Triangles

Revision: 2949

19 October 2025

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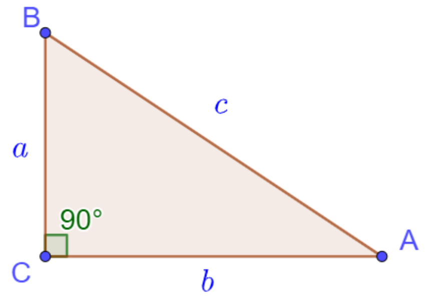
1. Pythagorean Theorem

1.1 Pythagorean Theorem

A. Statement

1.1: Right Angled Triangle: Terminology

- The side opposite the right angle in a right-angled triangle is called the hypotenuse. Hypotenuse is the longest side.
- The two sides other than the hypotenuse in a right-angled triangle are called the legs.



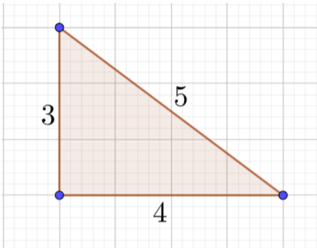
1.2: Pythagorean Theorem: Algebraic

$$a^2 + b^2 = c^2$$

In a right-angled triangle, the sum of the squares of the legs is equal to the square of the hypotenuse.

Example 1.3

3, 4 and 5 are the sides of a right-angled triangle. Verify that the Pythagorean Theorem holds.



$$3^2 + 4^2 = 9 + 16 = 25$$
$$5^2 = 25$$

1.4: Solving for the Hypotenuse

$$c = \sqrt{a^2 + b^2}$$

$$a^2 + b^2 = c^2$$

Take the square root both sides:

$$c = \sqrt{a^2 + b^2}$$

Example 1.5

Find the length of the missing side in each part below using the Pythagorean Theorem:

A. The legs of a right-angled triangle are 6 and 8.
B. The legs of a right-angled triangle are 5 and 12.
C. The legs of a right-angled triangle are 3 and 4.
D. The legs of a right-angled triangle are 8 and 15.
E. The hypotenuse is 13. And one of the legs is 12.

Parts A-D

$$c = \sqrt{a^2 + b^2}$$
$$c = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$$
$$c = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$$
$$c = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$
$$c = \sqrt{8^2 + 15^2} = \sqrt{289} = 17$$

Parts E-

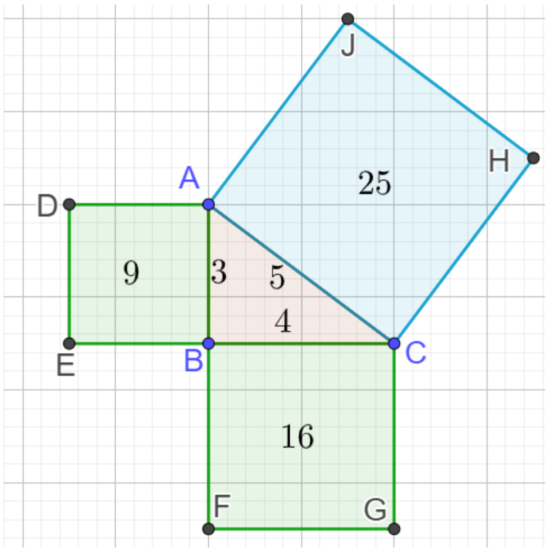
$$c^2 = a^2 + b^2 \Rightarrow a^2 = c^2 - b^2$$

We have the hypotenuse and a leg, and we want to find the other leg:

$$a^2 = c^2 - b^2 = 13^2 - 12^2 = 169 - 144 = 25 \Rightarrow a = 5$$

1.6: Pythagoras Theorem: Visual

The Pythagorean Theorem can be illustrated with a geometrical interpretation.



Draw a right-angled triangle with

$$AB = Leg_1 = 3$$
$$BC = Leg_2 = 4$$
$$Hypotenuse = AC = 5$$

as in the diagram alongside.

Draw a square

- With one side AB and area $3 \times 3 = 9$
- With one side BC and area $4 \times 4 = 16$
- With one side AC and area $5 \times 5 = 25$

From Pythagoras Theorem $a^2 + b^2 = c^2$. In case of this triangle:

$$3^2 + 4^2 = 5^2$$

$$\underbrace{9}_{\substack{Small\ Green\\ Square}} + \underbrace{16}_{\substack{Large\ Green\\ Square}} = \underbrace{25}_{\substack{Blue\\ Square}}$$

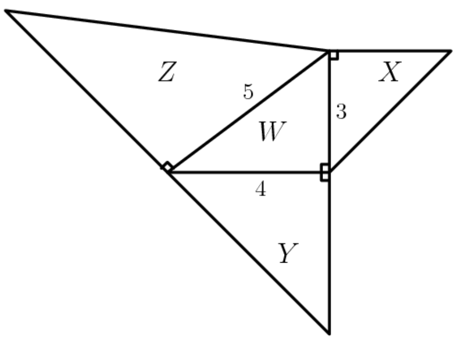
Hence,

$$Area\ of\ Two\ Green\ Squares = Area\ of\ Blue\ Square$$

Example 1.7

Choose the correct option

Right isosceles triangles are constructed on the sides of a $3-4-5$ right triangle, as shown. A capital letter represents the area of each triangle. Which one of the following is true? **(AMC 8 2002/16)**

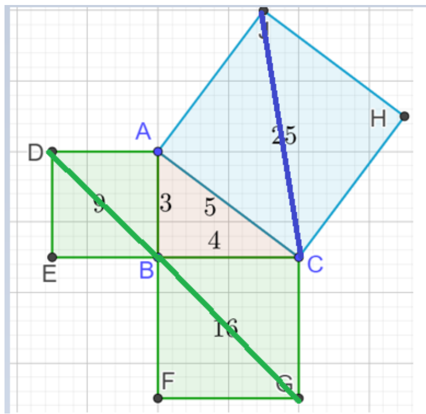


(A) $X + Z = W + Y$
(B) $W + X = Z$
(C) $3X + 4Y = 5Z$
(D) $X + W = \frac{1}{2}(Y + Z)$
(E) $X + Y = Z$

Method I:

From the visual representation in the previous property, we know that:

$$9 + 16 = 25$$



If we divide each square into two, we will get two right isosceles triangles, each with area half of the original square, and hence:

$$\frac{9}{2} + \frac{16}{2} = \frac{25}{2}$$

Method II:

$$Z = \frac{1}{2}hb = \frac{1}{2} \times 5 \times 5 = \frac{25}{2}$$
$$X = \frac{1}{2}hb = \frac{1}{2} \times 3 \times 3 = \frac{9}{2}$$
$$Y = \frac{1}{2}hb = \frac{1}{2} \times 4 \times 4 = \frac{16}{2}$$

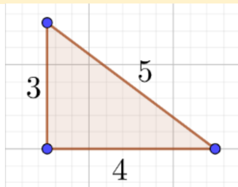
Hence,

$$X + Y = Z \Rightarrow Option\ E$$

Example 1.8: Hypotenuse

A. The legs of a right-angled triangle have length three and four, respectively. Find the perimeter of the triangle.
B. If a right-angled triangle has legs of length four and five, what is the length of the hypotenuse?
C. A garden is in the shape of a triangle with a first fence perpendicular to its second fence. If the first fence is 5 meters long, and the second fence is 6 meters long, then find the cost, in dollars, of fencing the garden at 40 cents per meter.

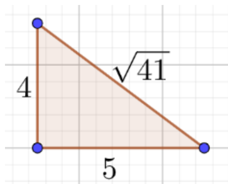
Part A



Substitute $a = 3, b = 4$:

$$c^2 = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$
$$P = a + b + c = 3 + 4 + 5 = 12$$

Part B



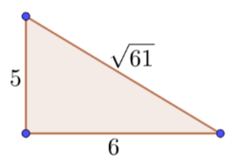
Substitute $a = 4, b = 5$:

$$c^2 = \sqrt{4^2 + 5^2} = \sqrt{16 + 25} = \sqrt{41}$$

Part C

This question has multiple steps. Make sure you do each step correctly:

$$Third\ Side = \sqrt{5^2 + 6^2} = \sqrt{25 + 36} = \sqrt{61}$$
$$Perimeter = 5 + 6 + \sqrt{61} = 11 + \sqrt{61}$$
$$Cost\ of\ fence = 0.4\left(11 + \sqrt{61}\right) = 4.4 + 0.4\sqrt{61} \text{ dollars}$$[1]



Example 1.9

For this question, you can use information from each part in subsequent parts.

A park is in the shape of a right-angled triangle. The shortest side of the park is 3 meters. The second shortest

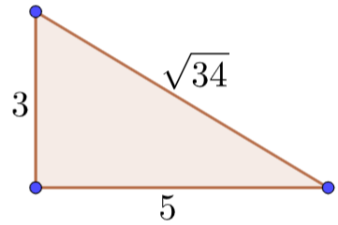
[1] Common mistake: Finding the answer in cents.

side is 5 meters. Abhay is driving his favorite tricycle from point A to point B (both are in the park, or on the sides) on a straight path.

A. Find the maximum possible distance between A and B.
B. If Abhay drives his tricycle at a speed of $0.5\frac{m}{s}$, find the time required.
C. If Abhay wants to take twenty seconds to travel the distance, what is the speed that he should travel at?

Hint: $Distance = Speed \times Time$

Draw a diagram with a right-angled triangle.



Part A

$$Max\ Distance = Hypotenuse = \sqrt{(3m)^2 + (5m)^2} = \sqrt{9m^2 + 25m^2}$$

Which simplifies to:

$$\sqrt{34m^2} = \sqrt{34}m$$

Part B

$$T = \frac{D}{S} = \frac{\sqrt{34}}{0.5} = \frac{\sqrt{34}}{\frac{1}{2}} = \sqrt{34} \div \frac{1}{2} = \sqrt{34} \times 2 = 2\sqrt{34}\,s$$

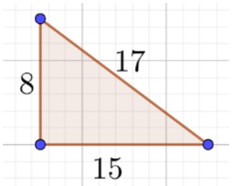
Part C

$$S = \frac{D}{T} = \frac{\sqrt{34}}{20}\frac{m}{s}$$

B. Finding a Leg

Example 1.10

A. The longest side of a right-angled triangle has length 17. The shortest side has length 8. Find the length of the third side.
B. Find the length of the other leg of a right-angled triangle, if the length of the hypotenuse is 9, and one leg is 5.



Part A

Substitute $a = 8, c = 17$ in:

$$a^2 + b^2 = c^2 \Rightarrow 8^2 + b^2 = 17^2 \Rightarrow 64 + b^2 = 289 \Rightarrow b^2 = 225 \Rightarrow b = 15$$

Part B

$$a^2 + b^2 = c^2 \Rightarrow 5^2 + b^2 = 9^2 \Rightarrow b^2 = 81 - 25 = 56 \Rightarrow b = \sqrt{56} = \sqrt{14} \times \sqrt{4} = 2\sqrt{14}$$

C. Pythagorean Triplets

1.11: Pythagorean Triplet

Sets of three integers that form the sides of a right-angled triangle are called Pythagorean Triplets.

Pythagorean Triplets are very useful to know and remember because they will often reduce your calculation time.

1.12: Primitive Pythagorean Triplet

If the numbers in a Pythagorean Triplet have $HCF = 1$, then the Triplet is called primitive.

Example 1.13

The table shows the 16 primitive Pythagorean Triplets with integers less than 100.

A. Memorize the shaded triplets
B. Verify the shaded triplets.

Primitive Pythagorean Triplets			
3, 4, 5	20, 21, 29	11, 60, 61	36, 77, 85
5, 12, 13	12, 35, 37	33, 56, 65	13, 84, 85
8, 15, 17	9, 40, 41	16, 63, 65	39, 80, 89
7, 24, 25	28, 45, 53	48, 55, 73	65, 72, 97

$$3^2 + 4^2 = 9 + 16 = 25 = 5^2$$
$$5^2 + 12^2 = 25 + 144 = 169 = 13^2$$
$$8^2 + 15^2 = 64 + 225 = 289 = 17^2$$
$$7^2 + 24^2 = 49 + 576 = 625 = 25^2$$

$$a^2 + b^2 = c^2$$

$$Ar$$

1.14: Multiple of a Pythagorean Triplet[2]

Multiplying a Pythagorean triplet by a positive natural number also results in a Pythagorean Triplet:

$$2(3,4,5) = (6,8,10)$$

Example 1.15

From the table above, multiply each of the four shaded Pythagorean Triplets by 2, 3 and 4 to get the corresponding Pythagorean Triplets.

	$\times 2$	$\times 3$	$\times 4$
3, 4, 5	6,8,10	9,12,15	12,16,20
5, 12, 13	10,24,25	15,36,39	20,48,52
8, 15, 17	16,30,34	24,45,51	
7, 24, 25	14,48,50		

Example 1.16

"Simplifying" a Pythagorean Triplet is possible by dividing each number in the Triplet by their HCF to get a smaller number. Simplify the following triplets

A. 30,40,50
B. 60,63,87

Part A

Divide each number by $HCF(30,40,50) = 10$ to get:

$$(30,40,50) = 10(3,4,5)$$

Part B

Divide each number by $HCF(60,63,87) = 3$ to get:

$$(60,63,88) = 3(20,21,29)$$

Example 1.17

A. Is (3,4,5) a primitive Pythagorean triplet
B. Is (6,8,10) a primitive Pythagorean triplet

[2] This video shows this property in action.

$$yes$$
$$no$$

Example 1.18
A primitive Pythagorean Triplet has $HCF = 1$ for its three sides. Using only the primitive Pythagorean triplets (3,4,5), (5,12,13), (8,15,17) and (7,24,25) or their integer multiples, find the number of right triangles that can be made with side lengths less than 100.

$$(3,4,5),(6,8,10),(9,12,15),\dots,(57,76,95) \Rightarrow 19\ Triangles$$
$$(5,12,13),(10,24,26),\dots,(35,84,91) \Rightarrow 7\ Triangles$$
$$(8,15,17),(16,30,34),\dots,(40,75,85) \Rightarrow 5\ Triangles$$
$$(7,24,25),(14,48,50),\dots,(21,72,75) \Rightarrow 3\ Triangles$$

Total
$$= 19 + 7 + 5 + 3 = 34$$

D. Isosceles Triangles

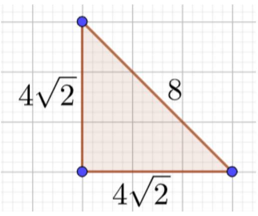
Example 1.19
A. The leg of an isosceles right-angled triangle is 3 units. Find the length of the hypotenuse?
B. In an isosceles right-angled triangle, the shortest side is 10 units. What is the length of the longest side?
C. In an isosceles right-angled triangle, the shortest side is $4\sqrt{2}$ units. What is the perimeter of the triangle?
D. If the hypotenuse of an isosceles right-angled triangle is 10, what are the lengths of its sides?

In an isosceles triangle, the two legs are equal.
$$c^2 = a^2 + a^2 = 2a^2$$

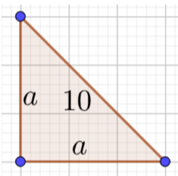
Part A
$$c^2 = 2(3^2) = 18 \Rightarrow c = \sqrt{18} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$$

Part B
$$c^2 = 2(10^2) = 200 \Rightarrow c = \sqrt{200} = \sqrt{100} \times \sqrt{2} = 10\sqrt{2}$$

Part C
Use the Pythagorean Theorem:
$$c^2 = 2a^2 + a^2 = 2\left(4\sqrt{2}\right)^2 = 2(32) = 64 \Rightarrow c = \sqrt{64} = 8$$
And hence the perimeter is:
$$P = 4\sqrt{2} + 4\sqrt{2} + 8 = 8\sqrt{2} + 8$$

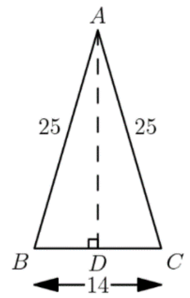


Part D
$$a^2 + a^2 = 10^2 \Rightarrow 2a^2 = 100 \Rightarrow a^2 = 50 \Rightarrow a = \sqrt{50} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$$



Example 1.20
In an isosceles triangle ABC with $AB = AC = 25$, the altitude AD bisects the base BC so that $BD = DC$. $BC = 14$. Determine the length of the altitude AD. **(CEMC 2011 Fryer, Adapted)**

$$BD = DC = 7$$
$$(7, x, 25) \Rightarrow x = 24$$

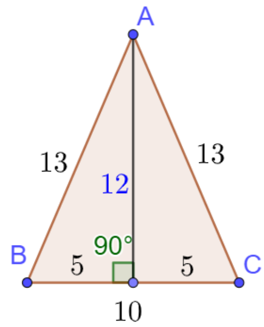


Example 1.21
An isosceles triangle has sides 13, 13 and 10. What is the area of the triangle?

Draw ΔABC with sides 13,13 and 10.
Drop a perpendicular from vertex A to side BC.

