Discrete Geometric Combinatorics

Revision: 1267

16 February 2025

Aziz Manva

azizmanva@gmail.com

Table of contents

1. 2D Discrete Geometry

Example 1.1
How many triangles can be formed from the vertices of a heptagon?

Overcounting
To make a triangle, you need to select three of the seven vertices. You can do this in

$$\underbrace{7}_{\substack{First\\Vertex}} \times \underbrace{6}_{\substack{Second\\Vertex}} \times \underbrace{5}_{\substack{Third\\Vertex}} \quad ways$$

But, this overcounts by the number of ways the vertices can be arranged among themselves. So, correct for this by dividing by 3!:

$$\frac{7 \times 6 \times 5}{3!} = \frac{7 \times \mathbf{6} \times 5}{\mathbf{6}} = 35$$

Combinations
From the seven vertices, we need to choose any three:

$$7\ Choose\ 3 = \binom{7}{3} = \frac{7 \times 6 \times 5}{3!} = \frac{7 \times \mathbf{6} \times 5}{\mathbf{6}} = 35$$

Example 1.2
Ten distinct points are identified on the circumference of a circle. How many different convex quadrilaterals can be formed if each vertex must be one of these 10 points? **(MathCounts 2005 Warm-Up 7)**

Overcounting
To make a quadrilateral, you need to select four of the ten vertices. You can do this in

$$\underbrace{10}_{\substack{First\\Vertex}} \times \underbrace{9}_{\substack{Second\\Vertex}} \times \underbrace{8}_{\substack{Third\\Vertex}} \times \underbrace{7}_{\substack{Third\\Vertex}} \quad ways$$

But, this overcounts by the number of ways the vertices can be arranged among themselves. So, correct for this by dividing 4!:

$$\frac{10 \times 9 \times 8 \times 7}{4!}$$

Combinations
From the ten vertices, we need to choose any four:

$$\binom{10}{4} = \frac{10!}{4!\,6!} = \frac{10 \times 9 \times 8 \times 7}{4!}$$

1.1 Counting Shapes

A. Counting Squares

There isn't much interesting about the geometry of squares that leads to interesting counting questions. So, we focus on standard counting topics to make the question more complicated.
These counting concepts can be mixed with others to create more complex questions.

1.3: Square
A square is a quadrilateral with all sides equal and all angles 90°.

Example 1.4
How many distinct shaped squares can you form with perimeter 100 units:

A. and integral side lengths?
B. or less and integral side lengths?
C. or more and integral side lengths?

$$P = 100 \Rightarrow 4s = 100 \Rightarrow s = 25$$
$$P \le 25 \Rightarrow 4s \le 100 \Rightarrow s \le 25 \Rightarrow s \in \{1,2, \dots ,24,25\}$$
$$P \ge 25 \Rightarrow 4s \ge 100 \Rightarrow s \ge 25 \Rightarrow s \in \{25,26, \dots\} \Rightarrow Infinite$$

Example 1.5

How many distinct shaped squares can you form with perimeter 16 units, if, for each square, you can decide:

A. The color of the line segment: Any one from green, blue, orange, and brown
B. The style of the line segment: Any one from *dashed*, *solid* and *dotted*

$$Original\ Answers \times \underbrace{4}_{Colors} \times \underbrace{3}_{Style} = Original\ Answers \times 12$$

Example 1.6

How many distinct shaped squares can you form with perimeter 16 units, if, for each side of the square, you can decide:

A. The color of the line segment: Any one from green, blue, orange, and brown
B. The style of the line segment: Any one from *dashed*, *solid* and *dotted*

$$Original\ Answers \times \underbrace{4}_{Sides} \times \underbrace{4}_{Colors} \times \underbrace{3}_{Style} = Original\ Answers \times 48$$

Example 1.7

How many distinct shaped squares can you form with perimeter 16 units, if for each vertex, you can decide:

A. The color of the vertex: Black, grey, yellow
B. The style of the vertex: Filled, empty, dotted, hatched

Example 1.8

How many distinct shaped squares can you form with perimeter 16 units, if for the area inside the square, you can decide if it is one of four options: plain, shaded, dotted, and hatched.

1.9: Order of Symmetry

The order of symmetry of an object is the number of times it looks the same when rotated around its center.

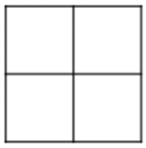
Example 1.10

A. What is the order of symmetry for a square?
B. If you rotate a square, after how many degrees does it look the same?

$$4$$
$$\frac{360}{4} = 90^\circ$$

1.2 Geometrical Counting

A. Counting Squares

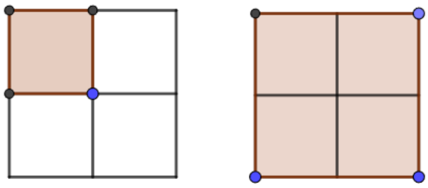


Example 1.11

Count the number of squares (of any size) in the diagram alongside.

Squares of size 1. We should be able to get 4 such squares.

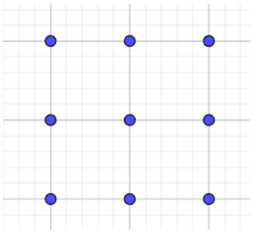
$$Top\ Left, Top\ Right, Bottom\ Left, Bottom\ Right$$



Squares of size 2. We will get a single such square.

Total number of squares is:

$$4+1=5$$

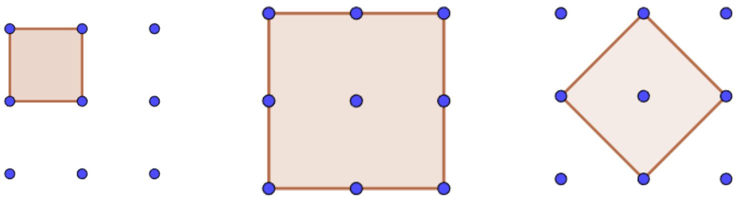


Example 1.12

The diagram alongside shown 9 lattice points. Lattice points means that they are equally spaced on a coordinate plane.
Count the number of squares that can be formed such that four of the lattice points form the four vertices of the square.

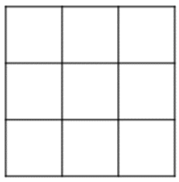
Squares of

$$Size\ 1 = 4$$
$$Size\ 2 = 1$$
$$Diagonal\ segments = 1$$



Total

$$= 4+1+1 = 6$$



Example 1.13

Count the number of squares (of any size) in the diagram alongside.

Squares of size 1

$$9$$

Squares of size 2:

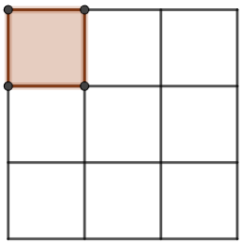
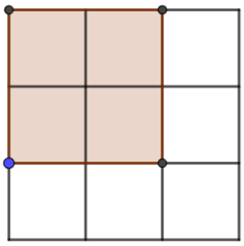
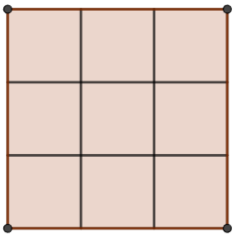
$$4$$

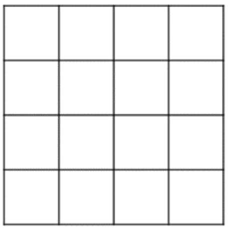
Squares of size 3:

$$1$$

Total

$$9 + 4 + 1 = 14$$

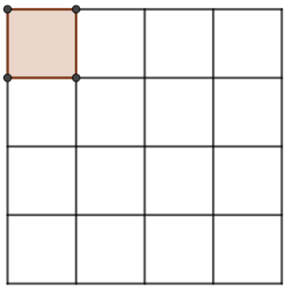
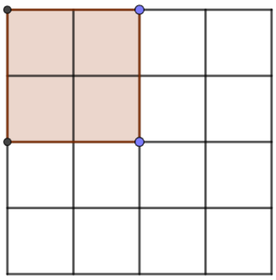
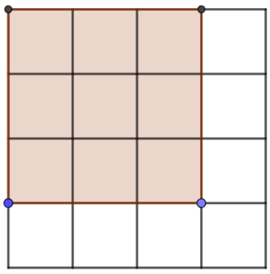


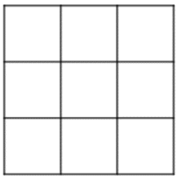
Example 1.14
Count the number of squares (of any size) in the diagram alongside.

Squares of size:

$$\textit{Size } 1: 16$$
$$\textit{Size } 2: 9$$
$$\textit{Size } 3: 4$$
$$\textit{Size } 4: 1$$

$$= 16 + 9 + 4 + 1 = 30$$



Example 1.15
Using the above examples generalize to find the number of squares in a for a $n \times n$ grid of the type alongside. The diagram has 3×3, but we want a general answer.

Squares of size n

$$1$$

Squares of size $n - 1$

$$2^2$$

Squares of size $n - 2$

$$3^2$$

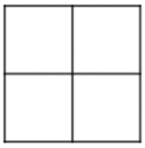
And so on until squares of size n

$$n^2$$

Add it all to get:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

B. Counting Rectangles



Example 1.16

Count the number of rectangles in the diagram alongside.

First, count the number of squares:

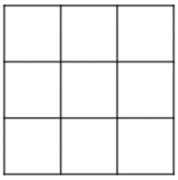
$$Size\ 1 \times 1 = 4$$
$$Size\ 2 \times 2 = 1$$

Then, count the number of rectangles:

$$Size\ 1 \times 2 = Size\ 2 \times 1 = 2$$

Total

$$= 4 + 1 + 2 + 2 = 9$$



Example 1.17

Count the number of rectangles in the diagram alongside.

First, count the number of squares:

$$Size\ 1 \times 1 = 9$$
$$Size\ 2 \times 2 = 4$$
$$Size\ 3 \times 3 = 1$$

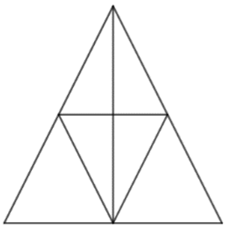
Then, count the number of rectangles:

$$Size\ 1 \times 2 = Size\ 2 \times 1 = 6$$
$$Size\ 1 \times 3 = Size\ 3 \times 1 = 3$$
$$Size\ 2 \times 3 = Size\ 3 \times 2 = 2$$

Total

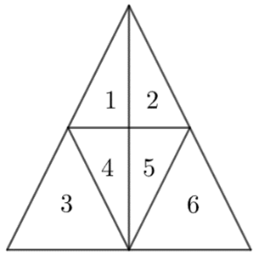
$$= 9 + 4 + 1 + 6(2) + 3(2) + 2(2) = 36$$

C. Counting Triangles



Example 1.18

The diagram alongside consists of a larger triangle split into smaller ones. Count the number of triangles of any size in the diagram.



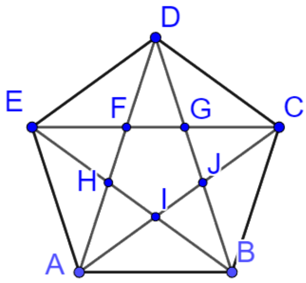
Triangles made of

$$One\ Number: 1,2,3,4,5,6 \Rightarrow 6\ Triangles$$
$$Two\ Numbers: 12, 14,25,45 \Rightarrow 4\ Triangle$$
$$Three\ Numbers: 143, 256 \Rightarrow 2\ Triangle$$
$$Four\ Numbers: 123456 \Rightarrow 1\ Triangle$$

The total number of triangles is:

$$6 + 4 + 2 + 1 = 13\ Triangles$$

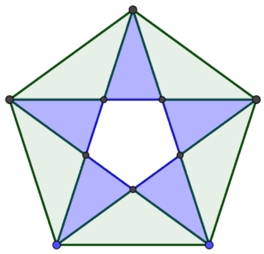
D. Geometrical Counting



Example 1.19

The diagram alongside shows a regular pentagon with its diagonals drawn. Determine the number of triangles in the figure.

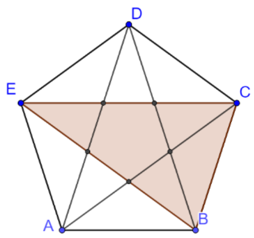
Case I: 5 Blue and 5 Green Triangles

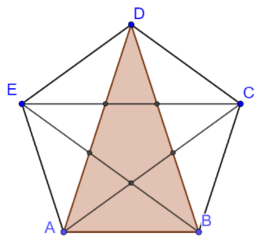


The blue triangles have their apex at a vertex of the pentagon, and their base is part of the diagonals. There are five of them (see diagram).
The green triangles have a side of the pentagon as their base, and the nearest vertex of the inner pentagon as the apex.

Case II: 5 Brown Triangles

These triangles have a side of the pentagon as a base, and a vertex of the pentagon as the apex.





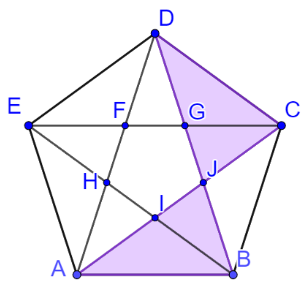
Two of them are shown in the diagram alongside. In all, these are:

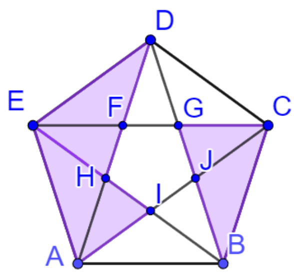
$$\Delta ECB, \Delta DAB, \Delta CEA, \Delta BDE, \Delta ADC$$

Case III: 5 Violet Triangles

These triangles have one side of the pentagon as the base, and one vertex of the inner pentagon $FGJIH$ as the third vertex.
The left diagram shows two of the five triangles, and the right diagram shows the other three.
In all, they are:

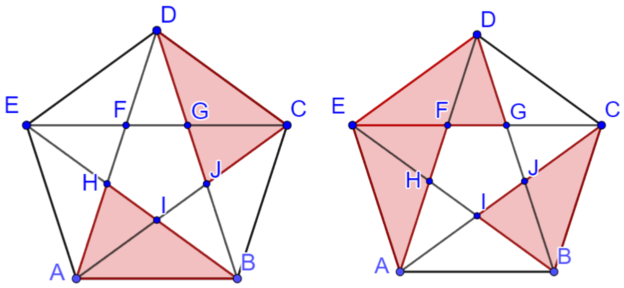
$$\Delta EIA, \Delta AJB, \Delta IBC, \Delta JCD, \Delta DGE$$





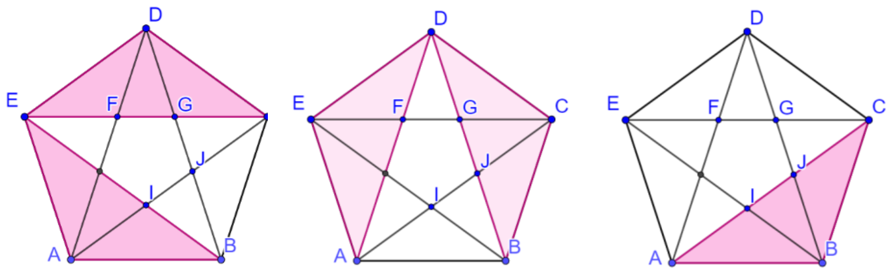
Case IV: 5 Orange Triangles

Like the violet triangles, these triangles have one side of the pentagon as the base, and one vertex of the inner pentagon $FGJIH$ as the third vertex. They are different from the violet triangles because they connect to the vertex on the left, rather than the vertex on their right. The diagram on the left shows two of the five triangles, and the one on the right shows the other three.



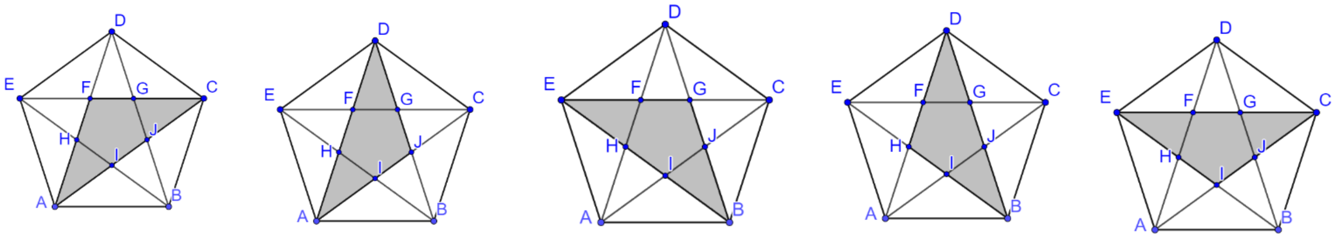
Case V: 5 Pink Triangles

There are 5 triangles with a diagonal of the pentagon as a base, and a vertex of the pentagon as the apex. These are shown in the diagram.



Case VI: 5 Grey Triangles

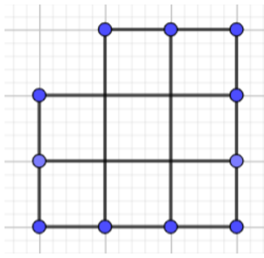
The grey triangles (like the pink triangles) also have a diagonal as the base. However, their apex is on a vertex of the inner pentagon.



$$10 \times 1 + 5 \times 5 = 10 + 25 = 35$$

Example 1.20

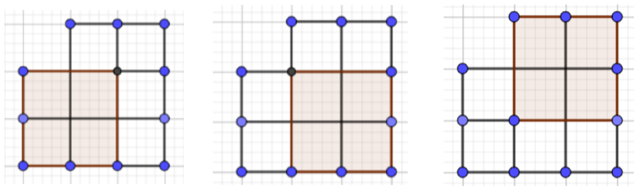
How many squares are there altogether in this diagram? **(NMTC Primary/Screening 2005/13)**



Case I: Squares of Side Length 1

There are 8 such squares.

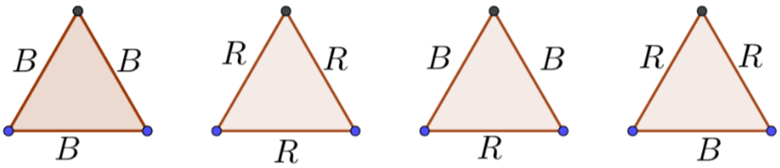
Case II: Squares of Side Length 2



The final answer is

$$8 + 3 = 11 \text{ } Squares$$

Example 1.21



If we have sticks of the same color and same length, we can make one triangle using them. If we have sticks of same length but two different colors, say blue and red, we can make four triangles as shown in the diagram. List out (and count) the triangles can be formed using sticks of same length, but three different colors, say Red, Blue and Green. **(NMTC Primary/Final, Primary-III)**

First, note from the diagram:

- All colors do not need to be used. The first triangle uses only blue colors.
- The third triangle BBR can be rotated to give a different triangle, but this is not counted. Hence, triangles that look the same under rotation are counted as one (and not two triangles).

We split this into cases:

Case I: Triangles using exactly one color:

$$\{BBB, GGG, RRR\} \Rightarrow 3\ Triangles$$

Case II: Triangles using exactly two colors:
One color has to occur twice, and one color has to occur once:

$$\{BBG, BBR, GGB, GGR, RRB, RRG\} \Rightarrow 6\ Triangles$$

Case III: Triangles using exactly three colors:

$$\{RGB\} \Rightarrow 1\ Triangle$$

$$3 + 6 + 1 = 10$$

Example 1.22: Coloring

Each square in a 2×2 table is colored either black or white. List out (and count) the different colorings of the table. **(NMTC Primary/Final/2004/11)**

Note that the table has four cells. We build up the solution by considering smaller cases first.

Case I:
Suppose there were only one cell:

$$\{B, W\} \Rightarrow 2\ Options$$

Case II:
If there are two cells, the first cell can be either black or white, and the second cell can make use of the options listed above.

$$\{BB, BW, WB, WW\} \Rightarrow 4\ Options$$

Case III:
The first cell can be black or white and the other two cells will have the same choices as Case II

$$\{BBB, BBW, BWB, BWW, WBB, WBW, WWB, WWW\} \Rightarrow 8\ Options$$

Finally, consider there are four cells. The first cell can be black, or can be white:

$$\{BBBB, BBBW, BBWB, BBWW, BWBB, BWBW, BWWB, BWWW\}$$
$$\{WBBB, WBBW, WBWB, WBWW, WWBB, WWBW, WWWB, WWWW\}$$

$$16\ Options$$

1.23: Rectangle

A rectangle is a quadrilateral with all angles 90°.

Example 1.24

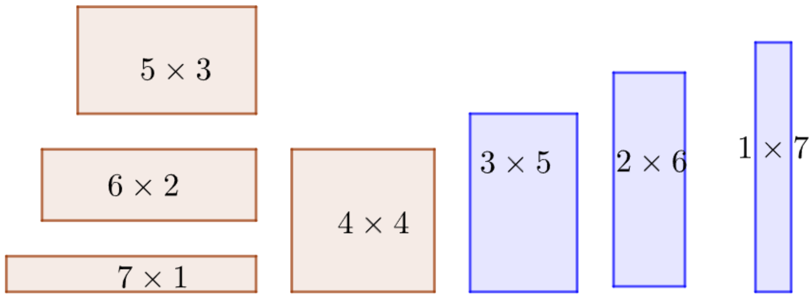
How many distinct rectangles can you make with perimeter 16 units and natural number side lengths? There are two cases here. Rectangles that look the same after rotation can be considered distinct. Or they can be considered the same. Answer both cases.

Case I: Rotations are distinct

Let

$$length = a, width = b$$

$$P = 16 \Rightarrow 2(a+b) = 16 \Rightarrow a + b = 8$$
$$(a, b) = (1,7), (2,6), (3,5), \ldots, (7,1) \Rightarrow 7\ Rectangles$$



Case II: Rotations are the same rectangle

$$(1,7), (2,6), (3,5), (4,4), \mathbf{(5,3), (6,2) (7,1)} \Rightarrow 7 - 3 = 4$$

Example 1.25

How many distinct rectangles can you make with perimeter 17 units and natural number side lengths? (Information regarding rotation is *missing*……Why so?)

$$P = 17 \Rightarrow 2(a+b) = 17 \Rightarrow a + b = \frac{17}{2}$$

$RHS = \frac{17}{2}$ is a fraction.
This means the LHS must also be a fraction.

If I add two natural numbers, the result is always a natural number.
Hence, there are no solutions to the above.

$$(a, b) = \phi \Rightarrow\ No\ Solutions$$

Example 1.26

Use the answer to the previous example to count the number of rectangles with integral sides for a perimeter of

n units, where n is a real number.
Note: Assume rotations are distinct.

Case I: n is not an even number

$$\textit{Not Possible}$$

Case II: n is even

$$P = n \Rightarrow 2(a+b) = n \Rightarrow a+b = \frac{n}{2}$$

If rotations are distinct, then:

$$\left(1, \frac{n}{2}-1\right)\left(2, \frac{n}{2}-2\right), \dots \left(\frac{n}{2}-1, 1\right) \Rightarrow \frac{n}{2}-1 \ Rectangles$$

E. Rectangles: Restrictions

Example 1.27

How many rectangles can you make with perimeter 20 units and side lengths odd integers?

Example 1.28

How many rectangles can you make with perimeter 24 units and side lengths even integers?

