Quadratics

08 January 2025

Revision: 2224

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1. Quadratic Equations

1.1 Formula Summary

A. Basics

Expansions

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

Different Forms

	Factored Form	Vertex Form	Standard Form
	$y = a(x-\alpha)(x-\beta)$	$y = a(x-h)^2 + k$	$y = ax^2 + bx + c$
Roots $x - intercept$ $Zeroes$	α, β	$Substitute\ y = 0$	$Substitute\ y = 0$
Vertex	$\frac{\alpha+\beta}{2}$	(h, k)	$x: -\frac{b}{2a}$ $y: substitute\ x = -\frac{b}{2a}$
$y - intercept$	$Substitute\ x = 0$	$Substitute\ x = 0$	c
Most useful for	$Roots$	$Vertex$	$y - intercept$

$$a > 0 \Rightarrow Upward\ Parabola$$
$$a < 0 \Rightarrow Downward\ Parabola$$

Completing the Square/Vertex

Used to convert from standard form into vertex form

Quadratic Formula/Discriminant

$$f(x) = ax^2 + bx + c \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$b^2 - 4ac > 0 \Rightarrow Two\ real, distinct\ roots \Rightarrow Two\ x - intercepts$$
$$b^2 - 4ac = 0 \Rightarrow Single\ real, repeated\ root \Rightarrow Single\ x\ intercept$$
$$b^2 - 4ac < 0 \Rightarrow Complex\ Roots \Rightarrow No\ x\ intercept$$

Interpretation

Interpretation of Vertex

$$y\ value\ of\ vertex: Max\ of\ function$$
$$x\ value\ of\ vertex: input\ to\ achieve\ max$$

Interpretation of Roots/Intercepts

$$x - intercept: Values\ that\ make\ the\ function\ zero$$
$$y - intercept: Value\ corresponding\ to\ x = 0$$

1.2 Binomial Expansions

A. Polynomial Basics

1.1: Types of Polynomials

An algebraic expression with

- one term when written in standard form is a monomial.
- two terms when written in standard form is a binomial.
- three terms when written in standard form is a trinomial.

Example 1.2

Classify the following polynomials as monomials, binomials, and trinomials:

A. $5x$
B. $24x^3 + 2x$
C. $4x - 5x$
D. $12x^2 - 12x^2 + 10x$

$$5x \to monomials$$
$$24x^3 + 2x \to binomial$$
$$4x - 5x \to binomial$$
$$12x^2 - 12x^2 + 10x \to trinomial$$

B. Distributive Property

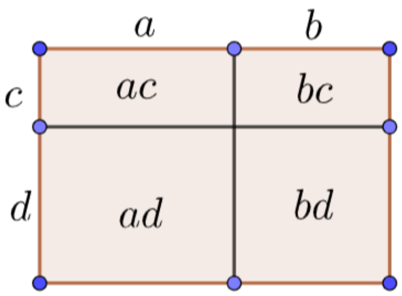
1.3: Distributive Property

$$a(x + y) = ax + ay$$

$$2(3x - 4y) = 6x - 8y$$
$$-\frac{1}{2}(x - 3y) = -\frac{1}{2}x + \frac{3}{2}y$$

1.4: Binomial Expansion

$$(a + b)(c + d) = ac + ad + bc + bd$$



Geometric Proof

Consider a rectangle (drawn alongside) with

$$Length = a + b$$
$$Width = c + d$$

The area of the rectangle is

$$Area = lw = (a + b)(c + d)$$

The area of the rectangle is also the sum of the four smaller rectangles that make it up:

$$Area = ac + ad + bc + bd$$

But the area of the rectangle has to be the same no matter which way you calculate it. Hence:

$$Area = (a + b)(c + d) = ac + ad + bc + bd$$

Algebraic Proof

$$(a + b)(c + d)$$

Apply the distributive property:

$$= a(c + d) + b(c + d)$$

Apply the distributive property one more time:

$$= ac + ad + bc + bd$$

Example 1.5

Expand using the distributive property:

A. $(5x+2)(3y+4)$
B. $(2x+6)(5y-3)$
C. $(2x+3)(3x+5)$

$$(5x+2)(3y+4) = 5x(3y+4) + 2(3y+4) = 15xy + 20x + 6y + 8$$
$$(2x+6)(5y-3) = 2x(5y-3) + 6(5y-3) = 10xy - 6x + 30y - 18$$
$$(2x+3)(3x+5) = 6x^2 + 10x + 9x + 15 = 6x^2 + 19x + 15$$

Example 1.6

A. $\left(3x - \frac{4}{x}\right)\left(2x + \frac{5}{x}\right)$

$$\left(3x - \frac{4}{x}\right)\left(2x + \frac{5}{x}\right) = 6x^2 + 15 - 8 - \frac{20}{x^2} = 6x^2 + 7 - \frac{20}{x^2}$$

Example 1.7

A rectangle has width $x+3$ and length $x-2$.

A. What is the minimum value of x?
B. What is the area of the rectangle, in terms of x?

Part A

The length and width must be positive.

$$x + 3 > 0 \Rightarrow x > -3$$
$$x - 2 > 0 \Rightarrow x > 2$$

We need the stronger of the two conditions. Hence:

$$x > 2$$

Part B

$$A = lw = (x+3)(x-2) = x^2 + x - 6$$

Example 1.8: Geometry

A. A right-angled triangle has length of one leg $x + \frac{1}{2}$, and length of the other leg as $x - \frac{2}{3}$. Find the area of the triangle.
B. A parallelogram has base $x - 5\ cm$ and height $x + 3\ cm$. What is the area of the parallelogram in square meters?
C. A trapezoid has one base $x + 3$ units long, and another base $x - 2$ units long. The height of the trapezoid is $x + 5$ units. Find the area of the trapezoid.
D. The diagonals of a rhombus are $\left(2x + \frac{2}{3}\right)$ and $\left(3x - \frac{1}{2}\right)$. Find the area of the rhombus.

$$Area\ of\ Triangle = \frac{1}{2}hb = \frac{1}{2}\left(x + \frac{1}{2}\right)\left(x - \frac{2}{3}\right) = \frac{1}{2}\left(x^2 - \frac{2}{3}x + \frac{1}{2}x - \frac{1}{3}\right) = \frac{1}{2}\left(x^2 - \frac{1}{6}x - \frac{1}{3}\right)$$
$$Area\ of\ parallelogram = hb = (x-5)(x+3) = x^2 - 2x - 15\ cm^2 = \frac{x^2 - 2x - 15}{10{,}000}m^2$$

$$Area\ of\ trapezoid = h \cdot \frac{b_1 + b_2}{2} = (x+5) \cdot \frac{2x+1}{2} = (x+5)\left(x + \frac{1}{2}\right) = x^2 + \frac{11}{2}x + \frac{5}{2}\ units^2$$
$$Area\ of\ a\ rhombus = \frac{d_1 d_2}{2} = \frac{1}{2}\left(2x + \frac{2}{3}\right)\left(3x - \frac{1}{2}\right) = \frac{1}{2}\left(6x^2 - x + 2x - \frac{1}{3}\right) = \frac{1}{2}\left(6x^2 + \frac{3}{2}x - \frac{1}{3}\right)$$

Example 1.9: Profit and Loss

I buy $4x + 5$ oranges at $x + 2$ Rupees per orange. I throw away 7 rotten oranges and sell the rest at $2x - 3$ Rupees per orange. How much money did I make?

$$Cost\ Price = CP = (4x+5)(x+2) = 4x^2 + 13x + 10$$
$$Selling\ Price = SP = (4x-2)(2x-3) = 8x^2 - 16x + 6$$
$$Profit = SP - CP = 8x^2 - 16x + 6 - (4x^2 + 13x + 10) = 4x^2 - 29x - 4$$

Example 1.10: Disguised Linear Equations

$x + 7$ students go for a team math competition. Each is to solve $x - 2$ problems. Two students fall ill, and each student gets one more problem to solve than expected to make up for the missing team members. If toffees are available in packs of 9, how many toffees will be left over if the math coach buys enough packs so that every student gets two toffees.

The total number of questions to be solved by the team remains the same in both cases:

$$(x+7)(x-2) = (x+5)(x-1)$$
$$x^2 + 5x - 14 = x^2 + 4x - 5$$
$$x = -5 + 14 = 9$$
$$x + 7 = 16$$
$$2(x+7) = 32$$

You need to buy 4 packs, which will give you 36 toffees. Number of toffees left over

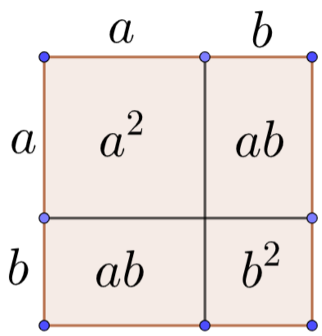
$$= 36 - 32 = 4$$

C. Square of a Sum

1.11: Square of a sum (Geometric)

The diagram alongside shows a square with side length $a + b$ and it has area:

$$(a+b)^2 = a^2 + 2ab + b^2$$



1.12: Square of a sum (Algebraic)

$$\boldsymbol{(a+b)^2 = a^2 + 2ab + b^2}$$

The square of a sum is given by the sum of:

- The square of the first term
- $+$ The square of the second term
- $+$ Twice the product of the two terms

$$(a+b)^2$$

Write it as a product of the expression with itself:

$$= (a+b)(a+b)$$

Apply the distributive property:

$$= a(a+b) + b(a+b)$$

Apply the distributive property again:

$$= a^2 + ab + ab + b^2$$

Simplify:

$$= a^2 + 2ab + b^2$$

Example 1.13

Expand directly using the formula:

$$(a+b)^2 = a^2 + 2ab + b^2$$

A. $(k+3)^2$
B. $(q+8)^2$
C. $(e+9)^2$
D. $(p+q)^2$
E. $(x+y)^2$
F. $(m+n)^2$

G. $(2e+7)^2$
H. $(3f+4)^2$
I. $(4z+2)^2$
J. $(3p+2q)^2$
K. $(2a+5\mathrm{b})^2$
L. $\left(\frac{z}{2}+\frac{x}{2}\right)^2$

M. $\left(\frac{a}{2}+\frac{b}{3}\right)^2$
N. $\left(\frac{p}{2}+\frac{2}{q}\right)^2$
O. $\left(\frac{2}{3}x+\frac{3}{4}y\right)^2$
P. $\left(\frac{2}{5}p+\frac{7}{11}q\right)^2$

Q. $(0.4u+0.7v)^2$
R. $\left(\frac{5}{13}g+\frac{11}{8}h\right)^2$

$$\left(\underbrace{k}_{a}+\underbrace{3}_{b}\right)^2 = \underbrace{k^2}_{a^2} + \underbrace{6k}_{2ab} + \underbrace{9}_{b^2}$$

$$(q+8)^2 = q^2 + 16q + 64$$

$$(e+9)^2 = e^2 + 18e + 81$$

$$(p+q)^2 = p^2 + 2pq + q^2$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(m+n)^2 = m^2 + 2mn + n^2$$

$$(2e+7)^2 = 4e^2 + 28e + 49$$

$$(3p+2q)^2 = 9p^2 + 12pq + 4q^2$$

$$\frac{z^2}{4} + 2\left(\frac{z}{2}\right)\left(\frac{x}{2}\right) + \frac{x^2}{4} = \frac{z^2}{4} + \frac{zx}{2} + \frac{x^2}{4}$$

$$\frac{a^2}{4} + 2\left(\frac{a}{2}\right)\left(\frac{b}{3}\right) + \frac{b^2}{9} = \frac{a^2}{4} + \frac{ab}{3} + \frac{b^2}{9}$$

$$\frac{p^2}{4} + 2\left(\frac{p}{2}\right)\left(\frac{2}{q}\right) + \frac{4}{q^2} = \frac{p^2}{4} + \frac{2p}{q} + \frac{4}{q^2}$$

$$\left(\frac{2}{3}x + \frac{3}{4}y\right)^2 = \frac{4}{9}x^2 + xy + \frac{9}{16}y^2$$

$$\left(\frac{2}{5}p + \frac{7}{11}q\right)^2 = \frac{4}{25}p^2 + \frac{28}{55}pq + \frac{49}{121}q^2$$

$$0.16u^2 + 0.56uv + 0.49v^2$$

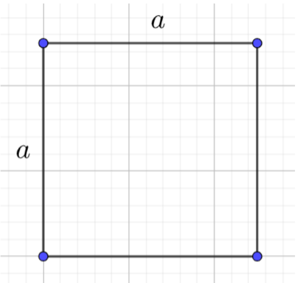
Example 1.14: Geometry

A. What is the area of a square with side length $2x + \frac{3}{4}$ units?
B. What is the area of a square with diagonal $x - 3$ units?
C. What is the area of a circle with diameter $x + 5$ units?

$$Area\ of\ square = s^2 = \left(2x + \frac{3}{4}\right)^2 = 4x^2 + 3x + \frac{9}{16}\ units^2$$
$$Area\ of\ square = \frac{d^2}{2} = \frac{(x-3)^2}{2} = \frac{x^2 - 6x + 9}{2}$$
$$Area\ of\ Circle = \pi r^2 = \pi\left(\frac{x+5}{2}\right)^2 = \pi\left(\frac{x^2 + 10x + 25}{4}\right)$$

Example 1.15

Milee is a year older than her brother, and four years younger than her sister. Milee's siblings are lifting weights. For each sibling, the number of sets is equal to their age, and the number of repetitions in each set is also equal to their age. If Milee's sister performed 105 more repetitions than her brother, find the product of the ages of all three.



Let

$$Milee's\ brother's\ age = a$$
$$Milee's\ sister's\ age = a + 5$$

$$(a+5)^2 - a^2 = 105$$
$$a^2 + 10a + 25 - a^2 = 105$$
$$10a = 80$$
$$a = 8$$
$$a + 1 = 9$$
$$a + 5 = 13$$

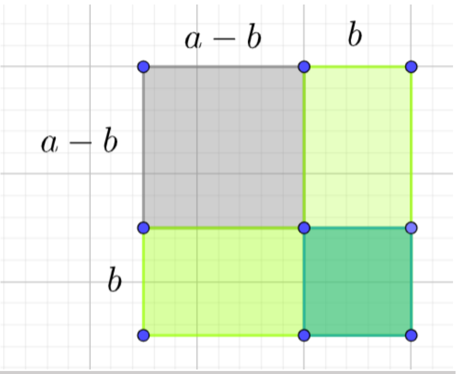
$$8 \times 9 \times 13 = 936$$

D. Square of a Difference

1.16: Square of a difference

$$(\boldsymbol{a} - \boldsymbol{b})^2 = \boldsymbol{a}^2 - 2\boldsymbol{ab} + \boldsymbol{b}^2$$

Start with a square of side length a. Let $b < a$ be a part of the side length of the square. Then, we are looking to find the area of the grey box in the diagram alongside.



$$\underbrace{(a-b)^2}_{Grey\ Box} = \underbrace{a^2}_{\substack{Original\\ Box}} - \underbrace{ab}_{\substack{Light\ Green\ Box\\ +Dark\ Green\ Box}} - \underbrace{ab}_{\substack{Light\ Green\ Box\\ +Dark\ Green\ Box}} + \underbrace{b^2}_{\substack{Dark\\ Green}}$$

We subtract the areas that we do not want from the large box.

$$ab\ is\ area\ of\ the\ light\ green\ box + dark\ green\ box$$

1.17: Square of a difference (Algebraic)

$$(\boldsymbol{a} - \boldsymbol{b})^2 = \mathbf{a}^2 - 2\boldsymbol{ab} + \boldsymbol{b}^2$$

The square of a difference is given by the sum of:

- The square of the first term
- + The square of the second term
- - Twice the product of the two terms

$$(a-b)^2 = a(a-b) - b(a-b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$$

$$\Big(\underbrace{p}_{First\ Term} - \underbrace{s}_{Second\ Term}\Big)^2 = \underbrace{p^2}_{(First\ Term)^2} - \underbrace{2ps}_{2\times FirstTerm\times SecondTerm} + \underbrace{s^2}_{(Second\ Term)^2}$$

Example 1.18

Box Method

$$(7x-1)^2$$

	$7x$	-1
$7x$	$49x^2$	$-7x$
-1	$-7x$	$+1$

$$49x^2 - 14x + 1$$

Example 1.19

Expand:

Basics

A. $(p-5)^2$
B. $(q-7)^2$
C. $(w-3)^2$
D. $(h-4)^2$

Variables

E. $(m-n)^2$
F. $(c-d)^2$
G. $(y-z)^2$
H. $(r-s)^2$

Integer Coefficients

I. $(2a-3)^2$
J. $(4p-5)^2$
K. $(2p-3q)^2$
L. $(4s-3r)^2$

Fractions

Fractional Coefficients

M. $\left(\frac{z}{2}-\frac{x}{2}\right)^2$
N. $\left(\frac{p}{2}-\frac{s}{3}\right)^2$
O. $\left(\frac{3x}{4}-\frac{4y}{5}\right)^2$
P. $\left(\frac{2}{3}p-\frac{5}{7}q\right)^2$

$$(p-5)^2 = p^2 - 10p + 25$$
$$(q-7)^2 = q^2 - 14q + 49$$
$$(w-3)^2 = w^2 - 6w + 9$$
$$(h-4)^2 = h^2 - 8h + 16$$
$$(m-n)^2 = m^2 - 2mn + m^2$$
$$(c-d)^2 = c^2 - 2cd + d^2$$
$$(y-z)^2 = y^2 - 2yz + z^2$$
$$(r-s)^2 = r^2 - 2rs + s^2$$
$$(2a-3)^2 = 4a^2 - 12a + 9$$
$$(4p-5)^2 = 16p^2 - 40p + 25$$
$$(2p-3q)^2 = 4p^2 - 12pq + 9q^2$$
$$(4s-3r)^2 = 16s^2 - 24rs + 9r^2$$

$$\left(\frac{3x}{4}-\frac{4y}{5}\right)^2 = \left(\frac{3x}{4}\right)^2 - (2)\left(\frac{3x}{4}\right)\left(\frac{4y}{5}\right) + \left(\frac{4y}{5}\right)^2 = \frac{9x^2}{16} - \frac{6}{5}xy + \frac{16}{25}y^2$$

Example 1.20

$$\left(\frac{p}{2}-\frac{s}{3}\right)^2 = \frac{p^2}{4} - 2\times\frac{p}{2}\times\frac{s}{3} + \frac{s^2}{9} =$$

E. Sum and Difference

1.21: Difference of Squares (Algebraic)

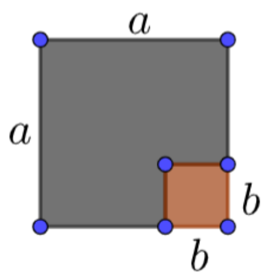
$$(a+b)(a-b)=a^2-b^2$$

Apply the Distributive Property:

$$(a+b)(a-b)=a(a-b)+b(a-b)=a^2-\boldsymbol{ab}+\boldsymbol{ab}-b^2=a^2-b^2$$

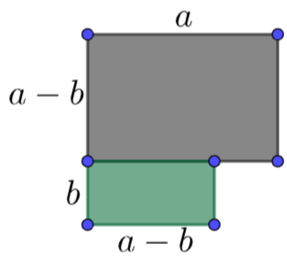
1.22: Difference of Squares (Geometric)

$$(a+b)(a-b)=a^2-b^2$$



Step I

Start with a square of side length a. Choose $b<a$, and remove an area equal to b^2 from the original square.



Step II

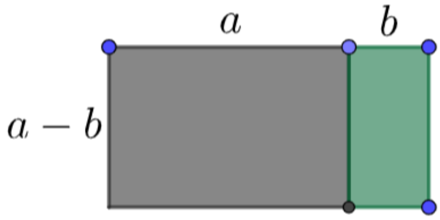
Divide the remaining area of the original square into two parts. Note the dimensions.

$$\underbrace{Length=a, Width=a-b}_{Grey\ Rectangle},\quad \underbrace{Length=a-b, Width=b}_{Green\ Rectangle}$$

Step III

Rotate the green rectangle, and place it alongside the grey rectangle. The larger rectangle created has dimensions

$$Length=a+b, Width=a-b$$



The area of the $\underbrace{polygon\ in\ Step\ I}_{a^2-b^2}$ must match the area of the $\underbrace{rectangle\ in\ Step\ III}_{(a+b)(a-b)}$

Hence, we have shown that:

$$(a+b)(a-b)=a^2-b^2$$

Example 1.23

Multiply

Coefficient of 1

A. $(p+3)(p-3)$
B. $(q+5)(q-5)$
C. $(j+7)(j-7)$
D. $(x+y)(x-y)$
E. $(p+q)(p-q)$

Integer Coefficients

F. $(2f-3)(2f+3)$
G. $(4r+7)(4r-7)$

Higher Powers

H. $(x^2+y^2)(x^2-y^2)$

Fractions

I. $\left(s+\frac{2}{3}\right)\left(s-\frac{2}{3}\right)$
J. $\left(r+\frac{4}{7}\right)\left(r-\frac{4}{7}\right)$
K. $\left(u+\frac{7}{11}\right)\left(u-\frac{7}{11}\right)$
L. $\left(2s-\frac{2}{3}\right)\left(2s+\frac{2}{3}\right)$

Fractional Coefficients

M. $\left(\frac{1}{2}s+\frac{3}{4}\right)\left(\frac{1}{2}s-\frac{3}{4}\right)$
N. $\left(\frac{2}{5}a+\frac{3}{7}b\right)\left(\frac{2}{5}a-\frac{3}{7}b\right)$

Decimal Coefficients

O. $(0.3a+0.7b)(0.3a-0.7b)$

$$(p+3)(p-3)=p^2-9$$
$$(q+5)(q-5)=q^2-25$$
$$(j+7)(j-7)=j^2-49$$
$$(x+y)(x-y)=x^2-y^2$$
$$(p+q)(p-q)=p^2-q^2$$
$$(x^2+y^2)(x^2-y^2)=x^4-y^4$$
$$(2f-3)(2f+3)=$$
$$(4r+7)(4r-7)=$$
$$\left(r+\frac{4}{7}\right)\left(r-\frac{4}{7}\right)=$$
$$\left(u+\frac{7}{11}\right)\left(u-\frac{7}{11}\right)=$$
$$\left(2s-\frac{2}{3}\right)\left(2s+\frac{2}{3}\right)=$$

$$\left(\underbrace{\frac{z}{2}}_{a}+\underbrace{\frac{x}{2}}_{b}\right)\left(\underbrace{\frac{z}{2}}_{a}-\underbrace{\frac{x}{2}}_{b}\right)=\underbrace{\frac{z^2}{4}}_{a^2}-\underbrace{\frac{x^2}{4}}_{b^2}=\frac{z^2-x^2}{4}$$

F. Numbers

Binomial expansion can be useful in applications. By expressing a number as a sum of two terms, or a difference of two terms, it can be squared easier using a formula. These calculations can also be a part of word problems, or more complicated questions.

Example 1.24

The side length of a square is 102 meters. Find the area of the square.

$$(a+b)^2=a^2+2ab+b^2$$
$$A=s^2=102^2=\left(\underbrace{100}_{a}+\underbrace{2}_{b}\right)^2=10{,}000+400+4=10404$$

Example 1.25

Use the same technique as above to find the squares of the following numbers:

A. 103
B. 101
C. 1001
D. 1003
E. 1002
F. 1017

$$103^2=\left(\underbrace{100}_{a}+\underbrace{3}_{b}\right)^2=\underbrace{10{,}000}_{a^2}+\underbrace{600}_{2ab}+\underbrace{9}_{b^2}=10{,}609$$
$$101^2=(100+1)(100+1)=10{,}000+100+100+1=10{,}201$$
$$1001^2=(1000+1)(1000+1)=1{,}000{,}000+1000+1000+1=1{,}002{,}001$$

$$1003^2 = (1000+3)(1000+3) = 1{,}000{,}000 + 3000 + 3000 + 9 = 1{,}006{,}009$$
$$1002^2 = (1000+2)(1000+2) = 1{,}000{,}000 + 2000 + 2000 + 4 = 1{,}004{,}004$$

$$1017^2 = (1000+17)(1000+17) = 1000000 + 34000 + 289 = 1{,}034{,}289$$

Example 1.26

Vinisha goes to the market to buy 99 bushels of corn priced at Rs. 99 each. She pays for the corn using only hundred-rupee notes. Let X equal the number of hundred rupees notes that Vinisha hands over. And Y equal the change, in rupees that Vinisha gets back. Find the product of X and Y.

$$Cost\ of\ Corn = \underbrace{99}_{\substack{No.of\\ Bushels}} \times \underbrace{99}_{\substack{Cost\\ per\ Bushel}} = 99^2$$

And now we can apply the formula for difference of a square to find the answer quickly:

$$(a-b)^2 = a^2 - 2ab + b^2$$
$$\Big(\underbrace{100}_{a} - \underbrace{1}_{b}\Big)^2 = \underbrace{10{,}000}_{a^2} - \underbrace{200}_{2ab} + \underbrace{1}_{b^2} = 9801\ Rs.$$

$$X = Number\ of\ hundred\ rupee\ notes = 99$$
$$Y = Change = 100 - 1 = 99$$
$$XY = 99^2 = 9801$$

Example 1.27

Use the same technique as above to find the squares of the following numbers:

A. 98
B. 97
C. 999
D. 998
E. 997

$$98^2 = (100-2)(100-2) = 10{,}000 - 400 + 4 = 9604$$
$$97^2 = (100-3)(100-3) = 10{,}000 - 600 + 9 = 9409$$
$$999^2 = (1000-1)(1000-1) = 1{,}000{,}000 - 2000 + 1 = 998{,}001$$
$$998^2 = (1000-2)(1000-2) = 1{,}000{,}000 - 4000 + 4 = 996{,}004$$
$$997^2 = (1000-3)(1000-3) = 1{,}000{,}000 - 6000 + 9 = 994{,}009$$

Example 1.28

A. $\frac{3.8^2 - 1.2^2}{3.8 - 1.2}$
B. $\frac{4.6^2 - 3.3^2}{4.6 - 3.3}$
C. $\frac{11.2^2 - 5.7^2}{11.2 + 5.7}$

We use the pattern $a^2 - b^2 = (a+b)(a-b)$ to factorize the numerator:

$$\frac{(3.8+1.2)(3.8-1.2)}{3.8-1.2} = 3.8 + 1.2 = 5$$

$$\frac{(4.6+3.3)(4.6-3.3)}{4.6-3.3} = 4.6+3.3 = 7.9$$
$$\frac{(11.2+5.7)(11.2-5.7)}{11.2+5.7} = 11.2-5.7 = 5.5$$

Example 1.29

Find:

A. $101^2 - 99^2$
B. $201^2 - 199^2$
C. $1001^2 - 999^2$

$$\underbrace{101^2}_{a^2} - \underbrace{99^2}_{b^2} = \left(\underbrace{101}_{a} + \underbrace{99}_{b}\right)\left(\underbrace{101}_{a} - \underbrace{99}_{b}\right) = 200 \times 2 = 400$$

$$201^2 - 199^2 = (400)(2) = 800$$
$$1001^2 - 999^2 = 2000 \times 2 = 4000$$

Example 1.30

Show that:

A. $63 \times 57 = 60^2 - 3^2$
B. $97 \times 103 = 100^2 - 3^2$
C. $995 \times 1005 = 1000^2 - 5^2$

$$63 \times 57 = (60+3)(60-3) = 60^2 - 3^2$$
$$97 \times 103 = (100-3)(100+3) = 100^2 - 3^2$$
$$995 \times 1005 = (1000-5)(1000+5) = 1000^2 - 5^2$$

Example 1.31

A regiment of soldiers walks in rectangular formation, and has 993 soldiers in a row. If there are 1007 rows, find the total number of soldiers.

$$993 \times 1007 = (1000-7)(1000+7) = 1000^2 - 7^2 = 1{,}000{,}000 - 49 = 999{,}951$$

Example 1.32

A rectangle has length 51 meters, and width 49 meters. Two squares are constructed outside the rectangle, one on its length, and one on its width. These squares are paved with square tiles of side length half a meter each. How many more tiles will be needed for the larger square as compared to the smaller square.

$$51^2 - 49^2 = (51+49)(51-49) = 100 \times 2 = 200\ \textit{square meters}$$

For a square tile:

$$\textit{Side length} = s = \frac{1}{2} \Rightarrow A = s^2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} m^2$$

And the number of additional tiles needed is:

$$\frac{200}{\frac{1}{4}} = 200 \div \frac{1}{4} = 200 \times 4 = 800\ \textit{tiles}$$

Example 1.33

Jay is j years old, and has j red boxes, each with j red cars. Vijay is v years old, and has v blue boxes, each with v blue cars. The difference in their ages is 3. They pair their cars so that each red car is paired with a blue car.

A. The cars that are left over are put in black boxes, with each box accommodating $j + v$ cars. How many boxes will they need?
B. If instead, the cars are packed into red and blue boxes, with no cars left over, and no space for more cars, then how many boxes will be needed?

Part A

The number of cars that Jay has is:

$$\underbrace{j}_{Boxes} \times \underbrace{j}_{Cars} = j^2$$

The number of cars that Vijay has is

$$\underbrace{v}_{Boxes} \times \underbrace{v}_{Cars} = v^2$$

Suppose, without loss of generality, that Jay is older. Then, the number of unpaired cars is:

$$j^2 - v^2 = (j - v)(j + v) = 3(j + v) = 3\ Black\ Boxes$$

Part B

$$3(j + v) = 3j + 3v = 3\ Red\ Boxes + 3\ Blue\ Boxes$$

G. Clever Algebraic Manipulations

- Having learnt binomial expansions, we can apply them to simplification in some situations.
- The properties that we learn here will get generalized when we learn Binomial Theorem.

1.34: Identity

$$(a + b)^2 - (a - b)^2 = 4ab$$

We can prove this:

$$LHS = (a + b)^2 - (a - b)^2 = a^2 + 2ab + b^2 - (a^2 - 2ab + b^2) = 4ab$$

The identity above can also be rearranged to get:

$$(a - b)^2 = (a + b)^2 - 4ab$$

Example 1.35

If $(a + b)^2 = 9$, and $ab = 2$, then find $(a - b)^2$.

Expand $(a + b)^2 = 9$ to get:

$$a^2 + 2ab + b^2 = 9$$

Subtract $4ab$ from both sides:

$$a^2 - 2ab + b^2 = 9 - 4ab$$

Factor the LHS, and substitute $4ab = 8$ in the RHS:

$$(a - b)^2 = 9 - 8 = 1$$

Example 1.36

When two positive integers are multiplied, the result is 24. When these two integers are added, the result is 11. When the smaller integer is subtracted from the larger integer, the result is: (CEMC Pascal 2020/4)

Method I

Let the larger number be a and the smaller number be b. Then $ab = 24$, and $\underbrace{a+b=11}_{Equation\ I}$:

Square both sides of Equation I:

$$a^2+2ab+b^2=121$$

Subtract $4ab=96$ from both sides:

$$a^2-2ab+b^2=25$$

Factor to get:

$$(a-b)^2=25$$

Take the square root both sides:

$$a-b=\pm 5$$

And since $a>b$, the answer must be positive:

$$a-b=5$$

Method II

$$Factor\ pairs\ of\ 24\ are\ \{(1,24),(2,12)(3,8)(4,6)\}$$

The factor that works is:

$$3+8=11$$

Difference

$$=8-3=5$$

Example 1.37

Expand:

A. $\left(x+\frac{1}{x}\right)^2$
B. $\left(a+\frac{1}{2a}\right)^2$
C. $\left(2b+\frac{1}{b}\right)^2$

$$\left(x+\frac{1}{x}\right)^2=x^2+2(x)\left(\frac{1}{x}\right)+\left(\frac{1}{x}\right)^2=x^2+2+\frac{1}{x^2}$$

$$\left(a+\frac{1}{2a}\right)^2=a^2+2(a)\left(\frac{1}{2a}\right)+\left(\frac{1}{2a}\right)^2=a^2+1+\frac{1}{4a^2}$$

$$\left(2b+\frac{1}{b}\right)^2=(2b)^2+(2)(2b)\left(\frac{1}{b}\right)+\left(\frac{1}{b}\right)^2=4b^2+4+\frac{1}{b^2}$$

Example 1.38

If $a-\frac{1}{a}=\frac{3}{7}$, find the value of $a^2-\frac{1}{a^2}$.

$$a-\frac{1}{a}=\frac{3}{7}$$

Square both sides of the above:

$$a^2-2+\frac{1}{a^2}=\frac{9}{49}$$

Add 4 to both sides:

$$a^2+2+\frac{1}{a^2}=\frac{107}{49}$$

$$\left(a+\frac{1}{a}\right)^2=\frac{107}{49}$$

$$a+\frac{1}{a}=\pm\frac{\sqrt{107}}{49}$$

$$a^2-\frac{1}{a^2}=\left(a-\frac{1}{a}\right)\left(a+\frac{1}{a}\right)=\left(\frac{3}{7}\right)\left(\pm\frac{\sqrt{107}}{49}\right)=\pm\frac{3\sqrt{107}}{343}$$

Example 1.39

A. Find the value of a^2-b^2 given that $a+b=\frac{7}{2},\ ab=\frac{5}{2}$.
B. Find the value of $a^2-\frac{1}{a^2}$ given that $a+\frac{1}{a}=5$.

Hint: Keep in mind that $a^2 - b^2 = (a+b)(a-b)$ is not the same as $(a-b)^2 = a^2 - 2ab + b^2$

Part A

Square both sides of $a + b = \frac{7}{2}$:

$$a^2 + 2ab + b^2 = \frac{49}{4}$$

Subtract $4ab = 10$ from both sides:

$$a^2 - 2ab + b^2 = \frac{49}{4} - 10$$

Factor the LHS, and simplify the RHS:

$$(a-b)^2 = \frac{9}{4}$$

Take square roots both sides:

$$a - b = \pm\frac{3}{2}$$

Finally:

$$a^2 - b^2 = (a+b)(a-b) = \left(\frac{7}{2}\right)\left(\pm\frac{3}{2}\right) = \pm\frac{21}{4}$$

Part B

Expand $\left(a + \frac{1}{a}\right)^2 = 5^2$:

$$a^2 + 2 + \frac{1}{a^2} = 25$$

Subtract 4 from both sides:

$$a^2 - 2 + \frac{1}{a^2} = 21$$

Factor:

$$\left(a - \frac{1}{a}\right)^2 = 21$$

Take the square root both sides:

$$a - \frac{1}{a} = \pm\sqrt{21}$$

Finally, the answer that we want:

$$a^2 - \frac{1}{a^2} = \left(a + \frac{1}{a}\right)\left(a - \frac{1}{a}\right) = \pm 5\sqrt{21}$$

1.40: Identity

$$(a+b)^2 = (a-b)^2 + 4ab$$

This is a re-arrangement of the identity above.

Example 1.41

Expand, simplify and factor:

$$1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2$$

Method I: Direct Method

Use the formula $(a-b)^2 = a^2 - 2ab + b^2$

$$1 + \frac{x^4}{16} - (2)\left(\frac{x^2}{4}\right)\left(\frac{1}{x^2}\right) + \frac{1}{x^4}$$

Simplify:

$$= 1 + \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4} = \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4}$$

Factor:

$$= \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

Method II: Change of Variable

If we let $a = \frac{x^2}{4}, b = \frac{1}{x^2}$, we get:

$$ab = \frac{1}{4} \Rightarrow 4ab = 1$$

Then:

$$4ab + (a-b)^2 = 4ab + a^2 - 2ab + b^2 = a^2 + 2ab + b^2 = (a+b)^2 = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)$$

1.42: Identity

$$(a+b)^2+(a-b)^2=2(a^2+b^2)$$
$$(a+b)^2-(a-b)^2=4ab$$

$$(a+b)^2+(a-b)^2=a^2+2ab+b^2+(a^2-2ab+b^2)=2(a^2+b^2)$$
$$(a+b)^2-(a-b)^2=a^2+2ab+b^2-(a^2-2ab+b^2)=4\text{ab}$$

Example 1.43

Simplify:

A. $(x+y)^2-(x-y)^2$
B. $(p+q)^2+(p-q)^2$
C. $(2j+3k)^2+(2j-3k)^2$
D. $\left(\frac{a}{2}+\frac{b}{3}\right)^2-\left(\frac{a}{2}-\frac{b}{3}\right)^2$

$$(x+y)^2-(x-y)^2=4xy$$
$$(p+q)^2+(p-q)^2=2(p^2+q^2)$$
$$(2j+3k)^2+(2j-3k)^2=2[(2j)^2+(3k)^2]=2[4j^2+9k^2]=8j^2+18k^2$$
$$\left(\frac{a}{2}+\frac{b}{3}\right)^2-\left(\frac{a}{2}-\frac{b}{3}\right)^2=(4)\left(\frac{a}{2}\right)\left(\frac{b}{3}\right)=\frac{2ab}{3}$$

1.44: Identity

$$a^2+b^2=(a+b)^2-2ab$$

$$a^2+b^2=a^2+2ab+b^2-2ab=(a+b)^2-2ab$$

Example 1.45

If $a^2+b^2=10$ and $ab=3$, then find the value of:

A. $(a+b)^2$
B. $(a-b)^2$

$$(a+b)^2=a^2+2ab+b^2=a^2+b^2+2ab=10+2(3)=10+6=16$$
$$(a-b)^2=a^2-2ab+b^2=a^2+b^2-2ab=10-2(3)=10-6=4$$

Example 1.46

The larger number among two numbers is greater than the smaller number by 7. The sum of the squares of both the numbers is 65. Determine the product of the two numbers.

Let the numbers be

$$Larger\ Number=a\ , Smaller\ Number=b$$

The sum of the square of the numbers is 65. Hence:

$$\underbrace{a^2+b^2=65}_{\textit{Equation I}}$$

The larger number is greater than the smaller number by 7. Hence $a-b=7$ and square both sides of this to get:

$$\underbrace{a^2-2ab+b^2=49}_{\textit{Equation II}}$$

Subtract Equation II from Equation I:

$$2ab=65-49=16\Rightarrow ab=8$$

Hence, the product of the two numbers

$$= ab = 8$$

Example 1.47

The larger number among two numbers is greater than the smaller number by 6. The sum of the squares of both the numbers is 64. Determine the product of the two numbers.

Let the numbers be

$$Larger\ Number = a\ , Smaller\ Number = b$$

The sum of the square of the numbers is 64. Hence:

$$\underbrace{a^2 + b^2 = 64}_{Equation\ I}$$

The larger number is greater than the smaller number by 7. Hence $a - b = 6$ and square both sides of this to get:

$$\underbrace{a^2 - 2ab + b^2 = 36}_{Equation\ II}$$

Subtract Equation II from Equation I:

$$2ab = 28 \Rightarrow ab = 14$$

Hence, the product of the two numbers

$$= ab = 14$$

1.48: Identity

$$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$$

Example 1.49

A. The product of three consecutive numbers is 7,980. Find the sum of the three numbers.
B. The product of three consecutive numbers is 15,600. Find the sum of the three numbers.

Part A

Let the middle number be x. Then,

$$Largest\ Number = x + 1, Smallest\ Number = x - 1$$

The product of the numbers

$$= (x-1)(x)(x+1)$$

Multiply the first and the last term using the difference of squares formula:

$$= (x^2 - 1)(x)$$

Multiply:

$$= x^3 - x = 7980$$

Now we can try with a single number.

$$x = 20 \Rightarrow x^3 = 8000 \Rightarrow x^3 - x = 7980 \Rightarrow Works$$

The sum of the numbers

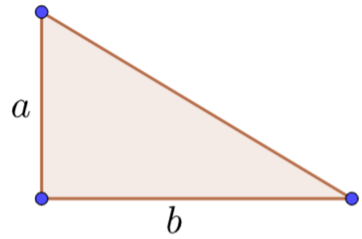
$$= x - 1 + x + x + 1 = 3x = 3(20) = 60$$

Part B

$$(x-1)(x)(x+1) = (x^2 - 1)(x) = x^3 - x = 15600$$
$$x = 25$$
$$3x = 75$$

H. Geometrical Applications

Algebra is very useful in calculations in geometry where you need to assume variables, or the question has variables.



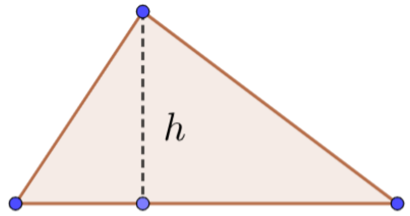
Example 1.50: Area of a Triangle

A. In a right triangle, the length of one leg is $3x+4$, and the length of the other leg is $2x-5$. Find the area of the triangle.
B. In a triangle, the height is $2x+5$ and the base is $7x-4$. What is the area of the triangle?

Part A

The area of a triangle:

$$=\frac{1}{2}hb=\frac{(3x+4)(2x-5)}{2}=\frac{6x^2-15x+8x-20}{2}=\frac{6x^2-7x-20}{2}$$



Part B

The area of a triangle

$$=\frac{1}{2}hb=\frac{(2x+5)(7x-4)}{2}=\frac{14x^2-8x+35x-20}{2}=\frac{14x^2+27x-20}{2}$$

Example 1.51

The length of a rectangle is $\left(\frac{1}{3}x+\frac{1}{2}\right)$, and its width is $2x+5$. Find the area of the rectangle.

$$A=lb=\left(\frac{1}{3}x+\frac{1}{2}\right)(2x+5)=\frac{2}{3}x^2+x+\frac{5}{3}x+\frac{5}{2}=\frac{2}{3}x^2+\frac{8}{3}x+\frac{5}{2}$$

Example 1.52

The length of a rectangle is $\left(\frac{1}{2}x-\frac{1}{3}\right)$, and its width is $3x-4$.

A. Find the area of the rectangle in terms of x.
B. The area of a rectangle is maximum when it is a square. Find the value of x that makes the rectangle a square.
C. Find the area of the square by squaring the value found in Part C.
D. Find the area of the rectangle by substituting in the expression found in Part A. (Calculator)
E. Are the answers in Part C and D the same? Why?

Part A

$$A=\left(\frac{1}{2}x-\frac{1}{3}\right)(3x-4)=\frac{3}{2}x^2-2x-x+\frac{4}{3}=\frac{3}{2}x^2-3x+\frac{4}{3}$$

Rather working with fractions, it is much easier to add inside the brackets, and then carry out the multiplication:

$$A=\left(\frac{3x-2}{6}\right)\left(\frac{3x-4}{1}\right)=\frac{9x^2-18x+8}{6}$$

Part B

In a square, the length is equal to the width:

$$\underbrace{\frac{1}{2}x-\frac{1}{3}}_{Length}=\underbrace{3x-4}_{Width}\Rightarrow 4-\frac{1}{3}=3x-\frac{1}{2}x\Rightarrow\frac{11}{3}=\frac{5}{2}x\Rightarrow x=\frac{22}{15}$$

Part C

$$x^2 = \left(\frac{22}{15}\right)^2 = \frac{484}{225}$$

Part D

Part E

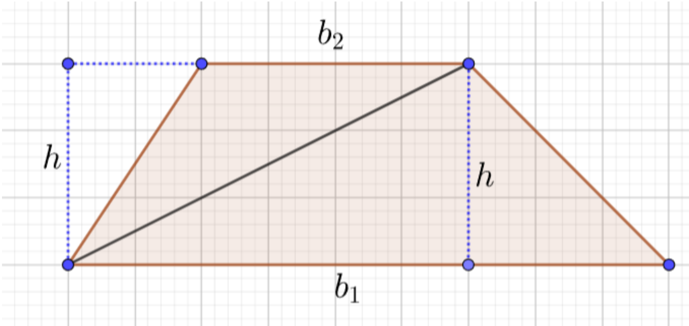
1.53: Area of a Trapezium

The area of a trapezium is given by:

$$A = \frac{1}{2}h(b_1 + b_2)$$

where

$$b_1 = base_1$$
$$b_2 = base_2$$
$$h = height$$



The area of a trapezium can be obtained by splitting the trapezium into two different triangles and adding their areas:

$$= \frac{1}{2}hb_1 + \frac{1}{2}hb_2$$

Factor our $\frac{1}{2}h$:

$$= \frac{1}{2}h(b_1 + b_2)$$

Example 1.54

Find the area of a trapezium with one base with length $3x + 4$, a second base with length $4x - 3$ and height $2x - \frac{1}{3}$.

$$A = \frac{1}{2}h(b_1 + b_2) = \frac{1}{2}[(3x + 4) + (4x - 3)]\left(2x - \frac{1}{3}\right) = \left(\frac{7x + 1}{2}\right)\left(\frac{6x - 1}{3}\right) = \frac{42x^2 - x - 1}{6}$$

(Continuation) Example 1.55

Suppose the trapezium above is a parallelogram. Find the numerical value of the area.

Since the trapezium is a parallelogram, the opposite sides must be equal:

$$3x + 4 = 4x - 3 \Rightarrow x = 7 \Rightarrow 3x + 4 = 3(7) + 4 = 21 + 4 = 25$$

And the height must equal

$$2x - \frac{1}{3} = 2(7) - \frac{1}{3} = 14 - \frac{1}{3} = \frac{41}{3}$$

And hence, the area is

$$A = bh = 25 \times \frac{41}{3} = \frac{1025}{3}$$

1.56: Area of a Parallelogram

The area of a parallelogram is given by

$$A = bh, \qquad b = base, h = height$$

Example 1.57

I. Rhombus

1.58: Area of a Rhombus

The area of a rhombus is half the product of its diagonals.

$$A = \frac{d_1 d_2}{2}, \qquad d_1 = First\ Diagonal, d_2 = Second\ Diagonal$$

Example 1.59

Find the area of a rhombus that has one diagonal $\left(\frac{2}{5}x - \frac{1}{2}\right)$ and a second diagonal that has area 1 unit more than the first diagonal.

$$A = \left(\frac{2}{5}x - \frac{1}{2}\right)\left(\frac{2}{5}x + \frac{1}{2}\right) = \left(\frac{2}{5}x\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{4}{25}x^2 - \frac{1}{4}$$

1.60: Concentric Circles and their Area

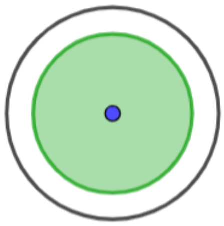
- Circles with the same center are called concentric circles.
- Area of a circle $= \pi r^2$, where r is the radius of the circle.
- Area of the region between two concentric circles is given by

Example 1.61

A. Two concentric circles have radius 3m and 4m respectively. Find the area of the region between the two circles.
B. Two concentric circles have radius $r\ meters$ and $R\ meters$ respectively, where $R > r$. Find the area of the region between the two circles, and write it in factored form

Part A

See diagram. We want the white portion, which is outside the inner circle, but inside the outer circle.



