Sets

05 June 2025

Revision: 56

Aziz Manva

azizmanva@gmail.com

Table of contents

1. Set Theory

1.1 Set Notation and Properties

A. Definition and Basics

1.1: Definition of Sets
A set is a collection of well-defined objects.

Example 1.2
Decide which of the following is a set:
A. Tall girls in Mumbai.
B. Girls in Mumbai taller than 5 feet 2 inches.
C. Expensive restaurants in Mumbai.

$$A\ is\ not\ a\ set\ because\ tall\ can\ be\ perceived\ differently.$$
$$B\ is\ a\ set.$$

1.3: Elements of a set
The objects which make up a set are called its elements, or also its members.

Example 1.4
What are the elements of the set $A = \{a, b, c\}$

There are three elements, which are a, b and c.

1.5: Roster Notation
In roster notation, we write the elements of the set within curly braces.

Example 1.6
Write the set with elements 1, 2, and 3 in roster notation. Call the set B.

$$B = \{1, 2, 3\}$$

Example 1.7
x is an integer that is greater than -3 and less than 4. Let X represent the set of values that x can take. Write X in roster notation.

$$X = \{-2, -1, 0, 1, 2, 3\}$$

1.8: Ellipsis
Three dots are used to indicate continuation to indicate continuation of a series.
has ten elements in it – the ten numbers from 1 to 10.

Dots lets us write larger sets without writing them all out.

Example 1.9
Write $\{1,2,3,4,5,6,7,8,9,10\}$ using ellipsis.

$$\{1, 2, \dots, 10\}$$

Example 1.10
Write the set of natural numbers from 1 to 500 using roster notation.

Writing them all out would be very time-consuming. We use ellipsis:

$$\{1, 2, \dots, 500\}$$

B. Properties of Sets

1.11: Equal Sets
If two sets have the same elements, they are called equal sets.

Example 1.12
Are the sets $H = \{A, B, C\}$ and $Z = \{A, B, C\}$ equal?

Yes

1.13: Repetition of Elements
Elements are not repeated in a set – they are written only once.

Example 1.14
Are the sets $\{1, 2, 3, 2\}$ and $\{1, 2, 3\}$ equal?

Yes

1.15: Order
Order in which elements are written is not important in a set.

Example 1.16
Are the sets $\{1, 2, 3\}$ and $\{1, 3, 2\}$ equal?

Since order is not important in a set, the two sets are equal:

$$\{1, 2, 3\} = \{1, 3, 2\}$$

1.17: Set Builder Notation
Here, a rule to decide the members of the set is written after a colon.

Example 1.18
Convert $\{1, 2, 3\}$ into set builder notation.

$$A = \{x : x \text{ } is \text{ } a \text{ } natural \text{ } number < 4\}$$

Example 1.19
List the sets below in roster form:

A. $\{x: x \text{ is a natural number} \le 4\}$
B. $\{x: x \text{ is a natural number} > 8\}$

C. $\{x: x \text{ is a natural number} \geq 8\}$

$$\{1, 2, 3, 4\}$$
$$\{9, 10, 11, \dots\}$$
$$8, 9, 10, \dots\}$$

C. Special Sets

Some sets are important enough for them to have names.

1.20: Universal Set

The set of all elements under consideration is called the Universal set.

1.21: Null Set / Empty Set

A set with no elements is called a null set.

ϕ, { } are ways of writing a null set
ϕ is the Greek Letter *Phi*.

Note: This is different from the Greek Letter π, which you might have already seen in Geometry.

1.22: Singleton Set

A set with only one element in it is called a singleton set.

Examples
$A = \{3\}$
$H = \{x: x \text{ is a prime number between 50 and 58}\}$

1.23: Finite and Infinite Sets

A set with a finite number of elements is called a finite set.
A set with an infinite number of elements is called an infinite set.

$$10^{15} = 1\ quadrillion$$
$$10^{18} = 1\ quintillion$$

Example 1.24

Classify the following sets as finite or infinite.

A. $H = \{x: x \text{ is a prime number between 50 and 100}\}$
B. $H = \{x: x \text{ is a prime number}\}$

1.25: Disjoint or Mutually Exclusive Sets

Are sets which have no common elements

It is important to distinguish between a large (but finite) set, and an infinite set.

Infinity (Advanced)
∞
The concept of infinity is related to sets. There are many sets with infinite elements.
Infinity does not follow the usual rules.

For example:

Addition		
$3 + 2 = 5$	$X + 2 = X + 2$	$\infty + 2 = \infty$

D. Cardinality

1.26: Cardinality
Cardinality is the number of elements of a set
Cardinality is written by using a small n before the name of the set.

Example 1.27
What is the cardinality of the set $A = \{1, 2, 3\}$?

$$n(A) = n\{1,2,3\} = 3$$

Example 1.28
Find the cardinality of each set.

A. $H = \{1, 2, \ldots, 12, 13\}$
B. $I = \{12, 13, \ldots 27, 28\}$
C. $J = \{32, 34, \ldots 76, 78\}$
D. $K = \{57, 59, \ldots 113, 115\}$

A. $H = \{1, 2, \ldots 12, 13\}$ has cardinality 13.
$$n(H) = 13$$

$$\{12, 13, \ldots 27, 28\} = \{\mathbf{1, 2, \ldots, 11}, 12, 13, \ldots 27, 28\} = 28 - 11 = 17$$

B. $n(I) = 28 - 12 + 1 = 17$
C. $n(J)$
$= \text{Cardinality of } \{16, 17, \ldots 38, 39\}$
$= 39 - 16 + 1 = 24$
D. $n(K)$
$= \text{Cardinality of } \{56, 58, \ldots 112, 114\}$
$= \text{Cardinality of } \{28, 29, \ldots 56, 57\}$
$= 57 - 28 + 1 = 30$

E. Belongs To

1.29: Belongs To
The symbol

$$\in$$

Which means "belongs to" is used to indicate that an element is a member of a set.

1.30: Does Not Belong To
The symbol

$$\notin$$

Which means "does not belong to" is used to indicate that an element is not a member of a set.

Example 1.31

$$X = \{x: x \text{ is a two digit odd number }\}$$
$$P = \{x: x \text{ is a prime number}\}$$

Decide if each statement below is true or false:

A. $4 \in P$
B. $2 \in P$
C. $1 \in P$
D. $7 \in X$
E. $9 \notin P$
F. $5 \notin X$

Type equation here.

F. Subsets and Supersets

1.32: Subsets

If every element of the set X is also an element of the set Y, then X is a subset of Y.

$$X \text{ is a subset of } Y \Leftrightarrow X \subseteq Y$$

1.33: Proper Subsets

If X is a subset of Y, and there is at least one element in Y which is not in X, then X is a proper subset of Y.

$$X \text{ is a proper subset of } Y \Leftrightarrow X \subset Y$$

1.34: Superset

If every element of the set X is also an element of the set Y, then Y is a superset of X.

$$Y \text{ is a superset of } X \Leftrightarrow Y \supseteq X$$

1.35: Proper Superset

If Y is a superset of X, and there is at least one element in Y which is not in X, then Y is a proper superset of X.

$$Y \text{ is a superset of } X \Leftrightarrow Y \supset X$$

Example 1.36

Consider the sets:

$$U = \text{Universal Set} = \{X: 1 \le X \le 10, \quad X \in \mathbb{N}\}$$
$$A = \{1,3,5\}$$
$$B = \{2,3,5\}$$
$$C = \{5,3,1\}$$
$$D = \{1,2,3,5\}$$

A. Is A a subset of D?
B. Is D a subset of A?
C. Is A a subset of C?
D. Is C a subset of A?
E. Is A a proper subset of D?
F. Is D a proper subset of A?
G. Is A a superset of D?
H. Is D a superset of A?
I. Is A a superset of C?

J. Is C a superset of A?
K. Is A a proper superset of C?

A = {x: x is a vowel in the English alphabet} = {A, E, I, O, U} has cardinality 5.
B= {y: y is a letter in the English alphabet} = {A, B, C,. . . , X, Y, Z} has cardinality 26.

Note that every member of set A is also a member of set B.
Hence, A is called a subset of B.
And B is called a superset of A.
Also, note that B has some elements which are not present in A.
Hence, A is called a proper subset of B.
And B is called a proper superset of A.

Definition - Subsets

-

If a set B has at least one element not in its subset A

- A is a proper subset of B.

1-I

Consider the sets {A: 1, 2, 3}, {B: 1, 2, 3, 4} and {C: 1, 2, 3, 4}
Q1: Identify subset and proper subset relationships among the sets above.
S1: A is a subset of B and C
A is also a proper subset of B
B is a subset of C, and C is a subset of B
Q2: State the cardinality of each set
S2: A, B and C have cardinality 3, 4 and 4 respectively.

1.2 Classifying Numbers

A. Natural Numbers / Counting Numbers

Natural numbers are numbers like

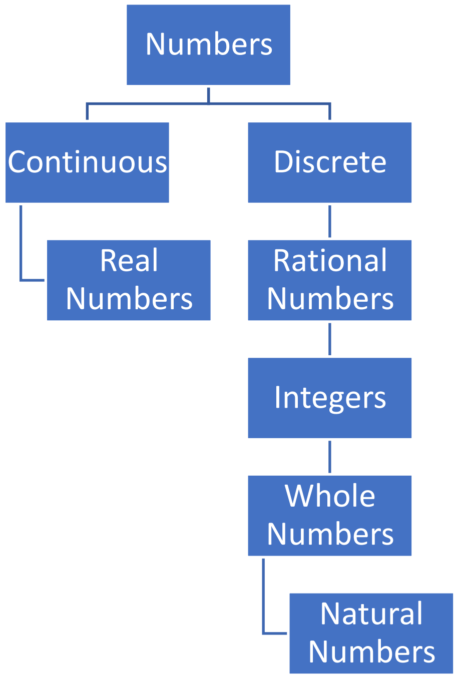
$$\mathbb{N} = \{1,2,3, \dots\}$$

Where:

$$\mathbb{N}\ is\ the\ name\ for\ the\ set\ of\ Natural\ Numbers$$

Think of adding 1 to any natural number, howsoever large. Since this is always possible, the number of natural numbers is infinite.

B. Whole Numbers



Adding zero to the natural numbers gives us the whole numbers.
While it may seem very simple:

- Zero adds properties that no natural number has. Specifically, zero satisfies the additive identity:
$$x + a = x \Rightarrow a = 0$$
The kind of equation above is only satisfied by zero.

Hence, whole numbers are numbers like
$$\mathbb{W} = \{0,1,2,3,\dots\}$$
Where:
$$\mathbb{W}\ is\ the\ name\ for\ the\ set\ of\ Whole\ Numbers$$

C. Integers

If you add the negative version of natural numbers to the whole numbers, you get integers.
The negative of zero is zero. Hence, there is no need to add a negative of zero.

Hence, positive whole numbers, negated whole numbers and zero together make up the integers:
$$\mathbb{I} = \{\dots,-3,-2,-1,0,1,2,3,\dots\}$$
Where:
$$\mathbb{I}\ is\ the\ name\ for\ the\ set\ of\ Whole\ Numbers$$

But there is a small problem. We are going to use the small letter
$$i$$
For complex numbers when we reach there. And to avoid potential confusion, mathematicians prefer to use:
$$\mathbb{Z} = \{\dots,-3,-2,-1,0,1,2,3,\dots\}$$

However, $\mathbb{I}$ is still correct. Just not preferred.

D. Rational Numbers

So far, we have not looked at fractions. None of the numbers that we talked about could handle fractions. Now, we introduce numbers that are defined as fractions.

A rational number is a number

- that can be written in the form $\frac{p}{q}$ (or, in other words, as a fraction)

However, there are some important restrictions:

- p and q are integers
- $q \neq 0$, which is very important since we cannot divide by zero, ever.

We can write out this definition in terms of sets, more formally, like this:

1.37: Rational Numbers

The set $\mathbb{Q}$ represents rational numbers. They are of the form:
$$\mathbb{Q} = \left\{ x \underbrace{|}_{\substack{such\\that}} x = \frac{p}{q}, \qquad p,q \in \mathbb{Z}, \qquad q \neq 0 \right\}$$

Example 1.38
Show that $\sqrt{2}$ is irrational.

Assume to the contrary that $\sqrt{2}$ is rational.
Then for some p and q:

$$\sqrt{2}=\frac{p}{q}, \qquad p,q\in\mathbb{Z}, \qquad q\neq 0$$

Square both sides:

$$2=\frac{p^2}{q^2}$$

In any perfect square, the power of any prime number must be even.

Numerator is a perfect square.
Denominator is also a perfect square.

Therefore, the power of 2 in $\frac{p^2}{q^2}$ must be even.
But, on the LHS, the power of 2 is odd.
Contradiction.
Hence, our original assumption was wrong.

Hence,

$$\sqrt{2}\ is\ irrational$$

E. Irrational Numbers
Numbers that are not rational are irrational.

$$\mathbb{P}=\mathbb{Q}'=\{x|x\ is\ not\ rational\}$$

For example, the number π which you might have seen in the formula for the area of a circle

$$Area\ of\ a\ Circle=\pi r^2$$

Is irrational.
Note that

$$\pi\approx 3.14, \pi\approx\frac{22}{7}$$

But π is not actually equal to any of these numbers:

$$\pi\neq 3.14, \pi\neq\frac{22}{7}$$

In fact, π cannot be represented as a fraction, with integer numerators and denominators.
This is precisely why it is an irrational number.

F. Real Numbers
Real numbers combine rational numbers and irrational numbers.
We use the letter $\mathbb{R}$ for the set of real numbers.

$$\mathbb{R}=\underbrace{\mathbb{Q}}_{\substack{Rational\\Numbers}}\cup\underbrace{\mathbb{Q}'}_{\substack{Irrational\\Numbers}}$$

1.39: Real Numbers
Real numbers are any number on the real number line.

There is no easy way to define real numbers other than this definition.

G. Summary

$$Set\ of\ Natural\ Numbers = \mathbb{N} = \{1, 2, 3, 4 \ldots\}$$
$$Set\ of\ Whole\ Numbers\ = \mathbb{W}\ = \{0, 1, 2, 3, 4 \ldots\}$$
$$Set\ of\ Integers\ = \mathbb{Z}\ = \{\ldots, -3\ , -2, -1, 0, 1, 2, 3, \ldots\}$$
$$Rational\ Numbers\ = \mathbb{Q} = \left\{x: x \in \frac{p}{q}, \qquad p, q \in \mathbb{Z}, q \neq 0\right\}$$
$$Irrational\ Numbers\ = \mathbb{Q}' = \mathbb{P}$$
$$Real\ Numbers = \mathbb{R} = \underbrace{\mathbb{Q}}_{\substack{Rational\\ Numbers}} \cup \underbrace{\mathbb{Q}'}_{\substack{Irrational\\ Numbers}}$$

H. Sets and Subsets

N is a subset of W or

$$\mathbb{N} \subset \mathbb{W}$$

W is a subset of Z or

$$\mathbb{W} \subset \mathbb{Z}$$

Z is a subset of Q or

$$\mathbb{Z} \subset \mathbb{Q}$$

Q is a subset of R or

$$\mathbb{Q} \subset \mathbb{R}$$

Combine the above:

$$\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Q is a superset of Z or $Q \supset Z$
Z is a superset of W or Z $\supset W$
W is a superset of N or $W\ \supset N$
Combine the above:

$$\mathbb{R} \supset \mathbb{Q} \supset \mathbb{Z} \supset \mathbb{W} \supset \mathbb{N}$$

I. Discrete and Continuous Values

The domain of a function can be of two types: discrete and continuous.

Values that are meaningful only in jumps are called discrete. Some values can take only whole numbers. Some can take integers. Some can take rational numbers. These are all discrete values.

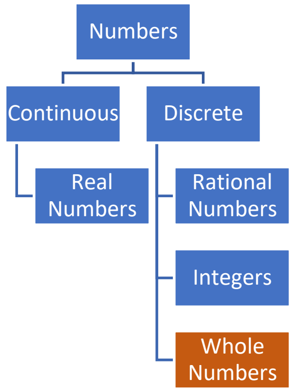
Values that can be any number on the real number line are called continuous.

Concept 1.40

Classify the following as discrete or continuous. If discrete, then classify as whole number, integer or rational

A. No. of Olympic Contestants in a sport
B. Money expressed in Rupees, to two decimal places

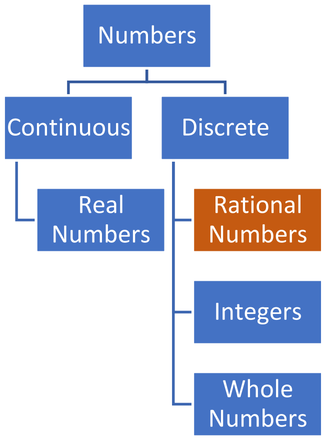
Part A



- Must be a whole number such as 42 or 5
- Cannot be fractional or decimal: $\frac{3}{5} = 0.6$ contestants does not make sense.
- Cannot be negative. -3 *contestants* in a sport does not make sense

Part B

- Is a discrete value
- Can take 33, or take 13.05
- But cannot take 23.001
- Hence, it is not continuous



1.41: Set Notation for Discrete Values

Discrete values are represented using set notation.

Concept 1.42

Write the odd numbers from 13 to 53 in set notation:

Roster Form

We can write all the numbers in a set, as below, and it would be correct:

$$\{13, 15,17,19,21,23,25,27,29,31,33,35,37,39,41,43,45,47,49,51,53\}$$

Roster Form with Ellipsis

But this is a really long way of writing it. So, a shorter way is to

- Write enough numbers that the pattern is clear
- Put three dots to indicate that the pattern continues
- Put the ending number so that we know where to stop

And this can be done as:

$$\{13,15, \dots ,51,53\}$$

The three dots … are called an ellipsis.

Set Builder Notation

We can also write a rule that the numbers in the set must satisfy, and this is given below:

$$\left\{ x \underbrace{|}_{\substack{such\\that}} 13 \le x \le 53, \quad x \underbrace{\in}_{\substack{belongs\\to}} \underbrace{\mathbb{Z}}_{\substack{Set\ of\\Integers}} \right\}$$

J. Rational Values

Concept Example 1.43

A bank calculates whether an account is in overdraft (negative balance) by using the function $f(b)$, where b is the balance in dollars. The bank system shows the value dollars and whole cents, which is used as in input for the function.

The minimum value that a bank account can have is 1 million dollars overdraft, while the maximum value of a single bank account is five million dollars.

Give the domain of $f(b)$.

Roster Form with Ellipsis

$$\{-1000000, -999999.99, \dots, 5000000\}$$

Set Builder Form

$$\left\{x \mid x = \frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0\right\}$$

We can use the definition of rational numbers to write the above set in set builder form, by using a similar style:

$$\left\{x \mid \frac{p}{100}, -100{,}000{,}000 \le p \le 500{,}000{,}000, p \in \mathbb{Z}\right\}$$

K. Continuous Values

Values like temperature, time, height of a person are continuous values. These values do not have restrictions, in that they can take all values on the number line.

For example, consider a time period between 10.00 am to 11.00 am. It will take all values on the number line between 10 am and 11 am.

Concept 1.44

In a cold day in Washington, it is 0° Celsius at 8 am in the morning. By 9 am, the temperate has increased to 10° Celsius. Was there some point of time between 8 am and 9 am, that the temperature was π° Celsius?

Yes. There has to be some point of time between 8 am and 9 am that the temperature was π° Celsius.
The temperature cannot go from 0 to 10, without passing through the value π.

Concept Example 1.45

$P(h)$ gives the air pressure at a height h feet above sea level in millibars. The height can range from 100 feet above sea level to 4000 feet above sea level. State the domain of $P(h)$.

$$[100, 4000]$$

L. Interval to Inequality Notation (Continuous Values)

Continuous values are represented using sets or intervals on the number line. In general, they are referred to as interval notation (and not set notation).

If the endpoints of an interval are included, then we use square brackets.
If the endpoints of the interval are not included, then we use round brackets.

Concept 1.46

Explain the following verbally. Also, write it in interval notation

A. $3 < x < 5$
B. $3 \le x \le 5$
C. $3 \le x < 5$
D. $3 < x \le 5$

$$3 < x < 5 \Leftrightarrow (3,5)$$
$$3 \le x \le 5 \Leftrightarrow [3,5]$$
$$3 \le x < 5 \Leftrightarrow [3,5)$$
$$3 < x \le 5 \Leftrightarrow (3,5]$$

M. Choosing between Discrete and Continuous Notations

Example 1.47

Shyam wants to purchase two chocolates for each student in his class for his birthday (including himself). Let s be the number of students present in the class of 20 students (including Shyam).

	Function	Domain
I	$f(s) = 2s$	$s \in (1,20)$
II	$f(s) = 2s + 1$	$s \in [1,20]$
III	$f(s) = 2s + 2$	$s \in \{1,2, \dots ,20\}$
IV	$f(s) = 2s - 1$	$s \in \{0,1, \dots ,19\}$

You need to write all valid pairs comprising a function and a domain from the table above. Use the row number to identify the functions and the domains.

