Work, Power, Energy & Momentum

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1. Energy, Work and Power

1.1 Kinetic Energy

A. Definition and Basics

1.1: Kinetic Energy

The kinetic energy of a moving body with mass m and velocity $\vec{v}$ is

$$\frac{1}{2}mv^2$$

Kinetic energy

- increases as velocity increases
- can never be negative
- is a scalar quantity
- has units of $J = Joule = kg\frac{m^2}{s^2}$

1.2: Kinetic Energy at Constant Mass

For a body with constant mass[1]:

$$KE \propto v^2$$

$$KE = \frac{1}{2}mv^2$$

1.3: Kinetic Energy at Constant Mass

If the kinetic energy is multiplied by a factor of x, then the velocity is multiplied by $\sqrt{x}$

$$xKE = \frac{1}{2}m\left(\sqrt{x}v\right)^2$$

$$KE = \frac{1}{2}mv^2$$

Multiply both sides by x, and rearrange:

$$xKE = x\left(\frac{1}{2}mv^2\right) = \frac{1}{2}m\left(\sqrt{x}v\right)^2$$

Example 1.4

A body is dropped on the ground from a height h_1 and after hitting the ground, it rebounds to a height h_2. If the ratio of velocities of the body just before and after hitting the ground is 4, then the percentage loss in kinetic energy of the body is $\frac{x}{4}$. The value of x is: (JEE Main, April 6, 2023-II)

Since $KE \propto v^2$ for equal masses:

$$\frac{KE_2}{KE_1} = \frac{v_2^2}{v_1^2} = \left(\frac{v_2}{v_1}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

[1] Usually, a moving will remain at constant mass unless there is some reason for the mass to increase. One reason could be that the speed of the body is an significant fraction of the speed of light, in which case relativistic effects will matter.

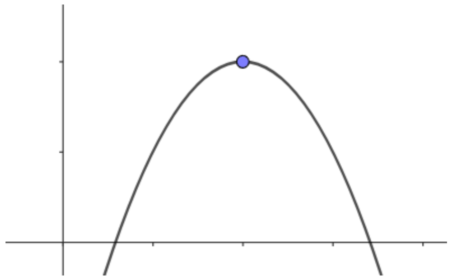
$$Percentage\ Loss = \frac{x}{4} = 1 - \frac{1}{16} = \frac{15}{16}$$
$$x = \frac{15}{4} = 3.75 = 375\%$$

B. Projectile Motion

1.5: Kinetic Energy at Vertex

If a projectile is launched with kinetic energy E, and launch angle θ, then its kinetic energy at the topmost point of the projectile motion is:

$$E\cos^2\theta$$



The original kinetic energy is:

$$E = \frac{1}{2}mv^2$$

The kinetic energy at the topmost point comes only from the horizontal velocity since the vertical component is zero:

$$\frac{1}{2}mv_H^2 = \frac{1}{2}m(v^2\cos^2\theta) = E\cos^2\theta$$

Where the horizontal component of velocity remains constant throughout the movement.

$$v_H^2 = (v\cos\theta)^2 = v^2\cos^2\theta$$

Example 1.6

A. A ball is projected with kinetic energy E at an angle of $60°$ to the horizontal. The kinetic energy of this ball at the highest point in its flight will become (in terms of E): **(JEE Main, July 29, 2022-I; JEE Main 2007)**
B. Repeat Part A for $45°$: **(JEE Main 2002; NEET 2001; NEET 1997)**

$$E\cos^2\theta = E\cos^2 60° = E\left(\frac{1}{2}\right)^2 = \frac{E}{4}$$
$$E\cos^2\theta = E\cos^2 45° = E\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{E}{2}$$

C. Kinematics

Example 1.7

A lift of mass $M = 500\ kg$ is descending with a speed of $2\frac{m}{s}$. Its supporting cable begins to slip thus allowing it to fall with a constant acceleration of $2\frac{m}{s^2}$. The kinetic energy of the lift at the end of the fall through to a distance of 6m will be (in kJ): **(JEE Main, Jan 31, 2023-I)**

$$v^2 = u^2 + 2as = 2^2 + 2(2)(6) = 28\frac{m^2}{s^2}$$
$$KE = \frac{1}{2}mv^2 = \frac{1}{2}(500)(28) = 7000J = 7kJ$$

Example 1.8

A spherical body of mass $2kg$ starting from rest acquires a kinetic energy of 10,000J at the end of the 5^{th} second. The force acted on the body (in N) is: **(JEE Main, Jan 24, 2023-I)**

$$\frac{1}{2}mv^2 = KE \Rightarrow \frac{1}{2}(2)v^2 = 10{,}000 \Rightarrow v = 100\frac{m}{s}$$

$$v = u + at \Rightarrow a = \frac{v-u}{t} = \frac{100-0}{t} = 20\frac{m}{s^2}$$

$$F = ma = (2)(20) = 40N$$

Example 1.9

73: An athlete in the Olympic games covers 100m in 10s. His kinetic energy can be estimated to be: **(JEE Main 2008)**

Assume that the athlete's weight is 80 kg. Then:

$$v = \frac{100}{10} = 10\frac{m}{s} \Rightarrow KE = \frac{1}{2}mv^2 = \frac{1}{2}(80)(10^2) = 4000J$$

Note: The exam question had option *ranges*. The correct option had the (reasonable) range for weight from

$$40\ kg - 100\ kg$$

Example 1.10

A ball of mass 2kg and another of mass 4kg are dropped together from a 60 feet tall building. After a fall of 30 feet each towards earth, their respective kinetic energies will be in the ratio of: **(NEET 2004)**

Since they are dropped together the common velocity is $v = u + at$. Ratio of kinetic energy

$$= \frac{\frac{1}{2}m_1v}{\frac{1}{2}m_2v} = \frac{m_1}{m_2} = \frac{2}{4} = \frac{1}{2}$$

1.11: Displacement

When a constant retarding force is applied to bring a body to a halt:

$$Displacement = \frac{KE}{F}$$

$$v^2 = u^2 + 2as$$

Substitute $acceleration = -a, final\ velocity = v = 0$

$$S = \frac{v^2 - u^2}{2(-a)} = \frac{0^2 - u^2}{2(-a)} = \frac{mu^2}{2ma} = \frac{KE}{F}$$

Where

$$KE = Initial\ kinetic\ energy\ of\ the\ body$$

Example 1.12

Mark the correct Statements

Statement I: A truck and a car moving with same kinetic energy are brought to a halt by applying brakes which apply equal equal retarding forces. Both come to rest in equal distances.
Statement II: A car moving towards east takes a turn and moves towards north, the speed remains unchanged.

The acceleration of the car is zero.

$$Displacement \propto \frac{KE}{F} \Rightarrow I \text{ } is \text{ } correct$$
$$Direction \text{ } changes \Rightarrow Velocity \text{ } changes \Rightarrow Acceleration \text{ } changes \Rightarrow II \text{ } is \text{ } not \text{ } correct$$

Example 1.13

A body of mass 5kg is moving with a momentum of 10 $kg\frac{m}{s}$. Now a force of 2N acts on the body in the direction of its motion for 5s. The increase in the kinetic energy of the body is: **(JEE Main, April 8, 2023-II)**

$$v_I = \frac{p}{m} = \frac{10}{5} = 2\frac{m}{s}$$
$$a = \frac{F}{m} = \frac{2}{5}\frac{m}{s^2}$$
$$v_F = v_I + at = 2 + \left(\frac{2}{5}\right)5 = 4\frac{m}{s}$$

$$\Delta KE = \frac{1}{2}m(v_F^2 - v_I^2) = \frac{1}{2}(5)(4^2 - 2^2) = \frac{1}{2}(5)(12) = 30J$$

D. Momentum

Example 1.14

A body of mass 5kg is moving with a momentum of 10 $kg\frac{m}{s}$. Now a force of 2N acts on the body in the direction of its motion for 5s. The increase in the kinetic energy of the body is: **(JEE Main, April 8, 2023-II)**

$$p_F = p_I + Impulse = p_I + F\Delta t = 10 + (2)5 = 20\ kg\frac{m}{s}$$
$$KE_F - KE_I = \frac{p_F^2}{2m} - \frac{p_I^2}{2m} = \frac{20^2 - 10^2}{2(5)} = \frac{400 - 100}{10} = \frac{300}{10} = 30J$$

Example 1.15

A. If the kinetic energy of a moving body becomes four times its initial kinetic energy, then the percentage change in its momentum will be: **(JEE Main, July 20, 2021-II)**
B. If kinetic energy of a body is increased by 300%, then percentage change in momentum will be: **(NEET 2002)**

Part A

$$v \propto \sqrt{KE} = \sqrt{4} = 2$$

Percentage increase from 1 to 2:

$$= \frac{2-1}{1} = 100\%$$

Part B

$$v \propto \sqrt{KE}$$
$$v_{new} = \sqrt{KE_{new}} = \sqrt{100\% + 300\%} = \sqrt{400\%} = \sqrt{4} = 2 \text{ } times = 1 + 1 = 100\% + 100\%$$
$$100\% \text{ } increase$$

1.16: Momentum

For a body with constant mass

$$p \propto v$$
$$kp = m(kv)$$

Example 1.17

A. The momentum of a body is increased by 50%. The percentage increase in the kinetic energy of the body is: **(JEE Main, April 8, 2023-II)**
B. If momentum of a body is increased by 20%, then its kinetic energy increases by (in percent): **(JEE Main, July 29, 2022-II)**

Part A

Since $p \propto v$:

$$New\ Velocity = V = 1.5v$$
$$V^2 = 2.25v$$
$$Increase\ in\ KE = \frac{2.25v - v}{v} = 1.25 = 125\%$$

$$Increase\ in\ KE = (1 + 25\%)^2 - 1 = 1.5^2 - 1 = 2.25 - 1 = 1.25 = 125\%$$

Part B

$$Increase\ in\ KE = (1 + 20\%)^2 - 1 = 1.2^2 - 1 = 1.44 - 1 = 0.44 = 44\%$$

1.18: Momentum and Kinetic Energy

$$KE = \frac{pv}{2} = \frac{p^2}{2m}$$

$$KE = \frac{mv^2}{2} = \frac{pv}{2}$$
$$KE = \frac{mv^2}{2} = \frac{m^2v^2}{2m} = \frac{p^2}{2m}$$

Example 1.19

A. Two bodies have kinetic energies in the ratio 16: 9. If they have same linear momentum, the ratio of their masses respectively is: **(JEE Main, April 13, 2023-I)**
B. Two bodies with kinetic energies in the ratio of 4:1 are moving with equal linear momentum. The ratio of their masses is: **(NEET 1999)**

Part A

$$\frac{9p^2}{2m_1} = \frac{16p^2}{2m_2} \Rightarrow \frac{m_1}{m_2} = \frac{9}{16}$$

Part B

$$\frac{p^2}{2m_1} = 4 \cdot \frac{p^2}{2m_2} \Rightarrow \frac{m_1}{m_2} = \frac{1}{4}$$

Example 1.20

A. A body of mass 8kg and another of mass 2kg are moving with equal kinetic energy. The ratio of their respective momenta will be: **(JEE Main, July 26, 2022-II)**
B. Two bodies of masses m and $4m$ are moving with equal kinetic energies. The ratio of the magnitudes of their linear momenta is: **(NEET 1998; NEET 1997; NEET 1989)**
C. Two masses of 1g and 9g are moving with equal kinetic energies. The ratio of the magnitudes of their

respective linear momenta is: (NEET 1993)

D. Two particles having masses 4g and 16g respectively are moving with equal kinetic energy. The ratio of the magnitudes of their momentum is $n: 2$. The value of n will be: (JEE Main, Feb 25, 2021-II)

Part A

Substitute $m_1 v_1^2 = m_2 v_2^2 \Rightarrow 8v_1^2 = 2v_2^2 \Rightarrow 2v_1 = v_2$ in:

$$\frac{p_1}{p_2} = \frac{m_1 v_1}{m_2 v_2} = \frac{8v_1}{2(2v_1)} = \frac{2}{1}$$

Part B

Substitute $mv_1^2 = 4mv_2^2 \Rightarrow v_1 = 2v_2$ in

$$\frac{p_1}{p_2} = \frac{mv_1}{4mv_2} = \frac{2v_2}{4v_2} = \frac{1}{2}$$

Part C

Substitute $(1)v_1^2 = (9)v_2^2 \Rightarrow v_1 = 3v_2$ in

$$\frac{p_1}{p_2} = \frac{(1)v_1}{(9)v_2} = \frac{3v_2}{(9)v_2} = \frac{1}{3}$$

Part D

Substitute $(4)v_1^2 = (16)v_2^2 \Rightarrow v_1 = 2v_2$ in

$$\frac{p_1}{p_2} = \frac{(4)v_1}{(16)v_2} = \frac{2v_2}{4v_2} = \frac{1}{2} \Rightarrow n = 1$$

Example 1.21

A. Two solids A and B of mass 1kg and 2kg respectively are moving with equal linear momentum. The ratio of their kinetic energies $(K.E.)_A : (K.E.)_B$ will be $\frac{A}{1}$, so the value of A will be: (JEE Main, Feb 24, 2011-II)

B. Two bodies of mass 5kg and 8kg are moving such that the momentum of body B is twice that of body A. The ratio of their kinetic energies will be: (JEE Main, June 30, 2022-I)

Part A

Substitute $1(v_A) = 2(v_B) \Rightarrow \frac{v_A^2}{v_B^2} = 4$ in

$$\frac{KE_A}{KE_B} = \frac{m_A}{m_B} \cdot \frac{v_A^2}{v_B^2} = \frac{1}{2} \cdot 4 = \frac{2}{1} \Rightarrow A = 2$$

Part B

$$m_B v_B = 2m_A v_A$$
$$\frac{v_A}{v_B} = \frac{m_B}{2m_A} = \frac{8}{2(5)} = \frac{4}{5}$$
$$\frac{v_A^2}{v_B^2} = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$$
$$\frac{m_A v_A^2}{m_B v_B^2} = \frac{5}{8} \cdot \frac{16}{25} = \frac{2}{5}$$

Example 1.22

An object explodes into two parts with masses m_1 and m_2 respectively, and velocities v_1 and v_2 respectively. The ratio of their kinetic energies $\frac{E_1}{E_2}$ in terms of their:

A. Velocities will be:

B. Masses will be: (NEET 2003)

Part A

$$\frac{E_1}{E_2}=\frac{\frac{1}{2}m_1v_1^2}{\frac{1}{2}m_2v_2^2}=\frac{\left(\frac{1}{2}m_1v_1\right)v_1}{\left(\frac{1}{2}m_2v_2\right)v_2}=\frac{v_1}{v_2}$$

Part B

$$\frac{E_1}{E_2}=\frac{\frac{1}{2}m_1v_1^2}{\frac{1}{2}m_2v_2^2}=\frac{\frac{1}{2}(m_1v_1)^2}{\frac{1}{2}(m_2v_2)^2}\cdot\frac{m_2}{m_1}=\frac{m_2}{m_1}$$

E. Rate of Kinetic Energy

Example 1.23

An engine pumps water continuously through a hose. Water leaves the hose with a velocity v and m is the mass per unit length of the water jet. What is the rate at which kinetic energy is imparted to the water? (In terms of m and v)(NEET 2009)

Mass per unit time

$$=mv$$

Kinetic Energy imparted per unit time

$$=\frac{1}{2}Mv^2=\frac{1}{2}(mv)v^2=\frac{1}{2}mv^3$$

F. Further Questions

Pending

Example 1.24

64: An object is dropped from a height h from the ground. Every time it hits the ground, it loses 50% of its kinetic energy. The total distance covered at $t\to\infty$ is: (JEE Main, April 8, 2017)

1.2 Gravitational Potential Energy

A. Conservation of Energy

1.25: Conservation of Energy

Energy can neither be destroyed nor created. It can only be converted from one form to another.

- We will focus on conversion of GPE to KE, and vice versa.
- Conversion of energy to heat energy will happen due to friction, and this would be the topic of thermodynamics.

1.26: Potential Energy

Potential energy is stored energy. It can come in various forms:

- Gravitational potential energy: Water at the top of a mountain, etc
- Elastic potential energy: A drawn bowstring, a pulled spring
- Chemical potential energy: The energy in a battery

1.27: Potential Energy Notation

$$Potential\ Energy = U(x)$$

Example 1.28

A particle with total energy E is moving in a potential energy region $U(x)$. Motion of the particle is restricted to the region where: (NEET 2013)

Particle has total energy E.
U cannot exceed E.

$$U(x) \le E$$

B. Gravitational Potential Energy

Example 1.29

Add in momentum notes: component breakup from projectile motion

1.30: Gravitational Potential Energy (GPE)

The gravitational potential energy of an object is the energy that is due to the position/height of an object:

$$mgh$$

Where

$$h = height\ of\ object\ from\ ground$$

- This assumes that the object on the ground has zero potential energy
-

1.31: Change in Gravitational Potential Energy (GPE)

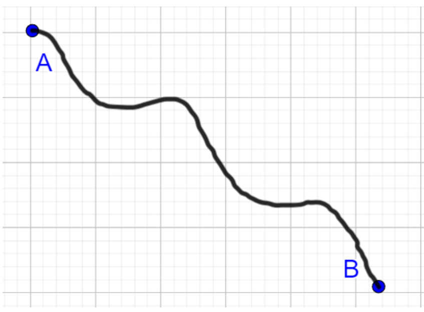
$$\Delta GPE = mg\Delta h$$

- In general, potential energy is measured as a change, and not as an absolute value.
- For questions with heights, you can introduce an origin which is not the ground, and assume that the energy is zero at the origin.

