LIMITS

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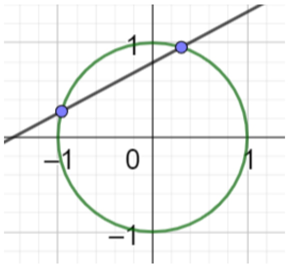
1. Limits & Continuity

1.1 Limits Basics

A. Average Rate of Change

1.1: Secant

- A line passing through two points on a function is a secant.
- A line cutting a circle in two distinct points is also a secant of the circle.



1.2: Average Rate of Change

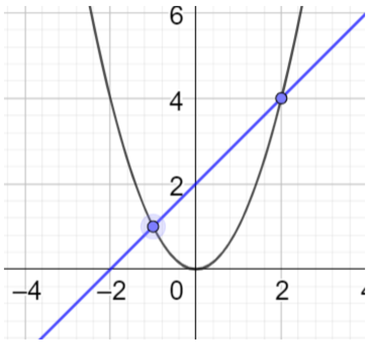
$$Avg.\,Rate\ of\ Change = Slope\ of\ Secant = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The average rate of change of a function between two points is the slope of the secant connecting the two points.

Example 1.3

A. Graph the secant line on the function $y = x^2$ from the point $x = -1$ to the point $x = 2$

B. Find the average rate of rate of the function over the interval $(-1,2)$.



Substitute $(x_1, y_1) = (-1,1)$ and $(x_2, y_2) = (2,4)$ in the formula for slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-1}{2-(-1)} = \frac{3}{3} = 1$$

1.4: Average Rate of Change-II

$$\text{Average rate of change of } f(x) \text{ over an interval } (x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$Avg.\,Rate\ of\ Change = Slope\ of\ Secant = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Substitute $y = f(x) \Rightarrow y_2 = f(x_2)\ and\ y_1 = f(x_1)$ to state the formula for in terms of functions.

$$Avg.\,Rate\ of\ Change = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Hence, our final definition is:

$$\text{Average rate of change of } f(x) \text{ over an interval } (x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example 1.5

Find the average rate of change of the function $y = f(x) = 2x^2 + 3x - 5$ over the interval $(-1,2)$.

The average rate of change is:

$$AVOR = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(2) - f(-1)}{2-(-1)} = \frac{9-(-6)}{3} = \frac{15}{3} = 5$$

1.6: Average Rate of Change-III

$$\text{Average rate of change of } f(x) \text{ over an interval } (x_1, x_1+h) = \frac{f(x+h)-f(x)}{h}$$

$$\text{Average rate of change of } f(x) \text{ over an interval } (x_1, x_2) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

Use a change of variables. Let $x_2 = x_1 + h \Rightarrow h = x_2 - x_1$:

$$\text{Average rate of change of } f(x) \text{ over an interval } (x_1, x_1+h) = \frac{f(x_1+h)-f(x_1)}{(x_1+h)-x_1} = \frac{f(x_1+h)-f(x_1)}{h}$$

The purpose of this change is to make the denominator simpler. In the chapter on derivatives, we will want to make the denominator close to zero, and this alternate definition will be useful in that context.

Example 1.7

Find the average rate of change of the following functions over the given intervals:

A. $f(x) = 3x + 5$ over the interval $(-7,9)$
B. $f(x) = x^2 + 3x + 7$ over the interval $(2,5)$

Use each of the three definitions, and confirm that the answers are the same for all three methods.

Part B

$$\frac{y_2-y_1}{x_2-x_1} = \frac{47-17}{5-2} = \frac{30}{3} = 10$$

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{f(5)-f(2)}{x_2-x_1} = \frac{47-17}{5-2} = \frac{30}{3} = 10$$

For the third method, $h = 5 - 2 = 3$:

$$\frac{f(x_1+h)-f(x_1)}{h} = \frac{f(2+3)-f(2)}{3} = \frac{f(5)-f(2)}{3} = \frac{47-17}{5-2} = \frac{30}{3} = 10$$

B. Instantaneous Rate of Change

The average rate of change gives us the change over an interval. But, we may be interested in the rate of change at a point. This rate of change at a point, or at the instant of time, is called the $\underbrace{\textit{instantaneous}}_{\textit{at that point}}$ rate of change.

This concept of instantaneous rate of change is a core idea in Calculus.

1.8: Instantaneous Rate of Change: Graphical

The slope of the tangent to a function at a point gives the $\underbrace{\textit{instantaneous}}_{\textit{at that point}}$ rate of change of the function at the point.

Example 1.9

Estimate average rate of change from graph

1.10: Instantaneous Rate of Change: Algebraic

The expression $\frac{f(x+h)-f(x)}{h}$ is not defined for values where h is zero, since that will make the denominator zero. However, we can find the value of the expression when h is close to zero.

Example 1.11

Tabulate the average rate of change of the following functions for increasing smaller absolute values of h:

$f(x) = x^2 + 3x + 7$ over the interval $(3,3+h)$

We want to find the average rate of change of $f(x) = x^2 + 3x + 7$ over the interval $(3,3+h)$

$$x = 3$$

x is Approaching 3 from the Right		x is 3 Approaching from the Left	
h	$\frac{f(x+h)-f(x)}{h}$	h	$\frac{f(x+h)-f(x)}{h}$
0.1	$\frac{25.91-25}{0.1} = \frac{0.91}{0.1} = 9.1$	-0.1	$\frac{24.11-25}{-0.1} = 8.9$
0.01	$\frac{25.0901-25}{0.01} = \frac{0.0901}{0.01} = 9.01$	-0.01	$\frac{24.9101-25}{-0.01} = 8.99$
0.001	$\frac{25.009001-25}{0.001} = 9.001$	-0.001	$\frac{24.991001-25}{-0.001} = 8.999$

We see from the table that as h gets closer and closer to zero, the value in the table gets closer and closer to 9

This idea is captured in the concept of a limit, which we discuss in greater detail in the next section.

Not only this, whether h is positive or h is negative does not matter. The same number 9 is being approached from both directions. This is important for a limit.

$$\lim_{x\to 3} x^2 + 3x + 7 = 9$$

Example 1.12

Estimate instantaneous rate of change 1.5^x at $x = 3$.

Find the tangent line at the point $x = 3$

$$\frac{1.5^{3.001} - 1.5^3}{0.001} = 1.368722 = m$$

$$x = 3 \Rightarrow y = 1.5^3 = 3.375$$

Substitute $(x_1, y_1) = (3,3.375)$

$$y - y_1 = m(x - x_1)$$
$$y - 3.375 = 1.369(x - 3)$$
$$y - 3.375 = 1.369x - 4.107$$
$$y = 1.369x - 0.732$$

1.13: Instantaneous Rate of Change: Algebraic

Example 1.14

A. Find the slope of the parabola $f(x) = x^2$ at the point when $x = 3$
B. Find the slope of the parabola $f(x) = x^2$

Part A

Substitute $x = 3$:

$$\frac{f(3+h)-f(3)}{h} = \frac{(3+h)^2-9}{h} = \frac{9+6h+h^2-9}{h} = \frac{6h+h^2}{h} = 6+h$$

Since h is getting close to zero, the slope is:

$$6$$

Part B

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2-x^2}{h} = \frac{x^2+2xh+h^2-x^2}{h} = \frac{2xh+h^2}{h} = 2x+h$$

Since h is getting close to zero, the slope is:

$$2x$$

C. Tables

Example 1.15

Consider

$$\frac{x^2}{x}$$

Case I: $x > 0$			Case II: $x < 0$		
x	x^2	$\frac{x^2}{x}$	X	x^2	$\frac{x^2}{x}$
10^{-1}	10^{-2}	10^{-1}	-10^{-1}	10^{-2}	-10^{-1}
10^{-2}	10^{-4}	10^{-2}	-10^{-2}	10^{-4}	-10^{-2}
10^{-3}	10^{-6}	10^{-3}	-10^{-3}	10^{-6}	-10^{-3}
10^{-4}	10^{-8}	10^{-4}	-10^{-4}	10^{-8}	-10^{-4}
➢ As x approaches zero from the right, the value of the expression decreases ➢ Informally, the magnitude of the limit keeps decreasing as x gets closer to zero			➢ As x approaches zero from the left, the value of the expression increases ➢ Informally, the magnitude of the limit keeps decreasing as x gets closer to zero		
Informally, for small x^2 is much less than x, and hence the value of the limit will approach zero.					

If we want a proof, we need to consider all cases, not just the informal arguments.
We can prove this by considering variables instead of numbers.

Informal Interpretation

The definition does not talk about the existence of the function at the value of the limit.
Rather, the value of the function can be made as close as desired to the value of the limit by adjusting the value of Delta.

Example 1.16

$$f(x) = \frac{1}{x}$$

As x approaches zero from the right on the number line:

x	0.1	0.01	0.001						

$\frac{1}{x}$	10	100	1000						

As x approaches zero from the left on the number line:

x	-0.1	-0.01	-0.001						
$\frac{1}{x}$	-10	-100	-1000						

1.17: Limit: Informal Definition

Let $\boldsymbol{f(x)}$ be defined on an open interval about $\boldsymbol{x_0}$ *except possibly at* $\boldsymbol{x_0}$ *itself*. If $\boldsymbol{f(x)}$ gets arbitrarily close to $\boldsymbol{L}$ (as close to $\boldsymbol{L}$ as we like) for all x sufficiently close to $\boldsymbol{x_0}$, we say that $\boldsymbol{f}$ approaches the limit L as x approaches $\boldsymbol{x_0}$ and we write:

$$\lim_{x \to x_0} f(x) = L$$

Note 1:

- $f(x)$ does not need to be defined at x_0, since the value of the limit does not depend on $f(x_0)$
- Rather, $f(x)$ depends on values of x near x_0. Also called the neighbourhood of x_0.

Note 2:

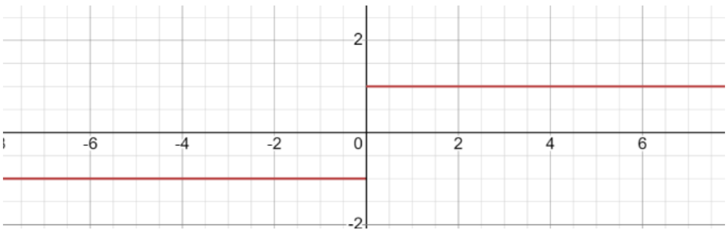
- It is necessary that $f(x)$ is defined on an interval about x_0. If it is not defined on an interval, we cannot check the behaviour of the function near x_0. Hence, $f(x)$ must be defined both to the left and the right of x_0.

Example 1.18

$$Signum\ Function: f(x) = \frac{x}{|x|}$$

$$\lim_{x \to 0} \frac{x}{|x|}$$

$$For\ x > 0, \frac{x}{|x|} = \frac{x}{x} = 1$$
$$For\ x = 0, the\ function\ is\ not\ defined$$
$$For\ x < 0, \frac{x}{|x|} = \frac{x}{-x} = -1$$



x	$f(x)$	x	$f(x)$
0.1	$\frac{0.1}{0.1} = 1$	-0.1	$\frac{-0.1}{0.1} = -1$
0.01	$\frac{0.01}{0.01} = 1$	-0.01	$\frac{-0.01}{0.01} = -1$

$$1 \neq -1$$

Behaviour from the left does not match behaviour from the right.

Hence,

$$Limit\ does\ not\ exist\ (DNE)$$

Example 1.19

$$y = \frac{x+2}{x-2}, x = 1$$

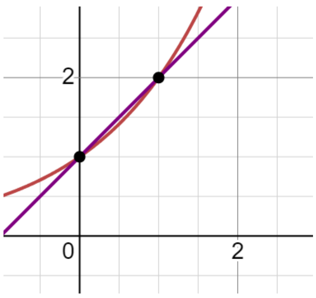
x	$f(x)$	$Average\ Rate\ of\ Change$
1.2	$\frac{1.2+2}{1.2-2} = -4$	$\frac{\Delta y}{\Delta x} = \frac{-4-(-3)}{1.2-1} = \frac{-1}{0.2} = -5$
1.1	$\frac{1.1+2}{1.1-2} = -3.\overline{4}$	$\frac{\Delta y}{\Delta x} = \frac{-3.\overline{4}-(-3)}{1.1-1} = -4.444444$
1.01	$\frac{1.01+2}{1.01-2} = -3.\overline{04}$	$\frac{\Delta y}{\Delta x} = \frac{-3.\overline{04}-(-3)}{1.01-1} = -4.040404$
1.001	$\frac{1.001+2}{1.001-2} = -3.\overline{004}$	$\frac{\Delta y}{\Delta x} = \frac{-3.\overline{004}-(-3)}{1.001-1} = -4.004004$
1.0001	$\frac{1.0001+2}{1.0001-2} = -3.\overline{0004}$	$\frac{\Delta y}{\Delta x} = \frac{-3.\overline{0004}-(-3)}{1.0001-1} = -4.00040004$
1	$\frac{1+2}{1-2} = -3$	
0.8	$\frac{0.8+2}{0.8-2} = -2.\overline{3}$	
0.9	$\frac{0.9+2}{0.9-2} = -2.\overline{63}$	
0.99	$\frac{0.99+2}{0.99-2} = -2.\overline{9603}$	
0.999	$\frac{0.999+2}{0.999-2} = -2.\overline{996003}$	
0.9999	$\frac{0.9999+2}{0.9999-2} = -2.\overline{99960003}$	

D. Exponential Functions

Example 1.20

Estimate the slope of $y = 2^x$ at the y-axis using successive secant lines closer and closer to $x = 0$.

$$Slope = \frac{y_2 - y_1}{x_2 - x_1}$$
$$(x_1, y_1) = (0,1)$$



As x approaches zero from the right on the number line:

x_2	1	$\frac{1}{2}$	$\frac{1}{3}$	
$y_2 = 2^x$	2	$\sqrt{2}$	$2^{\frac{1}{3}}$	
Slope	$\frac{2-1}{1-0} = 1$	$\frac{\sqrt{2}-1}{\frac{1}{2}-0} = 0.82$	$\frac{2^{\frac{1}{3}}-1}{\frac{1}{3}-0} = 0.779$	

As x approaches zero from the left on the number line:

x_2	-1	$-\frac{1}{2}$	
$y_2 = 2^x$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	
Slope	$\frac{\frac{1}{2}-1}{-1-0} = 0.5$	$\frac{\frac{1}{\sqrt{2}}-1}{-\frac{1}{2}-0} = 0.792$	

x_1	0									
y_1	1									
	1.000	2.000	3.000	4.000	5.000	6.000	10.000	100.000	1000.000	10000.000
x_2	1.000	0.500	0.333	0.250	0.200	0.167	0.100	0.010	0.001	0.000
y_2	2.000	1.414	1.260	1.189	1.149	1.122	1.072	1.007	1.001	1.000
Slope	1	0.828427	0.779763	0.756828	0.743492	0.734772	0.717735	0.695555	0.693387	0.693171
x_2	-1.000	-0.500	-0.333	-0.250	-0.200	-0.167	-0.100	-0.010	-0.001	0.000
y_2	0.500	0.707	0.794	0.841	0.871	0.891	0.933	0.993	0.999	1.000
Slope	0.5	0.585786	0.618898	0.636414	0.647247	0.654608	0.66967	0.69075	0.692907	0.693123

Example 1.21

Examples

E. Left Hand and Right-Hand Limits

The left-hand limit considers the behaviour of the function from the left-hand side only.
The right-hand limit considers the behaviour of the function from the right-hand side only.

The limit exists $if\ and\ only\ if$:

- both the left-hand limit and the right-hand limit exist and
- both are equal

$$\underbrace{\lim_{x\to a^-} f(x)}_{\substack{Left\ Hand\\ Limit}} = \underbrace{\lim_{x\to a^+} f(x)}_{\substack{Right\ Hand\\ Limit}} = l \Leftrightarrow \lim_{x\to a} f(x) = l$$

If the two limits exist, but are not equal, the overall limit does not exist

$$\underbrace{\lim_{x\to a^-} f(x)}_{\substack{Left\ Hand\\ Limit}} = p\,,\ \underbrace{\lim_{x\to a^+} f(x)}_{\substack{Right\ Hand\\ Limit}} = q, p \neq q \Rightarrow Overall\ limit\ does\ not\ exist$$

Example 1.22

Find the value of $3.\overline{9}$

Break this into two parts:

$$3.\overline{9} = 3 + 0.\overline{9}$$

Observe that

$$\frac{1}{3} = 0.\overline{3}$$

and multiply both sides by three to get:

$$1 = 0.\overline{9}$$

Conceptually, $0.\overline{9}$ has the 9 recurring till infinity.
Hence, 1 – $0.\overline{9}$ has an infinite number of 9's being subtracted, which makes the value of the answer keep getting smaller.
Hence, it is the same as 1.

Example 1.23

Consider a paradox thought of 2000 years ago, by Zeno, a Greek philosopher.
Part I: Imagine a person who wishes to walk a distance of 2 feet to reach a door.
In order to walk the two meters, he must walk half of that distance = 1 feet (which he does).
Part II: The person now has to walk to half of the remaining distance (2 – 1 = 1) = 0.5 (which he again does).
Part III: The person now has to walk to half of the remaining distance (1 – 0.5 = 0.5) = 0.25 (which he again does).
As you will realise this process continues *ad infinitum*. The paradox lies in the fact that two steps of a man are probably enough to cover two feet. However, seen this way it seems that the man will never cover the distance.
How does the person reach the door, if he follows this process?

S1: We can tabulate this process

Part Number	Distance at Start of Step	Distance Covered	Distance At End of Step
I	2	1	2 – 1 = 1
II	1	0.5	1 - 0.5 = 0.5
III	0.5	0.25	0.5 – 0.25 = 0.25
IV	0.25	0.125	0.25 – 0.125 = 0.125
IV	0.125		

The distances covered by the man are then

$$\underbrace{S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots}_{\substack{a = First\ Term = 1 \\ Common\ Ratio = r = \frac{1}{2}}} \Rightarrow S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$$

Example 1.24

Fill in the blanks in the numbered statements from the options (A,B,C) given. One option may be used in more than one blank (if necessary).

If $\lim_{x \to a} f(x) = b$ where a and b are real numbers, then the domain of $f(x)$

1. _____ contain a
2. _____ contain numbers in an open interval around a

A. *must*

B. may
C. $cannot$

$$1.\,Option\ B: may$$
$$1.\,Option\ B: must$$

Example 1.25

Mark the Correct Option

If $\lim_{x \to a} f(x) = b, f(a) = c$, then:

A. $b = c$
B. $b \neq c$
C. If a lies in the domain of f, then $b = c$
D. None of the above

$$None\ of\ the\ above$$

Example 1.26: Trigonometric Limits

Graphs
Tables
Substitution
Limits that do not exist via substitution
Factoring
Rationalization
Identities

1.2 Limit Laws

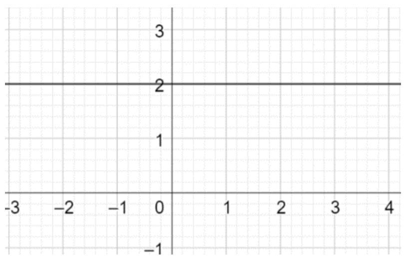
A. Basic Limits and Limit Laws

The value of a constant function does not change with the change in value of x. Hence, the limit of a constant function is the constant itself.

1.27: Limit of a constant function

If for some constant c, $f(x) = c$

$$\lim_{x \to a} f(x) = c$$



The constant function is a "nice", "well-behaved" function:

- No holes
- No places where the function is not defined
- No breaks in the continuity – you can draw the function without lifting your pencil.

Since the function is nice and well-behaved, at any point on the function:

$$\lim_{x \to a} f(x) = \lim_{x \to a} c = c$$

Example 1.28

A. If $f(x) = \frac{\pi}{e}$, find $\lim_{x \to \frac{e}{\pi}} f(x)$

B. Given that $f(x) = 4$, find $\lim\limits_{x \to 4} f(x) - \lim\limits_{x \to \frac{1}{4}} f(x) - \lim\limits_{x \to 0} f(x)$

Part A

$$\lim_{x \to \frac{e}{\pi}} f(x) = \frac{\pi}{e}$$

Part B

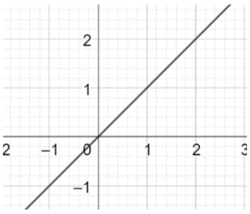
$$\lim_{x \to 4} f(x) - \lim_{x \to \frac{1}{4}} f(x) - \lim_{x \to 0} f(x) = 4 - 4 - 4 = -4$$

1.29: Limit of the identity function

For the identity function, $f(x) = x$

$$\lim_{x \to c} f(x) = c$$

$$\lim_{x \to c} f(x) = \lim_{x \to c} c = c$$



Example 1.30

Given that $f(x) = x$, find:

$$\lim_{x \to 3} f(x)$$

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} x = 3$$

1.31: Sum and Difference Rule

If $\lim\limits_{x \to c} f(x), \lim\limits_{x \to c} g(x)$ *exist* then:

The limit of a sum is the sum of the limits.

$$\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

The limit of a difference is the difference of the limits.

$$\lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$$

The above two properties let you distribute the operation of taking limits.

Example 1.32

A. Evaluate $\lim\limits_{x \to 3} x + 5$

B. (*Distribution*) Given that $\lim\limits_{x \to c} f(x) = 5$, and $\lim\limits_{x \to c} g(x) = -3$, find $\lim\limits_{x \to c} [f(x) + g(x)]$

C. (*Limit Equation*) Given that $\lim\limits_{x \to c} (f(x) + 3) = 5$, find $\lim\limits_{x \to c} f(x)$

Part A

$$\lim_{x \to 3} x + 5 = \lim_{x \to 3} x + \lim_{x \to 3} 5 = 3 + 5 = 8$$

Part B

$$\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = 5 - 3 = 2$$

Part C

$$\lim_{x\to c} f(x) + \lim_{x\to c} 3 = 5 \Rightarrow \lim_{x\to c} f(x) + 3 = 5 \Rightarrow \lim_{x\to c} f(x) = 2$$

Example 1.33

If $\lim_{x\to c} f(x), \lim_{x\to c} g(x)$ *exist* then

$$\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

This example illustrates why the condition that the individual limits exists is important.

1.34: Product Rule

If $\lim_{x\to c} f(x), \lim_{x\to c} g(x)$ *exist* then

$$\lim_{x\to c}[f(x) \times g(x)] = \lim_{x\to c} f(x) \times \lim_{x\to c} g(x)$$

The limit of a product is the product of the limits.

Example 1.35

A. Find $\lim_{x\to 2} 3x$
B. Given that $\lim_{x\to c} f(x) = 2$, and $\lim_{x\to c} g(x) = 12$, find $\lim_{x\to c}[f(x)g(x)]$
C. Given that $\lim_{x\to c} g(x)f(x) = 6$, and that $\lim_{x\to c} f(x) = 5$, find $\lim_{x\to c} g(x)$.
D. Given that $\lim_{x\to c} g(x)f(x) = 6$, and that $\lim_{x\to c} f(x) \in \mathbb{N}, \lim_{x\to c} g(x) \in \mathbb{N}$, find the ordered pairs that satisfy $\left(\lim_{x\to c} f(x), \lim_{x\to c} g(x)\right)$

Part A

$$\lim_{x\to 2} 3x = \lim_{x\to 2} 3 \times \lim_{x\to 2} x = 3 \times 2 = 6$$

Part B

$$\lim_{x\to c}[f(x)g(x)] = \lim_{x\to c} f(x) \times \lim_{x\to c} g(x) = 2 \times 12 = 24$$

Part C

$$\lim_{x\to c} f(x) \times \lim_{x\to c} g(x) = 6$$
$$5 \times \lim_{x\to c} g(x) = 6$$
$$\lim_{x\to c} g(x) = \frac{6}{5}$$

Part D

$$\lim_{x\to c} g(x)f(x) = 6$$
$$\lim_{x\to c} f(x) \times \lim_{x\to c} g(x) = 6$$
$$(1,6), (2,3), (6,1), (3,2)$$

1.36: Constant Multiple Rule

If $\lim_{x\to c} f(x)$ *exist* then

$$\lim_{x\to c} k \times f(x) = \lim_{x\to c} k \times \lim_{x\to c} f(x) = k \lim_{x\to c} f(x)$$

The limit of a constant times a function is the constant times the limit of the function

Example 1.37

A. Find $\lim_{x\to 2} 3x$

$$\lim_{x\to 2} 3x = 3\lim_{x\to 2} x = 3\times 2 = 6$$

1.38: Quotient Rule

If $\lim_{x\to c} f(x), \lim_{x\to c} g(x)$ $exist$ then

Limit of a quotient is the quotient of the limits:

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

Example 1.39

A. Evaluate $\lim_{x\to 1}\frac{x+5}{x}$

B. Given that $\lim_{x\to c} f(x) = 2$, and $\lim_{x\to c} g(x) = 12$, find $\lim_{x\to c}\frac{f(x)}{g(x)}$

C. Given that $\lim_{x\to c} f(x) = 2$, and $\lim_{x\to c}\frac{f(x)}{g(x)} = \pi$, find $\lim_{x\to c} g(x)$.

D. Evaluate $\lim_{x\to 0}\frac{\cos 5x}{\cos 4x}$

Part A

$$\frac{\lim_{x\to 1} x+5}{\lim_{x\to 1} x} = \frac{6}{1} = 6$$

Part B

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{2}{12} = \frac{1}{6}$$

Part C

$$\frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \pi \Rightarrow \frac{2}{\lim_{x\to c} g(x)} = \pi \Rightarrow \lim_{x\to c} g(x) = \frac{2}{\pi}$$

Part D

Use the quotient property:

$$\frac{\lim_{x\to 0}\cos 5x}{\lim_{x\to 0}\cos 4x}$$

Use Substitution:

$$\frac{\cos 0}{\cos 0} = \frac{1}{1} = 1$$

1.40: Power Rule

$$\lim_{x\to c}\left\{[f(x)]^{\frac{n}{m}}\right\} = \left[\lim_{x\to c} f(x)\right]^{\frac{n}{m}}$$

The limit of a power of a function is the power of the limit of the function.

In other words, the exponentiation and the limit operators can be interchanged.

Example 1.41

Evaluate the following limits:

A. $\lim_{x\to 2} 3x^3 + 4x^2$

B. $\lim_{x\to 0} 3x^2 - 5x + 6$

C. $\lim_{x\to 5}\sqrt[4]{(x^2-9)^3}$

Part A

Apply the sum rule:

$$\lim_{x\to 2} 3x^3 + \lim_{x\to 2} 4x^2$$

Apply the multiple rule:

$$3\lim_{x\to 2} x^3 + 4\lim_{x\to 2} x^2$$

Apply the power rule:

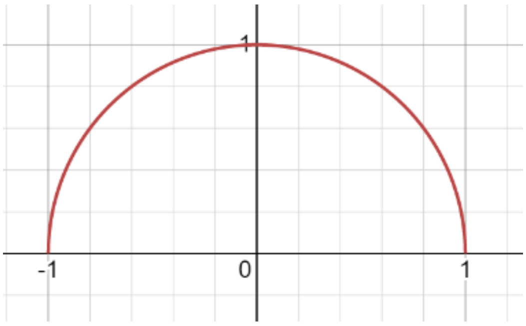
$$3\left(\lim_{x\to 2} x\right)^3 + 4\left(\lim_{x\to 2} x\right)^2 = 3\times 2^3 + 4\times 2^2$$
$$= 24 + 16 = 40$$

Part B

Use the sum and difference rules:
$$\lim_{x\to 0} 3x^2 + \lim_{x\to 0} -5x + \lim_{x\to 0} 6$$
Use the Constant Multiple Rules:
$$= 3\lim_{x\to 0} x^2 - 5\lim_{x\to 0} x + \lim_{x\to 0} 6$$
Evaluate by Substitution:
$$= 3(0) - 5(0) + 6 = 0 + 0 + 6 = 6$$
Part C
Rewrite the expression using exponents:
$$\lim_{x\to 5}(x^2-9)^{\frac{3}{4}}$$
Use the Power Rule to interchange the limit and the exponents:
$$\left[\lim_{x\to 5}(x^2-9)\right]^{\frac{3}{4}}$$
Use the difference rule:
$$\left[\lim_{x\to 5}(x^2) - \lim_{x\to 5}(9)\right]^{\frac{3}{4}}$$
Use the product rule and simplify:
$$[25-9]^{\frac{3}{4}} = (16)^{\frac{3}{4}} = 8$$

Example 1.42
Consider $f(x) = \sqrt{1-x^2}$. Algebraically, evaluate:
A. $\lim_{x\to 0} f(x)$
B. $\lim_{x\to \frac{3}{5}} f(x)$

We did the same question where the graph was given to us, and we found the value using the graph.
The value of the limit is just the value of the function at that point.



Part A
$$\lim_{x\to 0} f(x) = f(0) = 1$$
Part B
$$\lim_{x\to \frac{3}{5}} f(x) = f\left(\frac{3}{5}\right) = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

B. Limits of Polynomials and Rational Functions
The limit of polynomials and rational functions can be found using substitution.

