Calculus Topics

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1. Differential Equations

1.1 Definition and Basics

A. Differential Equations

1.1: Differential Equation

An equation that has a derivative in it is called a differential equation.

Example 1.2

A simple differential equation is $\frac{dy}{dx} = f(x)$. Solve it.

Which can be solved by integrating both sides with respect to x:

$$\frac{dy}{dx} = f(x) \Rightarrow \int \frac{dy}{dx} dx = \int f(x)\, dx \Rightarrow y = \int f(x)\, dx$$

1.3: Order and Degree

In a differential equation:

- Order is the order of the highest derivative in the equation
- Degree is the highest exponent of the highest order derivative

Notes:

1. Order and degree are always positive integers (if defined).
2. We will see cases where the degree is not defined in the next definition.

Example 1.4

Find the order and the degree of the following differential equations.

A. $x^2 \frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^4$ (CBSE 2019)

B. $x^3 \left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$ (CBSE 2013, 2019)

Part A

$$Order = 2$$
$$Degree = 1$$

Part B

$$Order = 2$$
$$Degree = 2$$

1.5: "Polynomial" versus Non-polynomial equations

- The definition of degree assumes that a differential equation can be written as a polynomial equation in terms of its derivatives.
- If a differential equation is not polynomial, then it does not have a degree.

$$\underbrace{y^2 + \frac{1}{2}x + x\frac{dy}{dx} = 0}_{Polynomial \Rightarrow Degree=1}, \quad \underbrace{y - x = \sin\left(\frac{dy}{dx}\right)}_{Not\ Polynomial \Rightarrow No\ Degree}$$

Example 1.6

Find the order and the degree (if defined) of the following differential equations

A. $\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = 2x^2 \log\left(\frac{d^2y}{dx^2}\right)$ **(CBSE 2019)**

B. $\frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^3\right\} = 0$ **(CBSE 2015)**

Part A

$$Order = 2$$

Since $\log\left(\frac{d^2y}{dx^2}\right)$ is not polynomial:

$$Degree\ is\ not\ defined$$

Part B

Differentiate using the chain rule:

$$\left(\frac{dy}{dx}\right)^{3-1}\left(\frac{d^2y}{dx^2}\right) = 0$$

$$Order = 2$$

$$Degree = 1$$

1.7: Linear Differential Equation

A differential equation that can be written in the form

B. Verifying Solutions

Before finding solutions, we learn the useful technique of validating solutions which have already been found. This requires differentiation, which is easier than integration.

Example 1.8

Verify that $y = \frac{1}{\sqrt{x}}, x > 0$ is a solution to $\frac{dy}{dx} + \frac{y}{2x} = 0$.

Differentiate both sides of $y = \frac{1}{\sqrt{x}}$:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}\left(x^{-\frac{1}{2}}\right) = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x^{\frac{3}{2}}}$$

$$\frac{y}{2x} = y \times \frac{1}{2x} = \frac{1}{\sqrt{x}} \times \frac{1}{2x} = \frac{1}{2x^{\frac{3}{2}}}$$

We can now verify that:

$$LHS = \frac{dy}{dx} + \frac{y}{2x} = -\frac{1}{2x^{\frac{3}{2}}} + \frac{1}{2x^{\frac{3}{2}}} = 0 = RHS$$

Example 1.9

Verify that $y = \frac{3}{t}$ is a solution of $\frac{dy}{dt} = -\frac{1}{3}y^2$.

Differentiate the given solution:

$$\frac{dy}{dt} = -\frac{3}{t^2}$$

Substitute $y = \frac{3}{t} \Rightarrow t = \frac{3}{y}$

$$\frac{dy}{dt} = -\frac{3}{\left(\frac{3}{y}\right)^2} = -\frac{3}{\frac{9}{y^2}} = -\frac{3}{y^2}$$

C. Forming Equations

1.10: General Solutions

Differential equations are solved by integration, with an arbitrary constant of integration.
Hence, we get, an infinite number of solutions.
Such a solution is a called a general solution.

1.11: Number of Constants
An equation of n^{th} order will have n arbitrary constants in its solution.

Example 1.12: Number of Constants in Solution

1.2 Separable Equations

A. Basics

1.13: Separable Equations
A differential equation that can be written in the form below is called a separable differential equation:

$$y' = f(x) \cdot g(y)$$

- The LHS contains only the derivative of y.
- The RHS is a product of a function of x, and a function of y.

A separable equation can be divided on both sides by the function of y to give only y variables on the LHS and only x variables on the RHS.

$$\underbrace{\frac{y'}{g(y)}}_{y\ variables} = \underbrace{f(x)}_{x\ variables}$$

Example 1.14: Classifying Equations
Are the following equations separable?

A. $y' = 2x + 3y$
B. $y' = (2x)(3y)$
C. $y' = f(x) + g(y)$

$$No$$
$$Yes$$
$$No$$

1.15: Solving Separable Equations

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

Separate the variables:

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

Integrate both sides with respect to x:

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx$$

Note that since the dx on the LHS cancels, you integrate with respect to y on the LHS:

$$\int \frac{1}{g(y)} dy = \int f(x)\, dx$$

Substitute $\int \frac{1}{g(y)} dy = G(y), \int f(x)\, dx = F(x)$, and add the constant of integration:

$$G(y) = F(x) + C$$

Example 1.16

$$\frac{dy}{dt} = -\frac{1}{3}y^2$$

Separate the variables:

$$-\frac{dy}{y^2} = \frac{1}{3}dt$$

Integrate:

$$\int -\frac{dy}{y^2} = \int \frac{1}{3}dt$$

Carry out the integration:

$$\frac{1}{y} = \frac{1}{3}t + C$$

Add the fractions:

$$\frac{1}{y} = \frac{t + 3C}{3}$$

Take the reciprocal on both sides:

$$y = \frac{3}{t + 3C}$$

Substitute $K = 3C$:

$$y = \frac{3}{t + K}$$

1.17: Constant of Integration

When integrating both sides of a differential equation, you will get a constant of integration on the LHS as well as the RHS.

However, the two constants can be replaced by a single constant.

Example 1.18

Integrate the equation $\frac{dy}{dx} = -\frac{x}{y}$, and determine the curve determined by the equation you get.

Separate the variables:

$$y\, dy = -x\, dx$$

Integrate:

$$\int y\, dy = \int -x\, dx$$

Carry out the integration:

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$y^2 = -x^2 + 2C$$
$$y^2 + x^2 = 2C$$

Substitute $2C = r^2$:

$$y^2 + x^2 = r^2$$

Which is the equation of a circle.

Example 1.19

$$\frac{dy}{dt} = y^2 \sin t$$

Separate the variables:

$$\frac{dy}{y^2} = \sin t \, dt$$

Integrate:

$$\int \frac{dy}{y^2} = \int \sin t \, dt$$

$$-\frac{1}{y} = -\cos t + C$$

Solve for y:

$$y = \frac{1}{\cos t - C}$$

Substitute $K = -C$

$$y = \frac{1}{\cos t + K}$$

Example 1.20

Solve the differential equation $\cos\left(\frac{dy}{dx}\right) = a, (a \in \mathbb{R})$ **(CBSE 2018)**

Take the cos inverse on both sides:

$$\frac{dy}{dx} = \cos^{-1} a$$

Separate the variables:

$$dy = \cos^{-1} a \, dx$$

Integrate both sides:

$$\int dy = \int \cos^{-1} a \, dx$$

Note that $\cos^{-1} a$ is simply a constant:

$$y = (\cos^{-1} a)(x) + C$$

Example 1.21

Solve for y:

$$\frac{dy}{dt} = -0.6y$$

General Solution

$$\frac{dy}{y} = -0.6\,dt$$
$$\int \frac{dy}{y} = \int -0.6\,dt$$

Integrate both sides to obtain the general solution:

$$\ln|y| = -0.6t + C$$

Solve for y

Exponentiate both sides to e:

$$|y| = e^{-0.6t+C}$$

Since the RHS is positive, the LHS is also positive and hence $|y| = y$:

$$y = e^{-0.6t+C}$$

Split the RHS:

$$y = e^{-0.6t}e^{C}$$

Substitute $k = e^{C}$:

$$y = ke^{-0.6t}$$

Example 1.22

Write the solution of the differential equation $\frac{dy}{dx} = 2^{-y}$ **(CBSE 2015)**

Separate the variables:

$$2^{y}dy = dx$$

Integrate both sides:

$$\int 2^{y}dy = \int 1 \cdot dx$$

Carry out the integration to get:

$$\frac{2^{y}}{\ln 2} = x + C_1$$

Explicit Solution for y

Solving explicitly for y is not easy (or required) in all questions. However, in this one, we can do (if we want).

Cross multiply:

$$2^{y} = x\ln 2 + C_1 \ln 2$$

Substitute $C = C_1 \ln 2$:

$$2^{y} = x\ln 2 + C$$

Take the natural log of both sides:

$$\ln 2^{y} = \ln(x\ln 2 + C)$$
$$y = \frac{\ln(x\ln 2 + C)}{\ln 2}$$

Example 1.23

$$\frac{dy}{dx} = e^{x-y}$$

$$\frac{dy}{dx} = \frac{e^{x}}{e^{y}}$$
$$e^{y}dy = e^{x}\,dx$$

$$\int e^y dy = \int e^x \, dx$$
$$e^y = e^x + C$$

1.24: Verifying Solutions

The solution of a differential equation can be verified by differentiating it.

Example 1.25

A. Find the solution of the differential equation $\frac{dy}{dx} = x^3 e^{-2y}$ (CBSE 2015)
B. Verify the solution.

Part A

Separate the variables:
$$e^{2y} dy = x^3 dx$$
Integrate both sides:
$$\int e^{2y} dy = \int x^3 \, dx$$
Carry out the integration:
$$\frac{e^{2y}}{2} = \frac{x^4}{4} + C_1$$
Multiply by 2 both sides to eliminate Fractions:
$$e^{2y} = \frac{x^4}{2} + 2C_1$$
Substitute $C = 2C_1$:
$$2e^{2y} = x^4 + C$$

Part B

To verify the solution, differentiate both sides of
$$2e^{2y} = x^4 + C$$
With respect to x.
$$\frac{d}{dx}(2e^{2y}) = \frac{d}{dx}(x^4 + C)$$
$$(2)(e^{2y})(2)\left(\frac{dy}{dx}\right) = 4x^3$$
$$e^{2y}\frac{dy}{dx} = 4x^3$$
$$\frac{dy}{dx} = x^3 e^{-2y}$$
And the above is the equation that we solved in Part A.
Hence, the solution has been verified.

Example 1.26

A. Find the general solution of the differential equation $\ln\left(\frac{dy}{dx}\right) = ax + by$ (CBSE 2022).
B. Verify the solution.

Part A

Exponentiate both sides:
$$\frac{dy}{dx} = e^{ax+by} = e^{ax}e^{by}$$
Separate the variables:
$$e^{-by} dy = e^{ax} dx$$
Integrate both sides:
$$\int e^{-by} dy = \int e^{ax} \, dx$$
Carry out the integration:
$$\frac{e^{-by}}{-b} = \frac{e^{ax}}{a} + C_1$$
Eliminate Fractions:
$$ae^{-by} = -be^{ax} - abC_1$$
Substitute $C = -abC_1$:
$$ae^{-by} + be^{ax} = C$$

Part B

Example 1.27

Find $\frac{dy}{dx}$ in terms of x given that $y = \frac{2}{2-e^{-0.8x}}$ and verify that $\frac{dy}{dx} = 0.8y(1-y)$

$$\frac{dy}{dx} = \left(\frac{2}{2-e^{-0.8x}}\right)' = \frac{-1.6e^{-0.8x}}{(2-e^{-0.8x})^2}$$

Substitute $y = \frac{2}{2-e^{-0.8x}}$ in $0.8y(1-y)$:

$$= 0.8\left(\frac{2}{2-e^{-0.8x}}\right)\left(1-\frac{2}{2-e^{-0.8x}}\right) = \left(\frac{1.6}{2-e^{-0.8x}}\right)\left(\frac{-e^{-0.8x}}{2-e^{-0.8x}}\right) = \frac{-1.6e^{-0.8x}}{(2-e^{-0.8x})^2}$$

$$y(0) = 1$$

B. Initial Value Problems

An initial value problem is a condition that gives you information about the function that is a solution to the differential equation. It lets you find a value of the constant in the differential equation.

1.28: Particular Solutions

A solution to a differential equation that has a specific value of the constant is called a particular solution.

Example 1.29

Find the particular solution to $\frac{dy}{dx} = x(y-2)^2, y(1) = 0$

Separate the variables, and integrate:

$$\int \frac{dy}{(y-2)^2} = \int x\,dx$$

Carry out the integration:

$$-\frac{1}{y-2} = \frac{x^2}{2} + C$$

Take the reciprocal, and move the minus sign to the RHS:

$$y - 2 = -\frac{2}{x^2+2C}$$

Solve for y:

$$y = \frac{-2+2x^2+4C}{x^2+2C} = \frac{2x^2-2+4C}{x^2+2C} = \frac{2(x^2-1+2C)}{x^2+2C} = \frac{2(x^2-1+K)}{x^2+K}$$

Substitute $(x,y) = (1,0)$

$$0 = \frac{2(1-1+K)}{1+K} \Rightarrow 0 = 2K \Rightarrow K = 0$$

Substitute $K = 0$:

$$y = \frac{2(x^2-1)}{x^2}$$

Example 1.30

Find the particular solution to:

$$\frac{dy}{dx} = 2xy^2, \qquad y(5) = 1$$

General Solution

$$\frac{dy}{y^2} = 2x\,dx$$

Integrate:

$$\int \frac{dy}{y^2} = \int 2x\,dx$$

$$\underbrace{-\frac{1}{y} = x^2 + C}_{\text{Equation I}}$$

Particular Solution

Solve it for C, and substitute $(x, y) = (5,1)$:

$$C = -\frac{1}{y} - x^2 = -\frac{1}{1} - 5^2 = -26$$

Solve Equation I for y, and substitute $C = -26$:

$$y = -\frac{1}{x^2 + C} = -\frac{1}{x^2 - 26} = \frac{1}{26 - x^2}$$

Example 1.31

Find the particular solution of the differential equation $\frac{dy}{dx} = 1 + x + y + xy$, given that $y = 0$ when $x = 1$. **(CBSE 2014)**

General Solution

The expression on the RHS is not separated, but we can separate it after factoring:

$$\frac{dy}{dx} = (1 + x)(1 + y)$$

Separate the variables:

$$\frac{dy}{(1 + y)} = (1 + x)\,dx$$

Integrate both sides:

$$\int \frac{dy}{1 + y} = \int (1 + x)\,dx$$

$$\ln|1 + y| = x + \frac{x^2}{2} + C$$

Particular Solution

Substitute $x = 1\ and\ y = 0$:

$$\ln|1 + 0| = 1 + \frac{1}{2} + C$$

$$C = -\frac{3}{2}$$

Substitute $C = -\frac{3}{2}$ in the general solution to get the particular solution:

$$\ln|1 + y| = x + \frac{x^2}{2} - \frac{3}{2}$$

Example 1.32

Find the particular solution of the differential equation $\ln\left(\frac{dy}{dx}\right) = 3x + 4y$, given that $y = 0\ when\ x = 0$ **(CBSE 2014)**

General Solution

Exponentiate both sides:

$$\frac{dy}{dx} = e^{3x+4y} = e^{3x} \cdot e^{4y}$$

Separate the variables:

$$\frac{1}{e^{4y}} dy = e^{3x}\, dx$$

Integrate both sides:

$$\int \frac{1}{e^{4y}} dy = \int e^{3x}\, dx \Rightarrow \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C$$

Particular Solution

Substitute $x = 0\ and\ y = 0$:

$$\frac{e^0}{-4} = \frac{e^0}{3} + C \Rightarrow C = -\frac{7}{12}$$

Substitute $C = -\frac{7}{12}$ in the general solution:

$$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} - \frac{7}{12} \Rightarrow 4e^{3x} + 3e^{-4y} - 7 = 0$$

Example 1.33

Find the particular solution of the differential equation $(1-y^2)(1+\ln|x|)dx + 2xy\, dy = 0$, given that $y = 0$ when $x = 1$. **(CBSE 2016)**

Separate the variables:

$$\frac{(1+\ln|x|)}{x} dx + \frac{2y}{1-y^2} dy = 0$$

Integrate both sides:

$$\int \left(\frac{1}{x} + \frac{\ln|x|}{x}\right) dx + \int \frac{2y}{1-y^2} dy = 0$$

$$\ln|x| + \frac{(\ln|x|)^2}{2} - \ln|1-y^2| = \ln C$$

Substitute $y = 0\ and\ x = 1$:

$$\ln 1 + \frac{(\ln 1)^2}{2} - \ln|1-0| = \ln C$$

$$\ln C = 0$$

Substitute $\ln C = 0$

$$\ln|x| + \frac{(\ln|x|)^2}{2} - \ln|1-y^2| = 0$$

C. Tangent Lines

Example 1.34

Find the tangent line to $\frac{dy}{dx} = x(y-2)^2$ at $(1,0)$ and use it to approximate $y(0.7)$

Substitute $(x, y) = (1,0)$ in the equation:

$$m = \frac{dy}{dx} = 1(0-2)^2 = 4$$

Substitute $(x, y) = (1,0), m = 4$ in the slope-point form of the equation of a line:

$$y - 0 = 4(x - 1)$$

Substitute $x = 0.7$ in the above equation:

$$y = 4x - 4 = 4(0.7) - 4 = 2.8 - 4 = -1.2$$

D. More Challenging Questions

Example 1.35

Solve the differential equation $(x+1)\frac{dy}{dx} = 2e^{-y} - 1; y(0) = 0$ (CBSE 2019)

Find the General Solution

$$(x+1)\frac{dy}{dx} = 2e^{-y} - 1 = \frac{2}{e^y} - 1 = \frac{2 - e^y}{e^y}$$

Separate the variables:

$$\frac{e^y}{2 - e^y}dy = \frac{1}{x+1}dx$$

Integrate both sides:

$$-\int \frac{e^y}{e^y - 2}dy = \int \frac{1}{x+1}dx$$

Let $u = e^y - 2 \Rightarrow du = e^y\,dy$:

$$-\int \frac{1}{u}du = \ln|x+1| + C_1$$

$$-\ln|u| = \ln|x+1| + C_1$$

$$\ln|x+1| + \ln|e^y - 2| = -C_1$$

Use $\log a + \log b = \log ab$:

$$\ln|(x+1)(e^y - 2)| = -C_1$$

Exponentiate both sides:

$$(x+1)(e^y - 2) = e^{-C_1}$$

Substitute $C = e^{-C_1}$

$$(x+1)(e^y - 2) = C$$

Find the particular solution

Substitute $(x, y) = (0,0)$:

$$C = (0+1)(1-2) = -1$$

Substitute $C = -1$:

$$(x+1)(e^y - 2) = -1$$

$$e^y - 2 = -\frac{1}{x+1}$$

$$y = \ln\left(2 - \frac{1}{x+1}\right)$$

Example 1.36

Find the particular solution of the differential equation $e^x \tan y\,dx + (2 - e^x)\sec^2 y\,dy = 0$, given that $y = \frac{\pi}{4}$ $when\ x = 0$ (CBSE 2018)

Find the General Solution

$$e^x \tan y\,dx = (e^x - 2)\sec^2 y\,dy$$

Separate the variables:

$$\frac{\sec^2 y}{\tan y}dy = \frac{e^x}{e^x - 2}dx$$

Integrate both sides:

$$\int \frac{\sec^2 y}{\tan y} dy = \int \frac{e^x}{e^x - 2} dx$$

Note that on both sides the numerator is the derivative of the denominator, and hence the integral is the natural log of the denominator:

$$\ln|\tan y| = \ln|e^x - 2| + C_1$$

Collate the log terms on one side:

$$\ln|\tan y| - \ln|e^x - 2| = C_1$$

Use the log property $\log a - \log b = \log \frac{a}{b}$:

$$\ln \left| \frac{\tan y}{e^x - 2} \right| = C_1$$

Exponentiate both sides:

$$\frac{\tan y}{e^x - 2} = e^{C_1}$$

Eliminate fractions and substitute $e^{C_1} = C$ to find the general solution:

$$\tan y = C(e^x - 2)$$

Find the value of the constant

Substitute $y = \frac{\pi}{4}$ *and* $x = 0$ in the general solution to find the value of C:

$$\tan \frac{\pi}{4} = e^C(e^0 - 2)$$
$$e^C = -1$$

Find the particular solution

Substitute $C = -1$ in the general solution to find the particular solution:

$$\tan y = 2 - e^x$$

E. Exponential Models

Example 1.37

Exponential Growth and Decay

Example 1.38

Newton's law of cooling gives the temperature for an object which is warmer than its surroundings.

$$\frac{dT_t}{dt} = -k(T_t - T_s)$$
$$T_t = Temperature\ at\ time\ t$$
$$T_s = Temperature\ of\ Surroundings = Constant$$

$$\frac{dT_t}{dt} = -k(T_t - T_s)$$

Separate the variables:

$$\frac{dT_t}{T_t - T_s} = -k\ dt$$

Integrate:

$$\int \frac{dT_t}{T_t - T_s} = \int -k\ dt$$

Carry out the integration:

$$\ln|T_t - T_s| = -kt + C_1$$

Exponentiate both sides:

$$T_t - T_s = e^{-kt+C} = e^{-kt}e^{C_1} = e^{-kt}C$$
$$T_t - T_s = e^{-kt}C$$

Use initial value conditions to find the value of C. Suppose the temperature at time $t = 0$ is T_0.
Substitute $t = 0 \Rightarrow Temperature = T_0$

$$T_0 - T_s = e^{-k\times 0}C$$
$$T_0 - T_s = e^0 C$$
$$T_0 - T_s = C$$

Finally, my equation is:

$$T_t - T_s = (T_0 - T_s)e^{-kt}$$

1.3 Slope Fields

A. Basics

We have seen how to solve simple differential equations that using direct integration, or separation of variables. However, certain differential equations may not have a closed-form solution[1] that is easy to find. In such cases, numerical integration is a way to understand the features of the differential equation.

