Sequences & Series

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1. Foundations

1.1 Summation Notation

A. Basics

Σ is the capital Greek letter sigma. We use it to write summation notation.

1.1: Sigma Notation

Sigma notation is a compact notation used to write the sum of a series.

$$\sum_{x=1}^{10} x$$

$$Index\ of\ Summation: Lower\ Limit\ x = 1$$

$$Index\ of\ Summation: Upper\ Limit\ x = 10$$

To evaluate an expression written in sigma notation, you replace the given variable by its first value, then its second value, and so on.

$$\sum_{x=1}^{10} x = \underbrace{1}_{x=1} + \underbrace{2}_{x=2} + \underbrace{3}_{x=3} + \cdots + \underbrace{10}_{x=10}$$

$$No.of\ Terms = 10$$

Notes:

- In this case, the number of terms is equal to the upper limit, but that need not always be the case.
- The variable used for summation is called a *dummy variable* because when you substitute the limits of summation, the variable disappears. You can use any letter for the dummy variable, and the value of the expression does not change.

$$\sum_{x=1}^{10} x = \sum_{n=1}^{10} n = 1 + 2 + 3 + \cdots + 10$$

Example 1.2: Sum of Natural Numbers

Write the summations below in summation notation:

A. $1 + 2 + 3 + \cdots + 100$
B. $1 + 2 + 3 + \cdots + x$
C. $e^x + e^{2x} + e^{3x} + \cdots + e^{2025x}$
D. $\cos\left(\frac{n\pi}{2}\right) + \cos(n\pi) + \cos\left(\frac{3n\pi}{2}\right) + \cdots + \cos(500n\pi)$

Part A

$$\sum_{n=1}^{100} n$$

Part B

$$\sum_{n=1}^{x} n$$

Part D

$$\cos\left(\frac{n\pi}{2}\right)+\cos\left(\frac{2n\pi}{2}\right)+\cos\left(\frac{3n\pi}{2}\right)+\cdots+\cos\left(\frac{1000n\pi}{2}\right)=\sum_{k=1}^{1000}\cos\left(\frac{kn\pi}{2}\right)$$

1.3: Alternating Series

An alternating series is a series whose terms alternate between positive and negative terms.

Note that:

$$n\ is\ an\ odd\ integer \Rightarrow (-1)^n=-1$$
$$n\ is\ an\ even\ integer \Rightarrow (-1)^n=1$$

Hence, we can use the parity of powers of (-1) to make the terms of a series alternate between negative and positive.

Example 1.4

Write the expressions below in summation notation:

A. $-1+2-3+4+\cdots+100$
B. $1-2+3-4+\cdots-100$
C. $\frac{1}{2}-1+\frac{3}{2}-2+\frac{5}{2}+\cdots+\frac{97}{2}$
D. $e^1+e^3+e^5+e^7+e^9+\frac{1}{e^2}+\frac{1}{e^4}+\frac{1}{e^6}+\frac{1}{e^8}$
E. $\log_2 3+\log_3 4+\log_4 5+\cdots+\log_{1000} 1001$
F. $\frac{\ln 2}{\cos(1)}+\frac{\ln 3}{\cos(2)}+\cdots+\frac{\ln 217}{\cos(216)}$
G. $x^1+x^2+x^3+\cdots$

$$\sum_{n=1}^{100}(-1)^n n=-1+2-3+\cdots+100$$

$$\sum_{n=1}^{100}(-1)^{n+1} n=1-2+3-4+\cdots-100$$

$$\frac{1}{2}-\frac{2}{2}+\frac{3}{2}-\frac{4}{2}+\frac{5}{2}+\cdots+\frac{97}{2}=\frac{1}{2}(1-2+3-4+\cdots+97)=\sum_{n=1}^{97}(-1)^{n+1}\frac{n}{2}$$

$$e^1+e^3+e^5+e^7+e^9+\frac{1}{e^2}+\frac{1}{e^4}+\frac{1}{e^6}+\frac{1}{e^8}=\sum_{n=1}^{9}e^{(-1)^{n+1}n}$$

$$\log_2 3+\log_3 4+\log_4 5+\cdots+\log_{1000} 1001=\sum_{n=2}^{1000}\log_n n+1$$

$$\frac{\ln 2}{\cos(1)}+\frac{\ln 3}{\cos(2)}+\cdots+\frac{\ln 217}{\cos(216)}=\sum_{n=1}^{216}\frac{\ln n+1}{\cos n}$$

$$x^1+x^2+x^3+\cdots=\sum_{n=1}^{\infty}x^n$$

1.5: Odd and Even Numbers

$$Even\ Number \rightarrow 2n$$
$$Odd\ Number \rightarrow 2n+1\ OR\ 2n-1$$

Example 1.6

A. $2+4+6+\cdots+20$
B. $1+3+5+\cdots+79$
C. $\ln 1+\ln 3+\ln 5+\cdots+\ln 2025$
D. $\log_3 1+\log_4 2+\log_3 3+\log_4 4+\cdots+\log_3 314$

$$\sum_{n=1}^{10} 2n$$
$$\sum_{n=0}^{39} 2n+1$$
$$\sum_{n=1}^{157} (\log_3(2n-1)+\log_4 2n)$$

Example 1.7

$$1+\frac{x^2}{2}+\frac{x^4}{6}+\frac{x^6}{24}+\cdots$$

$$1+\frac{x^2}{2}+\frac{x^4}{6}+\frac{x^6}{24}+\cdots=\sum_{n=0}^{\infty}\frac{x^{2n}}{(n+1)!}$$

Example 1.8

$$x+\frac{x^3}{2}+\frac{x^5}{6}+\frac{x^7}{24}+\cdots$$

$$x+\frac{x^3}{2}+\frac{x^5}{6}+\frac{x^7}{24}+\cdots=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(n+1)!}$$

1.9: Multiplication

If you can recognize multiplication in the series, it will naturally translate into multiplication in the summation notation.

Example 1.10

A. $10+13+16+\cdots+1000$

$$10+13+16+\cdots+1000=\sum_{n=0}^{330} 10+3n$$

Example 1.11

Expand each summation below:

$$\sum_{n=1}^{50}(1-2n)$$
$$\sum_{n=1}^{50}2n$$

$$-1-3-5-\cdots-99$$
$$2+4+8+\cdots+100$$

1.12: Summation independent of the variable

$$\sum_{n=1}^{k}x=kx$$

$$\sum_{n=1}^{k}x=\underbrace{x+x+\cdots+x}_{k\ times}=kx$$

Example 1.13

$$Evaluate\sum_{n=1}^{14}x$$

$$n=1\Rightarrow x$$
$$n=2\Rightarrow x$$
$$n=3\Rightarrow x$$
$$.$$
$$.$$
$$.$$
$$n=14\Rightarrow x$$

$$\underbrace{x}_{n=1}+\underbrace{x}_{n=2}+\underbrace{x}_{n=3}+\cdots+\underbrace{x}_{n=14}=\underbrace{x+x+\cdots+x}_{14\ times}=14x$$

Example 1.14

A grocery store clerk gets paid x dollars for every hour that she works. Last week, she worked for 14 hours. Write her total remuneration in summation notation, and then evaluate the expression that you wrote.

Let the salary of the grocery store clerk for the i^{th} hour be h_i:

$$Salary\ for\ 1st\ hour=h_1$$
$$Salary\ for\ 2^{nd}\ hour=h_2$$
$$Salary\ for\ i^{th}\ hour=h_i$$

$$Total\ Salary=h_1+h_2+h_3+\cdots+h_{14}=\sum_{i=1}^{14}h_i$$

But note that the salary for each hour:

$$= h_1 = h_2 = h_i = x$$

Substitute $h_i = x$

$$\sum_{i=1}^{14} x = \underbrace{x}_{i=1} + \underbrace{x}_{i=2} + \underbrace{x}_{i=3} + \cdots + \underbrace{x}_{i=14} = \underbrace{x + x + \cdots + x}_{14\ times} = 14x$$

1.15: Property I: Constant Property

A constant can be moved out of, or into the summation sign without changing its value.

$$\sum_{x=1}^{n} cx = c\sum_{x=1}^{n} x$$

Expand $\sum_{x=1}^{n} cx$ using the definition:

$$= c + 2c + \cdots + nc$$

Factor c from each term in the expression:

$$c(1 + 2 + \cdots + n)$$

Write the second term as a summation:

$$= c\sum_{x=1}^{n} x$$

Example 1.16

$$x - 2x + 3x - 4x + \cdots + kx, \qquad k\ is\ odd$$

$$\sum_{n=1}^{100} (-1)^{n+1} n = 1 - 2 + 3 - 4 + \cdots - 100$$

Multiply both sides of the above by x:

$$x\left(\sum_{n=1}^{100} (-1)^{n+1} n\right) = x(1 - 2 + 3 - 4 + \cdots - 100)$$

On the LHS, carry out the multiplication. On the RHS, move the constant inside the multiplication sign:

$$\sum_{n=1}^{n=k} (-1)^{n+1} nx = x - 2x + 3x - 4x + \cdots + kx, \qquad k\ is\ odd$$

1.17: Index of Summation

It is not necessary that the index of summation starts with 1 or 0. It can start with any number that you want.

Example 1.18

Write in summation notation:

A. $3 + 4 + \cdots + 17$
B. $50 - 51 + 52 - 53 + \cdots + 104$
C. $4 + 8 + 12 + \cdots + 396$
D. $\frac{25}{3} - \frac{30}{3} + \frac{35}{3} - \frac{40}{3} + \cdots + \frac{175}{3}$

$$3 + 4 + \cdots + 17 = \sum_{x=3}^{17} x$$

$$50-51+52-53+\cdots+104=\sum_{x=50}^{104}(-1)^x x=\sum_{x=1}^{55}(-1)^{x+1}(x+49)$$
$$4+8+12+\cdots+396=4$$

Example 1.19

1.20: Reindexing

Using a change of variable, you can change the index of summation of a series without changing the value of the expression. This is called reindexing.

Example 1.21

Write each of the above series using $n=1$ as the lower limit of summation:

A. $3+4+\cdots+17$
B. $50-51+52-53+\cdots+104$
C. $4+8+12+\cdots+396$
D. $\frac{25}{3}-\frac{30}{3}+\frac{35}{3}-\frac{40}{3}+\cdots+\frac{175}{3}$

$$3+4+\cdots+17=\sum_{x=3}^{17}x$$

Use a change of variable. Let $x=n+2 \Rightarrow n=x-2$:

$$=\sum_{n+2=3}^{n+2=17}n+2=\sum_{n=1}^{n=15}n+2$$

Since x is a dummy variable, we substitute $n=x$ without changing the value of the expression:

$$\sum_{n=1}^{n=15}n+2=\sum_{x=1}^{x=15}x+2$$

Hence, finally, we can say that:

$$\sum_{x=3}^{17}x=\sum_{x=1}^{x=15}x+2$$

Example 1.22: Changing the index of Multiplication

In the questions below, do not change the value of the final sum

$$\text{Change }\sum_{i=5}^{n}i\text{ to start from }i=0\Rightarrow\sum_{i=0}^{n-5}i+5$$
$$\text{Change }\sum_{i=7}^{n}2i+4\text{ to start from }i=-2\Rightarrow\sum_{i=-2}^{n-9}2(i-9)+4=\sum_{i=-2}^{n-9}2i-14$$

Example 1.23: Arithmetic Series

Write the following series in sigma notation

A. $1+3+\cdots+21$

B. $5 + 7 + \cdots + 101$
C. $3 - 6 + 9 - \cdots + 81$
D. $4 - 7 + 10 - \cdots + 82$

Part A

$$\sum_{n=0}^{10} 2n + 1$$
$$1 + 3 + \cdots + 21$$

Each term of the series form part of an arithmetic sequence with
$first\ term = a = 1, common\ difference = d = 2$

The n^{th} term of the sequence is given by:

$$1 + (n-1)2 = 1 + 2n - 2 = 2n - 1$$

Which we can convert into sigma notation as:

$$\sum_{n=1}^{11} 2n - 1$$

Part B

When the numbers do not start from 1, the lower limit of summation will generally not start from one:

$$5 + 7 + \cdots + 101 = \sum_{x=0}^{48} 2x + 5$$

$$\sum_{n=0}^{n=20} (n+1), n \in 2k, k \in \mathbb{W}$$
$$k = 0 \Rightarrow n = 2(0) = 0 \Rightarrow n + 1 = 0 + 1 = 1$$
$$k = 1 \Rightarrow n = 2(1) = 2 \Rightarrow n + 1 = 2 + 1 = 3$$
$$\vdots$$
$$k = 10 \Rightarrow n = 2(10) = 20 \Rightarrow n + 1 = 20 + 1 = 21$$

$$3 - 6 + 9 - \cdots + 81 = \sum_{x=1}^{27} (-1)^{x-1} 3x$$

$$4 - 7 + 10 - \cdots + 82 = \sum_{x=1}^{27} (-1)^{x-1}(3x + 1)$$

Example 1.24: Geometric Series

Write the following series in sigma notation:

A. $3 + 6 + 12 + 24$
B. $2 + 6 + 18 + 54 + \cdots + 1458$
C. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16}$

Part A

$$= 3(2^0 + 2^1 + 2^2 + 2^3) = 3\sum_{n=0}^{n=3} 2^n$$

Part B

$$= 2(3^0 + 3^1 + 3^2 + 3^3 + \cdots + 3^6) = 2\sum_{n=0}^{n=6} 3^n$$

B. Evaluating Sigma Notation

1.25: Arithmetic Series

The sum of n terms of an arithmetic series with $first\ term = a$, and $common\ difference = d$ is given by:

$$\sum_{k=1}^{n} a + (k-1)d = \frac{n}{2}[2a + (n-1)d]$$

1.26: Geometric Series

The sum of n terms of a geometric series with $first\ term = a$, and $common\ ratio = r$ is given by:

$$\sum_{x=1}^{n} ar^{x-1} = \frac{a(r^n - 1)}{r - 1}$$

Example 1.27

Determine the sum of the following summation:

$$\sum_{i=0}^{20} 3^{2i}$$

$$\sum_{i=0}^{20} 3^{2i} = 3^{2(0)} + 3^{2(1)} + \cdots + 3^{2(20)} = 3^0 + 3^2 + 3^4 + \cdots + 3^{20}$$

Geometric Series with $a = 1, r = 9, n = 21$:

$$S_{21} = \frac{a(r^n - 1)}{r - 1} = \frac{1(9^{21} - 1)}{9 - 1} = \frac{3^{42} - 1}{8}$$

Example 1.28: Geometric Series

Find the value of X, Y and Z given that:

$$X = \sum_{x=1}^{4} 2^x, \qquad Y = \sum_{x=4}^{7} 2^x, \qquad Z = \sum_{x=4}^{30} 2^x$$

Part A

$$\sum_{x=1}^{4} 2^x = 2^1 + 2^2 + 2^3 + 2^4 = 2 + 4 + 8 + 16 = 30$$

Part B

$$\sum_{x=4}^{7} 2^x = 2^4 + 2^5 + 2^6 + 2^7 = 16 + 32 + 64 + 128 = 240$$

This is a geometric series with $First\ Term = a = 2^4 = 16, Common\ Ratio = r = 2, Number\ of\ Terms = n = 4$
The sum of this series is:

$$= \frac{a(r^n - 1)}{r - 1} = \frac{2^4(2^4 - 1)}{2 - 1} = \frac{16(15)}{1} = 240$$

Part C

$$\sum_{x=4}^{30} 2^x = 2^4 + 2^5 + \cdots + 2^{30}$$

This is a geometric series with $First\ Term = a = 2^4 = 16, Common\ Ratio = r = 2, Number\ of\ Terms = n = 30 - 3 = 27$
The sum of this series is:

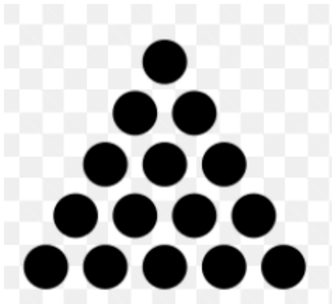
$$= \frac{a(r^n - 1)}{r - 1} = \frac{2^4(2^{27} - 1)}{2 - 1} = 2^4(2^{27} - 1)$$

1.29: Property II: Distribution of Summation over an Expression

You can distribute the summation operator over individual terms (for addition or subtraction)

$$\sum_{i=1}^{n} x + y = \sum_{i=1}^{n} x + \sum_{i=1}^{n} y$$

$$\sum_{i=1}^{n} x - y = \sum_{i=1}^{n} x - \sum_{i=1}^{n} y$$



1.30: Sum of Natural Numbers

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

- This is the sum of the first n natural numbers.
- It generates the triangular numbers.

Proof I: By forming Pairs

$$1 + 2 + \cdots + n$$
$$n + (n - 1) + \cdots + 1$$

Make pairs. Each pair has sum $n + 1$. And there are $\frac{n}{2}$ pairs, giving us a sum of:

$$\underbrace{\frac{n}{2}}_{\substack{No.of\\ Pairs}} \underbrace{(n+1)}_{\substack{Sum\ of\\ each\ pair}}$$

And this formula also works when n is odd, since the remaining number has value exactly half of the value of each pair. Hence, we can think of the remaining number as representing half a pair.

$$1 + 2 + 3 + 4 + 5$$
$$5 + 1 = 6$$
$$2 + 4 = 6$$
$$3 = \frac{6}{2}$$

Proof II: By forming Pairs

Substitute $f = First\ Term = 1, l = Last\ Term = n, n = No.of\ terms$ in:

$$S_n = n \times \left(\frac{f+l}{2}\right) = n \times \left(\frac{n+1}{2}\right)$$

Example 1.31

What is the sum of all integers from 80 through 90, inclusive? (MathCounts 2004 State Sprint)

Method I: Without Summation Notation

$$80 + 81 + 82 + \cdots + 90$$
$$= 80 + (80+1) + (80+2) + \cdots + 90$$
$$= (11)(80) + [0 + 1 + 2 + \cdots + 10]$$
$$= 880 + 55 = 935$$

Method I: With Summation Notation

$$\sum_{x=80}^{x=90} x$$

Reindex the sum:

$$= \sum_{x=0}^{x=10} 80 + x$$

Split the sum:

$$= \sum_{x=0}^{x=10} 80 + \sum_{x=0}^{x=10} x$$

Evaluate each sum:

$$= (11)(800) + \left(\frac{10 \times 11}{2}\right)$$
$$= 880 + 55$$
$$= 935$$

1.32: Sum of Squares of Natural Numbers

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Example 1.33

If $u_i = -3 + 4i$, and $v_i = 12 - 3i$, find $\sum_{i=1}^{10} u_i v_i$

Substitute the values of u_i and v_i, and carry out the multiplication:

$$\sum_{i=1}^{10} u_i v_i = \sum_{i=1}^{10} (-3+4i)(12-3i) = \sum_{i=1}^{10} -36 + 48i + 9i - 12i^2 = \sum_{i=1}^{10} -36 + 57i - 12i^2$$

Break up each summation term, and move the constant out of the summation:

$$\sum_{i=1}^{10} -36 + \sum_{i=1}^{10} 57i + \sum_{i=1}^{10} -12i^2 = -360 + 57\sum_{i=1}^{10} i - 12\sum_{i=1}^{10} i^2$$

Use the formula $1+2+3+\cdots+n=\frac{n(n+1)}{2}$ to evaluate the sum of the first n natural numbers:

$$57\sum_{i=1}^{10} i = 57\times\frac{10\times 11}{2} = 57\times\frac{5\times 11}{1} = 57\times 55 = 3135$$

Use the formula $1+2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$ to evaluate the sum of the squares of the first n natural numbers:

$$-12\sum_{i=1}^{10} i^2 = (-12)\,\frac{(10)(11)(21)}{6} = 4620$$

$$= -360+3135-4620 = -1845$$

C. Averages

Summation notation can be used in many different parts of Maths. We look at some applications.

1.34: Arithmetic Mean

The arithmetic mean of a set of n values $\{x_1, x_2, \ldots, x_n\}$ is

$$\bar{x} = \frac{x_1+x_2+\cdots+x_n}{n} = \frac{1}{n}\sum_{i=1}^{i=n} x_i$$

Note that

- The average of $\{x_1, x_2, \ldots, x_n\}$ is generally written as $\bar{x}$.
- We can write the summation of x values in a number of different ways:

$$\sum x_i = \sum_i x_i = \sum_{i=1}^{i=n} x_i$$

Example 1.35

The average weight of 14 students and their teacher in a class is 60 kg.

A. Find the sum of the weights of all 15 people in the class.
B. If the teacher weighs 65 kg, find the weight of all the students.

Part A

$$\frac{1}{15}\sum x = 60 \Rightarrow \sum x = 15\times 60 = 900$$

Part B

$$\left(\sum x\right) - x_{Teacher} = 900 - 65 = 935$$

1.36: Change of Scale

Example 1.37

Show that the average of a set of numbers is invariant to change of scale

Consider the numbers:	$x_1 + x_2 + \cdots + x_n$	$\sum_{i=1}^{n} x_i$

Divide each number by q. (Change of scale)	$\frac{x_1}{q}+\frac{x_2}{q}+\cdots+\frac{x_n}{q}$	$\frac{1}{q}\sum_{i=1}^{n} x_i$
Find the average	$\frac{x_1+x_2+\cdots+x_n}{nq}$	$\frac{1}{nq}\sum_{i=1}^{n} x_i$
Multiply the average by q: (Reverse change of scale)	$\frac{x_1+x_2+\cdots+x_n}{n}$	$\frac{1}{n}\sum_{i=1}^{n} x_i$

Example 1.38

Find the average of x_1, x_2, x_3, x_4, x_5 given that $x_i = 1000 + i$.

Method I: Work with the numbers

$$\frac{1001+1002+1003+1004+1005}{5}$$
$$=\frac{1000+1+1000+2+1000+3+1000+4+1000+5}{5}$$
$$=\frac{5\times 1000+1+2+3+4+5}{5}$$
$$=1000+\frac{15}{5}$$
$$=1003$$

Method II: Use a change of variable

Define a new variable

$$X_i = x_i - 1000 = 1000 + i - 1000 = i$$

Calculate the average of the new variable:

$$\bar{X} = \frac{1+2+3+4+5}{5} = \frac{15}{5} = 3$$

Change back to the original variable:

$$\bar{x} = \bar{X} + 1000 = 1000 + 3 = 1003$$

1.39: Change of Origin

Example 1.40

Show that the average of a set of numbers is invariant to change of origin.

Consider the sum of the numbers	$x_1 + x_2 + \cdots + x_n$	$\sum_{i=1}^{n} x_i$
Subtract p from each number. (Change of origin)	$x_1 - p, x_2 - p, \dots, x_n - p$	$\sum_{i=1}^{n} x_i - p$
Find the average and simplify it.	$\frac{(x_1-p)+(x_2-p)+\cdots+(x_n-p)}{n}$ $=\frac{x_1+x_2+\cdots+x_n-np}{n}$ $=\frac{x_1+x_2+\cdots+x_n}{n}-p$	$\frac{1}{n}\sum_{i=1}^{n}(x_i-p)$ Split the summation using the sum and difference property: $=\frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{n}\sum_{i=1}^{n} p$

		Use the constant property in the second term: $$=\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)-\left(\frac{1}{n}np\right)$$ $$=\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)-p$$
Add back p.	$$=\frac{x_1+x_2+\cdots+x_n}{n}=\bar{x}$$	$$\frac{1}{n}\sum_{i=1}^{n} x_i$$

Example 1.41

Find the average of

$$5700,5900,6400$$

$$\frac{5700+5900+6400}{3}$$

Use a change of scale:

$$=100\times\frac{57+59+64}{3}$$

Use a change of origin:

$$=100\times\frac{60\times 3-3-1+4}{3}$$
$$=100\times 60$$

Simplify:

$$=6000$$

Example 1.42

Show that the average of a set of numbers is invariant to change of origin and change of scale (provided the operations are performed in the correct order).

$$x_1,x_2,\ldots,x_n\to\underbrace{\frac{x_1-p}{q},\frac{x_2-p}{q},\ldots,\frac{x_n-p}{q}}_{\substack{\textit{Change of Origin: Subtract } p\\ \text{Change of Scale: Divide by } q}}\to\underbrace{\frac{x_1+x_2+\cdots+x_n-np}{nq}}_{\substack{\textit{Add all the numbers, and then}\\ \textit{divide by } n}}$$

To get our average, we will reverse the steps on the average:
Multiply by q, and add back p:

$$\left[\left(\frac{x_1+x_2+\cdots+x_n}{nq}-\frac{p}{q}\right)\times q\right]+p=\frac{x_1+x_2+\cdots+x_n}{n}-p+p=\frac{x_1+x_2+\cdots+x_n}{n}$$

Which is the same as the average of the original numbers.

Start with the sum of a set of numbers:

$$\sum_{i=1}^{n} x_i$$

Subtract p from each number (Change of origin)

$$\sum_{i=1}^{n} x_i-p$$

Divide each number by q. (Change of scale)

$$\sum_{i=1}^{n}\frac{x_i-p}{q}$$

Divide by n to find the average of this new set of numbers:

$$\sum_{i=1}^{n}\frac{x_i-p}{nq}$$

Move $\frac{1}{q}$ outside of the summation sign

$$\frac{1}{q}\sum_{i=1}^{n}\frac{x_i - p}{n}$$

Split the summation:

$$= \frac{1}{q}\left(\sum_{i=1}^{n}\frac{x_i}{n} - \sum_{i=1}^{n}\frac{p}{n}\right)$$

Use the constant multiple property:

$$= \frac{1}{q}\left[\left(\sum_{i=1}^{n}\frac{x_i}{n}\right) - p\right]$$

Reverse the operations we performed. Start by multiplying by q:

$$\left[\frac{1}{q}\left(\frac{1}{n}\sum_{i=1}^{n}x_i - p\right) \times q\right]$$

Continue by adding back p:

$$= \frac{1}{n}\sum_{i=1}^{n}x_i - p + p$$

$$= \frac{1}{n}\sum_{i=1}^{n}x_i$$

D. Number Theory

Example 1.43

Find the probability that the value of the units digit of the expression below is 7, for some random four-digit number n.

$$\sum_{x=1}^{x=n} x!$$

$$\sum_{x=1}^{x=n} x! = 1! + 2! + 3! + 4! + 5! + \cdots + n! = 1 + 2 + 6 + 24 + 120 + \cdots + n! = 153 + \cdots + n!$$

$$n \geq 5! \Rightarrow Units\ digit\ of\ n!\ is\ zero.$$

Hence, there is no change in the units digit after we reach 5!.
The units digit is always 3.

$$P(Units\ Digit = 7) = 0$$

E. Imaginary Numbers

1.44: Sum of four consecutive powers of i is zero

$$i = \sqrt{-1}$$
$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$

1.45: Sum of four consecutive powers of i is zero

The sum of any four consecutive, integral powers of i is zero

$$\boldsymbol{i^x + i^{x+1} + i^{x+2} + i^{x+3} = 0, \qquad x \in \mathbb{Z}}$$

$$\underbrace{i^x(1 + i + i^2 + i^3)}_{Factoring\ out\ i^x} = \underbrace{i^x(1 + i - 1 - i)}_{Substituting\ values} = i^x(0) = 0$$

Since x can be negative, this above identity is true for negative values of x as well.

Example 1.46

Find the sum of the first 2021 powers of iota. That is, evaluate:

$$\sum_{x=1}^{2021} i^x$$

Method I: Using the properties of *i*

$$i^1 + i^2 + \cdots + i^{2021}$$

Group the terms of the series, with each group having four consecutive powers of $\boldsymbol{i}$, and one $\boldsymbol{i}$ left over:

$$= \underbrace{(i^1 + i^2 + i^3 + i^4)}_{=0} + \underbrace{(i^5 + i^6 + i^7 + i^8)}_{=0} + \cdots + \underbrace{(i^{2017} + i^{2018} + i^{2019} + i^{2020})}_{=0} + i^{2021}$$

All groups have a value of zero, except for the lone i^{2021}, which can be simplified as:

$$i^{2021} = i^{2020} \times i = i$$

Method II: Using the properties of Geometric Series

Note that

$$\sum_{x=1}^{2021} i^x = i^1 + i^2 + \cdots + i^{2021}$$

Is a geometric series with $First\ term = a = i$ and $Common\ Ratio = i, No. of\ Terms = n = 2021$

Substitute $a = i$ and $r = i$ in the formula for the sum of a finite geometric series:

$$S = \frac{a(1 - r^n)}{1 - r} = \frac{i(1 - i^{2021})}{1 - i} = \frac{i(1 - i)}{1 - i} = i$$

F. Nested Sums

In a nested summation, you have a summation that is itself inside a summation sign.

Example 1.47

$$\sum_{b=1}^{b=5}\left(\sum_{a=1}^{a=5} ab\right)$$

Expand the inner summation operator:

$$\sum_{b=1}^{b=5} b + 2b + 3b + 4b + 5b = \sum_{b=1}^{b=5} 15b$$

Move the 15 outside of the summation sign:

$$= 15\sum_{b=1}^{b=5} b$$

Expand the inner summation operator:

$$= 15(1 + 2 + 3 + 4 + 5) = 15^2 = 225$$

1.48: Sum of Cubes of Natural Numbers

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example 1.49

Given that $\sum_{x=1}^{n} x^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$ find the value of

$$\sum_{x=1}^{x=n}\left(\sum_{a=1}^{a=x} a^3\right)$$

$$\sum_{x=1}^{x=n}\left(\frac{x(x+1)}{2}\right)^2$$

Carry out the multiplication:

$$\sum_{x=1}^{x=n}\frac{x^4+2x^3+x^2}{4}$$

Split the terms:

$$\sum_{x=1}^{x=n}\frac{x^4}{4}+\sum_{x=1}^{x=n}\frac{2x^3}{4}+\sum_{x=1}^{x=n}\frac{x^2}{4}$$

Substitute:

$$\frac{n(n+1)(2n+1)(3n^2+3n-1)}{120}+\frac{\left[\frac{n(n+1)}{2}\right]^2}{2}+\frac{n(n+1)}{8}$$

Example 1.50

Evaluate

$$\sum_{x}\left(\sum_{k=-3}^{k=3} x^{2k+1}+5\right), \qquad -5\le x\le 5, x\in\mathbb{Z}, x\neq 0$$

Split the inside summation:

$$\sum_{x}\left(\sum_{k=-3}^{k=3} x^{2k+1}+\sum_{k=-3}^{k=3} 5\right)$$

Distribute the outer summation to each inner summation:

$$\sum_{x}\sum_{k=-3}^{k=3} x^{2k+1}+\sum_{x}\sum_{k=-3}^{k=3} 5$$

Simplify the First Term

$$\sum_{k=-3}^{k=3} x^{2k+1}=\frac{1}{x^5}+\frac{1}{x^3}+\frac{1}{x}+x^1+x^3+x^5+x^7$$

Note that:

$$(-x)^{2k+1}=-x^{2k+1}\Rightarrow\sum_{k=-3}^{k=3}(-5)^{2k+1}=\sum_{k=-3}^{k=3}-5^{2k+1}=-\sum_{k=-3}^{k=3}5^{2k+1}$$

Since the terms cancel out, the final value is zero.

Simplify the Second Term

$$\sum_{x=-5}^{x=5}\sum_{k=-3}^{k=3} 5=\sum_{x=-5}^{x=5}35=11(35)=385$$

Example 1.51: Imaginary Numbers

The expression below can be written in the form $a+bi$ where a and b are real numbers and $i=\sqrt{-1}$. Find $a+b$.

$$\sum_{n=1}^{n=2022}\sum_{x=1}^{n} i^x$$

$$\sum_{x=1}^{1} i^x = \sum_{x=1}^{4n+1} i^x = i$$
$$\sum_{x=1}^{2} i^x = \sum_{x=1}^{4n+2} i^x = i + i^2 = i - 1$$
$$\sum_{x=1}^{3} i^x = \sum_{x=1}^{4n+3} i^x = i + i^2 + i^3 = i - 1 - i = -1$$
$$\sum_{x=1}^{4} i^x = \sum_{x=1}^{4n} i^x = i + i^2 + i^3 + i^4 = i - 1 - i + 1 = 0$$

$$n = 4 \Rightarrow \sum_{x=1}^{1} i^x + \sum_{x=1}^{2} i^x + \sum_{x=1}^{3} i^x + \sum_{x=1}^{4} i^x = i + i - 1 - 1 = 2i - 2$$

The entire summation is:

$$\sum_{n=1}^{n=2022}\sum_{x=1}^{n} i^x = \underbrace{\sum_{x=1}^{1} i^x + \sum_{x=1}^{2} i^x + \sum_{x=1}^{3} i^x + \sum_{x=1}^{4} i^x}_{2i-2} + \cdots + \underbrace{\sum_{x=1}^{2017} i^x + \sum_{x=1}^{2018} i^x + \sum_{x=1}^{2019} i^x + \sum_{x=1}^{2020} i^x}_{2i-2} + \sum_{x=1}^{2021} i^x + \sum_{x=1}^{2022} i^x$$

The number of groups formed is:

$$\left\lfloor\frac{2022}{4}\right\rfloor = 505$$
$$505(2i-2) + \sum_{x=1}^{2021} i^x + \sum_{x=1}^{2022} i^x$$
$$1010i - 1010 + i + i - 1$$
$$-1011 + 1012i$$

$$a + b = -1011 + 1012 = 1$$

1.2 Product and Other Notations

A. Evaluating Product Notation

Product notation is less frequently used as compared to summation notation. It does come up, however.

1.52: Product Notation

$$\underbrace{\prod_{i=1}^{n} i}_{Product\ over\ i} \quad = 1 \times 2 \times 3 \times \ldots \times n = n!$$

Example 1.53

Evaluate

A. Find the sum of digits of $\prod_{i=1}^{5} i$

B. Find the product of the digits of $\prod_{i=1}^{4} 2i$

C. Evaluate $\prod_{i=1}^{5}(2i-1)$

Part A

$$1 \times 2 \times \ldots \times 5 = 5! = 120 \Rightarrow 1 + 2 + 0 = 3$$

Part B

$$2 \times 4 \times 6 \times 8 = 2^4(4!) = 16 \times 24 = 384 \Rightarrow 3 \times 8 \times 4 = 96$$

Part C

$$1 \times 3 \times 5 \times 7 \times 9 = 945$$

1.54: Laws of Exponents

$$\prod_{n} x^n = x^{\sum n}$$

$$LHS = \prod_{n} x^n = x^{n_1} \times x^{n_2} \times \ldots \times x^{n_i} = x^{n_1+n_2+\cdots+n_i} = x^{\sum n} = RHS$$

Exponents convert multiplication into addition. Hence, we can convert the product operation into a summation operator over the exponents.

