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# PROBABILITY TOPICS

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# 1. PROBABILITY TOPICS

## 1.1 Some Background

### A. Creating Frequency Distributions

We first look at questions that involve frequency, and then see how symmetry can be applied to frequency.

#### Example 1.1

A school organizes a beach cleaning drive. It has a single fifth grader, two sixth graders and three seventh graders. Each fifth grader gathers one box of waste, each sixth grader gathers two boxes of waste, and each seventh grader gathers three boxes of waste. After the beach cleaning drive, I select a box at random. Find the probabilities regarding which grade the student belonged to.

#### Geometric Probability

Imagine the boxes assigned to the grades have colors. After collection, the boxes are stacked, and segregated grade-wise (see table).

The probabilities are then:

5th	6th	7th			

$$P(5th) = \frac{1}{14}, \quad P(6th) = \frac{4}{14} = \frac{2}{7}, \quad P(7th) = \frac{9}{14}$$

#### Frequency

We can get the same result as the previous answer by using a tabular method instead of a visual one. We need to count the number of boxes that each grade has collected, given by:

Grade	5 <sup>th</sup>	6 <sup>th</sup>	7 <sup>th</sup>	Total
Students per Grade	1	2	3	
Boxes collected per student per Grade	1	2	3	
No. of Boxes ( <i>Frequency</i> )	1	4	9	14
Probability = $\frac{\text{Frequency}}{\text{Total No. of Boxes}}$	$\frac{1}{14}$	$\frac{4}{14}$	$\frac{9}{14}$	$\frac{14}{14}$

### B. Using Frequency Tables

#### Example 1.2

Jean can't decide her favorite color. It keeps changing. Any time you ask her, she has a  $\frac{1}{3}$ rd probability of saying that red is her favorite color, and a  $\frac{2}{3}$ rd probability of saying blue.

- If you ask Jean her favorite color ten times, how many times would you expect to hear red?
- If you ask Jean her favorite color twenty times, how many times would you expect to hear blue?

#### Part A

$$P(\text{Red}) = \frac{1}{3} \Rightarrow \frac{1}{3} \times 10 = \frac{10}{3}$$

#### Part B

$$P(\text{Blue}) = \frac{2}{3} \Rightarrow \frac{2}{3} \times 20 = \frac{40}{3}$$

## C. Symmetry

### Example 1.3

AMC 10 – Sum greater than number picked from 1 to 10.

## D. Expected Value

### Example 1.4

An unfair coin when tossed, shows up heads with probability  $\frac{1}{4}$ . The coin is tossed twice. Find the expected number of heads.

### Example 1.5

Sarn has an urn containing three red and two green balls. He draws a ball from the urn, replaces it, and then draws another ball. Find the expected number of red balls.

### Example 1.6

Sarn has an urn containing three red and two green balls. He draws a ball from the urn, replaces it, and then draws another ball. Find the expected number of green balls.

## E. Winnings from a Game

### Example 1.7

I play a game where I roll an eight-sided die. If the number that comes up is prime, I get a dollar. If the number that comes up is composite, I lose a dollar. If the number is neither prime nor composite, I neither win nor lose anything.

- Is this game worth playing? How much do I expect to earn if I play this game once?
- How much do I expect to earn if I play this game 100 times?
- What is the maximum I should pay to play this game, if I do not want to expect a loss when playing?

### Part A

	<i>Prime</i>	<i>Composite</i>	<i>Neither</i>
Numbers	2,3,5,7	4,6,8	1
Frequency	4	3	1
Probability= $p(x)$	$\frac{4}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
Money = $x$	+1	-1	0
$x \cdot p(x)$	$\frac{4}{8}$	$-\frac{3}{8}$	0

$$\text{Expected Earnings} = \underbrace{\frac{4}{8}}_{(+1) \cdot p(\text{Prime})} + \underbrace{-\frac{3}{8}}_{(-1) \cdot p(\text{Composite})} + \underbrace{0}_{0 \cdot p(\text{Neither})} = \frac{1}{8} \text{ Dollars}$$

### Part B

$$\frac{1}{8} \times 100 = 12.5 \text{ Dollars}$$

### Part C

Every game you expect to earn

$$\frac{1}{8} \text{ Dollars}$$

Hence, the maximum that you would expect to pay per game is:

$$\frac{1}{8} \text{ Dollars}$$

### Example 1.8

A player chooses one of the numbers 1 through 4. After the choice has been made, two regular four-sided (tetrahedral) dice are rolled, with the sides of the dice numbered 1 through 4. If the number chosen appears on the bottom of exactly one die after it has been rolled, then the player wins 1 dollar. If the number chosen appears on the bottom of both of the dice, then the player wins 2 dollars. If the number chosen does not appear on the bottom of either of the dice, the player loses 1 dollar. What is the expected return to the player, in dollars, for one roll of the dice? (AMC 10B 2007/22)

	Both	Neither	Exactly One	Total
Probability	$\frac{1}{16}$	$\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$	$1 - \frac{1}{16} - \frac{9}{16} = \frac{6}{16}$	
Money	2	-1	1	
	$\frac{2}{16}$	$-\frac{9}{16}$	$\frac{6}{16}$	$-\frac{1}{16}$

### F. Fair Game

#### Example 1.9

A player pays \$5 to play a game. A die is rolled. If the number on the die is odd, the game is lost. If the number on the die is even, the die is rolled again. In this case the player wins if the second number matches the first and loses otherwise. How much should the player win if the game is fair? (In a fair game the probability of winning times the amount won is what the player should pay.) (AMC 10A 2006/13)

$$\frac{1}{2} \times \frac{1}{6} \times m = 5 \Rightarrow m = 60 \text{ dollars}$$

### G. Geometric Probability

#### Example 1.10

### H. Expected Profit

#### Example 1.11

A business forecasts three scenarios for the next year. In the optimistic scenario, which has 30% chance of happening, it will make a profit of \$5 million. In the base case scenario, which has a 60% chance of happening, it will make a profit of \$1 million. And in the pessimistic scenario, it will make a loss of \$1 million. What is the

expected value of next year's profit?

## I. Back Calculations

### Example 1.12

A bin has 5 green balls and  $k$  purple balls in it, where  $k$  is an unknown positive integer. A ball is drawn at random from the bin. If a green ball is drawn, the player wins 2 dollars, but if a purple ball is drawn, the player loses 2 dollars. If the expected amount won for playing the game is 50 cents, then what is  $k$ ? (AOPS Alcumus, Counting and Probability, Expected Value)

$$\begin{aligned} \left(\frac{5}{5+k}\right)(2) - \left(\frac{k}{5+k}\right)(2) &= \frac{1}{2} \\ \frac{10-2k}{5+k} &= \frac{1}{2} \\ 20-4k &= 5+k \\ k &= 3 \end{aligned}$$

## J. Contingency Tables

### Example 1.13

## 1.2 Expected Value

### A. Definition

#### 1.14: Expected Value

Expected value is given by:

$$n \cdot P(E)$$

Where

$n$  = Number of times the event will happen  
 $P(E)$  is the probability of the event happening

### Example 1.15

#### 1.16: Expected Value of Two Events

Expected value is given by:

$$E_1 \cdot P(E_1) + E_2 \cdot P(E_2)$$

Where

$E_1$  is the outcome if Event 1 happens  
 $E_2$  is the outcome if Event 2 happens  
 $P(E_1)$  is the probability that Event 1 happens  
 $P(E_2)$  is the probability that Event 2 happens

### Example 1.17

Bob rolls a fair six-sided die each morning. If Bob rolls a composite number, he eats sweetened cereal. If he rolls a prime number, he eats unsweetened cereal. If he rolls a 1, then he rolls again. In a non-leap year, what is the expected value of the difference between the number of days Bob eats unsweetened cereal and the number of

days he eats sweetened cereal? (MathCounts 2001 National Team)

### Outcomes

The outcomes when Bob rolls the die are:

$$\{1, 2, 3, 4, 5, 6\}$$

Out of the above:

$$\{4, 6\} \Rightarrow \text{Composite}, \quad \{2, 3, 5\} \Rightarrow \text{Prime}$$

If Bob he rolls a 1, he just rolls again.

Hence, ignore 1

### Sweetened Cereal

The expected number of days Bob will eat sweetened cereal is:

$$= 365 \times \underbrace{P(S)}_{\text{Sweetened}} = 365 \times \underbrace{P(C)}_{\text{Composite}} = 365 \times \frac{2}{5}$$

### Unsweetened Cereal

The expected number of days Bob will eat unsweetened cereal is:

$$= 365 \times \underbrace{P(U)}_{\text{Sweetened}} = 365 \times \underbrace{P(P)}_{\text{Composite}} = 365 \times \frac{3}{5}$$

### Difference

The difference between the two is:

$$365 \left( \frac{3}{5} \right) - 365 \left( \frac{2}{5} \right) = 365 \left( \frac{3}{5} - \frac{2}{5} \right) = 365 \left( \frac{1}{5} \right) = 73$$

## 1.18: Expected Value

Expected value is given by:

$$E_1 \cdot P(E_1) + E_2 \cdot P(E_2) + \cdots + E_n \cdot P(E_n)$$

Where

$E_i$  is the outcome if Event  $i$  happens

$P(E_i)$  is the probability that Event  $i$  happens

## B. Geometric Probability

### Example 1.19

The dartboard below has a radius of 6 inches. Each of the concentric circles has a radius two inches less than the next larger circle. If nine darts land randomly on the target, how many darts would we expect to land in a non-shaded region? (MathCounts 2004 State Sprint)

Single Dart

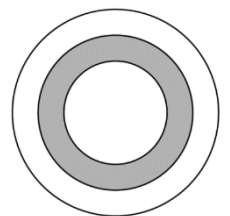
$$\text{Total Area} = \pi(r_{\text{outer circle}})^2 = \pi \times 6^2 = 36\pi$$

To calculate the shaded area, we substitute  $r = 2, R = 4$  in

$$\pi R^2 - \pi r^2 = 16\pi - 4\pi = 12\pi$$

$$P(\text{Shaded Region}) = \frac{\text{Successful Area}}{\text{Total Area}} = \frac{12\pi}{36\pi} = \frac{1}{3}$$

Expected Value



The expected number of darts (out of nine) to land in a shaded region is then:

$$= 9 \times \frac{1}{3} = 3$$

And then the number of darts expected to land in a non-shaded region

$$= 9 - 3 = 6$$

### Example 1.20

In the example above, suppose that:

- A dart landing in the innermost circle wins \$3
- A dart landing in the shaded region wins \$2
- A dart landing in the outer circle wins \$1

There is a 50% chance that a dart thrown at the dartboard hits the board. If the dart does not hit the board, the player pays \$2.

The areas of the circles are

$$\text{Inner Circle} = \pi r^2 = 4\pi$$

$$\text{Shaded Region} = \pi R^2 - \pi r^2 = 16\pi - 4\pi = 12\pi$$

$$\text{Outer Circle} = 36\pi - 16\pi = 20\pi$$

Hence, the ratio of the areas of these three regions is:

$$4\pi : 12\pi : 20\pi = 1 : 3 : 5$$

And the probability of hitting the dartboard is:

$$P(\text{Hit}) = 1 - P(\text{Do not Hit}) = 1 - \frac{1}{2} = \frac{1}{2}$$

And we then divide the probability of hitting in the ratio of the areas of the respective regions:

$$\frac{1}{2} \times \frac{1}{9} : \frac{1}{2} \times \frac{3}{9} : \frac{1}{2} \times \frac{5}{9} = \frac{1}{18} : \frac{3}{18} : \frac{5}{18}$$

$x$	Does not hit	Inner Circle	Shaded Circle	Outer Circle
$p(x)$	$\frac{1}{2}$	$\frac{1}{18}$	$\frac{3}{18}$	$\frac{5}{18}$
Money	-2	3	2	1

Expected Value

$$= \frac{1}{2} \times (-2) + \frac{1}{18} \times 3 + \frac{3}{18} \times 2 + \frac{5}{18} \times 1 = -1 + \frac{3}{18} + \frac{6}{18} + \frac{5}{18} = -1 + \frac{14}{18} = -\frac{4}{18} = -\frac{2}{9}$$

## C. Dungeons and Dragons

### Example 1.21

In a role-playing game, to determine the outcome of an attack roll, you roll a twenty-sided dice, and add attack modifiers. If the sum of the dice roll and the attack modifiers is greater than the target's Armor Class, the attack succeeds. If the outcome of the die roll is 20, the attack hits, and hits *critically*. What is the expected number of *critical hits* in 10 attacks for an attacker attacking a target with an Armor Class of 7, while using an attack modifier of 5?

There is a lot of extra information in this question: specifically, Armor Class, and attack modifiers.

The question only asks for critical hits, and the only condition required for a critical hit is that the die roll should be 20.

Hence, probability of Critical Hit in a single Attack

$$= \frac{1}{20}$$

Expected number of critical hits in 10 attacks

$$= 10 \times \frac{1}{20} = \frac{1}{2}$$

## D. Pairs

### Example 1.22

International checkers is played with twenty light and twenty dark pieces.

- If the pieces are mixed and arranged in a circle, without regard to their color, what is the expected value of the number of adjacent pairs of pieces that are both light.
- Answer the previous part, except that this time they are arranged in a line.

#### Part A: Circle

Imagine that the pieces are mixed and arranged. There are 20 light pieces.

#### Single Light Piece

Consider a single light piece. Apart from that light piece, there are 19 other light pieces and 20 dark pieces. We only consider the piece to its right, because when we generalize, all possible pairs will get taken into consideration.

#### Probability of a Pair

The probability of getting a pair is just the same as the probability that the adjacent piece is light, which is:

$$P(\text{Light}) = \frac{19}{39}$$

#### Expected Value: Pairs

And the expected value for 20 pieces is:

$$20 \left( \frac{19}{39} \right) = \frac{20 \times 19}{39} = \frac{380}{39}$$

#### Part B: Line

Start from leftmost.

Here, we have 20 pieces, but the 20<sup>th</sup> light piece can never have a light piece to its right, because the remaining 19 pieces are already on its left.

$$19 \left( \frac{19}{39} \times 1 + \frac{20}{39} \times 0 \right) = \frac{19 \times 19}{39} = \frac{361}{39}$$

## E. Maximum

### Example 1.23

Aarna picks, at random, a whole number of pizza slices that she is going to eat: from 1 to 3. Aarav picks, at random, a whole number of pizza slices that he is going to eat: from 1 to 4. What is the expected value of the larger of the two numbers? (If two numbers are equal, both are considered "larger").

	Aarav			
Aarna	1	2	3	4
1	1	2	3	4
2	2	2	3	4
3	3	3	3	4

$x$	1	2	3	4
$p(x)$	$\frac{1}{12}$	$\frac{3}{12}$	$\frac{5}{12}$	$\frac{3}{12}$

$$1 \times \frac{1}{12} + 2 \times \frac{3}{12} + 3 \times \frac{5}{12} + 4 \times \frac{3}{12}$$

## F. Infinite Series

### Example 1.24

Bhairav the Bat lives next to a town where 12.5% of the inhabitants have Type AB blood. When Bhairav the Bat leaves his cave at night to suck of the inhabitant's blood, chooses individuals at random until he bites one with type AB blood, after which he stops. What is the expected value of the number of individuals Bhairav the Bat will bite in any given night? (2015 CCA Math Bonanza Lightning Round #3.1)

### Example 1.25

Aman has a standard pack of 52 cards with two Jokers added. He selects a card at random from the 54 cards. If he selects a Joker, he replaces it, and selects again at random, till he selects a non-Joker card. What is the expected number of selections that Aman must make?

$$1 + \frac{2}{54} + \left(\frac{2}{54}\right)^2 + \dots \Rightarrow \text{Geometric Series with } a = 1, r = \frac{2}{54}$$

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{2}{54}} = \frac{1}{\frac{52}{54}} = \frac{54}{52} = \frac{27}{26}$$

## G. Recursion with Tree Diagrams

Let's relook at a couple of the examples that we did just to see a shorter, more elegant solution.

