QUADRILATERALS

REVISION: 3835

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1. Mensuration

1.1 Parallelogram

A. Definition

1.1: (Def) Parallelogram
A parallelogram is a quadrilateral with opposite sides parallel.

- Special cases of parallelograms include squares, rectangles, and rhombi.

1.2: Special Trapezoid
A parallelogram is a special case of a trapezoid, since a trapezoid has at least one pair of parallel sides, and a parallelogram has both pairs of sides parallel.

1.3: Properties of a Trapezoid/Quadrilateral.
Since a parallelogram is a
- special trapezoid, it inherits all the properties of a trapezoid.
- special trapezoid, it inherits all the properties of a quadrilateral.

B. Perimeter

Example 1.4
A parallelogram has adjacent sides with a total length of 12 cm. One pair of its opposite sides have a length of 10 cm. Find
A. The perimeter
B. Length of each side

The perimeter

$$2 \times Adjacent\ Sides = 2 \times 12 = 24\ cm$$

Since a pair of opposite sides = 10 cm hence:

$$One\ of\ them = 5\ cm$$

Adjacent Sides are

$$5\ cm\ and\ 12 - 5 = 7\ cm$$

Sides are:

$$5\ cm, \quad 7cm, \quad 5\ cm, \quad and\ 7\ cm$$

C. Height

1.5: (Def) Height of a Parallelogram
The height of a parallelogram is the distance between the opposite sides.

- The height is often convenient to calculate when originating from a vertex, but this is not required.

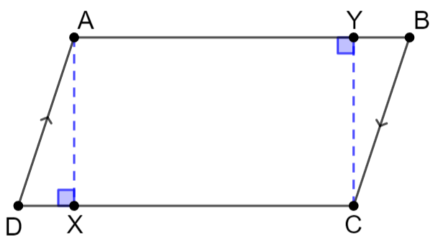
Example 1.6

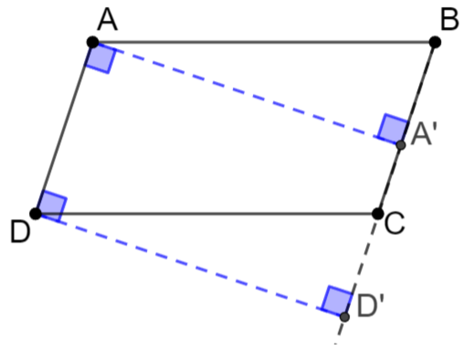
In parallelogram ABCD, draw:

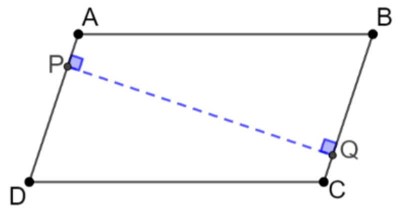
A. the height from AB to CD
B. the height from AD to BC

In parallelogram ABCD, the height from AB to CD

$$= Distance\ between\ AB\ and\ CD = AX = YC$$







The height from AD to BC is a little more difficult to visualize:

$$= Distance\ between\ AD\ and\ BC = AA' = DD' = PQ$$

Note that

$$PQ\ does\ not\ originate\ from\ a\ vertex$$

D. Area

1.7: Area of a Parallelogram

The area of a parallelogram is

$$Base \times Height$$

Example 1.8

The sides of a parallelogram, in order, are $4m$, $6m$, $4m$, and $6m$. The height from the longer side is $2m$. Determine the area of the parallelogram and the height from the shorter side.

The area of the parallelogram

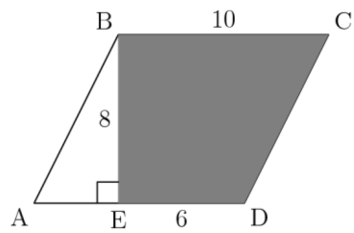
$$= b_1 h_1 = 6(2) = 12\ m^2$$

The area of the parallelogram is also:

$$b_2 h_2 = 4(h_2) = 12 \Rightarrow h_2 = 3\ m$$

Example 1.9

The area of the shaded region $BEDC$ in parallelogram ABCD is (AMC 8 1989/15)



The area of parallelogram $ABCD$ will be

$$A = hb = 10 \cdot 8 = 80$$

The area of ΔAEB

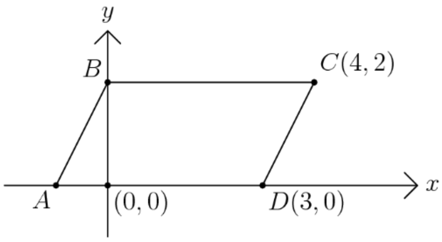
$$= \frac{1}{2}hb = \frac{1}{2}AE \cdot BE = \frac{1}{2}(AD - ED)BE = \frac{1}{2}(10 - 6)8 = 16$$

The area of the shaded region

$$= [ABCD] - [AEB] = 80 - 16 = 64$$

Example 1.10

The area in square units of the region enclosed by parallelogram $ABCD$ is: **(AMC 8 1991/10)**



The area of parallelogram $ABCD$ will be

$$A = hb = 2(4) = 8$$

E. Double Calculation of Area

1.11: Double Calculation of Area

The area of a parallelogram can be calculated in any way that we want and it will be the same always. In particular:

$$A = h_1 b_1 = h_2 b_2$$

If you know any three out of h_1, b_1, h_2, and b_2, then you can find the fourth quantity.

Example 1.12

Determine the number of values of x such that the adjacent sides of a parallelogram are $x + 3$ and $x + 4$. And the altitudes of the parallelogram have length $x + 5$ and $x + 6$.

We do not know which has which altitude. Hence, we consider cases:

Case I

$$(x + 3)(x + 6) = (x + 4)(x + 5)$$
$$x^2 + 9x + 18 = x^2 + 9x + 20$$
$$18 = 20$$
$$No\ Solutions$$

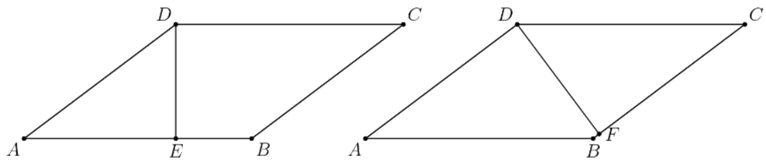
Case IU

$$(x + 3)(x + 5) = (x + 4)(x + 6)$$
$$x^2 + 8x + 15 = x^2 + 10x + 240$$
$$x = -4.5$$
$$x + 3 = -4.5 + 3 = -1.5 \Rightarrow Not\ Valid$$
$$No\ Solutions$$

Hence, there is no value of x that satisfies the above conditions.

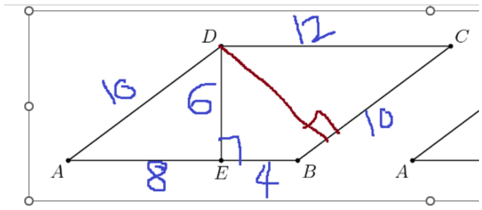
Example 1.13

In parallelogram $ABCD$, $\overline{DE}$ is the altitude to the base $\overline{AB}$ and $\overline{DF}$ is the altitude to the base $\overline{BC}$. [Note: Both pictures represent the same parallelogram.] If $DC = 12$, $EB = 4$, and $DE = 6$, then $DF =$ **(AMC 8 1995/24)**



The opposite sides of a parallelogram are equal.

$$AB = DC$$



Substitute $AB = AE + EB = AE + 4, DC = 12$

$$AE + 4 = 12 \Rightarrow AE = 8$$

DE is the altitude to the base AB. Therefore:

$$DE \perp AB \Rightarrow \Delta DAE \text{ is a right triangle}$$

In right triangle DAE, by Pythagorean Triplet:

$$(DA, DE, AE) = (10,6,8) \Rightarrow DA = 10$$

Since the opposite sides of a parallelogram are equal:

$$BC = 10$$

Calculate the area of parallelogram $ABCD$ in two different ways:

$$hb = DF \cdot BC = DE \cdot AB$$

Substitute $DE = 6, AB = 12, BC = 10$:

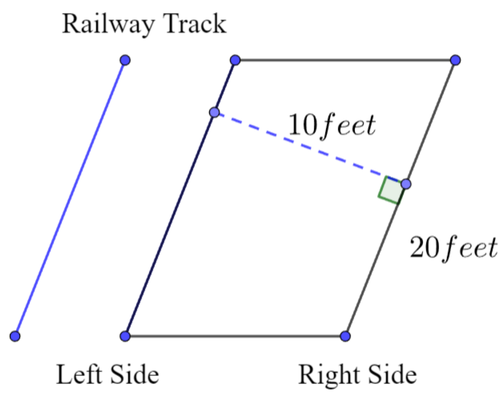
$$DF \cdot 10 = 6 \cdot 12 = 72 \Rightarrow DF = 7.2$$

Example 1.14

A non-rectangular road runs parallel to a straight railway track. The sidewalks of the road are 10 feet apart. The road is to be paved. What is the area to be paved if the road has a length of 20 feet.

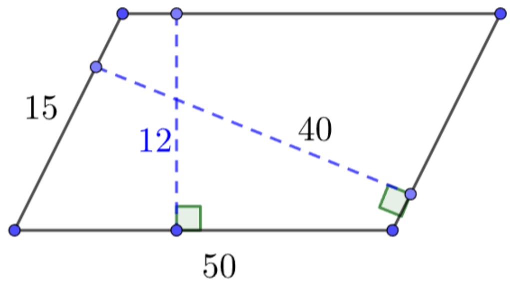
The area of the road

$$= Base \times Height = 20 \cdot 10 = 200\, ft^2$$



Example 1.15

A street has parallel curbs 40 feet apart. A crosswalk bounded by two parallel stripes crosses the street at an angle. The length of the curb between the stripes is 15 feet and each stripe is 50 feet long. Find the distance, in feet, between the stripes? **(AMC 10 2001/15)**



Calculate the area of the parallelogram in two different ways:

$$h(50) = 40(15)$$

$$h = 12$$

1.2 Rhombus

A. Definition and Perimeter

1.16: (Def) Rhombus
A rhombus is a quadrilateral that all sides equal.

- A rhombus is a special case of a parallelogram.
- Since it is a parallelogram, it is also a trapezoid.

1.17: (Def) Perimeter
The perimeter of a rhombus is four times any of the sides.

$$Perimeter = 4s, s = Side$$

Example 1.18
A rhombus has two sides which are $3x + 4$ and $2x + 7$. Determine the perimeter.

All sides of a rhombus are equal. Hence:

$$3x + 4 = 2x + 7$$
$$x = 3$$

Hence, one side is

$$3x + 4 = 3(3) + 4 = 9 + 4 = 13$$

The perimeter is

$$4s = 4(13) = 52$$

Example 1.19
What is the perimeter of a rhombus with

A. one side one foot, a second side 12 inches, and a third side 30.48 cm.
B. one of whose sides is $\frac{1}{2+\frac{2}{3+\frac{3}{4}}}$ units long?

Part A
Rhombuses have all sides equal
1 foot = 12 inches = (12 * 2.54) cm = 30.48 cm
P = 4 * 1 foot = 4 feet

Part B

$$4 \times \frac{1}{2+\frac{2}{3+\frac{3}{4}}} = \frac{4}{2+2\times\frac{4}{15}} = \frac{4}{\frac{38}{15}} = \frac{30}{19}$$

B. Area

1.20: Area (Diagonals)

If the diagonals of a rhombus have length d_1 and d_2, then the area of the rhombus is half the product of the diagonals:

$$Area = \frac{d_1 d_2}{2}$$

Example 1.21

What is the area of a rhombus with diagonals 4 cm?

$$Area = \frac{1}{2}d_1 d_2 = \frac{1}{2}(4)(4) = 8\ cm^2$$

Example 1.22

The area of a rhombus is $3\ ft^2$. And the length of one of its diagonals is $3\ ft$. Determine the length of the other diagonal.

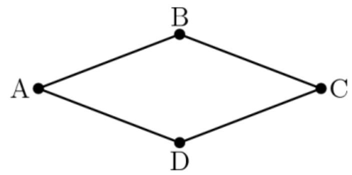
$$Area = \frac{d_1 d_2}{2} = \frac{3d_2}{2} = 3$$
$$d_2 = 2$$

1.23: Diagonals of a Rhombus

Diagonals of a rhombus are perpendicular bisectors of each other.

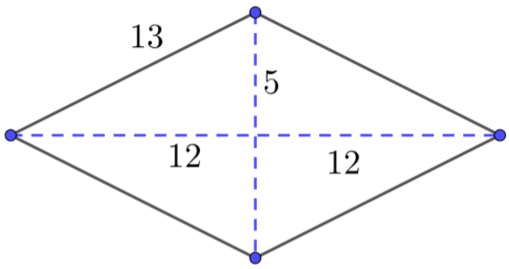
Example 1.24S

Quadrilateral $ABCD$ is a rhombus with perimeter 52 meters. The length of diagonal $\overline{AC}$ is 24 meters. What is the area in square meters of rhombus $ABCD$? **(AMC 8 2019/4)**



Area

$$= \frac{10 \cdot 24}{2} = 120$$



1.25: Area (Base and Height)

Since a rhombus is a parallelogram, the formula for area of a parallelogram is also applicable:

$$Area = Height \times Base$$

Example 1.26

A rhombus has an area of 108 square units. The lengths of its diagonals have a ratio of 3 to 2. What is the length of the longest diagonal, in units? **(MathCounts 2007 State Countdown)**

Write the given ratio, and use it to form a relation between the diagonals:

$$3:2 = 3x:2x$$

Create and simplify an expression for the area of the rhombus using the property that the area of a rhombus is half the product of its diagonals:

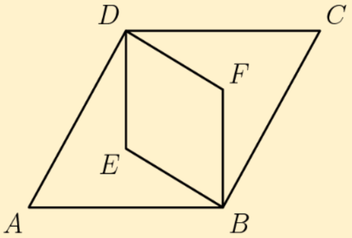
$$A = \frac{3x \times 2x}{2} = 3x^2$$

Equate this expression to the given value of the area, and solve the resulting equation:

$$3x^2 = 108 \Rightarrow x^2 = 36 \Rightarrow x = 6 \Rightarrow 3x = 18$$

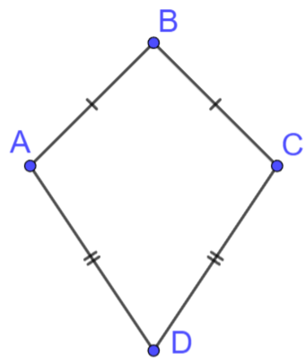
Example 1.27

Rhombus $ABCD$ is similar to rhombus $BFDE$. The area of rhombus $ABCD$ is 24 and $\angle BAD = 60°$. What is the area of rhombus $BFDE$? **(AMC 10B 2006/15)**



1.3 Kites

A. Definition and Perimeter



1.28: Definition

A kite has two pairs of adjacent equal sides.

In the diagram:

$$AB = BC, \qquad AD = DC$$

1.29: Perimeter

$$P = 2(s + l)$$

Where

$$s = Shorter\ side$$
$$l = longer\ side$$

$$2(AB + AD)$$

Example 1.30

Find the perimeter of a kite:

A. with shorter side 0.25 m and longer side 20% longer than the shorter side.
B. where the longer side is 20% more than the shorter side, and the shorter side is $\frac{3}{2-\frac{1}{1+\frac{3}{4}}}$

Part A

$$P = 2 \times \underbrace{\frac{1}{4}}_{Shorter\ Side} + 2 \times \underbrace{\frac{1}{4} \times \frac{5}{4}}_{Longer\ Side} = 2 \times \frac{1}{4}\left(1 + \frac{5}{4}\right) = \frac{1}{2} \times \frac{9}{4} = \frac{9}{8}$$

Part B

The shorter side

$$s = \frac{3}{2 - \frac{1}{1 + \frac{3}{4}}} = \frac{3}{1 - \frac{4}{\frac{7}{4}}} = \frac{3}{2 - 1 \times \frac{4}{7}} = \frac{3}{2 - \frac{4}{7}} = \frac{3}{\frac{10}{7}} = 3 \times \frac{7}{10} = \frac{21}{10}$$

$$P = 2(s + l) = 2\left(s + \underbrace{\frac{6s}{5}}_{20\%\ more}\right) = \frac{22s}{5} = \frac{22}{5} \times \frac{21}{10} = \frac{241}{25}$$

1.31: Diagonals of a Kite

Diagonals of a kite are perpendicular.

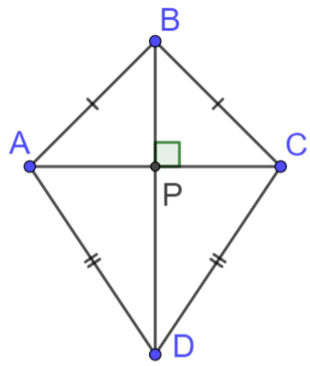
1.32: Area of a Kite

The area of a kite is half the product of the diagonals.

$$Area = \frac{1}{2} d_1 d_2$$

The area of the kite is the sum of the two triangles that make up the kite:

$$[ABCD] = [BAD] + [BCD]$$



Using the formula for the area of a triangle:

$$= \frac{1}{2}(AP)(BD) + \frac{1}{2}(CP)(BD)$$
$$= \frac{1}{2}(BD)[(AP) + (CP)]$$
$$= \frac{1}{2}(BD)(AC)$$

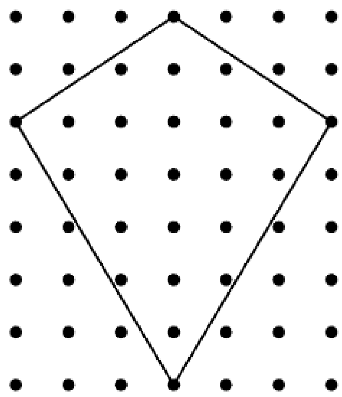
Substitute $BD = d_1, AC = d_2$

$$= \frac{1}{2} d_1 d_2$$

Example 1.33

Paragraph Based Question

To promote her school's annual Kite Olympics, Genevieve makes a small kite and a large kite for a bulletin board display. The kites look like the one in the diagram below. For her small kite Genevieve draws the kite on a one-inch grid. For the large kite she triples both the height and width of the entire grid.



A. What is the number of square inches in the area of the small kite? **(AMC 8 2001/7)**
B. Genevieve puts bracing on her large kite in the form of a cross connecting opposite corners of the kite. How many inches of bracing material does she need? **(AMC 8 2001/8)**
C. The large kite is covered with gold foil. The foil is cut from a rectangular piece that just covers the entire grid. How many square inches of waste material are cut off from the four corners? **(AMC 8 2001/9)**

Part A

$$=\frac{1}{2}d_1d_2=\frac{1}{2}(6)(7)=21\ in^2$$

Part B

The bracing material needed is equal to the length of the diagonals.

$$Length\ of\ Diagonals=D_1+D_2=3(6)+3(7)=18+21=39\ inches$$

Part C

Area of the grid:

$$=lw=[3(7)][3(6)]=9(42)$$

Area of the large kite

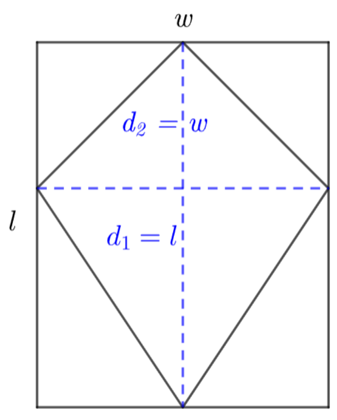
$$=\frac{1}{2}D_1D_2=\frac{1}{2}[3(7)][3(6)]=\frac{1}{2}\cdot\ 9(42)$$

Area of the waste material

$$=9(42)-\frac{1}{2}\cdot\ 9(42)=\frac{9(42)}{2}=9(21)=189\ in^2$$

Example 1.34

A kite is inscribed in a rectangle. Find the ratio of the area of the kite to the area of the rectangle.



The ratio that we want is:

$$\frac{Area\ of\ Kite}{Area\ of\ Rectangle}=\frac{\frac{1}{2}d_1d_2}{lw}$$

But note that $d_1=d_2=w$:

$$=\frac{\frac{1}{2}lw}{lw}=\frac{1}{2}$$

2. Squares and Rectangles

2.1 Area and Perimeter Basics

A. Perimeter

2.1: Rectangle

For a rectangle with length l and width w:

$$Perimeter = 2(l + w) = 2l + 2w$$
$$Area = lw$$

2.2: Square

For a square with side length s

$$Perimeter = 4s$$
$$Area = s^2$$

Example 2.3

A. A square has side 3.5 units, while a rectangle has length 2.25 units and breadth 5.25 units. What is the sum of their perimeters?
B. What is the perimeter of a square with area 64 units?
C. What is the perimeter of a rectangle with area 56 units, and one side 4 units?
D. A square and a triangle have equal perimeters. The lengths of the three sides of the triangle are 6.2 cm, 8.3 cm and 9.5 cm. The area of the square is: **(AMC 8 1985/12)**

Part A

$$3.5 \times 4 + 2(2.25 + 5.25) = 14 + 2 \times 7.5 = 14 + 15 = 29$$

Part B

$$\text{Perimeter} = \text{Side} \times 4 = \sqrt{64} \times 4 = 8 \times 4 = 32$$

Part C

The other side of the rectangle

$$= \frac{56}{4} = 14$$

The perimeter

$$= 2(14 + 4) = \ 2 \times 18 = 36$$

Part D

Perimeter of triangle

$$= 6.2 + 8.3 + 9.5 = 24$$

Side of square

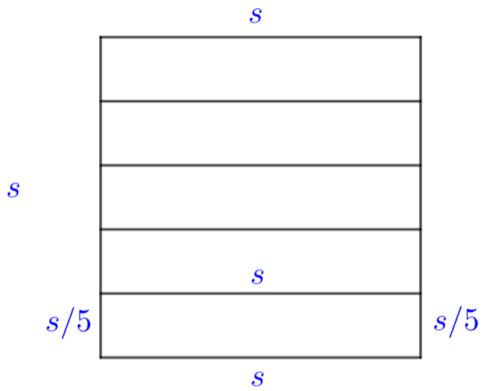
$$= \frac{24}{4} = 6$$

Area of square

$$= 6^2 = 36$$

Example 2.4

A square with side s is divided into five congruent rectangles stacked on top of each other. If the perimeter of each rectangle is 1 foot, how many inches is the perimeter of the square?



Substitute $l = s, w = \frac{s}{5}$ in $P = 2(l + w)$:

$$2\left(s + \frac{s}{5}\right) = 12 \text{ inches}$$

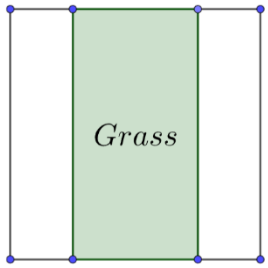
$$\frac{12s}{5} = 12$$

The side length

$$= s = 5$$

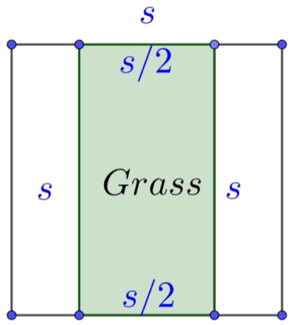
The perimeter of the square is:

$$4s = 20$$



Example 2.5

A square playground has a grassy area that is used to play football. The grassy area occupies half of the area of the playground. The perimeter of the grassy area is 100 *meters*. Find the perimeter and area of the entire playground.