1.39: $\frac{dy}{dx}$ as slope

Given a function $y = f(x)$, the quantity $\frac{dy}{dx}$ represents the slope of the function f at the point x.

1.40: Slope Field

A slope field gives the slope at any point in the coordinate plane due to a differential equation.

- A slope field is a way of visualizing the behavior of a differential equation qualitatively, rather than quantitatively.
- Every point in the coordinate plane is represented by a pair of coordinates (x, y), and the slope field associates with that pair (x, y), the slope of the differential equation at that point.
- Slope can be represented using a vector. Hence, the input is a point on the coordinate plane, and the output is a vector.

1.41: Single Variable Function

$$y = f(x)$$

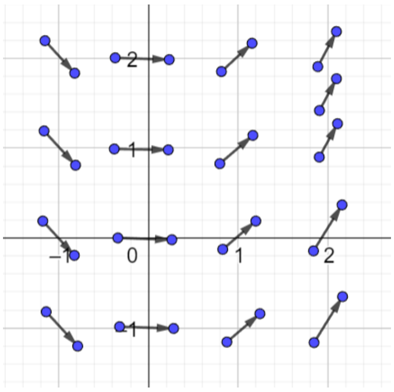
1.42: Vector Function

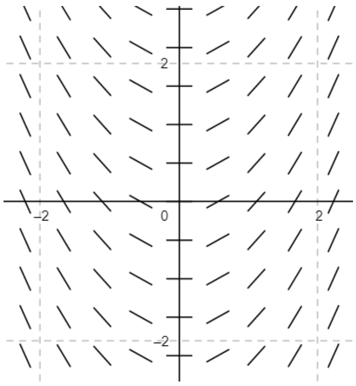
$$\vec{v} = f(x, y)$$

Example 1.43

Sketch a slope field for the differential equation $\frac{dy}{dx} = x$ over $-1 \le x \le 2, -1 \le y \le 2$[2].

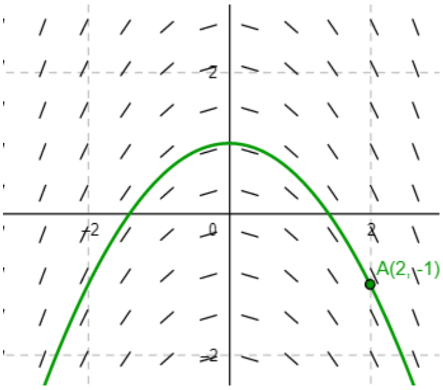
[1] A closed form solution is an explicit solution that connects the variables.
[2] https://www.geogebra.org/m/W7dAdgqc lets you plot slope fields.





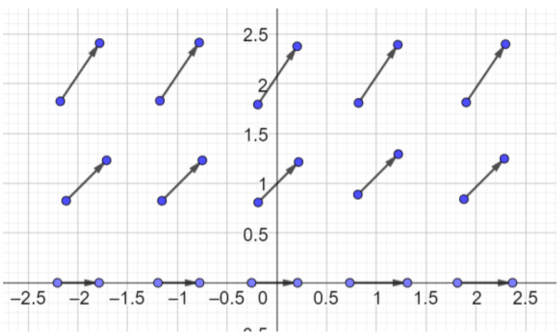
Example 1.44

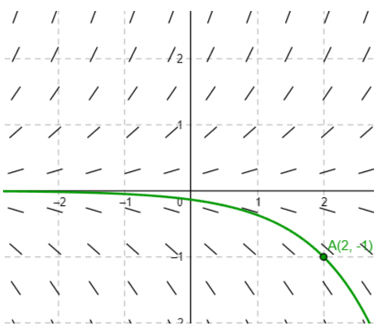
Sketch a slope field for the differential equation $\frac{dy}{dx} = -x$ over $-2 \le x \le 2, -2 \le y \le 2$.



Example 1.45

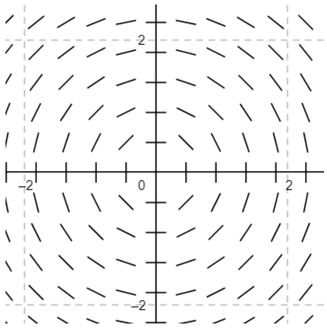
Sketch a slope field for the differential equation $\frac{dy}{dx} = y$ over $-2 \le x \le 2, -2 \le y \le 2$.





Example 1.46

Sketch a slope field for the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ over $-3 \le x \le 3, -3 \le y \le 3$.



Rather than attempting this numerically, we integrate the differential equation using separation of variables:

$$\int y\,dy = -\int x\,dx$$

Integrate the above to get:

$$y^2 = -x^2 + C$$

Substitute $C = r^2$, and collate all variables on the left to get the equation of a circle:

$$y^2 + x^2 = r^2$$

Hence, the slope field must have solution curve that are circles, and now drawing the slopes becomes much easier.

1.4 Euler's Method

A. Basics

1.47: Euler's Method

$$L(x) = y_0 + f(x_0, y_0)dx$$

Example 1.48

$$Differential\ Equation: \frac{df}{dx} = f - 2$$
$$Initial\ Value\ or\ Starting\ point: \ f(0) = 3$$
$$x_0 = 0$$
$$Increment: dx = 0.1$$

$$f(0.1) = y_1 = y_0 + f(x_0, y_0)dx = 3 + (3-2)(0.1) = 3.1$$
$$f(0.2) = y_2 = y_1 + f(x_1, y_1)dx = 3.1 + (3.1-2)(0.1) = 3.21$$

$$f(0.2) = y$$

Example 1.49

Solve the differential equation

$$\frac{df}{dx} = f - 2, \qquad f(0) = 3$$

Separate variables and integrate both sides:

$$\int \frac{df}{f-2} = \int dx$$
$$\ln|f-2| = x + C$$

Determine the value of C using the initial value condition $f(0) = 3$:

$$\ln|3-2| = 0 + C \Rightarrow C = 0$$

The particular solution to the differential equation is:

$$\ln|f-2| = x$$

Example 1.50
Compare the value found using Euler's Method for $x = 0.2$ and the actual value using the particular solution to determine the difference between the two.

$$\ln|f - 2| = 0.2$$
$$f - 2 = e^{0.2}$$
$$f = e^{0.2} + 2 = 3.22140275$$

Difference

$$= 3.22140275 - 3.21$$

1.5 Logistic Growth

Exponential Model
Assume a population P. The death rate in the population
=dP

The birth rate in the population
=bP

The net change in the population is
=bP-dP=kP

Since the above is the rate of change of population, we can write
dP/dt = kP
The above model results in constant growth or constant decay (exponential model).

Logistic Growth Model
Assume an existing population P, and a carrying capacity or limiting population M, where M is the long term value of the population.

k = r(M-P), r>0
Growth rate increases if the population is far below the carrying capacity, and decreases closer to (but still below) the carrying capacity.

If, for some reason, the population increases above the carrying capacity, then P>M, and M-P<0.
Hence, the rate is negative.

Second Derivative and Interpretations
For M/2 < P<M:
First Derivative is positive and second derivative is negative.
The population is increasing, but the rate of increase is decreasing.

For P<M/2:
The first and second derivative are both positive. The population is increasing, and the rate of increase is also

increasing.

For P=M/2:
The first is positive, but the second derivative is zero. The population is increasing, and the rate of increase is constant.

2. Calculus Topics

2.1 Newton-Raphson Method

A. Basics

2.1: Newton Raphson Method

Guess x_1 as an approximation to the solution of the equation $f(x) = 0$.
Use the recursive formula below to find further approximations:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, if\ f'(x_n) \neq 0$$

2.2: Convergence

Newton's method does not guarantee convergence.

2.3: Wrong Root

If you start too far away from the root, Newton's method may not give you the root you wanted.

When Newton's method converges to a root, it may not be the root you have in mind. Figure 4.52 shows two ways this can happen.

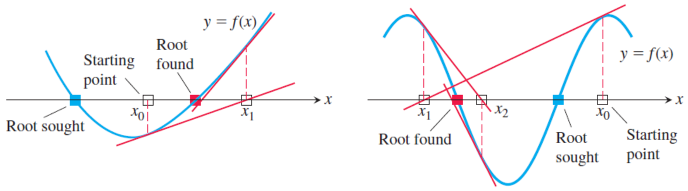


FIGURE 4.52 If you start too far away, Newton's method may miss the root you want.

2.2 Radius of Curvature

2.4: Radius of Curvature

$$R = \left| \frac{\left(1 + {y'}^2\right)^{\frac{3}{2}}}{y''} \right|,$$

Example 2.5

$$(0,1)$$
$$y = \cos x$$
$$f(x) = 1 - \frac{x^2}{2}$$

$$y' = -\sin x, \qquad y'' = -\cos x$$

`$+R|_{x=0} = \left|\frac{(1+y')^{\frac{3}{2}}}{y''}\right| = \left|\frac{(1-\sin^2 x)^{\frac{3}{2}}}{-\cos x}\right| = \left|\frac{(\cos^2 x)^{\frac{3}{2}}}{-\cos x}\right| = \left|\frac{\cos^3 x}{-\cos x}\right| = |-\cos^2 x| = 1$

$$f' = -x, \qquad f'' = -1$$

$$R|_{x=0} = \left|\frac{(1+y')^{\frac{3}{2}}}{y''}\right| = \left|\frac{(1-x^2)^{\frac{3}{2}}}{-1}\right| = \left|\frac{(1-0)^{\frac{3}{2}}}{-1}\right| = \left|\frac{1}{-1}\right| = 1$$

2.3 Parametric Differentiation

A. Differentiation

2.6: Parametric Differentiation

If a curve is defined parametrically with $x = f(t), y = g(t)$, then the rate of change of y with respect to x, which is the derivative of y with respect to x is given by:

$$\frac{dy}{dx} = \frac{dy/_{dt}}{dx/_{dt}}$$

We can motivate this $informally$ by thinking as below:

$$\frac{dy}{dx} = \frac{dy \times \frac{1}{dt}}{dx \times \frac{1}{dt}} = \frac{dy/_{dt}}{dx/_{dt}}$$

Example 2.7: Concept

Given $x = \cos t\,, y = \sin t$, find $\frac{dy}{dx}$

A. using the formula for parametric differentiation
B. by eliminating t and differentiating

Verify that the answers from both the parts are same.

Part A

Substitute $\frac{dy}{dt} = \cos t\,, \frac{dx}{dt} = -\sin t$ in

$$\frac{dy}{dx} = \frac{dy/_{dt}}{dx/_{dt}} = \frac{\cos t}{-\sin t} = -\cot t$$

Part B

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$
$$x^2 + y^2 = 1$$

Differentiating implicitly with respect to x:

$$2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x}{y} = \frac{\cos t}{-\sin t} = -\cot t$$

B. Tangent Lines

2.8: Tangent Line

To find the equation of a tangent line

- $m = \left.\frac{dy}{dx}\right|_{x=x_1}$ to find the slope of the tangent
- Substitute the coordinates of a point $P(x_1, y_1)$ and m into the point slope form of the equation of a line $(y - y_1 = m(x - x_1))$

Example 2.9

An object moves along the curve given by $x = 2t^2, y = 3t^3$ where t is in seconds. Determine the direction of movement of the object when $t = 2$.

The direction of movement will be given by the tangent line.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{9t^2}{4t} = \frac{9t}{4}$$

$$Slope = m = \left.\frac{dy}{dx}\right|_{t=2} = \frac{9 \cdot 2}{4} = \frac{9}{2}$$

$$(x_1, y_1) = (2t_1^2, 3t_1^3) = (2 \times 2^2, 3 \times 2^3) = (8{,}24)$$

Substitute into $y - y_1 = m(x - x_1)$

$$y - 24 = \frac{9}{2}(x - 8)$$

C. Parametric Curves

Example 2.10

Phillips Exeter Math 4

712. An object slides along the ellipse $x^2 - 2xy + 2y^2 = 10$. When it passes $(4, 3)$, the object's horizontal component of velocity is $\frac{dx}{dt} = 2$. What is the object's vertical component of velocity at that instant? In which direction is the object traveling around the ellipse, clockwise or counterclockwise?

Differentiate both sides with respect to t, and note that $x' = \frac{dx}{dt}, y' = \frac{dy}{dt}$:

$$(2x)x' - 2xy' - 2yx' + 4yy' = 0$$

Collate all y' terms on side:

$$y'(4y - 2x) = x'(2y - 2x)$$

Solve for y'

$$\frac{dy}{dt} = y' = \frac{x'(y - x)}{2y - x} = \frac{2(3 - 4)}{2(3) - 4} = \frac{2(-1)}{-2} = -1$$

$$\frac{dx}{dt} > 0, \frac{dy}{dy} < 0, First\ Quadrant \Rightarrow Clockwise$$

Example 2.11
Phillips Exeter Math 4

713. (Continuation) Calculate the size of the angle formed by the velocity vector and the radial vector $[x, y]$ at the point $(4, 3)$. At this instant, is the object getting closer to the origin or receding from it?

Substitute $\vec{\boldsymbol{v}} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (2, -1), \vec{\boldsymbol{r}} = (4,3)$ in

$$\cos\theta = \frac{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{r}}}{|\vec{\boldsymbol{v}}||\vec{\boldsymbol{r}}|} = \frac{(4)(2) + (3)(-1)}{\sqrt{2^2 + 1^2}\sqrt{4^2 + 3^2}} = \frac{8-3}{\sqrt{5}\sqrt{25}} = \frac{1}{\sqrt{5}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) = 63.43°$$

Example 2.12
Phillips Exeter Math 4

714. (Continuation) Let r be the distance from the origin to the point (x, y). When the object passes the point $(4, 3)$, what is the value of $\frac{dr}{dt}$? What does this tell you about the object's motion?

$$r = \sqrt{x^2 + y^2} \Rightarrow r^2 = x^2 + y^2$$

Differentiate both sides with respect to t:

$$2rr' = 2xx' + 2yy'$$

$$r' = \frac{xx' + yy'}{r} = \frac{4(2) + 3(-1)}{5} = \frac{8-3}{5} = \frac{5}{5} = 1$$

2.4 Parametric Integration

A. Basics

Integration with respect to a parameter can be done by using substitution.

2.13: Parametric Integration
Given the parametric equations $y = g(t), x = f(t)$:

$$\int f(x)\,dx = \int g(t)\,f'(t)\,dt$$

$$\int f(x)\,dx$$

Substitute $f(x) = y$

$$= \int y\,dx$$

Substitute $y = g(t), dx = f'(t)\,dt$:

$$\int g(t)\, f'(t)\, dt$$

Example 2.14: Area of a Circle

Basic Example

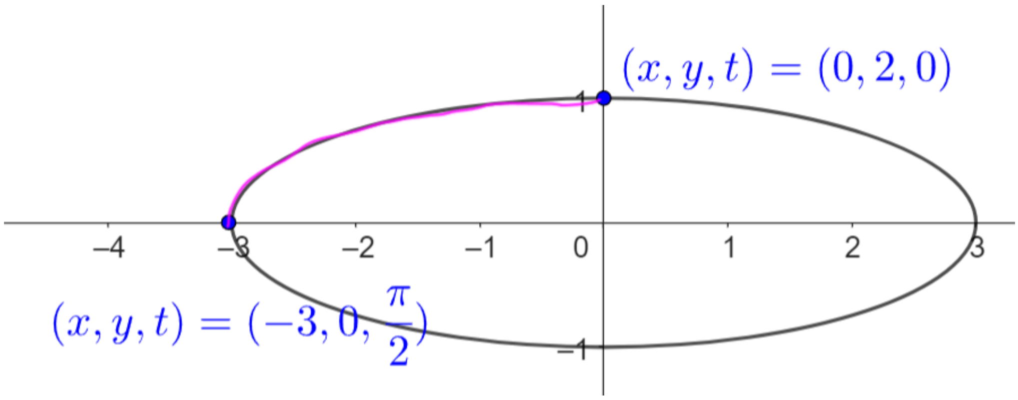
Example 2.15: Area of a Circle

Video

Example 2.16: Area of an Ellipse

A. Describe the motion of the parametrization $x = -3\sin t\,, y = 2\cos t$.
B. Use the parametrization to determine the area enclosed by the ellipse.

Part A



The motion begins at $(x, y) = (0,2)$, and goes counter clockwise around the ellipse.
The motion covers the entire ellipse from $t = 0$ to $t = 2\pi$.

Part B

To find the area of the ellipse, we use symmetry and multiply the area in the second quadrant by 4.

$$Limits\ of\ Integration: x = -3\,, x = 0$$

If $y = f(x)$ then we get:

$$4\int_{-3}^{0} f(x)\, dx = 4\int_{-3}^{0} y\, dx$$

Use a change of variables with the substitutions:

$$y = 2\cos t\,, dx = -3\cos t\, dt$$
$$x = -3:\ -3 = -3\sin t \Rightarrow \sin t = 1 \Rightarrow t = \frac{\pi}{2}$$
$$x = 0: 0 = -3\sin t \Rightarrow \sin t = 0 \Rightarrow t = 0$$

Which gives us, after substitution:

$$= 4\int_{\frac{\pi}{2}}^{0} (2\cos t)(-3\cos t)\, dt = -24\int_{\frac{\pi}{2}}^{0} \cos^2 t\ dt$$

Use the trigonometric identity $\cos^2 t = \frac{1+\cos 2t}{2}$ and integrate:

$$-24\int_{\frac{\pi}{2}}^{0}\frac{1+\cos 2t}{2}\,dt = -12\int_{\frac{\pi}{2}}^{0}1+\cos 2t\,dt = -12\left[t+\frac{\sin 2t}{2}\right]_{\frac{\pi}{2}}^{0}$$

Substitute the limits of integration:

$$-12\left[\left(0+\frac{\sin(2\cdot 0)}{2}\right)-\left(\frac{\pi}{2}+\frac{\sin\left(2\cdot\frac{\pi}{2}\right)}{2}\right)\right]$$

Substitute $\frac{\sin(2\cdot 0)}{2}=\frac{\sin 0}{2}=\frac{0}{2}=0$ and $\frac{\sin\left(2\cdot\frac{\pi}{2}\right)}{2}=\frac{\sin\pi}{2}=\frac{0}{2}=0$

$$=-12\left[(0+0)-\left(\frac{\pi}{2}+0\right)\right]=-12\left[-\frac{\pi}{2}\right]=6\pi$$

B. Velocity and Speed

2.17: Velocity

Given the parametric function $\big(x(t),y(t)\big)$, we have

$$Velocity=\vec{\boldsymbol{v}}=\frac{d\vec{\boldsymbol{r}}}{dt}=\left(\frac{dx}{dt},\frac{dy}{dt}\right)$$

The position vector is:

$$\vec{\boldsymbol{r}}=(x,y)=\big(x(t),y(t)\big)$$

The velocity is the rate of change of position with respect to time.
Differentiate the above to get:

$$Velocity=\vec{\boldsymbol{v}}=\frac{d\vec{\boldsymbol{r}}}{dt}=\big(x'(t),y'(t)\big)=\left(\frac{dx}{dt},\frac{dy}{dt}\right)$$

2.18: Speed

Given the parametric function $\big(x(t),y(t)\big)$, we have

$$Speed=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

$$\vec{\boldsymbol{v}}=\left(\frac{dx}{dt},\frac{dy}{dt}\right)$$

The magnitude of the above vector is:

$$|\vec{\boldsymbol{v}}|=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

Example 2.19

$$Speed=\sqrt{t^4-2t^2+1+4t^2}$$

Find $\big(x(t),y(t)\big)$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{t^4-2t^2+1+4t^2}$$

Square both sides:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = t^4 - 2t^2 + 1 + 4t^2$$

First Factorization:

$$\left(\frac{dx}{dt}\right)^2 = t^4 - 2t^2 + 1 = (t^2 - 1)^2 \Rightarrow \frac{dx}{dt} = t^2 - 1$$

$$\left(\frac{dy}{dt}\right)^2 = 4t^2 \Rightarrow \frac{dy}{dt} = 2t$$

Second Factorization:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = t^4 + 2t^2 + 1$$

$$\left(\frac{dx}{dt}\right)^2 = t^4 + 2t^2 + 1 = (t^2 + 1)^2 \Rightarrow \frac{dx}{dt} = t^2 + 1$$

$$\left(\frac{dy}{dt}\right)^2 = 0 \Rightarrow \frac{dy}{dt} = 0$$

C. Arc Length

2.20: Arc Length

Given the parametric function $(x(t), y(t))$, we have

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$Speed = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Integrating the speed over time gives us distance. Distance is the length of the curve travelled:

2.5 Polar Coordinates-I: Differentiation

A. Polar Coordinates

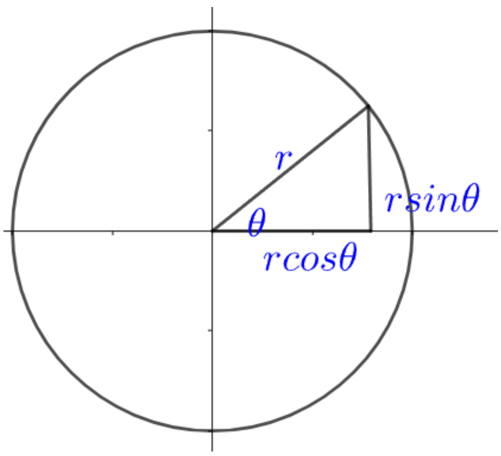
2.21: Polar Coordinates

Polar coordinates define a point in the form:

$$(r, \theta)$$

Where

$$r = directed\ distance$$
$$\theta = angle\ made\ by\ the\ point\ with\ polar\ axis$$
$$Polar\ axis = positive\ direction\ of\ x\ axis$$



2.22: Polar to Cartesian Coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$

2.23: Cartesian to Polar Coordinates

$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x}$$

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

- When solving for θ, you need to careful since the tan inverse has range only in quadrants I and IV, whereas the angle can be any quadrant.