### Example 1.26

Bhairav the Bat lives next to a town where 12.5% of the inhabitants have Type AB blood. When Bhairav the Bat leaves his cave at night to suck of the inhabitants blood, chooses individuals at random until he bites one with type AB blood, after which he stops. What is the expected value of the number of individuals Bhairav the Bat will bite in any given night? (2015 CCA Math Bonanza Lightning Round #3.1)

Let the expected number of bites be  $b$ .

$$b = 1/8 + (b+1)(7/8)$$

b=8

**Example 1.27**

Aman has a standard pack of 52 cards with two Jokers added. He selects a card at random from the 54 cards. If he selects a Joker, he replaces it, and selects again at random, till he selects a non-Joker card. What is the expected number of selections that Aman must make?

$$E = \frac{52}{54} + \frac{2}{54}(1 + E) \Rightarrow E = \frac{54}{52} = \frac{27}{26}$$

**Challenge 1.28**

Freddy the frog is jumping around the coordinate plane searching for a river, which lies on the horizontal line  $y = 24$ . A fence is located at the horizontal line  $y = 0$ . On each jump Freddy randomly chooses a direction parallel to one of the coordinate axes and moves one unit in that direction. When he is at a point where  $y=0$ , with equal likelihoods he chooses one of three directions where he either jumps parallel to the fence or jumps away from the fence, but he never chooses the direction that would have him cross over the fence to where  $y < 0$ . Freddy starts his search at the point  $(0, 21)$  and will stop once he reaches a point on the river. Find the expected number of jumps it will take Freddy to reach the river. (AIME I 2016/13)

**H. Scenarios****Example 1.29**

Suppose an AMC 10 question gives 6 marks for a correct answer, 1.5 marks for an unattempted question, and 0 marks for an incorrect answer. You eliminate two of the five answer choices and mark one of the remaining three answer choices randomly. What is the expected value of the number of marks that you will get?

$P(\text{Correct}) = 1/3$   
Marks = 6

Expected Value =  $(1/3) \cdot 6 = 2$  Marks

$P(\text{Correct}) = 1/5$   
Marks = 6

Expected Value =  $(1/5) \cdot 6 = 6/5 = 1.2 < 1.5$  Marks

**Example 1.30**

The daily demand distribution of a product is given below.

Day	1	2	3	4	5	6	7
Probability	0.2	0.1	0.15	0.25	0.2	0.06	0.04
Demand	1500	1200	1600	1400	1600	1000	1500

The company wants to produce ten percent more than the expected demand, so as to reduce stock-outs. What is the production target? (JMET 2011/63)

The expected demand is:

$$\begin{aligned} &= 0.2 \times 1500 + 0.1 + 1200 + 0.15 \times 1600 + 0.25 \times 1400 + 0.2 \times 1600 + 0.06 \times 1000 + 0.04 \times 1500 \\ &= 300 + 120 + 240 + 350 + 320 + 60 + 60 \\ &= 1450 \end{aligned}$$

Production Target

$$= 110\% \text{ of Demand} = 1450 + 10\% \text{ of } 1450 = 1450 + 145 = 1595$$

### Example 1.31

In the game of dungeons and dragons, a character playing the fighter class rolls a ten-sided die once per level to decide his hit points (life). What is the expected value of the number of hit points of a:

- A. level one fighter?
- B. level five fighter?

### Example 1.32

I deal thirteen cards from a standard fifty-two card pack. Consider *Ace* = 1, *Jack* = 11, *Queen* = 12, *King* = 13 and the numbered cards as having their usual values. What is the expected numeric value:

- A. Of a single card
- B. Of the sum of the numeric values of all thirteen cards

### Example 1.33

The contestants in a mixed martial arts fight are aged 22, 24 and 25 years respectively. I pick a contestant at random. What is the expected value of his age?

### Example 1.34

I have four 250-gram weights, two 500-gram weights, and one 1-kg weight. I pick a weight at random. What is the expected value of that weight?

### Example 1.35

A game of Bingo requires the caller to choose a number from 1 to 90. If the caller chooses a number, what is the expected value of the number of digits in the number.

### Example 1.36: Sum and Product of Expected Values

I roll an eight-sided dice, a ten-sided dice, and a twenty-sided dice. What is the expected value of the:

- A. sum of the dice rolls
- B. product of the dice rolls

$$4.5 + 5.5 + 10.5 = 20.5$$

## I. Symmetry

### Example 1.37

Two jokers are added to a 52-card deck and the entire stack of 54 cards is shuffled randomly. What is the expected number of cards that will be strictly between the two jokers? (2009 HMMT General 1)

The question, the way it is framed, invites us to think of placing the jokers *after* the cards.

However, the following is equivalent:

Add 52 cards randomly to 2 jokers, and determine the expected number of cards between the two jokers:

$\underbrace{\text{Before}}_{\text{1st Joker}}, \quad \underbrace{\text{1st Joker}}_{\square}, \quad \underbrace{\text{Between}}_{\text{the Jokers}}, \quad \underbrace{\text{2nd Joker}}_{\square}, \quad \underbrace{\text{After}}_{\text{2nd Joker}}$

Since we are considering only the positions of the jokers, by symmetry, each card has  $\frac{1}{3}$ rd probability of going to any of the spaces.

So, expected value

$$\frac{1}{3} \times 52 = \frac{52}{3}$$

### Example 1.38

I turn over 13 cards from a well shuffled deck of cards. What is the expected number of aces?

The probability of each card being an ace is:

$$\frac{4}{52} = \frac{1}{13}$$

The expected number of aces is:

$$13 \cdot \frac{1}{13} = 1$$

### Example 1.39

I turn over cards from a well shuffled deck of cards. What is the expected number of cards before the first ace is drawn? (Including the first ace in the number of cards drawn)

View the four aces as dividing the cards into five groups

$$- A_1 - A_2 - A_3 - A_4 -$$

Each of the remaining 48 cards can be in any of the five groups. So the number of cards (upto and including the first ace) is:

$$\frac{48}{5} + 1 = 10.6$$

### Example 1.40

Forty two cards are labeled with the natural numbers 1 through 42 and randomly shuffled into a stack. One by one, cards are taken off the top of the stack until a card labeled with a prime number is removed. How many cards are removed on average? (HMMT 2007, Individual, Combinatorics)

We can view the 13 prime numbers from 1 to 42 as dividing the cards into 14 groups.

Arrange the 13 primes first. The remaining 29 primes can be in any of the 14 groups, by symmetry.

The number of cards removed is the expected number of cards in the first group, which is:

$$\frac{29}{14} + 1 = \frac{43}{14}$$

## J. Economics/Finance (Optional)

### Example 1.41: Value of a Bond

The expected value of the income from a bond is given by

$$P(D) \times CF(D) + (1 - P(D))(CF(P))$$

Where:

$$\begin{aligned} P(D) &= \text{Probability of Default} \\ CF(D) &= \text{Cash Flow given default} \\ CF(P) &= \text{Promised Cash Flow} \end{aligned}$$

A junk bond has a promised cash flow of \$1000 at maturity. The probability of default is 10%, and the cash flow in case of default is estimated to 40% of the promised cash flow. Calculate the expected value of the cash flow from the bond at the time of maturity.

$$= 0.1 \times 0.4 \times 1000 + 0.9 \times 1000 = 40 + 900 = 940$$

## 1.3 Recursion

### A. Recursion with Induction

#### Example 1.42

Flora the frog starts at 0 on the number line and makes a sequence of jumps to the right. In any one jump, independent of previous jumps, Flora leaps a positive integer distance  $m$  with probability  $\frac{1}{2^m}$ . What is the probability that Flora will eventually land at 10? (AMC 12A 2023/17)

Let  $p(n)$  be the probability that Flora ever reaches or has reached  $n$ .

**Reaching 0:** The starting point for Flora is 0. Hence:

$$p(0) = 1$$

**Reaching 1:** From 0, there is only one way for Flora to reach 1 and hence:

$$p(1) = \frac{1}{2}p(0) = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

**Reaching 2:** Flora can jump from either 1 or 0. We add the probabilities that Flora reaches 2 directly from 1 and directly from 0.

(The probability that Flora goes from 0 to 1 can be bypassed because we multiply the probability of reaching 2 from 1 with the probability of reaching 1 itself.).

$$p(2) = \frac{1}{2}p(1) + \frac{1}{4}p(0) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot 1 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Similarly, we can calculate the probability of reaching 3, as being reached from any one of 0, 1 or 2:

$$p(3) = \frac{1}{2}p(2) + \frac{1}{4}p(1) + \frac{1}{8}p(0) = \frac{1}{2} \left[ p(2) + \frac{1}{2}p(1) + \frac{1}{4}p(0) \right] = \frac{1}{2} [p(2) + p(2)] = p(2) = \frac{1}{2}$$

**We can generalize this using induction.**

Base Case:  $n \geq 2$

$$p(2) = \frac{1}{2}, (\text{As shown above})$$

Inductive Case:

$$p(n) = \frac{1}{2}p(n-1) + \frac{1}{4}p(n-2) + \dots + \frac{1}{2^n}p(0)$$

Factor  $\frac{1}{2}$  from each term on the RHS:

$$p(n) = \frac{1}{2} \left[ p(n-1) + \frac{1}{2}p(n-2) + \dots + \frac{1}{2^{n-1}}p(0) \right]$$

Substitute  $\frac{1}{2}p(n-2) + \dots + \frac{1}{2^{n-1}}p(0) = p(n-1)$ :

$$p(n) = \frac{1}{2} [p(n-1) + p(n-1)] = p(n-1)$$

$$p(n) = \frac{1}{2}, n \geq 2$$

## B. Recursion: Self-Similarity

We begin by looking at symmetric recursion, which leads to a summation pattern involving geometric series. The geometric series exhibit self-similarity, which can we exploit to arrive at an answer.

### Example 1.43

Mary and Nick are playing a game. In round 1, Mary rolls a standard six-sided die. If she rolls a number that is not composite, she wins. If she doesn't win, then Nick gets his turn. He too, wins, if he rolls a number that is not composite. They alternate rolling the die till a winner is decided. Calculate Mary and Nick's chances of winning the game.<sup>1</sup>

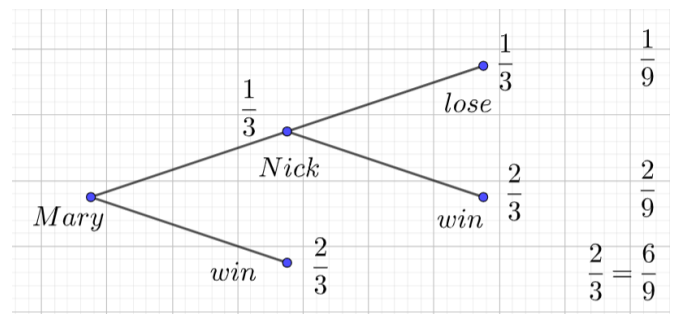
#### Method I

Use a tree diagram to calculate the probabilities in the first round. Note that:

$$P(\text{Mary Winning}) = \frac{6}{9}$$

$$P(\text{Nick Winning}) = \frac{2}{9}$$

$$P(\text{Neither Winning}) = \frac{1}{9}$$



<sup>1</sup> Using the shortcut formula developed later, we get  $P = \frac{1}{2-p} = \frac{1}{2-\frac{2}{3}} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$

Use these probabilities to create a tree diagram with the 1<sup>st</sup>, 2<sup>nd</sup> and 3<sup>rd</sup> rounds. From the tree diagram, we can see that the probability of Mary winning is given by:

$$\underbrace{\frac{2}{3}}_{1^{st} \text{ Round}} + \underbrace{\left(\frac{2}{3}\right)\left(\frac{1}{9}\right)}_{2^{nd} \text{ Round}} + \underbrace{\left(\frac{2}{3}\right)\left(\frac{1}{9^2}\right)}_{3^{rd} \text{ Round}} + \dots$$

While the probabilities have been calculated only till the third round, we can see that the process continues, and we recognize an infinite geometric series with  $a = \frac{2}{3}, r = \frac{1}{9}$ .

Substitute the above values in the formula for the sum of an infinite geometric series:

$$S = \frac{a}{1-r} = \frac{\frac{2}{3}}{1-\frac{1}{9}} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{2}{3} \times \frac{9}{8} = \frac{3}{4}$$

## Method II

1 <sup>st</sup> Round	Mary	Nick
Win	$\frac{2}{3}$	$\left(1 - \frac{2}{3}\right)\left(\frac{2}{3}\right) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$

$$P(\text{Mary}) = p$$

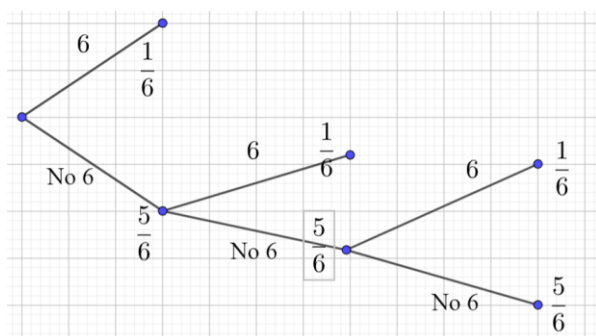
At each round, the probability that Nick will win game is  $\frac{1}{3}rd$  of the probability that Mary will the game:

$$\therefore P(\text{Nick}) = \frac{p}{3}$$

$$p + \frac{p}{3} = 1 \Rightarrow \frac{4p}{3} = 1 \Rightarrow p = \frac{3}{4}$$

## Example 1.44

Henrik and Zhao play a game in which they take turns successively rolling dice. The first person who rolls a 6 wins. If Zhao goes first, what is the probability that he wins the game? <sup>2</sup> (MA0, Combinations and Probability 2021/14)



## Method I: Geometric Series

The probability that Zhao wins in Round 1:

$$\text{Round 1} = \frac{1}{6}$$

The probability that Zhao wins in Round 2:

$$= \underbrace{\frac{1}{6}}_{\text{Winning 2nd Round}} \times \underbrace{\left(\frac{5}{6}\right)^2}_{\text{Both losing 1st Round}}$$

The probability that Zhao wins in Round 3:

$$= \underbrace{\frac{1}{6}}_{\text{Winning 3rd Round}} \times \underbrace{\left(\frac{5}{6}\right)^2}_{\text{Both losing 1st Round}} \times \underbrace{\left(\frac{5}{6}\right)^2}_{\text{Both losing 2nd Round}} = \frac{1}{6} \times \left(\frac{5}{6}\right)^4$$

<sup>2</sup> Using the shortcut formula developed later, we get  $P = \frac{1}{2-p} = \frac{1}{2-\frac{1}{6}} = \frac{1}{\frac{11}{6}} = \frac{6}{11}$

The final answer is given by a geometric series with

$$a = \frac{1}{6}, r = \left(\frac{5}{6}\right)^2 = \frac{25}{36}.$$

$$\frac{1}{6} + \left(\frac{1}{6}\right)\left(\frac{25}{36}\right) + \left(\frac{1}{6}\right)\left(\frac{25}{36}\right)^2 + \dots$$

Substitute the above values in the formula for the sum of an infinite geometric series:

$$S = \frac{a}{1-r} = \frac{\frac{1}{6}}{1-\frac{25}{36}} = \frac{1}{6} \times \frac{36}{11} = \frac{6}{11}$$

