Distinguishability

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1. Identical Objects

1.1 Equations: Enumerating Solutions

A. Background

1.1: Diophantine Equations[1]

- Diophantine Equations are a class of equations where the number of variables is more than the number of equations.
- The simplest case is two variables and one equation. This leads to an infinite number of solutions.

is a Diophantine Equation, named after the Greek mathematician of the 3rd Century, who studied equations.

Substituting any value of x in the equation gives a value of y. If $x = 4.1$, then

$$4.1 + y = 5 \quad \Rightarrow \quad y = 0.9$$

Substituting any value of y in the equation gives a value of y. If $x = 4.1$, then

$$4.1 + y = 5 \quad \Rightarrow \quad y = 0.9$$

However, if you introduce conditions, or constraints, on the equations, then the number of solutions is also restricted.

1.2: Ordered Pairs

Solutions to Diophantine Equations are written as ordered pairs.

Example 1.3

Consider the equation

$$a + b = 3$$

Is the solution $a = 2, b = 1$ the same as $a = 1, b = 2$.

$$(a, b) = (2,1)$$
$$(a, b) = (1,2)$$
$$(2,1) \neq (1,2)$$

No.

Example 1.4

$$x + y = 5$$

In the above equation:

A. Find x if y is 4.1
B. Find y is x is 4.1
C. Are the two solutions the same or different?

1.5: Integer Solutions

Traditionally, Diophantine Equations are solved for Integer Values only. If there are two variables, their solution can be given as a pair.

[1] Diophantine Equations are named the mathematician Diophantus of Greece.

$$(4,7) \neq (7,4)$$

B. Conditions on Solutions

1.6: Non-negative Integers

Non-negative integers are integers greater than or equal to zero.

$$x \text{ is non} - \text{negative} \Leftrightarrow x \geq 0$$

1.7: Positive Numbers

Positive numbers are integers greater than zero.

$$x \text{ is positive} \Leftrightarrow x > 0$$

Example 1.8

For the equation $a + b = 15$, find the number of:

A. *nonnegative* integral solutions
B. *positive* integral solutions

Part A

We obtain valid pairs (a, b) by substituting $a = 0, 1, 2, \ldots 15$:

$$(0, 15), (1, 14), \ldots, (15, 0) \Rightarrow 16 \text{ Values}$$

Part B

The smallest number a that satisfies the condition is $a = 1$.
We obtain valid pairs (a, b) by substituting $a = 1, 2, \ldots 14$:

$$(1, 14), (2, 13), \ldots (14, 1) \Rightarrow 14 \text{ Values}$$

Find the reason for the difference in the solutions to the above two parts.

Part A allows 0 as a solution. This allows the minimum solution to be 0, and the maximum solution to be 15.
Part B does not allow 0 as a solution.
Hence, number of solutions are reduced by 2.

Example 1.9

I did twelve homework problems over the weekend, and I did at least one problem each day. Find the number of ordered pairs that represent the number of problems I could have solved on Saturday and Sunday.

$$(Sat, Sun) = (1,11), (2,10), (3,9), (4,8), (5,7), (6,6), (7,5), (8,4), (9,3), (10,2), (11,1)$$
$$11 \text{ ordered pairs}$$

1.10: Number of Solutions

A two-variable Diophantine equation will have two fewer *positive* integral solutions as compared to *nonnegative* integral solutions.

The number of solutions to

$$x + y = c, \qquad c \in \mathbb{N}$$

x can take:

$$Non - negative\ Values: \{0, 1, 2 \dots, c - 1, c\} \Rightarrow c + 1\ values$$

For positive values, it cannot take the value zero, and the value c:

$$Positive\ Values: \{\mathbf{0}, 1, 2 \dots, c - 1, \boldsymbol{c}\} \Rightarrow \{1, 2 \dots, c - 1\} \Rightarrow c - 1\ values$$

1.11: Zero Solutions

If the conditions of a problem cannot be met, then the number of solutions will be zero.

1.12: Repeating Values in Ordered Pairs

If the values in an ordered pair are not repeated, then interchanging the elements results in a different ordered pair.

$$(a, b) \neq (b, a)$$

If the values in an ordered pair are the same, then interchanging the elements does not result in a different ordered pair:

$$(a, a) = (a, a)$$

1.13: Distinct Solutions

If we want distinct solutions, the value of both the variables cannot be the same. They must be different.