Example 1.32

An object is placed at point A. It is initially at rest. When it is released, it slides down the surface shown on the right diagram to reach point B on the ground. At what point during its journey does it have:

A. Maximum GPE
B. Maximum KE
C. Minimum GPE
D. Minimum KE



At the beginning, let

$$GPE = C$$

At the bottom

$$GPE = 0, KE = C$$

At the top

$$GPE = C, KE = 0$$
$$Maximum\ GPE = Minimum\ KE\ at\ the\ top$$
$$Minimum\ GPE = Maximum\ KE\ at\ the\ bottom$$

Example 1.33

Mark all correct options

For a body projected at an angle with the horizontal from the ground:

A. The vertical component of momentum is maximum at the highest point.
B. The kinetic energy is zero at the highest point of projectile motion.
C. Gravitational potential energy is maximum at the highest point.
D. The horizontal component of velocity is zero at the highest point.

At the highest point:

$$Vertical\ component\ is\ zero \Rightarrow Option\ A\ is\ not\ correct$$

Throughout the projectile motion, horizontal component remains constant. Hence, kinetic energy is nonzero throughout the projectile motion

$$Option\ B\ is\ not\ correct$$
$$Option\ D\ is\ not\ correct$$

GPE is minimum at the ground and maximum at the highest point

$$Option\ C\ is\ correct$$

C. Potential Energy as Difference

1.34: Zero Point of Potential Energy

- Potential energy is measured as a difference rather in absolute terms.
- The zero point is arbitrarily established.

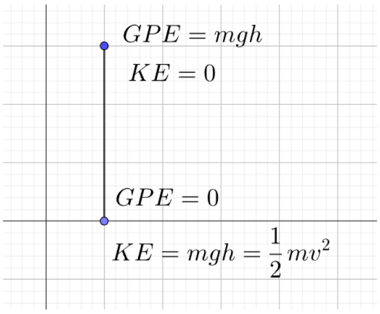
Note: We often use the ground as zero potential energy to simplify calculations, but this is not required. All answers will be correct even if we assume zero potential energy at a different height.

Example 1.35

A book has zero gravitational potential energy when it is on a table of height h. What is its gravitational potential energy when it is moved to the floor.

$$-mgh$$

D. Conversions



1.36: Gravitational to Kinetic Energy

A fall will convert gravitational energy to kinetic energy.
If the body bounces back, the rise will convert kinetic energy to potential energy.

1.37: Gravitational to Kinetic Energy

We can connect gravitational energy to kinetic energy using the relation:

$$v^2 = 2g\Delta h$$

Where

$$v = velocity\ of\ object\ released\ from\ rest\ falling\ height\ h$$

Method I: Kinematics

$$v^2 = u^2 + 2as$$

Substitute $u = initial\ velocity = 0, a = g, s = h$

$$v^2 = 2g\Delta h$$

Method II: By conservation of energy

$$KE_{Bottom} = GPE_{Top}$$
$$\frac{1}{2}mv^2 = mg\Delta h$$
$$v^2 = 2g\Delta h$$

This assumes that there is conversion only from gravitational to kinetic and vice versa.

Example 1.38

A. A child is sitting on a swing. Its minimum and maximum heights from the ground $0.75m$ and $2m$ respectively. Its maximum speed will be: **(NEET 2001)**
B. A rubber ball is dropped from a height of 5 m on a plane. On bouncing it rises to 1.8 m. The ball loses its velocity on bouncing by a factor of: (Express your answer as a fraction in simplest form) **(NEET 1998)**

Part A

Substitute $\Delta h = h_1 - h_2 = 2 - 0.75 = 1.25$

$$v = \sqrt{2g\Delta h} = \sqrt{2(10)(1.25)} = \sqrt{25} = 5\frac{m}{s}$$

Part B

$$\frac{v}{V} = \frac{\sqrt{2g\Delta h_1}}{\sqrt{2g\Delta h_2}} = \frac{\sqrt{\Delta h_1}}{\sqrt{\Delta h_2}} = \sqrt{\frac{1.8}{5}} = \sqrt{\frac{36}{100}} = \frac{6}{10} = \frac{3}{5}$$

Example 1.39

A ball is thrown vertically downwards from a height of 20 m with an initial velocity v_0. It collides with the ground, loses 50 percent of its energy in collision and rebounds to the same height. The initial velocity v_0 is: (take $g = 10\frac{m}{s^2}, \sqrt{2} \approx 1.4$) **(NEET 2015)**

The velocity with which it rebounds (using conservation between GPE and KE)

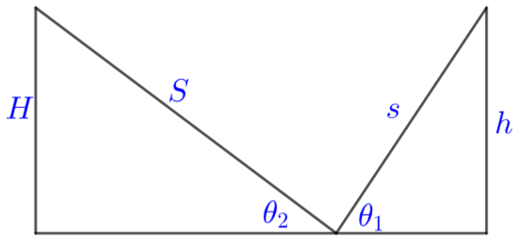
$$v = \sqrt{2g\Delta h} = \sqrt{2(10)(20)} = 20\frac{m}{s}$$

Since $KE = \frac{1}{2}mv^2 \Rightarrow v \propto \sqrt{KE}$ and the energy in v_0 is twice the energy in v:

$$v_0 = \sqrt{\frac{KE_2}{KE_1}}v = \sqrt{2}v = 1.4(20) = 28\frac{m}{s}$$

Example 1.40

A ball rolls down a smooth ramp that makes an angle θ_1 with the ground. The ball is initially at height h at rest. This ramp is next to another ramp which makes an angle θ_2 with the ground. Determine the maximum height reached by the body on the second ramp using kinematics concepts.



Speed at bottom of 1st ramp

Substitute $u = initial\ velocity = 0, a = \frac{F}{m} = \frac{mg\sin\theta_1}{m} = g\sin\theta_1, displacement = s$ in

$$v^2 = u^2 + 2as = 0^2 + 2(g\sin\theta_1)s$$

Substitute $\sin\theta_1 = \frac{h}{s} \Rightarrow h = s\ \sin\theta_1$:

$$v^2 = 2gh$$

Height up second ramp

$$S = displacement\ on\ left\ ramp$$
$$v^2 = u^2 + 2(-\,g\sin\theta_2)S$$

Substitute $\sin\theta_2 = \frac{H}{S} \Rightarrow H = S\sin\theta_2$:

$$0 = (2gh) + 2(-g)H$$
$$h = H$$

Example 1.41

A ball rolls down a smooth ramp from a height h that makes an angle θ_1 with the ground. This ramp is next to another ramp which makes an angle θ_2 with the ground.

A. Determine the maximum height reached by the body on the second ramp using conservation of energy concepts.
B. What is the assumption made in Part A.

Part A

When the ball rolls down the ramp

$$GPE\ is\ converted\ to\ KE$$

When the ball goes up the second ramp:

$$KE\ is\ converted\ to\ GPE$$

With conservation of energy, we know that

$$Old\ Height = New\ Height$$

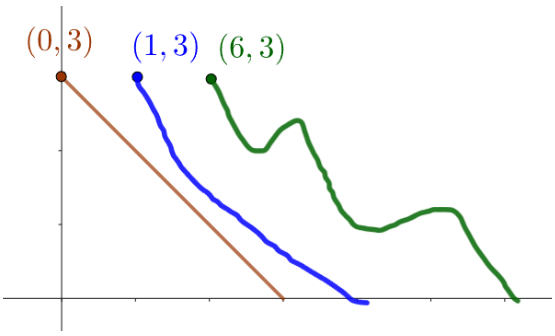
Part B

There is no conversion of energy into any form other than kinetic and gravitational.

Example 1.42

Mark the correct option

Of the three smooth ramps alongside, the first is a straight line. The second is a strictly decreasing function. The third is a "roller coaster." A ball is released from the topmost point of each ramp at time $t = 0$. Rank the balls in order of increasing velocity when they leave the ramps. Ignore friction.



A. $I > II > III$
B. $I < II < III$
C. $III > I < II$

D. $II > III < I$
E. None of these

By conservation of energy

$$I = II = III \Rightarrow Option\ E$$

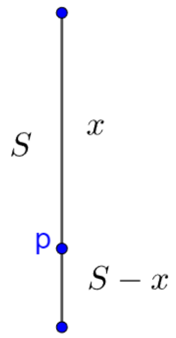
Example 1.43

A body of mass $1\ kg$ is thrown upwards with a velocity $20\frac{m}{s}$. It momentarily comes to rest after attaining a height of $18\ m$. How much energy is lost due to air friction? (take $g = 10\frac{m}{s^2}$) **(NEET 2009)**

$$\underbrace{\frac{1}{2}mv^2}_{\substack{Initial\\Kinetic\ Energy}} - \underbrace{mgh}_{\substack{Final\\Potential\\Energy}} = \frac{1}{2}\cdot 1\cdot 20^2 - 1\cdot 10\cdot 18 = 200 - 180 = 20J$$

Example 1.44

A particle is released from height S from the surface of the Earth. At a certain height, its kinetic energy is three times its potential energy. The height from the surface of Earth and the speed of the particle at that instant, are, respectively: (Answer in terms of g and the other values given in the question) **(NEET 2021)**



Let the displacement of the particle from its initial position be

$$x$$

Use the kinetic energy condition that $KE_p = 3GPE_p$:

$$\frac{1}{2}mv^2 = 3mg(S - x)$$

Cancel m and solve for x:

$$\frac{v^2}{6g} = S - x \Rightarrow \underbrace{x = S - \frac{v^2}{6g}}_{Equation\ I} = \frac{6gS - v^2}{6g}$$

Use the equation of motion $v^2 = u^2 + 2as$. Substitute $s = x = \frac{6gS - v^2}{6g}$:

$$v^2 = 2(g)\left(\frac{6gS - v^2}{6g}\right) \Rightarrow 3v^2 = 6gS - v^2 \Rightarrow v^2 = \frac{6gS}{4} = \frac{3gS}{2}$$

Substitute the value of v^2 from above into Equation I:

$$x = S - \frac{v^2}{6g} = S - \frac{\frac{3gS}{2}}{6g} = S - \frac{3gS}{2}\cdot\frac{1}{6g} = S - \frac{S}{4} = \frac{3S}{4}$$

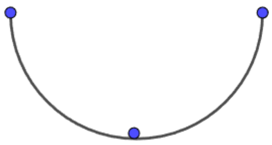
Determine the height from the ground as:

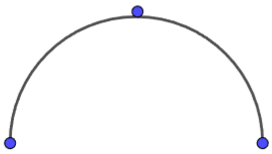
$$S - x = S - \frac{3S}{4} = \frac{S}{4}$$

E. Stable and Unstable Equilibrium

1.45: Stable and Unstable Equilibrium

- A system in a stable equilibrium tends to come back to the stable state.
- A system in an unstable equilibrium tends to go away from the equilibrium if it is disturbed.





- An important example of a system in a stable equilibrium is a spring at rest.

1.3 Work(1D)

A. Work: Change in Kinetic Energy

1.46: Work-Energy Theorem

Net work is the change in kinetic energy of an object due to the forces on it:

$$W = \Delta KE = \frac{1}{2}m(v^2 - u^2)$$

- Kinetic energy is a scalar quantity
- Work is a *signed* scalar quantity.
- *Net* force is important. Work is the change in kinetic energy due to the *resultant* of all forces.

Example 1.47

Mark the correct option

Work done:

A. can only be positive
B. can only be negative
C. can either be positive or negative
D. cannot be assigned a sign **(EAMCET 17 Sep 2020, Shift-I)**

Option C

Example 1.48

The kinetic energy acquired by a mass m in travelling distance d, starting from rest, under the action of a constant force is directly proportional to m^x. Determine the value of x. **(NEET 1994)**

$$\Delta KE = W = Fs$$

Since F is constant, and s (distance) is not dependent on mass

$$\Delta KE \propto m^0 \Rightarrow x = 0$$

1.49: Joule: Unit of Work

$$1\ Joule = 1\ kg\frac{m^2}{s^2}$$

- Units of work are same as units of kinetic energy

Example 1.50
Show that 1 $Joule = 1\ Nm$ and interpret it.

$$\left(kg\ \frac{m}{s^2}\right)m = Nm$$

The interpretation will be $1N$ of force applied over $1m$ results in 1 Joule of work.

Example 1.51
Show that the unit of work is dimensionally correct.

$$\frac{1}{2}m(v^2-u^2) = Mass \times Velocity = kg \times \frac{m^2}{s^2} = kg\frac{m^2}{s^2}$$

Example 1.52
A. A body of mass 0.5 kg travels on straight line path with velocity $v = (3x^2+4)\frac{m}{s}$. The net work done by the force during its displacement from $x = 0$ to $x = 2$ is: **(JEE Main, July 25, 2022-I)**

B. A particle of mass 500 gm is moving in a straight line with velocity $v = bx^{\frac{5}{2}}$. The work done by the net force during its displacement from $x = 0$ to $x = 4m$ is (in J) ______. (Take $b = 0.25m^{-\frac{3}{2}}s^{-1}$) **(JEE Main, June 29, 2022-I)**

Part A

$$W = \frac{1}{2}m(v^2-u^2) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(16^2-4^2) = \frac{1}{4}(240) = 60J$$

Part B

$$v = bx^{\frac{5}{2}} = \left(0.25m^{-\frac{3}{2}}s^{-1}\right)(x\ m)^{\frac{5}{2}} = \frac{x^{\frac{5}{2}}}{4}\frac{m}{s} \Rightarrow v^2 = \frac{x^5}{2^4}\frac{m^2}{s^2}$$

$$W = \frac{1}{2}m(v^2-u^2) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{4^5}{2^4}-0\right) = 2^4 = 16J$$

1.53: Work in terms of force
For a constant force parallel to the direction of movement:

$$W = Fs$$

where

$$Work = W, \qquad F = Force, \qquad s = displacement$$

$$\Delta KE = KE_F - KE_I$$

Substitute the definition of kinetic energy and note that mass is constant:

$$= \frac{1}{2}mv^2 - \frac{1}{2}mu^2 = (ma)\left(\frac{v^2-u^2}{2a}\right)$$

Substitute $F = ma, v^2 = u^2 + 2as \Rightarrow s = \frac{v^2-u^2}{2a}$

$$W = Fs$$

1.54: Zero Work
- If displacement is zero, then *net* work done is zero.
- Equivalently, if there is no movement, there is no change in kinetic energy and hence work done is zero.

Example 1.55

Mark the correct option

A man pushes the wall and fails to displace it. He does:

A. negative work
B. positive but not maximum work
C. no work at all
D. maximum work **(EAMCET 21 Sep 2020, Shift-II)**

$$No\ Work \Rightarrow Option\ C$$

Example 1.56

The work done by a force on an object is zero over an interval is zero. Can we conclude that the displacement of the object over that time interval is zero.

If displacement is zero, then work is zero.
The converse (if work is zero, then displacement is zero) may or may not be true.
We will see specific cases where this happens later in this section.

1.57: Sign of Work

If a force

- increases kinetic energy (accelerates a body), work is positive.
- decreases kinetic energy (deaccelerates a body), work is negative.

Example 1.58

A cyclist comes to a skidding stop in $10m$. During the process the force on the cycle due to the road is $200N$, and is directly opposite to the motion. How much work does the road do on the cycle? **(EAMCET 17 Sep 2020, Shift-II)**

$$W = Fs = (-200N)(10m) = -2000J$$

B. Conservative and Non-Conservative Forces

1.59: Conservative Force

Conservative force is a force such that the total work done in moving a particle between two points is path independent.

Conservative	Non conservative
Gravity	Friction, Air resistance, Drag
Elastic spring restoring force	Viscosity
Electrostatic force	Resistance to flow of electric current
Buoyancy	

1.60: Closed Loop

If a particle travels in a closed loop, the total work done by a conservative force is zero.

Work done by	$A\ to\ B$	$B\ to\ A$	$A\ to\ A$

Example 1.61

A vehicle moves in a straight line from A to B on a rough surface, and then from B to A. Fill in each cell in the table with one of $zero$, $positive$, or $negative$.

Friction			
Engine			

Work done by	A to B	B to A	A to A
Friction	$-ve$	$-ve$	$-ve$
Engine	$+ve$	$+ve$	$+ve$

Example 1.62

In the previous example, the work done by the engine from A to A is positive. Since displacement is zero, does this contradict the formula

$$W = Fs$$

Explain why or why not?

$$W = Fs$$

For a *constant* force.

1.63: Conservative Forces and Energy

With each conservative force, we associate potential energy.
We do not associate potential energy with non-conservative forces.

- Energy associated with gravity is gravitational potential energy
- Energy associated with electrostatic force is electric potential energy

Example 1.64

A. A man climbs a ladder. Explain in terms of work, the change in the potential energy of the man.
B. A ball is released from a cliff. Explain in terms of work, the change in the potential energy of the ball.

Part A

There is an increase in potential energy.
This is because the man has done positive work against the force of gravity.

Part B

There is a decrease in potential energy.
Gravity does positive work on the ball.

1.65: Conservative Forces and Energy

When a conservative force does work on a system, the potential energy decreases.
When a system does work on a conservative force, the potential energy increases.

Work here means *positive* work.

Example 1.66

The potential energy of a system increases if work is done:

A. Upon the system by a nonconservative force

B. By the system against a conservative force
C. By the system against a nonconservative force
D. Upon the system by a conservative force **(NEET 2011)**

$$Option\ B$$

C. Work due to Friction

1.67: Work due to kinetic friction

Work due to friction on a horizontal plane is:

$$-\mu_k mgs$$

Where

$$Displacement = s > 0$$
$$m = mass\ of\ object$$
$$\mu_k = coefficient\ of\ kinetic\ friction$$

For an object on a horizontal plane:

$$f_k = \mu_k F_N = \mu_k mg$$

$$W = Fs = (-\mu_k mg)(s)$$

Example 1.68

$$(-\mu_k mg)s = (\mu_k mg)(-s), \qquad s > 0$$

Assume general convention that right is positive direction and left is negative.

$$\underbrace{(-\mu_k mg)}_{\substack{-f_k=Friction \\ Leftward}} \underbrace{s}_{Displacement} = \underbrace{\mu_k mg}_{\substack{-f_k=Friction \\ Rightward}} \underbrace{-s}_{Displacement}$$

Example 1.69

Explain why work done by kinetic friction is always negative.

