STATISTICS

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1. Statistics

1.1 Charts and Tables

A. Continuous and Discrete Variables

1.1: Continuous and Discrete Variables

- A variable which can take all real numbers within a certain range is called a continuous variable.
- A variable which can take only specific values is called a discrete variable.

Example 1.2

Classify the following variables as discrete or continuous:

A. Height of a child
B. Weight of a child
C. No. of siblings of a child
D. Distance travelled by a ball which has been thrown
E. No. of doctors in a city

Height of a child – Continuous
Weight of a child – Continuous
No. of siblings of a child – Discrete
Distance travelled by a ball which has been thrown – Continuous
No. of doctors in a city – Discrete

B. Frequency Tables

1.3: Range

The difference between maximum and minimum value is called the range.

Example 1.4

1.5: Frequency

The number of times a data point occurs is called the frequency of the data point.

1.6: Cumulative Frequency

The frequency for data points less than or equal to the value is called the cumulative frequency.

Example 1.7

A doctor saw the following number of patients in his office for the first ten days of August.

8,10,7,8,10,9,7,10,5,10

A. Classify the data in an ungrouped frequency table
B. Find the cumulative frequency.
C. Find the range

Data	5	7	8	9	10
Frequency	1	2	2	1	4

Cumulative Frequency	1	3	5	6	10

$$Range = 10 - 5 = 5$$

Example 1.8

Number	1	2	3	4	5	6	7
Frequency	5	7	2	9	0	5	3
Cumulative Frequency	5	12	14	23	23	28	31

1.9: Class Intervals

Class intervals divide the data into intervals that make it easier to classify and observe patterns.

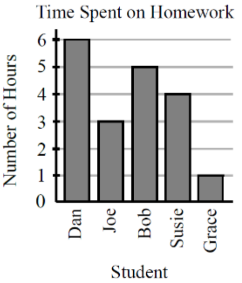
Example 1.10

A doctor saw the following number of patients in his office for the first ten days of August. Classify the data in a a grouped frequency table with intervals $1-5, 6-10$:

$$8,10,7,8,10,9,7,10,5,10$$

Class Interval	1-5	6-10
Frequency	1	9
Cumulative Frequency	1	10

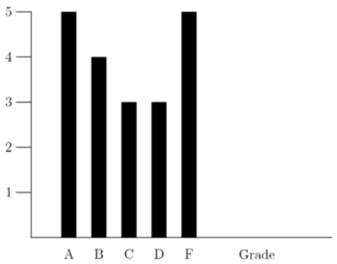
C. Bar Graphs



Example 1.11

The number of hours spent by five students on homework is shown on the graph. Which two students, adding their individual times together, spent the same amount of time on homework as Dan? (CEMC Gauss 2020/)

Dan=6
Bob+Grace=5+1=6

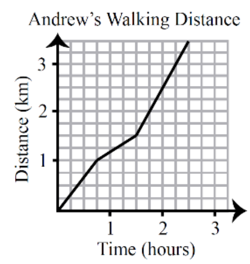


Example 1.12

The bar graph shows the grades in a mathematics class for the last grading period. If A, B, C, and D are satisfactory grades, what fraction of the grades shown in the graph are satisfactory? (AMC 8 1985/5)

$$\frac{5+4+3+3}{5+4+3+3+5} = \frac{15}{20} = \frac{3}{4}$$

Example 1.13
The line graph shows the distance that Andrew walked over time. How long did it take Andrew to walk the first 2 km? (CEMC Gauss 7 2020/8)



$1\ hour\ 45\ Minutes$

Example 1.14
The table below displays the grade distribution of the 30 students in a mathematics class on the last two tests. For example, exactly one student received a 'D' on Test 1 and a 'C' on Test 2 (see circled entry). What percent of the students received the same grade on both tests? (AMC 8 1986/12)

Example 1.15
A bar graph shows the number of hamburgers sold by a fast food chain each season. However, the bar indicating the number sold during the winter is covered by a smudge. If exactly 25% of the chain's hamburgers are sold in the fall, how many million hamburgers are sold in the winter? (AMC 8 1986/16)

Example 1.16
Assume the adjoining chart shows the 1980 U.S. population, in millions, for each region by ethnic group. To the nearest percent, what percent of the U.S. Black population lived in the South? (AMC 8 1986/23)

Example 1.17
The graph below shows the total accumulated dollars (in millions) spent by the Surf City government during 1988. For example, about .5 million had been spent by the beginning of February and approximately 2 million by the end of April. Approximately how many millions of dollars were spent during the summer months of June, July, and August? (AMC 8 1989/19)

Example 1.18
The grading scale shown is used at Jones Junior High. The fifteen scores in Mr. Freeman's class were:
89,72,54,97,77,92, 85,74,75,63,84,78,71,80,90.
In Mr. Freeman's class, what percent of the students received a grade of C? (AMC 8 1990/9)

Example 1.19
The graph relates the distance traveled [in miles] to the time elapsed [in hours] on a trip taken by an experimental airplane. During which hour was the average speed of this airplane the largest? (AMC 8 1990/23)

Example 1.20
The vertical axis indicates the number of employees, but the scale was accidentally omitted from this graph. What percent of the employees at the Gauss company have worked there for 5 years or more? (AMC 8 1991/18)

Example 1.21
The population of a small town is 480. The graph indicates the number of females and males in the town, but the vertical scale-values are omitted. How many males live in the town? (AMC 8 1992/9)

Example 1.22
The bar graph shows the results of a survey on color preferences. What percent preferred blue? (AMC 8 1992/11)

Example 1.23
Northside's Drum and Bugle Corps raised money for a trip. The drummers and bugle players kept separate sales

records. According to the double bar graph, in what month did one group's sales exceed the other's by the greatest percent? (AMC 8 1992/21)

Example 1.24
Which one of the following bar graphs could represent the data from the circle graph? (AMC 8 1993/5)

Example 1.25
This line graph represents the price of a trading card during the first 6 months of 1993. (AMC 8 1993/10)

Example 1.26
The pie charts below indicate the percent of students who prefer golf, bowling, or tennis at East Junior High School and West Middle School. The total number of students at East is 2000 and at West, 2500. In the two schools combined, the percent of students who prefer tennis is (AMC 8 1996/19)

Example 1.27
In 1960 only 5% of the working adults in Carlin City worked at home. By 1970 the "at-home" work force increased to 8%. In 1980 there were approximately 15% working at home, and in 1990 there were 30%. The graph that best illustrates this is (AMC 8 2000/4)

1.2 Mean, Median and Mode Basics

A. Mean of Two Numbers

1.28: Arithmetic Mean
Arithmetic Mean of two numbers is the sum of the numbers divided by 2.

$$\text{Mean}(a, b) = \frac{a+b}{2}$$

Example 1.29
Calculate the arithmetic mean:

A. 8,22
B. 7,14
C. $\frac{1}{2}$ and $\frac{1}{3}$

Part A
If there are two numbers, you add the numbers and divide by two:

$$\frac{8+22}{2} = \frac{30}{2} = 15$$

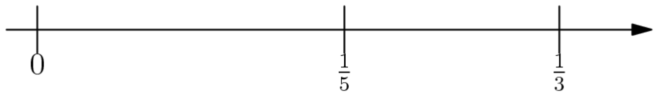
Part B

$$\frac{7+14}{2} = \frac{21}{2} = 10.5$$

Part C

$$\frac{\frac{1}{2}+\frac{1}{3}}{2} = \frac{5}{6} \div 2 = \frac{5}{6} \times \frac{1}{2} = \frac{5}{12}$$

Example 1.30



A. What is the number that lies halfway between 7 and 10 on the number line?
B. Find the number halfway between $\frac{1}{3}$ and $\frac{1}{4}$ on the number line.
C. The fraction halfway between $\frac{1}{5}$ and $\frac{1}{3}$ (on the number line) is **(AMC 8 1985/10)**

Part A

The number that lies halfway between the two numbers is their average

$$\frac{7+10}{2}=\frac{17}{2}=8\frac{1}{2}$$

Part B

The number halfway between $\frac{1}{3}$ and $\frac{1}{4}$ is the average of the two numbers

$$=\frac{\frac{1}{3}+\frac{1}{4}}{2}=\frac{7}{12}\div 2=\frac{7}{12}\times\frac{1}{2}=\frac{7}{24}$$

Part A

$$Avg\left(\frac{1}{5},\frac{1}{3}\right)=\frac{\frac{1}{5}+\frac{1}{3}}{2}=\frac{\frac{3+5}{15}}{2}=\frac{\frac{8}{15}}{2}=\frac{8}{15}\div 2=\frac{8}{15}\times\frac{1}{2}=\frac{4}{15}$$

Example 1.31

Molly has five blue pens, and six black pens. Her sister has three blue pens and seven black pens.

A. What is the average number of blue pens that the two sisters have? What is the average number of black pens that the two sisters have?
B. Is the fact that your answer in Part B is not a whole number an issue? Interpret your answer to parts A and B in terms of how you can share the pens between Molly and her sister.

Part A

$$Average\ no.of\ blue\ pens=\frac{5+3}{2}=\frac{8}{2}=4$$
$$Average\ no.of\ black\ pens=\frac{6+7}{2}=6.5$$

Part B

From the calculation of average point of view, an answer which is not a whole number is not an issue. However, if you wish to distribute the pens equally among Molly and her sister, then an equal division is not possible if you do not have a whole number answer.

1.32: Possible Values of Arithmetic Mean

The arithmetic mean of two numbers need not be a whole number.

Example 1.33

Mark the correct option

Monica counted the number of students she taught for two days. The average number of students taught was 3.5. Then, the average:

A. calculated is fine because you can teach half a student.
B. calculated is not fine, because you cannot teach half a student.
C. calculated needs to be interpreted correctly.
D. needs to be calculated again to correct it.

Option A
You cannot teach half a student, so option A is not correct.

Option B
You cannot teach half a student, but you can teach an odd number of students, making the average not be a whole number.
Hence, option B is not correct.

Option C
The calculation of average can result in 3.5 students. For example, one possibility is:

$$\frac{2+5}{2} = \frac{7}{2} = 3.5$$

Hence, we need to interpret the average correctly.

$$Option\ C\ is\ correct.$$

Answer the above question again if the average number of students taught was 4.25.

If Monica teaches 8 students in 2 days,

$$Average = \frac{8}{2} = 4$$

If she teaches 9 students in 2 days

$$Average = \frac{9}{2} = 4.5$$

Hence, an average of 4.5 students in 2 days is not possible. The calculations need to be checked.

$$Option\ D\ is\ correct.$$

1.34: Totals

$$Avg = \frac{a+b}{2} \Rightarrow a + b = 2(Avg)$$

Example 1.35
The average age of two students in a class is seven years. What is the sum of their ages?

Let the ages of the two children be x and y,

$$\frac{x+y}{2} = 7 \Rightarrow x + y = 14 \Rightarrow Sum\ of\ Ages = 14$$

Example 1.36

$$Mark\ the\ correct\ option$$

The average number of questions I did each day for two days was 27.5. Which of the following is not a possible value for my accuracy (percent of correct questions, rounded to two decimal places)?

A. 0%
B. 7%
C. 9.09%
D. 20%
E. 100%

The total number of questions I attempted would be

$$27.5 \times 2 = 55$$

$$\frac{11}{55}=\frac{1}{5}=20\%$$
$$\frac{5}{55}=\frac{1}{11}=0.\overline{09}=9.\overline{09}\%$$

$$Option\ B\ is\ not\ possible$$
$$\frac{4}{55}=7.\overline{27}\%$$

Example 1.37

A doctor met an average of 5 patients per day over the weekend – Saturday and Sunday. He charges Rs. 200 for meeting patients on Saturday, and Rs. 250 meeting them on a Sunday. He knows that he met at least one patient on each day. What is the difference between the minimum and the maximum amount he could have billed over the weekend?

Since the average is 5 patients per day, the total is 10 patients over the weekend.
The doctor met at least one patient each on Saturday, and Sunday, leaving 8 patients.

If he saw all 8 patients on Sunday, he would make
$$8\times 50=400\ more$$

Example 1.38

A. $(Puzzle-Type)$ Last year my brother and my sister had an average age of exactly 6 years. What is the maximum average age (rounded) they could have next year?
B. On 1st Jan of the year 2000, my two sisters had an average age of 12 years. On 1st Jan 2006, what will be the sum of their ages?
C. $(Puzzle-Type)$ On 1st Jan, 2000, my siblings were each an integer number of years old, and their average age was $5\ years$. Exactly two years later, my siblings were still each an integer number of years old, and their average age increased by $\frac{1}{3}$ years. Find the age of my youngest sibling on 1st Jan 2002.

Part A

Suppose today is $1st\ Jan, 2000$.

$1st\ Jan, 1999$	$1st\ Jan, 2000$	$1st\ Jan, 2001$	$31st\ Dec, 2001$
$Last\ Year$	Now	$Next\ Year$	$Next\ Year$
6	7	8	$8.99\approx 9$

$$Max\ Average\ Age=9\ Years$$

Part B

$$Average\ Age=12\Rightarrow Total=24\Rightarrow 24+12=\underbrace{36}_{Total}$$
$$24=12+12\Rightarrow \underbrace{36}_{Total}=18+18$$

Part C

$$Average\ Age=5\frac{1}{3}=\frac{16}{3}\Rightarrow No.of\ Siblings=3$$
$$On\ 1st\ Jan\ 2000\Rightarrow No.of\ Siblings=2\Rightarrow Total\ age=10$$
$$On\ 1st\ Jan\ 2002\Rightarrow Two\ Siblings\ will\ have\ total\ age=14$$
$$3rd\ Sibling\ will\ have\ age=16-14=2\ Years$$

Example 1.39

In July 1861, 366 inches of rain fell in Cherrapunji, India. What was the average rainfall in inches per hour during that month? (AMC 8 1986/1)

(A) $\frac{366}{31\times 24}$ (B) $\frac{366\times 31}{24}$ (C) $\frac{366\times 24}{31}$ (D) $\frac{31\times 24}{366}$ (E) $366\times 31\times 24$

$$\textit{Option A}$$

B. Mean of More than Two Numbers

1.40: Average of more than two numbers

The average of a set of n numbers is obtained by dividing the total of the numbers by the number of numbers.

$$\text{Average} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Example 1.41

A. Find the average of 7, 12, 15
B. Find the average of $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$.
C. The number of diners visiting a restaurant for breakfast (8:00 am-10:00 am), lunch (12:00-3:00 pm) and dinner (7:00 pm-11:00 pm) is 20, 15 and 35, respectively. What is the average number of diners visiting the restaurant in a single time period?
D. Ryan loves toy cars. He has seven yellow cars, twelve blue cars, ten green cars and one white car. What is the average number of cars that he has of each color?
E. Pari has 7 books on science fiction, 4 books on fantasy, 3 books on history on her bookshelf. What is the average number of books that she has of each genre?

Part A

Add the three numbers, and divide by three

$$\frac{7+12+15}{3} = \frac{34}{3}$$

Part B

$$\frac{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{3} = \frac{13}{12} \div 3 = \frac{13}{12} \times \frac{1}{3} = \frac{13}{36}$$

Part C

$$\frac{20+15+35}{3} = \frac{70}{3}$$

Part D

$$\frac{7+12+10+1}{4} = \frac{30}{4} = \frac{15}{2}$$

Part E

$$\frac{7+4+3}{3} = \frac{14}{3}$$

C. Median

1.42: Median

When numbers are arranged in either ascending order, or descending order, the median is the middlemost value.

- If there is an odd number of values in the data set, there will be one middlemost value.

- If there is an even number of values in the data set, then there will be two middle values, and the median is the arithmetic mean of those values.

Example 1.43
Find the median for the following:

A. 9,12,7,10,4
B. 10,4,7,9,12,15

The numbers are already arranged in ascending order. We just find the middle number:

$$4,7,\underbrace{9}_{\substack{Middle\\Number}},10,12 \Rightarrow Median = 9$$

$$4,7,\underbrace{9}_{\substack{First\ Middle\\Number}},\underbrace{10}_{\substack{Second\ Middle\\Number}},12,15 \Rightarrow Median = Avg(9,10) = \frac{9+10}{2} = \frac{19}{2} = 9.5$$

Example 1.44
A. The median of the three integers $3, 12, x$ is x. Find the number of possible values that x can take.
B. The median of the distinct positive integers $5, x, y$ is x. Find the number of possible values that x can take.

