Equations

26 January 2025

Revision: 3193

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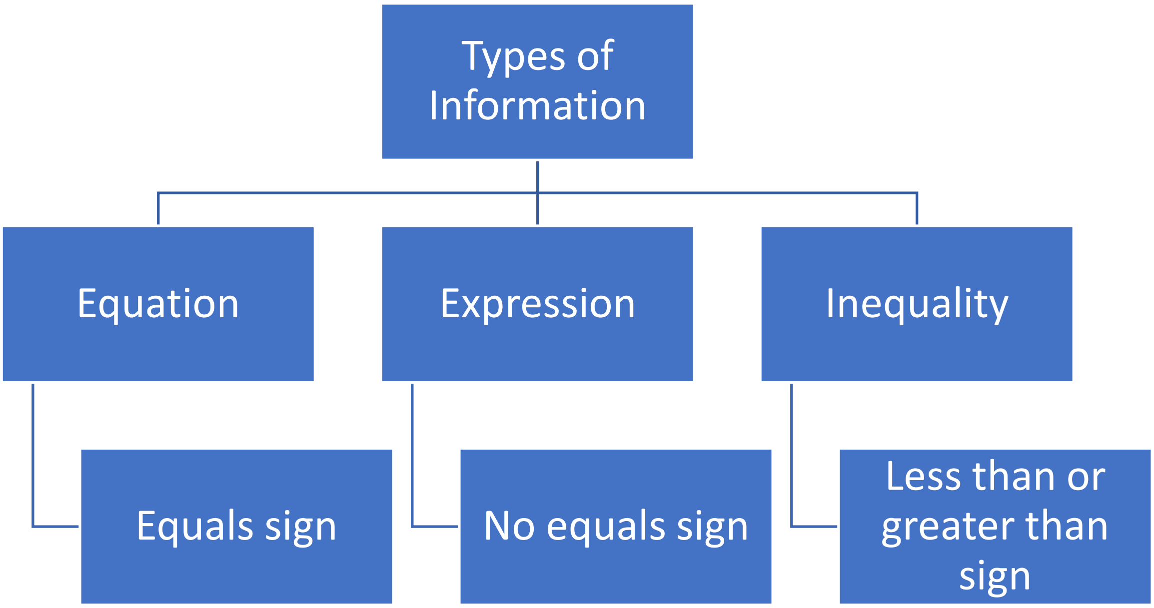
1. Single Variable Equations

1.1 Solving Simple Equations

A. Some terminology and notation

We begin with some important terminology related to algebra. This is useful as long as you are doing Algebra.



1.1: Expressions
- An algebraic expression is made of one or more algebraic terms.
- It does not have an equality or an inequality sign.

We have already looked at Expressions in the previous Note. Some of the kinds of questions which come with respect to expressions are:

- Evaluating an expression.
  - ✓ For example, evaluating $2t + \frac{t}{2}$, when $t = \frac{2}{3}$.
- Simplifying an expression.
  - ✓ For example, simplifying $2t + \frac{t}{2}$ in order to get a fraction with a single denominator
- Set up an expression that models the behavior of a quantity of interest.
  - ✓ For example, if the weekly wages for a paper delivery route consists of $20 plus $5 for each hour worked then the total wages for the week (assuming h hours worked) will be $5h + 20$.

Some concepts related to expressions include:

- Distributive Property used to open brackets in expressions
  - ✓ $2(x + 3y) = 2x + 6y$
- Factoring
  - ✓ $6x + 9y = 3(2x + 3y)$
- Binomial Expansions
  - ✓ $(a + b)^2 = a^2 + 2ab + b^2$
  - ✓ $(a - b)^2 = a^2 - 2ab + b^2$

- ✓ $(a+b)(a-b)=a^2-b^2$

1.2: Inequalities

An inequality has one of the following signs:

$$>: Greater\ Than$$
$$<: Less\ Than$$
$$\geq: Greater\ Than\ or\ Equal\ To$$
$$\leq: Less\ Than\ or\ Equal\ To$$

When one quantity is greater than another quantity, we get an inequality. For example:

- The number of people attending a party are more than 100. ($p>100$)

Similarly, one quantity may be greater than or equal to another quantity. For example:

- The passing marks for a subject are 35 or more out of 100. ($m\geq 35, m\leq 100$)

1.3: Equations

- An equation has a single equality sign.
- An equation is meant to be solved step by step to determine the value(s) of the variables that satisfy the equation.

Example 1.4

Explain why

A. $3x+9=7$ is an equation.
B. $2x<3$ is an inequality
C. $3x+6y+2z$ is an expression

$3x+9=7$ is an equation because it has a single equality sign.
$2x<3$ is an inequality because it has an inequality sign.
$3x+6y+2z$ is an expression because it has neither an equality sign nor an inequality sign.

Example 1.5

Classify the following as equations, expressions, or inequalities:

A. $3x+4$
B. $2x+4=7$
C. $2x<7$
D. $\frac{5}{9}x+\frac{3}{2}=\frac{7}{4}x-\frac{1}{3}$
E. $\frac{3}{2}x+\frac{4}{5}x-\frac{2}{3}+\frac{1}{2}x$
F. $4x+\frac{3}{4}\geq\frac{1}{2}$

$$3x+4: \text{Expression}$$
$$2x+4=7: \text{Equation}$$
$$2x<7: \text{Inequality}$$
$$\frac{5}{9}x+\frac{3}{2}=\frac{7}{4}x-\frac{1}{3}: Equation$$
$$\frac{3}{2}x+\frac{4}{5}x-\frac{2}{3}+\frac{1}{2}x: Expression$$
$$4x+\frac{3}{4}\geq\frac{1}{2}: Inequality$$

1.6: Notation

You solve an equation; you simplify an expression.

$$x + 2x = 3x\ (Simplified)$$
$$x + 5 = 7 \Rightarrow x = 2\ (Solved)$$

1.7: Parts of an Equation

An equation with an equality sign has

- An expression on the left side of the equality
  - This expression is called $LHS(Left\ Hand\ Side)$
- Another expression on the right hand of the equality
  - This expression is called $RHS(Right\ Hand\ Side)$

Example 1.8

Identify the left-hand side, and the right-hand side in the equations below:

A. $a + 3 = 7$
B. $5d + 5 = 3d + 2$
C. $\frac{2}{3}s + 3 = \frac{1}{2}s - \frac{5}{7}$
D. $2.3s - 4.1 = 3.2s + 6.5$
E. $\frac{1}{2}x - \frac{2}{3} = \frac{3}{2}x - \frac{1}{2}$

$$\underbrace{a + 3}_{\textbf{Left Hand Side}} = \underbrace{7}_{\textbf{Right Hand Side}}$$

$$\underbrace{5d + 5}_{\textbf{LHS}} = \underbrace{3d + 2}_{\textbf{RHS}}$$
$$\underbrace{\frac{2}{3}s + 3}_{LHS} = \underbrace{\frac{1}{2}s - \frac{5}{7}}_{RHS}$$
$$\underbrace{2.3s - 4.1}_{LHS} = \underbrace{3.2s + 6.5}_{RHS}$$
$$\underbrace{\frac{1}{2}x - \frac{2}{3}}_{LHS} = \underbrace{\frac{3}{2}x - \frac{1}{2}}_{RHS}$$

B. Verifying Solutions

1.9: Solution of an Equation

Given an equation, a value of the variable such that the equality is true is called a solution of the equation.

Example 1.10

Verify that the following solutions are valid:

A. $3x + 5 = 11, x = 2$
B. $2x - 4 = -8, x = -2$
C. $\frac{x}{2} + \frac{1}{2} = \frac{3}{2}, x = 2$

For each equation, we show that

$$LHS = RHS$$

Part A

$$LHS = 3x + 5 = 3(2) + 5 = 6 + 5 = 11 = RHS$$

Since the LHS is equal to the RHS, $x = 2$ is a valid solution

Part B

$$LHS = 2x - 4 = 2(-2) - 4 = -8 = RHS \Rightarrow Valid$$

Part C

$$LHS = \frac{x}{2} + \frac{1}{2} = \frac{2}{2} + \frac{1}{2} = \frac{3}{2} = RHS \Rightarrow Valid$$

Example 1.11

Verify that the following are not valid solutions:

A. $2x - 7 = 12, x = 5$
B. $4x + 2 = 3x, x = 1$

Part A

$$LHS = 2x - 7 = 2(5) - 7 = 10 - 7 = 3 \neq 12 \Rightarrow Not\ a\ valid\ solution$$

Part B

$$LHS = 4x + 2 = 4(1) + 2 = 4 + 2 = 6$$
$$RHS = 3x = 3(1) = 3$$
$$LHS \neq RHS$$

Example 1.12

Check whether the following are valid solutions or not:

A. $x + 6 = 12, x = -3$
B. $x - 6 = \frac{1}{2}, x = 6.5$

Part A

$$LHS = -3 + 6 = 3 \neq 12 = RHS \Rightarrow Not\ Valid$$

Part B

$$LHS = 6.5 - 6 = 0.5 = \frac{1}{2} = RHS \Rightarrow Valid$$

C. Nature of Solutions

1.13: Solving an Equation

- Solving an equation requires finding values such that the equality becomes true.
- For simple linear equations, there is only one solution.

For non-linear equations, or certain types of linear equations there can be:

- Single Solution
- Multiple solutions
- No solutions
- Infinite solutions

We will initially focus on questions that have a single solution, and talk the other cases later.

Example 1.14

$$x + 5 = 8 \Rightarrow 1\ Solution: x = 3$$

$$x + 5 = x + 3 \Rightarrow No\ Solution$$
$$x + 2 = x + 2 \Rightarrow Infinite\ Solutions$$

D. Single Step Equations

1.15: Solving Equations with Addition

$$x + a = b \Rightarrow x = b - a$$

$$x + 6 = 8$$

A value of 2 makes the equation true. We can this more formally by subtracting 6 from both sides:

$$x + 6 = 8 \Rightarrow x = 8 - 6 = 2$$

Example 1.16: Equations with Addition

Find the value of the letters so that the equation is true.

Basics

1. $a + 4 = 5$
2. $b + 7 = 12$
3. $z + 3 = 3$
4. $d + 4 = 9$
5. $e + 7 = 8$
6. $s + 4 = 2$
7. $z + 5 = 1$
8. $q + 12 = 0$
9. $w + 2020 = 1$

Fractions

10. $x + \frac{3}{5} = \frac{4}{5}$
11. $q + \frac{2}{3} = \frac{5}{3}$
12. $e + \frac{4}{5} = \frac{2}{5}$
13. $w + \frac{1}{3} = \frac{1}{2}$
14. $w + \frac{1}{2} = \frac{1}{3}$

Decimals

15. $a + 0.2 = 0.6$
16. $b + 0.31 = 0.4$
17. $c + 0.5 = 0.8$
18. $d + 0.8 = 0.5$
19. $e + 0.03 = 1$
20. $f + 0.3 = 1$

Basics

$$a + 4 = 5 \Rightarrow a = 1$$
$$b + 7 = 12 \Rightarrow b = 5$$
$$z + 3 = 3 \Rightarrow z = 0$$
$$d + 4 = 9 \Rightarrow d = 5$$
$$e + 7 = 8 \Rightarrow e = 1$$
$$s + 4 = 2 \Rightarrow s = -2$$
$$z + 5 = 1 \Rightarrow z = -4$$
$$q + 12 = 0 \Rightarrow q = -12$$
$$w + 2020 = 1 \Rightarrow w = -2019$$

Fractions

$$x + \frac{3}{5} = \frac{4}{5} \Rightarrow x = \frac{1}{5}$$
$$q + \frac{2}{3} = \frac{5}{3} \Rightarrow q = \frac{3}{3} = 1$$
$$e + \frac{4}{5} = \frac{2}{5} \Rightarrow e = -\frac{2}{5}$$
$$w + \frac{1}{3} = \frac{1}{2} \Rightarrow w = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$
$$w + \frac{1}{2} = \frac{1}{3} \Rightarrow w = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$$

Decimals

$$a + 0.2 = 0.6 \Rightarrow a = 0.4$$
$$b + 0.31 = 0.4 \Rightarrow b = 0.09$$
$$c + 0.5 = 0.8 \Rightarrow c = 0.3$$
$$d + 0.8 = 0.5 \Rightarrow d = -0.3$$
$$e + 0.03 = 1 \Rightarrow e = 0.97$$
$$f + 0.3 = 1 \Rightarrow f = 0.7$$

Example 1.17: Word Problems with Addition

Find the values of the numbers in each part below:

A. When five is added to a number, it becomes eight.
B. When twelve is added to a number, it becomes fifty-seven.
C. When three is added to a number, it remains three.
D. When three fifth is added to a number, it becomes four fifth.
E. A farmer had a certain number of acres of land. He then bought five more acres, and his total land holdings became 12 acres. How many acres of land did he originally have?
F. There is a number x such that, when it is added to any number, the value of the second number does not change. What is the number?

Part A

Let the number be a. Then, it must satisfy the equation:

$$5 + a = 8$$
$$a = 8 - 5 = 3$$

Part B

$$b + 12 = 57$$
$$b = 57 - 12 = 45$$

Part C

$$c + 3 = 3 \Rightarrow c = 0$$

The number is zero. Zero is the additive identity. Adding the additive identity to any number keeps the number the same.

Part D

$$\frac{3}{5} + d = \frac{4}{5} \Rightarrow d = \frac{1}{5}$$

Part E

$$e + 5 = 12 \Rightarrow e = 7$$

Part F

Consider a number y which can take any valid value.

$$x + y = y \Rightarrow x = 0$$

1.18: Solving Equations with Subtraction

$$x - a = b \Rightarrow x = b + a$$

$$x - 4 = 5$$
$$x = 5 + 4 = 9$$

Example 1.19

Find the value of the letters so that the equation is true:

Basics

1. $s - 9 = 1$
2. $d - 4 = 3$
3. $q - 2 = 23$
4. $r - 4 = 100$
5. $g - 100 = 1$
6. $h - 3 = 3$
7. $t - 4 = -2$
8. $y - 3 = -7$

Fractions

9. $u - \frac{3}{5} = \frac{1}{5}$
10. $k - \frac{6}{9} = \frac{2}{3}$

Decimals

11. $a - 0.2 = 0.8$
12. $b - 0.3 = 1$

Basics

$$s - 9 = 1 \Rightarrow s = 10$$

$$d - 4 = 3 \Rightarrow d = 7$$
$$q - 2 = 23 \Rightarrow q = 25$$

$$r - 4 = 100 \Rightarrow r = 104$$
$$g - 100 = 1 \Rightarrow g = 101$$
$$h - 3 = 3 \Rightarrow h = 6$$
$$t - 4 = -2 \Rightarrow t = 2$$
$$y - 3 = -7 \Rightarrow y = -4$$

Fractions

$$u - \frac{3}{5} = \frac{1}{5} \Rightarrow u = \frac{1}{5} + \frac{3}{5} = \frac{4}{5}$$
$$k - \frac{6}{9} = \frac{2}{3} \Rightarrow k - \frac{2}{3} = \frac{2}{3} \Rightarrow k = \frac{4}{3}$$

Decimals

$$a - 0.2 = 0.8 \Rightarrow a = 1$$
$$b - 0.3 = 1 \Rightarrow b = 1.3$$

Example 1.20: Word Problems with Subtraction

A. Anushka had a apples. She ate 3 three of them, and was left with nine. What is the value of a?
B. A family had f members. Then 9 of them left for the United States, and there were seven members left. What is the total number of members in the family?
C. An orchard has t trees in it, out of which 3 are cut to make way for a path. If the number of trees left is 100, what was is the value of t?
D. Rohan got a cake for his birthday. He ate one-fifth of the cake in the morning. He then ate x part of the cake at lunch, and was left with two-fifth of the cake. What is the value of x?
E. When four is subtracted from a number, it becomes twelve. Find the number.
F. When three is taken away from a number, it becomes three. Find the number.
G. When a number is subtracted from five, the answer is three. Find the number.
H. There is a number x such that, when it is subtracted from any number, the value of the second number does not change.

Part A

$$a - 3 = 9 \Rightarrow a = 12$$

Part B

$$f - 9 = 7 \Rightarrow f = 16$$

Part C

$$t - 3 = 100 \Rightarrow t = 103$$

Part D

$$\frac{4}{5} - x = \frac{2}{5} \Rightarrow x = \frac{2}{5}$$

Part E

$$x - 4 = 12 \Rightarrow x = 16$$

Part F

$$z - 3 = 3 \Rightarrow z = 6$$

Part G

$$5 - x = 3 \Rightarrow x = 2$$

Part H

The number is zero. Zero is the additive identity. Adding the additive identity to any number keeps the number the same.

1.21: Solving Equations with Multiplication

$$ax = b \Rightarrow x = \frac{b}{a}$$

We can write 3 times z as

$$3 \times z$$

In algebra, when a number is multiplied with a letter, we omit the multiplication sign, and hence, we write

$$3 \times z = 3z$$

$$5x = 30 \Rightarrow x = 6$$
$$5x = 22 \Rightarrow x = \frac{22}{5}$$

Example 1.22

Solve the equations

Basics

1. $3z = 9$
2. $2y = 12$
3. $7k = 7$
4. $8t = 0$
5. $4w = 20$
6. $6q = 18$

Fractional Answers

7. $5a = 7$
8. $6b = 9$
9. $4c = 5$
10. $6d = 11$
11. $7e = 13$

Basics

$$3z = 9 \Rightarrow z = \frac{9}{3} = 3$$
$$2y = 12 \Rightarrow y = 6$$
$$7k = 7 \Rightarrow k = 1$$
$$8t = 0 \Rightarrow t = 0$$
$$4w = 20 \Rightarrow w = 5$$
$$6q = 18 \Rightarrow q = 3$$

Fractional Answers

$$5a = 7 \Rightarrow a = \frac{7}{5}$$
$$6b = 9 \Rightarrow b = \frac{9}{6} = \frac{3}{2}$$
$$4c = 5 \Rightarrow c = \frac{5}{4}$$
$$6d = 11 \Rightarrow d = \frac{11}{6}$$
$$7e = 13 \Rightarrow e = \frac{13}{7}$$

1.23: Dividing a Number by a Fraction

We can divide the number a by the fraction $\frac{c}{d}$, and get a different fraction:

$$\frac{a}{\frac{c}{d}} = a \div \frac{c}{d} = a \times \frac{d}{c}$$

- $\frac{a}{\frac{c}{d}}$ is an example of a nested fraction, where the numerator, and the denominator of the fraction, or both are themselves fractions.

Example 1.24

Solve the equations

1. $\frac{3}{4}a = 7$
2. $\frac{2}{3}b = 6$
3. $\frac{5}{7}c = 1$
4. $\frac{7}{11}d = 3$

$$\frac{3}{4}a = 7 \Rightarrow a = \frac{7}{\frac{3}{4}} = 7 \div \frac{3}{4} = 7 \times \frac{4}{3} = \frac{28}{3}$$
$$\frac{2}{3}b = 6 \Rightarrow b = \frac{6}{\frac{2}{3}} = 6 \div \frac{2}{3} = 6 \times \frac{3}{2} = 9$$
$$\frac{5}{7}c = 1 \Rightarrow c = 1 \times \frac{7}{5} = \frac{7}{5}$$
$$\frac{7}{11}d = 3 \Rightarrow d = 3 \times \frac{11}{7} = \frac{33}{7}$$

1.25: Dividing a Fraction by a Fraction

We can divide the fraction $\frac{a}{b}$ by the fraction $\frac{c}{d}$, and get a different fraction:

$$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}$$

Example 1.26

Solve the equations

1. $\frac{2}{7}e=\frac{3}{5}$
2. $\frac{6}{5}f=\frac{1}{7}$
3. $\frac{2}{5}g=\frac{7}{9}$

$$\frac{2}{7}e=\frac{3}{5}\Rightarrow e=\frac{3/5}{2/7}=\frac{3}{5}\div\frac{2}{7}=\frac{3}{5}\times\frac{7}{2}=\frac{21}{10}$$
$$\frac{6}{5}f=\frac{1}{7}\Rightarrow f=\frac{1}{7}\times\frac{5}{6}=\frac{5}{42}$$
$$\frac{2}{5}g=\frac{7}{9}\Rightarrow g=\frac{7}{9}\times\frac{5}{2}=\frac{35}{18}$$

1.27: Multiplication with a negative number

$$-ax=b\Rightarrow x=-\frac{b}{a}$$

Start with

$$-ax=b$$

Divide both sides by $-a$:

$$\frac{-ax}{-a}=\frac{b}{-a}$$
$$x=-\frac{b}{a}$$

Example 1.28

Solve for the variable:

A. $-a=3$
B. $-b=4$
C. $-c=\frac{3}{2}$
D. $-d=\frac{2}{5}$
E. $-e=-2$
F. $-f=-\frac{3}{5}$

$$-a=3$$

Multiply both sides by -1:

$$(-1)(-a)=(-1)(3)$$

$$a = -3$$

Example 1.29

Solve for the variable:

Basics

G.

Coefficients

H. $-3g = 2$

I. $-2h = 7$

J. $-2i = 4$

K. $-4j = 3$

Fractional Coefficients

L. $-\frac{3}{4}k = \frac{2}{3}$

M. $-\frac{1}{3}l = \frac{4}{5}$

Basics

$$a = -3 \Rightarrow a = -\frac{3}{-1}$$
$$b = 4$$
$$c = -\frac{3}{2}$$
$$d = -\frac{2}{5}$$
$$e = 2$$
$$f = \frac{3}{5}$$

Coefficients

$$g = -\frac{2}{3}$$
$$h = -\frac{7}{2}$$
$$i = -\frac{4}{2} = -2$$
$$j = -\frac{3}{4}$$

Fractional Coefficients

$$k = \frac{2}{3} \times \left(-\frac{4}{3}\right) = -\frac{8}{9}$$
$$l = \frac{4}{5} \times \left(-\frac{3}{1}\right) = -\frac{12}{5}$$

Example 1.30

A. Disha had d dollars in her piggy bank. Then, her mother doubled the money in her piggy bank, and now she had 14 dollars. What is the value of d?
B. Ralph visits h houses on his regular newspaper route. Today, however, he is delivering three times the regular route, and he delivered 27 newspapers. What is the value of h?
C. When a number is multiplied by 6, the answer is 42. Find the number.
D. Four times of a number is 12. Find the number.
E. When a number is tripled, the answer is 18. Find the number.
F. Three times a number is twelve.
G. Six times a number is eighteen.
H. When a number is multiplied by four, it becomes sixteen.
I. Double of a number is 46.
J. Tripling a number results in fifteen. (Note: Tripling a number means multiplying it by three.)
K. Quadrupling a number results in thirty two. (Note: Quadrupling means multiplying it by four.)
L. There is a number x such that, when it is multiplied with any number, the value of the second number does not change. What is x?

Part A

$$2d = 14 \Rightarrow d = 7$$

Part B

$$3h = 27 \Rightarrow h = 9$$

Part C

$$6x = 42 \Rightarrow x = 7$$
$$4a = 12 \Rightarrow a = 3$$

$$3b = 18 \Rightarrow b = 6$$
$$3a = 12 \Rightarrow a = 4$$
$$6b = 18 \Rightarrow b = 3$$
$$4c = 16 \Rightarrow c = 4$$
$$2d = 46 \Rightarrow d = 23$$
$$3e = 15 \Rightarrow e = 5$$
$$4f = 32 \Rightarrow f = 8$$

The number is one. Zero is the multiplicative identity. Multiplying the multiplicative identity with any number keeps the number the same.

1.31: Solving Equations with Division

$$\frac{x}{a} = b \Rightarrow x = ab$$

Example 1.32: Division

Find the value of the letters so that the equation is true.

Basics

1. $\frac{a}{2} = 4$
2. $\frac{b}{2} = 6$
3. $\frac{c}{3} = 2$
4. $\frac{d}{3} = 3$
5. $\frac{e}{7} = 4$
6. $\frac{f}{3} = 12$
7. $\frac{g}{4} = 23$

Decimals

8. $\frac{a}{0.2} = 3$
9. $\frac{b}{3} = 0.7$
10. $\frac{c}{0.5} = 3$
11. $\frac{d}{0.1} = 9$
12. $\frac{e}{0.9} = 5$
13. $\frac{f}{0.3} = 0.2$
14. $\frac{g}{0.6} = 0.3$
15. $\frac{h}{0.8} = 0.6$

Fractions

16. $\frac{a}{\frac{3}{5}} = 2$
17. $\frac{b}{\frac{1}{4}} = 3$

Basics

$$\frac{a}{2} = 4 \Rightarrow a = 8$$
$$\frac{b}{2} = 6 \Rightarrow b = 12$$
$$\frac{c}{3} = 2 \Rightarrow c = 6$$
$$\frac{d}{3} = 3 \Rightarrow d = 9$$
$$\frac{e}{7} = 4 \Rightarrow e = 28$$
$$\frac{f}{3} = 12 \Rightarrow f = 36$$
$$\frac{g}{4} = 23 \Rightarrow g = 92$$

Decimals

$$\frac{a}{0.2} = 3 \Rightarrow a = 0.6$$
$$\frac{b}{3} = 0.7 \Rightarrow b = 2.1$$
$$\frac{c}{0.5} = 3 \Rightarrow c = 1.5$$
$$\frac{d}{0.1} = 9 \Rightarrow d = 0.9$$
$$\frac{e}{0.9} = 5 \Rightarrow e = 4.5$$
$$\frac{f}{0.3} = 0.2 \Rightarrow f = 0.06$$
$$\frac{g}{0.6} = 0.3 \Rightarrow g = 0.18$$
$$\frac{h}{0.8} = 0.6 \Rightarrow h = 0.48$$

Fractions

$$\frac{a}{\frac{3}{5}} = 2 \Rightarrow a = \frac{6}{5}$$
$$\frac{b}{\frac{1}{4}} = 3 \Rightarrow b = \frac{3}{4}$$

Example 1.33

A. A number divided by three results in seven.
B. A number when divided by seven, results in seven itself.
C. When a number is halved, it becomes five. (Note: Halving a number means dividing it by two.)
D. One-third of a number is three.
E. When a number is divided into four equal parts, the quotient is six, and the remainder is zero.

F. There is a number x such that, when any number is divided by x, the number does not change. What is x?

$$\underbrace{\frac{a}{3}=7\Rightarrow x=21}_{\textit{Method I: Fractions}},\qquad \underbrace{x\div 3=7\Rightarrow x=21}_{\textit{Method II: Division}}$$

$$\frac{b}{7}=7\Rightarrow b=49,\qquad b\div 7=7\Rightarrow b=49$$
$$\frac{c}{2}=5\Rightarrow c=10,\qquad c\div 2=5\Rightarrow c=10$$
$$\frac{d}{3}=3\Rightarrow d=9,\qquad d\div 3=3\Rightarrow d=9$$
$$\frac{e}{4}=6\Rightarrow e=24,\qquad e\div 4=6\Rightarrow e=24$$

The number is one. Dividing any number by one does not change the number.

Example 1.34: Mixed Practice

A. $a+7=3$
B. $4b=7$
C. $\frac{c}{7}=8$
D. $d-4=3$
E. $0.3e=0.2$
F. $f-0.8=1.2$
G. $\frac{f}{0.3}=0.02$

Part A
Subtract 7 from both sides:
$$a=3-7=-4$$
Part B
Divide both sides by 4:
$$b=\frac{7}{4}$$
Part C
Multiply both sides by 7:
$$c=7\times 8=56$$
Part D
Add 4 to both sides:
$$d=3+4=7$$
Part E
Divide both sides by 0.3
$$e=\frac{0.2}{0.3}=\frac{2}{3}$$
Part F
Add 0.8 to both sides:
$$f=1.2+0.8=2$$
Part G
Multiply both sides by 0.3:
$$g=0.3\times 0.02=0.006$$

1.35: Simplification

In some questions, you may need to simplify the expressions before finding the values.