The perimeter of the playground is:

$$s + s + \frac{s}{2} + \frac{s}{2} = 100$$

$$3s = 100$$

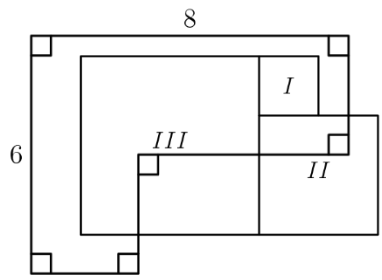
$$s = \frac{100}{3}$$

The perimeter of the entire playground

$$= 4s = \frac{400}{3}\ m$$

The area of the entire playground

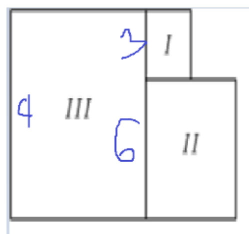
$$= s^2 = \left(\frac{100}{3}\right)^2 = \frac{10{,}000}{9} m^2$$



Example 2.6

Figures I, II, and III are squares. The perimeter of I is 12 and the perimeter of II is 24. The perimeter of III is **(AMC 8 1995/6)**

$$P = 4s_{III} = 4(s_I + s_{II}) = 4\left(\frac{12}{4} + \frac{24}{4}\right) = 4\left(\frac{36}{4}\right) = 36$$



Example 2.7: TSD

Betty and Ann are walking around a rectangular park with dimensions 600 m by 400 m, as shown. They both begin at the top left corner of the park and walk at constant but different speeds. Betty walks in a clockwise direction and Ann walks in a counterclockwise direction. Points P, Q,R,S, T divide the bottom edge of the park into six segments of equal length. When Betty and Ann meet for the first time, they are between Q and R. Which of the following could be the ratio of Betty's speed to Ann's speed? **(Gauss Grade 7 2017/20)**

A. 5:3
B. 9:4
C. 11:6
D. 12:5
E. 17:7

Ann's distance:

$$Min: 600, \quad Max: 700$$

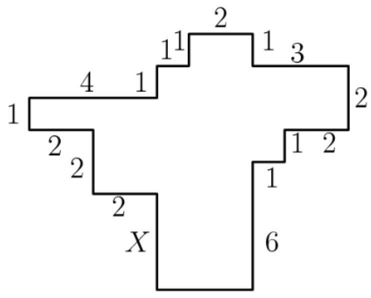
Betty' distance:

$$Min: 1300, \quad Max: 1400$$

Distance Ratios as Fractions:

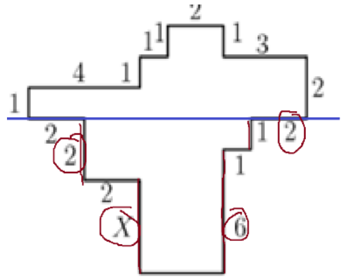
$$\frac{Min\ of\ Betty}{Max\ of\ Ann} = \frac{13}{7} = 1\frac{6}{7}$$
$$\frac{Max\ of\ Betty}{Min\ of\ Ann} = \frac{14}{6} = \frac{7}{3} = 2\frac{1}{3}$$

$$Option\ A: \frac{5}{3} = 1\frac{2}{3} < 1\frac{6}{7}$$
$$Option\ B: \frac{9}{4} = 2\frac{1}{4} \Rightarrow 1\frac{6}{7} < 2\frac{1}{4} < 2\frac{1}{3} \Rightarrow Correct$$

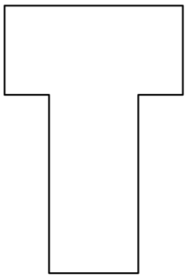


Example 2.8

In the diagram, all angles are right angles and the lengths of the sides are given in centimeters. Note the diagram is not drawn to scale. What is the length in X, in centimeters? **(AMC 8 2012/5)**

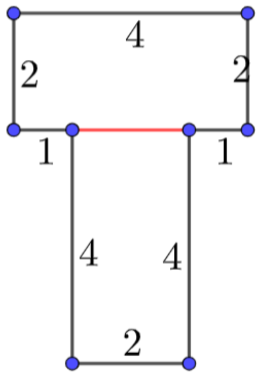


$$2 + X = 6 + 1$$
$$X = 5\ cm$$



Example 2.9

The letter T is formed by placing two 2×4 inch rectangles next to each other, as shown. What is the perimeter of the T, in inches? **(AMC 8 2006/6)**



Method I
Add the sides to get the perimeter

$$= 2 + 4 + 2 + 1 + 4 + 2 + 4 + 1 = 20$$

Method II
Note that the red line is missing from the perimeter of the shape. If the red line had been a part of the perimeter, then the perimeter would have been:

$$\underbrace{2}_{\substack{Double\ for\\ Two\ Rectangles}} \times \underbrace{2}_{\substack{Double\ for\\ the\ Sides}} \times \underbrace{(4+2)}_{\substack{Length\ plus\\ Width}} = 4 \times 6 = 24$$

However, the red line must be subtracted to give a final perimeter of:

$$24 - 2 \times 2 = 24 - 4 = 20$$

Example 2.10
Farmer Jane has a square grazing field with area 64 square units. She takes the fence from the field, and uses all of it to fence a rectangular area with length 4 units. What is the percentage change in the grazing area??

$$Side\ Length(Square) = \sqrt{64} = 8$$
$$Perimeter(Square) = Perimeter(Rectangle) = 32$$

$$2(l + w) = 32$$
$$l + w = 16$$
$$w = 16 - l = 16 - 4 = 12$$

$$Area(Rectangle) = (12)(4) = 48$$

$$\%\ change\ = \frac{64 - 48}{64} = \frac{16}{64} = \frac{1}{4} = 25\%\ reduction$$

Example 2.11
A rectangular garden 60 feet long and 20 feet wide is enclosed by a fence. To make the garden larger, while using the same fence, its shape is changed to a square. By how many square feet does this enlarge the garden? **(AMC 8 1999/5)**

$$Perimeter(Rectangle) = 2(60 + 20) = 2(80) = 160$$
$$Side(Square) = \frac{160}{4} = 40$$
$$Area(Square) - Area(Rectangle) = 40^2 - (60)(20) = 1600 - 1200 = 400$$

Example 2.12
A rectangular grazing area is to be fenced off on three sides using part of a 100-meter rock wall as the fourth side. Fence posts are to be placed every 12 meters along the fence including the two posts where the fence meets the rock wall. What is the fewest number of posts required to fence an area 36 m by 60 m? **(AMC 8 1986/18)**

To minimize the number of posts, make the side with length 60 m parallel to the rock wall. Then

$$No.of\ Fenceposts = \frac{Length\ of\ Fence}{Interval} + 1 = \frac{36 + 60 + 36}{6} + 1 = \frac{132}{12} + 1 = 11 + 1 = 12$$

Example 2.13

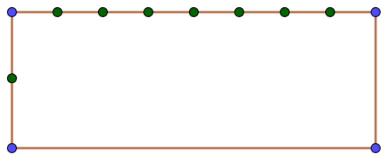
Carl decided to fence in his rectangular garden. He bought 20 fence posts, placed one on each of the four corners, and spaced out the rest evenly along the edges of the garden, leaving exactly 4 yards between neighboring posts. The longer side of his garden, including the corners, has twice as many posts as the shorter side, including the corners. What is the area, in square yards, of Carl's Garden? **(AMC 10B 2016/11)**

Out of the 20 fenceposts, 4 are at the corners, leaving for the sides:

$$20 - 4 = 16$$

These 16 fenceposts are equally divided among the opposite sides:

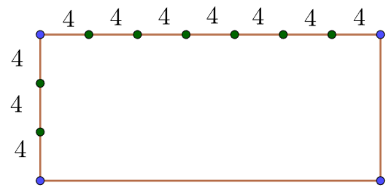
$$2(F_{Length} + F_{Width}) = 16$$
$$F_{Length} + F_{Width} = 8$$



Method I: Trial and Error

If the width has 1 fencepost not including the corners, then it has the length has

$$8 - 1 = 7\ fenceposts$$



Including the corners, we have:

$$Width = 1 + 2 = 3$$
$$Length = 7 + 2 = 9 \neq 2(3) = 6 \Rightarrow Not\ Valid$$

If we increase the width to have 2 fenceposts, them including the corners, we have

$$Width = 2 + 2 = 4$$
$$Length = 6 + 2 = 8 = 2(4) \Rightarrow Valid$$

Method II: Algebra

$$w, 8 - w$$

The total number of fenceposts on the width is:

$$w + 2$$

The total number of fencepossts on the lenght is:

$$8 - w + 2 = 10 + w$$
$$10 - w = 2(w + 2)$$
$$w = 2$$

Finding the Area

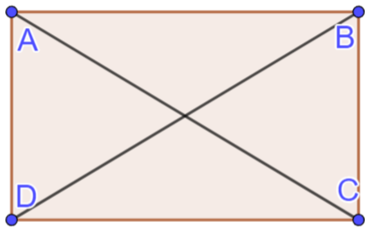
Whichever method, we use, the area is:

$$Area = lw = \underbrace{4(3)}_{Width} \times \underbrace{4(7)}_{Length} = (12)(28) = 336\ yards^2$$

B. Diagonals

2.14: Diagonals of a Rectangle

The diagonals of a rectangle have equal length.



Example 2.15

Milee went from the top left corner of a rectangular field to the bottom right corner, and counted 38 paces. Milee's friend takes paces which are half the size of Milee's. How many paces will she take to go from the bottom left corner to the top right corner?

$$Milee's\ friend = 2(38) = 76\ paces$$

2.16: Proving a Rectangle

If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

Example 2.17

Two sides of a parallelogram are 4 units and 5 units. The diagonals of the parallelogram are each $\sqrt{41}$ units. Find the area of the parallelogram.

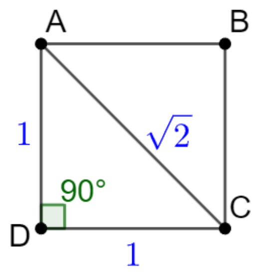
Since the diagonals are equal, the parallelogram is a rectangle. Hence:

$$A = lw = 4(5) = 20\ units^2$$

2.18: Diagonal of a Square

The diagonal (d) of a square with side length s is

$$d = \sqrt{2} \times s$$



Without loss of generality, let the length of the side of the square be 1.
Then, in right ΔADB, by Pythagoras Theorem:

$$AC^2 = 1^2 + 1^2 = 1 + 1 = 2$$

Take the square root both sides:

$$AC = \sqrt{2}$$

Example 2.19

A. What is the length of the diagonal of a square of side 3 units?
B. A farmer has a square field with a side length of $100m$. Determine the length of the diagonal in km.

Part A

Length of the diagonal

$$= \sqrt{2} \times Side = \sqrt{2} \times 3 = 3\sqrt{2}\ units$$

Part B

$$Side\ Length = 100m = \frac{1}{10} km$$

$$Diagonal = \frac{1}{10} \times \sqrt{2}\ km = \frac{\sqrt{2}}{10} km$$

Example 2.20

A. A boy walks three times around his square playground, and takes 120 paces to do so. Determine how many paces the boy will take to go from one corner to a non-adjacent corner. (Your answer need not be an integer).
B. A square field has area 16 square units. What is the distance from the top left corner of the field to the bottom right corner?

Part A

$$3P = 120\ Paces$$
$$P = 40\ paces$$

$$s = \frac{P}{4} = 10\ paces$$
$$d = \sqrt{2}s = \sqrt{2} \times 10 = 10\sqrt{2}$$

Part B

The area of the square is

$$A = s^2 = 16$$

Then, the side length is:

$$s = \sqrt{16} = 4$$

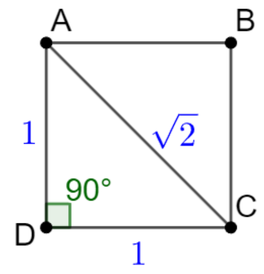
And the distance from top left to bottom right is the diagonal, which is:

$$d = 4 \times \sqrt{2} = 4\sqrt{2}$$

2.21: Side Length of a Square

The diagonal (d) of a square with side length s is

$$d = \sqrt{2} \times s \Rightarrow s = \frac{d}{\sqrt{2}}$$



Example 2.22

A. Find the perimeter of a square with diagonal $5\sqrt{2}\ m$.
B. What is the length of the side of a square, the sum of whose diagonals is 12 units?
C. Half the length of the diagonal of a square is 5. Determine the perimeter of the square.

Part A

$$s = \frac{5\sqrt{2}}{\sqrt{2}} = 5$$
$$P = 4s = 20$$

Part B

Since the sum of the diagonals is 12, first find the length of a single diagonal:

$$2 \times diagonal = 12 \Rightarrow Diagonal = d = 6$$

The side length is:

$$s = \frac{6}{\sqrt{2}} = \frac{6}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$

Part C

$$\frac{d}{2} = 5 \Rightarrow d = 10$$
$$s = \frac{10}{\sqrt{2}} = \frac{10}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$$
$$P = 4s = 4\left(5\sqrt{2}\right) = 20\sqrt{2}$$

Example 2.23

Calculate the cost of fencing a square field which has a distance of $5m$ between its nonadjacent corners if the cost of fencing $1m$ is $\sqrt{3}$ dollars.

The side length is:

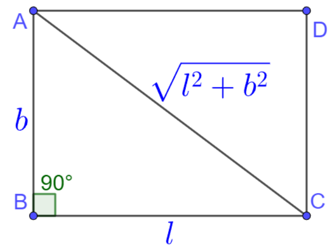
$$s = \frac{d}{\sqrt{2}} = \frac{5}{\sqrt{2}} = \frac{5}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2} m$$

The perimeter

$$P = 4s = 4 \times \frac{5\sqrt{2}}{2} = 10\sqrt{2}$$
$$Total\ Cost = Pc = (10\sqrt{2}m)\left(\sqrt{3}\frac{\$}{m}\right) = 10\sqrt{6}\ dollars$$

2.24: Diagonal of a Rectangle

Given a rectangle with length l and breadth b, the length of the diagonal is
$$d = \sqrt{l^2 + b^2}$$



Since $ABCD$ is a rectangle
$$\angle ABC = 90°$$
In right ΔABC, by Pythagoras Theorem:
$$d^2 = Diagonal^2 = AC^2 = AB^2 + BC^2 = l^2 + b^2$$
$$d = \sqrt{l^2 + b^2}$$

Example 2.25

A. What is the length of the diagonal of a rectangle of length 15 units and breadth 8 units?
B. Find the length of the longest possible straight path in a rectangular park of perimeter 34, and length 5.

Part A

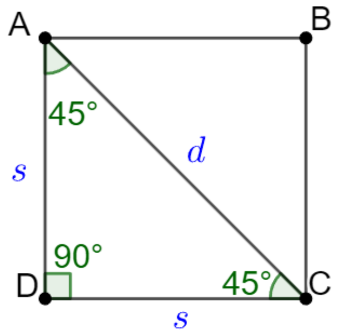
$$Diagonal = \sqrt{l^2 + b^2} = \sqrt{15^2 + 8^2} = \sqrt{225 + 64} = 17\ units$$
$$OR\ using\ Pythagorean\ Triplet\ (8,15,17) \Rightarrow Diagonal = 17$$

Part B

$$Breadth = \frac{34 - 5 \times 2}{2} = \frac{24}{2} = 12$$
$$Diagonal = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$$
$$OR\ using\ Pythagorean\ Triplet\ (5,12,13) \Rightarrow Diagonal = 13$$

2.26: Area of a Square in terms of its diagonal

The area of a square with diagonal d is
$$\frac{1}{2}d^2$$



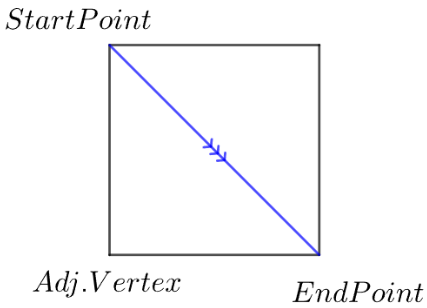
By Pythagoras Theorem in ΔADC
$$s^2 + s^2 = d^2 \Rightarrow s^2 = \frac{d^2}{2}$$

Example 2.27

It takes Paridhi twelve minutes to walk through the shortest path from one vertex to a non-adjacent vertex of a square field while walking at the rate of $6\frac{km}{hr}$. What is the area of the field, in square meters?

Determine the length of the path using

$$Diagonal = \underbrace{6\frac{km}{hr}}_{Speed} \cdot \underbrace{12\ min}_{Time} = 6\frac{km}{hr} \cdot \frac{1}{5}hr = \frac{6}{5}km = 1.2\ km$$



The area of the field is:

$$A = \frac{1}{2}d^2 = \frac{1}{2} \cdot 1.2^2 = \frac{1}{2} \cdot 1.2 \cdot 1.2 = 0.6 \cdot 1.2 = 0.72\ km^2$$
$$= 0.72\ (km)(km)$$

Convert from km^2 to m^2 by multiplying 1000:

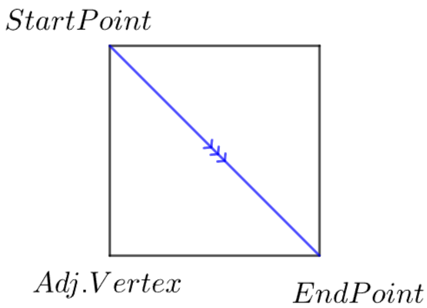
$$= 0.72\ (1000m)(1000m) = 720000\ m^2$$

Example 2.28

It takes Pari fifteen minutes to walk through the longest straight path from one vertex to a non-adjacent vertex of a square field. She walked at 4 $\frac{km}{hr}$. Determine the cost of planting the field with grass at \$0.25 per square meter.

Determine the length of the path using

$$Diagonal = \underbrace{4\frac{km}{hr}}_{Speed} \cdot \underbrace{15\ min}_{Time} = 4\frac{km}{hr} \cdot \frac{1}{4}hr = 1\ km$$



The area of the field is:

$$A = \frac{1}{2}d^2 = \frac{1}{2} \cdot 1^2 = \frac{1}{2}km^2 = \frac{1}{2}\ (km)(km)$$

Convert from km^2 to m^2 by writing $1\ km = 1000m$:

$$= \frac{1}{2}\ (1000m)(1000m) = 500{,}000\ m^2$$

Calculate the cost:

$$= \frac{1}{4} \times 500{,}000 = 125{,}000\$$$

2.29: Area of a Rectangle in terms of its diagonal

A rectangle *cannot* be uniquely identified based on the length of its diagonal.

$$Area \neq \frac{1}{2}d^2$$

This is proved by way of a counterexample in the next question.

Example 2.30

Consider a rectangle $R_1$with dimensions $(b, l) = (15,20)$. Consider another rectangle $R_2$with dimensions$(b, l) = (7,24)$. Show that the two rectangles have same diagonal, but different areas.

Using Pythagorean Triplets:

$$D_1 = (15,20) = 5(3,4,5) = (15,20,25)$$
$$D_2 = (7,24,25)$$

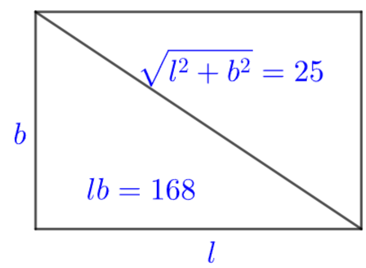
Calculate the area:

$$A_1 = 15 \times 20 = 300$$
$$A_2 = 7 \times 24 = 168$$

The diagonals are the same, but the areas are different.

Example 2.31

A rectangular parking lot has a diagonal of 25 meters and an area of 168 square meters. In meters, what is the perimeter of the parking lot? **(AMC 10B 2011/14)**



Square both sides of $\sqrt{\square} = 25$

$$\underbrace{l^2 + b^2 = 625}_{\text{Equation I}}$$

The area is:

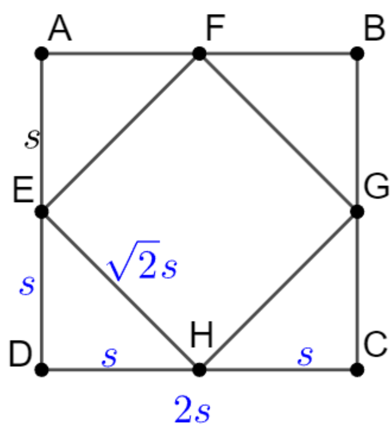
$$lb = 168 \Rightarrow 2A = \underbrace{2lb = 336}_{\text{Equation II}}$$

$$l^2 + 2lb + b^2 = 961$$
$$(l+b)^2 = 31^2$$
$$l + b = 31$$
$$P = 2(l+b) = 62$$

2.32: Midpoints form a Rectangle

The quadrilateral formed by joining the midpoints of the sides of a square with side length $2s$ in order forms a square with side length

$$\sqrt{2}s$$



Consider square ABCD with side length

$$2s$$

It has midpoints E, F, G and H:

$$ED = DH = s$$

By Pythagoras Theorem in right ΔEDH

$$EH = \sqrt{ED^2 + DH^2} = \sqrt{2s^2} = \sqrt{2}s$$

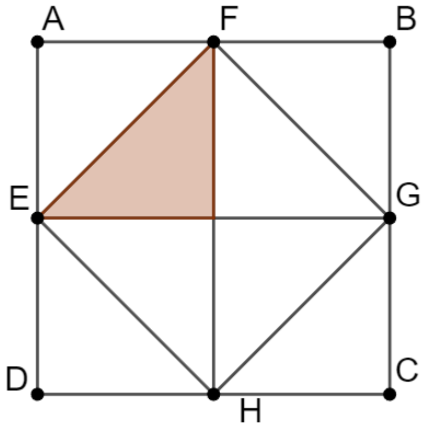
By similar logic:

$$EF = FG = HG = \sqrt{2}s$$

Example 2.33

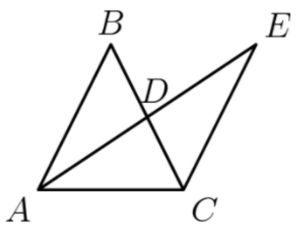
A square $ABCD$ has side length $\sqrt{13}$. P, Q, R and S are the midpoints of sides AB, BC, CD and DA. Determine the ratio of the area of quadrilateral $PQRS$ to the area of quadrilateral $ABCD$.

$$\left(\sqrt{2}s\right)^2 : (2s)^2 = 2s^2 : 4s^2 = 2:4 = 1:2$$



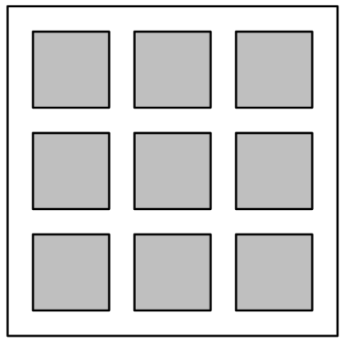
Example 2.34

Triangle ABC is an isosceles triangle with $\overline{AB} = \overline{BC}$. Point D is the midpoint of both $\overline{BC}$ and $\overline{AE}$, and $\overline{CE}$ is 11 units long. Triangle ABD is congruent to triangle ECD. What is the length of $\overline{BD}$? **(AMC 8 2006/19)**



Example 2.35

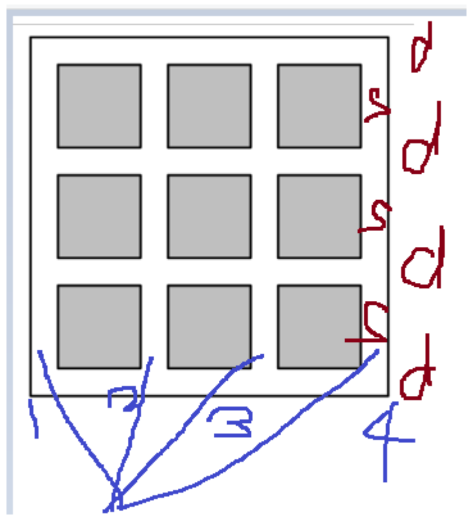
A large square region is paved with n^2 gray square tiles, each measuring s inches on a side. A border d inches wide surrounds each tile. The figure shows the case for $n = 3$. When $n = 24$, the 576 gray tiles cover 64% of the area of the large square region. What is the ratio $\frac{d}{s}$ for this larger value of n? **(AMC 8 2020/24)**



Area of the gray square tiles

The total area of the grey tiles

$$= \underbrace{24^2}_{\substack{\text{No. of}\\ \text{Tiles}}} \cdot \underbrace{s^2}_{\substack{\text{Area of}\\ \text{each tile}}} = 24^2 \cdot s^2$$



Length of Square

For 3 tiles, the length of the square will be:

$$3s + 4d$$

For 24 tiles, the total length of the gaps will be:

$$24s + 25d$$

Area of Square

Therefore, the area of the entire square will be:

$$Side^2 = (24s + 25d)^2$$

The ratio of the grey tiles to the larger square is $64\% = \frac{64}{100}$

$$\frac{24^2 \cdot s^2}{(24s + 25d)^2} = \frac{64}{100}$$

Take the square root both sides:

$$\frac{24s}{24s+25d}=\frac{8}{10}=\frac{4}{5}$$

Since the numerator and the denominator both have $24s$, take the reciprocal on both sides:

$$\frac{24s}{24s}+\frac{25d}{24s}=\frac{5}{4}$$

Substitute $\frac{24s}{24s}=1$:

$$1+\frac{25d}{24s}=\frac{5}{4}\Rightarrow\frac{25d}{24s}=\frac{1}{4}\Rightarrow\frac{25d}{6s}=1\Rightarrow\frac{d}{s}=\frac{6}{25}$$

C. Calculations

Example 2.36

Six rectangles each with a common base width of 2 have lengths of 1, 4, 9, 16, 25, and 36. What is the sum of the areas of the six rectangles? **(AMC 8 2014/6)**

The area that we want is:

$$2\cdot1+2\cdot4+2\cdot9+2\cdot16+2\cdot25+2\cdot36$$

Factor 2:

$$2(1+4+9+16+25+36)$$

Note that

$$1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

Substituting $n=6$:

$$2\cdot\frac{6\cdot7\cdot13}{6}=2(91)=182$$

Example 2.37

Karl's rectangular vegetable garden is 20 feet by 45 feet, and Makenna's is 25 feet by 40 feet. Which of the following statements are true?
(A)Karl's garden is larger by 100 square feet.
(B)Karl's garden is larger by 25 square feet.
(C)The gardens are the same size.
(D)Makenna's garden is larger by 25 square feet.
(E)Makenna's garden is larger by 100 square feet. **(AMC 8 2011/2)**

$$Area_{Karl}=20(45)=900$$
$$Area_{Makenna}=25(40)=1000$$

$$Option\ E$$

Example 2.38

How many square yards of carpet are required to cover a rectangular floor that is 12 feet long and 9 feet wide?

(There are 3 feet in a yard.) **(AMC 8 2015/1)**

Convert the dimensions from feet to yards, and then find the area:

$$\frac{12}{3}\times\frac{9}{3}=4\times 3=12$$

Shortcut

Find the area and then divide by $3^2=9$ to convert to square yards:

$$\frac{12\times 9}{3^2}=12$$

Example 2.39

At a store, when a length is reported as x inches that means the length is at least $x-0.5$ inches and at most $x+0.5$ inches. Suppose the dimensions of a rectangular tile are reported as 2 inches by 3 inches. In square inches, what is the

A. minimum area for the rectangle? **(AMC 10B 2011/3)**
B. maximum area for the rectangle?

Part A

The minimum dimensions are:

$$2-0.5=1.5=\frac{3}{2},\qquad 3-0.5=2.5=\frac{5}{2}$$

The area

$$=\frac{3}{2}\cdot\frac{5}{2}=\frac{15}{4}=3.75\ in^2$$

Part B

The maximum dimensions are:

$$2+0.5=2.5=\frac{5}{2},\qquad 3+0.5=3.5=\frac{7}{2}$$

The area

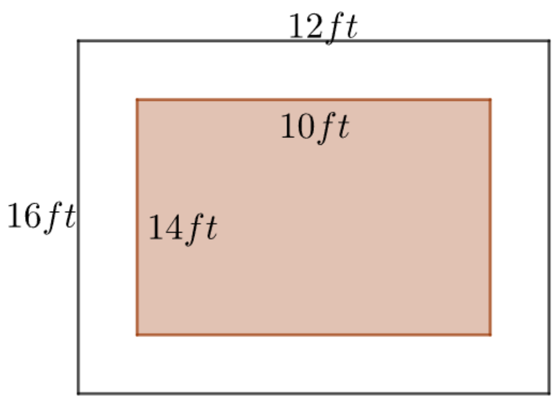
$$=\frac{5}{2}\cdot\frac{7}{2}=\frac{35}{4}=8.75\ in^2$$

Example 2.40

Tyler is tiling the floor of his 12-foot by 16-foot living room. He plans to place one-foot by one-foot square tiles to form a border along the edges of the room and to fill in the rest of the floor with two-foot by two-foot square tiles. How many tiles will he use? **(AMC 8 2018/9)**

The area of the border will be:

$$\underbrace{12\times 16}_{Total\ Area}-\underbrace{10\times 14}_{Inner\ Area}=192-140=52$$



Hence, the number of border tiles

$$=\frac{52}{1}=52$$

The number of inner tiles

$$\frac{Inner\ Area}{Area\ of\ each\ tile}=\frac{140}{2\times 2}=35$$

The total tiles

$$= 52 + 35 = 87$$

Example 2.41

Carrie has a rectangular garden that measures 6 feet by 8 feet. She plants the entire garden with strawberry plants. Carrie is able to plant 4 strawberry plants per square foot, and she harvests an average of 10 strawberries per plant. How many strawberries can she expect to harvest? **(AMC 8 2020/3)**

The area of the garden is:

$$6 \times 8 = 48\, ft^2$$

The number of strawberry plants is:

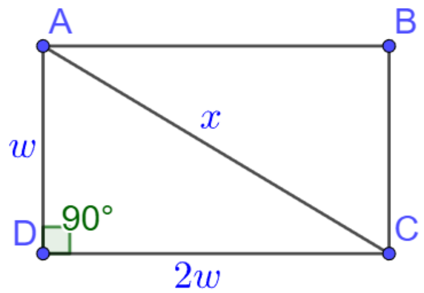
$$48 \times 4 = 192$$

The number of strawberries

$$= 192 \times 10 = 1920$$

Example 2.42

A rectangle with a diagonal of length x is twice as long as it is wide. What is the area of the rectangle? (Answer in terms of x) **(AMC 10A 2005/4)**



Draw the rectangle with width w and length $2w$. It has area:

$$w \times 2w = 2w^2$$

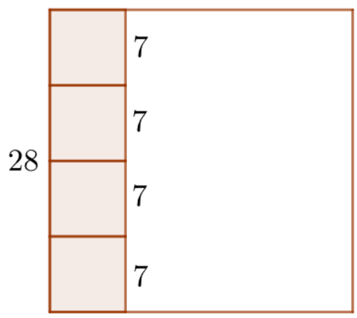
By Pythagoras Theorem in right ΔADC:

$$w^2 + (2w)^2 = x^2$$
$$w^2 + 4w^2 = x^2$$
$$5w^2 = x^2$$
$$w^2 = \frac{x^2}{5}$$

$$2w^2 = \frac{2}{5}x^2$$

Example 2.43

If four identical squares are stacked in a single row (such that the lowest square touches the bottom, and the highest square touches the top) in a larger square with area 784 units, what is the area of the larger square not covered by the smaller squares?