Method I: Mathematical

Frictional force is always opposite to the direction of motion.

$$f_k s$$

Hence, the two will have opposite signs, and hence work due to frictional force is always negative.

Method II: Conceptual

Frictional always reduces movement speed, and hence it reduces kinetic energy.
Hence, work due to frictional force is always negative.

1.70: Work due to static friction

Work due to static friction is zero.

Because displacement is zero.

Example 1.71

Work done by friction on an object on a ramp is zero. Can we conclude that the frictional force is zero?

No. It is possible that the object did not move, which meant displacement is zero.

$$W = Fs = F(0) = 0$$
$$F > 0 \text{ is possible}$$

Hence, frictional force can be non-zero.
In such a case, it would be static friction.

Example 1.72

Mark all correct options

Work done by friction can be:

A. Zero
B. Negative
C. Positive
D. None of the above

$$Options\ A, B$$

1.73: Work done by an engine

If a vehicle moves at a constant velocity over a rough surface, then:

$$W_{Engine} = -W_{Friction} = \mu_k mgs, s > 0$$

From Newton's First Law:$F_{Net} = 0$

$$F_{Net}(s) = 0$$
$$W = 0$$
$$W_{Engine} + W_{Friction} = 0$$
$$W_{Engine} = -W_{Friction}$$
$$W_{Engine} = -(-\mu_k mgs)$$
$$W_{Engine} = \mu_k mgs$$

Example 1.74

To maintain a speed of 80 $\frac{km}{h}$ by a bus of mass 500 kg on a plane rough road for 4 km distance, the work done by the engine of the bus will be (in KJ): (The coefficient of friction between the tire of bus and the road is 0.04) **(JEE Main, April 12, 2023-I)**

$$W_{Engine} = \mu_k mgs = (0.04)(500\ kg)\left(9.8\frac{m}{s}\right)(4000\ m) = 784\ kJ$$

Example 1.75

A small block starts slipping down form a point B on an inclined plane AB, which makes an angle θ with the horizontal section. BC is smooth and the remaining section CA is rough with a coefficient of friction μ. It is found that the block comes to rest as it reaches the bottom (point A) of the inclined plane. If $BC = 2AC$, the coefficient of friction is $\mu = k\tan\theta$. The value of k is: **(JEE Main, 2 Sep, 2020-I)**

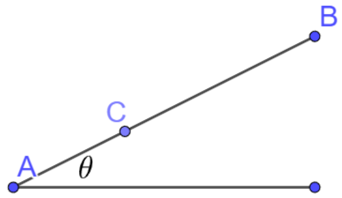
A. Solve the question using Kinematics.
B. Solve the question using Work.

C. Compare the two methods

Part A: Kinematics

Use the equation of motion twice:

$$v^2 = u^2 + 2as$$



Evaluate over the interval BC. Substitute $u^2 = 0, a = g\sin\theta, s = 2AC$

$$Velocity\ at\ C: v^2 = 0 + 2(g\sin\theta)(2AC) = 4(g\sin\theta)(AC)$$

Evaluate over the interval CA. Substitute $v^2 = 0, a = g\sin\theta - \mu_k g\cos\theta,\ u^2 = 4(g\sin\theta)(AC), s = AC$

$$0 = 4(g\sin\theta)(AC) + 2(g\sin\theta - \mu_k g\cos\theta)AC$$
$$0 = 6\sin\theta - 2\mu_k\cos\theta$$
$$\mu_k = \frac{6\sin\theta}{2\cos\theta} = 3\tan\theta \Rightarrow k = 3$$

Part B: Work

Note that the positive work due to gravity must equal the negative work due to friction since both the velocity at both the start point and the end point is zero:

$$W_{Gravity} = -W_{Friction}$$
$$F_{Gravity}s_{gravity} = F_{Friction}s_{Friction}$$
$$(mg\sin\theta)(3AC) = -(-\mu_k mg\cos\theta)AC$$

Divide both sides by $mg \cdot AC$

$$(\sin\theta)(3) = \mu_k\cos\theta$$
$$\mu_k = 3\tan\theta \Rightarrow k = 3$$

Part C: Work

If there are multiple rough and smooth parts, the method using Kinematics becomes even lengthier.
The method using Work is independent of the number of smooth and rough parts.

1.76: Air Resistance

Air resistance is friction caused by air resisting the movement of an object.

D. Work due to Gravity

1.77: Sign of Work due to gravity

Consider an object on which gravity does $work = W_{Gravity}$. If the object moves

closer to the Earth: $W_{Gravity} > 0$

away from the Earth: $W_{Gravity} < 0$

Example 1.78

Fill in the blanks

Fill in each of the four blanks below with one of $positive, negative, zero$. Options may be used more than once, or not at all.

A. A man takes a briefcase from the ground and lifts it up. The work done by him is ____. The work done by gravity is _____.

B. A man holding a briefcase gently lowers it to the ground. The work done by him is ____. The work done by gravity is _____.

$$A: positive, negative$$
$$B: negative, positive$$

Example 1.79

In each decide whether the work done is *positive*, *negative*, or *zero*:

A. By a man in lifting a bucket out of a well by means of a rope.
B. By the gravitational force in lifting a bucket out of a well by a rope tied to the bucket.
C. By friction on a body sliding down on an inclined plane

$$A: Positive$$
$$B: Negative$$
$$C: Negative$$

1.80: Work due to gravity

Work due to gravity on a falling object that falls a height h:

$$= Fs = mgh$$

$$W = Fs$$

Substitute $F = mg, s = h$:

$$W = mgh$$

1.81: Work done in raising an object against gravity

The work done in raising an object to a height h ais change in potential energy of the object:

$$W = -W_{Gravity} = mg\Delta h$$

$$W = Work\ Done = Kinetic\ Energy\ added = Change\ in\ Potential\ Energy = mg\Delta h$$

Example 1.82

300 J of work is done in sliding a 2 kg block up an inclined plane of height 10 m over a period of 3 minutes and 27 seconds. The plane is inclined at 32.7° to the horizontal. Work done against friction is: (take $g = 10\frac{m}{s}$)

(NEET, Adapted)

Work done against gravity

$$= mgh = 2(10)(10) = 200J$$

Work done against friction

$$= 300J - 200J = 100J$$

1.83: Work done in lowering an object at constant velocity

The work done in lowering an object over a height h at constant velocity is the work required to prevent gravity from accelerating the object:

$$W = -W_{Gravity} = -mgh$$

- The logic used here is analogous to the logic used for calculating the work done by an engine to move a vehicle at constant velocity against friction.

If you are lowering an object at constant velocity

$$F_{Net} = 0$$
$$Upward\ Force + Downward\ Force = 0$$
$$mg - mg = 0$$

Example 1.84
A porter lifts a heavy suitcase of mass 80 kg and at the destination lowers it down by a distance of 80 cm with a constant velocity. Calculate the work done by the porter in lowering the suitcase: (Given $g = 10\frac{m}{s^2}$) **(JEE Main, July 22, 2021-II)**

$$W_{Porter} = -W_{Gravity} = -mgh = -(80)(10)\left(\frac{80}{100}\right) = -640J$$

E. Non-Constant Force (Graphical)

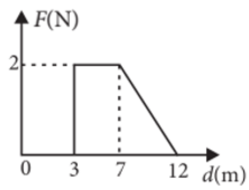
1.85: Non-Constant Force
The work done by a non-constant force is the area under the

$$F - x\ diagram$$

If a non-constant force is applied to an object, we want the total change in the kinetic energy that the force causes.

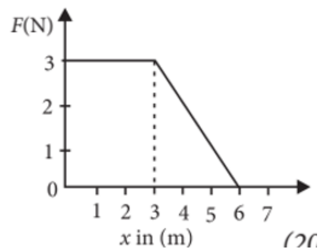
This can be obtained by calculating the average force, and multiplying it by the duration for which the force is applied.

$$Average\ Force = \frac{Total\ Force}{Distance}$$
$$Total\ Force = F_{Avg} \cdot s$$



Example 1.86
Force F on a particle moving in a straight line varies with distance d as shown in figure. The work done on the particle during its displacement of $12m$ is: **(NEET 2011)**

$$W = [ABCD] + [DCE] = 2(7-3) + \frac{1}{2}(2)(12-7) = 8 + 5 = 13J$$



Example 1.87
A force F acting on an object varies with distance x as shown here. The force is in N and x in m. The work done by the force in moving the object from $x = 0$ to $x = 6\ m$ is **(NEET 2005)**

$$W = Area = \left(\frac{6+3}{2}\right)(3) = 13.5J$$

Example 1.88
A person pushes a box on a rough horizontal platform surface. He applies a force of $200N$ over a distance of $15m$. Thereafter, he gets progressively tired and his applied force reduces linearly with distance to $100N$. The total distance through which the box has been moved is 30m. What is the work done by the person during the

total movement of the box? **(JEE Main, 4 Sep, 2020-II)**

$$0\ to\ 15: (200)(15) = 3000J$$
$$15\ to\ 30: \left(\frac{200+100}{2}\right)(15) = (150)(15) = 2250J$$

$$Total = 3000 + 2250 = 5250J$$

1.4 Work(2D/3D)

A. 2D Forces, Movement constrained to 1D

1.89: Only component in direction of movement matters
When calculating work, only the component of force in the direction of movement matters when calculating work.

Example 1.90

A. A car travels on a level, straight road for 3 km. Calculate the work done by gravity.
B. A body, constrained to move in y-direction, is subjected to a force given by $\vec{F} = \left(-2\hat{\imath} + 15\hat{\jmath} + 6\hat{k}\right) N$. The work done by this force in moving the body through a distance of $10\hat{\jmath}\ m$ along y-axis, is: **(NEET 1994)**

Part A

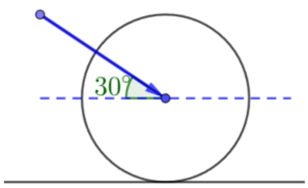
$$Zero$$

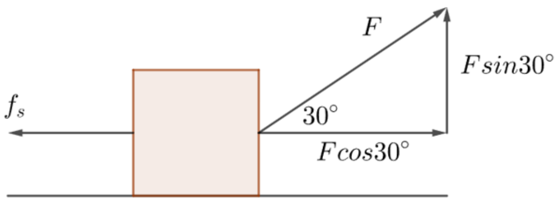
Part B

$$W = Fs = (10)(150) = 150J$$

Example 1.91

Give a practical example of an object constrained to move in x direction but the force (as commonly applied) has both x and y components.





- Pushing a garden roller (left diagram). The roller is being and the force has a downward component.
- Pulling an object on wheels via a rope. The force has an upward component.

1.92: Force perpendicular to displacement
If a force is perpendicular to the displacement of an object, the work done by the force is zero.

1.93: Force in 2D
For a constant force in any direction (but with movement in the x direction)

$$W = Fs\cos\theta$$

where

$$Work = W, \quad F = Force, \quad s = displacement$$
$$\theta = angle\ between\ force\ and\ direction\ of\ movement$$

Example 1.94

A body moves a distance of $10m$ along a straight line under the action of a $5N$ force. If the work done is $25J$, then the angle between the force and the direction of motion of the body is: **(NEET 1997)**

$$Fs\cos\theta = W \Rightarrow 5(10)\cos\theta = 25 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = 60°$$

Example 1.95

Two persons A and B perform same amount of work in moving a body through a certain distance d with application of force acting at angle 45° and 60° with the direction of displacement respectively. The ratio of force applied by person A to the force applied by person B is $\frac{1}{\sqrt{x}}$. The value of x is: **(JEE Main, Aug 27, 2021-I)**

$$F_1 d\cos\theta_1 = F_2 d\cos\theta_2$$

Solve for the ratio $\frac{F_1}{F_2}$:

$$\frac{F_1}{F_2} = \frac{\cos\theta_2}{\cos\theta_1} = \frac{\cos 60°}{\cos 45°} = \frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \Rightarrow x = 2$$

B. Work in 2D and 3D (Vectors)

1.96: Displacement

Displacement is the change in the final and the initial position:

$$Displacement = \vec{\boldsymbol{s}} = \vec{\boldsymbol{r}}_f - \vec{\boldsymbol{r}}_i$$

Where

$$\vec{\boldsymbol{r}}_f = final\ position$$
$$\vec{\boldsymbol{r}}_i = initial\ position$$

Example 1.97

An object particle moves to position $(5\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})$ from its initial position $(2\hat{\imath} + 3\hat{\jmath} - 4\hat{k})$

$$\vec{\boldsymbol{s}} = \vec{\boldsymbol{r}}_f - \vec{\boldsymbol{r}}_i = (5,-2,1) - (2,3,-4) = (5-2,-2-3,1+4) = (3,-5,5)$$

1.98: Work as Dot Product[2]

Work is the dot product of force and displacement:

$$Work = W = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{s}}$$

Where

$$Work = W$$
$$\vec{\boldsymbol{F}} = Force$$
$$\vec{\boldsymbol{s}} = displacement$$

This definition is useful when force and displacement are

[2] As opposed to the cross product

- available in component form
- or can be converted to component form.

- The dot product has a scalar output.

Example 1.99
Calculate work done when a force $\vec{F} = (f_x, f_y, f_z)$ displaces an object by $\vec{s} = (s_x, s_y, s_z)$

Use the definition of the dot product, and multiply the vectors component-wise:

$$W = \vec{F} \cdot \vec{s} = (f_x, f_y, f_z) \cdot (s_x, s_y, s_z) = f_x s_x + f_y s_y + f_z s_z$$

Example 1.100
A. A small particle moves to position $(5\hat{\imath} - 2\hat{\jmath} + \hat{k})$ from its initial position $(2\hat{\imath} + 3\hat{\jmath} - 4\hat{k})$ under the action of force $(5\hat{\imath} + 2\hat{\jmath} + 7\hat{k})\ N$. The value of work done will be (in J): **(JEE Main, Feb 1, 2023-I)**

B. A particle moves from a point $(-2\hat{\imath} + 5\hat{\jmath})$ to $(-4\hat{\jmath} + 3\hat{k})$ when a force of $(4\hat{\imath} + 3\hat{\jmath})$ is applied. How much work has been done by the force? **(NEET 2016)**

C. A uniform force of $(3\hat{\imath} + \hat{\jmath})$ newton acts on a particle of mass $2\ kg$. Hence, the particle is displaced from position $(2\hat{\imath} + \hat{k})$ meter to position $(4\hat{\imath} + 3\hat{\jmath} - \hat{k})$ meter. The work done by the force on the particle is: **(NEET 2013)**

Part A

Calculate the displacement vector as the difference of the final and initial position vectors:

$$\vec{s} = \vec{r}_f - \vec{r}_i = (5,-2,1) - (2,3,-4) = (5-2,-2-3,1+4) = (3,-5,5)$$

Calculate work as the dot product of the force and displacement vectors:

$$W = \vec{F} \cdot \vec{s} = (5,2,7) \cdot (3,-5,5) = 5(3) + 2(-5) + 7(5) = 15 - 10 + 35 = 40J$$

Part B

Use the same method as before, but with less writing:

$$\vec{s} = (0,4,3) - (-2,5,0) = (2,-1,3)$$
$$W = (4,3,0) \cdot (2,-1,3) = 8 - 3 + 0 = 5$$

Part C

$$\vec{s} = (4,3,-1) - (2,0,1) = (2,3,-2)$$
$$W = (3,1,0) \cdot (2,3,-2) = 6 + 3 + 0 = 9$$

1.5 Power

A. Power (1D)

1.101: Power as Rate of doing Work
The rate of doing

$$P = \frac{W}{t}$$

Example 1.102
The ratio of powers of two motors is $\frac{3\sqrt{x}}{\sqrt{x}+1}$, that are capable of raising 300 kg water in 5 minutes and 50 kg water

in 2 minutes respectively from a well of 100m deep. The value of x will be: **(JEE Main, April 13, 2023-I)**

$$P_1 : P_2 = \frac{m_1 gh}{t_1} : \frac{m_2 gh}{t_2} = \frac{m_1}{t_1} : \frac{m_2}{t_2} = \frac{300}{5} : \frac{50}{2} = 60 : 25 = 12 : 5 = \frac{12}{5} = \frac{3\sqrt{x}}{\sqrt{x}+1}$$
$$12\sqrt{x} + 12 = 15\sqrt{x}$$
$$\sqrt{x} = 4$$
$$x = 16$$

1.103: Unit of Power

The unit of power is Watt.

$$1W = \frac{1J}{s} = 1\frac{Nm}{s} = 1\frac{kg \cdot m^2}{s^3}$$

Example 1.104

Water falls from a height of 60 m at the rate of 15 $\frac{kg}{s}$ to operate a turbine. The losses due to frictional force are 10% of the input energy. How much power is generated by the turbine in kW? (take $g = 10\frac{m}{s^2}$) **(NEET 2021)**

$$W = \Delta KE = \Delta PE = mgh$$
$$P = 0.9\frac{W}{t} = 0.9\frac{mgh}{t} = 0.9\left(\frac{m}{t}\right)gh = 0.9 \cdot 15 \cdot 10 \cdot 60 = 8100\, W = 8.1\, kW$$

1.105: Power as Rate of doing Work

For a constant force

$$P = Fv$$

$$P = \frac{dW}{dt} = \frac{d}{dt}[Fs] = F \cdot \frac{ds}{dt} = Fv$$

- Power is a scalar quantity

Example 1.106

If a constant force is applied on a body and it moves with a velocity v, the power will be: **(EAMCET,18 Sep 2020, Shift-I)**

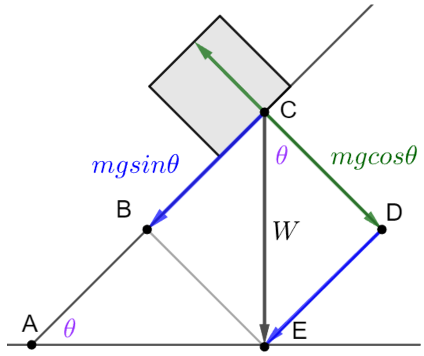
$$P = Fv$$

Example 1.107

85: A block of mass 5 kg starting from rest is pulled up on a smooth inclined plane making an angle of 30° with the horizontal with an acceleration of 1 $\frac{m}{s^2}$. The power delivered by the pulling force at $t = 10\, s$ from the start is: **(JEE Main, April 10, 2023-I)**

The net force on the object is

$$F_{Net} = F_{Total} - mg\sin\theta$$
$$ma = F_{Total} - mg\sin\theta$$
$$5(1) = F_{Total} - (5)(10)\left(\frac{1}{2}\right)$$
$$F_{Total} = 30N$$



Substitute $v = u + at = 0 + 1(10) = 10$ in

$$P = Fv = (30)(10) = 300\ W$$

Pending

Example 1.108

86: If the maximum load carried by an elevator is 1400 kg (600 kg passenger+800 kg elevator), which is moving up with a uniform speed of $3\frac{m}{s}$ and the frictional force acting on it is 2000N, then the maximum power used by the motor (in kW) is: **(JEE Main, April 10, 2023-II)**

$$F_{max} = mg + Friction$$
$$48\ kW$$

Example 1.109

93: A 60 HP electric motor lifts an elevator having a maximum total load capacity of 2000 kg. If the frictional force on the elevator is 4000 N, the speed of the elevator at full load is (approximated to the nearest tenth): $\left(1\ HP = 746\ W, g = 10\frac{m}{s^2}\right)$

A. $1.7\frac{m}{s}$
B. $1.9\frac{m}{s}$
C. $1.5\frac{m}{s}$
D. $2.0\frac{m}{s}$ **(JEE Main, 17 Jan, 2020-I)**

$$F_{Total} = mg + F_{Friction} = 2000(10) + 4000 = 24000\ N$$

$$v = \frac{P}{F} = \frac{60 \cdot 746}{24000} = \frac{746}{400} = \frac{373}{400} = 1\frac{173}{200} = 1\frac{86.5}{200} = 1.865 \approx 1.9$$

Pending

Example 1.110

96: A body of mass m accelerates uniformly from rest to v_1 in time t_1. The instantaneous power delivered to the body as a function of time t is: **(JEE Main 2004)**

B. Power in 2D/3D

1.111: Power as Rate of doing Work

For a constant force

$$P = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}$$

$$\vec{F} \cdot \vec{v} = (f_x, f_y, f_z) \cdot (v_x, v_y, v_z) = f_x v_x + f_y v_y + f_z v_z$$

➢ The dot product has a scalar output.