This area is given by

$$\underbrace{\pi(4)^2}_{\substack{Outer\\Circle}} - \underbrace{\pi(3)^2}_{\substack{Inner\\Circle}} = 16\pi - 9\pi = 7\pi$$

Part B

$$A(Circle_1) - A(Circle_2) = \pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi(R + r)(R - r)$$

Example 1.62: Parity

A. If n is an odd number, determine separately for each side of the identity $(n+1)(n-1) = n^2 - 1$ whether it is odd or even.
B. Prove that $n^2 - 1$ is odd, if n is an even number.

Part A

$$\underbrace{(n+1)}_{Even}\underbrace{(n-1)}_{Even} = \underbrace{n^2}_{Odd} - \underbrace{1}_{Odd}$$
$$Even = Even$$

Part B

The square of an even number is even

$$n^2 \textit{ is even}$$

Even number -1 is odd:

$$n^2 - 1 \textit{ is odd}$$

Example 1.63: Divisibility

A. Prove that $n^2 - 1$ is divisible by 4, if n is an odd number.
B. Prove that if n is an even number, $n^2 - 1$ is not divisible by 2^m, if m is a natural number, and divisible by a single value of m, if m is a whole number.

Part A

If n is odd, n^2 must be odd:

$$n^2 - 1 = \underbrace{(n+1)}_{Even}\underbrace{(n-1)}_{Even}$$

Hence,

$$\begin{aligned} n + 1 &= 2a, \qquad a \in \mathbb{Z} \\ n - 1 &= 2b, \qquad b \in \mathbb{Z} \end{aligned}$$

$$n^2 - 1 = (n+1)(n-1) = (2a)(2b) = 4ab \Rightarrow \textit{Divisible by } 4$$

Part B

$$\underbrace{n}_{Even} \rightarrow \underbrace{n^2}_{Even} \rightarrow \underbrace{n^2 - 1}_{Odd}$$

$$2^0 = 1 \Rightarrow 1 \textit{ divides } n^2 - 1$$

For $m > 0$:

$$2^m \textit{ is even}$$

An even number cannot divide an odd number.

$$\therefore \underbrace{2^m}_{Even} \textit{ does not divide } \underbrace{n^2 - 1}_{Odd}$$

Hence,

$$\begin{aligned} 2^m \textit{ divides } n^2 - 1 \ \textit{ for } m = 0 \\ 2^m \textit{ does not divide } n^2 - 1 \ \textit{ for } m = 1,2,3,\ldots \end{aligned}$$

J. Expressions

Example 1.64

A. $\frac{2p-3}{p^2-5p+6} - \frac{5}{p^2-9}$
B. $\dfrac{\frac{r+6}{r} - \frac{1}{r+2}}{\frac{r^2+4r+3}{r^2+2r}}$

Part A

$$\frac{2p-3}{(p-2)(p-3)} - \frac{5}{(p+3)(p-3)}$$

LCM is $(p-2)(p-3)(p+3)$:

$$\frac{(2p-3)(p+3) - 5(p-2)}{(p-2)(p-3)(p+3)}$$
$$\frac{2p^2 + 3p - 9 - 5p + 10}{(p-2)(p-3)(p+3)}$$
$$\frac{2p^2 - 2p + 1}{(p-2)(p-3)(p+3)}$$

Part B

$$\frac{\frac{(r+6)(r+2)-r}{r(r+2)}}{\frac{(r+1)(r+3)}{r(r+2)}}$$
$$\frac{(r+6)(r+2)-r}{r(r+2)} \times \frac{r(r+2)}{(r+1)(r+3)}$$
$$\frac{r^2+7r+12}{(r+1)(r+3)}$$
$$\frac{(r+4)(r+3)}{(r+1)(r+3)}$$
$$\frac{r+4}{r+1}$$

K. Review

Example 1.65

State True or False

A. The formula $(a-b)^2 = a^2 - 2ab + b^2$ is a special case of the formula $(a+b)^2 = a^2 + 2ab + b^2$
B. The formula $(a+b)(a-b) = a^2 - b^2$ is a special case of the formula $(a+b)^2 = a^2 + 2ab + b^2$

Example 1.66

State True or False for each option

We can prove the formula $(a-b)^2 = a^2 - 2ab + b^2$:

A. Algebraically
B. Geometrically
C. As a special case of the expansion of $(a+b)^2$

Option C is correct

Substitute $b = -b$ in $(a+b)^2 = a^2 + 2ab + b^2$:

$$\big(a+(-b)\big)^2 = a^2 + 2a(-b) + (-b)^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

Hence, option A, B and C are correct.

1.3 Equations Basics

A. Radicals

In this section, we learn how to solve equations that are perfect squares or perfect cubes. We can do this by taking square roots, or cube roots as necessary.

Example 1.67

A. $\sqrt{(-1)^2}$
B. $\sqrt{4^2}$

$$\sqrt{(-1)^2} = \sqrt{1} = 1$$
$$\sqrt{4^2} = \sqrt{16} = 4$$

1.68: Square Root of a Square

$$\sqrt{a^2} = a, a \geq 0$$
$$\sqrt{a^2} = -a, a < 0$$

When we square a number, and then take its square root:
- We get the original number back, if the number is positive or zero
- We get the negative of the number we started with, if the number is negative.

Example 1.69

I take the square of a negative quantity. I then take the square root of the result. The answer is:
A. The original quantity
B. The absolute value of the original quantity
C. The negative of the original quantity
D. Both B and C

$$y = -1 \Rightarrow y^2 = 1 \Rightarrow \sqrt{y^2} = \sqrt{1} = 1 = \underbrace{-(-1)}_{\text{Option } C} = \underbrace{|-1|}_{\text{Option } B}$$

Hence, option D is correct.

Example 1.70

I take the square of a quantity. I then take the square root of the result. The answer is neither positive nor negative. Then, the quantity that I started with:
A. Is Positive
B. Is Negative
C. Cannot be determined
D. Can be determined

Example 1.71

The square root of the square of a quantity is:
A. Always positive
B. Always negative
C. Sometimes positive and sometimes negative
D. Both positive and negative
E. None of the above

Let

$$x = 0 \Rightarrow x^2 = 0 \Rightarrow \sqrt{x^2} = 0$$

Option E is correct

Example 1.72

True or False

A. Squaring and taking the square root are opposite operations. Hence, the square root of the square of a number will result in the original number.

$$a = -1 \Rightarrow a^2 = 1 \Rightarrow \sqrt{a^2} = \sqrt{1} = 1$$

1.73: Square Roots are always non-negative

$$\sqrt{a} = x, x \geq 0$$

If you want the square root to be negative, you need to put a minus sign in front of it.

B. Perfect Squares

We can solve equations that have a square term on the left-hand side by taking square roots on both sides. It is important to put the $\pm$ sign before the square root since two values will have the same square.

1.74: Perfect Squares

For $c \geq 0$:

$$a^2 = c \Rightarrow a = \pm\sqrt{c}$$

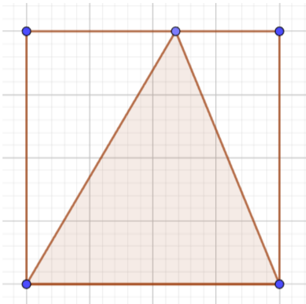
c must be non-negative because negative numbers do not have square roots in the real number system.

Example 1.75

Solve:

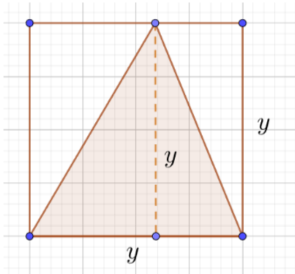
A. $x^2 = 4$
B. $z^2 = 16$
C. $s^2 = 25$

$$x^2 = 4 \Rightarrow x = \pm 2$$
$$z^2 = 16 \Rightarrow z = \pm 4$$
$$s^2 = 25 \Rightarrow s = \pm 5$$



Example 1.76

The area inside the square but outside the triangle is 72 square units. Find the side length of the square.



Let the side length of the square be y. Then the area is

$$y^2$$

The area of the triangle is

$$\frac{1}{2}y \times y = \frac{1}{2}y^2$$

Using complementary areas, the area inside the square, but outside the triangle is:

$$y^2 - \frac{1}{2}y^2 = 72$$
$$\frac{y^2}{2} = 72$$
$$y^2 = 144$$
$$y = \pm 12$$

But side length cannot be negative. Hence:

$$y = 12$$

Example 1.77

A. Solve $p = \sqrt{-8}$
B. Is the solution $z^2 = 4 \Rightarrow z = \sqrt{4}$ correct?

C. Does the equation $x^2 = -1$ have a solution in the real number system?

Part A

$$p = \sqrt{-8} \Rightarrow No\ Solutions\ in\ the\ \boldsymbol{Real}\ Number\ System$$

Part B

This solution is not correct because

$$\sqrt{4} = 2$$

What we actually want is

$$\pm\sqrt{4} = \pm 2$$

Part C

$$x^2 = -1 \Rightarrow x = \pm\sqrt{-1}$$

Negative numbers do not have a square root in the real number system, so the above equation does not have a solution.

Example 1.78

If a and c are real numbers, then the equation $-a^2 = c$ has:

A. No Solutions
B. Infinite Solutions
C. One Solution
D. Two Solutions

$$-(-3)^2 = -9$$

Option B.

Example 1.79

If a is a real number, and c is a non-negative real number then the equation $-a^2 = c$ has:

A. No Solutions
B. Infinite Solutions
C. One Solution
D. Two Solutions

$$-(-1)^{\ 2} = -(1) = -1 \Rightarrow Does\ not\ Work$$
$$a = c = 0 \Rightarrow -(0^2) = -0 = 0 \Rightarrow Works$$

The above solution where both variables are zero is the only solution that will work.

Option C

Example 1.80: Radicals

Solve. Simplify radicals, where possible.

A. $q^2 = 10$
B. $j^2 = 6$
C. $h^2 = 8$
D. $v^2 = 27$
E. $e^2 = 216$
F. $t^2 = 72$
G. $y^2 = 18$
H. $y^2 = 125$

$$q^2 = 10 \Rightarrow q = \pm\sqrt{10}$$
$$j^2 = 6 \Rightarrow j = \pm\sqrt{6}$$
$$h^2 = 8 \Rightarrow h = \pm\sqrt{8} = \pm\left(\sqrt{4} \times \sqrt{2}\right) = \pm 2\sqrt{2}$$
$$v^2 = 27 \Rightarrow v = \pm\sqrt{27} = \pm\left(\sqrt{9} \times \sqrt{3}\right) = \pm 3\sqrt{3}$$

Use the fact that $216 = 6^3 = 6^2 \times 6$:

$$e^2 = 216 \Rightarrow e = \pm\sqrt{216} = \pm\left(\sqrt{36} \times \sqrt{6}\right) = \pm 6\sqrt{6}$$

Use the fact that $72 = 6^2 \times 2$:

$$t^2 = 72 \Rightarrow t = \pm\sqrt{72} = \pm\left(\sqrt{36} \times \sqrt{2}\right) = \pm 6\sqrt{2}$$

Use the fact that $135 = 3 \times 3 \times 3 \times 5 = 3^2 \times 3 \times 5$

$$y^2 = 135 \Rightarrow y = \pm\sqrt{135} = \pm\left(\sqrt{9} \times \sqrt{15}\right) = \pm 3\sqrt{15}$$

Example 1.81

Solve the quadratic equations

A. $3x^2 + 5 = 152$
B. $5x^2 + 3 = 24$
C. $x^2 + 1 = 0$

$$3x^2 + 5 = 152 \Rightarrow 3x^2 = 147 \Rightarrow x^2 = 49 \Rightarrow x = \pm\sqrt{49} \Rightarrow x = \pm 7$$
$$5x^2 + 3 = 24 \Rightarrow x^2 = \frac{21}{5} \Rightarrow x = \pm\sqrt{\frac{21}{5}}$$
$$x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm\sqrt{-1} = \underbrace{\pm i}_{i=\sqrt{-1}}$$

C. Perfect Cubes

1.82: Perfect Cubes

$$a^3 = c \Rightarrow a = \sqrt[3]{c} = c^{\frac{1}{3}}$$

Note that:

- Unlike quadratics, it is not necessary that c must be non-negative.
- This is because all numbers have a cube root in the real number system.

Example 1.83

Solve:

A. $z^3 = 125$
B. $s^3 = 343$
C. $p^3 = -1$
D. $w^3 = 729$
E. $q^3 = -512$
F. $d^3 = 1000$

$$z^3 = 125 \Rightarrow z = 5$$

$$s^3 = 343 \Rightarrow s = 7$$
$$p^3 = -1 \Rightarrow p = -1$$
$$w^3 = 729 \Rightarrow w = 9$$
$$q^3 = -512 \Rightarrow q = -8$$
$$d^3 = 1000 \Rightarrow d = 10$$

Example 1.84

Solve. Give your answer in simplified radical form.

A. $r^3 = 16$
B. $h^3 = 32$
C. $j^3 = 128$
D. $k^3 = 256$

$$r^3 = 16 \Rightarrow r = \sqrt[3]{16} = \sqrt[3]{8} \times \sqrt[3]{2} = 2\sqrt[3]{2}$$
$$h^3 = 32 \Rightarrow h = \sqrt[3]{32} = \sqrt[3]{8} \times \sqrt[3]{4} = 2\sqrt[3]{4}$$
$$j^3 = 128 \Rightarrow j = \sqrt[3]{128} = \sqrt[3]{64} \times \sqrt[3]{2} = 4\sqrt[3]{2}$$
$$k^3 = 256 \Rightarrow j = \sqrt[3]{256} = \sqrt[3]{64} \times \sqrt[3]{4} = 4\sqrt[3]{4}$$

Example 1.85: Higher Powers

For even values of n:

$$a > 0, \qquad x^n = a \Rightarrow x = \pm\sqrt[n]{a}$$
$$a < 0, \qquad x^n = a \Rightarrow No\ Solutions$$

For odd values of n:

$$x^n = a \Rightarrow x = \sqrt[n]{a}$$

A. $z^4 = 81$
B. $s^4 = 32$
C. $x^5 = 32$

$$z^4 = 81 \Rightarrow z = \pm 3$$
$$s^4 = 32 \Rightarrow s = \pm\sqrt[4]{32} = \pm\left(\sqrt[4]{16} \times \sqrt[4]{2}\right) = \pm 2\sqrt[4]{2}$$

D. Difference of Squares

1.86: Perfect Squares

$$\underbrace{a^2}_{Square\ 1} - \underbrace{b^2}_{Square\ 2} = (a+b)(a-b)$$

$$(a+b)(a-b) = a(a-b) + b(a-b) = a^2 \mathbf{- ab + ab} - b^2 = \underbrace{a^2}_{Square\ 1} - \underbrace{b^2}_{Square\ 2}$$

The formula for a difference of squares can be used to factorize an expression. The direct application is for squares, but higher powers which are even can also be factorized.

Example 1.87

Factor Completely:

Variable with a Number

A. $z^2 - 25$
B. $a^2 - 36$
C. $36 - n^2$
D. $50 - 2n^2$

Two Variables

E. $x^2 - y^2$
F. $p^2 - q^2$
G. $u^2 - v^2$
H. $d^2 - e^2$
I. $2d^2 - 2e^2$

Integer Coefficients

J. $4n^2 - 9z^2$
K. $36p^2 - 49q^2$
L. $8l^2 - 18m^2$

Fractional Coefficients

M. $\frac{4}{9}a^2 - \frac{169}{225}b^2$
N. $\frac{50}{32}a^2 - \frac{75}{48}b^2$

Non-Perfect Square Coefficients

O. $3p^2 - 5q^2$
P. $7x^2 - 10y^2$
Q. $\frac{2}{3}a^2 - \frac{3}{4}b^2$
R. $27m^2 - 125n^2$
S. $8j^2 - 32k^2$

Fourth Powers

T. $x^4 - y^4$
U. $a^4 - b^4$
V. $p^4 - q^4$
W. $s^4 - 81$
X. $16x^4 - 81y^4$
Y. $x^4 - 64$

Eighth Powers

Z. $x^8 - y^8$

Variable with a Number

$$z^2 - 25 = (z+5)(z-5)$$
$$a^2 - 36 = (a+6)(a-6)$$
$$36 - n^2 = (6+n)(6-n)$$
$$50 - 2n^2 = 2(25 - n^2) = 2(5+n)(5-n)$$

Two Variables

$$x^2 - y^2 = (x+y)(x-y)$$
$$p^2 - q^2 = (p+q)(p-q)$$
$$u^2 - v^2 = (u+v)(u-v)$$
$$d^2 - e^2 = (d+e)(d-e)$$

Integer Coefficients

$$4n^2 - 9z^2 = (2n+3z)(2n-3z)$$
$$36p^2 - 49q^2 = (6p+7q)(6p-7q)$$
$$8l^2 - 18m^2 = \left(\sqrt{8}l + \sqrt{18}m\right)\left(\sqrt{8}l - \sqrt{18}m\right)$$
$$= \left(2\sqrt{2}l + 3\sqrt{2}m\right)\left(2\sqrt{2}l - 3\sqrt{2}m\right)$$

Fractional Coefficients

$$\frac{4}{9}a^2 - \frac{169}{225}b^2 = \left(\frac{2}{3}a + \frac{13}{15}b\right)\left(\frac{2}{3}a - \frac{13}{15}b\right)$$
$$\frac{50}{32}a^2 - \frac{75}{48}b^2 = \frac{25}{16}a^2 - \frac{25}{16}b^2$$
$$= \left(\frac{5}{4}\right)^2 (a+b)(a-b)$$

Non-Perfect Square Coefficients

$$3p^2 - 5q^2 = \left(\sqrt{3}p + \sqrt{5}q\right)\left(\sqrt{3}p - \sqrt{5}q\right)$$
$$7x^2 - 10y^2 = \left(\sqrt{7}x + \sqrt{10}y\right)\left(\sqrt{7}x - \sqrt{10}y\right)$$
$$\frac{2}{3}a^2 - \frac{3}{4}b^2 = \left(\sqrt{\frac{2}{3}}a + \frac{\sqrt{3}}{2}b\right)\left(\sqrt{\frac{2}{3}}a - \frac{\sqrt{3}}{2}b\right)$$
$$27m^2 - 125n^2 = \left(3\sqrt{3}m + 5\sqrt{5}n\right)\left(3\sqrt{3}m - 5\sqrt{5}n\right)$$
$$8j^2 - 32k^2 = 8(j-2k)(j+2k)$$

Fourth Powers

$$x^4 - y^4 = (x^2+y^2)(x^2-y^2)$$

But the second term can be further factorized, giving us:

$$(x^2+y^2)(x+y)(x-y)$$

$x^2 + y^2$cannot be factorized. Do not try to do anything further with this part of the expression.

$$x^2 + y^2 = x^2 + y^2 + 2ab - 2ab = (x+y)^2 - 2ab$$

$$a^4 - b^4 = (a^2+b^2)(a+b)(a-b)$$
$$p^4 - q^4 = (p^2+q^2)(p+q)(p-q)$$
$$s^4 - 81 = (s^2+9)(s+3)(s-3)$$
$$16x^4 - 81y^4 = (4x^2+9y^2)(2x+3y)(2x-3y)$$

$$x^4 - 64 = (x^2+8)(x^2-8)$$
$$= (x^2+8)\left(x+\sqrt{8}\right)\left(x-\sqrt{8}\right)$$
$$= (x^2+8)\left(x+2\sqrt{2}\right)\left(x-2\sqrt{2}\right)$$

Eighth Powers

$$x^8 - y^8 = (x^4+y^4)(x^2+y^2)(x+y)(x-y)$$

1.88: Zero Product Property

$$ab = 0 \Rightarrow a = 0\ OR\ b = 0$$

Example 1.89

Solve the equations below:

A. $n^2 - 49 = 0$

Part A

$$(n+7)(n-7) = 0$$

Apply the zero-product property:

$$n = -7 \; OR \; n = +7$$

Example 1.90

Factor $x^4 - (x-z)^4$

Use a difference of squares:

$$[x^2 - (x-z)^2][x^2 + (x-z)^2]$$

Expand:

$$[x^2 - (x^2 - 2xz + z^2)][x^2 + (x^2 - 2xz + z^2)]$$

Simplify:

$$[2xz - z^2][2x^2 - 2xz + z^2]$$

Factor z out of the first bracket:

$$z[2x - z][2x^2 - 2xz + z^2]$$

Example 1.91: Factorizing by using square roots

Recall that taking the square root of a number is the same as: Raising to the power $\frac{1}{2}$ or dividing the exponent of the number by 2.

Factorize the following completely using difference of squares:

A. $a^2 - 5$
B. $a^2 - \sqrt{5}$
C. $a^2 - 97^{\frac{1}{16}}$
D. $t^4 - 49$
E. $a^8 - 36$
F. $b^8 - 5$

$$a^2 - 5 = (a + \sqrt{5})(a - \sqrt{5})$$

$$a^2 - \sqrt{5} = (a + \sqrt[4]{5})(a - \sqrt[4]{5}), \qquad \because \left(\sqrt[4]{5}\right)^2 = 5^{\frac{2}{4}} = 5^{\frac{1}{2}} = \sqrt{5}$$

$$a^2 - 97^{\frac{1}{16}} = (a + \sqrt[32]{97})(a - \sqrt[32]{97}), \qquad \sqrt{97^{\frac{1}{16}}} = 97^{\frac{1}{16}\times\frac{1}{2}} = 97^{\frac{1}{32}}$$

$$(t^2 + 7)(t^2 - 7) = (t^2 + 7)(t + \sqrt{7})(t - \sqrt{7})$$

$$(a^4 + 6)(a^4 - 6) = (a^4 + 6)(a^2 + \sqrt{6})(a^2 - \sqrt{6}) = (a^4 + 6)(a^2 + \sqrt{6})(a + \sqrt[4]{6})(a - \sqrt[4]{6})$$

$$b^8 - 5 = (b^4 + \sqrt{5})(b^2 + \sqrt[4]{5})(b + \sqrt[8]{5})(b - \sqrt[8]{5})$$

E. Perfect Squares

1.92: Perfect Squares

$$a^2 + 2ab + b^2 = (a+b)^2$$
$$a^2 - 2ab + b^2 = (a-b)^2$$

$$\underbrace{x^2}_{\substack{\textit{First Term}\\\textit{Squared}}} + \underbrace{6x}_{\substack{\textit{Twice the product of}\\\textit{first and second term}}} + \underbrace{9}_{\substack{\textit{Second Term}\\\textit{Squared}}} = (x+3)^2$$

$$\underbrace{z^2}_{a^2} + \underbrace{10z}_{2ab} + \underbrace{25}_{b^2} = \left(\underbrace{z}_{a} + \underbrace{5}_{b}\right)^2$$

$$\underbrace{p^2}_{a^2} + \underbrace{22p}_{2ab} + \underbrace{121}_{b^2} = \left(\underbrace{p}_{a} + \underbrace{11}_{b}\right)^2$$

Example 1.93: Factoring Expressions

Factor each expression below:

Square of a Sum

A. $w^2 + 12w + 36$
B. $s^2 + 14s + 49$
C. $x^2 + 16x + 64$
D. $x^2 + 4x + 4$
E. $x^2 + 12x + 36$
F. $4x^2 + 20x + 25$
G. $9x^2 + 24x + 16$

Square of a Difference

H. $x^2 - 14x + 49$
I. $x^2 - 8x + 16$
J. $x^2 - 22x + 121$
K. $16p^2 - 40pq + 25q^2$
L. $3p^2 - 2\sqrt{21}pq + 7q^2$

Square of a Sum

$$w^2 + 12w + 36 = (w + 6)^2$$
$$s^2 + 14s + 49 = (s + 7)^2$$
$$x^2 + 16x + 64 = (x + 8)^2$$
$$x^2 + 4x + 4 = (x + 2)^2$$
$$x^2 + 12x + 36 = (x + 6)^2$$
$$4x^2 + 20x + 25 = (2x + 5)^2$$
$$9x^2 + 24x + 16 = (3x + 4)^2$$

Square of a Difference

$$x^2 - 14x + 49 = (x - 7)^2$$
$$x^2 - 8x + 16 = (x - 4)^2$$
$$x^2 - 22x + 121 = (x - 11)^2$$
$$16p^2 - 40pq + 25q^2 = (4p - 5q)^2$$
$$3p^2 - 2\sqrt{21}pq + 7q^2 = \left(\sqrt{3}p - \sqrt{7}q\right)^2$$

Example 1.94: Solving Equations

Solve the equation below:

$$x^2 + 6x = -9$$

Collate all terms on the LHS:

$$x^2 + 6x + 9 = 0$$

Write the left-hand side as a perfect square:

$$(x + 3)^2 = 0$$

Take the square root of both sides. The square root of the right-hand side is zero:

$$x + 3 = 0$$

Solve the equation for x:

$$x = -3$$

Solve the equations below:

A. $a^2 + 10a = -25$
B. $b^2 + 49 = -14b$
C. $\frac{c^2}{2} + 4c = -8$
D. $4d^2 + 8d + 4 = 0$
E. $h^2 - 16h + 64 = 0$
F. $l^2 - 8l + 16 = 0$
G. $o^2 - 18o + 81 = 0$

Part A

$$a^2 + 10a + 25 = 0$$
$$(a + 5)^2 = 0$$
$$a + 5 = 0$$

$22222a = -5$

Part B

$$b^2 + 14b + 49 = 0$$
$$(b + 7)^2 = 0$$

$$b+7=0$$
$$b=-7$$

Part C

Multiply both sides by 2:

$$c^2+8b=-16$$
$$c^2+8b+16=0$$
$$(c+4)^2=0$$
$$c+4=0$$
$$c=-4$$

Part E

Factor out 4:

$$4(d^2+2d+1)=0$$

Divide both sides by 4:

$$d^2+2d+1=0$$

Factor:

$$(d+1)^2=0$$
$$d+1=0$$
$$d=-1$$

Part F

$$(h-8)^2=0 \Rightarrow h-8=0 \Rightarrow h=8$$

Part G

$$(l-4)^2=0 \Rightarrow l-4=0 \Rightarrow l=4$$

Part H

$$(o-9)^2=0 \Rightarrow o-9=0 \Rightarrow o=9$$

Example 1.95: Solving Equations

Additional Factoring

A. $5x^2+70x+245=0$

Fractions

B. $u^2-u+\frac{1}{4}=0$

C. $c^2-\frac{4}{3}c+\frac{4}{9}=0$

D. $r^2-\frac{8}{5}r+\frac{16}{25}=0$

Part A

$$5(x+7)^2=0$$
$$x+7=0$$
$$x=-7$$

Part B

Note that $\sqrt{\frac{1}{4}}=\frac{1}{2}$, and that $2\times u\times -\frac{1}{2}=-u$, which is the middle term. Hence

$$\left(u-\frac{1}{2}\right)^2=0 \Rightarrow u-\frac{1}{2}=0 \Rightarrow u=\frac{1}{2}$$

Part C

Note that $\sqrt{\frac{4}{9}}=\frac{2}{3}$, and this works since $2\times c\times\frac{2}{3}=\frac{4}{3}$:

$$\left(c-\frac{2}{3}\right)^2=0 \Rightarrow c-\frac{2}{3}=0 \Rightarrow c=\frac{2}{3}$$

Part D

$$\left(r-\frac{4}{5}\right)^2=0 \Rightarrow r-\frac{4}{5}=0 \Rightarrow r=\frac{4}{5}$$

Example 1.96

The height h of an object dropped from an initial height h_0 after t seconds is given by $h=16t^2+h_0$. (Use the downward direction as positive, and the upward direction as negative).

A. Find the time that the object takes to hit the ground.
B. After how long will an object dropped from a height of 256 feet hit the ground

Part A

When the object hit the ground, the height is zero.
Substitute $h=0$ in $h=16t^2+h_0$:

$$0=16t^2+h_0$$
$$-\frac{h_0}{16}=t^2$$

Take the square root both sides:

$$t=\sqrt{-\frac{h_0}{16}}$$

Part B

$$t=\sqrt{-\frac{h_0}{16}}=\sqrt{-\frac{-256}{16}}=\sqrt{16}=4$$

Example 1.97

A. The area of a square is x^2-4x+8. If the side length of the square is 2, find the value of x.

B. The area of a circle is $x^2 + 10x + 25$. Find the circumference.
C. The area of a square is $4x^2 - 44x + 121$, where x is a positive integer. Find the smallest possible value of the perimeter.
D. The area of a square is $9x^2 - 42x + 49$, where x is a positive integer. Find the smallest possible value of the perimeter.

$$A = Area, s = side\ length, P = Perimeter$$
$$C = Circumference$$

Part A

$$\underbrace{x^2 - 4x + 8}_{Area} = \underbrace{2^2}_{Area}$$

Subtract 4 from both sides, factor and solve:

$$x^2 - 4x + 4 = 0 \Rightarrow (x-2)^2 = 0 \Rightarrow x = 2\ Units$$

Part B

$$\underbrace{(x^2 + 10x + 25)}_{Area} = \underbrace{\pi r^2}_{Area}$$

Divide both sides by π, and factor the LHS:

$$\frac{(x+5)^2}{\pi} = r^2 \Rightarrow r = \frac{x+5}{\sqrt{\pi}}$$

$$C = 2\pi r = 2\pi \frac{x+5}{\sqrt{\pi}} = \sqrt{\pi}(2x+10)\ Units$$

Part C

$$A = 4x^2 - 44x + 121 = (2x-11)^2$$
$$s = 2x - 11$$
$$2x - 11 > 0 \Rightarrow 2x > 11 \Rightarrow x > \frac{11}{2}$$
$$Smallest\ Integer\ Value\ of\ x = 6$$
$$Side\ Length = 2x - 11 = 12 - 11 = 1$$
$$Perimeter = 4\ Units$$

Part D

$$A = 9x^2 - 42x + 49 \Rightarrow s = 3x - 7$$
$$3x - 7 > 0 \Rightarrow x > \frac{7}{3}$$
$$Smallest\ Integer\ Value\ of\ x = 3$$
$$s = 3x - 7 = 9 - 7 = 2$$
$$P = 8\ Units$$

Example 1.98

A square with side length s has area $16x^2 - 40x + 25$ square meters. A rectangle has length $s + 2\ cm$, and width $\frac{1}{2}s\ cm$. Find, in meters, the perimeter and the area of the rectangle

For the square:

$$A = 16x^2 - 40x + 25 \Rightarrow s = 4x - 5$$

Perimeter of the rectangle:

$$P = 2\left(\underbrace{s+2}_{Length} + \underbrace{\frac{1}{2}s}_{Width}\right) = 2\left(\frac{3}{2}s + 2\right) = 3s + 4 = 3(4x-5) = 12x - 15\ cm = 0.12x - 0.15\ m$$

Area of the rectangle:

$$A = (s+2)\left(\frac{1}{2}s\right) = (4x-3)\left(2x - \frac{3}{2}\right) = 8x^2 - 6x - 6x + \frac{9}{2} = 8x^2 - 12x + 4.5\ cm^2$$

Divide by 10,000 to convert from cm^2 to m^2:

$$= 0.0008x^2 - 0.0012x + 0.00045\ cm^2$$

Example 1.99

The *Doyle Log Rule* is used in the lumber industry to estimate the number of board feet that a given log will yield. It is stated as $B = \frac{L}{16}(D^2 - 8D + 16)$, where B is the number of board feet, D is the diameter in inches of the log, and L is the length of the log in feet. For a log with a length of 32 feet, what is the diameter in feet that will yield 200 board feet?

Factor the expression in brackets:

$$B = \frac{L}{16}(D-4)^2$$

Substitute $B = 200, L = 32$:

$$200 = \frac{32}{16}(D-4)^2 \Rightarrow 100 = (D-4)^2 \Rightarrow \pm 10 = D - 4 \Rightarrow D \in \{-6, 14\}$$

Reject the negative value:

$$D = 14$$

1.4 Factoring Quadratics

A. Factoring

1.100: Trinomial

An algebraic expression with three terms is called a trinomial.

Here is an example of a trinomial:

$$\underbrace{a^2}_{First\ Term} + \underbrace{2ab}_{Middle\ Term} + \underbrace{b^2}_{Last\ Term}$$

Here is another trinomial where the last term is a number, and not a variable:

$$\underbrace{a^2}_{First\ Term} + \underbrace{2a}_{Middle\ Term} + \underbrace{1}_{Constant\ Term}$$

1.101: Quadratic Expression/Equation/Function

The standard of a quadratic is given below.

Quadratic Expression	Quadratic Equation	Quadratic Function
$ax^2 + bx + c, \qquad a \neq 0$	$ax^2 + bx + c = 0, a \neq 0$	$f(x) = ax^2 + bx + c = 0, a \neq 0$
$a \neq 0;\ a, b, c \in \mathbb{R}$		

- $a \neq 0$: a cannot be zero, since if a is zero, then the expression/equation/function becomes linear
- $a, b, c \in \mathbb{R}$: a, b, c are real numbers. This condition is relaxed when we study quadratic equations with complex co-efficients.
- a is the first coefficient when the quadratic is written in standard form. Hence, it is called the leading coefficient.

1.102: Sum and Product

$$x^2 + \underbrace{s}_{Sum} x + \underbrace{p}_{Product}$$

We can factorize by finding two numbers such that their:

- sum is the coefficient of the middle term
- their product is the constant term

We have earlier seen that:

$$(x+2)(x+3) = x(x+3) + 2(x+3) = x^2 + \underbrace{2x + 3x}_{5=2+3} + \underbrace{2 \times 3}_{6=2\times 3} = x^2 + \underbrace{5}_{Sum} x + \underbrace{6}_{Product}$$

The observation that:

$$5 = 2 + 3, \qquad 6 = 2 \times 3$$

Is crucial to the entire factoring process.

$$(x+a)(x+b) = x^2 + xa + xb + ab = x^2 + \underbrace{(a+b)}_{Sum} x + \underbrace{ab}_{Product}$$

Example 1.103

A. $x^2 + 7x + 10$
B. $x^2 + 10x + 21$
C. $x^2 + 9x + 20$

D. $x^2+8x+15$
E. $x^2+15x+44$
F. $x^2+10x+9$
G. $x^2+20x+100$
H. $x^2+12x+35$
I. $x^2+20x+91$

Part A

$$Product = 10, Sum = 7$$

Factor Pairs		Sum
1	10	11
2	5	7

Out of the factor pairs, we want the two numbers which add up to 7. Hence, we choose 2 and 5. Now we want to split:

$$x^2+7x+10 = x^2+\underbrace{2x+5x}_{7x}+10 = x(x+2)+5(x+2) = (x+2)(x+5)$$

Part B

$$\underbrace{x^2+10x+21}_{21=1\times 21=3\times 7} = (x+3)(x+7)$$

Part C

$$x^2+9x+20 = (x+4)(x+5)$$

Part D

$$x^2+8x+15 = (x+3)(x+5)$$

Part E

$$x^2+15x+44 = (x+4)(x+11*)$$

Part F

$$x^2+10x+9 = (x+1)(x+9)$$

Part G

$$x^2+20x+100 = (x+10)^2$$

Part H

$$x^2+12x+35 = (x+5)(x+7)$$

Part I

$$x^2+20x+91 = (x+13)(x+7)$$

After some practice with the splitting method, you start recognizing the numbers in the correct factor pair are found in the final answer. Hence, the intermediate steps can be skipped.

Example 1.104: Negative Values in Factoring

- If the constant term is negative, then one of the numbers is positive, and the other negative. Hence, the coefficient of the middle term gives the difference.
- If the constant term is positive, but the coefficient is negative, then both the numbers in the pair are negative.
- Use the above properties to factorize the expressions below:

A. x^2-x-6
B. $x^2-2x-143$
C. $x^2-5x-50$
D. $x^2-10x+21$
E. x^2+x-20

$$\underbrace{x^2-x-6}_{\substack{-6=-3\times 2\\ -3+2=1}} = (x-3)(x+2)$$
$$x^2-2x-143 = (x-13)(x+11)$$
$$x^2-5x-50 = (x-10)(x+5)$$
$$x^2-10x+21 = (x-3)(x-7)$$

	$Product = -6, Sum = -1$		
Product	Factor Pairs		Sum
−6	−1	6	5
−6	1	−6	−5
−6	2	−3	−1
−6	−2	3	1

$$x^2 + x - 20 = (x+5)(x-4)$$

You should be able to understand the effect of the sign of the coefficient on the factoring.

Type	Constant Term	Middle Term	Factors
I	$+ve$	$+ve$	$Both\ +ve$
II	$+ve$	$-ve$	$Both - ve$
III			
IV			

- If the middle term is negative, and the constant term is

Example 1.105

A. $x^2 - 7x + 12$
B. $x^2 + 7x + 12$
C. $x^2 + x - 12$
D. $x^2 - x - 12$

B. Solving Monic Quadratics

1.106: Factor Pairs

Factor Pairs are very important in solving Quadratics.

$$72 = \underbrace{1 \times 72}_{1+72=73} = \underbrace{2 \times 36}_{2+36=38} = \underbrace{3 \times 24}_{3+24=27} = \underbrace{4 \times 18}_{4+18=22} = \underbrace{6 \times 12}_{6+12=18} = \underbrace{8 \times 9}_{8+9=17}$$

Some tips to help you solve faster using the above example of factor pairs of 72:

- Make the factor pairs systematically. In the above case, we start with the smallest number (1), and then proceed higher.
- As the factors become closer, the sum becomes less. As the factors go further apart, their sum increases. Use this to guess the solutions faster.
    - ✓ (1,72) are far apart. Their sum is 73.

(8,9) are close together. Their sum is 17.

1.107: Zero Product Property

$$ab = 0 \Rightarrow a = 0\ OR\ b = 0\ OR\ Both$$

- If a product is zero, then it can only be zero when the numbers being multiplied have at least one zero.
- This is a very basic, and very important property. It is used very frequently in solving equations.

Example 1.108

Solve

A. $xy = 0$
B. $(x+4)(x-5) = 0$
C. $(3x+4)(2x-5) = 0$
D. $\left(3x - \frac{4}{5}\right)\left(2x + \frac{1}{2}\right) = 0$
E. $(2x + \sqrt{2})(5y - \sqrt[4]{3}) = 0$
F. $\left(\frac{2}{3}x - \frac{3}{4}\right)^2 = 0$

Part A

$$xy = 0 \Rightarrow x = 0\ OR\ y = 0$$

Part B

$$\underbrace{(x+4)}_{a}\underbrace{(x-5)}_{b}=0$$
$$x+4=0 \Rightarrow x=-4$$
$$x-5=0 \Rightarrow x=5$$

The final solution set is

$$x \in \{-4,5\}$$

Part C

$$3x+4=0 \Rightarrow x=-\frac{4}{3}$$
$$2x-5=0 \Rightarrow x=\frac{5}{2}$$

Part D

Use the zero-product property.

$$3x-\frac{4}{5}=0 \Rightarrow 3x=\frac{4}{5} \Rightarrow x=\frac{4}{15}$$
$$2x+\frac{1}{2}=0 \Rightarrow 2x=-\frac{1}{2} \Rightarrow x=-\frac{1}{4}$$

$$x \in \left\{-\frac{1}{4},\frac{4}{15}\right\}$$

Part E

Use the zero-product property:

$$2x+\sqrt{2}=0 \Rightarrow 2x=-\sqrt{2} \Rightarrow x=-\frac{\sqrt{2}}{2}$$
$$5y-\sqrt[4]{3}=0 \Rightarrow 5y=\sqrt[4]{3} \Rightarrow y=\frac{\sqrt[4]{3}}{5}$$

Part F

$$\left(\frac{2}{3}x-\frac{3}{4}\right)^2=0 \Rightarrow \left(\frac{2}{3}x-\frac{3}{4}\right)\left(\frac{2}{3}x-\frac{3}{4}\right)=0$$

Using the zero-product property:

$$\frac{2}{3}x-\frac{3}{4}=0\ OR\ \frac{2}{3}x-\frac{3}{4}=0$$

Since both the equations are the same, we only need to solve one:

$$\frac{2}{3}x=\frac{3}{4} \Rightarrow x=\frac{3}{4}\times\frac{3}{2}=\frac{9}{8}$$

Example 1.109: Monic Trinomials

Monic trinomials are the ones where the leading coefficient is one. Solve the following monic trinomials:

A. $x^2+5x+6=0$
B. $x^2-2x-15=0$
C. $x^2-4x-60=0$
D. $x^2+4x-96=0$
E. $x^2+x-182=0$
F. $x^2+3x-70=0$

Part A

$Product=6$ $Sum=5$		
Factor Pairs		Sum
1	6	7
2	3	5

We want to find numbers such that the product is 6 and the sum is 5:

$$(x+2)(x+3)=0$$

By the zero-product property, at least one of the expressions on the left-hand side must be zero.

$$x+2=0 \Rightarrow x=-2\ x+3=0 \Rightarrow x=-3 \Rightarrow x \in \{-2,-3\}$$

Part B

For each equation, factor and then apply the zero-product property:

$$x^2-2x-15=0 \Rightarrow (x+3)(x-5)=0 \Rightarrow x \in \{-3,5\}$$
$$x^2-4x-60=0 \Rightarrow (x+6)(x-10)=0 \Rightarrow x \in \{-6,10\}$$
$$x^2+4x-96=0 \Rightarrow (x+12)(x-8)=0 \Rightarrow x \in \{-12,8\}$$
$$x^2+x-182=0 \Rightarrow (x+14)(x-13)=0 \Rightarrow x \in \{-14,13\}$$
$$x^2+3x-70=0 \Rightarrow (x+10)(x-7)=0 \Rightarrow x \in \{-10,7\}$$

Example 1.110

Solve the quadratic equations

A. $x^2+5x+6=0$
B. $6x^2+17x+12=0$
C. $x^2+10x+25=0$
D. $x^2-14x+49=0$

Part A

$$x^2+\underbrace{5x}_{Sum}+\underbrace{6}_{Product}=0\Rightarrow Product=6, Sum=5\Rightarrow Factor\ Pairs\ of\ 6=\underbrace{1\times 6}_{1+6=7}=\underbrace{2\times 3}_{2+3=5}$$

$$Regroup: x^2+5x+6=\underbrace{x^2+2x}_{Factor\ this}+\underbrace{3x+6}_{Factor\ this}=\underbrace{x(x+2)+3(x+2)}_{Factor\ (x+2)}=(x+2)(x+3)$$

$$Zero-Product\ Property: \underbrace{(x+2)(x+3)=0}_{ab=0\Rightarrow a=0\ OR\ b=0}\Rightarrow \underbrace{x=-2}_{x+2=0}\ OR\ \underbrace{x=-3}_{x=-3}\Rightarrow \underbrace{x\in\{-2,-3\}}_{Set\ Notation}$$

Part B

$$Sum=17, Product=6\times 12=72$$

$$72=\underbrace{1\times 72}_{1+72=73}=\underbrace{2\times 36}_{2+36=38}=\underbrace{3\times 24}_{3+24=27}=\underbrace{4\times 18}_{4+18=22}=\underbrace{6\times 12}_{6+12=18}=\underbrace{8\times 9}_{8+9=17}$$

Now, factor by splitting the middle term:

$$\underbrace{6x^2+8x}_{Factor\ this}+\underbrace{9x+12}_{Factor\ this}=0\Rightarrow \underbrace{2x(3x+4)+3(3x+4)}_{Factor\ 3x+4}=0\Rightarrow (3x+4)(2x+3)=0$$

Use the zero-product property, and then solve each resulting equation:

$$3x+4=0\Rightarrow x=-\frac{4}{3}\ OR\ 2x+3=0\Rightarrow x=-\frac{3}{2}\Rightarrow x\in\left\{-\frac{4}{3},-\frac{3}{2}\right\}$$

Part C

We use the formula for the square of a sum: $(a+b)^2=a^2+2ab+b^2$

$$\underbrace{x^2}_{a^2}+\underbrace{10x}_{2ab}+\underbrace{25}_{b^2}=0\Rightarrow \underbrace{(x+5)^2=0}_{Factoring}\Rightarrow \underbrace{x+5=0}_{\substack{Take\ the\ Square\\ Root\ both\ sides}}\Rightarrow x=-5$$

Part D

We use the formula for the square of a difference: $(a-b)^2=a^2-2ab+b^2$

$$\underbrace{x^2}_{a^2}-\underbrace{14x}_{2ab}+\underbrace{49}_{b^2}=0\Rightarrow \underbrace{(x-7)^2=0}_{Factoring}\Rightarrow \underbrace{x-7=0}_{\substack{Take\ the\ Square\\ Root\ both\ sides}}\Rightarrow x=7$$

Example 1.111: Maximizing and Minimizing Parameters

A. Find the smallest positive integer b for which $x^2+bx+2008$ factors into a product of two binomials, each having integer coefficients. (Mandelbrot National Level Round 4)
B. Find the largest positive integer b for which $x^2+bx+2008$ factors into a product of two binomials, each having integer coefficients.
C. Find the smallest integer b for which $x^2+bx+2008$ factors into a product of two binomials, each having integer coefficients.