The side length of the larger square is

$$\sqrt{784} = 28$$

The side length of the smaller square

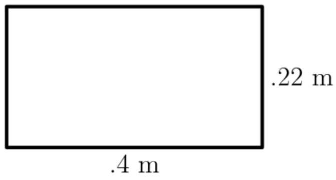
$$= \frac{28}{4} = 7$$

The area of each smaller square

$$= 7^2 = 49$$

The area that we want is:

$$784 - 49 \times 4 = 784 - 200 + 4 = 588$$



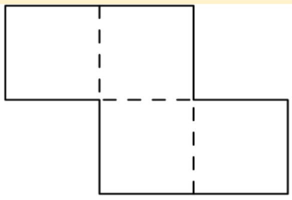
Example 2.44

The area of the rectangular region is (AMC 8 1987/5)

$$0.22 \times 0.4 = 22 \times 4 \times 0.001 = 0.088\ m^2$$

Example 2.45

The area of this figure is 100 cm^2. Its perimeter is (AMC 8 1990/15)
[figure consists of four identical squares]



The area of each square

$$= \frac{100}{4} = 25\ cm^2$$

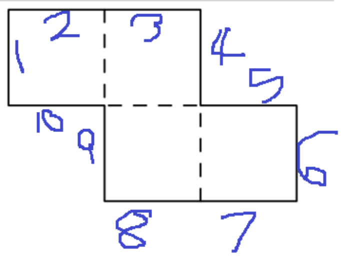
The side length of each square

$$= \sqrt{25} = 5\ cm$$

Method I

Since there are 10 sides (see diagram to the right), the perimeter

$$= 5 \times 10 = 50\ cm$$



Method II: Complementary Counting

If we had four complete squares, then they would have perimeter:

$$= \underbrace{5 \times 4}_{\substack{Perimeter \\ of\ 1\ square}} \times 4 = 80\ cm$$

But the dotted lines are not part of the perimeter. We have 3 dotted lines. But each dotted is a part of two squares. Hence, we must count it twice:

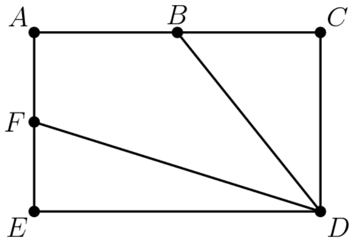
$$Dotted\ Lines = 2 \times 3 = 6$$

Hence, the length missing is:

$$= 5 \times 6 = 30\ cm$$

The final answer is:

$$80 - 30 = 50\ cm$$

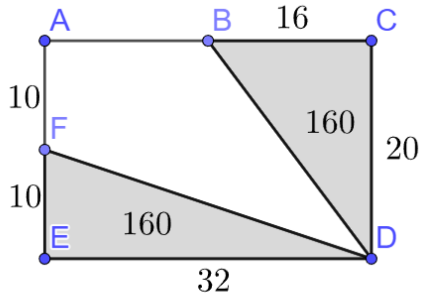


Example 2.46

The rectangle shown has length $AC = 32$, width $AE = 20$, and B and F are midpoints of $\overline{AC}$ and $\overline{AE}$, respectively. The area of quadrilateral $ABDF$ is: (AMC 8 1993/18)

The area that we want is:

$$
\begin{aligned}
&[ABCD]-[BCD]-[FED]\\
&=32\times 20-\frac{1}{2}\times 16\times 20-\frac{1}{2}\times 10\times 32\\
&=32\times 20-16\times 10-10\times 16\\
&=640-160-160\\
&=320
\end{aligned}
$$

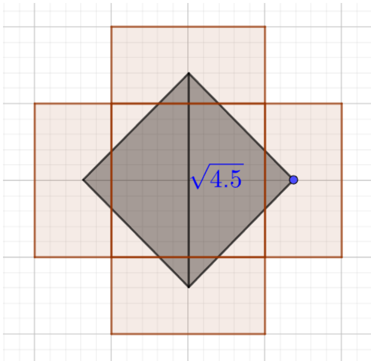


Example 2.47

A checkerboard consists of one-inch squares. A square card, 1.5 inches on a side, is placed on the board so that it covers part or all of the area of each of n squares. The maximum possible value of n is **(AMC 8 1993/25)**

Calculate the diagonal of the square card:

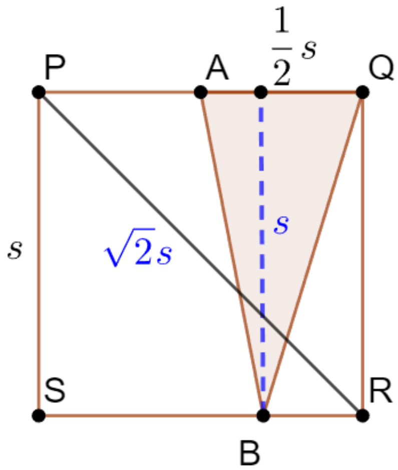
$$\sqrt{(1.5)^2+(1.5)^2}=\sqrt{2.25+2.25}=\sqrt{4.5}>\sqrt{4}=2$$



D. Triangles

Example 2.48: Diagonals

$PQRS$ is a square. A is the midpoint of side PQ. B is a point on RS such that RB is $\frac{1}{3}rd$ of RS. If the area of ΔAQB is t units, find the length of the diagonal of the square in terms of t.



Let the side length of the square be

$$s \Rightarrow Diagonal = \sqrt{2}s$$

The area of ΔAQB

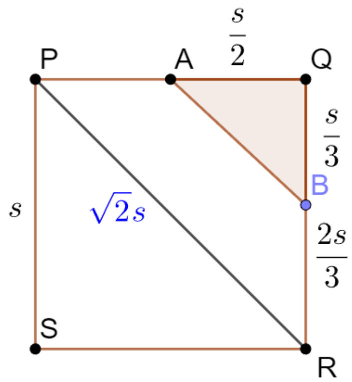
$$= \frac{1}{2}hb = \frac{1}{2}(PS)(AQ) = \frac{1}{2}s\left(\frac{1}{2}s\right) = \frac{1}{4}s^2$$

But the area of the triangle is t

$$\frac{1}{4}s^2 = t$$
$$s^2 = 4t$$
$$s = 2\sqrt{t}$$
$$\sqrt{2}s = 2\sqrt{2t}$$

Example 2.49: Diagonals

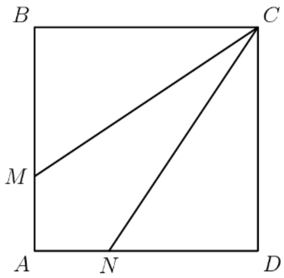
$PQRS$ is a square. A is the midpoint of side PQ. B is a point on RQ such that RB is $\frac{2}{3}rd$ of RQ. If the area of ΔAQB is t units, find the length of the diagonal of the square in terms of t.



Let the length of side of the square

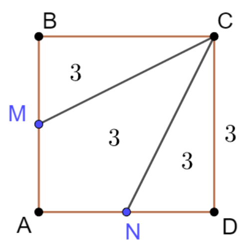
$$= s$$

$$\Delta AQB = \frac{1}{2}hb = \frac{1}{2}(AQ)(QB) = \frac{1}{2} \times \frac{s}{2}s \times \frac{s}{3}s = \frac{s^2}{12}$$
$$\frac{s^2}{12} = t \Rightarrow s^2 = 12t \Rightarrow s = \sqrt{12t}$$
$$Diagonal = \sqrt{2} \times \sqrt{12t} = \sqrt{24t} = \sqrt{4} \times \sqrt{6t} = 2\sqrt{6t}$$



Example 2.50

Square $ABCD$ has sides of length 3. Segments CM and CN divide the square's area into three equal parts. How long is segment CM? (AMC 8 1999/23)



The area of square $ABCD$

$$= 3^2 = 9$$

The area of each triangle

$$= \frac{9}{3} = 3$$

The area of ΔBCM

$$= \frac{1}{2}(BC)(BM) = \frac{1}{2}(3)(BM) = 3 \Rightarrow BM = 2$$

By Pythagoras Theorem in ΔBCM:

$$CM = \sqrt{BC^2 + BM^2} = \sqrt{3^2 + 2^2} = \sqrt{9+4} = \sqrt{13}$$

E. Percentage Change

Example 2.51

Each side of a square is increased by 30%. The percentage increase in area is:

Consider a square of side length s. The side is increased by:

$$30\%\ of\ s = 0.3s$$

New Area will be

$$(s + 0.3s)^2 = (1.3s)^2 = 1.69s^2$$

The percentage increase is:

$$\frac{Increase\ in\ Area}{Old\ Area} = \frac{New\ Area - Old\ Area}{Old\ Area} = \frac{1.69s^2 - s^2}{s^2} = \frac{0.69s^2}{s^2} = 0.69 = 69\%$$

2.52: Percentage change in side length

If the side length of a square increases by $p\%$, the area of the square does not depend on the side length of the square.

Example 2.53

Each side of a square with length 23.75 meters is increased by 30%. The percentage increase in area is:

Since the change does not depend on the side length, take a square of side length 1.

	Original	*New*	*Increase*
Side	1	1.3	
Area	1	1.69	0.69

$$\%\ increase = \frac{0.69}{1} = 0.69 = 69\%$$

Example 2.54

The dimensions of a square are changed so that the length is squared, and the breadth is halved. The old area as a percentage of the new area is:

$$\frac{2lb}{l^2b} = \frac{2}{l} = \frac{200}{l}\%$$

Example 2.55

If the length and width of a rectangle are each increased by 10%, then the perimeter of the rectangle is increased by what percentage? (AMC 8 1985/19)

Algebraic Method

Let the length and breadth be l and b respectively

$$Old\ Perimeter = 2(l + b)$$

$$New\ Length = l + 10\%\ of\ l = l + 0.1l = 1.1l$$
$$New\ Breadth = b + 10\%\ of\ b = b + 0.1b = 1.1b$$
$$New\ Perimeter = 2(1.1l + 1.1b) = 2.2(l + b)$$

$$\%\ increase = \frac{New\ P - Old\ P}{Old\ P} = \frac{2.2(l + b) - 2(l + b)}{2(l + b)} = \frac{0.2(\boldsymbol{l} + \boldsymbol{b})}{2(\boldsymbol{l} + \boldsymbol{b})} = \frac{0.2}{2} = 10\%$$

Shortcut Method

Note that in the final calculation, neither the length nor the width appeared. The violet terms above cancelled. Hence, the answer does not depend on the value of the length, or the value of the width.

Hence, we can take any length and width that we want. For ease of calculations, take

$$Length = Width = 1$$

$$Old\ P = 2(1 + 1) = 2(2) = 4$$
$$New\ Length = 1 + 10\%\ of\ 1 = 1 + 0.1 = 1.1$$
$$New\ Breadth = 1 + 10\%\ of\ 1 = 1 + 0.1 = 1.1$$

$$New\ Perimeter = 2(1.1 + 1.1) = 2(2.2) = 4.4$$

$$\%\ Increase = \frac{4.4 - 4}{4} = \frac{0.4}{4} = 0.1 = \frac{1}{10} = 10\%$$

Example 2.56

If the length of a rectangle is increased by 20% and its width is increased by 50%, then the area is increased by what percent (AMC 8 1993/21)

Let

$$Length = Width = 100$$

$$Old\ Area = 100^2 = 10{,}000$$
$$New\ Length = 100 + 20\%\ of\ 100 = 100 + 20 = 120$$
$$New\ Breadth = 100 + 50\%\ of\ 100 = 100 + 5 = 150$$

$$\%\ Increase = \frac{New\ Area - Old\ Area}{Old\ Area} = \frac{(120)(150) - 10{,}000}{10{,}000} = \frac{8{,}000}{10{,}000} = \frac{80}{100} = 80\%$$

Example 2.57

The length of a rectangle is increased by 10% and the width is decreased by 10%. What percent of the old area is the new area? (AMC 9 2009/8)

Let $Length = Width = 100$

$$Old\ Area = 100^2 = 10{,}000$$

$$New\ Length = 100 + 10\%\ of\ 100 = 100 + 10 = 110$$
$$New\ Breadth = 100 - 10\%\ of\ 100 = 100 - 10 = 90$$

$$\frac{New\ Area}{Old\ Area} = \frac{(110)(90)}{10{,}000} = \frac{9900}{10{,}000} = \frac{99}{100} = 99\%$$

F. Maximum and Minimum Area

A rectangle will have maximum area when

$$Length = Width \Leftrightarrow When\ it\ is\ a\ Square$$

If we allow the rectangle to only have integer values for its length and width, then the rectangle with

- minimum area will have maximum difference between the length and the width
- maximum area will have minimum difference between the length and the width:

Example 2.58

The perimeter of a rectangle with integer sides is 52 units. Find the difference between the maximum and the minimum possible area of the rectangle.

$$P = 2(l + w) = 52 \Rightarrow l + w = 26$$

Rectangle with maximum area will have minimum difference between length and width:

$$l = w = 13 \Rightarrow A = lw = 13^2 = 169$$

Rectangle with minimum area will have maximum difference between length and width:

$$l = 25, w = 1 \Rightarrow A = lw = 25 \times 1 = 25$$

Difference

$$169 - 25 = 144$$

Example 2.59

Rishi is making a rectangle using 50 matchsticks of equal length. He first makes a rectangle with minimum area. Then, he takes the same matchsticks, and a rectangle with maximum area. Find the difference in the areas of the two rectangles.

Suppose that each matchstick has length:

$$1\ unit$$

If you are making a rectangle, the perimeter of the rectangle is

$$P = 2(l + w)$$

From the condition given in the question, we know that:

$$2(l + w) = 50 \Rightarrow l + w = 25$$

The rectangle with minimum area will have maximum difference between the length and the width:

$$l = 24, w = 1 \Rightarrow A = lw = 24 \times 1 = 24$$

The rectangle with maximum area will have minimum difference between the length and the width:

$$l = 13, w = 12 \Rightarrow A = lw = 13 \times 12 = 156$$

And then, the difference between the two areas is:

$$156 - 24 = 132$$

Example 2.60

Ms. Osborne asks each student in her class to draw a rectangle with integer side lengths and a perimeter of 50 units. All of her students calculate the area of the rectangle they draw. What is the difference between the largest and smallest possible areas of the rectangles?**(AMC 8 2008/17)**

G. Paths and Borders

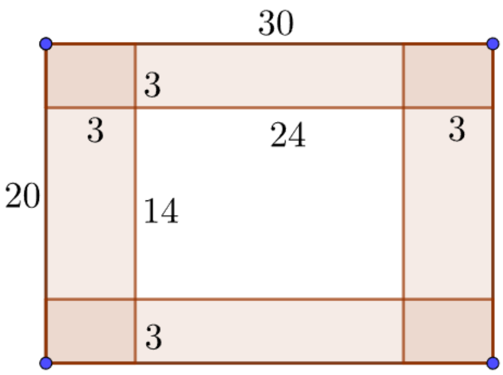
Identify a smaller figure inside a larger figure. Then:

$$A(Path) = A(Outside\ Figure) - A(Inside\ Figure)$$

Example 2.61: Path Inside

A square garden measuring 20 feet by 30 feet has a three feet marble walkway at its inside border, while the remaining area is covered by grass. What is the area of the walkway?

$$Outer\ Area = 20(30) = 600$$
$$Inner\ Area = (20-6)(30-6) = 14 \times 24 = 336$$



Area of the path

$$= [Outer] - [Inner] = 600 - 336 = 264\ sq.ft$$

Example 2.62: Path Outside

A rectangular fort(F) measuring 400 meters by 700 meters has a 3-meter moat(M) around it. What is the area of the moat?

A(F + M) – A(C) = 403 * 703 – 400 * 700 = 28000 + 2100 + 1200 + 9 – 28000 = 3309 sq. m.

Example 2.63: Costs

The page of a book which is 30 cm * 40 cm has a margin of 2 cm. The margin is included in the dimensions given. What percentage of the page is occupied by the margin?

(30*40)-(26*36)=1200-936=264

%=264/1200=22%

Example 2.64: Costs

Find the cost of adding a zari border of 1 inch on each side to a piece of cloth(C) measuring 18 inches by 36 inches, if zari(Z) is available at Rs. 288 per square foot.

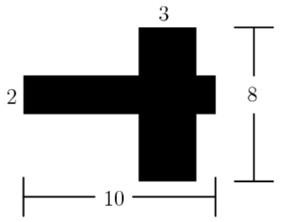
A(C + Z) – A(C) = 19 * 37 – 18 * 36 = 703 – 648 = 55 sq. inches
Cost of Zari = 288 * 55/144 = 110 Rs.

If more than one path is drawn through the centre, then the paths will overlap.

$$A(Path) = A(Path - I) + A(Path - II) - A(Overlap)$$

Example 2.65

The shaded region formed by the two intersecting perpendicular rectangles, in square units, is (AMC 8 1988/17)



Method I

The area of the longer rectangle is:

$$2(10) = 20$$

The area of the shorter rectangle is:

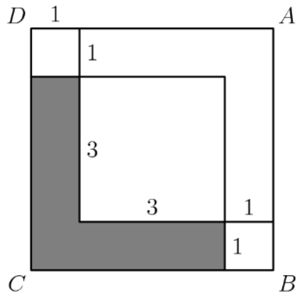
$$3(8) = 24$$

The total area

$$= 20 + 24 - 2(3) = 38$$

Method II

$$2(10) + 3(8 - 2) = 20 + 3(6) = 20 + 18 = 38$$

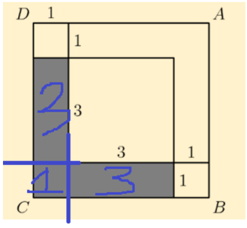


Example 2.66

Figure ABCD is a square. Inside this square three smaller squares are drawn with the side lengths as labeled. The area of the shaded L-shaped region is (AMC 8 2000/6)

We split the $L - shaped$ region into three rectangles as shown in the diagram.
The total area is:

$$1(3) + 1(3) + 1(1) = 3 + 3 + 1 = 7$$

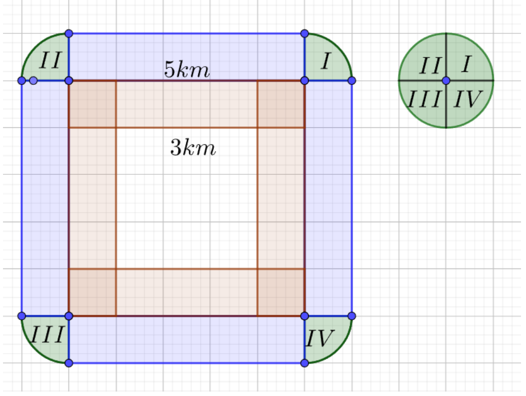


H. Applications

Example 2.67

Charlyn walks completely around the boundary of a square whose sides are each 5 km long. From any point on her path she can see exactly 1 km horizontally in all directions. What is the area of the region consisting of all points Charlyn can see during her walk, expressed in square kilometers, and rounded to the nearest whole number? (AMC 10 2000/18)

Draw the square with sides 5 km each. Note that Charlyn see 1 km around her in all directions, which means she can see in the shape of a circle.



Area of the brown region:

$$= Outer\ Square - Inner\ Square$$
$$5^2 - 3^2 = 25 - 9 = 16$$

Area of the blue region is four rectangles

$$= 4(5 \times 1) = 20$$

Area of the green region is four quarter circles that combine to make a complete circle which has area:

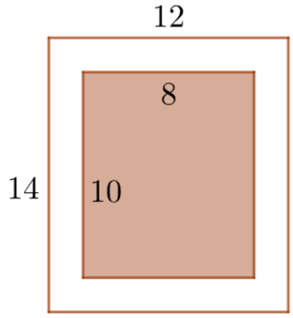
$$= \pi r^2 = \pi(1^2) = \pi \approx 3.14$$

Finally, the total area is

$$16 + 20 + 3.14 = 39.14 \approx 39$$

Example 2.68

A rectangular photograph is placed in a frame that forms a border two inches wide on all sides of the photograph. The photograph measures 8 inches high and 10 inches wide. What is the area of the border, in square inches? **(AMC 8 2012/6)**



The area of the border

$$\begin{aligned} &= Outer\ Area - Inner\ Area \\ &= 14 \cdot 12 - 10 \cdot 8 \\ &= 168 - 80 \\ &= 88\ in^2 \end{aligned}$$

2.69: Arithmetic Progression

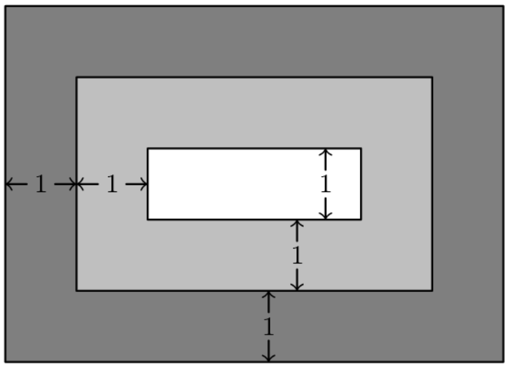
An arithmetic progression is a sequence where each successive term increases by the same amount

Examples of arithmetic progressions are:

$$5,10,15 \Rightarrow Increases\ by\ 5$$
$$3,10,17 \Rightarrow Increases\ by\ 7$$

Example 2.70

A rug is made with three different colors as shown. The areas of the three differently colored regions form an arithmetic progression. The inner rectangle is one foot wide, and each of the two shaded regions is 1 foot wide on all four sides. What is the length in feet of the inner rectangle? **(AMC 10A 2016/10)**



	Inner (white)	Middle (light gray)	Outer (dark gray)
Width	1	3	5
Length	1	3	5
Area of Rectangle	1	9	25
Area of Colored Region	1	$= 9 - 1 = 8$	$= 25 - 9 = 16$

$$1,8,16\ does\ not\ form\ an\ arithmetic\ progression$$

	Inner (white)	Middle (light gray)	Outer (dark gray)
Width	1	3	5
Length	2	4	6
Area of Rectangle	2	12	30
Area of Colored Region		$= 12 - 2 = 10$	$= 30 - 12 = 18$

$$2,10,18 \Rightarrow Arithmetic\ Progression$$

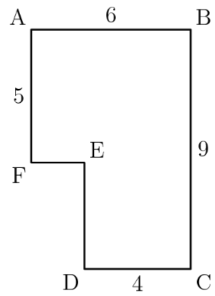
	Inner (white)	Middle (light gray)	Outer (dark gray)
Width	1	3	5
Length	x	$x+2$	$x+4$
Area of Rectangle	x	$3(x+2)=3x+6$	$5(x+4)=5x+20$
Area of Colored Region	x	$=2x+6$	$=2x+14$

$$x, \qquad 2x+6, \qquad 2x+14$$

The difference between the second and third terms is:
$$(2x+14)-(2x+6)=8$$

The difference between the first and second terms is:
$$(2x+6)-x=x+6=8 \Rightarrow x=2$$

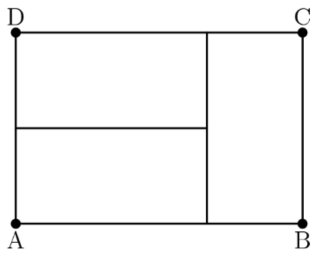
I. Composite Figures



Example 2.71

The area of polygon $ABCDEF$, in square units, is (AMC 8 1985/4)

$$5 \cdot 6 + 4 \cdot 5 = 30 + 16 = 46\ units^2$$



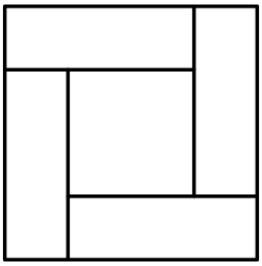
Example 2.72

Three identical rectangles are put together to form rectangle $ABCD$ as shown in the figure below. Given that the length of the shorter side of each of the smaller rectangles is 5 feet, what is the area in square feet of rectangle $ABCD$? (AMC 8 2019/2)