By a Pythagorean Triplet (5,12,13):

$$Height\ of\ \Delta ABC = 12$$



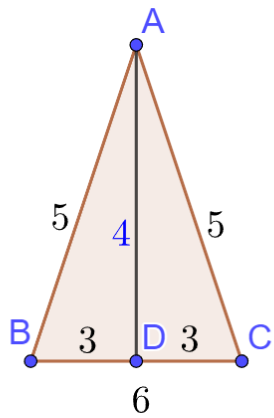
And then the area of ΔABC

$$= \frac{1}{2}hb = \frac{1}{2} \times 12 \times 10 = 60$$

Example 1.22
What is the number of square units in the area of a triangle whose sides measure 5, 5 and 6 units? **(MathCounts 2004 Chapter Countdown)**

By the Pythagorean Theorem:

$$AD^2 + BD^2 = AB^2$$
$$AD^2 + 3^2 = 5^2$$
$$AD^2 = 25 - 9 = 16$$
$$AD = 4$$



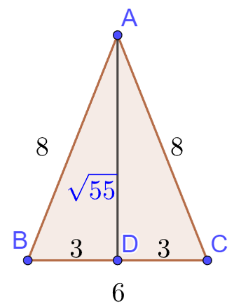
Area of ΔABC

$$= \frac{1}{2}hb = \frac{1}{2}(AD)(BC) = \frac{1}{2} \times 4 \times 6 = 12$$

Example 1.23
What is the number of square units in the area of a triangle whose sides measure 8, 8 and 6 units?

By the Pythagorean Theorem:

$$AD^2 + BD^2 = AB^2$$
$$AD^2 + 3^2 = 8^2$$
$$AD^2 = 64 - 9 = 55$$
$$AD = \sqrt{55}$$

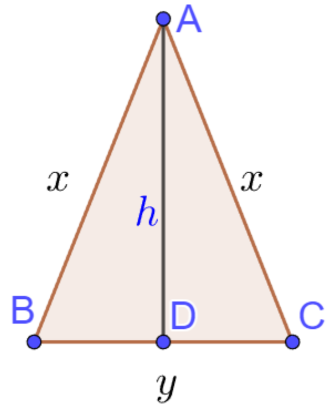


Area of ΔABC

$$=\frac{1}{2}hb=\frac{1}{2}(AD)(BC)=\frac{1}{2}\times\sqrt{55}\times 6=3\sqrt{55}$$

Example 1.24

An isosceles triangle has sides x, x and y. In terms of x and y, what is the area of the triangle?



By the Pythagorean Theorem:

$$x^2=h^2+\left(\frac{y}{2}\right)^2\Rightarrow h^2=x^2-\frac{y^2}{4}=\frac{4x^2-y^2}{4}$$

Taking square roots both sides:

$$h=\sqrt{\frac{4x^2-y^2}{4}}=\frac{\sqrt{4x^2-y^2}}{2}$$

Area of the triangle

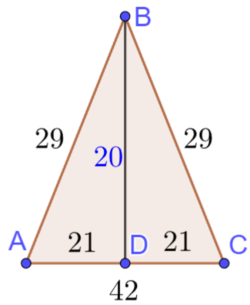
$$=\frac{1}{2}hb=\frac{1}{2}\times\frac{\sqrt{4x^2-y^2}}{2}\times y=\frac{y\sqrt{4x^2-y^2}}{4}$$

Example 1.25

In $\triangle ABC$, $AB=BC=29$, and $AC=42$. What is the area of ΔABC? **(AMC 8 2015/6)**

By Pythagorean Triplet (20,21,29):

$$BD=20$$



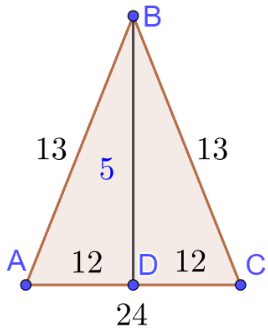
Area of ΔABC

$$=\frac{1}{2}hb=\frac{1}{2}(BD)(AC)=\frac{1}{2}\times 20\times 42=420$$

Example 1.26

The base of isosceles ΔABC is 24 and its area is 60. What is the length of one of the congruent sides? **(AMC 8 2007/14)**

$$AC=24\Rightarrow AD=12$$
$$[ABC]=\frac{1}{2}(BD)(AC)$$
$$60=\frac{1}{2}(BD)24$$
$$BD=\frac{60}{12}=5$$



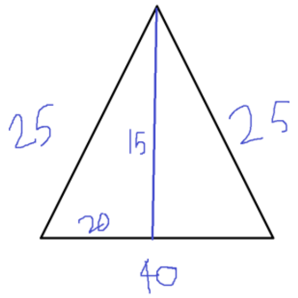
By Pythagorean Triplet (5,12,13)

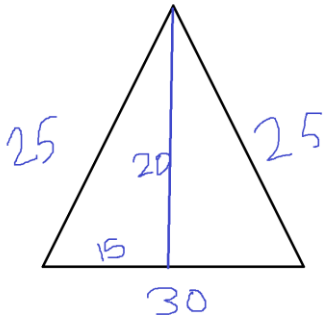
$$AB=13$$

Example 1.27

Let A be the area of the triangle with sides of length 25, 25, and 30. Let B be the area of the triangle with sides of length 25, 25, and 40. What is the relationship between A and B? **(AMC 8 2011/16)**

$(A) A=\frac{9}{16}B \quad (B) A=\frac{3}{4}B \quad (C) A=B \quad (D) A=\frac{4}{3}B \quad (E) A=\frac{16}{9}B$





$$Area\ of\ Triangle=\frac{1}{2}hb=\left(\frac{1}{2}b\right)h$$

$$3\times(3,4,5)=(15,20,25)$$

From the diagram, we see that the values are the same in both triangles (but interchanged).
Hence, the areas are equal.

$$Option\ C$$

Example 1.28

A triangle has side lengths 10, 10, and 12. A rectangle has width 4 and area equal to the area of the triangle. What is the perimeter of this rectangle? **(AMC 10B 2010/7)**

Area of the triangle

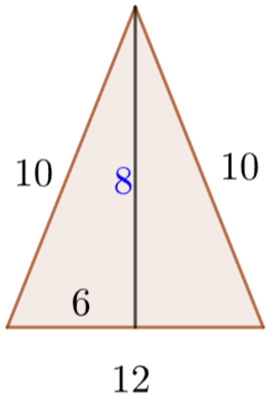
$$=\frac{1}{2}hb=\frac{1}{2}(8)(12)=(4)(12)$$

Length of the rectangle

$$=\frac{(4)(12)}{4}=12$$

Perimeter of the rectangle

$$=2(l+b)=2(4+12)=2(16)=32$$



E. Converse of Pythagorean Theorem

1.29: Converse of Pythagorean Theorem

A triangle with side lengths a, b, c is a right triangle only if:

$$a^2+b^2=c^2$$

- The converse of Pythagoras Theorem lets us check whether a triangle is right-angled based on its sides.
- If the condition is not met, then the triangle is not a right triangle.

Example 1.30

Check whether the lengths below are the sides of a right-angled triangle.

A. 5,6, and 7

B. 5, 11, 12
C. 8, 15, 17

Part A

$$5^2 + 6^2 = 25 + 36 = 61$$
$$7^2 = 49 \neq 61 \Rightarrow Not\ right - angled$$ [3]

Part B

$$5^2 + 11^2 = 25 + 121 = 146$$
$$12^2 = 144 \neq 146 \Rightarrow Not\ right\ angled$$

Part C

$$a^2 + b^2 = 8^2 + 15^2 = 64 + 225 = 289$$
$$c^2 = 17^2 = 289$$

Example 1.31

A triangle with sides $6, 8$ *and* 10 has one angle $30°$. Find the difference between the other two angles.

Note that:

$$(6,8,10) = 2(3,4,5) \Rightarrow Pythagorean\ Triplet \Rightarrow \Delta\ is\ right - angled$$

Hence,

$$One\ angle\ in\ the\ triangle\ is\ 90°$$
$$Second\ angle\ is\ given\ to\ be\ 30°$$

Using sum of angles of a triangle $= 180°$:

$$\therefore 3^{rd} Angle = 180 - 90 - 30 = 60°$$
$$Difference\ of\ two\ angles = 60 - 30 = 30°$$

Example 1.32

Triangle ABC has sides of length 5, 12 and 13 units, and triangle DEF has sides of length 8, 15 and 17 units. What is the ratio of the area of triangle ABC to the area of triangle DEF? Express your answer as a common fraction. **(MathCounts 2005 State Countdown)**

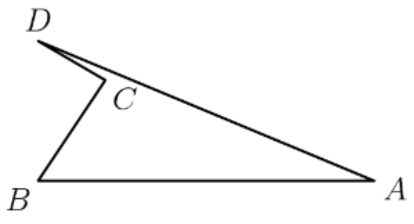
(5,12,13) and (8,15,17) are both Pythagorean Triplets. Hence, the triangles are right triangles.

For a right triangle, the area is half the product of the legs. The ratio of areas

$$= \frac{\frac{1}{2} \times 5 \times 12}{\frac{1}{2} \times 8 \times 15} = \frac{5 \times 12}{8 \times 15} = \frac{12}{8 \times 3} = \frac{4}{8} = \frac{1}{2}$$

Example 1.33

In the non-convex quadrilateral $ABCD$ shown below, $\angle BCD$ is a right angle, $AB = 12$, $BC = 4$, $CD = 3$, and $AD = 13$. What is the area of quadrilateral $ABCD$? **(AMC 8 2017/18)**



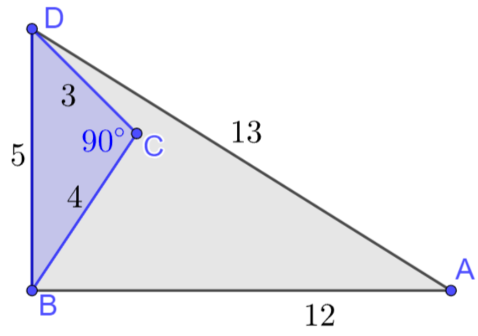
[3] Note that the solution is not

$$5^2 + 6^2 = 7^2 \Rightarrow 25 + 36 = 49 \Rightarrow 61 = 49 \Rightarrow Not\ Valid$$

We do not do it in this way because we cannot write an equality if we do not know whether it is true.

Since $\angle BCD$ is a right triangle, we notice that $(3,4,5)$ is a Pythagorean Triplet:

$$BD = 5$$



Notice that:

$$(BD, BA, AD) = 5,12,13 \Rightarrow Pythaogrean\ Triplet$$

The area that we want is

$$[ABCD] = [ABD] - [BCD] = \frac{1}{2}(5)(12) - \frac{1}{2}(3)(4) = 30 - 6 = 24$$

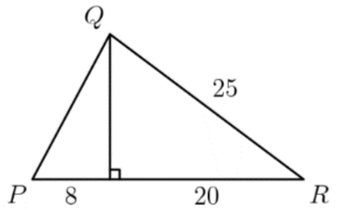
F. Multiple and Nested Triangles

Questions can be made more difficult by asking you to work with more than one triangle.

1.34: Mixing Pythagorean Triplets

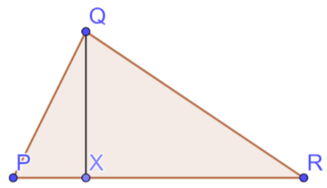
Pythagorean Triplets can be in such a way that the leg of one triangle is the hypotenuse of another triangle.

This is often used when making questions with multiple and nested triangles.



Example 1.35

In the diagram, what is the perimeter of ΔPQR? **(CEMC Pascal 2008)**



In right triangle QXR

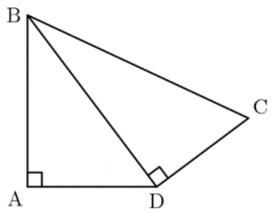
$$5(3,4,5) = (15,20,25) \Rightarrow QX = 15$$

In right triangle QXP

$$(8,15,PQ) \Rightarrow PQ = 17$$

Perimeter

$$= 25 + 28 + 17 = 70$$



Example 1.36

Triangles BAD and BDC are right triangles with $AB = 12\ units$, $BD = 15\ units$, and $BC = 17\ units$. What is the area, in square units, of quadrilateral ABCD? **(MathCounts 2005 State Countdown)**

In ΔBAD:

$$(x, 12,15) = 3\left(\frac{x}{3}, 4,5\right) \Rightarrow \frac{x}{3} = 3 \Rightarrow x = 9 \Rightarrow [BAD] = \frac{hb}{2} = \frac{9 \times 12}{2} = 54$$

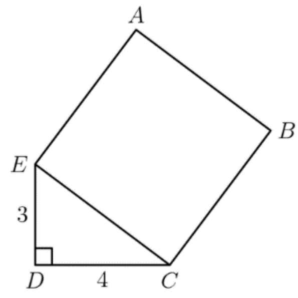
In ΔBDC:

Note that we have a Pythagorean Triple again:

$$(y, 15{,}17) = y = 8 \Rightarrow [BDC] = \frac{hb}{2} = \frac{8 \times 15}{2} = 60$$

The area of the quadrilateral is the sum of the areas of the two triangles:

$$[ABCD] = [BAD] + [BDC] = 54 + 60 = 114$$



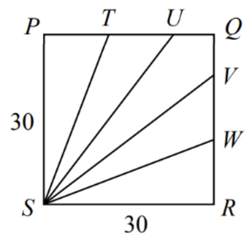
Example 1.37

As shown, a square is constructed on the hypotenuse of a right triangle whose legs have lengths 3 units and 4 units. What is the area of the pentagon $ABCDE$, in square units? **(MathCounts 2005 Chapter Countdown)**

The triangle has sides which are a Pythagorean Triplet:

$$(3{,}4, EC) \Rightarrow EC = 5$$
$$[EDC] = \frac{hb}{2} = \frac{3 \times 4}{2} = 6$$
$$[ABEC] = EC^2 = 5^2 = 25$$

$$[ABCDE] = [ABEC] + [EDC] = 25 + 6 = 31$$



(Calculator) Example 1.38

Square $PQRS$ has side length 30, as shown. The square is divided into 5 regions of equal area: ΔSPT, ΔSTU, ΔSVW, ΔSWR, and quadrilateral $SUQV$. The value of $\frac{SU}{ST}$ is closest to (rounded to two decimal places) **(Gauss Grade 8 2018/22)**

Use the condition that the square is divided into 5 regions of equal area:

$$[SPT] = \frac{1}{5}[PQRS] = \frac{1}{5} \cdot 30^2 = 30 \cdot 6 = 180$$

Use the formula for area of a triangle to work out the length of PT:

$$[SPT] = \frac{hb}{2} = \frac{(30)(PT)}{2} = 180 \Rightarrow 15PT = 180 \Rightarrow PT = 12$$
$$[SPU] = \frac{hb}{2} = \frac{(30)(PU)}{2} = 180 \cdot 2 \Rightarrow \ PU = 24$$

By Pythagoras Theorem, we can calculate the hypotenuse as:

$$In\ right\ \Delta SPU: SU = \sqrt{SP^2 + PU^2} = \sqrt{24^2 + 30^2} = \sqrt{1476}$$
$$In\ right\ \Delta SPT: ST = \sqrt{SP^2 + PT^2} = \sqrt{24^2 + 30^2} = \sqrt{1044}$$

Then the ratio we want:

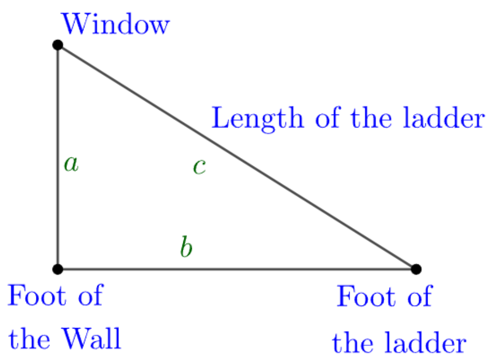
$$\frac{SU}{ST} = \frac{\sqrt{1476}}{\sqrt{1044}} \approx 1.189 = 1.19\ (Rounded\ to\ two\ decimal\ places)$$

G. Ladders

Applied questions on Pythagorean Theorem often introduce real life scenarios that require drawing a diagram. One such popular scenario is a ladder next to a wall.

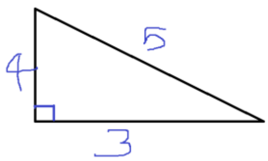
In the diagram alongside

- the wall is 90° with the ground, creating a right-angled triangle
- The window is a vertex of the triangle and gives extra flavor to the question.
- the length of the ladder gives the length of the hypotenuse
- the height of the wall is one length of a leg of the triangle
- the distance between the foot of the ladder, and the foot of the wall is another leg of the triangle.



Example 1.39

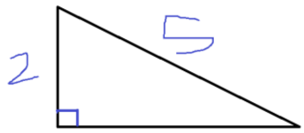
A ladder of length 5 meters is inclined to climb up to a church window that is 4 meters above the ground. Find the distance between the foot of the ladder and the church wall.



Refer diagram.
We see a Pythagorean Triplet

$$(3,4,5) \Rightarrow Required\ Distance = 3$$

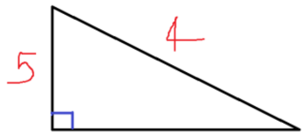
Repeat the above question if the church window is 2 meters above the ground.



Refer Diagram

$$2^2 + d^2 = 5^2 \Rightarrow d^2 = 25 - 4 = 21 \Rightarrow d = \sqrt{21}$$

Critical Thinking: In the first question, if the ladder is 4 meters, and the church window is 5 meters high, what can you say?



The hypotenuse should be the longest side in a right-angled triangle. However, that is not the case in the diagram alongside.

Hence, the data given is inconsistent, or logically impossible.