1.43: Limits of Polynomials
Given a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0,\ x \in \mathbb{R}, n \in \mathbb{N}$:
$$\lim_{x\to c} P(x) = P(c)$$

$$\lim_{x\to c} P(x) = \lim_{x\to c} a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$
Apply the sum rule to to get:
$$\lim_{x\to c} a_n x^n + \lim_{x\to c} a_{n-1}x^{n-1} + \cdots + \lim_{x\to c} a_0$$
Apply the power and multiple rules to get:
$$a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0$$
And note that this is just:
$$P(c)$$

Example 1.44
Evaluate the following limits

A. $\lim_{x\to 0} 3x^2 - 5x + 6$
B. $\lim_{x\to 2} x^3 - x$
C. $\lim_{x\to 4} x + 3$

Part A

We did the same above using the limit rules. Now, we do this directly using substitution:

$$\lim_{x\to 0} 3x^2 - 5x + 6 = 3(0)^2 - 5(0) + 6 = 6$$

Part B

$$\lim_{x\to 2} x^3 - x = 8 - 2 = 6$$

Part C

$$\lim_{x\to 4} x + 3 = 4 + 3 = 7$$

1.45: Limits of Rational Functions

If $\boldsymbol{P(x)}$ and $\boldsymbol{Q(x)}$ are polynomials, and $\boldsymbol{Q(c) \neq 0}$, then the limit of their quotient can be found using substitution.

$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$$

$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{\lim_{x\to c} P(x)}{\lim_{x\to c} Q(x)} = \frac{P(c)}{Q(c)}$$

C. Limits of a Composite Function

1.46: Limit of a Composite Function

$$\lim_{x\to c} f\big(g(x)\big) = f\left(\lim_{x\to c} g(x)\right)$$

For a composite function $f\big(g(x)\big)$, a limit can be found by evaluating the limit of the "inner function", and evaluating the outer function, so long as the outer function is continuous at the limit of the inner function.

1.3 Limits by Factoring

A. Limits by Factoring

1.47: Zero Denominators

If denominators are zero, substitution does not work.

Do not attempt to find a limit where the denominator is zero by substitution.

Example 1.48

1.49: Cancellation

$$g(c)=0 \Rightarrow \lim_{x\to c}\frac{f(x)g(x)}{g(x)}=\lim_{x\to c} f(x)$$

Suppose we wish to find

$$\lim_{x\to c}\frac{f(x)g(x)}{g(x)}, \qquad g(c)=0$$

The denominator evaluates to zero so we cannot calculate the limit by substitution. However, for $x\neq 0$, we can simplify:

$$\lim_{x\to c}\frac{f(x)g(x)}{g(x)}=\lim_{x\to c} f(x)$$

We can cancel only because we do not evaluate the limit at $x=c$, only for values in the neighbourhood of c.

Example 1.50

$$\lim_{x\to 0}\frac{3x^2}{x}+2$$

A. Try substitution.
B. Evaluate the limit using cancellation.

Part A

$$\lim_{x\to 0}\frac{3x^2}{x}=\frac{3(0^2)}{0}\Rightarrow \textit{Zero Denominator} \Rightarrow \textit{Not Defined}$$

The denominator becomes zero, and hence, we cannot find the value of the limit by substitution.

Part B

When finding $\lim_{x\to c} f(x)$, we are not concerned with $f(c)$. We are only concerned with values of $f(x)$ where x is near c.

Hence, we can manipulate the expression to simplify it:

$$\lim_{x\to 0}\frac{3x^2}{x}+2=\lim_{x\to 0}3x+2=2$$

Why were we able to cancel the x. Only because we are not evaluating the function at $x=0$, where the denominator is zero.

Example 1.51

Find

$$\lim_{x\to 0}\frac{x^2}{x}$$

We want to find the limit at $x=0$. Remember that we are not concerned with the behaviour at $x=0$.
Had we been so concerned, we would not have been able to divide by x.
However, since $x\neq 0$, we can cancel to get:

$$\lim_{x\to 0}\frac{x^2}{x}=\lim_{x\to 0}\frac{x}{1}=\lim_{x\to 0}x=\lim_{x\to 0}0=0$$

1.52: Some Identities

$$a^2-b^2=(a+b)(a-b)$$
$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

$$(a+b)(a^2-ab+b^2) = a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3 = a^3 + b^3$$
$$(a-b)(a^2+ab+b^2) = a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3 = a^3 - b^3$$

Example 1.53

Evaluate:

Difference of Squares

A. $\lim_{x\to 4}\frac{x-4}{x^2-16}$

B. $\lim_{x\to 1}\frac{x^2-1}{x-1}$

Quadratic Factorizations

C. $\lim_{x\to -3}\frac{x^2-2x-15}{x+3}$

D. $\lim_{x\to 2}\frac{x^2-3x+2}{x^2+x-6}$ (BITSAT 2007)

E. $\lim_{x\to\frac{1}{2}}\frac{4x^2+4x-3}{x-\frac{1}{2}}$

F. $\lim_{t\to 1}\frac{t+2\sqrt{t}-3}{\sqrt{t}-1}$

G. $\lim_{t\to 1}\frac{6t+7\sqrt{t}-3}{\sqrt{t}-\frac{1}{3}}$

Cubic Factorizations

H. $\lim_{x\to -1}\frac{x^3+1}{x+1}$

I. $\lim_{x\to 1}\frac{x^4-1}{x^3-1}$

J. $\lim_{y\to 1}\frac{y^2-1}{y^{\frac{3}{2}}-1}$

Part A

$$\lim_{x\to 4}\frac{x-4}{x^2-16} = \lim_{x\to 4}\frac{x-4}{(x-4)(x+4)} = \lim_{x\to 4}\frac{1}{x+4} = \frac{1}{8}$$

Part B

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1} x+1 = 1+1 = 2$$

Part C

$$\lim_{x\to -3}\frac{(x+3)(x-5)}{x+3} = \lim_{x\to -3} x-5 = -8$$

Part D

$$\lim_{x\to 2}\frac{x^2-3x+2}{x^2+x-6} = \lim_{x\to 2}\frac{(x-2)(x-1)}{(x-2)(x+3)} = \lim_{x\to 2}\frac{x-1}{x+3} = \frac{2-1}{2+3} = \frac{1}{5}$$

Part E

$$\lim_{x\to\frac{1}{2}}\frac{(2x+3)(2x-1)}{x-\frac{1}{2}} = \lim_{x\to\frac{1}{2}} 2(2x+3) = 2\left(2\times\frac{1}{2}+3\right) = 8$$

Part F

Use a change of variable. Let $x = \sqrt{t} \Rightarrow x^2 = t$:

$$\frac{t+2\sqrt{t}-3}{\sqrt{t}-1} = \frac{x^2+2x-3}{x-1} = \frac{(x-1)(x+3)}{x-1}$$

Change back to the original variable:

$$\frac{(\sqrt{t}-1)(\sqrt{t}+3)}{\sqrt{t}-1}$$

And, now we wish to find:

$$\lim_{x\to 1}\frac{(\sqrt{t}-1)(\sqrt{t}+3)}{\sqrt{t}-1} = \lim_{x\to 1}\sqrt{t}+3 = 1+3 = 4$$

Part G

$$\lim_{x\to\frac{1}{9}}\frac{(2\sqrt{t}+3)(3\sqrt{t}-1)}{\sqrt{t}-\frac{1}{3}}=\lim_{x\to\frac{1}{9}}3(2\sqrt{t}+3)=3\left(2\sqrt{\frac{1}{9}}+3\right)=11$$

Part H

$$\lim_{x\to-1}\frac{x^3+1}{x+1}=\lim_{x\to-1}\frac{(x+1)(x^2-x+1)}{x+1}=\lim_{x\to-1}x^2-x+1=1$$

Part I

$$\lim_{x\to1}\frac{(x^2+1)(x+1)(x-1)}{(x-1)(x^2+x+1)}=\lim_{x\to1}\frac{(x^2+1)(x+1)}{x^2+x+1}=\frac{2\times2}{3}=\frac{4}{3}$$

Part J

Use a change of variable. Let $\sqrt{y}=x\Rightarrow y=x^2$:

$$\lim_{x\to1}\frac{x^4-1}{x^3-1}=\lim_{x\to1}\frac{(x^2+1)(x+1)(x-1)}{(x-1)(x^2+x+1)}=\lim_{x\to1}\frac{(x^2+1)(x+1)}{x^2+x+1}=\frac{2\times2}{3}=\frac{4}{3}$$

Example 1.54

Evaluate

$$\lim_{x\to\frac{15}{2}}\frac{\frac{16x^4}{81}-625}{\frac{8x^3}{27}-125}$$

Example 1.55: Back Calculations

A. If $\lim_{x\to1}\frac{x^4-1}{x-1}=\lim_{x\to k}\frac{x^3-k^3}{x^2-k^2}$ then k is: **(JEE-M 2019)**

B. If $\lim_{x\to1}\frac{x^2-ax+b}{x-1}=5$ then $a+b$ is: **(JEE-M 2019)**

Part A

$$LHS=\lim_{x\to1}\frac{(x^2+1)(x+1)(x-1)}{x-1}=\lim_{x\to1}(x^2+1)(x+1)=(2)(2)=4$$

If $k=0$:

$$RHS=\lim_{x\to k}\frac{x^3-k^3}{x^2-k^2}=\lim_{x\to k}\frac{0^3-0^3}{0^2-0^2}=\lim_{x\to k}\frac{0}{0}\Rightarrow Not\ Defined$$
$$\Rightarrow 4=Not\ Defined\Rightarrow Not\ Valid\Rightarrow k\neq0$$

Hence, for $k\neq0$:

$$RHS=\lim_{x\to k}\frac{(x-k)(x^2+xk+k^2)}{(x-k)(x+k)}=\lim_{x\to k}\frac{x^2+xk+k^2}{x+k}=\frac{3k^2}{2k}=\frac{3k}{2}$$

Equate the two sides:

$$4=\frac{3k}{2}\Rightarrow k=\frac{8}{3}$$

Part B

Substituting $x-1$ gives us a zero denominator. Since the limit exists, the numerator must have a factor of $x-1$. That is, for some k,

$$x^2-ax+b=(x-1)(x+k)$$

Determine the value of k:

$$\lim_{x\to1}\frac{x^2-ax+b}{x-1}=\lim_{x\to1}\frac{(x-1)(x+k)}{x-1}=\lim_{x\to1}x+k=1+k=5\Rightarrow k=4$$

Substitute the value of k:

$$(x-1)(x+k)=(x-1)(x+4)=x^2+3x-4$$

Compare coefficients with the expression given in the question:

$$x^2-ax+b=x^2+3x-4$$

$$a=-3, b=-4 \Rightarrow a+b=-3-4=-7$$

Example 1.56

A. What is the implied domain of $f(x)=\frac{x^2-4}{x-2}$? Why is $x=2$ not in the domain?
B. Find $\lim_{x\to 2} f(x)$ by factoring.

Part A

$f(x)$ is not defined when the denominator is zero. That is:

$$x-2=0 \Rightarrow x=2 \Rightarrow D_f=\mathbb{R}-\{2\}$$

$f(x)$ is not defined at $x=2$, because that would make the denominator of the fraction zero.

Part B

We need to find

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x+2)(x-2)}{x-2}=\lim_{x\to 2} x+2=\underbrace{\lim_{x\to 2} 2+2}_{\text{Substitute } x=2}=\lim_{x\to 2} 4=4$$

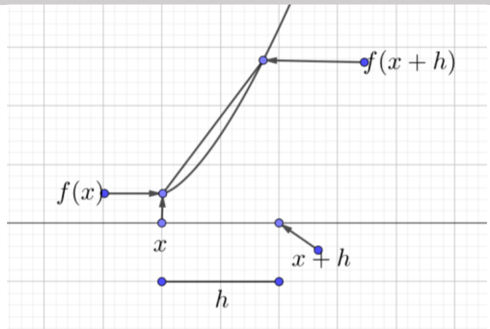
B. Derivatives by Factoring

1.57: Derivative

The derivative of $f(x)$ is

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

provided the limit exists.

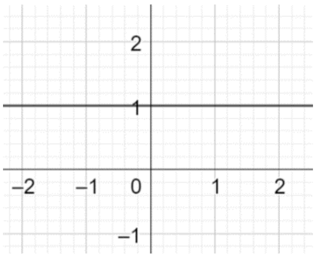


- The expression $\frac{f(x+h)-f(x)}{h}$ gives the slope between the points $(x, f(x))$ and $(x+h, f(x+h))$
- If we take the limit of the expression as $h\to 0$, we find the slope at a point.

Example 1.58: Derivative of Linear Functions

A. Find the derivative of $f(x)=c$
B. Given $f(x)=2x$, find $f'(x)$.
C. $f(x)=mx$, find $f'(x)$.
D. Given $y=mx+c$, find $\frac{dy}{dx}$.

Part A



$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{c-c}{h}=\lim_{h\to 0}\frac{0}{h}=0$$

Hence, slope of a constant function is zero.

Part B

Substitute $f(x)=2x, f(x+h)=2(x+h)=2x+2h$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{2x+2h-2x}{h}=\lim_{h\to 0}\frac{2h}{h}=\lim_{h\to 0} 2=2$$

Part C

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{m(x+h)-mx}{h}=\lim_{h\to 0}\frac{mx+mh-mx}{h}=\lim_{h\to 0}\frac{mh}{h}=\lim_{h\to 0}m=m$$

Part D

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{m(x+h)+c-(mx+c)}{h}=\lim_{h\to 0}\frac{mx+mh+c-[mx+c]}{h}=\lim_{h\to 0}\frac{mh}{h}=\lim_{h\to 0}m=m$$

Example 1.59: Derivatives of Powers

Find $f'(x)$ using the definition of the derivative:

A. $f(x)=x^2$.
B. $f(x)=x^3$
C. $f(x)=x^n, n\in\mathbb{N}$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Part A

Substitute $f(x)=x^2, f(x+h)=(x+h)^2$:

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}=\lim_{h\to 0}\frac{2xh+h^2}{h}=\lim_{h\to 0}\frac{h(2x+h)}{h}=\lim_{h\to 0}2x+h=2x$$

Part B

Apply the definition of the function, and then expand:

$$\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$$

Expand using the binomial theorem:

$$=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$

Cancel x^3:

$$=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$$

Factor and cancel h:

$$=\lim_{h\to 0}3x^2+3xh+h^2$$

Substitute $h=0$:

$$=3x^2$$

Part C

$$\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$

Expand using the binomial theorem:

$$=\lim_{h\to 0}\frac{\left(\binom{n}{0}\boldsymbol{x^n}+\binom{n}{1}x^{n-1}h+\binom{n}{2}x^{n-2}h^2+\cdots+\binom{n}{n}h^n\right)-\boldsymbol{x^n}}{h}$$

Factor out h and cancel it:

$$=\lim_{h\to 0}\frac{\boldsymbol{h}\left(\binom{n}{1}x^{n-1}+\binom{n}{2}x^{n-2}h\ldots+h^{n-1}\right)}{\boldsymbol{h}}=\lim_{h\to 0}\binom{n}{1}x^{n-1}+\binom{n}{2}x^{n-1}h+\cdots+h^{n-1}$$

Every term after the first term has an h in it, and hence as $h\to 0$, the terms tend to zero, leaving only the first term.

$$=nx^{n-1}$$

Example 1.60

Find $Q'(2)$ using the definition of the derivative:

$$Q(x) = \frac{3x+1}{5x-2}$$

Use the definition of $Q'(2) = \lim_{h\to 0} \frac{Q(2+h)-Q(2)}{h}$:

$$\lim_{h\to 0} \left[\frac{3(2+h)+1}{5(2+h)-2} - \frac{3(2)+1}{5(2)-2}\right]\frac{1}{h}$$

Simplify:

$$= \lim_{h\to 0} \left[\frac{6+3h+1}{10+5h-2} - \frac{6+1}{10-2}\right]\frac{1}{h} = \lim_{h\to 0} \left[\frac{7+3h}{8+5h} - \frac{7}{8}\right]\frac{1}{h}$$

Add the fractions:

$$= \lim_{h\to 0} \left[\frac{(56+24h)-(56+35h)}{(8+5h)8}\right]\frac{1}{h} = \lim_{h\to 0} \left[\frac{-11h}{(8+5h)8}\right]\frac{1}{h} = \lim_{h\to 0} \frac{-11}{(8+5h)8}$$

Substitute $h = 0$:

$$= \lim_{h\to 0} \frac{-11}{64}$$

Example 1.61

Find $Q'(x)$ using the definition of the derivative, and use it to evaluate $Q'(2)$:

$$Q(x) = \frac{3x+1}{5x-2}$$

Using the definition of $Q'(x) = \lim_{h\to 0} \frac{Q(x+h)-Q(x)}{h}$:

$$\lim_{h\to 0} \frac{\frac{3(x+h)+1}{5(x+h)-2} - \frac{3x+1}{5x-2}}{h}$$

Add the fractions:

$$= \lim_{h\to 0} \left[\frac{(3x+3h+1)(5x-2)-(3x+1)(5x+5h-2)}{(5x+5h-2)(5x-2)}\right]\frac{1}{h}$$

Use a change of variable. Let $3x+1 = y, 5x-2 = z$:

$$= \lim_{h\to 0} \left[\frac{(y+3h)(z)-(y)(z+5h)}{(z+5h)(z)}\right]\frac{1}{h} = \lim_{h\to 0} \left[\frac{yz+3hz-yz-5yh}{z^2+5hz}\right]\frac{1}{h} = \lim_{h\to 0} \left[\frac{h(3z-5y)}{z^2+5hz}\right]\frac{1}{h}$$

Cancel h:

$$\lim_{h\to 0} \frac{3z-5y}{z^2+5hz}$$

Substitute $h = 0$ to evaluate the limit, and change back to the original variable:

$$= \frac{3z-5y}{z^2} = \frac{3(5x-2)-5(3x+1)}{(5x-2)^2} = \frac{15x-6-15x-5}{(5x-2)^2} = \frac{-11}{(5x-2)^2}$$

$$Q'(2) = -\frac{11}{(10-2)^2} = -\frac{11}{8^2} = -\frac{11}{64}$$

C. Further Limits

Example 1.62

$\lim_{x\to b} \frac{x^n - b^n}{x-b}$ using factoring to get a finite geometric series.

$$\frac{x^n - b^n}{x-b} = \frac{b^n \frac{x^n}{b^n} - b^n}{b\left(\frac{x}{b} - 1\right)} = b^{n-1} \frac{\frac{x^n}{b^n} - 1}{x-b}$$

Since the second term is the sum of a finite geometric series, we can write

$$b^{n-1} \frac{\frac{x^n}{b^n} - 1}{x-b} = b^{n-1} \sum_{k=0}^{n-1} \left(\frac{x}{b}\right)^k$$

Compute the limit as $x \to b$:

$$\lim_{x\to b} b^{n-1} \sum_{k=0}^{n-1} \left(\frac{x}{b}\right)^k$$

Using the constant multiple rule $\lim cf(x) = c \lim f(x)$:

$$= b^{n-1} \lim_{x\to b} \sum_{k=0}^{n-1} \left(\frac{x}{b}\right)^k$$

Since it is not an indeterminate form, we can substitute $x = b$

$$= b^{n-1} \lim_{x\to b} \sum_{k=0}^{n-1} \left(\frac{b}{b}\right)^k = b^{n-1} \lim_{x\to b} \sum_{k=0}^{n-1} 1^k$$

Since the term inside the limit is a constant, the value of the limit is just the value of the expression:

$$= b^{n-1} \sum_{k=0}^{n-1} 1^k$$

Using the properties of summation notation, we get:

$$= b^{n-1} \times \underbrace{(1 + 1 + \cdots + 1)}_{n\ times} = nb^{n-1}$$

1.63: Limits by Factoring: Summary

Limits by factoring is usually applied to simplify a $\frac{0}{0}$ case.

You should know the underlying algebraic factoring techniques.

1.4 Limits by Rationalization

A. Zero Denominators: Limits by Rationalization

1.64: Rationalization

$$\frac{1}{\sqrt{a} + \sqrt{b}} \times \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}}$$

$$\frac{1}{\sqrt{a}+\sqrt{b}} \times \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$$

Example 1.65

Evaluate:

Fractions

A. $\lim_{x\to 0} \frac{\sqrt{x+25}-5}{x}$

B. $\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$

C. $\lim_{x\to 0} \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x^2}$ (BITSAT 2010)

In each question, substitution results in a zero denominator. Hence, we rationalize.

Part A

Multiply by the conjugate of the numerator:

$$\lim_{x\to 0} \frac{\sqrt{x+25}-5}{x} \times \frac{\sqrt{x+25}+5}{\sqrt{x+25}+5}$$

Simplify:

$$= \lim_{x\to 0} \frac{x}{x\left(\sqrt{x+25}+5\right)}$$

Evaluate using substitution:

$$= \frac{1}{\sqrt{25}+5} = \frac{1}{10}$$

Part B

Rationalize the numerator:

$$\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} \times \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$$

Simplify:

$$= \lim_{x\to -1} \frac{x^2-1}{(x+1)\left(\sqrt{x^2+8}+3\right)}$$

Factor the numerator and cancel:

$$= \lim_{x\to -1} \frac{x-1}{\sqrt{x^2+8}+3}$$

Now evaluate using substitution:

$$= \frac{-1-1}{\sqrt{(-1)^2+8}+3} = \frac{-2}{3+3} = -\frac{2}{6} = -\frac{1}{3}$$

Part C

Rationalize the numerator:

$$\lim_{x\to 0} \frac{(1+x^2)-(1-x^2)}{x^2\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}$$

Simplify:

$$= \lim_{x\to 0} \frac{2x^2}{x^2\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}$$

Cancel x^2 and evaluate by substitution:

$$= \frac{2}{\left(\sqrt{1+0}+\sqrt{1-0}\right)} = \frac{2}{2} = 1$$

Example 1.66

Find $f'(x)$ using the definition of the derivative:

A. $f(x) = \sqrt{x}$

B. $f(x) = \sqrt[3]{x}$

Part A

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

Rationalize in order to eliminate the zero denominator:

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Part B

Use the definition $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to get:

$$\lim_{h\to 0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$$

Substitute $a=\sqrt[3]{x+h}, b=\sqrt[3]{x}$ in identity $a^3-b^3=(a-b)(a^2+ab+b^2)$:

$$RHS = \left(\sqrt[3]{x+h}-\sqrt[3]{x}\right)\underbrace{\left(\left(\sqrt[3]{x+h}\right)^2+\sqrt[3]{x+h}\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2\right)}_{\text{Rationalizing Factor}}$$

$$LHS = \left(\sqrt[3]{x+h}\right)^3-\left(\sqrt[3]{x}\right)^3 = x+h-x = h$$

$$\lim_{h\to 0}\frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}\times\frac{\sqrt[3]{(x+h)^2}+\sqrt[3]{x+h}\sqrt[3]{x}+\sqrt[3]{x^2}}{\sqrt[3]{(x+h)^2}+\sqrt[3]{x+h}\sqrt[3]{x}+\sqrt[3]{x^2}}$$

Substitute the result from the identity above:

$$\lim_{h\to 0}\frac{h}{h\left(\sqrt[3]{(x+h)^2}+\sqrt[3]{x+h}\sqrt[3]{x}+\sqrt[3]{x^2}\right)}$$

Cancel h:

$$\lim_{h\to 0}\frac{1}{\sqrt[3]{(x+h)^2}+\sqrt[3]{x+h}\sqrt[3]{x}+\sqrt[3]{x^2}}$$

Substitute $h=0$:

$$\lim_{h\to 0}\frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x^2}+\sqrt[3]{x^2}} = \lim_{h\to 0}\frac{1}{3\sqrt[3]{x^2}}$$

1.5 Limit $\sin\theta$ over θ

A. Limit $\frac{\sin\theta}{\theta}$

1.67: An important limit

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Example 1.68

Evaluate the following limits

Basics

A. $\lim_{x\to 0}\frac{\sin\frac{\pi x}{e}}{\frac{\pi x}{e}}$

Creating the Limit

B. $\lim_{x\to 0}\frac{\sin 3x}{7x}$

C. $\lim_{x\to 0}\frac{\sin mx}{x}$

Infinite Limits

D. $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

Radicals

E. $\lim_{x\to 0}\sqrt{\frac{x-\sin x}{x+\sin^2 x}}$ (BITSAT 2018)

Part A

Use a change of variable. Let $\theta=\frac{\pi x}{e}$ and note that:

$$x\to 0 \ means\ \frac{\pi x}{e}\to 0$$

Hence, we get:

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

Part B

$$\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\frac{3x}{7x}=\lim_{x\to 0}\frac{3x}{7x}=\lim_{x\to 0}\frac{3}{7}=\frac{3}{7}$$

Part C

$$\lim_{x\to 0}\frac{\sin mx}{x}\cdot\frac{mx}{mx}=\lim_{x\to 0}\frac{mx}{x}=\lim_{x\to 0}\frac{mx}{x}=m$$

Part D

Rearrange:

$$\lim_{x\to\infty}x\sin\left(\frac{1}{x}\right)=\lim_{x\to\infty}\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

Use a change of variable. Let $\theta=\frac{1}{x}$ and note that:

$$x\to\infty\Rightarrow\theta\to 0$$

Make the substitutions:

$$\lim_{x\to\infty}\frac{1}{\theta}\sin(\theta)=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

Part E

The numerator is of the form $0-0$. We divide by x to convert to the form $\frac{0}{0}$:

$$\lim_{x\to 0}\sqrt{\frac{1-\frac{\sin x}{x}}{1+\frac{\sin x}{x}\cdot\sin x}}=\sqrt{\frac{1-1}{1+1\cdot 0}}=0$$

Example 1.69

$$\lim_{x\to 0}\frac{\sin 3x}{7x}=\lim_{x\to 0}\frac{\sin 3x}{x}\times\frac{3}{3}=\lim_{x\to 0}\frac{\sin 3x}{3x}\times 3$$

Let $\theta=3x$:

$$=\lim_{x\to 0}\frac{\sin\theta}{\theta}\times 3=3\lim_{x\to 0}\frac{\sin\theta}{\theta}=3\cdot 1=3$$

Example 1.70

$$\lim_{x\to 0}\frac{\sin 3x}{4x}=\lim_{x\to 0}\frac{\sin 3x}{\sin 4x}\cdot\frac{3x}{3x}\cdot\frac{4x}{4x}=\lim_{x\to 0}\frac{4x}{\sin 4x}\cdot\frac{\sin 3x}{3x}\cdot\frac{3x}{4x}$$

Let $\alpha=4x,\beta=3x$

$$=\lim_{x\to 0}\frac{\alpha}{\sin\alpha}\cdot\frac{\sin\beta}{\beta}\cdot\frac{3x}{4x}=\lim_{x\to 0}1\cdot 1\cdot\frac{3}{4}=\lim_{x\to 0}\frac{3}{4}=\frac{3}{4}$$

Example 1.71

A. $\lim_{\theta\to 0}\cos\theta$
B. $\lim_{\theta\to 0}\frac{\cos\theta}{1}$
C. $\lim_{\theta\to 0}\frac{1}{\cos\theta}$
D. $\lim_{\theta\to 0}\frac{\cos\theta}{\theta}$
E. $\lim_{\theta\to 0}\frac{\theta}{\cos\theta}$

$$\lim_{\theta\to 0}\cos\theta=1$$
$$\lim_{\theta\to 0}\frac{\cos\theta}{1}=1$$
$$\lim_{\theta\to 0}\frac{1}{\cos\theta}=1$$

$$\lim_{\theta\to 0}\frac{\cos\theta}{\theta}=\infty$$
$$\lim_{\theta\to 0}\frac{\theta}{\cos\theta}=0$$

1.72: An extension to an important limit

$$\lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1$$

$$\lim_{\theta\to 0}\frac{\tan\theta}{\theta}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta\cos\theta}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta}=1\cdot 1=1$$

Example 1.73

Evaluate $\lim_{\theta\to 0}\frac{\tan x}{2x}$:

A. Using the property $\lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1$

B. From first principles

Part A

$$\lim_{\theta\to 0}\frac{\tan x}{2x}=\lim_{\theta\to 0}\frac{\tan x}{x}\cdot\frac{1}{2}=\frac{1}{2}$$

Part B

$$\lim_{\theta\to 0}\frac{\tan x}{2x}=\lim_{\theta\to 0}\frac{\sin x}{2x\cos x}=\lim_{\theta\to 0}\frac{1}{2}\cdot\frac{1}{\cos x}\cdot\frac{\sin x}{x}=\frac{1}{2}\cdot 1\cdot 1=\frac{1}{2}$$

Example 1.74

$$\lim_{x\to 0}\frac{\cot 2x}{\cot x}$$

$$\lim_{x\to 0}\frac{\tan x}{\tan 2x}$$

$$\lim_{x\to 0}\frac{\tan x}{x}\times\frac{2x}{\tan 2x}\times\frac{x}{2x}$$

$$\lim_{x\to 0}1\cdot 1\times\frac{1}{2}$$

$$\lim_{x\to 0}\frac{1}{2}$$

$$\frac{1}{2}$$

Example 1.75

Evaluate:

A. $\lim_{x\to 0}\frac{\cot 3x}{\cot 5x}$

B. $\lim_{x\to 0}\frac{x\cot 4x}{\sin^2 x\cot^2 2x}$ (JEE-M 2019)

Part A

Rewrite in terms of *tan* by taking the reciprocal:

$$\lim_{x\to 0}\frac{\tan 5x}{\tan 3x}$$

Rewrite to satisfy the form $\frac{\tan\theta}{\theta}$:

$$\lim_{x\to 0}\frac{\tan 5x}{5x}\cdot\frac{3x}{\tan 3x}\cdot\frac{5x}{3x}=1\cdot 1\cdot\frac{5}{3}$$

Part B

$$\lim_{x\to 0}\frac{x\tan^2 2x}{\sin^2 x\tan 4x}=\lim_{x\to 0}\frac{x\cdot\frac{\tan^2 2x}{(2x)^2}\cdot(2x)^2}{x^2\cdot\frac{\sin^2 x}{x^2}\cdot\frac{\tan 4x}{4x}\cdot 4x}=\lim_{x\to 0}\frac{x\cdot(2x)^2}{x^2\cdot 4x}=\lim_{x\to 0}\frac{4x^3}{4x^3}=1$$

Example 1.76:

A. $\lim_{x\to 0}\frac{\cos x-1}{x}$

B. $\lim_{x\to 0}\frac{\sec x-1}{x}$

Part A

$$\lim_{x\to 0}\frac{\cos x-1}{x}\times\frac{\cos x+1}{\cos x+1}$$
$$=\lim_{x\to 0}\frac{\cos^2 x-1}{x(\cos x+1)}$$
$$=\lim_{x\to 0}\frac{-\sin^2 x}{x(\cos x+1)}$$
$$=\lim_{x\to 0}\frac{\sin x}{x}\times\frac{-\sin x}{\cos x+1}$$
$$=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{-\sin x}{\cos x+1}\right)$$
$$=(1)\left(\lim_{x\to 0}\frac{-0}{1+1}\right)=(1)\left(\lim_{x\to 0}0\right)=(1)(0)=0$$

Part B

$$\lim_{x\to 0}\frac{\sec x-1}{x}=\lim_{x\to 0}\frac{\frac{1}{\cos x}-1}{x}=\lim_{x\to 0}\frac{1-\cos x}{x\cos x}=\lim_{x\to 0}\frac{1-\cos x}{x}\cdot\frac{1}{\cos x}$$
$$=\lim_{x\to 0}\frac{1-\cos x}{x}\times\lim_{x\to 0}\frac{1}{\cos x}$$

From part A, the first limit is zero:

$$=0\times 1=0$$

Example 1.77:

Find the derivative of the functions below using the definition:

A. $f(x)=\sin x$

B. $f(x)=\sin\sqrt{x}$

Part A

Start with the definition:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Since $f(x)=\sin x$:

$$=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

Expand $\sin(x+h)$ using $\sin(a+b)=\sin a\cos b+\cos a\sin b$:

$$=\lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$

Take $\sin x$ common:

$$= \lim_{h\to 0}\frac{(\sin x)(\cos h - 1) + \cos x \sin h}{h}$$

Split the limits:

$$= \lim_{h\to 0}\frac{\sin x\,(\cos h - 1)}{h} + \lim_{h\to 0}\frac{\cos x \sin h}{h}$$

Move $\sin x$ and $\cos x$ outside since they are independent of h:

$$\sin x \lim_{h\to 0}\frac{(\cos h - 1)}{h} + \cos x \lim_{h\to 0}\frac{\sin h}{h}$$

Substitute $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0, \lim_{h\to 0}\frac{\sin h}{h} = 1$

$$= \sin x \times 0 + \cos x \times 1$$

Simplify:

$$\cos x$$

Part B

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

Example 1.78

$$\lim_{x\to 0}\frac{\sin(12x) + \tan(12x) + 12x}{3x}$$

If the individual limits exists, then we can split the limit:

$$\lim_{x\to 0}\frac{\sin(12x)}{3x} + \lim_{x\to 0}\frac{\tan(12x)}{3x} + \lim_{x\to 0}\frac{12x}{3x}$$

Using $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$:

$$\lim_{x\to 0}\frac{\sin(12x)}{3x}\times\frac{4}{4} = \lim_{x\to 0}\frac{4\sin(12x)}{12x} = 4\lim_{x\to 0}\frac{\sin(12x)}{12x} = 4\cdot 1 = 4$$

Using $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$:

$$\lim_{x\to 0}\frac{\tan(12x)}{3x} = \lim_{x\to 0}\frac{\sin(12x)}{3x\cos 3x} = \lim_{x\to 0}\frac{\sin(12x)}{3x}\cdot\frac{1}{\cos 3x}$$

Split the limits

$$= \lim_{x\to 0}\frac{\sin(12x)}{3x}\times\lim_{x\to 0}\frac{1}{\cos 3x} = 4\times 1 = 4$$

The third part can be done cancellation:

$$\lim_{x\to 0}\frac{12x}{3x} = \lim_{x\to 0} 4 = 4$$

The final answer is:

$$4 + 4 + 4 = 12$$

1.6 Limits at Infinity

A. Limits at Infinity / Horizontal Asymptotes

1.79: End Behavior of a Function

The end behavior of a function is the behavior as $x \to \infty$, or $x \to -\infty$

1.80: Behavior as $x \to \infty$

Given a constant c, and a variable x,

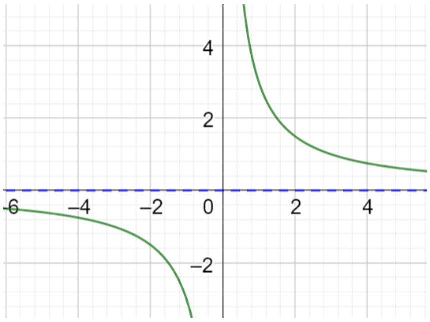
$$\lim_{x\to\infty} \frac{c}{x} = 0$$
$$\lim_{x\to\infty} \frac{x}{c} = \infty$$

Example 1.81: Basics

A. Find $\lim_{x\to\infty} \frac{3}{x}$
B. Find $\lim_{x\to\infty} \frac{x}{\pi}$
C. $\lim_{x\to\infty} \frac{2x}{x}$
D. $\lim_{x\to\infty} \frac{4x^3}{x}$

Part A

x	$\frac{3}{x}$
1	3
10	0.3
100	0.03
1000	0.003
10,000	0.0003



As x increases, the value of $\frac{3}{x}$ decreases. As $x \to \infty$, $\frac{3}{x} \to 0$

$$\therefore \lim_{x\to\infty} \frac{3}{x} = 0$$

Part B

$$\lim_{x\to\infty} \frac{x}{\pi} = \infty$$

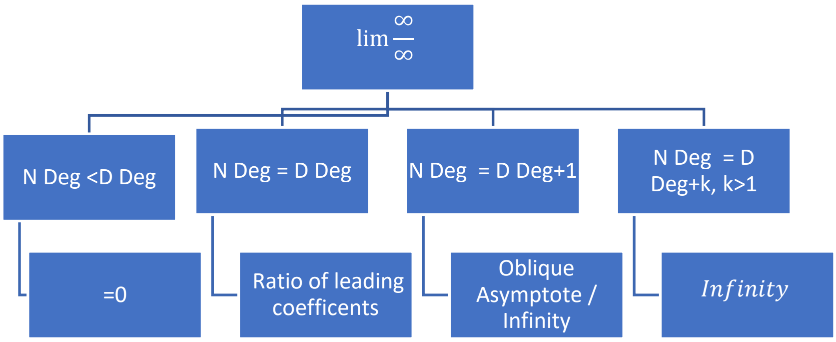
Part C

$$\lim_{x\to\infty} \frac{2x}{x} = \lim_{x\to\infty} 2 = 2$$

1.82: Indeterminate Form

$$\frac{\infty}{\infty}$$

- Divide both numerator and denominator by the highest power of x in the denominator.



1.83: Degree of Numerator = Degree of Denominator

Given polynomials f and g of equal degree:

- $\lim_{x\to\infty} \frac{f}{g}$ is the ratio of leading coefficients of the polynomials
- $y = \frac{f}{g}$ has a horizontal asymptote, which is equal to $\lim_{x\to\infty} \frac{f}{g}$

- For leading coefficients, the polynomials must be in standard form)

Example 1.84

Evaluate the following limits:

Equal Degree

A. $\lim_{x\to\infty} \frac{x+3}{2x+7}$

B. $\lim_{x\to\infty} \frac{ax+b}{cx+d}, a, c \neq 0$

C. $\lim_{n\to\infty} \frac{1+2+3+\cdots+n}{n^2+100}$ (BITSAT 2018)

All questions here are of the form $\frac{\infty}{\infty}$.

Part A

Factor out x in the numerator and denominator:

$$\lim_{x\to\infty} \frac{x\left(1+\frac{3}{x}\right)}{x\left(2+\frac{7}{x}\right)}$$

Cancel x in the numerator and the denominator:

$$= \lim_{x\to\infty} \frac{1+\frac{3}{x}}{2+\frac{7}{x}}$$

Use the quotient property:

$$= \frac{\lim_{x\to\infty} 1+\frac{3}{x}}{\lim_{x\to\infty} 2+\frac{7}{x}}$$

Use the sum rule to split the limits:

$$\frac{\lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{3}{x}}{\lim_{x\to\infty} 2 + \lim_{x\to\infty} \frac{7}{x}}$$

Evaluate each of the limits:

$$\frac{1+0}{2+0} = \frac{1}{2}$$

Part B

Divide by the highest power of x in the denominator, which is x:

$$\lim_{x\to\infty} \frac{\frac{ax}{x}+\frac{b}{x}}{\frac{cx}{x}+\frac{d}{x}} = \lim_{x\to\infty} \frac{a+\frac{b}{x}}{c+\frac{d}{x}} = \lim_{x\to\infty} \frac{a+0}{c+0} = \lim_{x\to\infty} \frac{a}{c}$$

Part C

Use the formula for the sum of the first n natural numbers:

$$\lim_{n\to\infty} \frac{\frac{n(n+1)}{2}}{n^2+100} = \lim_{n\to\infty} \frac{n^2+n}{2n^2+200}$$

Divide numerator and denominator by n^2:

$$\lim_{n\to\infty} \frac{1+\frac{n}{n^2}}{2+\frac{200}{n^2}} = \frac{1}{2}$$

1.85: A Shortcut

$$\lim_{x\to\infty}\frac{a_1x^n+a_2x^{n-1}+\cdots+a_{n+1}}{b_1x^n+b_2x^{n-1}+\cdots+b_{n+1}}=\frac{a_1}{b_1}$$

Divide both numerator and denominator by the highest power of x in the denominator, which is x^n:

$$\lim_{x\to\infty}\frac{\frac{a_1x^n}{x^n}+\frac{a_2x^{n-1}}{x^n}+\cdots+\frac{a_{n+1}}{x^n}}{\frac{b_1x^n}{x^n}+\frac{b_2x^{n-1}}{x^n}+\cdots+\frac{b_{n+1}}{x^n}}$$

Simplify:

$$=\lim_{x\to\infty}\frac{a_1+\frac{a_2}{x}+\cdots+\frac{a_{n+1}}{x^n}}{b_1+\frac{b_2}{x}+\cdots+\frac{b_{n+1}}{x^n}}$$

All terms except the term in each of the denominator and the numerator tend to zero:

$$=\lim_{x\to\infty}\frac{a_1}{b_1}=\frac{a_1}{b_1}$$

1.86: Degree of Numerator > Degree of Denominator

Given polynomials f and g, where degree of f is one more than degree of g:

- $\lim_{x\to\infty}\frac{f}{g}=\infty$ is the ratio of leading coefficients of the polynomials
- $y=\frac{f}{g}$ has an oblique asymptote, which is equal to $\lim_{x\to\infty}\frac{f}{g}$

Example 1.87

A. $\lim_{x\to\infty}\frac{3x^2+7x+3}{9x+2}$

Divide the numerator and the denominator by the highest power in the denominator, which is x:

$$\lim_{x\to\infty}\frac{\frac{3x^2}{x}+\frac{7x}{x}+\frac{3}{x}}{\frac{9x}{x}+\frac{2}{x}}=\lim_{x\to\infty}\frac{3x+7+\frac{3}{x}}{9+\frac{2}{x}}=\lim_{x\to\infty}\frac{3x+7}{9}=\infty$$

Example 1.88

If x is real and positive and grows beyond all bounds, then $\log_3(6x-5)-\log_3(2x+1)$ approaches: **(AHSME 1967/23)**

Combine using the quotient rule:

$$\lim_{x\to\infty}\left[\log_3\left(\frac{6x-5}{2x+1}\right)\right]$$

We want to understand the behavior as x grows very large. Divide both numerator and denominator by the highest power of x in the expression:

$$\lim_{x\to\infty}\left[\log_3\left(\frac{\frac{6x}{x}-\frac{5}{x}}{\frac{2x}{x}+\frac{1}{x}}\right)\right]=\lim_{x\to\infty}\left[\log_3\left(\frac{6-\frac{5}{x}}{2+\frac{1}{x}}\right)\right]$$

Interchange the limit operation and the log operation using the property $\lim_{x\to c}f\big(g(x)\big)=f\left(\lim_{x\to c}g(x)\right)$ so long as

f is continuous at $\lim_{x\to c} g(x)$:

$$= \log_3\left[\lim_{x\to\infty}\left(\frac{6-\frac{5}{x}}{2+\frac{1}{x}}\right)\right]$$

As $x \to \infty \Rightarrow -\frac{5}{x} \to 0,\ \frac{1}{x} \to 0$:

$$= \log_3\left[\lim_{x\to\infty}\left(\frac{6-0}{2+0}\right)\right] = \log_3\left[\lim_{x\to\infty}\left(\frac{6}{2}\right)\right] = \log_3\left[\lim_{x\to\infty} 3\right] = \log_3(3) = 1$$

Example 1.89: Rationalization

$$\lim_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right)$$

Try Substitution:

$$\lim_{x\to\infty} \infty - \infty \Rightarrow Indeterminate$$

Method I: Rationalization

Rationalize:

$$\lim_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x}\right)\times\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}$$
$$= \lim_{x\to\infty}\frac{x+1-x}{\sqrt{x+1}+\sqrt{x}}$$
$$= \lim_{x\to\infty}\frac{1}{\sqrt{x+1}+\sqrt{x}}$$
$$= 0$$

Example 1.90: Rationalization

$$\lim_{x\to\infty}\frac{x}{\sqrt{3x^2+2x+1}}$$

Factor x^2 inside the radical:

$$\lim_{x\to\infty}\frac{x}{\sqrt{x^2\left(3+\frac{2}{x}+\frac{1}{x^2}\right)}}$$

Since $x > 0 \Rightarrow \sqrt{x^2} = x$

$$= \lim_{x\to\infty}\frac{x}{x\sqrt{3+\frac{2}{x}+\frac{1}{x^2}}}$$
$$= \lim_{x\to\infty}\frac{1}{1\sqrt{3+\frac{2}{x}+\frac{1}{x^2}}}$$

Substitute $\lim_{x\to\infty}\frac{2}{x} = 0,\ \lim_{x\to\infty}\frac{1}{x^2} = 0$:

$$= \lim_{x\to\infty}\frac{1}{1\sqrt{3+0+0}} = \lim_{x\to\infty}\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

B. Limits at Infinity / Horizontal Asymptotes

1.91:

$$\infty - \infty$$

Example 1.92

$$\lim_{x\to\infty} \sqrt{x+3} - \sqrt{x+2}$$

Rationalize:

$$\lim_{x\to\infty} \sqrt{x+3} - \sqrt{x+2} \times \frac{\sqrt{x+3}+\sqrt{x+2}}{\sqrt{x+3}+\sqrt{x+2}}$$

Multiply the numerator:

$$= \lim_{x\to\infty} \frac{(x+3)-(x+2)}{\sqrt{x+3}+\sqrt{x+2}}$$

Simplify the numerator:

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x+3}+\sqrt{x+2}}$$

The denominator $\to \infty$

$$= 0$$

C. Further Questions

Challenge 1.93

When a line segment/square/cube of side length L is measured using a line segment/square/cube of side length l, the number N of line segments/squares/cubes covering the original line segment/square/cube is given by $\left(\frac{L}{l}\right), \left(\frac{L}{l}\right)^2, \left(\frac{L}{l}\right)^3$ respectively. The $n=0, n=1$, and $n=2$ steps of iteration of the process of obtaining the fractal curve called a Koch curve, of stretching and bending the middle one third of a line segment (of length L) into a triangle shape of side length $\frac{L}{3}$ is shown in the figure. In the second ($n=2$) iteration each of the 4 line segments of length $\frac{L}{3}$ will have their middle stretched and bent into triangle shapes of side $l = \frac{L}{3\times 3}$. After n iterations, determine the dimension, given by $D = -\frac{\log(N)}{\log(l)}$ as $n\to\infty$, $(l\to 0)$. (NSEA, 2023, Adapted)

$$\begin{aligned} n=0 &\Rightarrow N=1, & l &= L \\ n=1 &\Rightarrow N=4, & l &= \frac{L}{3} \\ n=2 &\Rightarrow N=16, & l &= \frac{L}{3^2} \end{aligned}$$

We need to find:

$$D = \lim_{\substack{n\to\infty \\ l\to 0}} -\frac{\log(N)}{\log(l)}$$

Substitute $N = 4^n$, $l = \frac{L}{3^n}$. Drop the condition for $l \to 0$ since the expression no longer has l in it:

$$= \lim_{n\to\infty} -\frac{\log(4^n)}{\log\left(\frac{L}{3^n}\right)}$$

Use the power log rule $\log x^n = n\log x$ and the quotient log rule $\log\left(\frac{a}{b}\right) = \log a - \log b$, and absorb the minus sign in the denominator:

$$= \lim_{n\to\infty} \frac{n\log(4)}{n\log 3 - \log L}$$

Divide numerator and denominator by n:

$$= \lim_{n\to\infty} \frac{\log(4)}{\log 3 - \frac{\log L}{n}}$$

As $n \to \infty, \frac{\log L}{n} \to 0$

$$= \lim_{n\to\infty} -\frac{\log(4)}{-\log 3} = \frac{\log 4}{\log 3}$$

1.7 Infinite Limits

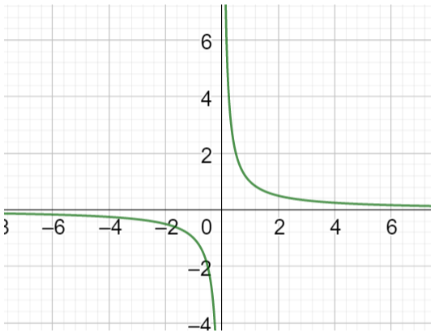
A. Infinite Limits/Vertical Asymptote

Example 1.94

Evaluate the limit:

$$\lim_{x\to 0} \frac{1}{x}$$

x	$\frac{1}{x}$	x	$\frac{1}{x}$
1	1	-1	-1
0.1	10	-0.1	-10
0.01	100	-0.01	-100
0.001	1000	-0.001	-1000



By observation:

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$
$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

Hence,

$$\lim_{x\to 0} \frac{1}{x}\ DNE$$

Example 1.95

Evaluate the limits:

$$\underbrace{\lim_{x\to 1^-} \frac{1}{x-1}}_{\textit{Left Hand Limit}}, \quad \underbrace{\lim_{x\to 1^+} \frac{1}{x-1}}_{\textit{Right Hand Limit}}, \quad \lim_{x\to 1} \frac{1}{x-1}$$

A. Using a Table (Informal)
B. Using properties of limits (Formal)

Part A: Using a Table

Left Hand Limit		Right Hand Limit	
From the Left		From the Right	
x	$\frac{1}{x-1}$	x	$\frac{1}{x-1}$

0.9	$\frac{1}{-0.1}=-10$	1.1	$\frac{1}{0.1}=10$
0.99	$\frac{1}{-0.01}=-100$	1.01	$\frac{1}{0.01}=100$
0.999	$\frac{1}{-0.001}=-1000$	1.001	$\frac{1}{0.001}=1000$
Approaches 1	Approaches $-\infty$	Approaches 1	Approaches $+\infty$

From we the table, we can guess that:

$$\underbrace{\lim_{x\to 1^-}\frac{1}{x-1}}_{\textit{Left Hand Limit}}=-\infty,\qquad \underbrace{\lim_{x\to 1^+}\frac{1}{x-1}}_{\textit{Right Hand Limit}}=+\infty$$

Note that neither the LHL nor the RHL exist, because ∞ is not a number.
Also, we can conclude that:

$$\lim_{x\to 1}\frac{1}{x-1}\ DNE$$

Part B: Using properties of limits

Use a change of variable. Substitute $x=1+h$:

$$\lim_{x\to 1}\frac{1}{x-1}=\lim_{(1+h)\to 1}\frac{1}{(1+h)-1}=\lim_{h\to 0}\frac{1}{h}$$

Now split the limit into left hand limit and right hand limit:

$$Right\ hand\ Limit=RHL=\underbrace{\lim_{h\to 0^+}\frac{1}{h}=+\infty}_{\text{As } h \text{ decreases, } \frac{1}{h} \text{ increases}}$$

$$Left\ Hand\ Limit=LHL=\lim_{h\to 0^-}\frac{1}{h}=-\infty$$

$$\lim_{h\to 0}\frac{1}{h}\ DNE$$

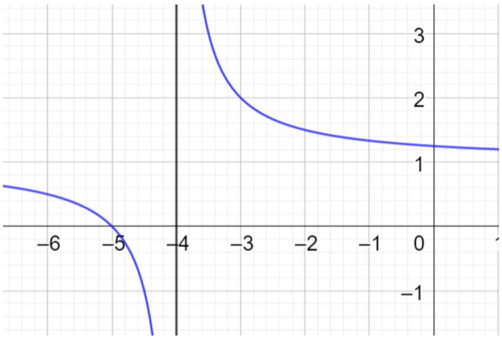
1.96: Step by Step Process for Infinite Limits/Vertical Asymptote

$$\lim_{x\to c}\frac{f(x)}{g(x)}$$

- Step I: Factor $f(x)$ and $g(x)$
- Step II: Cancel any common factors
- The remaining zeroes of $g(x)$ are the vertical asymptotes of the graph
- Step IIIA: If $g(c)\neq 0$, limit can be found by substitution
- Step IIIB: If $g(c)=0$, to check LHL substitute a number to the left of c
- Step IIIC: If $g(c)=0$, to check RHL substitute a number to the right of c

Example 1.97

$$\lim_{x\to -4}\frac{x^2+8x+15}{x^2+7x+12}$$



Step I

$$\lim_{x\to -4} \frac{(x+3)(x+5)}{(x+3)(x+4)}$$

Step II

$$\lim_{x\to -4} \frac{x+5}{x+4}$$

Step III

$$g(-4) = -4 + 4 = 0$$

To find RHL, substitute $x = -3$:

$$\frac{x+5}{x+4} = \frac{-3+5}{-3+4} = \frac{2}{1} = 2$$

Since $2 > 0$

$$RHL = +\infty$$

To find LHL, substitute $x = -4.5$:

$$\frac{x+5}{x+4} = \frac{-4.5+5}{-4.5+4} = \frac{0.5}{-0.5} < 0$$

Since $2 > 0$

$$LHL = -\infty$$

B. Review

Example 1.98

Select an appropriate neighborhood for finding the value of $\lim_{x\to 2} f(x)$

A. $\{-1,0,1,2,3,4,5\}$
B. $\{-2.001, -2.01, -2.1, 2.001, 2.01, 2.1\}$
C. $\{1.9, 1.99, 1.999, 2.001, 2.01, 2.1\}$
D. $\{1.9, 1.99, 1.99, 2, 2.001, 2.01, 2.1\}$

Option A:
The numbers are too far apart.

Option B
-2 is not at all close to 2. Hence, it is not a “neighbour”.

Option D
It includes the number itself.

Example 1.99

True or False

A. Limit of a function at a point depends on the value of the function at a point.
B. Limit of a function at a point depends on the value of the function in the neighbourhood of the point.
C. Limit of a function is always equal to the value of the function at a point.
D. If the limit of a function at a point is not defined, then the function is also not defined.
E. If the function is not defined at a point, then the limit at that point is also not defined

$$True: B$$
$$False: A, C,$$

Part A

$$False$$

Part B

$$True$$

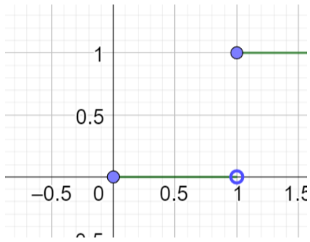
Part C

$$False$$

Part D

False

$$f(x) = \lfloor x \rfloor$$



Part E

False

$$Counterexample: f(x) = x, \qquad x \in \mathbb{R}, \qquad x \neq 2$$

1.8 Limit Properties and Sandwich Theorem

A. Basics

We use the squeeze theorem to find a limit where finding the limit by manipulating the function itself is not convenient. Hence, we "squeeze" the function between two limits.

Example 1.100

Aarohi has n marks in her Math test. The number of marks obtained in the Math test is always an integer. Aarohi's marks are more than or equal to Hasan's marks, and less than or equal to Mary's marks. Find the range of possible values that Aarohi could have got if:

A. Hasan got 3 marks, and Mary got 9 marks.
B. Hasan got 5 marks, and Mary got 5 marks.
C. Hasan got c marks, and Mary got c marks.

Part A

$$\{3,4,5,6,7,8,9\}$$

Part B

$$5$$

Part C

$$c$$

1.101: Sandwich/Squeeze Theorem

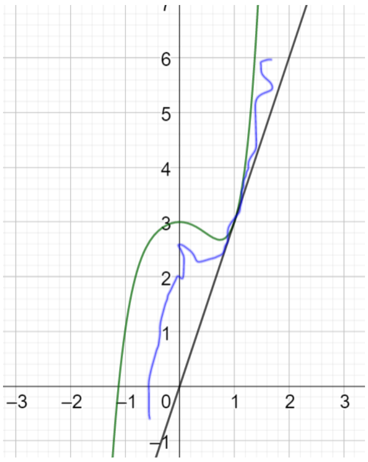
If $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at c itself, and

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

Then

$$\lim_{x\to c} f(x) = L$$

The key skill lies in finding $g(x)$ *and* $h(x)$



Example 1.102

Find $\lim\limits_{x\to 1} g(x)$ given that:

A. $g(x)$ is a function defined for all real numbers

B. $\underbrace{3x}_{\substack{Black\\Line}} \le \underbrace{g(x)}_{\substack{Blue\\Graph}} \le \underbrace{x^5 - x^2 + 3}_{\substack{Green\\Grapj}}$ for all x in the domain of g

Take the limit throughout the inequality:

$$\lim_{x\to 1} 3x \le \lim_{x\to 1} g(x) \le \lim_{x\to 1} x^5 - x^2 + 3$$

$$3 \le \lim_{x\to 1} g(x) \le 3$$

By the Sandwich Theorem:

$$\lim_{x\to 1} g(x) = 3$$

B. Trigonometry

1.103: Range of $\sin\theta$ and $cos\ \theta$

For functions with $\sin\theta$ and $\cos\theta$, a very useful property to remember is:

$$-1 \le \sin\theta \le 1$$

$$-1 \le \cos\theta \le 1$$

Example 1.104

Evaluate using the Sandwich Theorem:

$$\lim_{x\to\infty} \frac{\sin x}{x}$$

The range of the *sin* function is:

$$-1 \le \sin x \le 1$$

Divide throughout by x:

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

Take the limit throughout as $x \to \infty$:

$$\lim_{x\to\infty} -\frac{1}{x} \le \lim_{x\to\infty} \frac{\sin x}{x} \le \lim_{x\to\infty} \frac{1}{x}$$

$$0 \le \lim_{x\to\infty} \frac{\sin x}{x} \le 0$$

$$\lim_{x\to\infty}\frac{\sin x}{x}=0$$

Example 1.105

Evaluate using the Sandwich Theorem:

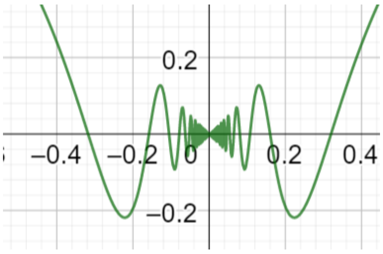
A. $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

B. $\lim_{x\to 0} x^2\cos\frac{1}{x}$

C. $\lim_{x\to\infty}\frac{x+\pi\sin x}{3x+9}$

Part A

We can plot the graph, which suggests that the limit is zero. (But we still need to prove it).