Part A

$$\{3,4,\ldots,12\} \Rightarrow 12 - 3 + 1 = 10$$

Part B

Minimum value of y is 1:

$$\{5,x,y\} \Rightarrow With\ minimum\ value\ of\ y = \{1,x,5\}$$
$$No.of\ possible\ values = n\{2,3,4\} = 3\ Values$$

Example 1.45
Find the position of the middlemost values if the data sets below are arranged in ascending order.

A. A data set with 100 data points.
B. A data set with 25 data points.
C. A data set with n data points.

Part A

The two middlemost values will be the ones in the

$$Avg(x_{50}, x_{51})$$

Part B

$$\frac{25+1}{2} = \frac{26}{2} \to x_{13}$$

Part C

If n is even:

$$Avg\left(x_{\frac{n}{2}}, x_{\frac{n}{2}+1}\right)$$

If n is odd:

$$x_{\frac{n+1}{2}}$$

1.46: Median of Two Numbers

Median of two numbers is equal to the average of the two numbers.

Consider the list of numbers:

$$a, b$$

Since there are an even number of numbers, the median is the average of the two middle numbers

$$= Avg(a, b) = \frac{a+b}{2}$$

D. Mode

1.47: Mode

In a data set, the mode is the number that occurs the maximum number of times.

Example 1.48

Consider the data set $\{4,2,7,2,9\}$. What is the mode?

$$2$$

1.49: Mode

Mode does not always mean a number. It can represent a data point which is not a number.

Example 1.50

Kristin recorded where she had her dinner each day of the week:

$$Monday: Joe's\ Diner$$
$$Tuesday: Home$$
$$Wednesday: Jane's\ Fine\ Dine$$
$$Thursday: Home$$
$$Friday: Home$$

What is the modal location where she had dinner?

1.51: More than One Mode

If two numbers have maximum frequency, then both numbers are the mode.

- If there is more than one mode in the data set, it is called bimodal.

Example 1.52

Find the mode in the data set:

$$3,2,5,1,3,2,7,9,8$$

$$2\ occurs\ twice, 3\ occurs\ twice \Rightarrow Mode = \{2,3\}$$

1.53: No Mode

If all numbers have equal frequency, then there is no mode.

Example 1.54

What is the mode in the data set $\{1,2,3,4,5\}$?

$$No\ Mode$$

Example 1.55: Mode from a Table
Consider the marks obtained in different subjects by a student in his school tests out of five marks. Find the modal marks.

Subject	Physics	Chemistry	Biology	Algebra	Geometry
Marks	3	4	3	5	3

$$3$$

Example 1.56
The number of students who got different marks are mentioned in the table alongside. Determine the modal marks.

No. of Students	5	2	4	7	12
Marks	1	2	3	4	5

The number of students who got 5 marks is more than the number of students who got any other value for marks.

$$Mode = 5$$

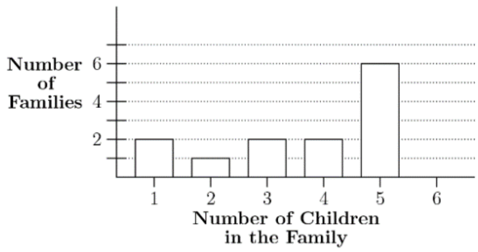
1.57: Mode from a Graph

E. Mean, Median Mode Basics

Example 1.58
The number halfway between 1/6 and 1/4 is (AMC 8 1994/13)

$$\frac{\frac{1}{6}+\frac{1}{4}}{2} = \frac{\frac{4+6}{24}}{2} = \frac{10}{24}\times\frac{1}{2} = \frac{5}{24}$$

Example 1.59
The graph shows the distribution of the number of children in the families of the students in Ms. Jordan's English class. The median number of children in the family for this distribution is (AMC 8 1995/19)



No. of Children		1	2	3	4	5	6
No. of Families	Frequency	2	1	2	2	6	0
	Cumulative Frequency	2	3	5	7	13	13

If you have 13 values, the median will be the middle value, which is:

$$\frac{13+1}{2} = \frac{14}{2} = 7th\ Value = 4$$

$$1,1,2,3,3,4,\underbrace{4}_{Median},5,5,5,5,5,5$$

In the above question, if we add a family with six children, what is the new median?

The number of families
$$= 13 + 1 = 14$$

The median of 14 values will be found by taking the average of the 7th and the 8th value:
$$1,1,2,3,3,4, \underbrace{4}_{7th\ Value}, \underbrace{5}_{8th\ Value}, 5,5,5,5,5$$
$$Median = Avg(4,5) = \frac{4+5}{2} = \frac{9}{2} = 4.5$$

Example 1.60

Alexis took a total of 243,000 steps during the 30 days in the month of April. What was her mean (average) number of steps per day in April? (CEMC Gauss Grade 7 2020/5)

$$\frac{243{,}000}{30} = 8100\ Steps$$

Example 1.61

The Gauss family has three boys aged 7, a girl aged 14, and a boy aged 15. What is the mean (average) of the ages of the children? (CEMC Gauss, Grade 8/2007/8)

$$\frac{21+14+15}{5} = \frac{50}{5} = 10$$

Example 1.62

The average (mean) of the numbers 6, 8, 9, 11, and 16 is (CEMC Gauss, Grade 7/2005/4)

$$\frac{6+8+9+11+16}{5} = \frac{50}{5} = 10$$

Example 1.63

The mean (average) of the numbers 12, 14, 16, and 18, is (CEMC Gauss, Grade 8/ 2003/11)

$$\frac{12+14+16+18}{4} = \frac{(15-3)+(15-1)+(15+1)+(15+3)}{4} = 15$$

Example 1.64

The average of 10, 4, 8, 7, and 6 is (CEMC Gauss, Grade 7/1999/8)

$$\frac{10+4+8+7+6}{5} = \frac{35}{5} = 7$$

Example 1.65

The average of – 5, – 2, 0, 4, and 8 is (CEMC Gauss, Grade 8/1999/6)

$$\frac{-5-2+0+4+8}{5} = \frac{5}{5} = 1$$

Example 1.66

The Severn Bridge has carried just over 300 million vehicles since it was opened in 1966. On average, roughly how many vehicles is this per day? **(UKMT JMC 2010/14)**

A. 600
B. 2000
C. 6000
D. 20000
E. 60000

Note: This question was asked in 2010, so answer accordingly.

No. of Years from 1966 to 2010

$$= 2010 - 1966 = 44$$

Average vehicles per day

$$= \frac{300{,}000{,}000}{44 \times 365} \approx \frac{300{,}000{,}000}{50 \times 400} = \frac{300{,}000}{20} = 15{,}000$$

Note that we underestimated the actual number, and hence, answer must be greater than 15000.

$$Option\ D$$

Example 1.67

What is the mean of 2/3 and 4/9? **(UKMT JMC 2011/13)**

$$\frac{1}{2}\left(\frac{6}{9} + \frac{4}{9}\right) = \frac{1}{2}\left(\frac{10}{9}\right) = \frac{5}{9}$$

F. Median

Example 1.68

The point totals that Mark scored in five basketball games were x, 11, 13, y, 12. How many different possible medians are there for his five point totals? **(CEMC Gauss, Grade 8/2020/17)**

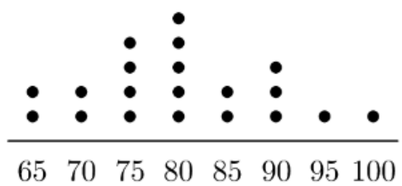
$$x, y, \underbrace{11}_{Median}, 12, 13,$$
$$x, 11, \underbrace{12}_{Median}, 13, y$$
$$11, 12, \underbrace{13}_{Median}, x, y$$

The possible values for the median are:

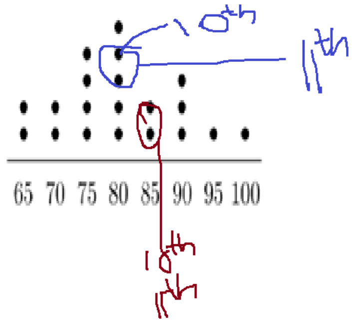
$$\{11, 12, 13\} \Rightarrow 3\ values$$

Example 1.69

Mr. Ramos gave a test to his class of 20 students. The dot plot below shows the distribution of test scores. Later Mr. Ramos discovered that there was a scoring error on one of the questions. He regraded the tests, awarding some of the students 5 extra points, which increased the median test score to 85. What is the minimum number of students who received extra points? (Note that the median test score equals the average of the 2 scores in the middle if the 20 test scores are arranged in increasing order.) **(AMC 8 2022/19)**



$$\frac{20}{2} = 10$$



The median in the dot plot given in the question is:

$$Avg\ of\ 10th\ Value\ and\ 11th\ Value = 80$$

There are two data points with a value of 85. They are currently the $14th$ and $15th$ data points.
For them to become the 10th and 11th data points, 4 scores must increase.

$$Answer = 4$$

G. Adding, Removing and Changing Data Points

Example 1.70

The number of pens that students in a class have is given below:

2,0,3,1,2,1

Find the new median if:

A. A new student with 4 pens joins the class
B. A new student with 0 pens joins the class

0,1,1,2,2,3

Example 1.71

The number of pens that students in a class have is given below:

2,2,0,3,1,2,1

Find the new median if:

A. A new student with 4 pens joins the class
B. A new student with 0 pens joins the class

0,1,1,2,2,2,3

H. Mean, Median Mode from Tables and Charts

1 Pending Graphs

Example 1.72

Which one of the sectors in the pie chart represents the mode? (UKMT JMC 2005/11)

Example 1.73

The graph shows the total distance that each of five runners ran during a one-hour training session. Which runner ran the median distance? (CEMC Gauss, Grade 8/2015/4)

Example 1.74

What is the median quiz score of the 25 scores shown on the bar graph? (CEMC Gauss, Grade 7/2004/11)

Example 1.75

Based on the graph shown, what is the mode for the amount of rainfall for the week? (CEMC Gauss, Grade 7/2019/5)

Example 1.76
Based on the graph shown, how many vehicles had an average speed of at least 80 km/h? (CEMC Gauss, Grade 8/2019/6)

Example 1.77
The graph shows points scored by Riley-Ann in her first five basketball games. The difference between the mean and the median of the number of points that she scored is (CEMC Gauss, Grade 7/2014/16)

Example 1.78
The graph shows the number of female students in five Grade 8 classes labelled 8A through 8E. The average (mean) number of female students in these five classes is

Example 1.79
Ben recorded the number of visits to his website from Monday to Friday as shown in the bar graph. The mean (average) number of visits per day to his website over the 5 days is
(A) less than 100
(B) between 100 and 200
(C) between 200 and 300
(D) between 300 and 400
(E) more than 400 (CEMC Gauss, Grade 7/2012/9)

Example 1.80
Mr. Garcia asked the members of his health class how many days last week they exercised for at least 30 minutes. The results are summarized in the following bar graph, where the heights of the bars represent the number of students. What was the mean number of days of exercise last week, rounded to the nearest hundredth, reported by the students in Mr. Garcia's class? (AMC 8 2018/8)

Example 1.81
The following bar graph represents the length (in letters) of the names of 19 people. What is the median length of these names? (AMC 8 2016/6)

Example 1.82
Consider this histogram of the scores for 81 students taking a test:
The median is in the interval labeled (AMC 8 1993/11)

Example 1.83
Jean writes five tests and achieves the marks shown on the graph. What is her average mark on these five tests? (CEMC Gauss, Grade 7/1998/4)

1.3 Mean

A. Basic Properties

1.84: Average of the same number
The average of two numbers which are the same is equal to the original numbers.

$$Avg(a,a) = \frac{a+a}{2} = \frac{2a}{2} = a$$

1.85: Average between Two Values

The average of two numbers x and y lies exactly between them. We can find the average by dividing the difference between the two numbers by 2, and adding it to the smaller number.

$$Avg(x,y) = x + \frac{y-x}{2}$$

- We can show that this expression is equivalent to our original definition of the average.

$$Avg(x,y) = x + \frac{y-x}{2} = \frac{2x+y-x}{2} = \frac{x+y}{2}$$

- For calculations, this formula is sometimes faster than the other one.

Example 1.86

A. The number of diners at a diner were 62 for lunch, and 72 for dinner. Find the average number of diners at the diner.
B. Find the average of 1005 and 1007.
C. The number of passengers at a railway station over a two-hour time frame were counted to be 2874. After some corrections, the count was revised to 2878. Find the average of the two values.

Part A

$$62 + \frac{72-62}{2} = 62 + \frac{10}{2} = 62 + 5 = 67$$

Part B

$$1005 + \frac{1007-1005}{2} = 1005 + 1 = 1006$$

Part C

$$2874 + \frac{2878-2874}{2} = 2874 + \frac{4}{2} = 2876$$

Example 1.87

A. What is the average of 8.4 and 9.6.
B. Find the average of 260.5 and 1290.25.

$$Avg\ (8.4, 9.6) = 8.4 + \frac{9.6 - 8.4}{2} = 9$$

$$1260.5 + \frac{90.25 - 60.25}{2} = 1260.5 + \frac{29.75}{2} = 1260.5 + 14.875 = 1275.375$$

1.88: Symmetric Values

For symmetric values in ascending or descending order:

$$Average = middle\ value$$

For example, the average of 51, 53, 55 is:

$$\frac{51+53+55}{3} = \frac{(53-2)+53+(53+2)}{3} = \frac{3 \times 53}{53} = 53$$

Example 1.89

For each part, calculate the average of the numbers. (Answer each part separately).

A. 75, 85, 65
B. 20, 35, 40, 45, 60

Values are symmetric on re-arranging.
Avg = middle value = 75
Avg = middle value = 40

1.90: Change of Origin

The average of $x_1, x_2, \dots, x_n$ is

$$y + Avg(x_1 - y, x_2 - y, \dots, x_n - y)$$

Use change of origin for multiple large values

Example 1.91

A. 5480, 5483, 5489
B. Calculate the average weight of a class when the weights of the students in it are 152, 154, and 157 pounds.

Part A

Take y as 5480:

$$5480 + \frac{0+3+9}{3} = 5480 + \frac{12}{3} = 5480 + 4 = 5484$$

Part B

Take y as 152:

$$152 + \frac{0+2+5}{3} = 152 + \frac{7}{3} = 152 + 2\frac{1}{3} = 154\frac{1}{3}$$

1.92: Equalizing

Example 1.93

A. 5480, 5483, 5489

Reduce the last number by 3, and increase the first number by 3:

$$Avg(5480{,}5483{,}5489) = Avg(5483, 5483, 5486)$$

Take 2 from the last number, and give it to the other two:

$$Avg(5483 + 1, 5483 + 1, 5486 - 2) = Avg(5484, 5484, 5484) = 5484$$

Example 1.94

A runner has finished 3 days of his six-day running trip. If he has completed $\frac{3}{7}^{th}$ of the trips total distance of 168 km, how many kilometers per day must he average for the rest of the trip?

$$Pending\ Km = 168\left(1 - \frac{3}{7}\right) = 168 \cdot \frac{4}{7} = 96$$

$$Average\ Km = \frac{96}{3} = 32$$

1.95: Difference of Averages is Average of the Difference

$$\bar{x} = Avg(x_1, x_2, \dots, x_n)$$
$$\bar{y} = Avg(y_1, y_2, \dots, y_n)$$

$$\bar{x} - \bar{y} = \frac{(x_1 - y_1) + (x_2 - y_2) + \dots + (x_n - y_n)}{n}$$

Example 1.96

Blake and Jenny each took four 100-point tests. Blake averaged 78 on the four tests. Jenny scored 10 points higher than Blake on the first test, 10 points lower than him on the second test, and 20 points higher on both the third and fourth tests. What is the difference between Jenny's average and Blake's average on these four tests? **(AMC 8 2003/7)**

Since difference of averages is the average of the difference:

$$\frac{10 - 10 + 20 + 20}{4} = \frac{40}{4} = 10$$

Example 1.97

The mean (average) of the numbers 20, 30, 40 is equal to the mean of the numbers

A. 28, 30, 31
B. 24, 30, 38
C. 22, 30, 39
D. 23, 30, 37
E. 25, 30, 34 **(CEMC Gauss, Grade 7, 2019/10)**

$$\frac{20 + 30 + 40}{3} = \frac{23 - 3 + 30 + 37 + 3}{3} = \frac{23 + 30 + 37}{3} \Rightarrow Option\ D$$

B. Adding a data point

Example 1.98

The mean age of the four members of 'All Sinners' boy band is 19. What is the mean age when an extra member who is 24 years old joins them? **(UKMT JMC 2003/9)**

