Ratio, Proportion and Variation

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1. Ratio and Proportion

1.1 Ratios as Fractions, Percents and Decimals

A. Definition and Terminology

1.1: Ratio

If for every *three* units of a first type, there are *five* units of a second type, the ratio between the two is:

$$\underbrace{3}_{antecedent} : \underbrace{5}_{consequent}$$

- The first term (3) is called the *antecedent*
- The second term(5) is called the *consequent*

- If the unit is different, then we need to mention the unit.
- In general, a ratio is a unitless quantity.

Example 1.2

I have five dolls and three cars.

A. What is the ratio of dolls to cars?
B. What is the ratio of cars to dolls?

Ratio of dolls to cars is:

$$\underbrace{5}_{Dolls} : \underbrace{3}_{Cars}$$

Ratio of cars to dolls is:

$$\underbrace{3}_{Cars} : \underbrace{5}_{Dolls}$$

Since the unit is different, we have to mention the unit.

Example 1.3

I have three action movies, five romantic movies, and four comedies that I really like. What is the ratio of:

A. Action movies to romantic movies that I like?
B. Romantic movies to comedies that that I like?
C. Comedies to action movies that I like

$$3:5$$
$$5:4$$
$$4:3$$

1.4: Equivalent Ratios

If we multiply, or divide each number of a ratio, by the same number, we get an equivalent ratio

To check if two ratios are equivalent, convert them to simplest form.

Example 1.5

Check if the following ratios are the same:

A. 6:4 and 18:12

B. $3:4$ and $10:13$

Part A

We simplify both the ratios, and find that both the ratios are equivalent to $3:2$. Hence, the ratios are the same.

$$6:4 = 3:2$$
$$18:12 = 3:2$$

Part B

$3:4$ *and* $10:13$ are already in simplest form.
Hence, they are not the same ratio.

B. Ratios as fractions

1.6: Ratios as Fractions

A ratio $x:y$ can be converted into a fraction $\frac{x}{y}$.

Example 1.7

Convert the following ratios to fractions:

A. $5:4$
B. $3:7$
C. $3:a$
D. $b:6$
E. $x:y$

Convert the following fractions to ratios:

F. $\frac{2}{3}$
G. $\frac{4}{5}$
H. $\frac{a}{4}$
I. $\frac{5}{b}$
J. $\frac{p}{q}$

$$5:4 = \frac{5}{4}$$
$$3:7 = \frac{3}{7}$$
$$3:a = \frac{3}{a}$$
$$b:6 = \frac{b}{6}$$
$$x:y = \frac{x}{y}$$

$$\frac{2}{3} = 2:3$$
$$\frac{4}{5} = 4:5$$
$$\frac{a}{4} = a:4$$
$$\frac{5}{b} = 5:b$$
$$\frac{p}{q} = p:q$$

1.8: Eliminating Fractions from Ratios

If one or both of the quantities in a ratio has fractions, the fractions can be eliminated by multiplying both sides of the ratio by the LCM of the denominators.

Example 1.9: Eliminating fractions from Ratios

Eliminate fractions from the following ratios

Equal Denominators

If the denominators are equal, multiply both sides by the denominator to eliminate fractions

A. $\frac{2}{5}:\frac{3}{5}$
B. $\frac{7}{12}:\frac{5}{12}$
C. $\frac{3}{11}:\frac{9}{11}$

Unequal Denominators-I

If one denominator is a multiple of another, multiply by the larger number.

D. $\frac{2}{3}:\frac{7}{9}$
E. $\frac{2}{5}:\frac{6}{25}$
F. $\frac{7}{4}:\frac{4}{12}$

Unequal Denominators-II

In the general case, multiply by the LCM of the two denominators:

G. $\frac{2}{3}:\frac{1}{4}$
H. $\frac{3}{5}:\frac{2}{7}$
I. $\frac{6}{7}:\frac{2}{9}$

Equal Denominators

$$\frac{2}{5}:\frac{3}{5}=\frac{2}{5}\times 5:\frac{3}{5}\times 5=2:3$$
$$\frac{7}{12}:\frac{5}{12}=7:5$$
$$\frac{3}{11}:\frac{9}{11}=3:9=1:3$$

Unequal Denominators-I

$$\frac{2}{3}:\frac{7}{9}=\frac{2}{3}\times 9:\frac{7}{9}\times 9=6:7$$
$$\frac{2}{5}:\frac{6}{25}=10:6=5:3$$
$$\frac{7}{4}:\frac{4}{12}=21:4$$

Unequal Denominators-II

$$\frac{2}{3}:\frac{1}{4}=8:3$$
$$\frac{3}{5}:\frac{2}{7}=21:10$$
$$\frac{6}{7}:\frac{2}{9}=54:14=27:7$$

Example 1.10

If $5:x=22.5:27.5$, then the value of x can be written in the form $a\frac{b}{c}$, where a, b and c are whole numbers and b and c have no common factor other than 1. Find $a+b+c$.

Write the given ratios in fraction form:

$$\frac{5}{x}=\frac{22.5}{27.5}$$

Take the reciprocal on both sides:

$$\frac{x}{5}=\frac{27.5}{22.5}$$

Multiply both sides by 5:

$$x=5\times\frac{275}{225}=\frac{5\times 11}{9}=\frac{55}{9}=6\frac{1}{9}$$
$$a+b+c=6+1+9=16$$

Example 1.11

Find the value of the variable in each case below:

A. $x: 4 = 4: 8$
B. $x: 2.5 = 1.5: 3.5$
C. $3: x = 7: 4$
D. $4: 2x = 1: 0.1$
E. $3: 8 = x: 3$
F. $1: \frac{1}{2} = 2x: 4\frac{1}{2}$
G. $4: 5 = 1: 3x$
H. $\frac{2}{3}: \frac{5}{4} = \frac{1}{2}: \frac{3}{2}x$

Part A

$$\frac{x}{4} = \frac{4}{8} \Rightarrow \frac{x}{4} = \frac{1}{2} \Rightarrow x = \frac{4}{2} = 2$$

Part B

$$\frac{x}{2.5} = \frac{1.5}{3.5} \Rightarrow \frac{10x}{25} = \frac{15}{35} \Rightarrow \frac{2x}{5} = \frac{3}{7} \Rightarrow x = \frac{3}{7} \times \frac{2}{5} = \frac{6}{35}$$

Part C

$$\frac{3}{x} = \frac{7}{4} \Rightarrow \frac{x}{3} = \frac{4}{7} \Rightarrow x = \frac{4}{7} \times 3 = \frac{12}{7}$$

Part D

$$\frac{4}{2x} = \frac{1}{0.1} \Rightarrow \frac{2}{x} = \frac{10}{1} \Rightarrow \frac{x}{2} = \frac{1}{10} \Rightarrow x = \frac{1}{5}$$

Part E

$$\frac{3}{8} = \frac{x}{3} \Rightarrow \frac{9}{8} = x$$

Part F

Part G

Part H

Example 1.12

A. The number of oranges I have is three fourths the number of apples. Find the ratio of oranges to apples.
B. One-third of the bumblebee population equals one-tenth of the honeybee population. Find the ratio of bumblebees to honeybees.
C. A small jug has two thirds the capacity of a large jug. I have four small jugs filled with milk, and five large jugs filled with water. Find the ratio of milk to water.
D. I have ten one-rupee coins, and my friend has twenty five-paise coins. Find the ratio of the money that I and my friend have.

Part A

$$\frac{3}{4}: 1 = 3: 4$$

$$O = \frac{3}{4}A \Rightarrow \frac{O}{A} = \frac{3}{4}O \Rightarrow O: A = 3: 4$$

Part B

Let the number of bumblebees be b. Let the number of honeybees be h. Then:

$$\frac{b}{3} = \frac{h}{10} \Rightarrow \frac{b}{h} = \frac{3}{10} \Rightarrow b: h = 3: 10$$

Part C

Let the capacity of a large jug be J, and the capacity of a small jug be j. Then:

$$j = \frac{2}{3}J$$

The ratio is:

$$\underbrace{4j}_{\substack{4\ Small\\ Jugs}} : \underbrace{5J}_{\substack{5\ Large\\ Jugs}} = 4\left(\frac{2}{3}J\right): 5J = \frac{8}{3}J: 5J = 8J: 15J$$

Part D

$$10(1\ Rupee): 20(5\ Paise)$$
$$1000\ Paise: 100\ Paise$$
$$10: 1$$

1.13: Comparing More than One Ratio

Example 1.14

I can cut vegetables at four fifths the speed that my mother does. And my mother can cut vegetables at twice the speed of my sister. Find the ratio of the speeds at which my sister and I can cut vegetables.

Suppose my sister cuts vegetables at the speed of s (for some unit which we do not mention).
Then, my mother cuts vegetables at the speed:

$$2s$$

And, I cut vegetables at the speed:

$$= \frac{4}{5} \times 2s = \frac{8}{5}s$$

Hence, the ratio of our speeds is:

$$\frac{8}{5}s: s = \frac{8}{5}: 1 = 8: 5$$

Example 1.15

The local animal shelter has three kittens for every cat, and four puppies for every dog. If they have five cats, and seven dogs, find the ratio of canines to felines. (Note: Cat and kittens are both feline, whereas dogs and puppies are canines).

The ratio of cats to kittens is $1: 3$. If they have five cats, they have $5 \times 3 = 15\ Kittens$.
The ratio of puppies to dogs is $1: 4$. If they have seven dogs, they have $7 \times 4 = 28\ puppies$

$$\underbrace{5 + 15}_{Felines} : \underbrace{7 + 28}_{Canines} = 20: 35 = 4: 7$$

Example 1.16

State True or False

I have a number of oranges and apples. The ratio of oranges to apples is $x: y$. If the number of apples increases by 1, and the number of oranges also increases by 1, then the ratio of apples to oranges:

A. Will remain same
B. Will increase
C. Can increase or decrease
D. None of the above

Ratio can increase

Suppose you have 1 orange and 2 apples. The ratio is:

$$1:2=\frac{1}{2}=\frac{3}{6}$$

If you increase the apples by one, and the oranges by 1, the ratio becomes:

$$2:3=\frac{2}{3}=\frac{4}{6}>\frac{3}{6}=1:2\Rightarrow \textit{Ratio can increase}$$

Ratio can decrease

Suppose you have 3 oranges and 2 apples. The ratio is:

$$3:2=\frac{3}{2}=\frac{9}{6}$$

If you increase the apples by one, and the oranges by 1, the ratio becomes:

$$4:3=\frac{4}{3}=\frac{8}{6}<\frac{9}{6}=3:2\Rightarrow \textit{Ratio can decrease}$$

Since the ratio can either increase or decrease,

$$\textit{Option C is correct}$$

C. Ratios as Percentages and Decimals

1.17: Ratios as Percentages

$$x:y=\frac{x}{y}=n\%$$

Example 1.18

Convert the ratios below to percentages

A. 1: 2
B. 4: 5
C. 7: 10
D. 2: 20
E. 3: 25
F. 7: 50
G. 12: 150
H. 11: 350

Convert the percentages below to ratios

I. 75%
J. 30%
K. 55%
L. 15%
M. 80%

Ratios to percentages

$$1:2=\frac{1}{2}=50\%$$

$$\frac{4}{5}=80\%$$

$$\frac{7}{10}=70\%$$

$$\frac{2}{20}=\frac{1}{10}=10\%$$

$$\frac{3}{25}=\frac{12}{100}=12\%$$

$$\frac{7}{50}=\frac{14}{100}=14\%$$

$$\frac{12}{150}=\frac{4}{50}=\frac{8}{100}=8\%$$

$$\frac{11}{350}=\frac{\frac{11}{7}}{50}=\frac{\frac{22}{7}}{100}=\frac{22}{7}\%$$

Percentages to ratios

$$75\%=\frac{75}{100}=\frac{3}{4}=3:4$$

$$30\%=\frac{3}{10}=3:10$$

$$55\%=\frac{55}{100}=\frac{11}{20}=11:20$$

$$15\%=\frac{15}{100}=\frac{3}{20}=3:20$$

$$80\% = \frac{80}{100} = \frac{4}{5} = 4:5$$

1.19: Ratios as Decimals

Example 1.20

Convert ratios to decimals

A. $2:10$
B. $3:100$
C. $5:25$
D. $7:20$
E. $8:50$

Convert decimals to ratios

F. 0.2
G. 0.7
H. 0.91
I. 0.35
J. 0.25

Convert ratios to decimals

$$2:10 = \frac{2}{10} = 0.2$$
$$3:100 = \frac{3}{100} = 0.3$$
$$5:25 = \frac{5}{25} = \frac{1}{5} = 0.2$$
$$7:20 = \frac{7}{20} = 0.35$$
$$8:50 = \frac{8}{50} = 0.16$$

Convert decimals to ratios

$$0.2 = \frac{2}{10} = 2:10 = 1:5$$
$$0.7 = \frac{7}{10} = 7:10$$
$$0.91 = \frac{91}{100} = 91:100$$
$$0.35 = \frac{35}{100} = \frac{7}{20} = 7:20$$
$$0.25 = \frac{25}{100} = \frac{1}{4} = 1:4$$

Example 1.21: Eliminating Decimals and Percentages from Ratios

A. $0.3:0.7$
B. $0.5:0.6$
C. $0.9:0.11$
D. $0.3:0.23$
E. $0.001:0.23$

$$0.3:0.7 = 3:7$$
$$0.5:0.6 = 5:6$$
$$0.9:0.11 = 90:11$$
$$0.3:0.23 = 30:23$$
$$0.001:0.23 = 1:230$$

D. Interchanging the antecedent and consequent

1.22: Interchanging the antecedent and the consequent

$$\underbrace{2}_{Antecedent} : \underbrace{3}_{Consequent} \neq \underbrace{3}_{Antecedent} : \underbrace{2}_{Consequent}$$

$$2:3=\frac{2}{3}\neq\frac{3}{2}=3:2$$

Example 1.23

Lucia adds one spoonful of sugar to every three cups of tea that she makes. Carlos adds three spoonsful of sugar to every cup of tea he makes. Compare the ratio of sugar to tea for Lucia and Carlos.

A. Are they the same?
B. If they are not the same, how many times is one ratio of the other.

$$1:3, \qquad 3:1 \Rightarrow Not\ the\ same$$

To compare, multiply the first ratio by 3:

$$\underbrace{3:9}_{Lucia}, \qquad \underbrace{3:1}_{Carlos} \Rightarrow Not\ the\ same$$

Carlos' tea has nine times the sugar that Lucia's tea has.

E. Comparing Ratios

To compare two ratios, we need to make one of the quantities the same. This then makes it easier to compare.

Example 1.24

A recipe calls for 2 spoons of vanilla essence in 3 kilos of dough. Simon creates a recipe with 3 spoons of vanilla essence in 4 kilos of dough. Compare the two and decide which has more vanilla.

Part A

If we make the dough the same, the vanilla is easy to compare.

$$LCM(3,4)=12$$

	Vanilla	*Dough*		*Vanilla*	*Dough*		*Vanilla*	*Dough*
	Spoons	Kilos		Spoons	Kilos		Spoons	Kilos
Recipe	2	3		8	12		6	9
Simon	3	4		9	12		6	8

$$Orginal\ Recipe\ has\ less\ vanilla\ compared\ to\ Simon's\ Recipe.$$

Part B

$$\frac{2}{3}?\frac{3}{4}=\frac{8}{12}?\frac{9}{12}=\frac{8}{12}<\frac{9}{12}\Rightarrow Simon's\ Recipe\ has\ more\ vanilla$$

Example 1.25

A recipe calls for one spoon of sugar in three cups of milk when making tea. Shridhar uses two spoons of sugar in seven cups of milk. Whose tea is sweeter, the recipe's or Sridhar's?

Part A

We need to compare

$$\underbrace{\underbrace{1}_{Spoon}:\underbrace{3}_{Cups}}_{Recipe}\ with\ \underbrace{\underbrace{2}_{Spoons}:\underbrace{7}_{Cups}}_{Sridhar}$$

To compare, we need to make either the spoons the same, or the cups the same.

$$\underbrace{1}_{Spoon}:\underbrace{3}_{Cups}=\underbrace{2}_{Spoon}:\underbrace{6}_{Cups}$$

Since the number for spoons are smaller, we choose to do it with spoons:

$$\underbrace{\underbrace{2}_{Spoons} : \underbrace{6}_{Cups}}_{Recipe} \ with\ \underbrace{\underbrace{2}_{Spoon} : \underbrace{7}_{Cups}}_{Sridhar}$$

Now we can compare. The recipe wants us to use 2 spoons in 6 cups. Sridhar has used 2 spoons in 7 cups. So, his tea will be less sweet than what the recipe wants.

Part B

$$\frac{1}{3} ? \frac{2}{7} \rightarrow \frac{7}{21} ? \frac{6}{21} \rightarrow \frac{7}{21} > \frac{6}{21}$$

Example 1.26

A recipe calls for one spoon of sugar in three cups of milk when making tea. Shivani uses five cups of milk with two spoons of sugar. Whose tea is sweeter, the recipe's or Shivani's?

Recipe		Shivani	
Spoon	Cups	Spoon	Cups
1	3	2	5
2	6	2	5

$$\frac{1}{3} ? \frac{2}{5} \rightarrow \frac{5}{15} < \frac{6}{15} \Rightarrow Shivani's\ recipe\ is\ sweeter$$

1.27: Comparing two ratios

We can compare two ratios to find how many times one ratio is a multiple of the other

Example 1.28

Jack has 12 pails for every 16 pails that Jill has. John has 16 marbles for every 12 marbles that Jane has.

A. Is the ratio of Jack's pails to Jill's pails the same as the ratio of John's marbles and Jane's marbles?
B. If they are not the same, how many times is one ratio of the other.

