Further Triangles

Revision: 2926

20 October 2025

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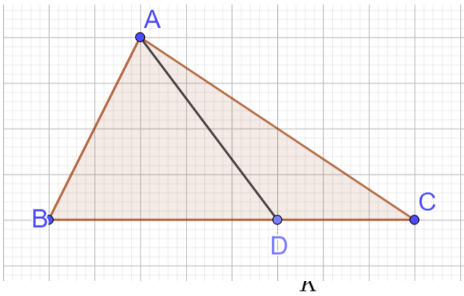
1. Angle Bisector and Incircle

1.1 Angle Bisectors

A. Definition

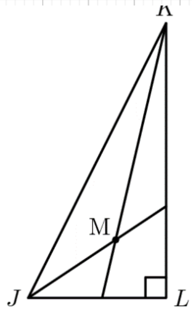


1.1: Angle Bisector
The line segment from the vertex of an angle that bisects the angle is called an angle bisector.

In the diagram (not drawn to scale), AD is the angle bisector if

$$\angle BAD = \angle CAD$$

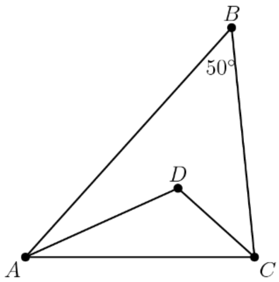
- Every triangle has three angle bisectors.
- An angle bisector never goes outside the triangle.



Example 1.2
In right triangle JKL, angle J measures 60 degrees and angle K measures 30 degrees. When drawn, the angle bisectors of angles J and K intersect at a point M. What is the measure of obtuse angle JMK? (MathCounts 2010 School Sprint)

$$180 - \angle KJM - \angle JKM = 180 - \frac{60}{2} - \frac{30}{2} = 180 - 30 - 15 = 135$$

B. Vertex Angle formed by Angle Bisectors



Example 1.3
The measure of angle ABC is $50°$, $\overline{AD}$ bisects angle BAC, and $\overline{DC}$ bisects angle BCA. The measure of angle ADC is (AMC 8 1996/24)

Algebraic Method
Let

$$\angle BAC = 2x \Rightarrow \angle DAC = x$$
$$\angle BCA = 2y \Rightarrow \angle DCA = y$$

By sum of angles of a triangle:

$$50 + 2x + 2y = 180 \Rightarrow 2x + 2y = 130 \Rightarrow x + y = 65$$

In ΔADC:

$$\angle ADC = 180 - x - y = 180 - (x + y) = 180 - 65 = 115$$

Logical Method
The base angles will add up to

$$180 - 50 = 130$$

After bisection, the inner angles will add up to

$$\frac{130}{2} = 65$$

The vertex angle ($\angle ADC$)

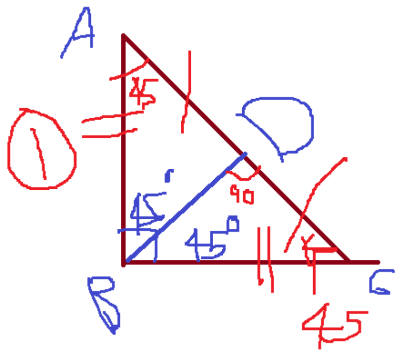
$$= 180 - 65 = 115$$

C. Right Triangle

Example 1.4

In isosceles right ΔABC, right-angled at B, the angle bisector from B intersects side AC at D. If the length of one leg of ΔABC is 1 unit:

A. Find the perimeter of ΔABD.
B. Find the area of ΔBDC.



Since ΔADB is a $45-45-90$ triangle:

$$AD = BD = \frac{1}{\sqrt{2}}$$

And it has perimeter

$$P = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 1 + \frac{2}{\sqrt{2}} = 1 + \sqrt{2}$$

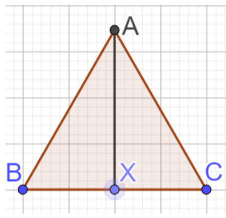
The area of ΔBDC

$$A = \frac{1}{2}hb = \frac{1}{2} \cdot BD \cdot DC = \frac{1}{2} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{4}$$

D. Equilateral Triangles

Example 1.5

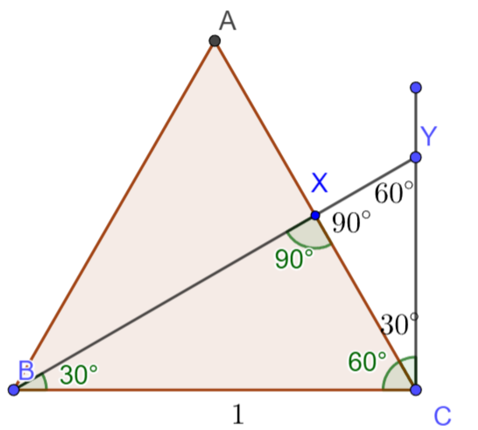
Show that the angle bisector of an equilateral triangle divides the triangle into two $30-60-90$ triangles.



$$\angle BAX = \angle \frac{BAC}{2} = \frac{60}{2} = 30°$$
$$\angle BXA = 180 - 30 - 60 = 90°$$

Example 1.6

Equilateral ΔABC has side length 1. The angle bisector from B intersects AC at X and YC at Y. YC is perpendicular to BC. Find the area of ΔXYC.



ΔBYC is a $30-60-90$ triangle.

$$YC = \frac{1}{\sqrt{3}}BC = \frac{1}{\sqrt{3}}$$

ΔXYC is a $30-60-90$ triangle.

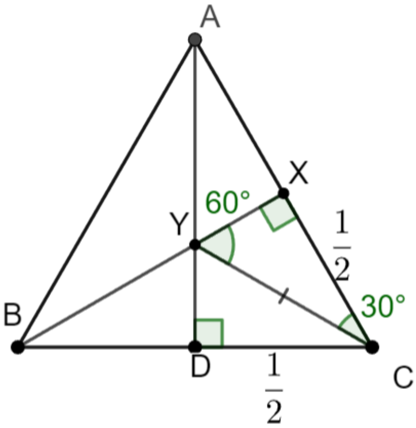
$$XY = \frac{1}{2} \times YC = \frac{1}{2} \times \frac{1}{\sqrt{3}} = \frac{1}{2\sqrt{3}}$$
$$XC = \frac{\sqrt{3}}{2} \times YC = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{3}} = \frac{1}{2}$$

$$[XYC] = \frac{1}{2}hb = \frac{1}{2} \cdot XY \cdot XC = \frac{1}{2} \cdot \frac{1}{2\sqrt{3}} \cdot \frac{1}{2} = \frac{1}{8\sqrt{3}} = \frac{\sqrt{3}}{24}$$

Example 1.7

Equilateral ΔABC has side length 1. The angle bisector from B intersects AC at X. The angle bisector from B

intersects the altitude from A at Y. Find the area of ΔXYC.



$$\Delta XYC \cong \Delta DYC$$
$$\angle XCY = \angle DCY = 30°$$
ΔXYC is a $30-60-90$ triangle.

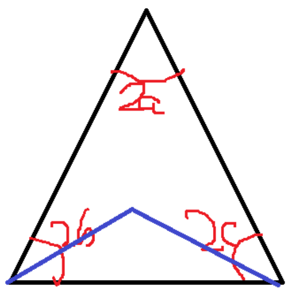
$$XY = \frac{XC}{\sqrt{3}} = \frac{1}{2} \times \frac{1}{\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

$$[XYC] = \frac{1}{2}hb = \frac{1}{2} \cdot XY \cdot XC = \frac{1}{2} \cdot \frac{1}{2\sqrt{3}} \cdot \frac{1}{2} = \frac{1}{8\sqrt{3}} = \frac{\sqrt{3}}{24}$$

E. Vertex Angle

1.8: Vertex Angle formed by Angle Bisectors

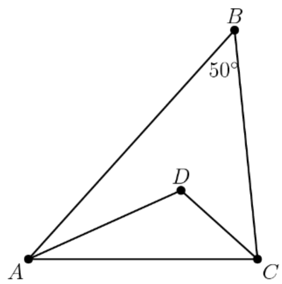
The vertex angle in the triangle formed by the base of a triangle and the angle bisectors of its base is equal to a right angle plus half the vertex angle of the original triangle.



In $\triangle ABC$ let $\angle$'s A, B and C have measures $2a$, $2b$ and $2c$ respectively.

By Sum of Angles of a Triangle:
$$2a + 2b + 2c = 180$$
Solve the above for a:
$$2a = 180 - 2b - 2c \Rightarrow a = 90 - b - c$$

The angle bisectors of $\angle B$ and $\angle C$ meet at X.
By Sum of Angles of a Triangle:
$$\angle X = 180 - b - c = 90 + 90 - b - c = 90 + a$$



Example 1.9

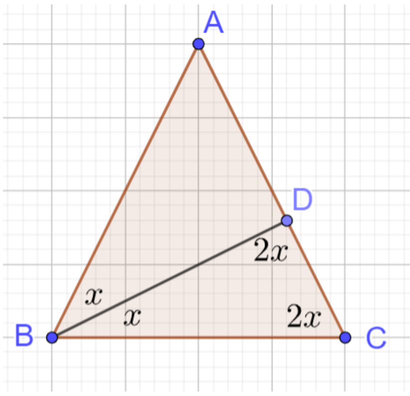
The measure of angle ABC is $50°$, $\overline{AD}$ bisects angle BAC, and $\overline{DC}$ bisects angle BCA. The measure of angle ADC is **(AMC 8 1996/24)**
(Redo the question using the property we just proved)

$$\angle ADC = 90 + \frac{1}{2} \cdot \angle ABC = 90 + \frac{1}{2} \cdot 50 = 90 + 25 = 115°$$

F. Challenging Problems

Example 1.10

In triangle ABC, $AB = AC$ and D is a point on $\overline{AC}$ so that $\overline{BD}$ bisects angle ABC. If $BD = BC$, what is the measure, in degrees, of angle A? **(MathCounts 2010 Chapter Sprint)**



In Isosceles ΔABC, let

$$\angle ABC = \angle ACB = 2x$$

In Isosceles ΔBDC,

$$\angle BDC = \angle BCD = 2x$$

By the definition of angle bisector

$$\angle DBC = \frac{2x}{2} = x$$

In ΔBDC, by sum of angles of a triangle:

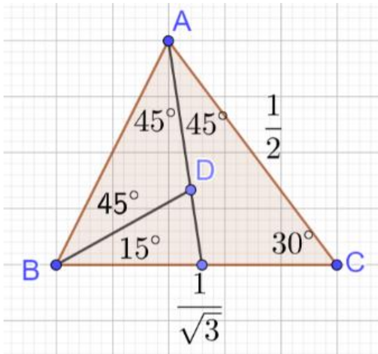
$$\underbrace{x}_{\angle DBC} + \underbrace{2x}_{\angle BCD} + \underbrace{2x}_{\angle BDC} = 180 \Rightarrow x = 36$$

In ΔABC, by sum of angles of a triangle:

$$\angle A = 180 - \underbrace{2x}_{\angle ABC} - \underbrace{2x}_{\angle ACB} = 180 - 144 = 36°$$

Challenge 1.11

In ΔABC, a line segment from B, drawn perpendicular to the angle bisector from A intersects it at D to form isosceles ΔABD. Also, BD divides angle B in the ratio 1:3. If the longest side of triangle ΔABC has length $\frac{1}{\sqrt{3}}$, find the second longest side.



If the perpendicular from B intersects AD, then

$$\angle ADB = 90°$$

ΔABD is isosceles right-angled triangle:

$$\angle BAD = \angle ABD = 45$$

Since AD is the angle bisector:

$$\angle DAC = \angle BAD = 45$$

Now, we are in a position to find $\angle DBC$ using the ratio $1:3$:

$$\angle DBC = 3\angle ABD = 135 \Rightarrow Not\ Possible$$

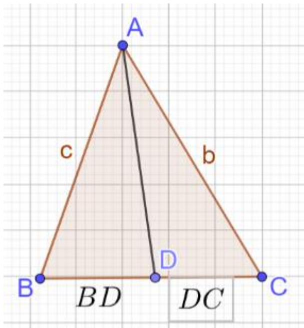
$$\angle DBC = \frac{1}{3}\angle ABD = 15$$

$$\angle ACB = 180 - 90 - 60 = 30 \Rightarrow \Delta BAC\ is\ 30 - 60 - 90\ triangle.$$

$$AC = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = \frac{1}{2}$$

1.2 Angle Bisector Theorem

A. Angle Bisector Theorem



1.12: Angle Bisector Theorem

An angle bisector of a triangle divides the side opposite it in the ratio of the other two sides of the triangle.

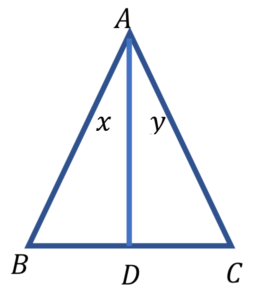
$$\frac{BD}{AB} = \frac{DC}{AC}$$

Example 1.13

In the diagram, AD is an angle bisector of the vertex angle A. Also,

$$AB = x = 24, AC = y = 36$$

A. Find the ratio $BD:DC$ in terms of x and y (as variables, not as numbers).
B. If $BC = 5$, find the length of BD and DC.
C. If $BD = 5$, find the length of DC.



Part A

By the angle bisector theorem:

$$BD:DC = x:y$$

Part B

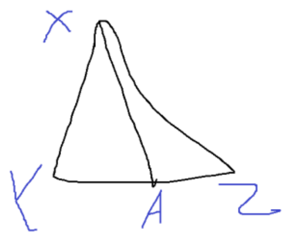
$$\frac{x}{y} = \frac{24}{36} = \frac{2}{3} \Rightarrow 2 + 3 = 5 \Rightarrow BC = \underbrace{2}_{BD} + \underbrace{3}_{DC} = 5$$

Part C

$$\frac{BD}{DC} = \frac{2}{3} \Rightarrow \frac{5}{DC} = \frac{2}{3} \Rightarrow DC = 5 \times \frac{3}{2} = 7.5$$

Example 1.14

In ΔXYZ, the angle bisector from X intersects side YZ at A. If the lengths of YA and AZ are x and y, respectively, and $XY = YA + YZ$, then find the length of XZ, in terms of x and y.



$$XY = YA + YZ = x + (x + y) = 2x + y$$

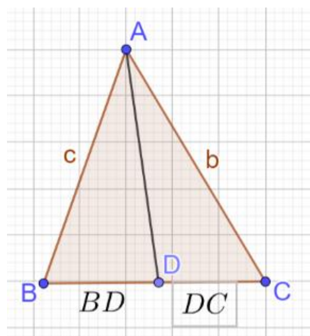
By the angle bisector theorem:

$$\frac{XY}{YA} = \frac{XZ}{AZ} \Rightarrow \frac{2x + y}{x} = \frac{XZ}{y} \Rightarrow XZ = \frac{y(2x + y)}{x} = \frac{2xy + y^2}{x}$$

1.15: Angle Bisector Theorem

The angle bisector from vertex A divides the base BC into two segments that have length in the ratio

$$AB:AC$$



$$\frac{BD}{AB} = \frac{DC}{AC} \Rightarrow \frac{BD}{DC} = \frac{AB}{AC} \Rightarrow BD:DC = AB:AC$$

Example 1.16

The sides of triangle CAB are in the ratio of $2:3:4$. Segment BD is the angle bisector drawn to the shortest side, dividing it into segments AD and DC. What is the length, in inches, of the longer subsegment of side AC if the length of side AC is 10 inches? Express your answer as a common fraction. **(MathCounts 2010 National Sprint)**

Without loss of generality, let

$$CB\ be\ the\ longer\ side \Rightarrow We\ need\ to\ find\ DC$$

AB and CD are in the ratio:

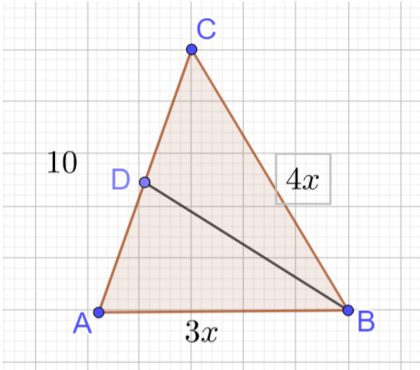
$$3x: 4x = 3: 4$$

Hence, by Angle Bisector Theorem:

$$AD: CD = 3: 4$$

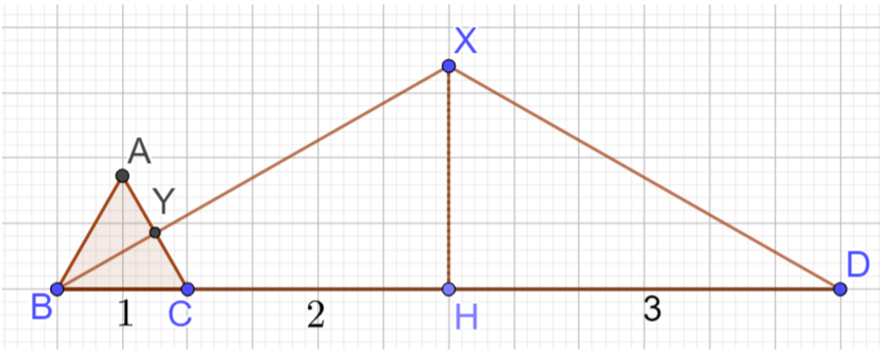
We know that length of AC is 10. Hence, we need to divide AC in the ratio 3: 4, and find the longer segment:

$$3 + 4 = 7 \Rightarrow DC = \frac{10}{7} \times 4 = \frac{40}{7}$$



Example 1.17

Equilateral ΔABC has side length 1. BC is extended to D such that $BD = 6$. The angle bisector from B intersects AC at Y and the perpendicular bisector of BD at X. Find the area of quadrilateral $CDXY$.



$$\angle ABC = 60° \Rightarrow \angle YBC = 30°$$

In $30 - 60 - 90\ \Delta XHB$:

$$XH = \frac{1}{\sqrt{3}} \cdot BH = \frac{1}{\sqrt{3}} \cdot 3 = \sqrt{3}$$

$$[XBD] = \frac{1}{2}hb = \frac{1}{2} \cdot XH \cdot BD = \frac{1}{2} \cdot \sqrt{3} \cdot 6 = 3\sqrt{3}$$

$$[BYC] = \frac{1}{2} \cdot ABC = \frac{1}{2} \cdot \frac{\sqrt{3}}{4} \cdot BC^2 = \frac{1}{2} \cdot \frac{\sqrt{3}}{4} \cdot 1^2 = \frac{\sqrt{3}}{8}$$

By complementary areas:

$$[CDXY] = [XBD] - [BYC] = 3\sqrt{3} - \frac{\sqrt{3}}{8} = \frac{24\sqrt{3}}{8} - \frac{\sqrt{3}}{8} = \frac{23\sqrt{3}}{8}$$

B. Checking for Angle Bisectors

Example 1.18

C. Right Triangles

In a right triangle, the angle bisector can be combined with the Pythagorean Theorem.

1 Pending

Example 1.19

A line bisecting the larger acute angle in a triangle with sides of length 33, 44 and 55 cm divides the opposite side into two segments. What is the length of the shorter segment of that side? Express your answer as a common fraction. (MathCounts 2019 Chapter Sprint Round/25)

The triangle is a right triangle since

$$(33,44,55) = 11(3,4,5)$$

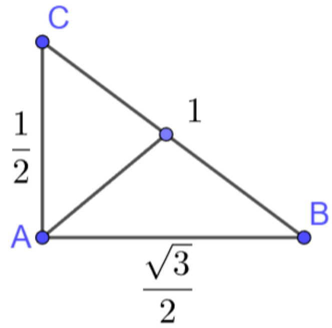
The side opposite the larger acute angle is 44.
By the angle bisector theorem, the shorter segment is:

$$44 \times \frac{3}{8} = \frac{33}{2}\ cm$$

Example 1.20

AD is the angle bisector from vertex A in right ΔABC, intersecting BC at D. ΔABC has $\angle B = 30°$, and hypotenuse $BC = 1$. Find the area of ΔABD and ΔADC.