Answer

The valid pairs are:

$$I \text{ and } III$$
$$III \text{ and } IV$$

1.3 Operations on Sets

A. Intersection and Union

1.48: Intersection of two Sets

The intersection of two sets X and Y is the set of the elements that belong to both X and Y.

$$\text{Intersection of } X \text{ and } Y = X \cap Y$$

1.49: Union of two Sets

The union of two sets X and Y is the set of the elements that belong to either X or Y.

$$\text{Union of } X \text{ and } Y = X \cup Y$$

Example 1.50

Let

$$X = \{x: -2 < x \le 4, \quad x \in \mathbb{Z}\}, \quad Y = \{x: 1 < x \le 5, \quad x \in \mathbb{Z}\}$$

List the elements of:

A. X

B. Y
C. $X \cup Y$
D. $X \cap Y$

$$X = \{-1,0,1,2,3,4\}$$
$$Y = \{2,3,4,5\}$$
$$X \cup Y = \{-1,0,1,2,3,4,5\}$$
$$X \cap Y = \{2,3,4\}$$

1.51: Union of Null Set

Union of null set with any set is always the other set

$$\phi \cup X = X$$

1.52: Intersection of Null Set

Intersection of null set with any set is always the null set:

$$\phi \cap X = \phi$$

1.53: Universal Set

The set of all objects under consideration in a particular scenario is called the universal set. The universal set can be:

- Explicitly defined or given in the question
- Understood from the context

1.54: Complement of a Set

All elements in the universal set, but not present in a set are the elements of the complement of the set.

$$Complement\ of\ X = X'$$
$$(Read: X\ complement\ OR\ X\ prime)$$

Example 1.55

Let the universal set be U = {1,2,3,4,5} and let the sets $X = \{1,2,3\}$ and $Y = \{2,4\}$ be defined. Find:

A. X'
B. Y'
C. $X' \cap Y'$
D. $(X \cap Y)$
E. $(X \cap Y)'$
F. $X' \cup Y'$

$$X' = \{4,5\}$$
$$Y' = \{1,3,5\}$$
$$X' \cap Y' = \{5\}$$
$$(X \cap Y) = \{2\}$$
$$(X \cap Y)' = \{1,3,4,5\}$$
$$X' \cup Y' = \{1,3,4,5\}$$

U = {A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z}
M = {Consonants}
N = {Letters that have at least one line of symmetry}

List the elements of:
A. M'
B. N'

1.56: Complement of Universal Set
The complement of the universal set is the null set
$$U' = \phi$$

1.57: Complement of Null Set
The complement of the null set is the universal set
$$\phi' = U$$

1.58: Union of Universal Set
Union of universal set with any set is always the Universal Set:
$$U \cup X = U$$

1.59: Intersection of Universal Set
Intersection of universal set with any set is always the other set
$$U \cap X = X$$

Challenge 1.60
Let Universal Set, U, be the set of Numbers from 1 to 25. Let
P be the set of primes.
O be the set of odd numbers.
E be the set of even numbers.
F be the set of factors of 60.
M be the set of multiples of 4.

State, in any appropriate form, the sets:

A. $P' \cap O'$
B. $P' \cap F'$
C. U'
D. $U' \cup (P' \cup O')$
E. $E' \cap M'$
F. $O' \cap E'$
G. $O' \cup E'$

$$P' = \{1,4,6,8,9,10,12,14,15,16,18,20,21,22,24,25\}$$
$$O' = \{2,4,6,8,10,12,14,16,18,20,22,24\}$$
$$F' = \{7,8,9,11,13,14,16,17,18,19,21,22,23,24,25\}$$
$$E' = \{1,3,5,7,9,11,13,15,17,19,21,23,25\}$$
$$M' = \{1,2,3,5,6,7,9,10,11,13,14,15,17,18,19,21,22,23,25\}$$

Part A
$$P' \cap O' = \{4,6,8,10,12,14,16,18,20,22,24\} = O' - \{2\}$$

Part B

$$\{8,9,14,16,18,21,22,24,25\}$$

Part C and D

The complement of the universal set is the null set.

$$Part\ C: \phi$$

The

$$P' \cup O' = \{1,2,4,6,8,9,10,12,14,15,16,18,20,21,22,24,25\}$$
$$\phi \cup (P' \cup O') = P' \cup O'$$

Part E

$$E' \cap M' = \{1,3,5,7,9,11,13,15,17,19,21,23,25\}$$

Example 1.61

Intersection of circle and line

Three cases

Example 1.62

Intersection of parabola and line

Three cases

B. AND, OR Notation

$$\boldsymbol{Union}: Elements\ in\ A\ \boldsymbol{OR}\ B \Leftrightarrow A \cup B$$
$$\boldsymbol{Intersection}: Elements\ in\ A\ \boldsymbol{AND}\ B \Leftrightarrow A \cap B$$

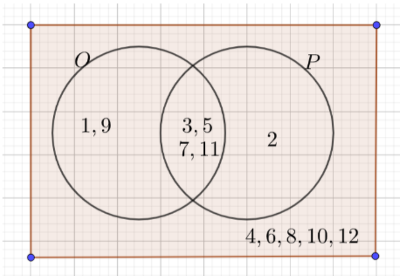
C. Venn Diagrams

Example 1.63

Represent the sets below on a Venn Diagram.

$$O = \{x: x\ is\ an\ odd\ number\ from\ 1\ to\ 12\}$$
$$P = \{x: x\ is\ a\ prime\ number\ from\ 1\ to\ 12\}$$
$$U = \{x: x\ is\ a\ whole\ numbers\ from\ 1\ to\ 12\}$$

$$O = \{1,3,5,7,9,11\}$$
$$P = \{2,3,5,7,11\}$$



Example 1.64

P = Primes from that are one more than a multiple of 4 and less than 50.
M = Numbers that are one less than a multiple of 6, and less than 50.

Represent the above sets on a Venn Diagram (without the Universal Set,)

Example 1.65

S = Two Digit Numbers, the sum of whose digits is a single digit prime number.
P = Two Digit Numbers, the product of whose digits is a single digit prime number
D = Two Digit Numbers, the difference of whose digits is an even prime number

Represent the above sets on a Venn Diagram.

$$S = \{20,11,30,21,12,50,41,32,23,41,70,61,52,43,34,25,16\}$$
$$P = \{21,12,31,13,51,15,17,71\}$$
$$D = \{13,20,24,35,46,57,68,79,31,42,53,64,75,86,97\}$$

Example 1.66

T = Numbers one more than a multiple of 3.
F = Numbers one more than a multiple of 4.
V = Numbers one more than a multiple of 5.

Universal set = Numbers from 40 to 70.

Example 1.67

$$A = Factors\ of\ 48$$
$$B = Factors\ of\ 60$$
$$Universal\ Set = U = \{Factors\ of\ 48\} \cup \{Factors\ of\ 60\} \cup \{Factors\ of\ 72\}$$

Represent the above sets on a Venn Diagram.

$$A = \{1,2,3,4,6,8,12,16,24,48\}$$
$$B = \{1,2,3,4,5,6,10,12,20,30,60\}$$
$$U = \{1,2,3,4,5,6,8,9,10,12,16,18,20,30,36,48,60,72\}$$

D. Shading Venn Diagrams

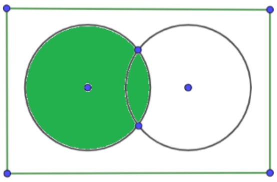
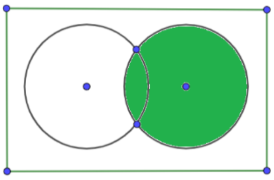
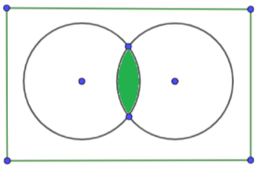
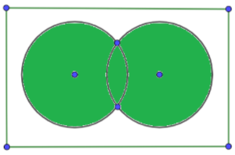
Example 1.68

Consider the Venn Diagram for two sets X and Y.
Shade:

A. X
B. Y
C. $X \cap Y$
D. $X \cup Y$
E. X'
F. Y'

Example 1.69

- First Diagram: A
- Second Diagram: B
- Third Diagram: $A \cap B \Leftrightarrow A\ AND\ B$
- Third Diagram: $A \cup B \Leftrightarrow A\ OR\ B$

Example 1.70

Consider the Venn Diagram for the intersection of three sets X, Y, Z

Shade:

A. X
B. Y
C. Z
D. $X \cap Y$
E. $Y \cap Z$
F. $X \cap Z$
G. $X \cap Y \cap Z$
H. X'
I. Y'
J. Z'
K. $X \cap (Y \cup Z)$
L. $Y \cap (X \cup Z)$
M. $Z \cap (X \cup Y)$

E. Algebra with Venn Diagrams

1.71: Complement of a Complement is the set itself

$$(A')' = A$$

Example 1.72

$$(A' \cup B')'$$

$$(A' \cup B')' = (A')' \cap (B')' = A \cap B$$

Example 1.73

$$(A \cup B)'$$

$$(A \cup B)' = A' \cap B'$$

Example 1.74

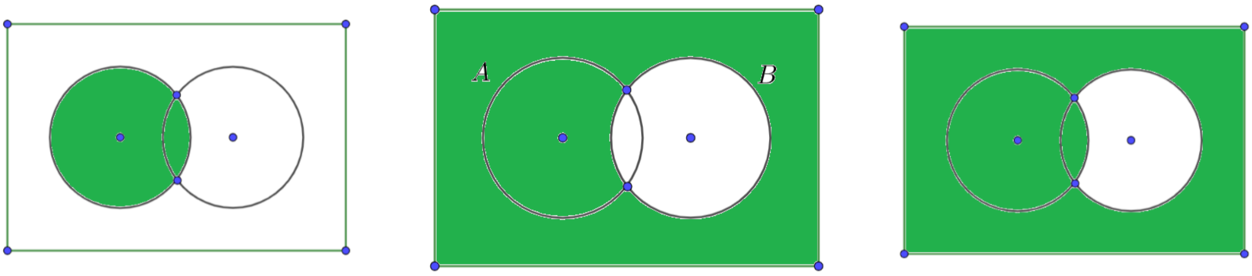
$$(A \cup B')'$$

$$(A \cup B')' = A' \cap B$$

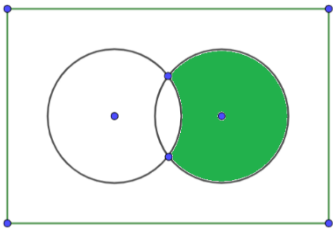
From left to right

- First Diagram: A

- Second Diagram: B’
- Third Diagram: $A \cup B'$



And now to find the final answer, we can graph the complement of the third diagram:



2. Counting with Sets

2.1 Single Set: Lists

A. Introduction

2.1: Lists

A single set has n elements. Sets are not ordered. But, we can order the elements of a set using a logical process to count the elements more easily.

Example 2.2

Count the number of positive integers less than 100.

We can list this out:

$$1,2, \dots ,99 \Rightarrow 99\ Numbers$$

2.3: Counting "Between"

When counting "between" two numbers, neither start number nor the end number is included.

Example 2.4

How many positive integers are between 1 and 100

$$1,2,3, \dots ,99,100 \Rightarrow 100\ Numbers$$
$$2,3, \dots ,99 \Rightarrow 98\ Numbers$$

Example 2.5

What is the number of positive integers between -4 and 14?

List out the integers:

$$-4,-3, \dots ,0,1,2 \dots ,14$$

Keep only positive integers:

$$1,2, \dots ,13 \rightarrow 13\ numbers$$

2.6: Counting "From" to "To"

When counting "from" to "to", the starting number and the ending number are both included.

Example 2.7

What is the number of even numbers from 10 to 100.

$$10,12, \dots ,100$$

Subtract 10 from each number:

$$0,2, \dots ,90$$

Divide each number by 2:

$$0,1, \dots ,45$$

Add 1 to each number:

$$1,2, \dots ,46$$

Example 2.8
What is the number of negative integers from −32 to 12?

$$n\{-32,-31,\dots\mathbf{0},\mathbf{1},\mathbf{2},\dots\mathbf{11},\mathbf{12}\}=n\{-32,-31,\dots,-2,-1\}=32$$

Example 2.9
What is the number of non-negative integers from -32 to 12?

$$n\{\mathbf{-32},\mathbf{-31},\dots\mathbf{-1},0,1,2,\dots 11,12\}=n\{0,1,2,\dots,12\}=13$$

Example 2.10
What is the positive difference in the number of elements of X and Y if:

A. X is the set consisting of all positive integers less than or equal to n, where n is some arbitrary but fixed natural number.
B. Y is the set consisting of all nonnegative integers less than or equal to n, where n is some arbitrary but fixed natural number

$$X=\underbrace{\{1,2,3\dots,n\}}_{\text{First n positive integers}},Y=\underbrace{\{0,1,2,3,\dots n\}}_{\text{First n+1 nonnegative Integers}}\Rightarrow(n+1)-n=1$$

B. Number of elements in a list
To count the numbers from 12 to 74, we can add a few numbers and then remove the ones we don't want:

$$\{12,13,14,\dots,73,74\}\Leftrightarrow n\left\{\underbrace{\mathbf{1},\mathbf{2},\mathbf{3},\dots\mathbf{10},\mathbf{11}}_{\text{Not Counted}},\underbrace{12,13,14,\dots,73,74}_{\text{Counted}}\right\}\Leftrightarrow 74-11=63$$

Since we know the first and the last element in the list when we start with the question, it becomes easier to write the formula in terms of those numbers:

$$\underbrace{74}_{\text{End Number}}-\underbrace{12}_{\text{Start Number}}+\underbrace{1}_{\text{Add one more}}=63$$

You want to count the numbers from 12 to 74.

- Imagine you have people standing in a row, with one person, each holding the numbers from 12 to 74.
- Suppose, you replace the number
  - ✓ 12 with $12-11=1$
  - ✓ 13 with $13-11=2$
  - ✓ The number $14-11=3$
  - ✓ .
  - ✓ .
  - ✓ .
  - ✓ The number 74 with $74-11=63$

	Start Number				End Number	
	12	13	14	...	74	***x* numbers**
Subtract 11 from each number	11	11	11	...	11	
	1	2	3	...	$74-11$	***x* numbers**

2.11: Bijection Principle
Subtracting 11 from the list above changes the values of the numbers, but not how many numbers it contains. This same idea is used in the proof below.

- A bijection is a one to one mapping between the elements of one set and the elements of another set.
- Each element of the first set is mapped to exactly one element of the second set.
- Hence, the number of elements in the two sets is exactly the same. In more technical terms, the two sets have the same cardinality.

2.12: Number of elements in a list starting from a and ending at b

The number of numbers in a list of integers starting from a, and ending at b is

$$\underbrace{b}_{\text{End Number}} - \underbrace{a}_{\text{Start Number}} + 1$$

You have a list

$$\{a, a+1, a+2, \dots, b\}$$

Where a is an integer.
Find the number of numbers in the list.

	Start Number				End Number
Subtract $a-1$ *from each number in the list*	a	$a+1$	$a+2$	...	b
	$a-1$	$a-1$	$a-1$	...	$a-1$
	1	2	3	...	$\underbrace{b-a+1}_{\text{Total Number of Numbers in the list}}$

$$a-(a-1) = a-a+1 = 1$$
$$a+1-(a-1) = a+1-a+1 = 2$$

$$b-(a-1) = b-a+1$$

Example 2.13

What is the number of positive integers from 7 to 82?

We want to find the number of numbers in the list:

$$n\{7, 8, \dots 82\}$$

We add a few numbers to make it complete, and then see which ones we don't want.

$$\underbrace{n\{1,2,3,\dots,6,7,8,9,\dots,81,82\}}_{82} - \underbrace{n\{1,2,\dots 7\}}_{7}$$

This process also eliminates one number that we do want (which is 7), so we add it back:

$$= \underbrace{n\{1,2,3,\dots,6,7,8,9,\dots,81,82\}}_{82} - \underbrace{n\{1,2,\dots 7\}}_{7} + 1 = 82 - 7 + 1 = 76$$

Shortcut

$$\underbrace{82}_{\text{End Number}} - \underbrace{7}_{\text{Start Number}} + 1 = 76$$

Example 2.14

Find the number of

A. Positive Integers from 8 to 54
B. Positive Integers between 8 and 54

Part A
Apply the formula

$$\underbrace{54}_{\substack{End\\Number}} - \underbrace{8}_{\substack{Start\\Number}} + 1 = 47$$

Part B
Using the previous answer
The number of numbers which is between 8 and 54 should be two less than the numbers from 8 to 54, and hence it should be:

$$47 - 2 = 45$$

Direct Calculation
For direct calculation, it is best to convert the $between$ into $from - to$:

$$Betweeen\ 8\ and\ 54 \Leftrightarrow From\ 9\ to\ 53$$

And then we can use the formula:

$$53 - 9 + 1 = 45$$

Example 2.15
Find the number of
A. Negative Integers from -72 to -33
B. Negative Integers between -72 and -33

Part A

$$-33 - (-72) + 1 = -33 + 72 + 1 = -33 + 73 = 40$$

Part B
Convert the $between$ into $from - to$:

$$Betweeen - 72\ and - 33 \Leftrightarrow From - 71\ to - 34$$

And then we can use the formula:

$$-34 - (-71) + 1 = -34 + 71 + 1 = -34 + 72 = 38$$

Example 2.16
Find the number of positive integers between -72 and -33

The range of integers which has been given is all negative. Hence, the number of positive integers in this range is

$$Zero$$

Example 2.17
Tom counted from the largest two-digit number to the smallest four-digit number (including both). How many numbers did he count in all?

$$1000 - 99 + 1 = 902$$

Review 2.18: Counting Negative Numbers, From and Between
Tweedledum and Tweedledee were counting integers. Tweedledum counted X numbers from -12 to 48. Tweedledee counted Y numbers between -13 and 49. They did not skip any number while counting. What is XY (where XY refers to the product of X and Y)?

Formula

$$X = \underbrace{48}_{\textit{Start Number}} - \underbrace{(-12)}_{\textit{End Number}} + 1 = 61 \Rightarrow XY = 61^2 = 3721$$

Logic

We want the numbers

$$\{-12,-11,-10,\dots,-1,0,1,..47,48\}$$
$$\underbrace{12}_{\textit{Negative Numbers}} + \underbrace{1}_{\textit{Zero}} + \underbrace{48}_{\textit{Positive Numbers}} = 61$$

Tweedledee

For him, we want the numbers

$$Between - 13\ and\ 49$$

Which is the same as the numbers

$$from - 12\ to\ 48$$

And hence

$$Y = X = 61$$

C. Real Life Scenarios

Example 2.19: Packing Strawberies

Randhir is packing strawberries in a box, with seven strawberries to each box. He starts with the box numbered seven. He continues packing boxes numbered eight, nine, and so on. The last box he packs is the box numbered twenty eight. What is the number of strawberries that Randhir that he packs?

$$\underbrace{7}_{\substack{\textit{Strawberries}\\ \textit{per Box}}} \times \underbrace{(28 - 7 + 1)}_{\textit{No.of Boxes}} = 7 \times 22 = 154$$

Example 2.20: Counting Seats in Rows

If rows 9 through 21 of a theatre (with each row having 12 seats) are filled, then how many people are seated in those rows? (Note: *Rows 9 through 21* means rows *from* 9 *to* 21 – *including* 9 and 21).

$$\underbrace{(21 - 9 + 1)}_{\textit{No.of Rows}} \times \underbrace{12}_{\textit{Seats per Row}} = 13 \times 12 = 156$$

D. Counting Number of Numbers

Example 2.21: Counting Numbers

What is the number of:

A. Two-digit numbers
B. Three-digit numbers
C. n-digit Numbers

Parts A and B

$$\{10,11,12,\dots,99\} = \left\{\underbrace{\mathbf{1,2,3,\dots,9}}_{\textit{Don't Count}}, \underbrace{10,11,12,\dots.,99}_{\textit{Count}}\right\} = 99 - 10 + 1 = 90$$

$$\{100,101,102,\dots.,999\} = \left\{\underbrace{\mathbf{1,2,3,\dots,99}}_{\textit{Don't Count}}, \underbrace{100,101,102,\dots.,999}_{\textit{Count}}\right\} = 999 - 100 + 1 = 900$$

Part C

$$2\ Digit\ Numbers: 90 = 9 \times 10^1 = 9 \times 10^{2-1}$$
$$3\ Digit\ Numbers: 900 = 9 \times 10^2 = 9 \times 10^{3-1}$$
$$4\ Digit\ Numbers: 9000 = 9 \times 10^3 = 9 \times 10^{4-1}$$
$$n\ Digit\ Numbers: 9 \times 10^{n-1}$$

2.22: Number of n digit Numbers

The number of n digit numbers is

$$9 \times 10^{n-1}$$

Example 2.23: Counting Numbers

What is the difference between the number of six-digit natural numbers and the number of two-digit natural numbers?

$$\underbrace{9{,}00{,}000}_{\text{No.of 6-digit numbers}} - \underbrace{90}_{\text{No.of 2-digit numbers}} = 899{,}910$$

E. Counting Possible Totals

Example 2.24: Sum of Dice Rolls

Shane rolls two six-sided dice, with faces numbered from one to six, and adds the numbers that come up on the two dice. How many such distinct totals can he get?

$$\underbrace{\{1,2,3,4,5,6\}}_{First\ Die}, \underbrace{\{1,2,3,4,5,6\}}_{Second\ Die} \Rightarrow Min\ Sum = 1 + 1 = 2, Max\ Sum = 6 + 6 = 12$$
$$We\ can\ \textbf{achieve}\ all\ sums\ in\ between\ as\ well$$

We need to count the number of numbers from 12 to 2, which is:

$$12 - 2 + 1 = 11$$

	Total					
B.	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2						8
3						9
4						10
5						11
6						12

Example 2.25: Sum of Dice Rolls

Shirley rolls a six-sided dice, with faces numbered from one to six, and then rolls a ten-sided dice, with faces numbered from one to ten, and adds the numbers on the top face of the two dice. What is the number of distinct totals that she can get?

$$\underbrace{\{1,2,3,4,5,6\}}_{First\ Dice}, \underbrace{\{1,2,3, \dots ,9,10\}}_{Second\ Dice} \Rightarrow Min\ Sum = 1 + 1 = 2, Max\ Sum = 6 + 10 = 16$$
$$We\ can\ get\ all\ sums\ in\ between\ as\ well$$

We need to count the number of numbers from 2 to 16, which is:

$$16 - 2 + 1 = 15$$

1	1	$1 + 1 = 2$
2	2	
3	3	
4	4	
5	5	
6	6	
	7	
	8	
	9	
	10	$10 + 6 = 16$

Example 2.26

Shilpa rolls three six-sided dice, each with faces numbered one to six. What is the number of possible totals that she can get?

$$Max = 6 \times 3 = 18$$
$$Min = 1 \times 3 = 3$$
$$Range = 18 - 3 + 1 = 16$$

Challenge 2.27

Shilpi has four dice. The first dice is tetrahedral, with faces numbered from one to four. The second dice is octahedral, with faces numbered from one to eight. The third dice is dodecahedral, with faces numbered from one to twelve. The fourth dice is icosahedral, with faces numbered from one to twenty.
She picks anywhere from two to four dice, at random, from the four, and rolls them. The number of totals that she can get on the rolls is X. What is the difference between the minimum and maximum value of X.