F. Rectangles: Inequalities

Example 1.29

How many rectangles can you make with perimeter 16 units or less and integral side lengths.

$$P = 4 \Rightarrow (1,1) \Rightarrow 1 \ Rectangle$$
$$P = 6 \Rightarrow (1,2), (2,1) \Rightarrow 2 \ Rectangle$$
$$P = 8 \Rightarrow (1,3), (2,2)(3,1) \Rightarrow 3 \ Rectangle$$
$$\vdots$$
$$P = 16 \Rightarrow (1,7), (2,6), \dots, (7,1) \Rightarrow 7 \ Rectangles$$

$$Total = 1 + 2 + \dots + 7 = \frac{7 \times 8}{2} = 28$$

$$a + b < 8$$
$$a' + b' < 7$$
$$a' + b' \le 6$$
$$a' + b' + c' = 6$$

$$\binom{8}{2} = \frac{7 \times 8}{2} = 28$$

G. Counting Triangles

Example 1.30

One angle of a triangle is 60°. The other two angles have integer measures. Find the number of such triangles if the angles cannot all have the same measure.

Note: Consider triangles that are the same under rotation as the same triangle.

We know that one angle of the triangle is 60°. Hence, the other two angles must add up to:

$$180 - 60 = 120°$$

If the angles had been equal, they would have been:

$$60 + 60 = 120$$

But the angles cannot be equal. Hence, the above is not a possibility.

Now, find the possibilities. An angle in a triangle cannot be 0°. Hence, the smallest value that an angle can take is:

$$1° \Rightarrow Other\ Angle = 119°$$

We can tabulate all the possibilities like this:

First Angle	1°	2°	3°	.	.	.	59°
Second Angle	119°	118°	117°	.	.	.	61°

And hence, the number of triangles is:

$$59$$

Note: Since two triangles that are the same under rotation are considered the same, we do not need to consider the cases where the second angle is less than 60.

Example 1.31

One angle of a triangle is 60°. The other two angles have integer measures. Find the number of such triangles if the triangle is acute.

Note: Consider triangles that are the same under rotation as the same triangle.

We cannot have a right or an obtuse angle in the triangle. Hence, the possibilities are:

First Angle	31°	32°	33°	.	.	.	60°
Second Angle	89°	88°	87°	.	.	.	60°

And hence, the number of triangles is

$$60 - 31 + 1 = 30$$

Example 1.32

One angle of a triangle is 60°. The other two angles have integer measures. Find the number of such triangles if the other two angles are both even.

Note: Consider triangles that are the same under rotation as the same triangle.

First Angle	2°	4°	.	.	.	60°
Second Angle	118°	116°	.	.	.	60°

And hence we have to count the number of numbers in the list:

$$2,4,6, \dots, 60$$

Divide each number by 2 to get:

$$1,2,3, \dots, 30 \Rightarrow 30\ Numbers \Rightarrow \mathbf{30\ Triangles}$$

Example 1.33

One angle of a triangle is 60°. The other two angles have integer measures. Find the number of such triangles if the other angles are both odd.

Note: Consider triangles that are the same under rotation as the same triangle.

Use complementary counting

$$\underbrace{60}_{\substack{Total\\Triangles}} - \underbrace{30}_{\substack{Even\\Angles}} = \mathbf{30\ Triangles\ with\ Two\ Odd\ Angles}$$

First Angle	1°	3°	.	.	.	59°
Second Angle	119°	117°	.	.	.	61°

And hence we have to count the number of numbers in the list:

$$1,3,5, \dots, 59$$

Add 1 to each number to get:

$$2,4,6, \dots, 60$$

Divide each number by 2 to get:

$$1,2,3, \dots, 30 \Rightarrow 30\ Numbers \Rightarrow \mathbf{30\ Triangles}$$

Example 1.34

One angle of a triangle is 60°. The other two angles have integer measures. Find the number of such triangles if the other two angles are both even, and all angles are acute.

Note: Consider triangles that are the same under rotation as the same triangle.

First Angle	32°	34°	.	.	.	60°
Second Angle	88°	86°	.	.	.	60°

And hence we have to count the number of numbers in the list:

$$32,34, \dots, 60$$

Divide each number by 2 to get:

$$16,17, \dots, 30 \Rightarrow 30 - 16 + 1 = 15\ Numbers \Rightarrow \mathbf{15\ Triangles}$$

H. Semiperimeter

1.35: Longest side less then semiperimeter

In a triangle if a is the longest side (and the other sides are b and c):

$$a < s, \qquad s = \frac{a+b+c}{2}$$

The longest side in a triangle is less than the semiperimeter.

$$a < \frac{a+b+c}{2}$$

Multiply both sides of the given inequality by 2:

$$2a < a + b + c$$

Subtract a from both sides:

$$a < b + c$$

And the above is the triangle inequality (which is true), and the steps are reversible.
Hence, the original inequality is also true.

Example 1.36

You want to make a triangle which uses exactly n matchsticks. Matchsticks are all the same. You cannot put a matchstick on top of another. How many different triangles can you make?
Write the answer for $n = 1,2,3, \ldots$

Note: Triangles which are the same under rotation are considered the same.

$$1 \text{ } or \text{ } 2 \text{ } or \text{ } 4 \text{ } Matchsticks: Zero \text{ } Triangles$$

With 3 matchsticks, you can make a triangle with each side length 1:

$$1 - 1 - 1 \Rightarrow 1 \text{ } Triangle$$

With 5 matchsticks, you can make a triangle with side lengths 2,2 and 1:

$$5 \text{ } Matchsticks: 2 - 2 - 1 \Rightarrow 1 \text{ } Triangle$$

Similarly:

$$6 \text{ } Matchsticks: 2 - 2 - 2 \Rightarrow 1 \text{ } Triangle$$
$$7 \text{ } Matchsticks: 3 - 2 - 2 \Rightarrow 1 \text{ } Triangle$$

1.37: Triangle Inequality

$$a < b + c$$

Example 1.38

How many different isosceles triangles have integer side lengths and perimeter 23? (2005 AMC8)

$$(1,1,21) \Rightarrow 1 + 1 = 2 < 21 \Rightarrow Not \text{ } Valid$$
$$\vdots$$
$$(5,5,13) \Rightarrow 5 + 5 = 10 < 13 \Rightarrow Not \text{ } Valid$$
$$(6,6,11) \Rightarrow 6 + 6 = 12 > 11, 11 - 6 = 5 < 6 \Rightarrow Valid$$
$$(7,7,11) \Rightarrow 6 + 6 = 12 > 11, 11 - 6 = 5 < 6 \Rightarrow Valid$$
$$(8,8,11) \Rightarrow 6 + 6 = 12 > 11, 11 - 6 = 5 < 6 \Rightarrow Valid$$
$$(9,9,11) \Rightarrow 6 + 6 = 12 > 11, 11 - 6 = 5 < 6 \Rightarrow Valid$$
$$(10,10,11) \Rightarrow 6 + 6 = 12 > 11, 11 - 6 = 5 < 6 \Rightarrow Valid$$
$$(11,11,1) \Rightarrow 6 + 6 = 12 > 11, 11 - 6 = 5 < 6 \Rightarrow Valid$$

1.3 Intersections

Our objective in this section is to apply counting techniques to geometrical objects. Along the way, we will learn/revise a few geometrical concepts, especially when dealing with three dimensional objects. The focus in this section is as much on the geometry as it is on the counting. The ability to visualize complex three-dimensional shapes is both useful and necessary for this section.

§1.3.A Counting with lines and line segments

1.39: Number of Line Segments formed by n distinct points

The number of line segments that can be formed by n distinct points on a line is:

$$\frac{n(n-1)}{2}$$

This is the same as the number of handshakes that can take place n people, since

- Every point can get to remaining $n-1$ points giving us
$$n-1 \textit{ line segments}$$
- Line AB is the same as line segment BA. Hence, to find the final answer, we need to divide by 2, giving us:
$$\frac{n(n-1)}{2}$$

Example 1.40: Counting Line Segments

A. Find the number of line segments that can be formed in a line that distinct three points on it.
B. Find the number of distinct line segments that can be formed in a line with ten points on it. A single point can be used in more than one line segment.
C. I have n points, which form 21 distinct line segments when they are connected to each other. Find the range of values that n can take.

Part A

Method I: Enumeration

Let the points on the line be

$$A, B, C$$

Then the line segments which can be formed are:

	AB	AC	BC
Repeat	BA	CA	CB

Method II: Counting

Each point can form a line segment with any of the other two points. By the multiplication principle this gives us

$$3 \times 2 = 6 \textit{ points}$$

But, this overcounts the number of line segments by a factor of two, since line segment AB is the same as line segment BA. Hence, the actual number of line segments is:

$$\frac{6}{2} = 3$$

Part B

Method I: Enumeration

Let the points on the line be $A, B, C, \dots, J$. Then the line segments which can be formed are:

$AB, AC, \dots, AJ$	$BC, BD \dots, BJ$	.	.	.	IJ
9	8	.	.	.	1

$$1 + 2 + \dots + 9 = \frac{9 \times 10}{2} = 45$$

Method II: Counting

Every point can be connected to every other point. Each of the ten points can be connected to remaining nine points:

$$9 \times 10 = 90$$

But, this overcounts the number of line segments by a factor of two, since line segment AB is the same as line segment BA. Hence, the actual number of line segments is

$$\frac{90}{2} = 45$$

Part C

The minimum number of points needed to form 21 line segments is:

$$\frac{n(n-1)}{2} = 21 \Rightarrow n(n-1) = 42 \Rightarrow n = 7$$

If the points are not distinct, then you can have more than 7 points. Hence, the range of values that n can take is:

$$\{7,8,9,\ldots\}$$

§1.3.B Number of Intersections

1.41: Number of Intersections formed by n lines

Maximum intersections among n lines is:

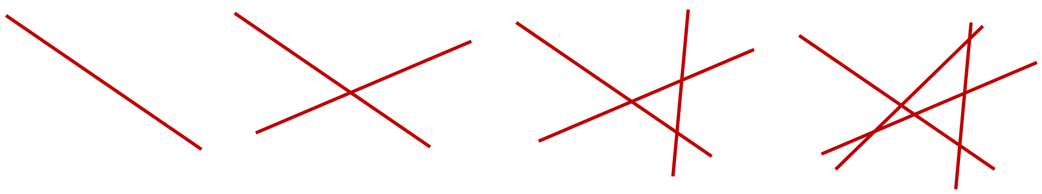
$$\frac{n(n-1)}{2}$$

The intersections will be maximum when:

- The lines are distinct
- No three lines are concurrent

Example 1.42

Arnav draws five lines in a plane. Find, at each stage of the drawing, the maximum number of intersections of these lines.



Finding a Pattern

		Handshakes / Intersections	
		Additional	Total
1st Line: Cannot intersect with any other line ⇒ No points of intersection	One Person: Cannot shake hands with anyone. Hence, there are no handshakes	0	0
2nd Line: Can intersect with the first line in a maximum of one place.	Two People: The second person can shake hands with the first person, and we are done, giving us one handshake	1	1
3rd Line: Can intersect with the earlier two lines in a maximum of two places.	Three People: The third person can shake hands with the earlier two people, and we are done, giving us two additional handshakes	2	3

4th Line: Can intersect with the earlier three lines in a maximum of three places.	Four People: The fourth person can shake hands with the earlier three people, and we are done, giving us three additional handshakes	3	6

Getting a Formula

$$1+2+3+\cdots+(n-1)=\frac{n(n-1)}{2}$$

1.43: Bijection Principle

Consider events X and Y. If the number of events in X is the same as the number of events in Y, then you can count either, and they will both be the same.

The bijection principle is useful in counting. If you wish to count something complicated, and you are able to relate to it as the same as something much easier to count, then your work is reduced.

Example 1.44: Comparing with Handshakes

Explain why the maximum number of intersections among n lines in a plane is also the maximum number of handshakes among n people in a room.

Lines	Intersections	People	Handshakes
1	0	1	0
2	1	2	1
3	3	3	3
4	6	4	6
5	10	5	10

If you have n people in a room, the handshakes you have are:

$$\underbrace{(n-1)}_{First\ Person}+\underbrace{(n-2)}_{Second\ Person}+\cdots+2+1+\underbrace{0}_{n^{th}\ Person}=\frac{n(n-1)}{2}$$

If you have n people in a room, the handshakes you have are:

$$\underbrace{(n-1)}_{First\ Person}+\underbrace{(n-2)}_{Second\ Person}+\cdots+2+1+\underbrace{0}_{n^{th}\ Person}=\frac{n(n-1)}{2}$$

Example 1.45

(For this question, if two lines are the same line, they will overlap completely. Do not count it as an intersection).

A. I have 10 lines. What is the number of maximum number of intersections of these lines? What is the minimum number of intersections of these lines?
B. Math City has eight streets, all of which are straight. No street is parallel to another street. One police officer is stationed at each intersection. What is the greatest number of police officers needed? **(MathCounts 2008 National Countdown)**
C. **I have n lines, which intersect at exactly 4 distinct places. What is the range of values that n can take? Draw a diagram illustrating one way this is possible?**

Part A

$$Max=\frac{10\times 9}{2}=45$$
$$Min=0$$

Part B

We need a police officer at each intersection, so we need to count the number of intersections.

At max, each street can intersect the remaining seven streets.

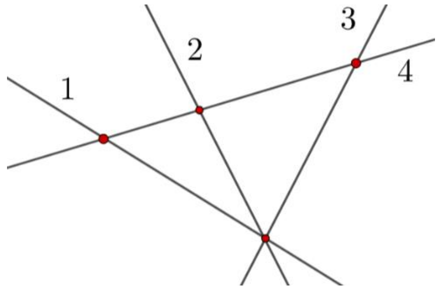
$$8 \times 7$$

But, Street A intersecting Street B is the same Street B intersecting Street A. Hence, we are overcounting by a factor of 2:

$$\frac{8 \times 7}{2} = 28$$

Part C

$$n = 3 \Rightarrow \frac{n(n-1)}{2} = \frac{3 \times 2}{2} = 3$$
$$n = 4 \Rightarrow \frac{n(n-1)}{2} = \frac{4 \times 3}{2} = 6$$



§1.3.C Number of Regions

1.46: Number of Regions that n lines divide a plane into

The maximum number of regions that n lines can divide a plane into is given by:

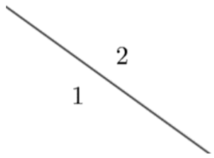
$$\frac{n(n+1)}{2} + 1 = n^{th}\ Triangular\ Number + 1$$

Example 1.47

Arnav draws five lines in a plane. Find, at each stage of the drawing, the maximum number of areas that the plane will be divided into.

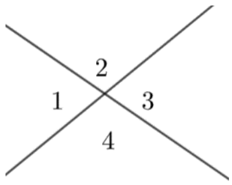
First Line

If you draw a single line, it divides the plane region into two distinct areas, numbered 1 and 2 in the diagram.



Second Line

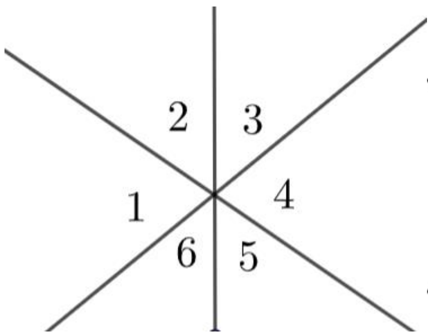
If the second line is the same as the first line, then the number of regions remains two. However, if the line is distinct from the first line, then we get four regions.

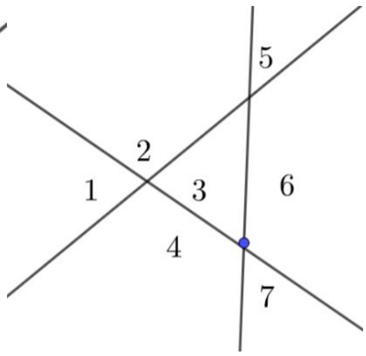


Third Line

There are two possibilities here.

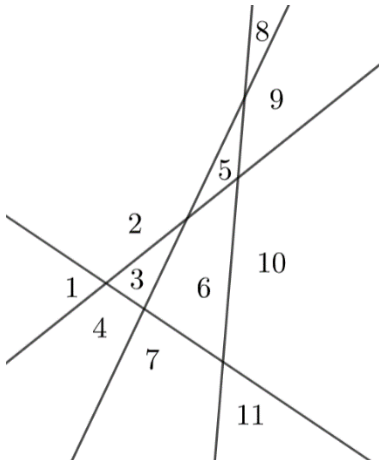
- ➢ The third line intersects at the point of intersection of the first two lines. This gives us six distinct areas. (Diagram on the left side).
- ➢ The third line does not intersect at the intersection point of the other two lines. This gives us seven distinct areas. (Diagram on the right side).





Fourth Line

Similar to the third line, we draw the fourth line so that it does not cross any of the intersections of the other two lines. This gives us four additional regions that the plane is divided into.



Getting a Formula

We can now form a pattern with what we have seen so far:

	Number of Areas	
Lines	Additional	Total
0	1	1
1	1	2
2	2	4
3	3	7
4	4	11
5	5	16

$$1 + (1 + 2 + 3 + 4 + \cdots + n) = \frac{n(n+1)}{2} + 1$$

Example 1.48

A. Find the maximum number of regions that can be formed by seven lines cutting a circle.
B. Find the difference between the minimum and the maximum number of regions formed by five lines in a plane.
C. n lines in a Cartesian plane divide it into 56 regions. Find the minimum and the maximum value of n.

Part A

$$n = 7 \Rightarrow \frac{n(n+1)}{2} + 1 = \frac{7 \times 8}{2} + 1 = 28 + 1 = 29$$

Part B

$$Max: n = 5 \Rightarrow \frac{n(n+1)}{2} + 1 = \frac{5 \times 6}{2} + 1 = 15 + 1 = 16$$

The minimum number of regions will be formed when the five lines are all the same line, and hence, we get:

$$Min: 2\ Regions$$
$$Difference = 16 - 2 = 14$$

Part C

Minimum number of lines:

$$\frac{n(n+1)}{2} + 1 = 56 \Rightarrow n(n+1) = 110 \Rightarrow n = 10$$

Maximum number of lines

$$= \infty$$

D. Points of Intersection

Example 1.49

What is the maximum number of points of intersection between 10 circles and 10 lines where the circles and lines are distinct? (MA Theta, 2018, Open, Counting and Probability/17)

The maximum number of intersection points created by n intersecting lines is $n(n-1)/2$ and the maximum number of intersection points created by m intersecting circles is $m(m-1)$. (These can be proved by induction.) The maximum number of intersection points among n lines and m circles is $2mn$, since each line falls across each circle twice. Thus the maximum number of intersection points is $n(n-1)/2 + m(m-1) + 2mn$. When $m = n = 10$, this becomes $45 + 90 + 200 = 335$

1.4 Circles; Diagonals in a Polygon

A. Points on a Circle

1.50: Points on a Circle

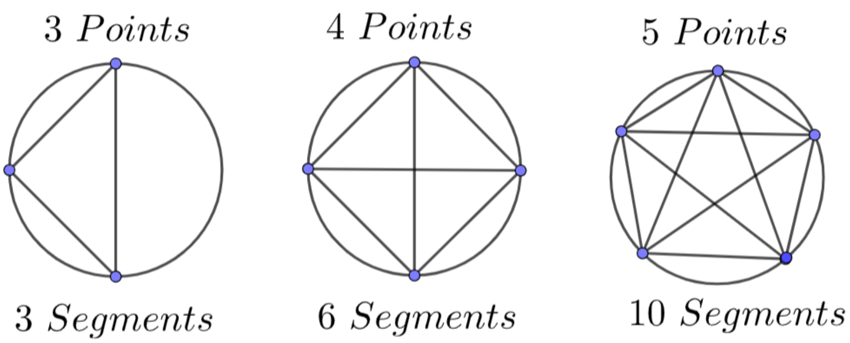
The maximum number of connecting line segments that can be formed by points on a circle is:

$$\frac{No.of\ Points \times No.of\ \frac{Connections}{Point}}{2} = \frac{n(n-1)}{2}$$

Example 1.51

Draw a circle, and count the maximum number of line segments that can be formed if the circle has:

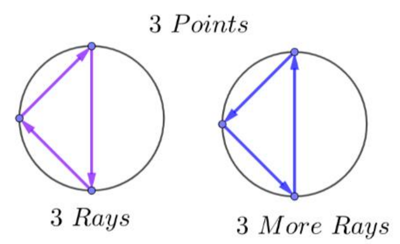
A. 3 distinct points
B. 4 distinct points
C. 5 distinct points



1.52: Rays on a Circle

$$n(n-1)$$

Rays have a direction. Hence, the number of line segments is exactly half the number of rays.



Example 1.53

A circle has seven points on it. Find the maximum number of

A. line segments that can be formed by joining any two points
B. rays that can be formed by joining any two points

Line Segments

Each of the seven points can be connected to any of the remaining six points. Total number of line segments should be

$$7 \times 6$$

But, this overcounts the number of line segments by two, since a line segment connecting Point A with Point B is the same as the line segment connecting Point B with Point A.
Therefore, actual number of line segments is

$$\frac{7 \times 6}{2} = 7 \times 3 = 21$$

$$\text{In general: } \underbrace{AB}_{Line\ Segment} = \underbrace{BA}_{Line\ Segment}$$

Rays

Rays are directed line segments. They have a start point, and another point which lies on the ray, and the ray continues till infinity.

$$\text{In general: } \underbrace{AB}_{Ray} \neq \underbrace{BA}_{Ray}$$

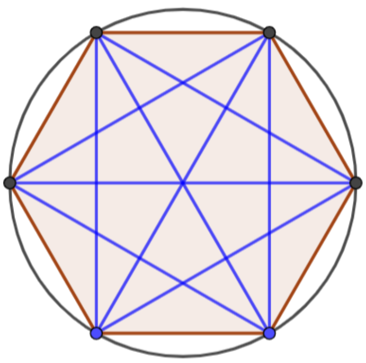
Total number of rays

$$\underbrace{7}_{Points} \times \underbrace{6}_{\substack{Each\ connected\ to \\ remaining\ 6\ points}} = 42$$

Example 1.54

A mystic rose is a pattern made by connecting dots in a circle via line segments. How many line segments will there be in a Mystic Rose with 9 distinct dots?

$$\frac{n(n-1)}{2} = \frac{9 \times 8}{2} = 36$$



1.55 Lines Connecting n vertices of a Polygon

Number of Lines connecting n vertices of a polygon $= \frac{n(n-1)}{2}$

Each vertex (of which there are n) can have a line drawn connecting to the remaining vertices (of which there are $n - 1$).
Proposed Number of Lines = $n(n - 1)$.
However, connecting the 1st point with the 2nd is the same as connecting the 2nd point with the first. Hence, we are overcounting the actual number of lines by a factor of two.

Example 1.56

What is the number of ways to choose two line segments from the 15 segments connecting the six vertices of regular hexagon ABCDEF such that the endpoints include each of A, B, C, D, E and F?

From points A, B, C, D, E and F, we need to choose three pairs that cover all the points.

Number of ways to choose the first pair:
6 Choose 2 = 6!/4!2! = 15

Number of ways to choose the second pair:
4 Choose 2 = 4!/2!2! = 6

Number of ways to choose the third pair:
2 Choose 2 = 1

15*6=90

1.57 Diagonals of a Polygon

A polygon with n sides has

$$\frac{n(n-3)}{2}\ diagonals$$

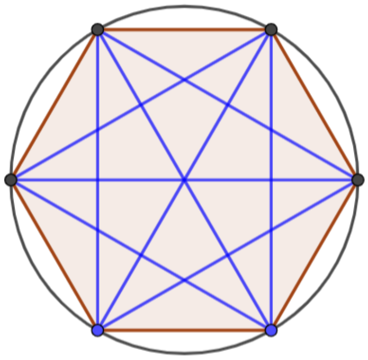
Using the formula above

The number of lines segment joining n vertices is $\frac{n(n-1)}{2}$.

The number of sides is n.