Example 1.14

A. Find the number of *nonnegative* integral solutions of $p + q = 7$
B. How many *positive* integral solutions of $x + y = 100$ exist?
C. What is the number of *positive* integral solutions of $r + s = 0$?
D. What is the number of solutions to $r + s = 0$ such that r and s are *nonnegative* integers?
E. If m and n are distinct positive integers, then what is the number of solutions of $m + n = 8$?
F. If m and n are distinct positive integers, then what is the number of solutions of $m + n = -8$?

$$p + q = 7 \Rightarrow (0,7)(1,6), \dots, (7,0) \Rightarrow 8\ Values$$
$$x + y = 100 \Rightarrow (1,99)(2,98), \dots, (99,1) \Rightarrow 99\ Values$$
$$r + s = 0 \Rightarrow No\ Solutions \Rightarrow Zero\ Solutions$$
$$r + s = 0 \Rightarrow (0,0) \Rightarrow 1\ Solution$$
$$m + n = 8 \Rightarrow (1,7)(2,6), \dots, (\mathbf{4, 4}), \dots, (7,1) \Rightarrow 6\ Solutions$$
$$m + n = -8 \Rightarrow Zero\ Solutions$$

C. Range Restrictions

If there are range restrictions, as before, you can find the minimum and the maximum, and use the concepts of counting lists in order to find the number of solutions that satisfy the constraints.

Example 1.15

Find the number of positive integral solutions to $y + z = 16$, such that

A. y and z are both less than 12.
B. y and z are both more than 5
C. y and z have a difference of more than two.
D. y and z have a difference of not more than five.
E. y is less than 12.
F. y is more than 5.

$$(5,11),(6,10),\dots,(11,5) \Rightarrow y \in \{5,6,\dots,11\} \Rightarrow 11-5+1 = 7\ pairs$$
$$(6,10),(7,9),\dots,(10,6) \Rightarrow y \in \{6,\dots,10\} \Rightarrow 10-6+1 = 5\ pairs$$
$$(1,15),(2,14),\dots,(15,1) - (7,9)(8,8),(9,7) \Rightarrow 15-3 = 12$$
$$(6,10)(7,9)(8,8),(9,7)(10,6) \Rightarrow 5\ Solutions$$
$$(y,z) = (5,11),(6,10),\dots,(15,1) \Rightarrow 15-5+1 = 11\ pairs$$
$$(y,z) = (6,10),\dots,(15,1) \Rightarrow 15-6+1 = 10\ pairs$$

D. Unordered Pairs

So we have been considering pairs, which can be ordered or unordered. However, if consider sets, order is not important in sets, and hence, we have to work accordingly.

Example 1.16

There are several sets of three different numbers whose sum is 15 which can be chosen from $\{1,2,3,4,5,6,7,8,9\}$. How many of these sets contain a 5? **(AMC 8 1991/11)**

Let the set of three numbers be:

$$\{a,b,5\}$$

We know that

$$a+b+5 = 15 \Rightarrow a+b = 10 \Rightarrow \{a,b\} \in \big\{\{1,9\}\{2,8\}\{3,7\}\{4,6\}\big\}$$

Note that

$$\{9,1\} = \{1,9\}$$

Since we are using sets and not ordered pairs.

E. Negative Integers

So far, the solutions that we have considered have been either positive, or non-negative. However, if we introduce restrictions, then negative numbers will also give a finite number of solutions.

Example 1.17

A. Find the number of integral solutions to $m+n=9$, such that both variables are more than -4.
B. Find the number of integral solutions to $a+b=24$, such that both variables are greater than -5.
C. Find the number of integral solutions to $s+t=13$, such that both variables are between -2 and 12.
➢ Find the number of integral solutions to $u+v=23$, such that u is greater than -12, and v is greater than -8.