The smallest number of totals is when she picks the tetrahedral and octahedral dice:
$$Min\ X = 12 - 2 + 1 = 11$$
If she picks all four dice:

$$Max\ X = 44 - 4 + 1 = 41$$
$$Diff = 41 - 11 = 30$$

Example 2.28

I have a spinner with the numbers 2,5 and 8 on it. I spin the spinner three times. What is the number of possible totals that I can get?

The minimum value that you can get is:
$$2 + 2 + 2 = 6$$

The maximum value that you can get is:
$$8 + 8 + 8 = 24$$

Notice that the numbers on the spinner have a difference of 3 between them:
$$8 - 5 = 5 - 2 = 3$$

Hence, every time you replace a smaller number with a larger number, you get a total which increases by 3. For example:

$$2+2+2=6$$
$$2+2+5=9$$
$$2+2+8=12$$
$$.$$
$$.$$
$$.$$
$$5+5+8=18$$
$$5+8+8=21$$
$$8+8+8=24$$

Hence, the possible totals are:

$$\{6,9,12,\dots,24\}$$
$$3\times\{2,3,4,\dots,8\}$$
$$3\times(\{1,2,3,\dots,7\}+1)$$
$$7\ Numbers$$

Example 2.29

Bag A has three chips labeled 1, 3, and 5. Bag B has three chips labeled 2, 4, and 6. If one chip is drawn from each bag, how many different values are possible for the sum of the two numbers on the chips? (AMC 8 2011/8)

3	$1+2$		
5	$1+4$	$3+2$	
7	$1+6$	$3+4$	$5+2$
9		$3+6$	$5+4$
11			$5+6$
5 Values			

$$O+E=O$$

Example 2.30

Bag A has four chips labeled 1, 3, and 5 and 7. Bag B has three chips labeled 2, 4, and 6 and 8. If one chip is drawn from each bag, how many different values are possible for the sum of the two numbers on the chips? (AMC 8 2011/8, Adapted)

3	$1+2$			
5	$1+4$	$3+2$		
7	$1+6$	$3+4$	$5+2$	
9	$1+8$	$3+6$	$5+4$	$7+2$
11		$3+8$	$5+6$	$7+4$
13			$5+8$	$7+6$
15				$7+8$
7 Values				

F. Counting on the Number Line

Example 2.31: Counting Natural Numbers on the Number Line

How many natural numbers lie between $\frac{5}{3}$ and 2π? **(AMC 8 2000/3)**

We want to find the natural numbers in the interval:

$$\left(\frac{5}{3}, 2\pi\right)$$

Write the numbers in a form which is easier to compare to integers. Write $\frac{5}{3}$ as a mixed number. Write the decimal value of 2π:

$$\left(1\frac{2}{3}, 6.28\right)$$

Now, it is easier to see that the smallest number that fit the condition is 2, and the largest number is 6:

$$n\{2, \dots .6\} \Rightarrow 6 - 2 + 1 = 5$$

Alternate Solution

Another way of arriving at the same solution using a little more mathematical notation is:

$$\underbrace{\lfloor 2\pi \rfloor}_{\text{Floor Function}} - \underbrace{\left\lceil 1\frac{2}{3} \right\rceil}_{\text{Ceiling Function}} + 1 = 6 - 2 + 1 = 5$$

The floor function gives us the first number that we will encounter on the left of 2π on the number line.
The ceiling functions gives the first number that we will encounter to the right of $1\frac{2}{3}$ on the number line.

Example 2.32: Counting Natural Numbers on the Number Line

The number $e \approx 2.71$ and $\pi \approx 3.14$. A frog starts on the real number line at π. At each hop it jumps to the natural number that is the nearest to its right. How many hops must it make before the number $3e$ lies to the left of the frog.

$$3e \approx 8.13$$
$$3.14 < x < 8.13 \Rightarrow 4,5,6,7,8,9 \Rightarrow 9 - 4 + 1 = 6$$

G. Multiple Lists

If we have multiple lists, each following a different set of rules, then we need to break the problem, and calculate the number of elements of each list separately.

Example 2.33: Counting Digits

Count the number of $\underbrace{digits}_{\text{not Numbers}}$ used to write the numbers from:

A. 1 to 9
B. 10 to 99
C. 100 to 999
D. 1 to 1000

Part A

We have nine numbers. Each number has one digit. Total number of digits is

$$1 \times 9 = 9$$

Part B

We have ninety numbers. Each number has two digits. Total number of digits is

$$2 \times 90 = 180$$

Part C

We have nine hundred numbers. Each number has three digits. Total number of digits is

$$3 \times 900 = 2700$$

Part D

$$= \underbrace{9}_{\substack{One\ Digit\\ Numbers}} + \underbrace{180}_{\substack{Two\ Digit\\ Numbers}} + \underbrace{2700}_{\substack{Three\ Digit\\ Numbers}} + \underbrace{4}_{\substack{Four\ Digit\\ Number}} = 2893$$

Example 2.34: Applying the Concept

Prekshak numbered houses from 35 to 784 using wooden digits bought at Rs. 2 per digit. Find the cost.

Prekshak numbered houses from

$$35, 36, \dots 99, 100, 101, \dots 784$$

We can classify the numbers into:

$$Two\ Digit\ Numbers: (35\ to\ 99) \Rightarrow 2\ digits\ in\ each\ number$$
$$Three\ Digit\ Numbers: (100\ to\ 784) \Rightarrow 3\ digits\ in\ each\ number$$

$$\underbrace{2}_{Digits} \times \underbrace{(99-35+1)}_{No.of\ Numbers} + \underbrace{3}_{Digits} \times \underbrace{(784-100+1)}_{No.of\ Numbers} = 2 \times 65 + 3 \times 685 = 130 + 2055 = 2185\ Digits$$

$$Cost = \underbrace{2}_{\substack{Cost\ per\\ Digit}} \times \underbrace{2185}_{No.of\ Digits} = \underbrace{4370}_{Total\ Cost}$$

Example 2.35

An auditorium with 20 rows of seats has 10 seats in the first row. Each successive row has one more seat than the previous row. If students taking an exam are permitted to sit in any row, but not next to another student in that row, then the maximum number of students that can be seated for an exam is (AMC 8 1991/13)

Seat No.	1	2	3	4	5	6	7	8	9	10	11	No. of Students
	S		S		S		S		S			$\frac{10}{2} = 5$
	S		S		S		S		S		S	$\frac{11}{2} = 6$

Note: $\frac{11}{2}$ is not 6, but in this case, we are rounding up the number.

Seats	10	11	12	13	14	.	.	.	28	29
Students	5	6	6	7	7	.	.	.	14	15

We need to add the following twenty numbers:

$$\mathbf{5} + \mathbf{6} + \mathbf{6} + \cdots + \mathbf{14} + \mathbf{14} + \mathbf{15}$$

We can make pairs, each adding up to twenty. Hence, we can make ten pairs, giving us a total of:

$$10 \times 20 = 200$$

Example 2.36

An auditorium with 30 rows of seats has 10 seats in the first row. Each successive row has one more seat than the previous row. If students taking an exam are permitted to sit in any row, but not next to another student in that row, what is the maximum number of students that can be seated for an exam? (MathCounts 2001 National Sprint)

Seat No.	1	2	3	4	5	6	7	8	9	10	11	No. of Students
	S		S		S		S		S			$\frac{10}{2}=5$
	S		S		S		S		S		S	$\frac{11}{2}=6$

Note: $\frac{11}{2}$ is not 6, but in this case, we are rounding up the number.

Seats	10	11	12	13	14	.	.	.	38	39
Students	5	6	6	7	7	.	.	.	19	20

We need to add the following twenty numbers:

$$5+6+6+\cdots+19+19+20$$

We can make pairs, each adding up to twenty five. Hence, we can make 15 pairs, giving us a total of:

$$15\times 25=375$$

H. Bijection to count Elements

Mapping each element of a set to an element of another set creates a bijection.
It is a very powerful tool in counting problems.

If you wish to count the elements of a set $A=\{a,b,c\ldots\}$, and it is difficult to count, you can map them to another set $B=\{1,2,3\ldots\}$, which is easier to count. If both the sets have the same number of elements (that is, the same cardinality), then counting the elements of one set is sufficient to find the number of elements in both sets.

$$A=\{a,b,c,\ldots.,z\}$$
$$B=\{1,2,3,\ldots.26\}$$

You can make pairs by mapping the first element of set A to the first element of set B:

$$(a,1),(b,2),(c,3),\ldots,(z,26)$$

This is also called 1-1 correspondence.

Example 2.37

Find the number of odd numbers from 1 to 75, not including either 1 or 75.

$$3,5,7,\ldots,73$$

Subtract 1 from each number in the list:

$$2,4,6,\ldots,72$$

Divide each number in the list by 2:

$$1,2,3,\ldots,36$$

Example 2.38

Timothy counted the odd numbers starting from 153 (including 153) and going up to 375 (including 375). How many numbers did he count?

$$153,155,\ldots,375$$

Add 1 to each number in the list:

$$154, 156, \dots ,376$$

Divide each number in the list by 2:

$$77,78, \dots ,188$$

Count:

$$188 - 77 + 1 = 112$$

Core Concept 2.39: Using Bijection

Find the number of elements in the list:

$$7.25, 7.75,8.25, \dots ,22.25,22.75$$

Shortcuts

$$\underbrace{7.25, 7.75}_{2\ 7's}, \underbrace{8.25,8.75}_{2\ 8's}, \underbrace{9.25,9.75}_{2\ 9's}, \dots \Rightarrow 2(22 - 7 + 1) = 2 \times 16 = 32$$

Bijection Method

Subtract 0.25 from each number. This changes the values, but does not change the number of elements:

$$7,7.5, 8,8.5, \dots ,22.5$$

Double each number to get rid of the fraction in every alternate number:

$$14,15, 16,17, \dots ,45$$

Apply the formula for counting lists:

$$45 - 14 + 1 = 32\ Numbers$$

Convert each term into a fraction, and then make the fraction improper:

$$\underbrace{7\frac{1}{4}, 7\frac{3}{4}, 8\frac{1}{4}, \dots ,22\frac{1}{4}, 22\frac{3}{4}}_{Mixed\ Numbers} \Leftrightarrow \underbrace{\frac{29}{4}, \frac{31}{4}, \frac{33}{4} \dots , \frac{89}{4}, \frac{91}{4}}_{Improper\ Fractions}$$

Multiply each term by 4, to get terms with a difference of 2. Since the terms are not divisible by 2, add 1 to each term to make it divisible by 2:

$$29,31,33, \dots ,89,91 \Rightarrow 30,32,34, \dots ,90,92$$

Divide each term by 2, and then subtract 14 from each term:

$$15,16,17, \dots ,45,46 \Rightarrow 1,2,3, \dots ,35,32$$

Example 2.40

Let

$$X = \{13,15, \dots . ,85\}, Y = \{22,24, \dots ,92\}$$

Take a number from Set X. Take a number Set Y. Add the two numbers so obtained to find Z. How many values can Z take?

$$Min = 13 + 22 = 35$$
$$Max = 85 + 92 = 177$$

$$O + E = O$$

All numbers that we can achieve are odd.
All odd numbers in the range 35 to 177 can be achieved.

Hence, we need to count numbers in the list:

$$n\{35,37, \dots ,177\}$$

Subtract 1 from each number in the list:

$$n\{34,36, \dots ,176\}$$

Divide each number by 2:

$$n\{17,18,\dots,88\} \Rightarrow 88-17+1=72$$

Example 2.41

A four-digit number can be made by repeating a two-digit number. For example, 1111 is made by repeating 11, and 1919 is made by repeating 19. How many such numbers are there between 2,000 and 10,000? **(Gauss 7 2020/21)**

$$2020,\quad 2121,\quad 2222,\quad \dots,\quad 9999$$

Once we choose the first two digits, the last two digits are automatically chosen. Hence, we only focus on counting the choices for the first two digits.

The smallest value that we can have for the first two digits is 20.
The largest value that we can have is 99
And we can have all numbers in between, giving us:

$$20,21,22,\dots,99 \Rightarrow 99-20+1=100-20=80\ Numbers$$

I. Converting numbers into the same form

If the numbers that needed to counted are not in the same format (decimal, percentage, fraction), then it is convenient to convert all numbers into the same form.
We see an example below

Example 2.42

Find the number of elements that are less than 100 in:

$$\frac{38}{5}, 8.2, \frac{44}{5}, 940\%, 10, \dots$$

Here, we need to find the last number in the list ourselves.

Method I:

- **Convert to Fractions with a Denominator of 5**
- **Use the bijection principle**

The numbers that we have are in different formats.

$$\underbrace{\frac{38}{5}}_{Fraction}, \underbrace{8.2}_{Decimal}, \frac{44}{5}, \underbrace{940\%}_{Percentage}, 10 \dots$$

Convert everything to fractions to see the pattern better:

$$\frac{38}{5}, \underbrace{\frac{41}{5}}_{8.2=8+0.2=8+\frac{1}{5}=\frac{41}{5}}, \frac{44}{5}, \underbrace{\frac{47}{5}}_{940\%=9.4=9+0.4=9+\frac{2}{5}=\frac{47}{5}}, \frac{50}{5}, \dots$$

All the denominators are now the same, making them easy to compare. Each numerator is one less than a multiple of three. $100=\frac{500}{5}$ fits the pattern, but is not to be included in the counting.

Add $\frac{497}{5}$ as the last number in the list.

$$\frac{38}{5}, \frac{41}{5}, \frac{44}{5}, \frac{47}{5}, \frac{50}{5}, \dots, \frac{497}{5}$$

Fractions are much harder to deal with. We don't want fractions. Multiply each number in the list by 5 to get rid of the denominator:

$$\underbrace{38,41}_{+3},\underbrace{44,47}_{+3},50\ldots,497 \Rightarrow \underbrace{39,42,45,\ldots,498}_{\substack{\text{Add 1 to make each}\\ \text{number a multiple of 3}}} \Rightarrow \underbrace{13,14,15\ldots166}_{\text{Divide by 3}} \Rightarrow \underbrace{1,2,\ldots,154}_{\text{Subtract 12}} \Rightarrow \underbrace{154}_{\text{Numbers}}$$

$$\frac{38}{5},\frac{41}{5},\ldots,\frac{497}{5}=\frac{12\times3+2}{5},\frac{13\times3+2}{5},\ldots,\frac{165\times3+2}{5}\Rightarrow 165-12+1=154$$

Method II:

- **Convert to Decimal Fractions**
- **Write the numbers as multiples of 5**

$$\underbrace{\frac{38}{5}}_{\text{Fraction}},\underbrace{8.2}_{\text{Decimal}},\frac{44}{5},\underbrace{940\%}_{\text{Percentage}},10\ \ldots\Rightarrow\frac{76}{10},\underbrace{\frac{82}{10}}_{8.2=\frac{82}{10}},\frac{88}{10},\underbrace{\frac{94}{10}}_{940\%=9.4=\frac{94}{10}},\frac{100}{10}\ \ldots$$

Successive numerators have a difference of 6, and are four more than a multiple of 6, so write them like that.

$$\frac{12\times6+4}{10},\frac{13\times6+4}{10},\ldots$$

The largest number must be less than $100=\frac{1000}{10}$. 1000 is more 4 more than a multiple of 6, so it has a valid form, but it too large. So, take $\frac{1000-6}{10}=\frac{994}{10}=$ as the largest number in the sequence:

$$\frac{12\times6+4}{10},\frac{13\times6+4}{10},\ldots,\frac{165\times6+4}{10}\Rightarrow 165-12+1=154$$

J. Finding Multiples

Multiples are important in Number Theory. We often want to find a multiple of a number.

Example 2.43: Finding Multiples

Consider the facts given below that

$$7\times11\times13=1001$$
$$3\times333=999$$

Using these, or otherwise, find the

A. seventh multiple of 17
B. largest multiple of 13 which is less than 1000
C. Smallest multiple of 7, which is more than 1000
D. Smallest multiple of 3, which is more than 1000.
E. Largest multiple of 17, which is less than 1000.

$$\textit{Direct Multiplication}: 7\times17=119$$

Shortcut Method

We will illustrate some shortcuts that involve finding a number that you know

- is a multiple of the number that you want
- is close to the number that you want

And then adding or subtracting as required in order to meet the other conditions.

$1001=13x\Rightarrow Ans=1001-13=988$
$1001=7x\Rightarrow Ans=1001$
$999=3x\Rightarrow Ans=999+3=1002$

Standard Method

The standard method for finding the largest multiple, or the smallest multiple, is to divide the larger number by the smaller:

$$\frac{1000}{17} = 58\frac{14}{17} \Rightarrow Ans = 1000 - 14 = 986$$

Strategy: Try to find numbers near the number that you want that can be calculated easily

Challenge 2.44: Finding Multiples

What is the number of three-digit multiples of 11 ending in 7?

The number must be of the form

$$ab7$$

Apply the test of divisibility of 11:

$$a + 7 - b = Multiple\ of\ 11$$

Range (a + 7) = 8 to 16
Range(b) = 0 to 9
Case I:

$$a + 7 - b = 0 \Rightarrow a + 7 = b \Rightarrow (a, b) = (1,8)(2,9)$$

187, 297
Case II: a + 7 = b + 11 implies a = b + 4

$$a + 7 - b = 11 \Rightarrow a = b + 4 \Rightarrow (a, b) = (4,0)(5,1)(6,2)(7,3)(8,4)(9,5)$$

407, 517, 627, 737, 847, 957

K. Counting Multiples

When counting multiples, a standard strategy is to:

- Find the smallest number and the largest number that meets the conditions
- Write out these numbers as multiples.
- Count the number of multiples using strategies from counting lists.

Example 2.45

How many multiples of 9 lie between 100 and 200?

Find the smallest multiple of 9 greater than 100:

$$\underbrace{108}_{\substack{Smallest\\Multiple}}$$

Find the largest multiple of 9 smaller than 200:

$$\underbrace{198}_{\substack{Largest\\Multiple}}$$

Write out a list of multiples that meets the requirements using the above two:

$$n\{\mathbf{22} \times 9, 21 \times 9, \dots, 13 \times 9, \mathbf{12} \times 9\}$$

Divide each number in the list by 9:

$$n\{22, 21, \dots, 12\} = 22 - 12 + 1 = 11$$

Example 2.46

How many multiples of

A. 9 lie between 100 and 200?
B. 6 lie between 100 and 200?
C. 7 lie between 100 and 200?

D. 8 lie between 100 and 200?

Part B, C and D

$$Part\ B: \{102, \dots, 198\} = \{6 \times 17, \dots, 6 \times 33\} \Rightarrow 33 - 17 + 1 = 17$$
$$Part\ C: \{105, \dots, 196\} = \{7 \times 15, \dots, 7 \times 28\} \Rightarrow 28 - 15 + 1 = 14$$
$$Part\ D: \{104, \dots, 192\} = \{8 \times 13, \dots, 8 \times 24\} \Rightarrow 24 - 13 + 1 = 12$$

Example 2.47

A number is called "alternative" if its digits alternate between two distinct values such that the preceding and succeeding digit of a number are necessarily the same. As an example, 3737 is an alternative number, but 2494 and 2222 is not. How many six-digit alternative numbers are divisible by 4 if numbers can begin with 0?
(JHMMC Grade 7 2022 R2/5)

Any alternating number has the form

$$ababab$$

For $ababab$ to be divisible by 4, ab must be divisible by 4:

$$\{00,04,08,\dots,96\} \rightarrow \{00,01,02,\dots,24\} \rightarrow 24 - 0 + 1 = 25\ Values$$

Numbers of the form $aaaaa$ are not allowed.

$$\{00,44,88\}: 3\ Values$$

The final answer is:

$$25 - 3 = 22\ Values$$

Example 2.48

Find the number of multiples of 16 between 100 and 200.

Listing Method

We first look at the listing method, which you would have already seen before. This requires finding the first and the last multiple that satisfy the conditions, converting it into a list, manipulating the list, and then finding the number of elements in the list:

$$\underbrace{\{112,128, \dots, 192\}}_{\text{List the multiples}} = \underbrace{\{16 \times 7, 16 \times 8, \dots 16 \times 12\}}_{\text{Wrote them as multiples}} = \underbrace{\{7,8, \dots, 12\}}_{\text{Divided by 16}} = \underbrace{12 - 7 + 1}_{\text{Counted the list}} = 6$$

Floor and Ceiling Functions

Ceiling Function

Instead of finding the smallest multiple manually, we can also divide 100 by 16

$$100 \div 16 = \frac{100}{16} = \frac{50}{8} = \frac{25}{4} = 6\frac{1}{4}$$

Now, the above tells us that the sixth multiple of 16 will be smaller than 100. Hence, we need the number after six, which is seven. In other words, we are rounding up.
Mathematically, this is represented using the ceiling function.

$$\underbrace{\left\lceil \frac{100}{16} \right\rceil = \left\lceil 6\frac{1}{4} \right\rceil = 7}_{Rounds\ up\ 6\frac{1}{4}\ to\ 7}$$

Floor Function

Similarly,

$$\left\lfloor\frac{200}{16}\right\rfloor = \left\lfloor 12\frac{1}{2}\right\rfloor = 12$$

L. Counting Multiple Lists

Till now, the examples that we have seen required us to find the number of elements of a single list.
However, it is possible to combine the two lists together. In such a scenario, it is important to separate out the lists and then find the numbers of elements in each list.