The length of each rectangle is
$$5 \times 2 = 10$$

The area of each rectangle is
$$5 \times 10 = 50$$

The total area is
$$50 \times 3 = 150\ ft^2$$

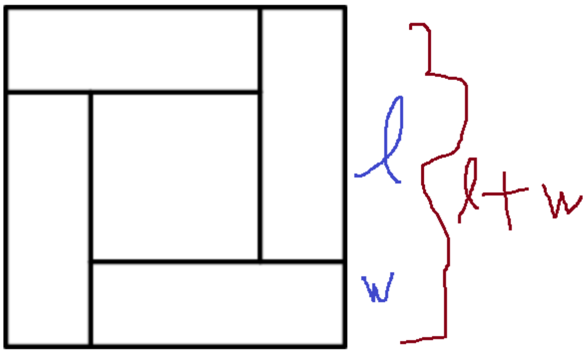


Example 2.73

Four congruent rectangles are placed as shown. If the perimeter of each rectangle is $\frac{\sqrt{3}}{5}$, then for the outer square, find the

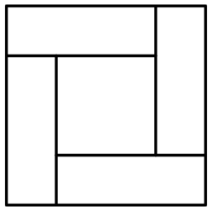
A. perimeter
B. area

$$Perimeter\ of\ Rectangle = 2(l+w) = \frac{\sqrt{3}}{5}$$
$$Perimeter\ of\ Outer\ Square = 4(l+w) = \frac{2\sqrt{3}}{5}$$
$$Area\ of\ outer\ square = (l+w)^2 = \frac{3}{100}$$



Example 2.74

Four congruent rectangles are placed as shown. The area of the outer square is 4 times that of the inner square. What is the ratio of the length of the longer side of each rectangle to the length of its shorter side? **(AMC 10A 2009/14)**



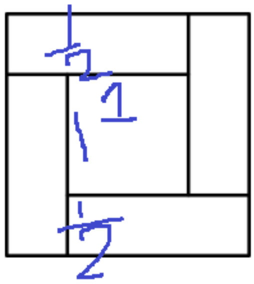
Method I

Since the question does not give lengths, we assume simple numbers. Let the inner square have

$$Side = 1 \Rightarrow Area = 1$$

Then, the outer square has area four times that of the inner square

$$Area(Outer) = 4 \times 1 = 4 \Rightarrow Side(Outer) = 2$$



The width of each rectangle

$$= \frac{2-1}{2} = \frac{1}{2}$$

The length of each rectangle

$$= 2 - \frac{1}{2} = \frac{3}{2}$$

The ratio is:

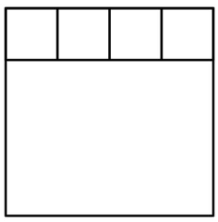
$$\frac{3}{2} : \frac{1}{2} = 3:1$$

Method II

$$(l+w)^2 = 4(l-w)^2$$
$$l+w = 2(l-w)$$
$$l+w = 2l-2w$$
$$l = 3w$$
$$l:w = 3:1$$

Example 2.75

Four identical squares and one rectangle are placed together to form one large square as shown. The length of the rectangle is how many times as large as its width? **(AMC 10A 2010/2)**



Let the side of each smaller square

$$= 1$$

The side of the larger square

$$= 4$$

The length of the rectangle

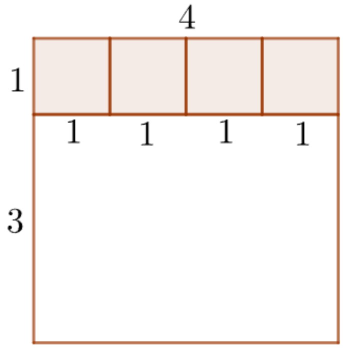
$$= 4$$

The width of the rectangle

$$= 4 - 1 = 3$$

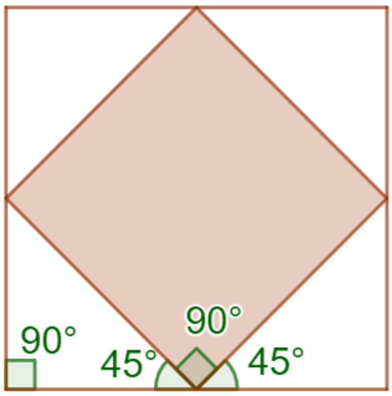
The length is:

$$\frac{4}{3} \textit{ times of the width}$$



J. Inscribed Figures

2.76: Joining the midpoints of a square
The quadrilateral formed by joining the midpoints of a square is itself a square.



2.77: Area of square formed by joining the midpoints of a square
The area of the square formed by joining the midpoints of the sides of a square is half of the larger square.

Method I

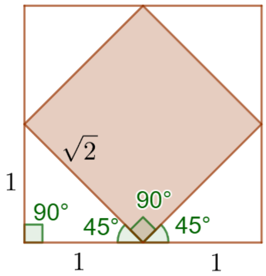
Area of larger square

$$= 2 \cdot 2 = 4$$

Area of smaller square

$$= \sqrt{2} \cdot \sqrt{2} = 2$$

Hence, area of smaller square is half of area of larger square.



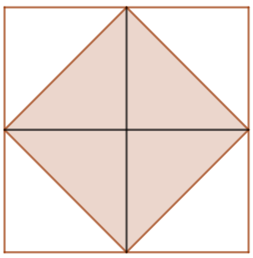
Method II

Draw a square and connect its midpoints to form another square.
Connect the opposite midpoints to form 4 brown triangles.

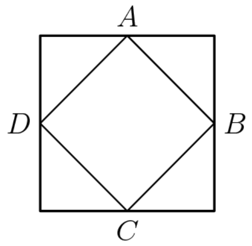
There are 4 brown triangles and 4 white triangles.
Each triangle has equal area.

Hence, the area of the brown square

$$= \frac{4}{8} = \frac{1}{2} \textit{ of the larger square}$$



Example 2.78



A. Points A, B, C and D are midpoints of the sides of the larger square. If the larger square has area 60, what is the area of the smaller square? (AMC 8 2006/5)
B. Each of the sides of a square S_1 with area 16 is bisected, and a smaller square S_2 is constructed using the bisection points as vertices. The same process is carried out on S_2 to construct an even smaller square S_3. What is the area of S_3? (AMC 10A 2008/10)

Part A

$$Area = \frac{60}{2} = 30$$

Part B

$$S_2 = \frac{S_1}{2} = 8 \Rightarrow S_3 = \frac{S_2}{2} = \frac{8}{2} = 4$$

Example 2.79: Geometric Sequence

Square S has side length s. Square S_1 is drawn by connecting the midpoints of S. Square S_2 is drawn by connecting the midpoint of S_1. The process continues. What is the area of S_{10}? S_n

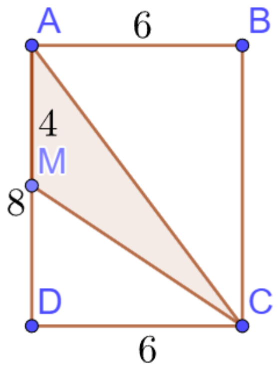
$$Area(S) = s^2$$
$$S_1 = \frac{s^2}{2}$$
$$S_2 = \frac{S_1}{2} = \frac{\frac{s^2}{2}}{2} = \frac{s^2}{2} \times \frac{1}{2} = \frac{s^2}{4} = \frac{s^2}{2^2}$$
$$S_3 = \frac{S_2}{2} = \frac{\frac{s^2}{2^2}}{2} = \frac{s^2}{2^3}$$
$$S_{10} = \frac{s^2}{2^{10}}$$
$$S_n = \frac{s^2}{2^n}$$

Example 2.80

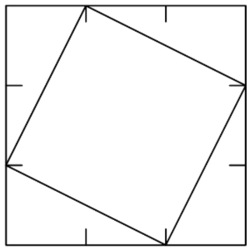
In rectangle $ABCD$, $AB = 6$ and $AD = 8$. Point M is the midpoint of $\overline{AD}$. What is the area of $\triangle AMC$? (AMC 8 2016/2)

The area of the triangle is

$$\frac{1}{2}hb = \frac{1}{2}(AM)(DC) = \frac{1}{2}(4)(6) = 12$$



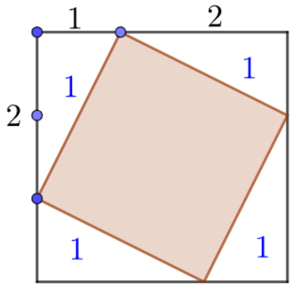
K. Trisection



Example 2.81

Each side of the large square in the figure is trisected (divided into three equal parts). The corners of an inscribed square are at these trisection points, as shown. The ratio of the area of the inscribed square to the area of the large square is **(AMC 8 1997/15)**

Method I



Area of each triangle

$$=\frac{1}{2}hb=\frac{1}{2}(1)(2)=1$$

Using complementary areas, area of the smaller square is

$$A(Smaller\ Square)=A(Larger\ Square)-A(Triangles)$$
$$=3^2-1(4)=9-4=5$$

The ratio is

$$\frac{5}{9}$$

Method II

By Pythagoras Theorem, the side length of the smaller square is:

$$=\sqrt{1^2+2^2}=\sqrt{5}$$

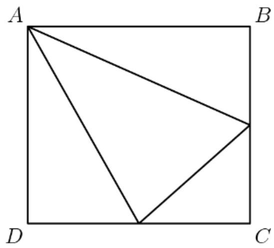
Area of the larger square is

$$\left(\sqrt{5}\right)^2=5$$

The ratio is

$$\frac{5}{3^2}=\frac{5}{9}$$

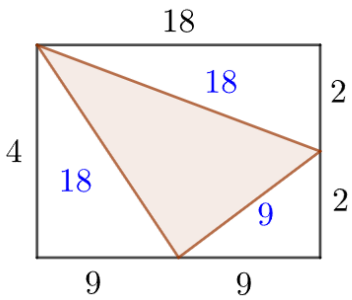
L. Complementary Areas

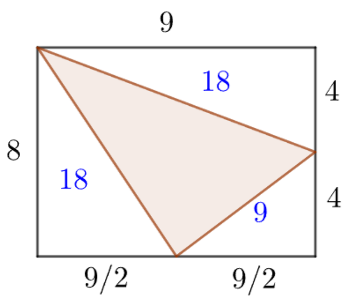


Example 2.82

The area of rectangle $ABCD$ is 72. If point A and the midpoints of $\overline{BC}$ and $\overline{CD}$ are joined to form a triangle, the area of that triangle is **(AMC 8 2000/25)**

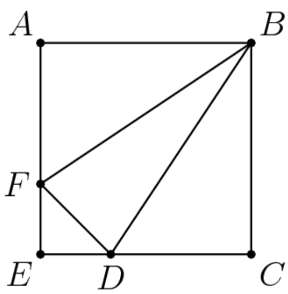
We can take numbers and do this. The area of the triangles is the same irrespective of the numbers:





The area is

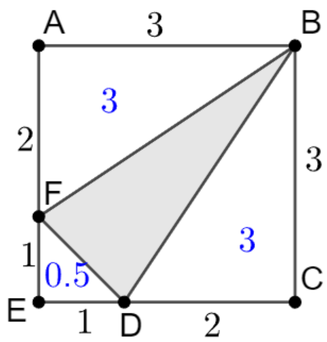
$$72-(18+18+9)=72-45=27$$

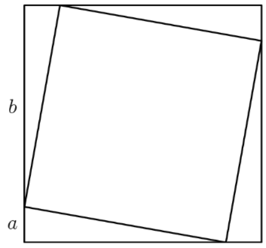


Example 2.83

In square $ABCE$, $AF=2FE$ and $CD=2DE$. What is the ratio of the area of ΔBFD to the area of square $ABCE$? **(AMC 8 2008/23)**

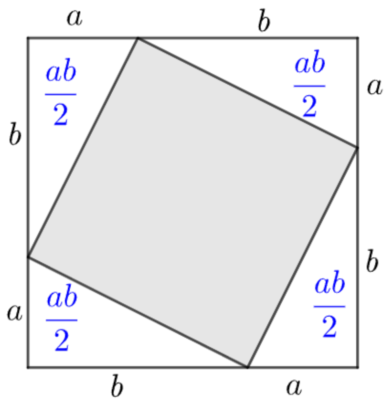
$$\frac{9-6.5}{9}=\frac{2.5}{9}=\frac{5}{18}=5:18$$





Example 2.84

A square with area 4 is inscribed in a square with area 5, with one vertex of the smaller square on each side of the larger square. A vertex of the smaller square divides a side of the larger square into two segments, one of length a, and the other of length b. What is the value of ab? **(AMC 8 2012/25)**



The area of the four white triangles is:

$$5-4=1$$

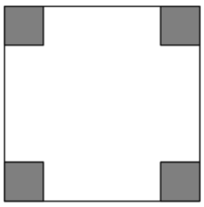
Area of each triangle

$$=\frac{ab}{2}$$

Area of four triangles

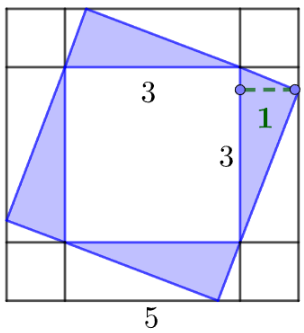
$$=4\left(\frac{ab}{2}\right)=2ab$$

$$2ab=1\Rightarrow ab=\frac{1}{2}$$



Example 2.85

One-inch squares are cut from the corners of this 5 inch square. What is the area in square inches of the largest square that can fit into the remaining space? **(AMC 8 2015/25)**



The area of the inner square

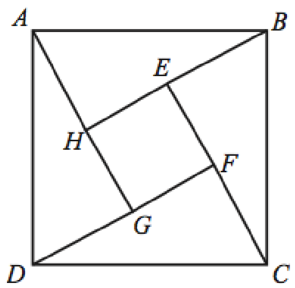
$$=3^2=9$$

The areas of the four triangles

$$=4\left(\frac{1}{2}hb\right)=4\left(\frac{1}{2}\cdot 1\cdot 3\right)=6$$

The total area is

$$=9+6=15$$



Example 2.86

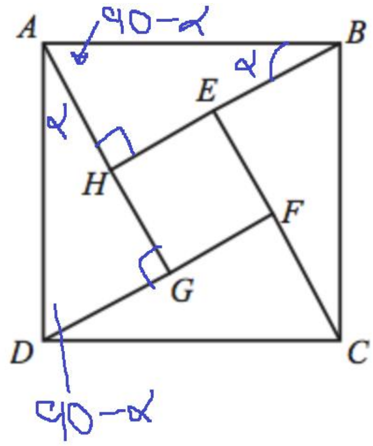
In the figure, the length of side AB of square $ABCD$ is $\sqrt{50}$ and $BE = 1$. What is the area of the inner square $EFGH$? **(AMC 10A 2005/8)**

Step I: Prove triangles congruent

$$\angle AHB = \angle AGD = 90°$$

Let

$$\angle ABH = \alpha \Rightarrow \angle BAH = 90 - \alpha$$



Since $\angle BAD = 90°$:

$$\angle BAH + \angle GAD = 90°$$
$$\angle GAD = 90 - \angle BAH = 90 - (90 - \alpha) = \alpha$$

By complementary angles in a right triangle:

$$\angle GDA = 90 - \angle GAD = 90 - \alpha$$

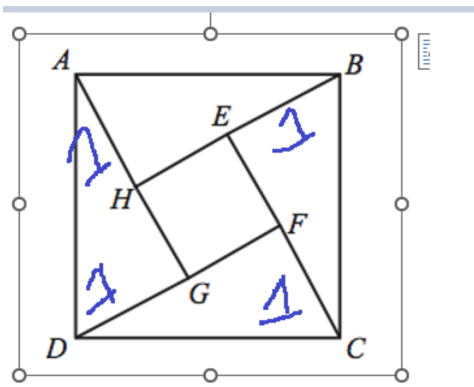
From the diagram, $\Delta BAH \cong \Delta DAG$ by $ASA\ Congruence$:

$$BA = DA = \sqrt{50}$$
$$\angle BAH = \angle GDA = 90 - \alpha$$
$$\angle ABH = \angle GAD = \alpha$$

By $CPCTC$:

$$AG = BH$$
$$GH + HA = HE + EB$$
$$HA = EB = 1$$

Step II: Use the Congruence



$$\angle AHB = 180 - 90 = 90°$$

By Pythagoras Theorem in ΔAHB

$$AH^2 + HB^2 = AB^2$$

Substitute $AH = 1, HB = HE + EB = HE + 1, AB = \sqrt{50}$

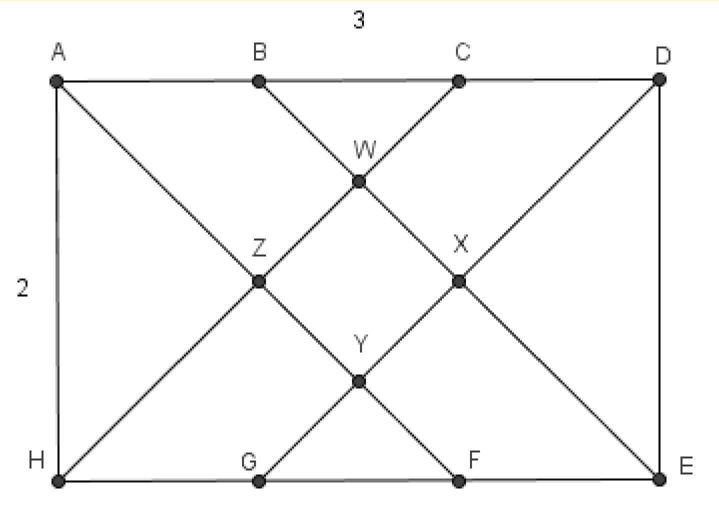
$$1^2 + (HE + 1)^2 = \left(\sqrt{50}\right)^2$$
$$1 + (HE + 1)^2 = 50$$
$$(HE + 1)^2 = 49$$

Take the square root both sides:

$$HE + 1 = 7$$
$$HE = 6$$
$$HE^2 = 36$$

Example 2.87

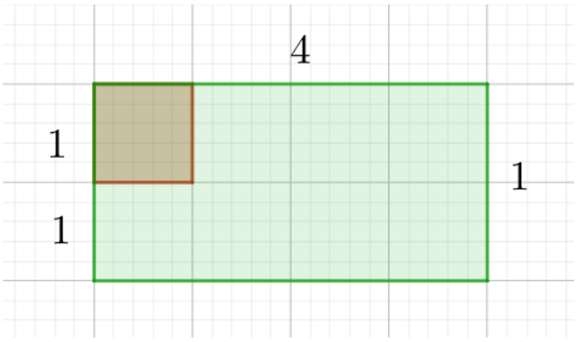
In rectangle $ADEH$, points B and C trisect $\overline{AD}$, and points G and F trisect $\overline{HE}$. In addition, $AH = AC = 2$, and $AD = 3$. What is the area of quadrilateral $WXYZ$ shown in the figure? **(AMC 10A 2006/17)**



M. Ratios

Example 2.88

A square is drawn inside a rectangle. The ratio of the width of the rectangle to a side of the square is 2:1. The ratio of the rectangle's length to its width is 2:1. What percent of the rectangle's area is in the square? **(AMC 10A 2008/2)**



Since the question uses ratios, we can take any side lengths that we want.

Take a square with side length 1. The rectangle will have width

$$1 \times 2 = 2$$

And it will have length

$$2 \times 2 = 4$$

The area of the rectangle will be

$$2 \times 4 = 8$$

And the percentage will be

$$\frac{1}{8} = 12.5\%$$

Example 2.89

Older television screens have an aspect ratio of 4: 3. That is, the ratio of the width to the height is 4: 3. The aspect ratio of many movies is not 4: 3, so they are sometimes shown on a television screen by "letterboxing" - darkening strips of equal height at the top and bottom of the screen, as shown. Suppose a movie has an aspect ratio of 2:1 and is shown on an older television screen with a 27-inch diagonal. What is the height, in inches, of each darkened strip? **(AMC 10A 2008/14)**

The TV has height and width in the ratio 4: 3. The movie has height which is half of its width.

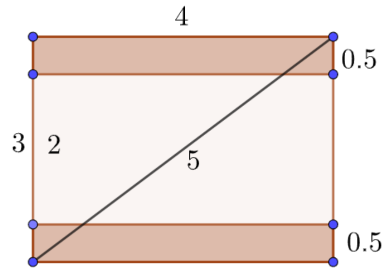
$$Height\ of\ movie = \frac{4}{2} = 2$$

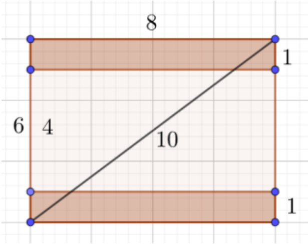
The height of the strip remaining has:

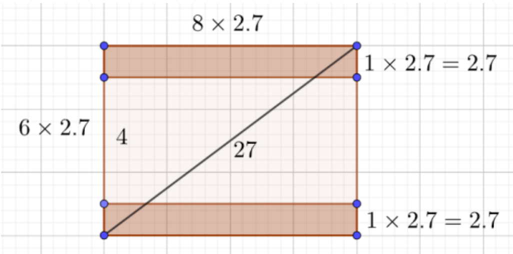
$$Height = \frac{4-2}{2} = \frac{2}{2} = 1$$

This gives us the diagram to the right, which has diagonal

$$(3,4,5) \Rightarrow Diagonal = 5$$







Double the dimensions to get a diagonal of 10, and then multiply by 2.7 to get a diagonal of 27.

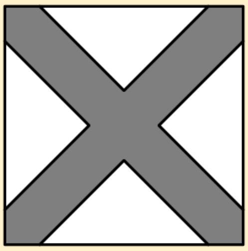
Method II

Scale the height of the side length by $\frac{27}{5}$ to get the height in the TV screen in the question:

$$\frac{1}{2} \times \frac{1}{5} \times 27 = \frac{27}{10} = 2.7\ inches$$

Example 2.90

A paint brush is swept along both diagonals of a square to produce the symmetric painted area, as shown. Half the area of the square is painted. What is the ratio of the side length of the square to the brush width? **(AMC 10A 2007/19)**



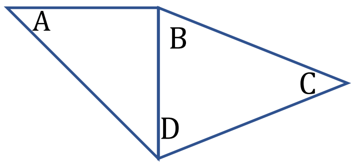
Example 2.91

In the question below, mark all options that are correct.

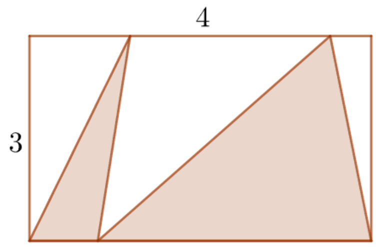
If two adjacent sides and a diagonal of a quadrilateral form a right-angled triangle, then that quadrilateral is definitely a:

A. Rhombus
B. Square
C. Rectangle
D. Parallelogram
E. Trapezium
F. None of these

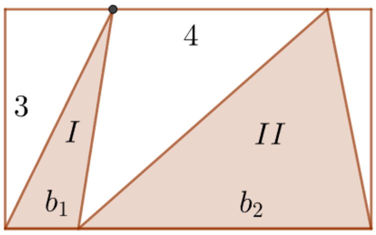
The quadrilateral in the diagram does not fall under Options A to E, though it meets the conditions in the question.
Hence, Option F.



N. Triangles in a Rectangle



Example 2.92
The diagram alongside shows a rectangle with dimension 3×4. If the triangle on the left has area which is one third of the triangle on the right, then determine the area of the smaller triangle.



Area of the Triangles

$$Area\ of\ I = \frac{1}{2}hb = \frac{1}{2}(3)b_1 = \frac{3}{2}b_1$$

$$Area\ of\ II = \frac{1}{2}hb = \frac{1}{2}(3)b_2 = \frac{3}{2}b_2$$

The total area is

$$\frac{3}{2}b_1 + \frac{3}{2}b_2 = \frac{3}{2}(b_1 + b_2) = \frac{3}{2}(4) = 6$$

Dividing the area
The ratio of the area of the triangle is:

$$1:3 \Rightarrow 1 + 3 = 4$$

The smaller triangle has area

$$Total\ area = \frac{1}{4} of\ 6 = \frac{1}{4} \times 6 = 1.5$$

2.2 Visual Techniques

A. Visual Techniques

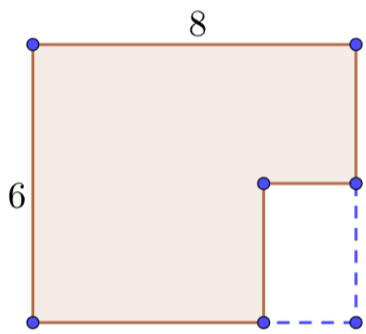
2.93: Rearrangement
In certain cases, a polygon can be rearranged without changing its perimeter.

Example 2.94
The perimeter of the polygon shown (i n the given units) is **(AMC 8 1986/13)**

Consider a rectangle that has the same perimeter as the given polygon. (Make sure you understand why the perimeters are the same.)

Hence, the perimeter of the polygon

$$= P(Rectangle) = 2(6 + 8) = 2 \times 14 = 28$$



2.95: Adding a Tile

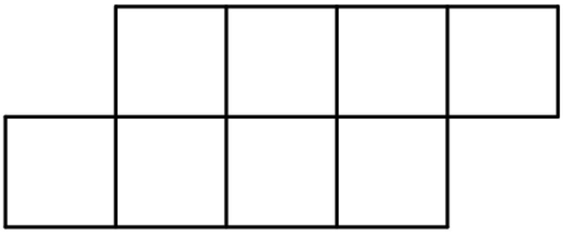
If you have a tiled shape made of squares with side length s, and you add a square to the shape, you will

- NOT change the perimeter if the new square shares two sides with the old shape
- Increase the perimeter by $2s$ if the new square shares a single side with the old shape

Example 2.96

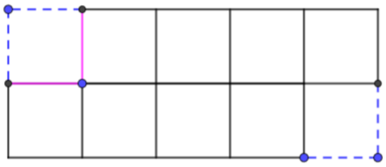
Mark all correct options

Eight 1×1 square tiles are arranged as shown so their outside edges form a polygon with a perimeter of 14 units. Two additional tiles of the same size are added to the figure so that at least one side of each tile is shared with a side of one of the squares in the original figure. Find all possible values of the perimeter of the new figure? **(AMC 8 1992/22, Adapted)**



Case I: Perimeter 14

If you add a tile at the top left and the bottom right, the perimeter of the new figure remains the same, since the blue dotted lines get added, and the purple lines get removed.