Part A

$$\underbrace{(-3,12)}_{\substack{Smallest\\Value\ of\ m}},(-2,11),\dots,(11,-2)\ \underbrace{(12,-3)}_{\substack{Largest\\Value\ of\ m}}$$

Use the method of counting lists:

$$\underbrace{12}_{\substack{Largest\\Value}} - \underbrace{(-3)}_{\substack{Smallest\\Value}} + \underbrace{1}_{Add\ 1} = 12+3+1 = 16\ solutions$$

Part B

$$(-4,28)(-3,27),\dots,(28,-4) \Rightarrow a \in \{-4,-3,\dots,28\} \Rightarrow 28-(-4)+1 = 33\ Pairs$$

Part C

$$(2,11)(3,10)\dots(11,2) \Rightarrow 11-2+1 = 10\ Pairs$$

Part D

If we had the restriction that both variables were greater than -12, we would have this solution:

$$(u,v) = (-11,34)(-10,33),\ldots,(30,-7),(31,-8),\ldots,(34,-11)$$

But note that the last few values are valid because of the additional restriction on the values of v:

$$(u,v) = (-11,34)(-10,33),\ldots,(30,-7),\mathbf{(31,-8)},\ldots,\mathbf{(34,-11)}$$

And hence, we only consider the valid values for u:

$$u \in \{-11,-10,\ldots,30\} \Rightarrow 30-(-11)+1 = 42\ Pairs$$

F. Restrictions on Solutions Set

Example 1.18

What is the number of solutions to $m+n=12$ for positive integers such that m is even?

Listing Method

We list out the valid pairs:

$$(2,10)(4,8)(6,6)(8,4)(10,2) \Rightarrow m \in \{2,4,6,8,10\} \Rightarrow \{1,2,3,4,5\} \Rightarrow 5\ Solutions$$

Change of Variable

Let

$$m\ be\ even \Rightarrow m = 2x$$
$$n\ is\ also\ even \Rightarrow n = 2y$$

Hence, our equation becomes:

$$2x+2y=12 \Rightarrow x+y=6 \Rightarrow x \in \{1,2,3,4,5\} \Rightarrow 5\ Solutions$$

Example 1.19

What is the number of solutions to $m+n=12$ for positive integers such that m is odd?

Again, we list out the valid pairs:

$$(1,11)(3,9)(5,7)(7,5)(9,3)(11,1) \Rightarrow m \in \{1,3,5,7,9,11\}$$

Add 1 to each element in the list:

$$\{2,4,6,8,10,12\}$$

Divide each element in the list by 2:

$$\{1,2,3,4,5,6\} \Rightarrow 6\ Solutions$$

Example 1.20

I have ten cars, which are either blue or red. The number of red cars I have is even and positive. The number of blue cars I have is also positive. Find the number of cars of each type I can have.

(Red, Blue) = (2,8),(4,6),(6,4),(8,2)

Example 1.21

Solve $s+t=20$, if both s and t are integers greater than -5 and (answer each separately):

A. s is even
B. s is odd

Part A

$$(-4,24),(-2,22),\ldots,(24,-4)$$

This means that:

$$s \in \{-4, -2, \dots, 24\}$$

Divide each number by 2:

$$\frac{s}{2} \in \{-2, -1, \dots, 12\} \Rightarrow 12 - (-2) + 1 = 15$$

Part B

$$(-3,23), (-1,21), \dots, (23, -3)$$

This means that:

$$s \in \{-3, -1, \dots, 23\}$$

Add 1 to each number:

$$s + 1 \in \{-2, 0, \dots, 24\}$$

Divide each number by 2:

$$\frac{s+1}{2} \in \{-1, 0, \dots, 12\} \Rightarrow 12 - (-1) + 1 = 14$$

G. Number Theory

Example 1.22

How many two-digit numbers have digits whose sum is a perfect square? (AMC 8 2006/11)

Strategy

Two-digit numbers range from:

$$10 \to Sum = 1, 99 \to Sum = 18$$

And hence the range of the sum of digits is:

$$Range = \{1,2,3,4, \dots 16,17,18\}$$

Out of the above, the following are perfect squares:

$$Perfect\ Squares = \{1,4,9,16\}$$

Let the digits of the number be

$$t, u, \qquad 1 \le t \le 9, \qquad 0 \le u \le 9$$

Casework

We do this using casework

$$Sum = 1: 10 \Rightarrow 1\ Numbers$$
$$Sum = 4 \Rightarrow t + u = 4 \Rightarrow \{13,22,31,41\} \Rightarrow 4\ Numbers$$
$$Sum = 9 \Rightarrow t + u = 9 \Rightarrow \{18,27, \dots, 90\} \Rightarrow 9\ Numbers$$
$$Sum = 16 \Rightarrow t + u = 16 \Rightarrow \{79,88,97\} \Rightarrow 3\ Numbers$$