Using complementary counting, the number of diagonals is:

$$\frac{n(n-1)}{2}-n=\frac{n^2-n-2n}{2}=\frac{n^2-3n}{2}=\frac{n(n-3)}{2}$$



Logic

When counting diagonals, we do not want to count sides. From any vertex of a polygon, we cannot get a diagonal by drawing a line segment to either vertex adjacent to it.

Hence, the number of diagonals is:

$$\underbrace{n}_{No.of\ Vertices}\times\underbrace{(n-3)}_{\substack{No.of\ Points\\ to\ draw\ Diagonals\ to}}=n(n-3)$$

But, as usual, we need to divide by 2 to account for the overcounting:

$$\frac{n(n-3)}{2}$$

Example 1.58

A. A diagonal of a polygon is a line segment that connect two vertices of the polygon, but is not a side of the polygon. What is the number of diagonals of a regular polygon with 12 sides?

B. 10 people are standing in a circle. They pass a ball to each other in every way possible, except that no one tosses the ball to the person to the right and the left of them in the circle. If the order in which the tosses is not important, count the number of distinct tosses.

C. How many diagonals does a Septagon have?

D. The number of diagonals that can be drawn in a polygon of 100 sides is **(AHMSE 1950/45)**

E. A polygon has the same number of diagonals as it has sides. Find the number of sides of the polygon.

F. A polygon has x vertices, and another polygon has $y=x-2$ vertices. What is the difference in the number of diagonals that the two polygons have? What is the minimum value of $x+y$?

Part A

Imagine that we put all the points on a circle. The total number of ways of obtaining line segments from these 12 points is

$$\frac{12\times 11}{2}=6\times 11=66$$

But, this overcounts the number of diagonals by the number of sides. (Since no side is a diagonal).

Hence, the number of diagonals

$$\underbrace{66}_{No.of\ Line\ Segments}-\underbrace{12}_{No.of\ Sides}=54$$

Part B

$$\frac{10\times 9}{2}-10=45-10=35$$

Part C

A heptagon has seven vertices. We count diagonals as follows:

$$n = 7 \Rightarrow \text{Diagonals} = \frac{n(n-3)}{2} = \frac{7 \times 4}{2} = 14$$

Part D

$$4850$$

Part E

$$\frac{n(n-3)}{2} = n \Rightarrow n - 3 = 2 \Rightarrow n = 5$$

Part F

$$\frac{x(x-3)}{2} - \frac{(x-2)(x-5)}{2} = \frac{x^2 - 3x - (x^2 - 7x + 10)}{2} = \frac{4x - 10}{2} = 2x - 5$$

$$\underbrace{Min(y = x - 2) = 3}_{\substack{Min.Vertices\ of \\ Polygon=3}} \Rightarrow Min(x) = 5 \Rightarrow y + x = 3 + 5 = 8$$

1.5 Tiling and Diagonals

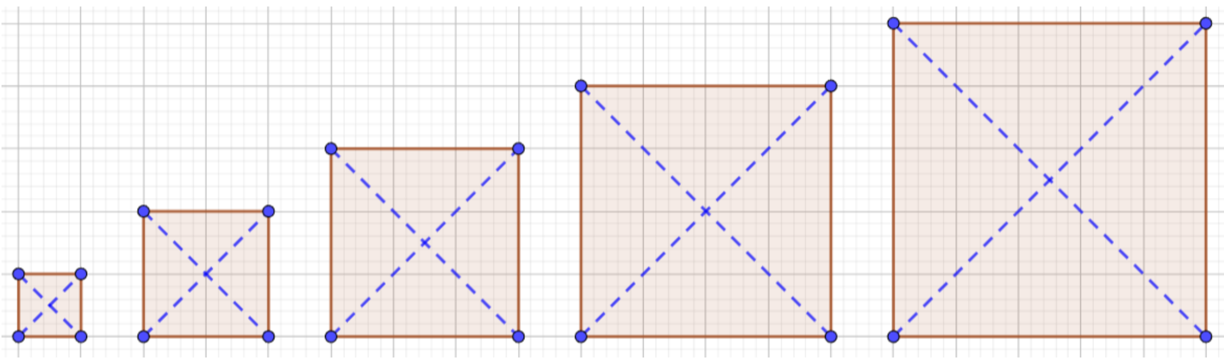
A. Tiling: Squares

1.59: Squares

Example 1.60

Consider squares of integer side length made by joining 1×1 tiles. For squares of length 1 to 5:

A. count the number of tiles that a diagonal passes through.
B. count the number of number of intersection points (where 4 tiles join together) that a diagonal passes through
C. Repeat Parts A and B for both diagonals



		Single Diagonal		Both Diagonals	
	Side Length	Tiles	Intersection Points	Tiles	Intersection Points
	1	1	0	1	0
	2	2	1	4	1
	3	3	2	5	4
	4	4	3	8	5
	5	5	4	9	8

Odd	n	n	$n-1$	$2n-1$	$2n-2$
Even	n	n	$n-1$	$2n$	$2n-3$

Example 1.61

A dance floor is 100 feet long by 100 feet wide. It is tiled using tiles that are each 2 feet in length and width. A blue line is drawn from the top left corner of the floor to the bottom right corner. A red line is drawn from the top right corner of the floor to the bottom left corner.

A. How many tiles will both lines together pass through
B. How many corner points (intersection of 4 tiles) will both lines together

The dance floor has

$$\underbrace{50}_{Length} \times \underbrace{50}_{Width} = 2500\, Tiles$$

Part A

$$Both\ Lines = \underbrace{50}_{Blue} + \underbrace{50}_{Red} = 100$$

Part B

$$\underbrace{50-1}_{Blue} + \underbrace{50-1}_{Red} - \underbrace{1}_{Overlap} = \underbrace{49}_{Blue} + \underbrace{49}_{Red} - \underbrace{1}_{Overlap} = 98 - 1 = 97$$

Answer the above question if the dance floor is 102 feet long by 102 wide instead (and everything else remains the same).

The number of tiles

$$= \frac{102}{2} = 51$$

Part A

$$\underbrace{51}_{Blue} + \underbrace{51}_{Red} - 1 = 51 + 50 = 101$$

Part B

$$\underbrace{51-1}_{Blue} + \underbrace{51-1}_{Red} = 50 + 50 = 100$$

Example 1.62

A square is divided into n^2 congruent squares by drawing lines parallel/perpendicular to the sides. Both diagonals of the squares pass through x of the smaller squares. Find the possible value of n if:

A. $x = 28$
B. $x = 30$

We do not know whether n is even or odd.

$$n\ is\ even \Rightarrow No.\ of\ tiles = 2n$$
$$n\ is\ odd \Rightarrow No.\ of\ tiles = 2n - 1$$

Part A

$$2n = 28 \Rightarrow n = 14 \Rightarrow Even \Rightarrow Valid$$
$$2n - 1 = 28 \Rightarrow 2n = 29 \Rightarrow n = 14.5 \Rightarrow Not\ an\ Integer \Rightarrow Not\ Valid$$

Part B

$$2n = 30 \Rightarrow n = 15 \Rightarrow Odd \Rightarrow Not\ Valid$$
$$2n - 1 = 30 \Rightarrow 2n = 31 \Rightarrow n = 15.5 \Rightarrow Not\ an\ Integer \Rightarrow Not\ Valid$$

There is no square that meets these conditions.

Example 1.63

A square-shaped floor is covered with congruent square tiles. If the total number of tiles that lie on the two diagonals is 37, how many tiles cover the floor? **(AMC 8 2017/11)**

$$2n - 1 = 37$$
$$2n = 38$$
$$n = 19$$

Example 1.64

The floor of a square room is covered with congruent square tiles. The diagonals of the room are drawn across the floor, and two diagonals intersect a total of 9 tiles. How many tiles are on the floor? **(MathCounts Chapter Sprint 2000/8)**

Let n be the side length of the room.

$$Even: 2n = 9 \Rightarrow n = \frac{9}{2} \Rightarrow Not\ Valid$$
$$Odd: 2n - 1 = 9 \Rightarrow n = 5 \Rightarrow n^2 = 25$$

Example 1.65

A square is divided into n^2 congruent squares by drawing lines parallel/perpendicular to the sides. Both diagonals of the squares pass through x intersections of 4 tiles. Find the possible value of n if:

A. $x = 28$
B. $x = 30$

We do not know whether n is even or odd.

$$n\ is\ even \Rightarrow No.of\ tiles = (n-1) + (n-1) - 1 = 2n - 3$$
$$n\ is\ odd \Rightarrow No.of\ tiles = (n-1) + (n-1) = 2n - 2$$

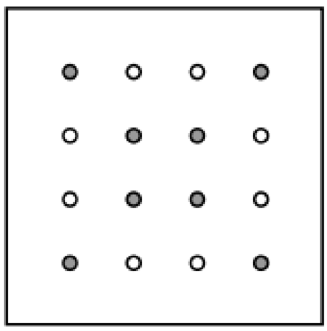
Part A

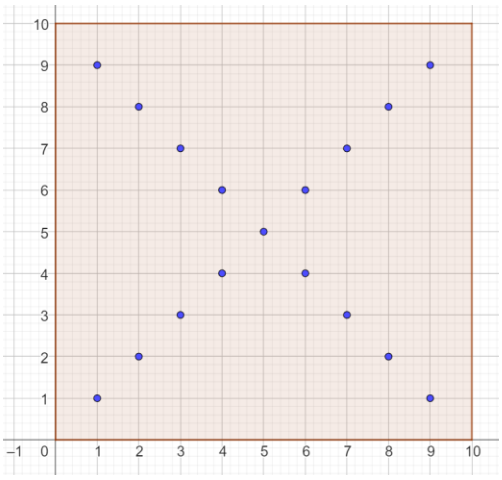
$$2n - 2 = 28 \Rightarrow 2n = 30 \Rightarrow n = 15 \Rightarrow Odd \Rightarrow Valid$$

Example 1.66

(CEMC Pascal 2022/21)

21. A 5 cm by 5 cm pegboard and a 10 cm by 10 cm pegboard each have holes at the intersection of invisible horizontal and vertical lines that occur in 1 cm intervals from each edge. Pegs are placed into the holes on the two main diagonals of both pegboards. The 5 cm by 5 cm pegboard is shown; it has 16 holes. The 8 shaded holes have pegs, and the 8 unshaded holes do not. How many empty holes does the 10 cm by 10 cm pegboard have?





B. Tiling: Rectangles

1.67: Number of Tiles

The number of tiles crossed by the diagonal of a rectangular floor with integral length l and integral width w is

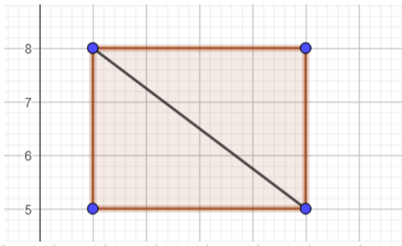
$$l + w - \mathrm{HCF}(l, w)$$

The rectangle to the right has:

$$length = 4, width = 3 \Rightarrow HCF(3,4) = 1$$

The diagonal crosses 3 vertical lines 2 horizontal lines:

$$\underbrace{3}_{\substack{Vertical\\Lines}} + \underbrace{2}_{\substack{Horizontal\\Lines}} + \underbrace{1}_{\substack{Start\\Tile}} = 6$$

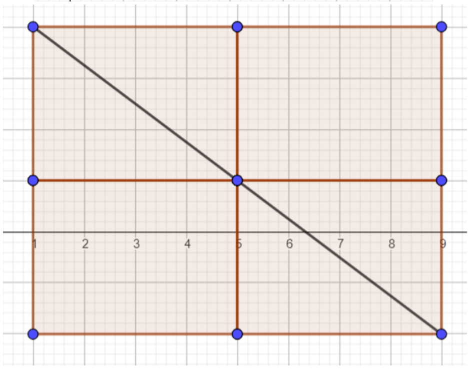


$$\underbrace{7}_{\substack{Vertical\\Lines}} + \underbrace{5}_{\substack{Horizontal\\Lines}} + \underbrace{1}_{\substack{Start\\Tile}} - \underbrace{2-1}_{\substack{Intersection\\Points}} = 12$$

$$\underbrace{7}_{\substack{Vertical\\Lines}} + \underbrace{5}_{\substack{Horizontal\\Lines}} + \underbrace{1}_{\substack{Start\\Tile}} - \underbrace{2}_{HCF(6,8)} + 1 = 12$$

$$\underbrace{7}_{\substack{Vertical\\Lines}} + \underbrace{5}_{\substack{Horizontal\\Lines}} + 2 - \underbrace{2}_{HCF(6,8)} = 12$$

$$\underbrace{8}_{Length} + \underbrace{6}_{Width} - \underbrace{2}_{HCF(6,8)} = 12$$



$$(l-1) + (w-1) + 1 - [HCF(l,w) - 1]$$

Example 1.68

A. A rectangular floor that is 10 feet wide and 17 feet long is tiled with 170 one-foot square tiles. A bug walks from one corner to the opposite corner in a straight line. Including the first and the last tile, how many tiles does the bug visit? (AMC 10A 2019/10)

B. A 24-foot by 72-foot rectangular dance floor is completely tiled with 1-foot by 1-foot square tiles. Two opposite corners of the dance floor are connected by a diagonal. This diagonal passes through the interior of exactly how many tiles? (MathCounts Chapter Sprint 2002/27)

Part A

$$l + w - \mathrm{HCF}(l, w) = 10 + 17 - 1 = 27 - 1 = 26$$

Part B

$$l + w - \gcd(l, w) = 72 + 24 - 24 = 72$$

Example 1.69

The diagonal of a rectangular floor tiled using 1×1 tiles passes through exactly 12 tiles. Find the sum of the possible values of the perimeter.

$$l + w - \text{HCF}(l, w)$$

The minimum value of $HCF(l, w)$ is 1.

Perimeter	Length	Width	$l + w$	$HCF(l, w)$	No. of Tiles
26	12	1	13	1	12
28	8	6	14	2	12
30	9	6	15	3	12
32	12	4	16	4	12
			17	5	12
36	12	6	18	6	12
			19	7	12
			20	8	12
152	**Total**				

1.70: Number of Tiles crossed by both diagonals

Both are even:
Overlap of 0

Both length and width are odd and equal:
Overlap of 1

One is odd, and the other is even:
Overlap of 2

Both length and width are odd and distinct:
Overlap of 3

Example 1.71

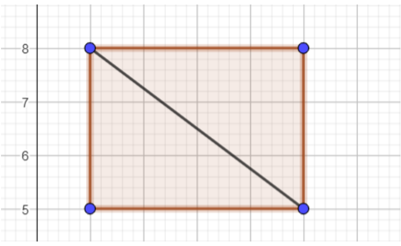
Julian tiled a 15 feet by 21 feet rectangular ballroom with one-foot square tiles. When he finished, he drew both diagonals on the floor connecting the opposite corners of the room. What is the total number of tiles that the diagonals pass through? **(2001 MathCounts Handbook Warm up 12 #6)**

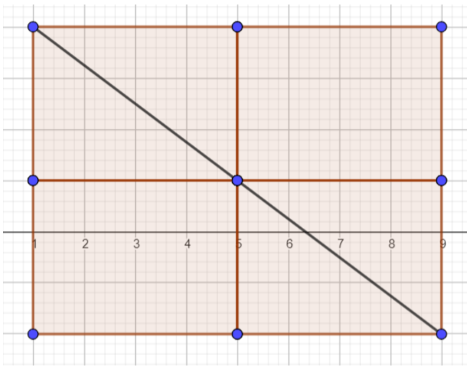
$$l + w - \gcd(l, w) = 21 + 15 - 3 = 33$$
$$33 + 33 - 3 = 63$$

1.72: Number of Intersection Points

The number of 4-corner intersection points crossed by the diagonal of a rectangular floor with integral length l and integral width w is

$$\text{HCF}(l,w)+1(\textit{Endpoints included})$$
$$\text{HCF}(l,w)-1(\textit{Endpoints excluded})$$





Example 1.73

A rectangular floor is tiled in a chalkboard fashion with square tiles of edge length 1. The dimensions of the floor are 321 by 123 units. An insect walks along the diagonal of the rectangle.

A. How many 4-corner intersection points does the insect cross on its walk between diagonally opposite corners? **(Michigan Mathematics Prize Competition 1998 #18)**
B. What are the coordinates of those intersection points?

Part A

$$\gcd(321,123)-1=3-1=2$$

Part B

$$(107,41),(214,82)$$

C. Tiling: Cuboids

1.74: Cuboids

For a cuboid with integer side lengths and length l, width w, and height h, the number of $1\times1\times1$ cuboids a diagonal passes through the interior is:

$$w+l+h-HCF(w,l)-HCF(l,h)-HCF(h,w)+HCF(w,l,h)$$

Example 1.75

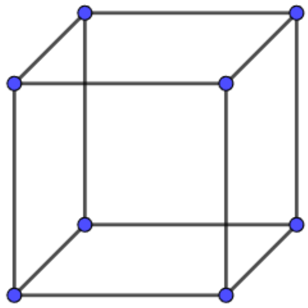
A $150\times324\times375$ rectangular solid is made by gluing together $1\times1\times1$ cubes. An internal diagonal of this solid passes through the interiors of how many of the $1\times1\times1$ cubes. **(AIME 1996/14)**

$$w=150=2\times3\times5^2$$
$$l=324=2^2\times3^4$$
$$h=375=3\times5^2$$
$$=\underbrace{150}_{w}+\underbrace{324}_{l}+\underbrace{375}_{h}-\underbrace{6}_{HCF(w,l)}-\underbrace{3}_{HCF(l,h)}-\underbrace{75}_{HCF(h,w)}+\underbrace{3}_{HCF(w,l,h)}=768$$

2. 3D Discrete Geometry

2.1 Faces, Edges and Vertices

A. Manual Counting

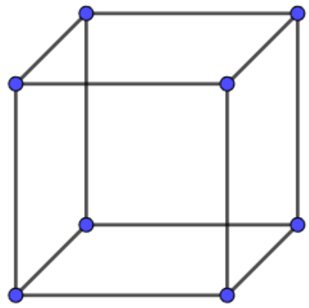


2.1: Faces, Edges and Vertices

- Vertex: A point which is at a corner of a 3D shape is a vertex.
- Edge: An edge is a line that connects two vertices.
- Face: Is a combination of edges that forms a single plane.

Example 2.2

Jamie counted the number of edges of a cube, Jimmy counted the numbers of corners, and Judy counted the number of faces. They then added the three numbers. What was the resulting sum? (AMC 8 2003/1)



A cube has a face each at the base, at the top, in front, on the left, on the right and on the back, giving

$$\underbrace{2}_{\substack{Top\ and\\ Bottom}} + \underbrace{2}_{\substack{Front\ and\\ Back}} + \underbrace{2}_{\substack{Left\ and\\ Right}} = 6$$

A cube has four vertices at the base. And four vertices at the top, giving us a total of

$$4 + 4 = 8\, Vertices$$

A cube has four edges at the base, four edges at the top, and four edges connecting the top with the base giving

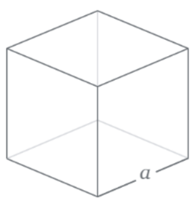
$$4 + 4 + 4 = 12\ edges$$

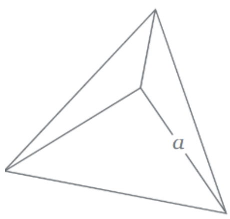
$$Total = \underbrace{6}_{Faces} + \underbrace{8}_{Vertices} + \underbrace{12}_{Edges} = 26$$

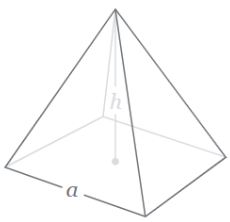
2.3: Convex Polyhedron

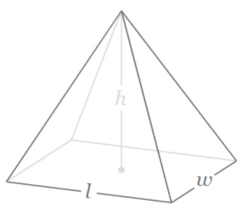
A polyhedron is convex if no angle in the polyhedron is more than 180 degrees.

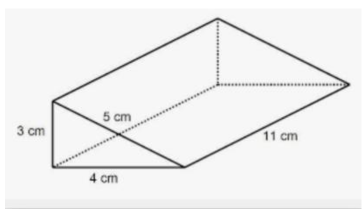
Some common convex polyhedra are given below:











2.4: Euler's Characteristic

Euler's Characteristic connects the number of faces, the number of vertices, and the number of edges of a $\underbrace{convex\ polygon}_{Two\ dimensions}$, or $\underbrace{convex\ polyhedron}_{Three\ Dimensions}$.

$$\underbrace{F}_{Faces} + \underbrace{V}_{Vertices} = \underbrace{E}_{Edges} + 2$$

Example 2.5

Verify that the following shapes satisfy Euler's Formula. A diagram of (most) of the shapes is above. Make a summary table once you have collated all the answers.

A. A cube is the three-dimensional version of a square. It has all sides the same.
B. Pyramids are of different types. A pyramid is named according to the base that it has. A triangular pyramid (tetrahedron) is the simplest pyramidal shape. It has the same shape for its base as for the rest of its faces.
C. A square pyramid has a square for a base. The rest of the faces are triangles.
D. A rectangular pyramid has a rectangle for a base. The rest of the faces are triangles.
E. A pentagonal pyramid has a pentagon for a base. The rest of the faces are triangles.
F. A triangular prism has a triangle for a base, and the remaining sides are at right angles to the base.

Part A: Cube

$$\underbrace{6}_{Faces} + \underbrace{8}_{Vertices} = \underbrace{12}_{Edges} + 2 = 14 \Rightarrow Verified$$

Part B: Triangular Pyramid

The base is a triangle, and each side of the base triangle is attached to a slant face leading towards the apex of the pyramid.

$$Total\ Faces = \underbrace{1}_{Base} + \underbrace{3}_{Slant\ Faces} = 4$$

The base has three vertices from the triangle, and one vertex for apex:

$$Total\ Vertices = \underbrace{3}_{Base} + \underbrace{1}_{Apex} = 4$$

Three edges from the base and three edges from the slant faces:

$$\underbrace{3}_{Base} + \underbrace{3}_{Edges} = 6$$

$$F + V = E + 2$$
$$LHS = F + V = 4 + 4 = 8$$
$$RHS = E + 2 = 6 + 2 = 8$$

Part C

$$F + V = E + 2$$
$$LHS = F + V = 5 + 5 = 10$$
$$RHS = E + 2 = 8 + 2 = 10$$

Part D

$$F + V = E + 2$$
$$LHS = F + V = 5 + 5 = 10$$
$$RHS = E + 2 = 8 + 2 = 10$$

Part E

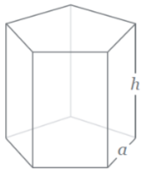
$$F + V = E + 2$$
$$LHS = F + V = 6 + 6 = 12$$
$$RHS = E + 2 = 10 + 2 = 12$$

Part F: Triangular Prism

$$F + V = E + 2$$
$$LHS = F + V = 5 + 6 = 11$$
$$RHS = E + 2 = 9 + 2 = 11$$

Part G: Pentagonal Prism

$$Faces = \underbrace{1}_{Base} + \underbrace{1}_{Top} + \underbrace{5}_{Sides} = 7$$
$$Vertices = \underbrace{5}_{Base} + \underbrace{5}_{Top} = 10$$
$$Edges = 5 + 5 + 5 = 15$$
$$LHS = F + V = 10 + 7 = 17$$
$$RHS = E + 2 = 15 + 2 = 17$$



Summary Table					
		Faces	Vertices	Edges	$F + V = E + 2$
Prisms	Cube/Cuboid/ Rectangular Prism	6	8	12	14
	Pentagonal Prism	7	10	15	17
	Octahedron	8	6	12	14
Pyramids	Triangular Pyramid (Tetrahedron)	4	4	6	8
	Square Pyramid	5	5	8	10

2.6: Curved Shapes do not follow Euler's Characteristic

Only shapes that have straight faces follow Euler's characteristic.