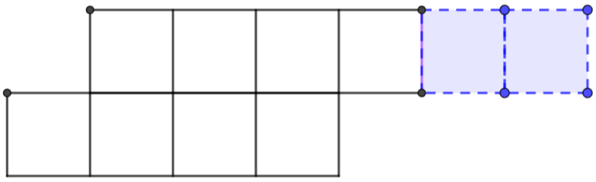


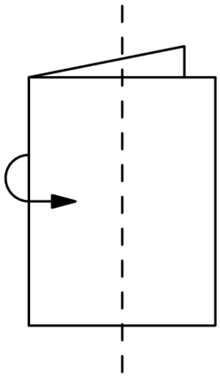
Case II: Perimeter 16

If you add a tile at the top left and top right, the perimeter increases by 2.

Case III: Perimeter 18

If you add two tiles at the right, the perimeter increases by 4.





Example 2.97

Suppose a square piece of paper is folded in half vertically. The folded paper is then cut in half along the dashed line. Three rectangles are formed-a large one and two small ones. What is the ratio of the perimeter of one of the small rectangles to the perimeter of the large rectangle? **(AMC 8 1989/24)**

Perimeter of small rectangle

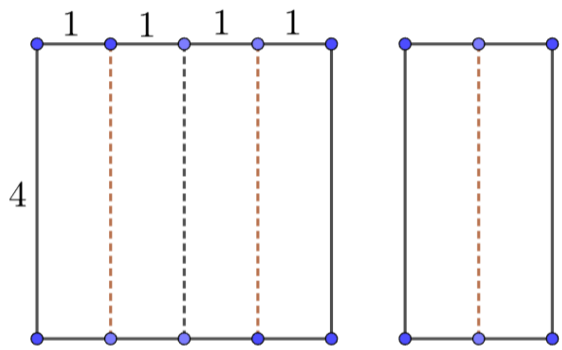
$$= 2(1 + 4) = 2(5)$$

Perimeter of large rectangle

$$= 2(2 + 4) = 2(6)$$

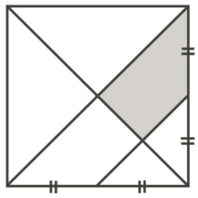
Ratio of perimeter of small rectangle to large rectangle

$$= 2(5): 2(6) = 5: 6$$



Example 2.98

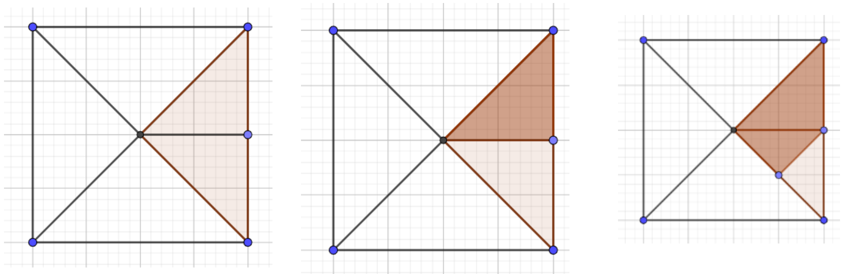
A square is divided, as shown. What fraction of the area of the square is shaded? **(Gauss Grade 7 2007/23)**



Method I

Divide the square into four equal parts using its diagonals. See 1st diagram on the left. The light brown triangle has area

$$\frac{1}{4} \text{ of the square}$$



Divide the light brown triangle into two equal parts, getting a dark brown triangle, which has area

$$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

Add a further dark brown triangle to get the shape that we want. The smaller dark brown triangle has area

$$\frac{1}{2} \times \frac{1}{8} = \frac{1}{16}$$

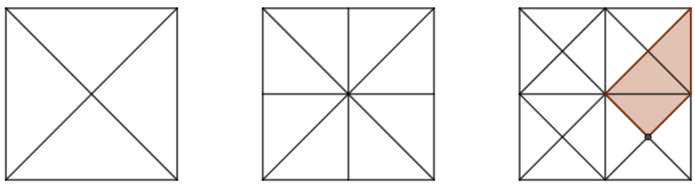
The total area is then:

$$\frac{1}{8} + \frac{1}{16} = \frac{3}{16}$$

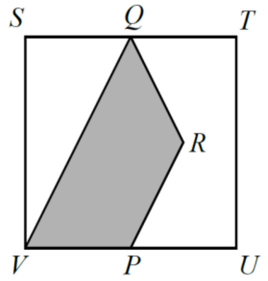
Method II:

Divide the square into 16 triangles of equal area as shown. The shaded area occupies 3 of the 16 triangles, and hence the required ratio is

$$\frac{3}{16}$$

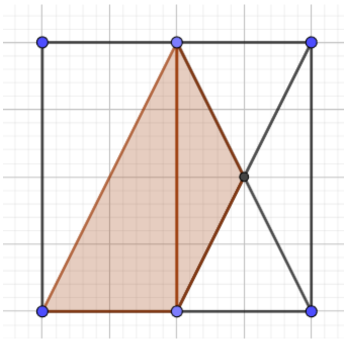


Example 2.99
In the diagram shown:
$STUV$ is a square
Q and P are the midpoints of ST and UV
$PR = QR$, and
VQ is parallel to PR
What is the ratio of the shaded area to the unshaded area? (Gauss Grade 7 2014/24)



Let the total area of $STUV$ be 1.

$$A(\Delta QVP) = \frac{1}{4}$$



Also,

$$PR \parallel VQ \Rightarrow PRT \text{ is a diagonal of } QTUP$$
$$QR = PR \Rightarrow \angle PQU = \angle QPT \Rightarrow QRU \text{ is a diagonal of } QTUP$$

Since diagonals of a rectangle divide it into four triangles with equal area:

$$A(QRP) = \frac{1}{4}A(QTUP) = \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$$

Total Shaded Area

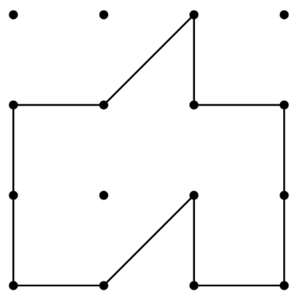
$$= \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

Total Unshaded Area

$$= 1 - \frac{3}{8} = \frac{5}{8}$$

Ratio of Unshaded to Shaded Area

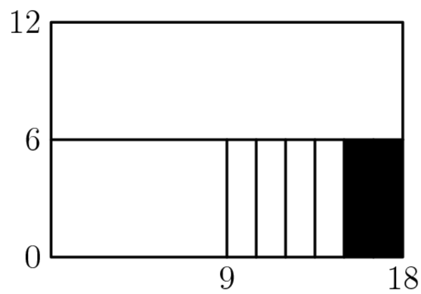
$$= \frac{3}{8} : \frac{5}{8} = 3:5$$



Example 2.100
Dots are spaced one unit apart, horizontally and vertically. The number of square units enclosed by the polygon is (AMC 8 1998/6)

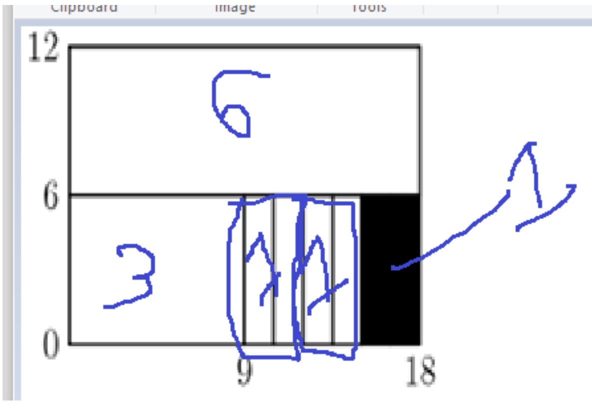
Move the top triangle and fit in the bottom space, giving a rectangle with area

$$2 \times 3 = 6$$



Example 2.101
What fraction of the large 12 by 18 rectangular region is shaded? (AMC 8 1987/12)

We can calculate the area of the region.



Let the given shaded region have area
$$1$$

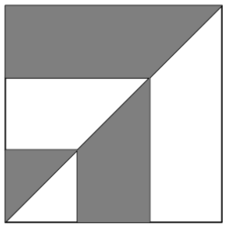
The bottom right rectangle will have area
$$3$$

The bottom left rectangle will also have area
$$3$$

And the top rectangle will have area
$$6$$

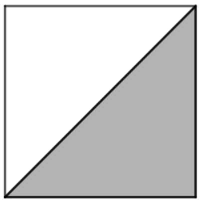
Total area
$$= 3 + 3 + 6 = 12$$

And the required fraction will be:
$$\frac{1}{12}$$



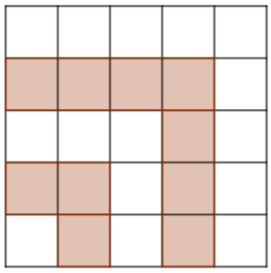
Example 2.102

What fraction of the square is shaded? (AMC 8 1990/3)



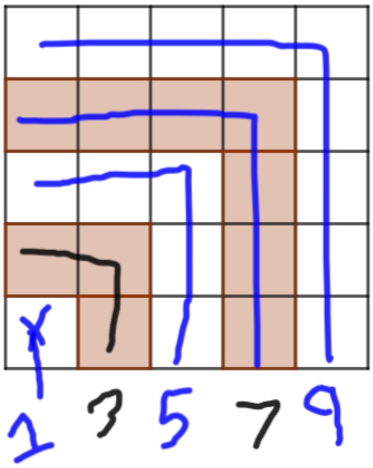
Reflect two of the grey shaded parts across the diagonal of the square.
Note the entire area to the right of the diagonal is now shaded.

By symmetry, the two areas are equal. The fraction is
$$\frac{1}{2}$$



Example 2.103

Determine the number of unshaded squares in the diagram alongside. All of the smallest squares have side length 1.



Count the squares as shown to the right. We get:

$$1,3,5,7,9$$

As the number of squares in each line.

And the number of unshaded squares is:

$$1+5+9=15$$

Example 2.104

What fraction of this square region is shaded? Stripes are equal in width, and the figure is drawn to scale. (AMC 8 1997/10)

Use the square at the bottom left as a unit square, and see the pattern:

$$1,3,5,7,9,11$$

Unshaded area:

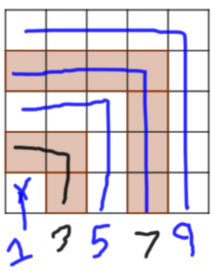
$$=\frac{1+5+9}{36}=\frac{15}{36}=\frac{5}{12}$$

Shaded Area

$$=1-\frac{5}{12}=\frac{7}{12}$$

Example 2.105

Suppose the figure at right is extended using the same pattern, but so that it is a square with a side length of 20. Find the shaded area.



$$1,3,5,\dots,39$$

Where

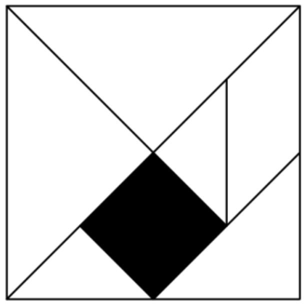
$$1\ is\ the\ 1st\ odd\ number=2(1)-1=1$$
$$3\ is\ the\ 2nd\ odd\ number=2(2)-1=4-1=3$$
$$39\ is\ the\ 20th\ odd\ number=2(20)-1=40-1=39$$

Out of these, we want the sum of alternate numbers:

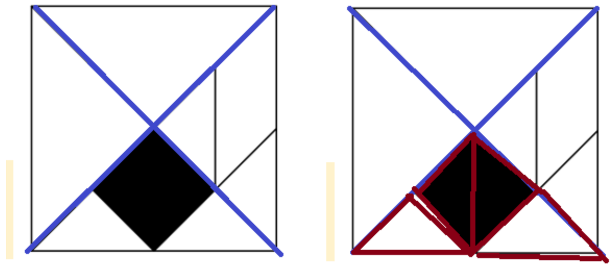
$$3+7+11+15+19+23+27+31+35+39$$

Arrange the numbers in pairs:

$$= (3 + 39) + (7 + 35) + (11 + 31) + (15 + 27) + (19 + 23)$$
$$= 42 + 42 + 42 + 42 + 42$$
$$= 5 \times 42$$
$$= 10 \times 21$$
$$= 210$$



Example 2.106
What is the ratio of the area of the shaded square to the area of the large square? (The figure is drawn to scale) **(AMC 8 1998/13)**

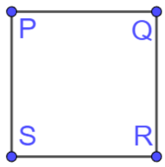


Divide the square into four equal parts by drawing the blue diagonals.

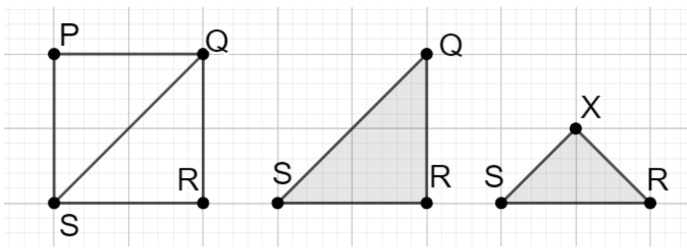
Further divide the bottom diagonal into four congruent brown triangles.

The required area is then:

$$\frac{2}{16} = \frac{1}{8}$$



Example 2.107
Let $PQRS$ be a square piece of paper. P is folded onto R and then Q is folded onto S. The area of the resulting figure is 9 square inches. Find the perimeter of square $PQRS$. **(AMC 8 1998/20)**



Fold the square in half to get ΔQSR. The makes the area half of what is was.
Fold ΔQSR in half to get a smaller triangle. This makes the area $\frac{1}{4}^{th}$ of the original square.

Area of square $PQRS$

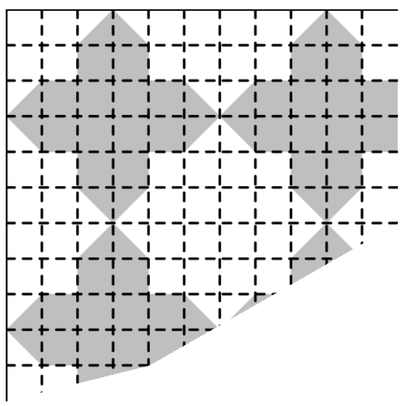
$$= 4(Area\ of\ resulting\ figure) = 4(9) = 36$$

Side length of $PQRS$

$$= \sqrt{36} = 6$$

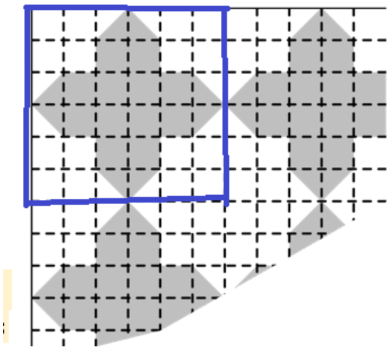
Perimeter of $PQRS$

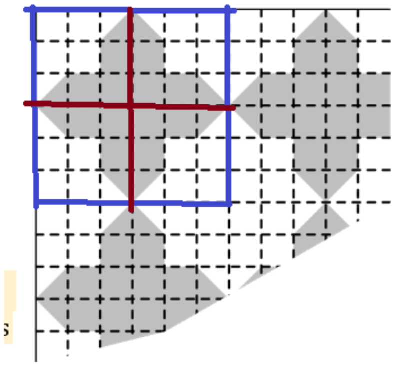
$$= 4 \cdot 6 = 24$$



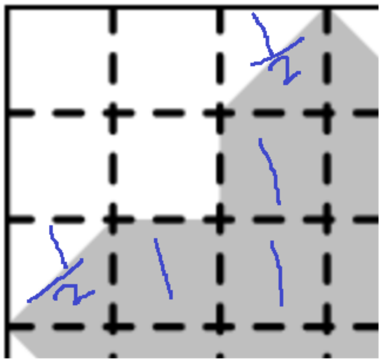
Example 2.108
A corner of a tiled floor is shown. If the entire floor is tiled in this way and each of the four corners looks like this one, then what fraction of the tiled floor is made of darker tiles? **(AMC 8 2002/23)**

The pattern repeats every six tiles. See the blue square created in the left diagram. Further, the blue square can be divided into four brown squares, each of which has equal shading.





Then the shading in the brown square



The required shading is

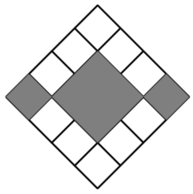
$$\frac{1}{2}+1+1+1+\frac{1}{2}=4$$

The total number of squares

$$3\times 3=9$$

The fraction is

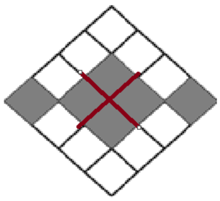
$$\frac{4}{9}$$



Example 2.109

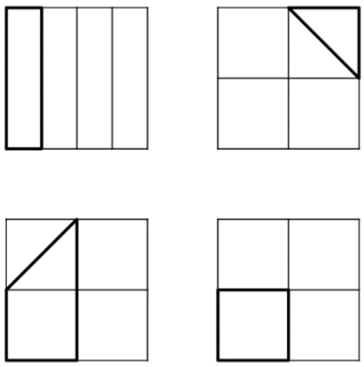
In the figure, what is the ratio of the area of the gray squares to the area of the white squares? (AMC 8 2008/6)

$$6:10=3:5$$



Example 2.110

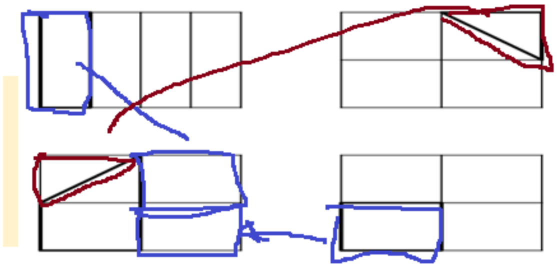
Each of the following four large congruent squares is subdivided into combinations of congruent triangles or rectangles and is partially bolded. What percent of the total area is partially bolded? (AMC 8 2011/7)

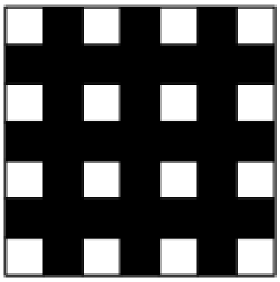


Visually rearrange the shapes.
Bottom right occupies a rectangle.
Top left occupies a rectangle.
Top right occupies a triangle at the top left.

Overall, we get

$$\frac{1}{4} = 25\%$$



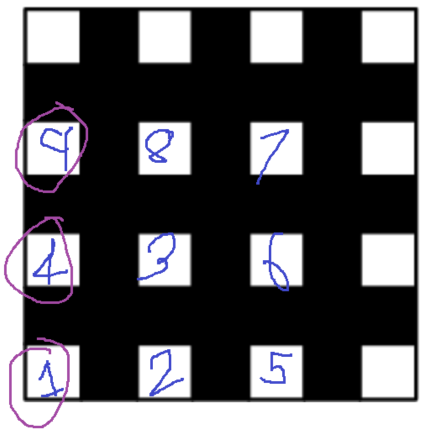


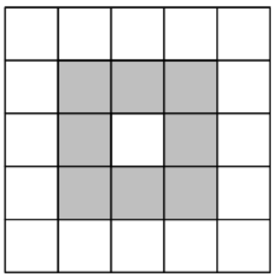
Example 2.111

The diagram represents a 7-foot-by-7-foot floor that is tiled with 1-square-foot black tiles and white tiles. Notice that the corners have white tiles. If a 15-foot-by-15-foot floor is to be tiled in the same manner, how many white tiles will be needed? (AMC 9 2009/18)

Size of floor

$$1\ Foot: 1\ White\ Tile$$
$$3\ Foot: 4 = 2^2\ White\ Tiles$$
$$5\ Foot: 9 = 3^2\ White\ Tiles$$
$$7\ Foot: 4^2\ White\ Tiles$$
$$9\ Foot: 5^2\ White\ Tiles$$
$$11\ Foot: 6^2\ White\ Tiles$$
$$13\ Foot: 7^2\ White\ Tiles$$
$$15\ Foot: 8^2\ White\ Tiles$$





Example 2.112

Extend the square pattern of 8 black and 17 white square tiles by attaching a border of black tiles around the square. What is the ratio of black tiles to white tiles in the extended pattern? (AMC 8 2011/3)

The number of black tiles which get added is:

$$5 \times 4 + 4 = 20 + 4 = 24$$

The total black tiles are now:

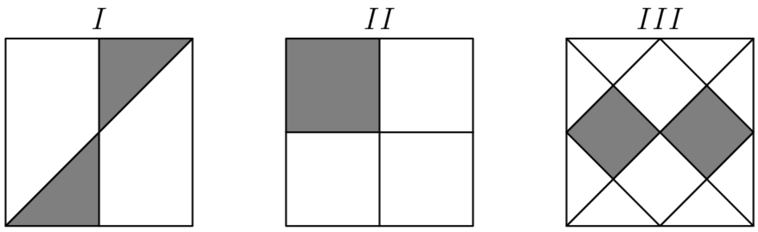
$$8 + 24 = 32$$

The ratio is:

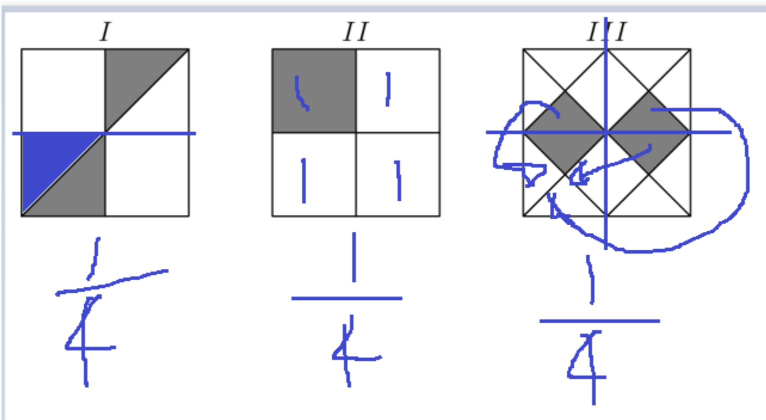
$$32: 17$$

Example 2.113

Each of the three large squares shown below is the same size. Segments that intersect the sides of the squares intersect at the midpoints of the sides. How do the shaded areas of these squares compare? **(AMC 8 1994/12)**



A. The shaded areas in all three are equal.
B. Only the shaded areas of I and II are equal.
C. Only the shaded areas of I and III are equal.
D. Only the shaded areas of II and III are equal.
E. The shaded areas of I, II and III are all different.



Example 2.114

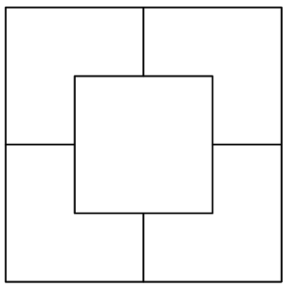
The perimeter of one square is 3 times the perimeter of another square. The area of the larger square is how many times the area of the smaller square? **(AMC 8 1994/16)**

Let

$$p_S = 4 \Rightarrow Side_S = 1 \Rightarrow Area = 1$$

$$p_L = 12 \Rightarrow Side_L = \frac{12}{4} = 3 \Rightarrow Area = 9$$

$$Ratio\ of\ areas = \frac{9}{1} \Rightarrow 9\ times$$

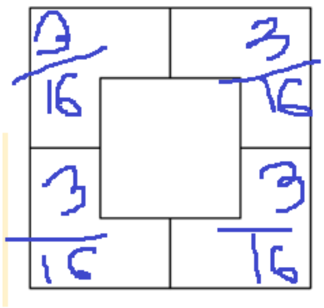


Example 2.115

The area of each of the four congruent L-shaped regions of this 100-inch by 100-inch square is $\frac{3}{16}$ of the total area. How many inches long is the side of the center square? **(AMC 8 1995/18)**

The total area of the $L-shaped$ regions is:

$$\frac{3}{16}\times 4=\frac{3}{4}$$



The area of the center square

$$=1-\frac{3}{4}=\frac{1}{4}\ of\ the\ entire\ square$$

The actual area

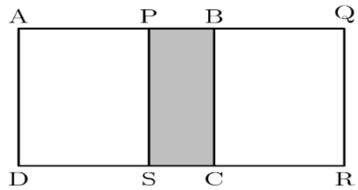
$$=\frac{1}{4}(100^2)=\frac{10{,}000}{4}=2500$$

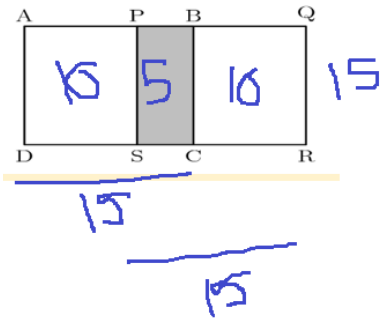
The side length is:

$$\sqrt{2500}=\sqrt{25}\times\sqrt{100}=5\times 10=50$$

Example 2.116

Two congruent squares, $ABCD$ and $PQRS$, have side length 15. They overlap to form the 15 by 25 rectangle $AQRD$ shown. What percent of the area of rectangle $AQRD$ is shaded? (AMC 8 2011/13)





The length of the shaded portion

$$=15+15-25=30-25=5$$

The percent of area that is shaded is in proportion to the length that is shaded because the height is the same throughout:

$$=\frac{5}{25}=20\%$$

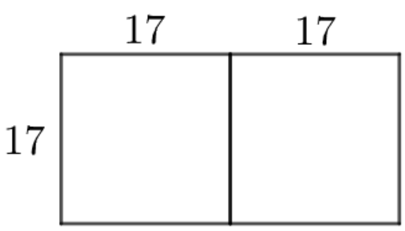
Method II: Algebra

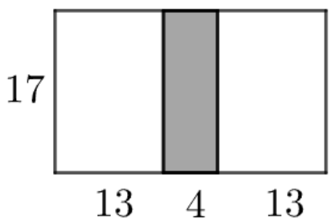
$$x+2(15-x)=25$$
$$x+30-2x=25$$
$$x=5$$

Example 2.117

Two squares $17\ cm\times 17\ cm$ overlap to form a rectangle $17\ cm\times 30\ cm$. The area of the overlapping region is: (NMTC Sub-Junior/Screening, 2011/3)

If you do not overlap the squares, and put them next to each you get:





If you overlap and move the right square to the left, the length will reduce.
To reduce the length by $34-30=4\ cm$, you move the right square $4cm$ to the left.