Example 1.36

Solve each question for the variable.

A. $4+a+2=10+3$
B. $b+2+3=7+1$
C. $c-5=10+2$
D. $2\times 3\times d=60$
E. $2\times e\times 7=98$

F. $\frac{f}{3} = 2 - 3$
G. $2 + 5 + g = 7 \times \frac{1}{2}$

Part A

$$6 + a = 13 \Rightarrow a = 13 - 6 = 7$$

Part B

$$b + 5 = 8 \Rightarrow b = 3$$

Part C

$$c - 5 = 12 \Rightarrow c = 17$$

Part D

$$6d = 60 \Rightarrow d = 10$$

Part E

$$14e = 98 \Rightarrow e = 7$$

Part F

$$\frac{f}{3} = -1 \Rightarrow f = -3$$

Part G

$$7 + g = \frac{7}{2} \Rightarrow g = \frac{7}{2} - 7 = \frac{7}{2} - \frac{14}{2} = -\frac{7}{2}$$

Example 1.37: Mixed Word Problems

A. Five times a number is 30. Find the number.
B. When three is added to a number, you get 31. Find the result when four is subtracted from the number.
C. When seven is subtracted from a number, you get 9. Find the result when two is added to the number.

$$5a = 30 \Rightarrow a = 6$$
$$b + 3 = 31 \Rightarrow b = 28 \Rightarrow b - 4 = 24$$
$$c - 7 = 9 \Rightarrow c = 16 \Rightarrow c + 2 = 18$$

Example 1.38: More Challenging Mixed Word Problems

Solve the word problems gives below. The quantities which are to be represented as variables have been marked. For example, in question with Farmer James, you should assume the number of apples to be a.

A. Farmer James had some $\underbrace{apples}_{a}$ from his farm. He found that half of them were rotten, and threw them away. He was then left with thirteen apples. How many apples did he have originally?
B. Stephanie buys bread rolls for her family and puts them in the bread basket. If the number of $\underbrace{bread\ rolls}_{b}$ is now twelve, and it was three before Stephanie added more, then how many bread rolls did she buy?
C. In 45 years, Ashley will be 4 times as old as he is now. Find Ashley's current $\underbrace{age}_{a}$.
D. Carlos bought some rope and cut it into two parts. Each part of the rope was 5 meters. What was the $\underbrace{length\ of\ the\ rope}_{r}$ that Carlos bought?
E. Sheela adds three gallons of gasoline to her tank, filling it up. If her tank has a capacity of seven gallons, then how much gasoline was $\underbrace{originally\ in\ the\ tank}_{g}$.
F. Percival has a grocery shop. He had no onions, so he bought a sack of onions to sell. After selling twelve kilograms of onions, he was left with fourteen kilograms of onions. What was the $\underbrace{weight\ of\ the\ onions}_{w}$ in the sack that he bought?

Part A

$$\frac{a}{2} = 13 \Rightarrow a = 13 \times 2 = 26$$

Part B

$$b + 3 = 12 \Rightarrow b = 9$$

Part C

	Current	45 Years later
Ashley	a	$a + 45$

Part D

$$a + 45 = 4a \Rightarrow 45 = 3a \Rightarrow a = 15$$

Part E

$$\frac{r}{2} = 5 \Rightarrow r = 10\ meters$$

Part F

$$\underbrace{g}_{\substack{Original\\Amount}} + \underbrace{3}_{\substack{Added\\Amount}} = \underbrace{7}_{\substack{Tank\\Capacity}} \Rightarrow g = 4$$

$$\underbrace{w}_{\substack{Original\ weight\\of\ Onions}} - \underbrace{12}_{\substack{Onions\\Sold}} = 14 \Rightarrow w = 26$$

Example 1.39

There are c customers at a café for breakfast. And the number of customers for lunch, which is twenty, is double the number of customers for breakfast. If the number of customers for dinner is half that of breakfast, find the number of customers at dinner?

$$Customers\ for\ Lunch = 20$$
$$Customers\ for\ Breakfast = \frac{1}{2}\ of\ Customers\ for\ Lunch = Half\ of\ 20 = 10$$
$$Customers\ for\ Dinner = Half\ of\ Customers\ for\ Breakfast = Half\ of\ 10 = 5$$

1.40: Literal Equations

- Equations where along with the unknown variable, there are other variables are called literal equations.
- The method for solving literal equations remains the same.

Literal Equations with Addition

Here is an example of the kind of equation that we have been solving so far:

$$a + 4 = 5 \Rightarrow a = 5 - 4 = 1$$

Here is a literal equation, which is similar, but the 4 has been replaced with a variable (b), that we do not know the value of:

$$a + b = c \Rightarrow a = c - b$$

Literal Equations with Multiplication

$$4x = 12 \Rightarrow x = 3$$

Example 1.41

A. If $a + b = c$, then find the value of b.
B. If $x + y = z$, then find the value of x. Also, find the value of y.
C. If $p + q = a$, then find the value of p. Also, find the value of q.
D. If $ab = c$, then find the value of a. Also, find the value of b.
E. If $pq = r$, then find the value of p. Also, find the value of q.

Part A

$$a + b = c \Rightarrow b = c - a$$

Part B

$$x = z - y, \qquad y = z - x$$

Part C

$$p = a - q, \qquad q = a - p$$

Part D

$$a = \frac{c}{b}, \qquad b = \frac{c}{a}$$

Part E

$$p = \frac{r}{q}, \qquad q = \frac{r}{p}$$

E. Multi-Step Questions

Two step equations are very important since one step equations are not enough to solve most of the questions that we encounter. A large number of questions in Algebra are on linear equations or make use of linear equations. For example, a quadratic equation is solved by factoring into two linear equations. Hence, these kinds of equations form the foundation for learning Algebra.

1.42: Multi-Step Equations

A multi-step equation combines two or more operations from BODMAS in the expression that is given.

- We will generally want to solve multi-step equations in the reverse order of BODMAS.

Example 1.43

A. Solve $2x + 5 = 11$ in the reverse order of BODMAS.
B. Solve $2x + 5 = 11$ by removing the 2 first, and then the 5.
C. Solve $3x - 7 = 2$ in the reverse order of BODMAS.
D. Solve $\frac{x}{3} + 2 = 4$ in the reverse order of BODMAS.

Part A

$$\underbrace{2x}_{Multiplication} + \underbrace{5}_{Addition} = 11$$

Since we are solving in the reverse order of BODMAS, we want to remove the addition first.

Subtract 5 from both sides of the equation:

$$2x + 5 - 5 = 11 - 5$$
$$2x = 6$$

This is a one-step equation. Divide both sides by 2:

$$\frac{2x}{2} = \frac{6}{2}$$

And, our final solution is:

$$x = 3$$

Part B

$$\underbrace{2x}_{Multiplication} + \underbrace{5}_{Addition} = 11$$

To eliminate the 2, divide both sides of the equation by 2:

$$\frac{2x + 5}{2} = \frac{11}{2}$$
$$x + \frac{5}{2} = \frac{11}{2}$$

Subtract $\frac{5}{2}$ from both sides:

$$x = \frac{11}{2} - \frac{5}{2} = \frac{6}{2} = 3$$

Part C

This question combines multiplication with subtraction. In reverse order of BODMAS, we remove the subtraction first, and then the multiplication:

Step I: Add 7 to both sides:

$$3x = 2 + 7$$
$$3x = 9$$

Step II: Divide both sides by 3:

$$\frac{3x}{3} = \frac{9}{3}$$
$$x = 3$$

Part D

This question combines division with addition. In reverse order of BODMAS, we remove the addition first, and then the division.

Step I: Subtract 2 from both sides:

$$\frac{x}{3} = 2$$

Step II: Multiply both sides by 3:

$$x = 6$$

Example 1.44

A number of equations are given below. Each of them is a two-step equation. Identify which number will you eliminate first from the equation based on *reverse order of BODMAS*. You do *not* need to solve the equation.

A. $7x - 2 = 4$
B. $9x + 3 = 5$
C. $\frac{x}{5} - 6 = 7$
D. $\frac{x}{12} + 6 = 8$
E. $\frac{3}{4}x - \frac{6}{7} = \frac{8}{9}$
F. $\frac{5x}{9} + \frac{7}{12} = -\frac{1}{9}$

$$Subtraction\ First: -2$$
$$Addition\ First: 3$$
$$Subtraction\ First: -6$$
$$Addition\ First: +6$$
$$Subtraction\ First: -\frac{6}{7}$$
$$Addition\ First: \frac{7}{12}$$

Example 1.45

In each part below, find the value of the variable.

A. $2x - 5 = 11$
B. $3x + 2 = 5$
C. $4x - 3 = 5$
D. $8x + 2 = 26$
E. $2x - 4 = 10$
F. $3x + 2 = 29$

Part A

Add 5 to both sides:
$$2x = 16$$
Divide both sides by 2:
$$x = 8$$

Part B

Subtract 2 from both sides:
$$3x = 3$$
Divide both sides by 3:
$$x = 1$$

Part C

Add 3 to both sides:
$$4x = 8$$
Divide both sides by 4:
$$x = 2$$

Part D

Subtract 2 from both sides:
$$8x = 24$$
Divide both sides by 8:
$$x = 3$$

Part E

Add 4 to both sides:
$$2x = 14$$
Divide both sides by 2:
$$x = 7$$

Part F

Subtract 2 from both sides:
$$3x = 27$$
Divide both sides by 3:
$$x = 9$$

1.46: Connection Between Equations and Problems

Just as a problem can be converted into an equation, an equation can also represent a problem.

Example 1.47: Framing Verbal Statements

For each equation below, convert it into a verbal statement

A. $x + 5 = 7$

B. $x - 3 = -9$
C. $4x = 7$
D. $\frac{x}{3} = 5$
E. $\frac{3}{x} = 8$
F. $3x + \frac{2}{3} = \frac{1}{3}$
G. $\frac{2}{3}x - \frac{1}{3} = \frac{4}{3}y$

A. When five is added to a number, the result is seven.
B. When three is subtracted from a number, the result is minus nine.
C. Four times a number is seven
D. When a number is divided by three, the result is five.
E. When three is divided by a number, the result is eight.
F. When three times a number is added to $\frac{2}{3}$, the result is $\frac{1}{3}$
G. When $\frac{1}{3}$ is subtracted from two-thirds of a number, then the result is four-thirds of a second number.

Example 1.48
Find the number in case:
A. When three times a number is added to five, the result is twelve.
B. When one-third of a number is subtracted from 3, the result

F. Variables on Both Sides

Example 1.49
A. Tom gets a cake for his birthday. He shares two-fifths of the cake with his family at home and takes the rest to school. At school, the cake is shared among five people, and each one gets 100 grams of cake. Find the weight of the original cake.

$$c - \frac{2}{5}c = \frac{3}{5}c$$
$$\frac{\frac{3}{5}c}{5} = 100$$
$$\frac{3}{5}c = 500$$
$$c = 500 \times \frac{5}{3} = \frac{2500}{3}$$

1.2 One Step Equations

A. Introduction
We can perform equal operations on both sides of an equation. To find the value of a variable, we remove all items except the variable.
To remove, we must perform the opposite operation:
- Addition gets converted to subtraction
- Subtraction gets converted to addition
- Multiplication gets converted to division

- Division gets converted to multiplication

B. Addition

1.50: Addition
To remove constants in addition from an equation, we use the opposite, that is, subtraction.

Example 1.51
$$x + 1 = 5$$

Subtract 1 from both sides of the equality:
$$x \underbrace{+1 - \mathbf{1}}_{Equals\ Zero} = 5 - \mathbf{1}$$

And, hence, the above simplifies to:
$$x = 4$$

Example 1.52
$$x + 6 = 4$$

Subtract 6 from both sides:
$$x + 6 - \mathbf{6} = 4 - \mathbf{6}$$
$$x = -2$$

Example 1.53: Addition
Solve the equations below:
$$x + \frac{1}{2} = \frac{1}{3}$$
$$x + 0.01 = 3$$

Part A
Subtract $\frac{1}{2}$ from both sides:
$$x + \frac{1}{2} - \frac{1}{2} = \frac{1}{3} - \frac{1}{2}$$
$$x = -\frac{1}{6}$$

Part B
Subtract 0.01 from both sides:
$$x + 0.01 - 0.01 = 3 - 0.01$$
$$x = 2.99$$

C. Subtraction

1.54: Subtraction
To remove constants in subtraction from an equation, we use the opposite, that is, addition.

Example 1.55
$$x - 7 = 12$$

$$x - 3 = 2$$

Part A
Add 7 to both sides of the equality:

$$x \underbrace{-7 + \mathbf{7}}_{Equals\ Zero} = 12 + \mathbf{7}$$

And, hence the equation simplifies to:

$$x = 19$$

Part B
Add 3 to both sides:

$$x - 3 + \mathbf{3} = 2 + \mathbf{3}$$
$$x = 5$$

Example 1.56: Subtraction

$$x - \frac{3}{2} = -\frac{2}{3}$$
$$x - 0.2 = -4$$

Part A
Add $\frac{3}{2}$ to both sides:

$$x - \frac{3}{2} + \frac{\mathbf{3}}{\mathbf{2}} = -\frac{2}{3} + \frac{\mathbf{3}}{\mathbf{2}}$$
$$x = -\frac{4}{6} + \frac{9}{6}$$
$$x = \frac{5}{6}$$

Part B
Add 0.2 to both sides:

$$x - 0.2 + \mathbf{0.2} = -4 + \mathbf{0.2}$$
$$x = -3.8$$

D. Multiplication

1.57: Multiplication

To remove constants in multiplication from an equation, we use the opposite, that is, division.

Example 1.58

$$8x = 32$$

Part A
Divide both sides by 8:

$$\frac{8x}{\mathbf{8}} = \frac{32}{\mathbf{8}}$$

And, hence the equation simplifies to:

$$x = 4$$

Example 1.59

$$3x = 21$$
$$7x = 20$$
$$5x = \frac{3}{4}$$

Part A

Divide both sides by 3

$$\frac{3x}{3} = \frac{21}{3}$$
$$x = 7$$

Part B

Divide both sides by 7:

$$\frac{7x}{7} = \frac{20}{7}$$
$$x = \frac{20}{7}$$

Part C

Divide both sides by 5:

$$\frac{5x}{5} = \frac{\frac{3}{4}}{5}$$
$$x = \frac{3}{4} \div 5 = \frac{3}{4} \times \frac{1}{5} = \frac{3}{20}$$

Example 1.60

$$0.2x = \frac{7}{11}$$

Divide both sides by 0.2

$$\frac{0.2x}{0.2} = \frac{\frac{7}{11}}{0.2}$$
$$x = \frac{7}{11} \div 0.2 = \frac{7}{11} \div \frac{2}{10} = \frac{7}{11} \div \frac{1}{5} = \frac{7}{11} \times 5 = \frac{35}{11}$$

1.61: Fractions

If you multiply a fraction by its reciprocal, it becomes one.

$$\frac{4}{5} \times \frac{5}{4} = 1$$
$$-\frac{7}{13} \times -\frac{13}{7} = 1$$

Example 1.62

$$\left(-\frac{3}{4}\right) f = \frac{5}{4}$$

Multiply both sides by the reciprocal of $-\frac{3}{4} = -\frac{4}{3}$:

$$\frac{4}{-3} \times -\frac{3}{4} f = \frac{5}{4} \times \frac{4}{-3}$$

Simplify:

$$f = \frac{5}{4} \times \frac{\mathbf{4}}{\mathbf{-3}} = \frac{5}{-3} = -\frac{5}{3}$$

Never write a fraction in standard form with the minus sign in the denominator.

Example 1.63

$$\frac{10}{3} = x\left(-\frac{5}{2}\right)$$

Multiply both sides by the reciprocal of $-\frac{5}{2} = -\frac{2}{5}$

$$\left(\frac{\mathbf{2}}{\mathbf{-5}}\right) \times \left(-\frac{5}{2}\right) x = \frac{10}{3}\left(\frac{\mathbf{2}}{\mathbf{-5}}\right)$$

$$x = \frac{2}{3} \times \frac{2}{-1} = -\frac{4}{3}$$

E. Division

1.64: Division

To remove constants in division from an equation, multiply both sides by that constant.

Example 1.65

$$\frac{x}{4} = 6$$

Multiply both sides by 4:

$$\mathbf{4} \times \frac{1}{4} x = 6 \times \mathbf{4}$$

And, hence the equation simplifies to:

$$x = 24$$

Example 1.66

$$\frac{x}{3} = 8$$

Multiply both sides by 3:

$$\mathbf{3} \times \frac{x}{3} = 8 \times \mathbf{3}$$

$$x = 24$$

Example 1.67

$$\frac{x}{0.2} = 5$$

Multiply both sides by 0.2:

$$0.2 \times \frac{x}{0.2} = 5 \times 0.2$$

$$x = 5 \times \frac{2}{10} = 5 \times \frac{1}{5} = 1$$

Example 1.68: Division

$$\frac{x}{\frac{2}{3}} = \frac{1}{3}$$

Multiply both sides by $\frac{2}{3}$:

$$\frac{x}{\frac{2}{3}} \times \frac{2}{3} = \frac{1}{3} \times \frac{2}{3}$$
$$x = \frac{2}{9}$$

Example 1.69

$$\frac{z}{-2.3} = 0.1$$

Multiply both sides by -2.3:

$$\frac{z}{-2.3} \times -2.3 = 0.1 \times -2.3$$
$$z = -0.23$$

Example 1.70

$$\frac{x}{-3.8} = 0.01$$

Multiply both sides by -3.8:

$$x = 0.01 \times -3.8 = -0.038$$

F. Percentage

Example 1.71

$$(35\%)x = 75$$

$$x = \frac{75}{35\%} = \frac{75}{\frac{35}{100}} = \frac{75}{\frac{7}{20}} = 75 \times \frac{20}{7} = \frac{1500}{7}$$

Example 1.72

$$\left(87\frac{1}{2}\%\right)x = 28$$

Calculate:

$$87\frac{1}{2}\% = 87.5\% = \frac{87.5}{100} = \frac{875}{1000} = \frac{7 \cdot 125}{1000} = 7 \cdot \frac{125}{1000} = 7 \cdot \frac{1}{8} = \frac{7}{8}$$

$$\frac{7}{8}x = 28$$

$$x = 28 \cdot \frac{8}{7} = 32$$

Example 1.73

$$\left(33\frac{1}{3}\%\right)x = 66$$

$$33\frac{1}{3}\% = \frac{100}{3}\% = \frac{100}{3} \times \frac{1}{100} = \frac{1}{3}$$

$$\frac{x}{3} = 66$$
$$x = 198$$

Example 1.74

$$\left(66\frac{2}{3}\%\right)x = 22$$

$$66\frac{2}{3}\% = \frac{200}{3}\% = \frac{200}{3} \times \frac{1}{100} = \frac{2}{3}$$

$$\frac{2}{3}x = 22$$
$$x = 22 \cdot \frac{3}{2} = 33$$

Example 1.75

$$\left(83\frac{1}{3}\%\right)x = 40$$

$$\frac{250}{3}\% = \frac{250}{3} \times \frac{1}{100} = \frac{250}{300} = \frac{25}{30} = \frac{5}{6}$$

$$\frac{5}{6}x = 40$$
$$x = 40 \cdot \frac{6}{5} = 48$$

G. Word Problems

Example 1.76

Farmer Jones gave his daughter $33\frac{1}{3}\%$ of the apples that he harvested today. She got 33 apples. How many apples did he harvest?

$$33\frac{1}{3}\% = \frac{100}{3}\% = \frac{100}{3} \cdot \frac{1}{100} = \frac{1}{3}$$
$$\frac{1}{3}x = 33$$
$$x = 99$$

H. Transposition

1.77: Transposition

This a shortcut that does not write the cancellation. We *transpose* (change sides) for the number we don't want.

Addition converts into subtraction

$$x + 4 = 3$$

Omit $x + 4 - 4 = 3 - 4$ and directly write:

$$x = 3 - 4$$

Subtraction converts into addition

$$y - 7 = 12$$

Move 7:

$$y = 12 + 7$$

Multiplication converts into division

$$8z = 24$$

Move 8 to the RHS:

$$z = \frac{24}{3}$$

Division converts into multiplication

$$\frac{a}{4} = 5$$

Move 4 to the RHS:

$$a = 5 \times 4$$

1.3 Two Step Equations

A. Multiplication with Addition and Subtraction

1.78: Reverse Order of Operations

To solve an equation, we don't want the variable to have any operations. To remove the operations, we will proceed in reverse order of regular order of operations.

Order of priority for operations:

- Brackets/Parentheses
- Exponentiation
- Of
- Division/Multiplication
- Addition/Subtraction

$$3 \times 4 + 2 = 12 + 2 = 14$$
$$3 \times (4 + 2) = 3 \times 6 = 18$$

Example 1.79

Solve:

$$3x + 2 = 6$$

$$\underbrace{3 \times x}_{\text{Multiplication}} \quad \underbrace{+2}_{\text{Addition}} \quad = 6$$

The reverse order is:

- Addition
- Multiplication

The opposite of addition is subtraction. The equation has 2 added to both sides. Hence, subtract 2 from both sides to get 0 on the left-hand side:

$$3x + 2 - 2 = 6 - 2$$
$$3x = 4$$

The opposite of multiplication is division. To find the value of x, divide both sides by 3:

$$\frac{3x}{\mathbf{3}} = \frac{4}{\mathbf{3}}$$

And, hence the final answer is:

$$x = \frac{4}{3}$$

Example 1.80

Solve $5x - 2 = 7$ by using

A. Addition first, followed by division
B. Division first, followed by addition

Part A

Subtraction has lower priority. We remove it first by adding 2 to both sides:

$$5x - 2 + 2 = 7 + 2$$
$$5x = 9$$

Divide both sides by 5:

$$x = \frac{9}{5}$$

Example 1.81

$$4z + 5 = 7$$

Subtract 5 from both sides:

$$4z + 5 - 5 = 7 - 5$$
$$4z = 2$$

Divide both sides by 4:

$$\frac{4z}{4} = \frac{2}{4}$$

Simplify:

$$z = \frac{1}{2}$$

Example 1.82

$$3s + 7 = 12$$

Subtract 7 from both sides:

$$3s + 7 - 7 = 12 - 7$$
$$3s = 5$$

Divide both sides by 3:

$$s = \frac{5}{3}$$

Example 1.83

$$3r + 5 = 4$$

Subtract 5 from both sides:

$$3r + 5 - 5 = 4 - 5$$
$$3r = -1$$

Divide both sides by 3:

$$r = \frac{-1}{3}$$

Example 1.84

$$6t + 2 = 12$$

Subtract 2 from both sides:

$$6t + 2 - 2 = 12 - 2$$
$$6t = 10$$

Divide both sides by 6:

$$t = \frac{10}{6} \Rightarrow t = \frac{5}{3}$$

B. Division with Addition and Subtraction

Example 1.85

$$\frac{d}{3} - 4 = 5$$

Add 4 to both sides:

$$\frac{d}{3} - 4 + 4 = 5 + 4$$
$$\frac{d}{3} = 9$$

Multiply by 3:

$$d = 27$$

Example 1.86

$$\frac{w}{5} + 2 = 6$$

Subtract 2 from both sides:

$$\frac{w}{5} = 4$$

Multiply both sides by 5:

$$w = 20$$

Example 1.87

$$\frac{s}{2} - 3 = 4$$

Add 3 to both sides:

$$\frac{s}{2} = 7$$

Multiply by 2 both sides:

$$s = 14$$

Example 1.88

$$\frac{q}{3} + 2 = 5$$

Subtract 2 from both sides:

$$\frac{q}{3} = 3$$

Multiply both sides by 3:

$$q = 9$$

C. Common Traps

1.89: Common Traps

If you do not follow the "reverse order of operations" rule, you can still get the correct answer, but you need to be more careful with the maths.

Example 1.90

Explain which step in the following "solution" to $2h + 6 = 8$ has the first mistake, and why?

$$Step\ I: h + 6 = \frac{8}{2}$$

$$h + 6 = 4$$

$$Step\ II: h = -2$$

Note that in Step I, the RHS is divided by 2. To keep the equation correct, we must divide the LHS also by 2.

$$\frac{2h + \mathbf{6}}{2} = \frac{8}{2}$$

Split the fraction:

$$\frac{2h}{2} + \frac{6}{2} = \frac{8}{2}$$

Simplify:

$$h + \mathbf{3} = 4$$

Example 1.91

Identify and fix the problem in the "*solution*" below:
Divide both sides by 5 to get $5x - 2 = 7$

$$x - 2 = \frac{7}{5}$$

$$x = \frac{7}{5} + 2 = \frac{7}{5} + \frac{10}{5} = \frac{17}{5}$$

$$\frac{5x-2}{5}=\frac{7}{5}$$
$$\frac{5x}{5}-\frac{2}{5}=\frac{7}{5}$$
$$x-\frac{2}{5}=\frac{7}{5}$$
$$x=\frac{9}{5}$$

Example 1.92

Is the following solution correct? If no, explain, why not?

$$Equation: 3r+\frac{1}{2}=21$$

Move the 3 to the RHS, changing it from multiplication to division, and getting $\frac{21}{7}=3$:

$$Step\ I: r+\frac{1}{2}=7$$
$$Step\ II: r=7-\frac{1}{2}=\frac{13}{2}$$

Step I is wrong because in Step I the RHS is divided by 3.
Hence, the LHS must also be divided by 3.

The correct Step I would be:

$$\frac{3r+\frac{1}{2}}{3}=\frac{21}{3}$$

D. Fractions and Decimals

Example 1.93

$$\frac{t}{2}+\frac{3}{5}=\frac{1}{4}$$

Subtract $\frac{3}{5}$ from both sides:

$$\frac{t}{2}+\frac{3}{5}-\frac{3}{5}=\frac{1}{4}-\frac{3}{5}$$
$$\frac{t}{2}=\frac{5}{20}-\frac{12}{20}$$

Multiply both sides by 2:

$$\frac{t}{2}=-\frac{7}{20}$$
$$\frac{t}{2}\times 2=-\frac{7}{20}\times 2$$

Example 1.94

$$\frac{3}{4}r-\frac{1}{2}=\frac{3}{4}$$

Add $\frac{1}{2}$ to both sides:

$$\frac{3}{4}r = \frac{3}{4} + \frac{1}{2}$$
$$\frac{3}{4}r = \frac{5}{4}$$

Multiply by $\frac{4}{3}$ on both sides:

$$r = \frac{5}{4} \times \frac{4}{3} = \frac{5}{3}$$

Example 1.95

$$\frac{2}{3}r + \frac{2}{3} = \frac{1}{2}$$

Subtract $\frac{2}{3}$ from both sides:

$$\frac{2}{3}r = \frac{1}{2} - \frac{2}{3}$$
$$\frac{2}{3}r = \frac{3}{6} - \frac{4}{6}$$
$$\frac{2}{3}r = -\frac{1}{6}$$

Multiply by $\frac{3}{2}$ on both sides:

$$r = -\frac{1}{6} \times \frac{3}{2} = -\frac{1}{4}$$

Example 1.96

$$0.01r - \frac{1}{2} = 0.55$$

Rewrite the equation using only decimals:

$$0.01r - 0.5 = 0.55$$

Add 0.5 to both sides:

$$0.01r = 0.55 + 0.5$$
$$0.01r = 1.05$$

Divide both sides by 0.01:

$$r = \frac{1.05}{0.01} = 105$$

E. Word Problems

Example 1.97

Farmer Harry and Farmer Jones have two piles of apples, each with an equal number of apples in them. They combined the piles into a single cart to take the market. When doing so they threw away five apples which were rotten. When they reached the market, they sold a total of 99 apples. What were the number of apples in each pile?