Example 2.7

Show that the following shapes do not satisfy Euler's Characteristic given above.

A. Cylinder
B. Sphere

Part A

$$LHS = \underbrace{3}_{Faces} + \underbrace{0}_{Vertices} = 3 \neq 4 = \underbrace{2}_{Edges} + 2 = RHS$$

Hence, a cylinder does not satisfy Euler's Characteristic.

Part B

$$LHS = \underbrace{1}_{Faces} + \underbrace{0}_{Vertices} = 1 \neq 2 = \underbrace{0}_{Edges} + 2 = RHS$$

2.8: Using Euler's Characteristic

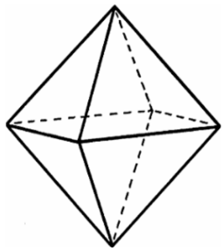
It can be used to find information on a 3D shape if some information regarding the shape is available.

Example 2.9

What is the number of edges that an octahedron (a convex three dimensional solid with 8 faces, and 6 vertices) must have?

Substitute $F = 8, V = 6$ in Euler's Formula:

$$F + V = E + 2$$
$$8 + 6 = E + 2$$
$$E = 12$$



Example 2.10

A regular dodecahedron is a Platonic solid (also a convex polyhedron) with pentagonal faces. It has 12 faces and 30 edges. Find the number of edges.

$$F + V = E + 2$$
$$12 + V = 30 + 2$$
$$V = 20$$

Example 2.11

A regular icosahedron which has 20 equilateral triangles for its faces, and 30 edges. Find the number of vertices.

$$20 + V = 30 + 2$$
$$V = 12$$

B. Counting Formulas

2.12: Vertices

If a fix number of faces meet at each vertex of a polyhedron, then:

$$No.of\ Vertices = \frac{No.of\ Faces \times No.of\,Sides\ per\ Face}{No.of\ Faces\ at\ each\ vertex}$$

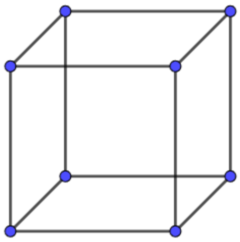
We divide by the number of faces meeting at each vertex because that many faces share a vertex.

Example 2.13

Use the formula above to calculate the number of vertices for:

A. A cube

$$No.of\ Vertices = \frac{No.of\ Faces \times No.of\,Sides\ per\ Face}{No.of\ Faces\ at\ each\ vertex} = \frac{6(4)}{3} = \frac{24}{3} = 8$$



2.14: Edges-I

If a polyhedron has the same two-dimensional shape for each of its faces, then:

$$No.of\ Edges = \frac{No.of\ Faces \times No.of\ Sides\ per\ Face}{2}$$

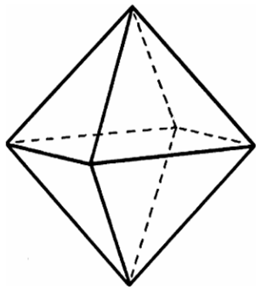
For edges, we divide by two because each edge is shared by two faces.

Example 2.15

Use the formula above to calculate the number of vertices and number of edges for:

A. Octahedron

Octahedron



Number of Edges

$$= \frac{No.of\ Faces \times No.of\ Sides\ per\ Face}{2} = \frac{8 \times 3}{2} = \frac{24}{2} = 12$$

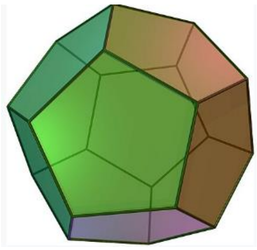
Number of Vertices

$$= \frac{No.of\ Faces \times No.of Sides\ per\ Face}{No.of\ Faces\ meeting\ at\ each\ vertex} = \frac{8 \times 3}{4} = \frac{24}{4} = 6$$

Example 2.16

A regular dodecahedron is made up of twelve regular pentagons. Find the number of edges.

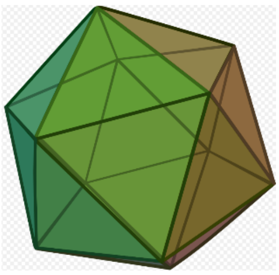
$$No.of\ Edges = \frac{No.of\ Faces \times No.of\ Sides\ per\ Face}{2} = \frac{12 \times 5}{2} = \frac{60}{2} = 30$$



Example 2.17

A regular icosahedron is a convex polyhedron made up of twenty equilateral triangles. Find the number of edges.

$$No.of\ Edges = \frac{No.of\ Faces \times No.of\ Sides\ per\ Face}{2} = \frac{20 \times 3}{2} = \frac{60}{2} = 30$$



Example 2.18

A triangular pyramid has four faces, with each face being a triangle, and three faces meeting at each vertex. Count the number of edges, and the number of vertices.

$$Edges = \frac{\underbrace{4}_{\substack{No.of\\ Faces}} \times \underbrace{3}_{\substack{Sides\\ per\ Face}}}{2} = \frac{12}{2} = 6$$

$$Vertices = \frac{\underbrace{4}_{\substack{No.of\\ Faces}} \times \underbrace{3}_{\substack{Sides\\ per\ Face}}}{3} = \frac{12}{3} = 4$$

2.19: Edges-II

If the faces of a polyhedron are not all the same type, then we account for each type of face separately.

$$Edges = \frac{f_1 \times s_1 + f_2 \times s_2 + \cdots + f_n s_n}{2} = \frac{1}{2}\sum_{i=1}^{n} f_i s_i$$

Where

$$f_1 = No.\,of\ Faces\ of\ Type\ 1$$
$$s_1 = No.\,of\ Sides\ in\ each\ of\ f_1$$
$$f_2 = No.\,of\ Faces\ of\ Type\ 2$$
$$s_2 = No.\,of\ Sides\ in\ each\ of\ f_2$$

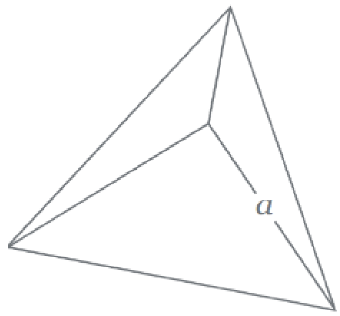
- The numerator will give the *uncorrected* value of the number of the edges.
- Dividing by 2 corrects for the fact that each edge is part of two faces.

Example 2.20

Count the number of edges, and the number of vertices.
A square pyramid has five faces, and the faces are of two types. The first type of face is a single square, each with four sides. The second type of face is a triangle (of which there are four), each with three sides

Count the number of edges:

$$Edges = \frac{\underbrace{1}_{\substack{No.of\\Squares}} \times \underbrace{4}_{\substack{Sides\\per\ Square}} + \underbrace{4}_{\substack{No.of\\Triangles}} \times \underbrace{3}_{\substack{Sides\\per\ Triangle}}}{2} = \frac{4+12}{2} = \frac{16}{2} = 8$$



Find the number of vertices using Euler's Characteristic:

$$F + V = E + 2$$
$$5 + V = 8 + 2$$
$$V = 5$$

2.21 Summary Table

	Shape of Face	Faces	Sides per Face	No. of Sides	No. of Edges	Faces at Vertex	
		f	s	fs	$\frac{fs}{2}$	n	
Cuboid	Rectangle	6	4	24	12		
Triangular Pyramid	Triangle	4	3	12	6		
Octahedron	Triangle	8	3	24	12		
Square Pyramid	Triangle	4	3	12	8		
	Square	1	4	4			
Dodecahedron	Pentagon	12	5	60	30		
Icosahedron	Triangles	20	3	60	30		

C. Diagonals of a Polyhedron

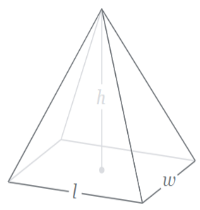
2.22 Polyhedron

- A polyhedron is a three-dimensional figure.
- It is the three-dimensional equivalent of a polygon.
- Examples of polyhedrons: Cubes, cuboids, pyramids, and prisms

2.23 Types of Line Segments in a Polyhedron

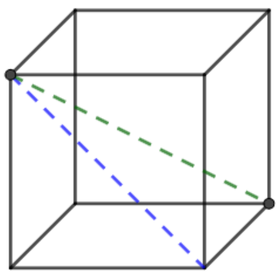
A line connecting two vertices of a polyhedron can be classified into one of three types:

- Edge: A line which is part of the boundary of the polyhedron, and part of two faces is an edge.
- Face Diagonals: A line lying completely on a face is called a face diagonal.
- Space Diagonals: A line which is not a part of any face is called a space diagonal.



Example 2.24

Find the number of edges, vertices, faces, space diagonals, and face diagonals for a cube.



$$Edges = 12$$

$Vertices = 8$

$$Faces = 6$$
$$Space\ Diagonals = 4$$

Face Diagonals
The number of face diagonals is

$$= 2 \times 6 = 12$$

2.25 Line Segments in a Polyhedron

A polyhedron with n vertices has

$$\frac{n(n-1)}{2}\ Line\ Segments$$

Joining these vertices

2.26 Classifying Line Segments in a Polyhedron

The line segments can be classified as:

$$Lines\ Segments\ joining\ n\ vertices = Edges + Face\ Diagonals + Space\ Diagonals$$

$$LS = E + FD + SD$$

Example 2.27

What is the number of line segments joining the vertices of a cube?

$$\frac{8 \times 7}{2} = 28$$
$$\underbrace{12}_{Edges} + \underbrace{4}_{\substack{Space\\Diagonals}} + \underbrace{12}_{\substack{Face\\Diagonals}} = 28$$

Example 2.28

Classify the line segments joining the vertices of an octahedron.
(Same as the example above).

D. Face Diagonals

2.29 No. of Face Diagonals in a Polyhedron

A polyhedron that has all faces of the same type has a number of face diagonals given by:

$$No.\,of\ Faces \times Diagonals\ per\ Face$$

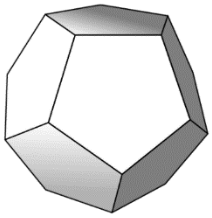
Also, recall that the number of diagonals of a n sided polygon is:

$$\frac{n(n-3)}{2}$$

Example 2.30

Find the number of face diagonals of:

A. An octahedron
B. A regular dodecahedron. It has 12 congruent regular pentagonal faces.



An octahedron consists of eight triangular faces, none of which has any diagonals. Hence, the number of face diagonals is:

$$8 \times 0 = 0$$

A regular dodecahedron has 12 congruent pentagonal faces, which of which has $\frac{n(n-3)}{2} = \frac{5(2)}{2} = 5$ diagonals:

$$5 \times 12 = 60$$

2.31 No. of Face Diagonals in a Polyhedron

If a polyhedron has different kinds of faces, then the number of face diagonals of the polyhedron is given by:

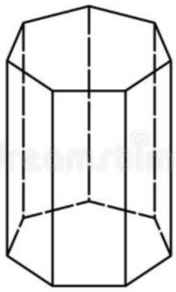
$$f_1 d_1 + f_2 d_2 + \cdots + f_n d_n$$

Where

$$f_i = No.\,of\ faces\ of\ type\ i$$
$$d_i = No.\ of\ diagonals\ in\ faces\ of\ type\ i$$

Example 2.32

Find the number of face diagonals of a heptagonal prism.



The number of diagonals in the two bases is:

$$2\left(\frac{n(n-3)}{2}\right) = n(n-3) = 7 \times 4 = 28$$

The number of diagonals in the rectangles is:

$$7 \times 2 = 14$$

The total is

$$28 + 14 = 42$$

Example 2.33

Consider a prism with a n sided convex polygon for its base and top with $4 \le n \le 10$. Let D be the number of face diagonals of the prism. Let $\tau(D)$ be the number of positive factors of D. Determine

$$Maximum[\tau(D)] - Minimum[\tau(D)]$$

The number of face diagonals of a n sided prism is:

$$2n + 2\left(\frac{n(n-3)}{2}\right) = 2n + n(n-3) = n(n-1)$$

$$n = 4: D = 4(3) = 12 \Rightarrow Factors \in \{1,2,3,4,6,12\} \Rightarrow 6\ Factors$$
$$n = 5: D = 5(4) = 20 \Rightarrow Factors \in \{1,2,4,5,10,20\} \Rightarrow 6\ Factors$$
$$n = 6: D = 6(5) = 30 \Rightarrow Factors \in \{1,2,3,5,6,10,15,30\} \Rightarrow 8\ Factors$$
$$n = 7: D = 7(6) = 42 \Rightarrow Factors \in \{1,2,3,7,14,21,42\} \Rightarrow 8\ Factors$$
$$n = 8: D = 8(7) = 56 \Rightarrow Factors \in \{1,2,4,7,8,14,28,56\} \Rightarrow 8\ Factors$$
$$n = 9: D = 9(8) = 72 \Rightarrow Factors \in \{1,2,3,4,6,8,9,12,18,24,36,72\} \Rightarrow 12\ Factors$$
$$n = 10: D = 10(9) = 90 \Rightarrow Factors \in \{1,2,3,5,6,9,10,15,18,30,45,90\} \Rightarrow 12\ Factors$$

$$Maximum[\tau(D)] - Minimum[\tau(D)] = 12 - 6 = 6$$

2.34 Diagonals of a Polyhedron

A polyhedron with $\boldsymbol{n}$ vertices has two types of diagonals: face diagonals and space diagonals. The two together are called just diagonals.

The number of diagonals can be counted using complementary counting as:

$$\underbrace{L}_{\substack{Line\\Segments}} - \underbrace{E}_{Edges}\quad diagonals$$

$$L = \frac{V(V-1)}{2}, \qquad V = No.of\ Vertices$$

The logical method that we used for counting number of diagonals of a polygon does not easily generalize to polyhedrons. But the other formula does.

Example 2.35

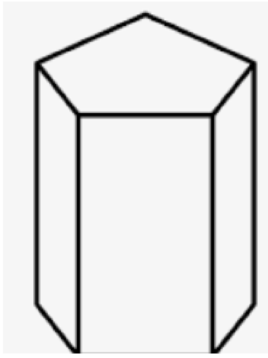
A cube has eight vertices (corners) and twelve edges. A segment, such as x, which joins two vertices not joined by an edge is called a diagonal. Segment y is also a diagonal. How many diagonals does a cube have? **(AMC 8 1997/17)**

$$\underbrace{\frac{8\times 7}{2}}_{Line\ Segments} - \underbrace{12}_{Edges} = 28 - 12 = 16$$

Example 2.36

A diagonal of a polyhedron is a line segment connecting two non-adjacent vertices. How many diagonals does a pentagonal prism have? **(MathCounts 2001 National Sprint)**

$$L - E = \frac{10 \times 9}{2} - 15 = 45 - 15 = 30$$



We can also count directly:

$$\underbrace{2 \times 5}_{\substack{Diagonals \\ from\ Rectangles}} + \underbrace{5 \times 2}_{\substack{Diagonals \\ from\ Pentagons}} = 10 + 10 = 20\ Face\ Diagonals$$

$$\underbrace{5}_{\substack{Vertices \\ at\ the\ top}} \times \underbrace{(5-3)}_{\substack{Vertices \\ at\ the\ bottom}} = 5 \times 2 = 10\ Space\ Diagonals$$

$$\underbrace{20}_{\substack{Face \\ Diagonals}} + \underbrace{10}_{\substack{Space \\ Diagonals}} = 30$$

2.37 Space Diagonals

Using complementary counting, a polyhedron with n vertices has number of:

$$\underbrace{SD}_{Space\ Diagonals} = \underbrace{Line\ Segments}_{Lines\ joining\ n\ Points} - \underbrace{E}_{Edges} - \underbrace{FD}_{Face\ Diagonals}$$

Example 2.38

An octahedron is formed by joining two congruent square pyramids at their base. Find the number of space diagonals of an octahedron.

Visualization

An octahedron is formed by joining two tetrahedrons.

Face Diagonals: Each face of the octahedron is a triangle. A triangle does not diagonals. Hence, there are no face diagonals in a tetrahedron.

Space Diagonals: Two diagonals will be formed by the four points forming a square that were a part of the base of both the tetrahedrons. One diagonal will be formed by a line going from one tip of the octahedron to the other tip of the octahedron.

Formula

$$Diagonals = \frac{6 \times 5}{2} - 12 - \left(\underset{Faces}{8} \times \frac{0}{\frac{Diagonals}{Face}}\right) = 15 - 12 = 3$$

Example 2.39

Find the number of space diagonals for a:

A. Cuboid: six rectangular faces, three of which meet at each vertex.
B. Regular icosahedron: 20 congruent equilateral triangle faces.
C. Regular dodecahedron: 12 congruent regular pentagonal faces.
D. Hexarhombic dodecahedron: 8 rhombic and 4 hexagonal faces
E.

Part A: Cuboid

$$No.of\ Edges = \frac{6 \times 4}{2} = \frac{24}{2} = 12$$

$$No.of\ Vertices = V = \frac{24}{3} = 8$$

Number of Face Diagonals

$$= 6 \times 2 = 12$$

Number of Space Diagonals

$$= \frac{8 \times 7}{2} - 12 - 12 = 4$$

Part B: Regular Icosahedron

$$Edges = \frac{20 \times 3}{2} = \frac{60}{2} = 30$$

Substitute $E = 30, F = 20$ in Euler's Formula:

$$20 + V = 30 + 2 \Rightarrow V = 12$$

Number of space diagonals:

$$= \underbrace{\frac{12 \times 11}{2}}_{\substack{Line \\ Segments}} - \underbrace{30}_{Edges} - \underbrace{0}_{\substack{Face \\ Diagonals}} = 66 - 30 = 36$$

Part C: Regular Dodecahedron

$$Edges = \frac{12 \times 5}{2} = \frac{60}{2} = 30$$

Substitute $E = 30, F = 12$ in Euler's Formula:

$$12 + V = 30 + 2 \Rightarrow V = 20$$

Total number of face diagonals:

$$= 12 \times \frac{n(n-3)}{2} = 12 \cdot \frac{5(2)}{2} = 12 \cdot 5 = 60$$

The number of space diagonals

$$= \underbrace{\frac{20 \times 19}{2}}_{\substack{Line \\ Segments}} - \underbrace{30}_{Edges} - \underbrace{60}_{\substack{Face \\ Diagonals}} = 190 - 90 = 100$$

Part D: Hexarhombic Dodecahedron

$$Edges = \frac{8 \times 4 + 4 \times 6}{2} = \frac{56}{2} = 28$$

Substitute $E = 30, F = 12$ in Euler's Formula:

$$12 + V = 28 + 2 \Rightarrow V = 18$$

The number of diagonals of a hexagon

$$= \frac{6(5)}{2} - 6 = 15 - 6 = 9$$

Total number of face diagonals:

$$= 2 \times 8 + 4 \times 9 = 52$$

The number of space diagonals

$$= \underbrace{\frac{18 \times 17}{2}}_{\substack{Line \\ Segments}} - \underbrace{28}_{Edges} - \underbrace{52}_{\substack{Face \\ Diagonals}} = 73$$

Example 2.40

Find the number of space diagonals for a convex polyhedron with 36 faces of which 24 are triangular and 12 are quadrilaterals. **(AIME I 2004/3, Modified)**

$$Edges = \frac{24 \times 3 + 12 \times 4}{2} = \frac{120}{2} = 60$$

Substitute $E = 60, F = 36$ in Euler's Formula:

$$36 + V = 60 + 2$$
$$V = 62 - 36 = 26$$

Number of Face Diagonals

$$= \underbrace{0 \times 24}_{\substack{Triangular \\ Faces}} + \underbrace{2 \times 12}_{\substack{Quadrilateral \\ Faces}} = 0 + 24 = 24$$

Number of Space Diagonals

$$= \frac{26 \times 25}{2} - 60 - 24 = 241$$

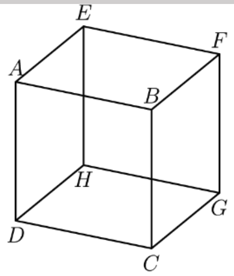
2.2 3D Shapes: Paths

A. Eulerian Paths

2.41 Eulerian Path

An Eulerian path is a path that travel along each edge of an object exactly once.

- If the start point and end point are different, it is an Eulerian trail.
- If the start point and end point are the same, it is an Eulerian cycle.



Example 2.42

$$ABCDA$$

The edges associated with A are:

$$AB, DA \Rightarrow 2$$

Example 2.43

An insect at vertex A of the cube wants to travel along the edges without repeating any edge. What is the maximum length of path possible? Give one such path.

It is possible to get a path of length nine edges. For example:

$$ABCDAEFGHE \Rightarrow 9\ Edges$$

The edges that remain unutilized are:

$$FB, DH, CG$$

It remains to demonstrate that you cannot have a path of more than nine edges.
Note that each vertex has exactly three edges joined together.

- To travel along an edge, without repeating an edge, you must arrive at a vertex via an edge, and leave via a different edge.

Consider two types of vertices:

- Type I: Where the path starts (and also ends).
- Type II: Which is not of Type I

$$ABCDAEFGHE \Rightarrow 9\ Edges$$

Consider the vertex A, which is of Type I:

$$AB, DA, AE \Rightarrow All\ Edges\ connected\ to\ A\ were\ utilized$$

Consider any vertex, which is of Type II:

- Any path that passes through a vertex of Type II must utilize an even number of edges connected to the vertex.

Hence, the maximum number of edges that can be traversed is

$$\underbrace{2}_{\substack{Type\ I\\Vertex}} \times \underbrace{3}_{\substack{Utilizable\\Edges}} + \underbrace{6}_{\substack{Type\ II\\Vertex}} \times \underbrace{2}_{\substack{Utilizable\\Edges}} = 6 + 12 = 18$$

However, this overcounts by a factor of 2, since an edge from A to B is the same as an edge from B to A.
Hence, the maximum length of path is:

$$\frac{18}{2} = 9$$

2.44 Type I Vertex

Type I Vertices are vertices where the path begins or ends (but not both at the same time). Each vertex of Type I in an Eulerian trail must have an odd number of edges in the path.

In a Type I vertex, in the beginning, you will start at the vertex without needing to arrive at it. Every time henceforth, when you arrive at the vertex, you must leave the vertex.
Hence, a Type I vertex will have an odd number of edges associated with it.

2.45 Type II Vertex

Type II Vertices are vertices in the middle of the path. Each vertex of Type II in an Eulerian path must have an even number of edges in the path.

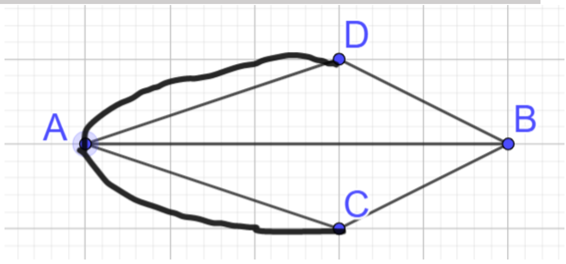
- If you want to arrive at a Type II vertex, you must also leave it. You must leave the vertex every time you arrive at it. Hence, the number of edges must be even.

2.46 Degree of A Vertex

The number of edges connected to a vertex is called the degree of the vertex.

2.47 Condition for Eulerian Path

- Any graph with an Eulerian path must have exactly zero or two vertices of odd degree.



Example 2.48: Seven Bridges of Konigsberg

This is a famous problem solved by Leonard Euler. Consider the graph alongside. Show that there is no path traverses all the edges exactly once.