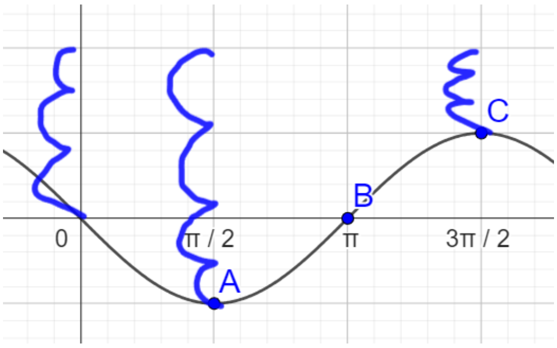
Example 1.112
The power utilized when a force of $(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k})N$ acts on a body for $4s$, producing a displacement of $(3\hat{\imath} + 4\hat{\jmath} + 5\hat{k})m$ is: (EAMCET 21 Sep 2020, Shift-I)

$$P = \vec{F} \cdot \vec{v} = \vec{F} \cdot \frac{\vec{s}}{t} = (2,3,4) \cdot \frac{(3,4,5)}{4} = 2 \times \frac{3}{4} + 3 \times \frac{4}{4} + 4 \times \frac{5}{4} = 1.5 + 3 + 5 = 9.5W$$

1.6 Springs
A. Springs

1.113: Ideal Spring
Ideal springs obey Hooke's Law.

1.114: Hooke's Law
$$F_x = F(x) = -kx$$
Where
$$k = spring\ constant$$
$$x = Displacement\ from\ equilibrium\ state$$



Example 1.115
A. F_x is called a "restoring" force. Why is that reasonable?
B. Is F_x variable or constant?
C. Is the equilibrium of a block attached to a spring a stable equilibrium, or unstable?
D. F_x is expressed as a function of x, what kind of function will it be?
E. What is the unit for k?

A. It works to restore the system back to its equilibrium state.
B. F_x is a function of x. Hence, it is variable.
C. Stable.
D. Linear
E. $k = \frac{F}{x} = \frac{N}{m}$

Example 1.116: Calculation of Spring Constant
A wire suspended vertically from one of its ends is stretched by attaching a weight of 200N to the lower end. The weight stretches the wire by 1 mm. Assuming the wire to be a spring, calculate the spring constant. (JEE-M 2003, Adapted)

$$k = \frac{F}{x} = \frac{200}{0.001} = 2{,}00{,}000 \frac{N}{m}$$

1.117: Work Done by the Spring Force

$$W_s = \frac{k}{2}\left(x_i^2 - x_f^2\right)$$

Where

$$x_i = Initial\ Position$$
$$x_f = Final\ position$$
$$k = spring\ force\ constant$$
$$W_s = Work\ done\ by\ Spring$$

Pending

Example 1.118

A block is attached to a spring. The system is parallel to the x axis, and the equilibrium position of the spring is at $x = 0$. Let $x_f = final\ position$, $x_i = initial\ position$. In each case below, decide whether the work done by the spring force is positive, negative, or zero, or cannot be determined.

A. $x_f > x_i, x_i > 0, x_f > 0$
B. $x_f > x_i, x_i < 0, x_f < 0$
C. $x_f > x_i, x_i < 0, x_f > 0$
D. $x_f > x_i, x_i > 0, x_f < 0$
E. $x_i > x_f$
F. $x_i = x_f$

Negative
Positive
Cannot be determined
Positive
Zero

1.119: Introducing a Coordinate System

When the displacement begins from the origin, the work done is:

$$W_s = -\frac{1}{2}kx^2$$

If the initial position is $x_i = 0$

$$W_s = \frac{k}{2}\left(-x_f^2\right) = -\frac{1}{2}kx^2$$

Where

$$x_f = x = Displacement\ from\ original\ position$$

Example 1.120

A vertical spring with force constant k is fixed on a table. A ball of mass m at a height h above the free upper end of the spring falls vertically on the spring so that the spring is compressed by a distance d. The net work done in the process (in terms of m, h, d, g) is: **(NEET 2009)**

The net work done is:

$$W_{Gravity} + W_{Spring} = mg(h + d) - \frac{1}{2}kd^2$$

B. Vertical Harmonic Oscillator

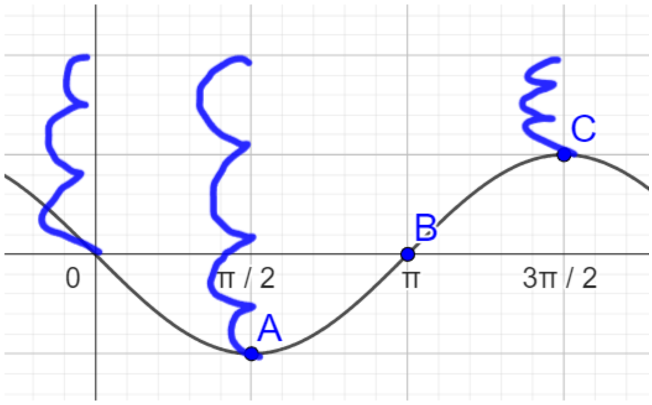
1.121: Oscillations

- We will study oscillations mathematically in the chapter on oscillations.
- Here, we will introduce terminology for a simple harmonic oscillator.

1.122: Simple Harmonic Oscillator (1D)

A simple harmonic oscillator consists of an object of mass m attached to a spring of negligible mass:

- The spring is vertical. Hence, gravity acts on the mass.
- y is the height (vertical axis). At $y = 0$, the spring is in equilibrium (gravity is counteracted by the spring force).
- t is time (horizontal axis). The mass is pulled down at time $t = 0$, and released from point A.
- The path generated is a sinusoidal curve. The one used in the diagram is $y = -\sin x$.
- We assume zero friction.



	KE	GPE	EPE	TOTAL	NATURE
A	0	GPE_{min}	EPE_{max}	$GPE_{min} + EPE_{max}$	Only potential
B	k_{max}	0	0	k_{max}	Only kinetic
C	0	GPE_{max}	EPE_{m}	$GPE_{max} + EPE_{max}$	Only potential

	KE	GPE	EPE	TOTAL	NATURE
A	0	-5	15	10	Only potential
B	10	0	0	10	Only kinetic
C	0	7	3	10	Only potential

Example 1.123

A block of mass M is attached to the lower end of a vertical spring. The spring is hung from a ceiling and has force constant value k. The mass is released from rest with the spring initially unstretched. The maximum extension produced in the length of the spring will be: **(NEET 2009)**

Gravitational potential energy of the mass converts to elastic potential energy. Both quantities must be equal:

$$Mgx = \frac{1}{2}kx^2 \Rightarrow x = \frac{2Mg}{k}$$

1.124: Simple Harmonic Oscillator: Horizontal

A horizontal harmonic oscillator is parallel to the x axis instead of the y axis.

- The spring is horizontal. Hence, gravity does not act on the mass.
- x (horizontal axis) is the displacement from the equilibrium position. At $x = 0$, the spring is in equilibrium.
- t is time (vertical axis). The mass is pulled (or pushed) at time $t = 0$, and released from point A.
- Graphing $x = f(t)$ generates a sinusoidal curve. Note that this interchanges the axes from the ones on

the physical diagram.
- The mass lies on a smooth surface. Hence, there is no friction.

	KE	EPE	Total	Nature
A	0	EPE_{max}	EPE_{max}	Only potential
B	k_{max}	0	k_{max}	Only kinetic
C	0	EPE_m	EPE_{max}	Only potential

C. Elastic Potential Energy

1.125: Elastic Potential Energy
Informally, this is the potential energy stored in an object that has been stretched.

- A drawn bowstring
- A pulled spring

1.126: Elastic Potential Energy
Elastic potential energy is stored if a spring is pulled or pushed from its equilibrium position. In the absence of friction, the stored energy is equal to the effort $(work)$ done to push or the spring.

$$EPE = -W_s = \frac{1}{2}kx^2$$

Where

$$W_s = Work\ done\ by\ spring$$

1.127: EPE versus displacement
The elastic potential energy is proportional to the square of the displacement.

$$EPE \propto x^2$$

Example 1.128
A. The potential energy of a long spring when tretched by 2 cm is U. If the spring is stretched by 8 cm, the potential energy stored in it is: **(NEET 2006)**
B. When a long spring is stretched by 2 cm, its potential energy is U. If the spring is stretched by 10 cm, the potential energy stored in will be: **(NEET 2006)**

Using $EPE \propto x^2$

$$Part\ A: EPE = \left(\frac{8}{2}\right)^2 U = 16U$$

$$Part\ B: EPE = \left(\frac{10}{2}\right)^2 U = 25U$$

Example 1.129
A mass of 0.5 kg moving with a speed of 1.5 $\frac{m}{s}$ on a horizontal smooth surface, collides with a nearly weightless spring of force constant $k = 50\frac{N}{m}$. The maximum compression of the spring would be: **(NEET 2004)**

At the point of maximum compression, all the kinetic energy is converted into elastic potential energy:

$$KE = EPE \Rightarrow \frac{1}{2}mv^2 = \frac{1}{2}kx^2$$

Substitute $m = 0.5, v = 1.5, k = 50$, and solve for x:

$$x = \sqrt{\frac{mv^2}{k}} = \sqrt{\frac{(0.5)(1.5^2)}{50}} = \sqrt{\frac{1.5^2}{100}} = \frac{1.5}{10} = 0.15\ m$$

Example 1.130

A 2 kg block slides on a horizontal floor with a speed of $4\frac{m}{s}$. It strikes an uncompressed spring, and compresses it till the block is motionless. The kinetic friction force is $15N$ and the spring constant is $10{,}000\frac{N}{m}$. The spring compresses by: (answer to one significant digit) **(JEE-M 2007)**

$$KE = EPE + E_{Kinetic\ Friction}$$
$$\frac{1}{2}mv^2 = \frac{1}{2}kx^2 + (15)(x)$$
$$\frac{1}{2}(2)(4^2) = \frac{1}{2}\cdot 10{,}000x^2 + (15)(x)$$
$$5000x^2 + 15x - 16 = 0$$
$$x = \frac{-15 \pm \sqrt{15^2 - (4)(5000)(-16)}}{2(5000)}$$
$$\approx \frac{\sqrt{320{,}000}}{10{,}000} \approx \frac{600}{10{,}000} = 0.06\ m$$

1.131: EPE in terms of force and extension

$$EPE = \frac{1}{2}\underbrace{F}_{Force}\underbrace{x}_{Extension}$$

$$EPE = \frac{1}{2}kx^2 = \frac{1}{2}\left(\frac{F}{x}\right)x^2 = \frac{1}{2}Fx$$

Example 1.132

A wire suspended vertically from one of its ends is stretched by attaching a weight of 200N to the lower end. The weight stretches the wire by 1 mm. Then the elastic energy in the wire is: **(JEE-M 2003)**

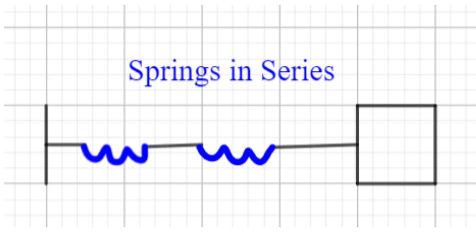
$$EPE = \frac{1}{2}Fx = \frac{1}{2}\cdot 200\cdot 0.001 = 0.1J$$

D. Series and Parallel Springs

1.133: Springs in Series

When two springs are connected in series, they are one after the other.

- Force is common
- Extension is sum of extensions



Extension need not be same. It will be same if spring constant is same.

Example 1.134
A rope hanging from the ceiling has a spring attached to it. The spring has a second rope attached to it. The second rope has a second spring attached to it. The second spring has an object of mass 1 kg attached to it. The first spring extends by 2 cm, and the second spring elongates by 3 cm from equilibrium. Determine the

A. tension in the first rope, the tension in the first spring and the tension in the second rope
B. extension of the rope/spring system as compared to equilibrium position of the springs.

Part A
Since force is common

$$1st\ Rope = 1st\ spring = 2nd\ rope = 10N$$

Part B
Extension is sum of extension

$$2\ cm + 3\ cm = 5\ cm$$

Example 1.135
Two springs of force constants $300\frac{N}{m}$ (Spring A), and $400\frac{N}{m}$ (Spring A) are joined together in series. The combination is compressed by 8.75 cm. The ratio of energy stored in A and B is $\frac{E_A}{E_B}$. Then $\frac{E_A}{E_B}$ is **(JEE-M 2013)**

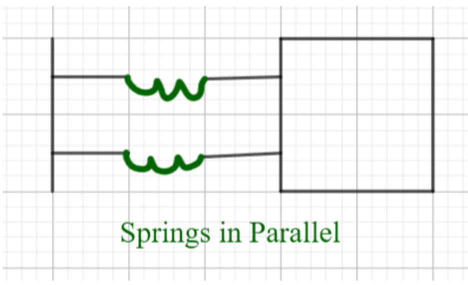
Force is common

$$300x = 400(8.75 - x)$$
$$300x = 3500 - 400x$$
$$700x = 3500$$
$$x = 5$$

$$\frac{E_A}{E_B} = \frac{\frac{1}{2}kx_A^2}{\frac{1}{2}kx_B^2} = \frac{\frac{1}{2}(300)(5^2)}{\frac{1}{2}(400)(3.75)^2} = \frac{4}{3}$$

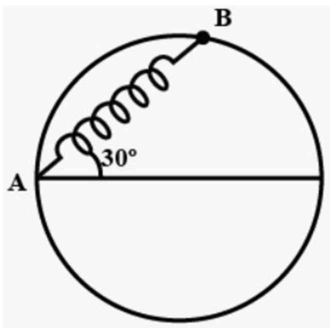
1.136: Springs in Parallel
When two springs are connected in parallel, they are together after the other.

- Extension is common
- Force is sum of forces



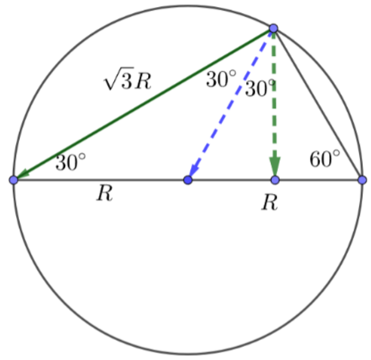
Force need not be same. It will be same if spring constant is same.