The angle bisector will divide the triangle into two triangles in the ratio of the sides of the bisected angle:



$$[ADB]:[ADC] = \frac{\sqrt{3}}{2}:\frac{1}{2} = \sqrt{3}:1$$

Area of ΔABC

$$= \frac{1}{2}hb = \frac{1}{2} \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8}$$

Divide the area of ΔABC in the above ratio

$$[ABD] = \frac{\sqrt{3}}{8} \times \frac{\sqrt{3}}{\sqrt{3}+1} = \frac{3(\sqrt{3}-1)}{8(2)} = \frac{3\sqrt{3}-3}{16}$$

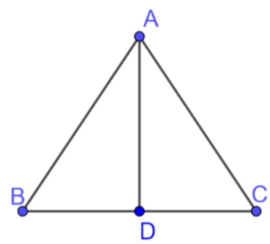
$$[ADC] = \frac{\sqrt{3}}{8} \times \frac{1}{\sqrt{3}+1} = \frac{\sqrt{3}(\sqrt{3}-1)}{8(2)} = \frac{3-\sqrt{3}}{16}$$

D. Isosceles and Equilateral Triangles

Example 1.21

In ΔABC, angle bisector AD divides side BC in the ratio $1:1$. If $AB = 5$, then find AC.

Method I



$$\frac{BD}{DC} = \frac{AB}{AC} = 1 \Rightarrow AB = AC = 5$$

Method II

If the angle bisector divides the opposite side in the ratio $1:1$, then it is also the median.

And if the angle bisector is the same as the median, the triangle is isosceles.
Hence,

$$AC = AB = 5$$

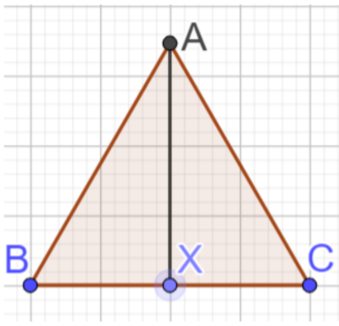
Proof 1.22

Show, using the properties of the angle bisector (and without using congruence), that the angle bisector in an equilateral triangle is also the median and the altitude.

Draw equilateral ΔABC, and construct the angle bisector from A.

$$\angle BAX = \angle\frac{BAC}{2} = \frac{60}{2} = 30°$$

$$\angle CXA = \angle BXA = 180 - 30 - 60 = 90° \Rightarrow AX\ is\ altitude$$

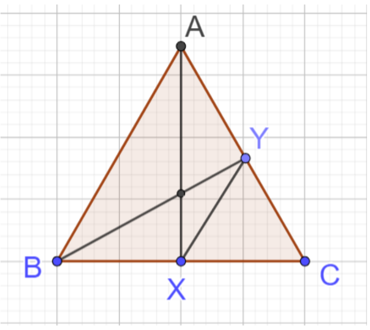


By Angle Bisector Theorem:

$$\frac{BX}{AB} = \frac{CX}{AC} \Rightarrow \frac{BX}{1} = \frac{CX}{1} \Rightarrow BX = CX \Rightarrow AX\ is\ median$$

Example 1.23

Equilateral ΔABC has side length 1. AX is an angle bisector for $\angle BAC$, intersecting BC at X. BY is the angle bisector for $\angle ABC$ intersecting AX at Y. Find XY and YA



E. Heronian Triangles

1.24: Heronian Triangle

Heron's formula for the area of a triangle with sides a, b and c and semi-perimeter $s = \frac{p}{2} = \frac{a+b+c}{2}$ is

$$\sqrt{s(s-a)(s-b)(s-c)}$$

A triangle with integer side lengths and integer area is a

$$Heronian\ Triangle$$

Example 1.25

A. ΔABC has side lengths $AB = 17, BC = 10, CA = 9$. AD is the bisector of $\angle A$, intersecting BC at D. Find the area of ΔADC.
B. ΔABC has side lengths $AB = 26, BC = 25, CA = 3$. AD is the bisector of $\angle A$, intersecting BC at D. Find the area of ΔADC.

Part A

Substitute $s = \frac{a+b+c}{2} = \frac{17+10+9}{2} = \frac{36}{2} = 18$ in :

$$A = \sqrt{18(1)(8)(9)} = \sqrt{9^2 \times 4^2} = 36 \Rightarrow [ADC] = 36 \times \frac{9}{26} = \frac{162}{13}$$

Part B

Substitute $s = \frac{a+b+c}{2} = \frac{26+25+3}{2} = \frac{54}{2} = 27$ in :

$$A = \sqrt{27(1)(2)(24)} = 36 \Rightarrow [ADC] = 36 \times \frac{3}{29} = \frac{108}{29}$$

Example 1.26

ΔABC has side lengths $AB = 15, BC = 13, CA = 4$. AD is the bisector of $\angle A$, intersecting BC at D. The area of $\Delta ABD = p + \frac{q}{r}$ where p, q, r are integers, $p < r$ and $HCF(q, r)$ is 1. Find $p + q + r$.

Substitute $s = \frac{a+b+c}{2} = \frac{15+13+4}{2} = \frac{32}{2} = 16$ in

$$A = \sqrt{16(1)(3)(12)} = \sqrt{16(36)} = \sqrt{16}\sqrt{36} = 4 \times 6 = 24$$

The area that we want is:

$$24 \times \frac{15}{15+4} = \frac{360}{19} = \frac{361}{19} - \frac{1}{19} = 19 - \frac{1}{19} = 18\frac{18}{19}$$

$$p + q + r = 18 + 18 + 19 = 55$$

1.3 Algebra and Ratios

A. Expressions

Some questions ask for a combination of two or more elements. It is sometimes possible to calculate two elements together (without calculating them separately).

Example 1.27

In ΔABC with perimeter 1 m, AD is the angle bisector from A, intersecting side BC at D. DC is 25% longer than BD. Find the sum of the length of AB and BD, in cm.

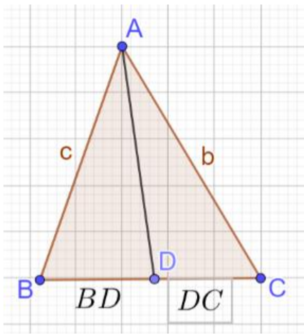
$$AB + AC + BC = 100$$

Substitute $AC = \frac{5}{4}AB, BC = BD + DC = BD + \frac{5}{4}BD = \frac{9}{4}BD$:

$$AB + \frac{5}{4}AB + \frac{9}{4}BD = 100$$

$$\frac{9}{4}(AB + BD) = 100$$

$$AB + BD = \frac{400}{9}cm$$



B. Algebra

Example 1.28

In right ΔABC, right-angled at B, $AB = 8$, and the hypotenuse is 2 units longer than BC. If AD is the angle bisector from A, then find the area of ΔADC.

$$8^2 + x^2 = (x+2)^2$$

$$64 + x^2 = x^2 + 4x + 4$$

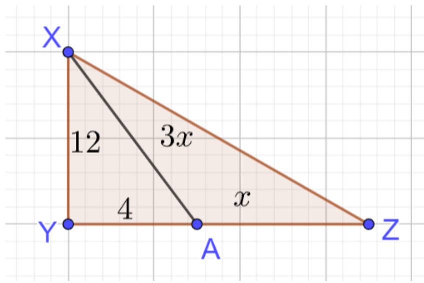
$$x = 15$$

$$\frac{AD}{8} = \frac{DC}{17} \Rightarrow \frac{AD}{DC} = \frac{8}{17} \Rightarrow AD:DC = 8:17$$

$$DC = 15 \times \frac{17}{25} = \frac{51}{5}$$
$$[ADC] = \frac{1}{2} \times 8 \times \frac{51}{5} = \frac{204}{5}$$

Example 1.29

XY has length 12 in right ΔXYZ. XA is the angle bisector from X, and intersects YZ at A. If $YA = 4$, find hypotenuse XZ.



Let $AZ = x$. By the angle bisector theorem:

$$\frac{XZ}{x} = \frac{XY}{YA} = \frac{12}{4} = 3 \Rightarrow XZ = 3x$$

By the Pythagorean Theorem in ΔXYZ, we know that

$$XY^2 + YZ^2 = XZ^2$$
$$144 + (4+x)^2 = (3x)^2$$
$$144 + 16 + 8x + x^2 = 9x^2$$

Collate all terms on one side to get a quadratic. Factor and solve:

$$8(x^2 - x - 20) = 0 \Rightarrow (x-5)(x+4) = 0$$

Take the positive solution:

$$x = 5 \Rightarrow XZ = 3x = 15$$

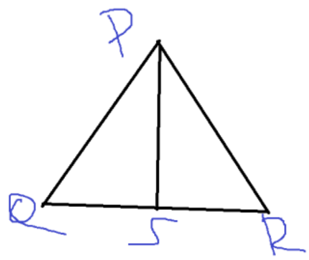
You can check that this forms a valid triangle, which is:

$$(9,12,15) = 3(3,4,5)$$

Example 1.30

In ΔPQR, the angle bisector from P intersects QR at S. What are the possible values of x, if:

$$PQ = x+5, QS = x-2, SR = x-3, PR = x+4$$



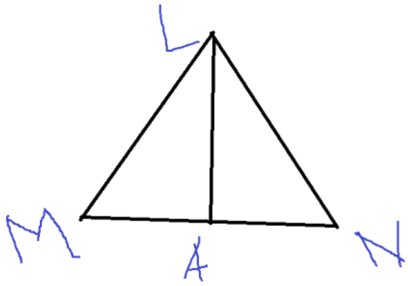
$$\frac{QS}{QP} = \frac{SR}{PR} \Rightarrow (QS)(PR) = (SR)(QP)$$
$$(x-2)(x+4) = (x-3)(x+5)$$
$$x^2 + 4x - 2x - 8 = x^2 - 3x + 5x - 15$$
$$2x - 8 = 2x - 15$$
$$-8 = -15$$
$$Not\ Possible$$
$$x \in \phi \Rightarrow No\ Solutions$$

Example 1.31

In ΔLMN, the angle bisector from L intersects MN at A. What are the possible values of x, if:

$$MA = x^2 - 2x - 15$$
$$AN = x^2 + 6x - 7$$
$$LM = x^2 - 3x - 10$$
$$LN = x^2 + 3x - 4$$

$$\frac{MA}{LM}=\frac{AN}{LN}$$
$$\frac{(x+3)(x-5)}{(x-5)(x+2)}=\frac{(x+7)(x-1)}{(x-1)(x+4)}$$
$$\frac{x+3}{x+2}=\frac{x+7}{x+4}$$
$$x^2+7x+12=x^2+9x+14$$
$$-2=2x$$
$$x=-1$$



$$MA=(x+3)(x-5)=(-1+3)(-1-5)=(2)(-6)=-12\Rightarrow Negative\Rightarrow Not\ Possible$$

Had the first value been positive, we would have needed to check remaining three quadratic expressions. If all values are positive, only then is x a valid solution.

C. Playing with Ratios

1.32: Ratios with Totals

When working with ratios, it is important to be able to work with totals. One specific property of totals is:

$$\frac{b}{g}=\frac{B}{G}\Leftrightarrow\frac{b}{b+g}=\frac{B}{B+G}$$

Here

$$b+g\ is\ an\ example\ of\ a\ total$$
$$B+G\ is\ another\ example\ of\ a\ total$$

Algebra

Take the reciprocal (*invertendo*) with $\frac{b}{g}=\frac{B}{G}$:

$$\frac{g}{b}=\frac{G}{B}$$

Add 1 to both sides (*componendo*):

$$\frac{g}{b}+1=\frac{G}{B}+1\Rightarrow\frac{b+g}{b}=\frac{G+B}{G}$$

Take the reciprocal again (*invertendo*)

$$\frac{b}{b+g}=\frac{B}{B+G}$$

k method

Prove the property in the forward direction. That is, if:

$$\frac{b}{g}=\frac{B}{G}\Rightarrow\frac{b}{b+g}=\frac{B}{B+G}$$

Let

$$\frac{b}{g}=\frac{B}{G}=k\Rightarrow b=gk,B=Gk$$

Then:

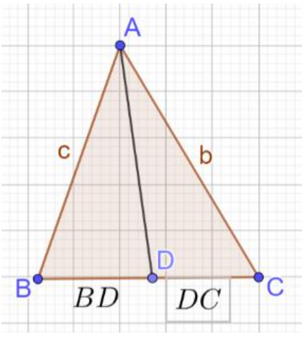
$$LHS = \frac{b}{b+g} = \frac{gk}{gk+g} = \frac{k}{k+1}$$
$$RHS = \frac{B}{B+G} = \frac{Gk}{Gk+G} = \frac{k}{k+1} = LHS$$

Example 1.33

If students in Kings Middle school have boys and girls in the ratio $b:g$, and students in Queen's Middle school have boys and girls in the ratio $B:G$, and we know that $\frac{b}{g} = \frac{B}{G}$ what can you conclude about the ratio of:

A. Boys to total students in both schools
B. Girls to total students in both schools

$$\frac{b}{b+g} = \frac{B}{B+G}$$
$$\frac{g}{b+g} = \frac{G}{B+G}$$



1.34: Angle Bisector Theorem (with Totals)

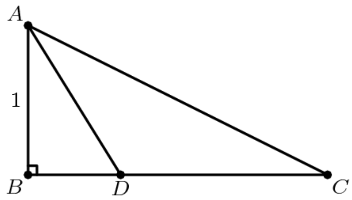
$$\frac{BD}{BC} = \frac{AB}{AB+AC}$$
$$\frac{DC}{BC} = \frac{AC}{AB+AC}$$

Using the angle bisector theorem:

$$\frac{BD}{DC} = \frac{AB}{AC}$$

Convert it to totals:

$$\frac{BD}{DC+BD} = \frac{AB}{AB+AC}$$
$$\frac{BD}{BC} = \frac{AB}{AB+AC}$$



Example 1.35

Triangle ABC has a right angle at B, $AB = 1$, and $BC = 2$. The angle bisector of $\angle A$ intersects side $\overline{BC}$ at D. What is BD? **(AMC 10B 2009/20)**

Angle Bisector Theorem (with Totals)

By the angle bisector theorem (with totals)

$$\frac{BD}{BC} = \frac{AB}{AB+AC}$$

Substitute $AB = 1, AC = \sqrt{AB^2 + BC^2} = \sqrt{1^2 + 2^2} = \sqrt{5}$:

$$BD = \frac{BC}{AB+AC} \times AB = \frac{2}{\sqrt{5}+1} \times 1 = \frac{2}{\sqrt{5}+1} \times \frac{\sqrt{5}-1}{\sqrt{5}-1} = \frac{2(\sqrt{5}-1)}{4} = \frac{\sqrt{5}-1}{2}$$

Angle Bisector Theorem (without Totals)

$$\frac{AB}{BD} = \frac{AC}{DC}$$
$$\frac{1}{x} = \frac{\sqrt{5}}{2-x}$$

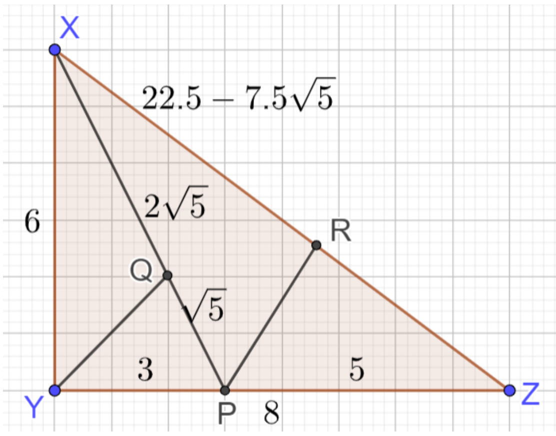
$$2 - x = \sqrt{5}x$$
$$\sqrt{5}x + x = 2$$
$$x(\sqrt{5} + 1) = 2$$
$$x = \frac{2}{\sqrt{5}+1} = \frac{\sqrt{5}-1}{2}$$

In the above question, what is the area of ΔADC?

$$[ADC] = [ABC] - [ABD] = \left(\frac{1}{2} \times 1 \times 2\right) - \left(\frac{1}{2} \times 1 \times \frac{\sqrt{5}-1}{2}\right) = 1 - \frac{\sqrt{5}-1}{4} = \frac{4-\sqrt{5}+1}{4} = \frac{5-\sqrt{5}}{4}$$

Example 1.36

In right ΔXYZ the side lengths are $XY = 6, YZ = 8$ and $XZ = 10$. XP is the angle bisector of $\angle YXZ$ intersecting YZ at P. YQ is the angle bisector of $\angle XYZ$, intersecting XP at Q. PR is the angle bisector of $\angle XPZ$ intersecting XZ at Q. Find $XP + XQ + XR$.



Step I: Find YP using the Angle Bisector Theorem

Use the angle bisector theorem in ΔXYZ:

$$\frac{YP}{PZ} = \frac{XY}{XZ} = \frac{6}{10} = \frac{3}{5} \Rightarrow YP = YZ \times \frac{3}{3+5} = 8 \times \frac{3}{8} = 3$$

Step II: Find XP using the Pythagorean Theorem

Use the Pythagorean Theorem in ΔXYP:

$$XP = \sqrt{XY^2 + YP^2} = \sqrt{6^2 + 3^2} = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$$

Step III: Find XQ using the Angle Bisector Theorem

Use the Angle Bisector Theorem in ΔXYP:

$$\frac{XQ}{QP} = \frac{XY}{YP} = \frac{6}{3} = 2$$

$$XQ = 3\sqrt{5} \times \frac{2}{3} = 2\sqrt{5}$$

Step IV: Find XR using the Angle Bisector Theorem (with Totals)

$$\frac{XR}{XZ} = \frac{XP}{XP + PZ}$$

Solve for XR:

$$XR = XZ \cdot \frac{XP}{XP+PZ} = 10 \cdot \frac{3\sqrt{5}}{3\sqrt{5}+5}$$

Rationalize:

$$= 10 \times \frac{3\sqrt{5}(3\sqrt{5}-5)}{45-25} = 10 \times \frac{45 - 15\sqrt{5}}{20} = 22.5 - 7.5\sqrt{5}$$

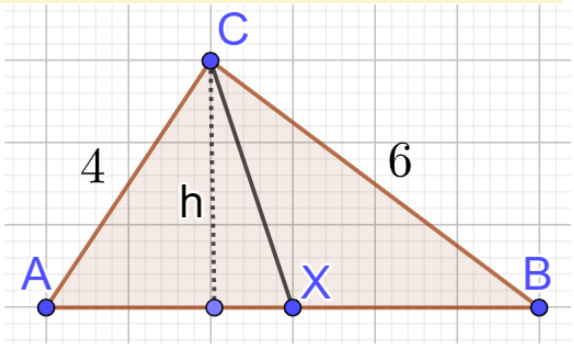
Step V: Find the Total

$$XP + XQ + XQ = 3\sqrt{5} + 2\sqrt{5} + 22.5 - 7.5\sqrt{5} = 22.5 - 2.5\sqrt{5}$$

D. Ratio of Areas

Example 1.37

In ΔACB, the angle bisector from C intersects AB at X. If $AC = 4, CB = 6$ and ΔACB has area π units, then find the area of ΔCAX.