Algebraic Method

Let the number of apples in each pile be

$$a$$

Then, the number of apples in two piles will be:

$$2a$$

After throwing away 5 apples, the number of apples will be:

$$2a - 5$$

And from the question, we know that

$$2a - 5 = 99$$

Add 5 to both sides:

$$2a - 5 + 5 = 99 + 5$$
$$2a = 104$$

Divide both sides by 2:

$$\frac{2a}{2} = \frac{104}{2}$$
$$a = 52$$

Logical Method

The total number of apples is:

$$99 + 5 = 104$$

The number of apples in each pile is:

$$\frac{104}{2} = 52\ apples\ per\ pile$$

F. Age Word Problems

Example 1.98

Kate has triplets. When the ages of all of her children are added, the sum is five less than Kate's age. Kate is thirty-five years. How old are her triplets?

Algebraic Method

Let the age of each triplet be

$$a$$

Then, the age of the three triplets together is:

$$3a$$

Then:

$$3a + 5 = Kate's\ age = 35$$
$$3a = 30$$
$$a = 10$$

Logical Method

The combined age of the triplets is:

$$35 - 3 = 30$$

The age of each triplet is:

$$\frac{30}{3} = 10$$

Example 1.99
John and Jane are siblings. John is a year older than Jane. Their combined age is 11 years. Determine their ages.

Algebraic Method
Let Jane's age

$$= j$$

John's age

$$= j + 1$$

Combined age is

$$j + j + 1 = 2j + 1$$

From the given information:

$$2j + 1 = 11$$
$$2j = 10$$
$$Jane's\ age = j = 5$$
$$John'sage = j + 1 = 6$$

Logical Method

$$11 = 10 + 1 = 5 + 5 + 1 = 5 + 6$$
$$Jane = 5\ years$$
$$John = 6\ years$$

Example 1.100
John and Jane are siblings. John is a year older than Jane. Their combined age is 23.6 years (where the years are written as a decimal). Determine their ages.

Algebraic Method
As before:

$$2j + 1 = 23.6$$
$$2j = 22.6$$
$$j = 11.3$$

Logical Method

$$23.6 = 22.6 + 1 = 11.3 + 11.3 + 1$$
$$Jane = 11.3\ years$$
$$John = 12.3\ years$$

G. Two Consecutive Numbers

Example 1.101
The sum of two consecutive numbers is 45. What are the numbers?

$$x + (x + 1) = 45$$
$$2x + 1 = 45$$

$$2x = 44$$
$$x = 22$$

$$45 = 22 + 23$$

Example 1.102
The sum of two consecutive odd numbers is 90. What are the numbers?

Let the numbers be
$$x, \qquad and\ (x + 2)$$

$$x + (x + 2) = 90$$
$$2x + 2 = 90$$
$$2x = 88$$
$$x = 44$$

Example 1.103
The sum of two consecutive even numbers is 84. What are the numbers?

Let the numbers be
$$x, \qquad and\ (x + 2)$$

$$x + (x + 2) = 84$$
$$2x + 2 = 84$$
$$2x = 82$$
$$x = 41$$

$$No\ Solution$$

$$40 + 42 = 82$$
$$42 + 44 = 86$$

H. Three Consecutive Numbers

Example 1.104
The sum of three consecutive numbers is 45. What are the numbers?

Method I
Let the three numbers be
$$x, \qquad x + 1, \qquad x + 2$$

$$x + (x + 1) + (x + 2) = 45$$
$$3x + 3 = 45$$
$$3x = 42$$
$$x = 14$$

Method II
Let the three numbers be
$$x - 1, \qquad x, \qquad x + 1$$

$$(x-1)+x+(x+1)=45$$
$$3x=45$$
$$x=15$$

Method III

$$\frac{45}{3}=15 \Rightarrow 14, \qquad 15, \qquad 16$$

Example 1.105

The sum of three consecutive odd numbers is 93. What are the numbers?

Method I

Let the three numbers be

$$x-2, \qquad x, \qquad x+2$$

$$(x-2)+x+(x+2)=93$$
$$3x=93$$
$$x=\frac{93}{3}=31$$

Method II

$$Middle\ Number=\frac{93}{3}=31 \Rightarrow 29, \qquad 31, \qquad 33$$

Example 1.106

The sum of three consecutive even numbers is 84. What are the numbers?

Method I

Let the three numbers be

$$x-2, \qquad x, \qquad x+2$$

$$(x-2)+x+(x+2)=84$$
$$3x=84$$
$$x=\frac{84}{3}=28$$

Method II

$$Middle\ Number=\frac{84}{3}=28 \Rightarrow 27, \qquad 28, \qquad 29$$

I. Percentages

Example 1.107

When 50% of x is added to the sum of 25% of x and 12.5% of x, the answer is the counting number n. If x is also a counting number, what is the smallest possible value

A. of n
B. of x

$$(50\%)x + (25\%)x + (12.5\%)x = n$$
$$\left(87\frac{1}{2}\%\right)x = n$$

Calculate:

$$87\frac{1}{2}\% = 87.5\% = \frac{87.5}{100} = \frac{875}{1000} = \frac{7\cdot 125}{1000} = 7\cdot\frac{125}{1000} = 7\cdot\frac{1}{8} = \frac{7}{8}$$

$$\frac{7}{8}x = n$$
$$x = n\cdot\frac{8}{7}$$

n should be divisible by 7. Smallest possible value of n is:

$$7$$

Smallest possible value of x is

$$56$$

1.4 Multiple Terms with Variables

A. Multiple Terms with Variables

Example 1.108

$$3c + 4c = 7$$

$$7c = 7$$
$$c = 1$$

Example 1.109

$$\frac{c}{3} - \frac{c}{2} = \frac{1}{4}$$

To add two fractions which do not have common denominator, we make the denominators common:

$$\frac{2c}{6} - \frac{3c}{6} = \frac{1}{4}$$
$$-\frac{c}{6} = \frac{1}{4}$$

Multiply both sides by -6:

$$c = \frac{1}{4}\cdot(-6) = -\frac{3}{2}$$

Example 1.110

$$\frac{3}{4}x + \frac{1}{4}x = 0.01$$

Then x can be written in the form $\frac{a}{b}$, where a and b are integers, and have no common factor. Find $a + b$.

$$\frac{3}{4}x + \frac{1}{4}x = \frac{4}{4}x = x = 0.01 = \frac{1}{100}$$

$$x = \frac{1}{100} = \frac{a}{b}$$
$$a + b = 1 + 100 = 101$$

B. Variables on both sides

1.111: Variables on both sides

If you have variables on both sides of an equation, bring all the variables to one side, all the numbers to the other side, and then simplify.

Example 1.112

Solve the equation:

$$4c = 3c + 7$$

Get all the variables on the LHS.
Subtract 3c from both sides:

$$4c \mathbf{- 3c} = 3c \mathbf{- 3c} + 7$$

Simplify:

$$c = 7$$

Example 1.113

$$3c = c + 13$$

Subtract c from both sides:

$$3c - c = c - c + 13$$
$$2c = 13$$

Divide both sides by 2:

$$c = \frac{13}{2}$$

Example 1.114

$$2z = 5z - 9$$

Subtract $5z$ from both sides:

$$2z - 5z = 5z - 5z - 9$$
$$-3z = -9$$

Divide both sides by -3:

$$z = 3$$

Example 1.115

$$-5s = 2s + 3$$

Subtract $2s$ from both sides:

$$-7s = 3$$

Divide both sides by -7:

$$s = -\frac{3}{7}$$

Example 1.116

$$-2p = 4p + 7$$

Subtract $4p$ from both sides:

$$-6p = 7$$

Divide both sides by -6:

$$p = -\frac{6}{7}$$

Example 1.117

$$4z + 2 = 2z - 10$$

Subtract 2z from both sides:

$$2z + 2 = -10$$

Subtract 2 from both sides:

$$2z = -12$$

Divide both sides by 2:

$$z = -6$$

1.118: Cancellation

If the same term occurs on both sides of an equation, it can be "cancelled" by subtracting

$$x + b = 2x + b$$

Example 1.119

$$\frac{1}{2}c + \frac{1}{3}c - 3 = \frac{1}{2}c - \frac{1}{3}c + 3$$

Subtract $\frac{1}{2}c$ from both sides:

$$\boldsymbol{\frac{1}{2}c - \frac{1}{2}c} + \frac{1}{3}c - 3 = \boldsymbol{\frac{1}{2}c - \frac{1}{2}c} - \frac{1}{3}c + 3$$
$$\frac{1}{3}c - 3 = -\frac{1}{3}c + 3$$

Now, move variables to the LHS, and numbers to the RHS:

$$\frac{1}{3}c + \frac{1}{3}c = 3 + 3$$
$$\frac{2}{3}c = 6$$
$$c = 6 \times \frac{3}{2} = 9$$

1.120: Direct Simplification

Keep in mind that subtracting the same thing from a quantity results in zero:

$$4 - 4 = 0$$
$$x - x = 0$$
$$c - c = 0$$

Example 1.121

$$4r + 2r - 4r = \frac{23}{17} + 12 - \frac{23}{17}$$

$$\mathbf{4r} + 2r - \mathbf{4r} = \mathbf{\frac{23}{17}} + 12 - \mathbf{\frac{23}{17}}$$

Simplify the LHS and the RHS:

$$2r = 12$$
$$r = 6$$

Example 1.122

$$\frac{x}{2} + \frac{x}{4} + \frac{1}{3} - \frac{x}{2} = \frac{2}{23} + \frac{1}{4} + \frac{1}{3} - \frac{2}{23}$$

$$\frac{x}{2} + \frac{x}{4} + \frac{1}{3} - \frac{x}{2} = \frac{2}{23} + \frac{1}{4} + \frac{1}{3} - \frac{2}{23}$$
$$\frac{x}{4} = \frac{1}{4}$$
$$x = 1$$

1.5 Parentheses

A. Parentheses

1.123: Opening Parentheses

Open the parenthesis by applying the distributive property. Then, solve the equation as usual.

$$a(x + y) = ax + ay$$

Example 1.124

$$2(x + 2) = 3$$

Open the parentheses by applying the distributive property:

$$2x + 4 = 3$$

Subtract 4 from both sides:

$$2x + 4 - 4 = 3 - 4$$
$$2x = -1$$

Divide both sides by 2:

$$x = -\frac{1}{2}$$

Example 1.125

$$3(2x - 4) = 2(-3x + 5)$$

$$6x - 12 = -6x + 10$$

$$12x = 22$$

$$x = \frac{22}{12} = \frac{11}{6}$$

Example 1.126

$$0.2(10 - 3x) = 0.3(9 - 4x)$$

Method I

Multiply both sides by 10:

$$10 \cdot 0.2(10 - 3x) = 10 \cdot 0.3(9 - 4x)$$
$$2(10 - 3x) = 3(9 - 4x)$$
$$20 - 6x = 27 - 8x$$
$$8x - 6x = 27 - 20$$
$$2x = 7$$
$$x = \frac{7}{2}$$

Method II

$$2 - 0.6x = 2.7 - 0.8x$$
$$0.8x - 0.6x = 2.7 - 2$$
$$0.2x = 0.7$$
$$2x = 7$$
$$x = \frac{7}{2}$$

Example 1.127

$$-3\left(x - \frac{1}{2}\right) = 5$$

Use the distributive property:

$$-3x + \frac{3}{2} = 5$$

Subtract $\frac{3}{2}$ from both sides:

$$-3x = 5 - \frac{3}{2} = \frac{7}{2}$$

Multiplying by $-\frac{1}{3}$ on both sides:

$$x = \frac{7}{2} \times \left(-\frac{1}{3}\right) = -\frac{7}{6}$$

1.128: Isolating Parentheses

Move the expression outside the parenthesis to the other side.

Example 1.129

$$2(x + 2) = 3$$

Divide both sides by 2:

$$\frac{2(x+2)}{2}=\frac{3}{2}$$
$$x+2=\frac{3}{2}$$

Subtract 2 from both sides:

$$x=-\frac{1}{2}$$

Example 1.130

$$4\left(x-\frac{1}{2}\right)=32$$

Method I

$$4x-2=32$$
$$4x=34$$
$$x=\frac{34}{4}=\frac{17}{2}$$

Method II

Divide both sides by 4:

$$x-\frac{1}{2}=8$$

Add $\frac{1}{2}$ to both sides:

$$x=8+\frac{1}{2}=\frac{17}{2}$$

Example 1.131: Parentheses

A. $3(x-2)=2(x+2)$
B. $4=4(x-7)$
C. $6=3(x+5)$
D. $12=2(x-1)$
E. $3(x+3)=6$
F. $-4n-8=4(-3n+2)$
G. $4(3+c)+c=c+4$
H. $3\left(q+\frac{4}{3}\right)=2$
I. $4\left(s+\frac{2}{3}\right)=7$

Part A

Use the distributive property on both sides:

$$3x-6=2x+4$$

Add 6 to both sides:

$$3x=2x+10$$

Subtracting $2x$ from both sides:

$$x=10$$

Part B

$$\frac{4}{4}=\frac{4(x-7)}{4}$$
$$1=x-7$$
$$8=x$$

Part C

Divide by 3 on both sides:

$$\frac{6}{3}=\frac{3(x+5)}{3}$$
$$2=x+5$$
$$x=-3$$

Part D

Divide both sides by 2:

$$6=x-1$$
$$x=7$$

Part E

$$3(x+3)=6$$

Divide both sides by 3

$$x+3=2$$

Subtract 3 from both sides:

$$x=-1$$

Part F

$$-4n-8=4(-3n+2)$$

Divide both sides by 4:

$$\frac{-4n-8}{4}=\frac{4(-3n+2)}{4}$$

Split the fraction on the LHS:

$$\frac{-4n}{4}-\frac{8}{4}=-3n+2$$
$$-n-2=-3n+2$$

Add $3n$ to both sides:

$$2n=4$$

Divide both sides by 2:

$$n=2$$

Part G

$$4(3+c)+c=c+4$$
$$4(3+c)=4$$
$$3+c=1$$
$$c=-2$$

Part H

$$3\left(q+\frac{4}{3}\right)=2$$

Divide both sides by 3

$$\frac{3\left(q+\frac{4}{3}\right)}{3}=\frac{2}{3}$$
$$q+\frac{4}{3}=\frac{2}{3}$$

Subtract $\frac{4}{3}$ on both sides:

$$q=\frac{2}{3}-\frac{4}{3}$$
$$q=-\frac{2}{3}$$

Part I

$$4\left(s+\frac{2}{3}\right)=7$$

Divide both sides by 4:

$$s + \frac{2}{3} = \frac{7}{4}$$

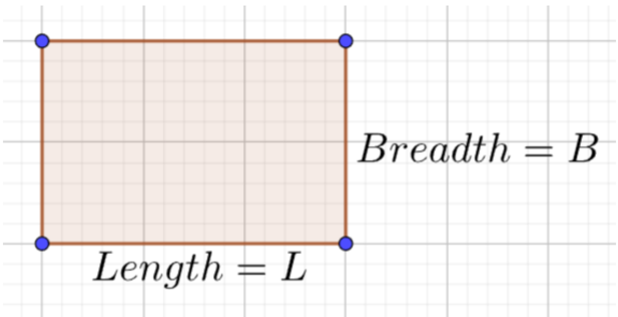
Subtract $\frac{2}{3}$ from both sides:

$$s = \frac{7}{4} - \frac{2}{3} = \frac{21}{12} - \frac{8}{12} = \frac{13}{12} = 1\frac{1}{12}$$

B. Word Problems

Example 1.132

Explain why $2(L + B)$ is an appropriate formula for the perimeter of a rectangle.



Example 1.133

The perimeter of a rectangle is 12 meters. The length of the rectangle is 2.5 m more than the width of the rectangle. Find the area of the rectangle.

$$Perimeter = \underbrace{2(L + B) = 12}_{Equation\ I}$$

Since the length of the rectangle is 2.5m more than the width of the rectangle:

$$L = B + 2.5$$

Make the substitution $L = 2.5 + B$ in Equation I:

$$2(B + 2.5 + B) = 12$$

Divide both sides by 2:

$$2B + 2.5 = 6$$

Subtract 2.5 from both sides:

$$2B = 3.5$$

Divide both sides by 2:

$$B = \frac{3.5}{2} = \frac{\frac{7}{2}}{2} = \frac{7}{4}$$

The length of the rectangle is then:

$$L = \frac{7}{4} + \frac{5}{2} = \frac{17}{4}$$

The area of the rectangle is:

$$Area = LB = \left(\frac{17}{4}\right)\left(\frac{7}{4}\right) = \frac{119}{16}$$

Example 1.134

A. Wood is available at d dollars per cord. Ana bought 5 cords of wood each week, for 20 weeks, in 2024. She paid 20 dollars every week for transportation. Her total cost of wood for 2024 was 1600 dollars. Write an equation for the cost of wood, and solve it to find the value of d.
B. $(Refer\ Part\ A\ first)$ In 2025, the cost of wood is expected to go up by one dollar per cord compared to 2024, and the cost of transportation by 2 dollars per trip. The expected number of weeks that wood will be purchased is 25. Determine the expected cost of purchasing wood in 2025.

Part A

Cost of wood

$$d\ dollars\ per\ cord$$

Cost of 5 cords of wood

$$= 5d$$

Cost of transportation per week

$$= 20\ dollars$$

Cost of wood per week

$$= 5d + 20$$

Cost of wood for 20 weeks

$$= 20(5d + 20)$$

From the given information:

$$20(5d + 20) = 1600$$

Divide both sides by 20:

$$5d + 20 = 80$$

Subtract 20 from both sides:

$$5d = 60$$

Divide both sides by 5

$$d = 12$$

Part B

$$25(5 \cdot 13 + 22) = 25(65 + 22) = 25 \cdot 87 = 2175$$

Example 1.135

Ronaldo's and his 5 classmates are going to share the cost of giving their three friends birthday gifts. The cost of the gift is the same for each friend. Their friends' parents will also put in money, and hence each person will have to contribute 30 dollars less than their "fair share". If the cost that each of Ronaldo and his five classmates will have to pay is 30 dollars, determine the cost of the gift to be bought for each friend.

Let the cost of one gift be

$$g$$

This cost is shared among $Ronaldo + 5\ classmates = 6\ students$. Hence, the fair share would be:

$$\frac{g}{6}$$

Since the parents' chip in money, the actual share is:

$$\frac{g}{6} - 30$$

Since there are three such friends, the total cost is:

$$3\left(\frac{g}{6} - 30\right)$$

$$3\left(\frac{g}{6} - 30\right) = 30$$

Divide by 3 both sides:

$$\frac{g}{6} - 30 = 10$$

Add 30 to both sides:

$$\frac{g}{6} = 40$$

Multiply by 6 on both sides:

$$g = 240\ dollars$$

1.6 Fractions

A. Revision

1.136: Fractions

Whenever we want to add or subtract fractions, they must have the same denominator.

Example 1.137

$$3\frac{3}{4} - x = x + \frac{1}{2}$$

$$\frac{15}{4} - \frac{2}{4} = 2x$$

$$\frac{13}{4} = 2x$$

$$x = \frac{13}{8}$$

B. Fractions with Variables in the numerator

Example 1.138

$$\frac{x}{3}+\frac{x}{5}=\frac{2}{7}$$

Make the denominator as $LCM(3,5)=15$:

$$\frac{5x}{15}+\frac{3x}{15}=\frac{2}{7}$$

Add the fractions:

$$\frac{8}{15}x=\frac{2}{7}$$

Multiply both sides by $\frac{15}{8}$:

$$\frac{15}{8}\times\frac{8}{15}x=\frac{2}{7}\times\frac{15}{8}$$
$$x=\frac{15}{28}$$

Example 1.139

$$\frac{x}{2}+\frac{x}{3}=-\frac{2}{7}$$

First, convert the LHS to have a common denominator of $\text{LCM}(2,3)=6$:

$$\frac{3x}{6}+\frac{2x}{6}=-\frac{2}{7}$$

Add the fractions:

$$\frac{5}{6}x=-\frac{2}{7}$$

Multiply both sides by $\frac{6}{5}$:

$$\frac{6}{5}\times\frac{5}{6}x=-\frac{2}{7}\times\frac{6}{5}$$
$$x=-\frac{12}{35}$$

Example 1.140

$$\frac{2x}{3}+\frac{3x}{4}=-\frac{1}{2}$$

$$\frac{8x}{12}+\frac{9x}{12}=-\frac{1}{2}$$
$$\frac{17}{12}x=-\frac{1}{2}$$
$$x=-\frac{1}{2}\times\frac{12}{17}=-\frac{6}{17}$$

Example 1.141

$$\frac{2x+1}{2}-\frac{x+4}{3}=4$$

Convert the fractions to have a denominator of $LCM(2,3)=6$:

$$\frac{3(2x+1)}{6}-\frac{2(x+4)}{6}=4$$

Simplify:

$$\frac{6x+3}{6}-\frac{2x+8}{6}=4$$

$$\frac{6x+3-2x-8}{6}=4$$

Add the fractions:

$$\frac{4x-5}{6}=4$$

Multiply both sides by 6:

$$4x-5=24$$

Add 5 to both sides:

$$4x=29$$

Divide both sides by 4:

$$x=\frac{29}{4}$$

Example 1.142

$$\frac{3x+5}{3}-\frac{2x-1}{5}=-\frac{2}{3}$$

$$\frac{5(3x+5)}{15}-\frac{3(2x-1)}{15}=-\frac{2}{3}$$
$$\frac{15x+25}{15}-\frac{6x-3}{15}=-\frac{2}{3}$$
$$\frac{15x+25-6x+3}{15}=-\frac{2}{3}$$
$$\frac{9x+28}{15}=-\frac{2}{3}$$

$$9x+28=-\frac{2}{3}\times 15=-10$$
$$9x=-10-28=-38$$
$$x=-\frac{38}{9}$$

C. Variables in the Denominator: Basics

Example 1.143

$$\frac{12}{x}=4$$

Multiply both sides by x:

$$x\cdot\frac{12}{x}=4\cdot x$$
$$12=4x$$

Divide both sides by 4:

$$\frac{12}{4} = \frac{4x}{4}$$
$$3 = x$$

Example 1.144

$$\frac{3}{x} = 4$$

Multiply both sides by x:

$$3 = 4x$$

Divide both sides by 4:

$$x = \frac{3}{4}$$

Example 1.145

$$\frac{27}{x} = 5$$

Multiply both sides by x:

$$\frac{27x}{x} = 5x$$
$$27 = 5x$$

Divide both sides by 5:

$$x = \frac{27}{5}$$

Example 1.146

$$\frac{32}{4x} = 64$$

Simplify:

$$\frac{8}{x} = 64$$

Multiply both sides by x:

$$8 = 64x$$

Divide both sides by 64:

$$x = \frac{8}{64} = \frac{1}{8}$$

Example 1.147

$$\frac{\frac{3}{4}}{2x} = \frac{1}{5}$$

Method I

Multiply both sides by $10x$:

$$10x \cdot \frac{\frac{3}{4}}{2x} = \frac{1}{5} \cdot 10x$$
$$5 \cdot \frac{3}{4} = 2x$$
$$\frac{15}{4} = 2x$$
$$x = \frac{15}{8}$$

Method II

Convert the fraction on the LHS into a division:

$$\frac{3}{4} \div 2x = \frac{1}{5}$$

Conver the division into multiplication by taking the reciprocal:

$$\frac{3}{4} \cdot \frac{1}{2x} = \frac{1}{5}$$
$$\frac{3}{8x} = \frac{1}{5}$$

Multiply both sides by x:

$$\frac{3}{8} = \frac{x}{5}$$

Multiply both sides by 5:

$$x = \frac{3}{8} \cdot 5 = \frac{15}{8}$$

D. Variables in the Denominator

Example 1.148

$$\frac{3}{x+3} = 2$$

Multiply both sides by $x + 3$:

$$(x+3) \cdot \frac{3}{x+3} = 2 \cdot (x+3)$$
$$3 = 2x + 6$$
$$-3 = 2x$$
$$x = -\frac{3}{2}$$

Example 1.149

$$\frac{7}{x-2} = 4$$

$$\frac{7}{4} = x - 2$$
$$\frac{7}{4} + 2 = x$$

$$x = \frac{15}{4}$$

Example 1.150

$$\frac{x-3}{x+4} = 2$$

Multiply both sides by the denominator:

$$\boldsymbol{(x+4)} \times \frac{x-3}{x+4} = 2\boldsymbol{(x+4)}$$
$$x - 3 = 2x + 8$$

Add 3 to both sides:

$$x = 2x + 11$$

Subtract $2x$ from both sides:

$$-x = 11$$

Multiply both sides by -1:

$$x = -11$$

Example 1.151

$$\frac{x+2}{x-3} = 5$$

$$x + 2 = 5(x-3)$$
$$x + 2 = 5x - 15$$
$$17 = 4$$
$$x = \frac{17}{4}$$

Example 1.152

$$\frac{x+7}{x-2} = \frac{1}{2}$$

$$2(x+7) = x - 2$$
$$2x + 14 = x - 2$$
$$x = -16$$

Example 1.153

$$\frac{x+3}{x+5} = \frac{3}{4}$$

$$4(x+3) = 3(x+5)$$
$$4x + 12 = 3x + 15$$
$$x = 3$$

Example 1.154

$$\frac{x-2}{x-7} = \frac{5}{3}$$

$$3(x-2)=5(x-7)$$
$$3x-6=5x-35$$
$$-2x=-29$$
$$x=\frac{-29}{-2}=\frac{29}{2}$$

Example 1.155

$$\frac{2x+3}{3x-4}=\frac{2}{3}$$

$$3(2x+3)=2(4x-4)$$
$$6x+9=8x-8$$
$$-2x=-17$$
$$x=\frac{-17}{-2}=\frac{17}{2}$$

Example 1.156

$$\frac{y+4}{y-2}=\frac{3}{4}$$

Example 1.157

$$\frac{r+2}{6r-1}=\frac{2}{3}$$

Example 1.158

$$\frac{q+3}{2q-1}=\frac{1}{4}$$

Example 1.159

$$\frac{e+3}{e-5}=\frac{2}{7}$$

Example 1.160

$$\frac{0.5x+3.5}{0.75x-4.5}=\frac{1}{2}$$

Example 1.161

$$\frac{1}{x+2}=\frac{1}{2x+3}$$

Example 1.162

$$\frac{3}{x+2}=\frac{x-3}{2x+4}$$

$$\frac{3}{x+2} = \frac{x-3}{2(x+2)}$$

LCD is $2(x+2)$. Multiply by the LCD:

$$6 = x - 3$$
$$x = 9$$

Example 1.163

$$\frac{5}{x} - \frac{1}{3} = \frac{1}{x}$$

LCM is 3x. Multiply both sides by 3x:

$$3x\left(\frac{5}{x} - \frac{1}{3}\right) = 3x\left(\frac{1}{x}\right)$$
$$3x\left(\frac{5}{x}\right) - 3x\left(\frac{1}{3}\right) = 3$$
$$15 - x = 3$$