Count the number of edges connected to each vertex:

$$A: 5, B: 3, C: 3, D: 3$$

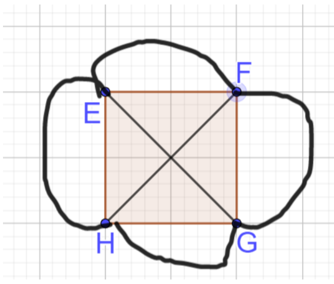
All the vertices have an odd number of edges. In a path that goes through the edges without repeating any edge, the maximum number of vertices that can have an odd number of edges is 2.
Hence, such a path is not possible.

$$1 \times 5 + 1 \times 3 + 2 \times 2 = 5 + 3 + 4 = 12$$
$$\frac{12}{2} = 6$$

$$AD, DA, AC, CA, AB, BD$$

Example 2.49

Show that there is no path that travels through each of the edges of the adjoining graph exactly once.



$$E, F, G, H: All\ 5\ have\ 5\ edges\ each$$

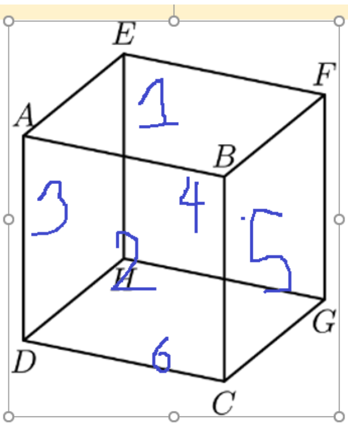
To have a path that travels through each edge exactly once, the number of vertices with an odd number of edges must be exactly zero or two.
In this case, it is 5. Hence, such a path is not possible.

B. Paths along Faces

Example 2.50

For each geometrical object below, how many ways are there to move from the top face to the bottom face via a sequence of adjacent faces so that each face is visited at most once and moves are permitted down, left, and right, but not up.

A. Cube
B. Hexagonal Prism standing on its base
C. Pentagonal Prism standing on its base
D. $n - sided$ prism standing on its base



Part A

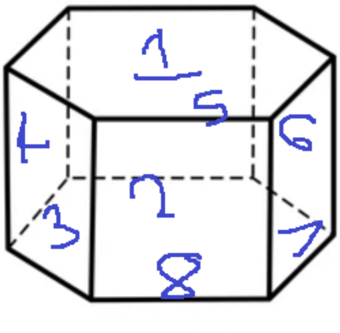
Any sequence of moves that meets the above condition must have the following parts:

- Step I: Move from top face to a side face:
$$4\ Ways$$
- Step II: Move along the side faces:
For example, if you pick 2 as your side face, then the options are:
$$2, \quad 23,234,2345, \quad 25,254,2543 \Rightarrow 7\ Ways$$
- Step III: From side face to bottom face:
$$1\ Ways$$

Hence, the final answer, by the multiplication principle
$$= 4 \times 7 = 28$$

Part B

Any sequence of moves that meets the above condition must have the following parts:



- Step I: Move from top face to a side face:

$$6\ Ways$$

- Step II: Move along the side faces:

$$\underbrace{2}_{1},\ \underbrace{23,234,2345,23456,234567}_{5},\ \underbrace{27,276,2765,27654,276543}_{5}$$

$$1+5+5=11\ Ways$$

- Step III: From side face to bottom face:

$$1\ Ways$$

Hence, the final answer, by the multiplication principle

$$=6\times 11=66\ Ways$$

Part C

$$5(1+4+4)=5(9)=45$$

Part D

$$n\big(1+(n-1)+(n-1)\big)=n(2n-1)$$

Example 2.51

Consider a $n-sided$ prism. The number of ways to travel from the top face to the bottom face via a sequence of adjacent faces so that each face is visited at most once and moves are permitted down, left, and right, but not up is 231. Find the number of space diagonals of the prism.

$$n(2n-1)=231=11(21)=11(2(11)-1)$$
$$n=11$$

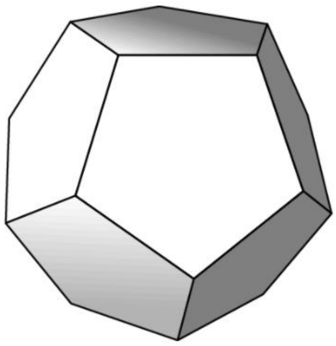
To find the number of space diagonals of a $11-sided$ prism, note:

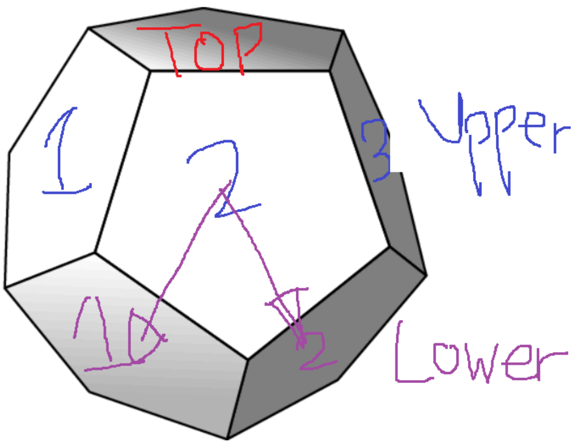
$$No.of\ choices\ in\ the\ base=11$$
$$No.of\ choices\ in\ the\ top=11-3=8$$

$$11(8)=88$$

Example 2.52

As shown in the figure below, a regular dodecahedron (the polyhedron consisting of 12 congruent regular pentagonal faces) floats in space with two horizontal faces. Note that there is a ring of five slanted faces adjacent to the top face, and a ring of five slanted faces adjacent to the bottom face. How many ways are there to move from the top face to the bottom face via a sequence of adjacent faces so that each face is visited at most once and moves are not permitted from the bottom ring to the top ring? (AMC 10A 2020/19)





Any sequence of moves that meets the above condition must have the following parts:

- Part A: Move from top face to top ring
$$5\ Ways$$

- Part B: Move along the top ring
$$1, 12, 123, 1234, 12345 \Rightarrow 5\ Options$$
$$15, 154, 1543, 15432 \Rightarrow 4\ Options$$
$$Total\ Options = 9\ Ways$$

- Part C: From top ring to bottom ring
$$2\ Ways$$

(For example, if are at **2** in the diagram, and want to move down, you can go to **2** or **10**.

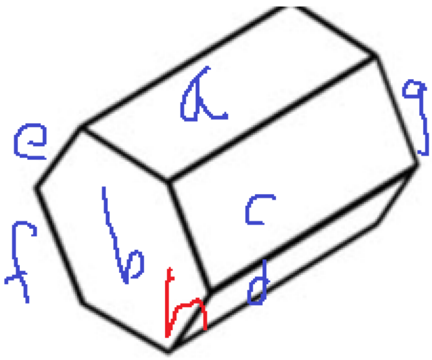
- Part D: Move along the bottom ring
The bottom ring is symmetrical to the top ring, and hence the number of ways to travel along the bottom is the same as the number of ways to travel along the top ring:
$$9\ Ways$$

- Part E: Move to bottom face
$$1\ Way$$

To move from top face to bottom face the above five parts must be done in sequence, and any of the choices can be combined in any way that you want.
Hence, the final answer, by the multiplication principle
$$= 5 \times 9 \times 2 \times 9 = 10 \times 81 = 810$$

Example 2.53

A hexagonal Prism is kept so that one of its six rectangular faces is the base. How many ways are there to move from the top face(a) to the bottom face(h) via a sequence of adjacent faces that starts from either b or g so that each face is visited at most once?



From the top base (a), you can go to:
- b or g
- c or d

By symmetry, the number of ways from
- b is the same as the number of ways from g.
- c is the same as the number of ways from d.

Paths Starting with ab
abh (1 Path)

Paths Starting with abd
abdh
abdgh
abdgfh

abdgefh
abdcgh
abdcgfh
abdcgefh
Paths above = 7
(Every path above can have its d replaced with an f, and we will get a new path.)

Total Paths = 7*2 =14

Paths Starting with abc
abcgh
abcgfh
abcgefh
abcdh
abcdgh
abcdgfh
abcdgefh
Paths above = 7
(Every path above can have its c replaced with an e, and we will get a new path.)
Total Paths = 7*2=16

Total of above is 1+14+8 = 31

And in each of the above 23 paths, we can interchange the b and the g (or if there is no g, replace the b with the g), giving an equivalent path:
31*2 =62

C. 2D Paths

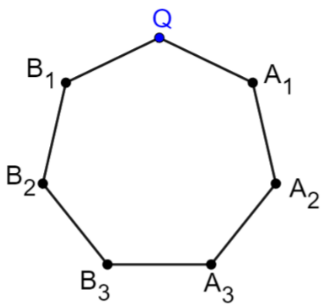
2.54 Length of a path: Polygon

If A and B are vertices of a polygon, define the length of the path from A to B to be the minimum number of edges of the polygon one must traverse in order to connect A and B.

- If AB is an edge of the polygon, then the length of the path from A to B is 1
- If AC and CB are edges and AB is not an edge, then $d(A, B) = 2$.

Example 2.55

Let Q, R, and S be randomly chosen distinct vertices of a convex heptagon (seven-sided polygon). What is the probability that the length of the path from Q to R is greater than the length of the path from R to S.



Total Outcomes:
Pick the vertex Q. Rotate the polygon so that Q is at the top (as shown in the diagram).

We can choose the remaining two vertices (with order matters) in

$$6 \cdot 5 = 30$$

Successful Outcomes:
The possible values of the length of the path between any two vertices of the

polygon are:

$$\{1,2,3\}$$

$$d(Q,R) > d(R,S)$$

Case I: $d(Q,R) = 1$
Not possible since $l(R,S)$ has minimum value 1.

Case II: $d(Q,R) = 2 > d(R,S) = 1$
$d(Q,R) = 2$. Q is on the apex. R can be

$$\{A_2, B_2\} = 2\ ways$$

$d(R,S) = 1$ can be achieved in

$$R = A_2 \Rightarrow S \in \{A_1, A_3\}$$
$$R = B_2 \Rightarrow S \in \{B_1, B_3\}$$

The total number of ways is:

$$2 \cdot 2 = 4$$

Case III: $d(Q,R) = 3 > d(R,S) = \{1,2\}$
$d(Q,R) = 2$. Q is on the apex. R can be

$$\{A_3, B_3\} = 2\ ways$$

$d(R,S) = 1$ can be achieved in

$$R = A_3 \Rightarrow S \in \{A_1, A_2, B_3, B_2\}$$
$$R = B_3 \Rightarrow S \in \{B_1, B_2, A_2, A_3\}$$

The total number of ways is:

$$2 \cdot 4 = 8$$

The probability is

$$\frac{Sucessful\ Outcomes}{Total\ Outcomes} = \frac{4+8}{30} = \frac{12}{30} = \frac{4}{10} = \frac{2}{5}$$

D. 3D Paths

2.56 Length of a path: Polyhedron

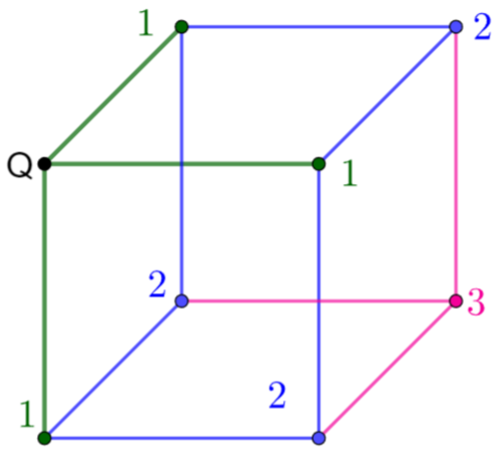
If A and B are vertices of a polygon, define the length of the path from A to B to be the minimum number of edges of the polyhedron one must traverse in order to connect A and B.

- If AB is an edge of the polyhedron, then the length of the path from A to B is 1
- If AC and CB are edges and AB is not an edge, then $d(A,B) = 2$.

Example 2.57

Let Q, R, and S be randomly chosen distinct vertices of a cuboid. What is the probability that the length of the path from Q to $d(Q,R) > d(R,S)$?

Rotate the cube so that the first selected point(Q) becomes the top left.



The number of ways to choose the remaining two distinct points out of the remaining 11 vertices is:

$$7 \cdot 6 = 42$$

Calculate distances (see diagram on the right):

- 3 points have distance 1
- 3 points have distance 2
- 1 point has distance 3
- 1 point has distance 0 (and is not relevant for this question)

The possible values for the distance are:

$$\{1,2,3\}$$

Consider cases for the LHS of the given inequality $d(Q,R) > d(R,S)$:

Case I: $\boldsymbol{d(Q,R) = 1}$
Not possible since $d(R,S)$ has minimum value 1.

Case II: $\boldsymbol{d(Q,R) = 2 > d(R,S) = 1}$
$d(Q,R) = 2$. A blue point can be chosen in

$$3\ ways$$

$d(R,S) = 1$. A point with a distance of 1 from R can be chosen in:

$$3\ ways$$

The total number of ways is:

$$3 \cdot 3 = 9$$

Case III: $\boldsymbol{d(Q,R) = 3 > d(R,S) = \{1,2\}}$
There is no choice for R since it must be the pink point.

S can be any point other than Q and R.

$$6\ choices$$

The total number of valid choices is:

$$0 + 9 + 6 = 15$$

The probability is

$$\frac{15}{42} = \frac{5}{14}$$

Example 2.58

If A and B are vertices of a polyhedron, define the distance $d(A,B)$ to be the minimum number of edges of the polyhedron one must traverse in order to connect A and B. For example, if AB is an edge of the polyhedron, then $d(A,B) = 1$, but if AC and CB are edges and AB is not an edge, then $d(A,B) = 2$. Let Q, R, and S be randomly chosen distinct vertices of a regular icosahedron (regular polyhedron made up of 20 equilateral triangles).

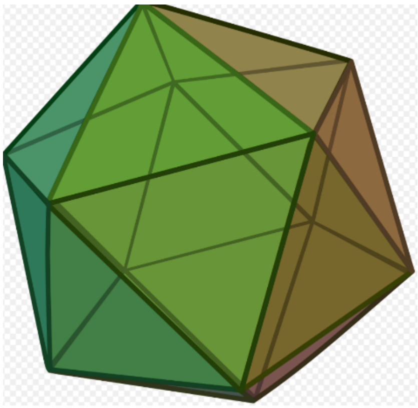
What is the probability that $d(Q,R) > d(R,S)$? **(AMC 2023 10A/25)**

Step I: Find the number of vertices

Substitute $F = 20\ (given), E = \frac{3F}{2} = \frac{3(20)}{2} = \frac{60}{2} = 30$ into the formula for Euler's Characteristic:

$$V - E + F = 2$$
$$V - 30 + 20 = 2$$
$$V = 12$$

Step II: Draw a diagram



A convex regular icosahedron has an apex, a pentagonal ring, another pentagonal ring below the first pentagonal ring, and a base.

Step III: Total Outcomes

Again, a convex regular icosahedron is symmetrical. Rotate the icosahedron so that the first selected point(Q) becomes the apex. The number of ways to choose the remaining two distinct points out of the remaining 11 vertices is:

$$11 \cdot 10 = 110$$

Step IV: Successful Outcomes

Consider distances from the apex.

$$Distance\ (Q - Apex\ to\ first\ pentagonal\ ring) = 1$$
$$Distance\ (Q - Apex\ to\ second\ pentagonal\ ring) = 2$$
$$Distance\ (Q - Apex\ to\ Base) = 3$$

The possible values for the distance are:

$$\{1,2,3\}$$

Consider cases for the LHS of the given inequality $d(Q,R) > d(R,S)$:

Case I: $\boldsymbol{d(Q,R) = 1}$

Not possible since $d(R,S)$ has minimum value 1.

Case II: $\boldsymbol{d(Q,R) = 2 > d(R,S) = 1}$

$d(Q,R) = 2$. Q is on the apex. A point on the second pentagonal ring can be chosen in

$$5\ ways$$

$d(R,S) = 1$ can be achieved in

$$5\ ways$$

The total number of ways is:

$$5 \cdot 5 = 25$$

Case III: $\boldsymbol{d(Q,R) = 3 > d(R,S) = \{1,2\}}$

$\boldsymbol{d(Q,R) = 3}$. There is no choice for R since it must be the base point.

S can be any point other than Q and R.

$$10\ choices$$

The total number of valid choices is:

$$0 + 25 + 10 = 35$$

The probability is

$$\frac{35}{110} = \frac{7}{22}$$

2.3 2D Cuts and Joins

A. Joins

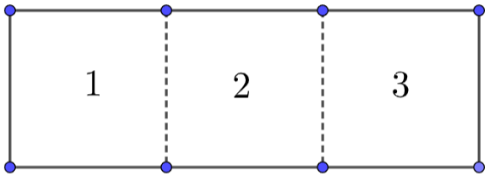
2.59 Cuts and Join

A cut is when a geometrical is split into parts.
A join is when two edges of a geometrical are joined together to make a single figure.

- In these kinds of questions, the visualization of the process is very important.
- If you are able to visualize the outcome accurately, you will be able to apply geometry, algebra and counting to get the information required in the question.

Example 2.60

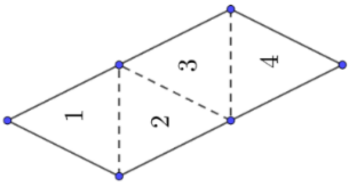
Three squares of side 1 m are joined end to end. Find the perimeter of the resulting polygon if, instead of three squares (but in the same pattern), there are 25 squares



$$\underbrace{25}_{Top} + \underbrace{25}_{Bottom} + \underbrace{2}_{Sides} = 52$$

Example 2.61

Triangles of side 1 m are joined end to end, from left to right, as shown in the diagram. Find the number of sides in the resulting polygon if there are 30 triangles



The first and the first triangle each have two sides which are not "inside" the diagram:

$$2 \times 2 = 4$$

The other triangles each have one side outside:

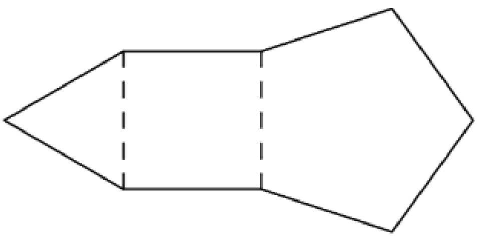
$$1 \times 28 = 28$$

Total

$$= 28 + 4 = 32$$

Example 2.62

Construct a square on one side of an equilateral triangle. On one non-adjacent side of the square, construct a regular pentagon, as shown. On a non-adjacent side of the pentagon, construct a hexagon. Continue to construct regular polygons in the same way, until you construct an octagon. How many sides does the resulting polygon have? (AMC 8 2009/9)



The Triangle and the Octagon have one side less. And the other shapes each have two sides less, giving a total of:

$$\underbrace{1}_{Triangle} + \underbrace{1}_{Octagon} + \underbrace{2 \times 4}_{Remaining\ Four\ Shapes} = 10$$

We can now subtract the 10 missing sides from the total sides, giving us

$$33 - 10 = 23$$

Triangle	Square	Pentagon	Hexagon	Heptagon	Octagon	Total	Repeated	Answer
3	4	5	6	7	8	33	10	23

B. Cuts

Example 2.63

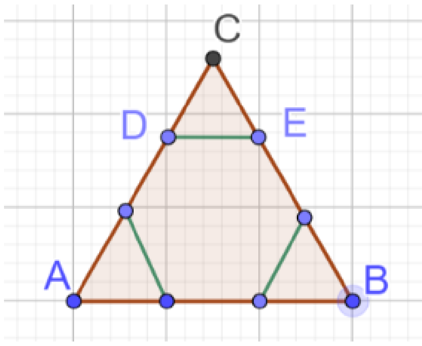
Find the number of sides, number of vertices, and the shape that remains when, from an equilateral triangle with length s, an equilateral triangle with length $\frac{s}{3}$ is chopped off

A. from a corner(II)
B. from each corner

Part A

Draw equilateral triangle ABC. Cut off equilateral CDE from the top. The remaining shape is DABE. It has 4 $corners$ and 4 $sides$.
DE is parallel to AB. So, it is a $Trapezium$.
Also, DA is equal to EB. Hence, it is an

$$Isosceles\ Trapezium$$



Part B

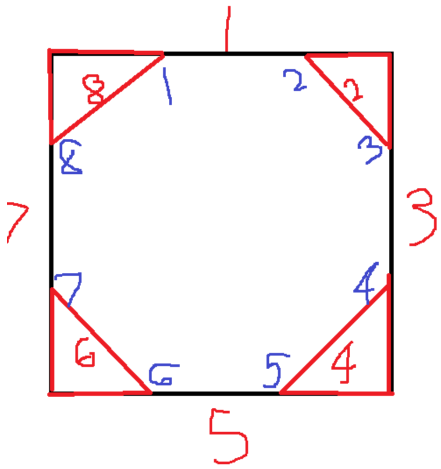
$$6\ corners, 6\ sides, Regular\ Hexagon$$

Example 2.64

Sally is playing with a cardboard square of side 12 inches. She cuts off four isosceles right-angled triangles, each with legs 12 cm, from each corner of the square. Find the number of sides, the number of vertices, and the name of the shape that remains. (For this question, you may use: 1 inch = 2.54 cm).

See the figure. The original square is in black, while the triangles which are cut are in red.
From the figure, we see that

$$No.of\ Sides = No.of\ Vertices = 8$$
$$Shape = Octagon$$



2.4 3D Cuts and Joins

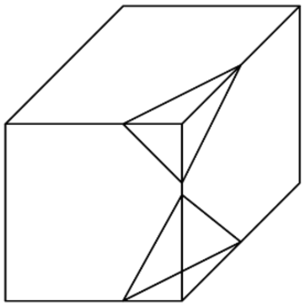
A. Edges and Vertices

Example 2.65

Each corner of a rectangular prism is cut off. Two (of the eight) cuts are shown.

A. How many edges does the new figure have? **(AMC 8 1990/18)**
B. How many vertices does the new figure have?

Note: Assume that the planes cutting the prism do not intersect anywhere in or on the prism.



Part A

The number of edges of a cube is 12. Each cut adds three edges, but does not take away any edge. Hence, the number of additional edges

$$\underbrace{3}_{\substack{Additional\\Edges}} \times \underbrace{8}_{\substack{No.of\\Vertices}} = 24$$

Hence, the total number of edges of the figure is

$$12 + 24 = 36$$

Part B

The number of vertices of a cube is 8. Each cut adds three vertices, and takes away one.

$$\underbrace{3-1}_{\substack{Additional\\Edges}} \times \underbrace{8}_{\substack{No.of\\Vertices}} = 2 \cdot 8 = 16$$

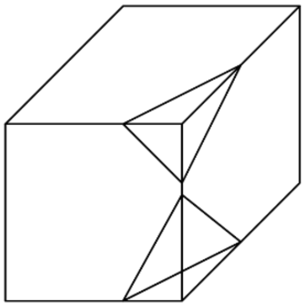
Hence, the total number of vertices of the figure is

$$8 + 16 = 24$$

Example 2.66

Each corner of a rectangular prism is cut off with a polygonal cut of n sides. Two (of the eight) cuts are shown assuming that the polygon is a triangle. Let E be the number of edges of the new figures, and V be the number of vertices of the new figure. Find $E + V$.

Note: Assume that the planes cutting the prism do not intersect anywhere in or on the prism.



