Number Systems

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1. Decimal System

1.1 Decimal Place Value System

A. Place Value System

Example 1.1
In the number 74982.1035, the value of the place occupied by the digit 9 is how many times as great as the value of the place occupied by the digit 3? (AMC 8 1997/6)

9 is in the thousand's place value
$$= 1000 = 10^3$$

3 is in the thousandth' place value
$$= \frac{1}{1000} = 10^{-3}$$

$$10^3 \div 10^{-3} = 10^{3-(-3)} = 10^6 = 100{,}000$$

Example 1.2
Let x be the number
$$0.\underbrace{0000 \ldots 0000}_{1996 \text{ zeros}}1,$$
where there are 1996 zeros after the decimal point. Which of the following expressions represents the largest number?

A. $3 + x$
B. $3 - x$
C. $3 \cdot x$
D. $\frac{3}{x}$
E. $\frac{x}{3}$ (AMC 8 1996/11)

We use estimation here

A is just over 3.
B is just under 3. This eliminates B, because A is bigger.
C is just over 0, since it is three times a tiny number. This eliminates C, because A is bigger.
D will give a huge number. $\frac{1}{x}$ will get very large when x gets close to zero. You can see this by examining the sequence:
$$\frac{3}{0.1} = 30, \qquad \frac{3}{0.01} = 300, \qquad \frac{3}{0.001} = 3000$$
D will be huge, and this eliminates A.
E will give a small number, since you're dividing a tiny number into thirds. This eliminates E.

Hence, option D is correct.

B. Expanded Notation

1.3: Two Digit Number

A two-digit number ab can be written in expanded notation as
$$ab = 10a + b$$

$$45 = 40 + 5 = 4 \times 10 + 5$$

Example 1.4
The two-digit number 27 is three times the sum of is digits $(2 + 7) \times 3 = 27$. Find all two-digit numbers each of which is 7 times the sum of its digits. **(NMTC Final Primary-III)**

Method I:
Any number that is 7 times the sum of its digits must be a multiple. Hence, check all the multiples of 7.

No.	14	21	28	35	42	49	56	63	70	77	84	91	98
Sum of Digits	5	3	10	8	6	13	11	9	7	14	12	10	17
	35	21	70	56	42	91	77	63	49	98	84	70	119

Method II: Algebra
$$tu = 10t + u$$

$$10t + u = 7(t + u)$$
$$10t + u = 7t + 7u$$
$$3t = 6u$$
$$t = 2u$$
This means that the ten's digit is the double of the units' digit.
$$(a, b) = (2,1)(4,2)(6,3)(8,4)$$

Example 1.5
How many two-digit numbers ab are there such that $a^2 + b^2 = 65$?

a	1	2	3	4	5	6	7	8	9
a^2	1	4	9	16	25	36	49	64	81
b^2	64	61	56	49	40	29	16	1	-16
b	8			7			4	1	

$$Numbers = 18,47,74,81$$

1.6: Two Digit Number
A three-digit number ab can be written in expanded notation as
$$abc = 100a + 10b + c$$

Example 1.7
ABC is a three-digit number in which the digit A is greater than the digits B and C. If the difference between ABC and CBA is 297, and the difference between ABC and BAC is 450, find all such possible three-digit numbers ABC and find their sum. **(NMTC Final Primary-III)**

Start with the first condition

$$\underbrace{100A+10B+C}_{ABC}-\underbrace{(100C+10B+A)}_{CBA}=297$$
$$99A-99C=297$$
$$A-C=3$$
$$C=A-3$$

Look at the second condition

$$\underbrace{100A+10B+C}_{ABC}-\underbrace{(100B+10A+C)}_{BAC}=450$$
$$90A-90B=450$$
$$A-B=5$$
$$B=A-5$$

The minimum value of A that will work is $A=5$, because that makes $B=0$.

$$A=5\Rightarrow B=A-5=5-5=0, C=5-3=2\Rightarrow 1st\ No=502$$
$$613$$
$$724$$
$$835$$
$$946$$

Example 1.8

The sum of two natural numbers is 17,402. One of the two numbers is divisible by 10. If the units digit of that number is erased, the other number is obtained. What is the difference of these two numbers? **(AMC 10A 2021/3)**

$$10a+a=17402\Rightarrow a=1582$$
$$Difference=10a-a=9a=14{,}238$$

Example 1.9

How many digits are in the product $4^5\cdot 5^{10}$? **(AMC 8 2011/15)**

$$4^5\cdot 5^{10}=2^{10}\cdot 5^{10}=10^{10}=1\ followed\ by\ 10\ zeroes\Rightarrow 11\ Digits$$

Example 1.10

How many zeros are at the end of the product $25\times\ 25\times\ 25\times 25\times 25\times 25\times 25\times 8\times 8\times 8$? **(AMC 8 1991/13)**

$$25^7\times 8^3=5^{14}\times 2^9=5^5\times 5^9\times 2^9=5^5\times 10^9\Rightarrow 9\ Zeroes$$

Example 1.11

The digit 1 is attached to the right of a 3 digit number making it a 4 digit number which is 7777 more than the given number. The sum of the digits of the number is: **(NMTC Sub-Junior/Screening 2006/11**)

$$\because No.=abc\Rightarrow \underbrace{abc1=abc+7777}_{As\ per\ given\ condition}\Rightarrow \underbrace{1000a+100b+10c+1=100a+10b+c+7777}_{Expand\ both\ numbers}$$
$$\underbrace{900a+90b+9c=7776}_{Collate\ variables\ on\ one\ side}\Rightarrow \underbrace{100a+10b+c=864}_{Divide\ both\ sides\ by\ 9}\Rightarrow abc=864\rightarrow Sum=18$$

Example 1.12

a, b, c are the digits of a nine digit number abcabcabc. The quotient when this number is divided by 1001001 is:

(NMTC Sub-Junior/Screening 2011/Part B/1)

$$abcabcabc = abc(1{,}000{,}000) + abc(1000) + abc = abc(1{,}000{,}000 + 1000 + 1) = abc(1{,}001{,}001)$$

Hence,

$$\frac{abcabcabc}{1{,}001{,}001} = \frac{abc(1{,}001{,}001)}{1{,}001{,}001} = abc$$

Example 1.13

If the digit 1 is placed after a two-digit number whose tens' digit is t, and units' digit is u, the new number is:

A. $10t + u + 1$
B. $100t + 10u + 1$
C. $1000t + 10u + 1$
D. $t + u + 1$
E. None of these answers (AHSME 1950/7)

$$Option\ B$$

C. Reversing the digits of a Numbers

Example 1.14

The hundreds digit of a three-digit number is 2 more than the units digit. The digits of the three-digit number are reversed, and the result is subtracted from the original three-digit number. What is the units digit of the result? **(AMC 8 2010/22)**

$$100(u+2) + 10t + u - (100u + 10t + u + 2) = 200 - 2 = 198$$

We can also do this using numbers:

$$200 - 2 = 198 \Rightarrow Units\ Digit: 8$$

Example 1.15

Determine how many two-digit numbers satisfy the following property: when the number is added to the number obtained by reversing its digits, the sum is 132. **(AMC 8 2016/11)**

Let the two-digit number be ab.

$$\underbrace{(10a + b)}_{Number} + \underbrace{(10b + a)}_{Reversed\ Number} = 132 \Rightarrow 11a + 11b = 132 \Rightarrow a + b = 12$$

$$(a, b) = \{(3,9)(4,8)(5,7)(6,6)(7,5)(8,4)(9,3)\} \Rightarrow 7\ Numbers$$

D. Sums of Squares of Digits

Sometimes numbers that meet a given condition may not exist. For example, as the next example shows three-digit numbers that are the sum of the squares of their digits do not exist.

Example 1.16

There are positive integers that have these properties:

- the sum of the squares of their digits is 50, and
- each digit is larger than the one to its left.

The product of the digits of the largest integer with both properties is **(AMC 8 1997/23)**

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55 \Rightarrow 5\ Digit\ Numbers\ will\ not\ work$$

Consider 4-digit numbers $abcd$.
Minimum value of

$$a^2 + b^2 + c^2 = 1^2 + 2^2 + 3^2 = 14$$

Hence:

$$d = 7 \Rightarrow d^2 = 49\ does\ not\ work$$
$$d = 6 \Rightarrow d^2 = 36\ works\ with\ minimum\ values\ of\ abc$$

Since we are going backwards from largest number possible, we have found the largest number, which is:

$$abcd = 1236$$

$$Product\ of\ Digits = 1 \times 2 \times 3 \times 6 = 36$$

Example 1.17

Let n be a 3-digit number such that n = sum of the squares of the digits of n. The number of such n is: **(NMTC Sub-Junior/Screening 2006/5, Modified)**

$$\underbrace{100a + 10b + c}_{Expanded\ Notation} = \underbrace{a^2 + b^2 + c^2}_{Sum\ of\ the\ Squares}$$

$$a^2 - 100a + b^2 - 10b + c^2 - c = 0 \Rightarrow \underbrace{a(a-100)}_{-ve} + \underbrace{b(b-10)}_{-ve} + \underbrace{c(c-1)}_{+ve} = 0$$
$$Largest\ value\ of\ a(a-100) = 1(1-100) = -99$$
$$Largest\ value\ of\ b(b-10) = 0(0-10) = 0$$
$$Largest\ value\ of\ c(1-c) = 9(9-1) = 72$$

Largest Value of $abc = -99 + 72 = -27$
This can never be zero. Hence, there are no solutions.

$$a^2 - 100a + b^2 - 10b + c^2 - c = 0$$

E. Sums of Cubes of Digits

Numbers that are the sum of the cubes of their digits are called Armstrong Numbers.

Example 1.18

F. Finding Numbers

Example 1.19

Rich chooses a 4-digit positive integer. He erases one of the digits of this integer. The remaining digits, in their original order, form a 3-digit positive integer. When Rich adds this 3-digit integer to the original 4-digit integer, the result is 6031. What is the sum of the digits of the original 4-digit integer? (Gauss Grade 8 2019/22)

abcd

abc, abd, bcd

abcd+abc
abcd+abd
abcd+bcd

In the last two cases, we have for the units digit
d+d=2d, which is even
but 6031 is odd

Hence, the only viable case is:
abcd+abc=6031
(1000a + 100b+10c+d) + 100a + 10b + c = 6031
1100a + 110b+11c+d = 6031

11(100a+10b+c)+d = 6031

The LHS is the sum of a multiple of 11 and a single digit number.

6031 = 548 3/11
Hence, the only possible value of d is 3.

11(100a+10b+c)= 6048
100a+10b+c= 548
abc=548
abcd=5483
a+b+c+d=5+4+8+3=20

G. Types of Numbers

Assume, if possible, to the contrary that $\sqrt{2}$ is a rational number.

$$\mathbb{Q} = \left\{x | x = \frac{p}{q}, \qquad p, q \in Z, q \neq 0\right\}$$

$For\ some\ p, q \in Z, q \neq 0$, we must have:

$$\sqrt{2} = \frac{p}{q} \Rightarrow 2 = \frac{p^2}{q^2} \Rightarrow 2q^2 = p^2$$

Now,

$$RHS = p^2 = Perfect\ Square \Rightarrow LHS = Perfect\ Square = 2q^2$$

Recall that the prime factorization of a perfect square will always have an even power for each distinct prime.
∴ It will have an even power of 2.
∴ $2q^2$ is not a perfect square.
Contradiction

Which means our assumption was wrong:

$$\sqrt{2} \notin \mathbb{Q}$$

1.2 Digital Sum, Digital Root and Digital Product

A. Basics

1.20: Digital Sum and Root

- The sum of the digits of a number is called its digital sum.
- Find a digital sum for a number (and repeat the process) till the answer is a single digit. This is the digital root.

Example 1.21

What are the digital sum, and the digital root of:

A. 3718
B. 19574?

	Digital Sum	Digital Root
3718	$3 + 7 + 1 + 8 = 19$	$1 + 9 = 10$ $1 + 0 = 1$
19574	$1 + 9 + 5 + 7 + 4 = 26$	$2 + 6 = 8$

Example 1.22: Enumerating Numbers

List all numbers with less than five digits which have digital sum 2, or digital sum 3.

Digital Sum	1 Digit	2 Digits	3 Digits	4 Digits
2	2	20,11	200, 101, 110	2000, 1001, 1010, 1100
3	3	30, 21, 12	300, 210, 201, 120, 102, 111	3000, 2100, 2010, 2001, 1200, 1020, 1002, 1110, 1011, 1101

Example 1.23

Find the sum of all three-digit numbers with digital sum:

A. 27
B. 26
C. 25

Part A

Only one case: 999

Part B

Three options: 998, 989, 899

Sum $= 2{,}886$

Part C

If all digits were 9, then the sum $= 27$.

Here, the sum of digits is 25, which is 2 less than 27.

I need to subtract two from 27.

Case I: 2 from a single digit: 7, 9, 9

Numbers: $799 + 979 + 997$
Case II: 2 from two different digits: 8, 8, 9
Numbers: $889 + 898 + 988$

Add all the numbers: 5550
Instead of thinking about ways to make 25 by adding digits, it is easier to think of ways to make 25 by subtracting from 27.

Example 1.24
What is the first digit of the smallest number whose sum of digits is 2007? **(NMTC Primary-Screening 2006/8)**

To make the number smallest, the digits should be as large as possible.
The largest digit possible is:

$$9$$

We want as many $9's$ as possible:

$$\frac{2007}{9}$$

Since we want only first digit, we should focus on the remainder, which is:

$$\frac{2+0+0+7}{9} = \frac{9}{9} \Rightarrow Remainder\ 0$$

Hence, we will have

$$\underbrace{999\ldots9}_{Large\ number\ of\ 9's} \quad \Rightarrow First\ Digit = 9$$

(Continuation) Example 1.25
In the above question, what is the number of digits?

Now, we want the quotient:

$$\frac{2007}{9} = 223$$

1.26: Single Digit Numbers
If the digital sum of a number is a single digit, then its digital root is also the same.

Example 1.27
Let

$$N = 274512$$
$$DS = Digital\ Sum\ of\ N$$
$$DR = Digital\ Root\ of\ N$$

Find the difference between the digital sum and the digital root of the number $(DS + DR)$

DS = $2 + 7 + 4 + 5 + 1 + 2 = 21$
DR = $2 + 1 = 3$
DS + DR = $21 + 3 = 24$
For 24, both digital sum and digital root are the same.
Hence, the difference is zero.

Example 1.28
Find the difference in the maximum and minimum values of the digital sum for a 23-digit even number that is

divisible by 5.

Last digit must be zero
Remaining digits can be 9 (for maximum) or 1 for the 1st digit (and zero for the remaining digits)
9 * 22 – 1 = 198 - 1 = 197

Example 1.29

Count the number of numbers with digital sum 2 that have:

A. Three Digits
B. Five Digits
C. Seven Digits
D. Ten Digits

3 Digits	200	101	110			3 Numbers
5 Digits	20000	10001	10010	10100	11000	5 Numbers
7 Digits	2,000,000	1,000,001	1,000,010	1000100	…	7 Numbers

For a number with n digits:
There are two values for the first digit: 2 and 1.

If the first digit is 2, all other digits must be zero = One Way

If the first digit is 1, there must another one, and the remaining digits must be zero.
The one which is not the leftmost place can occupy $n-1$ places.

$$n-1\ ways$$

Total Ways for n digits

$$1+n-1=n\ Numbers$$

Example 1.30

Let X be a non-zero natural number which has Y digits, DS be the Digital Sum of X, and DR be the Digital Root of X. Also, DS ≠ DR.
Y =

A. One
B. Two
C. More than One
D. More than two

For all single digit numbers, the digital sum is the same as the digital root.
Hence, Y > 1.
Hence, option C.

Min (DS) = ___________

The minimum value of the digital sum such that the digital root is different = 10.

Min (DR) = ___________

The minimum value of the digital root = 1
For example: 55 has digital root 1.
(0 is not possible, the question says that X is a non-zero number.

Max (DR) = __________

Since we repeat the process of adding the digits of the digital sum till we get a single digit, the maximum value of the digital root = 9.