Example 1.55

A. $\prod_{n=1}^{10} x^n = x^{p\times q}$ where p and q are prime numbers with $q > p$. Find $q - p$.

B. The product $\prod_{n=2}^{4} \sqrt[n]{x}$ can be written $x^{a\frac{b}{c}}$, where $a, b, c \in \mathbb{N}$ and $HCF(b, c) = 1$. Find $a + b + c$.

Part A

$$\prod_{n=1}^{10} x^n = x^{\sum_{n=1}^{10} n} = x^{\frac{n(n+1)}{2}} = x^{\frac{10(11)}{2}} = x^{55} = x^{5\times 11} \Rightarrow q - p = 11 - 5 = 6$$

Part B

$$\prod_{n=2}^{10} \sqrt[n]{x} = \prod_{n=2}^{4} x^{\frac{1}{n}} = x^{\sum_{n=2}^{4}\frac{1}{n}} = x^{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}} = x^{\frac{6+4+3}{12}} = x^{\frac{13}{12}} = x^{1\frac{1}{12}} \Rightarrow a + b + c = 1 + 1 + 12 = 14$$

Example 1.56

$$\sum_{a=1}^{n} \log x^a$$

Method I(Shorter)

Expand the summation:

$$\log x + \log x^2 + \cdots + \log x^n$$

Use the product rule for logarithms:

$$= \log(x \times x^2 \times \ldots x^n)$$

Use the product rule for exponents:

$$= \log(x^{1+2+\cdots+n})$$

Use the formula for the sum of the first n natural numbers:

$$= \log\left(x^{\frac{n(n+1)}{2}}\right)$$

Use the power rule for logarithms:

$$= \frac{n(n+1)}{2} \log x$$

Method II: Using Summation and Product Notation

Interchanging the operators results in the summation operator converting into a product operator:

$$\sum_{a=1}^{n}\log x^a = \log\left(\prod_{a=1}^{n} x^a\right)$$
Moving the product operator into the exponent

converts into a summation operator
$$\log\left(x^{\sum_{a=1}^{a=n} a}\right) = \log\left(x^{\frac{n(n+1)}{2}}\right) = \frac{n(n+1)}{2}\log x$$

Example 1.57

Write the finite value of the product below with infinite terms as a radical: (AMC 2022 12/8, Adapted) [1]

$$\prod_{n=1}^{\infty} \underbrace{\sqrt[3]{\dots\sqrt[3]{\sqrt[3]{10}}}}_{n\ radicals}$$

Expand the product

Write out the infinite product:

$$\sqrt[3]{10}\times\sqrt[3]{\sqrt[3]{10}}\times\sqrt[3]{\sqrt[3]{\sqrt[3]{10}}}\times\dots$$

Convert the radicals into fractional exponents:

$$=10^{\frac{1}{3}}\times 10^{\frac{1}{9}}\times 10^{\frac{1}{27}}\times\dots$$

Multiply the terms:

$$=10^{\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots}$$

The series in the exponent is a geometric series. Use the formula for the sum of a geometric series with $first\ term = a = \frac{1}{3}, common\ ratio = r = \frac{1}{3}$:

$$=10^{\frac{a}{1-r}} = 10^{\frac{\frac{1}{3}}{1-\frac{1}{3}}} = 10^{\frac{1}{3}\div\frac{2}{3}} = 10^{\frac{1}{3}\times\frac{3}{2}} = 10^{\frac{1}{2}}$$

Convert the fractional exponent into a radical:

$$=\sqrt{10}$$

Product Notation

We can recognize the pattern, and convert the product into a summation over the exponents to get:

$$=\prod_{n=1}^{\infty} 10^{\frac{1}{3^n}} = 10^{\sum_n \frac{1}{3^n}}$$

And then the solution can proceed as before.

B. Converting to Product Notation

Example 1.58: Converting to Product Notation

Write the following in Product Notation

A. $n(n-1)(n-2)\dots(n-k)$
B. $n(n+1)(n+2)\dots(n+k)$
C. $\underbrace{1\times 3\times 5\times\dots}_{n\ terms}$

Part A

This is called a falling factorial:

[1] The math remains the same from the original question, but it is presented in product notation.

$$\underbrace{\prod_{i=n-k}^{n} i}_{Falling\ Factorial}$$

Part B

This is called a rising factorial:

$$\underbrace{\prod_{i=n}^{n+k} i}_{Rising\ Factorial}$$

Part C

$$\underbrace{\prod_{i=1}^{n} 2i-1}_{Odd\ Numbers}$$

Example 1.59

A. The number of divisors $\tau(x)$ of a number x with prime factorisation $a^p b^q c^r$ is $\tau(x) = (p+1)(q+1)(r+1)$. Write $\tau(x)$ of a number x with prime factorization $p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_n^{a_n}$ in product notation.

B. A number x with prime factorization $p^a q^b r^c$ has sum of its factors $s(n) = (1 + p + p^2 + \dots + p^a)(1 + q + q^2 + \dots + q^b)(1 + r + r^2 + \dots + r^c)$. Write the expression for the sum of factors of $X = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_n^{a_n}$ using the formula for the sum of a finite geometric series.

Part A

$$\tau(x) = (a_1+1)(a_2+1)\dots(a_n+1) = \prod_{i=1}^{n}(a_i+1)$$

Part B

$$s(X) = \left(1 + p_1 + p_1^2 + \dots + p_1^{a_1}\right)\left(1 + p_2 + p_2^2 + \dots + p_2^{a_2}\right)\dots\left(1 + p_n + p_n^2 + \dots + p_n^{a_n}\right)$$

To find the sum of the first parentheses substitute $first\ term = a = 1, common\ ratio = r = p_1, number\ of\ terms = a + 1$ in the formula for the sum of a finite geometric series:

$$\frac{a(r^n-1)}{r-1} = \frac{p_1^{a_1+1}-1}{p_1-1}$$

And the entire expression is:

$$\left(\frac{p_1^{a_1+1}-1}{p_1-1}\right)\left(\frac{p_2^{a_2+1}-1}{p_2-1}\right)\dots\left(\frac{p_n^{a_n+1}-1}{p_n-1}\right) = \prod_{i=1}^{n}\frac{p_i^{a_i+1}-1}{p_i-1}$$

C. Properties

1.60: Product over a Constant

A product over a constant can be determined by exponentiating the constant by the number of terms.

$$\prod_{i=i}^{n} c = \underbrace{c \times c \times \dots \times c}_{n\ times} = c^n$$

Example 1.61

What is the minimum value of n for which the expression below is greater than 1000.

$$\prod_{i=i}^{n} 2$$

$$\prod_{i=i}^{n} 2 > 1000$$
$$2^n > 1000$$
$$n \geq 10$$

1.62: Moving a constant out

A constant can be moved out of a product sign by exponentiating it the number of terms.

$$\prod_{i=i}^{n} cf(i) = c^n \prod_{i=i}^{n} f(i)$$

$$\prod_{i=i}^{n} cf(i) = cf(1) \times cf(2) \times \dots cf(n) = c^n[f(1) \times f(2) \times \dots f(n)] = c^n \prod_{i=i}^{n} f(i)$$

Example 1.63

Prove that the product of the first

A. n even numbers is $2^n n!$.

B. n odd numbers is $\frac{(2n)!}{2^n n!}$

Part A

Without product notation

$$2 \times 4 \times 6 \times \dots \times 2n = 2^n(1 \times 2 \times 3 \times \dots \times n) = 2^n n!$$

With product notation

$$2 \times 4 \times 6 \times \dots \times 2n = \prod_{i=1}^{i=n} 2i = 2^n \prod_{i=1}^{i=n} i = 2^n n!$$

Part B

Without product notation

$$\underbrace{1 \times 3 \times 5 \times \dots}_{\substack{First\ n \\ Odd\ Numbers}} \times \frac{\underbrace{2 \times 4 \times 6 \times \dots}_{First\ n\ Even\ Numbers}}{\underbrace{2 \times 4 \times 6 \times \dots}_{First\ n\ Even\ Numbers}} = \frac{Product\ of\ First\ 2n\ Numbers}{\underbrace{2 \times 4 \times 6 \times \dots}_{First\ n\ Even\ Numbers}} = \frac{(2n)!}{2^n n!}$$

With product notation

Multiply and divide by the product of the first n even numbers:

$$\underbrace{1 \times 3 \times 5 \times \dots}_{n\ terms} = \underbrace{\prod_{i=1}^{n} 2i - 1}_{Odd\ Numbers} = \frac{\underbrace{\left(\prod_{i=1}^{n} 2i - 1\right)}_{Odd\ Numbers} \times \underbrace{\left(\prod_{i=1}^{n} 2i\right)}_{Even\ Numbers}}{\prod_{i=1}^{n} 2i} = \frac{\prod_{i=1}^{2n} i}{2^n n} = \frac{(2n)!}{2^n n}$$

Using Mathematical Induction

When $n = 1$:

$$\frac{(2n)!}{2^n n!} = \frac{(2)!}{2^1 1!} = \frac{2}{2} = 1 \Rightarrow Base\ case\ is\ proved$$

Now, if it is true for some integer k, then

$$\underbrace{1 \times 3 \times 5 \times \dots \times (2k-1)}_{k\ terms} = \frac{(2k)!}{2^k k!}$$

Multiply both sides of the above by $(2k+1)$:

$$LHS = \underbrace{1 \times 3 \times 5 \times \dots \times (2k+1)}_{k+1 \; terms}$$

$$RHS = \frac{(2k)!\,(2k+1)}{2^k k!} = \frac{(2k+1)!}{2^k k!} \times \frac{2k+2}{2k+2} = \frac{(2k+2)!}{2^k k!\,2(k+1)} = \frac{(2k+2)!}{2^{k+1}(k+1)!}$$

And we get the equality below which proves the inductive case:

$$\underbrace{1 \times 3 \times 5 \times \dots \times (2k+1)}_{k+1 \; terms} = \frac{\big(2(k+1)\big)!}{2^{k+1}(k+1)!}$$

1.64: Changing the Index of Multiplication

$$\prod_{k=x}^{k=y} k = \prod_{k=x-n}^{k=y-n} k+n = \prod_{k=x+n}^{k=y+n} k-n$$

Changing the index of multiplication for product notation is like changing the index of summation for summation notation

Example 1.65

Find x and y if:

$$\prod_{n=1}^{n=m} (n+k) = \prod_{n=x}^{n=y} n$$

$$\prod_{n=1}^{n=m} (n+k) = \prod_{n=1+k}^{n=m+k} n$$

$$x = 1+k, y = m+k$$

1.66: Interchanging Log and Product Operators[2]

$$\log\left(\prod_n x_n\right) = \sum_n \log x_n$$

Expand the LHS:

$$\log(x_1 \times x_2 \times \dots \times x_n)$$

Use the product rule for logarithms:

$$= \log x_1 + \log x_2 + \cdots + \log x_n$$

Write the expression as a summation:

$$= \sum_n \log x_n$$

Example 1.67

Evaluate

$$\sum_{i=1}^{i=100} \frac{1}{\log_i 100!} \quad (\textbf{\textit{XAT 2014}})$$

Use the change of base rule to move the logarithm from denominator to numerator:

[2] This generalizes the property $\log ab = \log a + \log b$

$$\sum_{i=1}^{i=100} \log_{100!} i$$

Interchange the summation and the log operators:

$$\log_{100!} \prod_{i=2}^{i=100} i$$

The product is a factorial:

$$= \log_{100!} 100! = 1$$

D. Applications

1.68: Vieta's Formulas

For a quadratic $ax^2 + bx + c, a \neq 0$

$$Sum\ of\ Roots = -\frac{b}{a}, \qquad Product\ of\ Roots = \frac{c}{a}$$

A quadratic has real roots when

$$Discriminant \geq 0 \Rightarrow b^2 - 4ac \geq 0$$

Example 1.69: Vieta's Formulas/Discriminant

Mark all correct options

If

$$f(x) = \prod_{k=1}^{999} (x^2 - 47x + k)$$

then the product of all real roots of $f(x) = 0$ is: (, Adapted)[3]

A. $\prod_{n=1}^{552} n$

B. $\prod_{n=1}^{552} n!$

C. $\prod_{n=1}^{999} n$

D. $\dfrac{\prod_{n=1}^{999} n}{\prod_{n=1}^{447} (n+552)}$

E. None of these

We have 999 quadratics, for a total for 999×2 roots.

Consider each quadratic. For the roots to be real, the discriminant must be non-negative.

$$b^2 - 4ac \geq 0 \Rightarrow 47^2 - 4k \geq 0 \Rightarrow k \leq 552.25 \Rightarrow k \in \{1, 2, \dots, 552\}$$

Using Vieta's Formulas, for a single quadratic:

$$Product\ of\ Roots = \frac{c}{a} = \frac{k}{1} = k$$

And hence the final answer is

$$1 \times 2 \times \dots \times 552 = 552! = \underbrace{\prod_{n=1}^{552} n}_{Option\ A} = \frac{\left(\prod_{n=1}^{552} n\right)\left(\prod_{n=553}^{999} n\right)}{\prod_{n=553}^{999} n} = \underbrace{\frac{\prod_{n=1}^{999} n}{\prod_{n=1}^{447} (n+552)}}_{Option\ D}$$

[3] The original question had options in terms of factorials only. This has options in terms of products.

Example 1.70

Let x_n be the n^{th} distinct root of:

$$f(x) = \prod_{k=1}^{100} (x^2 - k)$$

a is the number of zeroes at the end of $\prod_n x_n$. Let r be the number of roots of $f(x)$. Find $a + r$.

$$No.of\ Roots = 200$$

$$k = 1 \Rightarrow x^2 - 1 \Rightarrow Product\ of\ Roots = \frac{c}{a} = -\frac{1}{1} = -1$$

$$k = 2 \Rightarrow x^2 - 2 \Rightarrow Product\ of\ Roots = \frac{c}{a} = -\frac{2}{1} = -2$$

The product of roots of all the quadratics is:

$$(-1)(-2)(-3)\dots(-100) = (-1)^{100}100! = 100!$$

Any zero is basically generated by

$$10 = 2 \times 5$$

There are many $2's$ (for example, consider $64 = 2^6$), so the $5's$ are in short supply. Count the number of $5's$:

$$\{5,10,15,\dots,100\} = 5 \times \{1,2,3,\dots,20\} \Rightarrow 20\ Values$$

And we get two $5's$ each from the multiples of 25:

$$\{25, 50, 75, 100\} \Rightarrow 4\ Values$$

The total is:

$$20 + 4 = 24$$

E. Nested Products and Sums

1.71: Nested Products

$$\prod_n \prod_m x = \prod_n \left(\prod_m x \right)$$

- Like nested summations, nested products are evaluating by first finding the "inside" product.

Example 1.72

Find the sum of digits of

$$\log_2 \left(\prod_{m=1}^{m=100} \prod_{n=1}^{n=m} 2 \right)$$

Evaluate the inner product using the constant property:

$$\log_2 \left(\prod_{m=1}^{m=100} 2^m \right)$$

Interchange the log and the summation operators:

$$= \sum_{m=1}^{m=100} \log_2 2^m$$

Use the power rule for logarithms

$$= \sum_{m=1}^{m=100} m$$

Use the formula for the sum of the first n natural numbers:

$$= \frac{100(101)}{2} = 5050$$

Which has sum of digits:

$$10$$

Example 1.73

Find the number of zeros at the end of

$$\prod_{m=1}^{m=100}\prod_{n=1}^{n=m} n$$

The inside product is a factorial, giving us:

$$\prod_{m=1}^{m=100} m!$$

Count the number of $5's$:

5!	10!	15!	.	.	100!	Total
5	5	5	.		5	20
	10	10	.	.	10	19
		15	.	.	15	18
						.
						.
						.
						1

$$1+2+\cdots+20=\frac{20(21)}{2}=210$$

Count the number of $25's$:

25!	50!	75!	100!	Total
25	25	25	25	4
	50	50	50	3
		75	75	2
			100	1
				10

$$1+2+3+4=10$$

The final answer is

$$210+10=220$$

Example 1.74

A number x with prime factorization $p^a q^b r^c$ has sum of its factors $s(n)=(1+p+p^2+\cdots+p^a)(1+q+q^2+\cdots+q^b)(1+r+r^2+\cdots+r^c)$. Write the expression for the sum of factors of $X=p_1^{a_1}p_2^{a_2}p_3^{a_3}\ldots p_n^{a_n}$ using a summation nested in a product.

$$s(X)=\underbrace{\sum_{x=0}^{a_1} p_1^x}_{(1+p_1+p_1^2+\cdots+p_1^{a_1})}\times\underbrace{\sum_{x=0}^{a_2} p_2^x}_{(1+p_2+p_2^2+\cdots+p_2^{a_2})}\times\ldots\times\underbrace{\sum_{x=0}^{a_n} p_n^x}_{(1+p_n+p_n^2+\cdots+p_n^{a_n})}$$

And now we can write this using product notation:

$$\prod_{i=1}^{n}\sum_{x=0}^{a_1} p_i^x$$

F. Union and Intersection Notation

1.75: Union Notation

$$\bigcup_{i=1}^{n} S_i = S_1 \cup S_2 \cup \dots \cup S_n$$

- Union notation is used to indicate a union of intervals.
- It generalizes the union operator you would have seen in Set Theory.

1.76: Length of an Interval

The length of an interval $(a,b), [a,b], (a,b]\ or\ [a.b)$ is defined to be $|b-a|$.

For example

$$Length\ of\,(2,8) = |8-2| = 6$$

Example 1.77

A. Define the "length" of a union of intervals to be the sum of the length of the disjoint intervals that make up the union. Why is are the intervals *disjoint*?
B. Find the length of $\bigcup_{i=1}^{n}\left(2^i, 2^{i+1}\right)$
C. Find the length of $\bigcup_{i=0}^{n}\left(2^{2i}, 2^{2i+1}\right)$

Part A

$$(1,5) \cup (4,6) \Rightarrow Length = 4+2 = 6$$
$$(1,5) \cup (5,6) \Rightarrow Length = 4+1 = 5$$

We should get the same length in both cases since the:

$$(1,5) \cup (4,6) = (1,5) \cup (5,6) = (1,6) \Rightarrow Length = 6-1 = 5$$

Part B

$$(2^1, 2^2) \cup (2^2, 2^3) \cup \dots \cup (2^n, 2^{n+1})$$
$$= 1 +$$
$$= (2^1, 2^{n+1}) \Rightarrow Length = 2^{n+1} - 2$$

Part C

Write the union in set notation:

$$(2^0, 2^1) \cup (2^2, 2^3) \cup (2^4, 2^5) \cup \dots \cup (2^{2n}, 2^{2n+1})$$
$$= (1,2) \cup (4,8) \cup (16,32) \cup \dots \cup (2^{2n}, 2^{2n+1})$$

Determine the length of each interval, add the resulting geometric series:

$$= 1 + 4 + 16 + \dots + 4^n = \frac{1(4^{n+1}-1)}{4-1} = \frac{4^{n+1}-1}{3}$$

1.78: Intersection Notation

$$\bigcap_{i=1}^{n} S_i = S_1 \cap S_2 \cap \dots \cap S_n$$

- Union notation is used to indicate an intersection of intervals.
- It generalizes the intersection operator you would have seen in Set Theory.

Example 1.79
Determine the smallest value of n for which $\bigcap_{i=1}^{n} S_i$ is a null set.

G. "AND" Notation; "OR" Notation

1.80: AND Notation

Example 1.81

1.82: OR Notation

Example 1.83

1.3 Log Sequences; AP-GP General Revision

A. General Sequences

1.84: Sequence
A sequence $a_1, a_2, a_3, \dots, a_n$ is a set of terms.

1.85: General Term of a Sequence
- The expression for the n^{th} of a sequence, written a_n is called the general term.
- The general term can be written in explicit form, which is a direct formula for the n^{th} term.
- It can also be written in recursive form, where the n^{th} term depends on one or more prior terms.

Example 1.86
Each sequence below is a sequence of a standard type. Identify the type of sequence, and give its general term in explicit and recursive form.

A. $a, a+d, a+2d, a+3d, \dots, a+nd$
B. $a, ar, ar^2, \dots ar^n$
C. $a+(a+d)r+(a+2d)r^2+\cdots+[a+(n-1)d]r^{n-1}$
D. $1, 1+2, 1+2+3, \dots, 1+2+3+\cdots+n$
E. $1, 1+4, 1+4+9, \dots, 1+4+\cdots+n^2$
F. $1, 1+8, 1+8+27, \dots, 1+8+\cdots+n^3$
G. $2,4,6,8, \dots$
H. $1,3,5,7 \dots$

Part A
This is an arithmetic sequence, where each term increases by a common difference compared to the previous term.
In this sequence:

$$First\ Term = a, Common\ Difference = d$$
$$Explicit: a_n = a + (n-1)d$$
$$Recursive: a_n = a_{n-1} + d$$

Part B

This is a geometric sequence, where each term is multiplied by the common the ratio to get the next term,
In this sequence:

$$First\ Term = a, Common\ Ratio = r$$
$$Explicit: a_n = ar^{n-1}$$
$$Recursive: a_n = a_{n-1}r$$

Part C

An arithmetico-geometric sequence has ideas which combine arithmetic and geometric sequences. The first term is a. Each term increases a by the common difference d, and the increased value is multiplied by the common ratio r.

$$First\ Term = a, Common\ Difference = d, Common\ Ratio = r$$
$$Explicit: a_n = [a + (n-1)d]r^{n-1}$$

Part D

This is the sequence of triangular numbers, where the n^{th} triangular number is the sum of the first n numbers:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Part E

The n^{th} term of this sequence is the sum of the squares of the first n numbers.

$$a_n = 1 + 2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Part F

The n^{th} term of this sequence is the sum of the cubes of the first n numbers.

$$a_n = 1 + 8 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Part G

n^{th} even natural number:

$$Even\ Numbers = 2n$$

Part H

n^{th} odd natural number:

$$Odd\ Numbers = 2n - 1$$

1.87: Partial Sums

The sequence S_n formed by the sum of the first n terms of a sequence a_n is called the sequence of partial sums.

Example 1.88

A. $a_n = ar^{n-1}$

a_n	a	ar	ar^2	.	ar^{n-1}
S_n	a	$a + ar$	$a + ar + ar^2$	.	$a + ar + \cdots + ar^{n-1}$

1.89: Sequence Notation

$$\{a_n\}$$
$$\{a_n\}_1^\infty$$

1.90: Alternating Sequence

An alternating sequence is a sequence where the terms alternate between positive and negative.

$$a_1, -a_2, a_3, -a_4, a_5, \dots$$

Example 1.91

Given the sequence a_n write the general term of the alternating sequence

A. $b_1 = a_1, b_2 = -a_2, b_3 = a_3, b_4 = -a_4, b_5 = a_5, \dots$
B. $-a_1, +a_2, -a_3, +a_4, -a_5,$

Part A

We can define it piece-wise for $n \geq 1$:

$$b_{2n} = -a_{2n}$$
$$b_{2n-1} = a_{2n+1}$$

We can also define it in a single general term:

$$b_n = (-1)^{n+1} a_n$$

1.92: Limit

The end behavior of a sequence as it approaches infinity is called the limit of the sequence.

1.93: Convergent vs. Divergent Sequences

- If the sum of the terms of a sequence has a finite value as the number of terms goes to infinity, then the sequence is called convergent.
- If a sequence is not convergent, then it is called divergent.

Example 1.94

A. Is an arithmetic sequence S_n convergent?
B. Is a geometric sequence S_n convergent?

Part A

An arithmetic sequence either keeps increasing, or it keeps decreasing. It has no finite value.

$$\lim_{n \to \infty} S_n \; does\ not\ exist\ (DNE)$$

Part B

A geometric sequence $a, ar, \dots ar^{n-1}$ is convergent if and only if

$$\lim_{n \to \infty} S_n \; exists\ if\ and\ only\ if -1 < r < 1$$

Example 1.95

A. Investigate the convergence or divergence of $S_n = \underbrace{1 - 1 + 1 - 1 + \cdots}_{n\ terms}$
B. Even though the sequence is divergent, and hence does not have a sum, it has a $Cesaro - sum$.

Part A

$$S_1 = 1, \qquad S_2 = 0$$
$$S_{2n+1} = 1, \qquad S_{2n} = 0$$

$$\lim_{n \to \infty} S_n \; DNE \Rightarrow Divergent$$

Part B

$$S_{2n} = 0, \qquad S_{2n-1} = 1$$

$$Caesaro - Sum = \frac{0+1}{2} = \frac{1}{2}$$

Example 1.96

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

$$\underbrace{1}_{\substack{1st\\Term}} + \underbrace{1}_{\substack{3rd\\Term}} - \underbrace{1}_{\substack{2nd\\Term}} + \underbrace{1}_{\substack{5th\\Term}} + \underbrace{1}_{\substack{7th\\Term}} - \underbrace{1}_{\substack{4th\\Term}}$$

$$S_1 = 1$$
$$S_2 = 2$$
$$S_3 = 1$$
$$S_4 = 2$$
$$S_5 = 3$$
$$S_6 = 2$$

1.97: Increasing Sequence

An increasing sequence is a sequence where the next term is greater than the one before it.
a_(n+1)> a_n

Nondecreasing sequence
An nondecreasing sequence is a sequence where the next term is greater than or equal to the one before it.
a_(n+1) >= a_n

Decreasing Sequence
An decreasing sequence is a sequence where the next term is less than the one before it.
a_(n+1) < a_n

Nonincreasing Sequence
An nonincreasing sequence is a sequence where the next term is less than the one before it.
a_(n+1) <= a_n

Example 1.98

My profits have been increasing year after year.
(Increasing)

My marks in any test are never less than the test before it.
(Nondecreasing)

Example 1.99

A sequence is decreasing when the next term is greater than the previous term.

$$a_{n+1} < a_n$$

Rearrange to get the equivalent condition that a sequence is decreasing when the difference of successive terms is negative:

$$a_{n+1} - a_n < 0$$

B. Log Sequences

1.100: Log Rules

$$Product\ Rule: \log a + \log b = \log ab$$
$$Quotient\ Rule: \log a - \log b = \log \frac{a}{b}$$
$$Power\ Rule: \log a^b = b \log a$$
$$x^{\log_x a} = a$$

Example 1.101

Find the range of x for which S is a two-digit positive integer, if:

$$S_n = \log_x x + \log_x x^2 + \log_x x^3 + \cdots + \log_x x^x, x \in \mathbb{N}$$

Use the power rule on each term:

$$S_n = 1 \log_x x + 2 \log_x x + 3 \log_x x + \cdots + n \log_x x$$

Substitute $\log_x x = 1$

$$S = 1 + 2 + 3 + \cdots + x = \frac{x(x+1)}{2}$$
$$9 < \frac{x(x+1)}{2} < 100$$

Splitting the inequality and solving it will result in two quadratic inequalities, which we must solve and then find the intersection.

Instead, since S is an integer, x must also be an integer. Using trial and error, the smallest value that works is 4, and the largest value 13.

$$x \in \{4, 5, 6, \ldots, 13\}$$

Example 1.102

Show that:

$$\log x + \log 2x + \cdots + \log nx = \log n! + n \log x$$

A common method of handling series is to break them up logically into two different series that are easier to handle.

Use the product rule

$$LHS = \underbrace{\log 1 + \log x}_{First\ Term} + \underbrace{\log 2 + \log x}_{Second\ Term} + \cdots + \underbrace{\log n + \log x}_{n^{th} term}$$

Rearrange:

$$= \log 1 + \log 2 + \cdots + \log n + \underbrace{\log x + \cdots + \log x}_{n\ times}$$

Use the reverse of the product rule:

$$= \log 1 \times 2 \times \ldots \times n + \log x^n$$
$$= \log n! + n \log x = RHS$$

Example 1.103

If 1, log_9 (3^(1-x)+2), log_3 (4\bullet3^x -1) are in AP, then x can be written in the form 1- log_3 y. Find the value of y. (JEE Main 2002, Adapted)

Example 1.104

$$\sum_{i=1}^{n} \log i$$

$$\sum_{i=1}^{n} \log i = \log 1 + \log 2 + \cdots + \log n = \log(1 \times 2 \times \ldots \times n) = \log n!$$

Example 1.105

$$\sum_{n=1}^{i} \log \frac{n}{n+1}$$

Method I: Shorter Method

$$\sum_{n=1}^{i} \log \frac{n}{n+1} = \log\left(\frac{1}{2}\right) + \log\left(\frac{2}{3}\right) + \cdots + \log\left(\frac{i}{i+1}\right) = \log\left(\frac{1}{2} \times \frac{2}{3} \times \ldots \times \frac{i}{i+1}\right) = \log \frac{1}{i+1}$$

Method I

Split the term inside the summation sign:

$$\sum_{n=1}^{i} \log n - \log(n+1)$$

Split the summation:

$$= \sum_{n=1}^{i} \log n - \sum_{n=1}^{i} \log(n+1)$$

Change the index of summation

$$= \sum_{n=1}^{i} \log n - \sum_{n=2}^{i+1} \log(n)$$

$$= \log 1 + \sum_{n=2}^{i} \log n - \sum_{n=2}^{i} \log(n) - \log(i+1)$$

$$= \log 1 - \log(i+1) = \log\left(\frac{1}{i+1}\right)$$

C. Arithmetic Sequences

In the next section, we will study harmonic sequences, where each term is the reciprocals of an arithmetic sequence. Hence, we will revise the important properties of arithmetic sequences and series.

1.106: Arithmetic Sequence

An arithmetic sequence with first term a has a common difference d between successive elements.

$$a, a+d, a+2d, \ldots, a+(n-1)d$$

Some of examples of arithmetic sequences are:

$$12, 17, 22, 27, \ldots \Rightarrow Rule\ is: Add\ 5 \Rightarrow a = 12, d = 5$$

$$31, 28, 25, \ldots \Rightarrow Rule\ is: Subtract\ 3 \Rightarrow a = 31, d = -3$$

Example 1.107

If a, b, c form an arithmetic progression with $first\ term = a = 2$, and common difference$= 2$, then find $x: y$ given:

$$\log_a\{\log_b[\log_c x]\} = \log_c\{\log_b[\log_a y]\} = 0$$

$$\log_a\{\log_{a+2}[\log_{a+4} x]\} = \log_{a+4}\{\log_{a+2}[\log_a y]\}$$

This is very difficult to do anything with directly because the bases are not the same.

Key Idea: Create Simple Equations

Equating the first and third terms creates an easy equation. Then, equate the second and third terms.

Equating the first and the third term:	**Equating the second and the third term:**	**Substituting the values of x and y:**
$\log_2\{\log_4[\log_6 x]\} = 0$ $\log_4[\log_6 x] = 1$ $\log_6 x = 4$ $x = 6^4$	$\log_6\{\log_4[\log_2 y]\} = 0$ $\log_4[\log_2 y] = 1$ $\log_2 y = 4$ $y = 2^4$	$x: y = 6^4: 2^4 = 81: 1$

1.108: Sub Sequences

Given an arithmetic sequence

$$a_{n-k}, \dots, a_{n-1}, a_n, a_{n+1}, \dots, a_{n+k}$$

the terms

$$a_{n-k}, a_n, a_{n+k}$$

also form an arithmetic sequence

Given an arithmetic sequence with common difference 5, and middle term 50:

35,40,45, **50**, 55,60,65

The following symmetrical sequences are also arithmetic sequences:

$$45, 50, 55$$
$$40, 50, 60$$
$$35, 50, 65$$

1.109: General Term

For an arithmetic sequence, the general term (also the called the n^{th} term) is given by:

$$t_n = a + (n-1)d$$

$$First\ Term = a = a + 0d$$
$$Second\ Term = a + d = a + 1d$$
$$Third\ Term = a + 2d$$

And in general, d is added one less than the term number:

$$a + (n-1)d$$

1.110: Common Difference Property

If the difference between two terms of a sequence is constant, then the sequence is an arithmetic sequence.

43,45,47

$$47 - 45 = 2 = 45 - 43 \Rightarrow Arithmetic$$

If the sequence $x + 5, 2x + 4, 3x - 7$ is arithmetic:

$$(2x + 4) - (x + 5) = (3x - 7) - (2x + 4)$$
$$2(2x + 4) - (x + 5) = 3x - 7$$
$$4x + 8 - x - 5 = 3x - 7$$
$$3x + 3 = 3x - 7$$

1.111: Arithmetic Mean Property and its converse

Any term of an arithmetic sequence is the arithmetic mean of the terms that precede and follow it.

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}$$

$$RHS = \frac{a_{n-1} + a_{n+1}}{2} = \frac{[a + (n-2)d] + [a + nd]}{2} = \frac{2a + 2nd - 2d}{2} = a + nd - d = a + (n-1)d = LHS$$

Converse: If the general term of a sequence is the arithmetic mean of the terms that precede and follow it, then that sequence is an arithmetic sequence.

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}$$
$$2a_n = a_{n-1} + a_{n+1}$$
$$a_n - a_{n-1} = a_{n+1} - a_n$$

Since the common difference is the same, the sequence is arithmetic.

The arithmetic mean property has both theoretical and practical importance. We can apply it:

- To find missing values when we know that a sequence is arithmetic.
- To prove that a sequence is arithmetic

1.112: Middle term is average of all the terms (for odd number of Terms)

The average of an arithmetic sequence with an odd number of terms is equal to its middle term.

$$\frac{a_1 + a_2 + \cdots + a_n}{n} = Middle\ Term$$

Three Term Case

$$a_1 = a - d, a_2 = a, a_3 = a + d$$

Suppose we consider three terms:

$$\frac{a_1 + a_2 + a_3}{3} = \frac{(a - d) + a + (a + d)}{3} = \frac{3a}{3} = a = Middle\ Term$$

General Case

Let $a_{n-k}, \dots, a_{n-1}, a_n, a_{n+1}, \dots, a_{n+k}$ be an arithmetic sequence with middle term a and common difference d. Then, the terms are:

$$a - kd, \dots, a - d, a, a + d, \dots, a + kd$$
$$\frac{(a - kd) + \cdots + a + \cdots + (a + kd)}{2k + 1} = \frac{(2k + 1)a}{2k + 1} = a = Middle\ Term$$

1.113: Average for Even Number of Terms

The average of an arithmetic sequence with an even number of terms is equal to the average of its two middlemost terms:

$$\frac{a_1+\dots+a_{\frac{k}{2}}+a_{\frac{k}{2}+1}+\dots+a_{2k}}{k}=\frac{a_{\frac{k}{2}}+a_{\frac{k}{2}+1}}{2}$$

$$11,13,15,17$$

Average

$$=\frac{11+13+15+17}{4}=\frac{(14-3)+(14-1)+(14+1)+(14+3)}{4}=\frac{14\times 4}{4}=14$$

1.114: Symmetry in Average and Median

- When a sequence is arranged in ascending (or descending) order, the median is the middle term if there are an odd number of terms.
- An arithmetic sequence has a constant difference. Hence, its arithmetic mean is also its median.