Direct Calculation

$$Total\ Age\ of\ 4\ Members = 19 \times 4 = 76$$
$$Total\ Age\ of\ 5\ Members = 76 + 24 = 100$$
$$Average\ Age\ of\ 5\ Members = \frac{100}{5} = 20$$

Shortcut Method

We calculate the increase in the average by finding the difference between the new data point and the old average, and then dividing it by the number of data points:

$$\frac{New\ Value - Old\ Avg}{New\ No.of\ Data\ Points} = \frac{24 - 19}{5} = \frac{5}{5} = 1$$

Example 1.99

Mr. Gallop has two stables which each initially housed three ponies. His prize pony, Rein Beau, is worth £250,000. Usually, Rein Beau spends his day in the small stable, but when he wandered across into the large stable, Mr. Gallop was surprised to find that the average value of the ponies in each stable rose by £10,000. What is the total value of all six ponies? **(UKMT JMO 2012/B3)**

$$\frac{p_1+p_2}{2}-\frac{p_1+p_2+250{,}000}{3}=10{,}000$$
$$\frac{3(p_1+p_2)-2(p_1+p_2)-500{,}000}{6}=10{,}000$$
$$p_1+p_2-500{,}000=60{,}000$$
$$p_1+p_2=560{,}000$$

$$\frac{p_4+p_5+p_6+250000}{4}-\frac{p_4+p_5+p_6}{3}=10{,}000$$
$$\frac{3(p_4+p_5+p_6)+250000-4(p_4+p_5+p_6)}{4}=10{,}000$$
$$250000-(p_4+p_5+p_6)=40{,}000$$
$$p_4+p_5+p_6=210{,}000$$

Total
$$560000+250000+210000=1{,}002{,}000$$

C. Finding Missing Data Points

1.100: Finding Missing Data Points

$$x_1=\bar{x}n-(x_2+\cdots+x_n)$$

Consider a data set with n data points $x_1, x_2, \dots, x_n$, and average $\frac{x_1+x_2+\cdots+x_n}{n}$. If we know all the values other one data point we can solve for it:

$$\frac{x_1+x_2+\cdots+x_n}{n}=\bar{x}$$
$$x_1+x_2+\cdots+x_n=n\bar{x}$$
$$x_1=n\bar{x}-(x_2+\cdots+x_n)$$

Example 1.101

If your average score on your first six mathematics tests was 84 and your average score on your first seven mathematics tests was 85, then your score on the seventh test was **(AMC 8 1985/17)**

$$\underbrace{85\times 7}_{\substack{Total\ of\\ 7\ Tests}}-\underbrace{84\times 6}_{\substack{Total\ of\\ 6\ Tests}}=595-504=91$$

1.102: Symmetric Values

For symmetric values in ascending or descending order:

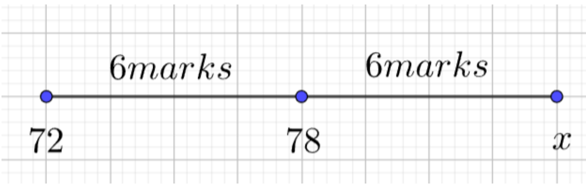
$$Average\ =\ middle\ value$$

Example 1.103

Karen was given a mark of 72 for Mayhematics. Her average mark for Mayhematics and Mathemagics was 78. What was her mark for Mathemagics? **(UKMT JMC 2009/14)**

Algebraic Method
Let the number of marks in Mathemagics be x. Then:

$$\frac{72+x}{2}=78$$
$$72+x=78+78$$
$$x=2+78=84$$



Symmetry
By symmetry, we know that the average is exactly between the two numbers.

$$Mathemagics=78+(78-72)=78+6=84$$

Example 1.104
The six-member squad for the Ladybirds five-a-side team consists of a 2-spot ladybird, a 10-spot, a 14-spot, an 18-spot, a 24-spot and a pine ladybird (on the bench). The average number of spots for members of the squad is 12. How many spots has the pine ladybird? **(UKMT JMC 2007/12)**[1]

$$2+10+14+18+24+p=12\times 6$$
$$68+p=72$$
$$p=4$$

Example 1.105
Andrea has finished the third day of a six-day canoe trip. If she has completed $\frac{3}{7}$ of the trip's total distance of 168 km, how many km per day must she average for the remainder of her trip? **(CEMC Gauss, Grade 7/2008/21, 8/2008/18)**

Remaining distance

$$=168\times\frac{4}{7}$$

Average

$$=\frac{168\times\frac{4}{7}\,km}{3\ days}=168\times\frac{4}{7}\times\frac{1}{3}=8\times 4=32\frac{km}{day}$$

Example 1.106
Vanessa set a school record for most points in a single basketball game when her team scored 48 points. The six other players on her team averaged 3.5 points each. How many points did Vanessa score to set her school record? **(CEMC Gauss, Grade 7/ 2009/17, 8/ 2009/14)**

$$3.5\times 6+V=48$$
$$21+V=48$$
$$V=27$$

Example 1.107
The mean (average) of the four integers 78, 83, 82, and x is 80. Which one of the following statements is true?

A. x is 2 greater than the mean
B. x is 1 less than the mean
C. x is 2 less than the mean

[1] A ladybird is a kind of insect, but we do not need to know this to answer the question.

D. x is 3 less than the mean
E. x is equal to the mean (CEMC Gauss, Grade 7/2017/17)

$$78+83+82+x=80\times 4$$

Rewrite in terms of 80:

$$(80-2)+(80+3)+(80+2)+x=80\times 4$$

Subtract 80×3 from both sides:

$$-2+3+2+x=80$$
$$3+x=80$$
$$x=77$$
$$Option\ D$$

Example 1.108

Theresa's parents have agreed to buy her tickets to see her favorite band if she spends an average of 10 hours per week helping around the house for 6 weeks. For the first 5 weeks she helps around the house for 8, 11, 7, 12 and 10 hours. How many hours must she work for the final week to earn the tickets? (AMC 8 2007/1)

	Week 1	Week 2	Week 3	Week 4	Week 5	Total
	8	11	7	12	10	
Difference from 10	-2	$+1$	-3	$+2$	0	-2

Since she has worked 2 hours less than what was needed over five weeks, she needs to work:

$$10+2=12\ Hours$$

In the last week

Example 1.109

Isabella must take four 100-point tests in her math class. Her goal is to achieve an average grade of 95 on the tests. Her first two test scores were 97 and 91. After seeing her score on the third test, she realized she can still reach her goal. What is the lowest possible score she could have made on the third test? (AMC 8 2012/7)

	Test 1	Test 2	Test 3	Test 4		
	97	91	92	100		95
Difference from 95	$+2$	-4	-3	$+5$		0

Example 1.110

Four students take an exam. Three of their scores are 70, 80, and 90. If the average of their four scores is 70, then what is the remaining score? (AMC 8 2016/3)

	70	80	90	40		70
	0	+10	+20	-30		

Remaining score

$$=70-30=40$$

Example 1.111

Shauna takes five tests, each worth a maximum of 100 points. Her scores on the first three tests are 76, 94, and 87. In order to average 81 for all five tests, what is the lowest score she could earn on one of the other two tests? (AMC 8 2019/7)

Since we want the lowest score on one of the tests, we maximize the score for Test 4 ($=100$), and hence minimize the score for Test 5.

Test 1	Test 2	Test 3	Test 4	Test 5
76	94	87	100	67
-5	$+13$	$+6$	$+19$	-33

D. Working with Totals

Example 1.112

The weight limit for an elevator is 1500 kilograms. The average weight of the people in the elevator is 80 kilograms. If the combined weight of the people is 100 kilograms over the limit, how many people are in the elevator? **(CEMC Gauss, Grade 8/1998/11)**

$$Total\ Weight = 1500 + 100 = 1600$$

$$No.of\ People = \frac{Total\ Weight}{Average\ Weight} = \frac{1600}{80} = 20$$

Example 1.113

37 sacks of flour with an average weight of 26 kg are loaded onto a flatbed with a capacity of one metric ton. If the loader (weighing 62 kg) must climb on the flatbed, for how many sacks will the flatbed exceed its capacity?

Total Weight $= 37 \times 26 + 62 = 1024$
$1024 - 26 = 998 < 1000$
One Sack

Example 1.114

Hannah scored 312 points during the basketball season. If her average (mean) was 13 points per game, how many games did she play? **(CEMC Gauss, Grade 7/2015/8)**

$$No.of\ Games = \frac{Total}{Avg\ per\ Game} = \frac{312}{13} = 24$$

Example 1.115

The average of four different positive whole numbers is 4. If the difference between the largest and smallest of these numbers is as large as possible, what is the average of the other two numbers? **(CEMC Gauss, Grade 7/2007/22)**

The total of the four numbers is

$$4 \times 4 = 16$$

Since we want to make the last number as large as possible, we make the other three numbers as small as possible. Then, the numbers are:

$$1,2,3,x$$

And the average is:

$$\frac{2+3}{2} = \frac{5}{2} = 2.5$$

Example 1.116

The average of the five numbers in a list is 54. The average of the first two numbers is 48. What is the average of the last three numbers? (AMC 8 2004/9)

$$48 + 48 + x_1 + x_2 + x_3 = 54 \times 5$$
$$x_1 + x_2 + x_3 = 270 - 96$$
$$\frac{x_1 + x_2 + x_3}{3} = \frac{174}{3} = 58$$

Example 1.117

The arithmetic mean (average) of four numbers is 85. If the largest of these numbers is 97, then the mean of the remaining three numbers is (AMC 8 1993/15)

$$x_1 + x_2 + x_3 + 97 = 85 \times 4$$

Subtract 97 from both sides, and split the 85×4:

$$x_1 + x_2 + x_3 = 85 \times 3 + 85 - 97$$

Simplify the RHS:

$$RHS = 85 \times 3 - 12 = 85 + 85 + 85 - 4 - 4 - 4 = 81 + 81 + 81 = 81 \times 3$$

Example 1.118

There is a list of seven numbers. The average of the first four numbers is 5, and the average of the last four numbers is 8. If the average of all seven numbers is $6\frac{4}{7}$, then the number common to both sets of four numbers is (AMC 8 2000/23)

Standard Method

Since the average of all seven number is $6\frac{4}{7}$:

$$\underbrace{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 6\frac{4}{7} \times 7 = \frac{46}{7} \times 7 = 46}_{\textit{Equation I}}$$

Average of first four numbers is 5, and the average of the last four numbers is 8:

$$\underbrace{x_1 + x_2 + x_3 + x_4 = 20}_{\textit{Equation II}}, \qquad \underbrace{x_4 + x_5 + x_6 + x_7 = 32}_{\textit{Equation III}}$$

Add Equations II and III

$$\underbrace{x_1 + x_2 + x_3 + 2x_4 + x_5 + x_6 + x_7 = 52}_{\textit{Equation IV}}$$

Subtract Equation I from Equation IV

$$x_4 = 6$$

Shortcut Method

$$x_4 = s_{1-4} + s_{4-7} - x_4 = 20 + 32 - 46 = 6$$

Example 1.119

Isabella had a week to read a book for a school assignment. She read an average of 36 pages per day for the first three days and an average of 44 pages per day for the next three days. She then finished the book by reading 10 pages on the last day. How many pages were in the book? (AMC 8 2014/3)

$$36 \times 3 + 44 \times 3 + 10 = 80 \times 3 + 10 = 240 + 10 = 250\ Pages$$

Example 1.120

Four numbers are written in a row. The average of the first two is 21, the average of the middle two is 26, and the average of the last two is 30. What is the average of the first and last of the numbers? (AMC 8 2022/16)

$$x_1 + x_2 = 42$$
$$x_3 + x_4 = 60$$

$$x_1 + x_2 + x_3 + x_4 = 102$$
$$x_2 + x_3 = 52$$
$$x_1 + x_4 = 50$$

E. Changing a data point

Example 1.121

A. The average weight of ten students in a class is 42 kilograms. If one student weighing 35 kg leaves the class, and their teacher weighing 60 kg enters the class, what is the new average weight?
B. The number of apples on each of the 25 trees in his orchard was recorded by a farmer. He calculated the average number of apples. After all the calculations were over, he noted that a tree with 32 apples had been mistakenly recorded to have 23 apples. If his old average was a, what was his new average in terms of a?[2]

Part A

Algebraic Method

$$\frac{w_1 + w_2 + \cdots + 60}{10} = \frac{w_1 + w_2 + \cdots + 35}{10} + \frac{25}{10} = Old\ Avg + \frac{25}{10} = 42 + 2.5 = 44.5$$

Shortcut Method

$$New\ Avg = Old\ Avg + \frac{Change}{10} = 42 + \frac{60 - 35}{10} = 42 + \frac{25}{10} = 42 + 2.5 = 44.5$$

Part B

$$New\ Avg = Old\ Avg + \frac{Change}{No.\ of\ Items} = a + \frac{32 - 23}{25} = a + \frac{9}{25} = a + 0.36$$

Example 1.122

Answer each part independently

My teacher takes a math test with only two scores, 0 and 10. The number of students in class is 20.

A. The average score is 2.5. After the marks are distributed, a few students go up and ask for their marks to be increased. If the new average is 4.5, for many students were the marks increased?
B. What is the number of distinct values that the average score for the class can take?

Part A

Method I

If the marks of 1 student are increased, then the change in average

$$= \frac{Change\ in\ Total}{20} = \frac{10}{20} = 0.5$$

The number of students

[2] This kind of error, where the digits in the number are interchanged, is called an error of *transposition*.

$$= \frac{Change\ in\ Avg}{Change\ per\ Student} = \frac{4.5-2.5}{0.5} = \frac{2}{0.5} = 4$$

Method II

$$Current\ Avg = 2.5 \Rightarrow Current\ Total = 2.5(20) = 50$$
$$New\ Avg = 4.5 \Rightarrow Current\ Total = 4.5(20) = 90$$
$$Increase = 90 - 50 = 40$$
$$No.of\ students = \frac{40}{10} = 4$$

Part B

The possible total scores are:

$$0,10,20,\dots,200$$

Each possible total score results in a distinct value of the average. The above scores correspond to x people getting a score of 10:

$$x \in \{0,1,2,\dots,20\} \Rightarrow 21\ Values$$

Example 1.123

I have green apples and red apples in my fruit store. The green apples weigh 300 g each, and the red apples weigh 350 g each. I currently have 7 apples with average weight $\frac{2300}{7}$ g. What is the minimum number of apples of one type I must replace with another type for the average weight in grams to be an integer?

If all 7 apples were green, the average would be:

$$300 = \frac{2100}{7}$$

If I replace a green apple with a red apple, the total weight increases by 50 grams, and the average weight increases by $\frac{50}{7}$.

If I were to replace x apples out of 7 green apples, the new weight would be

$$300 + \frac{50x}{7} = 300 + \frac{49x}{7} + \frac{x}{7} = \underbrace{300}_{Integer} + \underbrace{7x}_{Integer} + \frac{x}{7}$$

For $\frac{x}{7}$ to be an integer, x must be a multiple of 7. The smallest value of x that will work is 7. So, we can either have all green apples, or all red apples.

$$All\ green\ apples = 300 = \frac{2100}{7}$$
$$All\ red\ apples = 350 = \frac{2450}{7}$$

In the given question, it is easiest to replace 3 green apples with 3 red apples, making all of them red.

$$3\ apples$$

Note: The given scenario has

$$\frac{2300}{7} = \frac{2100}{7} + \frac{200}{7} \Rightarrow 3\ Green\ Apples + 4\ Red\ Apples$$

Example 1.124

The annual incomes of 1,000 families range from 8200 dollars to 98,000 dollars. In error, the largest income was entered on the computer as 980,000 dollars. The difference between the mean of the incorrect data and the mean of the actual data is **(AMC 8 1990/20)**

The average calculated was:

$$= \frac{x_1 + x_2 + \cdots + x_{1000} + (980{,}000 - 98{,}000)}{1000}$$

$$= \frac{x_1 + x_2 + \cdots + x_{1000} + 882000}{1000}$$

Split the fraction into two:

$$= \frac{x_1 + x_2 + \cdots + x_{1000}}{1000} + \frac{882000}{1000}$$

Note that the first term is the expression for the correct average:

$$= Correct\ Avg + 882$$

Shortcut Method

The average increases by

$$\frac{980{,}000 - 98{,}000}{1{,}000} = \frac{882000}{1000} = 882$$

F. Finding Missing Average

(add missing graph)Example 1.125

A scientist walking through a forest recorded as integers the heights of 5 trees standing in a row. She observed that each tree was either twice as tall or half as tall as the one to its right. Unfortunately some of her data was lost when rain fell on her notebook. Her notes are shown below, with blanks indicating the missing numbers. Based on her observations, the scientist was able to reconstruct the lost data. What was the average height of the trees, in meters? (AMC 8 2020/20)

G. Removing a Data point

Example 1.126

If 20 students, with an average weight of 42 kg, participating in a judo competition were eliminated, one per round (with weight of eliminated student: 40 kg, 38 kg, 36 kg. . .), then at the start of which round will the average weight of the remaining students be above 42.5?