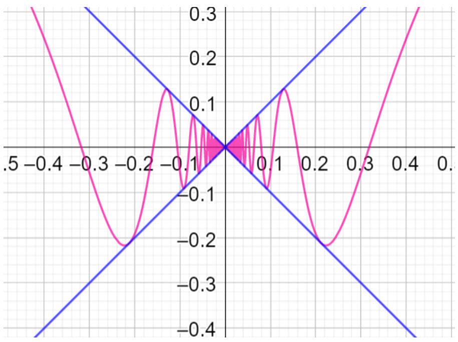
$$-1\le\sin\theta\le 1$$

Let $\theta=\frac{1}{x}$:

$$-1\le\sin\left(\frac{1}{x}\right)\le 1$$

Multiply throughout by x:

$$-x\le x\sin\left(\frac{1}{x}\right)\le x$$



Take the limit as $x\to 0$:

$$\underbrace{\lim_{x\to 0}-x\le\lim_{x\to 0}x\sin\left(\frac{1}{x}\right)\le\lim_{x\to 0}x}_{\textit{Inequality I}}$$

The first and the last limit evaluate to zero:

$$\lim_{x\to 0}-x=\lim_{x\to 0}x=0$$

Substitute the above in Inequality I:

$$0\le\lim_{x\to 0}x\sin\left(\frac{1}{x}\right)\le 0$$

By the Sandwich Theorem:

$$\lim_{x\to 0}x\sin\left(\frac{1}{x}\right)=0$$

Part B

$$-1\le\cos\left(\frac{1}{x}\right)\le 1$$

Multiply throughout by x^2:

$$-x^2\le x^2\cos\frac{1}{x}\le x^2$$

Take the limit of all parts of the inequality as $x\to 0$:

$$\lim_{x\to 0}-x^2\le\lim_{x\to 0}x^2\cos\frac{1}{x}\le\lim_{x\to 0}x^2$$

The first and the last limit evaluate to zero:

$$0\le\lim_{x\to 0}x^2\cos\frac{1}{x}\le 0$$

By the Sandwich Theorem:

$$\lim_{x\to 0}x^2\cos\frac{1}{x}=0$$

Part C

Multiply both sides of $-1\le\sin x\le 1$ by π:

$$-\pi\le\pi\sin x\le\pi$$

Add x:

$$x-\pi\le x+\pi\sin x\le x+\pi$$

Divide by $3x+9$:

$$\frac{x-\pi}{3x+9}\le\frac{x+\pi\sin x}{3x+9}\le\frac{x+\pi}{3x+9}$$

Take the limit of the first and the last part as $x\to\infty$:

$$\lim_{x\to\infty}\frac{x+\pi}{3x+9}=\lim_{x\to\infty}\frac{1+\frac{\pi}{x}}{3+\frac{9}{x}}=\frac{1}{3}$$

$$\frac{1}{3}\le\frac{x+\pi\sin x}{3x+9}\le\frac{1}{3}$$

By the Sandwich Theorem:

$$\lim_{x\to 0}\frac{x+\pi\sin x}{3x+9}=\frac{1}{3}$$

Example 1.106

Consider a circle with radius r. A regular polygon with n sides inscribed in the circle has area $\frac{1}{2}r^2 n\sin\left(\frac{2\pi}{n}\right)$, and a regular polygon with n sides circumscribed around the circle has area $r^2 n\tan\frac{\pi}{n}$.[1]
Use the sandwich theorem to show that the area of a circle is πr^2.

Let the area of the circle be A.
Note that the area of the circle is greater than the area of the inscribed polygon, but less than the area of the circumscribed polygon. That is, the area of the circle is sandwiched between the two areas:

$$\frac{1}{2}r^2 n\sin\left(\frac{2\pi}{n}\right) < A < r^2 n\tan\frac{\pi}{n}$$

As the value of n increases, the difference gets smaller and smaller. Take the limit of the inequality as $n \to \infty$:

$$\lim_{n\to\infty}\frac{1}{2}r^2 n\sin\left(\frac{2\pi}{n}\right) \le A \le \lim_{n\to\infty} r^2 n\tan\frac{\pi}{n}$$

Evaluate the limit on the left side:

$$\frac{1}{2}r^2\lim_{n\to\infty} n\sin\left(\frac{2\pi}{n}\right) = \frac{1}{2}r^2\lim_{n\to\infty} n\left(\frac{2\pi}{n}\right)\frac{\sin\left(\frac{2\pi}{n}\right)}{\left(\frac{2\pi}{n}\right)} = \frac{1}{2}r^2\lim_{n\to\infty} 2\pi = \frac{1}{2}r^2(2\pi) = \pi r^2$$

Evaluate the limit on the right side:

$$r^2\lim_{n\to\infty} n\frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}} = r^2\lim_{n\to\infty} n\sin\frac{\pi}{n} = r^2\lim_{n\to\infty} n\cdot\frac{\pi}{n}\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} = r^2\lim_{n\to\infty}\pi = \pi r^2$$

And by the Sandwich Theorem:

$$\lim_{n\to\infty} A = A = \pi r^2$$

C. Other Functions

Example 1.107

Let $f: R \to R$ be a positive increasing function with $\lim_{x\to\infty}\frac{f(3x)}{f(x)} = 1$. Then, $\lim_{x\to\infty}\frac{f(2x)}{f(x)}$ is equal to **(AIEEE 2010)**

Since $f(x)$ is an increasing function:

$$f(x) < f(2x) < f(3x)$$

Divide throughout by $f(x)$:

$$1 < \frac{f(2x)}{f(x)} < \frac{f(3x)}{f(x)}$$

Take the limit throughout as $x \to \infty$:

$$\lim_{x\to\infty} 1 \le \lim_{x\to\infty}\frac{f(2x)}{f(x)} \le \lim_{x\to\infty}\frac{f(3x)}{f(x)}$$

$$1 \le \lim_{x\to\infty}\frac{f(2x)}{f(x)} \le 1$$

By the Sandwich Theorem:

$$\lim_{x\to\infty}\frac{f(2x)}{f(x)} = 1$$

Example 1.108

[1] You can see a proof on these properties in the Note on Trigonometry.

$$\lim_{n\to\infty}\left\{\left(2^{\frac{1}{2}}-2^{\frac{1}{3}}\right)\left(2^{\frac{1}{2}}-2^{\frac{1}{5}}\right)\dots\left(2^{\frac{1}{2}}-2^{\frac{1}{2n+1}}\right)\right\}= \ (JEE\ Main\ 2023)$$

1.109: Range of Greatest Integer, Least Integer, and Fractional Part Functions

$$0\le\{x\}<1$$
$$x-1<\lfloor x\rfloor\le x$$
$$x\le\lceil x\rceil<x+1$$

- $\{x\}$ represents the fractional part function.
- $\lfloor x\rfloor$ represents the greatest integer function (GIF), that is the greatest integer less than or equal to x.
  - ✓ It is also called the floor function.
- $\lceil x\rceil$ represents the least integer function, that is the least integer greater than or equal to x.
  - ✓ It is also called the ceiling function.

Example 1.110

Evaluate by using the Sandwich Theorem:

A. $\lim_{n\to\infty}\frac{\{x\}}{x}$, where $[x]$ represents the greatest integer less than or equal to x.

B. $\lim_{n\to\infty}\frac{[x]}{x}$, where $[x]$ represents the greatest integer less than or equal to x.

Part A

$$0\le\{x\}\le 1$$

Divide throughout by x:

$$\frac{0}{x}\le\{x\}\le\frac{1}{x}$$

Take the limit as $x\to\infty$:

$$\lim_{x\to\infty}\frac{0}{x}\le\lim_{x\to\infty}\frac{\{x\}}{x}\le\lim_{x\to\infty}\frac{1}{x}$$
$$0\le\lim_{x\to\infty}\frac{\{x\}}{x}\le 0$$

By the Sandwich Theorem:

$$\lim_{x\to\infty}\frac{\{x\}}{x}=0$$

Part B

$$x-1<[x]\le x$$

Divide throughout by x:

$$\frac{x-1}{x}<\frac{[x]}{x}\le\frac{x}{x}$$
$$1-\frac{1}{x}<\frac{[x]}{x}\le 1$$

Take the limit as $x\to\infty$:

$$\lim_{x\to\infty}1-\frac{1}{x}\le\lim_{x\to\infty}\frac{[x]}{x}\le\lim_{x\to\infty}1$$
$$1\le\lim_{x\to\infty}\frac{[x]}{x}\le 1$$

By the Sandwich Theorem:

$$\lim_{x\to\infty}\frac{[x]}{x}=1$$

Example 1.111

The value of $\lim_{n\to\infty}\frac{[r]+[2r]+\dots+[nr]}{n^2}$, where r is a non-zero real number and $[r]$ denotes the greatest integer less than or equal to r, is equal to (JEE-M 2021)

$$r=[r]+\{r\}$$
$$[r]=r-\{r\},0\le\{r\}<1$$

$$r-1<[r]\le r$$
$$2r-1<[2r]\le 2r$$
$$3r-1<[3r]\le 3r$$
$$.$$

$$\vdots$$

$$nr - 1 < [nr] \le nr$$

Add all of the above inequalities:

$$r + 2r + \cdots + nr - n < [r] + [2r] + \cdots + [nr] \le r + 2r + \cdots + nr$$

Use the formula for the sum of the first n natural numbers:

$$\frac{n(n+1)r}{2} - n < [r] + [2r] + \cdots + [nr] \le \frac{n(n+1)r}{2}$$

Divide throughout by n^2, and split the fractions:

$$\frac{n^2r}{2n^2} + \frac{nr}{2n^2} - \frac{2n}{n^2} < \frac{[r] + [2r] + \cdots + [nr]}{n^2} \le \frac{n^2r}{2n^2} + \frac{nr}{2n^2}$$

Simplify:

$$\frac{r}{2} + \frac{r}{2n} - \frac{2}{n} < \frac{[r] + [2r] + \cdots + [nr]}{n^2} \le \frac{r}{2} + \frac{r}{2n}$$

Take the limit as $n \to \infty$ throughout:

$$\lim_{n\to\infty} \frac{r}{2} + \frac{r}{2n} - \frac{2}{n} < \lim_{n\to\infty} \frac{[r] + [2r] + \cdots + [nr]}{n^2} \le \lim_{n\to\infty} \frac{r}{2} + \frac{r}{2n}$$

Simplify:

$$\frac{r}{2} < \lim_{n\to\infty} \frac{[r] + [2r] + \cdots + [nr]}{n^2} \le \frac{r}{2}$$

By the Sandwich Theorem:

$$\lim_{n\to\infty} \frac{[r] + [2r] + \cdots + [nr]}{n^2} = \frac{r}{2}$$

Example 1.112

Logarithms

D. Even and Odd Functions

We have already seen left hand and right-hand limits graphically. Similarly, we have also seen continuity graphically. Continuity is $defined$ in terms of limits (and limits are defined using inequalities, not using continuity – that would be circular.). We now look at the limits and continuity in greater detail in this chapter.

1.113: Right Hand Limit

If $\boldsymbol{f(x)}$ is defined on an interval $(\boldsymbol{c}, \boldsymbol{b})$ where $\boldsymbol{c < b}$, and approaches arbitrarily close to $\boldsymbol{L}$ as $\boldsymbol{x}$ approaches $\boldsymbol{c}$ from within $(\boldsymbol{c}, \boldsymbol{b})$, then $\boldsymbol{f}$ has right-hand limit $\boldsymbol{L}$ at $\boldsymbol{c}$.

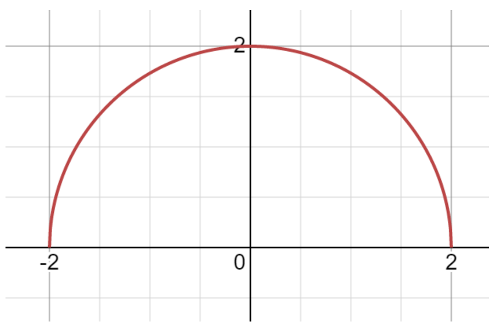
1.114: Left Hand Limit

If $\boldsymbol{f(x)}$ is defined on an interval $(\boldsymbol{a}, \boldsymbol{c})$ where $\boldsymbol{a < c}$, and approaches arbitrarily close to $\boldsymbol{L}$ as $\boldsymbol{x}$ approaches $\boldsymbol{c}$ from within $(\boldsymbol{a}, \boldsymbol{c})$, then $\boldsymbol{f}$ has left-hand limit $\boldsymbol{L}$ at $\boldsymbol{c}$.

Example 1.115

The function $f(x) = \sqrt{4 - x^2}$ is graphed alongside. Find:

A. $\lim_{x\to 2^-} f(x)$
B. $\lim_{x\to 2^+} f(x)$
C. $\lim_{x\to 2} f(x)$
D. $\lim_{x\to -2^-} f(x)$
E. $\lim_{x\to -2^+} f(x)$



F. $\lim_{x\to -2} f(x)$

Find the Domain

The square root cannot be negative. Hence it must be zero or positive. Hence, we need

$$4 - x^2 \geq 0$$

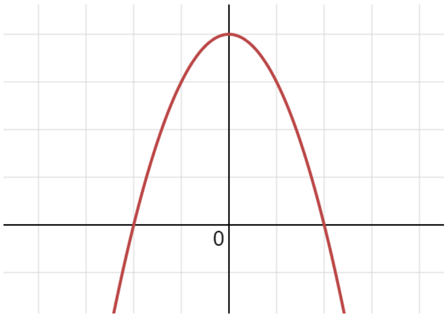
Solve the corresponding quadratic:

$$4 - x^2 = 0 \Rightarrow 4 = x^2 \Rightarrow x = \pm 2$$

When we write the above quadratic in standard form, we get:

$$-x^2 + 4 = 0$$

And since the leading coefficient is negative, this is a downward parabola.



And hence, the domain is:

$$[-2,2]$$

Parts A, B, and C

Part A

$$\lim_{x\to 2^-} f(x) = \sqrt{4 - 2^2} = \sqrt{4-4} = 0$$

Part B

Note that $f(x)$ is not defined to the right of 2.
Hence, we cannot check the behaviour of $f(x)$ to the right of 2.
Hence, the limit does not exist.

$$In\ Short: \lim_{x\to 2^+} f(x)\ does\ not\ exist\ since\ f(x)\ is\ not\ defined\ to\ the\ right\ of\ 2.$$

Part C

$$\lim_{x\to 2} f(x)\ \ does\ not\ exist\ because\ RHL\ is\ not\ defined$$

Parts D, E, and F

Part D

$$\lim_{x\to -2^-} f(x)\ does\ not\ exist\ since\ f(x)\ is\ not\ defined\ to\ the\ left\ of\ 2.$$

Part E

$$\lim_{x\to -2^+} f(x) = \sqrt{4 - (-2^2)} = \sqrt{4-4} = \sqrt{0} = 0$$

Part F

$$\lim_{x\to 2} f(x)\ \ does\ not\ exist\ because\ LHL\ is\ not\ defined$$

Example 1.116

If $f(x)$ is an odd function such that $\lim_{x\to 0^+} f(x) = 5$, find $\lim_{x\to 0^-} f(x)$.

x	$-x$
0.1	-0.1
0.01	-0.01
0.001	-0.001

From the table above, when $x \to 0^+$ we know that $-x \to 0^-$. Hence,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(-x)$$

Now, since $f(x)$ is an odd function, we have $f(-x) = -f(x)$ and hence:

$$\lim_{x\to 0^+} f(-x) = \lim_{x\to 0^+} -f(x) = -\lim_{x\to 0^+} f(x) = -5$$

Example 1.117

If $f(x)$ is an even function such that $\lim_{x\to 3^+} f(x) = 2$, what, if anything can we say about:

A. $\lim_{x\to -3^-} f(x)$
B. $\lim_{x\to -3^+} f(x)$

Use a change of variable. Let

$$h = x - 3 \Rightarrow x = 3 + h$$
$$As\ x \to 3^+, h = x - 3 \to 0^+$$

$$\lim_{x\to 3^+} f(x) = 2 \Rightarrow \lim_{h\to 0^+} f(3+h) = 2$$

Part A

$$\lim_{x\to -3^-} f(x)$$

Use a change of variable. Let:

$$h = x + 3 \Rightarrow x = -3 + h$$
$$\lim_{x\to -3^-} h = \lim_{x\to -3^-} x + 3 = 0^+$$

Make the substitutions:

$$= \lim_{h\to 0^+} f(-3-h) = \lim_{h\to 0^+} f\big(-(3+h)\big)$$

Now, since $f(x)$ is an even function, we have $f(x) = f(-x)$ and hence:

$$\lim_{h\to 0^+} f\big(-(3+h)\big) = \lim_{h\to 0^+} f(3+h) = 2$$

Part B

$$\lim_{x\to -3^+} f(x) = \lim_{h\to 0^-} f(-3-h)$$
$$\lim_{x\to -3^+} f(x) = \lim_{h\to 0^+} f(-3+h)$$

Hence, we cannot conclude anything about this expression.

1.9 Limits with *e*

A. Basics

B.

Example 1.118

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$$

We know that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$. Hence, let $n = 2x$

$$\lim_{x\to\infty}\left(1+\frac{2}{2x}\right)^{2x} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x} = \lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^x\right]^2$$

Using the Power Rule for limits, we can interchange the limit operator and the exponentiation:

$$= \left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right]^2 = e^2$$

Example 1.119

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n$$

We know that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$. Hence, let $n = rx$, where $r > 0$

$$\lim_{x\to\infty}\left(1+\frac{r}{rx}\right)^{rx} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{rx} = \lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^x\right]^r$$

Using the Power Rule for limits, we can interchange the limit operator and the exponentiation:

$$= \left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right]^r = e^r$$

1.10 Continuity

A. Continuous Graphs

1.120: Continuity: Informal Definition

If the graph of a function can be drawn without lifting your pencil, then it forms a single, smooth curve. It does not have any jumps, or jerks.

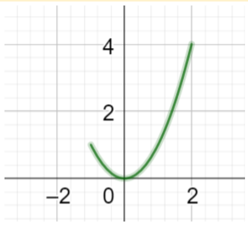
These kinds of graphs are called continuous graphs.

Example 1.121

True or False

A. For a function to be continuous, it must have a domain of all real numbers.

$$A: False$$
$$f(x) = x^2, \qquad -1 < x < 2$$



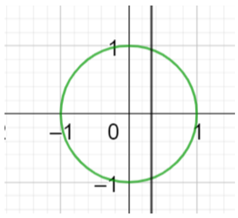
Example 1.122

Decide whether the following represent continuous functions:

A. $x^2 + y^2 = 1$
B. $f(x) = y = x^2, D_f = (0,3)$

Part A

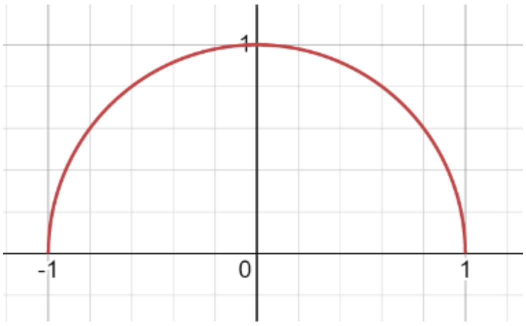
$x^2 + y^2 = 1$ is a circle and fails the vertical line test.



Part B

$f(x)$ is continuous over its domain, which is the open interval

$$(0,3)$$



Example 1.123

Consider $f(x) = \sqrt{1-x^2}$ graphed alongside. Evaluate:

A. $\lim_{x\to 0} f(x)$
B. $\lim_{x\to \frac{3}{5}} f(x)$

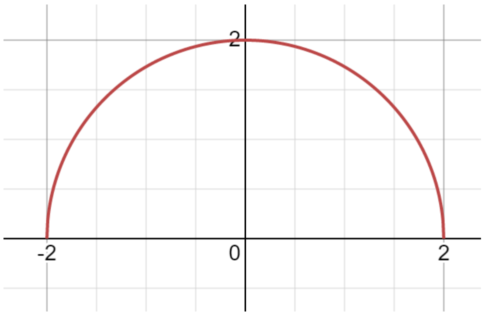
Part A

This is a continuous function. Hence, the value of the limit is just the value of the function at that point:

$$\lim_{x\to 0} f(x) = f(0) = 1$$

Part B

$$\lim_{x\to \frac{3}{5}} f(x) = \frac{4}{5}$$



Example 1.124

The function $f(x) = \sqrt{4-x^2}$ is graphed alongside.

A. Find the domain of $f(x)$.
B. Evaluate $\lim_{x\to 2} f(x)$, if it exists.
C. Evaluate $\lim_{x\to -2} f(x)$, if it exists
D. At what points, if any, does $\lim_{x\to a} f(x)$ exist?

Part A

From the graph, the domain of the function is:

$$[-2,2]$$

Part B

$$\lim_{x\to 2^-} f(x) = f(2) = 0$$

Note that $f(x)$ is not defined to the right of 2.
Hence, we cannot check the behaviour of $f(x)$ to the right of 2.
Hence, the right-hand limit does not exist.

$$In\ Short\text{:}\ \lim_{x\to 2^+} f(x)\ does\ not\ exist\ since\ f(x)\ is\ not\ defined\ to\ the\ right\ of\ 2.$$
$$\lim_{x\to 2} f(x)\ \ does\ not\ exist\ because\ RHL\ is\ not\ defined$$

Part C

$$\lim_{x\to -2^-} f(x)\ does\ not\ exist\ since\ f(x)\ is\ not\ defined\ to\ the\ left\ of\ 2.$$
$$\lim_{x\to -2^+} f(x) = f(-2) = 0$$
$$\lim_{x\to 2} f(x)\ \ does\ not\ exist\ because\ LHL\ is\ not\ defined$$

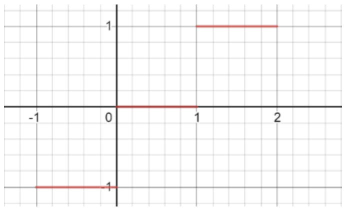
Part D

$$(-2,2)$$

B. Jump Discontinuity

1.125: Jump Discontinuity

The function jumps at a point, and suddenly takes another value.



Example 1.126

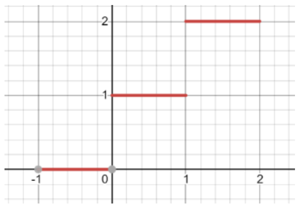
Consider the floor function

$$y = f(x) = \lfloor x \rfloor$$

Which is defined as the largest integer which is greater than or equal to x. Identify the kind of discontinuity which the floor function has.

The floor function jumps at multiple places from an integer to the next integer. It does not take any values which are not integers.
Hence, the floor function exhibits jump discontinuity.

Example 1.127

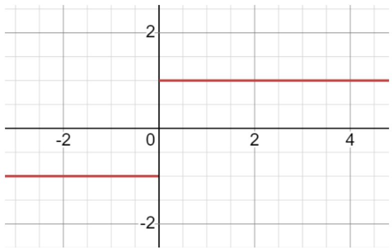


Consider the ceiling function

$$y = f(x) = \lceil x \rceil$$

Which is defined as the smallest integer which is larger than or equal to x. Identify the kind of discontinuity which the ceiling function has.

The ceiling function jumps at multiple places from an integer to the next integer. It does not take any values which are not integers.
Hence, the ceiling function exhibits jump discontinuity.



Example 1.128

The signum function $f(x) = \frac{x}{|x|}$ is used to get the sign of a number. Find

$$\lim_{x\to 0} f(x)$$

$$RHL = \lim_{x\to 0^+} f(x) = 1$$
$$LHL = \lim_{x\to 0^+} f(x) = 1$$
$$RHL \neq LHL$$
$$Limit\ DNE$$

1.129: Limits with Jump Discontinuities

A function does not have a limit at the point where it has a jump discontinuity.

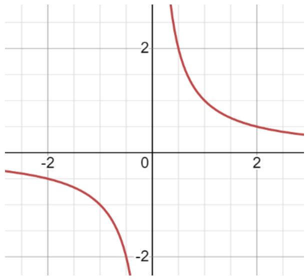
At a jump discontinuity, the RHL is not equal to the LHL.
Hence, the function does not a limit at a jump discontinuity.

C. Infinite Discontinuity

When the denominator of a fraction become zero at some points, then we have some interesting behaviour:

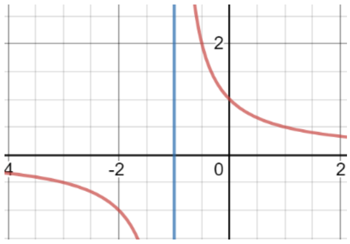
- Those points are not in the domain of the function
- The limit of the function at the point goes to positive infinity or negative infinity.

1.130: Infinite Discontinuity



- In an infinite discontinuity, the function approaches positive infinity or negative infinity or both.
- An infinite discontinuity occurs along with a vertical asymptote.
- A vertical asymptote is caused by the denominator of a function becoming zero at a point.
  - ✓ There is an exception (which we will consider when we removable discontinuity)

This will often happen because of a point which is not defined in the domain of f due to the denominator becoming zero.



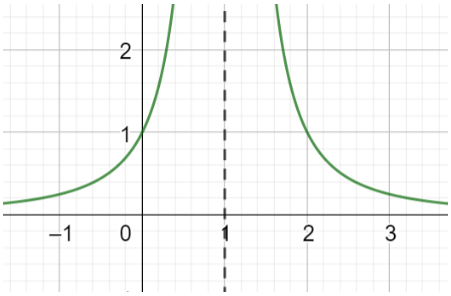
Example 1.131

The graph of $f(x)=\frac{1}{x+1}$ has a discontinuity at $x=-1$. Identify the kind of discontinuity.

As x approaches -1 from the left, the value of $f(x)$ approaches $-\infty$
As x approaches -1 from the right, the value of $f(x)$ approaches ∞

Hence, since the value of the function approaches infinity near the point of discontinuity, this function has an infinity discontinuity.

Example 1.132



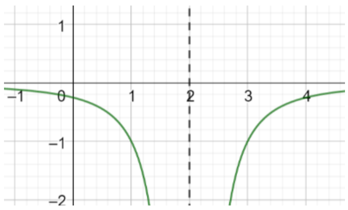
The graph of

$$y=f(x)=\frac{1}{(x-1)^2}$$

Has a discontinuity at $x=1$. Identify the kind of discontinuity.

As x approaches 1 from the left, the value of $f(x)$ approaches ∞
As x approaches 1 from the right, the value of $f(x)$ approaches ∞

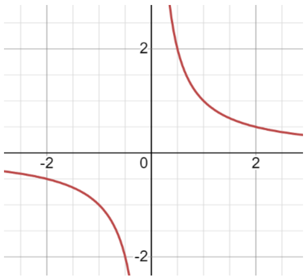
Example 1.133



The graph of $y=f(x)=\frac{-1}{(x-2)^2}$ has a discontinuity at $x=2$. Identify the kind of discontinuity.

As x approaches 2 from the left, the value of $f(x)$ approaches $-\infty$
As x approaches 2 from the right, the value of $f(x)$ approaches $-\infty$

Hence, it is an infinite discontinuity.



1.134: Infinite Limits are not limits

The limit does not exist.

$\lim_{x\to a} f(x)=\infty$ means that the value of the function increases without bound as x

approaches a.

Example 1.135

The function $f(x)=\frac{1}{x}$ is graphed alongside. Find $\lim_{x\to 0} f(x)$

Left Hand Limit

As x approaches 0 from the left, y approaches $-\infty$

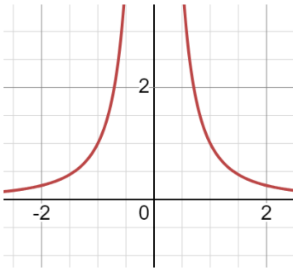
$$\lim_{x\to 0^-}\frac{1}{x}=-\infty$$

Right Hand Limit

As x approaches 0 from the right, y approaches ∞

$$\lim_{x\to 0^+}\frac{1}{x}=\infty$$

Approaching from the left		Approaching from the right	
x	$y=\frac{1}{x}$	x	$y=\frac{1}{x}$
-0.1	-10	0.1	10
-0.01	-100	0.01	100
-0.001	-1000	0.001	1000



Overall Limit

Since the two limits are different, the overall limit does not exist.

Example 1.136

The function $f(x)=\frac{1}{x^2}$ is graphed alongside. Find $\lim_{x\to 0} f(x)$.

Left Hand Limit

As x approaches 0 from the left, y approaches ∞

$$\lim_{x\to 0^-}\frac{1}{x^2}=\infty$$

Right Hand Limit

As x approaches 0 from the right, y approaches ∞

$$\lim_{x\to 0^+}\frac{1}{x^2}=\infty$$

Approaching from the left		Approaching from the right	
x	$y=\frac{1}{x^2}$	x	$y=\frac{1}{x^2}$
-0.1	100	0.1	100
-0.01	10,000	0.01	10,000
-0.001	1,000,000	0.001	1,000,000

Overall Limit

The overall limit is:

$$\lim_{x\to 0}\frac{1}{x^2}=\infty$$

D. Removable Discontinuity

1.137: Removable Discontinuity

If a function is otherwise continuous, but has a “hole”, which can be filled by defining the function suitably at the hole, then the function is said to have a removable discontinuity.

Example 1.138

Find the removable discontinuity in $f(x)=\frac{x^2-1}{x+1}$, and remove it by defining a piece-wise function. Do not change the definition of the function over its original domain.