Part A

The ratio of the number of pails that Jack has to the number of pails that Jill has is:

$$12:16 = \frac{12}{16} = \frac{3}{4}$$

The ratio of the number of marbles that John has to the number of marbles that Jane has is:

$$16:12 = \frac{16}{12} = \frac{4}{3}$$

Part B

$$\frac{3}{4} : \frac{4}{3} = \underbrace{3 : \frac{16}{3}}_{\substack{Multiply \\ by\ 3}} = \underbrace{1 : \frac{16}{9}}_{\substack{Divide \\ by\ 3}}$$

We can also do by setting up an equation and solving it:

$$\frac{3}{4}x = \frac{4}{3} \Rightarrow x = \frac{4}{3} \times \frac{4}{3} = \frac{16}{9}$$

Example 1.29

A. Suppose I have three oranges, five apples and seven grapes. Let A be the ratio of oranges to apples. Let B be the ratio of apples to grapes. What is the ratio of A to B? A is how many times of B?

Part A

$$A:B=\frac{3}{5}:\frac{5}{7}=\underbrace{21:25}_{Part\ A}=\frac{21}{25}:1$$

We can also find the multiple using algebra:

$$\frac{5}{7}x=\frac{3}{5}\Rightarrow x=\frac{3}{5}\times\frac{7}{5}=\frac{21}{25}$$

1.2 Using Ratios

A. Comparing Units

1.30: Making Units Comparable

If the units of comparison are not the same, we need to make them the same.

Example 1.31

A farmer has 30 apples and 50 oranges. What is the ratio of apples to oranges?

$$Apples:Oranges=30:50=3:5$$

We can also write:

$$3\ Apples:5\ Oranges$$

Example 1.32: Money

Person A has six rupees and twenty-five paise. Person has a quarter of a rupee. Find the ratio of the money that they have.

$$A=6\ Rs.+25\ paise=600\ paise+25\ paise=625\ paise$$
$$B=25\ paise$$

$$A:B$$
$$625\ paise:25\ paise$$
$$625:25$$
$$25:1$$

Example 1.33: Money

A. Find the ratio of 3 Rs to 50 paise.
B. Mark had 2 Rupees and Sixteen paise with him. He buys three dolls, each costing six paise. What is the ratio of the amount spent by him to the amount he originally had.
C. Nicholas has ten quarters (25 cent coins). Mike has five nickels (5 cent coins). What is the ratio of money that Nicholas has to the money that Mike has.

Part A

$$3\text{ Rs}:50\text{ Paise}=300\text{ paise}:50\text{ paise}=300:50=6:1$$
$$3\text{ Rs}:0.5\ Rs=3:0.5=30:5=6:1$$

Part B

$$625\ paise:25\ paise=25:1$$

Part C

$$216\ paise:18\ paise$$
$$216:18$$
$$108:9$$

$$12:1$$

Part D

$$250\ cents: 25\ cents = 10:1$$

Example 1.34: Time

A. Find the ratio of 24 minutes to 36 minutes.
B. What is ratio of one third of a minute to three fourths of a minute.
C. Ming can hold her breath for 1 minute and twenty seconds, while Grace is able to hold her breath for a minute more. What is the ratio of the time that the two can hold their breath?
D. A journey from City A to City B takes 2 hours ten minutes while a journey from City B to City C takes 1 hour 40 minutes. Find the ratio of the times taken for the journeys.
E. Walking up a hill takes 3 hours 15 minutes, while walking downhill takes 1 hour 45 minutes. Find the ratio of time taken when walking downhill to that when walking uphill.
F. A carpenter takes a quarter of an hour to polish a chair and half an hour to polish a table. He must polish a furniture set with three tables and ten chairs. Determine the ratio of the time taken for polishing chairs to that for polishing tables.
G. A direct plane flight from India to the US takes 18 hours, while one with a stopover takes one day and four hours. Determine the ratio of the two times.

Part A

$$24:36 = 2:3$$

Part B

Method I: Multiply the fractions by the $LCM(3,4)$:

$$\frac{1}{3}:\frac{3}{4} = 12 \times \frac{1}{3}:12 \times \frac{3}{4} = 4:9$$

Method II: Convert the minutes to seconds and then simplify:

$$\frac{1}{3}:\frac{3}{4} = 60 \times \frac{1}{3}:60 \times \frac{3}{4} = 20:45 = 4:9$$

Part C

$$1\ m\ 20\ s: 2\ m\ 20\ s$$
$$80\ s: 140\ s$$
$$8:14$$
$$4:7$$

Part D

$$2\ h\ 10\ m: 1\ h\ 40\ m$$
$$130\ m: 100\ m$$
$$13:10$$

Part E

$$1\ h\ 45\ m : 3\ h\ 15\ m$$
$$105\ m: 195\ m$$
$$21:39$$
$$7:13$$

Part F

$$10 \times \frac{1}{4}:3 \times \frac{1}{2} = \frac{5}{2}:\frac{3}{2} = 5:3$$

Part G

$$18\ hrs: 28\ hrs$$
$$9:14$$

1.35: Metric System: Prefixes

$$kilo = 1,000$$
$$Centi = \frac{1}{100}$$
$$Milli = \frac{1}{1000}$$
$$Micro = \frac{1}{1,000,000}$$

Example 1.36

A. A dog weighs 15 kg. Find the weight of the dog in grams.
B. A pen weighs 12 grams. Find its weight in kilograms, and micrograms.

$$Weight\ of\ Dog = 15\ kg = 15,000\ grams$$
$$Weight\ of\ Pen = 12\ grams = 0.012\ kg = 12,000,000\ micrograms$$

Example 1.37[1]

Gold costs approximately Rs. 5500 per gram. Find the

A. cost of gold in an ornament gilded with 20 milligrams of gold.
B. cost of gold in a necklace weighing 12 grams.
C. ratio of the cost of gold to the cost of silver if silver costs 75,000 per kg.

Part A

$$5500 \times \frac{20}{1000} = 110\ Rs.$$

Part B

$$Cost\ of\ 12\ grams = 12 \times 5,500 = 66,000\ Rs.$$

Part C

$$\frac{5500}{g} : \frac{75,000}{kg} = \frac{5500}{g} : \frac{75,000}{1000\ g} = 5,500: 75 = 1,100: 5 = 220: 3$$

Example 1.38

A. An important drug with essential ingredient 0.5 grams is being transported in a container weighing 2 kg. What is the ratio of the weight of the drug to the weight of the container?
B. Ants are amazing at transporting substances. Individual ants have been known to carry 20 times their own body weight. Find the maximum weight that an ant weighing 2.5 mg can carry, in grams.
C. If a human weighs 70 kg, and an ant weighs 3 mg, find the ratio of their weights.

Part A

$$0.5\text{ grams}: 2\text{ kg} = 0.5\text{ grams}: 2000\text{ grams} = 0.5: 2000 = 1: 4000$$

Part B

$$2.5\ mg \times 20 = 50\ mg = \frac{50}{1000} g = 0.05\ g$$

Part C

$$70\ kg: 3\ mg = 70,000,000\ mg: 3\ mg = 70,000,000: 3$$

Example 1.39: Speed

[1] As of Jan 2023

A. A man walks at a speed of $5\frac{km}{hr}$, and his car travels at a speed of $10\frac{m}{s}$. Find the ratio of the two speeds.
B. A tortoise can move at a speed of 5 meters per minute. A boy is able to chase after the tortoise at a speed of 0.5 meters per second. Find the ratio of the two speeds.
C. An aero plane can travel 700 kilometers per hour. A bullock cart can travel 0.5 meters per second. Find the ratio of the two speeds.
D. The speed of light is 300,000 kilometers per second. The speed of an ICBM missile in midcourse phase is 24000 km per hour. Find the ratio of the speed of light to that of the missile.[2]

Part A

$$5\frac{\text{km}}{\text{hr}}:10\frac{\text{m}}{\text{s}}=5\cdot\frac{1000\text{m}}{3600\text{s}}:10\frac{m}{s}=5\cdot\frac{5}{18}\frac{\text{m}}{\text{s}}:10\frac{\text{m}}{\text{s}}=\frac{25}{18}:10=\underbrace{\frac{5}{18}:2}_{Divide\ by\ 5}=\underbrace{5:36}_{Multiply\ by\ 18}$$

Part B

$$5\frac{m}{min}:0.5\frac{m}{sec}=5\frac{m}{60\ sec}:0.5\frac{m}{sec}=\frac{5}{60}:0.5=\frac{1}{12}:\frac{1}{2}=1:6$$

Part C

$$700\frac{km}{hr}:0.5\frac{m}{s}=700\cdot\frac{1000\ m}{3600\ s}:\frac{1}{2}\frac{m}{s}=700\cdot\frac{5}{18}:\frac{1}{2}=\frac{3500}{9}:1=3500:9$$

Part D

$$300{,}000\frac{km}{s}:24{,}000\frac{km}{hr}=300{,}000\frac{km}{s}:24{,}000\frac{km}{3600\ s}=300{,}000:\frac{240}{36}=300{,}000:\frac{20}{3}=45{,}000:1$$

B. Totals

Example 1.40

A. A school has 12 boys and 8 girls. Find the ratio of boys to girls, of boys to students and of girls to students. (Note: Your answer will be three different ratios for this and the next parts).
B. My favorite *maramari* juice has 3 cups of orange juice and 4 cups of *mosambi* juice. What is the ratio of orange juice to mosambi juice, of orange juice to total juice, and of *Mosambi* juice to total juice?
C. A class has seven students who learn French, and five students who learn Hindi. Each student learns exactly one language. Find the total number of students in the class. Then, find the ratio of students who learn French to those who learn Hindi, of students who learn French to total number of students, and of students who learn Hindi to total number of students.

Part A

$$No.of\ Students=12+8=20$$

$$Boys:Girls=12:8=3:2$$
$$Boys:Students=12:20=3:5$$
$$Girls:Students=8:20=2:5$$

Part B

$$Orange:Mosambi=3:4$$
$$Orange:Total=3:7$$
$$Mosambi:Total=4:7$$

Part C

$$French:Hindi=7:5$$
$$French:Total=7:12$$

[2] These are approximate values, of course.

$$Hindi: Total = 5:12$$

Example 1.41

A. The ratio of boys to girls in a dance class is 2:3. If the number of boys in the class is a single digit number, then find all possible values of the number of students in the class.
B. X is a two-digit number. The ratio of the ten's digit to the unit's digit of X is 2: 1. Let S be the sum of the digits of X. Find the sum of the possible values of X.

Part A

$$2:3 = \frac{2\ Boys}{\underbrace{3\ Girls}_{Students=5}} = \frac{4\ Boys}{\underbrace{6\ Girls}_{Students=10}} = \frac{6\ Boys}{\underbrace{9\ Girls}_{Students=15}} = \frac{8\ Boys}{\underbrace{12\ Girls}_{Students=20}}$$

From the above

$$Possible\ value\ of\ Boys = \{2,4,6,8\}$$
$$Possible\ values\ of\ Students = \{5,10,15,20\}$$

Part B

The ratio of the ten's digit to the unit's digit is:

Ratio	*Number*	*Sum*
2: 1	21	3
4: 2	42	6
6: 3	63	9
8: 4	84	12
		30

C. Distributing/Dividing in a Ratio

Example 1.42

Divide 24 apples between Harry and Hermione in the ratio 3: 5.

Method I

Distribute apples in the ratio as given:

$$\underbrace{3}_{Harry} : \underbrace{5}_{Hermione} : \underbrace{8}_{Total}$$

This is 8 apples, but we need to distribute 24. Since $24 = 8 \times 3$, multiply the entire ratio by 3:

$$\underbrace{3 \times 3}_{Harry} : \underbrace{5 \times 3}_{Hermione} : \underbrace{8 \times 3}_{24} = \underbrace{9}_{Harry} : \underbrace{15}_{Hermione} : \underbrace{24}_{Total}$$

Method II

Make sets: $3 + 5 = 8$ apples in each set.

$$\frac{24}{8} = 3 \text{ sets}$$

Harry: $3 \times 3 = 9$
Hermione: $3 \times 5 = 15$

$$\underbrace{9}_{Harry} : \underbrace{15}_{Hermione} : \underbrace{24}_{Total}$$

Example 1.43

A. Divide 30 books between A and B in the ratio 2:1.
B. A man divided his 99 silver coins in the ratio 5: 4 between his two daughters, with the elder daughter getting the larger share. How many silver coins will each get?

C. Portia's high school has 3 times as many students as Lara's high school. The two high schools have a total of 2600 students. How many students does Portia's high school have? **(AMC 10A 2021/2)**

Part A

A should get double of B.

$$A = 20, B = 10$$

$$2 + 1 = 3 \to \frac{30}{3} = 10 \Rightarrow A = 10 \times 2 = 20, B = 10 \times 1 = 10$$

Part B

$$5 \times \frac{99}{5+4} = 5 \times \frac{99}{9} = 5 \times 11 = 55$$
$$4 \times \frac{99}{5+4} = 4 \times \frac{99}{9} = 4 \times 11 = 44$$

Part C

We need to distribute students in the ratio 3: 1:

$$3 + 1 = 4 \Rightarrow \frac{2600}{4} \times 3 = 1{,}950$$

D. Distribution with Fractions

Example 1.44

A king distributed 4200 silver coins among his two favorite ministers A and B in the ratio $\frac{1}{3}:\frac{1}{4}$, respectively.

A. Without calculating how much each minister got, determine which minister got more.
B. How much will each get?

Part A

Compare the ratios to see which fraction is larger.

$$\frac{1}{3} > \frac{1}{4} \Rightarrow A\ got\ more\ than\ B$$

Part B

To eliminate fractions, multiply by $LCM(3,4) = 12$:

$$\frac{1}{3} : \frac{1}{4} = 4:3 \to 4 + 3 = 7$$

We need to divide Rs. 4200 into 7 parts. Each part will be:

$$\frac{4200}{7} = Rs. 600$$
$$A\ gets\ 600 \times 4 = 2400$$
$$B\ gets\ 600 \times 3 = 1800$$

Example 1.45

A farmer divides his orchard of 74 apple trees among his two daughters Josephine and Rosa in the ratio $\frac{1}{4}:\frac{1}{5}$, such that each daughter gets a whole number of trees.

A. How many trees will each daughter get?
B. How many more trees will Rosa get compared to Josephine.
C. How many trees will be left over after the distribution.
D. For the trees which are left over, the apples on the trees are distributed in the same ratio given above. A dozen boxes, each with a capacity of a dozen apples are filled from each tree. Now, find how many apples each sister gets from the leftover trees.

Part A

$$\frac{1}{4}:\frac{1}{5}=\frac{1}{4}\cdot 20:\frac{1}{5}\cdot 20=5:4\rightarrow 5+4=9$$
$$\frac{74}{9}=8\frac{2}{9}$$
$$Josephine=8\times 5=40$$
$$Rosa=8\times 4=32$$

Part B

$$32-40=-8$$

Part C

$$74-72=2\ Trees$$

Part D

$$\frac{12\times 12\times 2}{9}=4\times 4\times 2=32$$
$$Josephine=32\times 5=160$$
$$Rosa=32\times 4=128$$

E. Distribution with Percentages

Example 1.46

F. Minimum and Maximum

Example 1.47

I distribute chocolates to my brother and sister get them in the ratio 3: 4. What is the minimum number of chocolates I must buy, if I

A. I distribute a whole number of chocolates.
B. buy chocolates in packets of 8, and I distribute all the chocolates that I buy,

Part A

$$3:4\rightarrow 3+4=7\ Chocolates$$

Part B

$$LCM(7,8)=56$$

Example 1.48

A. A party has men and women in the ratio 2: 5. If the number of people attending is a two-digit number, find the difference between the minimum and the maximum possible number of people.
B. A school has boys and girls in the ratio 3: 5. If the number of students is a three-digit number, find the difference between the minimum and the maximum possible number of students?

Part A

The number of people at the party must be a multiple of

$$2:5\rightarrow 2+5=7$$
$$Smallest\ two\ digit\ multiple\ of\ 7=14$$
$$Largest\ two\ digit\ multiple\ of\ 7=98$$
$$Difference=98-14=84$$

Part B

$$3:5\rightarrow 3+5=8$$

The number of students must be a multiple of 8.

$$Smallest\ three\ digit\ multiple\ of\ 8 = 104$$
$$Largest\ three\ digit\ multiple\ of\ 8 = 992$$
$$Difference = 992 - 104 = 888$$

Example 1.49

The ratio of men to women at a dinner was 2: 3. The ratio of red chairs to blue chairs was 3: 5, and all the chairs were occupied.

A. What is the minimum number of seated people at the party, if all the attendees sat for the dinner?
B. If every table seated six people, what is the minimum number of seated people?