Number of Edges

Each cut adds n edges, but does not take away any edge. Hence, the number of additional edges

$$\underbrace{n}_{\substack{Additional\\Edges}} \times \underbrace{8}_{\substack{No.of\\Vertices}} = 8n$$

Hence, the total number of edges of the figure is

$$12 + 8n$$

Number of Vertices

The number of vertices of a cube is 8. Each cut adds n vertices, and takes away one.

$$\underbrace{n-1}_{\substack{Additional\\Vertices}} \times \underbrace{8}_{\substack{No.of\\Vertices}} = 8n - 8$$

Hence, the total number of vertices of the figure is

$$8 + 8n - 8 = 8n$$

$$E + V = 12 + 8n + 8n = 12 + 16n$$

B. Unequal Sized Cuts

Example 2.67

A cube of edge 3 cm is cut into N smaller cubes, not all the same size. If the edge of each of the smaller cubes is a whole number of centimeters, then $N =$ (AMC 8 1991/24)

Method I

Volume of the original cube

$$= 3^3 = 27$$

Cubes with edge smaller than 3 cm = {2 cm, 1 cm}

Since all the cubes are not of the same size, there must be at least one cube of edge length 2 cm, which has volume

$$8\ cm$$

The only possible cubes which can be accommodated now are cubes of edge length 1.

The number of such cubes will be

$$27 - 8 = 19\ Cubes$$
$$Total\ Cubes = 1 + 19 = 20$$

Method II

$$A + 8B = 27$$

Substitute $B = 1$:

$$A + 8 = 27$$
$$A = 19$$

$$1 + 19 = 20$$

Example 2.68

A cube of edge 4 cm is cut into N smaller cubes, not all the same size. If the edge of each of the smaller cubes is a whole number of centimeters, then find the sum of the possible values of $N =$

$$A + 8B + 27C = 64$$

Case I: $C = 1$

$$A + 8B + 27(1) = 64$$
$$A + 8B = 37$$

If $C = 1$, the other cubes can only be of size 1. Hence, $B = 0$

$$A = 37$$

$$(A, B, C) = (37,0,27) \Rightarrow N = 38$$

Case II: $C = 0$

$$A + 8B = 64$$

B	1	2	3	4	5	6	7
A	56	48	40	32	24	16	8
N	57	50	43	36	29	22	15

The sum of the possible values of N

$$= n\left(\frac{f+l}{2}\right) = 7\left(\frac{57+15}{2}\right) = \frac{7(72)}{2} = 7(36) = 252$$

$$38 + 252 = 290$$

C. Equal Sized Cuts

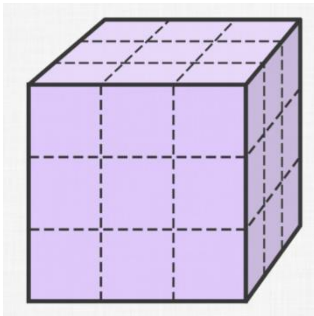
Example 2.69

A. A wooden cube is colored red. It has edge length 3. It is cut using 6 cuts to get a shape that looks like a Rubik's cube. It has 27 smaller cubes, each of edge length 1. Count the number of cubes with exactly n faces colored. $n = \{0,1,2,\dots\}$. Answer separately for each value of n.

B. The remaining faces which are not colored are now colored. Find the number of faces that need to be colored.

Part A

First, find the maximum value of n.



$$Max(n) = 3$$

The cubes which have three sides colored will be the corner cubes:

$$n_3 = No.of\ Vertices = 8$$

The cubes which have two sides colored will in the middle position in the edge:

$$n_2 = 4 \times 3 = 12$$

The cubes which have one sides colored will in the middle position in the face:

$$n_1 = 1 \times 6 = 6$$

There is one cube inside which has no face colored:

$$n_0 = 1$$

And we can verify by adding to check

$$8 + 12 + 6 + 1 = 27$$

Part B

Total Number of faces

$$= \underbrace{27 \times 6}_{\substack{Total\\Faces}} - \underbrace{9 \times 6}_{\substack{Colored\\Faces}} = 6(27-9) = 6(18) = 108$$

Example 2.70

A. A wooden cube is colored red. It has edge length 4. It is cut using 6 cuts to get a shape that has 64 smaller cubes, each of edge length 1. Count the number of cubes with exactly n faces colored. $n = \{0,1,2,\dots\}$. Answer separately for each value of n.

B. The remaining faces which are not colored are now colored. Find the number of faces that need to be colored.

$$n_3 = 8$$
$$n_2 = 2(4)(3) = 24$$
$$n_1 = 6(4) = 24$$
$$n_0 = 2^2 = 8$$

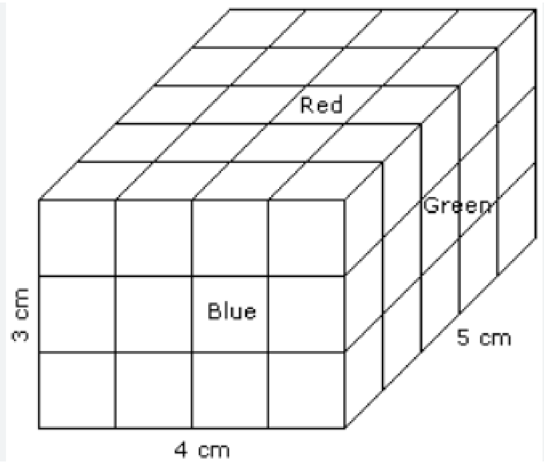
$$8 + 24 + 24 + 8 = 64$$

Total Number of faces

$$= \underbrace{64 \times 6}_{\substack{Total\\Faces}} - \underbrace{16 \times 6}_{\substack{Colored\\Faces}} = 6(48) = 288$$

Example 2.71: Making Cuts

A wooden cuboid of dimensions $3 \times 4 \times 5$ is colored red. It is cut to form 60 smaller cubes with edge length 1. Count the number of cubes with exactly n faces colored. $n = \{0,1,2, \dots\}$. Answer separately for each value of n.



The cubes which have three faces colored will be the corner cubes:

$$n_3 = No.of\ Vertices = 8$$

Cubes with two faces colored will be in the middle position in the edge:

$$n_2 = 4[(3-2) + (4-2) + (5-2)] = 4(1+2+3) = 4(6) = 24$$

The cubes which have one face colored will in the middle position in the face:

$$n_1 = 2\left(\underbrace{(1 \times 2)}_{3\times4\ Face} + \underbrace{(1 \times 3)}_{3\times5\ Face} + \underbrace{(2 \times 3)}_{4\times5\ Face}\right) = 2(2+3+6) = 2(11)$$
$$= 22$$

Cube inside which has no face colored:

$$n_0 = (3-2)(4-2)(5-2) = (1)(2)(3) = 6$$

And we can verify by adding to check:

$$8 + 24 + 22 + 6 = 60$$

Example 2.72: Making Cuts

A wooden cuboid is colored red. The cuboid has dimensions $l \times w \times h$ where l, w, h are natural numbers each greater than two. It is then cut to form lwh smaller cubes each with edge length 1. Count the number of cubes with exactly n faces colored. $n = \{0,1,2, \dots\}$. Answer separately for each value of n.

The cubes which have three faces colored will be the corner cubes:

$$n_3 = No.of\ Vertices = 8$$

The cubes which have two faces colored will in the middle position in the edge:

$$n_2 = 4[(l-2) + (w-2) + (h-2)] = 4(l+w+h-6) = 4l + 4w + 4h - 24$$

The cubes which have one face colored will in the middle position in the face:

$$n_1 = 2\left(\underbrace{(l-2)(w-2)}_{lw\ Face} + \underbrace{(w-2)(h-2)}_{wh\ Face} + \underbrace{(h-2)(l-2)}_{hl\ Face}\right)$$
$$= 2[(lw - 2l - 2w + 4) + (wh - 2w - 2h + 4) + (hl - 2h - 2l + 4)]$$
$$= 2[lw + wh + hl - 4l - 4w - 4h + 12]$$
$$= 2lw + 2wh + 2hl - 8l - 8w - 8h + 24$$

There is one cube inside which has no face colored:

$$n_0 = (l-2)(w-2)(h-2)$$
$$= (lw - 2l - 2w + 4)(h-2)$$
$$= lwh - 2hl - 2wh + 4h - 2lw + 4l + 4w - 8$$

Verification:
Add the numbers first:

$$8 - 24 + 24 - 8$$

Take terms which have exactly have one variable. Note that n_3 has no variable terms:

$$4l - 8l + 4l = 0$$
$$4w - 8w + 4w = 0$$
$$4h - 8h + 4h = 0$$

Take terms which have exactly two variables. Note that n_3 and n_2 have no two-variable terms:

$$2lw + 2wh + 2hl - 2hl - 2wh - 2lw = 0$$

And only n_1 has a three-variable term:

$$lwh$$

Which is what it should be.

2.73 Number of Cuboids

If

$$L = Lengthwise\ cuts, W = widthwise\ cuts, H = Heightwise\ cuts$$

the number of cuboids is:

$$(L+1)(W+1)(H+1)$$

2.74 Maximizing number of Cuboids

$H, W, and\ L$ should be as close to each other as possible, for a given value of

$$H + W + L$$

Example 2.75: Maximizing number of cuboids

I have a cuboid. I need to cut the cuboid into smaller cuboids by making cuts such that each cut is parallel to a face of the cuboid. Find the maximum number of cuboids that I can make if I make:

A. 6 Cuts
B. 7 Cuts
C. 9 Cuts
D. 11 Cuts
E. $n\ Cuts, n \in \mathbb{N}$

Part A

$$\underbrace{(1+1)}_{\substack{1\ Cut \\ Lengthwise}}\underbrace{(1+1)}_{\substack{1\ Cut \\ WidthWise}}\underbrace{(4+1)}_{\substack{4\ Cuts \\ Heightwise}} = 2 \times 2 \times 5 = 20$$

$$(2+1)(2+1)(2+1) = 3^3 = 27$$

Part B

$$(2+1)(2+1)(3+1) = 3^2 \times 4 = 36$$

Part C

$$(3+1)(3+1)(3+1) = 4^3 = 64$$

Part D

$$(3+1)(3+1)(5+1) = 4^2 \times 6 = 96$$
$$(3+1)(4+1)(4+1) = 4 \times 5^2 = 100$$

Part E

n is a multiple of 3: $n = 3x$

$$(x+1)(x+1)(x+1)$$

n is one more than a multiple of $3: n = 3x+1$

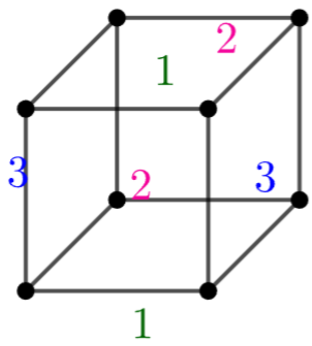
$$(x+1)(x+1)(x+2)$$

n is two more than a multiple of $3: n = 3x+2$

$$(x+1)(x+2)(x+2)$$

Example 2.76

Each face of a cube is painted with exactly one color. What is the smallest number of colours needed to paint a cube so that no two faces that share an edge are the same color? **(Gauss Grade 7 2014/22)**



Top& Bottom: Green
Front and Back: Pink
Right and Left: Blue

Hence, the final answer is 3.

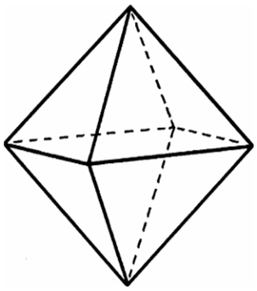
Example 2.77

A. Each face of an octahedron is numbered from 1 to 8. The top four faces are numbered from 1 to 4. The face below 1 is numbered 5, the face below 2 is numbered 6. What is the smallest number of colours needed to paint the faces so that no two faces that share an edge are the same color? Give the color scheme: $Face\ 1: Color\ X, Face\ 2: Color\ Y$.
B. Pyramid P_1 has a regular dodecagon for its base. It is joined to identical pyramid P_2 at the base. Each face of P_1 is numbered from 1 to 12 in clockwise order. Each face of P_2 is numbered from 13 to 24, with 13 below 1, 14 below 2 and so on. The shape is colored with the minimum number of colors such that no two faces that share an edge have the same color. If red is one of the colors used for face 2, find the sum of the face numbers that are colored red.

Hints: Dodecagon is a 12-sided polygon

Part A

$$Red: 1,3,6,8$$
$$Blue: 2,4,5,7$$



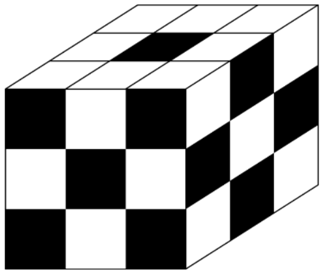
Part B

$$Blue: 1,3,5,7,9,11,14,16,18,20,22,24$$
$$Red: 2,4,6,8,10,12,13,15,17,19,21,23$$

$$2+4+6+8+10+12+13+15+17+19+21+23$$
$$=\frac{25\times 12}{2}=\frac{300}{2}=150$$

Shortcut:

$$\frac{1+2+\cdots+24}{2}=\frac{\frac{25\times 24}{2}}{2}=25\times 6=150$$



Example 2.78

The large cube shown is made up of 27 identical sized smaller cubes. For each face of the large cube, the opposite face is shaded the same way. The total number of smaller cubes that must have at least one face shaded is: **(AMC 8 1987/7)**

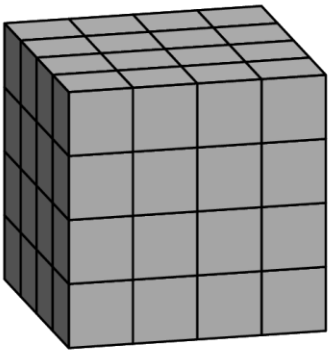
None of the smaller cubes have more than one face painted. We can see

$$10\ Faces$$

And we can see half the cube.

So, the total of number of cubes with exactly one face painted

$$10\times 2=20$$



Example 2.79

Akash's birthday cake is in the form of a $4\times 4\times 4$ inch cube. The cake has icing on the top and the four side faces, and no icing on the bottom. Suppose the cake is cut into 64 smaller cubes, each measuring $1\ \times 1\ \times 1$ inch, as shown below. How many small pieces will have icing on exactly two sides? **(AMC 8 2020/9)**

From the top layer

$$2+2+2+2=8$$

From the 2^{nd} and 3^{rd} layer:

$$2nd: 1+1+1+1=4$$
$$3rd: 1+1+1+1=4$$

From the bottom layer, we get the four corner cubes:

$$1+1+1+1=4$$

$$8+4+4+4=20$$

Example 2.80

A wooden cube n units on a side is painted red on all six faces and then cut into n^3 unit cubes. Exactly one-fourth of the total number of faces of the unit cubes are red. What is n? **(AMC 10A 2005/11)**

Surface area of all faces of the unit cubes

$$=\underbrace{6}_{\substack{Surface\ Area\\ of\ 1\ Cube}}\times\underbrace{n^3}_{\substack{No.\ of\\ Cubes}}=6n^3$$

Since the cubes are unit cubes, area of each face is 1. Surface area of all faces of the original cube (which is also equal to the number of faces of the unit cubes which are colored red):

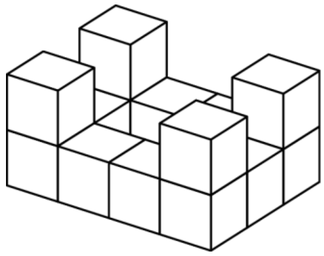
$$=6n^2$$

$$6n^2=\frac{1}{4}(6n^3)\Rightarrow 4n^2=n^3\Rightarrow 4=n$$

Example 2.81

Fourteen white cubes are put together to form the figure on the right. The complete surface of the figure, including the bottom, is painted red. The figure is then separated into individual cubes. How many of the individual cubes have

A. exactly four red faces? **(AMC 8 2003/13)**
B. exactly five red faces?
C. exactly three red faces?



$$\begin{gathered}\textit{Exactly } 4 \textit{ Red Faces}: 6\\ \textit{Exactly } 5 \textit{ Red Faces}: 4\\ \textit{Exactly } 3 \textit{ Red Faces}: 4\end{gathered}$$

Example 2.82

A $4 \times 4 \times 4$ cubical box contains 64 identical small cubes that exactly fill the box. How many of these small cubes touch a side or the bottom of the box? **(AMC 8 1998/21)**

Cubes that we do not want:

$$\begin{gathered}\textit{Top Layer}: 4 \textit{ Cubes}\\ 2^{nd} \textit{ Layer}: 4 \textit{ Cubes}\\ 3^{rd} \textit{ Layer}: 4 \textit{ Cubes}\end{gathered}$$

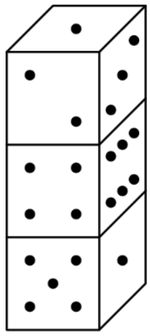
$$\textit{Total} = 4 + 4 + 4 = 12$$

Cubes that we do want

$$= 64 - 12 = 52$$

Example 2.83

Three dice with faces numbered 1 through 6 are stacked as shown. Seven of the eighteen faces are visible, leaving eleven faces hidden (back, bottom, between). The total number of dots NOT visible in this view is **(AMC 8 2000/8)**



The total number of dots on a die is

$$1 + 2 + 3 + 4 + 5 + 6 = 21$$

The total dots visible

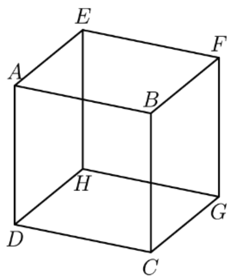
$$= 1 + 2 + 3 + 4 + 5 + 6 + 1 = 22$$

The non visible dots

$$= 21 \times 3 - 22 = 41$$

Example 2.84

How many pairs of parallel edges, such as $\overline{AB}$ and $\overline{GH}$ or $\overline{EH}$ and $\overline{FG}$, does a cube have? **(AMC 8 2015/12)**



Note that every edge is parallel to three other edges.
The total number of pairs is

$$\underbrace{12}_{\substack{No.of\\Edges}} \times \underbrace{3}_{\substack{Edges\ to\\pair\ with}} \times \underbrace{\frac{1}{2}}_{\substack{Overcounting\\Factor}} = 36 \times \frac{1}{2} = 18$$

3. Combinations in Geometry

3.1 Circles and Regular Polygons

A. Number of Triangles

3.1 Combinations

The number of ways of choosing k objects out of n distinct objects is:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

3.2 Triangle

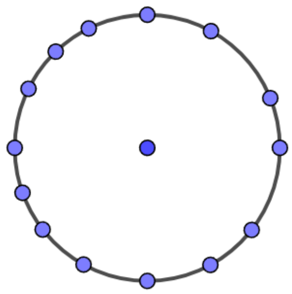
A triangle is formed from three non-collinear points.

Example 3.3

What is the number of triangles that can be made from

A. n distinct points on a circle?

B. n points, no three of which are collinear, in a plane?



Part A

Any three distinct points on a circle are non collinear. Hence, out of the available n points, we should choose 3, which can be done in:

$$\binom{n}{3}$$

Part B

Assuming the answer to the question is unique, then the n points could be arranged in a circle. And in that case, the answer is the same as the previous part

$$\binom{n}{3}$$

Example 3.4

Three (not necessarily distinct) points are chosen at random from n distinct points on a circle. Determine the probability that the three points form a triangle.

Total Outcomes

The total number of choices for the three points are:

$$\underbrace{n}_{\substack{First\\Point}} \times \underbrace{n}_{\substack{Second\\Point}} \times \underbrace{n}_{\substack{Third\\Point}} = n^3$$

Successful Outcomes

As discussed above, the number of triangles that can be formed is:

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$$

The probability

$$= \frac{Successful\ Outcomes}{Total\ Outcomes} = \frac{\frac{n(n-1)(n-2)}{6}}{n^3} = \frac{n(n-1)(n-2)}{6} \cdot \frac{1}{n^3} = \frac{(n-1)(n-2)}{6n^2}$$

Example 3.5

Three (not necessarily distinct) points are chosen at random from n distinct points on a circle. The probability that the three points form a triangle is $\frac{2}{25}$. Determine the number of triangles that can be formed on the circle.

Method I: Shortcut

$$\frac{2}{25}=\frac{2}{5^2}=\frac{12}{6\cdot 5^2}=\frac{(4)(3)}{6\cdot 5^2}=\frac{(5-1)(5-4)}{6\cdot 5^2}=\frac{(n-1)(n-2)}{6n^2}$$
$$n=5$$

Method II: Quadratic

$$\frac{(n-1)(n-2)}{6n^2}=\frac{2}{25}$$
$$13n^2-75n+50=0$$
$$(13n-10)(n-5)=0$$
$$n=5\ OR\ n=\frac{10}{13}$$
$$n=5$$
$$\binom{n}{3}=\binom{5}{3}=10$$

We can also calculate the number of triangles as:

$$\underbrace{125}_{\substack{No.\ of\\ Ways}}\cdot\underbrace{\frac{2}{25}}_{Probability}=10$$

B. Regular Polygons

3.6 Equally spaced points make a regular polygon

n equally spaced points on a circle can be connected to make a regular, convex polygon.

- Regular polygons are equiangular (*equal angles*) and equilateral (*equal sides*)
- Convex polygons are those where every internal angle is $< 180°$

For example:
- Three equally spaced points around a circle can be joined to form an equilateral triangle.
- Four equally spaced points around a circle can be joined to form a square.

3.7 Vertices of a regular polygon are concylic

A regular polygon is concyclic, which means its vertices lie on a circle

For example:
- The vertices of a regular hexagon all lie on a circle, with the vertices being equally spaced around the circle.

In some sense, this is the converse of the property above.

C. Number of Angles

Example 3.8

8 points are spaced equally on a circle. Two points are chosen at random. What is the probability that the angle

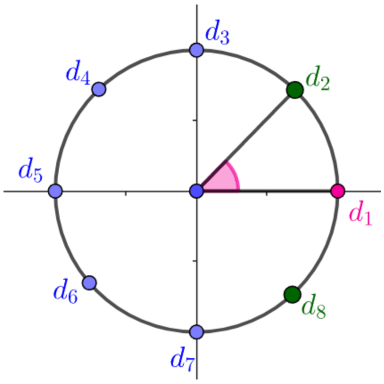
formed by the two points and the center of the circle is 45°?

Method I: Combinations
The number of ways to choose 2 points out of 8 is:

$$\binom{8}{2} = 28$$

For the angle to be 45°, we need to pick adjacent points:

$$d_1d_2, d_2d_3, \dots d_7d_8, d_8d_1 \Rightarrow 8\ pairs$$



The probability is:

$$= \frac{Successful\ Outcomes}{Total\ Outcomes} = \frac{8}{28} = \frac{2}{7}$$

Method II: Symmetry
Pick the first point, and rotate the circle so that the first point is rightmost.
There are two choices out of the remaining 7 points that result in an angle of 45°. Hence, the probability is:

$$\frac{2}{7}$$

Example 3.9

10 points are spaced equally on a circle. Two points are chosen at random. What is the probability that the angle formed by the two points and the center of the circle is less than 80°?

The angle between two

- adjacent points is 36°
- points that have one point between them is 72°
- points that have one point between them is 108°

Method I: Combinations
The number of ways to choose 2 points out of 10 is:

$$\binom{10}{2} = 45$$

For the angle to be 36°, we need to pick adjacent points:

$$d_1d_2, d_2d_3, \dots d_9d_{10}, d_{10}d_1 \Rightarrow 10\ pairs$$

For the angle to be 72°, we need to pick adjacent points:

$$d_1d_3, d_2d_4, \dots d_9d_{11}, d_{10}d_2 \Rightarrow 10\ pairs$$

The probability is:

$$= \frac{Successful\ Outcomes}{Total\ Outcomes} = \frac{20}{45} = \frac{4}{9}$$

Method II: Symmetry
We will do this using symmetry. Pick the first point, and rotate the circle so that the first point is rightmost.
Out of the remaining 9 points, 4 are valid.

$$\frac{Successful\ Outcomes}{Total\ Outcomes} = \frac{4}{9}$$

Example 3.10: Combinations

10 points are spaced equally on a circle. Three points are chosen at random. What is the probability that at least one angle formed by any two of the chosen points and the center of the circle is 36°?