The area of this overlap is

$$4 \times 17 = 68\ cm^2$$

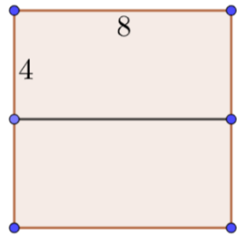
B. Simple Cutting

Example 2.118

A square with side length 8 is cut in half, creating two congruent rectangles. What are the dimensions of one of these rectangles? (AMC 10A 2012/2)

The dimensions are:

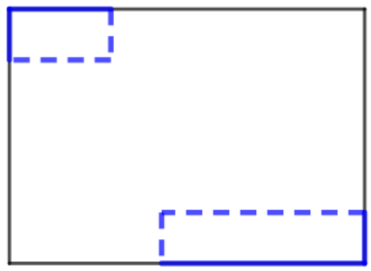
$$\underbrace{4}_{Breadth} \times \underbrace{8}_{Length}$$



2.119: Types of Angles

If you cut rectangular, non-overlapping regions from a rectangle, the perimeter does not change.

- Does not "change" can be written as does not "vary", and a single word for this is *invariant*.
- If the cuts overlap, then the perimeter does not remain the same.



Example 2.120

A. Tanmay bought a rectangular carpet. He cut a rectangle from the top left corner of the carpet. He cut another rectangle from the top right corner of the carpet. The cuts did not overlap. The carpet originally had dimensions 3×4. What was the perimeter of the carpet after the cuts were made.
B. If the cuts that Tanmay made had been overlapping, what would have been your answer to Part A.

Part A

The original carpet had

$$Perimeter = 2(3 + 4) = 2(7) = 14$$

The new carpet (after the cuts) will have the same perimeter

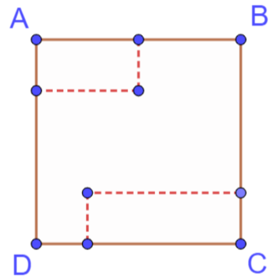
$$= 14$$

Part B

Cannot be determined without further information since the cuts are overlapping.

Example 2.121

Tulsi bought a square-shaped carpet. She removed two unequal, nonoverlapping rectangular regions from two of its corners. Now she could fit the carpet exactly into her study room. If the perimeter of the study room is 16 m, what is the area of the original carpet? (NMTC Primary-Screening, 2008/16)



The perimeter of the square

$$= Perimeter\ of\ Study\ Room = 16m$$

The side length of the square:

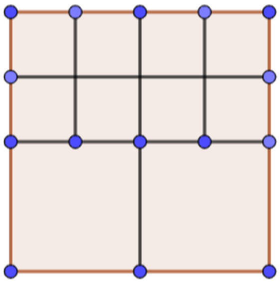
$$= \frac{Perimeter}{4} = \frac{16}{4} = 4m$$

The area of the square

$$= s^2 = 4^2 = 16m^2$$

Example 2.122

A square with integer side length is cut into 10 squares, all of which have integer side length and at least 8 of which have area 1. What is the smallest possible value of the length of the side of the original square? **(AMC 8 2012/17)**



There are 10 squares. Each square has integer side length. Hence, area of these 10 squares must be minimum:

$$10 \times 1 = 10$$

A square of side length will have area

$$3 \times 3 = 9 < 10 \Rightarrow Does\ not\ work$$

Let us try a square of side length 4. We make 8 squares of area 1, and 2 squares of area 4, and we are able to cover the whole square.

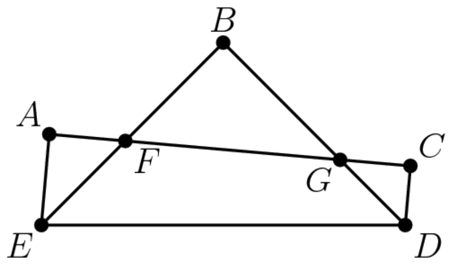
Hence,

$$Smallest\ possible\ value = 4$$

3. Quadrilaterals

3.1 Sum of Angles

Example 3.1
In the figure, $\angle A$, $\angle B$, and $\angle C$ are right angles. If $\angle AEB = 40°$ and $\angle BED = \angle BDE$, then $\angle CDE =$ (AMC 8 1995/13)



In Isosceles Right ΔBED

$$\angle BED = \angle BDE = 45°$$

Sum of angles of a Quadrilateral is 360°. Hence:

$$\angle CDE = 360 - \underbrace{90°}_{\angle A} - \underbrace{90°}_{\angle C} - \underbrace{40°}_{\angle AEB} - \underbrace{40°}_{\angle BED}$$

Example 3.2
The angles of quadrilateral $ABCD$ satisfy $\angle A = 2\angle B = 3\angle C = 4\angle D$. What is the degree measure of $\angle A$, rounded to the nearest whole number? (AMC 10B 2007/15)

$$a = 2b = 3c = 4d = k$$

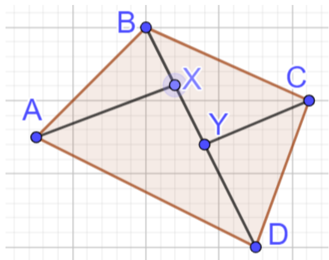
$$k + \frac{k}{2} + \frac{k}{3} + \frac{k}{4} = 360 \Rightarrow k = \frac{360 \times 12}{25} = 172.8 \approx 173$$

3.3: Area Formula
Consider quadrilateral $ABCD$ with diagonal BD. Also

$$AX \perp BD, \qquad CY \perp BD$$

Then, the area of quadrilateral $ABCD$ is:

$$\frac{1}{2}(BD)(AX + CY)$$



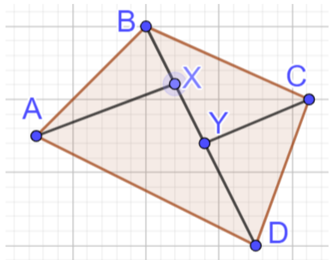
$$A(\Delta ABD) = \frac{1}{2}hb = \frac{1}{2}(BD)(AX)$$

$$A(\Delta CBD) = \frac{1}{2}hb = \frac{1}{2}(BD)(CY)$$

$$A(ABCD) = [ABD] + [CBD] = \frac{1}{2}(BD)(AX) + \frac{1}{2}(BD)(CY) = \frac{1}{2}(BD)(AX + CY)$$

Example 3.4
In quadrilateral ABCD, with diagonal BD, AX and CY are perpendicular to BD. Find the area of the quadrilateral if:

A. $BD = 13, AX = 5, CY = 4$
B. $BD = 12, AX = 4, CY = 7$



Part A

$$A(ABCD) = \frac{1}{2}(BD)(AX + CY) = \frac{1}{2}(12)(11) = 66$$

Part B

$$A(ABCD) = \frac{1}{2}(BD)(AX + CY) = \frac{1}{2}(13)(5+4) = \frac{117}{2}$$

Example 3.5

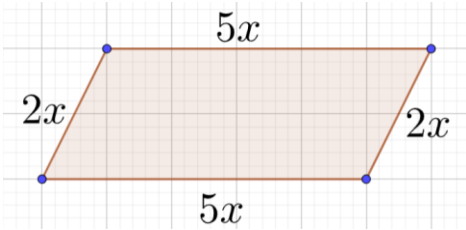
The perimeter of a parallelogram is 30 cm. If one of the sides measures 5 cm, find the length of the other three sides.

Example 3.6

The adjacent sides of a parallelogram are in the ratio 2: 5. If the perimeter of the parallelogram is 10, find the sides of the parallelogram.

$$2x + 5x + 2x + 5x = 10$$
$$14x = 10$$
$$x = \frac{10}{14} = \frac{5}{7}$$

$$2x = \frac{10}{7}, 5x = \frac{25}{7}$$



Example 3.7

A piece of wire 1 m long is used to make four squares of equal size. Each of the squares is then opened up and cut into four smaller, equal, squares. Find the side length of the smaller squares in m.

In all, we are making

$$4 \times = 16\ squares$$

Total length of wire

$$1\ m$$

Each square has

$$Perimeter = 1 \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{16} m \Rightarrow Side = \frac{1}{16} \times \frac{1}{4} = \frac{1}{64} m$$

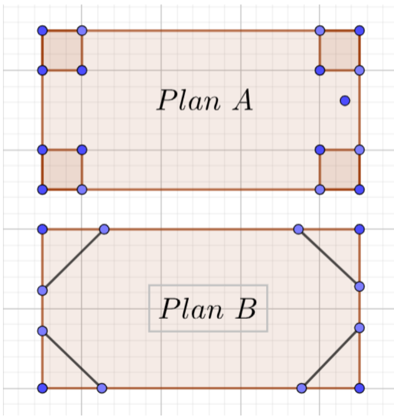
Example 3.8

A rectangular garden of dimensions $15m \times 25m$ has to have four flowerbeds added to its corners. The garden is currently all grass. Under plan

- A: The flowerbeds will be in the shape of rectangles of dimension $2m \times 2m$, planted with *germaniums*.
- B: The flowerbeds will be in the shape of isosceles right-angled triangles with length of leg $3m$, planted with *bougenville*.

Find under each plan:

A. the area of the remaining grass.
B. the cost of the flowers



Area of garden

$$= 15 \times 25 = 375\ m^2$$

Area of rectangular flowerbeds

$$= 4 \times 2 \times 2 = 16\ m^2$$

Area of triangular flowerbeds

$$= 4 \times \frac{1}{2} hb = 4 \times \frac{1}{2} \times 3 \times 3 = 18\ m^2$$

Part A

$$Plan\ A = 375 - 16 = 359\ m^2$$
$$Plan\ B = 375 - 18 = 357\ m^2$$

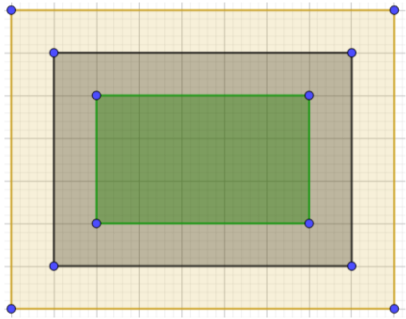
Part B

$$Plan\ A =$$

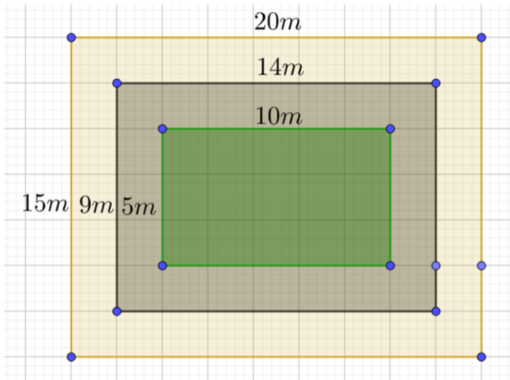
Example 3.9

In a rectangular park, the grass (green area) is surrounded by a walking track (2 $m\ width$, dark shaded area), which is surrounded by a jogging track (3 $m\ width$, light shaded area). The dimensions of the park (including both tracks) are $15m \times 20m$. Find the area of:

A. The jogging track
B. The walking track
C. The Grass



Area of jogging track
$$= (15 \times 20) - (9 \times 14) = 300 - 126 = 174$$
Area of walking track
$$14 \times 9 - 10 \times 5 = 126 - 50 = 76$$



Example 3.10

Rectangle ABCD has sides of length 3 and 4.

A. Find the length of the diagonal AC.
B. Find the length of the perpendicular from vertex B to the diagonal.

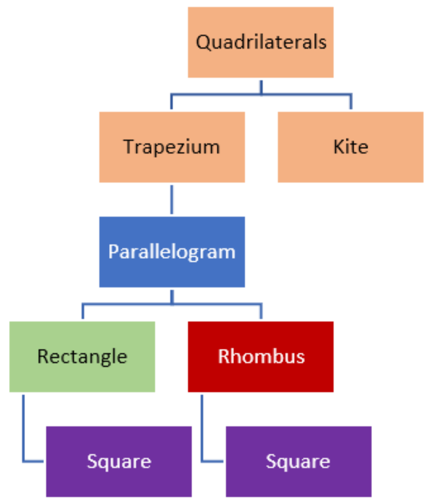
Similar Triangles

3.2 Trapezoids

A. Trapezoid

3.11: Definition

A quadrilateral with a pair of parallel sides is a trapezoid.



Example 3.12

Mark all correct options

Which of the following *must* be trapezoids:

A. Square
B. Rectangle
C. Rhombus
D. Kite
E. Parallelogram

A parallelogram is a quadrilateral with two pairs of parallel sides. Hence, it does have a pair of parallel sides.

Hence, a parallelogram is a special case of a trapezoid.

Squares, rectangles, and rhombuses are all special cases of parallelograms.
Hence, they are all trapezoids.

However, a kite is a quadrilateral that has two pairs of adjacent sides which are congruent. It may or may not have a pair of parallel sides.
Hence, it is not necessary that a kite is a trapezoid.

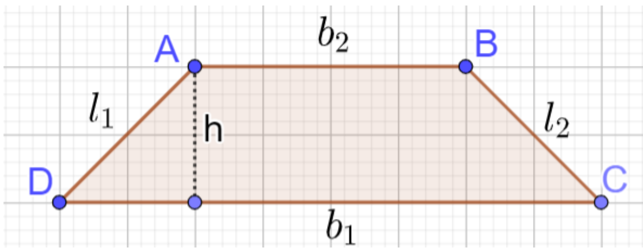
Hence, the correct answer is

$$\textit{Options } A, B, C, E$$

Example 3.13
What is the difference between a trapezoid and a trapezium?

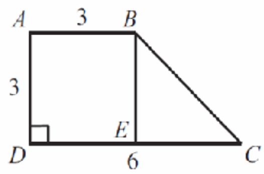
They are the same.

3.14: Terminology
- The two parallel sides are called the bases.
- The two other sides are called the legs.
- The distance between the bases is called the height.



Example 3.15
The quadrilateral drawn alongside is a trapezoid.
A. Identify the lines which are parallel to each other.
B. Identify the bases, the legs and the height.

$$\text{Line } AB \parallel \text{Line } CD$$
$$\text{Bases are } AB \text{ and } CD$$
$$\text{Legs are } AD \text{ and } BC$$
$$\text{Height} = h$$

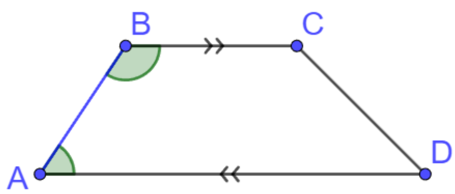


Example 3.167
In trapezoid $ABCD$, AD is perpendicular to DC, $AD = AB = 3$, and $DC = 6$. In addition, E is on DC, and BE is parallel to AD. Find the area of BEC. **(AMC 8 2007/8)**

$$BE = AD = 3$$
$$EC = DC - DE = DC - AB = 6 - 3 = 3$$

$$[BEC] = \frac{1}{2}hb = \frac{1}{2}(BE)(EC) = \frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2} = 4.5\ units^2$$

3.17: Angles on the same leg are supplementary
The two angles on the same leg of a trapezoid are supplementary



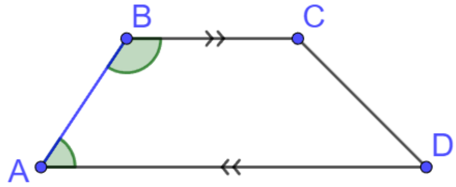
In trapezoid $ABCD$, let $BC \parallel AD$. Consider AB as the transversal of BC and AD.
By co-interior angles (angles on the same side of the transversal)

$$\angle CBA + \angle BAD = 180$$

Hence, the angles are supplementary.

Example 3.18

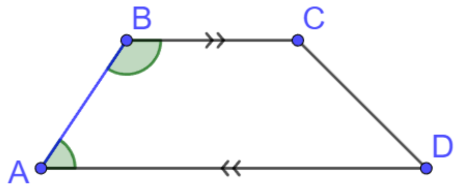
In trapezoid $ABCD$, $\angle A = 50°$. $\angle C = 110°$. Determine the measures of $\angle B$, and $\angle D$.



$$\angle B = 180 - \angle A = 180 - 50° = 130°$$
$$\angle D = 180 - \angle C = 180 - 110° = 70°$$

Example 3.19

In trapezoid $ABCD$, $\angle A = 40°$. $\angle C = \angle B + 10$. Determine the measure of $\angle D$.

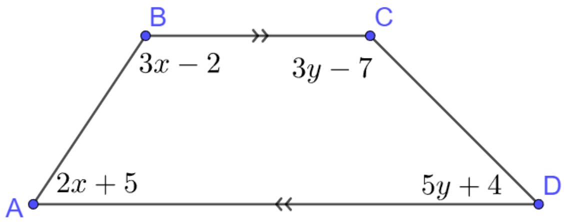


$$\angle B = 180 - \angle A = 180 - 40 = 140°$$
$$\angle C = \angle B + 10 = 140 + 10 = 150°$$
$$\angle D = 180 - \angle C = 180 - 150 = 30°$$

Example 3.20

In the quadrilateral drawn alongside, side $BC \parallel$ side AD. Determine the value of $x + y$ given that:

$$\angle CBA = 3x - 2$$
$$\angle BCD = 3y - 7$$
$$\angle BAD = 2x + 5$$
$$\angle CDA = 5y + 4$$

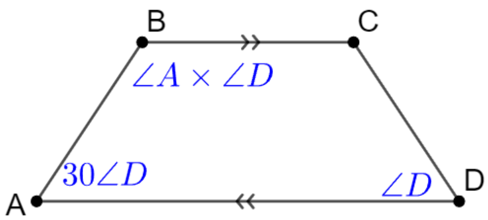


$$3x - 2 + 2x + 5 = 180 \Rightarrow 5x + 3 = 180 \Rightarrow x = \frac{177}{5}$$
$$3y - 7 + 5y + 4 = 180 \Rightarrow 8y - 3 = 180 \Rightarrow y = \frac{183}{8}$$

$$x + y = \frac{177}{5} + \frac{183}{8}$$

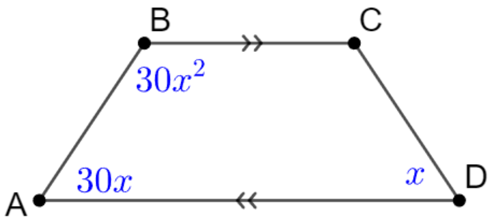
Example 3.21

Trapezoid $ABCD$ is shown in the diagram. Determine the ratio of $\angle C$ to $\angle D$ given that $\angle A$ is thirty times $\angle D$, and that $\angle B$ is $\angle A$ times $\angle D$.

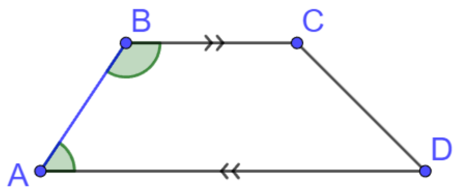


Let $\angle D = x$

$$30x^2 + 30x = 180$$
$$x^2 + x = 6$$
$$x^2 + x - 6 = 0$$
$$(x + 3)(x - 2) = 0$$
$$x \in \{2, -3\}$$
$$\angle D = 2$$



$$\angle C = 180 - 2 = 178$$
$$\angle C : \angle D = 178 : 2 = 89 : 1$$



3.22: Converse: Proving a Trapezoid

In quadrilateral $ABCD$, if $\angle A$ and $\angle B$ are supplementary, then the quadrilateral is a trapezoid.

Since the angles are supplementary:

$$\angle A + \angle B = 180$$

But, $\angle A$ and $\angle B$ are co-interior angles considering AB as the transversal of lines BC and AD.

$$\therefore AB \parallel CD$$

Hence,

$$Quadrilateral\ ABCD\ is\ a\ trapezoid$$

B. Perimeter

3.23: Perimeter

The sum of the lengths of the sides of a trapezoid is the perimeter.

Example 3.24

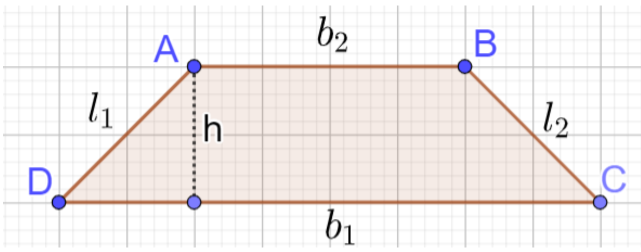
What is the perimeter of a trapezoid with bases 4 cm and 6 cm, and legs each 2 cm.

$$4 + 6 + 2 + 2 = 14$$

Example 3.25

Find the perimeter of the isosceles trapezoid drawn alongside (not to scale)

$$AB = \frac{1}{3}, \qquad CD = \frac{1}{2}, \qquad AD = \frac{1}{6}$$



$$AB + BC + CD + AD = \frac{1}{3} + \frac{1}{6} + \frac{1}{2} + \frac{1}{6} = \frac{2}{6} + \frac{1}{6} + \frac{3}{6} + \frac{1}{6} = \frac{7}{6} = 1\frac{1}{6}$$

C. Area

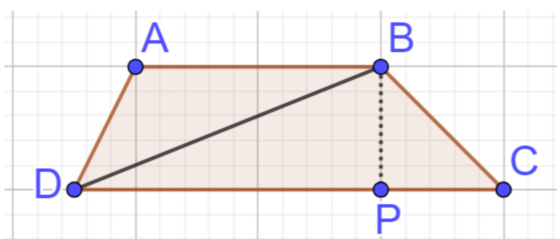
3.26: Area

The area of a trapezoid is given by:

$$Area = h \times \frac{b_1 + b_2}{2}$$

Where

$$h = Height$$
$$b_1 = Base\ 1$$
$$b_2 = Base\ 2$$



Consider Trapezoid $ABCD$ (drawn alongside). Draw

- Height BP
- Diagonal DB

The trapezoid is made of ΔABD and ΔBDC:

$$[ABCD] = [BDC] + [ABD] = \frac{1}{2}(BP)(CD) + \frac{1}{2}(BP)(AB) = \frac{1}{2}\underbrace{(BP)}_{Height}(CD + AB)$$

Example 3.27

Find the area of a trapezoid with height 3 cm, and bases 5 cm and 6 cm?

$$\frac{b_1 + b_2}{2} \times h = \frac{5+6}{2} \times 3 = \frac{11}{2} \times 3 = \frac{33}{2} = 16.5 \text{ cm}^2$$

Example 3.28

Find the area of a trapezoid with height 0.12 m, and bases $0.\overline{3}$ cm and $0.1\overline{6}$ cm

$$b_1 = 0.\overline{3} = 0.333333 \ldots = \frac{1}{3}\ cm$$
$$b_2 = 0.1\overline{6} = 0.16666 \ldots. = \frac{1}{6}\ cm$$
$$h = 0.12\ m = 12\ cm$$
$$Area = \frac{b_1 + b_2}{2} \times h = \frac{\frac{1}{3} + \frac{1}{6}}{2} \times 12 = \frac{3}{6} \times \frac{1}{2} \times 12 = 3 \text{ cm}^2$$

Example 3.29

Find the area of a trapezoid with height one-third of a foot, one base five-sixth of a foot, and the other base having $33\frac{1}{3}\%$ greater length than the first base.

$$h = \frac{1}{3} feet = \frac{1}{3} \times 12 = 4\ inches$$
$$b_1 = \frac{5}{6} feet = \frac{5}{6} \times 12 = 10\ inches$$

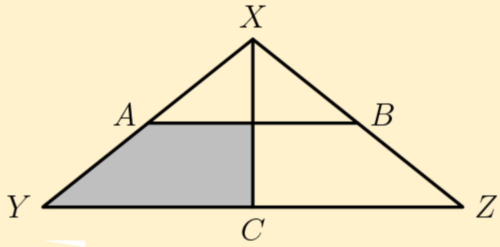
$$33\frac{1}{3}\% = \frac{100}{3}\% = \frac{100}{300} = \frac{1}{3}\ greater \Rightarrow b_2 = b_1 + \frac{1}{3} \times b_1 = b_1\left(1 + \frac{1}{3}\right) = b_1 \times \frac{4}{3} = 10 \times \frac{4}{3} = \frac{40}{3}$$

$$Areaa = \frac{b_1 + b_2}{2} \times h = \frac{10 + \frac{40}{3}}{2} \times 4 = \frac{70}{3} \times 2 = \frac{140}{3} \text{ inch}^2$$

D. Ratios

Example 3.30

The area of triangle XYZ is 8 square inches. Points A and B are midpoints of congruent segments $\overline{XY}$ and $\overline{XZ}$. Altitude $\overline{XC}$ bisects $\overline{YZ}$. The area (in square inches) of the shaded region is (AMC 8 2002/20)



Example 3.31

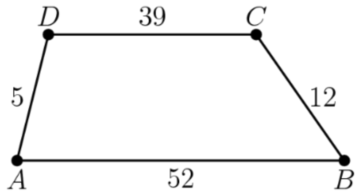
Trapezium ABCD has AB ∥ CD and area 18.

A. AB is half the length of CD. What is the ratio of the area of △ABD to that of the trapezium? Is it possible to answer this question without calculating the area of △ABD?
B. AB has length 6, and CD has length 8. What is Area(△ABD) : Area(△BCD)?
C. If AB has length p, and CD has length q, can you find Area(△ABD) : Area(△BCD)? Does your final answer increase or decrease if the area of the trapezium changes.

