

CALCULUS

TOPICS

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1. CALCULUS TOPICS

1.1 Newton-Raphson Method

A. Basics

1.1: Newton Raphson Method

Guess x_1 as an approximation to the solution of the equation $f(x) = 0$.

Use the recursive formula below to find further approximations:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ if } f'(x_n) \neq 0$$

1.2: Convergence

Newton's method does not guarantee convergence.

1.3: Wrong Root

If you start too far away from the root, Newton's method may not give you the root you wanted.

When Newton's method converges to a root, it may not be the root you have in mind.
 Figure 4.52 shows two ways this can happen.

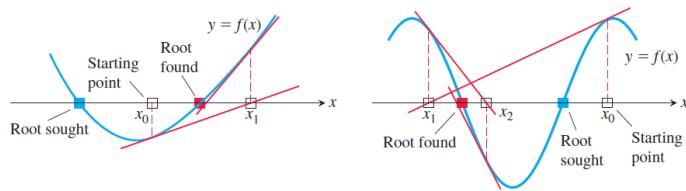


FIGURE 4.52 If you start too far away, Newton's method may miss the root you want.

1.2 Radius of Curvature

1.4: Radius of Curvature

Example 1.5

$$(0,1) \\ y = \cos x \\ f(x) = 1 - \frac{x^2}{2}$$

$$R = \left| \frac{(1 + y'^2)^{\frac{3}{2}}}{y''} \right|,$$

$$y' = -\sin x, \quad y'' = -\cos x$$

$$+R|_{x=0} = \left| \frac{(1+y')^{\frac{3}{2}}}{y''} \right| = \left| \frac{(1-\sin^2 x)^{\frac{3}{2}}}{-\cos x} \right| = \left| \frac{(\cos^2 x)^{\frac{3}{2}}}{-\cos x} \right| = \left| \frac{\cos^3 x}{-\cos x} \right| = |- \cos^2 x| = 1$$

$$f' = -x, \quad f'' = -1$$

$$R|_{x=0} = \left| \frac{(1+y')^{\frac{3}{2}}}{y''} \right| = \left| \frac{(1-x^2)^{\frac{3}{2}}}{-1} \right| = \left| \frac{(1-0)^{\frac{3}{2}}}{-1} \right| = \left| \frac{1}{-1} \right| = 1$$

1.3 Parametric Differentiation

A. Differentiation

1.6: Parametric Differentiation

If a curve is defined parametrically with $x = f(t), y = g(t)$, then the rate of change of y with respect to x , which is the derivative of y with respect to x is given by:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

We can motivate this *informally* by thinking as below:

$$\frac{dy}{dx} = \frac{dy \times \frac{1}{dt}}{dx \times \frac{1}{dt}} = \frac{dy/dt}{dx/dt}$$

Example 1.7: Concept

Given $x = \cos t, y = \sin t$, find $\frac{dy}{dx}$

- A. using the formula for parametric differentiation
- B. by eliminating t and differentiating

Verify that the answers from both the parts are same.

Part A

Substitute $\frac{dy}{dt} = \cos t, \frac{dx}{dt} = -\sin t$ in

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t$$

Part B

$$\begin{aligned} x^2 + y^2 &= \cos^2 t + \sin^2 t = 1 \\ x^2 + y^2 &= 1 \end{aligned}$$

Differentiating implicitly with respect to x :

$$\begin{aligned} 2x + 2y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y} = \frac{\cos t}{-\sin t} = -\cot t \end{aligned}$$

B. Tangent Lines

1.8: Tangent Line

To find the equation of a tangent line

- $m = \frac{dy}{dx} \Big|_{x=x_1}$ to find the slope of the tangent
- Substitute the coordinates of a point $P(x_1, y_1)$ and m into the point slope form of the equation of a line
 $(y - y_1 = m(x - x_1))$

Example 1.9

An object moves along the curve given by $x = 2t^2, y = 3t^3$ where t is in seconds. Determine the direction of movement of the object when $t = 2$.

The direction of movement will be given by the tangent line.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{9t^2}{4t} = \frac{9t}{4}$$

$$\text{Slope } m = \frac{dy}{dx} \Big|_{t=2} = \frac{9 \cdot 2}{4} = \frac{9}{2}$$

$$(x_1, y_1) = (2t_1^2, 3t_1^3) = (2 \times 2^2, 3 \times 2^3) = (8, 24)$$

Substitute into $y - y_1 = m(x - x_1)$

$$y - 24 = \frac{9}{2}(x - 8)$$

C. Parametric Curves

Example 1.10

Phillips Exeter Math 4

712. An object slides along the ellipse $x^2 - 2xy + 2y^2 = 10$. When it passes $(4, 3)$, the object's horizontal component of velocity is $\frac{dx}{dt} = 2$. What is the object's vertical component of velocity at that instant? In which direction is the object traveling around the ellipse, clockwise or counterclockwise?

Differentiate both sides with respect to t , and note that $x' = \frac{dx}{dt}, y' = \frac{dy}{dt}$:
$$(2x)x' - 2xy' - 2yx' + 4yy' = 0$$

Collate all y' terms on side:

$$y'(4y - 2x) = x'(2y - 2x)$$

Solve for y'

$$\frac{dy}{dt} = y' = \frac{x'(y - x)}{2y - x} = \frac{2(3 - 4)}{2(3) - 4} = \frac{2(-1)}{-2} = -1$$

$$\frac{dx}{dt} > 0, \frac{dy}{dt} < 0, \text{First Quadrant} \Rightarrow \text{Clockwise}$$

Example 1.11

Phillips Exeter Math 4

713. (Continuation) Calculate the size of the angle formed by the velocity vector and the radial vector $[x, y]$ at the point $(4, 3)$. At this instant, is the object getting closer to the origin or receding from it?

Substitute $\vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt} \right) = (2, -1)$, $\vec{r} = (4, 3)$ in

$$\cos \theta = \frac{\vec{v} \cdot \vec{r}}{|\vec{v}| |\vec{r}|} = \frac{(4)(2) + (3)(-1)}{\sqrt{2^2 + 1^2} \sqrt{4^2 + 3^2}} = \frac{8 - 3}{\sqrt{5} \sqrt{25}} = \frac{1}{\sqrt{5}}$$

$$\theta = \cos^{-1} \left(\frac{1}{\sqrt{5}} \right) = 63.43^\circ$$

Example 1.12

Phillips Exeter Math 4

714. (Continuation) Let r be the distance from the origin to the point (x, y) . When the object passes the point $(4, 3)$, what is the value of $\frac{dr}{dt}$? What does this tell you about the object's motion?

$$r = \sqrt{x^2 + y^2} \Rightarrow r^2 = x^2 + y^2$$

Differentiate both sides with respect to t :

$$2rr' = 2xx' + 2yy'$$

$$r' = \frac{xx' + yy'}{r} = \frac{4(2) + 3(-1)}{5} = \frac{8 - 3}{5} = \frac{5}{5} = 1$$

1.4 Parametric Integration

A. Basics

Integration with respect to a parameter can be done by using substitution.

1.13: Parametric Integration

Given the parametric equations $y = g(t)$, $x = f(t)$:

$$\int f(x) dx = \int g(t) f'(t) dt$$

$$\int f(x) dx$$

Substitute $f(x) = y$

$$= \int y dx$$

Substitute $y = g(t)$, $dx = f'(t) dt$:

$$\int g(t) f'(t) dt$$

Example 1.14: Area of a Circle

[Basic Example](#)

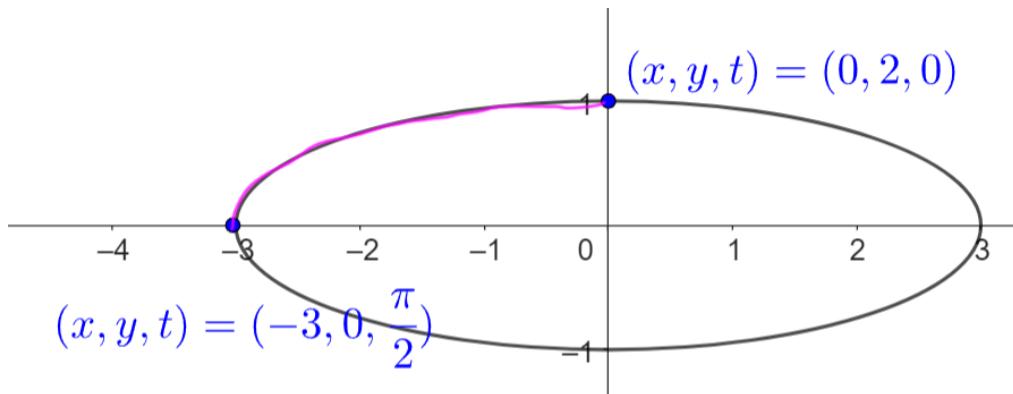
Example 1.15: Area of a Circle

[Video](#)

Example 1.16: Area of an Ellipse

- A. Describe the motion of the parametrization $x = -3 \sin t, y = 2 \cos t$.
- B. Use the parametrization to determine the area enclosed by the ellipse.

Part A



The motion begins at $(x, y) = (0, 2)$, and goes counter clockwise around the ellipse.
 The motion covers the entire ellipse from $t = 0$ to $t = 2\pi$.

Part B

To find the area of the ellipse, we use symmetry and multiply the area in the second quadrant by 4.

Limits of Integration: $x = -3, x = 0$

If $y = f(x)$ then we get:

$$4 \int_{-3}^0 f(x) dx = 4 \int_{-3}^0 y dx$$

Use a change of variables with the substitutions:

$$\begin{aligned} y &= 2 \cos t, dx = -3 \cos t dt \\ x = -3 &\Rightarrow -3 = -3 \sin t \Rightarrow \sin t = 1 \Rightarrow t = \frac{\pi}{2} \\ x = 0 &\Rightarrow 0 = -3 \sin t \Rightarrow \sin t = 0 \Rightarrow t = 0 \end{aligned}$$

Which gives us, after substitution:

$$= 4 \int_{\frac{\pi}{2}}^0 (2 \cos t)(-3 \cos t) dt = -24 \int_{\frac{\pi}{2}}^0 \cos^2 t dt$$

Use the trigonometric identity $\cos^2 t = \frac{1+\cos 2t}{2}$ and integrate:

$$-24 \int_{\frac{\pi}{2}}^0 \frac{1 + \cos 2t}{2} dt = -12 \int_{\frac{\pi}{2}}^0 1 + \cos 2t dt = -12 \left[t + \frac{\sin 2t}{2} \right]_{\frac{\pi}{2}}^0$$

Substitute the limits of integration:

$$-12 \left[\left(0 + \frac{\sin(2 \cdot 0)}{2} \right) - \left(\frac{\pi}{2} + \frac{\sin(2 \cdot \frac{\pi}{2})}{2} \right) \right]$$

$$\begin{aligned} \text{Substitute } \frac{\sin(2 \cdot 0)}{2} &= \frac{\sin 0}{2} = \frac{0}{2} = 0 \text{ and } \frac{\sin(2 \cdot \frac{\pi}{2})}{2} = \frac{\sin \pi}{2} = \frac{0}{2} = 0 \\ &= -12 \left[(0 + 0) - \left(\frac{\pi}{2} + 0 \right) \right] = -12 \left[-\frac{\pi}{2} \right] = 6\pi \end{aligned}$$

B. Velocity and Speed

1.17: Velocity

Given the parametric function $(x(t), y(t))$, we have

$$\text{Velocity} = \vec{v} = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt} \right)$$

The position vector is:

$$\vec{r} = (x, y) = (x(t), y(t))$$

The velocity is the rate of change of position with respect to time.

Differentiate the above to get:

$$\text{Velocity} = \vec{v} = \frac{d\vec{r}}{dt} = (x'(t), y'(t)) = \left(\frac{dx}{dt}, \frac{dy}{dt} \right)$$

1.18: Speed

Given the parametric function $(x(t), y(t))$, we have

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2}$$

$$\vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt} \right)$$

The magnitude of the above vector is:

$$|\vec{v}| = \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2}$$

Example 1.19

$$\text{Speed} = \sqrt{t^4 - 2t^2 + 1 + 4t^2}$$

Find $(x(t), y(t))$

$$\sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} = \sqrt{t^4 - 2t^2 + 1 + 4t^2}$$

Square both sides:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = t^4 - 2t^2 + 1 + 4t^2$$

First Factorization:

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 &= t^4 - 2t^2 + 1 = (t^2 - 1)^2 \Rightarrow \frac{dx}{dt} = t^2 - 1 \\ \left(\frac{dy}{dt}\right)^2 &= 4t^2 \Rightarrow \frac{dy}{dt} = 2t \end{aligned}$$

Second Factorization:

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= t^4 + 2t^2 + 1 \\ \left(\frac{dx}{dt}\right)^2 &= t^4 + 2t^2 + 1 = (t^2 + 1)^2 \Rightarrow \frac{dx}{dt} = t^2 + 1 \\ \left(\frac{dy}{dt}\right)^2 &= 0 \Rightarrow \frac{dy}{dt} = 0 \end{aligned}$$

C. Arc Length

1.20: Arc Length

Given the parametric function $(x(t), y(t))$, we have

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$Speed = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Integrating the speed over time gives us distance. Distance is the length of the curve travelled:

1.5 Polar Coordinates-I: Differentiation

A. Polar Coordinates

1.21: Polar Coordinates

Polar coordinates define a point in the form:

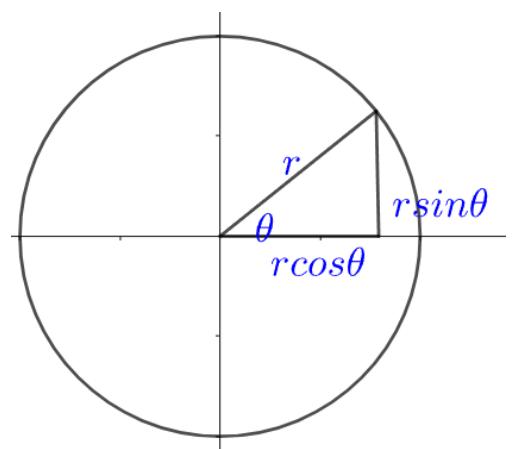
$$(r, \theta)$$

Where

r = directed distance

θ = angle made by the point with polar axis

Polar axis = positive direction of x axis



1.22: Polar to Cartesian Coordinates

$$x = r \cos \theta$$

$$y = r \sin \theta$$

1.23: Cartesian to Polar Coordinates

$$r = \sqrt{x^2 + y^2}$$
$$\tan \theta = \frac{y}{x}$$

$$x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2$$

- When solving for θ , you need to be careful since the \tan inverse has range only in quadrants I and IV, whereas the angle can be any quadrant.

B. Relation between polar and parametric

1.24: Parametric Equations

We defined parametric equations to be of the form $x = f(t), y = g(t)$, where t is a time parameter. Hence, the position of an object at time t was given by:

$$(x, y) = (f(t), g(t))$$

- Parametric equations with parameter t (by convention) refer to situations where the position of an object is a function of time.
- Apart from time, we can take other quantities as the parameter of interest.

1.25: Angle θ as a parameter

Instead of time parameter t , we can take angle θ as the parameter in a parametric setup. Given

$$x = f(\theta), y = g(\theta)$$

We can conclude

$$(x, y) = (f(\theta), g(\theta))$$

C. Differentiation

1.26: Parametric Differentiation

If a curve is defined parametrically with $x = f(t), y = g(t)$, then the rate of change of y with respect to x , which is the derivative of y with respect to x is given by:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

We can motivate this *informally* by thinking as below:

$$\frac{dy}{dx} = \frac{dy \times \frac{1}{dt}}{dx \times \frac{1}{dt}} = \frac{dy/dt}{dx/dt}$$

1.27: Polar Coordinates: Differentiation

Given $x = f(\theta), y = g(\theta)$, the rate of change of y with respect to x , which is the derivative of y with respect to x is given by:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

Example 1.28

What is the interpretation of

- A. $\frac{dy}{d\theta}$
- B. $\frac{dx}{d\theta}$

- A. Rate of change of y coordinate with respect to change in angle θ
- B. Rate of change of x coordinate with respect to change in angle θ

Example 1.29

$$r = 1 + \sin \theta$$

Find the rate of change of the y coordinate with respect to change in angle θ .

$$y = r \sin \theta$$

To differentiate with respect to θ we first express y as a function of θ by substituting $r = 1 + \sin \theta$

$$y = (1 + \sin \theta)(\sin \theta) = \sin \theta + \sin^2 \theta$$

Differentiate the above with respect to θ :

$$\frac{dy}{d\theta} = \cos \theta + 2 \sin \theta \cos \theta$$

Apply the double angle identity:

$$\frac{dy}{d\theta} = \cos \theta + \sin 2\theta$$

Example 1.30

$$r = 1 + \sin \theta$$

Find the rate of change of the x coordinate with respect to change in angle θ .

$$x = r \cos \theta = (1 + \sin \theta)(\cos \theta) = \cos \theta + \cos \theta \sin \theta = \cos \theta + \frac{1}{2} \sin 2\theta$$

$$\frac{dx}{d\theta} = -\sin \theta + \frac{1}{2} (\cos 2\theta)(2) = -\sin \theta + \cos 2\theta$$

Example 1.31

$$r = 1 + \sin \theta$$

Find the rate of change of the y coordinate with respect to change in x .