Example 2.49

What is the number of numbers in the sequence:

$$6,7,10,11,14,15,\dots,94,95,98\ (AOPS\ Alcumus)$$

This can be visualised as two arithmetic sequences:
6, 10, 14....
7, 11, 15....

Both the sequences have a difference of 4. We can rewrite them as:

$$4+2, 8+2, 12+2\dots\dots, 92+2, 96+2$$
$$4+3, 8+3, 12+3, \dots\dots 92+3$$

We can further rewrite it as:

$$4\times\mathbf{1}+2,\quad 4\times\mathbf{2}+2,\quad 4\times\mathbf{3}+2\dots\dots,\quad 4\times\mathbf{23}+2,\quad 4\times\mathbf{24}+2 \Rightarrow 24\ Numbers$$
$$4\times\mathbf{1}+3,\quad 4\times\mathbf{2}+3,\quad 4\times\mathbf{3}+3,\dots\dots 4\times\mathbf{23}+3 \Rightarrow 23\ Numbers$$

Total number of terms is:

$$24+23=47$$

M. Ceiling and Floor Functions (Optional)

The floor function rounds down a number.
The ceiling functions rounds up a number.

$$\text{Multiples of } x \text{ between } n \text{ and } m = \text{Floor}\left(\frac{m}{x}\right) - \text{Ceiling}\left(\frac{n}{x}\right) + 1 = \left\lfloor\frac{m}{x}\right\rfloor - \left\lceil\frac{n}{x}\right\rceil + 1$$

Example 2.50: Using the Ceiling and the Floor Functions

How many multiples of 9 lie between 100 and 200? Use the formula with ceiling and floor functions to solve the question.

$$\text{Multiples of 9 between 100 and 200} = \left\lfloor\frac{200}{9}\right\rfloor - \left\lceil\frac{100}{9}\right\rceil = \left\lfloor 22\frac{2}{9}\right\rfloor - \left\lceil 11\frac{1}{9}\right\rceil = 22-12+1 = 11$$

Example 2.51: Using the Ceiling and the Floor Functions

Find multiples of 1.5 between 100 and 200.

$$\left\lfloor\frac{200}{1.5}\right\rfloor - \left\lceil\frac{100}{1.5}\right\rceil + 1 = \left\lfloor 200\times\frac{2}{3}\right\rfloor - \left\lceil 100\times\frac{2}{3}\right\rceil + 1 = \left\lfloor 133\frac{1}{3}\right\rfloor - \left\lceil 66\frac{2}{3}\right\rceil + 1 = 133-67+1 = 67$$

2.2 Applications of Lists

A. Start Day and End Day

Example 2.52

A. Ajay travelled to Mexico between Monday and Saturday in the same week. How many days did he travel?

B. Alina travelled to Japan. Her trip was from Monday to Saturday. How many days did she travel?

$$Tuesday, Wednesday, Thursday, Friday \Rightarrow 4\ Days$$
$$Monday, Tuesday, Wednesday, Thursday, Friday, Saturday \Rightarrow 4\ Days$$

B. Start Date and End Date

We can use principles from counting lists to count the number of days that an event takes.

Example 2.53

Gauri's vacation was from 15th June to 29th June. Find the number of days that she had for vacation.

This can be directly converted into a list:

$$15\ June, 16, 17, \dots, 29\ June \Rightarrow 29 - 15 + 1 = 30 - 15 = 15$$

C. Splitting across Months

If you have an event that is split across different months, the standard technique is to count the days in different months separately.

Example 2.54

Shivam's vacation was from 17th June to 8th Aug. Find the number of days that he had for vacation.

$$Days\ in\ June: 17, 18, \dots, 30 \Rightarrow 30 - 17 + 1 = 14$$
$$Days\ in\ July = 31$$
$$Days\ in\ Aug = 1, 2, \dots 8 \Rightarrow 8$$

$$Total = 14 + 31 + 8 = 53$$

D. Figuring out the Day

Example 2.55

If in the month of March, Sunday, Monday, and Tuesday occurred five times, then in the month of September which days will occur five times?

Sun	Mon	Tue	Wed	Thu	Fri	Sat
1	2	3				
8						
15						
22						
29	30	31				

$$31 + 30 + 31 + 30 + 31 + 31$$

Each of the above has four complete weeks, which will not change the day:

$$3 + 2 + 3 + 2 + 3 + 3 = 16 = 2\ Weeks + 2\ Days$$

September has 30 days

Sun	Mon	Tue	Wed	Thu	Fri	Sat
		1	2			

		8	9			
		15				
		22				
		29	30			

E. Positions

We now look at positions in lists. We start with positions in numbers, and then move to positions in arrangements.

Finding an element: 7th multiple of 9 ⇔ 7th position in the list of multiples of seven.

Finding a position: When we find 729 is what multiple of 3, we are finding the position of 729 in the list of multiples of three.

2.56: n^{th} Even and Odd Numbers

$$n^{th} \text{ even natural number} = 2n$$
$$n^{th} \text{ odd natural number} = 2n - 1$$

To find the fifth odd number:

$$1,3,5,7,9 \Rightarrow 5th\ Number\ is\ 9$$
$$5 \times 2 - 1 = 10 - 1 = 9$$

To find the fourth even number:

$$2,4,6,8 \Rightarrow 4th\ Number\ is\ 8$$
$$4th\ Number = 4 \times 2 = 8$$

Example 2.57

A. Find the sum of the tenth even natural number and the twentieth odd natural number.
B. Find the sum of the 18^{th} odd number, and 35^{th} even number.

$2 \times 10 + (2 \times 20 - 1) = 20 + 39 = 59$

$$\underbrace{18 \times 2 - 1}_{18th\ Odd\ Number} + \underbrace{35 \times 2}_{35th\ Even\ Number} = 35 + 70 = 105$$

2.58: Back Calculations

Given a number n, it is in the

$$\left(\frac{n}{2}\right)^{th} \text{ even number (if even)}$$
$$\left(\frac{n+1}{2}\right)^{th} \text{ odd number (if odd)}$$

Example 2.59

If 237 is the n^{th} odd number, and 238 is the m^{th} even number, what is $3m - 2n$?

$238 = 119 \times 2 = 119^{th}$ even number
$237 = 119^{th}$ odd number
$3m - 2n = 3 \times 119 - 2 \times 119 = 119(3 - 2) = 119$

Example 2.60

What is the 1000^{th}:

A. Odd Integer
B. Integer with odd digits
C. Integer with odd number of digits

Part A

The 1000th odd integer is

$$1000 \times 2 - 1 = 2000 - 1 = 1999$$

Part B

$$1\ Digit\ Numbers: 1,3,5,7,9 \Rightarrow 5\ Numbers$$

$$2\ Digit\ Numbers: \underbrace{11,13,15,17,19}_{5\ Numbers}, \underbrace{31,33,35,37,39}_{5\ Numbers} \dots, \Rightarrow 5 \times 5 = 25$$

$$3\ Digit\ Numbers: \underbrace{111,113,115,117,119}_{5\ Numbers}, \underbrace{31,33,35,37,39}_{5\ Numbers} \dots, \Rightarrow 5 \times 5 \times 5 = 125$$

$$4\ Digit\ Numbers \Rightarrow 5^4 = 625$$

$$5 + 25 + 125 + 625 = 780$$

$$5\ Digit\ Numbers: 11111, \dots, 11999 \Rightarrow 125\ Numbers$$

$$780 + 125 = 905$$

$$13111, \dots, 13599 \Rightarrow 25 \times 3 = 75\ Numbers$$

$$13711, \dots.$$

Example 2.61

If Jayshri is fifth from the left and seventh from the right in a row of people seated at a table, what is the total number of people seated?

The main concept here is that Jayshri gets counted both from the left, and from the right. So, while you might think we should have:

$$5 + 7 = 12$$

We actually need to subtract 1.
Looking at diagram below will help visualize.

1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}						
				Jayshri						
				7^{th}	6^{th}	5^{th}	4^{th}	3^{rd}	2^{nd}	1^{st}
1	2	3	4	5	6	7	8	9	10	11

Hence, the final answer is:

$$5 + 7 - 1 = 12 - 1 = 11\ people$$

And note that this is very similar to our formula for counting lists.

Concept 2.62

There are one thousand people seated at a table. (It's a very long table). Rohini is x^{th} from the right, and y^{th} from the left. What is $x + y$?

		Leftmost							Rightmost
	$y^{th}Position$	1	2		y		998	999	1000
	$x^{th}Position$	1000	999		x		3	2	1
Total		1001	1001	…	1001	…	1001	1001	1001

From the table above, we can see that the sum of any left position, and the corresponding right position is:

$$1001$$

Hence, the answer, independent of the value of x is

$$1001$$

Example 2.63

Patrick Jane, the Mentalist, was tracking a murder suspect. He knew the suspect was among a group playing throwball, with the participants numbered consecutively, and spaced equally in a circle. When Patrick reached the group, the participant numbered 5 threw the ball to the participant standing directly opposite him, who was numbered 19. If Patrick sizes up the group (taking 10 seconds for each person), the process will take m minutes and s seconds (where $s\ < 60$). Find $m + s$?

Participants numbered from 6 to 18 are in between participants 5 to 19. This is:
18-5=13 people

The other side will also have
13 people

Total is:
13+13+2=28 people

Time taken:
=28*10=280 seconds=4 minutes 40 seconds
m+s=44

F. Exponents

Example 2.64: Perfect Squares and Cubes

Find the number of:

A. Perfect Squares from 50 to 300
B. Perfect Cubes from 100 to 1000

Part A

We don't want to list the squares. Instead, we find the smallest number that satisfies, and the largest number that satisfies

$$\left\{\underbrace{64}_{\substack{Smallest\\Square}}, 81, \dots, \underbrace{289}_{\substack{Largest\\Square}}\right\} \Rightarrow \{8^2, 9^2, \dots 17^2\}$$

Take square roots. The number of square roots is the same as the number of squares (bijection principle):

$$\{8,9, \dots 17\} \Rightarrow \underbrace{17 - 8 + 1}_{Count\ the\ Numbers} = 10$$

Part B

We follow a similar process with finding the number of cubes:

$$\left\{\underbrace{125}_{Smallest\ Cube}, 216, \dots \underbrace{1000}_{Largest\ Cube}\right\} \Rightarrow \{5^3, 6^3, \dots 9^3, 10^3\}$$

Take the cube roots, and then count the number of elements in the list:

$$\{5,6,\dots,9,10\} \Rightarrow 10 - 5 + 1 = 6$$

If you take a square root of a perfect square, then you will get a nonnegative number.

$$\sqrt{x} < \sqrt{x+1}$$

2.65: Perfect Square and Perfect Cube

A number is a perfect square and a perfect $if\ and\ only\ if$ it is a perfect sixth power.

$$n^6 = (n^3)^2 = (n^2)^3$$

Example 2.66: Perfect Squares and Cubes

The sequence 2, 3, 5,6,7,10,11, … contains all the positive integers from least to greatest that are neither squares nor cubes. What is the 400^{th} term of the sequence? (MathCounts 2007 Workout 2)

We use complementary counting. We count the numbers that we do not want, and add that numbers at the end of the list.

$$Perfect\ Cubes: 1^3 = 1, 2^3 = 8, \dots, 7^3 = 343, \dots$$
$$Perfect\ Square: 1^2 = 1, 2^2 = 4, \dots, 20^2 = 400, \dots$$

But there are some numbers which are both perfect squares and perfect cubes, which we need to subtract:

$$1^6 = 1, 2^6 = 64 \Rightarrow 2\ numbers$$

The final answer is:

$$400 + 7 + 20 - 2 = 425$$

As a check, note that:

$$21^2 = 441 > 425$$

Example 2.67

Identify the conditions when the equality

$$\sqrt{x^2} = x$$

Holds.

$$x = 5 \Rightarrow \sqrt{5^2} = 5 \Rightarrow \sqrt{x^2} = x$$
$$x = -5 \Rightarrow \sqrt{(-5)^2} = 5 \Rightarrow \sqrt{x^2} = -x$$

The above equality is true for $non-negative$ numbers.

Example 2.68

How many whole numbers are between $\sqrt{8}$ and $\sqrt{80}$? (AMC 8 1986/7)

Whole numbers can also be called non-negative integers.

$$\sqrt{8} < x < \sqrt{80} \Rightarrow x \in \{\sqrt{9}, \sqrt{16}, \dots \sqrt{64}\} \Rightarrow x \in \{3,4,5,6,7,8\} \Rightarrow 6\ Numbers$$

Example 2.69

How many natural numbers lie between the square root of the largest two-digit number, and the square root of the smallest four-digit number?

$$\sqrt{99} < x < \sqrt{1000} \Rightarrow \sqrt{99} < \{10,11, \dots ,31\} < \sqrt{1000} \Rightarrow 31 - 10 + 1 = 22$$

Example 2.70

How many between $\sqrt{8}$ and $\sqrt{80}$ are of the form $\frac{p}{3}$, where p is a natural number? **(AMC 8 1986/7, Adapted)**

$$\sqrt{8} < x < \sqrt{80}$$
$$\sqrt{8} = 2\sqrt{2} \approx 2 \times 1.41 = 2.82$$

Now, we need to identify the smallest number of the form $\frac{p}{3}$ that is greater than 2.82:

$$\frac{8}{3} = 2\frac{2}{3} = 2.\overline{6} < 2.82 < 3 = \frac{9}{3}$$

And, we also need to identify the largest number of the form $\frac{p}{3}$ that is smaller than $\sqrt{80}$:

$$\sqrt{80} \approx \sqrt{81} = 9 = \frac{27}{3} > 8.\overline{6}\frac{26}{3}$$

Hence, we are looking for the number of numbers in the list:

$$\frac{9}{3}, \frac{10}{3}, \dots, \frac{26}{3} \Rightarrow 9,10, \dots ,26 \Rightarrow 26 - 9 + 1 = 27 - 9 = 18$$

Example 2.71

How many whole numbers are between $\sqrt[3]{9}$ and $\sqrt[3]{999}$?

$$\sqrt[3]{9} < x < \sqrt[3]{999}$$
$$\sqrt[3]{8} = 2, \sqrt[3]{1000} = 10$$

We are looking for:

$$\sqrt[3]{27}, \sqrt[3]{64}, \dots, \sqrt[3]{729} \Rightarrow 3,4,5,6,7,8,9 \Rightarrow 7\ Numbers$$

Example 2.72

For how many integer values of b is $1 \le 2^b \le 1000$

$$\{1,2,4, \dots ,512\} = \{2^0, 2^1, 2^2, \dots, 2^9\} \Rightarrow \{0,1,2, \dots 9\} \Rightarrow 10\ Numbers$$

Example 2.73

For how many integer values of b is $1 < (-2)^b < 1000$

This will have the same answer as the one above, except that:

- When b is even, $(-2)^b$ is positive.
- When b is odd, $(-2)^b$ is negative.

$$\{1, -2, 4, -8, \ldots, 512\} = \{2^0, 2^1, 2^2, 2^3 \ldots, 2^9\} \Rightarrow \{0,2,4,6,8\} \Rightarrow 5\ Numbers$$

Example 2.74

For how many integer values of b is $\frac{1}{1000} < 2^b < 1$

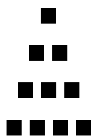
$$\left\{1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{512}\right\} = \{2^0, 2^{-1}, 2^{-2}, \ldots, 2^{-9}\} \Rightarrow \{0, -1, -2, \ldots, -9\} \Rightarrow 10\ Numbers$$

Example 2.75

For how many integer values of b is $\frac{1}{1000} < (-2)^b < 1$

G. Number Arrangements

Square
Triangular



If you arrange dots in rows, as above, with an increasing number of dots in each row, you get the shape of a triangle. Hence, the total number of dots in the arrangement is called a triangular number.

$$\therefore n^{th}\ Triangular\ Number = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Example 2.76

Lists and Positions
What number is directly above 142 in this array of numbers?

\[\begin{array}{cccccc}& & & 1 & &\\ & & 2 & 3 & 4 &\\ & 5 & 6 & 7 & 8 & 9\\ 10 & 11 & 12 &\cdots & &\\ \end{array}\]
(AMC 8 1993/24)
Ans = 120

H. Challenge: Partition of Sets

Example 2.77

There are many ways in which the list 0,1,2,3,4,5,6,7,8,9 can be separated into groups. For example, this list could be separated into the four groups 0,3,4,8 and 1,2,7 and 6 and 5,9. The sum of the numbers in each of these four groups is 15,10,6, and 14, respectively. In how many ways can the list 0,1,2,3,4,5,6,7,8,9 be separated into at least two groups so that the sum of the numbers in each group is the same? (Gauss 8 2019/24)

Find the total: 45

If we divide into two groups, then the sum of each group must be:
45/2, not possible

In general, we will only be able to divide into groups which are factors of 45.
Factors of 45 = 1,3,5,9,15,45

15 groups and above not possible, since we only have 9 numbers

45/9 = 5 Total
This is not possible, since the group with the number 9 will have total at least 9.

45/5 = 9 Total
(9)(1,8)(2,7)(3,6,)(4,5)
0 does not contribute.
So it can go in any group.
So, there are 5 ways to do it.

45/3 = 15 Total
9 has to be in one of the groups
Remaining numbers have to add up to 6.
6
5+1
4+2
1+2+3

(9,6)
Left: 1,2,3,4,5,7,8
To get total of 15:
8+7
8+5+2
8+4+3

(9,6)(8,7)(1,2,3,4,5)
(9,6)(8,5,2)(1,3,4,7)
(9,6)(8,4,3)(1,2,5,7)
(9,6)(8,4,2,1)(3,5,7)
4 Ways

(9,5,1)
Left: 2,3,4,6,7,8

(9,5,1)(7,8)(2,3,4,6)
(9,5,1)(3,4,8)(2,7,6)
2 Ways

(9,4,2)
Left: 1,3,5,6,7,8

(9,4,2)(7,8)(1,3,5,6)
(9,4,2)(1,6,8)(3,5,7)
2 Ways

(9,3,2,1)
Left: 4,5,6,7,8

(9,3,2,1)(7,8)(4,5,6)
1 Way

4+2+2+1=9
9*3=27

27+5=32

2.3 Visualizing Sets

A. Background

There are many different ways of thinking about sets. There are algebra-heavy methods that work, but may not be the easiest to do for a question. We look at a few different methods that visualizing sets in different ways. For an exam questions, the choice of method can be important in deciding the ease with which you can solve the question, and the time taken.
So, pay careful attention to when which method is preferable. This will be helpful in exam scenarios.

B. Contingency Tables

Contingency tables are a method of presenting information. Contingency tables can help us to understand information more quickly. They are also in calculating missing values, which are much more difficult to calculate otherwise.

2.78: Perfect Square and Perfect Cube

A simple two-way contingency table classifies something based on two parameters.
One parameter is presented in the columns, and the other parameter is presented in the rows.

Example 2.79: Creating a Contingency Table

A doctor is considering 50 patients based on whether they have heart disease and/or diabetes. It is also possible that a patient has neither. The following information is available:

A. 35 have Diabetes
B. 32 have heart disease
C. 18 do not have heart disease
D. 15 do not have Diabetes
E. 20 have both heart disease and Diabetes
F. 12 have heart disease, but not Diabetes
G. 15 have Diabetes, but not heart disease
H. 3 have neither heart disease, nor Diabetes

Present this information in the form of a table.

The parameters here are Diabetes and Heart Disease.
We put Diabetes in the columns, and heart disease in the rows:

		Diabetes		
		Yes	No	Total
Heart Disease	Yes	20	12	32
	No	15	3	18
	Total	35	15	50

We could also have done it the other way around, with heart disease in the columns, and diabetes in the rows.

Example 2.80: Interpreting a contingency table

Write the information in the table alongside on people visiting different countries as points.

		Visited Europe		
		Yes	No	Total
Visited USA	Yes	8	7	15
	No	33	32	65
	Total	41	39	80

- Total Number of People: 80
- Visited Europe: 41
- Visited USA: 15
- Not Visited Europe: 39
- Not Visited USA: 65
- Visited both Europe and USA: 8
- Visited Europe, but not USA: 33
- Visited USA, but not Europe: 7
- Visited neither Europe nor USA: 32

	VE	NVE	Total
VUS	8	7	15
NUS	33	32	65
Total	41	39	80

Example 2.81

I have fourteen children in my block. Five of the children play soccer. Seven of the children play baseball. Three of the children play both soccer and baseball. Show this information in a two-way contingency table and fill the rest of the table.

We have information on two things:
- Soccer
- Baseball

We can put Soccer in the columns and baseball in the rows.
Let $S = Soccer, NS = No\ Soccer, B = Baseball, NB = No\ Baseball$

	S	NS	Total
B	3		7
NB			
Total	5		14

	S	NS	Total
B	3	$\mathbf{4 = 7 - 3}$	7
NB	$\mathbf{2 = 5 - 3}$	$\mathbf{5 = 9 - 4}$ $\mathbf{5 = 7 - 2}$	$\mathbf{7 = 14 - 7}$
Total	5	9	14

	S	NS	Total
B	3	4	7
NB	2	5	7
Total	5	9	14

Self-Check in Contingency Tables

Playing neither soccer, nor baseball, is calculated twice. The number must be the same from both calculations. In this way, the table is self-checking. If the number is not the same from both the calculations, then:

- It is time to go back and check the calculations, and ensure that you have read the question correctly.
- It is also possible that the question requires you to establish a contradiction
- Finally, it is possible that the question has a mistake.