Find the discontinuity

When the denominator is not zero:

$$f(x) = \frac{x^2-1}{x+1} = \frac{(x+1)(x-1)}{x+1} = x-1$$

The denominator is zero when:

$$x+1=0 \Rightarrow x=-1$$

Hence, the function is not defined at:

$$x=-1$$

Hence, the implicit domain of the function is all real numbers except -1:

$$(-\infty,-1)\cup(-1,\infty)$$

There is a discontinuity at $x=-1$.

Remove the discontinuity

By defining it a piece-wise

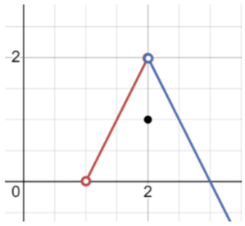
$$y=f(x)=\begin{cases}\dfrac{x^2-1}{x+1}, & x\in\mathbb{R}, x\neq -1\\ -3, & x=-3\end{cases}$$

Example 1.139

$$y=f(x)=\begin{cases}2x-2, & 1<x<2\\ -2x+6, & 2<x<3\\ 1, & x=2\end{cases}$$

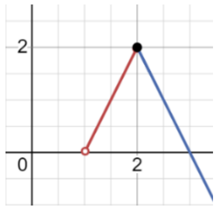
A. What kind of discontinuity does $f(x)$ have?

B. Change the function to remove the discontinuity.



Part A

Since this hole can be plugged by changing the definition of the function, this is a removable discontinuity.



Part B

We can remove the discontinuity by changing the definition of the function at $x=2$.

$$y=f(x)=\begin{cases}2x-2, 1<x<2\\ -2x+6, 2<x<3\\ 2, x=2\end{cases}$$

E. Cusps/Sharp Turns

1.140: Sharp Turn

A sharp turn is a point on a graph where the graph changes direction sharply (not smoothly).

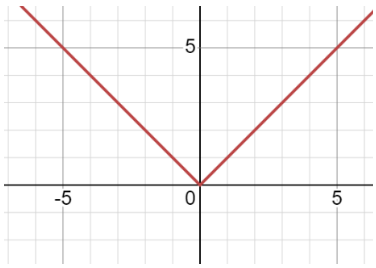
Example 1.141

The function $f(x)=|x|$ is graphed alongside. Find:

A. $\lim\limits_{x\to 0^-} f(x)$

B. $\lim\limits_{x\to 0^+} f(x)$

C. $\lim\limits_{x\to 0} f(x)$



$$LHL = \lim_{x\to 0^-} f(x) = 0$$

$$RHL = \lim_{x \to 0^+} f(x) = 0$$
$$\lim_{x \to 0} f(x) = 0$$

F. Algebraic Continuity

The value of a limit does not depend on the value of the function at the point. However, the value of a function at a point does matter for continuuity.

1.142: Right Continuity at a point

A function is right-continuous at a point c in its domain if:

$$\underbrace{f(c)}_{\substack{Value\ of\ the \\ Function}} = \underbrace{\lim_{x \to c+} f(x)}_{\substack{Value\ of \\ the\ RHL}}$$

Right continuity is of particular interest at:

- A left endpoint of a function
- A jump discontinuity

1.143: Left Continuity at a point

A function is left-continuous at a point c in its domain if:

$$\underbrace{f(c)}_{\substack{Value\ of\ the \\ Function}} = \underbrace{\lim_{x \to c-} f(x)}_{\substack{Value\ of \\ the\ LHL}}$$

1.144: Continuity at a point

A function is continuous at a point c in its domain if:

$$\underbrace{f(c)}_{\substack{Value\ of\ the \\ Function}} = \underbrace{\lim_{x \to c} f(x)}_{\substack{Value\ of \\ the\ Limit}}$$

Alternately, a function is continuous at a point c in its domain if it is both right-continous, and left-continous at that point.

And recall that a limit exists if and only if the corresponding right-hand limit and left hand limit both exist and are equal:

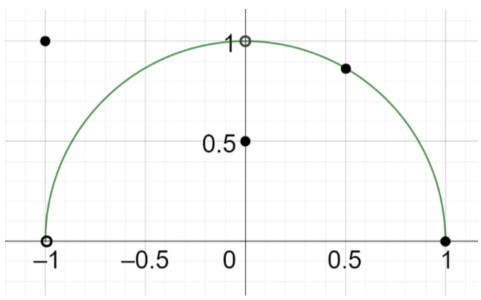
$$\lim_{x \to c} f(x) = \underbrace{\lim_{x \to c^-} f(x)}_{Left\ Hand\ Limit} = \underbrace{\lim_{x \to c^+} f(x)}_{Right\ Hand\ Limit}$$

Example 1.145

The function graphed alongside is given by the piecewise definition:

$$f(x) = \begin{cases} \sqrt{1 - x^2}, & -1 < x < 0, \quad 0 < x < 1 \\ 1, & x = -1 \\ 0.5, & x = 0 \end{cases}$$

Identify whether the following statements are true or false. If they are false, explain why they are false.



Continuity at the left endpoint: $x = 1$

A. $\lim_{x \to 1^+} f(x) = 1$

B. $\lim_{x\to 1^-} f(x) = 1$
C. $\lim_{x\to 1^-} f(x) = 0$
D. $\lim_{x\to 1} f(x) = 0$
E. $f(x)$ is continuous at $x = 1$.
F. $f(x)$ is left-continuous at $x = 1$.
G. $f(x)$ is right-continuous at $x = 1$.

Continuity at the left endpoint: $x = 1$

$$\lim_{x\to 1+} f(x)\text{ DNE} \Rightarrow A\ is\ False$$
$$\lim_{x\to 1-} f(x) = 0 \Rightarrow \text{B is False, C is True}$$

For the overall limit to be defined:

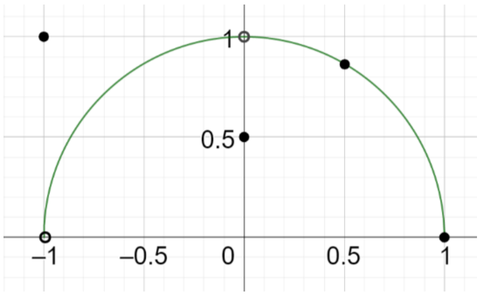
$$LHL = RHL, Both\ must\ exist$$
$$\lim_{x\to 1} f(x)\ DNE \Rightarrow D\ is\ False$$
$$\lim_{x\to 1} f(x)\ \ DNE \Rightarrow E\ is\ False$$
$$\lim_{x\to 1-} f(x) = f(1) = 0 \Rightarrow F\ is\ True$$
$$G\ is\ False$$

Example 1.146

The function graphed alongside is given by the piecewise definition:

$$f(x) = \begin{cases} \sqrt{1-x^2}, & -1 < x < 0, 0 < x < 1 \\ 1, & x = -1 \\ 0.5, & x = 0 \end{cases}$$

Identify whether the following statements are true or false. If they are false, explain why they are false.



Continuity at the left endpoint: $x = -1$

A. $\lim_{x\to -1^+} f(x) = 1$
B. $\lim_{x\to -1^+} f(x) = 0$
C. $\lim_{x\to -1^-} f(x) = 1$
D. $\lim_{x\to -1^-} f(x) = 0$
E. $\lim_{x\to -1} f(x) = 0$
F. $f(x)$ is continuous at $x = -1$.
G. $f(x)$ is left-continuous at $x = -1$.
H. $f(x)$ is right-continuous at $x = -1$.

Continuity at the right endpoint: $x = -1$

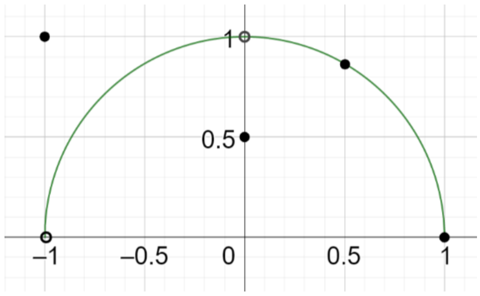
$$\lim_{x\to -1^+} f(x) = 0 \Rightarrow \text{A is false}$$
$$\lim_{x\to -1^-} f(x)\text{ DNE} \Rightarrow \text{B, C are false}$$
$$LHL \neq RHL \Rightarrow D\ is\ false$$
$$LHL\ DNE \Rightarrow E, F\ are\ false$$
$$f(-1) = 1$$
$$\lim_{x\to -1^+} f(x) = 0$$
$$f(-1) \neq \lim_{x\to -1^+} f(x) \Rightarrow G\ is\ False$$

Example 1.147

The function graphed alongside is given by the piecewise definition:

$$f(x) = \begin{cases} \sqrt{1-x^2}, & -1 < x < 0, 0 < x < 1 \\ 1, & x = -1 \\ 0.5, & x = 0 \end{cases}$$

Identify the following statements as true or false. If they are false, explain why they are false.



Continuity in the interior: $x = 0$

A. $\lim_{x\to 0+} f(x) = 1$
B. $\lim_{x\to 0+} f(x) = 0.5$
C. $\lim_{x\to 0-} f(x) = 1$
D. $\lim_{x\to 0-} f(x) = 0.5$
E. $\lim_{x\to 0} f(x) = 1$
F. $\lim_{x\to 0} f(x) = 0.5$
G. $f(x)$ is left-continuous at $x = 0$
H. $f(x)$ is right-continuous at $x = 0$
I. $f(x)$ is continuous at $x = 0$

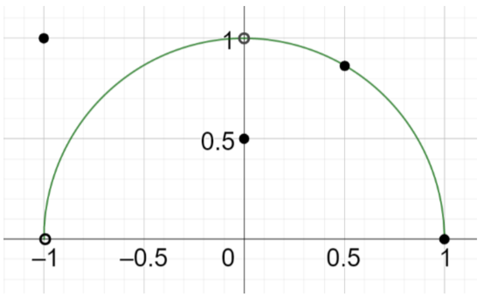
$$\begin{gathered} A: True \\ B: False \\ C: True \\ D: False \\ E: True \\ F: False \\ G: False \\ H: False \\ I: False \end{gathered}$$

Example 1.148

The function graphed alongside is given by the piecewise definition:

$$f(x) = \begin{cases} \sqrt{1-x^2}, & -1 < x < 0, 0 < x < 1 \\ 1, & x = -1 \\ 0.5, & x = 0 \end{cases}$$

Identify the following statements as true or false. If they are false, explain why they are false.



Continuity in the interior: $x = 0.5$

A. $\lim_{x\to 0.5+} f(x) = 0.5$
B. $\lim_{x\to 0.5^+} f(x) = \frac{\sqrt{3}}{2}$
C. $\lim_{x\to 0.5-} f(x) = \frac{\sqrt{3}}{2}$
D. $\lim_{x\to 0.5-} f(x) = 0.5$
E. $\lim_{x\to 0.5} f(x) = \frac{\sqrt{3}}{2}$

F. $\lim\limits_{x\to 0.5} f(x) = \frac{\sqrt{3}}{4}$
G. $f(x)$ is left-continuous at $x = 0.5$
H. $f(x)$ is right-continuous at $x = 0.5$
I. $f(x)$ is continuous at $x = 0.5$

$$f(0.5) = \sqrt{1-\left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \Rightarrow A\ is\ False$$
$$B: True$$
$$C: True$$
$$D: False$$
$$E: True$$
$$F: False$$
$$G: True$$
$$H: True$$
$$I: True$$

1.149: Continuity over an open interval

A function is continuous over an open interval (a, b) if:

- it is continuous at all points in (a, b)

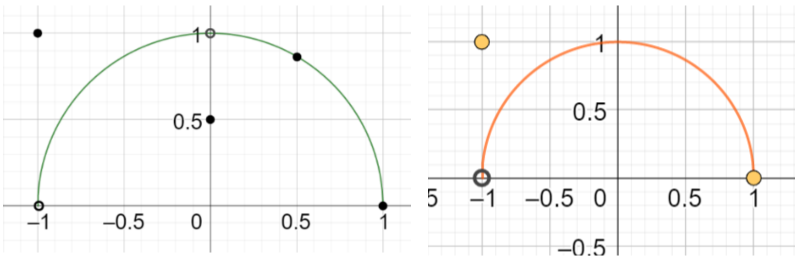
1.150: Continuity over a closed interval

A function is continuous over a closed interval $[a, b]$ if the following three conditions are met:

- It is right-continuous at a
- It is left-continuous at b
- it is continuous at all points in (a, b)

Example 1.151

The graphs of some functions are drawn. Each function is drawn in a different color. Identify the place(s) at which the functions are continuous?



Green Function

The function is continuous over:

$$(-1,0) \cup (0,1]$$

Orange Function

The function is continuous over:

$$(-1,1]$$

G. Jump Discontinuity

1.152: Jump Discontinuity

A jump discontinuity occurs when the lefthand limit and the right hand exist, but they are not equal to each other.

$$\underbrace{LHL}_{Left\ Hand\ Limit} \neq \underbrace{RHL}_{Right\ Hand\ Limit}$$

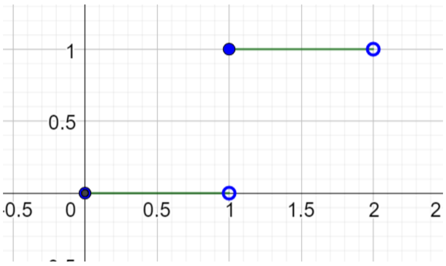
Here, the function jumps at a point, and suddenly takes another value.

Example 1.153

Consider the floor function

$$y = f(x) = \lfloor x \rfloor$$

Which is defined as the largest integer which is greater than or equal to x. Identify the kind of discontinuity which the floor function has using limits.



Determine the following:

A. $\lim_{x\to 1^-} f(x)$
B. $\lim_{x\to 1^+} f(x)$
C. $\lim_{x\to 1} f(x)$
D. $f(1)$
E. Is the function left-continuous at $x = 1$
F. Is the function right-continuous at $x = 1$
G. Is the function continuous at $x = 1$

$$\lim_{x\to 1^-} f(x) = 0$$
$$\lim_{x\to 1^+} f(x) = 1$$
$$RHL \neq LHL \Rightarrow \lim_{x\to 1} f(x)\ DNE$$
$$f(1) = 1$$
$$Not\ left-continuous$$
$$Right-continuous$$
$$Right-continuous, but\ not\ left-continuous \Rightarrow Overall\ not\ continuous$$

Example 1.154

$f(x)$ is a function with a domain of all real numbers, defined as follows:

$$f(x) = \begin{cases} x, & x < 0 \\ x + c, & x \geq 0 \end{cases}$$

A. Show that if $c = 1$, the function has a jump discontinuity.
B. Find the value of c that makes the function continuous.

Part A

$$f(x) = \begin{cases} x, x < 0 \\ x + 1, & x \geq 0 \end{cases}$$

Find the limit at $x = 0$:

$$LHL = \lim_{x\to 0^-} f(x) = 0$$
$$RHL = \lim_{x\to 0^+} f(x) = 1$$
$$LHL \neq RHL, but\ both\ exist$$
$$Jump\ Discontinuity$$

Part B

If the function is continuous, then

$$f(0) = \lim_{x\to 0} f(x)$$

Also, for the limit to exist, both

$$\lim_{x\to 0} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$

$$\lim_{x\to 0^-} x = \lim_{x\to 0^+} x + c$$
$$0 = 0 + c$$
$$c = 0$$

H. Infinite Discontinuity

1.155: Infinite Discontinuity

$$\lim_{x \to c^-} f(x) = \pm\infty \; OR \; \lim_{x \to c^+} f(x) = \pm\infty \; OR \; Both$$

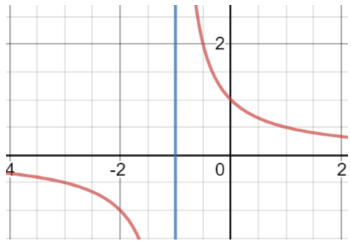
Here, the function approaches positive infinity or negative infinity or both. This will often happen because of a point which is not defined in the domain of f due to the denominator becoming zero.

Example 1.156

The graph of

$$y = f(x) = \frac{1}{x+1}$$

Has a discontinuity at $x = -2$. Identify the kind of discontinuity.



As x approaches -1 from the left, the value of $f(x)$ approaches $-\infty$

$$\lim_{x \to -1^-} f(x) = -\infty$$

As x approaches -1 from the right, the value of $f(x)$ approaches ∞

$$\lim_{x \to -1} f(x) = \infty$$

Example 1.157

A. Find the value of c such that $f(x) = \frac{1}{x+c}$ has a discontinuity at $x = 33$.

B. Find the range of c such that $f(x) = \frac{1}{x^2+cx+4}, x \in \mathbb{R}$ has no discontinuity.

Part A

$$33 + c = 0$$
$$c = -33$$

Part B

$$x^2 + cx + 4 = 0$$

Now, for there to be no discontinuity, the above equation must have no solutions in real numbers.

$$c^2 - 4(1)(1) < 0$$
$$c^2 < 4$$
$$-2 < c < 2$$
$$c \in (-2,2)$$

1.158: Infinite Discontinuity cannot be Removed

Infinite discontinuity cannot be removed without changing the nature of the function itself.

I. Removable Discontinuity

- If a function is otherwise continuous, but has a "hole", which can be filled by defining the function suitably at the hole, then the function is said to have a removable discontinuity at a point.
- This will often happen because of a factor in the denominator that approaches zero, but can be cancelled with a factor in the numerator for non-zero values.

1.159: Removable Discontinuity at a Point

A function has a removable discontinuity a point c in its domain if:

$$\lim_{x\to c} f(x) \neq f(c)$$

Notes:

- $\lim_{x\to c} f(x)$ exists which means that

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$$

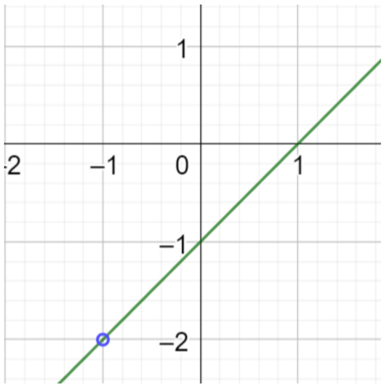
- The $\neq$ sign can be satisfied by one of two conditions:
  - ✓ The function is defined at c, but the value of the function is different from the value of the limit.
  - ✓ The function is not defined at the point c.

Example 1.160

A. Show that $f(x) = \frac{x^2-1}{x+1}$ has a removable discontinuity, and identify the (x, y) value for the hole.

B. Give a piecewise definition of $f(x)$ that removes the discontinuity.

Part A



First check, when is the denominator zero:

$$x + 1 = 0 \Rightarrow x = -1$$

Hence, the domain of the function is:

$$All\ real\ numbers\ except - 1$$

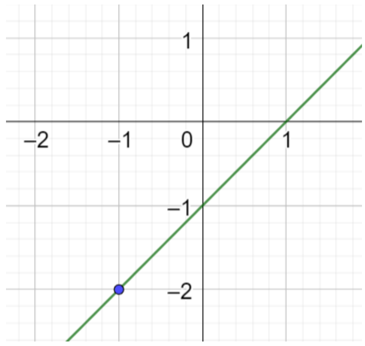
For any value in the domain of the function:

$$f(x) = \frac{x^2-1}{x+1} = \frac{(x+1)(x-1)}{x+1} = x - 1$$

$$(x, y) = (x, x-1) = (-1, -2)$$

Part B

$$f(x) = \begin{cases} \dfrac{x^2-1}{x+1}, x \in \mathbb{R}, x \neq -1 \\ \quad -2, \qquad x = -1 \end{cases}$$



Example 1.161

A. Identify and remove any points of discontinuity in $f(x) = \frac{9x^2-16}{3x-4}$

B. What is the value of a that makes $f(x)$ continuous if $f(x) = \begin{cases} \frac{9x^2-16}{3x-4} & if\ x \neq \frac{4}{3} \\ a & if\ x = \frac{4}{3} \end{cases}$

Part A

f is not defined when the denominator is zero:

$$3x - 4 = 0 \Rightarrow x = \frac{4}{3} \Rightarrow D_f = \mathbb{R} - \left\{\frac{4}{3}\right\}$$

For any number in the domain of f, we must have:

$$f(x)=\frac{9x^2-16}{3x-4}=\frac{(3x-4)(3x+4)}{3x-4}=3x+4$$

Hence, the graph of f has a hole in it at

$$x=\frac{4}{3}$$

To plug the hole, find the y-value of the "hole" (if it were in the domain of f):

$$f\left(\frac{4}{3}\right)=3\left(\frac{4}{3}\right)+4=4+4=8$$

To remove the hole at $x=\frac{4}{3}$, define the function piece-wise:

$$f(x)\begin{cases}\dfrac{9x^2-16}{3x-4} & if\ x\neq\dfrac{4}{3}\\ 8 & if\ x\neq\dfrac{4}{3}\end{cases}$$

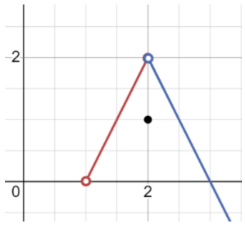
Part B

$$a=8$$

Example 1.162

Classify the nature of discontinuity of the piece-wise function graphed alongside given by

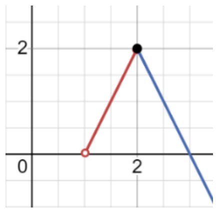
$$y=f(x)=\begin{cases}2x-2, 1<x<2\\ -2x+6, 2<x<3\\ 1, x=2\end{cases}$$



State how it can be removed, and provide an updated version of the function and its graph.

Part A

Since this hole can be plugged by changing the definition of the function, this is a removable discontinuity.



Part B

We can remove the discontinuity by changing the definition of the function at $x=2$.

$$y=f(x)=\begin{cases}2x-2, 1<x<2\\ -2x+6, 2<x<3\\ 2, x=2\end{cases}$$

$$\underbrace{\lim_{x\to c^-} f(x)}_{Left\ Hand\ Limit}=\underbrace{\lim_{x\to c^+} f(x)}_{Right\ Hand\ Limit}=2$$

$$f(2)=1\neq 2$$

Example 1.163

$$g(x)=\begin{cases}\dfrac{x^2-4}{x-2}, & x\neq 2\\ 4, & x=2\end{cases},\qquad h(x)=\begin{cases}\dfrac{x^2-4}{x-2}, & x\neq 2\\ 8, & x=4\end{cases}$$

A. What are the domains of $g(x)$ and $h(x)$
B. Find $\lim_{x\to 2} g(x)$ and $\lim_{x\to 2} h(x)$
C. What is the graphical difference between $g(x)$ and $h(x)$
D. Classify $g(x)$ and $h(x)$ based on continuity.

Part A

There are no restrictions on $g(x)$ or $h(x)$. They would have been undefined at $x = 2$, but the piece-wise definition circumvents that, and hence:

$$D_g = D_h = \mathbb{R}$$

Part B

The calculation of the limit does not depend on the value of the function at 2, and hence, the answer remains the same.

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} h(x) = 4$$

Part C

Graphically

- $g(x)$ can be drawn without lifting your pencil.
- $h(x)$ cannot be drawn without lifting your pencil.

Part D

$g(x)$ is continuous
$h(x)$ is not continuous

J. Ensuring Continuity

1.164: Ensuring Continuity

For a function to be continuous at a point:

$$LHL = RHL$$

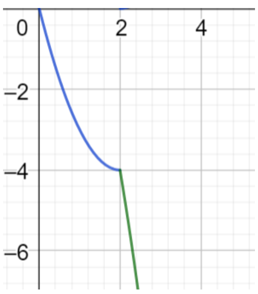
And, of course, for the limits to be equal, the limits must exist.

Example 1.165

$$f(x) = \begin{cases} (x-2)^2 - 4, & x \le 2 \\ 2 - ax^2, & x > 2 \end{cases}$$

$f(x)$ is a continuous piecewise function. Determine the value of a.

$$LHL = \lim_{x \to 2^-} (x-2)^2 - 4 = (2-2)^2 - 4 = 0 - 4 = -4$$
$$RHL = \lim_{x \to 2^+} 2 - ax^2 = 2 - a(2^2) = 2 - 4a$$



Since the function is continuous:

$$LHL = RHL \Rightarrow -4 = 2 - 4a \Rightarrow -6 = -4a \Rightarrow a = \frac{3}{2}$$

1.11 Intermediate Value Theorem

A. Statement

1.166: Graphical (Informal) Statement

If you draw a continuous graph (such as the one on the right), with $y = -2$ at its left endpoint, and $y = 3$ at its right endpoint, then

- the graph must take on all values between $y = -2$, and $y = 3$
- the graph must cross the x −axis, and hence it must have a root in the given interval

1.167: Intermediate Value Theorem (IVT)

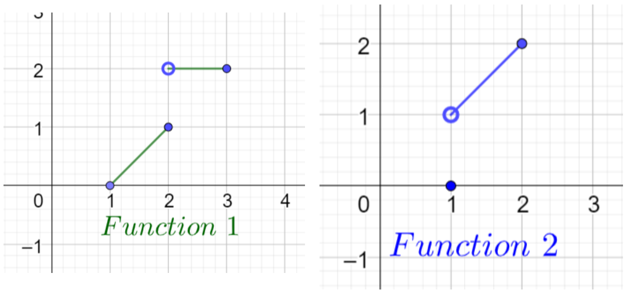
If a function f is continuous on the closed interval $[a, b]$, then for every value X between $f(a)$ and $f(b)$, there exists at least one number $c \in (a, b)$ such that $f(c) = X$.

Focus two important points:

- f is *continuous* on a *closed interval* $[a, b]$

As we see in the next example, both conditions are necessary.

Example 1.168



A. Explain why the adjacent graphs do not meet the conditions needed for the intermediate value theorem to hold over the interval which they are graphed.

B. In the context of the theorem, identify the endpoints of the given interval, that is a and b, and also $f(a)$ and $f(b)$.

C. Give a counterexample (a specific value of y that is not achieved) that shows the intermediate value theorem does not hold.

1.169: Root Finding

If a function f is continuous on a closed interv

If

Example 1.170

Show using th

2. Derivatives and Limits

2.1 Limit as a Derivative

A. The Idea

2.1: Derivative as a Limit

We have defined the derivative of the function $f(x)$ as the limit below (if it exists):

$$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$$

- We generally do not the use derivative definition to calculate derivatives (except for practice).
- Rather, the main purpose of the definition is to prove rules for differentiation, and theorems.

2.2: Limit as a Derivative

$$\lim_{x \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

- Once we have established the derivative of standard functions using a limit, we can use that knowledge to calculate (otherwise complicated) limits.

Example 2.3

$$\lim_{h \to 0} \frac{(2+h)^4 - 2^4}{h}$$

Consider $f = 2^x$:

$$f' = \frac{f(x+h) - f(x)}{h} = \frac{2^{x+h} - 2^x}{h}$$
$$f' = \ln 2\,(2^x)$$

(The above function is not useful in this context).

Consider $f = x^4$:

$$f' = \frac{f(x+h) - f(x)}{h} = \frac{(x+h)^4 - x^4}{h}$$

Let $x = 2$:

$$f'(2) = \frac{(2+h)^4 - 2^4}{h}$$
$$f' = 4x^3$$

$$x^8 - 4$$

Example 2.4

$$\lim_{h \to 0} \frac{\sin^2\left(\frac{\pi}{12} + h\right) - \sin^2\left(\frac{\pi}{12}\right)}{h}$$

Consider

$$f(\theta) = \sin^2\theta \Rightarrow f' = 2\sin\theta\cos\theta$$

$$f'\left(\frac{\pi}{12}\right)=\lim_{h\to 0}\frac{\sin^2\left(\frac{\pi}{12}+h\right)-\sin^2\left(\frac{\pi}{12}\right)}{h}=2\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)=\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$$

Example 2.5

$$\lim_{h\to 0}\frac{e^{e+h}-e^e}{h}$$

$$f(x)=e^x$$

$$f'(x)=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}=e^x$$

$$f'(e)=\lim_{h\to 0}\frac{e^{e+h}-e^e}{h}=e^e$$

2.2 Differentiability

A. Differentiability at a Point

We have learnt the differentiation rules related to:

- Polynomial, exponential, trigonometric and logarithmic functions
- Sum, difference, product and quotient of functions
- Composite functions (chain rule)

However, these rules are applicable only if the function is differentiable.

2.6: Differentiability at a point

A function is differentiable at a point if its left-hand derivative∗ (LHD) is equal to its to right hand derivative (RHD) at that point (and both exist).

$$LHD=RHD\Rightarrow Derivative\ exists$$

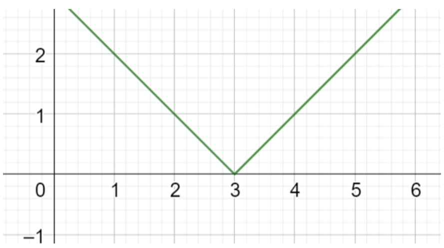
The derivative of the function $f(x)$ is given by the limit below, if it exists:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

For the limit to exist, the right-hand limit and the left-hand limit must both exist. In fact, they are important enough that we give them special names:

$$Left\ Hand\ Derivative=LHD=\lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h}$$

$$Right\ Hand\ Derivative=RHD=\lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}$$



Example 2.7

Show that the function $f(x)=|x-3|, x\in\mathbb{R}$, which is graphed alongside, is continuous, but not differentiable at $x=3$.

