Functions

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1. Function Basics

1.1 Functional Operations

A. Background

The common arithmetic operations are:

- Adding: $+$
- Subtracting: $-$
- Multiplication: $\times$
- Division: $\div$

We also learn about

- Exponentiation: $a^b \Rightarrow a \text{ multiplied by itself } b \text{ times}$
- Remainders: $5 \ (mod\ 3) = 2$

Each of the operations above is associated with a sign, which serves as convenient notation, or shorthand, for indicating the operation. It is also possible to define your own operations.

Example 1.1: Cost of Oranges

Nisha needs to buy oranges from the market. The vendor charges her 50 cents per orange, and $2 for the bag to carry the oranges in. She buys 5 Oranges on Monday, and 6 Oranges on Tuesday. Each day, she needs a separate bag. Find the money she has to pay the vendor on each day.

$5\ Oranges = 5 \times 0.5 + 2 = 2.5 + 2 = 4.5, \qquad 6\ Oranges = 6 \times 0.5 + 2 = 3 + 2 = 5$

Example 1.2: Defining a function for the cost of oranges

Nisha needs to buy oranges from the market. The vendor charges her 50 cents per orange, and $2 for the bag to carry the oranges in. Since she needs oranges frequently, she creates a symbol to denote the cost of oranges, including the cost of the bag.

$$\text{Cost of } n \text{ oranges} = \otimes\, n$$

A. Write $\otimes\, n$ in cents and in dollars.
B. Let x dollars be the cost of six oranges. Find x.
C. Using the value of x from the previous part, find the cost of x oranges.

$$Part\ A:\ \otimes\, n = (50n + 200)\ cents = (0.5n + 2)\ dollars$$
$$Part\ B: x = \otimes\, 6 = 0.5 \times 6 + 2 = 3 + 2 = 5$$
$$Part\ C:\ \otimes\, x = \otimes\, 5 = 0.5 \times 5 + 2 = 2.5 + 2 = 4.5\ dollars$$

B. Defining your own operations

1.3: Defining your own operations

Mathematically, you can define any operation that you wish, so as it is meaningful.

- Whether it makes sense, or is useful is secondary to the math.

Example 1.4

Define

$$a \otimes b = \frac{3a - 2b}{4}$$

A. $2 \otimes 4$

B. $3 \otimes 2$

$$2 \otimes 4 = \frac{3(2) - 2(4)}{4} = \frac{6-8}{4} = -\frac{2}{4} = -\frac{1}{2}$$
$$3 \otimes 2 = \frac{3(3) - 2(2)}{4} = \frac{9-4}{4} = \frac{5}{4}$$

1.5: Operation Expressions

If you have multiple defined operation, you can use order of operations to decide the priority. In particular, brackets will take the highest priority (unless otherwise specified)

Example 1.6

$$a \otimes b = \frac{3a - 2b}{4}$$

Given the definition above, find the value of:

$$\frac{4 \otimes 1}{1 \otimes 4}$$

Calculate the numerator and the denominator separately:

$$Numerator: 4 \otimes 1 = \frac{3(4) - 2(1)}{4} = \frac{12-2}{4} = \frac{10}{4} = \frac{5}{2}$$
$$Denominator: 1 \otimes 4 = \frac{3(1) - 2(4)}{4} = \frac{3-8}{4} = -\frac{5}{4}$$

The expression that we want is:

$$\frac{4 \otimes 1}{1 \otimes 4} = \frac{\frac{5}{2}}{-\frac{5}{4}} = \frac{5}{2} \div \left(-\frac{5}{4}\right) = \frac{5}{2} \times \left(-\frac{4}{5}\right) = -2$$

Example 1.7

Define $\ll x = \frac{x}{3} + \frac{3}{x}$. Find:

A. $\ll 3 + \ll 2$

B. $\frac{\ll 4}{\ll 1}$

Part A

$$\ll 3 = \frac{3}{3} + \frac{3}{3} = 1 + 1 = 2$$
$$\ll 2 = \frac{2}{3} + \frac{3}{2} = \frac{4}{6} + \frac{9}{6} = \frac{13}{6} = 2\frac{1}{6}$$
$$\ll 3 + \ll 2 = 2 + 2\frac{1}{6} = 4\frac{1}{6}$$

Part B

$$\ll 4 \Rightarrow \frac{4}{3} + \frac{3}{4} = \frac{16}{12} + \frac{9}{12} = \frac{25}{12}$$
$$\ll 1 = \frac{1}{3} + \frac{3}{1} = \frac{10}{3}$$
$$\frac{\ll 4}{\ll 1} = \ll 4 \div \ll 1 = \frac{25}{12} \div \frac{10}{3} = \frac{25}{12} \times \frac{3}{10} = \frac{5}{8}$$

Note that you cannot "cancel" the ≪ operator in numerator and denominator. In this case, it gives an incorrect answer.

$$\frac{\ll 4}{\ll 1} = \frac{4}{1} = 4$$

1.8: Commutative Property

The commutative property lets you interchange the order of operations.

Example 1.9

A. Is multiplication commutative?
B. Is subtraction commutative?
C. Is division commutative?
D. Is addition commutative?

$$Multiplication: Yes$$
$$Subtraction: No$$
$$Division: No$$
$$Addition: Yes$$

Example 1.10

$$a \otimes b = \frac{3a - 2b}{4}$$

Given the definition above, is $a \otimes b$ commutative?

We already saw that:

$$4 \otimes 1 = \frac{3(4) - 2(1)}{4} = \frac{12 - 2}{4} = \frac{10}{4} = \frac{5}{2}$$
$$1 \otimes 4 = \frac{3(1) - 2(4)}{4} = \frac{3 - 8}{4} = -\frac{5}{4}$$

Hence:

$$4 \otimes 1 \neq 1 \otimes 4$$

And in general, the operation defined is not commutative.

1.11: Equations

We can use defined operations to make equations. And those equations can be solved to find the value of variables.

Example 1.12

$$a \otimes b = \frac{3a - 2b}{4}$$

Given the definition above, find the value of x if

$$x \otimes 2 = -3$$

$$\frac{3x - 2(2)}{4} = -3$$
$$3x - 4 = -12$$

$$3x = -8$$
$$x = -\frac{8}{3}$$

Example 1.13

Define $a * b = 2a + 2b - ab,\ \ a, b$ are any two numbers. For example

$$4 * 3 = 2 \times 4 + 2 \times 3 - 4 \times 3 = 8 + 6 - 12 = 2$$

If $3 * x = 4$, then x is **(NMTC, Primary-Screening, 2006/12)**

Use the definition to evaluate the value of $3 * x$:

$$2 \times 3 + 2 \times x - 3 \times x = 4$$
$$6 + 2x - 3x = 4$$
$$-x = 4 - 6$$
$$-x = -2$$
$$x = 2$$

Example 1.14

Define $\triangle\, x = 3x + 2$. Find the value of $Max\left\{\triangle\, 2, \triangle\, 4, \triangle\, \frac{1}{2}, \triangle\, 0.3\right\} - Min\left\{\triangle\, 2, \triangle\, 4, \triangle\, \frac{1}{2}, \triangle\, 0.3\right\}$

$$\underbrace{\triangle\, 2 = 3(2) + 2}_{Substitute\ x=2} = 6 + 2 = 8$$
$$\underbrace{\triangle\, 4 = 3(4) + 2}_{Substitute\ x=4} = 12 + 2 = 14$$
$$\underbrace{\triangle\, 4 = 3\left(\frac{1}{2}\right) + 2}_{Substitute\ x=\frac{1}{2}} = 3\left(\frac{1}{2}\right) + 2 = \frac{3}{2} + 2 = \frac{7}{2}$$
$$\underbrace{\triangle\, 0.3 = 3(0.3) + 2}_{Substitute\ x=0.3} = 0.9 + 2 = 2.9$$

$$Max - Min = 14 - 2.9 = 11.1$$

Note that as x increases, Δx also increases, and hence, we only need to find the values of $\triangle\, 4$ and $\triangle\, 0.3$

1.15: Distributive Property

The distributive property holds if you are able to "distribute" an operation over brackets. For example:

$$a(b + c) = ab + ac \Rightarrow Multiplication\ is\ distributive\ over\ addition$$

Example 1.16

Is multiplication distributive over subtraction?

Substitute $c = -x$ in $a(b + c) = ab + ac$:

$$a\big(b + (-x)\big) = ab + a(-x)$$
$$a(b - x) = ab - ax$$

Hence, multiplication is distributive over subtraction.

Example 1.17

Define $\triangle\, x = 3x + 2$.

A. Is $\triangle x$ distributive over addition?
B. For any value of x, $\triangle(x_1 + x_2) = \triangle x_1 + \triangle x_2 + c$. Find the value of c.

Part A

The distributive properties would imply:

$$\triangle(x_1 + x_2) = \triangle x_1 + \triangle x_2$$

This does not hold in general because the LHS add 2 once, while the RHS adds 2 twice. For example, consider

$$x_1 = 0, x_2 = 0$$
$$\triangle(x_1 + x_2) = \triangle(0 + 0) = \triangle(0) = 0 + 2 = 2$$
$$\triangle 0 + \triangle 0 = (0 + 2) + (0 + 2) = 4$$

We can also do this algebraically:

$$\text{LHS} = \triangle(x_1 + x_2) = 3(x_1 + x_2) + 2$$
$$\text{RHS} = \triangle x_1 + \triangle x_2 = 3x_1 + 2 + 3x_2 + 2 = 3(x_1 + x_2) + 4 \neq LHS$$

Part B

From the above:

$$\triangle(x_1 + x_2) = 3(x_1 + x_2) + 2 = 3(x_1 + x_2) + 4 - 2 = \triangle(x_1 + x_2) - 2$$
$$\triangle(x_1 + x_2) = \triangle(x_1 + x_2) - 2$$
$$c = -2$$

C. Nested Operations

1.18: Nested Operations

We put one defined operation inside another defined operation using parentheses.

Example 1.19

Define $\boxtimes z$ to be the sum of the factors of an integer z. Evaluate:

$$\boxtimes 1 + \boxtimes 2 + \cdots + \boxtimes 2023$$

Note that any number has factors which are both positive and negative. For example:

$$\boxtimes 2 = -2 - 1 + 1 + 2 = 0$$

And this is true in general.

Hence:

$$\boxtimes 1 + \boxtimes 2 + \cdots + \boxtimes 2023 = 0 + 0 + \cdots + 0 = 0$$

Example 1.20

Define $\boxtimes y$ to be the sum of the positive factors of an integer y.

A. Find $\boxtimes 6$
B. Find $\boxtimes 9$
C. Find $\boxtimes 7$. Hence, find $\boxtimes(\boxtimes 7)$
D. Find $\boxtimes[\boxtimes(\boxtimes 5)]$
E. Find $\boxtimes(\boxtimes -2)$
F. Find the numbers $\boxtimes z$ such that $\boxtimes z \leq z$
G. Find the smallest value of y such that $\boxtimes y = 2y$

Parts A and B

$$\boxtimes 6 = 1 + 2 + 3 + 6 = 12$$

$$\boxtimes\, 9 = 1 + 3 + 9 = 13$$

Part C

$$\boxtimes\, 7 = 1 + 7 = 8$$
$$\boxtimes\,(\boxtimes\, 7) = \boxtimes\,(1 + 7) = \boxtimes\,(8) = 1 + 2 + 4 + 8 = 15$$

Part D

$$\boxtimes\,[\boxtimes\,(\boxtimes\, 5)] = \boxtimes\,[\boxtimes\,(6)] = \boxtimes\,[12] = 28$$

Part E

$$\boxtimes\,(\boxtimes\, -2) = \boxtimes\,(1 + 2) = \boxtimes\, 3 = 1 + 3 = 4$$

Part D:

$\boxtimes\, z$ for positive integers is always greater than the number.
$\boxtimes\, 1 = 1$
$\boxtimes\, z$ for negative integers is always positive. Hence, all negative numbers will satisfy the requirement.

$$x < 0\ OR\ x = 1$$

Example 1.21

The operation $\otimes$ is defined for all nonzero numbers by $a \otimes b = \frac{a^2}{b}$. Determine $[(1 \otimes 2) \otimes 3] - [1 \otimes (2 \otimes 3)]$. **(AMC 8 2000/17)**

$$\left[\left(\frac{1^2}{2}\right) \otimes 3\right] - \left[1 \otimes \left(\frac{2^2}{3}\right)\right] = \left[\frac{\left(\frac{1}{2}\right)^2}{3}\right] - \left[\frac{1^2}{\frac{4}{3}}\right] = \frac{1}{12} - \frac{3}{4} = \frac{1}{12} - \frac{9}{12} = -\frac{8}{12} = -\frac{2}{3}$$

D. Back-Calculations

If we are given the output to a defined operation, we can work backwards to find the input.

Example 1.22

Suppose that $a * b$ means $3a - b$. What is the value of x if $2 * (5 * x) = 1$? **(AMC 8 2016/11)**

$$\underbrace{2}_{a} * \underbrace{(5 * x)}_{b} = 1 \Rightarrow \underbrace{6}_{3\times 2} - \underbrace{(5 * x)}_{=15-x} = 1 \Rightarrow 6 - (15 - x) = 1 \Rightarrow 6 - 15 + x = 1 \Rightarrow x = 10$$

E. Multi-Variable Defined Operations

It is possible to define operations that have more than one input. This works the same way as a single input.

Example 1.23

Define $x \dagger y = \frac{x}{y} + \frac{y}{x}$. Calculate $\frac{(3\dagger 5)+(7\dagger 4)}{6\dagger 2}$

Example 1.24

Define $a \wedge b \wedge c = -$. Calculate $\frac{(3\dagger 5)+(7\dagger 4)}{6\dagger 2}$

F. Commutative Property

The commutative property says that interchanging the numbers does not matter in multiplication. For example

$$3 \times 5 = 5 \times 3$$

The commutative property may or may not hold in defined operations.

Example 1.25

Define the operation $*$ by $a * b = (a + b)b$. What is $(3 * 5) - (5 * 3)$? **(AMC 10 2007/2)**

The operation is not guaranteed to be commutative. So, we cannot say that:

$$(3 * 5) - (5 * 3) = 0$$

Rather, we work it out using the definition:

$$(3 * 5) - (5 * 3) = (3 + 5)5 - (5 + 3)3 = 8 \times 5 - 8 \times 3 = 8 \times 2 = 16$$

G. Associative Property

1.2 Function Definition: Input-Output Rules

A. Definitions and Basics

1.26: Definition

A function is a rule that associates *exactly one output* with each input.

The input is generally denoted x
The output is generally denoted $f(x)$

Example 1.27

A rule associates two outputs with each input. Is it a function?

$$No.$$

1.28: Notation

Functions are usually denoted using single letters. f and g are very common letters to use. Other letters can also be used.

$$\underbrace{y}_{Output} = f\left(\underbrace{x}_{Input}\right)$$

$$y = f(x) \Rightarrow Read: y\ is\ equal\ to\ f\ of\ x$$

- We commonly use the variable y to represent the output
- We commonly use the variable x to represent the input

Example 1.29

$$z = g(n)$$

$$Output: z$$
$$Input: n$$
$$Function: g$$

1.30: Inputs

The input for a function can be any kind of number, or sometimes not even a number. Some specific cases which are of interest are:

- Discrete: Eg: $\left\{-2, 3, 6, 9, 12, 2.5, 3\frac{1}{2}\right\}$
- Continuous. Eg: $-2 \le x \le 12$

Example 1.31

Let

$$\mathbb{R}, set\ of\ Real\ numbers$$
$$\mathbb{W}, set\ of\ Whole\ numbers$$
$$\mathbb{N}, set\ of\ Natural\ Numbers$$
$$\mathbb{C}, set\ of\ Complex\ Numbers$$
$$\mathbb{Q}, set\ of\ rational\ numbers$$

A. $p = f(n)$ where p is the number of pizzas I order for a party, and n is the number of people attending the party. What set does n belong to?
B. $s = f(h)$ where s is the clothing size for a school uniform, and h is the height of the student. What set does h belong to?

$$n \in \mathbb{W}$$
$$h \in \mathbb{R}$$

1.32: Rule

The rule can be:

- Mathematical
- Logical/verbal
- Tabular
- Graphical
- Ordered Pairs

Any form that precisely associates a single output with any valid input.

Example 1.33

Associate each form of the function below with one of the following ways of presentation:

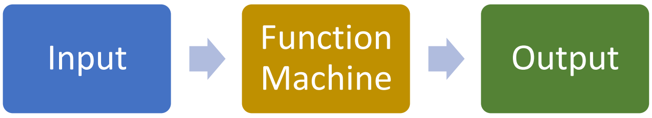
A. Logical/verbal
B. Mathematical
C. Specified in a table
D. Ordered Pairs

Form 1: $y = x - 5$
Form 2: y is the output obtained by subtracting 5 from the input.
Form 3:$(x, y) = \{(1, -4)(2, -3)(0, -5\ )(8,3)\}$
Form 4:

Input	1	2	0	8
Output	-4	-3	-5	3

$$Form\ 1: Mathematical$$
$$Form\ 2: \frac{Logical}{Verbal}$$
$$Form\ 3: Tabular$$

1.34: Functions as Input-Output Machine



A function can be thought of as an input-output machine. Every input the function gets has some process applied to it and this process then results in an output.
In the function machine analogy, we do not focus too much on the method by which the function is converted from the input into the output.

1.35: Mathematical Functions

Functions can be defined using mathematical rules. The maths that you have learnt, and are going to learn (linear, quadratic, polynomial, trigonometric, exponential, logarithmic) will be very useful here.

Example 1.36

A function has an output which is double its input. Write the function as a:

A. Mathematical Rule
B. Logical/Verbal Rule
C. Input-Output Table
D. Set of Ordered Pairs
E. Graph
F. Ordered Pairs on a Graph

Mathematical Rule

$$y = \underbrace{f}_{Function}\left(\underbrace{x}_{Input}\right) = \underbrace{2x}_{Output}$$

For example:

$$\underbrace{f}_{function}\left(\underbrace{6}_{Input}\right) = \underbrace{12}_{Output}, \qquad f(0.3) = 0.6$$

Logical or Verbal Rule:

$$f(x)\ is\ a\ function\ that\ doubles\ its\ input$$

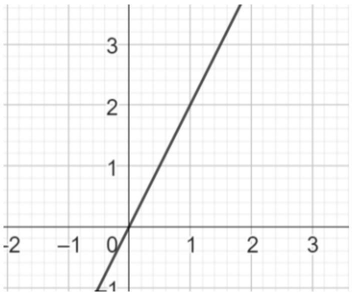
Input-Output Rule

Input − Output Rule						
Input	x	3	5	$\frac{3}{4}$	0.2	x
Output	$f(x)$	6	10	$\frac{6}{4}$	0.4	$2x$

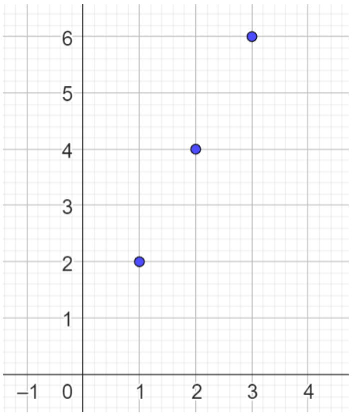
Set of Ordered Pairs

$$(x, y) = (1,2), (2,4), (3,6), \dots$$

Graph



Set of Ordered Pairs on a Graph



1.37: Logical Functions

A function does not have to be mathematical. It only needs to associate each input with an output. For instance, a function might be associated with the following

- Input: Student, Output: Grade
- Input: Name of Person, Output: City where the person lives
- Input: Person, Output: Aadhar Number of the Person

Example 1.38

$r(x)$ is a function that associates a student with a roll number. State this function and give the following data in an input-output table.

$$Students: \left\{\left(\underbrace{Teertha}_{Input}, \underbrace{A-23}_{Output}\right), \left(\underbrace{David}_{x}, \underbrace{B-14}_{y}\right)\left(\underbrace{Carlos}_{Student}, \underbrace{A-7}_{Roll\ No}\right)\right\}$$

$$Logical\ or\ Verbal\ Rule: r(student)\ is\ a\ function\ that\ gives\ the\ roll\ number\ for\ the\ student$$

Input-Output Table							
Input	Teertha	David	Carlos	.	.	.	$Student$
Output	$A-23$	$B-14$	$A-7$	.	.	.	$Roll\ No$

B. Checking for Functions

1.39: Checking for a Function

To be a function, a rule must meet two conditions:

- There must be no input that does not have an output.

- Every input must be associated with exactly one output

Things which are valid:
- Inputs can be repeated
- Outputs can be repeated (in other words, it is not necessary for the output of each input to be unique)
-

Example 1.40
Check whether the input-output table given represents a function.

Input – Output						
Input	3	2	5	6	1	5
Output	4	8	7	4	9	7.5

The table has 5 as an input twice (see below). And it has different outputs for the two 5's.

Input – Output Rule						
Input	3	2	5	6	1	5
Output	4	8	7	4	9	7.5

Hence, the above table does not represent a function.

Example 1.41
Does the given input-output table represent a function?

Input – Output Rule						
Input	3	2	5	6	1	5
Output	4	8	$\frac{3}{4}$	4	9	0.75

There is only one input that gets repeated, and that is

$$5 \to \frac{3}{4}, \qquad 5 \to 0.75$$

But note that

$$\frac{3}{4} = 0.75$$

And hence every input is mapped to precisely one output.
The fact that 5 as an input occurs twice is not a problem.

Example 1.42: Input-Output Table
Does this input-output table giving the favorite city of different children represent a function where the child is the input, and the city is the output?

Child	Roshni	Rishi	Radha	Jay	Vishal	Jyoti
City	London	New York	Toronto	London	Amsterdam	Geneva

Note that two different children have the same output, which is London. This is not an issue. Different inputs can have the same output.

Given that the table above represents a function, $City = f(Child)$. Also, find:

A. $f(Roshni)$
B. $f(Rishi)$

$$f(Roshni) = London$$
$$f(Rishi) = New\ York$$

Example 1.43: Input-Output Table
Does this input-output table giving the temperature at a

Time	3.00	9.00	12.00	3.00	6.00
Temperature	28	26	24	22	24

certain time of the day with the time of the day as an input, and the temperature as an output represent a function.

This table does not represent a function. The input

$$3.00$$

Occurs twice, and has two different outputs.

Example 1.44

Suggest a way to make the table from the prior example into a function.

One way to make this a function to check the real-world interpretation. If there is a different value of the output because one time is 3.00 pm, and the other is 3.00 am, then the table can be converted into a function by being precise about the input.

Method I

By using am and pm for the time:

Time	3.00 pm	9.00 pm	12.00 am	3.00 am	6.00 am
Temperature	28	26	24	22	24

Method II

By using a 24-hour clock instead of a 12-hour clock:

Time	3.00 pm	9.00 pm	12.00 am	3.00 am	6.00 am	
Temperature	28	26	24	22	24	

Then the table above is a function.

1.45: Equations

Equations are a standard way to represent a function.

- This does not mean that every equation is a function.
- Rather every equation which has a unique output for each input results in a function.

Example 1.46

Check if the following equations represents a function $\underbrace{y}_{Output} = \underbrace{f(x)}_{Input}$:

A. $y = 3x + 5$
B. $y = x^2$
C. $y = \sqrt{x}$
D. $x = y^2$
E. $x = y^3$

Parts A, B, C and E

In order to see whether the given equation is a function, we attempt to write it in the form

$$y = f(x)$$

Which we can successfully do for each of Parts A, B and C:

Linear Function

$$y = f(x) = 3x + 5$$

Part A is a linear function (a function that represents a line).
All lines are functions except for vertical lines.

Parabola

$$y = f(x) = x^2$$

For every input, there is precisely one output, and hence, this is a function.

Square Root Function

$$y = f(x) = \sqrt{x}$$

$$y = \sqrt{9} \Rightarrow y = \pm 3 \Rightarrow \mathbf{2\ Outputs} \Rightarrow \mathbf{Not\ a\ Function}$$

The above logic is not correct.

By convention, the output of the square root symbol only gives the positive square root as its output. Hence:

$$\sqrt{9} = +3$$

It does not give two outputs. Hence:

$$\sqrt{9} \neq \pm 3$$

If you want both the positive and the negative square root, then you must write:

$$\pm\sqrt{9}$$

$$y = f(x) = \sqrt[3]{x}$$

The above equations associates $\underbrace{exactly\ one\ value\ of\ y}_{\mathbf{important\ part}}$ with every value of x.

Hence, Parts A, B and C, and E are all functions.

Parts D

Suppose we attempt to write

$$y = f(x)$$

We must then solve the given equation for y:

$$x = y^2 \Rightarrow y = \pm\sqrt{x}$$

And this associates two values of y for each value of x (except when $x = 0$), and hence it is not a function.

Challenge 1.47

Check if the following equations represents a function $y = f(x)$:

A. $y = e^x, e \approx 2.71$
B. $y = \sin x$

1.48: Uniqueness of Output

If a function has a unique output for each input, then it is a special kind of functions.
Such functions are invertible functions.

Example 1.49

In the previous example, you decided that the following are both functions:

A. $f(x) = 3x + 5$
B. $f(x) = x^2$

Do each of the functions above associate a different output for each input? If yes, say why? If no, say why?
Hence, state whether it is necessary that a function associates a different output for each input.

Equation A gives a different output for each input.
Equation B does not give a different output for each input. Specifically:

$$f(-x) = f(x) = x^2$$

It is not necessary that a function associates a different output for each input.

C. Logical Rule

Example 1.50

Check whether the following rules mentioned below are functions:

A. A rule that associates the $\underbrace{Roll\ Number}_{Input}$ of each student in a college to $\underbrace{his\ name}_{Output}$.

B. A rule that associates the $\underbrace{name}_{Input}$ of each student in a college, to $\underbrace{his\ Roll\ Number}_{Output}$.

Part A

Every roll number is associated with precisely one name.
Hence, this rule represents a function.

Part B

Two students can have the same name.
But, they will have different roll numbers.

Hence, this rule does not represent a function.

Example 1.51

The PAN number is the unique identifying number issued by the Income Tax Department of the Government of India to every citizen of India who applies for one.

Check whether the following rules mentioned below are functions:

A. A rule that associates the $\underbrace{PAN\ number}_{Input}$ of each citizen of India to $\underbrace{his\ name}_{Output}$.

B. A rule that associates the $\underbrace{name}_{Input}$ of each citizen of India who has a PAN number, to $\underbrace{his\ PAN\ number}_{Output}$.

Part A

Every PAN number is associated with exactly one name.
Hence, the rule given in Part A is a function.

Part B

Every individual is associated with exactly one PAN Number.
But, it is possible that two individuals have the same name.

For same, if two people have the name

$$Mahesh\ Shah$$

They will still have different PAN Numbers.

Hence, the rule given in Part B is not a function.

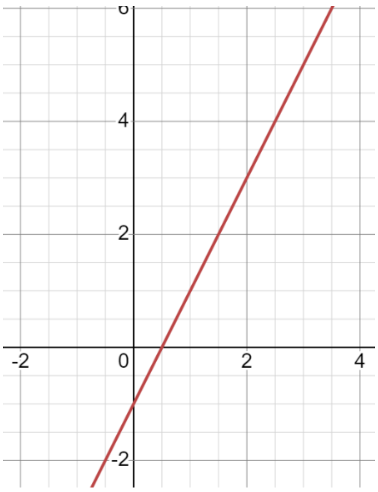
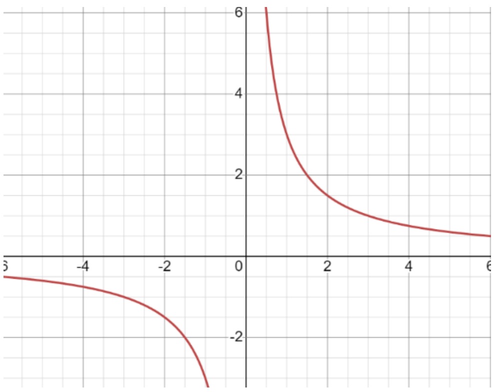
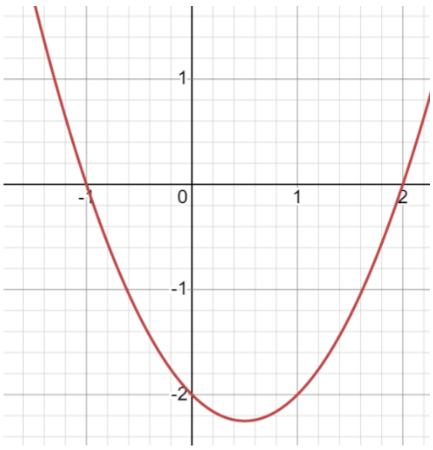
D. Graphing Functions

Functions can be represented as a graph by plotting them on the Cartesian coordinate plane.

Example 1.52

Use a graphing calculator to show the following functions in graphical form:

A. $y = 2x - 1$
B. $y = x^2 - x - 2$
C. $y = \frac{3}{x}$

1.53: Graphical Test: Vertical Line Test

For a rule to be a **function** it must associate a **single output** with every input.
Hence, any vertical line must cut the graph of a function in a maximum of one place.

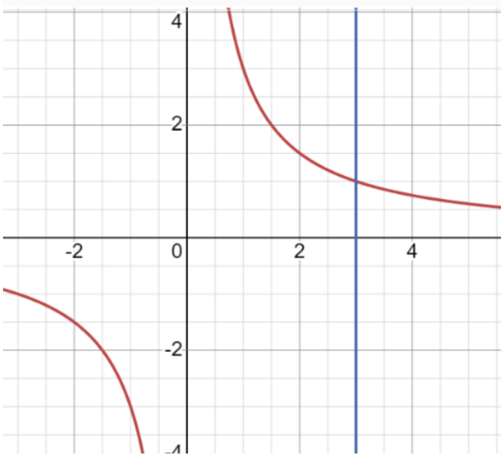
If you draw a vertical line through a function, you can identify the x coordinate and the y coordinate of the point where the line cuts the graph of the function.

The x coordinate represents the input. The y coordinate represents the output.

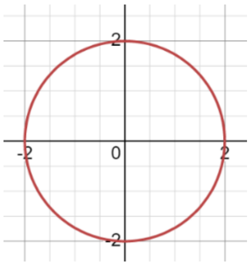
➢ If any vertical line cuts a graph in a maximum of one place, then every input is associated with exactly one output, and hence the graph is a function.
➢ If a vertical line cuts a graph in more than one place, then at least one input is associated with two or more outputs, and hence the graph is not a function.

Example 1.54

You used a graphing utility to enter the algebraic form of some functions above. Go back and have a look at those graphs, and confirm that any vertical line cuts the graph of these functions in a maximum of one place.



Any vertical line will cut the graphs in a maximum of one place.

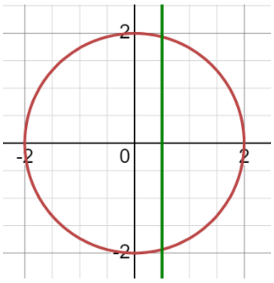


Example 1.55

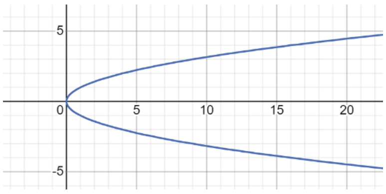
The adjoining figure shows the graph of a circle on the coordinate plane drawn using the equation

$$y^2 + x^2 = 4$$

Does this graph represent a function?



If you draw a vertical line through the graph of a circle, it cuts the graph in two places (except at the extremes).
The adjoining figure has a green line drawn, which cuts the graph in two places.
Hence, the graph of a circle is not a function.

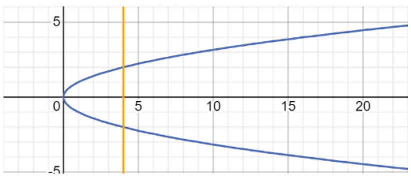


Example 1.56

The adjoining figure shows the graph of

$$x = y^2$$

Determine whether this is the graph of a function.



Any vertical line drawn to the right of

$$x = 0$$

Will cut the graph in more than one place.

This means that there is more than one output associated with a single input.
And hence, this is not the graph of a function.

E. Average Rate of Change

1.57: Average Rate of Change

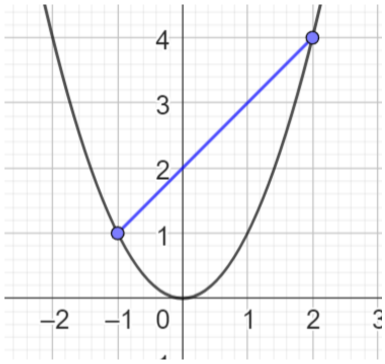
The average rate of change of a function is very important

$$Avg\ Rate\ of\ Change = \frac{Change\ in\ Output}{Change\ in\ Input} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example 1.58

Find the average rate of change of $y = f(x) = x^2$ over the interval: $\left(\underbrace{-1}_{x_1}, \underbrace{2}_{x_2}\right)$

$$y_1 = f(x_1) = f(-1) = (-1)^2 = 1$$
$$y_2 = f(x_2) = f(2) = (2)^2 = 4$$



We can tabulate this as follows:

x_1	x_2	y_1	y_2
-1	2	1	4

$$Avg\ Rate\ of\ Change = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{2^2 - (-1^2)}{2 - (-1)} = \frac{4 - 1}{2 + 1} = \frac{3}{3}$$
$$= 1$$

Example 1.59

Consider the function showing the number of lemonade glasses sold by Karuna over the period Jan to June 2020 given by the table alongside. Find the average rate of change of the number of glasses sold from Feb to June.

Month(m)	Jan	Feb	Mar	Apr	May	June
$Glasses\ sold = g(m)$	350	300	100	125	75	150

$$Avg.Rate\ of\ Change = \frac{150 - 300}{6 - 2} = -\frac{150}{4} = -37.5$$

The average rate of change of the number of glasses sold is

$$-37.5 \Rightarrow 37.5\ glasses\ less\ on\ average\ every\ month$$

F. Intercepts

1.60: Intercepts

The places where a graph cuts the x and the y axes are its intercepts.

1.61: Finding Intercepts Graphically

The graphical approach uses visual inspection to see where the graph of the functions intersects the x-axis. It is difficult to apply for approximation.

- The x intercept of a function is also called a *zero* of the function.

Example 1.62

Identify the zeroes of the function in the graph of on the right.

From the graph, we can see that the graph intersects the x-axis at

$$-4, -3, -2$$

And these are the zeroes of the function.

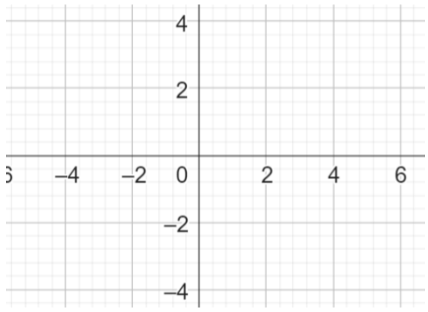
1.63: Finding Intercepts Algebraically

To find the x-intercept, substitute

$$y = 0$$

To find the y-intercept, substitute

$$x = 0$$



Example 1.64

Find the intercepts, if any, of each function below:

A. $f(x) = 5x + 4$
B. $f(x) = x^2 + 7x + 12$
C. $f(x) = \frac{1}{x}$
D. $g(x) = \frac{4}{x+2}$

Part A

We have

$$f(x) = 5x + 4 \Rightarrow y = 5x + 4$$

To find the y-intercept, substitute $x = 0$:

$$y = 4$$

To find the x-intercept, substitute $y = 0$:

$$5x + 4 = 0 \Rightarrow x = -\frac{4}{5}$$

$$A\ single\ zero, which\ is\ -\frac{4}{5}$$

Part B

y intercept

$$12$$

x intercept

$$x^2 + 7x + 12 = 0$$
$$(x+3)(x+4) = 0$$
$$x \in \{-3, -4\}$$
$$Two\ zeros\ which\ are -3\ and -4.$$

Part C

$$0 = \frac{1}{x} \Rightarrow 0 = 1$$

There is no solution.
Hence, there is no x intercept.

$$No\ Solution \Rightarrow No\ x - intercept$$

To find the y-intercept, substitute $x = 0$:

$$y = \frac{1}{0} \Rightarrow Not\ Defined \Rightarrow No\ y - intercept$$

Part D

To find the x-intercept, substitute $y = 0$

$$0 = \frac{4}{x+2} \Rightarrow 0 = 4 \Rightarrow No\ Solution \Rightarrow No\ x\ intercept$$

To find the y-intercept, substitute $x = 0$:

$$y = \frac{4}{x+2} \Rightarrow y = 2$$

G. Quadratic Functions

A quadratic function has complex behavior across its entire domain. But a lot of information can be meaningfully extracted by looking only the sign of a function.
A sign diagram does precisely this.

For example, consider the function:

$$f(x) = (x-a)(x-b) \Rightarrow Zeroes\ of\ f(x) \in \{a, b\},$$

Based on what we know about the behavior of a quadratic, we know that the function above represents an upward parabola, and hence:

x	$(-\infty, a)$	(a, b)	(b, ∞)
$f(x)$	$+ve$	$-ve$	$+ve$

Example 1.65

Draw a sign diagram for the following functions:

A. $f(x) = x^2 - 9x - 36$
B. $g(x) = -(x^2 - 9x - 36)$

Part A

First, we factor the function:

$$f(x) = (x-12)(x+3) \Rightarrow Zeroes \in \{-3, 12\}$$

This is the graph of a quadratic, which is always a parabola.
Since the leading coefficient is positive, we know that the graph represents an upward parabola.

x	$(-\infty, -3)$	$(-3, 12)$	$(12, \infty)$
$f(x)$	$+ve$	$-ve$	$+ve$

Part B

First, we factor the function:

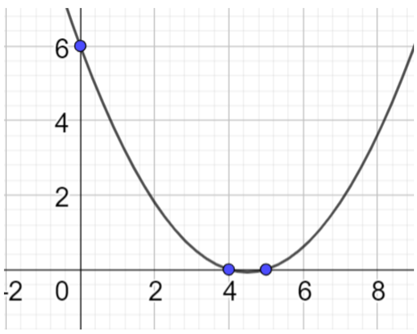
$$f(x) = -(x-12)(x+3) \Rightarrow Zeroes \in \{-3, 12\}$$

And now this is the same as the previous example, except that since the leading co-efficient is negative, it is a downward parabola.

x	$(-\infty, -3)$	$(-3, 12)$	$(12, \infty)$
$f(x)$	$-ve$	$+ve$	$-ve$

Example 1.66

A quadratic function has roots at 4 and 5, and has y-intercept 6. Find the function.



p and q are the roots, so

$$p = 4, \qquad q = 5$$

We also know that the y-intercept is 6. And, at the y-intercept, by definition, the value of x is:

$$0$$

Hence, substitute $f(x) = 6, x = 0, p = 4, q = 5$ into $f(x) = a(x-p)(x-q)$, giving us:

$$6 = a(0-4)(0-5) \Rightarrow 6 = a(-4)(-5) \Rightarrow 6 = 20a \Rightarrow a = \frac{6}{20} = \frac{3}{10} = 0.3$$

And hence the function is:

$$f(x) = \frac{3}{80}(x-4)(x-5)$$

H. Zeroes of Factored Functions

Earlier, we were working with quadratic functions, and we needed to factor the functions.
When working with higher-degree polynomials, the factoring itself can be a lengthy process, and hence, questions will often give the equation to us in factored form.

1.67: Zero of a Function

A zero of a function is the value of x (the input) which makes $y = f(x)$(output) zero.

Example 1.68

Find the zeros of the function:

$$f(x) = \left(3x - \frac{1}{2}\right)(2x - 7)$$

To find the zeroes, we substitute $f(x) = 0$:

$$\left(3x - \frac{1}{2}\right)(2x - 7) = 0$$

Use the zero-product property:

$$3x - \frac{1}{2} = 0 \Rightarrow 3x = \frac{1}{2} \Rightarrow x = \frac{1}{6}$$
$$2x - 7 = 0 \Rightarrow 2x = 7 \Rightarrow x = \frac{7}{2}$$
$$Zeros\ of\ x \in \left\{\frac{1}{6}, \frac{7}{2}\right\}$$

Example 1.69

Find the product of the zeros of the following function

$$(x - 27)(x - 26) \dots . (x + 26)(x + 27)$$

The product of the roots will be

$$(27)(26) \dots (0) \dots (-26)(27) = 0$$

Example 1.70

Find the zeroes of:

$$f(x) = (x - 3)(x - 4)(x - 5)$$

To find the zeroes, we let $f(x) = 0$:

$$(x - 3)(x - 4)(x - 5) = 0$$

By the zero-product property, we have:

$$x - 3 = 0 \Rightarrow x = 3$$
$$x - 4 = 0 \Rightarrow x = 4$$
$$x - 5 = 0 \Rightarrow x = 5$$

And we can combine the above three to get:

$$x \in \{3,4,5\} \Rightarrow Zeroes \in \{3,4,5\}$$

1.71: Multiplicity of a Zero

If a zero occurs in a function more than once due to the power of the expression, that is called the multiplicity of the zero.

$$(x - a)^n = 0 \Rightarrow a\ is\ a\ zero\ with\ multiplicity\ n, n \in \mathbb{N}$$

- Multiplicity of different zeroes need not be the same.

Example 1.72

Find the zeros of $f(x) = (x-3)^2$ and determine the multiplicity.

To find the zeroes, we let $f(x) = 0$:

$$(x-3)^2 = 0$$

Take square roots both sides:

$$x - 3 = 0$$
$$x = 3$$

$x = 3$ is a zero of the function. But the original function was

$$f(x) = (x-3)^2 = (x-3)(x-3)$$

Hence, 3 occurs twice as a zero in the function, and hence

$$Multiplicity\ of\ 3 = 2$$

Example 1.73

Determine the zeroes and their multiplicity in:

$$f(x) = \left(2x - \frac{3}{4}\right)^3 (x-5)(3x-5)^2 \left(7x - \frac{1}{2}\right)^{12}$$

$$\left(2x - \frac{3}{4}\right)^3 (x-5)(3x-5)^2 \left(7x - \frac{1}{2}\right)^{12} = 0$$

Use the zero-product property:

$$\left(2x - \frac{3}{4}\right)^3 = 0 \Rightarrow 2x - \frac{3}{4} = 0 \Rightarrow 2x = \frac{3}{4} \Rightarrow x = \frac{3}{8}, Multiplicity = 3$$
$$x - 5 = 0 \Rightarrow x = 5, Multiplicity = 1$$
$$(3x-5)^2 = 0 \Rightarrow 3x - 5 = 0 \Rightarrow x = \frac{5}{3}, Multiplicity = 2$$
$$\left(7x - \frac{1}{2}\right)^{12} = 0 \Rightarrow 7x - \frac{1}{2} = 0 \Rightarrow 7x = \frac{1}{2} \Rightarrow x = \frac{1}{14}, Multiplicity = 12$$

1.74: Parity of Multiplicity of a Zero

Parity is a technical term that refers to whether a number is even or odd. A root that occurs an

- odd number of times is said to have odd multiplicity
- even number of times is said to have even multiplicity

Example 1.75

Consider the function

$$h(x) = \left(x - \frac{3}{4}\right)\left(x - \sqrt{2}\right)^2 \left(x - \frac{\sqrt{5}}{\sqrt{7}}\right)^5 (x - 0.23)^{11}(x+54)^{53}$$

Find the zeroes of $h(x)$, find the multiplicity of each zero, and then the parity of the multiplicity.

$$x - \frac{3}{4} = 0 \Rightarrow x = \frac{3}{4}, \qquad Multiplicity = 1 \Rightarrow Parity = Odd$$
$$\left(x - \sqrt{2}\right)^2 = 0 \Rightarrow x = \sqrt{2}, \qquad Multiplicity = 2 \Rightarrow Parity = Even$$

$$\left(x-\frac{\sqrt{5}}{\sqrt{7}}\right)^5=0\Rightarrow x=\frac{\sqrt{5}}{\sqrt{7}}, \qquad Multiplicity=5\Rightarrow Parity=Odd$$
$$(x-0.23)^{11}=0\Rightarrow x=0.23, Multiplicty=11\Rightarrow Parity=Odd$$
$$(x+54)^{53}=0\Rightarrow x=-54, Multiplicity=53\Rightarrow Parity=Odd$$

Example 1.76

Mark all correct options

A function has a zero with odd multiplicity and is of the form $f(x)=(x-a)^b$. Then, which of the following is possible:

A. a is odd.
B. a is even.
C. a is neither even nor odd.
D. b is even.
E. b is odd.
F. b is neither even nor odd.

Since the multiplicity creates no restriction on a, it can be anything:

$$Options\ A, B, C$$

Since the zero is of odd multiplicity, b must be odd:

$$Option\ E$$

Final Answer:

$$Option\ A, B, C, E$$

I. End Behavior of Polynomial Functions

1.77: Polynomial Functions

Polynomial functions are of the form

$$P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0, \qquad n\in\mathbb{W}, a_n\neq 0$$

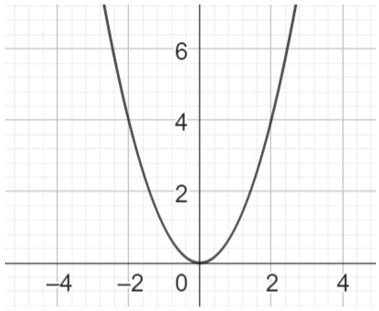
- Polynomials are written in standard form in descending order of n.
- The value of n is the degree of the polynomial

$$P(x)=3x+5\Rightarrow Degree=1$$
$$P(x)=2x^2+4x^1+1x^0\Rightarrow Degree=2$$
$$P(x)=\frac{1}{2}x^3-5x^2+3x^1+7\Rightarrow Degree=3$$
$$P(x)=\sqrt{x}+\sqrt[3]{x}\Rightarrow Not\ a\ polynomial$$
$$P(x)=5\Rightarrow Degree=0$$

1.78: End Behavior

The end behavior of polynomial functions is very important. End behavior refers to the value of the polynomial as

$$x \text{ approaches a very large postive number} \to \infty$$
$$x \text{ approaches a very large negative number} \to -\infty$$



1.79: End Behavior of Polynomial Functions

For polynomial function of even degree as

$$x \to \infty \Rightarrow P(x) \to \infty$$
$$x \to -\infty \Rightarrow P(x) \to \infty$$

For example, the diagram alongside shows the graph of

$$y = x^2$$

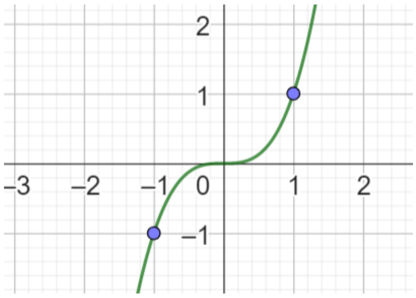
Which becomes

- very large when x becomes large
- very large even when x becomes very negative

1.80: End Behavior of Polynomial Functions

For polynomial function of odd degree as

$$x \to \infty \Rightarrow P(x) \to \infty$$
$$x \to -\infty \Rightarrow P(x) \to -\infty$$



Which becomes

- very large when x becomes large
- very negative when x becomes very negative

1.3 Evaluating and Finding Functions

A. Evaluating Functions

1.81: Evaluating Functions

Functions can be mathematical rules. Given the input, it is often required to calculate the associated output. This can be done by substituting the input into the function expression.

Example 1.82

Answer each part independently:

A. If $f(x) = 3x + 5$, then find the value of the function when $x = 7$.
B. Given that $g(x) = 4x^2 + \frac{7}{4}x + 2$, evaluate the function at $x = \frac{3}{4}$.
C. Consider $g(x) = 2x^3 + 3x^2 + 6x + 58$. When $x = 2$, what is $g(x)$?
D. What is the output of $s(x) = 3x^4 + 5x^3 + 2x + 1$ when $x = -1$.
E. If $f(x) = 8x^3 - 6x^2 + 12x - 0$, what is the value of $f\left(\frac{3}{4}\right)$? (MathCounts 2005 Workout 4)

Part A

$$f(7) = 3(7) + 5 = 21 + 5 = 26$$

Part B

$$g\left(\frac{3}{4}\right) = 4\left(\frac{3}{4}\right)^2 + \left(\frac{7}{4}\right)\left(\frac{3}{4}\right) + 2 = \frac{36}{16} + \frac{21}{16} + \frac{32}{16} = \frac{89}{16} = 5\frac{9}{16}$$

Part C

$$h(2) = 2(2)^3 + 3(2)^2 + 6(2) + 58 = 16 + 12 + 12 + 58 = 98$$

Part D

$$s(-1) = 3(-1)^4 + 5(-1)^3 + 2(-1) + 1 = 3 - 5 - 2 + 1 = -3$$

Part E

$$f\left(\frac{3}{4}\right) = 8\left(\frac{3}{4}\right)^3 - 6\left(\frac{3}{4}\right)^2 + 12\left(\frac{3}{4}\right) - 0$$

1.83: Variables as Inputs

Even if the input is a variable, we substitute and proceed as per usual.

Example 1.84

A. Evaluate $f(2x-5)$ given that $f(x) = 3x^2 - x + 6$.
B. Evaluate $f(6-x)$ given that $f(x) = -2x^2 - 3x + 5$

Part A

$$\begin{aligned} f(2x-5) &= 3(2x-5)^2 - (2x-5) + 6 \\ &= 3(4x^2 - 20x + 25) - 2x + 5 + 6 \\ &= 12x^2 - 60x + 75 - 2x + 11 \\ &= 12x^2 - 62x + 86 \end{aligned}$$

Part B

$$\begin{aligned} f(6-x) &= -2(6-x)^2 - 3(6-x) + 5 \\ &= -2(36 - 12x + x^2) - 18 + 3x + 5 \\ &= -72 + 24x - 2x^2 - 18 + 3x + 5 \\ &= -2x^2 + 27x - 85 \end{aligned}$$

Example 1.85

Use the function defined below to find the following, if they exist:

x	1	2	3	4	5	6	7
$f(x)$	7	6	5	4	3	2	1

A. $f(3)$
B. $f(8)$

$$f(3) = 5$$
$$f(8) = Not\ Defined$$

1.86: Nested Functions

It is possible to nest functions inside other functions.

Example 1.87

A. Let $f(x) = 2x - 3$ and $g(x) = x + 1$. What is the value of $f\big(1 + g(2)\big)$? (MathCounts 2003 Warm-Up 3)
B. In the above question, find $f(x) + [f(x)]^2$.

Part A

$$f\big(1 + g(2)\big) = f(1+3) = f(4) = 2(4) - 3 = 5$$

Part B

$$f(x) + [f(x)]^2 = (2x-3) + (2x-3)^2 = 2x - 3 + (4x^2 - 12x + 9) = 4x^2 - 10x + 6$$

1.88: Constant Function

A constant function has only a single output for all its valid inputs, and the output does not depend on its input.

$$f(x) = c, c = Some\ Constant$$

Example 1.89

Let $f(x) = 4$. Find:

A. $f(3)$
B. $f(4)$
C. $f\left(\frac{3}{7}\right)$
D. $f(\pi^2)$

$$f(3) = 4$$
$$f(4) = 4$$
$$f\left(\frac{3}{7}\right) = 4$$
$$f(\pi^2) = 4$$

Example 1.90

Akash is working with a function p such that $p(x) = 2$. Find $p(2)$.

$$p(x) = 2 \Rightarrow p(2) = 2$$

B. Back-Calculations

1.91: Back Calculations

Given an output for a function, it is possible to arrive at one or more values that could have led to the output.

- In practice, this often means solving a linear, quadratic or some other equation.

Example 1.92

A. If $f(x) = 3x + 5$ for all real numbers x, then find the value of x such that $f(x) = 12$.
B. If $f(x) = x^2$ for all real numbers x, then find the values(s) of x such that $f(x) = 73$
C. If $f(x) = x^2$ for all real numbers x, then find the values(s) of y such that $f(x) = y$ has only one solution for x.
D. Here are two functions: $f(x) = 3x^2 - 2x + 4$, $g(x) = x^2 - kx - 6$. If $f(10) - g(10) = 10$, what is the value of k? **(MathCounts 2007 National Countdown)**

Part A

$$f(x) = 12 \Rightarrow 3x + 5 = 12 \Rightarrow 3x = 7 \Rightarrow x = \frac{7}{3}$$

Part B

$$f(x) = 73 \Rightarrow x^2 = 73 \Rightarrow x = \pm\sqrt{73}$$

Part C

$$f(x) = y \Rightarrow x^2 = y \Rightarrow x = \pm\sqrt{y} \Rightarrow \sqrt{y} = -\sqrt{y} \Rightarrow y = 0$$

Part D

Before evaluating $f(10) - g(10)$, first simplify:

$$f(x) - g(x) = 3x^2 - 2x + 4 - (x^2 - kx - 6) = 2x^2 + (-2 + k)x + 10$$

Substitute 10 in the above expression:

$$200 - 20 + 10k + 10 = 10$$
$$180 + 10k = 0$$
$$k = -18$$

Example 1.93
A function f has the property that $f(3x - 1) = x^2 + x + 1$ for all real numbers x. What is $f(5)$? (AMC 12B 2007/9)

We need the input to be 5. This means that:
$$3x - 1 = 5 \Rightarrow 3x = 6 \Rightarrow x = 2$$

Substitute the value of $x = 2$ in the RHS of the function definition:
$$f(3x - 1) = x^2 + x + 1 = 4 + 2 + 1 = 7$$

C. Functional Equations

1.94: Functional Equation
A function equation is an equation where some of the terms are written in terms of functions.

- Functional equations can be difficult, and they are a subject of deep study.
- But we will look at some very basic equations.

Example 1.95
Let $f(x) = 3x - 4$. Let A be the solution to $2f(2x) = 3\left(\frac{x}{2}\right)$. Let B be the solution to $2f(2x) = 3f\left(\frac{x}{2}\right)$
Find A and B.

Finding A

$$LHS = 2f(2x) = 2[3(2x) - 4] = 12x - 8$$
$$12x - 8 = \frac{3x}{2} \Rightarrow \frac{21x}{2} = 8 \Rightarrow A = x = \frac{16}{21}$$

Finding B

$$2f(2x) = 3f\left(\frac{x}{2}\right)$$
$$2[3(2x) - 4] = 3\left[3\left(\frac{x}{2}\right) - 4\right]$$
$$12x - 8 = \frac{9x}{2} - 12$$
$$\frac{15x}{2} = -4$$
$$x = -\frac{8}{15}$$

Example 1.96
Suppose $f(x) = x^2 + 12$. If $m > 0$ and $f(3m) = 3(f(m))$, what is the value of m? (MathCounts 2005 Warm-Up 16)

$$f(3m) = 3\big(f(m)\big)$$

Substitute the definition of the function:

$$9m^2 + 12 = 3(m^2 + 12)$$
$$3m^2 + 4 = m^2 + 12$$

Simplify and solve:

$$2m^2 = 8$$
$$m^2 = 4$$
$$m = \pm 2$$

But reject the negative value since $m > 0$:

$$m = 2$$

Example 1.97

If $f(x) = 2x + 5$, solve $f(x) = 2f\left(\frac{6}{17}x\right)$. Note: $2f\left(\frac{6}{17}x\right)$ means $2 \times f\left(\frac{6}{17}x\right)$

$$f(x) = 2f\left(\frac{6}{17}x\right)$$

Substitute the definition of the function:

$$2x + 5 = 2\left[2\left(\frac{6x}{17}\right) + 5\right]$$

Simplify and solve:

$$2x + 5 = \frac{24x}{17} + 10$$
$$\frac{10x}{17} = 5$$
$$x = \frac{17}{2}$$

D. Recursive Functions

Example 1.98

A saint has a magic pot that doubles the number of gold coins that you put into it overnight. A disciple puts in some gold coins over the weekend (on Friday night), and collects 400 gold coins on Monday morning. How many gold coins did he put in to begin with?

$$Friday\ night: x\ gold\ coins$$
$$Saturday\ morning: 2x\ gold\ coins$$
$$Sunday\ morning: 4x\ gold\ coins$$
$$Monday\ morning: 8x\ gold\ coins$$

$$8x = 400 \Rightarrow x = \frac{400}{8} = 100$$

$$G(x) = 2G(x-1) = 2[2G(x-2)] = 8G(x-3)$$

Example 1.99: Factorial Function

$n!$ $(Read: n\ factorial)$ is defined for positive integer values of n as the product of the first n positive integers:

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$$

It can also be defined recursively as:

$$f(n) = n \times f(n-1), \qquad f(1) = 1$$

Find the value of $f(4)$ using the recursive definition. Show your working.

Build up the Function

$$f(2) = 2f(1) = 2 \times 1 = 2$$
$$f(3) = 3f(2) = 3 \times 2 = 6$$
$$f(4) = 4f(3) = 4 \times 6 = 24$$

Go Backwards

$$f(4) = 4f(3) = 4 \cdot 3f(2) = 4 \cdot 3 \cdot 2 \cdot f(1) = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Example 1.100: Fibonacci Function[1]

The Fibonacci sequence is defined recursively for integer inputs as:

$$f(n) = f(n-1) + f(n-2) \ \ for\ n \ge 3$$
$$f(1) = 1, f(2) = 1$$

Find the value of $f(4)$ using the recursive definition. Show your working.

Building it up

$$f(3) = f(2) + f(1) = 1 + 1 = 2$$
$$f(4) = f(3) + f(2) = 2 + 1 = 3$$

Recursive Method

Use the definition for $f(4)$:

$$f(4) = f(3) + f(2)$$

Use the definition for $f(3) = f(2) + f(1)$:

$$= [f(2) + f(1)] + f(2)$$

Simplify:

$$f(1) + 2f(2)$$

Substitute $f(1) = 1, f(2) = 1$

$$= 1 + 2(1) = 1 + 2 = 3$$

Example 1.101: Fibonacci-Style Function

The function $f(n) = 3f(n-2) - 2f(n-1)$, where $f(2) = 3$ and $f(1) = -1$. What is the value of $f(5)$?
(MathCounts 2001 National Target)

This looks difficult, until we recognize that $f(n)$ is dependent upon the values of precisely $n-1$ and $n-2$. We calculate in sequence:

$$f(3) = 3f(1) - 2f(2) = 3(-1) - 2(3) = -3 - 6 = -9$$
$$f(4) = 3f(2) - 2f(3) = 9\ + 18 = 27$$
$$f(5) = \ -27 - 54 = -81$$

E. Finding a formula for a function

[1] The Fibonacci function is based on the Fibonacci sequence. This is covered in more detail in the Note on Sequences and Series.

1.102: Finding a formula

To find $y = f(x)$

$$Solve\ for\ y$$

Example 1.103

State y as a function of x given that:

$$y + 4 = 3(x - 2)$$

Isolate y on the LHS and then simplify:

$$y = 3x - 6 - 4 \Rightarrow y = 3x - 10$$

State it as a function:

$$y = f(x) = 3x - 10$$

Example 1.104

State y as a function of x:

$$3x + 2y = 4$$

We want to isolate y. Subtract $3x$ from both sides:

$$2y = 4 - 3x$$

Divide both sides by 2:

$$y = 2 - \frac{3}{2}x$$

State it as a function:

$$y = f(x) = 2 - \frac{3}{2}x$$

Example 1.105

State y as a function of x:

$$3xy + x + 2 = 5 + y$$

We want to isolate y. Tale all the y terms to the RHS, and take all the non-y terms to LHS:

$$x - 3 = y - 3xy$$

Factor out y in the RHS:

$$x - 3 = y(1 - 3x)$$

Solve for y:

$$y = f(x) = \frac{x - 3}{1 - 3x}$$

Example 1.106

If a, b, c, d and e are unknown constants, find y as a function of x, if

$$3axy + bx + c = d + ey$$

We want to isolate y. Tale all the y terms to the RHS, and take all the non-y terms to LHS:

$$bx + c - d = ey - 3axy$$

Factor out y in the RHS:

$$bx + c - d = y(e - 3ax)$$

Solve for y:

$$y = \frac{bx + c - d}{e - 3ax}$$

$$3axy - ey = d - bx - c$$
$$y(3ax - e) = d - bx - c$$
$$y = \frac{d - bx - c}{3ax - e}$$

Example 1.107

Find y as a function of x:

$$xy^3 + ax = x^2y^3$$

If $x \neq 0$:

$$xy^3 - x^2y^3 = -ax$$
$$y^3 - xy^3 = -a$$
$$y^3(1 - x) = -a$$
$$y^3 = -\frac{a}{1 - x}$$
$$y = \sqrt[3]{\frac{a}{x - 1}}$$

F. Stating a formula as a function

A function can be used to state familiar formulas from algebra or geometry. This is not a very challenging concept by itself. Rather, it is a useful building block when manipulating geometrical functions.