1st Round can be ignored.
Total Weight at Start of:
2nd Round
$= (42 \times 20 - 40)/19 = 840 - 40$
$= 800 < 807.5 = 19 \times 42.5$
3rd Round
$= 800 - 38 = 762 < 765 = 18 \times 42.5$
4th Round (**Answer**)
$= 762 - 36 = 726 > 722.5 = 17 \times 42.5$

Example 1.127

There used to be 5 parrots in my cage. Their average value was ¤6000. One day while I was cleaning out the cage the most beautiful parrot flew away. The average value of the remaining four parrots was ¤5000. What was the value of the parrot that escaped? (UKMT Cayley 2003/A3)

The value of the parrot that escaped

$$Total\ Value\ of\ 5\ Parrots - Total\ Value\ of\ 4\ Parrots$$

$$= 5(6000) - 4(5000) = 30000 - 20000 = 10000$$

Example 1.128

What number should be removed from the list 1,2,3,4,5,6,7,8,9,10,11 so that the average of the remaining numbers is 6.1? (AMC 8 1996/12)

There are 11 numbers in this list. After removing one number, there will be 10 numbers with

$$Avg = 6.1 \Rightarrow Total = 6.1 \times 10 = 61$$

The current total is

$$1 + 2 + \cdots + 11 = \frac{11 \times 12}{2} = 11 \times 6 = 66$$

The number to be removed is:

$$66 - 61 = 5$$

Example 1.129

A. The average age of 5 people in a room is 30 years. An 18-year-old person leaves the room. What is the average age of the four remaining people? (AMC 8 2007/7)
B. The average age of 5 people in a room is x years. An 18-year-old person leaves the room. What is the increase in the average age of the four remaining people? (AMC 8 2007/7)

Part A

$$\frac{Total - Person\ Leaving}{Total\ People - 1} = \frac{5(30) - 18}{5 - 1} = \frac{150 - 18}{4} = 33$$

Part B

$$\frac{5x - 18}{4} - x = \frac{x - 18}{4}$$

You can use this as a shortcut for Part A:

$$30 + \frac{30 - 18}{4} = 30 + \frac{12}{4} = 30 + 3 = 33$$

Example 1.130

The mean (average) of a set of six numbers is 10. When the number 25 is removed from the set, the mean of the remaining numbers is (CEMC Gauss 7/2016/18, 8/2016/16)

Shortcut Method

$$10 - \frac{25 - 10}{5} = 10 - \frac{15}{5} = 10 - 3 = 7$$

Example 1.131

A teacher has a list of marks: 17, 13, 5, 10, 14, 9, 12, 16. Which two marks can be removed without changing the mean? (UKMT Grey Kangaroo 2011/12)

The current mean is:

$$\frac{17 + 13 + 5 + 10 + 14 + 9 + 12 + 16}{8} = \frac{96}{8} = 12$$

If we want to remove two numbers without changing the mean they should have:

$$Avg = 12 \Rightarrow Total = 24 = 10 + 14$$

H. Maximum and Minimum

Example 1.132

A. Nicky has to choose 7 different positive whole numbers whose mean is 7. What is the largest possible such number she could choose? **(UKMT JMC 2010/20)**
B. The average (arithmetic mean) of 10 different positive whole numbers is 10. The largest possible value of any of these numbers is: **(AMC 8 1991/18)**
C. A set of five different positive integers has an average (arithmetic mean) of 11. What is the largest possible number in this set? **(CEMC Gauss, Grade 8/2000/14)**

To make one number the largest, we make all other numbers the smallest:

Part A

$$\frac{1+2+\cdots+6+x}{7}=7$$
$$21+x=49$$
$$x=28$$

Part B

$$\frac{1+2+\cdots+9+x}{10}=10$$
$$45+x=100$$
$$x=55$$

Part C

$$\frac{1+2+3+4+x}{5}=11$$
$$10+x=55$$
$$45$$

Example 1.133

What is the smallest possible average of four distinct positive even integers? (AMC 8 2002/3)

$$\frac{2+4+6+8}{4}=\frac{20}{4}=5$$

(Calculator) Example 1.134

Naoki wrote nine tests, each out of 100. His average on these nine tests is 68%. If his lowest mark is omitted, what is his highest possible resulting average? (CEMC Gauss, Grade 7/2001/21)

Assuming 100 marks per test, his current total is:

$$Total=68\times 612$$

Assuming that his lowest score was zero, the new average

$$\frac{612}{8}=76.5$$

I. Arithmetic Mean: Challenging Questions

Example 1.135

A. You are told that one of the integers in a list of distinct positive integers is 97 and that their average value is 47. If the sum of all the integers in the list is 329, what is the largest possible value for a number in the list?
B. Suppose the sum of all the numbers in the list can take any value. What would the largest possible number in the list be then? **(UKMT Cayley 2007/6)**

Part A

Let the number of numbers in the list be n. Then:

$$Average = \frac{x_1 + x_2 + \cdots + x_n}{n} = 47$$
$$Total = x_1 + x_2 + \cdots + x_n = 47n$$

Since we know that the total is 329, hence

$$47n = 329$$
$$No.of\ Numbers = n = \frac{329}{47} = 7$$

To make one number have the largest possible value, we make the other numbers as small as possible.
We know that one number is 97.
Let the other number be x.

We need to decide the values of

$$7 - 1 - 1 = 5\ Numbers$$

Since the numbers also have to distinct and positive, the smallest numbers we can get are:

$$1,2,3,4,5 \Rightarrow Total = 1 + 2 + 3 + 4 + 5 = 15$$

Hence, the remaining number will have total:

$$329 - 97 - 15 = 217$$

Part B

We do not know the number of numbers in the list. Since we do not know the total (as we did in the previous part), we cannot find it either.

Instead, we work with maximum and minimum. To make one number as large as possible while the average the same, the other numbers have to be as small as possible.

A number:
- less than 47 will reduce the average
- equal to 47 will keep the average the same
- greater than 47 will increase the average

So, we want as numbers as possible which are less than 47. These also need to be distinct and positive:

$$\{1,2,3, \dots ,46\}$$
$$Total = \frac{46 \times 47}{2} = 23 \times 47 = 1081$$

We also have a 97 in the list, and an unknown (largest number= x) in the list, giving us a total of

$$46 + 1 + 1 = 48$$

Then, the sum of the numbers is

$$= 47 \times 48 = 2256$$

And the largest number is:

$$2256 - 1081 - 97 = 1078$$

Example 1.136

Which of the following sets of whole numbers has the largest average?
A. multiples of 2 between 1 and 101
B. multiples of 3 between 1 and 101
C. multiples of 4 between 1 and 101
D. multiples of 5 between 1 and 101
E. multiples of 6 between 1 and 101 **(AMC 8 1986/25)**

Part A

$$2 + 4 + \cdots + 100 = 2(1 + 2 + \cdots + 50) = 2\left(\frac{50 \times 51}{2}\right) = 50 \times 51 \Rightarrow Avg = \frac{50 \times 51}{50} = 51$$
$$3 + 6 + \cdots + 99 = 3(1 + 2 + \cdots + 33) = 3\left(\frac{33 \times 34}{2}\right) = 3(33 \times 17) \Rightarrow Avg = \frac{3(33 \times 17)}{33} = 51$$

$$4+8+\cdots+100=4(1+2+\cdots+25)=4\left(\frac{25\times 26}{2}\right)=2(25\times 26)\Rightarrow Avg=\frac{2(25\times 26)}{25}=52$$
$$5+10+\cdots+100=5(1+2+\cdots+20)=5\left(\frac{20\times 21}{2}\right)=5(10\times 21)\Rightarrow Avg=\frac{5(10\times 21)}{20}=\frac{105}{2}=52.5$$
$$6+12+\cdots+96=6(1+2+\cdots+16)=6\left(\frac{16\times 17}{2}\right)=3(16\times 17)\Rightarrow Avg=\frac{3(16\times 17)}{16}=51$$

$$Largest=52.5\Rightarrow Option\ D$$

Example 1.137

Five test scores have a mean (average score) of 90, a median (middle score) of 91 and a mode (most frequent score) of 94. The sum of the two lowest test scores is: **(AMC 8 1992/13)**

Let the five values arranged in ascending order be:

$$x_1,x_2,x_3,x_4,x_5$$

Median is the middle value. $x_3=91$

$$x_1,x_2,91,x_4,x_5$$

94 is greater than 91. So $x_4=x_5=94$

$$x_1,x_2,91,94,94$$

Hence, the average is:

$$\frac{x_1+x_2+91+94+94}{5}=90$$
$$x_1+x_2+90\times 3+9=90\times 5$$
$$x_1+x_2+9=90\times 2$$
$$x_1+x_2=180-9$$
$$x_1+x_2=171$$

Example 1.138

When the diagram below is complete, the number in the middle of each group of 3 adjoining cells is the mean of its two neighbors. What number goes in the right-hand end cell? **(UKMT JMC 2003/19)**

8			20	

8	a	b	20	c

Since a is the average of the first and the third cell:

$$a=\frac{8+b}{2}\Rightarrow \underbrace{2a=8+b}_{Equation\ I}$$

Since b is the average of a and 20:

$$b=\frac{a+20}{2}$$

Substitute $b=\frac{a+20}{2}$ in Equation I:

$$2a=8+\frac{a+20}{2}$$
$$2a=\frac{a+36}{2}$$
$$4a=a+36$$
$$3a=36$$
$$a=12$$

$$b = \frac{a+20}{2} = \frac{12+20}{2} = \frac{32}{2} = 16$$

$$20 = \frac{b+c}{2}$$
$$20 = \frac{16+c}{2}$$
$$40 = 16 + c$$
$$c = 24$$

Shortcut Method

The middle number will be the average of the other two, if the number increases by the same amount every time (they are in arithmetic progression).

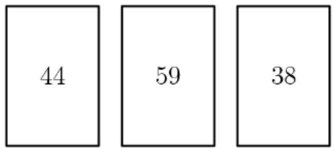
So, the jump is:

$$\frac{20-8}{3} = \frac{12}{3} = 4$$

And then we get the table below.

8	12	16	20	24

Example 1.139



Barry wrote 6 different numbers, one on each side of 3 cards, and laid the cards on a table, as shown. The sums of the two numbers on each of the three cards are equal. The three numbers on the hidden sides are prime numbers. What is the average of the hidden prime numbers? **(AMC 8 2006/25)**

59 is an odd number. And

$$Odd + Odd = Even \Rightarrow Not\ Valid$$

So, we must have

$$Odd + Even = Odd$$
$$59 + 2 = 61$$

Similarly:

$$61 - 44 = 17$$
$$61 = 23$$

The average is:

$$\frac{2+17+23}{3} = \frac{42}{3} = 14$$

Example 1.140

In a triangle with angles x, y and z the mean of y and z is x. What is the value of x? **(UKMT JMC 2010/8)**

$$\frac{y+z}{2} = x \Rightarrow y + z = 2x$$
$$x + y + z = x + 2x = 3x = 180 \Rightarrow x = 60$$

Example 1.141

There are five families living in my road. Which of the following could not be the mean number of children per family that live there?

A. 0.2
B. 1.2
C. 2.2
D. 2.4
E. 2.5 (UKMT Grey Kangaroo 2013/4)

$$Total\ Number\ of\ children = 5 \times Mean$$
$$0.2 \times 5 = 1$$
$$1.2 \times 5 = 6$$
$$2.2 \times 5 = 11$$
$$2.4 \times 5 = 12$$
$$2.5 \times 5 = 12.5 \Rightarrow Not\ Valid$$

Example 1.142

The number N is between 9 and 17. The average of 6, 10, and N could be

A. 8
B. 10
C. 12
D. 14
E. 16 (AMC 8 1989/17)

$$Min\ Total = 6 + 10 + 10 = 26$$
$$Max\ \ = 6 + 10 + 16 = 32$$

$$8 \times 3 = 24 < 26$$
$$10 \times 3 = 30$$
$$12 \times 3 = 36 > 32$$

$$Option\ B$$

Example 1.143

Laila took five math tests, each worth a maximum of 100 points. Laila's score on each test was an integer between 0 and 100, inclusive. Laila received the same score on the first four tests, and she received a higher score on the last test. Her average score on the five tests was 82. How many values are possible for Laila's score on the last test? (AMC 8 2018/13)

Suppose the scores that Laila got are
$$t, t, t, t, x\ with\ x > t$$

$$\frac{t + t + t + t + x}{5} = 82$$
$$4t + x = 410$$
$$t = \frac{410 - x}{4} = \frac{408}{4} + \frac{2 - x}{4}$$

x must be 2 more than a multiple of 4. The possible values are:
$$x \in \{98, 94, 90, 86\} \Rightarrow 4\ Values$$

Example 1.144

The list of integers $4, 4, x, y, 13$ has been arranged from least to greatest. How many different possible ordered pairs (x, y) are there so that the average (mean) of these 5 integers is itself an integer? **(CEMC Gauss, Grade 8/2015/23)**

$$\frac{4+4+x+y+13}{5} = \frac{21+x+y}{5} = \frac{20}{5} + \frac{1+x+y}{5}$$

$$4 \le x, y \le 13$$

$$x + y = 9 \Rightarrow (x, y) = (4,5)$$
$$x + y = 14 \Rightarrow (x, y) = (4,10)(5,9)(6,8)(7,7)$$
$$x + y = 19 \Rightarrow (6,13)(7,12)(8,11)(9,10)$$
$$x + y = 24 \Rightarrow (11,13)(12,12)$$

Example 1.145

Rishi got the following marks on four math tests: 71, 77, 80, and 87. He will write one more math test. Each test is worth the same amount and all marks are between 0 and 100. Which of the following is a possible average for his five math tests?

A. 88
B. 62
C. 82
D. 84
E. 86 **(CEMC Gauss, Grade 7/2008/18)**

$$Average < \frac{71+77+80+87+100}{5} = \frac{415}{5} = 83$$
$$Average > \frac{71+77+80+87}{5} = \frac{315}{5} = 63$$

$$63 < Avg < 83 \Rightarrow Option\ C$$

J. Averages of Consecutive Integers

Averages of consecutive integers come up often. The problems have certain properties that can be utilized to make the solutions faster.

Example 1.146

The average of three integers is fifteen.

A. Find them if they are consecutive.
B. Find them if they are consecutive and odd.
C. Is it possible for the three integers to be consecutive and even.

Part A

Start with numbers that have average (but may not meet all the conditions):

$$15, 15, 15$$

And we can convert this into three consecutive numbers that still have average 15:

$$14,15,16$$

Part B

$$13,15,17$$

Part C

$$\frac{x_1 + x_2 + x_3}{3} = 15 \Rightarrow x_1 + x_2 + x_3 = 45$$

Since the RHS is odd, the LHS must also be odd.
But the sum of three even numbers must be even, and hence we have a contradiction.

Hence, three consecutive even numbers which have an average of 15 is not possible.

Example 1.147

Four consecutive integers have a total of 34.

A. Find the difference between the largest and the smallest integer.
B. Find the smallest integer.

Part A

You do not need the value of 34. If you have any four consecutive integers, the difference between the largest and the smallest will always be:

$$3$$

Part B

We attempt to divide 34 by 4 to get an idea of what the numbers should be.

$$\frac{34}{4} = 8.5$$

8.5 is not an integer. But it gives us enough information to solve the question:

$$7,8, \quad \mathbf{8.5}, \quad 9,10$$

And hence, the numbers are:

$$7,8,9,10$$

Example 1.148

If the mean (average) of five consecutive integers is 21, the smallest of the five integers is (CEMC Gauss, Grade 7/2010/13, 8/2010/11)

Method I: Logic

Five numbers which are all 21 have average 21.

$$(21,21,21,21,21) \to Avg = 21$$

Subtracting x from one number and adding x to another number keeps the total, and hence the average the same.