Example 1.164

$$\frac{2}{x-1} - \frac{x}{x+3} = \frac{6}{x^2+2x-3}$$

Example 1.165

$$3 - \left\{1.6 - \left[3.2 - \left(3.2 + \frac{2.25}{x}\right)\right]\right\} = 0.65$$

Open the innermost bracket:

$$3 - \left\{1.6 - \left[3.2 - 3.2 - \frac{2.25}{x}\right]\right\} = 0.65$$
$$3 - \left\{1.6 - \left[-\frac{2.25}{x}\right]\right\} = 0.65$$

Open the innermost bracket:

$$3 - \left\{\frac{1.6}{1} + \frac{2.25}{x}\right\} = 0.65$$
$$\frac{3}{1} - \left\{\frac{1.6x + 2.25}{x}\right\} = 0.65$$
$$\frac{3x - 1.6x - 2.25}{x} = 0.65$$
$$1.4x - 2.25 = 0.65x$$
$$0.75x = 2.25$$
$$\frac{3}{4}x = \frac{9}{4}$$
$$x = \frac{9}{4} \times \frac{4}{3} = 3$$

E. Reciprocals

Example 1.166

Solve

$$\frac{2}{x}=\frac{3}{7}$$

Method I
Take the reciprocal both sides:

$$\frac{x}{2}=\frac{7}{3}$$

Multiply both sides by 2:

$$2\times\frac{x}{2}=\frac{7}{3}\times 2\Rightarrow x=\frac{14}{3}$$

Method II
Multiply both sides by x:

$$x\times\frac{2}{x}=\frac{3}{7}\times x\Rightarrow 2=\frac{3x}{7}$$

Multiply both sides by $\frac{7}{3}$:

$$\frac{7}{3}\times 2=\frac{3x}{7}\times\frac{7}{3}\Rightarrow\frac{14}{3}=x\Rightarrow x=\frac{14}{3}$$

Example 1.167

Solve $\frac{4}{y}=\frac{5}{6}$

Example 1.168

Solve

A. $\frac{5}{r}=\frac{2}{9}$
B. $\frac{2}{z}=\frac{5}{12}$
C. $\frac{12}{p}=\frac{7}{4}$
D. $\frac{0.5}{p}=\frac{0.25}{3}$
E. $\frac{0.4}{q}=\frac{0.25}{2}$

$$\frac{r}{5}=\frac{9}{2}\Rightarrow r=\frac{45}{2}$$

$$\frac{0.5}{p}=\frac{0.25}{3}\Rightarrow p=\frac{0.5\times 3}{0.25}=\frac{2\times 3}{1}=6$$

$$\frac{0.5}{p}=\frac{0.25}{3}\Rightarrow p=\frac{\frac{1}{2}\times 3}{\frac{1}{4}}=\frac{1}{2}\times 3\div\frac{1}{4}=\frac{1}{2}\times 3\times 4=6$$

$$\frac{0.4}{q}=\frac{0.25}{2}\Rightarrow q=\frac{0.4\times 2}{0.25}=\frac{0.8}{\frac{1}{4}}=0.8\div\frac{1}{4}=0.8\times 4=3.2$$

F. Addition and Subtraction

Example 1.169

Solve

A. $\frac{3}{x}+\frac{2}{5}=\frac{2}{x}$
B. $\frac{2}{x}-\frac{3}{5}+\frac{1}{x}=\frac{1}{5}$

Part A

Isolate x on the left-hand side by subtracting $\frac{2}{x}$ from both sides:

$$\frac{3}{x}-\frac{2}{x}+\frac{2}{5}=\frac{2}{x}-\frac{2}{x}\Rightarrow\frac{3-2}{x}+\frac{2}{5}=0$$

Subtract $\frac{2}{5}$ from both sides:

$$\frac{1}{x}+\frac{2}{5}-\frac{2}{5}=0-\frac{2}{5}\Rightarrow\frac{1}{x}=-\frac{2}{5}$$

Take the reciprocal both sides:

$$\frac{x}{1}=-\frac{5}{2}\Rightarrow x=-\frac{5}{2}$$

Part B

$$\frac{2}{x}+\frac{1}{x}=\frac{1}{5}+\frac{3}{5}\Rightarrow\frac{3}{x}=\frac{4}{5}\Rightarrow\frac{x}{3}=\frac{5}{4}\Rightarrow x=\frac{15}{4}$$

1.7 Nested and Continued Fractions

A. Nested Fractions

1.170: Nested Fractions

A fraction that a fraction for its numerator or its denominator or both is a nested fraction.

Example 1.171

A. The numerator of a fraction is $\frac{f}{g}$. The denominator is $\frac{x}{y}$. Write the fraction in simplified form.
B. The numerator of a fraction is $\frac{3}{4}$. The denominator is $\frac{2}{y}$. If the fraction is equal to $\frac{5}{7}$, find the value of the y.

Part A

$$\frac{N}{D}=\frac{\frac{f}{g}}{\frac{x}{y}}=\frac{f}{g}\times\frac{y}{x}=\frac{fy}{gx}$$

Part B

$$\frac{\frac{3}{4}}{\frac{2}{y}}=\frac{5}{7}\Rightarrow\frac{3}{4}=\frac{5}{7}\times\frac{2}{y}\Rightarrow y=\frac{5}{7}\times 2\times\frac{4}{3}=\frac{40}{21}$$

B. Continued Fractions

1.172: Continued Fraction

A continued fraction is a fraction where the numerator, or the denominator, or both are themselves expressions which contain fractions. And the fractions can be nested multiple times.

1.173: Reciprocal Property in Nested Fractions

$$\frac{1}{\frac{a}{b}} = 1 \div \frac{a}{b} = 1 \times \frac{b}{a} = \frac{b}{a}$$

Example 1.174

$$x + \frac{1}{\frac{3}{5}} = \frac{1}{\frac{3}{7}}$$

$$x + \frac{5}{3} = \frac{7}{3}$$

$$x = \frac{7}{3} - \frac{5}{3} = \frac{2}{3}$$

Example 1.175

$$x - \frac{1}{1 - \frac{1}{3}} = \frac{1}{2 + \frac{1}{3}}$$

$$x - \frac{1}{\frac{2}{3}} = \frac{1}{\frac{7}{3}}$$

$$x - \frac{3}{2} = \frac{3}{7}$$

$$x = \frac{3}{7} + \frac{3}{2} = \frac{6}{14} + \frac{21}{14} = \frac{27}{14}$$

Example 1.176

$$x + \frac{2}{1 + \frac{2}{3}} = \frac{3}{4 - \frac{2}{3}}$$

$$x + \frac{2}{\frac{5}{3}} = \frac{3}{\frac{10}{3}}$$

$$x + 2 \times \frac{3}{5} = 3 \times \frac{3}{10}$$

$$x + \frac{6}{5} = \frac{9}{10}$$

$$x = \frac{9}{10} - \frac{12}{10} = -\frac{3}{10} = -0.3$$

C. Variables in the Denominator

Example 1.177

$$1+\frac{1}{1+\frac{1}{x}}=5$$

$$LHS=1+\frac{1}{\frac{x+1}{x}}=1+\frac{x}{x+1}=\frac{2x+1}{x+1}$$

$$\frac{2x+1}{x+1}=5\Rightarrow 2x+1=5x+5\Rightarrow x=-\frac{4}{3}$$

Example 1.178

$$\frac{1}{1+\frac{1}{1-\frac{1}{x}}}=4$$

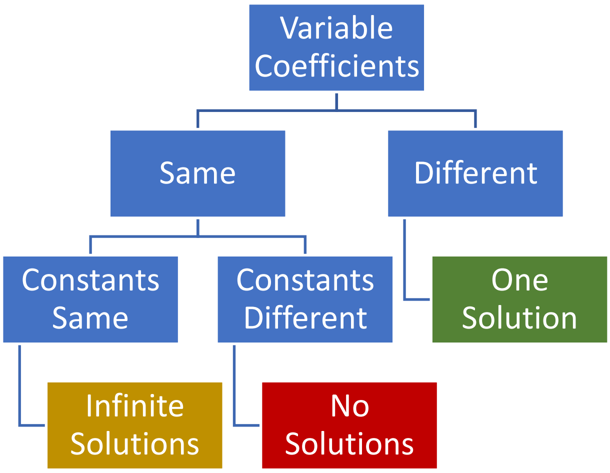
$$LHS=\frac{1}{1+\frac{1}{1-\frac{1}{x}}}=\frac{1}{1+\frac{1}{\frac{x-1}{x}}}=\frac{1}{1+\frac{x}{x-1}}=\frac{1}{\frac{x-1}{x-1}+\frac{x}{x-1}}=\frac{1}{\frac{2x-1}{x-1}}=\frac{x-1}{2x-1}$$

$$\frac{x-1}{2x-1}=4\Rightarrow x-1=8x-4\Rightarrow 3=7x\Rightarrow x=\frac{3}{7}$$

1.8 Number of Solutions

When solving a linear equation, we find three distinct cases:

A. Coefficients are Different



1.179: Case I: Coefficients are Different

Coefficients of the variable on both sides of the equation are not the same.

$$\underbrace{3x}_{LHS\ Coefficent=3}+5=\underbrace{2x}_{RHS\ Coefficent=2}+10\Leftrightarrow\underbrace{x=5}_{One\ Solution}$$

- This is the familiar case that we are used to. We can collect variables on one side, and constants on the other side.
- It results in exactly one solution.

B. Coefficients are Same

Coefficients of the variable on both sides of the equation are not the same. This has two sub-cases.

1.180: Case IIA: Coefficients are Same, Constants are same.

This results in infinite solutions.

$$\underbrace{3x+7=3x+7}_{\substack{Coefficents\ are\ same\\ Constants\ are\ also\ same}}\Leftrightarrow\underbrace{7=7\Leftrightarrow 0=0}_{Infinite\ Solutions}$$

1.181: Case IIA: Coefficients are Same, Constants are different

This results in no solutions

$$\underbrace{3x+3=3x+4}_{\substack{Coefficents\ are\ same\\ Constants\ are\ not\ same}} \Leftrightarrow \underbrace{3=4}_{No\ Solutions}$$

Example 1.182

Show that $3x+7=3x+7$ has infinite solutions.

Substitute $x=1$:

$$LHS=3x+7=3+7=10$$
$$RHS=3x+7=3+7=10$$

Substitute $x=0$:

$$LHS=3x+7=0+7=7$$
$$RHS=3x+7=0+7=7$$

In fact, since

$$LHS=RHS$$

Any value of x will work. Hence, there are infinite solutions.

C. Identifying Number of Solutions

Using the classification given above, the number of solutions to an equation can be identified.

Example 1.183

Find the number of solutions to each equation below, without actually solving the equation:

A. $2x+5=3x+9$
B. $22x+15=22x+19$
C. $14x+12=14x+12$
D. $\frac{2}{3}x+\frac{3}{4}=\frac{3}{2}x+0.75$
E. $\frac{2}{4}x+\frac{1}{2}=\frac{1}{2}x+0.5$

Part A

$$LHS\ x\ coefficient=2$$
$$RHS\ x\ coefficient=3$$
$$Coefficients\ are\ different \Rightarrow One\ Solution$$

Part B

$$LHS\ x\ coefficient=RHS\ x\ coefficient=22$$
$$Coefficients\ are\ same$$

Since the coefficients are same, we need to compare the constants:

$$LHS\ Constant=15$$
$$RHS\ Constant=19$$
$$Constants\ are\ different \Rightarrow No\ Solution$$

Part C

$$LHS\ x\ coefficient=RHS\ x\ coefficient=22$$

$$Coefficients\ are\ same$$

Since the coefficients are same, we need to compare the constants:

$$LHS\ Constant = RHS\ Constant = 12$$
$$Constants\ are\ different \Rightarrow Infinite\ Solutions$$

D. Simplifying Equations

If the equations are not simplified, they will need to be simplified before we can apply the rules that we have learnt.

Example 1.184

Find the number of solutions to each equation below:

A. $3x + 4 - 2x = 7 + 2 - x$
B. $7x + 9 - 4x = 3 + 2x - 5 + x$
C. $\frac{1}{2}x + \frac{1}{3}x = \frac{1}{2} - \frac{1}{3}$

Part A

$$x + 4 = 9 - x \Rightarrow 1\ Solution$$

Part B

$$3x + 9 = 3x - 2 \Rightarrow No\ Solutions$$

Part C

$$\frac{1}{2}x + \frac{1}{3}x = \frac{1}{2} - \frac{1}{3} \Rightarrow One\ Solution$$

E. Finding Values

If we are told the number of solutions of an equation, we can use it to identify the value of an unknown constant.

Example 1.185

Find the value of c in each case below:

A. $x + c = x + 4$ has infinite solutions
B. c is a natural number less than three. $x + c = x + 2$ has no solutions
C. $cx + 4 = 5x + 6$ has no solutions.
D. $cx + 4 = 4x + c$ has infinite solutions.
E. $\frac{1}{2}x + cx + 3 = \frac{5}{6}x + 3$ has infinite solutions

Example 1.186

If the equation $\frac{a}{4}x + 3 = \frac{b}{4}x + 3, a \neq 0, b \neq 0$ has infinite solutions, find $\frac{a}{b}$.

Since the number of solutions is infinite, the coefficients on both sides must be the same:

$$\frac{a}{4} = \frac{b}{4} \Rightarrow a = b$$

F. Establishing Restrictions

If we are told the number of solutions of an equation, we can use it to establish restrictions on the value of an unknown constant.

Example 1.187

A. If $\frac{5}{6}x + 3 = \frac{1}{2}x + \frac{1}{3}x + c$ has no solutions, then what is the value that c cannot take?

B. If the equation $\frac{a}{4}x + \frac{c}{2} + \frac{d}{3} = \frac{b}{4}x + 4, a \neq 0, b \neq 0$ has no solutions, find the value that $3c + 2d$ cannot have.

Part A

Constants must be different

$$c \neq 3$$

Part B

Since there are no solutions, the constant terms cannot be equal.

$$\frac{c}{2} + \frac{d}{3} \neq 4 \Rightarrow 3c + 2d \neq 24$$

G. Infinite Solutions

Example 1.188

Aditya has 9 apples with him, out of which x are rotten. Anil has 18 apples, of which cx are rotten. Aparna knows that the number of not rotten apples that Anil has is double of the number of not rotten apples that Aditya has. She also knows the value of c. She formed an expression for the number of not rotten apples that Aditya has, another expression for the number of not rotten apples that Anil has, and equated the two. She found, to her surprise and disgust, that x could take multiple values.

A. What is the value of c?

B. What are the values that x can take?

$$9 - x = 18 - cx \Rightarrow 9 - x = 2\left(9 - \frac{c}{2}x\right)$$

Since there are multiple solutions, the constants must be the same and so must be the coefficients:

$$-1 = -\frac{c}{2} \Rightarrow c = 2$$

H. No Solutions

Example 1.189

I. Single Solution

Example 1.190

J. Review

MCQ 1.191

A linear equation has sum of coefficients on the LHS the same as that on the RHS. The sum of constants on the LHS is different from the sum of constants on the RHS. The equation has:

A. A unique solution

B. No Solutions
C. Infinite Solutions
D. Cannot be determined

Challenge 1.192

If $ax + 4 = 3x + b$ has an infinite number of solutions and $cx + 3 = 7x + d$ has no solutions, then is it possible that $ax + b = dx + c$ has no solutions?

$$ax + 4 = 3x + b \Rightarrow a = 3, b = 4$$
$$cx + 3 = 7x + d \Rightarrow c = 7, d \neq 3$$

$$ax + b = dx + c \Rightarrow 3x + 4 = dx + 7 \Rightarrow d = 3 \Rightarrow Contradiction \Rightarrow Not\ Possible$$

1.9 Literal Constants

A. Background

So far, we have seen equations where the only unknown was a single variable. For example, we know that:

$$5x = 3 \Rightarrow x = \frac{3}{5}$$

If the equation that we are solving has coefficients that are not known, but will be substituted with numbers later, we can still solve the equation.

Example 1.193

A school is conducting a farewell party. It has invited 300 students to the party, and 600 parents. The principal of the school has also issued n invitations to some important guests.

A. Let p be the total number of people invited for the party. What is the value of p?
B. Does the principal know the value of n? Is n a constant or a variable for him?
C. What is the range of values that p can take?

Part A

Let the total number of people invited to the party be p.

$$p = 300 + 600 + n = 900 + n$$
$$p = 900 + n$$

Part B

$$n\ is\ a\ constant$$

Till you know the value of n, you can represent it using the letter n. In this case, it represents a literal constant.

Part C

$$p = \{900, 901, 902, \dots\}$$

Example 1.194

A. Solve the literal equation $x - c = 5$ for x.
B. What is the range of values that x can take.

Part A

Add c to both sides:

$$x = 5 + c$$

Part B

Once we substitute a value of c, then we can determine the value of x. c can be any real number:

$$c \in (-\infty, \infty)$$

Hence, $5 + c$ can also be any real number:

$$x \in (-\infty, \infty)$$

Example 1.195

A. Solve $ax = 5, \ a \neq 0, x \neq 0$ for a
B. Solve $ax = b$ for the variable x, considering that a and b are unknown constants.

Part A

$$ax = 5 \Rightarrow a = \frac{5}{x}$$

Part B

$$ax = b \Rightarrow x = \frac{b}{a}, \qquad a \neq 0$$

1.196: Variables on both sides

When you have the variable of interest on both sides

- Collate the terms with the variable on one side (usually the LHS)
- Factor the variable out, and divide to isolate the variable.

Example 1.197

A. Solve for y: $ay + d = cy + 4$
B. Solve for x: $3aw + 3ax = -4c + 4x$
C. Solve for x: $n(17 + x) = 34x - r$
D. Solve for c: $6z = 83 + zc + zt$

Part A

Take all the y terms on the LHS, and all the other terms to the RHS:

$$ay - cy = 4 - d$$

Factor out y:

$$y(a - c) = 4 - d$$

Solve for y:

$$y = \frac{4 - d}{a - c}, \qquad a - c \neq 0 \Rightarrow a \neq c$$

Part B

$$3aw + 3ax = -4c + 4x$$
$$3ax - 4x = -4c - 3aw$$
$$x(3a - 4) = -4c - 3aw$$
$$x = \frac{-4c - 3aw}{3a - 4} = \frac{4c + 3aw}{4 - 3a}$$

And we need to impose the restriction:

$$4 - 3a \neq 0 \Rightarrow 3a \neq 4 \Rightarrow a \neq \frac{4}{3}$$

Part E

$$17n + nx = 34x - r$$
$$nx - 34x = -r - 17n$$
$$x(n - 34) = -r - 17n$$

$$x=\frac{-r-17n}{n-34}=\frac{r+17n}{34-n}, 34-n\neq 0\Rightarrow n\neq 34$$

Part F

Solve for z:

$$6z-zc-zt=83$$
$$z(6-c-t)=83$$
$$z=\frac{83}{6-c-t}$$

And we need to impose the condition:

$$6-c-t\neq 0\Rightarrow c+t\neq 6$$

Example 1.198

A. Solve for q: $-\frac{1}{2}r(-p+q)=0.75t(s-q)$
B. Solve for x: $b(a+x)+c(x-a)+d(a+b)=0$
C. Solve for r: $S+r=I\left(1+\frac{r}{100}\right)$
D. Solve for E: $\frac{E}{L}+\frac{LE}{3n}=G-E$
E. Solve for a: $S=\frac{n}{2}[2a+(n-1)d]$
F. Solve for r: $S=\frac{a}{1-r}$
G. Solve for t_1: $H=\frac{KA(t_2-t_1)}{L}$
H. Solve for S_1: $V=\frac{1}{6}H(S_0+4S_1+S_2)$

Part A

$$-\frac{1}{2}r(-p+q)=\frac{3}{4}t(s-q)$$

The LHS has a fraction with a denominator of 2. The RHS has a fraction with a denominator of 4.
Multiply both sides by $LCM(2,4)=4$ to remove fractions:

$$-2r(-p+q)=3t(s-q)$$
$$2rp+2rq=3ts-3tq$$
$$3tq+2rq=3ts-2rp$$
$$q(3t+2r)=3ts-2rp$$
$$q=\frac{3ts-2rp}{3t+2r}, 3t+2r\neq 0$$

Part B

$$ba+bx+cx-ca+da+db=0$$
$$ba-ca+da+db=-bx-cx$$
$$ba-ca+da+db=x(-b-c)$$
$$\frac{ba-ca+da+db}{-b-c}=x$$
$$x=\frac{ca-ba-da-db}{b+c}, b+c\neq 0$$

Part C

Use the distributive property:

$$S+r=I+\frac{Ir}{100}$$

Collate all r terms on one side:

$$r-\frac{Ir}{100}=I-S$$
$$r\left(1-\frac{I}{100}\right)=I-S$$
$$r=\frac{I-S}{\left(1-\frac{I}{100}\right)}, \qquad 1-\frac{I}{100}\neq 0\Rightarrow I\neq 100$$

Part D

From the given equation, we know that:

$$L\neq 0, \qquad 3n\neq 0$$

$$\frac{E}{L}+\frac{LE}{3n}+E=G$$
$$E\left(\frac{1}{L}+\frac{L}{3n}+1\right)=G$$
$$E\left(\frac{3n+L^2+3nL}{3nL}\right)=G$$
$$E=\frac{G}{\frac{3n+L^2+3nL}{3nL}}=\frac{3nLG}{3n+L^2+3nL}$$

$$3n+L^2+3nL\neq 0\Rightarrow$$

Part E

$$S=an+n(n-1)d$$

$$S - n(n-1)d = an$$
$$\frac{S - n(n-1)d}{n} = a, \qquad n \neq 0$$

Part F

$$S(1-r) = a$$
$$S - Sr = a$$
$$S - a = Sr$$
$$\frac{S-a}{S} = r, \qquad S \neq 9$$

Part G

$$HL = KA(t_2 - t_1)$$
$$HL = KAt_2 - KAt_1$$
$$HL - KAt_2 = -KAt_1$$
$$= t_1 = \frac{HL - KAt_2}{-KA} = \frac{KAt_2 - HL}{+KA}$$

$$HL = KA(t_2 - t_1)$$
$$\frac{HL}{KA} = t_2 - t_1$$
$$\frac{HL}{KA} - t_2 = -t_1$$
$$t_1 = t_2 - \frac{HL}{KA} = \frac{KAt_2 - HL}{+KA}, \qquad KA \neq 0$$

Part H

$$6V = HS_0 + 4HS_1 + HS_2$$
$$6V - HS_0 - HS_2 = 4HS_1$$
$$\frac{6V - HS_0 - HS_2}{4H} = S_1$$
$$4H \neq 0 \Rightarrow H \neq 0$$

B. Perimeter

Example 1.199

The perimeter of a geometrical figure is given by the sum of the lengths of the sides. In each case below, find the formula for the perimeter of the given geometrical figure.

A. The sides of a triangle are x, y and z.
B. The length and width of a rectangle are l and w.

$$P = x + y + z$$
$$P = s + s + s + s = 4s$$
$$P = l + w + l + w = 2l + 2w = 2(l + w)$$

Example 1.200: Regular Polygons

A. The side length of a square is s.
B. An equilateral triangle has side length s.
C. A rhombus is a quadrilateral that has all sides equal. Find the perimeter of a rhombus with side length p.
D. A regular hexagon has six sides, each of equal length. Find the perimeter of a regular hexagon with side length y.

$$P = s + s + s = 3s$$
$$P = 6y$$

C. Area

Example 1.201

A. The area of a square is the square of the length of its side. Find the area of a square with side length x.
B. The area of a square is half the product of the length of its diagonals. Find the area of a square with length of diagonals d_1 and d_2.
C. The area of a rectangle is the product of its length and its width. Find the area of a rectangle with length l and width w.

D. The area of a triangle is half the product of a base and its corresponding height. Find the area of a triangle with base b and corresponding height h.
E. The area of a parallelogram is the product of its base and height. Find the area of a parallelogram with base b and corresponding height h.
F. The area of a trapezium is half the product of the sum of the two bases, the height, and the number $\frac{1}{2}$. Find the area of a trapezium with bases b_1 and b_2, and height h.

$$A = x^2$$
$$A = \frac{d_1 d_2}{2}$$
$$A = lw$$
$$A = \frac{bh}{2}$$
$$A = bh$$
$$A = \frac{h(b_1 + b_2)}{2}$$

D. Circles

Example 1.202

A. The diameter of a circle is twice the radius. Find the diameter of a circle with radius r.
B. The circumference of a circle is twice the product of its radius and the constant π. Find the circumference of a circle with radius r. $\left(\text{Note: } \pi \approx \frac{22}{7}, \pi \approx 3.14\right)$
C. The area of a circle is π times the square of its radius. Find the area of a circle with radius r.

$$D = 2r$$
$$C = 2\pi r$$
$$A = \pi r^2$$

E. Volume

Example 1.203

A. The volume of a cuboid is the product of its length, width and height. Find the volume (V), of a cuboid with length l, width w and height h.

$$V = lwh$$

F. Sphere

Example 1.204

A. The surface area of a sphere is 4π times the square of its radius. Find the surface area(SA) of a sphere with radius r.
B. The volume of a sphere is four-third the cube of its radius. Find the volume(V) of a sphere with radius r.

G. Perimeter to Side Length

Example 1.205

The perimeter of a square is given by the formula

$$P = 4s, \qquad P = Perimeter, \qquad s = side\ length$$

Find a formula for the side length in terms of the perimeter.

$$P = 4s \Rightarrow \frac{P}{4} = \frac{4s}{4} \Rightarrow \frac{P}{4} = s \Rightarrow s = \frac{P}{4}$$

Example 1.206

The perimeter of a equilateral triangle is given by the formula

$$P = 3s, \qquad P = Perimeter, \qquad s = side\ length$$

Find a formula for the side length in terms of the perimeter.

$$P = 3s \Rightarrow \frac{P}{3} = \frac{3s}{3} \Rightarrow \frac{P}{3} = s$$

H. Number of Solutions

Example 1.207

For what value of a is the equation $8x - 8 + 3ax = 5ax - 2a$ an identity

$$8x - 2ax = -2a + 8$$
$$x(8 - 2a) = -(8 - 2a)$$

If $a = 1, 8 - 2a = 6$

$$6x = -6 \Rightarrow x = -1 \Rightarrow Only\ 1\ solution$$

To make the equation an identity, we want a zero on both sides of the equality.

$$8 - 2a = 0 \Rightarrow 2a = 8 \Rightarrow a = 4$$

1.10 Word Problems

A. Basics

Example 1.208

Find the number in each case below:

A. Three times a number is five more than twice the number.
B. When two is added to a number, and then the result is multiplied by three, the answer is 7.
C. If one-third of a number is added to two-third of the same number, the answer is three-thirds.
D. If one-third of a number is added to one-fifth of the same number, the answer is one.

In each part, let the number be x.

Part A

$$3x = 2x + 5$$
$$3x - 2x = 5$$
$$x = 5$$

Part B

$$3(x + 2) = 7$$
$$3x + 6 = 7$$
$$3x = 1$$

$$x = \frac{1}{3}$$

Part C

$$\frac{1}{3}x + \frac{2}{3}x = \frac{3}{3}$$
$$\frac{3}{3}x = \frac{3}{3}$$
$$x = 1$$

Part D

$$\frac{x}{3} + \frac{x}{5} = 1$$
$$\frac{5x}{15} + \frac{3x}{15} = 1$$
$$\frac{8x}{15} = 1$$
$$x = \frac{15}{8}$$

Example 1.209

There are five boys in a class. And the girls are three-eights of the entire class. Find the number of girls.