B. Relation between polar and parametric

2.24: Parametric Equations

We defined parametric equations to be of the form $x = f(t), y = g(t)$, where t is a time parameter. Hence, the position of an object at time t was given by:

$$(x, y) = \big(f(t), g(t)\big)$$

- Parametric equations with parameter t (by convention) refer to situations where the position of an object is a function of time.
- Apart from time, we can take other quantities as the parameter of interest.

2.25: Angle $\boldsymbol{\theta}$ as a parameter

Instead of time parameter t, we can take angle θ as the parameter in a parametric setup. Given

$$x = f(\theta), y = g(\theta)$$

We can conclude

$$(x, y) = \big(f(\theta), g(\theta)\big)$$

C. Differentiation

2.26: Parametric Differentiation

If a curve is defined parametrically with $x = f(t), y = g(t)$, then the rate of change of y with respect to x, which is the derivative of y with respect to x is given by:

$$\frac{dy}{dx} = \frac{{dy}/{dt}}{{dx}/{dt}}$$

We can motivate this $informally$ by thinking as below:

$$\frac{dy}{dx} = \frac{dy \times \frac{1}{dt}}{dx \times \frac{1}{dt}} = \frac{{dy}/{dt}}{{dx}/{dt}}$$

2.27: Polar Coordinates: Differentiation

Given $x = f(\theta), y = g(\theta)$, the rate of change of y with respect to x, which is the derivative of y with respect to x is given by:

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}$$

Example 2.28

What is the interpretation of

A. $dy/d\theta$
B. $dx/d\theta$

A. Rate of change of y coordinate with respect to change in angle θ
B. Rate of change of x coordinate with respect to change in angle θ

Example 2.29

$$r=1+\sin\theta$$

Find the rate of change of the y coordinate with respect to change in angle θ.

$$y=r\sin\theta$$

To differentiate with respect to θ we first express y as a function of θ by substituting $r=1+\sin\theta$

$$y=(1+\sin\theta)(\sin\theta)=\sin\theta+\sin^2\theta$$

Differentiate the above with respect to θ:

$$\frac{dy}{d\theta}=\cos\theta+2\sin\theta\cos\theta$$

Apply the double angle identity:

$$\frac{dy}{d\theta}=\cos\theta+\sin 2\theta$$

Example 2.30

$$r=1+\sin\theta$$

Find the rate of change of the x coordinate with respect to change in angle θ.

$$x=r\cos\theta=(1+\sin\theta)(\cos\theta)=\cos\theta+\cos\theta\sin\theta=\cos\theta+\frac{1}{2}\sin 2\theta$$

$$\frac{dx}{d\theta}=-\sin\theta+\frac{1}{2}(\cos 2\theta)(2)=-\sin\theta+\cos 2\theta$$

Example 2.31

$$r=1+\sin\theta$$

Find the rate of change of the y coordinate with respect to change in x.
Find the rate of change of the y coordinate with respect to change in x when $\theta=\frac{\pi}{6}$.

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{\cos\theta+\sin 2\theta}{-\sin\theta+\cos 2\theta}$$

Substitute $\theta=\frac{\pi}{6}=30°\Rightarrow 2\theta=60°$ in the expression for $\frac{dy}{dx}$:

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}}=\frac{\cos 30^\circ+\sin 60^\circ}{-\sin 30^\circ+\cos 60^\circ}=\frac{\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}}{-\frac{1}{2}+\frac{1}{2}}=\frac{\sqrt{3}}{0}$$

Since the denominator is zero, the slope is

$$undefined\ or\ (informally)\infty$$

Hence, the function has a vertical tangent line at:

$$\theta=\frac{\pi}{6}$$

D. Slopes and Tangents

Example 2.32

Determine the equation of the tangent line to the graph $r=1+\sin\theta$ when $\theta=\frac{\pi}{2}$.

$$\frac{dy}{dx}=\frac{\cos\theta+\sin 2\theta}{-\sin\theta+\cos 2\theta}$$

$$Slope=m=\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}}=\frac{\cos\frac{\pi}{2}+\sin\pi}{-\sin\frac{\pi}{2}+\cos\pi}=\frac{0+0}{-1-1}=\frac{0}{-2}=0$$

Since the slope is 0,

$$Line\ is\ horizontal$$

$$r=1+\sin\theta=1+\sin\left(\frac{\pi}{2}\right)=1+1=2$$
$$x=r\cos\theta=2\cos\frac{\pi}{2}=2(0)=0$$
$$y=r\sin\theta=2\sin\frac{\pi}{2}=2(1)=2$$

Since the line passes through $y=2$, and it is a horizontal line, the equation is:

$$y=2$$

Example 2.33

Determine the equation of the tangent line to the graph $r=\sin 2\theta$ when $\theta=\frac{\pi}{4}$.

$$\frac{dy}{d\theta}=\frac{d}{d\theta}(r\sin\theta)=\frac{d}{d\theta}(\sin 2\theta\sin\theta)=2\cos 2\theta\sin\theta+\sin 2\theta\cos\theta$$
$$\frac{dx}{d\theta}=\frac{d}{d\theta}(r\cos\theta)=\frac{d}{d\theta}(\sin 2\theta\cos\theta)=2\cos 2\theta\cos\theta-\sin 2\theta\sin\theta$$
$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{2\cos 2\theta\sin\theta+\sin 2\theta\cos\theta}{2\cos 2\theta\cos\theta-\sin 2\theta\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}}=\frac{2\cos\frac{\pi}{2}\sin\frac{\pi}{4}+\sin\frac{\pi}{2}\cos\frac{\pi}{4}}{2\cos 2\theta\cos\frac{\pi}{4}-\sin\frac{\pi}{2}\sin\frac{\pi}{4}}=\frac{2(0)\sin\frac{\pi}{4}+(1)\left(\frac{1}{\sqrt{2}}\right)}{2(0)\cos\frac{\pi}{4}-(1)\left(\frac{1}{\sqrt{2}}\right)}=\frac{(1)\left(\frac{1}{\sqrt{2}}\right)}{-(1)\left(\frac{1}{\sqrt{2}}\right)}=-1$$

$$r = \sin 2\theta = \sin\frac{\pi}{2} = 1$$
$$x = r\cos\theta = (1)\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$
$$y = r\sin\theta = (1)\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$y - y_0 = m(x - x_0)$$
$$y - \frac{1}{\sqrt{2}} = -1\left(x - \frac{1}{\sqrt{2}}\right)$$
$$y = -x + \sqrt{2}$$

E. Some Further Topics

2.34: General Formula

If $x = r\cos\theta$, and $y = r\sin\theta$, such that $r = f(\theta)$ then:

$$\frac{dy}{dx} = \frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta}$$

Using the product rule since r is a function of θ:

$$\frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta$$
$$\frac{dx}{d\theta} = -r\sin\theta + \frac{dr}{d\theta}\cos\theta$$

Combining the above two gives us:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta}$$

Example 2.35

Why do we need to specify $r = f(\theta)$ in the formula:

$$\frac{dy}{dx} = \frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta}$$

Because if r is not a function of θ then we cannot differentiate it.

2.6 Polar Coordinates-II: Integration

A. Definition of Polar Integration

2.36: Sector of a Circle

A sector is a part of a circle enclosed by two radii and an arc.
Area of a sector is

$$\frac{\theta_{Degree}}{360}\pi r^2 = \frac{1}{2}r^2\theta$$

Where

$$\theta = Central\ angle$$

➢ Central angle is the angle subtended by the arc at the center of the circle.

2.37: Area under the Curve

The area of the region between the origin and the curve $r = f(\theta)$, for $\theta_L \le \theta \le \theta_U$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

Where

$$\alpha = Lower\ limit\ of\ integration$$
$$\beta = Upper\ limit\ of\ integration$$

Region OTS is the area between the origin and the curve $r = f(\theta)$.
It is bounded by the rays $\theta = \alpha$ and $\theta = \beta$.
In the case of integration in rectangular coordinates, we divide the region into rectangles. But rectangles will not work here. So, we will divide into sectors.

Substitute Radius of each sector $r_k = f(\theta_k)$, central angle $= \Delta\theta_k$ in $\frac{1}{2}r^2\theta$

$$Area\ of\ k^{th}\ Sector = \frac{1}{2}r_k^2\Delta\theta_k = \frac{1}{2}[f(\theta_k)]^2\Delta\theta_k$$

The total area of the sectors is:

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n}\frac{1}{2}[f(\theta_k)]^2\Delta\theta_k$$

And this area approximates the region that we want.

As the angle of each part approaches zero and the number of parts approaches infinity, we get:

$$A = \lim\left(\sum_{k=1}^{n}\frac{1}{2}[f(\theta_k)]^2\Delta\theta_k\right) = \int_{\alpha}^{\beta}\frac{1}{2}r^2 d\theta$$

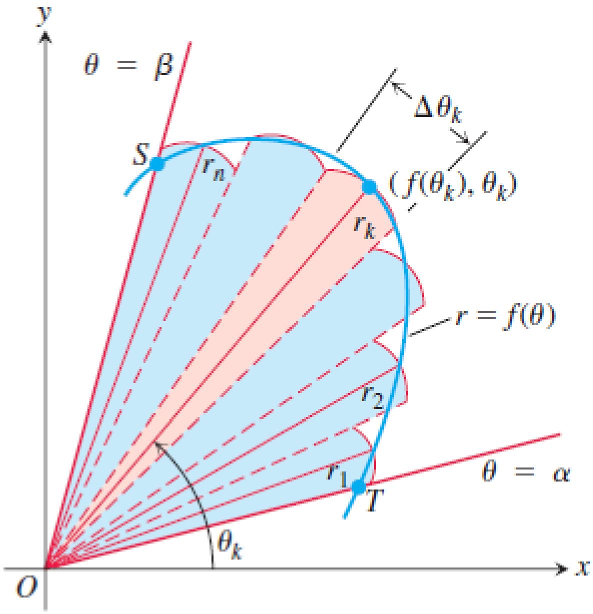


FIGURE 11.31 To derive a formula for the area of region OTS, we approximate the region with fan-shaped circular sectors.

B. Warm Up

Example 2.38: Area of a Circle

Calculate the area of a quarter circle in the first quadrant with center at the origin and radius R using polar integration.

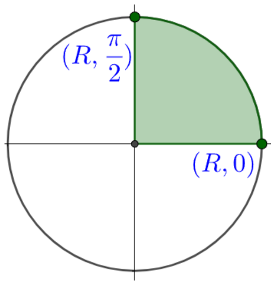
The equation of a circle with radius R and center at the origin is:

$$r = R, \qquad R = Radius$$

The limits of integration are the points where the circle intersects the x and the y axis.

$$Lower\ limit = \alpha = 0$$

$$Upper\ limit = \beta = \frac{\pi}{2}$$



Substitute the above into $\int_\alpha^\beta \frac{1}{2} r^2 d\theta$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} R^2 d\theta = \frac{R^2}{2}[\theta]_0^{\frac{\pi}{2}} = \frac{R^2}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi R^2}{4}$$

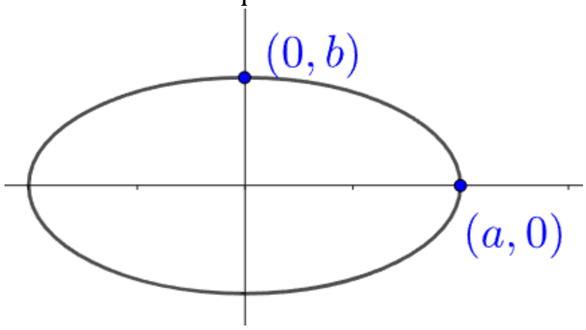
Example 2.39
Determine the area of an ellipse symmetrical about the origin.

In parametric form, the equation of an ellipse symmetrical about the origin is:

$$x = a\cos\theta\,, y = b\sin\theta$$

In rectangular coordinates, the area of the ellipse is four times the area in the first quadrant:

$$= 4\int_0^a f(x)\,dx$$



Convert to polar coordinates:

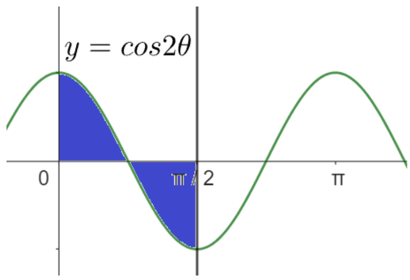
$$\alpha = 0, \beta = \frac{\pi}{2}$$

$$x = a\cos\theta \Rightarrow dx = -a\sin\theta\,d\theta$$

$$f(x) = y = b\sin\theta$$

$$= 4\int_0^{\frac{\pi}{2}} (b\sin\theta)(-a\sin\theta\,d\theta) = 4ab\int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta = 4ab\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= 2ab\left[\int_0^{\frac{\pi}{2}} d\theta - \int_0^{\frac{\pi}{2}} \cos 2\theta\,d\theta\right]$$



$$= 2ab\left[\int_{\frac{\pi}{2}}^{0} d\theta\right] = 2ab[\theta]_{\frac{\pi}{2}}^{0} = 2ab\left(\frac{\pi}{2} - 0\right) = \pi ab$$

C. Area of a Polar Region

Example 2.40:
Find the area of the region bounded by $r = \theta$ for $0 \le \theta \le \pi$

$$A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_0^\pi \theta^2 d\theta = \frac{1}{2}\left[\frac{\theta^3}{3}\right]_0^\pi = \frac{\pi^3}{6}$$

Example 2.41: Limacon Inner Loop

Find the area of the inner loop in the graph of $r = 1 + 2\cos\theta$.

$r = 3: 3 = 1 + 2\cos\theta \Rightarrow \cos\theta = 1 \Rightarrow \theta \in \{\mathbf{0}, \mathbf{2\pi}\}$

$r = 1: 1 = 1 + 2\cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$

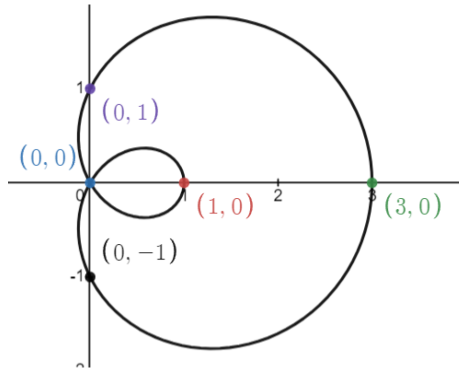
$r = 0: 0 = 1 + 2\cos\theta \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta \in \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$

$r = -1: -1 = 1 + 2\cos\theta \Rightarrow \cos\theta = -1 \Rightarrow \theta \in \{\boldsymbol{\pi}\}$

θ	**0**	$\boldsymbol{\frac{\pi}{2}}$	$\boldsymbol{\frac{2\pi}{3}}$	$\boldsymbol{\pi}$	$\boldsymbol{\frac{4\pi}{3}}$	$\boldsymbol{\frac{3\pi}{2}}$	$\boldsymbol{2\pi}$
r	**3**	**1**	**0**	**-1**	**0**	**1**	**3**

The function to be integrated is:

$$r^2 = (1 + 2\cos\theta)^2 = 1 + 4\cos\theta + 4\cos^2\theta$$



Substitute $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, and simplify:

$$= 1 + 4\cos\theta + 4 \cdot \frac{1 + \cos 2\theta}{2} = 3 + 4\cos\theta + 2\cos 2\theta$$

Substitute $Lower\ Limit = \alpha = \frac{2\pi}{3}$, $Upper\ Limit = \beta = \pi$ in

$$A = 2\int_\alpha^\beta \frac{1}{2} r^2 d\theta = \int_\alpha^\beta r^2 d\theta = \int_{\frac{2\pi}{3}}^{\pi} (3 + 4\cos\theta + 2\cos 2\theta) d\theta$$

Integrate, and substitute the limits of integration:

$$= [3\theta + 4\sin\theta + \sin 2\theta]_{\frac{2\pi}{3}}^{\pi} = (3\pi + 4\sin\pi + \sin 2\pi) - \left(3 \cdot \frac{2\pi}{3} + 4\sin\frac{2\pi}{3} + \sin\frac{4\pi}{3}\right)$$

Simplify:

$$= (3\pi + 4 \cdot 0 + 0) - \left(2\pi + 4 \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)$$

$$= 3\pi - \left(2\pi + \frac{3\sqrt{3}}{2}\right)$$

$$= \pi - \frac{3\sqrt{3}}{2}$$

Example 2.42: Limacon Outer Loop

Find the area of the outer loop in the graph of $r = 1 + 2\cos\theta$.

D. Area of a Region bounded by two Polar Curves

Example 2.43

Find the area of intersection of the polar curves $r = \cos\theta$ and $r = \sin\theta$.

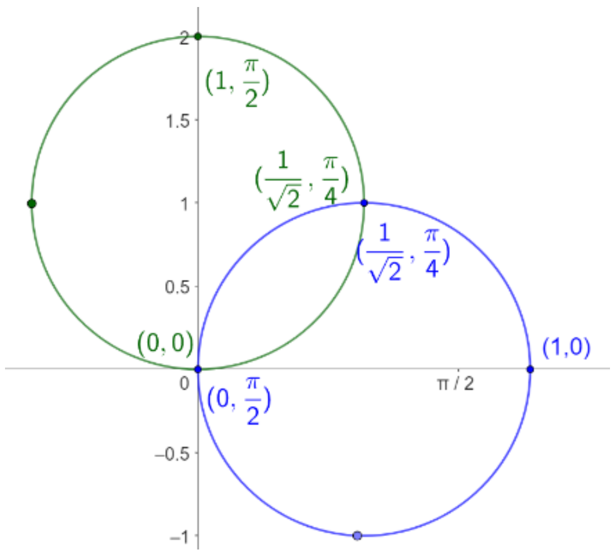
θ	$r = \sin\theta$	$r = \cos\theta$
0	0	1
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$

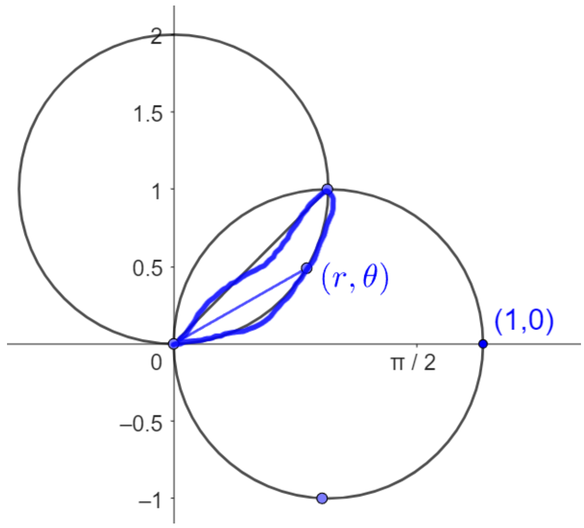
$\frac{\pi}{2}$	1	0

Note that each of the given curves is a circle that touches the origin. Determine the points on intersection of the two curves:

$$\sin\theta = \cos\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

Prepare a table of values.