Simplifying gives:

$$(19,20,21,22,23) \to Avg = 21$$

Method II: Algebra

Let x be the middle value. Then the numbers:

$$x-2, x-1, x, x+1, x+2$$

The average is:

$$\frac{(x-2)+(x-1)+x+(x+1)+(x+2)}{5} = 21$$

$$5x = 105$$

$$x = 21$$
$$x - 2 = 19$$

Example 1.149

A. The mean (average) of five consecutive even numbers is 12. The mean of the smallest and largest of these numbers is (CEMC Gauss, Grade 7/2013/16).
B. The mean (average) of five consecutive even numbers is x. The mean of the smallest and largest of these numbers is:
C. The mean (average) of n consecutive even numbers is x. n is an odd number. The mean of the smallest and largest of these numbers is:

Part A

$$8,10,12,14,16$$
$$Avg = \frac{16+8}{2} = \frac{24}{2} = 12$$

Part B

$$(x, x, x, x, x) \to Avg = x$$
$$(x-4, x-2, x, x+2, x+4) \to Avg = x$$
$$\frac{x-4+x+4}{2} = \frac{2x}{2} = x$$

Part C

Even if you have n consecutive even numbers, a similar argument as the part above holds. The algebra is a little more complicated.

K. Averages of Integers

Example 1.150

The sum of two numbers is 22. Their difference is 4. Find the two numbers.

$$x + y = 22 \Rightarrow Avg(x, y) = \frac{22}{2} = 11$$

Two numbers with average 11 are:

$$11,11 \Rightarrow Difference = 0$$

Subtract $\frac{4}{2} = 2$ from the first number, and add that the same 2 to the second number to maintain the average while creating a difference of 4:

$$11 - 2, 11 + 2 = 9,13 \Rightarrow Difference = 4$$

Numbers are:

$$9,13$$

L. Review

1.151: Adding or Removing a Data Point

	Add a Data Point	Remove a Data Point

More than the current average	$Avg\ will \uparrow$	$Avg\ will \downarrow$
Less than the current average	$Avg\ will \downarrow$	$Avg\ will \uparrow$
Same as the current average	Average will remain same.	Average will remain same.

Example 1.152

The average time that children in an apartment block spend on the internet is $12\ \frac{hrs}{week}$. Decide whether the average increases, decrease , or remains the same when a new child joins the building who

A. spends more than 12 hours per week on the internet
B. spends 12 hours per week on the internet
C. spends less than 12 hours per week on the internet
D. Answer the above questions if a child leaves instead of joining.

$$A: Increase, \quad B: Remain\ Same, \quad C: Decrease$$
$$D - A: Decrease, \quad D - B: Remain\ Same, \quad D - C: Increase$$

1.153: Changing a Data Point

If you change a data point, that is, replace with one data with another:

- The average will go up if the new data point is more than the earlier data point
- The average will go down if the new data point is less than the earlier data point

Example 1.154

Sushant wanted to find the average of X and Y. He mistakenly found the average of $X - 1000$, and Y. If the average that he found was x, find the average of X and Y.

Method I: Numbers

Since this works for X and Y, we can take any values for X and Y that we want. Let

$$X = Y = 1000$$
$$Avg(1000,1000) = 1000$$
$$Avg(0,1000) = \frac{1000}{2} = 500$$

The wrong average is 500 less than the actual average. So, if the wrong average was x, the actual average should be

$$x + 500$$

Method II: Algebra

$$Actual\ Avg = \frac{X+Y}{2}$$
$$Wrong\ Avg = \frac{X - 1000 + Y}{2} = \frac{X+Y}{2} - 500$$

1.155: Changing a Data Point

If among n numbers, a number is replaced by another, then:

$$Change\ in\ Avg = \frac{Difference}{n}$$

Example 1.156

The average of two numbers is x. If one of the numbers is reduced by 100, find the new average.

The new average will be

$$Old\ Avg - \frac{Diff}{n} = x - \frac{100}{2} = x - 50$$

1.157: Changing Multiple Data Points

If among n numbers, more than one number is replaced, calculate the net change due to the replacements, and then:

$$Change\ in\ Avg = \frac{Net\ Change}{n}$$

Example 1.158

The average of five numbers is 73. If one of the numbers is increased by 27, and another is decreased by 12, find the new average.

The new average will be

$$Old\ Avg - \frac{Diff}{n} = 73 + \frac{27-12}{2} = 73 + \frac{15}{5} = 73 + 3 = 76$$

Example 1.159

The average time that children in an apartment block spend on the internet is 12 $\frac{hrs}{week}$. Decide whether the average increases, decrease , or remains the same when a child with average

A. 11 hours joins the block, while a child with average 7 hours leaves the block.
B. 11 hours joins the block, while a child with average 14 hours leaves the block.
C. 22 hours leaves the block, while a child with average 14 hours joins the block.
D. 10 hours leaves the block, while a child with average 19 hours joins the block.

To decide whether the average increases or decreases, we check what happens to the total. If the net change is positive, the total, and hence the average increases. If the net change is negative, the total, and hence the average decreases.

$$11 - 7 = 4 \rightarrow +ve \rightarrow Increase$$
$$11 - 14 \rightarrow -ve \rightarrow Decrease$$
$$14 - 22 < 0 \Rightarrow Decrease$$
$$19 - 10 > 0 \Rightarrow Increase$$

Example 1.160

Answer the questions below. Information from one part can be in succeeding parts

A class of 15 students has an average weight of 52.

A. A student weighing 59 kg leaves the class to call the teacher. The average weight of the class is now:
B. The student and comes back with the teacher. The teacher weighs 68 kg. The average weight is now:
C. The teacher finds that a student weighing 84 kg had the digits of his weight transposed. Find the difference between the actual average and the wrong average for the 15 students?

Part A

$$Total\ Weight = 15 \times 52 = 780$$
$$\frac{780-59}{14} = \frac{721}{14} = 51\frac{1}{2}$$

Part B

$$\frac{780+68}{16} = \frac{848}{16} = 53$$

Part C

$$52+\frac{84-48}{15}=52+\frac{36}{15}=52+\frac{12}{5}=54.5$$

Example 1.161

Projected Consumption at a party				
	Alok	Tanmay	Nisha	Unit
Chocolate	2/3rd	1/4th	2/5th	Bars
Chips	0.3	0.4	0.5	Packet
Water	20%	65%	80%	Bottle

	Cost ($)	Quantity
Bar of chocolate	5	50 g
Packet of Chips	4	25 Chips
Bottle of Water	3	500 ml

Notes and Assumptions: Bars, Cans and Bottles can be shared / divided as required by party-goers. However, they can only be ordered or returned in whole number quantities.

Part I: For each item, find:
Q1: the average consumption
Q2: the number of units to be ordered.
Q3: the number of units to be ordered to have 20% more than projected consumption
Part II: If Actual Average consumption is only half of what is projected, and items were ordered without the 20% extra, then, for each item, find:
Q4: the number of units that can be returned
Q5: the quantity consumed, per person

1.4 Weighted Arithmetic Mean

Example 1.162

The mean (average) height of a group of children would be increased by 6 cm if 12 of the children in the group were each 8 cm taller. How many children are in the group? **(CEMC Gauss, Grade 8/2018/19)**

If 12 of the children were each 8 cm taller, the total increase in height

$$=12\times 8=96$$

If there are c children, the average would increase by:

$$\frac{96}{c}=6\Rightarrow c=\frac{96}{6}=16$$

Example 1.163

Celyna bought 300 grams of candy A for $5.00, and x grams of candy B for $7.00. She calculated that the average price of all of the candy that she purchased was $1.50 per 100 grams. What is the value of x? (**CEMC Gauss, Grade 7, 2020/22)**

$$Average\ Cost=\frac{Total\ Cost}{Total\ Weight}$$

Since the average price is given in terms of 100 grams, convert everything to 100 grams:

$$3(100)\ grams\ for\ \$5$$
$$y(100)\ grams\ for\ \$7$$

$$1.5 = \frac{5+7}{3+y}$$
$$4.5 + 1.5y = 12$$
$$1.5y = 7.5$$
$$y = \frac{7.5}{1.5} = 5$$

$$x = 100y = 100(5) = 500$$

A. Frequency

If an item occurs more than once, then that is the frequency of that item.

Example 1.164

Mark has a library with the following books. He decided to note down the color of the book covers one day, and made the following list:

$$Red, Blue, Yellow, Blue, Red, Red, Yellow, Red, Blue, Yellow, Red$$

A. Find the frequency of each color.
B. Find the color with the highest frequency.
C. Find the color with the lowest frequency.
D. Suppose a red book costs five dollars, a blue books costs four dollars, and a yellow book costs three dollars. Find the total cost of books of a particular color and cost of all books.
E. Find the average cost of a book in the library.
F. Find the mode in the frequency table that you made.

		Red	Blue	Yellow	Total
Frequency	f	5	3	3	11
Cost per book		5	4	3	
Total		25	12	9	46

$$Average = \frac{Total\ Cost}{No.of\ Books} = \frac{46}{11}$$

Mode

The color red occurs with maximum frequency. Hence, red is the modal frequency.

Example 1.165

No. of Families	f	2	4	3	1
No. of Movies		0	1	2	3

The frequency table shows the mean number of movies watched in a month by different families.

A. Find the average number of movies watched by a family.
B. Find the mode in the frequency table above.

No. of Families	f	2	4	3	1	10
No. of Movies		0	1	2	3	
Total		0	4	6	3	13

$$Average = \frac{Total\ No.of\ Movies}{No.of\ Families} = \frac{13}{10} = 1.3$$

Mode

Watching one movie a month has the highest frequency.
Hence, that is the mode.

Example 1.166

A group of students was asked their favorite prime number. The results are tabulated. Find the number of students who gave an answer which is not a prime number.

No. of Students	f	3	4	3	3
Prime Number		1	2	3	5

B. Weighted Arithmetic Mean

	Two Values	More than two values
Weighted Arithmetic Mean	$\frac{na+mb}{n+m}$	$\frac{m_1x_1+m_2x_2+\cdots+m_nx_n}{m_1+m_2+\cdots+m_n}$

Example 1.167

A. Calculate the average of 2 and 5.
B. Calculate the average of 3,3 *and* 5.

/

$$Average = \frac{2+5}{2} = \frac{7}{2} = 3.5$$

$$Average = \frac{3+3+5}{3} = \frac{2(3)+5}{3} = \frac{11}{3}$$

Example 1.168

Calculate the arithmetic mean of 55, 51, 53, 55, 55, 53

Method I

Arrange the six numbers in ascending order and add them:

$$51+53+53+55+55+55$$
$$= (50+1)+(50+3)+(50+3)+(50+5)+(50+5)+(50+5)$$
$$= 50\times 6 + (1+3+3+5+5+5)$$
$$= 50\times 6 + 22$$

Now divide by 6:

$$\frac{50\times \mathbf{6}}{\mathbf{6}} + \frac{22}{6} = 50 + \frac{22}{6} = 50 + \frac{11}{3} = 50 + 2\frac{2}{3} = 52\frac{2}{3}$$

Method II

		51	53	55
Frequency	f	1	2	3
Multiply		51×1	53×2	55×3

Add:

$$51 \times 1 + 53 \times 2 + 55 \times 3$$
$$= 1(51) + 2(53) + 3(53)$$
$$= (50 + 1) + 2(50 + 3) + 3(50 + 5)$$
$$= (50 + \mathbf{1}) + \mathbf{2}(50 + \mathbf{3}) + \mathbf{3}(50 + \mathbf{5})$$
$$= 50 \times 6 + \mathbf{1} + \mathbf{2} \times \mathbf{3} + \mathbf{3} \times \mathbf{5}$$
$$= \mathbf{50 \times 6 + 22}$$

Example 1.169

Juan organizes the stamps in his collection by country and by the decade in which they were issued. The prices he paid for them at a stamp shop were: Brazil and France, 6 cents each, Peru 4 cents each, and Spain 5 cents each. (Brazil and Peru are South American countries and France and Spain are in Europe.) The average price of his '70s stamps is closest to **(AMC 8 2002/10)**

A. 3.5 cents
B. 4 cents
C. 4.5 cents
D. 5 cents
E. 5.4 cents

Country	$50's$	$60's$	$70's$	$80's$
Brazil	4	7	12	8
France	8	4	12	15
Peru	6	4	6	10
Spain	3	9	13	9

Average Price

$$= \frac{Total\ Cost}{No.\,of\ Stamps} = \frac{12(6) + 12(6) + 6(4) + 13(5)}{12 + 12 + 6 + 13} = \frac{233}{43} \approx 5.4\ Cents \Rightarrow Option\ E$$

Example 1.170

Q6 (MCMC): 75 (3 data points), and 85(4 data points)

A. <75
B. Between 70 and 75
C. Between 75 and 85
D. Between 80 and 85
E. >85

S1: Options C and D.

Example 1.171

Q7: What is the minimum number of data points of 60 that must be added to 75(5 observations), and 85(8 observations) so that the average of all data points is below 73?

A. 6
B. 7
C. 8
D. 9

S2: $2 \times 5 + 8 \times 12 = 106$
Reduction must be more than 106.

$13 \times 8 = 104 < 106 < 117 = 13 \times 9$
Hence, option D.

C. Mixtures

1.172: Equal Quantities

When you mix two solutions in equal quantities, the resulting solution has concentration equal to average of the original concentrations.

Example 1.173

You mix equal quantities of two solutions consisting of 6% sodium chloride, and 9% sodium chloride respectively. What is the concentration of the resulting solution?

$$\frac{6+9}{2} = \frac{15}{2} = 7.5$$

1.174: Unequal Quantities

$$\frac{Quantity\ of\ Item}{Quantity\ of\ Solution}$$

Example 1.175

If you mix 4 gallons of 20% salt solution with 6 gallons of 30% salt solution. What is the

A. Quantity of the resulting solution?
B. Quantity of salt in the resulting solution?
C. Percentage of salt in the resulting solution?

The quantity of resulting solution

$$= 4\ gallons + 6\ gallons = 10\ gallons$$

Quantity of salt in the resulting solution

$$= 4 \times 20\% + 6 \times 30\% = 0.8 + 1.8 = 2.6$$

Percentage of salt in the resulting solution

$$= \frac{Quantity\ of\ Salt}{Quantity\ of\ Solution} = \frac{2.6}{10} = 26\%$$

Example 1.176

If you mix 3 quarts of 15% salt solution with 1 gallon of 20% salt solution. What is the

A. Quantity of the resulting solution?
B. Quantity of salt in the resulting solution?
C. Percentage of salt in the resulting solution?

Note: 1 Quart = 0.25 Gallons

$$0.25\ Gallons = 1\ Quart \Rightarrow 1\ Gallon = 4\ Quarts$$

The quantity of resulting solution

$$= 3\ quarts + 4\ quarts = 7\ quarts$$

Quantity of salt in the resulting solution

$$= 3 \times 15\% + 4 \times 20\% = 0.45 + 0.8 = 1.25$$

Percentage of salt in the resulting solution

$$= \frac{Quantity\ of\ Salt}{Quantity\ of\ Solution} = \frac{1.25\ Quarts}{7\ Quarts} = \frac{1.25}{7} = \frac{\frac{1.25}{7} \times 100}{100} = \frac{\frac{125}{7}}{100} = \frac{125}{7}\%$$

1.177: Mixtures

When you mix two solutions in any non-zero ratio of quantities, the resulting solution must have concentration greater than the smaller concentration, and smaller than the greater concentration.

Example 1.178

Mark all correct options

You mix quantities of two solutions consisting of 6% sodium chloride, and 9% sodium chloride respectively. The concentration of the resulting solution cannot be:

A. Greater than 9%
B. Greater than 10%
C. Less than 7%
D. Less than 6%

$$Options\ A, B, and\ D$$

Example 1.179

Find the concentration of the resulting solution if you mix:

A. 5 liters of 6% sodium chloride solution with 10 liters of 9% sodium chloride solution.
B. 12 beakers of 5% hydrochloric acid with 11 beakers of 3% hydrochloric acid solution.
C. 16 Tons of 4% brine with 10 tons of 5% brine.