Example 1.108: Squares and Cubes

State the formulae below in terms of functions

A. The $\underbrace{perimeter}_{p}$ of a square is four times the $\underbrace{side\ length}_{s}$ of the square. State p as a function of s. Also, state s as a function of p.
B. The $\underbrace{area}_{a}$ of a square is half the product of the length of its $\underbrace{diagonals}_{d}$. The diagonals of a square are congruent. State the area as a function of the diagonals. Also, state the length of the diagonals as a function of the area of the square.
C. The diagonal of a square is $\sqrt{2}$ times the side length of the square. State the diagonal as a function of the side length. Also, state the side length as a function of the diagonal.
D. The volume of a cube is the cube of its side length. State the volume of a cube as a function of its side length.
E. The longest diagonal of a cube is $\sqrt{3}$ times the side length of the cube. State the longest diagonal of a cube as a function of its side length.

Part A

$$p = f(s) = 4s, \qquad s = f(p) = \frac{p}{4}$$

Part B

$$a = f(d) = \frac{d^2}{2} \Rightarrow 2a = d^2 \Rightarrow d = \sqrt{2a} \Rightarrow d = f(a) = \sqrt{2a}$$

Part C

$$d = f(s) = s\sqrt{2} \Rightarrow s = f(d) = \frac{d}{\sqrt{2}}$$

Part D

$$V = f(s) = s^3 \Rightarrow s = f(V) = \sqrt[3]{s}$$

Part E

$$d = f(s) = s\sqrt{3} \Rightarrow s = f(d) = \frac{d}{\sqrt{3}}$$

Example 1.109: Circles

State the formulae below in terms of functions

A. The $\underbrace{circumference}_{C}$ of a circle is 2π times the $\underbrace{radius}_{r}$ of the circle. Write the circumference as a function of the radius. Also, write the radius as a function of the circumference.
B. The $\underbrace{area}_{a}$ of a circle is π times the square of the $\underbrace{radius}_{r}$ of the circle.

$$C = f(r) = 2\pi r^2 = f(C) = \frac{C}{2\pi}$$
$$a = f(r) = \pi r^2 \Rightarrow r = f(a) = \sqrt{\frac{a}{\pi}}$$

Example 1.110: Counting

State the number of diagonals of a polygon as a function of the number of vertices that it has, if the diagonals are given by the product of the following three quantities:

A. The number of vertices(n)
B. Three less than the number of vertices
C. Half

$$d = f(n) = n \times (n-3) \times \frac{1}{2} = \frac{n(n-3)}{2}$$

G. Functions with more than one Input

Example 1.111

Define $E(a,b,c) = ab^2 + c$. What value of a is the solution to the equation $E(a,4,5) = E(a,6,7)$? (AOPS Alcumus, Algebra, Evaluating Functions)

$$E(a,4,5) = E(a,6,7)$$

Substitute the definition of the function:

$$a \times 4^2 + 5 = a \times 6^2 + 7$$

Simplify and solve:

$$16a + 5 = 36a + 7 \Rightarrow -2 = 20a \Rightarrow a = -\frac{1}{10}$$

Example 1.112

State the formulae below in terms of functions

A. The $\underbrace{area}_{a}$ of a rectangle is the product of the $\underbrace{length}_{l}$, and the $\underbrace{width}_{w}$ of the rectangle. State the area of the rectangle as a function of the length and the width.

B. The $\underbrace{volume}_{v}$ of a cuboid is the product of the $\underbrace{length}_{l}$, $\underbrace{width}_{w}$ and $\underbrace{height}_{h}$ of the cuboid. State v as a function of l, w and h.

$$Part\ A: a = f(l, w) = lw$$
$$Part\ B: v = f(l, w, h) = lwh$$

H. Interpreting Functions

Example 1.113

A bar of chocolate costs 4 dollars. The function $m = c(b)$ gives the cost c of purchasing b bars of chocolate.

A. Identify the input variable, and its meaning
B. Identify the output variable and its meaning
C. Interpret the following statements
  I. $c(3) = 12$
  II. $c(a) = 4a$

Part A

The input variable is b.
b is the number of bars of chocolate purchased.

Part B

We apply the function c to the input b in order get the output m.
The output is m.
We can also write the output as $c(b)$

Part C

Sub-Part I

$$c(3) = 12$$
$$Input = 3 \Rightarrow 3\ bars\ of\ chocolate\ purchased$$
$$Output = 12 \Rightarrow 12\ Dollars\ is\ the\ cost\ of\ chocolate\ purchased$$

We rewrite the above in a single sentence:

$$The\ cost\ of\ 3\ bars\ of\ chocolate\ is\ 12\ dollars.$$

Sub-Part II

$$c(a) = 4a$$
$$Input = a \Rightarrow a\ bars\ of\ chocolate\ purchased$$
$$Output = 4a \Rightarrow 4a\ Dollars\ is\ the\ cost\ of\ chocolate\ purchased$$

We rewrite the above in a single sentence:

$$The\ cost\ of\ a\ bars\ of\ chocolate\ is\ 4a\ dollars.$$

Example 1.114

The function $f(x)$ models the height of the x^{th} floor of the building $AVALON$ from the ground in feet. Interpret the statement:

$$f(3) = H$$
$$Input = 3 \Rightarrow Input\ is\ 3rd\ floor\ of\ the\ building$$
$$Output = H \Rightarrow Height\ is\ H\ feet$$

We rewrite the above in a single sentence:

$$The\ height\ from\ the\ ground\ of\ the\ 3rd\ floor\ of\ AVALON\ is\ H\ feet.$$

I. Back Calculations

Example 1.115
Let f be a linear function for which $f(6) - f(2) = 12$. What is $f(12) - f(2)$? **(AMC 12 2003)**

Substitution Method
Use $\underbrace{f(x) = ax + b}_{the\ form\ of\ a\ linear\ function}$ in the given condition:

$$f(6) - f(2) = 12 \Rightarrow (6a + b) - (2a + b) = 12 \Rightarrow 4a = 12 \Rightarrow a = 3$$

We want to find:

$$f(12) - f(2) = (12x + b) - (2x + b) = 10a = 30$$

Slope from Coordinate Geometry

$$Slope = \frac{Rise}{Run} = \frac{12}{6-2} = \frac{12}{4} = 3 \Rightarrow 3 = \frac{Rise}{12-2} \Rightarrow Rise = 30$$

Example 1.116
If $f(1) = 5, f(2) = 8$ and $f(x) = ax + bx + 2$, what is the value of $f(3)$? **(MathCounts 1996 National Countdown)**

Note that $f(x)$ is a linear function in which we already know the constant term:

$$f(x) = ax + bx + 2 = (a + b)x + 2$$

And hence we only need one value of f to find the answer.

$$f(1) = 5 \Rightarrow (a + b) + 2 = 5 \Rightarrow a + b = 3$$

Substiute $a + b = 3, x = 3$ in $f(x) = (a + b)x + 2$:

$$3(3) + 2 = 11$$

1.4 Even and Odd Functions; Vertical Symmetry

A. Summary of Formulas

1.117: Even Function: Graphical Test
A function is an even function if:
- It has the $\boldsymbol{y}$-axis as a vertical line of symmetry.
- OR if reflecting it across the $\boldsymbol{y}$-axis does not change it.

1.118: Even Function: Algebraic Test
A function is an even function if:

$$f(x) = f(-x)\ for\ all\ x\ in\ the\ domain\ of\ f$$

1.119: Odd Function: Graphical Test
A function is an odd function if

- reflecting it across the $\boldsymbol{y}$-axis, and then reflecting it across the $\boldsymbol{x}$-axis does not change its value
- OR if it is symmetrical about the origin

1.120: Odd Function: Algebraic Test

$$f(x) = -f(-x)\ for\ all\ x\ in\ the\ domain\ of\ f$$

$$f(-x)\ is\ a\ reflection\ of\ f\ across\ the\ y\ axis$$
$$-f(x)\ is\ a\ reflection\ of\ f\ across\ the\ x\ axis$$

$$-f(-x)\ is\ a\ reflection\ of\ f\ across\ both\ the\ x\ axis\ and\ the\ y\ axis$$

B. Even Functions

There are two definitions of an even function. Both definitions are equivalent, and you should make sure that you understand why they are equivalent.

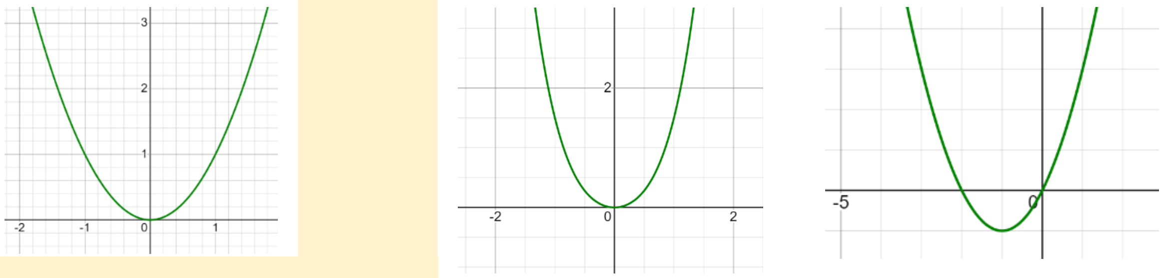
1.121: Even Function: Graphical Test

A function is an even function if:

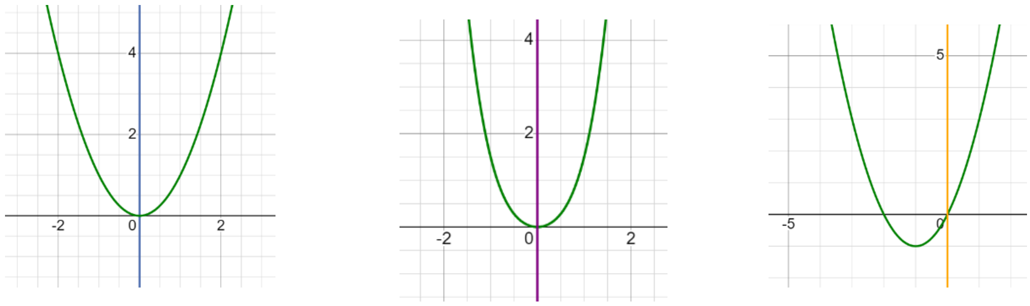
- It has the $\boldsymbol{y}$-axis as a vertical line of symmetry.
- OR if reflecting it across the $\boldsymbol{y}$-axis does not change it.

Example 1.122

Decide whether the graphs given below are the graphs of even functions.



Draw the y-axis. It is a line of symmetry for the first and the second graph, but not the third.



Example 1.123

A. The function $g(x) = x^2$ has slope 10 when $x = 5$. Determine the slope of $g(x)$ when $x = -5$.
B. An even function $f(x)$ has slope m when $x = x_1$. Determine the slope of $f(x)$ when $x = -x_1$.

Part A

$$-10$$

Part B

$$-m$$

1.124: Slope of an even function

For an even function,

$$Slope\ of\ f(x) = -Slope\ of\ f(-x)$$

1.125: Even Function: Algebraic Test

A function is an even function if:

$$f(x) = f(-x)\ for\ all\ x\ in\ the\ domain\ of\ f$$

The algebraic version of the graphical test relies on the same idea that the graphical version does. When you reflect a graph across the y-axis, at every point, you replace x with $-x$.

Hence, if you do the same thing algebraically, and the function does not change, then the function is an even function.

$$A\ function\ is\ an\ even\ function\ if\ replacing\ \boldsymbol{x}\ with\ -\boldsymbol{x}\ does\ not\ change\ a\ function$$

Example 1.126

Consider the function defined in the table below:

Domain	x	4	5	9	-4	-5	-9	12
Range	$f(x)$	7	8	9	7	8	9	6

$f(x)$ is:

A. An even function
B. Not an even function
C. Cannot be determined

$$f(12) = 6, f(-12)\ does\ not\ exist \Rightarrow Function\ is\ not\ even \Rightarrow Option\ B$$

Example 1.127

Some values of x and $f(x)$ are given for a function:

x	4	5	9	-4	-5	-9	12
$f(x)$	7	8	9	7	8	9	6

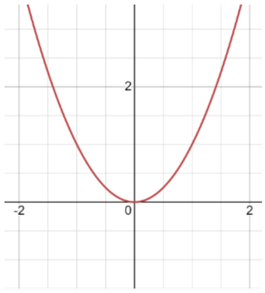
$f(x)$ is:

A. An even function
B. Not an even function
C. Cannot be determined

Here, we do not know $f(-12)$, and hence the answer is Option C.

Example 1.128

Check whether $f(x) = x^2$ is even.



Algebraic Method

Substitute $-x$ for x:

$$f(-x) = (-x)^2 = x^2 = f(x) \Rightarrow \textbf{\textit{Even}}$$

Graphical Method

We can do the same thing graphically. Draw the graph of

$$f(x) = x^2$$

Which is an upward parabola, with the vertex at the origin.

Since the graph is symmetrical about the y-axis, the function is even.

Example 1.129

Check whether $f(x) = 3x$ is even.

$$f(-x) = -3x \neq f(x) \Rightarrow \textbf{\textit{Not Even}}$$

In fact, this function is odd:

$$f(-x) = -3x = -f(x) \Rightarrow \textbf{\textit{Odd}}$$

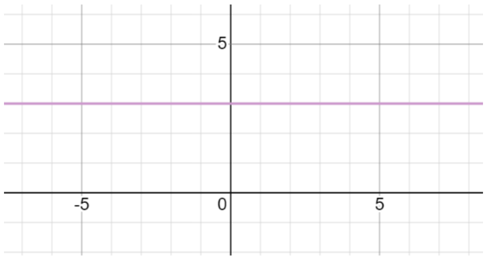
Example 1.130

Check whether $f(x) = x^2 + 5x + 3$ is even.

$$f(-x) = (-x)^2 + 5(-x) + 3 = x^2 - 5x + 3 \neq f(x) \Rightarrow \textbf{\textit{Not Even}}$$

Example 1.131

Check whether $f(x) = 3$ is even.



Algebraic Method

$$f(-x) = 3 = f(x) \Rightarrow \textbf{\textit{Even}}$$

You can do this graphically also. $f(x)$ is a constant function. The graph of

$$f(x) = 3$$

Is symmetrical about the y-axis, and hence the function is even.

Example 1.132

Check whether $f(x) = 2^x$ is even.

$$f(-x) = 2^{-x} = \frac{1}{2^x} \neq 2^2 \Rightarrow Not\ even$$

1.133: Powers

- Polynomial functions that have only even powers of the variable are even.
  - ✓ The constant term is x^0, and hence is an even power.
- Polynomial functions that have a non-zero coefficient of an odd power of the variable are not even.

Example 1.134

Check whether the following functions are even using the test given above.

A. $f(x) = x^2$
B. $f(x) = 3$
C. $f(x) = x^2 + 6x + 3$
D. $f(x) = 23x^{64} + 7$
E. $f(x) = 64x^{23} + 7$
F. $f(x) = x^{20} + x^{18} + x^{16} + \cdots + 4$

$$f(x) = \underbrace{x^2}_{Even} \Rightarrow Only\ Even \Rightarrow Function\ is\ even$$
$$f(x) = \underbrace{3x^0}_{Even} \Rightarrow Only\ Even \Rightarrow Function\ is\ even$$
$$f(x) = x^2 + 5x + 3 \Rightarrow Has\ 6x^1 \Rightarrow Power\ is\ odd \Rightarrow Function\ is\ not\ even$$
$$23x^{64} + 7 \Rightarrow Powers\ are\ Even \Rightarrow Function\ is\ even$$
$$63x^{23} + 7 \Rightarrow Powers\ are\ not\ even \Rightarrow Function\ is\ not\ even$$
$$x^{20} + x^{18} + x^{16} + \cdots + 4 \Rightarrow Powers\ are\ Even \Rightarrow Function\ is\ even$$

Example 1.135: Using Even Functions

If $f(x)$ is an even function and $f(3) = 7$, find $f(-3)$.

For an even function:

$$f(x) = f(-x)$$
$$f(-3) = f(3) = 7$$

1.136: Functions that are "partly even"

If a function is not even, but some part of the function is, the symmetry can still be exploited.

Example 1.137: Using Even Functions

A.
B. If $f(x) = x^4 + x^2 + 5x$, evaluate $f(5) - f(-5)$. **(MathCounts 2001 National Countdown)**
C. If $f(x) = x^4 + x^2 + 5x$, evaluate $f(x) - f(-x)$. **(MathCounts 2001 National Countdown)**
D. If $f(x) = x^{2020} + x$, find $f(7) - f(-7)$.
E. If $f(x) = x^{2020} + x^{2018} + \cdots + x^2 + x$, find $f(7) - f(-7)$
F. If $f(x) = ax^4 - bx^2 + x + 5$ and $f(-3) = 2$, then what is the value of $f(3)$? **(1995 AHSME)**

Part A

Part B

We break this function into two parts:

$$f(x) = \underbrace{x^4 + x^2}_{Even\ Part} + \underbrace{5x}_{\substack{Remaining\\ Part}}$$

For an even function:

$$f(x) = f(-x)$$
$$f(5) = f(-5)$$

The value of $x^4 + x^2$ will remain the same for both $f(5)$ and $f(-5)$. Therefore:

$$f(5)-f(-5)=\underbrace{5(5)-5(-5)}_{\substack{Ignore\ x^4+x^2\\ since\ it\ will\ cancel}}=25+25=50$$

Part C

$$f(x)-f(-x)=x^4+x^2+5x-(x^4+x^2-5x)=10x$$

And then we can find:

$$f(5)-f(-5)=10(5)=50$$

Part D

$$\underbrace{7^{2020}+7}_{f(7)}-\underbrace{[(-7)^{2020}-7]}_{f(-7)}=\underbrace{7^{2020}-7^{2020}}_{=0}+7+7=14$$

Part E

$$f(7)-f(-7)=7-(-7)=14$$

Part F

Logic

The difference between $f(3)$ and $f(-3)$ will only depend upon terms that have an odd power of x.

$f(-3)$ will have -3 term, while $f(3)$ will have 3 in it.
Hence,

$$f(3)=f(-3)+2(3)=2+6=8$$

Algebraic Method

$$f(x)-f(-x)=ax^4-bx^2+x+5-(ax^4-bx^2-x+5)=2x$$
$$f(x)-f(-x)=2x$$

Substitute $x=3$:

$$f(3)-f(-3)=2(3)\Rightarrow f(3)=8$$

Example 1.138

Check whether the following functions are even:

A. $f(x)=\frac{1}{2}(2^x+2^{-x})$

B. The hyperbolic cosh function defined as $f(x)=\cosh\frac{e^x+e^{-x}}{2}$

Part A

$$f(-x)=\frac{1}{2}(2^{-x}+2^x)=\frac{1}{2}(2^x+2^{-x})=f(x)\Rightarrow \textbf{Even}$$

Part B

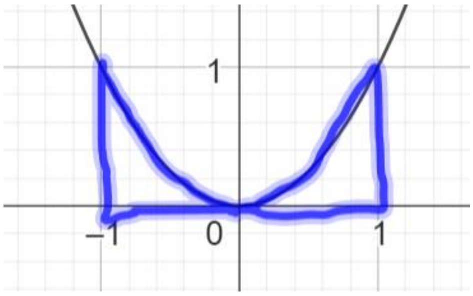
$$f(-x)=\frac{1}{2}(e^{-x}+e^x)=\frac{1}{2}(e^x+e^{-x})=f(x)\Rightarrow \textbf{Even}$$

1.139: Area under the Curve

The area between the graph of an even function and the x axis is also symmetrical.

Example 1.140

If the area between $y=x^2$ and the x axis is A for $0<x<1$, then determine the area between $y=x^2$and the $-1<x<0$.



C. Odd Functions

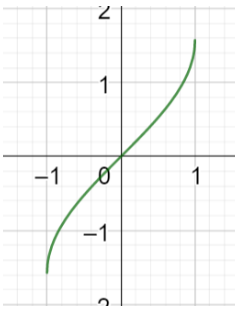
1.141: Odd Function: Graphical Test

A function is an odd function if

- reflecting it across the $\boldsymbol{y}$-axis, and then reflecting it across the $\boldsymbol{x}$-axis does not change its value
- OR if it is symmetrical about the origin

Example 1.142

Check whether the functions graphed below are odd:



$$First\ function = Odd$$

1.143: Odd Function: Algebraic Test

$$f(-x) = -f(-x)\ for\ all\ x\ in\ the\ domain\ of\ f$$

$$f(-x)\ is\ a\ reflection\ of\ f\ across\ the\ y\ axis$$
$$-f(x)\ is\ a\ reflection\ of\ f\ across\ the\ x\ axis$$

$$-f(-x)\ is\ a\ reflection\ of\ f\ across\ both\ the\ x\ axis\ and\ the\ y\ axis$$

1.144: Every odd function passes through the origin

Every odd function which includes the origin in its domain must pass through the origin.

Graphical Reason

Because it remains unchanged when reflected across the origin.

Algebraic Reason

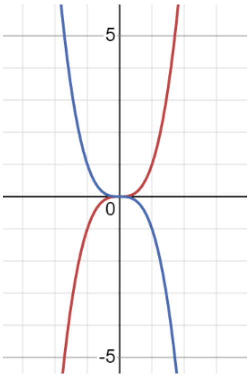
$$f(x) = -f(-x)$$
$$f(0) = -f(0)$$

$$f(0) = 0$$

Hence, if $f(0)$ is defined, it must pass through the origin.

Example 1.145

Check whether $f(x) = x^3$ is odd.



Algebraic Method

$$f(-x) = (-x)^3 = (-x^3) = -f(x) \Rightarrow \boldsymbol{Odd}$$

Graphical Method

$$Red\ Graph \Rightarrow y = x^3$$

Reflect across the y-axis

$$Blue\ Graph \Rightarrow y = (-x)^3$$

Reflect across the x-axis

$$Red\ Graph \Rightarrow y = -(-x^3) = x^3 = Original\ Graph$$

Example 1.146

A. Check whether $f(x) = x^3 + x$ is odd.
B. Check whether $f(x) = x^3 + x^2 + x$ is odd.

Part A

$$f(-x) = (-x)^3 + (-x) = -x^3 - x = -(x^3 + x) = -f(x) \Rightarrow \boldsymbol{Odd}$$

Part B

$$f(-x) = -x^3 + x^2 - x$$
$$-f(x) = -x^3 - x^2 - x \neq f(-x) \Rightarrow \boldsymbol{Not\ odd}$$

Example 1.147

Check whether $f(x) = \frac{1}{x}$ is odd.

$$f(-x) = -\frac{1}{x} = -f(x) \Rightarrow Odd\ Function$$

1.148: Odd Power Polynomials

Polynomials functions that have only odd powers of the variable are odd.

Example 1.149

Check whether the following function are even, odd or neither using the test given above.

A. $f(x) = x^3$
B. $f(x) = x^3 + x$
C. $f(x) = x^3 + x^2 + x$
D. $f(x) = 23x^{72} + 15$
E. $f(x) = x^{31} + x^{29} + \cdots + x + 7$
F. $f(x) = 37x^2 + 4$
G. $f(x) = 21x^{23} + x$
H. $f(x) = 21x^{23} + 7$
I. $f(x) = x^{27} + x^{26} + x^{25}$

$$f(x) = \underbrace{x^3}_{Odd} \Rightarrow Odd\ Function$$
$$f(x) = \underbrace{x^3}_{3\ is\ odd} + \underbrace{x}_{1\ is\ odd} \Rightarrow Odd\ Function$$
$$f(x) = \underbrace{x^3}_{3\ is\ odd} + \underbrace{x^2}_{2\ is\ even} + \underbrace{x}_{1\ is\ odd} \Rightarrow Not\ an\ Odd\ Function$$

Example 1.150

$$f(x) = \frac{1}{2}(2^x - 2^{-x})$$

$$f(-x) = \frac{1}{2}(2^{-x} - 2^x) = -\frac{1}{2}(-2^{-x} + 2^x) = -f(x) \Rightarrow Odd$$

MCQ 1.151

A graph fails the vertical line test, but is symmetrical about the origin (which means it passes the graphical test for an odd function):

A. The graph is an odd function.
B. The graph is a function, but not odd.
C. The graph is not a function.
D. The graph intersects the x axis an odd number of times.

Option C.

Challenge 1.152

D. Further Properties

1.153: Composition of an odd function with an odd function

$f(x)$ and $g(x)$ are odd functions, then

$$f\big(g(x)\big)\ is\ odd$$

$$f\big(g(-x)\big) = f\big(-g(x)\big) = -f\big(g(x)\big) \Rightarrow Odd$$

1.154: Product of two functions

$f(x)\ is\ even, g(x)\ is\ odd$:

Since $f(-x) = f(x), g(-x) = -g(x)$:

$$f(-x)g(-x) = -f(x)g(x)$$

E. Vertical Symmetry

1.155: Vertical Symmetry

A function is symmetric about the line $x = a$ if it satisfies

$$f(a + x) = f(a - x)$$

$f(a+x)$ is a transformation[2] of $f(x)$ shifted a units to the left
$f(a+x)$ is a transformation of $f(-x)$ shifted a units to the left
Let $g(x) = f(a+x)$. Then:

$$g(-x) = f(a-x) = g(x) \Rightarrow g(x) \text{ is even}$$

Since the shifted function $g(x)$ is symmetric about $x = 0$, the original function:

$$f(x) \text{ is symmetric about } x = a$$

Example 1.156

A quadratic function satisfies the functional equation: $f(3+x) = f(3-x)$. Given that the function has an x intercept at 5, what is the value of $\frac{f(13)}{f(7)}$

The function is symmetric about $x = 3$.

Method I: Change of origin/translation

Shift everything 3 units to the left, and call the new function $g(x)$. Use the property

$$g(x) = g(-x)$$
$$ax^2 + bx + c = a(-x)^2 + b(-x) + c$$
$$bx = -bx$$
$$b = 0$$

Use the given root in $g(x) = ax^2 + c$:

$$f(5-3) = f(2) = 4a + c = 0 \Rightarrow c = -4a$$
$$g(x) = ax^2 - 4a$$

$$\frac{f(13-3)}{f(7-3)} = \frac{f(10)}{f(4)} = \frac{100a - 4a}{16a - 4a} = \frac{96a}{12a} = 8$$

Method II: Vertex form of quadratic function: $f(x) = a(x-h)^2 + k$

Since the function is symmetric about $x = 3$, x coordinate of the vertex is 3.

$$h = 3$$

Since 5 is an x intercept:

$$f(5) = 0$$
$$a(5-3)^2 + k = 0$$
$$k = -4a$$

The required ratio is then:

$$\frac{f(13)}{f(7)} = \frac{a(13-3)^2 - 4a}{a(7-3)^2 - 4a} = \frac{96a}{12a} = 8$$

F. Even AND Odd Functions

Example 1.157

$$Even: f(x) = f(-x)$$
$$Odd: f(x) = -f(-x)$$

[2] If you need to learn transformation properties, see the section on transformations.

$$f(-x) = -f(-x)$$

Let $f(-x) = t$:

$$t = -t$$
$$t = 0$$
$$f(-x) = 0$$
$$f(x) = 0$$

The only function which is both even and odd is the function

$$f(x) = 0$$

G. Periodic Functions

Example 1.158

Determine for each of the six trigonometric functions, which are even, which are odd, and which are neither.

1.5 Domain

A. Definitions

1.159: Domain and Range

Domain is the set of acceptable input (or x-values) for the function.
Range is the set of output (or y-values) that the function can take.

The domain can be given explicitly when a function is defined.
If functions are given in tabular form, the domain and the range can be listed out from the table.

Example 1.160: Tabular Function

The following table gives the favorite subject as a function of the student name. Give the domain and range of that function.

Student	Pari	Nitish	Santosh	Arya
Subject	English	Humanities	Maths	English

Domain is the input values, which are all the students.

$$Domain = \{Pari, Nitish, Santosh, Arya\}$$

Range is the output values, which are the subjects:

$$Range = \{English, Humanities, Maths\}$$

Example 1.161: Numeric Values

State the domain and the range of the function below, which gives the shoe size of a student as a function of his roll number:

$$Shoe\ Size = f(Roll\ Number)$$

Roll Number	21	27	4	7
Shoe Size	4	6	5	6

Domain is the set of input values, which are the Roll Numbers:

$$Domain = \{4, 7, 21, 27\}$$

Range is the set of output values, which are the shoe sizes.

$$Range = \{4,5,6\}$$

Notes:

- We only write the number 6 once in the range. We are using set notation, and repetition is not meaningful in sets.
- We wrote domain and range in ascending order. This is to make things easy for us, but is not needed in set notation – sets do not need numbers to be in a particular order.

Example 1.162: Graphical Function

1.163: Explicit Domain

A function can be defined as a mapping from one set to another. The first set represents the domain, the second set represents the co-domain.

$$f: \underbrace{\mathbb{Q}}_{Domain} \to \underbrace{\mathbb{R}}_{Co-Domain}, \qquad y = f(x) = 2x$$

B. Co-Domain(Optional)

1.164: Co-Domain

Example 1.165: Identifying Domain, Co-Domain and Range

f is a function, defined as given below. Identify the domain, co-domain and the range for the function.

$$f: \mathbb{Z} \to \mathbb{Z}, \qquad y = f(x) = x^2$$

$$Domain = Set\ where\ function\ originates = \ \mathbb{Z} = Set\ of\ Integers$$
$$Co-domain = Set\ into\ which\ the\ functions\ goes = \ \mathbb{Z} = Set\ of\ Whole\ Numbers$$

Range: The values which the function actually achieves in its co-domain is called the range. Note that x^2 is always positive, and hence $f(x)$ will never be negative. Hence, the range of $f(x)$ is

$$Range = \mathbb{W} = Set\ of\ Integers$$

Example 1.166: Identifying Valid Inputs for the Domain

State which inputs are valid, and which are not for for f defined as $f: \mathbb{Z} \to \mathbb{Z},\ y = f(x) = x^2$.

x	-3	5	7	0.3
$f(x)$	9	25	49	0.09

The set of valid inputs for f is the set of integers. 0.3 is not an integer. Hence, it is not a valid input. Other inputs are valid.

Example 1.167: Making Range and Co-Domain the Same

Modify the function $f: \mathbb{Z} \to \mathbb{Z},\ y = f(x) = x^2$ to get a new function g such that its range and its co-domain are the same.

The range of f is W. The co-domain is currently $\mathbb{Z}$. If we change the co-domain from $\mathbb{Z}$ to $\mathbb{W}$, the co-domain and the range will be the same:

$$g: \mathbb{Z} \to \mathbb{W}, \qquad y = g(x) = x^2$$

C. Finding Range

Range is the set of valid y values for a function. Range can be found by:

- Graphical Methods: This requires looking at the graph of a function. This is usually the case for simple, concept-based questions.
- Algebraic Methods: These requires finding the inverse of the function and then finding the domain of the inverse.

Example 1.168

Find the range of the function below:

$$y = f(x) = 2x, \qquad x \in \mathbb{O}, \mathbb{O}\ is\ the\ set\ of\ odd\ integers$$

Consider some sample inputs and their corresponding outputs

$$\{\dots, -3, -1, 1, 3, \dots\} = \{\dots, -6, -4, -2, 2, 4, \dots\}$$

The inputs can be extended on the left as well as the right. This will cause the outputs to also get similarly extended.

A nice way to describe the outputs is then:

$$R_f = \mathbb{E} - \{0\}, \qquad E\ is\ an\ even\ number$$

Example 1.169

Find the range of the function below:

$$y = f(x) = \frac{x}{2}, \qquad D_f = \mathbb{E}, \qquad \mathbb{E}\ is\ the\ set\ of\ even\ integers$$

Consider some sample inputs and their corresponding outputs

$$\{\dots - 4, -2, 0, 2, 4 \dots\} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

The inputs can be extended on the left as well as the right. This will cause the outputs to also get similarly extended.

A nice way to describe the outputs is then:

$$R_f = \mathbb{Z}, \qquad \mathbb{Z}\ is\ an\ integer$$

D. Implicit Domain

If a function does not specify, its domain, then the domain is understood from the context. Numbers for which the function expression is not defined are excluded from the domain. In particular:

- Fractions cannot have denominator zero since division by zero is not defined.
- Expressions inside square roots cannot be negative since negative numbers do not have a square root in the real number system.

If asked to find the domain of a function, we will attempt to find the largest valid domain for the function.

E. Fractions / Division by Zero

Division by zero is not defined. Hence, the denominator of a fraction cannot be zero. If the denominator is an expression, then you need to separately check whether the denominator can be zero.

Example 1.170

Find the domain of the following functions:

A. $f(x) = 3x + 5$
B. $g(x) = \frac{3}{x}$
C. $h(x) = \frac{3x+5}{x}$
D. $g(x) = \frac{5}{x+2}$

E. $h(x) = \frac{x+\pi}{x+\pi}$
F. $s(x) = \frac{x+2}{2}$
G. $f(x) = \frac{3}{x+\frac{3}{4}} + \frac{5}{x+\frac{7}{2}} + \frac{5}{x+\frac{2}{3}}$

Part A
There are no restrictions.
$$Domain\ of\ f = D_f = \mathbb{R} = Set\ of\ Real\ Numbers$$
In interval notation as:
$$D_f = (-\infty, \infty)$$
Part B
The denominator of the expression $\frac{3}{x}$ cannot be zero. Therefore, the domain is the set of all real numbers, except zero, which can also be written:
$$D_g = \underbrace{\mathbb{R} - \{0\}}_{Set\ Notation} = \underbrace{(-\infty, 0) \cup (0, \infty)}_{Interval\ Notation}$$
Part C
Same as Part B
Part D
We cannot have:
$$x + 2 = 0 \Rightarrow x = -2$$
$$D_g = All\ real\ numbers\ except - 2 = \{x|\ x \neq -2\}$$
$$= \underbrace{(-\infty, -2) \cup (-2, \infty)}_{Interval\ Notation}$$
Part E
We cannot have:
$$x + \pi = 0 \Rightarrow x = -\pi$$
$$D_h = All\ real\ numbers\ except - \pi = \{x|\ x \neq -\pi\}$$
$$= \underbrace{(-\infty, -\pi) \cup (-\pi, \infty)}_{Interval\ Notation}$$

Part F
The denominator of the expression $\frac{x+2}{2}$ is a constant, and it is never zero. Hence, we have:
$$No\ Restrictions\ on\ the\ function \Rightarrow D_s = x \in \mathbb{R}$$
$$\Leftrightarrow x \in (-\infty, +\infty)$$
$$D_s = All\ real\ numbers = \{x|x \in \mathbb{R}\}$$
Part G
None of the denominators can be zero. Hence, we equate the denominators to zero separately:
$$x + \frac{3}{4} = 0 \Rightarrow x = -\frac{3}{4}$$
$$x + \frac{7}{2} = 0 \Rightarrow x = -\frac{7}{2}$$
$$x + \frac{2}{3} = 0 \Rightarrow x = -\frac{2}{3}$$
And hence none of these values can be in the domain of the function.
$$x \in \mathbb{R}, \qquad x \neq -\frac{7}{2}, -\frac{3}{4}, -\frac{2}{3}$$
And we can write this in interval notation:
$$\left(-\infty, -\frac{7}{2}\right) \cup \left(-\frac{7}{2}, -\frac{3}{4}\right) \cup \left(-\frac{3}{4}, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$$

Example 1.171
A continued fraction has multiple fractions. Each of the fractions in the continued fractions must have a non-zero denominator. Use this to find the domain of
$$f(x) = \cfrac{2}{1 + \cfrac{2}{3 + \cfrac{1}{x}}}$$

Innermost Fraction
The innermost fraction cannot have denominator zero, and hence:
$$0\ not\ in\ the\ domain\ of\ f$$
Second Level Fraction
$$3 + \frac{1}{x} = 0 \Rightarrow \frac{1}{x} = -3 \Rightarrow x = -\frac{1}{3} \Rightarrow -\frac{1}{3}\ not\ in\ the\ domain\ of\ f$$
Third Level Fraction
$$1 + \frac{2}{3 + \frac{1}{x}} = 0 \Rightarrow \frac{2}{3 + \frac{1}{x}} = -1 \Rightarrow -2 = 3 + \frac{1}{x} \Rightarrow -5 = \frac{1}{x} \Rightarrow x = -\frac{1}{5} \Rightarrow -\frac{1}{5}\ not\ in\ the\ domain\ of\ f$$

Write the Answer

$$x \text{ is any real number except } -\frac{1}{3}, -\frac{1}{5}, 0$$
$$\left\{x \mid x \in \mathbb{R}, \quad x \neq -\frac{1}{3}, -\frac{1}{5}, 0\right\}$$
$$\left(-\infty, -\frac{1}{3}\right) \cup \left(-\frac{1}{3}, -\frac{1}{5}\right) \cup \left(-\frac{1}{5}, 0\right) \cup (0, \infty)$$

F. Squares and Sum of Squares

If you have a square in the denominator, the only way for a square to be zero is when the expression inside the square is zero.

In other words
$$x^2 = 0 \Rightarrow x = 0$$

The sum of squares can only be zero if each square term is itself zero:
$$a^2 + b^2 = 0 \Rightarrow a = 0 \text{ AND } b = 0$$

Example 1.172

Find the domain of:

A. $f(t) = \frac{3}{(t+2)^2}$
B. $f(t) = \frac{3}{(t+4)^2} + \frac{3}{(t-4)^2}$
C. $f(t) = \frac{3}{(t+4)^2+(t-4)^2}$

Part A

Here, we equate the denominator to zero:
$$(t+2)^2 = 0 \Rightarrow t+2 = 0 \Rightarrow t = -2$$
$$D_f = \{t \mid t \neq -2\}$$

Part B

Equate each denominator to zero:
$$(t+4)^2 = 0 \Rightarrow t+4 = 0 \Rightarrow t = -4$$
$$(t-4)^2 = 0 \Rightarrow t-4 = 0 \Rightarrow t = 4$$
$$D_f = \{t \mid t \neq -4, 4\}$$

Part C: Using Logic

$$(t+4)^2 = 0 \Rightarrow t+4 = 0 \Rightarrow t = -4$$
$$(t-4)^2 = 0 \Rightarrow t-4 = 0 \Rightarrow t = 4$$

But t cannot -4 and 4 at the same time.
Hence, there are no values of t for which the denominator is zero.

Hence, the domain is
$$\text{All Real Numbers}$$

Part C (Alternate Method): Solving Quadratic Equations

The denominator of a fraction cannot be zero.
Hence, equate the denominator to zero and solve:
$$(t+4)^2 + (t-4)^2 = 0$$
$$t^2 + 8t + 16 + t^2 - 8t + 16 = 0$$
$$2t^2 + 32 = 0$$
$$2t^2 = -32$$
$$t^2 = -16$$
$$t = \sqrt{-16} \Rightarrow t \in \mathbb{C} \Rightarrow \text{No Real Solutions}$$

And since there are no real solutions, we must have
$$\text{Domain} = \text{All Real Numbers}$$

G. Square Roots

- Square roots of negative numbers are not defined for real numbers.
$$\sqrt{-4} \Rightarrow \text{Not defined in the Real Number System}$$
- Hence, to be defined in real numbers, expressions inside a square root sign must be non-negative.
- Validity of zero in a square root
  - ✓ In general, zero is acceptable in a square root, since the square root of zero is also zero, and hence a valid real number.
  - ✓ However, if a square root is the denominator of a fraction, then the square root cannot be zero.

Hence, in that case, the square root must be positive.

By default, the number system that we will use in assessing the domain will be the set of real numbers.
We will other sets of numbers (rational numbers, integers, whole numbers) only when there is a reason to not consider real numbers.

1.173: Domain with a Square Root

Since square roots are defined for nonnegative quantities, $\sqrt{x}$ has domain

$$x \geq 0$$

For example:

$$\sqrt{-3}\ is\ not\ defined$$
$$\sqrt{3}\ is\ defined$$

Example 1.174

Simplify and find the domain:

$$\sqrt[n]{27x^4y^2z^3}$$

A. $n = 2$
B. $n = 3$
C. $n\ is\ an\ odd\ integer$
D. $n\ is\ an\ even\ integer$

Part A

$$\sqrt{27} \times \sqrt{x^4} \times \sqrt{y^2} \times \sqrt{z^3}$$

The first term is just a number. No restrictions:

$$\sqrt{27} = 3\sqrt{3}$$

The second term has an even power of the variable.

$$4\ is\ even \Rightarrow x^4 \geq 0 \Rightarrow No\ restrictions$$
$$\sqrt{x^4} = x^2$$

The third term is a square:

$$y^2 \geq 0 \Rightarrow No\ Restrictions$$
$$\sqrt{y^2} = |y|$$

The fourth term has an odd power of z:

$$z > 0 \Rightarrow z^3 > 0$$
$$z = 0 \Rightarrow z^3 = 0$$
$$z < 0 \Rightarrow z^3 < 0 \Rightarrow \textbf{Not Allowed}$$
$$z \geq 0$$

Part B

We are taking a cube root. There are no restrictions on the expressions inside a cube root.

Part C

We are taking an odd root. There are no restrictions on the expressions inside an odd root.

Part D

The restrictions for an even root are the same as the restrictions for a square root. Hence, the answer is the same as the answer for Part A.

Example 1.175

Find the domain of:

A. $h(x) = \sqrt{x}$
B. $h(x) = \frac{1}{\sqrt{x}}$
C. $q(x) = \frac{5+x}{\sqrt{x-2}}$
D. $f(x) = \frac{3x+4}{\sqrt{7x+35}}$

Part A

Square roots cannot be negative. Hence, the expression inside the square root must be nonnegative. Therefore:

$$\underbrace{x \geq 0}_{\substack{Inequality\\Notation}} \Leftrightarrow \underbrace{x \in [0, \infty)}_{\substack{Interval\\Notation}} \Leftrightarrow \underbrace{x \in \mathbb{R}^+ + \{0\}}_{Set\ Notation}$$

And, we can also write

$$x\ is\ any\ nonnegative\ real\ number$$

Part B

Square roots cannot be negative. Hence, the expression inside the square root must be positive. Therefore:

$$x \geq 0$$

The expression in the denominator cannot be zero. Hence,

$$\sqrt{x} \neq 0 \Rightarrow x \neq 0$$

Combine the above two to get:

$$\underbrace{x > 0}_{\text{Inequality Notation}} \Leftrightarrow \underbrace{x \in (0, \infty)}_{\text{Interval Notation}} \Leftrightarrow \underbrace{x \in \mathbb{R}^+}_{\text{Set Notation}}$$

And, we can also write

$$x \text{ is any positive real number}$$

Part C

The denominator cannot be zero. Further, the expression inside the square root must be non-negative. We can combine the two conditions to get:

$$x - 2 > 0 \Rightarrow x > 2 \Rightarrow D_q = (2, \infty)$$

$$D_q = \text{All real numbers greater than } 2$$

Part D

$$7x + 35 > 0$$
$$7x > -35$$
$$x > -5$$
$$x \in (-5, \infty]$$

H. Integer Domain

Example 1.176

Find the domain of $g(x) = \sqrt{x+2}$

A. When considering only integers
B. When considering all real numbers

Part A: Integral Domain

The expression inside the square root cannot be negative. Hence, we must have:

$$x + 2 \geq 0 \Rightarrow x \geq -2 \Rightarrow x > -3 \Rightarrow x \in \{-2, -1, 0, 1, \dots\}$$
$$D_g = \text{All Integers greater than } -3$$

Part B: Real Number Domain

The expression inside the square root cannot be negative. Hence, we must have:

$$x + 2 \geq 0 \Rightarrow x \geq -2 \Rightarrow x \in [-2, \infty)$$
$$D_g = \text{All real numbers greater than or equal to } -2$$

Compare the answer for Part A with the answer for Part B.

$$\text{Part A}: D_g = \text{All Integers greater than } -3$$
$$\text{Part B}: D_g = \text{All real numbers greater than or equal to } -2$$

C. If in the answer for Part A, we replace Integers with Real Numbers, is the answer then correct for Part B.
D. If in the answer for Part B, we replace Real Numbers with Integers, is the answer then correct for Part A.

Part C

Let's make the changes asked in the question:

$$\text{Part A}: D_g = \text{All Real Numbers greater than } -3 \Leftrightarrow x \in (-3, \infty)$$

Draw a number line, and check:

We also get numbers between -3 *and* -2 (such as -2.5) which we just do not want.

Hence, if we make the change given in the question, the answer is not applicable for Part B.

Part D

$$Part\ B: D_g = All\ Integers\ greater\ than\ or\ equal\ to - 2 \Leftrightarrow \{-2, -1, 0, 1, \dots\}$$

Example 1.177

Consider $q(x) = \dfrac{5 + x}{\sqrt{10 - 3x}}$

A. Find the largest integer in the domain of $q(x)$
B. Find the smallest integer not in the domain of $q(x)$

Finding the Domain

The denominator cannot be zero. Further, the expression inside the square root must be non-negative. Combine the two conditions:

$$10 - 3x > 0 \Rightarrow 3x - 10 < 0 \Rightarrow x < \frac{10}{3} = 3\frac{1}{3} \Rightarrow D_q = \left(-\infty, 3\frac{1}{3}\right) \Rightarrow Largest\ Integer = 3$$

Part A

The largest integer that is in the domain of $q(x)$ is the rightmost integer that is covered by the red line. From the graph, this is 3.

Part B

The smallest integer not in the domain is the leftmost integer not covered by the red line.
From the graph, this is 4.

Example 1.178

Consider $f(x) = \dfrac{3x^2 + 5x}{\sqrt{\pi x - 2\pi}}$

A. Find the domain.
B. Find the smallest integer in the domain.
C. Find the largest integer not in the domain.

Part A: Find the Domain

$$\pi x - 2\pi > 0 \Rightarrow \pi x > 2\pi \Rightarrow x > 2 \Rightarrow x \in (2, \infty)$$

Part B

The smallest integer in the domain is the leftmost integer that is on the red line, which is 3.

Part C

The largest integer not in the domain is the rightmost integer not on the red line, which is 2.

I. Merging Solution Sets

Till now, the numerator did not play a role in the questions in this section, since there were no restrictions on the numerator. However, if the numerator has a square root, then

- The square root must be non-negative
- The denominator, as usual cannot be zero.

Example 1.179

Consider $a(x) = \dfrac{\sqrt{x+3}}{x-2}$

A. Find the domain
B. Find the largest integer not in the domain
C. Find the smallest integer in the domain.

Part A
Here, we have conditions from both the numerator and the denominator. We need to be more careful here. First, we find the conditions, then we simplify them, and finally we combine the results.
From the numerator

$$x + 3 \geq 0 \Rightarrow x \geq -3$$

From the denominator:

$$x \neq 2$$

Combine the two to get:
x is a real number greater than or equal to -3, but x cannot be equal to 2.

And write in mathematical notation as:

$$x \geq -3, x \neq 2 \Leftrightarrow x \in [-3,2) \cup (2, \infty)$$

Part B
2

Part C
-3

J. Cube Roots

Cube roots do not have any restriction on the domain, since cube roots exist for all real numbers. Hence, the domain of definition of a cube root is all real numbers.

Example 1.180

Find the domain of $f(x) = \sqrt[3]{x+5}$

$$D = (-\infty, \infty) = \mathbb{R}$$
$$\textit{Domain is all Real Numbers}$$

Example 1.181

Find the domain of $f(x) = \dfrac{1}{\sqrt[3]{x+5}}$

We do not have any restrictions because of the cube root.
But the expression inside the cube cannot be zero, since the cube root of zero is itself zero, and the denominator cannot be zero.

Hence, we solve

$$\sqrt[3]{x+5} = 0 \Rightarrow x+5 = 0 \Rightarrow x = -5$$

And hence the

$$Domain\ is\ all\ Real\ Numbers\ except -5.$$

K. n^{th} Roots

In general:

- There are no restrictions on cube roots, and any odd roots.
- Even roots must be non-negative.

Example 1.182

Find the domain of $f(x) = \sqrt[n]{x+5}$ if:

A. n is even
B. n is odd

Part A

$$x+5 \ge 0 \Rightarrow x \ge -5 \Rightarrow D_f = [-5, \infty)$$

Part B

There are no restrictions.

$$Domain\ is\ all\ Real\ Numbers$$

L. Absolute Value

Recall that the denominator of a fraction cannot be zero. Hence, an expression in absolute value cannot be zero either.

Example 1.183

Find the domain of the function $f(x) = \dfrac{|x+4|}{|x+2|}$

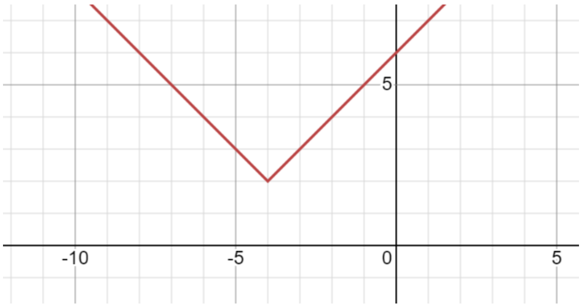
The denominator of the fraction cannot be zero. So, we equate the denominator to zero and solve to find the exceptions:

$$|x+2| = 0 \Rightarrow x+2 = \pm 0 \Rightarrow x = -2$$

$$Interval\ Notation: (-\infty, -2) \cup (-2, \infty)$$
$$Set: Set\ Builder: \{x | x \neq -2\}$$
$$Inequality: x < -2\ OR\ x > -2$$

Example 1.184

Find the domain of the function $g(x) = \dfrac{|x+4|}{|x+4|+2}$



Logical Method
Consider the denominator.

$$|x+4| > 0 \; for \; all \; values \; of \; x.$$
$$2 > 0$$

And hence, the denominator is always positive.
And hence, the domain of the function is:

$$All \; Real \; Numbers$$

Algebraic Method
Equate the denominator to zero:

$$|x+4| + 2 \Rightarrow |x+4| = -2$$

Since absolute value cannot be negative, there are no solutions to this equation. The domain of the function is

$$All \; Real \; Numbers$$

Example 1.185

Find the domain of the function $h(x) = \dfrac{|x+4|}{|x+5|-2}$

Equate the denominator to zero:

$$|x+5| - 2 = 0 \Rightarrow |x+5| = 2 \Rightarrow x+5 = \pm 2 \Rightarrow x = \pm 2 - 5 \Rightarrow x \in \{-7, -3\}$$

The domain is

$$All \; Real \; Numbers \; except \; \{-7, -3\}$$
$$Interval \; Notation: (-\infty, -7) \cup (-7, -3) \cup (-3, \infty)$$
$$Inequality \; Notation: x < -7 \; OR \; -7 < x < -3 \; OR \; x > -3$$

Example 1.186

Find the domain of the function $j(x) = \dfrac{5}{|x+4| - |x+3|}$

Equate the denominator to zero:

$$|x+4| - |x+3| = 0 \Rightarrow |x+4| = |x+3|$$

Use the property that $|a| = |b| \Leftrightarrow a = \pm b$:

$$x + 4 = x + 3 \Rightarrow 4 = 3 \Rightarrow \textbf{\textit{Not Possible}} \Rightarrow \textbf{\textit{No Solutions}}$$
$$x + 4 = -(x+3) \Rightarrow 2x = -7 \Rightarrow x = -\frac{7}{2}$$
$$Inequality \; Notation: x < -\frac{7}{2} \; OR \; x > -\frac{7}{2}$$

M. Nested Absolute Values

When the denominator has nested absolute values, we get an equation with nested absolute values.

- The standard method to solve such questions is to remove the absolute values, starting with the outermost.
- At each stage, the number of equations will double.

Absolute Value Nesting	Equations	Max. Solutions
1	2	2
2	4	4
3	8	8

Example 1.187

Find the domain of the function $j(x) = \dfrac{5}{\big||x+4|-5\big|-3}$

This has a nested absolute value. Standard methods to solve absolute values apply.

Isolate the outermost absolute value sign first:

$$\big||x+4|-5\big|-3=0 \Rightarrow \big||x+4|-5\big|=3$$

Use the property that $|a| = y \Leftrightarrow a = \pm y$:

$$|x+4|-5=\pm 3$$

Solve each case above individually:

$$\textbf{Case I}: |x+4|-5=3 \Rightarrow |x+4|=8 \Rightarrow x+4=\pm 8 \Rightarrow x=\pm 8-4 \Rightarrow x=\{-12,4\}$$
$$\textbf{Case II}: |x+4|-5=-3 \Rightarrow |x+4|=2 \Rightarrow x+4=\pm 2 \Rightarrow x=\pm 2-4 \Rightarrow x=\{-2,-6\}$$

$$\textit{All Real Numbers except } \{-12,-6,-2,4\}$$
$$(-\infty,-12)\cup(-12,-6)\cup(-6,-2)\cup(-2,4)\cup(4,\infty)$$

Challenge 1.188

Find the domain of the function $f(x) = \dfrac{5}{\big||x-1|-2\big|-3}$

Isolate the outermost absolute value sign first:

$$\big||x-1|-2\big|-3=0 \Rightarrow \big||x-1|-2\big|=3$$

Use the property that $|a| = y \Leftrightarrow a = \pm y$:

$$|x-1|-2=\pm 3$$

Solve each case above individually:

$$\textit{Case I}: |x-1|-2=+3 \Rightarrow |x-1|=5 \Rightarrow x-1=\pm 5 \Rightarrow x=\pm 5+1 \Rightarrow x\in\{6,-4\}$$
$$\textit{Case II}: |x-1|-2=-3 \Rightarrow |x-1|=-1 \Rightarrow \textit{No Solution}$$

Be careful with Case II, since absolute values can never be negative.

Hence, the final answer is

$$(-\infty,-4)\cup(-4,6)\cup(6,\infty)$$

Challenge 1.189

Find the domain of the function $f(x) = \dfrac{5}{\left|\left|x-\frac{1}{2}\right|-\frac{2}{3}\right|-\frac{3}{4}}$

Isolate the outermost absolute value sign first:

$$\left|\left|x-\frac{1}{2}\right|-\frac{2}{3}\right|=\frac{3}{4}$$

Use the property that $|a| = y \Leftrightarrow a = \pm y$:

$$\left|x-\frac{1}{2}\right|-\frac{2}{3}=\pm\frac{3}{4}$$

Solve each case above individually:

$$Case\ I: \left|x-\frac{1}{2}\right|-\frac{2}{3}=\frac{3}{4} \Rightarrow \left|x-\frac{1}{2}\right|=\frac{9}{12}+\frac{8}{12}=\frac{17}{12} \Rightarrow x-\frac{1}{2}=\pm\frac{17}{12} \Rightarrow x \in \left\{-\frac{11}{12},\frac{23}{12}\right\}$$

$$Case\ I: \left|x-\frac{1}{2}\right|-\frac{2}{3}=-\frac{3}{4} \Rightarrow \left|x-\frac{1}{2}\right|=\frac{-9}{12}+\frac{8}{12}=-\frac{1}{12} \Rightarrow No\ Solution$$

Hence, the final answer is:

$$\left(-\infty,-\frac{11}{12}\right)\left(-\frac{11}{12},\frac{23}{12}\right)\left(\frac{23}{12},\infty\right)$$

N. Variables

If you have variables, you have to be careful to check whether the expressions involving those variables are positive.

If they are not positive, then, as before, absolute value equations will not have a solution.

Challenge 1.190

If a, b, c are positive numbers such that $a < b < c$, find the domain of the function $f(x) = \frac{5}{||x-a|-b|-c}$

Isolate the outermost absolute value sign first:

$$||x-a|-b| = c$$

Use the property that $|a| = y \Leftrightarrow a = \pm y$:

$$|x-a|-b = \pm c$$

Solve each case above individually:

$$Case\ I: |x-a|-b=c \Rightarrow |x-a|=b+c \Rightarrow x-a=\pm(b+c) \Rightarrow x \in \{a-b-c, a+b+c\}$$

$$Case\ II: |x-a|-b=-c \Rightarrow |x-a|=b-c \Rightarrow\Rightarrow No\ Solution$$

Hence, the final answer is:

$$(-\infty, a-b-c) \cup (a-b-c, a+b+c) \cup (a+b+c, \infty)$$

O. Nested Roots

Expressions inside a square root sign must be non-negative. This same condition applies to nested square roots. However, the overall work increases because:

- Each square root must individually be non-negative.
- The intervals obtained from each square root must all be satisfied together. That is, we must take the intersection of the intervals.

Example 1.191

Find the domain of $f(x) = \sqrt{5-\sqrt{x-3}}$

Inner Inequality:

$$x-3 \ge 0 \Rightarrow x \ge 3$$

Outer Inequality:

$$5-\sqrt{x-3} \ge 0 \Rightarrow 5 \ge \sqrt{x-3} \Rightarrow 25 \ge x-3 \Rightarrow 28 \ge x \Rightarrow x \le 28$$

$$[3,28]$$

Example 1.192

Find the domain of $p(x) = \sqrt{2 - \sqrt{3 - x}}$

Each expression inside a square root must be positive. We start with the inner inequality:

$$3 - x \geq 0 \Rightarrow 3 \geq x \Rightarrow x \leq 3$$

Outer Inequality:

$$2 - \sqrt{3 - x} \geq 0 \Rightarrow 2 \geq \sqrt{3 - x} \Rightarrow 4 \geq 3 - x \Rightarrow x + 4 \geq 3 \Rightarrow x \geq -1$$

Number Line Method to Find the Intersection

And we need to meet both conditions, which is given by:

$$x \geq -1 \; AND \; x \leq 3 \Rightarrow -1 \leq x \leq 3 \Rightarrow x \in [-1,3]$$

Example 1.193

Find the domain of $f(x) = \sqrt{7 - \sqrt{x + 3}}$

$$x + 3 \geq 0 \Rightarrow x \geq -3$$

$$7 - \sqrt{x \pm 3} \geq 0 \Rightarrow 7 \geq \sqrt{x + 3} \Rightarrow 49 \geq x + 3 \Rightarrow 46 \geq x \Rightarrow x \leq 46$$

$$[-3,46]$$

Challenge 1.194

Find the domain of $f(x) = \sqrt{7 + \sqrt{x - 3}}$

Inner Inequality:

$$x - 3 \geq 0 \Rightarrow x \geq 3$$

Outer Inequality:

$$\textit{Bogus Solution}: 7 + \sqrt{x - 3} \geq 0 \Rightarrow 7 \geq -\sqrt{x - 3} \Rightarrow 49 \geq x - 3 \Rightarrow 52 \geq x \Rightarrow x \leq 52$$

$$\textit{Correct Solution}: 7 + \sqrt{x - 3} \geq 0$$

Note that each term of the left-hand side is individually greater than zero:

$$7 > 0, \qquad \sqrt{x - 3} \geq 0 \Rightarrow \underbrace{7}_{+ve} + \underbrace{\sqrt{x - 3}}_{non-negative} \geq 0 \; \textbf{\textit{is always true}}$$

Hence, the final answer is:

$$[3, \infty)$$

Challenge 1.195

Find the domain of $f(x) = \sqrt{1 - \sqrt{2 - \sqrt{3 - x}}}$

Innermost Inequality:

$$3 - x \geq 0 \Rightarrow 3 \geq x \Rightarrow x \leq 3$$

Middle Inequality:

$$2 - \sqrt{3-x} \geq 0 \Rightarrow 2 \geq \sqrt{3-x} \Rightarrow 4 \geq 3 - x \Rightarrow 4 + x \geq 3 \Rightarrow x \geq -1$$

Outer Inequality:
Remove the outermost radical:

$$1 - \sqrt{2 - \sqrt{3-x}} \geq 0 \Rightarrow 1 \geq \sqrt{2 - \sqrt{3-x}} \Rightarrow 1 \geq 2 - \sqrt{3-x}$$

Remove the second radical:

$$1 + \sqrt{3-x} \geq 2 \Rightarrow \sqrt{3-x} \geq 1 \Rightarrow 3 - x \geq 1$$

Solve the inequality for x:

$$3 \geq 1 + x \Rightarrow 2 \geq x \Rightarrow x \leq 2$$

$$[-1,2]$$

Example 1.196

Find the domain of $f(x) = \sqrt{3 - \sqrt{2 - \sqrt{1-x}}}$

P. Factored Expressions

If an expression is already given in factored form, then finding the numbers not in the domain of the function becomes much easier.

$$\boldsymbol{f(x) = \frac{1}{(x+\alpha)(x+\beta)(x+\gamma)} \Rightarrow Domain = All\ real\ numbers\ except\ \{-\alpha, -\beta, -\gamma\}}$$

Example 1.197

Find the domain of $p(x) = \dfrac{x}{(x+2)(x+3)(x-5)}$

$$x + 2 = 0$$
$$x + 3 = 0$$
$$x - 5 = 0$$

$$All\ Real\ Numbers\ except\ \{-2, -,3,5\}$$

Q. Back Calculations

Questions can give us the domain of a function, and require us to work out the values of constants in the functions. This requires an understanding of how the question would have been solved, if the values of the constants had been given.