$f(x)=|x-3|$ has two parts

$$x-3\geq 0\Rightarrow x\geq 3$$

$$x\geq 3\Rightarrow f(x)=x-3$$
$$x\leq 3\Rightarrow f(x)=3-x$$

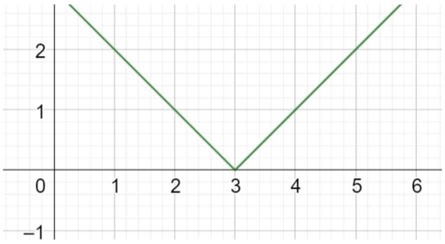
To check the differentiability graphically, we check the slope at $x = 0$

$$Slope\ from\ the\ left\ = -1 = -ve$$
$$Slope\ from\ the\ right\ = +1 = +ve$$

Hence, when you calculate the derivative, it is not defined.

Example 2.8

Show that the function $f(x) = |x - 3|, x \in \mathbb{R}$ is continuous, but not differentiable at $x = 3$. **(CBSE 2013)**



Continuity

$$\lim_{x\to 3^-} |x - 3| = \lim_{x\to 3^+} |x - 3| = 0$$

Since $RHL = LHL$, the overall limit exists.

$$f(3) = |x - 3| = 0$$

Since the value of the limit at $x = 3$ is equal to the value of the function at $x = 3$, the function is continuous:

$$\lim_{x\to 3} |x| = f(0) = 0 \Rightarrow f(x)\ is\ continuous$$

Differentiability

Evaluate $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ at $x = 3$:

$$LHD = \lim_{h\to 0^-} \frac{|3 + h - 3| - |3 - 3|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = -1$$
$$RHD = \lim_{h\to 0^+} \frac{|3 + h - 3| - |3 - 3|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = +1$$

Since

$$LHD \neq RHD \Rightarrow Derivative\ does\ not\ exist$$

2.9: Graphical meaning of differentiability

- If a function is differentiable, then it is smooth and continuous.
- Smooth means there are no sharp turns.
- If a function has a sharp turn, then it is not differentiable since the slope from the left is different from the slope from the right, which means that the left hand derivative is different from the right hand derivative.

2.10: Differentiability at a point

- Differentiability implies continuity.
- However, continuity does not imply differentiability.

Example 2.11

Find the values of a and b, if the function f defined below is differentiable at $x = 1$: **(CBSE 2016)**

$$f(x) = \begin{cases} x^2 + 3x + a, x \le 1 \\ bx + 2, x > 1 \end{cases}$$

Use $f(x) = x^2 + 3x + a$ for the left-hand derivative

$$LHD = Lf(x) = \frac{d}{dx} x^2 + 3x + a = 2x + 3$$

Evaluate the left-hand derivative at $x = 1$

$$LHD(1) = 2(1) + 3 = 5$$

Use $f(x) = bx + 2$ for the right-hand derivative

$$RHD = \frac{d}{dx}(bx + 2) = b$$

Since the function is differentiable at $x = 1$, the derivative exists at $x = 1$.
Hence, the left-hand derivative and the right-hand derivative must both exist and be equal at $x = 1$.

$$RHD = LHD$$
$$b = 5$$

Evaluate the right-hand limit of the function at $x = 1$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} bx + 2 = \lim_{x \to 1^+} 5x + 2 = 7$$

Evaluate the function at $x = 1$ to get:

$$f(1) = x^2 + 3x + a = (1)^2 + 3(1) + a = 4 + a$$

However, the function has a derivative at $x = 1$
Therefore, the function is continuous at $x = 1$.
Therefore, at $x = 1$, the value of the function is equal to the right-hand limit:

$$f(1) = \lim_{x \to 1^+} f(x)$$
$$4 + a = 7$$
$$a = 3$$

Substitute $x = 1$ in $\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$ to find $LHD(1)$:

$$= \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h}$$

Apply the definition of $f(x)$:

$$= \lim_{h \to 0^-} \frac{[(1+h)^2 + 3(1+h) + a] - (1^2 + 3 + a)}{h}$$

Expand:

$$= \lim_{h \to 0^-} \frac{1 + 2h + h^2 + 3 + 3h + a - (4 + a)}{h}$$

Simplify and factor:

$$= \lim_{h \to 0^-} \frac{h^2 + 5h}{h} = \lim_{h \to 0^-} \frac{h(h+5)}{h}$$

Cancel h in numerator and denominator:

$$= \lim_{h \to 0^-} h + 5$$

This limit is a well behaved that can be evaluated by substitution:

$$= 5$$

$$RHD(1) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h \to 0^+} \frac{b(1+h) + 2 - (b+2)}{h}$$
$$= \lim_{h \to 0^+} \frac{b + bh + 2 - b - 2}{h}$$

$$= \lim_{h \to 0^+} \frac{bh}{h} = \lim_{h \to 0^+} b = b$$

2.12: Differentiability at Endpoints

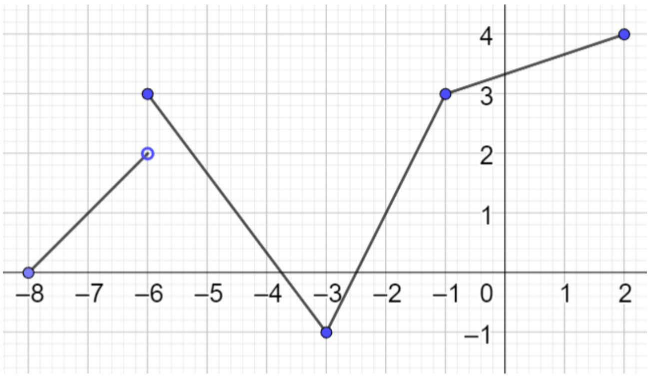
At its endpoints, a function is not differentiable because, at the:

- left endpoint, the left-hand limit (and hence the left-hand derivative) is not defined
- right endpoint, the right-hand limit (and hence the right-hand derivative) is not defined

Example 2.13

The function drawn alongside consists of straight line segments. Examine the graph, determine the points, if any(with reasons), where the function is

A. Not Differentiable
B. Differentiable
C. Not Continuous
D. Continuous



At the point:

$$LHD(-3) = \lim_{h \to 0^-} \frac{f(-3+h) - f(-3)}{h} = -\frac{4}{3}$$
$$RHD = \lim_{h \to 0^+} \frac{f(-3+h) - f(-3)}{h} = \frac{4}{2} = 2$$
$$LHD \neq RHD \Rightarrow f(x)\ is\ not\ differentiable\ at\ x = -3$$

Part A

At

$$x = -1, the\ function\ has\ a\ sharp\ turn$$
$$x = -3, the\ function\ has\ a\ sharp\ turn$$
$$x = -6, the\ function\ is\ not\ continous$$
$$x = -8, f(x)\ is\ not\ left\ continous \Rightarrow f(x)\ is\ not\ \text{continuous}$$
$$x = 2, f(x)\ is\ not\ right\ continous \Rightarrow f(x)\ is\ not\ \text{continuous}$$

Where ever the function is not continuous, it not differentiable. In summary, the points where the function is not differentiable are:

$$\{-8, -6, -3, -1, 2\}$$

Part B

What we want is exactly the complement of the answer to Part A over the domain of the function f, which is given by:

$$(-8, -6) \cup (-6, -3) \cup (-3, -1) \cup (-1, 2)$$

Part C

$$\{-8, -6, 2\}$$

Part D

What we want is exactly the complement of the answer to Part C over the domain of the function f, which is given by:

$$(-8, -6) \cup (-6, 2)$$

Example 2.14

Find whether the following function is differentiable at $x = 1$ and $x = 2$ or not. (CBSE 2015)

$$f(x) = \begin{cases} x, x < 1 \\ 2 - x, 1 \le x \le 2 \\ -2 + 2x - x^2, x > 2 \end{cases}$$

$x = 1$

$$LHL = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 - 1 = \lim_{x \to 1^+} 1 = 1$$
$$RHL = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = \lim_{x \to 1^+} 1 = 1 = LHL \Rightarrow Continous$$

$$LHD = 1$$
$$RHD = -1$$
$$LHD \ne RHD \Rightarrow Not\ Differentiable$$

$x = 2$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 2 - x = 0$$
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} -2 + 2x - x^2 = -2 + 2(2) - 2^2 = -2 + 4 - 4 = -2$$

Hence:

$$LHL \ne RHL \Rightarrow Not\ continuous \Rightarrow Not\ differentiable$$

Example 2.15

For what value of λ, the function defined by $f(x) = \begin{cases} \lambda(x^2 + 2), x \le 0 \\ 4x + 6, x > 0 \end{cases}$ is continuous at $x = 0$? Hence, check the differentiability of $f(x)$ at $x = 0$? (CBSE 2015)

Continuity

$$LHL = RHL$$
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$$
$$\lim_{x \to 0^+} \lambda(x^2 + 2) = \lim_{x \to 0^-} 4x + 6$$
$$\lambda(0^2 + 2) = 4(0) + 6$$
$$2\lambda = 6$$
$$\lambda = 3$$

Differentiability

$$LHD = \frac{d}{dx}\lambda(x^2 + 2) = 2\lambda x \Rightarrow LHD_{x=0} = 0$$
$$RHD = \frac{d}{dx}(4x + 6) = 4 \ne LHD$$

Hence,

$$Not\ differentiable$$

Example 2.16

$$f(x) = \begin{cases} 3x - 2, \quad 0 < x \le 1 \\ 2x^2 - x, 1 < x \le 2 \\ 5x - 4, x > 2 \end{cases}$$

Show that $f(x)$ is continuous at $x = 1, x = 2$ but not differentiable at $x = 2$. (CBSE 2010)

Continuity at $x = 1$

$$LHL = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 3x - 2 = \lim_{x\to 1^+} 3 - 2 = 1$$

$$RHL = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x^2 - x = \lim_{x\to 1^+} 2(1)^2 - 1 = 1 = LHL \Rightarrow Continuous$$

Continuity at $x = 2$

$$LHL = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2x^2 - x = 6$$

$$RHL = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 5x - 4 = 6 = LHL \Rightarrow Continuous$$

Differentiability at $x = 2$

$$LHD = \frac{d}{dx}(2x^2 - x) = 4x - 1 \Rightarrow LHD_{x=2} = 7$$

$$RHD = \frac{d}{dx}(5x - 4) = 5 \neq LHD \Rightarrow Not\ Differentiable$$

Example 2.17

$$f(x) = \begin{cases} x^2 + 1, & x \geq 0 \\ A\sin x + B\cos x, & x < 0 \end{cases}$$

$f(x)$ is differentiable. Find A and B.

Since $x^2 + 1$ and $A\sin x + B\cos x$ are continuous and differentiable at all points in their domain, we need to focus on the transition point.

Using the Differentiability Condition

Since the function is differentiable, it has the same left-hand derivative and right-hand derivative at $x = 0$.

$$RHD|_{x=0} = \frac{d}{dx}(x^2 + 1)\bigg|_{x=0} = 2x|_{x=0} = 2(0) = 0$$

Since the RHD derivative is zero, the LHD must also be zero at $x = 0$:

$$LHD = \frac{d}{dx}(A\sin x + B\cos x) = A\cos x - B\sin x$$

At $x = 0$:

$$A\cos x - B\sin x = 0 \Rightarrow \frac{A}{B} = \frac{\sin x}{\cos x} \Rightarrow \frac{A}{B} = \tan x$$

Substitute $x = 0$:

$$\frac{A}{B} = \tan 0 \Rightarrow \frac{A}{B} = 0 \Rightarrow\Rightarrow A = 0$$

Using the Continuity Condition

Since the function is continuous, the value from the left must equal the value from the right.

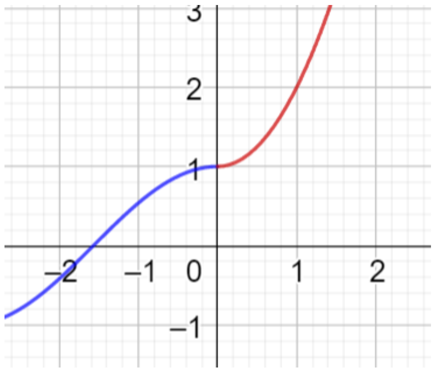
$$f(0) = 0^2 + 1 = 1$$

The limit from the right must equal 1.

$$\lim_{x\to 0^+} A\sin x + B\cos x = 1$$

This limit can be evaluated using substitution:

$$A\sin 0 + B\cos 0 = 1 \Rightarrow 0 + B(1) = 1 \Rightarrow B = 1$$



Hence, the final answer is:

$$A = 0, B = 1 \Rightarrow f(x) = \begin{cases} x^2 + 1, & x \ge 0 \\ \cos x, & x < 0 \end{cases}$$

2.3 LH Rule-I: 0 over 0, and ∞ over ∞

A. $\frac{0}{0}$ Case

2.18: L 'Hospital's Rule: $\frac{0}{0}$ Case

A limit of a quotient can be found using the rule:

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

The conditions to check are:

- $f(c) = g(c) = 0$
  - ✓ In other words, on substitution the limit results in the $\frac{0}{0}$ case.
- functions f and g are differentiable on an open interval I, except possibly at c. (where c lies in I)
- $g'(x) \neq 0$ on I if $x \neq c$
- The limit $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ exists

Example 2.19: Limits by Factoring

$$\lim_{x\to 3}\frac{x^2 - 9}{x - 3}$$

A. Show that, on substitution in the limit, we get a $\frac{0}{0}$ case.
B. Evaluate the limit by factoring.
C. Show that the conditions for using L 'Hospital's Rule apply.
D. Evaluate the limit by using L 'Hospital's Rule

Part A

$$\frac{x^2 - 9}{x - 3} = \frac{3^2 - 9}{3 - 3} = \frac{9 - 9}{3 - 3} = \frac{0}{0}\ Case$$

Part B

Factor the numerator:

$$\lim_{x\to 3}\frac{(x+3)(x-3)}{x-3}$$

Cancel:

$$= \lim_{x\to 3} x + 3$$

Substitute:

$$= 3 + 3 = 6$$

Part C

L 'Hospital's Rule applies because:

- The limit is the $\frac{0}{0}$ case
- Both the numerator and the denominator are differentiable functions over $\mathbb{R}$.

Part D

Apply L 'Hospital's Rule. Find the derivative of the numerator and the denominator (separately)

$$\lim_{x\to 3}\frac{\frac{d}{dx}(x^2-9)}{\frac{d}{dx}(x-3)}=\lim_{x\to 3}\frac{2x}{1}$$

Evaluate the limit by substitution:

$$=2(3)=6$$

$$\lim_{x\to 2}\frac{x^2-5x+6}{x-2}$$

Repeat Parts A-D using the limit given above.

Part A

$$\frac{x^2-5x+6}{x-2}=\frac{4-10+6}{2-2}=\frac{0}{0}\ Case$$

Part B

Factor the numerator:

$$\lim_{x\to 2}\frac{(x-2)(x-3)}{x-2}=\lim_{x\to 2}x-3=2-3=-1$$

Part C

L 'Hospital's Rule applies because:

- The limit is the $\frac{0}{0}$ case
- Both the numerator and the denominator are differentiable functions over $\mathbb{R}$.

Part D

Apply L 'Hospital's Rule. Find the derivative of the numerator and the denominator (separately)

$$\lim_{x\to 3}\frac{\frac{d}{dx}(x^2-5x+6)}{\frac{d}{dx}(x-2)}=\lim_{x\to 3}\frac{2x-5}{1}$$

Evaluate the limit by substitution:

$$=2(2)-5=4-5=-1$$

2.20: Checking for differentiability

If the functions f and g are differentiable over a specific domain, then it is important to confirm that the f and g are differentiable on an interval around c (except possibly at c).

In the first two examples that we did, the functions were differentiable over their entire domain. Hence, it was not difficult to confirm that they were differentiable on an interval around c (the number at which we wanted to calculate the limit). If, however, the functions are differentiable on a specific domain, then it becomes important to check for differentiability.

Example 2.21: Limits by Rationalization

$$\lim_{x\to 2}\frac{\sqrt{x+7}-3}{2x-4}$$

A. Show that, on substitution in the limit, we get a $\frac{0}{0}$ case.

B. Evaluate the limit by rationalization.
C. Evaluate the limit by using L 'Hospital's Rule.

Part A

$$\frac{\sqrt{x+7}-3}{2x-4}=\frac{\sqrt{2+7}-3}{2(2)-2}=\frac{\sqrt{9}-3}{4-2}=\frac{0}{0}$$

Part B

Rationalize the numerator:

$$\lim_{x\to 2}\frac{\sqrt{x+7}-3}{2x-4}\times\frac{\sqrt{x+7}+3}{\sqrt{x+7}+3}$$

Simplify:

$$=\lim_{x\to 3}\frac{x-2}{2(x-2)\left(\sqrt{x+7}+3\right)}$$

Cancel:

$$=\lim_{x\to 3}\frac{1}{2\left(\sqrt{x+7}+3\right)}$$

Substitute:

$$=\frac{1}{2\left(\sqrt{2+7}+3\right)}=\frac{1}{2\left(\sqrt{9}+3\right)}=\frac{1}{12}$$

Part C

Check that the conditions for L 'Hospital's Rule apply:

- The limit is the $\frac{0}{0}$ case
- The denominator is differentiable over all real numbers.
- The numerator has domain $[-7,\infty)$ and is differentiable everywhere except the left endpoint. Specifically, it is differentiable on an open interval around 2, such as $(-7,\infty)$.

Part D

Calculate the derivative of the numerator and the denominator:

$$=\lim_{x\to 3}\frac{\frac{d}{dx}\left(\sqrt{x+7}-3\right)}{\frac{d}{dx}(2x-4)}$$

Simplify:

$$=\lim_{x\to 3}\frac{\frac{1}{2\sqrt{x+7}}}{2}=\lim_{x\to 3}\frac{1}{4\sqrt{x+7}}$$

Substitute:

$$=\frac{1}{4\sqrt{2+7}}=\frac{1}{4\sqrt{9}}=\frac{1}{12}$$

$$\lim_{y\to 0}\frac{\sqrt{1+\sqrt{1+y^4}}-\sqrt{2}}{y^4}\ (\mathit{JEE\ Main\ 2019-9\ Jan})$$

Repeat Parts A-D using the limit given above.

Part A

$$\frac{\sqrt{1+1}-\sqrt{2}}{0}=\frac{\sqrt{2}-\sqrt{2}}{0}=\frac{0}{0}\ Case$$

Part B

Part C

Check that the conditions for L 'Hospital's Rule apply:

- The limit is the $\frac{0}{0}$ case
- The numerator and denominator are differentiable over all real numbers.

Part D

Apply LH Rule:

$$\lim_{y\to 0}\frac{\frac{1}{2\sqrt{1+\sqrt{1+y^4}}}\times\frac{1}{2\sqrt{1+y^4}}\times 4y^3}{4y^3}$$

Cancel the common factor:

$$\lim_{y\to 0}\frac{1}{2\sqrt{1+\sqrt{1+y^4}}}\times\frac{1}{2\sqrt{1+y^4}}$$

Substitute:

$$\frac{1}{2\sqrt{1+\sqrt{1}}} \times \frac{1}{2\sqrt{1}} = \frac{1}{2\sqrt{2}} \times \frac{1}{2} = \frac{1}{4\sqrt{2}}$$

$$\lim_{x \to a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}, \qquad a > 0 \; (\textbf{\textit{JEE Advanced 1978, Adapted}})$$

Repeat Parts A-D using the limit given above.

Part A

$$\frac{\sqrt{a+2a} - \sqrt{3a}}{\sqrt{3a+a} - 2\sqrt{a}} = \frac{\sqrt{3a} - \sqrt{3a}}{2\sqrt{a} - 2\sqrt{a}} = \frac{0}{0} Case$$

Part B

Part C

Check that the conditions for L 'Hospital's Rule apply:

- The limit is the $\frac{0}{0}$ case
- The numerator and denominator are differentiable over their entire domain except the left endpoint.

Part D

Apply LH Rule:

$$\lim_{x \to a} \frac{\frac{2}{2\sqrt{a+2x}} - \frac{3}{2\sqrt{3x}}}{\frac{1}{2\sqrt{3a+x}} - \frac{2}{2\sqrt{x}}} = \lim_{x \to a} \frac{\frac{1}{\sqrt{a+2x}} - \frac{3}{2\sqrt{3x}}}{\frac{1}{2\sqrt{3a+x}} - \frac{1}{\sqrt{x}}}$$

Substitute $x = a$, and convert the nested fraction into a single fraction:

$$= \frac{\frac{1}{\sqrt{3a}} - \frac{3}{2\sqrt{3a}}}{\frac{1}{2\sqrt{4a}} - \frac{1}{\sqrt{a}}} = \frac{\frac{2-3}{2\sqrt{3}\sqrt{a}}}{\frac{1-4}{4\sqrt{a}}} = \frac{-1}{2\sqrt{3}\sqrt{a}} \times \frac{4\sqrt{a}}{-3} = \frac{2}{3\sqrt{3}}$$

Example 2.22: Limits using $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

$$\lim_{x \to 0} \frac{\sin(12x) + \tan(12x) + 12x}{3x}$$

Method I: Using $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

If the individual limits exists, then we can split the limit:

$$\lim_{x \to 0} \frac{\sin(12x)}{3x} + \lim_{x \to 0} \frac{\tan(12x)}{3x} + \lim_{x \to 0} \frac{12x}{3x}$$

Using $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$:

$$\lim_{x \to 0} \frac{\sin(12x)}{3x} \times \frac{4}{4} = \lim_{x \to 0} \frac{4\sin(12x)}{12x} = 4 \lim_{x \to 0} \frac{\sin(12x)}{12x} = 4 \cdot 1 = 4$$

Using $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$:

$$\lim_{x \to 0} \frac{\tan(12x)}{3x} = \lim_{x \to 0} \frac{\sin(12x)}{3x\cos 3x} = \lim_{x \to 0} \frac{\sin(12x)}{3x} \cdot \frac{1}{\cos 3x}$$

Split the limits

$$= \lim_{x \to 0} \frac{\sin(12x)}{3x} \times \lim_{x \to 0} \frac{1}{\cos 3x} = 4 \times 1 = 4$$

The third part can be done cancellation:

$$\lim_{x\to 0}\frac{12x}{3x}=\lim_{x\to 0}4=4$$

The final answer is:

$$4+4+4=12$$

Method II: Using L 'Hospital's Rule

On Substitution

$$\lim_{x\to 0}\frac{0+0+0}{0}=\frac{0}{0}$$

Numerator and denominator are both differentiable on an interval around 0.

On differentiating numerator and denominator:

$$\lim_{x\to 0}\frac{12\cos(12x)+12\sec^2 12x+12}{3}$$

$$\lim_{x\to 0}4[\cos(12x)+\sec^2 12x+1]$$

Substitute

$$\lim_{x\to 0}4[1+1+1]=\lim_{x\to 0}4[3]=\lim_{x\to 0}12=12$$

Example 2.23: Exponential Functions[2]

$$\lim_{x\to 0}\frac{e^x-1}{x}$$

Calculate the derivative of the numerator and the denominator:

$$=\lim_{x\to 0}\frac{\frac{d}{dx}(e^x-1)}{\frac{d}{dx}(x)}=\lim_{x\to 0}\frac{e^x}{1}=\frac{1}{1}=1$$

Example 2.24

$$\lim_{x\to\infty}\frac{\ln(x^2)}{\sqrt{2x}}$$

$$\lim_{x\to\infty}\frac{\frac{2}{x}}{\frac{1}{2\sqrt{2x}}\times 2}=\lim_{x\to\infty}\frac{\frac{2}{x}}{\frac{1}{\sqrt{2x}}}=\lim_{x\to\infty}\frac{2}{x}\times\sqrt{2x}=\lim_{x\to\infty}\frac{2\sqrt{2}}{\sqrt{x}}=0$$

Example 2.25

Evaluate

$$\lim_{x\to 0}\frac{\log_e(3+x)-\log_e(3-x)}{x}\ (\textit{JEE Main 2003, Adapted})$$

$$\lim_{x\to 0}\frac{\ln(3)-\ln 3}{0}=\frac{0}{0}\ Case$$

The denominator is differentiable over $\mathbb{R}$

The numerator is defined and differentiable over $(-3,3)$.

[2] Questions on exponential, logarithmic, and trigonometric limits can also be done using in a different way (covered in the Section on Limits using Expansion).

Apply LH Rule:

$$\lim_{x\to 0}\frac{\frac{1}{3+x}-\left(\frac{-1}{3-x}\right)}{1}=\lim_{x\to 0}\frac{1}{3+x}+\frac{1}{3-x}$$

Substitute:

$$=\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$$

Challenge 2.26

The value of the limit $\lim_{\theta\to 0}\frac{\tan(\pi\cos^2\theta)}{\sin(2\pi\sin^2\theta)}$ is equal to (JEE-M 2021)

Substitute $\theta \to 0$:

$$\frac{\tan(\pi\cos^2 0)}{\sin(2\pi\sin^2 0)}=\frac{\tan(\pi)}{\sin(0)}=\frac{0}{0}\ Case$$

The denominator is differentiable over $\mathbb{R}$.
The numerator is differentiable over $(-\pi,\pi)$.

Apply LH Rule:

$$\lim_{\theta\to 0}\frac{\sec^2(\pi\cos^2\theta)(-2\pi\cos\theta\sin\theta)}{\cos(2\pi\sin^2\theta)(4\pi\cos\theta\sin\theta)}=\lim_{\theta\to 0}\frac{\sec^2(\pi\cos^2\theta)(-1)}{\cos(2\pi\sin^2\theta)(2)}$$

Substitute $\theta=0$:

$$\frac{\sec^2(\pi)(-1)}{\cos(0)(2)}=\frac{(-1)^2(-1)}{1(2)}=-\frac{1}{2}$$

B. $\frac{\infty}{\infty}$ Case

2.27: L 'Hospital's Rule for $\frac{\infty}{\infty}$ Case

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

The rule is also applicable when

$$\lim_{x\to c} f(x)=\lim_{x\to c} g(x)=\pm\infty$$

The other conditions remain the same as before.

Example 2.28: Limits at Infinity

$$\lim_{t\to\infty}\frac{6t+3}{7t-2}$$

A. Show that, on substitution in the limit, we get a $\frac{\infty}{\infty}$ case
B. Evaluate the limit by dividing by the highest power of the variable in the denominator
C. Evaluate the limit by using L 'Hospital's Rule

Part A

$$\frac{6t+3}{7t-2}=\frac{\infty+3}{\infty-2}=\frac{\infty}{\infty}$$

Part B

$$\lim_{t\to\infty}\frac{6t+3}{7t-2}=\lim_{t\to\infty}\frac{6+\frac{3}{t}}{7-\frac{2}{t}}=\frac{6+0}{7-0}=\frac{6}{7}$$

Part C

$$\lim_{t\to\infty}\frac{\frac{d}{dx}(6t+3)}{\frac{d}{dx}(7t-2)}=\lim_{t\to\infty}\frac{6}{7}=\frac{6}{7}$$

$$\lim_{t\to\infty}\frac{3t-5}{2-8t}$$

Repeat Parts A-C using the limit given above.

Part A

$$\frac{3t-5}{2-8t}=\frac{\infty-5}{2-\infty}=\frac{\infty}{\infty}$$

Part B

$$\lim_{t\to\infty}\frac{3t-5}{2-8t}=\lim_{t\to\infty}\frac{3-\frac{5}{\mathrm{t}}}{\frac{2}{\mathrm{t}}-8}=-\frac{3}{8}$$

Part C

$$\lim_{t\to\infty}\frac{3t-5}{2-8t}=\lim_{t\to\infty}\frac{3}{-8}=-\frac{3}{8}$$

C. Using L 'Hospital's Rule More Than Once

2.29: Multiple Applications of L 'Hospital's Rule

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}=\lim_{x\to c}\frac{f''(x)}{g''(x)}=\cdots$$

If, after the first application of LH Rule, we find that the conditions for LH Rule are still satisfied ($\frac{0}{0}$ *case or* $\frac{\infty}{\infty}$ *case*, differentiability), then you can apply LH Rule multiple times (as many times you want).

Example 2.30: Limits at Infinity

$$\lim_{t\to\infty}\frac{5t^3-7t^2+6t+3}{\frac{1}{2}t^3+4t^2-2t+11}$$

A. Show that, on substitution in the limit, we get a $\frac{\infty}{\infty}$ case.
B. Evaluate the limit by dividing by the highest power of the variable in the denominator.
C. Evaluate the limit by using L 'Hospital's Rule.