H. Average

Example 1.23

The list of integers 4, 4, x, y, 13 has been arranged from least to greatest. How many different possible ordered pairs (x; y) are there so that the average (mean) of these 5 integers is itself an integer? (Gauss Grade 8 2015/24)

For some integer n, we must have:

$$\frac{4+4+x+y+13}{5} = \frac{21+x+y}{5} = n \Rightarrow 21 + x + y = 5n$$

$$n = 6 \Rightarrow (x, y) = (4,5) \Rightarrow 1 \text{ Pair}$$
$$n = 7 \Rightarrow x + y + 21 = 35 \Rightarrow x + y = 14 \Rightarrow (x, y) = (4,10)(5,9)(6,8)(7,7) \Rightarrow 4 \text{ Pairs}$$
$$n = 8 \Rightarrow x + y + 21 = 40 \Rightarrow x + y = 19 \Rightarrow (x, y) = (6,13), (7,12), (8,11), (9,10) \Rightarrow 4 \text{ Pairs}$$

$$n = 9 \Rightarrow x + y + 21 = 45 \Rightarrow x + y = 24 \Rightarrow (x, y) = (11,13), (12,12) \Rightarrow 2 \text{ Pairs}$$

Final Answer:

$$1 + 4 + 4 + 2 = 11$$

I. Triangle Inequality

Example 1.24

In any triangle, the length of the longest side is less than half of the perimeter. All triangles with perimeter 57 and integer side lengths x, y, z, such that $x < y < z$ are constructed. How many such triangles are there? (Guass 2013/24)

Triangles cannot be Isosceles

$$P = x + y + z = 57$$

Maximum value of z is 57/2 =28.5
Since z is integer, maximum value is 28

z	$x + y$	$Pairs$	Number of Pairs
28	29	$(27,2)(26,3), \dots, (15,14)$	$27 - 15 + 1 = 13$
27	30	$(26,4)(25,5), \dots, (16,14)$	$26 - 14 + 1 = 11$
26	31	$(25,6)(25,5), \dots, (16,15)$	$25 - 16 + 1 = 10$
25	32	$(24,8)(23,9), \dots, (17,15)$	$24 - 17 + 1 = 8$
24	33	$(23,10)(22,11), \dots, (17,16)$	$23 - 17 + 1 = 7$
23	34	$(22,12)(21,13), \dots, (18,16)$	$22 - 18 + 1 = 5$

J. Pigeonhole Principle

Example 1.25

A subset B of the set of integers from 1 to 100, inclusive, has the property that no two elements of B sum to 125. What is the maximum possible number of elements in B? **(AMC 10B 2005/25)**

$$a + b = 125 \Rightarrow (a, b) = \{(1,124), (2,123), \dots, (62,63)\}$$

By the Pigeonhole Principle, from each of the above unordered pairs, we can take maximum one number. Hence, we can take 62 numbers.

(Continuation) Example 1.26

What is the number of ways to make the above selection such there are 62 elements in the subset?

We can only take the first number from the 24 pairs below, since the second number of each pair can never be a part of the subset:

$$(1,124), (2,123), \dots, (24,101) \Rightarrow 24\ Numbers \Rightarrow 1\ Choice\ Only$$

From the 38 pairs remaining, we can take exactly one number from each pair:

$$(25,100), (26,99), \dots (100,25)$$

For each pair, we have 2 choices. Total number of choices is:

$$2^{38}$$

K. Distinguishability

Example 1.27

7 indistinguishable prizes have to be distributed among 2 boys based on their performance in a sport competition. What is the number of ways in which the prizes can be distributed?

Let the number of awards given to the boys be
$$b_1 \text{ and } b_2$$
Now, by the condition given in the question:
$$b_1 + b_2 = 7$$

- Condition I: The minimum number of prizes that can be given is zero.
- Condition II: Further, the number of prizes given must be an integer.