Example 1.40

An 8.5-meter ladder is leaning against a vertical wall. How many meters is its base from the wall if the ladder reaches 7.5 meters up the wall? **(MathCounts 2007 State Countdown)**

The sides of the right triangle can be written:

$$(d, 7.5, 8.5)$$

Factor $\frac{1}{2}$ out of the sides:

$$= \frac{1}{2}(2d, 15, 17)$$

Now (8,15,17):

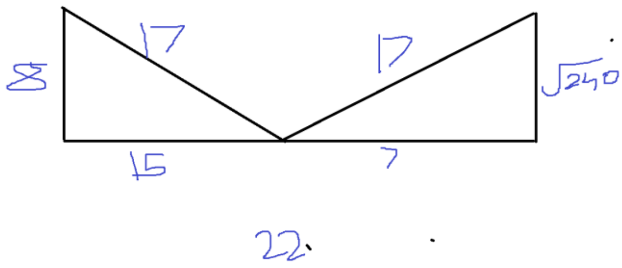
$$2d = 8 \Rightarrow d = 4$$

Example 1.41
A ladder in the street is inclined to climb up to a house window that is 8 feet above the ground. The distance from the foot of the ladder to the foot of the wall is 15 feet. Find the length of the ladder.

Repeat the above question if the window is 7 feet above the ground.

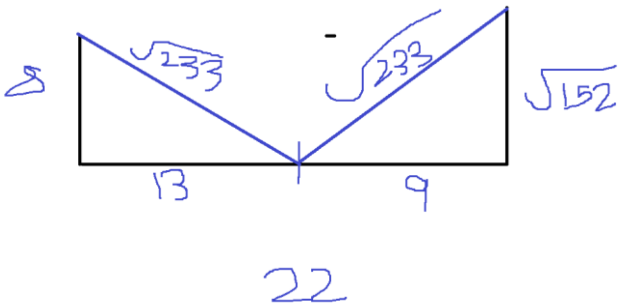
Example 1.42
A ladder is placed in the street to access a red house 15 meters away via a blue window that is 8 meters high. The ladder is then turned on the same point to access a blue house that is on the other side of the street via a red window. Find the height of the red window, if the width of the street is 22 meters.

$$\sqrt{240} = 4\sqrt{15}$$



Repeat the above question if the red house is 13 meters away.

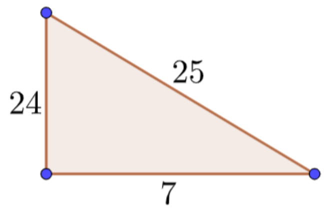
$$\sqrt{152} = 2\sqrt{38}$$



Example 1.43
A 25-foot ladder is placed against a vertical wall of a building. The foot of the ladder is 7 feet from the base of the building. If the top of the ladder slips 4 feet, then the foot of the ladder will slide: **(AHSME 1950/32)**

We get a Pythagorean triplet:

$$(7,24,25) \Rightarrow Height\ of\ Wall = 24$$



Ladder slips 4 feet:

$$(x, 20, 25) \Rightarrow 5\left(\frac{x}{5}, 4, 5\right) \Rightarrow \frac{x}{5} = 3 \Rightarrow x = 15$$

Slippage

$$= 15 - 7 = 8$$

H. Ramps

A ramp is an incline. People in wheelchairs cannot easily navigate stairs, but find ramps easier to use. Ramps can also be used to transport goods up an incline into a warehouse, etc.
When drawing a diagram with a ramp, remember that we can create a right-angle at the foot of where the ramp connects with the building:

- ➢ The length of the ramp will be the hypotenuse.

Example 1.44

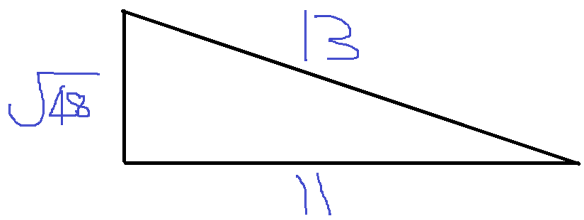
A ramp of length 13 meters travels a horizontal distance of 12 meters.

A. Find the vertical distance it travels.
B. Find the vertical distance travelled per unit of horizontal distance travelled.

$$5$$
$$\frac{5}{12}$$

Repeat Parts A. and B. if the ramp travels a horizontal distance of 11 meters.

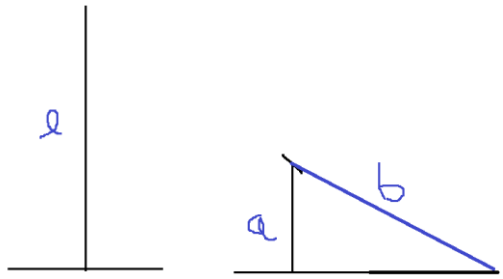
$$\sqrt{48} = 4\sqrt{3}$$
$$\frac{4\sqrt{3}}{11}$$



I. Flagpoles

Flagpoles are more interesting (and complex) compared to ladders, and ramps. This is because:

- Flagpoles cast shadows, which are often shorter or longer than the actual flagpole (creating similar triangles).
- If flagpoles snap or break, then we are essentially given the length of the flagpole, but which also represents the sum of two sides of a triangle.



Example 1.45

The bottoms of two vertical poles are 12 feet apart and are on a region of flat ground. One pole is 6 feet tall and the other is 15 feet tall. How long, in feet, is a wire stretched from the top of one pole to the top of the other pole? (MathCounts 2010 School Countdown)

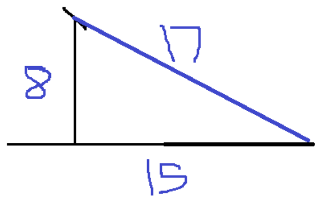
The sides are

$$(9,12,h) = 3\left(3,4,\frac{h}{3}\right) \Rightarrow \frac{h}{3} = 5 \Rightarrow h = 15$$

Example 1.46

In heavy winds, a flagpole snaps 8 feet above the ground to touches the ground 15 feet away from the base of the flagpole. What is the length of the flagpole?

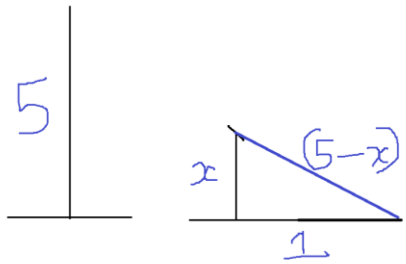
$$8 + 17 = 25$$



Example 1.47

A flagpole is originally 5 meters tall. A hurricane snaps the flagpole at a point x meters above the ground so that the upper part, still attached to the stump, touches the ground 1 meter away from the base. What is x? **(AMC 10B 2009/10)**

$$(5-x)^2 = x^2 + 1$$
$$25 - 10x + x^2 = x^2 + 1$$
$$24 = 10x$$
$$x = 2.4$$



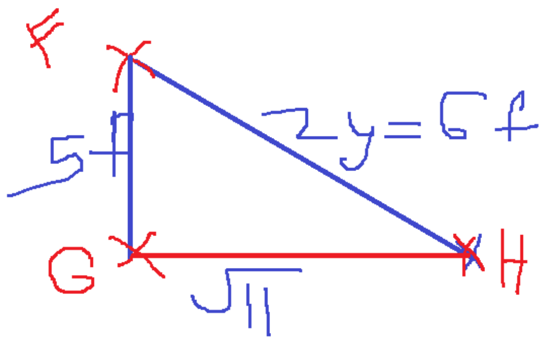
J. Flying Animals

Example 1.48

A hummingbird on the ground spots a flower and flies 2 yards in a straight line at an angle of 43° to reach the nectar. After drinking the nectar, the hummingbird notes that the height of the flower is 5 feet above the ground. Find the distance, in feet, between the point at which the hummingbird originally was, and the point on the ground exactly above the flower. ($1\ yard\ =\ 3\ feet$)

Consider ΔFGH with

$$H = Original\ Location\ of\ Hummingbird$$
$$F = Flower$$
$$G = Point\ on\ Ground\ directly\ above\ flower$$



Substitute $FH = 6\ feet, FG = 5\ feet$ in $FG^2 + GH^2 = FH^2$:

$$5^2 + GH^2 = 6^2 \Rightarrow GH^2 = 11 \Rightarrow GH = \sqrt{11}$$

K. Compass Directions

Questions that involve directions require use of the Pythagorean Theorem, particularly when the people travelling make turns that are 90°.

For this, it is important to know and be able to draw the compass directions, which are given alongside.

In particular:

- The four main directions are North, East, South and West.

There are also four ancillary directions:

- Exactly in between North and West lies North-West (NW)
- Exactly in between North and East lies North-East (NE)
- Exactly in between South and West lies South-West (SW)
- Exactly in between South and East lies South-East (SE)

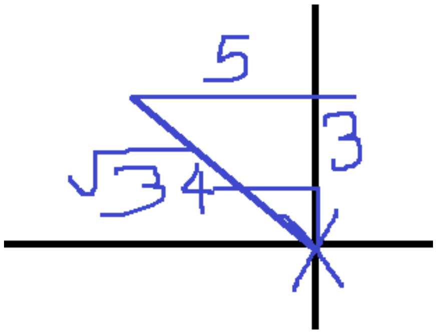
Example 1.49

A deer hunter travelled due north for 3 km, and then due west for 5 km.

A. What was his distance from his starting point?
B. What was his (approximate) compass direction from starting point.

$$Distance = \sqrt{3^2 + 5^2} = \sqrt{9 + 25} = \sqrt{34}$$

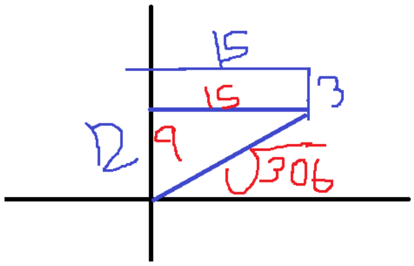
$$Direction = North - West$$



Example 1.50

A deer hunter travelled due north for 12 km, then due east for 15 km, and then due south for 3 km. What was his distance from his starting point?

$$D = \sqrt{15^2 + 9^2} = \sqrt{225 + 81} = \sqrt{306} = 3\sqrt{34}$$



Example 1.51

A travelled 5 km North-North-West, and then 12 km East-North-East. What is his distance from his starting point?

Apply Pythagoras (North-North-West is 90 degrees to the left of East-North-East):
Root ($5^2 + 12^2$) = Root (25 + 144) = Root (169) = 13 km.

Example 1.52

Aaron's school is north-west from his house, 5 km away. Aaron's library is south-west from his house, 12 km away. Aaron's math club is 3 km north-east from his house.

A. Draw a neatly labelled diagram showing the positions of Aaron's house, school, library and math club.
B. What is the straight-line distance between Aaron's school and his library?
C. Aaron drives in a straight line from his library to his math club. Then he remembers he forgot to issue the book that he wanted. So, he drives back. What is the distance that he drives?
D. A bird flies from Aaron's school to his math club. What is the distance that it flies?
E. Aaron drives from school to home, has an evening snack, and then goes to the math club. How much more (or less) does he travel compared to the bird.

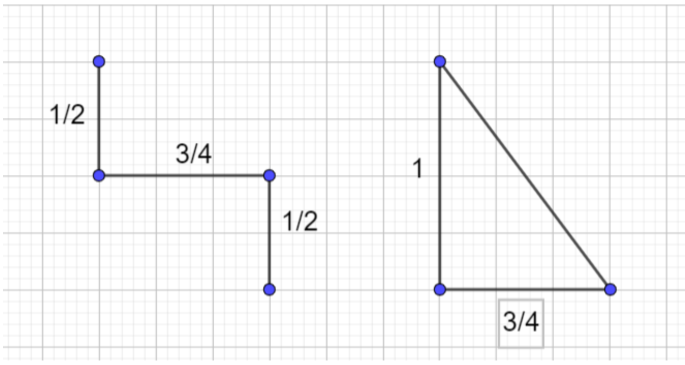
13
30
$\sqrt{34}$
$8 - \sqrt{34}$

Example 1.53

Bill walks $\frac{1}{2}$ mile south, then $\frac{3}{4}$ mile east, and finally $\frac{1}{2}$ mile south. How many miles is he, in a direct line, from his starting point? (AMC 8 2005/7)

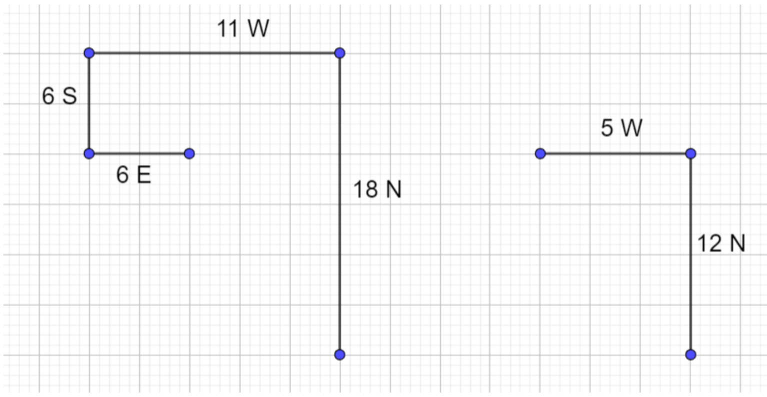
Bill makes the set of movements on the left diagram.
This is equivalent to the set of movements on the right diagram, giving us a right-angled triangle.
Hence, by the Pythagorean Theorem:

$$d=\sqrt{1^2+\left(\frac{3}{4}\right)^2}=\sqrt{1+\frac{9}{16}}=\sqrt{\frac{25}{16}}=\frac{5}{4}$$



Example 1.54

While walking on a plane surface, a traveler first headed 18 miles north, then 11 miles west, then 6 miles south and finally 6 miles east. How many miles from the starting point was the traveler after these four legs of the journey? (MathCounts 1996 Warm-Up 8)



The diagram on the left shows the movements of the traveler.

$$(5,12,x)\Rightarrow x=13$$

1.55: Orthogonal

An angle of 90° is called orthogonal.

Example 1.56

Minneapolis-St. Paul International Airport is 8 miles southwest of downtown St. Paul and 10 miles southeast of downtown Minneapolis. Which of the following is closest to the number of miles between downtown St. Paul and downtown Minneapolis? (AMC 10B 2004/8)
(A) 13 (B) 14 (C) 15 (D) 16 (E) 17

Southwest and southeast are orthogonal. Hence, we get a right triangle.

$$\sqrt{8^2+10^2}=\sqrt{64+100}=\sqrt{164}\approx\sqrt{169}=13$$

L. Boats

Example 1.57

M. Challenging Questions

Example 1.58

Example 1.59
In ΔABC, we have $AB = 1$, and $AC = 2$. Side BC and the median from A to BC have the same length. What is BC? (AMC 12 2002)

Step I: Determine the nature of the triangle
We do not know whether the triangle is a right, acute or obtuse. If the triangle is a right triangle:

$$AM = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + 1^2} = \sqrt{\frac{3}{4} + 1} = \frac{\sqrt{7}}{2} \neq \sqrt{3}$$

In fact, on comparing

$$\frac{\sqrt{7}}{2} \ ? \ \sqrt{3} \Rightarrow \sqrt{7} < \sqrt{12}$$

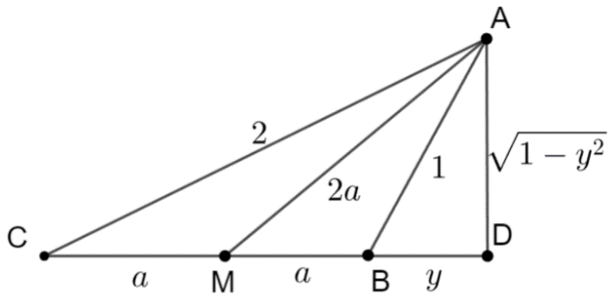
Hence, the triangle is not a right triangle.