E. Rearrangement

Example 3.32

In trapezoid $ABCD$ with bases AB and CD, we have $AB = 52$, $BC = 12$, $CD = 39$, and $DA = 5$ (diagram not to scale). The area of $ABCD$ is **(AMC 10A 2002/25)**



Example 3.33

In rectangle $ABCD$, $AB = 6$, $AD = 30$, and G is the midpoint of $\overline{AD}$. Segment AB is extended 2 units beyond B to point E, and F is the intersection of $\overline{ED}$ and $\overline{BC}$. What is the area of $BFDG$? **(AMC 10B 2012/19)**

$$GD \parallel BF \Rightarrow BGDF \text{ is a trapezoid} \Rightarrow b_1 = GD = 15, h = 6$$

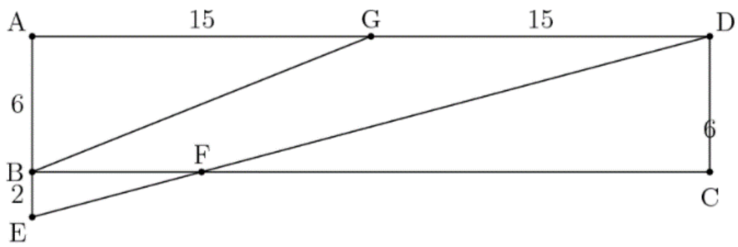
We need to find the other base, which is BF.
We can prove two triangles similar by AA Similarity:

$$\angle BAD = \angle EBF = 90°, \angle AED = \angle BEF \Rightarrow \Delta BEF \sim \Delta AED$$

W

$$\frac{BE}{AE} = \frac{2}{8} = \frac{1}{4} \Rightarrow \frac{BF}{AD} = \frac{1}{4} \Rightarrow BF = \frac{30}{4} = \frac{15}{2}$$

$$[BDFG] = \frac{15 + \frac{15}{2}}{2} \times 6 = \frac{45}{2} \times 3 = \frac{135}{2}$$



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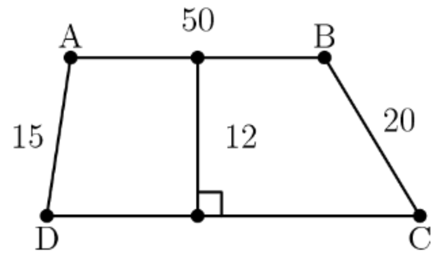
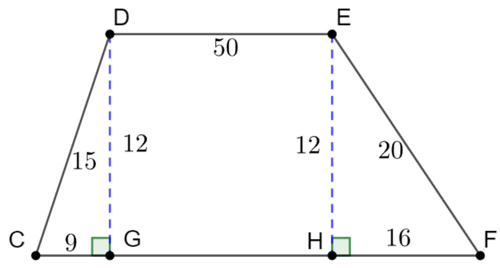
F. Pythagorean Theorem



Example 3.34

Quadrilateral $ABCD$ is a trapezoid, $AD = 15$, $AB = 50$, $BC = 20$, and the altitude is 12. What is the area of the trapezoid? **(AMC 8 2011/20)**



Draw $DG \perp CF$ and $EH \perp CF$. Since the distance between parallel lines is the same,

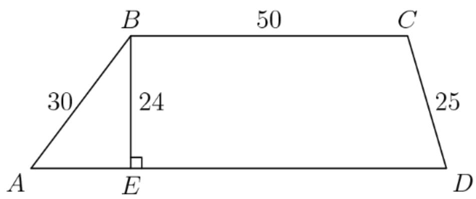
$$DG = EH = 12$$

By Pythagorean Triplets:

$$3(3,4,5) = (9,12,15) = (CG, DG, DC)$$
$$3(3,4,5) = (12,16,20) = (EH, HF, EF)$$

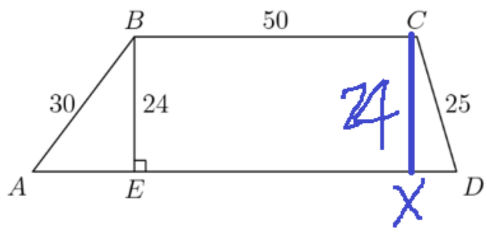
Then, the area of the trapezoid is:

$$[DCG] + [EHF] + [DEHG]$$
$$= \frac{1}{2}(12)(9) + \frac{1}{2}(12)(16) + (50)(12)$$
$$= 6(9) + 6(16) + (100)(6)$$
$$= 6(25) + 100$$
$$= 750$$



Example 3.35

What is the perimeter of trapezoid $ABCD$? **(AMC 8 2005/19)**



By Pythagorean Triplet:

$$6(3,4,5) = (18,24,30) = (AE, BE, AB) \Rightarrow AE = 18$$

By Pythagorean Triplet:

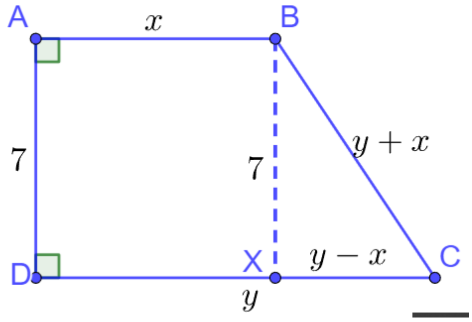
$$(7,24,25) = (XD, CX, CD) \Rightarrow XD = 7$$

The perimeter

$$= AB + BC + CD + DA = 30 + 50 + 25 + (7 + 50 + 18) = 180$$

Example 3.36

In trapezoid $ABCD$, $\overline{AB}$ and $\overline{CD}$ are perpendicular to $\overline{AD}$, with $AB + CD = BC$, $AB < CD$, and $AD = 7$. What is $AB \cdot CD$? **(AMC 10 2001/24)**



Let

$$AB = x, CD = y$$

Then

$$BC = AB + CD = x + y, \qquad AB \cdot CD = xy$$

Construct

$$BX \perp CD \quad \Rightarrow CX = y - x$$

By the Pythagorean Theorem in ΔBXC:

$$CX^2 + BX^2 = BC^2$$

$$(y-x)^2+7^2=(y+x)^2$$

Now, we have an equation, which we can simplify:

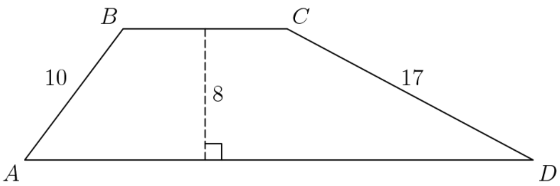
$$y^2-2xy+x^2+49=y^2+2xy+x^2$$

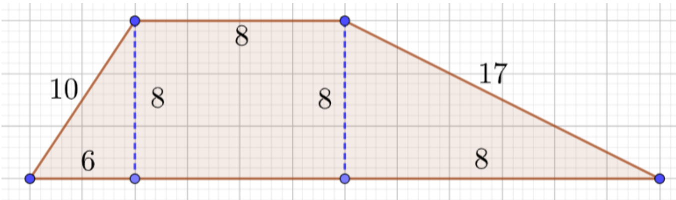
Simplify and collate like terms on each side:

$$4xy=49\Rightarrow xy=\frac{49}{4}=12\frac{1}{4}$$

Example 3.37

The area of trapezoid $ABCD$ is 164 cm^2. The altitude is 8 cm, AB is 10 cm, and CD is 17 cm. What is BC, in centimeters? **(AMC 8 2003/21)**





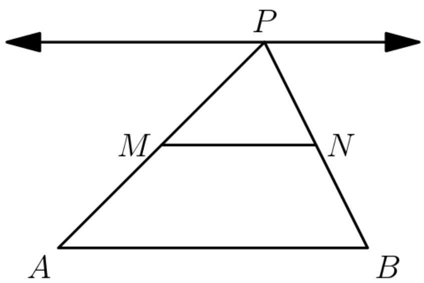
$$8\left(\frac{BC+AD}{2}\right)=164\Rightarrow BC+AD=41$$
$$BC+BC+6+15=41$$
$$BC=10\ cm$$

G. General

Example 3.38

Points M and N are the midpoints of sides PA and PB of ΔPAB. As P moves along a line that is parallel to side AB, how many of the four quantities listed below change? **(AMC 10 2000/5)**

(a) the length of the segment MN
(b) the perimeter of $\triangle PAB$
(c) the area of $\triangle PAB$
(d) the area of trapezoid $ABNM$



Analysis

Part A

Obviously, AB does not change in length.
Therefore, by the Midpoint Theorem, MN also does not change in length.

Part B

Consider an extreme scenario. P moves far to the left (or far to the right).
The perimeter will change.

Part C

$$A(\triangle PAB)=\frac{1}{2}hb$$

Neither the height, nor the base changes, so the area does not change.

Part D

$$A(Trapezoid\ ABNM)=\frac{AB+MN}{2}\times h$$

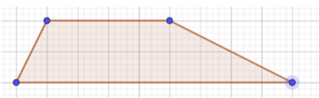
None of the above quantities change, so the area does not change.

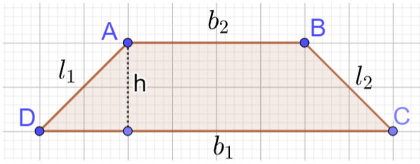
Final Answer
Out of the four given quantities, only one changes.

H. Isosceles Trapezoid

3.39: Isosceles Trapezoid
When the legs are a trapezoid are equal in length, it is an isosceles trapezoid.

Example 3.40
Decide whether each trapezoid drawn alongside is isosceles or not.



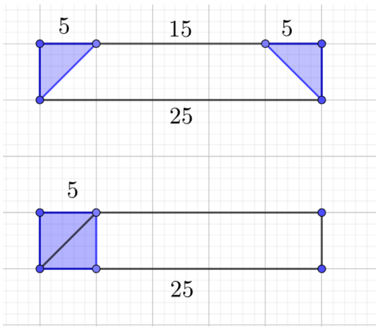


The one on the left is not Isosceles.
The one on the right is Isosceles.

Example 3.41
A rectangular yard contains two flower beds in the shape of congruent isosceles right triangles. The remainder of the yard has a trapezoidal shape, as shown. The parallel sides of the trapezoid have lengths 15 and 25 meters. What fraction of the yard is occupied by the flower beds? **(AMC 10B 2009/4)**

Draw a diagram of the rectangle.
The sides of the triangle are

$$\frac{25-15}{2}=\frac{10}{2}=5$$



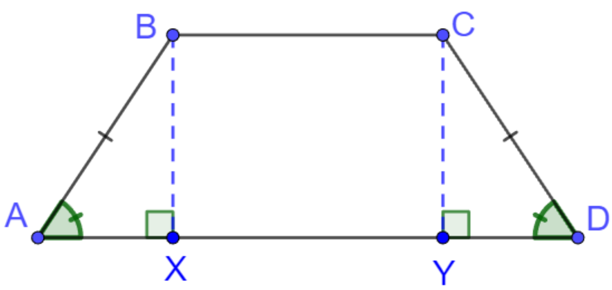
The right triangle can be moved to go to the left. The fraction of the yard occupied by the flower beds is

$$\frac{5}{25}=\frac{1}{5}$$

3.42: Isosceles Trapezoid
- The base angles of an isosceles trapezium are equal
- The "top angles" are also equal.

Base Angles
Draw the altitudes from B and C.

$$\angle BXA=\angle CYD=90°\ (Right\ Angle)$$



The distance between parallel lines is the same:

$$BX=CY\ (Side)$$

Since the trapezium is isosceles:

$$BA=CD\ (Hyp)$$

Combining the above three statements:

$$\Delta BXA = CYD\ (RHS)$$
$$\angle BAD = \angle CDA\ (CPCTC)$$

Top Angles

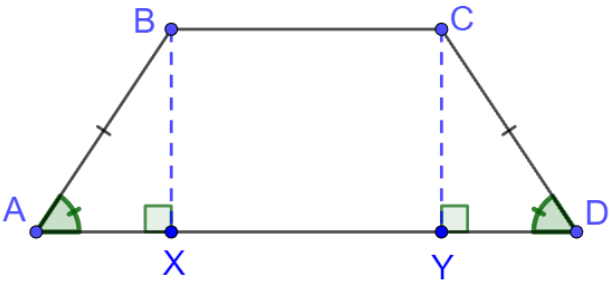
$$\angle ABX = \angle DCY (CPCTC)$$
$$\angle ABX + 90° = \angle DCY + 90°$$
$$\angle ABC = \angle DCB$$

3.43: Converse: Proving an Isosceles Trapezium

If the base angles of a trapezoid are congruent, the trapezoid is isosceles.

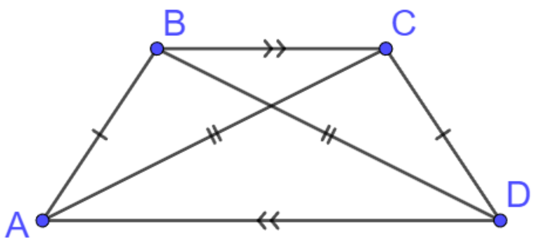
In trapezoid ABCD draw height BX and CY.
In ΔBAX and ΔCDY

$$\angle BAX = \angle CDY\ (Given)$$
$$BX = CY\ (Height\ is\ equal)$$
$$\angle BXA = \angle CYD = 90° (Def\ of\ Height)$$
$$\Delta BAX \cong \Delta CDY\ (SAA)$$
$$BA = CD\ (CPCTC)$$



3.44: Diagonals of an Isosceles Trapezium

The diagonals of an isosceles trapezium are congruent.



In ΔABC and ΔDCB:

$$AB = CD\ (Legs\ of\ isoceles\ trapezium)$$
$$BC = BC\ (Reflexive\ Property)$$
$$\angle ABC = \angle BCD\ (Top\ angles\ of\ isoceles\ trapezium)$$
$$\Delta ABC \cong \Delta DCB$$
$$BD \cong AC\ (CPCTC)$$

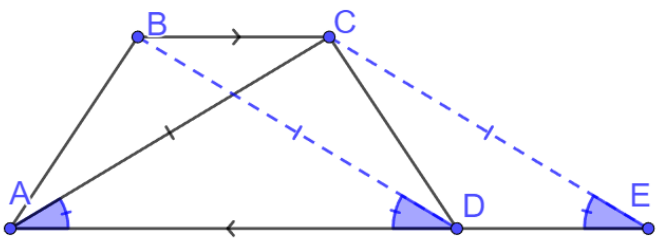
3.45: Converse

If the diagonals of a trapezoid are congruent, it is an isosceles trapezoid.

Consider trapezoid ABCD with $BC \parallel AD$. Construct $CE \parallel BD$, intersecting AD at E.

Consider BD and CE as transversals of BC and AD. Since they are corresponding angles:

$$\angle BDA = \angle CED$$



BD and CE are transversals to parallel lines BC and AD.
Hence,$BD = CE$. Since the diagonals are congruent, $AC = BD$. Combining the two statements:

$$AC = CE \Rightarrow \Delta\text{ACE is isoceles} \Rightarrow \angle CAD = \angle CEA$$

In ΔCAD and ΔBAD:

$$AC = AC\ (Side)$$
$$\angle CAD = \angle CEA\ (Angle)$$
$$CA = BD (Angle)\ (Given)$$
$$\Delta CAD \cong \Delta BAD (SAS\ Congruence)$$

$$AB \cong CD\ (CPCTC)$$

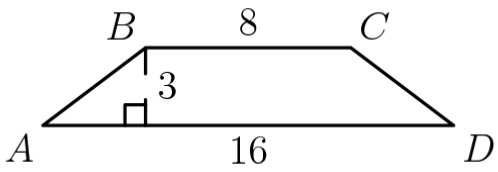
Example 3.46
What is the perimeter of an isosceles trapezium with height 12, shorter base 23, and longer base 32?

Longer side will have its extra length equally divided on both sides (isosceles trapezium)
Left-hand side triangle is right-angled. By Pythagoras Theorem:

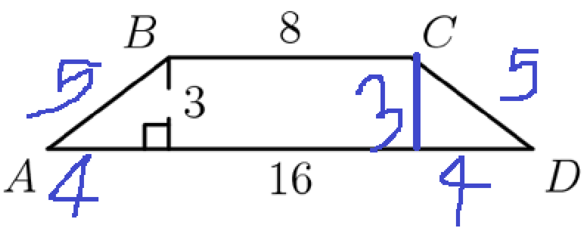
$$\text{Left Leg} = \text{Right Leg} = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$$

$$\text{Perimeter} = 23 + 32 + 13 * 2 = 81$$

Example 3.47
In trapezoid $ABCD$, the sides AB and CD are equal. The perimeter of $ABCD$ is (AMC 8 1999/14)



$$8 + 2(5) + 16 = 34\ units$$



I. Trapezoid Midsegment Theorem

3.48: Median of a Trapezoid
The median of a trapezoid is the line joining the midpoints of its two legs.

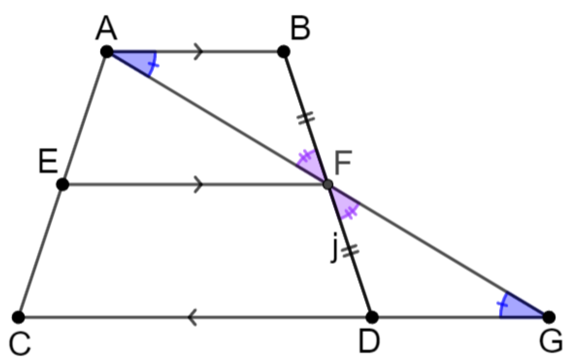
3.49: Median: Length and Parallel to Base Properties
The median of a trapezoid is parallel to its bases.

The length of the median of a trapezoid is the average of the two bases of the trapezoid.

$$Median = EF = \frac{AB + CD}{2}$$

Draw trapezoid $ABDC$ with $AB \parallel CD$.
Let E be the midpoint of AC, and F be the midpoint of BD.

Extend AF to meet CD at G.



$\Delta ABF \cong \Delta DFG$ by AAS since:

$$\angle BAF = \angle DGF\ (Alternate\ Interior\ Angles)$$
$$\angle AFB = \angle DFG\ (Vertically\ Opposite\ Angles)$$
$$BF = FD\ (Midpoint\ Definition)$$
$$AF = FG\ (CPCT) \Rightarrow F\ is\ midpoint\ of\ AG$$

By Midpoint Theorem: $EF \parallel CD$. Hence:

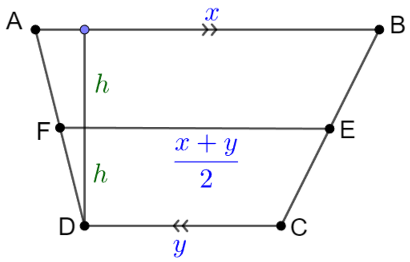
$$EF = \frac{CG}{2} = \frac{CD + DG}{2}$$

Substitute $DG = AG$ $(CPCT \text{ in } \Delta ABF \cong \Delta DFG)$

$$= \frac{CD + AB}{2}$$

Example 3.50

In trapezoid $ABCD$ we have $\overline{AB}$ parallel to $\overline{DC}$, E as the midpoint of $\overline{BC}$, and F as the midpoint of $\overline{DA}$. The area of $ABEF$ is twice the area of $FECD$. What is $\frac{AB}{DC}$? **(AMC 10B 2005/23)**



$$[ABEF] = 2[FECD]$$

Using $Area\ of\ trapezoid = \frac{1}{2}h(b_1 + b_2)$

$$\frac{1}{2}h\left[x + \left(\frac{x+y}{2}\right)\right] = 2\left[\frac{1}{2}h\left[\left(\frac{x+y}{2}\right) + y\right]\right]$$

$$x + \left(\frac{x+y}{2}\right) = 2\left[\left(\frac{x+y}{2}\right) + y\right]$$

$$\frac{3x+y}{2} = 2\left[\frac{x+3y}{2}\right]$$

$$3x + y = 2x + 6y$$

$$x = 5y$$

$$AB = 5DC$$

$$\frac{AB}{DC} = 5$$

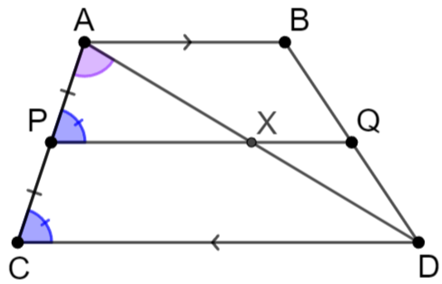
3.51: Median: Midpoint of Diagonals

The median of a trapezoid passes through the midpoints of its diagonals.
In the diagram,

$$PQ\ passes\ through\ Y\ and\ X$$

Where

$$Y\ is\ midpoint\ of\ CB$$
$$X\ is\ midpoint\ of\ AD$$



Draw trapezoid $ABDC$ with median:

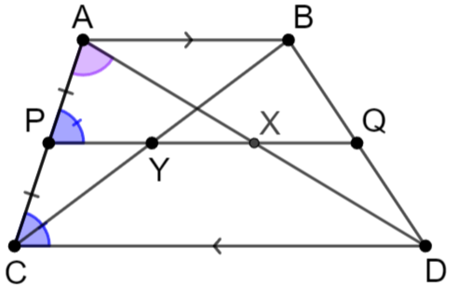
$$PQ \parallel AB \parallel CD$$

Since a parallel line divides the sides of a triangle in the same ratio:

$$\frac{AX}{XD} = \frac{AP}{PC} = 1$$

Hence,

$$X\ is\ the\ midpoint\ of\ Diagonal\ AD$$



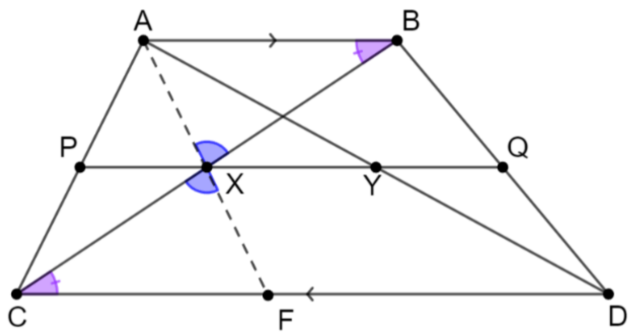
That PQ passes through the midpoint Y of the other diagonal can be proved similarly.

3.52: Midpoint of Diagonals

The line joining the mid points of the diagonals of a trapezoid is equal to half of the difference of the parallel sides.

In the diagram:

$$XY = \frac{1}{2}(CD - AB)$$



Draw trapezoid $ABDC$ with midpoints of diagonals X and Y as shown.
Let P and Q be the midpoints of AC and BD respectively.
From above, we know that the median PQ passes through X and Y.

Construction:
Extend AX till it intersects CD at F.

$\Delta BXA \cong \Delta FXC$ by ASA:

$$\angle ABC = \angle BCD \ (Alternate\ Interior\ Angles)$$
$$\angle BXA = \angle FXC \ (Vertically\ Opposite\ Angles)$$
$$BX = XC \ (X\ is\ midpoint\ of\ diagonal\ BC)$$

Hence, by CPCT:

$$\underbrace{AX = XF}_{Equation\ I}, \qquad \underbrace{CF = AB}_{Equation\ II}$$

From the first equality above, X and Y are midpoints of AF and AD respectively.
Also $XY \parallel FD$ since $PQ \parallel CD$.
Therefore, by Midpoint Theorem:

$$XY = \frac{1}{2}FD$$

By complementary lengths, $FD = CD - CF$

$$XY = \frac{1}{2}(CD - CF)$$

From Equation II, $CF = AB$

$$XY = \frac{1}{2}(CD - AB)$$

Which is what we wanted to prove.

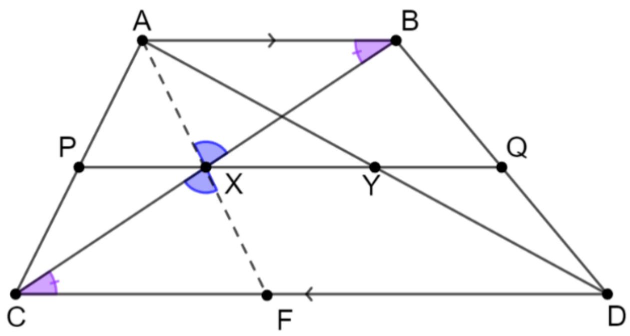
Example 3.53

The line joining the midpoints of the diagonals of a trapezoid has length 3. If the longer base is 97, then the shorter base is: (AHSME 1959/22)

$$XY = \frac{1}{2}(CD - AB)$$

Substitute $XY = 3, CD = 97$

$$3 = \frac{1}{2}(97 - AB)$$
$$6 = 97 - AB$$
$$AB = 97 - 6 = 91$$



3.3 Parallelograms

A. Definition

3.54: (Def) Parallelogram

A parallelogram is a quadrilateral with opposite sides parallel.

- Special cases of parallelograms include squares, rectangles, and rhombi.

3.55: Special Trapezoid

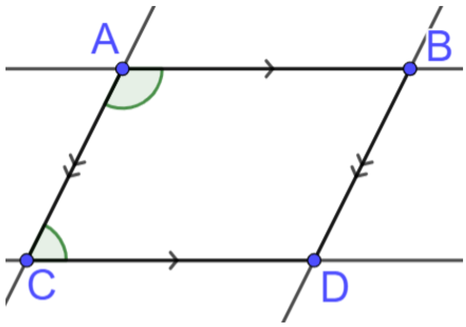
A parallelogram is a special case of a trapezoid, since a trapezoid has at least one pair of parallel sides, and a parallelogram has both pairs of sides parallel.

3.56: Properties of a Trapezoid/Quadrilateral.

Since a parallelogram is a

- special trapezoid, it inherits all the properties of a trapezoid.
- special trapezoid, it inherits all the properties of a quadrilateral.

B. Angles in a Parallelogram



Example 3.57

In parallelogram $ABCD$, shown alongside, identify the pairs of co-interior angles from the interior angles of the parallelogram.

$AB \parallel CB$ hence

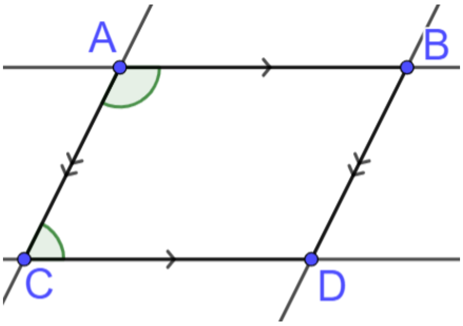
$$\angle BAC \ \& \ \angle ACD \ are \ co - interior$$
$$\angle ABD \ \& \ \angle CDB \ are \ co - interior$$

$AC \parallel BD$ hence

$$\angle BAC \ \& \ \angle ABD \ are \ co - interior$$
$$\angle ACD \ \& \ \angle CDB \ are \ co - interior$$

3.58: Adjacent and Opposite Angles

- Adjacent Angles of a parallelogram are supplementary.
- Opposite Angles of a parallelogram are congruent



Consider parallelogram $ABCD$

Adjacent Angles:
Since $AB \parallel CD$, $\angle C$ and $\angle A$ are co-interior angles:

$$\therefore \angle C \text{ and } \angle A \text{ are supplementary} \Rightarrow \angle A = 180 - \angle C$$

We can make a similar argument for:

$$\angle C \,\&\, \angle D, \qquad \angle D \,\&\, \angle B, \qquad \angle B \,\&\, \angle A$$

Opposite Angles:
Since $AD \parallel BC$, $\angle B$ and $\angle A$ are co-interior angles:

$$\therefore \angle B \text{ and } \angle A \text{ are supplementary}$$
$$\therefore \angle B = 180 - \angle A = 180 - (180 - \angle C) = \angle C$$

We can make a similar argument for:

$$\angle A \,\&\, \angle D$$

C. Height and Area

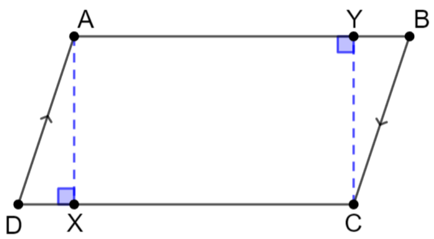
3.59: (Def) Height of a Parallelogram
The height of a parallelogram is the distance between the opposite sides.