Example 2.82

There are 40 students in my class. 12 of them learn French, and 17 of them learn Spanish. If 5 of them learn both French and Spanish, how many of them learn neither of the two languages.

Filling up the information in the question:

	S	NS	Total
F	5		12
NF			
Total	17		40

Filling in the rest of the table:

		Spanish		
		Yes	No	Total
French	Yes	5	7	12
	No	12	16	28
	Total	17	23	40

The number of people who speak neither of the two languages is 16.

Challenge 2.83

The Pythagoras High School band has 100 female and 80 male members. The Pythagoras High School orchestra has 80 female and 100 male members. There are 60 females who are members in both band and orchestra. Altogether, there are 230 students who are in either band or orchestra or both. The number of males in the band who are NOT in the orchestra is: **(AMC 8 1991/23)**

Create a contingency table, and add the information in the question:

	Band	Orchestra	Both	Total
Female	100	80	60	
Male	80	100		
Total	180	180		230

	Band	Orchestra	Both	Total
Female	100	80	60	

Male	80	100	70	
Total	180	180	130	230

The number of males in the band who are not in the orchestra

$$= 80 - 70 = 10$$

C. Applications of Contingency Tables

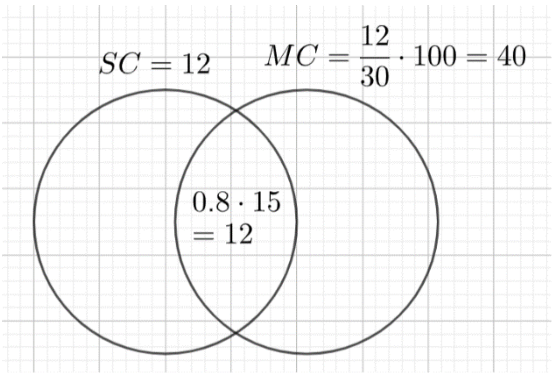
Example 2.84: Percentages

At Annville Junior High School, 30% of the students in the Math Club are in the Science Club, and 80% of the students in the Science Club are in the Math Club. There are 15 students in the Science Club. How many students are in the Math Club? **(AMC 8 1998/14)**

Note that the information that is most directly useful is given towards the end.

	Science Club	No Science Club	Total
Math Club	**Step I** $80\%\ of\ 15 = 12$		**Step II** $\frac{12}{30} \times 100 = 40$
No Math Club			
Total	**Given:** 15		

You can also do this using a Venn Diagram. The calculations are similar, but the presentation is different.



Example 2.85: Fractions and Percentages

There are 30 cars in my building's parking lot. All of the cars are red or white, and a car can have either 2 doors or 4 doors. $\frac{1}{3}$ of them are red, 50% of them are 4-door, and 8 of them are 2-door and white. How many of the cars are 4-door and red? **(AOPS Alcumus, Counting and Probability, Venn Diagrams)**

Start with a contingency table. The total number of cars is 30.

	Red	White	
2 Door			
4 Door			
			30

$\frac{1}{3}$ of them are red, 50% of them are 4-door, and 8 of them are 2-door and white.

	Red	White	
2 Door		8	
4 Door			15
	10		30

Now fill in the rest of the table:

	Red	White	
2 Door	7	8	15
4 Door	3	12	15
	10	20	30

Hence, the final answer is

$$3$$

2.86: Ratios

When information is presented in terms of ratios, it is often useful to introduce variables in your Venn Diagrams or Contingency Tables.

Example 2.87

On my ping-pong team there are four times as many $\underbrace{right-handed}_{RH}$ $\underbrace{boys}_{B}$ as $\underbrace{left-handed}_{LH}$ boys. Of the students on the team who are left-handed, there are twice as many $\underbrace{girls}_{G}$ as there are $\underbrace{boys}_{B}$. Half of the girls who are on the team are left-handed. If there are 36 people on the team, how many are right-handed boys? (Assume no player plays ping-pong equally well with both hands.) (AOPS Alcumus, Counting and Probability, Venn Diagrams)

$$Let\ x = No.of\ \underbrace{Left-Handed}_{RH}\ \underbrace{Boys}_{B}$$

Step I			
	LH	RH	
B	x	$4x$	
G	$2x$		
			36

Step II			
	LH	RH	
B	x	$4x$	
G	$2x$	$2x$	
			36

Step III			
	LH	RH	
B	x	$4x$	$5x$
G	$2x$	$2x$	$4x$
	$3x$	$6x$	$9x = 36$

$$9x = 36 \Rightarrow x = 4 \Rightarrow 4x = 16$$

Example 2.88

The number of boys in my school who play the guitar is twice the (non-zero) number of girls who play the guitar. The number of girls who play the flute is twice the number of boys who do so. Also, the number of girls who play the guitar is twice the number of girls who play the flute. No one in the school plays two instruments.

A. Find the minimum number of children in the school who can play at least one instrument.
B. If the number of children in the school who play the guitar is twenty-four, find the total number of children in the school.

Part A

The number of boys in my school who play the guitar is twice the number of girls who play the guitar.

	Boys	Girls
Flute		
Guitar	$2g$	g

The number of girls who play the flute is twice the number of boys who do so.

	Boys	Girls
Flute	f	$2f$
Guitar	$2g$	g

The number of girls who play the guitar is twice the number of girls who play the flute.

	Boys	Girls
Flute	f	$2f$
Guitar	$2g = 8f$	$g = 4f$

Now we can complete the table

	Boys	Girls	Total
Flute	f	$2f$	$3f$
Guitar	$8f$	$4f$	$12f$
Total	$9f$	$6f$	$15f$

The number of students who can play at least one instrument is

$$15f$$

And:

$$Min\ Value\ of\ f = 1 \Rightarrow Min\ Value\ of\ 15f = 15$$

Part B

No. of children who play the guitar

$$= 12f = 24 \Rightarrow f = 2$$

And hence the total number of children

$$= 15f = 30$$

Example 2.89: Averages

In a liberal arts college, out of 30 students taking math majors, 17 have decided to take a minor in logic, and 14 have decided to take a minor in philosophy. 10 students have decided to take both minors. The average annual salary of the students taking:

A. neither of the minors is 100,000/year
B. both minors is 97,000/year
C. exactly one minor is 101,000/year

A. Find the total number of students taking exactly one of the minors
B. Find the positive difference in the total salary of the students taking both minors as compared to the total salary of students taking neither of the minors.

	Philosophy	No Philosophy	Total
Logic	10		17
No			

Logic			
Total	14		30

	Philosophy	No Philosophy	Total
Logic	10	7	17
No Logic	4	9	13
Total	14	16	30

$$(4+7)\times 101{,}000 = 11\times 101{,}000 = 1{,}111{,}000$$
$$97000\times 10 - 100000\times 9 = 970{,}000 - 900{,}000 = 70{,}000$$

D. Tree Diagrams

Example 2.90: Fractions

The number of boys in a school is two-thirds of the total student population. One-fourth of the boys play football, while the rest play basketball. Two-fifths of the girls play football. Everyone in the school plays exactly one sport. Every classroom in the school has fifty-five children. All classrooms are fully occupied. The number of children in the school is a four-digit number. What is the minimum number of children in the school?

Tree Diagram

The total children are divided into

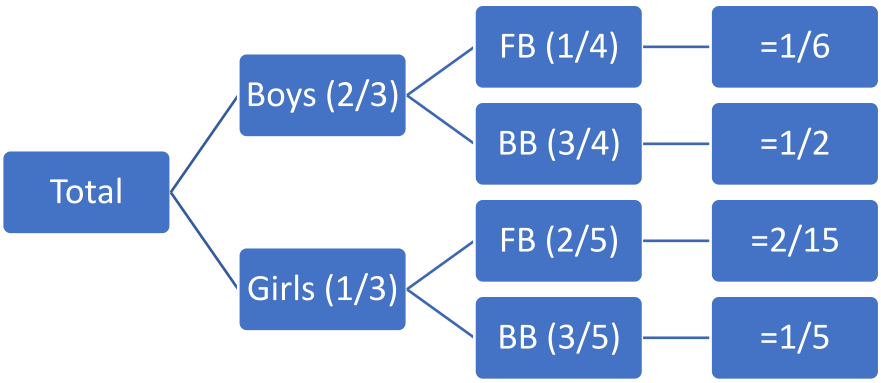
- boys $\left(\frac{2}{3}\right)$
- girls $\left(\frac{1}{3}\right)$

The boys are further divided into those who play

- Football $\left(\frac{1}{4}\right)$
- Basketball $\left(\frac{3}{4}\right)$

And the girls are further divided into those who play

- Football $\left(\frac{2}{5}\right)$
- Basketball $\left(\frac{3}{5}\right)$



Contingency Table

Suppose that the total number of children in the school is one whole.

	Boys	Girls	Total
Football			
Basketball			
Total	$\frac{2}{3}$	$=1-\frac{2}{3}=\frac{1}{3}$	1

	Boys	Girls	
Football	$=\frac{1}{4}\times\frac{2}{3}=\frac{1}{6}$	$=\frac{1}{3}\times\frac{2}{5}=\frac{2}{15}$	

Basketball	$= \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$	$= \frac{1}{3} \times \frac{3}{5} = \frac{1}{5}$	
Total	$\frac{2}{3}$	$= 1 - \frac{2}{3} = \frac{1}{3}$	1

Finding the Answer

	Boys	Girls	
Football	$\frac{1}{6}$	$\frac{2}{15}$	
Basketball	$\frac{1}{2}$	$\frac{1}{5}$	
Total	$\frac{2}{3}$	$\frac{1}{3}$	1

Finding the Answer

The number of children in the school must be a multiple of each of

$$\{6,15,2,5,3\}$$

And hence the minimum number of children in the school is

$$LCM(6,15,2,5) = LCM(6,15) = 30$$

Every classroom has fifty-five children, and all classrooms are fully occupied. Hence, the number of children must be a multiple of 55.

And hence the number of children in the school must be

$$LCM(30,55) = 330n$$

And the smallest multiple of 330 greater than 1000 is:

$$330,660,990, \mathbf{1320}$$

Example 2.91

The knights in a certain kingdom come in two colors. 2/7 of them are red, and the rest are blue. Furthermore, 1/6 of the knights are magical, and the fraction of red knights who are magical is 2 times the fraction of blue knights who are magical. What fraction of red knights are magical? **(AMC 2021 10B/9)**

Put into a table.

Red=2/7
Blue = 5/7
Magical = 1/6

Red and Magical be x
Red and Not Magical = 1/6 - x

x/(2/7) = 2*(1/6-x)/(5/7)

x = 2/27

Fraction of red knights who are magical
=(2/27) div (2/7)

= 2/27 * 7/2
= 7/27

E. Line Diagrams (Minimizing and Maximizing)

Sometimes questions are asked, not in terms of actual values for number of elements in the set, but in terms of the range that the values can take.
Specifically, questions can ask for the minimum number, or the maximum number that a particular region in a Venn Diagram, or a cell in a contingency table can take.

Example 2.92

Of the 10 students going for a camping trip, 5 students like Harry Potter and 3 students like Lords of the Rings. List the possible values of:

A. The number of students who like both.
B. The number of students who like at least one.
C. The number of students who like neither

Part A
It is possible that no student likes both. See Table Below:

	1	2	3	4	5	6	7	8	9	10
HP	Yes	Yes	Yes	Yes	Yes					
LOTR						Yes	Yes	Yes		

It is possible that a single student likes both. See Table Below

	1	2	3	4	5	6	7	8	9	10
HP	Yes	Yes	Yes	Yes	Yes					
LOTR					Yes	Yes	Yes			

It is also possible that two students like both:

	1	2	3	4	5	6	7	8	9	10
HP	Yes	Yes	Yes	Yes	Yes					
LOTR				Yes	Yes	Yes				

Finally, it is possible that three students like both:

	1	2	3	4	5	6	7	8	9	10
HP	Yes	Yes	Yes	Yes	Yes					
LOTR			Yes	Yes	Yes					

The final answer is:

$$\{0,1,2,3\}$$

Part B
Using the same tables as above:

$$5 + \{0,1,2,3\} = \{5,6,7,8\}$$

Part C
Using the same tables as above:

$$10 - \{5,6,7,8\} = 10 - \{5,4,3,2\}$$

Example 2.93
Of the 30 students selected for the MOP($Mathematical\ Olympiad\ Program$), 15 students can speak Mandarin Chinese, and 7 students can speak Hindi. List the possible values of:

A. The number of students who can speak both languages.
B. The number of students who can speak at least one of the two languages.
C. The number of students who can speak neither of the two languages.

Part A
Let M be the set of people who speak Mandarin Chinese, and H be the set of people who speak Hindi. The number of people who speak both languages is the intersection of the two sets. This can take values:

$$\{0,1, \ldots ,7\}$$

Part B
Consider cases from the possible values of the answers to Part A.

$$Both\ Languages = 0 \Rightarrow At\ least\ 1 = 15 + 7 - 0 = 22$$
$$Both\ Languages = 1 \Rightarrow At\ least\ 1 = 15 + 7 - 1 = 21$$
$$.$$
$$.$$
$$.$$
$$Both\ Languages = 7 \Rightarrow At\ least\ 1 = 15 + 7 - 7 = 15$$

Hence, the final answer is:

$$\{15,16, \ldots ,22\}$$

Part C
Consider cases from the possible values of the answers to Part A.

$$Both\ Languages = 0 \Rightarrow Neither = 30 - 22 = 8$$
$$Both\ Languages = 1 \Rightarrow At\ least\ 1 = 30 - 21 = 9$$
$$.$$
$$.$$
$$.$$
$$Both\ Languages = 7 \Rightarrow At\ least\ 1 = 30 - 15 = 15$$

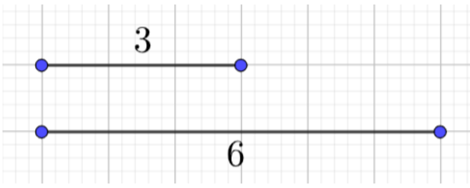
Hence, the final answer is:

$$\{8, 9, \ldots , 15\}$$

Example 2.94
Set A has 3 elements and Set B has 6 elements. What can be the minimum number of elements in the set $A \cup B$?
(JEE Advanced, 1980)

To determine the minimum, let the overlap between Set A and Set B be maximum.

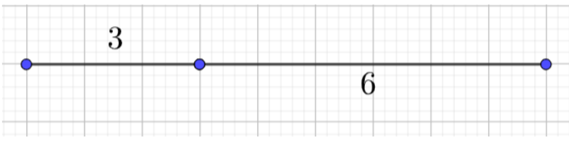
$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 3 + 6 - 3 = 6$$



In the above, what would be the maximum?

To determine the maximum, let the overlap between Set A and Set B be minimum.

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 3 + 6 - 0 = 9$$



Example 2.95

One hundred people were surveyed. Of these, 87 indicated they liked Mozart and 70 indicated they liked Bach. What is the minimum number of people surveyed who could have said they liked both Mozart and Bach? **(MathCounts 2008 School Countdown)**

$$87 + 70 - 100 = 57$$

Example 2.96

Twelve students in Mrs. Stephenson's class have brown eyes. Twenty students in the class have a lunch box. Of Mrs. Stephenson's 30 students, what is the least possible number of students who have brown eyes and a lunch box? **(MathCounts 2005 Chapter Countdown)**

2

Example 2.97

Ellen baked 2 dozen cupcakes of which half contained chocolate, two-thirds contained raisins, one-fourth contained chocolate chips, and one-sixth contained nuts. What is the largest possible number of cupcakes that had none of these ingredients? **(MathCounts 2008 School Countdown)**

8

Example 2.98

Eighty percent of adults drink coffee and seventy percent drink tea. What is the smallest possible percent of adults who drink both coffee and tea? **(MathCounts 2007 National Countdown)**

$$Min = 70 + 80 - 100 = 50$$

50

Example 2.99

Three-fourths of the students in Mr. Shearer's class have brown hair and six-sevenths of his students are right-handed. If Mr. Shearer's class has 28 students, what is the smallest possible number of students that could be both right-handed and have brown hair? **(MathCounts 2004 National Sprint)**

17

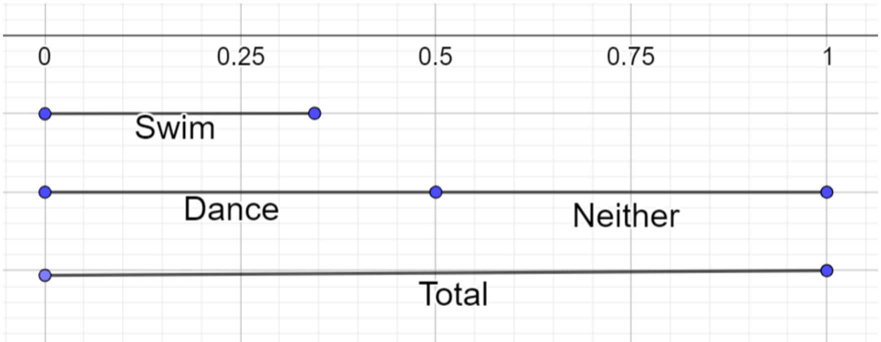
F. Fractions in Minimum and Maximum

Example 2.100

One-third of my school likes to swim. One-half of my school likes to dance. Find the maximum and the minimum fraction of people in my school who like neither.

Maximum

To maximize the fraction of students who like neither, the overlap between the fraction of people who swim and the fraction of people who dance should be maximum.
This is illustrated in the diagram below, where all people who swim, are people who dance.

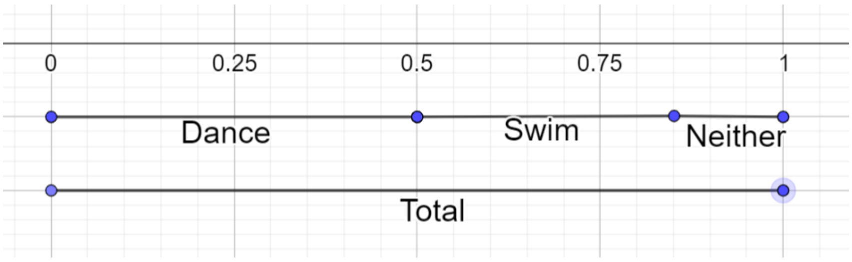


And hence, fraction of people who like neither is

$$1-\frac{1}{2}=\frac{1}{2}$$

Minimum

To minimize the fraction of students who like neither, there should be no overlap between the people who swim and the people who dance.
This is illustrated in the diagram below, where all people who swim, are people who dance.



Hence, the fraction of people who like neither is:

$$1-\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

In the question above, if the number of students in the school is 60, what is the

A. Maximum number of students who like neither?
B. Minimum number of students who like neither?

$$Maximum = 60\times\frac{1}{2}=30$$
$$Minimum = 60\times\frac{1}{6}=10$$

Example 2.101: (Alternate Method)

One-third of my school likes to swim. One-half of my school likes to dance. Find the maximum and the minimum fraction of people in my school who like neither.

Base Case

Let the total number of people in the school be one whole. We can then fill in the following table:

Base Case			
	Dance	Don't Dance	Total
Swim			$\frac{1}{3}$
Don't Swim			$\frac{2}{3}$
Total	$\frac{1}{2}$	$\frac{1}{2}$	1

Minimum Value

To minimize the green cell, maximize the cells above and to the left of it.

- The maximum that the cell above it will take $= Min\left(\frac{1}{3},\frac{1}{2}\right) = \frac{1}{3}$
- The maximum that the cell to the left of it can take will $= Min\left(\frac{1}{2},\frac{2}{3}\right) = \frac{1}{2}$

Minimum	Dance	Don't Dance	Total
Swim	0	$\frac{1}{3}$	$\frac{1}{3}$
Don't Swim	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}$
Total	$\frac{1}{2}$	$\frac{1}{2}$	1

Maximum Value

To maximize the green cell, minimize the cells above and to the left of it.

- The minimum that the cell above it will take $= 0$. This let us put the entire value of $\frac{1}{2}$ in the column for Don't Dance in the green cell.
- We now validate this by finding the value that cell to the left will take, which is

$$\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

Maximum	Dance	Don't Dance	Total	
Swim	$\frac{1}{3}$	0	$\frac{1}{3}$	
Don't Swim	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{2}{3}$	
Total	$\frac{1}{2}$	$\frac{1}{2}$	1	

Example 2.102

In a room, $\frac{2}{5}^{th}$ of the people are wearing gloves, and $\frac{3}{4}^{th}$ of the people are wearing hats. What is the minimum number of people in the room wearing both a hat and a glove? (AMC 8 2010/20)

Our contingency table will have fractions.
Consider the number of people in the room as 1 whole.