### Method II: Equation

Notice that at every stage, the probability of Zhao winning is greater than Henrik winning. In fact:

$$\underbrace{P(H) = \frac{5}{6}P(Z)}_{\text{Equation I}}$$

Since one of them must win, we have:

$$\underbrace{P(H) + P(Z) = 1}_{\text{Equation II}}$$

Substitute the value from Equation I into Equation II:

$$\begin{aligned} P(Z) + \frac{5}{6}P(Z) &= 1 \\ \frac{11}{6}P(Z) &= 1 \\ P(Z) &= \frac{6}{11} \end{aligned}$$

### Method III: Recursion

We have calculated that the probability of winning is:

$$p = \frac{1}{6} + \left(\frac{1}{6}\right)\left(\frac{25}{36}\right) + \left(\frac{1}{6}\right)\left(\frac{25}{36}\right)^2 + \dots$$

Factor  $\left(\frac{25}{36}\right)$  from the second and further terms:

$$p = \frac{1}{6} + \left(\frac{25}{36}\right) \underbrace{\left[\left(\frac{1}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{25}{36}\right) + \dots\right]}_{P(A)}$$

But note that after factoring, the expression that we get is itself  $p$ :

$$\begin{aligned} p &= \frac{1}{6} + \left(\frac{25}{36}\right)p \\ p - \left(\frac{25}{36}\right)p &= \frac{1}{6} \\ p \left[1 - \frac{25}{36}\right] &= \frac{1}{6} \\ p \left[\frac{11}{36}\right] &= \frac{1}{6} \\ p &= \frac{6}{11} \end{aligned}$$

### Example 1.45

Jack and Jill take turns tossing a fair coin. Whoever gets the first *tail* goes to get the water. Jack has the first turn. Find the probability that he is one to fetch the water.<sup>3</sup>

The probability that Jack wins is given by:

$$\underbrace{\frac{1}{2}}_{\text{Jack Wins 1st Round}} + \underbrace{\left(\frac{1}{4}\right)}_{\text{Both lose 1st Round}} \underbrace{\left(\frac{1}{2}\right)}_{\text{Both lose first 2 Rounds}} + \underbrace{\left(\frac{1}{16}\right)}_{\text{Both lose first 2 Rounds}} \underbrace{\left(\frac{1}{2}\right)}_{\text{Both lose first 2 Rounds}} + \dots = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots$$

The probability that Jill wins is given by:

$$\underbrace{\frac{1}{2}}_{\text{Jack Loses 1st Round}} \underbrace{\frac{1}{2}}_{\text{Jill Wins 1st Round}} + \underbrace{\left(\frac{1}{4}\right)}_{\text{Both lose 1st Round}} \underbrace{\frac{1}{2}}_{\text{Jack Loses 2nd Round}} \underbrace{\left(\frac{1}{2}\right)}_{\text{Both lose first 2 Rounds}} + \underbrace{\left(\frac{1}{16}\right)}_{\text{Both lose first 2 Rounds}} \underbrace{\frac{1}{2}}_{\text{Jack Loses 3rd Round}} \underbrace{\left(\frac{1}{2}\right)}_{\text{Both lose first 2 Rounds}} + \dots = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$$

We could add the geometric series and get the answers. But note that the geometric series for Jack is exactly double at each stage compared to Jill.

Hence:

$$P(\text{Jack}) = p \Rightarrow P(\text{Jill}) = \frac{p}{2}$$

<sup>3</sup> Using the shortcut formula developed later, we get  $P = \frac{1}{2 - \frac{1}{2}} = \frac{1}{2 - \frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$

$$p + \frac{p}{2} = 1 \Rightarrow \frac{3p}{2} = 1 \Rightarrow p = \frac{2}{3}$$

**Example 1.46**

Sam is playing a game of cards with his brother Dan. Sam goes first. If he gets a King, he wins. If he does not get a King, he puts the card back in the deck. Then it's Dan's turn. Same rules apply. Find the probability that:

- Sam wins two games in a row.
- Dan wins two games in a row.
- Sam and Dan each win one game (given that they play two games)

$$\begin{aligned} P(D) &= \frac{12}{13}P(S) \\ P(S) + P(D) &= 1 \\ P(S) + \frac{12}{13}P(S) &= 1 \\ P(S) &= \frac{13}{25} \\ P(D) &= 1 - P(S) = 1 - \frac{13}{25} = \frac{12}{25} \end{aligned}$$

**Part A****Part B**

$$P(SS) = \left(\frac{13}{25}\right)^2 = \frac{169}{625}$$

**Part C**

$$P(DD) = \left(\frac{12}{25}\right)^2 = \frac{144}{625}$$

$$P(DS) + P(SD) = 2 \left(\frac{12}{25}\right) \left(\frac{13}{25}\right) = \frac{312}{625}$$

**Example 1.47**

A and B are throwing darts at a dart-board, in turns, with A going first. The first person whose dart lands in the center ring wins. On their turn, A and B have equal probability of success,  $p$ . Find the probability that A wins.

$$P(A) = \underbrace{p}_{\text{Wins First Round}} + \underbrace{(1-p)^2}_{\text{A and B lose 1st Round}} \underbrace{p}_{\text{Wins Second Round}} + (1-p)^4 p + \dots$$

**Geometric Series**

Substitute  $a = p, r = (1-p)^2$  in  $S = \frac{a}{1-r}$ :

$$P(A) = \frac{p}{1 - (1-p)^2}$$

**Recursion Method**

$$\begin{aligned} P(A) &= p + (1-p)^2 P(A) \\ P(A)[1 - (1-p)^2] &= p \\ P(A) &= \frac{p}{1 - (1-p)^2} = \frac{p}{1 - (1 - 2p + p^2)} = \frac{p}{2p - p^2} = \frac{1}{2 - p} \end{aligned}$$

**Example 1.48**

Juan, Carlos and Manu take turns flipping a coin in their respective order. The first one to flip heads wins. What is the probability that Manu will win? Express your answer as a common fraction. ([MathCounts 2002 National Sprint](#))

TTH:  $(1/2)^3$

TTTTTH:  $(1/2)^6$

$$1/2^3 + 1/2^6 + \dots = 1/7$$

**Logical Method**

In every round, Carlos has half the probability that Juan has. Manu has half the probability that Carlos has. Let the probability that Manu wins be:

$$P(\text{Manu}) = m$$

$$\underbrace{4m}_{\text{Juan}} + \underbrace{2m}_{\text{Carlos}} + \underbrace{m}_{\text{Manu}} = 1$$

$$m = \frac{1}{7}$$

## C. Asymmetrical Recursion

### Example 1.49

Frieda the frog begins a sequence of hops on a  $3 \times 3$  grid of squares, moving one square on each hop and choosing at random the direction of each hop-up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she "wraps around" and jumps to the opposite edge. For example, if Frieda begins in the center square and makes two hops "up", the first hop would place her in the top row middle square, and the second hop would cause Frieda to jump to the opposite edge, landing in the bottom row middle square. Suppose Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops? (**AMC 10A 2021 Spring /23, AMC 12A 2021 Spring/23**)

### Strategy

Frieda begins at the center. We use complementary counting, and find the probability that Frieda visits only Center (C) or Edge (E) squares.

In Step I, she has four choices, all valid. The four choices also lead to symmetrical outcomes. Hence, we begin our analysis at Step 2.

### Tree Diagram

We want the sum of the probabilities:

$$3 \times \frac{1}{16} + 2 \times \frac{1}{64} = \frac{3}{16} + \frac{1}{32} = \frac{7}{32}$$

And, since we want the complementary probability, we calculate:

$$P(C) = 1 - P(C') = 1 - \frac{7}{32} = \frac{25}{32}$$

Establish a recurrence relation,<sup>4</sup> and verify the probability calculated in the example above.

### Base Cases

We know that Frieda begins in the center, and moves to an edge on the first step. Hence:

$$C_0 = E_1 = 1$$

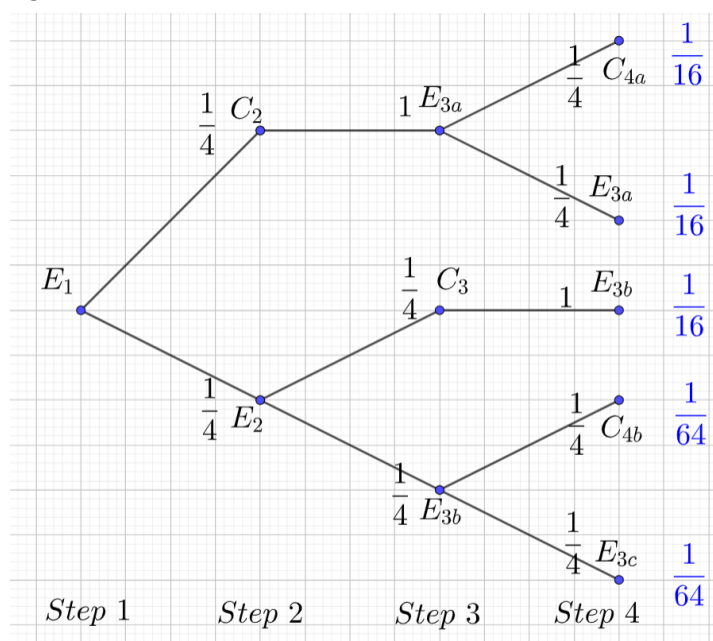
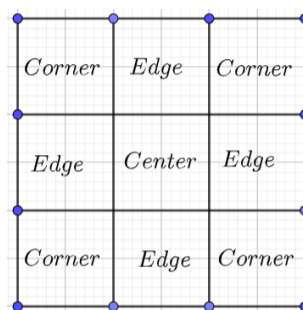
$$E_0 = C_1 = 0$$

Where

$C_0$  = Probability of being at Center at step 0

$E_0$  = Probability of being at Edge at step 0

### Recurrence Relation for $C_n$



<sup>4</sup> This is not an exam solution. It is meant to understand how recurrence relations work.

The probability of reaching the center at step  $n$  is  $\frac{1}{4}$  the probability of reaching an edge at step  $n - 1$ :

$$C_n = \frac{1}{4} E_{n-1}, n \geq 2$$

*Recurrence I*

### Recurrence Relation for $E_n$

The probability of reaching an edge on step  $n$  is:

- Same as the probability of reaching the center on step  $n - 1$
- $\frac{1}{4}$  the probability of reaching an edge on step  $n - 1$

Since both the above are valid ways, combine the two to get (for  $n \geq 2$ ):

$$E_n = C_{n-1} + \frac{1}{4} E_{n-1} = \frac{1}{4} E_{n-2} + \frac{1}{4} E_{n-1} = \frac{1}{4} (E_{n-2} + E_{n-1})$$

*Using Recurrence I*

Using the base cases, and the recurrence relation above, calculate

$$E_2 = \frac{1}{4} (E_1 + E_0) = \frac{1}{4} (1 + 0) = \frac{1}{4}$$

$$E_3 = \frac{1}{4} (E_2 + E_1) = \frac{1}{4} \left( \frac{1}{4} + 1 \right) = \frac{1}{4} \left( \frac{5}{4} \right) = \frac{5}{16}$$

### Recurrence Relation for $R_n$

Let the probability that Frieda does not visit a corner square upto (and including) step  $n$  be

$$R_n$$

In step  $n - 1$ , Frieda can be at

- the center, and must then visit an edge square with probability 1.
- An edge square, and must then visit an edge square with probability  $\frac{1}{4}$ , or the center square with probability  $\frac{1}{4}$ .

Hence,

$$R_n = C_{n-1} + \frac{1}{2} E_{n-1} = \frac{1}{4} E_{n-2} + \frac{1}{2} E_{n-1}$$

*Using Recurrence I*

We can calculate the probability of not visiting a corner square as:

$$R_4 = \frac{1}{4} E_2 + \frac{1}{2} E_3 = \frac{1}{4} \left( \frac{1}{4} \right) + \frac{1}{2} \left( \frac{5}{16} \right) = \frac{1}{16} + \frac{5}{32} = \frac{7}{32}$$

And, finally, the probability of visiting a corner square:

$$= 1 - R_4 = 1 - \frac{7}{32} = \frac{25}{32}$$

Calculate  $R_6$ . Would you prefer a tree diagram or the recurrence relation to do it?

$$E_2 = \frac{1}{4}, \quad E_3 = \frac{5}{16}$$

$$E_4 = \frac{1}{4} (E_2 + E_3) = \frac{1}{4} \left( \frac{1}{4} + \frac{5}{16} \right) = \frac{1}{4} \left( \frac{9}{16} \right) = \frac{9}{64}$$

$$E_5 = \frac{1}{4} (E_3 + E_4) = \frac{1}{4} \left( \frac{5}{16} + \frac{9}{64} \right) = \frac{1}{4} \left( \frac{29}{64} \right) = \frac{29}{256}$$

$$R_6 = \frac{1}{4} E_4 + \frac{1}{2} E_5 = \frac{1}{4} \left( \frac{9}{64} \right) + \frac{1}{2} \left( \frac{29}{256} \right) = \frac{18}{512} + \frac{29}{512} = \frac{47}{512}$$

For small values of  $n$ , the tree diagram is faster. As  $n$  increases, the tree diagram will be difficult to draw, and the recurrence relations will be better.

### Example 1.50

A bug starts at a vertex of a grid made of equilateral triangles of side length 1. At each step the bug moves in one of the 6 possible directions along the grid lines randomly and independently with equal probability. What is the probability that after 5 moves the bug never will have been more than 1 unit away from the starting position?  
(AMC 12B 2021 Fall/17)

The bug begins at the center. There are six paths for the bug, and all are valid and symmetrical. So, we ignore step I. The paths from Step II to Step V, using the multiplication principle (with repetition) is:

$$6 \times 6 \times 6 \times 6 = 6^4$$

Draw a tree diagram<sup>5</sup>. Let

- $C$  represent the center of the grid
- $V$  represent any other vertex of the grid,

Valid paths

$$= 36 + 12 + 24 + 12 + 24 + 24 + 8 + 16 = 156$$

Hence, the required probability is:

$$\frac{156}{6^4} = \frac{13}{108}$$

Calculate the probability that after 6 moves the bug never will have been more than 1 unit away from the starting position? Use a recurrence relation.

### Recurrence Relation for $C_n$

The same tree diagram that we drew earlier holds.

