical	For every movement of one unit in the x-direction, there is a change of 3 units in the y-direction.
Real World	For every additional hour that you use the internet, you will need to pay the ISP (Internet Service Provider), three dollars

E. Counting

Example 2.34
A lattice point is a point (x, y), with x and y both integers. For example, (2, 3) is a lattice point but (4, 1/3) is not. In the diagram, how many lattice points lie on the perimeter of the triangle? (CEMC Gauss 8 2007/24)

2.2 Slope Intercept Form

A. Types of Lines and Their Slope
A linear equation in x and y will give a line when you plot it on the coordinate plane.
Converse: The graph of a line on the coordinate plane is a linear equation.

Nature of Line	Nature of Slope $\left(\frac{y_2-y_1}{x_2-x_1}\right)$	Value of Slope	Example
Parallel to x-axis (Horizontal Line)	No change in y. The numerator of the formula is always zero.	$\therefore$ Slope $= 0$.	$y = 3$ Slope $= 0$
Parallel to y-axis (Vertical Line)	No change in x $\therefore$The denominator is always zero Division by zero is **not defined**.	$\therefore$ **No slope** (**not defined**).	$x = 4$ Slope is not defined
Going upward (from left to right)	Numerator will be positive for positive change in y.	$\therefore +ve$ Slope	$y = 2x + 3$ Slope $= 2\ (+ve)$
Going downward (from left to right)	Numerator will be negative for positive change in y.	$\therefore -ve$ Slope	$y = -3x + 4$ Slope $= -3\ (-ve)$

2.35: Slope-Intercept Form
The equation of a line in slope intercept form is:
$$y = mx + c$$
Where:
$$m = slope$$
$$c = \text{constant} = \ y - intercept$$

Let the y-intercept of a line with slope m be $(0, c)$.
Substitute the y-intercept in the definition of slope:
$$m = \frac{y_2 - y_1}{x_2 - x_1} \Rightarrow m = \frac{y_2 - c}{x_2 - 0} \Rightarrow mx_2 = y_2 - c \Rightarrow y_2 = mx_2 + c$$
But, we now have only one x variable and one variable in the equation. So, we do not need the subscript.
$$Let\ x = x_2, y = y_2 \Rightarrow y = mx + c$$

Example 2.36: Classifying Lines based on Slope

Classify the following lines into one of the following categories: Horizontal Line, Vertical Line, Line going up (from left to right), and line going down (from left to right). Also, write their slope.

A. $x = 4$
B. $y = -7$
C. $x = 0$
D. $y = 3x + 4$
E. $y = 0$
F. $y = -2x + 4$

A. Vertical, No Slope
B. Horizontal, Slope$= 0$
C. $y - axis$, Vertical Line, No Slope
D. Line going up when going from left to right, $Slope = 3$
E. $x - axis$, Horizontal, Slope $= 0$
F. Line going down when going from left to right, $Slope = -2$

B. Algebraic Form of a Line

Example 2.37

Verify that the table given below satisfies the equation

$$y = 2x + 3$$

x	-1	0	1	2	3	4
y	1	3	5	7	9	11

x	-1	0	1	2	3	4
y	$-1 \times 2 + 3 = 1$	3	5	7	9	11

Example 2.38

The cost to enter a theme park is three dollars. For every ride that you sit in the park, you need to pay a further two dollars.

A. Find the cost of:
   I. Entering the theme park, but not sitting in any ride.
   II. Sitting in one ride
   III. Sitting in two rides
   IV. Sitting in three rides
   V. Sitting in four rides
B. Plot the values obtained in part A on the coordinate plane using the x-axis for the number of rides, and the y-axis for the cost of those rides.
C. Find an equation that connects the number of rides with the cost of those rides.
D. Does the value $x = -1$ make sense in the context of the theme park?

	A	B	C	D	E	
Number of Rides	0	1	2	3	4	
$Cost$	3	5	7	9	11	

Let the number of rides be given by x.
Let the cost of x rides be y.

The equation connecting the two is

$$y = 2x + 3$$

C. x-intercept and y-intercept

At the x-intercept, the y coordinate is zero.
At the y-intercept, the x coordinate is zero.

Example 2.39

Sheetal has a hundred dollars in her bank account currently, which is Month Zero. She has been donating ten dollars every month to a charity since the last three months

A. Tabulate the amount of money in her bank account using a table.
B. Plot the point on the table using the coordinate plane. Plot the month number on the x-axis, and the amount of money in her bank account on the y-axis.
C. Use the table to write an equation connecting x and y.

Month	x	$x_1 = 0$	$x_2 = 1$	2	3	4
Money	y	$y_1 = 100$	$y_2 = 90$	80	70	60

Month	-1	-2	-3		
Money	110	120	130		

Slope

$$Slope = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{100 - 90}{0 - 1} = -\frac{10}{1} = -10$$
$$Slope = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{90 - 100}{1 - 0} = \frac{-10}{1} = -10$$

y-intercept

The y-intercept is the point on a line where it cuts the y-axis.
Recall that any point on the y-axis will have x coordinate zero.

Hence, the value of the y-intercept is the point where

$$x = 0$$

Equation

$$y = \underbrace{m}_{Slope} x + \underbrace{c}_{y-intercept} = 0$$

Example 2.40: Finding the equation of a line

Find the equation of each line below.

A. The slope of a line is 3, and the y-intercept is 7.
B. A line with y-intercept $\frac{3}{4}$ has slope $-\frac{2}{3}$.
C. The slope of a line has the same numerical value as the smallest prime number. The absolute value of the x-intercept of the line has the same numerical as the smallest odd prime number. The y-intercept of the line has a numerical value double of the absolute value of the x-intercept.

D. The point (5,7) lies on a line with y-intercept 4.
E. A line has the points (0,4) and (5,0) on it.
F. A line connecting the points (2,1) and $(5,-8)$.

Part A

$$y = mx + c \Rightarrow y = 3x + 7$$

Part B

$$y = mx + c \Rightarrow y = -\frac{2}{3}x + \frac{3}{4}$$

Part C

$$y = mx + c = 2x + 6$$

Part D

$$y - intercept = c = 4 = (0,4)$$

$$Slope = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-4}{5-0} = \frac{3}{5}$$

Substitute the values that we found of m and c into the equation of a line:

$$y = mx + c = \frac{3}{5}x + 4$$

Alternate Method:
Substitute $(x, y) = (5,7)$ in $y = mx + c$:

$$7 = 5m + 4 \Rightarrow m = \frac{3}{5}$$

Hence, the final answer is:

$$y = \frac{3}{5}x + 4$$

Part E

$$Slope = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-4}{5-0} = -\frac{4}{5}$$

The y-intercept is the point where the x coordinate is zero.

$$(0,4) \Rightarrow y - intercept = c = 4$$

$$y = mx + c \Rightarrow y = -\frac{4}{5}x + 4$$

Part F

$$m = \frac{-8-1}{5-2} = \frac{-9}{3} = \frac{-3}{1} = -3$$

Substitute $(x, y) = (2,1), m = -3$ in $y = mx + c$

$$1 = -3(2) + c$$

$$7 = c$$

Hence

$$Equation: y = -3x + 7$$

Example 2.41: Checking if Points lie on a Line

Given a point, we can check if the point lies on the line by substituting it into the equation.
If both sides of the equation have the same value, then the equation is satisfied, and the point lies on the line.

A. Does the point (3,13) lie on the line $y = 3x + 4$.
B. The equation of a line is $y = 6x + 5$. Does the point (3,23) lie on the line? Does the point (2,21) lie on the line?

Part A

$$LHS = y = 13$$

$$RHS = 3x + 4 = 3(3) + 4 = 13 = LHS$$

$$(3,13) \textit{ lies on the line}$$

Part B

$$y = 6x + 5 \Rightarrow 23 = 6(3) + 5 \Rightarrow 23 = 23 \Rightarrow Valid$$

Since the two sides of the equation match, the point (3,23) lies on the line.

$$y = 6x + 5 \Rightarrow 21 = 6(2) + 5 \Rightarrow 21 = 17 \Rightarrow Not\ Valid$$

Since the two sides of the equation do not match, the point (2,21) does not lie on the line.

Example 2.42: Finding coordinates

The equation of a line is $y = 6x + 5$.

A. Find the point on the line that has x coordinate 5.
B. If a point on the line has y coordinate 8, what is its x coordinate?

Substitute $x = 5$ in the equation of the line:

$$y = 6x + 5 = 6(5) + 5 = 30 + 5 = 35 \Rightarrow (x, y) = (5,35)$$

Substitute $y = 8$ in the equation of the line:

$$8 = 6x + 5 \Rightarrow 3 = 6x \Rightarrow x = \frac{1}{2} \Rightarrow (x, y) = \left(\frac{1}{2}, 8\right)$$

D. Function Notation[2]

Instead of writing the y coordinate, we can also write:

$$\underbrace{f(x)}_{\textit{Function}} = 6x + 5$$

Notice that the only change is that on the left-hand side, we have replaced y with $f(x)$.
Earlier, we used to write the coordinates of points as:

$$(x, y)$$

Now, since we replaced y with $f(x)$, we will do that for the coordinates of points as well, and write

$$(x, f(x))$$

This looks different, but works the same way as the earlier (x, y) notation. It is a different notation that is used frequently in higher grades.