Example 1.198

The domain of

$$f(x) = \frac{1}{x+a}$$

is all real numbers except -3. Find the value of a.

$$x + a = 0 \Rightarrow x = -a$$

$$-a = -3 \Rightarrow a = 3$$

Example 1.199

Find the value of a and b given that:

$$f(x) = \frac{1}{(x+a)(x+b)}, Domain\ of\ f = D_f = (-\infty, -7) \cup (-7, -4) \cup (-4, \infty)$$

The numbers not in the domain of $f(x)$ are:

$$-7, -4$$

$$(x+a)(x+b) = 0 \Rightarrow x \in \{-a, -b\}$$
$$\{-a, -b\} = \{-7, -4\} \Rightarrow \{a, b\} = \{4, 7\}$$

Example 1.200

Find the value of a and b given that:

$$f(x) = \frac{1}{(x+a)(x+b)}, Domain\ of\ f = D_f = x < -10\ OR - 10 < x < 5\ OR\ x > 5$$

Example 1.201

Find the value of a and b given that:

$$f(x) = \frac{1}{(x+a)(x+b)}, Domain\ of\ f = D_f = (-\infty, -7) \cup (-7, \infty)$$

R. Square Roots

Example 1.202

If $f(x) = \sqrt{x-a}$, then domain of f is:

A. (a, ∞)
B. $(-a, \infty)$
C. $[a, \infty]$
D. $[-a, \infty]$
E. Cannot be decided without knowing whether a is positive or negative.

Example 1.203

Find the value of a given that:

$$f(x) = \sqrt{x-a}, \qquad D_f = [7, \infty]$$

Example 1.204

$$f(x) = \frac{\sqrt{x+a}}{\sqrt{x+b}}, \qquad D_f = [-3, \infty]$$

Statement I: The value of a can be determined
Statement II: The value of b can be determined

Mark the correct option:

A. Only I is true

B. Only II is true
C. Both I and II are true
D. Neither I nor II is true

Example 1.205

Find, in terms of a and b, the domain of

$$f(x) = \sqrt{a - \sqrt{b - x}}$$

S. Applications

Example 1.206: Negative Numbers

Given that a, b, c are constants with $a < b < c < 0$, find the domain of

$$f(x) = \frac{3}{x+a} + \frac{5}{x-b} + \frac{7}{x+c}$$

Find the largest number not in the domain of $f(x)$.

$$\underbrace{a}_{Negative} < \underbrace{b}_{Negative} < \underbrace{c}_{Negative} < 0$$

$$x + a = 0 \Rightarrow x = -a$$
$$x - b = 0 \Rightarrow x = b$$
$$x + c = 0 \Rightarrow x = -c$$
$$\underbrace{b}_{Negative} < 0 < \underbrace{-c}_{Positive} < \underbrace{-a}_{Positive}$$
$$(-\infty, b) \cup (b, -c) \cup (-c, -a) \cup (-a, \infty)$$

Hence, the largest number not in the domain of $f(x)$ is $-a$.

Example 1.207: Counting

Consider the functions:

$$f_1(x) = \frac{1}{(x-x_1)(x-x_2)}, \qquad f_2(x) = \frac{1}{(x-x_3)(x-x_4)}, f_3(x) = \frac{1}{(x-x_5)(x-x_6)}$$

If the domain of each function is distinct, what is the minimum number of distinct values in

$$(x_1, x_2, \dots, x_6)$$

With two numbers, it is not possible.

$$\{0,1\}$$

But with three distinct numbers, it is possible. For example, let the three numbers be $\{0,1,2\}$:

$$(x_1, x_2) = (0,1)$$
$$(x_3, x_4) = (0,2)$$
$$(x_5, x_6) = (1,2)$$

If the above sequence of functions is continued

$$f_1, f_2, f_3, \dots f_n$$

what is the maximum value of n given that the number of distinct values in $(x_1, x_2, \dots, x_n)$ is v.

The number of ordered pairs that can be made is:

$$\frac{v(v-1)}{2}$$

And the number of functions, which is the value of n, is

$$\frac{v(v-1)}{2} \div 2 = \frac{v(v-1)}{4}$$

Example 1.208: Triangles

The three numbers not in the domain of

$$f(x) = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}, \qquad a < b < c, \qquad a, b, c \in \mathbb{N}$$

Are the sides of a triangle. Find, in terms of a and c, the largest value that b can take.

By triangle inequality

$$a + b < c \Rightarrow b < a - c$$

Example 1.209: Rectangles

The two numbers not in the domain of

$$f(x) = \frac{1}{x-l} + \frac{1}{x-w}, \qquad w < l, \qquad w, l \in \mathbb{N}$$

Are the adjacent sides of a rectangle with perimeter 76. Find the domain of $f(x)$ that maximizes the area of the rectangle.

$$2(w+l) = 76 \Rightarrow w + l = 38$$

To maximize the area, we want the numbers to be as close to each other as possible.
But, since $w < l$, we cannot have a square.

$$w = 18, l = 20 \Rightarrow A = 360$$

$$x = w \Rightarrow x - w = 0$$
$$x = l \Rightarrow x - l = 0$$

The two numbers not in the domain are

$$w, l$$

The required domain is

$$\mathbb{R} - \{18,20\}$$

Example 1.210: Series

Find the sum of the numbers not in the domain of

$$f(x) = \frac{1}{x+50} + \frac{1}{x+51} + \cdots + \frac{1}{x+59}$$

$$-50 - 51 - \cdots - 59$$
$$= -(50+0) + (-50-1) + \cdots + (-50-9)$$

$$= (-50)(10) - [1 + 2 + \cdots + 9] = -500 - \frac{9 \times 10}{2} = -500 - 45 = -545$$

Example 1.211: Telescoping Products

Find the product of all real numbers not in the domain of

$$p(x) = \frac{x}{\left(x+\frac{1}{2}\right)\left(x+\frac{2}{3}\right)\left(x+\frac{3}{4}\right)\dots\left(x+\frac{99}{100}\right)}$$

$$All\ Real\ Numbers\ except\ \left\{-\frac{1}{2}, -\frac{2}{3}, -\frac{3}{4}, \dots, -\frac{99}{100}\right\}$$

And hence their product will be:

$$\left(-\frac{1}{2}\right)\left(-\frac{2}{3}\right)\left(-\frac{3}{4}\right)\dots\left(-\frac{99}{100}\right) = -\frac{1}{100}$$

Note that the sign will be negative since we have fractions with denominator

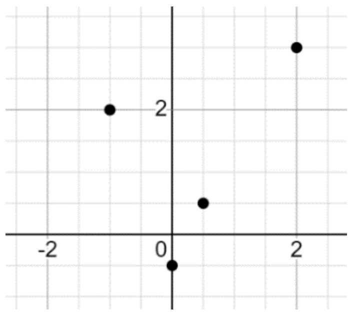
$$\{2,3,4,\dots,100\} \Rightarrow 99\ Fractions \Rightarrow 99\ Minus\ Signs \Rightarrow \boldsymbol{Negative\ Number}$$

1.6 Range[3]

A. Graphical Functions

1.212: Discrete Functions

A discrete function has a finite set of input-output values.



Hence, the domain and range for a function with discrete inputs and outputs will be written using set notation.

Example 1.213

Find the domain and range of the function graphed alongside.

Domain is the set of $x - values$:

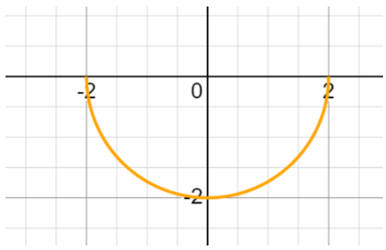
$$\left\{-1, 0, \frac{1}{2}, 2\right\}$$

Range is the set of $y - values$:

$$\left\{-\frac{1}{2}, \frac{1}{2}, 2, 3\right\}$$

1.214: Continuous Functions

A continuous function has an interval of values over which it will take an input.



Hence, the domain and range for a function with continuous inputs and outputs will be written using interval notation.

Example 1.215

Find the domain and range of the function graphed alongside.

Domain is the set of $x - values$:

$$[-2,2]$$

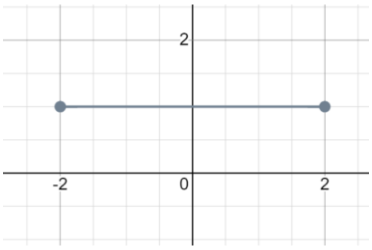
Range is the set of $y - values$:

[3] https://www.themathdoctors.org/finding-the-range-of-a-tricky-rational-function/

$$[-2,0]$$

Example 1.216

Is it possible for a function to have a domain expressed in interval notation, and a discrete range expressed in set notation? If yes, give an example of such a function and give its domain and range.



Example 1

We can do this using a constant function

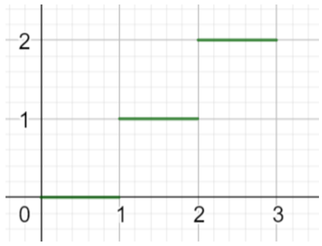
$$f(x) = 1$$

with a restricted

$$Domain = [-2,2]$$

And since the function is constant, its range is

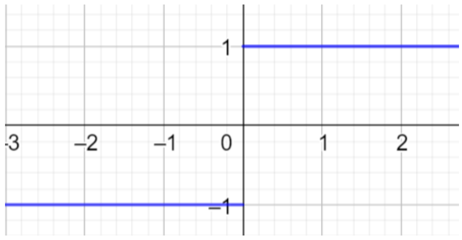
$$Range = \{1\}$$



Example 2

$$y = \lfloor x \rfloor$$
$$Domain = [0,3]$$
$$Range = \{0,1,2\}$$

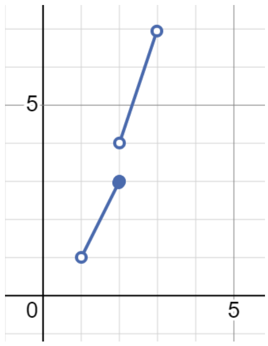
Example 3



$$y = sgn(x)$$
$$Domain = (-\infty, \infty)$$
$$Range = \{-1,1\}$$

1.217: Piece-Wise Graphs

A piece-wise graph has separate definitions for each "piece" that its domain is broken into.



Due to the flexibility of having more than one definition, these graphs allow for more complex shapes and behavior.

Example 1.218

Find the domain and range of the function graphed alongside.

Domain is the set of $x - values$:

$$(1,3)$$

Range is the set of y-values:

$$(1,3] \cup (4,7)$$

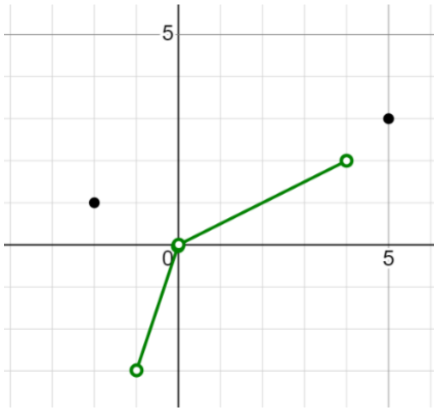
And the range can be written in inequality notation as:

$$1 < x \leq 3 \; OR \; 4 < x < 7$$

1.219: Mixed Function

In a mixed graph, you can have a mix of discrete and continuous parts of the function.

The usual rules for continuous and discrete functions will apply, but you will have to be careful when merging the final results.



Example 1.220

Consider the function graphed alongside. Find the domain and range of the function

Here, we need to be careful because we have both discrete and continuous parts to the function. The simplest method of attack is to consider these separately.

Domain

$$Discrete: \{-2,5\}$$
$$Continuous: (-1,0) \cup (0,4) \Leftrightarrow (-1,4), x \notin \{0\}$$

And we want both the parts:

$$(-1,0) \cup (0,4) \cup \{-2,5\}$$

Range

$$Discrete: \{1,3\}$$
$$Continuous: (-3,0) \cup (0,2)$$

And we want both the parts:

$$(-3,0) \cup (0,2) \cup \{3\}$$

And we do not need to write the $\{1\}$ separately, since it is already included in the interval $(0,2)$.

B. Linear Functions

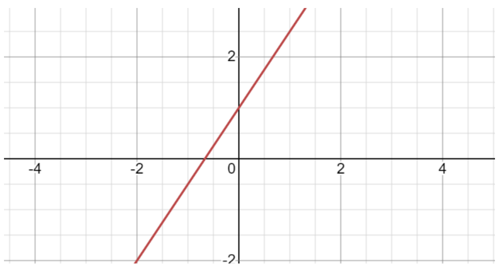
Linear functions represent a straight line, and have the form

$$f(x) = y = mx + c$$
$$m = slope, m \neq 0$$
$$c = y - intercept$$

Linear Function with Positive Slope

If you imagine extending the function to the

- left, the y-value will be as small (or as negative as we want).
- right, the y-value will be as large as we want.

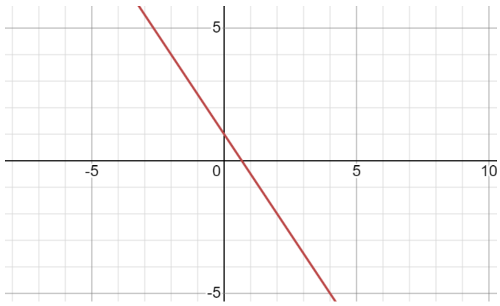


Hence, a linear function with positive slope

$$(-\infty, \infty) = All\ Real\ Numbers = \mathbb{R}$$

Linear Function with Negative Slope

For a linear function with negative slope, if you extend the function:
- To the left, the value of y increases without bound
- To the right, the value y decreases without bound.



Hence, a linear function with negative slope also has range
$$(-\infty, \infty) = All\ Real\ Numbers = \mathbb{R}$$

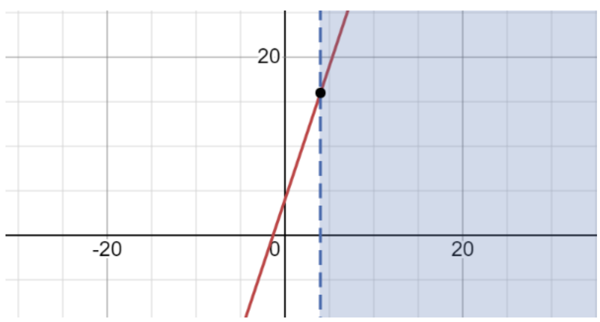
Linear Function with Zero Slope
Substitute $m = 0$ in $y = mx + c$ to get:
$$y = mx + c = 0x + c = c$$
This is a constant function, and it has range
$$c$$

Example 1.221
Find the range of the following functions:
A. $y = f(x) = 4$
B. $y = g(x) = 3x + 5$

Part A
This is a constant function, and its range has only a single value. Hence:
$$R_f = \{4\}$$
Part B
This is a non-constant linear function, and hence its range is:
$$R_g = (-\infty, \infty) = All\ Real\ Numbers = \mathbb{R}$$

C. Restricting the Domain
If we restrict the domain of a linear function, then the range will also be restricted.

Example 1.222
Find the range of the function:
$$f(x) = 3x + 4, \qquad D_f = (4, \infty)$$



Consider the graph of the function
$$f(x) = 3x + 4$$
Because of the restriction on the domain, we only consider the function in the blue shaded region.

If $x = 4$, then:
$$f(x) = 3(4) + 4 = 12 + 4 = 16$$

The value 16 is not part of the range of $f(x)$ since the interval
$$(4, \infty)$$
Does not include 4 and hence
$$(4,16)$$
Is not part of the graph of:
$$f(x) = 3x + 4, \qquad D_f = (4, \infty)$$

To the right of $x = 4$, $f(x)$ increases in value.

For $x = 4$, and to the left of $x = 4$, $f(x)$ is not defined.

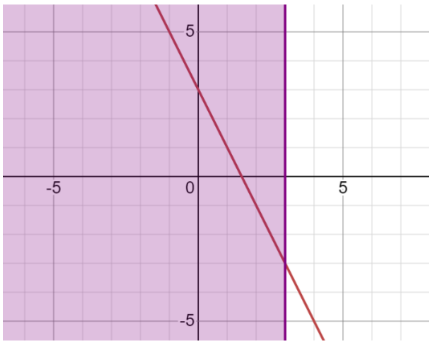
Hence, the range of $f(x) = 3x + 4$, $D_f = (4, \infty)$ is:

$$(16, \infty)$$

Example 1.223

Find the range of the function:

$$f(x) = -2x + 3, \qquad D_f = (-\infty, 3]$$



Since the function is only defined for

$$(-\infty, 3]$$

We are only interested in the shaded region.

The minimum value that the function takes is when $x = 3$, which is:

$$f(x) = -2x + 3 = -2(3) + 3 = -6 + 3 = -3$$

As x decreases, y increases.
Hence, the range of y is:

$$[-3, \infty)$$

D. Increasing Functions

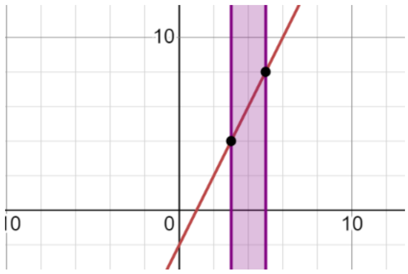
Recall that a function in which an increase in the value of x always results in an increase in the value of y is an increasing function.
A linear function with positive slope is an increasing function.

Hence, to find the range of such a function, we only need to evaluate the function at its endpoints.

Example 1.224

Find the range of the function:

$$f(x) = 2x - 2, \qquad D_f = [3,5]$$



Evaluate the function at its endpoints:

$$Left\ Endpoint: f(3) = 2(3) - 2 = 6 - 2 = 4$$
$$Rigtht\ Endpoint: f(5) = 2(5) - 2 = 10 - 2 = 8$$

Since the function is an increasing continuous function, the range is:

$$[4,8]$$

E. Decreasing Functions

Recall that a function in which an increase in the value of x always results in a decrease in the value of y is a decreasing function.
A linear function with negative slope is a decreasing function.

Example 1.225

Find the range of the function:

$$f(x) = -3x + 4, \qquad D_f = [1,2)$$

Example 1.226

Find the range of the function:

$$f(x) = -2x + 2, \qquad D_f = [e, \pi]$$

Where π and e are constants such that

$$\pi \approx 3.14, e \approx 2.71$$

(Give an exact answer)

Since we are asked for an exact answer, we must use the constants given in the question, and not approximate.

$$Left\ Endpoint: f(e) = -2e + 2$$
$$Right\ Endpoint: f(\pi) = -2\pi + 2$$

Which might make us think that the answer is:

$$[-2e + 2, -2\pi + 2]$$

But recall that this is a decreasing function. Also

$$e < \pi \Rightarrow -\pi < -e$$

Hence, in interval notation, we must write the smaller number on the left, giving us:

$$[-2\pi + 2, -2e + 2]$$

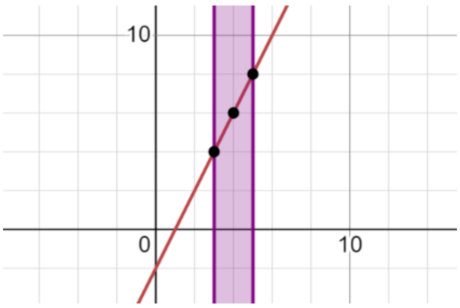
F. Discrete Domain

If the domain is discrete, then the range of the function will also be discrete.

$$y = f(x), \qquad D_f = \{x_1, x_2, \dots, x_n\} \Rightarrow R_f = \{f(x_1), f(x_2), \dots, f(x_n)\}$$

Example 1.227

A. Find the range of the function $f(x) = 2x - 2,\ D_f = \{3,4,5\}$
B. What is the connection between the function $f(x)$ and $g(x) = 2x - 2$



Part A

$$R_f = \{4,6,8\}$$

Part B

If we restrict the domain of $g(x)$, we get $f(x)$.
Hence, the three points that make up the graph of $f(x)$ all lie on the graph of $g(x)$.

Example 1.228

Find the range of the function:

$$y = f(x), \qquad D_f = \{x_1, x_2, x_3\}$$

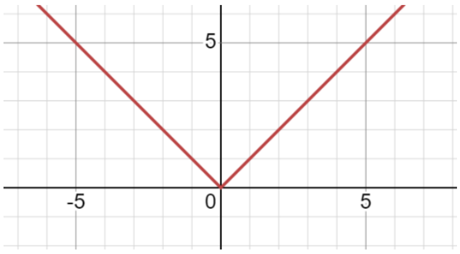
(Your answer will be in terms of variables)

$$R_f = \{f(x_1), f(x_2), f(x_3)\} = \{y_1, y_2, y_3\}$$

G. Absolute Value Function

The absolute value function gives the absolute value of a number.

If the number is:
- positive, there is no change
- negative, the minus sign is removed, but the number is otherwise unchanged.



Example 1.229
$y = |x|$ is graphed alongside. Find its domain and range.

$$Domain: (-\infty, \infty)$$

Since an absolute value can never be negative, the smallest value that $|x|$ can take is 0.
However, as we go to the right on the number line, or the left on the number line, $|x|$ becomes as large as want.
Hence, the range of $|x|$ is

$$Range: [0, \infty)$$

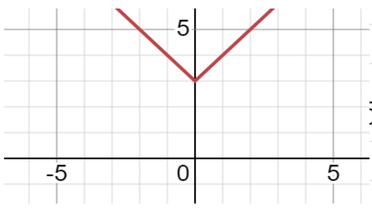
1.230: Vertical Shifts and Vertical Scales
- Vertically shifting the absolute value function does change its range.
- The absolute value function does not change its range when you scale it vertically, scale it horizontally or shift it horizontally.

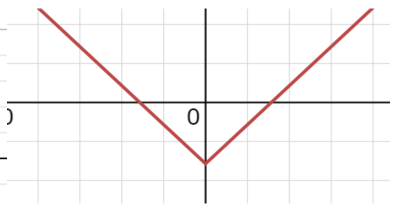
Example 1.231
Find the range of:

A. $|x| + 3$
B. $|x| - \pi$
C. $2|x|$
D. $\frac{1}{\pi}|x|$
E. $|x + 2|$
F. $|x - 2\pi|$
G. $|3x|$
H. $\left|\frac{x}{\pi}\right|$

Parts A and B

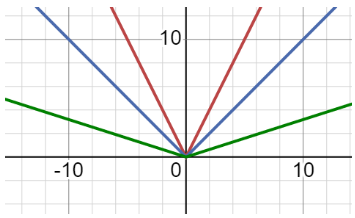
$$Range: [3, \infty)$$
$$Range: [-\pi, \infty)$$





Parts C and D

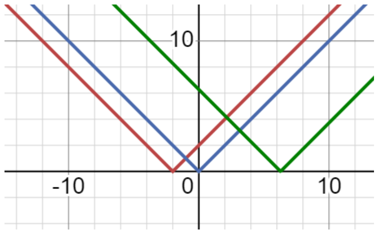
$$Blue\ Graph: y = |x|$$
$$Red\ Graph: y = 2|x|$$
$$Green\ Graph: y = \frac{1}{\pi}|x|$$



For all functions above, the range is

$$[0, \infty)$$

Parts E and F

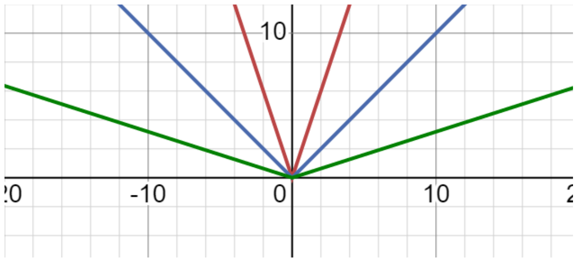


$$Blue\ Graph: y = |x|$$
$$Red\ Graph: y = |x + 2|$$
$$Green\ Graph: y = |x - 2\pi|$$

For all functions above, the range is

$$[0, \infty)$$

Parts G and H



$$Blue\ Graph: y = |x|$$
$$Red\ Graph: y = |3x|$$
$$Green\ Graph: y = \left|\frac{x}{\pi}\right|$$

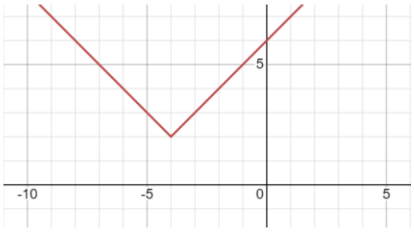
For all functions above, the range is

$$[0, \infty)$$

Example 1.232

Find the range of $y = |x + 4| + 2$

Transformations



$$|x| \rightarrow \underbrace{|x + 4|}_{\substack{Move\ left \\ by\ 4\ Units}} \rightarrow \underbrace{|x + 4| + 2}_{\substack{Move\ Up \\ by\ 2\ Units}}$$

Algebraically

The minimum value of $|x + 4| + 2$ occurs when

$$|x + 4| = 0 \Rightarrow x + 4 = 0 \Rightarrow x = -4$$

x increases without bound to the left of $x = -4$, and also to the right of $x = -4$.

Hence, the range of $y = |x + 4| + 2$ is

$$[2, \infty)$$

Example 1.233

If there are multiple absolute value terms, we see whether the expression inside the absolute value is positive or negative in different intervals.

Consider the expression $|x + 3| - |x - 4|$

A. Find the range
B. Hence, or otherwise, rewrite it as a piecewise function without using absolute value

Finding Critical Points

$|x + 3| - |x - 4|$ has two expressions inside the absolute value sign. We check where each expression is positive, and divide the number line accordingly.

$$|x + 3| = \begin{cases} (\boldsymbol{x} + \mathbf{3}) \text{ when } x + 3 > 0 \Rightarrow x \geq -3 \\ -(\boldsymbol{x} + \mathbf{3}) \text{ when } x < -3 \end{cases}$$

$$|x - 4| = \begin{cases} (\boldsymbol{x} - \mathbf{4}) \text{ when } x - 4 > 0 \Rightarrow x \geq 4 \\ -(\boldsymbol{x} - \mathbf{4}) \text{ when } x < 4 \end{cases}$$

We get three cases:

Case I: $(-\infty,-3)$:
$|x+3|-|x-4| = -x-3-(-x+4) = -x-3+x-4 = -7$

Case II: $(-3,4)$:
$|x+3|-|x-4| = x+3-(-x+4) = x+3+x-4 = 2x-1$
$2x-1_{x=-3} = 2(-3)-1 = -7, \qquad 2x-1_{x=4} = 7$

Case III: $(4,\infty)$:
$|x+3|-|x-4| = x+3-x+4 = 7$

Final Solution	$(-\infty,-3)$	$[-3,4]$	$(4,\infty)$
Intervals	-7	$[-7,7]$	7
$\boldsymbol{Range} = [-7,7]$			

Part B
From the behavior, we can conclude that

$$|x+3|-|x-4| = \begin{cases} y=-7, & x<-3 \\ y=2x-1, & x\in(-3,4) \\ y=+7, & x>4 \end{cases}$$

Challenge 1.234

A. Let $f(x)=|x-p|+|x-15|+|x-p-15|$, where $0<p<15$. Determine the minimum value taken by $f(x)$ for x in the interval $p\le x\le 15$. (AIME 1983/2)
B. Also, determine the range of $f(x)$ in the interval $p\le x\le 15$

Part A
Consider each term separately:

$$x-p>0 \Rightarrow x>p \Rightarrow Always\ True \Rightarrow |x-p| = x-p$$
$$x-15>0 \Rightarrow x>15 \Rightarrow Never\ True \Rightarrow |x-15| = 15-x$$
$$x-p-15>0 \Rightarrow x>p+15 \Rightarrow Never\ True \Rightarrow |x-p-15| = p+15-x$$

Hence,

$$f(x) = (x-p)+(15-x)+(p+15-x) = 30-x$$

This will reach minimum value when $x=15$:

$$30-x = 30-15 = 15$$

Part B

$$0<p<15$$

The values that x can take are

$$0<x\le 15 \Rightarrow f(x)\in[15,30)$$

H. Using Asymptotes

Example 1.235

Find the domain and range of the function:

A. $f(x)=\frac{1}{x}$
B. $f(x)=\frac{1}{3x+5}$

Part A

Equate the denominator to zero to find its vertical asymptotes

$$x = 0 \Rightarrow Vertical\ Asymptote\ at\ x = 0$$

The function will achieve all values going towards positive infinity and negative infinity.

Since the degree of the numerator is less than the degree of the denominator, the function has a horizontal asymptote at

$$y = 0$$

Hence, check if the function ever achieves its horizontal asymptote by substituting the value of y:

$$0 = \frac{1}{x} \Rightarrow 0 = 1 \Rightarrow Not\ Valid \Rightarrow 0 \notin R_f$$

Hence, the final answer is:

$$D_f: \mathbb{R} - \{0\}$$
$$R_f: \mathbb{R} - \{0\}$$

Part B

$$f(x) = \frac{1}{3x+5}$$

Horizontal Asymptote is

$$y = 0 \Rightarrow Cannot\ be\ achieved$$
$$D_f: \mathbb{R} - \left\{-\frac{5}{3}\right\}$$
$$R_f = \mathbb{R} - \{0\}$$

I. Inverse Functions to find Range

Example 1.236

Find the range of the following functions by finding the domain of the inverse function:

A. $f(x) = \frac{1}{x}$

B. $f(x) = \frac{1}{3x+5}$

Part A

Find the inverse function:

$$y = \frac{1}{x} \Rightarrow f^{-1}(x) = \frac{1}{x}$$

Find the domain of the inverse function and that is also equal to the range of the original function:

$$D_{f^{-1}}: \mathbb{R} - \{0\} \Rightarrow R_f: \mathbb{R} - \{0\}$$

Part B

Find the inverse function:

$$y = \frac{1}{3x+5} \Rightarrow f^{-1}(x) = \frac{1-5x}{3x}$$
$$R_f = D_{f^{-1}} = \mathbb{R} - \{0\}$$

J. Product of Functions

Example 1.237

$f(x)$ and $g(x)$ are functions with domain $\mathbb{R}$ such that $R_f = [-3,1]$, and $R_g = [-5,2]$. Let the range of $f(x)g(x)$ be $[a,b]$. What is the maximum possible value of b? Note: $f(x)g(x)$ represents the product of the two functions.

$$f(x_0) = -3, g(x_0) = -5$$

Maximum value of range = -3 * -5 = 15

[15,2]

K. Composite Functions

Example 1.238

Consider

$$g(x) = -2x + 4, \qquad f(x) = \sqrt{x}$$

A. Is $f(g(x))$ defined?
B. Can $f(g(x))$ be defined for a set of values?

Part A

$$D_f = [0, \infty)$$
$$R_g = (-\infty, \infty)$$

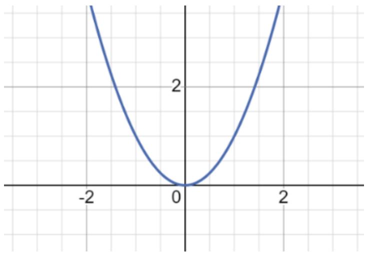
Since R_g is not a subset of D_f

L. Functional Expressions

Example 1.239

1.7 Quadratic Functions

A. Definition



1.240: Definition

The quadratic function

$$f(x) = ax^2 + bx + c, \qquad a \neq 0$$

Is a parabola

Example 1.241

What is the shape of

$$f(x) = ax^2 + bx + c, \qquad a = 0$$

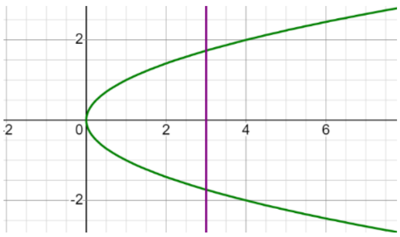
A line

1.242: Standard Form

When we write an expression in standard form, we write it in descending powers of x.

$$f(x) = ax^2 + bx + c$$

Is in standard form.



1.243: Sideways Parabola

The equation

$$x = ay^2 + by + c$$

Represents a parabola, but not a function.

Graphically

We can see in the graph alongside that every value of x is associated with two values of y.

Algebraically:

$$x = ay^2 + by + c$$

Take all variables to the left side:

$$ay^2 + by + c - x = 0$$

This is a quadratic in terms of y, which can be solved by using the quadratic formula.
Substitute $a = a, b = b, c = c - x$ in the quadratic formula:

$$y = \frac{-b \pm \sqrt{b^2 - 4(a)(c - x)}}{2a}$$

And this has two solutions:

- One with the positive square root
- One with the negative square root

Hence, y is not a function of x.
In the above x is a quadratic function of y. Hence, it is not a function of y in terms of x:

$$y = f(x) \text{ } does \text{ } not \text{ } hold \text{ } for \text{ } x = ay^2 + by + c$$

1.244: Zeros

The zeroes of a parabola (or a function in general) are the values at which the function takes the value zero. That is:

$$f(x) = 0 \Leftrightarrow y = 0 \Leftrightarrow x - intercepts \Leftrightarrow Roots$$

Remember that these are equivalent concepts:

- Finding the zeros of a function
- Equating the function to zero
- Finding the x-intercepts of a function
- Finding the roots of the corresponding equation, with $f(x) = 0$

Example 1.245

Given that $f(x) = x^2 + 5x + 6$, find:

A. $f(2)$
B. the zeros of the function
C. the roots of $y = x^2 + 5x + 6$
D. the x-intercepts of the function

Part A

$$f(2) = 2^2 + 5(2) + 6 = 4 + 10 + 6 = 20$$

Part B

To find the zeros, we equate the function to zero:

$$f(x) = 0 \Rightarrow x^2 + 5x + 6 = 0 \Rightarrow (x + 2)(x + 3) = 0 \Rightarrow x \in \{-3, -2\}$$

Part C

Roots are the same (since the equation is the same).

$$Roots = \{-3, -2\}$$

Part D

x-intercepts are the same as the roots

$$(-3,0) \text{ } and \text{ } (-2,0)$$

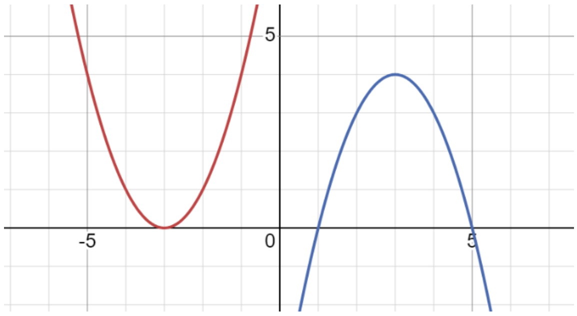
1.246: Upward and Downward Parabolas

The parabola

$$f(x) = ax^2 + bx + c$$

With leading co-efficient a when the parabola is written in standard form:

- Is an upward parabola if $a > 0$
- Is a downward parabola if $a < 0$



Remember the shape of an upward and a downward parabola.

- The red parabola is upward (concave up).
- The blue parabola is downward (concave down).

Example 1.247

What is the nature (upward, downward, or neither) of the following parabolas:

A. $f(x) = 3x^2 + 5x + 4$
B. $x = -3y^2 + 5y + 4$
C. $y = -3x^2 - 5x - 4$
D. $y = 3 + 5x - 4x^2$

$$A: Leading\ Coefficient = 3 \Rightarrow Upward\ parabola$$
$$B: Not\ a\ Function \Rightarrow Neither\ upward\ nor\ downward$$
$$C: Leading\ Coefficient = -3 \Rightarrow Upward\ parabola$$
$$D: Leading\ Coefficient\ in\ standard\ form\ -4 \Rightarrow Downward\ parabola$$

B. Graphs

Example 1.248: (Graphing Calculator Allowed)

Graph $y = x^2 + 5x + 6$. Then, for each part below, the on the same set of axes, graph:

A. $y = \frac{1}{x^2+5x+6}$
B. $y = \sqrt{x^2 + 5x + 6}$
C. $y = \frac{1}{\sqrt{x^2+5x+6}}$

Comment on the behavior of the function.

C. Domain

1.249: Domain of a Parabola

$$f(x) = ax^2 + bx + c$$

is a parabola with

$$Domain = \mathbb{R}$$

Domain is the set of values that x can take. There are no restrictions on the value of x, since there are no square roots, and no denominator to concern us.

Example 1.250

What is the domain of $f(x) = \pi x^2 + \pi^2 x + \pi^3$

$$Domain = \mathbb{R}$$

Example 1.251

For how many values of a does the quadratic $f(x) = ax^2 + bx + c$ have an implied domain of only positive real numbers.

Zero.

1.252: Quadratic Expression in the Denominator

The function

$$f(x) = \frac{1}{(x-a)(x-b)}$$

Has domain

$$\mathbb{R} \setminus \{a, b\}$$

That is

$$All\ real\ numbers\ except\ for\ a\ and\ b.$$

The denominator of a fraction cannot be zero. All solutions of the quadratic equation are not part of the domain of the function.

Example 1.253: Quadratics, and reducible to Quadratics

A. Find the domain of $f(x) = \frac{1}{(x+2)(x-3)}$
B. Find the domain of $f(x) = \frac{1}{x^2-x-12}$
C. Find the domain of $f(x) = \frac{1}{2x^2+5x+3}$
D. Find the domain of $f(x) = \frac{1}{2x^2+6x+3}$
E. Find the domain of $f(x) = \frac{1}{6x^3+x^2-12x}$
F. How many real numbers are not in the domain of $f(x) = \frac{1}{(x^2-2x-15)(x^2+8x+15)}$?
G. Find the domain of $f(x) = \frac{1}{y^8+8y^4+12}$
H. Find the domain of $f(x) = \frac{1}{y^8-8y^4+12}$
I. Find the domain of $f(x) = \frac{x+3}{(x+2)(x^2+10x+21)}$

Part A

$$(x+2)(x-3) = 0 \Rightarrow x \in \{-2, 3\}$$

And hence the domain is

$$\mathbb{R} \setminus \{-2, 3\} \Leftrightarrow (-\infty, -2) \cup (-2, 3) \cup (3, \infty)$$

Part B

Equate the denominator to zero and solve:

$$x^2 - x - 12 = 0 \Rightarrow (x-4)(x+3) = 0 \Rightarrow x \in \{-3, 4\}$$

Which means that the domain is:

$$All\ real\ numbers\ except -3\ and\ 4$$

Recall that two points not in the domain will divide the real number line into three valid intervals:

$$Domain = (-\infty, -3) \cup (-3, 4) \cup (4, \infty)$$

Part C

Equate the denominator to zero and solve to find the exceptions:

$$2x^2 + 5x + 3 = 0$$
$$2x^2 + 2x + 3x + 3 = 0$$
$$2x(x+1) + 3(x+1) = 0$$
$$(x+1)(2x+3) = 0$$
$$x \in \left\{-\frac{3}{2}, -1\right\}$$

And now the solutions above cannot be part of the domain. Hence, the domain is:

$$Domain = \left(-\infty, -\frac{3}{2}\right) \cup \left(-\frac{3}{2}, -1\right) \cup (-1, \infty)$$

Part D

Equate the denominator to zero and solve to find the exceptions:

$$2x^2 + 6x + 3 = 0$$

This cannot be factored easily, so apply the quadratic formula with $a = 2, b = 6, c = 3$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{36 - (4)(2)(3)}}{2(2)} = \frac{-6 \pm \sqrt{12}}{4} = \frac{-6 \pm 2\sqrt{3}}{4} = \frac{-3 \pm \sqrt{3}}{2}$$

And now the solutions above cannot be part of the domain.
Hence, the domain is:

$$Domain = \left(-\infty, \frac{-3-\sqrt{3}}{2}\right) \cup \left(\frac{-3-\sqrt{3}}{2}, \frac{-3+\sqrt{3}}{2}\right) \cup \left(\frac{-3+\sqrt{3}}{2}, \infty\right)$$

Part E

$$6x^3 + x^2 - 12x = x(2x+3)(3x-4) \Rightarrow x \in \left\{0, -\frac{3}{2}, \frac{4}{3}\right\}$$
$$\left(-\infty, -\frac{3}{2}\right) \cup \left(-\frac{3}{2}, 0\right) \cup \left(0, \frac{4}{3}\right) \cup \left(\frac{4}{3}, \infty\right)$$

Part F

$$(x-5)(x+3)(x+3)(x+5) = 0 \Rightarrow x \in \{-5, -3, 5\} \Rightarrow 3\ Numbers$$

Part G

Notice that the first term has exactly double the power of the second term:

$$y^8 = (y^4)^2$$

And hence, we are in a position to reduce this to a quadratic.

Let $x = y^4$:

$$y^8 + 8y^4 + 12 = x^2 + 8x + 12 = (x+2)(x+6) = 0 \Rightarrow x \in \{-2, -6\}$$
$$y^4 = -2\ OR\ y^4 = -6$$

Neither of the equations above has a solution in real numbers.
Hence, the domain is

$$(-\infty, \infty)$$

Part H

As before, substitute $x = y^4$:

$$y^8 - 8y^4 + 12 = x^2 - 8x + 12 = (x-2)(x-6) = 0 \Rightarrow x \in \{2, 6\}$$

Substitute $y^4 = x$ to solve the original equation:

$$y^4 = 2 \Rightarrow y = \pm\sqrt[4]{2}$$
$$OR\ y^4 = 6 \Rightarrow y = \pm\sqrt[4]{6}$$

Hence, the domain is:

$$\mathbb{R}\ \{\pm\sqrt[4]{2}, \pm\sqrt[4]{6}\}$$

Part I

Wrong Approach

Factor the quadratic in the denominator to get:

$$\frac{x+3}{(x+2)(x+3)(x+7)}$$

Now, you might feel like cancelling the $x+3$ in the numerator with the $x+3$ in the denominator to get:

$$\frac{x+3}{(x+2)(x+3)(x+7)}=\frac{1}{(x+2)(x+7)}\Rightarrow Domain\ is\ \mathbb{R}\backslash\{-2,-7\}$$

But we cannot divide by zero. And we do not know that $x+3$ is not zero. In fact, $x+3$ is zero precisely when $x=-3$.

Correct Approach

Factor the denominator and equate it to zero:

$$(x+2)(x+3)(x+7)=0\Rightarrow x\in\{-2,-3,-7\}$$

D. Back Calculations

Given the domain of a function with a quadratic in the denominator, we can use the factored form of the quadratic to determine the quadratic function.

Example 1.254

The domain of $f(x)=\frac{1}{\sqrt{x^2+5x+6}}$ is $(-\infty,-3)\cup(-2,\infty)$.

A. Use this to determine the domain of $g(x)=\sqrt{x^2+5x+6}$.
B. Find the points in the domain of $g(x)$, which are not a part of the domain of $f(x)$.

Part A

The domain of $g(x)$ includes its endpoints, while the domain of $f(x)$ does not. Otherwise, the domains are the same. Hence, the domain will be

$$(-\infty,-3]\cup[-2,\infty)$$

Part B: $\{-2,-3\}$

Example 1.255

Find $a+b+c$ given that

$$f(x)=\frac{1}{ax^2+bx+c},\qquad D_f=(-\infty,-4)\cup(-4,7)\cup(7,\infty)$$

From the domain, we can understand that the number which are not in the domain are:

$$-4\ and\ 7$$

These numbers should be the roots of the quadratic:

$$ax^2+bx+c=0\Leftrightarrow(x-\alpha)(x-\beta)=0\Leftrightarrow x\in\{-4,7\}$$

Substitute $\alpha=-4,\beta=7$ in $(x-\alpha)(x-\beta)=0$:

$$\big(x-(-4)\big)(x-7)=0\Rightarrow(x+4)(x-7)=0\Rightarrow x^2-3x-28=0\Rightarrow a=1,b=-3,c=-28$$
$$a+b+c=1-3-28=-30$$

Example 1.256

Find $g(x)$ given that:

$$f(x)=\frac{1}{g(x)},\qquad g(x)=ax^2+bx+c,\qquad D_f=\left(-\infty,3-\sqrt{2}\right)\cup\left(3-\sqrt{2},3+\sqrt{2}\right)\cup\left(3+\sqrt{2},\infty\right)$$

$$\left(x-\left(3+\sqrt{2}\right)\right)\left(x-\left(3-\sqrt{2}\right)\right)$$

Expand

$$= x^2 - x\left(3-\sqrt{2}\right) - x\left(3+\sqrt{2}\right) + \left(3+\sqrt{2}\right)\left(3-\sqrt{2}\right)$$

Expand further:

$$= x^2 - 3x + \sqrt{2}x - 3x - \sqrt{2}x + (9-2)$$
$$= x^2 - 6x + 7$$

Challenge 1.257

$$f(x) = \frac{1}{g(x)h(x)}, \qquad D_f = \mathbb{R} \setminus \{-6,-3,2,5\}$$

Then find $g(x)$ and $h(x)$ given that they are each of the form:

$$ax^2 + bx + c, Exactly\ one\ of\ a, b, c\ is\ negative$$

We know that the roots in some order are:

$$\{-6,-3,2,5\}$$

And hence we can write $g(x)h(x)$ as:

$$(x+6)(x+3)(x-2)(x-5)$$

Now, we need to decide which of a, b and c is negative. Since we used the convention that x is positive, when we multiply it out, we will get a as positive.
Hence, we need to decide whether b or c is negative.

Suppose we pair the positive values with the positive values, and the negative values with the negative values,then:

$$(x-2)(x-5) = x^2 - 7x + 10 \Rightarrow a\ is\ +ve, b\ is\ -ve, c\ is\ +ve \Rightarrow Works$$
$$(x+6)(x+3) = x^2 + 9x + 18 \Rightarrow a, b\ and\ c\ are\ both\ positive \Rightarrow \textbf{Does not work}$$

Hence, we must pair a positive value with a negative value:

$$(x+6)(x-2) = x^2 + 4x - 12$$
$$(x+3)(x-5) = x^2 - 2x - 15 \Rightarrow \textbf{Does not work}$$

Hence, we need to pair a positive value with a negative value in such a way that b is positive, but c is negative:

$$(x+6)(x-5) = x^2 + x - 30 = g(x)$$
$$(x+3)(x-2) = x^2 + x - 6 = h(x)$$

And the functions could also have been assigned the other way around.

Example 1.258

Find $a + b + c$ given that

$$f(x) = \frac{1}{ax^3 + bx^2 + cx}, \qquad D_f = D_f = \mathbb{R} \setminus \{0,2,5\}$$

The roots of $ax^3 + bx^2 + cx$ must be:

$$\{0,2,5\}$$

Hence, we must have the following three equations satisfied when the denominator is zero:

$$x = 0$$
$$x = 2 \Rightarrow x - 2 = 0$$

$$x = 5 \Rightarrow x - 5 = 0$$

Use the zero-product property:

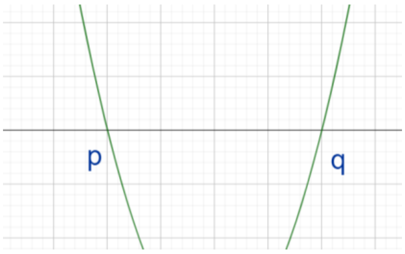
$$x(x-2)(x-5) = x(x^2 - 7x + 10) = x^3 - 7x^2 + 10x \Rightarrow a = 1, b = -7, c = 10$$
$$a + b + c = 1 - 7 + 10 = 4$$

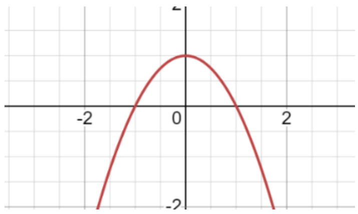
E. Square Roots

1.259: Upward and Downward Parabolas

Based on the leading coefficient, a parabola $y = ax^2 + bx + c$ can be classified as:

- An upward parabola if $Leading\ Coefficient = a > 0$
- A downward parabola if $Leading\ Coefficient = a < 0$
- Also, recall that if $a = 0$, the expression is not a quadratic.





1.260: Positive and Negative Intervals for an Upward Parabola

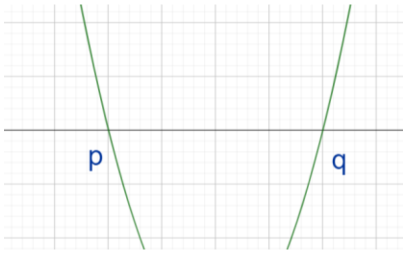
Positive Intervals

An upward parabola $y = ax^2 + bx + c$ with roots p and $q, p < q$ is positive in the interval

$$(-\infty, p) \cup (q, \infty)$$

Negative Intervals

$$(p, q)$$



Recall that square roots must be positive, and this also applies to quadratics. Hence, to determine the domain of

$$y = \sqrt{ax^2 + bx + c}$$

We must find the interval where:

$$ax^2 + bx + c > 0$$

Example 1.261

Find the domain of $f(x) = \sqrt{x^2 - 1}$

Find the roots of the corresponding quadratic equation:

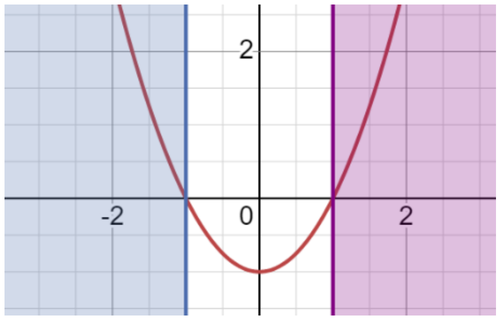
$$x^2 - 1 = 0 \Rightarrow x \in \{-1,1\}$$

The positive interval is:

$$(-\infty, -1) \cup (1, \infty)$$

And the domain includes both the positive interval, and the points where the expression is zero.
Hence, the domain is:

$$(-\infty, -1] \cup [1, \infty)$$



Example 1.262

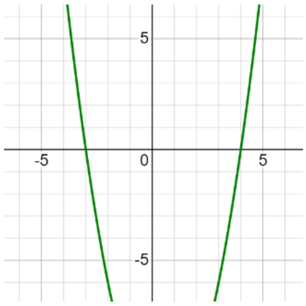
Find the domain of $f(x) = \sqrt{x^2 - x - 12}$

The expression in the square root must be positive, and hence we solve the inequality:

$$x^2 - x - 12 \geq 0 \Rightarrow (x-4)(x+3) \geq 0$$

Solve the corresponding quadratic equation to find the critical points:

$$(x-4)(x+3) = 0 \Rightarrow x \in \{-3, 4\}$$



Graph the upward parabola using the critical points. The intervals where the parabola is non-negative is also the domain of the function:

$$D_f = (-\infty, -3] \cup [4, \infty)$$

F. Square Roots in the Denominator

When you have a square root in the denominator

- The original condition that the square root must be non-negative remains in place
- The additional condition that the square root cannot be zero gets added

These two conditions combine to require that

$$f(x) = \frac{1}{\sqrt{x}} \Rightarrow x > 0$$

Example 1.263

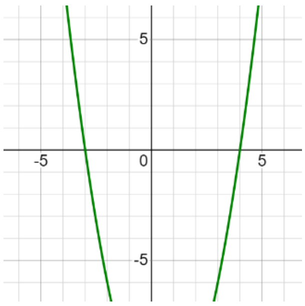
Find the domain of $f(x) = \dfrac{1}{\sqrt{x^2 - x - 12}}$

We have already solved this question when the expression was in the numerator, and obtained

$$D_f = (-\infty, -4] \cup [3, \infty)$$

Here, the only change is that the expression in the denominator cannot be zero, and hence we remove the points where the denominator is zero (which are the endpoints of the intervals):

$$D_f = (-\infty, -4) \cup (3, \infty)$$



Example 1.264

Steps	$f(x) = \frac{1}{\sqrt{x+4}}$	$f(x) = \frac{1}{\sqrt{x^2+5x+6}}$
Write and solve the inequality	$x + 4 > 0 \Rightarrow x > -4$	$x^2 + 5x + 6 > 0$ $(x+2)(x+3) > 0$
Set Builder (Rule)	$\left\{\begin{matrix} x: x \text{ is any real number greater than} \\ -4 \end{matrix}\right\}$	
Set Builder (Algebraic)	$\{x: x > -4, \quad x \in \mathbb{R}\}$	$x < -3\ OR\ x > -2$
Interval Notation	$[-4, \infty)$	$(-\infty, -3) \cup (-2, \infty)$

1.265: Positive and Negative Intervals for a Downward Parabola

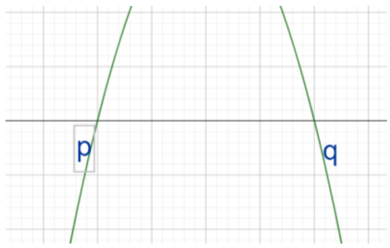
A downward parabola reverses the intervals as compared to an upward parabola.

The positive interval for a quadratic with $y = -ax^2 + bx + c, a > 0$ is:

$$(p, q)$$

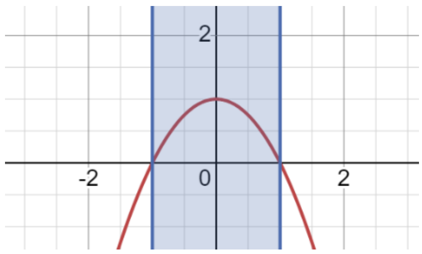
And the interval where it is negative is:

$$(-\infty, p) \cup (q, \infty)$$



Example 1.266

Find the domain of $f(x) = \sqrt{-x^2 + 1}$



Find the roots of the corresponding quadratic equation:

$$-x^2 + 1 = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x \in \{-1, 1\}$$

The positive interval is:

$$(-1, +1)$$

And the domain includes both the positive interval, and the points where the expression is zero.

Hence, the domain is:

$$[-1, 1]$$

G. Square Root in Numerator and Denominator

Example 1.267

Find the domain of the function

$$f(x) = \frac{\sqrt{x^2 - x - 20}}{\sqrt{x^2 + x - 20}}$$

Numerator

The expression in the numerator must be non-negative:

$$x^2 - x - 20 \geq 0 \Rightarrow (x-5)(x+4) \geq 0 \Rightarrow Roots\ are\ \{-4, 5\} \Rightarrow x \in (-\infty, -4] \cup [5, \infty)$$

Denominator

The expression in the denominator must be positive:

$$x^2 + x - 20 > 0 \Rightarrow (x+5)(x-4) > 0 \Rightarrow Roots\ are\ \{-5, 4\} \Rightarrow x \in (-\infty, -5) \cup (4, \infty)$$

And to find the domain, we must find the intersection of the above two intervals:

$$(-\infty, -5) \cup [5, \infty)$$

Find the largest integer not in the domain.

The domain is:

$$(-\infty,-5)\cup[5,\infty)$$

5 is in the domain, as are all integers larger than 5.
Hence, the largest integer not in the domain is

$$4$$

Find the smallest integer not in the domain

The domain is:

$$(-\infty,-5)\cup[5,\infty)$$

-6 is in the domain, and also all integers smaller than -6.
Hence, the smallest integer not in the domain is:

$$-5$$

Example 1.268

Find the domain of $g(x)=\dfrac{x^2+5x+6}{x^2+5x+6}$

Bogus Solution

One way to look at this is:

$$\frac{x^2+5x+6}{x^2+5x+6}=\frac{1}{1}=1$$

Hence, the domain of the function is all real numbers.

Correct Solution

There is a problem with the reasoning above.
Consider

$$x=-2\Rightarrow x^2+5x+6=(-2)^2+5(-2)+6=4-10+6=0$$

And then the expression becomes

$$\frac{x^2+5x+6}{x^2+5x+6}=\frac{0}{\mathbf{0}}\Rightarrow \boldsymbol{Not\ Defined}$$

In general, a denominator can never take the value zero.
And remember that:

$$\frac{0}{0}\neq 1,\qquad \frac{0}{0}\neq 0,\qquad \frac{0}{0}\,is\ \boldsymbol{Not\ Defined}$$

How do we know the values at which the denominator becomes zero.
This is as usual, by equating the denominator to zero.

$$x^2+5x+6=0\Rightarrow(x+2)(x+3)=0\Rightarrow x\in\{-2,-3\}$$

$$Domain\ is\ all\ real\ numbers\ except-2\ and-3$$

H. Reducible to Quadratics

Example 1.269

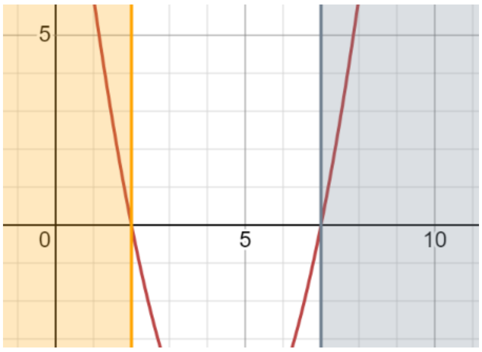
Find the domain of $f(x) = \sqrt{x^4 - 9x^2 + 14}$

Substitute $z = x^2$:

$$f(x) = \sqrt{z^2 - 9z + 14}$$

Solve the corresponding quadratic:

$$z^2 - 9z + 14 = 0 \Rightarrow (z-7)(z-2) = 0 \Rightarrow z \in \{2,7\}$$



$$z \in (-\infty, 2] \cup [7, \infty)$$

Substitute back $z = x^2$ to get:

$$x^2 \in (-\infty, 2] \cup [7, \infty)$$

Now split this into two parts:

$$x^2 \in (-\infty, 2] \Rightarrow x^2 \le 2 \Rightarrow -\sqrt{2} \le x \le \sqrt{2}$$
$$x^2 \in [7, \infty) \Rightarrow x^2 \ge 7 \Rightarrow x \ge \sqrt{7}\ OR\ x \le -\sqrt{7}$$

I. All Real Numbers in Domain

Example 1.270: Perfect Squares

Find the domain of $f(x) = \sqrt{x^2 + 6x + 9}$

Factoring and Cancellation

Notice that the expression inside the square root is a perfect square. Factor it:

$$f(x) = \sqrt{x^2 + 6x + 9} = \sqrt{(x+3)^2} = x + 3$$

And hence

$$Domain = All\ Real\ Numbers$$

Solving the corresponding Quadratic Equation

Solve the equation of the corresponding quadratic:

$$x^2 + 6x + 9 = 0 \Rightarrow (x+3)^2 = 0 \Rightarrow x + 3 = 0 \Rightarrow x = -3$$

And hence the solution should be:

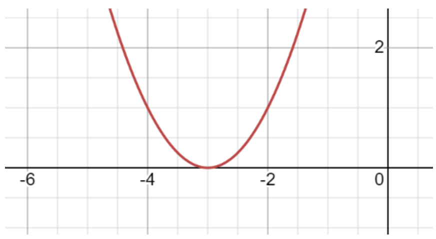
$$D_f = (-\infty, -3] \cup [-3, \infty)$$

But the endpoints of the interval are included and hence the two intervals can be merged to get:

$$(-\infty, \infty)$$

We can get a better picture of what is going on by graphing the quadratic. Note that when we solved it algebraically, we got only one answer to the equation. Technically, we have a single, repeated root, and this happens only when the parabola

$$touches\ but\ does\ not\ cross\ the\ x - axis$$



And, since the leading coefficient is positive, the parabola is an upward parabola, and hence

$$Domain = All\ Real\ Numbers$$

Example 1.271: Complex Solutions

Find the domain of

$$f(x) = \frac{1}{x^2 + 5x + 10}$$

Equate the denominator to zero:

$$x^2 + 5x + 10 = 0$$

Apply the quadratic formula with $a = 1, b = 5, c = 10$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{25 - (4)(1)(10)}}{2} = \frac{-5 \pm \sqrt{-15}}{2}$$

Example 1.272

Consider $f(x) = \frac{1}{ax^2+bx+c}$ *where* $a, b, c \in \mathbb{R}$ and one solution to $ax^2 + bx + c = 0$ is complex. Find all real numbers not in the domain of $f(x)$.