- $C_2$  is the paths that reach the center at Step 2.

$$C_2 = V_1 = 1$$

$$C_3 = V_2 = 2$$

- In some places, we have  $V_{3a}$  and  $V_{3b}$  since you can reach a vertex in multiple ways

$$C_4 = C_{4a} + C_{4b} = V_{3a} + V_{3b} = V_3$$

The paths to the center at step  $n$  are the same as the paths to reach a vertex at step  $n - 1$ :

$$\underbrace{C_n = V_{n-1}}_{\text{Recurrence I}}$$

### Recurrence Relation for $V_n$

We can reach a Vertex in two ways:

- From another Vertex in two ways:  $V_{n-\text{Vertex}} = 2V_{n-1}$
- From the center in six ways:  $V_{n-\text{Centre}} = 6C_{n-1}$

Since both the above ways are valid, the total number of ways to reach a vertex on the  $n^{\text{th}}$  step is given by:

$$\underbrace{V_n = 2V_{n-1} + 6C_{n-1}}_{\text{Recurrence II}}$$

We first need to calculate the base cases. There are actually six ways to reach a vertex on Step 1. All six are valid. Hence, we can ignore the first step and set:

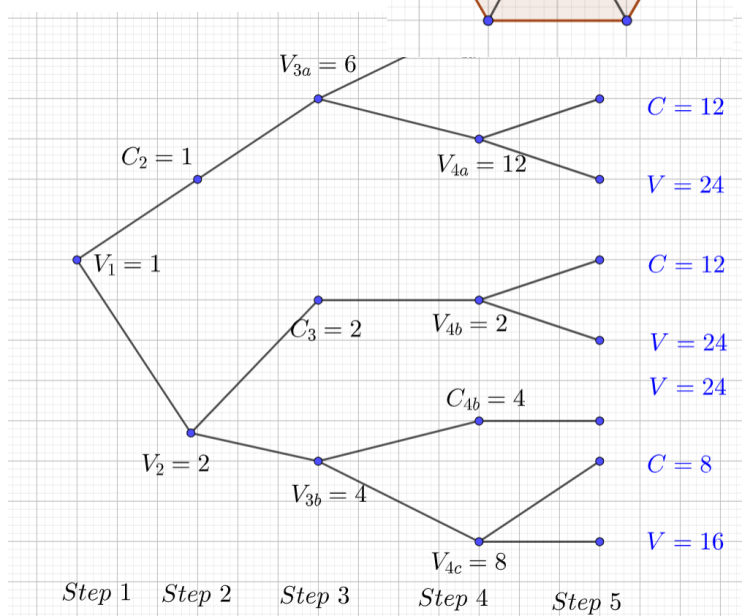
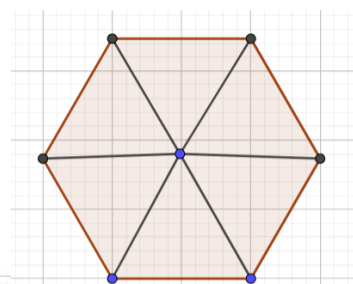
$$V_1 = 1, C_1 = 0$$

$$V_2 = 2V_1 + 6C_1 = 2 + 6(0) = 2$$

Substitute Recurrence I in Recurrence II to get a relation only in terms of  $V_n$ :

$$V_n = 2V_{n-1} + 6V_{n-2}, n \geq 2$$

Now, we use the recurrence:



<sup>5</sup> Compare the tree diagram here, which is in terms of probabilities with the one in the previous example (about the frog), which is terms of paths. The calculations are similar, but the method is slightly different.

$$\begin{aligned} V_3 &= 2V_2 + 6V_1 = (2)(2) + 6(1) = 4 + 6 = 10 \\ V_4 &= 2V_3 + 6V_2 = (2)(10) + (6)(2) = 20 + 12 = 32 \\ V_5 &= 2V_4 + 6V_3 = (2)(32) + (6)(10) = 64 + 60 = 126 \\ V_6 &= 2V_5 + 6V_4 = (2)(126) + (6)(32) = 252 + 192 = 444 \end{aligned}$$

### Recurrence Relation for $S_n$

The number of valid paths  $S_n$  for step  $n$  is given by:

- Valid ways to reach a vertex at step  $n$
- Valid ways to reach the center at step  $n - 1$

$$S_n = V_n + C_{n-1} = V_n + V_{n-1}$$

Hence:

$$S_6 = V_6 + V_5 = 444 + 126 = 570$$

### Calculating the Probability

Since we ignored the first step when calculating successful outcomes, we do the same when calculating total outcomes:

$$\text{Total Outcomes} = 6^5$$

$$\text{Probability} = \frac{\text{Successful Outcomes}}{\text{Total Outcomes}} = \frac{570}{6^5} = \frac{95}{1296}$$

### Example 1.51

Let  $A, B, C$  and  $D$  be the vertices of a regular tetrahedron, each of whose edges measures 1 meter. A bug, starting from vertex  $A$ , observes the following rule: at each vertex it chooses one of the three edges meeting at that vertex, each edge being equally likely to be chosen, and crawls along that edge to the vertex at its opposite end. Let  $p = \frac{n}{729}$  be the probability that the bug is at vertex  $A$  when it has crawled exactly 7 meters. Find the value of  $n$ . (AIME 1985/12)

### Example 1.52

A bug starts at a vertex of an equilateral triangle. On each move, it randomly selects one of the two vertices where it is not currently located, and crawls along a side of the triangle to that vertex. Given that the probability that the bug moves to its starting vertex on its tenth move is  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers, find  $m + n$ . (AIME II 2003/13)

### Recursive Formula (See Tree Diagram)

We can calculate the probability that the bug is at the starting vertex at step  $n$ :

$$= \underbrace{0}_{\text{Probability of reaching start in current step}} + \underbrace{P_{n-1}}_{\text{Bug is at start in previous step}} \cdot \underbrace{\frac{1}{2}}_{\text{Probability of reaching start in current step}} (1 - P_{n-1})$$

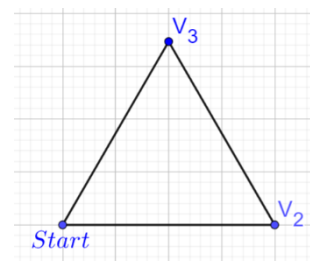
Simplify to get:

$$= \frac{1}{2}(1 - P_{n-1}), \quad n \geq 1$$

### Starting Condition

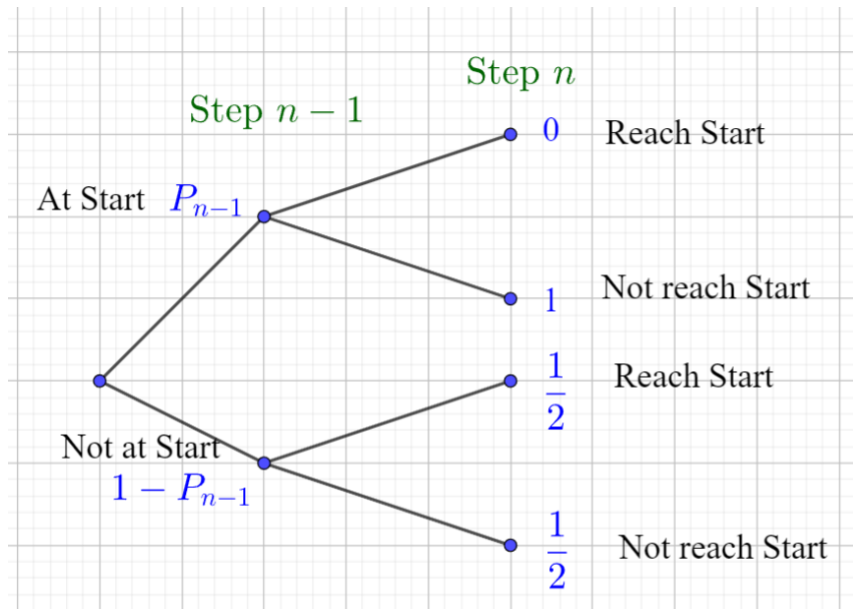
The bug begins at the starting vertex. Hence:

$$P_0 = 1$$



**Using the Formula**

$$\begin{aligned}
P_1 &= \frac{1}{2}(1 - P_0) = \frac{1}{2} \cdot 0 = 0 \\
P_2 &= \frac{1}{2}(1 - P_1) = \left(\frac{1}{2}\right)(1) = \frac{1}{2} \\
P_3 &= \frac{1}{2}(1 - P_2) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \\
P_4 &= \frac{1}{2}(1 - P_3) = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) = \frac{3}{8} \\
P_5 &= \frac{1}{2}(1 - P_4) = \left(\frac{1}{2}\right)\left(\frac{5}{8}\right) = \frac{5}{16} \\
P_6 &= \frac{1}{2}(1 - P_5) = \left(\frac{1}{2}\right)\left(\frac{11}{16}\right) = \frac{11}{32} \\
P_7 &= \frac{1}{2}(1 - P_6) = \left(\frac{1}{2}\right)\left(\frac{21}{32}\right) = \frac{21}{64} \\
P_8 &= \frac{1}{2}(1 - P_7) = \left(\frac{1}{2}\right)\left(\frac{43}{64}\right) = \frac{43}{128} \\
P_9 &= \frac{1}{2}(1 - P_8) = \left(\frac{1}{2}\right)\left(\frac{85}{128}\right) = \frac{85}{256} \\
P_{10} &= \frac{1}{2}(1 - P_9) = \left(\frac{1}{2}\right)\left(\frac{85}{128}\right) = \frac{171}{512}
\end{aligned}$$

**Example 1.53**

A moving particle starts at the point (4,4) and moves until it hits one of the coordinate axes for the first time. When the particle is at the point (a,b), it moves at random to one of the points (a-1,b), (a,b-1), or (a-1,b-1), each with probability  $\frac{1}{3}$ , independently of its previous moves. The probability that it will hit the coordinate axes at (0,0) is  $\frac{m}{3^n}$ , where m and n are positive integers. Find m + n. (AIME I 2019/5)

**1.4 State Spaces****A. Basics****Example 1.54**

Two players,  $P_1$  and  $P_2$  play a game against each other. In every round of the game, each player rolls a fair die once, where the six faces of the die have six distinct numbers. Let  $x$  and  $y$  denote the readings on the die rolled by  $P_1$  and  $P_2$ , respectively. If

- $x > y$ , then  $P_1$  scores 5 points, and  $P_2$  scores 0 points.
- $x = y$ , then each player scores 2 points.
- $x < y$ , then  $P_1$  scores 0 points and  $P_2$  scores 5 points.

Let  $X_i$  and  $Y_i$  be the total scores of  $P_1$  and  $P_2$  respectively, after playing the  $i^{th}$  round. (JEE-A, 2022/Paper-I/16, Adapted)

Find the probability that:

- A.  $X_2 \geq Y_2$
- B.  $X_2 = Y_2$
- C.  $X_3 = Y_3$
- D.  $X_3 > Y_3$

Note that the six distinct numbers have not been specified. But, for the winning condition, it is only important that the numbers be distinct. Hence, without loss of generality, let the numbers be:

$$\{1,2,3,4,5,6\}$$

The die is fair. Hence, the sample space is equiprobable.

	Total					
	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

$$P(X \text{ Wins}) = P(Y \text{ Wins}) = \frac{15}{36} = \frac{5}{12}$$

$$P(\text{Draw}) = \frac{6}{36} = \frac{2}{12}$$

### Part A

$$P(Y_2 > X_2) = \left(\frac{2}{12}\right)\left(\frac{5}{12}\right) + \left(\frac{5}{12}\right)\left(\frac{2}{12}\right) + \left(\frac{5}{12}\right)\left(\frac{5}{12}\right) = \frac{5}{12}\left(\frac{2}{12} + \frac{2}{12} + \frac{5}{12}\right) = \frac{5}{12}\left(\frac{9}{12}\right) = \frac{5}{16}$$

$$P(X_2 \geq Y_2) = 1 - \frac{5}{16} = \frac{11}{16}$$

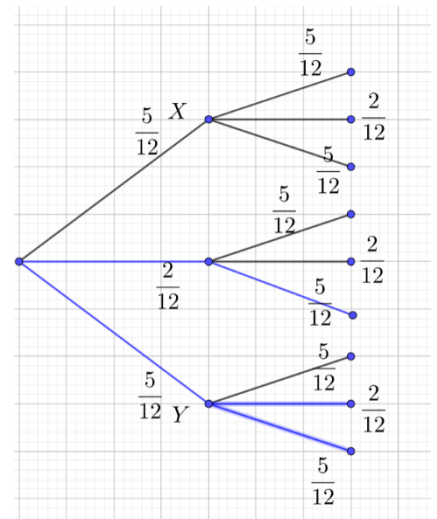
### Part B

By symmetry:

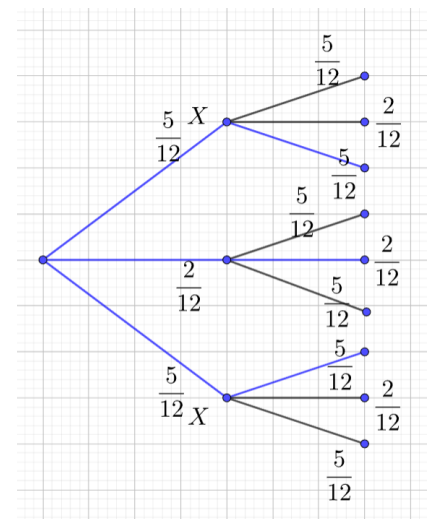
$$P(X_2 > Y_2) = P(Y_2 > X_2) = \frac{5}{16}$$

### Part C and D

The tree diagram with 2 rounds has 9 ending nodes. If you extend it to 3 rounds, it will have 27 ending nodes.



$$P(X_2 = Y_2) = \left(\frac{5}{12}\right)\left(\frac{5}{12}\right) + \left(\frac{2}{12}\right)\left(\frac{2}{12}\right) + \left(\frac{5}{12}\right)\left(\frac{5}{12}\right) = \frac{54}{144} = \frac{3}{8}$$



## 1.55: State Space

A state space uses variables to track the behavior of a system through time.

### Example 1.56

Two players,  $P_1$  and  $P_2$  play a game against each other. In every round of the game, each player rolls a fair die

once, where the six faces of the die have six distinct numbers. Let  $x$  and  $y$  denote the readings on the die rolled by  $P_1$  and  $P_2$ , respectively. If

- $x > y$ , then  $P_1$  scores 5 points, and  $P_2$  scores 0 points.
- $x = y$ , then each player scores 2 points.
- $x < y$ , then  $P_1$  scores 0 points and  $P_2$  scores 5 points.