Find the rate of change of the y coordinate with respect to change in x when $\theta = \frac{\pi}{6}$.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos \theta + \sin 2\theta}{-\sin \theta + \cos 2\theta}$$

Substitute $\theta = \frac{\pi}{6} = 30^\circ \Rightarrow 2\theta = 60^\circ$ in the expression for $\frac{dy}{dx}$:

$$\frac{dy}{dx} \Big|_{\theta=\frac{\pi}{6}} = \frac{\cos 30^\circ + \sin 60^\circ}{-\sin 30^\circ + \cos 60^\circ} = \frac{\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}}{-\frac{1}{2} + \frac{1}{2}} = \frac{\sqrt{3}}{0}$$

Since the denominator is zero, the slope is

undefined or (informally) ∞

Hence, the function has a vertical tangent line at:

$$\theta = \frac{\pi}{6}$$

D. Slopes and Tangents

Example 1.32

Determine the equation of the tangent line to the graph $r = 1 + \sin \theta$ when $\theta = \frac{\pi}{2}$.

$$\frac{dy}{dx} = \frac{\cos \theta + \sin 2\theta}{-\sin \theta + \cos 2\theta}$$

$$\text{Slope } m = \frac{dy}{dx} \Big|_{\theta=\frac{\pi}{2}} = \frac{\cos \frac{\pi}{2} + \sin \pi}{-\sin \frac{\pi}{2} + \cos \pi} = \frac{0 + 0}{-1 - 1} = \frac{0}{-2} = 0$$

Since the slope is 0,

Line is horizontal

$$\begin{aligned} r &= 1 + \sin \theta = 1 + \sin\left(\frac{\pi}{2}\right) = 1 + 1 = 2 \\ x &= r \cos \theta = 2 \cos \frac{\pi}{2} = 2(0) = 0 \\ y &= r \sin \theta = 2 \sin \frac{\pi}{2} = 2(1) = 2 \end{aligned}$$

Since the line passes through $y = 2$, and it is a horizontal line, the equation is:

$$y = 2$$

Example 1.33

Determine the equation of the tangent line to the graph $r = \sin 2\theta$ when $\theta = \frac{\pi}{4}$.

$$\begin{aligned} \frac{dy}{d\theta} &= \frac{d}{d\theta}(r \sin \theta) = \frac{d}{d\theta}(\sin 2\theta \sin \theta) = 2 \cos 2\theta \sin \theta + \sin 2\theta \cos \theta \\ \frac{dx}{d\theta} &= \frac{d}{d\theta}(r \cos \theta) = \frac{d}{d\theta}(\sin 2\theta \cos \theta) = 2 \cos 2\theta \cos \theta - \sin 2\theta \sin \theta \\ \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{2 \cos 2\theta \sin \theta + \sin 2\theta \cos \theta}{2 \cos 2\theta \cos \theta - \sin 2\theta \sin \theta} \end{aligned}$$

$$\frac{dy}{dx} \Big|_{\theta=\frac{\pi}{4}} = \frac{2 \cos \frac{\pi}{2} \sin \frac{\pi}{4} + \sin \frac{\pi}{2} \cos \frac{\pi}{4}}{2 \cos 2\theta \cos \frac{\pi}{4} - \sin 2\theta \sin \frac{\pi}{4}} = \frac{2(0) \sin \frac{\pi}{4} + (1) \left(\frac{1}{\sqrt{2}}\right)}{2(0) \cos \frac{\pi}{4} - (1) \left(\frac{1}{\sqrt{2}}\right)} = \frac{(1) \left(\frac{1}{\sqrt{2}}\right)}{-(1) \left(\frac{1}{\sqrt{2}}\right)} = -1$$

$$\begin{aligned} r &= \sin 2\theta = \sin \frac{\pi}{2} = 1 \\ x &= r \cos \theta = (1) \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \\ y &= r \sin \theta = (1) \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - \frac{1}{\sqrt{2}} &= -1 \left(x - \frac{1}{\sqrt{2}} \right) \\ y &= -x + \sqrt{2} \end{aligned}$$

E. Some Further Topics

1.34: General Formula

If $x = r \cos \theta$, and $y = r \sin \theta$, such that $r = f(\theta)$ then:

$$\frac{dy}{dx} = \frac{r \cos \theta + \frac{dr}{d\theta} \sin \theta}{-r \sin \theta + \frac{dr}{d\theta} \cos \theta}$$

Using the product rule since r is a function of θ :

$$\begin{aligned} \frac{dy}{d\theta} &= r \cos \theta + \frac{dr}{d\theta} \sin \theta \\ \frac{dx}{d\theta} &= -r \sin \theta + \frac{dr}{d\theta} \cos \theta \end{aligned}$$

Combining the above two gives us:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r \cos \theta + \frac{dr}{d\theta} \sin \theta}{-r \sin \theta + \frac{dr}{d\theta} \cos \theta}$$

Example 1.35

Why do we need to specify $r = f(\theta)$ in the formula:

$$\frac{dy}{dx} = \frac{r \cos \theta + \frac{dr}{d\theta} \sin \theta}{-r \sin \theta + \frac{dr}{d\theta} \cos \theta}$$

Because if r is not a function of θ then we cannot differentiate it.

1.6 Polar Coordinates-II: Integration

A. Definition of Polar Integration

1.36: Sector of a Circle

A sector is a part of a circle enclosed by two radii and an arc.

Area of a sector is

$$\frac{\theta_{\text{Degree}}}{360} \pi r^2 = \frac{1}{2} r^2 \theta$$

Where

$$\theta = \text{Central angle}$$

- Central angle is the angle subtended by the arc at the center of the circle.

1.37: Area under the Curve

The area of the region between the origin and the curve $r = f(\theta)$, for $\theta_L \leq \theta \leq \theta_U$ is

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$

Where

$$\alpha = \text{Lower limit of integration}$$

$$\beta = \text{Upper limit of integration}$$

Region OTS is the area between the origin and the curve $r = f(\theta)$.

It is bounded by the rays $\theta = \alpha$ and $\theta = \beta$.

In the case of integration in rectangular coordinates, we divide the region into rectangles. But rectangles will not work here. So, we will divide into sectors.

Substitute Radius of each sector $r_k = f(\theta_k)$, central angle $= \Delta\theta_k$ in $\frac{1}{2}r^2\theta$

$$\text{Area of } k^{\text{th}} \text{ Sector} = \frac{1}{2} r_k^2 \Delta\theta_k = \frac{1}{2} [f(\theta_k)]^2 \Delta\theta_k$$

The total area of the sectors is:

$$\sum_{k=1}^n A_k = \sum_{k=1}^n \frac{1}{2} [f(\theta_k)]^2 \Delta\theta_k$$

And this area approximates the region that we want.

As the angle of each part approaches zero and the number of parts approaches infinity, we get:

$$A = \lim \left(\sum_{k=1}^n \frac{1}{2} [f(\theta_k)]^2 \Delta\theta_k \right) = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

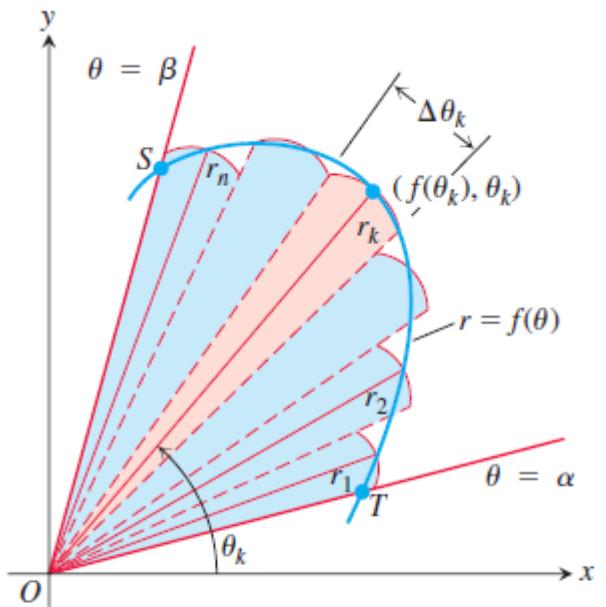


FIGURE 11.31 To derive a formula for the area of region OTS , we approximate the region with fan-shaped circular sectors.

B. Warm Up

Example 1.38: Area of a Circle

Calculate the area of a quarter circle in the first quadrant with center at the origin and radius R using polar integration.

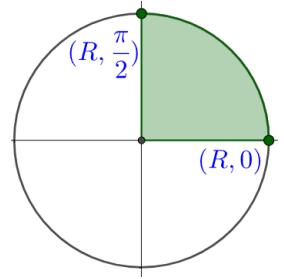
The equation of a circle with radius R and center at the origin is:

$$r = R, \quad R = \text{Radius}$$

The limits of integration are the points where the circle intersects the x and the y axis.

$$\text{Lower limit} = \alpha = 0$$

$$\text{Upper limit} = \beta = \frac{\pi}{2}$$



Substitute the above into $\int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} R^2 d\theta = \frac{R^2}{2} [\theta]_0^{\frac{\pi}{2}} = \frac{R^2}{2} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi R^2}{4}$$

Example 1.39

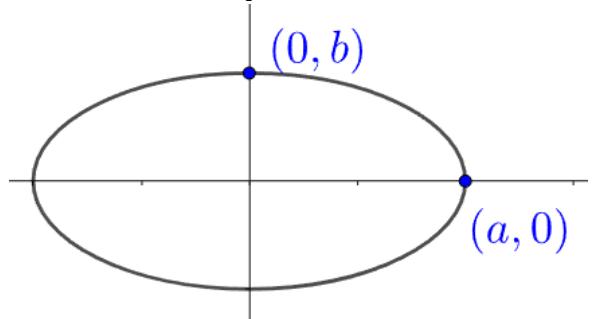
Determine the area of an ellipse symmetrical about the origin.

In parametric form, the equation of an ellipse symmetrical about the origin is:

$$x = a \cos \theta, y = b \sin \theta$$

In rectangular coordinates, the area of the ellipse is four times the area in the first quadrant:

$$= 4 \int_0^a f(x) dx$$

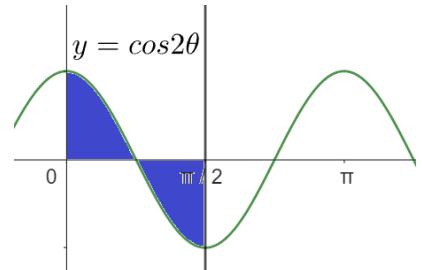


Convert to polar coordinates:

$$\alpha = 0, \beta = \frac{\pi}{2}$$

$$x = a \cos \theta \Rightarrow dx = -a \sin \theta d\theta \\ f(x) = y = b \sin \theta$$

$$= 4 \int_0^{\frac{\pi}{2}} (b \sin \theta)(-a \sin \theta d\theta) = 4ab \int_0^{\frac{\pi}{2}} \sin^2 \theta d\theta = 4ab \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta \\ = 2ab \left[\int_0^{\frac{\pi}{2}} d\theta - \int_0^{\frac{\pi}{2}} \cos 2\theta d\theta \right] \\ = 2ab \left[\int_{\frac{\pi}{2}}^0 d\theta \right] = 2ab[\theta]_{\frac{\pi}{2}}^0 = 2ab \left(\frac{\pi}{2} - 0 \right) = \pi ab$$



C. Area of a Polar Region

Example 1.40:

Find the area of the region bounded by $r = \theta$ for $0 \leq \theta \leq \pi$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_0^{\pi} \theta^2 d\theta = \frac{1}{2} \left[\frac{\theta^3}{3} \right]_0^{\pi} = \frac{\pi^3}{6}$$

Example 1.41: Limaçon Inner Loop

Find the area of the inner loop in the graph of $r = 1 + 2 \cos \theta$.

$$r = 3: 3 = 1 + 2 \cos \theta \Rightarrow \cos \theta = 1 \Rightarrow \theta \in \{0, 2\pi\}$$

$$r = 1: 1 = 1 + 2 \cos \theta \Rightarrow \cos \theta = 0 \Rightarrow \theta \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$$

$$r = 0: 0 = 1 + 2 \cos \theta \Rightarrow \cos \theta = -\frac{1}{2} \Rightarrow \theta \in \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$$

$$r = -1: -1 = 1 + 2 \cos \theta \Rightarrow \cos \theta = -1 \Rightarrow \theta \in \{\pi\}$$

| θ | 0 | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | π | $\frac{4\pi}{3}$ | $\frac{3\pi}{2}$ | 2π |
|----------|---|-----------------|------------------|-------|------------------|------------------|--------|
| r | 3 | 1 | 0 | -1 | 0 | 1 | 3 |

The function to be integrated is:

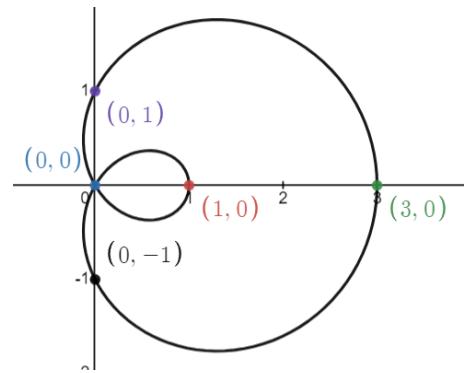
$$r^2 = (1 + 2 \cos \theta)^2 = 1 + 4 \cos \theta + 4 \cos^2 \theta$$

Substitute $\cos^2 \theta = \frac{1+\cos 2\theta}{2}$, and simplify:

$$= 1 + 4 \cos \theta + 4 \cdot \frac{1 + \cos 2\theta}{2} = 3 + 4 \cos \theta + 2 \cos 2\theta$$

Substitute Lower Limit = $\alpha = \frac{2\pi}{3}$, Upper Limit = $\beta = \pi$ in

$$A = 2 \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \int_{\alpha}^{\beta} r^2 d\theta = \int_{\frac{2\pi}{3}}^{\pi} (3 + 4 \cos \theta + 2 \cos 2\theta) d\theta$$



Integrate, and substitute the limits of integration:

$$= [3\theta + 4 \sin \theta + \sin 2\theta]_{\frac{2\pi}{3}}^{\pi} = (3\pi + 4 \sin \pi + \sin 2\pi) - \left(3 \cdot \frac{2\pi}{3} + 4 \sin \frac{2\pi}{3} + \sin \frac{4\pi}{3}\right)$$

Simplify:

$$\begin{aligned} &= (3\pi + 4 \cdot 0 + 0) - \left(2\pi + 4 \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) \\ &= 3\pi - \left(2\pi + \frac{3\sqrt{3}}{2}\right) \\ &= \pi - \frac{3\sqrt{3}}{2} \end{aligned}$$

Example 1.42: Limaçon Outer Loop

Find the area of the outer loop in the graph of $r = 1 + 2 \cos \theta$.

D. Area of a Region bounded by two Polar Curves

Example 1.43

Find the area of intersection of the polar curves $r = \cos \theta$ and $r = \sin \theta$.

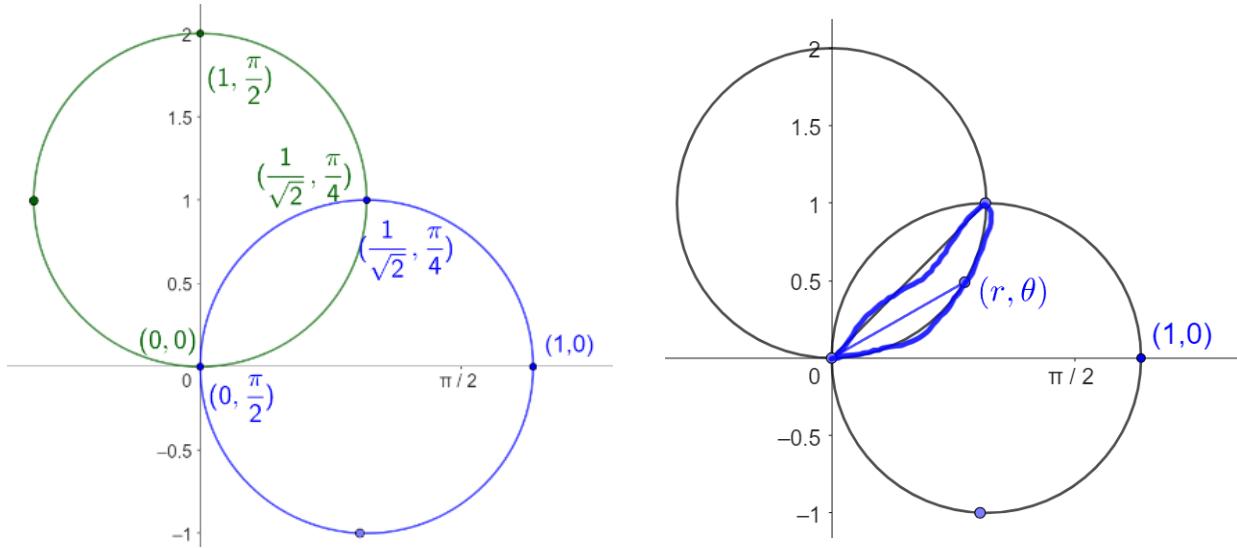
| θ | $r = \sin \theta$ | $r = \cos \theta$ |
|-----------------|----------------------|----------------------|
| 0 | 0 | 1 |
| $\frac{\pi}{4}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ |

| | | |
|-----------------|---|---|
| $\frac{\pi}{2}$ | 1 | 0 |
|-----------------|---|---|

Note that each of the given curves is a circle that touches the origin. Determine the points on intersection of the two curves:

$$\sin \theta = \cos \theta \Rightarrow \tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

Prepare a table of values.



$$\begin{aligned} \text{Substitute } \alpha = 0, \beta = \frac{\pi}{4}, r = \sin \theta \Rightarrow r^2 = \sin^2 \theta \text{ in } & \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta \\ &= 2 \int_0^{\frac{\pi}{4}} \frac{1}{2} \sin^2 \theta d\theta \end{aligned}$$

$$\begin{aligned} \text{Use a trigonometric substitution. Substitute } \sin^2 \theta = \frac{1-\cos 2\theta}{2}: \\ &= \int_0^{\frac{\pi}{4}} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta = \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{\frac{\pi}{4}} \end{aligned}$$

Substitute the limits of integration:

$$\left(\frac{\pi}{8} - \frac{\sin \frac{\pi}{2}}{4} \right) - (0 - 0) = \frac{\pi}{8} - \frac{1}{4}$$

E. Arc Length

1.44: Arc Length

The length of the curve $r = f(\theta)$ from α to β :

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} d\theta$$

Conditions to be met:

- $r(\theta)$ must have a continuous derivative for $\alpha \leq \theta \leq \beta$

- The point $P(r, \theta)$ traces the curve exactly once as θ ranges from α to β

Example 1.45

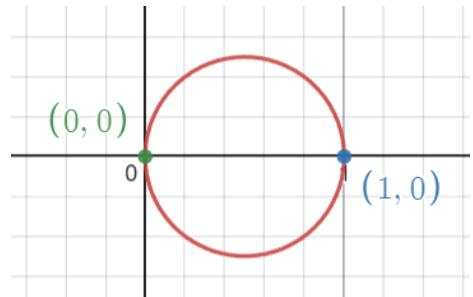
- A. $r = n \cos \theta$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$
- B. Plot the curve, determine the shape, and verify your answer from Part A by using a formula from high school geometry.