Odd Term Case

$$a-d,a,a+d\Rightarrow Median=a$$
$$a-d,a,a+d\Rightarrow Mean=a$$

Even Term Case

$$a-3d,a-d,a+d,a+3d\Rightarrow Median=Avg(a-d,a+d)=a$$
$$a-3d,a-d,a+d,a+3d\Rightarrow Mean=\frac{4a}{4}=a$$

1.115: Recursive Definition

An arithmetic sequence can be defined as:

$$a_n=a_{n-1}+d,\qquad a_1=c$$

An arithmetic sequence can be defined either recursively, or explicitly, and both definitions are equivalent. It is also possible to convert from one form to another, and this is useful.

1.116: Converting from Recursive Definition to Explicit Definition

An arithmetic sequence given in recursive form can be converted into explicit form

$$\underbrace{a_n=a_{n-1}+d,\qquad a_1=c}_{Recursive\ Definition}\Leftrightarrow\underbrace{a_n=c+(n-1)d}_{Explict\ Definition}$$

$$a_1=c$$
$$a_2=c+d$$
$$a_3=c+d+d=c+2d$$
$$a_n=c+(n-1)d$$

D. Arithmetic Series

1.117: Sum of an Arithmetic Series

The sum of an arithmetic series with $f=First\ Term, l=Last\ Term, n=No.of\ terms$ is:

$$S_n=n\times\left(\frac{f+l}{2}\right)=\text{No. of Terms}(Avg.of\ First\ and\ Last\ Term)$$

1.118: Sum of an arithmetic Series

$$S_n=\frac{n}{2}[2a+(n-1)d]$$

$$a = First\ Term, \quad n = No.of\ terms, \quad d = Common\ Difference$$

We write the sum twice, first in regular order, and then back to front.

	First Term	Second Term				n^{th} Term
S	a	$a+d$	.	.	.	$a+(n-1)d$
S	$a+(n-1)d$	$a+(n-2)d$				a
2S	$2a+(n-1)d$	$2a+(n-1)d$	.	.	.	$2a+(n-1)d$

$$2S = n[2a+(n-1)d] \Rightarrow S = \frac{n}{2}[2a+(n-1)d]$$

E. Geometric Sequences

When considering questions with harmonic sequences, they also require concepts from geometric sequences and series. Hence, we revise these as well.

1.119: Geometric Sequence

A geometric sequence with first term $a \neq 0$ has a common ratio $r \neq 0$ between successive elements, and a geometric sequence of n terms is represented by:

$$a, ar, ar^2, \dots, ar^{n-1}$$

For example

$$2,4,8, \dots$$
$$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots$$

1.120: Common Ratio: Consecutive Terms

Any two consecutive terms of a geometric sequence will be

$$ar^x, \qquad ar^{x+1}$$

If we divide any two consecutive terms of a geometric sequence, we get

$$\frac{ar^{x+1}}{ar^x} = \frac{r^{x+1-x}}{1} = r$$

- The common ratio of a geometric sequence can also be negative.
- If you are given terms $a_n,\ a_{n+k},\ k\ is\ even$, then you will get two values for the common ratio.

1.121: N^{th} Term of a geometric sequence

- The n^{th} term is given by ar^{n-1}.
- If we equate to the value of the term, we can solve for n.

To find the n^{th} term of a geometric sequence, substitute the values of a and r as constants, and leave n as n.

1.122: Subsequences

Consider a geometric sequence with middle term a and common ratio r

$$\dots, \frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2, \dots$$

If a_n is the middle term, then a_{n-k}, a_n, a_{n+k} are also in geometric sequence with common ratio r^k

$$k = 1 \Rightarrow \frac{a}{r}, a, ar$$
$$k = 2 \Rightarrow \frac{a}{r^2}, a, ar^2$$

1.123: Terms in GP ⇒Exponents/Logs in AP

Consider a geometric progression given by x^a, x^b, x^c. Then, a, b, c are in arithmetic progression.

A geometric progression x^a, x^b, x^c with common ratio $r = x^d$ can be written:

$$x^a, x^b, x^c = x^a, rx^a, r^2x^a = x^a, x^dx^a, x^{2d}x^a = x^a, x^{a+d}, x^{a+2d}$$

And taking the logs gives us:

$$\log x^a, \log x^{a+d}, \log x^{a+2d} = a \log x, (a+d) \log x, (a+2d) \log x$$

1.124: Logs of the Terms in AP ⇒Exponents are in GP

$$\ln a, \ln b, \ln c \to AP$$
$$2 \ln b = \ln a + \ln c$$
$$\ln b^2 = \ln ac$$
$$b^2 = ac$$
$$b = \sqrt{ac}$$

b is the geometric mean of a and c.

$$a, b, c \text{ are in geometric progression}$$

1.125: Properties

$\log a, \log b, \log c$ are in arithmetic progression if and only if a, b, c are in geometric progression.

Proving Forwards

$$\underbrace{\log a, \log b, \log c}_{in\ AP} \Leftrightarrow 2 \log b = \log a + \log c \Leftrightarrow \log b^2 = \log ac \Leftrightarrow b^2 = ac \Leftrightarrow \underbrace{b = \sqrt{ac}}_{in\ GP}$$

Proving Backwards

Consider quantities a, b, c in geometric progression with first term $\frac{q}{r}$ and common ratio r. Then:

$$a = \frac{q}{r}, b = q, c = qr$$

Take the log of each term above:

$$\log \frac{q}{r}, \log q, \log qr$$

Use the product and quotient rules:

$$\log q - \log r, \log q, \log q + \log r$$

This is an arithmetic progression with first term $\log q - \log r$ and common difference $\log r$.

$$\frac{\log\frac{q}{r} + \log qr}{2} = \frac{1}{2}\log q^2 = \log q$$

Example 1.126

$$\log_8 27\,,\log_8 p\,,\log_8 q\,,\log_8 125$$

Given that the above terms form an arithmetic sequence, show that $27, p, q$ and 125 form a geometric sequence.

Method I

$$3d = t_4 - t_1 = \log_8 125 - \log_8 27 = \log_8 \frac{5^3}{3^3} = 3\log_8 \frac{5}{3}$$
$$d = \log_8 \frac{5}{3}$$

$$t_2 = \log_8 3^3 + \log_8 \frac{5}{3} = \log_8 3^2 \times 5 \Rightarrow p = 3^2 \times 5$$
$$t_3 = \log_8 3^2 \times 5 + \log_8 \frac{5}{3} = \log_8 3 \times 5^2 \Rightarrow q = 3 \times 5^2$$

The terms that we need to show are:

$$27, p = 3^2 \times 5, q = 3 \times 5^2, 5^3$$

Which is a geometric series with

$$First\ Term = a = 27, Common\ ratio = r = \frac{5}{3}$$

Method II

$$t_2 = \log_8 27 + d = \log_8 p$$
$$\log_8 27 + \log_8 8^d = \log_8 p$$
$$\log_8 8^d \times 27 = \log_8 p$$
$$p = 8^d \times 27$$

$$\frac{T_2}{T_1} = \frac{8^d \times 27}{27} = 8^d$$

$$t_3 = \log_8 27 + 2d = \log_8 q$$
$$\log_8 27 + \log_8 8^{2d} = \log_8 q$$
$$\log_8 8^{2d} \times 27 = \log_8 q$$
$$q = 8^{2d} \times 27$$

$$\frac{T_3}{T_2} = \frac{8^{2d} \times 27}{8^d \times 27} = 8^d$$

Example 1.127

Show that $\log a\,, \log\frac{a^2}{b}\,, \log\frac{a^3}{b^2}\,, \ldots$ form an arithmetic progression.

Method I

Rewrite $\log a, \log\frac{a^2}{b}, \log\frac{a^3}{b^2}, \dots$ to get:

$$\log a, \log a\left(\frac{a}{b}\right), \log a\left(\frac{a}{b}\right)^2$$

$$\log a, \underbrace{\log a}_{\substack{First\\Term}} + \underbrace{\log\left(\frac{a}{b}\right)}_{\substack{Common\\Difference}}, \log a + 2\log\left(\frac{a}{b}\right)$$

Method II

$$\log\frac{a^2}{b} - \log a = \log\frac{a^2}{ab} = \log\left(\frac{a}{b}\right)$$

$$\log\frac{a^3}{b^2} - \log\frac{a^2}{b} = \log\frac{a^3}{b^2} \div \frac{a^2}{b} = \log\frac{a^3}{b^2} \times \frac{b}{a^2} = \log\frac{a}{b}$$

Method III

$$\frac{\log a + \log\frac{a^3}{b^2}}{2} = \frac{1}{2}\log\frac{a^4}{b^2} = \log\left(\frac{a^4}{b^2}\right)^{\frac{1}{2}} = \log\frac{a^2}{b}$$

Method IV

The general term is

$$a_n = \log\frac{a^n}{b^{n-1}}$$

We show that this sequence is an arithmetic progression by finding the difference between consecutive terms:

$$a_{n+1} - a_n = \log\frac{a^{n+1}}{b^n} - \log\frac{a^n}{b^{n-1}} = \log\frac{a^{n+1}}{b^n} \div \frac{a^n}{b^{n-1}} = \log\frac{a^{n+1}}{b^n} \times \frac{b^{n-1}}{a^n} = \underbrace{\log\frac{a}{b}}_{Constant} \Rightarrow AP$$

Since the difference between two terms is a constant, the sequence is an arithmetic progression.

Example 1.128

If a, b, and c are in geometric progression (G.P.) with $1 < a < b < c$ and $n > 1$ is an integer, then $\log_a n$, $\log_b n, \log_c n$ form a sequence

A. which is a G.P
B. which is an arithmetic progression (A.P)
C. in which the reciprocals of the terms form an A.P
D. in which the second and third terms are the n^th powers of the first and second respectively
E. none of these (AHSME 1973/28)

Since a, b, c are in geometric progression, for a ratio $r > 1$:

$$b = ar, c = ar^2$$

Hence, the sequence becomes

$$\log_a n, \log_{ar} n, \log_{ar^2} n$$

Use the property that $\log_a x = \frac{1}{\log_x a}$, and hence the reciprocals will interchange the number and the base:

$$\log_n a, \log_n ar, \log_n ar^2$$

Expand:

$$\log_n a, \log_n a + \log_n r, \log_n a + 2\log_n r$$

And this is now an arithmetic sequence with $first\ term = \log_n a$ and $common\ difference = d = \log_n r$

Option C.

Example 1.129

If $\log_3 2, \log_3(2^x - 5)$ and $\log_3\left(2^x - \frac{7}{2}\right)$ are in arithmetic progression, then determine the value of x. (JEE Adv.

1990)

Use a change of variable. Let $y = 2^x$.
Also, if the logs of the terms are in arithmetic progression, then the terms themselves must be in geometric progression:

$$(y-5)^2 = 2\left(y - \frac{7}{2}\right)$$

This is a quadratic. Expand and collate all terms on one side:

$$y^2 - 12y + 32 = 0$$

Change back to the original variable

$$y = \{4,8\} \Rightarrow 2^x = \{2^2, 2^3\}$$

However:

$$\log_3(2^x - 5) = \log_3(4-5) = \log_3(-1) \Rightarrow Not\ defined$$

$$x = 3$$

1.130: Geometric Mean Property-I

Any term of a geometric sequence is the geometric mean of the terms that precede and follow it:

$$a_{n-1}, a_n, a_{n+1}\ are\ a\ geometric\ sequence \Rightarrow a_n = \sqrt{a_{n-1} \times a_{n+1}}$$

$$\sqrt{a_{n-1} \times\ a_{n+1}} = \sqrt{ar^{n-2} \times ar^n} = \sqrt{a^2 r^{2n-2}} = ar^{n-1} = a_n$$

Equivalently:

$$(a_n)^2 = a_{n-1} \times a_{n+1}$$

Note: The geometric mean of two numbers a and b is given by the square root of the product of the two numbers. Geometric mean is usually not defined if the numbers for which we are taking the mean are negative.

1.131: Geometric Mean Property-II

If the general term of a sequence is the geometric mean of the terms that precede and follow it, then the sequence is a geometric sequence.

$$a_n = \sqrt{a_{n-1} \times a_{n+1}} \Rightarrow a_{n-1}, a_n, a_{n+1}\ are\ a\ geometric\ sequence$$

Suppose the equation given in the property above holds. Then:

$$a_n = \sqrt{a_{n-1} \times a_{n+1}}$$

Square both sides:

$$(a_n)^2 = a_{n-1} \times a_{n+1}$$

Let

$$a_{n-1} = a \Rightarrow a_n = ra_{n-1} = ra, \qquad for\ some\ r$$
$$(ra)^2 = a \times a_{n+1}$$
$$ar^2 = a_{n+1}$$

Hence, the terms are: $\underbrace{a}_{a_{n-1}}, \underbrace{ar}_{a_n}, \underbrace{ar^2}_{a_{n+1}} \Rightarrow Geometric\ Sequence$

1.132: Product of Terms

A geometric sequence $\dots, \underbrace{\frac{a}{r^2}}_{a_{n-2}}, \underbrace{\frac{a}{r}}_{a_{n-1}}, \underbrace{a}_{\substack{Middle\\Term}=a_n}, \underbrace{ar}_{\substack{Middle\\Term}=a_{n+1}}, \underbrace{ar^2}_{a_{n+2}}, \dots$ with an even number of terms has two middle terms. And each product below is equal to the product of the two middle terms.

$$a_n a_{n+1} = a \times ar = a^2 r$$
$$a_{n-1}a_{n+2} = \frac{a}{r} \times ar^2 = a^2 r$$
$$a_{n-k+1}a_{n+k} = a^2 r$$

1.133: Inserting Geometric Means

Given values x and y, the geometric mean property can be used to "insert" geometric means between them.

1.134: Geometric Sequence: Recursive Definition

A geometric sequence is defined as:

$$\underbrace{a_n = ra_{n-1}}_{\substack{Recursive\\Definition}}, \qquad \underbrace{a_1 = c}_{Base\ Case}$$

- A recursive definition is one that depends on the value that comes before it.
- That is, the n^{th} term of the sequence is r times the $(n-1)^{st}$ term of the sequence.
- And, we will also be given the starting value, which is the base case.

We can convert from the recursive definition to the explicit definition:

$$\underbrace{a_n = ra_{n-1}, \qquad a_1 = a}_{Recursive\ Definition} \Rightarrow \underbrace{Common\ ratio = r, n^{th} term = ar^{n-1}}_{Explicit\ Definition}$$

F. Infinite Geometric Series

1.135: Sum of Infinite Geometric Series

The sum S of an infinite geometric series $(a + ar + ar^2 + \cdots)$ with first term a, and common ratio r is given by:

$$S = \frac{a}{1-r}$$

$-1 < r < 1$ is the convergence condition. If the condition is not met, the series diverges (has an infinite value).

Example 1.136

$$\log_{10}\left(x^{\log_{10}\sqrt{x}}\right) + \log_{10}\left(x^{\log_{10}\sqrt{\sqrt{x}}}\right) + \log_{10}\left(x^{\log_{10}\sqrt{\sqrt{\sqrt{x}}}}\right) + \cdots \infty = 4$$

If $x \in S$, the sum of the elements of S can be written as $\frac{a}{b}$, where $a, b \in \mathbb{R}, HCF(a,b) = 1$, then find the sum of digits of $a + b$.

Change the radicals in the expression to fractional exponents:

$$\log_{10}\left(x^{\log_{10} x^{\frac{1}{2}}}\right) + \log_{10}\left(x^{\log_{10} x^{\frac{1}{4}}}\right) + \log_{10}\left(x^{\log_{10} x^{\frac{1}{8}}}\right) + \cdots \infty = 4$$

Use the product rule $\log a + \log b = \log ab$ in the given equation:

$$\log_{10}\left(x^{\log_{10} x^{\frac{1}{2}}} \cdot x^{\log_{10} x^{\frac{1}{4}}} \cdot x^{\log_{10} x^{\frac{1}{8}}} \dots\right) = 4$$

Use the laws of exponents $x^a x^b x^c \dots$

$$\log_{10}\left(x^{\log_{10} x^{\frac{1}{2}}+\log_{10} x^{\frac{1}{4}}+\log_{10} x^{\frac{1}{8}}+\cdots}\right)=4$$

Use the product rule one more time:

$$\log_{10}\left(x^{\log_{10} x^{\frac{1}{2}}\cdot x^{\frac{1}{4}}\cdot x^{\frac{1}{8}}+\cdots}\right)=4$$

Use the exponent rule one more time:

$$\log_{10}\left(x^{\log_{10} x^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots}}\right)=4$$

The exponent has an infinite geometric series with sum $a=\frac{1}{2}, r=\frac{1}{2} \Rightarrow Sum=\frac{a}{1-r}=\frac{\frac{1}{2}}{1-\frac{1}{2}}=\frac{\frac{1}{2}}{\frac{1}{2}}=1$

$$\log_{10}\left(x^{\log_{10} x}\right)=4$$

Use the power rule:

$$(\log_{10} x)=4$$
$$(\log_{10} x)^2=4$$

$$(\log_{10} x)=2 \Rightarrow x=10^2=100$$
$$(\log_{10} x)=-2 \Rightarrow x=10^{-2}=\frac{1}{100}$$

$$S \in \left\{100, \frac{1}{100}\right\}$$
$$Sum\ of\ elements\ of\ S=\frac{a}{b}=100+\frac{1}{100}=\frac{10001}{100}$$
$$a+b=10001+100=10101$$
$$Sum\ of\ Digits=1+1+1=3$$

Example 1.137

If $x, y \in R, x>0$ such that:

$$y=\log_{10} x+\log_{10} x^{\frac{1}{3}}+\log_{10} x^{\frac{1}{9}}+\cdots\infty$$

And

$$\frac{2+4+6+\cdots+2y}{3+6+9+\cdots+3y}=\frac{4}{\log_{10} x}$$

then the ordered pair (x, y) is equal to **(JEE 2021, 27 Aug, Shift-I)**

This looks like a system of equations in x and y, but y can be eliminated from the second equation directly.

$$\frac{2(1+2+\cdots+y)}{3(1+2+\cdots+y)}=\frac{4}{\log_{10} x}$$

Cancel $(1+2+\cdots+y)$ from numerator and denominator:

$$\frac{2}{3}=\frac{4}{\log_{10} x}$$

Cross-multiply:

$$\underbrace{\log_{10} x=6}_{\textit{Equation I}}$$
$$x=10^6$$

$$y=\log_{10} x^1+\log_{10} x^{\frac{1}{3}}+\log_{10} x^{\frac{1}{9}}+\cdots\infty$$

Use the power rule $\log x^a=a\log x$ in the first equation to get:

$$y=1\cdot\log_{10} x+\frac{1}{3}\log_{10} x+\frac{1}{9}\log_{10} x+\cdots$$

Factor $\log_{10} x$:

$$y = \log_{10} x \left(1 + \frac{1}{3} + \frac{1}{9} + \cdots\right)$$

Substitute Equation in the above:

$$y = 6\left(1 + \frac{1}{3} + \frac{1}{9} + \cdots\right)$$

The expression in parenthesis is an infinite geometric series. Substitute $a = 1, r = \frac{1}{3}$ in $\frac{a}{1-r}$:

$$y = 6\left(\frac{1}{1 - \frac{1}{3}}\right) = 6\left(\frac{1}{\frac{2}{3}}\right) = 6 \times \frac{3}{2} = 9$$

$$(x, y) = (10^6, 9)$$

Example 1.138

Find α^β given that

$$\alpha = 0.16, \qquad \beta = \log_{2.5}\left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\right) \ (JEE\ Main\ 2020, 3\ Sep, Shift - I)$$

Find the value of β

The expression in β is an infinite geometric series.

Substitute $a = \frac{1}{3}, r = \frac{1}{3}$ in $\frac{a}{1-r}$:

$$= \log_{\frac{10}{4}}\left(\frac{\frac{1}{3}}{1 - \frac{1}{3}}\right) = \log_{\frac{5}{2}}\left(\frac{\frac{1}{3}}{\frac{2}{3}}\right) = \log_{\frac{5}{2}}\left(\frac{1}{2}\right)$$

Rewrite using the rules of exponents:

$$= \log_{\frac{5}{2}}(2^{-1})$$

Use the power rule for logarithms:

$$= -\log_{\frac{5}{2}} 2$$

Find the value of α^β

$$\alpha^\beta = (0.16)^\beta = \left(\frac{16}{100}\right)^\beta = \left(\frac{4}{25}\right)^\beta$$

Substitute $\beta = -\log_{\frac{5}{2}} 2$:

$$= \left(\frac{4}{25}\right)^{-\log_{\frac{5}{2}} 2}$$

Use $\left(\frac{a}{b}\right)^{-x} = \left(\frac{b}{a}\right)^x$:

$$= \left(\frac{25}{4}\right)^{\log_{\frac{5}{2}} 2} = \left(\frac{5}{2}\right)^{2\log_{\frac{5}{2}} 2}$$

Use the power rule for logs:

$$= \left(\frac{5}{2}\right)^{\log_{\frac{5}{2}} 2^2} = \left(\frac{5}{2}\right)^{\log_{\frac{5}{2}} 4}$$

Use $a^{\log_a x} = x$:

$$= 4$$

Example 1.139

The product $2^{\frac{1}{4}} \cdot 4^{\frac{1}{16}} \cdot 8^{\frac{1}{48}} \cdot 16^{\frac{1}{128}} \cdot \ldots$ is equal to (JEE Main 2020, 9 Jan, Shift-I)

Convert all the bases to powers of 2:

$$2^{\frac{1}{4}} \cdot (2^2)^{\frac{1}{16}} \cdot (2^3)^{\frac{1}{48}} \cdot (2^4)^{\frac{1}{128}} \cdot \ldots$$

Use the exponent rule $(a^m)^n = a^{mn}$:

$$2^{\frac{1}{4}} \cdot 2^{\frac{1}{8}} \cdot 2^{\frac{1}{16}} \cdot 2^{\frac{1}{32}} \cdot \ldots$$

Use the exponent rule $a^m \cdot a^n = a^{m+n}$:

$$2^{\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}}$$

The exponent is a geometric series with $a = \frac{1}{4}, r = \frac{1}{2}$. Substitute in $\frac{a}{1-r}$:

$$2^{\frac{\frac{1}{4}}{1-\frac{1}{2}}} = 2^{\frac{\frac{1}{4}}{\frac{1}{2}}} = 2^{\frac{1}{2}} = \sqrt{2}$$

G. Finite Geometric Series

1.140: Sum of Infinite Geometric Series

The sum S of an finite geometric series $(a + ar + ar^2 + \cdots + ar^{n-1})$ with n terms, first term a, and common ratio r is given by:

$$S = \frac{a(1-r^n)}{1-r}$$

Example 1.141

The sum of the first n terms of a geometric sequence is given by

$$S_n = \sum_{r=1}^{n} \frac{2}{3}\left(\frac{7}{8}\right)^r.$$

(c) Find the least value of n such that $S_\infty - S_n < 0.001$.

First Term = 7/12
Sum to Infinity
= (7/12)/(1-7/8)
=(7/12)/(1/8)
=(7/12)*8
=14/3

$$\frac{14}{3} - S_n < 0.001$$

Substitute $S_n = \frac{a}{1-r}(1-r^n) = \frac{14}{3}\left(1-\left(\frac{7}{8}\right)^n\right)$:

$$\frac{14}{3} - \frac{14}{3}\left(1-\left(\frac{7}{8}\right)^n\right) < 0.001$$

Divide both sides by $\frac{14}{3}$:

$$1 - \left(1-\left(\frac{7}{8}\right)^n\right) < \frac{0.001}{\frac{14}{3}}$$

$$\left(\frac{7}{8}\right)^n < \frac{0.003}{14}$$

Take the natural log of both sides:

$$n \ln\frac{7}{8} < \ln\frac{0.003}{14}$$

$$n > \frac{\ln \frac{0.003}{14}}{\ln \frac{7}{8}}$$

Since we want the minimum of n, and the expression $\frac{14}{3} - S_n$ decreases as n increases, we need to round up the value of n that we get.
Hence, the final answer is:

$$n > \left\lceil \frac{\ln \frac{0.003}{14}}{\ln \frac{7}{8}} \right\rceil$$

Where $\lceil y \rceil$ is the smallest integer that is less than or equal to y.

1.142: Double Series

Suppose we have two geometric series given by:

$$a + ar + ar^2 + \cdots.$$
$$b + bq + bq^2 + \cdots$$

A double series consists of alternating terms from the above two series:

$$a + b + ar + bq + ar^2 + bq^2 + \cdots$$

1.143: Squares of Terms of Infinite Geometric Series

Squaring each term of the geometric series with $first\ term = a$, and $common\ ratio\ r$:

$$a + \ ar + ar^2 + \cdots + \ ar^{n-1}$$

Gives a geometric series with $first\ term = a^2$ and $common\ ratio = r^2$:

$$a^2 + \ a^2r^2 + a^2r^4 + \cdots + \ a^2r^{2(n-1)}$$

1.144: Odd and Even Power Decomposition

If I have an infinite geometric series with first term a, and common ratio r, then:

$$Sum\ of\ even\ powers = \frac{a}{1-r^2}$$
$$Sum\ of\ odd\ powers = \frac{ar}{1-r^2}$$

We can decompose (break up) the geometric series with first term a and common ratio r::

$$\boldsymbol{a} + ar + \boldsymbol{ar^2} + ar^3 + \boldsymbol{ar^4} + ar^5 + \cdots = \underbrace{(ar^0 + ar^2 + ar^4 + \cdots)}_{\text{Even Powers of } r} + \underbrace{(ar + ar^3 + ar^5 + \cdots)}_{\text{Odd Powers of } r}$$

The even powers form a geometric series with first term a, and common ratio r^2:

$$ar^0 + ar^2 + ar^4 + \cdots = \frac{a}{1-r^2}$$

The odd powers form a geometric series with first term ar, and common ratio r^2:

$$ar + ar^4 + ar^5 + \cdots = \frac{ar}{1-r^2}$$

1.145: Sum of a Finite Geometric Series

The sum of a finite geometric series $a + ar + ar^2 + \cdots + ar^{n-1}$ with first term a, common ratio r and n terms is given by:

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$$

We can define a finite geometric series with first term a, common ratio r and n terms as:

$$\underbrace{S = a + \boldsymbol{ar} + \boldsymbol{ar^2} + \cdots + \boldsymbol{ar^{n-1}}}_{Equation\ I}$$

Multiply both sides by r:

$$\underbrace{rS = \boldsymbol{ar} + \boldsymbol{ar^2} + \cdots + \boldsymbol{ar^{n-1}} + ar^n}_{Equation\ II}$$

Subtract Equation I from Equation II:

$$rS - S = ar^n - a$$
$$S(r-1) = a(r^n - 1)$$

Solve for S:

$$S = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$$

2. Special Sequences

2.1 Harmonic Sequences and Series[4]

A. Basics

2.1: Harmonic Sequence

If the reciprocals of the terms of a sequence form an arithmetic sequence, then that sequence is harmonic.

$$\underbrace{a-d, a, a+d}_{Arithmetic\ Sequence} \Leftrightarrow \underbrace{\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}}_{Harmonic\ Sequence}$$

A standard technique with harmonic sequences is

- Take the reciprocals of the terms. This results in an arithmetic sequence
- Use the properties of arithmetic sequences to manipulate the sequence as required.
- Take the reciprocal again to convert the sequence back to the original sequence.

Example 2.2

A. The first three terms of a harmonic sequence are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$. Find the fourth term.

B. Find the n^{th} term of the harmonic sequence: $\frac{1}{x}, \frac{1}{x+2}, \frac{1}{2x-2}$

C. A harmonic progression is a sequence of numbers such that their reciprocals are in arithmetic progression. Let S_n represent the sum of the first n terms of the harmonic progression; for example, S_3 represents the sum of the first three terms. If the first three terms of a harmonic progression are 3,4,6, then $S_4 =$ (AHSME 1959/33, Adapted)

Part A

Convert the given sequence into arithmetic by taking the reciprocal:

$$\underbrace{2,3,4}_{Arithmetic}$$

This has common difference $= d = 1$

$$t_4 = 5$$

Find the fourth term of the harmonic sequence by again taking the reciprocal:

$$\frac{1}{t_4} = \frac{1}{5}$$

Part B

Convert into an arithmetic sequence:

$$x, x+2, 2x-2$$

Apply the arithmetic mean property:

$$2(x+2) = x + 2x - 2$$

$$2x + 4 = 3x - 2$$

$$x = 6$$

The given terms are:

$$\frac{1}{6}, \frac{1}{8}, \frac{1}{10} \Rightarrow 6,8,10 \Rightarrow d = 2, a = 6$$

The general term is:

$$\frac{1}{6+(n-1)2} = \frac{1}{6+2n-2} = \frac{1}{4+2n}$$

Part C

$$HP = 3,4,6 \Rightarrow AP = \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$$

$$d = -\frac{1}{12}$$

$$t_4 = \frac{1}{12}$$

$$S_4 = 3 + 4 + 6 + 12 = 25$$

Example 2.3

Let $a_1, a_2, \dots, a_{10}$ be in AP, and $h_1, h_2, \dots, h_{10}$ be in HP. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then $a_4 h_7$ is: (JEE Adv. 1992)

[4] Many exam questions make use of AP/GP. The previous section has a summary of the properties of AP/GP.

Arithmetic Sequence

The common difference of the arithmetic sequence is:

$$d = \frac{a_{10} - a_1}{10 - 1} = \frac{3-2}{9} = \frac{1}{9}$$

The fourth term is:

$$a_4 = a_1 + (n-1)d = 2 + (4-1)\frac{1}{9} = 2\frac{1}{3} = \frac{7}{3}$$

Harmonic Sequence

The reciprocals of the harmonic sequence $h_1, h_2, \ldots, h_{10}$ form an arithmetic sequence $\frac{1}{h_1} = \frac{1}{2}, \frac{1}{h_2}, \ldots, \frac{1}{h_{10}} = \frac{1}{3}$, which has common difference:

$$d = \frac{\frac{1}{h_{10}} - \frac{1}{h_1}}{10 - 1} = \frac{\frac{1}{3} - \frac{1}{2}}{9} = \frac{2-3}{6} \times \frac{1}{9} = -\frac{1}{54}$$

And hence the seventh term of the arithmetic sequence is:

$$\frac{1}{h_7} = \frac{1}{h_1} + (n-1)d = \frac{1}{2} + (7-1)\left(-\frac{1}{54}\right) = \frac{1}{2} - \frac{1}{9} = \frac{7}{18}$$

The required product is then:

$$a_{4h_7} = \left(\frac{7}{3}\right)\left(\frac{18}{7}\right) = 6$$

Example 2.4

If a harmonic sequence of five terms has first term $= h_1 = 2$, and fifth term $= h_5 = 3$, then:

A. Find h_2, h_3, h_4

B. An infinite geometric sequence is formed by taking any two values of the harmonic sequence as its first two terms. Find the number of such sequences that are convergent.