Substitute $\alpha = 0, \beta = \frac{\pi}{4}, r = \sin\theta \Rightarrow r^2 = \sin^2\theta$ in $\int_\alpha^\beta \frac{1}{2} r^2 d\theta$

$$= 2\int_0^{\frac{\pi}{4}} \frac{1}{2}\sin^2\theta \, d\theta$$

Use a trigonometric substitution. Substitute $\sin^2\theta = \frac{1-\cos 2\theta}{2}$:

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1-\cos 2\theta}{2}\right) d\theta = \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{4}}$$

Substitute the limits of integration:

$$\left(\frac{\frac{\pi}{4}}{2} - \frac{\sin\frac{\pi}{2}}{4}\right) - (0 - 0) = \frac{\pi}{8} - \frac{1}{4}$$

E. Arc Length

2.44: Arc Length

The length of the curve $r = f(\theta)$ from α to β:

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

Conditions to be met:

- $r(\theta)$ must have a continuous derivative for $\alpha \le \theta \le \beta$

- The point $P(r, \theta)$ traces the curve exactly once as θ ranges from α to β

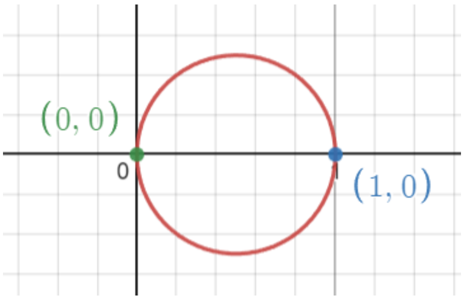
Example 2.45

A. $r = n\cos\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$

B. Plot the curve, determine the shape, and verify your answer from Part A by using a formula from high school geometry.

Substitute $r^2 = n^2\cos^2\theta, \frac{dr}{d\theta} = -n\sin\theta$:

$$\int_0^{\frac{\pi}{2}} \sqrt{n^2\cos^2\theta + n^2\sin^2\theta}\, d\theta = \int_0^{\frac{\pi}{2}} |n|\sqrt{1}\, d\theta = n[\theta]_0^{\frac{\pi}{2}} = \frac{\pi n}{2}$$



$$Radius = \frac{n}{2}$$

$$Arc\ Length\ of\ Semicircle = \frac{2\pi R}{2} = \pi R = \frac{\pi n}{2}$$

2.7 Vector Derivatives

A. Paths

2.46: Paths

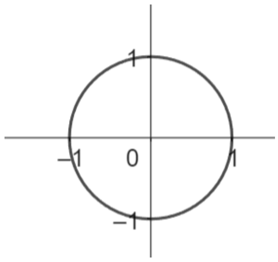
When a particle moves through space, the points through which the particle passes make up the path.

Example 2.47

If $x = \sin t, y = \cos t$ for $t \geq 0$, identify

A. the path travelled by the particle.

The path travelled is the unit circle.



2.48: Parametric Representation

The path of a particle over a time interval I can be represented parametrically as a function in three dimensions:

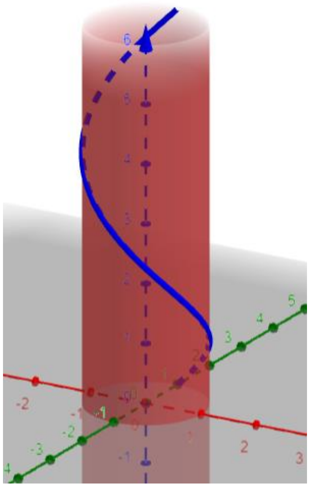
$$x = f(t), y = g(t), z = h(t), \qquad t \in I$$

Note that I represents a time interval, not the set of integers $\mathbb{I}$, which is usually represented as $\mathbb{Z}$.

Example 2.49

A particle travels along the path $x = \sin t, y = \cos t, z = t$ for $t \geq 0$ where t is in seconds and x, y and z are in meters. Find the:

A. shape of the path?
B. length of the path travelled in 1 second?
C. speed of the particle?



Part A

The particle travels along a right circular cylinder of radius 1 (graphed alongside).

Part B

$x = \sin t, y = \cos t$ would give a circle in the coordinate plane.

The time taken to complete 1 cycle around the circle

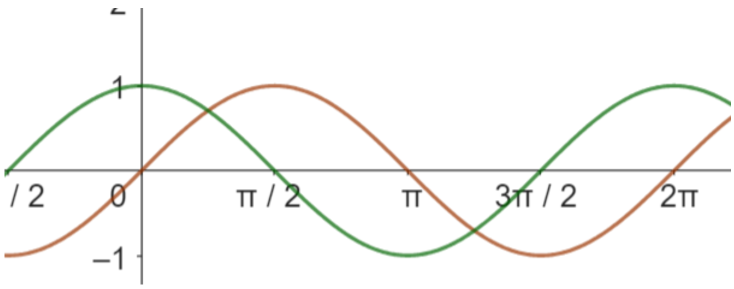
$$= 2\pi$$

Length of path around the circle

$$= 2\pi$$

Distance in 2D in 1 second

$$= \frac{Distance\ in\ 1\ Cycle}{Time\ for\ 1\ Cycle} = \frac{2\pi}{2\pi} = 1$$



The z component is separate. For every second that passes, the z coordinate increases by 1.

$$z = t$$

Length of path travelled in 1 second in the z direction

$$= 1$$

Total distance travelled in 1 second

$$= 1 + 1 = 2\frac{m}{s}$$

B. Paths as Vectors

2.50: Scalar Function

A function $y = f(t)$ assigns a scalar y to each valid scalar t in its domain.

The functions that you have seen earlier (for example, in introductory Calculus) are scalar functions.

$$y = \sqrt{t}$$

$$y = \sin t$$

2.51: Vector Function

A vector function $\vec{r}(t)$ is a function that assigns for each scalar t in its domain a vector $\vec{r}(t)$.

2.52: Paths in Vector Form

A path in vector form can be expressed by expressing the vector in component form:

$$\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$

Where

$$\hat{\imath}, \hat{\jmath}, \hat{k} \text{ are unit vectors in the } x, y \text{ and } z \text{ directions respectively}$$
$$f(t), g(t), h(t) \text{ are scalar functions}$$

2.53: Converting from Parametric to Vector Form

$$x = f(t), y = g(t), z = h(t) \Leftrightarrow \vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$

Example 2.54

A particle travels along the path $x = \sin t$, $y = \cos t$, $z = t$ for $t \geq 0$ where t is in seconds and x, y and z are in meters. Write this in vector form.

$$\vec{r}(t) = (\sin t)\hat{\imath} + (\cos t)\hat{\jmath} + t\hat{k}$$

C. Derivative Basics

2.55: Derivative

If $\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$ then its derivative is:

$$\vec{r}'(t) = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{r(t + \Delta t) - r(t)}{\Delta t} = \frac{d\vec{r}}{dt} = \frac{df}{dt}\hat{\imath} + \frac{dg}{dt}\hat{\jmath} + \frac{dh}{dt}\hat{k}$$

- To differentiate a vector, you can simply differentiate each of its components.
- This is what makes component form so useful and important.

2.56: Geometric Interpretation of the Derivative

- The derivative will give the vector tangent to the curve/path at the point where the derivative is calculated.
- This is the *direction vector*.

D. Finding Tangent Lines

2.57: Point-Vector Form

Given a point P with position vector $\vec{a}$ whose terminal point lies on a line, and a vector $\vec{b}$ parallel to the line, the vector form of the equation of the line is:

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

Consider an arbitrary point Q on the line with position vector $\vec{\boldsymbol{r}}$.
Since $\vec{\boldsymbol{b}}$ is parallel to the line:

$$\overrightarrow{\boldsymbol{PQ}} \parallel \vec{\boldsymbol{b}}$$

The two vectors above have the same direction. Hence, one must be a scalar multiple of the other:

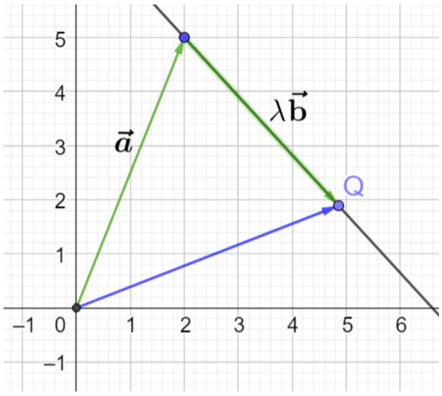
$$\overrightarrow{\boldsymbol{PQ}} = \lambda\vec{\boldsymbol{b}}$$

Write $\overrightarrow{\boldsymbol{PQ}}$ as a displacement vector:

$$\overrightarrow{\boldsymbol{PQ}} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{a}} = \lambda\vec{\boldsymbol{b}}$$

Solve for $\vec{\boldsymbol{r}}$:

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{a}} + \lambda\vec{\boldsymbol{b}}$$



2.58: Finding Vector Equation

To find vector equation of a line, you need:

$$\vec{\boldsymbol{r}} = Point\ on\ the\ Line = Position\ Vector$$

$$\vec{\boldsymbol{d}} = \frac{\boldsymbol{d}\vec{\boldsymbol{r}}}{\boldsymbol{dt}} = Direction\ Vector = Tangent\ Vector = Derivative$$

2.8 Vector Integration

A. Indefinite Integration

2.59: Integration of Scalar Functions

The indefinite integral of r with respect to t is the set of all antiderivatives of r, denoted by

$$\int r(t)\,dt = R(t) + C$$

2.60: Definition: Vector Integral

The indefinite integral of $\vec{\boldsymbol{r}}$ with respect to t is the set of all antitiderivatives of $\vec{\boldsymbol{r}}$, denoted by

$$\int \vec{\boldsymbol{r}}(t)\,dt = \vec{\boldsymbol{R}}(t) + \vec{\boldsymbol{C}}$$

Note that

$$\vec{\boldsymbol{r}}\ is\ the\ integrand\ vector$$
$$\vec{\boldsymbol{R}}\ is\ the\ integral\ vector$$
$$\vec{\boldsymbol{C}}\ is\ a\ constant\ vector$$

Each of the above is a *vector* quantity except for

$$t = time, which\ is\ scalar$$

2.61: Integrating Vectors Component-Wise

If $\vec{\boldsymbol{r}}(t) = f(t)\hat{\boldsymbol{\imath}} + g(t)\hat{\boldsymbol{\jmath}} + h(t)\widehat{\boldsymbol{k}}$ then its integral is:

$$\int \vec{\boldsymbol{r}}(\boldsymbol{t})\,dt = F(t)\hat{\boldsymbol{\imath}} + G(t)\hat{\boldsymbol{\jmath}} + H(t)\widehat{\boldsymbol{k}} + \vec{\boldsymbol{C}}$$

Where

$$F(t)\ is\ an\ antiderivative\ of\ f(t)$$

$$G(t)\ is\ an\ antiderivative\ of\ f(t)$$
$$H(t)\ is\ an\ antiderivative\ of\ f(t)$$

$$\int \vec{\boldsymbol{r}}(\boldsymbol{t})\,dt$$

Use the definition of $\vec{\boldsymbol{r}}(t)$:

$$= \int [f(t)\hat{\boldsymbol{\imath}} + g(t)\hat{\boldsymbol{\jmath}} + h(t)\hat{\boldsymbol{k}}]\,dt$$

Use the sum property of integration to split the vector:

$$= \int f(t)\hat{\boldsymbol{\imath}}\,dt + \int g(t)\hat{\boldsymbol{\jmath}}\,dt + \int h(t)\hat{\boldsymbol{k}}\,dt$$

Carry out the integration:

$$= F(t)\hat{\boldsymbol{\imath}} + G(t)\hat{\boldsymbol{\jmath}} + H(t)\hat{\boldsymbol{k}} + \vec{\boldsymbol{C}}$$

A vector can be integrated by integrating each of its components.

$$\int [f(t), g(t), h(t)]\,dt = \big(F(t), G(t), H(t)\big) + \vec{\boldsymbol{C}}$$

B. Definite Integration

2.62: Definite Integration Component Wise

C. Ideal Projectile Motion

2.9 Hyperbolic Function Derivatives

A. Definitions

2.63: Hyperbolic function derivations

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\operatorname{sech} x = \frac{1}{\cosh x}$$
$$\operatorname{csch} x = \frac{1}{\sinh x}$$
$$\coth x = \frac{1}{\tanh x}$$

B. Derivatives

There are no new rules that we need to calculate the derivatives of the hyperbolic functions. They can be calculated using the rules that we already know for finding derivatives.

2.64: Derivative of $\boldsymbol{sinh\ x}$ and $\boldsymbol{cosh\ x}$

$$\frac{d}{dt}(\sinh x) = \cosh x$$
$$\frac{d}{dt}(\cosh x) = \sinh x$$

Using $\frac{d}{dx}e^x = e^x, \frac{d}{dx}e^{-x} = -e^{-x}$

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x - (-)e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$
$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x$$

Note the difference with the trigonometric derivatives. The minus sign is missing.

2.65: Derivative of tanh x

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

Using the definition:

$$\frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$$

Using the quotient rule:

$$\frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$
$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

Use a change of variable. Let $e^x = a, e^{-x} = b$. Then, the numerator becomes:

$$(a+b)^2 - (a-b)^2 = (a^2 + 2ab + b^2) - (a^2 - 2ab + b^2) = 4ab = 4(e^x)(e^{-x}) = 4$$

Hence, the derivative that we want is:

$$\frac{4}{(e^x + e^{-x})^2} = \left(\frac{2}{e^x + e^{-x}}\right)^2 = \operatorname{sech}^2 x$$

2.66: Derivative of csch x

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

2.67: Derivative of sech x

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

2.68: Derivative of coth x

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

C. Inverse Hyperbolic Functions

2.69: Derivative of $sinh^{-1} x$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$

$$y = \sinh^{-1} x$$

Apply the function $\sinh x$ to both sides:

$$\sinh y = x$$

Differentiate both sides implicitly:

$$\cosh y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$$

But now our answer is in terms of y. Whereas we want our answer in terms of x. For this, we do some algebraic manipulation based on identities.

Note that:

$$\cosh^2 y = 1 + \sinh^2 y \Rightarrow \cosh y = \sqrt{1+\sinh^2 y}$$

(where we only need to take the positive square root since $\cosh x \geq 1$)

Substitute $\cosh y = \sqrt{1+\sinh^2 y}$:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+\sinh^2 y}}$$

Substitute $y = \sinh^{-1} x$:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+\sinh^2(\sinh^{-1} x)}} = \frac{1}{\sqrt{1+x^2}}$$

2.70: Derivative of $cosh^{-1}\ x$

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$$

2.71: Derivative of $tanh^{-1} x$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$$

2.72: Derivative of $csch^{-1} x$

$$\frac{d}{dx}(\operatorname{csch}^{-1} x) = \frac{1}{|x|\sqrt{x^2+1}}$$

2.73: Derivative of $sech^{-1} x$

$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$$

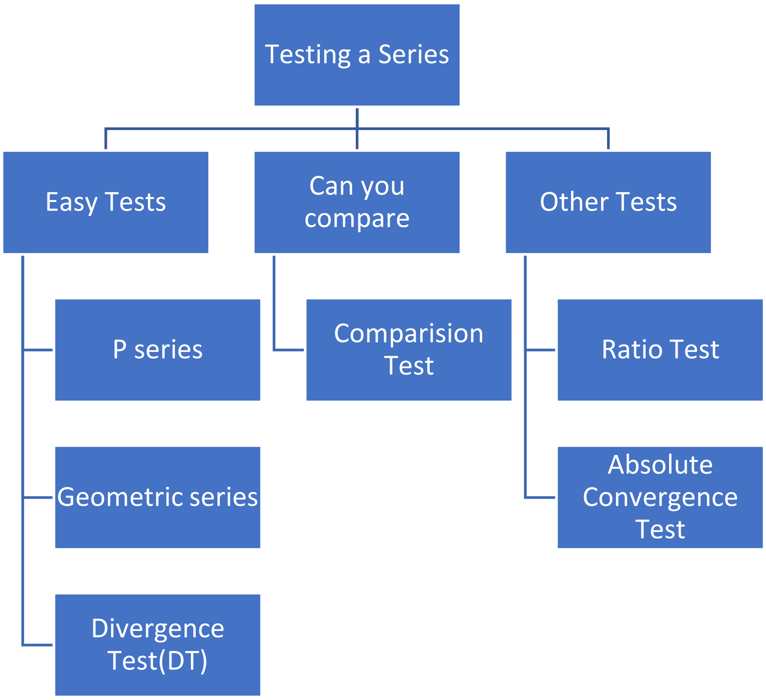
2.74: Derivative of $coth^{-1} x$

$$\frac{d}{dx}(\coth^{-1} x) = -\frac{1}{1-x^2}$$

3. Sequences and Series

3.1 Finding Maclaurin Series

A. Formula Summary and Resources



3.1: Important Maclaurin Series
In this section, we will learn the formulas below (and their derivations).
After completing this section, you should have these formulas memorized.

Exponential and Logarithmic

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} = \sum_{n=0}^{n=\infty} \frac{x^n}{n!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

Trigonometric

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots, \qquad |x| < \frac{\pi}{2}$$

Inverse Trigonometric

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \cdots$$

Video 3.2
Maclaurin and Taylor Series Lecture (UC Irvine)

B. Concept

3.3: Idea
You can construct a polynomial whose derivatives match the derivatives of the function of interest at a particular point.

Example 3.4
Approximate $f(x) = e^x$ at $x = 0$ using a fourth-degree polynomial.

Calculate the first four derivative of e^x:

$$y = y' = y'' = y''' = y'''' = e^x$$

Evaluate these four derivatives at $x = 0$:

$$y(0) = y'(0) = y''(0) = y'''(0) = y''''(0) = e^0 = 1$$

A fourth-degree polynomial must be of the form:

$$P(x) = ax^4 + bx^3 + cx^2 + dx + E$$
$$P'(x) = 4ax^3 + 3bx^2 + 2cx + d$$
$$P''(x) = 12ax^2 + 6bx + 2c$$
$$P'''(x) = 24ax + 6b$$
$$P''''(x) = 24ax$$

The polynomial should ideally equal the function of interest:

$$P(0) = f(0)$$
$$E = 1$$

The derivative of the polynomial should equal the derivative of the function of interest:

$$P'(0) = f'(0)$$
$$P'(0) = 4a(0) + 3b(0) + 2c(0) + d = 1 \Rightarrow d = 1$$

The second derivative of the polynomial should equal the second derivative of the function of interest:

$$P''(0) = f''(0)$$
$$P''(0) = 2c = 1 \Rightarrow c = \frac{1}{2}$$

The third derivative of the polynomial should equal the third derivative of the function of interest:

$$P'''(0) = f'''(0)$$
$$P'''(0) = 6b = 1 \Rightarrow b = \frac{1}{6}$$

The fourth derivative of the polynomial should equal the fourth derivative of the function of interest:

$$P''''(0) = 24a = 1 \Rightarrow a = \frac{1}{24}$$

$$P(x) = \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1$$

3.5: Warning

We are learning some mathematical techniques (which work), but not the conditions on when they work. Those conditions relate to the theory of infinite series (convergence, radius of convergence), which we will see later.

- Takeaway: You can apply these techniques to other situations, but not blindly.

C. Basics

3.6: Notation for Derivatives

Given a function f, we introduce some notation to represent its derivatives:

$$Function\ Itself = Zeroth\ Derivative = f^{(0)}$$
$$First\ Derivative = f' = f^{(1)}$$
$$Second\ Derivative = f'' = f^{(2)}$$
$$\vdots$$
$$N^{th}\ Derivative = f^{(n)}$$

3.7: Maclaurin Series

The Maclaurin series for a function f is an infinite series given by:

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{k!} x^k = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(2)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

The function evaluated at zero matches the Maclaurin series evaluated at zero:

$$S(0) = f(0)$$

The first derivative of the function evaluated at zero matches the first derivative of the Maclaurin series evaluated at zero:

$$S'(x) = f^{(1)}(0) + \frac{2f^{(2)}(0)}{2!}x + \frac{3f^{(2)}(0)}{3!}x^2 + \cdots + \frac{nf^{(n)}(0)}{n!}x^n + \cdots$$
$$= f^{(1)}(0) + f^{(2)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{nf^{(n)}(0)}{n!}x^n + \cdots$$
$$S'(0) = f^{(1)}(0)$$

The second derivative of the function evaluated at zero matches the second derivative of the Maclaurin series evaluated at zero:

$$S''^{(x)} = f^{(2)}(0) + \frac{2f^{(3)}(0)}{2!}x + \cdots$$
$$= f^{(2)}(0) + f^{(3)}(0)x + \cdots$$
$$S''(0) = f^{(2)}(0)$$

3.8: Maclaurin Series

The Maclaurin Series is a special case of the Taylor Series

D. Approximation

3.9: Approximation

An important use of a Maclaurin series is to approximate a function which is difficult to calculate directly.