Example 2.43

A function is given by

$$f(x) = 6x + 5$$

A. Identify the nature of the function
B. If $x = 5$, find $f(x)$. In other words, find the point $(5, f(5))$.
C. If $f(x) = 8$, what is x? In other words, find $(x, 8)$?

The highest power of x is 1. Hence, it is a linear function.

Substitute $x = 5$ in the equation of the line:

$$f(x) = 6x + 5 = 6(5) + 5 = 30 + 5 = 35 \Rightarrow (x, f(x)) = (5,35)$$

Substitute $y = 5$ in the equation of the line:

$$8 = 6x + 5 \Rightarrow 3 = 6x \Rightarrow x = \frac{1}{2} \Rightarrow (x, y) = \left(\frac{1}{2}, 8\right)$$

E. x- and y-intercepts

At the x-intercept, the value of y will be zero.
To get the $x -$ intercept, substitute $y = 0$ in the equation of a line

At the y-intercept, the value of x will be zero.
To get the $y -$ intercept. substitute $x = 0$ in the equation of a line

Example 2.44: Finding x and y intercepts

Find the x and the y intercepts of the following lines:

A. $y = -2x + 4$
B. $y = \frac{3}{5}x + 2$
C. $3y + 2x = 4$

Part A

[2] It is preferable if you have done some functions before you do this part.

Substitute $x = 0$ to find the y-intercept:

$$y = -2x + 4 \Rightarrow y = -2(0) + 4 = 4 \Rightarrow \underbrace{\mathbf{(0,4)}}_{y-intercept}$$

Substitute $y = 0$ to find the x-intercept:

$$0 = -2x + 4 \Rightarrow 2x = 4 \Rightarrow x = 2 \Rightarrow \underbrace{\mathbf{(2,0)}}_{x-intercept}$$

Part B

Substitute $x = 0$ to find the y-intercept:

$$y = \frac{3}{5}(0) + 2 = 0 + 2 = 2 \Rightarrow \underbrace{\mathbf{(0,2)}}_{y-intercept}$$

Substitute $y = 0$ to find the x-intercept:

$$0 = \frac{3}{5}x + 2 \Rightarrow -2 = \frac{3}{5}x \Rightarrow -2 \times \frac{5}{3} = x \Rightarrow x = -\frac{10}{3} \Rightarrow \underbrace{\left(-\frac{\mathbf{10}}{\mathbf{3}}, \mathbf{0}\right)}_{x-intercept}$$

Part C

Substitute $x = 0$ to find the y-intercept:

$$3y + 2x = 4 \Rightarrow 3y + 2(0) = 4 \Rightarrow 3y = 4 \Rightarrow y = \frac{4}{3} \Rightarrow y - intercept = \left(0, \frac{4}{3}\right)$$

Substitute $y = 0$ to find the x-intercept:

$$3y + 2x = 4 \Rightarrow 3(0) + 2x = 4 \Rightarrow 2x = 4 \Rightarrow x = 2 \Rightarrow x - intercept = (2,0)$$

F. x- and y-intercepts from co-ordinates of points

Example 2.45: x and y intercepts from a table

The table presents some points that all lie on the equation of a line. Find the x-intercept and the y-intercept of the line.

Line 1	x	4	5	6	7
	y	5	7	9	11

Method I:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-5}{5-4} = \frac{2}{1} = 2$$

If x reduces from 4 to zero, then $\Delta x = -4$

$$m = \frac{\Delta y}{\Delta x} = 2 \Rightarrow \frac{\Delta y}{-4} = 2 \Rightarrow \Delta y = -8$$

$$New\ y = 5 - 8 = -3$$

Method II: Extrapolate the table back

$$y - intercept = (0, -3)$$

$y = 0$ lies between the points in the table above where $y = 1$, and $y = -1$

x	0	1	2	3	4	5	6	7
y	-3	-1	1	3	5	7	9	11
	y-intercept							

$$x - intercept = (1.5,0)$$

x	1	1.5	2
y	-1	0	1
		x-intercept	

Example 2.46: x and y intercepts from a table

Subtracting 13	x	y	Adding 15
$22 = 35 - 13$	22	0	$0 = -15 + 15$
$35 = 48 - 13$	35	-15	$-15 = -30 + 15$
$48 = 61 \mathbf{- 13}$	48	-30	$-30 = -45 \mathbf{+ 15}$

$61 = 74 - \mathbf{13}$	61	-45	$-45 = -60 + \mathbf{15}$
	74	-60	

G. Converting word problems to equations

Example 2.47

A gymnasium charges a flat fee of \$500 for membership for the year. Any sessions with a personal trainer are charged at a cost of \$30 per session.

A. Identify the dependent and the independent variable
B. Identify the $y - intercept$
C. Identify the slope
D. Hence, write the equation of the line, and use it to find the cost of s sessions.

$$Dependent\ Variable = y = Cost\ of\ Gym$$
$$Independent\ Variable = x = No.of\ Sessions$$

y-intercept is the value when x, which is number of sessions is zero:

$$y - intercept = 500$$

Slope is the increase in the cost due to additional session

$$Slope = 30$$

The equation of a line in slope-intercept form is

$$y = mx + c$$

We know that y-intercept = 500, and that slope =30. Substitute:

$$y = 30x + 500 = 30s + 500$$

H. Collinearity of Points

2.48: Collinear Points

- Collinear points are points that lie on the same line.
- Three points A, B and C are collinear $if\ and\ only\ if$ Slope $AB =$ Slope BC

Example 2.49

A. Check if the points $A(2,3)$, $B(3,5)$ and $C(4,7)$ are collinear.
B. If points $A(1,5)$, $B(3, y)$ and $C(7,10)$ are collinear, find y.

Part A

Calculate the slopes:

$$m_{AB} = \frac{5-3}{3-2} = \frac{2}{1} = 2$$
$$m_{BC} = \frac{7-5}{4-3} = \frac{2}{1} = 2$$

Since the slopes are the same, the points are collinear:

$$m_{AB} = m_{BC} \Rightarrow \text{Points are collinear}$$

Part B

$$m_{AB} = m_{BC}$$
$$\frac{y-5}{3-1} = \frac{10-y}{7-3}$$
$$\frac{y-5}{2} = \frac{10-y}{4}$$
$$2y - 10 = 10 - y$$
$$3y = 20$$
$$y = \frac{20}{3}$$

I. Converting from Other Forms

2.50: Standard Form

The standard form of the equation of a line is:

$$Ax + By = C$$

Example 2.51

Rewrite the equation of a line in standard in the slope-intercept form.

$$Ax + By = C$$

In the slope-intercept form $y = mx + c$

$$By = -Ax + C \Rightarrow y = \underbrace{-\frac{A}{B}}_{m=Slope} x + \underbrace{\frac{C}{B}}_{y-intercept}$$

2.3 Slope Point Form

A. Forms of Equation of a Line

The equation of a line can be represented in different forms.

Each form has its own applications, and is useful in different contexts. Many exam questions come down to receiving information in one form, and converting it to another form.

Command over the different forms, their uses, and their interpretation is very important.

Form	Equation	Features	When to use
Special Cases			
Horizontal Line	$y = c$		Given a line with zero slope
Vertical Line	$x = c$		Given a line with no slope
General Cases			
Slope-Intercept	$y = mx + c$	$m =$ slope $c = y -$ intercept	Given the slope, and the y-intercept
Slope-Point	$y - y_1 = m(x - x_1)$	$m =$ slope (x_1, y_1) is a point	Given the slope, and a point. Can also be used with two points, by calculating slope first
Other Forms			
General	$Ax + By = C$		Useful in Linear Algebra
Two-point	$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$	$\frac{y_2 - y_1}{x_2 - x_1} =$ slope	Use when given two points

2.52: Slope-Point Form

The slope-point form of the equation of a line is given by:

$$y - y_1 = m(x - x_1)$$

- We can derive this equation from the formula for slope:

$$\underbrace{m = \frac{y_2 - y_1}{x_2 - x_1}}_{Definition\ of\ Slope} \Rightarrow \underbrace{y_2 - y_1 = m(x_2 - x_1)}_{Multiply\ both\ sides\ by\ x_2 - x_1} \Rightarrow \underbrace{y - y_1 = m(x - x_1)}_{Let\ y=y_2, x=x_2}$$

- This equation is useful for determining the equation of a line from two points.

Example 2.53: Lines with no and zero slope

Find the equation of the line with

A. slope zero, and y-intercept 3.
B. no slope, and x-intercept 7. Can you find it by substituting it in the slope-point form of the equation of a line?

A line with zero slope must be horizontal. Since, it has y-intercept 3, the equation of the line must be

$$y = 3$$

A line with no slope must be vertical. Since, it has x-intercept 7, the equation of the line must be

$$x = 7$$

If a line does not have slope, you cannot find the equation of the line by substituting it in the slope-point form.

Example 2.54: Determining the Equation of a line

Find the equation of the following lines:

A. Slope 5 and y-intercept 3
B. Slope 1.2 and $(7,12)$ lies on the line
C. The points $(2,3)$ and $(7,12)$ lie on the line.
D. The points $(-8,8)$ and $(1,-10)$ lie on the line.
E. A horizontal line passing though the point $(3,4)$.
F. A vertical line passing though the point $\left(\frac{2}{3},\frac{4}{7}\right)$.