Y + Min(DS) + Min(DR) + Max (DR) =

A. 0
B. 5
C. 9
D. Cannot be determined from the given information

There is no upper limit on Y, due to which the answer cannot be determined.
Hence, option D.

B. Repeating Numbers

Example 1.31

Consider the sequence where the n^{th} term is made of n occurring n times.

$$S = 1,22,333, \dots$$

Find t_{25}

$$t_{25} = \underbrace{252525 \dots}_{25\ Times}$$

For t_{122}, find:

A. The number of digits
B. Digital Sum
C. Digital Root
D. The number of times that each digit appears

$$t_{122} = \underbrace{122122122 \dots}_{122\ Times}$$
$$No.of\ Digits = 3 \times 122 = 366$$
$$Digital\ Sum = 122 \times (1 + 2 + 2) = 122 \times 5 = 610$$
$$Digital\ Root = Digital\ Sum(610) = 6 + 1 = 7$$

Tabulate the number of digits in each term of the sequence. Find a pattern.

1	1	$\underbrace{1010 \dots 10}_{10\ Tens}$	20	$\underbrace{100100 \dots 100}_{100\ Hundreds}$	300
22	2	$\underbrace{1111 \dots 11}_{11\ Elevens}$	22		
333	3	$\underbrace{1212 \dots 12}_{12\ Twelves}$	22		
.	.	.	.		
.	.	.	.		
.	.	.	.		

$\underbrace{999\ldots9}_{9\ Nine's}$	9	$\underbrace{9999\ldots99}_{99\ Ninety\ Niness}$	198		
$(No.of\ Digits\ of\ n)\times n$					

C. Harshad Numbers

1.32: Harshad Numbers

A number that is divisible by its digital sum is a Harshad Number or a Niven Number.
H_n represents Harshad numbers that have sum of digits *n.*

H_2 represents Harshad numbers with sum of digits 2.
Harshad numbers combine digital sums and divisibility.

Example 1.33

A. Find a pattern for H_2 by tabulating the first few terms. What is the logic behind the pattern?
B. What should be the number of twenty-two-digit H_2 numbers?
C. The number of values of n $(0 \le n \le 10^{23})$ which are even natural numbers with digital sum 2 is ______.

Part A

No. of Digits	Numbers that meet the condition	Count of Numbers	Numbers with Digital Sum 2 that are not even
1	2	1	-
2	20	1	11
3	200, 110	2	101
4	2000, 1100, 1010	3	1001
5	20000, 11000, 10100, 10010	4	10001

Pattern: If the number of digits is X, the count of numbers that satisfies H_2 is X – 1.
The exception to this pattern is when X = 1, and the count is also 1.
Logic: There are x numbers of x digits that have digital sum 2 (for $x > 1$).
Out of these, all except one (the one with the units' digit 1) are divisible by 2.

Part B

22 – 1 = 21

Part C

Minimum value of $n = 0$. Maximum value of $n = 10^{23}$
Neither the maximum value, nor the minimum value meet the condition of digital sum 2.

We need to check numbers with digits 1-22. Use the pattern obtained in prior questions:

$$1 + (1 + 2 + \cdots + 21) = 1 + \frac{21 \times 22}{2} = 1 + 231 = 232$$

Example 1.34

A. What numbers are included in H_3?
B. What numbers are included in H_9?
C. What numbers with three digits or less are included in H_6?

Part A

All numbers with digital sum 3, will be divisible by 3. Hence, we need to find numbers with digital sum 3.

No. of Digits	Numbers with Digital Sum 3	Count of Numbers
1	3	1
2	30, 21, 12	3
3	300, 210, 201, 120, 102, 111	6

Part B

All numbers with digital sum 9 will be divisible by 9:

No. of Digits	Numbers with Digital Sum 9	Count of Numbers
1	9	1
2	9, 18, 27, . . 81	9

Part C

All numbers with digital sum 6 will be divisible by 3.
Hence, we only need to check divisibility by 2.

No. of Digits	Numbers with Digital Sum 6	Count of Numbers
1	6	1
2	51, 42, 33, 24, 15	2
3	600 501, 510 402, 411, 420 303, 312, 321, 330 204, 213, 222, 231, 240 105, 114, 123, 132, 141, 150	12

Example 1.35

A. Find the first few terms of H_5
B. Find the first few terms of H_{15}

Part A

The number must end in 0 or 5 to be divisible by 5.
Hence, for all numbers greater than 9, the ending digit must be 0 (Why?).

No. of Digits	Numbers with Digital Sum 6	Count of Numbers
1	5	1
2	50	1
3	500, 410, 320, 230, 140	5

Part B

If the sum of digits is 15, then the number will *always be divisible by 3.*
We need to only check divisibility by 5. This means the last digit must be zero or five.

Single Digit Numbers:
Not possible. Maximum sum of digits is 9.

Two-digit numbers:
Not possible. If last digit is zero, or five, sum will not be 15.

Three-digit numbers:
Last digit five: 195, 285, 375, 465, 555, 645, 735, 825, 915
(9 Numbers)
Last digit zero: 690, 780, 870, 960
(4 Numbers)
Total: 13 Numbers

D. H_{11}

Example 1.36

A. Is it possible for single digit or two-digit numbers to belong to H_{11}?
B. Suppose a three-digit number ABC (where A, B and C represent digits not necessarily distinct) has digital sum 11, and is divisible by 11. List all possible values of ABC, and find their sum?

Part A
Single Digit Numbers: Not possible. Sum of digits cannot be 11

Two-digit numbers: Assume the number is XY.
The difference of X and Y must be zero or 11.
Not possible
Part B
As per test of divisibility by 11:
Case I: $A + C - B = 0$
$A + C = B$
$A + B + C = 2B$

Substitute: $A + B + C = 11$
$11 = 2B$
$B = 11/2$
Hence, no solutions.
Case II: $A + C - B = 11$
$A + C = B + 11$
$A + B + C = 2B + 11$
$11 = 2B + 11$
$2B = 0$
$B = 0$
$A + C = 11$
209, 308, 407, 506, 605, 704, 803, 902
(8 Numbers)

Example 1.37

- A Harshad number is divisible by the sum of its digits.
- A Harshad pair is an unordered pair of Harshad numbers, each of which has the same digits, but in a different order.
- A Harshad triplet is an unordered triplet of Harshad numbers, each of which has the same starting digit.

Let:
$$\{X: X \text{ } is \text{ } a \text{ } natural \text{ } number \text{ } with \text{ } three \text{ } digits \text{ } or \text{ } less\}$$
$$\{Y: Y \text{ } is \text{ } a \text{ } natural \text{ } number \text{ } with \text{ } sum \text{ } of \text{ } digits \text{ } 6\}$$

$$\{Z: Z\ is\ a\ natural\ number\ which\ is\ divisible\ by\ 6\}$$

Further

$$\{A: A\ is\ in\ X\ and\ Y\ and\ Z\}$$

List all Harshad pairs, and Harshad triplets in A.

In a previous question, we tabulated H_6, which will be useful here.

No. of Digits	Numbers with Digital Sum 6	Count of Numbers
1	6	1
2	51, 42, 33, 24, 15	2
3	600 501, 510 402, 411, 420 303, 312, 321, 330 204, 213, 222, 231, 240 105, 114, 123, 132, 141, 150	12

Single-Digit:

No pairs. There is only a single number.

No triplets.

Two-Digits:

No two numbers have the same starting digit.

No triplets.

Three Digits:

No triplets.

The pairs are:

501, 510
402, 420
303, 330
312, 321
201, 240
213, 231
105, 150
114, 141
123, 132

Example 1.38

Find the product of all single digit numbers that are divisible by the sum of their digits?

The digital sum of 1 is 1.
And 1 is divisible by 1.

The digital sum of 2 is 2.
And 2 is divisible by 2.

Continuing like this, there are nine single digit Harshad numbers.

$$\text{Sum} = 1 + 2 + \cdots + 9 = \frac{9 \times 10}{2} = 45$$

Zero does not meet the condition in the question since you cannot divide by zero.

Example 1.39
List the two-digit Harshad numbers?

10, 20, 40, 50, 70, 80 – for 1, 2, 4, 5, 7 and 8.
12, 21, and 30 – divisible by 3
24, 42 – divisible by 6
18 – 90 – divisible by 9
48, 84 – divisible by 12

DS	1	2	3	4	5	6	7	8	9
	10	11, 20	30,21, 12	40,31, 22,13	50,41,32, 23,14	60,51,42, 33,24,15	70,61,52,43, 34, 25, 16	80,71,62,53, 44, 35, 26, 17	90, 81, 72, 63, 54, 45, 36, 27, 18
	1	1	3	1	1	3	1	1	9
		2	5	6	7	10	11	12	21

	10						
	91,						

E. Digital Product

1.40: Digital Product
The product of the digits of a number is its digital product

Example 1.41
Find the digital product of each of the following numbers: 741, 89251, 6102

7 * 4 * 1 = 28
8 * 9 * 2 * 5 * 1 = 72 * 10 = 720
6 * 1 * 0 * 2 = 0

Example 1.42
In the equation below, A is a single digit number.

$$24 \times A = B$$

Let A_1 be the value of A that maximizes the digital product of B.
Let A_2 be the value of A that maximizes the digital product of B.
What is $A_1 - A_2$?
7

Example 1.43
X is a five-digit number with digital product 120. What is the difference between the largest and the smallest possible value of X?

$120 = 2^3 \times 3 \times 5$
The largest single digit that can be made from multiplying two or more of the above is 8.
Two digits are automatically 3, and 5 (since they are prime).
Finally, the remaining two digits must be 1.

Hence, Difference

= Largest – Smallest = 85311 – 11358 = 73953

1.3 Decimals

A. Classification

Decimals are not a very high frequency topic. Correspondingly, they do not have a very wide range of concepts to learn either. However, there are some important concepts to pick up related to:

1.44: Decimal Fractions

A decimal fraction is a fraction that has only powers of ten in its denominator. In other words, the denominator is one followed by one or more zeros.

Example 1.45

Classify the following as decimal fractions, or non-decimal fractions

$$\frac{3}{10} = 0.3, \qquad \frac{65}{100} = 0.65, \qquad \frac{3675}{100{,}000} = 0.003675$$

1.46: Decimal Fractions to Decimals

Converting a decimal fraction to a decimal number is a straightforward matter of dividing or multiplying (as the case may be) with the requisite power of ten.

Example 1.47

Remember that

$$5 \times 2 = 10, \qquad 25 \times 4 = 100, \qquad 125 \times 8 = 1000, \qquad 5^n \times 2^n = 10^n$$

This property is useful when converting fractions to decimal fractions.

1.48: Terminating versus Non-terminating Decimals

Decimals can be classified as either:

- Terminating: The decimal representation ends.
- Terminating: The decimal representation does not end.

Example 1.49

Classify the following numbers as terminating or non-terminating.

A. 2.54
B. 3.1111 ...
C. $3.\bar{1}$

$$A: Terminating$$
$$B, C: Non-terminating$$

1.50: Recurring versus Non-Recurring

Decimals can be classified as either:

- Recurring: The same digits are repeated, in the same order.
- Non-Recurring: The pattern does not repeat.

Example 1.51

Classify the following numbers as recurring or non-recurring.

A. 2.545454

B. 2.5153
C. $2.\overline{54}$

$$A, C: Recurring$$
$$B: Non\ recurring$$

1.52: Rational Numbers

The set of all rational numbers $\mathbb{Q}$ is given by:

$$\mathbb{Q} = \left\{x \mid x = \frac{p}{q}, \qquad p, q \in \mathbb{Z}, q \neq 0\right\}$$

A rational number is a number

- that can be written in the form $\frac{p}{q}$ (or, in other words, as a fraction)

However, there are some important restrictions:

- p and q are integers
- $q \neq 0$, which is very important since we cannot divide by zero, ever.

1.53: Irrational Numbers

A number which is not rational is irrational.

1.54: Surds

A square root of a prime number is called a surd. All surds are irrational.

Example 1.55

Classify the following as rational or irrational:

A. $\sqrt{4}$
B. $\sqrt{8}$
C. $\sqrt{\frac{32}{2}}$

$$\sqrt{4} = 2 \rightarrow Rational$$
$$\sqrt{8} = 2\sqrt{2} \rightarrow Irrational$$
$$\sqrt{\frac{32}{2}} = \sqrt{16} = 4$$

1.56: Irrational Numbers

A number which has a non-terminating, non-recurring decimal representation is irrational.

	Recurring	Non-Recurring
Terminating (Ends)	Rational $0.3434 = \frac{3434}{10000}$	Rational $0.34 = \frac{34}{100} = \frac{17}{50}$
Non-terminating (Does not end)	Rational $0.3434\ldots = 0.\overline{34} = \frac{34}{99}$	Irrational Numbers $\pi = 3.14\ldots$ $e = 2.71\ldots$ $\sqrt{2} = 1.41\ldots$

		$\sqrt{p}, \quad p \text{ is prime}$

We will put some effort into deciding whether a fraction has a terminating or a non-terminating decimal.

B. Terminating vs Non-terminating Decimals

Condition for a fraction to be a terminating decimal

If a fraction has an equivalent decimal fraction, then its decimal representation will be terminating.
A fraction will have an equivalent decimal fraction *if and only if* the prime factorization of the denominator of the fraction has only powers of 2 and 5.

$$\boldsymbol{a, b \in \mathbb{Z}, b \neq 0, \qquad \frac{a}{b} \text{ is a terminating decimal} \Leftrightarrow b = 2^x 5^y, \qquad x, y \in \mathbb{W}}$$

Example 1.57: Classifying by long division

Example 1.58: Classifying by converting into equivalent fractions

Determine, without doing the long division, whether the decimal representation of each numbers in

$$S = \left\{\frac{3}{5}, \frac{12}{40}, \frac{7}{6}, \frac{12}{256}, \frac{29}{320}, \frac{12}{75}\right\}$$

is terminating or non-terminating.

Part A

Multiply by $\frac{2}{2}$ to get denominator of 10

$$\frac{3}{5} = \frac{3}{5} \times \frac{2}{2} = \frac{6}{10} = 0.6 \Rightarrow Terminating$$

Part B

$$\underbrace{\frac{11}{40} = \frac{11}{40} \times \frac{25}{25} = \frac{275}{1000} = 0.275}_{\textit{Multiply by } \frac{25}{25} \textit{ to get denominator of } 10} \quad \Rightarrow Terminating$$

Part C

$$\frac{7}{6} = \frac{7}{2 \times \underbrace{3}_{3 \text{ is a problem}}} \quad \Rightarrow \text{Cannot be multiplied by any number to get a power of 10} \Rightarrow \textbf{Non - terminating}$$

Part D

$$\frac{12}{256} = \frac{12}{2^8} = \frac{12 \times 5^8}{2^8 \times 5^8} = \frac{12 \times 5^8}{10^8} = \textbf{Some Decimal}$$

Part E

$$\frac{29}{320} = \frac{29}{32 \times 10} = \frac{29}{2^5 \times 10} \times \frac{5^5}{5^5} = \frac{29 \times 5^5}{5^6 \times 2^6} = \textbf{Some Decimal}$$

Part F

$$\frac{12}{75} = \frac{12}{25 \times 3} = \frac{12}{5^2 \times 3} = \frac{4}{5^2} = \frac{4 \times 2^2}{5^2 \times 2^2} = \frac{4 \times 2^2}{10^2} = \textbf{Some Decimal}$$

1.59: Classifying by prime factorization of the denominator

If the fraction $\frac{N}{D}$, where N is the numerator and D is the denominator has a prime factorization of the form

$$D = 2^a \cdot 5^b$$

Then it is a terminating decimal

In other words, the denominator should only have 2 and 5 in its prime factorization.