Part A

$$2:3 \to 2+3=5 \Rightarrow No.of\ People\ must\ be\ a\ multiple\ of\ 5$$
$$3:5 \to 3+5=8 \Rightarrow No.of\ Chairs\ must\ be\ a\ multiple\ of\ 8$$
$$No.of\ People = No.of\ Chairs\ must\ be\ a\ multiple\ of\ LCM(5,8) = 40$$
$$Minimum\ Value = 40$$

Part B

$$No.of\ Chairs\ per\ Table = 6 \Rightarrow No.of\ People\ \ must\ be\ multiple\ of\ 6$$
$$No.of\ People\ must\ be\ multiple\ of\ LCM(5,8,6) = 120$$

Example 1.50

The ratio of men to women at a dinner was 2: 3. The ratio of red chairs to blue chairs was 3: 5, and one chair was not occupied. What is the minimum number of seated people at the party, if all the attendees sat for the dinner?

$$2:3 \to 2+3=5 \Rightarrow No.of\ People\ must\ be\ a\ multiple\ of\ 5 \in \{5,10,15,20,\dots\}$$
$$3:5 \to 3+5=8 \Rightarrow No.of\ Chairs\ must\ be\ a\ multiple\ of\ 8 \in \{8,16,24,\dots\}$$
$$No.of\ Occupied\ Chairs\ must\ be\ 1\ less\ than\ a\ multiple\ of\ 8$$

$$5 = 8 - 3$$
$$10 = 16 - 6$$
$$15 = 16 - 1 \Rightarrow Works$$

Example 1.51

A farmer was apportioning his flock of sheep. His elder son got all the Merino sheep. His elder daughter got 25% more sheep than his elder son, and 20% more than his younger daughter. His younger son got 25% more than his elder daughter. His wife got 20% more than his elder daughter. Find the minimum number of sheep the farmer owns.

$$S_e = Elder\ Son, \qquad S_y = Younger\ Son$$
$$D_e = Elder\ Daughter, \qquad D_y = Younger\ Daughter$$
$$W = Wife$$

$$S_e = 1$$
$$D_e = \frac{5}{4}$$
$$D_e = \frac{6}{5}D_y \Rightarrow D_y = \frac{5}{6}D_e = \frac{5}{6}\times\frac{5}{4} = \frac{25}{24}$$
$$S_y = \frac{5}{4}D_e = \frac{5}{4}\times\frac{5}{4} = \frac{25}{16}$$

$$W = \frac{6}{5}D_e = \frac{6}{5} \times \frac{5}{4} = \frac{6}{4} = \frac{3}{2}$$

Compare all the ratios together:

$$1:\frac{5}{4}:\frac{25}{24}:\frac{25}{16}:\frac{3}{2}$$

Since the number of sheep each gets much be an integer, multiply by $LCM(4,24,16,2) = 48$:

$$48: 60: 50: 75: 72$$

Smallest number of sheep

$$= 48 + 60 + 50 + 75 + 72 = 305$$

Example 1.52

Three vessels of sizes 3 liters, 4 liters, and 5 liters contain mixture of water and milk in the ratio 2:3, 3:7 and 4:11 respectively. The contents of all the three vessels are poured into a single vessel. What is the ratio of water to milk in the resultant mixture? **(NMTC Primary-III, Final)**

	No. of Parts	*Size of each* Part	Water	Milk
$3\ Liter\ Vessel$	$2 + 3 = 5$	$\frac{3}{5}$	$2 \times \frac{3}{5} = \frac{6}{5}$	$3 \times \frac{3}{5} = \frac{9}{5}$
$4\ liter\ Vessel$	$3 + 7 = 10$	$\frac{4}{10}$	$3 \times \frac{4}{10} = \frac{12}{10}$	$7 \times \frac{4}{10} = \frac{28}{10}$
$5\ Liter\ Vessel$	$4 + 11 = 15$	$\frac{5}{15}$	$4 \times \frac{5}{15} = \frac{20}{15}$	$11 \times \frac{5}{15} = \frac{55}{15}$

$$Total\ Water = \frac{6}{5} + \frac{12}{10} + \frac{20}{15} = \frac{36}{30} + \frac{36}{30} + \frac{40}{30} = \frac{112}{30}$$

$$Total\ Milk = \frac{9}{5} + \frac{28}{10} + \frac{55}{15} = \frac{54}{30} + \frac{84}{30} + \frac{110}{30} = \frac{248}{30}$$

The ratio of water to milk is:

$$\frac{112}{30}:\frac{248}{30} = 112: 248 = 56: 124 = 28: 62 = 14: 31$$

G. Word Problems

Example 1.53

A. If twenty percent of the wasp population in a forest equals 0.25 times the bumblebee population, then find the ratio of wasps to bumblebees.
B. If 0.05 times the number of mice in a forest equals thirty percent of the number of cats, then find the ratio of mice to cats. If each cat catches a mouse everyday (and no new mice are added), how many days will it take to catch all the mice?

Part A

$$20\%\ of\ W = 0.25 \times B$$
$$0.20W = 0.25B$$

Convert the decimals to fractions:

$$\frac{1}{5}\mathrm{W} = \frac{1}{4}B$$

Divide both sides by B:

$$\frac{W}{B} = \frac{5}{4}$$

Convert to a ratio:

$$W: B = 5: 4$$

Part B

$$0.05m = 30\%\ of\ c$$

$$\frac{m}{20}=\frac{3c}{10}$$ $$\frac{m}{c}=6$$	$$6\ Days$$

Example 1.54

The ratio of boys to girls in Mr. Brown's math class is 2:3. If there are 30 students in the class, how many more girls than boys are in the class? (AMC 8 1985/16)

$$2:3:5=12:18:30 \Rightarrow 18-12=6$$

Example 1.55

Pens are sold in packs. Every pack contains 3 blue pens and 4 black pens. I have packs of pens with me, with a total of 49 pens. If I want to have 12 more black pens than blue pens, how many more packs should I get?

$$One\ Pack\ has\ ratio\ 3:4 \Rightarrow Difference\ is\ 1$$
$$7\ packs\ will\ have\ difference\ of\ 7$$
$$12\ packs\ will\ have\ difference\ of\ 12$$

$$12-7=5\ Packs$$

Example 1.56

Answer each part independently

A. The batch at IIM L has 180 students, all of whom are studying Operations Research (OR). Each OR textbook is shared by three people. Determine the number of textbooks required.
B. The batch at IIM L has 180 students, all of whom are studying 7 subjects each. Each subject has 2 textbooks. Each textbook is shared by ten people. Determine the number of textbooks required.

Part A

$$\frac{180}{3}=60$$

Part B

$$\frac{180\times 7\times 2}{10}=18\times 14=180+72=252$$

Example 1.57

In a college of 300 students, every student reads 5 newspapers, and every newspaper is read by 60 students. Then, the number of newspapers is: (JEE Adv, 1998)

Every student reads 5 newspapers. Hence, the total number of newspapers should be:

$$=300\times 5\ Newspapers$$

However, every paper is read (shared) by 60 students. Hence, the actual number of newspaper copies is:

$$=\frac{300\times 5}{60}=5\times 5=25$$

Example 1.58

King Middle School has 1200 students. Each pupil takes 5 classes a day. Each teacher teaches 4 classes. Each class has 30 students and 1 teacher. How many teachers are there at King Middle School? (AMC 8 1985/23)

$$No.of\ Classes = \frac{1200 \times 5}{30} = 200$$
$$No.of\ Teachers = \frac{200}{4} = 50$$

1.3 Application: Length, Area, and Volume

A. Ratios with Lengths

1.59: Prefixes

Prefixes used in the metric system include:

$$Centi = \frac{1}{100} = One\ Hundredth$$
$$Milli = \frac{1}{1000} = One\ Thousandth$$
$$Kilo = 1000$$

1.60: Conversion Factors: Metric System

Conversion factors in the metric system for length include:

$$1\ cm = 10\ mm$$
$$1\ m = 100\ cm = 1{,}000\ mm$$
$$1\ km = 1{,}000\ m$$

Example 1.61

Find the ratio of:

A. a $kilometer$ to a cm
B. $12\ mm$ to $50\ cm$
C. $\frac{1}{10} m$ to $25\ cm$

$$1\ km: 1\ cm = 1000\ m: 1\ cm = 100{,}000\ cm: 1\ cm = 100{,}000: 1$$

Example 1.62

The sun is approximately 150 million km from the Earth. The speed of light is approximately 300,000 km per second. Determine the time taken for light from the sun to arrive at the Earth in minutes.

Determine the ratio of the total distance to the distance travelled in one second:

$$Total\ Distance: Distance\ per\ second$$
$$150\ million\ km: 300{,}000\ km$$
$$150 \times 1{,}000{,}000: 300{,}000$$
$$150 \times 10: 3$$
$$1500: 3$$
$$500: 1$$

Hence, the time should be approximately

$$500\ seconds = \frac{500}{60} = \frac{50}{6} = \frac{25}{3} = 8\frac{1}{3}\ minutes$$

Example 1.63: Word Problems

Rajesh ran ten laps around his house, and one fourth of a lap at the race track. If the distance around his house is 75 m, and the length of a lap at the race track is two-thirds of a kilometer, find the ratio of the distance run around the house to that run at the race track.

The distance that Rajesh ran around his house was:

$$75 \times 10 = 750\ m$$

The length that Rajesh ran at the race track:

$$= \frac{1}{4} \times \frac{2}{3} \times 1\ km = \frac{1}{4} \times \frac{2}{3} \times 1000\ m = \frac{2}{3} \times 250\ m = \frac{500}{3} m$$

The ratio of the two distances is:

$$750 : \frac{500}{3} = 750 \times 3 : \frac{500}{3} \times 3 = 2250 : 500 = 225 : 50 = 9 : 2$$

1.64: Conversion Factors: Imperial System

Conversion factors in the imperial system for length include

$$1\ foot = 12\ inches$$
$$1\ yard = 3\ feet$$

Example 1.65

Find the ratio of

A. $2\frac{1}{2}$ feet to 4 inches
B. $3\ yards$ to $33\ inches$

$$2\frac{1}{2}\text{feet} : 4\text{ inches} = 30\text{ inches} : 4\text{ inches} = 30 : 4 = 15 : 2$$

Example 1.66

In the imperial system, an inch comprises three barleycorns, and five and a half yards comprise a perch. Determine the number of barleycorns in one twelfth of a perch.

Example 1.67

If eight furlongs make a mile, 40 poles make a furlong, and five and a half yards make a pole, determine the ratio of the length of 176 yards to a mile.

1.68: Scale Factor

In the context of maps, and models, the ratio between the actual, and the map/model is called the scale factor.

Example 1.69: Finding Scale Ratios

A town planner is making a map. The distance between an amusement park, and a bus station is 510 meters. He shows it on the map using a distance of 17 cm. Find the scale factor used.

$$510\ m : 17\ cm = 51{,}000\ cm : 17\ cm = 3000 : 1$$

Example 1.70: Finding Lengths from Scale Factor

A. A map has a scale factor of $1:1000$. Find the actual length of a road with length 7 cm on the map.
B. A map has a scale factor of $1:200$. The distance between Komal's house and her school is 12 cm on the map. Find the actual distance between the house and the school.
C. The distance between two hill stations is 200 km. Pearl is making a map with a scale factor of $1:100{,}000$. How far apart will be the two hill stations be on her map? Is it a reasonable scale factor to use for the map.

Part A

$$7\ cm \times 1000 = 7000\ cm = 70\ m$$

Part B

$$12\ cm \times 200 = 2400\ cm = 24\ m$$

Part C

$$\frac{200 \times 1000\ m}{100{,}000} = 2\ m = 200\ cm$$

2 m will make an exceptionally large map.

Example 1.71: Using Scale Ratios

A. A drawing for a house uses a ratio of $1:40$. The distance between the kitchen and the bedroom is shown to be 15 cm. What is the actual distance?
B. An architect's drawing of a resort uses a scale of $1:100{,}000$. The entrance to the resort is 7 cm away from the reception on the map. What is the actual distance between the two points?

Part A

$$1:40 = 15\ cm:600\ cm = 15\ cm:6m$$

Part B

$$\underbrace{1}_{Map}:\underbrace{100{,}000}_{Actual} = 7\ cm:700{,}000\ cm = 7\ cm:7000\ m = 7\ cm:7\ km$$

Example 1.72: Applications

If a scale model of a castle uses 80 cm for the width (actual width: 1200 meter) and 120 cm for the length, then what is the length of the actual castle?

Method I

We can find the scale factor:

$$80\ cm:1200\ m = 80\ cm:120000\ cm = 1:1500$$

And we can use the scale factor to find the length:

$$= 120:180{,}000 = 120\ cm:1800\ m$$

Method II

Notice that $HCF(80,120) = 40$. We can avoid finding the scale factor and directly find the required length.

Use the fact that $80 = 40 \times 2, 120 = 40 \times 3 \Rightarrow 120 = 80 \times \frac{3}{2}$:

$$80\ cm:1200\ m = \frac{3}{2} \times 80\ cm:\frac{3}{2} \times 1200\ m = 120\ cm : 1800\ m$$

Example 1.73: Finding Scale Factor

A. The driving distance from New York to Philadelphia is approximately 150 kilometers. If the distance between the two cities on a map is 10 cm, find the scale factor used in the map.
B. The length and width of a park are in the ratio $4:3$. The length of a park is 40 m. The width of the park is shown on the map as 4 cm. Find the scale factor used in the map.

Part A

$$\frac{150\ km}{Scale\ Factor} = 10\ cm$$

$$Scale\ Factor = \frac{150\ km}{10\ cm} = \frac{150 \times 1000 \times 100\ cm}{10\ cm} = 1{,}500{,}000 = 1:1{,}500{,}000$$

Part B

The ratio of the length and the width is

$$4:3 = 4 \times 10 : 3 \times 10 = \underbrace{40}_{Length} : \underbrace{30}_{Width}$$

Hence, the width of the park is 30 m. This is represented on the map using 4 cm. The scale factor of the map is:

$$\underbrace{4\ cm}_{Map} : \underbrace{30\ m}_{Reality} = 4\ cm : 3000\ cm = 1:750$$

B. Ratios with Areas

Example 1.74: Ratio of Areas

A. A pond with an area of 100 square meters is next to a farm with an area of 500 square meters. Find the ratio of the area of the pond to that of the farm.
B. A square of size 2 cm is drawn inside a square of size 4 cm. Find the ratio of the area of the smaller square to the area inside the larger square but outside the smaller square.
C. A rectangle of length 2 cm and width 3 cm is drawn inside another rectangle of length 3 cm and width 4 cm. Find the ratio of the area of the smaller rectangle to the area inside the larger rectangle but outside the smaller rectangle.
D. A park with a side length of 10 meters has a paved walking path with a width of 1 m at its edge (inside the park). The rest of the park is grass. The paving is done using cobblestones that cost $20 per square meter to purchase and lay, while the cost of buying and planting the grass is $8 per square meter. Find the ratio of the cost of the walkway to the cost of the grass ground.

Part A

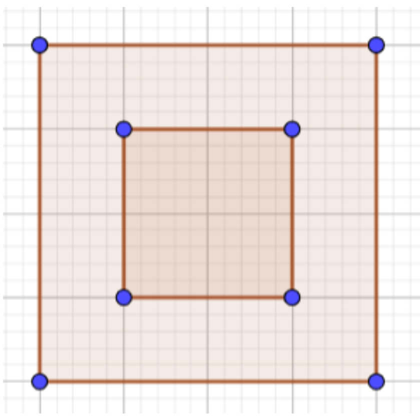
$$100:500 = 1:5$$

Part B

$$\underbrace{4\ cm^2}_{\substack{Smaller\\ Square}} : \underbrace{(16-4)\ cm^2}_{\substack{Inside\ larger,\\ but\ outside\ smaller}} = 4:12 = 1:3$$

Part C

$$\underbrace{6\ cm^2}_{\substack{Smaller\\ Rectangle}} : \underbrace{(12-6)\ cm^2}_{\substack{Inside\ larger,\\ but\ outside\ smaller}} = 6:6 = 1:1$$



Part D

$$Cost\ of\ Grass = \underbrace{8^2}_{Area} \times \underbrace{8}_{Cost} = \$8^3 = 8 \times 8 \times 8 = 16 \times 32$$

$$Cost\ of\ Paving = \underbrace{(100-64)}_{\substack{Area\ of\\ Cobblestones}} \times \underbrace{(20)}_{Cost} = 36 \times 20 = 9 \times 4 \times 4 \times 5$$

$$Ratio\ of\ the\ Two\ Costs = 9 \times 16 \times 5 : 16 \times 32 = 9 \times 5 : 32 = 45:32$$

Example 1.75

A. Two squares have side lengths in the ratio 2: 3. Find the ratio of their areas.
B. Find the ratio of the areas of two squares with side lengths in the ratio 4: 9.
C. Mohan's house has a square base. His friend's house also has a square base with lengths that are double those of Mohan's. Find the ratio of the area of Mohan's house to that of his friend.
D. Farmer Richard's farm is located on a hilltop, and is in the shape of a square. He has a square cabin next

to the farm. The side length of the farm is 100 times that of the cabin. Find the ratio of the area of the farm to that of the cabin.

Part A
The ratio of the areas will be the square of the ratio of the side lengths:

$$\underbrace{2:3}_{\substack{Ratio\\ of\ Lengths}} = 2^2:3^2 = \underbrace{4:9}_{\substack{Ratio\ of\\ Areas}}$$

Part B
The ratio of the areas will be the square of the ratio of the side lengths:

$$\underbrace{4:9}_{\substack{Ratio\\ of\ Lengths}} = 4^2:9^2 = \underbrace{16:81}_{\substack{Ratio\ of\\ Areas}}$$

Part C
We can take any length that we want for the side length of Mohan's house. The simplest length to take is

$$Ratio\ of\ Side\ Lengths = 1:2 \Rightarrow Ratio\ of\ Areas = 1:4$$

Even if we take variables, we get the same answer:

$$Ratio\ of\ Side\ Lengths = x:2x \Rightarrow Ratio\ of\ Areas = x^2:4x^2 = 1:4$$

Part D

$$Ratio\ of\ Side\ Lengths = 100:1 \Rightarrow Ratio\ of\ Areas = 10{,}000:1$$

Example 1.76
A blue square has a length of 1 meter, while a red square has a length of 10 cm. Find the ratio of the area of the blue square to that of the red square.

$$(1\ m)^2:(10\ cm)^2$$
$$(100\ cm)^2:(10\ cm)^2$$
$$10{,}000\ cm^2:100\ cm^2$$
$$100:1$$

Example 1.77
Find the ratio of

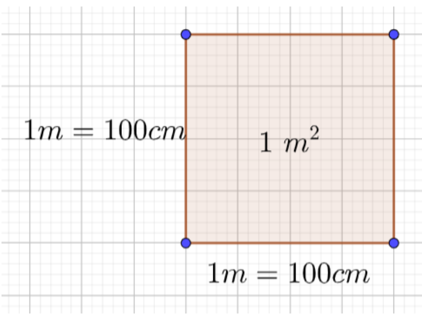
A. 1 square meter to 1 square centimeter?
B. 1 square meter to 1 square millimeter?
C. 1 square kilometer to 1 square meter?
D. 1 square kilometer to 1 square millimeter?