Part A

Since h_1, h_2, h_3, h_4, h_5 are in harmonic sequence, their reciprocals $\left(\frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, \frac{1}{h_4}, \frac{1}{h_5}\right)$ are in arithmetic sequence with common difference is:

$$d = \frac{\frac{1}{h_5} - \frac{1}{h_1}}{4} = \frac{\frac{1}{3} - \frac{1}{2}}{4} = \frac{-\frac{1}{6}}{4} = -\frac{1}{24}$$

$$\frac{1}{h_2} = \frac{1}{h_1} + d = \frac{1}{2} - \frac{1}{24} = \frac{12}{24} - \frac{1}{24} = \frac{11}{24} \Rightarrow h_2 = \frac{24}{11}$$

$$\frac{1}{h_3} = \frac{1}{h_1} + 2d = \frac{1}{2} - 2 \cdot \frac{1}{24} = \frac{12}{24} - \frac{2}{24} = \frac{10}{24} = \frac{5}{12} \Rightarrow h_3 = \frac{12}{5}$$

$$\frac{1}{h_4} = \frac{1}{h_1} + 3d = \frac{1}{2} - 3 \cdot \frac{1}{24} = \frac{12}{24} - \frac{3}{24} = \frac{9}{24} = \frac{3}{8} \Rightarrow h_3 = \frac{8}{3}$$

Part B

If we pick two terms h_a and h_b with $b > a$, then $h_b > h_a$ and the common ratio is greater than 1. the series is not convergent. For example:

$$a = 1, b = 5 \Rightarrow h_1 + h_5 = 2 + 3 + \cdots \Rightarrow a = 2, r = \frac{3}{2} > 1 \Rightarrow Not\ Convergent$$

If we pick two terms h_a and h_b with $a = b$, then the series is not convergent. For example:

$$a = b = 1 \Rightarrow h_1 + h_1 = 2 + 2 + \cdots \Rightarrow a = 2, r = 1 \Rightarrow Not\ Convergent$$

Hence, we need to pick two distinct terms with the second term having a smaller value than the first term. For example:

$$h_5 + h_1 = 3 + 2 + \cdots \Rightarrow a = 3, r = \frac{2}{3} < 1 \Rightarrow Convergent$$

This can be done in:

$$\binom{5}{2} = \frac{5!}{2!\,3!} = \frac{5 \times 4}{2} = 10\ Ways$$

Example 2.5

Let $a_1, a_2, a_3, \ldots$ be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is: (JEE Adv. 2012)

Taking the reciprocals of the given sequence gives us the arithmetic sequence:

$$\frac{1}{a_1} = \frac{1}{5}, \frac{1}{a_2}, \ldots, \frac{1}{a_{20}} = \frac{1}{20}$$

with common difference:

$$d = \frac{\frac{1}{a_{20}} - \frac{1}{a_1}}{20 - 1} = \frac{\frac{1}{20} - \frac{1}{5}}{19} = -\frac{4}{475}$$

We need $a_n < 0$ and since the reciprocal of a number does not change sign, we calculate the least n such that:

$$\frac{1}{a_n} < 0 \Rightarrow \frac{1}{a_1} + (n-1)d < 0$$

Substituting $\frac{1}{a_1} = \frac{1}{5}$ and $d = -\frac{4}{475}$:

$$\frac{1}{5} + (n-1)\left(-\frac{4}{475}\right) < 0$$

Clearing fractions:

$$\frac{95 - 4n + 4}{475} < 0$$

Multiplying both sides by 475:

$$95 - 4n + 4 < 0$$

Collating variables one side:

$$99 < 4n$$

Solving for n:

$$n > \frac{99}{4} = 24\frac{3}{4}$$

Hence the least value of n that works is:

$$n = 25$$

MCQ 2.6

Mark the correct option

If $a_1, a_2, \ldots, a_n$ are in HP, then the expression $a_1a_2 + a_2a_3 + \ldots a_{n-1}a_n$ is equal to

A. $n(a_1 - a_n)$
B. $(n-1)(a_1 - a_n)$
C. na_1a_n
D. $(n-1)(a_1a_n)$ (JEE Main 2002)

If this property is true in general, then it should be true for specific values. Recall that

$$40, 48, 60 = 4(10, 12, 15) \rightarrow HP$$
$$a_1a_2 + a_2a_3 = (10)(12) + (12)(15) = 120 + 180 = 300$$
$$(n-1)(a_1a_n) = (2)(10)(15) = 300 \Rightarrow Option\ D$$

2.7: Sum of Product of Consecutive Terms of a Harmonic Sequence

If $a_1, a_2, \ldots, a_n$ are in HP:

$$a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n = (n-1)(a_1a_n)$$

The reciprocals of the given HP $\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right)$ are in AP with common difference:

$$\frac{1}{a_2} - \frac{1}{a_1} = d \Rightarrow \frac{a_1 - a_2}{a_1a_2} = d \Rightarrow a_1a_2 = \frac{a_1 - a_2}{d}$$

In general

$$\frac{1}{a_n}-\frac{1}{a_{n-1}}=d\Rightarrow\frac{a_{n-1}-a_n}{a_na_{n-1}}=d\Rightarrow a_na_{n-1}=\frac{a_{n-1}-a_n}{d}$$

Hence:

$$LHS=\underbrace{\frac{a_1-a_2}{d}}_{a_1a_2}+\underbrace{\frac{a_2-a_3}{d}}_{a_2a_3}+\cdots+\underbrace{\frac{a_{n-1}-a_n}{d}}_{a_{n-1}a_n}$$

The expression above telescopes:

$$=\frac{a_1-a_n}{d}$$

Substitute $\frac{1}{a_n}-\frac{1}{a_1}=(n-1)d\Rightarrow\frac{a_1-a_n}{a_1a_n}=(n-1)d\Rightarrow a_1-a_n=a_1a_n(n-1)d$

$$=\frac{a_1a_n(n-1)d}{d}=(n-1)a_1a_n=RHS$$

Example 2.8

If $h_1,h_2,\dots,h_{10}$ are in harmonic sequence with first term 3 and tenth term five, then $h_2h_3+h_3h_4+\cdots+h_8h_9$ can be written in the form $\frac{2^a\times3^b\times5^c}{29^d\times43^e}$. Find $a+b+c+d+e$.

$$d=\frac{\frac{1}{h_{10}}-\frac{1}{h_1}}{10-1}=\frac{\frac{1}{5}-\frac{1}{3}}{9}=-\frac{2}{15}\times\frac{1}{9}=-\frac{2}{135}$$

$$\frac{1}{h_2}=\frac{1}{h_1}+d=\frac{1}{3}-\frac{2}{135}=\frac{45}{135}-\frac{2}{135}=\frac{43}{135}\Rightarrow h_2=\frac{135}{43}$$

$$\frac{1}{h_9}=\frac{1}{h_1}+8d=\frac{1}{3}-\frac{16}{135}=\frac{45}{135}-\frac{16}{135}=\frac{29}{135}\Rightarrow h_2=\frac{135}{29}$$

$$(9-1)h_2h_9=(8)\left(\frac{135}{43}\right)\left(\frac{135}{29}\right)=\frac{2^3\times3^6\times5^2}{29^1\times43^1}$$

$$a+b+c+d+e=3+6+2+1+1=13$$

B. Converting Series

MCQ 2.9

Mark the correct option

Let the positive numbers a,b,c,d be in AP. Then abc,abd,acd,bcd are:

A. NOT in AP/GP/HP
B. in AP
C. in GP
D. in HP (JEE Adv. 2001 Screening)

$$a,b,c,d\to AP$$

Divide each term by $abcd$, and note that it is still an arithmetic sequence:

$$\frac{1}{bcd},\frac{1}{acd},\frac{1}{abd},\frac{1}{abc}$$

Write the terms in reverse order, and note that it is still an arithmetic sequence:

$$\frac{1}{abc},\frac{1}{abd},\frac{1}{acd},\frac{1}{bcd}$$

Since the above terms are an arithmetic sequence, their reciprocals are a harmonic sequence.

Option D

MCQ 2.10

Mark the correct option

If $x > 1, y > 1, z > 1$ are in GP, then $\frac{1}{1+\ln x}, \frac{1}{1+\ln y}, \frac{1}{1+\ln z}$ are in:

A. AP
B. HP
C. GP
D. None of these **(JEE Adv. 1998)**

If x, y, z are in GP, then the logs of the terms are in AP:

$$\ln x, \ln y, \ln z$$

Adding 1 to each term keeps it an AP:

$$1 + \ln x, 1 + \ln y, 1 + \ln z$$

Taking the reciprocal of an AP makes it an HP:

$$\frac{1}{1 + \ln x}, \frac{1}{1 + \ln y}, \frac{1}{1 + \ln z} \Rightarrow \textit{Option B}$$

C. The Harmonic Series

2.11: The Harmonic Sequence

The reciprocals of the natural numbers form the harmonic sequence:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

2.12: The Harmonic Series

The sum of the terms of the harmonic sequence is called the harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

2.13: Divergence of the Harmonic Series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$
$$< \underbrace{\frac{1}{2} + \frac{1}{2}}_{Group\ 1} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{Group\ 2} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{Group\ 3} + \cdots$$
$$Group\ 1 = \frac{1}{2} + \frac{1}{2} = 1$$
$$Group\ 2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$
$$Group\ 3 = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$
$$Group\ 4_{8\ Terms} = \frac{1}{2}$$

D. Harmonic Mean

2.14: Harmonic Mean of Two Numbers

The harmonic mean of a and b is twice the reciprocal of the sum of the reciprocals of the two numbers.

$$Harmonic\ Mean(a,b) = \frac{2ab}{a+b}$$

The sum of the reciprocals

$$= \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$$

Twice the reciprocal of the sum of the reciprocals

$$= 2 \times \frac{ab}{a+b} = \frac{2ab}{a+b}$$

Challenge 2.15: Harmonic Mean

Given that a and b are constants, determine r in terms of the harmonic mean of p and q if:

$$a^p = b^q = (ab)^r$$

Equate the first part and the second to get an equation. We want the answer in terms of p and q. So, we solve for a (or $\log a$), in this case:

$$a^p = b^q \Rightarrow p \log a = q \log b \Rightarrow \underbrace{\log a = \frac{q}{p} \log b}_{Equation\ I}$$

Also, we can say:

$$b^q = (ab)^r \Rightarrow q \log b = r \log a + r \log b$$

Substitute the value of $\log a$ from Equation I:

$$q \log b = r \left(\frac{q}{p} \log b\right) + r \log b$$

Divide both sides by $\log b$:

$$q = r\frac{q}{p} + r \Rightarrow qp = rq + rp \Rightarrow r = \frac{qp}{q+p} = \frac{1}{2}\left(\frac{2qp}{q+p}\right) = \frac{1}{2}[Harmonic\ Mean(p,q)]$$

2.16: Harmonic Mean

Given n numbers $(y_1, y_2, \dots, y_n)$, their harmonic mean (HM) is n times the reciprocal of the sum of the reciprocals of the numbers.

$$HM(y_1, y_2, \dots, y_n) = \frac{n}{\frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n}}$$

The sum of the reciprocals

$$= \frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n}$$

The reciprocal of the sum of the reciprocals

$$= \frac{1}{\frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n}}$$

n times the reciprocal of the sum of the reciprocals

$$= \frac{n}{\frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n}}$$

Example 2.17: Harmonic Mean

Find the harmonic mean of:

$$\log_x x, \log_{x^2} x, \dots, \log_{x^n} x$$

$$HM(\log_x x, \log_{x^2} x, \dots, \log_{x^n} x) = HM\left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}\right) = \frac{n}{1+2+3+\dots+n} = \frac{n}{\frac{n(n+1)}{2}} = \frac{2}{n+1}$$

2.18: Harmonic Mean Property

If three numbers are in a harmonic progression, then the middle term is the harmonic mean of the other two.

$$a, b, c \text{ are in } HP \Rightarrow b = HM(a, c) = \frac{2ac}{a+c}$$

Let

$$\underbrace{a, b, c}_{\substack{Harmonic\\Progression}} \Leftrightarrow \underbrace{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}}_{\substack{Arithmetic\\Progression}}$$

Since the above is an arithmetic progression, it must satisfy the arithmetic mean property:

$$\frac{1}{b} = \frac{\frac{1}{a} + \frac{1}{c}}{2}$$

Take the reciprocals of the above:

$$b = \frac{2}{\frac{1}{a} + \frac{1}{c}} = \frac{2}{\frac{a+c}{ac}} = \frac{2ac}{a+c} = HM(a, c)$$

Example 2.19

A. Show that $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ form a harmonic sequence.
B. Show that 40, 48 and 60 form a harmonic sequence.

Part A

The reciprocals have a common difference:

$$d = 3 - 2 = 4 - 3$$

Since the reciprocals form an arithmetic sequence, the sequence itself is harmonic.

Part B

$$HM(40, 60) = \frac{2(40)(60)}{40+60} = \frac{4800}{100} = 48 = Middle\ Term$$

Since the middle term is the harmonic mean of the other two, the sequence is harmonic.

E. Applications of Harmonic Mean Property

MCQ 2.20

Mark the correct option

If a, b, c are the terms of a harmonic sequence, then the equality $2abc = ab^2 + b^2c$

A. Never holds.
B. Always holds.
C. Is true if and only if a, b, c are the terms of an arithmetic sequence.
D. Is true if and only if a, b, c are the terms of a geometric sequence.

Method I

Substitute $b = \frac{2ac}{a+c}$ in each side of the given equality:

$$LHS = 2abc = 2a \cdot \frac{2ac}{a+c} \cdot c = \frac{4a^2c^2}{a+c}$$

$$RHS = b^2(a+c) = \left(\frac{2ac}{a+c}\right)^2(a+c) = \frac{4a^2c^2}{a+c} = LHS$$

Method II

If a, b, c are the terms of a harmonic sequence, then their reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ form an arithmetic sequence with common difference:

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

Combine fractions:

$$\frac{a-b}{ab} = \frac{b-c}{bc}$$

Cross-multiply:

$$abc - b^2c = ab^2 - abc$$

Rearrange:

$$2abc = ab^2 + b^2c$$
$$\textit{Option B is correct}$$

Example 2.21

Find the number of ordered triplets (a, b, c) such that:

1. a, b, c are each two digit prime numbers
2. a, b, c (in that order) form a harmonic sequence
3. c, b, a (in that order) form an arithmetic sequence

If a, b, c are the terms of a harmonic sequence, then their reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ form an arithmetic sequence with common difference:

$$d_1 = \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b} \Rightarrow \frac{a-b}{ab} = \frac{b-c}{bc}$$

Since c, b, a from an arithmetic sequence, $d_2 = a - b = b - c$:

$$\frac{1}{a} = \frac{1}{c} \Rightarrow a = c$$

And we can find the value of b since b is the arithmetic mean of a and c:

$$b = \frac{a+c}{2} = \frac{a+a}{2} = a$$

Hence:

$$a = b = c \Rightarrow \textit{Constant Sequence}$$

The number of two-digit prime numbers is:

$$25 - 4 = 21$$

Hence, the answer is:

$$(11,11,11), (13,13,13), \dots, (97,97,97) \to 21\ \textit{Sequences}$$

Example 2.22

Mark all Correct Options

An n $(n \geq 3)$ term sequence which is both harmonic and arithmetic is:

A. a geometric sequence.
B. a constant sequence.
C. a non-constant sequence.
D. does not exist.

Consider terms T_1, T_2, T_3 of the sequence. By the logic of the previous example:

$$T_1 = T_2 = T_3$$

Similarly,

$$T_2 = T_3 = T_4$$

In general, by combining results:

$$T_1 = T_2 = \cdots = T_n \Rightarrow Constant\ Sequence \Rightarrow Option\ B$$

A constant sequence is also a geometric sequence with common ratio 1 $\Rightarrow$ *Option A*

2.23: Sequences which are both Harmonic and Arithmetic

If a sequence is both harmonic and arithmetic, then it is a constant sequence.

Example 2.24

Mark the correct option

If $\ln(a+c), \ln(a-c), \ln(a-2b+c)$ are in AP, then:

A. a, b, c are in AP.
B. a^2, b^2, c^2 are in AP.
C. a, b, c are in GP.
D. a, b, c are in HP. **(JEE Adv. 1994)**

$$\ln(a+c), \ln(a-c), \ln(a-2b+c) \to AP$$

If the natural logs of the terms form an arithmetic sequence, then the terms themselves form a geometric sequence:

$$a+c, a-c, a-2b+c \to GP$$

The above geometric sequence satisfies the geometric mean property:

$$(a-c)^2 = (a+c)(a-2b+c)$$

Expand:

$$a^2 - 2ac + c^2 = a^2 - 2ab + ac + ac - 2bc + c^2$$

Simplify:

$$-2ac = -2ab + 2ac - 2bc$$

Collate all b terms on LHS:

$$2ab + 2bc = 4ac \Rightarrow b(a+c) = 2ac \Rightarrow b = \frac{2ac}{a+c}$$

b is the harmonic mean of a and c.

Option D

Example 2.25

If x, y, z are in HP, then prove that $\log(x+z) + \log(x+z-2y) = 2\log(x-z)$ **(JEE Adv. 1978)**

If we let $x = a, y = b, z = c$, then this requires proving the converse of the previous example.
Each of the steps in the example above is reversible.

Since $\ln(a+c), \ln(a-c), \ln(a-2b+c)$ form an arithmetic sequence, the terms satisfy the arithmetic mean property, and hence:

$$2\log(a-c) = \log(a+c) + \log(a+c-2b)$$

Changing back to the original variables gives us the result we want.

F. Equality of Means

2.26: Equality of Means

$HM(a,b) = GM(a,b) = AM(a,b)$ if and only if $a = b$

If $a = b$:

$$\underbrace{\frac{2ab}{a+b} = \frac{2a^2}{2a} = a}_{Harmonic\ Mean}, \quad \underbrace{\sqrt{ab} = \sqrt{a^2} = a}_{Geometric\ Mean}, \quad \underbrace{\frac{a+b}{2} = \frac{2a}{a} = a}_{Arithmetic\ Mean}$$

If $HM = GM$:

$$\frac{2ab}{a+b} = \sqrt{ab} \Rightarrow \underbrace{\sqrt{ab} = \frac{a+b}{2}}_{\Rightarrow GM=AM}$$

Square both sides of the above:

$$4ab = a^2 + 2ab + b^2$$
$$0 = (a-b)^2$$
$$0 = a - b$$
$$a = b$$

Example 2.27[5]

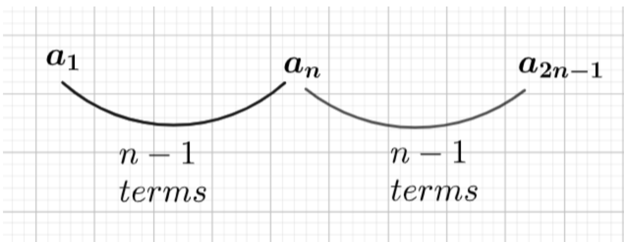
Mark all Correct Options

If the first and the $(2n-1)^{st}$ terms of an AP, a GP, and an HP (all of positive terms) are equal and their n^{th} terms are a, b and c respectively, then, for $n \geq 2$:

A. $a = b = c$
B. $a \geq b \geq c$
C. $a + c = b$
D. $ac - b^2 = 0$ (JEE Adv. 1988, Adapted)

Let

$$First\ Term = Last\ Term = x$$



Arithmetic Sequence

Let the terms of the arithmetic, geometric and harmonic sequences be:

$$AP: a_1 = x, a_2, \dots, a_n = a, \dots, a_{2n-1} = x$$
$$GP: b_1 = x, b_2, \dots, b_n = b, \dots, b_{2n-1} = x$$
$$HP: c_1 = x, b_2, \dots, b_n = b, \dots, c_{2n-1} = x$$

Notice the symmetry. By the subsequence property, which holds for each sequence above:

$$a_1 = x, a_n, a_{2n-1} = x \to AP$$
$$b_1 = x, b_n = b, b_{2n-1} = x \to GP$$
$$c_1 = x, c_n = x, c_{2n-1} = x \to HP$$

From the arithmetic, geometric and harmonic sequences, we use the property that the middle term is the arithmetic, geometric and harmonic mean respectively:

$$a = a_n = AM(x,x) = x$$
$$b = b_n = GM(x,x) = x$$

---

[5] The original question does not mention:

A. Positive, allowing $AP = HP = (1,1,1)$ and $GP = (1,-1,1)$, and then options A and B are incorrect. Several other sources mark options A, B correct without applying the positive restriction.
B. $n \geq 2$. If we consider $n = 1$ as valid, then the sequences have a single-term and Options A, B and D remain correct.

$$c = c_n = HM(x, x) = x$$

Hence:

$$a = b = c \Rightarrow Option\ A$$

Since option A is a special case of Option B, that is also correct:

$$a \geq b > c \Rightarrow Option\ B$$

And finally,

$$ac - b^2 = a^2 - a^2 = 0 \Rightarrow Option\ D$$

2.2 Fibonacci; Cyclic Sequences

A. Recursive Sequences

2.28: Recursive Sequences

- When any sequence is defined in terms of preceding terms of the sequence, the sequence is recursively defined.
- A recursive sequence also needs a bootstrap (or a starting point) to be completely defined.

Example 2.29

Each sequence below is given in terms of a recursive definition. Write an explicit definition for the sequence. If the sequence is of a common type, identify the nature and parameters and the n^{th} term of the sequence.

A. $a_n = na_{n-1}, n \geq 2, a_1 = 1$
B. $a_n = a_{n-1} + n, n \geq 2, a_1 = 1$
C. $a_n = a_{n-1} + 5, n \geq 2, a_1 = \frac{1}{2}$
D. $a_n = \frac{2}{3}a_{n-1}, n \geq 2, a_1 = \frac{81}{64}$
E. $a_n = a_{n-1} + 2n - 1, n \geq 2, a_1 = 1$
F. $f_n = f_{n-1} + f_{n-2}, n \geq 3, f_1 = 1, f_2 = 1$

Part A

$$a_1 = 1$$
$$a_2 = 2 \times 1 = 2$$
$$a_3 = 3(2) = 6$$
$$a_4 = 4(3) = 24$$
$$a_5 = 5(4) = 120$$
$$a_6 = 6(120) = 720$$

The explicit definition for factorials:

$$n! = n\underbrace{(n-1)(n-2)\dots 1}_{(n-1)!} = \underbrace{n(n-1)!}_{Recursive\ Defintion}$$

Hence, the n^{th} term is:

$$a_n = n!$$

Part B

$$a_1 = 1$$
$$a_2 = a_1 + 2 = 1 + 2 = 3$$
$$a_3 = a_2 + 3 = 3 + 3 = 6$$
$$a_4 = a_3 + 4 = 6 + 4 = 10$$

Triangular Numbers with explicit formula:

$$1 + 2 + 3 + \dots + (n-2) + (n-1) + n$$

The n^{th} term is

$$= n\left(\frac{n+1}{2}\right)$$

Part C

The first few terms are:

$$0.5, 5.5, 10.5, \dots$$

This is an arithmetic sequence with first term 0.5 and common difference 5. The n^{th} term is

$$= a + (n-1)d = 0.5 + (n-1)5 = 5n - 4.5$$

Part D

The first few terms are:

$$\frac{81}{64}, \frac{27}{32}, \frac{9}{16}, \dots$$

This is a geometric sequence with first term $\frac{81}{64} = \frac{3^4}{2^6}$, and common ratio $\frac{2}{3}$. It has n^{th} term

$$= ar^{n-1} = \frac{3^4}{2^6} \times \frac{2^{n-1}}{3^{n-1}} = \frac{2^{n-7}}{3^{n-5}}$$

Part E

The first few terms:

$$1, 4, 9, 16, 25, 36, \dots$$

These are the square numbers given by

$$n^2$$

Part F

The first few terms are:

$$1,1,2,3,5,8,13,21,34, \dots$$

This sequence is the famous Fibonacci sequence.

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

$$\varphi = \frac{1+\sqrt{5}}{2}, \psi = \frac{1-\sqrt{5}}{2}$$

These are the solutions to the quadratic

$$x^2 - x - 1 = 0$$

B. Cyclic Sequences

2.30: Cyclic Property

When a sequence repeats, it is said to be cyclic.

2.31: Self Inverse functions have cyclicity 2

Self-inverse functions result in a sequence that has cyclicity 2.

- A function that reverses the input of a function f is called the inverse function.
- There are certain functions that are their own inverse. Such functions are called self-inverse.

Example 2.32

Give examples of simple self-inverse functions.

A. Give them the input 5, and show that they have cyclicity 2.
B. Write them as recursive sequences and show that they are cyclical.

Reciprocal Function

$$f(n) = \frac{1}{n}, f(5) = \frac{1}{5}, f\left(\frac{1}{5}\right) = 5$$

$$a_1 = 5, a_n = \frac{1}{a_{n-1}} \Rightarrow a_1 = 5, a_2 = \frac{1}{5}, a_3 = 5, a_4 = \frac{1}{5}, \dots$$

Negation Function

$$f(n) = -n, f(5) = -5, f(-5) = 5$$

$$a_1 = 5, a_n = -a_{n-1} \Rightarrow a_1 = 5, a_2 = -5, a_3 = 5, a_4 = -5, \dots$$

Example 2.33

The first term of a sequence is $\frac{3}{8}$. If a is a term, the next term is $\frac{1-a}{1+a}$. The 2007^{th} term is: (NMTC Primary-Screening, 2007/11).

We calculate the second and the third terms:

$$\underbrace{t_2}_{\substack{\text{Second}\\\text{Term}}} = \frac{1-t_1}{1+t_1} = \frac{1-\frac{3}{8}}{1+\frac{3}{8}} = \frac{\frac{5}{8}}{\frac{11}{8}} = \frac{5}{11}$$

$$\underbrace{t_3}_{\substack{\text{Third}\\\text{Term}}} = \frac{1-t_2}{1+t_2} = \frac{1-\frac{5}{11}}{1+\frac{5}{11}} = \frac{\frac{6}{11}}{\frac{16}{11}} = \frac{6}{16} = \frac{3}{8}$$

The third term is the same as the first term. Hence, the sequence is cyclical. The pattern is:

$$\underbrace{\frac{3}{8}}_{\substack{Odd\ Term\\ Number}}, \quad \underbrace{\frac{5}{11}}_{\substack{Even\ Term\\ Number}}$$

$$2007\text{th term} \Rightarrow \text{Odd Term} \Rightarrow \frac{3}{8}$$

Example 2.34

Find the sum to 100 terms if:

$$t_1 = 1, \qquad t_2 = 2, \qquad t_n = \frac{t_{n-1}+1}{t_{n-2}}$$

(NMTC Primary-Final, 2006/2)

We use the definition to find the first few terms

$$t_3 = \frac{t_2+1}{t_1} = \frac{2+1}{1} = \frac{3}{1} = 3$$
$$t_4 = \frac{t_3+1}{t_2} = \frac{3+1}{2} = \frac{4}{2} = 2$$
$$t_5 = \frac{t_4+1}{t_3} = \frac{2+1}{3} = \frac{3}{3} = 1$$
$$t_6 = \frac{t_5+1}{t_4} = \frac{1+1}{2} = \frac{2}{2} = 1$$
$$t_7 = \frac{t_6+1}{t_5} = \frac{1+1}{1} = \frac{2}{1} = 2$$

$$t_3 = 3, t_4 = 2, t_5 = 1, t_6 = 1, t_7 = 2$$

This is a cyclical sequence which repeats every five terms, and the total of one cycle is:

$$1+2+3+2+1 = 9$$

To find the sum to 100 terms, note that the cycle repeats

$$\frac{100}{5} = 20\ times$$

And hence, the total is:

$$20 \times 9 = 180$$

Example 2.35

A. A sequence of numbers starts with 0,0,1 and every number of the sequence from the fourth term is the sum of the previous three numbers. The tenth number in the sequence is:
B. A sequence of numbers starts with 1, 2, and 3. The fourth number of the sequence is the sum of the previous three numbers in the sequence: 1+2+3=6. In the same way, every number after the fourth is the sum of the previous three numbers. What is the eighth number in the sequence? (AMC 2009 8/5)
C. A sequence of numbers starts with 1, 2, 3 and every number of the sequence from the fourth term is the sum of the previous three numbers. The tenth number in the sequence is: (NMTC Primary-III/Screening/8)

Part A

$$0,0,1,1,2,4,7,13,24,44$$

Part B

$$1,2,3,6,11,20,37,68$$

Part C

$$1,2,3,6,11,20,37,68,125,230$$

Example 2.36

A list of 8 numbers is formed by beginning with two given numbers. Each new number in the list is the product of the two previous numbers. Find the first number if the last three are shown:

$$-,-,-,-,-,16,64,1024$$

(AMC 1990 8/21)

$$t_1, t_2\ , t_3, t_4, t_5, 16,64,1024$$

$$Check: 16 \times 64 = 1024$$
$$t_1, t_2\ , t_3, t_4, 4,16,64,1024$$
$$t_1, t_2, t_3, 4,4,16,64,1024$$
$$t_1, t_2\ ,1,4,4,16,64,1024$$
$$t_1, 4\ ,1,4,4,16,64,1024$$
$$\frac{1}{4}, 4\ ,1,4,4,16,64,1024$$

C. Fibonacci Sequence (1, 1)

Recursive sequences have a distinct flavor compared to other mathematical topics. It is important to be understand and apply the concepts. In particular, the properties that we show for the Fibonacci sequence are applicable to Fibonacci-type sequences as well.

The Fibonacci sequence was first made popular by the Italian mathematician Fibonacci. It has many interesting properties, and hence many questions in competitions are based on this series:

2.37: Fibonacci Sequence

$$f_n = f_{n-1} + f_{n-2}, \qquad f_1 = 1, f_2 = 1, n \geq 3$$

$$\underbrace{2}_{f_3} = \underbrace{1}_{f_2} + \underbrace{1}_{f_1}$$

Example 2.38

Find the first few values of the Fibonacci Sequence

First two terms, by definition, of the Fibonacci series are 1 and 1:

- $f_1 = 1, f_2 = 1$

For all terms after the first two, any term is the sum of the two terms before it:

- $f_3 = f_2 + f_1 = 1 + 1 = 2$
- $f_4 = f_3 + f_2 = 2 + 1 = 3$
- $f_5 = f_4 + f_3 = 3 + 2 = 5$

$$1,1,2,3,5,8,13,21,34,55,89, \dots$$

Example 2.39

Calculate f_7. Also, find f_7 in terms of f_1 and f_2, where f_n represents the n^{th} term in the Fibonacci sequence. And verify that your answer both methods are the same via substitution.

$$f_6 = 8, f_7 = 13$$
$$f_7 = \underbrace{f_6 + f_5}_{f_6 = f_5 + f_4} = 2f_5 + f_4 = 3f_4 + 2f_3 = 5f_3 + 3f_2 = 8f_2 + 5f_1$$

Example 2.40

The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ starts with two 1s, and each term afterwards is the sum of its two predecessors. Which one of the ten digits is the last to appear in the units position of a number in the Fibonacci sequence? **(AMC 2000 10/6)**

We can do this by finding the terms of the Fibonacci sequence. But since we only want the unit's digit, we only add that, which reduces our work considerably:

$$1,1,2,3,5,8,3,1,4,5,9,4,3,7,0,7,7,4,1,5,6, \ldots.$$

Strike out each Units Digit as it appears:

0	1	2	3	4	5	**6**	7	8	9
9th	1st	2nd	3rd	6th	4th	**10th**	8th	5th	7th

D. Back Calculations in Fibonacci Sequence (1, 1)

Given terms in the Fibonacci Sequence, it can be extended backwards.

Example 2.41: Back Calculation

Suppose that

$$f_n = 3, f_{n+1} = 5, f_n = f_{n-1} + f_{n-2}$$

Find the terms

$$f_{n-1}, f_{n-2}, f_{n-3} \ldots. \text{ upto 10 terms}$$

And hence find a pattern connecting this sequence with the Fibonacci sequence.

The given formula is that of the Fibonacci sequence, as are the given terms. We need to calculate backwards.

$$f_{n+1} = f_n + f_{n-1} \Rightarrow f_{n-1} = f_{n+1} - f_n = 5 - 3 = 2$$

Tabulate the terms to see the pattern

		f_n	$\lvert f_n \rvert$
	$f_{n+1} - f_n = 5 - 3$	2	
f_{n-2}	$f_n - f_{n-1} = 3 - 2$	1	
f_{n-3}	$f_{n-1} - f_{n-2} = 2 - 1$	1	
f_{n-4}	$f_{n-2} - f_{n-3} = 1 - 1$	0	0
f_{n-5}	$f_{n-3} - f_{n-4} = 1 - 0$	1	1
f_{n-6}	$f_{n-4} - f_{n-5} = 0 - 1$	-1	1
f_{n-7}	$f_{n-5} - f_{n-6} = 1 - (-1)$	2	2
f_{n-8}	$f_{n-6} - f_{n-7} = -1 - 2$	-3	3
f_{n-9}	$f_{n-7} - f_{n-8} = 2 - (-3)$	5	5
f_{n-10}	$f_{n-8} - f_{n-9} = -3 - 5$	-8	8

Taking the absolute of f_n recovers the Fibonacci sequence, even if we go backwards in the sequence.

Consider the sequence defined as above. Let

$$f_m = 0$$

Find the sum of the twenty-one terms mentioned below:

$$f_{m-10}, f_{m-9}, \ldots, f_{m-1}, f_m, f_{m+1}, \ldots, f_{m+9}, f_{m+10}$$

m	$m+1$	$m+2$	$m+3$	$m+4$	$m+5$	$m+6$	$m+7$	$m+8$	$m+9$	$m+10$
0	1	1	2	3	5	8	13	21	34	55

m	$m-1$	$m-2$	$m-3$	$m-4$	$m-5$	$m-6$	$m-7$	$m-8$	$m-9$	$m-10$
0	1	-1	2	-3	5	-8	13	-21	34	-55
	2		4		10		26		68	

$$2+4+10+26+68=110$$

E. Fibonacci-Type Series (n, m)

You can generalize the Fibonacci sequence. The original sequence is:

$$f_n = f_{n-1} + f_{n-2}, f_1 = 1, f_2 = 1, n \geq 3$$

If we change the initial values, we get a different sequence, but one that still satisfies the recurrence relations that the Fibonacci sequence does.

Example 2.42

Find the first ten terms of the sequence given by:

A. $f_1 = 1, f_2 = 3, f_n = f_{n-1} + f_{n-2}, n \geq 3$
B. $f_1 = -1, f_2 = 0, f_n = f_{n-1} + f_{n-2}, n \geq 3$
C. $f_1 = \frac{1}{2}, f_2 = \frac{1}{3}, f_n = f_{n-1} + f_{n-2}, n \geq 3$

Part A

$$1,3,4,7,11,18,29,47,76,123$$

Part B

$$-1,0,-1,-1,-2,-3,-5,-8,-13$$

Part C

$$\frac{1}{2},\frac{1}{3},\frac{5}{6},\frac{7}{6},\frac{12}{6},\frac{19}{6},\frac{31}{6},\frac{50}{6}$$

Example 2.43

The increasing sequence of positive integers $a_1, a_2, a_3 \ldots$ has the property that

$$a_{n+2} = a_n + a_{n+1} \text{ for all } n \geq 1$$

If $a_7 = 120$, then $a_8 =$ (AHSME 1992/18)

Diophantine Equations

$$a_7 = \underbrace{8a_2}_{\text{Multiple of } 8} + \underbrace{5a_1}_{\text{Has to be multiple of } 8} = 120 = \underbrace{15 \times 8}_{\text{Multiple of } 8}$$

Try values:

$$a_1 > 0 \Rightarrow \boldsymbol{a_1 \neq 0}$$
$$a_1 = 8 \Rightarrow 5a_1 = 40 \Rightarrow 8a_2 = 80 \Rightarrow a_2 = 10$$
$$a_1 = 16 \Rightarrow 5a_1 = 80 \Rightarrow 8a_2 = 40 \Rightarrow a_2 = 5 \Rightarrow a_1 > a_2 \Rightarrow \textbf{\textit{Not Valid}}$$

$$a_3 = 8 + 10 = 18$$
$$a_4 = 10 + 18 = 28$$
$$a_5 = 18 + 28 = 46$$
$$a_6 = 28 + 46 = 74$$
$$a_7 = 46 + 74 = 120$$
$$a_8 = 74 + 120 = 194$$

System of Equations in Two Variables

Trying to create a system of equations in this question is not useful, since we have two variables, and only one equation.

$$a_6 = 3x + 5y$$
$$a_7 = 5x + 8y = 120$$
$$a_8 = 120 + 3x + 5y = 8x + 13y$$

Hence, the Diophantine equation methods used above are relevant for solving the question.