Total Outcomes

The number of ways to pick 3 points out of 10 is:

$$\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{6} = 120$$

Successful Outcomes

Case I: Three adjacent points

$$d_1 d_2 d_3, d_2 d_3 d_4, \dots, d_{10} d_1 d_2 \Rightarrow 10\ triplets$$

Case II: Exactly two adjacent points

Pick a pair of adjacent points in:

$$d_1 d_2, d_2 d_3, \dots d_9 d_{10}, d_{10} d_1 \Rightarrow 10\ ways$$

Pick a third point that is not adjacent to the first two. For each pair above, there are

$$6\ points$$

The total number of ways to achieve Case II is then, by the multiplication principle:

$$10 \cdot 6 = 60$$

Probability

The probability is then:

$$\frac{Successful\ Outcomes}{Total\ Outcomes} = \frac{10 + 60}{120} = \frac{70}{120} = \frac{7}{12}$$

Example 3.11: Symmetry

10 points are spaced equally on a circle. Three points are chosen at random. What is the probability that at least one angle formed by any two of the chosen points and the center of the circle is 36°?

Pick the first point (d_1), and rotate the circle so that the first point is rightmost.
Pick two points out of remaining nine points, which can be done in

$$\binom{9}{2} = 36\ ways$$

Complementary Probability

Picking d_2 or d_{10} results in angle of 36°. Hence, to not form an angle of 36°, we need to pick two points out of the remaining 7, which is:

$$\binom{9-2}{2} = \binom{7}{2} = 21\ ways$$

However, we do not want adjacent pairs since that will result in an angle of 36°:

$$d_3 d_4, d_4 d_5, \dots, d_8 d_9 \Rightarrow 6\ pairs$$

The final answer for the number of pairs that do not result in an angle of 36° is:

$$21 - 6 = 15$$

Hence, the probability is:

$$\frac{15}{36} = \frac{5}{12}$$

And the probability that we want is:

$$1 - \frac{5}{12} = \frac{7}{12}$$

D. Number of Right Triangles

3.12 Thales Theorem

If AB is a diameter of a circle, and C is a point on the circumference of the circle, then

$$\angle ABC = 90°$$

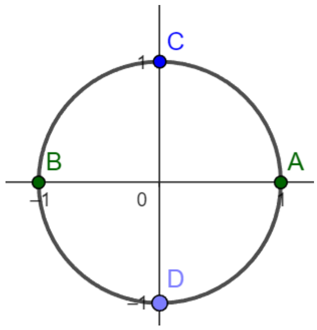
If A, B, C are points on the circumference of the circle and $\angle ABC = 90°$, then

$$BC \text{ is a diameter}$$

3.13 Antipodal Points

Points on a circle that are diametrically opposite each other are called

$$\textit{antipodal}$$



In other words, antipodal points are points that lie on the opposite ends of a diameter.

Example 3.14

Identify the antipodal points in the diagram alongside.

$$A, B \text{ and } C, D$$

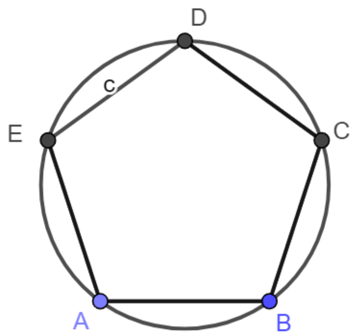
Example 3.15

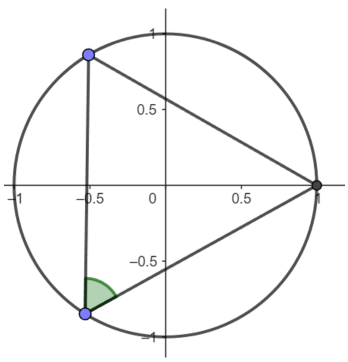
Explain why if the number of vertices of a regular polygon is odd, then you cannot form any right triangles by choosing points from among the vertices of that polygon.

If the number of vertices of the polygon is odd, the spacing between them is unequal, and hence, none of the vertices are antipodal.
None of them lie on opposite ends of a diameter.

Hence, no right triangles can be formed.

$$\textit{Not possible}$$





Example 3.16

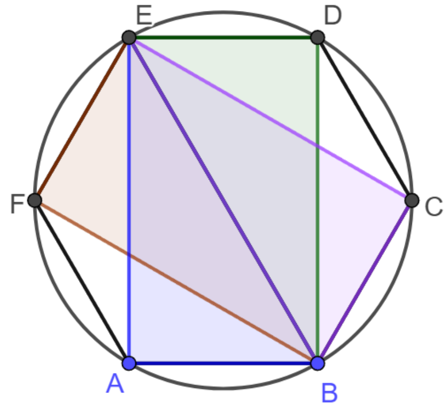
Square

There are two diagonals. Each diagonal results in two triangles.

$$2 \cdot 2 = 4$$

Example 3.17

Determine the number of right triangles that can be formed by choosing points from among the vertices of a regular hexagon



From the diameter/diagonal EB, we get four triangles:

$$\Delta EBF, \Delta EBA, \Delta EBD, \Delta EBC$$

Similarly, from the diagonals FC and DA, we get

$$Four\ triangles\ each$$

Hence, the total number of right triangles is:

$$3 \cdot 4 = 12$$

Example 3.18

Determine the number of right triangles that can be formed by choosing points from among the vertices of a regular $n-$sided polygons, where $n \le 12$.

Octagon

There are four diagonals that connect antipodal points. Each diagonal can be connected to six points:

$$4 \cdot 6 = 24$$

Decagon

There are five diagonals that connect antipodal points. Each diagonal can be connected to eight points:

$$5 \cdot 8 = 40$$

Dodecagon

$$6 \cdot 10 = 60$$

3.19 Function

A function provides an output for every valid input.

The square root function is $f(x) = \sqrt{x}$

$$f(4) = 2, f(2) = \sqrt{2}, f(-1)\ is\ not\ defined$$

A piecewise function is a function that splits its output into cases depending on the input:

$$f(x) = |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$$

Example 3.20

Determine the number of right triangles that can be made by choosing points from the vertices of a regular polygon.
Write your answer as a piecewise function.

Case I: n is odd

Not possible

Case II: n is even

The diameter can be selected in:

$$\frac{n}{2}$$

The third point can be selected in

$$n-2$$

The final answer is:

$$\frac{n}{2}(n-2)$$

Piecewise Function

$$n \geq 3: f(n) = \begin{cases} 0, & n \text{ is odd} \\ \frac{n}{2}(n-2), & n \text{ is even} \end{cases}$$

Example 3.21

Three distinct vertices of a regular 600-sided polygon are joined. Find the probability that the triangle formed has a right angle. **($MA\Theta$, 2019, Alpha, Combinatorics and Probability/27)**

The probability is:

$$\frac{Successful\ Outcomes}{Total\ Outcomes} = \frac{\frac{600}{2}(598)}{\binom{600}{3}} = \frac{\frac{600}{2}(598)}{\frac{600 \cdot 599 \cdot 598}{6}} = \frac{\frac{1}{2}}{\frac{599}{6}} = \frac{3}{599}$$

Example 3.22

Three distinct points are picked at random from the vertices of a n sided regular polygon. Determine the probability that the points form a right triangle. Write your answer as a piecewise function $p(n)$.

Case I: n is odd

Not possible

$$p(n) = 0$$

Case II: n is even

$$p(n) = \frac{\frac{n}{2}(n-2)}{\binom{n}{3}} = \frac{\frac{n}{2}(n-2)}{\frac{n(n-1)(n-2)}{6}} = \frac{\frac{1}{2}}{\frac{n-1}{6}} = \frac{3}{n-1}$$

Piecewise Function

$$For\ n \geq 3: p(n) = \begin{cases} 0, & n \text{ is odd} \\ \frac{3}{n-1}, & n \text{ is even} \end{cases}$$

E. Functional Equations

3.23 Functional Equation

An equation that has functions in it (instead of, or along with variables) is a functional equation.

$f(x) = 2^x$ satisfies:

$$f(x)f(y) = f(x+y)$$

$$LHS = f(x)f(y) = 2^x \cdot 2^y = 2^{x+y} = f(x+y)$$

Example 3.24

Let $f(n)$ be the number of right triangles that can be made by connecting the vertices of a n −sided regular polygon. Find the sum of all values of $n \le 1000$ such that

$$\frac{f(n)}{f(n+2)} = \frac{1}{2}$$

Use the definition:

$$\frac{\frac{n}{2}(n-2)}{\frac{n+2}{2} \cdot n} = \frac{1}{2}$$
$$\frac{n-2}{n+2} = \frac{1}{2}$$
$$2n - 4 = n + 2$$
$$n = 6$$

Sum is also:

$$6$$

3.25 No Solutions

It is possible that an equation (functional or otherwise) has no solutions.

Example 3.26

Let $f(n)$ be the number of right triangles that can be made by connecting the vertices of a n −sided regular polygon. Find the sum of all values of $n \le 1000$ such that

$$\frac{f(n)}{f(n+2)} = \frac{1}{6}$$

$$\frac{\frac{n}{2}(n-2)}{\frac{n+2}{2} \cdot n} = \frac{1}{6}$$
$$\frac{n-2}{n+2} = \frac{1}{6}$$
$$6n - 12 = n + 2$$
$$5n = 14$$
$$n = \frac{14}{5}$$

$$\textit{Not possible}$$

Hence:

$$n \in \{\phi\}$$

Sum is

$$0$$

Example 3.27

Let $f(n)$ be the number of right triangles that can be made by connecting the vertices of a n −sided regular polygon. Find the sum of all values of n such that

$$f(n) + f(n+4) = 84$$

Use the definition of $f(n)$ for even n:

$$\frac{n}{2}(n-2) + \frac{n+4}{2}(n+2) = 84$$

Eliminate fractions:

$$n(n-2) + (n+4)(n+2) = 168$$
$$n^2 - 2n + n^2 + 6n + 8 = 168$$
$$2n^2 + 4n - 160 = 0$$
$$n^2 + 2n - 80 = 0$$
$$(n+10)(n-8) = 0$$
$$n \in \{-10, 8\}$$

$$n = 8$$

Example 3.28

Let $f(n)$ be the number of right triangles that can be made by connecting the vertices of a n −sided regular polygon. For how many values of $n < 1000$ is $\sqrt{f(n) + f(n+2)}$ a positive integer?

Case I: n is odd

If n is odd, then you cannot form any triangles, and hence

$$\sqrt{f(n) + f(n+2)} = \sqrt{0+0} = 0$$

Case II: n is even

Use the definition of $f(n)$ for even n:

$$\sqrt{\frac{n}{2}(n-2) + \frac{n+2}{2}(n)}$$

Factor out $\frac{n}{2}$

$$= \sqrt{\frac{n}{2}[n+2+n-2]} = \sqrt{\frac{1}{2}n[2n]} = \sqrt{n^2} = n \in \mathbb{Z}^+$$

Hence, if n is even, $\sqrt{f(n) + f(n+2)}$ is always an integer. So, we need to count the number of valid values of n, keeping in mind that n cannot be odd, and n must be greater than 2:

$$n \in \{4, 6, 8, \dots, 998\}$$

To make it easier to count, divide each value by 2:

$$\{2, 3, 4, \dots, 499\} \Rightarrow 499 - 2 + 1 = 498$$

F. Number of Equilateral Triangles

Example 3.29

Determine the number of equilateral triangles that can be formed with regular $n-$ sided polygons, where $n \leq 12$

Square, Pentagon, Heptagon, Octagon, Decagon, and Undecagon

If the number of vertices of the polygon is not a multiple of three, then the vertices will be spaced in such a way that no three can be selected to be symmetrical, and hence the number of triangles is:

$$0$$

Triangle

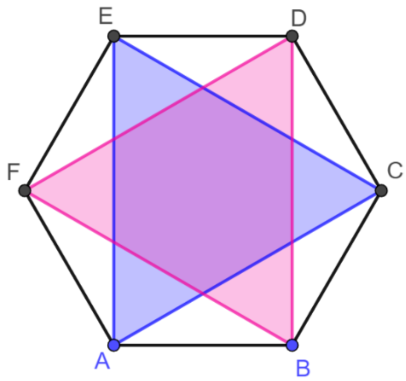
$$1$$

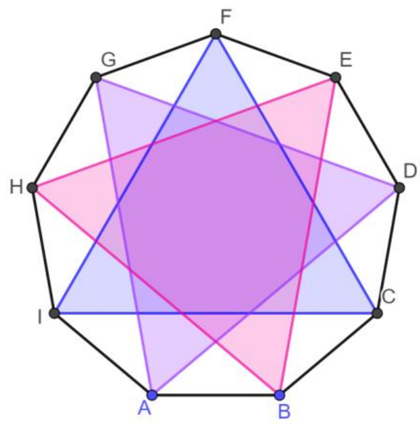
Hexagon

$$2$$

Nonagon

$$3$$





Example 3.30

Determine the number of equilateral triangles that can be formed with regular $n-$ sided polygons.
Write as a piecewise function.

$$n \geq 3: p(n) = \begin{cases} \frac{n}{3}, & n = 3k, k \in \mathbb{N} \\ 0, & otherwise \end{cases}$$

(Calculator) Example 3.31

An urn has balls numbered $3,4, \dots, 12$. A ball is chosen from the urn at random. The number on the ball is N. Three points are chosen at random from the vertices of a $N-$sided regular polygon. What is the probability that the three points from an equilateral triangle.

$$= \underbrace{\frac{1}{10} \cdot 1}_{Triangle} + \underbrace{\frac{1}{10} \cdot \frac{2}{\binom{6}{3}}}_{Hexagon} + \underbrace{\frac{1}{10} \cdot \frac{3}{\binom{9}{3}}}_{Nonagon} + \underbrace{\frac{1}{10} \cdot \frac{4}{\binom{12}{3}}}_{Dodecagon}$$

$$= \frac{1}{10}\left(1 + \frac{2}{20} + \frac{3}{72} + \frac{4}{660}\right) = \frac{101}{880}$$

G. Number of Isosceles Triangles

For creating right triangles, we found that the formula was different based on whether n was odd or even

$$n \geq 3: f(n) = \begin{cases} 0, & n \text{ is odd} \\ \frac{n}{2}(n-2), & n \text{ is even} \end{cases}$$

For equilateral triangles, the formula is based on whether the number of points is a multiple of 3:

$$n \geq 3: p(n) = \begin{cases} \frac{n}{3}, & n = 3k, k \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

Example 3.32

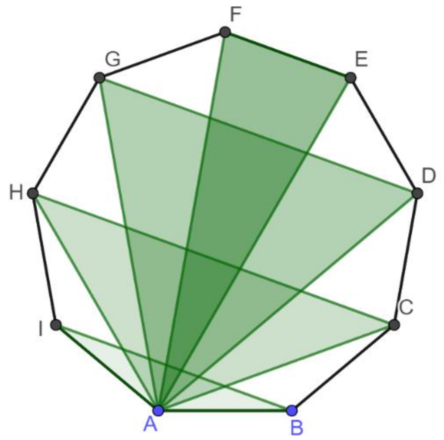
Determine the number of isosceles triangles that can be formed with regular $n-$ sided polygons.
Write as a piecewise function.

Cases based on parity: Odd and Even

Case I: n is odd

To make an isosceles triangle, we can first choose the vertex among any of the given n points in

$$n \text{ ways}$$



We can choose the length of the leg of the triangle in $n-1$ ways. But (see diagram alongside), choosing one point also chooses the point opposite it by symmetry. Hence, the number of ways to choose the legs is:

$$\frac{n-1}{2}$$

The total number of ways is:

$$\frac{n(n-1)}{2}$$

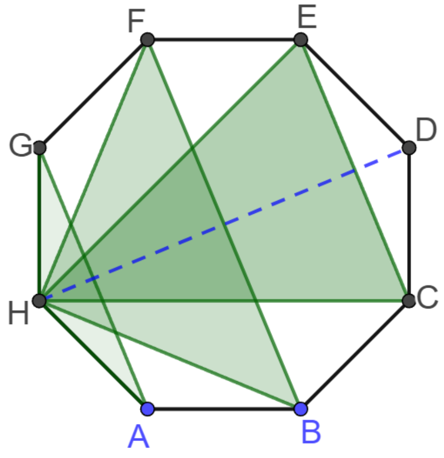
Note: The diagram has the case with $n=9$, and shows the $\frac{9-1}{2} = \frac{8}{2} = 4$ triangles originating from A, out of the total

$$9\left(\frac{9-1}{2}\right) = 9(4) = 36 \text{ triangles}$$

Case II: n is even

To make an isosceles triangle, we can first choose the vertex among any of the given n points in

$$n \text{ ways}$$



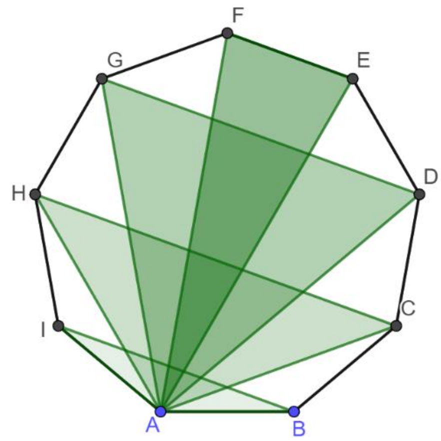
We can choose the length of the leg of the triangle in $n-1$ ways. But (see diagram alongside), choosing one point also chooses the point opposite it by symmetry. Hence, the number of ways to choose the legs is:

$$\frac{n-2}{2}$$

The total number of ways is:

$$\frac{n(n-2)}{2}$$

Cases based on multiple of 3



If n is not a multiple of 3, the number of equilateral triangles will be

$$0$$

If n is a multiple of 3, the number of equilateral triangles will be

$$\frac{n}{3}$$

ΔADG is counted among the triangles

$$\begin{gathered} with\ vertex\ A \\ with\ vertex\ D \\ with\ vertex\ G \end{gathered}$$

Hence, every equilateral triangle is counted thrice, for a total of:

$$3\left(\frac{n}{3}\right) = n\ triangles$$

Hence, we are overcounting:

$$n - \frac{n}{3} = \frac{2n}{3}\ triangles$$

Final Answer

Cases based on parity classified the numbers based on whether they were a multiple of 2 or not.
Cases based on equilateral triangles classified the numbers based on whether they were a multiple of 3 or not.

To combine the cases, we need to consider:

$$LCM(2,3) = 6$$

	$Not\ a\ multiple\ of\ 3$	$A\ multiple\ of\ 3$
Odd	$\{\mathbf{6k+1, 6k+5}\}$	$\{\mathbf{6k+3}\}$
Even	$\{\mathbf{6k+2, 6k+4}\}$	$\{\mathbf{6k}\}$

	$Not\ a\ multiple\ of\ 3$	$A\ multiple\ of\ 3$
Odd	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2} - \frac{2n}{3}$
Even	$\frac{n(n-2)}{2}$	$\frac{n(n-2)}{2} - \frac{2n}{3}$

For non-negative integers k:

$$For\ n \geq 3: f(n) = \begin{cases} \frac{n(n-2)}{2} - \frac{2n}{3}, & n = 6k \\ \frac{n(n-1)}{2}, & n = 6k+1\ or\ 6k+5 \\ \frac{n(n-2)}{2}, & n = 6k+2\ or\ 6k+4 \\ \frac{n(n-1)}{2} - \frac{2n}{3}, & n = 6k+3 \end{cases}$$

1 Pending

Example 3.33
Probability Question

H. Challenge Problem

Example 3.34
For each integer $n \geq 3$, let $f(n)$ be the number of 3-element subsets of the vertices of the regular n-gon that are the vertices of an isosceles triangle (including equilateral triangles). Find the sum of all values of n such that $f(n+1) = f(n) + 78$. **(AIME 2017/II/13)**

$$f(n) = \begin{cases} \frac{n(n-2)}{2} - \frac{2n}{3}, & n = 6k \\ \frac{n(n-1)}{2}, & n = 6k+1 \text{ or } 6k+5 \\ \frac{n(n-2)}{2}, & n = 6k+2 \text{ or } 6k+4 \\ \frac{n(n-1)}{2} - \frac{2n}{3}, & n = 6k+3 \end{cases}$$

Case I: $n = 6k \Rightarrow n + 1 = \mathbf{6k+1}$

$$f(n+1) = f(n) + 78$$
$$\frac{(n+1)(n)}{2} = \frac{n(n-1)}{2} + 78$$
$$n = 36 = 6(6) \Rightarrow Valid$$

Case II: $n = 6k + \mathbf{1} \Rightarrow n + 1 = \mathbf{6k+2}$

$$f(n+1) = f(n) + 78$$
$$\frac{(n+1)(n-1)}{2} = \frac{n(n-2)}{2} - \frac{2n}{3} + 78$$
$$n = 157 = 26(6) + 1 \Rightarrow Valid$$

Case III: $n = 6k + \mathbf{2} \Rightarrow n + 1 = \mathbf{6k+3}$

$$f(n+1) = f(n) + 78$$
$$\frac{(n+1)(n)}{2} - \frac{2(n+1)}{3} = \frac{n(n-2)}{2} + 78$$
$$n = \frac{472}{5}$$

Case IV: $n = 6k + \mathbf{3} \Rightarrow n + 1 = \mathbf{6k+4}$

$$f(n+1) = f(n) + 78$$
$$\frac{(n+1)(n-1)}{2} = \frac{n(n-1)}{2} - \frac{2n}{3} + 78$$

3.2 3D Shapes

A. Number of Triangles

When working with 3D shapes, visualization is a key skill.

Example 3.35

How many distinct triangles can be constructed by connecting three different vertices of a cube? (Two triangles are distinct if they have different locations in space.) **(MathCounts 1996 National Sprint)**

A cube has eight vertices, no three of which are collinear. Hence, we can form triangles by selecting any three of the eight vertices, which can be done in:

$$\binom{8}{3} = \frac{8 \times 7 \times 6}{6} = 56\,Triangles$$

Example 3.36

How many distinct triangles can be constructed by choosing points from the vertices of an octahedron? (Two triangles are distinct if they have different locations in space.)

An octahedron has six vertices, no three of which are collinear. Hence, we can form triangles by selecting any three of the six vertices, which can be done in:

$$\binom{6}{3} = \frac{6 \times 5 \times 4}{6} = 20\,Triangles$$

Example 3.37

Three points are chosen from the vertices of an octahedron. What is the probability that some part of the triangle so formed lies inside the octahedron?

We remove the triangles that lie on purely the surface of the octahedron. These are the 8 faces of the octahedron:

$$P = \frac{Successful\ Outcomes}{Total\ Outcomes} = \frac{\binom{6}{3} - 8}{\binom{6}{3}} = \frac{20 - 8}{20} = \frac{12}{20} = \frac{3}{5}$$

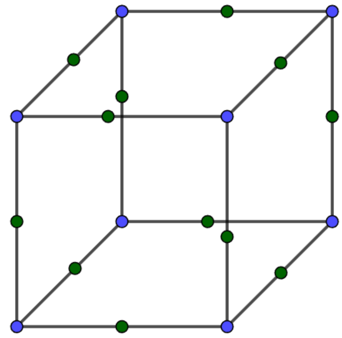
3.38 Degenerate Triangles

A “triangle” formed from three collinear points is a degenerate triangle.
Such a triangle has zero area.