Recall the solutions to quadratics are:

- Both Real
- OR Both Complex

(Specifically, if the solutions are complex, then one solution is the complex conjugate of the other)

Hence, there are no real solutions. Hence,

$$Domain = All\ Real\ Numbers$$

Example 1.273

What is the range of c if

$$f(x) = \frac{3x^2 + 2x + c}{x^2 + 4x + c}, D_f \in (-\infty, \infty)$$

Domain of f is all real numbers
$\Rightarrow$ Denominator $\neq 0$
$\Rightarrow x^2 + 4x + c$ has no real solutions.
$\Rightarrow x^2 + 4x + c$ has only complex solutions
$\Rightarrow$ Discriminant is negative
$\Rightarrow 16 - 4c < 0 \Rightarrow 4 < c \Rightarrow c \in (4, +\infty)$

Concept 1.274

Consider $f(x) = \frac{1}{ax^2+bx+c}$ *where* $a, b, c \in \mathbb{R}$ and the parabola $y = ax^2 + bx + c$ does not intersect the x axis. Find all real numbers not in the domain of $f(x)$.

Since the parabola $y = ax^2 + bx + c$ does not intersect the x axis, there are no places where the expression $ax^2 + bx + c$ is zero.
Hence, there are no real solutions to

$$ax^2 + bx + c = 0$$

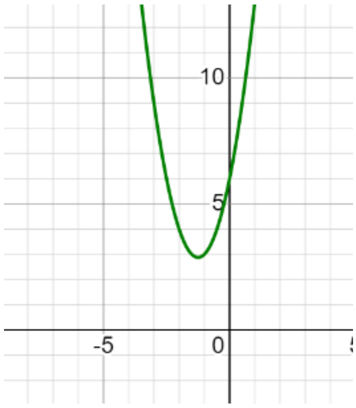
Hence,

$$Domain = All\ Real\ Numbers$$

Example 1.275
Find the domain of $f(x) = \frac{1}{2x^2+5x+6}$.

The denominator is a quadratic. Equate to zero, and attempt to solve:
$$2x^2 + 5x + 6 = 0$$

Factoring doesn't work for "nice" numbers. Switch to the quadratic formula and substitute $a = 2, b = 5, c = 6$ in
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{25 - (4)(2)(6)}}{2 \times 2} = \frac{-5 \pm \sqrt{-23}}{4} \Rightarrow x \in \mathbb{C}$$
The only solutions to the above quadratic are complex numbers.
Hence, there are no real solutions to the equation.
Hence, the denominator is never zero.
Hence, the domain is
$$All\ Real\ Numbers$$



We can also see this in the graph to the left, where quadratic does not intersect the x-axis at all.

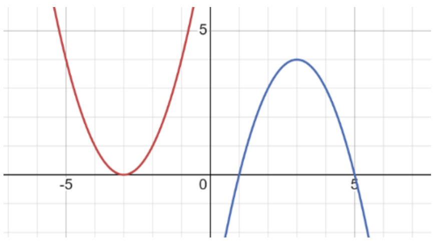
J. Back Calculations

Example 1.276
For what values of c will $f(x) = \frac{3x+5}{2x^2+3x+c}$ be defined for all values of x.

$f(x)$ will be defined for all values of x if the denominator is never zero.
Denominator will never be zero, if the expression in the denominator has no real roots. That is, if the expression has only complex roots.
This will happen when
$$D < 0 \Rightarrow 9 - 4(2)(c) < 0 \Rightarrow 9 - 8c < 0 \Rightarrow 9 < 8c \Rightarrow \frac{9}{8} < c \Rightarrow c > \frac{9}{8}$$

K. Vertex

1.277: Vertex
The lowest part of an upward parabola, and the topmost part of a downward parabola is its vertex.



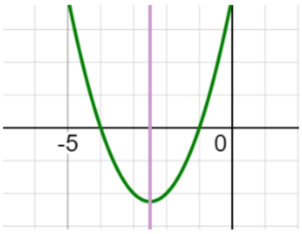
Example 1.278
Identify the coordinates of the vertex of each parabola drawn alongside.
Write your answer in the form
$$(x, y) = (x\ coordinate, y\ coordinate)$$

$$Red\ Parabola\ Vertex = (-3,0)$$
$$Blue\ Parabola\ Vertex = (3,4)$$

1.279: Vertex from Roots

If the roots of a quadratic are α and β, then the

- x coordinate of the vertex is the average of the roots.
- y coordinate of the vertex is $f(x\ coordinate)$.



We can see this from the symmetry of the parabola alongside.

Example 1.280

Find the coordinates of the vertex of $f(x) = x^2 + 5x + 6$

$$x^2 + 5x + 6 = 0 \Rightarrow x \in \{-3, -2\}$$

The x coordinate of the vertex is the

$$Avg\ of\ Roots = \frac{-3-2}{2} = -\frac{5}{2}$$

The y coordinate of the vertex is found by substituting the x coordinate in the function expression:

$$y = f\left(-\frac{5}{2}\right) = \left(-\frac{5}{2}\right)^2 + 5\left(-\frac{5}{2}\right) + 6 = \frac{25}{4} - \frac{25}{2} + 6 = \frac{25}{4} - \frac{50}{4} + \frac{24}{4} = -\frac{1}{4}$$

And hence the coordinates of the vertex are:

$$\left(-\frac{5}{2}, -\frac{1}{4}\right)$$

1.281: Finding the Vertex from Standard Form

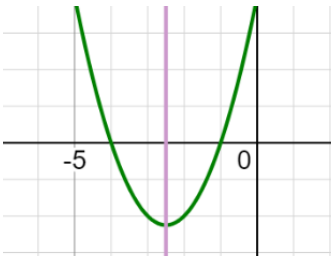
If it is not easy to find the roots, then we use this method below.

$$f(x) = ax^2 + bx + c$$

is a parabola with

$$x\ coordinate\ of\ vertex = \frac{-b}{2a}$$

$$y\ coordinate\ of\ vertex = f\left(-\frac{b}{2a}\right)$$



Due to the symmetry of a parabola, the vertex is the average of the roots. From the quadratic formula, the roots are:

$$\frac{-b-\sqrt{b^2-4ac}}{2a}, \frac{-b+\sqrt{b^2-4ac}}{2a}$$

Hence, the sum of the roots is:

$$\frac{-b-\sqrt{b^2-4ac}}{2a} + \frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$$

And hence the average of the roots is:

$$-\frac{b}{a} \div 2 = -\frac{b}{a} \times \frac{1}{2} = -\frac{b}{2a}$$

And, if we know the x coordinate, we can find the y coordinate by substituting the x coordinate:

$$y\ coordinate = f\left(-\frac{b}{2a}\right)$$

Example 1.282

$$f(x) = 2x^2 + 6x + 3$$

Find the coordinates of the vertex.

$$x\ coordinate = -\frac{b}{2a} = -\frac{6}{2 \times 2} = -\frac{3}{2}$$

$$y\ coordinate = f\left(-\frac{3}{2}\right) = 2\left(-\frac{3}{2}\right)^2 + 6\left(-\frac{3}{2}\right) + 3 = \frac{9}{2} - 9 + 3 = -1.5 = -\frac{3}{2}$$

And hence the coordinates of the vertex are:

$$\left(-\frac{3}{2}, -\frac{3}{2}\right)$$

L. Range

1.283: Range of a Parabola

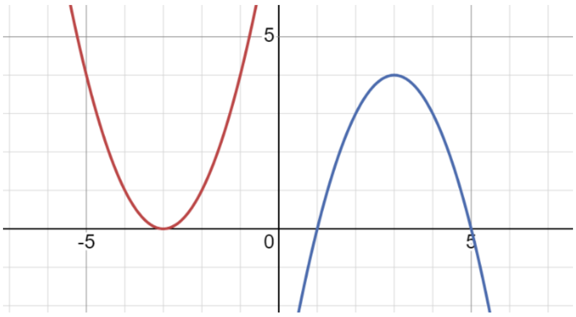
Given parabola $f(x) = ax^2 + bx + c$ with y coordinate of vertex y_1.

For an upward parabola, the vertex is the lowest point. Hence, the range is:

$$[y_1, \infty), a > 0$$

For a downward parabola, the vertex is the highest point. Hence, the range is

$$[-\infty, y_1), a < 0$$



2. Composite, Inverse and Piece-wise Functions

2.1 Composite Functions

A. Applying a Function to another function

If you apply a function to another function, you are nesting the function. This means that, for a function $f(x)$, you take the output of the function, and use it an input for another function.
This is called composition.

2.1: Function Composition

$f(g(x))$ means

- Apply the function g to x
- And then apply the function f to the output given by g.

B. Tabular

Example 2.2

C. Graphical

D. Algebraic

2.3: Composing a function with itself

$$f(f(x))$$

Example 2.4

Given that $f(x) = 3x + 4$, evaluate:

A. $f(f(1))$
B. $f(f(2))$
C. $f(f(x))$
D. $f\left(f\left(\frac{x}{2}\right)\right)$

$$f(f(1)) = f(3 + 4) = f(7) = 7(3) + 4 = 34$$
$$f(f(2)) = f(3(2) + 4) = f(10) = 10(3) + 4 = 34$$
$$f(f(x)) = f(3x + 4) = 3(3x + 4) + 4 = (9x + 12) + 4 = 9x + 16$$
$$f\left(f\left(\frac{x}{2}\right)\right) = f\left(3\left(\frac{x}{2}\right) + 4\right) = f\left(\frac{3x}{2} + 4\right) = 3\left(\frac{3x}{2} + 4\right) + 4 = \frac{9x}{2} + 16$$

Example 2.5

Given that $f(x) = x^2 + 2$, find

A. $f\left(f\left(\frac{1}{2}\right)\right)$
B. $f(f(x))$

In this question, it is easier to do Part A first, and then use the result to calculate Part B:

$$f\big(f(x)\big) = f(x^2+2) = (x^2+2)^2 + 2 = x^4 + 4x^2 + 6$$

$$f\left(f\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^2 + 6 = \frac{1}{16} + 1 + 6 = 7\frac{1}{16}$$

Example 2.6

Given that $f(x) = x^{-1} + \frac{x^{-1}}{1+x^{-1}}$, what is $f(f(-2))$? Express your answer as a common fraction **(MathCounts 2000 Warm-Up 17)**

$$f(-2) = -\frac{1}{2} + \frac{-\frac{1}{2}}{1+\frac{-1}{2}} = -\frac{1}{2} + \frac{-\frac{1}{2}}{\frac{1}{2}} = -\frac{1}{2} - 1 = -\frac{3}{2}$$

$$f\big(f(-2)\big) = f\left(-\frac{3}{2}\right) = -\frac{2}{3} + \frac{-\frac{2}{3}}{1+\frac{-2}{3}} = -\frac{2}{3} + \frac{-\frac{2}{3}}{\frac{1}{3}} = -\frac{2}{3} + \frac{-2}{1} = -\frac{8}{3}$$

2.7: Commutative Property

$$f\big(g(x)\big) \neq g\big(f(x)\big)$$

The order in which you apply functions matters
Function composition is not, in general, commutative.

Example 2.8

$$f(x) = 2x + 4, \qquad g(x) = 3x - 2$$

Find x such that $f\big(g(x)\big) = g\big(f(x)\big)$.

$$f\big(g(x)\big) = f(3x-2) = 2(3x-2) + 4 = 6x - 4 + 4 = 6x$$
$$g\big(f(x)\big) = g(2x+4) = 3(2x+4) - 2 = 6x + 12 - 2 = 6x + 10$$

$$6x = 6x + 10 \Rightarrow x \in \phi \Rightarrow No\ Solutions$$

2.9: Notation: Function Composition

$$f\big(g(x)\big) = (f \circ g)(x)$$

This has the same meaning, but has a different notation.

Example 2.10

Consider the functions:

$$f(x) = x^2, \qquad g(x) = x + 5$$

A. Find $(f \circ g)(x)$, and $(g \circ f)(x)$
B. Find the value of x for which $(f \circ g)(x) = (g \circ f)(x)$

Part A

$$f\big(g(x)\big) = f(x+5) = (x+5)^2 = x^2 + 10x + 25$$
$$g\big(f(x)\big) = g(x^2) = x^2 + 5$$

Part B

$$(f \circ g)(x) = (g \circ f)(x)$$
$$x^2 + 10x + 25 = x^2 + 5$$
$$10x + 25 = 5$$
$$10x = -20$$
$$x = -\frac{1}{2}$$

Example 2.11

If $f(x) = x + 2$ and $g(x) = x^2$, then for what value of x does $f(g(x)) = g(f(x))$? Express your answer as a common fraction. (MathCounts 2014 State Team)

Use the definition to start simplifying $f\big(g(x)\big) = g\big(f(x)\big)$:

$$f(x^2) = g(x+2) \Rightarrow x^2 + 2 = x^2 + 4x + 4 \Rightarrow 4x = -2 \Rightarrow x = -\frac{1}{2}$$

Example 2.12

Let $f(x) = Ax + B$ and $g(x) = Bx + A$, where $A \neq B$. If $f(g(x)) - g(f(x)) = B - A$, what is $A + B$? (AOPS Alcumus, Algebra, Function Composition)

We can't do much with the RHS, so begin with the LHS, and simplify $f\big(g(x)\big) - g\big(f(x)\big)$ by using the function definitions:

$$f(Bx + A) - g(Ax + B)$$
$$= A(Bx + A) + B - [B(Ax + B) + A]$$
$$= ABx + A^2 + B - [ABx + B^2 + A]$$
$$= A^2 + B - B^2 - A$$

And we know that $f(g(x)) - g(f(x)) = B - A$ giving us:

$$A^2 + B - B^2 - A = B - A$$
$$A^2 = B^2 \Rightarrow A = \pm B \Rightarrow A = -B \ (\textbf{\textit{Reject positive}})$$

And we wish to find:

$$A + B = -B + B = 0$$

2.13: Extending Function Composition

$f^n(x)$ means $\underbrace{f\Big(f\big(\ldots f(x)\big)\Big)}_{n\ times}$, where the function is applied to itself n times.

Note that

$$f^n(x) \neq f(x)^n$$

Where

$$f(x)^n = \underbrace{f(x) \cdot f(x) \cdot \ldots \cdot f(x)}_{n\ times}$$

Warning

$$\underbrace{f^n(x)}_{Composition} \neq \underbrace{[f(x)]^2}_{Exponentiation}$$

Example 2.14

Given that $f(x) = 2x$, find $f^5(1)$

$$f(1) = 2$$
$$f^2(1) = 4$$
$$f^3(1) = 8$$
$$f^4(1) = 16$$
$$f^5(1) = 32$$

Example 2.15

Given that $f(x) = x + 3$, find $f^4(1)$

$$f(1) = 4$$
$$f^2(1) = f(4) = 7$$
$$f^3(1) = f\left(f^2(1)\right) = f(7) = 10$$
$$f^4(1) = f\left(f^3(1)\right) = f(10) = 13$$

Example 2.16

Given that $f(x) = \frac{x}{2}$, find $f^5(96)$

$$f^5(96) = f^4(48) = f^3(24) = f^2(12) = f(6) = 3$$

Example 2.17

Given that $f(x) = \frac{x+1}{2}$, find $f^5(33)$

$$f^5(33) = f^4(17) = f^3(9) = f^2(5) = f(3) = 2$$

2.18: Composition of Three Functions

The composition of three functions can be written as:

$$h\left(g\left(f(x)\right)\right) = (h \circ g \circ f)(x)$$

Similarly, this notation can be extended to four or more functions being composed as well.

Example 2.19

Given that:

$$f(x) = \sqrt{x}, \qquad g(x) = x + 1, \qquad h(x) = x^2$$

A. Find $(h \circ g \circ f)(4)$
B. Simplify: $(h \circ g \circ f)(x)$
C. For what integer values of x will $(h \circ g \circ f)(x)$ have integer outputs?

Part A

$$h\left(g\left(f(4)\right)\right) = h\left(g(2)\right) = h(3) = 9$$

Part B

$$h\left(g\big(f(x)\big)\right) = h\left(g(\sqrt{x})\right) = h(\sqrt{x}+1) = (\sqrt{x}+1)^2 = x + 2\sqrt{x} + 1$$

Part C

We want:

$$h\left(g\big(f(x)\big)\right) = x + 2\sqrt{x} + 1$$
$$First\ Term = x\ is\ always\ an\ integer$$
$$Last\ Term = 1\ is\ always\ an\ integer$$
$$Middle\ Term = 2\sqrt{x}\ will\ be\ an\ integer, if\ x\ is\ a\ perfect\ square$$

Hence, $(h \circ g \circ f)(x)$ will have integer outputs when x is a perfect square.

E. Back Calculations

Example 2.20

Suppose $f(x) = x^2$, and $g(x)$ is a polynomial such that $f(g(x)) = 4x^2 + 4x + 1$. What are the possible values of $g(x)$? **(MathCounts 1993 Chapter Countdown)**

$$[g(x)]^2 = 4x^2 + 4x + 1$$
$$g(x) = \pm\sqrt{4x^2 + 4x + 1} = \pm(2x+1)$$

Example 2.21

Find the largest prime divisor of n given that $f(x) = x - 1$ and $f^n(2021) = 1$.

$$f(2021) = 2020$$
$$f^2(2021) = f\big(f(2021)\big) = f(2020) = 2019$$

In general, every time, we apply the function f, the number decreases by 1.
Hence, we can say that:

$$f(x) = x - 1$$
$$f^2(x) = f(x-1) = x - 2$$
$$.$$
$$.$$
$$.$$
$$f^n(x) = x - n$$

$$1 = 2021 - n \Rightarrow n = 2020 = 2^2 \times 5 \times 101 \Rightarrow 101\ is\ largest\ prime\ divisor$$

2.22: Method of Undetermined Coefficients

The method of undetermined coefficients lets us find the coefficients for an expression for which we

- Know the form (for example, linear, quadratic, etc)
- But do not know the values of the coefficients

Example 2.23

Find $g(x)$ given that:

$$f(x) = 2x + 5, \qquad f\big(g(x)\big) = 6x + 3$$

We have not been given the inside function. So, we will need to do back-calculation.

From the question, we can make out that $g(x)$ must also be a linear function.
Hence, let

$$g(x) = ax + b$$

Then,

$$f\big(g(x)\big) = f(ax+b) = 6x+3$$

$$\underbrace{\mathbf{2a}}_{\substack{Variable\\Coefficient}}\ \boldsymbol{x} + \underbrace{\mathbf{2b+5}}_{\substack{Constant\\Term}} = \underbrace{\mathbf{6}}_{\substack{Variable\\Coefficent}}\ \boldsymbol{x} + \underbrace{\mathbf{3}}_{\substack{Constant\\Term}}$$

Equate coefficients:

$$Variable\ Coefficient: 2ax = 6x \Rightarrow 2a = 6 \Rightarrow a = 3$$
$$2b + 5 = 3 \Rightarrow b = -1$$

$$g(x) = 3x - 1$$

Alternate Solution

$$f\big(g(x)\big) = 2g(x) + 5 = 6x + 3$$
$$2g(x) = 6x - 2$$
$$g(x) = 3x - 1$$

Example 2.24

Find $f(x)$ given that:

$$g(x) = 2x + 3, \qquad f\big(g(x)\big) = 7x - 4$$

We need to decide the form of $f(x)$. Since the output is linear (for a linear input), we can conjecture that $f(x)$ is linear.

$$f(x) = ax + b$$
$$f\big(g(x)\big) = 7x - 4$$
$$f(2x+3) = 7x - 4$$
$$a(2x+3) + b = 7x - 4$$
$$2ax + 3a + b = 7x - 4$$

Compare Coefficients:

$$2a = 7 \Rightarrow a = \frac{7}{2}$$
$$3a + b = -4 \Rightarrow b = -4 + 3a = -4 + 3\left(\frac{7}{2}\right) = -\frac{8}{2} + \frac{21}{2} = \frac{13}{2}$$

$$f(x) = ax + b = \frac{7}{2}x + \frac{13}{2}$$

Example 2.25

Find $g(x)$ given that $f(x) = x - 1$ and $g\big(f(x)\big) = x^2 - 2x + 4$.

We determine that $g(x)$ is quadratic.
Let $g(x) = ax^2 + bx + c$

$$g(f(x)) = a(x-1)^2 + b(x-1) + c = ax^2 + (b-2a)x + a - b + c = \underbrace{x^2 - 2x + 4}_{Given}$$

By the method of undetermined coefficients:

$$a = 1$$
$$b - 2a = -2 \Rightarrow b - 2 = -2 \Rightarrow b = 0$$
$$a - b + c = 4 \Rightarrow 1 - 0 + c = 4 \Rightarrow c = -3$$

F. Functions that "Cancel" and "Cycle"

2.26: Functions that "Cancel"

A function cancels if

$$f(f(x)) = x$$

Example 2.27

Let $f(x) = \frac{1}{x}$. Find:

A. $f^2(x)$
B. $f^3(x)$
C. $f^4(x)$

$$f(x) = \frac{1}{x}$$
$$f^2(x) = f(f(x)) = f\left(\frac{1}{x}\right) = x$$
$$f^3(x) = f(f^2(x)) = f(x) = \frac{1}{x}$$
$$f^4(x) = f(f^3(x)) = f\left(\frac{1}{x}\right) = x$$

What is the pattern in the above?

The pattern that when we apply the function to itself, the function repeats, and we get:

$$\underbrace{\frac{1}{x}, x}_{Cycle\ 1}, \underbrace{\frac{1}{x}, x}_{Cycle\ 2}, \frac{1}{x}, \dots$$

In other words, applying the function to itself cancels its previous application. And you get what you started with.

Using the above, determine

A. $f^n(x), n\ is\ odd$
B. $f^n(x), n\ is\ even$

$$For\ odd\ n: f^n(x) = \frac{1}{x}$$
$$For\ even\ n: f^n(x) = x$$

2.28: Functions that "Cycle"

A function cycles if

$$f^n(x) =$$

Example 2.29

If $f(x)=\frac{1}{1-x}$, and $f^n(x)$ means f is applied to itself n times, and $f^n(x)=x$, then find the value of n.

$$f^2(x)=f\left(\frac{1}{1-x}\right)=\frac{1}{1-\frac{1}{1-x}}=\frac{1}{\frac{1-x}{1-x}-\frac{1}{1-x}}=\frac{1}{\frac{-x}{1-x}}=\frac{1-x}{-x}=\frac{1-x}{-x}$$

$$f^3(x)=f\left(\frac{1-x}{-x}\right)=\frac{1}{1-\frac{1-x}{-x}}=\frac{1}{1+\frac{1-x}{x}}=\frac{1}{\frac{x+1-x}{x}}=\frac{1}{\frac{1}{x}}=x$$

Example 2.30

Let f be the function that applies the following rule to an input to get an output:

$$Subtract\ one\ from\ the\ input, and\ then\ divide\ the\ resulting\ number\ by\ the\ input$$

Find:

A. $f(x)$

B. $f\left(f\left(f(x)\right)\right)$

$$f(x)=\frac{x-1}{x}$$

$$f\left(f(x)\right)=f\left(\frac{x-1}{x}\right)=\frac{\left(\frac{x-1}{x}\right)-1}{\frac{x-1}{x}}=\frac{-\frac{1}{x}}{\frac{x-1}{x}}=\frac{-1}{x-1}=\frac{1}{1-x}$$

$$f\left(f\left(f(x)\right)\right)=f\left(\frac{1}{1-x}\right)=\frac{\left(\frac{1}{1-x}\right)-1}{\frac{1}{1-x}}=\frac{\frac{x}{1-x}}{\frac{1}{1-x}}=x$$

Example 2.31

For the reciprocal function, $f(x)=\frac{1}{x}$, find:

A. $f^{2021}(n)$

B. $f^{2022}(n)$

C. $f^n(2021)$

$$\frac{1}{n}$$

$$n$$

This question is not asking for a specific value. It is asking for the general solution applicable to all n. Hence, we divide into two cases, and present it using a piece-wise function:

$$f^n(2021)=\begin{cases}\frac{1}{2021}, n\in Odd\\ 2021, n\in Even\end{cases}$$

G. Decomposing Functions

The idea behind decomposing a function is break it into its constituent parts.

$$\frac{\sin x}{x}=sin$$

2.32: Decomposing Functions

Given a function, $f(x)$, if we can find two functions $h(x)$ and $g(x)$, such that

$$f(x) = h(g(x))$$

Then we have successfully decomposed the function.

Example 2.33

Algebraic

A. $f(x) = \sqrt{2x+4}$
B. $f(x) = \sqrt{x^2 - 5}$
C. $f(x) = (x^3 + 5)^4$

Trigonometric

D. $f(x) = \sin 4x$
E. $f(x) = \sin \pi x$

Exponential and Logarithmic

F. $f(x) = \log 7x$
G. $f(x) = \log ex$
H. $f(x) = e^{2x}$

Mixed

I. $f(x) = e^{\sin x}$
J. $f(x) = e^{\sin 4x}$
K. $f(x) = \sin(\log x)$
L. $f(x) = \log(\sin x)$
M. $f(x) = \sin(\sin x)$
N. $f(x) = \sin^2 x$

Algebraic

$$g(x) = 2x + 4, h(x) = \sqrt{x} \Rightarrow h(g(x)) = h(2x+4) = \sqrt{2x+4}$$
$$g(x) = x^2 - 5, h(x) = \sqrt{x}$$
$$g(x) = x^3 + 5, h(x) = x^4$$

Trigonometric

$$g(x) = 4x, h(x) = \sin x$$
$$g(x) = \pi x, h(x) = \sin x$$

Exponential and Logarithmic

$$g(x) = 7x, h(x) = \log x$$
$$g(x) = ex, h(x) = \log x$$

We can do $f(x) = e^{2x}$ in two ways:

$$g(x) = 2x, h(x) = e^x$$
$$g(x) = e^x, h(x) = x^2$$

Mixed

$$g(x) = \sin x, h(x) = e^x$$

$$g(x) = \log x, h(x) = \sin x$$
$$g(x) = \sin x, h(x) = \log x$$
$$g(x) = \sin x, h(x) = \sin x$$
$$g(x) = \sin x, h(x) = \sin x$$

$$g(x) = 4x, h(x) = \sin x, j(x) = e^x$$

H. Domain and Range

2.34: Composite Functions

For a composite function, $f(g(x))$ we need to first find $g(x)$ *and* then only can we find $f(g(x))$.

- $f(g(x))$ is defined only if the range of $g(x)$ is a subset of the domain of $f(x)$
- The domain of $f(g(x))$ *cannot* be larger than the domain of $g(x)$.

Example 2.35

$$f(x) = \sqrt{x}, \qquad g(x) = 2x + 3$$

A. Is $f(g(x))$ defined?
B. If we wish to define $f(g(x))$, determine its domain and range.

Part A

$$D_f: x \geq 0, \qquad R_g: \mathbb{R}$$

From the above,

$$R_g \textit{ is not a subset of } D_f \Rightarrow f(g(x)) \textit{ is not defined}$$

For instance, consider

$$f(g(-10)) = f(-17) = \sqrt{-17} \Rightarrow \textit{Not Defined}$$

Part B

Any output from $g(x)$ has to be a valid input for $f(x)$ in order for $f(g(x))$ to be defined.
Hence, the range of $g(x)$ must be a subset of the domain of $f(x)$.

$$D_f: x \geq 0 \Rightarrow R_g: y \geq 0 \Rightarrow 2x + 3 \geq 0 \Rightarrow x \geq -\frac{3}{2} \Rightarrow D_{f(g(x))}: x \in \left(-\frac{3}{2}, \infty\right)$$

Example 2.36

$$f(x) = \frac{1}{x^2 + 9x + 20}, \qquad g(x) = 5x + 4$$

A. Find the domain of $f(g(x))$.

Domain of f:

$$x^2 + 9x + 20 = 0 \Rightarrow x \in \{-5, -4\}$$
$$D_f: \mathbb{R} - \{-5, -4\}$$

Range of g:

$$\textit{Original } R_g: \textit{All real numbers}$$

We need to restrict:

$$R_g: \mathbb{R} - \{-5, -4\}$$
$$g(x) = -5 \Rightarrow 5x + 4 = -5 \Rightarrow 5x = -9 \Rightarrow x = -\frac{9}{5}$$
$$g(x) = -4 \Rightarrow 5x + 4 = -4 \Rightarrow 5x = -8 \Rightarrow x = -\frac{8}{5}$$
$$D_{f(g(x))}: \mathbb{R} - \left\{-\frac{9}{5}, -\frac{8}{5}\right\}$$

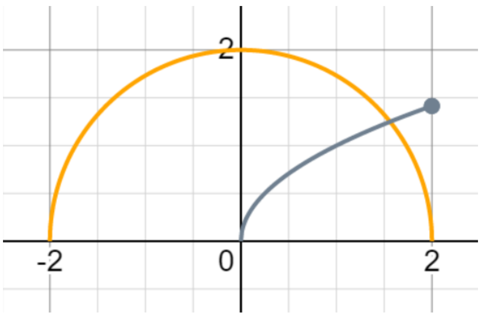
Example 2.37

Consider the functions

$$f(x) = \sqrt{4 - x^2}, \qquad g(x) = \sqrt{x}$$

Which are graphed alongside, with restricted domains.

A. Find the domain and range of $f(x)$
B. Find the domain and range of $g(x)$
C. Find the domain and range of $g(f(x))$
D. Find the domain and range of $f(g(x))$



$$f(g(x)) = \sqrt{4 - (\sqrt{x})^2} = \sqrt{4 - x}$$

Part A: $f(x)$

$$Domain:$$
$$Range:$$

Part B: $g(x)$

$$Domain:$$
$$Range:$$

Part C: $g(f(x))$

Part D: $f(g(x))$

I. Tabular Functions

Example 2.38

A. Find the domain and range of the function given below.
B. What is the number of values in $D_f \cap R_f$?

x	3	7	$\frac{3}{5}$	2	$\frac{1}{2}$	π
$f(x)$	6	1	9	0	$\frac{3}{4}$	e

Part A

$$D_f = \left\{\frac{1}{2}, \frac{3}{5}, 2,3,\pi,7\right\}$$
$$R_f = \left\{0, \frac{3}{4}, 1, e, 6,9\right\}$$

Part B

$$D_f \cap R_f = \phi$$

Example 2.39

Consider the invertible function given by the table below. Let

$$d = Number\ of\ values\ in\ D_f$$
$$r = Number\ of\ values\ in\ D_f$$

Find

$$d + r$$

x	3	3.1	3.2	.	.	.	9
$f(x)$	π	e	$\frac{3}{4}$	.	.	.	12

$$D_f = \{3,3.1,3.2, \dots, 9\} \to \{30,31, \dots, 90\} \to 90 - 30 + 1 = 61\ Values \Rightarrow d = 61$$

Since the function is invertible, the number of distinct outputs is the same as the number of distinct inputs, and hence:

$$r = d = 61$$

Hence, the final answer is:

$$d + r = 61 + 61 = 122$$

Example 2.40

Consider the functions below:

x	1	2	3	4
$g(x)$	5	9	0	2

x	5	9	0	3
$f(x)$	8	9	1	4

A. Find $\left(g(1)\right), f\left(g(2)\right), f\left(g(3)\right), f\left(g(4)\right)$
B. State whether $f\left(g(x)\right)$ is defined.
C. Determine the maximum domain of $g(x)$ such that $f\left(g(x)\right)$ is defined.

Part A

$$f\left(g(1)\right) = f(5) = 8$$
$$f\left(g(2)\right) = f(9) = 9$$
$$f\left(g(3)\right) = f(0) = 1$$
$$f\left(g(4)\right) = f(2) = \boldsymbol{Not\ Defined}$$

Part B

$$f\left(g(4)\right) = Not\ Defined$$

Hence, $f\left(g(x)\right)$ is not defined in general.

Part C

The maximum domain is

$$D_g = \{1,2,3\}$$

Example 2.41

Part A

$$f(x) = \sqrt{x+3} \Rightarrow D_f = [-3, \infty)$$
$$g(x) = -x \Rightarrow D_g = (-\infty, \infty)$$
$$p(x) = \frac{1}{x-3} \Rightarrow (-\infty, 3) \cup (3, \infty)$$
$$h(x) = \sqrt{2-x} \Rightarrow D_h = (-\infty, 2]$$
$$k(x) = x^2 + 1 \Rightarrow D_k = \mathbb{R}$$
$$m(x) = \sqrt{x} \Rightarrow D_m = [0, \infty)$$
$$j(x) = |x| \Rightarrow D_j = \mathbb{R}$$

Part B: $D_{f\circ g}$

$$Condition\ I: D_g = (-\infty, \infty)$$

We need to also ensure that:

$$R_g \subseteq D_f = [-3, \infty)$$
$$g(x)\ has\ range\ [-3, \infty)$$
$$g(x) \geq -3$$
$$-x \geq -3$$
$$x \leq 3$$
$$Condition\ II: D_g = (-\infty, 3\,]$$

Combine both the conditions:

$$D_{f\circ g} = (-\infty, 3\,]$$

Part C: $D_{p\circ h}$

$$R_h = [0, \infty)$$
$$R_h \subseteq D_p = (-\infty, 3) \cup (3, \infty)$$

We cannot have:

$$h(x) = 3$$
$$\sqrt{2-x} = 3$$
$$2 - x = 9$$
$$x = -7$$
$$D_{p\circ h} = (-\infty, -7) \cup (-7, -2)$$

Part D: $D_{k\circ m}$

$$k\left(m(x)\right)$$
$$D_m = [0, \infty)$$
$$R_m \subseteq D_k = \mathbb{R}$$

There is no restriction on the domain of k. Hence, we retain the original domain of $m(x)$.

$$D_{k\circ m} = [0, \infty)$$

Part E: $D_{h\circ j}$

$$h\left(j(x)\right)$$
$$Original\ Domain: D_j = \mathbb{R}$$
$$Original\ Range\ of\ j(x) = [0, \infty)$$

$$Condition: R_j \subseteq D_h = (-\infty, 2]$$

Combine the two:

$$Final\ R_j = [0,2]$$
$$0 \le j(x) \le 2$$
$$0 \le |x| \le 2$$
$$-2 \le x \le 2$$
$$D_{h\circ j} = [-2,2]$$

Part F: $D_{h\circ k}$

$$h\big(k(x)\big)$$
$$Original\ Domain\ \ of\ k(x) = \mathbb{R}$$
$$Original\ Range\ of\ k(x) = [1, \infty)$$
$$R_k \subseteq D_h = (-\infty, 2]$$

Combine the two conditions:

$$R_k = [1,2]$$
$$1 < x^2 + 1 < 2$$
$$0 < x^2 < 1$$
$$-1 < x < 1$$
$$D_{h\circ k} = [-1,1]$$

2.2 Inverse Functions: Tabular

Tabular Inverse Functions
Functional Equations
Self Inverse Functions

A. Logical Inverses

A function provides an output for every valid input. Sometimes, we want to follow the reverse process. Given an output, we want to determine the input that gave us the output.

$$\underbrace{y = f(x)}_{function} \Rightarrow \underbrace{x = f^{-1}(y)}_{Inverse\ Function}$$

$f(x)$		$f^{-1}(x)$	
Mathematical	Verbal	Verbal	Mathematical
$y = f(x) = 2x$	Doubles	Halves	$y = f^{-1}(x) = \frac{x}{2}$
$y = f(x) = x + 4$	Adds four	Subtracts four	$y = f^{-1}(x) = x - 4$
$y = f(x) = x \div \frac{2}{3} = \frac{3x}{2}$	Divides by $\frac{2}{3}$	Multiplies by $\frac{2}{3}$	$y = f^{-1}(x) = \frac{2x}{3}$
$y = f(x) = x - 5$	Subtracts 5	Add five	$y = f^{-1}(x) = x + 5$

A reverse function is called an inverse function, denoted:

$$\underbrace{y = f^{-1}(x)}_{Read:y\ is\ equal\ to\ f\ inverse}$$

An inverse function is also a function, which means it must satisfy all the rules of functions, including having only one output for every input. This creates restrictions on the functions that have inverses.

Example 2.42

Find the inverse of the following functions:

A. $y = f(x) = 4x$
B. $y = f(x) = 3x - 2$
C. $p = f(q) = \frac{q+2}{3}$
D. $y = f(x) = \frac{5}{2}x + 4$

Part A

$f(x)$ multiplies by 4.
Hence, $f^{-1}(x)$ divides by 4.

$$f^{-1}(x) = \frac{x}{4}$$

Part B

$f(x)$

- multiplies by 3
- subtracting 2

Hence, $f^{-1}(x)$

- adding 2
- divide by 3

$$f^{-1}(q) = 3q - 2$$

Part C

$f(q)$

- Adding 2
- Dividing by 3

Hence, $f^{-1}(x)$

- Multiplying by 3
- Subtracting 2

$$f^{-1}(x) = \frac{x+2}{3}$$

Part D

B. Inverses

Example 2.43: Expressions

Find the domain of the inverse of the function given by the ordered pairs below

$$Function: (-3,5)(4,8)(-6,9)(0,-3)$$
$$Inverse\ Function: (5,-3)(8,4)(9,-6)(-3,0)$$

$$Domain\ of\ Inverse\ Function: \{5,8,9,-3\}$$

Example 2.44: Expressions

The function $y = f(x)$ is given by the table below:

x	5	9	12.5	0	7	2π
$f(x)$	8	$\frac{1}{3}$	π	7	0	5

For the expressions below, find the values, if they exist:

A. $f(5)$, and hence, $f^{-1}(8)$
B. $f^{-1}\left(\frac{1}{3}\right)$
C. $f^{-1}(2\pi)$
D. $f^{-1}(\pi)$
E. $2f^{-1}(\pi)$

When finding the value of a function, we look for the output corresponding to the given input:

$$f\left(\underbrace{5}_{Input}\right) = \underbrace{8}_{Output}$$

$$f^{-1}\left(\underbrace{8}_{Output}\right) = \underbrace{5}_{Input}$$

When finding the value of an inverse function, we look for the input corresponding to the given output:

$$f^{-1}\left(\underbrace{\frac{1}{3}}_{Output}\right) = \underbrace{9}_{Input}$$

$$f^{-1}(2\pi) = Does\ Not\ Exist$$
$$f^{-1}(\pi) = 12.5$$
$$2f^{-1}(\pi) = 2 \times 12.5 = 25$$

C. One to One Functions and Invertibility

Recall that a function is a rule that assigns an output for every input that it has.
We will start looking at the reverse process, which is finding the input given the output of the function.

Before we start looking at how to reverse a function, we need to first check for its feasibility. Not all functions be reversed. This is related to the concept of one-to-one functions.

2.45: Test of One-to-One Function

A function is one-to-one, if for every input $\boldsymbol{x}$, there is a distinct output $\boldsymbol{f(x)}$.

OR

A function is one-to-one function, when the outputs are the same, the inputs are also the same.

$$\boldsymbol{f(a) = f(b) \Rightarrow a = b}$$

One-to-one functions are invertible. Functions which are not one-to-one are not invertible.

Example 2.46

Determine if the table below represents a one-to-one function. If not, determine the reasons why.

Candidate for x	2	5	11	0	$\frac{1}{2}$	π	$\frac{1}{2}$
Candidate for $f(x)$	3	7	3	9	12	0	4

Checking for Functions

Before we investigate whether it is a one-to-one function, we need to first see if the relation above represents a function.

$$x = \frac{1}{2} \Rightarrow f(x) = 12, \qquad x = \frac{1}{2} \Rightarrow f(x) = 4 \Rightarrow \boldsymbol{Not\ a\ Function}$$

If one of the columns above is removed, then the relation will be a function. Hence, there are two ways that the table above can be converted into a function.

Option I

Candidate for x	2	5	11	0	$\frac{1}{2}$	π
Candidate for $f(x)$	3	7	3	9	12	0

Option II

Candidate for x	2	5	11	0	π	$\frac{1}{2}$
Candidate for $f(x)$	3	7	3	9	0	4

Checking for One-to-One Functions

If you convert the table above into a function using either of the options above, then we can check that function whether it is one-to-one.

Suppose we consider Option I from above:

Candidate for x	2	5	11	0	$\frac{1}{2}$	π
Candidate for $f(x)$	3	7	3	9	12	0

There are two inputs that give an output of 3.
Hence, the function above is not one-to-one.

D. Finding Inverses

2.47: Graphical Inverse

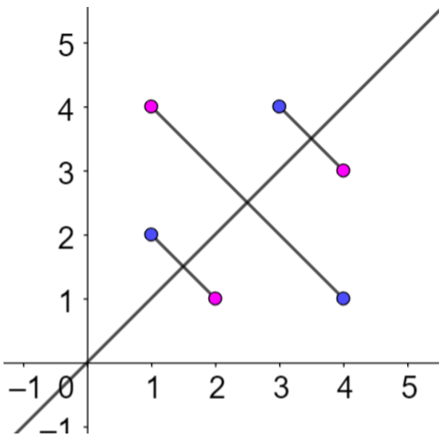
To find the inverse of a function, we swap its x and y values.

Example 2.48: Graphing

A. Graph the function $y = f(x)$ given by the table.
B. Graph its inverse $y = f^{-1}(x)$ on the same set of axes.

x	3	2	4
$y = f(x)$	4	1	2

$f(x)$ is graphed in blue.
$f^{-1}(x)$ is graphed in purple.



E. Equations and Nested Expressions

Example 2.49: Nested Expressions

The function $y = f(x)$ is given by the table below:

x	5	9	12.5	0	7	2π
$f(x)$	8	$\frac{1}{3}$	π	7	0	5

For the expressions below, find the values, if they exist:

$$f^{-1}\left(f^{-1}(7)\right)$$

$$f^{-1}\left(f^{-1}(7)\right) = f^{-1}(0) = 7$$

$$\frac{f^{-1}\left(f^{-1}\left(f^{-1}(8)\right)\right)}{2}$$

$$\frac{f^{-1}\left(f^{-1}\left(f^{-1}(8)\right)\right)}{2}=\frac{f^{-1}\left(f^{-1}(5)\right)}{2}=\frac{f^{-1}(2\pi)}{2}\Rightarrow Does\ not\ exist$$

$$f^{-1}\left(\frac{f^{-1}\left(f^{-1}(8)\right)}{2}\right)$$

$$f^{-1}\left(\frac{f^{-1}\left(f^{-1}(8)\right)}{2}\right)=f^{-1}\left(\frac{f^{-1}(5)}{2}\right)=f^{-1}\left(\frac{2\pi}{2}\right)=f^{-1}(\pi)=12.5$$

Define f^{-n} to be the inverse function of f, applied n times. Find $f^{-2021}(7)+\ f^{-2022}(7)$

We look for a pattern:

$$f^{-1}(7)=0$$
$$f^{-2}(7)=7$$
$$f^{-3}(7)=0\Rightarrow If\ n\ is\ odd, f^{-n}(7)=0$$
$$f^{-4}(7)=7\Rightarrow If\ n\ is\ even, f^{-n}(7)=7$$

$$\underbrace{f^{-2021}(7)}_{2021\ is\ odd}+\ \underbrace{f^{-2022}(7)}_{2022\ is\ even}=0+7=7$$

Challenge 2.50: Equations

The function $y=f(x)$ is given by the table below:

x	5	9	12.5	0	7	2π
$f(x)$	8	$\frac{1}{3}$	π	7	0	5

If $x<y$ and a is an integer, then find the ordered triplets (a,x,y) that satisfy (answer each separately):

A. $f^{-1}(x)=af^{-1}(y)$
B. $af^{-1}(x)=f^{-1}(y)$

Try to solve

$$5=9a\Rightarrow a=\frac{5}{9}\Rightarrow Not\ an\ Integer$$

Part A

Consider $f^{-1}(x)$:

$$Possible\ Inputs\ are\ \left\{8,\frac{1}{3},\pi,7,0,5\right\}\Rightarrow Possible\ Outputs\ are\ \{5,9,12.5,0,7,2\pi\}$$

The only solution that will work is if:

$$f^{-1}(x)=0\Rightarrow x=7$$
$$x<y,\qquad y=8, a=0$$
$$(a,x,y)=(0,7,8)$$

$$f^{-1}(x)=af^{-1}(y)$$

Substitute $x = 7, y = 8$

$$LHS = f^{-1}(7) = 0$$
$$RHS = af^{-1}(8) = 0 \times 5 = 0 = LHS \Rightarrow Valid$$

Part B

F. Self-Inverse Functions

2.3 Inverse Functions: Graphical

A. Horizontal Line Test

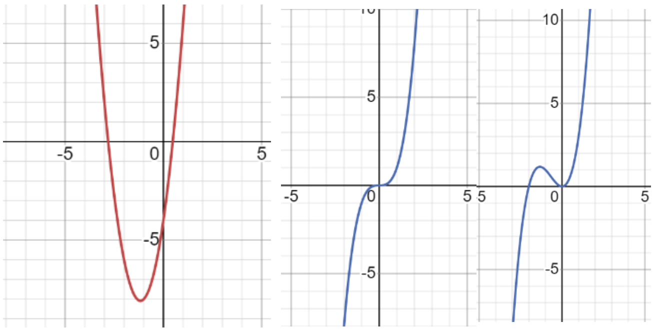
If you draw a horizontal line, the number of places in which it cuts the graph of a function tells the number of inputs that give that output.

2.51: One to One Function: Graphical Test

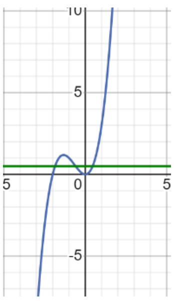
- For a function to be invertible, any horizontal line should cut the graph of a function in a maximum of one place.
- If a horizontal line cuts a graph of a function in more than one place, then, there are two or more inputs associated with an output, and hence the function is not invertible.

Example 2.52

Decide whether the graphs below represent invertible functions.



A. Not Invertible
B. Invertible
C. Not Invertible



B. Finding the Inverse Graphically

When we find the inverse of a function, we

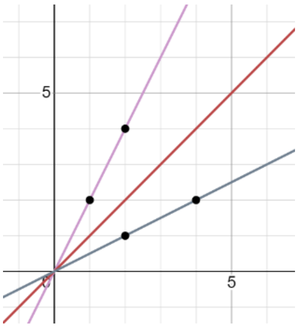
- Swap its input and its output
- In other words, we swap the x and the y values

Graphically we can switch the positions of the x and the y coordinates by reflecting the graph of the function across the line:

$$y = x$$

Example 2.53: Linear Functions

Find the inverse of $y = f(x) = 4x$



$$y = 4x$$
$$x = \frac{y}{4}$$

Swap x and y:

$$y = \frac{x}{4}$$
$$y = f^{-1}(x) = \frac{x}{4}$$

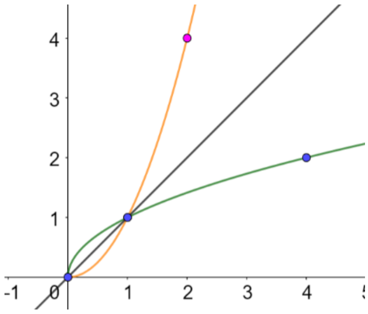
Find the inverse of $y = f(x) = 3x - 2$

Find the inverse of $p = f(q) = \frac{q+2}{3}$

Find the inverse of $y = f(x) = \frac{5}{2}x + 4$

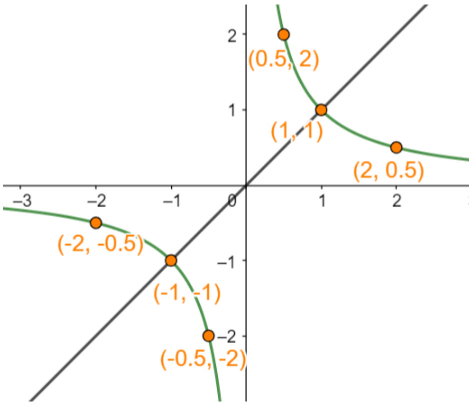
Example 2.54

Draw the graph of $f^{-1}(x)$, given that $f(x) = \sqrt{x}$



Example 2.55

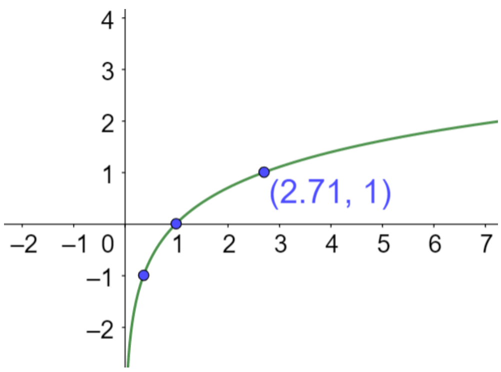
Draw the graph of $f^{-1}(x)$, given that $f(x) = \frac{1}{x}$.



Note that taking the reciprocal of a reciprocal gives back the original function.
We can see this graphically as well:

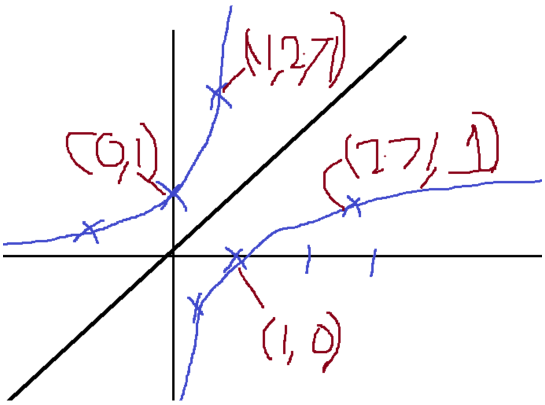
$$(2,0.5) \rightarrow (0.5,2)$$
$$(0.5,2) \rightarrow (2,0.5)$$

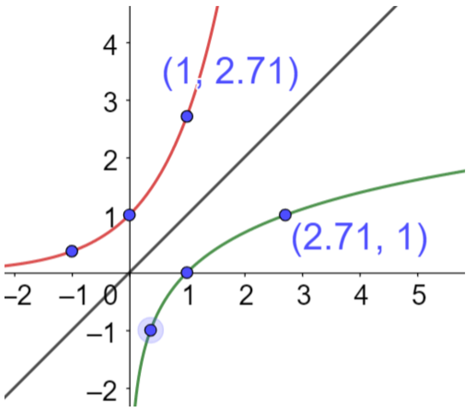
Hence, the function $f(x) = \frac{1}{x}$ is its own inverse.
Such a function is called a self-inverse function.



Example 2.56

The graph of $f(x) = \ln x$ is drawn alongside, and some points are marked. Draw the graph of $f^{-1}(x)$.





C. Odd and Even Functions

If the domain is not restricted, even functions are not invertible.
Even functions can be made invertible by restricting the domain.

It is not necessary that all odd functions are invertible.

Example 2.57

Consider the function

$$f(x) = x^2$$

A. Is the function even?
B. Is it invertible?
C. Can the function be made invertible by restricting its domain? If so, what are the possible options in restricting its domain?

$$f(-x) = x^2 = f(x) \Rightarrow Even\ Function$$

Since the function is even, the function is not invertible.

Yes, the function can be made invertible by restricting its domain.
The possible options are:

$$Option\ I: f(x) = x^2, x \geq 0, x \in \mathbb{R}$$
$$Option\ II: f(x) = x^2, x \leq\ 0, x \in \mathbb{R}$$

Example 2.58

Consider the function

$$f(x) = 0$$

A. Is the function odd?
B. Is it invertible?

D. Restricting the Domain

Before starting this section, you should have done the concept of domain of functions.

Example 2.59

Find $f^{-1}(x)$ for the function below. Note that the definition of the function restricts its domain.

$$y = f(x) = 2x, \qquad x \geq 5$$

And we must apply this restriction to the domain of the inverse function.

$$x > 5 \Rightarrow f(x) = 2x \geq 10$$

Hence, the final answer is:

$$y = f(x) = 2x, \qquad x > 5 \Rightarrow f^{-1}(x) = \frac{x}{2}, x \geq 10$$

Example 2.60

Jared is a re-packer who takes cartons and packs them in smaller boxes (of half the size) for grocery stores. Since he packs them in half the size, the $\underbrace{number\ of\ boxes}_{b}$ is double the $\underbrace{number\ of\ cartons}_{c}$. Jared has a minimum order size of five cartons.

A. Find a function $b = f(c)$ which gives the number of boxes as an output when given the number of cartons.
B. Find a function c= $f^{-1}(b)$ which gives the number of cartons as an output when given the number of boxes as an input.

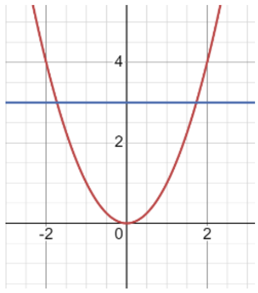
$$b = f(c) = 2c, \qquad c \geq 5, c \in \mathbb{Z}$$

$$c = f^{-1}(b) = \frac{b}{2}, \qquad b \geq 10, \qquad b\ is\ an\ even\ number$$

E. Restricting to make it one-to-one

Example 2.61

A. $y = f(x) = x^2$ (graphed alongside) does not have an inverse. Explain why graphically, as well as algebraically.
B. What is the domain of $f(x)$? Identify the largest subset of the domain of $f(x)$ such that a new function $y = g(x)$ defined on this domain has an inverse. (There are two possible answers).
C. Using the answer to Part B, find the inverses on the restricted domain, and graph them.



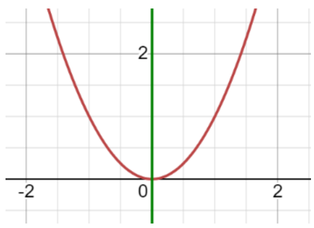
Part A
Graphically
The blue horizontal line shown cuts the graph of the function in more than one place. Hence, there are two inputs associated with one output.
Hence, the function does not have an inverse.
Algebraically

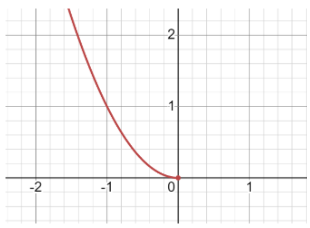
$$y = x^2 \Rightarrow y = \pm\sqrt{x} \Rightarrow 2\ values\ of\ x\ for\ 1\ value\ of\ y$$

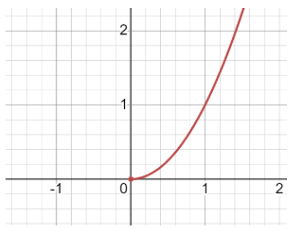
Part B
Domain
The domain of $f(x)$ is

$$x \in \mathbb{R}$$

Line of Symmetry
Note that $y = x^2$ has a line of symmetry at $x = 0$, and this gives us an idea for where we can restrict the domain of the function so that it passes the horizontal line test. We can "divide" this function into two separate functions, each of which individually pass the horizontal line test.







Restrictions on Domain
We can restrict the domain to be either

$$x \geq 0$$

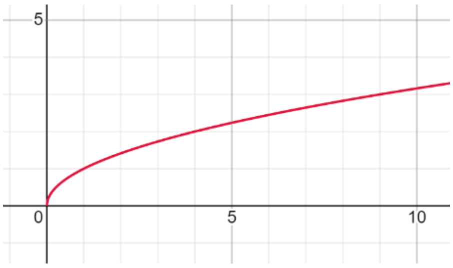
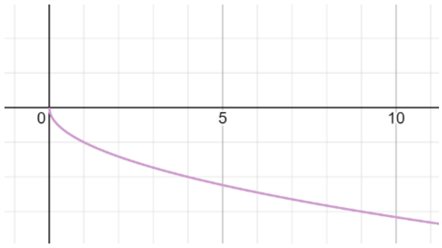
Or

$$x \leq 0$$

And both of these pass the horizontal line test, and are hence invertible.

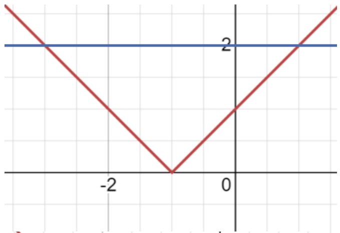
Part C

$$y = \sqrt{x}$$
$$y = -\sqrt{x}$$

Example 2.62

Find the inverse of $y = f(x) = |x+1|$, which is graphed alongside, on a suitable restricted domain. Use the same steps as in the previous example.



Restrictions on Domain

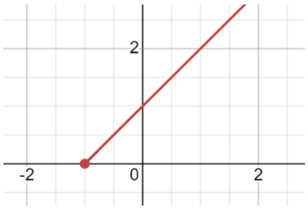
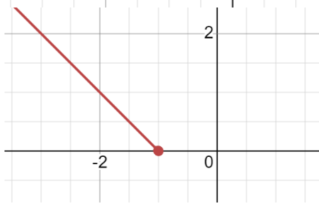
We can restrict the domain to be either

$$x \ge -1 \Rightarrow x \in [-1, \infty)$$

Or

$$x \le -1 \Rightarrow (-\infty, -1]$$

And both of these pass the horizontal line test, and are hence invertible.

Option I: $x \ge -1$

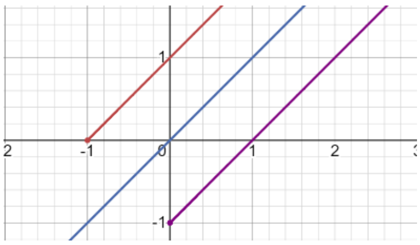
Since we know that $x \ge -1$, we must have:

$$\Rightarrow x + 1 \ge \mathbf{0} \Rightarrow |\boldsymbol{x} + \mathbf{1}| = \boldsymbol{x} + \mathbf{1}$$

And now we can find the inverse function.

$$\underbrace{y = x + 1}_{Function} \rightarrow \underbrace{x = y + 1}_{Swap\ x\ and\ y} \Rightarrow y = x - 1 \ \Rightarrow f^{-1}(x) = x - 1, x \ge 0$$

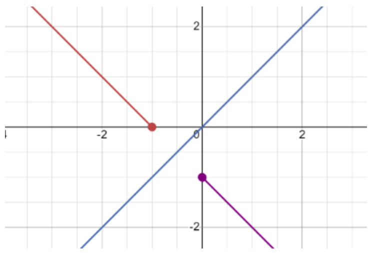
Where we need to put the condition that $x \ge 0$ since the absolute value function cannot have a negative output.



Option II: $x \le -1$

Case II: $x \le -1 \Rightarrow x + 1 \le \mathbf{0} \Rightarrow |\boldsymbol{x} + \mathbf{1}| = -(\boldsymbol{x} + \mathbf{1})$

$$\underbrace{y = -(x + 1)}_{Function} \rightarrow \underbrace{x = -(y + 1)}_{Swap\ x\ and\ y} \Rightarrow y = -x - 1 \ \Rightarrow f^{-1}(x) = -x - 1$$



2.4 Inverse Functions: Algebraic

A. Determine Invertibility Algebraically

We can look at the behaviour of a function to determine whether it is invertible.

Linear Functions

Linear Functions are functions of the form

$$f(x) = mx + c, \qquad m \ne 0$$

And such functions are always invertible.

Even Powers

Taking even powers is not an invertible operation.

Odd Powers

Taking odd powers is an invertible operation.

Example 2.63

Determine whether the following functions are one-to-one/invertible. In other words, given $f(x)$, can we determine the value of x:

A. $f(x) = 3x + 5$
B. $f(x) = x^2$
C. $f(x) = x^3$
D. $f(x) = |x|$
E. $f(x) = \sqrt{(|x|)^2}$

Part A

$$f(a) = f(b) \Rightarrow 3a + 5 = 3b + 5 \Rightarrow 3a = 3b \Rightarrow a = b \Rightarrow One - to - One$$

Part B

$$f(a) = f(b) \Rightarrow a^2 = b^2 \Rightarrow \pm\sqrt{a} = \pm\sqrt{b} \Rightarrow \boldsymbol{Not\ One - to - One}$$

Logically

$$4^2 = (-4)^2 = 16 \Rightarrow Two\ inputs\ with\ same\ output \Rightarrow \boldsymbol{Not\ One - to - One}$$

Part C

$$f(a) = f(b) \Rightarrow a^3 = b^3 \Rightarrow \sqrt[3]{a} = \sqrt[3]{b} \Rightarrow One - to - One$$

Part D

$$|4| = |-4| = 4 \Rightarrow Two\ inputs\ with\ same\ output \Rightarrow \boldsymbol{Not\ One - to - One}$$

Part E

$$f(a) = f(b) \Rightarrow \sqrt{(|a|)^2} = \sqrt{(|b|)^2} \Rightarrow |a| = |b| \Rightarrow a = \pm b \Rightarrow \boldsymbol{Not\ One - to - One}$$

2.64: One to One Function: Algebraic Test

A function is a one-to-one function, if for every input $\boldsymbol{x}$, there is a distinct output $\boldsymbol{f}(\boldsymbol{x})$.

It is important to understand that the definition of one-to-one functions is only applicable for functions. Hence, before checking for a one-to-one function, you need to check whether a function exists in the first place.