Part A

Substitute $m = 5$, and $c = 3$ in the slope intercept form to get:

$$y = 5x + 3$$

Part B

Substitute $m = 1.2, x_1 = 7, y_1 = 12$ in the slope point form $y - y_1 = m(x - x_1)$ to get:

$$y - 12 = 1.2(x - 7) \Rightarrow y - 12 = 1.2x - 8.4 \Rightarrow \underbrace{y = 1.2x + 3.6}_{\substack{\textit{Slope-Intercept Form} \\ \textit{Often most useful}}}$$

Part C

Find the slope:

$$m = \frac{12-3}{7-2} = \frac{9}{5}$$

Substitute $m = \frac{9}{5}, x_1 = 2, y_1 = 3$ in the slope point form $y - y_1 = m(x - x_1)$ to get:

$$y - 3 = \frac{9}{5}(x - 2) \Rightarrow y - 3 = \frac{9}{5}x - \frac{18}{5} \Rightarrow \underbrace{y = \frac{9}{5}x + \frac{17}{5}}_{\textit{Slope-Intercept Form}}$$

Part D

$$m = \frac{-10-8}{1-(-8)} = -\frac{18}{9} = -2$$

Substitute $m = -2, x_1 = -8, y_1 = 8$ in the slope-point form $y - y_1 = m(x - x_1)$:

$$y - 8 = -2\big(x - (-8)\big)$$
$$y - 8 = -2x - 16$$
$$y = -2x - 8$$

Part E

$$y = 4$$

Part F

$$x = \frac{2}{3}$$

Example 2.55: Converting to slope-intercept form

B. Graphing Lines

Example 2.56

A. Graph $y = 3x - 1$
B. Graph the line that goes through the points $(1,2)$ *and* $(2,3)$
C. Graph the line that goes through the point $(1,2)$ and has slope 3.

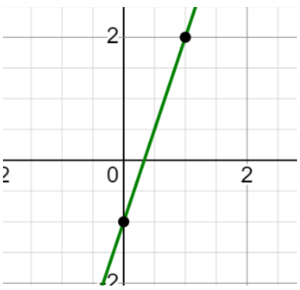
Part A

We have been given the slope-intercept form, so make use of it:

$$Slope = m = 3$$
$$y\ intercept = -1$$

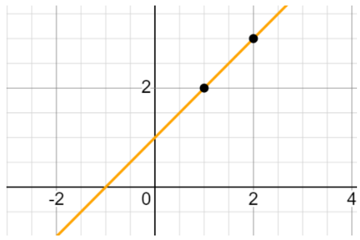
We can substitute $x = 1$ to get:

$$y = 3(1) - 1 = 3 - 1 = 2$$
$$(1,2)\ lies\ on\ the\ line$$



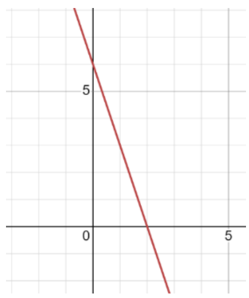
Part B

Plot the points on the coordinate plane and join



them.

Part C



C. Back Calculations

Example 2.57

Find the value of $a + b$ if the line given in the graph satisfies

$$3x + ay = b$$

Method I

Identify the intercepts:

$$(0,6)\ and\ (2,0)$$

Method II

Two points on the line are:

$$(0,6)\ and\ (2,0)$$
$$Slope = -3$$
$$y - intercept = 6$$

Substitute $m = -3, c = 6$

$$y = mx + c \Rightarrow\ y = -3x + 6 \Rightarrow 3x + y = 6$$

Example 2.58

Two points (1,3) and (4,5) lie on the line $ax + 4y = c$. Find the values of a and c.

Substitute (1,3) and (4,5)in the equation of the line (separately) to get two equations:

$$a + 12 = c, 4a + 20 = c$$

We get a system of equations.

$$a + 12 = 4a + 20 \Rightarrow 3a = -8 \Rightarrow a = -\frac{8}{3}$$

$$c = a + 12 = -\frac{8}{3} + 12 = \frac{-8 + 36}{3} = \frac{28}{3}$$

Example 2.59

A line with x-intercept 4 and y-intercept 7 has equation $by + 2x + c = 0$. Find bc.

$$8 + c = 0 \Rightarrow c = -8$$

$$7b + c = 0 \Rightarrow 7b + (-8) = 0 \Rightarrow 7b = 8 \Rightarrow b = \frac{8}{7}$$

$$bc = (-8)\left(\frac{8}{7}\right) = -\frac{64}{7}$$

2.4 Multiple Lines

A. Parallel Lines

2.60: Parallel Lines

- Case I: Parallel lines have the same slope
- Case II: Parallel lines are lines that have no slope.

Example 2.61: Checking for Parallel Lines

Determine which of the following lines, if any, are parallel.

$$y = 3x + 2, \qquad y = 7x + 5, \qquad y = 3x + 7$$

$$\underbrace{y = 3x + 2}_{Slope=3}, \qquad \underbrace{y = 7x + 5}_{Slope=7}, \qquad \underbrace{y = 3x + 7}_{Slope=3}$$

The lines with slope 3 are parallel

Example 2.62

What is the equation of the line parallel to 4x+2y=8 and passing through the point (0,1)? Write the equation in slope-intercept form. (MathCounts 1993 Warm-Up 9)

Example 2.63

A line through the points (2, -9) and (j, 17) is parallel to the line 2x + 3y = 21. What is the value of j? (MathCounts 2008 State Countdown)

Example 2.64

For what value of k are the lines 2x+3y=4k and x-2ky=7 parallel? Express your answer as a common fraction. (MathCounts 2004 School Sprint)

B. Perpendicular Lines

Case I: Horizontal and Vertical Lines

A line with zero slope(horizontal) is perpendicular to a line with no slope (vertical).

Case II: Everything Else

Two lines are perpendicular if their slopes are negative reciprocals of each other, or equivalently if the product of their slopes is negative unity.

$$\underbrace{l_1}_{\text{Line 1}} \underbrace{\perp}_{\text{Perpendicular}} \underbrace{l_2}_{\text{Line 2}} \Leftrightarrow m_1 m_2 = -1 \Leftrightarrow m_1 = -\frac{1}{m_2} \Leftrightarrow m_2 = -\frac{1}{m_1}$$

Example 2.65: Slopes of Perpendicular Lines

Find if the lines $y = \frac{2}{3}x + 6$, and $y = \frac{7}{4}x + 4$ are perpendicular.

$m_1 = \dfrac{2}{3}, m_2 = \dfrac{7}{4}$

You don't need to do the multiplication here because both slopes are positive, and the product of two positive numbers will never be negative.

Product of slopes $= m_1 m_2 = \frac{2}{3} \times \frac{7}{4} = \frac{7}{6}$

In order for two lines to be perpendicular, their slopes must be opposite in sign.

Example 2.66: Slopes of Perpendicular Lines

Q1: Line l_1 has positive slope. Line l_2 has negative slope. Line l_3 has zero slope. Line l_4 has no slope. Line l_5 has positive slope.

A. Which pair of lines must be perpendicular?
B. Which pair of lines can be perpendicular? (You can have more than one pair)
C. Which pair of lines cannot be perpendicular?

S1:

A. $l_3 \perp l_4$
B. $l_1 \perp l_2$ is possible. $l_2 \perp l_5$ is possible.
C. l_5 cannot be perpendicular to l_1, l_3 cannot be perpendicular to any line except l_4,

Example 2.67

Find the slope of the line that is perpendicular to the line $2x + 3y = 6$. (MathCounts 1993 State Countdown)

Example 2.68

What is the x-intercept of the line perpendicular to the line defined by 3x-2y=6 and whose y-intercept is 2? (MathCounts 2002 Workout 6)

Example 2.69

What is the intersection point of the line y=2x+5 and the line perpendicular to it that passes through the point (5, 5)? (MathCounts 2007 Warm-Up 15)

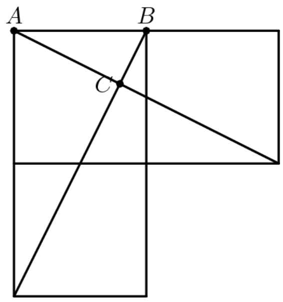
Example 2.70
What is the slope of a line perpendicular to the line containing the points (-1,2) and (1,-2)? Express your answer as a common fraction. (MathCounts 2005 State Countdown)

Example 2.71
Let line t be the line represented by 3x+4y = 5 and let line p be the line perpendicular to line t and containing the point (5,5). What is the x-coordinate of the point common to line t and line p? Express your answer as a common fraction. (MathCounts 2005 State Team)