Part A

$$m^2:cm^2 = m \times m:cm^2$$

Substitute $1\ m = 100\ cm$:

$$100\ cm \times 100\ cm : cm^2$$
$$= 10{,}000\ cm^2:1\ cm^2$$
$$= 10{,}000:1$$
$$= 10^4:1$$



Part B

$$m^2:mm^2 = m \times m:mm^2$$
$$= 1000\ mm \times 1000\ mm:mm^2$$
$$= 1{,}000{,}000\ mm^2:mm^2$$
$$= 10^6:1$$

Part C

$$1\ km^2 : 1\ m^2 = (1\ km)^2 : (1\ m)^2 = (1000\ m)^2 : (1\ m)^2 = 1{,}000{,}000 : 1 = 10^6 : 1$$

Part D

$$1\ km^2 : 1\ mm^2 = (1\ km)^2 : (1\ mm)^2 = (1{,}000{,}000\ m)^2 : (1\ mm)^2 = 10^{12} : 1$$

1.78: Scale Factor for Area

The scale factor for area is the square of the scale factor for length.

Example 1.79

A map has a scale factor of $1:10$. Find the ratio of areas on the map to areas in real life.

Method I

$$Area = Length \times Width$$

On the map, if I have a square with length 1 unit and width 1 unit, it has area in real life

$$\underbrace{10}_{Length} \times \underbrace{10}_{Width} = 100$$

The ratio of areas on the map to areas in real life will be:

$$1:100$$

Method I

Scale factor for areas is the square of scale factor for length:

$$1^2 : 10^2 = 1:100$$

Example 1.80

For the questions below, let A be the actual area, in cm^2. Let B be the actual area in m^2.

A. A house occupies an area of $5\ cm^2$ on a map with a scale factor $1:10$. Find AB.
B. A house on a map with a scale factor $1:100$ occupies an area of $10\ cm^2$. Find $A+B$.
C. A house on a map with a scale factor $1:20$ occupies an area of $4\ cm^2$. Find $\frac{A}{B}$.

The scale factor for area is the square of the scale factor for length

Part A

$$Scale\ Factor\ for\ Area = 1^2 : 10^2 = 1:100$$

$$Actual\ Area = \underbrace{500}_{=A}\ cm^2 = \frac{500}{10000} m^2 = \underbrace{\frac{5}{100}}_{=B}\ m^2$$

$$AB = 500 \times \frac{5}{100} = 25$$

Part B

$$Scale\ Factor\ for\ Area = 1^2 : 100^2 = 1:10{,}000$$

$$Actual\ Area = \underbrace{100{,}000}_{=A}\ cm^2 = \underbrace{10}_{=B}\ m^2$$

$$A + B = 100{,}000 + 10 = 100{,}010$$

Part C

$$Scale\ Factor\ for\ Area = 1^2 : 20^2 = 1:400$$

$$Actual\ Area = \underbrace{1600}_{=A}\ cm^2 = \frac{1600}{10{,}000}\ m^2 = \underbrace{\frac{16}{100}}_{=B} m^2$$

$$\frac{A}{B} = \frac{1600}{\frac{16}{100}} = 1600 \times \frac{100}{16} = 10{,}000$$

Example 1.81

A map has a scale factor of $1:300$. Find the actual area of, in m^2:

A. A park with length and width 5 cm and 7 cm respectively on the map.
B. A garden with area $10\ cm^2$ on the map.

Part A

We can apply the scale factor to the length and the width separately:

$$\underbrace{5 \times 300}_{Actual\ Length} \times \underbrace{7 \times 300}_{Actual\ Width} = 35 \times 9 \times 10{,}000 = 3{,}150{,}000\ cm^2$$

Divide by 10,000 to convert into m^2:

$$= 315\ m^2$$

We can do this faster by multiplying the area of the park by the square of the scale factor:

$$\underbrace{5 \times 7}_{Area} \times 300^2 = 3{,}150{,}000$$

Part B

$$\underbrace{10}_{Area} \times \underbrace{300^2}_{(Scale\ Factor)^2} = 900{,}000\ cm^2 = 90\ m^2$$

Example 1.82

A. A house has an area of 34 square meters, while the dining table in the house has an area of 340 square centimeters. Find the ratio of the two areas.

B. A field with an area of one fourth of a square kilometer is to be divided into plots of 250 square meters each. Find the ratio of the plots to the field. Also, find the number of plots that will be made.

Part A

$$34\ m^2 : 340\ cm^2 = 340{,}000\ cm^2 : 340\ cm^2 = 100 : 1$$

Part B

$$\frac{1}{4} km^2 : 250\ m^2 = \frac{1{,}000{,}000}{4} : 250 = 1{,}000{,}000 : 1{,}000 = 1{,}000 : 1$$

C. Ratios with Volumes

1.83: Liter

$$1\ L = 1000\ mL$$

- Liter is a standard unit of volume. Milk, petrol, and other commodities are sold by liter (which means, by volume).

Example 1.84

A. A small container has a capacity of 300 ml. A large container has a capacity of 2 Liters. Find the ratio of the capacities of the small container to that of the larger container.

B. Manya is measuring the quantities of two different chemical solutions. She notes that she has 0.7 liters of copper sulfate solution, and 30 ml of zinc sulfate solution. Find the ratio of copper sulfate solution to zinc sulfate solution.

Part A

$$300\ ml : 2\ Litres$$
$$300\ ml : 2000\ ml$$
$$3 : 20$$

Part B

$$0.7\ Litres : 30\ ml$$
$$700\ ml : 30\ ml$$
$$700 : 30$$
$$70 : 3$$

1.85: Connection between ml and cm^3

$$1\ mL = 1\ cm^3$$

- Lengths in the metric system are measured using cm. The corresponding unit of volume is then cm^3.
- The conversion factor between cm^3 and ml is simple and direct.

Example 1.86: Volume

A. What is the volume of half a liter of milk in cubic centimeters?
B. What is the volume of a petrol tank that can contain 40 liters of petrol in cubic centimeters?
C. A large jug of milk contains 200 ml of milk. It is diluted by adding 100 cm^3 of water. Find the ratio of milk to water.

Part A

$$0.5\ L = 500\ ml = 500\ cm^3$$

Part B

$$40\ L = 40{,}000\ cm^3$$

Part C

$$200\ ml: 100\ cm^3$$
$$200\ ml: 100\ ml$$
$$2:1$$

Example 1.87

Find the ratio of:

A. 1 cubic meter to 1 cubic centimeter?
B. 1 cubic meter to 1 cubic millimeter?
C. 1 cubic kilometer to 1 cubic meter?
D. 1 cubic kilometer to 1 cubic millimeter?

Part A

$$m^3: cm^3 = m \times m \times m: cm^3$$

Substitute $1\ m = 100\ cm$:

$$100\ cm \times 100\ cm \times 100\ cm : 1\ cm^3 = 10^6\ cm^3: 1\ cm^2 = 10^6: 1$$

Part B

$$m^3: mm^3 = m \times m \times m: mm^3$$

Substitute $1\ m = 100\ mm$:

$$1000\ mm \times 1000\ mm \times 1000\ mm : mm^3 = 10^9\ mm^3: 1\ mm^3 = 10^9: 1$$

Part C

$$= 10^9: 1$$

Part D

$$km^3: mm^3 = km \times km \times km: mm^3$$

Substitute $1\ km = 1000\ m = 1{,}000{,}000\ mm = 10^6\ mm$

$$10^6\ mm\ \times 10^6\ mm\ \times 10^6\ mm\ : mm^3 = 10^{18}\ mm^3: 1\ mm^3 = 10^{18}: 1$$

1.88: Scale Factor for Volume

The scale factor for volume is the cube of the scale factor for length.

Example 1.89

A. A scale model of a train uses a scale factor of $1: 30$. If the model has a wagon with a capacity of 200 ml, find the actual capacity of the wagon in m^3.
B. A replica of the Eiffel Tower is 60 cm high (actual Eiffel Tower is 300 m). The Eiffel Tower can fit inside a cylinder with an approximate volume of eight million m^3. Find the estimated volume of the scale model

in cm^3.

Part A

Convert from ml to cm^3:

$$200\ ml = 200\ cm^3$$

The scale factor for volume is the cube of the scale factor for length:

$$1^3 : 30^3 = 1 : 27000$$

The actual volume of the wagon:

$$5{,}400{,}000\ cm^3 = 5.4\ m^3$$

Part B

The scale factor for length is:

$$\frac{300\ m}{60\ cm} = \frac{30000\ cm}{60\ cm} = 1 : 500$$

The scale factor for volume is:

$$1^3 : 500^3 = 1 : 125\ million$$

The volume of the model is:

$$\frac{8\ Million\ m^3}{125\ million} = \frac{8}{125} m^3$$

To convert from m^3 into cm^3, we multiply by 1,000,000:

$$\frac{8{,}000{,}000}{125} = 64{,}000\ cm^3$$

Example 1.90: Word Problems

A. What is the ratio of the areas of a square measuring 3 yards each side to that measuring 2 feet each side? (1 Yard = 3 feet).

$$(3\ yards)^2 : (2\ feet)^2$$
$$(9\ feet)^2 : (2\ feet)^2$$
$$81\ ft^2 : 4\ ft^2$$
$$81 : 4$$

Example 1.91: Volume

A. A tank with capacity 3 m^3 is being filled by using a container that has capacity 5 liters. Find the ratio of the capacity of the tank to that of the container. How many containers will be needed to fill three-fourths of the tank.
B. Water tanks with capacity 0.5 m^3 are being transported. Since the water should not overflow, 20% of the tank is kept empty. How many containers of size 3 liters can be filled using two such tanks?
C. At a fuel station, a CNG tank has a capacity of 12 m^3. How many cars with fuel tanks with a capacity of 12 Liters can be filled from a full tank. (Assume that both car tank and fuel station tanks have the same pressure.)

Part A

$$Capacity\ of\ Container = 5\ Liters$$
$$Capacity\ of\ Tank = 3\ m^3 = 3{,}000{,}000\ cm^3 = 3{,}000{,}000\ ml = 3000\ L$$

The ratio between the two is:

$$3{,}000 : 5 = 600 : 1$$

$$600 \times \frac{3}{4} = 450\ Containers$$

Part B

$$Capacity\ of\ Container = 3\ L$$

$$Capacity\ of\ 80\%\ of\ 2\ tanks = 0.5\ m^3 \times \frac{4}{5} \times 2 = 500{,}000\ cm^3 \times \frac{8}{5} = 800{,}000\ ml = 800\ L$$

The number of containers that can be filled is:

$$\frac{800}{3} = 266\frac{2}{3}$$

Part C

$$Car\ Tank = 12\ L$$
$$Fuel\ Tank = 12\ m^3 = 12{,}000{,}000\ cm^3 = 12{,}000{,}000\ ml^3 = 12000\ L$$

$$\frac{12{,}000}{12} = 1000$$

1.4 Ratios of More Than Two Quantities

A. Ratios of more than two quantities

Just as we compare two quantities in a ratio, we can also compare more than two quantities.

Example 1.92

Hagrid and the Weasley twins (Fred and George) started a magic shop. Fred, George and Hagrid sold wands in the ratio 2:7:3. Also, they sold magic kits in the ratio 6:21:9. The wands sold by Fred, George and Hagrid were 0.5 m, 40 cm, and 250 mm in length respectively.

A. Is the ratio in which wands are sold by Fred, George and Hagrid the same as the ratio in which they sold magic kits?
B. If the total number of wands sold is 96, what is the number of wands sold by each one?
C. What is the ratio of the lengths of the wands?

$$(2:7:3)\ \times\ 3\ =\ 6:21:9$$

$$2 + 7 + 3 = 12 \Rightarrow \frac{96}{12} = 8 \Rightarrow 16:56:24$$
$$0.5\text{ m}: 40\text{ cm}: 250\text{ mm} = 50\text{ cm}: 40\text{ cm}: 25\text{ cm} = 50:40:25 = 10:8:5$$

B. Comparing ratios of more than two quantities

If ratios of three or more quantities cannot be directly compared, then they need to be made comparable.

Example 1.93

A and B have marbles in the ratio $2:3$. B and C have marbles in the ratio $4:2$. What is $A:B:C$?

If A has 2 marbles, then B has 3 marbles.
If B has 3 marbles, then C has 4 marbles.

In order to be able to compare, B should have the same number of marbles in both ratios. This is possible if we find the LCM of the marbles that B has in each ratio

$$LCM(3,4) = 12$$

And then make the number of marbles that B has equal to this LCM.
Multiply the first ratio by 4:

$$A:B\ = 2:3 = 2\cdot\mathbf{4}:3\cdot\mathbf{4} =\ 8:\mathbf{12}$$

Multiply the second ratio by 3:

$$B:C = 4:2 = 4\cdot 3:2\cdot 3 = 12:6$$

Now, note that B has the same number of marbles in both ratios. Hence, we can compare them directly:
Combined ratio of $A:B:C = 8:12:6 = 4:6:3$

Example 1.94

A factory has two meters of red wire for every 300 cm of green wire, and 40 cm of green wire for every 200 mm of yellow wire. What is the ratio of the wires of three different colors in the factory?

$$\text{Red}:\text{Green} = 2\text{ m}:300\text{ cm} = 200\text{ cm}:300\text{ cm} = 2:3 = 4:6$$
$$\text{Green}:\text{Yellow} = 40\text{ cm}:200\text{ mm} = 40\text{ cm}:20\text{ cm} = 2:1 = 6:3$$
$$\text{Red}:\text{Green}:\text{Yellow} = 4:6:3$$

Example 1.95

Ben has three times as many penguins as porcupines. Millicent has twice as many porcupines as penguins. In all, they have 17 animals. What is the number of birds that they have? **(UK Maths Contest)**

Penguins : Porcupines

$$Ben = 3:1\ (3 + 1 = 4)$$
$$Milly = 1:2(1 + 2 = 3)$$

We need to combine sets of 4 and 3 to get 17.

$$17 - 4 = 13 \Rightarrow \text{not a multiple of } 3$$
$$17 - 8 = 9 \Rightarrow \text{which is a multiple of } 3$$

Hence, we must have:

$$Ben = 6:2$$
$$Milly = 3:6$$

And hence the final answer is

$$6 + 3 = 9$$

C. Decimals in Ratios

Example 1.96

If you distribute Rs. 10,000 among A, B, C and D in the ratio 0.1: 0.02: 0.04: 0.32, how much will each one get?

$$0.1\times \mathbf{100}: 0.02\times \mathbf{100}: 0.04\times \mathbf{100}: 0.32\times \mathbf{100} = 10:2:4:32 = 5:1:2:16$$

$$5 + 1 + 2 + 16 = 25 \Rightarrow \frac{10{,}000}{25} = 40 \Rightarrow 5\times 40:1\times 40:2\times 40:16\times 40 = 200:40:80:480$$

D. Fractions in Ratios

To remove fractions from ratios, multiply both sides of the ratio by the LCM of the denominators of the fractions in the ratios.

Example 1.97

A stack of 45 dimes is divided into three piles in the ratio 1/6: 1/3: 1/4. How many dimes are in the pile with the least number of dimes. (MathCounts 2005 State Sprint)

Multiply the given ratio throughout by LCM(6,3,4)=12:
2: 4:3

2+4+3=9
45/9=5

Multiply the above ratio by 5:
10:20:15

Dimes in the pile with least number of dime =10

Example 1.98

If twenty percent of the wasp population in a forest equals [zero point two five] times the bumblebee population, and one-third of the bumblebee population equals one-tenth of the honeybee population, then the ratio of wasps to bumblebees to honeybees is:

$$20\% \text{ of } W = 0.25 \times B \Rightarrow 0.20W = 0.25B \Rightarrow \frac{1}{5}w = \frac{1}{4}B \Rightarrow 4W = 5B \Rightarrow \frac{W}{B} = \frac{5}{4} \Rightarrow W:B = 5:4 = \mathbf{15:12}$$

$1/3B = 1/10H \Rightarrow 10B = 3H \Rightarrow B/H = 3/10 \Rightarrow B:H = 3:10 = \mathbf{12:40}$
$W:B:H = 15:12:40$

Example 1.99

If you distribute Rs. 5700 among A, B, C and D in the ratio 1/3 : ¼ : 1/5 : 1/6, then twice the share of the largest minus thrice the share of the smallest is:

$\frac{1}{3}:\frac{1}{4}:\frac{1}{5}:\frac{1}{6} = \frac{1}{3} \times \mathbf{60} : \frac{1}{4} \times \mathbf{60} : \frac{1}{5} \times \mathbf{60} : \frac{1}{6} \times \mathbf{60} = 20:15:12:10$
$20 + 15 + 12 + 10 = 57$
$5700/57 = 100$
$2A - 3D = 2 \cdot 20 \cdot 100 - 3 \cdot 10 \cdot 100 = 4000 - 3000 = 1000$

Example 1.100

The *Flying Scotsman*, a famous locomotive originally built in 1923 was restored with a budgeted cost $93\frac{3}{4}:1$ of the original cost of making the locomotive. Due to cost escalation, the actual cost was $5\frac{1}{3}$ of the budgeted cost. If the actual cost of restoring the locomotive was approximately four million pounds, what is the original cost of the locomotive? (UK Maths Contest)

$$\frac{Actual}{Orginal} = \frac{Actual}{Budgeted} \times \frac{Budgeted}{Original} = 5\frac{1}{3} \times 93\frac{3}{4} = \frac{16}{3} \times \frac{375}{4} = 4 * 125 = 500$$

$$Original\ Cost = \frac{4{,}000{,}000}{500} = 8000$$

E. Converting Units

Sometimes ratios use different units from the ones that we need.
In such cases, convert them as required.

Example 1.101

Aman has fifty cent, 25 cent and 10 cent coins with him in the ratio 5:9:4. If the total money that he has is 206 dollars, what is the total number of coins with him?

	50 Cent coins	25 Cent Coins	10 Cent Coins	Total
Ratio of Coins	5	9	4	

Value of each Coin – Cents	50	25	10	
Ratio of Values	250	225	40	
Simplify the ratio above	50	45	8	103 cents
Multiply the ratio by 200	100	90	16	20600 cents
Coins in one dollars	2	4	10	
Coins	200	360	160	720

Note:
In this question, we need to convert ratio of *coins* into ratio of *value*.

1.5 Algebra with Ratios

A. Expressions

Sometimes it is not possible to find the value of some quantities with the given information, but we can still find the ratio between the quantities.