Example 1.60:

For each number below, decide whether the decimal representation of the number is terminating or non-terminating. Use prime factorization of the denominator and numerator

A. $\frac{401}{640}$
B. $\frac{101}{120}$
C. $\frac{41}{97}$
D. $\frac{65}{130}$

Convert the fractions into simplest form, and then find the prime factorization:

$$\frac{401}{640} \Rightarrow Denominator = 64 \times 10 = \underbrace{2^7 \times 5^1}_{\text{Only Powers of 2 and 5}} \Rightarrow \textbf{Terminating}$$

$$\frac{101}{120} = \frac{101}{2^3 \times \mathbf{3} \times 5} \Rightarrow Extra\ \mathbf{3} \Rightarrow \textbf{Non-terminating}$$

$$\frac{41}{\mathbf{97}} \Rightarrow Already\ Prime\ Factorized \Rightarrow \textbf{Non-terminating}$$

$$\frac{65}{130} = \frac{\mathbf{13} \times 5}{\mathbf{13} \times 2 \times 5} \Rightarrow The\ 13\ will\ cancel \Rightarrow \textbf{Terminating}$$

Challenge 1.61

Find the sum of the values of n for which $\frac{n}{120}$ has a non-terminating decimal representation.

$$\frac{n}{120} = \frac{n}{2^3 \times \mathbf{3} \times 5} = Terminating \Leftrightarrow n = 3x \Rightarrow n\ is\ a\ multiple\ of\ 3$$

$$\underbrace{(3 + 6 + \cdots + 120) = 3(1 + 2 + \cdots + 40) = 3\left(\frac{40 \times 41}{2}\right)}_{\text{Sum of all multiples of 3 from 1 to 120}}$$

$$\text{Sum of all } n = 1 + 2 + \cdots + 120 = \frac{120 \times 121}{2}$$

$$\sum Non-Terminating\ Values = \underbrace{\underbrace{\frac{120 \times 121}{2}}_{All\ Values} - \underbrace{3\left(\frac{40 \times 41}{2}\right)}_{Terminating\ Values}}_{Complementary\ Counting}$$

C. Terminating Decimals

Example 1.62

The last digit in the finite decimal representation of the number $\left(\frac{1}{5}\right)^{2004}$ is: (NMTC Sub-Junior/Screening

2004/25)

Split the power using the exponent rule $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

$$\left(\frac{1}{5}\right)^{2004} = \frac{1}{5^{2004}}$$

Multiply by $\frac{2^{2004}}{2^{2004}}$ to convert into a decimal fraction:

$$= \frac{1}{5^{2004}} \times \frac{2^{2004}}{2^{2004}} = \frac{2^{2004}}{10^{2004}}$$

$$Denominator = One\ followed\ by\ many\ zeros$$
$$Numerator = 2^{2004} \rightarrow Last\ Digit$$

$$Cyclicity\ of\ Powers\ of\ 2 \rightarrow (2, 4, 8, 16, 32, 64, 128, 256) \rightarrow Cycle\ is\ 4\ digits$$
$$2004\ is\ divisible\ by\ 4 \Rightarrow Cycle\ will\ complete\ \Rightarrow Last\ Digit = 6$$

Example 1.63: Number of Non-Zero Digits

Find the number of zero and non-zero digits in the decimal representation of $\frac{357}{3125\times 64}$

$$\frac{357}{3125 \times 64} = \frac{357}{5^5 \times 2^6} = \frac{357}{5^5 \times 2^6} \times \frac{5}{5} = \frac{357 \times 5}{10^6}$$
$$357 \times 5 \Rightarrow 4\ Digits, \quad \frac{357 \times 5}{10^6} \Rightarrow 6\ Decimal\ Places \Rightarrow 2\ Zero\ Digits, 4\ Non-zero\ digits$$

D. Conversion

1.64: Common Fractions

Certain fractions are common enough that they should be memorized. Some of these fractions are given below:

$$0.3333\ldots\ldots = \underbrace{0.\overline{3}}_{\substack{Complete\\Preferred}} = \underbrace{0.\overline{33}}_{\substack{Not\ necessary\\to\ repeat\ the\ 3}} = \frac{1}{3}$$
$$0.1666\ldots = 0.1\overline{6} = \frac{1}{6}$$
$$0.14857142857\ldots = 0.\overline{142857} = \frac{1}{7}$$
$$0.090909\ldots = 0.\overline{09} = \frac{1}{11}$$

Example 1.65

1.66: Conversion

Recurring decimals can be converted into fractions by using algebra, and treating the decimals as infinite series.

Example 1.67

Convert $0.\overline{3}$ from a recurring decimal into a fraction.

Method I

Let $x = 0.\overline{3}$

$$\underbrace{x = 0.33333\ldots}_{Equation\ I}$$

Multiply both sides by 10:

$$\underbrace{10x = 3.33333\ldots}_{Equation\ II}$$

Subtract Equation I from Equation II:

$$10x - x = 3.33\ldots - 0.33\ldots$$
$$9x = 3 \Rightarrow x = \frac{1}{3}$$

Method II: Recurring Notation

$$\underbrace{x = 0.\overline{3}}_{Equation\ I}$$

Multiply both sides by 10:

$$\underbrace{10x = 3.\overline{3}}_{Equation\ II}$$

Subtract Equation I from Equation II:

$$\underbrace{10x - x = 3.\overline{3} - 0.\overline{3}}_{Equation\ II - Equation\ I}$$

$$9x = 3 \Rightarrow x = \frac{3}{9} = \frac{1}{3}$$

Example 1.68

A. $0.\overline{4}$
B. $0.\overline{5}$

Part A

$$x = 0.44\ldots$$
$$10x = 4.44\ldots$$

$$9x = 4$$
$$x = \frac{4}{9}$$

1.69: Shortcut

If a is a single digit, then:

$$0.\bar{a} = \frac{a}{9}$$

$$x = 0.aa\ldots$$
$$10x = a.aa\ldots$$

$$9x = a$$

$$x = \frac{a}{9}$$

Example 1.70

1.71: Two recurring digits

If two digits are recurring, we multiply the recurring decimal with 100 (instead of 10)/

Example 1.72

Convert the following into fractions:

A. $0.\overline{23}$
B. $0.\overline{57}$

Part A

$$\underbrace{x = 0.\overline{23}}_{I}$$

Two digits are recurring, so we multiply both sides of the equation by 100:

$$\underbrace{100x = 23.\overline{23}}_{II}$$

Subtract Equation I from Equation I:

$$\underbrace{99x = 23}_{II-I}$$
$$x = \frac{23}{99}$$

Part B

$$x = \frac{57}{99}$$

Example 1.73

A. $0.0\overline{29}$
B. $0.00\overline{29}$

$$0.0\overline{29} = \frac{1}{10} \cdot 0.\overline{29} = \frac{1}{10} \cdot \frac{29}{99} = \frac{29}{990}$$
$$0.00\overline{29} = \frac{1}{100} \cdot 0.\overline{29} = \frac{1}{100} \cdot \frac{29}{99} = \frac{29}{9900}$$

1.74: Shortcut

If a and b are single digits, then:

$$0.\overline{ab} = \frac{ab}{99}$$

Example 1.75

Convert the following into fractions:

A. $0.\overline{01}$

B. $0.\overline{396}$
C. $0.\overline{142857}$

Part A

$$\underbrace{x = 0.\overline{\mathbf{01}}}_{I} \Rightarrow \underbrace{100x = 1.\overline{\mathbf{01}}}_{II} \Rightarrow \underbrace{99x = 1}_{II-I} \Rightarrow x = \frac{1}{99}$$

Two digits are recurring, so we multiply both sides of the equation by 100.

Part C

$$\underbrace{x = 0.\overline{396}}_{\textit{Equation I}}$$

Three digits are recurring, so we multiply both sides of the equation by 1000.

$$\underbrace{1000x = 396.\overline{396}}_{\textit{Equation II}}$$

Subtract Equation I from Equation II:

$$999x = 396$$
$$x = \frac{396}{999} = \frac{44}{111}$$

Part D

Six digits are recurring, so we multiply both sides of the equation by 1,000,000.

$$\underbrace{x = 0.\overline{\mathbf{142857}}}_{I} \Rightarrow \underbrace{1{,}000{,}000x = 142857.\overline{\mathbf{142857}}}_{II} \Rightarrow \underbrace{999{,}999x = 142857}_{II-I} \Rightarrow x = \frac{142{,}857}{999{,}999} = \frac{1}{7}$$

1.76: Shortcut

$$\underbrace{0.\overline{abc\ldots n}}_{n \text{ digits are recurring}} = \frac{abc\ldots n}{10^n - 1} = \frac{abc\ldots n}{\underbrace{999\ldots 9}_{n\ 9's}}$$

From the above examples, we can see that

- The recurring digits go in the numerator of the fraction
- The denominator has as many 9's as the number of recurring digits

It's equally important to understand the process by which the shortcut was arrived at. Because competition questions are not likely to be simple enough to be solved using just the shortcut. Application-based questions will check your understanding.

Example 1.77

Evaluate:

A. $0.\overline{4}$
B. $0.\overline{7}$
C. $0.\overline{9}$
D. $0.\overline{62}$
E. $0.\overline{38}$
F. $0.\overline{05}$
G. $0.\overline{214}$

$$0.\overline{4} = \frac{4}{9}$$
$$0.\overline{7} = \frac{7}{9}$$
$$0.\overline{9} = \frac{9}{9} = 1$$
$$0.\overline{62} = \frac{62}{99}$$
$$0.\overline{38} = \frac{38}{99}$$
$$0.\overline{05} = \frac{5}{99}$$
$$0.\overline{214} = \frac{214}{99}$$

Example 1.78: Breaking up numbers to use the Shortcut

Convert $3.\overline{41}$ into a fraction

$$3.\overline{41} = 3 + 0.\overline{41} = 3 + \underbrace{\frac{41}{99}}_{\substack{\textit{Using the}\\ \textit{Shortcut}}} = \frac{3}{1} + \frac{41}{99} = \frac{338}{99}$$

Example 1.79

Convert $0.\overline{3} + 0.\overline{12}$ into a fraction

$$0.\overline{3} + 0.\overline{12} = \frac{3}{9} + \frac{12}{99} = \frac{33}{99} + \frac{12}{99} = \frac{45}{99} = \frac{5}{11}$$

Example 1.80

When $0.\overline{2} + 0.\overline{08}$ is converted into a fraction, it can be written in the form $\frac{a}{b}$, where a and b are integers, and $HCF(a, b)$ is one. Find $a + b$.

$$0.\overline{2} + 0.\overline{08} = \frac{2}{9} + \frac{8}{99} = \frac{22+8}{99} = \frac{30}{99} = \frac{10}{33} \Rightarrow a + b = 10 + 33 = 43$$

E. Converting where the initial digit is not recurring

If the initial digit is not recurring, it is best to go back to the algebraic method, rather than use the short-cut. Also, you will quite possibly end up decimals in a fraction, so you have to be a little more careful with the calculations.

Example 1.81: Single Digit is Recurring

A. Convert $0.1\overline{6}$ into a fraction.
B. Convert $0.2\overline{5}$ into a fraction.

Part A

$$0.1\overline{6} = 0.1 + 0.0\overline{6} = \frac{1}{10} + \underbrace{\frac{2}{30}}_{\substack{From\\Below}} = \frac{3+2}{30} = \frac{5}{30} = \frac{1}{6}$$

$$\underbrace{x = 0.0\overline{6}}_{I} \Rightarrow \underbrace{10x = 0.6\overline{6}}_{II} \Rightarrow \underbrace{9x = 0.6}_{II-I} \Rightarrow x = \frac{0.6}{9} = \frac{6}{90} = \frac{2}{30}$$

We multiply by the number of digits that are recurring. This allows us to eliminate the recurring digits in an elegant way.

Part B

$$0.2\overline{5} = 0.2 + 0.0\overline{5} = \frac{2}{10} + \frac{5}{90} = \frac{18}{90} + \frac{5}{90} = \frac{23}{90}$$

$$\underbrace{x = 0.0\overline{5}}_{I} \Rightarrow \underbrace{10x = 0.5\overline{5}}_{II} \Rightarrow \underbrace{9x = 0.5}_{II-I} \Rightarrow x = \frac{0.5}{9} = \frac{5}{90}$$

Example 1.82: Two Digits are Recurring

Convert $0.0\overline{15}$ into a fraction

$$\underbrace{x = 0.0\overline{15}}_{I} \Rightarrow \underbrace{100x = 1.5\overline{15}}_{II} \Rightarrow \underbrace{99x = 1.5}_{II-I} \Rightarrow x = \frac{1.5}{99} = \frac{15}{990} = \frac{1}{66}$$

F. Cyclicity in Recurring Decimals

Any unit's digit under multiplication exhibits a regular pattern, and is hence, cyclical. Apart from the cyclicity in the unit's digit, a non-terminating, recurring decimal also exhibits repetition, and is hence, cyclical.

Example 1.83: Cyclicity

Consider the decimal expansion of $\frac{1}{7}$.

A. Find the cyclicity.
B. Let $\frac{1}{7}$ have decimal representation $0.\overline{abc \dots n}$, where $a, b, c \dots n$ are decimal digits. Find
C. $a + b + c + \cdots + n$
D. Find the face value of the digit which is at a position six million, six hundred thousand, six hundred and sixty-five digits to the right of the decimal point in the decimal expansion.
E. The sum of the face value (FV) and the place value (PV) of the 27^{th} digit to the right of the decimal point of $\frac{1}{7}$ can be written in the form $a.000 \dots 0a$ where a is a decimal digit, and there are x zeros separating the two a's. Find ax.

$$\text{Part A: } \frac{1}{7} = 0.\underbrace{\overline{142857}}_{Cycle\ of\ 6} = 0.\underbrace{142857}_{1st\ Cycle}\underbrace{142857}_{2nd\ Cycle}\dots$$

$$\text{Part B: } \frac{1}{7} = 0.\underbrace{\overline{142857}}_{Cycle\ of\ 6} \Rightarrow Required\ Sum = 1 + 4 + 2 + 8 + 5 + 7 = 27$$

$$\text{Part C: } \frac{1}{7} = 0.\underbrace{\underbrace{1}_{\substack{Pos\ 1\\Rem\ 1}}\ \underbrace{4}_{\substack{Pos\ 2\\Rem\ 2}}\ \underbrace{2}_{\substack{Pos\ 3\\Rem\ 3}}\ \underbrace{8}_{\substack{Pos\ 4\\Rem\ 4}}\ \underbrace{5}_{\substack{Pos\ 5\\Rem\ 5}}\ \underbrace{7}_{\substack{Pos\ 6\\Rem\ 0}}}_{Remainder\ on\ dividing\ position\ by\ 6}$$

$$\frac{6{,}600{,}665}{6} = \frac{6{,}600{,}660 + 5}{6} = Some\ Number + \frac{5}{6} \Rightarrow Remainder\ 5 \Rightarrow Digit\ 5$$

Part D: $\Rightarrow \underbrace{27^{th}\text{digit is } 2}_{27\div 6=4\ R3} \Rightarrow FV + PV = 2 + 2 \times 10^{-27} = 2.\underbrace{000\ldots0}_{26\ Zeros}2 \Rightarrow ax = 2 \times 26 = 52$

Example 1.84

What is the 100^{th} digit to the right of the decimal point in the decimal form of $\frac{4}{37}$? (AMC 8 1995/15)

$$\frac{4}{37} = 0.\overline{108} \Rightarrow Cyclicity\ of\ 3$$

$$\frac{100}{3} = 33\frac{1}{3} \Rightarrow Remainder = 1 \Rightarrow Digit = 1$$

	1	0	8	1	0	8	1	0	8
	1	2	3	4	5	6	7	8	9
Remainder	1	2	0	1	2	0	1	2	0

G. Properties of Numbers

1.85: Addition and Subtraction with Rational Numbers

Rational $\pm$ Rational $=$ Rational

$$5 + 6 = 11$$
$$\frac{22}{7} + \frac{2}{7} = \frac{24}{7}$$
$$0.123 + 3.006 = 3.129$$
$$\frac{12}{7} - \frac{5}{7} = \frac{7}{7} = 1$$

1.86: Multiplication and Division with Rational Numbers

Rational $\times$ Rational $=$ Rational

$$3 \times 4 = 12$$
$$\frac{6}{5} \times \frac{2}{7} = \frac{12}{35}$$
$$0.2 \times 0.3 = 0.06$$