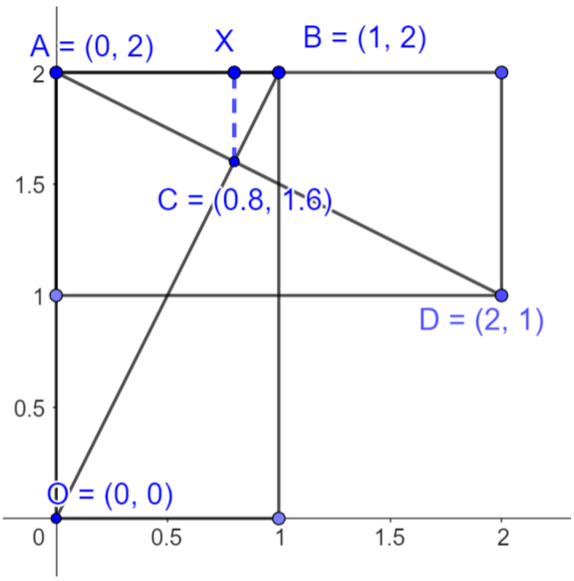
Example 2.72
The graph of the line x+y=b is a perpendicular bisector of the line segment from (1,3) to (5,7). What is the value of b? (MathCounts 2008 School Countdown)

C. Intersecting Lines

Example 2.73
The lines x=(1/4)y+a and y=(1/4)x+b intersect at the point (1,2). What is a+b? (AMC 10 2006, AMC 12 2006)



Example 2.74
Three unit squares and two line segments connecting two pairs of vertices are shown. What is the area of ΔABC? **(AMC 10A 2012/15)**



The line OB passes through the origin (and hence has $y-intercept$ 0). It has slope 2. Hence its equation is

$$y = 2x$$

The line AD has $y-intercept$ 02. It has slope $-\frac{1}{2}$. Hence, its equation is

$$y = -\frac{1}{2}x + 2$$

Point C is the point of intersection of OB and AD. Hence, we can solve the equations simultaneously:

$$2x = -\frac{1}{2}x + 2 \Rightarrow x = \frac{4}{5} \Rightarrow y = \frac{8}{5}$$

$$C = \left(\frac{4}{5}, \frac{8}{5}\right)$$

Draw altitude CX.

$$CX = 2 - \frac{8}{5} = \frac{2}{5}$$

The area of ΔABC

$$= \frac{1}{2}hb = \frac{1}{2}\left(\frac{2}{5}\right)(1) = \frac{1}{5}$$

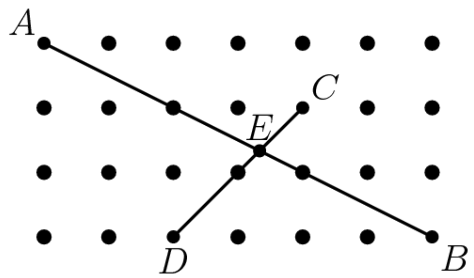
D. Other

Example 2.75

What are the coordinates of the point on the line 5x-9y=42 such that the x and y coordinates are the additive inverses of each other? Express your answer as an ordered pair. (MathCounts 2009 Chapter Countdown)

Example 2.76

The diagram shows 28 lattice points, each one unit from its nearest neighbors. Segment AB meets segment CD at E. Find the length of segment AE[3]. (AMC 10 2000/16)



Introduce a coordinate system with its origin at D.
Then, the line passing through points C and D has equation

$$\underbrace{y = x}_{Equation\ I}$$

We then find the equation of the line passing through AB.

$$Slope = m = \frac{\Delta y}{\Delta x} = \frac{-3}{6} = -\frac{1}{2}$$
$$y - intercept = b = 2$$

Substitute $m = -\frac{1}{2}, b = 2$ in $y = mx + b$:

$$\underbrace{y = -\frac{1}{2}x + 2}_{Equation\ II}$$

Point E is the point of intersection of the above two lines. Hence, its coordinates are the values that are the solution of Equation I and II.
Note that LHS of Equation I and II is the same. Hence, equate the RHS:

$$x = -\frac{1}{2}x + 2 \Rightarrow \frac{3}{2}x = 2 \Rightarrow x = \frac{4}{3} \Rightarrow \underbrace{y = \frac{4}{3}}_{From\ Eq.\ I}$$

Substitute $A(-2,3)$ and $E\left(\frac{4}{3},\frac{4}{3}\right)$ in the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{\left(-2 - \frac{4}{3}\right)^2 + \left(3 - \frac{4}{3}\right)^2} = \sqrt{\left(-\frac{10}{3}\right)^2 + \left(\frac{5}{3}\right)^2} = \sqrt{\frac{100}{9} + \frac{25}{9}} = \sqrt{\frac{125}{9}} = \frac{5\sqrt{5}}{3}$$

E. Quadrilateral Flowchart

Example 2.77: Quadrilaterals

Q2: Determine the nature of the quadrilateral in each of the below. Choose your answers from one or more of:

[3] An alternate solution using Similarity is available in the Note on Similarity/Polygons.

Parallelogram, Rhombus, Rectangle, Square, Kite, General.
ABCD

S2: U

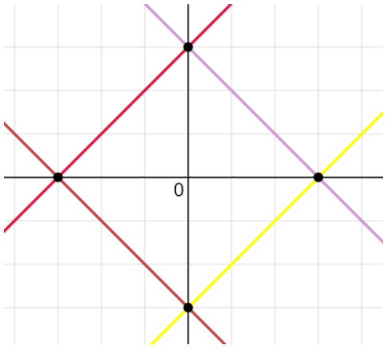
Q3: Show that the diagonals of a rectangle bisect each other.

S3: U

F. Area

Example 2.78

Line l has x −intercept and y-intercept 3 units away from the origin. There are four different possibilities that satisfy line l. These four possibilities together enclose a square. Find the area of the square.



Length of diagonal

$$= d = 3 - (-3) = 6$$

Area of Square

$$= \frac{1}{2} \times d^2 = \frac{1}{2} \times 6^2 = 18$$

Example 2.79

Find the area of the triangle enclosed by the x-axis, the y-axis, and the line with x-intercept 5, and y-intercept 4.

$$(5,0)\ and\ (0,4)$$

$$Area = \frac{1}{2} hb = \frac{1}{2} \times 5 \times 4 = 10$$

Example 2.80

Find the area of the triangle enclosed by the x-axis, the y-axis, and the line with equation $y = 3x + 7$.

$$(0,7)\ and\ \left(-\frac{7}{3}, 0\right)$$

$$\frac{1}{2} hb = \frac{1}{2} \times 7 \times \frac{7}{3} = \frac{49}{6}$$

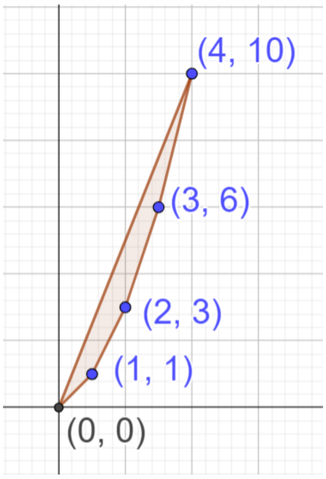
G. Polygons

Example 2.81

Let the slope between points A and B be written $m(AB)$. Vertex $V_n, 0 \le n \le 4$ of polygon P with five vertices is given by

$$(x, y) = \left(n, \frac{n(n+1)}{2}\right)$$

A. What is the slope of the line segment of P with maximum slope?
B. Find $\frac{m(V_0,V_1)+m(V_1,V_2)+m(V_2,V_3)+\ m(V_3,V_4)}{4}$
C. If the coordinates of the n^{th} vertex are given by $(x, y) = \left(n, (-1)^n \frac{n(n+1)}{2}\right)$, then repeat Parts A and B.



Part A

$$m(V_0,V_1)=\frac{y_1-y_0}{x_1-x_0}=\frac{1-0}{1-0}=1$$
$$m(V_1,V_2)=\frac{y_2-y_1}{x_2-x_1}=\frac{3-1}{2-1}=2$$
$$m(V_2,V_3)=\frac{y_3-y_2}{x_3-x_2}=\frac{6-3}{2-1}=2$$
$$m(V_3,V_4)=\frac{y_4-y_3}{x_4-x_3}=\frac{10-6}{4-3}=4\Rightarrow Max$$

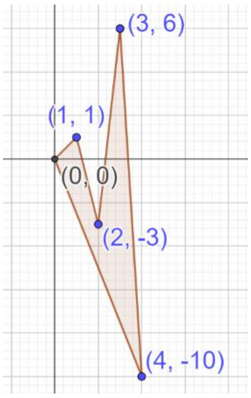
Part B

$$\frac{m(V_0,V_1)+m(V_1,V_2)+m(V_2,V_3)+\ m(V_3,V_4)}{4}=\frac{1+2+3+4}{4}=\frac{10}{4}=2.5$$

You can also calculate:

$$\frac{\frac{y_1-y_0}{x_1-x_0}+\frac{y_2-y_1}{x_2-x_1}+\frac{y_3-y_2}{x_3-x_2}+\frac{y_4-y_3}{x_4-x_3}}{4}=\frac{\frac{y_4-y_1}{1}}{4}=\frac{y_4-y_3}{x_4-x_1}=m(V_0,V_4)=\frac{10-0}{4-0}=\frac{10}{4}=2.5$$

Part C



Example 2.82

This question extends the previous example.