The angle bisector divides the triangle into two smaller triangles. Find the ratio of the areas of these two smaller triangles.

$$\frac{A(\Delta CAX)}{A(\Delta \mathrm{CAB})} = \frac{\frac{1}{2} \times h \times AX}{\frac{1}{2} \times h \times AB} = \frac{AX}{AB} = \frac{CA}{CA + CB} = \frac{4}{4+6} = \frac{4}{10} = \frac{2}{5}$$

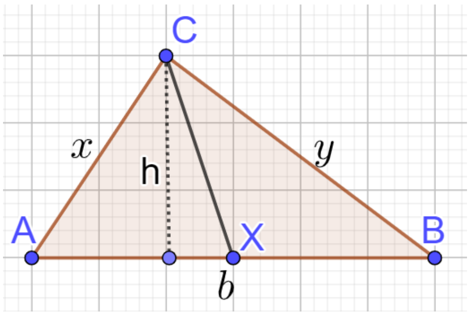
Hence, we need to divide $[ACB]$ in the ratio $2:5$:

$$[CAX] = [ACB] \times \frac{2}{5} = \pi \times \frac{2}{5} = \frac{2\pi}{5}$$

1.38: Ratio of Areas

Consider a triangle with base b and height h. The angle bisector to the base divides it into two smaller triangles. The ratio of the areas of the smaller triangles is the ratio of other two sides:

$$\frac{[CAX]}{[XCB]} = \frac{x}{y}$$



In ΔACB, the angle bisector from C intersects AB at X. If $AC = x, CB = y$, then:

$$\frac{[CAX]}{[XCB]} = \frac{\frac{1}{2} \times h \times AX}{\frac{1}{2} \times h \times XB} = \frac{AX}{XB} = \frac{CA}{CB} = \frac{x}{y}$$

In the same triangle, we can apply the properties of ratios to get:

$$\frac{[CAX]}{[CAB]} = \frac{x}{x+y}$$

Take the reciprocal of both sides of $\frac{[CAX]}{[XCB]} = \frac{x}{y}$ to get:

$$\frac{[XCB]}{[CAX]} = \frac{y}{x}$$

Add 1 to both sides:

$$\frac{[XCB]}{[CAX]} + 1 = \frac{y}{x} + 1 \Rightarrow \frac{[XCB] + [CAX]}{[CAX]} = \frac{y+x}{x} \Rightarrow \frac{[CAB]}{[CAX]} = \frac{y+x}{x}$$

Again, take the reciprocal of both sides:

$$\frac{[CAX]}{[CAB]} = \frac{x}{x+y}$$

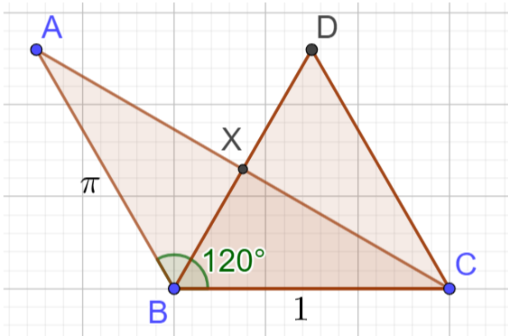
Example 1.39

$\angle ABC$ is $120°$, with $AB = \pi, BC = 1$. ΔDBC is equilateral with side length 1. Point X is on the intersection of BD and AC. Find the area of ΔDXC in terms of the area of ΔABC.

See diagram (not drawn to scale).

$$\angle DBC = 60° = \frac{1}{2} \times 120° = \frac{1}{2}\angle ABC$$

Hence, BD is the angle bisector of $\angle ABC$.



Since the angle bisector divides the triangle into two smaller triangles that have areas in the ratio of their sides:

$$\frac{[AXB]}{[XBC]} = \frac{AB}{BC} = \frac{\pi}{1}$$

Add 1 to both sides of the first and third parts:

$$\frac{[AXB]}{[XBC]} + 1 = \pi + 1$$

$$\frac{[AXB] + [XBC]}{[XBC]} = \pi + 1$$

$$\frac{[ABC]}{[XBC]} = \pi + 1$$

$$[XBC] = \frac{[ABC]}{1 + \pi}$$

Since ΔDBC is equilateral, its area is:

$$[DBC] = \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4} \times 1^2 = \frac{\sqrt{3}}{4}$$

And, finally,

$$[DXC] = [DBC] - [XBC] = \frac{\sqrt{3}}{4} - \frac{[ABC]}{1 + \pi}$$

E. $15 - 75 - 90$ Triangles (Optional)

We have seen how equilateral triangles can be used to derive the value of ratios in special triangles such as the $45 - 45 - 90$ triangle and the $30 - 60 - 90$ triangle. These two triangles are encountered frequently.
A less common triangle is the $15 - 75 - 90$ triangle. We now derive the ratios of the sides in such a triangle. (Warning: Heavy use of radicals).

Challenge 1.40

Equilateral ΔABC has side length 1. AX is the angle bisector for $\angle BAC$, intersecting BC at X. AZ is the angle bisector for $\angle XAC$ intersecting XC at Z. Find XZ.

Since ΔABC is equilateral:

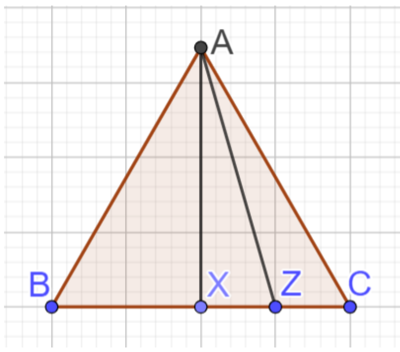
$$AX = Height = \frac{\sqrt{3}}{2}s = \frac{\sqrt{3}}{2}$$

By Angle Bisector Theorem:

$$XZ:ZC = AX:AC = \frac{\sqrt{3}}{2}:1$$

Since AX is median:

$$XC = \frac{1}{2}s = \frac{1}{2}$$



We need to divide XC in the ratio $XZ:ZC$:

$$XZ = \frac{1}{2} \times \frac{\frac{\sqrt{3}}{2}}{1 + \frac{\sqrt{3}}{2}} = \frac{\frac{\sqrt{3}}{4}}{\frac{2+\sqrt{3}}{2}} = \frac{\sqrt{3}}{4} \times \frac{2}{2+\sqrt{3}} = \frac{\sqrt{3}(2-\sqrt{3})}{2(4-3)} = \frac{2\sqrt{3}-3}{2}$$

(Continuation) Challenge 1.41

Show that

$$AZ = \frac{3-\sqrt{3}}{\sqrt{2}}$$

By the Pythagorean Theorem, in ΔAXZ:

$$AZ^2 = XZ^2 + AX^2 = \left(\frac{2\sqrt{3}-3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{12-12\sqrt{3}+9}{4} + \frac{3}{4} = \frac{24-12\sqrt{3}}{4} = 6-3\sqrt{3}$$

$$\left(\frac{3-\sqrt{3}}{\sqrt{2}}\right)^2 = \frac{9-6\sqrt{3}+3}{2} = \frac{12-6\sqrt{3}}{2} = 6-3\sqrt{3} = AZ^2 \Rightarrow AZ = \frac{3-\sqrt{3}}{\sqrt{2}}$$

(Continuation) Challenge 1.42

Find the ratios

$$\frac{XZ}{AZ}, \frac{AX}{AZ}, \frac{XZ}{AX}$$

$$\frac{XZ}{AZ} =$$
$$\frac{AX}{AZ} =$$
$$\frac{XZ}{AX} =$$

(Continuation) Challenge 1.43

Find the value of AZ by finding the square root of AZ^2 using an algebraic method (Not by squaring the given value, and not by trial and error).

$$\sqrt{6-3\sqrt{3}} = \left(a - b\sqrt{3}\right)^2$$

Square both sides:

$$6-3\sqrt{3} = a^2 - 2ab\sqrt{3} + 3b^2$$

By the method of undetermined coefficients:

$$-2ab = -3 \Rightarrow b = \frac{3}{2a}$$

$$a^2 + 3b^2 = 6 \Rightarrow a^2 + 3\left(\frac{3}{2a}\right)^2 = 6 \Rightarrow a^2 + \frac{27}{4a^2} = 6 \Rightarrow 4a^4 - 24a^2 + 27 = 0$$

This is a disguised quadratic. Use a change of variable. Let $x = a^2$:

$$4x^2 - 24x + 27 = 0$$
$$4x^2 - 6x - 18x + 27 = 0$$
$$2x(2x-3) - 9(2x-3) = 0$$
$$(2x-3)(2x-9) = 0$$

$$x \in \left\{\frac{3}{2}, \frac{9}{2}\right\} \Rightarrow a^2 \in \left\{\frac{3}{2}, \frac{9}{2}\right\} \Rightarrow a \in \left\{\pm\sqrt{\frac{3}{2}}, \pm\frac{3}{\sqrt{2}}\right\}$$

We now have four possible values to check. We could start with any of them, but note that one value has a $\sqrt{3}$ in it, and we already have a $\sqrt{3}$ in the coefficient of b.
So, start with the positive value of a in the other two options:

$$a = \frac{3}{\sqrt{2}} \Rightarrow \frac{3}{\sqrt{2}}b = \frac{3}{2} \Rightarrow b = \frac{1}{\sqrt{2}} \Rightarrow \left(a - b\sqrt{3}\right) = \left(\frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sqrt{3}\right) = \frac{3-\sqrt{3}}{\sqrt{2}}$$

Which we already know from above works.

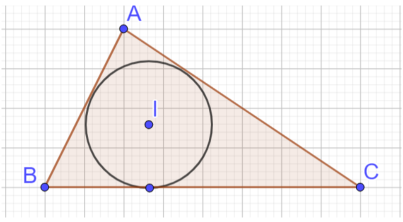
1.4 Incenter and Inradius

A. Definition

We first look at the definition of incircle, and incentre and inradius. Then, we use the definitions we just learned to do some angle chasing.

1.44: Incircle, incentre and inradius

- The circle internally tangent to the three sides of a triangle is called its **incircle**.
- The center of the incircle is called the **incenter**.
- The radius of the incircle is the **inradius**.



In the diagram alongside:

- The circle is the incircle for ΔABC.
- The center of the circle (I), is the incenter for ΔABC

1.45: Incenter and Angle Bisectors

The angle bisectors of a triangle are concurrent (meet) at its incenter.

By definition, the incenter of a triangle is the center of the circle that touches the three sides of the triangle.

In ΔQSB and ΔQRB:

$$QS = QR\ (Inradius)$$
$$\angle QRB = \angle QSB = 90^\circ\ \ (Distance\ Def)$$
$$QB = QB\ (Reflexive\ Property)$$
$$\Delta QSB \cong \Delta QRB\ (RHS\ Congruence)$$
$$\angle RBQ = \angle SBQ\ (CPCTC)$$

Similarly:

$$\angle RAQ = \angle QAT$$
$$\angle QCT = \angle QCS$$

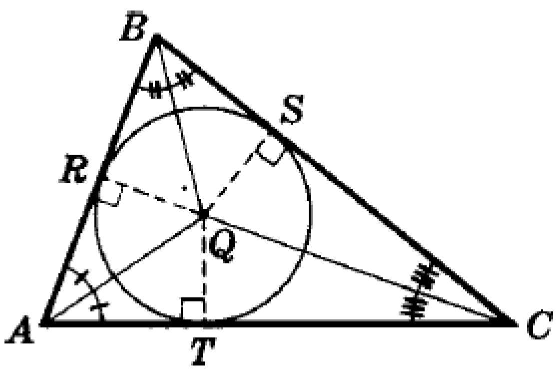


Fig. 4-28

Hence, the incenter of a circle is the point of concurrence of the angle bisectors of a triangle.

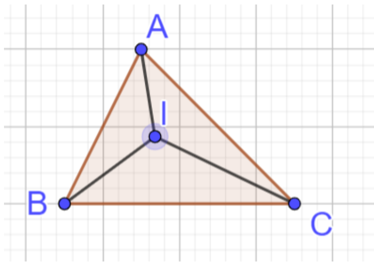
B. Angle Chasing

Example 1.46

ΔABC has incenter I. $\angle A = \angle ABI = 40°$. Find $\angle BIC$.

BI is the bisector of $\angle ABC$.

$$\angle ABC = 2 \times \angle ABI = 80°$$



By sum of angles of a triangle:

$$\angle ACB = 180 - \angle A - \angle ABC = 180 - 40 - 80 = 60$$

IC is the bisector of $\angle ACB$:

$$\angle ICB = \frac{60}{2} = 30$$

By sum of angles of a triangle:

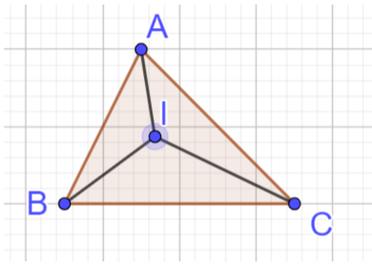
$$\angle BIC = 180 - \angle IBC - \angle ICB = 180 - 40 - 30 = 110$$

Example 1.47

ΔABC is an isosceles triangle with vertex angle $\angle A = 100°$ and incentre I. What is the measure of $\angle BIC$?

By Base Angles of an Isosceles Triangle:

$$\angle ABC = \angle ACB = \frac{180 - 100}{2} = \frac{80}{2} = 40$$



Since I is the incentre, IB and IC are angle bisectors of $\angle B$ and $\angle C$ respectively:

$$\angle IBC = \angle ICB = \frac{40}{2} = 20$$

By sum of angles of a triangle:

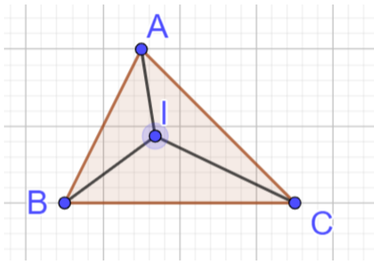
$$\angle BIC = 180 - 20 - 20 = 140$$

Example 1.48

In ΔABC with incentre I we are given $\angle BIA = 140°, \angle IAC = 30°$. Find $\angle BIC$.

IA is an angle bisector

$$\angle BAI = \angle IAC = 30°$$



By sum of angles of a triangle:

$$\angle ABI = 180 - 140 - 30 = 10$$

IB is a bisector:

$$\angle ABC = 2 \times ABI = 2 \times 10 = 20$$

By sum of angles of a triangle:

$$\angle ACB = 180 - 60 - 20 = 100$$

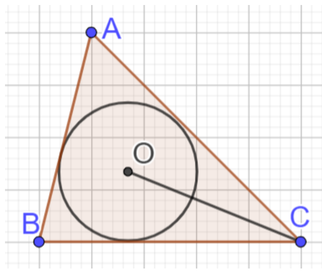
By sum of angles of a triangle:

$$\angle BIC = 180 - 10 - 50 = 120$$

C. More Angle Chasing

Example 1.49

O is the incenter of ΔABC with $\angle A = 40°$, and $\angle B = 30°$. Find $\angle ACO$.



By Angles in a Triangle:

$$\angle C = 180 - 40 - 30 = 110°$$

Since the incentre is the intersection of the angle bisectors, OC is the angle bisector of $\angle C$:

$$\angle OCO = \frac{110}{2} = 55°$$

Example 1.50

O is the incenter of isosceles ΔABC with $\angle A = 30°$. Find the supplement of the sum of the possible values of $\angle ACO$.

$$\angle B = 30° \Rightarrow \angle C = 120 \Rightarrow \angle ACO = \frac{120}{2} = 60°$$
$$\angle C = 30° \Rightarrow \angle ACO = \frac{30}{2} = 15°$$
$$\angle B = \angle C = 75° \Rightarrow \angle ACO = \frac{75}{2} = 37.5°$$

$$Sum = 60 + 15 + 37.5 = 112.5$$
$$Supplement = 180 - 112.5 = 67.5$$

Example 1.51

O is the incenter of isosceles ΔABC with $\angle A = x°$. The supplement of the sum of the possible values of $\angle ACO$ is $60°$. Find x.

$\angle C$ is the vertex angle:

$$\angle C = 180 - 2x \Rightarrow \angle ACO = 90 - x$$

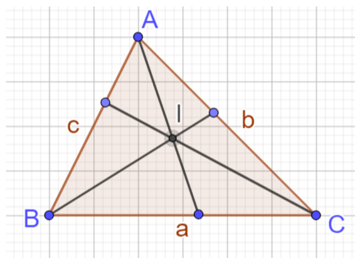
$\angle B$ is the vertex angle:

$$\angle C = x \Rightarrow \angle ACO = \frac{x}{2}$$

$\angle A$ is the vertex angle:

$$\angle C = \frac{180 - x}{2} = 90 - \frac{x}{2} \Rightarrow \angle ACO = \angle\frac{C}{2} = 45 - \frac{x}{4}$$
$$90 - 2 + \frac{x}{2} + 45 - \frac{x}{4} = 120 \Rightarrow 135 - \frac{3x}{4} = 120 \Rightarrow 15 = \frac{3x}{4} \Rightarrow x = 20$$

D. Incentre divides Angle Bisectors



1.52: Incentre divides Angle Bisectors

The incenter of a triangle divides the internal angle bisectors in the ratio of the sum of adjacent sides and the opposite side.

Given ΔABC with incentre I

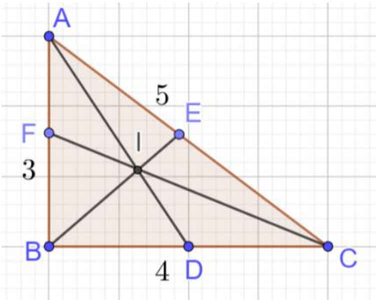
$$\frac{AI}{ID} = \frac{b + c}{a}, \quad \frac{BI}{IE} = \frac{c + a}{b}, \quad \frac{CI}{IF} = \frac{a + b}{c}$$

Example 1.53

Consider ΔABC, with sides $AB = 3, BC = 4$ and $CA = 5$. Angle bisectors AD, BE, and CF are drawn, intersecting sides BC, CA, and AB at D, E, and F respectively. Find:

A. $\frac{AI}{ID}$
B. $\frac{BI}{IE}$
C. $\frac{CI}{IF}$

$$\frac{AI}{ID} = \frac{b+c}{a} = \frac{5+3}{4} = \frac{8}{4} = 2$$
$$\frac{BI}{IE} = \frac{c+a}{b} = \frac{3+4}{5} = \frac{7}{5}$$
$$\frac{CI}{IF} = \frac{a+b}{c} = \frac{5+4}{3} = \frac{9}{3} = 3$$



Example 1.54
Back-Calculations

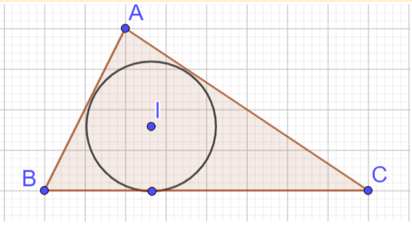
E. Inradius

1.55: Inradius
The radius of an incircle is called its inradius.

Example 1.56
In ΔABC with incentre I, the distance from I to side AB is $4\ cm$. What is the sum of the distances from I to BC and CA.

The circle is tangent to the three sides of the triangle. Hence, the distance from the center to the sides is equal to the radius.

Radius is always equal. Hence, required distance is:
$$4 + 4 = 8$$



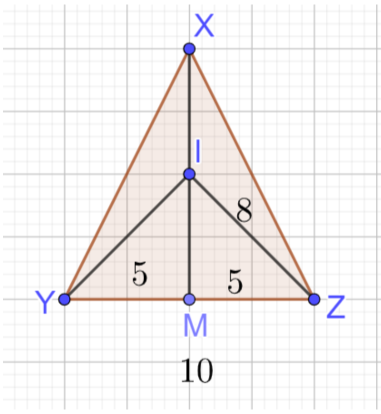
F. Isosceles Triangles

Example 1.57
In isosceles ΔXYZ, the base YZ has length 10, and the distance from the incentre to Z is 8. What is the inradius?

$$IZ = 8$$
In Isosceles ΔXYZ
$$\angle XYZ = \angle XZY$$

Since the incentre is the intersection of the angle bisectors:
$$\angle IYZ = \angle IZY = \frac{1}{2}\angle XYZ \Rightarrow \Delta IYZ\ is\ isosceles$$

In Isosceles ΔIYZ drop a perpendicular (which is also the median) to side YZ:
$$MZ = \frac{1}{2} \times YZ = \frac{1}{2} \times 10 = 5$$



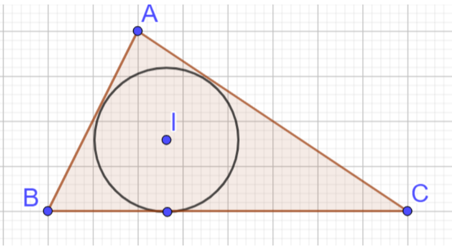
In Right Triangle IMZ, by the Pythagorean Theorem:

$$IM = \sqrt{IZ^2 - MZ^2} = \sqrt{8^2 - 5^2} = \sqrt{64 - 25} = \sqrt{39}$$

And since IM is perpendicular to YZ, it is also the inradius:

$$Inradius\ of\ \Delta XYZ = IM = \sqrt{39}$$

G. Equilateral Triangles



1.58: Incenter of Equilateral Triangle

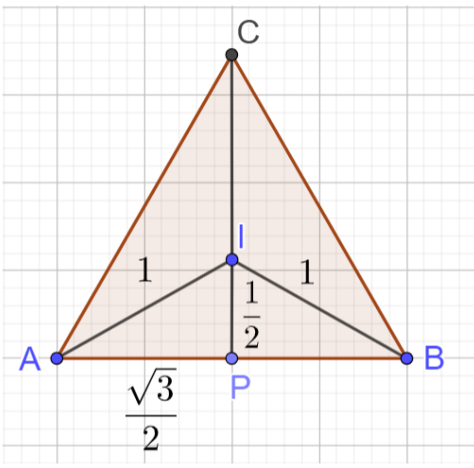
If the incenter of a triangle is equidistant from its vertices, then the triangle is an equilateral triangle.