Let c equal the number of people in the class:

$$\underbrace{c}_{Class} = \underbrace{5}_{Boys} + \underbrace{\frac{3}{8}c}_{Girls}$$

Subtract $\frac{3}{8}c$ from both sides:

$$c - \frac{3}{8}c = 5$$
$$\frac{5}{8}c = 5$$
$$c = 8$$

Example 1.210

Ronak is three years elder to Shyam. The sum of the ages of Ronak and Shyam is 45. Find the age of Ronak's sister, who is half his age.

Let Shyam's age be s. Then:

$$\underbrace{s}_{Shyam} + \underbrace{(s+3)}_{Ronak} = 45$$

$$2s = 42$$
$$Shyam's\ age = 21$$
$$Ronak's\ age = s + 3 = 24$$
$$Ronak's\ sister's\ age = \frac{s+3}{2} = 12$$

B. Percentage

Example 1.211

Twenty percent less than 60 is one-third more than what number?

Twenty percent less than 60 is

$$80\%\ of\ 60 = \frac{4}{5} \times 60 = 48$$

Hence:

$$48 = x + \frac{1}{3}x$$
$$48 = \frac{4}{3}x$$
$$x = 48 \cdot \frac{3}{4} = 36$$

Example 1.212

The state income tax where Kristin lives is levied at the rate of $p\%$ of the first \$28000 of annual income plus $(p+2)\%$ of any amount above \$28000. Kristin noticed that the state income tax she paid amounted to $(p+0.25)\%$ of her annual income. What was her annual income? **(AMC 12 2001/3, AMC 10 2001/9)**

Let her annual income be I.
On the first 28000 dollars of income, Kristin pays:

$$p\%\ of\ 28000 = p\%(28000)$$

The amount above 28000 is $I - 28000$, on which she pays $(p+2)\%$

$$(p+2)\%(I - 28000)$$

The income tax that she pays on her entire income is:

$$(p+0.25)\%I$$

$$\left(\frac{p}{100}\right)(28000) + \left(\frac{p+2}{100}\right)(I - 28000) = \left(\frac{p+0.25}{100}\right)I$$
$$\frac{28000p + Ip + 2I - 28000p - 56000}{100} = \frac{Ip + \frac{1}{4}I}{100}$$

Multiply by 100 both sides and simplify sides:

$$2I - 56000 = \frac{1}{4}I$$
$$\frac{7}{4}I = 56000$$
$$I = 56000 \times \frac{4}{7} = 32000$$

Note that while we got an equation in two variables, the p terms all cancelled, leaving us with an equation in a single variable.
Hence, this is an example of a linear equation in disguise.

Example 1.213

Let $R = gS - 4$. When $S = 8, R = 16$. When $S = 10$, R is equal to: **(AHSME 1950/2)**

Substitute $S = 8, R = 16$ in $R = gS - 4$ to find the value of g:

$$16 = 8g - 4 \Rightarrow g = 2.5$$

Substitute $g = 2.5, S = 10$ in $R = gS - 4$ to find the value of R:

$$R = 2.5 \times 10 - 4 = 21$$

Example 1.214: Sums of Consecutive Integers

Find the numbers in each case:

A. The sum of three consecutive integers is 18.
B. The sum of five consecutive odd integers is 135.
C. The sum of four consecutive even integers is 28.
D. The sum of four consecutive multiples of 5 is 110. What is the greatest of the four numbers? **(MathCounts 2002 School Target)**

Part A

Let the middle number be x.
Then, the largest number is $x + 1$
Then, the smallest number is $x - 1$

Then the sum of the numbers is

$$(x - 1) + x + (x + 1) = 3x$$

And we know that

$$3x = 18$$
$$x = 6$$

Part B

Let the middle number be x.

First Number	Second Number	Middle Number	Fourth Number	Fifth Number
$x - 4$	$x - 2$	x	$x + 2$	$x + 4$

Since the numbers add up to 135, we must have:

$$(x - \mathbf{4}) + (x - \mathbf{2}) + (x) + (x + \mathbf{2}) + (x + \mathbf{4}) = 135$$

The red numbers will add up to zero, meaning they will vanish:

$$x + x + x + x + x = 135 \Rightarrow 5x = 135 \Rightarrow x = \frac{135}{5} = 27$$

Shortcut:
When we have an odd number of numbers, the middle number will always be the average of the numbers. Hence, we can straightway do:

$$\frac{135}{5} = 27$$

To get the middle number.

Part C

Let the numbers be:

First Number	Second Number		Third Number	Fourth Number
$x - 3$	$x - 1$	x	$x + 1$	$x + 3$
4	6	7	8	10

$$(x - \mathbf{3}) + (x - \mathbf{1}) + (x + \mathbf{1}) + (x + \mathbf{3}) = 28$$

$$x + x + x + x = 28 \Rightarrow 4x = 28 \Rightarrow x = 7$$

Part D

$$(x - 15) + (x - 10) + (x - 5) + x = 110$$

$$4x - 30 = 110$$
$$4x = 140$$
$$x = 35$$

$$5[x + (x + 1) + (x + 2) + (x + 3)] = 110$$
$$4x + 6 = 22$$
$$4x = 16$$
$$x = 4$$
$$x + 3 = 7$$
$$5(x + 3) = 35$$

Example 1.215: Business Applications

A. (*Calculator*) The XYZ company has to pay \$5,000 for rent each month. In addition, their monthly electricity bill is 1.45\$ per kilowatt-hour of electricity used. If the total cost for both rent and electricity in January was \$16,520.25, how many kilowatt-hours of electricity did they use? **(MathCounts 2009 National Team)**

B. A school's band members raised money by selling magazine subscriptions and shirts. Their profit from selling shirts was \$5 per shirt minus a one-time \$40 set-up fee. Their profit from selling magazine subscriptions was \$4 per subscription. They made exactly the same profit from shirts as they did from magazines. They also sold the same number of shirts as magazine subscriptions. How many shirts did they sell? **(MathCounts 2008 School Sprint)**

C. Billy purchased 100 shirts at a cost of \$15 each and planned to resell them. The first week he sold some of them for \$25 each. The second week he sold the rest for \$20 each. If his total profit was \$600, how many shirts did he sell the first week? **(MathCounts 2008 School Team)**

D. After purchasing four mangoes, a man commented: "Well, if the price of a mango was Rs. 4 less, I would have got two more mangoes for the same amount of money I have now paid". What is the price of one mango? **(JMET 2010/74)**

Part A

$$5000 + 1.45x = 16520.25$$
$$1.45x = 11520.25$$
$$x = 7945$$

Part B

$$5x - 40 = 4x \Rightarrow x = 40$$

Part C

$$Revenue = Cost + Profit = 1500 + 600 = 2100$$
$$25x + 20(100 - x) = 2100$$
$$25x + 2000 - 20x = 2100$$
$$5x = 100$$
$$x = 20$$

Part C

$$4p = 6(p - 4) \Rightarrow p = 12$$

Example 1.216: Numbers

A. The sum of two numbers is 15. Four times the smaller number is 60 less than twice the larger number. What is the larger number? **(MathCounts 2010 School Countdown)**

B. Six times the reciprocal of a number is $\frac{9}{4}$. What is the number? Express your answer as a mixed number. **(MathCounts 1991 School Sprint)**

C. Joe multiplies a number by 4, adds 1, and then divides by 3, getting a result of 7. Sue divides the same original number by 3, adds 1, and multiplies by 4. What result does she get? Express your answer as a common fraction. **(MathCounts 2005 Chapter Countdown)**

D. When a positive number is multiplied by the sum of twice the number and half the number, the result is the original number. What is the number? Express your answer as a common fraction. **(MathCounts 2007 Chapter Countdown)**

Part A

Let the numbers be x and $15-x$

$$4x = 2(15-x) - 60$$
$$4x = 30 - 2x - 60$$
$$6x = -30$$
$$x = -5$$
$$15 - x = 15 - (5) = 20$$

Part B

$$\frac{6}{x} = \frac{9}{4} \Rightarrow x = \frac{8}{3} = 2\frac{2}{3}$$

Part C

$$\frac{4x+1}{3} = 7 \Rightarrow 4x = 20 \Rightarrow x = 5$$
$$\frac{x}{3} = \frac{5}{3} \Rightarrow 1 + \frac{x}{3} = \frac{8}{3} \Rightarrow 4\left(1 + \frac{x}{3}\right) = \frac{32}{3}$$

Part D

$$x\left(\frac{5x}{2}\right) = x$$

Divide by x:

$$\frac{5x}{2} = 1 \Rightarrow x = \frac{2}{5}$$

Example 1.217: Money

Dawn has \$1.20 in nickels, dimes and quarters. For each nickel she has one dime and for each dime she has one quarter. How many total coins does she have? **(MathCounts 1991 Chapter Sprint)**

$$n + d + q = 40$$
$$3n + 3d + 3q = 120$$
$$Coins = 3 + 3 + 3 = 9$$

Example 1.218: Fractions of Quantities

A. Dave has d pieces of candy and Ramon has r pieces. Dave gives Ramon $\frac{1}{3}$ of his candy. They then each have 20 pieces of candy. What is the value of d? **(MathCounts 2006 State Countdown)**

B. John divided his souvenir hat pins into two piles. The two piles had an equal number of pins. He gave his brother one-half of one-third of one pile. John had 66 pins left. How many pins did John originally have? **(MathCounts 2002 School Sprint)**

C. The gasoline gauge on a van initially read $\frac{1}{8}$ full. When 15 gallons of gasoline were added to the tank, the gauge then read $\frac{3}{4}$ full. How many more gallons would be needed to fill the tank? **(MathCounts 2003 State Sprint)**

Part A

$$d - \frac{d}{3} = 20 \Rightarrow \frac{2d}{3} = 20 \Rightarrow d = 30$$

Part B

John gave his brother

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{12} \ of\ his\ pins$$

Instead of working with fractions, let the original pins be $12p$.

$$12p - p = 11p = 66$$
$$p = 6 \Rightarrow 12p = 72$$

Part C

Let the capacity of the tank be x gallons.

Original gasoline in the tank

$$\frac{x}{8}\ gallons$$

After 15 gallons were added, the tank had

$$\frac{3x}{4}\ gallons$$

The difference was caused by adding 15 gallons. Hence:

$$\frac{3x}{4} - \frac{x}{8} = 15$$
$$\frac{6x}{8} - \frac{x}{8} = 15$$

$$\frac{5x}{8} = 15$$
$$\frac{x}{8} = 3$$
$$x = 24$$

Hence, the number of gallons needed to fill the tank is:

$$\underbrace{24}_{Capacity} - \underbrace{15}_{Added} - \underbrace{3}_{\substack{Already \\ there}} = 6$$

Example 1.219: Ratios

A. Billy Goats invested some money in stocks and bonds. The total amount he invested was $165,000. If he invested 4.5 times as much in stocks as he did in bonds, what was his total investment in stocks? (MathCounts 2011 School Sprint)

B. John ordered 4 pairs of black socks and some additional pairs of blue socks. The price of the black socks per pair was twice that of the blue. When the order was filled, it was found that the number of pairs of the two colors had been interchanged. This increased the bill by 50%. The ratio of the number of pairs of black socks to the number of pairs of blue socks in the original order was: (AHSME 1950/31)

Part A
5.5b=165000
b=30000
4.5b=135000
Part B
Let

$$cost\ of\ blue\ socks\ per\ pair = p$$
$$\therefore cost\ of\ black\ socks\ per\ pair = 2p$$

Also, let

$$No. of\ Blue\ socks = b$$

Then, by the given condition:

$$\frac{3}{2}(4 \times 2p + bp) = 4p + b \times 2p$$

Divide throughout by p:

$$\frac{3}{2}(8 + b) = 4 + 2b$$

Multiply by 2 both sides, and open the brackets:

$$24 + 3b = 8 + 4b$$
$$b = 16$$

Ratio of black to blue is:

$$4:16 = 1:4$$

Example 1.220: Fractions as Numbers

If the numerator of a fraction is increased by six, the value of the fraction will increase by one. If the denominator of the original fraction is increased by 36, the value of the original fraction will decrease by one. What is the original fraction? (MathCounts 2011 State Sprint)

n+6/d = n/d + 6/d
6/d = 1
d = 6

n/(d+36) = (n/d) - 1
n/(6+36) = (n/6) - 1
n/42 = n-6/6
n/7 = n-6

n =7n-42
n=7

C. Bank Balance

Example 1.221
John's bank account balance x months from now is given by the equation

$$20x + 100$$

A. What is the balance in John's account one year from now?
B. If Jake's current does not have a bank account, and he opens one now, how much money should he deposit in it every month to have the same balance as John at the end of one year. (Assume that he opens his account with a balance of zero dollars).

$$20x + 100 = 20(12) + 100 = 240 + 100 = 340$$

Jake will deposit money 12 times between now and one year from now.
The money he has to deposit each month will be:

$$\frac{340}{12} = \frac{170}{6}$$

Example 1.222: Geometry
The $\underbrace{length}_{l}$ of a rectangular playground exceeds twice its $\underbrace{width}_{w}$ by 25 feet, and the perimeter of the playground is 650 feet. What is the area of the playground in square feet. (MathCounts 2009 Warm-up 5)

$$\underbrace{w}_{width} \rightarrow \underbrace{2w}_{Twice\ of\ width} \rightarrow \underbrace{2w+25}_{\substack{Twice\ of\ width\\ plus\ 25}} = \underbrace{l}_{length}$$

$$P = 2l + 2w = 2(2w + 25) + 2w = 4w + 50 + 2w = 6w + 50$$

$$6w + 50 = 650 \Rightarrow 6w = 600 \Rightarrow w = 100 \Rightarrow l = 2w + 25 = 225 \Rightarrow A = lw = (225)(100) = 22500$$

D. Age Problems
Some systems can be solved faster by introducing only a single variable and stating everything in terms of that variable. If the system is complicated, then you might have to track many different expressions before.

1.223: Data Presentation Table

	Person A	Person B
$time = Now$		
$time = Some\ other\ time$		

Example 1.224
On Marika's birthday, in 2004, her father said, ''My age is now four times your age.'' In what year will Marika's father be able to say, ''My age is now three times your age,'' on Marika's birthday? (MathCounts 2005 Workout 1)

Example 1.225

Rishi age's is double of Radha's age. If Rishi is four years older than Radha, then find their ages.

Let Radha's age be r.

$$2r = r + 4$$
$$r = 4$$

	Rishi	*Radha*
One Way	$2r$	r
Second Way	$r + 4$	r

Example 1.226

Maria is three times as old as Bethany. Eighteen years ago, Maria was nine times as old as Bethany was then.

$$3b - 18 = 9(b - 18)$$
$$3b - 18 = 9b - 162$$
$$6b = 144$$
$$b = 24$$

	Maria	Bethany
Now	$3b$	b
18 Years	$3b - 18$	$b - 18$

Example 1.227

Mary is twelve years older than Botham. Seventeen years ago, Mary was four times as old as Botham was then. Find their ages.

$$b - 5 = 4(b - 17)$$
$$b - 5 = 4b - 68$$
$$63 = 3b$$
$$b = 21$$

	Now	*17 Years ago*
Mary	$b + 12$	$b - 5$
Botham	b	$b - 17$

Example 1.228

Jan is as old now as Gary was 15 years ago. Six years from now, Gary will be twice as old as Jan will be then. How old is Gary now? **(MathCounts 2010 State Sprint)**

$$g + 6 = 2(g - 9)$$
$$g + 6 = 2g - 18$$
$$24 = g$$

	Now	*Six Years Later*
Gary	g	$g + 6$
Jan	$g - 15$	$g - 9$

Example 1.229

Jerry is presently twice as old as his brother and six years older than his sister. How many years from now will Jerry's age be $\frac{2}{3}rd$ of the combined ages of his brother and sister at that time? **(MathCounts 2009 Chapter Sprint)**

	Jerry	Jerry's Brother	Jerry's Sister	Combined Age
Now	j	$\frac{j}{2}$	$j - 6$	
n years later	$j + n$	$\frac{j}{2} + n$	$j - 6 + n$	$\frac{3j}{2} + 2n - 6$

$$j + n = \left(\frac{2}{3}\right)\left(\frac{3j}{2} + 2n - 6\right)$$

$$j + n = j + \frac{4n}{3} - 4$$
$$0 = \frac{n}{3} - 4$$
$$12 = n$$

E. Disguised Linear Equations

Example 1.230

A playground is 9 meters longer than it is wide. In a redevelopment, the longer side of the playground decreases by 5 meters, and the shorter side increases by 3 meters. If the playground area remains the same after redevelopment, find the dimensions of the original and the redeveloped playground.

$$w(w+9) = (w+3)(w+4)$$
$$w^2 + 9w = w^2 + 3w + 4w + 12$$
$$9w = 7w + 12$$
$$2w = 12$$
$$w = 6$$

	Original	*New*
Width	w	$w+3$
Length	$w+9$	$w+4$
Area	$w(w+9)$	$(w+3)(w+4)$

Check:

$$6 \times 15 = 90$$
$$(9)(10) = 90$$

Example 1.231

The setup time for a painting process is 9 minutes, irrespective of batch size. The time taken to a paint a die is 0.1 minutes. Find the batch size so that we can average painting a die every minute.

Let the number of dice be d.

$$9 + 0.1d = d$$
$$9 = 0.9d$$
$$d = \frac{9}{0.9} = 10$$

F. Multi-Step Scenarios

Many popular puzzles describe scenarios in multiple steps.

- A mathematical solution requires assuming variables at the beginning, and following through with the description till an algebraic expression can be compared to a numerical quantity, hence forming an equation.
- If the question allows, working backwards or using logic can be simpler than forming and solving equations.

Example 1.232

Emma had just been given some coins by her parents. On the way to school, she lost exactly half of them, and then by retracing her steps she found exactly four-fifths of the coins she had lost. What fraction of the coins that she received from her parents were still missing after Emma retraced her steps? Express your answer as a common fraction. **(MathCounts 2008 School Target)**

$$\frac{1}{2}\times\frac{1}{5}=\frac{1}{10}$$

Example 1.233

A gentleman walking home was appealed to by three people in succession for assistance. To the first person he gave one dollar more than half the money he had. To the second person he gave two dollars more than half the money he then had. And to the third person he handed over three dollars more than half of what he had left. If this left him with exactly one dollar, how much money did he start with? **(Amusements in Mathematics, H. E. Dudeney, Adapted)**

Suppose he starts with x dollars. After giving money to the first person, he has:

$$\frac{x}{2}-1\ dollars$$

After giving money to the second person, he has:

$$\frac{\frac{x}{2}-1}{2}-2=\frac{x}{4}-\frac{1}{2}-2=\frac{x}{4}-\frac{5}{2}\ dollars$$

After giving money to the third person, he has:

$$\frac{\frac{x}{4}-\frac{5}{2}}{2}-3=\frac{x}{8}-\frac{5}{4}-3=\frac{x}{8}-\frac{17}{4}\ dollars$$

And we know that:

$$\frac{x}{8}-\frac{17}{4}=1\Rightarrow\frac{x}{8}=\frac{21}{4}\Rightarrow x=42\ Dollars$$

We can (and should!) check the answer:

$$First\ person\ gets\frac{42}{2}+1=21+1=22\ Dollars\Rightarrow Leaving\ 42-22=20\ Dollars$$

$$Second\ person\ gets\frac{20}{2}+2=10+2=12\ Dollars\Rightarrow Leaving\ 20-12=8\ Dollars$$

$$Third\ person\ gets\frac{8}{2}+3=4+3=7\ Dollars\Rightarrow Leaving\ 8-7=1\ Dollar$$

Example 1.234

A. Hui is an avid reader. She bought a copy of the best seller *Math is Beautiful*. On the first day, Hui read 1/5 of the pages plus 12 more, and on the second day she read 1/4 of the remaining pages plus 15 pages. On the third day she read 1/3 of the remaining pages plus 18 pages. She then realized that there were only 62 pages left to read, which she read the next day. How many pages are in this book? **(AMC 8 2010/21)**

B. Pustak Keeda of standard six bought a book. On the first day, he read 1/5 of the pages plus 12 pages. On the second day he read 1/4 of the remaining pages plus 15 pages and on the third day he read 1/3 of the remaining pages plus 20 pages. The fourth day which is the final day he read the remaining 60 pages of the book and completed reading. Find the total number of pages in the book, and the number of pages read on each day. **(NMTC Primary-Final, 2011/1)**

Part A

Let the number of pages be x. Start with the first day:

$$Pages\ Read=\frac{x}{5}+12\Rightarrow Pages\ Remaining=\frac{4}{5}x-12$$

Second Day:

$$Pages\ Read = \frac{1}{4}\left[\frac{4}{5}x - 12\right] + 15 = \frac{x}{5} + 12 \Rightarrow Pages\ Remaining = \frac{3}{5}x - 24$$

Third Day

$$Pages\ Read = \frac{1}{3}\left(\frac{3}{5}x - 24\right) + 18 = \frac{1}{5}x + 10 \Rightarrow Pages\ Remaining = \frac{2}{5}x - 34$$

And now on the fourth day, she reads the remaining pages, which are 62:

$$\frac{2}{5}x - 34 = 62 \Rightarrow \frac{2}{5}x = 96 \Rightarrow x = 96 \times \frac{5}{2} = 240$$

Part B

See Part A.

$$Pages\ read\ every\ day = \frac{x}{5} + 12$$

$$Total\ Pages = 240$$

2. Systems of Equations

2.1 Systems of Equations

A. Adding Equations

Example 2.1
I have three objects, called A, B and C.
Weight of A and B together is 3 kg. Weight of B and C together is 5 kg, and weight of C and A together is 4 kg.

$$A + B = 3$$
$$B + C = 5$$
$$C + A = 4$$

Add of the above:

$$2(A + B + C) = 12$$
$$A + B + C = 6$$
$$C = 3$$
$$A = 1$$
$$B = 2$$

Example 2.2
Three friends John, Jack and Jill decided to weigh themselves. The weight of John and Jack together is 57. The weight of John and Jill is 55. And the weight of Jack and Jill is 62. Find the weights of each.

$$John + Jack = 57$$
$$John + Jill = 55$$
$$Jack + Jill = 62$$

Add the equations:

$$2(John + Jack + Jill) = 174$$
$$John + Jack + Jill = 87$$

$$57 + Jill = 87 \Rightarrow Jill = 30$$
$$55 + Jack = 87 \Rightarrow Jack = 87$$
$$John + 62 = 87 \Rightarrow John = 25$$

B. Standard Solving Methods

Example 2.3: Elimination
Solve the system of equations below:

$$\underbrace{9x - 4y = 6}_{Equation\ I}, \qquad \underbrace{7x + y = 17}_{Equation\ II}$$

Multiply Eq II by 4:

$$\underbrace{28x + 4y = 68}_{Equation\ III}$$

Add Eq I and Eq III:

$$37x = 74 \Rightarrow x = 2$$

Substitute $x = 2$ in Eq I:

$$18 - 4y = 6 \Rightarrow 12 = 4y \Rightarrow 3 = y$$

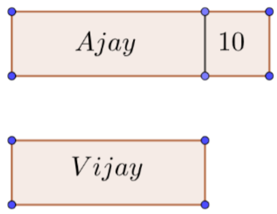
So, the solution is:

$$x = 2, y = 3$$

C. Models and Logic

Example 2.4: More and Less

A. Ajay and Vijay together have 60 cars. Ajay has 10 more cars than Vijay.
B. Monica has 22 marbles more than Rachel. If they together have 200 marbles, how many does each have?



From the model

$$Total = 60$$

If we subtract the extra cars that Ajay has, we are left with

$$60 - 10 = 50\ Cars$$

These 50 cars are equally divided between Ajay and Vijay and hence, Vijay has

$$\frac{50}{2} = 25\ Cars$$

And, finally, Ajay has

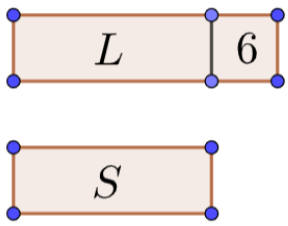
$$25 + 10 = 35\ Cars$$

Example 2.5: Sum and Difference

A. The sum of two numbers is 40. The difference between the two numbers is 6. Find the product of the two numbers.
B. Eva is thinking of two numbers. When she adds the numbers, she get a number one less than the smallest three digit number. When she subtracts the smaller number from the larger, she gets 13. Find the product of the two numbers.

$$\frac{40 - 6}{2} = \frac{34}{2} = 17$$
$$17 + 6 = 23$$

$$17 \times 23 = 391$$



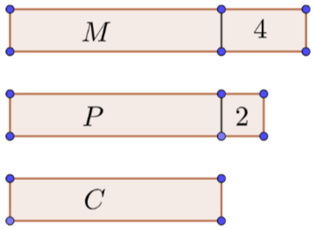
Example 2.6: Multiples / Ratios

Richard and William together have 150 books. The number of books that Richard has is four times the number of books that William has. How many more books must William buy so that he has the same number of books as Richard.

Example 2.7: Three Types of Objects

Four more people in my class marked their favorite subject as Maths (as compared to Chemistry). Two more people in my class marked their favorite subject as Physics (as compared to Chemistry). If everyone has a single favorite subject, and I sit with 23 other students in the class, find the number of students who like each subject?

$$Chemistry = \frac{24 - 4 - 2}{3} = \frac{18}{3} = 6$$
$$Physics = 6 + 2 = 8$$
$$Maths = 6 + 4 = 10$$



Example 2.8: Making Things Equal

Shalu has 26 drawings. Sheela has 16 drawings. How many drawings must Shalu give to Sheela so that they both have the same number?

Shalu has

$$26 - 16 = 10 \textit{ more drawings}$$

They will have an equal number of drawings if we split the difference

$$\frac{10}{2} = 5 \textit{ drawings that Shalu must give Sheela}$$

Example 2.9: Observation

Phoenix hiked the Rocky Path Trail last week. It took four days to complete the trip. The first two days she hiked a total of 22 miles. The second and third days she averaged 13 miles per day. The last two days she hiked a total of 30 miles. The total hike for the first and third days was 26 miles. How many miles long was the trail? **(MathCounts 1998 State Sprint)**

First two days $= 22$
Last two days $= 30$

Total Trip $= 52$

Example 2.10:Working with Rules

An octopus has 8 tentacles and 1 head. A jellyfish has 20 tentacles and no head. A cow has 4 legs and 1 head. Farmer Brown, who raises only octopi, jellyfish and cows on her farm, has animals with a total of 17 heads, 196 tentacles and 20 legs. How many animals does she have? **(MathCounts 2008 Chapter Team)**

Only cows have legs. Hence, the number of cows must be:

$$Cows = \frac{20}{4} = 5$$

This means that the number of octopi is:

$$\underbrace{17}_{\substack{Total\\Heads}} - \underbrace{5}_{\substack{Cow\\Heads}} = 12 \Rightarrow Octopi\ Tentacles = 12 \times 8 = 96$$

Hence, the number of jellyfish tentacles is:

$$Jellyfish\ Tentacles = 196 - 96 = 100 \Rightarrow Jellyfish = \frac{100}{20} = 5$$

Example 2.11: Using Change Rules

A. A group of children riding on bicycles and tricycles rode past Billy Bob's house. Billy Bob counted 7 children and 19 wheels. How many tricycles were there? **(AMC 8 2003/4)**
B. The Fort Worth Zoo has a number of two-legged birds and a number of four-legged mammals. On one visit to the zoo, Margie counted 200 heads and 522 legs. How many of the animals that Margie counted were two-legged birds? **(AMC 8 2012/9)**
C. A bakery sells croissants for \$4.5 and pancakes for \$2.5. One day, they sold 30 croissants and pancakes, and \$9 more from the croissants as compared to the pancakes. Find how many they sold of each?