Since we want to reduce the length of BC, and increase the length of the median, draw

$$\angle ABC \text{ as obtuse}$$

Step II: Apply Pythagoras Theorem
In ΔADB, by Pythagoras Theorem:

$$AD = \sqrt{1 - y^2}$$



In ΔADM, by Pythagoras Theorem:

$$AD^2 + DM^2 = AM^2$$
$$(1 - y^2) + (a + y)^2 = (2a)^2$$
$$(1 - y^2) + (a^2 + 2ay + y^2) = 4a^2$$
$$3a^2 - 2ay = 1$$
$$\underbrace{6a^2 - 4ay = 2}_{\textit{Equation I}}$$

In ΔADC, by Pythagoras Theorem:

$$AD^2 + DC^2 = AC^2$$
$$(1 - y^2) + (2a + y)^2 = 2^2$$
$$(1 - y^2) + (4a^2 + 4ay + y^2) = 4$$
$$\underbrace{4a^2 + 4ay = 3}_{\textit{Equation II}}$$

Add Equations I and II:

$$10a^2 = 5$$
$$a^2 = \frac{1}{2}$$
$$a = \frac{1}{\sqrt{2}}$$
$$2a = \frac{2}{\sqrt{2}} = \sqrt{2}$$

1.2 Applications

A. Triangle Inequality

Example 1.60
For what value of a is there a right triangle with sides $a+1$, $6a$, and $6a+1$? (MathCounts 2003 National Sprint)

$$6a > 0 \Rightarrow a > 0$$
$$a > 0 \Rightarrow 6a + 1 > a + 1$$

Hence, the hypotenuse must be

$$6a + 1$$

$$(a+1)^2 + (6a)^2 = (6a+1)^2$$
$$a^2 + 2a + 1 + 36a^2 = 36a^2 + 12a + 1$$
$$a^2 - 10a = 0 \Rightarrow a \in \{0,10\} \Rightarrow a = 10$$

B. Equations

Example 1.61
Twenty-nine is the shortest leg of a right triangle whose other leg and hypotenuse are consecutive whole numbers. What is the sum of the lengths of the other two sides?Eq **(MathCounts 2009 Workout 2)**

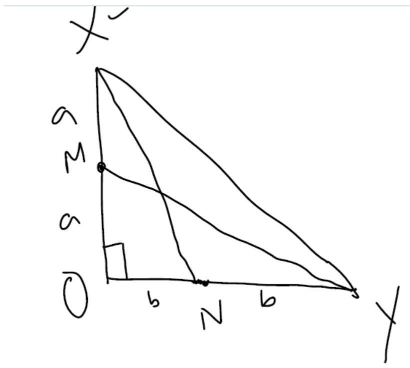
Let

$$Length\ of\ longer\ leg = n \Rightarrow Length\ of\ hypotenuse = n + 1 \Rightarrow Sum\ of\ the\ two = 2n + 1$$

By the Pythagorean Theorem:

$$29^2 + n^2 = (n+1)^2$$
$$29^2 + n^2 = n^2 + 2n + 1$$
$$841 = 2n + 1$$

Example 1.62
Let ΔXOY be a right-angled triangle with $m\angle XOY = 90°$. Let M and N be the midpoints of the legs OX and OY, respectively. Given $XN = 19$ and $YM = 22$, find XY. **(AMC 10B 2002/22)**



$$XY = \sqrt{(2a)^2 + (2b)^2} = \sqrt{4a^2 + 4b^2} = \sqrt{4(a^2 + b^2)} = 2\sqrt{a^2 + b^2}$$

In right ΔXON, by Pythagorean Theorem:

$$(2a)^2 + b^2 = 19^2 \Rightarrow 4a^2 + b^2 = 361$$

In right ΔMOY, by Pythagorean Theorem:

$$a^2 + (2b)^2 = 22^2 \Rightarrow a^2 + 4b^2 = 484$$

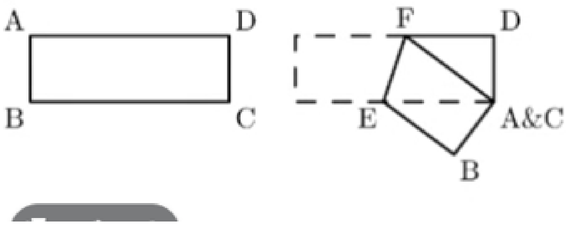
Add Equations I and II:

$$5a^2 + 5b^2 = 845 \Rightarrow a^2 + b^2 = 169 \Rightarrow 2\sqrt{a^2 + b^2} = 26$$

Example 1.63
In rectangle $ABCD$, $AB = 3$ and $BC = 9$. The rectangle is folded so that points A and C coincide, forming the pentagon $ABEFD$. What is the length of segment EF? Express your answer in simplest radical form. (MathCounts 2006 National Sprint)

By the Pythagorean Theorem in right ΔDFC:

$$FA^2 = DF^2 + DC^2$$



Substitute $DF = x \Rightarrow FA = 9 - x, DC = AB = 3$:

$$(9 - x)^2 = x^2 + 9$$
$$x^2 - 18x + 81 = x^2 + 9$$
$$18x = 72$$
$$x = 4$$

By the Pythagorean Theorem in right ΔABE:

$$EA^2 = BE^2 + AB^2$$

Substitute $BE = x \Rightarrow EA = 9 - x, AB = 3$:

$$(9 - x)^2 = x^2 + 9 \Rightarrow x = 4 \Rightarrow EA = 5$$

Drop a perpendicular from F intersecting EC at P.

$$EP = EC - PC = EC - FD = 5 - 4 = 1$$

By the Pythagorean Theorem in right ΔFPE:

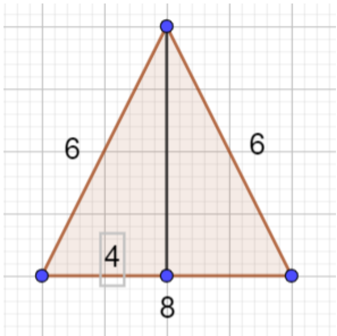
$$FE = \sqrt{FP^2 + EP^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$$

C. Isosceles Triangles: Area

Isosceles triangles have the property that the median, angle bisector and the altitude are all the same. We will use this to find the area of an Isosceles Triangle.

Example 1.64

A tent with a triangular cross-section (when taken perpendicular to the ground) has sloping sides of length 6 units, and a width of 8 units. Find the height of the central pole of the tent.



Substitute $a = 4, c = 6$ in $a^2 + b^2 = c^2$:

$$4^2 + b^2 = 6^2$$

Solve for b^2:

$$b^2 = 36 - 16 = 20$$

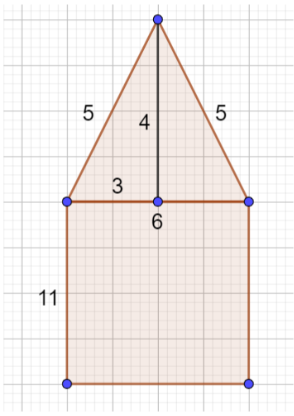
Simplify the radical:

$$b = \sqrt{20} = 2\sqrt{5}$$

Example 1.65

A house has a triangular roof. The sloping side of the roof has a length of side 5. The base of the house has a length of 20, and a width of 6. The height from the base of the house to the base of the roof is 11 feet. Take a cross-section of the house perpendicular to the base of the house. Find the area of:

A. Cross section of the house (not including the roof).
B. Cross section of the roof



Drawing a diagram

Here drawing a diagram is very important. The base of the roof cannot be of length 20 since

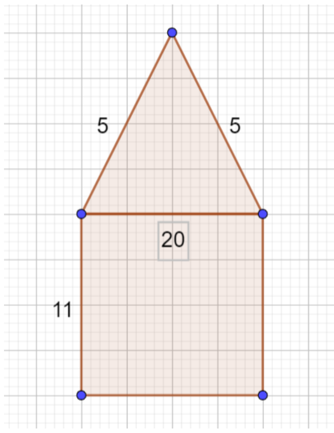
$$5 + 5 = 10 < 20$$

Which violates the triangle inequality.

Hence, draw the diagram so that the base of the roof has a length of

$$6$$

(Invalid diagram of cross-section shown on the right, valid on the left)



Now the roof gives an isosceles triangle with sides:

$$5 - 5 - 6$$

Part A

$$Area = 11 \times 6 = 66$$

Part B

Drop a perpendicular from the vertex of the triangle.
The base of the triangle gets divided into two equal parts, each with length 3.

Recognize a Pythagorean Triplet:

$$3 - 4 - 5$$

Hence, the area of the triangular cross-section is

$$\frac{1}{2}hb = \frac{1}{2} \times 4 \times 6 = 12$$

1.66: Area of Isosceles Triangle

If an isosceles triangle has two equal sides a and base b, then show that:

$$Area\ of\ the\ Triangle = \Delta = \frac{b}{4}\sqrt{4a^2 - b^2}$$

Note: Do *not* memorize this. Instead, apply the Pythagorean Theorem every time.

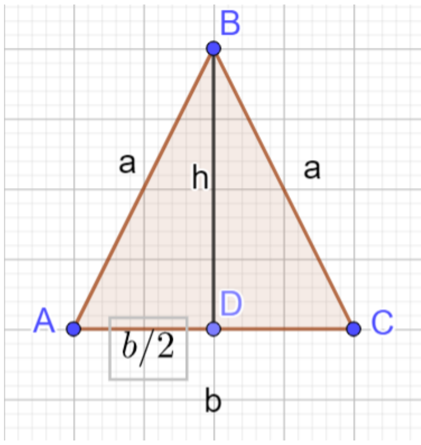
Draw a diagram.
Drop a perpendicular from Vertex B, intersecting base AC at D.

Since ΔABC is isosceles, the altitude to the base is also the median, and hence:

$$AD = DC = \frac{b}{2}$$

Also, let

$$length\ of\ BD = h$$



Then, in right ΔBAD, by the Pythagorean Theorem:

$$a^2 = h^2 + \left(\frac{b}{2}\right)^2$$

Solve for h^2:

$$h^2 = a^2 - \frac{b^2}{4} = \frac{4a^2 - b^2}{4}$$

Take square roots both sides:

$$h = \sqrt{\frac{4a^2 - b^2}{4}} = \frac{1}{2}\sqrt{4a^2 - b^2}$$

Substitute $h = \frac{1}{2}\sqrt{4a^2 - b^2}$ and $b = b$ in the formula for area of a triangle to get:

$$[ABC] = \frac{1}{2} \times hb = \frac{1}{2} \times \left(\frac{1}{2}\sqrt{4a^2 - b^2}\right) \times b = \frac{b}{4}\sqrt{4a^2 - b^2}$$

Which was the equality to be proved.

$$Q.E.D$$

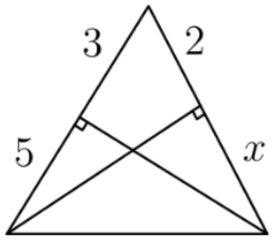
D. Isosceles Triangles: Perimeter

Example 1.67

E. Right Triangles

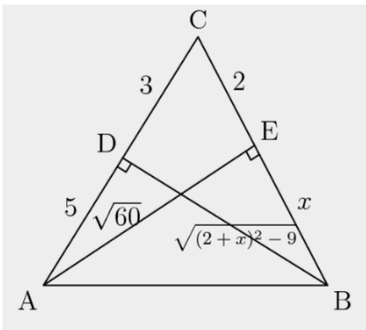
Example 1.68

Two of the altitudes of an acute triangle divide the sides into segments of lengths 5,3,2 and x units, as shown. What is the value of x? **(MathCounts 2010 National Countdown)**



Calculate the area of the triangle in two different ways[4]:

$$Area = \frac{1}{2}bh = \frac{1}{2} \cdot AC \cdot BD = \frac{1}{2} \cdot BC \cdot AE$$



Take the last two parts and square them:

$$\underbrace{AC^2 \cdot BD^2 = BC^2 \cdot AE^2}_{\textit{Equation I}}$$

Note that:

$$AC^2 = 8^2 = 64$$
$$BD^2 = BC^2 - 3^2$$
$$AE^2 = AC^2 - CE^2 = 8^2 - 2^2 = 60$$

Make the above substitutions in Equation I:

$$64 \cdot (BC^2 - 9) = BC^2 \cdot 60$$
$$64BC^2 - 64 \times 9 = BC^2 \cdot 60$$
$$4BC^2 = 64 \times 9$$
$$BC^2 = 16 \times 9$$

[4] This is solved in the Note on Similarity using Similar Right Triangles.

$$BC = 4 \times 3 = 12$$
$$2 + x = 12$$
$$x = 10$$

1.69: Geometric Mean

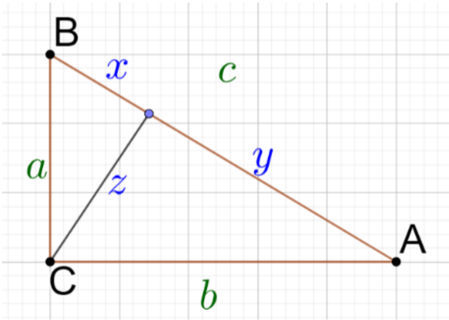
The geometric mean of x and y is the square root of their product

$$Geometric\ Mean(x, y) = \sqrt{xy}$$

1.70: Altitude of a Right Triangle

The length of the altitude of a right triangle is the geometric mean of the segments that it divides the hypotenuse into.

$$z = Geometric\ Mean(x, y) = \sqrt{xy}$$



Draw a right triangle (see diagram), with given notation:

$$c^2 = (x + y)^2 = x^2 + 2xy + y^2$$

Also, by the Pythagorean Theorem:

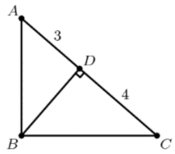
$$x^2 + z^2 = a^2, \quad y^2 + z^2 = b^2$$

Substitute the above values of a, b, c in $a^2 + b^2 = c^2$:

$$x^2 + z^2 + y^2 + z^2 = x^2 + 2xy + y^2$$
$$2z^2 = 2xy$$
$$z = \sqrt{xy}$$

Example 1.71

Triangle ABC has a right angle at B. Point D is the foot of the altitude from B, $AD = 3$, and $DC = 4$. What is the area of ΔABC? **(AMC 10A 2009/10)** [5]

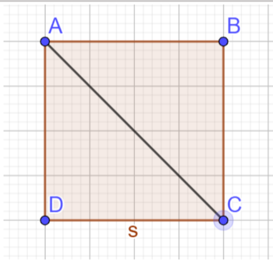


$$BD = \sqrt{3 \times 4} = 2\sqrt{3}$$
$$A(\Delta ABC) = \frac{1}{2}(BD)(AC) = \frac{1}{2}(2\sqrt{3})(3 + 4) = 7\sqrt{3}$$

F. Square

1.72: Diagonal of a square

Show that a square with side length s has diagonals with length $\sqrt{2}s$



Note that, in square $ABCD$, with side length s,

$$\Delta ADC\ is\ a\ right\ triangle$$

In right ΔADC, by the Pythagorean Theorem:

$$Diagonal = AC^2 = s^2 + s^2 = 2s^2 \Rightarrow AC = \sqrt{2s^2} = \sqrt{2}s$$

Example 1.73

Find the diagonal of a square with the following side lengths:

A. 4
B. $\sqrt{7}$

---

[5] For an AMC 10 question (that too at number 10), this is quite direct. It does require knowledge of the property. This is an *important* property.

C. $\sqrt[4]{7}$

Part A

$$From\ First\ Principles: d^2 = 4^2 + 4^2 = 32 \Rightarrow d = \sqrt{32} = 4\sqrt{2}$$
$$Using\ the\ formula: d = 4\sqrt{2}$$

Part B

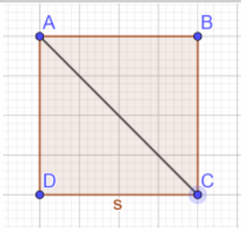
$$d = \sqrt{2} \times \sqrt{7} = \sqrt{14}$$

Part C

$$d = \sqrt{2} \times \sqrt[4]{7} = \sqrt{\sqrt{4}} \times \sqrt[4]{7} = \sqrt[4]{4} \times \sqrt[4]{7} = \sqrt[4]{28}$$

Property 1.74: Diagonal of a square

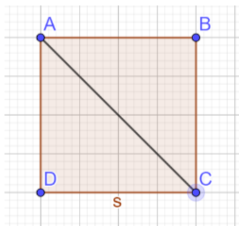
The longest line connecting any two points on a square is the diagonal of the square.



Example 1.75

What is the length of the longest straight path through a square field with side length $\sqrt{2}\ units$.

$$s = \sqrt{2} \Rightarrow d = \sqrt{2}s = \sqrt{2} \times \sqrt{2} = 2$$



Example 1.76

A farmer has a square field with area 400 $units^2$ where he plants alfalfa. He has his house and a well in the field. What is the longest possible distance between the house and the well?

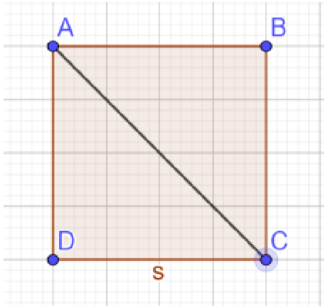
$$A = 400 \Rightarrow s = 20 \Rightarrow d = 20\sqrt{2}$$

Example 1.77

Shaun is playing at a square park with length 100 $units$. He tricycles down the shortest path from one corner to the opposite corner at a speed of $5\frac{units}{minute}$. At the other corner, his tricycle jams in the mud, and he walks back, crying, at a speed of $2\frac{units}{minute}$.

A. What is the time taken when tricycling?
B. What is the time taken when walking?
C. What is the difference in the time taken when tricycling, and walking?
D. Note that the difference in speed when going and coming is $5\frac{units}{minute} - 2\frac{units}{minute} = 3\frac{units}{minute}$. Find the time taken by someone to go from one corner to opposite corner via shortest path when travelling at a speed of $3\frac{units}{minute}$. Does this match your answer in Part C?

$$Length\ of\ Path = Diagonal = 100\sqrt{2}$$



Part A

$$T = \frac{D}{S} = \frac{100\sqrt{2}}{5} = 20\sqrt{2}\ minutes$$

Part B

$$T = \frac{D}{S} = \frac{100\sqrt{2}}{2} = 50\sqrt{2}\ minutes$$

Part C

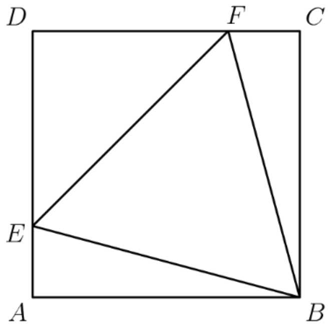
$$Difference = 50\sqrt{2} - 20\sqrt{2} = 30\sqrt{2}$$

Part D

$$T = \frac{D}{S} = \frac{100\sqrt{2}}{3} = \left(33\frac{1}{3}\right)\sqrt{2}\ minutes \neq 30\sqrt{2}$$

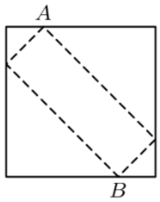
Example 1.78

Points E and F are located on square $ABCD$ so that $\triangle BEF$ is equilateral. What is the ratio of the area of $\triangle DEF$ to that of $\triangle ABE$? **(AMC 10A 2004/20)**



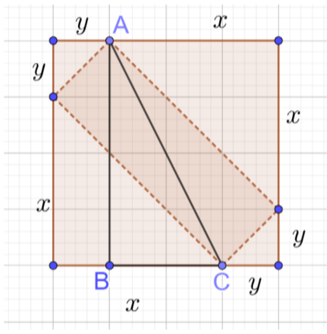
Example 1.79

An isosceles right triangle is removed from each corner of a square piece of paper, as shown, to create a rectangle. If AB = 12 units, what is the combined area of the four removed triangles, in square units? **(MathCounts 2009 State Sprint)**



Without loss of generality, assume $x > y$. In ΔABC, by the Pythagorean Theorem:

$$AB^2 + BC^2 = AC^2$$
$$(x+y)^2 + (x-y)^2 = 12^2$$
$$x^2 + 2xy + y^2 + x^2 - 2xy + y^2 = 144$$
$$2x^2 + 2y^2 = 144$$
$$x^2 + y^2 = 72$$



The area of the four triangles is:

$$\underbrace{2 \times \frac{1}{2}(x)(x)}_{Two\ Larger\ Triangles} + \underbrace{2 \times \frac{1}{2}(y)(y)}_{Two\ Smaller\ Triangles} = x^2 + y^2 = 72$$

G. Rectangle

Proof 1.80: Diagonal of a Rectangle

Show that a rectangle with length l and width w has diagonals with side length $\sqrt{l^2 + w^2}$

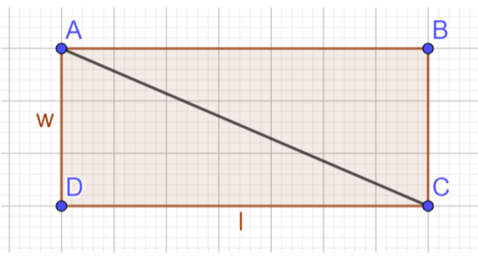
Note that, in rectangle $ABCD$, with length l and width w,

$$\Delta ADC\ is\ a\ right\ triangle$$

In right ΔADC, by the Pythagorean Theorem:

$$Diagonal = AC^2 = l^2 + w^2 \Rightarrow AC = \sqrt{l^2 + w^2}$$

Q.E.D.