Substitute $r^2 = n^2 \cos^2 \theta$, $\frac{dr}{d\theta} = -n \sin \theta$:

$$\int_0^{\frac{\pi}{2}} \sqrt{n^2 \cos^2 \theta + n^2 \sin^2 \theta} d\theta = \int_0^{\frac{\pi}{2}} |n| \sqrt{1} d\theta = n[\theta]_0^{\frac{\pi}{2}} = \frac{\pi n}{2}$$

$$\text{Radius} = \frac{n}{2}$$

$$\text{Arc Length of Semicircle} = \frac{2\pi R}{2} = \pi R = \frac{\pi n}{2}$$



1.7 Vector Derivatives

A. Paths

1.46: Paths

When a particle moves through space, the points through which the particle passes make up the path.

Example 1.47

If $x = \sin t$, $y = \cos t$ for $t \geq 0$, identify

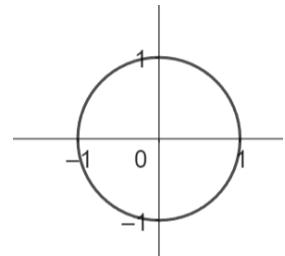
- A. the path travelled by the particle.

The path travelled is the unit circle.

1.48: Parametric Representation

The path of a particle over a time interval I can be represented parametrically as a function in three dimensions:

$$x = f(t), y = g(t), z = h(t), \quad t \in I$$



Note that I represents a time interval, not the set of integers \mathbb{I} , which is usually represented as \mathbb{Z} .

Example 1.49

A particle travels along the path $x = \sin t, y = \cos t, z = t$ for $t \geq 0$ where t is in seconds and x, y and z are in meters. Find the:

- A. shape of the path?
- B. length of the path travelled in 1 second?
- C. speed of the particle?

Part A

The particle travels along a right circular cylinder of radius 1 (graphed alongside).

Part B

$x = \sin t, y = \cos t$ would give a circle in the coordinate plane.

The time taken to complete 1 cycle around the circle

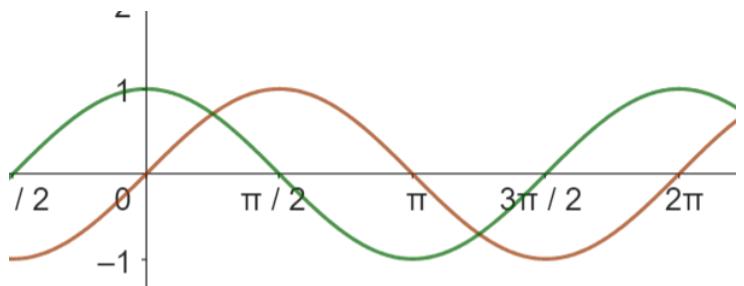
$$= 2\pi$$

Length of path around the circle

$$= 2\pi$$

Distance in 2D in 1 second

$$= \frac{\text{Distance in 1 Cycle}}{\text{Time for 1 Cycle}} = \frac{2\pi}{2\pi} = 1$$



The z component is separate. For every second that passes, the z coordinate increases by 1.

$$z = t$$

Length of path travelled in 1 second in the z direction

$$= 1$$

Total distance travelled in 1 second

$$= 1 + 1 = 2 \frac{m}{s}$$

B. Paths as Vectors

1.50: Scalar Function

A function $y = f(t)$ assigns a scalar y to each valid scalar t in its domain.

The functions that you have seen earlier (for example, in introductory Calculus) are scalar functions.

$$y = \sqrt{t}$$

$$y = \sin t$$

1.51: Vector Function

A vector function $\vec{r}(t)$ is a function that assigns for each scalar t in its domain a vector $\vec{r}(t)$.

1.52: Paths in Vector Form

A path in vector form can be expressed by expressing the vector in component form:

$$\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$

Where

$\hat{i}, \hat{j}, \hat{k}$ are unit vectors in the x, y and z directions respectively
 $f(t), g(t), h(t)$ are scalar functions

1.53: Converting from Parametric to Vector Form

$$x = f(t), y = g(t), z = h(t) \Leftrightarrow \vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$

Example 1.54

A particle travels along the path $x = \sin t, y = \cos t, z = t$ for $t \geq 0$ where t is in seconds and x, y and z are in meters. Write this in vector form.

$$\vec{r}(t) = (\sin t)\hat{i} + (\cos t)\hat{j} + t\hat{k}$$

C. Derivative Basics

1.55: Derivative

If $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ then its derivative is:

$$\vec{r}'(t) = \frac{d\vec{r}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \frac{df}{dt}\hat{i} + \frac{dg}{dt}\hat{j} + \frac{dh}{dt}\hat{k}$$

- To differentiate a vector, you can simply differentiate each of its components.
- This is what makes component form so useful and important.

1.56: Geometric Interpretation of the Derivative

- The derivative will give the vector tangent to the curve/path at the point where the derivative is calculated.
- This is the *direction vector*.

D. Finding Tangent Lines

1.57: Point-Vector Form

Given a point P with position vector \vec{a} whose terminal point lies on a line, and a vector \vec{b} parallel to the line, the vector form of the equation of the line is:

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

Consider an arbitrary point Q on the line with position vector \vec{r} .

Since \vec{b} is parallel to the line:

$$\overrightarrow{PQ} \parallel \vec{b}$$

The two vectors above have the same direction. Hence, one must be a scalar multiple of the other:

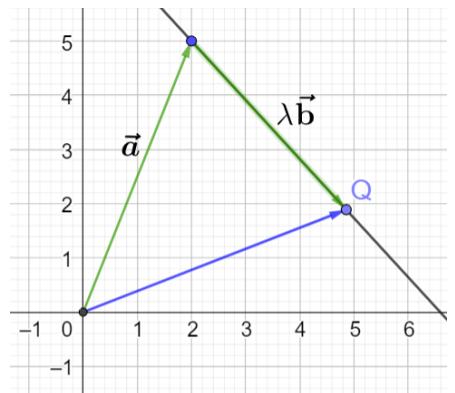
$$\overrightarrow{PQ} = \lambda \vec{b}$$

Write \overrightarrow{PQ} as a displacement vector:

$$\overrightarrow{PQ} = \vec{r} - \vec{a} = \lambda \vec{b}$$

Solve for \vec{r} :

$$\vec{r} = \vec{a} + \lambda \vec{b}$$



1.58: Finding Vector Equation

To find vector equation of a line, you need:

$\vec{r} = \text{Point on the Line} = \text{Position Vector}$

$\vec{d} = \frac{d\vec{r}}{dt} = \text{Direction Vector} = \text{Tangent Vector} = \text{Derivative}$

1.8 Vector Integration

A. Indefinite Integration

1.59: Integration of Scalar Functions

The indefinite integral of r with respect to t is the set of all antiderivatives of r , denoted by

$$\int r(t) dt = R(t) + C$$

1.60: Definition: Vector Integral

The indefinite integral of \vec{r} with respect to t is the set of all antitiderivatives of \vec{r} , denoted by

$$\int \vec{r}(t) dt = \vec{R}(t) + \vec{C}$$

Note that

\vec{r} is the integrand vector

\vec{R} is the integral vector

\vec{C} is a constant vector

Each of the above is a *vector* quantity except for

$t = \text{time, which is scalar}$

1.61: Integrating Vectors Component-Wise

If $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ then its integral is:

$$\int \vec{r}(t) dt = F(t)\hat{i} + G(t)\hat{j} + H(t)\hat{k} + \vec{C}$$

Where

$F(t)$ is an antiderivative of $f(t)$

$G(t)$ is an antiderivative of $f(t)$
 $H(t)$ is an antiderivative of $f(t)$

$$\int \vec{r}(t) dt$$

Use the definition of $\vec{r}(t)$:

$$= \int [f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}] dt$$

Use the sum property of integration to split the vector:

$$= \int f(t)\hat{i} dt + \int g(t)\hat{j} dt + \int h(t)\hat{k} dt$$

Carry out the integration:

$$= F(t)\hat{i} + G(t)\hat{j} + H(t)\hat{k} + \vec{C}$$

A vector can be integrated by integrating each of its components.

$$\int [f(t), g(t), h(t)] dt = (F(t), G(t), H(t)) + \vec{C}$$

B. Definite Integration

1.62: Definite Integration Component Wise

C. Ideal Projectile Motion

1.9 Hyperbolic Function Derivatives

A. Definitions

1.63: Hyperbolic function derivations

$$\begin{aligned}\sinh x &= \frac{e^x - e^{-x}}{2} \\ \cosh x &= \frac{e^x + e^{-x}}{2} \\ \tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ \operatorname{sech} x &= \frac{1}{\cosh x} \\ \operatorname{csch} x &= \frac{1}{\sinh x} \\ \coth x &= \frac{1}{\tanh x}\end{aligned}$$

B. Derivatives

There are no new rules that we need to calculate the derivatives of the hyperbolic functions. They can be calculated using the rules that we already know for finding derivatives.

1.64: Derivative of $\sinh x$ and $\cosh x$

$$\frac{d}{dt}(\sinh x) = \cosh x$$

$$\frac{d}{dt}(\cosh x) = \sinh x$$

Using $\frac{d}{dx} e^x = e^x$, $\frac{d}{dx} e^{-x} = -e^{-x}$

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x - (-)e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x$$

Note the difference with the trigonometric derivatives. The minus sign is missing.

1.65: Derivative of $\tanh x$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

Using the definition:

$$\frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$$

Using the quotient rule:

$$\frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

Use a change of variable. Let $e^x = a$, $e^{-x} = b$. Then, the numerator becomes:

$$(a + b)^2 - (a - b)^2 = (a^2 + 2ab + b^2) - (a^2 - 2ab + b^2) = 4ab = 4(e^x)(e^{-x}) = 4$$

Hence, the derivative that we want is:

$$\frac{4}{(e^x + e^{-x})^2} = \left(\frac{2}{e^x + e^{-x}}\right)^2 = \operatorname{sech}^2 x$$

1.66: Derivative of $\operatorname{csch} x$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

1.67: Derivative of $\operatorname{sech} x$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

1.68: Derivative of $\coth x$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

C. Inverse Hyperbolic Functions

1.69: Derivative of $\sinh^{-1} x$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$

$$y = \sinh^{-1} x$$

Apply the function $\sinh x$ to both sides:

$$\sinh y = x$$

Differentiate both sides implicitly:

$$\cosh y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$$

But now our answer is in terms of y . Whereas we want our answer in terms of x . For this, we do some algebraic manipulation based on identities.

Note that:

$$\cosh^2 y = 1 + \sinh^2 y \Rightarrow \cosh y = \sqrt{1 + \sinh^2 y}$$

(where we only need to take the positive square root since $\cosh x \geq 1$)

Substitute $\cosh y = \sqrt{1 + \sinh^2 y}$:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + \sinh^2 y}}$$

Substitute $y = \sinh^{-1} x$:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + \sinh^2(\sinh^{-1} x)}} = \frac{1}{\sqrt{1+x^2}}$$

1.70: Derivative of $\cosh^{-1} x$

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}$$

1.71: Derivative of $\tanh^{-1} x$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$$

1.72: Derivative of $\csch^{-1} x$

$$\frac{d}{dx}(\csch^{-1} x) = \frac{1}{|x|\sqrt{x^2 + 1}}$$

1.73: Derivative of $\sech^{-1} x$

$$\frac{d}{dx}(\sech^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$$

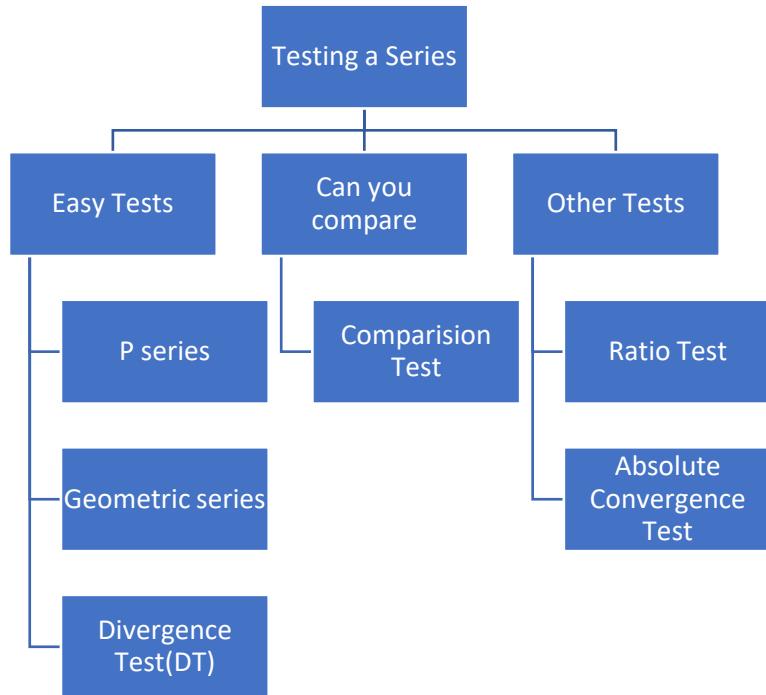
1.74: Derivative of $\coth^{-1} x$

$$\frac{d}{dx}(\coth^{-1} x) = -\frac{1}{1-x^2}$$

2. SEQUENCES AND SERIES

2.1 Finding Maclaurin Series

A. Formula Summary and Resources



2.1: Important Maclaurin Series

In this section, we will learn the formulas below (and their derivations).

After completing this section, you should have these formulas memorized.

Exponential and Logarithmic

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} = \sum_{n=0}^{n=\infty} \frac{x^n}{n!}$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

Trigonometric

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots, \quad |x| < \frac{\pi}{2}$$

Inverse Trigonometric

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \dots$$

Video 2.2

Maclaurin and Taylor Series Lecture (UC Irvine)

B. Concept

2.3: Idea

You can construct a polynomial whose derivatives match the derivatives of the function of interest at a particular point.

Example 2.4

Approximate $f(x) = e^x$ at $x = 0$ using a fourth-degree polynomial.

Calculate the first four derivatives of e^x :

$$y = y' = y'' = y''' = y'''' = e^x$$

Evaluate these four derivatives at $x = 0$:

$$y(0) = y'(0) = y''(0) = y'''(0) = y''''(0) = e^0 = 1$$

A fourth-degree polynomial must be of the form:

$$\begin{aligned} P(x) &= ax^4 + bx^3 + cx^2 + dx + E \\ P'(x) &= 4ax^3 + 3bx^2 + 2cx + d \\ P''(x) &= 12ax^2 + 6bx + 2c \\ P'''(x) &= 24ax + 6b \\ P''''(x) &= 24a \end{aligned}$$

The polynomial should ideally equal the function of interest:

$$\begin{aligned} P(0) &= f(0) \\ E &= 1 \end{aligned}$$

The derivative of the polynomial should equal the derivative of the function of interest:

$$\begin{aligned} P'(0) &= f'(0) \\ P'(0) = 4a(0) + 3b(0) + 2c(0) + d &= 1 \Rightarrow d = 1 \end{aligned}$$

The second derivative of the polynomial should equal the second derivative of the function of interest:

$$\begin{aligned} P''(0) &= f''(0) \\ P''(0) = 2c &= 1 \Rightarrow c = \frac{1}{2} \end{aligned}$$

The third derivative of the polynomial should equal the third derivative of the function of interest:

$$\begin{aligned} P'''(0) &= f'''(0) \\ P'''(0) = 6b &= 1 \Rightarrow b = \frac{1}{6} \end{aligned}$$

The fourth derivative of the polynomial should equal the fourth derivative of the function of interest:

$$P''''(0) = 24a = 1 \Rightarrow a = \frac{1}{24}$$

$$P(x) = \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1$$

2.5: Warning

We are learning some mathematical techniques (which work), but not the conditions on when they work. Those conditions relate to the theory of infinite series (convergence, radius of convergence), which we will see later.

- Takeaway: You can apply these techniques to other situations, but not blindly.