Combining the two conditions tells us that we are looking for nonnegative integral solution.
Hence, the number of solutions that we get is:
$$(0,7)(1,6), \dots, (7,0) \Rightarrow 7 - 0 + 1 = 8 \text{ Solutions}$$

Suppose that, in the question on distribution of prizes to boys above, the prizes were distinguishable. What would be your answer in this case? (This will need the Multiplication Principle from Counting)

Let's call the prizes
$$A, B, \dots, G$$
Prize A can be given either to the first boy, or the second boy. This gives us
$$2 \text{ choices}$$
Similarly, Prize B can be given either to the first boy, or the second boy. This also gives us:
$$2 \text{ choices}$$

In fact, we have two choices for each prize, giving us:
$$\underbrace{2}_{\substack{First\\Prize}} \times \underbrace{2}_{\substack{Second\\Prize}} \times \dots \times \underbrace{2}_{\substack{Seventh\\Prize}} = 2^7 = 128 \text{ Choices}$$

Example 1.28

Two leading scientists have been nominated for 9 indistinguishable awards at a scientific symposium. In order to not upset the competitive scientists, each scientist must receive at least one award. If they are the only ones who have been nominated for these awards, then in how many ways can the awards be distributed?

Let the number of awards given to the scientists be
$$s_1 \text{ and } s_2$$
Now, by the condition given in the question:
$$s_1 + s_2 = 9$$
And we know that each scientist must get at least one award. Further, the number of awards given must be an integer.
Hence, we are looking for positive integral solutions to the equation, which are:
$$(1,8)(2,7), \dots, (8,1) \Rightarrow 8 - 1 + 1 = 8 \text{ Solutions}$$

Example 1.29

Suppose that, in the question on distribution of awards to scientists above, the awards were distinguishable.

What would be your answer in this case?

Method I: Complementary Counting

If there are no restrictions, then, as in the previous example, the number of ways to distribute the awards is:

$$\underbrace{2}_{\substack{First\\Award}} \times \underbrace{2}_{\substack{Second\\Award}} \times \ldots \times \underbrace{2}_{\substack{Ninth\\Award}} = 2^9\ Choices$$

But, we have to give at least one award to each scientist. Hence, there are two ways in which we cannot distribute the awards

$$Way\ 1: All\ awards\ to\ first\ scientist$$
$$Way\ 2: All\ awards\ to\ second\ scientist$$

Hence, the final answer is:

$$2^9 - 2 = 512 - 2 = 510$$

Method II

We can do this using direct counting by splitting into cases. From the above, we know that the solutions for the distribution equation

$$s_1 + s_2 = 9$$

Are the following:

$$(1,8)(2,7), \ldots, (8,1) \Rightarrow 8 - 1 + 1 = 8\ Solutions$$

Case I: (1,8), (8,1)

We need to choose one award to give to one of the scientists and the remaining awards must be given to the other scientists:

$$2 \times \binom{9}{1} = 2 \times 9\ Ways = 18\ Ways$$

We can continue with the rest of the cases and get the same answer.

Example 1.30

Two tour guides are leading six tourists. The guides decide to split up. Each tourist must choose one of the guides, but with the stipulation that each guide must take at least one tourist. How many different groupings of guides and tourists are possible? **(AMC 10A 2004/12)**

Consider the number of choices for each tourist:

$$\underbrace{2}_{\substack{First\\Tourist:\\Two\ Choices}} \times \underbrace{2}_{\substack{Second\\Tourist:\\Two\ Choices}} \times \ldots \times \underbrace{2}_{\substack{Sixth\\Tourist:\\Two\ Choices}} = 2^6 = 64$$

$$64 - 2 = 62$$

Do the above question using combinations:

$$\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} = 6 + 15 + 20 + 15 + 6 = 62$$

1.31: Identity

Prove the identity below by counting in two different ways:

$$\boldsymbol{2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}}$$

I have n distinguishable objects. I want to select zero or more of these objects.