Part A

$$Product=2008=2^3\times 251$$
$$Sum=b$$

To make the sum minimum, the roots must be as close to each other as possible:

$$8+251=259$$

Part B

To make the sum maximum, the roots must be as far away from each other as possible:

$$1+2008=2009$$

Part C

To make the sum minimum, the roots must be as far away from each other as possible:

$$-1-2008=-2009$$

C. Forming Equations with Given Roots

1.112: Roots of a Quadratic

$\underbrace{\alpha, \beta}_{\text{Letters in the Greek Alphabet}}$ are generally the symbols used for the roots of a quadratic equation.

So far, we have been looking at solving quadratic equations. The objective has been, given a quadratic equation $ax^2 + bx + c = 0$, to find the roots α and β

We now look at the reverse process. That is, given a quadratic equation with roots α and β, we find the equation that leads to these roots.

Example 1.113

Why is not a good idea to take a and b as the roots of a quadratic?

A quadratic in standard form is written:

$$ax^2 + bx + c = 0, a \neq 0$$

If the roots are a *and* b, then a and b refers to the roots also, and to the coefficients of the quadratic also.

Example 1.114

Given that the roots of a quadratic equation are -2 and -3, find the quadratic equation.

We know that the roots are:

$$x = -2 \Rightarrow \underbrace{x + 2 = 0}_{\text{Equation I}}$$

$$x = -3 \Rightarrow \underbrace{x + 3 = 0}_{\text{Equation II}}$$

Multiply Equations I and II:

$$(x + 2)(x + 3) = 0$$

Expand to write in standard form:

$$x^2 + 5x + 6 = 0$$

1.115: Equation with roots α and β

The quadratic equation with roots $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is

$$(\boldsymbol{x} - \boldsymbol{\alpha})(\boldsymbol{x} - \boldsymbol{\beta}) = \mathbf{0} \Rightarrow \boldsymbol{x}^2 - \underbrace{(\boldsymbol{\alpha} + \boldsymbol{\beta})}_{\text{Sum of Roots}} \boldsymbol{x} + \underbrace{\boldsymbol{\alpha\beta}}_{\text{Product of the Roots}}$$

The solutions that satisfy the equation are α and β. Therefore, we must have:

$$x = \alpha \Rightarrow \underbrace{x - \alpha = 0}_{\text{Condition I}}, \qquad x = \beta \Rightarrow \underbrace{x - \beta = 0}_{\text{Condition II}},$$

We want to find an equation that satisfies both these conditions simultaneously. Hence, we multiply the two conditions:

$$(x - \alpha)(x - \beta) = 0 \Rightarrow x^2 - (\alpha + \beta)x + \alpha\beta$$

Example 1.116

The sum of the roots of a quadratic is -5, and the product of the roots is 6. Find the quadratic equation.

$$x^2 - \underbrace{(-5)}_{\text{Sum of Roots}} + \underbrace{6}_{\text{Product of Roots}} = 0 \Rightarrow x^2 + 5x + 6 = 0$$

1.117: General Equation with roots α and β

All quadratic polynomials with roots α and β are given by:

$$a[x^2 - (\alpha + \beta)x + \alpha\beta] = 0$$

We have already seen that the quadratic equation with roots α and β is given by:

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

However, we can multiply both sides by $a \neq 0$ to get:

$$a[x^2 - (\alpha + \beta)x + \alpha\beta] = 0$$

And this is the most general form of this equation.

Example 1.118

Given that the roots of a quadratic function are -2 and -3, find *all* quadratic functions that meet this requirement.

As before, we get:

$$x^2 + 5x + 6 = 0$$

Multiply both sides by a to get the final answer:

$$a(x^2 + 5x + 6) = 0, a \neq 0$$

Example 1.119

The equation $x^2 - (1A)x + A0 = 0$ has positive integer solutions where A is a positive single digit. How many such $A's$ exist? (Since A is representing a digit, if $A = 2$ then $A0$ represents the integer 20.) **(MathCounts 2009 Warm-up 9)3**

A	$A0 = Product$	$1A = Sum$
1	10	$11 = 10 + 1$
2	20	10+2
.	.	.
.	.	.
.	.	.
9	90	10+9

$$9\ Values$$

Example 1.120: Number of Solutions

A. Find the number of integer solutions to: $2y^2 + 2y - 3 = 0$

Sum$= 2$, Product $= 6 = (+3)(-1)$

$$2y^2 - y + 3y - 3 = 0$$
$$(2y - 1)(y + 3) = 0$$
$$y \in \left\{-3, \frac{1}{2}\right\} \Rightarrow 1\ Integer$$

Example 1.121

$$x^2 + ax + 60 = 0, \qquad x^2 + bx + 48 = 0$$

are quadratic equations with integer roots.

A. What is the difference between the smallest and largest value that a can take?
B. What is the difference between the smallest and largest value that b can take?
C. If $a = b$, what is the number of possible values for a?

$x^2 + ax + 60 = 0$				$x^2 + bx + 48 = 0$		
		$Values\ for\ a$				$Values\ for\ b$
1	60	± 61		1	48	± 49
2	30	± 32		2	24	± 26
3	20	± 23		3	16	**± 19**
4	15	**± 19**		4	12	**± 16**
5	12	± 17		6	8	± 14
6	10	**± 16**				

$$Diff = 61 - (-61) = 122$$
$$Diff = 49 - (-49) = 98$$
$$A \cap B = \pm\{16,19\} \Rightarrow n(A \cap B) = 4$$

D. Solving General Quadratics

In the quadratic $ax^2 + bx + c$, a is the leading coefficient.

- If $a = 1$, the quadratic is called a monic quadratic.
- If a is any value other than zero, then it is a general quadratic

If the leading coefficient is not one, the factorization process is similar, but the working is more involved. Further, the shortcut method does not apply, and hence more steps are needed.

Example 1.122

Expand $(2x + 3)(3x + 4)$ and determine the sum and the product.

$$2x(3x+4) + 3(3x+4) = 6x^2 + 8x + 9x + 12 = 6x^2 + \underbrace{17}_{Sum} x + \underbrace{12}_{12\times 6=72=Product}$$

1.123: Factoring General Trinomials using Sum and Product

$$ax^2 + bx + c \Rightarrow Sum = b, Product = ac$$

$$(ax+b)(cx+d) = acx^2 + adx + bcx + bd = acx^2 + (ad+bc)x + bd$$

Multiply ac and bd:

$$\underbrace{ac}_{Coefficient\ of\ First\ Term} \times \underbrace{bd}_{Constant\ Term} = abcd$$

Find the factor pairs of $abcd$. One of the factorizations must be:

$$abcd = ad \times bc$$

Once we get this using our factor pair table, we have found ad and bc, and we can use the method of splitting the middle term.

Example 1.124

$$6x^2 + 17x + 12$$

Set up the Factor Pairs

$$Product = 6 \times 12 = 72, Sum = 17$$

We are looking for factor pairs of 72 that add up to 17.

$Product = 6 \times 12 = 72$ $Sum = 17$			
72	1	72	73
72	2	36	38
72	3	24	27
72	4	18	22
72	6	12	18
72	9	8	17

Split the Middle Term

Split $17x = 9x + 8x$

$$6x^2 + 17x + 12 = 6x^2 + \underbrace{9x + 8x}_{17x} + 12$$

Factorize Pairs

Factor $3x$ from the first two terms, and 4 from the last two terms:

$$\underbrace{6x^2 + 9x}_{Factor\ Out\ 3x} + \underbrace{8x + 12}_{Factor\ Out\ 4} = 3x(2x+3) + 4(2x+3)$$

Factor out $2x + 3$ from each term:

$$= (2x+3)(3x+4)$$

Example 1.125

Factorize by splitting the middle term:

A. $7x^2 + 22x + 3$
B. $3y^2 + 8y + 4$
C. $3z^2 - 10z + 3$

Part A

$$S = 22, P = 21 = 1 \times 21$$
$$7x^2 + 21x + x + 3 = 7x(x+3) + 1(x+3) = (x+3)(7x+1)$$

Part B

$$S = 8, P = \underbrace{3 \times 4}_{3+4=7} = 12 = \underbrace{1 \times 12}_{1+12=13} = \underbrace{2 \times 6}_{2+6=8 \Rightarrow Works}$$
$$3y^2 + 6y + 2y + 4 = 3y(y+2) + 2(y+2) = (y+2)(3y+2)$$

Part C

$$S = -10, P = (3)(3) = 9 = (-1)(-9)$$
$$3z^2 - 9z - z + 3 = 3z(z-3) - 1(z-3) = (z-3)(3z-1)$$

Example 1.126

Factor and solve $2x^2 - 5x + 2 = 0$

$$P = 2 \times 2 = 4, S = -5$$

Note that the product is positive, but the sum is negative. Hence, we need two negative numbers that multiply to 4.
Hence, consider negative factor pairs of 4.

$Product = 2 \times 2 = 4$ $Sum = -5$		Sum
-1	-4	-5
-2	-2	-4

Split the middle term using $-5x = -4x - x$:

$$2x^2 - 4x - x + 2 = 0$$

Factor the first two and the last two terms. Pay careful attention to how -1 is factored out of the last two terms to match the first factoring:

$$2x(x-2) - 1(x-2) = 0$$

Factor $x - 2$ from both terms:

$$(2x-1)(x-2) = 0$$

Use the zero-product property $ab = 0 \Rightarrow a = 0\ OR\ b = 0$:

$$2x - 1 = 0 \Rightarrow x = \frac{1}{2}\ OR\ x - 2 = 0 \Rightarrow x = 2$$

Example 1.127

Factor and solve using the zero-product property:

A. $6x^2 - 13x + 6 = 0$
B. $10x^2 + 26x + 12 = 0$
C. $2x^2 - 5x - 12$
D. $6x^2 + 17x - 14$
E. $3x^2 + 29x + 18 = 0$
F. $5x^2 + 70x + 245 = 0$

Part A

$$P = 36, \qquad S = -4 - 9 = -13$$
$$6x^2 - 4x - 9x + 6 = 2x(3x-2) - 3(3x-2) = (3x-2)(2x-3) = 0$$
$$x = \frac{2}{3}\ OR\ x = \frac{3}{2}$$

Part B

$$P = 10 \times 12 = 120, Sum = 20 + 6 = 26$$
$$10x^2 + 20x + 6x + 12 = 10x(x+2) + 6(x+2) = (10x+6)(x+2) = 0$$
$$x = -\frac{6}{10} = -\frac{3}{5}\ OR\ x = -2$$

Part C

$$P = -24, Sum = -8 + 3 = -24$$
$$2x^2 - 8x + 3x - 12 = 2x(x-4) + 3(x-4) = (2x+3)(x-4) = 0$$
$$x = -\frac{3}{2}\ OR\ x = 4$$

Part D

$$P = 84, Sum = 21 - 4 = 17$$
$$6x^2 + 21x - 4x - 14 = 3x(2x+7) - 2(2x+7) = (2x+7)(3x-2) = 0$$
$$2x + 7 = 0 \Rightarrow x = -\frac{7}{2}\ OR\ 3x - 2 = 0 \Rightarrow x = \frac{2}{3}$$

Part E

Divide both sides by 5:

$$x^2 + 14x + 49 = 0$$
$$(x+7)^2 = 0$$
$$x + 7 = 0$$
$$x = -7$$

Example 1.128

In a two-digit number, the digits have a difference of 1. The sum of the squares of the digits is 5 more than the number itself. Find the number.

We have not been told which is the larger digit, so we have to use some casework.

Case I: Larger Digit is Ten's Digit

$$(a+1)^2 + a^2 = 10(a+1) + a + 5$$
$$a^2 + a^2 + 2a + 1 = 10a + 10 + a + 5$$
$$2a^2 - 9a - 16 = 0$$
$$Does\ not\ factor$$
$$No\ Solutions$$

Case II: Larger Digit is Unit's Digit

$$a^2 + (a+1)^2 = 10a + a + 1 + 5$$
$$a^2 + a^2 + 2a + 1 = 10a + a + 1 + 5$$
$$2a^2 - 9a - 5 = 0$$
$$2a^2 - 10a + a - 5 = 0$$
$$(2a+1)(a-5) = 0$$
$$a \in \left\{\frac{1}{2}, 5\right\}$$

Number is:

$$56$$

Example 1.129

If the difference of two real numbers is four, and their product is also four, then find the ratio of the cube of the sum of the numbers to the difference of their squares.

Let the numbers be x and y.

$$xy = 4, \qquad x - y = 4$$

Then, we need to find

$$\frac{(x+y)^3}{x^2 - y^2} = \frac{(x+y)^3}{(x+y)(x-y)} = \frac{(x+y)^2}{x-y}$$

Rewrite the numerator:

$$\frac{(x+y)^2}{x-y}=\frac{x^2+2xy+y^2}{x-y}=\frac{(x-y)^2+4xy}{x-y}$$

Substitute the given values:

$$\frac{4^2+4(4)}{4}=\frac{32}{4}=8$$

Example 1.130

Pipe A can fill a tank two hours slower than Pipe B. And they can fill the tank together in one hour and fifty-two and a half minutes. In how much time can Pipe A alone fill the tank?

$$Total\ Time=1\ hr\ 52.5\ min=1+\frac{52.5}{60}=1+\frac{525}{600}=1+\frac{105}{120}=1+\frac{7}{8}=\frac{15}{8}hours$$

$$\frac{1}{x}+\frac{1}{x+2}=\frac{8}{15}$$
$$\frac{2x+2}{x^2+2x}=\frac{8}{15}$$
$$30x+30=8x^2+16x$$
$$8x^2-14x-30=0$$
$$4x^2-7x-15=0$$
$$4x^2-12x+5x-15=0$$
$$(x-3)(4x-5)=0$$
$$x\in\left\{-\frac{5}{4},3\right\}$$

(Calculator Allowed) Example 1.131

At Pizza Perfect, Ron and Harold make pizza crusts. When they work separately Ron finishes the job of making 100 crusts 1.2 hours before Harold finishes the same job. When they work together they finish making 100 crusts in 1.8 hours. How many hours, to the nearest tenth of an hour, does it take Ron working alone to make 100 crusts? **(MathCounts 1994 Warm-Up 11)**

$$\frac{1}{x}+\frac{1}{x+1.2}=\frac{1}{1.8}\Rightarrow x=3.09\approx 3.1$$

Example 1.132

Find the number of integer values of a such that $6x^2+ax+12$ can be factorized as a product of two binomials.

$Product=6\times 12=72$ $Sum=17$			
72	1	72	73
72	2	36	38
72	3	24	27
72	4	18	22
72	6	12	18
72	9	8	17

$$a\in\{17,18,22,27,38,73\}\Rightarrow 6\ Values$$

E. Perfect Squares

Example 1.133

A. $x^2 + 6x + 9 = 0$
B. $4x^2 + 12x + 9 = 0$
C.

$$(x+3)^2 = 0$$

Compare with $(a+b)^2 = a^2 + 2ab + b^2$:

$$\underbrace{4}_{a^2}x^2 + \underbrace{12}_{2ab}x + \underbrace{9}_{b^2} = 0$$

$$a^2 = 4x^2 \Rightarrow a = 2x$$
$$b^2 = 9 \Rightarrow b = 3$$
$$2ab = 2(2x)(3) = 12x$$
$$(a+b)^2 = (2x+3)^2$$

Example 1.134

$$\frac{x^2}{4} + 3x + 9$$

$$\left(\frac{x}{2}\right)^2 + 2\left(\frac{x}{2}\right)(3) + 3^2 = \left(\frac{x}{2} + 3\right)^2$$

Example 1.135

$$\frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4}$$

$$\left(\frac{x^2}{4}\right)^2 + 2\left(\frac{x^2}{4}\right)\left(\frac{1}{x^2}\right) + \frac{1}{x^2} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

F. Back Calculations

Example 1.136

Find the sum of all integer values of k for which 2x^2+12x+k can be factored as the product of two binomials
=2(x^2+6x+k/2)

Let K=k/2
2(x^2+6x+K)

Sum=6
1+5, K=5, k=10
2+4, K=8, k=16
3+3, K=9,k=18

G. Two Variables

Example 1.137

$$9x^2 + 30xy + 25y^2$$

$$(3x)^2 + 2(3)(5)xy + (5y)^2 = (3x + 5y)^2$$

1.138: Two Variables

If we have two variables, we can

- Solve for one variable in terms of the other
- Determine the ratio of the two variables

Example 1.139

Determine all possible values of the ratio of a and b given that $12a^2 + 35b^2 - 43ab = 0$

$$Product = 12 \times 35 = 420 = (15)(28)$$
$$Sum = 43 = 15 + 28$$

Factor by grouping:

$$12a^2 - 15ab - 28ab + 35b^2 = 0$$
$$3a(4a - 5b) - 7b(4a - 5b) = 0$$
$$(3a - 7b)(4a - 5b) = 0$$

Use the zero-product property:

$$3a - 7b = 0 \Rightarrow 3a = 7b \Rightarrow \frac{a}{b} = \frac{7}{3}$$
$$4a - 5b = 0 \Rightarrow 4a = 5b \Rightarrow \frac{a}{b} = \frac{5}{4}$$

$$a : b \in \{7:3, 5:4\}$$

1.5 Factoring Word Problems

A. Numbers

Example 1.140: Products and Factoring

A. The product of two positive consecutive integers is 110. Find the sum of the integers.
B. The product of two consecutive positive even integers is 48. Find the quotient when the larger number is divided by the smaller.
C. The product of two consecutive odd integers is 143. Find the sum of all the digits in the two numbers.
D. The sum of the squares of three consecutive positive odd integers is 251. Find the sum of the numbers.

Part A

$$n(n + 1) = 110 = (10)(11)$$
$$n^2 + n - 110 = 0 \Rightarrow (n + 11)(n - 10) = 0 \Rightarrow n \in \{-11,10\}$$

The pairs that works are:

$$(10,11)$$

Part B

$$n(n+2) = 48 = 6 \times 8 \Rightarrow Quotient = \frac{8}{6} = \frac{4}{3}$$
$$n^2 + 2n - 48 = 0 \Rightarrow (n+8)(n-6) = 0 \Rightarrow n \in \{-8,6\}$$

Part C

$$n(n+2) = 143 = 11 \times 13 \Rightarrow Sum = 1 + 1 + 1 + 3 = 6$$

Part D

$$(2n-1)^2 + (2n+1)^2 + (2n+3)^2 = 251$$
$$(4n^2 - 4n + 1) + (4n^2 + 4n + 1) + (4n^2 + 12n + 9) = 251$$
$$12n^2 + 12n - 240 = 0$$
$$n^2 + n - 20 = 0$$
$$(n+5)(n-4) = 0$$
$$n = \{-5,4\}$$

$$2n - 1 = 2(4) - 1 = 8 - 1 = 7$$
$$2n + 1 = 9$$
$$2n + 3 = 11$$

$$7 + 9 + 11 = 27$$

Example 1.141: Squares

A. The sum of the squares of two consecutive integers is 113. Find all possible such pairs.
B. Three consecutive integers are such that the sum of the squares of the first two is equal to the square of the third. Find all possible values of the integers.

Part A

$$x^2 + (x+1)^2 = 113$$
$$x^2 + x^2 + 2x + 1 = 113$$
$$2x^2 + 2x - 112 = 0$$
$$x^2 + x - 56 = 0$$
$$(x+8)(x-7) = 0$$
$$\Rightarrow x \in \{-8,7\}$$

We get two possible pairs:

$$(-8,-7), (7,8)$$

Part B

$$(x-1)^2 + x^2 = (x+1)^2$$
$$x^2 - 2x + 1 + x^2 = x^2 + 2x + 1$$
$$x^2 - 4x = 0$$
$$x(x-4) = 0$$
$$x \in \{0,4\}$$

The possible values are:

$$(-1,0,1), (3,4,5)$$

Note that (3,4,5) is a Pythagorean Triplet.

B. Counting

Example 1.142: Counting – Sum of Natural Numbers

Vedika adds the natural numbers upto (and including) x and gets 45. Vayuna adds the natural numbers from 5 to y and gets 56. Given that the sum of the first n natural numbers is given by $\frac{n(n+1)}{2}$, what is the positive difference between the number of numbers that Vedika added, and the number of numbers that Vayuna added?

Vedika added the numbers

$$1 + 2 + \cdots + x = \frac{x(x+1)}{2} = 45$$
$$\frac{x(x+1)}{2} = 45 \Rightarrow \underbrace{x}_{9}\underbrace{(x+1)}_{10} = 90 = 9 \times 10 \Rightarrow x = 9$$

Vayuna added the numbers

$$\frac{y(y+1)}{2}+10=66 \Rightarrow \underbrace{y}_{11}\underbrace{(y+1)}_{12}=132=11\times 12$$

$$5,6,7,8,9,10,11 \Rightarrow 7\ Numbers$$

The difference that we want is:

$$9-7=2$$

Example 1.143: Counting – Number of Handshakes

The number of handshakes that happen at a gathering with n people where everyone shakes hands with everyone else is given by $\frac{n(n-1)}{2}$. If at a party there are 10 handshakes, find the number of people present.

$$\frac{n(n-1)}{2}=10 \Rightarrow \underbrace{n}_{5}\underbrace{(n-1)}_{4}=20=5\times 4 \Rightarrow n=5 \Rightarrow 5\ People$$

Example 1.144: Counting – Number of Diagonals

The number of diagonals in a convex polygon with n vertices is given by $\frac{n(n-3)}{2}$.

A. Find the number of vertices of a polygon with 14 diagonals.
B. A convex polygon has a single digit number of diagonals. Find the maximum possible number of sides it can have.

Part A

$$\frac{n(n-3)}{2}=14 \Rightarrow n(n-3)=28=7\times 4 \Rightarrow n=7$$

Part B

Try:

$$\frac{n(n-3)}{2}=9 \Rightarrow n(n-3)=18=6\times 3 \Rightarrow n=6$$

C. Decimal System

Questions based on the digits of a number rely on the knowledge of the decimal system. Many of them will require writing out numbers in expanded notation.

These questions rely as much on number theory as on algebra, and $\underbrace{some\ of\ them}_{\textbf{not all}!}$ can be solved much faster with number theory methods.

1.145: Expanded Form

$$ab\ is\ a\ two\ digit\ number \Rightarrow ab=10a+b$$
$$abc\ is\ a\ three\ digit\ number \Rightarrow abc=100a+10b+c$$

For example:

$$53=50+3=5(10)+3$$
$$257=200+50+7=2(100)+5(10)+7$$

Example 1.146

X is the sum of all possible values of a two digit number that is twenty-five greater than the sum of its unit's digit and the square of its ten's digit. Find the largest odd prime factor in the sum of digits of X.

Let the number be tu. Then

$$10t + u = t^2 + u + 25$$
$$t^2 - 10t + 25 = 0$$
$$(t-5)^2 = 0$$
$$t - 5 = 0$$
$$t = 5$$

While we get a single value of t, there are no restrictions on the value of u.

$$t = 5, u \in (0,1,2..,9)$$

Adding all the possible values of tu:

$$X = 50 + 51 + \cdots + 59 = \underbrace{(50 + 50 + \cdots + 50)}_{Ten\,Times} + 0 + 1 + \cdots + 9 = 500 + 45 = 545$$

The sum of the digits of X is:

$$5 + 4 + 5 = 14 = 2 \times 7$$

The largest odd prime factor

$$= 7$$

Example 1.147: Real Number Line

Line Segment

Example 1.148

Mr. Sanchez's students were asked to add two positive integers. Juan subtracted by mistake and got 2. Maria mistakenly multiplied and got 120. What was the correct answer? **(MathCounts 2002 State Countdown)**

Let the integers be x and y. Then, their difference is:

$$x - y = 2 \Rightarrow x = y + 2$$

The product of the two numbers is:

$$xy = (y+2)(y) = 120$$
$$y^2 + 2y - 120 = 0$$
$$(y+12)(y-10) = 0$$
$$y \in \{-12, 10\}$$

Since the integers are positive:

$$y = 10 \Rightarrow x = 12$$

The correct answer is

$$x + y = 10 + 12 = 22$$

Example 1.149

A two-digit number has a difference of 2 between its ten's digit and unit's digit. If the digits of the number are interchanged, the sum of the unit's digit and the square of the ten's digit is doubled. Find the product of the digits of the number.

Take an example to understand what the question is asking. Consider the number 21.

$$\underbrace{1}_{\substack{Unit's\\Digit}} + \underbrace{4}_{\substack{Square\ of\\Ten's\ Digit}} = 5$$

Interchanging the digits gives us 12.

$$\underbrace{2}_{\substack{Unit's\\Digit}} + \underbrace{1}_{\substack{Square\ of\\Ten's\ Digit}} = 3 \neq \frac{5}{2}$$

To solve this, take variables. Let the number be $\overline{t(t-2)}$.

$$2\left(\underbrace{t^2}_{\substack{Square\ of\\Ten's\ Digit}} + \underbrace{t-2}_{\substack{Unit's\\Digit}}\right) = \underbrace{(t-2)^2}_{\substack{Square\ of\\Ten's\ Digit}} + \underbrace{t}_{\substack{Unit's\\Digit}}$$

$$t^2 + t - 2 = \frac{1}{2}[t^2 - 4t + 4 + t]$$

$$t^2 + t - 2 = t^2 - 2t + 4$$
$$t - 2 = -2t + 4$$
$$3t = 6$$
$$t = 3$$

$$31 \to 9 + 1 = 10$$
$$13 \to 1 + 3 = 4$$

To solve this, take variables. Let the number be $\overline{t(t+2)}$.

$$\underbrace{t^2}_{\substack{Square\ of \\ Ten's\ Digit}} + \underbrace{t+2}_{\substack{Unit's \\ Digit}} = \frac{1}{2}\left[\underbrace{(t+2)^2}_{\substack{Square\ of \\ Ten's\ Digit}} + \underbrace{t}_{\substack{Unit's \\ Digit}}\right]$$

$$2t^2 + 2t + 4 = t^2 + 5t + 4$$
$$t^2 - 3t = 0$$
$$t(t-3) = 0$$
$$t \in \{0,3\}$$
$$t + 2 = 5$$

Number is:

$$35 \to 9 + 5 = 14$$
$$53 \to 25 + 3 = 28$$

D. Age Problems

Recall that age cannot be negative. Hence, we get a negative answer at any point of time, we will reject it.

Example 1.150

The Smiths have a son, Alexander and a daughter, Sophia with an age difference of six years. Three years from now, one-third the square of Alexander's age will be equal to Sophia's age then. Find Alexander's age now.

We don't know which one is older, but since we are taking a square, we guess that Alexander is younger and proceed. If it does not work out, we will have to redo (taking Sophia as younger).

	Alexander	Sophia
Current Age	x	$x+6$
After Three Years	$x+3$	$x+9$

$$\frac{1}{3}(x+3)^2 = x + 9$$
$$x^2 + 6x + 9 = 3x + 27$$
$$x^2 + 3x - 18 = 0$$
$$(x+6)(x-3) = 0$$
$$x \in \{-6,3\}$$

Since age cannot be negative:

$$x = 3$$

Example 1.151

Five years ago, a woman's age was the square of her son's age. Ten years from now, her age will be twice that of her son's age at that time. Find the sum of the present ages of the two. **(ICSE 2007, Adapted)**

	Son	Mother
Five years ago	x	x^2
Ten years from now	$x+15$	x^2+15

$$x^2 + 15 = 2(x + 15)$$
$$x^2 + 15 = 2x + 30$$
$$x^2 - 2x - 15 = 0$$
$$(x-5)(x+3) = 0$$
$$x \in \{-3,5\}$$

Since age cannot be negative,

$$x = 5$$

The current age of son and mother is

$$x + 5 + x^2 + 5 = (5+5) + (25+5) = 10 + 30 = 40$$

E. Ratio and Proportion

Example 1.152

Two numbers, both of which are two digits, have a sum of 21. The sum of their squares is the ratio 9: 16. Find the difference between the two numbers.

Let the numbers be x and y.

$$x + y = 21 \Rightarrow y = 21 - x$$

Hence, my numbers are

$$x \text{ and } 21 - x$$

Since the sum of the squares of the numbers is in the ratio 3: 4,

$$x: 21 - x = 3: 4$$

Convert the ratios to fractions:

$$\frac{x^2}{(21 - x)^2} = \frac{9}{16}$$

Take the square root both sides:

$$\frac{x}{21 - x} = \pm\frac{3}{4}$$

Consider both the cases.

Case I: Positive Case

$$\frac{x}{21 - x} = \frac{3}{4}$$
$$4x = 63 - 3x$$
$$7x = 63$$
$$x = 9$$
$$Not\ Valid$$

Case II: Negative Case

$$\frac{x}{21 - x} = -\frac{3}{4}$$
$$4x = -63 + 3x$$
$$x = -63$$
$$21 - x = 21 - (-63) = 84$$

The difference between the two numbers is:

$$84 - (-63) = 147$$

And we can check that -63 and 84 work:

$$(-63)^2: 84^2$$
$$(-7 \times 9)^2: (7 \times 12)^2$$
$$7^2 \times 9^2: 7^2 \times 12^2$$
$$9^2: 12^2$$
$$81: 144$$
$$9: 16$$

F. Time, Speed and Distance

1.153: Time, Speed and Distance Formula

$$T = \frac{D}{S}$$

Example 1.154

By increasing the speed of a car by 10 $\frac{km}{hr}$, a journey of 72 km takes 36 minutes less. Find the original speed of the car. **(ICSE 2005, Adapted)**

Let:

$$Original\ speed = s\ \frac{km}{hr} \Rightarrow New\ Speed = (s + 10)\ \frac{km}{hr}$$

Then, the time taken is:

$$T_{Original} = \frac{D}{S} = \frac{72}{s}, \qquad T_{New} = \frac{D}{S} = \frac{72}{s + 10}$$

Since there is a difference of 36 minutes between the original and the new time:

$$\frac{72}{s} - \frac{72}{s + 10} = \frac{36}{60}$$

Add the fractions:

$$\frac{72(s + 10) - 72s}{s(s + 10)} = \frac{3}{5}$$

$$\frac{240}{s(s+10)} = \frac{1}{5}$$

Cross-multiply:

$$1200 = s(s+10)$$
$$30 \times 40 = s(s+10)$$
$$s = 30$$

G. Real Life Applications

Example 1.155

If a farmer puts up a double fence around his field, it will cost him $200. He also has the option of putting up a single fence, which will cost him $2 per meter less. In that case, he will be able to put up the fence and be left over with enough fence to fence his garden as well, which is a square with sides of 1.25 m each. Find the perimeter of the farmer's field, and the cost of the double fence per meter.

Let the length of the double fence be d meters.

$$Length\ of\ double\ fence = d \Rightarrow Length\ of\ single\ fence = d + 4(1.25) = d + 5$$

Then, the cost of:

$$One\ meter\ of\ double\ fence = \frac{200}{d}, \qquad One\ meter\ of\ single\ fence = \frac{200}{d+5}$$

$$\frac{200}{d} - \frac{200}{d+5} = 2$$

$$\frac{200(d+5) - 200d}{d(d+5)} = 2$$
$$1000 = d^2 + 5d$$
$$d = \{-25, 20\}$$
$$d = 20$$

Example 1.156: Everyday Situations

Everyday situations can be the most difficult to convert into equations.

Each of n cats has $2n$ fleas. If two cats (and their fleas) are removed, and three fleas are removed from each remaining cat, the total number of fleas remaining would be half the original total number of fleas. If there is at least one cat, what is the value of n? **(MathCounts 2008 National Sprint)**

The original number of fleas was

$$\underbrace{n}_{Cats} \times \underbrace{2n}_{\substack{Fleas\\per\ Cat}} = 2n^2$$

The new number of fleas is:

$$\underbrace{n-2}_{\substack{Cats\\reduced\ by\ 2}} \times \underbrace{2n-3}_{\substack{Fleas\ per\ cat\\reduced\ by\ 3}} = 2n^2 - 7n + 6$$

After the reduction, the number of fleas becomes half of the original number of fleas:

$$\frac{1}{2}(2n^2) = 2n^2 - 7n + 6 \Rightarrow$$
$$0 = n^2 - 7n + 6$$
$$0 = (n-6)(n-1)$$
$$n \in \{\mathbf{1}, 6\}$$
$$n = \text{No. of Cats} = 6$$

Example 1.157

Cost

H. Geometry

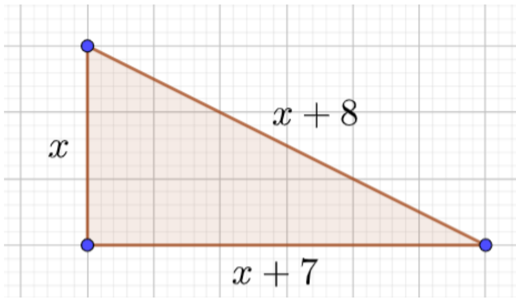
Example 1.158
A carpenter is designing a table. The table will be in the form of a rectangle whose length is 4 feet more than its width. How long should the table be if the carpenter wants the table to be 45 square feet. **(JMET 2011/72)**

$$l(l-4)=45$$
$$Factor\ pairs\ of\ 45=(1{,}45)(3{,}15)(5{,}9)\Rightarrow 5\ and\ 9\ have\ difference\ of\ 4\Rightarrow l=9$$

$$l^2-4l-45=0$$
$$P=45=(1{,}45)(3{,}15)(5{,}9)$$
$$(l-5)(l-9)=0$$
$$l\in\{5{,}9\}$$

1.159: Pythagorean Theorem
In a right-angled triangle with legs a and b and hypotenuse c, we have:
$$a^2+b^2=c^2$$

Example 1.160
In a right triangle, the hypotenuse is one unit longer than a leg, and the leg is itself seven units longer than the other leg. Find the area of the triangle.



Let the shorter leg be:
$$x\Rightarrow Other\ Leg=\ x+7\Rightarrow Hypotenuse=\ x+8$$
By the Pythagorean Theorem, we must have:
$$x^2+(x+7)^2=(x+8)^2$$
$$x^2+x^2+14x+49=x^2+16x+64$$
$$x^2-2x-15=0$$
$$(x-5)(x+3)=0$$
$$x\in\{-3{,}5\}$$
Reject the negative value since lengths cannot be negative:
$$x=5\Rightarrow Sides\ are\ \{5{,}12{,}13\}$$

Example 1.161
A certain circle's area is x square units, and its circumference is y units. The value of $x+y$ is 80π. What is the radius of the circle, in units? **(Maths 2007 State Countdown)**

$$x+y=80\pi$$
Let the circle have radius r. Substitute $x=2\pi r, y=\pi r^2$:
$$\underbrace{2\pi r}_{Circumference}+\underbrace{\pi r^2}_{Area}=80\pi$$
Divide both sides by π:
$$2r+r^2=80$$
Collate all terms on the LHS:
$$r^2+2r-80=0$$
Factor:
$$(r+10)(r-8)=0$$
Solve:
$$r\in\{-10,8\}$$

Reject the negative value since radius cannot be negative.

$$r = 8$$

Example 1.162

Circle A (with radius r) is touching a wall, and Circle B(with radius $3r$) is touching the point of Circle A farthest from the wall. The difference in their areas is $a\ cm^2$. The difference in their circumferences is $b\ cm$. If $a + 4b$ is $1{,}144\pi$, then find the distance between the wall and the point of Circle B farthest from the wall.

Difference in the areas

$$= a = \underbrace{9\pi r^2}_{Area\ of\ B} - \underbrace{\pi r^2}_{Area\ of\ A} = 8\pi r^2$$

Difference in the circumferences

$$= b = \underbrace{2\pi(3r)}_{\substack{Circumference\\ of\ B}} - \underbrace{2\pi r}_{\substack{Circumference\\ of\ A}} = 6\pi r - 2\pi r = 4\pi r$$

Substitute the values calculated above in $a + 4b = 572\pi$:

$$8\pi r^2 - 4(4\pi r) = 1{,}144\pi$$
$$8r^2 - 16r = 1144$$
$$r(r-2) = 143 = 13 \times 11$$

Simplify:

$$r^2 - 2r - 143 = 0 \Rightarrow (r-13)(r+11) = 0 \Rightarrow r = \{-11,13\} \Rightarrow r = 13$$

Example 1.163

A rectangular sheet of metal with length five inches greater than its width has squares of size 3 inches cut from its corners, and then the "flaps" of the resulting shape are folded to form an open-top box with a capacity of 450 units. If metal is available at \$0.25 per square inch, find the cost of converting the box from open-top to closed-top.

For the original sheet, let:

$$l = length \Rightarrow l - 5 = width$$

After the corners are cut, the length and width will reduce by $3 \times 2 = 6$ each, and hence the volume is:

$$V = lwh = (l-6)(l-11)(3) = 450$$

Divide both sides by three, collate all terms on one side, solve the resulting quadratic, and reject the negative value since length cannot be negative:

$$l^2 - 17l - 84 = 0 \Rightarrow (l-21)(l+4) = 0 \Rightarrow l \in \{-4,21\} \Rightarrow l = 21$$

Cost of top

$$= \frac{1}{4}(21-6)(16-6) = \frac{1}{4}(15)(10) = 75\ Dollars$$

I. Coordinate Geometry

Example 1.164: Distance Formula

A line segment begins at (1,3). It is 5 units long and ends at the point (4,x) and x>0. What is the value of x?
(MathCounts 2004 State Countdown)

$$\sqrt{(4-1)^2 + (x-3)^2} = 5$$
$$x^2 - 6x + 18 = 25$$
$$(x-7)(x+1) = 0$$
$$x = 7$$

Alternate:
$(1,3), (4,3), (4, x)$ form a right triangle.
Pythagorean Triplet

J. Factor Pairs

Example 1.165
If the positive integer n has positive integer divisors a and b with $n = ab$, then a and b are said to be complementary divisors of n. Suppose that N is a positive integer that has one complementary pair of divisors that differ by 20 and another pair of complementary divisors that differ by 23. What is the sum of the digits of N? (AMC 2023 10A/23)

From the conditions on factor pairs:

$$N = a(a - 20) = b(b - 23)$$

When the gap between the numbers increases, the smaller number decreases , and the larger number increases. Hence,

$$b > a$$
$$b - 23 < a - 20 \Rightarrow b < a + 3$$

Combining the above two:

$$a < b < a + 3 \Rightarrow b \in \{a + 1, a + 2\}$$

Case I: b=a+1

$$a(a - 20) = (a + 1)(a + 1 - 23)$$
$$a = -22$$

Case II: b=a+2

$$a(a - 20) = (a + 2)(a + 2 - 23)$$
$$a = 42$$
$$a(a - 20) = 42(22) = 924$$
$$Sum\ of\ the\ digits\ = 9 + 2 + 4 = 15$$

1.6 Quadratic Formula

A. Quadratic Formula

1.166: Roots of a quadratic equation in standard form
Given a quadratic in standard form $ax^2 + bx + c = 0$, the roots of the quadratic are given by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Divide throughout by a to make the leading coefficient one:[1]

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

[1] The proof of the Quadratic Formula requires knowledge of Completing the Square, which is covered in the section on completing the Square.

Add and subtract the square of half the second term in preparation to complete the square:

$$\left[x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right]-\left(\frac{b}{2a}\right)^2+\frac{c}{a}=0$$

Rewrite the terms inside the square brackets as a perfect square.

$$\left[x+\frac{b}{2a}\right]^2-\frac{b^2}{4a^2}+\frac{c}{a}=0$$

Isolate the perfect square term on the LHS:

$$\left[x+\frac{b}{2a}\right]^2=\frac{b^2}{4a^2}-\frac{c}{a}$$

Add the two fractions in the RHS by taking the LCM, and then take the square root on both sides:

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

Simplify the denominator in the RHS and isolate x on the LHS to arrive at the standard quadratic formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

B. Comparing Factoring with the Quadratic Formula

As you will see, the quadratic formula is lengthier, and takes more time as compared to factoring. Hence, if you can factor a quadratic, it is recommended to use factoring. However, certain quadratics cannot be factored (easily), and in those situations, the quadratic formula is handy.

Example 1.167

Find the root(s) of the equation

$$x^2+5x+6=0$$

A. By factoring
B. By using the quadratic formula

$$x^2+5x+6=0 \Rightarrow (x+2)(x+3)=0 \Rightarrow x\in\{-2,-3\}$$

Substitute $a=1, b=5, c=6$:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-5\pm\sqrt{25-4(1)(6)}}{2}=\frac{-5\pm\sqrt{1}}{2}=\frac{-5\pm 1}{2}$$

Take the two cases separately:

$$\frac{-5-1}{2}=-\frac{6}{2}=-3\,,\qquad \frac{-5+1}{2}=-\frac{4}{2}=-2\Rightarrow x\in\{-2,-3\}$$

Example 1.168

Consider the equation

$$x^2+4x+1=0$$

A. Is it easy to find the root(s) of the equation using factoring?
B. Find the roots using the quadratic formula.

Part A

$$P = 1, Sum = 4$$

Using our usual factoring methods, this is difficult to factor.

Part B

We identify the inputs for the quadratic formula which are the coefficients of the equation:

$$a = 1, b = 4, c = 1$$

Substitute the coefficients in the quadratic formula:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - 4(1)(1)}}{2(1)} = \frac{-4 \pm \sqrt{12}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = \frac{-2 \pm \sqrt{3}}{1} = -2 \pm \sqrt{3}$$

Example 1.169

Repeat the above question for

$$x^2 + 4x + 4 = 0$$

$$(x + 2)^2 = 0$$
$$x + 2 = 0$$
$$x = -2$$

Substitute $a = 1, b = 4, c = 4$

$$\frac{-4 \pm \sqrt{16 - 4(1)(4)}}{2(1)} = \frac{-4 \pm \sqrt{0}}{2} = \frac{-4}{2} = -2$$

Repeat the above question for

$$x^2 + 2x + 10 = 0$$

$$a = 1, b = 2, c = 10 \Rightarrow \frac{-2 \pm \sqrt{4 - 4(1)(10)}}{2(1)} = \frac{-2 \pm \sqrt{-36}}{2}$$

Example 1.170

$$3x^2 - 4x - 10 = 0$$

Substitute $a = 3, b = -4, c = -10$ in the quadratic formula:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 4(3)(-10)}}{6} = \frac{4 \pm \sqrt{136}}{6} = \frac{4 \pm 2\sqrt{34}}{6} = \frac{2 \pm \sqrt{34}}{3} = \frac{-2 \pm \sqrt{34}}{-3}$$

$$\frac{x - y}{a - 4} = \frac{x - y}{a - 4} \times \frac{-1}{-1} = \frac{-x + y}{-4 + a} = \frac{y - x}{4 - a}$$

Example 1.171: Fractional Answers

A. $4x^2 - 4x - 3 = 0$
B. $15x^2 + 19x + 6 = 0$

Part A

Substitute $a = 4, b = -4, c = -3$ into the quadratic formula:

$$x = \frac{4 \pm \sqrt{(-4)^2 - (4)(4)(-3)}}{(2)(4)} = \frac{4 \pm \sqrt{16 + 48}}{8} = \frac{4 \pm \sqrt{64}}{8} = \frac{4 \pm 8}{8} = \frac{1 \pm 2}{2}$$

The roots are:

$$x \in \left\{-\frac{1}{2}, \frac{3}{2}\right\}$$

Part B

Substitute $a = 15, b = 19, c = 6$ into the quadratic formula:

$$x = \frac{-19 \pm \sqrt{361 - (4)(15)(6)}}{(2)(15)} = \frac{-19 \pm \sqrt{1}}{30}$$

The roots are:

$$x \in \left\{\frac{-19 - 1}{30}, \frac{-19 + 1}{30}\right\} = \left\{\frac{-20}{30}, \frac{-18}{30}\right\} = \left\{-\frac{2}{3}, -\frac{3}{5}\right\}$$

1.172: Discriminant

In the quadratic formula, the quantity inside the square root is called the discriminant.

$$ax^2 + bx + c = 0 \Rightarrow Discriminant = b^2 - 4ac$$

1.173: Radical Roots

If the discriminant is not a perfect square, then the roots of the quadratic will have radicals.

Example 1.174: Radical Answers

A. $x^2 + 8x - 22 = 0$
B. $2x^2 + 3x - 4 = 0$

Part A

Identify the variables we need for the formula:

$$a = 1, b = 8, c = -22$$

Substitute the above into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{8^2 - (4)(1)(-22)}}{(2)(1)} = \frac{-8 \pm \sqrt{64 + 88}}{2} = \frac{-8 \pm \sqrt{152}}{2} = \frac{-8 \pm 2\sqrt{38}}{2} = -4 \pm \sqrt{38}$$

Part B

Substitute $a = 2, b = 3, c = -4$ into the quadratic formula:

$$x = \frac{-3 \pm \sqrt{9 - (-32)}}{(2)(2)} = \frac{-3 \pm \sqrt{41}}{4} = -\frac{3}{4} \pm \frac{\sqrt{41}}{4}$$

1.175: Complex Roots

If the discriminant is negative, then the roots of the quadratic will have a negative quantity inside the square root.

Such a number does not have a solution in the real number system. Such numbers are called *imaginary*.

Example 1.176: Imaginary Roots

A. $x^2 + x + 1 = 0$

Part A

Apply the quadratic formula: $a = 1, b = 1, c = 1$:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2}$$

Example 1.177: Eliminating the HCF

A.

Example 1.178: Rearranging

A. $6 + 2x^2 - 3x = 8x^2$
B. $4 + 2x - 7x^2 = 3 + x + 6x^2$

Part A

Collate all terms on one side:

$$0 = 6x^2 + 3x - 6$$
$$6x^2 + 3x - 6 = 0$$

Divide both sides by 3:

$$2x^2 + x - 2 = 0$$

Apply the quadratic formula: $a = 2, b = 1, c = -2$:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(2)(-2)}}{2(2)} = \frac{-1 \pm \sqrt{17}}{4} = -\frac{1 \pm \sqrt{17}}{4}$$

Part B

$$1 + x - x^2 = 0$$

Apply the quadratic formula: $a = -1, b = 1, c = 1$:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(-1)(1)}}{2(-1)} = \frac{1 \pm \sqrt{-5}}{2}$$

Example 1.179: Fractional Coefficents

Solve the equations below using the quadratic formula.