Part A

Base case: Everyone on bicycles.

$$\underbrace{14\ Wheels}_{All\ Bicycles} \to \underbrace{15\ Wheels}_{1\ Tricycle}$$

14 Wheels is too few, so convert one bicycle into a tricycle. This increases the number of wheels by 1. We need to add five wheels, so we convert 5 bicycles into tricycles. This is easy because of the small numbers, but let's also get a formula out of this:

$$No.\ of\ Tricycles = \frac{19-14}{3-2} = \frac{5}{1} = 5$$

Part B

If all the animals were birds, we would have:

$$\underbrace{400\ Legs}_{All\ Birds}$$

If we change one bird to one mammal, we :

$$402\ Legs$$

Changing one bird to one mammal increases the number of legs by 2. We need 522 $Legs$:

$$\frac{522-400}{2} = \frac{122}{2} = 61\ Mammals$$

Hence, the number of birds is:

$$200 - 61 = 139\ Birds$$

Part C

$$1\ Croissant = 1\ Pancake = \$2$$

If the bakery sold an equal number of each, then

$$15\ Croissants - 15\ pancakes = 15 \times 2 = \$30$$

We want a difference of \$9:

$$30 - 9 = \$21\ less$$

If we sell one croissant less, and one pancake more:

$$Money\ from\ Croissants\ reduces\ by\ \$4.5$$
$$Money\ from\ pancakes\ increases\ by\ \$2.5$$
$$Total\ Change = 4.5 + 2.5 = \$7\ less$$

No. of croissants to be replaced with pancakes is:

$$\frac{21}{7} = 3$$

Which means, we want

$$15 + 3 = 18\ Pancakes$$
$$15 - 3 = 12\ Croissants$$

Example 2.12

How many different combinations of pennies, nickels, dimes and quarters use 48 coins to total \$1.00? (Gauss Grade 7 2005/25)

$$No. of\ Coins = p + n + d + q = 48$$
$$Value\ of\ Coins = p + 5n + 10d + 25q = 100$$

Note that p must be a multiple of 5.

Cases that do not work

$$p = 0 \Rightarrow Min\ Value = 0 + 5(48) = 240$$
$$p = 5 \Rightarrow Min\ Value = 5 + 5(43) = 220$$
$$\vdots$$
$$p = 30 \Rightarrow Min\ Value = 30 + 5(18) = 120$$

35 Pennies

$$p = 35 \Rightarrow Min\ Value = 35 + 13(5) = 100 \Rightarrow Works \Rightarrow (p,n) = (35,13)$$

40 Pennies

$$p = 40 \Rightarrow Min\ Value = 40 + 12(5) = 80$$

Replace 4 nickels with 4 dimes:

$$(p,n,d) = (40,4,4)$$

Replace 1 nickel with 1 Quarter:

$$(p,n,d,q) = (40,7,0,1)$$

45 Pennies

$$p = 45 \Rightarrow Min\ Value = 40 + 3(5) = 55$$

Replace 2 nickels with 2 Quarters:

$$(p,n,d,q) = (45,1,0,2)$$

Example 2.13: Objects as Variables

A. Jack and Jill are filling water from their well using pails. Together, Jack and Jill fetched twelve pails of water. However, Jack had an ankle sprain when he fell down. Hence, he fetched two pails less than Jill. How many pails did each fetch?
B. I want to buy two pencils and a pen. They together cost Rs. 50. The pencil costs five rupees less than the pen. Find the cost of one pen and the cost of one pencil.
C. Two bags and a handbag cost $1275. The handbag is 75 dollars more expensive than the bag. Find the values of each.
D. A year's salary is 2000 and a shirt. Nine months salary is 1200 and a shirt. Find the cost of the shirt.
E. Turban
F. Knife

Part A

$$\underbrace{Jack + Jill = 12}_{Equation\ I}, \qquad \underbrace{Jill - 2 = Jack}_{Equation\ II}$$

Substitute the value of $Jack$ from Equation II into Equation I

$$Jill - 2 + Jill = 12$$
$$2 \times Jill - 2 = 12$$
$$2 \times Jill = 14$$
$$Jill = 7$$
$$Jack = 5$$

Part B

2 Pencils + Pen = 50
Pen - 5 = Pencil

2 Pens - 10 + Pen = 50
3 Pens - 10 = 50
3 Pens = 60
1 Pen = 60/3 = 20
Pencil = 15

Part C

We know that the handbag is 75 dollars more expensive than the bag. Hence:

$$Bag + 75 = Handbag$$

Also, two bags and a handbag cost $1275:

$$2 \times Bag + Handbag = 1275$$
$$2 \times Bag + Bag + 75 = 1275$$
$$3 \times Bag + 75 = 1275$$
$$3 \times Bag = 1200$$
$$Bag = 400$$
$$Handbag = 475$$

Part D

12 Months: 2000 + 1 Shirt
9 Months: 1200 + 1 Shirt

By working 3 months extra, he made 800 extra. Hence,
Salary for 3 Months = 800
Salary for 9 Months = 2400
Price of Shirt = 1200

D. Symmetry

Example 2.14:

The sum of two numbers is eleven. The difference of two numbers is three. Find the two numbers.

Part B

Let the numbers be x, y:

$$\underbrace{Sum: x + y = 11}_{Equation\ I}, \qquad \underbrace{Difference: \ x - y = 3}_{Equation\ II}$$

Add the two equations:

$$2x + 0 = 14 \Rightarrow 2x = 14 \Rightarrow x = 7 \Rightarrow y = 4$$

Method: Substitution

$$Sum = x + y = 11 \Rightarrow \underbrace{x = 11 - y}_{Equation\ I}$$
$$Difference = \underbrace{x - y = 3}_{Equation\ II}$$

$$11 - y - y = 3$$
$$11 - 2y = 3$$
$$8 = 2y$$
$$y = 4$$

Example 2.15: Symmetry

Adding equations to solve a system does not work in general. However, when the sum of coefficients of each variable add up to the same number, it creates a symmetry that can be exploited.

A. Solve the system of equations: $m + n = 96, m - n = 58$
B. The sum of two numbers is 5.6. The difference of the two numbers is 1.2. Find the two numbers.
C. The teacher asks Bill to calculate $a - b - c$, but Bill mistakenly calculates $a - (b - c)$ and gets an answer of 11. If the correct answer was 3, what is the value of $a - b$? **(MathCounts 2005 State Countdown)**

Part A

Add the two equations:

$$2m = 154 \Rightarrow m = \frac{154}{2} = 77$$
$$n = 96 - 77 = 19$$

Part C

Part D

$$a - b - c = 3$$
$$a - b + c = 11$$

Add the two equations

$$2a - 2b = 14 \Rightarrow a - b = 7$$

Example 2.16

A. Given the equations $3x + y = 17,\ 5y + z = 14,\ and\ 3x + 5z = 41$ what is the value of the sum $x + y + z$? **(MathCounts 2003 State Team)**
B. What is the value of $x + y + z$ when $6x + 5y - 2z = -4$ and $4x + 5y + 12z = 54$? **(MathCounts 2009 National Countdown)**
C. When twice a $\underbrace{number}_{a}$ is added to a $\underbrace{second\ number}_{b}$, the total is seventeen. When twice the second number is added to the first number, the total is nineteen. Find the numbers.
D. A, B and C are three friends. They each have some money. The money that A and B have together is fifteen dollars. The money that B and C have together is 20 dollars. The money that C and A have together is 25 dollars. Find the money that each of them has.
E. In a jar of $\underbrace{red}_{r}$, $\underbrace{green}_{g}$, and $\underbrace{blue}_{b}$ marbles, all but 6 are red marbles, all but 8 are green, and all but 4 are blue. How many marbles are in the jar? **(AMC 8 2012/19).** Find the number of red, green, and blue marbles.
F. Find the values of P, Q and R given that $P + Q + R = 75, Q + R + S = 90, P + R + S = 100, Q + S + P = 80$

Part A
Add the equations:
$$6x + 6y + 6z = 72 \Rightarrow x + y + z = 12$$
Part B
Add the two equations:
$$10x + 10y + 10z = 50 \Rightarrow x + y + z = 5$$
Part C
Let the first number be a.
Let the second number be b.
$$\underbrace{2a + b = 17}_{Equation\ I}, \qquad \underbrace{a + 2b = 19}_{Equation\ II}$$
Add the two equations, and divide both sides of the equation by three:
$$3a + 3b = 36 \Rightarrow \underbrace{a + b = 12}_{Equation\ III}$$
Subtract Eq III from Eq I:
$$a = 5$$
Subtract Eq III from Eq II:
$$b = 7$$
Part D
$$\underbrace{a + b = 15}_{Equation\ I}, \qquad \underbrace{b + c = 20}_{Equation\ II}, \qquad \underbrace{c + a = 25}_{Equation\ III}$$
Add the three equations:
$$2a + 2b + 2c = 60 \Rightarrow a + b + c = 30$$
Subtract Eq I from Eq IV:
$$c = 15$$
Subtract Eq II from Eq IV:
$$a = 10$$
Subtract Eq III from Eq IV:
$$b = 5$$
Part E
$$g + b = 6, , r + b = 8, r + g = 4$$
Add the three equations:
$$2g + 2b + 2r = 18$$
This means that double of all the marbles is 18. We want to find how many marbles are there, so we divide both sides by two to find that the total number of marbles is:
$$g + b + r = 9$$
$$r = 3, g = 1, b = 5$$
Part F
Add all the equations:
$$3P + 3Q + 3R + 3S = 345$$
$$P + Q + R + S = 115$$
Subtract each of the given equations from the equation above, one at a time, to find:
$$S = 40, P = 25, Q = 15, R = 35$$

E. Try Small Numbers

When presented with a question, rather than trying to solve the general case immediately, try to check the property or expression with the simplest numbers possible.
If the expression does not hold for small numbers, then it does not hold in general.
Some numbers to try:

- Any number: 0, 1, 2
- Any natural number: 1, 2
- Any even number: 2, 4
- Any odd number: 1, 3
- Any prime: 2, 3

Example 2.17

If a, b, and c are natural numbers (with $a < b < c$) such that $a + b + c = abc = n$, then the largest prime factor of $a^2bc + ab^2c + abc^2 + n^2$ is:

The inequality given is not required, since the expression we evaluate is cyclic. Try 1, 2 and 3 for the numbers (which works):
$$a^2bc + ab^2c + abc^2 + n^2 = abc(a + b + c) + n^2 = 6 \times 6 + 6^2 = 72$$
$$2^3 \times 3^2 \Rightarrow Largest\ Prime\ Factor = 3$$

F. Standard Solving Methods

Example 2.18: Equating

An Internet service provider allows a certain number of free hours each month and then charges for each additional hour used. Wells, Ted, and Vino each have separate accounts. This month the total hours used by Wells and Ted was 105, and each used all of their free hours. Their total cost was 10. Vino used 105 hours by himself and had to pay 26. What is the number of cents charged for each extra hour? **(MathCounts 2001 National Team)**

$$Wells\,\&\,Ted: c(105-2f) = 10 \Rightarrow 105c - 2fc = 10 \Rightarrow 105c = 10 + 2fc$$
$$Vino: c(105-f) = 26 \Rightarrow 105c - cf = 26 \Rightarrow 105c = 26 + fc$$

$$10 + 2fc = 26 + fc \Rightarrow fc = 16$$

$$105c - 16 = 26 \Rightarrow 105c = 42 \Rightarrow c = \frac{42}{105} = \frac{14}{35} = \frac{2}{5} = 40\ cents$$

Example 2.19

$$2x - 3y = -14$$
$$3x - 2y = -6$$

$$2x - 3y = -14 \Rightarrow \underbrace{6x - 9y = -42}_{Equation\ I}$$
$$3x - 2y = -6 \Rightarrow \underbrace{6x - 4y = -12}_{Equation\ II}$$

Subtract Equation II from Equation I:

$$-5y = -30 \Rightarrow y = 6$$

Substitute $y = 6$ in the second equation given in the question:

$$3x - 2(6) = -6$$
$$3x = 6$$
$$x = 2$$

Example 2.20: Elimination, Substitution

A. (*Solve by elimination*) Four burgers and three salads cost 17 dollars. Two burgers and five salads cost nineteen dollars. Find the cost of a burger. Also find the cost of a salad.

B. (*Solve by substitution*, *Solve alternately using equating*) Two dolls and a car cost 140 dollars. A car costs ten dollars less than a doll. Find the cost of one car and one doll.

C. I have a total of 16 red and blue cars. The number of red cars is two more than the number of blue cars. How many do I have of each?

D. From a group of boys and girls, 15 girls leave. There are then left two boys for each girl. After this 45 boys leave. There are then 5 girls for each boy. The number of girls in the beginning was: (AHSME 1950/30)

Part A

$$\underbrace{4b + 3s = 17}_{Equation\ I},\quad \underbrace{2b + 5s = 19}_{Equation\ II}$$

Multiply Equation II by 2:

$$\underbrace{4b + 10s = 38}_{Equation\ III}$$

Subtract Equation I from Equation III:

$$0b + 7s = 21 \Rightarrow s = 3$$

$$b = 2$$

Part B

The idea in substitution is to solve for one variable in one equation and substitute that value in the other equation. This reduces the number of variables in the equation.

Let the cost of a doll be d.

Let the cost of a car be c.

$$\underbrace{2d + c = 140}_{\textit{Equation I}}, \qquad \underbrace{d - 10 = c}_{\textit{Equation II}}$$

Substitute the value of c from Equation II into Equation I:

$$2d \underbrace{+d - 10}_{c} = 140$$
$$3d - 10 = 140$$
$$3d = 150$$
$$d = 50$$
$$c = 40$$

Part B: Alternate Method

We can also solve equations by finding the value of a single variable in both equations, and then equating the two.

$$d - 10 = c$$
$$2d + c = 140 \Rightarrow c = 140 - 2d$$

$$d - 10 = 140 - 2d \Rightarrow 3d = 150 \Rightarrow d = 50$$

Part C

$$\textit{Equation I}: \underbrace{r}_{\substack{Red\\Cars}} + \underbrace{b}_{\substack{Blue\\Cars}} = 16$$
$$\textit{Equation II}: r = b + 2$$

Equation I and Equation II are both currently in two variables. However, substitute the value of r as in Equation II into Equation I. This makes it an equation in one variable:

$$r + b = 16 \Rightarrow (b + 2) + b = 16 \Rightarrow b = 7$$

Part D

$$2(g - 15) = b$$
$$2g - 30 = b$$
$$g - 15 = 5(b - 45)$$
$$g - 15 = 5b - 225$$
$$\frac{g + 210}{5} = b$$
$$2g - 30 = \frac{g + 210}{5}$$
$$10g - 150 = g + 210$$
$$9g = 360$$
$$g = 40$$

G. More Variables

Example 2.21

Three Δ's and a ◊ will balance nine ▪. One $\underbrace{\Delta}_{a}$ will balance a $\underbrace{◊}_{b}$ and a $\underbrace{▪}_{c}$. How many ▪ will be needed to balance two ◊ **(AMC 8 1990/24)**

From the first condition:

$$\underbrace{3a + b = 9c}_{\textit{Equation I}}, \qquad \underbrace{a = b + c}_{\textit{Equation II}}$$

Substitute Equation II in Equation I:

$$3(b + c) + b = 9c \Rightarrow 3b + 3c + b = 9c \Rightarrow 4b = 6c \Rightarrow 2b = 3c$$

Example 2.22

A school trip has 75 people travelling, including 3 teachers. The number of boys is four more than the number of girls (teachers are not included in either boys or girls). Find the number of rooms required if:

A. Teachers stay together in one room.
B. Each room can accommodate a maximum of two boys or two girls.
C. Girls and boys stay separately.

Students + Teachers = 75
Teachers = 3
Students = 72
Boys + Girls = 72
(Boys,Girls) = (36,36), (37,35),(38,34)

Condition II:

Boys - 4 = Girls

$$Rooms\ for\ Teachers = 1$$
$$Rooms\ for\ Boys = \frac{38}{2} = 19$$
$$Rooms\ for\ Girls = \frac{34}{2} = 17$$

Total Rooms

$$= 1 + 19 + 17 = 37$$

Example 2.23

A. Five runners together complete a 100-mile endurance race by running separate, non-overlapping portions of the course. Runner B's portion is 1.5 times the length of Runner A's portion. The combined portion for Runners C and D is twice the length of the combined portion for Runners A and B. Runner E then runs the last 10 miles of the race. How many miles did Runner B run? **(MathCounts 2008 School Sprint)**

B. Each letter of the alphabet is assigned a random integer value, and the letter H is worth 10 points. The value of a word comes from the sum of its letters' values. If $MATH$ is worth 35 points, $TEAM$ is worth 42 points and $MEET$ is worth 38 points, what is the value of A? **(MathCounts 2006 School Team)**

C. The average of Amy's, Ben's, and Chris's ages is 6. Four years ago, Chris was the same age as Amy is now. In four years, Ben's age will be $\frac{3}{5}th$ of Amy's age at that time. How many years old is Chris now? **(AOPS Alcumus, Algebra, Advanced Systems of Equations)**

Part A

$$a + b + c + d + e = 100$$

Substitute $e = 10$:

$$a + b + c + d = 90$$

Substitute $c + d = 2(a + b)$:

$$a + b + 2(a + b) = 90 \Rightarrow a + b = 30$$

Substitute $b = 1.5a \Rightarrow b = \frac{3}{2}a \Rightarrow a = \frac{2}{3}b$:

$$\frac{2}{3}b + b = 30 \Rightarrow \frac{5}{3}b = 30 \Rightarrow b = 18$$

Part B

Start with MATH:

$$MATH = 35 \Rightarrow MAT + H = 35 \Rightarrow MAT = 25$$

Continue with TEAM:

$$TEAM = 42 \Rightarrow E + MAT = 42 \Rightarrow E + 25 = 42 \Rightarrow E = 17$$

Continue with MEET:

$$MEET = 38 \Rightarrow MT + EE = 38 \Rightarrow MT + 34 = 38 \Rightarrow MT = 4$$

Continue with MAT:

$$MAT = 25 \Rightarrow A + MT = 25 \Rightarrow A + 4 = 25 \Rightarrow A = 21$$

Part C

$$b + 4 = \frac{3}{5}(a + 4)$$
$$b + 4 = \frac{3}{5}c$$

$$b = \frac{3}{5}c - 4$$

$$a + b + c = 18 \Rightarrow \underbrace{2c + b = 22}_{Substitute\ a=c-4} \Rightarrow 2c + \frac{3}{5}c - 4 = 22 \Rightarrow 2c + \frac{3}{5}c = 26 \Rightarrow \frac{13}{5}c = 26 \Rightarrow c = 10$$

H. More Questions

Example 2.24

If $x + \frac{1}{y} = 1$ and $y + \frac{1}{z} = 1$, what is the value of the product xyz? **(MathCounts 2009 State Team)**

$$x + \frac{1}{y} = 1 \Rightarrow xy + 1 = y$$
$$y + \frac{1}{z} = 1 \Rightarrow yz + 1 = z$$

$$(xy + 1)z + 1 = z \Rightarrow xyz + z + 1 = z \Rightarrow xyz = -1$$

Example 2.25

In a sequence of positive integers each term after the first is $\frac{1}{3}rd$ of the sum of the term that precedes it and the term that follows it in the sequence. What is the 5^{th} term of this sequence if the 1st term is 2 and the 4^{th} term is 34? **(MathCounts 2009 Chapter Sprint)**

Let the five terms be:

$$2, \frac{2+b}{3}, b, 34, c$$

Using the property of the sequence, we must have:

$$b = \frac{\frac{2+b}{3} + 34}{3} \Rightarrow 3b = \frac{2 + b + 102}{3} \Rightarrow 9b = b + 104 \Rightarrow b = 13$$

Using the property of the sequence, we must have:

$$34 = \frac{13 + c}{3} \Rightarrow 102 \Rightarrow 102 = 13 + c \Rightarrow c = 89$$

I. Geometry

Example 2.26

The sum of the lengths of the first and second side of a triangle is 10. The sum of the lengths of the second and third side of a triangle is 12. The sum of the lengths of the third and the first side is 16. Find the lengths of the sides of the triangle.

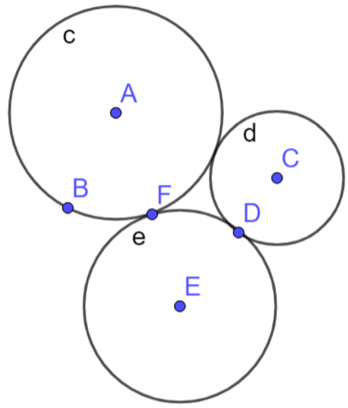
$$a + b = 10$$
$$b + c = 12$$
$$c + a = 16$$

Add the three equations above:

$$2a + 2b + 2c = 38$$

Divide by 2 both sides:

$$a + b + c = 19$$
$$c = 9$$



Example 2.27

Find the radii, if three circles touch each other and the distance between the center of the

$$first\ and\ the\ second\ circle = 5$$
$$first\ and\ the\ third\ circle = 6$$
$$second\ and\ the\ third\ circle = 7$$

Let the radii of the circles be

$$r_1, r_2, r_3$$

$$r_1 + r_2 = 5$$
$$r_1 + r_3 = 6$$
$$r_2 + r_3 = 7$$

$$r_1 + r_2 + r_3 = 9$$
$$r_1 = 2$$
$$r_2 = 3$$
$$r_3 = 4$$

Example 2.28

A. The incircle of ΔABC is drawn touching it at points X, Y and Z respectively. In other words, the circle is tangent to all three sides of the triangle. Given that $AB = 4, BC = 5, AC = 6$, find AX, XB, BY, YC, CZ, ZA.

B. From points A, B and C, tangents are drawn to circle O, meeting it at X, Y and Z respectively. Given that $AB = 4, BC = 5, AC = 6$, find AX, XB, BY, YC, CZ, ZA.

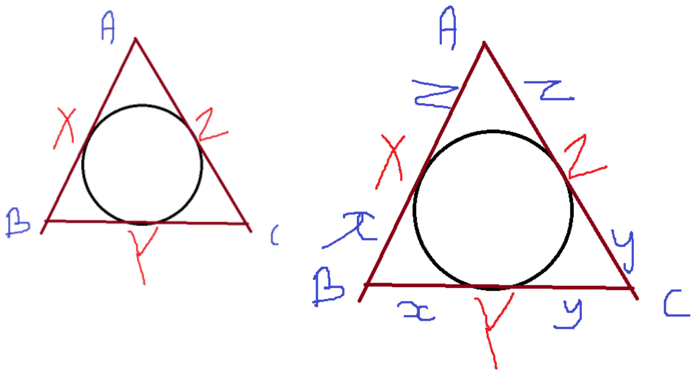
Hint: Tangents drawn from a point to a circle are congruent.

Part A

$$AB = x + z = 4$$
$$BC = x + y = 5$$
$$CA = y + z = 6$$

Add the three equations:

$$2(x + y + z) = 15 \Rightarrow x + y + z = 7.5$$
$$CZ = CY = y = 3.5$$
$$AX = AZ = z = 2.5$$
$$BX = BY = x = 1.5$$



Part B

Same as above

Example 2.29

Can the line segments joining the centers of three circles that each touch the other two externally form a 3 –

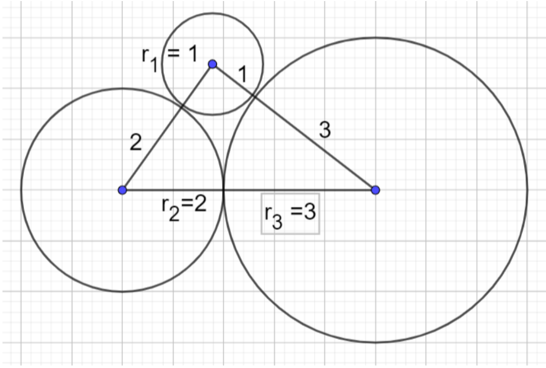
4 − 5 right triangle? Explain why, or why not, with proof. If the circles can be drawn, show it using a diagram.

Since the triangle formed by the centres of the three circles is 3 − 4 − 5 right triangle, we must have:

$$r_1 + r_2 = 3$$
$$r_2 + r_3 = 5$$
$$r_3 + r_1 = 4$$

Add the three equations:

$$2(r_1 + r_2 + r_3) = 12 \Rightarrow r_1 + r_2 + r_3 = 6$$
$$3 + r_3 = 6 \Rightarrow r_3 = 6 - 3 = 3$$
$$r_1 + 5 = 6 \Rightarrow r_1 = 6 - 5 = 1$$
$$r_2 + 4 = 6 \Rightarrow r_2 = 6 - 4 = 2$$



J. Upstream and Downstream

Example 2.30

A boat can travel upstream in a river at a speed of 7 miles per hour, and downstream at a speed of 10 miles per hour. Find:

A. The speed of the boat in still water
B. The speed of the water

K. Mixtures

Example 2.31: Mixtures

You have a solution of 6% hydrochloric acid and a solution of 9% hydrochloric acid. You want to make 5 litres of solution of 7% hydrochloric acid. How much should you use of each solution?

Example 2.32: Multiplication

A. a, b, c, d are natural numbers such that $a = bc, b = cd, c = da, d = ab$. Then, find the value of $(a + b)(b + c)(c + d)(d + a)$.
B. a, b, c, d are natural numbers such that $a = bcd, b = cda, d = abc$. Then find the value of $(a + b + c + d)^2$. (NMTC Sub-Junior/Screening 2005/6)
C. The volume of a rectangular solid each of whose side, front, and bottom faces are $12\ in^2$, $8\ in^2$, and $6\ in^2$ respectively is: (AHSME 1950/21)
D. If $3x = 8y$ and $5y = 15z$, what is the value of $\frac{x}{z}$? Express your answer in simplest form. (MathCounts 2009 Chapter Countdown)
E. If $\frac{x}{y} = \frac{3}{4}, \frac{y}{z} = \frac{2}{3}, \frac{z}{w} = \frac{5}{8}$, what is the value of $\frac{x+y+w}{w}$? Express your answer as a common fraction. (MathCounts 2003 National Target)

Part A

Multiply the four equations together:

$$abcd = (bc)(cd)(da)(ab) = a^2b^2c^2d^2 = (abcd)^2$$
$$abcd = (abcd)^2$$

Since a, b, c, d are natural numbers, $abcd \neq 0$.
Divide both sides of the equation by $abcd$:

$$abcd = 1$$

Since a, b, c, d are natural numbers, they must all be 1:

$$a, b, c, d = 1$$
$$\therefore (a + b)(b + c)(c + d)(d + a) = (2)(2)(2)(2) = 16$$

Part B

$$\underbrace{a = bcd}_{Equation\ I}, \quad \underbrace{b = cda}_{Equation\ II}, \quad \underbrace{d = abc}_{Equation\ III}$$

Multiply the three equations together:

$$abc = a^2b^2c^3d^2$$

Divide both sides by abc:

$$1 = abcd^2$$

Since a, b, c, d are natural numbers, they must all be 1:

$$a, b, c, d = 1$$
$$(a + b + c + d)^2 = (1 + 1 + 1 + 1)^2 = 4^2 = 16$$

Part C

Let the sides of the solid be x, y, z. Then:

$$xy = 12$$
$$yz = 8$$
$$zx = 6$$

Notice the symmetry in the above: each variable occurs twice.