1.87: Irrational and Rational: Addition and Subtraction

$$Rational \pm Irrational = Irrational$$

$$2 \pm \pi$$
$$e \pm 4$$

1.88: Irrational and Rational: Multiplication

$$Rational \times Irrational$$

Case I: Rational Number is Zero: Answer is Zero $\Rightarrow$ Rational

Case II: Rational Number is Non $-$ Zero: Answer is Irrational

Example 1.89

Classify each of the elements using the table above:

A. π
B. e
C. $\frac{22}{7}$
D. 2.71
E. $\pi-\frac{22}{7}$
F. 0.31
G. $0.\overline{31}$
H. 0.175175

	Recurring	Non-Recurring
Terminating (Ends)	$\{0.175175\}$	$\{2.71, 0.31\}$
Non-terminating (Does not end)	$\left\{\frac{22}{7}, 0.\overline{31}\right\}$	$\left\{\pi, e, \pi-\frac{22}{7}\right\}$

2. Non-Decimal Bases

2.1 Converting to Base 10

A. Real Life Bases

2.1: Converting Time

Days and Weeks

$$\underbrace{12\ weeks\ 3\ days}_{Convert\ to\ days} = 12 \times 7 + 3 = 84 + 3 = 87_{days}$$
$$\underbrace{39\ days}_{Convert\ to\ weeks\ and\ days} = 7 \times 5 + 4 = 5_{weeks}\ and\ 4_{days}$$

Hours and Days

$$\underbrace{3\ days\ and\ 4\ hours}_{Convert\ to\ hours} = 24 \times 3 + 4 = 72 + 4 = 76_{hours}$$
$$\underbrace{51\ hours}_{Convert\ to\ days\ and\ hours} = 48 + 3 = 24 \times 2 + 3 = 2_{days}\ and\ 3_{hours}$$

Seconds, minutes, hours and days

$$\underbrace{1\ hour}_{Convert\ to\ seconds} = 60\ m = 60 \times 60\ s = 3600\ s$$
$$\underbrace{4234}_{Convert\ to\ h:m:s} = \underbrace{3600}_{1\ Hour} + 634\ s = 1\ h\ 600\ s + 34\ s = 1h\ 10\ m\ 34s$$
$$\underbrace{1\ day}_{Convert\ to\ seconds} = 24\ hours = 1440\ minutes = 86{,}400\ seconds$$

2.2: Counting Feet and Inches

$$1\ Foot = 12\ Inches$$

Example 2.3

Convert the following measurements given in feet and inches into inches:

A. 2 feet 11 inches
B. 5 feet 3 inches

$$2_{ft}11_{inches} = 2 \times 12 + 11 = 24 + 11 = 35_{inches}$$
$$5_{ft}3_{inches} = 5 \times 12 + 3 = 60 + 3 = 63_{inches}$$

2.4: Counting Bananas

When counting bananas, we count in dozens (by 12), and not by 10.

$$1\ dozen\ dozen = 1\ Gross$$

Example 2.5

Convert the following numbers which are given in gross/dozen into decimal:

A. 1_{Gross}
B. $5\ Dozen + 4\ Units$
C. $3.5\ Dozen$

$$1_{Gross} = 12 \times 12 = 144_{Decimal}$$
$$5_{Dozen} \times 12 + 4_{Units} = 60 + 4 = 64$$
$$3.5_{dozen} \times 12 = 42$$

Example 2.6
A. $2_{ft}\ 10_{in} + 3_{ft}\ 9_{in}$
B. $8_{ft}\ 3_{in} - 5_{ft}\ 10_{in}$
C. One crate of bananas has 5 dozen bananas. If every crate has 4 spoilt bananas, what is the total number of edible bananas in eleven crates.
D. Divide 3 hours, and 7 minutes into two equal parts. Give your answer in hours, minutes and seconds.
E. Divide 2 hours, and 10 minutes into three equal parts. Give your answer in hours, minutes and seconds.

$$2_{ft}\ 10_{in} + 3_{ft}\ 9_{in} = 5_{ft}\ 19_{in} = 5_{ft} + \underbrace{12_{in}}_{1ft} + 7_{in} = 6_{ft}\ 7_{in}$$
$$8_{ft}\ 3_{in} - 5_{ft}\ 10_{in} = 3_{ft}\ 3_{in} - 10_{in} = 2_{ft}\ 15_{in} - 10_{in} = 2_{ft}\ 5_{in}$$

$$11(5_{dozen} - 4) = 55_{dozen} - 44 = 55_{dozen}\underbrace{-4_{dozen} + 4}_{-48+4=-44} = 51_{dozen} + 4$$
$$\frac{3h}{3} + \frac{7m}{3} = \frac{187m}{2} = 93\frac{1}{2}m = 1\,h\,33\,m\,30s$$

$$\frac{2h}{3} + \frac{10m}{3} = \frac{130m}{3} = 43\frac{1}{3}m = 43m\ 20s$$

Example 2.7
Convert the following measurements given in inches into feet and inches:
A. 29_{inches}
B. 71_{inches}

$$29_{inches} = 2 \times 12 + 5 = 2_{feet}\ 5_{inches}$$
$$71_{inches} = 5 \times 12 + 11 = 5_{feet}\ 11_{inches}$$

B. Non-Decimal Bases (Mathematical)
Decimal number system uses the ten digits that range from $0 - 9$. Number bases with a base less than 10 will use fewer digits, and numbers with a base greater than 10 will use greater digits.

2.8: Range of Digits in Number Bases
In base-b, there will be b digits, ranging from 0 to $b - 1$.

In decimal system, there are ten digits from $0 - 9$, and the largest valid digit is 9.
Digits greater than 9 are represented using letters:

$$A = 10, B = 11, C = 12, D = 13, E = 14, F = 15$$

	Base	No. of Digits	Largest Valid Digit
Hexadecimal	16	16	$F = 15$
Decimal	10	10	9
Octal	8	8	7
Base-22	22	22	$L = 21$
Ternary	3	3	2
Base b	b	b	$b - 1$

Example 2.9

A. What is the range of digits in base 5?
B. What is the range of digits in hexadecimal?

$$0-4 \Rightarrow 5\ Digits$$
$$0-9, A, B, C, D, E, F \Rightarrow 16\ Digits$$

C. Converting to Base 10

2.10: Decimal system

Expanded notation in the decimal system writes a number in terms of powers of ten. If abc is a three-digit number then:

$$abc = 100a + 10b + c = 10^2 \cdot a + 10 \cdot b + c$$

Example 2.11

Write 342_{10} in expanded notation.

$$342 = 300 + 40 + 2 = \underbrace{3}_{FV} \times \underbrace{10^2}_{PV} + \underbrace{4}_{FV} \times \underbrace{10^1}_{PV} + \underbrace{2}_{FV} \times \underbrace{10^0}_{PV}$$

2.12: Non-Decimal Bases

$$xyz_b = xb^2 + yb + z$$

Example 2.13

Write the following in expanded notation:

A. Three gross, and four dozen and two bananas.
B. 342_{12}
C. 123_{12}
D. 571_{12}

$$3 \times 144 + 4 \times 12 + 2 = 432 + 48 + 2 = 482\ Bananas$$
$$342_{12} = 3 \times 12^2 + 4 \times 12^1 + 2 = 432 + 48 + 2 = 482_{10}$$
$$123_{12} = 1 \times 144 + 2 \times 12 + 3 = 144 + 24 + 3$$
$$571_{12} = 5 \times 144 + 7 \times 12 + 1$$

Example 2.14

Write the following in expanded notation:

A. 342_5
B. 342_3

Part A

$$342_5 = 3 \times 5^2 + 4 \times 5 + 2 = 3 \times 25_{10} + 4 \times 5_{10} + 2 \times 1_{10} = 97_{10}$$

Part B

$$342_5 = 3 \times 3^2 + 4 \times 2 + 2$$

But, the largest valid digit in base−3 is 2.

Hence, this is not a valid number in base 3.

Example 2.15

Write 342_b in expanded notation. What is the valid range of digits for b?

$$342 = 3b^2 + 4b + 2, \qquad b \geq 5, \qquad b \in \mathbb{N}$$

Example 2.16

Convert to decimal:

A. 23_4
B. 123_5
C. 314_7
D. 101_2

$$23_4 = 2 \times 4 + 3 \times 1 = 8 + 3 = 11_{10}$$
$$123_5 = 1 \times 25 + 2 \times 5 + 3 \times 1 = 25 + 10 + 3 = 38_{10}$$
$$314_7 = 3 \times 7^2 + 1 \times 7 + 4 = 158_{10}$$
$$101_7 = 1 \times 2^2 + 0 \times 2 + 1 = 5_{10}$$

Example 2.17

Convert to decimal:

A. 2101_3
B. 3210_4

$$2101_3 = 2 \times 3^3 + 1 \times 3^2 + 0 \times 3 + 1 = 54 + 9 + 1 = 64$$
$$3210_4 = 3 \times 4^3 + 2 \times 4^2 + 1 \times 4 + 0 = 192 + 32 + 1 + 0 =$$

Example 2.18

The base of the decimal number system is ten, meaning, for example that $123 = 1 \cdot 10^2 + 2 \cdot 10 + 3$. In the binary system, which has base two, the first five positive integers are 1, 1/0, 11, 100, 101. The numeral 10011 in the binary system would then be written in the decimal system as: **(AHSME 1957/19)**

$$1 \cdot 2^4 + 1 \cdot 2^1 + 1 \cdot 2^0 = 16 + 2 + 1 = 19$$

(Calculator) Example 2.19

Write in expanded notation.

1. $F3A_{16}$
2. ABC_{13}
3. EAB_{17}

$$F3A_{16} = 15 \times 16^2 + 3 \times 16 + 10 \times 1 = 3840 + 48 + 10 = 3898_{10}$$
$$ABC_{13} = 10 \times 13^2 + 11 \times 13 + 12 \times 1 =$$
$$EAB_{17} = 14 \times 17 + 10 \times 17 + 11 \times 1 =$$

Example 2.20

A codebreaker has the code $X00$ in front of him. He thinks it represents a three-digit number ending in two zeroes in the number base b, $1 < b < 6, b \in \mathbb{N}$ and X is a valid digit in base b. The sum of all possible values of the code, written in decimal, is Y . Determine the sum of the digits of Y.

The valid numbers are:

$$100_2 + \underbrace{100_3 + 200_3}_{Base-3} + \underbrace{100_4 + 200_4 + 300_4}_{Base-4} + \underbrace{100_5 + 200_5 + 300_5 + 400_5}_{Base-5}$$

Converting to decimal:

$$4 + 9 + 18 + 16 + 32 + 48 + 25 + 50 + 75 + 100$$

Factoring:

$$= 4 + 9(1 + 2) + 16(1 + 2 + 3) + 25(1 + 2 + 3 + 4)$$

Simplifying:

$$= 4 + 9(3) + 16(6) + 25(10) = 4 + 27 + 96 + 250 = 377$$

$$Sum\ of\ digits\ 3 + 7 + 7 = 17$$

2.2 Converting to Other Bases

A. Converting to Other Bases

Example 2.21

Convert the following numbers which are given in decimal system into gross/dozen:

A. 23
B. 143
C. 144

$$23 = 12 \times 1 + 11 = 1_{dozen} + 11$$
$$143 = 144 - 1 = 12_{dozen} - 1 = 11_{dozen} + 11$$

Example 2.22

Convert 43 into base-12

$$43 \div 12 = 3\ Remainder\ 7 \Rightarrow 43 = 12 \times 3 + 7 = 3_{dozen} + 7$$

In Base 12, this is how we write. Convert 43_{10} to 43_{12}

$$43_{10} = 12 \times 3 + 7 = 30_{12} + 7 = 37_{12}$$

Example 2.23

Convert the following numbers into Base-12

A. 52
B. 33
C. 12
D. 60
E. 144

$$52 = 44_{12} = 4 \times 12 + 4 = 4\ Dozen + 4$$
$$33 = 29_{12} = 2 \times 12 + 9 \times 1 = 2\ Dozen + 9$$
$$12 = 10_{12} = 1 \times 12 + 0 \times 1 = 1\ Dozen + 0$$
$$60 = 50_{12} = 5 \times 12 + 0 \times 1 = 5\ Dozen + 0$$
$$144 = 1 \times 12^2 = 100_{12}$$

Example 2.24

In our number system the base is ten. If the base were changed to four, you would count as follows: 1,2,3,10,11,12,13,20,21,22,23,30, … The twentieth number would be: **(AHSME 1956/31)**

Method I

We write 20 as the sum of powers of

$$20 = 16 + 4 + 0 = 1 \cdot 16 + 1 \cdot 4 + 0 \cdot 1 = \mathbf{1} \cdot 4^2 + \mathbf{1} \cdot 4^1 + \mathbf{0} \cdot 4^0 = 110_4$$

Method II

$$20 \div 16 = 1R4$$
$$4 \div 4 = 1R0$$
$$0 \div 1 = 0R0$$

$$110_4$$

Example 2.25

In the numeration system with base 5, counting is as follows: 1,2,3,4,10,11,12,13,14,20, ... The number whose description in the decimal system is 69, when described in the base 5 system is a number with:

A. Two consecutive digits
B. Two non-consecutive digits
C. Three consecutive digits
D. Three non-consecutive digits
E. Four digits **(AHSME 1960/16)**

$$69 \div 25 = 2R19$$
$$19 \div 5 = 3R4$$
$$4 \div 1 = 4R0$$
$$234_5$$
$$Option\ C$$

Example 2.26

Convert 13_{10} to Binary

$$13_{10}$$

The largest number which is a power of two, which is less than 12, is 8.

$$13_{10} = 1 \times 8 + 5$$

The largest number which is a power of two, which is less than 5, is 4.

$$13_{10} = 1 \times 8 + 1 \times 4 + 1 = 1101_2$$

Example 2.27

Convert 999_{10} to Binary

Method I

$$999 \div 512 = \mathbf{1}\ \mathrm{R}487$$
$$487 \div 256 = \mathbf{1}\ \mathrm{R}231$$
$$231 \div 128 = \mathbf{1}\ \mathrm{R}103$$
$$103 \div 64 = \mathbf{1}\ \mathrm{R}39$$
$$39 \div 32 = \mathbf{1}\ \mathrm{R}7$$
$$7 \div 16 = \mathbf{0}\ \mathrm{R}7$$
$$7 \div 8 = \mathbf{0}\ \mathrm{R}7$$
$$7 \div 4 = \mathbf{1}\ \mathrm{R}3$$
$$3 \div 2 = \mathbf{1}\ \mathrm{R}1$$

$$1 \div 1 = \mathbf{1}\ R0$$

The red digits are the digits in Binary.
Bring them together to get the Binary Value:

$$1111100111_2$$

Example 2.28

Convert 27_{10} to Base 4:

$$27_{10} = 1 \times 16 + 11 = 1 \times 16 + 2 \times 4 + 3 \times 1 = 123_4$$

Convert to Hexadecimal:

$$17_{10} = 1 \times \mathbf{16} + 1 \times 1 = 11_{16}$$
$$255_{10} = 15 \times \mathbf{16} + 15 \times \mathbf{1} = FF_{16}$$
$$257_{10} = 1 \times \mathbf{16^2} + 0 \times \mathbf{16^1} + 1 \times \mathbf{16^0} = 101_{16}$$

Example 2.29

Convert 237 to binary.

Place Value	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
	256	128	64	32	16	8	4	2	1
	0	1	1	1	0	1	1	0	1

B. Conversion Drill Work

2.30: Place Value in Binary

	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
	256	128	64	32	16	8	4	2	1

Example 2.31

Write the numbers from 1 to 20 in Base-2.

Base 10	2^4	2^3	2^2	2^1	2^0	
		8	4	2	1	
0					0	$0 \times 2^0 = 0 \times 1 = 0$
1					1	$1 \times 2^0 = 1 \times 1 = 1$
2				1	0	$1 \times 2^1 + 0 \times 2^0 = 1 \times 2 + 0 \times 1 = 2$
3				1	1	
4			1	0	0	
5			1	0	1	
6			1	1	0	
7			1	1	1	
8		1	0	0	0	
9		1	0	0	1	
10		1	0	1	0	
11		1	0	1	1	
12		1	1	0	0	
13		1	1	0	1	
14		1	1	1	0	

15		1	1	1	1	
16	1	0	0	0	0	
17	1	0	0	0	1	
18	1	0	0	1	0	
19	1	0	0	1	1	
20	1	0	1	0	0	

Example 2.32

Write the numbers from 1 to 20 in Base-3.