A. $x^2 + \frac{5}{2}x - 1 = 0$
B. $\frac{1}{2}x^2 + \frac{3}{4}x - 1 = 0$

Part A

Multiply by 2 to eliminate fractions, and make the calculations easier:

$$2x^2 + 5x - 2 = 0$$

Apply the quadratic formula $a = 2, b = 5, c = -2$:

$$x = \frac{-5 \pm \sqrt{5^2 - 4(2)(-2)}}{2(2)} = \frac{-5 \pm \sqrt{41}}{4}$$

Example 1.180: Variables in the Denominator

A. $x + \frac{1}{x} = 1$

Multiply by x on both sides to eliminate fractions:

$$x^2 + 1 = x$$

Collate all terms one side:

$$x^2 - x + 1 = 0$$

Apply the quadratic formula: $a = 1, b = -1, c = 1$:

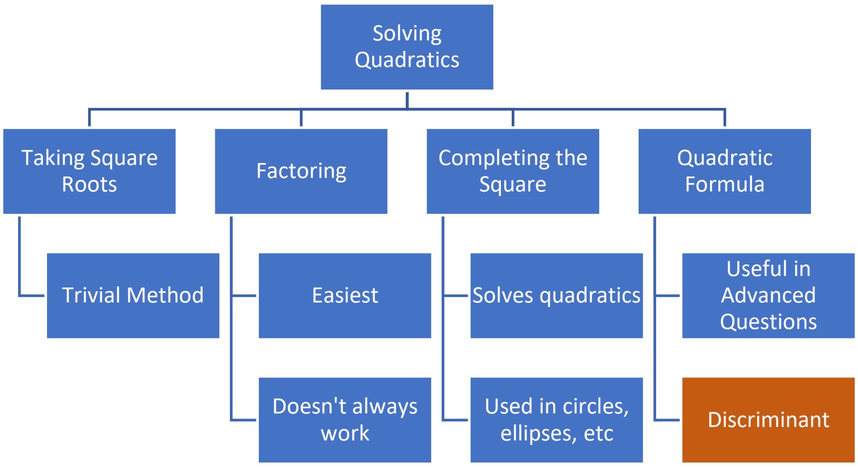
$$x = \frac{-1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2}$$

C. Methods of Solving a Quadratic

The three main methods of solving a quadratic are:

- Factoring
- Completing the Square
- Quadratic Formula

You can also solve quadratics by taking square roots. This is possible directly in the case of trivial equations, and requires some effort to set up in the case of other equations.



D. Golden Ratio

The golden ratio has applications in art since it is pleasing to the eye. (This is, of course, subjective, but the golden ratio is very commonly used). The golden ratio is used when designing buildings.

Example 1.181

A. If 1 is added to the reciprocal of a number, we get the number itself. Find the number.
B. Consider a line segment with non-zero length l. Divide the line into two parts such that the ratio of the line segment to the longer part is the same as the ratio of the longer part to the shorter part. Find this ratio.

Part A

We are going to use the variable ϕ because the number that we are referring to is the golden ratio, and the ϕ is often used to represent the golden ratio:

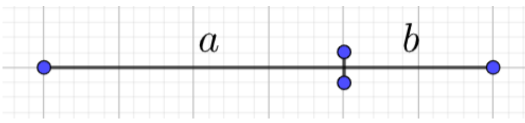
$$1+\frac{1}{\phi}=\phi \Rightarrow \phi+1=\phi^2 \Rightarrow \phi^2-\phi-1=0$$

Substitute $a=1, b=-1, c=-1$ in the quadratic formula:

$$\phi=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{1\pm\sqrt{(-1)^2-(4)(1)(-1)}}{2}=\frac{1\pm\sqrt{5}}{2}$$

Part B

$$\frac{Line\ Segment}{Longer\ Part}=\frac{Longer\ Part}{Shorter\ Part}$$



$$\frac{a+b}{a}=\frac{a}{b}=\phi \Rightarrow \frac{a}{a}+\frac{b}{a}=\frac{a}{b}=\phi$$

Substitute $\frac{a}{a}=1, \frac{a}{b}=\phi, \frac{b}{a}=\frac{1}{\phi}$:

$$1+\frac{1}{\phi}=\phi \Rightarrow \phi=\frac{1\pm\sqrt{5}}{2}$$

Example 1.182
Show that the three quantities below form the sides of a right triangle:

$$1, \sqrt{\frac{1+\sqrt{5}}{2}}, \frac{1+\sqrt{5}}{2}$$

$$1^2 + \left(\sqrt{\frac{1+\sqrt{5}}{2}}\right)^2 = 1 + \frac{1+\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$$

E. Geometry

Example 1.183
Two quarter circles are drawn in a square, using two adjacent vertices as the respective centers, and the radius r as the side length of the square. Find the distance from the intersection point of the circles to the side of the square, in terms of r.

$$r^2 = (r-l)^2 + \left(\frac{r}{2}\right)^2 = r^2 - 2rl + l^2 + \frac{r^2}{4}$$

$$\boldsymbol{r^2} = \boldsymbol{r^2} - 2rl + l^2 + \frac{r^2}{4}$$

$$0 = -2rl + l^2 + \frac{r^2}{4}$$

$$4l^2 - 8rl + r^2 = 0$$

Use the quadratic formula with $a = 4, b = -8r, c \Rightarrow r^2$

$$\frac{8r \pm \sqrt{64r^2 - 4(4)(r^2)}}{2\times 4} = \frac{8r \pm \sqrt{48r^2}}{2\times 4} = \frac{8r \pm 4r\sqrt{3}}{2\times 4} = \frac{2r \pm r\sqrt{3}}{2}$$

Since $l < r$, we choose the smaller solution:

$$\frac{2r - \sqrt{3}r}{2} = \frac{r}{2}(2-\sqrt{3})$$

F. Probability

Example 1.184
A box contains some green marbles and exactly four red marbles. The probability of selecting a red marble is x%. If the number of green marbles is doubled, the probability of selecting one of the four red marbles from the box is $(x - 15)$%. How many green marbles are in the box before the number of green marbles is doubled? **(MathCounts 2007 State Sprint)**

Suppose the number of green marbles is g.

$$\underbrace{\frac{4}{g+4} = x\%}_{\textit{Equation I}}, \qquad \underbrace{\frac{4}{2g+4} = (x-15)\%}_{\textit{Equation II}}$$

Subtract Equation II from Equation I:

$$\frac{4}{g+4}-\frac{4}{2g+4}=\frac{15}{100}$$

Add the fractions:

$$\frac{4(2g+4)-4(g+4)}{(g+4)(2g+4)}=\frac{3}{20}$$

Simplify:

$$\frac{4g}{2g^2+12g+16}=\frac{3}{20}$$

Cross-multiply:

$$80g\ =6g^2+36g+48$$

Simplify:

$$3g^2-22g+24=0$$

Use the quadratic formula with $a=3, b=-22, c=24$:

$$g=\frac{22\pm\sqrt{484-(4)(3)(24)}}{6}=\frac{22\pm\sqrt{196}}{6}=\frac{22\pm 14}{6}\in\left\{\frac{4}{3},6\right\}\Rightarrow g=6$$

1.7 Equations Reducible to Quadratics

A. Reciprocals

Example 1.185

The positive solution to the equation $y-\frac{1}{6}=\frac{15}{6y}$ can be written in the form $\frac{a}{b}$, where $a,b\in\mathbb{N}$ and $HCF(a,b)=1$. Find $a+b$.

Eliminate fractions by multiplying both sides by $LCM(6,6y)=6y$:

$$6y\left(y-\frac{1}{6}\right)=6y\left(\frac{15}{6y}\right)\Rightarrow 6y^2-y=15$$

Collate all terms on one side:

$$6y^2-y-15=0$$

Calculate the product and the sum:

$$P=-90=(-1)(90)=(-2)(45)=(-3)(30)=(-5)(18)=(-9)(10)=(-10)(9)$$
$$S=-1=-10+9$$

Split the middle term:

$$6y^2-10y+9y-15=0$$
$$2y(3y-5)+3(3y-5)=0$$
$$(3y-5)(2y+3)=0$$

Use the zero-product property:

$$3y-5=0\Rightarrow y=\frac{5}{3}\ OR\ 2y+3=0\Rightarrow y=-\frac{3}{2}$$
$$y=\frac{5}{3}\Rightarrow a+b=5+3=8$$

Example 1.186

A. A number is one greater than 12 times its reciprocal. Find the sum of the absolute value of all possible values of the number.
B. What is the smallest number which is one less than twice its reciprocal? (MathCounts 2007 State Countdown)

Part A

Let

$$No. = x \Rightarrow 12\ Times\ Reciprocal = \frac{12}{x}$$

Their difference is 1:

$$x - \frac{12}{x} = 1$$

Multiply both sides by x to eliminate fractions:

$$x^2 - x - 12 = 0$$
$$(x+3)(x-4) = 0$$
$$x \in \{-3,4\}$$

The sum of the absolute values is:

$$3 + 4 = 7$$

Part B

$$\frac{2}{x} - x = 1$$
$$2 - x^2 = x$$
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$
$$x \in \{-2,1\}$$

Smallest value is:

$$x = -2$$

B. Radicals

Example 1.187

The right-hand side of the equation below is an infinite nested radical. The expression contains an infinite number of parts. Find the value of y.

$$y = \sqrt{5\sqrt{5\sqrt{5\dots}}}$$

Since the radicals are repeated an infinite number of times, if you reduce the pattern by one, the value of the expression does not change:

$$y = \sqrt{5\underbrace{\sqrt{5\sqrt{5\dots}}}_{=y}} \Rightarrow y = \sqrt{5y}$$

To solve the equation above, square both sides:

$$y^2 = 5y$$

We would like to divide both sides of the equation by y. However, we cannot divide by y if it is zero. Hence, we do some casework:

Case I: $y = 0$

$$LHS = 0 = RHS \Rightarrow 0\ is\ a\ solution$$

Case II: $y \neq 0$

Divide both sides by y:

$$\frac{y^2}{y} = \frac{5y}{y} \Rightarrow y = 5$$

Example 1.188

A. $y = \sqrt{2\sqrt{2\sqrt{2\dots}}}$
B. $y = \sqrt{7\sqrt{7\sqrt{7\dots}}}$

C. Fourth Degree Equations

D. Higher Degree Equations

Example 1.189

E. One Term is the Square of the Other

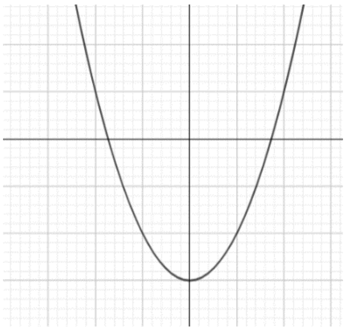
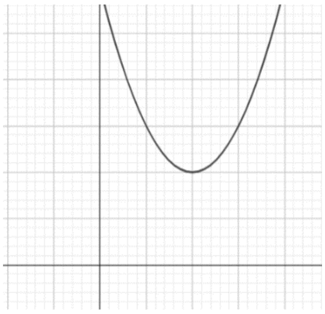
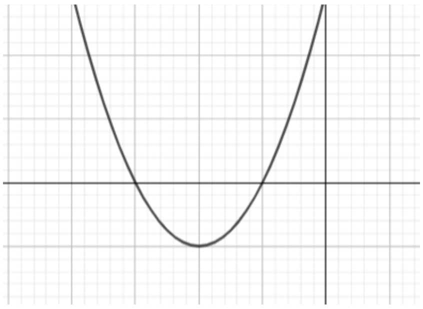
Example 1.190

2. Graphing

2.1 Graphing Basics

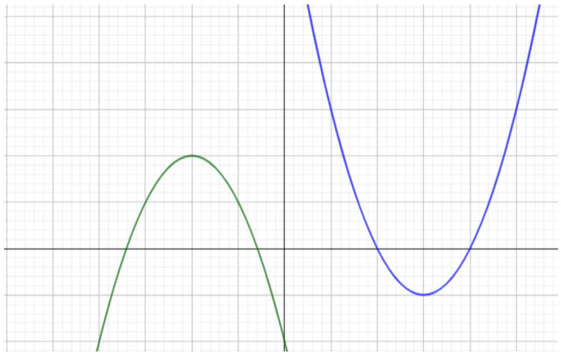
A. Graph of a Quadratic

The graph of a quadratic has an important shape. Some graphs of quadratics are shown below:

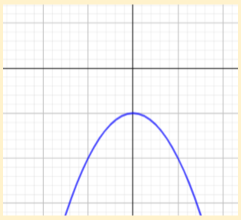
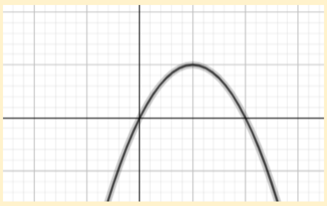
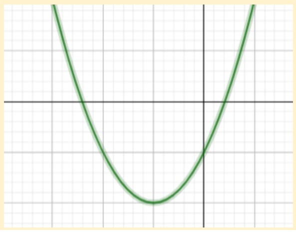
2.1: Upward Parabola and Downward Parabola

- A parabola which "opens up" is an upward parabola. In the diagram alongside, the **blue graph** is that of an upward parabola.
- A parabola which "opens down" is a downward parabola. In the diagram alongside, the **green graph** is that of a downward parabola.



Example 2.2

Identify each parabola in the diagram below as an upward or a downward parabola.

$$\text{Blue Graph: Downward}$$
$$\text{Black Graph: Downward}$$
$$\text{Green Graph: Upward}$$

2.3: Different Forms of a Quadratic

For $a \neq 0$:

$$Standard\ Form: y = ax^2 + bx + c$$
$$Factored\ Form: y = a(x - \alpha)(x - \beta)$$
$$Vertex\ Form: y = a(x - h)^2 + k$$

2.4: Upward and Downward Parabolas

All three forms of a quadratic have an a at the beginning of the expression. If

$$a > 0 \Rightarrow Upward\ Parabola$$
$$a < 0 \Rightarrow Downward\ Parabola$$

Example 2.5

Basics

Identify the form of the quadratic and whether it represents an upward or a downward parabola.

A. $y = 3x^2 + 5x - 9$
B. $y = -3(x-2)(x-4)$
C. $y = -5\left(x - \frac{1}{2}\right)^2 - \frac{2}{3}$

Applications

Convert the given quadratic into one of the three forms of a quadratic. Then determine whether it is an upward or a downward parabola.

D. $y = 5x - 3x^2 - 9$
E. $y = -\frac{2}{3}(7-x)(x-4)$

Part A

$$Standard\ Form \Rightarrow a = 3 \Rightarrow Upward\ Parabola$$

Part B

$$Factored\ Form \Rightarrow a = -3 \Rightarrow Downward\ Parabola$$

Part C

$$Vertex\ Form \Rightarrow a = -5 \Rightarrow Downward\ Parabola$$

Part D

The given parabola is not in standard form, so first convert it to standard form:

$$y = -3x^2 + 5x - 9$$
$$a = -3 \Rightarrow Downward\ Parabola$$

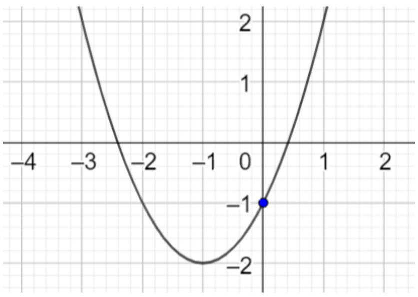
Part E

Factor -1 from the first parentheses to make the coefficient of x positive:

$$y = \frac{2}{3}(x-7)(x-4)$$
$$a = \frac{2}{3} \Rightarrow Upward\ Parabola$$

2.6: y intercept of a Quadratic

y intercept of a quadratic is the point where the quadratic intersects the y axis.



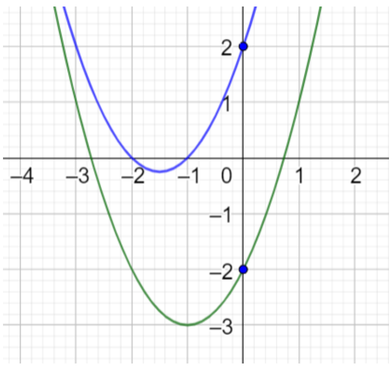
- At the y-intercept, the x coordinate is zero.
- Hence, to find the y −intercept of a quadratic, substitute $x = 0$ in the equation of the quadratic.
-

For example, in the quadratic graphed, the y-intercept is:

$$(0, -1)$$

Example 2.7

Identify the y-intercept for each parabola as specified below.



A. For the parabolas drawn in the diagram alongside.
B. $(Standard\ Form)\ y = 2x^2 + 6x - 4$
C. $(Standard\ Form)\ y = ax^2 + bx - c$
D. $(Factored\ Form)\ y = 3(x-2)\left(x - \frac{1}{2}\right)$
E. $(Factored\ Form)\ y = 2(x+4)(x-2)$
F. $(Vertex\ Form)\ y = 2(x-3)^2 + 4$
G. $(Vertex\ Form)\ y = 2\left(x - \frac{1}{2}\right)^2 + \frac{1}{3}$

Part A

We check the graph and write the coordinates of the y-intercept.

$$Blue = (0,2)$$
$$Green = (0,-2)$$

Part B

For this (and succeeding parts), we substitute $x = 0$ in the equation of the quadratic:

$$(0,-4)$$

Part C

$$(0,-c)$$

Part D

$$y = 3(-2)\left(-\frac{1}{2}\right) = 3$$
$$(0,3)$$

Part E

$$y = 2(4)(-2) = -16$$
$$(0,-16)$$

Part F

$$y = 2(-3)^2 + 4 = 22$$
$$(0,22)$$

Part G

$$y = 2\left(-\frac{1}{2}\right)^2 + \frac{1}{3} = \frac{5}{6}$$
$$\left(0,\frac{5}{6}\right)$$

2.8: x intercept of a Quadratic

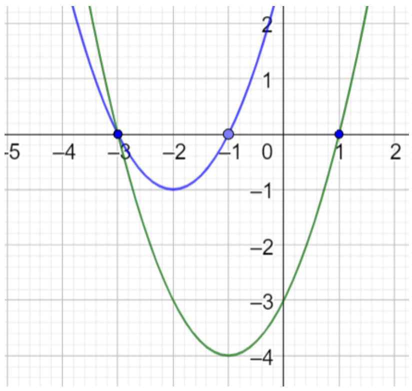
x intercept of a quadratic is the point where the quadratic intersects the x axis.

- At the x-intercept, the y coordinate is zero. Hence, in order to find the x intercept, you will substitute $y = 0$ in the equation.
- In other words, the x-intercepts are also the *roots* of the quadratic equation.
- If you look at this from the point of view of functions, the x intercepts are called the *zeroes* of the function.

Example 2.9

Identify the x-intercept(s) for each parabola as specified below.

A. For the parabolas drawn in the diagram alongside, find the sum of the distinct roots of the two parabolas.
B. $(Standard\ Form)\ y = x^2 + 5x + 6$
C. $(Standard\ Form)\ y = x^2 - x - 56$
D. $(Factored\ Form)\ y = 3(x-2)\left(x - \frac{1}{2}\right)$
E. $(Factored\ Form)\ y = 2(x+4)(x-2)$
F. $(Vertex\ Form)\ y = 2(x-3)^2 - 8$
G. $(Vertex\ Form)\ y = 2\left(x - \frac{1}{2}\right)^2 - \frac{1}{3}$



Part A

For roots, we only want the x coordinate of the x intercept:

$$\{-3,-1,1\}$$
$$-3 - 1 + 1 = -3$$

Part B

$$x^2 + 5x + 6 = 0$$
$$Product = 6 = (2)(3), Sum = 5$$
$$(x+2)(x+3) = 0$$

The roots are:

$$x \in \{-2,-3\}$$

The x intercepts are:

$$(-2,0),(-3,0)$$

Part C

$$x^2 - x - 56 = 0$$
$$Product = -56 = (-8)(7), Sum = -1$$
$$(x-8)(x+7) = 0$$
$$(8,0)(-7,0)$$

Part D

$$3(x-2)\left(x - \frac{1}{2}\right) = 0$$
$$x - 2 = 0 \Rightarrow x = 2$$
$$x - \frac{1}{2} = 0 \Rightarrow x = \frac{1}{2}$$
$$(2,0)\left(\frac{1}{2},0\right)$$

Part E

$$2(x+4)(x-2) = 0$$
$$(-4,0)(2,0)$$

Part F

$$2(x-3)^2-8=0$$
$$2(x-3)^2=8$$
$$(x-3)^2=4$$
$$x-3=\pm 2$$
$$x=\pm 2+3$$
$$x\in\{1,5\}$$
$$(1,0),(5,0)$$

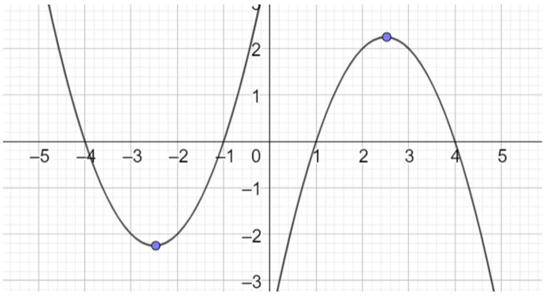
Part G

$$2\left(x-\frac{1}{2}\right)^2-\frac{1}{3}=0$$

$$\left(x-\frac{1}{2}\right)^2=\frac{1}{6}$$
$$x-\frac{1}{2}=\pm\sqrt{\frac{1}{6}}$$
$$x=\pm\sqrt{\frac{1}{6}}+\frac{1}{2}$$
$$\left(-\sqrt{\frac{1}{6}}+\frac{1}{2},0\right),\left(\sqrt{\frac{1}{6}}+\frac{1}{2},0\right)$$

2.10: Vertex of a Quadratic

The topmost or the bottom most part of a quadratic is called its vertex.



- For an upward parabola, the vertex is the minimum value achieved by y.
- For a downward parabola, the vertex is the maximum value achieved by y.

Example 2.11

Questions from Graph

2.12: Vertex of a Quadratic

The vertex of a quadratic in vertex form $y=a(x-h)^2+k$ is

$$(h,k)$$

- Note that the minus sign is a part of the formula. If you get an expression with a plus sign, the minus sign must be created.

Example 2.13

A. $y=3(x-5)^2+1$
B. $y=3(x+5)^2+1$
C. $y=2\left(x-\frac{1}{2}\right)^2+\frac{2}{3}$
D. $y=\frac{1}{3}(x+3)^2-\frac{1}{2}$

Part A

$$(5,1)$$

Part B

$$y=3[x-(-5)]^2+1$$
$$(-5,1)$$

Part C

$$\left(\frac{1}{2},\frac{2}{3}\right)$$

Part D

$$y = \frac{1}{3}\left(x - (-3)\right)^2 - \frac{1}{2}$$
$$\left(-3, -\frac{1}{2}\right)$$

2.2 Graphing Standard Form

A. Standard Form of a Quadratic

$$\underbrace{ax^2 + bx + c = 0}_{\textit{Standard Form}} \Rightarrow \underbrace{bx = -c - ax^2}_{\textit{Not in Standard Form}}$$

Quadratics have a standard form which satisfies the following conditions

- RHS (Right-Hand Side) is zero
- The quadratic is written in decreasing powers of x.

It is not necessary that a quadratic should be written only as
$$ax^2 + bx + c = 0$$
It can also be written as
$$px^2 + qx + r = 0$$
However, the expression $ax^2 + bx + c = 0$ is far more common.

B. Roots of a Quadratic

The values of x which satisfy the equation are called the roots of the quadratic equation.
The values of x at which the function becomes zero, are called the zeroes of the quadratic function.

$$\textit{Roots of Quadratic Equation} \Leftrightarrow \textit{Zeros of Quadratic Function}$$

C. Review

Example 2.14

State True or False. If False, give a counter-example, and correct the statement.

A. The roots of a quadratic cannot both be zero.
B. The leading coefficient of a quadratic must not be zero.
C. The roots of a quadratic equation are the same as the zeroes of the corresponding quadratic function.
D. A quadratic equation of the form $ax^2 + bx + c, a \neq 0$ will either be an upward parabola, or a downward parabola, or a parabola pointing right, or a parabola pointing left.

Statement A:
$$x^2 = 0 \Rightarrow \underbrace{x = 0}_{\textit{Repeated Roots}} \Rightarrow \textit{Both roots can be zero}$$

Statement B: Correct
If the leading coefficient is zero, then the equation reduces to
$$bx + c = 0$$
Which is a line, not a parabola

Statement C: Correct

Statement D: Incorrect
It will only be an upward or a downward parabola. To get a parabola, which points right or left, we need invert the role of x and y to get:
$$x = ay^2 + by + c$$

D. $y = x^2$

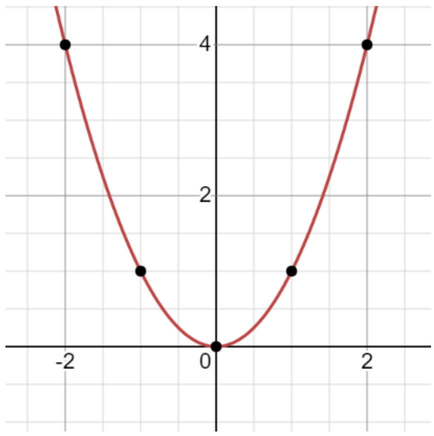
Example 2.15

Graph

$$y = x^2$$

Plot a table of values starting from $x = 0$, and going in both directions.

x	-2	-1	0	1	2
y	4	1	0	2	4



- x^2 can never be negative. This is reflected in the graph as well, where the lowest point of the graph is 0.
- The lowest point of the graph is called the vertex.
- The graph is symmetric about the y-axis. Hence, this is the graph of an even function.
- The graph is curved, not straight.
- The curvature is less near the origin, and increases as you move away from the origin. (One way to measure curvature is slope).
- This is the graph of a parabola. Parabolas are very important in Maths.

2.16: Axis of Symmetry

2.17: Vertex

2.18: Scale Factor

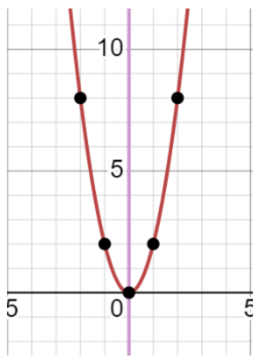
We now introduce a scale factor. A scale factor is a value by which all $y - values$ on the graph are multiplied.

- If the scale factor is more than 1, the graph becomes steeper.
- If the scale factor is between zero and 1, the graph becomes less steep.

Example 2.19

Graph

$$y = 2x^2$$



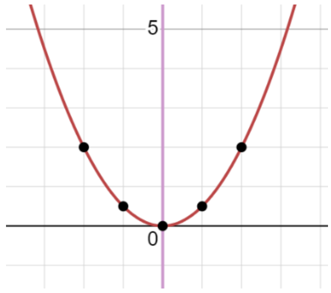
Plot a table of values starting from $x = 0$, and going in both directions.

x	-2	-1	0	1	2
y	8	2	0	4	8

Example 2.20

Graph

$$y = \frac{1}{2}x^2$$



Plot a table of values starting from $x = 0$, and going in both directions.

x	-2	-1	0	1	2
y	8	2	0	1	8

2.21: Vertical Stretch and Shrink

A scale factor greater than 1 is a vertical stretch.
A scale factor between zero and 1 is a vertical shrink.

Example 2.22

Identify the scale factor in each parabola below. Decide whether it is a vertical stretch or shrink.

A. $y = 3x^2$
B. $y = \frac{3}{2}x^2$
C. $y = \frac{1}{3}x^2$

2.23: Scale Factor: $y = ax^2, a < 0$

If $a < 0$, the graph will "flip". In other words, it will get reflected across the x-axis.

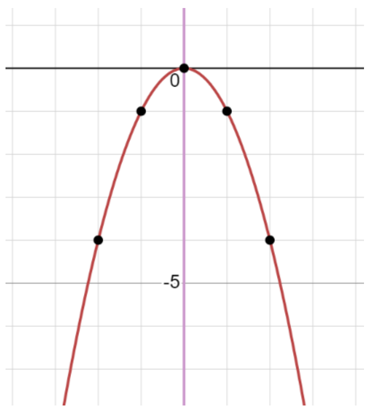
Here as well, it's meaningful to consider two cases:

- $-1 < a < 0$: The graph becomes less steep
- $a < -1$: The graph becomes more steep

Example 2.24

Graph

$$y = -x^2$$



Plot a table of values starting from $x = 0$, and going in both directions.

x	-2	-1	0	1	2
y	-4	-1	0	-2	-4

Example 2.25

Graph

$$y = -\frac{1}{2}x^2$$

Plot a table of values starting from $x = 0$, and going in both directions.

x	-2	-1	0	1	2
y					

Example 2.26

Graph

$$y = -2x^2$$

Plot a table of values starting from $x = 0$, and going in both directions.

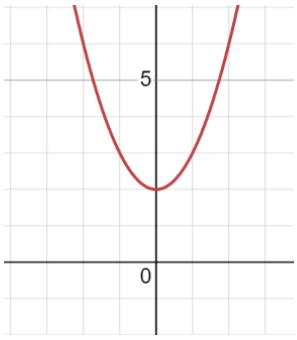
x	-2	-1	0	1	2
y					

2.27: Vertical Shift: Moving Up and Down

Example 2.28

Graph

$$y = x^2 + 2$$

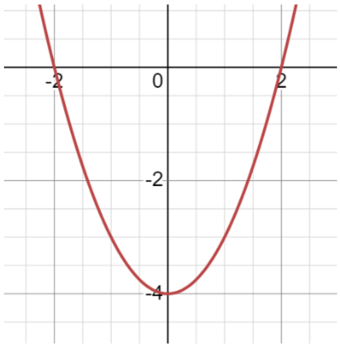


This is the same as the graph of $y = x^2$ moved up by 2 units.

Example 2.29

Graph

$$y = x^2 - 4$$



We need to identify the x-intercepts. The x-intercepts are where the graph cuts the x-axis.

Recall that any point on the x-axis has y value zero. Hence, to find the x-intercepts, we substitute $y = 0$ in the equation of the graph, which is $y = x^2 - 4$

$$0 = x^2 - 4 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

E. $y = ax^2 + bx + c$

Vertex

The tip of the graph of a quadratic function is called its vertex. Vertices are important because they are examples of turning points, a place where a function changes direction. A vertex represent either the maximum or the minimum y-value of the parabola across all x-values

2.30: Standard Form

$$y = ax^2 + bx + c, \qquad a \neq 0$$

The form above is called the standard form of a quadratic.
The condition $a \neq 0$ is important so that the expression does not become linear (as opposed to quadratic)

2.31: Features of Standard Form

For a quadratic

$$y = ax^2 + bx + c, \qquad a \neq 0$$

- The parabola opens up when $a > 0$. The parabola opens down when $a < 0$. If $a = 0$, then it won't be a parabola: it will be a line.

- The y-intercept is given by c.
- The axis of symmetry is given by $x = -\frac{b}{2a}$.
- The x coordinate of the vertex is given by $-\frac{b}{2a}$. The y coordinate of the vertex is given by substituting $x = -\frac{b}{2a}$ in the equation of the quadratic.
- The x-intercepts/roots are found by substituting $y = 0$, converting to factored form $0 = (x - \alpha)(x - \beta)$ and solving for α and β.

Step by Step Procedure 2.32

Consider the quadratic

$$y = x^2 + 2x - 8$$

A. Does it open up or down?
B. What is the y-intercept?
C. What is the axis of symmetry?
D. Use the axis of symmetry and the y-intercept to find one more point on the graph by reflecting the y-intercept across the axis of symmetry.
E. What is the vertex?
F. What are the x-intercept/roots?
G. Plot the y-intercept, the vertex, the roots, and the additional point that you found on graph paper.
H. Connect the points you plotted in the shape of a parabola to complete your sketch of the parabola.

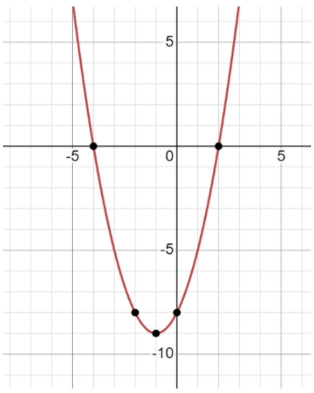
$$Opens\ Up$$
$$y - intercept: -8$$
$$Axis\ of\ Symmetry:\ -1$$

Find the coordinates of the vertex:

$$x\ coordinate = -1$$
$$y = x^2 + 2x - 8 = 1 - 2 - 8 = -9$$
$$Vertex = (-1, -9)$$

Find the x-intercepts/roots:

$$(x + 4)(x - 2) = 0 \Rightarrow x \in \{-4, 2\}$$



Example 2.33

Graph the quadratic $y = x^2 + 5x + 6$ using the same process as used you earlier.

Part A: Opens up or down

$$Opens\ Up$$

Part B: y-intercept

$$y - intercept = (0,6)$$

Part C: Axis of Symmetry

$$Axis\ of\ Symmetry = -\frac{b}{2a} = -\frac{5}{2}$$

Part D: One More Point on the Graph

Reflecting $(0,6)$ across the axis of symmetry $x = -\frac{5}{2}$

$$Does\ not\ change\ the\ y\ coordinate$$
$$Gives\ x\ coordinate\ double\ of\ -\frac{5}{2} = -\frac{10}{2} = -5$$

$$One\ more\ point = (-5,2)$$

Part E: Vertex

$$Vertex: x\ coordinate = -\frac{5}{2}$$

To find the y coordinate of the vertex, substitute the x coordinate of the vertex in the graph:

$$y = x^2 + 5x + 6 = \left(-\frac{5}{2}\right)^2 + 5\left(-\frac{5}{2}\right) + 6 = \frac{25}{4} - \frac{25}{2} + 6 = -\frac{25}{4} + 6 = -6\frac{1}{4} + 6 = -\frac{1}{4}$$

Part F: x-intercept/Roots

$$x^2 + 5x + 6 = 0 \Rightarrow (x+2)(x+3) = 0 \Rightarrow x \in \{-3, -2\}$$

Example 2.34

Graph the quadratic $y = x^2 - 7x + 12$ using the same process as used you earlier.

F. Roots versus Zeroes

For a quadratic function:

- zeros represent the x-values where the function intersects the x-axis, $y = 0$
- zeroes are also the x-intercepts

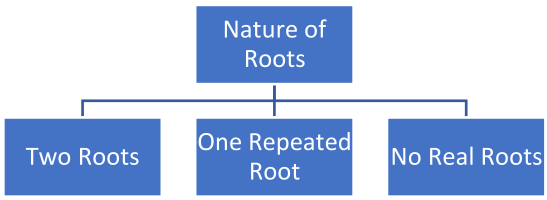
For a quadratic equation

- the roots represent the values where the expression is zero

$$Equation: \underbrace{x^2 - 9 = 0 \Leftrightarrow x = \pm 3}_{Roots=\pm 3}, \qquad Function: \underbrace{y = f(x) = x^2 - 9}_{Zeros\ are \pm 3}$$

For practical purposes, the concepts of roots, zeroes, and x-intercepts are interchangeable. This is very important to remember when solving questions.

Parabolas can be of three types with respect to their zeroes/roots/x-intercepts:



- They have two zeros – meaning they cut the x-axis in two places.
  - ✓ They will always cross the x-axis in this case.
- They have only one zero. They will touch the x-axis but not cross it.
  - ✓ In this case, the vertex will touch the x-axis.
  - ✓ The zero/root is called a repeated zero/root.
- They have no zero. They will neither cross the x-axis, nor touch it. This can be of two types:
  - ✓ If it is pointing upwards, it will always be above the x-axis.
  - ✓ If it is pointing downwards, it will always be below the x-axis.

$$Two\ Solutions = Two\ Roots: x^2 - 4 = 0 \Rightarrow x = \pm 2$$
$$One\ Solutions = Two\ Repeated\ Roots: x^2 = 0 \Rightarrow x = 0$$
$$No\ Real\ Solutions: x^2 + 4 = 0 \Rightarrow x^2 = -4 \Rightarrow x = \sqrt{-4}$$

G. Real Life Scenarios

Vertices have real life applications:

- **Physics:** Path of a ball thrown upwards is a parabola. The maximum height of the ball will be given by the y-coordinate of the vertex.
- **Economics:** Economic variables are sometimes modelled using quadratic functions. In such cases, we are

often interested in the maximum/minimum values, and such values will be given by the vertex.

Example 2.35

When will a ball hit the ground: intercepts
When will a ball reach maximum height: Vertex
What is the time a ball will remain in the air: Distance between intercepts

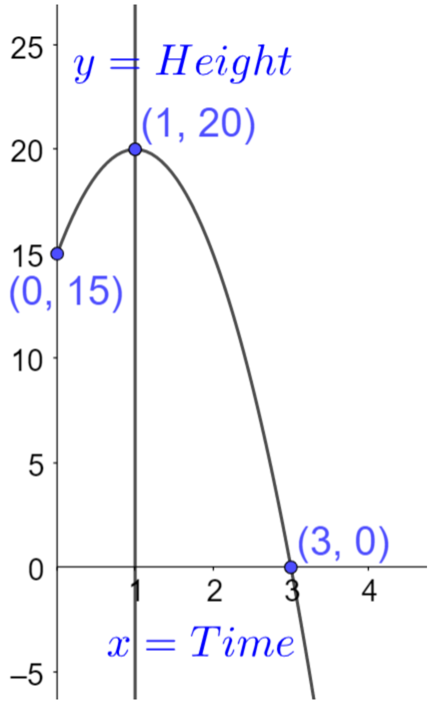
Example 2.36

A stone is thrown from a cliff. The height of the stone in meters, x seconds after being thrown, is given by

$$h(x) = -5x^2 + 10x + 15$$

Graph the given quadratic function. Then answer each question below first using the graph, and then verify it algebraically.

A. What is on the x-axis? What is on the y-axis?
B. What is the height when the ball is first thrown?
C. What is the time when the ball hits the ground?
D. At what time does the ball reach the maximum height?
E. What is the maximum height that the ball reaches?



Part A

$$x - axis = Time$$
$$y = axis = Height$$

Part B

When the ball is thrown, it is time $x = 0$. Hence, we want to find the value at this time. This is given by:

$$y - intercept = 15$$

Algebraically, substitute $x = 0$ in the function:

$$h(0) = -5(0^2) + 10(0) + 15 = 15$$

Part C

From the graph, at time $x = 3$, the ball hits the ground. This is also the $x - intercept$ or the root of the graph.

Algebraically, when the ball hits the ground, the height of the ball is 0. Hence, substitute $h(x) = 0$

$$h(x) = -5x^2 + 10x + 15 = 0$$

Separate the last two parts as an equation:

$$-5x^2 + 10x + 15 = 0$$
$$x^2 - 2x + 3 = 0$$
$$(x - 3)(x + 1) = 0$$
$$x \in \{-1, +3\} \Rightarrow Reject\ Negative\ Value \Rightarrow x = 3$$

Part D

From the graph, the ball reaches its maximum height at

$$x = 1$$

Algebraically, we know that $h(x) = -5x^2 + 10x + 15$ is a downward parabola. The ball will reach its maximum height at the x coordinate of the vertex. This is given, for a quadratic in standard form, by

$$x = -\frac{b}{2a} = \frac{-10}{2 \times (-5)} = 1$$

Part E

From the graph, the maximum height that the ball reaches is

$$20\ meters$$

Algebraically, we want the y coordinate of the vertex. Hence, substitute the x coordinate of the vertex, which is $x = 1$ into the equation of the parabola:

$$h(1) = -5(1^2) + 10(1) + 15 = 20$$

Example 2.37

TechX is a company that makes widgets. The profit p for the company, in millions of dollars, if it makes (and sells) w million widgets is given by:

$$p(w) = -w^2 + 7w - 12$$

A. Graph the quadratic function. What is on the x-axis? What is the on the y-axis.
B. What is the profit if the company does not make any widgets?
C. What is the maximum profit that the company can make?
D. What is the number of widgets that the company is making if it is breaking even? (Breaking even means there is not profit, and no loss).
E. What is the number of widgets at which the company makes its maximum profit?

Part A

Profits on the y-axis
No. of Widgets on x-axis.

Part B

When the company does not make any widgets:

$$w = 0 \Rightarrow p(w) = -0^2 + 7(0) - 12 = -12$$

Part C

The maximum profit that the company can make is the y coordinate of the vertex.
We first find the x coordinate of the vertex:

$$-\frac{b}{2a} = -\frac{7}{2(-1)} = \frac{7}{2}$$

Then, we substitute the x coordinate of the vertex into the function, to find the y coordinate of the vertex:

$$p\left(\frac{7}{2}\right) = -\left(\frac{7}{2}\right)^2 + 7\left(\frac{7}{2}\right) - 12 = -\frac{49}{4} + \frac{49}{2} - 12 = -\frac{49}{4} + \frac{98}{4} - \frac{48}{4} = \frac{1}{4}$$

Part D

When you break even

$$Profit = 0 \Rightarrow p(w) = 0 \Rightarrow 0 = -w^2 + 7w - 12 \Rightarrow w \in \{3,4\}$$

Part E

We want the x coordinate of the vertex, which we already found to be:

$$\frac{7}{2}$$

Example 2.38

$$\frac{1}{2}(y+2) = (x-3)^2$$

$$(y+2) = 2(x^2 - 6x + 9)$$
$$y = 2x^2 - 12x + 16$$

Find the vertex:

$$Vertex_x = \frac{-b}{2a} = \frac{12}{2(2)} = 3$$
$$Vertex_y = y = 2(3)^2 - 12(3) + 16 = 18 - 36 + 16 = -2$$

Axis of Symmetry

$$x = 3$$

Find the roots:

$$2x^2 - 12x + 16 = 0$$
$$x^2 - 6x + 8 = 0$$
$$(x-4)(x-2) = 0$$
$$x \in \{2,4\}$$

Find the y-intercept:

$$y = 16$$

H. Graphical Transformations

We look at various changes to the graph of the "square" function. These will let us graph a parabola of any shape, and location.

2.39: Parabola: Base Case

$$y = x^2$$

$$Vertex = (0,0)$$

2.40: Horizontal Shift

$$y = (x-h)^2$$
$$h > 0 \Rightarrow Shift\ Left$$
$$h < 0 \Rightarrow Shift\ Right$$

Example 2.41

Graph the equations below.

A. $y = (x+3)^2$

2.42: Vertical Stretch/Shrink

$$y = ax^2$$
$$Vertical\ Stretch: a > 1$$
$$Vertical\ Shrink: 0 < a < 1$$
$$Vertical\ Flip: a < 0$$

$$Vertex = (0,0)$$

2.43: Vertical Shift

$$y = x^2 + k$$
$$k > 0 \Rightarrow Moved\ up\ k\ units$$
$$k < 0 \Rightarrow Moved\ down\ k\ units$$

2.3 Graphing Vertex Form

A. Vertex Form

We learnt about three different transformations:

- Vertical Stretch/Shrink
- Horizontal Shift
- Vertical Shift

We can combine these three into a single equation, that will let us graph any parabola.

2.44: Vertex Form

A quadratic in vertex form has the format:

$$y = a(x - h)^2 + k$$

Some properties of the quadratic are:

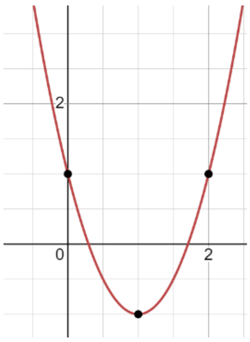
$$Vertex = (h, k)$$
$$Axis\ of\ Symmetry\ is\ x = h$$
$$a > 1 \Rightarrow Vertical\ Stretch, 0 < a < 1 \Rightarrow Vertical\ Shrink$$
$$a > 0 \Rightarrow Upward\ Parabola, a < 0 \Rightarrow Downward\ Parabola$$
$$h \rightarrow Horizontal\ Shift$$
$$k \rightarrow Vertical\ Shift$$

$$f(x) = -3(x + 1)^2 + 5$$
$$f(x) = -3(x - (-1)\,)^2 + 5$$

Example 2.45

Graph:

$$y = 2(x - 1)^2 - 1$$



Identify the vertex:

$$(h, k) = (1, -1)$$

Plot a few more points near the vertex:

$$x = 0 \Rightarrow y = 2(0 - 1)^2 - 1 = 2 - 1 = 1 \Rightarrow (0,1)$$

(Note that this point also is the y-intercept)

$$x = 2 \Rightarrow y = 2(2 - 1)^2 - 1 = 2 - 1 = 1 \Rightarrow (2,1)$$

B. Basics

2.46: Finding the y-intercept

At the y-intercept, the value of the x-coordinate is zero. Hence, to find the y-intercept,

$$Substitute\ x = 0$$

Example 2.47

Identify the y-intercept in the following equations:

A. $y = 2(x - 3)^2 + 5$
B. $y = 3(x - 4)^2 + 2$
C. $y = 3(x + 4)^2 - 2$

Substitute $x = 0$:

$$y = 3(x-4)^2 + 2 = 3(-4)^2 + 2 = 3 \times 16 + 2 = 48 + 2 = 50$$

Substitute $x = 0$:

$$y = 3(x+4)^2 - 2 = 3(4)^2 - 2 = 3(4)^2 + 2 = 46$$

C. Vertex

Example 2.48

Identify the vertex in the following equations:

A. $y = 3(x-4)^2 + 2$
B. $y = -2(x-5) + 20$
C. $y = 5(x+4)^2 - 2$
D. $y = \frac{1}{2}\left(x - \frac{2}{3}\right)^2 - \frac{17}{5}$
E. $y = -\frac{2}{5}\left(x + \frac{5}{7}\right)^2 + \frac{1}{9}$

$$y = a(x-h)^2 + k$$

Part A

$$y = 3(x - \mathbf{4})^2 + \mathbf{2}$$
$$x = \mathbf{4}, y = \mathbf{2}$$

Part B

$$y = -2(x-5) + 20$$

Part C

$$y = 3[x - (\mathbf{-4})]^2 + (\mathbf{-2})$$
$$x = \mathbf{-4}, y = \mathbf{-2}$$

D. x-intercepts

E. Putting it all together

Example 2.49:

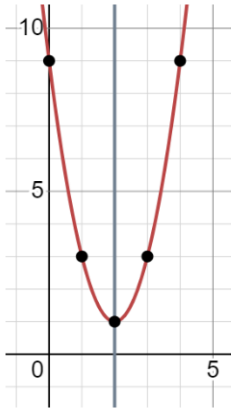
Consider the quadratic

$$y = 2(x-2)^2 + 1$$

A. Is the parabola an upward parabola or a downward parabola?
B. Identify the vertex, and the axis of symmetry.
C. Identify the intercepts
D. Identify few more points on the graph
E. Reflect the y-intercept and the other non-vertex points across the axis of symmetry.