Example 1.81

The length of diagonal of a rectangle with sides $\sqrt[4]{18}$ and $\sqrt[4]{72}$ can be written in the form $a\sqrt{b}$ where b has no perfect square terms. Find $a+b$.

$$Diagonal^2 = \left(\sqrt[4]{18}\right)^2 + \left(\sqrt[4]{72}\right)^2 = \sqrt{18} + \sqrt{72} = 3\sqrt{2} + 6\sqrt{2} = 9\sqrt{2}$$

$$Diagonal = \sqrt{9\sqrt{2}} = 3\sqrt{\sqrt{2}} \Rightarrow a = 3, b = \sqrt{2} \Rightarrow a + b = 3 + \sqrt{2}$$

Example 1.82

The perimeter of a rectangle is 56 meters. The ratio of its length to its width is 4:3. What is the length in meters of a diagonal of the rectangle? (MathCounts 2005 State Team)

$$2(3k + 4k) = 56 \Rightarrow k = 4$$

$$Diagonal = \sqrt{(3k)^2 + (4k)^2} = \sqrt{9k^2 + 16k^2} = \sqrt{25k^2} = 5k = 20$$

H. Perimeter of a Trapezoid

Example 1.83

Draw a right triangle at the vertex

I. Cube

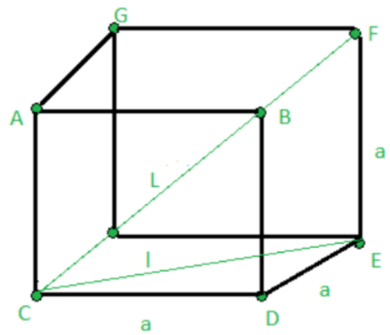
1.84: Face and Space Diagonals

- A face diagonal of a cube is a diagonal that lies entirely on a face.
- A space diagonal of a cube is a diagonal that does not lie entirely on a face.

1.85: Face Diagonal of a Cube

A cube with side length s has face diagonals with side length

$$\sqrt{2}s$$



In right ΔCDE, by the Pythagorean Theorem:

$$CE^2 = s^2 + s^2 \Rightarrow CE = \sqrt{2}s$$

1.86: Space Diagonal of a Cube

A cube with side length s has space diagonals with side length

$$\sqrt{3}s$$

In right ΔCEF, by the Pythagorean Theorem:

$$CF^2 = 2s^2 + s^2 \Rightarrow CF = \sqrt{3s^2} = \sqrt{3}s$$

Example 1.87

Find the length of the space diagonal of a cube with side length:

A. 3
B. $\sqrt{2}$
C. $\sqrt[4]{27}$

$$d = 3\sqrt{3}$$

$$d = \sqrt{2} \times \sqrt{3} = \sqrt{6}$$
$$d = \sqrt{3} \times \sqrt[4]{27} = \sqrt[4]{9} \times \sqrt[4]{27} = \sqrt[4]{243} = \sqrt[4]{81} \times \sqrt[4]{3} = 3\sqrt[4]{3}$$
$$d = \sqrt{3} \times \sqrt[4]{27} = 3^{\frac{1}{2}} \times 3^{\frac{3}{4}} = 3^{\frac{5}{4}}$$

Property 1.88: Longest Diagonal of a Cube
The longest diagonal of a cube is its space diagonal. All four space diagonals have the same length.

Example 1.89
A cubical container has a volume of 216 units. Find the length of the longest straight rod that can be kept in the cubical container.

$$V = 216 \Rightarrow s = \sqrt[3]{216} = 6 \Rightarrow d = 6\sqrt{3}$$

Property 1.90: Finding Side given Diagonal

$$d = \sqrt{3}s \Rightarrow s = \frac{d}{\sqrt{3}}$$

Example 1.91
The *Starship Enterprise* spots an enemy spaceship. The ship engineers explains that if they position the Enterprise at the bottom right, front, corner of an imaginary cube, and the enemy spaceship at the top, left, back corner, then the shortest distance between the two spaceships is 300 *units*. What is the side length of the imaginary cube that the engineer has in mind?

$$s = \frac{d}{\sqrt{3}} = \frac{300}{\sqrt{3}} = \frac{300\sqrt{3}}{3} = 100\sqrt{3}$$

J. Cuboid

1.92: Space Diagonal and Face Diagonals
A face diagonal of a cube is a diagonal that lies entirely on a face. A space diagonal of a cube is a diagonal that does not lie entirely on a face.

1.93: Length of Space Diagonal
A cuboid with length l, width w and height h has space diagonals with side length $\sqrt{l^2 + w^2 + h^2}$.

Calculate the length of the base diagonal of the cuboid.
In right ΔDCG, by the Pythagorean Theorem:

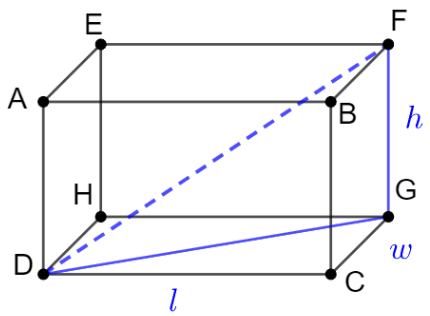
$$DG^2 = l^2 + w^2$$

In right ΔDGF, by the Pythagorean Theorem:

$$DF^2 = DG^2 + GF^2 = l^2 + w^2 + h^2$$

Take square roots both sides:

$$DF = \sqrt{l^2 + w^2 + h^2}$$



1.94: Longest Diagonal of a Cuboid

The longest diagonal of a cuboid is its space diagonal. All four space diagonals have the same length.

Example 1.95

Finding height given space diagonal, length, and width

K. Triangular Prisms

Example 1.96: Triangular Prisms

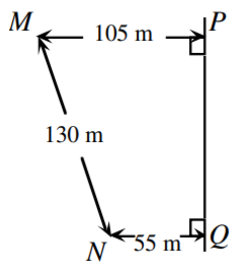
Height given slant edge and base
Slant edge given height and base
Base given slant edge and height

Example 1.97: Pyramid

Find Height
Find Diagonal of Base
Find Slant Edge

Example 1.98: Cutting a Cylinder

Find the length of the longest rod.



L. Finding the Minimum

(Calculator) Example 1.99

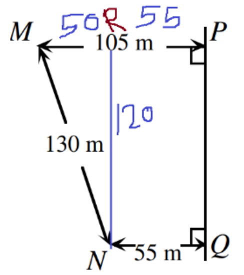
A main gas line runs through P and Q. From some point T on PQ, a supply line runs to a house at point M. A second supply line from T runs to a house at point N. What is the minimum total length of pipe required for the two supply lines? **(CEMC Cayley 1999/22)**

Drop a perpendicular from N to MP. Then,

$$MR = 105 - 55 = 50$$

By a Pythagorean Triplet

$$NR = 120$$



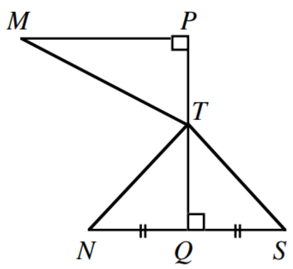
Reflect N across PQ to get S.

$$NQ = QS, \angle TQN = \angle TQS \Rightarrow \Delta TQN \cong TQS \Rightarrow TN = TS$$

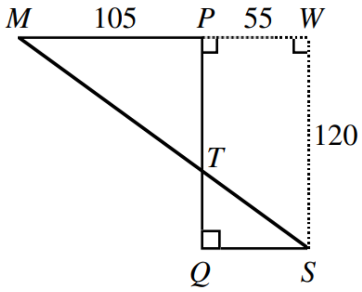
We need to minimize

$$MT + TN = MT + TS$$

Which will get minimized when M, T and S are collinear.



$$MS = \sqrt{(105+55)^2 + 120^2} = 200$$



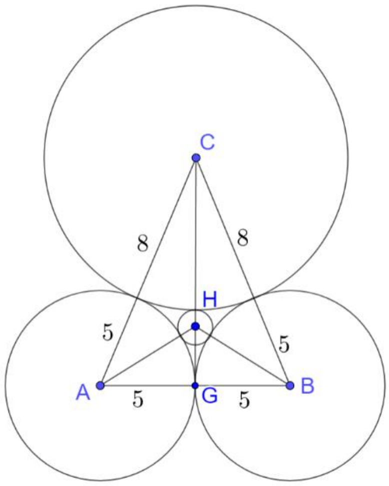
M. Circles

1

Example 1.100

Circles of radii 5, 5, 8 and $\frac{m}{n}$ are mutually externally tangent, where m and n are relatively prime positive integers. Find $m + n$. **(AIME 1997/4)**

Since the fourth circle is externally tangent, we want the circle inside the three circles (with center H and radius r).



In ΔCAB, draw the altitude to G, and note that it passes through H because of symmetry.

Using Pythagorean Triplets:

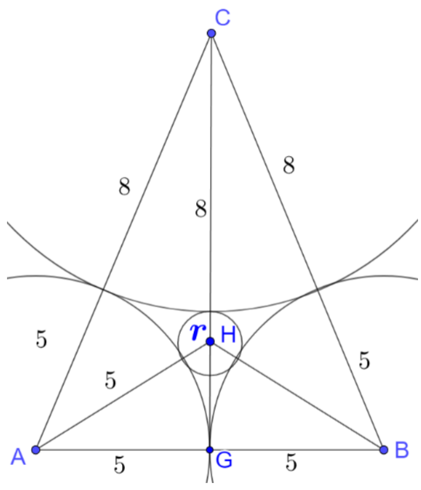
$$CA = 8 + 5 = 13, AG = 5 \Rightarrow CG = 12$$

In ΔHAG, by the Pythagorean Theorem:

$$HG^2 + AG^2 = AH^2$$
$$HG^2 + 5^2 = (5+r)^2$$
$$HG = \sqrt{10r + r^2}$$

In ΔCAB, by the Pythagorean Theorem:

$$CG^2 + AG^2 = CA^2$$
$$\left(8 + r + \sqrt{10r + r^2}\right)^2 + 5^2 = 13^2$$
$$\left(8 + r + \sqrt{10r + r^2}\right)^2 = 12^2$$
$$8 + r + \sqrt{10r + r^2} = 12$$
$$\sqrt{10r + r^2} = 4 - r$$
$$10r + r^2 = 16 - 8r + r^2$$
$$18r = 16$$
$$r = \frac{16}{18} = \frac{8}{9}$$

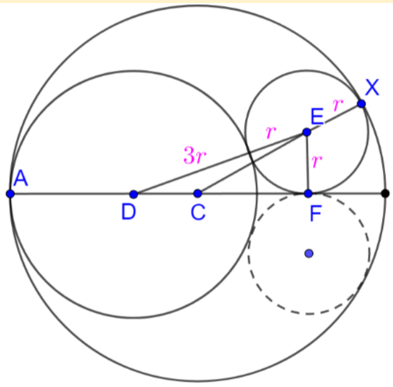


2

Example 1.101

Circle C with radius 2 has diameter AB. Circle D is internally tangent to circle C at A. Circle E is internally tangent to circle C, externally tangent to circle D, and tangent to AB. The radius of circle D is three times the radius of circle E, and can be written in the form $\sqrt{m} - n$, where m and n are positive integers. Find $m + n$.

(AIME II 2014/8)[6]



Draw $EF \perp AC$. In right ΔDEF, by the Pythagorean Theorem:

$$DF^2 + EF^2 = DE^2$$

Substitute $EF = r \Rightarrow DE = 4r$:

$$(DC + CF)^2 + r^2 = (4r)^2$$

Substitute:

$$CD = CA - AD = 2 - 3r$$

$$CF = \sqrt{CE^2 - EF^2} = \sqrt{(CX - EX)^2 - EF^2} = \sqrt{(2 - r)^2 - r^2}$$
$$= \sqrt{4 - 4r}\ (Pythagoras\ in\ \Delta CEF)$$

To get:

$$\left(2 - 3r + \sqrt{4 - 4r}\right)^2 = 15r^2$$

Take square roots:

$$2 - 3r + \sqrt{4 - 4r} = \sqrt{15}r$$

Isolate the radical:

$$\sqrt{4 - 4r} = r\left(\sqrt{15} + 3\right) - 2$$

Square both sides:

$$4 - 4r = r^2\left(\sqrt{15} + 3\right)^2 - 4r\left(\sqrt{15} + 3\right) + 4$$

Cancel the 4, and divide by $r \neq 0$ on both sides:

$$-4 = r\left(15 + 6\sqrt{15} + 9\right) - 4\left(\sqrt{15} + 3\right)$$

Collate all r terms on one side and numbers on the other:

$$r\left(6\sqrt{15} + 24\right) = -4 + 4\left(\sqrt{15} + 3\right) = 8 + 4\sqrt{15}$$

Divide both sides by 2, and rationalize:

$$r = \frac{4 + 2\sqrt{15}}{3\sqrt{15} + 12} \cdot \frac{3\sqrt{15} - 12}{3\sqrt{15} - 12}$$

Multiply:

$$r = \frac{12\sqrt{15} - 48 + 6 \cdot 15 - 24\sqrt{15}}{9 \cdot 15 - 144}$$

Simplify, and multiply both sides by 3:

$$3r = 3 \cdot \frac{42 - 12\sqrt{15}}{-9} = \sqrt{240} - 14$$

$$m + n = 240 + 14 = 254$$

N. Arc Length[7]

Example 1.102

[6] Two solutions to this question using Law of Cosines, and Descartes' Theorem are in the Note on Trigonometry.
[7] Section 6.3 (Arc Length), Thomas's Calculus (14th Edition)

2. Special Right Triangles

2.1 45 – 45 – 90 Triangles

A. 45 – 45 – 90 Triangle

Special right triangles occur frequently in geometry, and are very useful in trigonometry and vectors. The ratios for these triangles should be memorized.

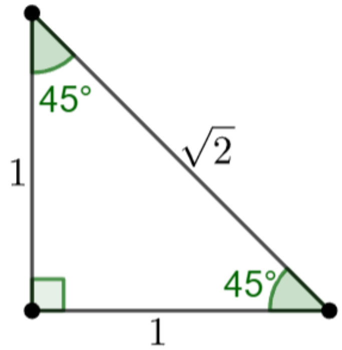
2.1: $45° - 45° - 90°$ Triangle

A $45° - 45° - 90°$ triangle is an isosceles right-angled triangle.

2.2: Sides in a $45° - 45° - 90°$ Triangle

The hypotenuse of an isosceles right-angled triangle is $\sqrt{2}$ times either of the congruent legs.

$$Hypotenuse = \sqrt{2} \times Leg$$

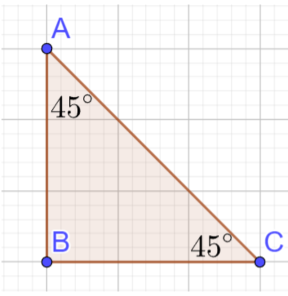


Consider an isosceles right-angled triangle with:

$$length\ of\ leg = 1$$

By Pythagoras Theorem:

$$Hyp^2 = 1^2 + 1^2 = 2 \Rightarrow Hyp = \sqrt{2}$$



Example 2.3: Finding Lengths

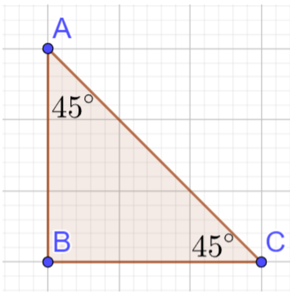
Use the diagram alongside to answer each part below.