Let  $X_i$  and  $Y_i$  be the total scores of  $P_1$  and  $P_2$  respectively, after playing the  $i^{th}$  round. (JEE Advanced, 2022/Paper-I/16, Adapted)

Find the probability that:

- A.  $X_2 \geq Y_2$
- B.  $X_2 = Y_2$
- C.  $X_3 = Y_3$
- D.  $X_3 > Y_3$

Note that instead of tracking the probabilities on a tree diagram, we can simply track

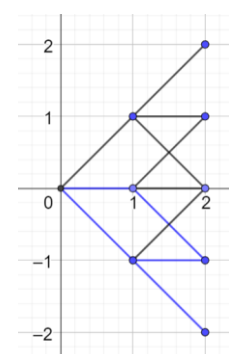
$g = \text{game number}$

$W = \text{No. of Wins by } X - \text{No. of Wins by } Y$

**Part A**

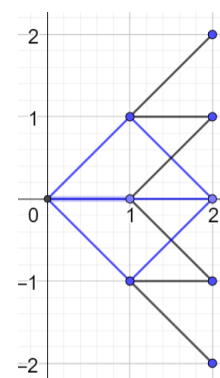
$$P(Y_2 > X_2) = \left(\frac{2}{12}\right)\left(\frac{5}{12}\right) + \left(\frac{5}{12}\right)\left(\frac{2}{12}\right) + \left(\frac{5}{12}\right)\left(\frac{5}{12}\right) = \frac{5}{12}\left(\frac{2}{12} + \frac{2}{12} + \frac{5}{12}\right) = \frac{5}{12}\left(\frac{9}{12}\right) = \frac{5}{16}$$

$$P(X_2 \geq Y_2) = 1 - \frac{5}{16} = \frac{11}{16}$$



**Part B**

$$P(X_2 = Y_2) = \left(\frac{5}{12}\right)\left(\frac{5}{12}\right) + \left(\frac{2}{12}\right)\left(\frac{2}{12}\right) + \left(\frac{5}{12}\right)\left(\frac{5}{12}\right) = \frac{54}{144} = \frac{3}{8}$$



### Part C

$$P(D, D, D) = \left(\frac{2}{12}\right)^3$$

$$P(X, D, Y) = P(Y, D, X) = \left(\frac{5}{12}\right)^2 \left(\frac{2}{12}\right)$$

$$P(X, Y, D) = P(Y, X, D) = \left(\frac{5}{12}\right)^2 \left(\frac{2}{12}\right)$$

$$P(D, X, Y) = P(D, Y, X) = \left(\frac{5}{12}\right)^2 \left(\frac{2}{12}\right)$$

If you add all the cases, you get:

$$= \frac{8}{1728} + \frac{300}{1728} = \frac{308}{1728} = \frac{77}{432}$$

### Part D

Case I: Y has 2 or 3 wins:

*Not possible*

Case I: Y has 1 win:

$$(Y, X, X), (X, Y, X), (X, X, Y) = \left(\frac{5}{12}\right)^3 (3)$$

Case II: Y has 0 wins, 0 Draws

$$(X, X, X) = \left(\frac{5}{12}\right)^3 (1)$$

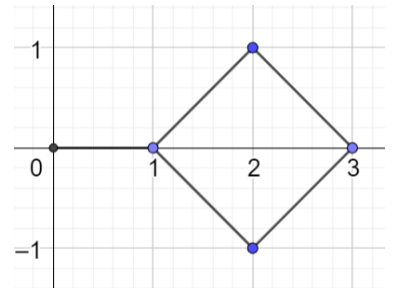
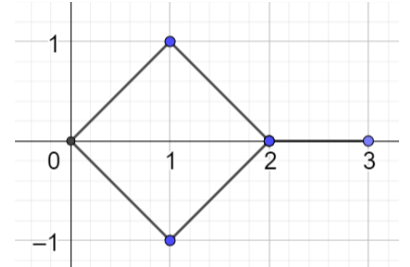
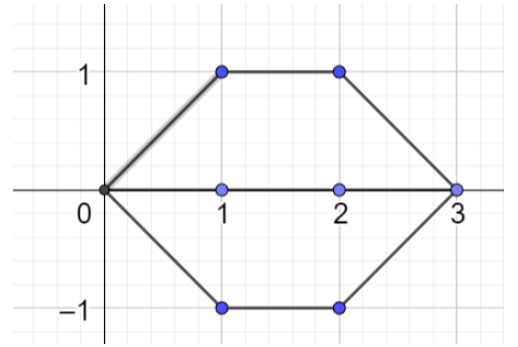
Case III: Y has 0 wins, 1 Draws

$$(X, X, D), (X, D, X), (D, X, X) = \left(\frac{5}{12}\right)^2 \left(\frac{2}{12}\right) (3)$$

Case IV: Y has 0 wins, 2 Draws

$$(X, D, D), (D, X, D), (D, D, X) = \left(\frac{2}{12}\right)^2 \left(\frac{5}{12}\right) (3)$$

$$\left(\frac{5}{12}\right)^3 (4) + \left(\frac{5}{12}\right)^2 \left(\frac{2}{12}\right) (3) + \left(\frac{2}{12}\right)^2 \left(\frac{5}{12}\right) (3)$$



$$= \frac{500}{1728} + \frac{150}{1728} + \frac{60}{1728} = \frac{710}{1728} = \frac{355}{864}$$

## 1.5 Geometric Probability

### A. Area as Probability

#### Example 1.57

Rachel and Robert run on a circular track. Rachel runs counterclockwise and completes a lap every 90 seconds, and Robert runs clockwise and completes a lap every 80 seconds. Both start from the same line at the same time. At some random time between 10 minutes and 11 minutes after they begin to run, a photographer standing inside the track takes a picture that shows one-fourth of the track, centered on the starting line. What is the probability that both Rachel and Robert are in the picture? (AMC 10B 2009/23, AMC 12B 2009/18)

After 10 minutes:

$$\frac{600}{90} = \frac{60}{9} = \frac{20}{3} = 6\frac{2}{3}$$

$$18.75 < x < 41.25 \text{ AND } 30 < x < 50 \Rightarrow 30 < x < 41.25$$

$$P = \frac{41.25 - 30}{60} = \frac{11.25}{60} = \frac{45}{240} = \frac{3}{16}$$

#### Example 1.58

On a checkerboard composed of 64 unit squares, what is the probability that a randomly chosen unit square does not touch the outer edge of the board? (AMC 8 2009/10)

Ans = 9/16

#### Warmup 1.59

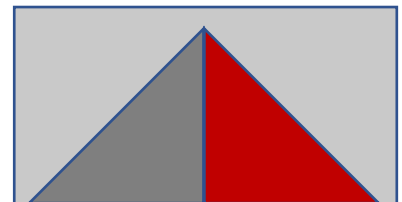
Kunal throws a dart at the dartboard in the diagram. If the dart has equal probability of any part of the dartboard, find the probability of hitting each colored region. (Note: Areas that look equal are actually equal).



$$p + p = 1 \Rightarrow 2p = 1 \Rightarrow p = \frac{1}{2}$$

#### Example 1.60

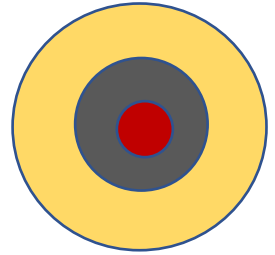
A surveyor releases a survey probe from a balloon at a mountain (side view shown in the diagram). If it is known that the probe hits the mountain, and it has equal probability of reaching any part of the diagram, find the probability of hitting each colored region.



$$p + p = 1 \Rightarrow 2p = 1 \Rightarrow p = \frac{1}{2}$$

**Example 1.61**

A dart is thrown at the circular dartboard shown alongside. The circles are concentric, with the innermost circle having a radius of two units, the middle circle a radius of two units, and the largest circle a radius of four units. Find the probability of hitting each color.

**Total Area**

$$A = \pi r^2 = 16\pi$$

**Coloured Areas**

$$\text{Red Area} = \pi r^2 = \pi$$

$$\text{Gray Area} = 4\pi - \pi = 3\pi$$

$$\text{Yellow Area} = 16\pi - 4\pi = 12\pi$$

**Probability**

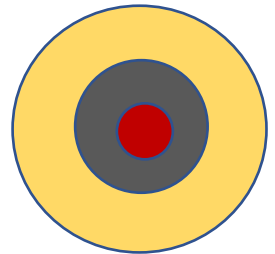
$$P(\text{Red}) = \frac{\pi}{16\pi} = \frac{1}{16}$$

$$P(\text{Gray}) = \frac{3\pi}{16\pi} = \frac{3}{16}$$

$$P(\text{Yellow}) = \frac{12\pi}{16\pi} = \frac{12}{16} = \frac{3}{4}$$

**Example 1.62**

A dart is thrown at the circular dartboard shown alongside. The circles are concentric. The probability of hitting the gray region is half, and the probability of hitting the yellow region is seven-sixteenth. Find



- The probability of hitting the red region.
- The ratio of the radii of the respective circular regions.

**Part A**

We use complementary probability

$$P(\text{Red}) = 1 - P(\text{Yellow}) - P(\text{Gray}) = 1 - \frac{7}{16} - \frac{1}{2} = \frac{1}{16}$$

**Part B**

$$P(\text{Red}) = \frac{1}{16} \Rightarrow \frac{\text{Area}(\text{Red})}{\text{Total Area}} = \frac{1}{16}$$

Recall that radius is a linear measure, while area is squared. Hence, the radii of the circles will be proportional to the square root of the areas of the circles.

Hence, take square roots to find the ratio of the radii:

$$\frac{\text{Radius}(\text{Red})}{\text{Radius}(\text{Largest Circle})} = \sqrt{\frac{1}{16}} = \frac{1}{4}$$

**Example 1.63**

A point is chosen at random from within a circular region. What is the probability that the point is closer to the center of the region than it is to the boundary of the region? (AMC 8 1996/25)

Ans =  $\frac{1}{4}$

**Example 1.64**

On the dart board shown in the figure below, the outer circle has radius 6 and the inner circle has radius 3.

Three radii divide each circle into three congruent regions, with point values shown. The probability that a dart will hit a given region is proportional to the area of the region. When two darts hit this board, the score is the sum of the point values in the regions. What is the probability that the score is odd? (AMC 8 2007/25)

Ans = 35/72

## B. Real Number Line

### Example 1.65

Nick picks a real number at random from the interval (0,5). What is the probability that the number lies in the interval (1,3)?

$$\text{Probability} = \frac{\text{Length of Valid Interval}}{\text{Total Interval}} = \frac{2}{5}$$



### Example 1.66

Britney picks a random number on the real number line between 0 and 1 and then rounds it off to the nearest integer. Find the probability of each possible integer that she can get.



Suppose the number Britney picks is  $x$ .

Apply the rounding off rules:

$$\text{Round}(x) = 1 \Rightarrow 0.5 \leq x \leq 1$$

$$\text{Round}(x) = 0 \Rightarrow 0 \leq x \leq 0.5$$

$$\text{Probability} = \frac{\text{Length of Valid Interval}}{\text{Total Interval}} = \frac{0.5}{1} = \frac{1}{2}$$

### Example 1.67

Lee picks a random number on the real number line between 0 and 2 and then rounds it off to the nearest integer. Find the probability:



- Of each possible integer that Lee can get.
- That, before rounding, the number that Lee picks is exactly one.

## C. Coordinate Plane

### Example 1.68s

Sean picks a random non-zero number. Sally picks another random non-zero number distinct from the one that Sean picks. Stephen makes an ordered pair from the numbers that Sean and Sally picked, writing them in the form

$$(x, y)$$

## D. Triangles

### Example 1.69

A point is selected at random from  $\triangle ABC$ . What is the probability that it lies inside the region formed by joining the midpoints of  $AB$ ,  $BC$  and  $CA$ .

Connect the midpoints of  $AB$ ,  $BC$  and  $CA$  to form a triangle, and let them be  $X, Y$  and  $Z$  respectively

By the Midpoint Theorem (which is a special case of the Basic Proportionality Theorem),

$$XY = \frac{1}{2}BC, \quad XZ = \frac{1}{2}CA, \quad YZ = \frac{1}{2}AB$$

Hence,

$$\triangle XYZ \sim \triangle ABC$$

By similarity, the ratio of the areas are proportional to the square of the ratios of the sides, and this is the same as the probability of the point being inside the region:

$$\frac{P(\triangle XYZ)}{P(\triangle ABC)} = \frac{A(\triangle XYZ)}{A(\triangle ABC)} = \frac{\left(\frac{1}{2}\right)^2}{1} = \frac{1}{4}$$

## E. Triangle Inequality

### Example 1.70

Two sides of a nondegenerate triangle measure 2" and 4" and the third side measures some whole number of inches. If a cube with faces numbered 1 through 6 is rolled, what is the probability, expressed as a common fraction, that the number showing on top could be the number of inches in the length of the third side of the triangle? (**MathCounts 1995 Chapter Target**)

By the triangle inequality:

$$4 - 2 < x < 4 + 2 \Rightarrow 2 < x < 6 \Rightarrow x \in (3, 4, 5) \Rightarrow P = \frac{3}{6} = \frac{1}{2}$$

## F. Coordinate Geometry

### Example 1.71

$x$  and  $y$  are each random values between 0 and 2 (exclusive).

- What is the probability that  $x$  is greater than  $y$ ?
- If  $x$  and  $y$  are changed to be random values between 0 and 2 (inclusive), will your answer change? Your calculations?

#### Part A

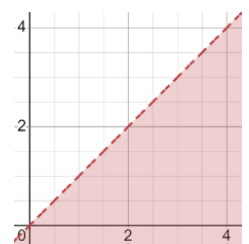
##### By Symmetry

$$p + p = 1 \Rightarrow 2p = 1 \Rightarrow p = \frac{1}{2}$$

##### By Coordinate Geometry

Shade the region on the coordinate where  $x$  is greater than  $y$ . By symmetry, the probability is

$$\frac{1}{2}$$



## Part B

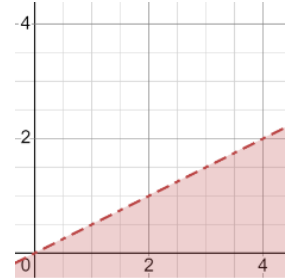
Neither answers nor calculations will change.

### Example 1.72

$x$  is a random value between 0 and 4.  $y$  is also a random value between 0 and 4. What is the probability that  $x$  is greater than  $2y$ ?

Shade the region on the coordinate plane where  $x$  is greater than  $y$ . We want to find the area of the triangle divided by the area of the square, which is given by:

$$\frac{A(\text{Triangle})}{A(\text{Square})} = \frac{4 \times 2 \times \frac{1}{2}}{4 \times 4} = \frac{1}{4}$$



## G. Triangle Inequality

### Challenge 1.73

A one-meter long stick is cut at two random places. What is the probability that the three sticks so formed can be used to form a triangle?

#### Total Outcomes / Area

Place a real number line along the one-meter stick (see diagram below). Let the cuts be made at the number  $x$  and the number  $y$ , with



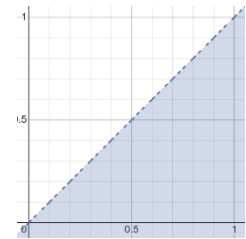
As shown in the number line, we assume that

$$x > y$$

Hence, the lengths of the three sticks obtained are:

$$y, \quad x - y, \quad 1 - x$$

Also, we can calculate the total area that satisfies  $x > y$  as the **blue region** shown to the right.



#### Successful Outcomes / Area

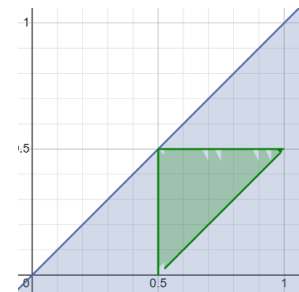
The three sides of the triangle must satisfy the triangle inequality:

$$(1 - x) + (x - y) > y \Rightarrow 1 - y > y \Rightarrow y < \frac{1}{2}$$

$$y + (x - y) > 1 - x \Rightarrow x > 1 - x \Rightarrow x > \frac{1}{2}$$

$$(1 - x) + y > x - y \Rightarrow y > x - \frac{1}{2}$$

The region that satisfies all three inequalities is shown in the **green region** to the right.