Part A

$$\frac{Qty.\,of\ Sodium\ Chloride}{Qty.\,of\ Solution} = \frac{5 \times 6\% + 10 \times 9\%}{6 + 9} = \frac{120\%}{15} = 8\%$$

Part B

$$\frac{Qty.\,of\ Acid}{Qty.\,of\ Solution} = \frac{12(5\%) + 11(3\%)}{12 + 11} = \frac{93}{23}\%$$

Part C

$$\frac{Qty.\,of\ Brine}{Qty.\,of\ Solution} = \frac{16(4\%) + 10(5\%)}{16 + 10} = \frac{114}{26} = \frac{57}{13}\%$$

Item	Kg / Pack	Rs. / Box
Figs	3	120
Raisins	2	140
Walnuts	5	200

D. Costs in Mixtures

Pine Nuts	1	300

Example 1.180
If you mix 2 boxes of figs, 1 box of raisins, 3 boxes of walnuts, and 4 boxes of pine nuts, what is the average cost of the resulting mixture?

Example 1.181
If you mix 4 gallons of 20% salt solution with 4 gallons of 30% salt solution, what is the percentage of salt in the resulting solution?

(20+30)/2=50/2=25

Gap:
30-20=10

This gap is equally divided.
10/2=5

Final Answer is 20+5=25

Example 1.182
A. Find the average of 20,20,30.
B. If you mix 8 gallons of 20% salt solution with 4 gallons of 30% salt solution, what is the percentage of salt in the resulting solution?
C. What is the connection between Part A and Part B?

Part A
(20+20+30)/3=70/3=23 1/3

Part B
Gap: 30-20=10

This is divided in the ratio 8:4=2:1.
However, it will be closer to 20, then to 30.
So, we need to take the reciprocal of the ratio, giving us 1:2.

(10/3)*1=10/3=3 1/3

Final Answer:
=20+3 1/3= 23 1/3

Example 1.183
Wheat worth Rs. 6.70 per kg must be mixed with wheat worth Rs. 8.20 per kg in what ratio to get wheat worth Rs. 7.45?

7.45 is exactly between 6.7 and 8.2.

Hence, 50%:50%, or 1:1

Alternative Method:
8.2 – 7.45: 7.45 – 6.7 = 0.75:0.75 = 1:1

6.70		7.45		8.2
	8.2-7.45		7.45-6.7	

Distance between 7.45 and 6.7 = 0.75
Distance between 8.2 and 7.45 = 0.75

Final Ratio will be

$$D(8.2 - 7.45): D(7.45 - 6.7) = 0.75: 0.75 = 1$$

6.70		7.45		8.2
	8.2-7.45		7.45-6.7	

Example 1.184

$$Percentage = \frac{total\ vinegar}{total\ solution} = \frac{30 \times 0.3 + 50 \times 0.2}{30 + 50} = \frac{9 + 10}{80} = \frac{9}{80}$$

$$750 \times \frac{7.5}{100} \times$$

E. Percentages and Quantities

Example 1.185

2.2% sodium thiopental solution is available in a factory. To make it more concentrated, it is mixed with 3.4% solution of the same chemical (bought at Rs. 8/litre) to get 15 litres of 2.75% solution.

A. What percent of the 15 litres must be 2.2% solution?
B. What is the amount of 2.2% solution that must be used?
C. What is the cost of the 3.4% solution that must be used?

Part A
3.4 – 2.75:2.75 – 2.2 = 0.65:0.55 = 65:55
Percentage of 2.2% solution = 65/120 × 100 = 65/6 × 5 = 325/6 = 54.16

Part B
15 × 65/120 = 65/8 = 8.125 Litres

Part C
(15 × 55/120) × 8 = 55/8 × 8 = Rs. 55

F. Costs while mixing: Containers and Costs

Example 1.186

20-litre casks of wine worth \$60/litre are mixed with 30-litre casks of wine worth \$80/litre to get wine worth \$72/litre, which is then stored in 15-litre casks that must be purchased for \$5 each. The 20-litre and 30-litre casks can be discarded for \$1 and \$1.5 each, respectively.

A. If 10 20-litre casks are used, the number of 30-litre casks that must be used is closest to:
B. What is the average cost of the casks per litre of mixed wine?

Part A
\$60: \$80 = 80 – 72: 72 - 60: = 8:12 = 2:3
3 casks of 20 Litres = 2 casks of 30 Litres
Ratio of casks = 2:3 × 3:2 = 6:6 = 1:1

Part B

Cost of Casks (for 150 Litres)
= Ten 15-L + Three 20-L + Three 30-L = 5 × 10 – 1 × 3 – 1.5 × 3 = 50 – 3 – 4.5= 42.5
Cost per litre = 42.5/150 = 83/300

Replacements

x% replaced with water	% tage left:
1st Round	$(100 - x)$%
Nth Round	$(100 - x)^n$ %

Example 1.187

A tanker with 300 litres of milk has 30 litres withdrawn and replaced with water. If this process happens thrice, what is the amount of milk, in litres, withdrawn in all?

After 1st Round: 300 – 30 = 270
After 2nd Round: 270 × 9/10 = 243
After 3rd Round: 243 × 9/10 = 218.7
Milk withdrawn = 300 – 218.7 = 81.3

Shortcut for Calculations

$300 \times (9/10)^3 = 300 \times 729/1000 = 2187/10 = 218.7$

Example 1.188

Wine of 70 ml from a cask of 3 Litres is replaced with water. Then, 10% of the cask spills. How much wine is lost due to the spill?

(3000 – 70) × 0.1 = 2930 × 0.1 = 293 ml

Example 1.189

In what ratio must wine from two casks (which have wine and water in the ratio 2:3 and 4:3 respectively) be mixed to get a mixture that is half water?

Wine in 1st = 2/5 = 14/35
Wine in 2nd = 4/7 = 20/35

Required = 50% = 17.5/35
Ratio = 20 – 17.5:17.5 – 14 = 2.5:3.5 = 5:7

G. Profit and Loss

Example 1.190

In what ratio must a dishonest grocer mix wheat with chaff (available free) in order to make 20% by selling the wheat at cost price.

Gain = 20% of CP = 20/120 of SP = 1/6 of SP
Cost = SP – P = 1 – 1/6 = 5/6 SP
Wheat: Chaff = 5:1
Hence, option C.

Example 1.191

A dishonest milkman purchases milk on which he pays 10% tax on the marked price. What percent of the milk must be added as water so that he makes 15% by selling the milk at the marked price?

Marked Price = 100
CP = 100 + 10 = 110
SP = 110 × 1.15 = 126.5
Mixing Required = 26.5: 100

Example 1.192

If a dishonest milkman purchases 97 litres of milk, how many 1-litre bottles of water (available free) must he keep ready to add so that he makes 15% by selling the milk at 10% above the cost price?

Gain= 15% of Mixture Cost = 15/115 of SP = 3/23 of SP

Mixture Cost= SP – P = 1 – 3/23 = 20/23 SP = **200/230 SP**
CP = 9/10 SP = **207/230 SP**
Ratio of water = 7:200 = 3.5:100

H. Finance: Average Returns

Example 1.193

Aarna has a million dollars. She invests 40% in Stock X, and the rest in Stock Y. At the end of the year, Stock X generates a return of 10%, and Stock Y generates a return of 5%. Calculate the average return on her money.

$$Investment\ in\ X = X_{Inv} = 40\% = 0.4$$
$$Investment\ in\ Y = Y_{Inv} = 1 - 0.4 = 0.6$$
$$Return\ on\ X = X_R = 10\% = 0.1$$
$$Return\ on\ Y = Y_R = 5\% = 0.05$$

$$Avg\ Return = \frac{X_{Inv}X_R + Y_{Inv}Y_R}{Total\ Invesment} = \frac{(0.4)(0.1) + (0.6)(0.05)}{0.4 + 0.6} = \frac{0.04 + 0.03}{1} = 0.07 = 7\%$$

I. Weighted Arithmetic Mean: Basics

The regular average for two numbers a and b is:

$$\frac{a+b}{2}$$

And the regular average for n numbers $x_1, x_2, \dots, x_n$ is:

$$\frac{x_1+x_2+\cdots+x_n}{n}$$

1.194: Weighted Arithmetic Mean: Two Numbers

The weighted arithmetic mean of two numbers a and b where a has a weight of m, and b has a weight of n is:

$$\frac{ma+nb}{m+n}$$

A weighted average is equivalent to have the value a repeat m times, and the value b repeat n times. Hence, it is equivalent to finding the average of:

$$\left\{\underbrace{a, a, \dots, a}_{m\ times}, \underbrace{b, b, \dots, b}_{n\ times}\right\}$$

Example 1.195

The average weight of 6 boys is 150 pounds and the average weight of 4 girls is 120 pounds. The average weight of the 10 children is: (AMC 8 1988/18)

Substitute $a = 150, b = 120, m = 6, n = 4$ in the formula for the weighted average:

$$\frac{ma+nb}{m+n} = \frac{6(150)+4(120)}{6+4} = \frac{900+480}{10} = \frac{1380}{10} = 138\ Pounds$$

Example 1.196

The average age of the 6 people in Room A is 40. The average age of the 4 people in Room B is 25. If the two groups are combined, what is the average age of all the people? (AMC 8 2008/10)

Substitute $a = 40, b = 25, m = 6, n = 4$ in the formula for the weighted average:

$$\frac{ma+nb}{m+n} = \frac{6(40)+4(25)}{10} = \frac{240+100}{10} = \frac{340}{10} = 34\ Years$$

Example 1.197

In a set of five numbers, the average of two of the numbers is 12 and the average of the other three numbers is 7. The average of all five numbers is (CEMC Gauss, Grade 8/1999/16)

Substitute $a = 12, b = 7, m = 2, n = 3$ in the formula for the weighted average:

$$\frac{ma+nb}{m+n} = \frac{2(12)+3(7)}{2+3} = \frac{24+21}{5} = \frac{45}{5} = 9$$

Example 1.198

In a group of seven friends, the mean (average) age of three of the friends is 12 years and 3 months and the mean age of the remaining four friends is 13 years and 5 months. In months, the mean age of all seven friends is (CEMC Gauss, Grade 8/2017/19)

Convert the ages to months to make the calculations easier:

$$12\ years\ \&\ 3\ months = 12(12)+3 = 144+3 = 147\ Months$$

$$13\ years\ \&\ 5\ months = 13(12) + 5 = 156 + 5 = 161\ Months$$

Substitute $a = 147, b = 161, m = 3, n = 4$ in the formula for the weighted average:

$$\frac{ma + nb}{m + n} = \frac{3(147) + 4(161)}{7} = \frac{441 + 644}{7} = \frac{1085}{7} = 155\ Months$$

1.199: Weighted Arithmetic Mean: Three Numbers

The weighted arithmetic mean of three numbers x, y, z with weights p, q, r is:

$$Avg = \frac{px + qy + rz}{p + q + r}$$

Example 1.200

Antonette gets 70% on a 10-problem test, 80% on a 20-problem test and 90% on a 30-problem test. If the three tests are combined into one 60-problem test, what percent is her overall score? **(AMC 8 2006/12)**

$$\frac{10(70\%) + 20(80\%) + 30(90\%)}{10 + 20 + 30} = \frac{7 + 16 + 27}{60} = \frac{50}{60} = \frac{5}{6} = 83\frac{1}{3}\%$$

Example 1.201

Ryan got 80% of the problems correct on a 25-problem test, 90% on a 40-problem test, and 70% on a 10-problem test. What percent of all the problems did Ryan answer correctly? **(AMC 8 2010/9)**

$$\frac{25(80\%) + 40(90\%) + 10(70\%)}{10 + 20 + 30} = \frac{20 + 36 + 7}{75} = \frac{63}{75} = \frac{21}{25} = \frac{84}{100} = 84\%$$

1.202: Weighted Arithmetic Mean: More than Two Numbers

The weighted arithmetic mean of n numbers $x_1, x_2, \ldots, x_n$ where x_1 has a weight of w_1, x_2 has a weight of w_2, and so up to x_n has a weight of w_n is:

$$\frac{w_1x_1 + w_2x_2 + \cdots w_nx_n}{w_1 + w_2 + \cdots + w_n}$$

Example 1.203

A quiz has three questions, with each question worth one mark. If 20% of the students got 0 questions correct, 5% got 1 question correct, 40% got 2 questions correct, and 35% got all 3 questions correct, then the overall class mean (average) mark was **(CEMC Gauss, Grade 7/2012/22)**

$$Avg = \frac{px + qy + rz}{p + q + r}$$

From the question:

$$Numbers: x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 3$$
$$Weights\ are: w_1 = 20\%, w_2 = 5\%, w_3 = 40\%, w_4 = 35\%$$

Substitute the above in the formula for the weighted average:

$$\frac{20\%(0) + 5\%(1) + 40\%(2) + 35\%(3)}{20\% + 5\% + 40\% + 35\%} = \frac{0.05 + 0.8 + 1.05}{100\%} = \frac{1.9}{1} = 1.9$$

J. Weighted Arithmetic Mean: Back Calculations

1.204: Back Calculations

$$W.Avg = \frac{ma + nb}{m + n}$$

The above equation for weighted average has five variables:

$$W.Avg, a, b, m, n$$

If we are given any four of these five values, then we can find the fifth by substituting into the equation and solving for the unknown variable.

Example 1.205

The heights of 12 boys and 10 girls in a class are recorded. The average height of all 22 students in the class is 103 cm. If the average height of the boys is 108 cm, then the average height of the girls is **(CEMC Gauss, Grade 8/2011/20)**

Substitute $Avg = 103, m = 12, n = 10, a = 108$ in $Avg = \frac{ma+nb}{m+n}$

$$103 = \frac{(12)(108) + 10b}{12 + 10}$$

$$b = \frac{(103)(22) - (12)(108)}{10} = \frac{2296 - 1296}{10} = \frac{970}{10} = 97$$

Example 1.206

The average age of the 40 members of a computer science camp is 17 years. There are 20 girls, 15 boys, and 5 adults. If the average age of the girls is 15 and the average age of the boys is 16, what is the average age of the adults? **(AMC 8 1999/13)**

$$Avg = \frac{px + qy + rz}{p + q + r}$$

The numbers from the question are:

$$Avg = 17$$

$$Age\ of\ Girls = x = 15, \qquad No.of\ Girls = p = 20$$

$$Age\ of\ Boys = 16, \qquad No.of\ Boys = q = 15$$

$$Age\ of\ Adults = z = ?, \qquad No.of\ Adults = r = 5$$

$$17 = \frac{(15)(20) + (16)(15) + 5z}{40}$$

$$5z = (17)(40) - 300 - 240 = 140$$

$$z = \frac{140}{5} = 28$$

Example 1.207

In 2003, the average monthly rainfall in Mathborough was 41.5 mm. In 2004, the average monthly rainfall in Mathborough was 2 mm more than in 2003. The total amount of rain that fell in Mathborough in 2004 was **(CEMC Gauss, Grade 8/2005/9)**

The average monthly rainfall in 2004

$$= 41.5 + 2 = 43.5$$

The total rainfall

$$= 43.5 \times 12 = 522\ mm$$

Example 1.208

A purse contains a collection of quarters, dimes, nickels, and pennies. The average value of the coins in the purse is 17 cents. If a penny is removed from the purse, the average value of the coins becomes 18 cents. How many nickels are in the purse? **(CEMC Gauss, Grade 8/2005/25)**

Example 1.209

Before the last of a series of tests, Sam calculated that a mark of 17 would enable her to average 80 over the series, but that a mark of 92 would raise her average mark over the series to 85. How many tests were there in the series? **(UKMT Cayley 2007/2)**

Let

$$Total\ so\ far = T\ marks$$
$$No.of\ tests = t$$

Then:

$$\frac{T+17}{t} = 80 \Rightarrow \underbrace{T + 17 = 80t}_{Equation\ I}$$
$$\frac{T+92}{t} = 85 \Rightarrow \underbrace{T + 92 = 85t}_{Equation\ II}$$

Subtract Equation I from Equation II:

$$75 = 5t \Rightarrow t = 15$$

Example 1.210

Chloe and Zoe are both students in Ms. Demeanor's math class. Last night, they each solved half of the problems in their homework assignment alone and then solved the other half together. Chloe had correct answers to only 80% of the problems she solved alone, but overall 88% of her answers were correct. Zoe had correct answers to 90% of the problems she solved alone. What was Zoe's overall percentage of correct answers? (AMC 8 2017/14)

In this question, it is faster to assume a number of questions in the assignment. Suppose

$$No.of\ Questions = Q = 100$$

Then, the question solved

$$Alone = Half\ of\ 100 = 50$$
$$Together = 100 - 50 = 50$$

Correct Questions from those solved alone

$$For\ Chloe = 80\%\ of\ 50 = 40$$
$$For\ Zoe = 90\%\ of\ 50 = 45$$

For Chloe, total correct questions

$$Total\ Correct\ Questions = 88\%\ of\ 100 = 88$$
$$Correct\ Questions\ solved\ together = 88 - 40 = 48$$

For Zoe, total correct Questions

$$= 45 + 45 = 93$$

K. Weighted Arithmetic Mean: Mixtures

Example 1.211

A mixture of 30 liters of paint is 25% red tint, 30% yellow tint and 45% water. Five liters of yellow tint are added to the original mixture. What is the percent of yellow tint in the new mixture? (AMC 8 2007/17)

The current amount of yellow tint

$$= 30\%\ of\ 30 = \frac{30}{100} \times 30 = 9$$

The amount of yellow tint in the new mixture

$$= 9 + 5 = 14$$

The percent of yellow tint in the new mixture

$$= \frac{14}{35} = \frac{2}{5} = 40\%$$

1.5 Range and More

A. Definition

1.212: Range

Range is the difference between the maximum and the minimum value in a data set.

$$Range = Max - Min$$

Example 1.213

A. What is the range of 25,35,17.
B. What is the range of 12?

$$Range = 35 - 17 = 18$$
$$Range = 12 - 12 = 0$$

1.214: Mid-Range

Mid-range is the value between the max and the min.

$$Mid - range = \frac{Max + Min}{2}$$

Example 1.215

The weights of 7 students in a class are given by the data set: $\{25,18,29,34,42,33,17\}$.