Multiply the three equations:

$$x^2y^2z^2 = 12 \times 8 \times 6$$

Take Square Roots:

$$Volume = xyz = 24$$

Part D

$$3x = 8y \Rightarrow \underbrace{\frac{x}{y} = \frac{8}{3}}_{Equation\ I}, \quad 5y = 15z \Rightarrow \underbrace{\frac{y}{z} = 3}_{Equation\ II}$$

Multiply the two equations to get

$$\frac{x}{y} \times \frac{y}{z} = \frac{8}{3} \times 3 \Rightarrow \frac{x}{z} = 8$$

Part E

We need to find

$$\frac{x + y + w}{w} = \frac{x}{w} + \frac{y}{w} + \frac{w}{w} = \frac{x}{w} + \frac{y}{w} + 1$$
$$\frac{x}{w} = \frac{x}{y} \times \frac{y}{z} \times \frac{z}{w} = \frac{3}{4} \times \frac{2}{3} \times \frac{5}{8} = \frac{5}{16}$$
$$\frac{y}{w} = \frac{y}{z} \times \frac{z}{w} = \frac{2}{3} \times \frac{5}{8} = \frac{5}{12}$$

Example 2.33: k Method

The k method introduces a new variable in an equality. It is particularly useful when there are more than two variables.

A. The sum of three numbers x, y, z is 165. When the smallest number x is multiplied by 7, the result is n. The value n is obtained by subtracting 9 from the largest number y. This number n also results by adding 9 to the third number z. What is the product of the three numbers? (MathCounts 2001 National Team)

B. Four positive integers A, B, C and D have a sum of 64. If $A + 3 = B - 3 = C \times 3 = D \div 3$, what is the value of the product $A \times B \times C \times D$? (AOPS Alcumus, Algebra, Advanced Systems of Equations)

Part A

$$7x = n \Rightarrow x = \frac{n}{7}$$
$$y - 9 = n \Rightarrow y = n + 9$$
$$z + 9 = n \Rightarrow z = n - 9$$

Substitute the above three in $x + y + z = 165$ giving:

$$\frac{n}{7} + (n + 9) + (n - 9) = 165 \Rightarrow \frac{15n}{7} = 165 \Rightarrow n = 77$$
$$x = 11, y = 86, z = 68 \Rightarrow xyz = 68{,}328$$

Part B

We introduce a new variable k, and equate the existing expressions to k:

$$A + 3 = B - 3 = C \times 3 = \frac{D}{3} = k$$

Now, rewrite all the existing variables in terms of k:

$$A = k - 3, \qquad B = k + 3, \qquad C = \frac{k}{3}, \qquad D = 3k$$

Substitute the above values in $A + B + C + D = 64$ giving us:

$$(k-3)+(k+3)+\left(\frac{k}{3}\right)+(3k)=64 \Rightarrow \frac{16k}{3}=64 \Rightarrow k=12$$

And now find the values of the variables:

$$A=k-3=9, \qquad B=k+3=15, \qquad C=\frac{k}{3}=4, \qquad D=3k=36$$

$$A\times B\times C\times D=9\times 15\times 4\times 36=19{,}440$$

2.2 Systems of Equations: Special Cases

A. No Solutions

We can represent a linear equation in two variables as a line on the coordinate plane. To find the solution of two equations in two variables, we look for their intersection point.
So far, we have been looking at the most "useful" cases where there is exactly one solution. However, this is not the only case.

2.34: No Solutions: Graphical Approach

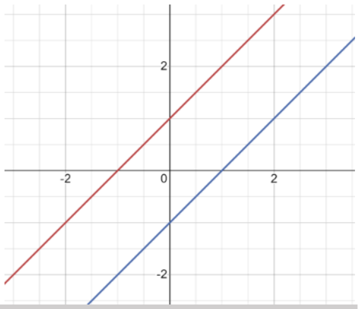
Parallel lines do not intersect. Hence, two equations do not have a solution when the lines representing them are parallel.

Example 2.35

Show, graphically and algebraically, that the system of equations $y=x+1, y=x-1$ has no solution

Part A: Graphical

Graph the two equations. The two lines have the same slope, but different y-intercepts, and will never intersect. Hence, the system has no solutions.



Part B: Algebraic

The LHS of both equations is equal. Hence, the RHS must also be equal.

$$x+1=x-1 \Rightarrow +1=-1 \Rightarrow \textit{No Solution}$$

2.36: No Solutions: Algebraic Approach

The equations of two parallel lines, when solved simultaneously, have no solution.

Slopes will be same, but y intercepts will be different. Hence, let the equations be:

$$y=mx+c_1, \qquad y=mx+c_2, \qquad c_1\neq c_2$$

The left-hand side of both the equations is the same. Hence, the right-hand side must also be the same.

$$mx+c_1=mx+c_2 \Rightarrow c_1=c_2 \Rightarrow Contradicion$$

Example 2.37

For the simultaneous equations

$$2x-3y=8$$
$$6y-4x=9$$

A. $x=4, y=0$
B. $x=0, y=\frac{3}{2}$
C. $x=0, y=0$
D. There is no solution
E. There are an infinite number of solutions (AHSME 1950/14)

B. Infinite Solutions

2.38: Infinite Solutions: Graphical Approach

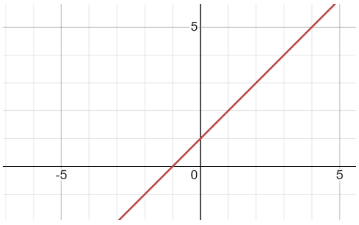
Two equations when an infinite number of solutions when they both represent the same line.

Example 2.39

Show, graphically that the system of equations below has an infinite number of solutions:

$$y = x + 1$$
$$2y = 2x + 2$$

Graph the two equations.
Both equations represent the same line.



Hence, the two equations intersect in an infinite number of points, and hence they have an infinite number of solutions.

2.40: Infinite Solutions: Algebraic Approach

- Two equations have an infinite number of solutions when one equation is a multiple of the other equation.
- Equivalently, they both represent the same line.

Example 2.41

Show, graphically that the system of equations below has an infinite number of solutions:

$$y = x + 1$$
$$2y = 2x + 2$$

Multiplying the first equation by 2 gives us the second equation.
This gives us the second equation.

Multiplying the equation does not change the equation (as we already know), and hence it is actually equivalent to the earlier equation. We have not added any new information about the variables.

Hence, there are an infinite number of solutions.

C. Review

Example 2.42

$$3x + 2y = 7$$
$$3y + 4x = 5$$

Solve the above system of equations by:

A. Elimination
B. Graphing
C. Substitution

Question 2.43

A system of equation can have no solutions, a single solution, or an infinite number of solutions. What are the conditions when each of this happens?

A. Answer graphically.
B. Answer algebraically.

Question 2.44

In each system of equations below, identify whether the system has no solutions, a single solution, or an infinite number of solutions.

A. $System\ I:\ 3x+2y=7,\ 6x+4y=8$
B. $System\ II: 3x+2y=7,\ 4x+6y=7$
C. $System\ III: 3x+2y=7, 6x+4y=14$

2.3 Advanced Systems of Equations

A. Symmetry

2.45: Addition Based

Example 2.46: Three Variables, Three Equations:

$$a+b=3$$
$$b+c=7$$
$$c+a=9$$

$$2(a+b+c)=19$$
$$a+b+c=\frac{19}{2}$$

2.47: Multiplication Based

Example 2.48: Three Variables, Three Equations:

$$xy=a$$
$$yz=b$$
$$zx=c$$
$$(where\ a,b,c\ are\ constants)$$

$$x^2y^2z^2=abc$$
$$xyz=\sqrt{abc}$$

Example 2.49: Three Variables, Three Equations:

$$Scenario\ leading\ to\ multiplication\ based:$$
$$Cuboid:$$
$$Front\ surface\ area=a$$
$$Side\ surface\ area=b$$
$$Top\ surface\ area=c$$

$$xy = a$$
$$yz = b$$
$$zx = c$$
$$(where\ a, b, c\ are\ constants)$$

$$x^2y^2z^2 = abc$$
$$xyz = \sqrt{abc}$$

2.4 AMC Questions

A. Fractions

Example 2.50

The equality $\frac{1}{x-1} = \frac{2}{x-2}$ is satisfied by: **(AHSME 1955/4)**

By observation, zero works:

$$LHS = \frac{1}{0-1} = -1, RHS = \frac{2}{0-2} = \frac{2}{-2} = -1 = LHS$$

And since this is a linear equation, we only need solution.

Or, we can solve using cross multiplication:

$$x - 2 = 2x - 2$$
$$x = 2x$$
$$x = 0$$

Example 2.51

If $\frac{1}{x} - \frac{1}{y} = \frac{1}{z}$, then z equals (answer in terms of x and y): **(AHSME 1958/2)**

$$\frac{1}{z} = \frac{1}{x} - \frac{1}{y} = \frac{y-x}{xy} \Rightarrow z = \frac{xy}{y-x}$$

Example 2.52

A farmer divides his herd of n cows among his four sons so that one son gets one-half the herd, a second son, one-fourth, a third son, one-fifth, and the fourth son, 7 cows. Then n is: **(AHSME 1959/9)**

$$\frac{n}{2} + \frac{n}{4} + \frac{n}{5} + 7 = n$$
$$\frac{10n + 5n + 4n}{20} + 7 = n$$
$$7 = n - \frac{19n}{20}$$
$$\frac{n}{20} = 7$$
$$n = 140$$

Example 2.53

If one minus the reciprocal of $(1 - x)$ equals the reciprocal of $(1 - x)$, then x equals **(AHSME 1976/1)**

First Solution

$$1-\frac{1}{1-x}=\frac{1}{1-x}$$

Add the fractions on the left-hand side:

$$\frac{1-x}{1-x}-\frac{1}{1-x}=\frac{1}{1-x}$$

$$\underbrace{\frac{-x}{1-x}=\frac{1}{1-x}}_{\textit{Equation I}}$$

$$-x=1$$

$$x=-1$$

Second Solution

Start with Equation I and cross multiply:

$$(-x)(1-x)=1-x$$
$$-x+x^2=1-x$$
$$x^2=1$$
$$x=\pm 1$$

But, note that if $x=1$

$$1-x=1-1=0\Rightarrow \textbf{\textit{Not Valid}}$$

Example 2.54

The distance light travels in one year is approximately 5,870,000,000,000 miles. The distance light travels in 100 years is: (write your answer in scientific notation): (AHSME 1956/3)

$$x=5{,}870{,}000{,}000{,}000=587\times 10^{10}=5.87\times 10^{12}$$

Multiply both sides of the above by 100:

$$100x=5.87\times 10^{14}$$

Example 2.55

The three-digit number $2a3$ is added to the number 326 to give the three-digit number $5b9$. If $5b9$ is divisible by 9, then $a+b$ equals (AHSME 1967/8)

$5b9$ is divisible by 9:

$$5+b+9=14+b\Rightarrow b=4\Rightarrow 5b9=549$$

$$2a3+326=549$$
$$2a3=549-326=223$$
$$a=2$$

Example 2.56

An altitude h of a triangle is increased by a length m. How much must be taken from the corresponding base b so that the area of the new triangle is one-half that of the original triangle? (AHSME 1961/16)

The increased height

$$=h+m$$

If the decrease is x, then the decreased base

$$=b-x$$

The area of the new triangle

$$=\frac{1}{2}(h+m)(b-x)$$

The area of the original triangle

$$=\frac{1}{2}hb$$

By the condition given in the question, the area of the new triangle is half the area of the original triangle:

$$\frac{1}{2}(h+m)(b-x)=\left(\frac{1}{2}\right)\left(\frac{1}{2}hb\right)$$

Cancel $\frac{1}{2}$ on both sides, and divide by $(h+m)$:

$$b-x=\frac{hb}{2(h+m)}$$

Solve for x:

$$x=b-\frac{hb}{2(h+m)}$$

To subtract the fractions, make the denominators equal:

$$=b\times\frac{2(h+m)}{2(h+m)}-\frac{hb}{2(h+m)}$$

Simplify:

$$=\frac{2bh+2bm}{2(h+m)}-\frac{hb}{2(h+m)}$$

Carry out the subtraction:

$$=\frac{bh+2bm}{2(h+m)}$$

Factor out b in the numerator:

$$= \frac{b(h+2m)}{2(h+m)}$$

Example 2.57

Let P units be the increase in circumference of a circle resulting from an increase in π units in the diameter. Then P equals: **(AHSME 1968/1)**

$$Original\ diameter = d$$
$$Original\ Circumference = \pi d$$

$$New\ Diameter = d + \pi$$
$$New\ Circumference = \pi(d+\pi) = \pi d + \pi^2$$

$$P = \pi d + \pi^2 - \pi d = \pi^2$$

Example 2.58

When x is added to both the numerator and denominator of the fraction $\frac{a}{b}, a \neq b, b \neq 0$, the value of the fraction is changed to $\frac{c}{d}$. Then x equals (answer in terms of a, b, c, d): **(AHSME 1969/1)**

$$\frac{a+x}{b+x} = \frac{c}{d}$$

Cross multiply:

$$ad + xd = cb + xc$$

Collate all x terms on the LHS:

$$xd - xc = cb - ad$$

Factor out x on the LHS:

$$x(d-c) = cb - ad$$

Solve for x:

$$x = \frac{cb-ad}{d-c} = \frac{ad-bc}{c-d}$$

Example 2.59

If a dealer could get his goods for 8% less while keeping his selling price fixed, his profit, based on cost, would be increased to $(x+10)\%$ from his present profit of $x\%$, which is: **(AHSME 1972/2)**

Original Selling Price

For the sake of simplicity, let

$$Original\ Cost\ Price = 100\ dollars$$

The original profit is given to be

$$x\%\ of\ 100 = x\ dollars$$

Hence, the original selling price:

$$(100 + x\ )\ dollars$$

New Selling Price

Consider the price reduction.

$$New\ Cost\ Price = 100 - 8 = 92\ dollars$$

The new profit will be:

$$(x+10)\%\ of\ 92 = \frac{92x+920}{100} = 0.92x + 9.2$$

The new selling price

$$= 92 + 0.92x + 9.2 = (101.2 + 0.92x)\ dollars$$

But the selling price is the same in both cases:

$$100 + x = 101.2 + 0.92x$$
$$0.08x = 1.2$$
$$8x = 120$$
$$x = 15$$

Profit is:

$$15\%\ of\ cost\ price$$

Example 2.60

A man has 2.73 in pennies, nickels, dimes, quarters and half dollars. If he has an equal number of coins of each kind, then the total number of coins he has is **(AHSME 1977/3)**

Suppose he has one coin of each type. Then, the money he has will be:

$$1 + 5 + 10 + 25 + 50 = 91 \ cents$$

But the actual money with him is:

$$\frac{2.73}{0.91} = 3 \ times \Rightarrow 3 \ coins \ of \ each \ type$$

The total number of coins

$$= 3 \times 5 = 15$$

Example 2.61

If four times the reciprocal of the circumference of a circle equals the diameter of the circle, then the area of the circle is **(AHSME 1978/2)**

$$\frac{4}{C} = d \Rightarrow \frac{4}{2\pi r} = 2r \Rightarrow 4 = 4\pi r^2 \Rightarrow \pi r^2 = 1$$

Example 2.62

If $\frac{x}{x-1} = \frac{y^2+2y-1}{y^2+2y-2}$, then x equals **(AHSME 1981/6)**

Cross-multiply:

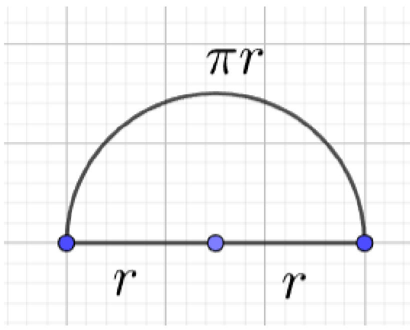
$$x(y^2 + 2y - 2) = (x - 1)(y^2 + 2y - 1)$$

Use the distributive property:

$$\boldsymbol{xy^2} + \boldsymbol{2xy} - 2x = \boldsymbol{xy^2} + \boldsymbol{2xy} - x - y^2 - 2y + 1$$
$$-2x = -x - y^2 - 2y + 1$$
$$-x = -y^2 - 2y + 1$$
$$x = y^2 + 2y - 1$$

Example 2.63

The perimeter of a semicircular region, measured in centimeters, is numerically equal to its area, measured in square centimeters. The radius of the semicircle, measured in centimeters, is **(AHSME 1982/4)**



By the condition given in the question:

$$\underbrace{\pi r + 2r}_{Perimeter} = \underbrace{\frac{\pi r^2}{2}}_{Area}$$

Clearing fractions:

$$2\pi r + 4r = \pi r^2$$

Divide by πr both sides:

$$r = \frac{2\pi r}{\pi r} + \frac{4r}{\pi r} = 2 + \frac{4}{\pi}$$

Example 2.64

Alice sells an item at 10 less than the list price and receives 10% of her selling price as her commission. Bob sells the same item at 20 less than the list price and receives 20% of his selling price as his commission. If they both get the same commission, then the list price is **(AHSME 1983/7)**

Let the list price be p. Then, by the condition given in the question:

$$\frac{10}{100}(p-10)=\frac{20}{100}(p-20)$$
$$p-10=2p-40$$
$$p=30$$

Example 2.65

If $2x+1=8$, then $4x+1=$ **(AHSME 1985/1)**

$$2x=7$$
$$4x=14$$
$$4x+1=15$$

Example 2.66

A large bag of coins contains pennies, dimes and quarters. There are twice as many dimes as pennies and three times as many quarters as dimes. The minimum amount of money which could be in the bag is **(AHSME 1985/4, Adapted)**

Suppose there are p pennies. Then,

$$No.\ of\ Dimes=2p$$
$$No.of\ Quarters=6p$$

$$p+20p+150p=171p$$

Smallest value that p can take is 1.

$$Minimum\ amount=171p=171\ cents=1.71\ dollars$$

Example 2.67

Which terms must be removed from the sum $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}$ if the sum of the remaining terms is to equal 1? **(AHSME 1985/5)**

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}-x=1$$

Convert all the fractions to a denominator of $LCM(2,4,6,8,10,12)$

$$\frac{60+30+20+15+12+10}{120}-x=1$$
$$\frac{147}{120}-x=1$$
$$x=\frac{147}{120}-1=\frac{27}{120}=\frac{9}{40}$$

The denominator of the fraction which is removed must be a factor of 40.

$$40=2^3\times 5$$

The only fraction with a 5 its denominator is $\frac{1}{10}$. Hence, check:

$$\frac{9}{40}-\frac{1}{10}=\frac{9}{40}-\frac{4}{40}=\frac{5}{40}=\frac{1}{8}$$

Hence, the terms which are removed must be:

$$\frac{1}{8}+\frac{1}{10}=\frac{9}{40}$$

Example 2.68
If $a-1=b+2=c-3=d+4$, which of the four quantities a,b,c,d is the largest? (AHSME 1987/7)

$$b=a-3$$
$$c=a+2$$
$$d=a-5$$

$$a, a-3, a+2, a-5 \Rightarrow a+2 \ is \ largest \Rightarrow c \ is \ largest$$

Example 2.69
If $3(4x+5\pi)=P$ then $6(8x+10\pi)$, in terms of P is equal to, (AHSME 1992/1)

$$6(8x+10\pi)=12(4x+5\pi)=4\times 3(4x+5\pi)=4P$$

Example 2.70
Pat intended to multiply a number by 6 but instead divided by 6. Pat then meant to add 14 but instead subtracted 14. After these mistakes, the result was 16. If the correct operations had been used, the value produced would have been: (AHSME 1994/5)

$$\frac{x}{6}-14=16 \Rightarrow \frac{x}{6}=30 \Rightarrow x=180 \Rightarrow 6x=1080 \Rightarrow 6x+14=1094$$

Example 2.71
If $\angle A$ is four times $\angle B$, and the complement of $\angle B$ is four times the complement of $\angle A$, then $\angle B=$ (AHSME 1994/9)

$$\angle B=y \Rightarrow \angle A=4\angle B=4y$$
$$Complement \ of \ \angle B=90-y$$
$$Complement \ of \ \angle A=90-4y$$

$$90-y=4(90-4y)$$
$$90-y=360-16y$$
$$15y=270$$
$$y=\frac{270}{15}=\frac{90}{5}=18^\circ$$

Example 2.72
Each day Walter gets 3 dollars for doing his chores or 5 dollars for doing them exceptionally well. After 10 days of doing his chores daily, Walter has received a total of 36 dollars. On how many days did Walter do them exceptionally well? (AHSME 1996/2)

$$5x+3(10-x)=36$$
$$5x+30-3x=36$$
$$2x=6$$
$$x=3$$

Example 2.73
At the end of 1994 Walter was half as old as his grandmother. The sum of the years in which they were born is 3838. How old will Walter be at the end of 1999? (AHSME 1999/8)

$$1994 - w + 1994 - 2w = 3838$$
$$3988 - 3838 = 3w$$
$$w = 50$$
$$w + 5 = 55$$

Example 2.74

A picture 3 feet across is hung in the center of a wall that is 19 feet wide. How many feet from the end of the wall is the nearest edge of the picture? **(AMC 8 1986/10)**

$$x + x + 3 = 19$$
$$x = 8$$

Example 2.75

The glass gauge on a cylindrical coffee maker shows that there are 45 cups left when the coffee maker is 36% full. How many cups of coffee does it hold when it is full? **(AMC 8 1988/20)**

$$0.36x = 45$$
$$x = \frac{45}{0.36} = \frac{4500}{36} = \frac{1500}{12} = 125$$

Basics

Example 2.76

Margie bought 3 apples at a cost of 50 cents per apple. She paid with a 5-dollar bill. How much change did Margie receive? **(AMC 8 2011/1)**

$$5 - 3 \times 0.5 = 5 - 1.5 = 3.5\ Dollars = 3\ Dollars\ 50\ Cents$$

Example 2.77

If $991 + 993 + 995 + 997 + 999 = 5000 - N$, then $N =$ **(AMC 8 1991/4)**

$$LHS = (1000 - 9) + (1000 - 7) + (1000 - 5) + (1000 - 3) + (1000 - 1) = 5000 - 25$$
$$N = 25$$

Example 2.78

Theresa's parents have agreed to buy her tickets to see her favorite band if she spends an average of 10 hours per week helping around the house for 6 weeks. For the first 5 weeks she helps around the house for 8, 11, 7, 12 and 10 hours. How many hours must she work for the final week to earn the tickets? **(AMC 8 2007/1)**

$$8 + 11 + 7 + 12 + 10 = (10 - 2) + (10 + 1) + (10 - 3) + (10 + 2) + (10) = 50 - 2 = 48$$
$$Required\ Hours = 6 \times 10 = 60$$
$$Hours\ for\ Final\ Week = 60 - 48 = 12$$

Example 2.79

Chandler wants to buy a 500 dollar mountain bike. For his birthday, his grandparents send him 50 dollars, his aunt sends him 35 dollars and his cousin gives him 15 dollars. He earns 16 dollars per week for his paper route.

He will use all of his birthday money and all of the money he earns from his paper route. In how many weeks will he be able to buy the mountain bike? (AMC 8 2007/5)

$$\frac{500-100-35-15}{16}=\frac{400}{16}=25$$

Example 2.80
Susan had 50 dollars to spend at the carnival. She spent 12 dollars on food and twice as much on rides. How many dollars did she have left to spend? (AMC 8 2007/20)

$$50-12-24=50-36=14$$

Example 2.81
If $\frac{3}{5}=\frac{M}{45}=\frac{60}{N}$ what is $M+N$? (AMC 8 2008/8)

$$\frac{3}{5}=\frac{27}{45}=\frac{60}{100}\Rightarrow M+N=27+100=127$$

Example 2.82
A teacher tells the class,
"Think of a number, add 1 to it, and double the result. Give the answer to your partner. Partner, subtract 1 from the number you are given and double the result to get your answer."
Ben thinks of 6, and gives his answer to Sue. What should Sue's answer be? (AMC 8 1995/4)

Substitute $x=6$ in $2[2(x+1)-1]$

$$2[2(6+1)-1]=2[13]=26$$

Example 2.83
Bridget bought a bag of apples at the grocery store. She gave half of the apples to Ann. Then she gave Cassie 3 apples, keeping 4 apples for herself. How many apples did Bridget buy? (AMC 8 2009/1)

$$\frac{a}{2}-3=4$$
$$\frac{a}{2}=7$$
$$a=14$$

Example 2.84
The sum of two numbers is S. Suppose 3 is added to each number and then each of the resulting numbers is doubled. What is the sum of the final two numbers in terms of S? (AMC 10 2001/3)

$$a+b=S$$
$$(a+3)+(b+3)=S+6$$
$$2(a+3)+2(b+3)=2(S+6)=2S+12$$

Example 2.85
Socks cost \$4 per pair and each T-shirt costs \$5 more than a pair of socks. Each member needs one pair of socks and a shirt for home games and another pair of socks and a shirt for away games. If the total cost is \$2366, how many members are in the League? (AMC 10A 2003/2, AMC 12A 2003/2)

$$Socks = 4, Tshirt = 9$$

$$2(4+9) = 2 \times 13 = 26$$
$$\frac{2366}{26} = 91$$

Example 2.86
Al gets the disease algebritis and must take one green pill and one pink pill each day for two weeks. A green pill costs $1 more than a pink pill, and Al's pills cost a total of 546$ for the two weeks. How much does one green pill cost? **(AMC 10B 2003/2, AMC 12B 2003/2)**

$$Green\ pill = g, Pink\ pill = g - 1$$
$$Per\ day = g + g - 1 = 2g - 1$$

For 2 Weeks:

$$14(2g-1) = 546$$
$$2g - 1 = 39$$
$$2g = 40$$
$$g = 20$$

Example 2.87
You and five friends need to raise 1500 dollars in donations for a charity, dividing the fundraising equally. How many dollars will each of you need to raise? **(AMC 10A 2004/1)**

$$\frac{1500}{6} = 250$$

Example 2.88
On the AMC 12, each correct answer is worth 6 points, each incorrect answer is worth 0 points, and each problem left unanswered is worth 2.5 points. If Charlyn leaves 8 of the 25 problems unanswered, how many of the remaining problems must she answer correctly in order to score at least 100? **(AMC 12A 2004/2)**

$$2.5 \times 8 = 5 \times 4 = 20$$
$$\frac{100-20}{6} = \frac{80}{6} = \frac{40}{3} = 13\frac{1}{3} \Rightarrow 14\ Questions$$

Money

Example 2.89
If the value of 20 quarters and 10 dimes equals the value of 10 quarters and n dimes, then $n =$ **(AMC 8 1989/7)**

$$(20)(25) + (10)(10) = (10)(25) + 10n$$
$$600 = 250 + 10n$$
$$350 = 10n$$
$$35 = n$$

Example 2.90
Patty has 20 coins consisting of nickels and dimes. If her nickels were dimes and her dimes were nickels, she would have 70 cents more. How much are her coins worth? **(AMC 10B 2004/15)**

$$5n + 10(20 - n) + 70 = 5(20 - n) + 10n$$
$$5n + 200 - 10n + 70 = 100 - 5n + 10n$$
$$-5n + 270 = 100 + 5n$$
$$170 = 10n$$
$$n = 17$$
$$d = 20 - n = 20 - 17 = 3$$

$$(17)(5) + (3)(10) = 85 + 30 = 115\ cents = 1.15\ dollars$$

Example 2.91
On a trip from the United States to Canada, Isabella took d U.S. dollars. At the border she exchanged them all, receiving 10 Canadian dollars for every 7 U.S. dollars. After spending 60 Canadian dollars, she had d Canadian dollars left. What is the sum of the digits of d? **(AMC 10B 2004/7, AMC 12B 2004/5)**

$$\frac{10}{7}d - 60 = d$$
$$\frac{3}{7}d = 60$$
$$d = 140$$
$$Sum\ of\ digits = 1 + 4 + 0 = 5$$