	Gloves	No Gloves	Total
Hats			$\frac{3}{4}$
No Hats			
Total	$\frac{2}{5}$		1

Two values can be calculated directly;

	Gloves	No Gloves	Total
Hats			$\frac{3}{4}$
No Hats			$\frac{1}{4}$
Total	$\frac{2}{5}$	$\frac{3}{5}$	1

And since we want to minimize the number of people wear both a hat and a glove, we maximise the number of people

- with no gloves
- with no hats

	Gloves	No Gloves	Total
Hats	$\frac{3}{20}$	$\frac{3}{5}$	$\frac{3}{4}$
No Hats	$\frac{1}{4}$	0	$\frac{1}{4}$
Total	$\frac{2}{5}$	$\frac{3}{5}$	1

This lets us calculate the minimum number of people with both hats and gloves two ways

$$\frac{3}{4}-\frac{3}{5}=\frac{15}{20}-\frac{12}{20}=\frac{3}{20}$$
$$\frac{2}{5}-\frac{1}{4}=\frac{8}{20}-\frac{5}{20}=\frac{3}{20}$$
$$\frac{3}{20}\rightarrow Minimum\ people=3$$

2.4 Venn Diagrams: Two Sets

A. Revision: Basics of Sets

Sets are collections of objects. For example, the set of prime numbers can be written as:

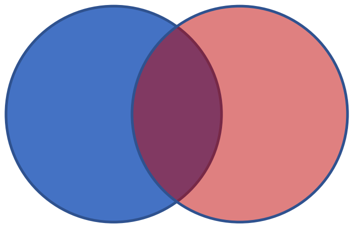
$$\underbrace{P=\{2,3,5,7,11,13,17,19,23,\ldots\}}_{\textit{Roster Form}}\Leftrightarrow\underbrace{P=\{x\mid x\ is\ a\ prime\}}_{\textit{Set Builder Form}}$$

Some terminology related to sets:

- A member of a set is called an element of the set.
- The number of elements of a set is called its cardinality. The cardinality of the set S is written $n(S)$.
- The set consisting of all elements not in a set is called the complement of the set.

- ✓ Let there be a set S.
- ✓ The complement of S is denoted S'.
- ➢ The set consisting of all elements presently under consideration is called the Universal set
  - ✓ The universal set is usually denoted U.

Terminology:



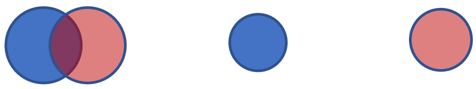
- ➢ The set consisting of elements in either of the sets is called union of the two sets.
  - ✓ Union of Set A and Set B is written $A \cup B$
- ➢ The set consisting of elements that are present in both the sets is called the intersection of the sets.
  - ✓ Intersection of Set A and Set B is written $A \cap B$

B. Union of Two Sets

Questions on Venn Diagrams can be solved by multiple methods. The choice of method is a matter of preference. However, certain questions can be solved much more easily using a specific method.
Hence, it is good to know (and practice!) multiple methods of arriving at the same answer.

2.103: Union of Two Sets[1]

$$\underbrace{n(A \cup B)}_{\text{Union of } A \text{ and } B} = \underbrace{n(A)}_{\text{Elements in } A} + \underbrace{n(B)}_{\text{Elements in } B} - \underbrace{n(A \cap B)}_{\substack{\text{Elements in Intersection} \\ \text{of } A \text{ and } B}}$$



Example 2.104

Given that

$$n(A) = 5, \qquad n(B) = 13, \qquad n(A \cap B) = 3, \qquad n(A \cup B) = 15$$

Verify that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Wrong Method

$$15 = 5 + 13 - 3$$
$$15 = 15$$

You cannot use the equals sign before you have established equality.

Right Method

Hence, calculate the LHS ($Left\ Hand\ Side$) and the RHS ($Right\ Hand\ Side$) separately

$$LHS = 15$$
$$RHS = 5 + 13 - 3 = 15$$
$$LHS = RHS = 15$$

Example 2.105

I have fourteen children in my class. Twelve of the children speak English. Seven of the students speak Hindi. If everyone speaks at least one language, how many students speak both English and Hindi?

[1] This same formula is used with slightly different notation in probability.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$5 \text{ speak both English and Hindi}$$

$$n(E \cup H) = n(E) + n(H) - n(E \cap H)$$
$$14 = 12 + 7 - n(E \cap H)$$
$$n(E \cap H) = 19 - 14 = 5$$

	Eng	No Eng	Total
Hindi	5	2	7
No Hindi	7	0	7
Total	12	2	14

Example 2.106

I have fourteen children in my block. Five of the children play soccer. Seven of the children play baseball. Three of the children play both soccer and baseball. Find the number of children who play at least one of the two sports.

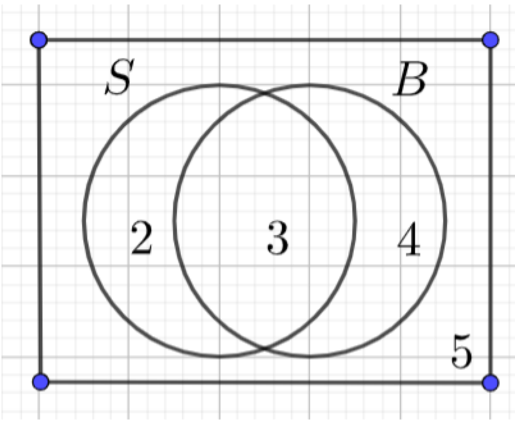
At least one of the two sports means children who play

- **Only Soccer**
- **Only Baseball**
- **Both soccer and Baseball**

It does not include children who do not play either of the sports.

$$S = Set\ of\ Students\ who\ play\ Soccer,$$
$$B = Set\ of\ Students\ who\ play\ Baseball$$
$$n(S) = 5, \quad n(B) = 7, \quad n(S \cap B) = 3, \quad U = 14$$

$$n(S \cup B) = n(S) + n(B) - n(S \cap B) = 5 + 7 - 3 = 9$$



In further questions, we will try and use letters that are meaningful. Sets will not be explicitly defined.

We presented the same information in a contingency table earlier. Can the same question be answered using the contingency table? If so, how?

Method I

If we add up the boxes, we will get the total number of students who play either baseball or soccer

	S	NS	Total
B	3	4	7
NB	2	5	7
Total	5	9	14

$$\underbrace{3}_{\text{Play Baseball and Soccer}} + \underbrace{4}_{\text{Play Baseball, but not soccer}} + \underbrace{2}_{\text{Play Soccer, but not Baseball}} = 9$$

Method II

If we add subtract the maroon box from the blue box, we get the total number of students who play either baseball or soccer:

$$\underbrace{14}_{Total\ Students} - \underbrace{5}_{\substack{Play\ neither\\ soccer\ nor\ baseball}} = 9$$

	S	NS	Total
B	3	4	7
NB	2	5	7
Total	5	9	14

Method III

We subtract the red box from the sum of the blue boxes:

$$\underbrace{5}_{Play\ Soccer} + \underbrace{7}_{Play\ Baseball} - \underbrace{3}_{Play\ both} = 9$$

	S	NS	Total
B	3	4	7
NB	2	5	7
Total	5	9	14

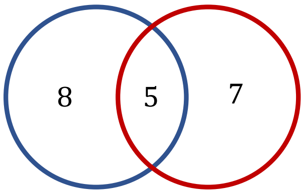
Example 2.107

Thirteen students in my class learn Math. Twelve students in my class learn Science. All students in my class take at least one subject. If the number of students who learn both Math, and Science is five, what is the number of students:

A. who only learn Maths?
B. who only learn Science?
C. In all in my class
D. Who learn either Math, or Science, but not both.

Venn Diagrams

Let the blue circle be the people who learn Math, and the red circle be the people who learn Science.



Number of people who only learn Math:

$$The\ number\ in\ the\ Blue\ circle, but\ outside\ the\ Red\ Circle = 8$$

Number of people who only learn Science:

$$The\ number\ in\ the\ Red\ circle, but\ outside\ the\ Blue\ Circle = 7$$

People in all in my class:

$$8 + 5 + 7 = 20$$

People who learn Math or Science, but not both:

$$\underbrace{8}_{Only\ Math} + \underbrace{7}_{Only\ Science} = 15$$

Algebra

Number of people who only learn Math:

$$n(M \cap S') = n(M) - n(M \cap S) = 13 - 5 = 8$$

Number of people who only learn Science:

$$n(M' \cap S) = n(S) - n(M \cap S) = 12 - 5 = 7$$

The total number of students in the class:

$$n(M \cup S) = n(Only\ M) + n(Only\ S) + n(M\ and\ S) = 8 + 7 + 5 = 20$$
$$n(M \cup S) = n(M) + n(S) - n(M \cap S) = 13 + 12 - 5 = 25 - 5 = 20$$

Contingency Table

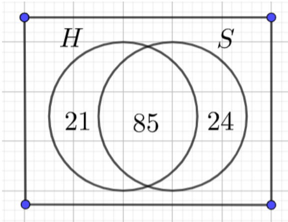
	Science	No Science	Total
Math	5		13

No Math		0	
Total	12		

	Science	No Science	Total
Math	5	8	13
No Math	7	0	7
Total	12	8	20

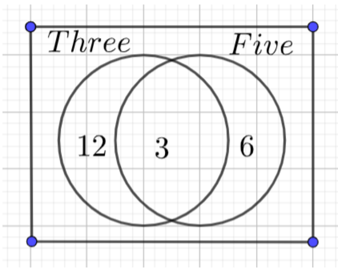
Example 2.108
Every student in the senior class is taking history or science and 85 of them are taking both. If there are 106 seniors taking history and 109 seniors taking science, how many students are in the senior class? **(MathCounts 2002 State Sprint)**

$$n(H \cup S) = n(H) + n(S) - n(H \cap S) = 106 + 109 - 85 = 130$$



A. Number Theory
Many questions in number theory are related to counting the union of two sets.

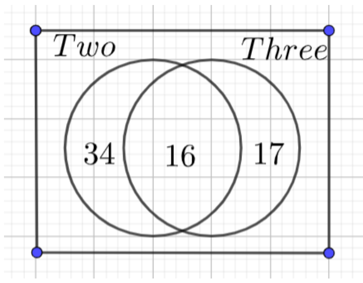
Example 2.109
How many whole numbers from 1 through 46 are divisible by either 3 or 5 or both? **(AMC 8 1991/9)**



$$n\{3,6,..,45\} \Rightarrow 15$$
$$n\{5,10,..,45\} \Rightarrow 9$$
$$n\{15,30,45\} \Rightarrow 3$$

$$n(3 \cup 5) = n(3) + n(5) - n(3 \cap 5) = 15 + 9 - 3 = 21$$

Example 2.110
How many numbers from 1 to 100 are divisible by 2, or 3, or both?



The numbers divisible by are 2 are:
$$\{2,4,\dots,100\} \Rightarrow \{1 \times 2, 2 \times 2, \dots, 50 \times 2\} \Rightarrow 50\ Numbers$$

The numbers divisible by are 3 are:
$$\{3,6,\dots,99\} \Rightarrow \{1 \times 3, 2 \times 3, \dots, 33 \times 3\} \Rightarrow 33\ Numbers$$

The numbers divisible by $LCM(2,3) = 6$ are:
$$\{6,12,\dots,96\} \Rightarrow \{1 \times 6, 2 \times 6, \dots, 16 \times 6\} \Rightarrow 16\ Numbers$$

$$n(Two \cup Three) = n(Two) + n(Three) - n(Two \cap Three) = 50 + 33 - 16 = 67$$

Example 2.111
If all multiples of 3 and all multiples of 4 are removed from the list of whole numbers 1 through 100, then how many whole numbers are left? **(MathCounts 1991 State Countdown)**

Numbers which are multiples of 3 or 4 are:
$$33 + 25 - 8 = 50$$

And after the number are removed, the numbers which are left are:

$$100 - 50 = 50$$

Example 2.112

How many numbers between 2000 and 3000 are not multiples of three or four?

Step I: Count the multiples

$$n(3) = \left\lfloor \frac{2999}{3} \right\rfloor - \left\lceil \frac{2001}{3} \right\rceil + 1 = 999 - 667 + 1 = 333$$
$$n(4) = \left\lfloor \frac{2999}{4} \right\rfloor - \left\lceil \frac{2001}{4} \right\rceil + 1 = 749 - 501 + 1 = 249$$
$$n(12) = \left\lfloor \frac{2999}{12} \right\rfloor - \left\lceil \frac{2001}{12} \right\rceil + 1 = 249 - 167 + 1 = 83$$
$$n(3 \cup 4) = n(3) + n(4) - n(3 \cap 4) = 333 + 249 - 83 = 499$$

Step II: Numbers from 2001 to 2999

$$2999 - 2001 + 1 = 999$$

Step III: Complementary Counting

We want the numbers which are not multiples of three and four, so subtract them from 999:

$$999 - 499 = 500 \text{ numbers}$$

Example 2.113

How many numbers between 100 and 1000 are either $\underbrace{perfect\ squares}_{PS}$ or $\underbrace{perfect\ cubes}_{PC}$?

A number which is a perfect square is of the form

$$n^2, \qquad n \in \mathbb{N}$$
$$n(PS) = n\{121, \dots 961\} = n\{11, \dots 31\} = 31 - 11 + 1 = 21$$

A number which is a perfect cube is of the form

$$n^3, \qquad n \in \mathbb{N}$$

$$n(PC) = n\{125, \dots 729\} = n\{5, 6, 7, 8, 9\} = 5$$

And, we need to find the numbers which are both perfect squares and perfect cubes, and hence they are of both forms

$$n^2, n^3 \Rightarrow n^{LCM(2,3)} = n^6, \qquad n \in \mathbb{N}$$
$$n(PS \cap PC) = \{729\} = 1$$

$$n(PS \cup PC) = n(PS) + n(PC) - n(PS \cap PC) = 21 + 5 - 1 = 25$$

Challenge 2.114

The increasing sequence 2,3,5,6,7,10,11, . .. consists of all positive integers that are neither the square nor the cube of a positive integer. Find the 500th term of this sequence. **(AIME 1990/1)**

528

Challenge 2.115

Find between 1/1000 and 1000, how many numbers are integer powers of 2 or 3?

$$P(2) = n\left(\frac{1}{512}, \frac{1}{256} \dots, 256, 512\right) = n(2^{-9}, 2^{-8} \dots, 2^8, 2^9) = 9 - (-9) + 1 = 19$$
$$P(3) = n\left(\frac{1}{729}, \frac{1}{243} \dots, 243, 729\right) = n(3^{-6}, 3^{-5} \dots, 3^5, 3^6) = 6 - (-6) + 1 = 13$$
$$P(2 \cap 3) = 1 \Leftrightarrow 2^x = 3^x \; iff \; x = 1$$
$$P(2 \cup 3) = P(2) + P(3) - P(2 \cap 3) = 19 + 13 - 1$$

B. Single Set

We do need a separate formula for this. Rather, the focus of the questions can be on counting the number of elements of a single set, given other information.

2.116: Elements belonging only to a set

$$n(Only\ A) = n(A) - n(A \cap B)$$

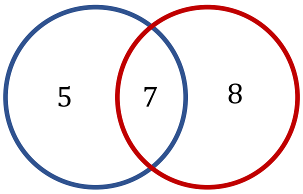
Example 2.117

Twelve students of the twenty students in my class play basketball. Seven students play both basketball and football. If every person in my class plays least one sport, what is:

A. The number of students who play football
B. The number of students who play only football

Venn Diagrams

Let the blue circle be basketball, and the red circle be football.



Algebra

Substitute the known values in the formula for the union of a set:

$$\underbrace{20}_{n(F \cup B)} = n(F) + \underbrace{12}_{n(B)} - \underbrace{7}_{n(F \cap B)} \Rightarrow n(F) = 15$$

Number of people who only football is given by:

$$n(only\ F) = \underbrace{15}_{n(F)} - \underbrace{7}_{n(F\ and\ B)} = 8$$

Contingency Tables

Fill in the information given in the table below.

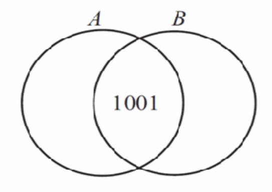
	BB	NBB	Total
FB	7		
NFB		0	
Total	12		20

And then the rest of the table can be filled up:

	BB	NBB	Total
FB	7	8	15
NFB	5	0	5
Total	12	8	20

Example 2.118

Sets A and B, shown in the Venn diagram, have the same number of elements. Their union has 2007 elements and their intersection has 1001 elements. Find the number of elements in A. (AMC 8 2007/13)



Use the formula for union of two sets:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Substitute the known values, and $n(A) = n(B)$:

$$2007 = 2 \times n(A) - 1001 \Rightarrow n(A) = 1504$$

Example 2.119

Sets A and B, shown in the Venn diagram, are such that the total number of elements in set A is twice the total number of elements in set B. Altogether, there are 3011 elements in the union of A and B, and their intersection has 1000 elements. What is the total number of elements in set A? (MathCounts 2011 Chapter Sprint)

Use the formula for union of two sets:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Substitute the known values, and also use the fact that $n(A) = 2 \times n(B)$

$$3011 = 2 \times n(B) + n(B) - 1000$$

Solve the above linear equation:

$$n(B) = 1337$$

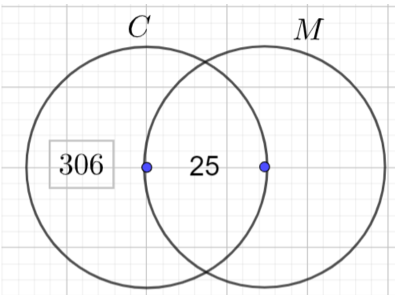
Multiply by 2 to get the value of $n(A)$

$$n(A) = 2n(B) = 2674$$

Example 2.120

In a town of 351 adults, every adult owns a car, motorcycle, or both. If 331 adults own cars and 45 adults own motorcycles, how many of the car owners do not own a motorcycle? (AMC 8 2011/6)

Algebra / Venn Diagram Method



$$n(C \cup M) = n(C) + n(M) - n(C \cap M)$$
$$351 = 331 + 45 - n(C \cap M)$$
$$n(C \cap M) = 331 + 45 - 351 = 25$$

..

Car owners do not own a motorcycle

$$= n(Only\ C) = n(C) - n(C \cap M) = 331 - 25 = 306$$

Contingency Table Method

	Car	No Car	Total
MB			45
No MB		0	
Total	331		351

	Car	No Car	Total
MB			45
No MB	306	0	306

Total	331		351

C. Intersection of Two Sets

2.121: Intersection of Two Sets

If in the formula for the union of a set, one element is missing, it can be found by substituting the known values.

The formula for the union of two sets is:

$$\underbrace{n(A\cup B)}_{\text{Union of } A \text{ and } B} = \underbrace{n(A)}_{\text{Elements in } A} + \underbrace{n(B)}_{\text{Elements in } B} - \underbrace{n(A\cap B)}_{\text{Elements in Intersection of } A \text{ and } B}$$

Example 2.122

Each of the 39 students in the eighth grade at Lincoln Middle School has one dog or one cat or both a dog and a cat. Twenty students have a dog and 26 students have a cat. How many students have both a dog and a cat? (AMC 8 2008/11)

Algebraic Method

$$\underbrace{39}_{n(C\cup D)} = \underbrace{20}_{n(D)} + \underbrace{26}_{n(C)} - n(C\cap D) \Rightarrow n(C\cap D) = 20 + 26 - 39 = 7$$

Contingency Table

	Dog	No Dog	Total
Cat			26
No Cat		0	
Total	20		39

	Dog	No Dog	Total
Cat	7	19	26
No Cat	13	0	13
Total	20	19	39

Example 2.123

In a class of 50 students, 28 participate in MATHCOUNTS, 21 participate in science club, and 6 students participate in neither. How many students participate in both MATHCOUNTS and science club? (MathCounts 2003 National Countdown)

21+28=49
50-6=44

49-44=5

Example 2.124

Fifty students were surveyed about their participation in hockey and baseball. The results of the survey were:

- 33 students played hockey
- 24 students played baseball

- 8 students played neither hockey nor baseball

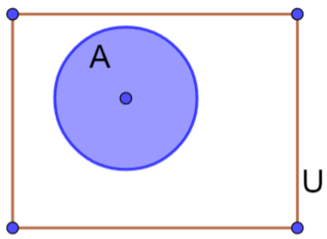
How many of the students surveyed played both hockey and baseball? **(CEMC 2005 Gauss 8)**

15

Example 2.125

There are 200 students enrolled at Memorial Middle School. Seventy of the students are in band and 95 are in chorus. If only 150 students are in band and/or chorus, how many students are in both band and chorus? **(MathCounts 2009 National Sprint)**

$$n(B \cup C) = n(B) + n(C) - n(B \cap C)$$
$$150 = 70 + 95 - n(B \cap C)$$
$$n(B \cap C) = 15$$



2.126: Complement of a Set

The complement of a set is the set of all elements not in the set.

$$n(A) + n(A') = n(U)$$

Rather than proving algebraically, the diagram alongside should help you understand.

$$A = Blue\ Region$$
$$A' = White\ region\ outside\ the\ circle\ but\ inside\ the\ box$$

$$A + A'\ add\ up\ to\ the\ universal\ set$$

Example 2.127

Recall that the universal set is the set of all elements (numbers) under consideration. Also, recall that the complement of a S, written S' is the set of all elements not in the set. A universal set U with $n(U) = 30$ has subsets X and Y such that $n(X) = 18$, and $n(Y) = 16$. If $n(X \cup Y)' = 2$, find $n(X \cap Y)$.

Use the property that $n(A) + n(A') = n(U)$:

$$n(X \cup Y) + n(X \cup Y)' = n(U)$$
$$n(X \cup Y) + 2 = 30$$
$$n(X \cup Y) = 28$$

Use the formula for the union of a set:

$$n(X) + n(Y) - n(X \cap Y) = 28$$
$$18 + 16 - n(X \cap Y) = 28$$
$$n(X \cap Y) = 6$$

Example 2.128

Let S be the set of the 2005 smallest positive multiples of 4, and let T be the set of the 2005 smallest positive multiples of 6. How many elements are common to S and T? **(AMC 10A 2005/22)**

List the multiples of 4 and the multiples of 6:

4,8, **12**, 16,20, **24**, 28,32, **36**, 40, …
6, **12**, 18, **24**, …

The numbers which are common to both lists are multiples of $LCM(4,6) = 12$

We count the number of multiples of 12, which are in the first 2005 multiples of 4.
Every 3rd multiple of 4 is a multiple of 12:

$$\frac{2005}{3} = 668\frac{1}{3}$$

Drop the fractional part to get:

$$668$$

Example 2.129

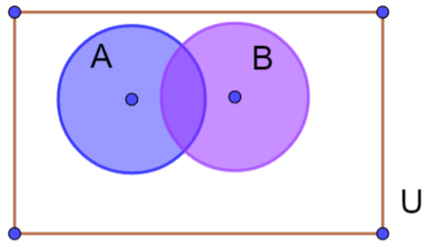
In the above example, explain what is wrong with the "solution" below:

Every 2nd multiple of 6 is a multiple of 12.
Hence, the number of multiples of 12 to be found in the list 6, **12**, 18, **24**, … **is:**

$$\frac{2005}{12} = 167$$

6 is a bigger number than 4.
The first 2005 multiples of 6 will include multiples of 12 that are not among the first 2005 multiples of 4.