3.10: Maclaurin Polynomial

The n^{th} Maclaurin polynomial is obtained by calculating the Maclaurin series upto

$$x^n$$

Example 3.11

Find the third Maclaurin polynomial for $f(x) = \sqrt{1+x}$, and use it to approximate $\sqrt{1.8}$.

Note that:

$$f(0.8) = \sqrt{1+0.8} = \sqrt{1.8}$$

Calculate the first three derivatives and evaluate them at $x = 0$:

$$f(0) = \sqrt{1+0} = \sqrt{1} = 1$$

$$f'(x) = \frac{1}{2\sqrt{1+x}} \Rightarrow f'(0) = \frac{1}{2\sqrt{1+0}} = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4(1+x)^{\frac{3}{2}}} \Rightarrow f''(0) = -\frac{1}{4(1+0)^{\frac{3}{2}}} = -\frac{1}{4}$$

$$f'''(x) = \frac{3}{8(1+x)^{\frac{5}{2}}} \Rightarrow f'''(0) = \frac{3}{8(1+0)^{\frac{5}{2}}} = \frac{3}{8}$$

Substitute the above into the formula for the Maclaurin Series with $x = 0.8$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 1 + \frac{1}{2}(0.8) + \frac{-\frac{1}{4}}{2!}(0.64) + \frac{\frac{3}{8}}{6}(0.512)$$

$$\approx 1.352$$

E. e^x

Example 3.12

Show that the Maclaurin series expansion of $f(x) = e^x$ is

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{k!}x^k = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

The derivatives of all orders of $f(x)$ are

$$f^{(0)}(x) = f^{(1)}(x) = f^{(2)}(x) = f^{(n)}(x) = e^x$$

Evaluate the derivatives at the value $x = 0$:

$$f^{(0)}(0) = f^{(1)}(0) = f^{(2)}(0) = f^{(n)}(0) = e^0 = 1$$

Substitute the above into the formula, all the f terms become 1, and we get:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

3.13: Substitution

You can use the expansion of e^x to find the expansion of $e^{f(x)}$.

Example 3.14

Find the Maclaurin series expansions below using the formula for e^x.

A. e^{2x}
B. e^{-x}

Part A

$$e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \cdots + \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

Part B

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \cdots + \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

We can write this a little better by getting the minus sign out of the brackets:

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3} + \cdots + \frac{(-x)^n}{n!} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

Example 3.15

Show that the Maclaurin series expansion of $\ln(1 + x)$ is:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$\sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{k!} x^k = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$f(0) = \ln(1 + 0) = \ln 1 = 0$$

We will need the derivatives of all orders of $f(x)$.

$$f^{(1)}(x) = \frac{1}{1 + x} \Rightarrow f^{(1)}(0) = \frac{1}{1 + 0} = 1$$

Using the formula $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$

$$f^{(2)}(x) = -\frac{1}{(1 + x)^2} \Rightarrow f^{(2)}(0) = -\frac{1}{(1 + 0)^2} = -1$$

$$f^{(3)}(x) = \frac{2(1 + x)}{(1 + x)^4} = \frac{2}{(1 + x)^3} \Rightarrow f^{(3)}(0) = \frac{2}{(1 + 0)^3} = 2 = 2!$$

$$f^{(4)}(x) = -\frac{2(3)(1+x)^2}{(1+x)^6} = \frac{-6}{(1+x)^4} \Rightarrow f^{(3)}(0) = -\frac{6}{(1+0)^4} = -6 = -3!$$

$$\ln(1+x) = 0 + 1 \cdot x - \frac{1}{2!} \cdot x^2 - \frac{2!}{3!} \cdot x^3 - \frac{3!}{4!} \cdot x^4 + \cdots$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

3.16: Maclaurin Series of a Polynomial

The Maclaurin series of a polynomial is the polynomial itself.

Video 3.17

Calculating the Maclaurin series of a polynomial **(MIT OCW 18.01 Single Variable Calculus)**

$$3 = 3$$
$$0 = 2$$
$$3 = 5$$

F. Trigonometric Functions

Example 3.18

Show that the Maclaurin series expansion of $f(x) = \sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{k!} x^k = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots$$

Calculate the derivatives of the function f, and evaluate them at $x = 0$.

$$f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$$
$$f^{(1)}(x) = \cos x \Rightarrow f^{(1)}(0) = \cos 0 = 1$$
$$f^{(2)}(x) = -\sin x \Rightarrow f^{(2)}(0) = -\sin 0 = -0 = 0$$
$$f^{(3)}(x) = -\cos x \Rightarrow f^{(3)}(0) = -\cos 0 = -1$$
$$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(0) = \sin 0 = 0$$
$$f^{(5)}(x) = \cos x \Rightarrow f^{(5)}(0) = \cos 0 = 1$$

Every alternate term is zero, we can simplify and rewrite as follows:

$$f^{(1)}(0)x + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(5)}(0)}{5!} x^3 \ldots + \frac{f^{(n)}(0)}{n!} x^n + \cdots$$

Substitute $f^{(1)}(0) = 1, f^{(3)}(0) = -1, f^{(5)}(0) = 1$:

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

We want to write this in summation notation.

- The terms alternate between positive and negative. This can be achieved using
$$(-1)^n \Rightarrow 1 \text{ for even } n, \text{and } (-1) \text{ for odd } n$$
- We only want the terms that have the odd powers of x. This can be achieved using the expression
$$2n+1$$
Which generates the n^{th} odd number
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

Example 3.19

Determine the exact value of:
$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{(2n+1)!}$$

Substitute $x=\pi$ in $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$:
$$\sin\pi=\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{(2n+1)!}$$
$$\sin\pi=0$$

Example 3.20

Show that the Maclaurin series expansion of $f(x)=\cos x$ is
$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

Our formula for the Maclaurin Series expansion is:
$$f(0)+f^{(1)}(0)x+\frac{f^{(2)}(0)}{2!}x^2+\cdots$$

Calculate the derivatives of the function f, and evaluate them at $x=0$.
$$f(x)=\cos x \Rightarrow f(0)=\cos 0=1$$
$$f^{(1)}(x)=-\sin x \Rightarrow f^{(1)}(0)=-\sin 0=0$$
$$f^{(2)}(x)=-\cos x \Rightarrow f^{(2)}(0)=-\cos 0=-1$$
$$f^{(3)}(x)=\sin x \Rightarrow f^{(3)}(0)=\sin 0=0$$
$$f^{(4)}(x)=\cos x \Rightarrow f^{(4)}(0)=\cos 0=1$$
$$f^{(5)}(x)=-\sin x \Rightarrow f^{(5)}(0)=-\sin 0=0$$

Since we know that every alternate term is zero, we can simplify and rewrite as follows:
$$f(0)+\frac{f^{(2)}(0)}{2!}x^2++\frac{f^{(4)}(0)}{2!}x^2+\cdots$$
$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

3.2 Operations with Maclaurin Series

A. Adding Maclaurin Series

3.21: Adding Maclaurin Series

You can add/subtract two Maclaurin series term by term if they both converge.

Example 3.22

The hyperbolic function $\sinh x$ is defined as:

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$$

Use the Maclaurin series for e^x and e^{-x} to determine the Maclaurin series for $\sinh x$.

$$\underbrace{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots + \frac{x^n}{n!} + \cdots}_{Identity\ I}$$

$$\underbrace{e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \dots + \frac{(-x)^n}{n!} + \cdots}_{Identity\ II}$$

Subtracting Identities II from Identity I:

$$e^x - e^{-x} = 0 + 2x + 0 + \frac{2x^3}{3!} + \cdots + 0 \cdot x^{2n} + \frac{2x^{2n+1}}{(2n+1)!} + \cdots$$

Since the zero terms do not change the value, we do not need to write them:

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \cdots + \frac{2x^{2n+1}}{(2n+1)!} + \cdots$$

Divide both sides by 2:

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

We can check convergence by using the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{a^{2(n+1)+1}}{[2(n+1)+1]!}}{\frac{a^{2n+1}}{(2n+1)!}} = \frac{a^{2(n+1)+1}}{[2(n+1)+1]!} \times \frac{(2n+1)!}{a^{2n+1}} = \frac{a^3}{(2n+3)(2n+1)}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{a^3}{(2n+3)(2n+1)} = 0$$

B. Differentiating Maclaurin Series

3.23: Differentiating Term by Term

Differentiating a Maclaurin series expansion can help us derive new identities.

Example 3.24

The Maclaurin

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Differentiate both sides of the above to determine the Maclaurin series expansion for $\cos x$.

$$\cos x = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \cdots = \sum_{n=0}^{\infty}(2n+1)\frac{(-1)^n}{(2n+1)!}x^{2n}$$

Simplify:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

Example 3.25

A. Find the Maclaurin series for $\cos 2x$
B. Use a trig identity to find the Maclaurin Series for $\sin^2 x$
C. Use a trig identity to find the Maclaurin Series for $\cos^2 x$
D. Use the Maclaurin series results from Parts B and C to show that $\sin^2 x + \cos^2 x = 1$.

Part A

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots$$

Part B

$$\sin^2 x = \frac{1-\cos 2x}{2} = \frac{1 - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots\right)}{2} = \frac{1}{2}\left[\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} - \cdots\right]$$

Part C

$$\cos^2 x = \frac{\cos 2x + 1}{2} = \frac{\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots\right) + 1}{2} = 1 + \frac{1}{2}\left[-\frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots\right]$$

Part D

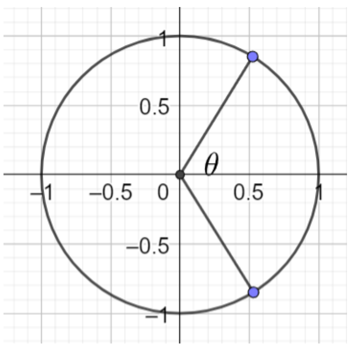
$$\sin^2 x + \cos^2 x = 1 + \frac{1}{2}\left[-\frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots\right] + \frac{1}{2}\left[\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} - \cdots\right] = 1$$

3.26: Odd and Even Functions

- An even function is a function where $f(x) = f(-x)$. The graph of an even function remains unchanged if you reflect it across the y −axis.
- An odd function is a function where $-f(x) = f(-x)$. The graph of an odd function remains unchanged if you reflect it across the origin.

Example 3.27

A. Show that $\cos x$ is an even function.
B. Show that $\sin x$ is an odd function.



$$\cos(-\theta) = \cos\theta \Rightarrow \cos x \ \textit{is an even function}$$
$$\sin(-\theta) = -\sin\theta \Rightarrow \sin x \ \textit{is an odd function}$$

3.28: Polynomials as Odd and Even Functions

- A polynomial with only odd powers is an odd function.
- A polynomial with only even powers is an even function.

Polynomial with Odd Powers

$$P(x) = x^{2n+1} + x^{2n-1} + \cdots + x^1, n \in \mathbb{N}$$
$$P(-x) = (-x)^{2n+1} + (-x)^{2n-1} + \cdots + (-x)^1 = -(x^{2n+1} + x^{2n-1} + \cdots + x^1) = -P(x)$$

Polynomial with Even Powers

$$P(x) = x^{2n} + x^{2n-2} + \cdots + x^0, n \in \mathbb{N}$$
$$P(-x) = (-x)^{2n} + (-x)^{2n-2} + \cdots + (-x)^0 = x^{2n} + x^{2n-2} + \cdots + x^0 = P(x)$$

Example 3.29

Show that

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots, \qquad |x| < \frac{\pi}{2}$$

Using the Maclaurin Series Expansion:

Video 3.30

Show that

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$$

Using polynomial long division on $\frac{\sin x}{\cos x}$, by expanding each out as a series and

C. Geometric Series

Look at some expansions using the geometric series.

Example 3.31

Expand each of the below as a geometric series:

A. $\frac{1}{1+x}$
B. $\frac{1}{1+x^2}$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \cdots, |x| < 1$$
$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots$$

Example 3.32

Find the Maclaurin series of

$$\frac{1}{(1+x)^2}$$

We know that:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots, |x| < 1$$

Differentiate both sides of the above:

$$-\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 +$$
$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - \cdots$$

D. Integration

3.33: Integration
We can integrate a Maclaurin series term by term to get a Maclaurin series for a different expression.

Example 3.34
Show that the Maclaurin series for $\tan^{-1} x$ is:

$$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

Differentiate both sides of $f(x) = \tan^{-1} x$:

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

Expand the RHS as a geometric series with $r = -x^2$:

$$f'(x) = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots$$

Integrate both sides. Since $f(0) = 0 \Rightarrow C = 0$ (where C is the constant of integration):

$$f(x) = \tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

Example 3.35
Find the series expansion of $\ln(\cos x)$

Example 3.36
Determine the Maclaurin series for:

$$f(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Use the quotient rule $\log\left(\frac{a}{b}\right) = \log a - \log b$:

$$f = \frac{1}{2}[\ln(1+x) - \ln(1-x)]$$

Use the chain rule to differentiate. Then, expand as a geometric series:

$$f' = \frac{1}{2}\left[\frac{1}{1+x} + \frac{1}{1-x}\right] = \frac{1}{2}\left[\frac{2}{1-x^2}\right] = \frac{1}{1-x^2} = 1 + x^2 + x^4 + \cdots$$

Integrate:

$$f(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots$$

Where the constant of integration is zero since:

$$C = f(0) = \frac{1}{2}\ln\left(\frac{1+0}{1-0}\right) = \frac{1}{2}\ln(1) = \frac{1}{2}\cdot 0 = 0$$

Example 3.37

$$f(x) = \frac{1}{(1-2x)(1-3x)}$$

Method I: Addition

Split using partial fraction decomposition:

$$\frac{1}{(1-2x)(1-3x)} = \frac{-2}{1-2x} + \frac{3}{1-3x}$$

Using the formula for a geometric series: $\frac{a}{1-r} = a + ar + ar^2 + \cdots$

$$a = -2, r = 2x: A = \frac{-2}{1-2x} = (-2) + (-2)(2x) + (-2)(2x)^2 + (-2)(2x)^3 + \cdots$$

$$a = 3, r = 3x: B = \frac{3}{1-3x} = 3 + 3(3x) + 3(3x)^2 + 3(3x)^3 + \cdots$$

$$\frac{-2}{1-2x} = -2 - 4x - 8x^2 - 16x^3 + \cdots$$
$$\frac{3}{1-3x} = 3 + 9x + 27x^2 + 81x^3 + \cdots$$

$$T_1 = -2 + 3 = 1$$
$$T_2 = -4x + 9x = 5x$$
$$T_3 = -8x^2 + 27x^2 = 19x^2$$
$$T_4 = -16x^3 + 81x^3 = 65x^3$$

$$\frac{1}{(1-2x)(1-3x)} = 1 + 5x + 19x^2 + 65x^3 + \cdots + (3^n - 2^n)x^{n-1} + \cdots$$

Method I: Multiplication

$$\left(\frac{1}{1-2x}\right)\left(\frac{1}{1-3x}\right) = (1 + 2x + 4x^2 + 8x^3 \ldots)(1 + 3x + 9x^2 + 27x^3 + \cdots)$$

$$\begin{aligned} = 1 \quad &+ 3x \quad + 9x^2 \quad + 27x^3 + \cdots \\ &+2x \quad + 6x^2 \quad + 18x^3 + \cdots \\ &\qquad\quad +4x^2 \quad + 12x^3 + \cdots \\ &\qquad\qquad\quad +8x^3 + \cdots \end{aligned}$$

$$= 1 + 5x + 19x^2 + 65x^3 + \cdots$$

E. Approximation

Example 3.38

Approximate $\int_0^1 e^{-x^2}dx$ using a Maclaurin series to three decimal places.

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$f(x) = e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \frac{(-x^2)^5}{5!} + \cdots$$
$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \cdots$$

$$F(x) = \int e^{-x^2}\, dx \ = x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{11}}{11 \cdot 5!}$$

We need accuracy to three decimal places.
Since we will round off at the fourth decimal place
Hence, the error must be

$$less\ than\ 0.0005$$

$$\int_0^1 e^{-x^2} dx = F(1) - F(0) \approx 0.7467 - 0$$
$$F(1) = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} \approx 0.7467$$
$$F(0) = 0$$

Example 3.39

Approximate $\int_0^1 e^{-\frac{x^2}{2}} dx$ using a Maclaurin series to three decimal places.

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$f(x) = e^{-\frac{x^2}{2}} = 1 + \left(-\frac{x^2}{2}\right) + \frac{\left(-\frac{x^2}{2}\right)^2}{2!} + \frac{\left(-\frac{x^2}{2}\right)^3}{3!} + \frac{\left(-\frac{x^2}{2}\right)^4}{4!} + \frac{\left(-\frac{x^2}{2}\right)^5}{5!} + \cdots$$
$$= 1 - \frac{x^2}{2} + \frac{x^4}{2^2} - \frac{x^6}{2^3 \cdot 3!} + \frac{x^8}{2^4 \cdot 4!} - \frac{x^{10}}{2^5 \cdot 5!} + \cdots$$

$$F(x) = \int e^{-x^2}\, dx \ = x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{5 \cdot 2^2 \cdot 2!} - \frac{x^7}{7 \cdot 2^3 \cdot 3!} + \frac{x^9}{9 \cdot 2^4 \cdot 4!} - \frac{x^{11}}{11 \cdot 2^5 \cdot 5!}$$
$$= x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336} + \frac{1}{3456}$$

Substitute $x = 1$ and evaluate the first four terms:

$$F(1) = 1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} \approx 0.8553$$

The maximum error is:

$$Fifth\ Term = \frac{1}{3456} \approx 0.0002$$

Hence, the range is:

$$(0.8553, 0.8553 + 0.0002) = (0.8553, 0.8555)$$

Since the answer to three decimal places is different at the lower and upper end of the range, we add one more term:

$$F(1) = 1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456} \approx 0.8556$$

The maximum error is now:

$$\frac{1}{11 \cdot 2^5 \cdot 5!} = \frac{1}{42240} \approx 2.36 \times 10^{-5}$$

And hence the answer will not change even at the maximum range after rounding.

3.3 Taylor Series/Convergence

A. Taylor Series

A Taylor series is a generalization of a Maclaurin series. A Maclaurin series is always centered at 0, but this need not always be the case.

3.40: Taylor Series

If a function $f(x)$ has derivatives of all orders in an open interval I containing a, then the Talyor series of $f(x)$ is given by:

$$T(x) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots = \sum_{k=0}^{k=\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

- The Taylor series is centered at a.
- This means that the approximation is very good at a. It may or may not become worse as we move away from a.

3.41: Taylor Series versus Function Derivatives

The derivatives of a Talyor series are equal to the derivatives of the function itself at

$$x = a$$

Evaluate the Taylor series at $x = a$, and we get the original function value:

$$T(a) = f(a)$$

Differentiate $T(x)$ once:

$$T'(x) = f^{(1)}(a) + \frac{2f^{(2)}(a)}{2!}(x-a) + \cdots$$

Evaluate $T^{(1)}(x)$ at $x = a$:

$$T'(a) = f^{(1)}(a)$$

Differentiate $T^{(2)}(x)$ twice:

$$T''(x) = f^{(2)}(a) + \cdots$$

Evaluate $T^{(2)}(x)$ at $x = a$:

$$T''(a) = f^{(2)}(a)$$

This process can be continued, and at every derivative we will get the same result.

Example 3.42

Show that the Taylor series at $a = 0$ is the Maclaurin series:

Substitute $a = 0$ in the Taylor series

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{k!} x^k = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots$$

Example 3.43

Find an equation for the fourth-degree polynomial $p(x)$ that has the following properties:

$$p(1) = 1, \quad p^{(1)}(1) = \frac{1}{2}, \quad p^{(2)}(1) = -\frac{1}{4}, \quad p^{(3)}(1) = \frac{3}{8}, \quad p^{(4)}(1) = -\frac{15}{16}$$

Verify by differentiation and evaluation that it does have those derivatives.

Finding the Polynomial

The formula for the Taylor series is:

$$f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots$$

Substitute $a = 1$:

$$f(1) + f^{(1)}(1)(x-1) + \frac{f^{(2)}(1)}{2!}(x-1)^2 + \cdots$$

Substitute the given values in the fourth degree Taylor Polynomials:

$$p(x) = 1 + \frac{1}{2}(x-1) - \frac{\frac{1}{4}}{2}(x-1)^2 + \frac{\frac{3}{8}}{6}(x-1)^3 - \frac{\frac{15}{16}}{24}(x-1)^4$$

Verification

$$p'(x) = \frac{1}{2} - \frac{1}{4}(x-1) + \frac{\frac{3}{8}}{2}(x-1)^2 - \frac{\frac{15}{16}}{6}(x-1)^3$$

$$p''(x) = -\frac{1}{4} + \frac{3}{8}(x-1) - \frac{\frac{15}{16}}{2}(x-1)^2$$

$$p'''(x) = \frac{3}{8} - \frac{15}{16}(x-1)$$

$$p^{(4)}(x) = -\frac{15}{16}$$

Example 3.44

Find the fourth degree Taylor polynomial for $f(x) = \sqrt{x}$ at $x = 1$.