Example 1.137
A bead of mass m is attached to one end of a spring of natural length R and spring constant $K = \frac{(\sqrt{3}+1)mg}{R}$. The other end of the spring is fixed at point A on a smooth vertical ring of radius R. The normal reaction at B just after it is released to move is: **(EAMCET, Adapted)**



Point B forms a $30 - 60 - 90$ triangle with hypotenuse as the diameter and hence, the length of the spring is

$$\frac{\sqrt{3}}{2}(2R) = \sqrt{3}R$$



Extension of the spring is

$$\sqrt{3}R - R = (\sqrt{3}-1)R$$

$$F = kx = \left(\frac{(\sqrt{3}+1)mg}{R}\right)(\sqrt{3}-1)R = 2mg$$

$$W = mg$$

$$N = F\cos 30° + W\cos 30° = (2mg)\cos 30° + mg\cos 30° = 3\,mg\cos 30° = \frac{3\sqrt{3}}{2}mg$$

1.7 Calculus-Based Problems

A. Non-Constant Force (Differentiation)

Example 1.138

A body of mass 3 kg is under a constant force which causes a displacement s in meters in it, given by the relation $s = \frac{1}{3}t^2$, where t is in seconds. Work done by the force in 2 seconds is: **(NEET 2006)**

The velocity is:

$$v = \frac{ds}{dt} = \frac{2}{3}t$$

By the work-energy theorem:

$$W = \Delta KE = \frac{1}{2}m(v^2 - u^2) = \frac{1}{2}(3)\left[\left(\frac{2}{3}t\right)^2 - 0\right] = \frac{2}{3}t^2 = \frac{2}{3}t^2 = \frac{2}{3}(2^2) = \frac{8}{3}J$$

Example 1.139

A force acts on a $3g$ particle in such a way that the position of the particle as a function of time is given by $x = 3t - 4t^2 + t^3$, where x is in meters and t is in seconds. The work done during the first 4 seconds is (in mJ): **(NEET 1998)**

The velocity is:

$$v = \frac{dx}{dt} = 3 - 8t + 3t^2$$

By the work-energy theorem:

$$W = \frac{1}{2}m(v_4^2 - v_0^2) = \frac{1}{2}(3)(19^2 - 3^2) = 528\,mJ$$

B. Energy (Differentiation)

Example 1.140

A. The potential energy of a particle in a force field is $U = \frac{A}{r^2} - \frac{B}{r}$ where A and B are positive constants and r is the distance of the particle from the center of the field. For stable equilibrium, the distance of the particle is: **(NEET 2012)**

B. The potential energy between two atoms, in a molecule is given by $U(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$, where a and b are positive constants and x is the distance between the atoms. The atom is in stable equilibrium, when $x =$**(NEET 1995)**[3]

Part A

$$\frac{dU}{dr} = -\frac{2A}{r^3} + \frac{B}{r^2} = 0 \Rightarrow \frac{B}{r^2} = \frac{2A}{r^3} \Rightarrow r = \frac{2A}{B}$$

Note: There is one only solution. Hence, it must be a minimum (since the question has told us there is one).

Part B

$$\frac{dU}{dr} = -\frac{12a}{x^{13}} + \frac{6b}{x^7} = 0 \Rightarrow \frac{6b}{x^7} = \frac{12a}{x^{13}} \Rightarrow x^6 = \frac{2a}{b} \Rightarrow x = \left(\frac{2a}{b}\right)^{\frac{1}{6}}$$

C. Non-Constant Force (Integration)

1.141: Work with a non-constant force

$$W = \int F\,dx$$

Example 1.142

A. A block of mass 10 kg is moving along the x −axis under the action of force $F = 5x\ N$. The work done by the force in moving the block from $x = 2$ to $4m$ will be: **(JEE Main, April 15, 2023-I)**

B. A position dependent force $F = (7 - 2x + 3x^2)N$ acts on a small body of mass $2kg$ and displaces it from $x = 0$ to $x = 5$. The work done in joule is: **(NEET 1994, NEET 1992)**

Using $W = \int F\,dx$ in each case:

$$Part\ A: \int_2^4 5x\,dx = \left[\frac{5x^2}{2}\right]_2^4 = \left(\frac{5}{2}\right)(4^2 - 2^2) = \left(\frac{5}{2}\right)(16 - 4) = \left(\frac{5}{2}\right)(12) = 5(6) = 30J$$

$$Part\ B: \int_0^5 (7 - 2x + 3x^2)\,dx = [7x - x^2 + x^3]_0^5 = 7(5) - 5^2 + 5^3 = 135J$$

1.143: Displacement in the y direction

$$W = \int F\,dy$$

Example 1.144

A. A force $F = 20 + 10y$ acts on a particle in the y −direction where F is in newton and y in meter. Work done by this force to move the particle from $y = 0$ to $y = 1m$ is: **(NEET 2019)**

B. A force $F = (5 + 3y^2)$ acts on particle in the y-direction where F is in newton and y is in meter. The work done by the force during a displacement from $y = 2m$ to $y = 5m$ is (in J): **(JEE Main, Feb 1, 2023-I)**

C. A force of $F = (5y + 20)\hat{j}\ N$ acts on a particle. The work done by this force when the particle is moved from $y = 0$ to $y = 10m$ is: **(JEE Main, July 25, 2021-II; JEE Main 2014)**

Using $W = \int F\,dy$ in each case:

[3] The potential energy function here has the form of a Lennard-Jones potential.

$$Part\ A: \int_0^1 (20 + 10y)\ dy = \left[20y + \frac{10}{2}y^2\right]_0^1 = 20 + 5 = 25J$$

$$Part\ B: \int_2^5 (5 + 3y^2)\ dy = [5y + y^3]_2^5 = (25 + 125) - (10 + 8) = 150 - 18 = 132J$$

$$Part\ C: \int F\ dy = \int_0^{10} (5y + 20)\ dy = \left[\frac{5y^2}{2} + 20y\right]_0^{10} = 20(10) + 250 = 450J$$

1.145: Work Done by the Spring Force

$$W_s = \frac{k}{2}\left(x_i^2 - x_f^2\right)$$

$$W = \int_{x_i}^{x_f} -F_x\ dx = \int_{x_i}^{x_f} -kx\ dx = -\frac{k}{2}[x^2]_{x_i}^{x_f} = -\frac{k}{2}\left(x_f^2 - x_i^2\right) = \frac{k}{2}\left(x_i^2 - x_f^2\right)$$

1.146: Displacement in multiple directions

$$W = \int F_x\ dx + \int F_y\ dy + \int F_z\ dz$$

D. Power (Integration)

Example 1.147

A body of mass 1kg begins to move under the action of a time dependent force $\vec{\boldsymbol{F}} = (t\hat{\boldsymbol{\imath}} + 3t^2\hat{\boldsymbol{\jmath}})N$. The power developed by above force at time $t = 2s$ will be: (JEE Main, Jan 24, 2023-II)

$$m\vec{\boldsymbol{a}} = 1 \cdot \vec{\boldsymbol{a}} = (t, 3t^2) \Rightarrow \vec{\boldsymbol{v}} = \left(\frac{t^2}{2}, t^3\right)$$

Calculate power as the dot product of force and velocity vectors:

$$P = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} = (t, 3t^2) \cdot \left(\frac{t^2}{2}, t^3\right) = \frac{t^3}{2} + 3t^5 = \frac{2^3}{2} + 3 \cdot 2^5 = 4 + 96 = 100W$$

Example 1.148

A body of mass 2 kg is initially at rest. It starts moving unidirectionally under the influence of a source of constant power P. Its displacement in $4s$ is $\frac{1}{3}\alpha^2\sqrt{P}m$. The value of α will be: (JEE Main, Jan 30, 2023-II)

$$Pt = W = \frac{1}{2}mv^2 = \frac{1}{2}(2)v^2 = v^2$$

$$v = \sqrt{Pt} = \sqrt{P} \cdot \sqrt{t}$$

Integrate both sides:

$$s = \sqrt{P} \cdot \left(\frac{2}{3}t^{\frac{3}{2}}\right) = \sqrt{P}\left(\frac{2}{3}4^{\frac{3}{2}}\right) = \frac{16\sqrt{P}}{3}$$

$$\alpha^2 = 16 \Rightarrow \alpha = 4$$

Example 1.149

A body of mass 2 kg is driven by an engine delivering constant power of $1\frac{J}{s}$. The body starts from rest and moves in a straight line. After 9 seconds, the body has moved a distance (in m): **(JEE Main, 5 Sep, 2020-II)**

$$P = Fv = (ma)v = m \cdot \frac{dv}{dt}v$$

Substitute $m = 2, P = 1$

$$2 \cdot \frac{dv}{dt}v = 1$$
$$2v\,dv = dt$$
$$v^2 = t$$
$$v = \sqrt{t}$$
$$s = \frac{2}{3}t^{\frac{3}{2}} = \frac{2}{3}9^{\frac{3}{2}} = 18\,m$$

2. Impulse and Momentum

2.1 Impulse

A. Momentum

2.1: Momentum
Momentum is the product of the mass and the velocity of an object

$$\vec{p} = m\vec{v}$$

- Momentum is a vector quantity since velocity is a vector quantity and mass is a scalar quantity.
- $m\vec{v}$ is a scalar product
- Momentum has the same direction as velocity

2.2: Assumptions
When working with momentum, you can make certain assumptions which simplify formulas.
Recognizing which assumptions is crucial in determining which formulas are valid.

Assumption of constant mass:

- Valid if a ball dropped from a tower
- Not valid if a rocket is accelerated in space because the rocket loses mass as it ejects fuel
- Not valid if a body is moving at speeds which are close to speed of light since the mass of the body will be different from its rest mass.

Assumption of constant force:

- Valid if a uniform force is applied to an object
- Valid if average force is being calculated
- Not Valid if force applied is a function of time

B. Impulse at Constant Mass

2.3: Impulse
Impulse is the change in momentum:

$$Impulse = \Delta\vec{p}$$

2.4: Impulse at constant mass
At constant mass, impulse scales linearly with velocity:

$$Impulse = \Delta\vec{p} = m\Delta\vec{v}$$
$$Impulse \propto m\Delta\vec{v}$$

$$Impulse = \Delta\vec{p}$$

Substitute the definition of momentum:

$$= \Delta(m\vec{v})$$

Since the mass is constant, it does not change. Move it out of the Δ operator:

$$= m\Delta\vec{v}$$

2.5: Unit of Impulse

In the metric system, the

$$Unit\ of\ impulse = kg \cdot \frac{m}{s} = Ns$$

Example 2.6

A rocket is fired into the air vertically upward. Is the formula below applicable:

$$Impulse = m\Delta\vec{\boldsymbol{v}}$$

No. Because as the fuel burns, it is ejected from the rocket, and the mass reduces.

Example 2.7

A particle of mass m is moving with uniform velocity v_1 in a particular direction. It is given an impulse such that its velocity becomes v_2 in the same or opposite direction. The magnitude of impulse, in terms of m, v_1 and v_2 is equal to: **(NEET 1990, Adapted)**

$$Impulse = m\Delta v = m(v_2 - v_1)$$

2.8: Normal Collisions with a Wall

In normal collisions, an object hits a wall, and bounces back along the same path.

- Normal is a technical word that means perpendicular.
- For example, the normal to a curve is perpendicular to the curve at that point.

Example 2.9

A stone is dropped from a height h. It hits the ground with a certain momentum P. If the same stone is dropped from a height 100% more than the previous height, the momentum when it hits the ground will change by: **(NEET 2012)**

For the ball dropped from height h:

$$v^2 = u^2 + 2as \Rightarrow v^2 = 2gh \Rightarrow v = \sqrt{2gh}$$

For the ball dropped from height $2h$:

$$V = \sqrt{2g(2h)} = \sqrt{2gh}\sqrt{2} = \sqrt{2}v$$

Since mass of both the balls is the same, the percentage change depends only on the velocity:

$$\%\ change = \frac{V - v}{v} = \frac{\sqrt{2}v - v}{v} = \frac{\sqrt{2} - 1}{1} \approx 1.41 - 1 = 0.41 = 41\%$$

Example 2.10

A. A body of mass M hits normally a rigid wall with velocity V and bounces back with the same velocity. The impulse experienced by the body is: **(NEET 2011)**

B. A ball of mass 0.15 kg is dropped 10m, strikes the ground and rebounds to the same height. The magnitude of impulse imparted to the ball is nearly (take $g = 10\frac{m}{s^2}$) **(NEET 2021)**

C. A batsman hits back a ball of mass 0.4 kg straight in the direction of the baller without changing its speed of $15\frac{m}{s}$. The impulse imparted to the ball is: (Answer in Ns, where $N = Newton, s = second$) **(JEE Main, June 26, 2022-II)**

Part A

$$Impulse = m\Delta v = m(v_2 - v_1) = m[V - (V)] = 2MV$$

Part B

Substitute $m = 0.15, v = \sqrt{2gh}$ in $Impulse = 2mV$

$$= 2(0.15)\sqrt{2(10)(10)} = 3 \times 1.4 = 4.2\ kg\frac{m}{s}$$

Part C

$$Impulse = 2MV = 2(0.4)(15) = 12\ Ns$$

C. Impulse at Constant Mass and Constant Force

2.11: Impulse at Constant Mass and Constant Force

If a constant force is applied to a constant mass for a time Δt, then:

$$Impulse = \vec{\boldsymbol{F}}\Delta t$$

$$Impulse = \Delta\vec{\boldsymbol{p}} = \Delta m\vec{\boldsymbol{v}} = m\Delta\vec{\boldsymbol{v}} = m\frac{\Delta\vec{\boldsymbol{v}}}{\Delta t}\cdot\Delta t = m\vec{\boldsymbol{a}}\cdot\Delta t = \vec{\boldsymbol{F}}\Delta t$$

Example 2.12

Statement I: If a force is applied to a constant mass, then $\underbrace{Impulse = m\Delta\vec{\boldsymbol{v}}}_{Equation\ I}$

Statement II: If a constant force is applied to a constant mass, then $\underbrace{Impulse = \vec{\boldsymbol{F}}\Delta t}_{Equation\ II}$

A. Use Equations I and II to solve for $\vec{\boldsymbol{F}}$ in terms of m, Δt and $\Delta\vec{\boldsymbol{v}}$.
B. Do the dimensions on the equation that you arrived in Part A validate?

Part A

$$\vec{\boldsymbol{F}}\Delta t = m\Delta\vec{\boldsymbol{v}} \Rightarrow \vec{\boldsymbol{F}} = \frac{m\Delta\vec{\boldsymbol{v}}}{\Delta t}$$

Part B

$$LHS = Force = kg\frac{m}{s^2}$$

$$RHS = \frac{m\Delta\vec{\boldsymbol{v}}}{\Delta t} = \frac{(\boldsymbol{kg})\left(\frac{\boldsymbol{m}}{\boldsymbol{s}}\right)}{s} = kg\frac{m}{s^2} = RHS$$

2.13: Average Force

If a body with constant mass experiences acceleration for a time Δt, then the average force applied is:

$$\vec{\boldsymbol{F}} = \frac{m\Delta\vec{\boldsymbol{v}}}{\Delta t} = \frac{\Delta\vec{\boldsymbol{p}}}{\Delta t}$$

Example 2.14

A. A cricketer catches a ball of mass 150 gm in 0.1 sec moving with speed $20\frac{m}{s}$. He experiences a force with a magnitude of: (NEET 2001)
B. A ball of mass 0.15 kg hits the wall with its initial speed of $12\frac{m}{s}$, and bounces back without changing its initial speed. If the force applied by the wall on the ball during the contact is 100N, calculate the time duration of the contact of the ball with the wall. (JEE Main, July 26, 2022-II)

Part A

$$\boldsymbol{F} = \frac{m\Delta\vec{\boldsymbol{v}}}{\Delta t} = \frac{(0.15)(20)}{0.1} = 30N$$

Part B

$$\Delta t = \frac{m\Delta v}{F} = \frac{(0.15)(2 \times 12)}{100} = \frac{3.6}{100} = 0.036 \text{ } seconds$$

Example 2.15

Choose the correct option

In two different experiments, an object of mass 5 kg moving with a speed of 25 $\frac{m}{s}$ hits two different walls and comes to rest within (i) 3 seconds, (ii) 5 seconds respectively.

A. Impulse and average force acting on the object will be the same for both the cases.
B. Impulse will be same for both the cases, but the average force will be different.
C. Average force will be same for both the cases, but the impulse will be different.
D. Average force and impulse will be different for both the cases. **(JEE Main, July 28, 2022-I)**

In the absence of information, assume that the direction of movement is the same in both the cases.
(We cannot assume that it is different).

Impulse will be same.

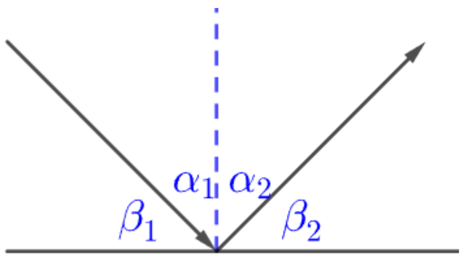
$$\Delta p = \Delta p$$

Average force $= \frac{\Delta p}{\Delta t}$

$$\frac{\Delta p}{3} \neq \frac{\Delta p}{5}$$

Option B

D. Collisions at an Angle



2.16: Symmetry in collisions at an angle

Exam questions often assume symmetry in collisions.
So, in the diagram alongside

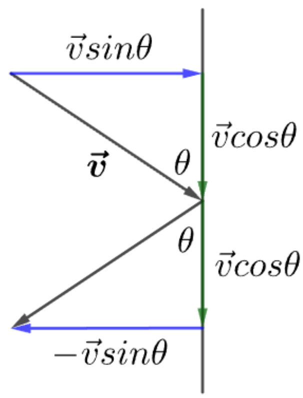
$$\alpha_1 = \alpha_2$$
$$\beta_1 = \beta_2$$

2.17: Impulse in collisions at an angle

When an object strikes a wall at an angle, and gets reflected symmetrically without loss of speed:

- Component of impulse parallel to the wall is zero.
- Magnitude of impulse perpendicular to the wall is double the horizontal component of the velocity.
- Direction of impulse perpendicular to the wall is opposite to the direction of initial velocity.