Example 2.44

The list 11, 20, 31, 51,82 is an example of an increasing list of five positive integers in which the first and second integers add to the third, the second and third add to the fourth, and the third and fourth add to the fifth. How many such lists of five positive integers have 124 as the fifth integer? **(Gauss 7 2020/23)**

Let the first two numbers in the list be a and b. Then, we must have:

$$a, b, a+b, a+2b, 2a+3b$$

$$\underbrace{2a}_{Even} + \underbrace{3b}_{Even} = \underbrace{124}_{Even}, \qquad b > a$$

$$b = \frac{124-2a}{3} = \frac{123-2a+1}{3} = \frac{123}{3} - \frac{2a-1}{3}$$

$2a$ should be 1 more than a multiple of 3:

$$a = 2 \Rightarrow b = \frac{124-2a}{3} = \frac{124-4}{3} = \frac{120}{3} = 40$$
$$a = 5 \Rightarrow b = \frac{124-2a}{3} = \frac{124-10}{3} = \frac{120}{3} = 38$$

$$b \in \{40,38,36,34,32,30,28,26\} \Rightarrow 8\ Values$$

Example 2.45

Consider the sequence of numbers:

$$4, 7, 1, 8, 9, 7, 6$$

For $n > 2$, the n^{th} term of the sequence is the units digit of the sum of the two previous terms. Let S_n denote the sum of the first n terms of this sequence. The smallest value of n for which $S_n > 10{,}000$ is: **(AMC 2002 12A/21)**

The given sequence has the formula:

$$a_n = \text{Units Digit}(a_{n-1} + a_{n-2})$$

Calculate the first few terms of the sequence in order to try and get a cycle:

$$\underbrace{\mathbf{4, 7}, 1, 8, 9, 7, 6, 3, 9, 2, 1, 3}_{Cycle=12\ numbers}, \mathbf{4, 7}$$

The sum of the cycle is:

$$4+7+1+8+9+7+6+3+9+2+1+3 = 60$$

Calculate the number of times the cycle must be repeated:

$$\frac{10{,}000}{60} = 166\frac{40}{60}$$

Hence, we need 166 complete cycles, giving us

$$60 \times 166 = 9960$$

And the remaining 40 will be made up by:

$$4+7+1+8+9+7+6 = 42 \Rightarrow 7\ Numbers$$

And the total number of numbers needed is:

$$= 166 \times 12 + 7 = 1999$$

Erdos was trying the question above. He attempted to calculate the cyclicity of the sequence, which in this case, for the starting two numbers 4 and 7 is a cycle of 12. However, considering the decimal number system, what is the worst-case scenario for a sequence defined like the one above with positive integers a and b:

$$a, b, \dots$$

a and b must be a units digit in the decimal system. Hence, by the multiplication principle, the possible number of values that two consecutive numbers in the above sequence can take is:

$$10 + 10 = 10^2 = 100$$
$$Max\ length\ of\ cycle = 100$$

F. Fibonacci-Type Series

Example 2.46

Let the sequence S, T, U and V be defined as

$$S = 12, 21, 2112, 211221$$

S_n: the n^{th} term of s – the digits of S_{n-1} followed by the digits of S_{n-2} for $n \geq 3$
T_n: the number of 1's in S_n
U_n: the number of 2's in S_n
V_n: the sum of the digits of S_n

Tabulate the first few values of each of T_n, U_n and V_n. **(Charm of Problem Solving, Sequences/18)**

n	S_n	T_n	U_n	V_n
1	12	1	1	3
2	21	1	1	3
3	2112	2	2	6
4	211221	3	3	9
5		5	5	15
6		8	8	24

Example 2.47

Define S as follows

$$S = 1, \quad 2, \quad 221, \quad , 2212212 \dots$$
$$S_n = the\ digits\ of\ S_{n-1}\ repeated\ twice\ in\ that\ order\ followed\ by\ the\ digits\ of\ S_{n-2}\ for\ n \geq 3$$

Write the first 10 terms. Generate three new sequences as in the previous example, and hence find the number of 2's and 1's used in the 15th term of S by extending the terms of T, U, V. Also, find the digital sum in the 15th term of S.

n	S_n	T_n	U_n	V_n
1	1	1	0	
2	2	0	1	
3	221	1	2	
4	(221)(221)(2)	2	5	
5	[(221)(221)(2)][(221)(221)(2)][221]	5	12	
6	{[(221)(221)(2)][(221)(221)(2)][221]} {[(221)(221)(2)][(221)(221)(2)][221]} {(221)(221)(2)}	12	29	

7		29	70	
8		70	169	
9		169	408	
10		408	985	

T_n satisfies a linear recurrence relationship:

$$T_n = 2T_{n-1} + T_{n-2}$$

G. Fibonacci-Type Series: Difference

The Fibonacci sequence can also be changed, more fundamentally, by making it the difference of the prior two terms:

$$f_n = f_{n-1} - f_{n-2}$$

This is a $\underbrace{\textit{cyclical sequence}}_{\textit{sequence repeats}}$, as we will see.

Example 2.48

Show that the sum of any six consecutive terms of the sequence below is zero:

$$f_n = f_{n-1} - f_{n-2}, \qquad f_1 = 3, f_2 = 5$$

$$f_1 = 3$$
$$f_2 = 5$$
$$f_3 = 5 - 3 = 2$$
$$f_4 = 2 - 5 = -3$$
$$f_5 = -3 - 2 = -5$$
$$f_6 = -5 - (-3) = -5 + 3 = -2$$
$$f_7 = -2 - (-5) = -2 + 5 = 3$$
$$f_8 = 3 - (-2) = 3 + 2 = 5$$

The sequence is only defined by the previous two terms, and these are exactly the same as the two starting terms.
Hence, the sequence will repeat with a cyclicity of six.

Any six terms of the sequence must be, in some order:

$$\mathbf{3}, \quad \mathbf{5}, \quad \mathbf{2}, \quad \mathbf{-3}, \quad \mathbf{-5}, \quad \mathbf{-2}$$

Note that terms in the same color add up to zero, and hence the entire set of six terms adds up to zero.

2.49: Cyclicity of Fibonacci-type sequence

For the sequence $f_n = f_{n-1} - f_{n-2}, \ n \geq 3$

- The sum of any six terms is zero
- The sequence is cyclical with a cyclicity of six.

Assume that the first two terms of the sequence are f_1 and f_2.

$$f_3 = f_2 - f_1$$
$$f_4 = f_2 - f_1 - f_2 = -f_1$$
$$f_5 = -f_1 - (f_2 - f_1) = -f_2$$
$$f_6 = -f_2 + f_1$$
$$f_7 = -f_2 + f_1 - (-f_2) = f_1$$
$$f_8 = f_1 - (-f_2 + f_1) = f_2$$

As before, note that the seventh and the eighth term are exactly the same as the first and the second term.

Hence, the sequence repeats with cyclicity six.

Add any six terms of the sequence to get:

$$\boldsymbol{f_1} + \boldsymbol{f_2} + (f_2 - f_1) + (-f_1) + (-f_2) + (-f_2 + f_1)$$

Example 2.50

A sequence of integers $a_1, a_2, a_3, \ldots$ is chosen so that $a_n = a_{n-1} - a_{n-2}$ for each $n \geq 3$. What is the sum of the first 2001 terms of this sequence if the sum of the first 1492 terms is 1985, and the sum of the first 1985 terms is 1492? **(AIME 1985/5)**

Let the first term be x. Let the second term be y.
Sum of any six consecutive terms of $a_n = a_{n-1} - a_{n-2}$ is zero. Therefore, we can ignore multiples of six when finding the value of the sum of the terms

$$R\left(\frac{1492}{6}\right) = 4 \Rightarrow Sum\ of\ first\ 4\ terms = x + y + (y - x) - x = \underbrace{2y - x = 1985}_{\textit{Equation I}}$$

$$R\left(\frac{1985}{6}\right) = 5 \Rightarrow Sum\ of\ first\ 5\ terms = \underbrace{y - x = 1492}_{\textit{Equation II}}$$

Subtract the second equation from the first:

$$2y - x - (y - x) = 1985 - 1492 \Rightarrow y = 493$$
$$Sum\ of\ first\ 3\ terms = x + y + y - x = 2y = 986$$

Example 2.51

A function f is defined recursively by $f(1) = f(2) = 1$ and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \geq 3$. What is $f(2018)$? **(AMC 2018 10B/20)**

The definition is:

$$f(n) = f(n-1) - f(n-2) + n$$

Use the definition in $f(n-1)$, and simplify:

$$= [f(n-2) - f(n-3) + n - 1] - f(n-2) + n$$
$$= -f(n-3) + 2n - 1$$

Use the definition in $f(n-3)$, and simplify:

$$= -[f(n-4) - f(n-5) + n - 3] + 2n - 1$$
$$= -f(n-4) + f(n-5) + n + 2$$

Use the definition in $f(n-4)$, and simplify:

$$= -[f(n-5) - f(n-6) + n - 4] + f(n-5) + n + 2$$
$$= f(n-6) + 6$$

Use the simpler recursive definition that we found above:

$$f(2018) = f(2012) + 6$$
$$= f(2006) + 12$$
$$= f(2000) + 18$$
$$\vdots$$

$$= f(2) + 2016$$

2.3 Recursive Sequences

A. General Recursive Sequences

Example 2.52

Let $a_0 = 1$ and $a_n = \sqrt{(n+2)a_{n-1}}$ for $n \geq 1$. Find a_{2014}. **(JHMMC Grade 7 2014/38)**

$$a_1 = \sqrt{(1+2)a_0 + 1} = \sqrt{(1+2)\cdot 1 + 1} = \sqrt{4} = 2$$
$$a_2 = \sqrt{(2+2)a_1 + 1} = \sqrt{4\cdot 2 + 1} = \sqrt{9} = 3$$
$$a_3 = \sqrt{(3+2)a_2 + 1} = \sqrt{5\cdot 3 + 1} = \sqrt{16} = 4$$

The pattern is

$$a_n = n + 1 \Rightarrow a_{2014} = 2014 + 1 = 2015$$

Use mathematical induction. The base case for a_1 is shown above. For the inductive case, assume $a_n = n + 1$ is true for some k. Then:

$$a_{k+1} = \sqrt{(k+1+2)a_k + 1} = \sqrt{(k+3)(k+1)+1} = \sqrt{k^2 + 4k + 4} = \sqrt{(k+2)^2} = k + 2$$

Hence, the inductive case is proved.

$$a_n = n + 1, \qquad n \in \mathbb{Z}^+$$

Example 2.53

A grocer stacks oranges in a pyramid-like stack whose rectangular base is 5 oranges by 8 oranges. Each orange above the first level rests in a pocket formed by four oranges below. The stack is completed by a single row of oranges. How many oranges are in the stack? **(AMC 2004 10A/8)**

Each orange rests in a pocket formed by four oranges below. Hence, each successive level above the first level will reduce the length and the width each of the stack by one. We calculate the total number of oranges below.

Width	Length	
5	8	40
4	7	28
3	6	18
2	5	10
1	4	4
	Total	100

Example 2.54

Oranges are stacked in a pyramid format. A single orange is stacked in the dip or gap between four oranges in the layer below it. If the number of oranges in the topmost layer is one, then

A. find an explicit formula for the number of oranges in the n^{th} layer,
B. find a recursive formula for the number of oranges in the n^{th} layer,

Part A

$$T_n = n^2$$

Part B

$$1,4,9,16,25$$
$$1, 2^2, 3^2, 4^2, 5^2$$
$$T_n = \left(\sqrt{T_{n-1}} + 1\right)^2$$

Example 2.55

Let $x_1 = 97$, and for $n > 1$, let $x_n = \frac{n}{x_{n-1}}$. Calculate the product $x_1x_2x_3x_4x_5x_6x_7x_8$. **(AIME 1985/1)**

Using the Definition

Let $x_1 = a, x_2 = \frac{2}{a}$. Then

$$x_3 = \frac{3}{\frac{2}{a}} = \frac{3a}{2}, x_4 = \frac{4}{\frac{3a}{2}} = \frac{8}{3a}, x_5 = \frac{5}{\frac{8}{3a}} = \frac{15a}{8}, x_6 = \frac{6}{\frac{15a}{8}} = \frac{48}{15a}, x_7 = \frac{7}{\frac{48}{15a}} = \frac{105a}{48}, x_8 = \frac{8}{\frac{105a}{48}} = \frac{384}{105a}$$

$$x_1x_2x_3x_4x_5x_6x_7x_8 = a \times \frac{2}{a} \times \frac{3a}{2} \times \frac{8}{3a} \times \frac{15a}{8} \times \frac{48}{15a} \times \frac{105a}{48} \times \frac{384}{105a} = 384$$

Simplifying the Formula

$$x_n = \frac{n}{x_{n-1}}$$

Solve the above for n:

$$x_n \cdot x_{n-1} = n$$

Substitute $n = 2$:

$$2 = x_2 \cdot x_1$$

Substitute $n = 4$:

$$4 = x_4 \cdot x_3$$

Putting it all together, we get:

$$x_1x_2 \cdot x_3x_4 \cdot x_5x_6 \cdot x_7x_8 = \underbrace{2}_{x_1x_2} \cdot \underbrace{4}_{x_3x_4} \cdot \underbrace{6}_{x_5x_6} \cdot \underbrace{8}_{x_7x_8} = 384$$

Example 2.56

In the sequence $2001, 2002, 2003, \ldots$, each term after the third is found by subtracting the previous term from the sum of the two terms that precede that term. For example, the fourth term is $2001 + 2002 - 2003 = 2000$. What is the 2004^{th} term in this sequence? **(AMC 2004 10B/19)**

We can generalize the calculation for the fourth term as:

$$\underbrace{2001}_{a_n} + \underbrace{2002}_{a_{n+1}} - \underbrace{2003}_{a_{n+2}} = \underbrace{2000}_{a_{n+3}}$$

Which we can write as:

$$a_n + a_{n+1} - a_{n+2} = a_{n+3}$$
$$a_n + a_{n+1} = a_{n+2} + a_{n+3}$$

And now note that we can form pairs:

$$\underbrace{2001,2002}_{Pair\ 1}, \quad \underbrace{2003,2000}_{Pair\ 2}, \quad \underbrace{2005,1998}_{Pair\ 3}$$

$$a_{2n} = 2002 - (2n - 2)$$

$$a_2 = 2002$$
$$a_4 = 2002 - 2$$
$$a_6 = 2002 - 4$$
$$\vdots$$
$$a_{2004} = 2002 - 2002 = 0$$

Example 2.57: Finding the General Term

Question on finding the general term[6]

B. Adding

Example 2.58

If the sequence a_n is defined by:

$$a_1 = 2, \qquad a_{n+1} = a_n + 2n$$

where $n \geq 1$, then a_{100} equals **(AHSME 1984/12)**

Using Recursion

Substitute $n = 100$ in $a_{n+1} = a_n + 2n$:

$$a_{100} = a_{99} + 2(99)$$

Substitute $a_{99} = a_{98} + 2(98)$:

$$= a_{98} + 2(98) + 2(99)$$

Substitute $a_{98} = a_{97} + 2(97)$:

$$= a_{97} + 2(97) + 2(98) + 2(99)$$

We can keep repeating this until we get:

$$= a_1 + 2(1) + 2(2) + \cdots + 2(98) + 2(99)$$

Factor 2 from all the terms except the first term:

$$= a_1 + 2[1 + 2 + \cdots + 99]$$

Use the formula for the sum of the first n natural numbers $= \frac{n(n+1)}{2}$:

$$= a_1 + 2\left[\frac{99 \times 100}{2}\right] = a_1 + 99 \times 100$$

Substitute $a_1 = 2$:

$$= 2 + 9900 = 9902$$

Using the Pattern

We can generalize the difference of successive terms in the sequence:

$$a_{n+1} - a_n = a_n + 2n - [a_{n-1} + 2(n-1)] = a_n + 2n - a_{n-1} - 2n + 2 = (a_n - a_{n-1}) + 2$$

Hence, the difference of successive terms follows an arithmetic sequence:

$$a_2 - a_1 = 4 - 2 = 2 = 2(1)$$
$$a_3 - a_2 = 8 - 4 = 4 = 2(2)$$
$$a_4 - a_3 = 14 - 8 = 6 = 3(2)$$
$$\vdots$$
$$a_{99} - a_{98} = 2(98)$$

[6] By blackpenredpen

$$a_{100} - a_{99} = 2(99)$$

Add all the equations above:

$$(a_{100} - a_{99}) + (a_{99} - a_{98}) + \cdots + (a_3 - a_2) + (a_2 - a_1) = 2(1) + 2(2) + \cdots + 2(99)$$

$$a_{100} - a_1 = 2\left[\frac{99 \times 100}{2}\right]$$

$$a_{100} = a_1 + 9900 = 2 + 9900 + 9902$$

Use the Pattern

	a_1	a_2	a_3	a_4	.	.	.	a_{100}
Sequence	2	4	8	14	.	.	.	
Successive Differences		2	4	6	.	.	.	99×2
Formula		$2+2$	$2+2+4$	$2+2+4+6$				

Note that successive differences from an arithmetic sequence:

$$2 + (2 + 4 + 6 + \cdots + 198) = 2 + 2(1 + 2 + \cdots + 99) \;= 2 + (2)\left(\frac{99 \times 100}{2}\right) \;= 2 + 9900 \;= 9902$$

Quadratic Sequence

	a_1	a_2	a_3	a_4	.	.	.
Sequence	2	4	8	14	.	.	.
Successive Differences		2	4	6	.	.	.

Note that successive differences form an arithmetic sequence. And hence the sequence is the sum of terms of an arithmetic sequence, which results in a quadratic sequence.

$$y = ax^2 + bx + c$$

$$a_1 = \underbrace{a + b + c = 2}_{Equation\ I}, \quad a_2 = \underbrace{4a + 2b + c = 4}_{Equation\ II}, \quad a_3 = \underbrace{9a + 3b + c = 8}_{Equation\ III}$$

Subtract Equation I from Equation II, and Equation II from Equation III:

$$\underbrace{3a + b = 2}_{Equation\ IV}, \underbrace{5a + b = 4}_{Equation\ V}$$

Subtract Equation IV from Equation V:

$$2a = 2 \Rightarrow a = 1$$

Substitute $a = 1$ in Equation IV

$$3(1) + b = 2 \Rightarrow b = -1$$

Substitute $a = 1, b = -1$ in Equation I

$$1 + (-1) + c = 2 \Rightarrow c = 2$$

$$(a, b, c) = (1, -1, 2) \Rightarrow ax^2 + bx + c = x^2 - x + 2$$

$$a_{100} = 100^2 - 100 + 2 = 10{,}000 - 100 + 2 = 9902$$

Example 2.59

Let $\{a_k\}$ be a sequence of integers such that $a_1 = 1$ and $a_{m+n} = a_m + a_n + mn$, for all positive integers m and n. Then a_{12} is (AMC 2002 10B/23)

Use the pattern

$$a_1 = 1$$
$$a_2 = a_{1+1} = a_1 + a_1 + (1)(1) = 1 + 1 + 1 = 3$$
$$a_3 = a_{1+2} = a_1 + a_2 + (1)(2) = 1 + 3 + 2 = 6$$

These are the triangular numbers.

$$a_{12} = 78$$

Add Equations

Substitute $m = 1$ in the given definition:

$$a_{1+n} = a_1 + a_n + n = a_n + n + 1$$

Start with the definition:

$$a_1 = 1$$
$$a_2 = a_{1+1} = a_1 + 2$$
$$a_3 = a_{1+2} = a_2 + 3$$
$$\vdots$$
$$a_{12} = a_{1+11} = a_{11} + 12$$

Add up the equations:

$$a_1 + a_2 + \cdots + a_{12} = a_1 + a_2 + \cdots + a_{11} + 1 + 2 + \cdots + 12$$
$$a_{12} = 78$$

C. Estimation

Example 2.60

Henry decides one morning to do a workout, and he walks $\frac{3}{4}$ of the way from his home to his gym. The gym is 2 kilometers away from Henry's home. At that point, he changes his mind and walks $\frac{3}{4}$ of the way from where he is back home. When he reaches that point, he changes his mind again, and walks $\frac{3}{4}$ of the distance from there back towards the gym. If Henry keeps changing his mind when he has walked $\frac{3}{4}$ of the distance towards either the gym or home from the point where he last changed his mind, he will get very close to walking back and forth between a point A kilometers from home and a point B kilometers from home. What is $|A - B|$. **(AMC 2019 10B/ 18)**

The question has told us that Henry gets very close to A and very close to B after changing his mind many times. Hence, we assume/simplify that A and B are fixed points.

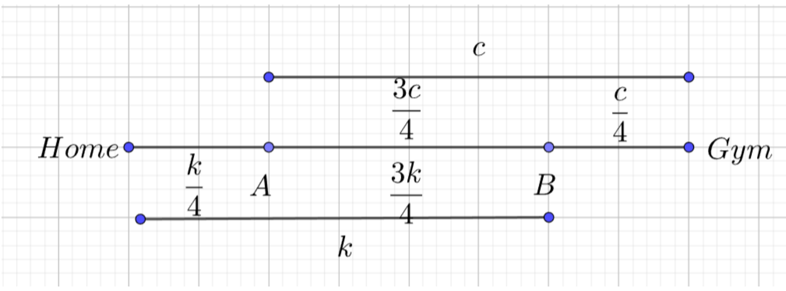
Step I

When Henry is at point B, he will decide to go back to his home. Let the distance to his home from B:

$$= k$$

Once Henry starts back in the direction of home, he goes $\frac{3}{4}$ of the way, which is $\frac{3k}{4}$, and then decides to return. Hence, the distance from A to his home is:

$$\textit{Distance from A to his home} = \frac{k}{4}$$
$$\textit{Distance from A to B is} = \frac{3k}{4}$$



Step II

Let the distance from A to his gym be c

Similarly, at point B, Henry decides to go back to the gym, and therefore:

$$\text{Distance from } B \text{ to his gym} = \frac{c}{4}$$

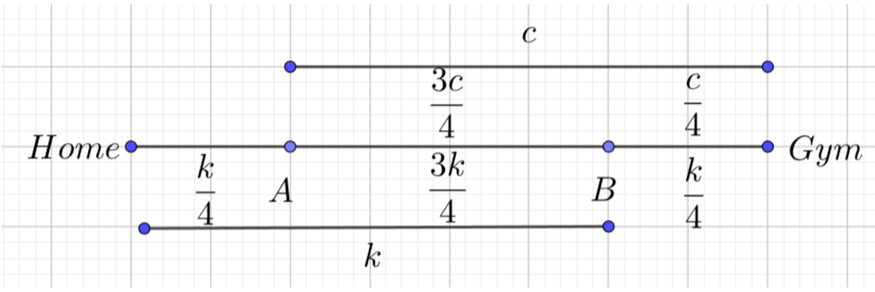
$$\text{Distance from } A \text{ to } B \text{ is} = \frac{3c}{4}$$

Step III

The distance from A to B is the same no matter which way we calculate it, and hence:

$$\frac{3c}{4} = \frac{3k}{4} \Rightarrow c = k$$

Update the diagram to indicate the distance from B to the gym in terms of k (rather than in terms of c).



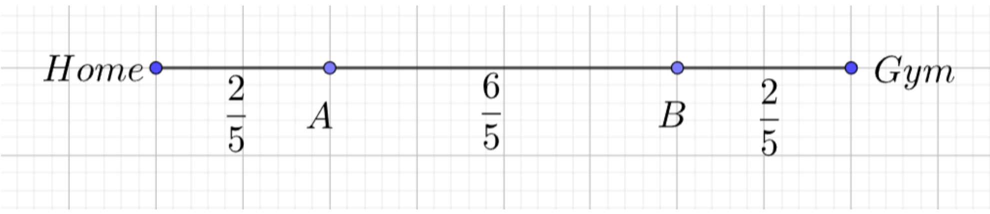
Now we can calculate the ratio of the three distances to be:

$$\frac{k}{4} : \frac{3k}{4} : \frac{k}{4} = k : 3k : k = 1 : 3 : 1$$

We now need to find $\frac{3}{1+3+1} = \frac{3}{5}$ of the distance between his home and his gym, which is:

$$\frac{3}{5} \times 2 = \frac{6}{5}$$

Hence, the final diagram on the distances will be:



Hence,

$$\text{Distance from } A \text{ to home} = \frac{2}{5}$$

$$\text{Distance from } B \text{ to home} = \frac{2}{5} + \frac{6}{5} = \frac{8}{5}$$

Example 2.61

Henry decides one morning to do a workout, and he walks $\frac{3}{4}$ of the way from his home to his gym. The gym is 2 kilometers away from Henry's home. At that point, he changes his mind and walks $\frac{3}{4}$ of the way from where he is back home. When he reaches that point, he changes his mind again, and walks $\frac{3}{4}$ of the distance from there back towards the gym. If Henry keeps changing his mind when he has walked $\frac{3}{4}$ of the distance towards either the gym or home from the point where he last changed his mind, he will get very close to walking back and forth between a point A kilometers from home and a point B kilometers from home. What is $|A-B|$. **(AMC 2019 10B/18)**

If the question had not mentioned that Henry gets very close to specific points A and B, you cannot directly make that conclusion.
Solve the question using a process that does not rely on A and B being known to be fixed points.

Introduce an $x-axis$ and place Henry's home at 0, and his gym at 2.

Iteration 1

Start with Henry at his home.
Let b_1 be the point at which Henry decides to return to his home (for the first time):

$$\textit{Position on the } x-axis\text{: } b_1=\frac{3}{4}\times 2=\frac{3}{2}$$

Let a_1 be the point at which Henry decides to return to his gym (for the first time):

$$a_1=\frac{1}{4}b_1$$

Iteration 2

Let b_2 be the point at which Henry decides to return to his home (for the second time).

The starting point for Henry for Iteration 2 is:

$$a_1=\frac{1}{4}b_1$$

The distance between a_1 and the gym is:

$$2-a_1$$

Henry will travel $\frac{3}{4}$ of the distance from a_1 to the gym:

$$\frac{3}{4}(2-a_1)$$

Hence, the final position is:

$$b_2=\underbrace{\frac{3}{4}(2-a_1)}_{\substack{Distance\\ Travelled}}+\underbrace{a_1}_{\substack{Original\ Position\\ at\ beginning\ of\ iteration}}$$

General Term: b_n

Rather than calculating specific values, we will generalize:

$$b_n = \underbrace{\frac{3}{4}(2 - a_{n-1})}_{\substack{Distance \\ Travelled}} + \underbrace{a_{n-1}}_{\substack{Original\ Position \\ at\ beginning\ of\ iteration}} = \frac{3}{2} - \frac{3}{4}a_{n-1} + a_{n-1} = \frac{3}{2} + \frac{1}{4}a_{n-1}$$

Substitute $a_{n-1} = \frac{1}{4}b_{n-1}$:

$$b_n = \frac{3}{2} + \frac{1}{4}\left(\frac{1}{4}b_{n-1}\right)$$

$$b_n = \frac{3}{2} + \frac{1}{16}b_{n-1}$$

Hence, we have b_n in terms of b_{n-1}, which means we have a recursive formula in terms of b only.

Substitute $b_{n-1} = \frac{3}{2} + \frac{1}{16}b_{n-2}$

$$b_n = \frac{3}{2} + \frac{1}{16}\left(\frac{3}{2} + \frac{1}{16}b_{n-2}\right) = \frac{3}{2} + \left(\frac{1}{16}\right)\left(\frac{3}{2}\right) + \left(\frac{1}{16}\right)^2 b_{n-2}$$

Substitute $b_{n-2} = \frac{3}{2} + \frac{1}{16}b_{n-3}$

$$= \frac{3}{2} + \left(\frac{1}{16}\right)\left(\frac{3}{2}\right) + \left(\frac{1}{16}\right)^2\left(\frac{3}{2} + \frac{1}{16}b_{n-3}\right) = \frac{3}{2} + \left(\frac{1}{16}\right)\left(\frac{3}{2}\right) + \left(\frac{1}{16}\right)^2\left(\frac{3}{2}\right) + \left(\frac{1}{16}\right)^3 b_{n-3}$$

The general pattern is:

$$\frac{3}{2} + \left(\frac{1}{16}\right)\left(\frac{3}{2}\right) + \left(\frac{1}{16}\right)^2\left(\frac{3}{2}\right) + \cdots + \left(\frac{1}{16}\right)^k b_{n-k}$$

As the number of terms becomes very large, k becomes very large, the last term becomes very small and can be ignored:

$$k \to \infty \Rightarrow \left(\frac{1}{16}\right)^k b_{n-k} \to 0$$

Hence, we are left with:

$$\frac{3}{2} + \left(\frac{1}{16}\right)\left(\frac{3}{2}\right) + \left(\frac{1}{16}\right)^2\left(\frac{3}{2}\right) + \cdots$$

Which is an infinite geometric series with $first\ term = a = \frac{3}{2}$, $common\ ratio = \frac{1}{16}$, and sum

$$B = \lim_{n\to\infty} b_n = \frac{a}{1-r} = \frac{\frac{3}{2}}{1 - \frac{1}{16}} = \frac{\frac{3}{2}}{\frac{15}{16}} = \frac{3}{2}\cdot\frac{16}{15} = \frac{8}{5}$$

General Term: a_n

$$a_n = \frac{1}{4}b_n$$

Substitute $b_n = \frac{3}{2} + \frac{1}{4}a_{n-1}$

$$a_n = \frac{1}{4}\left[\frac{3}{2} + \frac{1}{4}a_{n-1}\right] = \frac{3}{8} + \frac{1}{16}a_{n-1}$$

$$a_n = \frac{3}{8} + \frac{1}{16}\left[\frac{3}{2} + \frac{1}{16}a_{n-2}\right] = \frac{3}{8} + \left(\frac{1}{16}\right)\left(\frac{3}{2}\right) + \left(\frac{1}{16}\right)^2 a_{n-2}$$

$$a_n = \frac{3}{8} + \left(\frac{1}{16}\right)\left(\frac{3}{2}\right) + \left(\frac{1}{16}\right)^2\left[\frac{3}{8} + \frac{1}{16}a_{n-3}\right] = \frac{3}{8} + \left(\frac{1}{16}\right)\left(\frac{3}{2}\right) + \left(\frac{1}{16}\right)^2\left(\frac{3}{8}\right) + \left(\frac{1}{16}\right)^3 a_{n-3}$$

$$a_n = \frac{3}{8} + \left(\frac{1}{16}\right)\left(\frac{3}{2}\right) + \left(\frac{1}{16}\right)^2\left(\frac{3}{8}\right) + \cdots + \left(\frac{1}{16}\right)^k a_{n-k}$$

As the number terms becomes large, we can ignore the last term:

$$a_n = \frac{3}{8} + \left(\frac{1}{16}\right)\left(\frac{3}{2}\right) + \left(\frac{1}{16}\right)^2\left(\frac{3}{8}\right) + \cdots$$

Which is an infinite geometric series with $first\ term = a = \frac{3}{8}$, $common\ ratio = \frac{1}{16}$, and sum

$$A = \lim_{n\to\infty} a_n = \frac{a}{1-r} = \frac{\frac{3}{8}}{1-\frac{1}{16}} = \frac{\frac{3}{8}}{\frac{15}{16}} = \frac{3}{8}\cdot\frac{16}{15} = \frac{2}{5}$$

And finally:

$$|A-B| = \left|\frac{2}{5} - \frac{8}{5}\right| = \left|-\frac{6}{5}\right| = \frac{6}{5}$$

Example 2.62

Define a sequence recursively by $x_0 = 5$ and $x_{n+1} = \frac{x_n^2+5x_n+4}{x_n+6}$ for all nonnegative integers n. Let m be the least positive integer such that $x_m \le 4 + \frac{1}{2^{20}}$. In which of the following intervals does m lie?

A. [9,26]
B. [27,80]
C. [81,242]
D. [243,728]
E. $[729, \infty]$ **(AMC 2019 10B/24)**

$$x_m \le 4 + \frac{1}{2^{20}} \Rightarrow x_m - 4 \le \frac{1}{2^{20}}$$

Part A: Property I: $x_n > 4$, for all n (Proved using induction)

$x_0 = 5 > 4$ is the base case. The inductive case is that if $x_n > 4$, then :

$$x_{n+1} - 4 = \frac{x_n^2+5x_n+4}{x_n+6} - 4 = \frac{x_n^2+5x_n+4-4(x_n+6)}{x_n+6} = \frac{x_n^2+x_n-20}{x_n+6} = \frac{\underbrace{(x_n-4)}_{+ve}\underbrace{(x_n+5)}_{+ve}}{\underbrace{x_n+6}_{+ve}} \to Positive$$

$$\Rightarrow x_{n+1} - 4\ is\ positive \Rightarrow x_{n+1} - 4 > 0 \Rightarrow x_{n+1} > 4$$

Part B: Property II: x_n is decreasing

$$\underbrace{\frac{x_n^2+5x_n+4}{x_n+6}}_{x_{n+1}} - x_n = \frac{x_n^2+5x_n+4-x_n(x_n+6)}{x_n+6} = \frac{x_n^2+5x_n+4-x_n^2-6x_n}{x_n+6} = \frac{4-x_n}{x_n+6}$$

Given that $x_n > 4$:

$$\underbrace{4-x_n}_{-ve} \div \underbrace{x_n+6}_{+ve} \to Negative$$

Since the difference between successive terms is negative, the sequence is decreasing.

Part C: Range of Next Term

x_n is a decreasing sequence always greater than 4. Compare the next term of the sequence to the previous term.

$$x_{n+1}-4=\frac{(x_n-4)(x_n+5)}{x_n+6}=(x_n-4)\cdot\frac{x_n+5}{x_n+6}$$

From Property I, we know that $x_n > 4$. From Property II and $x_0=5$, we know that $x_n \le 5$

$$\frac{x_n+5}{x_n+6}\ for\ 4\le x_n\le 5\in\left[\frac{4+5}{4+6},\frac{5+5}{5+6}\right]=\left[\frac{9}{10},\frac{10}{11}\right]$$

Part D: Range of n^{th} term

$$(x_n-4)\cdot\frac{x_n+5}{x_n+6}=x_{n+1}-4=(x_n-4)\cdot\frac{x_n+5}{x_n+6}$$

Use the range from part C for $\frac{x_n+5}{x_n+6}$. The lower value is $\frac{9}{10}$, and the upper value is $\frac{10}{11}$.

$$(x_n-4)\cdot\frac{9}{10}<x_{n+1}-4\le(x_n-4)\cdot\frac{10}{11}$$

The above expression is for a jump from one term to the next. If the jump is from x_0 to x_n, which is n terms, then the bounds are multiplied by themselves n times:

$$(x_0-4)\cdot\left(\frac{9}{10}\right)^n<x_n-4\le(x_0-4)\cdot\left(\frac{10}{11}\right)^n$$

Substitute $x_0=5$:

$$\left(\frac{9}{10}\right)^n<x_n-4\le\left(\frac{10}{11}\right)^n$$

Part E: Lower Bound

The above is a compound inequality which can be solved by breaking into separate inequalities:

$$\left(\frac{9}{10}\right)^n<x_n-4\le\frac{1}{2^{20}}\Rightarrow\left(\frac{9}{10}\right)^n\le\frac{1}{2^{20}}$$

$$\left(\frac{9}{10}\right)^3\approx 0.73,\qquad 0.9^4\approx 0.66,\qquad 0.9^5\approx 0.6,\qquad 0.9^6=0.54,\qquad 0.9^7=0.48\Rightarrow 0.9^{6.5}\approx\frac{1}{2}$$

Exponentiate both sides of the last approximation to the power 20:

$$(0.9^{6.5})^{20}\approx\left(\frac{1}{2}\right)^{20}\Rightarrow 0.9^{130}\approx\frac{1}{2^{20}}\Rightarrow n>130\Rightarrow 130\in[81,243]\Rightarrow Option\ C\ is\ correct$$

D. Geometry

$$S_1=1$$
$$S_2=1+\frac{1}{2}$$
$$S_3=1+\frac{1}{2}+\frac{1}{4}$$
$$S_n=\underbrace{1+\frac{1}{2}+\frac{1}{4}+\cdots}_{n\ terms}$$
$$\lim_{n\to\infty}S_n=2$$

Example 2.63

A. Zeno's frog is on the number line. It starts at the number 4, and hops $\frac{2}{3}$ of a unit to the right. Its n^{th} hop is k times of the $(n-1)^{st}$ hop, where $n\ge 2$. Zeno has calculated that if the frog hops an infinite number of times, it will reach the number 7. Find the value of k.