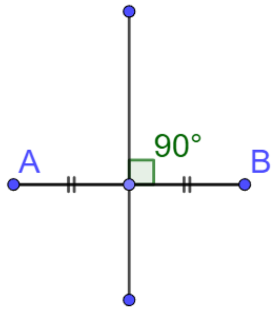
Let the slope between points A and B be written $m(AB)$. Vertex $V_n, 0\le n\le 2020$ of polygon P with 2021 vertices is given by:

$$(x,y)=\left(-2021+n,-2021+(-1)^n\frac{n(n+1)}{2}\right)$$

A. What is the slope of the line segment of P with maximum slope?
B. Find $m(V_0,V_1)+m(V_1,V_2)+\cdots+m(V_{2019},V_{2020})$

2.5 Perpendicular Bisectors

A. Perpendicular Bisector

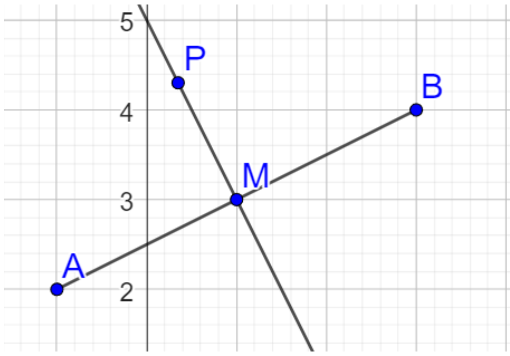


2.83: Perpendicular Bisector

The perpendicular bisector of a line segment AB passes through the midpoint M of AB, and is perpendicular to AB.

Example 2.84

Find the equation of the perpendicular bisector of $A(-1,2)$ and $B(3,4)$



Using the midpoint formula, Midpoint of AB

$$Midpoint\ of\ AB = \left(\frac{-1+3}{2}, \frac{2+4}{2}\right) = (1,3)$$

The slope of AB:

$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{3+1} = \frac{2}{4} = \frac{1}{2}$$

Let P be a point on the perpendicular bisector of AB. It will intersect AB at M. Slope of PM is the negative reciprocal of the slope of AB

$$= -\frac{2}{1} = -2$$

Since PM passes through M, M is a point on PM. Use point-slope of equation of a line with $m = -2, M(1,3)$

$$y - 3 = -2(x-1)$$
$$y - 3 = -2x + 2$$
$$y = -2x + 5$$

2.85: Process of finding Equation of Perpendicular Bisector

To find the equation of perpendicular bisector $A(x_1, y_1)$ and $B(x_2, y_2)$

- Find the midpoint M of AB using the midpoint formula $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
- Find the slope of AB using the slope formula $m_1 = \frac{y_2-y_1}{x_2-x_1}$
- Let P be a general point on the perpendicular bisector. Slope of $PM = m_2 = -\frac{1}{m_1}$ (negative reciprocal of slope of perpendicular line)
- Find the equation of the line using $y - y_1 = m(x - x_1)$

2.86: Equation of Perpendicular Bisector

The equation of the perpendicular bisector of $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by:

$$(y_2 - y_1)y + (x_2 - x_1)x = \frac{(x_2^2 + y_2^2) - (x_1^2 + y_1^2)}{2}$$

Midpoint of AB

$$= M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Slope of AB

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

Let P be a point on the perpendicular bisector of AB. Slope of PM

$$= \text{Negative reciprocal of slope of } AB = -\frac{x_2 - x_1}{y_2 - y_1}$$

Since PM passes through M, M is a point on PM. Use point-slope of equation of a line with $m = -\frac{x_2 - x_1}{y_2 - y_1}, M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$$y - \frac{y_1 + y_2}{2} = -\frac{x_2 - x_1}{y_2 - y_1}\left(x - \frac{x_1 + x_2}{2}\right)$$

Multiply both sides by $y_2 - y_1$:

$$(y_2 - y_1)y - \frac{(y_2 + y_1)(y_2 - y_1)}{2} = -(x_2 - x_1)x + \frac{(x_2 - x_1)(x_2 + x_1)}{2}$$

Rearrange to get positive signs on both sides of the equation:

$$(y_2 - y_1)y + (x_2 - x_1)x = \frac{x_2^2 - x_1^2}{2} + \frac{y_2^2 - y_1^2}{2}$$

Rearrange the RHS:

$$(y_2 - y_1)y + (x_2 - x_1)x = \frac{(x_2^2 + y_2^2) - (x_1^2 + y_1^2)}{2}$$

2.87: Circumcenter

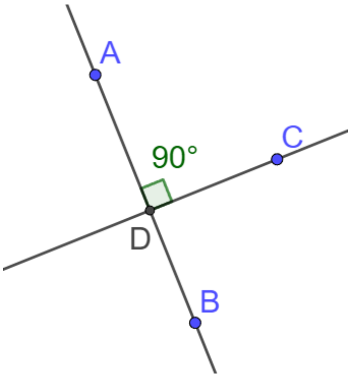
- The circumcenter is the center of the circle that passes through the vertices of a triangle.
- The circumcenter is the point of concurrency of the perpendicular bisectors of a triangle.

2.6 Distance from a point to a line

A. Distance

2.88: Distance from a point to a line (Definition)

The distance from a point to a line is the length of the perpendicular segment from the point to the line.



In the diagram alongside, the distance from C to line AB is

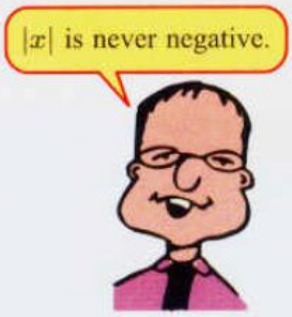
$$Length\ of\ CD$$

2.89: Distance from a point to a line (Formula)

Given a line l in two dimensions $Ax + By + C = 0$ and a point $X(h, k)$ not on the line, the distance d from the point to the line is given by:

$$d = \frac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}}$$

b The **modulus** of x is $|x|$. It is the *size* of x, ignoring its sign, and can be defined by $|x| = \sqrt{x^2}$.
A property of modulus is that $|xy| = |x||y|$ for all real numbers x, y.

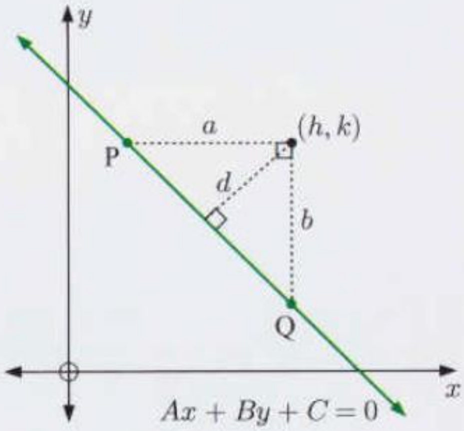


Consider the shortest distance d from a point (h, k) to the line $Ax + By + C = 0$.
Point P is the point on the line with y-coordinate k.
Point Q is the point on the line with x-coordinate h.
Show that:

i the distance $a = \dfrac{|Ah + Bk + C|}{|A|}$

ii the distance $b = \dfrac{|Ah + Bk + C|}{|B|}$

iii the distance $d = \dfrac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}}$.



Finding the distance a

Let P be the point of intersection of the line l and the line passing through (h, k) and parallel to the x axis. Note that P has the same y coordinate as (h, k)

$$P_y = k$$

The x coordinate of point P can be found by substituting k in the equation of the line $Ax + By + C = 0$:

$$Ax + B(k) + C = 0$$
$$x = \frac{-C - Bk}{A}$$

The coordinates of point P are then:

$$P\left(\frac{-C - Bk}{A}, k\right)$$

Since P and (h, k) have the same y coordinate, the distance between them is the absolute value of the difference of their x coordinates:

$$|x_2 - x_1| = \left|h - \frac{-C - Bk}{A}\right| = \left|\frac{Ah + Bk + C}{A}\right| = \frac{|Ah + Bk + C|}{|A|}$$

Substitute $E = Ah + Bk + C$:

$$\frac{|E|}{|A|}$$

Finding the distance b

Let Q be the point of intersection of the line l and the line passing through (h, k) and parallel to the y axis.