Part A

$$a > 0 \Rightarrow Upward\ Parabola$$



Part A

$$Vertex = (2,1)$$
$$Axis\ of\ Symmetry: x = 2$$

Part B

To find the y-intercept, substitute $x = 0$:

$$y = 2(2)^2 + 1 = 8 + 1 = 9$$

To find the x-intercept, substitute $y = 0$:

$$0 = 2(x-2)^2 + 1 \Rightarrow -\frac{1}{2} = (x-2)^2$$

Take square roots both sides:

$$\pm\sqrt{-\frac{1}{2}} = x - 2$$

We get negative numbers inside a square root, which leads to complex solutions. Hence, there are no x-intercepts.

Part C

Substitute $x = 1$:

$$y = 2(1-2)^2 + 1 = 2 + 1 = 3$$

Substitute $x = -1$ (leads to a very large number, not a good idea!!):

$$y = 2(-1-2)^2 + 1 = 2 \times 9 + 1 = 19$$

Part D

$$(1,3) \to (3,3)$$
$$(0,9) \to (4,9)$$

2.50: Graphing in Vertex Form

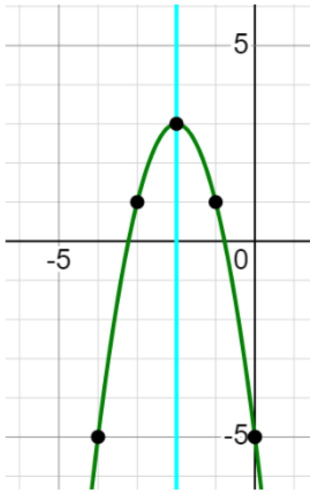
- Find the vertex
- Find the axis of symmetry
- Find the y-coordinate of a few other x-coordinates, and then reflect them.

Example 2.51:

Consider the quadratic

$$y = -2(x+2)^2 + 3$$

A. Is the parabola an upward parabola or a downward parabola?
B. Identify the vertex, and the axis of symmetry.
C. Identify the intercepts
D. Identify few more points on the graph
E. Reflect the y-intercept and the other non-vertex points across the axis of symmetry.



Part A

$$a < 0 \Rightarrow Downward\ Parabola$$

Part B

Vertex form has a minus sign in front of the x coordinate, so we need to create a minus here:

$$y = -2[x - (-\mathbf{2})]^2 + \mathbf{3} \Rightarrow \text{Vertex} = (x, y) = (-2, 3)$$

$$Vertex = (-2,3)$$

$$Axis\ of\ Symmetry: x = -2$$

Part C

To find the y-intercept, substitute $x = 0$:

$$y = -2(0+2)^2 + 3 = -8 + 3 = -5$$

To find the x-intercept, substitute $y = 0$:

$$0 = -2(x+2)^2 + 3 \Rightarrow \frac{3}{2} = (x+2)^2$$

Take square roots:

$$\pm\sqrt{\frac{3}{2}} = x + 2 \Rightarrow x = -2 \pm \sqrt{\frac{3}{2}} \Rightarrow x \in \left\{-2 - \sqrt{\frac{3}{2}}, -2 + \sqrt{\frac{3}{2}}\right\}$$

Part D

Substitute $x = -1$:

$$y = -2(-1+2)^2 + 3 = 1 \Rightarrow (-1,1)$$

Part E

$$(-1,1) \to (-3,1)$$

$$(0,-5) \to (-4,-5)$$

2.52: Graphing a Quadratic in Standard Form

- Find the Roots. Those are also the x-intercepts
- x coordinate of vertex is average of x-intercepts
- y coordinate of vertex is f(x) evaluated at x coordinate of vertex

The constant term gives you the y-intercept of the quadratic.
y = ax^2 + bx +c
Substitute x = 0
y = a(0) + b(0) +c
y = c

A quadratic that passes through the origin has constant term zero.

F. Standard Form

The graph of a parabola is symmetrical about its vertex. Therefore, the *roots* are equidistant from the vertex.

$$Vertex_x = Average(x\ coordinates\ of\ roots)$$

$$= \frac{1}{2}\left(\underbrace{\frac{-b + \sqrt{b^2 - 4ac}}{2a}}_{1st\ Root} + \underbrace{\frac{-b - \sqrt{b^2 - 4ac}}{2a}}_{2nd\ Root}\right) = \frac{1}{2}\left(\frac{-2b}{2a}\right) = \frac{1}{2}\left(\frac{-b}{a}\right) = -\frac{b}{2a}$$

Example 2.53

Graph

$$x^2 + 5x + 6$$

x-intercept

The solutions to a quadratic equation (essentially, the roots of a quadratic equation) are also its x-intercepts.

$$x^2 + 5x + 6 = 0 \Rightarrow (x+2)(x+3) = 0 \Rightarrow x \in \{-2, -3\} \Rightarrow x - intercepts = \{-2, -3\}$$

Finding the Vertex: x coordinate

$$Using\ the\ Formula: b = 5, a = 1 \Rightarrow -\frac{b}{2a} = -\frac{5}{2} = -2.5$$

$$Average\ of\ Roots:\ x\ coordinate = \frac{x_1 + x_2}{2} = \frac{-2-3}{2} = -\frac{5}{2} = -2.5$$

Finding the Vertex: y coordinate

The y coordinate of the vertex can be found by substituting the x coordinate of the vertex in the equation of the parabola:

$$y = x^2 + 5x + 6 = (-2.5)^2 + 5(-2.5) + 6 = 6.25 - 12.5 + 6 = -0.25$$

Example 2.54

Graph

$$y = \frac{1}{2}x^2 + 4x + 3$$

Vertex

x coordinate of the vertex

$$= -\frac{b}{2a} = -\frac{4}{2\left(\frac{1}{2}\right)} = -4$$

To find the y coordinate, substitute $x = -4$ in the equation:

$$y = \frac{1}{2}x^2 + 4x + 3 = \frac{1}{2}(4^2) + 4(-4) + 3 = 8 - 16 + 3 = -5$$

y intercept

To find the y intercept, substitute $x = 0$:

$$y = \frac{1}{2}x^2 + 4x + 3 = 3$$

2.4 Graphing Factored Form

A. Factored Form

2.55: Factored Form

A quadratic in vertex form has the format:

$$y = a(x-\alpha)(x-\beta)$$

Some properties of the quadratic are:

$$Roots\ are\ \{\alpha, \beta\}$$

$$Vertex_x = \frac{\alpha+\beta}{2}, \qquad Vertex_y: Substitute\ \frac{\alpha+\beta}{2}\ into\ the\ equation$$

$$Axis\ of\ Symmetry\ is\ x =$$

$$a > 0 \Rightarrow Upward\ Parabola, a < 0 \Rightarrow Downward\ Parabola$$

$$h \rightarrow Horizontal\ Shift$$

$$k \to Vertical\ Shift$$

Example 2.56: Using the Factored Form

Find the vertex of $y = (x+4)(x-2)$ by using the factored form.

$$y = (x+4)(x-2) \Rightarrow x \in \{-4, 2\}$$
$$Vertex_x = \frac{-4+2}{2} = -\frac{2}{2} = -1$$
$$Vertex_y = y = \underbrace{(x+4)(x-2)}_{x=-1} = (-1+4)(-1-2) = (3)(-3) = -9$$

Example 2.57: Converting it to Vertex Form

Find the vertex of $y = (x+4)(x-2)$ by converting the equation into vertex form

Expand the RHS, and then complete the square:

$$(x+4)(x-2) = x^2 + 2x - 8 = x^2 + 2x + \mathbf{1} - \mathbf{1} - 8 = (x+1)^2 - 9 \Rightarrow Vertex = (-1, -9)$$

B. Review

Example 2.58

For each quadratic function below, find the intercepts, the coordinates of the vertex, and the axis of symmetry.

Across all forms,

- Substitute $x = 0$, to find the y −intercept
- Substitute $y = 0$, to find the x −intercept
- The axis of symmetry is $x = h$, where h is the x coordinate of the vertex.

Standard Form $(y = ax^2 + bx + c)$

To find the x coordinate of the vertex, calculate:

$$x = -\frac{b}{2a}$$

To find the y coordinate of the vertex, substitute the x coordinate of the vertex in the equation.

A. $y = x^2 - 4x - 21$
B. $y = 2x^2 + 5x + 2$

Vertex Form $(y = a(x-h)^2 + k)$

The vertex is given by

$$(x, y) = (h, k)$$

C. $y = 2(x-4)^2 - 7$
D. $y = 4\left(x - \frac{2}{3}\right)^2 - \frac{3}{7}$
E. $y = 2(x+2)^2 + 3$

Factored Form

To find the x coordinate of the vertex, calculate:

$$\frac{\alpha+\beta}{2}, where\ \alpha, \beta\ are\ the\ roots$$

To find the y coordinate of the vertex, substitute the x coordinate of the vertex in the equation.

F. $y = 2(x-7)(x+2)$

Part A

To find the y-intercept, substitute $x = 0$:

$$y - intercept = 21$$

To find the x-intercept, substitute $y = 0$:

$$x^2 - 4x - 21 = 0$$
$$(x-7)(x+3) = 0$$
$$x \in \{-3, 7\}$$

Vertex:

$$x = -\frac{b}{2a} = -\frac{-4}{2(1)} = 2$$

Axis of symmetry

$$x = 2$$

Substitute the x value of the vertex in the function to find the y-value:

$$y = f(2) = (2)^2 - 4(2) + 21 = 17$$

Part B

To find the y-intercept, substitute $x = 0$:

$$y - intercept = 2$$

To find the x-intercept, substitute $x = 0$:

$$2x^2 + 5x + 2 = 0$$

$$Sum = 5, Product = 4 \Rightarrow 4 = (1)(4)$$

$$2x^2 + 4x + x + 2 = 0$$

$$2x(x+2) + 1(x+2) = 0$$

$$(x+2)(2x+1) = 0$$

$$x \in \left\{-2, \frac{1}{2}\right\}$$

Vertex:

$$x = -\frac{b}{2a} = -\frac{5}{2(2)} = -\frac{5}{4}$$

Axis of Symmetry

$$x = -\frac{5}{4}$$

Substitute the x value of the vertex in the function to find the y-value:

$$y = f\left(-\frac{5}{4}\right) = 2\left(-\frac{5}{4}\right)^2 + 5\left(-\frac{5}{4}\right) + 2$$

Simplify:

$$= \frac{25}{8} - \frac{50}{8} + \frac{16}{8} = -\frac{9}{8}$$

Part C

$$y = 2(x-4)^2 - 7$$

To find the y-intercept, substitute $x = 0$:

$$y = 2(0-4)^2 - 7 = 32 - 7 = 25$$

To find the x-intercept, substitute $y = 0$:

$$2(x-4)^2 - 7 = 0$$

$$(x-4)^2 = \frac{7}{2}$$

$$(x-4) = \pm\sqrt{\frac{7}{2}}$$

$$x = 4 \pm \sqrt{\frac{7}{2}}$$

The vertex can be identified directly from the equation:

$$y = 2(x-4)^2 - 7$$

$$Vertex = (4, -7)$$

The axis of symmetry:

$$x = 4$$

Part D

$$y = 4\left(x - \frac{2}{3}\right)^2 - \frac{3}{7}$$

To find the y-intercept, substitute $x = 0$:

$$y = 4\left(-\frac{2}{3}\right)^2 - \frac{3}{7} = \frac{16}{9} - \frac{3}{7} = \frac{85}{63}$$

To find the x-intercept, substitute $y = 0$:

$$4\left(x - \frac{2}{3}\right)^2 = \frac{3}{7}$$

$$\left(x - \frac{2}{3}\right)^2 = \frac{3}{7} \times \frac{1}{4} = \frac{3}{28}$$

$$\left(x - \frac{2}{3}\right) = \pm\sqrt{\frac{3}{28}} = \pm\frac{\sqrt{3}}{2\sqrt{7}}$$

$$x = \frac{2}{3} \pm \frac{\sqrt{3}}{2\sqrt{7}}$$

The vertex can be identified directly from the equation:

$$Vertex = \left(\frac{2}{3}, -\frac{3}{7}\right)$$

Part E

$$y = 2(x+2)^2 + 3$$

To find the y-intercept, substitute $x = 0$:

$$y = 11$$

To find the x-intercept, substitute $y = 0$:

$$2(x+2)^2 + 3 = 0$$

$$(x+2)^2 = -\frac{3}{2}$$

$$(x+2) = \sqrt{-\frac{3}{2}}$$

On the RHS, we get the square of a negative number, which has no real solutions, and hence the equation has no x-intercepts.

The vertex can be identified directly from the equation:

$$Vertex = (-2,3)$$

The axis of symmetry

$$x = -2$$

Part F

To find the y-intercept, substitute $x = 0$:

$$y = 2(-7)(2) = -28$$

To find the x-intercept, substitute $y = 0$:

$$2(x-7)(x+2) = 0$$

$$x \in \{-2,7\}$$

x coordinate of the vertex is the average of the roots:

$$\frac{-2+7}{2} = \frac{5}{2}$$

Axis of Symmetry

$$x = \frac{5}{2}$$

Substitute the x value of the vertex in the function to find the y-value:

$$y = 2\left(\frac{5}{2} - 7\right)\left(\frac{5}{2} + 2\right) = 2\left(-\frac{9}{2}\right)\left(\frac{9}{2}\right) = -\frac{81}{2}$$

C. Equation from Graphs

The equation of a quadratic is

$$y = ax^2 + bx + c$$

There are three coefficients which need to be determined in order to uniquely identify a quadratic. These coefficients can be determined in various ways:

- The conceptually simplest is to determine any three points that lie on the quadratic, and substitute in the general equation of a quadratic.
  - ✓ This will give you three equations in three variables.
  - ✓ Solving this can be both tedious, and error-prone.
- A faster way is to find a point where one of the coefficients is zero
  - ✓ At the y-intercept, x is zero, and hence the constant term(c), becomes easy to find.

Value of a+b+c for the equation ax^2 + bx + c is given by substituting x=1
Value of a-b+c for the equation ax^2 + bx + c is given by substituting x=-1

Example 2.59: Three Points Given

Example 2.60: Vertex and Additional Point

Find the product of the zeroes of $f(x) = -|a|x^2 + bx + c$ if it has a maximum at $(2,1)$ and (p, q) lies on the graph with $|p| = 4, |q| = 3, pq = 12$.

Step I: Determine the value of (p, q)

Since $f(x)$ has a maximum at $(2,1)$, it is a downward parabola. Hence, all points on the graph will be below $(2,1)$.

$$pq = 12 \Rightarrow \underbrace{(p,q) = (4,3)}_{p,q\ are\ both+ve}\ OR\ \underbrace{(p,q) = (-4,-3)}_{p,q\ are\ both-ve}$$

Step II: Find a

$$\underbrace{f(x) = y = a(x-h)^2 + k}_{Use\ Vertex\ Form} \Rightarrow \underbrace{y = a(x-2)^2 + 1}_{Substitute\ (h,k)=(2,1)} \Rightarrow \underbrace{-3 = a(-4-2)^2 + 1}_{Substitute\ (p,q)=(x,y)=(-4,-3)} \Rightarrow \underbrace{-3 = 36a + 1 \Rightarrow -\frac{1}{9} = a}_{Simplify\ and\ Solve}$$

Step III: Find the equation of the parabola by substituting (a, h, k)

$$\underbrace{y = a(x-h)^2 + k}_{Vertex\ Form} \Rightarrow y = -\frac{1}{9}(x-2)^2 + 1$$

Step IV: Find the roots of the parabola by setting the equation equal to zero:

$$0 = -\frac{1}{9}(x-2)^2 + 1 \Rightarrow 9 = (x-2)^2 \Rightarrow \pm 3 = x - 2 \Rightarrow x \in \{-1,5\} \Rightarrow (-1) \times 5 = -5$$

D. Equation from Tables

Example 2.61

The formula which expresses the relationship between x and y as shown in the accompanying table is:

A. $y = 100 - 10x$

B. $y = 100 - 5x^2$
C. $y = 100 - 5x - 5x^2$
D. $y = 20 - x - x^2$
E. None of these **(AHSME 1950/17)**

x	0	1	2	3	4
y	100	90	70	40	0

The first differences tell us the relationship is not linear. This eliminates Option A.
Option C is correct.

x	0	1	2	3	4
y	100	90	70	40	0
		10	20	30	40

E. Reversing x and y

$$\underbrace{y = ax^2 + bx + c}_{\substack{\textit{Standard Equation}\\ \textit{of a Parbola}}} \Rightarrow \underbrace{x = ay^2 + by + c}_{\textit{Switching x and y}}$$

The graph of the second equation is:
- The graph of the first equation rotated 90 degrees to the right, and possibly also shifted
- Not a function

$$x = ay^2 + by + c$$
$$x - intercept\ = \ constant\ term$$
$$Discriminant = D = b^2 - 4ac \Rightarrow Does\ not\ change$$
$$y - intercept\ = \ Roots \Rightarrow \underbrace{D > 0}_{\textit{Two y-intercepts}},\ \underbrace{D = 0}_{\textit{One y-intercept}},\ \underbrace{D < 0}_{\textit{No y-intercept}}$$

3. Further Topics

3.1 Completing the Square

A. Completing the Square

3.1: Completing the Square

The vertex form of a quadratic is given by:

$$y = a(x-h)^2 + k$$

wheren

$$h \to x\ coordinate\ of\ vertex$$
$$k \to y\ coordinate\ of\ vertex$$

Completing the square lets us convert a quadratic into the vertex form.

We will first review how we factor using perfect squares. This same method will then be extended to factoring expressions which are not perfect squares

Example 3.2

Factor $x^2 + 10x + 25$, showing how we know that the first term and the second term can be derived.

Use the **method of undetermined coefficients** to compare coefficients with the formula for the square of a sum:

$$\underbrace{\boldsymbol{x^2}}_{x^2} + \underbrace{\boldsymbol{10x}}_{2xb} + \underbrace{\boldsymbol{25}}_{b^2} \equiv \boldsymbol{x^2} + \boldsymbol{2xb} + \boldsymbol{b^2} = (\boldsymbol{x} + \boldsymbol{b})^2$$

$$\underbrace{2xb = 10x \Rightarrow 2b = 10 \Rightarrow b = 5}_{Middle\ Term}, \qquad \underbrace{b^2 = 25 \Rightarrow b = \pm 5}_{Last\ Term}$$

We prefer factorizations that have $+x$ in them because they are easier to manipulate, and feel more natural:

$$(-x-5)^2 = [(-1)(x+5)]^2 = (-1)^2(x+5)^2 = (x+5)^2$$

Example 3.3

If an expression is not a perfect square, it can be re-written as a perfect square. Complete the square in the following expressions:

Basics

A. $x^2 + 8x + 1$
B. $x^2 + 6x + 4$
C. $x^2 - 2x + 12$
D. $x^2 - 4x + 1$

Fractions

E. $x^2 + 3x + 1$
F. $x^2 + 5x + 1$
G. $x^2 + \frac{7x}{2} + 1$
H. $x^2 + 0.6x + 1$

Middle Term Missing

I. $5x^2 + 5$
J. $3x^2 + 7$
K. $\frac{1}{3}x^2 + \frac{5}{7}$

Basics

$$\underbrace{x^2 + 8x + \boldsymbol{16}}_{Perfect\ Square} - \boldsymbol{16} + 1 = (x+4)^2 - 15$$
$$x^2 + 6x + 9 - 9 + 4 = (x+3)^2 - 5$$
$$x^2 - 2x + 1 - 1 + 12 = (x-1)^2 + 11$$
$$x^2 - 4x + 4 - 4 + 1 = (x-2)^2 - 3$$

Fractions

It is possible (common, in fact) to encounter fractions when completing the square. This does not pose a problem so long the calculations are done carefully.

$$x^2+3x+1=x^2+3x+\frac{9}{4}-\frac{9}{4}+1=\left(x+\frac{3}{2}\right)^2-\frac{5}{4}$$
$$x^2+5x+\frac{25}{4}-\frac{25}{4}+1=\left(x+\frac{5}{2}\right)^2-\frac{21}{4}$$
$$x^2+\frac{7x}{2}+\frac{49}{16}-\frac{49}{16}+1=\left(x+\frac{7}{4}\right)^2-\frac{33}{16}$$
$$x^2+0.6x+0.09-0.09+1=(x+0.3)^2+0.91$$

Middle Term Missing

If the middle term in a quadratic expression is missing, it can still be factored as a perfect square in the standard way. This is important for some questions related to graphing/vertex that we will do later.

$$5x^2+5\ =5(x-0)^2+5$$
$$3(x-0)^2+7$$
$$\frac{1}{3}(x-0)^2+\frac{5}{7}$$

Example 3.4: General Quadratics (Leading coefficient $\neq 1$)

Factoring general quadratics (where the leading coefficient) is not one requires an additional step. This often leads to fractions in the calculations. Complete the square in the following expressions

A. $2x^2+3x-2$
B. $2x^2+3x+4$

Part A

Factor out the leading coefficient:

$$2\left(x^2+\frac{3}{2}x-1\right)$$

Find the last term:

Divide the coefficient of the middle term $\left(\frac{3}{2}\right)$ by 2 to find the last term:

$$Last\ Term=\frac{3}{2}\div 2\ =\frac{3}{2}\times\frac{1}{2}=\frac{3}{4}$$

Then, square the last term:

$$Square\ of\ Last\ Term=\left(\frac{3}{4}\right)^2=\frac{3}{4}\times\frac{3}{4}=\frac{9}{16}$$

Complete the Square:

Add and subtract the square of the last term$\left(\frac{9}{16}\right)$:

$$=2\left(x^2+\frac{3}{2}x\ +\frac{9}{16}-\frac{9}{16}-\frac{16}{16}\right)$$

Factor the first three terms, and add the last two terms:

$$=2\left[\left(x+\frac{3}{4}\right)^2-\frac{25}{16}\right]$$

Move $-\frac{25}{16}$ outside the bracket by multiplying it by 2:

$$=2\left[\left(x+\frac{3}{4}\right)^2\right]-\frac{25}{8}$$

Part B

Factor out the leading coefficient:

$$\underbrace{2\left(x^2+\frac{3}{2}x+2\right)}_{\substack{\text{Factor out the}\\ \text{leading coefficient}}}$$

Add and subtract $\frac{3}{2}\underbrace{\rightarrow}_{\substack{\text{Divide}\\ \text{by 2}}}\frac{3}{4}\underbrace{\rightarrow}_{\text{Square}}\frac{9}{16}$:

$$2\left(x^2+\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}+2\right)$$

Factor and Simplify:

$$2\left[\left(x+\frac{3}{4}\right)^2+\frac{23}{16}\right]$$

Move the constant term outside the brackets

$$=2\left(x+\frac{3}{4}\right)^2+\frac{23}{8}$$

Example 3.5: Solving Equations

Solve the following equations by completing the square:

A. $x^2+6x+9=0$
B. $x^2+10x+1=0$
C. $x^2+12x+1=0$
D. $3x^2-27=0$
E. $4x^2-12=0$
F. $\frac{1}{3}x^2-\frac{5}{7}=0$
G. $3x^2+20x+15=0$

Part A

Factor the LHS as a perfect square:

$$(x+3)^2=0$$

Take Square Roots both sides:

$$x+3=0$$

Solve for x:

$$x=-3$$

Part B

$$\underbrace{x^2+10x+25}_{\text{Perfect Square}}-25+1=0$$

Factor the LHS as a perfect square:

$$(x+5)^2-24=0$$

Isolate the square on the LHS:

$$(x+5)^2=24$$

Take Square Roots

$$x+5=\pm\sqrt{24}\Rightarrow x=-5\pm\sqrt{24}$$

Part C

$$\underbrace{x^2+12x+36}_{\text{Perfect Square}}-36+1=0$$

Factor the LHS as a perfect square:

$$(x+6)^2-35=0$$

Isolate the square on the LHS:

$$(x+6)^2=35$$

Take Square Roots

$$x+6=\pm\sqrt{35}\Rightarrow x=-6\pm\sqrt{35}$$

Part D

$$3x^2-27=0\Rightarrow 3x^2=27\Rightarrow x^2=9\Rightarrow x=\pm\sqrt{9}=\pm 3$$

Part E

$$4x^2-12=0\Rightarrow 4x^2=12\Rightarrow x^2=3\Rightarrow x=\pm\sqrt{3}$$

Part F

$$\frac{1}{3}x^2-\frac{5}{7}=0\Rightarrow\frac{1}{3}x^2=\frac{5}{7}\Rightarrow x^2=\frac{15}{7}\Rightarrow x=\pm\sqrt{\frac{15}{7}}$$

Part G

Divide both sides by 3:

$$x^2+\frac{20}{3}x=-5$$

Complete the square on the LHS and factor it:

$$x^2+\frac{20}{3}x+\left(\frac{10}{3}\right)^2=-5+\left(\frac{10}{3}\right)^2$$

Factor the LHS:

$$\left(x+\frac{10}{3}\right)^2=\frac{55}{9}$$

Take square roots and solve for x:

$$x+\frac{10}{3}=\pm\sqrt{\frac{55}{9}} \Rightarrow x=-\frac{10}{3}\pm\sqrt{\frac{55}{9}}$$

3.6: Minimum Value of an Expression

$$Min\left(\underbrace{Perfect\ Square}_{Minimum\ Value:Zero}+\underbrace{Constant}_{Value\ does\ not\ change}\right)=Constant$$

Recall the trival inequality, which tell us that the minimum value of a square is zero

$$a^2\ge 0 \Leftrightarrow A\ square\ is\ always\ nonnegative$$

Suppose we rewrite an expression as the sum of a perfect square and a constant:

$$Expression=\underbrace{Perfect\ Square}_{Minimum\ Value:Zero}+\underbrace{Constant}_{Value\ does\ not\ change}$$

Since the constant does not change value, the minimum value of the expression is achieved when the square is zero.

3.7: Minimum Value of an Expression

$$y=a(x-h)^2+k$$

Has

$$Minimum\ Value=k$$

Example 3.8

Find the minimum value of $y=x^2+8x+1$ as it ranges from positive infinity to negative infinity. Also, find the x-value that achieves this minimum y-value.

$$Min\ (x^2+8x+1)=Min\left(\underbrace{(x+4)^2-15}_{See\ prior\ question\ for\ working}\right)=Min((-4+4)^2-15)=Min(-15)$$

Minimum value of $y=-15$ achieved at $x+4=0\Rightarrow x=-4$

Example 3.9: Finding the Vertex

Completing the square in a quadratic puts it into vertex form. This lets us identify the vertex directly. Find the coordinates of the vertex of each quadratic below by rewriting it as a perfect square.

A. $y=2x^2+11x+15$

$$2\left(x^2+\frac{11x}{2}+\frac{15}{2}\right)=2\left(x^2+\frac{11x}{2}+\frac{121}{16}-\frac{121}{16}+\frac{15}{2}\right)=2\left[\left(x^2+\frac{11}{4}\right)^2-\frac{1}{16}\right]=2\left(x+\frac{11}{4}\right)^2-\frac{1}{8}$$

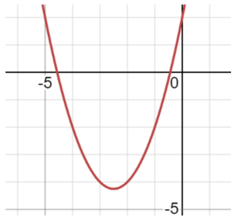
$$\text{y}-\text{coordinate}=Min(y)=-\frac{1}{8}$$

$$x\text{ coordinate}=\text{Value of }x\text{ at which }y\text{ is minimised}=-\frac{11}{4}$$

3.2 Quadratic Optimization

A. Basics

Finding the minimum or maximum of a given quadratic expression is something that can be done with the techniques that we already. This is very important for applications in geometry, physics, economic and many other topics.



3.10: Upward Parabola

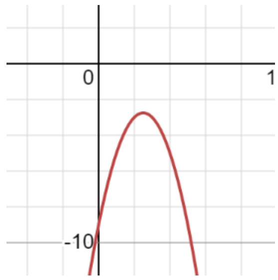
For an upward parabola, the minimum is given by

$$y \ coordinate \ of \ the \ vertex$$

There is no maximum for an upward parabola.

As you can see from the graph, the minimum is achieved at the vertex, and as you go:

- Left from the vertex, the y-value keeps increasing
- Right from the vertex, the y-value keeps increasing



3.11: Downward Parabola

For an downward parabola, the maximum is given by

$$y \ coordinate \ of \ the \ vertex$$

There is no minimum for a downward parabola.

As you can see from the graph, the maximum is achieved at the vertex, and as you go:

- Left from the vertex, the y-value keeps decreasing
- Right from the vertex, the y-value keeps decreasing

3.12: Factored Form

$$y = a(x-\alpha)(x-\beta)$$

In factored form, the x intercepts are the same as the roots, which are:

$$\alpha, and \ \beta$$

By symmetry, the vertex is exactly in between the x intercepts, and hence, the x coordinate of the vertex is the average of the roots:

$$\frac{\alpha+\beta}{2}$$

To find the y coordinate of the vertex, substitute the x coordinate of the vertex in the equation.

Example 3.13

A. For the quadratic $y = 3(x-3)(x+4)$, find the minimum value of y.

A. Show that for a quadratic $y = a(x-\alpha)(x-\beta)$, the y coordinate of the vertex is: $-a\left(\frac{\alpha^2-2\alpha\beta+\beta^2}{4}\right)$

Part A

Part B

To find the y coordinate of the vertex, substitute the x coordinate of the vertex into the equation of the parabola:

$$y = a(x-\alpha)(x-\beta) = a\left(\frac{\alpha+\beta}{2}-\alpha\right)\left(\frac{\alpha+\beta}{2}-\beta\right) = a\left(\frac{\beta-\alpha}{2}\right)\left(\frac{\alpha-\beta}{2}\right) = -a\left(\frac{\alpha^2-2\alpha\beta+\beta^2}{4}\right)$$

3.14: Standard Form

$$y = ax^2 + bx + c$$

The roots of the above expression are:

$$\frac{-b+\sqrt{b^2-4ac}}{2a}, \frac{-b-\sqrt{b^2-4ac}}{2a}$$

The x coordinate of the vertex is given by the average of the two roots:

$$\left(\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}\right) \times \frac{1}{2} = -\frac{2b}{4a} = -\frac{b}{2a}$$

To find the y coordinate of the vertex, substitute the x coordinate of the vertex in the equation.

Example 3.15

$$y = x^2 + 5x + 6$$

Identify the minimum value of the above expression, by the formula for standard form.

$$x = -\frac{b}{2a} = -\frac{5}{2}$$

$$y = x^2 + 5x + 6 = \left(-\frac{5}{2}\right)^2 + 5\left(-\frac{5}{2}\right) + 6 = -\frac{1}{4}$$

3.16: Vertex Form

$$y = a(x-h)^2 + k$$

$$Coordinates\ of\ Vertex = (h, k)$$

$$Minimum\ Value = k$$

Example 3.17

Consider the expression:

$$y = (x+2)(x+3)$$

A. By completing the square, find the minimum value of the expression
B. Also, find the value of x when the expression is minimum

$$x^2 + 5x + 6 = x^2 + 5x + \frac{25}{4} - \frac{25}{4} + 6 = \left(x + \frac{5}{2}\right)^2 - \frac{1}{4}$$

Now, the minimum value of a square is 0.
Hence,

$$Minimum\ Value\ of\ \left(x + \frac{5}{2}\right)^2\ is\ zero \rightarrow Achieved\ when\ x = -\frac{5}{2}$$

And, the overall minimum value of the expression is:

$$\left(-\frac{5}{2} + \frac{5}{2}\right)^2 - \frac{1}{4} = 0^2 - \frac{1}{4} = -\frac{1}{4}$$

Example 3.18

Find the minimum of the following quadratic expression:

$$y = \frac{3}{4}\left(x - \frac{7}{2}\right)^2 + 27.12$$

Example 3.19

Find the minimum of

$$y = (x+2)(x+3)$$

By converting it into a form where the minimum can be directly read from the form.

$$x^2 + 5x + 6 = x^2 + 5x + \frac{25}{4} - \frac{25}{4} + 6 = \left(x + \frac{5}{2}\right)^2 - \frac{1}{4}$$

Note that this is now in vertex form.
Comparing the above equation to $y = a(x-h)^2 + k$:

$$Scaling\ Factor = a = 1$$

$$x\ coordinate\ of\ vertex = \ h = -\frac{5}{2}$$

$$Minimum = y\ coordinate\ of\ vertex = k = -\frac{1}{4}$$

B. Range

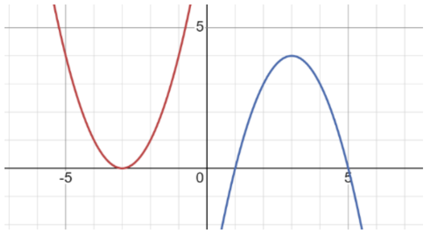
3.20: Range of a Parabola

Given parabola $f(x) = ax^2 + bx + c$ with y coordinate of vertex y_1.
For an upward parabola, the vertex is the lowest point. Hence, the range is:

$$[y_1, \infty), a > 0$$

For a downward parabola, the vertex is the highest point. Hence, the range is

$$[-\infty, y_1), a < 0$$



Example 3.21

Find the range of the following functions:

$$y = x^2 + 7x + 12$$
$$y = -x^2 - 5x - 6$$

$$y = x^2 + 7x + 12 = \left(x + \frac{7}{2}\right)^2 - \frac{1}{4}$$

$$y\ coordinate\ of\ vertex = -\frac{1}{4}$$

$$Range = \left[-\frac{1}{4}, \infty\right)$$

$$y = -(x^2 + 5x + 6) = -\left[\left(x + \frac{5}{2}\right)^2 - \frac{1}{4}\right]$$

$$y\ coordinate\ of\ vertex = -\frac{1}{4}$$

$$Range = \left(-\infty, -\frac{1}{4}\right]$$

C. Percentage

Example 3.22

In a certain city the rate of taxation is the following: $x\%$ tax is collected for an income of x thousand dollars. What income, in dollars, will yield the greatest take home pay? (Take-home pay is the income minus the tax on that income.) **(MathCounts 1996 Chapter Team)**

$$Tax\ Collected = \frac{d}{100} \times 1000d = 10d$$
$$Take\ Home\ Pay = 1000d - 10d^2 = -10d^2 + 1000d$$

And since the above is now in standard form:

$$At\ Max, \qquad x = -\frac{b}{2a} = -\frac{1000}{2 \times 10} = 50$$
$$Income = 50{,}000\ Dollars$$

Alternate Solution

$$x - \frac{x}{100} of\ x = x - \frac{x^2}{100} = \left(-\frac{1}{100}\right)x^2 + x$$
$$x = -\frac{b}{2a} = -\frac{1}{2 \times \left(-\frac{1}{100}\right)} = 50$$

D. Geometry

Example 3.23

Length of Wire=40 cm

$$S_1 = x\ cm \Rightarrow P_1 = 4x\ cm$$
$$P_2 = 40 - 4x\ cm \Rightarrow S_2 = 10 - x$$

Combined area of two squares

$$= x^2 + (10 - x)^2 = x^2 + 100 - 20x + x^2 = 2x^2 - 20x + 100$$

Minimum area of the sum of the two squares is given by x coordinate of vertex (Side Length)

$$= -\frac{b}{2a} = 20/4 = 5$$

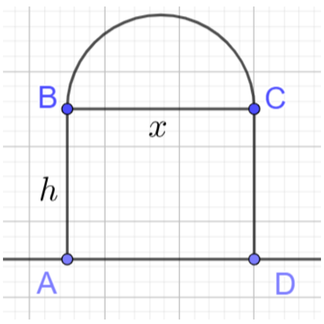
y coordinate of the vertex (Area)

$$= 2(5)^2 - 20(5) + 100 = 50 - 100 + 100 = 50$$

Example 3.24

An enclosure next to a mountain (AD) consists of semi-circle BC sitting atop rectangle $ABCD$. AD does not need to be fenced and nor does the length BC. Show that if the length of the available fence is 60 units, then:

A. Area of the enclosure is given by $30x - \frac{\pi x^2}{8}$
B. The maximum area is attained when $h = 0$.



The perimeter of the enclosure is given by:

$$\underbrace{h}_{\substack{Left\\ Side}} + \underbrace{h}_{\substack{Right\\ Side}} + \underbrace{\frac{\pi x}{2}}_{\substack{Semi\\ Circle}} = 60 \Rightarrow 2h = 60 - \frac{\pi x}{2} \Rightarrow h = 30 - \frac{\pi x}{4}$$

Total Area

$$= \underbrace{hx}_{\substack{Area\ of\\ Rectangle}} + \underbrace{\frac{\pi r^2}{2}}_{\substack{Area\ of\\ Semi-Circle}} = \left(30 - \frac{\pi x}{4}\right)(x) + \frac{\pi \left(\frac{x}{2}\right)^2}{2} = 30x - \frac{\pi x^2}{4} + \frac{\pi x^2}{8} = 30x - \frac{\pi x^2}{8}$$

The maximum of quadratic $ax^2 + bx + c$ is found at the vertex. Substitute $b = 30, a = -\frac{\pi}{8}$:

$$Vertex = -\frac{b}{2a} = -\frac{30}{2\left(-\frac{\pi}{8}\right)} = \frac{120}{\pi}$$

Now, the minimum of h can be zero.
Then, the value of x will be

$$C = \frac{\pi D}{2} \Rightarrow 60 = \frac{\pi D}{2} \Rightarrow D = \frac{120}{\pi}$$

Example 3.25

A rectangle having integer length and width has a perimeter of 100 units. What is the number of square units in the least possible area? **(MathCounts 2002 National Countdown)**

Let the dimensions of the rectangle be

$$a \text{ and } b, \qquad a, b \in \mathbb{N}$$

Then:

$$2(a+b) = 100 \Rightarrow a + b = 50 \Rightarrow b = 50 - a$$

We can introduce some constraints on the values of a and b:

$$0 < a < 50 \Rightarrow 1 \le a \le 49$$

Also,

$$A = ab = a(50-a) = 50a - a^2 = -a^2 + 50a$$

This is a downward parabola, with vertex:

$$x \text{ coordinate} = \frac{-50}{-2} = 25$$
$$y \text{ coordinate} = -(25)^2 + 50(25) = -625 + 1250 = 625$$
$$Vertex \text{ is } (25,625)$$

Since it is a downward parabola, the further away you go from the vertex, the smaller value.
The smallest value in the domain is

$$(\boldsymbol{a} \cdot \boldsymbol{b})^2 \le (\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{b})$$

E. Challenging Applications

Example 3.26

Economics: Revenue and Profits

Example 3.27: Cauchy-Schwarz Inequality[2]

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \le (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$$

Consider the expression:

$$X = (a_1x - b_1)^2 + (a_2x - b_2)^2 + \cdots + (a_nx - b_n)^2$$

Expand:

---

[2] For vectors $\boldsymbol{a} = (a_1, a_2, \dots, a_n), \boldsymbol{b} = (b_1, b_2, \dots b_n)$, this inequality can be written in dot product notation as $(\boldsymbol{a} \cdot \boldsymbol{b})^2 \le (\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{b})$.

$$(a_1^2x^2 - 2xa_1b_1 + b_1^2) + (a_2^2x^2 - 2xa_2b_2 + b_2^2) + \cdots + (a_n^2x^2 - 2xa_nb_n + b_n^2)$$

Collate all like terms together:

$$x^2(a_1^2 + a_2^2 + \cdots + a_n^2) - 2x(a_1b_1 + a_2b_2 + \cdots + a_nb_n) + (b_1^2 + b_2^2 + \cdots + b_n^2)$$

The above is now a quadratic in x.

Since X is the sum of perfect squares, we must have:

$$X \geq 0$$

Which means that the quadratic has:

$$Zero\ x\ intercept, or\ single\ x\ intercept$$

Which means that

$$Discriminant \leq 0 \Rightarrow b^2 - 4ac \leq 0 \Rightarrow b^2 < 4ac$$

Substitute the values from the quadratic above:

$$[-2(a_1b_1 + a_2b_2 + \cdots + a_nb_n)]^2 \leq 4(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$$
$$4(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq 4(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$$
$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$$

Example 3.28: Regression

Consider points $P_1(x_1, y_1), P_2(x_2, y_2), \ldots P_n(x_n, y_n)$ and a line $y = mx$ with perpendicular distances from the points to the line given by:

$$D_1, D_2, \ldots D_n$$

Define the $line\ of\ best\ fit$ to be the line that minimizes:

$$D_1^2 + D_2^2 + \cdots + D_n^2$$

Show that, for the line of best fit:

$$m = \frac{x_1y_1 + x_2y_2 + \cdots + x_ny_n}{x_1^2 + x_2^2 + \cdots + x_n^2}$$

$$D_1^2 + D_2^2 + \cdots + D_n^2$$
$$= (y_1 - mx_1)^2 + (y_2 - mx_2)^2 + \cdots + (y_n - mx_n)^2$$
$$= [y_1^2 - 2mx_1y_1 + m^2x_1^2] + [y_2^2 - 2mx_2y_2 + m^2x_2^2] + \cdots + [y_n^2 - 2mx_ny_n + m^2x_n^2]$$
$$= m^2[x_1^2 + x_2^2 + \cdots + x_n^2] - 2m[x_1y_1 + x_2y_2 + \cdots + x_ny_n] + [y_1^2 + y_2^2 + \cdots + y_n^2]$$

The above is a quadratic in m, with minimum given by:

$$= -\frac{b}{2a} = -\frac{-2[x_1y_1 + x_2y_2 + \cdots + x_ny_n]}{2[x_1^2 + x_2^2 + \cdots + x_n^2]} = \frac{x_1y_1 + x_2y_2 + \cdots + x_ny_n}{x_1^2 + x_2^2 + \cdots + x_n^2}$$

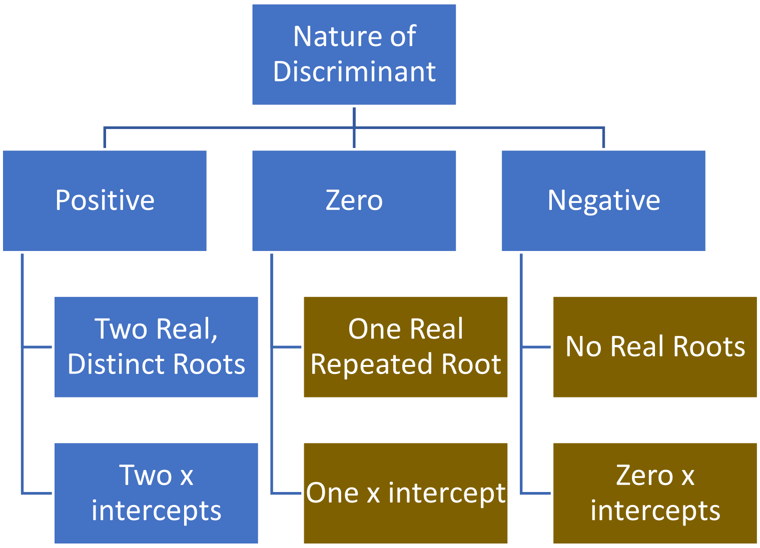
3.3 Discriminant

A. Discriminant and Nature of Roots

$\underbrace{b^2 - 4ac}_{\text{Discriminant}}$ determines the nature of the roots of the quadratic.

$$b^2 - 4ac = D = Discriminant$$

- Discriminant does not have a square root sign as a part of its definition.
- The chart on the right should be memorized.



B. Real and Distinct Roots: $D > 0$:

If D is greater than 0, the roots are real, and distinct:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

3.29: Classifying D

- D is a perfect square $\Rightarrow$ Roots are rational.
- D is not a perfect square $\Rightarrow$ Roots are irrational. If one root is $p + \sqrt{q}$, the other root is $p - \sqrt{q}$. That is, roots are conjugates of each other.
- $D = 0 \Rightarrow$ Substitute $D = 0$ in the quadratic formula. Since the square root $\underbrace{vanishes}_{becomes\ zero}$, we are left with only one value.
- $D < 0 \Rightarrow$Roots are complex conjugates of each other. We define $i = \sqrt{-1} \Rightarrow \sqrt{-35} = \sqrt{-1} \times \sqrt{35} = i\sqrt{35}$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a}$$

Example 3.30

Show that the roots of

A. $x^2 + 5x + 6 = 0$ are rational.
B. $x^2 + 5x + 5 = 0$ are irrational.
C. $x^2 + 6x + 9 = 0$ are real and repeated.
D. $x^2 + x + 9 = 0$ are complex conjugates of each other

Part A

$$x^2 + 5x + 6 = 0 \Rightarrow x = \frac{-5 \pm \sqrt{25 - (4)(1)(6)}}{2 \times 1} = \frac{-5 \pm \sqrt{1}}{2} \Rightarrow x \in \{-3, -2\}$$

Because 1 is a perfect square, we can take the square root to get an integer answer, and hence we are guaranteed rational roots.