Consider an object which makes an angle θ with a wall, and collides with it. With no loss of speed, it bounces at angle θ to the wall.
$(Diagram\ alongside)$



Impulse is:

$$\Delta\vec{\boldsymbol{p}} = m\boldsymbol{\Delta}\vec{\boldsymbol{v}}$$

Write the velocity vector in component form:

$$m\Delta v_x + m\Delta v_y$$

Using trigonometry, write the components of the velocity vector:

$$v_{x_1} = \vec{v}\sin\theta\,, \qquad v_{x_2} = -\vec{v}\sin\theta\,, \qquad v_{y_1} = v_{y_2} = \vec{v}\cos\theta$$

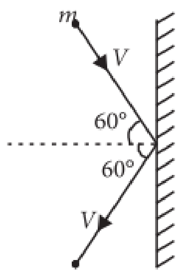
$$\Delta v_y = v_{y_2} - v_{y_1} = 0$$
$$\Delta v_x = -\vec{v}\sin\theta - \vec{v}\sin\theta = -2\vec{v}\sin\theta$$

Calculate momentum:

$$= m\Delta v_x + m\Delta v_y = -2m\vec{\boldsymbol{v}}\sin\theta$$

Example 2.18

A rigid ball of mass m strikes a rigid wall at 60° and gets reflected without loss of speed as shown in the figure. The absolute value of impulse imparted by the wall on the ball will be: **(NEET 2016)**



The angle is 60° with the normal. The angle with the wall is:

$$90° - 60° = 30°$$

Substitute $velocity = V\ (from\ the\ diagram), \theta = 30°$ in:

$$Impulse = -2m\vec{\boldsymbol{v}}\sin\theta = m\left(2V \cdot \frac{1}{2}\right) = mV\ N$$

Example 2.19

A 0.5 kg ball moving with a speed of 12 $\frac{m}{s}$ strikes a hard wall at an angle of 30° with the wall. It is reflected with the same speed at the same angle. If the ball is in contact with the wall for 0.25 seconds, the average force acting on the wall is: **(NEET 2006)**

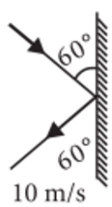
$$\Delta\vec{\boldsymbol{v}}_{\boldsymbol{x}} = 2v\sin\theta = 2(12)\left(\frac{1}{2}\right) = 12$$

Substitute $\Delta t = 0.25, m = 0.5,\ \Delta\vec{\boldsymbol{v}}_{\boldsymbol{x}} = 12$

$$\vec{\boldsymbol{F}} = \frac{m\Delta\vec{\boldsymbol{v}}}{\Delta t} = \frac{0.5 \times 12}{0.25} = 24N$$

Example 2.20

A body of mass 3 kg hits a wall at an angle of 60° and returns at the same angle. The impact time was 0.2s. The force exerted on the wall is: **(NEET 2000)**



Substitute $v = 10\frac{m}{s}$ from the diagram into:

$$\Delta v_x = 2v\sin\theta = 2(10)\left(\frac{\sqrt{3}}{2}\right) = 10\sqrt{3}$$

Substitute $\Delta t = 0.2, m = 3, \Delta v_x = 10\sqrt{3}$

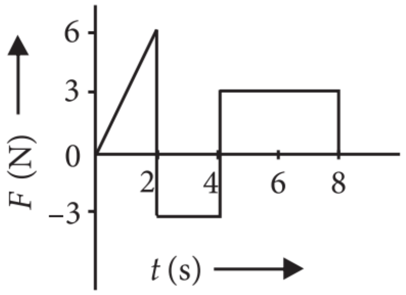
$$\boldsymbol{F} = \frac{m\Delta\vec{\boldsymbol{v}}}{\Delta t} = \frac{3\times 10\sqrt{3}}{0.2} = 150\sqrt{3}N$$

E. Non-Constant Force

2.21: Impulse from Graphical Integration

Impulse is the area under the curve of a force-time graph.

- Area above the area is positive.
- Area above the area is negative.



Example 2.22

The force F acting on a particle of mass m is indicated by the force-time graph as shown. The change in momentum of the particle over the time interval from zero to $8s$ is: **(NEET 2014)**

$$(t_0, t_2) = \frac{1}{2}\cdot 2\cdot 6 = 6$$
$$(t_2, t_4) = (-3)\cdot 2 = -6$$
$$(t_4, t_8) = 3\cdot 4 = 12$$

Total

$$= 12$$

2.2 Conservation of Momentum: 1D

A. Collisions

2.23: Law of conservation of energy

Energy can be neither be created nor be destroyed. It can only be converted from one form into another.

2.24: Conservation of momentum

The law of conservation of momentum says that the momentum of a system is conserved during a collision or an explosion:

$$\underbrace{\vec{\boldsymbol{p}}_i}_{\text{Initial Momentum}} = \underbrace{\vec{\boldsymbol{p}}_f}_{\text{Final Momentum}}$$

This is not a formula. Rather, this is an idea from which formulas that are useful will be derived.

Example 2.25

Choose the correct option

Consider the following two statements:

A. Linear momentum of a system of particles is zero.
B. Kinetic energy of a system of particles is zero.

a. A does not imply B and B does not imply A

b. A implies B but B does not imply A.
c. A does not imply B, but B implies A.
d. A implies B and B implies A. (JEE Main 2003)

Kinetic energy is a scalar $\frac{1}{2}mv^2$, and because of the square it is additive.
Hence, if the kinetic energy is zero, that is because each individual particle of the system has zero velocity.
Hence,

$$B \text{ implies } A$$

Momentum is a vector mv, and the direction can be positive or negative.
Hence, the momentum of a system can be zero even if the momentum of the individual particle is nonzero.

For example, a gun-bullet system has zero momentum both before and after the bullet is fired.

2.26: Object not initially at rest

$$m_1v_1 + m_2v_2 = mv$$

$$Initial\ Momentum = mv$$
$$Final\ Momentum = m_1v_1 + m_2v_2$$

$$Momentum\ of\ First\ Part = m_1v_1$$
$$Momentum\ of\ Second\ Part = m_2v_2$$

- v_1 and v_2 are in one dimension. Direction is indicated using positive and negative.

Example 2.27

A. A particle of mass m moving with velocity v collides with a stationary particle of mass $2m$. After collision, they stick together and continue to move together with velocity: (JEE Main, April 10, 2023-I; NEET 1996)

B. A man of 60 kg is running on the road and suddenly jumps into a stationary trolley car of mass 120 kg. Then the trolley starts moving with a velocity of $2\frac{m}{s}$. The velocity of the running man was: (JEE Main, June 28, 2022-I)

Part A

Substitute $m_1 = m, m_2 = m + 2m = 3m$ in $m_1v = m_2v_F$:

$$mv = (3m)v_F \Rightarrow v_F = \frac{v}{3}$$

Part B

Substitute $m_1 = 60kg, m_2 = 120kg, v_F = 2\frac{m}{s}$ in $m_1v_I = m_2v_F$

$$(60)v_I = (120 + 60)(2) \Rightarrow v_I = \frac{180 \cdot 2}{60} = 6\frac{m}{s}$$

B. Explosions

2.28: Object initially at rest

$$m_1v_1 = m_2v_2$$

As explained below, we can take

$$v_1 = |V_{Gun}| = Magnitude\ of\ velocity\ of\ gun$$
$$v_2 = |V_{Bullet}| = Magnitude\ of\ velocity\ of\ bullet$$

If a system is at rest

$$Initial\ Momentum = 0$$

(For example, a system could consist of $gun + bullet$)

If a bullet is fired from the gun, the bullet has a certain velocity, the gun has a certain velocity, but the momentum of the system is conserved:

$$m_{Gun}V_{Gun} + m_{Bullet}V_{Bullet} = 0$$
$$m_{Gun}V_{Gun} = -m_{Bullet}V_{Bullet}$$

m is always positive. Hence, $V_{Gun}$and V_{Bullet} must be opposite in sign.

If we take $v_1 = V_{Gun}$ and $v_2 = -V_{Bullet}$, then both have the same sign and:

$$m_1v_1 = m_2v_2$$

Example 2.29

An ice skater pushes his partner away at a speed of $3\frac{m}{s}$. The ice skater has a mass of m. The partner has a mass of $2m$. Determine the speed of the ice skater.

$$m(v) = (2m)(3)$$
$$v = 6$$

1 Pending (Part B Only)

Example 2.30

A. A man fires a bullet of mass $200g$ at a speed of $5\frac{m}{s}$. The gun is of one kg mass. By what velocity does the gun rebound backwards? **(NEET 1996)**

B. A bomb of mass 16 kg at rest explodes into two pieces of masses 4 kg and 12 kg. The velocity of the 12 kg mass is $4\frac{m}{s}$. The kinetic energy of the other mass is: **(JEE Main 2006)**

Part A

$$v_1 = \frac{m_2v_2}{m_1} = \frac{(0.2)(5)}{1} = 1\frac{m}{s}\ backwards$$

Part B

$$v_1 = \frac{m_2v_2}{m_1} = \frac{(12)(4)}{4} = 12\frac{m}{s}$$
$$KE = \frac{1}{2}mv^2 = \left(\frac{1}{2}\right)(4)(12^2) = 288J$$

Example 2.31

A bullet $10g$ leaves the barrel of gun with a velocity of $600\frac{m}{s}$. If the barrel of gun is 50 cm long, and mass of gun is 3kg, then value of impulse supplied to the gun will be: **(JEE Main, April 23, 2023-I)**

The initial momentum of the system is zero.

Hence, the impulse for the gun is same in magnitude, but opposite in sign as the impulse for the bullet:

$$Impulse_{Gun} = Impulse_{Bullet} = \Delta p = mv = (10 \times 10^{-3})(600) = 6Ns$$

2.32: Muzzle Velocity

Muzzle velocity is the velocity with which the bullet leaves the gun.

Example 2.33

A person holding a rifle (mass of person together with rifle is 100 kg) stands on a smooth surface and fires 10 shots horizontally, in $5s$. Each bullet has a mass of $10g$ with muzzle velocity of $800\frac{m}{s}$. The final velocity acquired by the person and the average force exerted on the person are: **(NEET 2013)**

$$m_1v_1 = m_2v_2$$

Substitute $m_1 = 100\ kg, m_2 = 10 \times 10g = 100g = 01.kg$:

$$(100)v_1 = (0.1)(800)$$

$$v_1 = 0.8\frac{m}{s^2}$$

$$\boldsymbol{F} = \frac{m\Delta\vec{\boldsymbol{v}}}{\Delta t} = \frac{100 \times 0.8}{5} = 16N$$

2.34: *n* bullets

If a gun fires more than one bullet, it experiences force/recoil equivalent to the total momentum of all the bullets.

$$m_1v_1 = nm_2v_2$$

Where

$$n = no. of\ bullets$$

$$m = mass\ of\ 1\ bullet$$

If you are calculating velocity, for metric units, if mass is in kg, and velocity is in $\frac{m}{s}$, then

$$n\ is\ in\frac{bullets}{sec}$$

Example 2.35

A machine gun of mass 10 kg fires $20g$ bullets at the rate of 180 bullets per minute with a speed of $100\frac{m}{s}$ each. The recoil velocity of the gun is: **(JEE Main, Jan 30, 2023-II)**

$$m_1v_1 = nm_2v_2$$

Substitute $n = 180\frac{bullets}{min} = 3\frac{bullets}{sec}$:

$$v_1 = \frac{nm_2v_2}{m_1} = \frac{(3)(20 \times 10^{-3})(100)}{10} = \frac{(3)(20 \times 10^{-3})(100)}{10} = 0.6\frac{m}{s}$$

Example 2.36

An average force of 125N is applied on a machine gun firing bullets each of mass $10g$ at the speed of $250\frac{m}{s}$ to keep it in position. The number of bullets fired per second by the gun is: **(JEE Main, April 11, 2023-I)**

System is initially at equilibrium. Over 1 second:

$$Force\ on\ machine\ gun = Recoil\ from\ bullets$$

Hence:

$$F = nmv \Rightarrow n = \frac{F}{mv} = \frac{125}{(10 \times 10^{-3})(250)} = \frac{125}{2.5} = \frac{250}{5} = 50$$

C. Explosions with Kinematics

Example 2.37

A mass of 1 kg is thrown up with a velocity of 100 $\frac{m}{s}$. After 5 seconds, it explodes into two parts. One part of mass 400g comes down with a velocity of 25 $\frac{m}{s}$. The velocity of other part is: (take $g = 10\frac{m}{s^2}$). **(NEET 2000)**

The velocity of the body at $time\ t = 5s$:

$$v = u + at = 100 + (-10)(5) = 50\frac{m}{s}$$

Using the law of conservation of momentum:

$$m_1v_1 + m_2v_2 = mv$$

Substitute $m_1 = 0.4\ kg, v_1 = -25, m_2 = 0.6\ kg, m = 1\ kg, v = 50\frac{m}{s}$:

$$(0.4)(-25) + (0.6)v_2 = (1)(50)$$
$$-100 + 6v_2 = 500$$
$$v_2 = 100\frac{m}{s}$$

The other part is going upwards.

Example 2.38

Mark the correct option

A projectile moving vertically upwards with a velocity of 200 $\frac{m}{s}$ breaks into two equal parts at a height of 490m. One part starts moving vertically upwards with a velocity of 400 $\frac{m}{s}$. How much time will it take, after the breakup with the other part to hit the ground? **(JEE Main, May 12, 2012)**

A. $2\sqrt{10}s$
B. $5s$
C. $10s$
D. $\sqrt{10}s$

Let

$$mass\ of\ projectile = 2m$$

Let velocity of:

$$part\ going\ up = v_1 = 400, \qquad other\ part = v_2, \qquad original\ projectile = v = 200$$

By conservation of momentum:

$$mv_1 + mv_2 = (2m)v \Rightarrow 400 + v_2 = 2 \cdot 200 \Rightarrow v_2 = 0$$

Substitute $Initial\ velocity = v_2 = u = 0, a = g$ in $s = ut + \frac{1}{2}at^2$

$$s = (0)t + \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2s}{g}}$$

$$t_{g=10} = \sqrt{\frac{2(490)}{10}} = \sqrt{98} = 7\sqrt{2}$$
$$t_{g=9.8} = \sqrt{\frac{2(490)}{9.8}} = \sqrt{10} \Rightarrow Option\ D$$

D. Kinetic Energy

Example 2.39

A. A bomb of mass 30 kg at rest explodes into two pieces of mass 18 kg and 12 kg. The velocity of the 18 kg mass is $6\frac{m}{s}$. The kinetic energy of the other mass is: **(NEET 2005)**

B. A shell of mass $200g$ is ejected from a gun of mass $4kg$ by an explosion that generates 1.05 kJ of energy. The initial velocity of the shell is: **(NEET 2008)**

Part A

By conservation of momentum:

$$m_1v_1 = -m_2v_2 \Rightarrow v_2 = -\frac{m_1v_1}{m_2} = -\frac{(18)(6)}{12} = -9\frac{m}{s}$$
$$KE = \frac{1}{2}m_2v_2^2 = \frac{1}{2}(12)(-9)^2 = 486J$$

Part B

By conservation of momentum:

$$m_sv_s = -m_gv_g \Rightarrow 0.2v_s = -4v_g \Rightarrow v_g = -\frac{v_s}{20}$$
$$KE = \frac{1}{2}(0.2)v_s^2 + \frac{1}{2}(4)\left(-\frac{v_s}{20}\right)^2 = \frac{1}{10}v_s^2 + \frac{v_s^2}{200} = \frac{21v_s^2}{200} = 1050J \Rightarrow v_s^2 = 10{,}000 \Rightarrow v_s = 100\frac{m}{s}$$

E. Loss of Kinetic Energy

Example 2.40

A metal ball of mass 2 kg moving with a speed of 36 km/hr has a head on collision with a stationary ball of mass 3 kg. If, after collision, both the balls move as a single mass, then the loss in kinetic energy due to collision is: **(NEET 1997)**

$$m_1v_1 = (m_1 + m_2)v_2$$

Substitute $m_1 = 2\ kg, m_2 = 3\ kg, v_1 = 36\frac{km}{hr} = 36 \cdot \frac{1000}{3600}\frac{m}{s} = 36 \cdot \frac{5}{18}\frac{m}{s} = 10\frac{m}{s}$:

$$2(10) = 5v_2 \Rightarrow v_2 = 4$$

Loss in kinetic energy

$$= \frac{1}{2}m_1v_1^2 - \frac{1}{2}(m_1 + m_2)v_2^2 = \frac{1}{2} \cdot 2 \cdot 10^2 - \frac{1}{2} \cdot 5 \cdot 4^2 = 100 - 40 = 60J$$

2.41: Loss of Kinetic Energy

A bag of sand of mass m_2 is suspended by a rope. A bullet of mass m_1 travelling with speed u gets embedded in it, resulting in speed v for the $(bag + bullet)$. The loss of kinetic energy will be:

$$Loss\ of\ Kinetic\ Energy = \Delta KE = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)u^2$$

By conservation of momentum $p_I = p_F$:

$$m_1 u = (m_1 + m_2)v \Rightarrow v = \frac{m_1 u}{m_1 + m_2} \Rightarrow v^2 = \frac{m_1^2 u^2}{(m_1 + m_2)^2}$$

Loss in kinetic energy $= \Delta KE = KE_I - KE_F$

$$= \left(\frac{1}{2}\right)m_1 u^2 - \frac{1}{2}(m_1 + m_2)\frac{m_1^2 u^2}{(m_1 + m_2)^2} = \left(\frac{1}{2}\right)m_1 u^2 - \frac{1}{2}\frac{m_1^2 u^2}{m_1 + m_2}$$

Factor $\frac{1}{2}m_1 u$:

$$= \frac{1}{2}m_1 u^2\left[1 - \frac{m_1}{m_1 + m_2}\right] = \frac{1}{2}m_1 u^2\left[\frac{m_1 + m_2 - m_1}{m_1 + m_2}\right] = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)u^2$$

Example 2.42

A bag of sand of mass 9.8 kg is suspended by a rope. A bullet of mass $200g$ travelling with speed $10\frac{m}{s}$ gets embedded in it, then the loss of kinetic energy will be: **(JEE Main, July 26, 2022-II)**

$$\frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)u^2 = \frac{1}{2}\left(\frac{0.2 \times 9.8}{10}\right)10^2 = \frac{1}{2}\left(\frac{0.2 \times 9.8}{10}\right)10^2 = 9.8\,J$$

F. Elastic Collisions

2.43: Elastic Collisions

- If a collision is elastic, then kinetic energy before the collision is equal to the kinetic energy after the collision.
- In other words, kinetic energy is conserved in elastic collisions.