Note that Q has the same x coordinate as (h, k)

$$Q_x = h$$

The y coordinate of point Q can be found by substituting h in the equation of the line $Ax + By + C = 0$:

$$Ah + By + C = 0 \Rightarrow y = \frac{-C - Ah}{B}$$

Hence, the coordinates of point Q are:

$$Q\left(h, \frac{-C - Ah}{B}\right)$$

Since Q and (h, k) have the same x coordinate, the distance between them is $|y_2 - y_1|$ given by:

$$\left|k - \frac{-C - Ah}{B}\right| = \left|\frac{Ah + Bk + C}{B}\right| = \frac{|Ah + Bk + C|}{|B|}$$

Substitute $E = Ah + Bk + C$:

$$\frac{|E|}{|B|}$$

Finding the distance d

Area of triangle $= \frac{1}{2}hb$

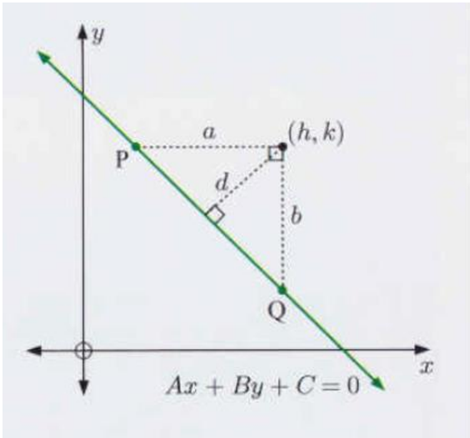
$$\frac{1}{2}ab = \frac{1}{2}PQ(d)$$

Substitute $PQ = \sqrt{a^2 + b^2}$:

$$ab = \sqrt{a^2 + b^2}d$$

Solve for d:

$$d = \frac{ab}{\sqrt{a^2 + b^2}}$$



$$ab = \frac{|E|}{|A|} \times \frac{|E|}{|B|} = \frac{E^2}{|AB|}$$

$$\sqrt{a^2 + b^2} = \sqrt{\left(\frac{|E|}{|A|}\right)^2 + \left(\frac{|E|}{|B|}\right)^2} = \sqrt{\frac{E^2}{A^2} + \frac{E^2}{B^2}} = \sqrt{\frac{B^2E^2 + A^2E^2}{A^2B^2}} = \frac{|E|}{|AB|}\sqrt{(A^2 + B^2)}$$

$$d = \frac{ab}{\sqrt{a^2 + b^2}} = \frac{\frac{E^2}{|AB|}}{\frac{|E|}{|AB|}\sqrt{(A^2 + B^2)}} = \frac{|E|}{\sqrt{A^2 + B^2}} = \frac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}}$$

2.7 AMC Questions

A. Lines

Example 2.90

The lines with equations $ax - 2y = c$ and $2x + by = -c$ are perpendicular and intersect at $(1, -5)$. What is c? **(AMC 10B 2017/10)**

Write each equation in slope intercept form:

$$y = \frac{a}{2}x - \frac{c}{2} \Rightarrow m_1 = \frac{a}{2}$$
$$y = -\frac{2}{b}x - \frac{c}{b} \Rightarrow m_2 = -\frac{2}{b} \Rightarrow -\frac{1}{m_2} = \frac{b}{2}$$

Since the lines are perpendicular, the slopes are negative reciprocals of each other:

$$m_1 = -\frac{1}{m_2}$$
$$\frac{a}{2} = \frac{b}{2} \Rightarrow a = b$$

Substitute $(1, -5)$ in $ax - 2y = c$

$$a(1) - 2(-5) = c \Rightarrow \underbrace{a + 10 = c}_{\textit{Equation I}}$$

Substitute $(1, -5), a = b$ in $2x + by = -c$

$$2(1) + a(-5) = -c \Rightarrow \underbrace{5a - 2 = c}_{\textit{Equation II}}$$

From Equations I and II:

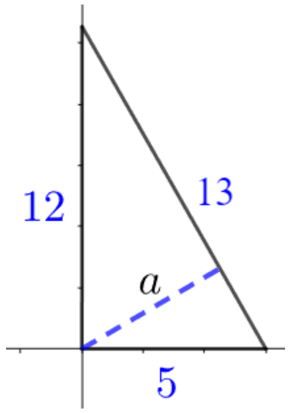
$$a + 10 = 5a - 2 \Rightarrow a = 3 \Rightarrow c = a + 10 = 13$$

Example 2.91

The line $12x + 5y = 60$ forms a triangle with the coordinate axes. What is the sum of the lengths of the altitudes of this triangle? (AMC 10B 2015/13)

Find the intercepts:

$$x - intercept: 12x = 60 \Rightarrow x = 5$$
$$y - intercept: 5y = 60 \Rightarrow y = 12$$



The triangle is

$$5 - 12 - 13$$

Calculate the area of the triangle in two different ways:

$$\left(\frac{1}{2}\right) 13a = \left(\frac{1}{2}\right) 5 \times 12 \Rightarrow a = \frac{60}{13}$$

$$5 + 12 + \frac{60}{13} = 17 + \frac{60}{13} = \frac{221 + 60}{13} = \frac{281}{13}$$

Example 2.92

A line with slope 3 intersects a line with slope 5 at point $(10,15)$. What is the distance between the x-intercepts of these two lines? (AMC 10B 2003/11)

$$y - y_1 = m(x - x_1)$$
$$y - 15 = 3(x - 10) \Rightarrow -15 = 3x - 30 \Rightarrow x = 5$$
$$y - 15 = 5(x - 10) \Rightarrow -15 = 5x - 50 \Rightarrow x = 7$$

Difference

$$= 7 - 2 = 5$$

Example 2.93

Positive integers a and b are such that the graphs of $y = ax + 5$ and $y = 3x + b$ intersect the x-axis at the same point. What is the sum of all possible x-coordinates of these points of intersection? (AMC 10A 2014/21)

Example 2.94

A line that passes through the origin intersects both the line $x = 1$ and the line $y = 1 + \frac{\sqrt{3}}{3}x$. The three lines create an equilateral triangle. What is the perimeter of the triangle? (AMC 10A 2014/17)

Example 2.95

A lattice point in an xy-coordinate system is any point (x, y) where both x and y are integers. The graph of $y = mx + 2$ passes through no lattice point with $0 < x \le 100$ for all m such that $\frac{1}{2} < m < a$. What is the maximum possible value of a? (AMC 10B 2011/24)

We want to find the maximum value of a.

$$y = mx + \underbrace{2}_{Integer} \notin \mathbb{Z}$$

Since 2 is an integer, we must have

$$mx \notin \mathbb{Z}$$

This means that $m \neq 1$, since that will make mx an integer for integral values of x.

$$a < 1$$

We need to minimize the value of m for $m = \frac{n}{d}$, $n, d \in \mathbb{N}$ between $\frac{1}{2}$ and 1 such that

$$\frac{n}{d}x \in \mathbb{Z} \Rightarrow d \in \{1,2, \dots ,100\}$$

is an integer. This is given by:

$$\frac{n}{d} = \frac{2}{3}, \frac{3}{4}, \frac{3}{5}, \dots, \frac{50}{99}, \frac{51}{100}$$

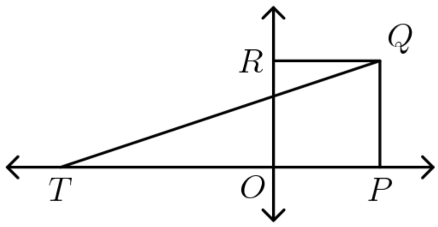
The minimum value , out of these is:

$$\frac{50}{99}$$

B. Basics

Example 2.96

Figure $OPQR$ is a square. Point O is the origin, and point Q has coordinates (2,2). What are the coordinates for T so that the area of triangle PQT equals the area of square $OPQR$?(AMC 8 1996/17)



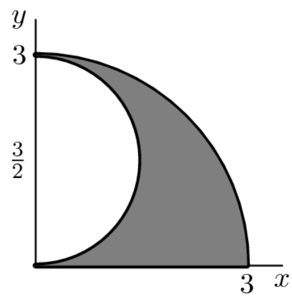
C. Area

Example 2.97

What is the area of the region enclosed by the graph of the equation $x^2 + y^2 = |x| + |y|$? (AMC 10B 2016/21)

Example 2.98

The shaded region below is called a shark's fin falcata, a figure studied by Leonardo da Vinci. It is bounded by the portion of the circle of radius 3 and center (0,0) that lies in the first quadrant, the portion of the circle with radius $\frac{3}{2}$ and center (0,3/2) that lies in the first quadrant, and the line segment from (0,0) to (3,0). What is the area of the shark's fin falcata? (AMC 10B 2015/9)



Example 2.99

Two circles of radius 2 are centered at (2,0) and at (0,2). What is the area of the intersection of the interiors of the two circles? **(AMC 10B 2007/13)**

Example 2.100

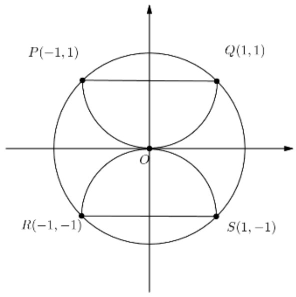
In rectangle $ABCD$, we have $A = (6, -22)$, $B = (2006,178)$, $D = (8, y)$, for some integer y. What is the area of rectangle $ABCD$? **(AMC 10B 2006/20)**

Example 2.101

A square in the coordinate plane has vertices whose y-coordinates are 0, 1, 4, and 5. What is the area of the square? **(AMC 10A 2014/18)**

Example 2.102

Semicircles POQ and ROS pass through the center O. What is the ratio of the combined areas of the two semicircles to the area of circle O? **(AMC 8 2010/23)**



D. Area with Lines

Example 2.103

Two lines with slopes $\frac{1}{2}$ and 2 intersect at (2,2). What is the area of the triangle enclosed by these two lines and the line $x + y = 10$? **(AMC 10A 2019/8)**

Example 2.104

What is the area of the triangle formed by the lines $y = 5$, $y = 1 + x$, and $y = 1 - x$? **(AMC 8 2019/21)**

$$A = \frac{1}{2} \times h \times b = \frac{1}{2} \times 4 \times 8 = 16$$

Example 2.105

A triangle with vertices (6, 5), (8, -3), and (9, 1) is reflected about the line $x = 8$ to create a second triangle. What is the area of the union of the two triangles? **(AMC 10A 2013/16)**