#### Probability

$$\text{Probability} = \frac{\text{Green Area}}{\text{Blue Area}} = \frac{\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}}{1 \times 1 \times \frac{1}{2}} = \frac{1}{4}$$

## H. Time Arrivals

### Example 1.74

Romeo and Juliet have a date at a given time, and each will arrive at the meeting place with a delay between 0

and 1 hour, with all pairs of delays being equally likely. The first to arrive will wait for 15 minutes and will leave if the other has not yet arrived. What is the probability that they will meet? (MIT OCW)

### [Solution](#)

## I. 3D Probability

### Example 1.75

A cube of side length  $\frac{7}{3}$  is dropped into a random place on the 3D lattice in the coordinate space (a coordinate plane with an extra dimension), with the sides of the square parallel/perpendicular to the axes. The expected number of lattice points (points with integer coordinates) in the interior of the cube is  $\frac{a}{b}$  in simplest terms, what's  $a + b$ ? (JHMMC Grade 2020 R1/31)

$$\frac{7}{3} = 2\frac{1}{3}$$

Consider the x dimension:

With probability  $\frac{1}{3}$ , you will have 3 points

With probability  $\frac{2}{3}$ , you will have 2 points

The expected number of lattice points is:

$$\frac{1}{3} \times 3 + \frac{2}{3} \times 2 = 1 + \frac{4}{3} = \frac{7}{3}$$

By symmetry, for the y dimension, the expected number of lattice points is:

$$\frac{7}{3}$$

And also for the z dimension:

$$\frac{7}{3}$$

Since the three values are independent, by the multiplication principle, the final answer is:

$$\left(\frac{7}{3}\right)^3 = \frac{343}{27}$$

## 1.6 Symmetry: Reducing Counting

### A. Basics

#### Example 1.76: Basics

A five-digit integer will be chosen at random from all possible positive five-digit integers. What is the probability that the number's units digit will be less than 5? Express your answer as a common fraction. (MathCounts 2005 School Countdown)

If we have a one-digit integer:

$$\text{Probability} = \frac{\text{Successful Outcomes}}{\text{Total Outcomes}} = \frac{5}{10} = \frac{1}{2}$$

Even if we have a five-digit integer, the question is only concerned about the last digit, and hence, the probability is still

$$\frac{\text{Successful Outcomes}}{\text{Total Outcomes}} = \frac{5}{10} = \frac{1}{2}$$

### Example 1.77: Tossing Coins

A fair coin is tossed twice. What is the probability of getting:

- A. Different outcomes in the two tosses
- B. Same outcomes in the two tosses

#### Part A

There are no restrictions on the first roll. Hence, we accept the first roll with probability

$$1$$

We want the second roll to be different from the first roll, which will happen with probability

$$\frac{1}{2}$$

Overall probability is

$$1 \times \frac{1}{2} = \frac{1}{2}$$

#### Part B

There are no restrictions on the first roll. Hence, we accept the first roll with probability

$$1$$

We want the second roll to be the same as the first roll, which will happen with probability

$$\frac{1}{2}$$

Overall probability is

$$1 \times \frac{1}{2} = \frac{1}{2}$$

### Example 1.78: Drawing Cards

Cybil and Ronda are sisters. The 10 letters from their names are placed on identical cards so that each of 10 cards contains one letter. Without replacement, two cards are selected at random from the 10 cards. What is the probability that one letter is from each sister's name? Express your answer as a common fraction. (**MathCounts 2008 State Team**)

#### Method I

Let the first pick be any letter. It will be from some sister's name. The second pick should be from a different sister's name, which has probability:

$$= \frac{5}{9}$$

#### Method II

The first letter could be from Cybil, and the second letter from Rhonda, which has probability:

$$\frac{5}{10} \times \frac{5}{9}$$

Or, the first letter could be from Rhonda, and the second letter from Cybil, which also has probability:

$$\frac{5}{10} \times \frac{5}{9}$$

Hence, the required probability:

$$= \left( \frac{5}{10} \times \frac{5}{9} \right) + \left( \frac{5}{10} \times \frac{5}{9} \right) = 2 \left( \frac{1}{2} \times \frac{5}{9} \right) = \frac{5}{9}$$

### Example 1.79: Rolling Dice

- A. Beatrice is playing Dungeons and Dragons with her sister Rose. First, she rolls an attack roll using an eight-sided dice, and then Rose rolls a defense roll using the same eight-sided dice. If the values on both the rolls are the same, then it's a tie, and they roll again. What is the probability that must roll more than once?
- B. Two cubical dice each have removable numbers 1 through 6. The twelve numbers on the two dice are removed, put into a bag, then drawn one at a time and randomly reattached to the faces of the cubes, one number to each face. The dice are then rolled and the numbers on the two top faces are added. What is the probability that the sum is 7? (AMC 10A 2009/22)

#### Part A

There is no restriction on the first roll. Hence, we simply ignore the first roll.

However, the second roll must match the first roll, and this will happen with probability

$$\frac{1}{8}$$

#### Part B

We are drawing two numbers from the twelve numbers:

$$\{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6\}$$

We can get a sum of 7 in the following ways

$$1 + 6, 2 + 5, 3 + 4 \Rightarrow \text{The two numbers are never equal}$$

Suppose we draw the first number and it is

$$n$$

For the second draw, the number of numbers remaining

$$= \text{Total Outcomes} = 11$$

For the sum to be 7, the second number must be

$$7 - n \Rightarrow 2 \text{ Ways}$$

And hence the probability is

$$\frac{2}{11}$$

### Example 1.80

A standard six-sided die is rolled five times. What is the probability that the five rolls are all the same or all different?

#### All the same

The first roll can be any number. The second and further rolls must then match the first number, which will happen with probability

$$\underbrace{\left(\frac{1}{6}\right)}_{2^{\text{nd}} \text{ Roll}} \times \underbrace{\left(\frac{1}{6}\right)}_{3^{\text{rd}} \text{ Roll}} \times \underbrace{\left(\frac{1}{6}\right)}_{4^{\text{th}} \text{ Roll}} \times \underbrace{\left(\frac{1}{6}\right)}_{5^{\text{th}} \text{ Roll}} = \left(\frac{1}{6}\right)^4 = \frac{1}{1296}$$

**All different**

Again, the first roll can be any number. The second roll cannot repeat the first, and has only five numbers that work for us with probability:

$$\frac{5}{6}$$

Similarly, the third roll cannot repeat the first, or the second, and will be successful with probability

$$\frac{4}{6}$$

Continuing the pattern, we get:

$$\frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} \times \frac{2}{6} = \frac{120}{1296}$$

**Final Probability**

The final probability is then:

$$\frac{120}{1296} + \frac{1}{1296} = \frac{121}{1296}$$

**Example 1. 81: Seating Arrangements**

Angie, Bridget, Carlos, and Diego are seated at random around a square table, one person to a side. What is the probability that Angie and Carlos are seated opposite each other? (AMC 8 2011/12)

**Method I: Symmetry**

We are not bothered about opposite till we seat both of them. Only seat Angie. He can sit where ever he wants.

	Seat 2	
Seat 1		Seat 3
	Angie	

Now let Carlos sit. He has to select one of three seats, out of which one is opposite Angie. Hence, the probability is:

$$\frac{1}{3}$$

**Combinations**

Angie and Carlos are selecting two seats out of four seats, which can be done in

$$\binom{4}{2} = 6 \text{ ways}$$

Out of the 6 ways of being seated in a four-seater square table, the number of ways where Angie and Carlos are opposite each are

2 ways

Hence, required probability is

$$\frac{2}{6} = \frac{1}{3}$$

**Example 1.82**

A standard six-sided die is rolled twice. Find the probability that the two numbers thrown are different.

### Method I: Complementary Probability

$$P(\text{Different}) = 1 - P(\text{Same}) = 1 - \frac{1}{6} = \frac{5}{6}$$

### Method II: Casework

$$\begin{aligned} &P(1, \text{Not } 1) + P(2, \text{Not } 2) + P(3, \text{Not } 3) + P(4, \text{Not } 4) + P(5, \text{Not } 5) + P(6, \text{Not } 6) \\ &= \left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = 6\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = \frac{5}{6} \end{aligned}$$

### Method III: Symmetry

The first number can be any number. The second number will not match the first with probability

$$\frac{5}{6}$$

### Example 1.83

- The 600 students at King Middle School are divided into three groups of equal size for lunch. Each group has lunch at a different time. A computer randomly assigns each student to one of three lunch groups. The probability that three friends, Al, Bob, and Carol, will be assigned to the same lunch group is approximately **(AMC 8 1986/24)**
- Three houses are available in a locality. Three persons apply for the houses. Each applies for one house without consulting others. The probability that all the three apply for the same house is: **(JEE Main 2005)**

#### Part A

The probability of being assigned to each lunch group is the same. Hence, without loss of generality, assign Al to any lunch group. This happens with probability

$$1$$

The probability that Bob gets assigned to the same lunch group as Al is:

$$\frac{1}{3}$$

The probability that Carol gets assigned to the same lunch group as Al is also:

$$\frac{1}{3}$$

The probability then that all three get assigned to the same lunch group is:

$$1 \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

#### Part B

Note that the scenario is different, but the question is same (including the numbers).

$$P = \frac{1}{9}$$

## B. Dealing Cards

### Example 1.84

A dealer deals a well-shuffled pack of cards to four people. Consider Ace to be worth 1 point, Jack to be worth 11, Queen to be worth 12, and King to be worth 13. What is the probability that:

- The Ace of Hearts is dealt before the Seven of Spades?
- The Ace of Spades is the first of the Aces to be dealt?
- A king is the first among all the face cards which are dealt?
- The first card dealt has a *different* "value" from the second card.
- The first card dealt has the *same* "value" from the second card. (Here, consider Ace to be worth 1 point, Jack to be worth 11, Queen to be worth 12, and King to be worth 13).

### Strategy: Narrow the Problem Down

Here, there is a lot of information being thrown at us. 52 cards are being dealt. Taking all of them into consideration makes the problem more difficult.

Instead, we only focus on the part of the pack of cards that we need to answer our specific question.

#### Part A

We only consider two cards out of 52, since those are the only ones we are concerned with. This gives us two arrangements:

$$(Ace, Seven), (Seven, Ace) \Rightarrow P = \frac{1}{2}$$

#### Part B

We only consider the four Aces. They have to be in the deck in some sequence:

$$\begin{array}{cccc} \square & \square & \square & \square \\ \downarrow & \downarrow & \downarrow & \downarrow \\ Ace\ 1 & Ace\ 2 & Ace\ 3 & Ace\ 4 \end{array}$$

We are interested in placing the Ace of Spades in this sequence. We have four choices, of which, only in the first choice will the Ace of Spades be dealt first.

$$P = \frac{1}{4}$$

#### Part C

We only consider the twelve face cards, which are arranged in the deck in some sequence

$$\begin{array}{ccc} \square & \square & \dots & \square \\ \downarrow & \downarrow & & \downarrow \\ Face & Face & & Face \\ Card\ 1 & Card\ 2 & & Card\ 12 \end{array}$$

For the first place, we want a King, so out of the twelve face cards, we will pick a King with probability:

$$\frac{4}{12} = \frac{1}{3}$$

(Compare the logic used in Part C with the logic used in Part B.)

#### Part D

There are no restrictions on the value of the first card.

Having dealt the first card, we no longer have a 52-card deck. Instead, we have a 51-card deck.

Whatever be the value of the card that we dealt, there will be

- 3 Cards with the same value

➤ 48 Cards with a different value

Hence, the probability of getting a card with a different value is

$$\frac{48}{51} = \frac{16}{17}$$

**Part E**

$$1 - \frac{16}{17} = \frac{1}{17}$$

**Example 1.85**

A card sharp is playing cards. He deals the cards for bridge to four people, including himself. The cards are shuffled, and he deals them randomly. But from long years of experience, and secret raised markings on the cards, he knows which cards are being dealt. What is the probability that, out of the exactly

- A. thirteen cards that comprise the suit of hearts, the two of hearts is dealt second.
- B. twelve face cards, the Queen of Clubs is dealt third.

**Example 1.86**

Two cards are dealt from a deck of four red cards labeled  $A, B, C, D$  and four green cards labeled  $A, B, C, D$ . A winning pair is two of the same color or two of the same letter. What is the probability of drawing a winning pair? (AMC 8 2007/21)

**Logic**

A	B	C	D
A	B	C	D

Choose the first card (on which there are no restrictions), leaving seven cards.

A	B	C	D
A	B	C	D

Of these, three have the same color, and one has the same letter. Hence, the probability is:

$$\frac{4}{7}$$

**Permutations**

By multiplication principle, total outcomes for drawing two cards is:

$$\text{Total Outcomes}(TO) = 8 \times 7$$

To get a winning pair, we don't have any restrictions on the first card, giving us 8 choices. Once we do choose the first card, the second card must either be same letter (1 Choice) or same color (3 choices):

$$\text{Successful Outcomes}(SO) = 8 \times (1 + 3) = 8 \times 4$$

The required probability is

$$P = \frac{SO}{TO} = \frac{8 \times 4}{8 \times 7} = \frac{4}{7}$$

**Combinations**

The number of ways of choosing two cards from eight is:

$$\binom{8}{2} = \frac{8 \times 7}{2} = 28$$

Out of these, the ways that we want are:

$$\text{Both Letters Same} = 4 \binom{2}{2} = 4 \text{ Ways}$$

$$\text{Both Colours Same} = 2 \binom{4}{2} = 2 \times \frac{4 \times 3}{2} = 12$$

Probability of Success

$$\frac{4 + 12}{28} = \frac{16}{28} = \frac{4}{7}$$

## C. 3D Shapes

### Example 1.87

I pick two vertices of a cube, at random. What is the probability that they form:

- An edge
- A face diagonal (a diagonal that lies completely on a face of the cube)
- A space diagonal (a diagonal that lies completely in space)

Note that a cube has

8 Vertices

Because of the symmetry of the cube, it does not matter which is the first point to get picked. For all parts below, suppose, without loss of generality, that the first point to be picked is

A

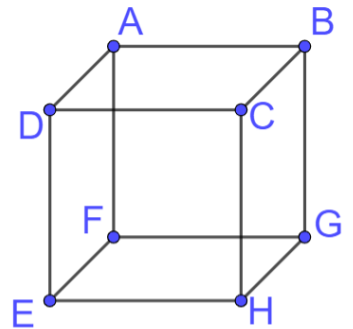
We can get edges, face diagonals and space diagonals from:

$$\underbrace{AB, AD, AF}_{\text{Edges}}, \underbrace{AC, AG, AE}_{\text{Face Diagonals}}, \underbrace{AH}_{\text{Space Diagonals}}$$

Hence, the required probabilities are:

$$\frac{3}{\binom{7}{1}}, \frac{3}{\binom{7}{1}}, \frac{1}{\binom{7}{1}}$$

$P(\text{Edge}) \quad P(\text{Face Diagonal}) \quad P(\text{Space Diagonal})$



## 1.7 Symmetry

### A. Basics

Read the information below and answer the questions that follow

### Example 1.88

Ashani is tossing coins.

- He tosses the coin once
- He tosses the coin thrice
- He tosses the thirty-three times.

For each part, what is the probability that the number of heads is equal to the number of tails.

Since the number of coins tosses is always odd, the number of heads can never be equal to the number of tails.  
Hence:

$$P = 0$$

For each part above, what is the probability that the number of heads is more than the number of tails?

## Parts A and B

$$H, T \Rightarrow \frac{1}{2}$$

$$(HHH)(HHT)(HTH)(HTT)(THH)(THT)(TTH)(TTT) \Rightarrow \frac{4}{8} = \frac{1}{2}$$

## Part C

Parts A and B can be done easily using enumeration. However, Part C involves really large numbers, and enumeration is not recommended.