Part B

$$x^2 + 5x + 5 = 0 \Rightarrow x = \frac{-5 \pm \sqrt{25 - (4)(1)(5)}}{2 \times 1} = \frac{-5 \pm \sqrt{5}}{2} \Rightarrow x \in \left\{\frac{-5 + \sqrt{5}}{2}, \frac{-5 - \sqrt{5}}{2}\right\}$$

Because 5 is not a perfect square, we cannot find the square root in rational numbers, and hence the roots are irrational.

Part C

$$x^2 + 6x + 9 = 0 \Rightarrow x = \frac{-6 \pm \sqrt{36 - (4)(1)(9)}}{2 \times 1} = \frac{-6}{2} = -3$$

Because the square root vanishes, we only get one root. We call this a repeated root.

Part D

$$x = \frac{-1 \pm \sqrt{1 - (4)(1)(9)}}{2 \times 1} = \frac{-1 \pm \sqrt{-35}}{2}$$

Because the discriminant is negative, we cannot take the square root in real numbers, though we can do so in complex numbers. Then, our roots are:

$$x = \frac{-1 + i\sqrt{35}}{2}, \frac{-1 - i\sqrt{35}}{2}$$

Roots are complex conjugates of each other because they are same except for the sign, where one is positive and the other is negative.

Example 3.31

The real factors of $x^2 + 4$ are:

A. $(x^2 + 2)(x^2 + 2)$
B. $(x^2 + 2)(x^2 - 2)$
C. $x^2(x^2 + 4)$
D. $(x^2 - 2x + 2)(x^2 + 2x + 2)$
E. Non-existent (AHSME 1950/15)

$$x^2 + 4 = 0 \Rightarrow x^2 = -4 \Rightarrow x = \sqrt{-4} \Rightarrow No\ Real\ Solutions$$

Hence, Option E

C. Classifying Roots

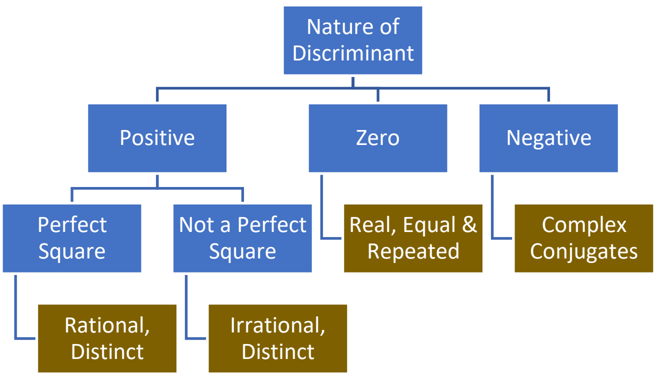
We will now study the rules that lead to roots being real (or complex) in greater detail. This will let us determine the nature of the roots of a quadratic without actually solving it.

Classifying the nature of the roots of a quadratic equation is an important building block in solving complex questions. The examples below provide practice in this skill.

Example 3.32

What is the nature of the roots of the following quadratic equations:

A. $y = 2x^2 + 5x + 4$
B. $y = 3x^2 + 7x + 1$
C. $y = x^2 + 7x + \frac{13}{4}$
D. $y = x^2 + 4x + 4$



Calculate the discriminant $b^2 - 4ac$ for each:

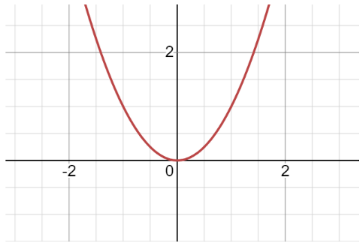
A. $25 - (4)(2)(4) = 25 - 32 = -7 \Rightarrow$
Roots are complex, and distinct (Complex Conjugates)
B. $49 - (4)(3)(1) = 49 - 12 = 37 \Rightarrow$ Roots are irrational and distinct
C. $49 - (4)(1)\left(\frac{13}{4}\right) = 49 - 13 = 36 \Rightarrow$ Roots are rational and distinct
D. $16 - (4)(1)(4) = 16 - 16 = 0 \Rightarrow$ Roots are real, equal and repeated

Example 3.33

Calculate the discriminant for each of the below expressions. Hence, write what can be concluded about the number (*one or two*), and nature(*real or complex*) of the roots of the corresponding quadratic equations.

A. x^2
B. $x^2 + 1$
C. $x^2 - 1$

Part A



$$x^2 + 0x + 0 \Rightarrow a = 1, b = 0, c = 0$$

Substitute the above in $b^2 - 4ac$:

$$0 - (4)(1)(0) = 0 - 0 = 0$$

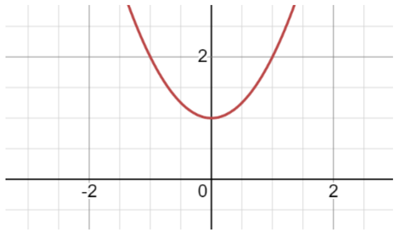
Since the discriminant is zero, the quadratic has one real, repeated root.
This can be seen in the diagram alongside, where the graph intersects the x-axis at exactly one place.

Part B

$$x^2 + 0x + 1 \Rightarrow a = 1, b = 0, c = 1$$

Substitute the above in $b^2 - 4ac$:

$$0 - (4)(1)(1) = 0 - 4 = -4$$



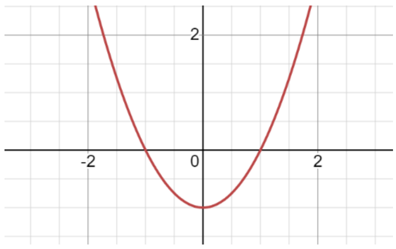
Since the discriminant is negative, the quadratic has no real roots.
This can be seen in the diagram alongside, where the graph does not intersect the x-axis.

Part C

$$x^2 + 0 \cdot x - 1 \Rightarrow a = 1, b = 0, c = -1$$

Substitute the above in $b^2 - 4ac$:

$$0 - (4)(1)(-1) = 0 + 4 = 4$$



Since the discriminant is positive, the quadratic has two real roots.
This can be seen in the diagram alongside, where the graph intersects the x-axis at exactly one place.

Example 3.34

If the sum of the coefficients of a quadratic equation in standard form is zero, then show that the quadratic equation passes through the point $(1, a + b + c)$.

Substituting $x = 1$ in the standard form $(y = ax^2 + bx + c)$ yields:

$$y = a(1)^2 + b(1) + c = a + b + c$$

Hence, if $x = 1$, then $y = a + b + c$

D. Graphs based on Nature of Roots

Nature of Roots	Solution Set: a > 0			
	$y > 0$ when:	$y < 0$ when:		
D > 0: Real and Distinct Graph cuts the x-axis in two distinct places	$x < \alpha \cup x > \beta$ $(-\infty, \alpha) \cup (\beta, \infty)$	$\alpha < x < \beta$		
D = 0: Real and Repeated Graphs touches the x-axis,	$x < \alpha \cup x > \alpha$ $(-\infty, \alpha) \cup (\alpha, \infty)$			

but does not cross it. Root is repeated	$\mathcal{R}-\{\alpha\}$			
D < 0: Complex Graph does not cut the x-axis at all, but is some distance away	$(-\infty,\infty)$ $\forall x\in\mathcal{R}$			
The table uses inequality notation, interval notation, and set notation to show the region satisfied. These are equivalent, and interchangeable.				

E. Concept Questions

Example 3.35

State True or False:

A. A quadratic equation can have three roots.
B. A quadratic equation cannot have zero roots
C. The coefficient of the middle term does not affect the number of real roots that a quadratic has.
D. If a quadratic equation has a real, repeated root, then the square of the coefficient of the middle term is equal to four times the product of the leading coefficient and the constant term.
E. If a quadratic has complex roots, then the roots are complex conjugates of each other.
F. If a quadratic has rational roots, then the roots are conjugates of each other.

Statement A: Incorrect
The number of roots of an equation is the same as the degree of the highest power. A quadratic equation will always have two roots.

Statement B: Correct
A quadratic equation will always have two roots (whether real or complex).

Statement C: Incorrect
The number of roots that a quadratic has is decided by the discriminant, which is
$$b^2-4ac$$
If b is large enough, the discriminant is positive. If b is not large enough, the discriminant is negative. Hence, the coefficient of the middle term does matter when deciding the number of real roots.

Statement D: Correct
$$Real\ Repeated\ Root \Rightarrow D=0\Rightarrow b^2-4ac=0\Rightarrow b^2=4ac$$

Statement E: Correct
$$Complex\ Roots \Rightarrow \frac{-b+\sqrt{D}}{2},\frac{-b-\sqrt{D}}{2}\Rightarrow Complex\ Conjugates$$
Statement F: Incorrect
$$x^2+5x+6\Rightarrow(x+2)(x+3)=0\Rightarrow x\in\{-2,-3\}\Rightarrow Roots\ are\ not\ conjgates$$
Roots will be conjugates if the roots are irrational.

MCMC 3.36

If a quadratic has a discriminant which is a perfect square, then the roots will be:

A. Irrational
B. Rational

C. Integers
D. Complex Conjugates
E. Real

Answer Options: B, E

Rational Numbers (and hence Real Numbers) are guaranteed

$$x^2 + 5x + 6 = 0 \Rightarrow (x+2)(x+3) = 0 \Rightarrow x \in \{-2, -3\} \Rightarrow x \in \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Since x is an integer, it not irrational, or complex.

Integers are not guaranteed

$$D = b^2 - 4ac = 25 - 4(1)(6) = 25 - 24 = 1 \Rightarrow Perfect\ Square$$

$$x^2 + x + \frac{2}{9} = 0 \Rightarrow = \left(x + \frac{2}{3}\right)\left(x + \frac{1}{3}\right) = 0 \Rightarrow x \in \left\{-\frac{2}{3}, -\frac{1}{3}\right\} \Rightarrow x \in \mathbb{Q} \subset \mathbb{R}$$

$$D = b^2 - 4ac = 1 - 4(1)\left(\frac{2}{9}\right) = 1 - \frac{8}{9} = \frac{1}{9} = \left(\frac{1}{3}\right)^2 \Rightarrow Perfect\ Square$$

Algebraic Method

$$Roots = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{p^2}}{2a} = \frac{-b \pm p}{2a} \Rightarrow Definitely\ Rational$$

Will be an integer, if the a in the denominator cancels with the numerator, else not.

3.4 Quadratic Inequalities

A. Trivial Inequality

3.37: Trivial Inequality

For any real number a, we must have:

$$a^2 \geq 0$$

We can prove this by considering three distinct cases that cover all possible values of a.

$$a > 0 \Rightarrow a^2 > 0$$
$$a < 0 \Rightarrow a^2 > 0$$
$$a = 0 \Rightarrow a^2 = 0$$

- The minimum value of a square is zero.
- Squares can never be negative.

The trivial inequality is a critical property is used in many non-trivial questions and proofs. It is very important and definitely not trivial.

Example 3.38: Applying the property of minimum value of a square

It is given that $a^2 \geq 0$. Hence, find the minimum value of a^2, if a is an

A. integer
B. odd integer
C. even integer

Part A

$$a = 0 \Rightarrow a^2 = 0$$

Part B

$$a = 1 \Rightarrow a^2 = 1$$

Part C

$$a = 0 \Rightarrow a^2 = 0$$

3.39: Sum of Squares cannot be negative

For real numbers $a, b, c, \ldots$, if the sum of their squares is zero, then each of the numbers must be individually zero.

$$a^2 + b^2 + c^2 + \cdots = 0 \Rightarrow a, b, c, .. \in \{0\}$$

Example 3.40

A. If the sum of the squares of two quantities adds up to zero, then find the product of the quantities added to the difference of the quantities.
B. If $a^2 + b^2 + c^2 = 0$, then find the value of the sum obtained when the products of a, b, and c taken one at a time, then two at a time and finally three together are all added to unity.
C. If $(a-1)^2 + (b-2)^2 + (c-3)^2 + (d-4)^2 = 0$, then $a \times b \times c \times d + 1$ is: (NMTC Primary/Screening 2004/19)

Part A

$$a^2 + b^2 = 0$$
$$a = 0, b = 0$$
$$ab + (a - b) = 0 - 0 = 0$$

Part B

$$a^2 + b^2 + c^2 = 0$$

Using the zero-sum of squares property,

$$a = b = c = 0$$

Hence, all the terms except the last of the required expression are zero.

$$\therefore a + b + c + ab + bc + ca + abc + 1 = 1$$

Part C

If the squares are themselves expressions, the method of solving does not change, but each square expression needs to be equated to zero and solved independently.

$$\underbrace{(a-1)^2}_{=0} + \underbrace{(b-2)^2}_{=0} + \underbrace{(c-3)^2}_{=0} + \underbrace{(d-4)^2}_{=0} = 0$$

$$(a-1)^2 = 0 \Rightarrow a - 1 = 0 \Rightarrow a = 1$$
$$(b-2)^2 = 0 \Rightarrow b - 2 = 0 \Rightarrow b = 2$$
$$(c-3)^2 = 0 \Rightarrow c - 3 = 0 \Rightarrow c = 3$$
$$(d-4)^2 = 0 \Rightarrow d - 4 = 0 \Rightarrow d = 4$$

$$a \times b \times c \times d + 1 = (1)(2)(3)(4) + 1 = 24 + 1 = 25$$

3.41: Less Than Quadratic Inequality

A less than quadratic inequality is equivalent to an absolute value less-than inequality, which has one interval for its solution.

$$\underbrace{x^2 < a}_{Inequality} \Leftrightarrow \underbrace{-\sqrt{a} < x < \sqrt{a}}_{Interval} \Leftarrow |x| < \sqrt{a}$$

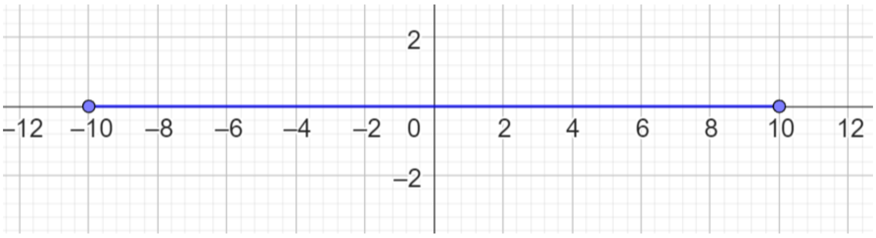
Where

$$a > 0$$

$$x^2 < 100 \Leftrightarrow -10 < x < 10 \Leftrightarrow |x| < 10$$

For example

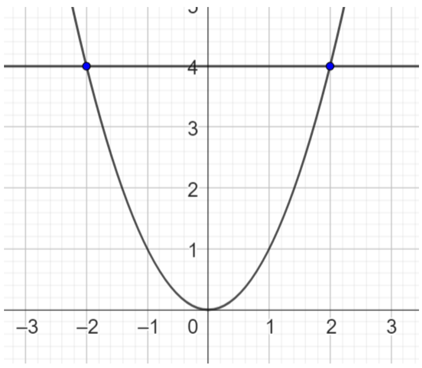
$$12^2 = 144 > 100$$
$$5^2 = 25 < 100$$
$$(-1.5)^2 = 2.25 < 100$$



Example 3.42

Solve

$$x^2 < 4$$

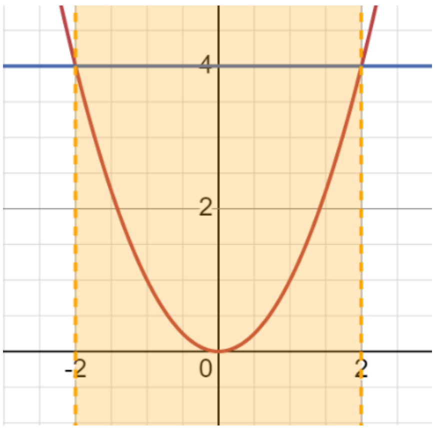


Since we see where $x^2 < 4$, graph

$$y = x^2$$

Solving

$$x^2 < 4 - 2 < x < 2$$



This is not very surprising since we are taking a square of x, so we get the square root in the answer.

The final solution set can be shaded as shown alongside.

And we can also write the answer in interval notation as:

$$x^2 < 4 \Rightarrow x \in (-2,2)$$

3.43: Trivial Quadratic Inequalities

$$a^2 < c \Leftrightarrow a \in \left(-\sqrt{c}, \sqrt{c}\right)$$

$$a^2 < c$$

A quadratic inequality has a corresponding absolute value inequality that has the same solution set:

$$a^2 < c \Leftrightarrow |a| < \sqrt{c}$$

And the solution to the absolute value inequality is:

$$-\sqrt{c} < a < \sqrt{c}$$

Which we can write this in interval notation as:

$$a \in \left(-\sqrt{c}, \sqrt{c}\right)$$

Example 3.44

Solve

$$x^2 \le 9$$

$$x^2 \le 9 \Rightarrow x \in [-3,3]$$

3.45: Trivial Quadratic Inequalities

$$a^2 \le c \Leftrightarrow a \in \left[-\sqrt{c}, \sqrt{c}\right]$$

Example 3.46

Mark all correct options

The solution set of $x^2 < 25$ is:

A. $(-5,5)$
B. $-5,5$
C. $\{-5,5\}$
D. $[-5,5]$

Option A

Option A is the correct option.

Option B and C

These options represent two values, rather than a range of values.
Hence, they are not correct.

As a counter-example

$$x = 0 \Rightarrow x^2 = 0 \Rightarrow 0 < 25 \Rightarrow x^2 < 25$$

But options B and C do not include 0 among their solutions.

Option D

When we have square brackets, the endpoints are included. However, since this is a less than inequality, the endpoints are not included.
Hence, option D is incorrect.

Example 3.47

Solve each of the following inequalities, and write the answer in interval notation.

A. $x^2 \le 36$
B. $y^2 < 49$
C. $z^2 \le 10$
D. $p^2 \le \pi$
E. $x^2 \le \sqrt{\frac{22}{7}}$

$$|x| \le 6 \Rightarrow -6 \le x \le 6 \Rightarrow a \in [-6,6]$$
$$|y| \le 7 \Rightarrow -7 \le y \le 7 \Rightarrow a \in (-7,7)$$
$$|z| \le \sqrt{10} \Rightarrow -\sqrt{10} \le z \le \sqrt{10} \Rightarrow z \in \left[-\sqrt{10}, \sqrt{10}\right]$$
$$|p| \le \sqrt{\pi} \Rightarrow -\sqrt{\pi} \le a \le \sqrt{\pi} \Rightarrow p \in \left[-\sqrt{\pi}, \sqrt{\pi}\right]$$

We have a square root already. But that does not change the property we want to apply:

$$x^2 \le \sqrt{\frac{22}{7}} \Leftrightarrow x \in \left[-\sqrt[4]{\frac{22}{7}}, \sqrt[4]{\frac{22}{7}}\right]$$

Example 3.48

Mark all correct options

The solution set for the inequality $x^2 < \pi^2$ is

A. $(-\pi, \pi)$
B. $(-3.14, 3.14)$
C. $\left(-\frac{22}{7}, \frac{22}{7}\right)$
D. $[-\pi, \pi]$

Option A is correct.

$$\pi \approx 3.14 \; but \; \pi \ne 3.14$$
$$\pi \approx \frac{22}{7} \; but \; \pi \ne \frac{22}{7}$$

Options B and C are incorrect.

Example 3.49

Mark all correct options

$$\sqrt{x^2}$$

A. x
B. $-x$
C. $|x|$
D. 1

$$\sqrt{x^2} = |x|$$
$$x = -1 \Rightarrow x^2 = 1 \Rightarrow \sqrt{x^2} = 1 = |x|$$
$$x = 1 \Rightarrow x^2 = 1 \Rightarrow \sqrt{x^2} = 1 = |x|$$

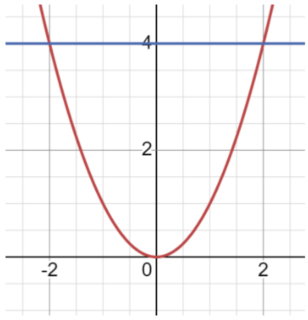
B. Greater Than Quadratic Inequality

This is equivalent to an absolute value greater-than inequality, which has two intervals for its solution.

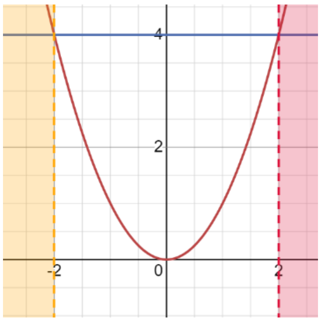
Example 3.50

Solve

$$x^2 > 4$$



We already graphed $y = x^2$ when solving the less-than version of the inequality. Note that $-2 < x < 2$ does not work. And all other values do.



Hence, the final answer is

$$x < -2 \; OR \; x > 2$$

Which can be written in interval notation as:

$$(-\infty, -2) \cup (2, \infty)$$

3.51: Trivial Quadratic Inequalities

$$a^2 \geq c \Leftrightarrow a \in (-\infty, -\sqrt{c}] \cup [\sqrt{c}, \infty]$$

$$a^2 \geq c \Rightarrow |a| > \sqrt{c} \Rightarrow a > \sqrt{c} \; OR \; a < -\sqrt{c} \Rightarrow a \in (-\infty, -\sqrt{c}] \cup [\sqrt{c}, \infty]$$

Example 3.52

Solve the following inequalities, and write the final answer in interval notation.

A. $x^2 > 16$
B. $y^2 \geq 32$
C. $z^2 > 27$
D. $p^2 > e$

$$|x| > 4 \Rightarrow x < -4 \; OR \; x > 4 \Rightarrow x \in (-\infty, -4) \cup (4, \infty)$$
$$|y| \geq \sqrt{32} = 4\sqrt{2} \Rightarrow y \leq -4\sqrt{2} \; OR \; y \geq 4\sqrt{2} \Rightarrow y \in (-\infty, -4\sqrt{2}) \cup (4\sqrt{2}, \infty)$$
$$|z| > \sqrt{27} = 3\sqrt{3} \Rightarrow z < -3\sqrt{3} \; OR \; z > 3\sqrt{3} \Rightarrow z \in (-\infty, -3\sqrt{3}) \cup (3\sqrt{3}, \infty)$$
$$|p| > \sqrt{e} \Rightarrow p < -\sqrt{e} \; OR \; p > \sqrt{e} \Rightarrow p \in (-\infty, -\sqrt{e}) \cup (\sqrt{e}, \infty)$$

Example 3.53

A. $a^2 < 9$
B. $x^2 \geq 64$

$$a^2 < 9 \Rightarrow |a| < 3 \Rightarrow -3 < a < 3 \Rightarrow a \in (-3,3)$$
$$x^2 \geq 64 \Rightarrow |x| \geq 8 \Rightarrow x \leq -8 \; OR \; x \geq 8 \Rightarrow x \in [-\infty, -8] \cup [8, \infty]$$

C. Perfect Square Inequalities

Example 3.54

$$(x+2)^2 < 20$$

Use a change of variable. Let $y = x + 2$:

$$y^2 < 20$$
$$-\sqrt{20} < y < \sqrt{20}$$

Change back to the original variable:

$$-\sqrt{20} < x + 2 < \sqrt{20}$$

$$-\sqrt{20}-2<x<\sqrt{20}-2$$
$$x \in \left(-\sqrt{20}-2, \sqrt{20}-2\right)$$

D. General Solution: Algebraic Method

While we discuss the algebraic method, this is not commonly recommended to use as the method for solving a quadratic inequality. The main reason is that it requires casework. Here we had only two cases, but if we want to solve an inequality with a third degree equation, does not scale well when we have higher degree inequalities.

Example 3.55

Solve:

A. $x^2+15x+36<0$
B. $z^2-40z+336 \le 0$

Part A

$$(x+3)(x+12)<0$$

We want to divide here by one of the factors, but we do not know whether it is positive or negative.

Case I: $x+12>0 \Rightarrow x>-12$

$$x+3<0 \Rightarrow x<-3$$

Both of the above conditions need to be met.
And hence, for this case, the final answer is:

$$\textit{Final Answer}: \ -12<x<-3$$

Case II: $x+12<0 \Rightarrow x<-12$

$$x+3>0 \Rightarrow x>-3$$

Both of the above conditions need to be met.
And hence, for this case, the final answer is:

$$\textit{Intersection is null set}$$

Case III: $x=-12$

$$(0+3)(-12+12)=3(0)=0$$

Part B

$$(z-12)(z-28) \le 0$$

Case I: $z-28>0 \Rightarrow z>28$

$$z-12 \le 0 \Rightarrow z \le 12$$
$$\textit{Intersection is null set}$$

Case I: $z-28<0 \Rightarrow z<28$

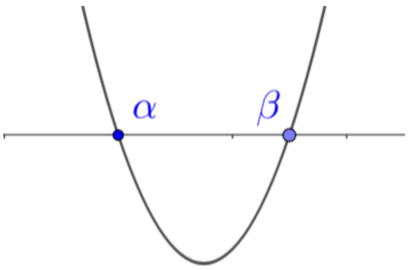
$$z-12 \ge 0 \Rightarrow z \ge 12$$

E. General Solution: Graphical

3.56: Critical Points

The roots of the corresponding equation are the critical points for the inequality.

Because of the connection between the equation and the corresponding inequality, a standard method is to instead solve the equation.



3.57: Graph of an upward parabola

The equation ax^2+bx+c is the graph of an upward parabola.

$$Roots \in \{\alpha, \beta\}$$

Example 3.58

Draw a graph and write the solution set to the following inequalities:

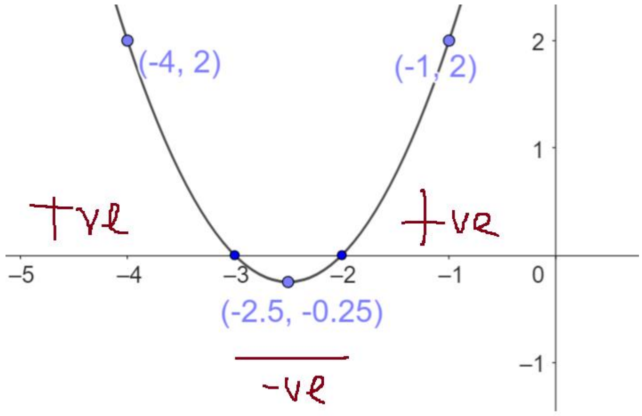
$$x^2+5x+6>0$$
$$x^2+5x+6<0$$

Solve the corresponding equation

$$x^2 + 5x + 6 = 0$$
$$(x+3)(x+2) = 0$$
$$Roots: x \in \{-3, -2\}$$

Graph the function

Graph $y = x^2 + 5x + 6$



Read the graph:

$$x^2 + 5x + 6 > 0 \Rightarrow x \in (-\infty, -3) \cup (-2, \infty)$$
$$x^2 + 5x + 6 < 0 \Rightarrow x \in (-3, -2)$$

3.59: Including the endpoints

For a $\geq$ or a $\leq$ inequality, the endpoints are included, and hence the intervals will use square brackets for the finite values.

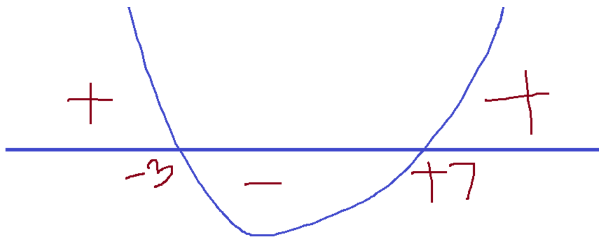
Example 3.60

Draw a graph and write the solution set to the following inequalities:

$$y = x^2 - 4x - 21 \geq 0$$
$$y = x^2 - 4x - 21 \leq 0$$

Solve the corresponding equation

$$x^2 - 4x - 21 = 0$$
$$(x-7)(x+3) = 0$$
$$x \in \{-3, 7\}$$



Read the graph:

$$y \geq 0 \Rightarrow x \in (-\infty, -3] \cup [7, \infty)$$
$$y \leq 0 \Rightarrow x \in [-3, 7]$$

Example 3.61

Draw a graph and write the solution set to the following inequalities:

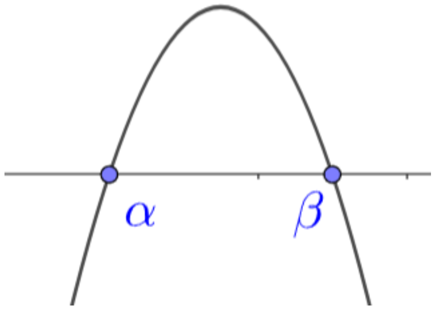
$$y = x^2 + 3x - 4 \geq 0$$
$$y = x^2 + 3x - 4 \leq 0$$

Solve the corresponding equation

$$x^2 + 3x - 4 = 0$$
$$(x+4)(x-1) = 0$$
$$x \in \{-4, 1\}$$

Read the graph:

$$y \geq 0 \Rightarrow x \in (-\infty, -4] \cup [1, \infty)$$
$$y \leq 0 \Rightarrow x \in [-4,1]$$



3.62: Graph of an upward parabola

The equation $ax^2 + bx + c$ is the graph of an downward parabola.

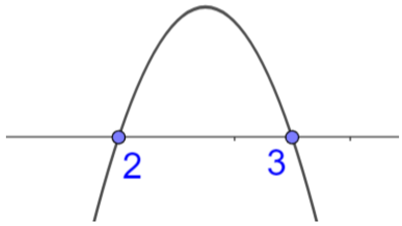
$$Roots \in \{\alpha, \beta\}$$

Example 3.63

$$y = -x^2 + 5x - 6 > 0$$

Solve the corresponding equation

$$-x^2 + 5x - 6 = 0$$
$$x^2 - 5x + 6 = 0$$
$$(x-2)(x-3) = 0$$
$$x \in \{2,3\}$$



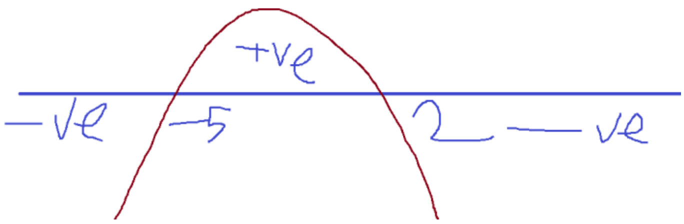
Read the graph

$$y > 0 \Rightarrow x \in (2,3)$$

Example 3.64

$$y = -x^2 - 3x + 10 < 0$$
$$y = -x^2 - 3x + 10 > 0$$

$$-x^2 - 3x + 10 = 0$$
$$x^2 + 3x - 10 = 0$$
$$(x+5)(x-2) = 0$$
$$x \in \{-5,2\}$$



$$y < 0 \Rightarrow x \in (-\infty, -5) \cup (2, \infty)$$
$$y > 0 \Rightarrow x \in (-5,2)$$

Example 3.65

Solve the following inequalities for integers. Note that the only difference between the four parts lie in the condition. The expression is the same for all parts.

A. $x^2 + 13x + 22 > 0$
B. $x^2 + 13x + 22 \geq 0$
C. $x^2 + 13x + 22 < 0$
D. $x^2 + 13x + 22 \leq 0$

Find the Critical Points

Solve the corresponding equation, found by replacing the inequality sign with equations.

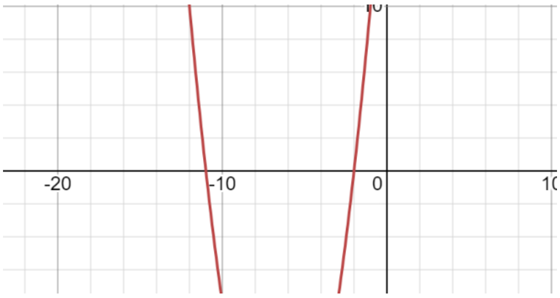
$$x^2 + 13x + 22 = 0 \Rightarrow (x+2)(x+11) = 0$$

Hence, the critical points are:

$$x \in \underbrace{\{-11,-2\}}_{\text{Set,not an interval}}$$

Make a Sign Diagram

The critical points divide the number line into three intervals.



Consider $x^2 + 13x + 22$:

$$\Rightarrow \underbrace{Leading\ Coefficient}_{\text{Coefficient of } x^2} = 1 \Rightarrow +ve \Rightarrow Upward\ Parabola$$

Since the equation represents an upward parabola, the first interval will be positive. After the first interval, the signs will alternate between positive and negative.

$(-\infty,-11)$	$(-11,-2)$	$(-2,\infty)$
$+ve$	$-ve$	$+ve$

Part A: Find the positive intervals

The first interval and the last interval are positive. Write the solution:

$$\underbrace{x < -11\ OR\ x > -2}_{\text{Inequality Notation}} \Leftrightarrow \underbrace{(-\infty,-11) \cup (-2,\infty)}_{\text{Interval Notation}}$$

Part B: Find the positive intervals, and include zero

The only difference between Part A and Part B is that zero values are acceptable for Part B, but not acceptable for Part A.

Therefore, the only change that we need to make from the answer for Part A is to include the endpoints of the interval that we got for Part A.

$$\underbrace{(x \le -11\ OR\ x \ge -2)}_{\text{Inequality Notation}} \Leftrightarrow \underbrace{(-\infty,-11] \cup [-2,\infty)}_{\text{Interval Notation}}$$

Part C: Find the negative intervals

Use the table to find the intervals where the expression is negative

$$\underbrace{-11 < x < -2}_{\text{Inequality Notation}} \Leftrightarrow \underbrace{(-11,-2)}_{\text{Interval Notation}}$$

Part D: Find the negative intervals, and include zero

This is the same as Part C, except that we also add zero:

$$\underbrace{-11 \le x \le -2}_{\text{Inequality Notation}} \Leftrightarrow \underbrace{[-11,-2]}_{\text{Interval Notation}}$$

Example 3.66

Find the values of x where the expression $x^2 + 5x + 6$ is positive, zero, and negative. Hence, solve

A. $x^2 + 5x + 6 > 0$
B. $x^2 + 5x + 6 < 0$
C. $x^2 + 5x + 6 = 0$

$$y = x^2 + 5x + 6 = 0 \Rightarrow (x+2)(x+3) = 0 \Rightarrow x \in \{-2,-3\}$$

- Roots are -2 and -3
- Coefficient of $x^2 = 1 \Rightarrow +ve \Rightarrow$ Upward parabola

$$Interval\ of\ x\ values\ where\ y\ is + ve = (-\infty, -3) \cup (-2, \infty)$$
$$Interval\ of\ x\ values\ where\ y\ is - ve = (-3, -2)$$
$$Zeroes\ of\ y = \{-3, -2\}$$

$$x^2 + 5x + 6 > 0 \Rightarrow x \in (-\infty, -3) \cup (-2, \infty)$$
$$x^2 + 5x + 6 < 0 \Rightarrow x \in (-3, -2)$$
$$x^2 + 5x + 6 = 0 \Rightarrow x \in \{-3, -2\}$$

Example 3.67

Answer the following as a statement, in inequality notation and in interval notation.
When is $-(x\text{^}2 + 5x + 6)$:

A. Positive
B. Negative
C. Non-negative
D. Non-negative

$$x^2 + 5x + 6 = 0 \Rightarrow (x + 2)(x + 3) = 0 \Rightarrow x \in \{-3, -2\}$$

Part A: Positive

$$x\ is\ between\ -3\ and\ -2$$
$$-3 < x < -2$$
$$(-3, -2)$$

Part B: Negative

$$x\ is\ less\ than -3\ OR\ x\ is\ greater\ than\ -2$$
$$x < -3\ OR\ x > -2$$
$$(-\infty, -3) \cup (-2, \infty)$$

Part C: Non-negative

$$x\ is\ between\ -3\ and\ -2\ (including\ the\ endpoints)$$
$$-3 \le x \le 2$$
$$[-3, -2]$$

Part D: Not Positive

$$x\ is\ less\ than\ or\ equal\ to\ -3\ OR\ x\ is\ greater\ than\ or\ equal\ to\ -2$$
$$x \le -3\ OR\ x \ge -2$$
$$(-\infty, -3] \cup [-2, \infty)$$

Example 3.68

When a number is multiplied by five more than itself, the result is greater than -6. Find the range of values that the number can take.

$$x(x + 5) > -6 \Rightarrow x^2 + 5x + 6 > 0 \Rightarrow (x + 2)(x + 3) > 0$$

To get the critical points, equate the LHS to zero:

$$(x + 2)(x + 3) = 0 \Rightarrow \underbrace{x = \{-2, -3\}}$$

Solution Set: $\underbrace{x \in (-\infty, -3) \cup (-2, \infty)}_{\text{Interval Notation}} \Leftrightarrow \underbrace{x < -3\ OR\ x > -2}_{\text{Inequality Notation}} \Leftrightarrow \underbrace{\{x |\ x \in \mathbb{R}, x < -3\ OR\ x > -2\}}_{\text{Set Builder Notation}}$

F. Back Calculations

Example 3.69

Find a if $x^2 + ax + 12 > 0$ when $x \in (-3, -4)$

Roots are $(-3, -4)$, and the parabola opens upwards.

$$x \in (-3, -4) \Rightarrow (x + 3)(x + 4) > 0 \Rightarrow x^2 + 7x + 12 > 0 \equiv x^2 + ax + 12 > 0 \Rightarrow a = 7$$

Example 3.70

The square of a number lies between 7 and 27.

A. Find the range of values that it can take.
B. If it is an integer, find its solution set.

$$7 < x^2 < 27$$
$$\sqrt{7} < x < \sqrt{27} \textbf{ OR } -\sqrt{27} < x < -\sqrt{7}$$
$$x \in \left(\sqrt{7}, \sqrt{27}\right) \cup \left(-\sqrt{27}, -\sqrt{7}\right)$$
$$x = \{\pm 3, \pm 5\}$$

Example 3.71

Find the condition for the quadratic $x^2 + bx + 8$ to have:

A. Two Real Roots
B. One Root
C. No Roots

Part A: Two Real Roots

For a quadratic $ax^2 + bx + c = 0$ to have two real roots, the discriminant must be greater than zero:

$$b^2 - 4ac > 0$$

Substitute $a = 1, c = 8$ in the above inequality, and isolate b:

$$b^2 - 32 > 0 \Rightarrow b^2 > 32$$

Convert the square into absolute values:

$$|b| > \sqrt{32} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$$

Remove the absolute sign using properties of absolute value. If you remember the properties of squares, you can directly write the step below without writing the step above.:

$$b < -4\sqrt{2} \text{ } OR \text{ } b > 4\sqrt{2} \Rightarrow b \in \left(-\infty, -4\sqrt{2}\right) \cup \left(4\sqrt{2}, \infty\right)$$

Part B: One Real Root

For a quadratic to have one root, the discriminant must be zero:

$$D = b^2 - 4ac = 0$$

Substitute $a = 1, c = 8$ in the above equation, and solve for b:

$$b^2 - 32 = 0 \Rightarrow b^2 = 32 \Rightarrow b = \pm 4\sqrt{2}$$

Part C: No Roots

For a quadratic to have two complex roots, the discriminant must be less than zero:

$$b^2 - 4ac < 0$$

Substitute $a = 1, c = 8$ in the above inequality, and isolate b:

$$b^2 - 32 < 0 \Rightarrow b^2 < 32$$

Convert the square into absolute values:

$$|b| < \sqrt{32} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$$

Remove the absolute sign using properties of absolute value. If you remember the properties of squares, you can directly write the step below without writing the step above.:

$$-4\sqrt{2} < b < 4\sqrt{2} \Rightarrow b \in \left(-4\sqrt{2}, 4\sqrt{2}\right)$$

Example 3.72

Find the condition for the quadratic $x^2 + bx + 12$ to have:

A. Two Real Roots
B. One Root
C. No Roots

$$b \in \left(-\infty, -4\sqrt{3}\right) \cup \left(4\sqrt{3}, \infty\right)$$
$$b = \pm 4\sqrt{3}$$
$$b \in \left(-4\sqrt{3}, 4\sqrt{3}\right)$$

Example 3.73: Double Quadratics

Let $f(x) = x^2 - kx + (k - 1)^2$ for some constant k. What is the largest possible real value of k such that f has at least one real root? **(CCA Math Bonanza, Individual Round, 2020/5)**

$$D = b^2 - 4ac \ge 0$$

Substitute $a = 1, b = -k, c = (k-1)^2$

$$(-k)^2 - 4(k-1)^2 \ge 0$$
$$k^2 - 4(k^2 - 2k + 1) \ge 0$$
$$-3k^2 + 8k - 4 \ge 0$$

Multiply by -1:

$$3k^2 - 8k + 4 \le 0$$

$$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{8 \pm \sqrt{64 - (4)(3)(4)}}{2(3)} = \frac{8 \pm \sqrt{16}}{6} = \frac{8 \pm 4}{6} = \left\{\frac{2}{3}, 2\right\}$$

Since the leading coefficient of the quadratic is positive, it is an upward parabola.
Hence, the region that satisfies the above inequality is:

$$\left[\frac{2}{3}, 2\right]$$

And the largest real value is

$$2$$

Example 3.74

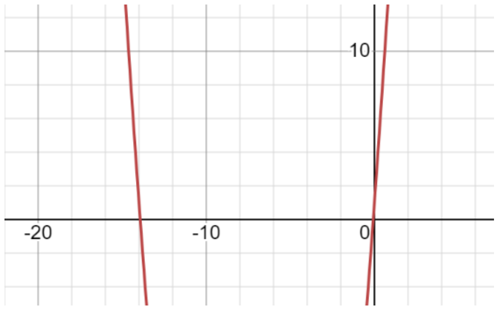
Find the value(s) of k such that $\frac{x^2+4x+3}{x^2-4x+3} = k$ has:

A. Two solutions
B. A single solution
C. No solutions

Eliminate fractions:

$$x^2 + 4x + 3 = kx^2 - 4kx + 3k$$

Collate all terms on one side, and write in standard form:
$$x^2(k-1)+x(-4k-4)+3k-3=0$$



The above expression is not a quadratic when $k=1$:
Substitute $k=1$:
$$x^2(1-1)+x(-4-4)+3-3=0 \Rightarrow -8x=0 \Rightarrow x=0$$

When $k \neq 1$, the expression is a quadratic with discriminant:
$$\underbrace{(-4)^2(k+1)^2}_{b^2}-\underbrace{4}_{4}\underbrace{(k-1)}_{a}\underbrace{(3k-3)}_{c}$$
Expand:
$$16(k^2+2k+1)-4(3k^2-3k-3k+3)$$
Factor 4 and simplify:
$$4(k^2+14k+1)$$
$k^2+14k+1$ is an upward parabola. To determine the zeroes, we use the quadratic formula:
$$k=\frac{-14\pm\sqrt{14^2-(4)(1)(1)}}{2}=\frac{-14\pm\sqrt{192}}{2}=\frac{-14\pm 8\sqrt{3}}{2}=-7\pm 4\sqrt{3}$$
Hence,
$$No\ Real\ Solutions: k \in \left(-7-4\sqrt{3},-7+4\sqrt{3}\right)$$
$$Single\ Solutions: k \in \left\{-7-4\sqrt{3},-7+4\sqrt{3}\right\} \cup \{1\}$$
$$Two\ Solutions: k \in \left(-\infty,-7-4\sqrt{3}\right) \cup \left(-7+4\sqrt{3},\infty\right)-\{1\}$$

G. Probability

Example 3.75
Integers a and b are chosen randomly and uniformly from the set $\{1,2,3,4,5\}$, but a and b are not necessarily distinct. What is the probability that the polynomial x^2+ax+b has two distinct real roots? **(P_Groudon's Mock AMC 12#5)**[3]

Total Outcomes
By the multiplication principle,
$$Total\ Outcomes = 5 \times 5 = 25$$
Successful Outcomes
We want
$$D>0 \Rightarrow a^2-4b>0 \Rightarrow a^2>4b$$
First, calculate the values of b and 4b:
$$b \in \{1,2,3,4,5\} \Rightarrow 4b \in \{4,8,12,16,20\}$$
Then, check the values of b that work for each a (see table).

a	a^2	$4b$	No. of Values
1	1		
2	4		
3	9	4, 8	2
4	16	4.8, 12	3
5	25	4,8,12,16,20	5

Probability
$$Probability=\frac{Sucessful\ Outcomes}{Total\ Outcomes}=\frac{2+3+5}{25}=\frac{10}{25}=\frac{2}{5}$$

H. Constraints/Integer Solutions
A standard number theory application is focus on the integers in a quadratic inequality. This can happen in a variety of ways:
- Constraining the solution set to include only integers
- Counting the number of distinct solutions

[3] https://artofproblemsolving.com/community/q2h1915699p13127637

- Asking for the greatest integer, or the smallest integer that satisfies an inequality

If the question has certain restrictions, then you need to ensure that the restrictions are met. Examples of restrictions:
- Ordering a number of pizzas from a restaurant (p): $p > \mathbb{N} \Leftrightarrow 0\ p \in \mathbb{Z}^+ \ \Leftrightarrow p > 0, p \in \mathbb{Z}$
- Question needs t to have integer solutions: $t \in \mathbb{Z}$

Example 3.76

$$x^2 - x \geq 0$$

$$x(x-1) \geq 0 \Rightarrow x = \{0,1\}$$

Example 3.77

$$You\ solved\ this\ same\ question\ for\ real\ numbers\ before.$$

Solve the following inequalities for integers. Note that the only difference between the four parts lie in the condition. The expression is the same for all parts.

A. $x^2 + 13x + 22 > 0$
B. $x^2 + 13x + 22 \geq 0$
C. $x^2 + 13x + 22 < 0$
D. $x^2 + 13x + 22 \leq 0$

Part A

We need to take the solution from the earlier example:

$$\underbrace{(x < -11\ OR\ x > -2)}_{Inequality\ Notation}$$

And rewrite it to only allow integers for solutions. We can do this by writing in set theory notation and imposing the condition that x must be an integer.

$$\left\{x \underbrace{|}_{such\ that} x < -11, x \in \mathbb{Z}\right\} \cup \left\{x \underbrace{|}_{such\ that} x > -2, x \in \mathbb{Z}\right\}$$

Writing it without set theory notation

Set theory provides an elegant way of writing the answer, but it is not mandatory. For example, the same answer as written as also be correctly written as:

$$x\ is\ an\ integer\ less\ than\ -11, or\ greater\ than\ -2.$$

Part B

$$Type\ I: \left\{x \underbrace{|}_{such\ that} x \leq -11, x \in \mathbb{Z}\right\} \cup \left\{x \underbrace{|}_{such\ that} x \geq -2, x \in \mathbb{Z}\right\}$$

Because we only allow integer solutions, this can be written as:

$$Type\ II: \left\{x \underbrace{|}_{such\ that} x < -10, x \in \mathbb{Z}\right\} \cup \left\{x \underbrace{|}_{such\ that} x > -3, x \in \mathbb{Z}\right\}$$

We could not have written the $Type\ II$ answer if we had been considering real numbers.