In the diagram, if:

$$IA = IB = IC \Rightarrow \Delta ABC\ is\ equilateral$$

Example 1.59

The distance of the incentre of a triangle from each of its vertices is 1 cm. What is the area of the triangle?



Since the distance to each of its vertices is equal, the triangle is equilateral. Draw the triangle (see diagram).

$$IA = IB = 1$$

Since the above distances are equal, we get an isosceles triangle. The thought process needed here is similar to the one used when we find the area of an isosceles triangle.

In Isosceles ΔIAB:
Draw the altitude from I intersecting side AB at P

In ΔIAP:

$$\angle IAP = \frac{60}{2} = 30, \qquad \angle IPA = 90, \qquad \angle AIP = 180 - 90 - 30 = 60$$

Hence,

$$\Delta IAP\ is\ a\ 30 - 60 - 90\ triangle$$

In a $30 - 60 - 90$ *triangle*, the side opposite 60° is $\frac{\sqrt{3}}{2}$ times the hypotenuse

$$AP = \frac{\sqrt{3}}{2} \times AI = \frac{\sqrt{3}}{2} \times 1 = \frac{\sqrt{3}}{2}$$

In an Isosceles triangle, the altitude is also the median:

$$AB = 2 \times AP = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

And, finally, now that we have the side, we can find the area using the formula for the area of an equilateral triangle:

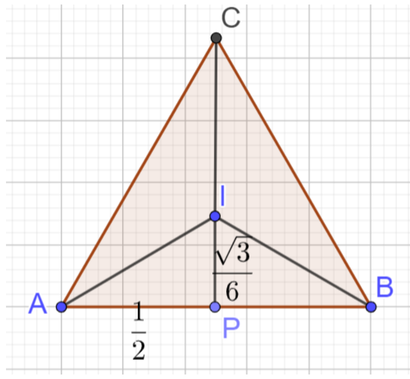
$$A = \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}\left(\sqrt{3}\right)^2 = \frac{3\sqrt{3}}{4}$$

1.60: Incentre of Equilateral Triangle

Converse of Previous: If a triangle is equilateral, the incentre is equidistant from its vertices.

Example 1.61

Dropping a perpendicular from the incentre of an equilateral triangle to each of the three sides creates six smaller triangles. What is the area of each smaller triangle if the side of the equilateral triangle is 1 cm?



Draw ΔABC with incenter I. Connect I with the vertices.
In ΔIAB, draw altitude IP intersecting AB at P.

$$AP = \frac{1}{2}AB = \frac{1}{2} \times 1 = \frac{1}{2}$$

In ΔIAP:

$$hyp \times \frac{\sqrt{3}}{2} = AP \Rightarrow hyp \times \frac{\sqrt{3}}{2} = \frac{1}{2} \Rightarrow hyp = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

In ΔIAP:

$$\text{Side opposite } 30° = IP = \frac{1}{2} \times hyp = \frac{1}{2} \times \frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{6}$$

Area of ΔIAP

$$= \frac{1}{2}hb = \frac{1}{2} \times \frac{1}{2} \times \frac{\sqrt{3}}{6} = \frac{\sqrt{3}}{24}$$

1.62: Combining Statements and Converses

The incentre of a triangle is equidistant from its vertices $if\ and\ only\ if$ the triangle is equilateral.

If both a statement, and its converse are true, they can be combined into a single statement using $if\ and\ only\ if$.

H. Calculating Inradius

1.63: Inradius

In a triangle with lengths of sides $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$:

$$Inradius = r = \frac{\Delta}{s}$$

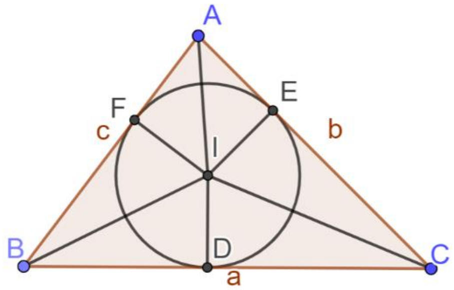
Recall that when we learnt Heron's Formula, we had the following:

$$\text{Area of a Triangle} = \text{Delta} = \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$s = \text{Semiperimeter} = \frac{a+b+c}{2}$$

Consider ΔABC with incentre I. We must have:

$$\underbrace{Area(\Delta ABC) = Area(\Delta IBC) + Area(\Delta IAC) + Area(\Delta IAB)}_{\textit{Equation I}}$$



Segment BC is tangent to the incircle, and since a radius is perpendicular to the tangent at the point of tangency, we know that $BD \perp ID$, which tells us that:

$$Area(\Delta IBC) = \frac{1}{2}hb = \frac{1}{2} \cdot ID \cdot BC = \frac{1}{2}ra$$

Similarly:

$$Area(\Delta IAC) = \frac{1}{2}rb, \qquad Area(\Delta IAB) = \frac{1}{2}rc$$

Substitute the above three in I:

$$Area(\Delta ABC) = \Delta = \frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc = r\left(\frac{a+b+c}{2}\right) = rs \Rightarrow \Delta = rs \Rightarrow r = \frac{\Delta}{s}$$

Example 1.64
Calculation of area given inradius and lengths of sides

Example 1.65
The sides of a triangle are 13, 14 and 15. Find its inradius.

$$s = \frac{a+b+c}{2} = \frac{13+14+15}{2} = \frac{42}{2} = 21$$
$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21(8)(7)(6)} = \sqrt{3 \times 7(2^3)(7)(2 \times 3)} = \sqrt{3^2 \times 7^2(2^4)} = 84$$
$$r = \frac{\Delta}{s} = \frac{84}{21} = 4$$

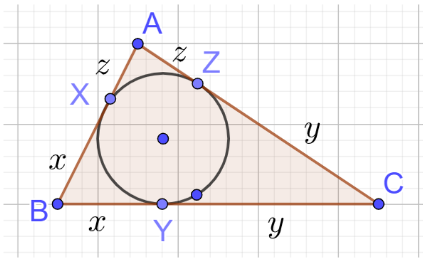
I. Lengths of tangents to incircle

Given a triangle with an incircle, we can consider the length of the segments to the circle as tangents. Given the lengths of the sides, it is possible to calculate the lengths of the segments using the properties of tangents.

1.66: Tangent from a point to a circle
Tangents from a point to a circle are congruent.

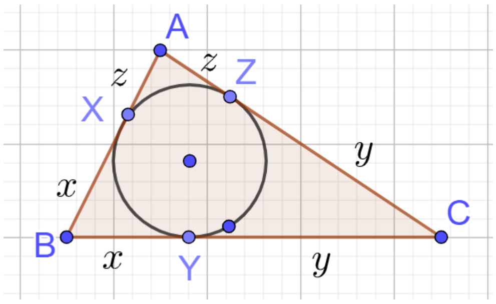
Example 1.67
Tangents are drawn to a circle from points A, B and C. The tangents from point A touch the circle at X and Z. The tangents from point B touch the circle at X and Y. The tangents from point C touch the circle at Y and Z. Draw a diagram and identify the congruent tangents.

$$BX = BY = x$$
$$CY = CZ = y$$
$$AX = AZ = z$$



Example 1.68
The incircle of ΔABC is drawn touching it at points X, Y and Z respectively. In other words, the circle is tangent to all three sides of the triangle. Given that $AB = 4, BC = 5, AC = 6$, find the lengths of the tangents drawn to the circle from the vertices of ΔABC.

Tangents drawn from a point to a circle are congruent.



$$AB = x + z = 4$$
$$BC = x + y = 5$$
$$CA = y + z = 6$$

Add the three equations:

$$2(x + y + z) = 15 \Rightarrow x + y + z = 7.5$$

$$CZ = CY = y = 3.5$$
$$AX = AZ = z = 2.5$$
$$BX = BY = x = 1.5$$

1.69: Lengths of Tangents

In ΔABC with sides in the usual notation:

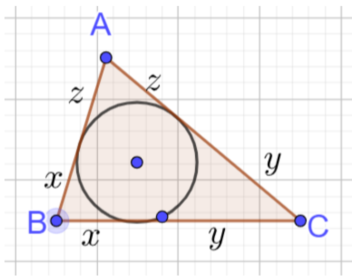
$$BC = a, \qquad AC = b, \qquad AB = c$$

The tangents drawn from points A, B and C have length:

$$z = s - a$$
$$y = s - c$$
$$x = s - b$$

Where

$$s = semiperimeter = \frac{a+b+c}{2}$$



Tangents drawn from a point to a circle are congruent.

$$\underbrace{x+y=a}_{Equation\ I}, \qquad \underbrace{z+y=b}_{Equation\ II}, \qquad \underbrace{x+z=c}_{Equation\ III}$$

Add the three equations:

$$2(x+y+z) = a+b+c$$
$$x+y+z = \frac{a+b+c}{2} \Rightarrow \underbrace{x+y+z=s}_{Equation\ IV}$$

Subtract Equation I from Equation IV:

$$z = s - a$$

Subtract Equation II from Equation IV:

$$x = s - b$$

Subtract Equation III from Equation IV:

$$y = s - c$$

J. Triangle formed by Incircle at Points of Tangency

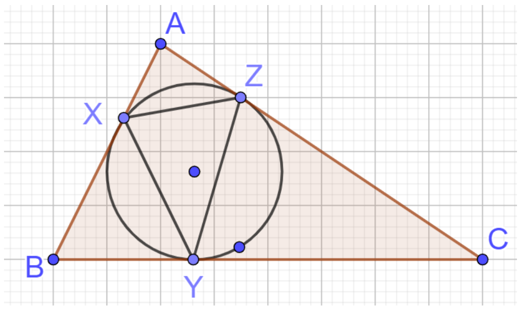
The incircle touches the triangle at three points, where the sides are tangent to the circle. These three points can be joined to form a triangle.

Example 1.70

The incircle to ΔABC is tangent to AB at X, BC at Y and AC at Z. Find the angles of ΔXYZ if $\angle A = 20, \angle B = 40$.

1.71: Angles in "Inner" Triangle

Let the incircle of ΔABC be tangent to the sides at X, Y and Z respectively. Then the angles of ΔXYZ are the averages of the angles of ΔABC, taken two at a time.



$$\angle AXZ = \angle AZX = \frac{180 - \angle A}{2}$$
$$\angle BXY = \angle BYX = \frac{180 - \angle B}{2}$$
$$\angle CYZ = \angle CZY = \frac{180 - \angle C}{2}$$

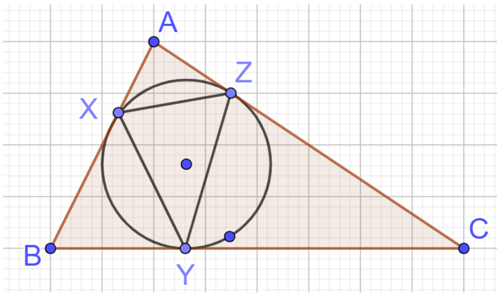
$$\angle ZXY = 180 - \frac{180 - \angle A}{2} - \frac{180 - \angle B}{2} = \frac{\angle A + \angle B}{2}$$
$$\angle ZYX = 180 - \frac{180 - \angle B}{2} - \frac{180 - \angle C}{2} = \frac{\angle B + \angle C}{2}$$
$$\angle XZY = 180 - \frac{180 - \angle A}{2} - \frac{180 - \angle C}{2} = \frac{\angle A + \angle C}{2}$$

Example 1.72

The incircle to ΔABC is tangent to AB at X, BC at Y and AC at Z. If $\angle A = 30, \angle B = 50$. Find the angles of ΔXYZ.

$$\angle C = 180 - 30 - 50 = 100$$

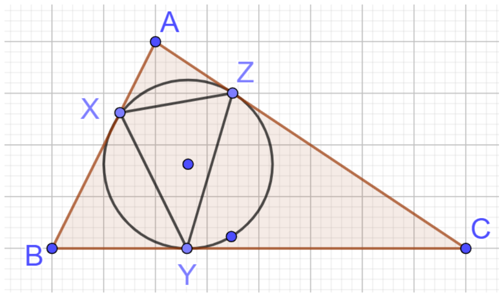
$$\angle ZXY = \frac{30 + 50}{2} = \frac{80}{2} = 40$$
$$\angle XYZ = \frac{50 + 100}{2} = \frac{150}{2} = 75$$
$$\angle XZY = \frac{30 + 100}{2} = \frac{130}{2} = 65$$



Example 1.73

The incircle to ΔABC is tangent to AB at X, BC at Y and AC at Z. If $\angle A = 20$, find $\angle XYZ$.

$$\angle XYZ = \frac{\angle B + \angle C}{2} = \frac{180 - \angle A}{2} = \frac{160}{2} = 80$$



K. Back Calculations

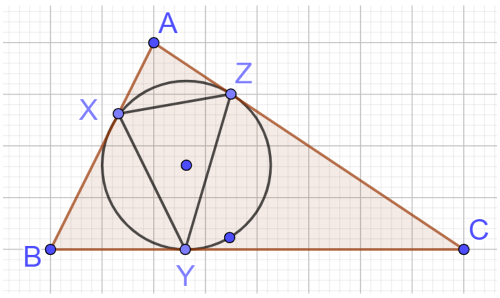
1.74: Circumcircle

Circumcircle of a triangle is the circle that passes through the vertices of the triangle.

Example 1.75

The circumcircle of ΔXYZ is inscribed in ΔABC. If $\angle X = 30, \angle Y = 70$, find the angles of ΔABC.

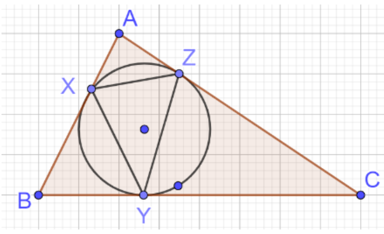
By angles in a triangle:
$$\angle Z = 180 - 30 - 70 = 80$$
Let:
$$\angle A = a, \angle B = b, \angle C = c \Rightarrow \underbrace{a + b + c = 180}_{\textit{Equation I}}$$
Then, by using Equation I:
$$\frac{a+b}{2} = 30 \Rightarrow a + b = 60 \Rightarrow c = 120$$
$$\frac{b+c}{2} = 70 \Rightarrow b + c = 140 \Rightarrow a = 40$$
$$\frac{a+c}{2} = 80 \Rightarrow a + c = 160 \Rightarrow b = 20$$



Example 1.76

Checking Feasbility

Is it possible for the circumcircle of a right-angled triangle to be inscribed in a triangle?



Suppose

$$\angle ZXY = 90 \Rightarrow \frac{\angle A + \angle B}{2} = 90 \Rightarrow \angle A + \angle B = 180 \Rightarrow Not\ Possible$$

L. Arithmetic Sequences (Optional)[1]

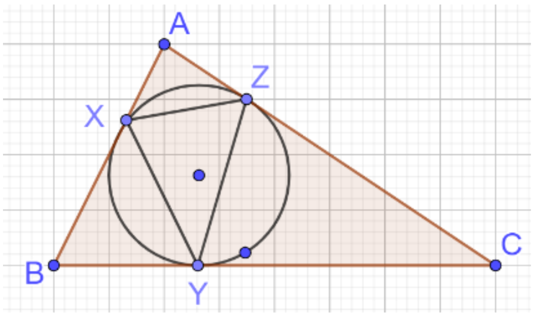
1.77: Arithmetic Sequence

If the difference between two consecutive elements of a sequence is constant, then it is an arithmetic sequence. It is of the form:

$$\dots, a-2d, a-d, a, a+d, a+2d, \dots$$
$$a = middle\ term$$
$$d = common\ difference$$

Example 1.78

In ΔABC, the measures of the angles form an arithmetic sequence such that the measure of the smallest angle is equal to the common difference of the arithmetic sequence. Also, $\angle A < \angle B < \angle C$. The incircle to ΔABC is tangent to AB at X, BC at Y and AC at Z. Find the angles of ΔXYZ.



For any triangle with angles in arithmetic sequence, we must have:

$$a - d + a + a + d = 180 \Rightarrow 3a = 180 \Rightarrow a = 60$$

For the triangle in this question, we have:

$$a - d = d \Rightarrow a = 2d \Rightarrow 2d = 60 \Rightarrow d = 30$$

Hence, the angles are:

$$\angle A = 30°, \qquad \angle B = 60°, \qquad \angle C = 90°$$

$$\angle ZXY = \frac{30+60}{2} = \frac{90}{2} = 45$$
$$\angle XYZ = \frac{60+90}{2} = \frac{150}{2} = 75$$
$$\angle XZY = \frac{30+90}{2} = \frac{120}{2} = 60$$

1.79: Arithmetic Sequence

The incircle to ΔABC is tangent to AB at X, BC at Y and AC at Z.

If in ΔABC, the measures of the angles form an arithmetic sequence with common difference d, then in ΔXYZ, the angles form an arithmetic sequence with common difference $\frac{d}{2}$.

That is,

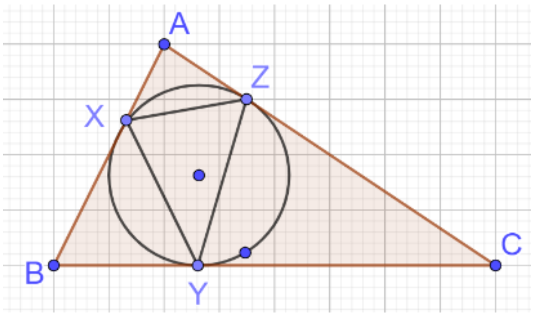
$$If\ \angle A = a - d, \qquad \angle B = a, \qquad \angle C = a + d, \qquad d > 0$$

Then,

$$\angle ZXY = a - \frac{d}{2}, \qquad \angle ZYX = a + \frac{d}{2}, \qquad \angle XZY = a$$

[1] You should know Arithmetic Sequences. If you need to learn/revise, they are covered in the Note on Sequences and Series.

$$\angle ZXY = \frac{\angle A + \angle B}{2} = \frac{a-d+a}{2} = a - \frac{d}{2}$$
$$\angle ZYX = \frac{\angle B + \angle C}{2} = \frac{a+a+d}{2} = a + \frac{d}{2}$$
$$\angle XZY = \frac{\angle A + \angle C}{2} = \frac{a-d+a+d}{2} = a$$



Example 1.80

Let triangle T_{n+1} have its vertices at the points where the incircle of triangle T_n is tangent to its sides. Let the triplet t_n represent the angles of T_n. Given that $t_1 = (12,60,108)$, find

A. t_2
B. t_3
C. t_n

$$t_1 = (12,60,108) = (60-48,60,60+48)$$
$$t_2 = (60-24,60,60+24)$$
$$t_3 = (60-12,60,60+12)$$
$$\vdots$$
$$t_n = \left(60 - \frac{48}{2^{n-1}}, 60, 60 + \frac{48}{2^{n-1}}\right)$$

Example 1.81

$$59.9375, 60$$

Go outward forming triangles from incircle.
Angles of outermost triangle

$$t_1 = \left(60 - \frac{1}{2^4}, 60, 60 + \frac{1}{2^4}\right)$$
$$t_2 = \left(60 - \frac{1}{2^3}, 60, 60 + \frac{1}{2^3}\right)$$
$$t_3 = \left(60 - \frac{1}{2^2}, 60, 60 + \frac{1}{2^2}\right)$$
$$\vdots$$
$$t_n = \left(60 - \frac{1}{2^{5-n}}, 60, 60 + \frac{1}{2^{5-n}}\right)$$

$$60 - \frac{1}{2^{5-n}} > 0 \Rightarrow 60 > \frac{1}{2^{5-n}} \Rightarrow 60 > 2^{n-5}$$
$$32 < 60 < 64 \Rightarrow 2^5 < 60 < 2^6$$
$$n - 5 < 6 \Rightarrow n < 11 \Rightarrow n \le 10$$

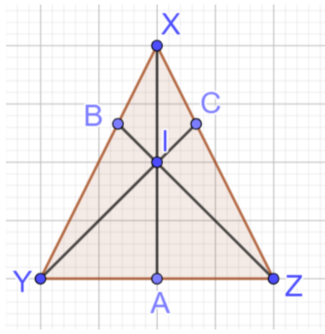
1.5 Special Triangles

A. Isosceles Triangles

1.82: Incentre of an Isosceles Triangle

The altitude to the base of an isosceles triangle passes through the incentre of the triangle.