Base -10	3^2	3^1	3^0
	9	3	1
0			0
1			1
2			2
3		1	0
4		1	1
5		1	2
6		2	0
7		2	1
8		2	2
9	1	0	0
10	1	0	1
11	1	0	2
12	1	1	0
13	1	1	1
14	1	1	2
15	1	2	0
16	1	2	1
17	1	2	2
18	2	0	0
19	2	0	1
20	2	0	2

Example 2.33

Write the numbers from 1 to 20 in Base-4.

Base -10	4^2	4^1	4^0
	16	4	1
0			0
1			1
2			2
3			3
4		1	0
5		1	1
6		1	2
7		1	3
8		2	0

9		2	1
10		2	2
11		2	3
12		3	0
13		3	1
14		3	2
15		3	3
16	1	0	0
17	1	0	1
18	1	0	2
19	1	0	3
20	1	1	0

C. Mixed Conversions

Mixed conversions involve require converting from one non-decimal base into another non-decimal base. The most direct way to do this is to use the decimal system as an intermediate step. A mixed conversion question then simplifies to two conversion questions:

- Converting from a non-decimal system in base y to base 10
- Converting from base 10 to a non-decimal system in base x

Example 2.34

Find the variables for each element in the set: $\{34_6 = x_8, 234_{11} = y_9\}$. x_8 means a number in base-8.

$$34_6 = 22_{10} = 26_8, \qquad 234_{11} = 279_{10} = 340_9$$

D. Conversion Properties

2.35: Converting b^n

In base b, b^n will be written as

$$\underbrace{100\dots0}_{n+1\ zero's}$$

$$343_7 = (7^3)_{Base\ 7} = 1000$$

Example 2.36

Convert 729_{10} to Base 3 and Base 9

$$729_{10} = 1 \times 9^3 + 0 \times 9^2 + 0 \times 9^1 + 0 \times 9^0 = 1{,}000_3$$
$$729_{10} = 1 \times 3^6 = 1{,}000{,}000_3$$

2.37: Converting $b^n - 1$

In base b,

$b^n - 1$ will be written as $\underbrace{(b-1)(b-1)\dots(b-1)}_{n\ times}$

$$3_{10} = 11_2$$
$$7_{10} = 111_2$$

Example 2.38

$$15_{10} = 16_{10} - 1 = 2^4 - 1 = 10000 - 1 = 1111_2$$

Example 2.39

Convert 999_{10} to Binary

$$\begin{gathered} 999 \\ = 1023 - 24 \\ = 1111111111 - 16_{10} - 8_{10} \\ = 1111100111 \end{gathered}$$

Example 2.40

A. Convert 256 to binary.
B. Convert 255 to binary.

$$\begin{gathered} 100{,}000{,}000_2 \\ 11{,}111{,}111 \end{gathered}$$

Example 2.41

Convert the following numbers, which are written in the decimal system, into the bases indicated below the numbers.

A. 24 in Base-5
B. 26 in Base 3
C. 63 in base-4 and also base-8
D. 624 in base 5
E. 79 in base 9

$$, \underbrace{63}_{\substack{Base-4 \\ Base-8}}, \underbrace{624}_{Base-5}, \underbrace{79}_{Base-9}, \underbrace{214}_{Base-6}, \underbrace{2046}_{Base-2}, \underbrace{2400}_{Base-7}$$

$$\begin{aligned} 24_{10} &= 25_{10} - 1_{10} = 100_5 - 1_5 = 44_5 \\ 26_{10} &= 27_{10} - 1_{10} = 1000_3 - 1_3 = 222_3 \\ 63_{10} &= 64_{10} - 1_{10} = 1000_4 - 1_4 = 333_4 \\ 63_{10} &= 64_{10} - 1_{10} = 100_4 - 1_4 = 77_4 \\ 624_{10} &= 625_{10} - 1_{10} = 10000_5 - 1_5 = 4444_5 \\ 79_{10} &= 81_{10} - 2_{10} = 100_9 - 2_9 = 87_9 \\ 214_{10} &= 216_{10} - 2_{10} = 1000_6 - 2_6 = 554_6 \\ 2046_{10} &= 2048_{10} - 2_{10} = 11111111110_2 \\ 2400_{10} &= 2401_{10} - 1_{10} = 10000_7 - 1_7 = 6666_7 \end{aligned}$$

2.42: Converting from base b^2 to b and vice versa

When converting from base $\boldsymbol{b}$ to $\boldsymbol{b}^2$:

- Each two digits in base $\boldsymbol{b}$ (starting from the rightmost) correspond to a single digit in base $\boldsymbol{b}^2$.

When converting from base $\boldsymbol{b}^2$ to $\boldsymbol{b}$, the reverse applies.

- Each single digit in base b^2 corresponds to two digits in base $\boldsymbol{b}$.

When converting from base x into base x^n, n digits in base x will directly convert into a single digit in base x^n.

In the field of computers, this optimization is particularly important for converting from binary to base-4, octal and hexadecimal and also vice versa.

Consider base b^2:
The range of digits is:

$$0 \text{ to } (b^2 - 1)$$

Hence, the maximum value of the rightmost digit is:

$$\boldsymbol{b^2 - 1}$$

Consider base b:
The maximum value of any digit in base b is

$$b - 1$$

Hence, the maximum value that a two-digit number can take is when both digits are

$$b - 1$$

Write out the maximum value two-digit number in expanded notation

$$\underbrace{(b-1)}_{Face\ Value} \times \underbrace{b}_{Place\ Value} + \underbrace{(b-1)}_{Face\ Value} \times \underbrace{1}_{Place\ Value} = b^2 - b + b - 1 = \boldsymbol{b^2 - 1}$$

Example 2.43

A. How many digits will xyz_{b^2} have in base b.

xyz_{b^2} has three digits. xyz_b will have six digits.

B. Write xyz_{b^2} in expanded notation.

$$xyz_{b^2} = \underbrace{x \times b^4}_{\substack{Max\ Value = b^7 - 1 \\ Can\ be\ written\ as\ z_b0000}} + \underbrace{y \times b^2}_{\substack{Max\ Value = b^5 - 1 \\ Can\ be\ written\ as\ y_b00}} + \underbrace{z}_{\substack{Max\ Value = b^2 - 1 \\ Needs\ exactly\ two\ digits\ in\ base\ b}}$$

$$\underbrace{xyz_{b^2}}_{\substack{Each\ digit\ here\ represents \\ two\ digits\ in\ base\ b}} \equiv \underbrace{x_b y_b z_b}_{\substack{Find\ the\ value\ of\ each \\ and\ concatenate}}$$

To convert from Base-2 to Base-4, find the Base-4 value of two digits at a time (starting from the right), and then concatenate.

Example 2.44

Convert the following numbers into the indicated bases:

$$\underbrace{0100011110_2}_{Convert\ to\ Base-4} = \underbrace{01}_{1}\,\underbrace{00}_{0}\,\underbrace{01}_{1}\,\underbrace{11}_{3}\,\underbrace{10}_{2} = 10132_4$$

$$\underbrace{011101110010_2}_{Convert\ to\ Octal} = \underbrace{011}_{3}\,\underbrace{101}_{5}\,\underbrace{110}_{6}\,\underbrace{010}_{2} = 3562_8 = \underbrace{35_8}_{29_{10}}\,\underbrace{62_8}_{50_{10}} = (29_{10})(50_{10})_{64}$$

$$\underbrace{100000001111_2}_{Convert\ to\ Hexadecimal} = \underbrace{1000}_{8}\,\underbrace{0000}_{0}\,\underbrace{1111}_{F} = 80F_{16}$$

Example 2.45

A. 212_3 to base 9.

$$0200_3 + 12_3 = 02 \cdot 9 + 5 \cdot 1 = 25_9$$

Example 2.46

The base three representation of x is
$$12112211122211112222$$
The first digit (on the left) of the base nine representation of x is: **(AHSME 1981/16)**

$$12,11,22,11,12,22,11,11,22,22$$

$$12_3 = 5_{10} = 5_9$$

2.3 Expressions and Equations

A. Expressions

2.47: Expanded notation in base b

$$xyz_b = x^2b + yb + z$$

Example 2.48

The number 121_b written in the integral base b, is the square of an integer. Determine the solution set for b, given that b is a positive integer. **(AHSME 1962/22)**

$$121_b = b^2 + 2b + 1 = (b+1)^2$$

The above expression is a perfect square.
Now, we have to ensure valid range of digits. Since 2 is used in the number, the smallest valid b is 3. Hence:
$$b \in \{3,4,5,\dots\}$$

2.49: Factorization of a prime number

The only factorization possible for a prime number p is:
$$p = p \times 1$$

Example 2.50: Factorizing to check for Primality

For how many natural numbers $n \geq 2$ is 1001_n a prime number.

Write 1001_n in expanded notation, and factor
$$n^3 + 1 = (n+1)(n^2 - n + 1)$$

For n to be a prime number, one of the above must be 1, and the other must be a prime.

Case I:
If the second term is 1, then:
$$n^2 - n + 1 = 1$$
$$n^2 - n = 0$$
$$n(n-1) = 0$$
$$n \in \{0,1\} \Rightarrow Not\ Valid$$
Case II:
If the first term is 1, then:
$$n + 1 = 1 \Rightarrow n = 0 \Rightarrow Not\ Valid$$

Hence, there are no valid solutions and hence there are no such numbers that meet the required condition.

Example 2.51: Factorizing to check for Primality

For how many natural numbers $n \geq 2$ is 1010_n a prime number.

$$n^3 + n = n(n^2 + 1)$$
$$\textit{Zero Solutions}$$

$$n = 1 \Rightarrow \textit{Not Valid}$$
$$n^2 + 1 = 1 \Rightarrow n = 0 \Rightarrow \textit{Not Valid}$$

Hence, there are no valid solutions and hence there are no such numbers that meet the required condition.

B. Linear Equations

2.52: Two Digit Numbers

Questions based on two-digit numbers will generally result in linear equations since the highest power of the base, when expanded, will be 1.

2.53: Valid Range of Digits

The valid range of digits for base b is:

$$1,2,3, \dots, (b-1)$$

When solving equations, the valid range of digits must be taken into account.

Example 2.54:

Find the valid positive integral bases b for which $34_b + 21_b = 55_b$.

Write it in expanded notation:

$$(3b + 4) + (2b + 1) = (5b + 5)$$
$$5 = 5 \Rightarrow \textit{Always True}$$

Hence, b can be any valid base.

$$b \geq 6, b \in \mathbb{N}$$

Example 2.55:

The symbol 25_b represents a two-digit number in the base b. If the number 52_b is double the number 25_b, then b is: **(AHSME 1965/15)**

$$2(25_b) = 52_b$$
$$2(2b + 5) = 5b + 2$$
$$4b + 10 = 5b + 2$$
$$b = 8$$

C. Quadratic Equations

2.56: Three Digit Numbers

Questions based on three-digit numbers will generally result in linear equations since the highest power of the base, when expanded, will be 1.

Example 2.57

Find the value of positive integer number b given that $23_b + 12_b = 101_b$.

Write in expanded notation:

$$23_b + 12_b = 101_b$$
$$2b + 3 + b + 2 = b^2 + 1$$

Collate all terms on the LHS to get a quadratic:

$$b^2 - 3b - 4 = 0$$

Solve the quadratic to get the values of b:

$$b \in \{4, -1\}$$

Hence, the final answer is:

$$b = 4$$

Verify:

$$23_b + 12_b = 8 + 3 + 4 + 2 = 17$$
$$101_b = 16 + 0 + 0 = 17$$

Example 2.58

If $131_b = 55_{10}$, find the possible values of 55_b in base-10.

Method I:

Write the LHS of the given equation to expanded notation

$$b^2 + 3b + 1 = 55$$

Solve the resulting quadratic to find b:

$$(b + 9)(b - 6) = 0$$
$$b = \{6, -9\}$$
$$b = 6 \Rightarrow 55_6 = 6 \times 5 + 5 = 35$$
$$b = -9 \Rightarrow 55_{-9} = -45 + 5 = -40$$

Method II (Logic):

$$\underbrace{5}_{55_b\ exists} < b < \underbrace{10}_{131<55} \Rightarrow \underbrace{131_6 = 36 \times 1 + 3 \times 6 + 1 = 55 \Rightarrow b = 6}_{\text{Trying } 6}$$

Example 2.59

A number N in base 10, is 503 in base b, and 305 in base $b + 2$. What is the product of the digits of N? **(IOQM 2021/2)**

$$503_b = 305_{b+2}$$

Formulate a quadratic and expand:

$$5b^2 + 3 = 3(b + 2)^2 + 5$$

Expand:

$$5b^2 + 3 = 3b^2 + 12b + 12 + 5$$

Collate like terms on one side and solve the quadratic:

$$2b^2 - 12b - 14 = 0$$
$$b^2 - 6b - 7 = 0$$

$$(b-7)(b+1)=0$$
$$b \in \{7,1\}$$

$$N = 503_7 = 5 \times 7^2 + 3 = 245 + 3 = 248 \Rightarrow Product = 64$$

Example 2.60

In the base ten number system the number 526 means $5 \times 10^2 + 2 \times 10 + 6$. In the land of Mathesis, however, numbers are written in the base r. Jones purchases an automobile there for 440 monetary units (abbreviated m.u). He gives the salesman a 1000 m.u bill and receives, in change 340 m.u. The base r is: **(AHSME 1961/17)**

$$440_r + 340_r = 1000_r$$
$$4r^2 + 4r + 3r^2 + 4r = r^3$$
$$r^3 - 7r^2 - 8r = 0$$

Divide by r, since $r \neq 0$:

$$r^2 - 7r - 8 = 0$$
$$(r-8)(r+1) = 0$$
$$r = 8$$

Example 2.61

If 554 is the base b representation of the square of the number whose base b representation is 24, then b, when written in base 10 equals: **(AHSME 1973/6)**

Using the given condition:

$$(24_b)^2 = 554_b$$

Writing in expanded notation:

$$(2b+4)^2 = 5b^2 + 5b + 4$$

Expanding:

$$4b^2 + 16b + 16 = 5b^2 + 5b + 4$$

Collate all terms on one side:

$$b^2 - 11b - 12 = 0$$

Factor:

$$(b-12)(b+1) = 0$$

Solve:

$$b \in \{-1,12\} \Rightarrow b = 12$$

2.4 Number Theory

A. Diophantine Equations

Example 2.62

Suppose that $XYZ_4 + 200_{10} = XYZ_9$, where X, Y, and Z are valid digits in base 4 and 9. What is the sum when you add all possible values of X, all possible values of Y, and all possible values of Z?

$X, Y, Z \in \underbrace{\{0,1,2,3\}}_{\text{Valid Digits in Base 4}}$

Expand $XYZ_4 + 200_{10} = XYZ_9$:

$$16X + 4Y + Z + 200 = 81X + 9Y + Z \Rightarrow 65X + 5Y = 200 \Rightarrow \underbrace{13X + Y = 40}_{\substack{\text{Diophantine} \\ \text{Two variables} \\ \text{One Equation}}}$$

$$X = 3 \Rightarrow Y = 1, X = 2 \Rightarrow Y = \mathbf{14}, X = 1 \Rightarrow Y = \mathbf{27}$$

$XYZ_4 + 200_{10} = XYZ_9$ simplifies to:

$$31Z_4 + 200_{10} = 31Z_9$$

The value of Z_4 is the same as Z_9. Hence, this equation, will hold for all valid values of Z:

$$Z = \{0,1,2,3\}$$

$$\underbrace{\sum X + \sum Y + \sum Z}_{\substack{\text{Summation over all} \\ \text{possible values}}} = 3 + 1 + (0 + 1 + 2 + 3) = 10$$

Example 2.63

If $n_{10} = ABC_7 = CBA_{11}$, then find the largest possible value of n.