Part A
Part B
Part C

Apply LH Rule:

$$\lim_{t\to\infty}\frac{15t^2-14t+6t}{\frac{3}{2}t^2+8t-2}$$

Apply LH Rule 2nd time:

$$=\lim_{t\to\infty}\frac{30t-14}{3t+8}$$

Apply LH Rule 3rd time:

$$=\lim_{t\to\infty}\frac{30}{3}=10$$

Example 2.31

$$\lim_{x\to 0}\frac{\frac{e^x}{a}-\frac{1}{a}-\frac{x}{a}}{\frac{x^2}{b}},\qquad a\neq 0, b\neq 0$$

Substitute:

$$\frac{\frac{1-1-0}{a}}{\frac{0}{b}}=\frac{\frac{0}{a}}{\frac{0}{b}}=\frac{0}{0}\ Case$$

Factor $\frac{b}{a}$ out of the expression to simplify:

$$\lim_{x\to 0}\left(\frac{b}{a}\right)\frac{e^x-1-x}{x^2}$$

Use the constant multiple rule to move $\frac{b}{a}$ outside of the limit:

$$=\left(\frac{b}{a}\right)\lim_{x\to 0}\frac{e^x-1-x}{x^2}$$

Apply LH Rule:

$$=\left(\frac{b}{a}\right)\lim_{x\to 0}\frac{e^x-1}{2x}$$

Apply LH Rule again:

$$=\left(\frac{b}{a}\right)\lim_{x\to 0}\frac{e^x}{2}$$

Substitute:

$$=\left(\frac{b}{a}\right)\left(\frac{e^0}{2}\right)=\frac{b}{2a}$$

Example 2.32

$$\lim_{x\to 0}\frac{x-x\cos x}{x-\sin x}$$

$\frac{0}{0}$ Case. Apply LH Rule:

$$\lim_{x\to 0}\frac{1+x\sin x-\cos x}{1-\cos x}=\lim_{x\to 0}1+\frac{x\sin x}{1-\cos x}=1+\lim_{x\to 0}\frac{x\sin x}{1-\cos x}$$

The fraction is still $\frac{0}{0}$. Apply LH Rule 2^{nd} time:

$$1+\lim_{x\to 0}\frac{\sin x+x\cos x}{\sin x}=2+\lim_{x\to 0}\frac{x\cos x}{\sin x}$$

The fraction is still $\frac{0}{0}$. Apply LH Rule 3^{rd} time:

$$\begin{aligned}&2+\lim_{x\to 0}\frac{\cos x-x\sin x}{\sin x}\\&=3+\lim_{x\to 0}\frac{x\sin x}{\sin x}\\&=3+0\\&=3\end{aligned}$$

Example 2.33

$$\lim_{x\to\infty}x^3e^{-x}$$

$$\lim_{x\to\infty} x^3 e^{-x} = \lim_{x\to\infty} \frac{x^3}{e^x}$$

Apply LH Rule three times:

$$\lim_{x\to\infty} \frac{3x^2}{e^x} = \lim_{x\to\infty} \frac{6x}{e^x} = \lim_{x\to\infty} \frac{6}{e^x} = 0$$

Example 2.34

$$\lim_{x\to\infty} x^n e^{-x}$$

$$\lim_{x\to\infty} x^n e^{-x} = \lim_{x\to\infty} \frac{x^n}{e^x}$$

Consider cases.
Case I: $n = 0$

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

Case II: $n < 0$

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{1}{x^{-n} e^x}$$

Since $n < 0 \Rightarrow -n > 0$
Let $m = -n, m > 0$

$$= \lim_{x\to\infty} \frac{1}{x^m e^x} = 0$$

Case III: $n > 0$

$$\lim_{x\to\infty} x^n e^{-x} = \lim_{x\to\infty} \frac{x^n}{e^x}$$

Apply LH Rule multiple times. The numerator becomes successively:

$$x^n \to nx^{n-1} \to n(n-1)x^{n-2} \dots.$$

This will then reduce to either Case I or Case II.

If $n \in \mathbb{N}$. For example, if $n = 3$

$$\lim_{x\to\infty} \frac{x^3}{e^x} = \lim_{x\to\infty} \frac{3x^2}{e^x} = \lim_{x\to\infty} \frac{6x}{e^x} = \lim_{x\to\infty} \frac{6}{e^x} = 0$$

If $n \in \mathbb{Q}. For\ example, if\ n = 3.75$

$$\lim_{x\to\infty} \frac{x^{3.75}}{e^x} = \lim_{x\to\infty} \frac{3.75x^{2.75}}{e^x} = \lim_{x\to\infty} \frac{(3.75)(2.75)x^{1.75}}{e^x} = \lim_{x\to\infty} \frac{(3.75)(2.75)(1.75)x^{0.75}}{e^x}$$

$$= \lim_{x\to\infty} \frac{(3.75)(2.75)(1.75)(0.75)x^{-0.25}}{e^x}$$

And now the above is an example of Case I. Hence, the limit is 0.

D. Defined Functions

2.35: Defined Functions

Some questions give functions whose explicit definition is not known. Rather, the values of the functions, and their derivatives at specific inputs are given.

Example 2.36

A. Given that $f(1) = 1, f'(1) = 2$, calculate $\lim_{x\to1}\frac{\sqrt{f(x)}-1}{\sqrt{x}-1}$ **(JEE-M 2002)**

B. Given that $f(9) = 9, f'(9) = 4$, calculate $\lim_{x\to9}\frac{\sqrt{f(x)}-3}{\sqrt{x}-3}$ **(JEE-A 1982)**

Part A

On substitution, we get:

$$\frac{\sqrt{f(1)}-1}{\sqrt{1}-1} = \frac{\sqrt{1}-1}{\sqrt{1}-1} = \frac{0}{0}$$

Apply LH Rule:

$$\lim_{x\to1}\frac{\frac{1}{2\sqrt{f(x)}}\times f'(x)}{\frac{1}{2\sqrt{x}}} = \lim_{x\to1}\frac{\sqrt{x}}{\sqrt{f(x)}}\times f'(x)$$

Substitute $x = 1, f(1) = 1, f'(1) = 2$,

$$\frac{\sqrt{1}}{\sqrt{1}}\times 2 = 2$$

Part B

On substitution, we get:

$$\frac{\sqrt{f(9)}-3}{\sqrt{9}-3} = \frac{\sqrt{9}-3}{3-3} = \frac{0}{0}$$

Apply LH Rule:

$$\lim_{x\to1}\frac{\frac{1}{2\sqrt{f(x)}}\times f'(x)}{\frac{1}{2\sqrt{x}}} = \lim_{x\to1}\frac{\sqrt{x}}{\sqrt{f(x)}}\times f'(x)$$

Substitute $x = 9, f(1) = 1, f'(1) = 2$,

$$\frac{\sqrt{9}}{\sqrt{9}}\times 4 = 4$$

Example 2.37

A. Given that $f(2) = 4, f'(2) = 4$, calculate $\lim_{x\to2}\frac{xf(2)-2f(x)}{x-2}$ **(JEE-M 2002, Adapted)**

B. Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that $f(2) = 4$ and $f'(2) = 1$. Then, the value of $\lim_{x\to2}\frac{x^2f(2)-4f(x)}{x-2}$ is equal to **(JEE-M 2021)**

Part A

On substitution, we get:

$$\frac{2(4)-2(4)}{2-2} = \frac{0}{0}$$

Apply LH Rule:

$$\lim_{x\to2}\frac{f(2)-2f'(x)}{1} = \frac{4-2(4)}{1} = -4$$

Part B

E. Composite Functions

2.38: Limit of a Composite Function

$$\lim_{x\to\alpha} f(g(x)) = f\left(\lim_{x\to\alpha} g(x)\right)$$

- We can interchange the function and limit operators, so long as $f(x)$ is continuous at $\lim_{x\to\alpha} g(x)$.
- This property is useful in evaluating the limit of a composite function.

Example 2.39

$$\lim_{x\to0}\frac{(e^x-1)^2}{x^2}$$

Apply LH Rule:

$$\lim_{x\to 0}\frac{2(e^x-1)e^x}{2x}=\lim_{x\to 0}\frac{e^{2x}-e^x}{x}$$

On substitution, it is $\frac{0}{0}$. Apply LH Rule once more:

$$=\lim_{x\to 0}\frac{2e^{2x}-e^x}{1}$$

Split the limits:

$$=2\lim_{x\to 0}e^{2x}-\lim_{x\to 0}e^x$$

Use the property for limit of a composite function:

$$=2e^{\lim_{x\to 0}2x}-e^{\lim_{x\to 0}x}$$
$$=2e^{2\lim_{x\to 0}x}-e^{\lim_{x\to 0}x}$$
$$=2e^0-e^0$$
$$=2-1$$
$$=1$$

Challenge 2.40

Let α be a positive real number. Let $f:\mathbb{R}\to\mathbb{R}$ and $g:(\alpha,\infty)\to\mathbb{R}$ be the functions defined by:

$$f(x)=\sin\left(\frac{\pi x}{12}\right),\qquad g(x)=\frac{2\log_e\left(\sqrt{x}-\sqrt{\alpha}\right)}{\log_e\left(e^{\sqrt{x}}-e^{\sqrt{\alpha}}\right)}$$

Then, the value of $\lim_{x\to\alpha^+}f\big(g(x)\big)$ is: **(JEE-A 2022/Paper-I/2)**

We wish to find the limit of a composite function. Since the outer function ($\sin x$) is continuous throughout its domain, we can use the property for limit of a composite function, and take the limit "inside":

$$\lim_{x\to\alpha^+}f\big(g(x)\big)=f\left(\lim_{x\to\alpha^+}g(x)\right)=f\left(\lim_{x\to\alpha^+}\frac{2\ln\left(\sqrt{x}-\sqrt{\alpha}\right)}{\ln\left(e^{\sqrt{x}}-e^{\sqrt{\alpha}}\right)}\right)$$

Let $y=\sqrt{x}-\sqrt{\alpha}$ and note that as $x\to\alpha^+, y\to 0^+$:

$$\lim_{x\to\alpha^+}\ln\left(\sqrt{x}-\sqrt{\alpha}\right)=\lim_{y\to 0^+}\ln(y)=-\infty$$

Let $z=e^{\sqrt{x}}-e^{\sqrt{\alpha}}$ and note that as $x\to\alpha^+, z\to 0^+$

$$=\lim_{x\to\alpha^+}\ln\left(e^{\sqrt{x}}-e^{\sqrt{\alpha}}\right)=\lim_{z\to 0^+}\ln z=-\infty$$

Hence, we have the case $\frac{\infty}{\infty}$. Focus only on the limit for now.
Apply L 'Hospital's Rule and differentiate the numerator and the denominator (using the chain rule):

$$\lim_{x\to\alpha^+}\frac{\frac{2}{\sqrt{x}-\sqrt{\alpha}}\times\frac{1}{2\sqrt{x}}}{\frac{1}{e^{\sqrt{x}}-e^{\sqrt{\alpha}}}\times e^{\sqrt{x}}\times\frac{1}{2\sqrt{x}}}=\lim_{x\to\alpha^+}\frac{2}{e^{\sqrt{x}}}\times\frac{e^{\sqrt{x}}-e^{\sqrt{\alpha}}}{\sqrt{x}-\sqrt{\alpha}}$$

The first factor has a limit which can be easily evaluated.
Separate it using $\lim_{x\to c}f(x)g(x)=\lim_{x\to c}f(x)\lim_{x\to c}g(x)$ (provided the limits exist)

$$=\lim_{x\to\alpha^+}\frac{2}{e^{\sqrt{x}}}\times\lim_{x\to\alpha^+}\frac{e^{\sqrt{x}}-e^{\sqrt{\alpha}}}{\sqrt{x}-\sqrt{\alpha}}$$

The second limit, on substitution is $\frac{0}{0}$.
Apply L 'Hospital's Rule again on the second limit, and evaluate the first limit by substitution:

$$=\frac{2}{e^{\sqrt{\alpha}}}\times\lim_{x\to\alpha^+}\frac{e^{\sqrt{x}}\times\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{x}}}=\frac{2}{e^{\sqrt{\alpha}}}\times\lim_{x\to\alpha^+}e^{\sqrt{x}}$$

The remaining limit can also be evaluated using substitution:

$$= \frac{2}{e^{\sqrt{\alpha}}} \times e^{\sqrt{\alpha}} = 2$$

$$\lim_{x\to\alpha^+} f(2) = \lim_{x\to\alpha^+} \sin\left(\frac{2\pi}{12}\right) = \lim_{x\to\alpha^+} \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

F. One Sided Limits

2.41: L 'Hospital's Rule with One Sided Limits

L 'Hospital's Rule is applicable to one sided limits as well.

Example 2.42

$$\lim_{x\to 0} \frac{\sin x}{x^2}$$

Part A

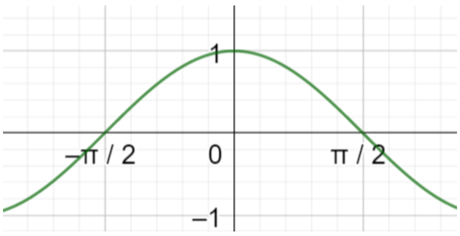
$$\lim_{x\to 0} \frac{\sin x}{x^2} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{x} = \lim_{x\to 0} \frac{\sin x}{x} \times \lim_{x\to 0} \frac{1}{x}$$

While $\lim_{x\to 0} \frac{\sin x}{x} = 1$, the second limit does not exist since:

$$\lim_{x\to 0^+} \frac{1}{x} = \infty,\ \lim_{x\to 0^-} \frac{1}{x} = -\infty \Rightarrow \lim_{x\to 0} \frac{1}{x}\ DNE$$

$$\lim_{x\to 0} \frac{\sin x}{x^2}\ DNE$$

Part B



- $\lim_{x\to 0} \frac{\sin x}{x^2}$ on substitution gives $\frac{0}{0}$ case.
- Numerator and denominator are differentiable over $\mathbb{R}$.

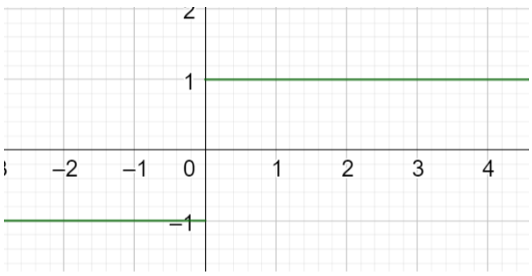
Apply L 'Hospital's Rule:

$$\lim_{x\to 0} \frac{\sin x}{x^2} = \lim_{x\to 0} \frac{\cos x}{2x}$$

$$RHL = \lim_{x\to 0^+} \frac{\cos x}{2x} = \infty, \qquad LHL = \lim_{x\to 0^-} \frac{\cos x}{2x} = -\infty \Rightarrow \lim_{x\to 0} \frac{\sin x}{x^2}\ DNE$$

Example 2.43

$$\lim_{x\to 0} \frac{x}{|x|}$$



$$|x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

$$x = -3 \Rightarrow |x| = 3 = -x$$

Method I: Without LH Rule

$$LHL = \lim_{x\to 0^-} \frac{x}{-x} = -1$$

$$RHL = \lim_{x\to 0^+} \frac{x}{x} = 1$$

Method II: LH Rule

Substitution gives $\frac{0}{0}$ *Case*

Numerator is differentiable over $\mathbb{R}$
Denominator is differentiable over $\mathbb{R}$ except at 0. However, at 0, the left-hand derivative exists, and also the right-hand derivative exists.

Hence, we can calculate the left-hand and the right-hand limit separately.

$$LHL = \lim_{x\to 0^-}\frac{x}{-x} = \lim_{x\to 0^-}\frac{\frac{d}{dx}x}{\frac{d}{dx}(-x)} = \lim_{x\to 0^-}\frac{1}{-1} = -1$$

$$RHL = \lim_{x\to 0^+}\frac{x}{x} = \lim_{x\to 0^+}\frac{\frac{d}{dx}x}{\frac{d}{dx}(x)} = \lim_{x\to 0^-}\frac{1}{1} = 1$$

G. Theory

2.44: Checking for Conditions

- It is important to ensure that the conditions for LH Rule have been met before actually applying it.
- If the conditions are not met, but the rule is still applied, you can get incorrect answers.

The conditions to check are:

- On substitution the limit results in $\frac{0}{0}$ OR $\frac{\infty}{\infty}$.
- c lies in some open interval I
- functions f and g are differentiable on I, except possibly at c.
- $g'(x) \neq 0$ on I if $x \neq c$

Example 2.45

Identify the flaw in the reasoning used to calculate the given limit and fix it.

$$\lim_{x\to 0}\frac{e^x-1}{x^2} = \underbrace{\lim_{x\to 0}\frac{e^x}{2x}}_{LH\ Rule} = \underbrace{\lim_{x\to 0}\frac{e^x}{2}}_{\substack{LH\ Rule \\ (Again)}} = \underbrace{\frac{1}{2}}_{Substitution}$$

Consider the following statements with respect to the above, and mark correct or incorrect for each one. If you mark incorrect, write the corrected steps.

A. Application of LH Rule(1st Time).
B. Application of LH Rule(2nd Time).
C. Substitution

Statement A is correct.

$$\lim_{x\to 0}\frac{e^x-1}{x^2} = \underbrace{\lim_{x\to 0}\frac{e^x}{2x}}_{LH\ Rule}, \qquad \frac{1}{0}\ Case$$

Hence, LH is neither applicable nor necessary. Statement B is incorrect. The corrected version is:

$$\lim_{x\to 0^+}\frac{e^x}{2x} = \infty,\ \lim_{x\to 0^-}\frac{e^x}{2x} = -\infty$$
$$\lim_{x\to 0}\frac{e^x-1}{x^2}\ DNE$$

The substitution is correct, but the intermediate step is incorrect. Hence, the final answer is incorrect.

Example 2.46

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=\underbrace{\lim_{\theta\to 0}\frac{\cos\theta}{1}}_{\substack{\text{Using}\\ \text{L 'Hospital's Rule}}}=\underbrace{\lim_{\theta\to 0}\frac{1}{1}}_{\substack{\text{By}\\ \text{Substitution}}}=1$$

Consider the following statements with respect to the above, and mark True or False for each one. If you mark False, explain why it is false.

A. The step making use of L'Hospital's Rule is correct.
B. The substitution step is correct.
C. The final answer is correct.
D. The above is a correct proof of $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$

The conditions for L 'Hospital's Rule apply.
However, the steps makes use of $\frac{d}{d\theta}(\sin\theta)=\cos\theta$,and the proof of that derivative makes use of

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

Hence, the proof is an example of circular reasoning.

2.4 LH Rule-II: More Indeterminate Forms

A. Indeterminate Forms

We looked at the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ and saw how to handle them using LH Rule.
However, there are other indeterminate forms, and we now look at how to handle some of these indeterminate forms.

2.47: Indeterminate Forms

$$\infty\cdot 0$$
$$\infty-\infty$$

2.48: Change of Variable

- Using a change of a variable

Example 2.49: Change of Variable

$$\lim_{x\to\infty} x\sin\frac{1}{x}$$

Note that as $x\to\infty$ this is of the form:

$$x\sin\frac{1}{x}=\infty\cdot 0$$

Use a change of variable. Let

$$h=\frac{1}{x}\Rightarrow x=\frac{1}{h}$$

As

$$x\to\infty, h=\frac{1}{x}\to 0.$$

Hence, making the change of variable, we get:

$$\lim_{h\to 0}\frac{1}{h}\sin h=\lim_{h\to 0}\frac{\sin h}{h}=1$$

Note: This solution did not require L 'Hospital's Rule. The change of variable was sufficient.

$$\lim_{x\to\infty} 2x\sin\frac{1}{x}$$

$$\lim_{x\to\infty} 2x\sin\frac{1}{x} = 2\lim_{x\to\infty} x\sin\frac{1}{x} = 2\cdot 1 = 2$$

2.50: Change of Form

By converting it into a form where LH Rule can be applied, such as $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form.

Example 2.51: Change of Form

$$\lim_{x\to 0^+} x\ln x$$

Note that this is currently in:

$$\underbrace{x}_{0}\underbrace{\ln x}_{-\infty} = 0\cdot\infty\ Form$$

We convert it to $\frac{\infty}{\infty}$ form by moving the x term to the denominator:

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln(x)}{\frac{1}{x}}$$

Numerator is differentiable on an interval to the right of 0.
Denominator is differentiable on an interval to the right of 0.

Apply L 'Hospital's Rule:

$$\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}\frac{1}{x}\cdot\left(-\frac{x^2}{1}\right) = \lim_{x\to 0^+} -x = 0$$

Example 2.52: Change of Form

$$\lim_{x\to 0^+} x^2\ln(x^3)$$

Note that this is currently in:

$$\underbrace{x^2}_{0}\underbrace{\ln(x^3)}_{-\infty} = 0\cdot\infty\ Form$$

We convert it to $\frac{\infty}{\infty}$ form by moving the x^2 term to the denominator:

$$\lim_{x\to 0^+} x^2\ln(x^3) = \lim_{x\to 0^+}\frac{3\ln(x)}{\frac{1}{x^2}}$$

Numerator is differentiable on an interval to the right of 0.
Denominator is differentiable on an interval to the right of 0.

Apply L 'Hospital's Rule:

$$\lim_{x\to 0^+}\frac{\frac{3}{x}}{\frac{-2}{x^3}}=\lim_{x\to 0^+}\frac{3}{x}\cdot\frac{x^3}{-2}=\lim_{x\to 0^+}-\frac{3x^2}{2}=0$$

Evaluate

$$\lim_{x\to 0^+}\sqrt[n]{x}\ln(x^m),\qquad n>1,m>1$$

Note that this is currently in:

$$\underbrace{\sqrt[n]{x}}_{0}\underbrace{\ln(x^m)}_{-\infty}=\infty\cdot 0\ Form$$

We convert it to $\frac{\infty}{\infty}$ form by moving the $\sqrt[n]{x}$ term to the denominator:

$$\lim_{x\to 0^+}\sqrt[n]{x}\ln(x^m)=\lim_{x\to 0^+}\frac{m\ln(x)}{\frac{1}{\sqrt[n]{x}}}$$

Numerator is differentiable on an interval to the right of 0.
Denominator is differentiable on an interval to the right of 0.

Apply L 'Hospital's Rule:

$$\lim_{x\to 0^+}\frac{\frac{m}{x}}{\left(-\frac{1}{n}\right)\left(x^{-\frac{1}{n}-1}\right)}$$

Moving the x term to the denominator:

$$=\lim_{x\to 0^+}\frac{\frac{m}{x}}{-\frac{1}{nx^{\frac{1}{n}+1}}}$$

Eliminate the nested fraction by multiplying by the reciprocal of the denominator:

$$=\lim_{x\to 0^+}\frac{m}{x}\left(-\frac{nx^{\frac{1}{n}+1}}{1}\right)$$

Simplify:

$$=\lim_{x\to 0^+}-mnx^{\frac{1}{n}}$$

Use the constant multiple rule:

$$=mn\lim_{x\to 0^+}-x^{\frac{1}{n}}=0$$

Example 2.53: Change of Form

$$\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)$$

Note that the limit is currently:

$$\infty-\infty\ Form$$

We convert this into $\frac{0}{0}$ form by adding the fractions:

$$\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)=\lim_{x\to 0}\left(\frac{x-e^x+1}{x(e^x-1)}\right)$$

Numerator and denominator are differentiable over $\mathbb{R}$. Apply L 'Hospital's Rule:

$$\lim_{x\to 0}\left(\frac{1-e^x}{e^x+xe^x-1}\right)$$

This is still $\frac{0}{0}$ form. Also, numerator and denominator are still differentiable over $\mathbb{R}$. Apply L 'Hospital's Rule (2nd Time):

$$\lim_{x\to 0}\left(\frac{-e^x}{e^x+e^x+xe^x}\right)$$

Substitute $x=0$:

$$\frac{-1}{1+1+0}=-\frac{1}{2}$$

B. Indeterminate Forms-II

2.54: Log of the Limit

$$1^\infty$$

- finding the log of the limit, and then exponentiating the final answer.

Example 2.55

$$\lim_{x\to 0}(1-x)^{\frac{1}{x}}$$

$$f(x)=(1-x)^{\frac{1}{x}} \Rightarrow \ln f(x)=\frac{\ln(1-x)}{x}$$

Take the limit both sides:

$$\lim_{x\to 0}\ln f(x)=\lim_{x\to 0}\frac{\ln(1-x)}{x}$$

Apply LH Rule:

$$\lim_{x\to 0}\ln f(x)=\lim_{x\to 0}\frac{-\frac{1}{1-x}}{1}=-1$$

$$\lim_{x\to 0}f(x)=e^{-1}=\frac{1}{e}$$

Example 2.56

$$\lim_{x\to 0^+}\left(\frac{1}{x}\right)^x$$

$$f(x)=\left(\frac{1}{x}\right)^x=x^{-x}$$

Take the limit of the natural log of $f(x)$:

$$\lim_{x\to 0^+}\ln f(x)=\lim_{x\to 0^+}(-x)(\ln x)$$

This is of the form $0\cdot\infty$. Rewrite in the form $\frac{\infty}{\infty}$ by moving the x to the denominator:

$$\lim_{x\to 0^+}\ln f(x)=\lim_{x\to 0^+}\frac{-\ln x}{\frac{1}{x}}$$

Apply LH Rule:

$$\lim_{x\to 0^+}\ln f(x)=\lim_{x\to 0^+}\frac{-\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0^+}\frac{1}{x}\times\frac{x^2}{1}=\lim_{x\to 0^+}x=0$$

Exponentiate both sides:

$$e^{\lim_{x\to 0^+}\ln f(x)}=e^0$$
$$\lim_{x\to 0^+}e^{\ln f(x)}=1$$
$$\lim_{x\to 0^+}f(x)=1$$

$$\lim_{x\to 0^+}x^{-x}=e^{\lim_{x\to 0^+}\ln x^{-x}}=e^{\lim_{x\to 0^+}\ln(-x)(\ln x)}=e^{\lim_{x\to 0^+}\frac{-\ln x}{\frac{1}{x}}}=e^{\lim_{x\to 0^+}\frac{-\frac{1}{x}}{-\frac{1}{x^2}}}=e^{\lim_{x\to 0^+}\frac{x^2}{x}}=e^0=1$$

2.57: Log of the Limit

$$0^0$$

- finding the log of the limit, and then exponentiating the final answer.

Example 2.58

$$\lim_{x\to 0}\left(|x|^{\sin x}\right)$$

Try Substitution

On substitution we get:

$$0^0$$

Which is not defined.

Convert into log form

Use the property that $x = e^{\ln x}$:

$$\lim_{x\to 0}\left(e^{\ln\left(|x|^{\sin x}\right)}\right) = \lim_{x\to 0}\left(e^{\sin x\ln(|x|)}\right)$$

Use the property for the limit of a composite function to interchange the exponentiation and the limit operator

$$e^{\lim_{x\to 0}\sin x\ln(|x|)}$$

We evaluate

$$\lim_{x\to 0}\sin x\ln(|x|) \Rightarrow \infty\cdot 0\ Form$$

Rewrite this in $\frac{\infty}{\infty}$ form:

$$= \lim_{x\to 0^+}\left(\frac{\ln|x|}{\csc x}\right)$$

Apply the definition of the absolute value function and split the limit into RHL and LHL:

Right Hand Limit

$$= \lim_{x\to 0^+}\left(\frac{\ln x}{\csc x}\right)$$

Apply L 'Hospital's Rule:

$$= \lim_{x\to 0^+}\left(\frac{\frac{1}{x}}{-\csc x\cot x}\right)$$

Remove the nested fraction:

$$= \lim_{x\to 0^+}\left(\frac{1}{x}\times\frac{1}{-\csc x\cot x}\right)$$

Move the trigonometric terms to the numerator:

$$= \lim_{x\to 0^+}\left(-\frac{\sin x\tan x}{x}\right)$$

Write it as the product of two limits

$$= \lim_{x\to 0^+}\left(-\frac{\sin x}{x}\right)\times\lim_{x\to 0^+}\tan x$$

By substitution:

$$= (-1)(0) = 0$$

Left Hand Limit

$$= \lim_{x\to 0^-}\left(\frac{\ln(-x)}{\sin x}\right)$$

Apply L 'Hospital's Rule:

$$= \lim_{x\to 0^-}\left(\frac{-\frac{1}{x}}{-\csc x\cot x}\right) = \lim_{x\to 0^-}\left(\frac{\sin x\tan x}{x}\right)$$

Write it as the product of two limits

$$= \lim_{x\to 0^-}\left(\frac{\sin x}{x}\right)\times\lim_{x\to 0^-}\tan x$$

By substitution:

$$= (1)(0) = 0$$

Hence:

$$RHL = LHL = 0$$

Hence, the final answer is:

$$e^{\lim_{x\to 0}\sin x\ln(|x|)} = e^0 = 1$$

2.59: Log of the Limit

- finding the log of the limit, and then exponentiating the final answer.