Example 1.102

A. Given that $0.2(x+y) = 0.7(x-y)$, find the ratio of x to y.
B. Given that $3(2x+3y) = 11(2x-4y)$, find the ratio of x to y.

Part A
Use the distributive property to open the parentheses:
$$0.2x + 0.2y = 0.7x - 0.7y$$
Get the y terms on the LHS, and the x terms on the RHS:
$$0.9y = 0.5x$$
Divide both sides by y:
$$\frac{x}{y} = \frac{0.5}{0.9} = \frac{5}{9}$$
Convert from a fraction into a ratio:
$$x:y = 5:9$$
Part B
Use the distributive property to open the parentheses:
$$6x + 9y = 22x - 44y$$
Get the y terms on the LHS, and the x terms on the RHS:
$$53y = 16x$$
Divide both sides by y:
$$\frac{x}{y} = \frac{53}{16}$$
Convert from a fraction into a ratio:
$$x:y = 53:16$$

Example 1.103

A farmer has two types of apples in his apple orchard. He makes a huge pile of all the apples together. Then, he divides them into three piles, one with green apples, a second with the same number of red apples as in the first pile, and a third with red apples. He notices that three times the number of apples in the huge pile is equal to fourteen times the number of apples in the third pile (from when he divided it into three piles). Find the ratio of green apples to red apples.

Let the number of

$$Green\ apples = g, Red\ apples = r$$

The number of apples in

$$Huge\ Pile = g + r$$
$$1st\ Pile = 2nd\ Pile = g$$
$$3rd\ Pile = r - g$$

$$3(g + r) = 14(r - g)$$
$$3g + 3r = 14r - 14g$$
$$17g = 11r$$
$$\frac{g}{r} = \frac{11}{17}$$
$$g : r = 11 : 17$$

MCMC 1.104

Mark all correct options

Alok counted the total number of apples and oranges and the difference between them. He found that 50% of one was equal to 10% of the other. What of the following could be the ratio of apples to oranges?

A. 3:2
B. 2:3
C. 3: 5
D. Can't be determined

Let the number of apples and oranges be A and O. However, we do not know which is larger, and hence which one is A, and which one is O.

$$\frac{1}{10}(A + O) = \frac{5}{10}(A - O)$$
$$A + O = 5A - 5O$$
$$4A = 6O$$
$$\frac{A}{O} = \frac{6}{4} = \frac{3}{2}$$
$$A : O = 3 : 2$$

$$Option\ A, Option\ B$$

Hence, option E (since we don't know which one is more).

Check with $A = 3, O = 2$

$$\frac{1}{10}(A + O) = \frac{1}{10}(3 + 2) = \frac{5}{10} = \frac{1}{2}$$
$$\frac{1}{2}(A - O) = \frac{1}{2}(3 - 2) = \frac{1}{2} \times 1 = 1$$

Explain why the equation below is an incorrect way of solving:

$$\frac{5}{10}(A + O) = \frac{1}{10}(A - O)$$

$$Total\ No. of\ Fruits > Difference\ of\ Two\ Fruits$$
$$A + O > A - O$$

Taking 50% of $A + O$ will not work since 50% of a larger number cannot be equal to 10% of a smaller number.

If you solve the above equation, you will get negative values of A and O, which does not make sense since they represent the number of apples and oranges, which must be a positive integer.

B. Equations

Example 1.105

I have three blue marbles for every two red marbles that I have. I get a gift of fifteen blue marbles and four red marbles. I now have two blue marbles for every red marble. Find the total number of marbles I now have.

Single Variable Solution

The original ratio of blue marbles to red marbles is:

$$3:2=\underbrace{3x}_{Blue}:\underbrace{2x}_{Red}$$

If we add the fifteen blue marbles, and the four red marbles, the ratio becomes:

$$\underbrace{3x+15}_{Blue}:\underbrace{2x+4}_{Red}=2:1$$

Convert the ratios into fractions:

$$\frac{3x+15}{2x+4}=\frac{2}{1}\Rightarrow 3x+15=4x+8\Rightarrow 7=x$$

Hence, the total number of marbles after the gift is:

$$=3x+2x+19=5x+19=35+19=54$$

Two Variable Solution

Let the number of blue marbles be b, and the number of red marbles be r. Before the gift:

$$\frac{b}{r}=\frac{3}{2}\Rightarrow b=\frac{3}{2}r$$

After the gift:

$$\frac{b+15}{r+4}=2\Rightarrow b+15=2r+8$$

Substitute $b=\frac{3}{2}r$ from above

$$\frac{3}{2}r+15=2r+8$$

$$7=\frac{r}{2}$$

$$r=14$$

$$b=\frac{3}{2}r=21$$

$$b+r+19=21+14+19=54$$

Example 1.106

Two numbers are in the ratio 3:5. If 9 is subtracted from each, then they are in the ratio, 12:23. What is the larger number? (JMET 2011/83)

Algebra Method

Let the numbers be:

$$3x:5x$$

Then:

$$\frac{3x-9}{5x-9}=\frac{12}{23}$$

Cross-multiply:

$$69x-9(23)=60x-9(12)$$

Collate like terms on one side:

$$9x=-9(12)+9(23)$$

Solve for x:

$$x=11$$

Since we want the larger number

$$5x=55$$

Trial and Error Method

$$12:23\rightarrow 21:32$$
$$24:46\rightarrow 33:55=3:5$$

C. Inequalities

Example 1.107

At the beginning of the winter, there were at least 66 students registered in a ski class. After the class started, eleven boys transferred into this class and thirteen girls transferred out. The ratio of boys to girls in the class was then 1:1. Which of the following is not a possible ratio of boys to girls before the transfers? (**CEMC Gauss Grade 8 2013/25)**

(A) 4:7
(B) 1:2
(C) 9:13
(D) 5:11
(E) 3:5

Let the original number of students be

$$Boys = b, \qquad Girls = g$$

After the class started:

$$\underbrace{b}_{Boys} + 11 = \underbrace{g}_{Girls} - 13 \Rightarrow g = b + 24$$

$$b + g \geq 66 \Rightarrow b + b + 24 \geq 66 \Rightarrow 2b \geq 42 \Rightarrow b \geq 21$$

$$Ratio\ of\ Boys\ to\ girls = b: g = \frac{b}{g} = \frac{b}{b+24}$$

$$Option\ A: \frac{b}{b+24} = \frac{4}{7} \Rightarrow 7b = 4b + 96 \Rightarrow 3b = 96 \Rightarrow b = 32$$
$$Option\ B: \frac{b}{b+24} = \frac{1}{2} \Rightarrow 2b = b + 24 \Rightarrow b = 24$$
$$Option\ C: \frac{b}{b+24} = \frac{9}{13} \Rightarrow 13b = 9b + 216 \Rightarrow 4b = 216 \Rightarrow b = 54$$
$$Option\ D: \frac{b}{b+24} = \frac{5}{11} \Rightarrow 11b = 5b + 120 \Rightarrow 6b = 120 \Rightarrow b = 20$$
$$Option\ E: \frac{b}{b+24} = \frac{3}{5} \Rightarrow 5b = 3b + 72 \Rightarrow 2b = 72 \Rightarrow b = 36$$

But since $b \geq 21$, 20 is not a valid answer.

$$Option\ D$$

Example 1.108

Ram and Shyam have monthly incomes in the ratio 3: 4, and monthly expenses in the ratio 4: 5. If monthly savings for Shyam are 400, determine the maximum value of savings for Ram.

Let

$$Income\ for\ Ram\ and\ Shyam\ be\ I_R\ and\ I_S$$
$$Expenses\ for\ Ram\ and\ Shyam\ be\ E_R\ and\ E_S$$

Then, savings for Ram

$$= I_R - E_R$$

Substitute $I_R = \frac{3}{4}I_S, E_R = \frac{4}{5}E_S$:

$$= \frac{3}{4}I_s - \frac{4}{5}E_S$$

Substitute $I_S = E_S + 400$:

$$= \frac{3}{4}(E_S + 400) - \frac{4}{5}E_S = \frac{3}{4}E_S + 300 - \frac{4}{5}E_S = 300 - \frac{E_S}{20}$$

Since $E_s > 0$, the maximum value of the savings for Ram is 300.

D. K method

The K method is a standard, especially important method.
It converts relations between quantities into ratios.

Example 1.109

Rishabh's collection of 63 marbles is such that yellow marbles and red marbles are in the ratio $3:4$. Find the number of marbles of each color.

Short Method

$$\underbrace{3}_{Yellow} : \underbrace{4}_{Red} : \underbrace{7}_{Total} = 27:36:63$$

K method

Let the number of yellow marbles be y, and the number of red marbles be r.

$$y:r = 3:4$$

Convert both sides of the above into fractions:

$$\frac{y}{r} = \frac{3}{4}$$

Eliminate fractions by multiplying both sides by $4r$:

$$4y = 3r$$

Introduce a new constant K which is equal to the above:

$$4y = 3r = K$$

Solve for each variable in terms of K:

$$4y = K \Rightarrow y = \frac{K}{4}, \qquad 3r = K \Rightarrow r = \frac{K}{3}$$

Now, we know the sum of the yellow and the red marbles is 63. Hence:

$$y + r = 63$$

Substitute $y = \frac{K}{4}$ and $r = \frac{K}{3}$ in the above:

$$\frac{K}{4} + \frac{K}{3} = 63$$

$$\frac{3K + 4K}{12} = 63$$

$$\frac{7K}{12} = 63$$

$$K = 63 \times \frac{12}{7} = 108$$

$$y = \frac{K}{4} = \frac{108}{4} = 27$$

$$r = \frac{K}{3} = \frac{108}{3} = 36$$

Example 1.110

Two-thirds of my sister's age is the same as three-fourth of my age.

A. Without finding the ratio, who is older? Me or my sister?
B. Find the ratio of my sisters age to mine.

$$\frac{2}{3}s = \frac{3}{4}m \Rightarrow 8s = 9m \Rightarrow \frac{s}{m} = \frac{9}{8} \Rightarrow s:m = 9:8$$

Example 1.111

The angles of a triangle $\angle A$, $\angle B$ and $\angle C$ are such that $3m\angle A = 4m\angle B = 6m\angle C$. Find the angles of the triangle.

Short Method

$$3A = 4B \Rightarrow \frac{A}{B} = \frac{4}{3} \Rightarrow A:B = 4:3$$
$$4B = 6C \Rightarrow \frac{B}{C} = \frac{6}{4} = \frac{3}{2} \Rightarrow B:C = 3:2$$

$$A:B:C = 4:3:2:\underbrace{9}_{Total} = 80:60:40:\underbrace{180}_{Total}$$

***K* method**

Introduce a new constant K which is equal to the above:

$$3A = 4B = 6C = K$$

Solve for the values of each variable:

$$\underbrace{A = \frac{K}{3}, \quad B = \frac{K}{4}, \quad C = \frac{K}{6}}_{Relation\ I}$$

Substitute the values from Relation I in $A + B + C = 180$ to get:

$$\underbrace{\frac{K}{3}}_{A} + \underbrace{\frac{K}{4}}_{B} + \underbrace{\frac{K}{6}}_{C} = 180$$

Add the fractions on the LHS by taking the $LCM(3,4,6) = 12$:

$$\frac{4K}{12} + \frac{3K}{12} + \frac{2K}{12} = 180$$

Simplify:

$$\frac{9K}{12} = 180 \Rightarrow K = 180 \times \frac{12}{9} = 20 \times 12 = 240$$

Find the value of A, B and C:

$$A = \frac{K}{3} = 80, \quad B = \frac{K}{4} = 60, \quad C = \frac{K}{6} = 40$$

Example 1.112

If $3X = 4Y = 7Z$, then $Z:Y:X =$

$$3X = 4Y = 7Z \underbrace{= K}_{Assume}$$

Solve for each variable:

$$X = \frac{K}{3}, Y = \frac{K}{4}, Z = \frac{K}{7}$$

Now we find the required ratio:

$$Z:Y:X = \frac{K}{7}:\frac{K}{4}:\frac{K}{3}$$

Divide throughout by K:

$$= \frac{1}{7}:\frac{1}{4}:\frac{1}{3}$$

Multiply throughout by $LCM(7,4,3) = 84$:

$$= 12:21:28$$

Example 1.113

25% percent of the scorpion population equals 12% of the cactus population, which equals 33% of the desert rat population. The ratio of the scorpion population to the cactus population to the desert rat population is:

$$\frac{25}{100}S = \frac{12}{100}C = \frac{33}{100}R = K$$
$$\frac{S}{4} = \frac{3C}{25} = \frac{33R}{100} = K$$
$$S = 4K, C = \frac{25K}{3}, R = \frac{100K}{33}$$

$$S:C:R = 4K:\frac{25K}{3}:\frac{100K}{33} = 4:\frac{25}{3}:100:33 = 132:275:100$$

Explain what is wrong with the following "solution":

$$\frac{25}{100}P = \frac{12}{100}P = \frac{33}{100}P \Rightarrow \frac{25}{100} = \frac{12}{100} = \frac{33}{100} \Rightarrow 25:12:33$$

This solution assumes that the population of scorpions, cactuses and desert rats is each equal to P, which is not the case.

Example 1.114

25 cobblers, 20 tailors, 18 hatters, and 12 glovers spent a total of 133 shillings. Five cobblers spent as much as four tailors, twelve tailors spent as much as nine hatters, and six hatters spent as much as eight glovers. Find out how much each of the four types of people spent. **(Amusements in Mathematics, H. E. Dudeney, Adapted)**

Let the amount spent by each cobbler be c, each tailor be t, each hatter be h, and each glover be g.
Then, the total amount spent by all of them together is:

$$25c + 20t + 18h + 12g = 133$$

The K method is difficult to introduce since we do not have a relation that gives an equality between all four quantities. Hence, we look at each individual constraint given in the question.
Five cobblers spent as much as four tailors:

$$5c = 4t \Rightarrow 25c = 20t$$

Twelve tailors spent as much as nine hatters:

$$12t = 9h \Rightarrow 24t = 18h$$

Six hatters spent as much as eight glovers:

$$6h = 8g \Rightarrow 9h = 12g \Rightarrow 12t = 9h = 12g$$

$$\underbrace{20t}_{25c} + 20t + \underbrace{24t}_{18h} + \underbrace{12t}_{12g} = 133 \Rightarrow t = \frac{133}{76} = \frac{7}{4}$$

Alternate Method

$$5c = 4t \Rightarrow c = \frac{4}{5}t$$

$$12t = 9h \Rightarrow h = \frac{3}{4}t$$

$$6h = 8g \Rightarrow g = \frac{6h}{8} = \frac{3}{4}h = \frac{3}{4}\left(\frac{3}{4}t\right) = \frac{9}{16}t$$

E. Factorization

If the number of variables is more than the number of equations:

- we cannot solve them completely
- we can still find the ratio between the variables

Example 1.115

Find $a:b$, if $4a^2 - 13ab + 3b^2 = 0$

$4a^2 - 12ab - ab + 3b^2 = 0$
$4a(a - 3b) - b(a - 3b) = 0$
$(4a - b)(a - 3b) = 0$
$4a - b = 0 \Rightarrow 4a = b \Rightarrow a:b = 1:4$
$a - 3b = 0 \Rightarrow a = 3b \Rightarrow a:b = 3:1$

Example 1.116

(MCMC): At the parade ground, while inspecting his troops, arranged in square battalions, with their commanding officer at the head of the formation, a general noticed that if the number of battalions and the number of soldiers to a side were interchanged, he could still get square formations and the commanding officer at their head. The number of people at the parade ground, assuming no one other than the ones discussed were present, could have been:

A. 126
B. 218
C. 64
D. 345

x = soldiers/side
y = no. of battalions

As per condition:
$x^2y = y^2x$

We can't solve this equation, but we can find the ratio of x and y.
$x^2y - y^2x = 0$
$xy(x - y) = 0$
$x = 0$ OR $y = 0$ OR $x = y$

Discard zero values: $x = y$

Hence, the number of soldiers in formation must be a cube plus two people (commanding officer, and the inspecting general).
Hence, options B and D.

1.6 Proportions

A. Checking for Proportions

When we have pairs of numbers which are in the same ratio, these are called a proportion. Proportions have lot of application in maths in higher grades. For example, in Geometry, etc.

1.117: Proportions

Consider the four numbers a, b, c, d. If the ratio of the first two numbers is equal to the ratio of the last numbers. That is, if:

$$a: b = c: d$$

Then we say that the four numbers are in proportion.

For example, since

$$2: 4 = 6: 12$$

We say that the ratios are in proportion

1.118: Writing a proportion

If $a: b = c: d$, then we say that a, b, c, d are in proportion, and we write:

$$\underbrace{2}_{1^{st}\,Term} : \underbrace{4}_{2^{nd}\,Term} :: \underbrace{6}_{3^{rd}\,Term} : \underbrace{12}_{4^{th}\,Term}$$

- Note that the ratios are separated by a colon, as they usually are.
- The first ratio and the second ratio are separated by a double colon.

$$\underbrace{2}_{1^{st}\,Term} : \underbrace{4}_{2^{nd}\,Term} :: \underbrace{6}_{3^{rd}\,Term} : \underbrace{12}_{4^{th}\,Term}$$

Is read as

$$2 \text{ is to } 4 \text{ is proportional to } 6 \text{ is to } 12$$
$$OR$$
$$2 \text{ is to } 4 \text{ as to } 6 \text{ is to } 12$$

Example 1.119

Are the numbers 2, 3, 4, 6 in proportion?

Method I: Compare the simplified ratios

Find the ratio of the first two numbers and note that it is already simplified:

$$2: 3$$

Find the ratio of the next two numbers, and simplify it:

$$4: 6 = 2: 3$$

Since both pairs of ratios are the same, the numbers are in proportion.

Method II: Convert into fractions and compare the fractions

Write the ratio of the first two numbers as a fraction

$$\frac{2}{3}$$

Write the ratio of the last two numbers as a fraction

$$\frac{4}{6} = \frac{2}{3}$$

Since the two fractions are equal, we know that the four numbers form a proportion.