C. Basics

2.6: Notation for Derivatives

Given a function f , we introduce some notation to represent its derivatives:

Function Itself = Zeroth Derivative = $f^{(0)}$

First Derivative = $f' = f^{(1)}$

Second Derivative = $f'' = f^{(2)}$

\vdots
 N^{th} Derivative = $f^{(n)}$

2.7: Maclaurin Series

The Maclaurin series for a function f is an infinite series given by:

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{k!} x^k = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

The function evaluated at zero matches the Maclaurin series evaluated at zero:

$$S(0) = f(0)$$

The first derivative of the function evaluated at zero matches the first derivative of the Maclaurin series evaluated at zero:

$$\begin{aligned} S'(x) &= 0 + f^{(1)}(0) + \frac{2f^{(2)}(0)}{2!}x + \frac{3f^{(3)}(0)}{3!}x^2 + \cdots + \frac{nf^{(n)}(0)}{n!}x^n + \cdots \\ &= f^{(1)}(0) + f^{(2)}(0)x + \frac{f^{(3)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{(n-1)!}x^n + \cdots \\ S'(0) &= f^{(1)}(0) \end{aligned}$$

The second derivative of the function evaluated at zero matches the second derivative of the Maclaurin series evaluated at zero:

$$\begin{aligned} S''(x) &= f^{(2)}(0) + \frac{2f^{(3)}(0)}{2!}x + \cdots \\ &= f^{(2)}(0) + f^{(3)}(0)x + \cdots \\ S''(0) &= f^{(2)}(0) \end{aligned}$$

2.8: Maclaurin Series

The Maclaurin Series is a special case of the Taylor Series

D. Approximation

2.9: Approximation

An important use of a Maclaurin series is to approximate a function which is difficult to calculate directly.

2.10: Maclaurin Polynomial

The n^{th} Maclaurin polynomial is obtained by calculating the Maclaurin series upto x^n

Example 2.11

Find the third Maclaurin polynomial for $f(x) = \sqrt{1+x}$, and use it to approximate $\sqrt{1.8}$.

Note that:

$$f(0.8) = \sqrt{1+0.8} = \sqrt{1.8}$$

Calculate the first three derivatives and evaluate them at $x = 0$:

$$\begin{aligned} f(0) &= \sqrt{1+0} = \sqrt{1} = 1 \\ f'(x) &= \frac{1}{2\sqrt{1+x}} \Rightarrow f'(0) = \frac{1}{2\sqrt{1+0}} = \frac{1}{2} \\ f''(x) &= -\frac{1}{4(1+x)^{\frac{3}{2}}} \Rightarrow f''(0) = -\frac{1}{4(1+0)^{\frac{3}{2}}} = -\frac{1}{4} \\ f'''(x) &= \frac{3}{8(1+x)^{\frac{5}{2}}} \Rightarrow f'''(0) = \frac{3}{8(1+0)^{\frac{5}{2}}} = \frac{3}{8} \end{aligned}$$

Substitute the above into the formula for the Maclaurin Series with $x = 0.8$

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ = 1 + \frac{1}{2}(0.8) + \frac{-\frac{1}{4}}{2!}(0.64) + \frac{\frac{3}{8}}{6}(0.512) \\ \approx 1.352 \end{aligned}$$

E. e^x

Example 2.12

Show that the Maclaurin series expansion of $f(x) = e^x$ is

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{k!} x^k = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

The derivatives of all orders of $f(x)$ are

$$f^{(0)}(x) = f^{(1)}(x) = f^{(2)}(x) = f^{(n)}(x) = e^x$$

Evaluate the derivatives at the value $x = 0$:

$$f^{(0)}(0) = f^{(1)}(0) = f^{(2)}(0) = f^{(n)}(0) = e^0 = 1$$

Substitute the above into the formula, all the f terms become 1, and we get:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2.13: Substitution

You can use the expansion of e^x to find the expansion of $e^{f(x)}$.

Example 2.14

Find the Maclaurin series expansions below using the formula for e^x .

- A. e^{2x}
- B. e^{-x}

Part A

$$e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \cdots + \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

Part B

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \cdots + \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

We can write this a little better by getting the minus sign out of the brackets:

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3} + \cdots + \frac{(-x)^n}{n!} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

Example 2.15

Show that the Maclaurin series expansion of $\ln(1 + x)$ is:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$\sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{k!} x^k = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots$$

$$f(0) = \ln(1 + 0) = \ln 1 = 0$$

We will need the derivatives of all orders of $f(x)$.

$$f^{(1)}(x) = \frac{1}{1+x} \Rightarrow f^{(1)}(0) = \frac{1}{1+0} = 1$$

Using the formula $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$

$$f^{(2)}(x) = -\frac{1}{(1+x)^2} \Rightarrow f^{(2)}(0) = -\frac{1}{(1+0)^2} = -1$$

$$f^{(3)}(x) = \frac{2(1+x)}{(1+x)^4} = \frac{2}{(1+x)^3} \Rightarrow f^{(3)}(0) = \frac{2}{(1+0)^3} = 2 = 2!$$

$$f^{(4)}(x) = -\frac{2(3)(1+x)^2}{(1+x)^6} = \frac{-6}{(1+x)^4} \Rightarrow f^{(3)}(0) = -\frac{6}{(1+0)^4} = -6 = -3!$$

$$\begin{aligned}\ln(1+x) &= 0 + 1 \cdot x - \frac{1}{2!} \cdot x^2 - \frac{2!}{3!} \cdot x^3 - \frac{3!}{4!} \cdot x^4 + \dots \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}\end{aligned}$$

2.16: Maclaurin Series of a Polynomial

The Maclaurin series of a polynomial is the polynomial itself.

Video 2.17

[Calculating the Maclaurin series](#) of a polynomial ([MIT OCW 18.01 Single Variable Calculus](#))

$$\begin{array}{l} 3 = 3 \\ 0 = 2 \\ 3 = 5 \end{array}$$

F. Trigonometric Functions

Example 2.18

Show that the Maclaurin series expansion of $f(x) = \sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{k!} x^k = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!} x^2 + \dots + \frac{f^{(n)}(0)}{n!} x^n + \dots$$

Calculate the derivatives of the function f , and evaluate them at $x = 0$.

$$\begin{aligned} f(x) &= \sin x \Rightarrow f(0) = \sin 0 = 0 \\ f^{(1)}(x) &= \cos x \Rightarrow f^{(1)}(0) = \cos 0 = 1 \\ f^{(2)}(x) &= -\sin x \Rightarrow f^{(2)}(0) = -\sin 0 = -0 = 0 \\ f^{(3)}(x) &= -\cos x \Rightarrow f^{(3)}(0) = -\cos 0 = -1 \\ f^{(4)}(x) &= \sin x \Rightarrow f^{(4)}(0) = \sin 0 = 0 \\ f^{(5)}(x) &= \cos x \Rightarrow f^{(5)}(0) = \cos 0 = 1 \end{aligned}$$

Every alternate term is zero, we can simplify and rewrite as follows:

$$f^{(1)}(0)x + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(5)}(0)}{5!} x^5 \dots + \frac{f^{(n)}(0)}{n!} x^n + \dots$$

Substitute $f^{(1)}(0) = 1, f^{(3)}(0) = -1, f^{(5)}(0) = 1$:

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

We want to write this in summation notation.

- The terms alternate between positive and negative. This can be achieved using
 $(-1)^n \Rightarrow 1 \text{ for even } n, \text{ and } (-1) \text{ for odd } n$
- We only want the terms that have the odd powers of x . This can be achieved using the expression
 $2n + 1$

Which generates the n^{th} odd number

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Example 2.19

Determine the exact value of:

$$\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{(2n+1)!}$$

Substitute $x = \pi$ in $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$:

$$\begin{aligned}\sin \pi &= \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} \\ \sin \pi &= 0\end{aligned}$$

Example 2.20

Show that the Maclaurin series expansion of $f(x) = \cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Our formula for the Maclaurin Series expansion is:

$$f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \dots$$

Calculate the derivatives of the function f , and evaluate them at $x = 0$.

$$\begin{aligned}f(x) &= \cos x \Rightarrow f(0) = \cos 0 = 1 \\ f^{(1)}(x) &= -\sin x \Rightarrow f^{(1)}(0) = -\sin 0 = 0 \\ f^{(2)}(x) &= -\cos x \Rightarrow f^{(2)}(0) = -\cos 0 = -1 \\ f^{(3)}(x) &= \sin x \Rightarrow f^{(3)}(0) = \sin 0 = 0 \\ f^{(4)}(x) &= \cos x \Rightarrow f^{(4)}(0) = \cos 0 = 1 \\ f^{(5)}(x) &= -\sin x \Rightarrow f^{(5)}(0) = -\sin 0 = 0\end{aligned}$$

Since we know that every alternate term is zero, we can simplify and rewrite as follows:

$$\begin{aligned}f(0) + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(4)}(0)}{2!}x^4 + \dots \\ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\end{aligned}$$

2.2 Operations with Maclaurin Series

A. Adding Maclaurin Series

2.21: Adding Maclaurin Series

You can add/subtract two Maclaurin series term by term if they both converge.

Example 2.22

The hyperbolic function $\sinh x$ is defined as:

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$$

Use the Maclaurin series for e^x and e^{-x} to determine the Maclaurin series for $\sinh x$.

$$\begin{aligned} e^x &= \underbrace{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots + \frac{x^n}{n!} + \dots}_{\text{Identity I}} \\ e^{-x} &= \underbrace{1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \dots + \frac{(-x)^n}{n!} + \dots}_{\text{Identity II}} \end{aligned}$$

Subtracting Identity II from Identity I:

$$e^x - e^{-x} = 0 + 2x + 0 + \frac{2x^3}{3!} + \dots + 0 \cdot x^{2n} + \frac{2x^{2n+1}}{(2n+1)!} + \dots$$

Since the zero terms do not change the value, we do not need to write them:

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \dots + \frac{2x^{2n+1}}{(2n+1)!} + \dots$$

Divide both sides by 2:

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

We can check convergence by using the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{a^{2(n+1)+1}}{[2(n+1)+1]!}}{\frac{a^{2n+1}}{(2n+1)!}} = \frac{a^{2(n+1)+1}}{[2(n+1)+1]!} \times \frac{(2n+1)!}{a^{2n+1}} = \frac{a^3}{(2n+3)(2n+1)}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{a^3}{(2n+3)(2n+1)} = 0$$

B. Differentiating Maclaurin Series

2.23: Differentiating Term by Term

Differentiating a Maclaurin series expansion can help us derive new identities.

Example 2.24

The Maclaurin

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

Differentiate both sides of the above to determine the Maclaurin series expansion for $\cos x$.

$$\cos x = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \dots = \sum_{n=0}^{\infty} (2n+1) \frac{(-1)^n}{(2n+1)!} x^{2n}$$

Simplify:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Example 2.25

- A. Find the Maclaurin series for $\cos 2x$
- B. Use a trig identity to find the Maclaurin Series for $\sin^2 x$
- C. Use a trig identity to find the Maclaurin Series for $\cos^2 x$
- D. Use the Maclaurin series results from Parts B and C to show that $\sin^2 x + \cos^2 x = 1$.

Part A

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots$$

Part B

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1 - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots\right)}{2} = \frac{1}{2} \left[\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} - \dots \right]$$

Part C

$$\cos^2 x = \frac{\cos 2x + 1}{2} = \frac{\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots\right) + 1}{2} = 1 + \frac{1}{2} \left[-\frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots \right]$$

Part D

$$\sin^2 x + \cos^2 x = 1 + \frac{1}{2} \left[-\frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots \right] + \frac{1}{2} \left[\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} - \dots \right] = 1$$

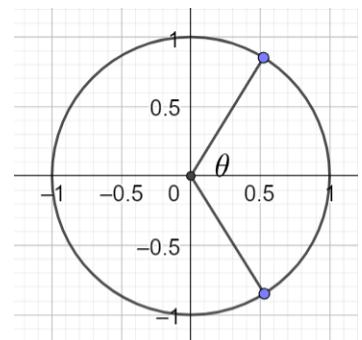
2.26: Odd and Even Functions

- An even function is a function where $f(x) = f(-x)$. The graph of an even function remains unchanged if you reflect it across the y -axis.
- An odd function is a function where $-f(x) = f(-x)$. The graph of an odd function remains unchanged if you reflect it across the origin.

Example 2.27

- A. Show that $\cos x$ is an even function.
- B. Show that $\sin x$ is an odd function.

$$\begin{aligned}\cos(-\theta) &= \cos \theta \Rightarrow \cos x \text{ is an even function} \\ \sin(-\theta) &= -\sin \theta \Rightarrow \sin x \text{ is an odd function}\end{aligned}$$



2.28: Polynomials as Odd and Even Functions

- A polynomial with only odd powers is an odd function.
- A polynomial with only even powers is an even function.

Polynomial with Odd Powers

$$P(x) = x^{2n+1} + x^{2n-1} + \cdots + x^1, n \in \mathbb{N}$$

$$P(-x) = (-x)^{2n+1} + (-x)^{2n-1} + \cdots + (-x)^1 = -(x^{2n+1} + x^{2n-1} + \cdots + x^1) = -P(x)$$

Polynomial with Even Powers

$$P(x) = x^{2n} + x^{2n-2} + \cdots + x^0, n \in \mathbb{N}$$

$$P(-x) = (-x)^{2n} + (-x)^{2n-2} + \cdots + (-x)^0 = x^{2n} + x^{2n-2} + \cdots + x^0 = P(x)$$

Example 2.29

Show that

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots, \quad |x| < \frac{\pi}{2}$$

Using the [Maclaurin Series Expansion](#):

Video 2.30

Show that

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$$

Using polynomial long division on $\frac{\sin x}{\cos x}$, by expanding each out as a series and

C. Geometric Series

Look at some expansions using the geometric series.

Example 2.31

Expand each of the below as a geometric series:

- A. $\frac{1}{1+x}$
- B. $\frac{1}{1+x^2}$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \cdots, |x| < 1$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots$$

Example 2.32

Find the Maclaurin series of

$$\frac{1}{(1+x)^2}$$

We know that:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots, |x| < 1$$

Differentiate both sides of the above:

$$-\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 +$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - \cdots$$

D. Integration

2.33: Integration

We can integrate a Maclaurin series term by term to get a Maclaurin series for a different expression.

Example 2.34

Show that the Maclaurin series for $\tan^{-1} x$ is:

$$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Differentiate both sides of $f(x) = \tan^{-1} x$:

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

Expand the RHS as a geometric series with $r = -x^2$:

$$f'(x) = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 + \dots$$

Integrate both sides. Since $f(0) = 0 \Rightarrow C = 0$ (where C is the constant of integration):

$$f(x) = \tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Example 2.35

Find the series expansion of $\ln(\cos x)$

Example 2.36

Determine the Maclaurin series for:

$$f(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$

Use the quotient rule $\log\left(\frac{a}{b}\right) = \log a - \log b$:

$$f = \frac{1}{2} [\ln(1+x) - \ln(1-x)]$$

Use the chain rule to differentiate. Then, expand as a geometric series:

$$f' = \frac{1}{2} \left[\frac{1}{1+x} + \frac{1}{1-x} \right] = \frac{1}{2} \left[\frac{2}{1-x^2} \right] = \frac{1}{1-x^2} = 1 + x^2 + x^4 + \dots$$

Integrate:

$$f(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2n+1}}{2n+1} + \dots$$

Where the constant of integration is zero since:

$$C = f(0) = \frac{1}{2} \ln\left(\frac{1+0}{1-0}\right) = \frac{1}{2} \ln(1) = \frac{1}{2} \cdot 0 = 0$$

Example 2.37

$$f(x) = \frac{1}{(1-2x)(1-3x)}$$

Method I: Addition

Split using partial fraction decomposition:

$$\frac{1}{(1-2x)(1-3x)} = \frac{-2}{1-2x} + \frac{3}{1-3x}$$

Using the formula for a geometric series: $\frac{a}{1-r} = a + ar + ar^2 + \dots$

$$a = -2, r = 2x: A = \frac{-2}{1-2x} = (-2) + (-2)(2x) + (-2)(2x)^2 + (-2)(2x)^3 + \dots$$

$$a = 3, r = 3x: B = \frac{3}{1-3x} = 3 + 3(3x) + 3(3x)^2 + 3(3x)^3 + \dots$$

$$\begin{aligned}\frac{-2}{1-2x} &= -2 - 4x - 8x^2 - 16x^3 + \dots \\ \frac{3}{1-3x} &= 3 + 9x + 27x^2 + 81x^3 + \dots\end{aligned}$$

$$\begin{aligned}T_1 &= -2 + 3 = 1 \\ T_2 &= -4x + 9x = 5x \\ T_3 &= -8x^2 + 27x^2 = 19x^2 \\ T_4 &= -16x^3 + 81x^3 = 65x^3\end{aligned}$$

$$\frac{1}{(1-2x)(1-3x)} = 1 + 5x + 19x^2 + 65x^3 + \dots + (3^n - 2^n)x^{n-1} + \dots$$

Method I: Multiplication

$$\begin{aligned}\left(\frac{1}{1-2x}\right)\left(\frac{1}{1-3x}\right) &= (1 + 2x + 4x^2 + 8x^3 \dots)(1 + 3x + 9x^2 + 27x^3 + \dots) \\ &= 1 + 3x + 9x^2 + 27x^3 + \dots \\ &\quad + 2x + 6x^2 + 18x^3 + \dots \\ &\quad + 4x^2 + 12x^3 + \dots \\ &\quad + 8x^3 + \dots \\ &= 1 + 5x + 19x^2 + 65x^3 + \dots\end{aligned}$$

E. Approximation

Example 2.38

Approximate $\int_0^1 e^{-x^2} dx$ using a Maclaurin series to three decimal places.