D. Elements outside both Sets



2.130: Complement of a Set

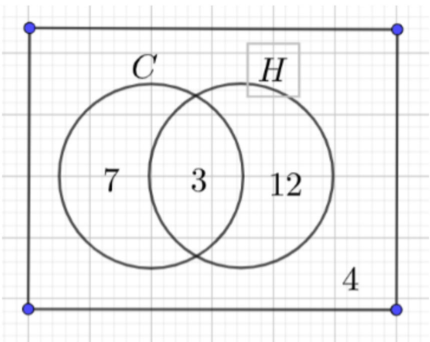
The number of elements not in A, and not in B is those elements which are in the universal set, but outside of both A and B.
They can also be written

$$n[(A \cup B)']$$

Example 2.131

Out of 26 students in a class, 15 students have gone hiking, 3 students have gone both camping and hiking, and 7 students have only gone camping. How many students have gone:

A. Camping
B. for at least one activity
C. for neither camping nor hiking



Define

$$C = Camping, H = Hiking$$

Part A

Then, the number of students who have only camping:

$$n(C) = n(Only\ C) + n(C \cap H) = 7 + 3 = 10$$

Part B

Number of students who went at least one activity means that number of activities can 1 or 2, but not 0.
This means we want the union of camping and hiking:

$$n(C \cup H) = 10 + 15 - 3 = 25 - 3 = 22$$

Part C

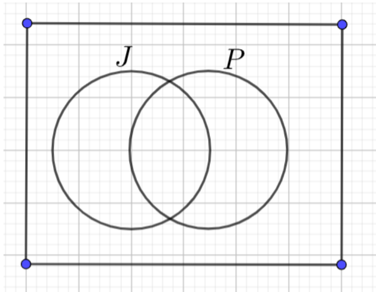
This is the number of people outside both the set:

$$n[(C \cup H)'] = n(U) - n(C \cup H) = 26 - 22 = 4$$

Example 2.132

Out of 50 programmers in an office, 24 program in Python, while 18 program in Java. If the number of people who program in both is one-third the number of people who program in neither, what is the number of people who program in neither?

Algebra



$$n(J) + n(P) - n(J \cap P) + n(J \cap P)' = 50$$
$$24 + 18 + \frac{2}{3}n(J \cap P)' = 50$$
$$\frac{2}{3}n(J \cap P)' = 8$$
$$n(J \cap P)' = 12$$

Contingency Table

	Python	No Python	
Java	x		18
No Java		$3x$	
	24		50

	Python	No Python	
Java	x	$18 - x$ $26 - 3x$	18
No Java		$3x$	
	24	26	50

$$18 - x = 26 - 3x \Rightarrow 2x = 8 \Rightarrow x = 4 \Rightarrow 3x = 12$$

Example 2.133

In a group of 30 high school students, 8 take French, 12 take Spanish and 3 take both languages. How many students of the group take neither French nor Spanish? (MOEMS 1 Olympiad 6)

30-(8+12-3)=30-17=13

Example 2.134

Everyone in a class of 30 students takes math and history. Seven students received an A in history and 13 received an A in math, including four that received an A in both courses. How many students did not receive an A in any of these two courses? (MathCounts 2003 State Countdown)

30 - (7+13-4)=30-16=14

Example 2.135

In a class of 30 students, exactly 7 have been to Mexico and exactly 11 have been to England. Of these 30 students, 4 have been to both Mexico and England. How many students in this class have not been to Mexico or England? (CEMC 2009 Gauss 7)

16

Example 2.136

A survey of 120 teachers determined the following:
70 had high blood pressure
40 had heart trouble
20 had both high blood pressure and heart trouble
What percent of the teachers surveyed had neither high blood pressure nor heart trouble? **(MathCounts 1991 School Sprint)**

25

Example 2.137

Roslyn has ten boxes. Five of the boxes contain pencils, four of the boxes contain pens, and two of the boxes contain both pens and pencils. How many boxes contain neither pens nor pencils? **(MathCounts 2005 Chapter Sprint)**

3

Example 2.138

In a class of 30 students, exactly 7 have been to Mexico and exactly 11 have been to England. Of these students, 4 have been to both Mexico and England. How many students in this class have not been to Mexico or England? (Gauss Grade 7 2009/11)

	VE	NVE	Total
VM	4	3	7
NVM	7	16	23
Total	11	19	30

Example 2.139

How many numbers from 1 to one million are not perfect squares or perfect cubes?

We use complementary counting.

$$Perfect\ Squares: 1^2, 2^2, \ldots, 1000^2 \Rightarrow 1000$$
$$Perfect\ Cubes: 1^3, 2^3, \ldots, 100^2 \Rightarrow 100$$
$$Both\ Perfect\ Squares\ and\ Cubes: 1^6, 2^6, \ldots, 10^6 \Rightarrow 10$$

The total number of numbers which are perfect squares or perfect cubes, or both are:

$$1000 + 100 - 10 = 1090$$

And the number which are not is:

$$1{,}000{,}000 - 1090 = 998{,}910$$

Example 2.140

Out of 200 people in New York, 150 had used the metro, and 80 had used the bus. Twice as many people had used both as had used neither. What is the number of people who had used both?

$$U = n(M) + n(B) - n(M \cap B) + n(M \cup B)'$$
$$200 = 150 + 80 - 2n + n$$
$$n = 150 + 80 - 200 = 30$$
$$2n = 60$$

Challenge 2.141

A Dungeons and Dragons club with 99 members (including the Dungeon Master (DM)) had the following participation in the campaigns for the year:

- Orc Campaign: 61
- Troll Campaign: 51
- Get Together: 31

Each of the above figures includes the DM. The Get Together was mandatory for those not in any campaigns. From the members for whom the Get Together was voluntary, the number of members who attended was exactly double of those who participated in both campaigns. Find the number of people who did not participate in any of the campaigns.

Let

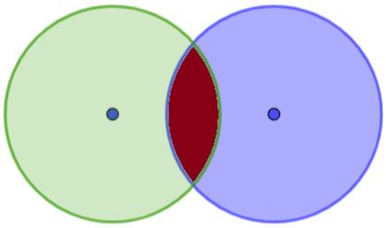
$$O = Orc\ Campaign, T = Troll\ Campaign$$

$$n(O) + n(T) - n(O \cap T) + n(O \cup T)' = n(U)$$
$$60 + 50 - n(O \cap T) + [30 - 2n(O \cap T)] = 98$$
$$140 - 98 = 3n(O \cap T)$$
$$n(O \cap T) = 14$$
$$30 - 2(O \cap T) = 30 - 28 = 2$$

E. Exclusive OR

Normally when we use the word OR, we mean to say that we want the union. Which means that elements which belong to both sets should be counted.

But sometimes, we want to count the number of elements that belong to *exactly one set*. This is called the *exclusive OR*.



2.142: Exclusive OR (XOR)

$$Exclusive\ OR(A, B) = n(Only\ A) + n(Only\ B)$$

Example 2.143

Ten students are taking both algebra and drafting. There are 24 students taking algebra. There are 11 students who are taking drafting only. How many students are taking algebra or drafting but not both? (**MathCounts 2001 Workout 2**)

Number of students taking only algebra

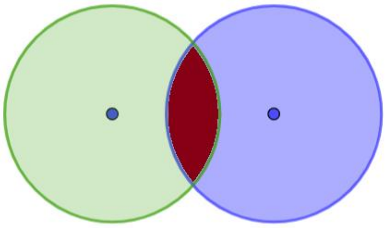
$$= 24 - 10 = 14$$

Number of students taking drafting only

$$= 11$$

Students are taking algebra or drafting but not both

$$= \underbrace{14}_{\substack{Only\\Algebra}} + \underbrace{11}_{\substack{Only\\Drafting}} = 25$$



2.144: Exclusive OR (XOR)

$$Exclusive\ OR(A,B) = \underbrace{n(A)}_{Elements\ in\ A} + \underbrace{n(B)}_{Elements\ in\ B} - \underbrace{2n(A\cap B)}_{\substack{Elements\ in\ Intersection\\ of\ A\ and\ B}}$$

In the diagram we want the region shaded green and the region shaded blue.
We do not want the region shaded maroon.

$$\underbrace{n(A) = Green + Maroon}_{Equation\ I}, \qquad \underbrace{n(B) = Blue + Maroon}_{Equation\ II}$$

Add Equations I and II:

$$n(A) + n(B) = Green + Blue + 2\cdot Maroon$$

Move the $2\cdot Maroon$ to the LHS:

$$n(A) + n(B) - 2\cdot Maroon = Green + Blue$$

But note that $Maroon = n(A\cap B)$

$$n(A) + n(B) - 2n(A\cap B) = Green + Blue$$

And finally, note that $Green + Blue = Exclusive\ OR(A,B)$

$$n(A) + n(B) - 2n(A\cap B) = Exclusive\ OR(A,B)$$

Example 2.145

Let A be the set of integers from 1 to 1000. Find the number of elements of A that are multiples of:

A. 3, but not 4
B. 4, but not 3
C. Either 3, or 4, but not both

First calculate the multiples of 3, 4 and $LCM(3,4) = 12$

$$\{3,6,\dots,999\} \Rightarrow M(3) = \frac{999}{3} = 333$$

$$\{4,8,\dots,1000\} \Rightarrow M(4) = \frac{1000}{4} = 250$$

$$\{12,24,\dots,996\} \Rightarrow M(12) = \frac{996}{12} = 83$$

Then, use the formula:

$$n(Only\ A) = n(A) - n(A\cap B)$$

Part A

$$M(3, but\ not\ 4) = \underbrace{M(3)}_{Multiples\ of\ 3} - \underbrace{M(12)}_{Multiples\ of\ 3\ and\ 4} = 333 - 83 = 250$$

Part B

$$M(4, but\ not\ 3) = \underbrace{M(4)}_{Multiples\ of\ 4} - \underbrace{M(12)}_{Multiples\ of\ 3\ and\ 4} = 250 - 83 = 167$$

Part C

$$M(3, but\ not\ 4) + M(4, but\ not\ 3) = 250 + 167 = 417$$

Calculate the number of elements that are multiple of either 3, or 4, but not both using the direct formula

$$333 + 250 - 2 \times 83 = 417$$

If A is the set of integers between 1 and 1000, then will there be any change in the answer?

The numbers 1 and 1000 are no longer to be considered.
1 is not a multiple of 3, 4, or 12. Hence, no change there.
1000 is not a multiple of 3 or 12. But it is a multiple of 4. Hence, the number of multiples of 4 reduces by one:

$$M(4) = 250 - 1$$

$M(3, but\ not\ 4)$ remains the same.
$M(4, but\ not\ 3)$ reduces by one to be 166.
$M(3\ or\ 4\ but\ not\ both)$ reduces by 1 to be 416

Example 2.146

How many numbers between 1 and 2005 are integer multiples of 3 or 4 but not 12? **(AMC 10B 2005/13)**

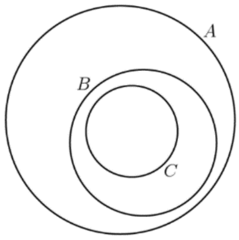
We use the exclusive OR formula:

$$Exclusive\ OR(A, B) = \underbrace{n(A)}_{Elements\ in\ A} + \underbrace{n(B)}_{Elements\ in\ B} - \underbrace{2n(A \cap B)}_{\substack{Elements\ in\ Intersection \\ of\ A\ and\ B}}$$

$$= n(3) + n(4) - 2n(3 \cap 4)$$
$$= 668 + 501 - 2(167) = 835$$

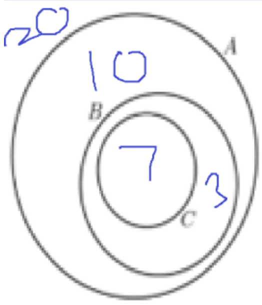
F. Subsets

If one set under consideration is a subset of another set, this can be diagrammed in multiple ways. The choice of diagram depends on preference. You should be aware of the different ways the diagrams can be made.



Example 2.147

A, B and C are circular regions as shown. There are 7 items in circle C. There are exactly 20 items in A and 10 of those items are not in B. How many items are in B, but not in C? **(MathCounts 2011 Chapter Sprint)**



The number of elements in B

$$= n(A) - n(A \cap B') = 20 - 10$$

The number of elements in B but not in C

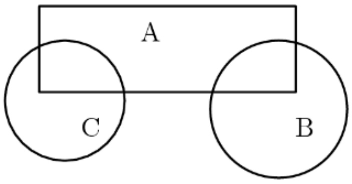
$$= n(B) - n(C) = 10 - 7 = 3$$

2.5 Venn Diagrams: Three Sets

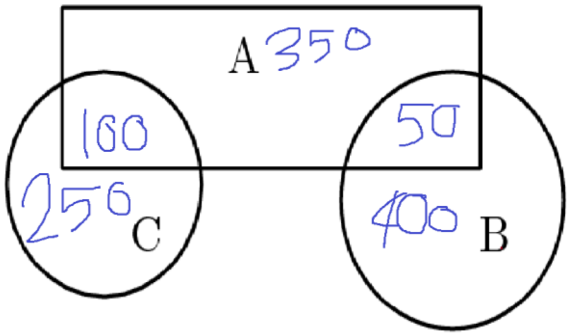
A. Basics

Example 2.148

Three flower beds overlap as shown. Bed A has 500 plants, bed B has 450 plants, and bed C has 350 plants. Beds A and B share 50 plants, while beds A and C share 100. The total number of plants is: **(AMC 8 1999/9)**



$$350 + 100 + 250 + 50 + 400 = 1150$$

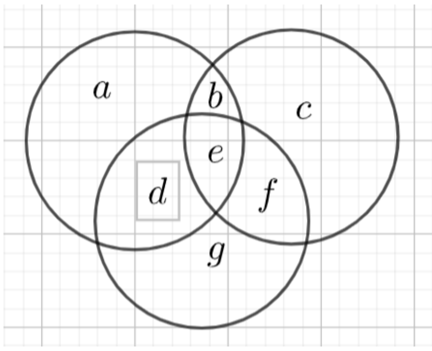


B. Number of Elements in the Union

The formula for the three set Venn Diagram is more complex than the two set Venn Diagram.

2.149: Number of Elements in Three Sets

$$n(A \cup B \cup C) = \underbrace{n(A) + n(B) + n(C)}_{\textit{One at at a Time}} - \underbrace{[n(A \cap B) + n(B \cap C) + n(A \cap C)]}_{\textit{Two at a Time}} + \underbrace{n(A \cap B \cap C)}_{\textit{Three at a time}}$$



Left Hand Side

$$= n(A \cup B \cup C) = a + b + c + d + e + f + g$$

Right Hand Side

$$\begin{aligned} &= (a + b + e + d) + (b + c + e + f) + (d + e + f + g) \\ &\quad - [(b + e) + (e + f) + (d + e)\,] + e \\ &= (a + 2b + c + 2d + 3e + 2f + g) - (b + 3e + d + f) + e \\ &= a + b + c + d + e + f \\ &= RHS \end{aligned}$$

Example 2.150

How many integers between 1 and 280, inclusive, are not divisible by 2, 5 or 7? **(MathCounts 2022 State Sprint)**

We use complementary counting. Count the numbers which are divisible.
Div by $2 = 140$
Div by $5 = 56$
Div by $7 = 40$
Div by 2 and 5: 28
Div by 2 and 7: 20
Div by 5 and 7: 8
Div by 2, 5 and 7: 4

$$\begin{aligned} n(A \cup B \cup C) &= n(2) + n(5) + n(7) - [n(2 \cap 5) + n(5 \cap 7) + n(2 \cap 7)] + n(2 \cap 5 \cap 7) \\ &= 140 + 56 + 40 - [28 + 8 + 20] + 4 = 96 \end{aligned}$$

Hence, the numbers which are not divisible are:

$$280 - 96 = 184$$

Example 2.151

The summary of a survey of 100 students listed the following totals:
59 students did math homework
49 students did English homework
42 students did science homework
20 students did English and science homework
29 students did science and math homework
31 students did math and English homework
12 students did math, science and English homework
How many students did no math, no English and no science homework? (MathCounts 2007 Chapter Sprint)

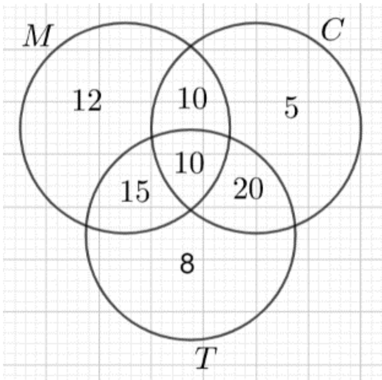
$$n(M \cup E \cup S) = 59 + 49 + 42 - (20 + 29 + 31) + 12 = 82$$
$$n[(M \cup E \cup S)'] = n(U) - n(M \cup E \cup S) = 100 - 82 = 18$$

Example 2.152

An investigator interviewed 100 students to determine their preferences for the three drinks: milk, coffee and tea. He reported the following:

- 10 students had all the three drinks
- 20 had milk and coffee
- 30 had coffee and tea
- 25 had milk and tea
- 12 had milk only
- 5 had coffee only
- 8 had tea only

How many did not take any of the three drinks? (JEE-A 1978)



Draw a diagram and fill in the information:

$$n(M \cap C \cap T) = 10$$

Number of people drinking milk and coffee, but not tea

$$= n(M \cap C) - n(M \cap C \cap T) = 20 - 10 = 10$$

Number of people drinking coffee and tea, but not milk

$$= n(C \cap T) - n(M \cap C \cap T) = 30 - 10 = 20$$

Number of people drinking milk and tea, but not coffee

$$= n(M \cap T) - n(M \cap C \cap T) = 25 - 10 = 15$$

$$n(Only\ M) = 12$$
$$n(Only\ C) = 5$$
$$n\ (Only\ T) = 8$$

$$n(M \cup C \cup T) = 12 + 10 + 5 + 15 + 10 + 20 + 8 = 80$$
$$n[(M \cup C \cup T)'] = n(U) - n(M \cup C \cup T) = 100 - 80 = 20$$

Example 2.153

Alexio has 100 cards numbered 1-100, inclusive, and places them in a box. Alexio then chooses a card from the box at random. What is the probability that the number on the card he chooses is a multiple of 2, 3 or 5? Express your answer as a common fraction. (MathCounts 2002 State Sprint)

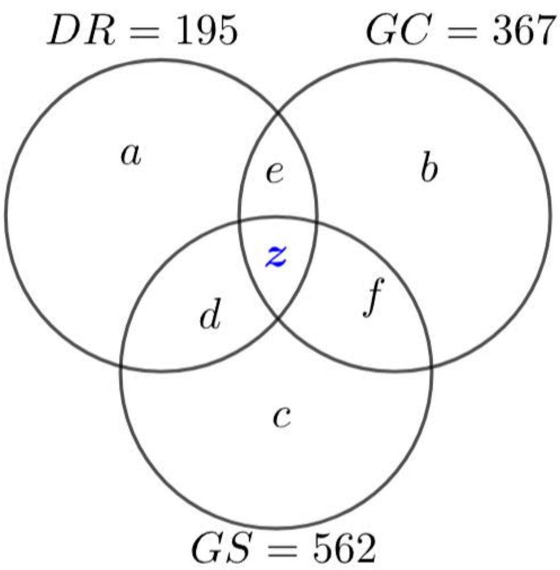
$$n(A \cup B \cup C) = n(2) + n(3) + n(5) - [n(2 \cap 3) + n(2 \cap 5) + n(3 \cap 5)] + n(2 \cap 3 \cap 5)$$
$$= 50 + 33 + 20 - [16 + 10 + 6] + 3$$

$$= 74$$

$$P = \frac{74}{100}$$

Example 2.154

Among the 900 residents of Aimeville, there are 195 who own a diamond ring, 367 who own a set of golf clubs, and 562 who own a garden spade. In addition, each of the 900 residents owns a bag of candy hearts. There are 437 residents who own exactly two of these things, and 234 residents who own exactly three of these things. Find the number of residents of Aimeville who own all four of these things. **(AIME 2024/II/1)**



There are 437 residents who own exactly two of these things:

$$a + b + c = 437$$

234 residents who own exactly three of these things:

$$d + e + f = 234$$

195 who own a diamond ring 367 who own a set of golf clubs, and 562 who own a garden spade:

$$a + e + d + z = 195$$
$$b + e + f + z = 367$$
$$c + d + f + z = 562$$

Add the above three equations:

$$(a + b + c) + 2(d + e + f) + 3z = 195 + 367 + 562$$

Substitute $a + b + c = 437, d + e + f = 234$:

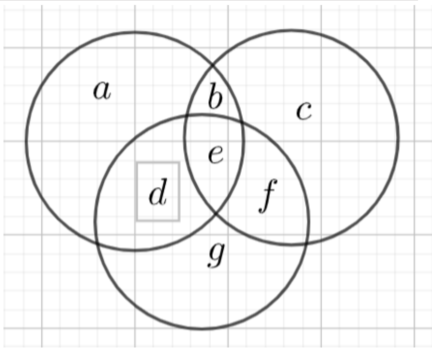
$$437 + 2(234) + 3z = 1124$$

$$z = \frac{1124 - 905}{3} = \frac{219}{3} = 73$$

C. Number of Elements in the Intersection

2.155: Number of Elements in the Intersection

$$\underbrace{[n(A \cap B) + n(B \cap C) + n(A \cap C)]}_{Two\ at\ a\ Time} - 2\underbrace{n(A \cap B \cap C)}_{Three\ at\ a\ time}$$



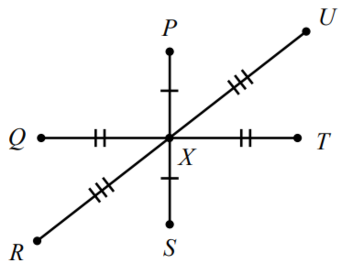
$$b + e + d + f$$

$$n(A \cap B) + n(B \cap C) + n(A \cap C)$$
$$= (b + e) + (e + f) + (d + e)$$
$$= b + e + d + 3e$$

$$n(A \cap B) + n(B \cap C) + n(A \cap C) - n(A \cap B \cap C)$$
$$= b + d + 3e - 2e$$
$$= b + e + d + f$$

Example 2.156

Every 12 minutes, Bus A completes a trip from P to X to S to X to P. Every 20 minutes, Bus B completes a trip from Q to X to T to X to Q. Every 28 minutes, Bus C completes a trip from R to X to U to X to R. At 1:00 p.m., Buses A, B and C depart from P, Q and R, respectively, each driving at a constant speed, and each turning around instantly at the endpoint of its route. Each bus runs until 11:00 p.m. At how many times between 5:00 p.m. and 10:00 p.m. will two or more buses arrive at X at the same time? **(CEMC Gauss 8/2020/24, 7/2020/25)**



Bus A and B

Bus A starts from P at 1:00 pm and reaches X at 1:03 pm.
It travels to S and comes back to X at 1:09 pm.