And we can also write it in verbal form like this:

$$x\ is\ an\ integer\ less\ than\ or\ equal\ to\ -11, or\ greater\ than\ or\ equal\ to\ -2.$$

Part C

$$\left\{x \underbrace{|}_{such\ that} -11 < x < -2, x \in \mathbb{Z}\right\}$$

Part D

$$\left\{x \underbrace{|}_{such\ that} -11 \le x \le -2, x \in \mathbb{Z}\right\}$$

Example 3.78

Find the number of integers that satisfy $6 - 8x(1 - x) > 0$

$$6 - 8x + 8x^2 > 0 \Rightarrow 4x^2 - 4x + 3 > 0$$

Calculate the discriminant of the quadratic

$$b^2 - 4ac = 16 - (4)(4)(3) = 16 - 48 < 0$$

Hence, the quadratic is never less than zero.
Hence, an infinite number of integers satisfy the inequality.

I. Smallest and Largest Integer

Example 3.79

Find the smallest integer that is not a valid solution for

A. $x^2 + 13x + 22 > 0$
B. $x^2 + 13x + 22 \ge 0$

Part A

The solution set for the inequality is:

$$\left\{x \underbrace{|}_{such\ that} x < -11, x \in \mathbb{Z}\right\} \cup \left\{x \underbrace{|}_{such\ that} x > -2, x \in \mathbb{Z}\right\}$$

The question is asking the smallest integer that is not a valid solution. Hence, we need to find all integers which do not satisfy the above solution set, which is given by:

$$\left\{x \underbrace{|}_{such\ that} -11 \le x \le -2, x \in \mathbb{Z}\right\}$$

And from the above, we want the smallest integer, which is

$$-11$$

Part B

The solution set for the inequality is:

$$Type\ I: \left\{x \underbrace{|}_{such\ that} x \le -11, x \in \mathbb{Z}\right\} \cup \left\{x \underbrace{|}_{such\ that} x \ge -2, x \in \mathbb{Z}\right\}$$

We want the smallest integer that does not satisfy this inequality and hence we reverse the solution set, giving us:

$$\left\{x \underbrace{|}_{such\ that} -11 < x < 2, x \in \mathbb{Z}\right\}$$

Hence, the smallest number is

$$-10$$

Practice 3.80

For each part below, find the smallest integer that is not a valid solution.

A. $x^2 - 6x - 91 > 0$
B. $x^2 - 6x - 91 \geq 0$

$$x^2 - 6x - 91 = 0 \Rightarrow (x - 13)(x + 7) = 0 \Rightarrow x \in \{-7,13\}$$

$$\underbrace{(x < -7\ OR\ x > 13)}_{Inequality\ Notation} \Leftrightarrow \underbrace{(-\infty, -7) \cup (13, \infty)}_{Interval\ Notation}$$

$$\underbrace{(x \leq -7\ OR\ x \geq 13)}_{Inequality\ Notation} \Leftrightarrow \underbrace{(-\infty, -7] \cup [13, \infty)}_{Interval\ Notation}$$

Example 3.81

Find the largest integer that is a valid solution for

A. $x^2 + 13x + 22 < 0$
B. $x^2 + 13x + 22 \leq 0$

Part A

Find the valid solutions, which are:

$$\left\{x \underbrace{|}_{such\ that} -11 < x < -2, x \in \mathbb{Z}\right\}$$

And the largest integer which satisfies this inequality is:

$$-3$$

Part B

Find the valid solutions, which are:

$$\left\{x \underbrace{|}_{such\ that} -11 \leq x \leq -2, x \in \mathbb{Z}\right\}$$

And the largest integer which satisfies this inequality is:

$$-2$$

Practice 3.82

For each part below, find the largest integer that is not a valid solution.

A. $x^2 - 6x - 91 > 0$
B. $x^2 - 6x - 91 \geq 0$

$$\underbrace{-7 < x < 13}_{Inequality\ Notation} \Leftrightarrow \underbrace{(-7,13)}_{Interval\ Notation}$$

$$\underbrace{-7 \leq x \leq 13}_{Inequality\ Notation} \Leftrightarrow \underbrace{[-7,13]}_{Interval\ Notation}$$

Example 3.83

Find the sum of the smallest and the largest integer that satisfies $x^2 + 5x - 14 > 0$

$(x+7)(x-2) > 0 \Rightarrow x \in (-7,2) \Rightarrow \text{Max}\{(-7,2)\} + \text{Min}\{(-7,2)\} = 1 - 6 = -5$

Example 3.84

Find the largest whole number n such that

$$(1+2+\cdots+n)^2 < 1^3 + 2^3 + 3^3 + \cdots + 2021^3$$

$$\left(\frac{n(n+1)}{2}\right)^2 \le \left(\frac{2021(2021+1)}{2}\right)^2 \Rightarrow n < 2021 \Rightarrow \max(n) = 2020$$

J. Systems of Inequalities

Example 3.85

For the universal set $U \in [-100,100]$ find $(A \cup B) \cap (A \cap B)'$:

$$A: 6x^2 + 23x + 24 < 4$$
$$B: 5x^2 - 23x - 10 > 0$$

Condition A

$$(3x+4)(2x+5) < 0$$
$$\textit{Critical Points are } x \in \left\{-\frac{4}{3}, -\frac{5}{2}\right\}$$
$$A: x \in \left(-\frac{5}{2}, -\frac{4}{3}\right)$$

Condition B

$$(x-5)(5x+2) > 0$$
$$\textit{Critical Points are } x \in \left\{-\frac{2}{5}, 5\right\}$$
$$B: \left[-100, -\frac{2}{5}\right) \cup (5,100]$$

Final Answer

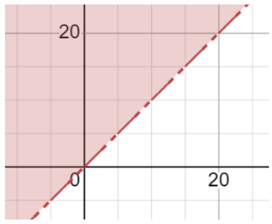
$$A \cup B = B$$
$$A \cap B = A$$
$$(A \cap B)' = A' = \left[-100, -\frac{5}{2}\right] \cup \left[-\frac{4}{3}, 100\right]$$
$$(A \cup B) \cap (A \cap B)' = B \cap A' = \left[-100, -\frac{5}{2}\right] \cup \left[-\frac{4}{3}, -\frac{2}{5}\right) \cup (5,100]$$

K. Geometry

Challenge 3.86

A rectangle has area more than 40, but perimeter less than 40. Find the range of values that the width can take. (As is usual, assume that the width of a rectangle must be less than its length).

Condition I: Width



Let

$$width = w = x, length = l = y$$

Then, we must have

$$x < y$$

Which is graphed alongside

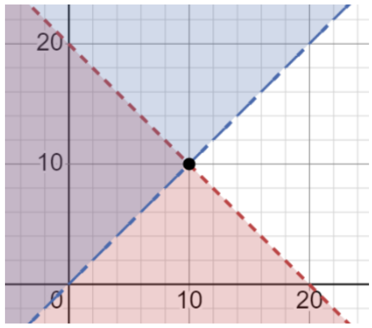
Condition II: Perimeter

The perimeter must be less than 40, which gives us:

$$p < 40 \Rightarrow 2(x + y) < 40 \Rightarrow x + y < 20$$

Hence, we know that the sum of the length and the width must be less than 40.

Combine Condition I and II



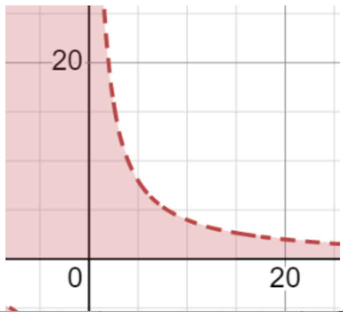
If

$$x > 10, y > x > 10 \Rightarrow x + y > 10 + 10 > 20 \Rightarrow \textbf{\textit{Contradiction}}$$

Hence, the width must be less than

$$10$$

Condition on Area

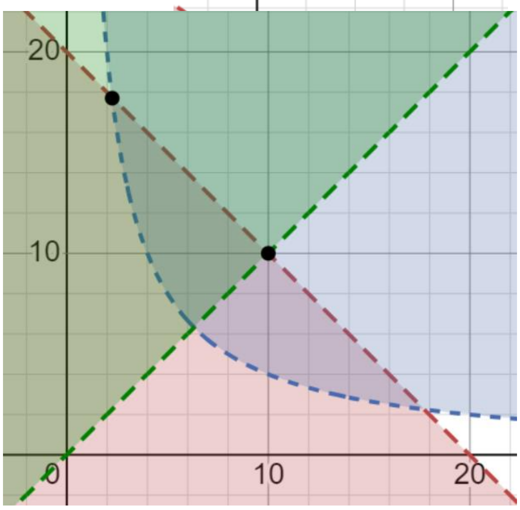


The area must be greater than 40, which tells us that:

$$y > 40 \Rightarrow x > \frac{40}{y}$$

This is the graph of a hyperbola, which we have not studied so far.

Combining the Three Conditions



We can combine the three conditions to get the graph alongside.

Consider the line corresponding to the inequality $x + y < 20$:

$$x + y = 20 \Rightarrow x = 20 - y$$

Consider the hyperbola corresponding to the inequality $xy < 40$:

$$xy = 40 \Rightarrow x = \frac{40}{y}$$

At the point of intersection of the hyperbola, and the line both x values must be equal:

$$\frac{40}{y} = 20 - y \Rightarrow 40 = 20y - y^2 \Rightarrow y^2 - 20y + 40 = 0$$

Solve using the quadratic formula to get the value of y:

$$y = 10 \pm 2\sqrt{15}$$

And the smaller value is the lowest possible value for the width

$$Width = x = 10 - 2\sqrt{15}$$

Final Answer
And hence the final answer is

$$10 - 2\sqrt{10} < w < 10$$

L. Real Life Scenarios

Example 3.87

At least 221 soldiers are marching in a rectangular formation. The number of rows is four less than the number of columns. Find the range that the number of rows can take.

$$x(x+4) \geq 221 \Rightarrow x^2 + 4x - 221 \geq 0 \Rightarrow (x-13)(x+17) > 0$$

From the Inequality: $x \in (-\infty, -\mathbf{17}] \cup [13, \infty)$

Reject negative numbers since the numbers of rows cannot be negative.
Solution Set for the question: $\underbrace{x \in [13, \infty), x \in \mathbb{Z}}_{\text{Interval Notation}} \Leftrightarrow \underbrace{x \geq 13, x \in \mathbb{Z}}_{\text{Inequality Notation}} \Leftrightarrow \underbrace{\{x \mid x \in \mathbb{Z}, \quad x \geq 13\}}_{\text{Set Builder Notation}}$

M. Wavy Curve Method

Example 3.88

$$\frac{x+2}{x-4} \geq 0$$

Find the Critical Points:

$$x + 2 = 0 \Rightarrow x = -2, \qquad x - 4 = 0 \Rightarrow x = 4 \Rightarrow \textit{Critical points}: x \in \{-2, 4\}$$

The critical points divide the number line into three regions:

$$\underbrace{(-\infty, -2)}_{\textit{Region I}} \cup \underbrace{(-2, 4)}_{\textit{Region II}} \cup \underbrace{(4, \infty)}_{\textit{Region III}}$$

Check Region I by substituting $x = -3$:

$$\frac{-3+2}{-3-4} = \frac{-1}{-7} = \frac{1}{7} = +ve$$

Check Region II by substituting $x = 0$:

$$\frac{2}{-4} = -\frac{1}{2} = -ve$$

Check Region III by substituting $x = 5$:

$$\frac{5+2}{5-4} = \frac{7}{1} = 7 = +ve$$

If you see consecutive regions, they alternate between positive and negative. This is not a coincidence.

3.5 Vieta's Formulas

A. Sum and Product of Roots

The sum and product of roots of a quadratic are important functions by themselves.

3.89: Sum of Roots(Factored Form)

For a quadratic $ax^2 + bx + c, a \neq 0$ and roots α and β:

$$\textit{Sum of Roots} = \alpha + \beta = -\frac{b}{a}$$

$$Product\ of\ Roots = \alpha\beta = \frac{c}{a}$$

Expand the factored form of a quadratic $a(x-\alpha)(x-\beta)=0$ to get:

$$a[x^2 - \alpha x - \beta x + \alpha\beta] = 0$$

Factor:

$$ax^2 - a(\alpha+\beta)x + a(\alpha\beta) = 0$$

From the above, we can see that:

$$Sum\ of\ Roots = \alpha + \beta = \frac{Negative\ of\ the\ x\ term\ coefficient}{Square\ term\ coefficient}$$

$$Product\ of\ Roots = \alpha\beta = \frac{Constant\ Term}{Square\ term\ coefficient}$$

3.90: Sum of Roots (Quadratic Formula)

For a quadratic $ax^2 + bx + c, a \neq 0$ and roots α and β:

$$\alpha + \beta = -\frac{b}{a}$$

$$\alpha\beta = \frac{c}{a}$$

Sum of Roots

The quadratic formula gives us the roots of a quadratic. Add the roots:

$$\text{Sum of Roots} = \underbrace{\frac{-b + \sqrt{\boldsymbol{b^2 - 4ac}}}{2a}}_{1st\ Root} + \underbrace{\frac{-b - \sqrt{\boldsymbol{b^2 - 4ac}}}{2a}}_{2nd\ Root} = -\frac{2b}{2a} = -\frac{b}{a}$$

Product of Roots

Multiply the roots

$$\left(\frac{\boldsymbol{-b} + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{\boldsymbol{-b} - \sqrt{b^2 - 4ac}}{2a}\right)$$

The terms in the numerator follow the pattern $(\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{a} - \boldsymbol{b}) = a^2 - \boldsymbol{b^2}$:

$$= \frac{(-b)^2 - \left(\sqrt{b^2 - 4ac}\right)^2}{4a^2}$$

The square and the square root cancel:

$$\frac{b^2 - b^2 + 4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

Example 3.91

Find the sum and product of the roots of $x^2 + 7x + 10 = 0$ by:

A. solving the equation to find the roots, and then calculating the sum and product of those roots.
B. using the formula for the sum, and the formula for the product

Part A

$$x^2 + 7x + 10 = 0$$
$$(x+2)(x+5) = 0$$
$$x \in \{-2, -5\}$$
$$Sum = -7$$

$$Product = 10$$

Part B

Using the formula with $a = 1, b = 7, c = 10$:

$$Sum = -\frac{b}{a} = -\frac{7}{1} = -7$$

$$Product = \frac{c}{a} = \frac{10}{1} = 10$$

Example 3.92

Consider the equation

$$x^2 + 3x + 4 = 0$$

A. Use the discriminant to show that the roots are complex.
B. Use the quadratic formula to find the roots, and hence find the sum and the product of the roots.
C. Use the sum and product formulas established above to find the sum and product of the roots
D. Which method is easier?

Part A

The discriminant of the quadratic is:

$$b^2 - 4ac = 3^2 - 4(1)(4) = 9 - 16 = -7$$

Since the discriminant is negative, the roots are complex.

We can calculate the roots using the quadratic formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{-7}}{2}$$

The sum of the roots:

$$= \frac{-3 + \sqrt{-7}}{2} + \frac{-3 - \sqrt{-7}}{2} = -\frac{6}{2} = -3$$

The product of the roots

$$= \left(\frac{-3 + \sqrt{-7}}{2}\right)\left(\frac{-3 - \sqrt{-7}}{2}\right) = \frac{9 - \left(\sqrt{-7}\right)^2}{4} = \frac{9 + 7}{4} = \frac{16}{4} = 4$$

Part B

$$Sum = -\frac{b}{a} = -\frac{3}{1} = -3$$

$$Product = \frac{c}{a} = \frac{4}{1} = 4$$

Example 3.93

What is the sum of all possible solutions of $\frac{9x}{13} + \frac{13}{9x} = \frac{250}{117x}$? (MathCounts 2008 National Countdown)

Multiply throughout by the LCM of the three denominators which is $117x$:

$$81x^2 + 169 - 250 = 0$$

$$Sum\ of\ Roots = -\frac{b}{a} = -\frac{0}{81} = 0$$

Example 3.94

The sum of the roots of the equation $4x^2 + 5 - 8x = 0$ is equal to: (AHSME 1950/3)

Write the equation first in standard form

$$4x^2 - 8x + 5 = 0$$

The values are;

$$a = 4, b = -8$$

Use the formula to calculate the sum of roots:

$$= -\frac{b}{a} = -\frac{-8}{4} = 2$$

Example 3.95

Let f be a function for which $f\left(\frac{x}{3}\right) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$? **(AMC 10 2000/24, AMC 12 2000/15)**

Method I

Solve the equation

$$x^2 + x + 1 = 7 \Rightarrow x^2 + x - 6 = 0 \Rightarrow Sum\ of\ Roots = -\frac{b}{a} = -\frac{1}{1} = -1$$

Hence, find the relation between $3z$ and $\frac{x}{3}$:

$$3z = \frac{x}{3} \Rightarrow z = \frac{x}{9} \Rightarrow Sum\ of\ Roots = -\frac{1}{9}$$

Method II

$$3z = \frac{x}{3} \Rightarrow x = 9z$$

$$f(3z) = (9z)^2 + 9z + 1 = 81z^2 + 9z + 1 = 0$$

$$Sum\ of\ Roots = -\frac{b}{a} = -\frac{9}{81} = -\frac{1}{9}$$

$$f\left(\frac{x}{3}\right) = x^2 + x + 1$$

$$f\left(\frac{x}{3}\right) = x^2 + x + 1$$

3.96: Sum of Integral Roots

The formula for the sum of roots is applicable irrespective of the nature of roots: real, integral, rational or complex.

If there are conditions on the roots, these need to be checked separately.

Example 3.97

Find the sum of all integral solutions[4] to:

$$4x^2 + 8x + 3 = 0$$

The sum of the roots is

---

[4] Where integral solutions means integers, and not integrals from calculus.

$$= -\frac{b}{a} = -\frac{4}{2} = -2$$

But this is not the sum of the integral solutions.

In fact, when we find the solutions using the quadratic formula, we get two roots, none of which is an integer:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{64 - (4)(4)(3)}}{2(4)} = \frac{-8 \pm 4}{8} = -1 \pm \frac{1}{2} = \left\{-\frac{1}{2}, -\frac{3}{2}\right\}$$

Example 3.98

The question "The square of an integer is 182 greater than the integer itself. What is the sum of all integers for which this is true? **(MathCounts 2005 National Countdown)**" is solved as:

Step I: Frame an equation: $x^2 = x + 182$

Step II: Collate all terms on one side: $x^2 - x - 182 = 0$

Step III: $Required\ Sum = Sum\ of\ Roots = -\frac{b}{a} = -\frac{-1}{1} = 1$

With respect to the question and solution above, choose True or False for each statement below:

A. Step I is correct, with correct reasoning
B. Step II is correct, with correct reasoning
C. Step III is correct, with correct reasoning
D. Final answer of 1 is correct

Step I is correct, with correct reasoning. True.
Step II is correct, with correct reasoning. True.

Step III uses the formula for sum of roots. It does not check whether the roots are integers, which a complete solution should.

$$(x - 13)(x + 14) = 0 \Rightarrow Both\ roots\ are\ integers$$

Hence, we can use the formula for sum of roots.

The final answer of 1 is correct, since the formula for sum of roots can be used since both the roots are integers.

Example 3.99: System of Equations

Suppose that a and b are nonzero real numbers, and that the equation $x^2 + ax + b = 0$ has solutions a and b. What is the pair (a, b)? **(AMC 10B 2002/10)**

The sum of roots

$$a + b = -\frac{a}{1} \Rightarrow b = -2a$$

The product of roots

$$ab = \frac{b}{1} \Rightarrow a = 1$$

$$b = -2a = -2$$

Example 3.100: Back Calculations

If both roots of the quadratic equation: $x^2 - 63x + k = 0$ are

A. prime numbers, how many possible values of k are there? **(AMC 10 2002/14)**
B. natural numbers, how many possible values of k are there?
C. integers, how many possible positive values of k are there?

D. Integers, how many possible values of k are there?
E. positive fractions of the form $\frac{p}{q}$, where p is a natural number, and q is either one or two, how many possible values of k are there?

Part A

$$Sum\ of\ Roots = -(-63) = 63$$

The problem reduces to finding the ways that 63 can be written as the sum of two prime numbers p_1, and p_2.

$$\underbrace{63}_{Odd} = \underbrace{p_1}_{Odd} + \underbrace{p_2}_{Even}$$

Apply parity to the above equation. One prime must be odd, and the other must be even.

$$63 = 61 + 2 \Rightarrow \boldsymbol{One\ Solution\ (Answer)}$$

Since, the only even prime is 2, we get only one solution.

Part B

$$63 = p_1 + p_2$$

$$(1,62), (2,61), (3,60), \dots, (62,1) \Rightarrow 62\ Pairs$$

But note the pairs start to repeat since:

$$k = 1 \times 62 = 62 \times 1$$

Is the same pair.

$$No.\ of\ possible\ values\ of\ k = \frac{62}{2} = 31$$

Part C

For k to be positive:

$$p_1 p_2 > 0 \Rightarrow Both\ are\ +ve, or\ both\ are\ -ve$$

Both p_1 and p_2 cannot both be negative, since the sum would then be negative. Hence, the answer is the same
as the earlier solution

$$31\ Possible\ Values$$

Part D

Since p_1 and p_2 are integers, there is no restriction on their value. So, we can have

$$(-2,65), (-1,64), (0,63), (1,62), \dots.$$

$$\Rightarrow Infinite\ Values\ for\ k$$

Part E

3.101: Linear Term Zero

If the linear term of a quadratic is zero, then the sum of the roots is also zero.

$$ax^2 + bx + c$$

Since the linear term is zero, $b = 0$, and we have

$$Sum\ of\ Roots = -\frac{b}{a} = -\frac{0}{a} = 0$$

Example 3.102: Ratio of Roots

Ratio of roots

Example 3.103: Domain

Sum of values not defined

Example 3.104: Symmetric Functions

Symmetric functions of roots are those where replacing α with β does not change the expression. These expressions have importance in higher level maths, and hence it is worth studying them.
Some expressions are given below. State whether the expressions are symmetric or not.

A. $\alpha + \beta$
B. $a^2 + \beta^2$
C. $\alpha + \frac{1}{\beta}$
D. $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$
E. $a^2 + \beta^2 + \alpha\beta$

$$Symmetric: A, B, D, E$$
$$Not\ Symmetric: C$$

$$\alpha + \frac{1}{\beta} \neq \beta + \frac{1}{\alpha}$$

Example 3.105: Reciprocals

Find the sum of the reciprocals of the roots of the equation:

A. $3x^2 + 4x + 2 = 0$
B. $x^2 + 4x + 2 = 0$
C. $ax^2 + bx + c = 0$

Part A

$$Sum\ of\ Roots = -\frac{b}{a} = -\frac{4}{3} = -\frac{4}{3}$$
$$Product\ of\ Roots = \frac{c}{a} = \frac{2}{3} = \frac{2}{3}$$
$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta}{\alpha\beta} + \frac{\alpha}{\alpha\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{Sum\ of\ Roots}{Product\ of\ Roots} = -\frac{4}{3} \div \frac{2}{3} = -\frac{4}{3} \times \frac{3}{2} = -\frac{4}{2} = -2$$

Part B

Part C

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta}{\alpha\beta} + \frac{\alpha}{\alpha\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{Sum\ of\ Roots}{Product\ of\ Roots} = -\frac{b}{a} \div \frac{c}{a} = -\frac{b}{a} \times \frac{a}{c} = -\frac{b}{c}$$

Example 3.106: Sum of Squares

A. Show that for a quadratic with roots α and β, we must have $\alpha^2 + \beta^2 = \left(\underbrace{\alpha + \beta}_{SOR}\right)^2 - 2\underbrace{\alpha\beta}_{POR}$
B. Hence, find the sum of the squares of the roots of the equation: $2x^2 + 5x + 2 = 0$

Part A

We want to find:

$$\alpha^2 + \beta^2$$

Add and subtract $2\alpha\beta$:

$$\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta$$

The first three terms form a perfect square:

$$(\alpha + \beta)^2 - 2\alpha\beta$$

To find the sum of the squares of the roots, we add and subtract $2\alpha\beta$, which gives us the expression for the square of the sum of the roots, and the residual is twice the product of the roots.

Part B

$$(\alpha + \beta)^2 - 2\alpha\beta = \left(-\frac{5}{2}\right)^2 - 2\left(\frac{2}{2}\right) = \frac{25}{4} - 2 = \frac{17}{4}$$

Example 3.107: Reciprocals of Squares

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$
$$\frac{1}{\alpha} + \frac{1}{\beta} \to \left(\frac{1}{\alpha}\right)^2 + \left(\frac{1}{\beta}\right)^2 = \frac{1}{\alpha^2} + \frac{1}{\beta^2}$$

So, this is the same as **the sum of the reciprocals of the squares of the roots**.

B. Perfect Powers

If an expression can be factorized into an expression that only has $\alpha\beta$ and $\alpha+\beta$ then the value can be determined using the formulas for sum and product of roots.

Example 3.108

Given that p and q are the roots of the equation $3x^2+4x+7$, find the value of:

A. $p^2+2pq+q^2$
B. $p^3+3p^2q+3pq^2+q^3$

$$p^2+2pq+q^2=(p+q)^2=\left(-\frac{b}{a}\right)^2=\left(-\frac{4}{3}\right)^2=\frac{16}{9}$$
$$p^3+3p^2q+3pq^2+q^3=(p+q)^3=\left(-\frac{b}{a}\right)^3=\left(-\frac{4}{3}\right)^3=-\frac{64}{27}$$

Example 3.109

Given that p and q are the roots of the equation $5x^2-9x+7$, find the value of:

$$\sqrt[8]{p^3+4p^3q+6p^2q^2+4pq^3+q^4}$$

$$\sqrt[8]{p^3+4p^3q+6p^2q^2+4pq^3+q^4}=\sqrt[8]{(p+q)^4}=\sqrt{p+q}=\sqrt{-\frac{-9}{5}}=\frac{3}{\sqrt{5}}=\frac{3\sqrt{5}}{5}$$

C. Other Factorizations

Example 3.110

D. More Expressions

Example 3.111

Convert the expressions below into equivalent expressions into terms of the sum of roots $(\alpha+\beta)$, and the product of roots $(\alpha\beta)$ of a quadratic equation:

A. $\alpha^2\beta+\beta^2\alpha$
B. $(\alpha-1)(\beta-1)$
C. $\frac{1}{\alpha-1}+\frac{1}{\beta-1}$

$$\alpha^2\beta+\beta^2\alpha=(\alpha\beta)(\alpha+\beta)$$

$$(\alpha-1)(\beta-1)=\alpha\beta-\alpha-\beta+1=\alpha\beta-(\alpha+\beta)+1$$

$$\frac{1}{\alpha-1}+\frac{1}{\beta-1}=\frac{\beta-1+\alpha-1}{(\alpha-1)(\beta-1)}=\frac{\beta+\alpha-2}{\alpha\beta-(\alpha+\beta)+1}$$

E. Multiple Equations

Example 3.112

Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + \left(\frac{1}{b}\right)$ and $b + \left(\frac{1}{a}\right)$ are the roots of the equation $x^2 - px + q = 0$. What is q? **(AMC 10B 2006/14)**

Consider $x^2 - px + q = 0$:

$$POR = \frac{c}{a} = \frac{q}{1} = q$$
$$POR = \left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) = ab + 2 + \frac{1}{ab}$$

Consider $x^2 - mx + 2 = 0$:

$$POR = \frac{c}{a} = \frac{2}{1} = 2$$

Substitute 2 in $ab + 2 + \frac{1}{ab}$ to get:

$$2 + 2 + \frac{1}{2} = \frac{9}{2}$$

F. Answering in Terms of Variables

Apart from the expressions above, there can be a large variety of expressions, which can be reduced to sum and product of roots. We look at a few examples below.

Example 3.113

Let the quadratic equation $ax^2 + bx + c = 0$ have roots α and β. Find in terms of the coefficients of the given quadratic:

A. $(\alpha + 1)(\beta + 1)$
B. $\frac{1}{\alpha+1} + \frac{1}{\beta+1}$

The coefficients of the quadratic are a, b and c. Hence, our answer must be in terms of these variables only.

$$(\alpha + 1)(\beta + 1) = \underbrace{\alpha\beta}_{POR} + \underbrace{\alpha + \beta}_{SOR} + 1 = \frac{c}{a} - \frac{b}{a} + 1 = \frac{c - b + 1}{a}$$

$$\frac{1}{\alpha + 1} + \frac{1}{\beta + 1} = \frac{(\beta + 1) + (\alpha + 1)}{(\alpha + 1)(\beta + 1)} = \frac{\overbrace{\alpha + \beta}^{SOR} + 2}{\underbrace{\alpha\beta}_{POR} + \underbrace{\alpha + \beta}_{SOR} + 1} = \frac{-\frac{b}{a} + 2}{\frac{c - b + 1}{a}}$$

G. Review

Review 3.114

Find the following for $\frac{3}{4}x^2 - \frac{2}{3}x + \frac{7}{2}$:

A. Sum of Roots
B. Product of Roots
C. Sum of reciprocals of roots
D. Sum of squares of roots
E. Sum of reciprocals of squares of roots

	Memorize this	

$\alpha+\beta$	$-\frac{b}{a}$	$\frac{2}{3}\div\frac{3}{4}=\frac{2}{3}\times\frac{4}{3}=\frac{8}{9}$
$\alpha\beta$	$\frac{c}{a}$	$\frac{7}{2}\div\frac{3}{4}=\frac{14}{3}$
$\frac{1}{\alpha}+\frac{1}{\beta}$	$\frac{\alpha+\beta}{\alpha\beta}$	$\frac{8}{9}\div\frac{14}{3}=\frac{8}{9}\times\frac{3}{14}=\frac{4}{21}$
$\alpha^2+\beta^2$	$(\alpha+\beta)^2-2\alpha\beta$	$\left(\frac{8}{9}\right)^2-2\left(\frac{14}{3}\right)=\frac{64}{81}-\frac{28}{3}=-\frac{692}{81}$
$\frac{1}{\alpha^2}+\frac{1}{\beta^2}$	$\frac{\alpha^2+\beta^2}{(\alpha\beta)^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$	

3.6 Transformations of Roots

A. Introduction

We know a few methods to find the roots of a quadratic. We also know certain formulas regarding the sum and product of the roots of a quadratic.
In this chapter, we look at quadratics that have roots that are related in some way to the roots of another quadratic.

B. Shifting: Algebraic Approach

3.115: Roots increased by k

Consider the quadratic $ax^2+bx+c=0$ with roots α,β. The quadratic with roots $\alpha+k,\beta+k$ is given by:

$$a(x-k)^2+b(x-k)+c=0$$

Note:

$$\textit{Roots increase by } k \Rightarrow \textit{Substitute } x-k$$

Consider the original equation. It has roots α and β and hence, it must be of the form:

$$\underbrace{(x-\alpha)(x-\beta)=0}_{\textit{Original Equation}}$$

The new equation has roots $\alpha+k$ and $\beta+k$ and hence it must be of the form:

$$[x-(\alpha+k)][x-(\beta+k)]=0$$
$$[x-\alpha-k][x-\beta-k]=0$$
$$\underbrace{[(x-k)-\alpha][(x-k)-\beta]=0}_{\textit{New Equation}}$$

Note that replacing x with $x-k$ in the original equation changes the roots from α,β to $\alpha+k,\beta+k$:

$$\underbrace{(x-\alpha)(x-\beta)=0}_{\textit{Original Equation}}\rightarrow\underbrace{[(x-k)-\alpha][(x-k)-\beta]=0}_{\textit{New Equation}}$$

Making the substitution $x-k$ for x gives us:

$$a(x-k)^2+b(x-k)+c=0$$

Example 3.116

Expand

$$a(x-k)^2+b(x-k)+c=0$$

$$a(x^2 - 2kx + k^2) + bx - bk + c = 0$$
$$ax^2 - 2akx + ak^2 + bx - bk + c = 0$$
$$ax^2 - (2ak - b)x + ak^2 - bk + c = 0$$

Example 3.117

Find the quadratic equation each of whose roots are 3 greater than the roots of the quadratic equation:

$$x^2 + 5x + 6 = 0$$

Method I: Finding the Roots and then the New Equation

$$(x + 2)(x + 3) = 0 \Rightarrow x \in \{-2, -3\}$$

Add 3 to each of the roots:

$$x \in \{-2 + 3, -3 + 3\} = \{0,1\}$$

And hence the equation must be:

$$(x - 1)(x - 0) = 0 \Rightarrow (x - 1)(x) = 0 \Rightarrow x^2 - x = 0$$

Method II: Using the property

The roots are increased by three, and hence, we need to substitute $x - 3$ for x in the original equation:

$$(x - 3)^2 + 5(x - 3) + 6 = 0$$
$$x^2 + 6x + 9 + 5x - 15 + 6 = 0$$
$$x^2 - x = 0$$

3.118: Same Transformation for both Roots

Note that the same transformation is applied to both roots. This is a feature of all questions under this topic. It is possible to have a different transformation for each root, but then the properties discussed here will not apply.

Example 3.119

Let the roots of the quadratic equation $x^2 + x + 1 = 0$ by ω and ω^2. Find the quadratic equation with roots $\omega + 2$ and $\omega^2 + 2$.

Method I: Finding the Roots and then the New Equation

$$x^2 + x + 1 = 0$$

Apply the quadratic formula with $a = 1, b = 1, c = 1$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$$

We could add 2 to each of the roots, and find the equation using the new roots, but the calculations will be quite cumbersome.

Also, we want to avoid calculations with complex numbers, if possible. The second method does all this.

Method II: Using the property

Since the roots are being increased by 2, substitute $x - 2$ for x in the original equation:

$$(x - 2)^2 + (x - 2) + 1 = 0$$
$$x^2 - 4x + 4 + x - 2 + 1 = 0$$
$$x^2 - 3x + 3 = 0$$

Example 3.120

Let the roots of the quadratic equation $x^2 + x + 1 = 0$ by ω and ω^2. Find the quadratic equation with roots $\omega - 4$ and $\omega^2 - 4$.

Since the roots are being increased by 2, substitute $x-(-4)=x+4$ for x in the original equation:

$$(x+4)^2+(x+4)+1=0$$
$$x^2+8x+16+x+4+1=0$$
$$x^2+9x+21=0$$

C. Shifting: Graphical Approach

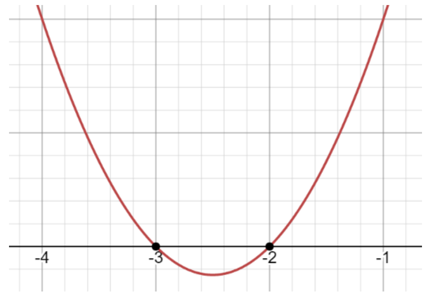
3.121: Roots increased by k

Consider the parabola

$$f(x)=ax^2+bx+c \text{ with zeroes} \in \{\alpha,\beta\}$$

For some constant k, the parabola which has zeroes $\{\alpha+k,\beta+k\}$ is given by:

$$f(x-k)=a(x-k)^2+b(x-k)+c=0$$



If k is positive, the roots move left.
If k is negative, the roots move right.

Example 3.122

Answer each part separately

Consider the graph of a quadratic shown alongside with equation $f(x)=px^2+qx+r$. Determine the equation of the new graph if, the shape of the graph is not changed, but the roots of the quadratic (α,β) are

A. moved left by π units
B. moved right by e units

$$f(x)=p(x+\pi)^2+q(x+\pi)+r$$
$$f(x)=p(x-e)^2+q(x-e)+r$$

D. Scaling: Algebraic Approach

Example 3.123

Find the quadratic equation each of whose roots are double the roots of the quadratic equation:

$$x^2+5x+6=0$$

$$(x+2)(x+3)=0 \Rightarrow x\in\{-3,-2\}$$

Double the roots:

$$x\in\{2(-3),2(-2)\}=\{-6,-4\}$$

And hence the equation must be:

$$(x+6)(x+4)=0 \Rightarrow x^2+10x+24=0$$

3.124: Roots multiplied by k

Consider the quadratic $ax^2+bx+c=0$ with roots α,β. The quadratic with roots $k\alpha,k\beta$ is given by:

$$a\left(\frac{x}{k}\right)^2+b\left(\frac{x}{k}\right)+c=0$$

Note:

$$\text{Roots increase by } k \Rightarrow \text{Substitute } \frac{x}{k}$$

Consider the original equation. It has roots α and β and hence, it must be of the form:

$$\underbrace{(x-\alpha)(x-\beta)=0}_{\textit{Original Equation}}$$

The new equation has roots $k\alpha$ and β and hence it must be of the form:

$$[x-k\alpha][x-k\beta]=0$$

We want to know how x is going to change, not how the roots are going to change. So, divide both sides of the above equation by k^2:

$$\frac{[x-k\alpha]}{k}\cdot\frac{[x-k\beta]}{k}=\frac{0}{k^2}$$

$$\underbrace{\left[\frac{x}{k}-\alpha\right]\left[\frac{x}{k}-\beta\right]=0}_{\textit{New Equation}}$$

Note that replacing x with $\frac{x}{k}$ in the original equation changes the roots from α,β to $k\alpha,k\beta$:

$$\underbrace{(x-\alpha)(x-\beta)=0}_{\textit{Original Equation}}\to\underbrace{\left[\frac{x}{k}-\alpha\right]\left[\frac{x}{k}-\beta\right]=0}_{\textit{New Equation}}$$

Making the substitution $\frac{x}{k}$ for x gives us:

$$a\left(\frac{x}{k}\right)^2+b\left(\frac{x}{k}\right)+c=0$$

Example 3.125

Find the quadratic equation each of whose roots are double the roots of the quadratic equation:

$$x^2+5x+6=0$$

Substitute $\frac{x}{2}$ for x:

$$\left(\frac{x}{2}\right)^2+5\left(\frac{x}{2}\right)+6=0$$

$$\frac{x^2}{4}+\frac{5x}{2}+6=0$$

$$x^2+10x+24=0$$

(Alternate Formula) 3.126: Roots multiplied by k

Consider the quadratic $ax^2+bx+c=0$ with roots α,β. The quadratic with roots $k\alpha,k\beta$ is given by:

$$ax^2+bkx+k^2c=0$$

Consider the original equation. It has roots α and β and hence, it must be of the form:

$$\underbrace{(x-\alpha)(x-\beta)=0}_{\textit{Original Equation}}\Rightarrow x^2-(\alpha+\beta)x+\alpha\beta=0$$

The new equation has roots $k\alpha,k\beta$ and hence it must be of the form:

$$[x-k\alpha][x-k\beta]=0\Rightarrow x^2-k(\alpha+\beta)x+k^2\alpha\beta=0$$

Example 3.127

Find the quadratic equation each of whose roots are double the roots of the quadratic equation:

$$x^2+5x+6=0$$

Multiply the middle term by 2, and the constant by $2^2=4$:

$$x^2+2(5x)+4(6)=0$$

$$x^2+10x+24=0$$

E. Scaling: Graphical Approach

3.128: Roots multiplied by k

Consider the parabola

$$f(x) = ax^2 + bx + c, \qquad Zeroes \in \alpha, \beta$$

If we want to find the parabola with roots $k\alpha, k\beta$ that is equivalent to a horizontal scale by a scale factor of k, which is achieved by substituting $\frac{x}{k}$ for x:

$$f\left(\frac{x}{k}\right) = ax^2 + bx + c, Zeroes \in k\alpha, k\beta$$

F. Reciprocal Roots

Example 3.129

Find the quadratic equation each of whose roots are the reciprocal of the roots of the quadratic equation:

$$x^2 + 5x + 6 = 0$$

$$(x+2)(x+3) = 0 \Rightarrow x \in \{-3, -2\}$$

And hence the new equation with reciprocal roots must be:

$$\left(x + \frac{1}{2}\right)\left(x + \frac{1}{3}\right) = 0 \Rightarrow x^2 + \frac{1}{3}x + \frac{1}{2}x + \frac{1}{6} = 0$$

$$x^2 + \frac{5x}{6}x + \frac{1}{6} = 0$$

Multiply by 6 to eliminate fractions:

$$6x^2 + 5x + 1 = 0$$

3.130: Reciprocal Roots

Consider the quadratic $ax^2 + bx + c = 0$ with roots α, β. The quadratic with roots $\frac{1}{\alpha}, \frac{1}{\beta}$ is given by:

$$cx^2 + bx + a = 0$$

Consider the original equation. It has roots α and β and hence, it must be of the form:

$$\underbrace{(x-\alpha)(x-\beta) = 0}_{Original\ Equation} \Rightarrow x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

The new equation has roots $k\alpha, k\beta$ and hence it must be of the form:

$$\left[x - \frac{1}{\alpha}\right]\left[x - \frac{1}{\beta}\right] = 0 \Rightarrow x^2 - \frac{1}{\alpha}x - \frac{1}{\beta}x + \frac{1}{\alpha\beta} = 0$$

$$x^2 - \frac{(\alpha+\beta)x}{\alpha\beta} + \frac{1}{\alpha\beta} = 0$$

Multiply by $\alpha\beta$ to eliminate fractions:

$$\alpha\beta x^2 - (\alpha+\beta)x + 1 = 0$$

From comparing the original equation and the new equation, we can see that the coefficients have been interchanged.
Hence, the new equation is:

$$cx^2 + bx + a = 0$$

Example 3.131
Find the quadratic equation each of whose roots are the reciprocal of the roots of the quadratic equation:
$$x^2+5x+6=0$$

Use the property, and interchange the equations:
$$6x^2+5x+1=0$$

Example 3.132
Find all quadratic equation(s) $y=ax^2+bx+c=0$ such that the roots of $z=cx^2+bx+a=0$ are the same as the roots of y.

Let the roots of $ax^2+bx+c=0$ be
$$\alpha \text{ and } \beta$$
Hence, the roots of $cx^2+bx+a=0$
$$\frac{1}{\alpha} \text{ and } \frac{1}{\beta}$$
$$\alpha=\frac{1}{\alpha}\Rightarrow\alpha^2=1\Rightarrow\alpha=\pm1$$
$$\beta=\frac{1}{\beta}\Rightarrow\beta^2=1\Rightarrow\beta=\pm1$$

$$(\alpha,\beta)=(1,1)(-1,-1)(1,-1)(-1,1)$$

$$(x-1)(x-1)=x^2-2x+1$$
$$(x+1)(x+1)=x^2+2x+1$$
$$(x+1)(x-1)=x^2-x$$
$$(x-1)(x+1)=x^2-x$$
Hence, there are three such possibilities.