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$$\begin{aligned} f(x) = e^{-x^2} &= 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \frac{(-x^2)^5}{5!} + \dots \\ &= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \dots \end{aligned}$$

$$F(x) = \int e^{-x^2} dx = x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{11}}{11 \cdot 5!}$$

We need accuracy to three decimal places.
 Since we will round off at the fourth decimal place
 Hence, the error must be

less than 0.0005

$$\begin{aligned} \int_0^1 e^{-x^2} dx &= F(1) - F(0) \approx 0.7467 - 0 \\ F(1) &= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} \approx 0.7467 \\ F(0) &= 0 \end{aligned}$$

Example 2.39

Approximate $\int_0^1 e^{-\frac{x^2}{2}} dx$ using a Maclaurin series to three decimal places.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \\ f(x) = e^{-\frac{x^2}{2}} &= 1 + \left(-\frac{x^2}{2}\right) + \frac{\left(-\frac{x^2}{2}\right)^2}{2!} + \frac{\left(-\frac{x^2}{2}\right)^3}{3!} + \frac{\left(-\frac{x^2}{2}\right)^4}{4!} + \frac{\left(-\frac{x^2}{2}\right)^5}{5!} + \dots \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{2^2} - \frac{x^6}{2^3 \cdot 3!} + \frac{x^8}{2^4 \cdot 4!} - \frac{x^{10}}{2^5 \cdot 5!} + \dots \\ F(x) = \int e^{-x^2} dx &= x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{5 \cdot 2^2 \cdot 2!} - \frac{x^7}{7 \cdot 2^3 \cdot 3!} + \frac{x^9}{9 \cdot 2^4 \cdot 4!} - \frac{x^{11}}{11 \cdot 2^5 \cdot 5!} \\ &= x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336} + \frac{1}{3456} \end{aligned}$$

Substitute $x = 1$ and evaluate the first four terms:

$$F(1) = 1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} \approx 0.8553$$

The maximum error is:

$$\text{Fifth Term} = \frac{1}{3456} \approx 0.0002$$

Hence, the range is:

$$(0.8553, 0.8553 + 0.0002) = (0.8553, 0.8555)$$

Since the answer to three decimal places is different at the lower and upper end of the range, we add one more term:

$$F(1) = 1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456} \approx 0.8556$$

The maximum error is now:

$$\frac{1}{11 \cdot 2^5 \cdot 5!} = \frac{1}{42240} \approx 2.36 \times 10^{-5}$$

And hence the answer will not change even at the maximum range after rounding.

2.3 Taylor Series/Convergence

A. Taylor Series

A Taylor series is a generalization of a Maclaurin series. A Maclaurin series is always centered at 0, but this need not always be the case.

2.40: Taylor Series

If a function $f(x)$ has derivatives of all orders in an open interval I containing a , then the Taylor series of $f(x)$ is given by:

$$T(x) = f(a) + f^{(1)}(a)(x - a) + \frac{f^{(2)}(a)}{2!}(x - a)^2 + \dots = \sum_{k=0}^{k=\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k$$

- The Taylor series is centered at a .
- This means that the approximation is very good at a . It may or may not become worse as we move away from a .

2.41: Taylor Series versus Function Derivatives

The derivatives of a Taylor series are equal to the derivatives of the function itself at

$$x = a$$

Evaluate the Taylor series at $x = a$, and we get the original function value:

$$T(a) = f(a)$$

Differentiate $T(x)$ once:

$$T'(x) = f^{(1)}(a) + \frac{2f^{(2)}(a)}{2!}(x - a) + \dots$$

Evaluate $T^{(1)}(x)$ at $x = a$:

$$T'(a) = f^{(1)}(a)$$

Differentiate $T^{(2)}(x)$ twice:

$$T''(x) = f^{(2)}(a) + \dots$$

Evaluate $T^{(2)}(x)$ at $x = a$:

$$T''(a) = f^{(2)}(a)$$

This process can be continued, and at every derivative we will get the same result.

Example 2.42

Show that the Taylor series at $a = 0$ is the Maclaurin series:

Substitute $a = 0$ in the Taylor series

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{k!} x^k = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Example 2.43

Find an equation for the fourth-degree polynomial $p(x)$ that has the following properties:

$$p(1) = 1, \quad p^{(1)}(1) = \frac{1}{2}, \quad p^{(2)}(1) = -\frac{1}{4}, \quad p^{(3)}(1) = \frac{3}{8}, \quad p^{(4)}(1) = -\frac{15}{16}$$

Verify by differentiation and evaluation that it does have those derivatives.

Finding the Polynomial

The formula for the Taylor series is:

$$f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots$$

Substitute $a = 1$:

$$f(1) + f^{(1)}(1)(x-1) + \frac{f^{(2)}(1)}{2!}(x-1)^2 + \cdots$$

Substitute the given values in the fourth degree Taylor Polynomials:

$$p(x) = 1 + \frac{1}{2}(x-1) - \frac{\frac{1}{4}}{2}(x-1)^2 + \frac{\frac{3}{8}}{6}(x-1)^3 - \frac{\frac{15}{16}}{24}(x-1)^4$$

Verification

$$\begin{aligned} p'(x) &= \frac{1}{2} - \frac{1}{4}(x-1) + \frac{\frac{3}{8}}{2}(x-1)^2 - \frac{\frac{15}{16}}{6}(x-1)^3 \\ p''(x) &= -\frac{1}{4} + \frac{\frac{3}{8}}{8}(x-1) - \frac{\frac{15}{16}}{2}(x-1)^2 \\ p'''(x) &= \frac{\frac{3}{8}}{8} - \frac{\frac{15}{16}}{16}(x-1) \\ p^{(4)}(x) &= -\frac{\frac{15}{16}}{16} \end{aligned}$$

Example 2.44

Find the fourth degree Taylor polynomial for $f(x) = \sqrt{x}$ at $x = 1$.

Calculate the first four derivatives and evaluate them:

$$\begin{aligned} f(x) &= \sqrt{x} \Rightarrow f(1) = 1 \\ f^{(1)}(x) &= \frac{1}{2}x^{-\frac{1}{2}} \Rightarrow f^{(1)}(1) = \frac{1}{2} \\ f^{(2)}(x) &= -\frac{1}{4}x^{-\frac{3}{2}} \Rightarrow f^{(2)}(1) = -\frac{1}{4} \\ f^{(3)}(x) &= -\frac{3}{8}x^{-\frac{5}{2}} \Rightarrow f^{(3)}(1) = -\frac{3}{8} \\ f^{(4)}(x) &= -\frac{15}{16}x^{-\frac{7}{2}} \Rightarrow f^{(4)}(1) = -\frac{15}{16} \end{aligned}$$

Substitute the above values into the formula for the Taylor series:

$$p(x) = 1 + \frac{1}{2}(x-1) - \frac{\frac{1}{4}}{2}(x-1)^2 + \frac{\frac{3}{8}}{6}(x-1)^3 - \frac{\frac{15}{16}}{24}(x-1)^4$$

Example 2.45

Find the third-degree Taylor polynomial for $f(x) = \ln x$ at $x = 1$.

$$\begin{aligned} f(x) &= \ln x \Rightarrow f(1) = \ln 1 = 0 \\ f^{(1)}(x) &= x^{-1} \Rightarrow f^{(1)}(1) = 1 \\ f^{(2)}(x) &= -x^{-2} \Rightarrow f^{(2)}(1) = -1 \\ f^{(3)}(x) &= 2x^{-3} \Rightarrow f^{(3)}(1) = 2 \\ f^{(4)}(x) &= -6x^{-4} \Rightarrow f^{(4)}(1) = -6 \end{aligned}$$

$$\begin{aligned} \text{Taylor Series} &= \sum_{k=0}^{k=\infty} \frac{f^k(1)}{k!} (x-1)^k = f(1) + f^{(1)}(1)(x-1) + \frac{f^{(2)}(1)}{2!}(x-1)^2 + \dots \\ &= 0 + 1(x-1) + \frac{-1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \dots \end{aligned}$$

Example 2.46

Taylor series for $\ln x$ at $x = 1$

$$T(x) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \dots$$

$$\begin{aligned} f(1) &= \ln(1) = 0 \\ f'(x) &= \frac{1}{x} \Rightarrow f'(1) = \frac{1}{1} = 1 \\ f''(x) &= -\frac{1}{x^2} \Rightarrow f''(1) = -\frac{1}{1} = -1 \\ f'''(x) &= \frac{2}{x^3} \Rightarrow f'''(1) = \frac{2}{1} = 2 \end{aligned}$$

$$\begin{aligned} T(x) &= 0 + 1(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \dots \\ \ln x &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots \end{aligned}$$

Video 2.47

Find the Taylor series for [ln x at x=2](#).

B. Lagrange Remainder Formula

2.48: Taylor Series

We can compare a Taylor series to a finite Taylor polynomial by adding a $R_n(b)$ to take care of the missing terms:

$$T(b) = T_n(b) + R_n(b)$$

Where

x is where the Taylor polynomial or series is being evaluated

The Taylor series is given by the infinite series:

$$T(b) = f(a) + f^{(1)}(a)(b-a) + \frac{f^{(2)}(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \cdots$$

If we consider only n terms of the series, we get the n^{th} Taylor polynomial:

$$T_n(b) = f(a) + f^{(1)}(a)(b-a) + \frac{f^{(2)}(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n$$

We can connect the n^{th} Taylor polynomial and the Taylor series (for a specific function) by putting the value of the remaining terms in a single remainder term $R_n(x)$

$$T(b) = T_n(b) + R_n(b)$$

2.49: Lagrange Remainder Formula

The Lagrange remainder formula for a n^{th} degree Taylor polynomial is based on the $(n+1)^{st}$ derivative of the function of interest:

$$R_n(b) = \frac{f^{n+1}(c)}{(n+1)!}(b-a)^{n+1}$$

Where

$$a < c < b$$

- We know that c lies strictly between a and b , but we do not know the value of c unless we have some information on the value of b .
- Hence, we often consider a worst scenario, which is the value of c that maximizes $f^{n+1}(c)$, and consider the remainder must be bounded by the maximum.

Example 2.50

Consider the Taylor series centered at 0, for the function $f(x) = e^x$ and used to evaluate e^4 . For the second degree and third-degree Taylor polynomials, calculate the

- A. value of c in Lagrange's Remainder Formula for $f(x)$
- B. error when compared to the value of the function

We are working with Maclaurin series since the Taylor series is centered at $a = 0$.

We can write the series as the sum of a Taylor polynomial and a Lagrange remainder term:

$$f(b) = T_n(b) + R_n(b)$$

Substitute $b = 4$, $n = 2$ and solve for $R_n(4)$:

$$\underbrace{R_n(4) = f(4) - T_2(4)}_{\text{Equation I}}$$

Case I: $n = 2$

Here, we want the second-degree Taylor polynomial:

$$T_2(4) = 1 + b + \frac{b^2}{2!} = 1 + 4 + \frac{4^2}{2!} = 13$$

$$R_2(4) = \frac{f^{n+1}(c)}{(n+1)!} (b)^{n+1} = \frac{e^c}{3!} \cdot 4^3 = \frac{64e^c}{6} = \frac{32e^c}{3}$$

Substitute the values calculated above in Equation I, and solve the resulting exponential equation:

$$\frac{32e^c}{3} = e^4 - 13 \Rightarrow c = \ln \left[\frac{3}{32} (e^4 - 13) \right] \approx 1.36$$

$$\text{Error} = e^4 - 13 \approx$$

Case II: $n = 3$

$$T_3(4) = 1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} = 1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!} = \frac{71}{3}$$

$$R_3(4) = \frac{f^{n+1}(c)}{(n+1)!} (b)^{n+1} = \frac{e^c}{4!} \cdot 4^4 = \frac{32e^c}{3}$$

Substitute the values calculated above in Equation I:

$$\frac{32e^c}{3} = e^4 - \frac{71}{3} \Rightarrow c = \ln \left[\frac{3}{32} \left(e^4 - \frac{71}{3} \right) \right] \approx 1.06$$

$$\text{Error} = e^4 - \frac{71}{3} \approx$$

C. Convergence Test

2.51: Convergence of Taylor Polynomials

If

$$R_n(b) \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for all } x \in I$$

Then we say that

$$T_n(x) \rightarrow f \text{ on } I \text{ as } n \rightarrow \infty$$

- If the *remainder term tends to zero* as the number of terms becomes large, then the difference between the function and the Taylor polynomial also tends to zero.
- Hence, the Taylor series converges to the function.

Example 2.52

Prove that the Maclaurin series for $f(x) = e^x$ converges to e^x when $x = 4$.

The Taylor expansion of e^b centered at $a = 0$ is:

$$e^b = 1 + b + \frac{b^2}{2!} + \cdots + \frac{b^n}{n!} + \cdots$$

The Remainder term for the n^{th} Taylor polynomial at $x = 4$:

$$R_n(4) = \frac{f^{n+1}(c)}{(n+1)!} b^{n+1} = \frac{e^4}{(n+1)!} 4^{n+1}$$

Take the limit as $n \rightarrow \infty$:

$$\lim_{n \rightarrow \infty} \frac{e^4}{(n+1)!} 4^{n+1}$$

Since e^c is a constant, we can move it out of the limit:

$$e^4 \lim_{n \rightarrow \infty} \frac{4^{n+1}}{(n+1)!}$$

Expand the numerator and the denominator:

$$e^4 \lim_{n \rightarrow \infty} \frac{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \dots \cdot 4 \text{ (n+1 times)}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot (n+1)}$$

Rearrange:

$$= e^4 \lim_{n \rightarrow \infty} \frac{4 \cdot 4 \cdot 4 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{4}{5} \times \frac{4}{6} \times \frac{4}{7} \times \dots \times \frac{4}{n+1}$$

Since each term after the first fraction is less than 1 and each term is less than the prior term hence, as $n \rightarrow \infty$, the limit $\rightarrow 0$:

$$= e^4 \times 0 = 0$$

D. Determining Maximum Value of Derivative

2.53: Convergence of Taylor Polynomials

If

$$|f^{n+1}(t)| \leq M \text{ for all } a < t < b$$

Then

$$|R_n(b)| \leq \frac{M|b-a|^{n+1}}{(n+1)!}$$

The Lagrange remainder formula has a term for the $(n+1)^{st}$ derivative, evaluated at c .

- In certain situations, it can be difficult to determine the value of c , or to determine an exact expression for the derivative itself.
- However, by considering a bound on the value of $f^{n+1}(t)$, we can still establish convergence.

Example 2.54

Show that the Maclaurin series for $\sin x$ converges to $\sin x$ for every value of x .

$$\begin{aligned} f(x) &= \sin x \\ f^{(1)}(x) &= \cos x \\ f^{(2)}(x) &= -\sin x \\ f^{(3)}(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \end{aligned}$$

Hence, the derivatives of $\sin x$ are cyclical with a cyclicity of 4.

$$\text{Max}(|f^{n+1}(t)|) = M = 1$$

$$|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!} = \frac{(1)|x|^{n+1}}{(n+1)!} = \frac{|x|^{n+1}}{(n+1)!}$$

Take the limit as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!}$$

Expand the numerator and the denominator:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{x \cdot x \cdot x \cdot x \cdot x \cdots x \text{ (n+1 times)}}{1 \cdot 2 \cdot 3 \cdot 4 \cdots (n+1)} \\ &= \lim_{n \rightarrow \infty} \frac{x \cdot x \cdot x \cdot x \text{ (x times)}}{1 \cdot 2 \cdot 3 \cdot 4 \cdots x} \times \frac{x}{x+1} \times \frac{x}{x+2} \times \frac{x}{x+3} \times \cdots \times \frac{x}{n+1} \end{aligned}$$

Since each term after the first fraction is less than 1 and each term is less than the prior term hence, as $n \rightarrow \infty$, the limit $\rightarrow 0$:

$$= 0$$

Example 2.55

Show that the Maclaurin series for $\cos x$ converges to $\cos x$ for every value of x .

$$\begin{aligned} f(x) &= \cos x \\ f^{(1)}(x) &= -\sin x \\ f^{(2)}(x) &= -\cos x \\ f^{(3)}(x) &= \sin x \\ f^{(4)}(x) &= \cos x \end{aligned}$$

Hence, the derivatives of $\cos x$ are cyclical with a cyclicity of 4.

$$\max(|f^{n+1}(t)|) = M = 1$$

$$|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}$$

Which is the same limit as in the previous example.

2.4 Infinite Series

A. Sequences and Series

2.56: Sequences

A sequence is a list of numbers.

Fibonacci Sequence: 1, 1, 2, 3, 5, 8, ...

2.57: Infinite Series

An infinite series is the sum of an infinite sequence of numbers.

2.58: Partial Sums

The n^{th} partial sum of an infinite series is given by the sum of the first n terms of the series.

Example 2.59

Determine the first four partial sums, and the n^{th} partial sum of the series:

$$1 + 2 + 3 + 4 + \dots$$

$$\begin{aligned}S_1 &= 1 \\S_2 &= 1 + 2 = 3 \\S_3 &= 1 + 2 + 3 = 6 \\S_4 &= 1 + 2 + 3 + 4 = 10\end{aligned}$$

$$S_n = \frac{n(n+1)}{2}$$

2.60: Convergence and Divergence

- If the sequence of partial sums of a series converges to a limit L , we say that the series converges, and that its limit is L .
- If the sequence of partial sums does not converge, we say that the series diverges.

Example 2.61

Does the sequence below converge?

$$1 + 1 + 1 + 1 + 1 + \dots$$

The n^{th} partial sum of the series

$$= S_n = n$$

The limit of the sequence of partial sums as the number of terms becomes very large is:

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} n = \infty \Rightarrow \text{Limit DNE} \Rightarrow \text{Diverges}$$

Example 2.62

Does the sequence below converge?

$$1 - 1 + 1 - 1 + 1 - \dots$$

Sum to n terms

$$= S_n = \begin{cases} 1, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

Since the above sum alternates between 0 and 1:

$$\begin{aligned}\lim_{n \rightarrow \infty} S_n &= \text{DNE} \\&\text{Series diverges}\end{aligned}$$

B. Geometric Series

You might have seen geometric series before. Common questions relate to:

- Finding the general term = ar^{n-1}
- Finding the value of a specific term
- Finding the first term and common ratio directly from the series or from given conditions

We will mostly focus here on the sum, and more so on radius of convergence¹.