B. Zeno constructs a square of area a on a line. Adjoining the first square (and still on the line), he constructs a second square that has area $\sqrt{a}$ times the first square. If Zeno constructs an infinite number of such squares, and the total area of the squares is 2 units, then find the value of a.

Part A
The recursive formula is:
$$h_n = kh_{n-1}, n \geq 2$$
The length of the hops is:
$$h_1 = \frac{2}{3}, h_2 = \frac{2}{3}k, h_3 = \frac{2}{3}k^2$$
The total length of n hops:
$$= \underbrace{\frac{2}{3} + \frac{2}{3}k + \frac{2}{3}k^2 + \cdots}_{n\ terms}$$
This is a geometric series with $first\ term = a = \frac{2}{3}$, and $common\ ratio = r = k$, and it has sum
$$\frac{a}{1-r} = \frac{\frac{2}{3}}{1-k} = 7 - 4$$
$$\frac{2}{3} = 3 - 3k$$
$$3k = \frac{7}{3}$$
$$k = \frac{7}{9}$$

Part B
The area of the n^{th} square is:
$$A_n = \sqrt{a}\, A_{n-1}, n \geq 2$$
$$a + (\sqrt{a})a + (\sqrt{a})^2 a + \cdots$$

This is a geometric series with $first\ term = a$, and $common\ ratio = \sqrt{a}$, and it has sum
$$\frac{a}{1-\sqrt{a}} = 2$$
$$a = 2 - 2\sqrt{a}$$
$$a + 2\sqrt{a} - 2 = 0$$
Use a change of variable. Let $x = \sqrt{a}$:
$$x^2 + 2x - 2 = 0$$
Use the quadratic formula:
$$x = \sqrt{3} - 1$$
$$a = x^2 = \left(\sqrt{3} - 1\right)^2 =$$

E. Triangle Inequality

Example 2.64: Triangle Inequality

A triangle can be formed having side lengths 4, 5 and 8. It is impossible, however, to construct a triangle with side lengths 4, 5 and 9. Ron has eight sticks, each having an integer length. He observes that he cannot form a triangle using any three of these sticks as side lengths. The shortest possible length of the longest of the eight sticks is (Gauss Grade 7 2001/25)

Three Sticks
Start with two sticks with the smallest length possible:
$$(1,1)$$
Convert the third stick to have lengths equal to the sum of the first two:
$$(1,1,2)$$
By the triangle inequality, the above three lengths do not form a triangle.

Four Sticks
Add a fourth stick, which has length equal to the longest sticks:
$$(1,1,2,3)$$
Note that, again because of the triangle inequality, no three of the sticks form a triangle.

Generalize:
Note that the sequence of sticks obtained is exactly the Fibonacci sequence.

Extend the sequence to eight terms:
$$(1,1,2,3,5,8,13,21)$$

Shortest possible length
$$= 21$$

Example 2.65

Let T_1 be a triangle with sides 2011, 2012, and 2013. For $n \geq 1$, if $T_n = \triangle ABC$ and D, E, and F are the points of tangency of the incircle of $\triangle ABC$ to the sides AB, BC and AC, respectively, then T_{n+1} is a triangle with side lengths AD, BE, and CF, if it exists. What is the perimeter of the last triangle in the sequence (T_n)? **(AMC 2011 10B/25)**

F. Number of Factors

2.66: Number of Factors[7]

The number of divisors $\boldsymbol{\tau(x)}$ of a number $\boldsymbol{x}$ with prime factorisation $\boldsymbol{a^p b^q c^r}$ is given by:

$$x = a^p b^q c^r \Rightarrow \tau(x) = (p+1)(q+1)(r+1)$$

Example 2.67

If $\tau(z)$ is the number of positive factors of z, determine the smallest possible value of a_n and the value of n at which it is first achieved given that:

$$a_n = \tau(a_{n-1}), \qquad n \geq 2$$
$$a_1 = \tau\left(30^{\tau(x)}\right), \qquad x = 2^3 \cdot 3^4 \cdot 7^{10}$$

$$\tau(x) = \tau(2^3 \cdot 3^4 \cdot 7^{10}) = (3+1)(4+1)(10+1) = 220$$
$$a_1 = \tau\left(30^{\tau(x)}\right) = \tau(30^{220}) = \tau(2^{220} \cdot 3^{220} \cdot 5^{220}) = (220+1)^3 = 221^3 = (13 \cdot 17)^3$$
$$a_2 = \tau(a_1) = \tau(221^3) = \tau(13^3 \times 17^3) = (3+1)(3+1) = 16 = 2^4$$
$$a_3 = \tau(a_2) = \tau(2^4) = 4 + 1 = 5$$
$$a_4 = \tau(a_3) = \tau(5) = 2$$

$$Minimum = 2$$
$$Acheived\ at\ n = 4$$

2.4 Arithmetico-Geometric Sequences/Series

A. Further Resources

Example 2.68

Bandana Gupta on AGP

B. Arithmetico-Geometric Sequences

2.69: Arithmetico-Geometric Sequence

An arithmetico-geometric sequence has the form:

$$a, (a+d)r, (a+2d)r^2, \ldots, [a+(n-1)d]r^{n-1}$$

As the name implies, arithmetico-geometric sequences combine features of both arithmetic and geometric sequences:

- One component of the term increases by d each time. This is the common difference.
- One component of the term gets multiplied by r each time. This the common ratio.

Example 2.70: General and Specific terms

Consider the infinite arithmetico-geometric sequence with first term 1, common difference 2 and common ratio

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[7] The derivation of this formula and a detailed discussion can be found in the Notes on N1:Basics (Number Theory).

3.

A. Write the first few terms
B. Write the general term
C. Evaluate the tenth term. Write your answer in exponent notation.

Part A

$$T_1 = 1$$
$$T_2 = [1 + 2(1)]3^1 = 3(3) = 9$$
$$T_3 = [1 + 2(2)]3^2 = 5(9) = 45$$

Part B

$$T_n = [1 + (n-1)(2)]3^{n-1} = [1 + 2n - 2]3^{n-1} = [2n - 1]3^{n-1}$$

Part C

$$T_{10} = [2(10) - 1]3^{10-1} = [19]3^9$$

Example 2.71

Find the common difference and the common difference for the following arithmetico-geometric sequences. Show that they are correct by calculating T_2 and T_3.

A. $\frac{1}{10}, \frac{2}{10^2}, \frac{3}{10^3}, \dots$
B. $\frac{3}{2}, \frac{2}{3}, \frac{5}{18}, \dots$
C. $\frac{1}{3}, \frac{5}{30}, \frac{4}{75}, \dots$

Part A

Notice that the numerator is an AP, and the denominator is a GP

$$Numerators\ form\ an\ AP \to 1,2,3 \Rightarrow d = \frac{1}{10}$$
$$Denominators\ form\ an\ GP \to 10, 10^2, 10^3 \Rightarrow r = \frac{1}{10}$$
$$T_1 = a = \frac{1}{10}$$
$$T_2 = (a + d)r = \left(\frac{1}{10} + \frac{1}{10}\right)\frac{1}{10} = \left(\frac{2}{10}\right)\left(\frac{1}{10}\right) = \frac{2}{10^2}$$
$$T_3 = (a + 2d)r^2 = \left(\frac{1}{10} + \frac{2}{10}\right)\frac{1}{10^2} = \left(\frac{3}{10}\right)\left(\frac{1}{10^2}\right) = \frac{3}{10^3}$$

Part B

This becomes much easier to decode if you rewrite the sequence as below:

$$\frac{3}{2}, \frac{4}{6}, \frac{5}{18}, \dots \Rightarrow T_1 = a = \frac{3}{2}$$
$$Numerators\ form\ an\ AP \to 3,4,5 \Rightarrow d = \frac{1}{2}$$
$$Denominators\ form\ an\ GP \to 2,6,18 \Rightarrow r = \frac{1}{3}$$
$$T_2 = \left(\frac{3}{2} + \frac{1}{2}\right)\left(\frac{1}{3}\right) = \left(\frac{4}{2}\right)\left(\frac{1}{3}\right) = \frac{4}{6} = \frac{2}{3}$$
$$T_3 = \left(\frac{3}{2} + \frac{2}{2}\right)\left(\frac{1}{3^2}\right) = \left(\frac{5}{2}\right)\left(\frac{1}{9}\right) = \frac{5}{18}$$

Part C

$$\frac{1}{3}, \frac{5}{30}, \frac{4}{75}, \dots \Rightarrow d = \frac{1}{2}, r = \frac{1}{5}$$
$$T_1 = \frac{1}{3}$$

$$T_2=\left(\frac{1}{3}+\frac{1}{2}\right)\left(\frac{1}{5}\right)=\left(\frac{5}{6}\right)\left(\frac{1}{5}\right)=\frac{5}{30}$$
$$T_3=\left(\frac{1}{3}+1\right)\left(\frac{1}{5^2}\right)=\left(\frac{4}{3}\right)\left(\frac{1}{25}\right)=\frac{4}{75}$$

$$\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\cdots\Rightarrow a=\frac{1}{2},d=1,r=\frac{1}{2}$$
$$T_1=\frac{1}{2}$$
$$T_2=\left(\frac{1}{2}+1\right)\left(\frac{1}{2}\right)=\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)=\frac{3}{4}$$
$$T_3=\left(\frac{1}{2}+2\right)\left(\frac{1}{4}\right)=\left(\frac{5}{2}\right)\left(\frac{1}{4}\right)=\frac{5}{8}$$

Example 2.72
Back Calculations
Word Problems

C. Infinite Arithmetico-geometric Series

Example 2.73
Prove the formula for the sum of a geometric series.[8]

For $-1<r<1$, assign a value to the sum of an infinite geometric series:
$$\underbrace{S=a+ar+ar^2+\cdots}_{Equation\ I}$$
Multiply both sides by r:
$$\underbrace{rS=ar+ar^2+\cdots}_{Equation\ II}$$

Subtract Equation II from Equation I, factor S on the LHS, and then solve for S:
$$S-rS=a\Rightarrow\ S(1-r)=a\Rightarrow S=\frac{a}{1-r}$$

2.74: Infinite Arithmetico-geometric Series
$$a+(a+d)r+(a+2d)r^2+\cdots+[a+(n-1)d]r^{n-1}+\cdots$$

The process to find the sum of an infinite arithmetic-geometric uses the same process as that to prove the formula for the sum of an infinite geometric series.

Example 2.75
JEE 2024

Example 2.76

[8] We will assume convergence of the series for $-1<r<1$. That is, we assume that it has a finite sum.

The limiting sum of the infinite series, $\frac{1}{10}+\frac{2}{10^2}+\frac{3}{10^3}+\cdots$ whose n^{th} term is $\frac{n}{10^n}$ is: **(AHSME 1962/40)**

Multiply both sides of $\underbrace{S=\frac{1}{10}+\frac{2}{10^2}+\frac{3}{10^3}+\frac{4}{10^4}+\cdots}_{\textit{Equation I}}$ by $r=\frac{1}{10}$:

$$\underbrace{\frac{S}{10}=0+\frac{1}{10^2}+\frac{2}{10^3}+\frac{3}{10^4}+\cdots}_{\textit{Equation II}}$$

Subtract Equation II from Equation I:

$$S-\frac{1}{10}S=\left(\frac{1}{10}-0\right)+\left(\frac{2}{10^2}-\frac{1}{10^2}\right)+\left(\frac{3}{10^3}-\frac{2}{10^3}\right)+\left(\frac{4}{10^4}-\frac{3}{10^4}\right)+\cdots$$

Simplify both sides to get:

$$\frac{9}{10}S=\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+\cdots$$

The RHS is an infinite geometric series. Substitute $a=\frac{1}{10}, r=\frac{1}{10}$ in $\frac{a}{1-r}=\frac{\frac{1}{10}}{1-\frac{1}{10}}=\frac{\frac{1}{10}}{\frac{9}{10}}=\frac{1}{9}$:

$$\frac{9}{10}S=\frac{1}{9}\Rightarrow S=\frac{10}{81}$$

Alternate Solution

Split up the terms:

$$\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+\cdots\Rightarrow a=\frac{1}{10}, r=\frac{1}{10}\Rightarrow S_1=\frac{\frac{1}{10}}{1-\frac{1}{10}}=\frac{\frac{1}{10}}{\frac{9}{10}}=\frac{1}{9}$$

$$+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+\cdots\Rightarrow a=\frac{1}{10^2}, r=\frac{1}{10}\Rightarrow S_2=\frac{\frac{1}{10^2}}{1-\frac{1}{10}}$$

$$+\frac{1}{10^3}+\frac{1}{10^4}+\cdots\Rightarrow a=\frac{1}{100^3}, r=\frac{1}{10}\Rightarrow S_3=\frac{\frac{1}{10^3}}{1-\frac{1}{10}}$$

We need to find the sum:

$$S_1+S_2+S_3+\cdots=\frac{\frac{1}{10}}{1-\frac{1}{10}}+\frac{\frac{1}{10^2}}{1-\frac{1}{10}}+\frac{\frac{1}{10^3}}{1-\frac{1}{10}}+\cdots=\frac{\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\cdots}{\frac{9}{10}}=\frac{S_1}{\frac{9}{10}}=\frac{1}{9}\times\frac{10}{9}=\frac{10}{81}$$

Example 2.77

Find $\alpha^{2\beta}$ given that $\alpha=1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\cdots, \beta=\log_{0.25}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right)$. **(JEE Main 2021, 25 July, Shift-I, Adapted)**

Let

$$\underbrace{\alpha=1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\cdots}_{\textit{Equation I}}$$

Note if you ignore the first term, α has an arithmetic series in the numerator, and a geometric series in the denominator. Hence, multiply α by the common ratio $=r=\frac{1}{3}$:

$$\underbrace{\frac{\alpha}{3}=0+\frac{1}{3}+\frac{2}{3^2}+\frac{6}{3^3}+\cdots}_{\textit{Equation II}}$$

Subtract Equation II from Equation I, and simplify to get a geometric series with $a=\frac{4}{3}, r=\frac{4}{3}$, which we substitute in $\frac{a}{1-r}$:

$$\frac{2\alpha}{3}=1+\frac{1}{3}+\frac{4}{3^2}+\frac{4}{3^3}+\cdots=\frac{4}{3}+\frac{4}{3^2}+\frac{4}{3^3}+\cdots=\frac{\frac{4}{3}}{1-\frac{1}{3}}=\frac{\frac{4}{3}}{\frac{2}{3}}=\frac{4}{2}=2\Rightarrow\alpha=3$$

Also note that β has a geometric series with $a=\frac{1}{3}, r=\frac{1}{3}$:

$$\beta=\log_{0.25}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right)=\log_{\frac{1}{4}}\left(\frac{\frac{1}{3}}{1-\frac{1}{3}}\right)=\log_{\frac{1}{4}}\left(\frac{\frac{1}{3}}{\frac{2}{3}}\right)=\log_{\frac{1}{4}}\left(\frac{1}{2}\right)=\frac{1}{2}$$

Finally,

$$\alpha^{2\beta}=3^{2\times\frac{1}{2}}=3^1=3$$

Example 2.78

Find the value of $2^{\frac{1}{4}}\cdot 4^{\frac{1}{8}}\cdot 8^{\frac{1}{16}}\cdot\ldots\infty$ (JEE Main 2002)

Convert each term to have a prime factor base:

$$(2^1)^{\frac{1}{4}}\cdot(2^2)^{\frac{1}{8}}\cdot(2^3)^{\frac{1}{16}}\cdot\ldots\infty$$

Use the exponent law $(a^m)^n=a^{mn}$:

$$2^{\frac{1}{4}}\cdot 2^{\frac{2}{8}}\cdot 2^{\frac{3}{16}}\cdot\ldots\infty$$

Use the exponent law $a^{x_1}\cdot a^{x_2}\cdot a^{x_3}\cdot\ldots=a^{x_1+x_2+x_3+\cdots}$

$$2^{\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\cdots}$$

$$\textit{Numerators form an arithmetic series}: 1,2,3,\ldots$$
$$\textit{Denominators form a geometric series}: 4,8,16,\ldots$$
$$\textit{The overall series is arithmetico}-\textit{geometric}.$$

We want to find the value of the exponent:

$$\underbrace{S=\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\cdots}_{\textit{Equation I}}$$

Multiply both sides of the above by $\frac{1}{2}$:

$$\underbrace{\frac{1}{2}S=0+\frac{1}{8}+\frac{2}{16}+\frac{3}{32}+\cdots}_{\textit{Equation II}}$$

Subtract Equation II from Equation I:

$$\frac{1}{2}S=\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

The RHS is a geometric series with $a=\frac{1}{4},\ r=\frac{1}{2}$ and $sum=\frac{a}{1-r}=\frac{\frac{1}{4}}{1-\frac{1}{2}}=\frac{\frac{1}{4}}{\frac{1}{2}}=\frac{1}{4}\times 2=\frac{1}{2}$:

$$\frac{1}{2}S=\frac{1}{2}\Rightarrow S=1\Rightarrow 2^{\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\cdots}=2^S=2^1=2$$

Example 2.79: Probability

The probability distribution for discrete variable $x=0,1,2,\ldots$ is given by $P(X=x)=k(x+1)3^{-x}$. The probability of $P(X \geq 2)$ is given by: **(JEE Main 2023)**

The sum of *mutually exclusive and collectively exhaustive* events is 1. Therefore:

$$\underbrace{1 = P(0) + P(1) + P(2) + P(3) + \cdots}_{\textit{Equation I}}$$

Using $P(X = x) = k(x + 1)3^{-x}$:

$$\underbrace{1 = k\left[1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \cdots\right]}_{\textit{Equation II}}$$

Multiply both sides by the *common ratio* $= \frac{1}{3}$:

$$\underbrace{\frac{1}{3} = k\left[0 + \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \cdots\right]}_{\textit{Equation III}}$$

Subtract Equation III from Equation II:

$$\frac{2}{3} = k\left[1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\right]$$

The above is a geometric series with $first\ term = a = 1, common\ ratio = \frac{1}{3} \Rightarrow Sum = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$:

$$\frac{2}{3} = k\left[\frac{3}{2}\right] \Rightarrow k = \frac{4}{9}$$

Rearrange Equation I to get the required probability, and use $P(X = x) = \frac{4}{9}(x + 1)3^{-x}$:

$$P(X \geq 2) = 1 - P(0) - P(1) = 1 - \frac{4}{9}\left(\frac{1}{3^0}\right) - \frac{4}{9}\left(\frac{2}{3^1}\right) = 1 - \frac{4}{9}\cdot\frac{5}{3} = 1 - \frac{20}{27} = \frac{7}{27}$$

Example 2.80: Statistics Application

You conduct independent trials, with only two possible outcomes: success or failure. The probability of success is p. Show that the:

A. range of values that p can take is $0 \leq p \leq 1$.
B. probability of failure is $1 - p$.
C. probability that the first success will occur on the x^{th} trial is given by $p(x) = (1 - p)^{x-1}p$.
D. the mean number of trials required is $\frac{1}{p}$. Use the formula $Mean = E[X] = \sum_{x\in\mathbb{N}} xp(x)$ to calculate the mean.

Part A
Part B
Part C
Part C

The formula for the mean is:

$$E[X] = \sum_{x\in\mathbb{N}} xp(x)$$

From Part C, substitute $p(x) = (1 - p)^{x-1}p$:

$$E[X] = \sum_{x\in\mathbb{N}} x(1 - p)^{x-1}p$$

Move the p out of the summation sign since it is a constant:

$$E[X] = p\sum_{x\in\mathbb{N}} x(1 - p)^{x-1}$$

Expand the summation:

$$\underbrace{E[X] = p[1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + \cdots]}_{\textit{Equation I}}$$

The RHS is an AGP with $common\ ratio = r = 1 - p$. Multiply both sides of the above by the common ratio. Add an extra zero:

$$\underbrace{(1-p)E[X] = p[0 + (1-p) + 2(1-p)^2 + 3(1-p)^3 + \cdots]}_{\textit{Equation II}}$$

Subtract Equation II from Equation I:

$$E[X] - E[X] + pE[X] = p[1 + (1-p) + (1-p)^2 + (1-p)^3 + \cdots]$$

Simplify and divide both side by p:

$$E[X] = 1 + (1-p) + (1-p)^2 + (1-p)^3 + \cdots$$

Use the formula for the sum of an infinite geometric series with $a = 1, r = 1 - p$:

$$E[X] = \frac{a}{1-r} = \frac{1}{1-(1-p)} = \frac{1}{p}$$

2.81: Sum of Infinite Arithmetico-geometric Series

D. Finite Arithmetico-geometric Series

2.82: Finite Arithmetico-geometric Series

$$a + (a+d)r + (a+2d)r^2 + \cdots + [a + (n-1)d]r^{n-1}$$

The method for finding the sum of a finite arithmetico-geometric series is similar that if the series is infinite.

- Multiply both sides by the common ratio.
- Find the difference
- There will be one extra term compared to the original series. Be careful not to lose track of this.
- Sum the resulting finite geometric series.

Example 2.83

Find the value of:

$$\left(\sum_{k=1}^{20} k\frac{1}{2^k}\right) - 2$$

(JEE Main 2019, 8 April, Shift-I, Adapted)

Let $S = \sum_{k=1}^{20} k\frac{1}{2^k}$, and expand the summation:

$$\underbrace{S = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{20}{2^{20}}}_{\textit{Equation I}-10\ \textit{Terms}}$$

This is a finite arithmetico-geometric series. Multiply the above equation by the common ratio:

$$\underbrace{\frac{S}{2} = 0 + \frac{1}{2^2} + \frac{2}{2^3} + \cdots + \frac{19}{2^{20}} + \frac{20}{2^{21}}}_{\textit{Equation II}-11\ \textit{Terms}}$$

Subtract Equation II from Equation I. Note that because Equation II has 11 terms, and Equation I has 10 terms, their difference has 11 terms:

$$\frac{S}{2}=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{20}}-\frac{20}{2^{21}}$$

Now the first ten terms are a finite geometric series with $a=\frac{1}{2}, r=\frac{1}{2}$:

$$\frac{S}{2}=\frac{\frac{1}{2}\left(1-\frac{1}{2^{20}}\right)}{1-\frac{1}{2}}-\frac{20}{2^{21}}=1-\frac{1}{2^{20}}-\frac{20}{2^{21}}=1-\frac{1}{2^{20}}-\frac{10}{2^{20}}=1-\frac{11}{2^{20}}$$

Multiply by 2 both sides:

$$S=2-\frac{11}{2^{19}}$$

Subtract 2 from both sides:

$$S-2=-\frac{11}{2^{19}}$$

2.84: Common Ratio greater than 1

If the common ratio of an AGP is greater than 1, the method to solve it remains the same.

Example 2.85

If $10^9+2(11)^1(10)^8+3(11)^2(10)^7+\ldots+10(11)^9=k(10)^9$ then k is equal to: (JEE Main 2014)

Divide both sides by 10^9:

$$\underbrace{k=1+2\left(\frac{11}{10}\right)^1+3\left(\frac{11}{10}\right)^2+\ldots+10\left(\frac{11}{10}\right)^9}_{Equation\ I-10\ Terms}$$

This is an AGP with common ratio $r=\frac{11}{10}$. Multiply both sides by $r=\frac{11}{10}$:

$$\underbrace{\frac{11}{10}k=0+1\left(\frac{11}{10}\right)+2\left(\frac{11}{10}\right)^2+3\left(\frac{11}{10}\right)^3+\cdots+10\left(\frac{11}{10}\right)^{10}}_{Equation\ II-11\ Terms}$$

Subtract Equation II from Equation I:

$$-\frac{1}{10}k=1+\left(\frac{11}{10}\right)^1+\left(\frac{11}{10}\right)^2+\cdots+\left(\frac{11}{10}\right)^9-10\left(\frac{11}{10}\right)^{10}$$

Note that the first ten terms of the RHS form a finite geometric series with $a=1, r=\frac{11}{10}$ in $\frac{a(r^n-1)}{r-1}$:

$$-\frac{1}{10}k=\frac{1\left[\left(\frac{11}{10}\right)^{10}-1\right]}{\frac{11}{10}-1}-10\left(\frac{11}{10}\right)^{10}$$

Simplify:

$$-\frac{1}{10}k=10\left(\frac{11}{10}\right)^{10}-10-10\left(\frac{11}{10}\right)^{10}$$

Note that the first term and the last term on the RHS cancel:

$$-\frac{1}{10}k=-10\Rightarrow k=100$$

2.86: Sum of Finite Arithmetico-geometric Series

E. More Uses of the Method

Sometimes a series may not be an AGP, but the method used in AGP can still be useful. However, this may require additional steps.

Example 2.87

Sum of first n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\cdots$ is equal to (JEE Adv. 1988)

Let:

$$\underbrace{S=\frac{1}{2^1}+\frac{3}{2^2}+\frac{7}{2^3}+\frac{15}{2^4}+\cdots+\frac{2^n-1}{2^n}}_{Equation\ I}$$

Note that the numerators do not form an arithmetic progression, but the denominators do form a geometric progress. We can try our AGP method.

Divide both sides of the above by the common ratio$=r=2$:

$$\underbrace{\frac{S}{2}=0+\frac{1}{2^2}+\frac{3}{2^3}+\frac{7}{2^4}+\cdots+\frac{2^n-1}{2^{n+1}}}_{Equation\ II}$$

Subtract Equation II from Equation I:

$$\frac{S}{2}=\underbrace{\left(\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\cdots\right)}_{n\ terms}-\frac{2^n-1}{2^{n+1}}$$

Each of the n terms simplifies to $\frac{1}{2}$:

$$\frac{S}{2}=\underbrace{\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\right)}_{n\ terms}-\frac{2^n-1}{2^{n+1}}$$

Simplify:

$$\frac{S}{2}=\frac{n}{2}-\frac{2^n-1}{2^{n+1}}$$

Multiply by 2 both sides

$$S=n-\frac{2^{n+1}-2}{2^{n+1}}$$

Split the fraction and simplify:

$$S=n-1+\frac{1}{2^n}=n-1+2^{-n}$$

F. Quadratic-Geometric Sequence

2.88: Quadratic-Geometric Sequence

A quadratic-geometric sequence is a combination of a quadratic sequence, and a geometric sequence.

- The first difference of a quadratic sequence is an arithmetic sequence.

Example 2.89

If $S=\frac{7}{5}+\frac{9}{5^2}+\frac{13}{5^3}+\frac{19}{5^4}+\cdots$ then $160S$ is equal to (JEE Main 2021, 31 Aug, Shift-II)

Quadratic-Geometric Sequence

We know that:

$$\underbrace{S=\frac{7}{5}+\frac{9}{5^2}+\frac{13}{5^3}+\frac{19}{5^4}+\cdots}_{Equation\ I}$$

This is a quadratic-geometric sequence with common ratio $= r = \frac{1}{5}$. Multiply both sides by the common ratio:

$$\underbrace{\frac{S}{5} = 0 + \frac{7}{5^2} + \frac{9}{5^3} + \frac{13}{5^4} + \cdots}_{Equation\ II}$$

Subtract Equation II from Equation I:

$$\frac{4}{5}S = \frac{7}{5} + \frac{2}{5^2} + \frac{4}{5^3} + \frac{6}{5^4} + \cdots$$

Arithmetico-Geometric Sequence

Subtract $\frac{7}{5}$ from both sides, and let K equal the new quantity:

$$\underbrace{K = \frac{4}{5}S - \frac{7}{5} = \frac{2}{5^2} + \frac{4}{5^3} + \frac{6}{5^4} + \frac{8}{5^5} + \cdots}_{Equation\ III}$$

Note that K is an arithmetico geometric series. Hence, multiply both sides by the common ratio $= r = \frac{1}{5}$:

$$\underbrace{\frac{K}{5} = 0 + \frac{2}{5^3} + \frac{4}{5^4} + \frac{6}{5^5} + \cdots}_{Equation\ IV}$$

Subtract Equation IV from Equation III:

$$\frac{4}{5}K = \frac{2}{5^2} + \frac{2}{5^3} + \cdots$$

The RHS is an infinite geometric series with $a = \frac{2}{25}, r = \frac{1}{5}$:

$$\frac{4}{5}K = \frac{\frac{2}{25}}{1 - \frac{1}{5}} = \frac{2}{25} \times \frac{5}{4} = \frac{1}{10}$$

Hence:

$$K = \frac{1}{10} \times \frac{5}{4} = \frac{1}{8}$$

Value of S

Finally,

$$K = \frac{4}{5}S - \frac{7}{5} = \frac{1}{8}$$

Add $\frac{7}{5}$ to both sides:

$$\frac{4}{5}S = \frac{1}{8} + \frac{7}{5} = \frac{61}{40}$$

Multiply both sides by $\frac{5}{4}$:

$$S = \frac{61}{40} \times \frac{5}{4} = \frac{61}{32}$$

Multiply both sides by 160:

$$160S = \frac{61}{32} \times 160 = 305$$

2.5 Telescoping

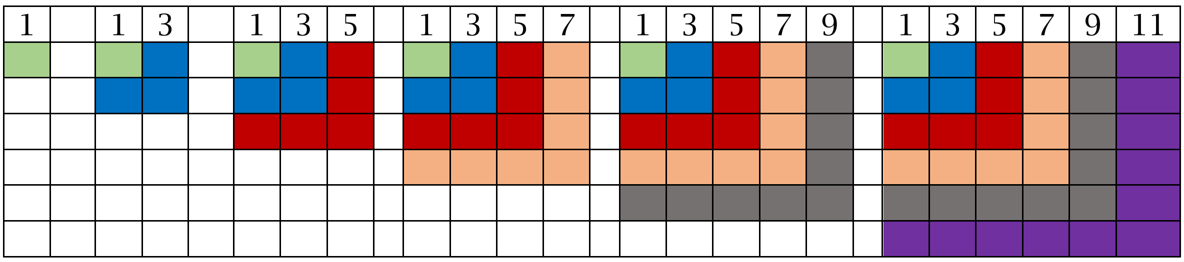
A. Sum of Odd Numbers

2.90: Sum of First n odd Numbers: Visual Proof

The sum of the first n odd numbers:

$$\underbrace{1 + 3 + 5 + \cdots}_{n\ terms} = n^2$$

Consider the "proof without words" below which shows that every extra term added increases the size of square figure by exactly the number needed to make it one larger.



$$1 + 3 = 4 = 2^2$$
$$1 + 3 + 5 = 9 = 3^2$$
$$1 + 3 + 5 + 7 = 16 = 4^2$$

$$1 + 3 + 5 + 7 + 9 = 25 = 5^2$$
$$1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2$$

2.91: Sum of First n odd Numbers: Algebraic Proof

The sum of the first n odd numbers:

$$\underbrace{1 + 3 + 5 + \cdots}_{n\ terms} = n^2$$

We prove this using mathematical induction.

Base Case: $n = 1$:

$$LHS = 1\ 3 + \cdots + (2n - 1) = 1$$
$$RHS = n^2 = 1$$
$$LHS = RHS \Rightarrow Base\ Case\ is\ valid$$

Inductive Case:

Let the statement be true for k:

$$1 + 3 + \cdots + (2k - 1) = k^2$$

Add $(2(k + 1) - 1) = 2k + 1$ to both sides:

$$\text{LHS} = 1\ + 3 + \cdots + (2k - 1) + (2k + 1)$$
$$RHS = k^2 + 2k + 1 = (k + 1)^2$$

Hence, it is true for $k + 1$. This completes the proof.

2.92: A useful identity

$$2n + 1 = (n + 1)^2 - n^2$$

$$2n + 1 = n^2 + 2n + 1 - n^2 = (n + 1)^2 - n^2$$

There is an interesting interpretating of this identity:

$$1 + 3 + 5 + \cdots + (2n + 1) = (n + 1)^2$$
$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Subtract Equation II from Equation I:

$$2n + 1 = (n + 1)^2 - n^2$$

2.93: Another useful identity

$$\frac{2n + 1}{n^2(n + 1)^2} = \frac{1}{n^2} - \frac{1}{(n + 1)^2}$$

Going from the RHS to the LHS is simple addition.