G. Squares

3.133: Roots which are Squares
Consider the quadratic $ax^2+bx+c=0$ with roots α,β. The quadratic with roots α^2,β^2 is given by:
$$a^2x^2+(b^2-2ac)x+c^2=0$$

Consider the original equation. It has roots α and β and hence, it must be of the form:
$$\underbrace{(x-\alpha)(x-\beta)=0}_{\textit{Original Equation}}\Rightarrow x^2-(\alpha+\beta)x+\alpha\beta=0$$

The new equation has roots α^2,β^2 and hence it must be of the form:
$$[x-\alpha^2][x-\beta^2]=0\Rightarrow x^2-(\alpha^2+\beta^2)x+(\alpha\beta)^2=0$$

$$\alpha+\beta=-\frac{b}{a}$$
$$\alpha\beta=\frac{c}{a}\Rightarrow(\alpha\beta)^2=\frac{c^2}{a^2}$$

$$\alpha^2+2\alpha\beta+\beta^2-2\alpha\beta=(\alpha+\beta)^2-2\alpha\beta=\left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)=\frac{b^2}{a^2}-\frac{2c}{a}=\frac{b^2-2ac}{a^2}$$

$$x^2 + \left(\frac{b^2 - 2ac}{a^2}\right)x + \frac{c^2}{a^2} = 0$$

Multiply both sides by a^2:

$$a^2x^2 + (b^2 - 2ac)x + c^2 = 0$$

Example 3.134

H. Square Roots

3.135: Roots which are Square Roots

Example 3.136

I. Combining Transformations

Example 3.137

Consider the equation $x^2 + 5x + 6 = 0$ with roots α and β. Determine the equation which has roots:

$$2\alpha + 6, 2\beta + 6$$

Multiply the roots by 2 to give the equation with roots 2α and 2β:

$$\left(\frac{x}{2}\right)^2 + 5\left(\frac{x}{2}\right) + 6 = 0$$
$$\frac{x^2}{4} + \frac{5x}{2} + 6 = 0$$
$$x^2 + 10x + 24 = 0$$

Replace x with $x - 6$ to get the equation with roots $2\alpha + 6$ and $2\beta + 6$

$$(x - 6)^2 + 10(x - 6) + 24 = 0$$
$$x^2 - 12x + 36 + 10x - 60 + 24 = 0$$
$$x^2 - 2x = 0$$

Check:

$$x^2 + 5x + 6 \Rightarrow x \in \{-2, -3\}$$
$$2\alpha + 6 = 2$$
$$2\beta + 6 = 0$$

$$(x - 2)(x) = x^2 - 2x$$

Example 3.138

Wrong Method

Multiply the roots by 2:

$$2\alpha, 2\beta \Rightarrow x \to \frac{x}{2}$$

Add 6 to the roots:

$$2\alpha + 6, 2\beta + 6 \Rightarrow \frac{x}{2} \to \frac{x}{2} - 6 = \frac{x - 12}{2}$$

Substitute $\frac{x-12}{2}$ for x in the original equation:

$$\left(\frac{x-12}{2}\right)^2+5\left(\frac{x-12}{2}\right)+6=0$$
$$\frac{x^2-24x+144}{4}+\frac{5x-60}{2}+6=0$$
$$x^2-24x+144+10x-120+24=0$$
$$x^2-14x+48=0$$

Example 3.139

Consider the equation $x^2+5x+6=0$ with roots α and β. Determine the equation which has roots:

$$\frac{1}{4\alpha},\frac{1}{4\beta}$$

First find the equation with roots 4α and 4β by substituting $\frac{x}{4}$:

$$\left(\frac{x}{4}\right)^2+5\left(\frac{x}{4}\right)+6=0$$
$$\frac{x^2}{16}+\frac{5x}{4}+6=0$$
$$x^2+20x+96=0$$

Find the equation with roots $\frac{1}{4\alpha}$ and $\frac{1}{4\beta}$ by interchanging the coefficients:

$$96x^2+20x+1=0$$

Example 3.140

Consider the equation $x^2+5x+6=0$ with roots α and β. Determine the equation which has roots:

$$\frac{1+4\alpha}{2\alpha},\frac{1+4\beta}{2\beta}$$

$$\frac{1+4\alpha}{2\alpha}=\frac{1}{2\alpha}+2$$

Find the equation with roots 2α and 2β by substituting $\frac{x}{2}$:

$$\left(\frac{x}{2}\right)^2+5\left(\frac{x}{2}\right)+6=0$$
$$\frac{x^2}{4}+\frac{5x}{2}+6=0$$
$$x^2+10x+24=0$$

Find the equation which roots which are reciprocal by interchanging the coefficients:

$$24x^2+10x+1=0$$

Substitute $x-2$ to increase the roots by 2:

$$24(x-2)^2+10(x-2)+1=0$$
$$24(x^2-4x+4)+1\,0(x-2)+1=0$$
$$24x^2-96x+96+10x-20+1=0$$
$$24x^2-86x+77=0$$

Example 3.141

Consider the equation $x^2 + 5x + 6 = 0$ with roots α and β. Determine the equation which has roots:

$$\frac{1}{3\alpha + 2}, \frac{1}{3\beta + 2}$$

Determine the equation with roots 3α, and 3β:

$$\left(\frac{x}{3}\right)^2 + 5\left(\frac{x}{3}\right) + 6 = 0$$

$$\frac{x^2}{9} + \frac{5x}{3} + 6 = 0$$

$$x^2 + 15x + 54 = 0$$

Determine the equation with roots $3\alpha + 2$, and $3\beta + 2$:

$$(x - 2)^2 + 15(x - 2) + 54 = 0$$

$$x^2 - 4x + 4 + 15x - 30 + 54 = 0$$

$$x^2 + 11x + 28 = 0$$

Determine the equation with roots $\frac{1}{3\alpha+2}$, and $\frac{1}{3\beta+2}$:

$$28x^2 + 11x + 1 = 0$$

J. Change of Variables

We can use a change of variables to find the new equation. This method is mechanical and less conceptual, but it works for a large variety of transformations. It is especially useful where the transformation is complicated, and must be thought of as several simpler transformations combined.

We first see it in the context of some simple transformations which we already know how to solve, and then extend it to more complicated transformations.

(Increasing roots) Example 3.142

Let the equation $x^2 + 5x + 6 = 0$ have roots α, β. Find using the change of variable method, the equation with roots $\alpha + 2, \beta + 2$

We introduce a new variable t that is equal to what we wish to find:

$$\alpha + 2 = t \Rightarrow \alpha = t - 2$$

Since α is a root of $ax^2 + bx + c = 0$, it must satisfy the equation. So, we can write:

$$(1)\alpha^2 + 5\alpha + 6 = 0$$

Substitute $\alpha = t - 2$ as found above:

$$(t - 2)^2 + 5(t - 2) + 6 = 0$$

$$(t^2 - 4t + 4) + (5t - 10) + 6 = 0$$

$$t^2 + t = 0$$

Check:

$$x^2 + 5x + 6 = 0 \Rightarrow x \in \{-3, -2\}$$

$$t^2 + t = 0 \Rightarrow t \in \{-1, 0\}$$

Which does satisfy the requirement that the roots be increased by 2.

(Continuation) Example 3.143

Generalize what your solution for the above question to the equation $ax^2 + bx + c = 0$ with roots α, β.

Since α is a root of $ax^2 + bx + c = 0$, it must satisfy the equation. So, we can write:

$$a\alpha^2 + b\alpha + c = 0$$

Substitute $\alpha = t - 2$ as found above:

$$a(t-2)^2 + b(t-2) + c = 0$$
$$a(t^2 - 4t + 4) + b(t-2) + c = 0$$
$$at^2 - 4at + 4a + bt - 2b + c = 0$$
$$at^2 - (4a - b)t + 4a - 2b + c = 0$$

(Decreasing roots) Example 3.144

Let the equation $x^2 + 5x + 6 = 0$ have roots α, β.

A. Find using the change of variable method, the equation with roots $\alpha - 3, \beta - 3$.
B. Use the new equation to find $\alpha + \beta - 6$.

We introduce a new variable t that is equal to what we wish to find:

$$\alpha - 3 = t \Rightarrow \alpha = t + 3$$

Substitute $\alpha = t - 3$ as found above:

$$(t+3)^2 + 5(t+3) + 6 = 0$$
$$t^2 + 6t + 9 + 5t + 15 + 6 = 0$$
$$t^2 + 11t + 30 = 0$$

$$\alpha + \beta - 6 = (\alpha - 3) + (\beta - 3) = Sum\ of\ Roots = -\frac{b}{a} = -\frac{11}{1} = -11$$

Check

$$x^2 + 5x + 6 = 0 \Rightarrow \alpha + \beta = -5 \Rightarrow \alpha + \beta - 6 = -11$$

(Multiplying roots) Example 3.145

Let the equation $x^2 + 5x + 6 = 0$ have roots α, β.

A. Find using the change of variable method, the equation with roots $2\alpha, 2\beta$.
B. Use the new equation to find $4\alpha\beta$.

We introduce a new variable t that is equal to what we wish to find:

$$2\alpha = t \Rightarrow \alpha = \frac{t}{2}$$

Substitute $\alpha = \frac{t}{2}$ as found above:

$$\left(\frac{t}{2}\right)^2 + 5\left(\frac{t}{2}\right) + 6 = 0$$
$$t^2 + 10t + 24 = 0$$

$$4\alpha\beta = (2\alpha)(2\beta) = Product\ of\ Roots = \frac{c}{a} = \frac{24}{1} = 24$$

Check

$$x^2 + 5x + 6 = 0 \Rightarrow \alpha\beta = 6 \Rightarrow 4\alpha\beta = 24$$

(Dividing Roots) Example 3.146

Let the equation $x^2 + 5x + 6 = 0$ have roots α, β.

A. Find using the change of variable method, the equation with roots $\frac{\alpha}{5}, \frac{\beta}{5}$.

We introduce a new variable t that is equal to what we wish to find:

$$\frac{\alpha}{5} = t \Rightarrow \alpha = 5t$$

Substitute $\alpha = 5t$ as found above:

$$(5t)^2 + 5(5t) + 6 = 0$$
$$25t^2 + 25t + 6 = 0$$

(Reciprocal Roots) Example 3.147

Let the equation $x^2 + 5x + 6 = 0$ have roots α, β.

A. Find using the change of variable method, the equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}$.

B. Use the new equation to find $\frac{\alpha+\beta}{\alpha\beta}$

We introduce a new variable t that is equal to what we wish to find:

$$\frac{1}{\alpha} = t \Rightarrow \alpha = \frac{1}{t}$$

Substitute $\alpha = \frac{1}{t}$ as found above:

$$\left(\frac{1}{t}\right)^2 + 5\left(\frac{1}{t}\right) + 6 = 0$$
$$1 + 5t + 6t^2 = 0$$

$$\frac{\alpha+\beta}{\alpha\beta} = \frac{1}{\alpha} + \frac{1}{\beta} = Sum\ of\ Roots = -\frac{b}{a} = -\frac{5}{6}$$

Check

$$x^2 + 5x + 6 = 0 \Rightarrow \alpha\beta = 6, \alpha + \beta = -5 \Rightarrow \frac{\alpha+\beta}{\alpha\beta} = --\frac{5}{6}$$

K. Combining Transformations with Change of Variables

Example 3.148

Let the equation $x^2 + 5x + 6 = 0$ have roots α, β. Find using the change of variable method, the equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}$.

Example 3.149

Consider the equation $x^2 + 5x + 6 = 0$ with roots α and β. Determine the equation which has roots:

$$\frac{1}{4\alpha}, \frac{1}{4\beta}$$

We introduce a new variable t that is equal to what we wish to find:

$$\frac{1}{4\alpha} = t \Rightarrow \alpha = \frac{1}{4t}$$

Since α is a root of $ax^2 + bx + c = 0$, it must satisfy the equation. So, we can write:

$$a\alpha^2 + b\alpha + c = 0$$

Substitute $\alpha = \frac{1}{4t}$ as found above:

$$\left(\frac{1}{4t}\right)^2 + 5\left(\frac{1}{4t}\right) + 6 = 0$$
$$\frac{1}{16t^2} + \frac{5}{4t} + 6 = 0$$
$$1 + 20t + 96t^2 = 0$$
$$96t^2 + 20t + 1 = 0$$

Example 3.150

Let the equation $x^2 + 5x + 6 = 0$ have roots α, β. Find using the change of variable method, the equation with roots $2\alpha + 6, 2\beta + 6$.

We introduce a new variable t that is equal to what we wish to find:

$$2\alpha + 6 = t \Rightarrow 2\alpha = t - 6 \Rightarrow \alpha = \frac{t-6}{2}$$

Since α is a root of $ax^2 + bx + c = 0$, it must satisfy the equation. So, we can write:

$$a\alpha^2 + b\alpha + c = 0$$

Substitute $\alpha = \frac{t-6}{2}$ as found above:

$$\left(\frac{t-6}{2}\right)^2 + 5\left(\frac{t-6}{2}\right) + 6 = 0$$
$$\frac{t^2 - 12t + 36}{4} + \frac{5t - 30}{2} + 6 = 0$$
$$t^2 - 12t + 36 + 10t - 60 + 24 = 0$$
$$t^2 - 2t + 36 + 10t - 60 + 24 = 0$$
$$t^2 - 2t = 0$$

Example 3.151

Consider the equation $x^2 + 5x + 6 = 0$ with roots α and β. Determine the equation which has roots:

$$\frac{1+4\alpha}{2\alpha}, \frac{1+4\beta}{2\beta}$$

We introduce a new variable t that is equal to what we wish to find:

$$\frac{1+4\alpha}{2\alpha} = t \Rightarrow \frac{1}{2\alpha} + 2 = t \Rightarrow \frac{1}{2\alpha} = t - 2 \Rightarrow \alpha = \frac{1}{2t-4}$$

Since α is a root of $ax^2 + bx + c = 0$, it must satisfy the equation. So, we can write:

$$a\alpha^2 + b\alpha + c = 0$$

Substitute $\alpha = \frac{1}{2t-4}$ as found above:

Example 3.152

Consider the equation $x^2 + 5x + 6 = 0$ with roots α and β. Determine the equation which has roots:

$$\frac{1}{3\alpha+2}, \frac{1}{3\beta+2}$$

We introduce a new variable t that is equal to what we wish to find:

$$\frac{1}{3\alpha+2} = t \Rightarrow \frac{1}{t} = 3\alpha + 2 \Rightarrow \frac{1-2t}{t} = 3\alpha \Rightarrow \frac{1-2t}{3t} = \alpha$$

Since α is a root of $ax^2 + bx + c = 0$, it must satisfy the equation. So, we can write:

$$a\alpha^2 + b\alpha + c = 0$$

Substitute $\alpha = \frac{1-2t}{3t}$ as found above:

3.7 Quadratic Equations Topics

A. Undefined Expressions

Expressions are not defined when their denominator is zero.

Example 3.153

Find the values when the expression is not defined.

B. Absolute Value Equations

Example 3.154

Solve

$$|(x+3)(x+5)| = 1$$

$$-4-\sqrt{2}, -4+\sqrt{2}, -4$$

C. Simultaneous Equations

We look at solving a simultaneous system consisting of one or more quadratics.

$$Equation\ of\ a\ Line: y = mx + c_1$$
$$Standard\ Form:\ y = ax^2 + bx + c$$
$$Factored\ Form: y = a(x-\alpha)(x-\beta)$$
$$Vertex\ Form: y = a(x-h)^2 + k$$

Example 3.155

Find the points of intersection of the two functions given below:

$$p(x) = 2x^2 + 5x + 6, \qquad r(x) = -2x^2 + 5x + 6$$

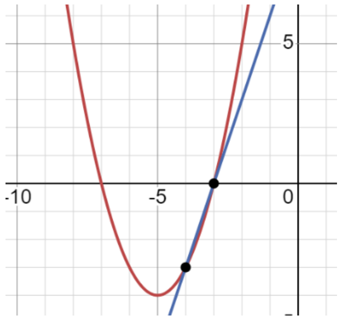
$$p(x) = r(x) \Rightarrow 2x^2 + 5x + 6 = -2x^2 + 5x + 6 \Rightarrow 4x^2 = 0 \Rightarrow x = 0$$

Example 3.156

$$y = x^2 + 10x + 21$$
$$y = 3x + 9$$

A. Find the solutions by equating the two equations.
B. (Graphing Calculator allowed) Sketch the graph of the quadratic and line. Highlight the points of intersection, and shade the region “inside” the two graphs.
C. Find the distance between the two points of intersection

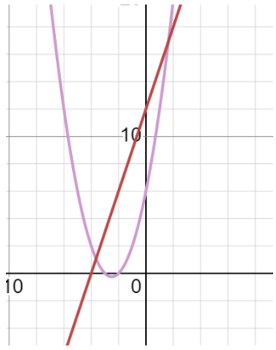
$$x^2 + 10x + 21 = 3x + 9$$
$$x^2 + 7x + 12 = 0$$
$$(x+3)(x+4) = 0$$
$$x \in \{-3, -4\}$$



$$x = -3 \Rightarrow y = 3x + 9 = 3(-3) + 9 = -9 + 9 = 0 \Rightarrow (x, y) = (-3, 0)$$
$$x = -4 \Rightarrow y = 3x + 9 = 3(-4) + 9 = -12 + 9 = -3 \Rightarrow (x, y) = (-4, -3)$$

Example 3.157

$$y = x^2 + 5x + 6$$
$$y = 3x + 12$$



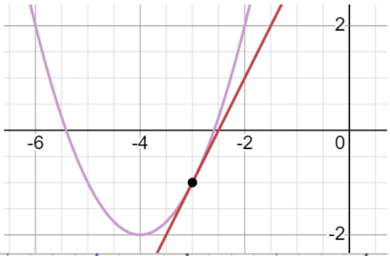
$$x^2 + 5x + 6 = 3x + 12$$
$$x^2 + 2x - 6 = 0$$

$$D = b^2 - 4ac = 2^2 - (4)(1)(-6) = 4 - (-24) = 28$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{28}}{2} = \frac{-2 \pm 2\sqrt{7}}{2} = -1 \pm \sqrt{7}$$
$$x = -1 - \sqrt{7} \Rightarrow y = 3\left(-1 - \sqrt{7}\right) + 12 = -3 - 3\sqrt{7} + 12 = 9 - 3\sqrt{7}$$
$$x = -1 + \sqrt{7} \Rightarrow y = 3\left(-1 + \sqrt{7}\right) + 12 = -3 + 3\sqrt{7} + 12 = 9 + 3\sqrt{7}$$

Example 3.158

$$y = x^2 + 8x + 14$$
$$y = 2x + 5$$

$$(-3, -1)$$

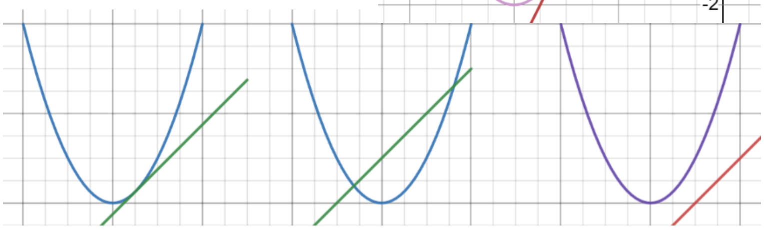


3.159: Three Possible Cases

$$D > 0 \Rightarrow Two\ Intersections$$
$$D = 0 \Rightarrow One\ Intersection$$
$$D < 0 \Rightarrow No\ Intersections$$

Where

$$D = b^2 - 4ac$$



The graph alongside shows the three possible cases when finding the points of intersection of a quadratic and a line. An important concept is to connect the algebraic conditions with the graphs they lead to.

Example 3.160

Identify the number of solutions to the following:

Example 3.161

$$p(x) = 2x^2 + 5x + 6$$
$$q(x) = 3x + 12$$

Find the point(s) of intersection of $q(x)$ and $p(x)$

$$p(x) = q(x) \Rightarrow 2x^2 + 5x + 6 = 3x + 12 \Rightarrow 2x^2 + 2x - 6 = 0 \Rightarrow x^2 + x - 3 = 0 \Rightarrow x = -$$

Example 3.162

$$q(x) = 3x + 12$$
$$r(x) = -2x^2 + 5x + 6$$

Find the point(s) of intersection of $q(x)$ and $r(x)$.

$$r(x) = q(x)$$
$$3x + 12 = -2x^2 + 5x + 6$$
$$2x^2 - 2x + 6 = 0$$

Calculate the discriminant

$$D = b^2 - 4ac = (-2)^2 - 4(2)(6) = -44 < 0$$

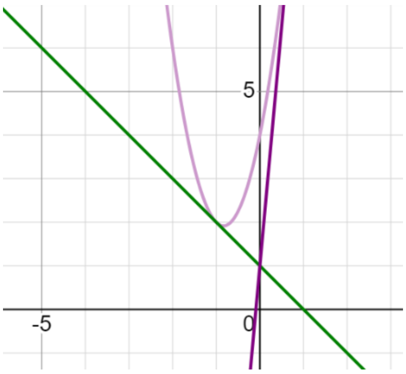
Example 3.163: Tangents to Quadratics

The curve $3x^2 + 5x + 4$ has two lines tangent to it, both with y-intercept 1. Find the equations of these lines.

$$3x^2 + 5x + 4 = mx + 1$$
$$3x^2 + (5 - m)x + 3 \ = \ 0$$

This has only one solution, because a tangent will intersect the quadratic in only one place. So, discriminant has to be zero.

$$(5 - m)^2 - (4)(3)(3) = 0$$
$$(5 - m)^2 = 2^2 \times 3^2$$
$$5 - m = \pm 6$$
$$m = \{11, -1\}$$



Example 3.164: Back Calculations

The line $y = 1 - 4x$ meets the curve $y = 4x^2 - 6x - 5$ at the points A and B. The tangent to the curve at B (where $x_B > x_A$) meets the horizontal line through A at the point C. Find the point C.

The line will meet the point when their y coordinates are equal:

$$1 - 4x = 4x^2 - 6x - 5$$

Solve the above quadratic:

$$x = -1, y = 1 - 4(-1) = 1 + 4 = 5 \Rightarrow \mathrm{A} = (-1,5)$$
$$x = \frac{3}{2}, y = 1 - 4\left(\frac{3}{2}\right) = 1 - 6 = -5 \Rightarrow B = \left(\frac{3}{2}, -5\right)$$

Since the tangent line passes through B, it must have equation (in point-slope form):

$$y + 5 = m\left(x - \frac{3}{2}\right)$$

Solve for y:

$$y = mx - \frac{3m}{2} - 5$$

The tangent line intersects the quadratic. At the point of intersection, the x and y coordinates must be the same:

$$4x^2 - 6x - 5 = mx - \frac{3m}{2} - 5$$

Collate all terms on one side:

$$4x^2 + (-6 - m)x + \frac{3m}{2} = 0$$

Since the tangent line to a quadratic intersects the quadratic at exactly one point, the equation above has exactly one solution.
Hence, the discriminant must be zero.

$$a = 4, b = -(6 + m), c = 3/2m$$

$$b^2 - 4ac = 0$$
$$\left(-(6+m)\right)^2 - 4(4)\left(\frac{3m}{2}\right) = 0$$
$$36 + 12m + m^2 - 24m = 0$$
$$m^2 - 12m + 36 = 0$$
$$(m-6)^2 = 0$$
$$m = 6$$

Hence, the equation of the line that we want is:

$$y = mx - \frac{3m}{2} - 5 = 6x - 9 - 5 = 6x - 14$$

And the intersection of the above line with $y = 5$ happens at:

$$6x - 14 = 5$$
$$6x = 19$$
$$x = \frac{19}{6}$$

Example 3.165: Back Calculations

Find the set of values of k for which the line $y = 2x + k$ cuts the curve y=x^2 + kx +5 at two distinct points.

x^2 + kx +5 = 2x +k
x^2+(k-2)x+5-k =0

(k-2)^2 - (4)(5-k)>0
k^2 -4k +4 -20 +4k >0
k^2 -16 >0
k^2>16
k<-4 OR k>4

Example 3.166: Working with Variables

Show that any linear function passing through $P(0,3)$ will meet the curve $f(x) = 2x^2 - x - 2$ twice.

$$2x^2 - x - 2 = mx + 3 \Rightarrow 2x^2 - (1+m)x - 5 = 0$$

$$D = [-(1+m)]^2 - (4)(2)(-5) = \underbrace{[-(1+m)]^2}_{+ve} + \underbrace{(4)(2)(5)}_{+ve} = (+ve)$$

Therefore, the equation always has two solutions.
Therefore, the line will always intersect the quadratic in two places.

Consider the vertical line the origin. Does this violate what we showed above?

If we have a vertical line passing through $P(0,3)$, then:

- It passes through the origin
- It intersects the parabola in exactly one place

However,

- It is not a function (m is not defined)
- It does not violate what we showed above.

Example 3.167: General Solution

Find the condition

If a point lies on two shapes, it must satisfy the equations of both shapes. Therefore, equate the two:

$$ax^2 + bx + c_2 = mx + c_1 \Rightarrow ax^2 + (b-m)x + c_2 - c_1$$

The value of its discriminant is

$$(b-m)^2 - 4(a)(c_2 - c_1)$$

What if:

A. The constant term is the same
B. $b = m$

Example 3.168: Finding Values of Expressions

Find the value of $\frac{a^2}{2} - ab + \frac{b^2}{2}$ if

$$a^2 + 2a + b = 5, \qquad a^2 - a + 4b = -4$$

Simplify the expression that we want to find:

$$\frac{a^2}{2} - ab + \frac{b^2}{2} = \frac{1}{2}(a^2 - 2ab + b^2) = \frac{(a-b)^2}{2}$$

We have been given the equations:

$$a^2 + 2a + b = 5, \qquad a^2 - a + 4b = -4$$

Add 9 to both sides of the second equation:

$$a^2 - a + 4b + 9 = 5$$

Equate the LHS of the first and the third equation:

$$a^2 + 2a + b = a^2 - a + 4b + 9 \Rightarrow 3a - 3b = 9 \Rightarrow a - b = 3 \Rightarrow \frac{(a-b)^2}{2} = \frac{9}{2}$$

Example 3.169: Substitution

The values of y which will satisfy the equations

$$2x^2 + 6x + 5y + 1 = 0$$
$$2x + y + 3 = 0$$

may be found by solving:

A. $y^2 + 14y - 7$
B. $y^2 + 8y + 1$
C. $y^2 + 10y - 7 = 0$
D. $y^2 + y - 12 = 0$
E. None of these equations (AHSME 1950/6)

$$2x + y + 3 = 0 \Rightarrow x = -\frac{y+3}{2}$$

Substitute $x = -\frac{y+3}{2}$ in $2x^2 + 6x + 5y + 1 = 0$:

$$2\left(-\frac{y+3}{2}\right)^2 + 6\left(-\frac{y+3}{2}\right) + 5y + 1 = 0$$
$$y^2 + 10y - 7 = 0$$

Hence, Option C.

Example 3.170:

Absolute Value

Example 3.171: Completing the Square

A. Three real numbers x, y and z are such that $x^2 + 6y = -17$, $y^2 + 4z = 1$, and $z^2 + 2x = 2$. What is the value of $x^2 + y^2 + z^2$? **(IOQM 2013/11)**
B. Let a, b, and c be positive integers with $a \geq b \geq c$ such that $a^2 - b^2 - c^2 + ab = 2011$ *and* $a^2 + 3b^2 + 3c^2 - 3ab - 2ac - 2bc = -1997$. What is a? **(AMC 10A 2012/24)**

Part A

Add the three equations and rearrange:

$$x^2 + 2x + y^2 + 6y + z^2 + 4z = -14$$

Complete the square three times:

$$x^2 + 2x + \mathbf{1} + y^2 + 6y + \mathbf{9} + z^2 + 4z + \mathbf{4} = -14 + \mathbf{1} + \mathbf{9} + \mathbf{4}$$
$$(x+1)^2 + (y+3)^2 + (z+2)^2 = 0$$

$$(x+1)^2 = 0 \Rightarrow x + 1 = 0 \Rightarrow x = -1$$
$$(y+3)^2 = 0 \Rightarrow y + 3 = 0 \Rightarrow y = -3$$
$$(z+2)^2 = 0 \Rightarrow z + 2 = 0 \Rightarrow z = -2$$

$$x^2 + y^2 + z^2 = (-1)^2 + (-3)^2 + (-2)^2 = 1 + 9 + 4 = 14$$

Part B

Add the two equations:

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2ac - 2bc = 14$$

Rearrange:

$$(a^2 - 2ab + b^2) + (a^2 - 2ac + c^2) + (b^2 - 2bc + c^2) = 14$$

Factor:

$$(a-b)^2 + (a-c)^2 + (b-c)^2 = 14$$

$$9+4+1=14$$
$$(a-c)^2=9\Rightarrow a-c=3$$

We get the following unordered pair:
$$(a-b,b-c)=(2,1)$$
Case I: $a-b=1, b-c=2\Rightarrow b=a-1, c=b-2=a-3$

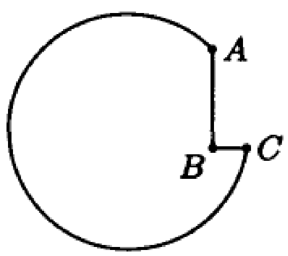
Substitute the above in $a^2-b^2-c^2+ab=2011$:
$$a^2-(a-1)^2-(a-3)^2+a(a-1)=2011$$
$$a^2-(a^2-2a+1)-(a^2-6a+9)+a^2-a=2011$$
$$2a-1+6a-9-a=2011$$
$$7a-10=2011$$
$$7a=2021$$
$$a\notin\mathbb{Z}$$
Case II: $a-b=2, b-c=1\Rightarrow b=a-2, c=b-1=a-3$
Substitute the above in $a^2-b^2-c^2+ab=2011$:
$$a^2-(a-2)^2-(a-3)^2+a(a-2)=2011$$
$$a^2-(a^2-4a+4)-(a^2-6a+9)+a^2-2a=2011$$
$$4a-4+6a-9-2a=2011$$
$$8a-13=2011$$
$$8a=2024$$
$$a=253$$

Example 3.172: Geometrical Applications

A. Let $\triangle XOY$ be a right-angled triangle with $m\angle XOY=90^0$ Let M and N be the midpoints of the legs OX and OY, respectively. Given $XN=19$ and $YM=22$, find XY. (AMC 10B 2002/22)

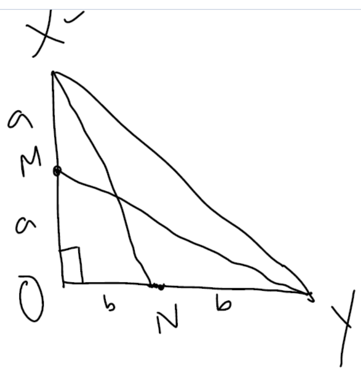
B. A wire of length 90 cm is divided into two parts. The first part is bent into an equilateral triangle. And the other part is bent into a rectangle which has length triple of its width. If the numeric value of the area of the rectangle is the square of the numeric value of the area of the triangle, find the dimensions of the triangle and the rectangle.

C. (*Challenge*) A machine-shop cutting tool has the shape of a notched circle, as shown. The radius of the circle is $\sqrt{50}$ cm, the length of AB is 6 cm and that of BC is 2 cm. The angle ABC is a right angle. Find the square of the distance (in centimeters) from B to the center of the circle. (AIME 1983/4)



Part A

$$XY=\sqrt{(2a)^2+(2b)^2}=\sqrt{4a^2+4b^2}=\sqrt{4(a^2+b^2)}=2\sqrt{a^2+b^2}$$
In right ΔXON, by Pythagorean Theorem:
$$(2a)^2+b^2=19^2\Rightarrow 4a^2+b^2=361$$
$$a^2+(2b)^2=22^2\Rightarrow a^2+4b^2=484$$
Add Equations I and II:
$$5a^2+5b^2=845\Rightarrow a^2+b^2=169\Rightarrow 2\sqrt{a^2+b^2}=26$$



Part B

Let
$$Side\ Length\ of\ Triangle=s$$
$$Width\ of\ rectangle=w\Rightarrow Length\ of\ Rectangle=3w$$

The perimeter of the two shapes must equal the length of the wire:

$$3s + 8w = 90 \Rightarrow w = \frac{90 - 3s}{8}$$

From the condition given for area:

$$A_{Rectangle} = (A_{Triangle})^2 \Rightarrow 3w^2 = \left(\frac{\sqrt{3}}{4}s^2\right)^2 \Rightarrow \sqrt{3}w = \frac{\sqrt{3}}{4}s^2 \Rightarrow w = \frac{s^2}{4}$$

Equate the two expressions for the value of w:

$$\frac{s^2}{4} = \frac{90 - 3s}{8} \Rightarrow 2s^2 + 3s - 90 = 0$$

Apply the quadratic formula with $a = 2, b = 3, c = -90$:

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9 + 720}}{2 \times 2} = \frac{-3 \pm 27}{4} \Rightarrow s \in \left\{-\frac{15}{2}, 6\right\} \Rightarrow s = 6$$

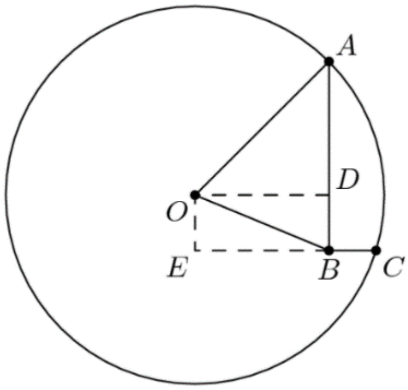
$$w = \frac{90 - 3s}{8} = \frac{90 - 3(6)}{8} = \frac{72}{8} = 9 \Rightarrow 3w = 27$$

Part C

Shortcut

We want to find $OB^2 = OE^2 + EB^2$. By observation:

$$x = OE = 1, \qquad y = EB = 5$$



Algebraic Method

Set up equations using the Pythagorean Theorem:

$$In\ \Delta AOD: OA^2 = OD^2 + AD^2 \Rightarrow \underbrace{\left(\sqrt{50}^2\right) = y^2 + (6 - x)^2}_{Equation\ I}$$

$$In\ \Delta OEC: OC^2 = OE^2 + EC^2 \Rightarrow \underbrace{\left(\sqrt{50}^2\right) = x^2 + (y + 2)^2}_{Equation\ II}$$

Since the LHS of both Eq. I and II is equal, equate the RHS of both equations:

$$y^2 + 36 - 12x + x^2 = x^2 + y^2 + 4y + 4 \Rightarrow 32 - 12x = 4y \Rightarrow \underbrace{8 - 3x = y}_{Equation\ III}$$

Substitute the value of y from Equation III in Equation I:

$$\left(\sqrt{50}^2\right) = (8 - 3x)^2 + (6 - x)^2$$
$$50 = 64 - 48x + 9x^2 + 36 - 12x + x^2$$
$$0 = 10x^2 - 60x + 50$$
$$0 = x^2 - 6x + 5$$
$$0 = (x - 6)(x - 1)$$
$$x \in \{1,5\}$$

If $x = 5$:

$$y = 7\ from\ Eq. I, y = 3\ from\ Eq. II \Rightarrow \boldsymbol{Contradiction}$$

If $x = 1$:

$$y = 5\ from\ both\ Eq. I\ and\ Eq. II \Rightarrow \boldsymbol{Works}$$

Hence, the final answer is:

$$x^2 + y^2 = 1^2 + 5^2 = 1 + 25 = 26$$

D. Quadratic Type Equations

Example 3.173

Find the values of r that satisfy:

$$2r^2 \cdot e^{rx} + r \cdot e^{rx} - e^{rx} = 0$$

$$e^{rx}(2r^2+r-1)=0$$

$e^{rx}>0$ for all values of r and x. Hence, divide by e^{rx}:

$$2r^2+r-1=0\Rightarrow(2r-1)(r+1)=0\Rightarrow r\in\left\{-1,\frac{1}{2}\right\}$$

Example 3.174: Infinite Continued Fractions

$$x=1+\frac{1}{x}\,is\ an\ important\ pattern\ that\ leads\ to\ a\ quadratic.$$

Solve for x:

$$x: x=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

$$x: x=\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cdots}}}$$

Part A

The RHS denominator is the same as the RHS. Hence, we get an equation, clear fractions to get a quadratic, and solve using the quadratic formula:

$$x=1+\frac{1}{x}\Rightarrow x^2=x+1\Rightarrow x^2-x-1=0\Rightarrow x=\frac{1\pm\sqrt{5}}{2}\Rightarrow\underbrace{x=\frac{1+\sqrt{5}}{2}}_{\because x>0}$$

Part B

Substitute x in the infinite pattern:

$$x=\cfrac{2}{2+\cfrac{\mathbf{2}}{\mathbf{2}+\cfrac{\mathbf{2}}{\mathbf{2}+\cdots}}}\Rightarrow x=\frac{2}{2+\boldsymbol{x}}$$

Clear fractions to get a quadratic:

$$2x+x^2=2$$

Complete the square:

$$x^2+2x+1=3\Rightarrow(x+1)^2=3$$

Take the positive square root only (since x must be positive):

$$x+1=\sqrt{3}\Rightarrow x=\sqrt{3}-1$$

Challenge 3.175: Infinite Continued Fractions

Find A^2, where A is the sum of the absolute values of all roots of the following equation: (AIME 1991/7)

$$x=\sqrt{19}+\cfrac{91}{\sqrt{19}+\cfrac{91}{\sqrt{19}+\cfrac{91}{\sqrt{19}+\cdots}}}$$

x is an infinite continued fraction, giving us an equation from which we can clear fractions, and then apply the quadratic formula with $a=1, b=-\sqrt{19}, c=-91$ to get the roots as:

$$x=\sqrt{19}+\frac{91}{x}\Rightarrow x^2=\sqrt{19}x+91\Rightarrow x^2-\sqrt{19}x-91=0\Rightarrow x=\frac{\sqrt{19}\pm\sqrt{19-4(1)(-91)}}{2}=\frac{\sqrt{19}\pm\sqrt{383}}{2}$$

$$A=\left|\frac{\sqrt{19}+\sqrt{383}}{2}\right|+\left|\frac{\sqrt{19}-\sqrt{383}}{2}\right|$$

Using $a-b<0 \Rightarrow |a-b| = b-a$ in the second term:

$$A = \frac{\sqrt{19}+\sqrt{383}}{2} + \frac{\sqrt{383}-\sqrt{19}}{2} = \frac{2\sqrt{383}}{2} = \sqrt{383}$$
$$A^2 = 383$$

Example 3.176: Infinite Nested Radicals

Solve for x:

A. $x = \sqrt{5\sqrt{5\sqrt{5\ldots}}}$

B. $x = \sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$

Part A

The expression is self-similar. Note that we can divide by x both sides (only because $x>0$):

$$x = \sqrt{5x} \Rightarrow x^2 = 5x \Rightarrow x = 5$$

Part B

The expression is self-similar. Proceed as before, squaring both sides, solving the resulting quadratic and discarding the negative value:

$$x = \sqrt{6+x} \Rightarrow x^2 = 6+x \Rightarrow x = \{-2,3\} \Rightarrow x = 3$$

Example 3.177

An equation reducible into a quadratic has a maximum of three non-zero terms: square term, linear term, and constant term. When one term is the square of another term, the first term can be written as $\boldsymbol{x^{2n}}$, if the second term is $\boldsymbol{x^n}$. We can then make the substitution $y = x^n$.

Solve for x using a change of variable:

A. $(2x+3)^2 - 14x - 21 = -6$

B. $4^{x-1} = 2^x + 8$

C. $4^x + 2^{x+2} = 3$

D. $x^6 + 2x^3 + 4 = 0$

E. $(x^2 - 6)^2 = -3x^2 + 18$

F. $\frac{1}{x^{-\frac{2}{3}}} + \frac{19}{x^{-\frac{1}{3}}} + 34 = 0$

G. What is the sum of the solutions to the equation $\sqrt[4]{x} = \frac{12}{7-\sqrt[4]{x}}$? (AIME 1986/1)

Part A

Use a change of variable. Let

$$y = 2x+3 \Rightarrow -7y = -14x - 21$$

Make the substitutions to get:

$$y^2 - 7y + 6 = 0 \Rightarrow y = \{6,1\}$$
$$2x+3 = \{6,1\} \Rightarrow x = \left\{\frac{3}{2}, -1\right\}$$

The above technique was useful, but not necessary in this question. We could have done this by usual methods (simplification). However, the change of variable technique is very useful in more complicated questions.

Part B

$$2^{2x-2} - 2^x - 8 = 0$$
$$\frac{2^{2x}}{4} - 2^x - 8 = 0$$
$$2^{2x} - 4\cdot 2^x - 32 = 0$$

Let $a = 2^x$:

$$a^2 - 4a - 32 = 0 \Rightarrow (a-8)(a+4) = 0$$
$$a \in \{-4,8\}$$

Reject the negative value since $2^x > 0$:

$$a = 8 \Rightarrow 2^x = 8 \Rightarrow x = 3$$

Part C

Let $a = 2^x$:

$$2^{2x} + 4\cdot 2^x - 3 = 0 \Rightarrow a^2 + 4a - 3 = 0$$

Quadratic formula with $a = 1, b = 4, c = -3$:

$$a=\frac{-4\pm\sqrt{16-(4)(1)(-3)}}{2}$$
$$=\frac{-4\pm\sqrt{28}}{2}=-2\pm\sqrt{7}$$
$$2^x=-2-\sqrt{7}\Rightarrow Negative\ Value\ not\ possible$$
$$2^x=-2+\sqrt{7}\Rightarrow Valid\ Solution$$

Part D: Odd Degree Substitution

If you make a substitution of odd degree, the number of solutions will remain the same as that of the changed variable.

Substitute $y=x^3 (\because x^6=(x^3)^2)$ to obtain the quadratic:

$$y^2+2y+4=0\Rightarrow y=\pm 2$$
$$x^3=\pm 2\ \Rightarrow\ x=\pm\sqrt[3]{2}$$

Part E: Odd Degree Substitution

If you make a substitution of even degree, the number of solutions can increase.

Let

$$y=x^2-6\Rightarrow -3y=-3x^2+18$$

Make the substitutions to get:

$$y^2=-3y$$

We now have two cases:

Case I: $y=0$

This satisfies the equation above. Hence:

$$x^2-6=0\Rightarrow x=\pm\sqrt{6}$$

Case I: $y\neq 0$

We can then divide both sides by y:

$$y=-3\Rightarrow x^2-6=-3\Rightarrow x=\pm\sqrt{3}$$
$$Final\ Answer: x=\pm\{\sqrt{3},\sqrt{6}\}$$

Part F: Substituting for fractional exponents

Note that $x^{-\frac{2}{3}}=x^{-\frac{1}{3}\times 2}$.

Hence, let $y=x^{\frac{1}{3}}\Rightarrow y^{-1}=x^{-\frac{1}{3}}$:

$$\frac{1}{y^{-2}}+\frac{19}{y^{-1}}+34=0\Rightarrow y^2+19y+34=0$$
$$y=2,17\Rightarrow x^{\frac{1}{3}}=2,17\Rightarrow x=8,17^3$$

Part G

Moving the denominator of the RHS to the LHS leads to a quadratic in $\sqrt[4]{x}$.

Let $y\ =\sqrt[4]{x}$:

$$y(7\ -\ y)=\ 12\Rightarrow y^2-\ 7y+12=0$$
$$(y-3)(y-4)=\ 0\Rightarrow\ y=\{3,4\}$$

Change back to the original variable:

$$\sqrt[4]{x}=\{3,4\}\Rightarrow x=\{3^4,4^4\}$$
$$Sum=4^4+3^4=337$$

Example 3.178: Reciprocals

$$y+\frac{1}{y}=c\Rightarrow y^2+1-c=0$$

Use the pattern above to solve the equations:

A. $\frac{35}{\sqrt{x}}+\sqrt{x}=12$

B. $6\sqrt{\frac{x}{1-x}}-2\sqrt{\frac{1-x}{x}}=-1$

C. $\sqrt{x}+\sqrt{3}+\frac{6\sqrt{x}-6\sqrt{3}}{x-3}+5=0$

Part A

Use a change of variable. Let $y=\sqrt{x}$. Then multiply both sides by y, rearrange, factor and solve:

$$\frac{35}{y}+y=12\Rightarrow y^2-12y+35=0\Rightarrow(y-5)(y-7)=0\Rightarrow y=\sqrt{x}\in\{5,7\}\Rightarrow x=\{25,49\}$$

Part B

Use a change of variable. Let $y=\sqrt{\frac{x}{1-x}}$:

$$6y-\frac{2}{y}=-1\Rightarrow 6y^2+y-2=0$$

Product is $6(-2)=-12$, Sum is 1. Split the terms as $4\ and\ -3$:

$$6y^2+4y-3y-2=0\Rightarrow(3y+2)(2y-1)=0\Rightarrow y\in\left\{-\frac{2}{3},\frac{1}{2}\right\}$$

Reject the negative value since the expression inside a square cannot be positive for real number solutions:

$$y=\frac{1}{2}$$

Change back to the original variable:

$$\sqrt{\frac{x}{1-x}}=\frac{1}{2}\Rightarrow\frac{x}{1-x}=\frac{1}{4}\Rightarrow 4x=1-x\Rightarrow x=\frac{1}{3}$$

Part C

Sometimes, it may be necessary to recognize the pattern and convert to the form that we want before we can make a substitution. Instead of clearing fractions, rationalize the numerator in the second term:

$$\frac{6(\sqrt{x}-\sqrt{3})}{x-3}\times\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}}=\frac{6(\boldsymbol{x-3})}{(\boldsymbol{x-3})(\sqrt{x}+\sqrt{3})}=\frac{6}{(\sqrt{x}+\sqrt{3})}$$

This changes the original equation to:

$$(\sqrt{\boldsymbol{x}}+\sqrt{\boldsymbol{3}})+\frac{6}{(\sqrt{\boldsymbol{x}}+\sqrt{\boldsymbol{3}})}+5=0$$

Note that this is in the form $\boldsymbol{y}+\frac{\boldsymbol{n}}{\boldsymbol{y}}$ for $\boldsymbol{y}=\sqrt{\boldsymbol{x}}+\sqrt{\boldsymbol{3}}$

Hence, substitute $\boldsymbol{y}=\sqrt{\boldsymbol{x}}+\sqrt{\boldsymbol{3}}$ to obtain:

$$y+\frac{6}{y}+5=0$$

Multiply both sides by y to clear fractions, and get:

$$y^2+5y+6=0\Rightarrow y=2,3\Rightarrow x^3=\pm 2\ \Rightarrow x=\pm\sqrt[3]{2}$$

E. Quadratic Type Equations-II: More Challenging Questions

Example 3.179

Find the positive solution to

$$\frac{1}{x^2-10x-29}+\frac{1}{x^2-10x-45}-\frac{2}{x^2-10x-69}=0\ \textbf{(AIME 1990/4)}$$

Substitute $a=x^2-10x-29$, and then eliminate fractions:

$$\frac{1}{a}+\frac{1}{a-16}-\frac{2}{a-40}=0\Rightarrow(a-16)(a-40)+a(a-40)-2(a)(a-16)=0$$

Simplify:

$$-64a+\ 40\times 16=0\Rightarrow a=10$$

Resubstitute:

$$x^2-10x-29=10\Rightarrow(x-13)(x+3)=0\Rightarrow x=\{-3,13\}$$

The positive solution is

$$13$$

Example 3.180: Radicals

What is the product of the real roots of the equation $x^2+18x+30=2\sqrt{x^2+18x+45}$? **(AIME 1983/3)**

Use a change of variable. Let $y=x^2+18x+30$:

$$y=2\sqrt{y+15}\Rightarrow y^2=4(y+15)\Rightarrow(y-10)(y+6)=0\Rightarrow y\in\{-6,10\}$$

But note that:

$$RHS=2\sqrt{y+15}>0\Rightarrow y>0\Rightarrow y=10$$

Hence:

$$x^2+18x+30=10\Rightarrow x^2+18x+20=0$$

Check that the equation above has real roots by calculating the discriminant:

$$b^2-4ac=18^2-(4)(1)(20)>0\Rightarrow 2\ Real\ Roots$$

Since the above equation has two real roots, we can calculate:

$$\text{Product of roots} = \frac{c}{a} = \frac{20}{1} = 20$$

F. Cyclic Polynomials and Quadratics

Definition

A cyclic polynomial is one which is invariant (does not change) under cyclic permutation of the variables.

Solving

To find the roots of a cyclic quadratic, try substituting 1, or (-1) for x.

G. Proving Variables in AP/GP/HP

3.8 Further Topics

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