Average
Example 2.92
The number N is an integer between 9 and 17. The sum of the values that the average of 6, 10, and N could be is: (AMC 8 1989/17, Adapted)

$$\frac{6 + 10 + N}{3} = \frac{16}{3} + \frac{N}{3}$$
$$\left(\frac{16}{3} + \frac{10}{3}\right) + \left(\frac{16}{3} + \frac{11}{3}\right) + \cdots + \left(\frac{16}{3} + \frac{16}{3}\right)$$
$$= 7 \times \frac{16}{3} + \frac{10 + 11 + 12 + 13 + 14 + 15 + 16}{3}$$
$$= \frac{112}{3} + \frac{91}{3}$$
$$= \frac{203}{3}$$

Example 2.93
The average (arithmetic mean) of 10 different positive whole numbers is 10. The largest possible value of any of these numbers is: (AMC 8 1991/19)

$$\frac{n_1 + n_2 + \cdots + n_{10}}{10} = 10$$
$$n_1 + n_2 + \cdots + n_{10} = 100$$
To make n_{10} maximum, we make the others as small as possible:
$$1 + 2 + \cdots + 9 + n_{10} = 100$$
$$n_{10} = 100 - 45$$
$$n_{10} = 55$$

Example 2.94
Four students take an exam. Three of their scores are 70, 80, and 90. If the average of their four scores is 70, then what is the remaining score? (AMC 8 2016/8)

$$\frac{70+80+90+x}{4}=70$$
$$240+x=280$$
$$x=40$$

Example 2.95
The average value of all the pennies, nickels, dimes, and quarters in Paula's purse is 20 cents. If she had one more quarter, the average value would be 21 cents. How many dimes does she have in her purse? (AMC 10A 2004/14, AMC 12A 2004/11)

Suppose she has n coins. Since the average value of a coin is 20 cents, the total value of all the coins
$$=20n\ cents$$

Adding a quarter increases the number of coins by one (from $20n$ to $21n$), and the value of the coins from $20n$ to $20n+25$. Also, adding a quarter makes the average value 21 $cents$.
Combine all the above:
$$20n+25=21(n+1)$$
$$20n+25=21n+21$$
$$n=4$$
$$20n=80$$
To get 80 cents from 4 coins, we must have:
$$3\ quarters\ and\ 1\ nickel \Rightarrow 0\ dimes$$

Age
Example 2.96
Jose is 4 years younger than Zack. Zack is 3 years older than Inez. Inez is 15 years old. How old is Jose? **(AMC 8 1995/2)**

$$Z=I+3=15+3=18$$
$$J=Z-4=18-4=14$$

Example 2.97
Anna and Bella are celebrating their birthdays together. Five years ago, when Bella turned 6 years old, she received a newborn kitten as a birthday present. Today the sum of the ages of the two children and the kitten is 30 years. How many years older than Bella is Anna? **(AMC 8 2022/5)**

$$b=6+5=11$$
$$k=5$$

$$11+5+a=30$$
$$a=14$$

Geometry
Example 2.98
If the length of a rectangle is increased by 20% and its width is increased by 50%, then the area is increased by

(AMC 8 1993/21)

$$\frac{1.2l \times 1.5w}{lw} = 1.8 = 180\% \Rightarrow 80\%\ increase$$

Capacity

Example 2.99

When four gallons are added to a tank that is one-third full, the tank is then one-half full. The capacity of the tank in gallons is (AMC 8 1992/14)

$$\frac{1}{3}c + 4 = \frac{1}{2}c$$
$$4 = \frac{3c}{6} - \frac{2c}{6}$$
$$4 = \frac{c}{6}$$
$$c = 24$$

Example 2.100

When Walter drove up to the gasoline pump, he noticed that his gasoline tank was $\frac{1}{8}$ full. He purchased 7.5 gallons of gasoline for 10 dollars. With this additional gasoline, his gasoline tank was then $\frac{5}{8}$ full. The number of gallons of gasoline his tank holds when it is full is (AMC 8 1996/10)

$$\frac{5}{8}c - \frac{1}{8}c = 7.5$$
$$\frac{4}{8}c = 7.5$$
$$c = 15$$

Change/Slope

Example 2.101

For every 3 rise in temperature, the volume of a certain gas expands by 4 cubic centimeters. If the volume of the gas is 24 cubic centimeters when the temperature is 32, what was the volume of the gas in cubic centimeters when the temperature was 20? (AMC 8 1991/21)

$$V_{Final} = V_{Org} + \frac{4}{3}\Delta T = V_{Org} + \frac{4}{3}\Delta T = 24 + \left(\frac{4}{3}\right)(20 - 32) = 24 + \left(\frac{4}{3}\right)(-12) = 8$$

Example 2.102

One proposal for new postage rates for a letter was 30 cents for the first ounce and 22 cents for each additional ounce (or fraction of an ounce). The postage for a letter weighing 4.5 ounces was (AMC 8 1990/13)

$$p = 30 + 22(4) = 30 + 88 = 113\ cents$$

Example 2.103

The taxi fare in Gotham City is \$2.40 for the first $\frac{1}{2}$ mile and additional mileage charged at the rate \$0.20 for each additional 0.1 mile. You plan to give the driver a $2 tip. How many miles can you ride for $10? (AMC 8 2011/10)

$$(0.2)(10m) + 2.40 + 2 = 10$$
$$2m = 5.6$$
$$m = 2.8$$
$$m + 0.5 = 3.3$$

Multi-Step

Example 2.104

Cindy was asked by her teacher to subtract 3 from a certain number and then divide the result by 9. Instead, she subtracted 9 and then divided the result by 3, giving an answer of 43. What would her answer have been had she worked the problem correctly? (**AMC 10A 2002/6, AMC 12A 2002/2**)

$$\frac{x-9}{3} = 43 \Rightarrow x - 9 = 129 \Rightarrow x = 138$$
$$x - 3 = 135 \Rightarrow \frac{x-3}{9} = 15$$

Example 2.105

Eight friends ate at a restaurant and agreed to share the bill equally. Because Judi forgot her money, each of her seven friends paid an extra $2.50 to cover her portion of the total bill. What was the total bill? **(AMC 8 2013/4)**

$$\frac{B}{7} - \frac{B}{8} = \frac{5}{2}$$
$$\frac{B}{56} = \frac{5}{2}$$
$$B = \frac{5}{2} \times 56 = 140$$

Example 2.106

Three generous friends, each with some money, redistribute the money as followed: Amy gives enough money to Jan and Toy to double the amount each has. Jan then gives enough to Amy and Toy to double their amounts. Finally, Toy gives enough to Amy and Jan to double their amounts. If Toy had 36 dollars at the beginning and 36 dollars at the end, what is the total amount that all three friends have? **(AMC 8 1998/25)**

Step Zero:
$$Toy = 36$$
Step I:
Amy doubles the money for Jan and Toy
$$Toy = 36 \times 2 = 72$$
Step II:
Jan doubles the money that Amy and Toy:
$$Toy = 72 \times 2 = 144$$

Step III:
At the end of Step III, Toy has 36 dollars. So, he gave away:
$$144 - 36 = 108$$

Before Step III, Jan and Amy must have had 108 dollars for their money to get doubled after Toy gives them 108 dollars.

The total money with the three of them:
$$\underbrace{108}_{\substack{With\ Jan\\ and\ Amy}} + \underbrace{108}_{\substack{Given\ by\\ Toy\ to\ Jan\\ and\ Amy}} + \underbrace{36}_{\substack{Left\\ with\\ Toy}} = 252$$

Example 2.107: K Method

Loki, Moe, Nick and Ott are good friends. Ott had no money, but the others did. Moe gave Ott one-fifth of his money, Loki gave Ott one-fourth of his money and Nick gave Ott one-third of his money. Each gave Ott the same amount of money. What fractional part of the group's money does Ott now have? **(AMC 8 2002/25)**

$$O = 0$$
$$\frac{m}{5} = \frac{l}{4} = \frac{n}{3} = k$$
$$m = 5k$$
$$l = 4k$$
$$n = 3k$$
$$m + l + n = 5k + 4k + 35 = 12k$$

$$\frac{3k}{12k} = \frac{1}{4}$$

Suppose that the amount of money with
$$Moe = 5, Loki = 4, Nick = 3$$
Then, Otto gets:
$$1 + 1 + 1 = 3$$
And his share of the total money is:
$$\frac{3}{5 + 4 + 3} = \frac{3}{12} = \frac{1}{4}$$

Example 2.108

In a far-off land three fish can be traded for two loaves of bread and a loaf of bread can be traded for four bags of rice. How many bags of rice is one fish worth? **(AMC 8 1999/22)**

$$3F = 2B, \qquad B = 4R$$
$$3F = 2(4R)$$
$$3F = 8R$$
$$F = \frac{8}{3}R = 2\frac{2}{3}R$$

Example 2.109

Granny Smith has \$63. Elberta has \$2 more than Anjou and Anjou has one-third as much as Granny Smith. How many dollars does Elberta have? **(AMC 8 2001/3)**

$$Anjou = \frac{1}{3}Granny = \frac{1}{3} \times 63 = 21$$
$$Elberta = 2 + 21 = 23$$

Example 2.110

Connie multiplies a number by 2 and gets 60 as her answer. However, she should have divided the number by 2 to get the correct answer. What is the correct answer? **(AMC 8 2005/1)**

$$2x = 60 \Rightarrow \frac{x}{4} = 15$$

Example 2.111

Al, Bert, and Carl are the winners of a school drawing for a pile of Halloween candy, which they are to divide in a ratio of 3:2:1, respectively. Due to some confusion, they come at different times to claim their prizes, and each assumes he is the first to arrive. If each takes what he believes to be the correct share of candy, what fraction of the candy goes unclaimed? (AMC 12A 2003/10)

	Before District Play	After District Play
Play	g	$g+8$
Won	$\frac{9g}{20}$	$\frac{9g}{20}+6, \frac{g}{2}+4$

	Al	Bert	Carl
Take	$\frac{3}{6}=\frac{1}{2}$	$\frac{2}{6}=\frac{1}{3}$	$\frac{1}{6}$
Leave	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$

$$\frac{1}{2}\times\frac{2}{3}\times\frac{5}{6}=\frac{5}{18}$$

Percent

Example 2.112

After Sally takes 20 shots, she has made 55% of her shots. After she takes 5 more shots, she raises her percentage to 56%. How many of the last 5 shots did she make? (AMC 8 2004/6)

Currently, Sally has made

$$55\%\ of\ 20=\frac{55}{100}\times 20=\frac{55}{5}=11\ Shots$$

Her final tally is:

$$56\%\ of\ 25=\frac{56}{100}\times 25=\frac{56}{4}=14\ Shots$$

The number of shots that she made after 20 is:

$$14-11=3\ Shots$$

Example 2.113

Suppose 15% of x equals 20% of y. What percentage of x is y? (AMC 8 2020/15)

$$\frac{15x}{100}=\frac{20y}{100}\Rightarrow 3x=4y\Rightarrow\frac{y}{x}=\frac{3}{4}=75\%$$

Ratios

Example 2.114

Before the district play, the Unicorns had won 45% of their basketball games. During district play, they won six more games and lost two, to finish the season having won half their games. How many games did the Unicorns play in all? (AMC 8 2007/20)

$$\frac{g}{2}+4=\frac{9g}{20}+6$$
$$\frac{10g}{20}-\frac{9g}{20}=2$$

$$\frac{g}{20} = 2$$
$$g = 40$$
$$g + 8 = 48$$

Fractions

Example 2.115
All of Marcy's marbles are blue, red, green, or yellow. One third of her marbles are blue, one fourth of them are red, and six of them are green. What is the smallest number of yellow marbles that Macy could have? **(AMC 8 2017/9)**

$$b + r + g + y = m$$
$$\frac{m}{3} + \frac{m}{4} + 6 + y = m$$
$$y = m - \frac{7m}{12} - 6 = \frac{5m}{12} - 6 = \frac{5m}{12} - 6$$

Note that m must be a multiple of 12.

$$Try\ m = 12 \times 1 \Rightarrow \frac{5m}{12} - 6 = 5 - 6 = -1 \Rightarrow Not\ possible$$
$$Try\ m = 12 \times 2 \Rightarrow \frac{5m}{12} - 6 = 10 - 6 = 4$$

Example 2.116
If $\frac{2+3+4}{3} = \frac{1990+1991+1992}{N}$, then $N =$ **(AMC 8 1991/12)**

$$N = \frac{1990 + 1991 + 1992}{3} = 1991$$

Example 2.117
At Clover View Junior High, one half of the students go home on the school bus. One fourth go home by automobile. One tenth go home on their bicycles. The rest walk home. What fractional part of the students walk home? **(AMC 8 1995/7)**

$$1 - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{10}\right) = 1 - \left(\frac{10 + 5 + 2}{20}\right) = 1 - \frac{17}{20} = \frac{3}{20}$$

Example 2.118
If 5 times a number is 2, then 100 times the reciprocal of the number is **(AMC 8 1996/9)**

$$5x = 2$$
$$x = \frac{2}{5}$$
$$\frac{1}{x} = \frac{5}{2}$$

$$\frac{100}{x} = \frac{500}{2} = 250$$

Example 2.119

Many calculators have a reciprocal key $\frac{1}{x}$ that replaces the current number displayed with its reciprocal. For example, if the display is 0004 and the $\frac{1}{x}$ key is depressed, then the display becomes 000.25. If 00032 is currently displayed, what is the fewest number of times you must depress the $\frac{1}{x}$ key so the display again reads 00032? **(AMC 8 1989/18)**

Press the reciprocal key

$$\frac{1}{32}$$

Press the reciprocal key again

$$32$$
$$Two\ Times$$

Example 2.120

Two-thirds of the people in a room are seated in three-fourths of the chairs. The rest of the people are standing. If there are 6 empty chairs, how many people are in the room? **(AMC 8 2004/20)**

The empty chairs are

$$c-\frac{3}{4}c=\frac{1}{4}c=6$$
$$c=24$$
$$\frac{3}{4}c=18$$
$$\frac{2}{3}p=18$$
$$p=27$$

Example 2.121

A number x is 2 more than the product of its reciprocal and its additive inverse. Find the number. **(AMC 10 2001/2, Adapted)**

$$x=\left(\frac{1}{x}\right)(-x)+2=-1+2=1$$

Example 2.122

Starting with some gold coins and some empty treasure chests, I tried to put 9 gold coins in each treasure chest, but that left 2 treasure chests empty. So instead I put 6 gold coins in each treasure chest, but then I had 3 gold coins left over. How many gold coins did I have? **(AMC 8 2017/17)**

$$9(t-2)=6t+3$$
$$9t-18=6t+3$$
$$3t=21$$
$$t=7$$

$$c=6t+3=6(7)+3=45$$

Example 2.123

The 5-digit number $2018U$ is divisible by 9. What is the remainder when this number is divided by 8? (AMC 8

2018/7)

If a number is divisible by 9, the sum of digits will be a multiple of 9.

$$2 + 0 + 1 + 8 + U = U + 11 \Rightarrow U = 7$$

We want the remainder when 20187 is divided by 8. We need check only the last three digits:

$$187 - 160 = 27 - 24 = 3$$
$$Remainder\ 3$$

Example 2.124: k method

Let $a + 1 = b + 2 = c + 3 = d + 4 = a + b + c + d + 5$. What is $a + b + c + d$? (AMC 10A 2002/16)

Subtract 5 from all parts of the equality:

$$a - 4 = b - 3 = c - 2 = d - 1 = a + b + c + d = k$$

Solve for each variable individually:

$$a - 4 = k \Rightarrow a = k + 4$$
$$b - 3 = k \Rightarrow b = k + 3$$
$$c - 2 = k \Rightarrow c = k + 2$$
$$d - 1 = k \Rightarrow d = k + 1$$

Then, the required expression is:

$$a + b + c + d = k + 4 + k + 3 + k + 2 + k + 1 = 4k + 10$$
$$4k + 10 = k$$
$$3k = -10$$
$$k = -\frac{10}{3}$$

Example 2.125: Number of Solutions

For which value of k does the equation $\frac{x-1}{x-2} = \frac{x-k}{x-6}$ have no solution for x? (AMC 10B 2002/12, Adapted)

Cross Multiply

$$(x - 6)(x - 1) = (x - 2)(x - k)$$
$$x^2 - 7x + 6 = x^2 - 2x - kx + 2k$$
$$-7x + 6 = (-2 - k)x + 2k$$

The equation will not have a solution if the x coefficient on the LHS is the same as the x coefficient on the RHS:

$$-7 = -2 - k$$
$$k = -2 + 7 = 5$$

Consecutive Integers

Example 2.126

The sum of six consecutive positive integers is 2013. What is the largest of these six integers? (AMC 8 2013/8)

$$x + (x - 1) + (x - 2) + (x - 3) + (x - 4) + (x - 5) = 2013$$
$$6x - 15 = 2013$$
$$6x = 2028$$
$$x = 338$$

Example 2.127

The sum of 5 consecutive even integers is 4 less than the sum of the first 8 consecutive odd counting numbers. What is the smallest of the even integers? (AMC 10B 2003/3)

$$1+3=4$$
$$1+3+5=9$$
$$\underbrace{1+3+5+\cdots}_{8\ Numbers}=8^2=64$$

$$x+x+2+x+4+x+6+x+8=64-4$$
$$5x+20=60$$
$$5x=40$$
$$x=8$$

Percents

Example 2.128

A. While eating out, Mike and Joe each tipped their server 2 dollars. Mike tipped 10% of his bill and Joe tipped 20% of his bill. What was the difference, in dollars between their bills? (AMC 10A 2005/1)
B. Two is 10% of x and 20% of y. What is $x-y$? (AMC 12A 2005/1)

$$0.1x=2\Rightarrow x=\frac{2}{0.1}=20$$
$$0.2y=2\Rightarrow y=\frac{2}{0.2}=10$$
$$x-y=20-10=\$10$$

Example 2.129

A positive number x has the property that x% of x is 4. What is x? (AMC 10B 2005/2, AMC 12B 2005/2)

$$x\times\frac{x}{100}=4\Rightarrow x^2=400\Rightarrow x=\pm20\Rightarrow Reject-20\Rightarrow x=20$$

Example 2.130

Mary is 20% older than Sally, and Sally is 40% younger than Danielle. The sum of their ages is 23.2 years. How old will Mary be on her next birthday? (AMC 12A 2006/7)

$$Danielle's\ age=d$$
$$Sally's\ Age=0.6d$$
$$Mary's\ Age=(1.2)(0.6d)=0.72d$$

$$d+0.6d+0.72d=23.2$$
$$2.32d=23.2$$
$$d=\frac{23.2}{2.32}=10$$

$$Mary=0.72d=7.2$$
$$8\ Years$$

Multi Step

Example 2.131

Last year Mr. Jon Q. Public received an inheritance. He paid 20% in federal taxes on the inheritance, and paid 10% of what he had left in state taxes. He paid a total of 10500 for both taxes. How many dollars was his inheritance? (AMC 12A 2007/5)

Let his inheritance be I.
His money after paying 20% federal taxes will be

$$0.8I$$

His money after paying 10% in state taxes

$$= 90\%\ of\ 0.8I = (0.9)(0.8I) = 0.72I$$

Total Tax

$$= I - 0.72I = 0.28I = 10500$$

$$I = \frac{10500}{0.28} = \frac{1{,}050{,}000}{28} = 37500$$

Example 2.132

A gallon of paint is used to paint a room. One third of the paint is used on the first day. On the second day, one third of the remaining paint is used. What fraction of the original amount of paint is available to use on the third day? (AMC 10B 2005/3)

Let the original paint be p.
The paint remaining after the first day will be

$$p - \frac{1}{3}p = \frac{2}{3}p$$

The paint remaining after the second day will be

$$\frac{2}{3}p - \left(\frac{1}{3}\right)\left(\frac{2}{3}p\right) = \frac{6}{3}p - \frac{2}{9}p = \frac{4}{9}p$$

$$\frac{4}{9}\ of\ the\ original\ paint$$

Example 2.133

The equations $2x + 7 = 3$ and $bx - 10 = -2$ have the same solution. What is the value of b? (AMC 10A 2005/3, AMC 12A 2005/2)

$$2x + 7 = 3 \Rightarrow 2x = -4 \Rightarrow x = -2$$

Substitute $x = -2$ in $bx - 10 = -2$

$$b(-2) - 10 = -2 \Rightarrow -2b = 8 \Rightarrow b = -4$$

Fractions

Example 2.134

Brianna is using part of the money she earned on her weekend job to buy several equally-priced CDs. She used one fifth of her money to buy one third of the CDs. What fraction of her money will she have left after she buys all the CDs? (AMC 10B 2005/5, AMC 12B 2005/3)

Let the money be m. Let the number of CDs be c.

$$\frac{1}{5}m \leftrightarrow \frac{1}{3}c$$

Multiply both sides by 3:

$$\frac{3}{5}m \leftrightarrow c$$

Money that she will have left

$$m - \frac{3}{5}m = \frac{2}{5}m$$

Rates

Example 2.135

Josh and Mike live 13 miles apart. Yesterday, Josh started to ride his bicycle toward Mike's house. A little later Mike started to ride his bicycle toward Josh's house. When they met, Josh had ridden for twice the length of time as Mike and at four-fifths of Mike's rate. How many miles had Mike ridden when they met? **(AMC 10A 2005/7, AMC 12A 2005/6)**

Let the distance that Mike travels be m.
The distance that Josh travels will be:

$$\underbrace{(2)}_{\substack{Time\\Factor}}\underbrace{\left(\frac{4}{5}\right)}_{\substack{Speed\\Factor}} m = \frac{8}{5}m$$

But, the total distance travelled by the two is the distance between their houses.

$$m + \frac{8}{5}m = 13 \Rightarrow \frac{5m + 8m}{5} = 13 \Rightarrow 13m = 13 \times 5 \Rightarrow m = 5$$

Basics

Example 2.136

Sandwiches at Joe's Fast Food cost \$3 each and sodas cost \$2 each. How many dollars will it cost to purchase 5 sandwiches and 8 sodas? **(AMC 10A 2006/1, AMC 12A 2006/1)**

$$5 \times 3 + 8 \times 2 = 15 + 16 = \$31$$

Example 2.137

Doug and Dave shared a pizza with 8 equally-sized slices. Doug wanted a plain pizza, but Dave wanted anchovies on half the pizza. The cost of a plain pizza was 8 dollars, and there was an additional cost of 2 dollars for putting anchovies on one half. Dave ate all the slices of anchovy pizza and one plain slice. Doug ate the remainder. Each paid for what he had eaten. How many more dollars did Dave pay than Doug? **(AMC 10A 2006/5, AMC 12A 2006/5)**

The cost of each plain slice

$$= \frac{8}{8} = \$1$$

Doug ate 3 plain slices. His cost was

$$= 3 \times 1 = \$3$$

Dave's cost

$$= (8 + 2) - 3 = 7$$

Their difference

$$= 7 - 3 = 4$$

Example 2.138

A football game was played between two teams, the Cougars and the Panthers. The two teams scored a total of 34 points, and the Cougars won by a margin of 14 points. How many points did the Panthers score? **(AMC 10B 2006/3)**

Let the Panthers score p points.
Then, the Cougars will score $p + 14$.

$$p + p + 14 = 34$$
$$2p = 20$$
$$p = 10$$

Example 2.139

The 2007 AMC 10 will be scored by awarding 6 points for each correct response, 0 points for each incorrect response, and 1.5 points for each problem left unanswered. After looking over the 25 problems, Sarah has decided to attempt the first 22 and leave only the last 3 unanswered. How many of the first 22 problems must she solve correctly in order to score at least 100 points? **(AMC 10B 2007/6, AMC 12B 2007/5)**

Since Sarah will leave the last three questions unanswered, she will get

$$3 \times 1.5 = 4.5\ points$$

Hence, she needs to score an additional

$$100 - 4.5 = 95.5\ points$$

$$16 \times 6 = 96 > 95.5$$
$$She\ needs\ 16\ Questions$$

$$\left\lceil \frac{100 - 3 \times 1.5}{6} \right\rceil = \left\lceil \frac{100 - 4.5}{6} \right\rceil = \left\lceil \frac{95.5}{6} \right\rceil = 16$$

Ratios

Example 2.140

The ratio of Mary's age to Alice's age is 3:5. Alice is 30 years old. How many years old is Mary? **(AMC 10A 2006/3, AMC 12A 2006/3)**

$$3:5 = 18:30$$

Example 2.141

Some boys and girls are having a car wash to raise money for a class trip to China. Initially 40% of the group are girls. Shortly thereafter two girls leave and two boys arrive, and then 30% of the group are girls. How many girls were initially in the group? **(AMC 10B 2007/14, AMC 12B 2007/10)**

A reduction of two girls without a change in the number of people in the group results in a reduction of 10% of the girl population.

$$10\%\ of\ the\ group = 2\ Girls = 2\ People$$
$$100\%\ of\ the\ group = 20\ People$$
$$40\%\ of\ the\ group = 8\ Girls$$

Let the number of people in the group initially be p.
The number of girls in the group initially

$$= 40\%\ of\ the\ group = 0.4p$$

After two girls leave, the number of girls

$$= 0.4p - 2$$

Since two boys arrive instead of the two girls, the total:

$$\frac{0.4p - 2}{p} = \frac{3}{10}$$

Example 2.142

At Frank's Fruit Market, 3 bananas cost as much as 2 apples, and 6 apples cost as much as 4 oranges. How many oranges cost as much as 18 bananas? **(AMC 12B 2007/4)**

$$3B = 2A, \qquad 6A = 4O$$
$$9B = 6A = 4O$$
$$18B = 12A = 8O$$

Consecutive Integers

Example 2.143

How many sets of two or more consecutive positive integers have a sum of 15? **(AMC 10A 2008/9, AMC 12A 2006/8)**

Let x be the smallest of the consecutive positive integers:

$$x + (x+1) = 15 \Rightarrow x = 7$$

$$x + (x+1) + (x+2) = 15 \Rightarrow x = 4$$

$$x + (x+1) + (x+2) + (x+3) = 15 \Rightarrow x = \frac{9}{4}$$

$$x + (x+1) + (x+2) + (x+3) + (x+4) = 15 \Rightarrow x = 1$$

Example 2.144

The larger of two consecutive odd integers is three times the smaller. What is their sum? (AMC 10A 2007/4, AMC 12A 2007/3)

Average

Example 2.145

The Dunbar family consists of a mother, a father, and some children. The average age of the members of the family is 20, the father is 48 years old, and the average age of the mother and children is 16. How many children are in the family? (AMC 10A 2007/10)

Example 2.146

A teacher gave a test to a class in which 10% of the students are juniors and 90% are seniors. The average score on the test was 84. The juniors all received the same score, and the average score of the seniors was 83. What score did each of the juniors receive on the test? (AMC 10B 2007/16, AMC 12B 2007/12)

Literal Equations

Example 2.147

Tom's age is T years, which is also the sum of the ages of his three children. His age N years ago was twice the sum of their ages then. What is T/N? (AMC 10B 2007/12, AMC 12B 2007/8)

148 Examples