Calculate the first four derivatives and evaluate them:

$$f(x) = \sqrt{x} \Rightarrow f(1) = 1$$

$$f^{(1)}(x) = \frac{1}{2} x^{-\frac{1}{2}} \Rightarrow f^{(1)}(1) = \frac{1}{2}$$

$$f^{(2)}(x) = -\frac{1}{4} x^{-\frac{3}{2}} \Rightarrow f^{(2)}(1) = -\frac{1}{4}$$

$$f^{(3)}(x) = -\frac{3}{8} x^{-\frac{5}{2}} \Rightarrow f^{(3)}(1) = -\frac{3}{8}$$

$$f^{(4)}(x) = -\frac{15}{16} x^{-\frac{7}{2}} \Rightarrow f^{(4)}(1) = -\frac{15}{16}$$

Substitute the above values into the formula for the Taylor series:

$$p(x) = 1 + \frac{1}{2}(x-1) - \frac{\frac{1}{4}}{2}(x-1)^2 + \frac{\frac{3}{8}}{6}(x-1)^3 - \frac{\frac{15}{16}}{24}(x-1)^4$$

Example 3.45

Find the third-degree Taylor polynomial for $f(x) = \ln x$ at $x = 1$.

$$f(x) = \ln x \Rightarrow f(1) = \ln 1 = 0$$
$$f^{(1)}(x) = x^{-1} \Rightarrow f^{(1)}(1) = 1$$
$$f^{(2)}(x) = -x^{-2} \Rightarrow f^{(2)}(1) = -1$$
$$f^{(3)}(x) = 2x^{-3} \Rightarrow f^{(3)}(1) = 2$$
$$f^{(4)}(x) = -6x^{-4} \Rightarrow f^{(4)}(1) = -6$$

$$Taylor\ Series = \sum_{k=0}^{k=\infty} \frac{f^k(1)}{k!}(x-1)^k = f(1) + f^{(1)}(a)(x-1) + \frac{f^{(2)}(1)}{2!}(x-1)^2 + \cdots$$

$$= 0 + 1(x-1) + \frac{-1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots$$

Example 3.46

Taylor series for $\ln x$ at $x = 1$

$$T(x) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots$$

$$f(1) = \ln(1) = 0$$
$$f'(x) = \frac{1}{x} \Rightarrow f'(1) = \frac{1}{1} = 1$$
$$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -\frac{1}{1} = -1$$
$$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = \frac{2}{1} = 2$$

$$T(x) = 0 + 1(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \cdots$$
$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$

Video 3.47

Find the Taylor series for [$\ln x$ at x=2](#).

B. Lagrange Remainder Formula

3.48: Taylor Series

We can compare a Taylor series to a finite Taylor polynomial by adding a $R_n(b)$ to take care of the missing terms:

$$T(b) = T_n(b) + R_n(b)$$

Where

$$x \text{ is where the Talyor polynomial or series is being evaluated}$$

The Taylor series is given by the infinite series:

$$T(b) = f(a) + f^{(1)}(a)(b-a) + \frac{f^{(2)}(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \cdots$$

If we consider only n terms of the series, we get the n^{th} Taylor polynomial:

$$T_n(b) = f(a) + f^{(1)}(a)(b-a) + \frac{f^{(2)}(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n$$

We can connect the n^{th} Taylor polynomial and the Taylor series (for a specific function) by putting the value of the remaining terms in a single remainder term $R_n(x)$

$$T(b) = T_n(b) + R_n(b)$$

3.49: Lagrange Remainder Formula

The Lagrange remainder formula for a n^{th} degree Taylor polynomial is based on the $(n+1)^{st}$ derivative of the function of interest:

$$R_n(b) = \frac{f^{n+1}(c)}{(n+1)!}(b-a)^{n+1}$$

Where

$$a < c < b$$

- We know that c lies strictly between a and b, but we do not know the value of c unless we have some information on the value of b.
- Hence, we often consider a worst scenario, which is the value of c that maximizes $f^{n+1}(c)$, and consider the remainder must be bounded by the maximum.

Example 3.50

Consider the Taylor series centered at 0, for the function $f(x) = e^x$ and used to evaluate e^4. For the second degree and third-degree Taylor polynomials, calculate the

A. value of c in Lagrange's Remainder Formula for $f(x)$
B. error when compared to the value of the function

We are working with Maclaurin series since the Taylor series is centered at $a = 0$.

We can write the series as the sum of a Taylor polynomial and a Lagrange remainder term:

$$f(b) = T_n(b) + R_n(b)$$

Substitute $b = 4, n = 2$ and solve for $R_n(4)$:

$$\underbrace{R_n(4) = f(4) - T_2(4)}_{Equation\ I}$$

Case I: $n = 2$

Here, we want the second-degree Taylor polynomial:

$$T_2(4) = 1 + b + \frac{b^2}{2!} = 1 + 4 + \frac{4^2}{2!} = 13$$

$$R_2(4) = \frac{f^{n+1}(c)}{(n+1)!}(b)^{n+1} = \frac{e^c}{3!} \cdot 4^3 = \frac{64e^c}{6} = \frac{32e^c}{3}$$

Substitute the values calculated above in Equation I, and solve the resulting exponential equation:

$$\frac{32e^c}{3} = e^4 - 13 \Rightarrow c = \ln\left[\frac{3}{32}(e^4 - 13)\right] \approx 1.36$$

$$Error = e^4 - 13 \approx$$

Case II: $n = 3$

$$T_3(4) = 1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} = 1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!} = \frac{71}{3}$$

$$R_3(4) = \frac{f^{n+1}(c)}{(n+1)!}(b)^{n+1} = \frac{e^c}{4!} \cdot 4^4 = \frac{32e^c}{3}$$

Substitute the values calculated above in Equation I:

$$\frac{32e^c}{3} = e^4 - \frac{71}{3} \Rightarrow c = \ln\left[\frac{3}{32}\left(e^4 - \frac{71}{3}\right)\right] \approx 1.06$$

$$Error = e^4 - \frac{71}{3} \approx$$

C. Convergence Test

3.51: Convergence of Taylor Polynomials

If

$$R_n(b) \to 0 \ as \ n \to \infty, for \ all \ x \in I$$

Then we say that

$$T_n(x) \to f \ on \ I \ as \ n \to \infty$$

- If the *remainder term tends to zero* as the number of terms becomes large, then the difference between the function and the Taylor polynomial also tends to zero.
- Hence, the Taylor series converges to the function.

Example 3.52

Prove that that the Maclaurin series for $f(x) = e^x$ converges to e^x when $x = 4$.

The Taylor expansion of e^b centered at $a = 0$ is:

$$e^b = 1 + b + \frac{b^2}{2!} + \cdots + \frac{b^n}{n!} + \cdots$$

The Remainder term for the n^{th} Taylor polynomial at $x = 4$:

$$R_n(4) = \frac{f^{n+1}(c)}{(n+1)!}b^{n+1} = \frac{e^4}{(n+1)!}4^{n+1}$$

Take the limit as $n \to \infty$:

$$\lim_{n\to\infty} \frac{e^4}{(n+1)!} 4^{n+1}$$

Since e^c is a constant, we can move it out of the limit:

$$e^4 \lim_{n\to\infty} \frac{4^{n+1}}{(n+1)!}$$

Expand the numerator and the denominator:

$$e^4 \lim_{n\to\infty} \frac{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \dots \cdot 4\ (n+1\ times\ )}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot (n+1)}$$

Rearrange:

$$= e^4 \lim_{n\to\infty} \frac{4 \cdot 4 \cdot 4 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{4}{5} \times \frac{4}{6} \times \frac{4}{7} \times \dots \times \frac{4}{n+1}$$

Since each term after the first fraction is less than 1 and each term is less than the prior term hence, as $n \to \infty$, the limit $\to 0$:

$$= e^C \times 0 = 0$$

D. Determining Maximum Value of Derivative

3.53: Convergence of Taylor Polynomials

If

$$|f^{n+1}(t)| \le M\ for\ all\ a < t < b$$

Then

$$|R_n(b)| \le \frac{M|b-a|^{n+1}}{(n+1)!}$$

The Lagrange remainder formula has a term for the $(n+1)^{st}$ derivative, evaluated at c.

- In certain situations, it can be difficult to determine the value of c, or to determine an exact expression for the derivative itself.
- However, by considering a bound on the value of $f^{n+1}(t)$, we can still establish convergence.

Example 3.54

Show that the Maclaurin series for *sin* converges to $\sin x$ for every value of x.

$$f(x) = \sin x$$
$$f^{(1)}(x) = \cos x$$
$$f^{(2)}(x) = -\sin x$$
$$f^{(3)}(x) = -\cos x$$
$$f^{(4)}(x) = \sin x$$

Hence, the derivatives of $\sin x$ are cyclical with a cyclicity of 4.

$$Max(|f^{n+1}(t)|) = M = 1$$

$$|R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!} = \frac{(1)|x|^{n+1}}{(n+1)!} = \frac{|x|^{n+1}}{(n+1)!}$$

Take the limit as $n \to \infty$

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!}$$

Expand the numerator and the denominator:

$$\lim_{n\to\infty} \frac{x \cdot x \cdot x \cdot x \cdot x \dots \cdot x \ (n+1 \ times\ )}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot (n+1)}$$

$$= \lim_{n\to\infty} \frac{x \cdot x \cdot x \cdot x \ (x \ times\ )}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot x} \times \frac{x}{x+1} \times \frac{x}{x+2} \times \frac{x}{x+3} \times \dots \times \frac{x}{n+1}$$

Since each term after the first fraction is less than 1 and each term is less than the prior term hence, as $n \to \infty$, the limit $\to 0$:

$$= 0$$

Example 3.55

Show that the Maclaurin series for $\cos x$ converges to $\cos x$ for every value of x.

$$f(x) = \cos x$$
$$f^{(1)}(x) = -\sin x$$
$$f^{(2)}(x) = -\cos x$$
$$f^{(3)}(x) = \sin x$$
$$f^{(4)}(x) = \cos x$$

Hence, the derivatives of $\cos x$ are cyclical with a cyclicity of 4.

$$Max(|f^{n+1}(t)|) = M = 1$$

$$|R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!}$$

Which is the same limit as in the previous example.

3.4 Infinite Series

A. Sequences and Series

3.56: Sequences

A sequence is a list of numbers.

$$Fibonacci\ Sequence: 1,1,2,3,5,8, \dots$$

3.57: Infinite Series

An infinite series is the sum of an infinite sequence of numbers.

3.58: Partial Sums

The n^{th} partial sum of an infinite series is given by the sum of the first n terms of the series.

Example 3.59

Determine the first four partial sums, and the n^{th} partial sum of the series:

$$1+2+3+4+\cdots$$

$$S_1 = 1$$
$$S_2 = 1+2 = 3$$
$$S_3 = 1+2+3 = 6$$
$$S_4 = 1+2+3+4 = 10$$

$$S_n = \frac{n(n+1)}{2}$$

3.60: Convergence and Divergence

- If the sequence of partial sums of a series converges to a limit L, we say that the series converges, and that its limit is L.
- If the sequence of partial sums does not converge, we say that the series diverges.

Example 3.61

Does the sequence below converge?

$$1+1+1+1+1+\cdots$$

The n^{th} partial sum of the series

$$= S_n = n$$

The limit of the sequence of partial sums as the number of terms becomes very large is:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty \Rightarrow Limit\ DNE \Rightarrow Diverges$$

Example 3.62

Does the sequence below converge?

$$1-1+1-1+1-\cdots$$

Sum to n terms

$$= S_n = \begin{cases} 1, n\ is\ odd \\ 0, n\ is\ even \end{cases}$$

Since the above sum alternates between 0 and 1:

$$\lim_{n\to\infty} S_n = DNE$$
$$Series\ diverges$$

B. Geometric Series

You might have seen geometric series before. Common questions relate to:

- Finding the general term $= ar^{n-1}$
- Finding the value of a specific term
- Finding the first term and common ratio directly from the series or from given conditions

We will mostly focus here on the sum, and more so on radius of convergence[3].

3.63: Finite Geometric Series

The geometric series with $first\ term = a$ and $common\ ratio = r$ is:

[3] To revise geometric series, refer that chapter in the Note on Sequences and Series

$$a, ar, ar^2, \dots ar^{n-1}, \dots$$

It has sum

$$\frac{a(1-r^n)}{1-r}$$

$$S = a + ar + \cdots + ar^{n-1}$$
$$rS = ar + ar + \cdots + ar^n$$

$$rS - S = ar^n - a$$
$$S(r-1) = a(r^n - 1)$$
$$S = \frac{a(1-r^n)}{1-r}$$

Example 3.64

Determine the n^{th} partial sum of the series below and hence check its convergence or divergence:

$$2 - 1 + \frac{1}{2} - \frac{1}{4} + \cdots$$

This is a geometric series with $a = 2, r = \frac{1}{2}$. The n^{th} partial sum of the series is:

$$\frac{a(1-r^n)}{1-r} = \frac{2\left[1 - \left(-\frac{1}{2}\right)^n\right]}{1 - \left(-\frac{1}{2}\right)}$$

Since we do know whether n is odd or even, we apply a $\pm$ sign:

$$= \frac{2\left[1 \pm \frac{1}{2^n}\right]}{\frac{3}{2}} = \frac{4}{3}\left(1 \pm \frac{1}{2^n}\right)$$

The limit of the sequence of partial sums as the number of terms becomes very large is:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{4}{3}\left(1 \pm \frac{1}{2^n}\right) = \frac{4}{3}(1) = \frac{4}{3}$$

3.65: Infinite Geometric Series

A infinite geometric series with $first\ term = a$ and $common\ ratio = r$ has sum

$$\frac{a}{1-r}$$

Where

$$Radius\ of\ Convergence: -1 < r < 1$$

The n^{th} partial sum of an infinite geometric series with $first\ term = a, common\ ratio = r$ is:

$$\frac{a(1-r^n)}{1-r}$$

As the number of terms becomes very large:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

This limit has a finite value if:

$$-1 < r < 1$$

And does not exist if

$$r \le -1 \; or \; r \ge 1$$

Example 3.66

Determine the radius of convergence and the sum for that radius of convergence:

$$2 + \frac{4}{3}(x+5) + \frac{8}{9}(x+5)^2 + \cdots$$

This is a geometric series with $first\ term = a = 2, common\ ratio = r = \frac{2}{3}(x+5)$
The radius of convergence is $-1 < r < 1$:

$$-1 < \frac{2}{3}(x+5) < 1 \Rightarrow -\frac{3}{2} - 5 < x < \frac{3}{2} - 5$$

It has sum

$$\frac{a}{1-r} = \frac{2}{1 - \frac{2}{3}(x+5)}, \qquad -\frac{13}{2} < x < -\frac{7}{2}$$

Example 3.67

Determine the radius of convergence and the sum for that radius of convergence:

$$\sum_{n=0}^{\infty} \left(\frac{x-2}{3}\right)^n$$

$$\sum_{n=0}^{\infty} \left(\frac{x-2}{3}\right)^n = 1 + \left(\frac{x-2}{3}\right) + \left(\frac{x-2}{3}\right)^2 + \cdots$$

This is a geometric series with $a = 1, r = \left(\frac{x-2}{3}\right)$ and sum:

$$\frac{a}{1-r} = \frac{1}{1 - \frac{x-2}{3}} = \frac{1}{\frac{3-x+2}{3}} = \frac{3}{5-x}$$

For the series to converge, we must have $-1 < r < 1$:

$$-1 < \frac{x-2}{3} < 1$$
$$-3 < x - 2 < 3$$
$$-1 < x < 5$$

3.68: Radius of Convergence

Radius of convergence is the interval which results in convergence of the series. If the variable r lies

- in the interval of convergence, the series converges.
- outside the interval of convergence, the series diverges.

3.69: Determining the radius of convergence

Determining the radius of convergence requires solving the inequality

$$-1 < r < 1$$

Example 3.70
Determine the radius of convergence and the sum for that radius of convergence:
$$\sum_{n=0}^{\infty} \cos^n x = 1 + \cos x + \cos^2 x + \cdots + \cos^n x + \cdots$$

The above is a geometric series with $a = 1, r = \cos x$.
The radius of convergence is $-1 < r < 1$:
$$-1 < \cos x < 1$$

Eliminate all values where
$$\cos x = 1 \text{ } or \cos x = -1 \Rightarrow \{x : x = k\pi, k \in \mathbb{Z}\}$$

It has sum
$$\frac{a}{1-r} = \frac{1}{1-\cos x}, x \neq k\pi, k \in \mathbb{Z}$$

Example 3.71
Determine the radius of convergence and the sum for that radius of convergence:
$$\sum_{n=1}^{\infty} (\log_a x)^n = \log_a x + (\log_a x)^2 + \cdots + (\log_a x)^n + \cdots, a > 1$$

The above is a geometric series with $a = \log_a x, r = \log_a x$.
The radius of convergence is $-1 < r < 1$:
$$-1 < \log_a x < 1$$
$$-1 < \frac{\ln x}{\ln a} < 1$$
If $a > 1 \Rightarrow \ln a > 0$
$$-\ln a < \ln x < \ln a \Rightarrow \frac{1}{a} < x < a$$

If $0 < a < 1 \Rightarrow \ln a < 0$
$$-\ln a > \ln x > \ln a$$
$$\ln a^{-1} > \ln x > \ln a$$
$$\frac{1}{a} > x > a$$
$$a < x < \frac{1}{a}$$

It has sum
$$\frac{a}{1-r} = \frac{\log_a x}{1-\log_a x}, \qquad \frac{1}{a} < x < a, \qquad a > 1$$
$$\frac{a}{1-r} = \frac{\log_a x}{1-\log_a x}, \qquad a < x < \frac{1}{a}, \qquad 0 < a < 1$$

C. Telescoping Series

3.72: Telescoping Series
A telescoping series is one where we can cancel all terms except the first and last terms.

3.73: Splitting a Fraction

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$LHS = \frac{1}{n} - \frac{1}{n+1} = \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} = \frac{1}{n(n+1)} = RHS$$

Example 3.74

$$\frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \frac{1}{6\cdot 7} + \cdots +$$

A. Calculate the n^{th} partial sum.
B. Determine the convergence or divergence of the series.

$$\left(\frac{1}{4}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{6}\right) + \left(\frac{1}{6}-\frac{1}{7}\right) + \cdots + \left(\frac{1}{n+2}-\frac{1}{n+3}\right) + \left(\frac{1}{n+3}-\frac{1}{n+4}\right)$$

The n^{th} partial sum of the series is:

$$S_n = \frac{1}{4} - \frac{1}{n+4}$$

As the number of terms becomes very large:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{4} - \frac{1}{n+4} = \frac{1}{4}$$

D. n^{th} term test for divergence

3.75: Convergence Requirement

If the series $\sum_{n=1}^{\infty} a_n$ converges then

$$a_n \to 0$$

For a series to be convergent, it is a requirement that as $n \to \infty$, the value of the n^{th} term tends to zero.

3.76: n^{th} term test for divergence

If $\lim\limits_{n\to\infty} a_n \neq 0$ then

$$\sum_{n=1}^{\infty} a_n \ diverges$$

Note: If the limit does not exist, then also it is not equal to zero

Since the limit as $n \to \infty$ of the sequence must tend to zero, if this requirement is not met, then the series must diverge.

Example 3.77

$$\sum_{n=1}^{\infty} \sqrt{n+1} - \sqrt{n}$$

A. Apply the divergence test to the above series.
B. Use telescoping to determine the n^{th} partial sum, and hence test for convergence
C. What is the conclusion from Parts A and B?

Part A

Rationalize the numerator of the expression:

$$\sqrt{n+1}-\sqrt{n}\times\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\sqrt{n+1}+\sqrt{n}}$$

Take the limit:

$$=\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}=0$$

$$Divergence\ test\ does\ not\ apply$$
$$Divergence\ test\ is\ inconclusive$$

Part B

The n^{th} partial sum is:

$$\sum_{n=1}^{\infty}\sqrt{n+1}-\sqrt{n}=\left(\sqrt{2}-\sqrt{1}\right)+\left(\sqrt{3}-\sqrt{2}\right)+\left(\sqrt{4}-\sqrt{3}\right)+\cdots+\left(\sqrt{n+1}-\sqrt{n}\right)=\sqrt{n+1}-\sqrt{1}$$

Take the limit as $n\to\infty$

$$\lim_{n\to\infty}\sqrt{n+1}-\sqrt{1}=\infty$$
$$Series\ diverges$$

Example 3.78

Apply the divergence test to the harmonic series:

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

$$\lim_{n\to\infty}\frac{1}{n}=0\Rightarrow Inconclusive$$

Note: The harmonic series diverges

Example 3.79

Apply the divergence test to the series below:

$$\sum_{n=1}^{\infty}\frac{e^n}{e^n+2n}$$

$$\lim_{n\to\infty}\frac{e^n}{e^n+2n}=\lim_{n\to\infty}\frac{e^n}{e^n+2}=\lim_{n\to\infty}\frac{1}{1+\frac{2}{e^n}}=1\neq 0\Rightarrow Diverges$$

E. Combining Series

3.80: Combining Series

THEOREM 8 If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum ka_n = k\sum a_n = kA$ (any number k).