Consider the probability of getting more heads than tails, and let this probability be

$$p$$

By symmetry, the probability of getting more tails than heads is also:

$$p$$

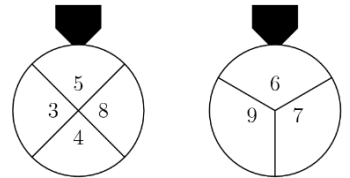
Since the sum of mutually exclusive and exhaustive outcomes is 1, we must have:

$$p + p = 1 \Rightarrow 2p = 1 \Rightarrow p = \frac{1}{2}$$

## B. Parity

### Example 1.89

Every time these two wheels are spun, two numbers are selected by the pointers. What is the probability that the sum of the two selected numbers is even? (AMC 8 1989/25)



We tabulate the probabilities of the outcomes:

	Odd	Even
First Spinner	$\frac{1}{2}$	$\frac{1}{2}$
Second Spinner	$\frac{2}{3}$	$\frac{1}{3}$

### Multiplication Rule

The probability that both are even is:

$$\underbrace{\frac{1}{2} \times \frac{2}{3}}_{\text{Both are odd}} + \underbrace{\frac{1}{2} \times \frac{1}{3}}_{\text{Both are even}} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

### Symmetry

We are looking for

$$\underbrace{\text{Odd}}_{\text{Second Spinner}} \times \underbrace{\text{Odd}}_{\text{First Spinner}} + \underbrace{\text{Even}}_{\text{Second Spinner}} \times \underbrace{\text{Even}}_{\text{First Spinner}}$$

Observe the first row of the table. It is symmetrical since both odd and even have probability  $\frac{1}{2}$ . Substitute those probabilities:

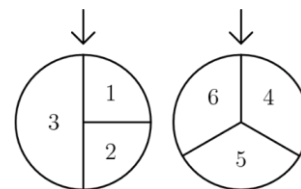
$$\underbrace{\text{Odd}}_{\text{Second Spinner}} \times \frac{1}{2} + \underbrace{\text{Even}}_{\text{Second Spinner}} \times \frac{1}{2}$$

Factor out  $\frac{1}{2}$ :

$$\frac{1}{2} \left( \underbrace{\text{Odd}}_{\text{Second Spinner}} + \underbrace{\text{Even}}_{\text{Second Spinner}} \right)$$

But  $P(\text{Odd}) + P(\text{Even}) = 1$ , and hence, we must have

$$\frac{1}{2} \left( \underbrace{\text{Odd}}_{\text{Second Spinner}} + \underbrace{\text{Even}}_{\text{Second Spinner}} \right) = \frac{1}{2} (1) = \frac{1}{2}$$



### Example 1.90: Even Parity

- Two standard six-sided dice are tossed. One die is red and the other die is blue. What is the probability that the number appearing on the red die is greater than the number appearing on the blue die? (**Gauss 7, 2014/23**)
- Diana and Apollo each roll a standard die obtaining a number at random from 1 to 6. What is the probability that Diana's number is larger than Apollo's number? (**AMC 8 1995/20**)
- A dice is rolled twice. What is the probability that the number in the second roll will be higher than that in the first? (**XAT 2007/52**)
- A fair 6-sided die is rolled twice. What is the probability that the first number that comes up is greater than or equal to the second number? (**AMC 8 2011/18**)

#### Part A

##### Enumeration

Write the outcomes in the table alongside as follows:

$$\left( \overset{\text{Red Die}}{\underset{\text{Red Die}}{1}}, \overset{\text{Blue Die}}{\underset{\text{Blue Die}}{1}} \right)$$

Shade the region which is the valid region for us, and count the valid outcomes. Hence, the final probability is:

$$\frac{15}{36} = \frac{5}{12}$$

	1	2	3	4	5	6	
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)	5
2	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)	4
3	(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)	3
4	(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)	2
5	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)	1
6	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)	0
							15

##### Symmetry

When we roll a red die and a blue die, there are three possible cases:

- $\text{Red} > \text{Blue}$
- $\text{Red} = \text{Blue}$
- $\text{Red} < \text{Blue}$

There is nothing to distinguish between the red and the blue dice. Hence, by symmetry

$$P(\text{Red} > \text{Blue}) = P(\text{Red} < \text{Blue}) = p(\text{say})$$

Also, we can calculate that there are exactly six outcomes where red is equal to blue, and hence,

$$P(\text{Red} = \text{Blue}) = \frac{6}{36} = \frac{1}{6}$$

And we know that:

$$P(\text{Red} > \text{Blue}) + P(\text{Red} < \text{Blue}) + P(\text{Red} = \text{Blue}) = 1$$

Substitute  $P(\text{Red} > \text{Blue}) = P(\text{Red} < \text{Blue}) = p$ , and  $P(\text{Red} = \text{Blue}) = \frac{1}{6}$ :

$$p + p + \frac{1}{6} = 1 \Rightarrow 2p = \frac{5}{6} \Rightarrow p = \frac{5}{12}$$

**Parts B and C**

For both the parts, by the same logic as above, the answer is

$$\frac{5}{12}$$

**Part D**

From the above, we know that

$$P(\text{1st Number} > \text{2nd Number}) = \frac{5}{12}$$

$$P(\text{1st Number} = \text{2nd Number}) = \frac{1}{6}$$

Hence, the probability that we are looking for

$$= \frac{5}{12} + \frac{1}{6} = \frac{5}{12} + \frac{2}{12} = \frac{7}{12}$$

**Example 1.91**

Harold tosses a coin four times. The probability that he gets at least as many heads as tails is: (AMC 8 2002/21)

**Example 1.92**

Alan, Jason, and Shervin are playing a game with MafsCounts questions. They each start with 2 tokens. In each round, they are given the same MafsCounts question. The first person to solve the MafsCounts question wins the round and steals one token from each of the other players in the game. They all have the same probability of winning any given round. If a player runs out of tokens, they are removed from the game. The last player remaining wins the game. If Alan wins the first round but does not win the second round, what is the probability that he wins the game? (CCA Math Bonanza, Individual Round, 2020/4)

	Alan	Jason	Shervin
Start	2	2	2
After 1 <sup>st</sup> Round	4	1	1
After 2 <sup>nd</sup> Round	3	3	0

Note that Shervin could have won the 2<sup>nd</sup> round either. It will not affect the answer.

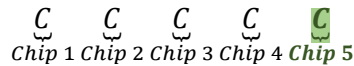
By symmetry, both players have equal chance of winning, and hence

$$P(\text{Alan wins}) = \frac{1}{2}$$

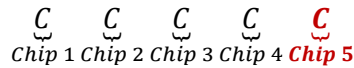
**Example 1.93**

A box contains 3 red chips and 2 green chips. Chips are drawn randomly, one at a time without replacement, until all 3 of the reds are drawn or until both green chips are drawn. What is the probability that the 3 reds are drawn? (AMC 8 2016/21)

The question asks for the probability that the three reds are drawn first. Imagine continuing to draw even after the three reds are drawn. In such a case, if the three reds are drawn first, the last chip drawn must be green, which gives us the arrangement below:


  
 $\overset{C}{\text{Chip 1}} \overset{C}{\text{Chip 2}} \overset{C}{\text{Chip 3}} \overset{C}{\text{Chip 4}} \overset{G}{\text{Chip 5}}$

An arrangement like the one below will not work since the three reds will not be drawn first:

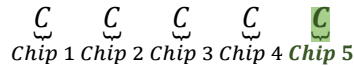

  
 $\overset{C}{\text{Chip 1}} \overset{C}{\text{Chip 2}} \overset{C}{\text{Chip 3}} \overset{C}{\text{Chip 4}} \overset{G}{\text{Chip 5}}$

Hence, what we really need to find is the probability that:

$$P(\text{Last chip is Green})$$

### Symmetry

Think of the drawing the chips and arranging them in a line:

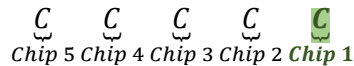

  
 $\overset{C}{\text{Chip 1}} \overset{C}{\text{Chip 2}} \overset{C}{\text{Chip 3}} \overset{C}{\text{Chip 4}} \overset{G}{\text{Chip 5}}$

Now, remove the numbers from the chips:

CCCCG

The green chip is last from the left, but first from the right.

Consider renumbering the line so that the green chip is now numbered first, the other four chips are numbered two to five:


  
 $\overset{G}{\text{Chip 5}} \overset{C}{\text{Chip 4}} \overset{C}{\text{Chip 3}} \overset{C}{\text{Chip 2}} \overset{C}{\text{Chip 1}}$

Then, by symmetry:

$$\therefore P(\text{Last chip is Green}) = P(\text{First chip is Green}) = \frac{2}{5}$$

From an arrangements point of view:

Successful Outcomes: We want to put the green chip last, and we can arrange the remaining green chip and the three red chips in any way we want.

Total Outcomes: We want to arrange three red and two green chips in any way we want.

By symmetry, we can make the green chip first (instead of last) since the number of arrangements of the remaining chips do not change.

$$P = \frac{\text{Successful Outcomes}}{\text{Total Outcomes}} = \frac{CCCCG}{CCCCC} = \frac{GCCCC}{CCCCC}$$

### Permutations

#### Successful Outcomes

The last chip is green. Hence, we are left with three red and one green chip

The only choice is the position of the green chip, for which there are

$$\{GCCC, CGCC, CCGC, CCGC\} \Rightarrow 4 \text{ Ways}$$

#### Total Outcomes

There are five chips (3 red, 2 green), which can be arranged (using identical objects with repetition) in

$$\frac{5!}{2!3!} = \frac{5 \times 4}{2} = 10$$

### Probability

$$\text{Probability} = \frac{\text{Successful Outcomes}}{\text{Total Outcomes}} = \frac{4}{10} = \frac{2}{5}$$

### Combinations

#### Successful Outcomes

Since the last chip is green, we need to arrange three red and one green chip, which can be done in the following ways

CCCC

From the combination point of view, we can choose the position of the green chip out of four places, which can be done in:

$$4 \text{ Choose } 1 = \binom{4}{1} = \frac{4!}{1!3!} = 4 \text{ Ways}$$

### Total Outcomes

We choose the location of the two green chips (or the three red chips) in

$$\binom{5}{2} = \frac{5!}{2!3!} = 10 \text{ Ways}$$

$$P = \frac{\text{Successful Outcomes}}{\text{Total Outcomes}} = \frac{4}{10} = \frac{2}{5}$$

### Example 1.94

A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white? (AMC 10 2001/23, AMC 12 2001/11)

Make sure you read and see the similarity between this example and the previous one. Also, a small but important difference – this question is asking for the complement of the probability that was asked in the previous example:<sup>6</sup>

$$P(\text{White}) = 1 - P(\text{Red}) = 1 - \frac{2}{5} = \frac{3}{5}$$

## C. 2D Paths

### Example 1.95

A frog is on the coordinate plane. It jumps up, down, left, right with equal probability to a lattice point that is one unit away at each step. What is the probability that it reaches (0,1) on step 1 given that it is at the origin on step 0.

### Method I: Equations

The probability of reaching any of the highlighted points is equal

$$P(0,1) = P(1,0) = P(-1,0) = P(0,-1) = p$$

Since the above events cover the entire sample space:

$$P(0,1) + P(1,0) + P(-1,0) + P(0,-1) = 1$$

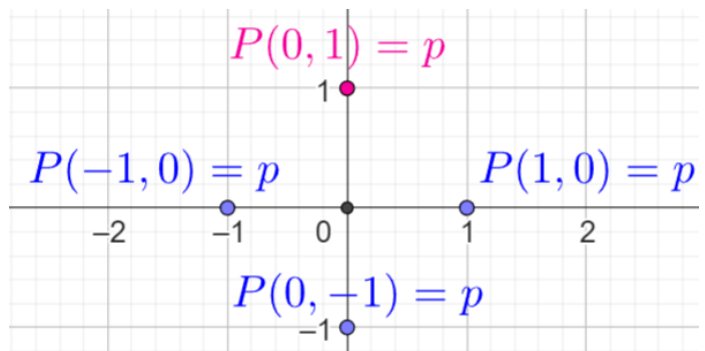
And as shown above, each of these has probability  $p$ .

Hence:

$$p + p + p + p = 1$$

$$4p = 1$$

$$p = \frac{1}{4}$$



<sup>6</sup> You can also refer a [detailed solution online](#).

**Method II: Symmetry**

The scenario is symmetrical, and the probability of reaching any of the given points is equal. Hence, the required probability is:

$$\frac{1}{4}$$

**Example 1.96**

A frog is on the coordinate plane. It jumps up, down, left, right with equal probability to a lattice point that is one unit away at each step. What is the probability that it reaches a horizontal side of the “box” given by the coordinate points  $(2,2)$ ,  $(-2,-2)$ ,  $(2,-2)$ ,  $(-2,2)$  given that at step 0 it is on  $(-1,1)$ .

By symmetry

$$P(\text{Horizontal}) = P(\text{Vertical}) = \frac{1}{2}$$

In general, for any point on the diagonal:

$$P(\text{Horizontal}) = P(\text{Vertical}) = \frac{1}{2}$$

**Example 1.97**

A frog sitting at the point  $(1, 2)$  begins a sequence of jumps, where each jump is parallel to one of the coordinate axes and has length 1, and the direction of each jump (up, down, right, or left) is chosen independently at random. The sequence ends when the frog reaches a side of the square with vertices  $(0,0)$ ,  $(0,4)$ ,  $(4,4)$ , and  $(4,0)$ . What is the probability that the sequence of jumps ends on a vertical side of the square? (AMC 10A 2020/13)

The frog begins at the green point (marked F)

We consider cases.

Case I: The frog jumps straight left with probability

$$p_{\text{left}} = \frac{1}{4}$$

Given that the frog jumps left, the probability of vertical is

$$1$$

Using the multiplication rule, the probability from this case is:

$$1 \cdot \frac{1}{4} = \frac{1}{4}$$

Case II: The frog does not jump straight left with probability

$$p_{\text{not-left}} = 1 - p_{\text{left}} = 1 - \frac{1}{4} = \frac{3}{4}$$

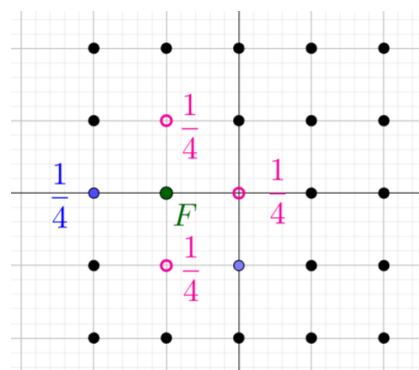
Given that the frog does not jump left, the probability of vertical is

$$\frac{1}{2}$$

Using the multiplication rule, the probability from this case is:

$$\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$$

Using the addition rule, the final probability is:



$$\frac{1}{4} + \frac{3}{8} = \frac{2}{8} + \frac{3}{8} = \frac{5}{8}$$

## D. Games

### Example 1.98

Two players,  $P_1$  and  $P_2$  play a game against each other. In every round of the game, each player rolls a fair die once, where the six faces of the die have six distinct numbers. Let  $x$  and  $y$  denote the readings on the die rolled by  $P_1$  and  $P_2$ , respectively. If

- $x > y$ , then  $P_1$  scores 5 points, and  $P_2$  scores 0 points.
- $x = y$ , then each player scores 2 points.
- $x < y$ , then  $P_1$  scores 0 points and  $P_2$  scores 5 points.

I.	Probability of $(X_2 \geq Y_2)$ is	(P) $\frac{3}{8}$
II.	Probability of $(X_2 > Y_2)$ is	(Q) $\frac{11}{16}$
III.	Probability of $(X_3 = Y_3)$ is	(R) $\frac{5}{16}$
IV.	Probability of $(X_3 > Y_3)$ is	(S) $\frac{355}{864}$
		(T) $\frac{77}{432}$

Let  $X_i$  and  $Y_i$  be the total scores of  $P_1$  and  $P_2$  respectively, after playing the  $i^{th}$  round. (JEE-A, 2022/I/16)

The correct option is:

- A. (I)  $\rightarrow$  (Q); (II)  $\rightarrow$  (R); (III)  $\rightarrow$  (T); (IV)  $\rightarrow$  (S)  
 B. (I)  $\rightarrow$  (Q); (II)  $\rightarrow$  (R); (III)  $\rightarrow$  (T); (IV)  $\rightarrow$  (T)  
 C. (I)  $\rightarrow$  (P); (II)  $\rightarrow$  (R); (III)  $\rightarrow$  (Q); (IV)  $\rightarrow$  (S)  
 D. (I)  $\rightarrow$  (P); (II)  $\rightarrow$  (R); (III)  $\rightarrow$  (Q); (IV)  $\rightarrow$  (T)

Since all options have  $II \rightarrow R$

$$P(X_2 > Y_2) = \frac{5}{16}$$

By symmetry rules are same for  $X$ , and  $Y$ .

$$P(X_2 \geq Y_2) = 1 - P(Y_2 > X_2) = 1 - P(X_2 > Y_2) = 1 - \frac{5}{16} = \frac{11}{16} \Rightarrow I \rightarrow Q$$

Since  $I \rightarrow Q$ , options C and D are eliminated, leaving only options A and B

Hence,  $III \rightarrow T$

$$\begin{aligned} P(X_3 > Y_3) &= P(Y_3 > X_3) = p \\ p + p + \frac{77}{432} &= 1 \\ 2p &= 1 - \frac{77}{432} = \frac{355}{432} \\ p &= \frac{355}{864} \\ \text{Option A} \end{aligned}$$

## E. Symmetry with Recursion

### Example 1.99

Flora the frog starts at 0 on the number line and makes a sequence of jumps to the right. In any one jump, independent of previous jumps, Flora leaps a positive integer distance  $m$  with probability  $\frac{1}{2^m}$ . What is the probability that Flora will eventually land at 10? (AMC 12A 2023/17)

Let Flora be  $k$  units to the left of 10 when she lands either on 10, or to the right of 10. (We ignore the event that she lands on a number before 10 because then we can Flora to jump one more time.)