Method III: Check if the ratio of the first two numbers is equivalent to the ratio of the last two numbers

$$\underbrace{2:3}_{\substack{Ratio\ of \\ First\ Two\ Numbers}} = (2:3)\times 2 = \underbrace{4:6}_{\substack{Ratio\ of \\ Last\ Two\ Numbers}}$$

Since the ratio of the first two numbers is equivalent to the ratio of the last two numbers, the four numbers form a proportion.

Method IV: Convert into fractions and use the cross-multiplication method to compare
Write the ratio of the first two numbers as a fraction. Write the ratio of the last two numbers as a fraction. Compare the two fractions using the cross-multiplication method:

$$\frac{2}{3} ? \frac{4}{6} \Rightarrow 12 ? 12$$

Example 1.120
Are the numbers 4, 10, 6, 15 in proportion?

We need to check if 4,10,6,15 are in proportion.
Find the ratio of the first two numbers.

$$4:10 = 2:5$$

Similarly, find the ratio of the next two numbers:

$$6:15 = 2:5$$

Since both the ratios are the same, the numbers are in proportion.

Example 1.121
King High School has 3 volleyball players for every 7 basketball players. Queen High School has 6 volleyball players for every 14 basketball players. Do the ratios of volleyball to basketball players in King High School and Queen High School form a proportion?

King High School:

$$Volleyball: Basketball = 3:7$$

Queen High School:

$$Volleyball: Basketball = 6:14 = 3:7$$

B. Means and Extremes

1.122: Means and Extremes
If a, b, c, d are in proportion, then the two numbers on the outside are called the extremes. The two numbers in the middle are called the means.

$$\underbrace{a}_{Extreme}, \underbrace{b}_{Mean}, \underbrace{c}_{Mean}, \underbrace{d}_{Extreme}$$

Example 1.123
If $3:4 :: 9:12$, then identify the means and the extremes.

$$Extremes\ are\ \{3,12\}$$

$$Means\ are\ \{4,9\}$$

1.124: Test using Product of Means and Extremes

If four numbers a, b, c, d are in proportion, then the product of the means (bc) is equal to the product of the extremes (ad):

$$ad = bc$$

If four numbers a, b, c, d are in proportion, then

$$a:b = c:d$$

Convert the ratios into fractions:

$$\frac{a}{b} = \frac{c}{d}$$

Multiply both sides by bd:

$$bd \times \frac{a}{b} = \frac{c}{d} \times bd$$

Simplifying gets us the result we want:

$$ad = bc$$

All of the above steps are reversible, and hence, if

$$ad = bc$$

Then the four numbers are in proportion.

Example 1.125

A. Using the test of product of means and extremes, show that that the numbers 2,4,6,12 are in proportion.
B. Check if the numbers 2,3,4,5 are in proportion.
C.

Part A

$$\underbrace{\underbrace{2 \times 12}_{Product\ of\ Means} \quad = \quad \underbrace{4 \times 6}_{Product\ of\ Extremes}}_{\substack{Holds\ for\ all\ proportions \\ Converse\ is\ also\ true}}$$

Part B

Find the product of the means:

$$3 \times 4 = 12$$

Find the product of the extremes:

$$2 \times 5 = 10$$

Check if the means are equal to the extremes:

$$10 \neq 12$$

Hence, the numbers are not in proportion.

Example 1.126

Disha wanted to check if the numbers 2,3,4,5 are in proportion. She used the steps given below:

$$\frac{2}{3} = \frac{4}{5} \Rightarrow 2 \times 5 = 4 \times 3 \Rightarrow 10 = 12$$

She concluded that the numbers are not in proportion.

A. Is her answer correct?
B. Is her process correct? If her process is not correct, identify the correct process.

Part A
Yes, her answer is correct.

$$\frac{2}{3} \stackrel{?}{=} \frac{4}{5} \Rightarrow 10 \neq 12$$

Part B
Her process is not correct, because you cannot use an equality sign for two quantities that are not equal. The correct version is given alongside.

Example 1.127
Each of the parts below has four numbers. Decide if the four numbers are in proportion?

A. 0.2, 0.4, 8, 16
B. $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{7}$
C. $\frac{1}{27}, \frac{1}{6}, \frac{1}{9}, \frac{1}{2}$

Part A

$$\frac{0.2}{0.4} = \frac{2}{4} = \frac{1}{2}, \frac{8}{16} = \frac{1}{2} \Rightarrow Frm$$

Part B

$$\frac{1}{3} \times \frac{1}{7} = \frac{1}{21}$$
$$\frac{1}{4} \times \frac{1}{5} = \frac{1}{20} \neq \frac{1}{20}$$

Part C

$$\frac{1}{27} \times \frac{1}{2} = \frac{1}{54} = \frac{1}{6} \times \frac{1}{9}$$

Example 1.128
A shoe store noted that there were 40 customers on Monday, which increased to 50 on Tuesday. Sales on Monday were Rs. 10,000, while those on Tuesday were Rs. 12,000.
Decide if:

$$Ratio\ 1: Customers\ \ On\ Monday: Customers\ on\ Tuesday$$
$$Ratio\ 2: Sales\ on\ Monday: Sales\ on\ Tuesday$$

Forms a proportion

$$Ratio\ 1 = 40\ customers: 50\ customers = 40:50 = 4:5$$
$$Ratio\ 2 = 10{,}000\ Rs.: 12{,}000\ Rs. = 10:12 = 5:6$$

Since the two ratios are not equivalent, they do not form a proportion.

C. Finding A Single Missing Term
One of the most important applications of proportions is setting up a proportion, and then finding a missing term. This is useful in geometry also when working with rates.

1.129: Setting up a Proportion
If the numbers a, b, c, d are in proportion, then we know that:

$$\frac{a}{b} = \frac{c}{d}$$

Writing this equation is called setting up a proportion.

Example 1.130

A. The numbers $2, 5, p, 12$ are proportional. Set up a proportion.
B. Formula an equation connecting $1, -2, 7, r$ given that the numbers are in proportion.

Part A

$$\frac{2}{5} = \frac{9}{12}$$

Part B

$$-\frac{1}{2} = \frac{7}{r}$$

Example 1.131

A. The first, second and fourth terms of a proportion are 2, 3 and 5. Find the third term. Show the term that you have found is correct by applying the test of a proportion.
B. The first three terms of a proportion are 15, 20 and 30. What is the fourth term? Show the term that you have found is correct by applying the test of a proportion.

Part A

Let the third term be x. Then, $2, 3, x, 5$ are in proportion. Set up the proportion:

$$\frac{2}{3} = \frac{x}{5} \Rightarrow x = \frac{2}{3} \times 5 = \frac{10}{3}$$

The terms are $2, 3, \frac{10}{3}, 5$:

$$Product\ of\ Extremes = 2 \times 5 = 10$$
$$Product\ of\ Means = 3 \times \frac{10}{3} = 10$$

Part B

Let the fourth term be x. Then, since the terms are in proportion, we must have:

$$\frac{15}{20} = \frac{30}{x} \Rightarrow \frac{20}{15} = \frac{x}{30} \Rightarrow x = 30 \times \frac{20}{15} = 40$$

Check

$$15 \times 40 = 600$$
$$20 \times 30 = 600$$

Example 1.132

If each set of numbers below are in proportion, then find the variables.

A. $3, 5, a, 25$
B. $21, b, 3, 5$
C. $\frac{1}{2}, \frac{1}{7}, \frac{1}{3}, c$
D. $d, 0.3, 0.2, 0.6$

$$5a = 3 \times 25 \Rightarrow a = 15$$
$$3b = 21 \times 5 \Rightarrow b = 35$$
$$\frac{1}{2} \times c = \frac{1}{7} \times \frac{1}{3} \Rightarrow \frac{c}{2} = \frac{1}{21} \Rightarrow c = \frac{2}{21}$$
$$0.6d = 0.06 \Rightarrow d = 0.1$$

MCMC 1.133

Mark all correct options

$\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{7}$ are four terms. What replacements should we make for the first and the last term so that the terms are in proportion?

A. 1, and $\frac{1}{20}$
B. 1 and 2
C. 2 and 3
D. 3 and 4

$$\frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$$
$$Option\ A: 1 \times \frac{1}{20} = \frac{1}{20}$$

The other options will have integers for answers. So, they are not correct. The final answer is:
$$Option\ A$$

$\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{7}$ are four terms. What replacement should we make for the second term so that the terms are in proportion?

$x \times 1/5 = 1/21$
$x = 1/21 \div 1/5 = 1/21 \times 5 = 5/21$

$\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{7}$ are four terms. What should be subtracted from the third term so that the terms are in proportion?

$\frac{1}{4} \times x = 1/21$
$x = 1/21 \div \frac{1}{4} = 4/21$
$1/5 - 4/21 = (21 - 20)/105 = 1/105$

Challenge 1.134[3]

If $A = 0.11\ldots$, $B = x$, $C = \frac{1}{27}$, and $D = 0.1 + 0.0666\ldots$, and A, B, C and D are in proportion then $x =$?

$1/9 \times 1/6 = x \times 1/27$
$x = \frac{1}{2}$

D. Continued Proportion

1.135: Continued Proportions

If the second term, and the third term in a proportion are the same, then these terms are called the mean proportional, and the proportion is called a continued proportion.

Example 1.136

Suppose the numbers $3, x, x, 12$ form a proportion. Find the value of x.

Set up an equation with the proportion:

[3] You should know recurring decimals before attempting this example.

$$\frac{3}{x}=\frac{x}{12}$$

Multiply both sides $12x$ to eliminate fractions:

$$36=x^2$$

Take the square root both sides, and keep in mind that we will get two answers:

$$x=\pm\sqrt{36}=\pm 6$$

1.137: Continued Proportions (Alternate)

If we have three terms

$$a,b,c$$

And the ratio of the first and second terms is equal to the ratio of the second and the third terms, then the terms are said to be in continued proportion.

$$\frac{a}{b}=\frac{b}{c}\Rightarrow b^2=ac$$

Here
a is called the first proportional.
b is called the mean proportional.
c is called the third proportional.

$$\frac{a}{b}=\frac{b}{c}$$

Multiply both sides by bc:

$$bc\cdot\frac{a}{b}=\frac{b}{c}\cdot bc$$
$$ca=b^2$$

Rearrange:

$$b^2=ac$$

Example 1.138

A. Consider the numbers 2, 8, 8, 32, which are in proportion. What is the mean proportional?

$$Mean\ Proportional=8$$

Example 1.139

Show that 5, 10,20 are in continued proportion.

Method I

$$Ratio\ of\ first\ and\ second\ term=\frac{5}{10}=\frac{1}{2}$$
$$Ratio\ of\ second\ and\ third\ term=\frac{10}{20}=\frac{1}{2}$$

Since both the ratios are equal, the terms are in continued proportion.

Method II

Since the ratios are in continued proportion, we must have:

$$\frac{5}{10}?\frac{10}{20}\Rightarrow 100?100\Rightarrow 100=100\Rightarrow Vertified$$

Example 1.140

The numbers below are in proportion. Find the variable.

A. $3, a, a, 27$
B. $61, b, b, 7$
C. $13, c, c, 13$
D.

Part A

$$a^2 = (3)(27) = 81$$
$$a = \pm\sqrt{81} = \pm 9$$

Part B

$$b^2 = 7 \times 61 \Rightarrow b = \pm\sqrt{61 \cdot 7} = \pm\sqrt{427}$$

Part C

$$c^2 = 13^2 \Rightarrow c = \pm 13$$

Example 1.141

The numbers below are in proportion. Find the variable.

A. $\frac{2}{3}, a, a, \frac{27}{32}$
B. $\frac{3}{5}, b, b, \frac{11}{7}$
C. $\frac{343}{5}, c, c, \frac{125}{7}$
D. $\frac{2}{11}, d, d, \frac{3}{5}$

Part A

$$a^2 = \frac{2}{3} \times \frac{27}{32} = \frac{9}{16} \Rightarrow a = \pm\sqrt{\frac{9}{16}} = \pm\frac{\sqrt{9}}{\sqrt{16}} = \pm\frac{3}{4}$$

Part B

$$b^2 = \frac{3}{5} \cdot \frac{11}{7} = \frac{33}{35} \Rightarrow b = \pm\sqrt{\frac{33}{35}}$$

Part C

$$c^2 = \frac{343}{5} \cdot \frac{125}{7} = \frac{7^3}{5} \cdot \frac{5^3}{7} = 7^2 \cdot 5^2$$

Take the square root both sides:

$$c = \pm\sqrt{7^2 \cdot 5^2} = \pm(7 \cdot 5) = \pm 35$$

Part D

$$d^2 = \frac{2}{11} \cdot \frac{3}{5} = \frac{6}{55} \Rightarrow d = \pm\sqrt{\frac{6}{55}}$$

Example 1.142

If x, y, z are in continued proportion, then write $[(x+y)+z][(x-y)+z]$ in terms of x^2, y^2 and z^2.

$$[(x+y)+z][(x-y)+z]$$
$$= (x+y)(x-y) + z(x+y) + z(x-y) + z^2$$
$$= x^2 - y^2 + z(x+y+x-y) + z^2$$
$$= x^2 - y^2 + 2zx + z^2$$

Substitute $\frac{x}{y} = \frac{y}{z} \Rightarrow xz = y^2$:

$$= x^2 - y^2 + 2y^2 + z^2$$

$$= x^2 + y^2 + z^2$$

1.143: Geometric Mean

When three numbers are in continued proportion, the middle term is said to be the geometric mean of the other two.

If a, b, c are in continued proportion, then b is the geometric mean of a and c, and hence:

$$b^2 = ac$$

$$\frac{a}{b} = \frac{b}{c} \Rightarrow b^2$$

- This does not introduce any new math for now. Rather, it is a different way of looking at continued proportions.

Example 1.144

Find the geometric mean of

A. 2 and 32
B. 5 and 125
C. 3 and 12
D. 4 and 5

Part A

$$b^2 = 2 \cdot 32 \Rightarrow b = \pm\sqrt{64} = \pm 8$$

Part B

$$b^2 = 5 \cdot 125 \Rightarrow b = \pm\sqrt{625} = \pm 25$$

Part C

$$b^2 = 3 \cdot 12 \Rightarrow b = \pm\sqrt{36} = \pm 6$$

Part D

$$b^2 = 4 \cdot 5 \Rightarrow b = \pm\sqrt{20}$$

Example 1.145

If 64 is divided into three parts proportional to 2, 4, and 6, the smallest part is: (AHSME 1950/1)
(Answer as a mixed number)

$$2 + 4 + 6 = 12$$
$$\frac{2}{12} \times 64 = \frac{32}{3} = 10\frac{2}{3}$$

1.7 Direct Proportion

A. Unitary Method

The unitary method is not necessary for the simple question below. However, answers to questions in many contexts can be conceptualized using the unitary method.

Example 1.146

A dozen apples cost \$24. How much must Sohan pay for five apples?

Step by Step Method

Note that

$$12\ apples = \$24$$

Divide both sides by 12:

$$1\ apple = \frac{\$24}{12} = \$2$$

Multiply both sides by 5:

$$5\ apple = \$2 \times 5 = \$10$$

Shortcut Method

Since $12\ apples = \$24$, the cost of:

$$5\ Apples = \frac{24}{12} \times 5 = 2 \times 5 = 10$$

Example 1.147

A stack of paper containing 500 sheets is 5 cm thick. Approximately how many sheets of this type of paper would there be in a stack 7.5 cm high? **(AMC 8 1985/6)**

$$500\ sheets = 5\ cm$$

Divide both sides by 5:

$$100\ sheets = 1\ cm$$

Multiply both sides by 7.5:

$$750\ sheets = 1\ cm$$

B. Fractions versus Decimals

Conceptually, fractions and decimals will lead to the same answers. However, without a calculator, decimals can be substantially more difficult to work with. Hence, fractions are often preferable.

1.148: Fractions versus Decimals

It is not necessary that the numbers that we work with always result in "nice" decimals. In such scenarios, it is often better to work with fractions.

$$\frac{2}{4} = \frac{1}{2} = 0.5 \rightarrow Nice$$

$$\frac{3}{7} = 0.\overline{428571} \rightarrow Recurring\ Decimal, not\ \text{nice}$$

If you get recurring decimals, it is better to work with fractions.

Example 1.149

Seven apples cost \$11. Find the cost of eight apples.

$$7\ apples\ cost\ \$11$$

Divide both sides by 7:

$$1\ apple\ costs\ \$\frac{11}{7}$$

Multiply both sides by 8:

$$8\ apples\ cost\ \frac{11}{7} \times 8 = \$\frac{88}{7}$$

1.150: Converting Decimals to Fractions

If the question gives us decimals, and those decimals lead to recurring decimals in the calculation, it is better to convert the decimals to fractions before proceeding with the calculations.

Example 1.151

Arnav can purchase seven candies for five dollars and fifty cents. Find the cost of six candies.

Convert the dollars given into a fraction:
5 dollars and fifty cents

$$\text{5 dollars and fifty cents} = 5.50 = 5 + 0.5 = 5 + \frac{1}{2} = \frac{11}{2}\$$$

$$7\ candies\ cost\ \frac{11}{2}\$$$
$$1\ candies\ costs\ \frac{11}{2} \times \frac{1}{7} = \frac{11}{14}\$$$
$$6\ candies\ cost\ \frac{11}{14} \times 6 = \frac{33}{7}\$$$

C. Rates

1.152: Rates

Rates refers to the speed at which something is getting done.

1.153: Unit Rates

Example 1.154

A. A construction company can make 25 houses in one year. Find the number of houses that can be made in a month. Also, find the number of months needed to make a house.
B. A circus artist can do 23 backflips in 7 minutes. How many backflips can the artist do in a minute? How many minutes does the artist take to do a backflip?
C. A quick service restaurant serves 2300 burgers in eight hours. What is the number of burgers that it can serve in a minute? What is the time, in minutes, needed to serve one burger?