Example 2.65

Read the information below and then answer the questions that follow:

$f(x) = x^n$, and $f(a) = f(b)$

If n is an even number, then how many of the following can hold true:

A. $a > b$
B. $a = b$
C. $a < b$

Try some small numbers:

$$a = b = 1 \Rightarrow f(a) = f(b)$$
$$a = 1, b = -1 \Rightarrow a > b \Rightarrow f(a) = f(b)$$
$$a = -1, b = 1 \Rightarrow a < b \Rightarrow f(a) = f(b)$$

Hence, all three options can be true.
Answer is 3.

If n is an odd number, then how many of the following can hold true:

A. $a > b$

B. $a = b$
C. $a < b$

Again, try some numbers:

$$a = b = 1 \Rightarrow f(a) = f(b)$$
$$a = b = -1 \Rightarrow f(a) = f(b)$$

From the above, if the inputs are the same, the outputs are also same.
We also that that this is a one-to-one or invertible function.

Hence, only option B is correct.

$$a > b \Rightarrow a^n > b^n, \qquad Odd\ values\ of\ n$$

B. Finding Inverses

Review:

When we wanted to find the inverse of a function graphically, we swapped the value of x and y. This can be achieved by reflecting the graph of the function across the line

$$y = x$$

Algebraic Method:

The algebraic method is similar to the graphical method. We start by swapping x with y.
Then we solve for the new y.

2.66: Finding an Inverse Function

The method to find an inverse function is:

- Write $y = f(x)$ and then substitute the definition of $f(x)$
- Swap the values of x and y
- Solve for y
- Write the new y as a function of the new x

Example 2.67

A. Find the inverse of $y = f(x) = 4x$
B. Find the inverse of $y = f(x) = 3x - 2$
C. Find the inverse of $p = f(q) = \frac{q+2}{3}$
D. Find the inverse of $y = f(x) = \frac{5}{2}x + 4$
E. The inverse of $f(x) = \sqrt{x+1}$ can be written in the form $(x + a)(x + b)$ where a and b are non-zero constants. Find $a + b$.
F. Given that $f(x) = \frac{3x-8}{x-3}, x \neq 3$, find $f^{-1}(x)$ and comment on anything special.

Part A

Write $y = f(x)$ and then substitute the definition of $f(x)$

$$y = f(x) = 4x \Rightarrow \underbrace{y = 4x}_{\substack{Use\ the \\ definition}}$$

Swap the values of x and y:

$$x = 4y$$

Solve for y:

$$y = \frac{x}{4}$$

Write the new y as a function of the new x:

$$f^{-1}(x)=\frac{x}{4}$$

Part B

$$\underbrace{y=3x-2}_{\text{Use the definition}}\Rightarrow\underbrace{x=3y-2}_{\text{Swap } x \text{ and } y}\Rightarrow\underbrace{y=\frac{x+2}{3}}_{\text{Solve for } y}\Rightarrow f^{-1}(x)=\frac{x+2}{3}$$

Part C

$$\underbrace{p=\frac{q+2}{3}}_{\text{Use the definition}}\Rightarrow\underbrace{q=\frac{p+2}{3}}_{\text{Swap } p \text{ and } q}\Rightarrow\underbrace{p=3q-2}_{\text{Solve for } p}\Rightarrow f^{-1}(q)=3q-2$$

Part D

$$\underbrace{y=\frac{5}{2}x+4}_{\text{Use the definition}}\Rightarrow\underbrace{x=\frac{5}{2}y+4}_{\text{Swap } x \text{ and } y}\Rightarrow\underbrace{\frac{2}{5}(x-4)=y}_{\text{Solve for } y}\Rightarrow f^{-1}(x)=\frac{2}{5}(x-4)$$

Part E

$$y=\sqrt{x+1}\Rightarrow x=\sqrt{y+1}\Rightarrow x^2=y+1\Rightarrow y=x^2-1=(x+1)(x-1)\Rightarrow f^{-1}(x)=(x+1)(x-1)$$

$$a=1,b=-1\Rightarrow a+b=0$$

Part F

Write the function, and swap x and y:

$$y=\frac{3x-8}{x-3}\to x=\frac{3y-8}{y-3}$$

Eliminate fractions, and collate all y terms on the RHS:

$$xy-3x=3y-8\Rightarrow 8-3x=3y-xy$$

Factor y in the RHS, and solve for y:

$$8-3x=y(3-x)\Rightarrow y=\frac{8-3x}{3-x}=\frac{3x-8}{x-3}$$

$$f^{-1}(x)=\frac{3x-8}{x-3}$$

We note that the inverse function is equal to the original function.

$$f^{-1}(x)=f(x)$$

C. Reciprocals versus Inverses

2.68: Reciprocals versus Inverses

Be careful of the difference between reciprocals and inverses.

$$f^{-1}(x)\Rightarrow Inverse\ Function$$

$$[f(x)]^{-1}=\frac{1}{f(x)}\Rightarrow Reciprocal$$

$$[f^{-1}(x)]^{-1}=\frac{1}{f^{-1}(x)}\Rightarrow Reciprocal\ of\ the\ Inverse$$

Example 2.69

If $f(x)=\frac{4x+1}{3}$ what is the value of $[f^{-1}(1)]^{-1}$? (Mathcounts 1999 National Sprint)

First, find the inverse function:

$$y=\frac{4x+1}{3}\Rightarrow \underbrace{x=\frac{4y+1}{3}}_{Swap\ x\ and\ y}\Rightarrow y=\frac{3x-1}{4}\Rightarrow f^{-1}(x)=\frac{3x-1}{4}$$

$$f^{-1}(1)=\frac{3-1}{4}=\frac{2}{4}=\frac{1}{2}\Rightarrow\frac{1}{f^{-1}(1)}=2$$

D. Cancelling Functions

2.70: Functions and Inverses Cancel

$$f^{-1}\big(f(x)\big)=x$$

Let

$$y=f(x)$$

Apply the inverse function of f to both sides:

$$f^{-1}(y)=f^{-1}\big(f(x)\big)=x$$

In other words, the function f and the inverse function f^{-1} cancel each other.

Example 2.71

Given that

A. $f(x)=3x+5$, and $f^{-1}(x)=g(x)$, find $f\big(g(5)\big)$

B. $f(x)=\frac{a}{b}, a=\sqrt[23]{x^{21}}, b=\frac{7x}{4}$ find $z=f^{-1}\left(f\left(\dots f^{-1}\left(f\left\{f^{-1}\left[f\left(\frac{23}{21}\right)\right]\right\}\right)\right)\right)$

Part A

$$f\big(g(5)\big)=f\big(f^{-1}(5)\big)=5$$

Part B

The expression has precisely matched pairs of $f(x)$ and $f^{-1}(x)$. Hence

$$z=\frac{23}{21}$$

Example 2.72

Without finding the inverse function, find b given that:

$$f(x)=2x+4,\qquad f^{-1}(b)=f^{-1}(2)+3$$

Let

$$f^{-1}(2)=a$$

Apply f to both sides:

$$f\big(f^{-1}(2)\big)=f(a)\Rightarrow 2=f(a)\Rightarrow 2=2a+4\Rightarrow a=-1$$

We know that:

$$f^{-1}(b)=f^{-1}(2)+3$$

Substitute $f^{-1}(2)=-1$:

$$f^{-1}(b)=-1+3$$
$$f^{-1}(b)=2$$

Apply to both sides:

$$b = f(2) = 2(2) + 4 = 8$$

E. Proving Inverses

2.73: Proving Inverses

If

$$f^{-1}(f(x)) = x, for\ all\ x\ in\ the\ domain\ of\ f$$

then f^{-1} is the inverse of the function f.

The definition of inverse functions can also be used to show that two functions are inverses of each other.

Example 2.74

A. Find $f^{-1}(x)$ given that $f(x) = 3x - 7$
B. Prove that it is the inverse.

Part A

$$f^{-1}(x) = \frac{x+7}{3}$$

Part B

$$f^{-1}(f(x)) = f^{-1}(3x-7) = \frac{(3x-7)+7}{3} = \frac{3x}{3} = x \Rightarrow \textbf{\textit{Proved}}$$

Example 2.75

A. Find $f^{-1}(x)$ given that $f(x) = \frac{x+6}{3x-4}$
B. Prove that it is the inverse.

Part A

$$f(x) = \frac{x+6}{3x-4} \Rightarrow y = \frac{x+6}{3x-4}$$

Swap x and y and solve for y:

$$x = \frac{y+6}{3y-4} \Rightarrow 3yx - 4x = y + 6 \Rightarrow 3yx - y = 4x + 6 \Rightarrow y(3x-1) = 4x + 6 \Rightarrow y = \frac{4x+6}{3x-1}$$

And this last is our inverse function:

$$f^{-1}(x) = \frac{4x+6}{3x-1}$$

Part B

$$f^{-1}(f(x)) = f^{-1}\left(\frac{x+6}{3x-4}\right) = \frac{4\left(\frac{x+6}{3x-4}\right)+6}{3\left(\frac{x+6}{3x-4}\right)-1} = \frac{\frac{4x+\mathbf{24}+18x-\mathbf{24}}{\mathbf{3x-4}}}{\frac{\mathbf{3x}+18-\mathbf{3x}+4}{\mathbf{3x-4}}} = \frac{22x}{22} = x$$

F. Existence

Example 2.76

G. Even and Odd Functions

Example 2.77

Even Functions
Only possible when domain is $x = 0$

2.78: Inverses of Odd Functions

If f is an odd invertible function, then its inverse is also odd.

Let:

$$y = f(x) \Rightarrow x = f^{-1}(y)$$

Substitute $y = f(x)$ in $f^{-1}(-y)$:

$$f^{-1}\big(-f(x)\big)$$

Since f is odd, substitute $-f(x) = f(-x)$, and note that the function and its inverse cancel:

$$f^{-1}\big(f(-x)\big) = -x$$

Substitute $x = f^{-1}(y)$:

$$-f^{-1}(y)$$

Now, equate the first and the last step to get:

$$f^{-1}(-y) = -f^{-1}(y) \Rightarrow f \textit{ is odd}$$

H. Composite Functions

Example 2.79

Suppose, in $g\big(f(x)\big)$, we know the "inner" function $f(x)$, and also the composite function $g\big(f(x)\big)$. We can make use of the cancelling property of inverse functions to find $g(x)$.

A. Find $g(x)$ given that $f(x) = x - 1$ and $g\big(f(x)\big) = x^2 - 2x + 4$.

$$f^{-1}(x) = x + 1$$

Use the fact that $g\left(f\big(f^{-1}(x)\big)\right) = g(x)$:

$$g\left(f\big(f^{-1}(x)\big)\right) = g\big(f(x+1)\big) = (x+1)^2 - 2(x+1) + 4 = x^2 + 3 = g(x)$$

Check:

$$g\big(f(x)\big) = g(x-1) = (x-1)^2 + 3 = x^2 - 2x + 4 \Rightarrow Works$$

Example 2.80

Given $f(x) = 3x + 7$, and $g(x) = \sqrt{x}$, show that $(g \circ f)^{-1}(x) = (f^{-1} \circ g^{-1})(x)$

$$(g \circ f)(x) = \sqrt{3x + 7}$$

Find the inverses and their composition:

$$f(x) = 3x + 7 \Rightarrow f^{-1}(x) = \frac{x - 7}{3}$$

$$g(x) = \sqrt{x} \Rightarrow g^{-1}(x) = x^2$$

$$RHS = (f^{-1} \circ g^{-1})(x) = f^{-1}(x^2) = \frac{x^2 - 7}{3}$$

Find the inverse of the composition:

$$y = \sqrt{3x + 7} \Rightarrow x = \sqrt{3y + 7} \Rightarrow x^2 = 3y + 7 \Rightarrow \frac{x^2 - 7}{3} = y \Rightarrow LHS = (g \circ f)^{-1}(x) = \frac{x^2 - 7}{3}$$

$$LHS = RHS \Rightarrow Hence\ proved$$

2.81: Shoes and Socks Theorem (Informal)

Let f and g be invertible functions, and their composition be defined. Then:

$$(g \circ f)^{-1}(x) = (f^{-1} \circ g^{-1})(x)$$

$$f = Putting\ on\ socks$$
$$g = Putting\ on\ shoes$$

$$(g \circ f) = Putting\ on\ socks, and\ then\ shoes$$
$$(g \circ f)^{-1} = Removing\ shoes\ and\ socks$$

This would consist of:

$$g^{-1} = Removing\ Shoes$$
$$f^{-1} = Removing\ socks$$

Note that the inverse process proceeds in the reverse order as the original process.

2.82: Inverse of a Composite Function

Let f and g be invertible functions, and their composition be defined. Then the inverse of the composite function $(g \circ f)(x)$, denoted $(g \circ f)^{-1}(x)$ is the composition of the inverse of the functions f and g, in the reverse order of the original composition. That is:

$$(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$$

I. Back Calculations

Example 2.83

If $f(x) = 3x + 7$, then find the value of x for which

$$2f(x) + 3\left(f^{-1}(x)\right) = x - 2$$

$$2(3x + 7) + 3\left(\frac{x - 7}{3}\right) = x - 2$$
$$6x + 14 + x - 7 = x - 2$$
$$7x + 7 = x - 2$$
$$6x = -9$$
$$x = -\frac{3}{2}$$

Example 2.84

A. Graph $y = f(x) = \begin{cases} 2x + k, x \geq 4 \\ 3x - 2, x < 4 \end{cases}$, where k is an unknown constant.

B. Find the smallest value of k such that $f(x)$ has an inverse.

Example 2.85

A. Find all linear functions $f(x)$ such that $f(x) = 4f^{-1}(x) + 27$

B. Find all linear functions $f(x)$ such that $f(x) = cf^{-1}(x) + d$, where c, d are real numbers.

C. Substitute $c = 4, d = 27$ into the answer that you found in part B.

Part A

Substitute $f(x) = ax + b \Rightarrow f^{-1}(x) = \frac{x-b}{a}$ in the given condition:

$$ax+b=4\left(\frac{x-b}{a}\right)+27=\frac{4x-4b+27a}{a}$$
$$a^2x+ab=4x-4b+27a$$

Use the method of undetermined coefficients.
Equating coefficients:

$$a^2=4\Rightarrow a=\pm 2,\qquad ab=27a-4b$$
$$Case\ I: a=2\Rightarrow 2b=54-4b\Rightarrow 6b=54\Rightarrow b=9\Rightarrow f_1(x)=2x+9$$
$$Case\ II: a=-2\Rightarrow -2b=-54-4b\Rightarrow 2b=-54\Rightarrow b=-27\Rightarrow\ f_2(x)=-2x-27$$

Part B

Substitute $f(x)=ax+b, f^{-1}(x)=\frac{x-b}{a}$ in the given condition:

$$ax+b=c\left(\frac{x-b}{a}\right)+d=\frac{cx-cb+da}{a}$$
$$a^2x+ab=cx-cb+da$$

Equating coefficients:

$$a^2=c\Rightarrow a=\pm\sqrt{c},\qquad ab=da-cb$$
$$Case\ I: a=\sqrt{c}\Rightarrow\sqrt{c}b=d\sqrt{c}-cb\Rightarrow b(\sqrt{c}+c)=d\sqrt{c}\Rightarrow b=\frac{d\sqrt{c}}{\sqrt{c}+c}\Rightarrow f_1(x)=\sqrt{c}x+\frac{d\sqrt{c}}{c+\sqrt{c}}$$
$$Case\ II: a=-\sqrt{c}\Rightarrow -\sqrt{c}b=-d\sqrt{c}-cb\Rightarrow b(c-\sqrt{c})=d\sqrt{c}\Rightarrow b=\frac{d\sqrt{c}}{c-\sqrt{c}}\Rightarrow f_2(x)=-\sqrt{c}x+\frac{d\sqrt{c}}{c-\sqrt{c}}$$

And we can combine the two functions in one expression:

$$f(x)=\pm\sqrt{c}x+\frac{d\sqrt{c}}{c\pm\sqrt{c}}$$

Example 2.86

If $f(x)=ax+b$ and $f^{-1}(x)=bx+a$ with a and b real, what is the value of $a+b$? **(AMC 12B 2004/13)**

$$f(x)=ax+b\Rightarrow f^{-1}(x)=\frac{x-b}{a}=\frac{x}{a}-\frac{b}{a}$$
$$f^{-1}(x)=bx+a$$

$$\frac{x}{a}-\frac{b}{a}=bx+a$$

Use the method of undetermined coefficients.
Equating coefficients, we get:

$$x\ term: \frac{1}{a}=b$$
$$Constant\ Term: -\frac{b}{a}=a\Rightarrow b=-a^2$$

Equate the two expressions for b:

$$\frac{1}{a}=-a^2\Rightarrow a^3=-1\Rightarrow a=-1$$

Substitute $a=-1$ in

$$\frac{1}{a}=b\Rightarrow\frac{1}{-1}=b$$

Hence,

$$a+b=-1-1=-2$$

Example 2.87

Suppose that the inverse of the function $f(x) = \sqrt{ax+b}$ is given by $f^{-1}(x) = -(a+2)x^2 + (2b+1)$ for all x satisfying $ax + b \geq 0$. Find $a + b$. (AOPS Alcumus, Intermediate Algebra, Inverse Functions)

$$y = \sqrt{ax+b} \to x = \sqrt{ay+b} \Rightarrow x^2 = ay + b \Rightarrow x^2 - b = ay \Rightarrow y = \frac{1}{a}x^2 - \frac{b}{a}$$

Hence:

$$f^{-1}(x) = \frac{1}{a}x^2 - \frac{b}{a} = -(a+2)x^2 + (2b+1)$$

Use the method of undetermined coefficients.
Comparing coefficients in the x^2 term:

$$\frac{1}{a} = -(a+2) \Rightarrow 1 = -a^2 - 2a \Rightarrow a^2 + 2a + 1 = 0 \Rightarrow (a+1)^2 = 0 \Rightarrow a = -1$$

Compare the constant term, and substitute $a = -1$:

$$-\frac{b}{a} = 2b + 1 \Rightarrow -\frac{b}{-1} = 2b + 1 \Rightarrow b = 2b + 1 \Rightarrow b = -1$$

$$a + b = -1 - 1 = -2$$

Example 2.88

Define the function $f(x) = \frac{a}{1-x}$. If $f(-1) = f^{-1}(4a+1)$, find the product of all possible values of a. (AOPS Alcumus, Intermediate Algebra, Inverse Functions)

Find the inverse of $f(x) = \frac{a}{1-x}$:

$$y = \frac{a}{1-x} \Rightarrow x = \frac{a}{1-y} \Rightarrow 1 - y = \frac{a}{x} \Rightarrow 1 - \frac{a}{x} = y \Rightarrow y = \frac{x-a}{x} \Rightarrow f^{-1}(x) = \frac{x-a}{x}$$

By the given condition:

$$f(-1) = f^{-1}(4a+1) \Rightarrow \frac{a}{1-(-1)} = \frac{4a+1-a}{4a+1} \Rightarrow \frac{a}{2} = \frac{3a+1}{4a+1}$$

Eliminate fractions:

$$4a^2 + a = 6a + 2 \Rightarrow 4a^2 - 5a - 2 = 0$$

By Vieta's formulas, the product of the roots (solutions) for the above quadratic is:

$$P = \frac{c}{a} = \frac{-2}{4} = -\frac{1}{2}$$

2.5 Domain and Range

Example 2.89

Find the domain of $f\big(g(x)\big)$

A. $f(x) = \sqrt{x+4}, g(x) = x^2 - x + 2$

$$f\big(g(x)\big) = f(x^2 - x + 2) = \sqrt{x^2 - x + 6}$$

The range of the inside function has to be equal to, or a subset of the domain of the outside function.

$$x + 4 > 0 \Rightarrow x \geq -4$$

$$g(x) = x^2 - x + 2 \geq -4$$
$$x^2 - x + 6 \geq 0$$
$$(x-3)(x+2) \geq 0$$

$$Roots\ are\ \{-2,3\}$$
$$x < -2\ \ OR\ x > 3$$

A. Basics

2.90: Domain and Range of Inverse Functions

Given a function $f(x)$ with Domain D_f and Range R_f, the inverse function $f^{-1}(x)$ has:

$$Domain: R_f, Range: D_f$$

In other words:

- The domain of an inverse function is the range of the original function.
- The range of an inverse function is the domain of the original function.

$$f(x), Domain: x \geq 3, Range: y \leq 0$$
$$f^{-1}(x): Domain: x \leq 0, y \geq 3$$

Example 2.91

For each function below, find the domain and range. Then, check for invertibility, find the inverse function, and its domain and range.

A. $f(x) = 4x + 3, x > 5$
B. $f(x) = -, x > 5$

Part A

Original Function:

$$Domain\ is\ given: x > 5$$
$$Range: y = 4x + 3 > 4(5) + 3 = 20 + 3 = 23 \Rightarrow Range\ is\ y > 23$$

Invertibility:
$f(x)$ is a linear function, and all linear functions are invertible.
Hence, $f(x)$ is invertible.

Inverse Function:

$$y = 4x + 3 \Rightarrow x = 4y + 3 \Rightarrow y = \frac{x-3}{4} \Rightarrow f^{-1}(x) = \frac{x-3}{4}$$

Domain and Range:
For the inverse function, we need to interchange domain and range:

$$Domain: x > 23$$
$$Range: y > 5$$

Part B

For th

Example 2.92

$$f(x) = \frac{x^2 + 4x + 3}{x^2 - 4x + 3}$$

A. Find the domain of $f(x)$
B. Find the domain such that $f(x)$ is invertible.

Part A
To find the domain, determine the places where the denominator is not defined.

$$x^2 - 4x + 3 = 0 \Rightarrow (x-3)(x-1) = 0 \Rightarrow x \in \{1,3\}$$
$$D_f = (\infty, 1) \cup (1,3) \cup (3, \infty)$$

Part B
We will determine for the number of solutions for

$$\frac{x^2 + 4x + 3}{x^2 - 4x + 3} = k$$

Eliminate fractions, then collate all terms on one side, and write in standard form:

$$x^2 + 4x + 3 = kx^2 - 4kx + 3k$$
$$x^2(k-1) + x(-4k-4) + 3k - 3 = 0$$

The above expression is linear (not quadratic) when $k = 1$:
Substitute $k = 1$:

$$x^2(1-1) + x(-4-4) + 3 - 3 = 0 \Rightarrow -8x = 0 \Rightarrow x = 0$$

When $k \neq 1$, the expression is a quadratic with discriminant:

$$\underbrace{(-4)^2(k+1)^2}_{b^2} - \underbrace{4}_{4}\underbrace{(k-1)}_{a}\underbrace{(3k-3)}_{c}$$

Expand, factor out 4 and simplify

$$16(k^2 + 2k + 1) - 4(3k^2 - 3k - 3k + 3) = 4(k^2 + 14k + 1)$$

$k^2 + 14k + 1$ is an upward parabola. To determine the zeroes, we use the quadratic formula:

$$k = \frac{-14 \pm \sqrt{14^2 - (4)(1)(1)}}{2} = \frac{-14 \pm \sqrt{192}}{2} = \frac{-14 \pm 8\sqrt{3}}{2} = -7 \pm 4\sqrt{3}$$

Hence,

$$\textit{No Real Solutions}: k \in \left(-7 - 4\sqrt{3}, -7 + 4\sqrt{3}\right)$$
$$\textit{Single Solutions}: k \in \left\{-7 - 4\sqrt{3}, -7 + 4\sqrt{3}, 1\right\}$$
$$\textit{Two Solutions}: k \in \left(-\infty, -7 - 4\sqrt{3}\right) \cup \left(-7 + 4\sqrt{3}, \infty\right) - \{1\}$$

Hence, the range of $f(x)$ is the set of values of k which have a solution (either one or two):

$$\left(-\infty, -7 - 4\sqrt{3}\right] \cup \left[-7 + 4\sqrt{3}, \infty\right) \cup \{1\}$$

Part C

$$\left\{-7 - 4\sqrt{3}, -7 + 4\sqrt{3}, 0\right\}$$

B. Restricted Domains / Inverses of Quadratics

2.93: Restricted Domain

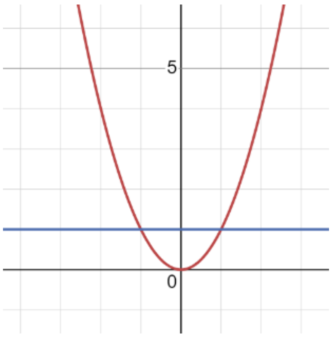
If a function is not invertible, then it may be invertible on a restricted domain.
When looking to restrict the domain, the usual objective is to preserve as much of the original domain as possible.

Example 2.94

Consider the equation $y = x^2$.

A. Is it a function?

B. Is the function invertible?
C. Restrict the domain of the function so that it is invertible. Provide at least three ways to do so. For each way, find the inverse function.



Part A

The graph of $y = x^2$ is a parabola, and passes the vertical line test. Algebraically, there is exactly one value of y associated with any value of x, and hence, $y = x^2$ is a function.

Part B

Any horizontal line $y = c, c > 0$ will intersect the graph at two places. Hence, the function is not invertible.

$$y = x^2 \Rightarrow x = \pm\sqrt{y}$$

For any value of $y > 0$, we get two values of x.
Hence, the function fails the test of invertibility.

Part C

$$Option\ 1: D_f: x \geq 0 \Rightarrow f^{-1}(x) = \sqrt{x}$$
$$Option\ 2: D_f: x \leq 0 \Rightarrow f^{-1}(x) = -\sqrt{x}$$

$$Option\ 3: (-\infty, -a) \cup [0, a], \qquad a > 0$$

Suppose, we take $a = 5$:

Example 2.95: Restricting the Domain of a Quadratic

A. Find the maximum possible domain on which the quadratic $y = x^2 + 5x + 7$ has an inverse, if 0 is a part of the domain.
B. Find the maximum possible domain on which the quadratic $y = -\frac{11}{13}x^2 + \frac{2}{3}x + \frac{3}{4}$ has an inverse, if 0 is a part of the domain.
C. The maximum possible domain on which the quadratic $f(x) = ax^2 + bx + 4$ has an inverse is $\left[-\frac{3}{7}, \infty\right)$. If $a + b$ is 4, then $f(2)$ can be written in the form $p\frac{q}{r}$ where p, q, r are integers and $HCF(q, r) = 1$. Find $p + q + r$.

Part A
Part B
Part C

$$-\frac{b}{2a} = -\frac{3}{7} \Rightarrow b = \frac{6}{7}a$$

Substitute the above

$$a + b = 4 \Rightarrow a + \frac{6}{7}a = 4 \Rightarrow \frac{13}{7}a = 4 \Rightarrow a = \frac{28}{13}$$
$$b = 4 - \frac{28}{13} = \frac{52}{13} - \frac{28}{13} = \frac{24}{13}$$

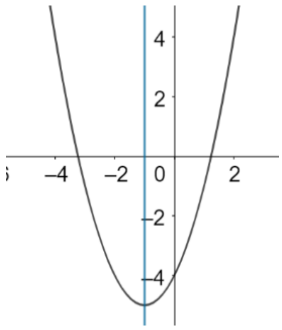
$$f(2) = 4a + b + 4 = 4\left(\frac{28}{13}\right) + \frac{24}{13} + 4 = \frac{136}{13} + 4 = 10\frac{6}{13} + 4 = 14\frac{6}{13}$$
$$p + q + r = 14 + 6 + 13 = 33$$

Example 2.96

Find the inverse of $f(x) = x^2 + 2x - 4$, and find its domain and range.

A. By using the quadratic formula
B. By completing the square

Restricting the domain



$f(x)$ is an upward parabola with a bilateral line of symmetry at its vertex, which is:

$$x = -\frac{b}{2a} = -\frac{2}{2(1)} = -1$$

Hence, to make the function one-to-one/invertible, we restrict the domain to:

$$D_f = [-1, \infty)$$

Finding the range of $f(x)$

Substitute $x = -1$ in $f(x)$ to find the y coordinate of the vertex:

$$y = (-1)^2 + 2(-1) - 4 = 1 - 2 - 4 = -5 \Rightarrow R_f = [-5, \infty)$$

Finding the inverse

Swap x and y in the original function:

$$x = y^2 + 2y - 4$$

Solve it for y by using the quadratic formula. First, write it in standard form:

$$y^2 + 2y - 4 - x = 0$$

Apply $a = 1, b = 2, c = -4 - x$:

$$y = \frac{-2 \pm \sqrt{4 - (4)(1)(-4 - x)}}{2(1)}$$
$$= \frac{-2 \pm 2\sqrt{1 - (-4 - x)}}{2}$$
$$= -1 \pm \sqrt{x + 5}$$

Because we have taken the domain $[-1, \infty)$, we only take the positive square root:

$$y = -1 + \sqrt{x + 5}$$

$$D_{f^{-1}} = R_f = [-5, \infty)$$
$$R_{f-1} = D_f = [-1, \infty)$$

Part B

$$x = y^2 + 2y + 1 - 1 - 4$$
$$x + 5 = (y + 1)^2$$
$$\sqrt{x + 5} = y + 1$$
$$y = \sqrt{x + 5} - 1$$

Example 2.97

Find the inverse of $y = x^2 + x + 1$

$$Vertex_x = -\frac{b}{2a} = -\frac{1}{2(1)} = -\frac{1}{2}$$
$$Restrict\ Domain\ to \left[-\frac{1}{2}, \infty\right)$$

Swap x and y in the original function:

$$x = y^2 + y + 1$$

Complete the square:

$$x = y^2 + 2y + \frac{1}{4} - \frac{1}{4} + 1$$

$$x - \frac{3}{4} = \left(y + \frac{1}{2}\right)^2$$

$$\sqrt{x - \frac{3}{4}} = y + \frac{1}{2}$$

$$y = \sqrt{x - \frac{3}{4}} - \frac{1}{2}$$

Example 2.98

Find the inverse of $f(x) = 3x^2 + 2x + 1$

$$f(x) = 3x^2 + 2x + 1$$
$$Vertex_x = -\frac{b}{2a} = -\frac{2}{2(3)} = -\frac{1}{3}$$
$$Restrict\ Domain\ to\ \left[-\frac{1}{3}, \infty\right)$$

Swap x and y in the original function:

$$x = 3y^2 + 2y + 1$$

Factor out 3 in the first two terms on the RHS:

$$/x = 3\left(y^2 + \frac{2}{3}y + \frac{1}{3}\right)$$

Add and subtract $\frac{1}{9}$ in preparation to complete the square:

$$x = 3\left(y^2 + \frac{2}{3}y + \frac{1}{9} - \frac{1}{9} + \frac{3}{9}\right)$$

Complete the square, move $-\frac{1}{9}$ out of the brackets,

$$x = 3\left(y + \frac{1}{3}\right)^2 - \frac{2}{3}$$

Move the constant to the LHS:

$$x - \frac{2}{3} = 3\left(y + \frac{1}{3}\right)^2$$

Divide both sides by 3:

$$\frac{x}{3} - \frac{2}{9} = \left(y + \frac{1}{3}\right)^2$$

Take square roots both sides:

$$\sqrt{\frac{x}{3} - \frac{2}{9}} = \left(y + \frac{1}{3}\right)$$

Solve for y by subtracting $\frac{1}{3}$ from both sides:

$$y = \sqrt{\frac{x}{3} - \frac{2}{9}} - \frac{1}{3}$$

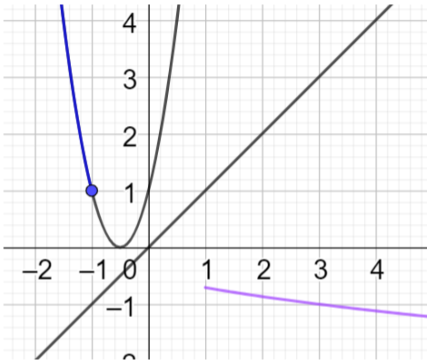
C. Given Domains

Example 2.99: Inverses on given Domains

A function cannot have an inverse on a domain where it does not pass the invertibility test. However, the domain can be further restricted beyond what is required for it to pass the invertibility test.

Find the inverse of $y = 4x^2 + 4x + 1, x < -1$, and state its domain and range.

Domain and Range of $f(x)$



The above function is an upward parabola with x coordinate of vertex

$$x = -\frac{b}{2a} = -\frac{4}{2(4)} = -\frac{1}{2}$$

The domain of the function could have been $x \le -\frac{1}{2}$. However, it has been further restricted to start from left of the vertex.

Range of $f(x)$
Substitute $x = -1$ in $f(x)$

$$f(-1) = 4(-1)^2 + 4(-1) + 1 = 4 - 4 + 1 = 1$$
$$R_f = [1, \infty)$$

Inverse Function

Swap x and y in $f(x)$:

$$x = 4y^2 + 4y + 1 \Rightarrow x = (2y+1)^2$$

Since the domain of $f(x)$ is to the left of the vertex, take the negative square root only:

$$-\sqrt{x} = 2y + 1 \Rightarrow y = \frac{-\sqrt{x} - 1}{2} = -\frac{\sqrt{x} + 1}{2}$$

Recall that the domain of $f^{-1}(x)$ is the range of $f(x)$

$$D_{f^{-1}(x)} = R_f = (1, \infty)$$
$$R_{f^{-1}} = D_f = (-\infty, -1]$$

2.6 Self Inverse Functions

A. Basics

2.100: Self-Inverse Functions

If a function is its own inverse, then such a function is called a self-inverse function. In other words,

$$f(x) = f^{-1}(x)$$

We can apply the function f to both sides of the above equality to get an equivalent version that is sometimes useful:

$$f\big(f(x)\big) = x$$

That is, applying a self-inverse function twice to an input gives us the original input.

Example 2.101

Show that the following functions are self-inverse functions

A. $f(x) = x$
B. $f(x) = \frac{1}{x}$
C. $f(x) = -x$
D. $f(x) = -\frac{1}{ax}, a \neq 0$

Part A

$$f\big(f(x)\big) = f(x) = x \to Self - Inverse$$

The constant function is a (trivial) self-inverse function.

Part B

$$f\big(f(x)\big) = f\left(\frac{1}{x}\right) = x \to Self - Inverse$$

The reciprocal function is a self-inverse function.

Part C

$$f\big(f(x)\big) = f(-x) = x \to Self - Inverse$$

Part D

$$f\big(f(x)\big) = f\left(-\frac{1}{ax}\right) = -\frac{1}{a\left(-\frac{1}{ax}\right)} = 1 \div \frac{1}{x} = x \to Self - Inverse$$

B. Graphical Test

2.102: Reflection across $y = x$

Reflecting a point (x, y) across the line $y = x$ results in the x and y coordinates being interchanged:

$$(x, y) \xLeftrightarrow{reflect\ across\ y=x} (y, x)$$

2.103: Self-Inverse Functions: Graphical Test

Graphically, to find the inverse of a function, we reflect it across the line:

$$\boldsymbol{y = x}$$

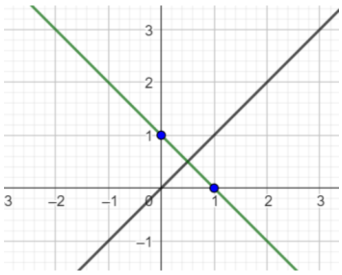
If this reflection does not change the graph, then the function is a self-inverse function.

Example 2.104

Show that the function below is a self-inverse function:

$$y = -x + c, \qquad c \in \mathbb{R}$$

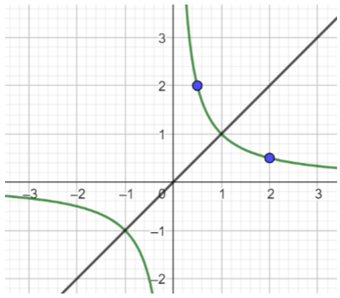
If you draw the graph, it remains the same when reflected across the line $y = x$. Hence, it is a self-inverse function.



Example 2.105

Show that the function below is a self-inverse function:

$$y = \frac{1}{x}$$



The graph of $y = \frac{1}{x}$ is a rectangular hyperbola, which is symmetrical about the line
$$y = x$$

Hence, it is a self-inverse function.

C. Linear Functions

Example 2.106
If f is a linear function that is its own inverse, find f given that $f(3) = 5$.

Since f is a linear function:
$$f(x) = ax + b$$

Since $f(3) = 5$:
$$f(3) = \underbrace{3a + b = 5}_{\textit{Equation I}}$$

Since the function is a self-inverse function:
$$f(5) = \underbrace{5a + b = 3}_{\textit{Equation II}}$$

Solving Equations I and II:
$$2a = -2 \Rightarrow a = -1$$
$$b = 8$$

$$f(x) = -x + 8$$

Example 2.107
Find all functions $f(x) = ax$, such that $f^{-1}(x) = f(x)$.

Definition of Self-Inverse Function
Because the function is a self-inverse function, it must satisfy:
$$f\big(f(x)\big) = x \Rightarrow f(ax) = x \Rightarrow a^2 x = x$$
If $x = 0$:
$$\textit{Any value of } a \textit{ will work}$$
If $x \neq 0$:
$$a^2 = 1 \Rightarrow a = \pm 1$$

Equating the Inverse with the function
$$y = f(x) = ax \Rightarrow y = f^1(x) = \frac{x}{a}$$

$$f^{-1}(x) = f(x) \Rightarrow ax = \frac{x}{a} \Rightarrow a^2 x = x$$
And now the solution can proceed as in the previous method.

Example 2.108

Find all functions $f(x) = ax + b$, such that $f^{-1}(x) = f(x)$.

Inverse of a Generic Linear Function

We want to find all linear functions that are self-inverses. Hence, the function must be of the form:

$$y = ax + b$$

Find the inverse of the function above by swapping x and y, and then solving for y:

$$x = ay + b \Rightarrow y = \frac{x-b}{a} \Rightarrow f^{-1}(x) = \frac{x-b}{a} = \frac{x}{a} - \frac{b}{a} = \frac{1}{a}x - \frac{b}{a}$$

Comparing inverse with the original

By the condition given in the question $f(x) = f^{-1}(x)$:

$$ax + b = \frac{1}{a}x - \frac{b}{a}$$

By the **method of undetermined coefficients**, both sides must have the same coefficients for the same terms because of the equality:

The coefficient of the x terms on the LHS has to equal the coefficient of the x term on the RHS:

$$a = \frac{1}{a} \Rightarrow a^2 = 1 \Rightarrow a = \pm 1$$

The constant term on the LHS has to equal the constant term on the RHS:

$$b = -\frac{b}{a}$$

There are two cases:

Case I: $b = 0$

$$0 = -\frac{0}{a} \Rightarrow 0 = 0, a \neq 0 \Rightarrow Satisfy$$

Case II: $b \neq 0$

We can divide by b both sides get:

$$1 = -\frac{1}{a} \Rightarrow a = -1$$

The final answer is

$$If\ a = -1, then\ any\ value\ of\ b\ is\ valid$$
$$If\ a = 1, then\ b = 0$$

D. $\frac{Linear}{Linear}$ Functions[4]

Example 2.109

Show that $f(x)$ is a self-inverse function.

$$f(x) = \frac{4x-3}{8x-4}$$

[4] This topic is more challenging than the ones prior.

$$f(f(x)) = \frac{4\left(\frac{4x-3}{8x-4}\right)-3}{8\left(\frac{4x-3}{8x-4}\right)-4} = \frac{\frac{4x-3}{8x-4}-3}{\frac{8x-6}{8x-4}-4} =$$

Example 2.110

Find the value of a given that:

$$f(x) = \frac{2x-5}{3x-a}, \qquad f^{-1}(x) = f(x)$$

For a self-inverse function:

$$f(f(x)) = x, \qquad \forall x \in D_f$$

We compose f to find $f(f(x))$:

$$f\left(\frac{2x-5}{3x-a}\right) = \frac{2\left(\frac{2x-5}{3x-a}\right)-5}{3\left(\frac{2x-5}{3x-a}\right)-a} = \frac{\frac{4x-10-15x+5a}{3x-a}}{\frac{6x-15-3ax+a^2}{3x-a}} = \frac{-11x-5(2+a)}{(6-3a)x-15+a^2}$$

Since $f(f(x)) = x$:

$$\frac{-11x-5(2+a)}{(6-3a)x-15+a^2} = x$$

$$-11x-5(2+a) = x^2(6-3a) - x(15-a^2)$$

Example 2.111

Find the condition such that the function below is self-inverse.

$$f(x) = \frac{ax+b}{cx+d}$$

For a self-inverse function:

$$f(f(x)) = x, \qquad \forall x \in D_f$$

We compose f to find $f(f(x))$:

$$f((x)) = f\left(\frac{ax+b}{cx+d}\right) = \frac{a\left(\frac{ax+b}{cx+d}\right)+b}{c\left(\frac{ax+b}{cx+d}\right)+d} = \frac{\frac{a^2x+ab+bcx+bd}{cx+d}}{\frac{acx+bc+dcx+d^2}{cx+d}} = \frac{(a^2+bc)x+b(a+d)}{(a+d)cx+bc+d^2}$$

We want the above composite function to equal x:

$$\frac{(a^2+bc)x+b(a+d)}{(a+d)cx+bc+d^2} = x$$

$$(a^2+bc)x+b(a+d) = (a+d)cx^2+(bc+d^2)x$$

Collating terms on one side:

$$(a+d)cx^2+(d^2-a^2)x-b(a+d) = 0$$

Factor $a+d$:

$$(a+d)[cx^2+(d-a)x-b] = 0$$

Either the first term must equal zero, or the second term must equal zero.

The second term is a quadratic in x, and will vary with x.

Hence, we must have:

$$a + d = 0 \Rightarrow a = -d$$

Example 2.112

If $f(x) = \frac{ax+b}{cx+d}$, $abcd \neq 0$ and $f(f(x)) = x$ for all x in the domain of f, what is the value of $a + d$? **(MathCounts 2014 National Sprint)**

We already know algebraically that

$$a = -d$$

We now establish the same graphically.[5]
$f(x)$ has vertical asymptote when the denominator is zero:

$$cx + d = 0 \Rightarrow x = -\frac{d}{c}$$

Since the numerator and the denominator of the function have the same degree, $f(x)$ has horizontal asymptote which is the ratio of the leading coefficients:

$$y = \frac{a}{c}$$

Since the function is its own inverse, the function must remain same when reflected over the line $y = x$, and so must the asymptotes.
This will happen only when the asymptotes intersect at the line $y = x$.
Hence, we must have

$$y = x \Rightarrow -\frac{d}{c} = \frac{a}{c} \Rightarrow -d = a$$

Example 2.113

Find the domain of f:

$$p^2 + p\left(\frac{\pi\sqrt{\pi} + e}{\sqrt{\pi}}\right) + e\sqrt{\pi} = 0, \qquad f(x) = \frac{ex - \pi}{p^2x - e}, \qquad f^{-1}(x) = f(x)$$

Find the value of p from the quadratic:

$$p^2 + p\left(\pi + \frac{e}{\sqrt{\pi}}\right) + e\sqrt{\pi} = 0$$

$$Sum = \pi + \frac{e}{\sqrt{\pi}}, \qquad Product = e\sqrt{\pi} = (\pi)\left(\frac{e}{\sqrt{\pi}}\right)$$

$$\left(p + \frac{e}{\sqrt{\pi}}\right)(p + \pi) = 0 \Rightarrow p \in \left\{-\frac{e}{\sqrt{\pi}}, -\pi\right\}$$

Find the value of p from the function:

We compose f to find $f(f(x))$:

$$f\left(\frac{ex - \pi}{p^2x - e}\right) = \frac{e\left(\frac{ex - \pi}{p^2x - e}\right) - \pi}{p^2\left(\frac{ex - \pi}{p^2x - e}\right) - e} = \frac{\frac{e^2x - e\pi - \pi p^2x + e\pi}{p^2x - e}}{\frac{p^2ex - \pi p^2 - p^2ex + e^2}{p^2x - e}} = \frac{(e^2 - \pi p^2)x}{e^2 - \pi p^2} = x$$

[5] You can look at the chapter on Graphing for properties and examples that show how to find the asymptote of a function.

With the condition that the denominator cannot be zero:

$$e^2 - \pi p^2 = 0 \Rightarrow \pi p^2 = e^2 \Rightarrow p^2 = \frac{e^2}{\pi} \Rightarrow p = \pm\frac{e}{\sqrt{\pi}}$$

Find the value of p that satisfies both conditions:

$$p \neq -\frac{e}{\sqrt{\pi}} \Rightarrow p = -\pi$$

Find the domain of f:

$$f(x) = f^{-1}(x) = \frac{ex - \pi}{p^2 x - e}$$

$$p^2 x - e = 0 \Rightarrow x = \frac{e}{p^2} = \frac{e}{(-\pi)^2} = \frac{e}{\pi^2}$$

$$D_f = \left(-\infty, \frac{e}{\pi^2}\right) \cup \left(\frac{e}{\pi^2}, \infty\right)$$

2.7 Piece-Wise Functions

A. Definition

So far, we have seen functions defined in terms of tables, or terms of a single formula. But sometimes, a single formula or a single type of behavior is not enough to define a function.
In such a situation, the function can be defined piece-wise.

2.114: Piece-Wise Function
A piece-wise has different behavior depending upon the type of input.

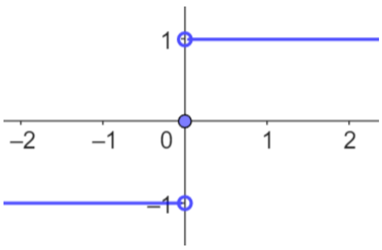
You need to evaluate different cases, based on the definition given.

B. Signum Function

The signum function is of interest to engineers. It gives the sign of a quantity, and is defined as follows:

2.115: Signum Function
The signum function is defined as follows:

$$f(x) = \begin{cases} +1, x \geq 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases}$$



The function has only three outputs:
- $+1$, if the number is positive
- 0, if the number is zero
- -1, if the number is zero

Example 2.116

Given that $f(x) = \begin{cases} +1, x \geq 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases}$, find:

A. $f(3)$
B. $f(-3)$
C. $f(\pi)$
D. $f(-\pi)$

E. $f^2(-\pi)$
F. $f^{2021}(4)$

$$f^2(-\pi)f(-\pi)$$
$$f(3) = 1$$
$$f(-3) = -1$$
$$f(\pi) = 1$$
$$f(-\pi) = -1$$
$$f^2(-\pi) = f(-1) = -1$$
$$f^{2021}(4) = 1$$

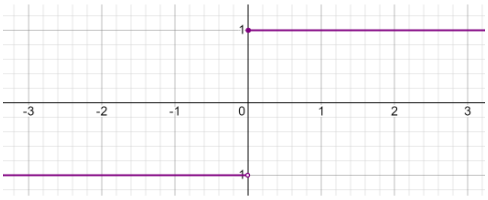
Example 2.117

Given that $f(x) = \begin{cases} +1, x \geq 0 \\ -1, x < 0 \end{cases}$, graph:

A. $f(x)$
B. $f(x) + 2$
C. $f(x) - 3$
D. $f(x + 1)$
E. $f(x - 5)$
F. $2f(x)$
G. $\frac{1}{3}f(x)$

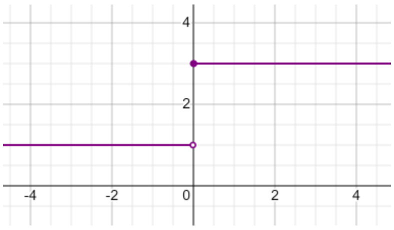
Part A

This is the graph of the signum function.



Part B

Here, the graph moves up by 2 units.



Part C

Here, the graph moves down by 3 units.

Part D

Here, the graph moves left by 1 unit.

Part E

Here, the graph moves right by 5 units.

Part F

The graph is vertically scaled by a factor of 2.

Part G

The graph is vertically scaled by a factor of $\frac{1}{3}$.

C. Absolute Value Function

When learning negative numbers, we encounter absolute value. For example:

$$|5| = 5, \qquad |-5| = 5$$

Example 2.118

Find:

A. $|8|$
B. $|-8|$
C. $\left|-\frac{4}{3}\right|$
D. $\left|\frac{4}{3}\right|$

$$8$$
$$8$$
$$\frac{4}{3}$$
$$\frac{4}{3}$$

Example 2.119

Generalize the principle in the above example to define absolute value as a piece-wise function.

Strategy

We consider two cases:

Case I: The number is positive or zero.

The absolute value function does not change the number. Hence,

$$|x| = x \ if\ x \geq 0 \ (Eg: |4| = 4 = 4$$

Case II: The number is negative

$$|x| = -x \ if\ x < 0 \ \left(Eg: |-4| = -\underbrace{(-4)}_{x} = 4\right)$$

Write a Function

We can combine the two cases above into a single piece-wise function as below:

$$y = f(x) = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

The above notation using a single curly brace to bifurcate the behaviour based on the value of x is standard notation for piece-wise notation.

Equivalent Definition

Equivalently, since

$$0 = -0$$

We could have defined the function as

$$y = f(x) = \begin{cases} x, x > 0 \\ -x, x \leq 0 \end{cases}$$

2.120: Absolute Value Function

The absolute value function is defined as follows:

$$f(x) = |x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

Example 2.121

Given that $f(x) = |x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$, evaluate:

A. $f(7)$
B. $f(-7)$

$$7$$
$$7$$

Example 2.122

Multiple Choice Multiple Correct: Mark all options that are correct.

Simplify: $y = f(\pi) + f(-\pi)$.

A. $\frac{44}{7}$
B. 6.28
C. 2π
D. 0

$$f(\pi) + f(-\pi) = \pi + \pi = 2\pi$$

In everyday work/ school work, we use approximations to π. Two common approximations are

$$\pi \approx \frac{22}{7}, \qquad \pi \approx 3.14$$

However, π is an irrational number and is hence, a

$$non - terminating, non - repeating\ decimal$$

Hence, options A and B are incorrect.

Only Option C is correct.

D. Composition

Example 2.123

If $f(x) = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$, find $f^{2021}(-0.3)$

$$f(x) = 0.3$$
$$f^2(x) = f\big(f(x)\big) = f(0.3) = 0.3$$

After you apply the absolute function once, the number becomes positive.
No matter how many times you apply it, it remains positive.

$$f^{2021}(-0.3) = 0.3$$

Example 2.124

Given the definition $f(x) = |x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$, Vishal made the simplification below:

$$f(y) + f(-y) = y + y = 2y$$

State whether his steps were correct. If his steps were not correct, identify the mistake, and give the correct answer.

The absolute value function is defined piece-wise. To analyze, we need to break it into cases:

Case I: $y > 0$

Take an example: $y = 5 > 0$

$$f(5) + f(-5) = 5 + 5 = 10 \Rightarrow 10 = 2 \times 5$$

The logic above is correct:

$$f(y) + f(-y) = y + y = 2y$$

Case II: $y < 0$

Take an example: $y = -5$

$$f(-5) + f(5) = 5 + 5 = 10 \Rightarrow 10 \neq 2 \times (-5), 10 = -\big(2 \times (-5)\big)$$

The logic above is not correct. The corrected version is:

$$f\left(\underbrace{y}_{-ve}\right) + f\left(\underbrace{-y}_{+ve}\right) = -y - y = -2y$$

Example 2.125

Given the definition $f(x) = |x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$, graph

A. $f(x)$
B. $f(x + \pi)$
C. $f(x - 3)$
D. $f(x) + 4$
E. $f(x) - 5$
F. $7f(x)$
G. $\frac{1}{2}f(x)$

Part A

This is the graph of the absolute value function.

Part B

Here, the graph moves left by π units.

Part C

Here, the graph moves right by 3 units.

Part D

Here, the graph moves up by 4 unit.

Part E

Here, the graph moves down by 5 units.

Part F

The graph is vertically scaled by a factor of 7.

Part G

The graph is vertically scaled by a factor of $\frac{1}{2}$.

E. Collatz Conjecture

Example 2.126

The Collatz conjecture gives the process given below. State this process as a piece-wise function.

A. Start with any positive integer n
B. If n is even, divide it by 2
C. If n is odd, multiply it by 3, and then add 1

$$f(x) = \begin{cases} \frac{x}{2}, & x \text{ is even} \\ 3x+1, & x \text{ is odd} \end{cases}$$

Example 2.127

Let $f(x) = \begin{cases} \frac{x}{2}, & x \text{ is even} \\ 3x+1, & x \text{ is odd} \end{cases}$.

Find:

A. $f^8(5)$
B. $f^{2021}(5)$

$$f^8(5) = f^7(16) = f^6(8) = \underbrace{f^5(4) = f^4(2) = f^3(1)}_{Cycle} = \underbrace{f^2(4) = f(2) = 1}_{Length\ of\ 3}$$

The above has a cycle of three:

	$f^8(5) = 1$	$f^9(5) = 4$	$f^{10}(5) = 2$
No. of Times function is applied	8	9	10
Remainder when divided by 3	$R\left(\frac{8}{3}\right) = 2$	$R\left(\frac{9}{3}\right) = 0$	$R\left(\frac{10}{3}\right) = 1$

$$R\left(\frac{2021}{3}\right) = 2 \Rightarrow f^{2021}(5) = 1$$

Example 2.128

Given $f(x) = \begin{cases} \frac{x}{2}, x \text{ is even} \\ 3x+1, x \text{ is odd} \end{cases}$,

graph

A. $f(x)$
B. $f(x+1)$
C. $f(x-2)$
D. $f(x)+3$
E. $f(x)+4$
F. $3f(x)$
G. $\frac{1}{2}f(x)$

F. Linear Piece-Wise Functions

Till now, we looked at some mathematical functions that are important, in and of themselves. But, piece-wise functions can be defined in whichever way you want. A function does not have to be important to be asked in question.

Example 2.129

Consider the following piece-wise function, both of whose parts are themselves linear functions.

$$f(x)=\begin{cases}-3x+10, & x\geq 1\\ 2x+4, & x<1\end{cases}$$

Find:

A. $f(5)$
B. $f(-3)$
C. $f(f(6))$

$$f(5)=-3(5)+10=-15+10=-5$$
$$f(-3)=2(-3)+4=-6+4=-2$$
$$f(f(6))=f(-3(6)+10)=f(-8)=-16+4=-12$$

Example 2.130

Let $f(x)=\begin{cases}-3x+10, x\geq 1\\ 2x+4, x<1\end{cases}$. Graph

A. $f(x)$
B. $f(x+5)$
C. $f(x-1)$
D. $f(x)+2$
E. $f(x)-4$
F. $2f(x)$
G. $\frac{1}{3}f(x)$

Example 2.131

Given that $f(x)=\begin{cases}-3x+10, x\geq 1\\ 2x+4, x<1\end{cases}$, find $f^{2021}\left(\frac{8}{3}\right)$

$$f^1\left(\frac{8}{3}\right)=-3\left(\frac{8}{3}\right)+10=-8+10=2$$
$$\underbrace{f^2\left(\frac{8}{3}\right)=f(2)=-3(2)+10=4}_{\textbf{2 has Remainder 2 when divided by 3}}$$
$$\underbrace{f^3\left(\frac{8}{3}\right)=f(4)=-3(4)+10=-12+10=-2}_{\textbf{3 has Remainder 0 when divided by 3}}$$
$$\underbrace{f^4(-2)=f(4)=2(-2)+4=-4+4=0}_{\textbf{4 has Remainder 1 when divided by 3}}$$
$$f^5\left(\frac{8}{3}\right)=f(0)=2(0)+4=0+4=4$$

We have a cyclicity of three (established using f^2, f^3 and f^4). f^1 is not part of the cyclicity.
Hence, we want:

$$Rem\left(\frac{2021}{3}\right)=2\Rightarrow f^{2021}\left(\frac{8}{3}\right)=f^2\left(\frac{8}{3}\right)=4$$

Example 2.132

Given that $f(x) = \begin{cases} -3x + 10, x \geq 1 \\ 2x + 4, x < 1 \end{cases}$, find all values of that x that satisfy $f(f(x)) = 1$

One-Stage Equation
We start by solving a simpler question

$$f(x) = 1$$

We do not know which of the above two cases for the piece-wise function was applicable to x. Hence, we must solve both.

Case I: $x \geq 1$

$$-3x + 10 = 1 \Rightarrow -3x = -9 \Rightarrow x = \frac{-9}{-3} = 3 \Rightarrow \textbf{Valid because } \boldsymbol{x \geq 1}$$

Case II: $x < 1$

$$2x + 4 = 1 \Rightarrow 2x = -3 \Rightarrow x = -\frac{3}{2} \Rightarrow \textbf{Valid because } \boldsymbol{x < 1}$$

Part B
We know that

$$f(3) = 1, \qquad f\left(-\frac{3}{2}\right) = 1$$

Hence, to solve the nested function, we know that the value in the inside part must be one of the two values above:

$$f(\boldsymbol{f(x)}) = 1 \Rightarrow \boldsymbol{f(x)} \in \left\{-\frac{3}{2}, 3\right\}$$

Case I: $f(x) = -\frac{3}{2}$

$$-3x + 10 = -\frac{3}{2} \Rightarrow -3x = -\frac{3}{2} - \frac{20}{2} = -\frac{23}{2} \Rightarrow x = \frac{-23}{-6} = \frac{23}{6} \Rightarrow \textbf{Valid}, \boldsymbol{x > 1}$$

$$2x + 4 = -\frac{3}{2} \Rightarrow 2x = -\frac{3}{2} - \frac{8}{2} = -\frac{11}{2} \Rightarrow x = -\frac{11}{4} \Rightarrow \textbf{Valid because } \boldsymbol{x < 1}$$

Case II: $f(x) = 3$

$$-3x + 10 = 3 \Rightarrow -3x = -7 \Rightarrow x = \frac{7}{3} \Rightarrow \textbf{Valid}, \boldsymbol{x > 1}$$

$$2x + 4 = 3 \Rightarrow 2x = -1 \Rightarrow x = -\frac{1}{2} \Rightarrow \textbf{Valid because } \boldsymbol{x < 1}$$

Hence, combine the above to get:

$$f(f(x)) = 1 \Rightarrow f(x) \in \left\{-\frac{3}{2}, 3\right\} \Rightarrow x \in \left\{-\frac{11}{4}, -\frac{1}{2}, \frac{7}{3}, \frac{23}{6}\right\}$$

Example 2.133

A partner in a business has remuneration based on the following conditions:

A. Base Salary: \$3000
B. A percentage of Net Profit (NP):
   a. 0% of Net Profit upto \$20,000
   b. 20% of Net Profit(for NP \$20,000 but below \$40,000)
   c. 25% of Net Profit, and 1% of square of NP (for NP above \$40,000)

Assume that profit will always be greater than or equal to zero and state the remuneration of the partner as a function of the Net Profit of the business.

The behavior of this function changes depending upon the Net Profit.
Hence, a single definition will not do.

Part of Domain			
Starting Point (included)	Ending Point (not included)	Value of function	Type of Function
Negative infinity	Zero	Zero	Not Defined
Zero	20,000	3000	Constant
20,000	40,000	0.2NP + 3000	Linear
40,000	Infinity	$0.01NP^2 + 0.2NP + 3000$	Quadratic

We can state the above in the more usual mathematical form as below

$$y = f(x)\begin{cases} 3{,}000 & \text{if } 0 \le x < 20{,}000 \\ 0.2x + 3000 & \text{if } 20{,}000 \le x < 40{,}000 \\ 0.01x^2 + 0.2x + 3000 & \text{if } 20{,}000 \le x < 40{,}000 \end{cases}$$

G. Inverse Functions

Example 2.134

Find f^{-1} given that $y = f(x) = \begin{cases} x^2, x \ge 0 \\ -x^2, x < 0 \end{cases}$

$$f(x) = \begin{cases} \sqrt{x}, x \ge 0 \\ -\sqrt{-x}, x < 0 \end{cases}$$

Example 2.135: Absolute Value Functions

Find the inverse of $y = x|x|$

$$f(x) = \begin{cases} \sqrt{x}, & x \ge 0 \\ -\sqrt{-x}, & x < 0 \end{cases}$$

3. Graphing Functions

3.1 Horizontal Asymptotes

A. Polynomial Functions: Behaviour at $\pm\infty$

3.1: End Behavior
The end behavior of a function tracks the behavior of the function as it grows very large (either positive, or negative).
In general, there are three things that can happen:
- y value becomes very large (positive)
- y value becomes very negative
- y value approaches a particular value.