Example 2.44

Mark the correct option

When two bodies collide elastically, then

A. Kinetic energy of the system is conserved
B. Only momentum is conserved
C. Both kinetic energy and momentum is conserved
D. Neither kinetic energy nor momentum is conserved **(EAMCET, 18 Sep 2020, Shift-I)**

$$Option\ C$$

2.45: Elastic Collisions with equal mass

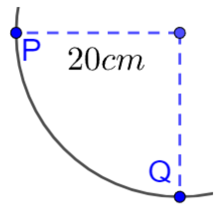
When objects with equal mass collide elastically, their velocities are interchanged.

$$Derivation: Resnick\ Halliday, 10E\ Section\ 9 - 7$$

Example 2.46

A. Two equal masses m_1 and m_2 moving along the same straight line with velocities $+3\frac{m}{s}$ and $-5\frac{m}{s}$ respectively collide elastically. Their velocities after the collision will be: **(NEET 2016; NEET 1998; NEET 1994; NEET 1991)**

B. A small ball P slides down the quadrant of a circle and hits the other ball Q of equal mass which is initially at rest. Neglect the effect of friction and assume the collision to be elastic. The velocity of ball Q after collision will be: **(JEE Main, Jan 30, 2023-I)**



Since masses are equal and collision is elastic, velocities will interchange:

$$Part\ A: Mass\ m_1: -5\frac{m}{s}, \qquad Mass\ m_2: 3\ \frac{m}{s}$$

$$Part\ B: v_{Q-After\ Collision} = v_{P-Before\ Collision} = \sqrt{2gh} = \sqrt{2(10)(0.2)} = \sqrt{4} = 2\frac{m}{s}$$

Example 2.47

Body A of mass $4m$ moving with speed u collides with another body B of mass $2m$, at rest. The collision is head on and elastic in nature. After the collision, the (non-zero) fraction of energy lost by the colliding body is: **(NEET 2019)**

	$Mass$	Speed-Before	p_{Before}	Speed	p_{After}
A	$4m$	u	$4mu$	v	$4mv$
B	$2m$	0	0	v_B	$2mv_B$
			$4mu$		$4mv + 2mv_B$

By conservation of momentum

$$4mu = 4mv + 2mv_B \Rightarrow \underbrace{v_B = 2(u - v)}_{Equation\ I}$$

Since the collision is elastic, kinetic energy is conserved. Hence:

$$KE_{Before} = KE_{After} \Rightarrow \frac{1}{2}(4m)u^2 = \frac{1}{2}(4m)v^2 + \frac{1}{2}(2m)v_B^2 \Rightarrow \underbrace{v_B^2 = 2(u^2 - v^2)}_{Equation\ II}$$

$$4(u - v)^2 = 2(u^2 - v^2)$$
$$2(u - v)^2 = u^2 - v^2$$
$$2u^2 - 4uv + 2v^2 = u^2 - v^2$$
$$u^2 - 4uv + 3v^2 = 0$$
$$u^2 - uv - 3uv + 3v^2 = 0$$
$$u(u - v) - 3v(u - v) = 0$$
$$(u - 3v)(u - v) = 0$$
$$u = v \Rightarrow Rejected, because\ KE\ lost\ is\ zero$$
$$u = 3v \Rightarrow v = \frac{u}{3} \Rightarrow v^2 = \frac{u}{9}$$
$$\Rightarrow Fraction\ of\ energy\ lost\ is\ \frac{8}{9}$$

G. Inelastic Collisions

2.48: Coefficient of Restitution

The coefficient of restitution is the ratio of the *relative* velocity before collision to the *relative* velocity after collision.

$$e = \frac{|v_{relative}\ of\ separation|}{|v_{relative}\ of\ approach|}$$

- Note that the denominator can never be zero since if
$$v_{relative}\ of\ approach = 0 \Rightarrow Collision\ will\ not\ occur$$
- In an elastic collision, the coefficient of restitution is 1.
- In an inelastic collision, the coefficient of restitution is less than 1.

Example 2.49
The coefficient of restitution e for a perfectly elastic collision is: **(NEET 1988)**

1

Example 2.50
Mark all correct options

In inelastic collisions, you can apply the concept of:

A. Coefficient of restitution
B. Conservation of momentum
C. Conservation of kinetic energy
D. Conservation of potential energy

A, B

Example 2.51
A moving block having mass m, collides with another stationary block having mass $4m$. The lighter block comes to rest after collision. When the initial velocity of the lighter block is v, then the value of coefficient of restitution will be: **(NEET 2018)**

By conservation of momentum:

$$mv = (4m)V \Rightarrow V = \frac{v}{4}$$
$$e = \frac{|v_{sep}|}{|v_{app}|} = \frac{\frac{v}{4}}{v} = \frac{v}{4} \times \frac{1}{v} = \frac{1}{4} = 0.25$$

Example 2.52
A ball moving with velocity $2\frac{m}{s}$ collides head on with another stationary ball of double the mass. If the coefficient of restitution is 0.5, then their velocities (in $\frac{m}{s}$) after collision will be: (answer as an ordered pair in $\frac{m}{s}$) **(NEET 2010)**

From the coefficient of restitution:

Body	*Initial*	*Final*
1	$u_1 = 2$	v_1
2	$u_2 = 0$	v_2

$$e = \frac{v_2 - v_1}{u_2 - u_1} \Rightarrow 0.5 = \frac{v_2 - v_1}{2 - 0} \Rightarrow \underbrace{v_2 - v_1 = 1}_{Equation\ I}$$

From conservation of momentum:

$$m_1u_1 = m_1v_1 + (2m_1)v_2 \Rightarrow \underbrace{2 = v_1 + 2v_2}_{Equation\ II}$$

Add Equations I and II:

$$3v_2 = 3 \Rightarrow v_2 = 1 \Rightarrow v_1 = 0 \Rightarrow (v_1, v_2) = (0,1)$$

Example 2.53
In the previous example, explain what is wrong with the solution:

$$|v_2 - v_1| = 1 \Rightarrow v_2 - v_1 = \pm 1, 2 = v_1 + 2v_2 \Rightarrow (v_1, v_2) \in \left\{(0,1), \left(\frac{4}{3}, \frac{1}{3}\right)\right\}$$

$$v_1 = \frac{4}{3} > \frac{1}{3} = v_2 \Rightarrow Contradiction$$

Example 2.54
A body of mass 1 kg collides head on elastically with a stationary body of mass 3 kg. After collision, the smaller body reverses its direction of motion and moves with a speed of $2\frac{m}{s}$. The initial speed of the smaller body before collision is (in $\frac{m}{s}$) **(JEE Main, Jan 25, 2023-II)**

Let

$$Body\ of\ mass\ 1\ kg = Body\ 1$$
$$Body\ of\ mass\ 3\ kg = Body\ 2$$

By conservation of momentum:

$$m_1 v_{1I} + m_2 v_{2I} = m_1 v_{1F} + m_2 v_{2F}$$

Substitute $m_1 = 1, m_2 = 3, v_{2I} = 0, v_{1F} = -2$:

$$(1)v_{1I} + m_2(0) = (1)(-2) + (3)v_{2F} \Rightarrow \underbrace{v_{1I} = -2 + 3v_{2F}}_{Equation\ I}$$

Since the collision is elastic:

$$e = \frac{v_{2F} - (-2)}{v_{1I} - 0} = 1 \Rightarrow \underbrace{v_{1I} = v_{2F} + 2}_{Equation\ II}$$

From Equation I and Equation II:

$$-2 + 3v_{2F} = v_{2F} + 2 \Rightarrow v_{2F} = 2$$

$$v_{1I} = v_{2F} + 2 = 2 + 2 = 4\frac{m}{s}$$

H. Gravitational Potential Energy

2.55: Height for rebounding ball
For a ball dropped from height H, colliding with coefficient of restitution e, the maximum height after rebounding is:

$$H = e^2 h$$

Let $B = Before\ collision,\ A = After\ collision$, from the coefficient of restitution:

$$\frac{|v_A|}{|v_B|} = e \Rightarrow |v_B| = \frac{|v_A|}{e}$$

Substitute $v_B^2 = \left(\frac{v_A}{e}\right)^2 = \frac{v_A^2}{e^2}$ in $v_B^2 = 2gh$:

$$\frac{v_A^2}{e^2} = 2gh$$

We can now solve for v_A^2:

$$v_A^2 = 2g(e^2h) = 2gH$$

Where we used a change of variable:

$$Rebound\ Height = H = e^2h$$

Example 2.56

A ball is dropped from a height of $20m$. If the coefficient of restitution for the collision between ball and floor is 0.5, after hitting the floor, the ball rebounds to a height of: **(JEE Main, Jan 31, 2023-II)**

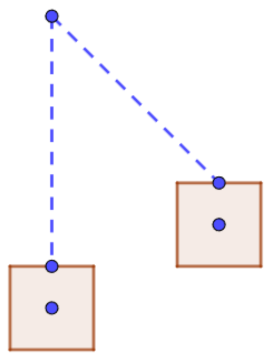
$$H = e^2h = \left(\frac{1}{2}\right)^2 \cdot 20 = 5m$$

Example 2.57

A bullet of mass 10 g moving horizontally with a velocity of $400\frac{m}{s}$ strikes a wooden block of mass 2 kg which is suspended by light inextensible string of length 5 m. As a result, the center of gravity of the block is found to rise a vertical distance of 10 cm. The speed of the bullet after it emerges out horizontally from the block will be: (take $g = 10\frac{m}{s^2}, \sqrt{2} \approx 1.4$) **(NEET 2016)**

Using conservation of energy

$$KE_{Block} = GPE_{Block} = \sqrt{2g\Delta h} = \sqrt{2(10)(0.1)} = \sqrt{2}$$



By conservation of momentum just after the impact of the bullet with the block:

$$m_{Bullet}v_{Bullet-Before} = m_{Block}v_{Block} + m_{Bullet}v_{Bullet-After}$$

$$0.01(400) = 2\sqrt{2} + 0.01v$$

$$v = 400 - 200\sqrt{2} = 100(4 - 2\sqrt{2}) = 100(4 - 2.8) = 120\frac{m}{s}$$

I. Elastic Potential Energy

2 Pending

Example 2.58

A block of mass M moving on a frictionless horizontal surface collides with a spring of spring constant k and compresses it by length L. The maximum momentum of the block after collision is: (JEE Main 2005)

$$\frac{1}{2}Mv^2 = \frac{1}{2}kL^2$$

$$v^2 = \frac{kL^2}{M}$$

$$v = \sqrt{\frac{k}{M}}L$$

$$Mv = \sqrt{kML}$$

2.3 Conservation of Momentum: 2D/3D

A. Explosions

Example 2.59

An explosion breaks a rock into three parts. Two of them go off at right angles to each other. The first part of mass 1 kg moves with a speed of $12\frac{m}{s}$ and the second part of mass 2 kg moves with $8\frac{m}{s}$ speed. If the third part flies off with $4\frac{m}{s}$ speed, then its mass is: **(NEET 2009; NEET 2013)**

Method I

The object is initially at rest. Hence, the final momentum must also sum to zero.
Draw a diagram and use Pythagoras.

$$p_1 = (1)(12) = 12, \qquad p_2 = (2)(8) = 16$$

By Pythagorean Triplet $4(3,4,5) = (12,16,20)$:

$$Combined\ momentum = m_3 v_3 = 20 \Rightarrow m_3 = \frac{20}{4} = 5\ kg$$

Method II

$$Initial\ Momentum = Final\ Momentum = 0$$
$$\vec{\boldsymbol{p}}_1 + \vec{\boldsymbol{p}}_2 + \vec{\boldsymbol{p}}_3 = 0$$
$$\vec{\boldsymbol{p}}_3 = -(\vec{\boldsymbol{p}}_1 + \vec{\boldsymbol{p}}_2) = -[(12,0) + (0,16)] = -[12,16]$$
$$|\vec{\boldsymbol{p}}_3| = \sqrt{12^2 + 16^2} = 20$$
$$m_3 v_3 = 20 \Rightarrow m_3 = \frac{20}{4} = 5\ kg$$

Example 2.60

A 1 kg stationary bomb is exploded in three parts having mass $1:1:3$ respectively. Parts having same mass move in perpendicular direction with velocity $30\frac{m}{s}$. The magnitude of velocity of bigger part will be: **(NEET 2001; NEET 1989)**

Draw a diagram and use Pythagoras.

$$p_1 = p_2 = (1)(30) = 30$$

By $45 - 45 - 90$ triangle:

$$p_3 = m_3 v_3 = 30\sqrt{2} \Rightarrow v_3 = \frac{30\sqrt{2}}{3} = 10\sqrt{2}\frac{m}{s}$$

Example 2.61

A body of mass M at rest explodes into three pieces in the ratio of masses 1:1:2. Two smaller pieces fly off perpendicular to each other with velocities of $30\frac{m}{s}$ and $40\frac{m}{s}$ respectively. The magnitude of velocity of the third piece will be: **(JEE Main, June 29, 2022-I)**

Let the masses have values $1,1\ and\ 2$. Then:

$$\vec{\boldsymbol{p}}_1 = (1)(30) = 30\hat{\boldsymbol{\imath}}, \qquad \vec{\boldsymbol{p}}_2 = (1)(40) = 40\hat{\boldsymbol{\jmath}}$$

By Pythagorean Triplet (3,4,5):

$$|\vec{\boldsymbol{p}}_1 + \vec{\boldsymbol{p}}_2| = 50 \Rightarrow p_3 = 2v = 50 \Rightarrow v = 25$$

Example 2.62
An object flying in air with velocity $\left(20\hat{\boldsymbol{\imath}} + 25\hat{\boldsymbol{\jmath}} - 12\hat{\boldsymbol{k}}\right)$ suddenly breaks in two pieces whose masses are in the ratio $1:5$. The smaller mass flies off with a velocity $\left(100\hat{\boldsymbol{\imath}} + 35\hat{\boldsymbol{\jmath}} + \mathbf{8}\hat{\boldsymbol{k}}\right)$. The velocity of the larger piece will be: **(NEET 2019)**

Without loss of generality, let the masses be 1 *and* 5. By conservation of momentum:

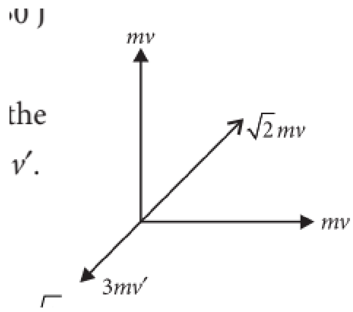
$$(1)(100,35,8) + 5\vec{\boldsymbol{v}}_2 = 6(20,25,-12)$$
$$5\vec{\boldsymbol{v}}_2 = (120 - 100, 150 - 35, -72 - 8)$$
$$\vec{\boldsymbol{v}}_2 = \left(\frac{20}{5}, \frac{115}{5}, -\frac{80}{5}\right) = (4,23,-16)$$

Pending

Example 2.63
44: A block moving horizontally on a smooth surface with a speed of $40\frac{m}{s}$ splits into two equal parts. If one of the parts moves at $60\frac{m}{s}$ in the same direction, then the fractional change in the kinetic energy will be $x:4$ where $x =$ **(JEE Main, Aug 31, 2021-I)**

B. Kinetic Energy

Example 2.64
A particle of mass $5m$ at rest suddenly breaks on its own into three fragments. Two fragments of mass m each move along mutually perpendicular directions with speed v each. The energy released during the process is: (Answer in terms of m and v). **(NEET 2019)**



The resultant momentum for the two smaller fragments is:

$$\sqrt{2}mv$$

The momentum for the fragment of mass $3m$ is:

$$3mV = -\sqrt{2}mv \Rightarrow V = -\frac{\sqrt{2}}{3}v$$

The kinetic energy released is the kinetic energy of the fragments

$$= 2\left(\frac{1}{2}mv^2\right) + \frac{1}{2}(3m)\left(-\frac{\sqrt{2}}{3}v\right)^2 = mv^2 + \frac{3m}{2}\left(-\frac{2}{9}v^2\right) = mv^2 + \frac{1}{3}mv^2 = \frac{4}{3}mv^2$$

C. Inelastic Collisions

2.65: Conservation of Linear Momentum
Two bodies A and B with mass m_A and m_B collide. They have initial velocity $\vec{\boldsymbol{u}}_A$ and $\vec{\boldsymbol{u}}_B$. Their velocity after collision is $\vec{\boldsymbol{v}}_A$ and $\vec{\boldsymbol{v}}_B$

Then, by law of conservation of momentum:

$$m_A\vec{\boldsymbol{u}}_A + m_B\vec{\boldsymbol{u}}_B = m_A\vec{\boldsymbol{v}}_A + m_B\vec{\boldsymbol{v}}_B$$

- Conservation of momentum applies in multiple dimensions, as in one dimension.
- In one dimension, we indicated direction using sign, and hence a scalar equation was sufficient.
- In two dimensions, we need a vector equation.
- Equivalently, we can consider components, and equate the components.