Example 3.81

$$\sum_{k=1}^{k=\infty} \frac{(-1)^{k+1}}{k}(x-1)^k$$

Determine the convergence or divergence of the series above when $x = 0$

Substitute $x = 0$:

$$\sum_{k=1}^{k=\infty} \frac{(-1)^{k+1}}{k}(-1)^k$$

$$\sum_{k=1}^{k=\infty} \frac{(-1)^{2k+1}}{k} = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots = -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right) = -\sum_{n=1}^{n=\infty} \frac{1}{k}$$

F. Adding and Deleting Terms

3.82: Adding and Deleting Terms

We can add a finite number of terms, or delete a finite number of terms from a series without altering the convergence or divergence of the series.

Example 3.83

$$\sum_{n=1}^{\infty} \frac{1}{n+4}$$

Expand the given series gives us:

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots$$

Add $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots$$

Which is now the harmonic series, which diverges.

G. Tails

3.84: Tail of a Series

The tail of a series is the series obtained by dropping a finite number of terms from the beginning of the series.

Example 3.85

What is the tail of the harmonic series obtained by dropping the first three terms?

$$\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots$$

3.86: Convergence and Divergence of Tails

A series converges $if\ and\ only\ if$ its tails converge.

Example 3.87

$$\sum_{n=1}^{\infty}\frac{1}{n+4}$$

Expanding the given series gives us:

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots$$

And the above is a tail of the harmonic series, which diverges.

Hence, the given series diverges.

3.5 Integral Test

A. Integral Test

3.88: Integral Test

Let $a_1, a_2, \dots, a_n$ be a sequence of positive terms. If $a_n = f(n)$, where $f(x)$ is a continuous, positive, and decreasing function for all $x \geq N,\ \ N \in \mathbb{N}$ then the:

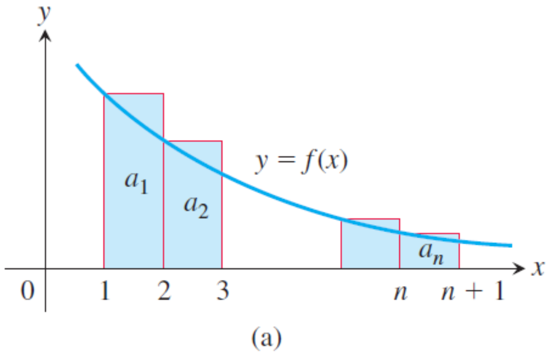
$$series\ \sum_{n=N}^{\infty} a_n\ and\ the\ integral\ \int_{N}^{\infty} f(x)$$

Both $converge$ or both $diverge$

Divergence Case

Consider rectangles with width 1 and height a_n to the right of the continuous, positive, decreasing function $f(x)$. Since the function is decreasing, the area of each rectangle is greater than the area under the curve.

$$a_1 > \int_1^2 f(x)$$
$$a_2 > \int_2^3 f(x)$$
$$\vdots$$
$$a_n > \int_n^{n+1} f(x)$$



(a)

Add the above inequalities:

$$a_1 + a_2 + \cdots + a_n > \int_1^2 f(x) + \int_2^3 f(x) + \cdots + \int_n^{n+1} f(x)$$

Using integral properties:

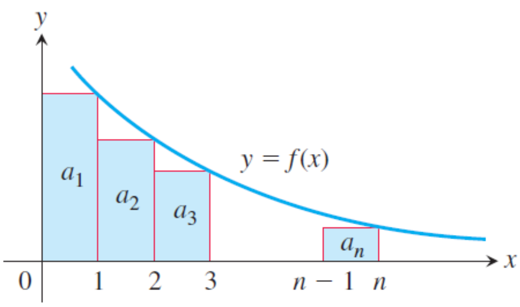
$$\sum_{n=1}^{n} a_i > \int_1^{n+1} f(x)$$

Hence, if the integral diverges, the series diverges too.

Convergence Case

The convergence is similar, in that we take rectangles of width 1. However, the rectangles are to the instead of to the right. Hence,

$$a_2 < \int_1^2 f(x)$$
$$a_3 < \int_2^3 f(x)$$



$$\vdots$$
$$a_n < \int_{n-1}^{n+1} f(x)$$

Add the above:

$$a_2 + \cdots + a_n < \int_1^2 f(x) + \int_2^3 f(x) + \cdots + \int_{n-1}^{n} f(x)$$

Add a_1 to both sides of the above:

$$a_1 + a_2 + \cdots + a_n < a_1 + \int_1^2 f(x) + \int_2^3 f(x) + \cdots + \int_{n-1}^{n} f(x)$$

Using integral properties:

$$\sum_{n=1}^{n} a_i < a_1 + \int_1^n f(x)$$

Hence, if the integral converges, the series converges too.

B. Harmonic Series

3.89: Divergence of Harmonic Series

The harmonic series (given below) diverges:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}$$

Method I: Comparison

We can do this without the integral test. Successive groups of terms can be added to be greater than half. Hence, the result can be increased to any value without limit:

$$1+\frac{1}{2}+\underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{>\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2}}+\underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}}+\underbrace{\left(\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}\right)}_{>\frac{1}{16}+\frac{1}{16}+\cdots+\frac{1}{16}=\frac{8}{16}=\frac{1}{2}}+\cdots\Rightarrow Diverges$$

Method II: Integral Test

Apply the integral test, and check:

$$\int_1^t \frac{1}{x}\,dx=[\ln|x|]_1^t=\ln t-\ln 1=\ln t$$

Take as the limit as

$$\lim_{t\to\infty}\ln t=\infty\Rightarrow Diverges$$

C. p series

3.90: p-series

A series of the form below is called a p series, where p is any real number:

$$\frac{1}{1^p}+\frac{1}{2^p}+\cdots+\frac{1}{n^p}+\cdots$$

The harmonic series is a p series with

$$p=1$$

Example 3.91

Write out the expanded form of each p series. Then determine its convergence or divergence, using the integral test.

A. $p=3$
B. $p=\frac{2}{3}$

Part A

$$\frac{1}{1^3}+\frac{1}{2^3}+\cdots+\frac{1}{n^3}+\cdots$$

$$\int_1^t \frac{1}{x^3}dx=\int_1^t x^{-3}\,dx=-\frac{1}{2}[x^{-2}]_1^t=-\frac{1}{2}\left(\frac{1}{t^2}-1\right)$$

$$\lim_{t\to\infty} = -\frac{1}{2}\left(\frac{1}{t^2}-1\right) = \frac{1}{2}$$
$$Converges$$

Part B

$$\frac{1}{1^{\frac{2}{3}}} + \frac{1}{2^{\frac{2}{3}}} + \cdots + \frac{1}{n^{\frac{2}{3}}} + \cdots$$
$$\int_1^t \frac{1}{x^{\frac{2}{3}}}\,dx = \int_1^t x^{-\frac{2}{3}}\,dx = 3\left[x^{\frac{1}{3}}\right]_1^t = 3\left(t^{\frac{1}{3}}-1\right)$$
$$\lim_{t\to\infty} 3\left(t^{\frac{1}{3}}-1\right) = \infty$$

$$Diverges$$

Example 3.92

Fill in the blanks with the correct option

A p series is _____ a geometric series.

A. Always
B. Never
C. Sometimes
D. None of the above

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

If the above series is a geometric series, then it must have a common ratio given by

$$r = \frac{\frac{1}{2^p}}{\frac{1}{1^p}} = \frac{\frac{1}{3^p}}{\frac{1}{2^p}}$$
$$\frac{1^p}{2^p} = \frac{2^p}{3^p}$$
$$\left(\frac{1}{2}\right)^p = \left(\frac{2}{3}\right)^p$$
$$p = 0$$

$$Sometimes$$

Example 3.93

$$\lim_{t\to\infty} \frac{1}{t^{p-1}}$$

Evaluate the above limit when

A. $p = 0.5$
B. $p = -1$
C. $p = 3$

$$p = 0.5 \Rightarrow \lim_{t\to\infty} t^{1-0.5} = \lim_{t\to\infty} t^{0.5} = \lim_{t\to\infty} \sqrt{t}$$
$$p = -1 \Rightarrow \lim_{t\to\infty} t^{1-(-1)} = \lim_{t\to\infty} t^2$$

3.94: p series test

$$\frac{1}{1^p}+\frac{1}{2^p}+\cdots+\frac{1}{n^p}+\cdots$$

The series above

- Converges when $p>1$
- Diverges when $p=1$
- Diverges when $p<1$

Use the integral test:

$$\int_1^t \frac{1}{x^p}\,dx=\int_1^t x^{-p}\,dx=\left[\frac{1}{(-p+1)(x^{p-1})}\right]_1^t=\frac{1}{-p+1}\left(\frac{1}{t^{p-1}}-1\right)=\frac{1}{p-1}\left(1-\frac{1}{t^{p-1}}\right)$$

$$\lim_{t\to\infty}\frac{1}{p-1}\left(1-\frac{1}{t^{p-1}}\right)=\frac{1}{p-1}\lim_{t\to\infty}\left(1-\frac{1}{t^{p-1}}\right)$$

$$Converges$$

If $p>1$

$$\lim_{t\to\infty}\left(1-\frac{1}{t^{p-1}}\right)=1$$

If $p<1\Rightarrow p-1<0\Rightarrow 1-p>0$:

$$\lim_{t\to\infty}\left(1-\frac{1}{t^{p-1}}\right)=\lim_{t\to\infty}\left(1-t^{1-p}\right)=\infty$$

Example 3.95

Consider the following infinite series. How many of them converge?

$$\sum_{n=1}^{\infty}\frac{1}{n^{-\frac{17}{3}}},\quad \sum_{n=1}^{\infty}\frac{1}{n^{-\frac{13}{3}}},\cdots,\quad \sum_{n=1}^{\infty}\frac{1}{n^{\frac{23}{3}}}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^p}\Rightarrow p\in\left\{-\frac{17}{3},-\frac{13}{3},\ldots\frac{3}{3},\frac{7}{3},\frac{11}{3},\frac{15}{3},\frac{19}{3},\frac{23}{3}\right\}$$

$$5\ Terms$$

D. Logarithmic p series

3.96: Logarithmic p series

The series obtained by dividing the harmonic series by $(\ln x)^p$ is a *logarithmic p series*

$$\frac{1}{x\,(\ln x)^p}$$

Example 3.97

Determine the convergence or divergence of:

$$\frac{1}{2\ln 2}+\frac{1}{3\ln 3}+\frac{1}{4\ln 4}+\cdots$$

$$\int_2^t\frac{1}{x\ln x}dx$$

Substitute $u = \ln x \Rightarrow du = \frac{1}{x}dx$, and change the limits of integration by taking the logs of the limits:

$$\int_{\ln 2}^{\ln t} \frac{1}{u} du = [\ln|u|]_{\ln 2}^{\ln t} = \ln|\ln t| - \ln|\ln 2|$$

As

$$\lim_{t\to\infty} \ln|\ln t| - \ln|\ln 2| = \infty$$
$$Diverges$$

Example 3.98

Determine the convergence or divergence of:

$$\frac{1}{2(\ln 2)^2} + \frac{1}{3(\ln 3)^2} + \frac{1}{4(\ln 4)^2} + \cdots$$

$$\int_2^t \frac{1}{x(\ln x)^2} dx$$

Substitute $u = \ln x \Rightarrow du = \frac{1}{x}dx$, and change the limits of integration by taking the logs of the limits:

$$\int_{\ln 2}^{\ln t} \frac{1}{u^2} du = \left[-\frac{1}{u}\right]_{\ln 2}^{\ln t} = -\frac{1}{\ln t} + \frac{1}{\ln 2}$$

As

$$\lim_{t\to\infty} \left(-\frac{1}{\ln t} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}$$

E. Further Examples

Example 3.99

$$\sum_{n=1}^{\infty} \frac{1}{n+4}$$

$$\int_1^t \frac{1}{x+4} dx = [\ln|x+4|]_1^t$$

$$\lim_{t\to\infty} (\ln|x+4|) = \infty$$

$$Diverges$$

Example 3.100

Check the convergence or divergence of:

$$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$$

Use the integral test and compare with $f(x)$ which is continuous, positive, and decreasing for $x > 1$:

$$f(x) = \frac{1}{x+\sqrt{x}} = \frac{1}{\sqrt{x}(\sqrt{x}+1)}$$

Find the associated indefinite integral. Substitute $u = \sqrt{x} + 1 \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx$:

$$\int \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx = 2\int \frac{1}{u}\,du = 2\ln|u| + C = 2\ln\left|\sqrt{x}+1\right| + C$$

By the integral test:

$$\int_1^\infty \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx = 2\left[\ln\left|\sqrt{x}+1\right|\right]_1^\infty = \infty \Rightarrow Diverges$$

F. Error Estimation

3.101: Error Estimation

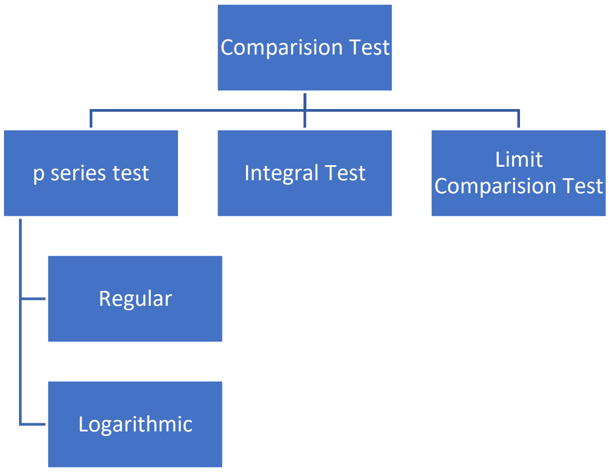
G. Applications

Example 3.102

3.6 Comparison Test

A. Comparison Test

The comparison test let us compare a series whose convergence or divergence we do not know or find difficult to determine, with a series whose convergence or divergence we do know.



This means that we need
- a series to compare with
- a test to apply to that series

The chart alongside shows some basic options for comparison:
- The p series / logarithmic p test is straightforward to apply. You should also know the proof of the p series test, in case a subjective question asks to prove tests that you use.
- The integral test was covered in the previous section.
- The limit comparison test compares the limits of the n^{th} terms of two sequences. You will often need to use the limit comparison test in conjunction with other tests.

B. Basics

3.103: Comparison Test

If $\sum a_n, \sum b_n$ and $\sum c_n$ are each series with $non-negative$ terms. If

$$a_n \le b_n \le c_n \qquad for\ all\ n > N, N \in \mathbb{N}$$

- To prove convergence, find something that is larger, and prove convergence for that series

$$c_n\ converges, b_n\ also\ converges$$

- To prove divergence, find something that is smaller, and prove divergence for that series

$$a_n\ diverges, b_n\ also\ diverges$$

3.104: Checking for Smaller and Larger

Taking the reciprocal of an inequality with positive expressions on both sides reverses the inequality. For positive x and y:

$$x > y \Rightarrow \frac{1}{x} < \frac{1}{y}$$

$$3 > 2 \Rightarrow \frac{1}{3} < \frac{1}{2}$$

Note this does not for work for x and y both negative:

$$-3 > -2\ but\ \frac{1}{-3} < \frac{1}{-2}\ is\ not\ true$$

Example 3.105

The integral $\int \frac{1}{\ln x}$ is not elementary. It has an integral, but that integral cannot be expressed in terms of known functions. Hence, use the comparison test to determine the convergence or divergence of:

$$\frac{1}{\ln 2} + \frac{1}{\ln 3} + \cdots + \frac{1}{\ln n} + \cdots = \sum_{n=2}^{\infty} \frac{1}{\ln n}$$

For $x \geq 2$, compare with

$$\frac{1}{x \ln x} < \frac{1}{\ln x}$$

$$\frac{1}{x \ln x}\ diverges \Rightarrow \frac{1}{\ln x}\ also\ diverges$$

Example 3.106

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2}$$

By the p series test with $p = 2$:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\ converges$$

Since $\frac{1}{n^2+3n+2} < \frac{1}{n^2}$:

$$the\ given\ series\ converges$$

Example 3.107

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^3+\sqrt{2}}$$

By the p series test with $p=3$:

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \ converges$$

Since

$$\frac{1}{n^3+\sqrt{2}} < \frac{1}{n^3} \Rightarrow the\ given\ series\ converges$$

Example 3.108

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{n-e}{n^4+\sqrt{\pi}}$$

By the p series test with $p=3$:

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \ converges$$

Since

$$\frac{n-e}{n^4+\sqrt{\pi}} = \frac{1-\frac{e}{n}}{n^3+\frac{\sqrt{\pi}}{n}} < \frac{1}{n^3}, the\ given\ series\ converges$$

Example 3.109

Determine the convergence or divergence of

$$\sum_{n=10}^{\infty} \frac{1}{\sqrt{n}-3}$$

$$\frac{1}{\sqrt{n}-3} > \frac{1}{\sqrt{n}} \ which\ diverges\ by\ the\ p\ series\ test$$
$$given\ series\ also\ diverges$$

Example 3.110

Check the convergence or divergence of:

$$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$$

For $n \ge 1$, compare with:

$$n+\sqrt{n} < 2n \Rightarrow \frac{1}{n+\sqrt{n}} \ge 2n$$

$$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n} \to Harmonic\ Series \Rightarrow Diverges$$

By the comparison test:

$$\sum_{n=1}^{\infty}\frac{1}{n+\sqrt{n}}\ diverges$$

C. Limit Comparison Test

- The limit comparison test requires you to find the ratio of two sequences as the term number tends to infinity.
- This is different from the divergence test, where you find the value of the sequence as the term number tends to infinity.

3.111: Limit Comparison Test

Consider sequences $a_n > 0$ and $b_n > 0$ for all $n \geq N$ $(N\ is\ an\ integer)$. If

$$\lim_{n\to\infty}\frac{a_n}{b_n} = c > 0\ then\ both\ converge\ or\ both\ diverge$$
$$\lim_{n\to\infty}\frac{a_n}{b_n} = 0\ and\ \sum b_n\ converges, then\ \sum a_n\ also\ converges$$
$$\lim_{n\to\infty}\frac{a_n}{b_n} = \infty\ and\ \sum b_n\ diverges, then\ \sum a_n\ also\ diverges$$

Example 3.112

Determine the convergence or divergence of:

$$\sum_{n=1}^{\infty}\frac{1}{n^2+3n+2}$$

$$\lim_{n\to\infty}\frac{\frac{1}{n^2+3n+2}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2}{n^2+3n+2} = \lim_{n\to\infty}\frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2}+\frac{3n}{n^2}+\frac{2}{n^2}} = \frac{1}{1+0+0} = 1 > 0$$

By the p series test with $p = 2$:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\ converges$$

By the limit comparison test:

$$since\ \sum_{n=1}^{\infty}\frac{1}{n^2}\ converges, \qquad \sum_{n=1}^{\infty}\frac{1}{n^2+3n+2}\ also\ converges$$

Example 3.113

Determine the convergence or divergence of:

$$\sum_{n=10}^{\infty}\sqrt{\frac{n^2+1}{n^3+5}}$$

$$\lim_{n\to\infty}\frac{\sqrt{\frac{n^2+1}{n^3+5}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\sqrt{\frac{n^2+1}{n^3+5}}\cdot\sqrt{n} = \lim_{n\to\infty}\sqrt{\frac{n^3+n}{n^3+5}} = \sqrt{\lim_{n\to\infty}\frac{n^3+n}{n^3+5}} = 1$$

$$\textit{Since} \sum_{n=10}^{\infty} \frac{1}{\sqrt{n}} \textit{ diverges}, \sum_{n=10}^{\infty} \sqrt{\frac{n^2+1}{n^3+5}} \textit{ also diverges}$$

Example 3.114

Determine the convergence or divergence of:

$$\sum_{n=1}^{\infty} \frac{1}{3\sqrt{n}+\sqrt[4]{n}}$$

Compare with $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, which is a divergent p series:

$$\lim_{n\to\infty} \frac{\frac{1}{3\sqrt{n}+\sqrt[4]{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{3\sqrt{n}+\sqrt[4]{n}} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{\sqrt{n}}}{\frac{3\sqrt{n}}{\sqrt{n}}+\frac{\sqrt[4]{n}}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{3+\frac{1}{\sqrt[4]{n}}} = \lim_{n\to\infty} \frac{1}{3+\frac{1}{\sqrt[4]{n}}} = \frac{1}{3} > 0 \Rightarrow \textit{Diverges}$$

D. Comparison using Telescoping

Example 3.115

Determine the convergence or divergence of:

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots$$

For $n \geq 2$:

$$\frac{1}{n(n-1)} > \frac{1}{n!}$$

Check the convergence of $\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$:

$$\sum_{n=2}^{m} \frac{1}{n(n-1)} = \sum_{n=2}^{m} \left[\frac{1}{n-1} - \frac{1}{n}\right]$$

Expanding gives:

$$= \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \cdots + \left(\frac{1}{m-1}-\frac{1}{m}\right)$$
$$= \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \cdots + \left(\frac{1}{m-1}-\frac{1}{m}\right)$$
$$= 1 - \frac{1}{m}$$

$$\lim_{m\to\infty} 1 - \frac{1}{m} = 1$$

Hence, since

$$\sum_{n=2}^{m} \frac{1}{n(n-1)} \textit{ converges}, \sum_{n=2}^{\infty} \frac{1}{n!} \textit{ also converges}$$

And

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \sum_{n=2}^{\infty} \frac{1}{n!} \text{ also converges}$$

3.7 Absolute Convergence; Ratio Test

A. Absolute Convergence Test

3.116: Absolute Convergence

If a series is absolutely convergent then it is convergent.