Part A

$$25\ houses\ in\ 12\ months$$
$$\frac{25}{12}\ houses\ in\ 1\ month$$
$$1\ house\ in\ \frac{12}{25}\ months$$

Part B

$$23\ backflips\ in\ 7\ minutes$$
$$\frac{23}{7}\ backflips\ in\ 1\ minute$$
$$1\ backflip\ in\ \frac{7}{23}\ minutes$$

Part C

$$2300\ burgers\ in\ 8\ Hours$$
$$2300\ burgers\ in\ 480\ Minutes$$

$$\frac{2300}{480}=\frac{230}{48}=\frac{115}{24}\ burgers\ in\ 1\ minute$$
$$1\ burger\ in\ \frac{480}{2300}=\frac{24}{115}\ minutes$$

1.155: Comparing Rates

When comparing rates, we need to make one of the two parameters the same.

Example 1.156

A kangaroo family goes to a hopping competition. The adult kangaroos can hop 7 times in two minutes, while the baby kangaroos can hop 13 times in three minutes.

A. Compare the hopping speeds for the two.
B. If the adult kangaroos hop 9 meters at a time, and the baby kangaroos hop 4 meters at a time, compare the speed at which they travel.

Part A

$$Adults: 7\ hops\ in\ 2\ minutes \Rightarrow \frac{7}{2}\ hops\ per\ minute$$
$$Babies: 13\ hops\ in\ 3\ minutes \Rightarrow \frac{13}{3}\ hops\ per\ minute$$

Since the time is the same, we can compare the number of hops.

$$\frac{7}{2}?\frac{13}{3} \to \frac{21}{\underbrace{6}_{Adults}} < \frac{26}{\underbrace{6}_{Babies}} \Rightarrow Babies\ hop\ faster$$

Part B

Distance travelled in a minute

$$Adults=\frac{7}{2}\times 9=\frac{63}{2}\ meters$$
$$Babies=\frac{13}{3}\times 4=\frac{52}{3}\ meters$$

$$\frac{63}{2}?\frac{52}{3} \to \frac{189}{6} > \frac{104}{6} \Rightarrow Adults\ travel\ faster$$

Example 1.157

In a chilly eating contest, the lead competitor eats 25 chillies in 4 minutes. Find the rate at which the competitor eats in:

A. Chillies per minute
B. Chillies per second
C. Chillies per hour

$$25\ chillies\ in\ 4\ minutes$$
$$\frac{25}{4}\ chillies\ in\ 1\ minutes=\frac{25}{4}\frac{chillies}{minute}$$
$$\frac{25}{4}\times\frac{1}{60}=\frac{5}{4}\times\frac{1}{12}\ chillies\ in\ 1\ minutes=\frac{5}{48}\frac{chillies}{minute}$$

$$\frac{25}{4}\times 60\ chillies\ in\ 1\ hour=375\ \frac{chillies}{hour}$$

1.158: Changing a Rate

Example 1.159

A woodcutter can split 20 logs in 40 minutes. He needs to split 40 logs in 40 minutes. By what factor should he increase his speed to complete the task?

We can potentially do this orally. The speed is just double. But, look at the method, which is useful in more complicated questions:

$$20 \ logs\ in\ 40\ minutes$$
$$\frac{1}{2}\log in\ 1\ minutes$$

$$40\ logs\ in\ 40\ minutes$$
$$1\ log\ in\ 1\ minutes$$

$$1 \div \frac{1}{2} = 1 \times \frac{2}{1} = 2\ times$$

Example 1.160

A carpenter can make seven chairs in five minutes.

A. How long will he take to make five chairs?
B. By what factor must he change his speed to make five chairs in seven minutes?

Part A

$$7\ chairs\ in\ 5\ minutes$$

Divide both sides by 7:

$$\frac{7}{7} = 1\ chair\ in\ \frac{5}{7}\ minutes$$

Multiply both sides by 5:

$$1 \times 5 = 5\ chairs\ in\ \frac{5}{7} \times 5 = \frac{25}{7}\ chairs$$

Part B

The carpenter's current speed is:

$$7\ chairs\ in\ 5\ minutes \Rightarrow \frac{7}{5}\ chairs\ in\ 1\ minutes$$

The required speed is:

$$5\ chairs\ in\ 7\ minutes \Rightarrow \frac{5}{7}\ chairs\ in\ 1\ minute$$

Now that the time is the same in both speeds, we can compare the number of chairs.

$$\frac{5}{7} \div \frac{7}{5} = \frac{5}{7} \times \frac{5}{7} = \frac{25}{49}$$

Hence, the carpenter should change his speed to be $\frac{25}{49}$ of his original speed.

D. Using Proportions

Example 1.161
A stack of paper containing 500 sheets is 5 cm thick. Approximately how many sheets of this type of paper would there be in a stack 7.5 cm high? **(AMC 8 1985/6)**

Proportions

Note that we have a proportion.

$$500:5 = n:7.5$$

Convert the ratios into fractions:

$$\frac{500}{5} = \frac{n}{7.5}$$

Solve for n:

$$n = \frac{500}{5} \times 7.5$$

Unitary Method

$$5\ cm \to 500\ sheets$$
$$1\ cm \to 100\ sheets$$
$$7.5\ cm \to 750\ sheets$$

E. Proportionality

1.162: Proportional Relationships
If x is directly proportional to y, then we write:

$$\underbrace{x \propto y}_{\substack{Proportional\\ Relationship}} \quad \Leftrightarrow \quad \underbrace{x = ky}_{\substack{Proportional\\ Equation}}$$

- The double arrow $\Leftrightarrow$ indicates the LHS ($proportional\ relationship$) and the RHS ($proportional\ equation$) represent the same idea.
- k is an unknown constant to be determined.
- In this case, k is called the constant of proportionality.

Example 1.163
A dozen apples cost $24.

A. Set up a proportional relationship between the cost of apples, C, and the number of apples, N.
B. Convert the proportional relationship into an equation.
C. Find the value of the constant of proportionality.
D. Write the equation from Part B using the constant from Part C.
E. Use the equation from Part D to find the cost of five apples?

Part A

Cost of apples is proportional to the number of apples:

$$C \propto N$$

Part B

Convert the proportional relationship into an equation by introducing a constant of proportionality:

$$C \propto N \Leftrightarrow C = kN$$

k is a constant whose value we do not know.

Part C

To find the value of k, substitute the known values of C and K into the equation.
When $C = 24, N = 12$

$$24 = k(12) \Rightarrow k = 2$$

Here, k will be the cost of one apple.

Part D

$$C = 2N$$

Part E

$$C = 2(5) = 10$$

F. Joint Proportion

1.164: More than Two Variables

When you have more than two variables, you have to be especially careful with your calculations.

- In particular, you can only manipulate two variables at a time, while keeping the third constant.

Example 1.165: Logical Method

Three people can dine at three seats in three hours. How many people can dine at nine seats in nine hours?

The given information is:

$$3\ people\ at\ 3\ seats\ in\ 3\ hours$$

The key idea that we will change the number of people, and one more variable at a time (not two).

Divide people by 3, and correspondingly divide number of seats by 3. Leave number of hours unchanged:

$$1\ person\ at\ 1\ seats\ in\ 3\ hours$$

Multiply people by 9, and correspondingly multiply number of seats by 9. Leave number of hours unchanged:

$$9\ people\ at\ 9\ seats\ in\ 3\ hours$$

Now, we have matched the people required, and also the seats required. Match the number of hours by multiplying the number of hours by 3. Multiply the number of people also by 3, and leave the number of seats unchanged:

$$27\ people\ at\ 9\ seats\ in\ 9\ hours$$

Example 1.166: Constant of Proportionality

Three people can dine at three seats in three hours. How many people can dine at nine seats in nine hours?

Introduce some variables. Let

$$p = no.of\ people, \qquad s = no.of\ seats, \qquad t = time\ in\ hours$$

If the time remains the same, but we have more seats, the number of people who can dine increases. Hence:

$$\underbrace{p \propto s, if\ t\ is\ constant}_{Relation\ I}$$

If the number of seats remains the same, but we have more time, the number of people who can dine increases. Hence:

$$\underbrace{p \propto t, if\ s\ is\ constant}_{\textit{Relation II}}$$

We can combine the above to get a joint proportional relationship:

$$p \propto st$$

Convert the proportional relationship into an equation:

$$\underbrace{p = Kst}_{\textit{Equation I}}$$

Substitute the known values $p = 3, s = 3, t = 3$ in the above equation:

$$3 = k(3)(3) \Rightarrow K = \frac{1}{3}$$

Substitute the constant of proportionality $K = \frac{1}{3}$ in Equation I to get:

$$\underbrace{p = \frac{1}{3}st}_{\textit{Equation II}}$$

Substitute $s = 9, t = 9$ in Equation II:

$$p = \frac{1}{3} \cdot 9 \cdot 9 = 27$$

Example 1.167

Five cubes take five minutes to melt when taken out of the fridge, and put into five different plates. How long will ten cubes take to melt when:

A. put into ten different plates outside of the fridge?
B. you have only one plate, and take the cubes out from the fridge one at a time.

Part A

Each cube melts at its own pace, in its own plate.

$$5\ Minutes$$

Part B

One cube takes 5 minutes to melt. In total, 10 cubes take

$$5 \times 10 = 50\ minutes$$

Example 1.168

Three cats can catch three rats in three minutes. How many rats can nine cats catch in nine minutes?

$$3\ cats\ can\ catch\ 3\ rats\ in\ 3\ minutes$$

Imagine that each cat is assigned to one rat.

$$\underbrace{C \to R}_{\textit{3 Minutes}}, \underbrace{C \to R}_{\textit{3 Minutes}}, \underbrace{C \to R}_{\textit{3 Minutes}}$$

$$1\ cat\ can\ catch\ 1\ rat\ in\ 3\ minutes$$

Multiply the number of cats by 9, and keep the minutes the same:

$$9 \text{ } cats \text{ } can \text{ } catch \text{ } 9 \text{ } rats \text{ } in \text{ } 3 \text{ } minutes$$

Multiply the number of minutes by 3, and keep the cats the same:

$$9 \text{ } cats \text{ } can \text{ } catch \text{ } 27 \text{ } rats \text{ } in \text{ } 9 \text{ } minutes$$

Example 1.169

Three cats can catch three rats in three minutes. How much time will five cats take to catch five rats?

$$3 \text{ } Cats \text{ } can \text{ } catch \text{ } 3 \text{ } rats \text{ } in \text{ } 3 \text{ } minutes$$

Imagine you have three cats. One is black, one is grey, and one is brown.
Also, you have three rats, numbered 1, 2 and 3.

Black Cat	Grey Cat	Brown Cat
Rat 1	Rat 2	Rat 3
3 Minutes	3 Minutes	3 Minutes

Hence,

$$1 \text{ } Cat \text{ } can \text{ } catch \text{ } 1 \text{ } rat \text{ } in \text{ } three \text{ } minutes$$

And if you have five cats to catch five rats, then:

$$5 \text{ } Cats \text{ } can \text{ } catch \text{ } 5 \text{ } rats \text{ } in \text{ } three \text{ } minutes$$

1.8 Inverse Proportion

A. Basics

Direct proportion means that an increase in one quantity results in an increase in another quantity. For example

- $Time\ and\ Distance$: If I can run 10 km in one hour, then I can run $10x$ km in x hours
- $Costs$: If an apple costs d dollars, then a apples will cost ad dollars. Specifically, if an apple costs 5 dollars, then 3 apples will cost $5 \times 3 = 15\ dollars$
- $Thickness\ of\ Paper$: If one sheet of paper is 1 mm thick, then a stack of 500 sheets will be $1 \times 500 = 500\ mm = 50cm = 0.5\ m$ thick

However, sometimes quantities are inversely related. An inverse relation means that an increase in one quantity results in a decrease in another quantity. For example, suppose I divide 12 apples among my friends (not including me).

- If I have two friends, each friend 6 apples.
- If I have three friends, each friend gets four apples.
- If I have six friends, each friend gets two apples. And so.

The important point is that increase in number of friends results in decrease in number of apples per friend.

1.170: Inverse Proportion

Two quantities are inversely related when increase in one quantity results in a decrease in the other quantity. Specifically, if x and y are inversely related, we write

$$x \propto \frac{1}{y}$$

Example 1.171

A. Convert the inverse relationship $x \propto \frac{1}{y}$ into an equation by introducing a constant of proportionality.
B. I have 12 apples to be divided among my friends. Analyze this in terms of the equation from Part A.
C. Use this to determine the number of apples that each friend gets if you have 9 friends.

Part A
It is traditional (but not compulsory) to use the letter k for a constant of proportionality.

$$x \propto \frac{1}{y} \Leftrightarrow x = \frac{k}{y}$$

Part B
Let a be the number of apples. Let f be the number of friends. Then:

$$a = \frac{k}{f}$$

When $apples = a = 6, friends = f = 2$

$$6 = \frac{k}{2} \Rightarrow k = 12$$

So, my equation becomes:

$$a = \frac{12}{f}$$

Part C
Substitute $f = 9$ in:

$$a = \frac{12}{f} = \frac{12}{9} = \frac{4}{3}$$

Example 1.172

I have a ditch to be dug. A single worker can finish digging the ditch in 20 days. To speed up the work, I put three additional workers on the task. Find the number of days in which the task will be completed.

Let w be the number of workers. Let d be the number of days each worker works.

$$w = \frac{k}{d} \Rightarrow wd = k$$

Substitute $w = 1, d = 20$

$$k = wd = (1)(20) = 20$$

Substitute $w = 4$ in $wd = 20$:

$$4d = 20 \Rightarrow d = 5$$

B. Time, Speed and Distance

$$T \propto D$$
$$T \propto \frac{1}{S}$$

If you have the correct units, the relation is:

$$T = \frac{D}{S}$$

Example 1.173

A centipede can crawl $3\frac{3}{4}$ cm in $5\frac{1}{4}$ $seconds$. How long will the centipede take to cross a $2\frac{2}{3}$ cm long twig.

$$Speed = \frac{Distance}{Time} = \frac{3\frac{3}{4}\ cm}{5\frac{1}{4}\ seconds} = \frac{\frac{15}{4}\ cm}{\frac{21}{4}\ seconds} = \frac{15}{4} \times \frac{4}{21}\frac{cm}{second} = \frac{5}{7}\frac{cm}{second}$$

$$Time = \frac{Distance}{Speed} = \frac{2\frac{2}{3}\ cm}{\frac{5}{7}\frac{cm}{second}} = \frac{\frac{8}{3}\ cm}{\frac{5}{7}\frac{cm}{second}} = \frac{8}{3} cm \times \frac{7}{5}\frac{second}{cm} = \frac{56}{15}\ seconds$$

C. Rates

Example 1.174

The students in Mrs. Reed's English class are reading the same 760-page novel. Three friends, Alice, Bob and Chandra, are in the class. Alice reads a page in 20 seconds, Bob reads a page in 45 seconds and Chandra reads a page in 30 seconds. Chandra and Bob, who each have a copy of the book, decide that they can save time by "team reading" the novel. In this scheme, Chandra will read from page 1 to a certain page and Bob will read from the next page through page 760, finishing the book. When they are through they will tell each other about the part they read. What is the last page that Chandra should read so that she and Bob spend the same amount of time reading the novel? **(AMC 8 2006/15)**

The ratio of time taken is:

$$\underbrace{45}_{Bob} : \underbrace{30}_{Chandra} = 3:2$$

Since $LCM(3,2) = 6$, find the reading that can be done in 6 seconds.

$$Bob = 3\ sec \times 2\ units$$
$$Chandra = 2\ sec \times 3\ units$$

Hence, the ratio of pages read is:

$$2:3$$

Chandra must read:

$$\frac{760}{5} \times 3 = 456$$

1.9 Componendo – Dividendo

A. Ratio Revision

1.175: Ratio as a Fraction

$$a:b = c:d: \Leftrightarrow \frac{a}{b} = \frac{c}{d}$$

1.176: Reciprocal of Ratios (Invertendo)

$$\frac{a}{b}=\frac{c}{d} \Leftrightarrow \frac{b}{a}=\frac{d}{c}$$

$$\frac{x}{y}=\frac{2}{3} \Rightarrow \frac{y}{x}=\frac{3}{2}$$

1.177: Rearrangement Property (Alternendo)

$$\frac{a}{b}=\frac{c}{d} \Leftrightarrow \frac{a}{c}=\frac{b}{d}$$

B. Componendo

Example 1.178

If $3a-5b=0$, then find $\frac{a}{b}$.

$$3a=5b \Rightarrow \frac{a}{b}=\frac{5}{3}$$

Example 1.179

If $7a-12b=0$, then find $\frac{a+b}{b}$.

$$7a=12b$$
$$\frac{a}{b}=\frac{12}{7}$$

Add 1 to both sides:

$$\frac{a}{b}+1=\frac{12}{7}+1$$
$$\frac{a+b}{b}=\frac{19}{7}$$

1.180: Componendo

$$\frac{a}{b}=\frac{c}{d} \Leftrightarrow \frac{a+b}{b}=\frac{c+d}{d}$$

Add 1 to both sides of the LHS:

$$\frac{a}{b}+1=\frac{c}{d}+1 \Rightarrow \frac{a+b}{b}=\frac{c+d}{d}$$

Example 1.181

Apply Componendo to

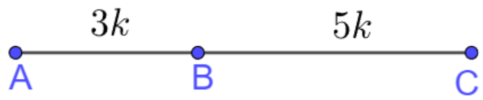
$$\frac{N}{D}=\frac{n}{d}$$

$$\frac{N+D}{D}=\frac{n+d}{d}$$

Example 1.182

Point B lies on AC such that $AB:BC=3:5$. Find $AC:BC$.

$$\frac{AB}{BC} = \frac{3}{5}$$



Add 1 to both sides:

$$\frac{AB}{BC} + 1 = \frac{3}{5} + 1 \Rightarrow \frac{AB + BC}{BC} = \frac{3+5}{5}$$

$$\frac{AC}{BC} = \frac{8}{5}$$

C. Dividendo

Example 1.183

If $5a - 2b = 0$, then find $\frac{a-b}{b}$.