Example 2.66

A mass m moving horizontally (along the x-axis) with velocity v collides and sticks to a mass of $3m$ moving vertically upward (along the y-axis) with velocity $2v$. The final velocity of the combination is: (answer in component form) **(NEET 2011)**

$$Initial\ Momentum = \vec{\boldsymbol{p}}_I = (mv, 0) + (0, 6mv) = (mv, 6mv)$$
$$Final\ Momentum = \vec{\boldsymbol{p}}_F = (4m)\overrightarrow{\boldsymbol{v_F}}$$

By conservation of momentum:

$$(4m)\overrightarrow{\boldsymbol{v_F}} = (mv, 6mv) \Rightarrow \overrightarrow{\boldsymbol{v_F}} = \left(\frac{v}{4}, \frac{3v}{2}\right)$$

Example 2.67

A mass m moves with a velocity v and collides inelastically with another identical mass. After collision, the $1st$ mass moves with velocity $\frac{v}{\sqrt{3}}$ in a direction perpendicular to the initial direction of motion. Find the speed of the 2nd mass after collision. **(JEE Main 2005)**

By conservation of momentum

$$\vec{\boldsymbol{p}}_F = \vec{\boldsymbol{p}}_I \Rightarrow m\left(0, \frac{v}{\sqrt{3}}\right) + m(v_x, v_y) = m(v, 0) + (0,0)$$

Dividing by m and isolating (v_x, v_y) to get the velocity of the second mass:

$$(v_x, v_y) = \left(v, -\frac{v}{\sqrt{3}}\right)$$

And the speed is the magnitude of the velocity:

$$|\boldsymbol{v}| = \sqrt{v^2 + \left(-\frac{v}{\sqrt{3}}\right)^2} = \sqrt{\frac{4}{\sqrt{3}}v^2} = \frac{2}{\sqrt{3}}v$$

Example 2.68

Two spheres A and B of masses m_1 and m_2 respectively collide. A is at rest initially B is moving with velocity B along x −axis. After collision B has a velocity $\frac{v}{2}$ in a direction perpendicular to the original direction. The mass A moves after collision in the direction $\theta = \tan^{-1} C$ (with respect to the x −axis). Find C: **(NEET 2012)**

	Mass	Initial Velocity	$p_{Initial}$	$Final\ Velocity$	p_{Final}
A	m_1	0	0	(v_x, v_y)	$m_1(v_x, v_y)$
B	m_2	$(v, 0)$	$m_2(v, 0)$	$\left(0, \frac{v}{2}\right)$	$m_2\left(0, \frac{v}{2}\right)$
Total			$m_2(v, 0)$		$m_1(v_x, v_y) + m_2\left(0, \frac{v}{2}\right)$

By conservation of momentum:

$$m_2(v, 0) = m_1(v_x, v_y) + m_2\left(0, \frac{v}{2}\right)$$
$$m_2\left(v, -\frac{v}{2}\right) = m_1(v_x, v_y)$$

$$\left(\frac{m_2}{m_1}\right)\left(v, -\frac{v}{2}\right) = (v_x, v_y)$$
$$\theta = \tan^{-1}\left(-\frac{1}{2}\right) \Rightarrow C = -\frac{1}{2}$$

Example 2.69

A ball of mass $200g$ rests on a vertical post of height $20m$. A bullet of mass $10g$, travelling in horizontal direction, hits the center of the ball. After collision, both travel independently. The ball hits the ground at a distance of $30m$ and the bullet at a distance of $120m$ from the foot of the post. The value of initial velocity of the bullet will be: (take $g = 10\frac{m}{s^2}$) **(JEE Main, Jan 30, 2023-I)**

Focus only on the vertical component. Calculate the time taken by the ball and the bullet to hit the ground
Substitute $s = 20, u = 0, a = g = 10\frac{m}{s}$ in $s = ut + \frac{1}{2}at^2$
$$20 = 0(t) + \frac{1}{2}(10)t^2 \Rightarrow t^2 = \frac{20}{5} = 4 \Rightarrow t = 2$$

The horizontal component of the velocities is constant during the motion:
$$v_{Ball} = v_B = \frac{Displacement}{Time} = \frac{30}{2} = 15\frac{m}{s}$$
$$v_{bullet} = v_b = \frac{Displacement}{Time} = \frac{120}{2} = 60\frac{m}{s}$$

By conservation of momentum ($m_B v_B + m_b v_b = m_b v$):
$$(0.2)(15) + (0.01)(60) = (0.01)v$$
$$3 + 0.6 = \frac{v}{100} \Rightarrow v = 360\frac{m}{s}$$

D. Elastic Collisions

3 Pending

Example 2.70

On a frictionless surface, a block of mass M moving at speed v collides elastically with another block of same mass M which is initially at rest. After collision, the first block moves at an angle θ to its initial direction and has a speed $\frac{v}{3}$. The

A. second block's speed after the collision, in terms of v is: **(NEET 2015)**
B. angle between the velocity vectors of the two blocks is:
C. value of θ is:

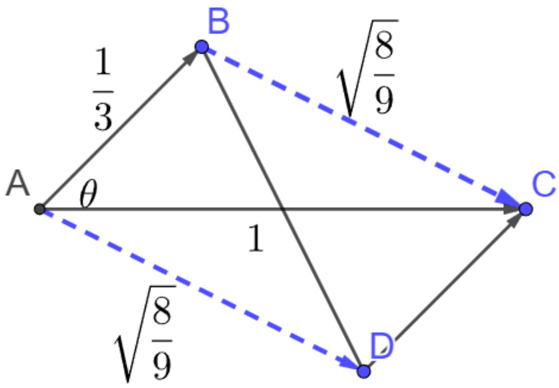
Part A
By conservation of kinetic energy:
$$\frac{1}{2}Mv^2 = \frac{1}{2}M\left(\frac{v}{3}\right)^2 + \frac{1}{2}Mv_F^2$$
$$v^2 = \frac{v^2}{9} + v_F^2$$
$$v_F = \frac{2\sqrt{2}}{3}v$$

Part B

In the diagram, seht

$$Velocity\ of\ first\ block = \overrightarrow{AB} \Rightarrow |\overrightarrow{AB}| = \frac{v}{3}$$

$$Velocity\ of\ second\ block = \overrightarrow{AD} \Rightarrow |\overrightarrow{AD}| = \sqrt{\frac{8}{9}}v$$



Note that ΔABD is a right triangle since:

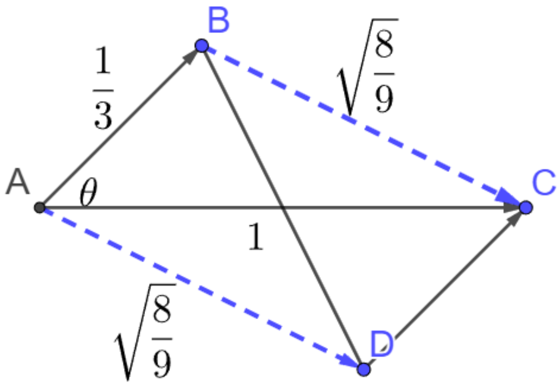
$$AB^2 + AD^2 = \left(\frac{v}{3}\right)^2 + \left(\sqrt{\frac{8}{9}}v\right)^2 = \frac{v^2}{9} + \frac{8v^2}{9} = v^2 = BD^2$$

Hence, the angle between the velocity vectors of the two blocks is:

$$90° = \frac{\pi}{2}$$

Part C

In the diagram, move AD so that it is parallel to itself and its tail is at the tip of B.



By vector addition,

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Since

$$\cos\theta = \frac{\frac{v}{3}}{v} = \frac{1}{3} \Rightarrow \theta = \cos^{-1}\frac{1}{3}$$

And θ is the angle between $\overrightarrow{AB}$ and the original direction $\overrightarrow{AC}$.

E. Collisions with Kinetic Energy

Example 2.71

A. A body A with mass m_A moving with velocity $\vec{\boldsymbol{u}}_A$ strikes a motionless body B with mass m_B. If the velocity after the collision of body A is $\vec{\boldsymbol{v}}_A$, find the velocity vector $\vec{\boldsymbol{v}}_B$ for body B after the collision.

B. ($Calculator$) Body A moving due East at $32\frac{m}{s}$ on a frictionless, horizontal surface strikes body B, and rebounds at a speed of $8.0\frac{m}{s}$ at an angle 29° north of west. If the masses of A and B are 2.1 kg and 3.6 kg respectively, find the percentage of the original kinetic energy of the system that is converted into other forms of energy during the collision.

Part A

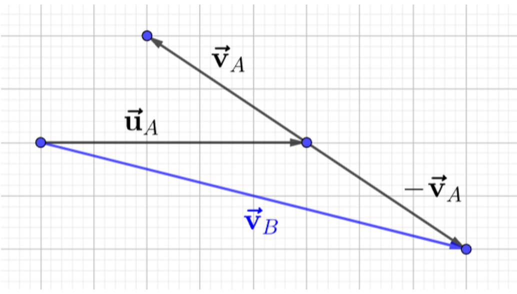
By conservation of momentum:

$$m_A\vec{\boldsymbol{u}}_A + m_B\vec{\boldsymbol{u}}_B = m_A\vec{\boldsymbol{v}}_A + m_B\vec{\boldsymbol{v}}_B$$

Since body B is initially at rest, substitute $m_B\vec{\boldsymbol{u}}_B = 0$, and solve for $\vec{\boldsymbol{v}}_B$:

$$m_A\vec{\boldsymbol{u}}_A = m_A\vec{\boldsymbol{v}}_A + m_B\vec{\boldsymbol{v}}_B \Rightarrow \vec{\boldsymbol{v}}_B = \left(\frac{m_A}{m_B}\right)(\vec{\boldsymbol{u}}_A - \vec{\boldsymbol{v}}_A)$$

Part B



$$\vec{\boldsymbol{v}}_B = \left(\frac{m_A}{m_B}\right)(\vec{\boldsymbol{u}}_A - \vec{\boldsymbol{v}}_A)$$

Substitute $m_A = 2.1, m_B = 3.6, \vec{\boldsymbol{u}}_A = (32,0), \vec{\boldsymbol{v}}_A = (-8\cos 29°, 8\sin 29°)$:

$$\vec{\boldsymbol{v}}_B = \left(\frac{2.1}{3.6}\right)(32 + 8\cos 29°, -8\sin 29°)$$

Take the magnitude on both sides:

$$|\vec{\boldsymbol{v}}_B| = \left(\frac{2.1}{3.6}\right)\sqrt{(32 + 8\cos 29°)^2 + (8\sin 29°)^2} \approx 22.860$$

Initial Kinetic Energy

$$= \frac{1}{2}mu_A^2 + \frac{1}{2}mu_B^2 = \left(\frac{1}{2}\right)(2.1)(32^2) + 0 = 1075.5$$

Final Kinetic Energy

$$= \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 = \left(\frac{1}{2}\right)(2.1)(8^2) + \left(\frac{1}{2}\right)(3.6)(22.860^2) \approx 1007.84$$

Percentage of kinetic energy converted into other forms:

$$= \frac{1075.5 - 1007.84}{1075.5} = \frac{67.66}{1075.5} \approx 6.29\%$$

2.4 Calculus

A. Differentiation

2.72: Momentum and Net Force-I

Net force is the derivative of momentum.
That is, the rate of change of momentum of a body is equal to the net force on it.

$$F_{Net} = \frac{d}{dt}\vec{\boldsymbol{p}}$$

$$F_{Net} = ma = m\left(\frac{d}{dt}\boldsymbol{v}\right) = \left(\frac{d}{dt}m\boldsymbol{v}\right) = \frac{d}{dt}\vec{\boldsymbol{p}}$$

2.73: Constant Velocity

$$F_{Net} = v\frac{dm}{dt}$$

$$F_{Net} = \frac{d}{dt}\vec{\boldsymbol{p}} = \frac{d}{dt}(mv) = v\frac{dm}{dt}$$

Some examples of where this equation is applicable:

- Air escaping a balloon at constant velocity
- A rocket with an exhaust at constant velocity

2.74: Dyne

$$1\ Dyne = 1\ \frac{g \cdot cm}{s^2} = 10^{-5}N$$

Example 2.75

A balloon has mass of $10g$ in air. The air escapes from the balloon at a uniform rate with velocity $4.5\ \frac{cm}{s}$. If the balloon shrinks in $5s$ completely, then the average force acting on that balloon will be in (in dyne): **(JEE Main, July 28, 2022-I)**

$$\frac{dm}{dt} = \frac{10\ g}{5\ s} = 2\frac{g}{s}$$
$$F_{Net} = v\frac{dm}{dt} = \left(4.5\ \frac{cm}{s}\right)\left(2\frac{g}{s}\right) = 9\ \frac{g \cdot cm}{s^2} = 9\ dyne$$

2.76: Mass Flow Rate

An equivalent way of thinking about the rate of change of mass in a system is that it is the mass flow rate.

$$Mass\ Flow\ Rate = \frac{dm}{dt}$$

Example 2.77

A block of metal weighing $2\ kg$ is resting on a frictionless plane. It is struck by a jet releasing water at a rate of 1 $\frac{kg}{s}$ and at a speed of $10\ \frac{m}{s}$. Then, the initial acceleration of the block, in $\frac{m}{s^2}$, will be: **(JEE Main, June 29, 2022-I)**

$$F_{Net} = v\frac{dm}{dt} = (10)(1) = 10N$$
$$a = \frac{F}{m} = \frac{10}{2} = 5\frac{m}{s^2}$$

B. Rockets

2.78: Rockets

Mass will vary in a rocket with working engine which uses combustion engine.

Rocket will work on:

- Newton's Third Law of Motion
- Principle of conservation of momentum

2.79: Thrust

The net force generated by an exhaust leaving a rocket is called thrust. If the exhaust has constant velocity v, then:

$$Net\ Force = Thrust = v\frac{dm}{dt}$$

Example 2.80

A. In a rocket, fuel burns at the rate of $1\frac{kg}{s}$. This fuel is ejected from the rocket with a velocity of $60\frac{km}{s}$. This exerts a force on the rocket equal to: **(NEET 1994)**
B. If the force on a rocket, moving with a velocity of $300\frac{m}{s}$ is $210N$, then the rate of combustion of the fuel is: **(NEET 1999)**

$$A: F_{Net} = v\frac{dm}{dt} = \left(60 \times 10^3 \frac{m}{s}\right)\left(1\frac{kg}{s}\right) = 60{,}000\ N$$
$$B: \frac{dm}{dt} = \frac{F_{Net}}{v} = \frac{210}{300} = 0.7\frac{kg}{s}$$

2.81: Force on a Rocket/Elevator

If a is the magnitude of acceleration of a rocket/elevator, then the force on the rocket/elevator is:

$$m(g+a), a \text{ in upward direction}$$
$$m(g-a), a \text{ in downward direction}$$

The formulas are the same as discussed for the reading of a weighing scale in a rocket/elevator.
This is because a weighing scale measures the force/acceleration on an object.

2.82: Thrust against gravity

$$F_{Net} = Thrust = m(g+a) = v\frac{dm}{dt}$$

To move a rocket upwards, we must counteract gravity, and we must also provide enough force to create the acceleration that we want.

Example 2.83

A. A 5000 kg rocket is set for vertical firing. The exhaust speed is $800\frac{m}{s}$. To give an initial upward acceleration of $20\frac{m}{s^2}$, the amount of gas ejected per second to supply the needed thrust will be (take $g = 10\frac{m}{s^2}$). **(NEET 1998)**

B. A 600 kg rocket is set for vertical firing. If the exhaust speed is $1000\frac{m}{s}$, the mass of the gas ejected per second to supply the thrust needed to overcome the weight of the rocket is: **(NEET 1990)**

C. The initial mass of a rocket in vertical firing position on the ground is 1000 kg. Calculate at what rate the fuel should be burnt so that the rocket is given an acceleration of $20\frac{m}{s^2}$. The gases come out at a relative speed of $500\frac{m}{s}$ with respect to the rocket. **(JEE Main, Aug 26, 2021-I)**

Part A

$$\frac{dm}{dt} = \frac{m(g+a)}{v} = \frac{5000(10+20)}{800} = \frac{50 \times 30}{8} = 187.5\frac{kg}{s}$$

Part B

$$\frac{dm}{dt} = \frac{m(g+a)}{v} = \frac{600(10+0)}{1000} = 6\frac{kg}{s}$$

Part C

$$\frac{dm}{dt} = \frac{m(g+a)}{v} = \frac{1{,}000(10+20)}{500} = 60\frac{kg}{s}$$

C. Integration

2.84: Momentum and Net Force-II

Impulse (change of momentum) is the integral of net force.

$$p(t) = \left(\int F\,dt\right) + C$$

$$F_{Net} = \frac{d}{dt}p$$

Integrate both sides:

$$\int F\,dt = p + C$$

Example 2.85

A bullet is fired from a gun. The force on the bullet is given by $F = 600 - 2 \times 10^5 t$ where F is in Newton and t in seconds. The force on the bullet becomes zero as soon as it leaves the barrel. What is the average impulse imparted to the bullet? **(NEET 1998)**

Determine the time when the force is zero:

$$0 = 600 - 2 \times 10^5 t$$
$$2 \times 10^5 t = 600$$
$$t = 300 \times 10^{-5} = 3 \times 10^{-3}\ seconds$$

The impulse is then

$$\int F\,dt = \int (600 - 2 \times 10^5 t)\,dt = 600t - 10^5 t^2 + C$$

Since initial momentum is 0,

$$C = 0$$

Substitute $t = 3 \times 10^{-3}$:

$$600(3 \times 10^{-3}) - 10^5(3 \times 10^{-3})^2 = 1800 \times 10^{-3} - 10^5(9 \times 10^{-6}) = 1.8 - 0.9 = 0.9N$$

2.86: Momentum and Net Force-II

Impulse (change of momentum) is the integral of net force.

$$\int_{t_1}^{t_2} F\,dt = \Delta p$$

If the mass is constant:

$$\int_{t_1}^{t_2} F\,dt = \int_{t_1}^{t_2} ma\,dt = m\int_{t_1}^{t_2} a\,dt = m(\Delta v) = \Delta mv = \Delta p$$

Example 2.87

A particle of mass m is moving in a straight line with momentum p. Starting at time $t = 0$, a force $F = kt$ acts in the same direction on the moving particle during time interval T so that its momentum changes from p to $3p$. Here k is a constant. The value of T in terms of k and p is: **(JEE Main, 11 Jan, 2019-II)**

$$\int_{t_1}^{t_2} F\,dt = \Delta p$$

Substitute $\int_0^T kt\,dt = \left[\frac{kt^2}{2}\right]_0^T = \frac{kT^2}{2}, \Delta p = 2p$

$$\frac{kT^2}{2} = 2p \Rightarrow T = 2\sqrt{\frac{p}{k}}$$

2.5 Further Topics

88 Examples