Example 1.83
In Isosceles ΔXYZ, with $XY = XZ$, the incentre is at the point of concurrency of the angle bisectors XA, YC and ZB.
Explain why the incentre lies on the altitude and the median from vertex X.



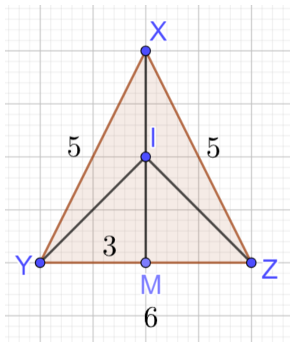
In an isosceles triangle when drawn from congruent sides to the base:

$$Median = Altitude = Angle\ Bisector$$

Hence,

$$I\ lies\ on\ altitude\ as\ well\ as\ median.$$

Example 1.84
What is the area of a circle inscribed in a triangle with side lengths 5, 5 and 6?



In Isosceles ΔIYZ drop a perpendicular (which is also the median) to side YZ:

$$MZ = \frac{1}{2} \times YZ = \frac{1}{2} \times 6 = 3$$

$$r = \frac{\Delta}{s} = \frac{\frac{1}{2} \times 6 \times 4}{\frac{1}{2} \times (5+5+6)} = \frac{24}{16} = \frac{3}{2}$$

$$Area\ of\ Incircle = \pi r^2 = \pi \left(\frac{3}{2}\right)^2 = \frac{9}{4}\pi = 2.25\pi$$

B. Equilateral Triangle

1.85: Inradius of equilateral triangle
An equilateral triangle with side length x has

$$Inradius = r = \frac{\sqrt{3}x}{6}$$

$$Area\ of\ Incircle = \frac{\pi x^2}{12}$$

$$Inradius = r = \frac{\Delta}{s} = \frac{\frac{\sqrt{3}}{4}x^2}{\frac{3x}{2}} = \frac{\sqrt{3}}{4}x^2 \times \frac{2}{3x} = \frac{\sqrt{3}x}{6}$$

$$Area\ of\ Incircle = \pi r^2 = \pi \left(\frac{\sqrt{3}x}{6}\right)^2 = \pi \left(\frac{3x^2}{36}\right) = \frac{\pi x^2}{12}$$

Example 1.86
The number of inches in the perimeter of an equilateral triangle equals the number of square inches in the area of its inscribed circle. What is the radius, in inches, of the circle?

The relation between the side of an equilateral triangle, and the inradius is:

$$r = \frac{\sqrt{3}x}{6}$$

Find the perimeter of the triangle in terms of the radius:

$$x = \frac{6r}{\sqrt{3}} = \frac{6\sqrt{3}r}{3} = 2\sqrt{3}r \Rightarrow P = 3x = 6\sqrt{3}r$$

$$A = P \Rightarrow \pi r^2 = 6\sqrt{3}r \Rightarrow r = \frac{6\sqrt{3}}{\pi}$$

C. Right Triangles

The formula for the inradius of a right triangle can be stated in terms of the hypotenuse. Knowing this result can greatly simplify "advanced" problems, as we will see.

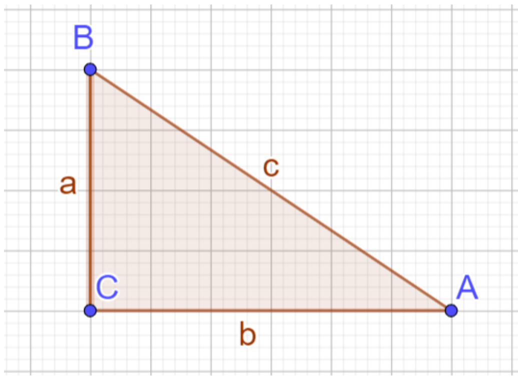
1.87: Inradius for a right triangle

The inradius of a right triangle is:

$$r = s - h = \frac{a+b-c}{2}$$

Where

$$h = c = hypotenuse$$



Substitute $\Delta = \frac{1}{2}ab,\ s = \frac{a+b+c}{2}$ to get:

$$r = \frac{\Delta}{s} = \frac{\frac{1}{2}ab}{\frac{a+b+c}{2}} = \frac{ab}{a+b+c}$$

We need $a+b-c$ in the numerator, so create it in the numerator:

$$\frac{ab}{a+b+c} \times \frac{a+b-c}{a+b-c}$$

Apply the formula for difference of squares in the denominator:

$$\frac{ab(a+b-c)}{(a+b)^2 - c^2} = \frac{ab(a+b-c)}{a^2+2ab+b^2-c^2}$$

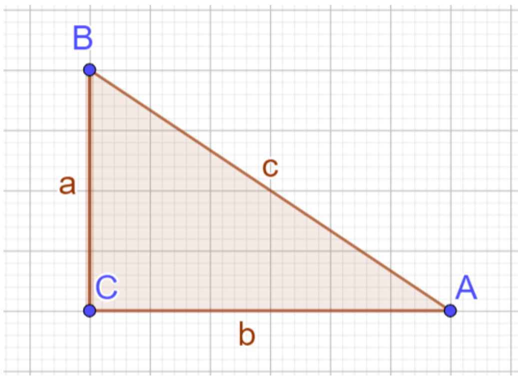
But $a^2+b^2=c^2 \Rightarrow a^2+b^2-c^2=0$:

$$\frac{ab(a+b-c)}{2ab} = \frac{a+b-c}{2}$$

Example 1.88

A right triangle has perimeter 32 and area 20. What is the length of its hypotenuse? **(AMC 10A 2008/18)**

Method I: Pythagoras Theorem



From the diagram, since the perimeter is 32:

$$a+b+c = 32 \Rightarrow c = 32 - (a+b)$$

Since the area is 20:

$$\frac{1}{2}ab = 20 \Rightarrow ab = 40$$

By Pythagoras Theorem:

$$a^2+b^2 = [32-(a+b)]^2$$
$$a^2+b^2 = 1024 - 64(a+b) + (a+b)^2$$
$$a^2+b^2 = 1024 - 64(a+b) + a^2 + 2ab + b^2$$
$$0 = 1024 - 64(a+b) + 2ab$$

Substitute $ab = 40$:

$$64(a+b)=1024+80$$
$$a+b=17.25$$

$$c=32-(a+b)=32-17.25=14.75$$

Method II: Inradius

$$r=\frac{\Delta}{s}=\frac{20}{16}=\frac{20}{32/2}=\frac{20}{16}=\frac{5}{4}$$
$$r=s-h\Rightarrow h=s-r=16-\frac{5}{4}=\frac{59}{4}=14.75$$

Example 1.89

Calculate the inradius of a triangle with sides 3,4, 5.

$$r=\frac{3+4-5}{2}=\frac{7-5}{2}=\frac{2}{2}=1$$

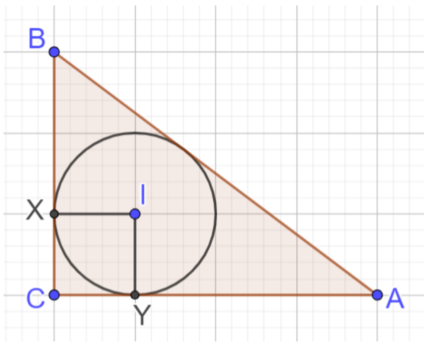
1.90: Inradius for a right triangle

The inradius of a right triangle is:

$$r=s-h=\frac{a+b-c}{2}$$

Where

$$h=c=hypotenuse$$



We can calculate the formula for inradius using the length of tangent formula calculated above.

$$\Delta BCA\ is\ right-angled\Rightarrow\angle BCA=90°$$

Since the radius is perpendicular to tangent at point of tangency:

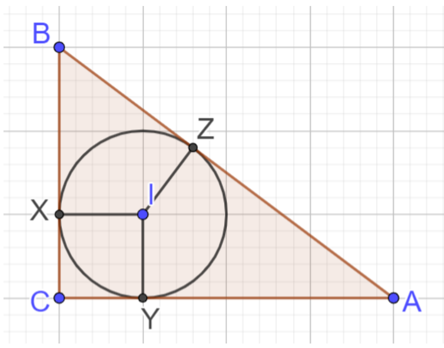
$$\angle IXC=\angle IYC=90°$$

In Quadrilateral $IXCY$, by sum of angles:

$$\angle XIC=360-90-90-90=90°\Rightarrow IXCY\ is\ a\ rectangle$$

In Quadrilateral $IXCY$,

$$IX=IY\Rightarrow Adjacent\ sides\ are\ equal\Rightarrow IXCY\ is\ a\ square.$$



As already shown when we calculated the formula for lengths of tangents to incircle:

$$r=IX=CY=s-h$$

Example 1.91

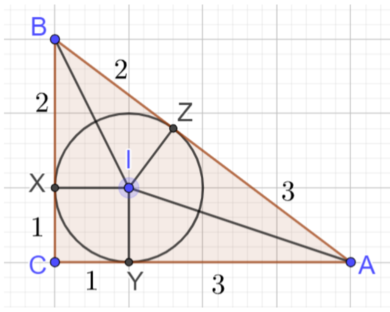
ΔBCA, drawn alongside, is a $3-4-5$ right triangle. I is the incenter. Find the lengths of the tangents to the incircle.

Find the semi-perimeter

$$s=\frac{a+b+c}{2}=\frac{3+4+5}{2}=\frac{12}{2}=6$$

Use the formula to find the lengths of the tangents:

$$CY = CX = s - AB = 6 - 5 = 1$$
$$AY = AZ = s - BC = 6 - 3 = 3$$
$$BX = BZ = s - CA = 6 - 4 = 2$$



Find the area of quadrilateral:

A. $IXCY$
B. $IZAY$
C. $IZBX$

$$[IXCY] = s^2 = 1 \times 1 = 1$$
$$[IZAY] = 2[IYA] = 2\left[\frac{1}{2} \times 1 \times 3\right] = 3$$
$$[IZBX] = 2[IXB] = 2\left[\frac{1}{2} \times 1 \times 2\right] = 2$$

Example 1.92

Find the lengths of the tangents to the incircle in a $30 - 60 - 90$ right triangle with length of hypotenuse 2.

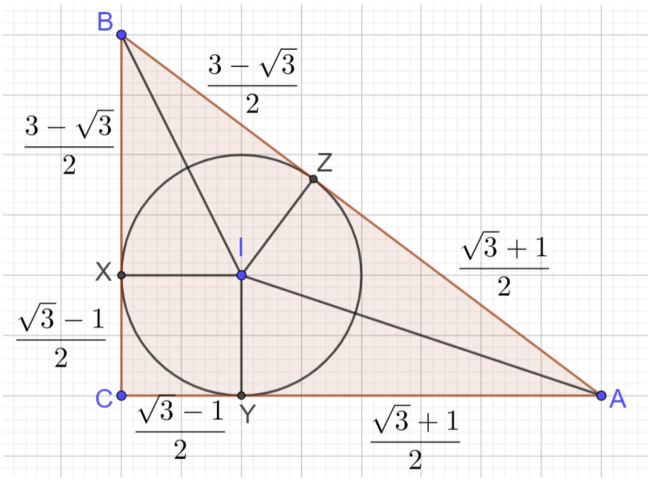
Let

$$BA = 2 \Rightarrow BC = 1, CA = \sqrt{3}$$

Find the semi-perimeter

$$s = \frac{2 + 1 + \sqrt{3}}{2} = \frac{3 + \sqrt{3}}{2}$$

$$CX = CY = \frac{3 + \sqrt{3}}{2} - 2 = \frac{\sqrt{3} - 1}{2}$$
$$AY = AZ = \frac{3 + \sqrt{3}}{2} - 1 = \frac{\sqrt{3} + 1}{2}$$
$$BX = BZ = \frac{3 + \sqrt{3}}{2} - \sqrt{3} = \frac{3 - \sqrt{3}}{2}$$



Find the area of quadrilateral:

A. $IXCY$
B. $IZAY$
C. $IZBX$

$$[IXCY] = s^2 = \left(\frac{\sqrt{3} - 1}{2}\right)^2 = \frac{3 - 2\sqrt{3} + 1}{4} = \frac{4 - 2\sqrt{3}}{4} = \frac{2 - \sqrt{3}}{2}$$
$$[IZAY] = 2[IYA] = 2\left[\frac{1}{2}\left(\frac{\sqrt{3} + 1}{2}\right)\left(\frac{\sqrt{3} - 1}{2}\right)\right] = \frac{3 - 1}{4} = \frac{2}{4} = \frac{1}{2}$$
$$[IZBX] = 2[IXB] = 2\left[\frac{1}{2}\left(\frac{3 - \sqrt{3}}{2}\right)\left(\frac{\sqrt{3} - 1}{2}\right)\right] = \frac{3\sqrt{3} - 3 - 3 + \sqrt{3}}{4} = \frac{4\sqrt{3} - 6}{4}$$

Find the area of the incircle of the triangle.

$$A = \pi r^2 = \pi \left(\frac{\sqrt{3} - 1}{2} \right)^2 = \pi \frac{2 - \sqrt{3}}{2}$$

The area inside the triangle, but outside the incircle is divided into three distinct parts. Find the area of each part.

2. Further Topics

2.1 Perpendicular Bisector & Circumcircle

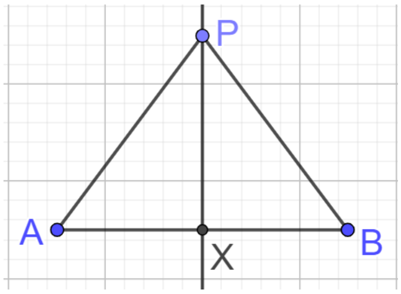
A. Perpendicular Bisectors

2.1: Perpendicular Bisector Theorem
Any point on the perpendicular bisector of a line segment is equidistant from the endpoints of the segment.

To be precise, if P lies on the perpendicular bisector of AB, then

$$PA = PB$$

Consider line segment AB. Draw perpendicular bisector of AB, perpendicular to AB at X.



Case I: The point is on Segment AB
X is the midpoint of AB, and hence

$$AX = XB \Rightarrow Proved$$

Case II: The point is not on Segment AB
Draw PA and PB.

In ΔPAX and ΔPBX:

$$PX = PX\ (Common\ Side)$$
$$AX = XB\ (X\ is\ midpoint\ of\ AB)$$
$$\angle PXA = \angle PBX = 90°\ (PX \perp AB)$$

∴ By SAS Congruence:

$$\Delta PAX \cong \Delta PBX$$

By CPCTC:

$$PA \cong PB \Rightarrow Proved$$

Case II: Using Pythagorean Theorem
Since X is the midpoint of AB:

$$AX = XB$$

By the Pythagorean Theorem

$$In\ \Delta PAX: PA^2 = PX^2 + AX^2$$
$$In\ PBX: PB^2 = PX^2 + XB^2 = PX^2 + AX^2$$

$$PA^2 = PB^2 \Rightarrow PA = PB$$

Example 2.2
Points A and B have a distance of 3 units between them. Find all points that are equidistant from A and B.

All points that lie on the perpendicular bisector of A and B.

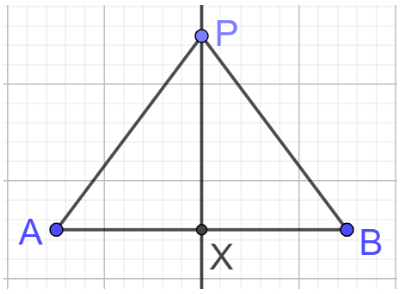
2.3: Converse of Perpendicular Bisector Theorem
If a point is equidistant from the endpoints of a line segment, then it lies on the perpendicular bisector of that

segment.

To be precise, if $PA = PB$, then P lies on the perpendicular bisector of AB.

$$PA = PB \Rightarrow \Delta PAB \text{ is Isosceles}$$

Draw $PX \perp AB$:



Since $PX \perp AB$

$$\underbrace{\angle PXA = \angle PXB = 90°}_{\textit{Statement I}}$$

$$\underbrace{AX = XB}_{\textit{Statement II}} \quad (\textit{In Isosceles } \Delta, \textit{altitude is also the median})$$

Hence, by combining Statement I and II:

$$P \textit{ lies on the perpendicular bisector of } AB$$

Example 2.4
Point X lies on the perpendicular bisector of Points A and B. The distance from Point X to Point A is 4 units. What is the distance from Point X to Point B?

Also 4 units

B. Perpendicular Bisectors of a Triangle

2.5: Perpendicular Bisectors of a Triangle
The lines perpendicular to the sides of a triangle, and bisecting it, are the perpendicular bisectors of the triangle.

Example 2.6
What is the number of perpendicular bisectors of a triangle?

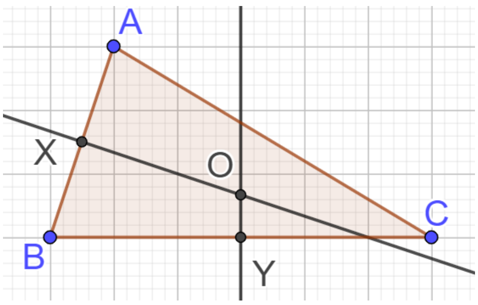
3

2.7: Concurrent Lines
If a set of lines intersects at a single point, they are said to be concurrent.

2.8: Circumcenter
- Part A: The three perpendicular bisectors of a triangle are concurrent at the circumcenter.
- Part B: The circumcenter is the center of the circle that passes through the three vertices of the triangle.

Draw ΔABC.
Draw the perpendicular bisectors of AB, and BC intersecting

$$AB \text{ at } X, BC \text{ at } Y$$



Since the sides of the triangle cannot be parallel, the lines perpendicular also cannot be parallel, and hence the lines must intersect.

Let O be the point of intersection of the two perpendicular bisectors drawn.