$ABC_7 = CBA_{11} \Rightarrow 49A + 7B + C = 121C + 11B + A \Rightarrow 0 = 30C + B - 12A$

$$\underbrace{12A = 30C + B}_{\text{Consider Values of } C} \Rightarrow \begin{cases} C = 0 \Rightarrow \mathbf{A = 1, B = 12} \\ C = 1 \Rightarrow A = 3, B = 6 \\ C = 2 \Rightarrow A = 5, B = 0 \\ C = 3 \Rightarrow \mathbf{A = 8}, B = 6 \end{cases}$$

$$B = 12A - 30C = 6(2A - 5C)$$

Example 2.64: Recurring Decimals

In base R_1 the expanded fraction F_1 becomes $.373737\ldots$, and the expanded fraction F_2 becomes $.737373\ldots$. In base R_2 fraction F_1 when expanded, becomes $.252525\ldots$ while the fraction F_2 becomes $.525252\ldots$. The sum of R_1 and R_2, each written in the base ten, is: (AHSME 1966/39)

Example 2.65

The numeral 47 in base a represents the same number as 74 in base b. Assuming that both bases are positive integers, the least possible value of $a + b$ written as a Roman numeral, is: (AHSME 1971/11)

Expand $47_a = 74_b$ to get:

$$4a + 7 = 7b + 4 \Rightarrow \underbrace{4a + 3 = 7b}_{\text{Equation I}}$$

Solve for b:

$$b = \frac{4a + 3}{7} = \frac{4a - 4 + 7}{7} = \frac{4(a - 1)}{7} + 1$$

$a - 1$ is a multiple of 7:

$$a - 1 = 0 \Rightarrow a = 1 \Rightarrow Not\ Valid$$

$$a - 1 = 7 \Rightarrow a = 8 \Rightarrow b = \frac{4a + 3}{7} = \frac{4(8) + 3}{7} = \frac{35}{7} = 5 \Rightarrow Not\ Valid$$

$$a - 1 = 14 \Rightarrow a = 15 \Rightarrow b = \frac{4a + 3}{7} = \frac{4(15) + 3}{7} = \frac{63}{7} = 9 \Rightarrow Valid$$

The sum of

$$a + b = 8 + 15 = 24 = XXIV$$

Alternate Method with Mod Arithmetic

Apply mod 7 to both sides of Equation I:

$$4a + 3 \equiv 0 \ (mod\ 7)$$

Add 4 to both sides:

$$4a \equiv 4 \ (mod\ 7)$$
$$a \equiv 1 \ (mod\ 7)$$
$$a \in \{1,8,15\}$$

And the rest of the calculations will follow the method above.

Example 2.66

In base 10, the number 2013 ends in the digit 3. In base 9, on the other hand, the same number is written as $(2676)_9$ and ends in the digit 6. For how many positive integers b does the base-b-representation of 2013 end in the digit 3? (AMC 10A 2013/19)

$$2013 = x_n b^n + x_{n-1} b^{n-1} + \cdots + x_1 b + 3$$

Subtract 3 from both sides:

$$2010 = x_n b^n + x_{n-1} b^{n-1} + \cdots + x_1 b$$

Factor b in the RHS:

$$2010 = b(x_n b^{n-1} + x_{n-1} b^{n-2} + \cdots + x_1)$$

From the RHS, we can see that b must be a factor of 2010.
Using the formula for the number of factors of a number:

$$\tau(2010) = \tau(2^1 \times 3^1 \times 5^1 \times 67^1) = 2 \times 2 \times 2 \times 2 = 16$$

However, we must also apply the condition that $b > 3$, and since the factors of 2010 include 1,2 and 3, the final answer is:

$$16 - 3 = 13$$

B. Number of Digits in base b

2.67: Valid Range of Digits

A number written in the decimal system as x_{10} has $y + 1$ digits when converted to base b

$$where\ b^y\ is\ the\ largest\ power\ of\ b\ that\ is\ smaller\ than\ x_{10}.$$

Example 2.68

Let x be the number of digits in 1000_{10}, when it is written in base b, $2 \le b \le 9$. Find the sum of all possible distinct values of x. Also, find how many times each value of x is repeated.

Base	*Lowest Power of b* > 1000	*Highest Power of b* < 1000	Number of Digits in Base-2
2	$2^{10} = 1024$	$2^9 = 512$	$9 + 1 = 10$
3	$3^7 = 2187$	$3^6 = 729$	7

4	$4^5 = 1024$	$4^4 = 256$	5
5	$5^5 = 3125$	$5^4 = 625$	5

Example 2.69

Find the number of digits in the following numbers when expressed in the given bases

A. 343_{10} in base 8 **3**
B. 512_{10} in base-3 **6**
C. 235_{10} in base 7 **3**
D. 510_{10} in base 5 **4**
E. 111_{10} in base 11 **2**
F. 10000_{10} in base 2 **14**

Example 2.70

What is the sum of all positive integers that have twice as many digits when written in base 2 as they have when written in base 3? Express your answer in base 10.

Decimal	Base-3	Base-2	
1	1	1	
2	2	10	Largest 2-digit Number in Base-3 $= 22_3 = 1000_4$
3	10	11	
4	11	100	
5	12	101	
6	20	110	
7	21	111	
8	22	1000	
9	100	1001	Largest 3-digit Number in Base-3 $= 222_3 = 11010_2$ This has only five digits.
.	.	.	
26	222	11010	

Largest Number	Base 3		Base 2
1 Digit	3-1=2	2 Digits	2
2 Digits	9-1=8	4 Digits	8
3 Digits	27-1=26	6 Digits	32
4 Digits	81-1=80	8 Digits	128

C. Palindromes

Example 2.71: Palindromes

Find the sum, in decimal, of the first four integers that are palindromes when written in Base-2, as well as Base-4.

Any binary palindrome will be odd, since the first digit must be one, and hence, the last digit must also be one:

$$1_4 = 1_2 = 1_{10}$$
$$11_4 = \underbrace{101_2}_{Palindrome} = \mathbf{5_{10}}$$
$$33_4 = \underbrace{1111_2}_{Palindrome} = \mathbf{15_{10}}$$

$$101_4 = \underbrace{10001_2}_{Palindrome} = \mathbf{17_{10}}$$
$$Total - 1 + 5 + 15 + 17 = 38$$

D. Divisibility

2.72: Divisibility by $b - 1$

A number X_b in base b has the digits
$$a_n a_{n-1} \dots a_2 a_1$$

We can write it in expanded notation, and get:
$$P(x) = b^{n-1}a_n + b^{n-2}a_{n-1} + \dots + ba_2 + a_1$$

Method I: Polynomial/Remainders Theorem

A number X_b in base b has the digits
$$a_n a_{n-1} \dots a_2 a_1$$

We can write it in expanded notation, and get:
$$P(x) = b^{n-1}a_n + b^{n-2}a_{n-1} + \dots + ba_2 + a_1$$

The above is a polynomial in b with coefficients $a_n, a_n - 1, \dots, a_2, a_1$. Make use of the remainder theorem:
$$b - 1 = 0 \Rightarrow b = 1$$

Substitute $b = 1$ in $P(x)$ to find the remainder when it is divided by $b - 1$:
$$P(1) = 1^{n-1}a_n + 1^{n-2}a_{n-1} + \dots + (1)a_2 + a_1 = a_n + a_{n-1} + \dots + a_1$$

Method I: Mod Arithmetic
$$b^{n-1}a_n + b^{n-2}a_{n-1} + \dots + ba_2 + a_1\{\,(mod\ b - 1)\}$$

Substitute $b \equiv 1(mod\ b - 1)$
$$1^{n-1}a_n + 1^{n-2}a_{n-1} + \dots + (1)a_2 + a_1\{\,(mod\ b - 1)\}$$
$$a_n + a_{n-1} + \dots + a_2 + a_1\{\,(mod\ b - 1)\}$$

Proof IB: Divisibility by factors of $b - 1$

Digital sum of $\boldsymbol{X_b}$ = Remainder $\left(\frac{X_b}{Any\ factor\ of\ b-1}\right) \Rightarrow \boldsymbol{a_n + a_{n-1} + \dots + a_1 \equiv X_b\{\mathbf{mod}\ (\mathbf{Any\ factor\ of}\ b - 1)\}}$

Step I: From Proof IA, we know that the digital sum gives us the remainder when dividing by $b - 1$.
Step II: If this remainder is divisible by the factor of $b - 1$, then so is X_b.
But dividing X_b directly by a factor of $b - 1$, is equivalent to the two-step process above.

2.73: Divisibility by $b - 1$

Write $\boldsymbol{X_b}$ in expanded notation. The sum of the digits that use odd powers of $\boldsymbol{b}$, subtracted from the sum of the digits that use even powers of $\boldsymbol{b}$, gives the remainder when the number is divided by $\boldsymbol{b + 1}$.

Case I: X has odd number of digits (which means n is odd).

To Prove: $a_n - a_{n-1} + a_{n-2} - a_{n-3} \dots + a_1 \equiv X_b \big(\text{mod } (b+1)\big)$
Substitute $b = -1$ in $P(x)$ to find the remainder when it is divided by $b + 1$:
$P(-1) = (-1)^{n-1}a_n + (-1)^{n-2}a_{n-1} + (-1)^{n-3}a_{n-2} + (-1)^{n-4}a_{n-3} \dots + (-1)a_2 + a_1$
$= a_n - a_{n-1} + a_{n-2} - a_{n-3} + \cdots + a_1$

Writing Assignment **2.1**
Prove the case where n is even.

Example 2.74
Find the sum of all numbers x, whose divisibility in base-37 can be checked using sum of digits.

In base-37, we can check divisibility by 36 (and all factors of 36) using the sum of digits.
Hence, the question is asking for sum of factors of 36.

$$s(36) = s(2^2 \times 3^2) = (1+2+4)(1+3+9) = (7)(13) = 91$$

E. Pythagorean Triplets

Example 2.75
Find the sum of all possible values of the base b, if the following sets of numbers are Pythagorean Triplets

$$3_b, 10_b, 11_b$$

$$3^2 + b^2 = (b+1)^2 \Rightarrow b = 5$$

Example 2.76
Find b, if the following sets of numbers are Pythagorean Triplets

$$12_b, 110_b, 111_b$$

$$(12_b)^2 = (111_b)^2 - (110_b)^2$$

Apply $a^2 - b^2$ on the RHS to rewrite

$$RHS = (111_b + 110_b)(111_b - 110_b) = (221_b)(1_b) = (221_b)$$
$$(12_b)^2 = (221_b)$$
$$(b+2)^2 = 2b^2 + 2b + 1$$
$$b^2 + 4b + 4 = 2b^2 + 2b + 1$$
$$0 = b^2 - 2b - 3$$
$$b = \{3, -1\}$$

F. Divisibility Tests

Example 2.77
Find the number of numbers that satisfy the following conditions for natural number X:

- When X is written in base $b, 1 < b < 20, b \in \mathbb{N}$, there is at least one b for which X becomes a three-digit numbers ending in two zeroes
- When X is written in base $B, 1 < B < 4, B \in \mathbb{N}$, the rightmost digit is a zero for all bases B.

Condition I: The rightmost digit is a zero in base 2

$$Multiple\ of\ 2$$

Condition II: The rightmost digit is a zero in base 3

$$Multiple\ of\ 3$$

Condition III: Any three-digit number ending in two digits is of the form:

$$x00_b = x \cdot b^2$$

The numbers which meet condition III are for $2 \le b \le 9$ are:

	Value of b									
x	2	3	4	5	6	7	8	9	10	
1	4	9	16	25	36	49	64	81		
2		$2 \cdot 9$	$2 \cdot 16$	$2 \cdot 25$	$2 \cdot 36$	$2 \cdot 49$	$2 \cdot 64$	$2 \cdot 81$		
3			$3 \cdot 16$	$3 \cdot 25$	$3 \cdot 36$	$3 \cdot 49$	$3 \cdot 64$	$3 \cdot 81$		
4				$4 \cdot 25$	$4 \cdot 36$	$4 \cdot 49$	$4 \cdot 64$	$4 \cdot 81$		
5					$5 \cdot 36$	$5 \cdot 49$	$5 \cdot 64$	$5 \cdot 81$		
6						$6 \cdot 49$	$6 \cdot 64$	$6 \cdot 81$		
7							$7 \cdot 64$	$7 \cdot 81$		
8								$8 \cdot 81$		
9										
		1	1		5	1	2	4		14

Looking at the above table, we come up with the following set of rules that valid numbers must meet:

- b is a multiple of neither 2 nor 3, x must be multiple of 6 ($example: b = 7$)
- b is a multiple of 2, x must be multiple of 3 ($example: b = 8$)
- b is a multiple of 3, x must be multiple of 2 ($example: b = 9$)
- b is a multiple of 6, x can be any valid number ($example: b = 6$)

$$X00_{10} \Rightarrow 300_{10}, 600_{10}, 900_{10} \Rightarrow 3\ Number$$
$$X00_{11} \Rightarrow 600_{11} \Rightarrow 1\ Number$$
$$X00_{12} \Rightarrow 100_{12}, 200_{12}, \dots, A00_{12}, B00_{12} \Rightarrow 11\ Numbers$$
$$X00_{13} \Rightarrow 600_{13}, C00_{13} \Rightarrow 2\ Numbers$$
$$X00_{14} \Rightarrow 300_{14}, 600_{14}, \dots, C00_{14} \Rightarrow 4\ Numbers$$
$$X00_{15} \Rightarrow x \in \{2,4,6,8,10,12,14\} \Rightarrow 7\ Numbers$$
$$X00_{16} \Rightarrow x \in \{3,6,9,12,15\} \Rightarrow 5\ Numbers$$
$$X00_{17} \Rightarrow x \in \{6,12\} \Rightarrow 2\ Numbers$$
$$X00_{18} \Rightarrow x \in \{1,2,\dots,17\} \Rightarrow 17\ Numbers$$
$$X00_{19} \Rightarrow x \in \{6,12,18\} \Rightarrow 3\ Numbers$$

$$14 + 3 + 1 + 11 + 2 + 4 + 7 + 5 + 2 + 17 + 3 = 69$$

G. Remainders/Last Digit of a Perfect Square

Example 2.78

If the base 8 representation of a perfect square is $ab3c$, where $a \neq 0$, then c equals: (AHSME 1982/26)

A. Formulate an equation using the conditions given in the question.
B. Explain why working mod 8 makes it difficult to answer this question.
C. Answer the question using mod 16.
D. Explain the learning from the question.

Part A: Formulate an Equation

Expanding the number using the base 8 representation gives:

$$ab3c_8 = 512a + 64b + 24 + c$$

Since we want the last digit, c, consider a general number in base 8 which is the square root of $ab3c_8$. We can write it as

$$(8k + r), r \in \{0,1,2,\dots,7\}$$

Square it:

$$(8k + r)^2 = 64k^2 + 16kr + r^2$$

From the condition given in the question, we know that the two are equal:

$$\underbrace{512a + 64b + 24 + c = 64k^2 + 16kr + r^2}_{\textit{Equation I}}$$

Part B: Working *mod* 8

Apply mod 8 to both sides of Equation I

$$c \equiv r^2 \ (mod\ 8)$$

Note that $r \in \{0,1,2,\dots,7\}$. The possible values of r^2 are

r	0	1	2	3	4	5	6	7
r^2	0	1	4	9	16	25	36	49
$r^2 (mod\ 8)$	0	1	4	1	0	1	4	1

$$c \in \{0,1,4\} \Rightarrow \textit{Inconclusive}$$[1]

When we divided by 8, the 24 in the original expression left no remainder. And hence we did not make use of that information.

Part C: Working *mod* 16

Apply mod 16 to both sides of Equation I

$$8 + c \equiv r^2 \ (mod\ 16)$$

Note that $r \in \{0,1,2,\dots,7\}$. The possible values of r^2 are

r	0	1	2	3	4	5	6	7
r^2	0	1	4	9	16	25	36	49
$r^2 (mod\ 16)$	0	1	4	9	0	9	4	1

$$c \in \{0,1,4,9\}$$

Since $c > 0$:

$$8 + c > 8 \Rightarrow 8 + c = 9 \Rightarrow c = 1$$

Part D:

mod 16 worked since it removed the variables while still keeping a remainder from the 24 (digit 3) given in the question.
The choice of remainder (mod) is key in solving a question.