- When you apply the determinant formula for area of a triangle, it is possible to get a value of zero.
- This indicates that the points are collinear, and the triangle is degenerate.

Example 3.39

Let V be the set of vertices of a cube. Let M be the set of midpoints of edges of the same cube. Let P be the union of V and M. Determine the number of distinct triangles (with non-zero area) that can be formed by choosing points from P. (Two triangles are distinct if they have different locations in space.)



The number of ways in which we can select three points out of $8 + 12 = 20$ is

$$\binom{20}{3} = \frac{20 \cdot 19 \cdot 18}{6} = 1140$$

Out of these, the 12 edges are the choices where the points are collinear.
Hence, the final answer is:

$$1140 - 12 = 1128$$

Example 3.40

Let V be the set of vertices of a cube. Let M be the set of midpoints of edges of the same cube. Let P be the union of V and M. Three points from P that form a non-degenerate triangle are chosen. Determine the probability that some of the triangle lies inside the cube.

First determine the number of triangles that lie completely on a face of the cube:

$$\binom{8}{3} - 4 = \frac{8 \cdot 7 \cdot 6}{6} - 4 = 56 - 4 = 52$$

The number of triangles that lie on all six faces of the cube is:

$$52 \cdot 6 = 312$$

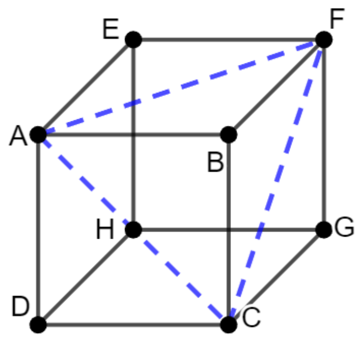
The probability of getting a triangle that lies inside the cube is:

$$= 1 - \frac{312}{1128} =$$

B. Number of Equilateral Triangles

Example 3.41

How many distinct equilateral triangles can be constructed by connecting three different vertices of a cube? (Two triangles are distinct if they have different locations in space.)



Case I: Sides are edges of the cube

All three sides must be edges of the cube. And this is not possible. For example, if you choose

$$First\ Side = AB, Second\ side = BC \Rightarrow No\ valid\ third\ side$$

Case II: Sides are space diagonals of the cube

This is also not possible.

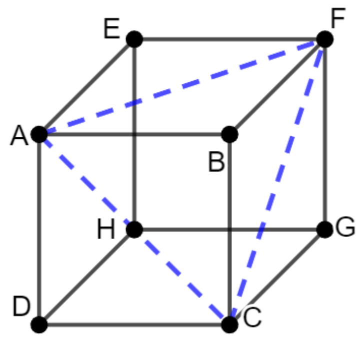
Case III: Sides are face diagonals of the cube

$$\Delta AFC\ is\ equilateral$$

$$\begin{gathered} Diagonal\ AF: AFC, AFH \\ Diagonal\ EB: EBD, EBG \\ Diagonal\ HC: HCA, HCF \\ Diagonal\ DG: DGE, DGB \end{gathered}$$

$$8\ Triangles$$

Example 3.42
Three different vertices of a cube are connected. What is the probability that an equilateral triangle is formed?



Method I

$$=\frac{Successful\ Outcomes}{Total\ Outcomes}=\frac{8}{56}=\frac{1}{7}$$

Method II

Because of the symmetry of the cube, we can select the first point without loss of generality to be

$$A$$

For the second point:

$$\frac{Successful\ Outcomes}{Total\ Outcomes}=\frac{\{H,F,C\}}{\{B,C,D,E,F,G\}}=\frac{3}{7}$$

For the third point:

$$\frac{Successful\ Outcomes}{Total\ Outcomes}=\frac{\{H,F,C\}-1}{\{B,C,D,E,F,G\}-1}=\frac{2}{6}=\frac{1}{3}$$

The final probability is:

$$1\cdot\frac{3}{7}\cdot\frac{1}{3}=\frac{1}{7}$$

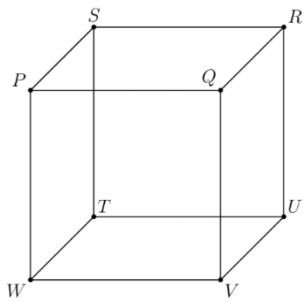
Example 3.43
How many distinct equilateral triangles can be constructed by connecting three different vertices of a cube? (Two triangles are distinct if they have different locations in space.)

Answer this using:

A. Number of triangles formed
B. Probability of forming an equilateral triangle

$$56\cdot\frac{1}{7}=8\ Triangles$$

Example 3.44
Any three vertices of the cube PQRSTUVW, shown in the figure below, can be connected to form a triangle. (For example, vertices P, Q, and R can be connected to form isosceles ΔPQR.) How many of these triangles are equilateral and contain P as a vertex? **(AMC 2024 8/20)**

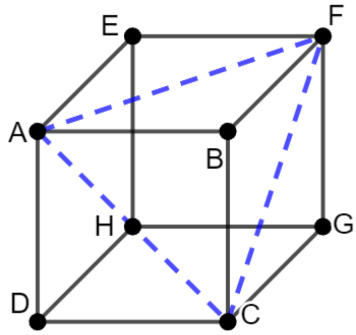


$$\Delta PRV, \Delta PRT, \Delta PVT$$

Example 3.45
Find the ratio of the number of distinct scalene triangles to the ratio of non-isosceles right triangles that can be

constructed by connecting three different vertices of a cube? (Two triangles are distinct if they have different locations in space.)

$$\text{Isosceles Right Triangles: Eg: } \Delta ADC$$
$$Equilateral\ Triangles: Eg: \Delta AFC$$
$$Scalene\ Triangles: \Delta AEG$$
$$Non\ Isosceles\ Right\ Triangles: \Delta AEG$$



Every scalene triangle in the 56 triangles made in the cube is also a non-isosceles right triangle.

$$Ratio\ is\ 1:1$$

Example 3.46: Making Triangles[1]

Three distinct vertices of a cube are chosen at random. What is the probability that the plane determined by these three vertices contains points inside the cube? **(AMC 2009 10A/24)**

Complementary Probability

Calculate the probability that the vertices do not contain points inside the cube. Hence, the three points must lie on a single face of the cube.

The number of total outcomes is:

$$\binom{8}{3} = \frac{8 \times 7 \times 6}{6} = 56$$

Successful outcomes are those where the three points lie on a single face of the cube:

$$\underbrace{\binom{4}{3}}_{\substack{Choose\ 3\ points \\ from\ 4}} \cdot \underbrace{6}_{\substack{Number \\ of\ Faces}} = 6 \cdot 4 = 24$$

The probability of all the points being on a single face is:

$$\frac{24}{56} = \frac{3}{7}$$

The probability that we want is:

$$1 - \frac{3}{7} = \frac{4}{7}$$

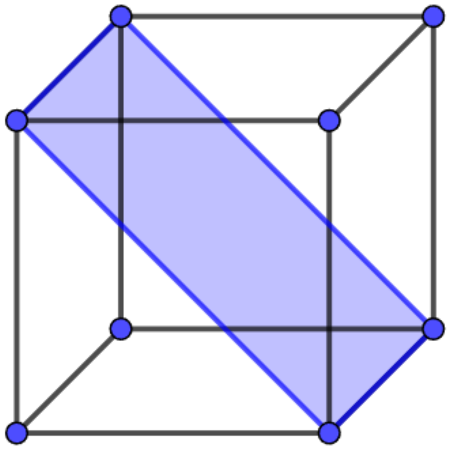
C. Other Shapes

Example 3.47

How many quadrilaterals can be formed by choosing vertices from a cube. (Two quadrilaterals are distinct if they have different locations in space.)

[1] A solution that relies more on the geometry is found in the section on Symmetry (in the Note on Probability Topics)

6 Faces
6 Cross (1 from each edge)



Total =6+6=12

Example 3.48
How many tetrahedrons can be formed by choosing vertices from a cube. (Two tetrahedrons are distinct if they have different locations in space.)

(8C4)=70
70-12=58

Example 3.49
Probability

3.3 Lattice Points

A. Basics

2 Pending

Example 3.50: Making Triangles
How many distinct triangles can be drawn using three of the dots below as vertices? (AMC 8 2005/21)
Ans = 18

3 Pending

How many non-congruent triangles have vertices at three of the eight points in the array shown below? (AMC 8 2009/20)
Ans = 8

B. Dots on Two Parallel Lines

4 Pending

Example 3.51
In the diagram, line A has seven points:
(1,2)(2,2), ... (7,2)
And line B has nine points:
(1,1)(2,1), ... (9,1)
Not all of the points are shown. Using the points, how many

A. Distinct triangles with non-zero area can be made?
B. Squares can be made?
C. Rectangles can be made?
D. Parallelograms can be made?
E. Quadrilaterals can be made?

Part A
We need three vertices to make a triangle.
Case I: If all three vertices are on the same line, then we get a triangle with zero area, which does not meet the condition given in the question.
Case II: If one vertex is on line A, and the other two vertices are on line B. This gives us:

$$\binom{7}{1}\binom{9}{2} = 7 \times 36 = 252$$

Case III: One vertex is on line B, and the other two vertices are on line A. This gives us:

$$\binom{7}{2}\binom{9}{1} = 21 \times 9 = 189$$

Total Ways

$$= 252 + 189 = 441$$

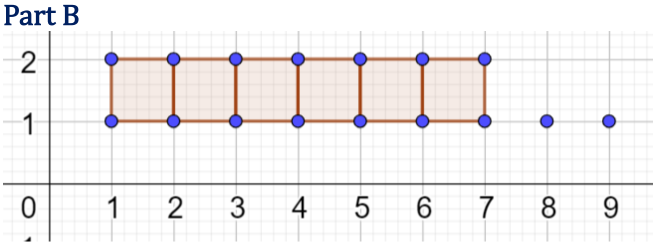
We can also get the same answer using complementary counting
The total number of ways to choose 3 points out of $7 + 9 = 16$ points is:

$$\binom{16}{3}$$

But, we do not want the $\binom{7}{3}$ ways where the three vertices are all on line A, and the $\binom{9}{3}$ where the three vertices are all on line B. And, hence, the final answer is:

$$\binom{16}{3} - \binom{7}{3} - \binom{9}{3}$$

Part B



The only square that you can make is a square of side length 1.
In total, you can make

$$6$$

Part C
If you choose a point on the top row to be the top left corner of the rectangle, there has to be a 90° angle created with the bottom row. This is only possible when the point on the bottom row is exactly below the point on the top row.
Mathematically, this means that the x coordinate of the bottom point is decided as soon as you pick the top

point.

You also need to choose the top right corner of the rectangle.
Hence, the final numbers of choices

$$= \binom{7}{2} = \frac{7 \times 6}{2} = 21$$

Part D

Parallelograms of side length 1 for the top and bottom row will be:

$$\underbrace{6 \times 8}_{\substack{Side \\ Length\ 1}} = 48$$

Similarly, parallelograms of other side lengths for the top and bottom row will be

$$\underbrace{5 \times 7}_{\substack{Side \\ Length\ 2}} + \underbrace{4 \times 6}_{\substack{Side \\ Length\ 3}} + \underbrace{3 \times 5}_{\substack{Side \\ Length\ 4}} + \underbrace{2 \times 4}_{\substack{Side \\ Length\ 5}} + \underbrace{1 \times 3}_{\substack{Side \\ Length\ 6}}$$

Simplifying and adding, we get the total as

$$= 48 + 35 + 24 + 15 + 8 + 3 = 133$$

Part E

Direct Method

$$\binom{7}{2}\binom{9}{2}$$

Complementary Counting

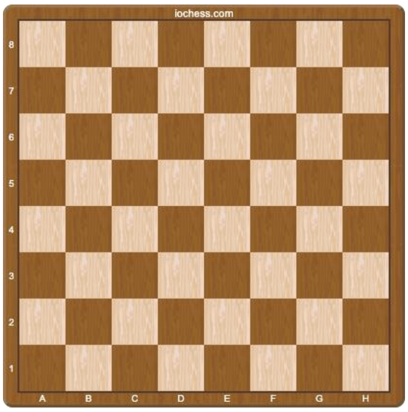
$$\binom{16}{4} - \binom{7}{4} - \binom{9}{4} - \binom{9}{1}\binom{7}{3} - \binom{7}{1}\binom{9}{3}$$

C. Chessboard Squares

A standard chessboard consists of a grid of squares comprising eight rows and eight columns. Many questions in counting and probability are based on chessboards. (The questions below do not need knowledge of chess, they just make use of the "board").

5 Pending

Example 3.52: Making Squares



See the grid on the chessboard. The largest square that can be formed is

$$8 \times 8 \Rightarrow The\ entire\ chessboard \Rightarrow 1\ Square$$

The smallest square that can be formed is

$$1 \times 1 \Rightarrow 64\ Squares$$

In such a manner, we can make squares:

$$(1 \times 1), (2 \times 2), \dots (8 \times 8)$$

A. Find the number of squares of size 8, 7 and 6 respectively.
B. Find a pattern for the number of squares that can be formed.
C. What is the total number of squares that can be formed?

(Squares must be aligned to the grid points. They cannot be placed halfway along a square from the existing grid).

Part A

Squares of Size 8

This is straightforward. It's the maximum size, and you can fit only one square of that size on the chessboard.

Squares of Size 7

Consider the bottom left corner of the square

$$Columns: Column\ A, or\ Column\ B \Rightarrow 2\ Choices$$
$$Rows: Row\ 1\ or\ Row\ 2 \Rightarrow 2\ Choices$$
$$Total\ Choices = 2 \times 2 = 4$$

Squares of Size 6

Consider the bottom left corner of the square

$$Columns: Column\ A, Column\ B, Column\ C \Rightarrow 3\ Choices$$
$$Rows: Row\ 1, Row\ 2\ or\ Row\ 3 \Rightarrow 3\ Choices$$
$$Total\ Choices = 3 \times 3 = 9$$

Part B

$$1 + 4 + 9 + 16 + \cdots + 64$$

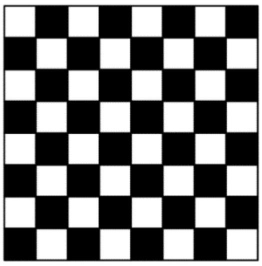
Part C

We want to find

Recall that $1 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ and we can use it to find:

$$n = 8 \Rightarrow 1 + 2^2 + 3^2 + 4^2 + \cdots + 8^2 = \frac{8(9)(17)}{6} = 204$$

6 Pending



Example 3.53

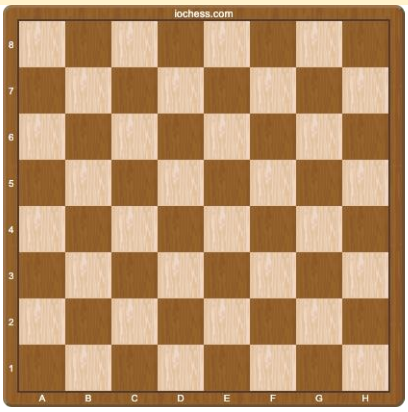
An 8 by 8 checkerboard has alternating black and white squares. How many distinct squares, with sides on the grid lines of the checkerboard (horizontal and vertical) and containing at least 4 black squares, can be drawn on the checkerboard? (MathCounts 2003 Chapter Team)

$$n = 6 \Rightarrow 1 + 2^2 + \cdots + 6^2 = \frac{6(7)(13)}{6} = 91$$

D. Chessboard Rectangles

7 Pending

Example 3.54: Making Rectangles



Just as we can form squares on a chessboard, so we can form rectangles also. Count the number of rectangles that can be formed on a chessboard (Rectangles must be aligned to the grid points. They cannot be placed halfway along a square from the existing grid).

The chessboard is

$$8\ by\ 8$$

And hence it actually has

$$9\ Lines\ by\ 9\ Lines$$

We can choose two distinct horizontal lines in:

$$\binom{9}{2} = 36\ Ways$$

And we can choose two distinct vertical lines in:

$$\binom{9}{2} = 36\ Ways$$

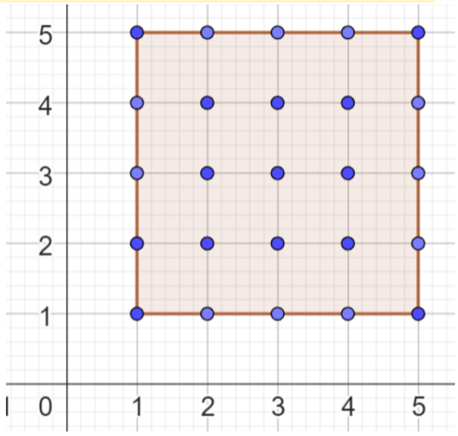
And to make a rectangle, we need to choose two distinct horizontal lines, and two distinct vertical lines. These will intersect at four places, giving us the corners of a rectangle. This can be done in:

$$36 \times 36 = 6^2 \times 6^2 = 6^4 = 1296\ Ways$$

E. Lattice Grids

Example 3.55: Making Triangles

AMC 10A 2017/23



It's not possible for a triangle to have negative area.

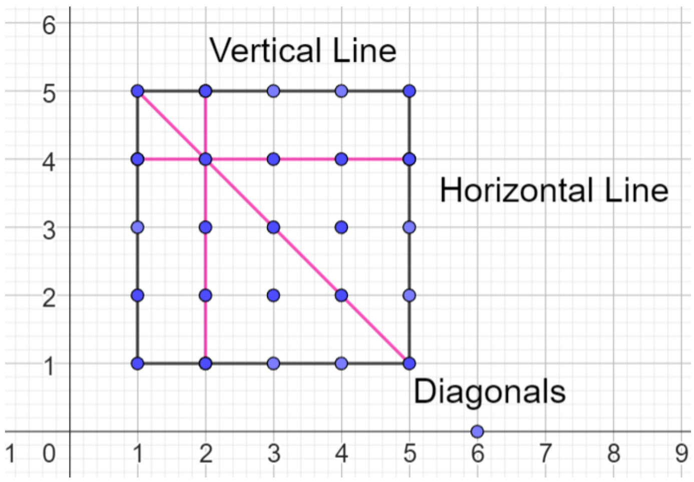
A degenerate triangle is a triangle with zero area. This happens when the points lie in a straight line.

The region has a total of:
5\times5=25 points

The number of ways to make triangle with nonnegative area is the number of ways to choose 3 points that all lie in the region shown:

$$\binom{25}{3} = 2300$$

Case I: Lines that pass through five points



The line can be:

$$Horizontal: 5\ cases$$
$$Vertical: 5\ cases$$
$$Diagonal: 2\ Cases$$
$$Total = 5 + 5 + 2 = 12$$

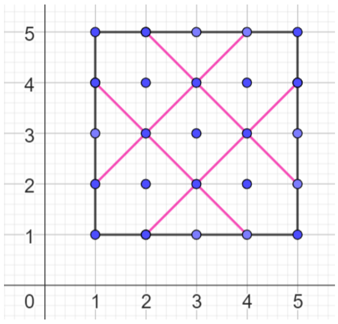
In each case, the number of ways of selecting 3 points out of 5 is:

$$\binom{5}{3} = 10$$

By the multiplication principle, the number of values to eliminate resulting from this case is:

$$12 \times 10 = 120$$

Case II: Lines that pass through four points



We can have four lines that pass four points (shown in the diagram alongside).

In each of those four lines, we can choose 3 points out of 4 in

$$\binom{4}{3} = 4\ ways$$

By the multiplication principle, the number of values to eliminate resulting from this case is:

$$4 \times 4 = 16$$

Case III: Lines that pass through three points

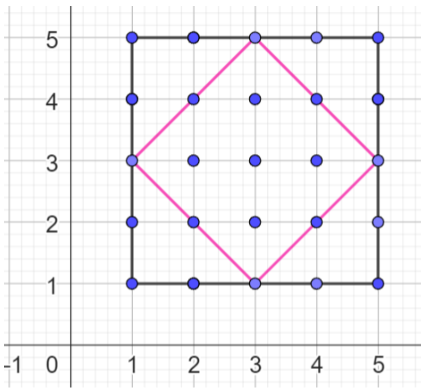
We can choose three points on a line segment that passes through three in

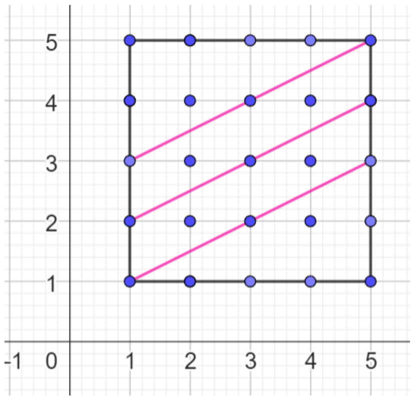
$$\binom{3}{3} = 1\ way$$

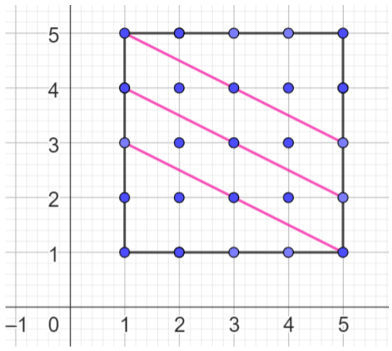
And we have 16 such lines as shown below.

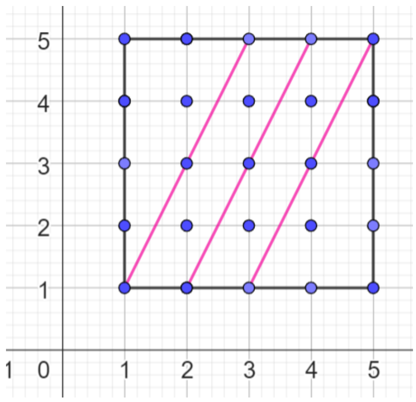
By the multiplication principle, the number of values to eliminate resulting from this case is:

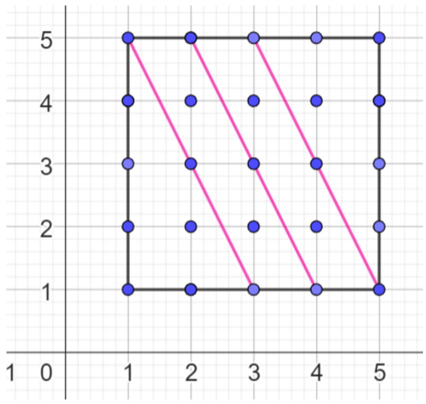
$$1 \times 16 = 16$$











Final Answer

$$2300 - 120 - 16 - 16 = 2148$$

3.4 Further Topics

Example 3.56

How many different patterns can be made by shading exactly two of the nine squares? Patterns that can be matched by flips and/or turns are not considered different. For example, the patterns shown below are not considered different. (AMC 8 1990/25)

Ans = 8

Example 3.57: Triangle Inequality

How many different isosceles triangles have integer side lengths and perimeter 23? (AMC 8 2005/15)

Ans = 6

Example 3.58: Intersections

A circle and two distinct lines are drawn on a sheet of paper. What is the largest possible number of points of intersection of these figures? (AMC 8 2002/1)

Ans = 5

Example 3.59: Tiling/Parity

A "domino" is made up of two small square. Which of the "checkerboards" illustrated below CANNOT be covered exactly and completely by a whole number of non-overlapping dominoes? (AMC 8 1991/5)

Ans = 3 * 5

Example 3.60

Points R, S and T are vertices of an equilateral triangle, and points X, Y and Z are midpoints of its sides. How many noncongruent triangles can be drawn using any three of these six points as vertices? (AMC 8 2001/23)

Ans = 4

61 Examples