Since point O lies on line XO, by the Perpendicular Bisector Theorem:

$$\therefore O \text{ is equidistant from } A \text{ and } B \Rightarrow \underbrace{OA = OB}_{\textbf{Equation I}}$$

Since point O lies on line YO, by the Perpendicular Bisector Theorem:

$$\therefore O \text{ is equidistant from } B \text{ and } C \Rightarrow \underbrace{OB = OC}_{\textbf{Equation II}}$$

Combine Equations I and II into a single equation, giving us:

$$\underbrace{OA = OB = OC}_{\textbf{Equation III}} \Rightarrow O \text{ is equidistant from } A, B \text{ and } C$$

From the above, we can say that:

$$\therefore O \text{ is equidistant from } A \text{ and } C \Rightarrow \underbrace{OA = OC}_{\textbf{Equation IV}}$$

Part A

Hence,

$$O \text{ lies on the perpendicular bisector of } AC$$

We already knew that O lies on XO and YO. Now, since it lies on the perpendicular bisector of AC as well:

$$O \text{ lies on all three } \perp \text{ bisectors} \Rightarrow \text{All three } \perp \text{ bisectors are concurrent}$$

Part B

From Equation III, we know that

$$\underbrace{OA = OB = OC}_{\textbf{Equation III}} \Rightarrow O \text{ is equidistant from } A, B \text{ and } C$$

This means that

$$O \text{ is the center of the circle that passes through } A, B \text{ and } C$$

Example 2.9

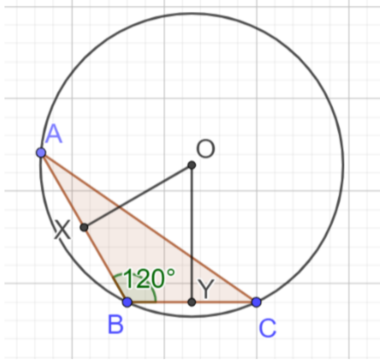
Consider points X, Y and Z. Find all points that are equidistant from X, Y and Z.

Points equidistant from X and Y are those that lie on the perpendicular bisector of X and Y.
Points equidistant from X and Z are those that lie on the perpendicular bisector of X and Z.
Points equidistant from Y and Z are those that lie on the perpendicular bisector of Y and Z.

To meet all three conditions, they must lie on all three perpendicular bisectors above.
This is the circumcenter of ΔXYZ.

Example 2.10

ΔABC is an obtuse triangle with $\angle ABC = 120°$, and circumcenter O. X and Y are the midpoints of AB and BC. Find $\angle XOY$.



Draw a diagram.
In Quadrilateral $XOYB$, by angle sum property of a quadrilateral:

$$\angle XOY = 360 - \underbrace{90}_{\angle OXB} - \underbrace{90}_{\angle OYB} - \underbrace{120}_{\angle ABC} = 60°$$

C. Circumcenter and Circumcircle

2.11: Circumcircle, Circumcenter, and Circumradius

- The unique circle that passes through the three vertices of a triangle is called its circumcircle.
- The center of the circumcircle is called circumcenter.
- The radius of the circumcircle is called the circumradius.

Example 2.12

What is the maximum number of circumcircles that a triangle can have? The minimum?

1
1

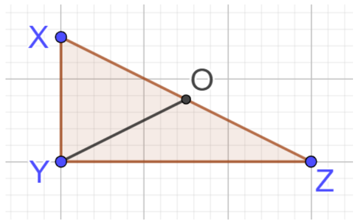
D. Angle Chasing

Example 2.13

In right-triangle ΔXYZ, with Y as the right-angle, with O as circumcenter, $\angle XOY = 36°$. Find $\angle XOZ$ and $\angle YOZ$.

Draw a diagram.
O is the circumcenter and is the midpoint of hypotenuse XZ.

$$\angle XOZ = 180°$$
$$\angle YOZ = 180 - 36 = 144°$$



Example 2.14

In acute triangle ΔXYZ with O as circumcenter, $\angle XOY = 140°$, and $\angle YOZ = 130°$.

A. Find $\angle XOZ$
B. If XZ is 3 units, find XO.

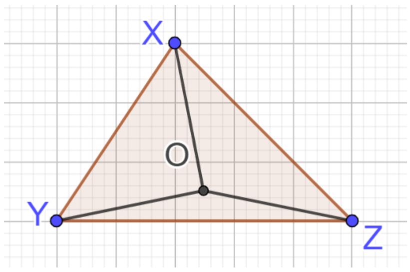
Part A
By angles around a point:

$$\angle XOZ = 360 - \underbrace{140}_{\angle XOY} - \underbrace{130}_{\angle YOZ} = 90°$$

Part B

$$OX = OZ \Rightarrow \Delta XOZ \text{ is Isosceles} \Rightarrow \Delta XOZ \text{ is } 45-45-90 \text{ triangle}$$

$$XO = XZ \times \frac{1}{\sqrt{2}} = 3 \times \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

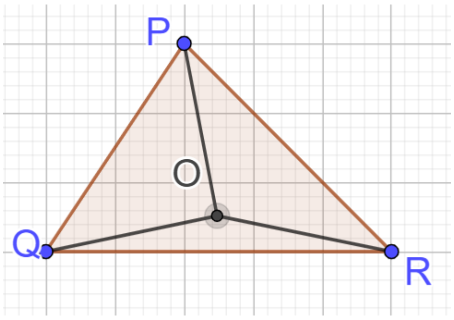


Example 2.15

In acute-angled ΔPQR, O is the point of intersection of the perpendicular bisectors of the sides. Find $\angle POR$ if:

$$\angle OQR = 30°, \angle ORQ = 50°, \angle PQR = 85°, \angle QPO = 40°$$

$$\angle QOR = 180 - 30 - 50 = 100°$$
$$\angle PQO = 85 - 30 = 55°$$
$$\angle POQ = 180 - 55 - 40 = 85$$
$$\angle POR = 360 - 100 - 85 = 175°$$



E. Location of Circumcenter

2.16: Location of Circumcenter

- For an acute-angled triangle, the circumcenter is inside the triangle.
- For a right-angled triangle, the circumcenter is the midpoint of the hypotenuse.
- For an obtuse-angled triangle, the circumcenter is outside the triangle.

MCQ 2.17

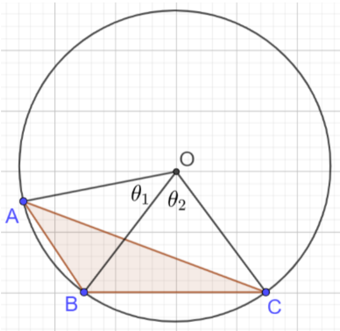
In ΔABC with circumcenter O:

$$\angle AOB + \angle BOC + \angle AOC = 360°$$

Then, ΔABC cannot be:

A. Acute Angled
B. Obtuse Angled
C. Right Angled
D. Isosceles
E. Scalene

We evaluate this using cases.



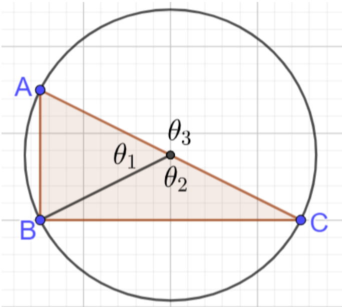
Case I: ΔABC is obtuse

For an obtuse-angled ΔABC (see diagram):

$$\angle AOB + \angle BOC + \angle AOC = \theta_1 + \theta_2 + (\theta_1 + \theta_2) = 2\theta_1 + 2\theta_2$$

And the above does not add up 180° for an obtuse angled triangle.

Option B is correct.



Case II: ΔABC is right-angled

For an right-angled ΔABC (see diagram):

$$\angle AOB + \angle BOC + \angle AOC = \theta_1 + \theta_2 + \theta_3$$

And, by angles around a point:

$$\theta_1 + \theta_2 + \theta_3 = 360°$$

Option C is not correct

Example 2.18

In obtuse-angled triangle ΔXYZ with O as circumcenter, $\angle XOY = 70°$, and $\angle YOZ = 100°$. Then $\angle XOZ$:

A. Is 90°
B. Is 170°
C. Is 30°
D. Either B or C

Case I: ΔXYZ is obtuse-angled at $\angle Y$

$$\angle XOZ = \angle XOY + \angle YOZ = 70 + 100 = 170°$$

This case gives us a solution, which is Option B.

Case II: ΔXYZ is obtuse-angled at $\angle Z$

$\angle YOX = \angle YOZ + \text{ZOX} \Rightarrow 70 = 80 + \text{XOZ} \Rightarrow \text{XOZ}\ is - ve \Rightarrow No\ Solution$

This case does not give us any solutions.

Case III: ΔXYZ is obtuse-angled at $\angle X$

$$\angle YOZ = \angle XOY + \angle XOZ \Rightarrow 100 = 70 + \angle XOZ \Rightarrow \angle XOZ = 30°$$

This case gives us a solution, which is Option C.

Since both Option B or Option C can be correct, the final answer is Option D.

F. Circle Properties

In this section, we focus on the circle properties of the circumcircle. If you have not seen these properties before, you can refer the Note on Circles.

G. Inscribed Angles

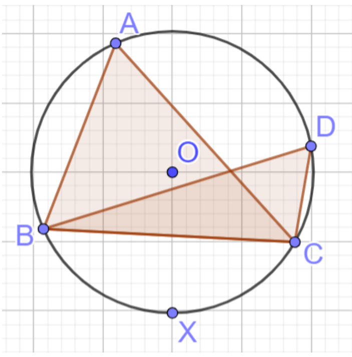
2.19: Angles in the same Arc

Two angles inscribed in the same arc are congruent.

Example 2.20

O is the circumcenter of ΔABC with $\angle A = 40$. O is also the circumcenter of ΔDBC. Find the sum of the possible values of $\angle BDC$.

There are two cases to consider.



Case I: D is on the same side of the chord as A

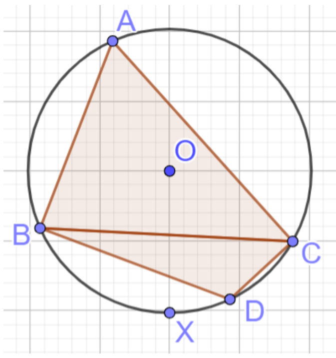
See Diagram.

$\angle BAC$ is inscribed in $\widehat{BXC}$.

$\angle BDC$ is inscribed in $\widehat{BXC}$.

Hence,

$$\angle BAC \cong \angle DAC = 40°$$



Case I: D is on the opposite side of the chord as A

See Diagram.

$\angle BAC$ is inscribed in $\widehat{BXC}$.

$\angle DAC$ is inscribed in $\widehat{BXC}$.

Hence,

$$\angle BAC \cong \angle DAC = 40°$$

2.21: Angles in congruent Arcs
Two angles inscribed in congruent arcs are congruent.

Example 2.22
O is the circumcenter of ΔABC with $\angle A = 40$. P is the circumcenter of ΔXYZ. The circumradius of ΔABC is equal to the circumradius of ΔXYZ.

2.23: Congruent Inscribed Angles
If two inscribed angles have the same measure, the arcs they intercept are congruent.

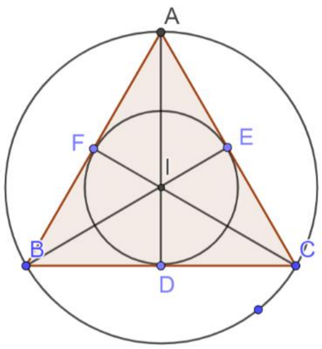
H. Central Angle

2.24: Inscribed Angle
An inscribed angle in a circle is one-half of the corresponding central angle.

Example 2.25
O is the circumcenter of ΔABC with $\angle A = 40$ and $\angle B = 70$. Find

I. Equilateral Triangles
An equilateral triangle has all three sides equal. This gives it symmetry. And the symmetry results in many elegant and powerful properties, which do not otherwise hold for general triangles.
One of these properties is that the incenter and the circumcenter of an equilateral triangle are the same. We first prove this property, and then we put it to work for us.

2.26: Incenter and Circumcenter coincide in an equilateral triangle
The incentre of an equilateral triangle is also its circumcenter.



Draw equilateral ΔABC and construct incentre I as the intersection of:
$$Angle\ Bisectors\ IA, IB, IC$$
Since ΔABC is equilateral:
$$\angle ABC = \angle ACB = 60°$$
Since IB and IC are angle bisectors:
$$\angle IBD = \angle ICB = \frac{60}{2} = 30°$$
In Isosceles ΔIBC,
$$Median\ ID = Altitude\ ID \Rightarrow ID\ is \perp bisector\ of\ BC$$
Similarly,
$$IE\ is \perp bisector\ of\ AC$$
$$IF\ is \perp bisector\ of\ AB$$
Hence, point I is intersection of perpendicular bisectors of the sides of ΔABC.
Hence, it is also the circumcenter.

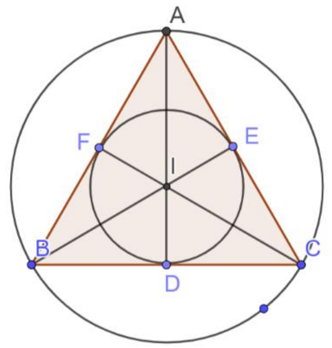
2.27: Angle Bisector and Perpendicular Bisectors
In an equilateral triangle, the angle bisectors and perpendicular bisectors are concurrent.
$Angle\ Bisectors(IA, IB, IC)$ and Perpendicular Bisectors of the sides $(ID, IE\ and\ IF)$ are concurrent.

This should be clear from the proof of the previous property, but is important enough to state as a property on its own.
Since the circumcenter and the incenter in an equilateral triangle are the same, that common point is the point of concurrency of both its angle bisectors and the perpendicular bisectors of its sides.

2.28: $30-60-90$ Triangles formed by Incenter/Circumcenter

The common incenter/circumcenter of an equilateral triangle divides it into 6 congruent $30-60-90$ triangles.
In the diagram, these are:

$$\Delta IBD, \Delta ICD, \Delta ICE, \Delta IAE, \Delta IAF, \Delta IBF$$



Draw equilateral ΔABC with incenter/circumcenter I.

Since ΔABC is equilateral:

$$\angle ABC = 60°$$

Since IB is an angle bisector:

$$\angle IBD = \frac{60}{2} = 30$$

Since ID is a perpendicular bisector:

$$\angle IDB = 90°$$

By sum of angles in a triangle, in ΔIBD:

$$\angle BIC = 180 - 30 - 90 = 60$$

Hence,

$$\Delta IBD \text{ } is \text{ } a \text{ } 30-60-90 \text{ } triangle$$

Similarly, the following are all $30-60-90$ $triangles$:

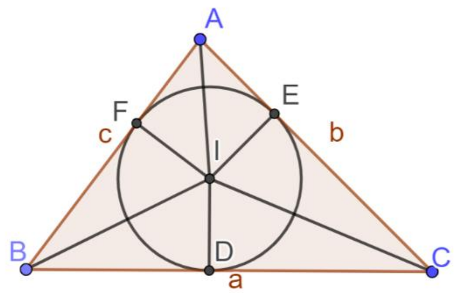
$$\Delta ICD, \Delta ICE, \Delta IAE, \Delta IAF, \Delta IBF$$

2.29: Circumradius of an equilateral triangle

The circumradius of an equilateral triangle is $\frac{1}{\sqrt{3}}$ times the side length

Note: Pay attention to the properties of equilateral triangles, being shown here, not just the answer. In particular, how can we make use of the fact that

$$Circumcenter = Incenter$$



Draw ΔABC with side length s with I as circumcenter.
The circumradius is the distance from the circumcenter to the vertices.

$$Circumradius = IA = IB = IC$$

Since the triangle is equilateral, the circumcenter is also the incenter.
Hence, I is the intersection of angle bisectors:

$$IA, \quad IB, \quad IC$$

2.30: Circumradius and Area of Circumcircle of an equilateral triangle

The circumradius of an equilateral triangle with side length x is is $\frac{1}{\sqrt{3}}x$:

$$Circumradius = \frac{1}{\sqrt{3}}x$$

$$Area\ of\ Circumcircle = \frac{\pi x^2}{3}$$

$$Circumradius = R = \frac{abc}{4\Delta} = \frac{x^3}{4\left(\frac{\sqrt{3}}{4} \times x^2\right)} = \frac{x}{\sqrt{3}}$$

$$Area\ of\ Circumcircle = \pi R^2 = \pi\left(\frac{x}{\sqrt{3}}\right)^2 = \frac{\pi x^2}{3}$$

Example 2.31

The number of inches in the perimeter of an equilateral triangle equals the number of square inches in the area of its circumscribed circle. What is the radius, in inches, of the circle? **(AMC 10A 2003/17)**

$$R = \frac{x}{\sqrt{3}} \Rightarrow x = R\sqrt{3} \Rightarrow P = 3x = 3\sqrt{3}R$$

$$A = P \Rightarrow \pi R^2 = 3\sqrt{3}R \Rightarrow R = \frac{3\sqrt{3}}{\pi}$$

Example 2.32

What is the ratio of the inradius and the circumradius of an equilateral triangle?

Example 2.33

What is the ratio of the area of the circumcircle and the area of the incircle of an equilateral triangle?

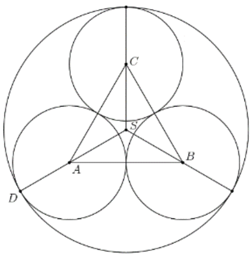
Example 2.34

A circle is circumscribed about an equilateral triangle with side lengths of 9 units each. What is the area of the circle, in square units? Express your answer in terms of π. **(MathCounts 2008 School Countdown)**

$$A = \pi R^2 = \pi\left(\frac{abc}{4\Delta}\right)^2 = \pi\left(\frac{9^3}{4\left(\frac{\sqrt{3}}{4} \times 9^2\right)}\right)^2 = \pi\left(\frac{9}{\sqrt{3}}\right)^2 = \pi\left(\frac{81}{3}\right) = 27\pi$$

Example 2.35

Three circles of radius 1 are externally tangent to each other and internally tangent to a larger circle. What is the radius of the large circle? **(AMC 10B 2004/16)**



Connect the centers of the three circles, getting an equilateral triangle with side length 2.

By symmetry, the circumcenter of the circle passing through CAB is also the center of the larger circle.

Use the form ula for circumcenter to find

$$SA = 2 \times \frac{\sqrt{3}}{3}$$

$$AD = 1$$

Add the two and you are done.

J. Circumradius

2.36: Circumradius

Let the circumradius of a triangle be $\boldsymbol{R}$. And let the lengths of the three sides of the triangle be $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$.

$$Circumradius = R = \frac{abc}{4\Delta} = \frac{abc}{4rs}$$

Draw Diameter CD passing through center O in the circumcircle of ΔABC.[2]

$$\therefore \angle CAD = 90°$$

Draw altitude CE in ΔCAB with length h.

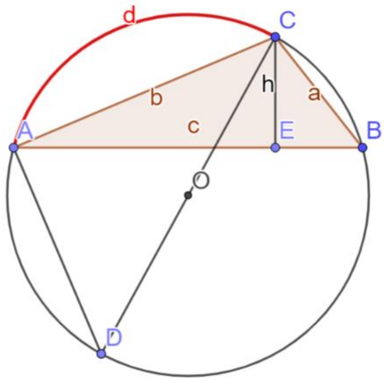
$$\angle CAD = \angle CEB = 90° \Rightarrow \mathbf{I}$$

Since both $\angle ADC$ and $\angle ABC$ subtend the same arc $\boldsymbol{d}$:

$$\angle ADC = \angle ABC \Rightarrow \mathbf{II}$$

Combine **I** and **II** above to show that we have two similar triangles:

$$\Rightarrow \Delta CAD \sim \Delta CEB \ (AA\ Similarity)$$



Hence:

$$\frac{h}{a} = \frac{b}{2R} \Rightarrow R = \frac{ab}{2h}$$

And if we substitute $\Delta = \frac{1}{2}ch \Rightarrow h = \frac{2\Delta}{c}$:

$$R = \frac{ab}{2\left(\frac{2\Delta}{c}\right)} = \frac{abc}{4\Delta}$$

K. General Triangles

Example 2.37

The sides of a triangle are 13, 14 and 15. Find its:

A. Semi-perimeter
B. Area
C. Circumradius.

$$\text{Semiperimeter} = \frac{a+b+c}{2} = \frac{13+14+15}{2} = \frac{42}{2} = 21$$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21(8)(7)(6)} = \sqrt{3^2 \times 7^2(2^4)} = 84$$

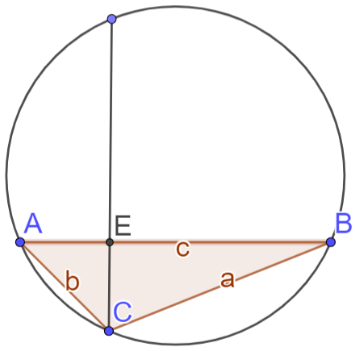
$$Circumradius = R = \frac{abc}{4\Delta} = \frac{13 \times 14 \times 15}{4 \times 84} = \frac{13 \times 15}{4 \times 6} = \frac{195}{24} = 8\frac{1}{8}$$

Challenge 2.38

The diagram alongside has two perpendicular chords in a circle. Given that

$$AE = 2, EB = 9, EC = 3$$

find twice the square of the radius of the given circle. **(NMTC Sub-Junior 2021, Adapted)**



[2] This link from Khan Academy has a video with the same proof.