[1] The original AMC question had an option for "cannot be determined conclusively".

2.5 Counting

A. Enumeration

Example 2.79

Determine the number of numbers that:

- are divisible by 8
- have nine digits when written in base-9
- have exactly one zero
- have no digit which is 3 or larger

$$Sum\ of\ digits = 8$$

Eight 1's and one zero. The position of the zero can be in

$$8\ positions \Rightarrow 8\ numbers$$

$$Sum\ of\ digits = 16$$

Eight 2's and one zero. The position of the zero can be in

$$8\ positions \Rightarrow 8\ numbers$$

Total Numbers

$$= 16$$

B. Counting Arguments

Example 2.80

a_1 is a number that is written as 10 in base a_2. a_2 is a number written as 10 in base a_3.

$$a_1 = 10_{a_2}$$
$$a_2 = 10_{a_3}$$
$$a_3 = 10a_4$$

Example 2.81

The number 10! (10 is written in base 10), when written in the base 12 system, ends in exactly k zeroes. The value of k is (AHSME 1970/23)

We need to find the highest power of 12 in 10!.

C. Permutations and Combinations

Example 2.82

Determine the number of numbers that are divisible by 8, and when written in base 9, have nine digits, out of which exactly one digit is zero, and the largest digit 3.

- are divisible by 8
- have nine digits when written in base 9
- have exactly one zero
- have largest digit 3

Case I: Sum of Digits $= 8$

$$Not\ possible \Rightarrow 0\ numbers$$

Case II: Sum of Digits $= 24$

Eight 3's and one zero. The position of the zero can be in

$$8\ positions \Rightarrow 8\ numbers$$

Case III: Sum of Digits $= 16$

One 3, 6 2's, one 1, and one zero:

$$Example: 322222210$$
$$Position\ of\ zero: 8\ Ways$$
$$Position\ of\ 1: 8\ Ways$$
$$Position\ of\ 3: 7\ Ways$$

Total

$$= 8 \times 8 \times 7 = 448$$

Case III-B: Two 3's, 4 2's, two 1's, and one zero:

$$Example: 332222110$$
$$= \underbrace{\binom{8}{1}}_{\substack{\text{Choices}\\ \text{for } 0}} \times \underbrace{\binom{8}{2}}_{\substack{\text{Choices}\\ \text{for } 3}} \times \underbrace{\binom{6}{2}}_{\substack{\text{Choices}\\ \text{for } 1}} = 3360$$

Case III-C: Three 3's, 2 2's, three 1's, and one zero:

$$Example: 333221110$$
$$= \underbrace{\binom{8}{1}}_{\substack{\text{Choices}\\ \text{for } 0}} \times \underbrace{\binom{8}{3}}_{\substack{\text{Choices}\\ \text{for } 3}} \times \underbrace{\binom{5}{3}}_{\substack{\text{Choices}\\ \text{for } 1}} = 4480$$

Case III-D: Four 3's, four 1's, and one zero:

$$Example: 333221110$$
$$= \underbrace{\binom{8}{1}}_{\substack{\text{Choices}\\ \text{for } 0}} \times \underbrace{\binom{8}{4}}_{\substack{\text{Choices}\\ \text{for } 3}} = 560$$

Total:

$$= 0 + 8 + 448 + 3360 + 4480 + 560 = 8856$$

D. Stars and Bars

Example 2.83: Stars and Bars

For positive integers n, denote $D(n)$ by the number of pairs of different adjacent digits in the binary (base two) representation of n. For example,

$$D(3) = D(11_2) = 0$$
$$D(21) = D(10101_2) = 4$$
$$D(97) = D(1100001_2) = 2$$

For how many positive integers less than or equal to 97, does $D(n) = 2$? **(AHSME 1997/30)**

$$n \leq 97$$

Case I: $D(n)$ has seven digits in binary, and two leftmost digits are 1, then only one number is possible

$$D(97) = D(1100001_2) = 2 \Rightarrow 1\ Number$$

Case II: $D(n)$ has seven digits in binary, and second digit is zero

First digit must be 1. Second digit is 0. The possible numbers are:

$$1011111_2, \quad 1001111_2, \quad 1000111_2, \quad 1000011_2, \quad 1000001_2 \Rightarrow 5\ Numbers$$

Case III: $D(n)$ has up to six digits in binary

The number must be of the form:

$$\underbrace{0 \ldots.}_{\substack{Zero\ or \\ more\ 0's}} \quad \underbrace{1 \ldots}_{\substack{Some\ 1's \\ a}} \quad \underbrace{0 \ldots}_{\substack{Some\ 0's \\ b}} \quad \underbrace{1 \ldots}_{\substack{Some\ 1's \\ c}}$$

Let

$$Number\ of\ 1's\ in\ the\ leftmost\ block = a$$
$$Number\ of\ 0's\ in\ the\ middle\ block = b$$
$$Number\ of\ 1's\ in\ the\ rightmost\ block = c$$

$$a + b + c \le 6, \qquad a, b, c \in \mathbb{N}$$

For example:

$$000101 = 101 \Rightarrow a = 1, b = 1, c = 1$$

Use a change of variable to convert from positive solutions to non-negative solutions.
Let $A = a - 1, B = b - 1, C = c - 1$:

$$A + 1 + B + 1 + C + 1 \le 6, \qquad A, B, C \in \mathbb{W}$$
$$A + B + C \le 3, \qquad A, B, C \in \mathbb{W}$$

Convert this from a Diophantine inequality into an equality by introducing an additional (slack) variable:

$$A + B + C + D = 3, \qquad A, B, C, D \in \mathbb{W}$$

Now, this is a Diophantine equality. Solve using Stars and Bars.

$$3\ objects = 3\ stars$$
$$3\ plus\ signs = 3\ bars\ or\ dividers$$

Choose the number of ways that 3 dividers can be arranged in six positions:

$$\binom{6}{3} = \frac{6!}{3!\,3!} = 20 \Rightarrow 20\ Numbers$$

And the final answer is:

$$1 + 5 + 20 = 26\ Numbers$$

E. Recursive Sequences

Example 2.84

Find the zeroes required to write the numbers from 1 to 111 in binary.

Define $f(n)$ to be the number of zeroes in all binary numbers that exactly n digits.

One Digit Numbers	0	0	$f(1)=1$
	1	1	
Two Digit Numbers	2	10	$f(2)=1$
	3	11	
Three Digit Numbers	4	100	$f(3)=\mathbf{2}+2\boldsymbol{f}(\mathbf{2})=2+2=4$
	5	101	
	6	110	
	7	111	
Four Digit Numbers	8	1000	$f(4)=\mathbf{2^{4-2}}+2\boldsymbol{f}(\mathbf{3})=4+2(8)=12$
	9	1001	
	10	1010	
	11	1011	
	12	1100	
	13	1101	
	14	1110	
	15	1111	

The recursion that we have established is:
$$f(n)=2^{n-2}+2f(n-1)$$

$$f(5)=2^{5-2}+2f(4)=8+2(12)=32$$
$$f(6)=2^{6-2}+2f(5)=16+2(32)=80$$
$$f(7)=2^{7-2}+2f(6)=32+2(80)=192$$
$$f(8)=2^{8-2}+2f(7)=64+384=448$$

$$127_{10}=1111111$$
From 1 to 127, the number of zeroes used is:
$$\sum_{i=2}^{7}f(i)=1+4+12+32+80+192=321$$

Since
$$127=112+15=1110000_2+0001111_2$$
The number of zeroes used from 112 to 127 is precisely
$$f(5)=32$$

Hence, the final answer is:
$$321-32=289$$

2.6 Arithmetic

A. Addition and Subtraction

2.85: Addition (No Carryover)

Example 2.86

Place Value	32	16	8	4	2	1
Carryover						

Base 2	1	0	1	0	1	1
-						1
	1	0	1	0	1	0

2.87: Base-10 Addition with Carryover

Place Value	100	10	1
Place	Hundreds	Tens	Units
Carryover	1	1	
Base 10	3	7	9
+		8	5
	4	6	4

Example 2.88

Place Value	32	16	8	4	2	1
Carryover			1	1	1	
Base 2	1	1	0	1	1	1
+						1
	1	1	1	0	0	0

Value of Carryover	$100_8 = 64$	$10_8 = 8$	
Carryover	1	1	
Base 8	3	3	7
+		7	3
	4	3	2

Example 2.89

Base-5 Arithmetic					Base-b Arithmetic			
							$\cancel{10}$ $(b-1)$ 10	10
Borrowing		$\cancel{10}$ 4	10			$\cancel{1}$0	0	0
	$\cancel{1}$ 0	0	0		-			1
-			1				$b-1$	$b-1$
		4	4					

Similarly,

$$1000 - 1 = (b-1)(b-1)(b-1)_b$$

Example 2.90

If N, written in base 2, is 11000, the integer immediately preceding N, written in base 2, is:
(AHSME 1969/3)

$$11000 - 1 = 10000 + 1000 - 1 = 10000 + 111 = 10111$$

Example 2.91: Cryparthmetic

A cryptographer devises the following method encoding positive integers. First, the integer is expressed in base 5. Second a 1-to-1 correspondence is established between the digits that appear in the expressions in base 5 and the elements of the set $\{V, W, X, Y, Z\}$. Using this correspondence, the cryptographers find that three consecutive integers in increasing order are coded as VYZ, VYX, VVW respectively. What is the base-10 expression for the integer coded as XYZ? **(AHSME 1987/16)**

The digits used in base-5 are:

$$\{0,1,2,3,4\}$$

Since $VYX + 1 = VVW$, there must have been a carry forward to change the digit with value 5;

$$\{W = 0, \quad 1,2,3, \quad X = 4\}$$

Since $VYZ + 1 = VYX$

$$Z + 1 = X \Rightarrow Z = 3$$
$$\{W = 0, \quad 1,2, \quad Z = 3, X = 4\}$$

Finally, $VYX + 1 = VVW$

$$Y + 1 = V \Rightarrow Y = 1, V = 2$$
$$\{W = 0, Y = 1, V = 2, Z = 3, X = 4\}$$

$$XYZ = 413_5 = 4 \cdot 5^2 + 1 \cdot 5 + 3 = 100 + 5 + 3 = 108$$

B. Multiplication and Division

Example 2.92: Multiplication

Base-9					
	Carryover (Nines Digit)		4	4	
	Carryover (Ones Digit)		1	2	
			4	5	6
				7	3
		1	4	8	0
+	3	5	3	6	0
Carryover (Addition)			1		
	3	6	8	5	0

Example 2.93: Division

Base-9		1	5	2	1
	5	7	8	1	8

		5			
		2	8		
		2	7		
			1	1	
			1	1	
			0	0	8
					5
					3

Example 2.94
Explain, by converting to base-5, why $5^n - 1$ is always divisible by 4.

Method I: Number Bases

$$5^n_{10} - 1 = 1000\ldots0_5 - 1 = 4444\ldots.4_5 = 4_5(11111\ldots1_5)$$

Method II: Mod Arithmetic

$$5^n \equiv 1^n \equiv 1 (mod\ 4)$$

Subtract 1 from both sides:

$$5^n - 1 \equiv 0\ (mod\ 4)$$

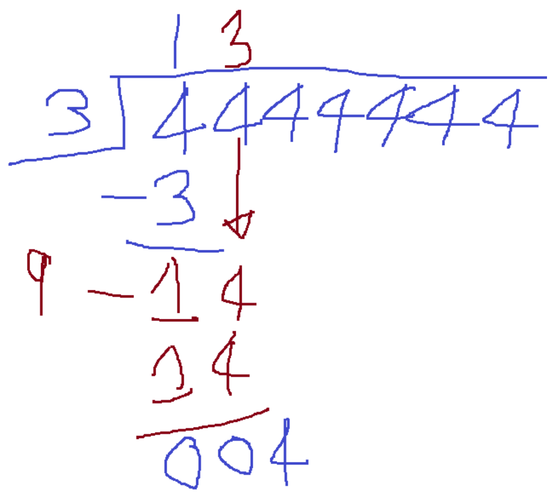
Example 2.95
Explain, by converting to base-5, why $5^n - 1$ is divisible by 3 when n is even and has remainder 1 when divided by 3 when n is odd.

Method I: Number Bases

$$5^n_{10} - 1 = 1000\ldots0_5 - 1 = 4444\ldots.4_5$$
$$4(11111\ldots1_5)$$

Method II: Mod Arithmetic

$$5^n \equiv 2^n \equiv (-1)^n\ (mod\ 3)$$
$$n\ is\ even: 5^n \equiv 1 \Rightarrow\ 5^n - 1 \equiv 0 (mod\ 3) \Rightarrow Divisible$$
$$n\ is\ odd: 5^n \equiv -1 \Rightarrow\ 5^n - 1 \equiv -2 \equiv 1 (mod\ 3)$$
$$\Rightarrow Remainder\ 1\ when\ divided\ by\ 3$$



2.7 Fractions and Decimals

A. Basics

2.96: Working with 10^n in Base-10
The decimal system makes working with powers of 10 easy when we want to multiply and divide. To
- Divide by 10^n, we move the decimal point n times to the left.
- Multiply by 10^n, we move the decimal point n times to the right

$$\frac{54}{10} = 5.4$$
$$0.23 \times 10^2 = 23$$

2.97: Decimal versus Radix

A non-decimal base system does not have a decimal point. Rather, the point is called a radix. However, while the name is different, the purpose is the same.

2.98: Working with b^n in base-b

To

- divide by b^n, move the radix n times to the left.
- multiply by b^n, move the radix n times to the right

Example 2.99: Fractions with powers of base-b in the denominator

Write the following fractions as decimals in the base that the numerator is written in:

$$\frac{34_7}{7_{10}}, \quad \frac{23_5}{25_{10}}, \quad \frac{112_9}{729_{10}}, \quad \frac{45_6}{1296_{10}}, \quad \frac{1_7}{2401_{10}}, \quad \frac{12345_{16}}{256_{10}}$$

Convert the denominator to base 7 and divide by moving the radix:

$$\frac{34_7}{7} = \frac{34_7}{10_7} = 3.4_7$$

We can calculate the other fractions using the same method:

$$\frac{23_5}{25} = \frac{23_5}{5^2} = \frac{23_5}{100_5} = 0.23_5$$

$$\frac{112_9}{729} = \frac{112_9}{9^3} = \frac{112_9}{100_3} = 1.12_9$$

$$\frac{45_6}{1296} = \frac{45_6}{6^4} = \frac{45_6}{10000_6} = 0.0045_6$$

$$\frac{1_7}{2401} = \frac{1_7}{1000_7} = 0.001_7$$

$$\frac{12345_{16}}{256} = \frac{12345_{16}}{100_{16}} = 123.45_{16}$$

B. Conversions

Example 2.100

Find $\frac{5}{16}$ in base-2, base-4, and base-16.

$$\frac{5_{10}}{16_{10}} = \frac{101_2}{10000_2} = 0.0101_2$$

$$\frac{5_{10}}{16_{10}} = \frac{11_4}{100_4} = 0.11_4$$

$$\frac{5_{10}}{16_{10}} = \frac{5_{16}}{10_{16}} = 0.5_{16}$$

Example 2.101

Convert $\frac{12}{7}$ to base 7

$$\frac{12}{7} = \frac{15_7}{10_7} = 1.5_7$$

C. Recurring Decimals

Example 2.102

$$x = 0.\bar{5}_8$$

$$\underbrace{x = 0.\bar{5}_8}_{Equation\ I}$$

Multiply both sides by 8:

$$\underbrace{10_8 x = 5.\bar{5}_8}_{Equation\ II}$$

Subtract Equation I from Equation II:

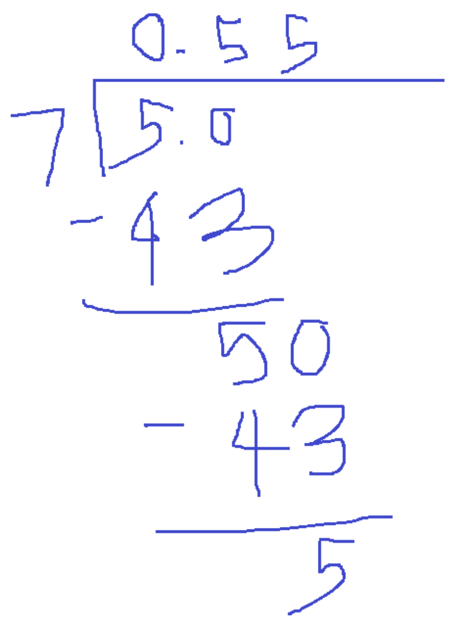
$$7x = 5 \Rightarrow x = \frac{5_8}{7_8}$$

Example 2.103

$$\frac{5_8}{7_8}$$

From the long division alongside, the value of

$$\frac{5_8}{7_8} = 0.5555\ \ldots. = 0.\bar{5}$$



$$\frac{5}{7} = \frac{5}{8_{10} - 1}$$

Divide numerator and denominator by 8:

$$\frac{\frac{5}{8}}{1 - \frac{1}{8}}$$

The above is the sum of an infinite geometric series with:

$$first\ term = a = \frac{5}{8}, common\ ratio = \frac{1}{8}$$

$$= \frac{5}{8_{10}} + \frac{5}{8_{10}^2} + \frac{5}{8_{10}^3} + \cdots$$

$$= \frac{5}{10_8} + \frac{5}{100_8} + \frac{5}{1000_8} + \cdots$$

$$= 0.5_8 + 0.05_8 + 0.005_8 + \cdots$$

$$= 0.555\ \ldots_8$$

$$= 0.\bar{5}_8$$

Example 2.104

Convert to a fraction:

$$x = 0.\bar{5}_b$$

$$x = \frac{5}{b-1}, b \ge 6$$

Example 2.105

Convert to a fraction:

$$x = 0.\overline{51}_b$$

$$x = \frac{51}{b^2 - 1}$$

D. Expanded Notation with Decimals

2.106: Expanded Notation

Consider the number $l_3 l_2 l_1 . r_1 r_2 r_3 \ldots$ in base b

The digits to the left of the radix, as usual, expand to:

$$l_3 l_2 l_1 = l_3 \cdot b^2 + l_2 \cdot b + l_1$$

And the digits to the right of the radix expand to:

$$r_1 r_2 r_3 = \frac{r_1}{b} + \frac{r_2}{b^2} + \frac{r_3}{b^3} + \cdots$$

Example 2.107

Convert 0.3_7 to a fraction

$$0.3_7 = \frac{3_7}{10_7} = \frac{3}{7}$$

E. Problem Solving

Example 2.108

Shenlar has a sum S that is initially set equal to 0. For each integer n from 1 to 100 inclusive, Shenlar adds the value of $\frac{n}{2^n}$ to S if and only if 2^n is divisible by n. When S is written in binary (base-2), what is the sum of the digits of S after the point?[2]

The denominator is a perfect power of 2. It will be divisible by the numerator only when the numerator is also a perfect power of 2. Hence, we must have:

$$\sum_{m=0}^{6} \frac{2^m}{2^{(2^m)}} = \frac{2^0}{2^1} + \frac{2^1}{2^2} + \frac{2^2}{2^4} + \frac{2^3}{2^8} + \frac{2^4}{2^{16}} + \frac{2^5}{2^{32}} + \frac{2^6}{2^{64}}$$

Which simplifies to:

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^{12}} + \frac{1}{2^{27}} + \frac{1}{2^{58}}$$

On adding the first two terms:

$$1 + \frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^{12}} + \frac{1}{2^{27}} + \frac{1}{2^{58}}$$

[2] OTSS TMC 12B 2020/5, 10B 2020/8

For the digits after the point, we need the last five terms only:

$$\frac{1}{2^2}+\frac{1}{2^5}+\frac{1}{2^{12}}+\frac{1}{2^{27}}+\frac{1}{2^{58}}$$

And every term in this expression above converts to single digit when converted to binary.

$$\frac{1}{2^2}=\frac{1}{100_2}=0.01$$

$$\frac{1}{2^5}=\frac{1}{100000_2}=0.00001$$

$$\frac{1}{2^{12}}=\frac{1}{\underbrace{100\ldots0_2}_{12\ Zeroes}}=\underbrace{0.00\ldots1}_{11\ Zeroes}$$

$$\frac{1}{2^{27}}=\frac{1}{\underbrace{100\ldots0_2}_{27\ Zeroes}}=\underbrace{0.00\ldots1}_{26\ Zeroes}$$

$$\frac{1}{2^{58}}=\frac{1}{\underbrace{100\ldots0_2}_{58\ Zeroes}}=\underbrace{0.00\ldots1}_{57\ Zeroes}$$

Each term results in a single digit in the binary system, and there is no overlap.
Hence the sum of the digits is

$$1+1+1+1+1=5$$

2.8 Fractional and Other Bases

A. Factorial Base

2.109: Factorial Base

$$a_1+a_2\times 2!+a_3\times 3!+\cdots+a_n\times n!$$

$$0\le a_k\le k$$

Example 2.110

The number 695 is to be written with a factorial base on numeration, that is

$$695=a_1+a_2\times 2!+a_3\times 3!+\cdots+a_n\times n!$$

where $a_1, a_2, a_3, \ldots a_n$ are integers such that $0\le a_k\le k$ and $n!$ means $n(n-1)(n-2)\ldots 2\times 1$. Find the RHS of the above equation. **(AHSME 1961/35, Adapted)**

$$5!=120<695<6!=720$$

$$695\div(5!=120)=5\ Remainder\ 95$$
$$95\div(4!=24)=3\ Remainder\ 23$$
$$23\div(3!=6)=3\ Remainder\ 5$$
$$5\div(2!=2)=2\ Remainder\ 1$$
$$1\div(1!=1)=1$$

$$1\cdot 1!+2\cdot 2!+3\cdot 3!+3\cdot 4!+5\cdot 5$$

1 Pending

2.111: Rank of a Permutation

2 Pending

2.112: Lehmer Code

3 Pending

Example 2.113
Rank of a permutation example

B. Fractional/Decimal Bases

2.114: Fractional Base

	Base $\frac{1}{2}$								
Digit	5^{th}	4^{th}	3^{rd}	2^{nd}	1^{st}				
	$(-2)^4$	$(-2)^3$	$(-2)^2$	$\left(\frac{1}{2}\right)^1$	$\left(\frac{1}{2}\right)^0$				
Decimal	16	-8	$\frac{1}{4}$	$\frac{1}{2}$	1	2			
1					1				
2			1	1	0				
3			1	1	1				
4			1	0	0				
5			1	0	1				
6	1	1	0	1	0				
7	1	1	0	1	1				
8	1	1	0	0	0				
9	1	1	0	0	1				
10	1	1	1	1	0				

C. Negative Number Bases

It is possible, though slightly more esoteric, to represent numbers using a negative number base. The usual trend in such number bases is that base $(-b, b \in N)$ uses digits from 1 to $1 - b$.
For example,

Base (-2) will use digits 0 and 1.
Base (-3) will use digits 0, 1 and 2.

Just as we started non-decimal number bases by listing the numbers from one onwards, we do the same here for negative number bases.

	Base (-2)					Base (-3)			Base (-4)		
Digit	5^{th}	4^{th}	3^{rd}	2^{nd}	1^{st}	3^{rd}	2^{nd}	1^{st}	3^{rd}	2^{nd}	1^{st}
	$(-2)^4$	$(-2)^3$	$(-2)^2$	$(-2)^1$	$(-2)^0$	$(-3)^2$	$(-3)^1$	$(-3)^0$	$(-4)^2$	$(-4)^1$	$(-4)^0$
Decimal	**16**	**-8**	**4**	**-2**	**1**	**9**	**-3**	**1**	**16**	**-4**	**1**
1					1			1			1
2			1	1	0			2			2
3			1	1	1	1	2	0			3
4			1	0	0	1	2	1	1	3	0
5			1	0	1	1	2	2	1	3	1
6	1	1	0	1	0	1	1	0	1	3	2
7	1	1	0	1	1	1	1	1	1	3	3
8	1	1	0	0	0	1	1	2	1	2	0
9	1	1	0	0	1	1	0	0	1	2	1
10	1	1	1	1	0	1	0	1	1	2	2

Example 2.115

In base –2 notation, digits are 0 and 1 only and the places go up in powers of -2. For example, 11011 stands for and equals number 7 in base 10. If the decimal number 2019 is expressed in base – 2, how many non-zero digits does it contain? (PRMO Aug 25 2019/15)

$$2019 = 2048 - 29 =$$
$$2048 = 4096 - 2048 = 2^{11} = 2^{12} - 2^{11} = (-2)^{12} + (-2)^{11}$$
$$+\left(-32 + \underbrace{3}_{4-2+1}\right)$$

We use the above to get:

$$2019_{-2} = 1100000100111 \Rightarrow 6 \text{ non} - \text{zero digits}$$

Example 2.116

In base $-x$ notation, for a natural number x greater than the one, the digits used are from 1 to $(x-1)$. The places go up in powers of $-x$. For example, 10_{10} written in base -2 equals 1110_{-2}. If $y = 2$, write each of the terms below in the indicated bases:

$$(y + 100)_{-y} + (y^2 + 200)_{-(y+1)} + (y^3 + 300)_{-(y+2)}$$

Substitute $y = 2$, to get:

$$102_{-2} + 204_{-3} + 292_{-4}$$
$$102 =$$

2.9 Under/overdetermined Systems; Recursion

A. Underdetermined Systems

2.117: Exactly determined systems

A number system is exactly determined if there is only one way of expression a number.
The decimal system is exactly determined.

$$387 = 300 + 80 + 7$$

2.118: Underdetermined Systems

When a number base has more digits than required, the system is underdetermined.

Example 2.119

$$a_1 + a_2 \times 2! + a_3 \times 3! + \cdots + a_n \times n!$$
$$0 \le a_k \le k$$

Write the numbers from 1 to 6 in the factorial number system.

Using the factorial number system:

$$1 = 1$$
$$2 = 10$$
$$3 = 11$$
$$4 = 20$$
$$5 = 21 = 2 \times 2! + 1 \times 1! = 4 + 1$$
$$6 = 100$$

Example 2.120

Write the number of ways to write the numbers from 1 to 6 if

$$a_1 + a_2 \times 2! + a_3 \times 3! + \cdots + a_n \times n!$$
$$0 \le a_k \le k + 1$$

$$a_1 \in \{0,1,2\}$$
$$a_2 \in \{0,1,2,3\}$$

$$1 = 1$$
$$2 = 10_{Factorial} = 2_{Factorial}$$

$$3 = 11_{Factorial}$$

$$4 = 20_{Factorial} = 12_{Factorial}$$
$$5 = 21_{Factorial}$$
$$6 = 100_{Factorial} = 22_{Factorial}$$

Example 2.121

Write the number of ways to write the numbers from 1 to 6 if

$$a_0 + a_1 + a_2 \times 2! + a_3 \times 3! + \cdots + a_n \times n!$$
$$a_0 \in \{0,1\}$$
$$0 \le a_k \le k, \qquad k \ge 1$$

$$1 = 1$$
$$1 = 10$$

$$2 = 100\ (a_0 = 0, a_1 = 0, a_2 = 0)$$
$$2 = 11\ (a_0 = 1, a_1 = 1)$$

2.122: Quasidecimal Systems

A *quasidecimal* system has places values based on powers of ten, but instead of using the digits $0 - 9$ uses the digits $0 - A$, where $A = 10$.

$$5A_{QD} = 5 \times 10^1 + A = 50 + 10 = 60_{10}$$

Example 2.123

Answer each part separately

How many numbers have more than one representation in *quasidecimal*, and

A. have exactly two digits in both the decimal and *quasidecimal* systems.
B. have exactly two digits when written in decimal
C. have exactly two digits when written in *quasidecimal*.

Part A

The valid numbers are:

$$\{, 1A = 20, 2A = 30, \ldots, 8A = 90\} \Rightarrow 8\ Numbers$$

Part B

The 8 solutions from Part A are also valid for Part B.

We also have:

$$A = 10 \Rightarrow 1\ Number$$

Total

$$= 8 + 1 = 9$$

Part C

The 8 solutions from Part A are also valid for Part C.

$$\{9A = 100\} \Rightarrow 1\ Number$$
$$\{A0 = 100, A1 = 101, A2 = 102, \ldots, AA = 110\} \Rightarrow 11\ Numbers$$

Total

$$= 8 + 1 + 11 = 20\ Numbers$$

B. Trinary

2.124: Binary with digit 2 (Trinary)

The binary system uses the number base 2. If you keep the place values in terms of powers of 2, but allow the binary system to use the digit 2, you get the trinary system.

Example 2.125

Write the numbers from 1 to 16 in the trinary system in all possible ways.

n	$f(n)$	Solutions	n	$f(n)$	Solutions
1	1	1	9	3	1001, 201, 121
2	2	2, 10	10	5	1010, 1002, 210, 202, 122
3	1	11	11	2	1011, 211
4	3	100, 12, 20	12	5	1100, 1020, 220, 1012, 212
5	2	101, 21	13	3	1101, 1021, 221
6	3	110, 102, 22	14	4	1110, 1102, 1022, 222
7	1	111	15	1	1111
8	4	1000, 200, 120, 112	16	5	10000, 2000, 1200, 1120, 1112

4 Pending

2.126: Recursion Property 1

5 Pending

2.127: Recursion Property 2

6 Pending

2.128: Recursion Property 3

7 Pending

2.129: Recursion Property 4

$$1: f(2n) = f(n) + f(n-1)$$

append a zero at the end of any valid representation of
on of $n-1$ to get $2n$)

$$2: f(2n+1) = f(n)$$

append a 1 at the end of any valid representation of n to

$$3: f(2^k - 2) = k$$

$$4: f(4n) = f(n) + 2f(n-1)$$

8 Pending

Example 2.130

$$f(2020)$$

2.10 Problem Solving; Further Problems

A. Converting a Problem to a Number Base

Example 2.131

The increasing sequence 1,3,4,9,10,12,13 … consists of all those positive integers which are powers of 3, or sums of distinct powers of 3. Find the 100^{th} term of this sequence. **(AIME 1986/7)**

We write the given numbers as sums, and also in base-3:

$$3^0 = 1 = 1_3$$
$$3^1 = 3 = 10_3$$
$$3^0 + 3^1 = 4 = 11_3$$
$$3^2 = 9 = 100_3$$
$$3^2 + 3^0 = 10 = 101_3$$
$$3^2 + 3^1 = 12 = 110_3$$
$$3^2 + 3^1 + 3^0 = 13 = 111_3$$

Note that in the above, we are only using 1's and 0's.
This is because the question considers only sums of distinct powers of 3. Hence, each power of 3 can be included in the sum at most once.

Hence, we are working in base-3, but we are counting in binary.

$$3^0 = 1 = 1_3 \Rightarrow 1_2 = 1$$
$$3^1 = 3 = 10_3 \Rightarrow 10_2 = 2$$

$$3^0 + 3^1 = 4 = 11_3 \Rightarrow 11_2 = 3$$
$$3^2 = 9 = 100_3 \Rightarrow 100_2 = 4$$
$$3^2 + 3^0 = 10 = 101_3 \Rightarrow 101_2 = 5$$
$$3^2 + 3^1 = 12 = 110_3 \Rightarrow 110_2 = 6$$
$$3^2 + 3^1 + 3^0 = 13 = 111_3 \Rightarrow 111_2 = 7$$

We want the 100^{th} term which is

$$100 = 64 + 32 + 4 = 1100100_2$$
$$1100100_3 = 1 \cdot 3^6 + 1 \cdot 3^5 + 1 \cdot 3^2 = 729 + 243 + 9 = 981$$

B. Olympiad Problems

Example 2.132
IMO 1988/3

System of functional equations that converts to finding number of palindromes in binary.

2.11 To Do Next

A. Recurring Decimals

9 Pending

Example 2.133
Suppose n is a positive integer and d is a single digit in base 10. Find n if

$$\frac{n}{810} = 0.d25d25d25\ldots$$ **(AIME 1989/3)**

134 Examples