A. If $AB = 3$, find the length of AC.
B. If $BC = 4$, find the length of AB.
C. If $AC = 2$, find the length of AB.

$$AC = \sqrt{2} \times Leg = \sqrt{2} \times AC = 3\sqrt{2}$$

$$AB = BC = 4$$

$$AC = \frac{Hyp}{\sqrt{2}} = \frac{AC}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$



Example 2.4

Use the diagram alongside to answer each part below. Find the length of AC if:

A. $AB = 5$
B. $AB = 2\sqrt{2}$
C. $AB = \sqrt[3]{2} - \sqrt[4]{2}$

$$AC = 5\sqrt{2}$$

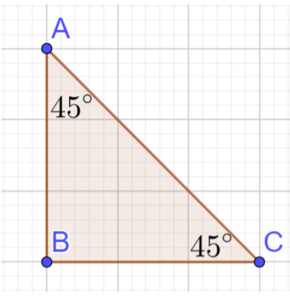
$$2\sqrt{2} \times \sqrt{2} = 2 \times 2 = 4$$

$$(\sqrt{2})(\sqrt[3]{2} - \sqrt[4]{2}) = \left(2^{\frac{1}{2}}\right)\left(2^{\frac{1}{3}} - 2^{\frac{1}{4}}\right) = 2^{\frac{1}{2}+\frac{1}{3}} - 2^{\frac{1}{2}+\frac{1}{4}} = 2^{\frac{5}{6}} - 2^{\frac{3}{4}}$$

Example 2.5

Use the diagram alongside to answer each part below. Find the length of AB if:

A. $AC = 3\sqrt{2}$
B. $AC = 9\sqrt{2}$
C. $AC = 4$
D. $AC = 10$
E. $AC = 7$



$$AB = \frac{Hyp}{\sqrt{2}} = \frac{3\sqrt{2}}{\sqrt{2}} = 3$$

$$AB = 4 \times \frac{1}{\sqrt{2}} = \frac{4}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

$$AB = \frac{10}{\sqrt{2}} = \frac{10}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$$

Example 2.6

A. A garden is in the shape of an isosceles right triangle. If one of its smaller sides is 12 meters, find the length of its longest side?
B. A garden is in the shape of an isosceles right triangle. If its longest side is 12 meters, find the length of its shortest side?

Part A

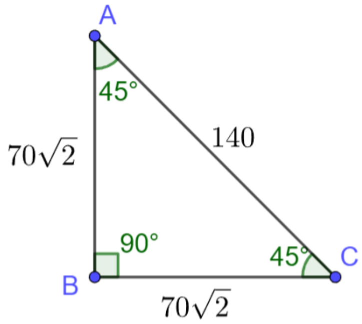
$$Hypotenuse = \sqrt{2} \times Leg = \sqrt{2} \times 12 = 12\sqrt{2}$$

Part B

$$Leg = \frac{Hyp}{\sqrt{2}} = \frac{12}{\sqrt{2}} = \frac{12}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{12\sqrt{2}}{2} = 6\sqrt{2}$$

Example 2.7

A field that is in the shape of an isosceles right triangle is to be fenced. Find the cost of the fence at \$5 per meter, if the length of one leg of the triangle is $70\sqrt{2}$ meters.



Length of two legs

$$= AB + BC = 70\sqrt{2} + 70\sqrt{2} = 140\sqrt{2}$$

Length of the hypotenuse

$$= AC = \sqrt{2} \times 70\sqrt{2} = 70\left(\sqrt{2}\right)^2 = 70 \times 2 = 140$$

Perimeter

$$= AB + BC + AC = 140\sqrt{2} + 140$$

Cost

$$= \underbrace{5}_{\substack{Cost \\ per\ Meter}} \times \underbrace{\left(140\sqrt{2} + 140\right)}_{No.of\ Meters} = \left(700\sqrt{2} + 700\right)\$$$

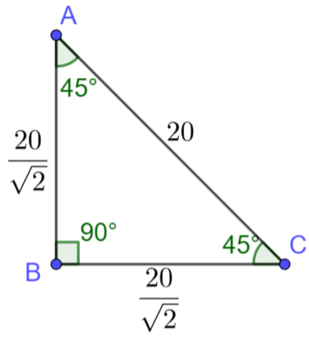
Example 2.8

What is the area, in square units, of an isosceles right triangle with a hypotenuse of 20 units? (MathCounts 2009

State Countdown)

The area

$$= \frac{1}{2}hb = \frac{1}{2}(AB)(BC)$$



Substitute $AB = BC = \frac{AC}{\sqrt{2}} = \frac{20}{\sqrt{2}}$

$$= \frac{1}{2} \times \frac{20}{\sqrt{2}} \times \frac{20}{\sqrt{2}} = \frac{400}{4} = 100\ units^2$$

Example 2.9

A. Point B is due east of point A. Point C is due north of point B. The distance between points A and C is $10\sqrt{2}$, and $\angle BAC = 45°$. Point D is 20 meters due north of point C. The distance AD is between which two integers? (AMC 10B 2012/12)

B. Prove that the area of an isosceles right triangle with hypotenuse hyp is $\frac{hyp^2}{4}$.

Part A

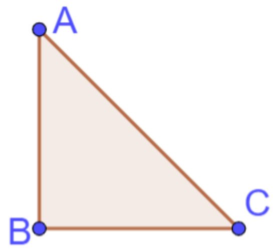
$$AB = \frac{10\sqrt{2}}{\sqrt{2}} = 10$$
$$AD = \sqrt{10^2 + 30^2} = \sqrt{1000}$$
$$31 < AD < 32$$

Part F

$$Leg = \frac{hyp}{\sqrt{2}} \Rightarrow A = \frac{1}{2}hb = \frac{1}{2} \times \frac{hyp}{\sqrt{2}} \times \frac{hyp}{\sqrt{2}} = \frac{hyp^2}{4}$$

Example 2.10

ΔABC, drawn alongside, is an isosceles right-angled triangle, with a right angle at B. Find the area and the perimeter for each part below (independently):



A. $AB = 4$
B. $AC = 4$
C. $AB = \sqrt{2}$
D. $AB = \sqrt{8}$
E. $AB = 1 + \sqrt{2}$
F. $AB = \frac{1}{1+\sqrt{2}}$

Part A

$$BC = AB = 4, \qquad AC = 4\sqrt{2}$$
$$Perimeter = 4 + 4 + 4\sqrt{2} = 8 + 4\sqrt{2}$$
$$\text{Area} = \frac{1}{2}hb = 2 \times 4 \times 4 = 8$$

Part B

$$AB = BC = \frac{4}{\sqrt{2}} = \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$
$$Perimeter = 4 + 2\sqrt{2} + 2\sqrt{2} = 4 + 4\sqrt{2}$$
$$Area = \frac{1}{2}hb = \frac{1}{2} \times 2\sqrt{2} \times 2\sqrt{2} = 4$$

Part C

$$BC = \sqrt{2}, \qquad AC = 2$$
$$AC = \sqrt{2} \times \sqrt{2} = 2$$

Part D

$$AB = \sqrt{2} \times \sqrt{8} = \sqrt{16} = 4$$

Part E

$$AC = \sqrt{2}(1 + \sqrt{2}) = \sqrt{2} + 2$$

Part F

Rationalize:

$$AC = \sqrt{2}\left(\frac{1}{1+\sqrt{2}}\right)$$

$$= \frac{\sqrt{2}}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}} = \frac{\sqrt{2}-2}{1-2} = 2-\sqrt{2}$$

Example 2.11

In a right triangle the square of the hypotenuse is equal to twice the product of the legs. One of the acute angles of the triangle is: **(AHSME 1959/15)**

Let the legs be x and y. Then:

$$x^2 + y^2 = 2xy \Rightarrow (x-y)^2 = 0 \Rightarrow x - y = 0 \Rightarrow x = y$$

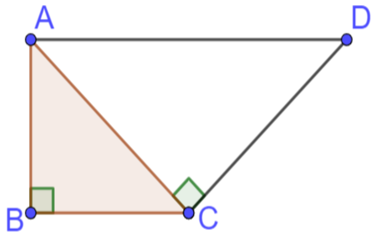
Hence, the triangle is an isosceles triangle. And we know that it is right-angled. Combine the two to get:

$$Right - angled\ isoceles\ triangle \Rightarrow 45 - 45 - 90$$

B. Sequence of Triangles

2.12: Sequence of Triangles

A sequence of triangles consists of more than one triangle adjacent to each other.
If the triangles are all special right triangles, the properties of those triangles can be applied.



Example 2.13

$AB = BC$ and $\angle ABC = 90°$. ΔACD is an isosceles right-angled triangle. If $AB = 3$, find the perimeter and area of ΔACD.

ΔABC is a $45-45-90$ triangle and so is ΔACD:

$$AC = CD = AB(\sqrt{2}) = 3\sqrt{2}$$

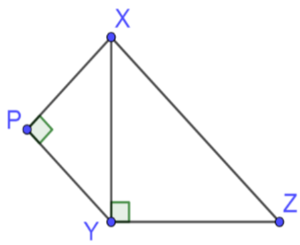
$$[ACD] = \frac{1}{2}hb = \frac{1}{2}(AC)(CD) = \frac{1}{2}(3\sqrt{2})^2 = \frac{1}{2} \times 9 \times 2 = 9$$

In $45-45-90$ triangle ACD:

$$AD = AC(\sqrt{2}) = 3\sqrt{2}(\sqrt{2}) = 3 \times 2 = 6$$

The perimeter is:

$$AC + CD + AD = 3\sqrt{2} + 3\sqrt{2} + 6 = 6 + 6\sqrt{2}$$



Example 2.14

ΔXYZ is an isosceles right-angled triangle. Also, $XZ = 4, XP = PY$ and $\angle XPY = 90°$. Find the perimeter and area of ΔXPY.

In $45-45-90$ triangle XYZ:

$$XY = \frac{XZ}{\sqrt{2}} = \frac{4}{\sqrt{2}} = \frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

In $45-45-90$ triangle XPY:

$$XP = \frac{2\sqrt{2}}{\sqrt{2}} = 2$$

The perimeter

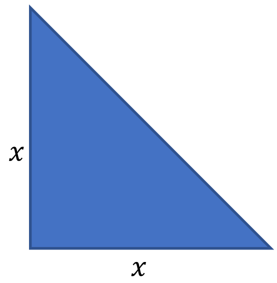
$$= XP + PY + XY = 2 + 2 + 2\sqrt{2} = 4 + 2\sqrt{2}$$

The area is

$$[XPY] = \frac{1}{2}hb = \frac{1}{2}(XP)(PY) = \frac{1}{2}(2^2) = \frac{1}{2} \times 4 = 2$$

*Example 2.15

The perimeter of an isosceles right-angled triangle is 2012. Find its area in simplified and rationalized form. (NMTC Sub-Junior/Screening 2012/A/14)



Let:

$$Leg = x \Rightarrow Hypotenuse = \sqrt{2}x$$

Hence:

$$Perimeter = x + x + \sqrt{2}x = 2x + \sqrt{2}x = x(2 + \sqrt{2})$$

As given in the question:

$$Perimeter = x(2 + \sqrt{2}) = 2012$$

$$x = \frac{2012}{2 + \sqrt{2}}$$

Hence:

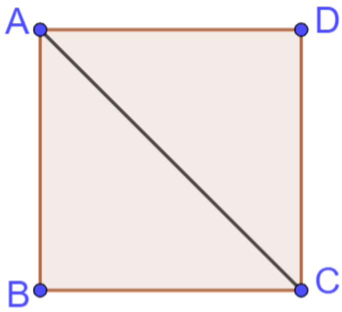
$$Area = \frac{1}{2}hb = \frac{1}{2} \times \left(\frac{2012}{2 + \sqrt{2}}\right)^2 = \frac{1}{2} \times \frac{2012^2}{4 + 4\sqrt{2} + 2} = \frac{1}{2} \times \frac{2012^2}{6 + 4\sqrt{2}}$$

To rationalize the denominator, multiply numerator and denominator by $6 - 4\sqrt{2}$:

$$\frac{1}{\mathbf{2}} \times \frac{2012^2}{6 + 4\sqrt{2}} \times \frac{\mathbf{6 - 4\sqrt{2}}}{6 - 4\sqrt{2}} = \frac{2012^2(3 - 2\sqrt{2})}{36 - 32} = \frac{2012^2(3 - 2\sqrt{2})}{4} = 1006^2(3 - 2\sqrt{2})$$

2.16: $45° - 45° - 90°$ Triangle in Squares

The diagonal of a square divides it into two $45 - 45 - 90$ triangles.



Construct square $ABCD$, and draw diagonal AC.

In ΔABC:

$$\angle ABC = 90° \ (Angle\ of\ a\ Square)$$

Since $ABCD\ is\ a\ square \Rightarrow AB = AC \Rightarrow \Delta ABC\ is\ Isosceles$:

$$\angle BAC = \angle BCA = \frac{180 - 90}{2} = \frac{90}{2} = 45°$$

$$\Delta ABC\ \ is\ a\ 45 - 45 - 90\ triangle.$$

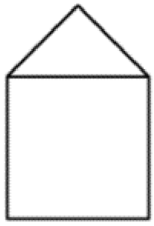
Similarly:

$$\Delta ADC\ is\ a\ 45 - 45 - 90\ triangle$$

Example 2.17

A square piece of paper is folded once so that one pair of opposite corners coincide. When the paper is unfolded, two congruent triangles have been formed. Given that the area of the original square is 49 square inches, what is the number of inches in the perimeter of one of these triangles? Express your answer in simplest radical form. (MathCounts 1998 State Countdown)

$$Area\ of\ Square = 49\ in^2$$
$$Side\ of\ Square = 7\ in$$
$$Length\ of\ Diagonal = 7\sqrt{2}$$
$$Perimeter = 7 + 7 + 7\sqrt{2} = 14 + 7\sqrt{2}$$



Example 2.18
A pentagon is drawn by placing an isosceles right triangle on top of a square as pictured. What percent of the area of the pentagon is the area of the right triangle? (MathCounts 1997 Warm-Up 17)

Method I
Let the square have $Side\ Length = 1 \Rightarrow Area = 1$
The triangle is an isosceles right-angled triangle: $45 - 45 - 90$ triangle

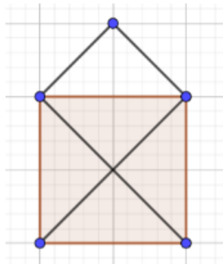
$$Leg\ Length = \frac{1}{\sqrt{2}}$$
$$Area = \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4}$$

The total area is

$$1 + \frac{1}{4} = \frac{5}{4}$$

The percent is:

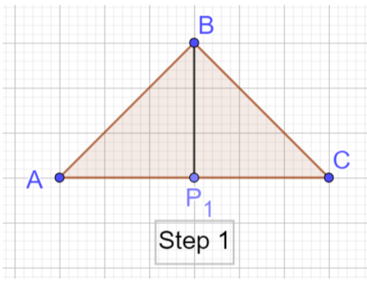
$$\frac{Area(Right\ Triangle)}{Area(Pentagon)} = \frac{\frac{1}{4}}{\frac{5}{4}} = \frac{1}{4} \times \frac{4}{5} = \frac{1}{5} = 20\%$$



Method II
Draw the diagonals of the square. Note that the four triangles so formed are congruent to the top triangle. Hence, the percent is:

$$\frac{1}{5} = 20\%$$

2.19: Perpendicular in a $45° - 45° - 90°$ Triangle
The perpendicular in a $45 - 45 - 90$ triangle is half the length of the hypotenuse.



Consider $45 - 45 - 90$ ΔABC, right-angled at $\angle B$.
Draw perpendicular BP_1.

Since ΔABC is isosceles:

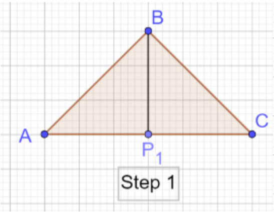
$$BP_1\ is\ the\ perpendicular\ bisector\ of\ AC$$

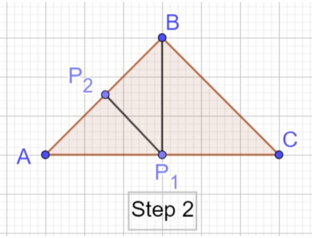
Also, ΔAP_1B is isosceles:

$$BP_1 = AP_1 = \frac{1}{2} \times AC = \frac{1}{2} \times Hypotenuse$$

Example 2.20
ΔABC is an isosceles right triangle, right-angled at B with $AC = 1$.