Consider the circle passing through points

$$A, B, and\ C$$

Recall that three points define a circle. Hence, if we find the circumradius of the triangle, we are done.

Sides of ΔABC

Using Pythagorean Theorem in right ΔECB:

$$a = \sqrt{EB^2 + EC^2} = \sqrt{9^2 + 3^2} = \sqrt{81 + 9} = \sqrt{90} = 3\sqrt{10}$$

Using Pythagorean Theorem in right ΔAEC:

$$b = \sqrt{AE^2 + EC^2} = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$$

$$c = AE + EB = 2 + 9 = 11$$

Four times the area of ΔABC

$$4\Delta = 4[ABC] = 4\left(\frac{1}{2}hb\right) = 4\left(\frac{1}{2} \times EC \times AB\right) = 4\left(\frac{1}{2} \times 3 \times 11\right) = 6 \times 11$$

Substitution

Substitute above values in $R = \frac{abc}{4\Delta}$ to get:

$$R = \frac{abc}{4\Delta} = \frac{3\sqrt{10} \times \sqrt{13} \times 11}{6 \times 11} = \frac{\sqrt{10 \times 13}}{2} = \frac{\sqrt{130}}{2} \Rightarrow 2R^2 = 2 \times \frac{130}{4} = 65$$

L. Right Triangles

Example 2.39

The sides of a triangle are 3, 4, and 5. Find its circumradius.

$$R = \frac{abc}{4\Delta} = \frac{3 \times 4 \times 5}{\frac{1}{2} \times 4 \times 3 \times 4} = \frac{5}{2} = 2.5$$

2.40: Circumradius of a Right Triangle

The circumradius of a right triangle is half the length of its hypotenuse.

The sides of a right triangle are a, b, and c (with c as the hypotenuse). Find its circumradius.

$$R = \frac{abc}{4\Delta} = \frac{abc}{\frac{1}{2} \times 4 \times ab} = \frac{c}{2}$$

Example 2.41

The sides of a triangle are 3, 4, and 5. Find its circumradius.

(3,4,5) is a Pythagorean Triplet. Hence, the triangle is right-angled.

$$\therefore\ R = \frac{hyp}{2} = \frac{5}{2} = 2.5$$

Example 2.42

Two sides of a right triangle are 3 and 4. Find the sum of all possible values of the area of the circle that passes through the vertices of the triangle. (Answer in terms of π).

Case I: 3 and 4 are the legs

$$\therefore\ R = \frac{hyp}{2} = \frac{5}{2} = 2.5 \Rightarrow A = \pi R^2 = \pi \times 2.5^2 = 6.25\pi$$

Case II: 3 is the hypotenuse

$$Hyp\ is\ longest\ side \Rightarrow 3\ is\ longest\ side \Rightarrow 3 > 4 \Rightarrow Contradiction \Rightarrow Not\ Possible$$

Case III: 4 is the hypotenuse

$$\therefore\ R = \frac{hyp}{2} = \frac{4}{2} = 2 \Rightarrow A = \pi R^2 = \pi \times 2^2 = 4\pi$$

$$Total = 6.25\pi + 4\pi = 10.25\pi$$

Example 2.43

A triangle with side lengths in the ratio 3: 4: 5 is inscribed in a circle with radius 3. What is the area of the triangle? **(AMC 10A 2007/14)**

$$3{:}\,4{:}\,5\ is\ a\ Pythagorean\ Triplet \Rightarrow \Delta\ is\ right-angled$$
$$\therefore\ Circumradius\ is\ half\ of\ the\ hypotenuse.$$

Hence, hypotenuse is double of the circumradius:

$$Hyp = 2R = 6$$

The triangle has side lengths in the ratio $\underbrace{3}_{Leg\ 1} : \underbrace{4}_{Leg\ 2} : \underbrace{5}_{Hypotenuse}$, and $hypotenuse = 6$, so its other sides are

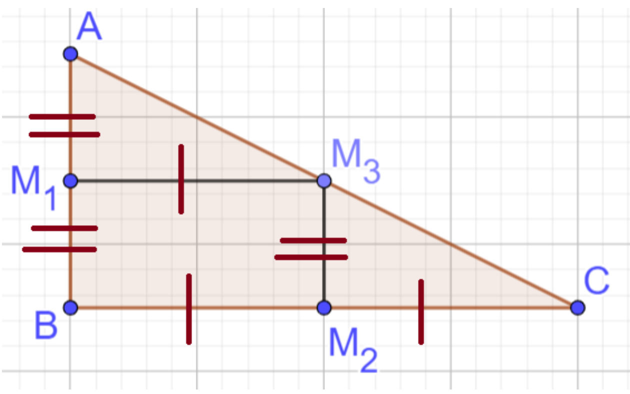
$$3 \times \frac{6}{5} = \frac{18}{5}, \qquad 4 \times \frac{6}{5} = \frac{24}{5}$$

Finally, area of the triangle:

$$= \frac{1}{2}hb = \frac{1}{2} \times \frac{18}{5} \times \frac{24}{5} = \frac{216}{25}$$

2.44: Circumcenter of a Right Triangle

The circumcenter of a right triangle is the midpoint of its hypotenuse.



Draw right ΔABC.
Draw perpendicular bisector of BC and let it intersect the hypotenuse at M_3, and BC at M_2.

From M_3, draw line $\perp$ to M_3M_2 and let it intersect the leg AB at M_1.

ΔAM_1M_3 and ΔM_3M_2C are similar because:

$$\angle AM_1M_3 = \angle M_3M_2C = 90°$$
$$\angle AM_3M_1 = \angle M_3CM_2$$

Also,

$$M_1M_3 = BM_2 = M_2C \Rightarrow \Delta AM_1M_3 \cong \Delta M_3M_2C$$

By CPCT:

$$AM_1 = M_3M_2 = M_1B \Rightarrow M_1\ is\ midpoint\ of\ AB$$

Also, by CPCT:

$$AM_3 = M_3C \Rightarrow M_3\ is\ midpoint\ of\ AC$$

Example 2.45
A right triangle with circumradius 1 has its legs in the ratio 3: 5. Determine the inradius of the triangle.

$$Circumradius = R = 1 \Rightarrow Hyp = 2$$

The legs are in the ratio:

$$3:5 = 3x:5x$$

Hence, by the Pythagorean Theorem:

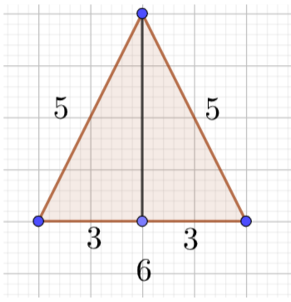
$$(3x)^2 + (5x)^2 = 2^2 \Rightarrow 34x^2 = 4 \Rightarrow x = \sqrt{\frac{2}{17}} \Rightarrow 3x = 3\sqrt{\frac{2}{17}} \Rightarrow 5x = 5\sqrt{\frac{2}{17}}$$

$$Inradius = r = \frac{\Delta}{s} = \frac{\frac{1}{2}\times 3\sqrt{\frac{2}{17}}\times 5\sqrt{\frac{2}{17}}}{\frac{3\sqrt{\frac{2}{17}}+5\sqrt{\frac{2}{17}}+2}{2}} = \frac{\frac{15}{2}\times\frac{2}{17}}{\frac{8\sqrt{\frac{2}{17}}+2}{2}} = \frac{15}{17}\times\frac{2}{8\sqrt{\frac{2}{17}}+2} = \frac{30}{136\sqrt{\frac{2}{17}}+34}$$

M. Isosceles Triangles

Example 2.46
An isosceles triangle with equal sides of 5 inches and a base of 6 inches is inscribed in a circle. What is the radius, in inches, of the circle? Express your answer as a mixed number. **(MathCounts 1992 State Target)**

$$R = \frac{abc}{4\Delta} = \frac{5\times 5\times 6}{4(4\times 3)} = \frac{50}{16} = \frac{25}{8} = 3\frac{1}{8}$$



N. Applications

2.47: Circles tangent internally
The distance between the centers of two circles tangent internally is the difference of their radii.

Example 2.48
Circles tangent internally
Coordinate Geometry: Distance Formula

Introduce an origin at the bottom left of the right triangle. The vertices are:

$$B(0,1), A(0,0), C(3,0)$$

Center of smaller circle

$$= (r, r)$$

Center of circumcircle

$$= Midpoint\ of\ Hypotenuse = \left(\frac{0+3}{2}, \frac{1+0}{2}\right) = \left(\frac{3}{2}, \frac{1}{2}\right)$$

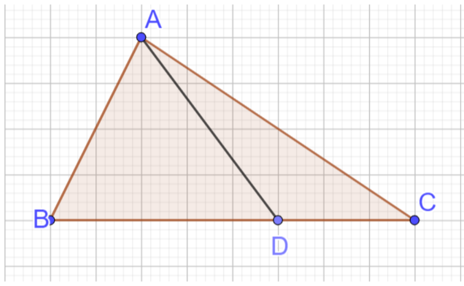
The radius of cir

For circles tangent internally, the distance between the centers of the circles is the difference of their radii. Use distance formula:

$$\left(r-\frac{3}{2}\right)^2+\left(r-\frac{1}{2}\right)^2=r-\frac{\sqrt{10}}{2}$$

2.2 Medians and Altitudes

A. Median



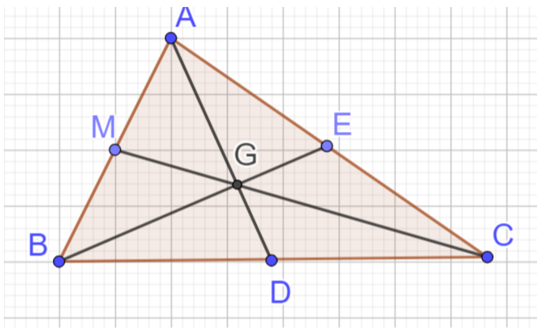
The line segment from the vertex of a triangle to the midpoint of the opposite side is called its median.

- By definition, a median bisects the side opposite.
- Every triangle has three medians.
- The medians never go outside the triangle.

In the diagram alongside (not drawn to scale), AD is the median drawn from vertex A. This means that

$$BD=DC$$

Intersection of Medians



The three medians of a triangle intersect at the centroid.
The centroid divides each median in the ratio 2: 1.

Physical Interpretation of Centroid

Centroid has applications in physics as well. It represents the point at which an object will balance.

B. Centroid

The centroid is the point at which an object will balance. For example, consider the see-saw drawn alongside. The weight at the left is larger than the weight on the right.
Hence, to balance the see-saw, the pole on which it is balanced is to the left of the center.

In general, the point at which an object can be balanced on a pole is the centroid of the object.

2.49: Centroid divides medians in the ratio $2:1$

The centroid of a triangle divides medians in the ratio 2: 1, with the longer part towards the vertex, and the smaller part towards the base.

C. Equilateral Triangles

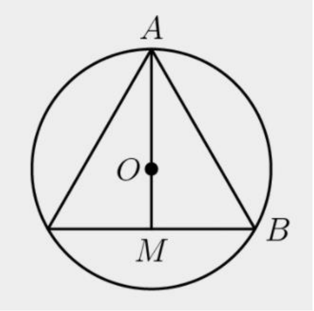
2.50: Points in an Equilateral Triangle

In an equilateral triangle

$$Incentre=Circumcenter=Centroid$$

Example 2.51

A circle is circumscribed about an equilateral triangle with side lengths of 9 units each. What is the area of the circle, in square units? Express your answer in terms of π. **(MathCounts 2008 School Countdown)**



$$Height = AM = \frac{\sqrt{3}}{2} \times 9 = \frac{9\sqrt{3}}{2}$$
$$Circumradius = \frac{2}{3} \times AM = \frac{2}{3} \times \frac{9\sqrt{3}}{2} = 3\sqrt{3}$$
$$Area = \pi r^2 = \pi(3\sqrt{3})^2 = 27\pi$$

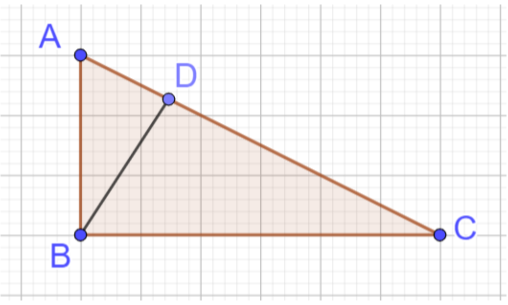
Example 2.52

In triangle ABC, medians AD and CE intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$? **(AMC 10B 2013/16)**

D. Altitude

The line segment from the vertex of a triangle which is perpendicular to the opposite side is called its altitude.

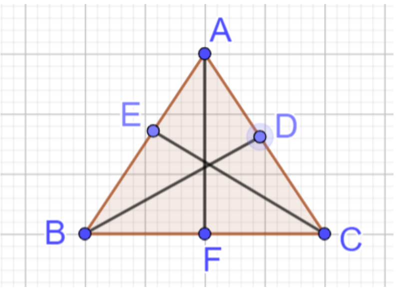
- The length of altitude is called its height. Make sure you understand the difference between the two.
- Every triangle has three altitudes.



Right-Angled Triangle:

In the diagram alongside, ΔABC is right angled at $\angle B$.
The altitude to side AB is BC.
The altitude to side BC is AB.
The altitude to side AC is AB.

In a right-angled triangle, the altitudes are the legs of the triangle.

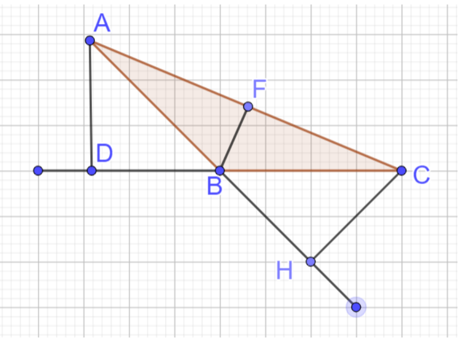


Acute Triangle

In the diagram alongside, ΔABC is an acute -angled triangle, with altitudes AF, BD and CE.
Note that all the altitudes fall within the triangle.

Obtuse-Angled Triangle:

In the diagram alongside, obtuse triangle ABC has altitudes BF, AD and CH.
BF, which is the the altitude opposite the obtuse angle lies inside the triangle.
AD and CH lie outside the triangle and sides BC and AB must be "produced" (extended) for them to intersect the altitudes.



Orthocenter

The three altitudes of a triangle intersect at the orthocenter.

- In an acute triangle, the orthocenter lies inside the triangle.
- In a right triangle, the orthocenter lies at the intersection of the legs.

In an obtuse triangle, the orthocenter lies outside the triangle.

Example 2.53

Right triangle ABC has leg lengths $AB = 20$ and $BC = 21$. Including $\overline{AB}$ and $\overline{BC}$, how many line segments with integer length can be drawn from vertex B to a point on hypotenuse $\overline{AC}$? **(AMC 10A 2018/16)**

Primitive Pythagorean Triplet:

$$(20,21,29) \Rightarrow AC = 29$$

$$\frac{1}{2} \times 20 \times 21 = \frac{1}{2} \times 29 \times h \Rightarrow h = \frac{420}{29} \in (14,15)$$

To the left, going towards AB, we get

$$\{15,16, \ldots ,20\} \Rightarrow 6\ Numbers$$

To the left, going towards BC, we get

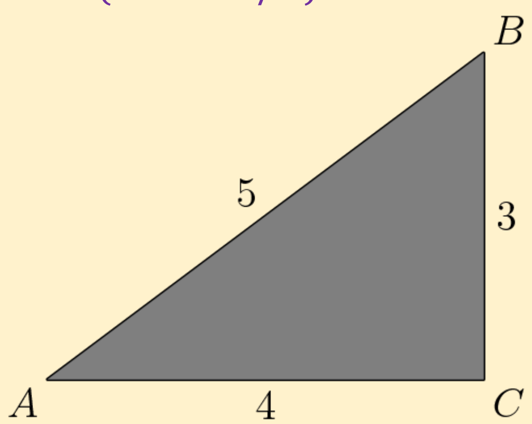
$$\{15,16, \ldots ,21\} \Rightarrow 7\ Numbers$$

Total

$$= 6 + 7 = 13\ Values$$

Example 2.54

A paper triangle with sides of lengths 3, 4, and 5 inches, as shown, is folded so that point A falls on point B. What is the length in inches of the crease? **(lie on 2018/13)**



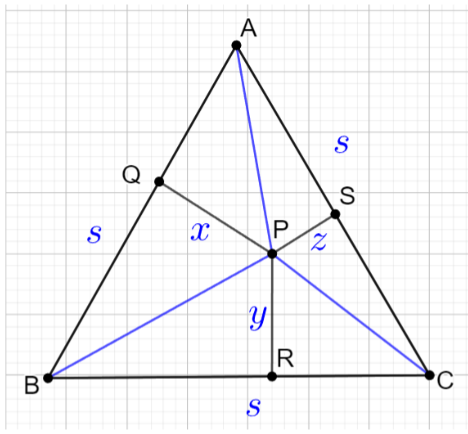
E. Equilateral Triangles

2.55: Altitudes, Medians, Angle Bisectors and Perpendicular Bisectors

2.56: Viviani's Theorem

For any equilateral triangle, the sum of the altitudes from any point in the triangle is equal to the altitude from a vertex of the triangle to the other side.

- Consider equilateral ΔABC with side length s
- Let point P be (anywhere) inside ΔABC
- Draw altitudes PQ, PR and PS to AB, BC and CA



Since ΔABC is made of three smaller triangles, its area must be the sum of those triangles:

$$[ABC] = \underbrace{\frac{sx}{2}}_{[APB]} + \underbrace{\frac{sy}{2}}_{[BPC]} + \underbrace{\frac{sz}{2}}_{[CPA]} = \frac{s}{2}(x+y+z)$$

Since ΔABC is an equilateral triangle, we must have:

$$\frac{s}{2}(x+y+z) = \frac{s}{2} \times Height$$

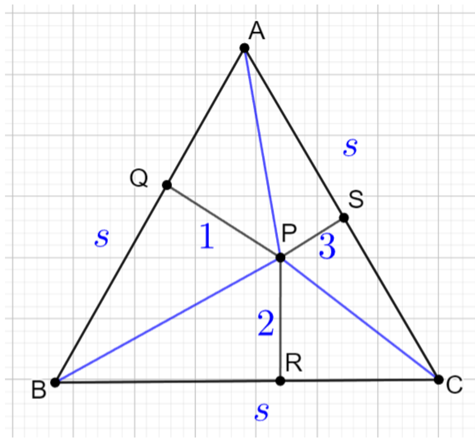
$$x+y+z = Height$$

Since height is the length of the altitude, and all three altitudes of an equilateral triangle are congruent, the property is proved.