Let  $p(n)$  be the probability that she lands on  $n$ .

$$p(10) = \frac{1}{2^k}, \quad p(11) = \frac{1}{2^{k+1}}, \quad p(12) = \frac{1}{2^{k+2}}, \dots$$

Summing up the probabilities other the  $p(10)$  gives:

$$p(11) + p(12) + \dots = \frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \dots$$

The above is an infinite geometric series with  $a = \frac{1}{2^{k+1}}$ ,  $r = \frac{1}{2}$  and sum:

$$S = \frac{a}{1-r} = \frac{\frac{1}{2^{k+1}}}{\frac{1}{2}} = \frac{1}{2^{k+1}} \cdot 2 = \frac{1}{2^k}$$

Hence

$$p(10) = p(11) + p(12) + \dots$$

Let

$$p = P(10) = p(11) + p(12) + \dots$$

But Flora must eventually land on either 10 or a point to the right of 10. Hence:

$$p + p = 1 \Rightarrow p = \frac{1}{2}$$

## F. Reframing the Question

### Example 1.100

Kamal picks two points on a circle. Harris picks two more points on the circle. What is the probability that the line segments formed by the two points that Kamal picked, and the two points that Harris picked intersect?

Consider the quadrilateral ABCD formed by the four points. Suppose, without loss of generality, that A is the first point picked.

If C is the next point picked, the quadrilateral will be concave, and the line segments will intersect.

If B or D is the next point picked, the quadrilateral will be convex, and the line segments will not intersect.

Hence, the probability is

$\frac{1}{3}$

### Example 1.101

Four distinct points, A, B, C, and D, are to be selected from 1996 points evenly spaced around a circle. All quadruples are equally likely to be chosen. What is the probability that the chord AB intersects the chord CD? (AHSME 1996)

The same solution as the previous example applies. In fact, this is a specific case of the previous example.

**Example 1.102<sup>7</sup>**

- A. What is the probability that the triangle formed by three random points picked on a circle include the center of the circle?
- B. What is the probability that the tetrahedron formed by picking four points on a sphere contains the center of the sphere?

**1.8 Sum of Dice Rolls****A. Basics****Example 1.103**

Ava rolls two standard, fair, six-sided dice. The probability of getting a value of  $n$ ,  $2 \leq n \leq 12$  for the total of the two dice is given by the piecewise function:

$$P(n) = \begin{cases} \frac{a}{36}, & 2 \leq n \leq 7 \\ \frac{b}{36}, & 8 \leq n \leq 12 \end{cases} = \begin{cases} \frac{a}{36}, & 1 \leq n \leq 6 \\ \frac{b}{36}, & 7 \leq n \leq 12 \end{cases}$$

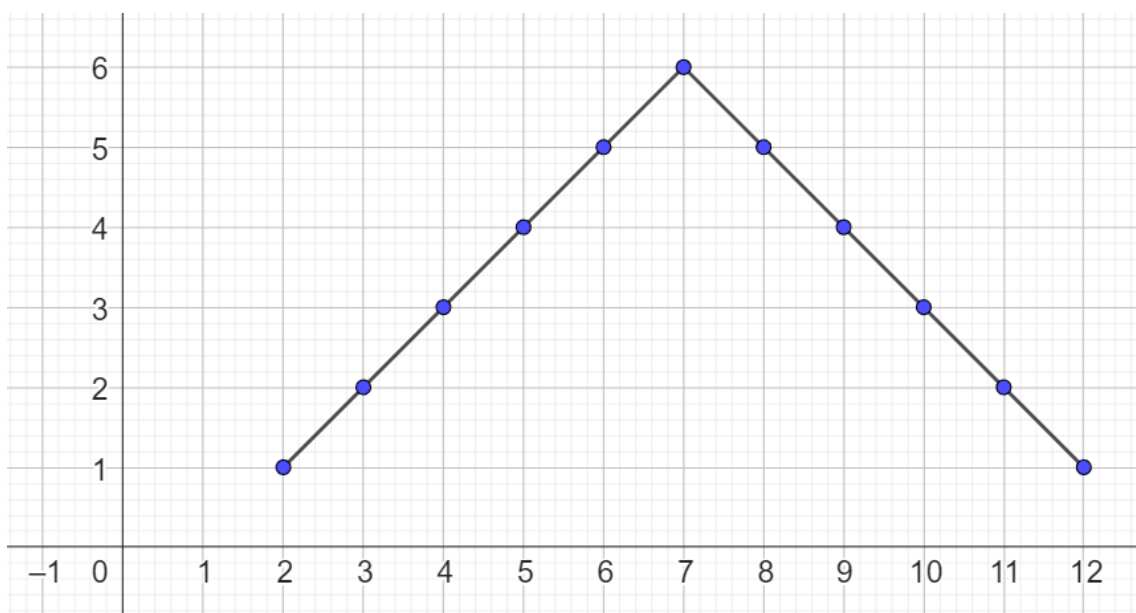
Determine the value of:

$$a + b = k, \quad k \in \mathbb{N}$$

	Total					
	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Sum = n	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

<sup>7</sup> Refer to [this video](#) for an animated solution of these two questions.



$$P(n) = \begin{cases} \frac{n-1}{36}, & 1 \leq n \leq 7 \\ \frac{13-n}{36}, & 8 \leq n \leq 12 \end{cases}$$

$$a + b = (n - 1) + (13 - n) = 12$$

### Example 1.104

Ava rolls two standard, fair dice, one of which has 4 sides and the other has 5 sides. The probability of getting a value of  $n$ ,  $2 \leq n \leq 9$  for the total of the two dice is given by the piecewise function:

$$P(n) = \begin{cases} a, & 2 \leq n \leq c \\ b, & c \leq n \leq 9 \end{cases}$$

Determine the value of

$$a + b + c = k, \quad k \in \mathbb{N}$$

	1	2	3	4	5
1	2	3	4	5	6
2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9

Sum = n	2	3	4	5	6	7	8	9
Probability	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{4}{20}$	$\frac{4}{20}$	$\frac{3}{20}$	$\frac{2}{20}$	$\frac{1}{20}$

$$P(n) = \begin{cases} \frac{n-1}{20}, & 2 \leq n \leq 5 \\ \frac{10-n}{20}, & 5 \leq n \leq 9 \end{cases}$$

### Example 1.105

Ava rolls two standard, fair dice, one of which has 4 sides and the other has 6 sides. The probability of getting a

value of  $n$ ,  $2 \leq n \leq 9$  for the total of the two dice is given by the piecewise function:

$$P(n) = \begin{cases} a, & 2 \leq n \leq 4 \\ b, & 5 \leq n \leq 7 \\ c, & 8 \leq n \leq 10 \end{cases}$$

Determine the value of

$$a + b + c = k, \quad k \in \mathbb{Q}$$

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10

Sum = n	2	3	4	5	6	7	8	9	10
Probability	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{3}{20}$	$\frac{4}{20}$	$\frac{4}{20}$	$\frac{4}{20}$	$\frac{3}{20}$	$\frac{2}{20}$	$\frac{1}{20}$

$$P(n) = \begin{cases} \frac{n-1}{24}, & 2 \leq n \leq 4 \\ \frac{4}{24}, & 5 \leq n \leq 7 \\ \frac{11-n}{24}, & 8 \leq n \leq 10 \end{cases}$$

$$a + b + c = \frac{n-1}{24} + \frac{4}{24} + \frac{11-n}{24} = \frac{14}{24} = \frac{7}{12}$$

## 1.9 Approaches to Probability

Probability is of three types:

1. Theoretical Probability
2. Empirical Probability
3. Bayesian Probability

### A. Naive Probability / Theoretical Probability

#### 1.106: Theoretical Probability

For an equiprobable finite sample space, the probability of an event is given by:

$$P(H) = \frac{\text{Successful Outcomes}}{\text{Total Outcomes}}$$

Typical useful for theoretical models:

- Toss of a coin
- Roll of a die
- Pack of cards

#### Example 1.107

- A. If I flip a coin, what is the probability of heads?

B. What is the probability that aliens exist on Mars?

### Part A

Justify by  $\frac{1}{2}$ :

$$\{H, T\} \Rightarrow P(H) = \frac{\text{Successful Outcomes}}{\text{Total Outcomes}} = \frac{1}{2}$$

The assumption in the formula above:

*Sample space is equiprobable*

Hence, the answer is wrong unless

*You know the coin is fair*

### Part B

The sample space is:

*{Aliens exist, Aliens do not exist}*

And the sample space is obviously not equiprobable.

## 1.108: Empirical Probability

The probability of an event is determined by past experience and frequency of that event.

- If a doctor is looking at a patient with a high fever, and he knows that 2 out of 10 such patients have been hospitalized in the past, he can conclude that the probability of hospitalization for the patient in front of him is  $\frac{2}{10} = 0.2$

## 1.109: Bayesian Probability

### Example 1.110

I have a coin to flip. If I flip the coin, what is the probability of heads?

Now, you actually flip the coin a few times, say 10 times, and you get

*4 Heads and 6 Tails*

## 1.10 AMC Questions

Start from 10A 2005

### A. Decimal System

The sum of the digits of a two-digit number is subtracted from the number. The units digit of the result is 6. How many two-digit numbers have this property? (AMC 10A 2005/16)

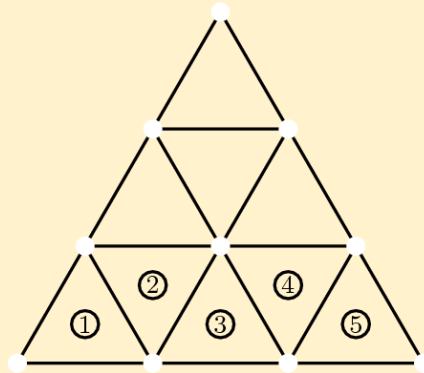
### B. Lists

A game is played with tokens according to the following rule. In each round, the player with the most tokens

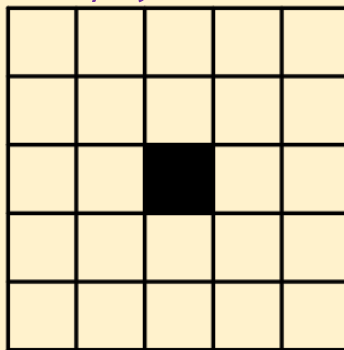
gives one token to each of the other players and also places one token in the discard pile. The game ends when some player runs out of tokens. Players  $A$ ,  $B$ , and  $C$  start with 15, 14, and 13 tokens, respectively. How many rounds will there be in the game? (AMC 10A 2004/8)

### C. Geometrical

A large equilateral triangle is constructed by using toothpicks to create rows of small equilateral triangles. For example, in the figure we have 3 rows of small congruent equilateral triangles, with 5 small triangles in the base row. How many toothpicks would be needed to construct a large equilateral triangle if the base row of the triangle consists of 2003 small equilateral triangles? (AMC 10A 2003/23)

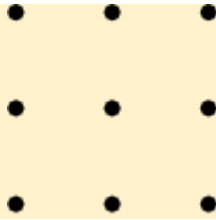


The  $5 \times 5$  grid shown contains a collection of squares with sizes from  $1 \times 1$  to  $5 \times 5$ . How many of these squares contain the black center square? (AMC 10A 2004/16)



### D. Coordinate Geometry

A set of three points is randomly chosen from the grid shown. Each three-point set has the same probability of being chosen. What is the probability that the points lie on the same straight line? (AMC 10A 2004/5)



## E. Strategies

There are 5 yellow pegs, 4 red pegs, 3 green pegs, 2 blue pegs, and 1 orange peg to be placed on a triangular peg board. In how many ways can the pegs be placed so that no (horizontal) row or (vertical) column contains two pegs of the same color? (AMC 10 2000/13)

We have a problem with restrictions. It usually makes sense to apply the most restrictive condition first. There are five rows, and five columns and five yellow pegs, so we need to fit in those first.

Y				
	Y			
		Y		
			Y	
				Y

Y				
G	Y			
	G	Y		
		G	Y	
			G	Y

Y				
R	Y			
	R	Y		
		R	Y	
			R	Y

Y				
R	Y			
G	R	Y		
	G	R	Y	
		G	R	Y

Y				
R	Y			
G	R	Y		
B	G	R	Y	
	B	G	R	Y

Y				
---	--	--	--	--

R	Y			
G	R	Y		
B	G	R	Y	
O	B	G	R	Y

What is the maximum number for the possible points of intersection of a circle and a triangle? (AMC 10 2001/4)

Six

How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits? (AMC 10A 2005/14)

## F. Multiplication Rule

Henry's Hamburger Heaven offers its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties, and any collection of condiments. How many different kinds of hamburgers can be ordered? (AMC 10A 2004/12)

The sum of the digits of a two-digit number is subtracted from the number. The units digit of the result is 6. How many two-digit numbers have this property? (AMC 10A 2005/16)

## G. Combinations

Three tiles are marked  $X$  and two other tiles are marked  $O$ . The five tiles are randomly arranged in a row. What is the probability that the arrangement reads  $XOXOX$ ? (AMC 10A 2005/9)

## H. Counting: Other Topics

### Problem 12

Figures 0, 1, 2, and 3 consist of 1, 5, 13, and 25 nonoverlapping unit squares, respectively. If the pattern were continued, how many nonoverlapping unit squares would there be in figure 100? (AMC 10 2000/12)



Figure  
0



Figure  
1

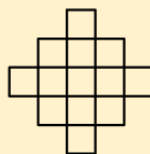


Figure  
2

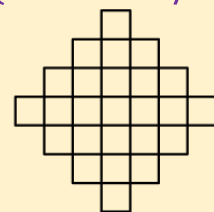


Figure  
3

## I. Diophantine/Distribution

Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible? (AMC 10 2001/19)

Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected? (AMC 10A 2003/21)

## J. Probability: Frequency and Distributions

Tina randomly selects two distinct numbers from the set  $\{1, 2, 3, 4, 5\}$ , and Sergio randomly selects a number from the set  $\{1, 2, \dots, 10\}$ . What is the probability that Sergio's number is larger than the sum of the two numbers chosen by Tina? (AMC 10A 2002/24)

Coin  $A$  is flipped three times and coin  $B$  is flipped four times. What is the probability that the number of heads obtained from flipping the two fair coins is the same? (AMC 10A 2004/10)

## K. Geometric Probability

A point  $(x, y)$  is randomly picked from inside the rectangle with vertices  $(0,0)$ ,  $(4,0)$ ,  $(4,1)$ , and  $(0,1)$ . What is the probability that  $x < y$ ? (AMC 10A 2003/12)

## 1.11 Further Topics

### 111 Examples