A. Find the range.
B. Find the mid-range.

$$Range = Max - Min = 42 - 17 = 25$$
$$Mid = Range = \frac{42 + 17}{2} = 29.5$$

Example 1.216

Hershey's is a bakery that sells pastries. They write down the number of pastries sold each day. The range of the number of pastries sold in the first week of 2023 was 7. The minimum number of pastries sold was on Wednesday, which was 13 pastries. Find the maximum number of pastries sold.

$$Range = Max - Min$$
$$7 = Max - 13$$
$$Max = 20$$

Example 1.217

Mark all correct options

Hershey's is a bakery that sells pastries. They write down the number of pastries sold each day. The range of the pastries sold in the year 2022 was 25. Then, which statement can be true regarding the minimum and maximum number of pastries sold in a particular day during the year:

A. The max was odd, and the min was even
B. The min was odd, and the max was even
C. The min and the max were both odd
D. The min and the max were both even

$$Range = Max - Min$$
$$25 = Max - Min$$
$$Min = Max - 25$$

We can take this with examples:

$$If\ max = 25 \to Min = 25 - 25 = 0 \Rightarrow Option\ A\ is\ valid$$
$$If\ max = 26 \to Min = 26 - 25 = 1 \Rightarrow Option\ B\ is\ valid$$

Check Option C:

$$Option\ C: Min = Odd - Odd = Even \Rightarrow Not\ valid$$
$$Option\ D: Min = Even - Odd = Odd \Rightarrow Not\ valid$$

Hence, the final answer is

$$Options\ A, B$$

B. Basics

Example 1.218

The range of a list of integers is 20, and the median is 17. What is the smallest possible number of integers in the list? **(UKMT JMC 2017/20)**

If have a single number in the list, the range will zero, since

$$Max = Min$$

If we do this with two numbers, the numbers must be:

$$x, x + 20$$

The median of two numbers is the average of the two middle numbers, which in this case means the average of the two numbers.

$$Median = Avg(x, x + 20)$$
$$17 = \frac{x + x + 20}{2}$$

$$34 = 2x + 20$$
$$14 = 2x$$
$$x = 7$$
$$x + 20 = 27$$

The numbers 7,20 work. Smallest possible number of integers
$$= 2$$

Example 1.219

Billy's basketball team scored the following points over the course of the first 11 games of the season: 42, 47, 53, 53, 58, 58, 58, 61, 64, 65, 73. If his team scores 40 in the 12th game, which of the following statistics will show an increase?

A. range
B. median
C. mean
D. mode
E. mid-range **(AMC 8 2015/5)**

$$Old\ Range = 64 - 42$$
$$New\ Range = 64 - 40$$

Hence, the range changes.

$$Option\ A$$

Example 1.220

Two integers are inserted into the list 3,3,8,11,28 to double its range. The mode and median remain unchanged. What is the maximum possible sum of the two additional numbers? **(AMC 8 2023/20)**

$$Current\ Range = 28 - 3 = 25$$
$$New\ Range = 25 \times 2 = 50$$

$$Current\ Mode = 3\ (occurs\ twice)$$

The mode must remain unchanged, so no number can occur twice in the new list.

$$3,3, \underbrace{8}_{Middle\ Value}, 11,28$$

The median must remain unchanged. So, one number must be less than 8, and the other number must be more than 8.

$$Option\ 1: x, 3,3, \underbrace{8}_{Middle\ Value}, y, 11,28$$
$$Option\ 2: 3,3, x, \underbrace{8}_{Middle\ Value}, 11,28, y$$

Since we want the sum of the two possible numbers to be greatest, we choose Option 2.

$$x = 7, y = 3 + 50 = 53$$
$$Sum = 7 + 53 = 60$$

Example 1.221

On coach Wooden's basketball team:

• Meghan is the tallest player,
• Meghan's height is 188 cm, and
• Avery is the shortest player.
When used with the information above, which of the following single statements is enough to determine Avery's height?
(A) The median of the players' heights is 170 cm
(B) The mode of the players' heights is 160 cm
(C) The mean of the players' heights is 165 cm
(D) The range of the players' heights is 33 cm
(E) There are 10 players on the team **(CEMC Gauss, Grade 8/2017/17)**

Substitute $Max = 188, Min = 33$ in $Range = Max - Min$

$$33 = 188 - Min \Rightarrow Can\ be\ solved$$

$$Option\ D$$

C. Mean and Median

Example 1.222

The number of pens that students in a class have is given below:

2,0,3,1,2,1

Is the median less than, more than or equal to the mean?

$$Mean = \frac{0+1+1+2+2+3}{6} = \frac{9}{6} = 1.5$$

$$0,1,1,2,2,3$$

$$Median = Avg(1,2) = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

Example 1.223

The mean score of five students on a test is equal to their median score. If the student with median marks has m marks, find the sum of the marks of the other four students.

$$a, b, m, c, d$$

$$\frac{a+b+m+c+d}{5} = m$$
$$a+b+m+c+d = 5m$$
$$a+b+c+d = 4m$$

Example 1.224

Five students take a test on which any integer score from 0 to 100 inclusive is possible. What is the largest possible difference between the median and the mean of the scores? **(IOQM 2021/10)**

$$0,25,50,75,100 \Rightarrow Mean = 50, Median = 50 \Rightarrow Diff = 0$$

To make the difference the maximum, we want make the median the minimum, and the mean the maximum.

$$Median = 3rd\ Value$$

Hence, the 1st and 2nd values must be less than or equal to median.
The last values can be maximum.

Finally, we get:

$$0,0,0,100,100 \Rightarrow \frac{200}{5} - 0 = 40$$

D. Mixing

Example 1.225
A set of five different positive integers has a mean (average) of 20 and a median of 18. What is the greatest possible integer in the set? (Gauss Grade 7 2012/19)

The middle number is the median. Then, the integers arranged in ascending order are:

$$a, b, 18, d, e$$

$$\frac{a+b+18+d+e}{5} = 20 \Rightarrow a+b+18+d+e = 100 \Rightarrow a+b+d+e = 82$$

The smallest value each integer can take is given below

$$a = 1, b = 2, d = 19$$

$$e = 82 - 1 - 2 - 19 = 60$$

Example 1.226
A fifth number, n, is added to the set $\{3,6,9,10\}$ to make the mean of the set of five numbers equal to its median. The number of possible values of n is: (AMC 8 1988/21)

$$Avg\ of\ five\ numbers = \frac{3+6+9+10+n}{5} = \frac{28+n}{5}$$

Arranging the values in the set in ascending order after adding n gives us the following cases:

$$\underbrace{\{3,6,9,10,n\}, \{3,6,9,n,10\}}_{Median=9}, \quad \underbrace{\{3,6,n,9,10\}}_{Median=n}, \quad \underbrace{\{3,n,6,9,10\}, \{n,3,6,9,10\}}_{Median=6}$$

These cases give us the following values for n:

$$\underbrace{\frac{28+n}{5} = 9 \Rightarrow n = 17}_{Median=9}, \quad \underbrace{\frac{28+n}{5} = n \Rightarrow n = 7}_{Median=n}, \quad \underbrace{\frac{28+n}{5} = 6 \Rightarrow n = 2}_{Median=6}$$

Challenge 1.227
When the mean, median, and mode of the list $10, 2, 5, 2, 4, 2, x$ are arranged in increasing order, they form a non-constant arithmetic progression. What is the sum of all possible real values of x ? (AMC 10 2000/23)

Mode and Mean
The value 2 occurs thrice. Hence, independent of the value of x, we must have:

$$Mode = Highest\ Frequency = 2$$

$$Mean = \frac{10+2+5+2+4+2+x}{7} = \frac{25+x}{7}$$

Median

For the median, arrange the numbers in ascending order:

$$2,2,2,4,5,10$$

The value of the median will depend on the value of x:

$$\underbrace{x,2,2,2,4,5,10}_{Median=2},\quad \underbrace{2,2,2,x,4,5,10}_{Median=x},\quad \underbrace{2,2,2,4,5,10,x}_{Median=4} \Rightarrow Median \in \{2,x,4\}$$

Case I: $Median = 2$

$$Mode = Median = 2 \Rightarrow AP\ is\ constant \Rightarrow \boldsymbol{Contradiction}$$

Case II: $Median = 4, x > 4$

$$Mode = 2, Median = 4 \Rightarrow AP = \{0,2,4\}\{2,3,4\}\{2,4,6\}$$

$$\underbrace{\frac{25+x}{7} = 0 \Rightarrow x = -25}_{x>4\Rightarrow Contradiction},\quad \underbrace{\frac{25+x}{7} = 3 \Rightarrow x = -4}_{x>4\Rightarrow Contradiction},\quad \underbrace{\frac{25+x}{7} = 6 \Rightarrow x = 17}_{No\ Contradiction}$$

Case III: $Median = x, 2 < x < 4$

$$Mode = 2, Median = x, Mean = \frac{25+x}{7}$$

Using the property that the middle term in an arithmetic progression is the mean of the terms that come before and after it:

$$\frac{2+\frac{25+x}{7}}{2} = x \Rightarrow 2 + \frac{25+x}{7} = 2x \Rightarrow x = 3$$

Sum of all values

$$= 3 + 17 = 20$$

Example 1.228

One day the Beverage Barn sold 252 cans of soda to 100 customers, and every customer bought at least one can of soda. What is the maximum possible median number of cans of soda bought per customer on that day? **(AMC 8 2014/24)**

If the number of cans sold is arranged in ascending order:

$$Median = Avg(50th, 51st)$$

To maximize the median, we want to minimize the first 49 numbers, which each have to be at least 1.

$$First\ 49\ Numbers = 1\ each \Rightarrow Sum = 49$$

The remaining cans are:

$$252 - 49 = 203$$

Divide 203 among the remaining 51 customers:

$$\underbrace{3}_{One\ 3}, \underbrace{4,4,\ldots 4}_{Fifty\ 4's}$$

The median will be:

$$Avg(3,4) = 3.5$$

E. Mixing Mean, Median, Mode

Example 1.229

The mean of four positive integers is 5. The median of the four integers is 6. What is the mean of the largest and smallest of the integers? **(UKMT JMC 2020/13)**

Let the numbers arranged in ascending order be:

$$a, b, c, d$$

The median

$$\frac{b+c}{2} = 6 \Rightarrow b + c = 12$$

The mean is:

$$\frac{a+b+c+d}{4} = 5 \Rightarrow a + 12 + d = 20 \Rightarrow a + d = 8 \Rightarrow \frac{a+d}{2} = 4$$

Example 1.230

George wrote seven tests and each was marked out of 100. No two of his marks were the same.
He recorded the seven marks to do a statistical analysis. He accidentally recorded his highest mark higher than it actually was. How many of the following are altered because of his mistake?
- Mean
- Median
- Minimum test score
- Range **(CEMC Gauss, Grade 8/2003/14)**

Let the numbers arranged in ascending order be:

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7$$

Suppose the highest mark gets recorded as $x_7 + 1$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 + 1$$

$$Mean\ will\ change$$
$$Median = x_4\ is\ unchanged$$
$$Min = x_1\ is\ unchanged$$
$$Range = x_7 + 1 - x_1\ will\ change$$

$$2\ of\ them\ will\ change$$

Example 1.231

Veronica has 6 marks on her report card.
The mean of the 6 marks is 74.
The mode of the 6 marks is 76.
The median of the 6 marks is 76.
The lowest mark is 50.
The highest mark is 94.
Only one mark appears twice and no mark appears more than twice.
Assuming all of her marks are integers, the number of possibilities for her second lowest mark is (CEMC Gauss, Grade 8/2002/14)

Let the numbers arranged in ascending order be $x_1, x_2, x_3, x_4, x_5, x_6$.

$$Min = x_1 = 50, \qquad Max = x_6 = 94, \qquad Median = \frac{x_3 + x_4}{2} = 76$$

But mode is also 76. And only one mark appears twice. Hence

$$x_3 = x_4 = 76$$

The average of $50, x_2, 76, 76, x_5, 94$ is

$$\frac{50 + x_2 + 76 + 76 + x_5 + 94}{6} = 74 \Rightarrow x_2 + x_5 = 148$$

Since $x_2 > x_1 = 50$, try

$$x_2 = 51 \Rightarrow 51 + x_5 = 148 \Rightarrow x_5 = 148 - 51 = 97 \Rightarrow Not\ Valid$$
$$x_5 = 93 \Rightarrow x_2 + 93 = 148 \Rightarrow x_2 = 148 - 93 = 55 \Rightarrow Valid$$

$$x_2 = 75 \Rightarrow 75 + x_5 = 148 \Rightarrow x_5 = 73 \Rightarrow Not\ Valid$$
$$x_5 = 77 \Rightarrow x_2 + 77 = 148 \Rightarrow x_5 = 71 \Rightarrow Valid$$

Finally:

$$x_2 \in \{55, 56, \dots, 71\} \Rightarrow 71 - 55 + 1 = 17\ Values$$

Example 1.232

In which set of scores is the median greater than the mean?

A. 10, 20, 40, 40, 40
B. 40, 50, 60, 70, 80
C. 20, 20, 20, 50, 80
D. 10, 20, 30, 100, 200
E. 50, 50, 50, 50, 100 (CEMC Gauss, Grade 8/2014/13)

Try Option A

$$Median = 40$$
$$Mean(10,20,40,40,40) < 40$$

$$Option\ A\ is\ correct$$

Example 1.233

The mean (average), the median and the mode of the five numbers 12, 9, 11, 16, x are all equal. What is the value of x? (CEMC Gauss, Grade 8/2019/13)

The mode must be x, since all the other numbers appear once.

We use casework:

$$x = 9 \Rightarrow Numbers\ are\ 9,9,11,12,16 \Rightarrow Mean \neq 9$$
$$x = 16 \Rightarrow Numbers\ are\ 9,11,12,16,16 \Rightarrow Mean \neq 16$$
$$x = 11 \Rightarrow 9,11,11,12,16 \Rightarrow Mean = \frac{9 + 11 + 11 + 12 + 16}{5} = \frac{59}{5} \neq 11$$
$$x = 12 \Rightarrow 9,11,12,12,16 \Rightarrow Mean = \frac{9 + 11 + 12 + 12 + 16}{5} = \frac{60}{5} = 12 \Rightarrow Valid$$

Example 1.234

The scores of eight students on a quiz are 6, 7, 7, 8, 8, 8, 9, and 10. Which score should be removed to leave seven scores with the same mode and range as the original eight scores, but with a higher average (mean)? (CEMC Gauss, Grade 7/2004/18)

We want to remove the lowest score to increase the average.
If we remove 6, the range will change, so 6 does not work.

If we remove 7, mode remains 8, and range remains 4.

$$Answer = 7$$

Example 1.235

Five students wrote a quiz with a maximum score of 50. The scores of four of the students were 42, 43, 46, and 49. The score of the fifth student was N. The average (mean) of the five students' scores was the same as the median of the five students' scores. The number of values of N which are possible is **(CEMC Gauss, Grade 7/2006/25)**

$$42,43,46,49, N$$

We will do this using casework.