3.2: Polynomial of Even Degree
For a polynomial of even degree with positive leading coefficient as:

$$x \to \infty, y = \infty$$
$$x \to -\infty, y = \infty$$

- If the leading coefficient is negative, the results are reversed:

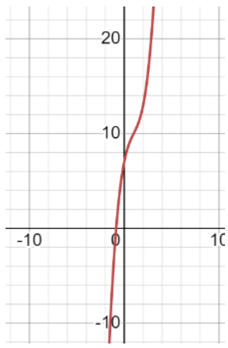
$$\infty \ becomes - \infty$$

3.3: Polynomial of Odd Degree
For a polynomial of even degree with positive leading coefficient as:

$$x \to \infty, y = \infty$$
$$x \to -\infty, y = -\infty$$

Example 3.4
Determine the end behavior of the following polynomials:
A. $y = x^3 - 3x^2 + 5x + 7$
B. $y = -2x^3 + 4x^2 + 15x + 27$
C. $y = 4x^4 - 2x^3 + 4x^2 + 15x + 27$
D. $y = -4x^4 - 2x^3 + 4x^2 + 15x + 27$

Part A



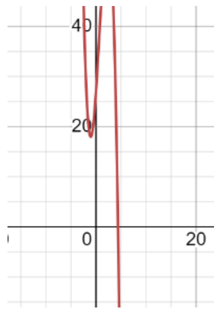
$$Degree = Odd$$
$$Leading\ Coefficient = +ve$$

Behavior as it goes to positive ∞:

$$x = 1\ in\ the\ first\ term: x^3 = 1^3 = 1 = +ve$$
$$x = -1\ in\ the\ first\ term: x^3 = (-1)^3 = -1 = -ve$$

$$As\ x \to \infty, y \to +\infty$$
$$As\ x \to -\infty, y \to -\infty$$

Part B



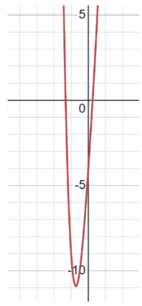
$$Degree = Odd$$

$$Leading\ Coefficient = -ve$$

Behaviour as it goes to positive ∞:

$$x = 1 \text{ in the first term}: -2x^3 = 2 \times 1^3 = -ve$$
$$x = -1 \text{ in the first term}: -2x^3 = 2(-1)^3 = +ve$$
$$As\ x \to \infty, y \to -\infty$$
$$As\ x \to -\infty, y \to +\infty$$

Part C



$$Degree = Even$$
$$Leading\ Coefficient = +ve$$

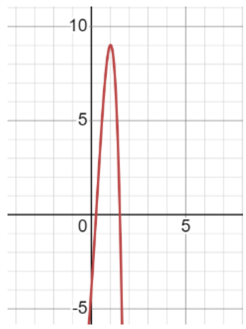
Behaviour as it goes to positive ∞:

$$x = 1 \text{ in the first term}: 4x^3 = 4 \times 1^4 = +ve$$
$$x = -1 \text{ in the first term}: 4x^3 = 4 \times (-1)^4 = +ve$$

$$As\ x \to \infty, y \to \infty$$
$$As\ x \to -\infty, y \to \infty$$

Part D

$$Degree = Even$$
$$Leading\ Coefficient = -ve$$



Behaviour as it goes to positive ∞:

$$x = 1 \text{ in the first term}: -4x^4 = -4(1)^4 = -ve$$
$$x = -1 \text{ in the first term}: -4x^4 = -4(-1)^4 = -4(1) = -ve$$

$$As\ x \to \infty, y \to -\infty$$
$$As\ x \to -\infty, y \to -\infty$$

B. Rational Function: Horizontal Asymptotes

3.5: Rational Function

A rational function is a ratio of two polynomials.

3.6: Horizontal Asymptotes: Meaning

Horizontal asymptotes represent the behavior of a function as it approaches positive infinity, and negative infinity.

3.7: Denominator has higher degree than Numerator

If the degree of the denominator is greater than the degree of the numerator, the asymptote will

$$always\ be\ y = 0$$

Horizontal asymptotes represent the end behaviour of the graph of a function. That is, they represent the values that the function will take at very large values, and very small values.

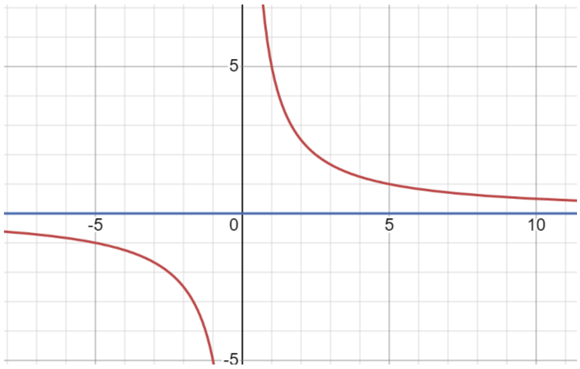
Example 3.8

Find the horizontal asymptotes of:

$$f(x) = \frac{5}{x}$$

End Behaviour

x	$f(x)$	x	$f(x)$
5	1	-5	-1
50	0.1	-50	-0.1
500	0.01	-500	-0.01



Asymptote Rule

$$\underbrace{0}_{\substack{Degree\ of\\ Numerator}} < \underbrace{1}_{\substack{Degree\ of\\ Denominator}}$$

Since the degree of the denominator is greater than the degree of the numerator, the asymptote will be

$$y = 0$$

Example 3.9

Identify the end behavior of the following:

A. $\frac{x}{x^2}$
B. $3x^2$

$$\frac{x}{x^2} \to Horizontal\ Asymptote\ 0$$
$$3x^2\ goes\ to\ \infty\ as\ x \to \pm\infty \Rightarrow No\ Horizontal\ Asymptote$$

3.10: Horizontal Asymptotes of $\frac{n}{x}$

The horizontal asymptote of $\frac{n}{x}$ is 0.

For example, see the graphs of

$$\frac{1}{x} \to graphed\ in\ red$$
$$\frac{2}{x} \to graphed\ in\ blue$$
$$\frac{10}{x} \to graphed\ in\ green$$

No matter how large the numerator, if it is a constant,

$$\frac{n}{x}\ will\ always\ have\ a\ horizontal\ asymptote\ of\ zero$$

Example 3.11

Compare the graph of $\frac{1}{x^2}$ with the graph of $\frac{1}{x}$, and find the horizontal asymptotes of $\frac{1}{x^2}$.

Check the graph:

- Red Graph: $y = \frac{1}{x}$
- Purple Graph: $y = \frac{1}{x^2}$

If x is negative:

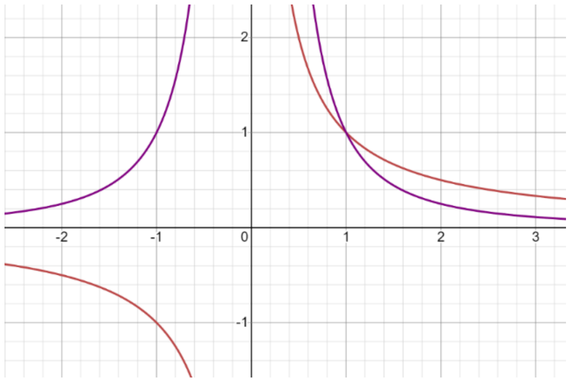
$$x^2 \ is\ positive$$

Hence, while the graph of $\frac{1}{x}$ lies below the $x - axis$ to the left of the origin, the graph of $\frac{1}{x^2}$ lies above the x-axis.

If x is positive:

For large values of x, x^2 is much larger than x, and hence the value of $\frac{1}{x^2}$ is smaller than $\frac{1}{x}$.

This can be seen in the graph since the purple graph is closer to the x-axis, as compared to the red graph.



3.12: $\frac{f(x)}{g(x)}$, *Degree of* $f(x)$ < *Degree of* $g(x)$

Given polynomial functions $f(x)$ and $g(x)$, such that $Degree\ of\ f(x) < Degree\ of\ g(x)$, then the horizontal asymptote of:

$$\frac{f(x)}{g(x)} \ is\ y = 0$$

$$\frac{f(x)}{g(x)} = \frac{a_2x^{n-1} + \cdots}{b_1x^n + b_2x^{n-1} + \cdots}$$

Divide numerator and denominator by the highest power in the denominator, which is x^n:

$$\frac{f(x)}{g(x)} = \frac{\frac{a_2x^{n-1}}{x^n} + \cdots}{\frac{b_1x^n}{x^n} + \frac{b_2x^{n-1}}{x^n} + \cdots} = \frac{\frac{a_2}{x} + \cdots}{b_1 + \frac{b_2}{x} + \cdots}$$

For large values of x, we can ignore all terms except the first term in the numerator, and the first term in the denominator, giving us:

$$\frac{f(x)}{g(x)} = \frac{0}{b_1} = 0$$

Example 3.13

Consider the function:

$$f(x) = \frac{2x + 5}{x^2 + 8x + 2}$$

A. Graph the function and find the horizontal asymptotes.
B. Can the graph of a function cross its horizontal asymptote?

Part A

$$\underbrace{1}_{\substack{Degree\ of \\ Numerator}} < \underbrace{2}_{\substack{Degree\ of \\ Denominator}}$$

Since the degree of the denominator is greater than the degree of the numerator, the asymptote will be

$$y = 0$$

$$f(x) = \frac{2x + 5}{x^2 + 8x + 2}$$

Multiply both numerator and denominator by $\frac{1}{x^2}$:

$$f(x) = \frac{\frac{2x}{x^2} + \frac{5}{x^2}}{\frac{x^2}{x^2} + \frac{8x}{x^2} + \frac{2}{x^2}} = \frac{\frac{2}{x} + \frac{5}{x^2}}{1 + \frac{8}{x} + \frac{2}{x^2}}$$

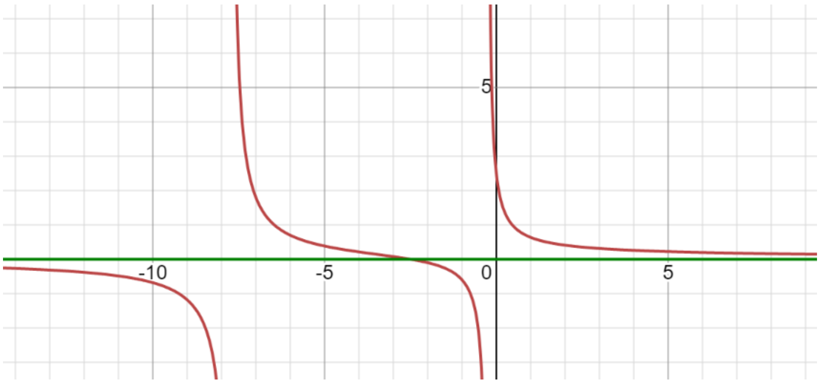
As we saw in the previous example, if n is a constant:

$$\frac{n}{x} \textit{ is close to zero when } x \textit{ is large}$$

$$\frac{n}{x^2} \textit{ is close to zero when } x \textit{ is large}$$

Hence, for large values of x, $f(x)$ is close to[6]:

$$f(x) = \frac{\frac{2}{x} + \frac{5}{x^2}}{1 + \frac{8}{x} + \frac{2}{x^2}} = \frac{0+0}{1+0+0} = \frac{0}{1} = 0$$



Part B

As the graph shows, a function can indeed cross its horizontal asymptote.

Example 3.14

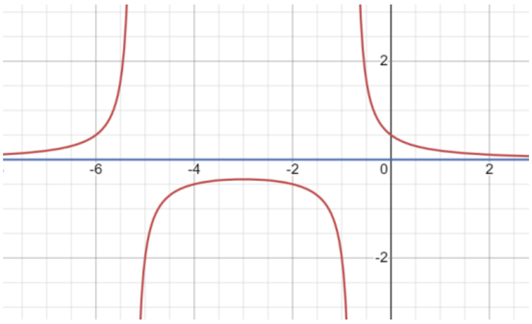
Find the horizontal asymptotes of:

$$f(x) = \frac{2}{x^2 + 6x + 4}$$

$$\underbrace{0}_{\substack{Degree\ of\\ Numerator}} < \underbrace{2}_{\substack{Degree\ of\\ Denominator}}$$

Since the degree of the denominator is less than the degree of the numerator, the asymptote will be

$$y = 0$$



Example 3.15: Denominator has same degree as Numerator[7]

Find the horizontal asymptotes of:

$$f(x) = \frac{3x+5}{2x+7}$$

[6] This is not a formal treatment. We will study this properly when doing Limits in Calculus

[7] In Calculus, this is equivalent to finding the limit as the variable goes towards positive infinity and negative infinity.

To find the equation of the horizontal asymptote, we divide throughout by x:

$$f(x)=\frac{\frac{3x}{x}+\frac{5}{x}}{\frac{2x}{x}+\frac{7}{x}}=\frac{\mathbf{3}+\frac{5}{x}}{\mathbf{2}+\frac{7}{x}}$$

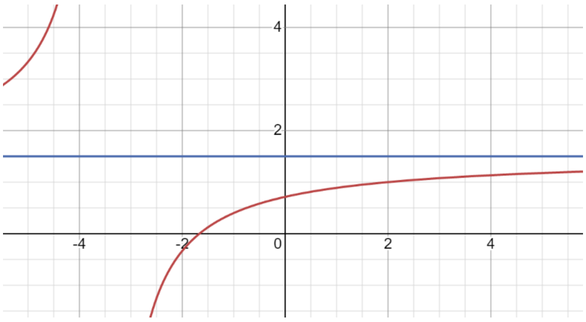
We can ignore

$$\frac{5}{x},\frac{7}{x}$$

Because as x gets very large, these values will become very close to zero.

Hence, the horizontal asymptote is:

$$y=\frac{3}{2}$$



3.16: $\frac{f(x)}{g(x)}$, *Degree of* $f(x)$ = *Degree of* $g(x)$

Given polynomial functions $f(x)$ and $g(x)$, such that $Degree\ of\ f(x)=Degree\ of\ g(x)$, then the horizontal asymptote of

$$\frac{f(x)}{g(x)}\ is\ \frac{a}{b}$$

Where:

$$a=leading\ coefficent\ of\ f(x)$$
$$b=leading\ coefficient\ of\ g(x)$$

$$\frac{f(x)}{g(x)}=\frac{a_1x^n+a_2x^{n-1}+\cdots}{b_1x^n+b_2x^{n-1}+\cdots}$$

Divide numerator and denominator by the highest power in the expression, which is x^n:

$$\frac{f(x)}{g(x)}=\frac{\frac{a_1x^n}{x^n}+\frac{a_2x^{n-1}}{x^n}+\cdots}{\frac{b_1x^n}{x^n}+\frac{b_2x^{n-1}}{x^n}+\cdots}=\frac{a_1+\frac{a_2}{x}+\cdots}{b_1+\frac{b_2}{x}+\cdots}$$

For large values of x, we can ignore all terms except the first term in the numerator, and the first term in the denominator, giving us:

$$\frac{f(x)}{g(x)}=\frac{a_1}{b_1}$$

Example 3.17

Find the horizontal asymptotes of:

$$f(x)=\frac{5x+3}{8x+4}$$

Ratio of leading coefficients:

$$=\frac{5}{8}$$

Example 3.18: Denominator has lower degree than Numerator

Find the horizontal asymptotes, if any, of

$$f(x)=\frac{x^2+5x+7}{x+2}$$

This case will not give any horizontal asymptotes. It will give oblique/slant asymptotes, which are considered in the next section.

$$f(x) = \frac{\frac{x^2}{x^2} + \frac{5x}{x^2} + \frac{7}{x^2}}{\frac{x}{x^2} + \frac{2}{x^2}}$$

As the value of x becomes large, all terms expect the first term in the numerator becomes zero:

$$f(x) = \frac{1+0+0}{0+0} = \frac{1}{0}$$

(Note that we cannot divide by zero, so this is technically incorrect. Rather this is supposed to indicate that we are dividing a number by a very small quantity, which gives us a very large answer).

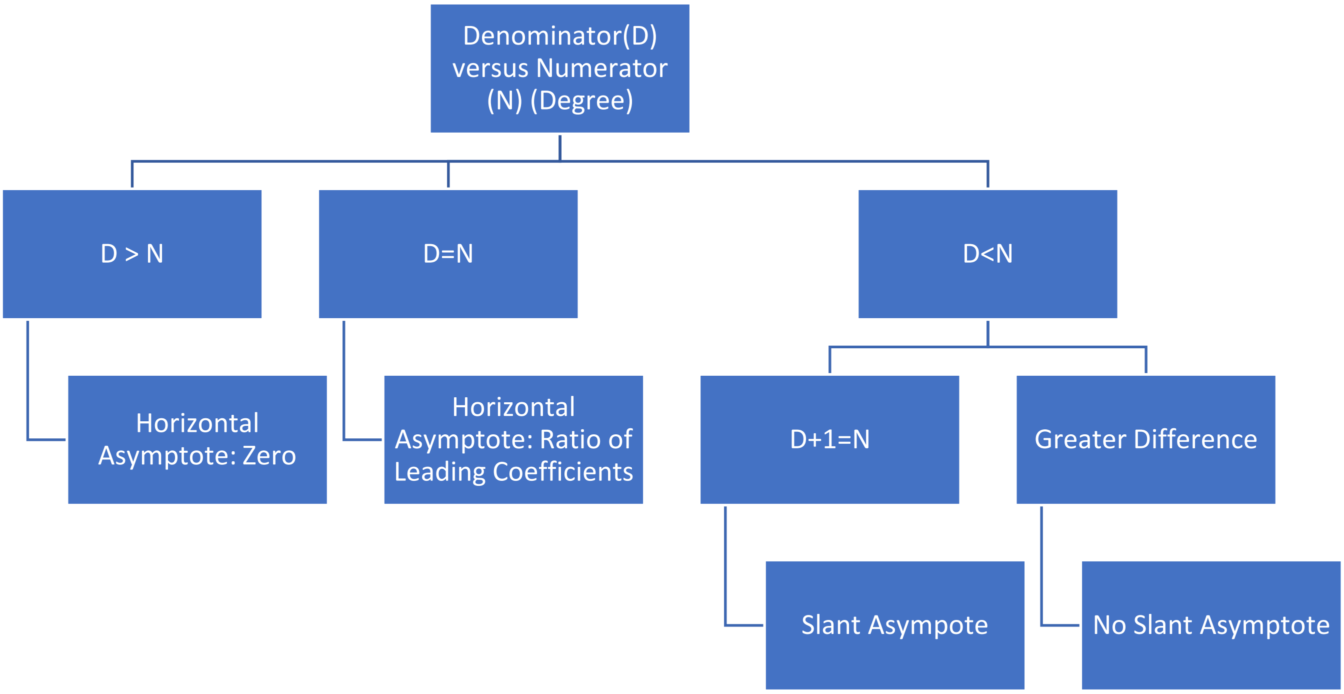
3.19: $\frac{f(x)}{g(x)}$, $\boldsymbol{Degree\ of\ f(x) > Degree\ of\ g(x)}$

Given polynomial functions $f(x)$ and $g(x)$, such that $Degree\ of\ f(x) > Degree\ of\ g(x)$, then

$$\frac{f(x)}{g(x)}$$

Has no Horizontal asymptote

3.20: Summary



When finding the horizontal asymptote for a function, we need to compare the degree of the numerator with the degree of the denominator. There are three possible cases:

Denominator(D) has higher degree than Numerator(N)

As the value of x approaches ∞ or $-\infty$, the overall expression becomes closer and closer to zero.

Hence, the horizontal asymptote in this case is always zero.

Denominator(D) has same degree as Numerator(N)

As the value of x approaches ∞ or $-\infty$, the overall expression is controlled by the leading coefficients. Since both have the same degree,

the value is determined by the ratio of the leading coefficients.

Denominator(D) has lower degree compared to Numerator(N)

As the value of x approaches ∞ or $-\infty$, the overall expression becomes does not become close to any single value. It can potentially become ∞ or $-\infty$.

If the degree of the denominator is one less than the degree of the denominator, than the function will have a slant asymptote. (A slant asymptote is also called an oblique asymptote).

3.2 Vertical Asymptotes

A. Vertical Asymptotes and Intercepts

Asymptotes are lines which a graph approaches but does not touch. They represent one way to understand the behavior of a function and its graph.

3.21: Vertical Asymptotes

The vertical asymptotes of a function $\frac{f(x)}{g(x)}$ are given by the zeros of $g(x)$

In other words, if

$$g(x) = 0 \Rightarrow x = \{x_1, x_2, \dots, x_n\}$$

Then the vertical asymptotes are

$$x = x_1, x = x_2, \dots, x = x_n$$

Vertical asymptotes occur when the denominator is not defined. Hence, to find the vertical asymptotes

- equate the denominator to zero
- The solution to the equations are the vertical asymptotes.

Example 3.22

Evaluate $f(x) = \frac{1}{x}$ for values of x very near to 0.

Negative Values		Positive Values	
x	$f(x) = \frac{1}{x}$	x	$f(x) = \frac{1}{x}$
-0.1	$-\frac{1}{0.1} = -\frac{1}{\frac{1}{10}} = -1 \times 10 = -10$	0.1	$\frac{1}{0.1} = \frac{1}{\frac{1}{10}} = 1 \times 10 = 10$
-0.01	$-\frac{1}{0.01} = -\frac{1}{\frac{1}{100}} = -1 \times 100 = -100$	0.01	$\frac{1}{0.01} = \frac{1}{\frac{1}{100}} = 1 \times 100 = 100$
-0.001	$-\frac{1}{0.01} = -\frac{1}{\frac{1}{100}} = -1 \times 100 = -100$	0.001	$\frac{1}{0.01} = \frac{1}{\frac{1}{100}} = 1 \times 100 = 100$
-0.0001	$-\frac{1}{0.001} = -\frac{1}{\frac{1}{1000}} = -1 \times 1000 = -1000$	0.0001	$\frac{1}{0.001} = \frac{1}{\frac{1}{1000}} = 1 \times 1000 = 1000$

Example 3.23: Finding Asymptotes

Find all vertical asymptotes for the following graphs.

A. $f(x) = \frac{1}{x}$

B. $f(x) = \frac{4}{x+2}$

C. $f(x) = \frac{3}{3x+\frac{1}{2}}$

D. $f(x) = \frac{3}{\frac{5}{4}x-\frac{2}{7}}$

E. $f(x) = \frac{2}{x^2+5x+6}$

F. $f(x) = \frac{6}{x^2+10x+21}$

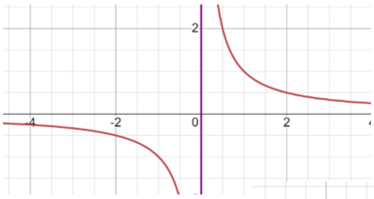
G. $f(x) = \frac{2}{x^2+9x+14}$

H. $f(x) = \frac{2}{3x^3+27x^2+60x}$

Part A

Equate the denominator to zero:

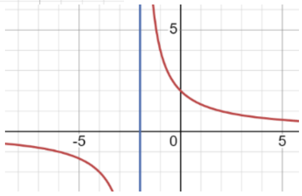
$$x = 0 \Rightarrow Vertical\ Asympote\ at\ x = 0$$



Part B

Equate the denominator to zero:

$$x + 2 = 0 \Rightarrow x = -2 \Rightarrow Vertical\ Asympote\ at\ x = -2$$

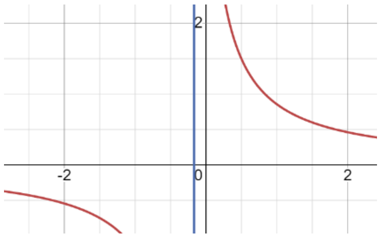


Part C

Equate the denominator to zero:

$$3x + \frac{1}{2} = 0 \Rightarrow 6x + 1 = 0 \Rightarrow 6x = -1$$

$$x = -\frac{1}{6} \Rightarrow Vertical\ Asympote\ at\ x = -\frac{1}{6}$$

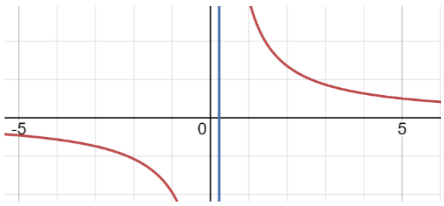


Part D

Equate the denominator to zero:

$$\frac{5}{4}x - \frac{2}{7} = 0 \Rightarrow x = \frac{8}{35}$$

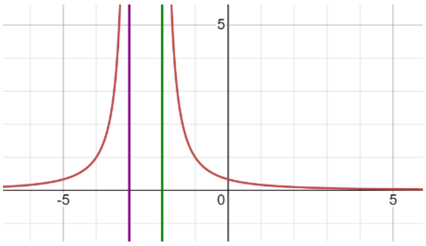
$$Vertical\ Asympote\ at\ x = \frac{8}{35}$$



Part E

Equate the denominator to zero:

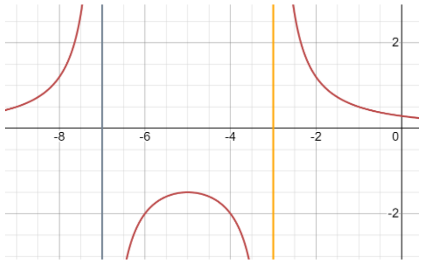
$$x^2 + 5x + 6 = 0$$
$$x^2 + 2x + 3x + 6 = 0$$
$$x(x + 2) + 3(x + 2) = 0$$
$$(x + 3)(x + 2) = 0$$
$$x = \{-2, -3\}$$
$$Vertical\ Asymptotes: x = -2, x = -3$$



Part F

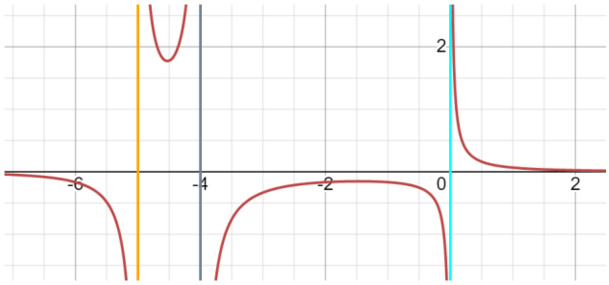
Equate the denominator to zero:

$$x^2 + 10x + 21 = 0$$
$$(x + 7)(x + 3) = 0$$
$$x = \{-3, -7\}$$
$$Vertical\ Asymptotes: x = -3, x = -7$$



Part G

Part H



Equate the denominator to zero:

$$3x^3 + 27x^2 + 60x = 0$$

Factor out the HCF from the LHS:

$$(3x)(x^2 + 9x + 20) = 0$$

Factor the quadratic:

$$(3x)(x + 5)(x + 4) = 0 \Rightarrow x = \{-5, -4, 0\}$$

Example 3.24: Modelling

Population of bacteria

B. Extended Example: Holes versus Vertical Asymptotes

Hole: Function is otherwise continuous, but has a gap, which can be filled by defining the function at a value.
Vertical Asymptote: A gap created because the denominator of the function is equal to zero at a value.

Example 3.25

$$f(x) = \frac{x + 2}{x^2 + 5x + 6}$$

For $x \neq 0$:

$$f(x) = \frac{x + 2}{x^2 + 5x + 6} = \frac{x + 2}{(x + 2)(x + 3)} = \frac{1}{x + 3}$$

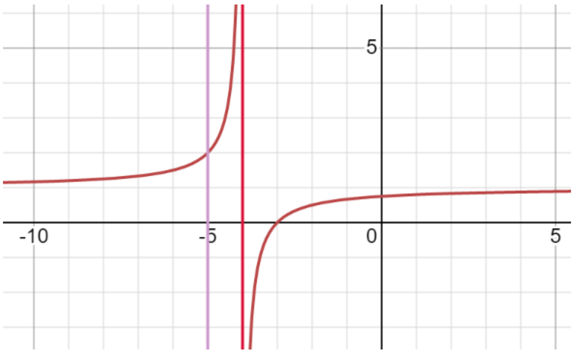
$$x^2 + 5x + 6 = 0 \Rightarrow x \in \{-2, -3\}$$

$$x = -3 \Rightarrow Vertical\ Asymptote$$
$$x = -2 \Rightarrow Hole$$

Example 3.26

Find holes, and vertical asymptotes, if any, in

$$f(x) = \frac{x^2 + 8x + 15}{x^2 + 9x + 20}$$



Domain

We can get the domain by equating the denominator:

$$x^2 + 9x + 20 = 0 \Rightarrow x \in \{-4, -5\} \Rightarrow D_f = \mathbb{R} - \{-4, -5\}$$

However, we can also simplify:

$$f(x) = \frac{(x+3)(x+5)}{(x+4)(x+5)} = \frac{x+3}{x+4}, \qquad x \neq -4, -5$$

Vertical Asymptote

$x = -4$ is a vertical asympote. Even after we simplify the function, $x = -4$ is still a zero of the denominator.

Holes

$x = -5$ is a hole. The factor/zero $x = 5$ can be simplify cancelled out from the expression, so long as $x \neq -5$.

In the above example, if the value $x = 7$ is excluded, while defining the function, from the domain then at $x = 7$ will you have:

A. Vertical Asymptote
B. Horizontal Asymptote
C. Hole
D. None of the above

$$Hole$$

Plug the holes in the function given in the prior example by defining the function piece-wise.

$$f(x) = \begin{cases} \dfrac{x^2 + 8x + 15}{x^2 + 9x + 20}, x \in \mathbb{R} - \{-5, -4, 7\} \\ \dfrac{x+3}{x+4}, x = -5 \\ \dfrac{x+3}{x+4}, x = 7 \end{cases}$$

C. Oblique/Slant Asymptotes

Oblique asymptotes occur when the degree of the numerator is one more than the degree of the denominator.

Example 3.27

Find the oblique asymptotes of

$$f(x) = \frac{3x^2 + 5}{x}$$

Break up the fraction

$$f(x) = \frac{3x^2}{x} + \frac{5}{x} = \underbrace{3x}_{\substack{Oblique \\ Asymptote}} + \underbrace{\frac{5}{x}}_{\substack{Residual \\ Term}}$$

For very large values of x, the second term will be less and less important, since it will successively smaller values.
Hence, the oblique asymptote of the function is

$$y = 3x$$

3.28: Polynomial Long Division

$$f(x) = \frac{P(x)}{Q(x)}$$

Using polynomial long division, we can rewrite the function as:

$$\frac{P(x)}{Q(x)} = L(x) + \frac{c}{Q(x)}, L(x) = Some\ Linear\ Function, c = constant$$

As the value of x becomes very large (infinity), or very small (negative infinity), $Q(x)$ also has very large absolute value. And hence,

$$\frac{c}{Q(x)} \approx 0$$

Which means that the behaviour of $f(x)$ for very large and very small values depends only on

$$L(x)$$

And this is the oblique asymptote.

Example 3.29

Find the oblique asymptotes of:

$$f(x) = \frac{3x^2 + 6}{x + 4}$$

Use polynomial long division

$$\frac{3x^2 + 6}{x + 4} = 3x - 12 + \frac{54}{x + 4}$$

For very large values of x, the last term will be less and less important, since it will successively smaller values.
Hence, the oblique asymptote of the function is

$$y = 3x - 12$$

3.3 Intercepts and Sign Diagrams

A. Intercepts

3.30: Some common rational functions

$$\frac{Linear}{Linear}, \frac{Linear}{Quadratic}, \frac{Quadratic}{Linear}, \frac{Quadratic}{Quadratic}$$

While there is an infinite variety of rational function, the above four types are important, and worth knowing separately.

Example 3.31

Classify each function if it is a common rational function. If it is not one of the common functions, say so.

A. $y = \frac{2x-5}{3x^2+5x+12}$

B. $y = \frac{\sin x}{2x-12}$
C. $y = \frac{\sqrt{x}}{5x-7}$
D. $y = \frac{ax+b}{cx-d}, a \neq 0, c \neq 0$
E. $y = \frac{ax^2+bx+c}{px+r}, a \neq 0, p \neq 0$
F. $y = \frac{3x^2+5x-9}{2x^2+4x}$

$$A: \frac{Linear}{Quadratic}$$
$$B: Not\ rational, \sin x\ \ is\ not\ polynomial$$
$$C: \sqrt{x}\ is\ not\ polynomial$$
$$D: \frac{Linear}{Linear}$$
$$E: \frac{Quadratic}{Linear}$$
$$F: \frac{Quadratic}{Quadratic}$$

3.32: Finding the Intercepts

To find the

$$x - intercept: Substitute\ y = 0$$
$$y - intercept: Substitute\ x = 0$$

Example 3.33

Classify each function below as a common linear function. Find its intercepts and asymptotes:

A. $y = \frac{3x+5}{2x-7}$
B. $y = \frac{x^2+5x+6}{x^2+8x+7}$

Part A

The function is $\frac{Linear}{Linear}$.

The asymptotes are:

$$HA: y = \frac{3}{2}, \qquad VA: x = \frac{7}{2}$$

The intercepts are:

$$\frac{3x+5}{2x-7} = 0 \Rightarrow 3x + 5 = 0 \Rightarrow x = -\frac{5}{3}$$
$$y = \frac{3(0)+5}{2(0)-7} = -\frac{5}{7}$$

Part B

The function is $\frac{Quadratic}{Quadratic}$.

The asymptotes are:

$$HA: y = 1, \qquad VA: x = -7, x = -1$$

The intercepts are:

$$y = \frac{6}{7}$$

$$\frac{x^2+5x+6}{x^2+8x+7}=0 \Rightarrow x^2+5x+6=0 \Rightarrow x=-2, x=-3$$

B. Sign Diagram

3.34: Critical Points

Sign diagram is based on critical points. Critical points include:

- x intercept
- Vertical Asymptote

To make a sign diagram, divide the real number line into distinct intervals based on the critical points.

3.35: Intervals

Example 3.36

Find the critical points, and hence the intervals in each case.

A. The horizontal asymptote of a function is $y=4$. The x intercepts occur at -7 and 7. The vertical asymptote occurs at 3.
B. $\{\pm\sqrt{2}, 2\}$

Part A

The critical points are:

$$\{\pm 7, 3\}$$

The intervals are:

$$(-\infty,-7)\cup(-7,3)\cup(3,7)\cup(7,\infty)$$

Part B

$$\left(-\infty,-\sqrt{2}\right)\cup\left(-\sqrt{2},\sqrt{2}\right)\cup\left(\sqrt{2},2\right)\cup(2,\infty)$$

3.37: Sign Diagram

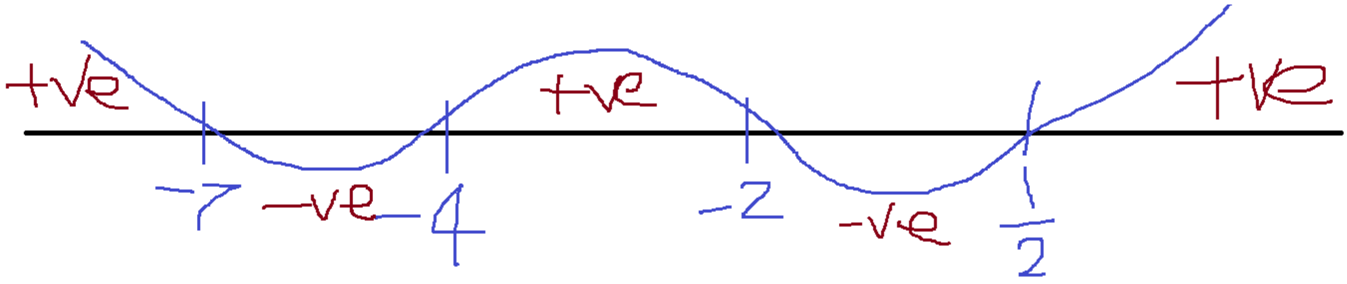
Example 3.38

$$y=\frac{(x+2)(x+4)}{\left(x-\frac{1}{2}\right)(x+7)}$$

Critical Points:

$$\left\{-7,-4,-2,\frac{1}{2}\right\}$$

$$HA: y=1>0$$



C. Graphing

Example 3.39

Classify the function below as a common linear function. Find its intercepts and asymptotes. Use a sign diagram to determine where it is positive or negative. Then draw a rough sketch of the graph.

$$y = \frac{x^2 + 5x + 6}{x^2 + 8x + 7}$$

The function is $\frac{Quadratic}{Quadratic}$.

The asymptotes are:

$$HA: y = 1, \qquad VA: x = -7, x = -1$$

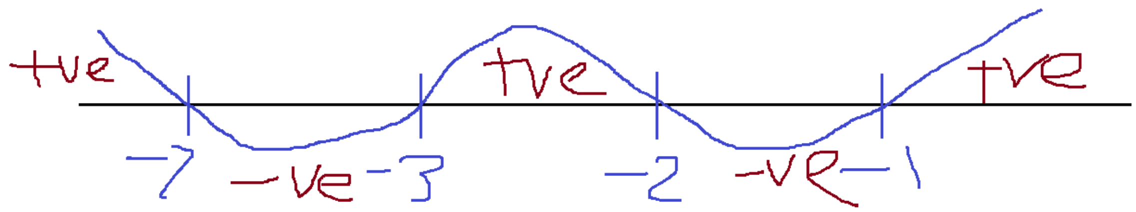
The intercepts are:

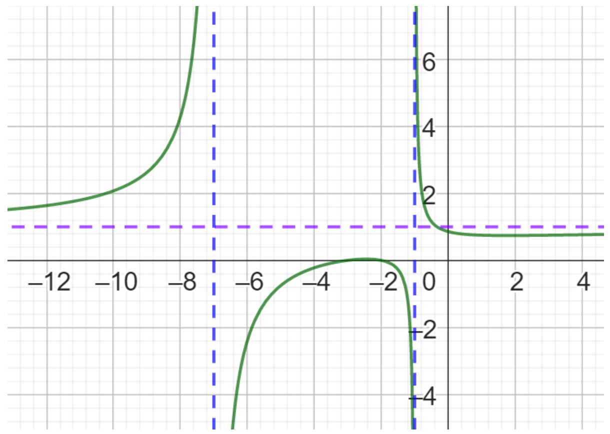
$$y = \frac{6}{7}$$

$$\frac{x^2 + 5x + 6}{x^2 + 8x + 7} = 0 \Rightarrow x^2 + 5x + 6 = 0 \Rightarrow x = -2, x = -3$$

Critical Points are:

$$\{-7, -3, -2, -1\}$$





D. Graphing Functions

If a function belongs to one of the function categories specifically discussed in a separate chapter, such as:

- Linear
- Quadratic
- Cubic
- Polynomial
- Absolute Value
- Rational
- Exponential
- Logarithmic
- Trigonometric

Then you will already have some idea as how to graph it based on its properties.

When we to draw the graph of a function we look at a few important things:

Intercepts	This tells you where the function cuts the axes. This gives specific points that must lie on the graph of the function.	$\boldsymbol{x - intercept}$ To find the x-intercept, substitute $y = 0$ $\boldsymbol{y - intercept}$ To find the y-intercept, substitute $x = 0$
Asymptotes	Vertical Asymptotes: These are created by points that are not defined in the domain of a graph. A function will go near positive infinity or negative or both near its vertical asymptotes. Horizontal Asymptotes They tell us the end behaviour of a function. In other words, how does a function behave as the input value approaches positive infinity or negative infinity. Oblique Asymptotes Oblique asymptotes occur when the end behaviour does not approach a particular value, but behaves like a linear function.	
Dominant Terms	Dominant terms tell us the term in the function that contributes the most to its behaviour over a particular range of input values.	
Sign Diagram	Sign Diagram tells us the places where the function is positive and negative.	

3.4 Transformations

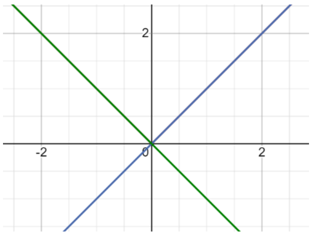
A. Absolute Value Function

When considering transformations, we want to consider

- Shifting: Moving a graph without changing its shape
- Scaling: Changing the shape of a graph with reference to an axis
- Reflection: Taking a mirror image of the function across the x-axis or the y-axis or both.

Look at the graph to the right. It has

$$Blue\ Line: y = x$$
$$Green\ Line: y = -x$$

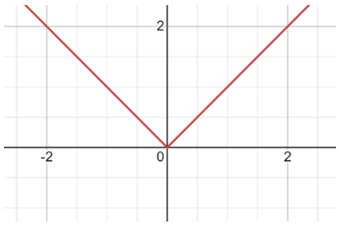


If we take:

- ➢ The blue to the right of zero on the x-axis,
- ➢ The green line to the left of zero on the x-axis,

Then we get the graph of

$$y = |x|$$



We can write the definition of the absolute value function as a piece-wise function:

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

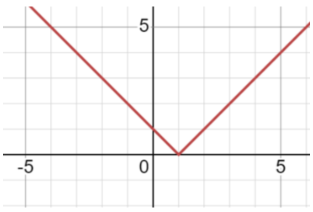
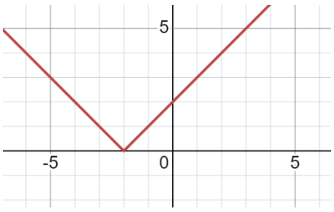
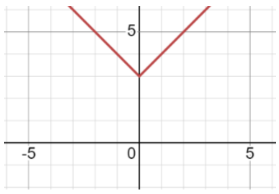
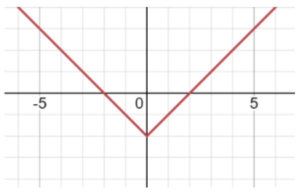
Example 3.40

Let

$$y = f(x) = |x|$$

Draw the graph of $f(x)$:

A. If it is moved to the right by 1 unit
B. If it is moved to the left by 2 units
C. If it is moved up by 3 units
D. If it is moved down by 2 units

Four equations are given below. These match the graphs that you drew above. Match each equation to one of the graphs using the properties of transformation of graphs:

1. $y = |x - 1|$
2. $y = |x + 2|$
3. $y = |x| + 3$
4. $y = |x| - 2$

Example 3.41: Translation

Identify the transformations applied to $f(x) = |x|$ to get the following graphs:

A. $y = |x + 1|$
B. $y = |x| + 2$
C. $y = |x - 3|$
D. $y = |x| - 2$

A. Moving left by 1 unit
B. Moving up by 2 Units
C. Moving right by 3 Units

D. Moving down by 2 Units

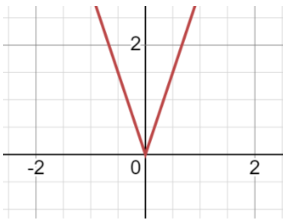
$y = k\lvert x\rvert$	$\underbrace{Vertical\ Stretch}_{k>1}$
	OR $\underbrace{Vertical\ Shrink}_{0<k<1}$ by a factor of k

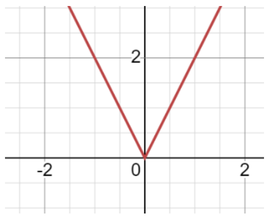
$y = \lvert kx\rvert$	$\underbrace{Horizontal\ Shrink}_{k>1}$
	OR $\underbrace{Horizontal\ Stretch}_{0<k<1}$

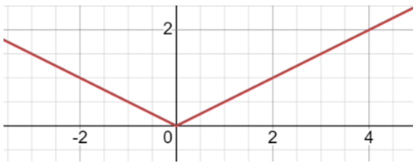
Example 3.42: Scaling and Shifting

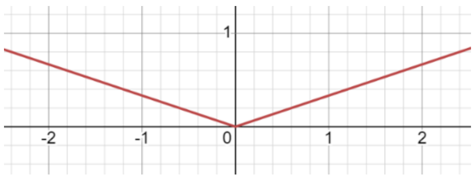
Draw the graph of $|x|$:

A. Scaled vertically by a factor of 3
B. Scaled horizontally by a factor of $\frac{1}{2}$
C. Scaled vertically by a factor of $\frac{1}{2}$
D. Scaled horizontally by a factor of 3









The equations of the graphs of the above functions are given below. Match each equation to its graph:

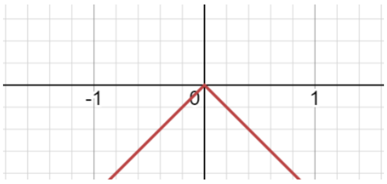
A. $y = 3|x|$
B. $y = |2x|$
C. $y = \frac{1}{2}|x|$
D. $y = \left|\frac{1}{3}x\right|$

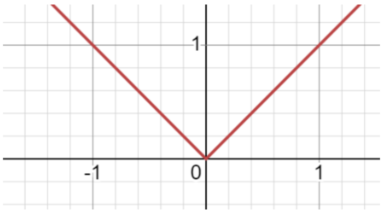
$y = f(-x)$	Reflection across the y-axis
$y = -f(x)$	Reflection across the x-axis
$y = -f(-x)$	Reflection across the origin

Example 3.43

Draw the graph of $y = |x|$ if it is:

A. Reflected across the y-axis
B. Reflected across the x-axis
C. Reflected across the origin



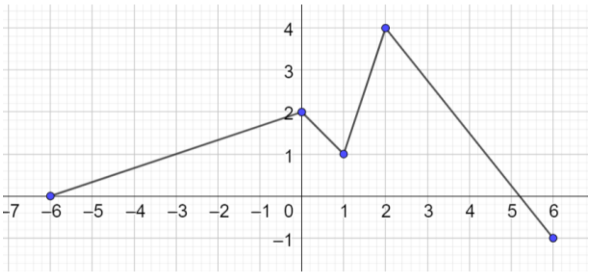


B. Piece-Wise Functions

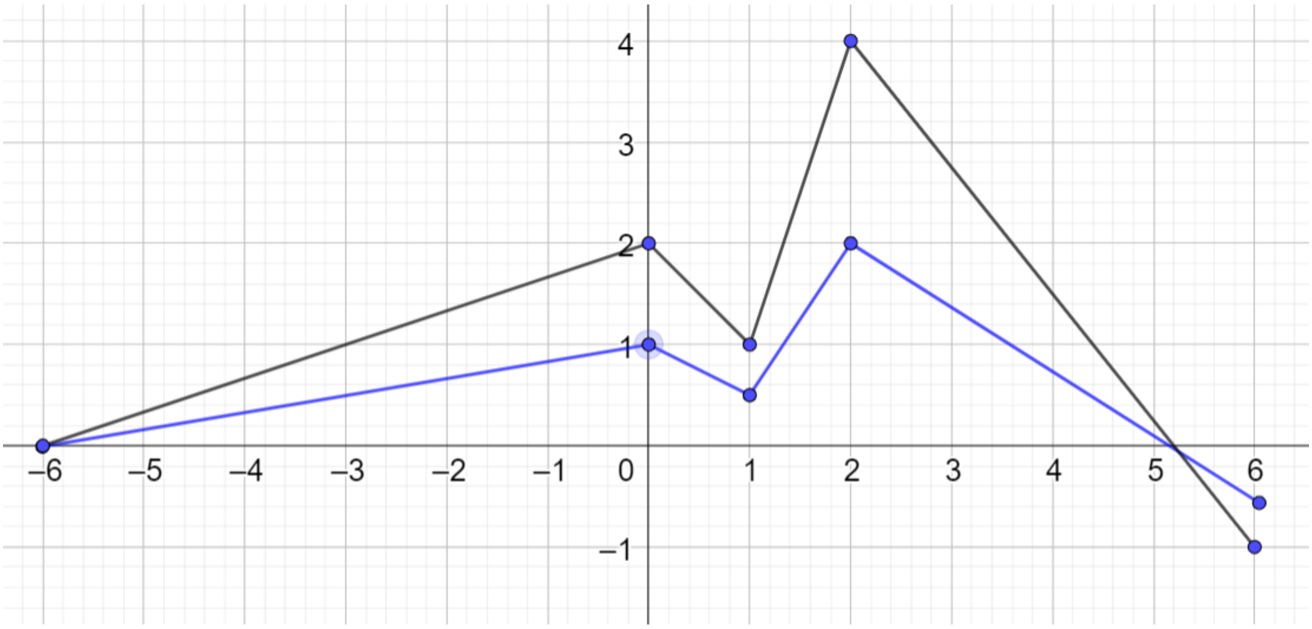
Example 3.44

The graph of $f(x)$ is given. The blue dots are points on the graph, and are all lattice points. The rest of the graph comprises straight line segments. Use this information to graph:

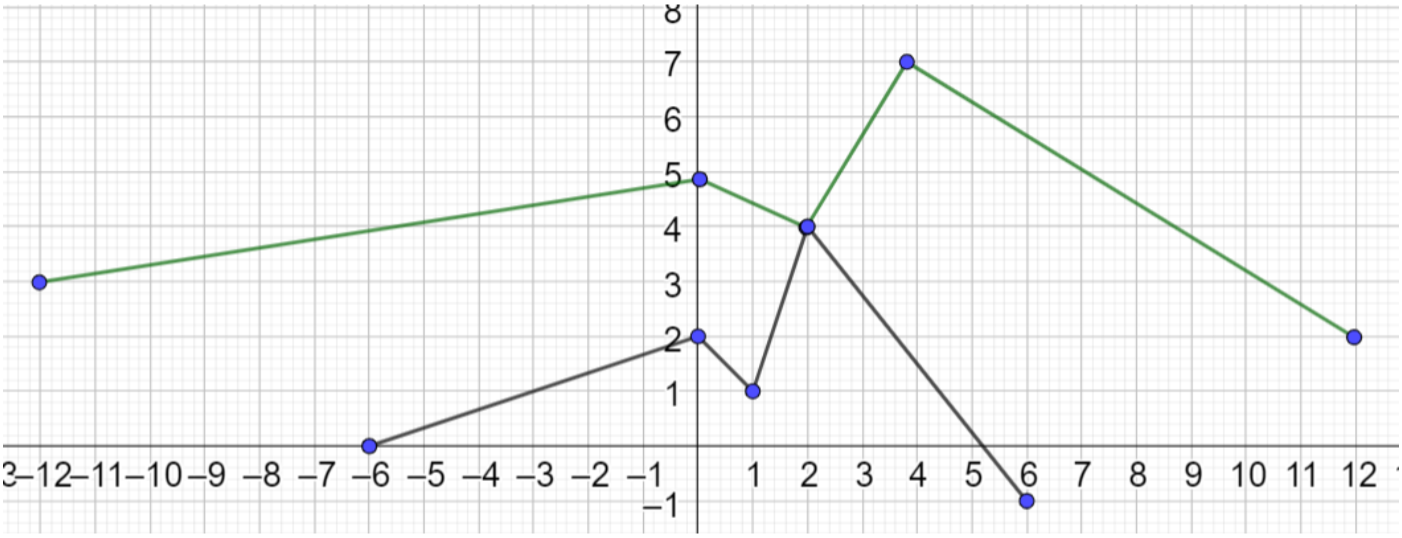
A. $0.5f(x)$
B. $f\left(\frac{x}{2}\right) + 3$



Part A



Part B



3.5 Shifts, Scales and Reflections

A. Summary

3.45: Vertical Shifting

Given $y = f(x)$

$$y = f(x) + k, \qquad k > 0 \Rightarrow Moves\ up\ by\ k\ units$$
$$y = f(x) - k, \qquad k > 0 \Rightarrow Moves\ down\ by\ k\ units$$

- add k to each y-value, you are effectively moving the graph up by k units.
- Subtract k from each y-value, you are effectively moving the graph down by k units.

3.*46:* Horizontal Shifting

Given $y = f(x)$

$$y = f(x + k), \qquad k > 0 \Rightarrow Shifts\ f(x)\ left\ by\ k\ units$$
$$y = f(x - k), \qquad k > 0 \Rightarrow Shifts\ f(x)\ right\ by\ k\ units$$

Note the difference between vertical shifting and horizontal shifting.

- In vertical shifting, you add or subtract k after the function has been calculated. The function input does not change.
- In horizontal shifting, you increase or decrease the input of the function.

3.47: Vertical Scaling

Given $y = f(x)$

$$y = kf(x), \qquad k > 1 \Rightarrow Vertical\ Stretch\ by\ a\ factor\ of\ k$$
$$y = kf(x), \qquad 0 < k < 1 \Rightarrow Vertical\ Shrink\ by\ a\ factor\ of\ k$$

The terminology can be confusing. Each transformation below is equivalent:

- Vertical Scale by 2$\Leftrightarrow$Vertical Stretch by 2$\Leftrightarrow$Vertical Shrink by $\frac{1}{2}$
- Vertical Scale by $\frac{3}{4}$ $\Leftrightarrow$Vertical Stretch by $\frac{3}{4}$ $\Leftrightarrow$Vertical Shrink by $\frac{4}{3}$

3.48: Horizontal Scaling

Given $y = f(x)$:

$$y = f(kx), \qquad k > 0 \Rightarrow Horizontal\ Scaling\ by\ factor\ of\ \frac{1}{k}$$

To scale by k, we replace every instance of x in the function with $\frac{1}{k}x$

3.49: Reflections

Given $y = f(x)$:

$$y = -f(x) \Rightarrow Reflect\ across\ the\ x - axis$$
$$y = f(-x) \Rightarrow Reflect\ across\ the\ y - axis$$

In order to reflect across the

- x-axis, negate the entire expression
- y-axis, replace every instance of x with $-x$.

B. Background

The graph of a quadratic can be moved around, and can change shape while still remaining a parabola. We are going to consider the following changes:

- Shifting: Moving up, down, left and right
- Scaling: Changing size without changing shape
- Reflection: Which, informally, is a mirror image

We will consider these changes on the

- y-axis: Vertical
- x-axis: Horizontal

The properties below are applicable to all functions, but we will learn them here in the context of quadratics.

3.50: Vertical Shifting

Given $y = f(x)$

$$y = f(x) + k, \qquad k > 0 \Rightarrow Moves\ up\ by\ k\ units$$
$$y = f(x) - k, \qquad k > 0 \Rightarrow Moves\ down\ by\ k\ units$$

If you:

- add k to each y-value, you are effectively moving the graph up by k units.
- Subtract k from each y-value, you are effectively moving the graph down by k units.

Example 3.51

Consider the graph of $f(x) = x$. What is the translation that will give you each graph:

A. $f(x) = x + 3$
B. $f(x) = x - \frac{1}{2}$
C. $f(x) = x + \pi$

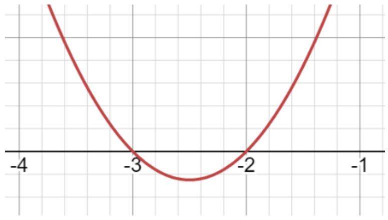
$$A: Move\ up\ by\ 3\ Units$$

$$B: Move\ down\ by\ \frac{1}{2}\ Units$$
$$C: Move\ up\ by\ \pi\ Units$$

Example 3.52

Find the new graph, if $f(x) = x^2 + 5x + 6$ is:

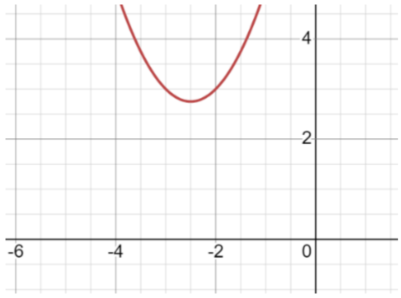
A. Shifted up 3 units
B. Shifted down 4 units



Part A

To move the graph up by 3 units, we add 3 to $f(x)$ giving us:

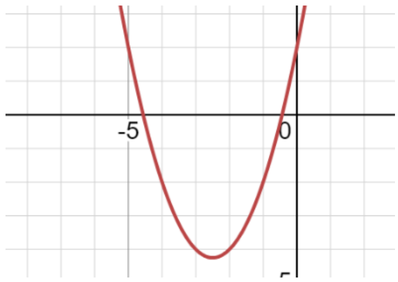
$$f(x) + 3 = x^2 + 5x + 6 + 3 = x^2 + 5x + 9$$



Part B

To move the graph up by 3 units, we add d to $f(x)$ giving us:

$$x^2 + 5x + 6 - 4 = x^2 + 5x + 2$$



C. Vertical Scaling

3.53: Vertical Scaling

Given $y = f(x)$

$$y = kf(x), \quad k > 1 \Rightarrow Vertical\ Stretch\ by\ a\ factor\ of\ k$$
$$y = kf(x), \quad 0 < k < 1 \Rightarrow Vertical\ Shrink\ by\ a\ factor\ of\ k$$

The terminology can be confusing. Each transformation below is equivalent:

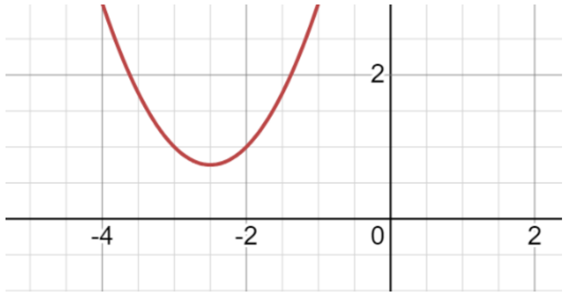
- Vertical Scale by 2⇔Vertical Stretch by 2⇔Vertical Shrink by $\frac{1}{2}$
- Vertical Scale by $\frac{3}{4}$ ⇔Vertical Stretch by $\frac{3}{4}$ ⇔Vertical Shrink by $\frac{4}{3}$

Example 3.54

Identity the transformation applied to the graph of $f(x) = x$ to get the graphs below:

A. $f(x) = 3x$
B. $f(x) = \frac{x}{6}$
C. $f(x) = \frac{\pi x}{6}$
D. $f(x) = \frac{2x}{e}$

$$A: Vertical\ Stretch\ by\ a\ factor\ of\ 3$$
$$B: Vertical\ Stretch\ by\ a\ factor\ of\ \frac{1}{6}$$
$$C: Vertical\ Stretch\ by\ a\ factor\ of\ \frac{\pi}{6}$$
$$D: Vertical\ Stretch\ by\ a\ factor\ of\ \frac{2}{e}$$



Example 3.55

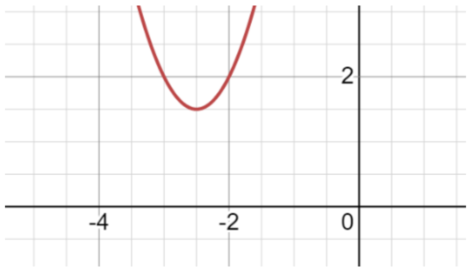
Answer each part independently. Let $y = f(x) = x^2 + 5x + 7$.
Find the new graph, if $f(x)$ is:

A. Stretched vertically by a factor of 2
B. Shrunk vertically by a factor of 3

Part A

To stretch the graph vertically by a factor of 2, we multiply the function by 2:

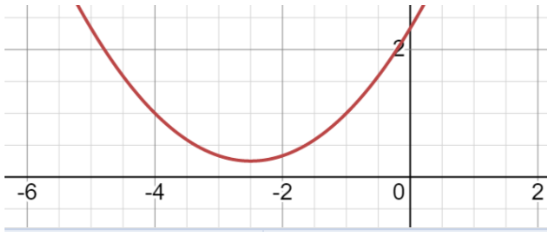
$$2f(x) = 2(x^2 + 5x + 7) = 2x^2 + 10x + 14$$



Part B

To shrink the graph vertically by a factor of 3, we multiply the function by $\frac{1}{3}$:

$$\frac{1}{3}(x^2 + 5x + 7) = x^2 + \frac{5}{3}x + \frac{7}{3}$$



D. Horizontal Shifting and Scaling

3.56: Horizontal Shifting

Given $y = f(x)$

$$y = f(x + k), \qquad k > 0 \Rightarrow Shifts\ f(x)\ left\ by\ k\ units$$
$$y = f(x - k), \qquad k > 0 \Rightarrow Shifts\ f(x)\ right\ by\ k\ units$$

Note the difference between vertical shifting and horizontal shifting.

- In vertical shifting, you add or subtract k after the function has been calculated. The function input does not change.
- In horizontal shifting, you increase or decrease the input of the function.