Going from the LHS to the RHS requires a little more creativity. We use the identity from the previous point. The LHS is:

$$\frac{2n + 1}{n^2(n + 1)^2}$$

Split the fraction

$$= \frac{(n + 1)^2 - n^2}{n^2(n + 1)^2} = \frac{(n + 1)^2}{n^2(n + 1)^2} - \frac{n^2}{n^2(n + 1)^2}$$

Simplify:

$$= \frac{1}{n^2} - \frac{1}{(n + 1)^2}$$

Example 2.94

The sum of 10 terms of the series:

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \cdots$$

Is **(JEE Main 2021, 21 Aug, Shift-I)**

Write out the first ten terms of the series. Note that the first numerator is the second odd number, and hence the 10th numerator is the 11th odd number:

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \cdots + \frac{21}{10 \times 11}$$

Using the identity $2n + 1 = (n+1)^2 - n^2$:

$$\frac{2^2 - 1^2}{1^2 \times 2^2} + \frac{3^2 - 2^2}{2^2 \times 3^2} + \frac{4^2 - 3^2}{3^2 \times 4^2} + \cdots + \frac{11^2 - 10^2}{10 \times 11}$$

Separate each fraction into two:

$$= \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) + \cdots + \left(\frac{1}{10^2} - \frac{1}{11^2}\right)$$

Telescope. In each parentheses, the second term cancels with the first term of the next parentheses. We are only left with the first term of the first parentheses, and the last term of the last parenthesis:

$$= \frac{1}{1} - \frac{1}{11^2} = \frac{1}{1} - \frac{1}{121} = \frac{120}{121}$$

B. Telescoping: Splitting Fractions

2.95: Breaking $\frac{1}{n(n+1)}$

$$\frac{1}{n} - \frac{1}{n+1} = \underbrace{\frac{n+1}{n(n+1)} - \frac{n}{n(n+1)}}_{Take\ the\ LCM} = \frac{1}{n(n+1)}$$

$$\frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6} = \frac{1}{2 \times 3}$$
$$\frac{1}{4} - \frac{1}{5} = \frac{5}{20} - \frac{4}{20} = \frac{1}{20} = \frac{1}{4 \times 5}$$

Example 2.96: Breaking

A. Evaluate $\frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \cdots + \frac{1}{n(n-1)}$

B. Evaluate $1677\left(\frac{1}{1560} + \frac{1}{1640} + \frac{1}{1722} + \frac{1}{1806}\right)$

Part A

Rewrite the denominators:

$$\frac{1}{5 \times 6} + \frac{1}{6 \times 7} + \cdots + \frac{1}{n(n-1)}$$

Apply the formula to break each term into two:

$$\left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

Everything cancels, except the first and the last term:

$$\frac{1}{5}-\frac{1}{n}=\frac{n-5}{5n}$$

Part B

$$1677\left(\frac{1}{39\times 40}+\frac{1}{40\times 41}+\frac{1}{41\times 42}+\frac{1}{42\times 43}\right)$$

Split the fractions:

$$=1677\left[\left(\frac{1}{39}-\frac{1}{40}\right)+\left(\frac{1}{40}-\frac{1}{41}\right)+\left(\frac{1}{41}-\frac{1}{42}\right)+\left(\frac{1}{42}-\frac{1}{43}\right)\right]$$

Telescope:

$$=1677\left[\frac{1}{39}-\frac{1}{43}\right]=1677\left[\frac{43}{39}-\frac{39}{43}\right]=1677\left(\frac{4}{1677}\right)=4$$

2.97: Breaking $\frac{1}{n(n+k)}$

$$\frac{1}{k}\left(\frac{1}{n}-\frac{1}{n+k}\right)=\frac{1}{n(n+k)}$$

$$\frac{1}{n}-\frac{1}{n+k}=\frac{n+k}{n(n+k)}-\frac{n}{n(n+k)}=\frac{k}{n(n+k)}$$

$$x=\frac{1}{2}-\frac{1}{5}=\frac{5}{10}-\frac{2}{10}=\frac{3}{10}\Rightarrow\frac{x}{3}=\left(\frac{1}{3}\right)\left(\frac{1}{2}-\frac{1}{5}\right)=\frac{1}{2\times(2+3)}$$

Example 2.98:

A. Evaluate $\frac{1}{8}+\frac{1}{24}+\frac{1}{48}+\frac{1}{80}$
B. Evaluate $\frac{1}{10}+\frac{1}{40}+\frac{1}{88}+\frac{1}{154}$
C. Evaluate $\frac{1}{3}+\frac{1}{15}+\cdots+\frac{1}{n(n+2)}$

Part A

Rewrite the denominators:

$$\frac{1}{2\times 4}+\frac{1}{4\times 6}+\frac{1}{6\times 8}+\frac{1}{8\times 10}$$

Apply the formula to break each term into two:

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{8}-\frac{1}{10}\right)$$

Factor $\frac{1}{2}$ from all the terms:

$$\left(\frac{1}{2}\right)\left[\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{8}-\frac{1}{10}\right)\right]$$

Within the brackets, cancel all except the first and the last terms:

$$\left(\frac{1}{2}\right)\left[\frac{1}{2}-\frac{1}{10}\right]=\frac{1}{2}\left[\frac{5}{10}-\frac{1}{10}\right]=\frac{1}{2}\left[\frac{4}{10}\right]=\frac{2}{10}=\frac{1}{5}$$

Part B

Rewrite the denominators:

$$\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\frac{1}{11\times 14}$$

Apply the formula to break each term into two:

$$\left(\frac{1}{3}\right)\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{3}\right)\left(\frac{1}{5}-\frac{1}{8}\right)+\left(\frac{1}{3}\right)\left(\frac{1}{8}-\frac{1}{11}\right)+\left(\frac{1}{3}\right)\left(\frac{1}{11}-\frac{1}{14}\right)$$

Factor $\frac{1}{3}$ from all the terms:

$$\left(\frac{1}{3}\right)\left[\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{8}\right)+\left(\frac{1}{8}-\frac{1}{11}\right)+\left(\frac{1}{11}-\frac{1}{14}\right)\right]$$

Telescope. Within the brackets, cancel all except the first and the last terms:

$$\left(\frac{1}{3}\right)\left[\frac{1}{2}-\frac{1}{14}\right]=\frac{1}{3}\left[\frac{7}{14}-\frac{1}{14}\right]=\frac{1}{3}\left[\frac{6}{14}\right]=\frac{1}{7}$$

Part C

Rewrite the denominators:

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{n(n+2)}$$

Apply the formula to break each term into two:

$$\left(\frac{1}{2}\right)\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots\left(\frac{1}{2}\right)\left(\frac{1}{n}-\frac{1}{n+2}\right)$$

Factor $\frac{1}{2}$ from all the terms:

$$\left(\frac{1}{2}\right)\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots\left(\frac{1}{n}-\frac{1}{n+2}\right)\right]$$

Within the brackets, cancel all except the first and the last terms:

$$\left(\frac{1}{2}\right)\left[1-\frac{1}{n+2}\right]=\frac{1}{2}\left[\frac{n+2-1}{n+2}\right]=\frac{n+1}{2(n+2)}$$

Challenge 2.99

Evaluate:

$$\frac{1}{2^2-1}+\frac{1}{4^2-1}+\frac{1}{6^2-1}+\cdots+\frac{1}{20^2-1}= \quad (\textbf{\textit{CAT}}\ \mathbf{2000})$$

Strategy: Factor

We need to do a little additional work here before we see the series in the form that will let us use the formula. First, factor the denominators using $a^2-b^2=(a+b)(a-b)$:

$$\frac{1}{(2-1)(2+1)}+\frac{1}{(4-1)(4+1)}+\frac{1}{(6-1)(6+1)}+\cdots+\frac{1}{(20-1)(20+1)}$$

Then simplify the brackets:

$$\frac{1}{(1)(3)}+\frac{1}{(3)(5)}+\frac{1}{(5)(7)}+\cdots+\frac{1}{(19)(21)}$$

Telescope

Now the series is in the form that we need:

$$\frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{19}-\frac{1}{21}\right)\right]$$

Telescope and simplify:

$$\frac{1}{2}\left[1-\frac{1}{21}\right]=\frac{1}{2}\left[\frac{20}{21}\right]=\frac{10}{21}$$

Example 2.100

Show that the sum of the reciprocals of the first n triangular numbers is less than 2.

$$n^{th}\text{ Triangular No.}=\frac{n(n+1)}{2}\Rightarrow\text{Reciprocal}=\frac{2}{n(n+1)}$$

Sum of the reciprocals of the first n triangular numbers:

$$=\frac{2}{1\times 2}+\frac{2}{2\times 3}+\frac{2}{3\times 4}+\cdots+\frac{2}{n(n+1)}$$

Factor 2 out of every term in the series:

$$2\left[\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n(n+1)}\right]$$

Use fraction decomposition to re-write each term and then telescope:

$$2\left[\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]=2\left(1-\frac{1}{n+1}\right)=2\left(\frac{n}{n+1}\right)<2$$

C. A "Popular" Problem

The problem below has been asked in at least three different competitions (see below) in recent years.

Example 2.101

Find the sum:

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\cdots+\sqrt{1+\frac{1}{2007^2}+\frac{1}{2008^2}}$$ **(CAT 2008)**

Identifying the Pattern

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}=\sqrt{\frac{9}{4}}=\frac{3}{2}=1\frac{1}{2}$$

$$\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}=\sqrt{\frac{49}{36}}=\frac{7}{6}=1\frac{1}{6}$$

$$\sqrt{1+\frac{1}{3^2}+\frac{1}{4^2}}=\sqrt{\frac{169}{144}}=\frac{13}{12}=1\frac{1}{12}$$

We have perfect squares every time inside the radical. But, we don't know what the sum will be. Switch over to Algebra.

Find the nth term

The nth term of the series is

$$\sqrt{1+\frac{1}{n^2}+\frac{1}{(n+1)^2}}=\sqrt{\frac{n^2(n+1)^2+(n+1)^2+n^2}{n^2(n+1)^2}}=\frac{\sqrt{n^4+2n^3+3n^2+2n+1}}{n^2+n}$$

Since the numerator in the numbers above is always more than the denominator, try:

$$(n^2+n+1)^2=n^4+2n^3+3n^2+2n+1$$

Simplify the nth term

$$\sqrt{\frac{n^4+2n^3+3n^2+2n+1}{n^2(n+1)^2}}=\sqrt{\frac{(n^2+n+1)^2}{n^2(n+1)^2}}=\frac{n^2+n+1}{n^2+n}=1+\frac{1}{n(n+1)}$$

Find the sum of the series

Substitute the value of n. We want to find

$$\underbrace{1+\frac{1}{(2)}}_{n=1}+\underbrace{1+\frac{1}{2(3)}}_{n=2}+\cdots+\underbrace{1+\frac{1}{2007(2008)}}_{n=2007}$$

Separate out the 1's:

$$\underbrace{1+1+\cdots+1}_{2007\ times=1}+\underbrace{\frac{1}{1(2)}+\frac{1}{2(3)}+\cdots+\frac{1}{2007(2008)}}_{y}=2007+\underbrace{1-\frac{1}{2008}}_{y}=2008-\frac{1}{2008}$$

Factor the Numerator

If we had wanted to factor the numerator as a perfect square without a guess as to what it was, we could still have done it using the method below:

- The square root of a fourth-degree equation must be a second-degree equation.
- The first and last coefficient must be 1, since the first and last coefficient of $n^4+2n^3+3n^2+2n+1$ is 1.

Hence, we are looking for an expression of the form:

$$(n^2+an+1)^2$$

Expand the above, and equate it to the numerator:

$$n^4+2an^3+(2+a^2)n^2+2an+1=n^4+2n^3+3n^2+2n+1$$

By equating coefficients, we get three equations:

$$n^3: 2a=2\Rightarrow a=1$$
$$n^2: 2+a^2=3\Rightarrow a=\pm 1$$

Hence, $a=1$ satisfies all the above equations. This tells us that:

$$n^4+2n^3+3n^2+2n+1=(n^2+n+1)^2$$

Example 2.102

(TAMU 2023/Best Student/20)

Problem 20. Determine the value of

$$S=\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\cdots+\sqrt{1+\frac{1}{22^2}+\frac{1}{23^2}}.$$

Example 2.103

If

$$\sum_{k=1}^{40}\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}=a+\frac{b}{c}$$

Where $a,b,c\in\mathbb{N}, b<c, \gcd(b,c)=1$ then find the value of $a+b$. (IOQM 2021/18)

This is the same question as one we did above, except that it is written in scarier notation. If we substitute $k=1,2,3,\ldots,40$, we get:

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\cdots+\sqrt{1+\frac{1}{40^2}+\frac{1}{41^2}}$$

Apply the same method as above to get:

$$\underbrace{1+1+\cdots+1}_{40\ times=1}+\underbrace{\frac{1}{1(2)}+\frac{1}{2(3)}+\cdots+\frac{1}{40(41)}}_{y}=40+\underbrace{1-\frac{1}{41}}_{y}=40\frac{40}{41}\Rightarrow a+b=40+40=80$$

D. Rationalizing the denominator

2.104: Rationalize the Denominator

If you have a series with radicals in the denominator, one of the options to telescope it is to rationalize the denominator.

This will not work with all series. Only the ones which can be telescoped.

Example 2.105

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{8}+\sqrt{9}}=$$

Multiply each term by the *conjugate surd* of the denominator:

$$\frac{1}{\sqrt{1}+\sqrt{2}}\cdot\frac{\sqrt{2}-\sqrt{1}}{\sqrt{2}-\sqrt{1}}+\frac{1}{\sqrt{2}+\sqrt{3}}\cdot\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}+\cdots+\frac{1}{\sqrt{8}+\sqrt{9}}\cdot\frac{\sqrt{9}-\sqrt{8}}{\sqrt{9}-\sqrt{8}}$$

Use the formula $(a+b)(a-b)=a^2-b^2$ in the denominator, and then telescope:

$$\frac{\sqrt{2}-\sqrt{1}}{1}+\frac{\sqrt{3}-\sqrt{2}}{1}+\frac{\sqrt{4}-\sqrt{3}}{1}+\cdots+\frac{\sqrt{9}-\sqrt{8}}{1}=\frac{\sqrt{9}-\sqrt{1}}{1}=\frac{3-1}{1}=\frac{2}{1}=2$$

Example 2.106

Find the sum to n terms of

$$\frac{1}{\sqrt{7}+\sqrt{10}}+\frac{1}{\sqrt{10}+\sqrt{13}}+\frac{1}{\sqrt{13}+\sqrt{16}}+\cdots$$

Recognize that the expression inside the first term and the second term of the denominator is each an arithmetic progression:

$$7,10,13,\ldots\Rightarrow a=4,d=3\Rightarrow n^{th}\ term=4+3n$$
$$10,13,16\ldots\Rightarrow a=7,d=3\Rightarrow n^{th}\ term=7+3n$$

Rewrite the series to include the n^{th} term:

$$\frac{1}{\sqrt{7}+\sqrt{10}}+\frac{1}{\sqrt{10}+\sqrt{13}}+\frac{1}{\sqrt{13}+\sqrt{16}}+\cdots+\frac{1}{\sqrt{4+3n}+\sqrt{7+3n}}$$

Rationalize the denominators:

$$=\frac{\sqrt{10}-\sqrt{7}}{3}+\frac{\sqrt{13}-\sqrt{10}}{3}+\frac{\sqrt{16}-\sqrt{13}}{3}+\cdots+\frac{\sqrt{7+3n}-\sqrt{4+3n}}{3}$$

Telescope

$$=\frac{\sqrt{7+3n}-\sqrt{7}}{3}$$

E. Completing the Square

Example 2.107

If $\sum_{k=1}^{N} \frac{2k+1}{(k^2+k)^2} = 0.9999$ then determine the value of N. (IOQM 2021/3)

Consider the expression inside the summation sign.

$$\frac{2k+1}{(k^2+k)^2}$$

The denominator is already a perfect square. Complete the square in the numerator. Add and subtract k^2:

$$\frac{k^2+2k+1-k^2}{(k^2+k)^2}$$

Factor the numerator and the denominator:

$$=\frac{(k+1)^2-k^2}{k^2(k+1)^2}$$

Split the fraction into two:

$$=\frac{1}{k^2}-\frac{1}{(k+1)^2}$$

Hence, the LHS becomes:

$$LHS=\sum_{k=1}^{N}\frac{2k+1}{(k^2+k)^2}=\sum_{k=1}^{N}\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right)$$

Write as the summation:

$$\left(\frac{1}{1}-\frac{1}{2^2}\right)+\left(\frac{1}{2^2}-\frac{1}{3^2}\right)+\cdots+\left(\frac{1}{N}-\frac{1}{(N+1)^2}\right)$$

Note that the series telescopes. We are left with the first and the last term only. Equate this to the RHS:

$$1-\frac{1}{(N+1)^2}=0.9999$$

Rearrange to get:

$$\frac{1}{(N+1)^2}=1-0.9999=0.0001=\frac{1}{10000}$$

Take the reciprocal of the first and the last quantity:

$$(N+1)^2=10{,}000$$

Take the square root on both sides, reject the negative root, and solve for N:

$$N+1=100\Rightarrow N=99$$

Telescoping in Summation Notation[9]

Split the summation:

$$\sum_{k=1}^{N}\frac{1}{k^2}-\sum_{k=1}^{N}\frac{1}{(k+1)^2}$$

Split:

[9] This method is relevant more for learning summation notation then for its use in solving the question.

$$\sum_{k=1}^{1}\frac{1}{k^2}+\sum_{k=2}^{N}\frac{1}{k^2}-\sum_{k=1}^{N-1}\frac{1}{(k+1)^2}-\sum_{k=N}^{N}\frac{1}{(k+1)^2}$$

Simplify the first and last term by substitution:

$$1+\sum_{k=2}^{N}\frac{1}{k^2}-\sum_{k=1}^{N-1}\left(\frac{1}{(k+1)^2}\right)-\frac{1}{(N+1)^2}$$

Reindex the third term:

$$1+\sum_{k=2}^{N}\frac{1}{k^2}-\sum_{k=2}^{N}\frac{1}{k^2}-\frac{1}{(N+1)^2}$$

Simplify:

$$1-\frac{1}{(N+1)^2}$$

F. Introducing a New Quantity

Completing the square requires adding and subtracting the same quantity that lets us then write part of an expression as a perfect square. However, there can be other ways of rewriting an expression, some of which can be very creative, or difficult to think of when first seeing a problem.

Example 2.108: Rewriting the expression

Find the sum to n terms of:

$$1(2)+2(3)+\cdots+n(n+1)$$

Let T_n be the n^{th} term. Then we can write:

$$T_n=n(n+1)$$

Multiply both sides by 3:

$$3T_n=3n(n+1)$$

Now make a substitution whose reason is not immediately obvious.
Substitute $3=(n+2)-(n-1)$ in the above expression:

$$3T_n=n(n+1)\underbrace{[(n+2)-(n-1)]}_{=3}$$

Expand:

$$3T_n=n(n+1)(n+2)-\boldsymbol{(n-1)n(n+1)}$$
$$3T_{n-1}=\boldsymbol{(n-1)(n)(n+1)}-\boldsymbol{(n-2)(n-1)(n)}$$
$$3T_{n-2}=\boldsymbol{(n-2)(n-1)(n)}-(n-3)(n-2)(n-1)$$
$$\vdots$$
$$3T_1=\cancel{(1)(2)(3)}-\boldsymbol{(0)(1)(2)}$$

Add up all the terms above, and note that they telescope:

$$3T_n+3T_{n-1}+\cdots+3T_1=n(n+1)(n+2)$$
$$3(T_n+T_{n-1}+\cdots+T_1)=n(n+1)(n+2)$$
$$T_n+T_{n-1}+\cdots+T_1=\frac{n(n+1)(n+2)}{3}$$

Method II: With Summation Notation

Write out the given series as a summation:

$$X=\sum_{x=1}^{x=n} x(x+1)$$

Multiply both sides by 3:

$$3X=\sum_{x=1}^{x=n} 3x(x+1)$$

Substitute $3=(x+2)-(x-1)$:

$$3X=\sum_{x=1}^{x=n} x(x+1)[(x+2)-(x-1)]$$

Use the distributive property:

$$3X=\sum_{x=1}^{x=n} x(x+1)(x+2)-(x-1)(x)(x+1)$$

Split the summation:

$$3X=\sum_{x=1}^{x=n} x(x+1)(x+2)-\sum_{x=1}^{x=n}(x-1)(x)(x+1)$$

Reindex the second summation by substituting $u=x-1 \Rightarrow x=u+1$:

$$\sum_{u+1=1}^{u+1=n}(u+1-1)(u+1)(u+1+1)=\sum_{u=0}^{u=n-1}(u)(u+1)(u+1+1)$$

Since the index of summation is a dummy variable, change it back to x:

$$3X=\sum_{x=1}^{x=n} x(x+1)(x+2)-\sum_{x=0}^{x=n-1}(x)(x+1)(x+2)$$

Note that substituting $x=0$ in the second summation results in a zero term, and hence we can ignore it:

$$3X=\sum_{x=1}^{x=n} x(x+1)(x+2)-\sum_{x=1}^{x=n-1}(x)(x+1)(x+2)$$

All the terms cancel, since both the summation terms have the same formula, and the same index of summation, except that the first summation term has an extra n^{th} term, and we are left with:

$$3X=n(n+1)(n+2)+0$$

$$X=\frac{n(n+1)(n+2)}{3}$$

Writing Assignment 2.109

Show that

$$1(2)(3)(4)+2(3)(4)(5)+\cdots+n(n+1)(n+2)(n+3)=\frac{n(n+1)(n+2)(n+3)(n+4)}{5}$$

Example 2.110

Find the sum to n terms of

$$1(2)(3)\dots(1+k)+2(3)(4)\dots(2+k)+\cdots+n(n+1)\dots(n+k)$$

Write the expression in summation notation:

$$S_n = \sum_{i=1}^{n} T_i = \sum_{i=1}^{n} i(i+1) \times \dots \times (i+k)$$

Multiply both sides of the above by $k+2 = (\boldsymbol{i}+\boldsymbol{k}+\mathbf{1}) - (\boldsymbol{i}-\mathbf{1})$:

$$(k+2)Sn = \sum_{i=1}^{n} i(i+1) \times \dots \times (i+k)\,[(\boldsymbol{i}+\boldsymbol{k}+\mathbf{1}) - (\boldsymbol{i}-\mathbf{1})]$$

Use the distributive property on the RHS:

$$(k+2)Sn = \sum_{i=1}^{n} i(i+1) \times \dots \times (i+k)\,(\boldsymbol{i}+\boldsymbol{k}+\mathbf{1}) - \boldsymbol{i}(\boldsymbol{i}+\mathbf{1}) \times \dots \times (\boldsymbol{i}+\boldsymbol{k})(\boldsymbol{i}-\mathbf{1})$$

Split up the sum on the RHS:

$$(k+2)Sn = \sum_{i=1}^{n} i(i+1) \times \dots \times (i+k)\,(i+k+1) - \sum_{i=1}^{n} (i-1)i(i+1) \times \dots \times (i+k)$$

To telescope, make the terms the same form by reducing the index of summation on the red term:

$$(k+2)Sn = \sum_{i=1}^{n} i(i+1) \times \dots \times (i+k)\,(i+k+1) - \underbrace{\sum_{i=1}^{n-1} (i)(i+1) \times \dots \times (i+k+1)}_{\text{Change the index of summation and drop the } i=0 \text{ term since it vanishes}}$$

Simplify:

$$(k+2)S_n = n(n+1) \times \dots \times (n+k)(n+k+1)$$

Solve for S_n:

$$S_n = \frac{n(n+1) \times \dots \times (n+k)(n+k+1)}{(k+2)}$$

A. Find the sum to n terms:

$A.\ 1 \times 1 + 2 \times 2 + 3 \times 6 + 4 \times 10 + \cdots.$

$B.\ 3 \times 8 + 6 \times 11 + 9 \times 14 \dots$

$$C.\ 3 + 7 + 13 + 21 + 31 + \cdots$$

$$E.\ 1 + \frac{1^2+2^2}{1+2} + \frac{1^2+2^2+3^2}{1+2+3}$$

G. Comparing

Example 2.111

Determine without a calculator, which is larger:

$$1.005^{200} \text{ or } 2$$

Con

$$P = (1+0.005)^{200} = \left(1+\frac{5}{1000}\right)^{200} = \left(1+\frac{1}{200}\right)^{200} = \left(1+\frac{1}{200}\right)\left(1+\frac{1}{200}\right)\dots\left(1+\frac{1}{200}\right)$$

$$Q=2=\frac{400}{200}=\left(\frac{201}{200}\right)\left(\frac{202}{201}\right)\left(\frac{203}{202}\right)\dots\left(\frac{399}{398}\right)\left(\frac{400}{399}\right)$$

Which has

$$399-200+1=400-200=200\ terms$$

$$=\left(1+\frac{1}{200}\right)\left(1+\frac{1}{201}\right)\left(1+\frac{1}{202}\right)\dots\left(1+\frac{1}{398}\right)\left(1+\frac{1}{399}\right)$$

Compare term by term:

$$1st\ Term: Equal$$
$$2nd\ Term: 1+\frac{1}{200}>\ 1+\frac{1}{201}$$
$$3rd\ Term: 1+\frac{1}{200}>\ 1+\frac{1}{202}$$
$$\vdots$$
$$200th\ \ Term: 1+\frac{1}{200}>\ 1+\frac{1}{399}$$

Overall, since one term in P is equal, and other terms in P are greater than Q, we must have:

$$P>Q$$

2.6 Miscellaneous Sequences

A. Rearranging

Example 2.112

Sum of first n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\cdots$ is equal to (JEE Adv. 1988)

The general term is:

$$\frac{2^n-1}{2^n}=\frac{2^n}{2^n}-\frac{1}{2^n}=1-\frac{1}{2^n}$$

Use this idea to rewrite the series as a sum of two series:

$$\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{4}\right)+\left(1-\frac{1}{8}\right)+\left(1-\frac{1}{16}\right)+\cdots$$

Separate the two series out:

$$=\underbrace{1+1+\cdots+1}_{n\ times}-\underbrace{\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}\right)}_{Geometric\ Series}$$

The first series is easy to sum. The second is a finite geometric series with $a=\frac{1}{2}, r=\frac{1}{2}$ and $Sum=\frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}$:

$$=n-\left[\frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}\right]=n-\left[\frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{\frac{1}{2}}\right]=n-1+\frac{1}{2^n}=n-1+2^{-n}$$

B. Product

2.113: Product of Consecutive Numbers

$$1(2)+2(3)+\cdots+n(n+1)=\frac{n(n+1)(n+2)}{3}$$
$$1(2)(3)+2(3)(4)+\cdots+n(n+1)(n+2)=\frac{n(n+1)(n+2)(n+3)}{4}$$
$$1(2)(3)(4)+2(3)(4)(5)+\cdots+n(n+1)(n+2)(n+3)=\frac{n(n+1)(n+2)(n+3)(n+4)}{5}$$

Example 2.114

Evaluate:

$$\sum_{k=1}^{9} k(k+1)$$
$$\sum_{k=0}^{7} k(k+1)$$

$$\sum_{k=1}^{9} k(k+1)=\frac{9\times 10\times 11}{3}=330$$
$$\sum_{k=0}^{7} k(k+1)=\sum_{k=1}^{7} k(k+1)=\frac{7\times 8\times 9}{3}=168$$

Example 2.115

$$a_1+a_2+\cdots+a_n=\frac{n^2+3n}{(n+1)(n+2)}=S_n,\qquad 28\sum_{k=1}^{k=10}\frac{1}{a_k}=P_1\times P_2\times\ldots\times P_m$$

Where P_m are first prime numbers, then find m. **(JEE Main 2023)**

Use $a_k=S_k-S_{k-1}$:

$$=\frac{k(k+3)}{(k+1)(k+2)}-\frac{(k-1)(k+2)}{k(k+1)}=\frac{(k^3+3k^2)-(k-1)(k+2)(k+2)}{k(k+1)(k+2)}$$

Multiply and simplify to get:

$$=\frac{(k^3+3k^2)-(k^3+3k^2-4)}{k(k+1)(k+2)}=\frac{4}{k(k+1)(k+2)}$$

Substitute $\frac{1}{a_k}=\frac{k(k+1)(k+2)}{4}$:

$$28\sum_{k=1}^{k=10}\frac{1}{a_k}=28\sum_{k=1}^{k=10}\frac{k(k+1)(k+2)}{4}=\frac{28}{4}\sum_{k=1}^{k=10}k(k+1)(k+2)$$

Use the formula: $1(2)(3)+2(3)(4)+\cdots+n(n+1)(n+2)=\frac{n(n+1)(n+2)(n+3)}{4}$

$$=\frac{28}{4}\times\frac{10\times 11\times 12\times 13}{4}$$

$$= 7 \times 10 \times 11 \times 3 \times 13$$
$$= 2 \times 3 \times 5 \times 7 \times 11 \times 13$$

Since the above expression has the first six prime numbers:
$$m = 6$$

C. Triangular Numbers

2.116: Sum of First n natural Numbers

$$\sum_{x=1}^{n} x = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

- These numbers are also called the triangular numbers.

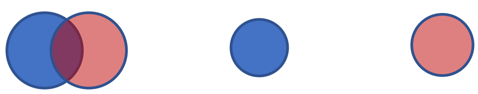
Example 2.117

If the sum of the first 20 terms of the series $\log_{7^{\frac{1}{2}}} x + \log_{7^{\frac{1}{3}}} x + \log_{7^{\frac{1}{4}}} x + \cdots$ is 460, then the value of x is: (JEE Main 2020, 5 Sep, Shift-II)

By the power rule extension (or by using change of base), the LHS is
$$2\log_7 x + 3\log_7 x + 4\log_7 x + \cdots + 21\log_7 x = 460$$
Factor $\log_7 x$:
$$\log_7 x\,(2 + 3 + \cdots + 21)$$
Use the formula for the sum of the first n natural numbers:
$$\log_7 x\left(\frac{21 \times 22}{2} - 1\right) = 460$$
$$\log_7 x\,(230) = 460$$
Divide both sides by 230:
$$\log_7 x = 2$$
Convert from logarithmic from to exponential form:
$$x = 7^2 = 49$$

2.118: Union of Two Sets

$$\underbrace{n(A \cup B)}_{\text{Union of } A \text{ and } B} = \underbrace{n(A)}_{\text{Elements in } A} + \underbrace{n(B)}_{\text{Elements in } B} - \underbrace{n(A \cap B)}_{\text{Elements in Intersection of } A \text{ and } B}$$



Example 2.119

The sum of all natural numbers n such that $100 < n < 200$, and $HCF(91, n) > 1$ is: (JEE Main, 8 April, Shift-I)

$$HCF(91, n) > 1 \Rightarrow HCF(13 \times 7, n) > 1$$
Hence, n must be a multiple of 7, or a multiple of 13, or both.

Multiples of 7

The multiples of 7 from 100 to 200 have sum:

$$= 105 + 112 + \cdots + 196$$
$$= 7(15 + 16 + \cdots + 28)$$

Use complementary counting:

$$= 7[(1 + 2 + \cdots + 28) - (1 + 2 + \cdots + 14)]$$

Use the formula for the sum of the first n natural numbers:

$$= 7\left[\frac{28 \times 29}{2} - \frac{14 \times 15}{2}\right] = 7[14 \times 29 - 7 \times 15]$$

Simplify:

$$= 49[2 \times 29 - 15] = 49[43] = 2107$$

Multiples of 13

The multiples of 13 from 100 to 200 have sum:

$$= 13(8 + 9 + \cdots + 15)$$

Use complementary counting:

$$= 13[(1 + 2 + \cdots + 15) - (1 + 2 + \cdots + 7)]$$

Use the formula for the sum of the first n natural numbers:

$$= 13\left[\frac{15 \times 16}{2} - \frac{7 \times 8}{2}\right] = 13[15 \times 8 - 7 \times 4]$$

Simplify:

$$= 52[15 \times 2 - 7] = 52[23] = 1192$$

Multiples of 91

There is only a single number that is a multiple of 91 from 100 to 200:

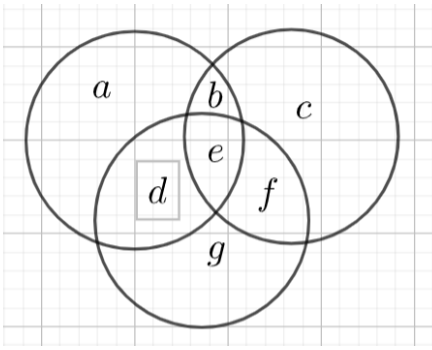
$$= 182$$

Hence, the final answer is:

$$2107 + 1196 - 182 = 3{,}121$$

2.120: Number of Elements in Three Sets

$$n(A \cup B \cup C) = \underbrace{n(A) + n(B) + n(C)}_{\textit{One at at a Time}} - \underbrace{[n(A \cap B) + n(B \cap C) + n(A \cap C)]}_{\textit{Two at a Time}} + \underbrace{n(A \cap B \cap C)}_{\textit{Three at a time}}$$



Left Hand Side

$$= n(A \cup B \cup C) = a + b + c + d + e + f + g$$

Right Hand Side

$$\begin{gathered} = (a + b + e + d) + (b + c + e + f) + (d + e + f + g) \\ - [(b + e) + (e + f) + (d + e)\,] + e \\ = a + 2b + 3e + 2d + 2f + g - b - 3e - d - f + e \\ = a + b + e + d + f \end{gathered}$$

1

Example 2.121

How many positive integers not exceeding 2001 are multiples of 3 or 4 but not 5? (AMC 10 2001/25)

$$\textit{Multiples of } 3 = \{\quad\}$$

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

(Calculator) Example 2.122

Ashley writes out the first 2017 positive integers. She then underlines any of the 2017 integers that is a multiple of 2, and then underlines any of the 2017 integers that is a multiple of 3, and then underlines any of the 2017 integers that is a multiple of 5. Finally, Ashley finds the sum of all the integers which have not been underlined. What is this sum? (Gauss Grade 7 2017/25)

We will use complementary counting. Find the sum of all the numbers from 1 to 2017, and then subtract the sum of the numbers which have been underlined.

The sum of the first n natural numbers is given by $\frac{n(n+1)}{2}$. Hence:

$$1+2+\cdots+2017=\frac{2017\times 2018}{2}=2{,}035{,}153$$

$$n(S_2\cup S_3\cup S_5)=n(S_2)+n(S_3)+n(S_5)-n(S_6)-n(S_{15})-n(S_{10})+n(S_{30})$$

$$S_2=2+4+\cdots+2016=2(1+2+\cdots+1008)=2\left(\frac{1008\times 1009}{2}\right)=1{,}017{,}072$$
$$S_3=3+6+\cdots+2016=3(1+2+\cdots+672)=3\left(\frac{672\times 673}{2}\right)=678{,}384$$
$$S_5=5+10+\cdots+2015=5(1+2+\cdots+403)=5\left(\frac{403\times 404}{2}\right)=407{,}030$$
$$S_6=6+12+\cdots+2016=6(1+2+\cdots+336)=6\left(\frac{336\times 337}{2}\right)=339{,}696$$
$$S_{15}=15+30+\cdots+2010=15(1+2+\cdots+134)=15\left(\frac{134\times 135}{2}\right)=135{,}675$$
$$S_{10}=10+20+\cdots+2010=10(1+2+\cdots+201)=10\left(\frac{201\times 202}{2}\right)=203{,}010$$

$$S_{30}=30+60+\cdots+2010=30(1+2+\cdots+67)=30\left(\frac{67\times 68}{2}\right)=68{,}340$$

$$n(S_2\cup S_3\cup S_5)=\underbrace{1017072}_{n(S_2)}+\underbrace{678{,}384}_{n(S_3)}+\underbrace{407{,}030}_{n(S_5)}-\underbrace{339{,}696}_{n(S_6)}-\underbrace{135{,}675}_{n(S_{15})}-\underbrace{203{,}010}_{n(S_{10})}+\underbrace{68{,}340}_{n(S_{30})}=1{,}492{,}445$$

$$2{,}035{,}153-1{,}492{,}445=$$

D. Sum of Squares

2.123: Sum of Squares

$$\sum_{x=1}^{n}x^2=1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

Example 2.124

Find A if: (JEE 2019, 12 Jan, Shift-II)

$$S_k=\frac{1+2+\cdots+k}{k},\qquad S_1^2+S_2^2+\cdots+S_{10}^2=\frac{5}{12}A$$

Use the formula for the sum of the first k numbers:

$$S_k=\frac{k(k+1)}{2k}=\frac{k+1}{2}\Rightarrow S_k^2=\frac{(k+1)^2}{4}$$

The LHS of the given equality is:

$$S_1^2+S_2^2+\cdots+S_{10}^2=\frac{2^2}{4}+\frac{3^2}{4}+\cdots+\frac{11^2}{4}=\frac{1}{4}[2^2+3^2+\cdots+11^2]$$

Use the formula for the sum of the squares of the first n numbers:

$$=\frac{1}{4}\left[\frac{11(12)(23)}{6}-1\right]=\frac{1}{4}[506-1]=\frac{1}{4}[505]$$

$$\frac{5}{12}A=\frac{1}{4}[505]$$
$$A=\frac{1}{4}[505]\times\frac{12}{5}=303$$

Example 2.125

The sum of the series below upto 11th term is: (JEE Main 2019, 9 April, Shift-II)

$$1+2\times 3+3\times 5+4\times 7+\ldots$$

$$=1+2\times 3+3\times 5+4\times 7+\ldots$$

It is critical in this question to get the general term correct:

$$\left(N^{th}Number\right)\left(N^{th}Odd\ Number\right)=r(2r-1)=2r^2-r$$

Hence, the summation that we are looking for

$$\sum_{r=1}^{11}2r^2-r=2\sum_{r=1}^{11}r^2-\sum_{r=1}^{11}r$$

Substitute $r=11$, and use the corresponding formulas:

$$=2\left[\frac{11(12)(23)}{6}\right]-\left(\frac{11\times 12}{2}\right)$$

Simplify:

$$=11(4)(23)-11\times 6$$

Factor out 11, and simplify:

$$=11[92-6]$$
$$=11[86]$$
$$=946$$

Example 2.126

The sum of 20 terms of the series $1+(1+2)+(1+2+3)+\cdots+(1+2+3+\cdots+k)$ is (JEE Main 2020, 8 Jan, Shift-I, Adapted)

Write as a summation:

$$\sum_{k=1}^{20}(1+2+\cdots+k)$$

Use the formula for the sum of the first n natural numbers:

$$\sum_{k=1}^{20}\frac{k(k+1)}{2}$$

Move $\frac{1}{2}$ out of the summation sign:

$$=\frac{1}{2}\sum_{k=1}^{20}(k^2+k)$$

Split the summation:

$$=\frac{1}{2}\left[\sum_{k=1}^{20}k^2+\sum_{k=1}^{20}k\right]$$

Use the formulas:

$$=\frac{1}{2}\left[\frac{20(20+1)(40+1)}{6}+\frac{20(20+1)}{2}\right]$$

$$=\frac{1}{2}[10(7)(41)+10(21)]$$

$$=\frac{10}{2}[287+21]=\frac{10}{2}[308]=\frac{3080}{2}=1540$$

Example 2.127

A. Find the sum of the first n triangular numbers.
B. Use the formula to evaluate $\sum_{k=1}^{20}(1+2+\cdots+k)$
C. Is it a good idea to memorize this "formula".