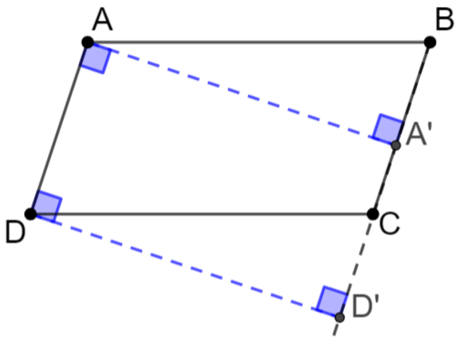
- The height is often convenient to calculate when originating from a vertex, but this is not required.

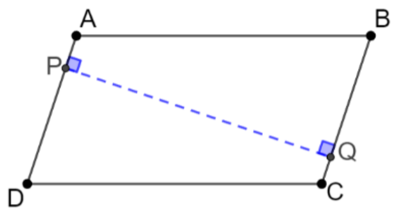
Example 3.60
In parallelogram $ABCD$, draw:
C. the height from AB to CD
D. the height from AC to BD

In parallelogram ABCD, the height from AB to CD

$$= Distance\ between\ AB\ and\ CD = AX = YC$$







The height from AD to BC is a little more difficult to visualize:

$$= Distance\ between\ AD\ and\ BC = AA' = DD' = PQ$$

Note that

$$PQ\ does\ not\ originate\ from\ a\ vertex$$

3.61: Area of a Parallelogram

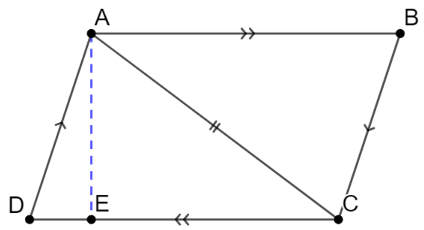
The area of a parallelogram is the product of its height and base.

$$A = hb$$

Method I: Congruence of Triangles

The diagonal of a parallelogram divides it into two congruent triangles. Hence:

$$[ABCD] = [ADC] + [ABC] = 2(A\Delta ADC) = 2\left(\frac{1}{2}hb\right) = (AE)(DC)$$



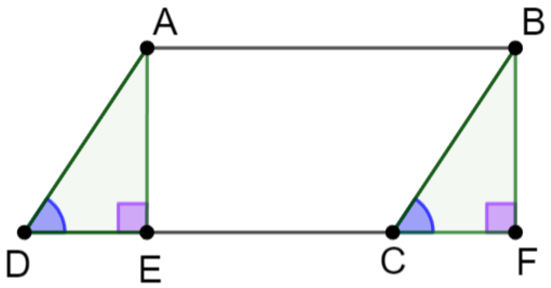
Method II: Comparing to a Rectangle

In parallelogram $ABCD$, create rectangle $ABFE$ by drawing

$$AE \perp CD, BF \perp CD$$

$\Delta DAE \cong \Delta CBF$ by AAS:

$$DA = CB\ (Opp.\ sides\ of \parallel -gram)$$
$$\angle ADE = \angle BCF\ (Corresponding\ Angles)$$
$$\angle AED = \angle BFC = 90°$$



Hence:

$$[ABCD] = [ABCE] + [DAE] = [ABCE] + [CBF] = [ABFE] = AB \cdot AE$$

Method III: Cutting and Moving

In parallelogram $ABCD$, draw

$$AE \perp CD$$

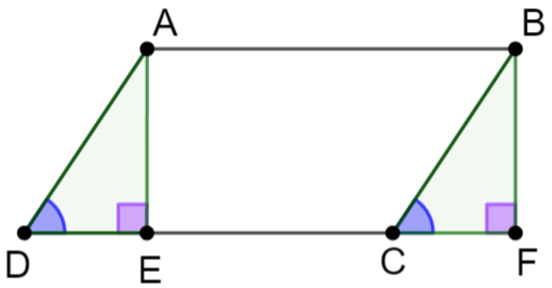
Translate ΔADE right till AD is located over BC, giving:

$$\Delta BCF$$

$$AE \perp CD \Rightarrow BF \perp CD \Rightarrow ABCF\ is\ a\ rectangle$$

Hence:

$$[ABCD] = [ABFE] = AB \cdot AE$$



Method IV: Special Case of Trapezoid

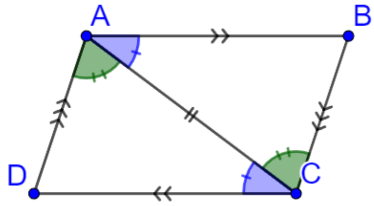
A parallelogram is a special case of a trapezoid. Hence, use the formula for area of a trapezoid with $b = b_1 = b_2$

$$A = \frac{b_1 + b_2}{2}h = \frac{b + b}{2}h = hb$$

D. Diagonals

Example 3.62

Diagonal AC is drawn in parallelogram $ABCD$. Identify pairs of congruent angles. Identify pairs of congruent angles.



Since the opposite angles of a parallelogram are congruent:

$$\angle ABC = \angle ADC, \angle BAC = \angle DCB \; (Opp.\ angles\ of \parallel -gram)$$

Consider parallel lines AD and BC. AC is the transversal. Hence:

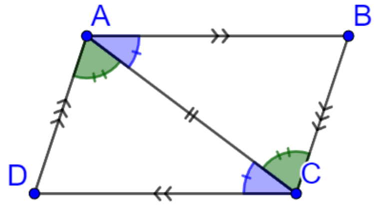
$$\angle DAC = \angle ACB \; (Alternate\ Interior\ Angles)$$

Consider parallel lines AB and CD. AC is the transversal. Hence:

$$\angle ACD = \angle BAC \;\; (Alternate\ Interior\ Angles)$$

3.63: Triangles formed by Diagonals / Opp sides are ≅

- A diagonal of a parallelogram divides it into two congruent triangles.
- Hence, opposite sides of a parallelogram are congruent.



$\Delta ABC \cong \Delta ADC$ by ASA:

$$\angle DAC = \angle ACB, \qquad \angle ACD = \angle BAC \; (Previous\ Example)$$
$$AC = AC \; (Reflexive)$$
$$AB = CD \; (CPCT)$$

Similarly, we can show

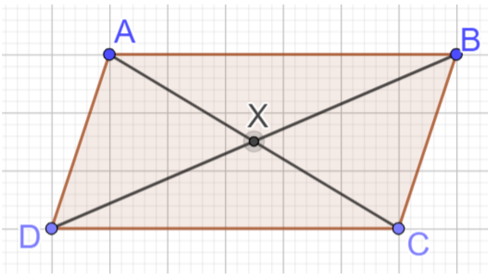
$$AD = BC\ in \cong \Delta's\ ADB\ \&\ CDB$$

3.64: Diagonals Bisect

The diagonals of a parallelogram bisect each other.

That is, if X is the intersection point of the diagonals, then:

$$X\ is\ the\ midpoint\ of\ AC$$
$$X\ is\ the\ midpoint\ of\ BD$$



In parallelogram $ABCD$:
$\Delta AXD \cong \Delta BXC$ by ASA:

$$AD = BC \; (Opp\ sides\ of\ a \parallel gram)$$
$$AD \parallel BC \Rightarrow AC\ is\ transversal \Rightarrow \; Alternate\ Interior: \angle DAC = \angle ACB$$
$$AB \parallel CD \Rightarrow AC\ is\ transversal \Rightarrow Alternate\ Interior: \angle ADB = \angle DBC$$
$$XD = XB, \qquad XA = XC \; \text{(CPCT)}$$

E. Proving a Quadrilateral to be a Parallelogram

3.65: Opposite Sides are parallel

If the opposite sides of a quadrilateral are parallel, the quadrilateral is a parallelogram.

This is the definition of a parallelogram.

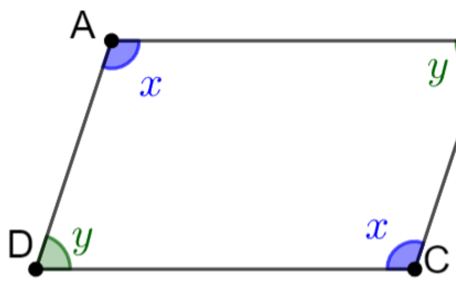
3.66: Opposite Angles are congruent

If the opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

In quadrilateral $ABCD$, if

$$\angle A = \angle C, \angle B = \angle D \Rightarrow ABCD\ is\ a\ parallelogram$$

Given: Opposite angles of a quadrilateral are congruent.
To be proved: The quadrilateral is a parallelogram.



Draw quadrilateral $ABCD$ with opposite angles congruent.

Step I: Establish a relation between the angles

By the sum of the angles of a quadrilateral:

$$2x + 2y = 360°$$
$$x + y = 180°$$

Step II: Show that $AC \parallel BD$

$$\angle BAD + \angle ADC = 180° \Rightarrow Angles\ are\ supplementary$$

$\angle BAC$ and $\angle ADC$ are co-interior angles with transversal AD of lines AB and CD. If co-interior angles are supplementary, then the lines are parallel:

$$\therefore AC \parallel BD$$

Step III: Show that $AD \parallel BC$

$$\angle DAB + \angle ABC = 180° \Rightarrow Angles\ are\ supplementary$$

$\angle DAB$ and ∠ABC are co-interior angles with transversal AB of lines AD and BC. If co-interior angles are supplementary, then the lines are parallel:

$$\therefore AD \parallel BC$$

Since the quadrilateral has both pairs of sides parallel, by definition

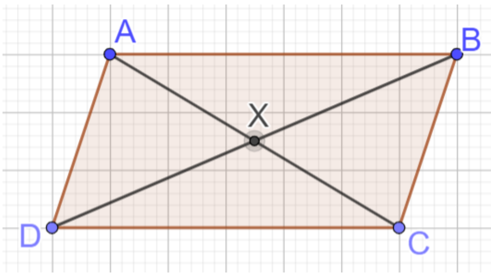
$$It\ is\ a\ parallelogram$$

3.67: Diagonals Bisect

If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.

That is, in quadrilateral $ABCD$, if X is the intersection point of the diagonals and

$$XA = XB, XB = XD \Rightarrow ABCD\ is\ a\ parallelogram$$



Given: The diagonals of a quadrilateral bisect each other.
To be proved: The quadrilateral is a parallelogram.

In quadrilateral $ABCD$:
$\Delta AXD \cong \Delta BXC$ by SAS:

$$XD = XB, \qquad XA = XC\ (Given)$$
$$\angle AXD = \angle BXC\ (Vertically\ opposite\ angles)$$
$$AD = BC\ (CPCT)$$

$\Delta AXB \cong \Delta DXC$ by SAS:

$$XD = XB, \qquad XA = XC\ (Given)$$
$$\angle AXB = \angle DXC\ (Vertically\ opposite\ angles)$$

$$AB = DC\ (CPCT)$$

Hence, the opposite sides of quadrilateral $ABCD$ are congruent.
Hence, the quadrilateral is a parallelogram.

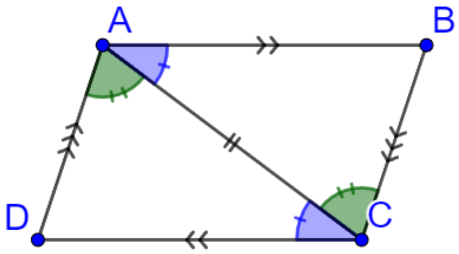
3.68: Opposite Sides are congruent

If the opposite sides of a quadrilateral are congruent, the quadrilateral is a parallelogram.

That is, in quadrilateral $ABCD$

$$AD = BC, \qquad AB = CD \Rightarrow ABCD\ is\ a\ parallelogram$$

Given: Opposite sides of a quadrilateral are equal
To be proved: Quadrilateral is a parallelogram



$\Delta ABC \cong \Delta ADC$ by SSS:

$$AD = BC, \qquad AB = CD\ \ (Given)$$
$$AC = AC\ (Reflexive)$$
$$\angle DAC = \angle ABC, \qquad \angle DCA = \angle BAC\ (CPCT)$$

Consider AC as transversal of AD and CB:

$$Alternate\ Interior\ Angles\ (DAC\ \&\ ACB)\ are \cong \Rightarrow AD \parallel CB$$

Consider AC as transversal of AB and CD:

$$Alternate\ Interior\ Angles\ (DCA\ \&\ BAC)\ are \cong \Rightarrow AB \parallel CD$$

In Quadrilateral ABCD:

$$AD \parallel CB, AB \parallel CD \Rightarrow Opp. Sides\ are\ parallel \Rightarrow Parallelogram$$

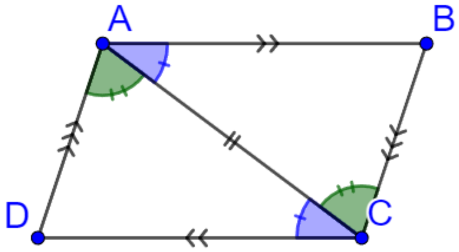
3.69: One pair of opposite sides is congruent and parallel

If one pair of opposite sides of a quadrilateral is congruent and parallel, the quadrilateral is a parallelogram.

That is, in quadrilateral $ABCD$

$$AB = CD\ \&\ AB \parallel CD \Rightarrow ABCD\ is\ a\ parallelogram$$

Given: One pair of opposite sides of a quadrilateral is congruent and parallel.
To be proved: The quadrilateral is a parallelogram.



$\Delta ABC \cong \Delta ADC$ by SAS:

$$AB = CD\ \ (Given)$$
$$\angle BAC = \angle DCA\ (Alternate\ Interior\ Angles)$$
$$AC = AC\ (Reflexive)$$

Hence, in the given quadrilateral:

$$AB = CD\ (Given), AD = BC\ (CPCT)$$

Hence, both pairs of opposite sides are congruent.
As shown above, if both pairs of opposite sides are congruent, then the quadrilateral is a parallelogram.

F. Joining Midpoints

We can use the methods that we have learnt of proving a parallelogram to prove further properties of parallelograms.

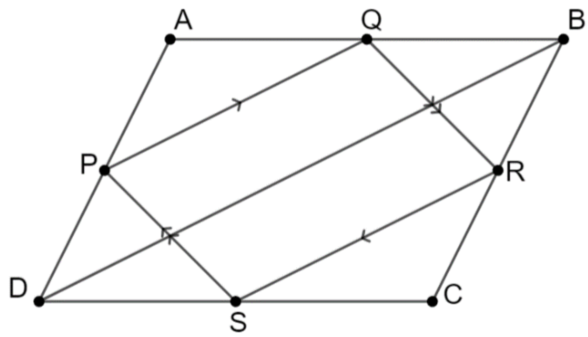
3.70: Joining Midpoints gives a Parallelogram

The quadrilateral formed by joining the midpoints of a parallelogram *in order* is also a parallelogram.

That is, in parallelogram $ABCD$ join the midpoints to get quadrilateral $PQRS$. Then:

$$PQRS \text{ is a parallelogram}$$

Given: A quadrilateral is formed by joining the midpoints of a parallelogram.
To be proved: The quadrilateral by joining the midpoints is a parallelogram.



Draw diagonal DB.
P is midpoint of AD, and Q is midpoint of AB. Hence, by Midpoint Theorem, in ΔADBL

$$\underbrace{PQ \parallel DB, \qquad PQ = \frac{1}{2}DB}_{\textit{Result 1}}$$

R is midpoint of BC, and S is midpoint of CD. Hence, by Midpoint Theorem, in ΔCBD:

$$\underbrace{SR \parallel DB, \qquad SR = \frac{1}{2}DB}_{\textit{Result 2}}$$

From Result 1 and 2:

$$\underbrace{PQ \parallel SR, \qquad PQ = SR}_{\textit{Result 3}}$$

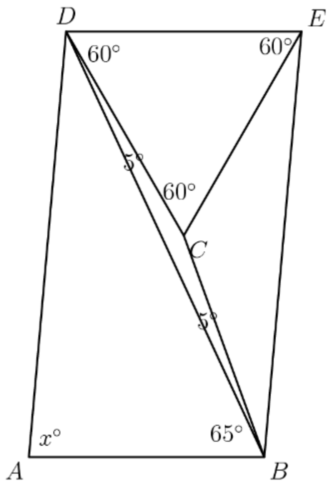
If one pair of opposite sides in a quadrilateral is parallel and congruent, the quadrilateral is a parallelogram. Hence:

$$PQRS \text{ is a parallelogram}$$

G. Constructing a Parallelogram

Example 3.71

Quadrilateral $ABCD$ has $AB = BC = CD$, angle $ABC = 70$ and angle $BCD = 170$. What is the measure of angle BAD? **(AMC 10B 2008/24)**



Draw

- Diagonal DB
- Line DE congruent to DC and parallel to AB to create parallelogram ABED

In Isosceles ΔDCB

$$\angle CDB = \angle CBD = \frac{180 - \angle DCB}{2} = \frac{180 - 170}{2} = 5$$
$$\therefore \angle ABD = \angle ABC - \angle CBD = 70 - 5 = 65$$

In parallelogram $ABED$:

$$\angle BDE = 65 \ (Alternate\ Interior\ Angles)$$
$$\angle EDC = \angle BDE - \angle BDC = 65 - 5 = 60$$

In Isosceles ΔDCB

$$\angle EDC = 60 \Rightarrow \angle DEC = \angle DCE = 60° \Rightarrow \Delta ECD\ is\ equilateral.$$

In parallelogram $ABED$, using parallelogram properties:

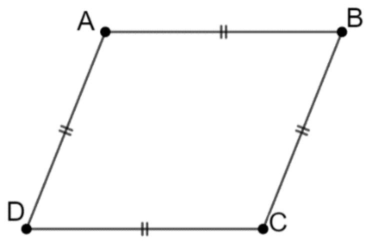
$$\angle BAD = \angle DEB = x$$
$$\angle CEB = x - 60$$
$$\angle ABE = 180 - x$$
$$\angle CBE = \angle ABE - \angle ABC = 110 - x$$

Since ΔECB is isosceles:

$$x - 60 = 110 - x \Rightarrow x = 85$$

3.4 Rhombi

A. Definition



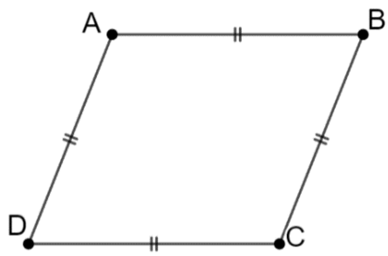
3.72: Equilateral Quadrilateral

- A rhombus is an *equilateral* quadrilateral.
- Equilateral means all the sides are equal.

- Plural of rhombus is rhombuses or rhombi

3.73: A rhombus is also a parallelogram/kite

- Every rhombus is a parallelogram.
- Every rhombus is a parallelogram.



Step I: Show that a rhombus is a parallelogram.

In Rhombus $ABCD$

$$AB = CD, \qquad AD = BC$$

Since the opposite sides of the quadrilateral are equal:

$$It\ is\ a\ parallelogram$$

Step I: Show that a rhombus is a kite.

$$AB = AD, \qquad CB = CD$$

Since two pairs of adjacent sides are equal, by definition:

$$It\ is\ a\ Kite$$

Example 3.74

$$Mark\ all\ correct\ options$$

It is true that every:

A. rhombus is a parallelogram.
B. parallelogram is a rhombus.
C. rhombus is a quadrilateral
D. rhombus is a trapezoid
E. trapezoid is a rhombus

$$Options\ A, C, D$$

B. Diagonals

1 Pending

3.75: Diagonals of a Rhombus
The diagonals of a rhombus are perpendicular to each other.

2 Pending

3.76: Diagonals of a Rhombus
The diagonals of a rhombus bisect the vertex angles.

3 Pending

3.77: Diagonals of a Rhombus
The diagonals of a rhombus divide it into four congruent triangles.

4 Pending

3.78: Diagonals of a Rhombus
The diagonals of a rhombus are perpendicular bisectors of each other

C. Proving a Rhombus from a Parallelogram

5 Pending

3.79: Parallelogram with perpendicular diagonals
If the diagonals of a parallelogram are perpendicular to each other, the parallelogram is a rhombus.

6 Pending

3.80: Parallelogram with Diagonal that bisects a vertex angle
If the diagonal of a parallelogram bisects a vertex angle, the parallelogram is a rhombus.

D. Midpoints

7 Pending

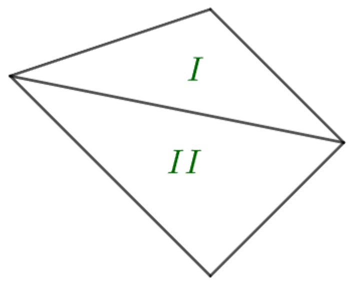
3.81: Midpoints form a Rectangle
The quadrilateral formed by joining the midpoints of the sides of a rhombus in order form a rectangle. And this rectangle has area half of the rhombus.

3.5 General Quadrilaterals

A. Angles in a Quadrilateral

3.82: Sum of Angles
The sum of angles of a quadrilateral is

$$360°$$



A quadrilateral can be divided into two triangles which each have sum $180°$. Hence, the total of the angles of a quadrilateral is

$$180 \times 2 = 360°$$

Example 3.83
The angles of a quadrilateral are in the ratio $1: 2: 3: 4$. Determine the values of the angles.

$$1 + 2 + 3 + 4 = 10$$
$$\frac{360}{10} = 36$$
$$36 \times 1 = 36$$
$$36 \times 2 = 72$$
$$36 \times 3 = 108$$
$$36 \times 4 = 144$$

Example 3.84
The largest angle in a quadrilateral is $30°$ more than the smallest angle. The other angles each are right angles. Determine the values of the angles.

Suppose the smallest angle is x. Then the largest is $x + 30$

$$x + 90 + 90 + x + 30 = 360$$
$$2x = 360 - 210$$
$$2x = 150$$
$$x = \frac{150}{2} = 75$$

$$x + 30 = 105$$

$$\{75,90,90,105\}$$

B. Perimeter

Example 3.85

A. What is the perimeter (in feet) of a regular pentagon with side 6 inches?
B. What is the perimeter (in inches) of a regular pentagon with side 6 feet?
C. A pentagonal field with each side five meters is to be fenced using fencing that costs five dollars per meter. Fenceposts that cost five dollars each are to be placed at a distance one meter apart. What is the total cost?

Side * 6 = 6 * 6 = 36 inches = 3 feet
Side * 5 = 6 * 5 = 30 feet = 360 inches feet
Cost = Fencing + Fenceposts = 5 * 5 * 5 + 5 * 5 * 5 = 125 + 125 = 250

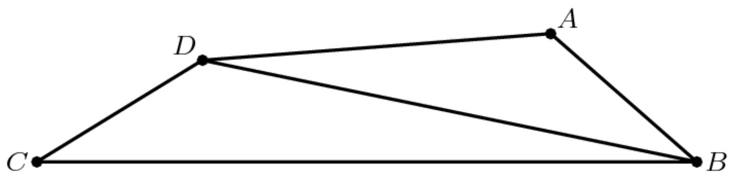
C. Triangle Inequality

3.86: Triangle Inequality

Sum of lengths of any two sides of a triangle is greater than the third side.

Example 3.87

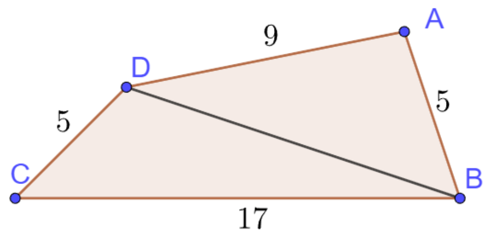
In quadrilateral $ABCD$, $AB = 5$, $BC = 17$, $CD = 5$, $DA = 9$, and BD is an integer. What is BD? **(AMC 10A 2009/12; AMC 12A 2009/10)**



In ΔADB, by the triangle inequality:

$$BD < DA + AB = 9 + 5 = 14$$

$$\underbrace{BD < 14}_{\text{Inequality I}}$$



In ΔCDB, by the triangle inequality:

$$BD + CD > BC$$

$$BD + 5 > 17$$

$$\underbrace{BD > 12}_{\text{Inequality II}}$$

Combining Inequalities I and II:

$$12 < BD < 14$$

Since BD is an integer:

$$BD = 13$$

D. Quadrilateral Inequality

3.88: Quadrilateral Inequality

Sum of lengths of any three sides of a quadrilateral is greater than the fourth side

Example 3.89
Joy has 30 thin rods, one each of every integer length from 1 cm through 30 cm. She places the rods with lengths 3 cm, 7 cm, and 15 cm on a table. She then wants to choose a fourth rod that she can put with these three to form a quadrilateral with positive area. How many of the remaining rods can she choose as the fourth rod?
(AMC 10A 2017/10)

Since the quadrilateral must have positive area, it cannot be a degenerate quadrilateral. That is, the rods should form a "proper quadrilateral."

Let the length of the fourth rod be r. By the quadrilateral inequality:
$$r < 3 + 7 + 15$$
$$r < 25$$

The longest(given) side must be less than the sum of the other three sides:
$$15 < r + 3 + 7$$
$$15 < r + 10$$
$$r > 5$$

$$5 < r < 25$$
$$\{6, \mathbf{7}, 8,9,10,11,12,13,14, \mathbf{15}, 16,17,18,19,20,21,22,23,24\} \Rightarrow 19\ rods$$

Remove 7 and 15 since those rods have already been used:
$$19 - 2 = 17\ rods$$

3.6 Further Topics

90 Examples