A. Drop a perpendicular from point B, meeting AC at P_1. Find the length of BP_1.
B. Draw the perpendicular from P_1, meeting AB at P_2. Find the length of P_1P_2
C. Draw the perpendicular from P_2, meeting AC at P_3. Find the length of P_2P_3





Part A
ΔBP_1C is $45-45-90$

$$Hence, it\ is\ isosceles \Rightarrow BP_1 = CP_1$$

Also, BP_1 is the perpendicular bisector of AC:

$$CP_1 = \frac{1}{2}\times AC = \frac{1}{2}\times 1 = \frac{1}{2}$$

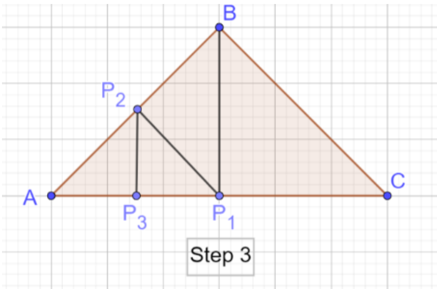
Part B
Method I: Consider ΔABP_1

$$AB = Leg\ of\ \Delta ABC = \frac{Hyp}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$P_1P_2 = AP_2 = \frac{1}{2}\ of\ AB = \frac{1}{2}\times\frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

Nesting Method: Consider ΔBP_1P_2

$$P_1P_2 = \frac{Hyp}{\sqrt{2}} = \frac{\frac{1}{2}}{\sqrt{2}} = \frac{1}{2}\times\frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

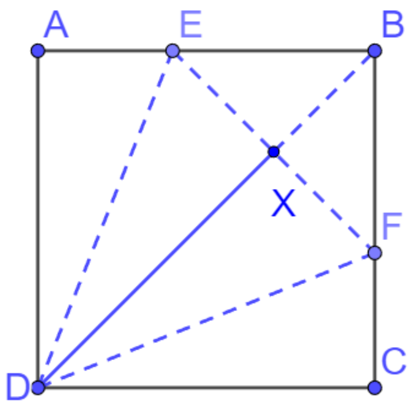


Part C

$$P_2P_3 = \frac{1}{2}\ of\ AP_1 = \frac{1}{2} of \frac{1}{2} of\ AC = \frac{1}{2}\times\frac{1}{2}\times 1 = \frac{1}{4}$$

Example 2.21

Square $ABCD$ has side length 1 unit. Points E and F are on sides AB and CB, respectively, with $AE = CF$. When the square is folded along the lines DE and DF, sides AD and CD coincide and lie on diagonal BD. The length of segment AE can be expressed in the form $\sqrt{k}-m$ units. What is the integer value of $k+m$? (MathCounts 2011 Chapter Target)



ΔAED is folded over DE to get ΔDEX.
DE is a line of symmetry for quadrilateral $ADXE$.

$$AD = DX$$
$$AE = EX$$
$$ED = ED$$
$$\Delta AED \cong \Delta XED$$

Similarly,

$$\Delta CFD \cong \Delta XFD$$

Once the square is folded, points A and C lie on the diagonal at point X.

$$DX = 1$$
$$DB = Diagonal\ of\ a\ Square = \sqrt{2}$$
$$BE = 1 - AE$$

Since ΔEBF is a $45-45-90$ triangle:

$$XB = Altitude = \frac{1-AE}{\sqrt{2}}$$

Since $DX + XB = DB$:

$$\underbrace{\sqrt{2}}_{DB} = \underbrace{1}_{DX} + \underbrace{\frac{1-AE}{\sqrt{2}}}_{XB}$$

$$2 = \sqrt{2} + 1 - AE$$

$$AE = \sqrt{2} - 1$$

$$k + m = 2 + 1 = 3$$

C. Equilateral Triangles

Equilateral triangles can be split into two $30 - 60 - 90$ triangles by dropping a perpendicular from any vertex to the opposite base. This is especially useful, as we see now.

2.22: Height of an Equilateral Triangle

The height of an equilateral triangle is $\frac{\sqrt{3}}{2}$ times the side length.

$$h = \frac{\sqrt{3}}{2}s$$

Construct equilateral ΔABC with side length s.
Draw height AX $\perp BC$:

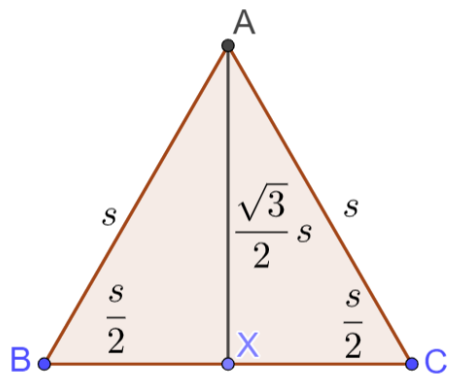
$$\angle AXB = \angle AXC = 90°$$

An equilateral triangle is also an isosceles triangle. In an isosceles triangle, the altitude to the base bisects the base. Hence:

$$BX = XC = \frac{s}{2}$$

In right ΔABX, by the Pythagorean Theorem:

$$AB^2 = BX^2 + AX^2$$



Substitute $AB = s, BX = \frac{s}{2}, AX = h$:

$$s^2 = \left(\frac{s}{2}\right)^2 + h^2$$

$$h^2 = s^2 - \left(\frac{s}{2}\right)^2 = s^2 - \frac{s^2}{4} = \frac{3s^2}{4}$$

Take square roots:

$$h = \sqrt{\frac{3}{4}s^2} = \frac{\sqrt{3}}{2}s$$

Example 2.23: Working with Equilateral Triangles

A. If an equilateral triangle has perimeter 12, then find the height of the triangle.
B. If an equilateral triangle has perimeter 9, then find the height of the triangle.
C. If the height of an equilateral triangle is 4, then find the perimeter of the triangle.
D. If the height of an equilateral triangle is 3, then find the perimeter of the triangle.

$$P = 12 \Rightarrow s = \frac{12}{3} = 4 \Rightarrow h = \frac{\sqrt{3}}{2} \times s = \frac{\sqrt{3}}{2} \times 4 = 2\sqrt{3}$$

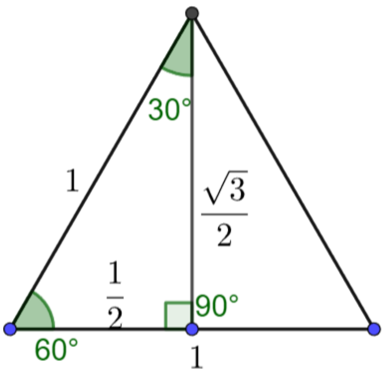
$$P = 9 \Rightarrow s = \frac{9}{3} = 3 \Rightarrow h = \frac{\sqrt{3}}{2} \times 3 = \frac{\sqrt{3}}{2} \times 3 = \frac{3\sqrt{3}}{2}$$

$$h = \frac{\sqrt{3}}{2}s \Rightarrow 4 = \frac{\sqrt{3}}{2}s \Rightarrow s = 4 \times \frac{2}{\sqrt{3}} = \frac{8}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{8\sqrt{3}}{3} \Rightarrow P = 8\sqrt{3}$$

2.24: Area of an Equilateral Triangle

The area of an equilateral triangle is $\frac{\sqrt{3}}{4}$ times the square of the side length s.

$$A = \frac{\sqrt{3}}{4}s^2$$



We start with the height calculated above.

$$A = \frac{1}{2} \times Height \times Base = \frac{1}{2} \times \left(\frac{\sqrt{3}}{2}s\right) \times s = \frac{\sqrt{3}}{4}s^2$$

Example 2.25

A. An equilateral triangle has perimeter 4 units. What is its area?
B. What is the ratio of the numerical value of the area, in square units, of an equilateral triangle of side length 4 units to the numerical value of its perimeter, in units? Express your answer as a common fraction in simplest radical form. **(MathCounts 2007 Warm-Up 9)**
C. The altitude of an equilateral triangle is $\sqrt{6}$ units. What is the area of the triangle, in square units? Express your answer in simplest radical form. **(MathCounts 2008 School Countdown)**
D. An equilateral triangle has an area of $64\sqrt{3}\ cm^2$. If each side of the triangle is decreased by 4 cm, by how many square centimeters is the area decreased? (MathCounts 1991 State Sprint)
E. The altitude of an equilateral triangle is h units. Express, in simplest radical form, the perimeter of the triangle (in terms of h), and the area of the triangle, (in terms of h^2).

Part A

$$P = 4 \Rightarrow s = \frac{4}{3}$$

$$A = \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}\left(\frac{4}{3}\right)^2 = \frac{\sqrt{3}}{4} \times \frac{16}{9} = \frac{4\sqrt{3}}{9}$$

Part B

Ratio of area to perimeter is:

$$\frac{\sqrt{3}}{4}s^2 : 3s = 4^2 \times \frac{\sqrt{3}}{4} : 3 \times 4 = \sqrt{3} : 3 = \frac{\sqrt{3}}{3}$$

Part C

Find the side length:

$$h = \frac{\sqrt{3}}{2}s \Rightarrow \sqrt{6} = \frac{\sqrt{3}}{2}s \Rightarrow s = 2\sqrt{2}$$

Find the area:

$$A = \frac{1}{2}hb = \frac{1}{2} \times \sqrt{6} \times 2\sqrt{2} = \sqrt{12} = 2\sqrt{3}$$

Part D

Current Triangle:

$$A = \frac{\sqrt{3}}{4}s^2 = 64\sqrt{3}$$

$$s^2 = 256$$
$$s = 16$$

Reduced Triangle:

$$s = 16 - 4 = 12$$

$$A = \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}(12)^2 = 36\sqrt{3}$$

Reduction

$$= 64\sqrt{3} - 36\sqrt{3} = 28\sqrt{3}$$

Part E

Find the side length:

$$h = \frac{\sqrt{3}}{2}s \Rightarrow s = \frac{2h}{\sqrt{3}}$$

Find the perimeter:

$$P = 3s = \frac{6h}{\sqrt{3}} = \frac{6h\sqrt{3}}{3} = 2\sqrt{3}h$$

Find the Area:

$$A = s^2 \times \frac{\sqrt{3}}{4} = \left(\frac{2h}{\sqrt{3}}\right)^2 \times \frac{\sqrt{3}}{4} = \frac{4h^2}{3} \times \frac{\sqrt{3}}{4} = \frac{h^2\sqrt{3}}{3}$$

Example 2.26

A square and an equilateral triangle have equal perimeters. The area of the triangle is $2\sqrt{3}$ square inches. What is the number of inches in the length of the diagonal of the square? **(MathCounts 2004 State Team)**

The side length of the triangle:

$$A = \frac{\sqrt{3}}{4}s^2 = 2\sqrt{3} \Rightarrow s = \sqrt{8} = 2\sqrt{2}$$

Find the perimeter of the triangle:

$$P = 3s = 6\sqrt{2}$$

Find the side length of the square:

$$S = \frac{P}{4} = \frac{6\sqrt{2}}{4} = \frac{3\sqrt{2}}{2}$$

Find the length of the diagonal:

$$D = \frac{3\sqrt{2}}{2} \times \sqrt{2} = \frac{3 \times 2}{2} = 3$$

Example 2.27

A square and an equilateral triangle have equal perimeters. The area of the triangle is $16\sqrt{3}$ square centimeters. How long, in centimeters, is a diagonal of the square? Express your answer in simplest radical form.
(MathCounts 2010 School Sprint)[8]

Note that:

- All squares are similar
- Are equilateral triangles are similar

$$Area = 2\sqrt{3} \Rightarrow Diagonal = 3$$
$$Area = 16\sqrt{3} \Rightarrow Diagonal = x$$

Ratio of areas:

$$16\sqrt{3}: 2\sqrt{3} = 8: 1$$

Ratio of lengths is the square root of the ratio of areas:

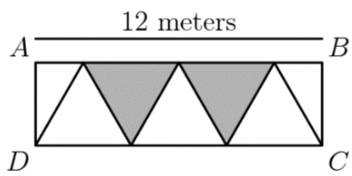
$$\sqrt{8}: \sqrt{1} = 2\sqrt{2}: 1$$

Hence, the diagonal of the square in the current question:

$$3 \times 2\sqrt{2} = 6\sqrt{2}$$

Example 2.28

Side CD of rectangle $ABCD$ measures 12 meters, as shown. Each of the three triangles with a side along segment CD is an equilateral triangle. What is the total area of the shaded regions? Express your answer in simplest radical form. **(MathCounts 2009 Warm-up 7)**



The three equilateral triangles along side CD are congruent to the shaded triangles.
Each triangle has side length

[8] This is the same question as the one above with numbers changed. We solve this using similarity instead of the earlier approach.

$$= \frac{12}{3} = 4$$

The area of the triangles is:

$$Area = 2\left(\frac{\sqrt{3}}{4} \times 4^2\right) = 8\sqrt{3}$$

2.29: Triangle with minimum perimeter / maximum area

- The triangle with minimum perimeter for a given area is an equilateral triangle.
- The triangle with maximum area for a given perimeter is an equilateral triangle.

Example 2.30

A. What is the minimum perimeter of a triangle with area 1 unit?
B. What is the maximum area of a triangle with perimeter 1 unit?
C. The numerical value of the perimeter of a triangle is equal to the numerical value of its area. Find this numerical value.

Part A

Triangle with minimum perimeter must be equilateral.

$$A = \frac{\sqrt{3}}{4}s^2 = 1 \Rightarrow s^2 = \frac{4}{\sqrt{3}} \Rightarrow s = \frac{2}{\sqrt[4]{3}}$$

$$P = 3s = \frac{6}{\sqrt[4]{3}} = 2 \times 3^{\frac{3}{4}}$$

Part B

Triangle with maximum area must be equilateral.

$$P = 1 \Rightarrow s = \frac{1}{3}$$

$$A = \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4} \times \left(\frac{1}{3}\right)^2 = \frac{\sqrt{3}}{36}$$

Part C

$$\underbrace{\frac{\sqrt{3}}{4}s^2}_{Area} = \underbrace{3s}_{Perimeter}$$

$$s = \frac{3 \times 4}{\sqrt{3}} = \sqrt{3} \times 4$$

$$P = 3s = 12\sqrt{3}$$

2.2 30 – 60 – 90 Triangles

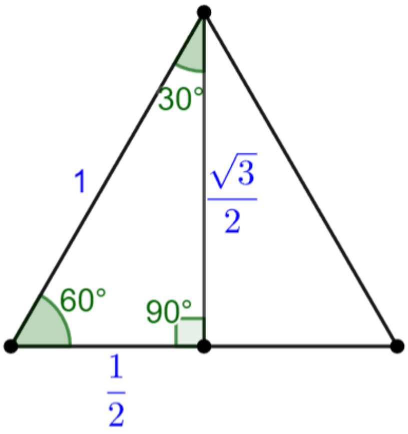
A. 30 – 60 – 90 Triangles

Just like $45 - 45 - 90$ triangles, $30 - 60 - 90$ triangles are very important. They have applications in geometry, and the ratios that are created in these triangles are a very convenient way to get ratios needed in trigonometry and vectors. The triangle below (or its derivation) should be committed to memory.

2.31: $30° - 60° - 90°$

The sides of a $30° - 60° - 90°$ triangle are in the ratio:

$$\underbrace{1}_{Hypotenuse} : \underbrace{\frac{1}{2}}_{\substack{Side\ opp\\ 30°}} : \underbrace{\frac{\sqrt{3}}{2}}_{\substack{Side\ opp\\ 60°}}$$



Draw an equilateral triangle with side length s.
Drop a perpendicular from the vertex to get a 30-60-90 triangle.
Then, the sides of the triangle are:

$$\underbrace{s}_{Hypotenuse} : \underbrace{\frac{1}{2}s}_{\substack{Side\ opp\\ 30°}} : \underbrace{\frac{\sqrt{3}}{2}s}_{\substack{Side\ opp\\ 60°}}$$

Divide the ratio above by s, to get the ratio of the sides of a $30 - 60 - 90\ triangle$.

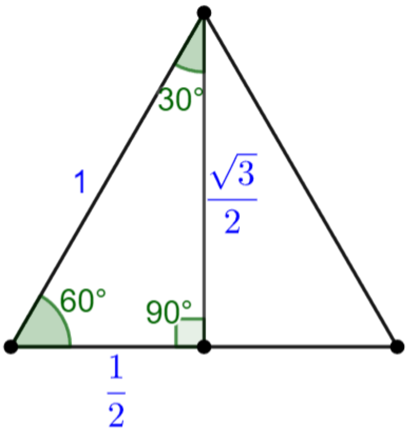
Example 2.32

A. The hypotenuse of a $30 - 60 - 90$ triangle is $\sqrt{3}$. Find the length of the other two sides.
B. The shortest side of a $30 - 60 - 90$ triangle is 3. Find the length of the other two sides.
C. One leg of a right triangle is 12 inches, and the measure of the angle opposite that leg is 30. What is the number of inches in the hypotenuse of the triangle? (MathCounts 2002 State Countdown)

Part A

$$Side\ opposite\ 30° = \sqrt{3} \times \frac{1}{2} = \frac{\sqrt{3}}{2}$$

$$Side\ opposite\ 60° = \sqrt{3} \times \frac{\sqrt{3}}{2} = \frac{3}{2}$$



Part B

Shortest side is side opposite 30°.

$$Hypotenuse = 3 \times 2 = 6$$

$$Side\ opposite\ 60° = 3 \times \sqrt{3} = 3\sqrt{3}$$

Part C

$$Hyp = 12 \times 2 = 24$$

Example 2.33

D. The hypotenuse of a $30 - 60 - 90$ triangle is $\sqrt{3}$. Find the length of the other two sides.
E. The shortest side of a $30 - 60 - 90$ triangle is 3. Find the length of the other two sides.
F. One leg of a right triangle is 12 inches, and the measure of the angle opposite that leg is 30. What is the number of inches in the hypotenuse of the triangle? (MathCounts 2002 State Countdown)
G. The ratio of the measures of the angles of a triangle is 3:2:1. Given that the shortest side of the triangle is 12 meters long, what is the number of meters in the longest side of the triangle? (MathCounts 1997 Chapter Countdown)
H. The hypotenuse of a $30 - 60 - 90$ triangle has length $\frac{1}{\sqrt{\pi}}$ units. Considering $\pi = \frac{22}{7}$, the area of the triangle can be written in the form $\frac{a\sqrt{b}}{c}, a, b, c \in \mathbb{N}, b$ does not have any perfect square factors, and $HCF(a, c) = 1$. Find $a + b + c$.
I. The hypotenuse of a $30 - 60 - 90$ triangle has length h units. Find the area of the triangle, in terms of h.
J. A $30 - 60 - 90$ triangle has area $\sqrt{3}$ unit. Find the perimeter of the triangle.

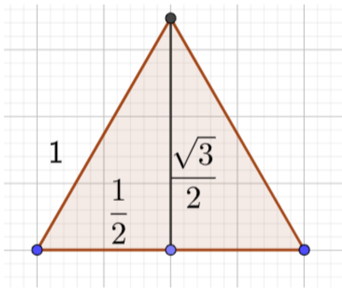
K. A $30-60-90$ triangle has area a units. Find the perimeter of the triangle.