Example 3.57

Given the graph of $f(x) = x$, identify the transformation applied to the graph to obtain the function given below:

A. $f(x) = x + 3$
B. $f(x) = x - \frac{1}{2}$
C. $f(x) = x + 2\pi$

Example 3.58

Compare the three functions:

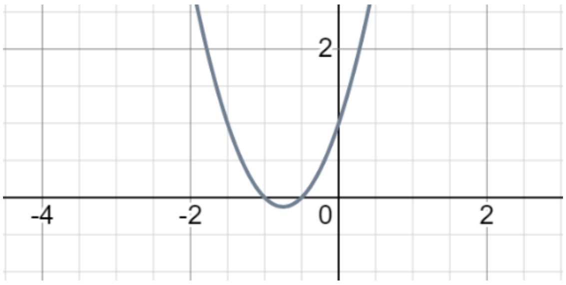
$$f(x) = x^2$$
$$g(x) = f(x+1) = (x+1)^2$$
$$h(x) = f(x-1) = (x-1)^2$$

x	-2	-1	0	1	2	
$f(x)$	4	1	0	1	4	
$x+1$	-1	0	1	2	3	**Shifted to the Right**
$g(x)$	1	0	1	4	9	
$x-1$	-3	-2	-1	0	1	**Shifted to the Left**
$g(x)$	9	4	1	0	1	

Example 3.59

Answer each part independently. Let $y = g(x) = 2x^2 + 3x + 1$. Find the graph of $g(x)$ if it is:

A. Shifted right 1 unit
B. Shifted left 2 units

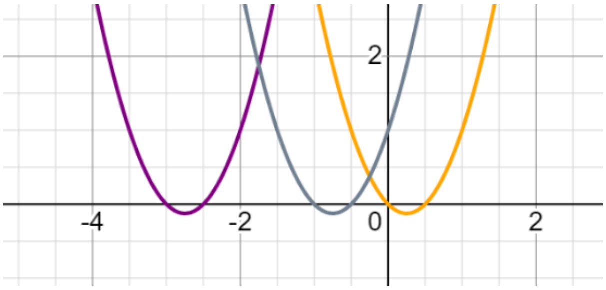


Part A

$2(x-1)^2 + 3(x-1) + 1$
$= 2x^2 - 4x + 2 + 3x - 3 + 1$
$= 2x^2 - x$

Part B

$2(x+2)^2 + 3(x+2) + 7$
$= 2x^2 + 8x + 8 + 3x + 6 + 7$
$= 2x^2 + 11x + 15$



3.60: Horizontal Scaling

Given $y = f(x)$:

$$y = f(kx), \qquad k > 0 \Rightarrow Horizontal\ Scaling\ by\ factor\ of\ \frac{1}{k}$$

In order to scale by k, we replace every instance of x in the function with $\frac{1}{k}x$

Example 3.61

Given the graph of $f(x) = x$, identify the transformation applied to the graph to obtain the function given below:

A. $f(x) = 3x$
B. $f(x) = \frac{x}{2}$
C. $f(x) = \pi x$

Example 3.62

Compare the three functions

$$f(x) = x^2$$
$$g(x) = f(2x) = 4x^2$$
$$h(x) = f\left(\frac{x}{2}\right) = \frac{x^2}{4}$$

x	-2	-1	0	1	2	
$f(x)$	4	1	0	1	4	
$2x$	-4	-2	0	2	4	SHRUNK BY A FACTOR OF TWO
$g(x)$	16	4	0	4	16	
$\frac{x}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	STRETCHED BY A FACTOR OF TWO
$h(x)$	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	

Example 3.63

Answer each part independently. Let $y = g(x) = 4x^2 + 2x + 2$. Find the equation of the new graph, if $g(x)$ is:

A. Horizontally stretched by a factor of 2
B. Horizontally shrunk by a factor of 3

$$\boldsymbol{Part\ A}: 4\left(\frac{1}{2}x\right)^2 + 2\left(\frac{1}{2}x\right) + 2 = x^2 + x + 2$$
$$\boldsymbol{Part\ B}: 4(3x)^2 + 2(3x) + 2 = 36x^2 + 6x + 2$$

E. Reflections

3.64: Reflections

Given $y = f(x)$:

$$y = -f(x) \Rightarrow Reflect\ across\ the\ x - axis$$
$$y = f(-x) \Rightarrow Reflect\ across\ the\ y - axis$$

In order to reflect across the

- x-axis, negate the entire expression
- y-axis, replace every instance of x with $-x$.

Example 3.65

Answer each part independently. Let $y = p(x) = 2x^2 + 3x + 5$. Find the graph of $p(x)$ if it is:

A. Reflected across the x-axis
B. Reflected across the y-axis

$$\boldsymbol{Part\ A}: -(2x^2 + 3x + 5) = -2x^2 - 3x - 5$$
$$\boldsymbol{Part\ B}: 2(-x)^2 + 3(-x) + 5 = 2x^2 - 3x + 5$$

F. Order of Transformations

Order of transformations

- does not matter for shifting

Example 3.66

Answer each question independently. Let $y = s(x) = x^2 + 2x + 7$. Find the new graph, if the graph of $s(x)$ t is:

A. First moved to the right by 3 units, and then down by 2 units
B. First moved down by 2 units, and then moved right by 3 units

Is your answer to both the parts the same? Can you justify graphically?

$$Part\ A: \underbrace{(x-3)^2 + 2(x-3) + 7 = x^2 - 6x + 9 + 2x - 6 + 7}_{\textbf{Move Right by 3 Units}} = x^2 - 4x + 10 \to \underbrace{x^2 - 4x + 8}_{\textbf{Move Down by 2 Units}}$$

$$Part\ B: \underbrace{x^2 + 2x + 5}_{Move\ Down\ by\ 2\ Units} \to \underbrace{(x-3)^2 + 2(x-3) + 5}_{Move\ Right\ by\ 3\ Units} = x^2 - 6x + 9 + 2x - 6 + 5 = x^2 - 4x + 8$$

3.6 Preparing for Calculus

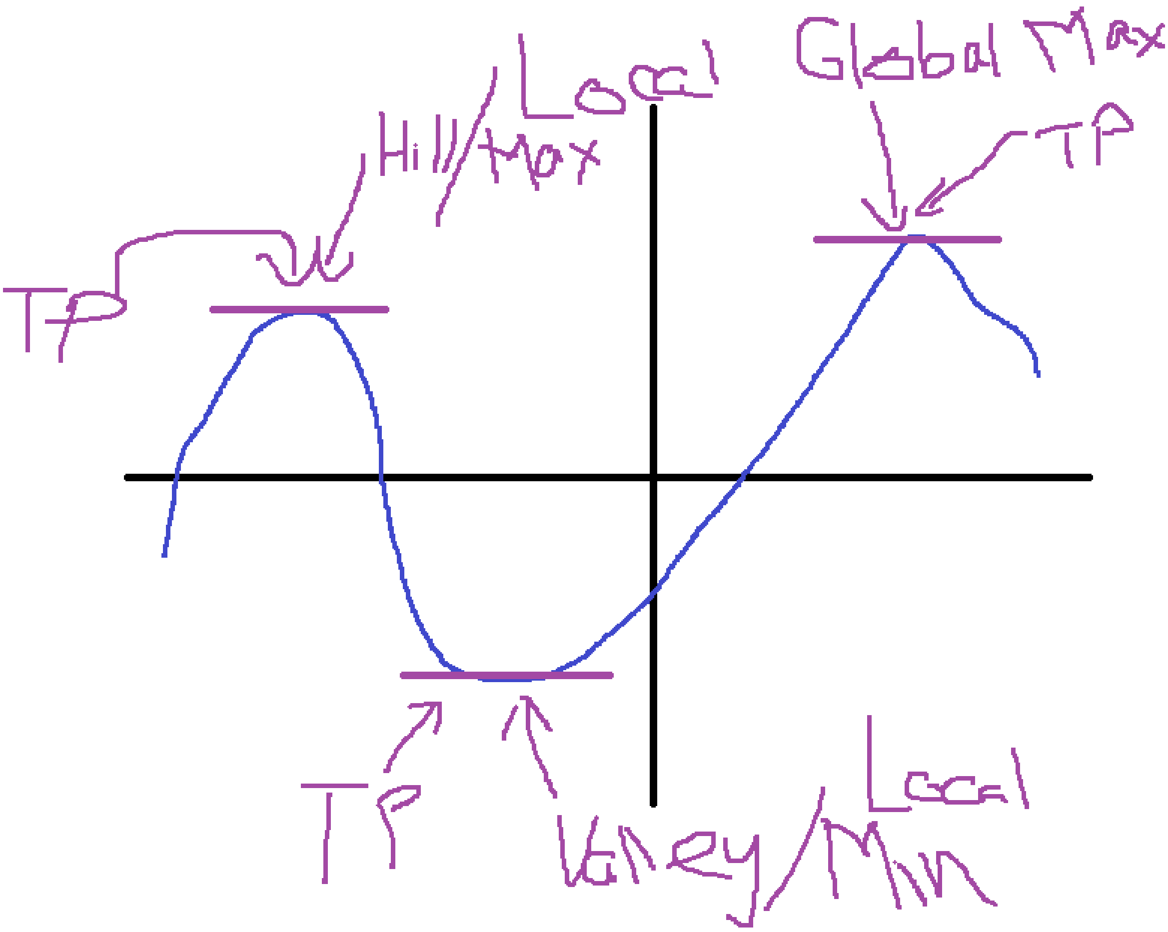
A. Turning Points

B. Positive and Negative Intervals

C. Maxima and Minima

D. Endpoints

E. Local versus Global Maxima and Minima



F. Bringing it together

4. Rational Functions

4.1 Rational Functions via Transformations

A. Parent Function: $\frac{1}{x}$

Mark
- Vertical Asymptotes
- Horizontal Asymptotes
- X intercept
- Y intercept

4.1: Graph of $f(x) = \frac{1}{x}$

Example 4.2

A. Graph $f(x) = \frac{1}{x}$

4.3: Horizontal Shift

$$f(x+k),$$
$$k > 0, Shifted\ Left$$
$$k < 0, Shifted\ Right$$

Example 4.4

Graph $f(x) = \frac{1}{x+2}$

4.5: Vertical Shift

$$f(x) + k$$
$$k > 0, Shift\ up$$
$$k < 0, Shift\ down$$

Example 4.6

$$\frac{1}{x} + 2$$

4.7: Horizontal Dilation

4.8: Vertical Dilation

$$kf(x)$$
$$k > 1 \Rightarrow Vertical\ Stretch$$
$$0 < k < 1 \Rightarrow Vertical\ Shrink$$

Example 4.9

$$\frac{2}{x}$$

B. Combining Transformations

Example 4.10

A. $f(x) = \frac{2}{x} + 1$
B. $f(x) = \frac{3}{x-2} + 1$

4.2 Rational Functions: Algebra

A. Types of Rational Functions

4.11: Rational Function

A rational function is a function of the form

$$\frac{P(x)}{Q(x)}$$

Where $P(x)$ and $Q(x)$ are polynomial functions

Hence, a rational function is a function of the form:

$$\frac{Polynomial}{Polynomial}$$

Hence, trigonometric, exponential, logarithmic functions and their combinations are not rational functions.

4.12: Important Rational Functions

There are four important types of rational functions:

$$\frac{Linear}{Linear}, \quad \frac{Linear}{Quadratic}, \quad \frac{Quadratic}{Linear}, \quad \frac{Quadratic}{Quadratic}$$

Beyond this, you can have functions with cubic polynomials, fourth degree polynomials, etc.

4.13: Dominant Term

Given a rational function of the form $\frac{P(x)}{Q(x)}$, we can carry out the division and get an expression of the form:

$$\frac{Dividend}{Divisor} = Quotient + \frac{Remainder}{Divisor}$$

$$\frac{x+1}{x+2} = \frac{(x+2)-1}{x+2} = 1 - \frac{1}{x+2}$$

Depending on the value of x, the "dominant term" is different.

When

$$x \to \pm\infty \Rightarrow \frac{1}{x+2} \to 0 \Rightarrow 1 \text{ is the dominant term}$$

When

$$x \to -2^+ \Rightarrow \frac{1}{x+2} \to +\infty$$

$$x \to -2^- \Rightarrow \frac{1}{x+2} \to -\infty$$

Hence,

$$At \pm \infty, \quad 1 \; is \; the \; dominant \; term$$
$$Near -2, \quad \frac{1}{x+2} \; is \; the \; dominant \; term$$

B. $\frac{\textit{Linear}}{\textit{Linear}}$

Example 4.14

$$f(x) = \frac{x+3}{x+2}$$

Determine, for the function:

A. Intercepts
B. Asymptotes (horizontal, vertical, oblique)
C. Dominant Terms
D. Sign Diagram

Hence, draw a sketch of the graph, labelling intercepts, asymptotes, and any other points of interest.

Intercepts

$$\text{Substitute } y = 0: 0 = \frac{x+3}{x+2} \Rightarrow x - intercept = -3$$
$$\text{Substitute } x = 0: y - intercept = \frac{0+3}{0+2} = \frac{3}{2} = 1.5$$

Dominant Terms

$$f(x) = \frac{x+3}{x+2} = \frac{(x+2)+1}{x+2} = 1 + \frac{1}{x+2}$$

Horizontal Asymptote

At ∞ and $-\infty$, the first term will dominate, which gives us the expression

$$f(x) = \underbrace{1}_{\substack{Dominant\ Term\\ Corresponds\ to\\ Horizontal\ Asymptote}} + \frac{1}{x+2}$$
$$x \to \pm\infty \Rightarrow y = 1$$

Vertical Asymptote

At values near 2, the second term will dominate, which gives us the expression:

$$x \to -2 \Rightarrow y \to \frac{1}{x+2}$$
$$f(x) = 1 + \underbrace{\frac{1}{x+2}}_{\substack{Dominant\ Term\\ Corresponds\ to\\ Vertical\ Asymptote}}$$

To find the vertical asymptote, equate the denominator to zero:

$$x + 2 = 0 \Rightarrow x = -2$$

$$x \to -2^+ \Rightarrow \frac{1}{x+2} \to +\infty$$
$$x \to -2^- \Rightarrow \frac{1}{x+2} \to -\infty$$

Sign Diagram/Wavy Curve Method

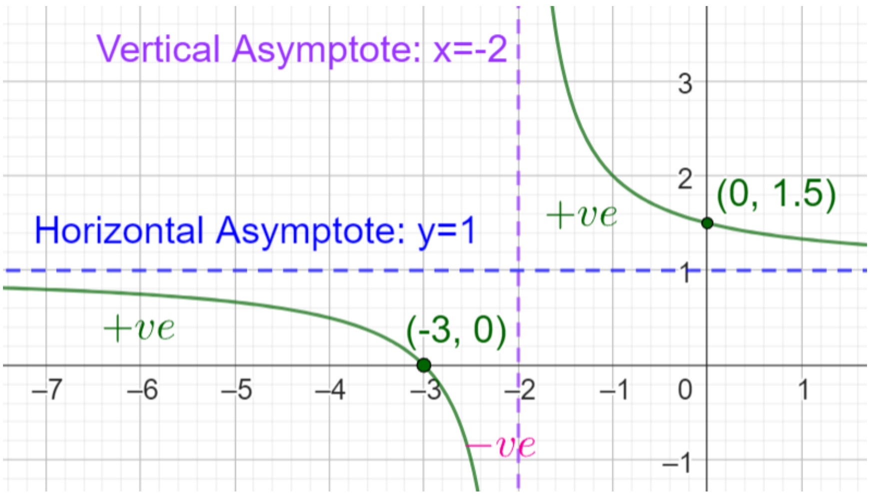
$$\frac{x+3}{x+2}$$

To find the sign diagram, equate each term of the expression above to zero:

$$x+2=0 \Rightarrow x=-2$$
$$x+3=0 \Rightarrow x=-3$$

Split the graph into different intervals based on its roots.

	$(-\infty,-3)$	$(-3,-2)$	$(-2,\infty)$
Value	-4	-2.5	0
	$\frac{-4+3}{-4+2}=\frac{-ve}{-ve}=+ve$	$\frac{-4+3}{-4+2}=\frac{-ve}{-ve}$ $=+ve$	
	$+ve$	$-ve$	$+ve$



C. $\frac{Quadratic}{Linear}$

4.15: Removable Discontinuity

If a $\frac{Quadratic}{Linear}$ function has a "cancelling factor" in the numerator and denominator, it will be a

- Linear Function
- With a "hole" or a "removable discontinuity"

Example 4.16

$$f(x)=\frac{x^2+5x+6}{x+2}$$

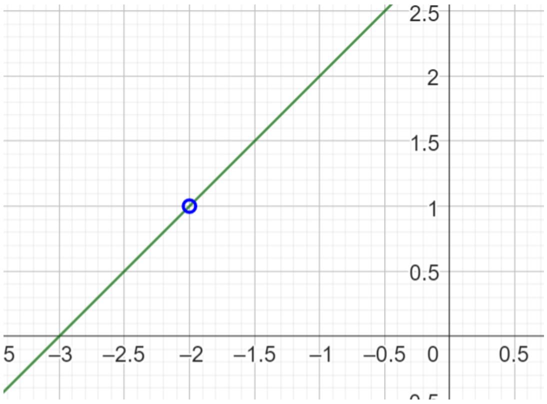
When $x \neq -2$

$$f(x) = \frac{x^2 + 5x + 6}{x + 2} = \frac{(x+2)(x+3)}{(x+2)} = x + 3$$
$$\Rightarrow Linear\ Function$$

When $x = -2$

$$f(x)\ is\ not\ defined$$

Hence, the graph of $f(x) = \frac{x^2+5x+6}{x+2}$ is a linear function with a "hole" or a "removable discontinuity".



4.17: Vertical Asymptote

If a $\frac{Quadratic}{Linear}$ function does not have a "cancelling factor" in the numerator and denominator, it will have

- a vertical asymptote where the denominator is not defined
- No horizontal asymptote since the function is unbounded both above and below. In other words, it tends to infinity (or negative infinity) when $x \to \pm\infty$
- An oblique asymptote

Example 4.18

$$f(x) = \frac{x^2 + 5x + 6}{x + 4}$$

$$f(x) = \frac{x^2 + 5x + 6}{x + 4} = x + 1 + \frac{2}{x + 4}$$

When $x \to \pm\infty$, the dominant term is $x + 1$. Hence, the function has no horizontal asymptote. As

$$x \to \infty \Rightarrow f(x) \to \infty$$
$$x \to - \Rightarrow f(x) \to -\infty$$

To find the vertical asymptotes, equate the denominator to zero:

$$x + 4 = 0 \Rightarrow x = -4$$

$$x \to -4^+ \Rightarrow \frac{2}{x+4} \to +\infty$$
$$x \to -4^+ \Rightarrow \frac{2}{x+4} \to -\infty$$

$$x = 0: y - intercept = \frac{6}{4} = \frac{3}{2} = 1.5$$
$$y = 0: x^2 + 5x + 6 = 0 \Rightarrow x \in \{-2, -3\} \Rightarrow x - intercepts\ are - 2\ and - 3$$

Use the wavy curve method

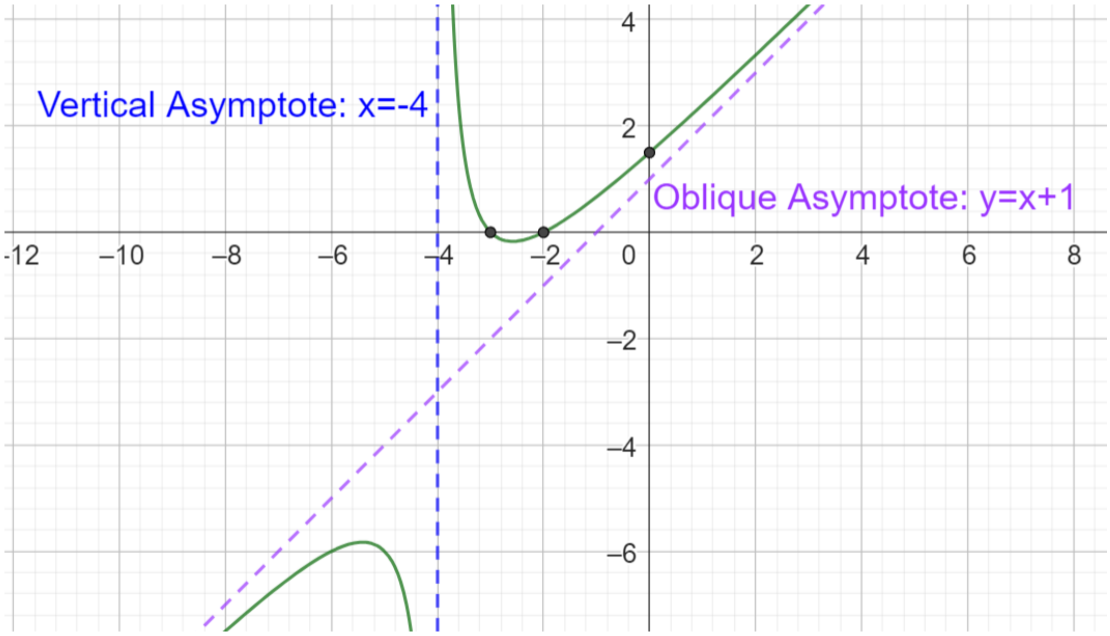
$$Roots\ are - 2, -3, -4$$

Since the function is has

$$Degree\ Of\ Numerator - Degree\ of\ Denominator = 2 - 1 = 1 \Rightarrow Odd$$

Hence, at $-\infty$ the function is negative.

	$(-\infty,-4)$	$(-4,-3)$	$(-3,-2)$	$(-2,\infty)$
Sign	$-ve$	$+ve$	$-ve$	$+ve$



D. $\frac{\textit{Linear}}{\textit{Quadratic}}$

4.19: Classification based on the discriminant

If the denominator of a function is a quadratic, there will be three cases based on the nature of roots. We can classify on the basis of the discriminant:

- Discriminant is positive, there will be two real roots
- Discriminant is zero, there will be a real, repeated root
- Discriminant is negative, there will be only complex roots

4.20: Complex Roots

If the discriminant is negative, the roots are complex, and hence the denominator is never zero. Hence, the function has a domain of all real numbers, and no vertical asymptotes.

Example 4.21: Quadratic with Complex Roots

Graph

$$f(x)=\frac{x+1}{x^2+2x+2}$$

Vertical Asymptotes

Vertical asymptotes occur when the denominator is zero:

$$x^2+2x+2=0\Rightarrow Discriminant=2^2-4(1)(2)=4-8=-4<0$$

Since the denominator is never zero, there are no vertical asymptotes.

Domain

Since the denominator is never zero, the domain is all real numbers.

Horizontal Asymptotes

This is the behaviour when $x \to \infty$, or $x \to -\infty$

$$x \to \infty \Rightarrow y \to 0$$
$$x \to -\infty \Rightarrow y \to 0$$

The horizontal asymptote is

$$y = 0$$

Intercept

To find the y intercept, substitute $x = 0$ in the given equation:

$$y = \frac{x+1}{x^2+2x+2} = \frac{0+1}{0^2+2(0)+2} = \frac{1}{2}$$

To find the x intercept, substitute $y = 0$ in the given equation:

$$0 = \frac{x+1}{x^2+2x+2} \Rightarrow 0 = x+1 \Rightarrow x = -1$$

Sign Diagram/Wavy Curve

The denominator is always positive. The sign of the numerator decides the sign of the function:

$$x+1 > 0 \Rightarrow x > -1$$
$$(-\infty, -1) \Rightarrow -ve$$
$$(-1, \infty) \Rightarrow +ve$$

Range

To get the range, we convert the function into a quadratic in x, and apply properties of the discriminant:

$$y = \frac{x+1}{x^2+2x+2}$$

Cross multiply:

$$yx^2 + 2yx + 2y = x + 1$$

Collate all terms on LHS:

$$yx^2 + (2y-1)x + 2y - 1 = 0$$

The above is a quadratic, and x takes on only real values, which means the discriminant must be non-negative:

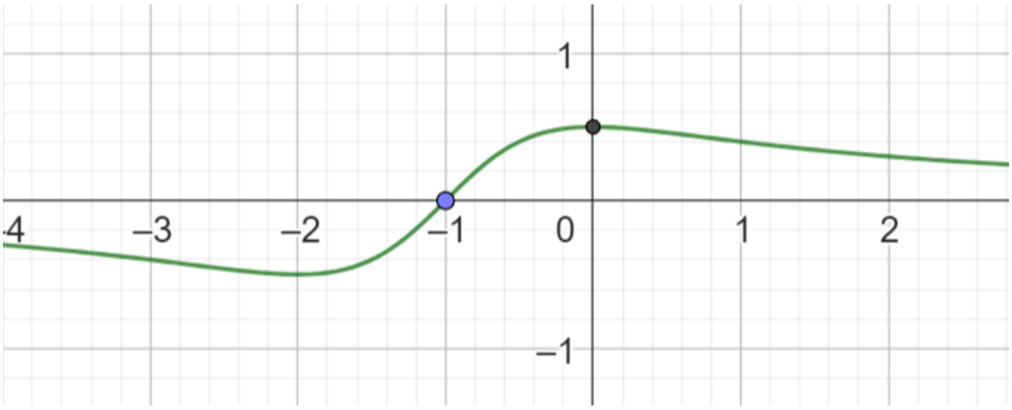
$$b^2 - 4ac \ge 0$$
$$(2y-1)^2 - 4(y)(2y-1) \ge 0$$
$$4y^2 - 4y + 1 - 8y^2 + 4y \ge 0$$
$$-4y^2 + 1 \ge 0$$
$$4y^2 - 1 \le 0$$
$$y^2 \le \frac{1}{4}$$
$$-\frac{1}{2} \le y \le \frac{1}{2}$$

The x value corresponding to the maximum is at the solution for:

$$\frac{1}{2} = \frac{x+1}{x^2+2x+2}$$

The x value corresponding to the minimum is at the solution for:

$$-\frac{1}{2}=\frac{x+1}{x^2+2x+2}$$

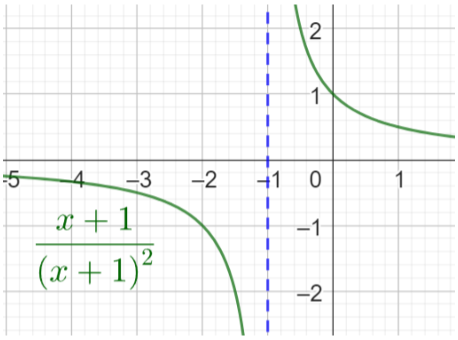


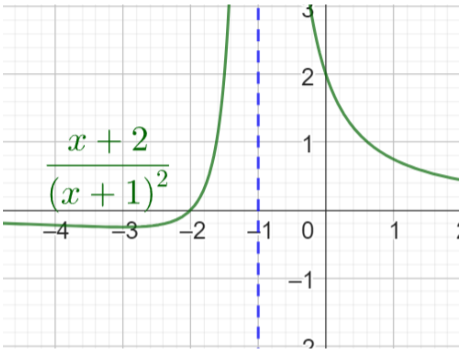
4.22: Real, repeated Roots

If a quadratic has a zero discriminant, then it will have a single repeated root.

The associated $\frac{Linear}{Quadratic}$ function will have a vertical asymptote at the repeated root.

- If there is a cancelling factor in the numerator $\left(Eg: f(x)=\frac{x+1}{(x+1)^2}\right)$, then the function will be unbounded in two different directions when $x\to -1$
- If the numerator does not have a cancelling factor $\left(Eg: f(x)=\frac{x+2}{(x+1)^2}\right)$, then the function will be unbounded in the same direction when $x\to -1$





Example 4.23: Quadratic with real, repeated roots

$$f(x)=\frac{x+1}{(x+1)^2}$$

Vertical Asymptote

$$(x+1)^2=0\Rightarrow x=-1$$

For $x\neq -1$, the function simplifies to:

$$f(x)=\frac{x+1}{(x+1)^2}=\frac{1}{x+1}$$

Consider the behaviour to the right of the vertical asymptote:

$$f(-0.9) = \frac{1}{-0.9+1} = \frac{1}{0.1} = 10 \to +ve$$
$$f(-0.99) = \frac{1}{-0.99+1} = \frac{1}{0.01} = 100 \to +ve$$

Consider the behaviour to the left of the vertical asymptote:

$$f(-1.1) = \frac{1}{-1.1+1} = \frac{1}{-0.1} = -10 \to -ve$$
$$f(-1.01) = \frac{1}{-1.01+1} = \frac{1}{-0.01} = -100 \to -ve$$

In general:

$$x \to -1 \; from\ the\ right \Rightarrow f(x) \to \infty$$
$$x \to -1 \; from\ the\ left \Rightarrow f(x) \to -\infty$$

Domain

The function is valid for inputs except where the denominator is not defined (which is $x = -1$).

$$Domain_f = D_f = (-\infty, -1) \cup (-1, \infty) = \mathbb{R} \setminus \{-1\}$$

Horizontal Asymptote

$$x \neq -1 \Rightarrow f(x) = \frac{1}{x+1}$$
$$x \to \infty \Rightarrow y \to 0$$
$$x \to -\infty \Rightarrow y \to 0$$

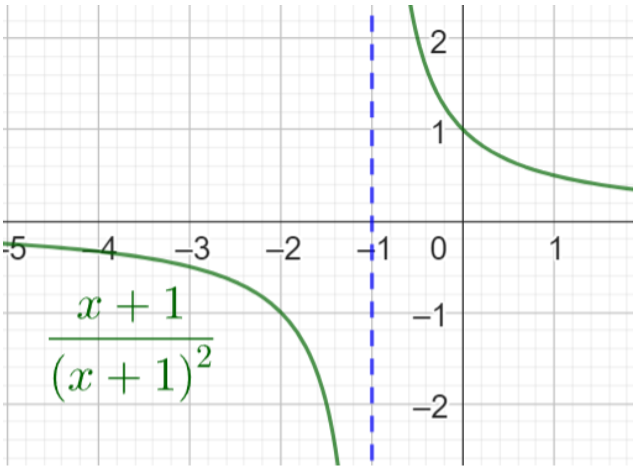
Intercepts

To find the y −intercept, substitute $x = 0$

$$y = \frac{1}{0+1} = 1 \Rightarrow y - intercept = 1$$

To find the x −intercept, substitute $y = 0$

$$0 = \frac{1}{x+1} \Rightarrow 0 = 1 \Rightarrow No\ Solutions \Rightarrow No\ x\ intercept$$



Range

$$y = \frac{1}{x+1}$$
$$yx + y - 1 = 0$$
$$yx = 1 - y$$
$$x = \frac{1-y}{y}$$

x cannot take values where the denominator is zero:

$$y \neq 0$$

And when calculating the vertical asymptote, we saw that the function is unbounded both to positive infinity and to negative infinity near the vertical asymptote.

Hence, the range is all real numbers except 0:

$$Range\ of\ f = R_f = \mathbb{R}\setminus\{0\}$$

4.24: Real, and unequal Roots

If a quadratic has a positive discriminant, then it will have distinct real roots.

The associated $\frac{Linear}{Quadratic}$ function will have

- Two vertical asymptotes at the two roots if there is no cancelling factor in the numerator $\left(Eg: f(x) = \frac{x-4}{(x-3)(x+2)}\right)$
- If there is a "cancelling factor" in the numerator, there will be a vertical asymptote (VA) at the non-cancelled root, and a removable discontinuity (RD) at the cancelled root.

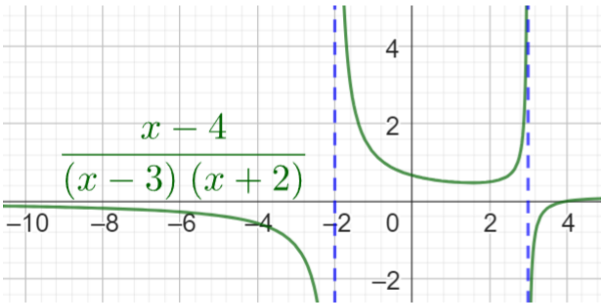
$$\left(Eg: f(x) = \frac{x-3}{(x-3)(x+2)}, Vertical\ Asymptote\ at\ -2, Removable\ discontinuity\ at\ 3\right)$$

Example 4.25: Quadratic with real, and unequal roots

$$f(x) = \frac{x-4}{(x-3)(x+2)}$$

Vertical Asymptotes

$$(x-3)(x+2) = 0 \Rightarrow x \in \{-2,3\}$$



E. $\frac{Quadratic}{Quadratic}$

4.3 Method of Undetermined Coefficients

A. Solving Equations to Find Variables

Usually, algebra questions ask us to find the value of a variable given an equation. Equations can be of various types:

- Linear Equation: The highest power of the variable is one.

$$x + 5 = 7 \Rightarrow x = 2$$

- Quadratic Equation: The highest power of the variable is two.

$$x^2 + 5x + 6 = 0 \Rightarrow x \in \{-2, -3\}$$

- Cubic Equation: The highest power of the variable is three.

$$(x+1)(x+2)(x+3) = 0 \Rightarrow x \in \{-1, -2, -3\}$$

- Biquadratic Equation: The highest power of the variable is four.

$$(x^2 + 5x + 7)(2x^2 + 3x + 4) = 0$$

Example 4.26

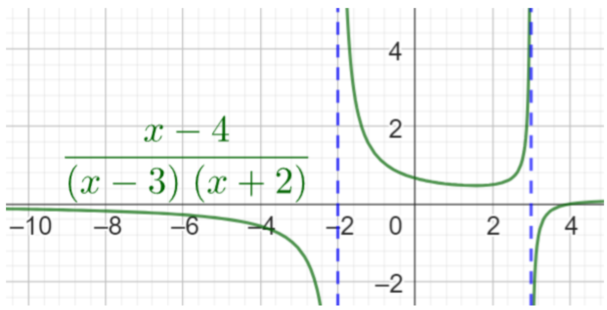
Find the sum of the roots of $x^3 - 4x^2 - 21x = 0$.

Solution:

Divide both sides by x:

$$\underbrace{x^2 - 4x - 21 = 0}_{Step\ I} \Rightarrow \underbrace{(x+3)(x-7) = 0}_{Step\ II} \Rightarrow \underbrace{x \in \{-3,7\}}_{Step\ III}$$

$$\Rightarrow -3 + 7 = 4$$



The answer is correct. Explain whether the method is also correct. If the method is not correct, then state the incorrect step and the flaw in that incorrect step.

Mistake in the Above Solution

In Step I of the solution, we divide by x. However, we do not know the value of x.
In particular, if $x = 0$, then we cannot divide 0, which means we cannot divide by x unless we ensure that $x \neq 0$.

Correct Solution

We ensure that $x \neq 0$ by taking up some casework.

$$x^3 - 4x^2 - 21x = 0$$

Consider two cases:
Case I: $x = 0$

$$x^3 - 4x^2 - 21x = 0^3 - 4 \cdot 0^2 - 21 \cdot 0 = 0 \Rightarrow 0 = 0 \Rightarrow Satisfies \Rightarrow 0\ is\ a\ root$$

Case II: $x \neq 0$
Now we are in a position to divide by zero, and the solution given in the question is now correct (for this case):

$$\underbrace{x^2 - 4x - 21 = 0}_{Step\ I} \Rightarrow \underbrace{(x+3)(x-7) = 0}_{Step\ II} \Rightarrow \underbrace{x \in \{-3,7\}}_{Step\ III} \Rightarrow -3 + 7 = 4$$

And hence the roots are:

$$Roots \in \{-3,0,7\} \Rightarrow Sum = 3 + 0 + 7 = 4$$

Example 4.27

Find the sum of the roots of $x^3 - 4x^2 - 21x = 0$ using factoring and the zero-product property.

Factor:

$$x(x^2 - 4x - 21) = 0 \Rightarrow x(x+3)(x-7) = 0$$

Apply the zero-product property:

$$x = 0\ OR\ x + 3 = 0 \Rightarrow x = -3\ OR\ x - 7 = 0 \Rightarrow x = 7$$

And hence the roots are:

$$x \in \{-3,0,7\}$$

Example 4.28

A. I have zero apples, which I share among five people. How many apples will each person get?
B. I have five apples, which I share among zero people. How many apples will each person get?
C. I have zero apples, which I share among zero people. How many apples will each person get?

$$\frac{0}{5} = 0$$

$$\frac{5}{0}\ is\ not\ defined\ (Cannot\ divide\ by\ zero)$$

$$\frac{0}{0}\ is\ not\ defined$$

B. Determining Coefficients

In certain situations, our objective is not to find the value of a variable by solving an equation. Instead, we want to find the values of the coefficients of an expression.
This is often done using the method of undetermined coefficients, which is explained next. This method is the

focus of this entire chapter.

4.29: Method of Undetermined Coefficients

The method of undetermined coefficients lets us find the values of coefficients in two expressions that we know are equal.

Suppose we know that the equality below is true for all values of x:

$$Ax^4 + Bx^3 + Cx^2 + Dx + E = Px^4 + Qx^3 + Rx^2 + Sx + T$$

This will only happen, when the coefficients of each term are equal. Then we can conclude:

$$Ax^4 = Px^4 \Rightarrow A = P$$
$$Bx^3 = Qx^3 \Rightarrow B = Q$$
$$Cx^2 = Rx^2 \Rightarrow C = R$$
$$Dx = Sx \Rightarrow D = S$$
$$E = T$$

What we have done is equated the coefficients on each side of the equality.

- This is a broad technique and has many powerful uses. We will use it in partial fraction decomposition, for example, but its use is not restricted to any chapter.
- Learning to recognize and use this technique is key to many "difficult" questions where it plays a part.

Example 4.30

Find the values of A and B, given that the equality below is true for all values of x:

$$Ax + B = 5x + 4$$

The coefficient of the x term on the LHS must equal the coefficient of the x term on the RHS:

$$Ax = 5x \Rightarrow A = 5$$

Similarly, the constant term on the LHS must equal the constant term on the RHS:

$$B = 4$$

So, the final answer is

$$A = 5, \qquad B = 4$$

4.31: Applicability of Method of Undetermined Coefficients

The method is applicable when we want to find coefficients such that the equality is always true, independent of the value of the variable.

This can be expressed in a number of ways, which are all equivalent, and mean the same thing:

- Given that the equation $Ax + B = 5x + 4$ is true for all values of x.
- Given that the equation $Ax + B = 5x + 4$ holds independent of the value of x.
- Given that $Ax + B = 5x + 4$ is an identity. (Identity means always true)
- Given that $Ax + B \equiv 5x + 4$. (where $\equiv$ is the symbol for identity)

Example 4.32

Find the values of A and B, given that the below equality is true for all values of x:

$$(A + B)x + B = 2x - 6$$

Equate the coefficients on both sides:

$$\text{Linear Term}: A + B = 2$$
$$\text{Constant Term}: B = -6$$

$$A + B = 2 \Rightarrow A - 6 = 2 \Rightarrow A = 8$$

Example 4.33

The equality below holds independent of the value of x:

$$Ax + Bx + A - B = 3x + 2$$

A. Write the above in a form that makes it possible to compare coefficients
B. Compare coefficients to find the values of A, and B.

$$(\boldsymbol{A} + \boldsymbol{B})x + (\boldsymbol{A} - \boldsymbol{B}) = \mathbf{3}x + \mathbf{2}$$

$$A + B = 3$$
$$A - B = 2$$

Add the two equations:

$$A = \frac{5}{2} \Rightarrow B = \frac{1}{2}$$

4.34: Number of Solutions

In a system of equations,

- if the number of variables equals the number of "useful" equations, then we will have
- if the number of variables is more than the number of "useful" equations, then we will have infinite solutions.
- if the number of variables is less than the number of "useful" equations, then we will have no solutions.

The term useful equation here means that equations do not repeat information. For example:

$$x + y = 3 \Rightarrow 2x + 2y = 6$$

Are essentially the same equation since multiplying the first equation by two gives the second equation. Hence, we have only one useful equation.

As the above shows, the existence of solutions is not guaranteed. Hence, in a question, it is your responsibility to ensure that a solution can be found.

$$x + y = 3 \Rightarrow \text{Infinite Solutions}$$
$$x + 2 = 3 \Rightarrow \text{Single Solution}$$
$$x + 2 = 3, \qquad x + 2 = 5 \Rightarrow \text{No Solution}$$

Example 4.35

Determine the number of solutions for A, B, and C given that:

$$Ax + Bx + A + B + C \equiv 7x + 2$$

Note: $\equiv$ here represents an identity and not mod arithmetic.

$$(A + B)x + A + B + C \equiv 7x + 2$$

$$\text{Constant Term}: A + B + C = 2$$
$$A + B = 7$$

Substitute $A + B = 7$ in $A + B + C = 2$:

$$7 + C = 2 \Rightarrow C = -5 \Rightarrow Single\ Solution$$

But

$$A + B = 7 \Rightarrow A = 7 - B \Rightarrow Infinite\ Solutions\ for\ A\ and\ B$$

(Continuation) Example 4.36

In the above, if we add the restriction that A, B and C are positive integers, then what are the solutions for (A, B, C). Write each set of solutions as an ordered triplet, or write ϕ if there are no solutions.

From the above:

$$C = -5 \Rightarrow No\ Solutions$$

(Continuation) Example 4.37

In the above, if we add the restriction that A and B are positive integers, then what are the solutions for (A, B, C). Write each set of solutions as an ordered triplet.

$$C = -5$$
$$A + B = 7$$

$$\{1,6,-5\}, \{2,5,-5\}, \{3,4,-5\}, \{4,3,-5\}, \{5,2,-5\}, \{6,1,-5\}$$

C. Finding Functions

The method of undetermined coefficients has great use in finding functions that meet certain conditions. This is often very important in modelling situations, though we will see it in the context of mathematical problems.

Example 4.38

Find $f(x)$ given that $f(x)$ is a linear function such that:

$$f(2) = 4, \qquad f(4) = 7$$

Since $f(x)$ is a linear function, it must be of the form:

$$f(x) = Ax + B$$

Substitute the given values and compare:

$$f(2) = \underbrace{2A + B = 4}_{\textit{Equation I}}$$
$$f(4) = \underbrace{4A + B = 7}_{\textit{Equation II}}$$

We now have two equations in two variables, which we solve.
Multiply Equation I by 2 giving us, and then subtract Equation II from it:

$$4A + 2B = 8$$
$$4A + B = 7$$
$$B = 1$$

Substitute $B = 1$ in Equation I:

$$2A + B = 4 \Rightarrow 2A + 1 = 4 \Rightarrow A = \frac{3}{2}$$

Now, we can find the function by substituting the values of A and B in the form of the equation:

$$f(x) = Ax + B = \frac{3}{2}x + 1$$

D. Higher Powers

It is not necessary that the function be linear. If you know that the expression only has two non-zero coefficients, then also you can

Example 4.39

Given that:

$$f(x) = Ax^2 + B, \qquad f(1) = 2, \qquad f(2) = 3$$

$f(3)$ can be written in the form $\frac{a}{b}$ where a and b are natural numbers, and $HCF(a, b) = 1$. Find $a + b$.

$$f(1) = A + B = 2$$
$$f(2) = 4A + B = 3$$

$$4A + 4B = 8$$
$$4A + B = 3$$
$$3B = 5 \Rightarrow B = \frac{5}{3}$$

$$A + B = 2 \Rightarrow A = 2 - B = 2 - \frac{5}{3} = \frac{1}{3}$$

Now that we have found A and B, we can substitute to find the function:

$$f(x) = Ax^2 + B = \frac{1}{3}x^2 + \frac{5}{3}$$

And

$$f(3) = \frac{1}{3}(3^2) + \frac{5}{3} = \frac{9}{3} + \frac{5}{3} = \frac{14}{3}$$

Example 4.40

Find $f(2)$, given that:

$$f(x) = Ax^3 + B, \qquad f(3) = 1, \qquad f(1) = 3$$

$$f(3) = 27A + B = 1$$
$$f(1) = A + B = 3$$

$$26A = -2 \Rightarrow A = -\frac{1}{13}$$

$$B = 3 - A = 3 - \left(-\frac{1}{13}\right) = \frac{40}{13}$$

$$f(x) = Ax^3 + B = -\frac{1}{13}x^3 + \frac{40}{13}$$
$$f(2) = -\frac{1}{13} \times 8 + \frac{40}{13} = \frac{32}{13}$$

E. Quadratics: Revision

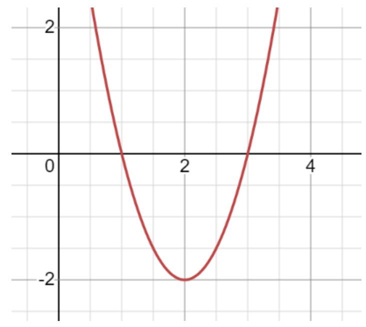
4.41: Factored Form

$$y = a(x - \alpha)(x - \beta)$$

a and β are the roots.
a is the scaling factor

$$x - coordinate\ of\ vertex = Average\ of\ Roots = \frac{a+\beta}{2}$$

Example 4.42



Consider the graph drawn alongside of:

$$y = 2(x-3)(x-1)$$

A. What are the x-intercepts?
B. What is the x coordinate of the vertex?
C. What is the y coordinate of the vertex?
D. What is the y-intercept?
E. When $x = 4$, what is y?
F. When $y = 1$, what is x?
G. Is it an upward parabola or downward parabola?

Part A

To find the x-intercept, equate $y = 0$:

$$2(x-3)(x-1) = 0 \Rightarrow x \in \{1,3\}$$

Part B

$$x\ coordinate\ of\ vertex = Average\ of\ Roots = \frac{1+3}{2} = \frac{4}{2} = 2$$

Part C

To find the y coordinate of the vertex, substitute the x coordinate of the vertex in the equation of the graph:

$$y = 2(2-3)(2-1) = 2(-1)(1) = -2$$

Part D

To find the y-intercept, substitute $x = 0$ in the equation:

$$y = 2(x-3)(x-1) = 2(0-3)(0-1) = 2(-3)(-1) = 6$$

Part E

Substitute $x = 4$ in the equation of the graph:

$$y = 2(x-3)(x-1) = 2(4-3)(4-1) = 2(1)(3) = 6$$

Part F

Substitute $y = 1$ in the equation of the graph:

$$y = 2(x-3)(x-1) \Rightarrow 1 = 2(x-3)(x-1)$$

This needs to solved as a quadratic in x:

$$0 = 2(x^2 - 4x + 3) - 1$$
$$2x^2 - 8x + 5 = 0$$

Substitute $a = 2, b = -8, c = 5$ in the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{8 \pm \sqrt{64 - 4(2)(5)}}{4} = \frac{8 \pm \sqrt{24}}{4} = \frac{8 \pm 2\sqrt{6}}{4} = \frac{4 \pm \sqrt{6}}{2}$$

4.43: Vertex Form

$$y = a(x-h)^2 + k$$

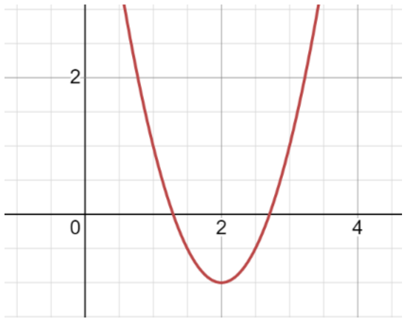
$$Vertex = (h, k)$$
$$Scale\ Factor\ is\ a$$

Example 4.44

Consider the graph drawn alongside of:

$$y = 2(x-2)^2 - 1$$

A. What are the x-intercepts?
B. What are the coordinates of the vertex?
C. What is the y-intercept?
D. When $x = 4$, what is y?
E. When $y = 1$, what is x?
F. Is it an upward parabola or downward parabola?



Part A: Using the Quadratic Formula

To find the x-intercepts, substitute $y = 0$ in the equation of the graph:

$$0 = 2(x-2)^2 - 1$$
$$0 = 2(x^2 - 4x + 4) - 1$$
$$0 = 2x^2 - 8x + 7$$

Substitute $a = 2, b = -8, c = 7$ in the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{8 \pm \sqrt{64 - 4(2)(7)}}{4} = \frac{8 \pm \sqrt{8}}{4} = \frac{8 \pm 2\sqrt{2}}{4} = \frac{4 \pm \sqrt{2}}{2}$$

Part A: Method II: Completing the Square

$$0 = 2(x-2)^2 - 1$$
$$2(x-2)^2 = 1$$
$$(x-2)^2 = \frac{1}{2}$$
$$x - 2 = \pm\sqrt{\frac{1}{2}}$$
$$x = \pm\sqrt{\frac{1}{2}} + 2 = \frac{\pm\sqrt{1}}{\sqrt{2}} + 2 = \frac{\pm\sqrt{2}}{2} + 2 = \frac{4 \pm \sqrt{2}}{2}$$

Part B

$$y = 2(x-2)^2 + (-1)$$
$$y = a(x-h)^2 + k$$
$$Vertex = (2, -1)$$

Part C

To find the y-intercept, substitute $x = 0$ in the equation:

$$y = 2(x-2)^2 - 1 = 2(0-2)^2 - 1 = 8 - 1 = 7$$

Part D

Substitute $x = 4$ in the equation:

$$y = 2(x-2)^2 - 1 = 2(4-2)^2 - 1 = 8 - 1 = 7$$

Part E
Substitute $y = 1$ in the equation:

$$1 = 2(x-2)^2 - 1$$

4.45: Standard Form

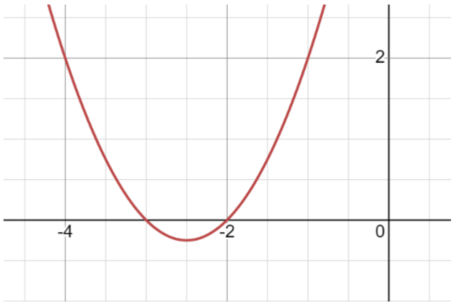
$$y = ax^2 + bx + c$$

To find the y-intercept, substitute $x = 0$ in the equation:

$$y = ax^2 + bx + c = a(0)^2 + b(0) + c = c$$

Example 4.46
Consider the graph drawn alongside of:

$$y = x^2 + 5x + 6$$



A. What are the x-intercepts?
B. What are the coordinates of the vertex?
C. What is the y-intercept?
D. When $x = 4$, what is y?
E. When $y = 1$, what is x?
F. Is it an upward parabola or downward parabola?

Part A
Substitute $y = 0$:

$$x^2 + 5x + 6 = 0 \Rightarrow (x+2)(x+3) = 0 \Rightarrow x \in \{-2, -3\}$$

Part B

$$x\ coordinate\ of\ vertex = Average\ of\ Roots = \frac{-2-3}{2} = -\frac{5}{2}$$

Substitute $x = -\frac{5}{2}$ in the equation:

$$y\ coordinate\ of\ vertex = x^2 + 5x + 6 = \left(-\frac{5}{2}\right)^2 + 5\left(-\frac{5}{2}\right) + 6 = \frac{25}{4} - \frac{20}{4} + 6 = 7\frac{1}{4}$$

Part C

$$y = x^2 + 5x + 6 = 0^2 + 5 \times 0 + 6 = 6$$

Shortcut:

$$y - intercept = 6\ by\ observation$$

F. Determining Quadratic Equations using Given Equations
In this section, we look at how to uniquely identify the equation of a quadratic given algebraic information about in terms of equations.

Example 4.47
Find the values of A, B and C, given that:

$$Ax^2 + Bx + C \equiv 2x^2 + 3x - 12$$

$$A = 2, \qquad B = 3, \qquad C = -12$$

Example 4.48

$$A(x^2+1)+B(x^2+x)+C(x+1)=3x^2+5x-4$$

1. Write the above in a form that makes it possible to compare coefficients
2. Use the method of undetermined coefficients to find the values of A, B and C.

$$Ax^2+A+Bx^2+Bx+Cx+C=3x^2+5x-4$$
$$(A+B)x^2+(B+C)x+C+A=3x^2+5x-4$$

Equate coefficients:

$$\underbrace{A+B=3}_{Equation\ I}$$
$$\underbrace{B+C=5}_{Equation\ II}$$
$$\underbrace{C+A=-4}_{Equation\ III}$$

We can exploit the symmetry/cyclicity in the equations by adding the three of them:

$$2(A+B+C)=4\Rightarrow\underbrace{A+B+C=2}_{Equation\ IV}$$

Now, subtract Equation I from IV:

$$C=-1$$

Now, subtract Equation II from IV:

$$A=-3$$

Now, subtract Equation III from IV:

$$B=6$$

G. Determining Quadratic Equations using Graphs

In this section, we look at how to uniquely identify the equation of a quadratic given

- points on the graphs
- the graph itself
- information about the graph
- any combination of the above

Given information about the graph of a quadratic recall that there are 3 forms of the quadratic that you can use to determine the equation:

$$Factored\ Form: y=a(x-\alpha)(x-\beta)\Rightarrow\underbrace{a}_{Scaling\ Factor},\underbrace{\alpha,\beta}_{Roots}$$
$$Vertex\ Form: y=a(x-h)^2+k\Rightarrow a,\underbrace{h,k}_{Vertex}$$
$$Standard\ Form: y=ax^2+bx+c\Rightarrow a,b,\underbrace{c}_{y-intercept}$$

	Factored Form	Vertex Form	Standard
Roots / x –intercept	α,β	Convert to Standard Form	Factor Use Quadratic Formula Complete the Square
y $-$ *intercept*	Substitute $x=0$	Substitute $x=0$	c

Vertex	$x: \frac{\alpha+\beta}{2}$ $y: Substitute\ \frac{\alpha+\beta}{2}\ in\ equation$	(h,k)	$x: -\frac{b}{2a}$ $y: Substitute\ -\frac{b}{2a}\ in\ equation$
Upward or Downward	$a>0 \Rightarrow Upward$ $a<0 \Rightarrow Downward$	$a>0 \Rightarrow Upward$ $a<0 \Rightarrow Downward$	$a>0 \Rightarrow Upward$ $a<0 \Rightarrow Downward$

If we know the intercepts that a parabola makes with the axes, that is sufficient to find the equation of the parabola.

Example 4.49

The y-intercept of a quadratic function is 1, and the x-intercepts are 2 and 3. Find the function using the factored form.

Strategy

x intercepts are also the roots. Hence, we already know 2 out of 3 coefficients that we need for the factored form. If we are able to find the third coefficient, then we have found the equation:

Execute

Substitute $\alpha = 2, \beta = 3, (x,y) = (0,1)$ in $y = a(x-\alpha)(x-\beta)$:

$$1 = a(0-2)(0-3) \Rightarrow 1 = 6a \Rightarrow a = \frac{1}{6}$$

$$y = a(x-\alpha)(x-\beta) = \frac{1}{6}(x-2)(x-3)$$

In the above, find the function using the standard form.

Substitute $c = 1$ in the standard form $y = ax^2 + bx + c$:

$$y = ax^2 + bx + 1$$

Now we have two x-intercepts given by $(2,0)$ and $(3,0)$.

Substitute $(2,0)$:

$$0 = 4a + 2b + 1$$

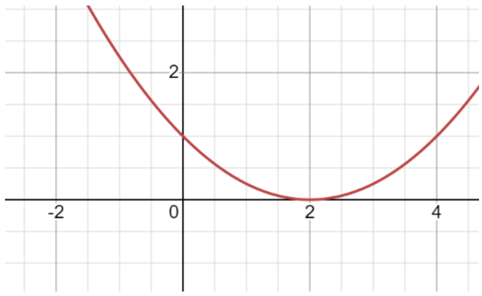
Substitute $(3,0)$:

$$0 = 9a + 3b + 1$$

As you can see, this method is lengthier than the other method.
So, choice of method is important.

Example 4.50

The y-intercept of a quadratic function is 1, and the x-intercept is 2. Find the function using the factored form.



Root of multiplicity 2.
The factored form of a quadratic is

$$y = a(x-\alpha)(x-\beta)$$

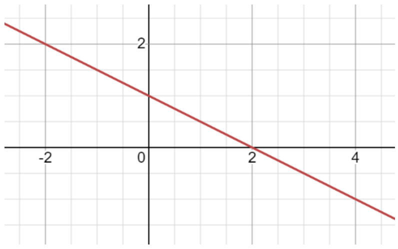
Substitute $\alpha = \beta = 2, (x,y) = (0,1)$ in the above:

$$a(0-2)(0-2) = 1 \Rightarrow 4a = 1 \Rightarrow a = \frac{1}{4}$$

And, hence the final answer is:

$$y = \frac{1}{4}(x-2)^2$$

The y-intercept of a function is 1, and the x-intercept is 2. Find the simplest form of the function.

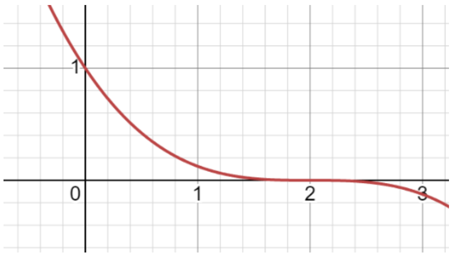


The simplest form of a function that passes through two points is a line.
Hence, we use the linear function

$$y = mx + c$$

$$y = -\frac{1}{2}x + 1$$

The y-intercept of a cubic function is 1, and the x-intercept is 2. Find the function.



Substitute $\alpha = \beta = \gamma = 2, (x,y) = (0,1)$ in the factored form of a quadratic $y = a(x-\alpha)(x-\beta)(x-\gamma)$:

$$a(0-2)(0-2)(0-2) = 1 \Rightarrow -8a = 1 \Rightarrow a = -\frac{1}{8}$$

And, hence the final answer is:

$$y = -\frac{1}{8}(x-2)^3$$

The y-intercept of a polynomial of degree n function is 1, and the x-intercept is 2. Find the function.

$$y = a(x-\alpha)^n \Rightarrow 1 = a(0-2)^n \Rightarrow a = \frac{1}{(-2)^n}$$

Example 4.51: Vertex and x-intercept

The vertex of a parabola is at $(3,5)$. The x-intercept is 5. Find the equation of the parabola.

Substitute $(h,k) = (3,5)$ in the vertex form of a parabola:

$$y = a(x-h)^2 + k \Rightarrow y = a(x-3)^2 + 5$$

Substitute $(x,y) = (5,0)$ in the above:

$$0 = a(5-3)^2 + 5 \Rightarrow 0 = 4a + 5 \Rightarrow a = -\frac{5}{4}$$

$$y = -\frac{5}{4}(x-3)^2 + 5$$

Example 4.52: Vertex and additional point

The vertex of a parabola is at (3,5). (1,6) lies on the parabola. Find the equation of the parabola.

Substitute $(x, y) = (1,6)$ in $y = a(x-3)^2 + 5$:

$$6 = a(1-3)^2 + 5 \Rightarrow 6 = 4a + 5 \Rightarrow a = \frac{1}{4}$$

$$y = \frac{1}{4}(x-3)^2 + 5$$

H. Cubic Equations

I. Self-Inverse Functions

4.4 Partial Fractions

A. Basics

Partial fraction decomposition is an Algebra technique.

4.53: Partial Fraction Decomposition

Writing a function with a denominator of quadratic (or higher degree) so that it is expressed as the sum of fractions of lower degree is partial fraction decomposition.

For example:

$$\frac{px+q}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

The above is abstract, so let's look at an example of what we mean.

4.54: Simultaneous Equations Method

To find values A and B such that:

$$\frac{px+q}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

- Multiply by $(x+a)(x+b)$
- Expand and simplify the RHS
- Equate coefficients
- Solve the resulting system of equations to find the values of A and B.

Example 4.55

Split using partial fraction decomposition:

$$\frac{1}{2x^2 - 3x - 5}$$

Factor the quadratic in the denominator $\left(Sum = -3, Product = -10 = (-5)(+2)\right)$:

$$\frac{1}{2x^2 - 5x + 2x - 5} = \frac{1}{x(2x-5) + 1(2x-5)} = \frac{1}{(2x-5)(x+1)}$$

We wish to write the above expression as a sum in the following way. There are some constants A and B for which the equation below is an identity:

$$\underbrace{\frac{1}{(2x-5)(x+1)} = \frac{A}{2x-5} + \frac{B}{x+1}}_{Identity\ I}$$

To determine A and B, begin by eliminating fractions. Multiply both sides by $(2x-5)(x+1)$:

$$1 = A(x+1) + B(2x-5)$$

Use the distributive property

$$1 = Ax + A + 2Bx - 5B$$

Collate like terms and factor:

$$0x + 1 = (A+2B)x + A - 5B$$

Use the method of undetermined coefficients. Since we know (from Identity I) that the equation is always true, equate coefficients on the left with coefficients on the right:

$$Equate\ x\ coefficient: \underbrace{A + 2B = 0}_{Equation\ I}$$

$$Equate\ Constant\ Coefficient: \underbrace{A - 5B = 1}_{Equation\ II}$$

We now have a system of equations with two variables and two equations. Subtract Equation II from Equation I:

$$7B = -1 \Rightarrow B = -\frac{1}{7}$$

Substitute the above value of B in Equation I:

$$A + 2\left(-\frac{1}{7}\right) = 0 \Rightarrow A = \frac{2}{7}$$

Substitute $A = \frac{2}{7}, B = -\frac{1}{7}$ in Identity I

$$\frac{1}{(2x-5)(x+1)} = \frac{\frac{2}{7}}{2x-5} + \frac{-\frac{1}{7}}{x+1}$$

Example 4.56

$$\frac{5x+1}{x^2-x-12} = \frac{A}{x+a} + \frac{B}{x+b}$$

If the above is an identity, find constants A, B, a, and b.