2.63: Finite Geometric Series

The geometric series with *first term* = a and *common ratio* = r is:

¹ To revise geometric series, refer that chapter in the Note on Sequences and Series

$$a, ar, ar^2, \dots ar^{n-1}, \dots$$

It has sum

$$\frac{a(1 - r^n)}{1 - r}$$

$$S = a + ar + \dots + ar^{n-1}$$

$$rS = ar + ar + \dots + ar^n$$

$$rS - S = ar^n - a$$

$$S(r - 1) = a(r^n - 1)$$

$$S = \frac{a(1 - r^n)}{1 - r}$$

Example 2.64

Determine the n^{th} partial sum of the series below and hence check its convergence or divergence:

$$2 - 1 + \frac{1}{2} - \frac{1}{4} + \dots$$

This is a geometric series with $a = 2, r = \frac{1}{2}$. The n^{th} partial sum of the series is:

$$\frac{a(1 - r^n)}{1 - r} = \frac{2 \left[1 - \left(-\frac{1}{2} \right)^n \right]}{1 - \left(-\frac{1}{2} \right)}$$

Since we do know whether n is odd or even, we apply a \pm sign:

$$= \frac{2 \left[1 \pm \frac{1}{2^n} \right]}{\frac{3}{2}} = \frac{4}{3} \left(1 \pm \frac{1}{2^n} \right)$$

The limit of the sequence of partial sums as the number of terms becomes very large is:

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{4}{3} \left(1 \pm \frac{1}{2^n} \right) = \frac{4}{3} (1) = \frac{4}{3}$$

2.65: Infinite Geometric Series

A infinite geometric series with *first term* = a and *common ratio* = r has sum

$$\frac{a}{1 - r}$$

Where

$$\text{Radius of Convergence: } -1 < r < 1$$

The n^{th} partial sum of an infinite geometric series with *first term* = a , *common ratio* = r is:

$$\frac{a(1 - r^n)}{1 - r}$$

As the number of terms becomes very large:

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{a(1 - r^n)}{1 - r}$$

This limit has a finite value if:

$$-1 < r < 1$$

And does not exist if

$$r \leq -1 \text{ or } r \geq 1$$

Example 2.66

Determine the radius of convergence and the sum for that radius of convergence:

$$2 + \frac{4}{3}(x+5) + \frac{8}{9}(x+5)^2 + \dots$$

This is a geometric series with *first term* = $a = 2$, *common ratio* = $r = \frac{2}{3}(x+5)$

The radius of convergence is $-1 < r < 1$:

$$-1 < \frac{2}{3}(x+5) < 1 \Rightarrow -\frac{3}{2} - 5 < x < \frac{3}{2} - 5$$

It has sum

$$\frac{a}{1-r} = \frac{2}{1 - \frac{2}{3}(x+5)}, \quad -\frac{13}{2} < x < -\frac{7}{2}$$

Example 2.67

Determine the radius of convergence and the sum for that radius of convergence:

$$\sum_{n=0}^{\infty} \left(\frac{x-2}{3}\right)^n$$

$$\sum_{n=0}^{\infty} \left(\frac{x-2}{3}\right)^n = 1 + \left(\frac{x-2}{3}\right) + \left(\frac{x-2}{3}\right)^2 + \dots$$

This is a geometric series with $a = 1, r = \left(\frac{x-2}{3}\right)$ and sum:

$$\frac{a}{1-r} = \frac{1}{1 - \frac{x-2}{3}} = \frac{1}{\frac{3-x+2}{3}} = \frac{3}{5-x}$$

For the series to converge, we must have $-1 < r < 1$:

$$\begin{aligned} -1 &< \frac{x-2}{3} < 1 \\ -3 &< x-2 < 3 \\ -1 &< x < 5 \end{aligned}$$

2.68: Radius of Convergence

Radius of convergence is the interval which results in convergence of the series. If the variable r lies

- in the interval of convergence, the series converges.
- outside the interval of convergence, the series diverges.

2.69: Determining the radius of convergence

Determining the radius of convergence requires solving the inequality

$$-1 < r < 1$$

Example 2.70

Determine the radius of convergence and the sum for that radius of convergence:

$$\sum_{n=0}^{\infty} \cos^n x = 1 + \cos x + \cos^2 x + \cdots + \cos^n x + \cdots$$

The above is a geometric series with $a = 1, r = \cos x$.

The radius of convergence is $-1 < r < 1$:

$$-1 < \cos x < 1$$

Eliminate all values where

$$\cos x = 1 \text{ or } \cos x = -1 \Rightarrow \{x: x = k\pi, k \in \mathbb{Z}\}$$

It has sum

$$\frac{a}{1-r} = \frac{1}{1-\cos x}, x \neq k\pi, k \in \mathbb{Z}$$

Example 2.71

Determine the radius of convergence and the sum for that radius of convergence:

$$\sum_{n=1}^{\infty} (\log_a x)^n = \log_a x + (\log_a x)^2 + \cdots + (\log_a x)^n + \cdots, a > 1$$

The above is a geometric series with $a = \log_a x, r = \log_a x$.

The radius of convergence is $-1 < r < 1$:

$$\begin{aligned} -1 &< \log_a x < 1 \\ -1 &< \frac{\ln x}{\ln a} < 1 \end{aligned}$$

If $a > 1 \Rightarrow \ln a > 0$

$$-\ln a < \ln x < \ln a \Rightarrow \frac{1}{a} < x < a$$

If $0 < a < 1 \Rightarrow \ln a < 0$

$$\begin{aligned} -\ln a &> \ln x > \ln a \\ \ln a^{-1} &> \ln x > \ln a \\ \frac{1}{a} &> x > a \\ a &< x < \frac{1}{a} \end{aligned}$$

It has sum

$$\begin{aligned} \frac{a}{1-r} &= \frac{\log_a x}{1-\log_a x}, \quad \frac{1}{a} < x < a, \quad a > 1 \\ \frac{a}{1-r} &= \frac{\log_a x}{1-\log_a x}, \quad a < x < \frac{1}{a}, \quad 0 < a < 1 \end{aligned}$$

C. Telescoping Series

2.72: Telescoping Series

A telescoping series is one where we can cancel all terms except the first and last terms.

2.73: Splitting a Fraction

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$LHS = \frac{1}{n} - \frac{1}{n+1} = \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} = \frac{1}{n(n+1)} = RHS$$

Example 2.74

$$\frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \frac{1}{6 \cdot 7} + \dots +$$

- A. Calculate the n^{th} partial sum.
- B. Determine the convergence or divergence of the series.

$$\left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \dots + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \left(\frac{1}{n+3} - \frac{1}{n+4}\right)$$

The n^{th} partial sum of the series is:

$$S_n = \frac{1}{4} - \frac{1}{n+4}$$

As the number of terms becomes very large:

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{4} - \frac{1}{n+4} = \frac{1}{4}$$

D. n^{th} term test for divergence

2.75: Convergence Requirement

If the series $\sum_{n=1}^{\infty} a_n$ converges then

$$a_n \rightarrow 0$$

For a series to be convergent, it is a requirement that as $n \rightarrow \infty$, the value of the n^{th} term tends to zero.

2.76: n^{th} term test for divergence

If $\lim_{n \rightarrow \infty} a_n \neq 0$ then

$$\sum_{n=1}^{\infty} a_n \text{ diverges}$$

Note: If the limit does not exist, then also it is not equal to zero

Since the limit as $n \rightarrow \infty$ of the sequence must tend to zero, if this requirement is not met, then the series must diverge.

Example 2.77

$$\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$$

- A. Apply the divergence test to the above series.
- B. Use telescoping to determine the n^{th} partial sum, and hence test for convergence
- C. What is the conclusion from Parts A and B?

Part A

Rationalize the numerator of the expression:

$$\sqrt{n+1} - \sqrt{n} \times \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

Take the limit:

$$= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

*Divergence test does not apply
 Divergence test is inconclusive*

Part B

The n^{th} partial sum is:

$$\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n} = (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + (\sqrt{4} - \sqrt{3}) + \dots + (\sqrt{n+1} - \sqrt{n}) = \sqrt{n+1} - \sqrt{1}$$

Take the limit as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \sqrt{n+1} - \sqrt{1} = \infty$$

Series diverges

Example 2.78

Apply the divergence test to the harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0 \Rightarrow \text{Inconclusive}$$

Note: The harmonic series diverges

Example 2.79

Apply the divergence test to the series below:

$$\sum_{n=1}^{\infty} \frac{e^n}{e^n + 2n}$$

$$\lim_{n \rightarrow \infty} \frac{e^n}{e^n + 2n} = \lim_{n \rightarrow \infty} \frac{e^n}{e^n + 2} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{2}{e^n}} = 1 \neq 0 \Rightarrow \text{Diverges}$$

E. Combining Series

2.80: Combining Series

THEOREM 8 If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum k a_n = k \sum a_n = kA$ (any number k).

Example 2.81

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x-1)^k$$

Determine the convergence or divergence of the series above when $x = 0$

Substitute $x = 0$:

$$\begin{aligned} & \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (-1)^k \\ & \sum_{k=1}^{\infty} \frac{(-1)^{2k+1}}{k} = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \dots = -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right) = -\sum_{n=1}^{\infty} \frac{1}{n} \end{aligned}$$

F. Adding and Deleting Terms

2.82: Adding and Deleting Terms

We can add a finite number of terms, or delete a finite number of terms from a series without altering the convergence or divergence of the series.

Example 2.83

$$\sum_{n=1}^{\infty} \frac{1}{n+4}$$

Expand the given series gives us:

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \dots$$

Add $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \dots$$

Which is now the harmonic series, which diverges.

G. Tails

2.84: Tail of a Series

The tail of a series is the series obtained by dropping a finite number of terms from the beginning of the series.

Example 2.85

What is the tail of the harmonic series obtained by dropping the first three terms?

$$\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots$$

2.86: Convergence and Divergence of Tails

A series converges if and only if its tails converge.

Example 2.87

$$\sum_{n=1}^{\infty} \frac{1}{n+4}$$

Expanding the given series gives us:

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \dots$$

And the above is a tail of the harmonic series, which diverges.

Hence, the given series diverges.

2.5 Integral Test

A. Integral Test

2.88: Integral Test

Let a_1, a_2, \dots, a_n be a sequence of positive terms. If $a_n = f(n)$, where $f(x)$ is a continuous, positive, and decreasing function for all $x \geq N$, $N \in \mathbb{N}$ then the:

series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x) dx$

Both converge or both diverge

Divergence Case

Consider rectangles with width 1 and height a_n to the right of the continuous, positive, decreasing function $f(x)$. Since the function is decreasing, the area of each rectangle is greater than the area under the curve.

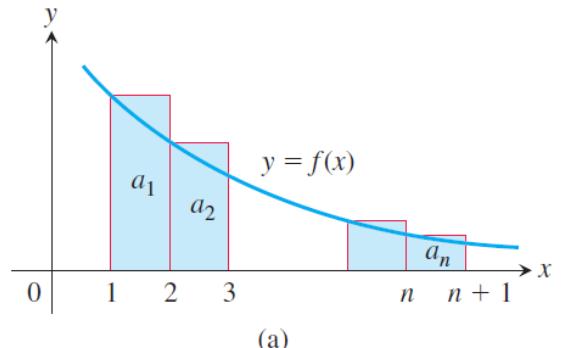
$$a_1 > \int_1^2 f(x) dx$$

$$a_2 > \int_2^3 f(x) dx$$

.

.

$$a_n > \int_n^{n+1} f(x) dx$$



Add the above inequalities:

$$a_1 + a_2 + \dots + a_n > \int_1^2 f(x) dx + \int_2^3 f(x) dx + \dots + \int_n^{n+1} f(x) dx$$

Using integral properties:

$$\sum_{i=1}^n a_i > \int_1^{n+1} f(x) dx$$

Hence, if the integral diverges, the series diverges too.

Convergence Case

The convergence is similar, in that we take rectangles of width 1. However, the rectangles are to the left instead of to the right.

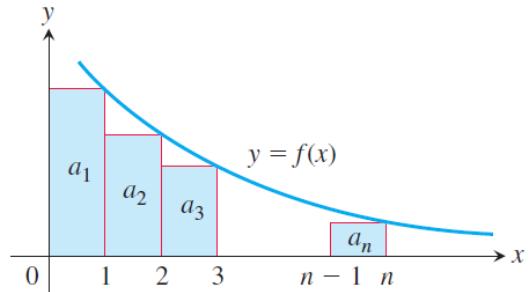
Hence,

$$a_2 < \int_1^2 f(x) dx$$

$$a_3 < \int_2^3 f(x) dx$$

.

$$a_n < \int_{n-1}^{n+1} f(x) dx$$



Add the above:

$$a_2 + \dots + a_n < \int_1^2 f(x) dx + \int_2^3 f(x) dx + \dots + \int_{n-1}^n f(x) dx$$

Add a_1 to both sides of the above:

$$a_1 + a_2 + \dots + a_n < a_1 + \int_1^2 f(x) dx + \int_2^3 f(x) dx + \dots + \int_{n-1}^n f(x) dx$$

Using integral properties:

$$\sum_{i=1}^n a_i < a_1 + \int_1^n f(x) dx$$

Hence, if the integral converges, the series converges too.

B. Harmonic Series

2.89: Divergence of Harmonic Series

The harmonic series (given below) diverges:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots = \sum_{n=1}^{\infty} \frac{1}{n}$$

Method I: Comparison

We can do this without the integral test. Successive groups of terms can be added to be greater than half. Hence, the result can be increased to any value without limit:

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4} \right)}_{>\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right)}_{>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16} \right)}_{>\frac{1}{16}+\frac{1}{16}+\dots+\frac{1}{16}=\frac{8}{16}=\frac{1}{2}} + \dots \Rightarrow \text{Diverges}$$

Method II: Integral Test

Apply the integral test, and check:

$$\int_1^t \frac{1}{x} dx = [\ln|x|]_1^t = \ln t - \ln 1 = \ln t$$

Take as the limit as

$$\lim_{t \rightarrow \infty} \ln t = \infty \Rightarrow \text{Diverges}$$

C. p series

2.90: p -series

A series of the form below is called a p series, where p is any real number:

$$\frac{1}{1^p} + \frac{1}{2^p} + \dots + \frac{1}{n^p} + \dots$$

The harmonic series is a p series with

$$p = 1$$

Example 2.91

Write out the expanded form of each p series. Then determine its convergence or divergence, using the integral test.

A. $p = 3$

B. $p = \frac{2}{3}$

Part A

$$\begin{aligned} & \frac{1}{1^3} + \frac{1}{2^3} + \dots + \frac{1}{n^3} + \dots \\ \int_1^t \frac{1}{x^3} dx &= \int_1^t x^{-3} dx = -\frac{1}{2}[x^{-2}]_1^t = -\frac{1}{2}\left(\frac{1}{t^2} - 1\right) \end{aligned}$$

$$\lim_{t \rightarrow \infty} = -\frac{1}{2} \left(\frac{1}{t^2} - 1 \right) = \frac{1}{2}$$

Converges

Part B

$$\frac{\frac{1}{2}}{1^3} + \frac{\frac{1}{2}}{2^3} + \cdots + \frac{\frac{1}{2}}{n^3} + \cdots$$

$$\int_1^t \frac{1}{x^{\frac{2}{3}}} dx = \int_1^t x^{-\frac{2}{3}} dx = 3 \left[x^{\frac{1}{3}} \right]_1^t = 3 \left(t^{\frac{1}{3}} - 1 \right)$$

$$\lim_{t \rightarrow \infty} 3 \left(t^{\frac{1}{3}} - 1 \right) = \infty$$

Diverges

Example 2.92

Fill in the blanks with the correct option

A p series is ____ a geometric series.