Part D

Since the angles are in the ratio

$$3:2:1 = 90:60:30$$
$$\Rightarrow Triangle\ is\ a\ 30-60-90\ triangle.$$

In a $30-60-90\ triangle$, the shortest side is half of the longest side.



Hence, the longest side is double of the shortest side

$$= 12 \times 2 = 24$$

Part E

$$\frac{1}{2}hb = \frac{1}{2} \times \underbrace{\frac{1}{2\sqrt{\pi}}}_{height} \times \underbrace{\frac{1}{\sqrt{\pi}} \times \frac{\sqrt{3}}{2}}_{base}$$

Simplify:

$$\frac{\sqrt{3}}{8\pi} \times \frac{\sqrt{3}}{8 \times \frac{22}{7}} = \frac{\sqrt{3}}{8} \times \frac{7}{22} = \frac{7\sqrt{3}}{176}$$
$$a+b+c = 7+3+176 = 186$$

Part G

$$Area = \frac{1}{2}\left(\frac{1}{2}h\right)\left(\frac{\sqrt{3}}{2}h\right) = \frac{\sqrt{3}}{8}h^2$$

Part G

Let the hypotenuse be h.

$$\frac{\sqrt{3}}{8}h^2 = \sqrt{3}$$
$$h^2 = 8$$
$$h = \sqrt{8} = 2\sqrt{2}$$
$$P = 2\sqrt{2} + \frac{2\sqrt{2}}{2} + 2\sqrt{2}\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{4\sqrt{2}+2\sqrt{2}+2\sqrt{6}}{2} = \frac{6\sqrt{2}+2\sqrt{6}}{2}$$

Part H

Let the hypotenuse be h.

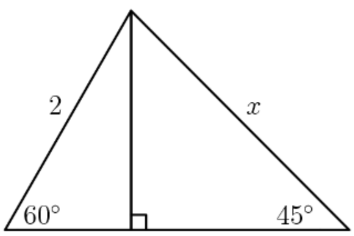
$$\frac{\sqrt{3}}{8}h^2 = a$$
$$h^2 = \frac{8}{\sqrt{3}}a$$
$$h = \frac{\sqrt{8}}{\sqrt[4]{3}} = \frac{2\sqrt{2}}{\sqrt[4]{3}}$$
$$P = \frac{2\sqrt{2}}{\sqrt[4]{3}}a + \frac{2\sqrt{2}}{2\sqrt[4]{3}}a + \frac{2\sqrt{2}}{\sqrt[4]{3}}a\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{4\sqrt{2}a+2\sqrt{2}a+2\sqrt{6}a}{2\sqrt[4]{3}} = \frac{6\sqrt{2}a+2\sqrt{6}a}{2\sqrt[4]{3}}$$

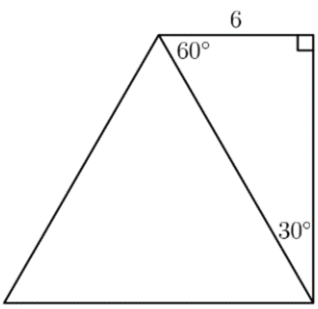
B. Multiple Triangles

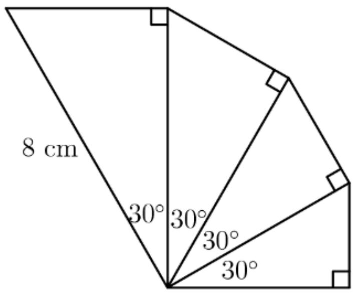
Example 2.34

The isosceles right triangle ABC has right angle at C and area 12.5. The rays trisecting $\angle ACB$ intersect AB at D and E. What is the area of $\triangle CDE$? **(AMC 10A 2015/19)**

Example 2.35: Multiple Triangles







A. ($Left\ Diagram$) What is the value of x in the diagram? **(MathCounts 1994 State Countdown)**
B. ($Right\ Diagram$) Each triangle is a 30-60-90 triangle, and the hypotenuse of one triangle is the longer

leg of an adjacent triangle. The hypotenuse of the largest triangle is 8 centimeters. What is the number of centimeters in the length of the longer leg of the smallest triangle? Express your answer as a common fraction. **(MathCounts 1999 School Target)**

C. $(Middle\ Diagram)$ A 30-60-90 triangle is drawn on the exterior of an equilateral triangle so the hypotenuse of the right triangle is one side of the equilateral triangle. If the shorter leg of the right triangle is 6 units, what is the distance between the two vertices that the triangles do not have in common? Express your answer in simplest radical form. **(MathCounts 2008 National Target)**

Part A

$$Side\ opp\ 60° = \sqrt{3}$$
$$x = \sqrt{6}$$

Part B

All four triangles are $30-60-90$ triangles. The longer leg of the largest triangle is:

$$8 \times \frac{\sqrt{3}}{2}$$

The longer leg of the second-largest triangle is:

$$8 \times \left(\frac{\sqrt{3}}{2}\right)^2$$

The longer leg of the smallest triangle is:

$$8 \times \left(\frac{\sqrt{3}}{2}\right)^4 = 8 \times \frac{9}{4} = \frac{9}{2}$$

Note that this forms a geometric sequence with $a = 8, r = \frac{\sqrt{3}}{2}$:

$$\underbrace{8 \times \frac{\sqrt{3}}{2}}_{\substack{Largest \\ Triangle}}, \underbrace{8 \times \left(\frac{\sqrt{3}}{2}\right)^2}_{\substack{2nd\ Largest \\ Triangle}}, \dots, \underbrace{8 \times \left(\frac{\sqrt{3}}{2}\right)^n}_{\substack{n^{th}\ Largest \\ Triangle}}$$

Part C

$$Hyp = Side\ of\ equilateral\ triangle = 12$$
$$Longer\ Leg = 6\sqrt{3}$$

By Pythagoras:

$$Distance = \sqrt{12^2 + \left(6\sqrt{3}\right)^2} = 6\sqrt{7}$$

2.36: Similar Right Triangles

The two triangles formed by dropping a perpendicular from the vertex of a right-angled triangle are similar to each other, and to the original triangle.

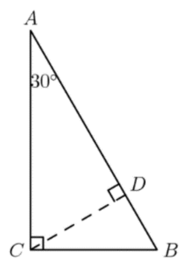
This is a very important property. Consider ΔABC which is $30-60-90$. Drop a perpendicular from C to intersect AB at D.

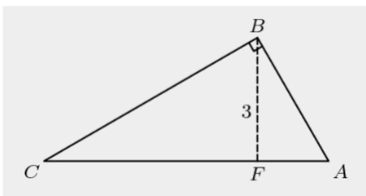
$$\angle ACD = 180 - 90 - 30 = 60 \Rightarrow \Delta ACD\ is\ 30-60-90$$
$$\angle DCB = 180 - 90 - 60 = 30 \Rightarrow \Delta DCB\ is\ 30-60-90$$

Example 2.37

A. $(Left\ Diagram)$ If altitude CD is $\sqrt{3}$ centimeters, what is the number of square centimeters in the area of ΔABC? **(MathCounts 1999 School Team)**

B. $(Right\ Diagram)$The altitude to the hypotenuse of a triangle with angles of 30 and 60 degrees is 3 units. What is the area of the triangle, in square units? Express your answer in simplest radical form. **(MathCounts 2009 State Countdown)**





Part A

$$AC = 2 \times CD = 2\sqrt{3}$$
$$BC = (2\sqrt{3})\left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{2}\right) = 2$$

Part B

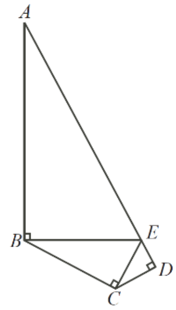
$$A = \frac{1}{2}hb = \frac{1}{2}(2\sqrt{3})(2) = 2\sqrt{3}$$

$6\sqrt{3}$

Example 2.38

In the diagram, ΔABE, ΔBCE and ΔCDE are right-angled, with $\angle AEB = \angle BEC = \angle CED = 60$, and $AE = 24$. (CEMC 2006 Hypatia)

A. Find the length of CE.
B. Determine the perimeter of quadrilateral $ABCD$.
C. Determine the area of quadrilateral $ABCD$.



Part A

$$BE = 24 \times \frac{1}{2} = 12$$
$$CE = 12 \times \frac{1}{2} = 6$$

Part B

$$AB = AE \times \frac{\sqrt{3}}{2} = 24 \times \frac{\sqrt{3}}{2} = 12\sqrt{3}$$
$$BC = BE \times \frac{\sqrt{3}}{2} = 12 \times \frac{\sqrt{3}}{2} = 6\sqrt{3}$$
$$CD = CE \times \frac{\sqrt{3}}{2} = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$$
$$DE = CE \times \frac{1}{2} = 6 \times 2 = 3$$
$$P = \underbrace{12\sqrt{3}}_{AB} + \underbrace{6\sqrt{3}}_{BC} + \underbrace{3\sqrt{3}}_{CD} + \underbrace{3}_{DE} + \underbrace{24}_{AE} = 27 + 21\sqrt{3}$$

Part C

Area of quadrilateral ABCD:

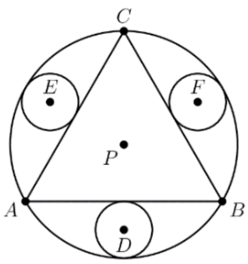
$$= \frac{1}{2}\left[(12)(12\sqrt{3}) + (6)(6\sqrt{3}) + (3)(3\sqrt{3})\right]$$

Factor 9 from each term:

$$= \frac{9}{2}\left[16\sqrt{3} + 4\sqrt{3} + \sqrt{3}\right] = \frac{9}{2}\left[21\sqrt{3}\right] = \frac{189\sqrt{3}}{2}$$

Example 2.39

Equilateral triangle ABC has side length 6 cm and is inscribed in circle P. Congruent smaller circles centered at D,E and F are inscribed in the three regions between an arc of circle P and a side of ΔABC, as shown. If segments AF, BE and CD all intersect P what is the area of ΔDEF? Express your answer as a common fraction in simplest radical form. (MathCounts 2022 National Sprint)



Draw $PM \perp AB$

$$MB = \frac{AB}{2} = \frac{6}{2} = 3$$

In $30 - 60 - 90$ ΔPMB

$$PB = 3 \times \frac{2}{\sqrt{3}} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$
$$PM = \frac{PB}{2} = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

$$MD = \frac{2\sqrt{3} - PM}{2} = \frac{2\sqrt{3} - \sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

$$PD = PM + MD = \sqrt{3} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

Draw $PZ \perp DF$

$$DZ = \frac{3\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} = \frac{9}{4}$$
$$DF = \frac{9}{2}$$
$$[DEF] = \frac{\sqrt{3}}{4} \times \left(\frac{9}{2}\right)^2 = \frac{81\sqrt{3}}{16}$$

Alternatively:
The ratio of lengths in ΔCAB and ΔDEF is:

$$PB{:}\,PD = \frac{3\sqrt{3}}{2} : 2\sqrt{3} = \frac{3}{2} : 2 = 3:4$$

The ratio of areas of ΔCAB and ΔDEF

$$= 3^2 : 4^2 = 9:16$$

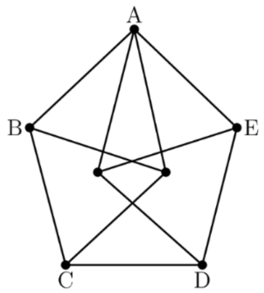
The area of ΔDEF

$$= [CAB] \times \frac{9}{16} = \frac{\sqrt{3}}{4} \times 6^2 \times \frac{9}{16} = \frac{81\sqrt{3}}{16}$$

C. Pentagons

Example 2.40

The figure alongside is an example of a Moser spindle. It is constructed from 11 congruent line segments. Is $ABCDE$ a regular pentagon?

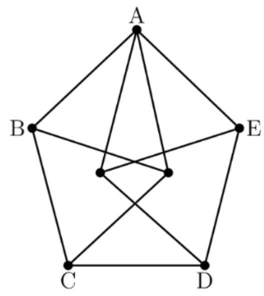


A regular pentagon is equilateral (*equal sides*) and equiangular (*equal angles*). The question has told us:

$$Equal\ sides \Rightarrow Equilateral\ Pentagon$$

This does not mean that the pentagon is *equiangular*. In fact, the pentagon (as we will see next)

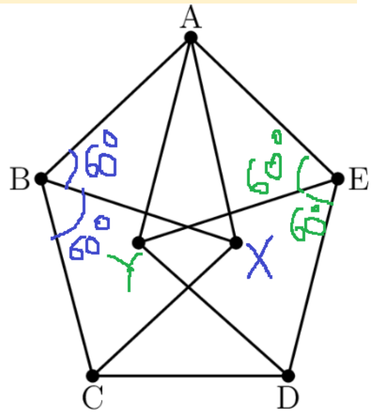
$$\textit{is not equiangular} \Rightarrow \textit{not equilateral}$$



Example 2.41

The figure is constructed from 11 line segments, each of which has length 2. The area of pentagon $ABCDE$ can be written as $\sqrt{m}+\sqrt{n}$, where m and n are positive integers. What is $m+n$? **(AMC 2021 10B/20; 12B/15)**

Step I: Figure out the angles



Name the vertices shown in the diagram as X and Y. Since all the segments are equal:

$$\Delta ABX \ \&\ \Delta CBX \textit{ are equilateral} \Rightarrow \angle ABX = \angle CBX = 60°$$

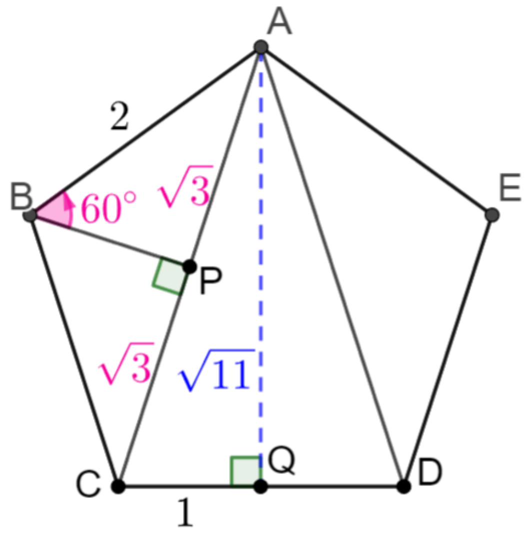
Hence:

$$\angle ABC = \angle ABX + \angle AEY = 120°$$

Step II: Use the angles

Redraw the pentagon. Abstract out the inner lines. Instead draw diagonals

$$AC \textit{ and } AD$$



Draw a perpendicular from vertex B to diagonal AC:

$$\angle ABP = \angle\frac{ABC}{2} = \frac{120°}{2} = 60° \Rightarrow \Delta ABP \textit{ is } 30-60-90$$

Hence:

$$BP = AB\cdot\frac{1}{2} = 2\cdot\frac{1}{2} = 1$$
$$AP = AB\cdot\frac{\sqrt{3}}{2} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$$
$$AC = 2\cdot AC = 2\sqrt{3}$$

Draw $AQ \perp CD$ in isosceles ΔACD

$$CQ = QD = \frac{2}{2} = 1$$

By Pythagorean Theorem in right ΔACQ:

$$AQ = \sqrt{AC^2 - CQ^2} = \sqrt{\left(2\sqrt{3}\right)^2 - 1^2} = \sqrt{12-1} = \sqrt{11}$$

Step III: Calculate the areas

$$[ABC] = \frac{1}{2}(BP)(AC) = \frac{1}{2}\cdot 1\cdot 2\sqrt{3} = \sqrt{3}$$
$$[ACD] = \frac{1}{2}(AQ)(CD) = \frac{1}{2}\cdot\sqrt{11}\cdot 2 = \sqrt{11}$$

Finally, the area of the pentagon is:

$$2\cdot[ABC] + [ACD] = 2\sqrt{3}+\sqrt{11} = \sqrt{12}+\sqrt{11} = \sqrt{m}+\sqrt{n}$$
$$m+n = 12+11 = 23$$

D. Review and Challenge

Concept Check 2.42

Question: In an isosceles right-angled triangle, the leg is $(\sqrt[4]{2}-\sqrt[3]{2})$. Find the hypotenuse.

$$Solution: (\sqrt{2})(\sqrt[4]{2}-\sqrt[3]{2}) = \left(2^{\frac{1}{2}}\right)\left(2^{\frac{1}{4}}-2^{\frac{1}{3}}\right) = 2^{\frac{1}{2}+\frac{1}{4}} - 2^{\frac{1}{2}+\frac{1}{3}} = 2^{\frac{3}{4}} - 2^{\frac{5}{6}}$$

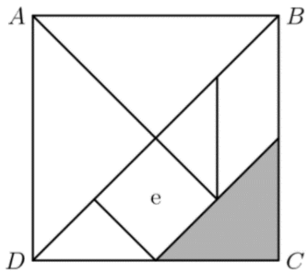
Analyze.

$$\sqrt[4]{2} < \sqrt[3]{2} \Rightarrow \sqrt[4]{2}-\sqrt[3]{2} < 0$$

Length cannot be negative.
Hence, the question is not meaningful.

Example 2.43

Quadrilateral $ABCD$ is a square with area 16 square inches. The figure represents the pieces of a Chinese tangram in which all the triangles are isosceles and piece e is a square. What is the area of the gray piece, in square inches? (MathCounts 2008 Chapter Countdown)



$$A = 16 \Rightarrow s = 4$$
$$A = \frac{1}{2}s^2 = \frac{1}{2}\times 2\times 2 = 2$$

2.3 Further Topics

44 Examples