The n^{th} triangular number is the sum of the first n natural numbers.

$$\sum_{x=1}^{x=n}\frac{x(x+1)}{2}=\sum_{x=1}^{x=n}\frac{x^2+x}{2}$$

Split the terms in the summation expression:

$$\left(\sum_{x=1}^{x=n}\frac{x^2}{2}\right)+\left(\sum_{x=1}^{x=n}\frac{x}{2}\right)$$

Use the formula for the sum of the squares of the first n natural numbers in the first term, and the formula for the sum of the first n natural numbers in the second term:

$$\frac{n(n+1)(2n+1)}{12}+\frac{n(n+1)}{4}$$

Convert the second fraction to have a denominator of 12:

$$\frac{n(n+1)(2n+1)}{12}+\frac{3n(n+1)}{12}$$

Factor $n(n+1)$:

$$\frac{n(n+1)[2n+1+3]}{12}$$

Simplify:

$$\frac{n(n+1)[2n+4]}{12}$$

Cancel:

$$\frac{n(n+1)(n+2)}{6}$$

2.128: Sum of Cubes of Natural Numbers

The sum of the cube of the first n natural numbers is the square of the sum of the first n natural numbers. That is:

$$\sum_{x=1}^{n}x^3=1^3+2^3+\cdots+n^3=(1+2+\cdots+n)^2=\left[\frac{n(n+1)}{2}\right]^2$$

Example 2.129

The sum below is equal to:

$$\sum_{n=1}^{7}\left[\frac{3}{2}\cdot 1^2+\frac{3}{2}\cdot 2^2+\cdots+\frac{3}{2}\cdot n^2\right]\ \boldsymbol{(JEE\ Main\ 2020, 8\ Jan, Shift-II)}$$

Factor out $\frac{3}{2}$, and use the sum of squares of the natural numbers:

$$\sum_{n=1}^{7}\left[\frac{3}{2}(1^2+2^2+\cdots+n^2)\right]=\sum_{n=1}^{7}\left[\frac{3}{2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$$

Multiply, and move the $\frac{1}{4}$ outside, and split the summation:

$$=\frac{1}{4}\sum_{n=1}^{7}[2n^3+3n^2+n]=\frac{1}{4}\left[\sum_{n=1}^{7}2n^3+\sum_{n=1}^{7}3n^2+\sum_{n=1}^{7}n\right]$$

Use the formula for the corresponding sums:

$$=\frac{1}{4}\left[2\left(\frac{7(8)}{2}\right)^2+3\left(\frac{7(8)(15)}{6}\right)+\frac{7(8)}{2}\right]$$

Simplify, cancel the 4

$$=\frac{1}{4}[2(7\times 4)^2+28(15)+7(4)]$$

Cancel the 4

$$=[7^2\cdot 8+7(15)+7]$$

Factor out 7, and simplify:

$$=7[56+15+1]=7[72]=504$$

Example 2.130

If the sum of the first 15 terms of the series below is equal to $225k$, then k is: **(JEE Main 2019, 12 Jan, Shift-II)**

$$\left(\frac{3}{4}\right)^3+\left(1\frac{1}{2}\right)^3+\left(2\frac{1}{4}\right)^3+3^3+\left(3\frac{3}{4}\right)^3+\cdots$$

Rewrite the given series:

$$\left(\frac{3}{4}\right)^3+\left(\frac{6}{4}\right)^3+\left(\frac{9}{4}\right)^3+\left(\frac{12}{4}\right)^3+\left(\frac{15}{4}\right)^3+\cdots$$

Factor out :

$$\left(\frac{3}{4}\right)^3[1^3+2^3+\cdots+15^3]$$

Use the corresponding formula:

$$\left(\frac{3}{4}\right)^3\left[\frac{15\times 16}{2}\right]^2=\frac{27}{64}\times 15^2\times 8^2=27\times 225$$

Compare with the equality:

$$27\times 225=222k\Rightarrow k=27$$

Example 2.131

The sum of the following series up to 15 terms is: **(JEE 2019, 9 Jan, Shift-II)**

$$1+6+9\left(\frac{1^2+2^2+3^2}{7}\right)+12\left(\frac{1^2+2^2+3^2+4^2}{9}\right)+15\left(\frac{1^2+2^2+\cdots+5^2}{11}\right)+\cdots$$

We have two hanging terms at the beginning of the series. But, we see that they also follow the pattern, and we can rewrite the given series as:

$$3\left(\frac{1^2}{3}\right)+2(3)\left(\frac{1^2+2^2}{5}\right)+3(3)\left(\frac{1^2+2^2+3^2}{7}\right)+3(4)\left(\frac{1^2+2^2+3^2+4^2}{9}\right)+3(5)\left(\frac{1^2+2^2+\cdots+5^2}{11}\right)+\cdots$$

Write the general term, and use the formula for the sum of the first r squares:

$$T_r=3r\left(\frac{1^2+2^2+\cdots+r^2}{2r+1}\right)=\frac{3r}{2r+1}\left(\frac{r(r+1)(2r+1)}{6}\right)=r\left(\frac{r(r+1)}{2}\right)=\frac{1}{2}(r^3+r^2)$$

Hence, we need to find the sum:

$$\sum_{r=1}^{15} T_r = \sum_{r=1}^{15} \frac{1}{2}(r^3 + r^2) = \frac{1}{2}\left(\sum_{r=1}^{15} r^3 + \sum_{r=1}^{15} r^2\right)$$

Use the corresponding formulas:

$$= \frac{1}{2}\left\{\left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{6}\right\}_{n=15}$$

Factor $\frac{n(n+1)}{2}$ out of each term:

$$= \frac{1}{2}\left\{\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{2n+1}{3}\right]\right\}_{n=15}$$

Get to a common denominator, and add:

$$= \frac{1}{2}\left\{\frac{n(n+1)}{2}\left[\frac{3n^2+3n}{6} + \frac{4n+2}{6}\right]\right\}_{n=15} = \frac{1}{2}\left\{\frac{n(n+1)}{2}\left[\frac{3n^2+7n+2}{6}\right]\right\}_{n=15}$$

Substitute $n = 15$, and simplify:

$$= \frac{1}{24}\{15(16)[3(15)^2 + 7(15) + 2]\} = 10[675 + 105 + 2] = 10[782] = 7820$$

2

Example 2.132

111:

Example 2.133

If $1 + (1 - 2^2 \cdot 1) + (1 - 4^2 \cdot 3) + (1 - 6^2 \cdot 5) + \cdots + (1 - 20^2 \cdot 19) = \alpha - 220\beta$, then an ordered pair (α, β) is equal to: (JEE Main 2020, 4 Sep, Shift-I)

A. (11,97)
B. (10,103)
C. (10,97)
D. (11,103)

Rearrange the LHS:

$$\underbrace{1 + 1 + \cdots + 1}_{11\ Terms} - (2^2 \cdot 1 + 4^2 \cdot 3 + 6^2 \cdot 5 + \ldots + 20^2 \cdot 19)$$

Note each term in the parentheses is of the form $(2n)^2(2n-1)$:

$$= 11 - \sum_{n=1}^{10} (2n)^2(2n-1)$$

Multiply:

$$= 11 - \sum_{n=1}^{10} (8n^3 - 4n^2)$$

Split the summation:

$$= 11 - 8\sum_{n=1}^{10} n^3 + 4\sum_{n=1}^{10} n^2$$

Use the formula for the sum:

$$= 11 - 8\left(\frac{10 \cdot 11}{2}\right)^2 + 4\frac{(10)(11)(21)}{6}$$

Factor $10 \cdot 11$ from the second and the third terms:

$$= 11 - 10 \cdot 11\left[8\frac{10 \cdot 11}{4} - 4\frac{(21)}{6}\right]$$

Simplify:

$$= 11 - 110[2 \cdot 10 \cdot 11 - 14] = 11 - 110[206] = 11 - 220[103]$$

Hence, the correct answer is:

$$Option\ D$$

2.7 Polynomial Sequences

A. Linear Sequences

2.134: x vs n

By convention

- n is used for term numbers in a sequence where n is a natural number.
- A variable that can take any real number would usually be $x, y, z, \ldots$

2.135: Linear Sequence

A sequence which is also a linear function is a linear sequence.

For example:

- $T_n = 2n$ is a linear sequence
- $T_n = 4n + 9$ is a linear sequence

In general:

$$T_n = an + b$$

Is a linear sequence.

2.136: Linear Sequences are Arithmetic

$$T_n = an + b \Rightarrow T_n\ is\ arithmetic$$

$$T_n = an + b$$
$$T_{n-1} = a(n-1) + b = an - a + b$$

In order to be an arithmetic sequence, the difference of successive terms should be constant. That is:

$$T_n - T_{n-1} = (an + b) - (an - a + b) = a = Constant$$

2.137: Arithmetic Sequences are Linear

The n^{th} term of an arithmetic sequence with first term a and common difference d is:

$$T_n = a + (n-1)d = a + nd - d = nd + a - d$$

Example 2.138
Write the sequence below as a linear function:

$$7, 9, 11, \ldots$$

The general term is:

$$a + (n-1)d = 7 + (n-1)2 = 7 + 2n - 2 = 2n + 5$$

$$f(n) = 2n + 5, n \in \mathbb{N}$$

B. Quadratic Sequences

2.139: Sum of an Arithmetic Series
For an arithmetic sequence with first term a and common difference d, the sum to n terms is:

$$S_n = \frac{n}{2}[2a + (n-1)d]$$
$$a = first\ term$$
$$n = number\ of\ terms$$
$$d = common\ difference$$

Example 2.140
A. Show that the sum to n terms of the series $3 + 7 + 11 + 15 + \cdots$ is a quadratic expression.
B. What are the coefficients?

Part A
Substitute $a = 3, d = 4$ in $S_n = \frac{n}{2}[2a + (n-1)d]$:

$$S_n = \frac{n}{2}[2(3) + (n-1)4] = \frac{n}{2}[6 + 4n - 4] = \frac{n}{2}[4n + 2] = 2n^2 + n$$

Part B

$$an^2 + bn + c = 2n^2 + n + 0$$
$$a = 2, b = 1, c = 0$$

2.141: Sum of Arithmetic Sequence is Quadratic
The sum of an arithmetic sequence is a quadratic sequence.

Consider an arithmetic sequence with

$$First\ Term = a, \qquad No.of\ terms = n, \qquad Common\ Difference = d$$

The arithmetic sequence has sum:

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Distribute the RHS to get:

$$= \frac{2an}{2} + \frac{n(n-1)d}{2} = \frac{2an}{2} + \frac{dn^2 - nd}{2}$$

Simplify:

$$= an + \frac{d}{2}n^2 - \frac{dn}{2}$$

Rearrange:

$$= \frac{d}{2}n^2 + an - \frac{dn}{2}$$

Factor n from the last two terms:

$$= \frac{d}{2}n^2 + \left(a - \frac{1}{2}d\right)n$$

Note that the first term is quadratic, and the second term is linear:

$$\underbrace{\frac{d}{2}n^2}_{\substack{Quadratic\\Term}} + \underbrace{\left(a - \frac{1}{2}d\right)n}_{\substack{Linear\\Term}}$$

2.142: Sum of Arithmetic Sequence is Quadratic

The sum to n terms of an arithmetic sequence can be written:

$$S_n = An^2 + Bn$$

From the previous:

$$A = \frac{d}{2}, \qquad B = a - \frac{1}{2}d$$

Example 2.143

Consider the parabola

$$y = Ax^2 + Bx$$

A. What is the nature of the expression?
B. What is the y intercept of the above?
C. Interpret the y intercept that you found in Part A.

Part A

$$Quadratic\ Expression$$

Part B

$$y\ intercept = 0$$

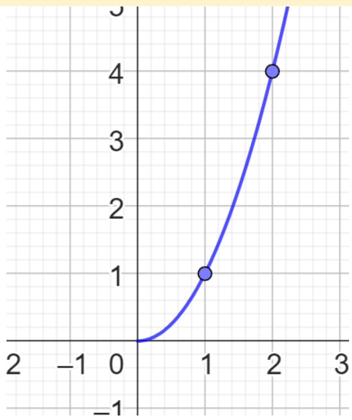
Part C

The graph passes through the origin.

Example 2.144

Consider the sum of the first n odd numbers.

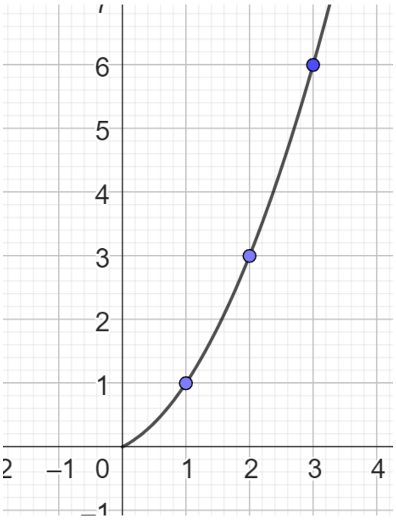
A. What is the sum?
B. Plot the parabola



Example 2.145

Consider the sum of the first n natural numbers.

A. What is the sum?
B. Plot the parabola



Example 2.146

A. What is the definition of an upward, and a downward parabola?
B. What kind of arithmetic sequence generates an upward parabola? What kind generates a downward parabola?

Part A

A parabola is given by

$$Ax^2 + Bx + C, A \neq 0$$
$$Upward\ Parabola: A > 0$$
$$Downward\ Parabola: A < 0$$

Part B

$$\underbrace{\frac{d}{2}n^2}_{\substack{Quadratic\\Term}} + \underbrace{\left(a - \frac{1}{2}d\right)}_{\substack{Linear\\Term}} n = Ax^2 + Bx + C$$

Comparing, we see that:

$$A = \frac{d}{2}$$
$$A > 0 \Rightarrow \frac{d}{2} > 0 \Rightarrow d > 0$$

If the common difference is positive, then it is an upward parabola.
If the common difference is negative, then it is a downward parabola.

Example 2.147

Find the sum to n terms of the series $3 + 7 + 11 + 15 + \cdots$ using the polynomial method

The sum is:

$$S_1 = 3$$
$$S_2 = 3 + 7 = 10$$

Let the sum to n terms be given by $f(n) = An^2 + Bn$:

$$f(1) = A + B = 3 \Rightarrow \underbrace{2A + 2B = 6}_{\textit{Equation I}}$$
$$f(2) = \underbrace{4A + 2B = 10}_{\textit{Equation II}}$$

Subtract Equation I from Equation II:

$$2A = 4 \Rightarrow A = 2 \Rightarrow B = 1$$

$$An^2 + Bn = 2n^2 + n$$

2.148: Difference Method

Given a sequence $T_1, T_2, T_3, \dots$

$$First\ difference = d_n = T_{n+1} - T_n$$
$$Second\ difference = D_n = d_{n+1} - d_n$$

Example 2.149

Find the first and the second difference of the sequence below:

$$T = \{2,6,13,23,36,52\}$$

2		6		13		23		36		52
	4		7		10		13		16	
		3		3		3		3		

C. Quadratic Expression for Sums

2.150: Connecting Terms with Sums

For any sequence, $T_n = S_n - S_{n-1}$

$$S_n - S_{n-1} = (T_1 + T_2 + \dots + T_n) - (T_1 + T_2 + \dots + T_{n-1}) = T_n$$

Example 2.151

A. If the sum to 12 terms of a sequence is 134, and the sum to 11 terms is -112, then find the twelfth term.

B. For every n the sum of n terms of an arithmetic progression is $2n + 3n^2$. The rth term is: (AHSME 1965/20)

Part A

$$T_{12} = S_{12} - S_{11} = 246$$

Part B

The sum to r terms is:

$$S_r = 2r + 3r^2$$

The sum to $r-1$ terms is:

$$\begin{aligned} S_{r-1} &= 2(r-1) + 3(r-1)^2 \\ &= 2r - 2 + 3(r^2 - 2r + 1) \\ &= 2r - 2 + 3r^2 - 6r + 3 \\ &= 3r^2 - 4r + 1 \end{aligned}$$

$$T_r = S_r - S_{r-1} = 2r + 3r^2 - (3r^2 - 4r + 1) = 6r - 1$$

Example 2.152

A. Show that the sequence with sum $S_n = 3n^2 - 11n$ is an arithmetic sequence.

$$S_{n-1} = 3(n-1)^2 - 11(n-1) = 3n^2 - 6n + 3 - 11n + 11 = 3n^2 - 17n + 14$$
$$t_n = S_n - S_{n-1} = 3n^2 - 11n - 3n^2 + 17n - 14 = 6n - 14$$

$$t_{n-1} = 6(n-1) - 14 = 6n - 20$$
$$t_n - t_{n-1} = 6n - 14 - 6n + 20 = 6$$

2.153: Finding common difference from Sum

The sum to n terms of an arithmetic progression can be written as:

$$S = \frac{n}{2}[2a + (n-1)d] = \frac{2an + n^2d - nd}{2} = \frac{n^2d + n(2a-d)}{2} = \underbrace{\left(\frac{d}{2}\right)n^2 + \left(\frac{2a-d}{2}\right)n}_{\substack{\text{Quadratic with constant term zero} \\ \text{Goes through the origin}}}$$

The common difference is twice the coefficient of the square term in a quadratic expression that represents the sum of n terms of an arithmetic progression.

Example 2.154

If the sum of n terms of an arithmetic progression is $pn + qn^2$, where p and q are constants, find the common difference. (NCERT)

The common difference is twice the coefficient of the square term in a quadratic expression that represents the sum of n terms of an arithmetic progression.

$$d = 2q$$

We can also do this using the method of undetermined coefficients:

$$qn^2 + pn \equiv \left(\frac{d}{2}\right)n^2 + \left(\frac{2a-d}{2}\right)n \Rightarrow q = \frac{d}{2} \Rightarrow d = 2q$$

D. Graphing Quadratics

2.155: Definition

Example 2.156

Show that if, for an arithmetic progression, the sum to m terms is the same as the sum to n terms, then the sum to $m+n$ terms is zero.

S_n of an AP is a parabola passing through the origin.

$S_m = S_n \Rightarrow$ By symmetry, S_{n+m} is a root of the quadratic $\Rightarrow S_{n+m} = 0$

E. General Quadratic Sequences

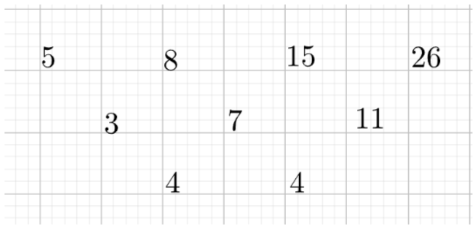
Example 2.157

Let a sequence u_n be defined by $u_1 = 5$ and the relationship $u_{n+1} - u_n = 3 + 4(n-1), n = 1,2,3 \dots$.

A. If u_n is expressed as a polynomial in n, the algebraic sum of its coefficients is: **(AHSME 1969/32)**
B. **Find u_n as a polynomial in n.**
C. The question has given a recursive definition for u_{n+1} in terms of u_n. Find u_{n+1} in terms of u_{n-1}.

Part A

$$u_1 = 5$$
$$u_2 = u_1 + 3 + 4(n-1) = 5 + 3 + 0 = 8$$
$$u_3 = u_2 + 3 + 4(n-1) = 8 + 3 + 4 = 15$$
$$u_4 = u_3 + 3 + 4(n-1) = 15 + 3 + 8 = 26$$



$$f(n) = An^2 + Bn + C$$
$$\underbrace{f(1) = A + B + C = 5}_{Equation\ I}$$

The sum of coefficients is also

$$A + B + C = 5$$

Part B

$$\underbrace{f(2) = 4A + 2B + C = 8}_{Equation\ II}$$
$$\underbrace{f(3) = 9A + 3B + C = 15}_{Equation\ III}$$

Subtract Eq. I from Eq II, and Eq. II from Eq. III

$$\underbrace{3A + B = 3}_{II-I=IV}, \qquad \underbrace{5A + B = 7}_{III-II=V}$$

Subtract Eq. IV from Eq. V:

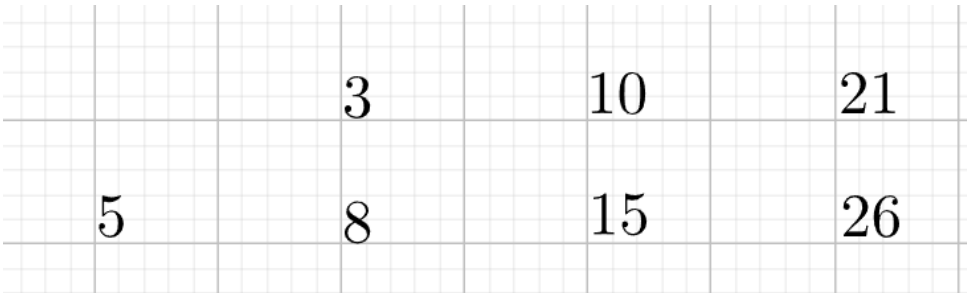
$$2A = 4 \Rightarrow A = 2$$
$$B = -3$$
$$C = 6$$
$$An^2 + Bn + C = 2n^2 - n + 6$$

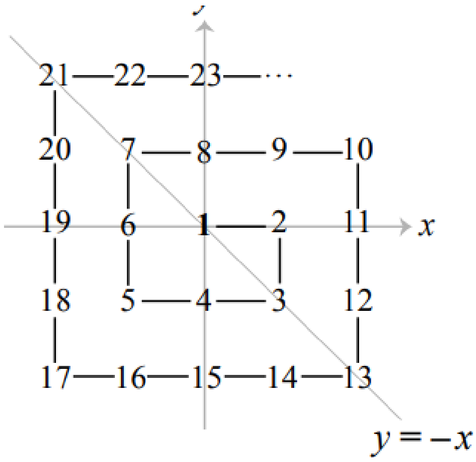
Part C

Rearrange $u_{n+1} - u_n = 3 + 4(n-1)$ to get:

$$u_{n+1} = u_n + 3 + 4(n-1)$$
$$u_{n+1} = u_{n-1} + 3 + 4(n-2) + 3 + 4(n-1)$$
$$u_{n+1} = u_{n-1} + 6 + 4(2n-3)$$

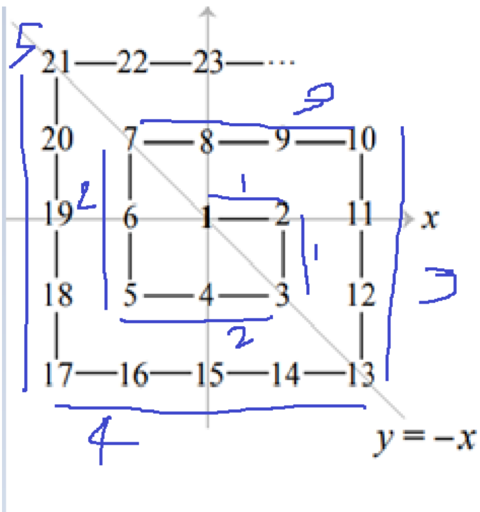
Example 2.158





(Calc) Example 2.159

The rectangular spiral shown in the diagram is constructed as follows. Starting at $(0, 0)$, line segments of lengths $1, 1, 2, 2, 3, 3, 4, 4, \ldots$ are drawn in a clockwise manner, as shown. The integers from 1 to 1000 are placed, in increasing order, wherever the spiral passes through a point with integer coordinates (that is, 1 at $(0, 0)$, 2 at $(1, 0)$, 3 at $(1, -1)$, and so on). What is the sum of all of the positive integers from 1 to 1000 which are written at points on the line $y = -x$? (CSMC 2017/A/5)



Note that from the diagram each time the length of the line segments increases by 2, and hence the pattern will also continue.

The given pattern is:

$$1,3,7,13,21,\ldots$$

Find the common difference

$$3-1=2$$
$$7-3=4$$
$$13-7=6$$
$$21-13=8$$

The sequence of common differences is

$$2,4,6,8,\ldots$$

$$T_0 = 1 = 1$$
$$T_1 = 3 = 1 + \{2\}$$
$$T_2 = 7 = 1 + \{2+4\}$$
$$T_3 = 13 = 1 + \{2+4+6\}$$

$$T_n = 1 + (2 + 4 + 6 + \cdots + 2n)$$
$$T_n = 1 + 2(1 + 2 + 3 + \cdots + n)$$
$$T_n = 1 + 2\left(\frac{n(n+1)}{2}\right)$$
$$T_n = 1 + n(n+1)$$
$$T_n = n^2 + n + 1$$

$$n^2 + n + 1 < 1000$$

We know that $32^2 = 1024$. Check $n = 31$:

$$31^2 + 31 + 1 = 993$$

$$T_0 + T_1 + T_2 + \cdots + T_{31}$$

$$\sum_{n=0}^{n=31} n^2 + n + 1 = \sum_{n=0}^{n=31} n^2 + \sum_{n=0}^{n=31} n + \sum_{n=0}^{n=31} 1$$

$$\sum_{n=0}^{n=31} 1 = 32$$

$$\sum_{n=0}^{n=31} n = \frac{n(n+1)}{2} = \frac{31 \cdot 32}{2} = 496$$

$$\sum_{n=0}^{n=31} n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{31(32)(63)}{6} = 10416$$

$$32 + 496 + 10416 = 10944$$

F. Polynomial Sequences and Cubic Sequences

Example 2.160

G. Exponential Sequences

Example 2.161

2.8 Series Expansions

A. *e*

2.162: Factorials

Example 2.163

2.164: Expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} = \sum_{n=0}^{n=\infty} \frac{x^n}{n!}$$

Example 2.165

2.9 Further Topics

Example 2.166

Let S_n be the sum of the first n terms of an arithmetic progression. If $S_{3n} = 3S_{2n}$, then the value of S_n (as a number) is:

Method I: Quadratic Sequence

Substitute $S_n = An^2 + Bn = n(An + B)$ in $S_{3n} = 3S_{2n}$:

$$3n[A(3n) + B] = 3 \cdot 2n[A(2n) + B]$$

Divide by $3n$ both sides:

$$A(3n) + B = 2[A(2n) + B]$$

Simplify and distribute:

$$3An + B = 4An + 2B$$

Collate all terms on one side:

$$\underbrace{An + B = 0}_{\textit{Equation I}}$$

Substitute Equation I in S_n:

$$S_n = n(An + B) = n(0) = 0$$

Hence, the sum to n terms is zero.

Method II: Sum to n terms

$$S_{3n} = 3S_{2n}$$

Use the formula $S_n = \frac{n}{2}[2a + (n - 1)d]$:

$$\frac{3n}{2}[2a + (3n - 1)d] = 3 \cdot \frac{2n}{2}[2a + (2n - 1)d]$$

Cancel $\frac{3n}{2}$ on both sides:

$$2a + (3n - 1)d = 2[2a + (2n - 1)d]$$
$$2a + (3n - 1)d = 4a + (4n - 2)d$$
$$\underbrace{(1 - n)d = 2a}_{\textit{Equation II}}$$

Substitute Equation II in $S_n = \frac{n}{2}[2a + (n - 1)d]$:

$$S_n = \frac{n}{2}[(1 - n)d + (n - 1)d] = \frac{n}{2}[d(1 - n + n - 1)] = \frac{n}{2}[d(0)] = 0$$

Example 2.167

6: Let S_n be the sum of the first n terms of an arithmetic progression. If $S_{3n} = 3S_{2n}$, then the value of $\frac{S_{4n}}{S_{2n}}$ (as a number) is: (JEE Main 2021, 25 July, Shift-I)

$$\frac{S_{4n}}{S_{2n}} = \frac{4n[A(4n) + B]}{2n[A(2n) + B]} = \frac{2[4An + B]}{[2An + B]} = \frac{2[3An + An + B]}{[An + An + B]} = \frac{2[3An + 0]}{[An + 0]} = \frac{2[3An]}{[An]} = 6$$

$$\frac{S_{4n}}{S_{2n}} = \frac{\frac{4n}{2}[(1 - n)d + (4n - 1)d]}{\frac{2n}{2}[(1 - n)d + (2n - 1)d]} = \frac{2[3nd]}{[nd]} = 6$$

Example 2.168

$$1 + 2 + 3 + \cdots$$

Is $S_2 = 2S_1$

$$S_1 = 1, S_2 = 3$$

Example 2.169

$$-1,0,1$$
$$S_3 = 0$$
$$S_6 = 4$$

$$-2,-1,0,1,2$$

Example 2.170

7: Let S_n denote the sum of first n terms of an arithmetic progression. If $S_{10} = 530, S_5 = 140$, then $S_{20} - S_6 =$ **(JEE Main 2021, 22 July, Shift-II)**

Using $S_n = An^2 + Bn$

$$S_{10} = \underbrace{100A + 10B = 530}_{\textit{Equation I}}$$
$$S_5 = 25A + 5B = 140 \Rightarrow \underbrace{50A + 10B = 280}_{\textit{Equation II}}$$

$$A = 5, B = 3 \Rightarrow S_n = 5n^2 + 3n$$

$$S_{20} - S_6 = 5(20^2) - 3(20) - [5(6^2) - 3(6)] = 1862$$

Example 2.171

19: Five numbers are in AP, whose sum is 25, and product is 2520. If one of these five numbers in $-\frac{1}{2}$, then the greatest number among them is: **(JEE Main 2020, 7 Jan, Shift-I)**

Let the five terms be:

$$a - 2d, a - d, a, a + d, a + 2d$$

$$Sum = 5a = 25 \Rightarrow a = 5$$

Substitute $a = 5$ in $Product = a(a^2 - d^2)(a^2 - 4d^2) = 2520$:

$$5(25 - d^2)(25 - 4d^2) = 2520$$

Divide both sides by 5 and expand:

$$4d^4 - 125d^2 + 625 = 504$$

Collate all terms on one side:

$$4d^4 - 125d^2 + 121 = 0$$

This is a disguised quadratic. Substitute $y = d^2$:

$$4y^2 - 125y + 121 = 0$$
$$4y^2 - 4y - 121y + 121 = 0$$
$$(y - 1)(4y - 121) = 0$$
$$y = 1 \ OR \ y = \frac{121}{4}$$

$$d \in \left\{\pm 1, \pm \frac{11}{2}\right\}$$

If $d = \pm 1$:

$$Sequence = \ 3,4,5,6,7 \Rightarrow Reject$$

If $d = +\frac{11}{2}$:

$$-6, -0.5, 5, 10.5, 16 \Rightarrow Largest\ Term = 16$$

If $d = -\frac{11}{2}$:

$$16, 10.5, 5, -0.5, -6 \Rightarrow Largest\ Term = 16$$

Example 2.172

20: If the sum of the first 40 terms of the series $3 + 4 + 8 + 9 + 13 + 14 + 18 + 19 + \cdots$ is $102(m)$, find the value of m:

$$7, 17, 27, \ldots \Rightarrow a = 7, d = 10$$

$$S_{20} = \frac{20}{2}[14 + 19 \times 10] = 10[14 + 190] = 2040 = 20(102) \Rightarrow m = 20$$

Example 2.173

Find the number of terms common to the sequences below:

$$Sequence: 3, 7, 11, \ldots, 407$$
$$Sequence: 2, 9, 16, \ldots, 709$$

$$3, 7, 11, 15, 19, 23, \ldots, 407 \Rightarrow a_1 = 3, d_1 = 4$$
$$2, 9, 16, 23, \ldots, 709 \Rightarrow a_2 = 2, d_2 = 7$$

The sequence of terms common to both sequences will an arithmetic sequence with

$$common\ difference = d = LCM(4,7) = 28$$
$$first\ term = 23$$

The sequence is:

$$23, 23 + 28, 23 + 2(28), \ldots \Rightarrow n^{th}\ term = 23 + (n-1)28$$

$$23 + (n-1)28 \le 407$$
$$23 + 28n - 28 \le 407$$
$$28n \le 412$$
$$n \le \frac{412}{28}$$
$$n \in \{1, 2, 3, \ldots, 14\}$$
$$14\ Terms$$

Example 2.174

24: Some identical balls are arranged in rows to form an equilateral triangle. The first row consists of one ball, the second row consists of two balls and so on. If 99 more identical balls are added to the total number of balls used in forming the equilateral triangle, then all these balls can be arranged in a square whose each side contains exactly 2 balls less than number of balls each side of the triangle contains. Then, the number of balls used to form the equilateral triangle is **(JEE Main 2019, 9 April, Shift-II)**

The number of balls in the equilateral triangle is:

$$1+2+\cdots+n=\frac{n(n+1)}{2}$$

The number of balls in the square is:

$$(n-2)^2$$

Since from

$$\frac{n(n+1)}{2}+99=(n-2)^2$$
$$n^2+n+198=2(n^2-4n+4)$$
$$n^2-9n-190=0$$
$$n\in\{19,-10\}$$
$$n=19$$
$$\frac{n(n+1)}{2}=\frac{19\times 20}{2}=190$$

Example 2.175

8:The sum of all the elements in the set {n\in(1,2,...,100)}: HCF of n and 2040 is 1} is equal to:

176 Examples