- A. Always
- B. Never
- C. Sometimes
- D. None of the above

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

If the above series is a geometric series, then it must have a common ratio given by

$$r = \frac{\frac{1}{2^p}}{\frac{1}{1^p}} = \frac{\frac{1}{3^p}}{\frac{2^p}{1^p}}$$

$$\frac{1^p}{2^p} = \frac{2^p}{3^p}$$

$$\left(\frac{1}{2}\right)^p = \left(\frac{2}{3}\right)^p$$

$$p = 0$$

Sometimes

Example 2.93

$$\lim_{t \rightarrow \infty} \frac{1}{t^{p-1}}$$

Evaluate the above limit when

- A. $p = 0.5$
- B. $p = -1$
- C. $p = 3$

$$p = 0.5 \Rightarrow \lim_{t \rightarrow \infty} t^{1-0.5} = \lim_{t \rightarrow \infty} t^{0.5} = \lim_{t \rightarrow \infty} \sqrt{t}$$

$$p = -1 \Rightarrow \lim_{t \rightarrow \infty} t^{1-(-1)} = \lim_{t \rightarrow \infty} t^2$$

2.94: p series test

$$\frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p} + \cdots$$

The series above

- Converges when $p > 1$
- Diverges when $p = 1$
- Diverges when $p < 1$

Use the integral test:

$$\int_1^t \frac{1}{x^p} dx = \int_1^t x^{-p} dx = \left[\frac{1}{(-p+1)(x^{p-1})} \right]_1^t = \frac{1}{-p+1} \left(\frac{1}{t^{p-1}} - 1 \right) = \frac{1}{p-1} \left(1 - \frac{1}{t^{p-1}} \right)$$

$$\lim_{t \rightarrow \infty} \frac{1}{p-1} \left(1 - \frac{1}{t^{p-1}} \right) = \frac{1}{p-1} \lim_{t \rightarrow \infty} \left(1 - \frac{1}{t^{p-1}} \right)$$

Converges

If $p > 1$

$$\lim_{t \rightarrow \infty} \left(1 - \frac{1}{t^{p-1}} \right) = 1$$

If $p < 1 \Rightarrow p-1 < 0 \Rightarrow 1-p > 0$:

$$\lim_{t \rightarrow \infty} \left(1 - \frac{1}{t^{p-1}} \right) = \lim_{t \rightarrow \infty} (1 - t^{1-p}) = \infty$$

Example 2.95

Consider the following infinite series. How many of them converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^{-\frac{17}{3}}}, \quad \sum_{n=1}^{\infty} \frac{1}{n^{-\frac{13}{3}}}, \dots, \quad \sum_{n=1}^{\infty} \frac{1}{n^{\frac{23}{3}}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \Rightarrow p \in \left\{ -\frac{17}{3}, -\frac{13}{3}, \dots, \frac{3}{3}, \frac{7}{3}, \frac{11}{3}, \frac{15}{3}, \frac{19}{3}, \frac{23}{3} \right\}$$

5 Terms

D. Logarithmic p series

2.96: Logarithmic p series

The series obtained by dividing the harmonic series by $(\ln x)^p$ is a *logarithmic p series*

$$\frac{1}{x (\ln x)^p}$$

Example 2.97

Determine the convergence or divergence of:

$$\frac{1}{2 \ln 2} + \frac{1}{3 \ln 3} + \frac{1}{4 \ln 4} + \cdots$$

$$\int_2^t \frac{1}{x \ln x} dx$$

Substitute $u = \ln x \Rightarrow du = \frac{1}{x} dx$, and change the limits of integration by taking the logs of the limits:

$$\int_{\ln 2}^{\ln t} \frac{1}{u} du = [\ln|u|]_{\ln 2}^{\ln t} = \ln|\ln t| - \ln|\ln 2|$$

As

$$\lim_{t \rightarrow \infty} \ln|\ln t| - \ln|\ln 2| = \infty$$

Diverges

Example 2.98

Determine the convergence or divergence of:

$$\frac{1}{2(\ln 2)^2} + \frac{1}{3(\ln 3)^2} + \frac{1}{4(\ln 4)^2} + \dots$$

$$\int_2^t \frac{1}{x(\ln x)^2} dx$$

Substitute $u = \ln x \Rightarrow du = \frac{1}{x} dx$, and change the limits of integration by taking the logs of the limits:

$$\int_{\ln 2}^{\ln t} \frac{1}{u^2} du = \left[-\frac{1}{u} \right]_{\ln 2}^{\ln t} = -\frac{1}{\ln t} + \frac{1}{\ln 2}$$

As

$$\lim_{t \rightarrow \infty} \left(-\frac{1}{\ln t} + \frac{1}{\ln 2} \right) = \frac{1}{\ln 2}$$

E. Further Examples

Example 2.99

$$\sum_{n=1}^{\infty} \frac{1}{n+4}$$

$$\int_1^t \frac{1}{x+4} dx = [\ln|x+4|]_1^t$$

$$\lim_{t \rightarrow \infty} (\ln|x+4|) = \infty$$

Diverges

Example 2.100

Check the convergence or divergence of:

$$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$$

Use the integral test and compare with $f(x)$ which is continuous, positive, and decreasing for $x > 1$:

$$f(x) = \frac{1}{x + \sqrt{x}} = \frac{1}{\sqrt{x}(\sqrt{x} + 1)}$$

Find the associated indefinite integral. Substitute $u = \sqrt{x} + 1 \Rightarrow du = \frac{1}{2\sqrt{x}} dx$:

$$\int \frac{1}{\sqrt{x}(\sqrt{x} + 1)} dx = 2 \int \frac{1}{u} du = 2 \ln|u| + C = 2 \ln|\sqrt{x} + 1| + C$$

By the integral test:

$$\int_1^{\infty} \frac{1}{\sqrt{x}(\sqrt{x} + 1)} dx = 2 \left[\ln|\sqrt{x} + 1| \right]_1^{\infty} = \infty \Rightarrow \text{Diverges}$$

F. Error Estimation

2.101: Error Estimation

G. Applications

Example 2.102

2.6 Comparison Test

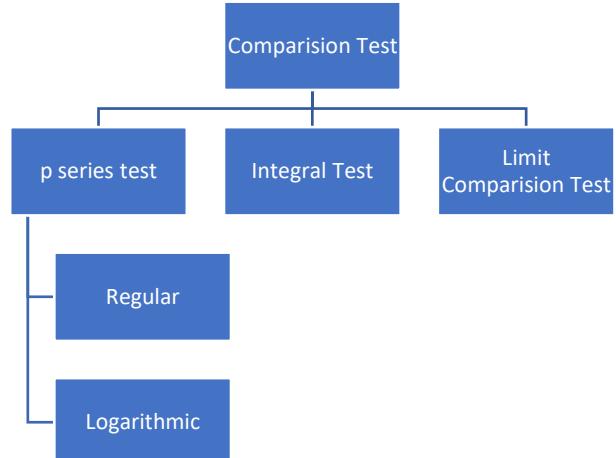
A. Comparison Test

The comparison test let us compare a series whose convergence or divergence we do not know or find difficult to determine, with a series whose convergence or divergence we do know.

This means that we need

- a series to compare with
- a test to apply to that series

The chart alongside shows some basic options for comparison:



- The p series / logarithmic p test is straightforward to apply. You should also know the proof of the p series test, in case a subjective question asks to prove tests that you use.
- The integral test was covered in the previous section.
- The limit comparison test compares the limits of the n^{th} terms of two sequences. You will often need to use the limit comparison test in conjunction with other tests.

B. Basics

2.103: Comparison Test

If $\sum a_n$, $\sum b_n$ and $\sum c_n$ are each series with *non-negative* terms. If

$$a_n \leq b_n \leq c_n \quad \text{for all } n > N, N \in \mathbb{N}$$

- To prove convergence, find something that is larger, and prove convergence for that series
 c_n converges, b_n also converges
- To prove divergence, find something that is smaller, and prove divergence for that series
 a_n diverges, b_n also diverges

2.104: Checking for Smaller and Larger

Taking the reciprocal of an inequality with positive expressions on both sides reverses the inequality.

For positive x and y :

$$x > y \Rightarrow \frac{1}{x} < \frac{1}{y}$$

$$3 > 2 \Rightarrow \frac{1}{3} < \frac{1}{2}$$

Note this does not work for x and y both negative:

$$-3 > -2 \text{ but } \frac{1}{-3} < \frac{1}{-2} \text{ is not true}$$

Example 2.105

The integral $\int \frac{1}{\ln x} dx$ is not elementary. It has an integral, but that integral cannot be expressed in terms of known functions. Hence, use the comparison test to determine the convergence or divergence of:

$$\frac{1}{\ln 2} + \frac{1}{\ln 3} + \cdots + \frac{1}{\ln n} + \cdots = \sum_{n=2}^{\infty} \frac{1}{\ln n}$$

For $x \geq 2$, compare with

$$\frac{1}{x \ln x} < \frac{1}{\ln x}$$

$$\frac{1}{x \ln x} \text{ diverges} \Rightarrow \frac{1}{\ln x} \text{ also diverges}$$

Example 2.106

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2}$$

By the p series test with $p = 2$:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges}$$

Since $\frac{1}{n^2 + 3n + 2} < \frac{1}{n^2}$:

the given series converges

Example 2.107

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + \sqrt{2}}$$

By the p series test with $p = 3$:

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \text{ converges}$$

Since

$$\frac{1}{n^3 + \sqrt{2}} < \frac{1}{n^3} \Rightarrow \text{the given series converges}$$

Example 2.108

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{n - e}{n^4 + \sqrt{\pi}}$$

By the p series test with $p = 3$:

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \text{ converges}$$

Since

$$\frac{n - e}{n^4 + \sqrt{\pi}} = \frac{1 - \frac{e}{n}}{n^3 + \frac{\sqrt{\pi}}{n}} < \frac{1}{n^3}, \text{ the given series converges}$$

Example 2.109

Determine the convergence or divergence of

$$\sum_{n=10}^{\infty} \frac{1}{\sqrt{n} - 3}$$

$$\frac{1}{\sqrt{n} - 3} > \frac{1}{\sqrt{n}} \text{ which diverges by the } p \text{ series test}$$

given series also diverges

Example 2.110

Check the convergence or divergence of:

$$\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$$

For $n \geq 1$, compare with:

$$n + \sqrt{n} < 2n \Rightarrow \frac{1}{n + \sqrt{n}} \geq \frac{1}{2n}$$

$$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \rightarrow \text{Harmonic Series} \Rightarrow \text{Diverges}$$

By the comparison test:

$$\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}} \text{ diverges}$$

C. Limit Comparison Test

- The limit comparison test requires you to find the ratio of two sequences as the term number tends to infinity.
- This is different from the divergence test, where you find the value of the sequence as the term number tends to infinity.

2.111: Limit Comparison Test

Consider sequences $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N is an integer). If

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= c > 0 \text{ then both converge or both diverge} \\ \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= 0 \text{ and } \sum b_n \text{ converges, then } \sum a_n \text{ also converges} \\ \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \infty \text{ and } \sum b_n \text{ diverges, then } \sum a_n \text{ also diverges} \end{aligned}$$

Example 2.112

Determine the convergence or divergence of:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2}$$

$$\lim_{n \rightarrow \infty} \frac{1}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 3n + 2} = \lim_{n \rightarrow \infty} \frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2} + \frac{3n}{n^2} + \frac{2}{n^2}} = \frac{1}{1 + 0 + 0} = 1 > 0$$

By the p series test with $p = 2$:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges}$$

By the limit comparison test:

$$\text{since } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges, } \sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2} \text{ also converges}$$

Example 2.113

Determine the convergence or divergence of:

$$\sum_{n=10}^{\infty} \sqrt{\frac{n^2 + 1}{n^3 + 5}}$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{\frac{n^2 + 1}{n^3 + 5}}}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \sqrt{\frac{n^2 + 1}{n^3 + 5}} \cdot \sqrt{n} = \lim_{n \rightarrow \infty} \sqrt{\frac{n^3 + n}{n^3 + 5}} = \sqrt{\lim_{n \rightarrow \infty} \frac{n^3 + n}{n^3 + 5}} = 1$$

Since $\sum_{n=10}^{\infty} \frac{1}{\sqrt{n}}$ diverges, $\sum_{n=10}^{\infty} \sqrt{\frac{n^2+1}{n^3+5}}$ also diverges

Example 2.114

Determine the convergence or divergence of:

$$\sum_{n=1}^{\infty} \frac{1}{3\sqrt{n} + \sqrt[4]{n}}$$

Compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a divergent p series:

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{3\sqrt{n} + \sqrt[4]{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{3\sqrt{n} + \sqrt[4]{n}} = \lim_{n \rightarrow \infty} \frac{\frac{\sqrt{n}}{\sqrt{n}}}{\frac{3\sqrt{n}}{\sqrt{n}} + \frac{\sqrt[4]{n}}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{1}{3 + \frac{1}{\sqrt[4]{n}}} = \lim_{n \rightarrow \infty} \frac{1}{3 + \frac{1}{\sqrt[4]{n}}} = \frac{1}{3} > 0 \Rightarrow \text{Diverges}$$

D. Comparison using Telescoping

Example 2.115

Determine the convergence or divergence of:

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

For $n \geq 2$:

$$\frac{1}{n(n-1)} > \frac{1}{n!}$$

Check the convergence of $\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$:

$$\sum_{n=2}^m \frac{1}{n(n-1)} = \sum_{n=2}^m \left[\frac{1}{n-1} - \frac{1}{n} \right]$$

Expanding gives:

$$\begin{aligned} &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\ &= 1 - \frac{1}{m} \end{aligned}$$

$$\lim_{m \rightarrow \infty} 1 - \frac{1}{m} = 1$$

Hence, since

$\sum_{n=2}^m \frac{1}{n(n-1)}$ converges, $\sum_{n=2}^{\infty} \frac{1}{n!}$ also converges

And

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \sum_{n=2}^{\infty} \frac{1}{n!} \text{ also converges}$$

2.7 Absolute Convergence; Ratio Test

A. Absolute Convergence Test

2.116: Absolute Convergence

If a series is absolutely convergent then it is convergent.

2.117: Absolute Convergence

A series $\sum a_n$ is absolutely convergent if the corresponding series of absolute values

$$\sum |a_n|$$

Converges.

Example 2.118

Check the convergence of the series using the absolute convergence test:

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} + \dots$$

The corresponding series of absolute values is

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

Which converges since it is a geometric series with $r = \frac{1}{2} < 1$

Hence, the original series converges.

Example 2.119

Check the convergence of the series using the absolute convergence test

$$\frac{\cos 1}{1^3} + \frac{\cos 2}{2^3} + \dots + \frac{\cos n}{n^3}$$

Use the absolute convergence test to check the corresponding series of absolute values:

$$|a_n| = \left| \frac{\cos n}{n^3} \right|$$

Use the comparison test to compare with $\frac{1}{n^3}$

$$\left| \frac{\cos n}{n^3} \right| \leq \frac{1}{n^3} \text{ since } |\cos n| \leq 1$$

$\left| \frac{\cos n}{n^3} \right|$ converges since $\frac{1}{n^3}$ converges by p series test

Since the series is absolutely convergent, it is convergent.

B. Ratio Test

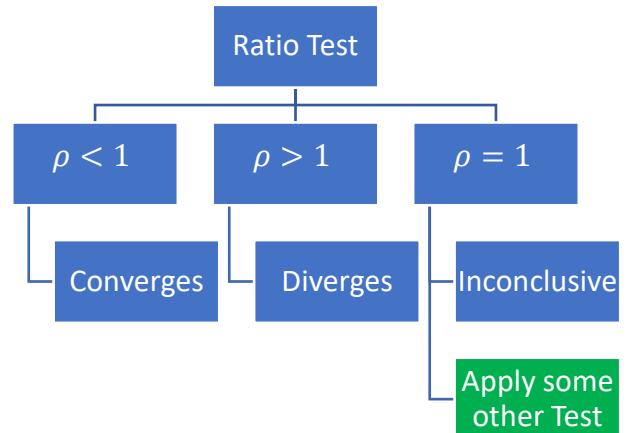
Absolute convergence is the basis of the ratio test, which is an important test.

2.120: Ratio Test

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| a_{n+1} \times \frac{1}{a_n} \right| = \rho$$

For:

$$\begin{aligned}\rho < 1 &\text{ series converges absolutely} \\ \rho = 1 &\text{ test is inconclusive} \\ \rho > 1 &\text{ series diverges}\end{aligned}$$



- The ratio test compares consecutive terms of a sequence.
- If the limit of the absolute value of the ratio of consecutive terms of the sequence as $n \rightarrow \infty$ is less than 1, then the series converges absolutely (and hence it converges).

Example 2.121

Check the convergence of the series below using the ratio test:

$$\sum_{n=1}^{\infty} \frac{(0.45)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{(1.1)^n}{n}$$

$$\lim_{n \rightarrow \infty} \left| a_{n+1} \times \frac{1}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(0.45)^{n+1}}{n+1} \times \frac{n}{(0.45)^n} = \lim_{n \rightarrow \infty} \frac{0.45n}{n+1} = 0.45 < 1 \Rightarrow \text{Converges}$$

$$\lim_{n \rightarrow \infty} \left| a_{n+1} \times \frac{1}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(1.1)^{n+1}}{n+1} \times \frac{n}{(1.1)^n} = \lim_{n \rightarrow \infty} \frac{1.1n}{n+1} = 1.25 > 1 \Rightarrow \text{Diverges}$$

2.122: Recursive Definition of Factorial

$$n! = n(n-1)(n-2) \dots (n-k)!$$

Example 2.123

Check the convergence of the series below:

$$\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$$

Apply the ratio test:

$$\left| a_{n+1} \times \frac{1}{a_n} \right| = \frac{[(n+1)!]^3}{(3n+3)!} \times \frac{(3n)!}{(n!)^3}$$

Expand using the recursive definition of the factorial:

$$= \frac{[(n+1)(n!)]^3}{(3n+3)(3n+2)(3n+1)(3n)!} \times \frac{(3n)!}{(n!)^3}$$

$$= \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)}$$

The numerator when expanded is a third-degree polynomial with leading coefficient 1.

The denominator when expanded is a third-degree polynomial with leading coefficient $3 \times 3 \times 3 = 27$.

The limit of the above expression as $n \rightarrow \infty$ is the ratio of the leading coefficients, which is

$$\frac{1}{27} < 1 \Rightarrow \text{Series Converges}$$

Example 2.124

Given that $\lim_{n \rightarrow \infty} \frac{a_n}{a_{n-1}} = p$, determine the values of p for which the series below converges

$$\sum_{n=0}^{\infty} a_n 2^n$$

The ratio of consecutive terms of the sequence $\sum_{n=0}^{\infty} 2^n a_n$ is:

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1} a_{n+1}}{2^n a_n} = \frac{2 a_{n+1}}{a_n}$$

Take the limit as $n \rightarrow \infty$, and move the 2 outside since it is a constant:

$$\lim_{n \rightarrow \infty} \left| \frac{2 a_{n+1}}{a_n} \right| = 2 \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Since a_{n+1} and a_n are consecutive terms, so are a_n and a_{n-1} . Since we take the limit as n is very large, subtracting 1 from the term number does not matter. (The term number will not go negative).

$$= 2 \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n-1}} \right|$$

And the limit is given in the question as p . Hence:

$$\lim_{n \rightarrow \infty} \left| \frac{2 a_{n+1}}{a_n} \right| = 2|p|$$

For the series to converge, we must have

$$\begin{aligned} 2|p| &< 1 \\ |p| &< \frac{1}{2} \\ -\frac{1}{2} &< p < \frac{1}{2} \end{aligned}$$

Example 2.125

Check the convergence of:

$$\sum_{n=1}^{\infty} \frac{4^{n+2}}{\ln(n+1)}$$

Apply the ratio test:

$$\left| a_{n+1} \times \frac{1}{a_n} \right| = \frac{4^{n+3}}{\ln n + 2} \times \frac{\ln n + 3}{4^{n+2}} = \frac{4 \ln(n+3)}{\ln(n+2)}$$

Take the limit as $n \rightarrow \infty$ and apply LH rule:

$$\lim_{n \rightarrow \infty} \frac{4 \ln n + 3}{\ln n + 2} = 4 \lim_{n \rightarrow \infty} \frac{\frac{4}{n} + \frac{3}{n^2}}{\frac{1}{n} + \frac{2}{n^2}} = 4 \lim_{n \rightarrow \infty} \frac{n+2}{n+3} = 4 \Rightarrow \text{Diverges}$$

2.8 Alternating Series; Conditional Convergence

A. Alternating Series

2.126: Alternating Series

A series which has terms which alternate between positive and negative terms is an alternating series.