One way to think about this is that for each object, I have a choice:

$$\text{To pick or not to pick} \Rightarrow 2 \text{ Choices}$$

$$\underbrace{2}_{\substack{First\\Object}} \times \underbrace{2}_{\substack{Second\\Object}} \times \ldots \times \underbrace{2}_{\substack{Nth\\Object}} \Rightarrow 2^n = LHS$$

Another way to think about this, I can pick

$$0\ Objects, 1\ Object, \ldots, n\ objects$$

And the number of ways to do this is:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = RHS$$

But the number of ways to choose the objects has to be the same, irrespective of which method we use to count it.
Hence,

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

L. Casework

Example 1.32

Three friends have a total of 6 identical pencils, and each one has at least one pencil. In how many ways can this happen? **(AMC 8 2004/17)**

$$a + b + c = 6, \qquad a, b, c \geq 1$$

Since every friend has at least one pencil, give them one pencil:

$$A + B + C = 3, \qquad A, B, C \geq 0$$

$$A = 3 \Rightarrow B + C = 0 \Rightarrow (B, C) = (0,0) \Rightarrow 1\ Solution$$
$$A = 2 \Rightarrow B + C = 1 \Rightarrow (B, C) = (1,0)(0,1) \Rightarrow 2\ Solutions$$
$$A = 1 \Rightarrow B + C = 2 \Rightarrow (B, C) = (0,2)(1,1)(2,0) \Rightarrow 3\ Solutions$$
$$A = 0 \Rightarrow B + C = 3 \Rightarrow (B, C) = (0,3)(1,2)(2,1)(3,0) \Rightarrow 4\ Solutions$$

Total number of solutions

$$= 1 + 2 + 3 + 4 = 10\ Solutions$$

Example 1.33

Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible? **(AMC 10 2001/19)**

$$a + b + c = 4$$

Total number of solutions

$$= 1 + 2 + 3 + 4 + 5 = 15\ Solutions$$

Example 1.34

Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected? **(AMC 10A 2003/21)**

$$a+b+c=6$$

Total number of solutions

$$=1+2+3+4+5+6+7=\frac{7\times 8}{2}=28$$

Example 1.35

Alice has 24 apples. In how many ways can she share them with Becky and Chris so that all of them have at least 2 apples? **(AMC 8 2019/25)**

$$\underbrace{a}_{Alice}+\underbrace{b}_{Becky}+\underbrace{c}_{Chris}=24$$

Each one needs to have at least two apples. So, give them the required two apples, and define new variables:

$$a'=a-2,\qquad b'=b-2,\qquad c'=c-2\Rightarrow a'+b'+c'=18$$

In order to solve this equation, fix the value of a':

$Value\ of\ a'$		Pairs for $(b'+c')$	Number of Pairs
18	$b'+c'=0$	(0,0)	1
17	$b'+c'=1$	(0,1)(1,0)	2
16	$b'+c'=2$	(0,2)(1,1)(2,0)	3
.		.	.
.		.	.
.		.	.
0			19

$$1+2+3+\cdots+19=\frac{19\times 20}{2}=190$$

Example 1.36

Find the number of solutions to:

$$x+y+z=20,\qquad x\ is\ an\ even\ number$$

We do casework based on x, which can only take values from 0 to 10.

x	$y+z$	$No.of\ Solutions$
0	20	21
1	18	19
2	16	17
.	.	.
.	.	.
.	.	.
10	0	1

The tota number of solutions is:

$$1+3+5+\cdots+21=11^2=121$$

Note:
In the above, (where we used the result for the sum of odd numbers

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Example 1.37

Find the number of solutions to:

$$x + y + z = 20, x, y \text{ are even numbers}$$

We do casework based on z, which can take values from 0 to 20.

z	$x + y$	$\frac{x+y}{2}$	$No. of\ Solutions$
0	20	10	11
2	18	9	10
4	16	8	9
.	.	.	.
.	.	.	.
.	.	.	.
20	0	0	1

The tota number of solutions is:

$$1 + 2 + 3 + \cdots + 11 = \frac{11 \times 12}{2} = 66$$

M. Further Examples

Example 1.38

Brady is stacking 600 plates in a single stack. Each plate is colored black, gold, or red. Any black plates are always stacked below any gold plates, which are always stacked below any red plates. The total number of black plates is always a multiple of two, the total number of gold plates is always a multiple of three, and the total number of red plates is always a multiple of six. For example, the plates could be stacked with:

- 180 black plates below 300 gold plates below 120 red plates, or
- 450 black plates below 150 red plates, or
- 600 gold plates