$$\frac{a}{b} = \frac{2}{5}$$

Subtract 1 from both sides:

$$\frac{a}{b} - 1 = \frac{2}{5} - 1$$
$$\frac{a-b}{b} = \frac{-3}{5}$$

1.184: Dividendo

$$\frac{a}{b} = \frac{c}{d} \Leftrightarrow \frac{a-b}{b} = \frac{c-d}{d}$$

Subtract 1 from both sides of the LHS:

$$\frac{a}{b} - 1 = \frac{c}{d} - 1 \Rightarrow \frac{a-b}{b} = \frac{c-d}{d}$$

Example 1.185

Apply Componendo to

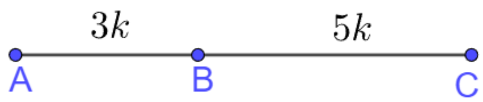
$$\frac{N}{D} = \frac{n}{d}$$

$$\frac{N-D}{D} = \frac{n-d}{d}$$

Example 1.186

Point B lies on AC such that $AC: BC = 8: 5$. Find $AB: BC$.

$$\frac{AC}{BC} = \frac{8}{5}$$



Subtract 1 to both sides:

$$\frac{AB}{BC} - 1 = \frac{8}{5} - 1$$

$$\frac{AC - BC}{BC} = \frac{8-5}{5}$$
$$\frac{AB}{BC} = \frac{3}{5}$$

D. Componendo-Dividendo

1.187: Componendo

$$\frac{a}{b} = \frac{c}{d} \Leftrightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d}$$

Part A: Prove the statement from left to right:

$$From\ Componendo: \underbrace{\frac{a+b}{b} = \frac{c+d}{d}}_{Equation\ I}$$
$$From\ Dividendo: \underbrace{\frac{a-b}{b} = \frac{c-d}{d}}_{Equation\ II}$$

Divide Equation by Equation II:

$$\frac{\frac{a+b}{b}}{\frac{a-b}{b}} = \frac{\frac{c+d}{d}}{\frac{c-d}{d}}$$

Cancel and simplify to get:

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}$$

Part B: Prove the statement from right to left:

$$\frac{(a+b)+(a-b)}{(a+b)-(a-b)} = \frac{(c+d)+(c-d)}{(c+d)-(c-d)}$$
$$\frac{2a}{2b} = \frac{2c}{2d}$$
$$\frac{a}{b} = \frac{c}{d}$$

Example 1.188

Prove that:

$$S = \frac{6pq}{p-q} \Rightarrow \frac{S+3q}{S-3q} + \frac{S-3p}{S+3p} = 2$$

Verification

We can verify that small values of p and q:

$$p = 2, q = 1 \Rightarrow S = \frac{6(2)(1)}{2-1} = \frac{12}{1} = 12$$
$$\frac{S+3q}{S-3q} + \frac{S-3p}{S+3p} = \frac{12+3}{12-3} + \frac{12-6}{12+6} = \frac{15}{9} + \frac{6}{18} = \frac{5}{3} + \frac{1}{3} = \frac{6}{3} = 2$$

Verification

$$\frac{S+3q}{S-3q} = k$$

Example 1.189

Prove that

$$\frac{x}{y}=\frac{p}{q}\Rightarrow\frac{x^2+y^2}{p^2+q^2}=\frac{xy}{pq}$$

$$\frac{x}{y}=\frac{p}{q}$$

Apply Componendo-Dividendo:

$$\frac{x+y}{x-y}=\frac{p+q}{p-q}$$

Square both sides:

$$\frac{x^2+2xy+y^2}{x^2-2xy+y^2}=\frac{p^2+2pq+q^2}{p^2-2pq+q^2}$$

Apply Componendo-Dividendo:

$$\frac{(x^2+2xy+y^2)+(x^2-2xy+y^2)}{(x^2+2xy+y^2)-(x^2-2xy+y^2)}=\frac{(p^2+2pq+q^2)+(p^2-2pq+q^2)}{(p^2+2pq+q^2)-(p^2-2pq+q^2)}$$

$$\frac{2(x^2+y^2)}{4xy}=\frac{2(p^2+q^2)}{4pq}$$

$$\frac{x^2+y^2}{xy}=\frac{p^2+q^2}{pq}$$

$$\frac{x^2+y^2}{p^2+q^2}=\frac{xy}{pq}$$

Example 1.190

Prove that

$$\frac{x}{y}=\frac{p}{q}\Rightarrow\frac{x^2+y^2}{p^2+q^2}=\frac{x^2-y^2}{p^2-q^2}$$

$$\frac{x^2}{y^2}=\frac{p^2}{q^2}$$

$$\frac{x^2+y^2}{x^2-y^2}=\frac{p^2+q^2}{p^2-q^2}$$

$$\frac{x^2+y^2}{p^2+q^2}=\frac{x^2-y^2}{p^2-q^2}$$

Example 1.191

Prove that

$$\frac{x}{y}=\frac{p}{q}\Rightarrow\frac{x^2+y^2}{x^2-y^2}=\frac{p^2+q^2}{p^2-q^2}$$

$$\frac{x^2}{y^2}=\frac{p^2}{q^2}$$

$$LHS=\frac{x^2+y^2}{x^2-y^2}=\frac{\frac{x^2}{y^2}+1}{\frac{x^2}{y^2}-1}=\frac{\frac{p^2}{q^2}+1}{\frac{p^2}{q^2}-1}=\frac{\frac{p^2+q^2}{q^2}}{\frac{p^2-q^2}{q^2}}=\frac{p^2+q^2}{p^2-q^2}$$

Example 1.192

Prove that

$$\frac{x}{y}=\frac{p}{q}\Rightarrow\frac{x^2+y^2}{p^2+q^2}=\frac{y^2}{q^2}$$

$$\frac{x}{y}=\frac{p}{q}=k\Rightarrow\frac{x^2}{y^2}=\frac{p^2}{q^2}=k^2\Rightarrow x^2=y^2k^2,p^2=q^2k^2$$

$$\frac{x^2+y^2}{p^2+q^2}=\frac{y^2k^2+y^2}{q^2k^2+q^2}=\frac{y^2(k^2+1)}{q^2(k^2+1)}=\frac{y^2}{q^2}$$

1.193: k method

If $\frac{x}{y}=\frac{p}{q}$, then:

$$\frac{x^2+y^2}{p^2+q^2}=\frac{x^2-y^2}{p^2-q^2}=\frac{xy}{pq}=\frac{y^2}{q^2}$$

E. k Method

1.194: k method

The k method introduces a constant k that is equal to the value of the ratio:

$$\frac{a}{b}=\frac{c}{d}=k\Rightarrow a=bk,\qquad c=dk$$

Example 1.195

Prove the statement below using the k method:

$$\frac{a}{b}=\frac{c}{d}\Leftrightarrow\frac{a+b}{a-b}=\frac{c+d}{c-d}$$

$$LHS=\frac{a+b}{a-b}=\frac{bk+b}{bk-b}=\frac{b(k+1)}{b(k-1)}=\frac{k+1}{k-1}$$
$$RHS=\frac{c+d}{c-d}=\frac{dk+d}{dk-d}=\frac{d(k+1)}{d(k-1)}=\frac{k+1}{k-1}=LHS$$

Example 1.196

Prove that:

$$\frac{a}{b}=\frac{c}{d}\Rightarrow\frac{a+2b}{b}=\frac{c+2d}{d}$$

$$\frac{a}{b}+2=\frac{c}{d}+2$$
$$\frac{a+2b}{b}=\frac{c+2d}{d}$$

2. Variation

2.1 Direct Variation

Definition

If X increases when Y increases, and decreases when Y decreases then
$X \propto Y^n$
It can be converted into an equation:
$X = kY^n$
where:

- k is a constant (does not change)
- k is called the **constant of proportionality**

Example 2.1

$$P \propto \frac{1}{V} \Rightarrow P = \frac{k}{V} \Rightarrow PV = k$$

Example 2.2

1 apple costs Rs. 2. Frame a relationship between the number of apples and their cost.

Let number of apples be A
Let the cost of apples be $R\ rupees$

$$C \propto A \Rightarrow C = kA \Rightarrow C = 2k$$

Example 2.3

$$D \propto T \Rightarrow D = T \times S$$
$$D \propto S \Rightarrow D = T \times S$$

Example 2.4

x and y are in direct proportion.

A. Frame an equation that connects x and y.
B. Are $x - 1$ and $y - 1$ also in direct proportion.

If x and y are in direct proportion, then for some constant k:

$$x = ky \Rightarrow k = \frac{x}{y} \Rightarrow \underbrace{k = 2}_{Suppose}$$

$x-1$	0	1	2		x	1	2	3
$y-1$	1	3	5		y	2	4	6
$\frac{x-1}{y-1}$	$\frac{0}{1}$	$\frac{1}{3}$	$\frac{2}{5}$		$\frac{x}{y}$	$\frac{1}{2}$	$\frac{2}{4} = \frac{1}{2}$	$\frac{3}{6} = \frac{1}{2}$

We can check by framing an equation that must hold if the two quantities are in direct proportion:

$$x - 1 = k_1(y - 1) \Rightarrow k_1 = \frac{x - 1}{y - 1}$$

$x - 1$ and $y - 1$ are NOT in direct proportion

Example 2.5

x and y are in inverse proportion.

A. Frame an equation that connects x and y.
B. Are $x+2$ and $y+2$ also in inverse proportion.

$$x = \frac{k}{y} \Rightarrow k = xy \Rightarrow \underbrace{k=2}_{Suppose}$$

$x+2$	3	5	6		x	1	2	4
$y+2$	4	3	$\frac{5}{2}$		y	2	1	$\frac{1}{2}$
$k=xy$	12	15	15		$k=xy$	2	2	2

$x+2$ and $y+2$ are NOT in inverse proportion.

Example 2.6: Determining an equation

x varies directly with y^3. Also, $x = 6$ when $y = 4$

A. State the relationship between x and y in the form of a proportional relationship.
B. State the relationship between x and y in the form of an equation.
C. Find k.
D. State the relationship between x and y in the form of an equation that does not include variables other than x and y.

$$x \propto y^3$$
$$x = ky^3$$

Substitute $x = 6, y = 4$ in $x = ky^3$:

$$6 = 64k \Rightarrow k = \frac{3}{32}$$
$$x = \frac{3}{32}y$$

Method to determine an equation:

- State the proportional relationship
- State an equation connecting the variables (with k)
- Find k (using the information given)
- State the equation with the value of k

Example 2.7

A. a varies directly with b
B. a varies directly with the square of b
C. a varies directly with the cube of b

Convert the above statements into

Q1: Proportional relationships.

Q2: Equations.

S1:

A. $A \propto b$
B. $A \propto b^2$
C. $A \propto b^3$

S2:

A. $A = kb$
B. $A = kb^2$
C. $A = kb^3$

Example 2.8

The cost of a diamond (DC) varies directly as the square of its weight(W), while the cost of transporting (TC) it varies directly as its weight.

Q3 (MCMC): The relationship between the cost of transporting a diamond and its weight is correctly represented by:

A. $TC = W$
B. $TC = kW$
C. $TC \propto W$
D. $TC \propto kW$

S3: Options B and C

Q4 (MCMC): The relationship between the cost of a diamond and its weight is correctly represented by:

A. $DC = W^2$
B. $DC = kW^2$
C. $DC \propto W^2$
D. $DC \propto kW^2$

S4: Options B and C

Example 2.9

Answer the following questions based on the information below. Information from one question may be used in other questions.

The period of oscillation of a pendulum is governed by the relationship

$$T \propto \sqrt{\frac{L}{g}}$$

where:
L is the length of the pendulum
g is acceleration due to gravity

Q5: If

$$T = 2\pi\sqrt{\frac{L}{g}}$$

and g is a constant, then, the constant of proportionality is:

A. $2\pi/g$
B. L/g
C. $2\pi/L$
D. $2\pi/\sqrt{g}$
E. $2\pi L\sqrt{g}$

S5: Option D.

Note 1

This question can be answered using maths (no need for science). It is confusing because it presents information that may be unfamiliar.

A. 2π is a constant.
B. If g is a constant, we do not need to do anything further with it. It is a part of the constant of proportionality.

Example 2.10

The cost of a diamond varies directly as the square of its weight.

A. A diamond that weighs 1 gram costs a million dollars. Find the cost of a diamond that weighs 2 grams.
B. A diamond that weighs 3 grams costs 2 million dollars. Find the cost of a diamond that weighs 2 grams.

$$c \propto w^2 \Rightarrow c = kw^2$$

Part A

We work in millions of dollars for cost, and grams for weight.
Substitute $c = 1, w = 1$ in $c = kw^2$:

$$1 = kw \Rightarrow k = 1$$

Substitute $w = 2$ in $c = w^2$

$$c = 2^2 = 4 \Rightarrow 4\ Million\ Dollars$$

Part B

We work in millions of dollars.
Substitute $c = 2, w = 3$ in $c = kw^2$:

$$2 = k(3^2) \Rightarrow k = \frac{2}{9}$$

Substitute $w = 2$ in $c = \frac{2}{9}w^2$

$$c = \frac{2}{9}w^2 = \frac{2}{9} \times 4 = \frac{8}{9} \Rightarrow \frac{8}{9}\ Million\ Dollars$$

Example 2.11

The cost of a diamond varies directly as the square of its weight. A diamond that broke into four pieces with weights in the ratio 1:2:3:4 lost Rs. 70,000 of value in the process. What is the price of the original diamond?

A. 10,000
B. 50,000
C. 1,00,000
D. 5,00,000

S6: Let the weights of the pieces be x, $2x$, $3x$, and $4x$.
Then, the weight of the unbroken diamond $= 10x$

Proportionality Relationship and Equation:

$C \propto W^2 \quad \Rightarrow \quad C = kW^2$

By the information given in the question

$k(10x)^2 = kx^2 + k(2x)^2 + k(3x)^2 + k(4x)^2 + 70000$
$100kx^2 = kx^2 + 4kx^2 + 9kx^2 + 16kx^2 + 70000$
$70kx^2 = 70000$
$kx^2 = 1000$
$100kx^2 = 100000$

2.2 Inverse Variation

Linear Inverse Variation

Definition

If X increases when Y decreases, and X decreases when Y increases then

$$X \propto \frac{1}{Y^n}$$

It can be converted into an equation:

$$X = \frac{k}{Y^n}$$

where:

- k is a constant (does not change)
- k is called the **constant of proportionality**

Example 2.12

A violin string one foot long vibrates at a frequency of 32 cycles per one-tenth of a second. If length varies inversely as frequency, how many complete cycles will happen in a second with a 9-inch string?

$$L \propto \frac{1}{F} \Rightarrow L = \frac{k}{F} \Rightarrow LF = k$$

Substitute $F = 320$ / second, $L = 12$ inches:

$$k = LF = 12 \times 320$$

Substitute $L = 9, k = 12 \times 320$:

$$LF = k$$
$$9F = 12 \times 320$$
$$3F = 4 \times 320$$

Example 2.13

Answer the following three questions based on the information below.
The volume of a gas (which varies inversely with the pressure) is 5.1×10^{-5} m^3 at a pressure of 1,00,000 g/cm^2

Q6: If the volume increases, the pressure:

A. increases
B. decreases
C. remains constant
D. can increase or decrease

S7: Option B

Q7: If the pressure decreases, the volume:

A. increases
B. decreases
C. remains constant
D. can increase or decrease

S8: Option A

Q8: What pressure (in kg/cm^2) has to be applied to have a volume of 340 cm^3?

A. 12
B. 15
C. 17
D. 18

S9: Substitute $V = 51$ cm³, $P = 100$ kg/cm²:
$PV = k$
$k = 51 \times 100$

When V = 340:
P × 340 = 51 × 100
P = 5100 / 340 = 15

Non-Linear Inverse Variation

Example 2.1
Word Problems: Non-Linear Inverse Variation

2.3 Joint Variation

Varying with more than one variable

If X varies with more than one variable (A, B, C . . .), then X is said to jointly vary with these variables.

Example 2.14

Answer the following questions based on the information below.

X ∝ a if b is constant, and X ∝ b if a is constant

Q9 (MCMC): Which of the following is / are correct (where k_x is the constant of proportionality for x)?

A. $X = k_a a$
B. $X = k_b b$
C. $X = k_{ab} ab$
D. $X = k_a k_b ab$

S10: Options C and D

Q10 (MCMC): Which of the following is / are correct?

A. $X \propto ab$
B. $X \propto a/b$
C. $X \propto a^2 b^2$
D. $X \propto kab$

S11: Option A only

Q11 (Match the Column): A 10% increase in a, with b remaining constant, will result in a:

Variable	Change
A. Change in X	I. 10% Increase
B. Change in k_a	II. No change
C. Change in k_b	III. 10% Decrease
D. Change in k_{ab}	

S12: A: I, B: II, C: II, D: II

Example 2.15

Answer the following questions based on the information below.
The volume of an object with shape X varies jointly as its height and the square of its radius. An object with a radius of 9 feet and a height of 8 feet has $(1/\pi)$ times the volume as 6^6 cubes with a side length of 2 inches.

Q12: What is shape X?

A. Circle
B. Cone
C. Sphere
D. Cylinder

Q13: What is the volume of an object with a radius of 15 inches and a height of 7 inches.

S13: Volume of 6^6 cubes
$= 6^6 \times (2 \text{ inches})^3$
$= 6^6 \times (1/6 \text{ feet})^3$
$= 6^6 \times 1/6^3 \text{ feet}^3$
$= 6^3 \text{ feet}^3$
$= 216 \text{ feet}^3$

Substitute $V = 216$, $r = 9$, $h = 8$:
$V/\pi = kr^2h$
$216 = k\pi \times 81 \times 8$
$k = 1/3$

Hence, $V = (1/3)\pi r^2 h$
This is the formula for volume of a cone.
Hence, option B

S14: $V = (1/3)\ \pi \times 15^2 \times 7 = 525\ \pi$

Once you solve the first question, the second question takes hardly any time

16 Examples