Example:

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$$

2.127: Alternating Series Test

An alternating series

$$u_1 - u_2 + u_3 - u_4 + \dots$$

Converges if:

- $u_n > 0$
- $u_n \rightarrow 0$ as $n \rightarrow \infty$
- u_n are eventually nonincreasing $u_n \geq u_{n+1}$

B. Alternating Series Error Bound

The error bound of a series is the absolute value of the difference between the n^{th} partial sum and the “value” of the series.

2.128: Error Bound

$L = \text{Limit of partial sums as number of terms tends to infinity}$

$s_n = \text{partial sum upto } n \text{ terms}$

$$\text{Error bound} = |L - s_n|$$

2.129: Overestimation and Underestimation

$L - s_n < 0$ if $t_{n+1} < 0 \Rightarrow s_n$ overestimates L

$L - s_n > 0$ if $t_{n+1} > 0 \Rightarrow s_n$ underestimates L

t_{n+1} is the first unused term

Example 2.130

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$$

$$L = \frac{a}{1-r} = \frac{1}{1 - \left(-\frac{1}{2}\right)} = \frac{2}{3}$$

$$s_1 = 1 \Rightarrow L - s_1 = \frac{2}{3} - 1 = -\frac{1}{3} \Rightarrow \text{Overestimate by } \frac{1}{3} \text{ because } t_{n+1} = -\frac{1}{2} < 0$$

$$s_2 = 1 - \frac{1}{2} = \frac{1}{2} \Rightarrow L - s_2 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \Rightarrow \text{Underestimate by } \frac{1}{6} \text{ because } t_{n+1} = \frac{1}{4} > 0$$

$$s_3 = 1 - \frac{1}{2} + \frac{1}{4} = \frac{1}{2} = \frac{3}{4} \Rightarrow L - s_3 = \frac{2}{3} - \frac{3}{4} = -\frac{1}{12} \Rightarrow \text{Overestimate by } \frac{1}{12} \text{ because } t_{n+1} = -\frac{1}{8} < 0$$

Example 2.131

If $L = \ln 2 \approx 0.69$, then calculate the error when the first partial sum is used to estimate the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

$$s_1 = 1$$

First unused term = $s_2 = -\frac{1}{2}$

$$L - s_1 = \ln 2 - 1 \approx -0.31$$

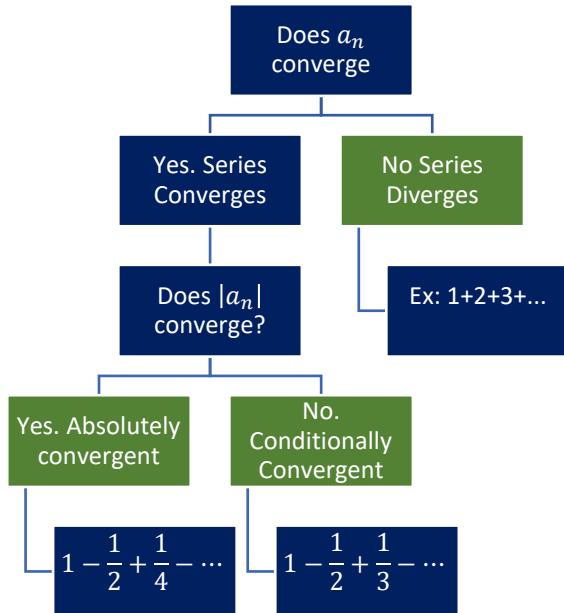
Error is negative

$$\text{Overestimation } -\frac{1}{2} < 0$$

C. Absolute Versus Conditional Convergence

2.132: Conditional Convergence

If a series converges, but does not converge absolutely, then it is conditionally convergent.



Case I:

Using any test, if the series diverges, then you get Case I. For example:

$$1 + 2 + 3 + 4 + \dots$$

Case II:

If the series converges, there are two possibilities. It can be absolutely convergent, or conditionally convergent. We will see both the cases.

Case II-A: Absolutely convergent

A series absolute terms converge is absolutely convergent. An absolutely convergent series is automatically convergent.

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$$

The absolute values are

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \Rightarrow \text{geometric series with } r = \frac{1}{2} \Rightarrow \text{Converges}$$

Case II-B: Conditional Convergence

If the series converges, but the series with absolute values of the terms does not converge, then the series is conditionally convergent.

$$\begin{aligned} \text{Alternating Harmonic Series: } & 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \Rightarrow \text{Converges by alternating series test} \\ \text{Harmonic Series: } & 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \Rightarrow \text{Diverges (See example before)} \end{aligned}$$

Hence, the alternating harmonic series is conditionally convergent.

Example 2.133

$$\sum_{n=2}^{\infty} \frac{\cos n}{n(\ln n)^2}$$

Consider the corresponding series with absolute values. The denominator is always positive:

$$\sum_{n=2}^{\infty} \left| \frac{\cos n}{n(\ln n)^2} \right| = \sum_{n=2}^{\infty} \frac{|\cos n|}{n(\ln n)^2}$$

Since $|\cos n| \leq 1$ use the Comparison Test. Compare with:

$$\sum_{n=2}^{\infty} \frac{|\cos n|}{n(\ln n)^2} \leq \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

Which is a logarithmic p series with

$$n = 2 \Rightarrow \text{Converges}$$

By the logarithmic p series test $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges

By the comparison test: $\sum_{n=2}^{\infty} \frac{|\cos n|}{n(\ln n)^2}$

Since $\sum_{n=2}^{\infty} \frac{\cos n}{n(\ln n)^2}$ converges absolutely, it converges

2.134: Rearranging Conditionally Convergent Series

This is not allowed, since the sum will change.

2.9 Radius and Interval of Convergence

A. Radius of Convergence

2.135: Radius of Convergence

The radius of convergence of a power series $\sum c_n(x - a)^n$ is the number R such that the series converges for $|x - a| < R$

- The series will diverge when $|x - a| > R$
- The endpoints $|x - a| \leq R$ must be checked separately, but this is not required to determine the value of R .

Example 2.136

Given that $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, determine the radius of convergence R and the value of a of the following series:

$$\sum_{n=0}^{\infty} a_n (-3)^n x^{2n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}(-3)^{n+1}x^{2n+2}}{a_n(-3)^n} \right| = \lim_{n \rightarrow \infty} |-3x^2| = 3x^2$$

$$\begin{aligned} 3x^2 &< 1 \\ x^2 &< \frac{1}{3} \\ |x| &< \frac{1}{\sqrt{3}} \\ R &= \frac{1}{\sqrt{3}} \\ -\frac{1}{2} &< x < \frac{1}{2} \end{aligned}$$

B. Interval of Convergence

2.137: Endpoint of Radius of Convergence

The endpoints of the radius of convergence must be checked separately

$$\begin{aligned} |x - a| &< R \\ -R &< x - a < R \\ a - R &< x < a + R \end{aligned}$$

2.138: Interval of Convergence

The interval over which a power series $\sum c_n(x - a)^n$ converges is called its interval of convergence. The interval can be:

Open: $(a - R, a + R)$
Half-open: $[a - R, a + R]$ or $(a - R, a + R]$
Closed: $[a - R, a + R]$

Example 2.139

$$\sum_{n=1}^{\infty} \frac{c^n}{n}$$

- A. Check the convergence of the series above.
- B. Show that the series is a geometric series. Determine the common ratio of the series.
- C. Use your results from Parts A and B to determine when a geometric series converges.

Part A

$$\lim_{n \rightarrow \infty} \left| a_{n+1} \times \frac{1}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{c^{n+1}}{n+1} \times \frac{n}{c^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{cn}{n+1} \right| = |c|$$

If

$$\begin{aligned} |c| > 1 &\Rightarrow \text{Diverges} \\ |c| < 1 &\Rightarrow \text{Converges} \end{aligned}$$

When

$$c = 1 \Rightarrow \text{Test is inconclusive}$$

Hence, we need to use some other test.

Substitute $c = 1$. Gives us the

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \Rightarrow \text{Harmonic Series} \Rightarrow \text{Diverges}$$

Part B

$$\frac{c}{n} + \frac{c^2}{n} + \frac{c^3}{n}$$

\Rightarrow Geometric series with first term $= \frac{c}{n}$ and common ratio $= c$

It converges when

$$|c| < 1$$

Example 2.140: Tan Inverse Series

The Maclaurin series for $\tan^{-1} x$ is:

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \dots$$

Use the ratio test:

$$\left| a_{n+1} \times \frac{1}{a_n} \right| = \left| \frac{x^{2n+1}}{2n+1} \times \frac{2n-1}{x^{2n-1}} \right| = \left| \frac{x^2(2n-1)}{2n+1} \right|$$

As $n \rightarrow \infty$, the above expression tends to:

$$\begin{aligned} |x^2| &< 1 \\ x^2 &< 1 \\ -1 &< x < 1 \end{aligned}$$

Check the endpoints:

$$x = 1 \Rightarrow \text{Series} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \sum_{n=0}^{\infty} \frac{1}{2n+1}$$

$$\text{For } n = 0 \text{ onwards, } b_n = \frac{1}{2n+1}$$

$$b_n \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$b_{n+1} \leq b_n \text{ term}$$

Converges by Alternating Series Test

$$x = 1 \Rightarrow \text{Series} = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \dots = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2n+1}$$

$$b_n \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$b_{n+1} \leq b_n \text{ term}$$

Converges by Alternating Series Test

Example 2.141

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$$

Use the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(x-3)^{n+1}}{2n+3}}{\frac{(x-3)^n}{2n+1}} \right| = \lim_{n \rightarrow \infty} \frac{(x-3)(2n+1)}{2n+3} = |x-3|$$

$$|x-3| < 1$$

$$-1 < x-3 < 1$$

$$2 < x < 4$$

Check the endpoints:

$$x = 4: \sum_{n=0}^{\infty} (-1)^n \frac{(1)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \text{Converges by Alternating Series Test}$$

$$x = 2: \sum_{n=0}^{\infty} (-1)^n \frac{(-1)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{2n+1} = \sum_{n=0}^{\infty} \frac{1}{2n+1}$$

Use the Integral Test for the above ($f(x) = \frac{1}{2x+1}$) is continuous, positive, and decreasing

$$\int_0^{\infty} \frac{1}{2x+1} dx = \left[\frac{1}{2} \ln|2x+1| \right]_0^{\infty} \Rightarrow \text{Diverges} \Rightarrow \text{Series Diverges}$$

Example 2.142: Natural Logarithm

Determine the interval of convergence of the Taylor series:

$$\ln(x+1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} -$$

Use the ratio test:

$$\lim_{n \rightarrow \infty} \left| a_{n+1} \times \frac{1}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+1} \times \frac{n}{x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{xn}{n+1} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{1 + \frac{1}{n}} \right| = |x|$$

Series Converges: $-1 < x < 1$

Series Diverges: $x < -1$ OR $x > 1$

Inconclusive: $x \in \{-1, 1\}$

Check Endpoints

$$x = 1: 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \Rightarrow \text{Alternating Harmonic Series} \Rightarrow \text{Converges}$$

$$x = -1: -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \dots \Rightarrow - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \right) \Rightarrow \text{Harmonic Series} \Rightarrow \text{Diverges}$$

Hence, the interval of convergence of the power series is:

$$-1 < x \leq 1$$

Example 2.143: Natural Logarithm

Determine the interval of convergence of the Taylor series for $\ln x$ centered at $x = 1$ given below:

$$\ln x = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \dots$$

Use the ratio test:

$$\lim_{n \rightarrow \infty} \left| a_{n+1} \times \frac{1}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x - 1)^{n+1}}{n + 1} \times \frac{n}{(x - 1)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x - 1)n}{n + 1} \right| = |x - 1|$$

Convergence requires:

$$|x - 1| < 1 \Rightarrow -1 < x - 1 < 1 \Rightarrow 0 < x < 2$$

Check the endpoints:

$$x = 2: 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \Rightarrow \text{Alternating Harmonic Series} \Rightarrow \text{Converges}$$

$$x = 0: -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \dots \Rightarrow \text{Negative of Harmonic Series} \Rightarrow \text{Diverges}$$

Hence, the series converges when the above condition is met.

Example 2.144

Check the convergence of the series

$$\frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2n+1}}{2n+1} + \dots$$

Use the ratio test:

$$\lim_{n \rightarrow \infty} \left| a_{n+1} \times \frac{1}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{2n+3} \times \frac{2n+1}{x^{2n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^2(2n+1)}{2n+3} \right| = |x^2| = x^2 < 1$$

$$-1 < x < 1$$

Check the endpoints:

$$x = \pm 1: a_n = \frac{\pm 1}{2n+1} \Rightarrow \text{Integral Test: } \int_1^\infty \frac{\pm dx}{2x+1} = \pm \left[\frac{1}{2} \ln|2x+1| \right]_1^\infty \Rightarrow \text{Diverges}$$

C. Special Cases

2.145: Converges Everywhere

$R = \infty$: This means that the series converges everywhere

2.146: Converges Only at $x = a$

$R = 0$

- The power series has only one term.
- It gives the value of the function at that point.

3. SERIES TOPICS

3.1 Review

A. n^{th} term test for divergence

3.1: n^{th} term test for divergence

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} - \sqrt{n}} : \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} - \sqrt{n}} \times \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n \rightarrow \infty} \sqrt{n+1} + \sqrt{n} = \infty \neq 0 \Rightarrow \text{Diverges}$$

B. Geometric Series

3.2: Geometric Series

$$\sum_{n=0}^{\infty} r^n \text{ is convergent iff } |r| < 1$$

$$\sum_{n=0}^{\infty} \frac{2^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \Rightarrow r = \frac{2}{3} < 1 \Rightarrow \text{Cgt}$$

$$\sum_{n=0}^{\infty} \frac{e^n}{2.5^n} \Rightarrow r < 1 \Rightarrow \text{Dgt}$$

C. Harmonic and Related Series

3.3: Integral Test

$$\text{Harmonic Series: } \sum_{n=1}^{\infty} \frac{1}{n} : \int_1^{\infty} \frac{1}{n} = [\ln n]_1^{\infty} = \ln n - \ln 1 = \ln n \Rightarrow \text{Dgt}$$

3.4: Alternating Series Test

The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the Alternating Series Test.
 Using the expansion of $\ln(1 + x)$:

$$\ln 2 = \ln(1 + 1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

3.5: Comparison Test

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{Dgt}} < \underbrace{\sum_{n=1}^{\infty} \frac{\ln n}{n}}_{\text{Also dgt}}$$

3.6: Dirichlet Test

$$\sum_{n=1}^{\infty} \frac{\sin n}{n}$$

Sum $[\sin(n)/n]$ is convergent by Dirichlet's Test because:

1. Sum $[\sin(n)]$ is bounded by considering n as an AP, and using product to sum formula, and then telescoping.
2. $1/n$ has limit tending to zero as n tends to infinity.

D. p series

3.7: p series test

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

- Diverges for $p \leq 1$
- Converges for $p > 1$

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2 + 2024}}_{\text{Also Cgt}} < \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{Cgt(p series)}}$$

3.8: Asymptotic Comparison Test

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2 + 2024}}_{\text{Also Cgt}} \sim \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{Cgt(p series)}}$$

$$\underbrace{\sum_{n=1}^{\infty} \frac{n^2 + 5}{n^5 + 2}}_{\text{Also Cgt}} \sim \underbrace{\sum_{n=1}^{\infty} \frac{n^2}{n^5}}_{\text{Cgt(p series)}} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^3}}_{\text{Cgt(p series)}}$$

3.9: Limit Comparison Test

Limit Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n \geq 0, b_n > 0$ for all n . Define,

$$c = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$$

If c is positive (i.e. $c > 0$) and is finite (i.e. $c < \infty$) then either both series converge or both series diverge.

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} \frac{n^2 + 5}{n^5 + 2} = \sum_{n=1}^{\infty} \frac{n^2}{n^5} \cdot \frac{\left(\frac{n^2}{n^5} + \frac{5}{n^2}\right)}{\frac{n^5}{n^5} + \frac{2}{n^5}} = \sum_{n=1}^{\infty} \frac{1}{n^3} \cdot \frac{1 + \frac{5}{n^2}}{1 + \frac{2}{n^5}} \\ \sum_{n=1}^{\infty} b_n &= \sum_{n=1}^{\infty} \frac{1}{n^3} \end{aligned}$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^3} \cdot \frac{1 + \frac{5}{n^2}}{1 + \frac{2}{n^5}}}{\frac{1}{n^3}} = \lim_{n \rightarrow \infty} \frac{1 + \frac{5}{n^2}}{1 + \frac{2}{n^5}} = 1 \Rightarrow \text{Finite}$$

$$\sum_{n=1}^{\infty} b_n \text{ cgt} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ cgt}$$

E. Euler-Mascheroni Constant

3.10: Euler-Mascheroni Constant

$$\gamma_{\substack{\text{Finite} \\ \text{Convergent}}} = \left(\sum_{n=1}^{\infty} \frac{1}{n} \right) - \ln n$$

F. Basel Problem

3.11: Sum of Reciprocals of Squares

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

3.2 Further Topics

12 Examples