Example 2.106

A rectangular region is bounded by the graphs of the equations $y = a, y = -b, x = -c$, and $x = d$, where a, b, c, and d are all positive numbers. Which of the following represents the area of this region? **(AMC 10A 2011/9)**

Example 2.107

The y-intercepts, P and Q, of two perpendicular lines intersecting at the point $A(6,8)$ have a sum of zero. What is the area of $\triangle APQ$? **(AMC 10A 2014/14)**

Example 2.108

Let points $A = (0,0)$, $B = (1,2)$, $C = (3,3)$, and $D = (4,0)$. Quadrilateral $ABCD$ is cut into equal area pieces by a line passing through A. This line intersects $\overline{CD}$ at point $\left(\frac{p}{q}, \frac{r}{s}\right)$, where these fractions are in lowest terms. What is $p + q + r + s$? **(AMC 10A 2013/18)**

Example 2.109

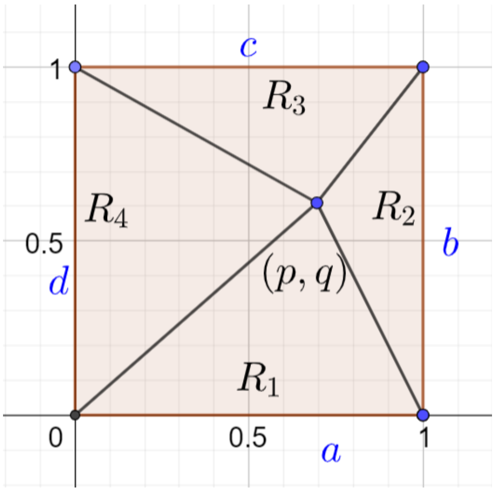
Let R be a unit square region and $n \geq 4$ an integer. A point X in the interior of R is called n-*ray* partitional if there are n rays emanating from X that divide R into n triangles of equal area. How many points are 100-ray partitional but not 60-ray partitional? **(AMC 10A 2011/25)**

Draw a unit square with its bottom left vertex located at the origin.[4]
Choose a point in the interior of R with coordinates

$$(p, q)$$

If a ray is not drawn to the vertex, then the vertex will be part of a quadrilateral-shaped region, and not a triangle. Hence, draw rays to the vertices, getting four triangular regions:

$$R_1, R_2, R_3, R_4$$



Using the formula for area of a triangle $A = \frac{1}{2}hb$, the area of these regions is:

$$[R_1] = \frac{q}{2}, [R_2] = \frac{1-p}{2}, [R_3] = \frac{1-q}{2}, [R_4] = \frac{p}{2}$$

The number of triangles in each region must be an integer. Let:

$$R_1, R_2, R_3, R_4 \textit{ have } a, b, c, d \textit{ triangles respectively}$$

Hence, the area of a triangle in regions:

$$R_1, R_2, R_3, R_4 \textit{ must be, respectively, } \quad \frac{q}{2a}, \frac{1-p}{2b}, \frac{1-q}{2c}, \frac{p}{2d}$$

However, no matter which region the triangle is in, the area must be equal. Hence,

$$\frac{q}{2a} = \frac{1-q}{2c} \Rightarrow 2cq = 2a - 2aq \Rightarrow q(2c + 2a) = 2a \Rightarrow q = \frac{a}{a+c}$$
$$\frac{p}{2d} = \frac{1-p}{2b} \Rightarrow 2bp = 2d - 2dp \Rightarrow p(2b + 2d) = 2d \Rightarrow p = \frac{d}{b+d}$$

Also, note that

$$[R_1] + [R_3] = \frac{q}{2} + \frac{1-q}{2} = \frac{1}{2}, \qquad [R_2] + [R_4] = \frac{1-p}{2} + \frac{p}{2} = \frac{1}{2}$$

Since the areas are equal, the number of triangles in them must also be equal.

$$a + c = b + d$$

[4] The solution here uses the same logic as this video.

100-Ray Partitional

$$a+b+c+d=100 \Rightarrow a+c=b+d=50$$

The coordinates of a 100-ray partitional point must be of the form:

$$(p,q)=\left(\frac{d}{b+d},\frac{a}{a+c}\right)=\left(\frac{d}{50},\frac{a}{50}\right), \qquad d,a\in\mathbb{Z}$$

Since we want a point in the interior of the square, $0<p,q<1$, and we must have:

$$1\le d\le 49 \Rightarrow 49\ Possibilities$$
$$1\le a\le 49 \Rightarrow 49\ Possibilities$$

And, by the multiplication principle, we can count the number of values as:

$$49\times 49=49^2=2{,}401$$

60-Ray Partitional

$$a+b+c+d=60 \Rightarrow a+c=b+d=30$$

The coordinates of a 60-ray partitional point must be of the form:

$$(p,q)=\left(\frac{d'}{30},\frac{a'}{30}\right), \qquad d',a'\in\mathbb{Z}$$

100-Ray, but not 60-Ray Partitional

We now count and remove points which are both 100-ray partitional and 6-ray partitional.

For a number to be both 100-ray partitional, and 60-ray partitional, its x coordinate must satisfy both of the conditions below simultaneously:

$$\frac{d}{\underbrace{50}_{100\ Ray}}=\frac{d'}{\underbrace{30}_{60\ Ray}} \Rightarrow 3d=5d'$$

This is a Diophantine equation, and has solutions:

$$(d,d')=(5a,3a)=(5,3),(10,6),\dots(45,27)\Rightarrow 9\ Choices$$

Similarly, its y coordinate must satisfy both of the conditions below simultaneously:

$$\frac{a}{\underbrace{50}_{100\ Ray}}=\frac{a'}{\underbrace{30}_{60\ Ray}} \Rightarrow 3a=5a'$$

This is a Diophantine equation, and has solutions:

$$(a,a')=(5a,3a)=(5,3),(10,6),\dots(45,27)\Rightarrow 9\ Choices$$

And, again by the Multiplication Principle, we can count the number of values that are both 100-ray partitional and 60-ray partitional as:

$$\underbrace{9}_{x}\times\underbrace{9}_{y}=81$$

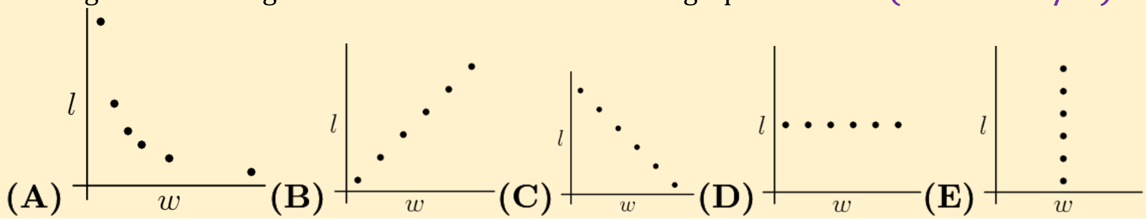
Hence, the final answer is:

$$2401-81=2{,}320$$

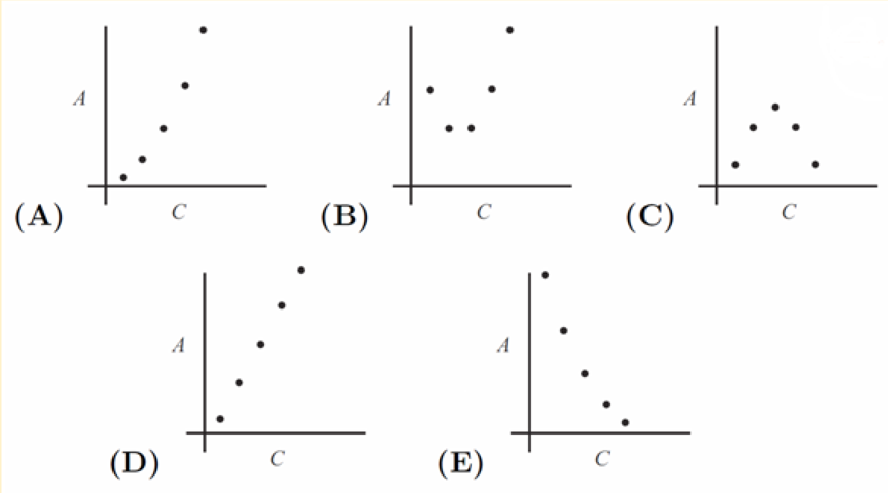
E. Graphs

Example 2.110

Jorge's teacher asks him to plot all the ordered pairs (w,l) of positive integers for which w is the width and l is the length of a rectangle with area 12. What should his graph look like? **(AMC 8 2006/10)**



Example 2.111
Amanda draws five circles with radii 1, 2, 3, 4 and 5. Then for each circle she plots the point (C, A), where C is its circumference and A is its area. Which of the following could be her graph? (AMC 8 2007/16)



F. Midpoint Formula

Example 2.112
Let points $A = (0,0,0)$, $B = (1,0,0)$, $C = (0,2,0)$, and $D = (0,0,3)$. Points E, F, G, and H are midpoints of line segments $\overline{BD}$, $\overline{AB}$, $\overline{AC}$, and $\overline{DC}$ respectively. What is the area of $EFGH$? (AMC 10A 2012/21)