3.117: Absolute Convergence

A series $\sum a_n$ is absolutely convergent if the corresponding series of absolute values

$$\sum |a_n|$$

Converges.

Example 3.118

Check the convergence of the series using the absolute convergence test:

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} + \cdots$$

The corresponding series of absolute values is

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

Which converges since it is a geometric series with $r = \frac{1}{2} < 1$

Hence, the original series converges.

Example 3.119

Check the convergence of the series using the absolute convergence test

$$\frac{\cos 1}{1^3} + \frac{\cos 2}{2^3} + \cdots + \frac{\cos n}{n^3}$$

Use the absolute convergence test to check the corresponding series of absolute values:

$$|a_n| = \left|\frac{\cos n}{n^3}\right|$$

Use the comparison test to compare with $\frac{1}{n^3}$

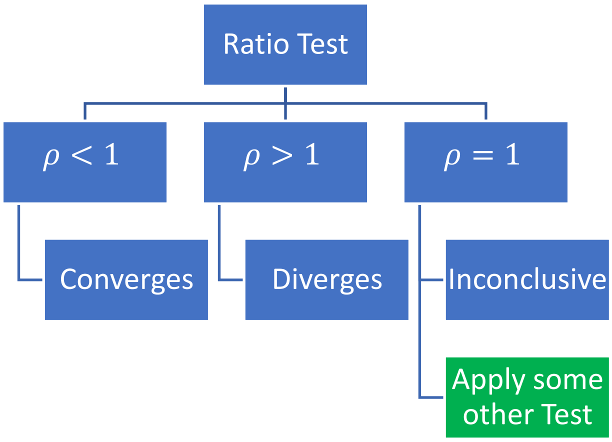
$$\left|\frac{\cos n}{n^3}\right| \le \frac{1}{n^3} \text{ since } |\cos n| \le 1$$

$$\left|\frac{\cos n}{n^3}\right| \text{ converges since } \frac{1}{n^3} \text{ converges by } p \text{ series test}$$

Since the series is absolutely convergent, it is convergent.

B. Ratio Test

Absolute convergence is the basis of the ratio test, which is an important test.



3.120: Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\times\frac{1}{a_n}\right| = \rho$$

For:

$$\rho < 1 \textit{ series converges absolutely}$$
$$\rho = 1 \textit{ test is inconclusive}$$
$$\rho > 1 \textit{ series diverges}$$

- The ratio test compares consecutive terms of a sequence.
- If the limit of the absolute value of the ratio of consecutive terms of the sequence as $n \to \infty$ is less than 1, then the series converges absolutely (and hence it converges).

Example 3.121

Check the convergence of the series below using the ratio test:

$$\sum_{n=1}^{\infty}\frac{(0.45)^n}{n}$$
$$\sum_{n=1}^{\infty}\frac{(1.1)^n}{n}$$

$$\lim_{n\to\infty}\left|a_{n+1}\times\frac{1}{a_n}\right| = \lim_{n\to\infty}\frac{(0.45)^{n+1}}{n+1}\times\frac{n}{(0.45)^n} = \lim_{n\to\infty}\frac{0.45n}{n+1} = 0.45 < 1 \Rightarrow Converges$$

$$\lim_{n\to\infty}\left|a_{n+1}\times\frac{1}{a_n}\right| = \lim_{n\to\infty}\frac{(1.1)^{n+1}}{n+1}\times\frac{n}{(1.1)^n} = \lim_{n\to\infty}\frac{1.1n}{n+1} = 1.25 > 1 \Rightarrow Diverges$$

3.122: Recursive Definition of Factorial

$$n! = n(n-1)(n-2)\ldots(n-k)!$$

Example 3.123

Check the convergence of the series below:

$$\sum_{n=1}^{\infty}\frac{(n!)^3}{(3n)!}$$

Apply the ratio test:

$$\left|a_{n+1}\times\frac{1}{a_n}\right| = \frac{[(n+1)!]^3}{(3n+3)!}\times\frac{(3n)!}{(n!)^3}$$

Expand using the recursive definition of the factorial:

$$= \frac{[(n+1)(n!)]^3}{(3n+3)(3n+2)(3n+1)(3n)!}\times\frac{(3n)!}{(n!)^3}$$

$$= \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)}$$

The numerator when expanded is a third-degree polynomial with leading coefficient 1.
The denominator when expanded is a third-degree polynomial with leading coefficient $3 \times 3 \times 3 = 27$.

The limit of the above expression as $n \to \infty$ is the ratio of the leading coefficients, which is

$$\frac{1}{27} < 1 \Rightarrow Series\ Converges$$

Example 3.124

Given that $\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = p$, determine the values of p for which the series below converges

$$\sum_{n=0}^{\infty} a_n 2^n$$

The ratio of consecutive terms of the sequence $\sum_{n=0}^{\infty} 2^n a_n$ is:

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1} a_{n+1}}{2^n a_n} = \frac{2a_{n+1}}{a_n}$$

Take the limit as $n \to \infty$, and move the 2 outside since it is a constant:

$$\lim_{n\to\infty} \left|\frac{2a_{n+1}}{a_n}\right| = 2 \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

Since a_{n+1} and a_n are consecutive terms, so are a_n and a_{n-1}. Since we take the limit as n is very large, subtracting 1 from the term number does not matter. (The term number will not go negative).

$$= 2 \lim_{n\to\infty} \left|\frac{a_n}{a_{n-1}}\right|$$

And the limit is given in the question as p. Hence:

$$\lim_{n\to\infty} \left|\frac{2a_{n+1}}{a_n}\right| = 2|p|$$

For the series to converge, we must have

$$2|p| < 1$$
$$|p| < \frac{1}{2}$$
$$-\frac{1}{2} < p < \frac{1}{2}$$

Example 3.125

Check the convergence of:

$$\sum_{n=1}^{\infty} \frac{4^{n+2}}{\ln(n+1)}$$

Apply the ratio test:

$$\left|a_{n+1} \times \frac{1}{a_n}\right| = \frac{4^{n+3}}{\ln n + 2} \times \frac{\ln n + 3}{4^{n+2}} = \frac{4\ln(n+3)}{\ln(n+2)}$$

Take the limit as $n \to \infty$ and apply LH rule:

$$\lim_{n\to\infty}\frac{4\ln n+3}{\ln n+2}=4\lim_{n\to\infty}\frac{\frac{4}{n+3}}{\frac{1}{n+2}}=4\lim_{n\to\infty}\frac{n+2}{n+3}=4\Rightarrow Diverges$$

3.8 Alternating Series; Conditional Convergence

A. Alternating Series

3.126: Alternating Series

A series which has terms which alternate between positive and negative terms is an alternating series.

Example:

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$$

3.127: Alternating Series Test

An alternating series

$$u_1-u_2+u_3-u_4+\cdots$$

Converges if:

- $u_n>0$
- $u_n\to 0\ as\ n\to\infty$
- u_n are eventually nonincreasing $u_n\geq u_{n+1}$

B. Alternating Series Error Bound

The error bound of a series is the absolute value of the difference between the n^{th} partial sum and the "value" of the series.

3.128: Error Bound

$$L=Limit\ of\ partial\ sums\ as\ number\ of\ terms\ tends\ to\ infinity$$
$$s_n=partial\ sum\ upto\ n\ terms$$

$$Error\ bound=|L-s_n|$$

3.129: Overestimation and Underestimation

$$L-s_n<0\ if\ t_{n+1}<0\ \Rightarrow s_n\ overestimates\ L$$
$$L-s_n>0\ if\ t_{n+1}>0\Rightarrow s_n\ underestimates\ L$$

$$t_{n+1}\ \ is\ the\ first\ unused\ term$$

Example 3.130

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$$

$$L=\frac{a}{1-r}=\frac{1}{1-\left(-\frac{1}{2}\right)}=\frac{2}{3}$$

$$s_1=1\Rightarrow L-s_1=\frac{2}{3}-1=-\frac{1}{3}\Rightarrow Overestimate\ by\ \frac{1}{3}\ because\ t_{n+1}=-\frac{1}{2}<0$$
$$s_2=1-\frac{1}{2}=\frac{1}{2}\Rightarrow L-s_2=\frac{2}{3}-\frac{1}{2}=\frac{1}{6}\Rightarrow Underestimate\ by\ \frac{1}{6}\ because\ t_{n+1}=\frac{1}{4}>0$$
$$s_3=1-\frac{1}{2}+\frac{1}{4}=\frac{1}{2}=\frac{3}{4}\Rightarrow L-s_3=\frac{2}{3}-\frac{3}{4}=-\frac{1}{12}\Rightarrow Overestimate\ by\ \frac{1}{12}\ because\ t_{n+1}=-\frac{1}{8}<0$$

Example 3.131

If $L=\ln 2\approx 0.69$, then calculate the error when the first partial sum is used to estimate the series
$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

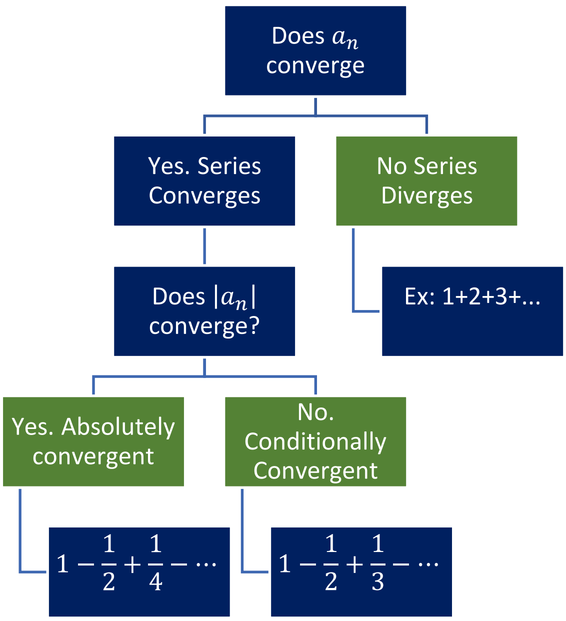
$$s_1=1$$
$$First\ unused\ term=s_2=-\frac{1}{2}$$
$$L-s_1=\ln 2-1\approx -0.31$$

$$Error\ is\ negative$$
$$Overestimation\ -\frac{1}{2}<0$$

C. Absolute Versus Conditional Convergence

3.132: Conditional Convergence

If a series converges, but does not converge absolutely, then it is conditionally convergent.



Case I:
Using any test, if the series diverges, then you get Case I. For example:

$$1+2+3+4+\cdots$$

Case II:
If the series converges, there are two possibilities. It can be absolutely convergent, or conditionally convergent. We will see both the cases.

Case II-A: Absolutely convergent
A series absolute terms converge is absolutely convergent. An absolutely convergent series is automatically convergent.

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$$

The absolute values are

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots \Rightarrow \textit{ geometric series with } r=\frac{1}{2} \Rightarrow \textit{Converges}$$

Case II-B: Conditional Convergence
If the series converges, but the series with absolute values of the terms does not converge, then the series is conditionally convergent.

$$\textit{Alternating Harmonic Series}: 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4} \Rightarrow \textit{Converges by alternating series test}$$

$$\textit{Harmonic Series}: 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \Rightarrow \textit{Diverges (See example before)}$$

Hence, the alternating harmonic series is conditionally convergent.

Example 3.133

$$\sum_{n=2}^{\infty} \frac{\cos n}{n(\ln n)^2}$$

Consider the corresponding series with absolute values. The denominator is always positive:

$$\sum_{n=2}^{\infty} \left|\frac{\cos n}{n(\ln n)^2}\right| = \sum_{n=2}^{\infty} \frac{|\cos n|}{n(\ln n)^2}$$

Since $|\cos n| \le 1$ use the Comparison Test. Compare with:

$$\sum_{n=2}^{\infty} \frac{|\cos n|}{n(\ln n)^2} \le \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

Which is a logarithmic p series with

$$n=2 \Rightarrow \textit{Converges}$$

$$\textit{By the logarithmic p series test} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \textit{ converges}$$

$$\textit{By the comparision test}: \sum_{n=2}^{\infty} \frac{|\cos n|}{n(\ln n)^2}$$

$$\textit{Since } \sum_{n=2}^{\infty} \frac{\cos n}{n(\ln n)^2} \textit{ converges absolutely, it converges}$$

3.9 Radius and Interval of Convergence

A. Radius of Convergence

3.134: Radius of Convergence

The radius of convergence of a power series $\sum c_n(x-a)^n$ is the number R such that the series converges for

$$|x-a| < R$$

- The series will diverge when $|x-a| > R$
- The endpoints $|x-a| \le R$ must be checked separately, but this is not required to determine the value of R.

Example 3.135

Given that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, determine the radius of convergence R and the value of a of the following series:

$$\sum_{n=0}^{\infty} a_n(-3)^n x^{2n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(-3)^{n+1}x^{2n+2}}{a_n(-3)^n}\right| = \lim_{n\to\infty}|-3x^2| = 3x^2$$

$$3x^2 < 1$$
$$x^2 < \frac{1}{3}$$
$$|x| < \frac{1}{\sqrt{3}}$$
$$R = \frac{1}{\sqrt{3}}$$
$$-\frac{1}{2} < x < \frac{1}{2}$$

B. Interval of Convergence

3.136: Endpoint of Radius of Convergence

The endpoints of the radius of convergence must be checked separately

$$|x-a| < R$$
$$-R < x-a < R$$
$$a-R < x < a+R$$

3.137: Interval of Convergence

The interval over which a power series $\sum c_n(x-a)^n$ converges is called its interval of convergence. The interval can be:

$$Open: (a-R, a+R)$$
$$Half-open: [a-R, a+R) \; or \; (a-R, a+R]$$
$$Closed: [a-R, a+R]$$

Example 3.138

$$\sum_{n=1}^{\infty} \frac{c^n}{n}$$

A. Check the convergence of the series above.
B. Show that the series is a geometric series. Determine the common ratio of the series.
C. Use your results from Parts A and B to determine when a geometric series converges.

Part A

$$\lim_{n\to\infty} \left| a_{n+1} \times \frac{1}{a_n} \right| = \lim_{n\to\infty} \left| \frac{c^{n+1}}{n+1} \times \frac{n}{c^n} \right| = \lim_{n\to\infty} \left| \frac{cn}{n+1} \right| = |c|$$

If

$$|c| > 1 \Rightarrow Diverges$$
$$|c| < 1 \Rightarrow Converges$$

When

$$c = 1 \Rightarrow Test\ is\ inconclusive$$

Hence, we need to use some other test.
Substitute $c = 1$. Gives us the

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \Rightarrow Harmonic\ Series \Rightarrow Diverges$$

Part B

$$\frac{c}{n} + \frac{c^2}{n} + \frac{c^3}{n}$$

⇒Geometric series with $first\ term = \frac{c}{n}$ and common ratio $= c$

It converges when

$$|c| < 1$$

Example 3.139: Tan Inverse Series

The Maclaurin series for $\tan^{-1} x$ is:

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \cdots$$

Use the ratio test:

$$\left| a_{n+1} \times \frac{1}{a_n} \right| = \left| \frac{x^{2n+1}}{2n+1} \times \frac{2n-1}{x^{2n-1}} \right| = \left| \frac{x^2(2n-1)}{2n+1} \right|$$

As $n \to \infty$, the above expression tends to:

$$|x^2| < 1$$
$$x^2 < 1$$
$$-1 < x < 1$$

Check the endpoints:

$$x = 1 \Rightarrow Series = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2n+1}$$

$$For\ n = 0\ onwards, \quad b_n = \frac{1}{2n+1}$$

$$b_n \to 0\ as\ n \to \infty$$

$$b_{n+1} \le b_n term$$

$$Converges\ by\ Alternating\ Series\ Test$$

$$x = 1 \Rightarrow Series = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2n+1}$$

$$b_n \to 0\ as\ n \to \infty$$

$$b_{n+1} \le b_n term$$

$$Converges\ by\ Alternating\ Series\ Test$$

Example 3.140

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$$

Use the ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{2n+3}}{\frac{(x-3)^n}{2n+1}} \right| = \lim_{n\to\infty} \frac{(x-3)(2n+1)}{2n+3} = |x-3|$$

$$|x-3| < 1$$

$$-1 < x - 3 < 1$$

$$2 < x < 4$$

Check the endpoints:

$$x = 4: \sum_{n=0}^{\infty} (-1)^n \frac{(1)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = Converges\ by\ Alternating\ Series\ Test$$

$$x = 2: \sum_{n=0}^{\infty} (-1)^n \frac{(-1)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{2n+1} = \sum_{n=0}^{\infty} \frac{1}{2n+1}$$

Use the Integral Test for the above $\left(f(x) = \frac{1}{2x+1}\right)$ is continuous, positive, and decreasing

$$\int_0^{\infty} \frac{1}{2x+1} dx = \left[\frac{1}{2}\ln|2x+1|\right]_0^{\infty} \Rightarrow Diverges \Rightarrow Series\ Diverges$$

Example 3.141: Natural Logarithm

Determine the interval of convergence of the Taylor series:

$$\ln(x+1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} -$$

Use the ratio test:

$$\lim_{n\to\infty}\left|a_{n+1}\times\frac{1}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\times\frac{n}{x^n}\right|=\lim_{n\to\infty}\left|\frac{xn}{n+1}\right|=\lim_{n\to\infty}\left|\frac{x}{1+\frac{1}{n}}\right|=|x|$$

$$Series\ Converges:\ -1<x<1$$
$$Series\ Diverges: x<-1\ OR\ x>1$$
$$Inconclusive: x\in\{-1,1\}$$

Check Endpoints

$$x=1: 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\Rightarrow Alternating\ Harmonic\ Series\Rightarrow Converges$$
$$x=-1:\ -1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\cdots\Rightarrow-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)\Rightarrow Harmonic\ Series\Rightarrow Diverges$$

Hence, the interval of convergence of the power series is:

$$-1<x\le 1$$

Example 3.142: Natural Logarithm

Determine the interval of convergence of the Taylor series for $\ln x$ centered at $x=1$ given below:

$$\ln x=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\cdots$$

Use the ratio test:

$$\lim_{n\to\infty}\left|a_{n+1}\times\frac{1}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{n+1}\times\frac{n}{(x-1)^n}\right|=\lim_{n\to\infty}\left|\frac{(x-1)n}{n+1}\right|=|x-1|$$

Convergence requires:

$$|x-1|<1\Rightarrow-1<x-1<1\Rightarrow 0<x<2$$

Check the endpoints:

$$x=2: 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\Rightarrow Alternating\ Harmonic\ Series\Rightarrow Converges$$
$$x=0: -1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\cdots\Rightarrow Negative\ of\ Harmonic\ Series\Rightarrow Diverges$$

Hence, the series converges when the above condition is met.

Example 3.143

Check the convergence of the series

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)=x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots+\frac{x^{2n+1}}{2n+1}+\cdots$$

Use the ratio test:

$$\lim_{n\to\infty}\left|a_{n+1}\times\frac{1}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{2n+3}}{2n+3}\times\frac{2n+1}{x^{2n+1}}\right|=\lim_{n\to\infty}\left|\frac{x^2(2n+1)}{2n+3}\right|=|x^2|=x^2<1$$

$$-1<x<1$$

Check the endpoints:

$$x=\pm1: a_n=\frac{\pm1}{2n+1}\Rightarrow Integral\ Test: \int_1^\infty\frac{\pm dx}{2x+1}=\pm\left[\frac{1}{2}\ln|2x+1|\right]_1^\infty\Rightarrow Diverges$$

C. Special Cases

3.144: Converges Everywhere

$R = \infty$: This means that the series converges everywhere

3.145: Converges Only at $x = a$

$$R = 0$$

- The power series has only one term.
- It gives the value of the function at that point.

3.10 Further Topics

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