Example 2.60: Defined Functions

Let $f: \mathbb{R} \to \mathbb{R}$ such that $f(1) = 3$ and $f'(1) = 6$. Then $\lim_{x\to 0}\left[\frac{f(1+x)}{f(1)}\right]^{\frac{1}{x}}$ is (JEE-A 2002-Screening)

On substitution, we get:

$$\left[\frac{f(1)}{f(1)}\right]^{\frac{1}{0}} = 1^{\infty}\ Form$$

Let $y = \lim_{x\to 0}\left[\frac{f(1+x)}{f(1)}\right]^{\frac{1}{x}}$. Then:

$$\ln y = \ln \lim_{x\to 0}\left[\frac{f(1+x)}{f(1)}\right]^{\frac{1}{x}} = \lim_{x\to 0}\frac{1}{x}\ln\left[\frac{f(1+x)}{f(1)}\right] = \lim_{x\to 0}\frac{\ln f(1+x) - \ln f(1)}{x}$$

Apply L 'Hospital's Rule, and substitute $x = 0$:

$$= \lim_{x\to 0}\frac{\frac{1}{f(1+x)}\times f'(1+x)}{1} = \frac{f'(1)}{f(1)} = \frac{6}{3} = 2$$

Finally:

$$\ln y = 2 \Rightarrow y = \lim_{x\to 0}\left[\frac{f(1+x)}{f(1)}\right]^{\frac{1}{x}} = e^2$$

2.5 LH Rule-III: Further Examples

A. Continuity

2.61: Continuity

Example 2.62

B. Limit of a Composite Function

C. Quadratics

Example 2.63

If α, β are the distinct roots of $ax^2 + bx + c = 0$, then, in terms of a, b, c, α, β:

$$\lim_{x\to\alpha}\frac{1 - \cos(ax^2 + bx + c)}{(x-\alpha)^2}$$

is: (JEE-M 2005)

Substitute $x = \alpha$:

$$\frac{1-\cos(0)}{(\alpha-\alpha)^2}=\frac{1-1}{0^2}=\frac{0}{0}\ Case$$

The numerator and the denominator are differentiable over $\mathbb{R}$.

Rewrite the given expression in terms of its roots α and β, and the vertical stretch parameter a:

$$\lim_{x\to\alpha}\frac{1-\cos[a(x-\alpha)(x-\beta)]}{(x-\alpha)^2}$$

Apply LH Rule:

$$\lim_{x\to\alpha}\frac{\sin[a(x-\alpha)(x-\beta)][a(x-\alpha)+a(x-\beta)]}{2(x-\alpha)(1)}$$

$$=\lim_{x\to\alpha}\frac{\sin[a(x-\alpha)(x-\beta)][a(2x-\alpha-\beta)]}{2(x-\alpha)(1)}$$

Substitute

$$\frac{0}{0}\ Case$$

Apply LH Rule (2nd Time):

$$\lim_{x\to\alpha}\frac{-\cos[a(x-\alpha)(x-\beta)][a(2x-\alpha-\beta)]^2+\sin[a(x-\alpha)(x-\beta)](2a)}{2}$$

Substitute $x=\alpha$:

$$\frac{-\cos[0][a(2x-\alpha-\beta)]^2+\sin[0](2a)}{2}$$

$$=\frac{-[a(2x-\alpha-\beta)]^2+0(2a)}{2}$$

$$=\frac{-[a(2\alpha-\alpha-\beta)]^2}{2}$$

$$=-\frac{a^2}{2}(\alpha-\beta)^2$$

Example 2.64

If α,β are the distinct roots of $x^2+bx+c=0$, then, in terms of b and c:

$$\lim_{x\to\beta}\frac{e^{2(x^2+bx+c)}-1-2(x^2+bx+c)}{(x-\beta)^2}$$

is: (JEE-Main 2021)

Since β is a root of the given quadratic, substituting β in x^2+bx+c results in zero. On Substitution:

$$\frac{e^{2(0)}-1-2(0)}{(0)^2}=\frac{1-1-0}{0}=\frac{0}{0}$$

Apply LH Rule:

$$\lim_{x\to\beta}\frac{e^{2(x^2+bx+c)}(2)(2x+b)-2(2x+b)}{2(x-\beta)(1)}=\lim_{x\to\beta}\frac{e^{2(x^2+bx+c)}(2x+b)-(2x+b)}{(x-\beta)}$$

Substituting $x=\beta$ gives us $\frac{e^{2(0)}(2x+b)-(2x+b)}{0}=\frac{(2x+b)-(2x+b)}{0}=\frac{0}{0}$.

Apply LH Rule one more time:

$$\lim_{x\to\beta}e^{2(x^2+bx+c)}(2x+b)^2+(2)e^{2(x^2+bx+c)}-2$$

Substituting $x=\beta$ gives us:

$$=e^{2(0)}(2x+b)^2+(2)e^{2(0)}-2=(1)(2x+b)^2+(2)(1)-2=(2x+b)^2$$

To calculate the last expression, use Vieta's Formulas:

$$b = -\alpha - \beta \Rightarrow 2x + b = 2\beta + (-\alpha - \beta) = \beta - \alpha$$

Now, we need the difference of the roots.

$$\alpha + \beta = -b \Rightarrow \alpha^2 + 2\alpha\beta + \beta^2 = b^2$$

Subtract $4\alpha\beta = 4c$ from both sides:

$$\alpha^2 - 2\alpha\beta + \beta^2 = (\beta - \alpha)^2 = \underbrace{b^2 - 4c}_{Final\ Answer}$$

D. Fundamental Theorem of Calculus

2.6 Relative Rates of Growth

A. Basics

2.65: Little-Oh Notation

$$f = o(g) \Leftrightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

- Given functions f and g, f is of smaller order than g if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$
- The notation for "smaller order" is $f = o(g)$

Example 2.66

2.67: Big-Oh Notation

3. Limits with Series

3.1 Limits with Maclaurin Series

A. Limits using e^x

The Maclaurin expansion of the standard function can be substituted into expressions to simplify the calculations of limits that would otherwise often need L'Hospital's Rule.

We revisit L 'Hospital's Rule and redo a few questions using series expansions.

Video 3.1

This fast-paced video covers the Taylor Theorem, its proof, important expansions using the Taylor Theorem, and also the application of expansion to limits.

3.2: L 'Hospital's Rule

A limit of a quotient can be found using the rule:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

The conditions to check are:

- $f(c) = g(c) = 0$
- OR $f(c) = g(c) = \pm\infty$
  - ✓ In other words, on substitution the limit results in the $\frac{0}{0}$ case or the the $\pm\frac{\infty}{\infty}$ case
- functions f and g are differentiable on an open interval I, except possibly at c. (where c lies in I)
- $g'(x) \neq 0$ on I if $x \neq c$
- The limit $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists

3.3: e^x expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

Example 3.4

$$\lim_{x \to 0} \frac{e^x - 1}{x}$$

Evaluate the limit using:

A. LH Rule
B. Expansions

Part A: L 'Hospital's Rule

Calculate the derivative of the numerator and the denominator:

$$= \lim_{x\to 0} \frac{\frac{d}{dx}(e^x - 1)}{\frac{d}{dx}(x)} = \lim_{x\to 0} \frac{e^x}{1} = \frac{1}{1} = 1$$

Part B: Series Expansions

Substitute the expansion for e^x in the limit:

$$\lim_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \cdots\right) - 1}{x} = \lim_{x\to 0} \frac{x + \frac{x^2}{2!} + \cdots}{x}$$

Divide the numerator and denominator by x:

$$\lim_{x\to 0} \frac{1 + \frac{x}{2!} + \cdots}{1} = \lim_{x\to 0} 1 + \frac{x}{2!} + \cdots = 1$$

3.5: Shortcut for Expansions

We only need to write the series upto the highest power of x in the denominator.

In the previous example, the denominator only had powers upto x.
Hence, we divided by x.

Powers beyond x, such as x^2, x^3, etc tend to zero, and hence they can be ignored.

Example 3.6

$$\lim_{x\to 0} \frac{e^x - 1}{x}$$

$$\lim_{x\to 0} \frac{(1 + x + \cdots) - 1}{x} = \lim_{x\to 0} \frac{x + \cdots}{x} = \lim_{x\to 0} \frac{1 + \cdots}{1} = \lim_{x\to 0} 1 + \cdots = 1$$

Example 3.7

$$\lim_{x\to 0} \frac{\frac{e^x}{a} - \frac{1}{a} - \frac{x}{a}}{\frac{x^2}{b}}$$

Given that a and b are non-zero real numbers:

A. Evaluate the limit using L 'Hospital's Rule
B. Evaluate the limit using series expansions

Part A: L 'Hospital's Rule

Substitute:

$$\frac{\frac{1 - 1 - 0}{a}}{\frac{0}{b}} = \frac{\frac{0}{a}}{\frac{0}{b}} = \frac{0}{0}\ Case$$

Factor $\frac{b}{a}$ out of the expression to simplify:

$$\lim_{x\to 0} \left(\frac{b}{a}\right) \frac{e^x - 1 - x}{x^2}$$

Use the constant multiple rule to move $\frac{b}{a}$ outside of the limit:

$$= \left(\frac{b}{a}\right) \lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$

Apply LH Rule:

$$= \left(\frac{b}{a}\right) \lim_{x \to 0} \frac{e^x - 1}{2x}$$

Apply LH Rule again:

$$= \left(\frac{b}{a}\right) \lim_{x \to 0} \frac{e^x}{2}$$

Substitute:

$$= \left(\frac{b}{a}\right)\left(\frac{e^0}{2}\right) = \frac{b}{2a}$$

Part B: Series Expansions

$$\frac{b}{a} \lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$

Substitute e^x and simplify

$$= \frac{b}{a} \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2} + \cdots\right) - 1 - x}{x^2} = \frac{b}{a} \lim_{x \to 0} \frac{\frac{x^2}{2} + \cdots}{x^2}$$

Convert the denominator to division:

$$= \frac{b}{a} \lim_{x \to 0} \left(\frac{x^2}{2!} + \cdots\right) \frac{1}{x^2}$$

Simplify

$$\frac{b}{a} \lim_{x \to 0} \frac{1}{2} = \frac{b}{2a}$$

B. Limits using $\sin x$ and $\cos x$

3.8: Maclaurin series for $\sin x$ and $\cos x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Example 3.9

$$\lim_{x \to 0} \frac{x \cos x - \sin x}{x - \sin x}$$

$$Numerator = x \cos x - \sin x = x\left(1 - \frac{x^2}{2!} + \cdots\right) - \left(x - \frac{x^3}{3!} + \cdots\right) = -\frac{x^3}{3} + \cdots$$
$$Denominator = x - \left(x - \frac{x^3}{3!} + \cdots\right) = \frac{x^3}{3!} - \cdots$$

Substitute the above to get:

$$\lim_{x\to 0}\frac{-\frac{x^3}{3}+\cdots}{\frac{x^3}{3!}-\cdots}=\lim_{x\to 0}\frac{2x^3+\cdots}{x^3-\cdots}=\lim_{x\to 0}\frac{2+\cdots}{1-\cdots}=2$$

C. Limits using log functions

3.10: Maclaurin Series for log functions

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

Example 3.11

The value of the limit below is: **(JAM MS 2018)**

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{n^2}e^{-2n}$$

Let

$$y=\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{n^2}e^{-2n}$$

Take the natural log both sides:

$$\ln y=\ln\left[\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{n^2}e^{-2n}\right]$$

Since the expression is continuous as $n\to\infty$, use the property $\lim_{x\to\alpha}f\big(g(x)\big)=f\left(\lim_{x\to\alpha}g(x)\right)$ to interchange the log and limit operators:

$$\ln y=\lim_{n\to\infty}\left[\ln\left[\left(1+\frac{2}{n}\right)^{n^2}e^{-2n}\right]\right]$$

Use the power rule and product rule of logarithms to simplify:

$$\ln y=\lim_{n\to\infty}\left[n^2\ln\left(1+\frac{2}{n}\right)-2n\ln e\right]$$
$$\ln y=\lim_{n\to\infty}\left[n^2\ln\left(1+\frac{2}{n}\right)-2n\right]$$

Use $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$ to expand $\ln\left(1+\frac{2}{n}\right)$:

$$\ln y=\lim_{n\to\infty}\left[n^2\left\{\frac{2}{n}-\left(\frac{2}{n}\right)^2\frac{1}{2}+\left(\frac{2}{n}\right)^3\frac{1}{3}-\cdots\right\}-2n\right]$$

Multiply n^2 with the Maclaurin series:

$$\ln y=\lim_{n\to\infty}\left[\left\{2n-2+\frac{8}{3n}-\cdots\right\}-2n\right]$$

Cancel the $2n$ with the $-2n$:

$$\ln y=\lim_{n\to\infty}\left[-2+\frac{8}{3n}-\cdots\right]=\lim_{n\to\infty}[-2]=-2$$

Exponentiate both sides:

$$y = e^{-2}$$

Note:
We could also have done this using:

$$e^{\ln\left[\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{n^2} e^{-2n}\right]}$$

D. Differentiation and Integration

These limits require further manipulation compared to the earlier ones.

3.12: Maclaurin Series for *tan x*

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots, \qquad |x| < \frac{\pi}{2}$$

Example 3.13

$$\lim_{x\to 0} \frac{[\ln(\cos x)]^2}{x^5 - x^4}$$

Series expansion

We need a series expansion for $\ln(\cos x)$. Differentiate:

$$\frac{d}{dx}[\ln(\cos x)] = -\tan x = -\left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots\right)$$

Integrate throughout:

$$\ln(\cos x) = \int -\tan x = \int \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots\right) dx$$

Integrate term by term:

$$\ln(\cos x) = \frac{x^2}{2} + \frac{x^4}{12} + \cdots$$

Square both sides of the above:

$$[\ln(\cos x)]^2 = \left(\frac{x^2}{2} + \frac{x^4}{12} + \cdots\right)^2$$

The highest power of x in the denominator is x^5.
Hence, terms beyond x^5 can be ignored. So, we only need the first term in the numerator, because the second term is of the form x^6.

$$[\ln(\cos x)]^2 = \frac{x^4}{4} + \cdots$$

Calculate the limit

Hence, the limit becomes:

$$= \lim_{x\to 0} \frac{\frac{x^4}{4} + \cdots}{x^5 - x^4}$$

Divide by the lowest power of x in the denominator:

$$= \lim_{x\to 0} \frac{\frac{1}{4}+\cdots}{x-1} = -\frac{1}{4}$$

E. Back Calculations

Example 3.14

$$\beta = \lim_{x\to 0} \frac{\alpha x - (e^{3x}-1)}{\alpha x(e^{3x}-1)}$$

for some $\alpha \in \mathbb{R}$. Then the value of $\alpha + \beta$ is: **(JEE-M 2021, 2022)**

Substitute $e^{3x}-1 = \left(1+3x+\frac{(3x)^2}{2!}+\cdots\right)-1 = 3x+\frac{9x}{2}+\cdots$ in the given limit:

$$\beta = \lim_{x\to 0} \frac{\alpha x - \left(3x+\frac{9x}{2}+\cdots\right)}{\alpha x\left(3x+\frac{9x}{2}+\cdots\right)}$$

Simplify in the numerator, and multiply in the denominator:

$$\beta = \lim_{x\to 0} \frac{(\alpha-3)x - \frac{9x}{2} - \cdots}{\alpha\left[3x^2+\frac{9x^3}{2}+\cdots\right]}$$

Divide the numerator and denominator by the lowest power of x in the denominator, which is x^2:

$$\beta = \lim_{x\to 0} \frac{\frac{\alpha-3}{x} - \frac{9}{2} - \cdots}{\alpha\left[3+\frac{9x}{2}+\cdots\right]}$$

Since the limit has a finite value, the first term must have a zero numerator to prevent that term from going to infinity:

$$\alpha - 3 = 0 \Rightarrow \alpha = 3$$

Substitute $\alpha = 3$, and eliminate all powers of x:

$$\beta = \lim_{x\to 0} \frac{-\frac{9}{2}-\cdots}{3[3]} = -\frac{9}{2}\times\frac{1}{9} = -\frac{1}{2}$$

$$\alpha + \beta = 3 - \frac{1}{2} = \frac{5}{2}$$

3.15: Terms forced to zero

$$\lim_{x\to 0} \frac{ax+bx^2+cx^3+dx^4 \ldots}{x^3} = k,$$

If the limit above has a finite value then it is only possible if:

$$a = b = 0$$

Split the fraction:

$$\lim_{x\to 0} \frac{ax}{x^3}+\frac{bx^2}{x^3}+\frac{cx^3}{x^3}+\frac{dx^4}{x^3}\ldots$$

Simplify:

$$= \lim_{x\to 0} \frac{a}{x^2} + \frac{b}{x} + c + dx + \cdots$$

If $a = b = 0$, then we get:

$$= \lim_{x\to 0} \frac{0}{x^2} + \frac{0}{x} + c + dx + \cdots = \lim_{x\to 0} c + dx + \cdots = c$$

If either of a or b (or both) are non-zero, then the limit tends to infinity.

Hence:

$$a = b = 0$$

Example 3.16

Find the value of α, β, γ given that: **(JEE-M 2021, 2022, Adapted)**

$$\lim_{x\to 0} \frac{\alpha e^x + \beta e^{-x} + \gamma \sin x}{x \sin^2 x} = \frac{2}{3}, \qquad \alpha, \beta, \gamma \in \mathbb{R}$$

The series expansion for the denominator is:

$$x \sin^2 x = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = x\left(x^2 - \frac{x^4}{6} - \frac{x^4}{6} + \cdots\right) = x^3 - \frac{x^5}{6} - \frac{x^5}{6} + \cdots$$

Substitute the standard expansions in the numerator:

$$\lim_{x\to 0} \frac{\alpha\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) + \beta\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right) + \gamma\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x^3 - \frac{x^5}{6} - \frac{x^5}{6} + \cdots} = \frac{2}{3}$$

Collate like terms.

$$\lim_{x\to 0} \frac{(\alpha + \beta)x^0 + (\alpha - \beta + \gamma)x + \left(\frac{\alpha}{2} + \frac{\beta}{2}\right)x^2 + \left(\frac{\alpha}{6} - \frac{\beta}{6} - \frac{\gamma}{6}\right)x^3 + \cdots}{x^3 - \frac{x^5}{6} - \frac{x^5}{6} + \cdots} = \frac{2}{3}$$

Since the lowest power of x in the denominator is x^3, the coefficient of all lower powers must be zero.

Substitute $\alpha + \beta = 0 \Rightarrow \beta = -\alpha$ from the constant term into[3]:

$$\boldsymbol{x\ term}: \alpha - \beta + \gamma = \alpha - (-\alpha) + \gamma = \underbrace{2\alpha + \gamma = 0}_{Equation\ I}$$

$$\boldsymbol{x^3\ term}: \frac{\alpha}{6} - \frac{\beta}{6} - \frac{\gamma}{6} = \frac{2}{3} \Rightarrow \alpha - \beta - \gamma = 4 \Rightarrow \underbrace{2\alpha - \gamma = 4}_{Equation\ II}$$

Add I and II

$$4\alpha = 4 \Rightarrow \alpha = 1 \Rightarrow \beta = -1$$
$$2\alpha + \gamma = 2 + \gamma = 0 \Rightarrow \gamma = -2$$

The final answer is:

$$(\alpha, \beta, \gamma) = (1, -1, -2)$$

---

[3] The coefficient of x^2 has the same information as the constant term: $\frac{\alpha}{2} + \frac{\beta}{2} = 0 \Rightarrow \beta = -\alpha$

3.2 Limits with Taylor Series

A. Basics

Example 3.17: Trig Identities

$$\lim_{x\to\pi}\frac{\sin x}{e^\pi - e^x}$$

$$\frac{\sin\pi}{e^\pi - e^\pi} = \frac{0}{0}\ Case$$

Part A: Using L'Hospital's Rule:

$$\lim_{x\to\pi}\frac{\sin x}{e^\pi - e^x} = \lim_{x\to\pi}\frac{\cos x}{-e^x} = \frac{-1}{-e^\pi} = \frac{1}{e^\pi}$$

Part B: Using Series Expansions

Substitute the Taylor series expansion for $\sin x = \sin(\pi - x)$ in the numerator and $e^x\ at\ x = \pi$ in the denominator:

$$\lim_{x\to\pi}\frac{(\pi - x) - \frac{(\pi - x)^3}{3!} + \cdots}{e^\pi - \left(e^\pi + e^\pi(x-\pi) + \frac{e^\pi(x-\pi)^2}{2!} + \cdots\right)}$$

Simplify the denominator, and factor out e^π:

$$\lim_{x\to\pi}\frac{(\pi - x) - \frac{(\pi - x)^3}{3!} + \cdots}{-e^\pi\left((x-\pi) + \frac{(x-\pi)^2}{2!} + \cdots\right)}$$

Divide by $\pi - x$ in the numerator and denominator. The fact that $\pi - x = (-1)(x-\pi)$ is useful here.

$$\lim_{x\to\pi}\frac{1 - \frac{(\pi - x)^2}{3!} + \cdots}{-e^\pi\left(-1 + \frac{\pi - x}{2!} + \cdots\right)}$$

Note that the higher order terms tend to zero:

$$= \frac{1-0}{-e^\pi(-1+0)} = \frac{1}{e^\pi}$$

3.3 Definition of Limits

A. Challenge and Response

Example 3.18

A. Consider the function $y = f(x) = 3x + 4$. Evaluate $\lim_{x\to 2} 3x + 4$

B. The limit that you found in Part A is 10. We want to check for what values of x is $f(x)$ close to 10. For this, begin by finding for what values of x is $f(x)$ within 2 units of 10. You will find the property below useful:

$$|x| < a \Rightarrow -a < x < a$$

C. Check the endpoints of the answer that you found in Part B by substitution.

Part A

$$\lim_{x\to 2} 3x+4 = 6+4 = 10$$

Part B

$$10-2 \le f(x) \le 10+2$$
$$8 \le f(x) \le 12$$
$$-2 \le f(x)-10 \le 2$$
$$|f(x)-10| \le 2$$
$$|3x+4-10| \le 2$$
$$|3x-6| \le 2$$
$$-2 \le 3x-6 \le 2$$
$$4 \le 3x \le 8$$
$$\frac{4}{3} \le x \le \frac{8}{3}$$
$$1\frac{1}{3} \le x \le 2\frac{2}{3}$$

Part C

$$x = 1\frac{1}{3} \Rightarrow 3x+4 = 3\left(\frac{4}{3}\right)+4 = 4+4 = 8$$
$$x = 2\frac{2}{3} \Rightarrow 3x+4 = 3\left(\frac{8}{3}\right)+4 = 8+4 = 12$$

B. Epsilon-Delta Arguments

3.19: Definition

The limit $\lim_{x\to a} f(x) = L$ exists, if, for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$ then

$$|f(x)-L| < \epsilon$$

3.20: Absolute Value Inequalities

$$|x| < p \Leftrightarrow -p < x < p$$

This can be understood using the "distance interpretation".
$|x| < 3$ is the same as x is less than 3 units away from zero on the number line, which we can write

$$-3 < x < 3$$

Example 3.21

$$\left|\frac{2n-1}{n+1}-2\right| < 0.1$$
$$\left|\frac{2n-1}{n+1}-2\right| < p$$

$$\left|\frac{2n-1}{n+1}-2\right| = \left|\frac{2n-1}{n+1}-\frac{2n+2}{n+1}\right| = \left|\frac{-3}{n+1}\right| = \frac{3}{n+1}$$

$$\frac{3}{n+1} < 0.1$$

$$3 < 0.1n + 0.1$$
$$2.9 < 0.1n$$
$$29 < n$$
$$n > 29$$

$$\frac{3}{n+1} < p$$
$$3 < pn + p$$
$$3 - p < pn$$
$$\frac{3-p}{p} < n$$
$$n > \frac{3-p}{p}$$

Example 3.22

$$\left|\frac{2^n - 1}{2^n + 1} - 1\right| < p$$

$$\left|\frac{2^n - 1}{2^n + 1} - 1\right| = \left|\frac{2^n - 1 - 2^n - 1}{2^n + 1}\right| = \left|\frac{-2}{2^n + 1}\right| = \frac{2}{2^n + 1}$$

Hence, the given inequality becomes:

$$\frac{2}{2^n + 1} < p$$
$$2 < p2^n + p$$
$$2 - p < p2^n$$
$$p2^n > 2 - p$$
$$2^n > \frac{2-p}{p}$$

Take the log both sides:

$$\ln 2^n > \ln \frac{2-p}{p}$$
$$n > \frac{\ln \frac{2-p}{p}}{\ln 2}$$

Example 3.23

$$\left|\frac{2^n - 1}{2^n + 1} - 1\right| < p$$

Remove the absolute value sign:

$$-p < \frac{2^n - 1}{2^n + 1} - 1 < p$$

Case I

$$-p < \frac{2^n - 1}{2^n + 1} - 1$$

Add 1 to both sides:

$$-p + 1 < \frac{2^n - 1}{2^n + 1}$$

Multiply both sides by $2^n + 1$:

$$(1 - p)(2^n + 1) < 2^n - 1$$

Expand on the LHS:

$$2^n - p2^n + 1 - p < 2^n - 1$$

Keep only the 2^n term on the LHS. Move all other terms to the RHS:

$$-p2^n < -2 + p$$

Divide both sides by $-p$, and reverse the sign of the inequality:

$$2^n > \frac{-2 + p}{-p}$$

$$2^n > \frac{2-p}{p}$$

Take the log both sides:

$$\ln 2^n > \ln \frac{2-p}{p}$$

$$n > \frac{\ln \frac{2-p}{p}}{\ln 2}$$

Case II

$$\frac{2^n - 1}{2^n + 1} - 1 < p$$

$$\frac{2^n - 1}{2^n + 1} < 1 + p$$

$$2^n - 1 < (1+p)(2^n + 1)$$

$$2^n - 1 < 2^n + p2^n + 1 + p$$

$$-p2^n < 2 + p$$

$$2^n > \frac{2+p}{-p}$$

$$+ve > -ve$$

$$Always\ True$$

Example 3.24

Verify that $|x^2 - 4| < 0.0001$ is satisfied by x close to 2. How close is enough?

Remove the absolute value sign using the property $|y| < c \Leftrightarrow -c < y < c$:

$$-0.0001 < x^2 - 4 < 0.0001$$

Add 4 throughout:

$$3.9999 < x^2 < 4.0001$$

Take square roots:

$$\sqrt{3.9999} < x < \sqrt{4.0001}\ OR\ -\sqrt{4.0001} < x < -\sqrt{3.9999}$$

We only need the positive values.

$$\sqrt{3.9999} < x < \sqrt{4.0001}$$

Subtract 2 throughout:

$$\sqrt{3.9999} - 2 < x - 2 < \sqrt{4.0001}$$

Use a calculator to approximate:

$$-2.50 \times 10^{-5} < x - 2 < 2.49 \times 10^{-5}$$

To convert to absolute value, use the greater of the values:

$$|x - 2| < 2.50 \times 10^{-5}$$

3.25: Definition

Example 3.26

A. Calculate $\lim_{n\to\infty} \left(\frac{5}{6}\right)^n$

B. $f(n) = \left(\frac{5}{6}\right)^n$. Prove that $\lim_{n\to\infty} f(n) = 0$

Part A

$$\lim_{n\to\infty} \left(\frac{5}{6}\right)^n = 0$$

Part B

We need to find N such that:

$$N > n \Rightarrow \left(\frac{5}{6}\right)^n < p, \qquad 0 < p < 1$$

Exponentiate both sides to the power $\frac{1}{n}$:

$$\frac{5}{6} < p^{\frac{1}{n}} \Rightarrow p^{\frac{1}{n}} > \frac{5}{6}$$

Take the log both sides:

$$\ln p^{\frac{1}{n}} > \ln \frac{5}{6} \Rightarrow \frac{1}{n} \ln p > \ln \frac{5}{6}$$

Divide by $\ln p$ both sides. Since p is a small number, $0 < p < 1 \Rightarrow \ln p < 0$. Hence, flip the sign of the inequality:

$$\frac{1}{n} < \frac{\ln \frac{5}{6}}{\ln p}$$

Take the reciprocal both sides. Note that $\frac{x}{y} < \frac{a}{b} \Rightarrow \frac{y}{x} > \frac{b}{a}$. Hence, flip the sign of the inequality:

$$n > \frac{\ln p}{\ln \frac{5}{6}} \Rightarrow N = \frac{\ln p}{\ln \frac{5}{6}}$$

Example 3.27

$$\left|\arctan x - \frac{\pi}{2}\right| < p$$

Note that the range of $\arctan x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Hence, for all x:

$$\arctan x < \frac{\pi}{2} \Rightarrow \arctan x - \frac{\pi}{2} < 0$$

Use the property that $|x| = -x$ for $x < 0$:

$$\left|\arctan x - \frac{\pi}{2}\right| = \frac{\pi}{2} - \arctan x$$

$$\frac{\pi}{2} - \arctan x < p$$
$$-\arctan x < p - \frac{\pi}{2}$$
$$\arctan x > \frac{\pi}{2} - p$$
$$x > \tan\left(\frac{\pi}{2} - p\right)$$

3.4 Further Topics

A. Proofs of Limits

Example 3.28

Proofs of limits by Blackpenredpen

29 Examples