G. Counting

Example 2.113
For how many integers x is the point $(x, -x)$ inside or on the circle of radius 10 centered at $(5,5)$? (AMC 10B 2015/12)

H. Conics

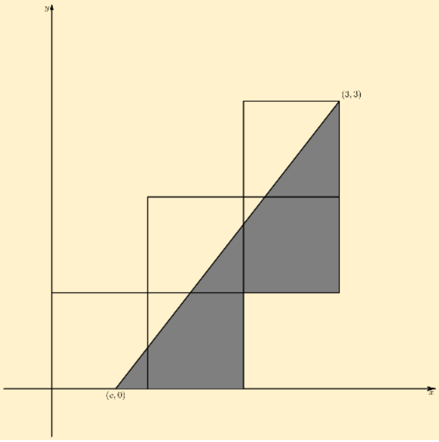
Example 2.114
The vertices of an equilateral triangle lie on the hyperbola $xy = 1$, and a vertex of this hyperbola is the centroid of the triangle. What is the square of the area of the triangle?(AMC 10B 2017/24)

Example 2.115
All three vertices of $\triangle ABC$ are lying on the parabola defined by $y = x^2$, with A at the origin and $\overline{BC}$ parallel to the x-axis. The area of the triangle is 64. What is the length of BC? All three vertices of $\triangle ABC$ are lying on the parabola defined by $y = x^2$, with A at the origin and $\overline{BC}$ parallel to the x-axis. The area of the triangle is 64. What is the length of BC? (AMC 10B 2016/9)

I. Mix

Example 2.116
Points $A(11,9)$ and $B(2,-3)$ are vertices of $\triangle ABC$ with $AB = AC$. The altitude from A meets the opposite side at $D(-1,3)$. What are the coordinates of point C? (AMC 10B 2017/8)

Example 2.117
Five unit squares are arranged in the coordinate plane as shown, with the lower left corner at the origin. The slanted line, extending from $(c, 0)$ to $(3,3)$, divides the entire region into two regions of equal area. What is c? (AMC 10B 2009/17)



Example 2.118
Let S be a set of points (x, y) in the coordinate plane such that two of the three quantities $3, x + 2$, and $y - 4$ are equal and the third of the three quantities is no greater than this common value. Which of the following is a correct description for S? (AMC 10A 2017/12)
(A) a single point
(B) two intersecting lines
(C) three lines whose pairwise intersections are three distinct points
(D) a triangle
(E) three rays with a common endpoint

Example 2.119
Distinct points P, Q, R, S lie on the circle $x^2 + y^2 = 25$ and have integer coordinates. The distances PQ and RS are irrational numbers. What is the greatest possible value of the ratio $\frac{PQ}{RS}$? (AMC 10A 2017/17)

Example 2.120
Two points B and C are in a plane. Let S be the set of all points A in the plane for which $\triangle ABC$ has area 1. Which of the following describes S? (AMC 10B 2007/10)
(A)two parallel lines (B)a parabola (C)a circle (D)a line segment (E)two points

Example 2.121
Points $A(6,13)$ and $B(12,11)$ lie on circle ω in the plane. Suppose that the tangent lines to ω at A and B intersect at a point on the x-axis. What is the area of ω? (AMC 10B 2019/23)

Example 2.122
Andy the Ant lives on a coordinate plane and is currently at $(-20,20)$ facing east (that is, in the positive x-

direction). Andy moves 1 unit and then turns 90° degrees left. From there, Andy moves 2 units (north) and then turns 90° degrees left. He then moves 3 units (west) and again turns 90° degrees left. Andy continues his progress, increasing his distance each time by 1 unit and always turning left. What is the location of the point at which Andy makes the 2020th left turn? **(AMC 10B 2020/13)**

Example 2.123

Points A and B are 10 units apart. Points B and C are 4 units apart. Points C and D are 3 units apart. If A and D are as close as possible, then the number of units between them is (AMC 8 1996/8)

Example 2.124

Points A, B, C, D, E and F lie, in that order, on $\overline{AF}$, dividing it into five segments, each of length 1. Point G is not on line AF. Point H lies on $\overline{GD}$, and point J lies on $\overline{GF}$. The line segments $\overline{HC}, \overline{JE}$, and $\overline{AG}$ $are\ parallel. Find\ HC/JE$ **(AMC 10A 2002/20)**

Example 2.125

Points B and C lie on AD. The length of AB is 4 times the length of BD, and the length of AC is 9 times the length of CD. The length of BC is what fraction of the length of AD? **(AMC 10B 2008/6)**

Example 2.126

Four distinct points are arranged on a plane so that the segments connecting them have lengths a, a, a, a, $2a$, and b. What is the ratio of b to a? **(AMC 10B 2012/21)**

3. Further Topics

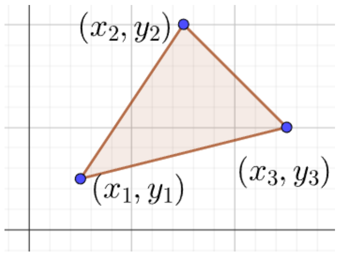
3.1 Rotation

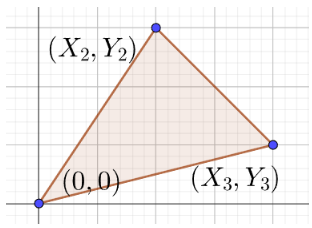
A. Rotation

Example 3.1

The medians of a triangle are concurrent at the centroid.

Prove the statement above using coordinate geometry.





($See\ left\ diagram$) Consider a general ΔABC with coordinates

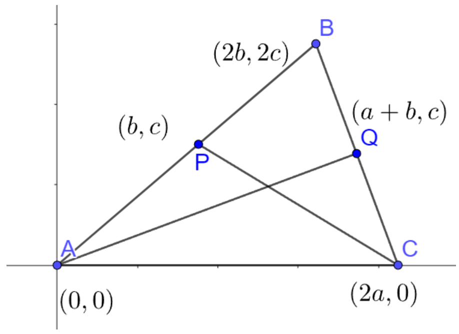
$$A(x_1, y_1), B(x_2, y_2)\ and\ C(x_3, y_3)$$

($See\ right\ diagram$) Use a change of origin. Let (x_1, y_1) be the origin. The new coordinates:

$$A(0,0), B(x_2 - x_1, y_2 - y_1)\ and\ C(x_3 - x_1, y_3 - y_1)$$

Use a change of variable.

$$A(0,0), B(X_2, Y_2)\ and\ C(X_3, Y_3)$$



Use a rotation. Rotate the triangle so that point C lies on the x axis.

$$A(0,0), B(2b, 2c)\ and\ C(2a, 0)$$

Let point P be the midpoint of AB, and point Q be the midpoint of BC:

$$P = \left(\frac{2b+0}{2}, \frac{2c+0}{2}\right) = (b, c)$$

$$Q = \left(\frac{2b+2a}{2}, \frac{2c+0}{2}\right) = (a+b, c)$$

Type equation here.

Lines AQ and CP:

We calculate slope, use a point on the line and make use of the equation below to determine the equation of each line:

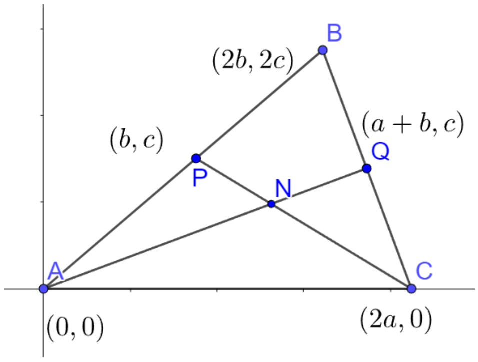
$$y - y_1 = m(x - x_1)$$

Line AQ has $m = \frac{c-0}{a+b-0} = \frac{c}{a+b}$, $(x_1, y_1) = (0,0)$:

$$y - 0 = \left(\frac{c}{a+b}\right)(x - 0) \Rightarrow \underbrace{\frac{y}{c} = \frac{x}{a+b}}_{Equation\ I}$$

Line CP has $y - y_1 = m(x - x_1)$ using $m = \frac{c-0}{b-2a} = \frac{c}{b-2a}$, $(x_1, y_1) = (2a, 0)$:

$$y - 0 = \left(\frac{c}{b-2a}\right)(x - 2a) \Rightarrow \underbrace{\frac{y}{c} = \frac{x-2a}{b-2a}}_{Equation\ II}$$



From Equations I and II:

$$\frac{x}{a+b} = \frac{(x-2a)}{b-2a}$$
$$xb - 2ax = xa + xb - 2a^2 - 2ab$$

Cancel xb on both sides, and divide $-a$ both sides:

$$2x + x = 2a + 2b \Rightarrow x = \frac{2a+2b}{3}$$
$$y = \frac{cx}{a+b} = \frac{c\left(\frac{2a+2b}{3}\right)}{a+b} = \frac{2c}{3}$$

$$N\left(\frac{2a+2b}{3}, \frac{2c}{3}\right)$$

Lines AQ and CP: $\boldsymbol{y - y_1 = m(x - x_1)}$

3.2 Rotation of Axes

2 Examples