Factor the denominator:

$$x^2 - x - 12 = (x+3)(x-4)$$

If we use the values on the RHS in the denominators, then our denominators will match when we add by taking the LCM. We want:

$$\frac{5x+1}{(x+3)(x-4)} = \frac{A}{x+3} + \frac{B}{x-4}$$

Eliminate fractions by multiplying both sides by $(x+3)(x-4)$:

$$5x + 1 = A(x-4) + B(x+3)$$

Expand and simplify the right-hand side:

$$5x + 1 = Ax - 4A + Bx + 3B$$

Collate like terms together:

$$5x + 1 = (A + B)x - 4A + 3B$$

Use the method of undetermined coefficients. Since the value on the left must equal the value on the right, the coefficients must be equal:

$$5 = A + B$$
$$1 = -4A + 3B$$

This is a system of equations in two variables. Multiply the first equation by 4, and add it to the second equation:

$$20 = 4A + 4B$$
$$1 = -4A + 3B$$

We get:

$$21 = 7B \Rightarrow B = 3$$

Substituting $B = 3$ in $5 = A + B$, we get:

$$5 = A + 3 \Rightarrow A = 2$$

And hence, the final answer is:

$$\frac{5x + 1}{(x + 3)(x - 4)} = \frac{2}{x + 3} + \frac{3}{x - 4}$$

4.57: Substitution Method

To find values A and B such that:

$$\frac{px + q}{(x + a)(x + b)} = \frac{A}{x + a} + \frac{B}{x + b}$$

- Multiply by $(x + a)(x + b)$
- Substitute Values to make some terms zero
- Simplify the remaining equations

Example 4.58

$$\frac{5x + 1}{x^2 - x - 12} = \frac{A}{x + a} + \frac{B}{x + b}$$

If the above is an identity, find constants A, B, a, and b.

We did this same question earlier, so we know that:

$$5x + 1 = A(x - 4) + B(x + 3)$$

If we substitute $x = 4$, then the term with A will "vanish":

$$5(4) + 1 = A(4 - 4) + B(4 + 3)$$
$$21 = 7B$$
$$B = 3$$

Similarly, if we substitute $x = -3$, then the term with B will "vanish":

$$5x + 1 = A(x - 4)$$
$$-14 = -7A$$
$$A = 2$$

And hence the values are

$$B = 3, \qquad A = 2$$

4.59: Degree of Numerator and Denominator

For partial fraction decomposition to work

$$Degree\ of\ Denominator > Degree\ of\ Numerator$$

If this is not the case, we polynomial long division[8] to reduce the fraction so that the degree of the denominator is less than the degree of the numerator.

Example 4.60

Find the partial fraction decomposition of:

$$f(x) = \frac{2x+13}{x^2+13x+30}$$

$$\frac{2x+13}{x^2+13x+30} = \frac{x+3+x+10}{(x+3)(x+10)} = \frac{1}{x+3} + \frac{1}{x+10}$$

Partial Fractions Method:

$$\frac{2x+13}{(x+3)(x+10)} = \frac{A}{x+3} + \frac{B}{x+10}$$

Multiply both sides by $x+3$:

$$\frac{2x+13}{(x+10)} = A + \frac{B(x+3)}{x+10}$$

Substitute $x = -3$ in the above:

$$\frac{2(-3)+13}{(-3+10)} = A + \frac{B(-3+3)}{x+10}$$

The term with B will become zero:

$$\frac{7}{7} = A + \frac{B(0)}{x+10}$$

Hence, we can get the value of A:

$$A = 1$$

Shortcut

To find the value of A, substitute $x+3=0 \Rightarrow x=-3$ after covering up the $x+3$ term:

$$\frac{2x+13}{(\boldsymbol{x+3})(x+10)} \to \frac{2(-3)+13}{(\boldsymbol{x+3})(-3+10)} = \frac{7}{7} = 1$$

To find the value of B, substitute $x+10=0 \Rightarrow x=-3$ after covering up the $x+3$ term:

$$\frac{2x+13}{(x+3)(\boldsymbol{x+10})} \to \frac{2(-10)+13}{(-10+3)} = \frac{-7}{-7} = 1$$

B. Three Linear Factors

If the number of factors increases, the method does not change, though the calculations do increase.

Example 4.61

Find the partial fraction decomposition for:

$$\frac{3}{(x+3)(x+4)(x-5)}$$

[8] Recall that Synthetic Division only works if the divisor is linear.

We are looking for constants A, B and C such that:

$$\frac{3}{(x+3)(x+4)(x-5)}=\frac{A}{(x+3)}+\frac{B}{(x+4)}+\frac{C}{(x-5)}$$

Multiply both sides by $(x+3)(x+4)(x-5)$:

$$3=A(x+4)(x-5)+B(x+3)(x-5)+C(x+3)(x+4)$$

Substitute $x=-3$. The B term become zero. So does the C term:

$$3=A(x+4)(x-5)$$
$$3=A(-3+4)(-3-5)=3=A(1)(-8)$$
$$A=-\frac{3}{8}$$

Substitute $x=-4$. The C term becomes zero. So does the A term:

$$3=B(x+3)(x-5)$$
$$3=B(-4+3)(-4-5)\Rightarrow 3=B(-1)(-9)$$
$$B=\frac{1}{3}$$

Substitute $x=5$. The A term become zero. So does the B term:

$$3=C(5+3)(5+4)\Rightarrow 3=C(8)(9)$$
$$C=\frac{1}{24}$$

And hence we can say that:

$$\frac{3}{(x+3)(x+4)(x-5)}=\frac{-3}{8(x+3)}+\frac{1}{3(x+4)}+\frac{1}{24(x-5)}$$

$$3=A(x+4)(x-5)\Rightarrow 3=A(-3+4)(-3-5)=3=A(1)(-8)\Rightarrow A=-\frac{3}{8}$$

Shortcut Method

$$\frac{3}{\boldsymbol{(x+3)}(-3+4)(-3-5)}=\frac{3}{(1)(-8)}=-\frac{3}{8}$$
$$\frac{3}{(-4+3)\boldsymbol{(x+4)}(-4-5)}=\frac{3}{(-1)(-9)}=\frac{3}{9}=\frac{1}{3}$$
$$\frac{3}{(5+3)(5+4)\boldsymbol{(x-5)}}=\frac{3}{8(9)}=\frac{1}{24}$$

Example 4.62

Find the partial fraction decomposition of

$$\frac{6x^2-7x-1}{(x-1)(x+1)(x-2)}$$

$$\frac{6x^2-7x-1}{(x-1)(x+1)(x-2)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x-2}$$

Example 4.63

Let $p, q, and\ r$ be the distinct roots of the polynomial $x^3 - 22x^2 + 80x - 67$. It is given that there exist real numbers $A, B, and\ C$ such that

$$\frac{1}{s^3 - 22s^2 + 80s - 67} = \frac{A}{s-p} + \frac{B}{s-q} + \frac{C}{s-r}$$

for all $s \notin \{p, q, r\}$. What is $\frac{1}{A} + \frac{1}{B} + \frac{1}{C}$? **(AMC 10A 2019/24)**

Multiply both sides by $(s-p)(s-q)(s-r)$:

$$1 = A(s-q)(s-r) + B(s-p)(s-r) + C(s-p)(s-q)$$

Substitute $s = p$:

$$1 = A(p-q)(p-r) \Rightarrow \frac{1}{A} = (p-q)(p-r) = p^2 - \boldsymbol{pr} - \boldsymbol{pq} + \boldsymbol{qr}$$

Substitute $s = q$:

$$1 = B(q-p)(q-r) \Rightarrow \frac{1}{B} = (q-p)(q-r) = q^2 - \boldsymbol{qr} - \boldsymbol{pq} + \boldsymbol{pr}$$

Substitute $s = r$

$$1 = C(r-p)(r-q) \Rightarrow \frac{1}{C} = (r-p)(r-q) = r^2 - \boldsymbol{qr} - \boldsymbol{pr} + \boldsymbol{pq}$$

Add the above three:

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} = p^2 + q^2 + r^2 - (pr + pq + qr)$$

Make the substitution $p^2 + q^2 + r^2 = (p+q+r)^2 - 2(pq + pr + rq)$

$$= (p+q+r)^2 - 3(pq + pr + rq)$$

$$Sum\ of\ Roots = p + q + r = -\frac{b}{a} = -\frac{-22}{1} = 22$$

$$pr + pq + qr = \frac{c}{a} = \frac{80}{1} = 80$$

$$= 22^2 - 3(80) = 484 - 240 = 244$$

C. Linear Repeated Factors

When you have a factor repeated twice, you will need to find a constant for:

- the factor repeated once
- the factor repeated twice

4.64: Numerator for Factor

$$\frac{Ax + B}{(ax+b)^2} = \frac{C}{ax+b} + \frac{D}{(ax+b)^2}$$

$$\frac{C}{ax+b} + \frac{D}{(ax+b)^2} = \frac{C(ax+b)}{(ax+b)^2} + \frac{D}{(ax+b)^2} = \frac{Cax + b + d}{(ax+b)^2}$$

Example 4.65

Find the partial fraction decomposition for:

$$\frac{3}{(x+3)(x+4)^2}$$

$$\frac{3}{(x+3)(x+4)^2}=\frac{A}{(x+3)}+\frac{B}{(x+4)}+\frac{C}{(x+4)^2}$$

Multiply both sides by $(x+3)(x+4)^2$:

$$\frac{3(x+3)(x+4)^2}{(x+3)(x+4)^2}=\frac{A(x+3)(x+4)^2}{(x+3)}+\frac{B(x+3)(x+4)^2}{(x+4)}+\frac{C(x+3)(x+4)^2}{(x+4)^2}$$

Simplify:

$$3=A(x+4)^2+B(x+3)(x+4)+C(x+3)$$

Substitute $x=-3$:

$$3=A(x+4)^2\Rightarrow 3=A(-3+4)^2\Rightarrow 3=A$$

Substitute $x=-4$. The A and B terms become zero:

$$3=C(x+3)\Rightarrow 3=C(-4+3)\Rightarrow C=-3$$

Now we don't have any more values to substitute, and hence we must fall back on the simultaneous equations approach.

$$3=A(x+4)^2+B(x+3)(x+4)+C(x+3)$$

Expand the terms to get:

$$0x+3=A(x^2+8x+\mathbf{16})+B(x^2+7x+\mathbf{12})+C(x+\mathbf{3})$$

Expanding the entire RHS would be a task, but we don't need to. Use the method of undetermined coefficients and compare *only the constant term*:

$$Constant\ Term\ on\ LHS=Constant\ Term\ on\ RHS$$

$$3=16A+12B+3C\Rightarrow B=\frac{3-16A-3C}{12}=\underbrace{\frac{3-16(3)-3(-3)}{12}}_{\text{Substitute } A=3, C=-3}=\frac{-36}{12}=-3$$

And hence we can say that:

$$\frac{3}{(x+3)(x+4)^2}=\frac{3}{(x+3)}-\frac{3}{(x+4)}-\frac{3}{(x+4)^2}$$

4.66: Factor repeated n times

When you have a factor repeated n times, you will need to find a constant for:

- the factor repeated once
- the factor repeated twice
- and so on till
- the factor repeated n times

Example 4.67

Write the split form of the partial fraction decomposition of the following fractions. (Do not find the actual constants, just write the form).

A. $\frac{2x+5}{(x+1)(x-5)^2(x+2)^3}$

$$\frac{2x+5}{(x+1)(x-5)^2(x+2)^3}=\frac{A}{x+1}+\frac{B}{x-5}+\frac{C}{(x-5)^2}+\frac{D}{x+2}+\frac{E}{(x+2)^2}+\frac{F}{(x+2)^3}$$

D. Irreducible Quadratic Factors

4.68: Irreducible Quadratic Factors

Quadratic expressions that cannot be factored over the real numbers are irreducible for us.
They can be factored over the complex numbers, but that is not useful for us.

4.69: Degree of Numerator and Denominator

For partial fraction decomposition to work

$$\textit{Degree of Denominator} > \textit{Degree of Numerator}$$

Example 4.70

$$\frac{x^2}{x^4-1}$$

$$\frac{x^2}{(x+1)(x-1)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}$$

Multiply by the denominator of the LHS:

$$x^2 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x+1)(x-1)$$

Substitute $x = 1$:

$$1 = B(2)(2) \Rightarrow B = \frac{1}{4}$$

Substitute $x = -1$:

$$1 = A(-1-1)(1+1) \Rightarrow A = -\frac{1}{4}$$

Substituting $x = 0, A = -\frac{1}{4}, B = \frac{1}{4}$:

$$0 = -\frac{1}{4}(-1)(1) + \frac{1}{4}(1)(1) + (D)(1)(-1)$$

$$0 = \frac{1}{4} + \frac{1}{4} - D$$

$$D = \frac{1}{2}$$

Use the method of undetermined coefficients, and compare the coefficients of the x^3 term:

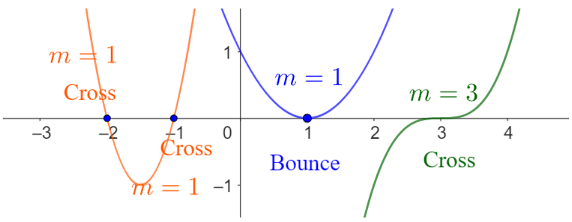
$$0 = A + B + C \Rightarrow 0 = -\frac{1}{4} + \frac{1}{4} + C \Rightarrow C = 0$$

4.5 Wavy Curve: Polynomials

A. Basics

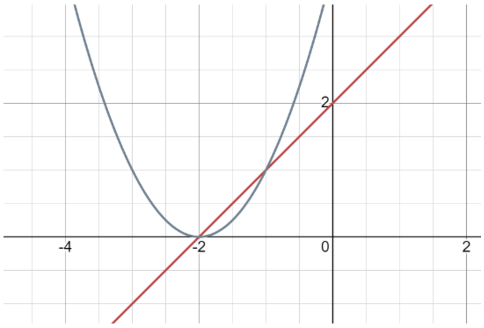
4.71:Roots

- At a root, the function has value zero.
- At a root, the function *crosses* the x-axis OR the function *bounces* at the x-axis



It will:

- *cross* if the root has odd multiplicity
- *bounce* if it has even multiplicity.



Example 4.72

In the diagram alongside, classify each graph on whether it cuts, or it bounces at the $x - axis$.

$$Red\ Graph: Cuts$$
$$Grey\ Graph: Bounces$$

4.73: Change of Sign

- A *continuous* function can only change sign if it crosses the x axis.
- If it crosses the x axis, then the function has a root at the x axis.
- Hence, we only need to check the intervals created by the roots.

- This is a consequence of the intermediate value theorem.

4.74: Test Points in Intervals

- The roots divide the real number line into intervals. We use a test point in each interval.
- The value of the point does not matter, so long as it within the interval. Hence, we usually the point to be easy for calculations.

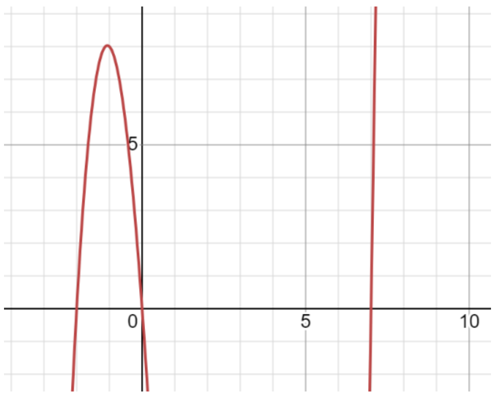
Example 4.75

Solve $x(x+2)(x-7) \le 0$

Use the zero-product property to determine the roots of the LHSL

$$x = 0$$
$$x + 2 = 0 \Rightarrow x = -2$$
$$x - 7 = 0 \Rightarrow x = 7$$
$$Roots \in \{-2,0,7\}$$

	$(-\infty,-2)$	$(-2,0)$	$(0,7)$	$(7,\infty)$
	$x = -3$	$x = -1$	$x = 1$	$x = 8$
$x + 2$	$-ve$	$+ve$	$+ve$	$+ve$
x	$-ve$	$-ve$	$+ve$	$+ve$
$x - 7$	$-ve$	$-ve$	$-ve$	$+ve$
	$-ve$	$+ve$	$-ve$	$+ve$
No. of Negative Signs	3	2	1	0



We want the regions where the expression is less than zero. This means we want the negative regions:

$$\underbrace{(-\infty,-2) \cup (0,7)}_{Interval\ Notation} \Leftrightarrow \underbrace{x < -2\ OR\ 0 < x < 7}_{Inequality\ Notation}$$

We also want the values where the expression is zero, which are:

$$\{-2,0,7\}$$

And, we incorporate these values in our final answer:

$$\underbrace{(-\infty,-2]\cup[0,7]}_{\textit{Interval Notation}} \Leftrightarrow \underbrace{x\le -2\ OR\ 0\le x\le 7}_{\textit{Inequality Notation}}$$

4.76: Wavy Curve / Snake Method

The wavy curve focuses *only on the sign* of the expression and uses the multiplicity to determine the regions where the expression is positive and negative.

Instead of using test points, we can use the properties related to even and odd multiplicity:

- Odd multiplicity: Cross
- Even Multiplicity: Bounce

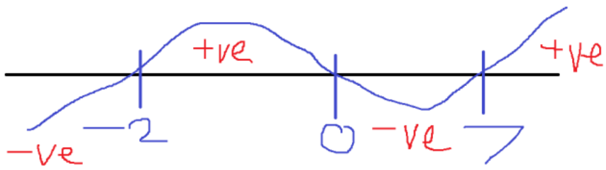
Example 4.77

Solve $x(x+2)(x-7)\le 0$

$$Roots \in \{-2,0,7\}$$

We still need one test point, say in the leftmost interval.
Substitute $x=-3$ in $x(x+2)(x-7)$

$$-3(-3+2)(-3-7)=(-3)(-1)(-10)<0$$



Example 4.78

$$(x-2)(x+5)(x-9)>0$$

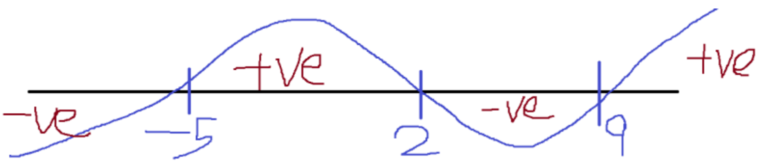
$$Roots: x\in\{-5,2,9\}$$

The intervals are:

$$(-\infty,-5)\cup(-5,2)\cup(2,9)\cup(9,\infty)$$

The leftmost interval is:

$$x=-6\Rightarrow \underbrace{(-6-2)}_{-ve}\underbrace{(-6+5)}_{-ve}\underbrace{(-6-9)}_{-ve}$$

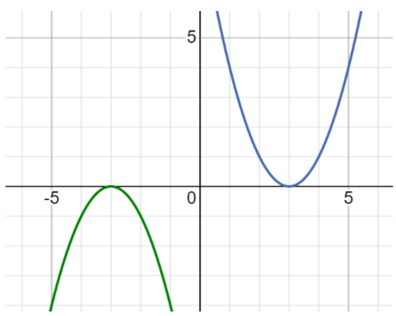


The final answer is the union of the positive intervals:

$$(-5,2)\cup(9,\infty)\Rightarrow -5<x<2\ OR\ x>9$$

B. Polynomial Properties

The leading coefficient determines the behavior of the polynomial for very large values (∞) and very negative values ($-\infty$).



4.79: Polynomials of even degree

If the leading coefficient is

- positive, then the polynomial is positive at $\pm\infty$
- negative, then the polynomial is negative at $\pm\infty$

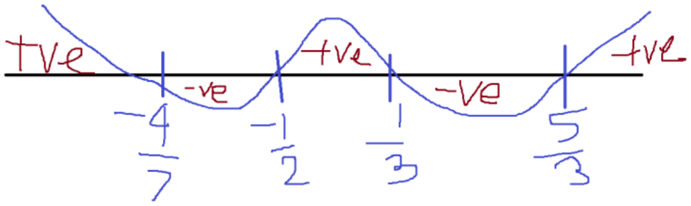
For convenience, you can multiply both sides by -1, and flip the sign of the inequality to make the leading coefficient positive.

Example 4.80

$$(2x+1)(3x-5)(7x+4)(3x-1) \geq 0$$

$$Roots: x \in \left\{-\frac{4}{7}, -\frac{1}{2}, \frac{1}{3}, \frac{5}{3}\right\}$$

Polynomial of degree four, with positive leading coefficient, hence it is positive at $\pm\infty$.



The final answer is the union of the positive intervals:

$$\left(-\infty, -\frac{4}{7}\right) \cup \left(-\frac{1}{2}, \frac{1}{3}\right) \cup \left(\frac{5}{3}, \infty\right)$$

Since it is a $\geq$ inequality, add the roots $\left\{-\frac{4}{7}, -\frac{1}{2}, \frac{1}{3}, \frac{5}{3}\right\}$:

$$\left(-\infty, -\frac{4}{7}\right] \cup \left[-\frac{1}{2}, \frac{1}{3}\right] \cup \left[\frac{5}{3}, \infty\right)$$

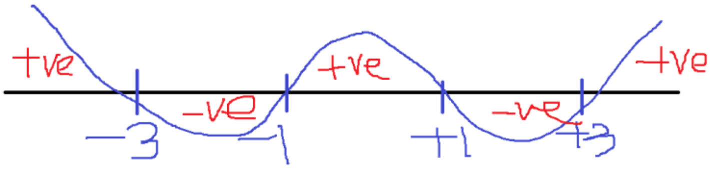
4.81: Factoring

If the expression given to you is not factored, you will need to factor it before you proceed with the rest of the solution.

Example 4.82

$$(x^2+4x+3)(x^2-4x+3) \leq 0$$

$$(x+3)(x+1)(x-1)(x-3) \leq 0$$
$$Roots: x \in \{-3, -1, +1, +3\}$$



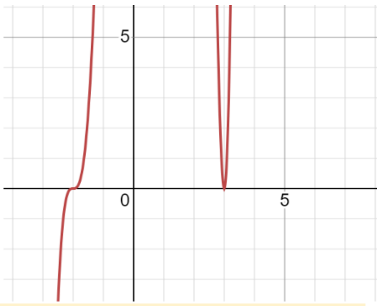
$$(-3,-1) \cup (1,3) \Leftrightarrow -3 < x < -1 \; OR \; 1 < x < 3$$

Add the roots $\{-3, -1, +1, +3\}$

$$[-3,-1] \cup [1,3] \Leftrightarrow -3 \leq x \leq -1 \; OR \; 1 \leq x \leq 3$$

4.83: Polynomials of odd degree

If the leading coefficient is
- positive, then the polynomial is positive at ∞, and negative at $-\infty$
- negative, then the polynomial is negative at ∞, and positive at $-\infty$



For convenience, you can multiply both sides by -1, and flip the sign of the inequality to make the leading coefficient positive.

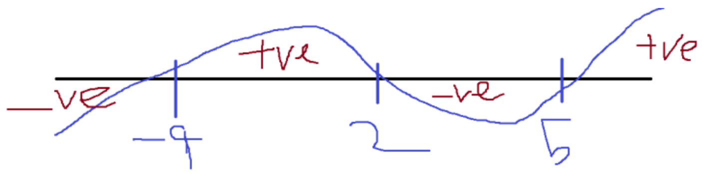
Example 4.84

$$(-x+2)(-x+5)(-x-9)>0$$

Rather than dealing with negative coefficients for x, factor out the negative sign:

$$(-1)(x-2)(-1)(x-5)(-1)(x+9)>0$$
$$(-1)(x-2)(x-5)(x+9)>0$$
$$(x-2)(x-5)(x+9)<0$$

$$Roots \in \{-9,2,5\}$$



We want the negative intervals, and hence the final answer is:

$$(-\infty,-9)\cup(2,5)$$

Example 4.85

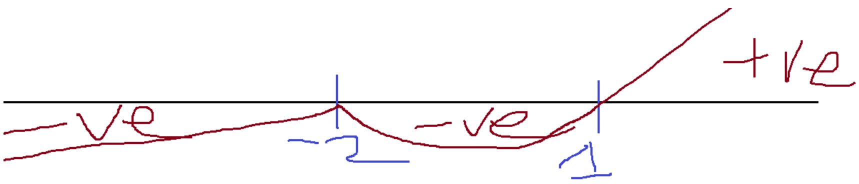
$$(x+2)^2(x-1)^3\geq 0$$

Find the critical points by equating each factor to zero:

$$(x+2)^2=0\Rightarrow x+2=0\Rightarrow x=-2\Rightarrow Even\ multiplicity\Rightarrow Bounce$$
$$(x-1)^3=0\Rightarrow x-1=0\Rightarrow x=1\Rightarrow Odd\ Multiplicity\Rightarrow Cut$$

Check the sign by substituting $x=-3$:

$$(-3+2)^2(-3-1)^3=(-3+2)^2(-4)^3\Rightarrow -ve$$



The answer, from the above number line is:

$$[1,\infty)$$

$$\frac{x^5}{x^3}=x^2$$
$$\frac{(x-5)^5}{(x-5)^3}=(x-5)^2$$

C. Quadratic Formula

If a quadratic does not factor easily, then it may be necessary to use the quadratic formula to factor it.

Example 4.86

Find the solution set for the inequality:

$$-\frac{2x^2}{3} > -\frac{4}{3} + x$$

$$\frac{2x^2}{3} < \frac{4}{3} - x$$
$$2x^2 < 4 - 3x$$
$$2x^2 + 3x - 4 < 0$$

Substitute $a = 2, b = 3, c = -4$ into the quadratic formula to find the roots of the quadratic:

$$x = \frac{-3 \pm \sqrt{9 - (-32)}}{2(2)} = \frac{-3 \pm \sqrt{41}}{2(2)}$$

This is a polynomial of even degree with leading positive coefficient, and hence the first region will be positive.

$$\left(\frac{-3 - \sqrt{41}}{4}, \frac{-3 + \sqrt{41}}{4}\right)$$

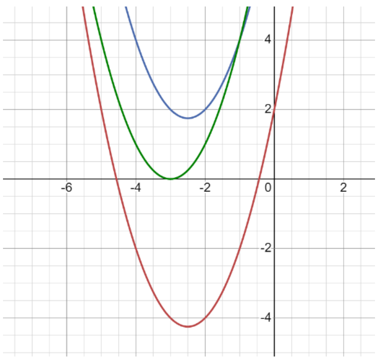
Example 4.87

The expression $x^2 + bx + c$ is never negative. Find, in terms of c, the interval in which b must lie.

Consider quadratics with positive leading coefficient:

If the discriminant of a quadratic is positive, then:

- Refer Red Graph: It has two real roots. Hence, it will intersect the x-axis in two different places. And, hence, it will be negative at some point in its domain.
- Refer Green Graph: It has a single, repeated, real root, and the discriminant is zero.
- Refer Blue Graph: It has no real roots, and its discriminant is negative.



For the expression above:

- Red graph is ruled out. Hence, discriminant cannot be positive.
- Green graph is possible. Discriminant can be zero.
- Blue Graph is possible. Discriminant can be negative.

Consider the discriminant of the quadratic $x^2 + bx + c$:

$$D \le 0$$
$$b^2 \le 4(1)(c)$$
$$b^2 \le 4c$$
$$-2\sqrt{c} \le b \le 2\sqrt{c}$$

$$b \in \left[-2\sqrt{c}, 2\sqrt{c}\right]$$

D. Irreducible Quadratics

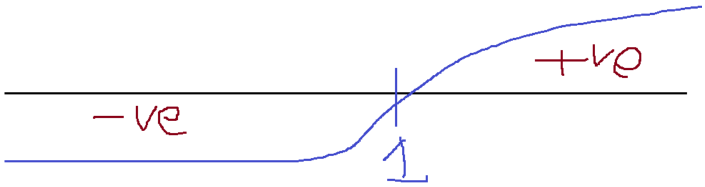
4.88: Irreducible Quadratics
When the discriminant of a quadratic is less than zero, it has no real solutions. Such a quadratic will not have any real zeroes.

Since an irreducible quadratic will not have any real zeros, it will not result in any critical points/zeroes of the expression.

Example 4.89

$$x^3 < 1$$

You may already know the graph of $y = x^3$. But we will solve it using the techniques of this chapter.



$$x^3 - 1 < 0$$
$$(x-1)(x^2+x+1) < 0$$

For the quadratic $x^2 + x + 1$, calculate

$$b^2 - 4ac = 1 - (4)(1)(1) = 1 - 4 = -3 \Rightarrow -ve \Rightarrow No\ Real\ Solutions$$

Critical Points:

$$1$$

And hence, the final solution is

$$(-\infty, -1)$$

E. Challenge Questions

Challenge 4.90

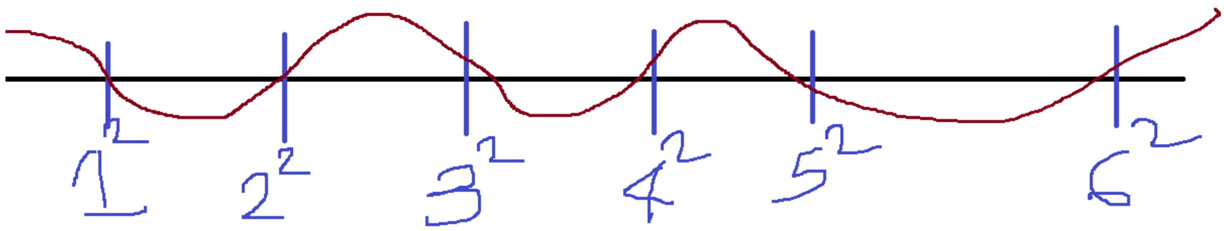
Define $P(x) = (x-1^2)(x-2^2)\dots(x-100^2)$. How many integers n are there such that $P(n) \le 0$? **(AMC 10A 2020/17)**

$$(x-1^2)(x-2^2)\dots(x-100^2) \le 0$$

Find the critical points:

$$x - 1^2 = 0 \Rightarrow x = 1^2$$
$$x - 2^2 = 0 \Rightarrow x = 2^2$$
$$x \in \{1^2, 2^2, \dots, 100^2\} \Rightarrow 100\ Numbers$$

If we multiply, this will give me a 100-degree polynomial.
Since this is a polynomial of even degree, with leading positive coefficient, it will be positive at $-\infty$.



From the number line, we can pick up the intervals where the polynomial is less than zero:

$$(1^2, 2^2) \cup (3^2, 4^2) \cup (5^2, 6^2) \cup \dots \cup (99^2, 100^2) \Rightarrow 50\ Intervals$$

Add the roots:

$$[1^2, 2^2] \cup [3^2, 4^2] \cup [5^2, 6^2] \cup \dots \cup [99^2, 100^2]$$

Find the number of integers in this solution set:

$$[1^2, 2^2] \Rightarrow \{1,2,3,4\} \Rightarrow 4 - 1 + 1 = 4$$
$$[3^2, 4^2] \Rightarrow \{9,10, \dots ,16\} \Rightarrow 16 - 9 + 1 = 8$$
$$[5^2, 6^2] \Rightarrow \{25,26, \dots ,36\} \Rightarrow 36 - 25 + 1 = 12$$

$$n^2 - (n-1)^2 + 1 = n^2 - n^2 + 2n - 1 + 1 = 2n$$

And hence the above is an arithmetic sequence:

$$4 + 8 + 12 + \dots + 200 = 4(1 + 2 + \dots + 50) = 4\left(\frac{50 \times 51}{2}\right) = 5100$$

4.6 Wavy Curve: Rational

A. Factors in the Denominator

4.91: Roots of the Denominator

- When calculating roots, include the roots of the denominator also.
- Expressions change sign (as per multiplicity rules) for denominator roots also
- Remove the denominator roots from the final answer.

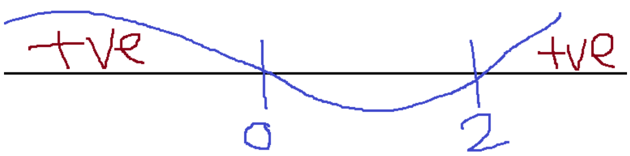
Example 4.92

$$\frac{(x-2)^3}{x} > 0$$

Find the zeroes of the numerator and the denominator:

$$Roots \in \{0,2\}$$

Degree is 2. Hence, it is positive at $-\infty$.



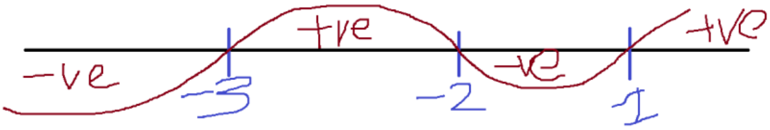
$$(-\infty, 0) \cup (2, \infty)$$

Example 4.93

Find the region where the expression $\frac{x^2+5x+6}{x-1}$ is non-negative.

$$\frac{(x+3)(x+2)}{x-1} \geq 0 \Rightarrow Roots: x \in \{-3, -2, 1\}$$

Degree is 1(odd). At $-\infty$, it will be $-ve$.



$$\underbrace{(-3,-2)}_{+ve} \cup \underbrace{(1,\infty)}_{+ve}$$

$$At\ 1, the\ denominator\ is\ not\ defined \Rightarrow Exclude\ 1$$

$$At - 3\ and - 2, the\ expression\ is\ zero \Rightarrow Include - 2\ and - 3$$

$$[-3,-2] \cup (1,\infty)$$

B. "Cancelling" Factors

Example 4.94

Solve:

$$\frac{x^2+4x-5}{x^2+6x+5} \geq 0$$

$$\frac{(x+5)(x-1)}{(x+5)(x+1)} \geq 0 \Rightarrow Roots: x \in \{-5,-1,1\}$$

$$At\ -5\ function\ is\ not\ defined: Remove$$

$$At\ 1\ the\ function\ is\ defined. Inequality\ is \geq : Include$$

$$(-\infty,-5) \cup (-5,-1) \cup [1,\infty]$$

4.95: Cancelling Factors

- If you have a common factor in the numerator and the denominator, you can "cancel".
- However, the expression is not defined for any value that makes the denominator zero. Hence, you need to remove the zeros of the "cancelled" expression from the final solution set.

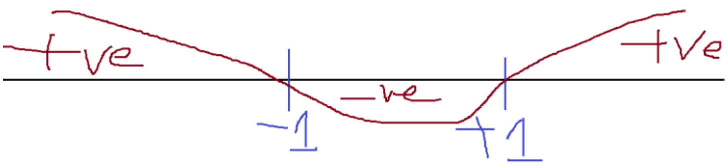
Example 4.96

Solve:

$$\frac{x^2+4x-5}{x^2+6x+5} \geq 0$$

For $x \neq -5,-1$, we get:

$$\frac{(x+5)(x-1)}{(x+5)(x+1)} \geq 0 \Rightarrow \frac{x-1}{x+1} \geq 0 \Rightarrow Roots: x \in \{-1,1\}$$



$$(-\infty,-1) \cup (1,\infty)$$

Remove the roots from the denominator since the function is not defined at those points. Include $+1$ since the function is defined there, and the inequality is $\geq$

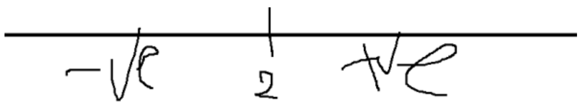
$$(-\infty, -5) \cup (-5, -1) \cup [1, \infty]$$

Example 4.97

$$\frac{x^2 - 2x + 1}{x - 2} > x$$

$$\frac{x^2 - 2x + 1}{x - 2} - x > 0 \Rightarrow \frac{1}{x - 2} > 0$$

$$Roots: x = 2$$



This is of odd degree.

$$(2, \infty) \Leftrightarrow x > 2$$

C. Repeated Zeroes

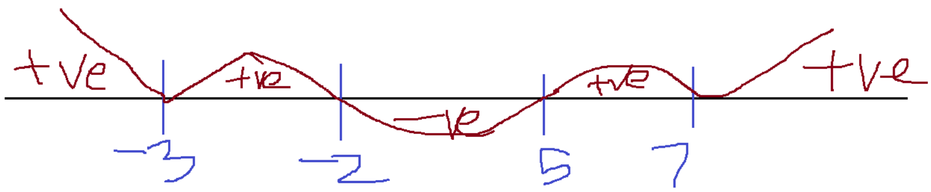
Example 4.98

Find the interval that satisfies

$$\frac{(x-5)^5(x+2)^7}{(x+3)^4(x-7)^2} \leq 0$$

$$Roots: x \in \left\{ \underbrace{-3}_{Bounce}, \underbrace{-2}_{Cross}, \underbrace{5}_{Cross}, \underbrace{7}_{Bounce} \right\}, \qquad Not\ defined\ at\ \{-3, 7\}$$

$$Degree = 5 + 7 - 4 - 2 = 6 \Rightarrow Even \Rightarrow +ve\ at - \infty$$



$$(-2, 5)$$

We want to exclude $\{-3, 7\}$, and they are not in the above region.
We want to include -2 and 5, and hence, the final answer is:

$$[-2, 5]$$

5. Relations

5.1 Relations

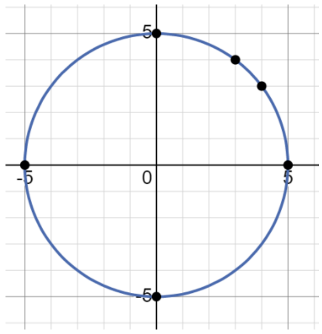
A. Relation versus Functions

Example 5.1
Consider the relation between x and y (and r is a constant)

$$y^2 + x^2 = 25$$

Determine whether the relation is a function:

A. By drawing a graph of the relation
B. Algebraically



Graphically

x	0	0	5	-5	3
y	5	-5	0	0	4

Algebraically

$$y^2 + x^2 = 25$$

Isolate $y^2$and take square roots:

$$y^2 = 25 - x^2$$

$$y = \pm\sqrt{25 - x^2}$$

Example 5.2
Consider the relation

$$y = \sqrt{x}$$

$$x = 9 \Rightarrow y = \sqrt{9} = \pm 3$$

Hence, the above relation is not a function.
Is the above argument correct, or incorrect? If incorrect, find the flaw.

$$x = 9 \Rightarrow y = \sqrt{9} = 3$$

Example 5.3
Consider the relation

$$x = \sqrt{y}$$

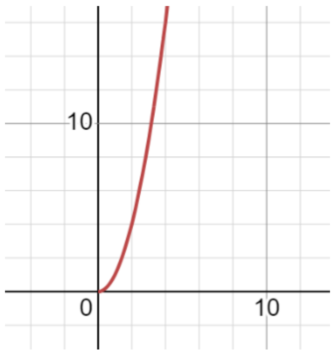
Square the relations:

$$x^2 = y$$

Take square roots

$$x = \pm\sqrt{y}$$

Hence, the above relation is not a function.
Is the above argument correct, or incorrect? If incorrect, find the flaw and the step at which it occurred.



When you square an equation, you have to be careful about introducing extraneous roots.

For example:

$$x = \pm\sqrt{y} \Rightarrow y = 9 \Rightarrow x = \pm 3$$

If you substitute $x = -3$ in the original equation, it is not satisfied:

$$-3 \neq \sqrt{9}$$

The problem happened because squaring the equation introduced an extra root.

Example 5.4

Is the statement

$$\sqrt{x^2} = x$$

Always true?

No. Consider the counterexample below:

$$x = -5 \Rightarrow x^2 = 25 \Rightarrow \sqrt{x^2} = 5$$

$$x \text{ is } +ve \Rightarrow \sqrt{x^2} = x$$
$$x \text{ is } -ve \Rightarrow \sqrt{x^2} = -x$$

5.2 Relations

A. Ordered Pair

A pair of elements is called an ordered pair. It is called an ordered pair because the order is important.

$$\underbrace{(Pari, English)}_{\textbf{Pari is studying English}} \neq \underbrace{(English, Pari)}_{\textbf{English is studying Pari}}$$

Example 5.5

$$A \text{ is a set of four students} = \left\{\underbrace{Pari}_{P}, \underbrace{Nitish}_{N}, \underbrace{Santosh}_{S}, \underbrace{Arya}_{A}\right\}$$

$$B \text{ is a set of four subjects} = \left\{\underbrace{English}_{E}, \underbrace{Maths}_{M}, \underbrace{Science}_{S}, \underbrace{Humanities}_{H}\right\}$$

We can write the combination of $(\boldsymbol{Student}, \boldsymbol{Subject})$ as a pair. For example, Pari studying English can be written:

$$(Pari, English) \rightarrow \underbrace{(P, E)}_{\textbf{Shortform}}$$

Represent the information below as a set of ordered pairs:

- Pari is studying English and Maths
- Nitish is studying all subjects except Humanities
- Santosh is not studying any subject
- Arya is studying only English.

S1: $\underbrace{(P, E)(P, M)}_{\textbf{2 Subjects}} \underbrace{(N, E)(N, M)(N, S)}_{\textbf{3 Subjects}} \underbrace{(A, E)}_{\textbf{1 Subject}}$

5.6: Cartesian Product

The Cartesian product of two sets A and B is the set of all ordered pairs (a, b) such that $a \in A, b \in B$.

Example 5.7
Find the Cartesian product of the sets $A = \{1,2,3\}$ and $B = \{x, y\}$.

$$\{(1, x)(2, x)(3, x)(1, y)(2, y)(3, y)\}$$

Example 5.8
Set A has m elements. And Set B has n elements. Find the number of elements in the Cartesian product of sets A and B.

Picking an element from set A
$$= m \text{ ways}$$
Picking an element from set B
$$= n \text{ ways}$$
And the total number of ways
$$= m \times n$$

5.9: Relation
Any subset of the Cartesian Product of two sets is a relation between the two sets.

5.10: Domain of Relation
The set of all first elements of a relation is called the domain.

5.11: Range of Relation
The set of all second elements of a relation is called the range.

Example 5.12: Using Sets
Find the domain and range of the Cartesian product of the sets $A = \{1,2,3\}$ and $B = \{x, y\}$.

$$\{(1, x)(2, x)(3, x)(1, y)(2, y)(3, y)\}$$

Set of first elements of the Cartesian product is:
$$\{1,2,3,1,2,3\}$$
But since repetition does not matter in a set, we have:
$$Domain = \{1,2,3\}$$

Set of second elements of the Cartesian product is:
$$\{x, y, x, y\}$$
But since repetition does not matter in a set, we have:
$$Range = \{x, y\}$$

B. Linear Relationships

Example 5.13: Using a Linear Relationship
A relation is defined by the rule $y = 2x + 4$, for $x \in \{0,1,2,4\}$.

A. Find the range of the relation.

B. Express the relation as a set of ordered pairs.

C. Quadratic Relationships

Example 5.14: Using a Quadratic Relationship

A relation is defined by the rule $y = x^2 + 5x + 6$

A. Find the domain of the relation.
B. Find the range of the relation.

There are no restrictions on the value of x. Hence,

$$Domain = \mathbb{R}$$

The graph of the given relation is an upward parabola with

$$x - value\ of\ vertex = Avg.of\ Roots = \frac{-2-3}{2} = -\frac{5}{2}$$
$$y - value\ of\ vertex = y_{x=-\frac{5}{2}} = -\frac{1}{4}$$
$$Range = \left[-\frac{1}{4}, \infty\right)$$

D. Cubic Relationships

Example 5.15: Using a Cubic Relationship

A relation is defined by the rule $y = 3x^3 + 4x^2 + 5x + 5$

A. Find the domain of the relation.
B. Find the range of the relation.

There are no restrictions on the value of x. Hence,

$$Domain = \mathbb{R}$$

The main thing to understand is that the value of x is dominated by the x^3 for values which are very large, and very small.

$$x \to +\infty, x^3 is + ve\ and\ large \Rightarrow y \to +\infty$$
$$x \to -\infty, x^3 is - ve\ and\ |x^3|\ is\ large \Rightarrow y \to -\infty$$

Since the given relation is continuous, it will take all values in between. Hence

$$Range = \mathbb{R}$$

E. Exponential Relationships

Example 5.16: Using a Exponential Relationship

A relation is defined by the rule $y = 3e^x + 4$

A. Find the domain of the relation.
B. Find the range of the relation.

F. Logarithmic Relationships

Example 5.17: Using a Logarithmic Relationship

A relation is defined by the rule $y = \ln x + 4$

A. Find the domain of the relation.
B. Find the range of the relation.

G. Circle

Example 5.18

Consider the relationship

$$x^2 + y^2 = 7$$

Find the domain

Isolate y^2:

$$y^2 = 7 - x^2$$

Take the square root of both sides:

$$y = \pm\sqrt{7 - x^2}$$

Find the domain:

$$7 - x^2 \geq 0 \Rightarrow 7 \geq x^2 \Rightarrow x^2 \leq 7 \Rightarrow -\sqrt{7} \leq x \leq \sqrt{7}$$

Find the range by considering the relation in the form $y = \pm\sqrt{7 - x^2}$ derived above.

We know that

$$-\sqrt{7} \leq x \leq \sqrt{7}$$

Also, $x^2 \geq 0$ which means that

$$as\ x\ increases, \sqrt{7 - x^2}\ decreases$$

The value of $\sqrt{7 - x^2}$ will be

$$Maximum\ when\ x = 0 \Rightarrow \sqrt{7 - x^2} = \sqrt{7}$$
$$Minimum\ when\ x = \sqrt{7} \Rightarrow \sqrt{7 - x^2} = \sqrt{0} = 0$$
$$Range\ of\ \sqrt{7 - x^2} = (0,7)$$

Similarly, the value of $-\sqrt{7 - x^2}$ will be

$$Minimum\ when\ x = 0 \Rightarrow -\sqrt{7 - x^2} = -\sqrt{7}$$
$$Maximum\ when\ x = \sqrt{7} \Rightarrow \sqrt{7 - x^2} = \sqrt{0} = 0$$
$$Range\ of - \sqrt{7 - x^2} = (-7,0)$$

Combining the two, we get:

$$Range\ of \pm \sqrt{7 - x^2} = (-7,7)$$

Find the range of $x^2 + y^2 = 7$ by solving for x.

In general, if we solve for x, and find the set of values that y can take, that gives us the range of y.

Isolate x^2:

$$x^2 = 7 - y^2$$

Take the square root of both sides:

$$x = \pm\sqrt{7 - y^2}$$

Find the domain:

$$7 - y^2 \geq 0 \Rightarrow 7 \geq y^2 \Rightarrow y^2 \leq 7 \Rightarrow -\sqrt{7} \leq y \leq \sqrt{7}$$

Alternate Method:

$$x^2 + y^2 = 7$$

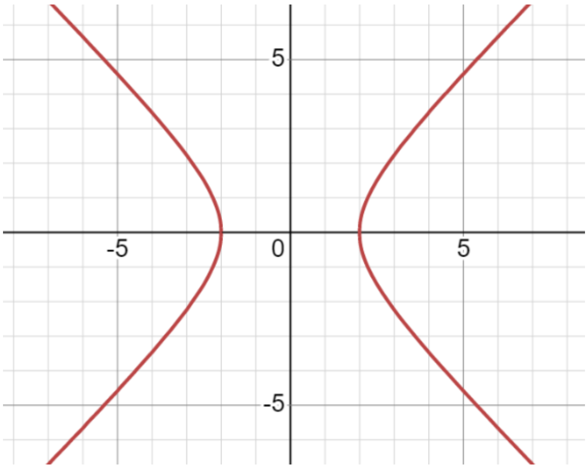
Note that the above expression is symmetric in x and y. What we mean by symmetric is that replacing x with y, and y with x does not change the relationship.

Hence, the same process that we followed for y can also be done for x.
Hence, in general, the domain and the range are the same.

H. Hyperbola

Example 5.19

Find the domain of the hyperbola $x^2 - y^2 = a^2,\ a\ is\ a\ constant$



Solve for y:

$$y^2 = \sqrt{x^2 - a^2} \Rightarrow y = \pm\sqrt{x^2 - a^2}$$

The expression inside the square root must be non-negative:

$$x^2 - a^2 \geq 0 \Rightarrow x^2 \geq a^2 \Rightarrow x > a\ OR\ x < -a$$

And hence the domain of x is:

$$(-\infty, a] \cup [a, \infty)$$

Find the range of the hyperbola in the example above.

To find the range, solve for x:

$$x^2 = y^2 + a^2 \Rightarrow x = \pm\sqrt{y^2 + a^2}$$

We know that:

$$y^2 \geq 0, a^2 \geq 0$$

Hence, the expression inside the square root is never negative.
Hence, there are no restrictions on the value of y.

Hence, the range is

$$Range = \mathbb{R}$$

Example 5.20

Find the domain of $axy + y - x = a$

Collate all y terms on one side:

$$axy + y = a + x$$

Solve for y:

$$y(ax+1) = a + x \Rightarrow y = \frac{a+x}{ax+1}$$

The denominator cannot be zero

$$ax + 1 = 0 \Rightarrow x = -\frac{1}{a}$$

The domain is

$$\mathbb{R}\backslash\left\{-\frac{1}{a}\right\}$$

Find the range of $axy + y - x = a$

Solve for x:

$$axy - x = a - y$$
$$x(ay-1) = a - y$$
$$x = \frac{a-y}{ay-1}$$

$$ay - 1 = 0 \Rightarrow y = \frac{1}{a}$$

$$\mathbb{R}\backslash\left\{\frac{1}{a}\right\}$$

5.3 Relations on the Coordinate Plane

A. Basics

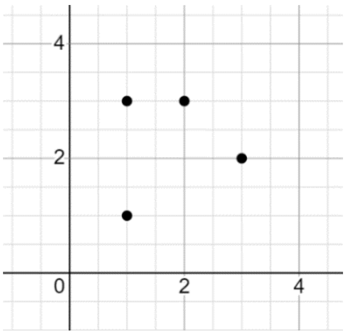
When graphing relations on the coordinate we assume, by default, that

$$Domain = Set\ of\ x\ values$$
$$Range = Set\ of\ y\ values$$

Example 5.21

A relation is defined by the rule $y = 2x + 4$, for $x \in \{0,1,2,4\}$. Express this relation on the coordinate plane.

B. Discrete Relations



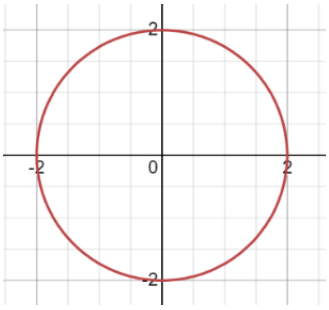
Example 5.22

Find the domain and range of the relation graphed alongside.

$$Domain = \{1,2,3\}$$
$$Range = \{1,3,2\} = \{1,2,3\}$$

C. Continuous Relations

So far, we have been looking at sets with discrete values. If we create a relation between the variable x, and the variable y, then we can get continuous values.



Example 5.23

The relation $y^2 + x^2 = 4$ is graphed alongside. Find the domain and range of the relation.

$$Domain = [-2,2]$$
$$Range = [-2,2]$$

For example, the association of students with subjects that we looked at above is an example of a relation.

Properties of Relations

- An element of Set A can be associated with zero or more elements of Set B.
  - ✓ Pari is associated with two subjects. Nitish is associated with three subjects. Arya is associated with one subject.
- It is not necessary that all elements of Set A have an association.
  - ✓ Santosh is not studying any subject .
- It is not necessary that all elements of Set B have an association.
  - ✓ No one is studying Humanities.

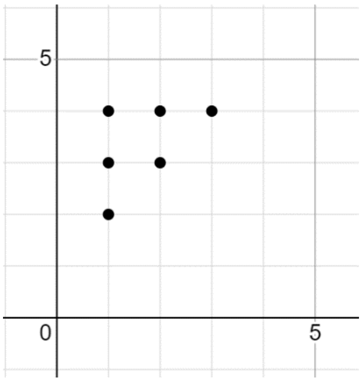
Domain

The set of values of the originating set is called the domain

Range

The set of values of the destination is called the range.

D. Relations using Inequalities



Example 5.24

Consider the relation $y > x,\ \ x, y \in Natural\ Numbers$ and both x and y are less than 5. Find the set of ordered pairs that satisfy this relation.

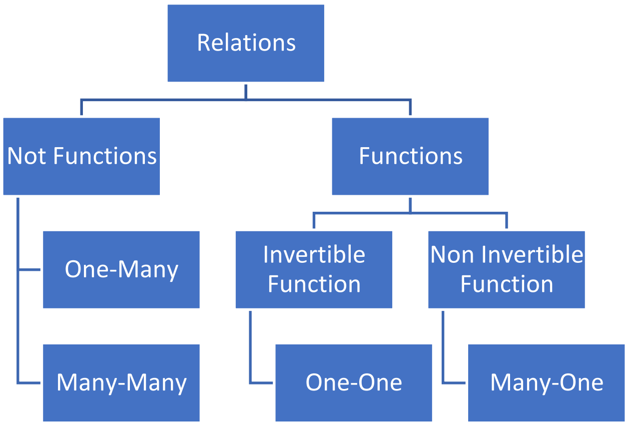
5.25: Types of Relations

One-One: Each element of Set A is mapped to exactly one element of Set B

Many-One: Many elements of Set A are mapped to one element of set B.

One-Many: At least one element of Set A is mapped to many elements of Set B.

Many-Many: At least one element of Set A is mapped to more than one element of Set B. And at least one element of Set B is mapped to more than one element of Set A.



5.26: Relation: Definition 2

A relation is a set of ordered pairs from set A to set B. Each element of A can be associated with zero or more elements of B.

5.4 Counting Arguments

5.1: Number of Ordered Pairs

Let Set A have m elements and Set B have n elements.

$$\textbf{No. of ordered pairs in the Cartesian Product of A and B} = \underbrace{\boldsymbol{m \times n = mn}}_{\substack{\textit{Direct application of} \\ \textit{multiplication principle}}}$$

Example 5.27: Counting Ordered Pairs

Example 5.28: Back Calculations

5.2: Number of Relations from A to B

$$\textbf{No. of Relations from A to B} = \underbrace{1st\ Pair}_{\textbf{2 Choices}} \times \underbrace{2nd\ Pair}_{\textbf{2 Choices}} \times \ldots \times \underbrace{mn^{th}\ Pair}_{\textbf{2 Choices}} = \boldsymbol{2^{mn}}$$

Example 5.29: Counting Number of Relations

Example 5.30: Back Calculations

5.3: Number of Functions $f: A \to B$

An arbitrary element in A can be associated with any of the n elements in B, giving us n choices.
Each of the elements in A is independent of the other elements in terms of its mapping.
Hence, total number of $f: A \to B$

$$= \underbrace{n \times n \times \ldots \times n}_{m\ times} = n^m$$

5.5 Function as a type of Relation

A. Definition

A relation between A and B is called a function if every element of Set A is associated with $\underbrace{\textit{exactly one}}_{\text{neither more nor less}}$ element of Set B.

Some important parts of the definition:

- There can be not an element of set A which is not associated with an element of Set B.
- An element of Set A cannot be associated with more than one element of Set B.

Example 5.31

$$A = \left\{\underbrace{Pari}_{P}, \underbrace{Nitish}_{N}, \underbrace{Santosh}_{S}, \underbrace{Arya}_{A}\right\}, B = \left\{\underbrace{English}_{E}, \underbrace{Maths}_{M}, \underbrace{Science}_{S}, \underbrace{Humanities}_{H}\right\}$$

$$Student - Subject\ Combinations: \underbrace{(P,E)(P,M)}_{2\ Subjects}, \underbrace{(N,E)(N,M)(N,S)}_{3\ Subjects}, \underbrace{(A,E)}_{1\ Subject}$$

Is the relation above also a function? Give reasons why or why not?

Let's write the relation as an input-output table

Input	Pari	Nitish	Santosh	Arya
Output 1	English	English		English
Output 2	Maths	Math		
Output 3		Science		
No. of Outputs	2	3	0	1
	>1	>1	<1	$=1$

- Pari and Nitish do not meet the conditions. They have more than one output associated with the input.
- Santosh does not meet the conditions. He has no output associated with the input.
- Arya meets the conditions. She has exactly one output associated with the input.

Hence, the above relation is not a function.
Even one input that does not meet the rules means the relation is not a function.

Example 5.32

$$A = \left\{\underbrace{Pari}_{P}, \underbrace{Nitish}_{N}, \underbrace{Santosh}_{S}, \underbrace{Arya}_{A}\right\}, B = \left\{\underbrace{English}_{E}, \underbrace{Maths}_{M}, \underbrace{Science}_{S}, \underbrace{Humanities}_{H}\right\}$$

$$Student - Subject\ Combinations: \underbrace{(P,E)}_{1\ Subjects}, \underbrace{(N,E)}_{1\ Subject}, \underbrace{(S,M)}_{1\ Subject}, \underbrace{(A,E)}_{1\ Subject}$$

Is the relation above also a function? Give reasons why or why not?

Let's write the relation as an input-output table

Input		Pari	Nitish	Santosh	Arya
Output 1		English	English	Maths	English

- Each of the above inputs (students) is associated with precisely one output.
- All inputs have outputs

Hence, the above relation is a function.

B. Relations as Functions

The following relations are functions

- One-One
- Many-One

The following relations are not functions:

- One-Many
- Many-Many

5.6 Types of Function

A. Surjective

A surjective function is one where the range of the function is the same as its co-domain.
A surjective function is one where every element of Set B is mapped to at least one element of Set A.

B. Injective

An injective function is one where every element of Set B has at most one corresponding element of Set A.
An injective is a one-one function.

C. Bijective

If a function is both surjective and injective, then it is bijective. A bijective function is invertible. This is because: Every element in Set B is mapped to exactly one element in Set A. Hence, we have a single output for every input.

5.7 More about Relations

A. Equivalence Relation

Reflexive: A relation is reflexive if it relates every element to itself.

$$a \sim a$$
$$\forall x \in X: xRx$$

Transitive: A relation is transitive if for all elements a, b and c in X, whenever R relates a to b and a to c, it relates a to c.

$$\text{If } a \sim b \text{ and } b \sim c, then\ a \sim c$$

This can also be written as:

$$a \sim b \text{ and } b \sim c \Rightarrow a \sim c$$

Symmetric A relation R is symmetric if for all elements $a,$ and b X, whenever R relates a to b, it also relates b to a

$$a \sim b \text{ if and only if } b \sim a$$
$$a \sim b \Leftrightarrow b \sim a$$

B. Equivalence Classes

Equivalence Relations can be used to split a set into equivalence classes. Elements a and b belong to the same equivalence class if and only if they are equivalent.

Algebraic Operations on Functions

$$(f + g)(x) = f(x) + g(x)$$
$$(f + g)(x) \neq (f + g) \cdot (x)$$

Domain resulting from Algebraic Operations of Functions

$$Dom(f \pm g) = Dom(f) \cap Dom(g)$$

Example 5.33

Let $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x}$. Then, find the domain of $(f+g)(x)$.

$$(f+g)(x) = f(x) + g(x) = \sqrt{x} + \frac{1}{x}$$
$$Dom(f) = \{x : x \geq 0, x \in \mathbb{R}\}$$
$$Dom(g) = \{x : x \in \mathbb{R}, x \neq 0\}$$
$$Dom(f+g) = \{x : x > 0\}$$

Example 5.34

Let $f(x) = \frac{1}{2x}$ and $g(x) = \frac{x-\frac{1}{2}}{x}$. Then, find the domain of $(f+g)(x)$.

$$(f+g)(x) = f(x) + g(x) = \frac{1}{2x} + \frac{x-\frac{1}{2}}{x} = \frac{1+2x-1}{2x} = \frac{2x}{2x} = 1$$
$$Dom(f) = \{x : x \in \mathbb{R}, x \neq 0\}$$
$$Dom(g) = \{x : x \in \mathbb{R}, x \neq 0\}$$
$$Dom(f+g) = \{x : x \in \mathbb{R}, x \neq 0\}$$

Be careful when evaluating a sum/difference, etc of functions. If the sum/difference is defined, but the original functions are not, then the evaluation is not valid.

Composition of Function

In composition of functions, we evaluate the inner function, and then substitute it in the inner function.

$$(f \circ g)(x) = f\big(g(x)\big)$$

Composition of functions is not commutative in general.

$$(f \circ g)(x) \neq (g \circ f)(x)$$

Example 5.35

Let $f(x) = x^2$, $g(x) = x + 1$, and $h(x) = 2x$.

A. Find $h\big(g(f(x))\big)$

Find and compare$(f \circ g)(x)$ with $(g \circ f)(x)$. Are they the same?

36 Examples