Example 2.57

Point P is inside equilateral ΔABC. Points Q, R, and S are the feet of the perpendiculars from P to $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. Given that $PQ = 1$, $PR = 2$, and $PS = 3$, what is AB? **(AMC 10B 2007/17)**

Method I: Viviani's Theorem



Let s be the side length of the triangle. Then, its altitude is:

$$\frac{\sqrt{3}}{2}s$$

By Viviani's Theorem:

$$\frac{\sqrt{3}}{2}s = \underbrace{1}_{PQ} + \underbrace{2}_{PR} + \underbrace{3}_{PS} \Rightarrow s = \frac{12}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$$

Method II: First Principles

$$\underbrace{\frac{s}{2}}_{[APB]} + \underbrace{\frac{2s}{2}}_{[BPC]} + \underbrace{\frac{3s}{2}}_{[CPA]} = \frac{\sqrt{3}}{4}s^2 \Rightarrow 3s = \frac{\sqrt{3}}{4}s^2 \Rightarrow s = 4\sqrt{3}$$

2.58: Nine-Point Circle

F. Review

Terminology 2.59

Define the following terms, which were introduced in this chapter:

A. Angle Bisector
B. Incircle
C. Incentre
D. Inradius
E. Perpendicular bisector of a side of a triangle
F. Circumcircle
G. Circumcenter
H. Circumradius

Refer the definitions in the chapter

Formulas 2.60

A. BD is an angle bisector in ΔABC. If $\angle B = b°$, then what is the measure of $\angle ABD$?
B. ΔABC has sides a, b and c. What is the inradius of the triangle?

$$\angle ABD = \frac{b}{2}$$
$$r = \frac{\Delta}{s}, \qquad \Delta = \sqrt{s(s-a)(s-b)(s-c)}, \qquad s = \frac{a+b+c}{2}$$

G. Concepts

Concept 2.61

In how many places does the incircle of a triangle touch it?

3

Concept 2.62

At the point where an incircle touches the sides of a triangle, what is the angle between the radius and the side? Why?

90° because the sides are tangent to the circle, and a radius is perpendicular to the tangent at the point of tangency.

Concept Check 2.63

What can you say about a triangle where the
A. inradius is the same as the circumradius.
B. circumcircle is the same as the incircle.
C. incentre is the same as the circumcenter.

Parts A and B

Such a triangle does not exist.

Part C

Triangle is equilateral. The two circles are concentric

True or False 2.64

A. The incenter of a triangle is also its circumcenter if and only if the triangle is isosceles.
B. If a triangle is equilateral, then its incentre is the same as its circumcenter, and hence its inradius is the same as its circumradius.

$$False. Should\ be\ equilateral$$
$$False.$$

MCQ 2.65

Consider isosceles triangle ABC with incentre I. Out of IA, IB and IC (these represent lengths):
A. Exactly two are the same
B. At most two are the same
C. At least two are the same

D. None of the above

Option C

MCQ 2.66

In ΔABC construct the perpendicular bisectors of the sides and find their point of intersection O. Then, the distance from O to the:

A. sides is the inradius.
B. vertices is the inradius.
C. sides is the circumradius.
D. vertices is the circumradius.

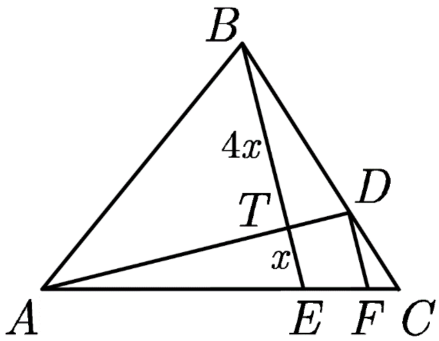
Option D

H. AMC Questions

Example 2.67

In $\triangle ABC$, points D and E lie on BC and AC, respectively. If AD and BE intersect at T so that $\frac{AT}{DT} = 3$ and $\frac{BT}{ET} = 4$, what is $\frac{CD}{BD}$? **(AMC 10B 2004/20)**

(Official Solution, Adapted)

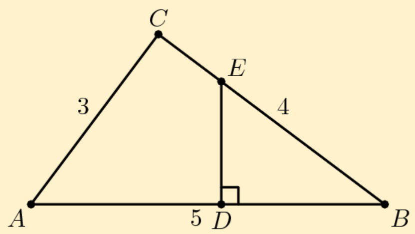


Example 2.68

In $\triangle ABC$, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is BD? **(AMC 10B 2005/10)**

Example 2.69

The area of $\triangle EBD$ is one third of the area of $3 - 4 - 5$ $\triangle ABC$. Segment DE is perpendicular to segment AB. What is BD? **(AMC 10B 2011/9)**



Example 2.70

In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$? **(AMC 10B 2013/23)**

Example 2.71
In a given plane, points A and B are 10 units apart. How many points C are there in the plane such that the perimeter of $\triangle ABC$ is 50 units and the area of $\triangle ABC$ is 100 square units? **(AMC 10B 2019/10)**

Example 2.72
Right triangles T_1 and T_2 have areas 1 and 2, respectively. A side of T_1 is congruent to a side of T_2, and a different side of T_1 is congruent to a different side of T_2. What is the square of the product of the other (third) sides of T_1 and T_2 ?**(AMC 10B 2019/15)**

Example 2.73
In $\triangle ABC$ with a right angle at C, point D lies in the interior of $\overline{AB}$ and point E lies in the interior of $\overline{BC}$ so that $AC = CD$, $DE = EB$, and the ratio $AC: DE = 4: 3$. What is the ratio $AD: DB$? **(AMC 10B 2019/16)**

Example 2.74
Points P and Q lie in a plane with $PQ = 8$. How many locations for point R in this plane are there such that the triangle with vertices P, Q, and R is a right triangle with area 12 square units? **(AMC 10B 2020/8)**

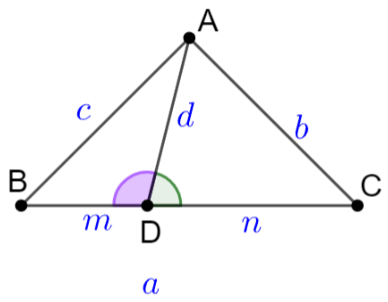
2.3 Cevians

A. Stewart's Theorem

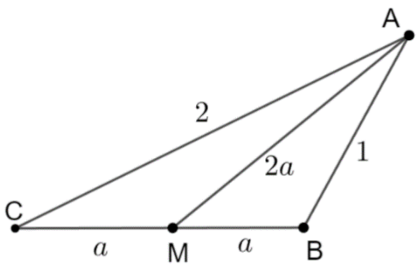
2.75: Stewart's Theorem

$$man + dad = bmb + cnc$$

Which can be remembered using the mnemonic:
A *man* and his *dad* put a *bomb* in the *sink*



Example 2.76
In ΔABC, we have $AB = 1$, and $AC = 2$. Side BC and the median from A to BC have the same length. What is BC? (AMC 12 2002)



$$man + dad = bmb + cnc$$

For this question, we want to find

$$BC = AM = d$$

Substitute $m = \frac{d}{2}, n = \frac{d}{2}, c = 2, b = 1$

$$\left(\frac{d}{2}\right)(d)\left(\frac{d}{2}\right) + ddd = (1)\left(\frac{d}{2}\right)(1) + (2)\left(\frac{d}{2}\right)(2)$$

$$\frac{d^3}{4} + d^3 = \frac{d}{2} + 2d$$

$$\frac{5d^3}{4} = \frac{5d}{2}$$

$$d^2 = 2$$
$$d = \sqrt{2}$$

2.4 Triangle Inequality

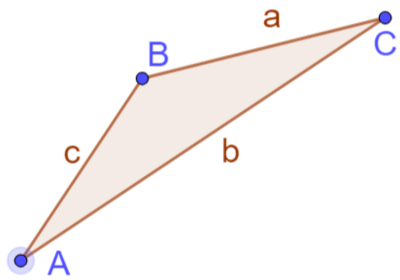
A. Basics

The triangle inequality combines geometry with one of the more difficult topics in Algebra – Inequalities. It leads to some important restrictions on the length of values that triangle sides can take.

2.77: Maximum Length of Side

In a triangle, any side is less than the sum of the other two sides. In a triangle with sides a, b and c:

$$a < b + c$$
$$b < a + c$$
$$c < a + b$$



We take it as an axiom in geometry that

A straight line is the shortest distance between two points.

And the property above directly follows from the axiom.

If an ordered triple of lengths does not satisfy the triangle inequality, then the lengths cannot form a triangle. This is a powerful tool, since it shows there is *no* triangle that satisfies the given conditions.

Example 2.78

Show that a triangle with side lengths $3, 4$ *and* 5 is a valid triangle.

$$3 + 4 = 7 > 5$$
$$3 + 5 = 8 > 4$$
$$4 + 5 = 9 > 3$$

Example 2.79

Show that a triangle with side lengths $3, 4$ *and* 6 is a valid triangle.

$$3 + 4 = 7 > 6$$
$$3 + 6 = 9 > 4$$
$$4 + 6 = 10 > 3$$

Example 2.80

Show that a triangle with side lengths $3, 4$ *and* 7 is not a valid triangle.

$$3 + 4 = 7$$

Sum of two sides is equal to the third side.
Hence, the triangle inequality is not satisfied.

This is not a valid triangle.

Example 2.81: Checking for Validity

Is a triangle with side lengths $5, 8$ and 12 a valid triangle?

$$5 + 8 = 13 > 12$$
$$8 + 12 = 20 > 5$$
$$5 + 12 = 17 > 8$$

Example 2.82

Is a triangle with side lengths 4, 7 and 13 a valid triangle?

$$4 + 7 = 11 < 13$$

This is not a valid triangle.

Example 2.83

Given that x is an integer, how many triangles exist with side lengths $3x, 4x$ *and* $8x$?

$$3x + 4x > 8x \Rightarrow 7x > 8x \Rightarrow Not\ Possible \Rightarrow \boldsymbol{No\ Triangles}$$

The number of such triangles is zero.

(Continuation) Example 2.84

Harry solved the above example, and he wrote

$$There\ are\ no\ values\ of\ x\ that\ satisfy\ the\ inequality\ 7x > 8x$$

A. State the exceptions to Harry's statement?
B. Can you make a valid triangle with those exceptions?

$$Exceptions\ are\ when\ x \leq 0$$
$$No, because\ length > 0$$

2.85: Distance is never negative

The distance between two points is never negative.
The smallest value that that distance can have is zero.

2.86: Difference of Sides

In a triangle, the difference of any two sides is less than the third side. In a triangle with sides a, b and c:

$$a - b < c$$
$$b - a < c$$
$$a - c < b$$
$$c - a < b$$
$$b - c < a$$
$$c - b < a$$

The properties above can be directly derived from the earlier properties on the sum of sides.
For example:

$$a < b + c \Rightarrow a - b < c$$
$$b < a + c \Rightarrow b - a < c$$

Example 2.87

A triangle has two sides $5\ feet\ and\ 6\ feet$. What are the values that the third side can be? (Answer in inequality form).

Maximum Length
Third side must be less than $5 + 6 = 11$. That is:

$$Third\ Side < 11\ feet$$

Minimum Length
And the difference of the two given sides must be less than the third side

$$6 - 5 = 1 < Third\ Side$$

$$1 < Third\ Side < 11$$

Example 2.88
A triangle has two sides which $7\ inches\ and\ 1\ foot$, respectively. What are the values that the third side can be? (Answer in inequality form).

$$Third\ Side < 7 + 12 \Rightarrow Third\ Side < 19$$
$$12 - 7 < Third\ Side \Rightarrow 5 < Third\ Side$$

$$5 < Third\ Side < 19$$

Example 2.89
A triangle has two sides which $\frac{1}{2}\ inches\ and \frac{1}{3}\ inches$, respectively. What are the values that the third side can be? (Answer in inequality form).

Third side has to be less than the sum of the other two sides:

$$Third\ Side < \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

Third side has to be more than the difference the other two sides:

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{6} < Third\ Side$$

$$\frac{1}{6} < Third\ Side < \frac{5}{6}$$

B. Integer Lengths

Example 2.90
Robin has made a triangle out of matchsticks, placed end to end. One side of the triangle is made up of 12 matchsticks. The second side needs 15 matchsticks. Let the total number of matchsticks used by Robin be t. What is the difference between the maximum possible value of t, and the minimum possible value of t?

Because Robin is using matchsticks, he cannot use a fractional number of matchsticks.

$$t < 12 + 15 = 27 \Rightarrow t \leq 26$$
$$15 - 12 < t \Rightarrow 3 < t \Rightarrow 4 \leq t$$

$$Difference = 26 - 4 = 22$$

Example 2.91

Two sides of a triangular plot to grow flowers are 30 inches and 39 inches, respectively.

A. If the third side of the plot is an integral number of feet long, what is its maximum possible length?
B. If the perimeter of the plot is an integral number of feet, what is its maximum possible value?

$$Sum\ of\ Two\ Sides = \underbrace{30+42}_{Inches} = \underbrace{69}_{Inches} = \frac{69}{\underbrace{12}_{Feet}} = \underbrace{5\frac{9}{12}}_{Feet}$$

$$Third\ Side < 5\frac{9}{12} \Rightarrow Max = 5$$

C. Degenerate Triangles

A special case of invalid triangles important enough to have their own name are degenerate triangles. These "triangles" are actually lines, but occur, for example, in the formula for the area of a triangle (using coordinate geometry. If the area of the triangle is zero, then it is a degenerate triangle.

2.92: Degenerate "Triangles"-I

If point B lies on line segment AC, the three points are collinear. However, it is also a "triangle".
Such triangles are called degenerate triangles.

Degenerate triangles do not satisfy the triangle inequality. For most questions, you will not consider degenerate triangles as triangles. (If you need to, then the question will specify).

Example 2.93

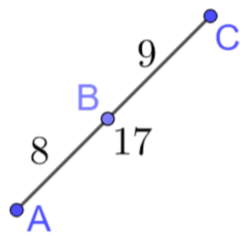
If a triangle has side lengths $AB = 8\ feet, BC = 9\ feet\ and\ AC = 17\ feet$, then

A. What kind of triangle is it?
B. What can be said about the points A, B and C?

Part A

$$8 + 9 = 17 \Rightarrow a + b = c$$

Sum of two sides is equal to the third side.
This is not possible in any triangle other than a degenerate triangle.



Part B

Hence, the three points are collinear.

2.94: Degenerate "Triangles"-II

Degenerate triangles have zero area.

Example 2.95

A triangle has side lengths $AB = 4\ feet, BC = 5\ feet\ and\ AC = 9\ feet$. What is the area of the triangle?

$$4 + 5 = 9 \Rightarrow Triangle\ is\ degenerate \Rightarrow Area\ is\ Zero$$

D. Semiperimeter

2.96: Semiperimeter

Semiperimeter of a geometrical figure is half the perimeter.

Example 2.97

A triangle has side lengths of 3, 4 and 6. Find the semiperimeter.

$$s = \frac{P}{2} = \frac{3+4+6}{2} = \frac{13}{2} = 6.5$$

2.98: Semiperimeter is greater than any side

$$s > a$$
$$s > b$$
$$s > c$$

$$a + b > c$$
$$a + b + c > 2c$$
$$\frac{a+b+c}{2} > c$$
$$s > c$$

Example 2.99

A triangle having integer sides has a perimeter of 23. What is the largest possible length of any one of its sides?

$$Perimeter = 23 \Rightarrow s = \frac{P}{2} = \frac{23}{2} = 11.5$$

$$Largest\ possible\ side < 11.5$$

Try 11:

$$6,6,11 \Rightarrow Valid\ Triangle$$
$$6 + 6 = 12 > 11$$

Example 2.100

In any triangle, the length of the longest side is less than half of the perimeter. All triangles with perimeter 57 and integer side lengths x, y, z, such that $x < y < z$ are constructed. How many such triangles are there? (Gauss 2013/24)

Triangles cannot be Isosceles

$$P = x + y + z = 57$$

Maximum value of z is 57/2 =28.5
Since z is integer, maximum value is 28

z	$x+y$	$Pairs$	Number of Pairs
28	29	$(27,2)(26,3), \dots, (15,14)$	$27 - 15 + 1 = 13$
27	30	$(26,4)(25,5), \dots, (16,14)$	$26 - 14 + 1 = 11$
26	31	$(25,6)(25,5), \dots, (16,15)$	$25 - 16 + 1 = 10$
25	32	$(24,8)(23,9), \dots, (17,15)$	$24 - 17 + 1 = 8$
24	33	$(23,10)(22,11), \dots, (17,16)$	$23 - 17 + 1 = 7$
23	34	$(22,12)(21,13), \dots, (18,16)$	$22 - 18 + 1 = 5$

E. Average

F. Max and Min

Example 2.101

Consider a triangle with sides lengths 3 and 4. Find the:

A. Maximum value of the length of the third side
B. Minimum value of the length of the third side
C. Range of values that the third side can take

Let the length of the third side be x.

We can find the maximum value of the length of the third side:

$$3 + 4 > x \Rightarrow x < 7$$

Similarly, we can find the minimum value of the length of the third side:

$$4 - 3 < x \Rightarrow x > 1$$

We can combine the above two inequalities to get the valid range of values that the third side can take:

$$1 < x < 7 \Rightarrow x \in (1,7)$$

Example 2.102

In a triangle with side lengths 5, 6 and x, what is the sum of all possible integral values of x? **(MathCounts 1997 Warm-Up 12)**

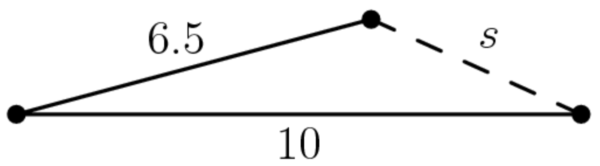
$$6 - 5 < x < 11 \Rightarrow 1 < x < 11 \Rightarrow x \in \{2,3,4, \dots ,10\}$$

The sum of these values will be:

$$2 + 3 + \cdots + 10 = (1 + 2 + 3 + \cdots + 10) - 1 = \frac{10 \times 11}{2} - 1 = 55 - 1 = 54$$

Example 2.103

The sides of a triangle have lengths 6.5, 10, and s, where s is a whole number. What is the smallest possible value of s? **(AMC 8 1992/17)**



Example 2.104

What is the smallest whole number larger than the perimeter of any triangle with a side of length 5 and a side of length 19? **(AMC 8 2015/8)**

Example 2.105

Nondegenerate $\triangle ABC$ has integer side lengths, $\overline{BD}$ is an angle bisector, $AD = 3$, and $DC = 8$. What is the smallest possible value of the perimeter? **(AMC 10A 2010/16)**

Example 2.106

In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 15. What is the greatest possible perimeter of the triangle? **(AMC 10B 2006/10)**

G. Range

Example 2.107

The sides of a triangle with positive area have lengths 4, 6, and x . The sides of a second triangle with positive area have lengths 4, 6, and y . What is the smallest positive number that is not a possible value of $|x-y|$? **(AMC 10 2000/10)**

H. Geometrical Inequalities

We can use the triangle inequality to establish other inequalities that triangles must fulfill. Some of these inequalities can require a high degree of creativity to establish.

2.108: Medians and Semiperimeter

The sum of the lengths of the three medians of a triangle is not greater than the triangle's perimeter. **(NMTC Primary/Final 2004/15)**

Let M, P and Q be the midpoints of sides BC, AC and AB in Triangle ABC. (Construction)
Extend AM such that AM = MD. (Construction)
TPT: AM + BP + CQ < AB + AC + BC

In ΔABD:

$$AD < AB + BD$$
$$AD < AB + AC$$
$$2AM < AB + AC\ (Inequality\ I)$$

By similar argument as above, we can show that:
2BP < AB + BC (Inequality II)
2CQ < AC + BC (Inequality III)

Add Inequality I, II and III:
2(AM + BP + CQ) < 2(AB + AC + BC)
AM + BP + CQ < AB + AC + BC

MCQ 2.109

Mark all correct options.

A triangle with zero area:

A. Does not exist
B. Can be a line segment
C. Can be a single point
D. Is a degenerate triangle

Options A, B, and C

MCQ 2.110

Mark the correct option

The longest side of a triangle:

A. Is its hypotenuse
B. Must be its hypotenuse
C. Must be greater than the sum of the other two sides

D. Must be less than the sum of the other two sides.

We do not know whether the triangle is right-angled. So, we put aside Options A and B, and look for other options.

Option C is incorrect, and violates the triangle inequality

$$a + b < c$$

Option D states the triangle inequality correctly.
Option D is correct.

2.5 Further Topics

111 Examples