Case I: If $N \le 43$

$$N, 42,43,46,49 \Rightarrow Median = 43$$

$$\frac{N + 42 + 43 + 46 + 49}{5} = 43$$
$$N + 42 + 43 + 46 + 49 = 43 + 43 + 43 + 43 + 43$$
$$N = 1 + 0 - 3 - 6 + 43 = 35 < 43 \Rightarrow Valid$$

Case II: If $N \ge 46$

$$42,43,46, N, 49 \Rightarrow Median = 46$$

$$\frac{N + 42 + 43 + 46 + 49}{5} = 46$$
$$N + 42 + 43 + 46 + 49 = 46 + 46 + 46 + 46 + 46$$
$$N = 4 + 3 + 0 - 3 + 43 = 50 > 46 \Rightarrow Valid$$

Case III: If $43 < N < 46$

$$42,43, N, 46, 49 \Rightarrow Median = N$$

$$\frac{N + 42 + 43 + 46 + 49}{5} = N$$
$$N + 180 = 5N$$
$$4N = 180$$
$$N = 45 \Rightarrow Valid$$

$$3\ Values$$

Example 1.236

Three numbers have a mean (average) of 7. The mode of these three numbers is 9. What is the smallest of these three numbers? (CEMC Gauss, Grade 8/2012/13)

$$\frac{a+b+c}{3} = 7 \Rightarrow a+b+c = 21$$

$$(b,c) = (9,9)$$
$$a+9+9 = 21 \Rightarrow a = 3 = Smallest$$

Example 1.237

What is the sum of the mean, median, and mode of the numbers 2,3,0,3,1,4,0,3? (AMC 8 2010/4)

$$0,0,1,2,\underbrace{3,3,3}_{Mode},4 \Rightarrow Mode = 3$$
$$0,0,1,\underbrace{2,3}_{Median},3,3,4 \Rightarrow Avg(2,3) = \frac{2+3}{2} = \frac{5}{2} = 2.5$$
$$Mean = \frac{0+0+1+2+3+3+3+4}{8} = \frac{1+2+9+4}{8} = \frac{16}{8} = 2$$

$$Sum = 3 + 2.5 + 2 = 7.5$$

Example 1.238

The mean, median, and unique mode of the positive integers 3, 4, 5, 6, 6, 7, and x are all equal. What is the value of x? (AMC 8 2012/11)

$$3,4,5,6,6,7,x \Rightarrow Mode = 6$$
$$Mean = \frac{3+4+5+6+6+7+x}{7} = 6$$
$$31 + x = 42$$
$$x = 11$$

Example 1.239

Hammie is in the 6^{th} grade and weighs 106 pounds. Her quadruplet sisters are tiny babies and weigh 5, 5, 6, and 8 pounds. Which is greater, the average (mean) weight of these five children or the median weight, and by how many pounds? (AMC 8 2013/5)

$$5,5,6,8,106$$
$$Median = 6$$
$$Mean = \frac{5+5+6+8+106}{5} = 26$$

$$Mean - Median = 26 - 6 = 20$$

Example 1.240

There is a set of five positive integers whose average (mean) is 5, whose median is 5, and whose only mode is 8. What is the difference between the largest and smallest integers in the set? (AMC 8 1997/14)

$$\frac{a+b+c+d+e}{5}=5 \Rightarrow a+b+c+d+e=25$$

$$Median = c = 5$$
$$Mode = d = e = 8$$

$$a+b+5+8+8=25$$
$$a+b=4$$
$$a \neq b \text{ since } 8 \text{ is the only mode}$$
$$a=1, b=3$$

Difference
$$=8-1=7$$

Example 1.241
The mean of a set of five different positive integers is 15. The median is 18. The maximum possible value of the largest of these five integers is (AMC 8 2001/21)

Example 1.242
The numbers -2, 4, 6, 9 and 12 are rearranged according to these rules:
The largest isn't first, but it is in one of the first three places.
The smallest isn't last, but it is in one of the last three places.
The median isn't first or last.
What is the average of the first and last numbers? (AMC 8 2004/11)

Example 1.243
Here is a list of the numbers of fish that Tyler caught in nine outings last summer: 2,0,1,3,0,3,3,1,2. Which statement about the mean, median, and mode is true?
median $<$ mean $<$ mode
mean $<$ mode $<$ median
mean $<$ median $<$ mode
median $<$ mode $<$ mean
mode $<$ median $<$ mean (AMC 8 2011/4)

Example 1.244
The diagram shows the number of students at soccer practice each weekday during last week. After computing the mean and median values, Coach discovers that there were actually 21 participants on Wednesday. Which of the following statements describes the change in the mean and median after the correction is made?
The mean increases by 1 and the median does not change.
The mean increases by 1 and the median increases by 1.
The mean increases by 1 and the median increases by 5.
The mean increases by 5 and the median increases by 1.
The mean increases by 5 and the median increases by 5. (AMC 8 2019/10)

Example 1.245
The heights of 4 athletes on a team are 135 cm, 160 cm, 170 cm, and 175 cm. Laurissa joins the team. On the new team of 5 athletes, the mode height of the players is equal to the median height which is equal to the mean (average) height. How tall is Laurissa? (CEMC Gauss, Grade 7/2019/10)

1.6 Harmonic Mean

A. Definition

	For two values	For n values
Harmonic Mean (n times the sum of the reciprocals of n numbers)	$\text{HM}(a,b) = \dfrac{2}{1/a + 1/b} = \dfrac{2}{\frac{a+b}{ab}} = \dfrac{2ab}{a+b}$	$\dfrac{n}{1/x_1 + 1/x_2 \dots + 1/x_n}$
Weighted Harmonic Mean	$\text{HM}(na, mb) = \dfrac{n+m}{n/a + m/b}$	

Uses

There is an inverse relationship between

- Speed and time
- Cost and number of units

The arithmetic mean does not give the correct answer when averaging speed and cost.

Hence, the harmonic mean is used when averaging:
Speed: When same distances are travelled
Cost of purchase: When same number of units are purchased

B. Calculations

$$HM(x,y) = 2 \times \frac{1}{\frac{1}{x} + \frac{1}{y}} = \frac{2}{\frac{x+y}{xy}} = \frac{2xy}{x+y}$$

Example 1.246

Calculate the harmonic mean:

A. 20, 40

$$HM(20,40) = \frac{2 * 20 * 40}{20 + 40} = \frac{1600}{60} = \frac{80}{3} = 26\frac{2}{3}$$

Example 1.247

The harmonic mean of two numbers can be calculated as twice their product divided by their sum. The harmonic mean of 1 and 2016 is closest to which integer? **(AMC 12B 2016/2)**

$$\frac{2xy}{x+y} = \frac{(2)(1)(2016)}{1 + 2016} = \frac{2 \times 2016}{2017} = \frac{4032}{2017} = 2 - \frac{2}{2017} \Rightarrow Very\ close\ to\ 2$$

Example 1.248

Mark True or False

The harmonic mean of two numbers is a good approximation for the arithmetic mean of the two numbers.

Consider the numbers 1 and 2016:

$$Arithmetic\ Mean = \frac{x+y}{2} = \frac{2017}{2} = 1008.5$$

$$Harmonic\ Mean = \frac{2xy}{x+y} = \frac{(2)(1)(2016)}{1+2016} \approx 2$$

Hence, the statement is false.

C. Average Speed

1.249: Averaging Rates

When averaging rates, the correct mean to apply is the harmonic mean, and not the arithmetic mean.

Some examples of rates are:

- Speed at which you travel ($a\ Time, Speed, Distance\ Concept$)
- Cost at which you purchase a quantity
- The speed at which you complete a task ($for\ example, baking\ cakes, or\ making\ a\ chair$).

Example 1.250

A car travels from Mumbai to Pune at an average speed of 40 km/hr, and back at an average speed of 60 km/hr. What is the average speed for the entire journey?

Assume:
Distance = LCM (40, 60) = 240 km
Then:
Forward journey = 240/40 = 6 hours
Return journey = 240/60 = 4 hours
Speed
= Total Distance / Total Time = 480 / 10 = 48

Example 1.251

You travel from Mumbai to Pune at $x\frac{km}{hr}$, and travel back at $y\frac{km}{hr}$. Find the average speed for the entire journey.

$$Distance\ from\ Mumbai\ to\ Pune = Distance\ from\ Pune\ to\ Mumbai = D$$

$$S = \frac{Total\ Distance}{Total\ Time} = \frac{2D}{\frac{D}{x}+\frac{D}{y}} = \frac{2D}{\frac{Dy+Dx}{xy}} = 2D \times \frac{xy}{D(x+y)} = \frac{2xy}{x+y}$$

Note that this requires the distances travelled to be equal.
We will remove this restriction further on in the Notes.

1.252: Averaging Speed over Equal Distances

If you travel equal distances at different speeds, the average speed over the entire journey is the harmonic mean of the speeds.

Example 1.253

A car travels 120 miles from A to B at 30 miles per hour but returns the same distance at 40 miles per hour. The average speed for the round trip is closest to which integer? **(AHSME 1950/27)**

$$T = \frac{D}{S} = \frac{120 \times 2}{\frac{120}{30} + \frac{120}{40}} = \frac{240}{7} \approx 34\ hr$$

(Alternate Solution) Example 1.254

For this question, 30 and 40 both divide 120, but this is not necessary, since we can cancel the 120 from the expression. This is done below.

$$T = \frac{D}{S} = \frac{120 \times 2}{\frac{120}{30} + \frac{120}{40}} = \frac{120 \times 2}{\frac{120(40+30)}{30 \times 40}} = \frac{2 \times 30 \times 40}{70} = \frac{240}{7}$$

Example 1.255

Samia set off on her bicycle to visit her friend, traveling at an average speed of 17 kilometers per hour. When she had gone half the distance to her friend's house, a tire went flat, and she walked the rest of the way at 5 kilometers per hour. In all it took her 44 minutes to reach her friend's house. In kilometers rounded to the nearest tenth, how far did Samia walk? **(AMC 10B 2017/7)**

Method I: Harmonic Mean

Let the total distance be D, and note that distance travelled by cycling and walking are equal.
Samia travelled $\frac{D}{2}\,km$ at an average speed of $17\frac{km}{hr}$, and $\frac{D}{2}\ km$ at an average speed of $5\frac{km}{hr}$.

$$Average\ Speed\ of\ entire\ journey = HM(5,17) = \frac{2(5)(17)}{5+17} = \frac{2(85)}{22} = \frac{85}{11}\frac{km}{hr}$$

$$Distance = D = \frac{85}{11}\frac{km}{hr} \times \frac{44}{60}hr = \frac{85}{11} \times \frac{11}{15} = \frac{17}{3}\ km$$

The distance that Samia walked is:

$$\frac{D}{2} = \frac{17}{3} \times \frac{1}{2} = \frac{17}{6} = 2.8\overline{3} \approx 2.8$$

Method II: TSD

Suppose

$$Distance\ Walked = Distance\ Cycled = x$$

	$Cycling$	$Walking$	$Total$
Distance	x	x	$2x$
Time	$\frac{D}{S} = \frac{x}{17}$	$\frac{D}{S} = \frac{x}{5}$	$\frac{22}{85}x$

$$\frac{22}{85}x = \frac{44}{60}$$

$$x = \frac{44}{60} \times \frac{85}{22} = \frac{17}{6} = 2.8\overline{3} \approx 2.8$$

Example 1.256

George walks 1 mile to school. He leaves home at the same time each day, walks at a steady speed of 3 miles per hour, and arrives just as school begins. Today he was distracted by the pleasant weather and walked the first $\frac{1}{2}$ mile at a speed of only 2 miles per hour. At how many miles per hour must George run the last $\frac{1}{2}$ mile in order to arrive just as school begins today? **(AMC 8 2014/17)**

Note that $Distance\ Walked = Distance\ Run = \frac{1}{2}$

$$Average\ Speed = HM(2, x) = 3\ Miles$$
$$\frac{2(2)(x)}{2 + x} = 3$$
$$4x = 6 + 3x$$
$$x = 6\frac{miles}{hr}$$

Example 1.257
Mr. Earl E. Bird gets up every day at 8:00 AM to go to work. If he drives at an average speed of 40 miles per hour, he will be late by 3 minutes. If he drives at an average speed of 60 miles per hour, he will be early by 3 minutes. How many miles per hour does Mr. Bird need to drive to get to work exactly on time? **(AMC 10A 2002/12, AMC 12A 2002/11)**

$$Average\ Speed = \frac{Total\ Distance}{Total\ Time} = \frac{2D}{\frac{D}{40} + \frac{D}{60}} = HM(40,60) = 48$$

D. Average Rate

Example 1.258
Ray's car averages 40 miles per gallon of gasoline, and Tom's car averages 10 miles per gallon of gasoline. Ray and Tom each drive the same number of miles. What is the cars' combined rate of miles per gallon of gasoline? **(AMC 12B 2014/4)**

$$\frac{(2)(40)(10)}{10 + 40} = \frac{800}{50} = 16$$

E. Average Cost

Example 1.259
If 1782 quintals of grains were bought at a rate of 72 Rs., and another 1782 quintals were bought at a rate of Rs. 92, what is the cost of 205 quintals at the average rate?

$$\text{HM}\ (72, 92)\ \times\ 205 = \frac{2 * 72 * 92}{72 + 92} \times 205 = \frac{2 * 72 * 92}{164} \times 205 = \frac{36 * 92}{41} \times 205 = 16560$$

F. Estimation

Example 1.260

$$x = \frac{1}{\frac{1}{71} + \frac{1}{72} + \frac{1}{73} + \frac{1}{74} + \frac{1}{75}}$$

A. Estimate x. (No Calculator)
B. Use a calculator to find the value of x.
C. Find the difference between the two.
D. What is the percentage error in your estimation?

Estimate:

$$x \approx \frac{1}{\frac{1}{73}+\frac{1}{73}+\frac{1}{73}+\frac{1}{73}+\frac{1}{73}} = \frac{1}{\frac{5}{73}} = \frac{73}{5} = 14\frac{3}{5} = 14.6 \Rightarrow Integer\ part\ is\ 14$$

$$x = 14.59451$$
$$14.6 - 14.59451 = 0.00549$$
$$\frac{0.00549}{14.59451} = 0.037\%$$

G. Weighted Harmonic Mean

Example 1.261
A car travels from Delhi to Bangalore at an average speed of 50 km/hr. On the return trip, it makes a detour, travelling twice the distance at an average speed of 25 km/hr. What is the average speed for the entire journey?

$$\text{WHM}(50, 25 \times 2) = \frac{3}{\frac{1}{50}+\frac{2}{25}} = \frac{3}{\frac{1+4}{50}} = 3 \times \frac{50}{5} = 30$$

Alternative Method:
Assume Distance from Delhi to Bangalore = LCM(25, 50) = 50
Time(D-B) = D/S = 50/50 = 1
Time(B-D) = D/S = 100/25 = 4

$$Avg\ Speed = \frac{Total\ Distance}{Total\ Speed} = \frac{150}{5} = 30$$

Example 1.262
If you walk for 45 minutes at a rate of 4 mph and then run for 30 minutes at a rate of 10 mph how many miles will you have gone at the end of one hour and 15 minutes? (AMC 8 1985/13)

1.7 Frequency Distributions

A. Basics

Example 1.263
Consider the following distribution showing the number of burgers eaten in a year by a set of 35 students from a school.

No. of Burgers	0-10	11-20	21-30	31-40	41-50
Frequency	5	7	12	6	5

A. Find the cumulative frequency, and state what it represents.
B. Find the interval in which the median lies.
C. Which is the modal interval?

Part A

- Cumulative frequency for a particular class represents the number of students who eaten that many burgers or less.
- For example, the cumulative frequency for the class $21-30$ is 24. This means that there are

No. of Burgers	0-10	11-20	21-30	31-40	41-50
Frequency	5	7	12	6	5
Cumulative Frequency	5	12	24	30	35

24 students who have eaten 30 burgers or less.

Part B

There are 35 students. If we arrange the students according to number of burgers eaten, the median value will be

$$\frac{35+1}{2}=\frac{36}{2}=18^{th}\ Student \Rightarrow 11-20\ Burgers$$

Part C

The modal interval is the interval with highest frequency. Hence, it is $21-30$ burgers, which has frequency 12.

B. Rates

Example 1.264

The elevation of Lake Ontario is 75.00 m and the elevation of Lake Erie is 174.28 m. A ship travels between the two lakes, passing through the locks of the Welland Canal. If the ship takes 8 hours to travel between the lakes, the average (mean) change in elevation per hour is (CEMC Gauss, Grade 7/2004/12)