Times for A to reach X are every six minutes:

$$\{1:03, 1:09, \dots, 2:03, \dots, 3:03, \dots, 9:57\}$$

Times for B to reach X are every ten minutes:

$$\{1:05, 1:15, \dots, 2:05, \dots, 3:05, \dots, 9:55\}$$

A and B reach X together are every $LCM(6,10) = 30\ minutes$ starting $1:15$:

$$\{1:15, 1:45, \dots, 9:45\} \Rightarrow 5 \times 2 = \underbrace{10\ times}_{n(A \cap B)}$$

Bus B and C

C reaches X at intervals of fourteen minutes.

$$\{1:07, 1:21, \dots\}$$

$$14n + 7 > 240 \Rightarrow Smallest\ n = 17$$
$$14(17) + 7 = 245 \Rightarrow C\ reaches\ X\ at$$
$$\{5:05, 5:19, 5:33, 5:47, \dots\}$$

Since:

$$LCM(10,14) = 70$$

Bus B and C reach X together every 70 minutes after $5:05$:

$$\{5:05, 6:15, 7:25, 8:35, 9:45\} \Rightarrow \underbrace{5\ times}_{n(B \cap C)}$$

Bus A and C

$$LCM(6,14) = 42$$

Bus A and C reach X together every 42 minutes after $5:33$:

$$\{5:33, 6:15, 6:57, 7:39, 8:21, 9:03, 9:45\} \Rightarrow \underbrace{7\ times}_{n(A \cap C)}$$

Bus A, B and C

All three buses reach together at:

$$\{6:15, 9:45\} \Rightarrow \underbrace{2\ times}_{n(A \cap B \cap C)}$$

$$\underbrace{10}_{n(A \cap B)} + \underbrace{5}_{n(B \cap C)} + \underbrace{7}_{n(A \cap C)} - 2 \times \underbrace{2}_{n(A \cap B \cap C)} = 22 - 4 = 18$$

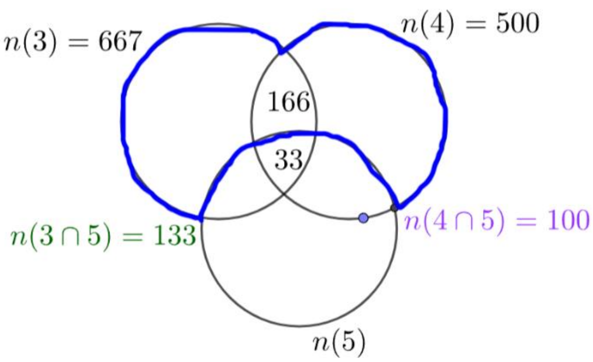
D. Complements

Example 2.157

How many positive integers not exceeding 2001 are multiples of 3 or 4 but not 5? **(AMC 10 2001/25)**

Define, for positive integers not exceeding 2001

$$n(3) = Multiples\ of\ 3$$
$$n(4) = Multiples\ of\ 4$$



Then:

$$n(3) = \left\lfloor \frac{2001}{3} \right\rfloor = 667,$$
$$n(4) = \left\lfloor \frac{2001}{4} \right\rfloor = 500$$
$$n(3 \cap 4) = \left\lfloor \frac{2001}{LCM(3,4)} \right\rfloor = \left\lfloor \frac{2001}{12} \right\rfloor = 166$$

Step I: Multiples of 3 or 4

This can be calculated using the two set Venn Diagram formula for a union of sets:

$$n(3 \cup 4) = n(3) + n(4) - n(3 \cap 4) = 667 + 500 - 166 = 1001$$

Step II: Multiples of 3 or 4 but not 5

From the above we need to subtract the numbers which are multiples of 5:

$$n(3 \cap 5) = \left\lfloor \frac{2001}{LCM(3,5)} \right\rfloor = \left\lfloor \frac{2001}{15} \right\rfloor = 133$$
$$n(4 \cap 5) = \left\lfloor \frac{2001}{LCM(4,5)} \right\rfloor = \left\lfloor \frac{2001}{20} \right\rfloor = 100$$
$$n(3 \cap 4 \cap 5) = \left\lfloor \frac{2001}{LCM(3,4,5)} \right\rfloor = \left\lfloor \frac{2001}{60} \right\rfloor = 33$$

$$n(3 \cup 4 \cap 5') = n(3 \cup 4) - n(3 \cap 5) - n(4 \cap 5) + n(3 \cap 4 \cap 5)$$
$$n(3 \cup 4 \cap 5') = 1001 - 133 - 100 + 33$$

Example 2.158

Ashley writes out the first 2017 positive integers. She then underlines any of the 2017 integers that is a multiple of 2, and then underlines any of the 2017 integers that is a multiple of 3, and then underlines any of the 2017 integers that is a multiple of 5. Finally, Ashley finds the sum of all the integers which have not been underlined. What is this sum? **(Gauss Grade 7 2017/25)**

We will use complementary counting. Find the sum of all the numbers from 1 to 2017, and then subtract the sum of the numbers which have been underlined.

The sum of the first n natural numbers is given by $\frac{n(n+1)}{2}$. Hence:

$$1 + 2 + \cdots + 2017 = \frac{2017 \times 2018}{2} = 2{,}035{,}153$$

$$n(S_2 \cup S_3 \cup S_5) = n(S_2) + n(S_3) + n(S_5) - n(S_6) - n(S_{15}) - n(S_{10}) + n(S_{30})$$

$$S_2 = 2 + 4 + \cdots + 2016 = 2(1 + 2 + \cdots + 1008) = 2\left(\frac{1008 \times 1009}{2}\right) = 1{,}017{,}072$$
$$S_3 = 3 + 6 + \cdots + 2016 = 3(1 + 2 + \cdots + 672) = 3\left(\frac{672 \times 673}{2}\right) = 678{,}384$$
$$S_5 = 5 + 10 + \cdots + 2015 = 5(1 + 2 + \cdots + 403) = 5\left(\frac{403 \times 404}{2}\right) = 407{,}030$$

$$S_6 = 6 + 12 + \cdots + 2016 = 6(1 + 2 + \cdots + 336) = 6\left(\frac{336 \times 337}{2}\right) = 339{,}696$$
$$S_{15} = 15 + 30 + \cdots + 2010 = 15(1 + 2 + \cdots + 134) = 15\left(\frac{134 \times 135}{2}\right) = 135{,}675$$
$$S_{10} = 10 + 20 + \cdots + 2010 = 10(1 + 2 + \cdots + 201) = 10\left(\frac{201 \times 202}{2}\right) = 203{,}010$$

$$S_{15} = 30 + 60 + \cdots + 2010 = 30(1 + 2 + \cdots + 67) = 30\left(\frac{67 \times 68}{2}\right) = 68{,}340$$

$$n(S_2 \cup S_3 \cup S_5) = \underbrace{1017072}_{n(S_2)} + \underbrace{678{,}384}_{n(S_3)} + \underbrace{407{,}030}_{n(S_5)} - \underbrace{339{,}696}_{n(S_6)} - \underbrace{135{,}675}_{n(S_{15})} - \underbrace{203{,}010}_{n(S_{10})} + \underbrace{68{,}340}_{n(S_{30})} = 1{,}492{,}445$$

$$2{,}035{,}153 - 1{,}492{,}445$$

E. Algebra

Example 2.159

A Gourmet Club asked its 52 members which cuisine they liked:

- Clue 1: The number of members who liked Chinese was 5 more than those who liked Thai and double of those who liked Mexican. The ratio of members who liked Chinese, Thai and Mexican was 6: 5: 3.
- Clue 2: 25 members liked Thai
- Clue 3: 7 members liked both Chinese and Thai while 6 members liked both Thai and Mexican.
- Clue 4: The number of members who liked only Thai was one more than the number of members who liked Mexican.
- Clue 5: The number of members who did not like any of Chinese, Thai and Mexican is 4.

Determine the number of members who liked each cuisine

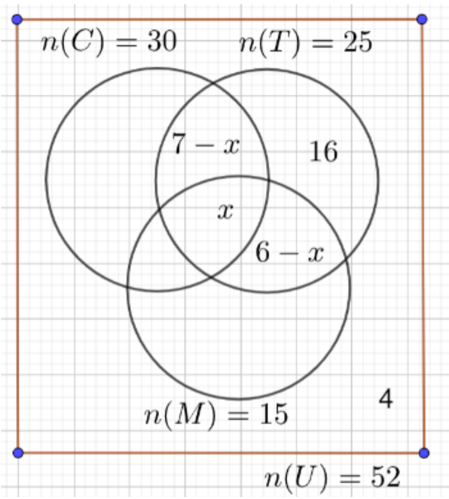
The ratio of members who liked Chinese, Thai and Mexican was

$$6: 5: 3 = 6x: 5x: 3x$$

Also, the number of members who liked Chinese was 5 more than those who liked Thai:

$$6x = 5 + 5x$$
$$x = 5$$

$$\underbrace{30}_{Chinese} : \underbrace{25}_{Thai} : \underbrace{15}_{Mexican}$$



Determine the number of members who liked all three cuisines

From Clue 4, the number of members who liked only Thai

$$= \underbrace{15}_{Liked\ Mexican} + 1 = 16$$

Out of the 25 members who liked Thai, the members who also liked Chinese, or Mexican or both are given by

$$25 - 16 = 9$$

And from the diagram:

$$n(C \cap T) + n(T \cap M) - n(C \cap T \cap M) = 9$$
$$(7 - x) + (x) + (6 - x) = 9$$
$$13 - x = 9$$

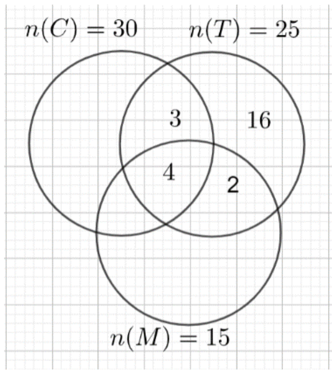
$$x = 4, \qquad 7 - x = 3, \qquad 6 - x = 2$$

Alternate Method

$$n(C \cup T) + n(T \cup M) - n(C \cap T \cap M) = 9$$
$$7 + 6 - n(C \cap T \cap M) = 9$$
$$n(C \cap T \cap M) = 4$$

Determine the number of members who liked at least one of the three cuisines

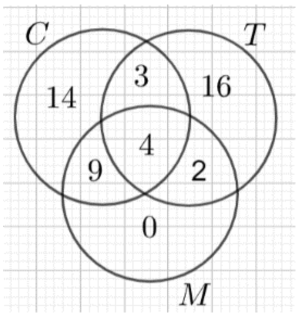
Number of members who like at least one of Chinese, Thai or Mexican is
$$n(C \cup T \cup M) = n(U) - n(C \cup T \cup M)' = 52 - 4 = 48$$
We can now draw an updated diagram, focusing only on the three sets.



Determine the number of members who liked Chinese and Mexican

$n(C \cup T \cup M)$ is given by:
$$n(C) + n(T) + n(M) - n(C \cap T) - n(T \cap M) - n(C \cap M) + n(C \cap T \cap M)$$

Substituting the known values:
$$48 = 30 + 25 + 15 - 7 - 6 - n(C \cap M) + 4$$
$$n(C \cap M) = 13$$

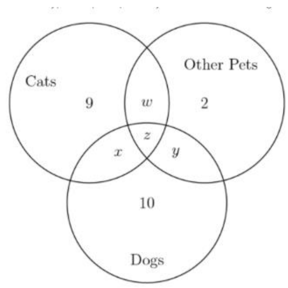


Determine the number of elements in the other regions in the Venn Diagram.

$$n(C \cap M \cap T') = 13 - 4 = 9$$
$$n(M \cap C' \cap T') = 15 - 9 - 4 - 2 = 0$$
$$n(C \cap T' \cap M') = 30 - 3 - 4 - 9 = 14$$

Challenge 2.160

Jeremy made a Venn diagram showing the number of students in his class who own types of pets. There are 32 students in his class. In addition to the information in the Venn diagram, Jeremy knows half of the students have a dog, 3/8 have a cat, six have some other pet and five have no pet at all. How many students have all three types of pets (i.e. they have a cat and a dog as well as some other pet)? **(MathCounts 2006 State Sprint)**



Students in the lettered regions are:
$$w + x + y + z = 32 - 5 - 9 - 10 - 2 = 6$$
But, students in the region $x + y + z$ are also the same:
$$\underbrace{x + y + z = 16 - 10 = 6}_{\textit{Equation I}} \Rightarrow w = 0$$

Therefore,
$$\underbrace{x + z = 12 - 9 = 3}_{\textit{Equation II}}, \qquad \underbrace{z + y = 6 - 2 = 4}_{\textit{Equation III}}$$

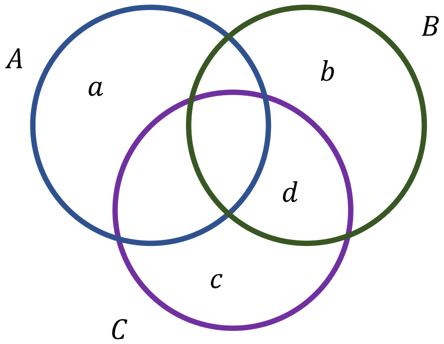
$$\underbrace{y = 3}_{\textit{Eq.I}-\textit{Eq II}}, \qquad \underbrace{x = 2}_{\textit{Eq.I}-\textit{Eq III}} \Rightarrow z = 1$$

Challenge 2.161

$(Warning: Difficult. Feel\ free\ to\ skip\ and\ move\ on)$

Problems A, B and C were posed in a mathematical contest. 25 competitors solved at least one of the three. Amongst those who did not solve A, twice as many solved B as C. The number solving only A was one more than the number solving A and at least one other. The number solving just A equalled the number solving just B plus the number solving just C. How many solved just B? (IMO 1966 A/1)

Set up a Venn Diagram



Let

$$a\ competitors\ solve\ just\ A$$
$$b\ competitors\ solve\ just\ B$$
$$c\ competitors\ solve\ just\ C$$
$$d\ competitors\ solve\ B\ and\ C\ but\ not\ A$$

Set up Equations

The number solving just A equaled the number solving just B plus the number solving just C:

$$\underbrace{a = b + c}_{Equation\ I}$$

Amongst those who did not solve A, twice as many solved B as C. Hence:

$$b + d = 2(c + d) \Rightarrow \underbrace{d = b - 2c}_{Equation\ II}$$

Since the earlier equation is in terms of b and c, find the value of d in terms of b and c:

The number who solved A and at least one of B or C:

$$25 - a - b - c - d$$

The number solving only A was one more than the number solving A and at least one other. Hence:

$$a = 1 + 25 - a - b - c - d$$
$$2a - b - c - d = 26$$

Solve

Substitute the value of d from Equation II, and the value of a from Equation I:

$$2(b + c) + b + c + (b - 2c) = 26$$
$$4b + c = 26$$

Since $c \geq 0, b \geq 0$, we have a Diophantine equation.

$$c = 1 \Rightarrow b = \frac{25}{4} \Rightarrow Not\ Valid$$
$$c = 2, b = 6$$
$$c = 6, b = 4$$

But we also know that:

$$d = b - 2c \geq 0 \Rightarrow b \geq 2c \Rightarrow b = 6, c = 2.$$

Hence:

$$b = 6$$

F. Number Theory

Example 2.162:

How many numbers between 1000 to 2000 are:

A. Divisible by 3, 4 and 5
B. Divisible by 3 and 4, but not 5
C. Divisible by 3 and 5, but not 4
D. Divisible by 5 and 4, but not 3
E. Divisible by 3, but not 4 and 5
F. Divisible by 4, but not 3 and 5
G. Divisible by 5 but not 3 and 4

G. Rules

Example 2.163

If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then:
A. $A = C$
B. $B = C$
C. $A \cap B = \phi$
D. $A = B$ (JEE Main, 2009)

$$x \in A \text{ and } x \in B \Rightarrow x \in A \cap B \Rightarrow x \in A \cap C \Rightarrow x \in C \Rightarrow B = C \Rightarrow Option\ B$$

H. Carpets

Example 2.164

Three rugs have a combined area of 200 m^2. By overlapping the rugs to cover a floor area of 140 m^2, the are which is covered by exactly two layers of rug is 24 m^2. What area of floor is covered by three layers of rug? (Cayley 1998)

$$x + 2(24) + 3z = 200$$
$$x + 24 + z = 140$$

$$24 + 2z = 60$$
$$z = 18$$

I. Probability

Example 2.165

A (JEE-A 2025/PI/2)

Three students S_1, S_2, and S_3 are given a problem to solve. Consider the following events:

U: At least one of S_1, S_2, and S_3 can solve the problem,
V: S_1 can solve the problem, given that neither S_2 nor S_3 can solve the problem,
W: S_2 can solve the problem and S_3 cannot solve the problem,
T: S_3 can solve the problem.

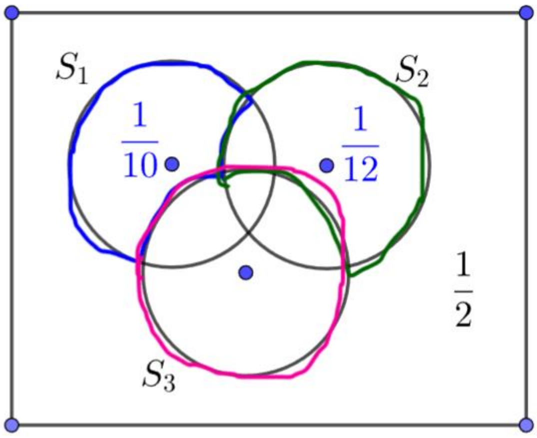
For any event E, let $P(E)$ denote the probability of E. If

$$P(U) = \frac{1}{2}, \qquad P(V) = \frac{1}{10}, \qquad \text{and} \quad P(W) = \frac{1}{12},$$

then $P(T)$ is equal to

Method I: 3 Set Venn Diagrams

$$1-\frac{1}{2}-\frac{1}{10}-\frac{1}{12}=\frac{30}{60}-\frac{6}{60}-\frac{5}{60}=\frac{19}{60}$$



Method II: Algebra
Let

$$P(Student\ i\ solves)=S_i$$

Set up three equations using the given information:

- Equation I: The probability that none of the students can solve the problem is $\frac{1}{2}$
- Equation II: The probability that S_1 can solve, while S_2 and S_3 cannot is $\frac{1}{10}$
- Equation III: The probability that S_2 can solve, while S_3 cannot is:

$$\underbrace{S_1^C S_2^C S_3^C=\frac{1}{2}}_{Equation\ I},\qquad \underbrace{S_1 S_2^C S_3^C=\frac{1}{10}}_{Equation\ II},\qquad \underbrace{S_2 S_3^C=\frac{1}{12}}_{Equation\ III}$$

Divide Equation I by Equation II:

$$\frac{S_1^C}{S_1}=\frac{\frac{1}{2}}{\frac{1}{10}}\Rightarrow\frac{1-S_1}{S_1}=5\Rightarrow 1-S_1=5S_1\Rightarrow S_1=\frac{1}{6}$$

Divide Equation II by Equation III:

$$\frac{S_1S_2^C}{S_2}=\frac{\frac{1}{10}}{\frac{1}{12}}\Rightarrow\frac{\left(\frac{1}{6}\right)(1-S_2)}{S_2}=\frac{6}{5}\Rightarrow\frac{(1-S_2)}{6S_2}=\frac{6}{5}\Rightarrow 5-5S_2=36S_2\Rightarrow 5=41S_2$$

Substitute $S_2=\frac{5}{41}$ in Equation III:

$$S_2S_3^C=\left(\frac{5}{41}\right)(1-S_3)=\frac{1}{12}$$
$$1-S_3=\frac{1}{12}\times\frac{41}{5}=\frac{41}{60}$$
$$S_3=\frac{19}{60}$$

2.6 Other Topics

166 Examples