In how many different ways could Brady stack the plates **(Gauss 8/2017/25)**

Let the number of:

$$black\ plates = 2b, \qquad gold\ plates = 3g, \qquad red\ plates = 6r$$

We want to determine the number of nonnegative integer solutions to

$$2b + 3g + 6r = 600$$

Rearrange and note that since the RHS is even, the LHS must also be even:

$$\underbrace{2b}_{Even} + 3g = \underbrace{600 - 6r}_{Even}$$

Since $2b$ is even $3g$ must also be even:

$$3g\ is\ even \Rightarrow g\ is\ a\ multiple\ of\ 2 \Rightarrow g = 2G \Rightarrow 3g = 6G$$

Again, look at the equation:

$$2b + 6G = 600 - 6r$$

$$b = 3(100 - r - G)$$

Since the RHS is a multiple of 3, the LHS must also be a multiple of 3.

$$b\ is\ a\ multiple\ of\ 3 \Rightarrow b = 3B \Rightarrow 2b = 6B$$

Hence, with a further substitution $r = R$ our equation becomes:

$$6B + 6G + 6R = 600$$
$$B + G + R = 100$$

We need non-negative integer solutions for the above:

$$B = 100 \Rightarrow G + R = 0 \Rightarrow 1\ Solution$$
$$B = 99 \Rightarrow G + R = 1 \Rightarrow 2\ Solutions$$
$$B = 98 \Rightarrow G + R = 2 \Rightarrow 3\ Solutions$$
$$\vdots$$
$$B = 0 \Rightarrow G + R = 100 \Rightarrow 101\ Solutions$$

Total number of solutions is:

$$1 + 2 + 3 + \cdots + 101 = \frac{101(102)}{2} = 5151$$

N. Using Combinations

Example 1.39

6 doughnuts—2 jelly-filled, 2 chocolate, and 2 glazed—are to be divided up among Adisa, Batu, and Catalina so that everyone receives exactly 2 doughnuts. If two doughnuts of the same flavor are identical, in how many ways can the treats be distributed if each person refuses to get both of the glazed doughnuts?**(JHMMC 2022 R1/27)**

Step I: Glazed Doughnuts

Begin with the most restrictive condition. The number of ways to distribute the glazed doughnuts is:
3C2 = 3
Sample Distribution: A=1,B=1,C=0

Step II: Chocolate Doughnuts

Number of ways to distribute the chocolate doughnuts:
Case I: Give both doughnuts to a single person, which is only possible with C:1
A=1,B=1,C=2

Case II: Give one doughnut to each:3C2
A=1,B=2,C=1
A=2,B=1,C=1
A=2,B=2,C=1
(where we have continued the sample distribution above)

Total ways

$$= 1 + 3 = 4$$

Step III: Jelly filled Doughnuts

Given the above, there is no freedom with respect to the jelly filled doughnuts. They must be distributed to the remaining people who still have space to accept them.

$$1 \; way$$

By the multiplication principle, the final answer is:

$$3 \cdot 4 \cdot 1 = 12$$

Example 1.40

An urn has 4 red balls, 3 green balls, 2 blue balls, and no other balls. Three of the balls are selected from the urn, one by one, without replacement. There are k distinct, unordered outcomes possible for the drawing, with probabilities $p_1, p_2, \dots, p_k$. The product of these probabilities can be written in the form

$$\frac{a^b \cdot c^d}{84^k}$$

Where a, b are prime numbers, and c, d are positive integers. Find $a + b + c + d + k$.

Total Outcomes

If you are drawing three balls out of $4 + 3 + 2 = 9$, the number of total outcomes (where the order of the balls is not important):

$$= \binom{9}{3} = \frac{9 \cdot 8 \cdot 7}{2 \cdot 3} = 3 \cdot 4 \cdot 7 = 84$$

Successful Outcomes

We can consider the outcomes as:

$$Red + Green + Blue = 3$$

Which we can write in shorter notation as:

$$R + G + B = 3, B \le 2$$

Consider cases on the number of blue balls. $B = 3$ is not possible because the number of blue balls is only two.

Case I: $B = 2$

$$B = 2 \Rightarrow R + G = 1$$

$$(R, G) = (1,0): \underbrace{\binom{4}{1}}_{Red} \underbrace{\binom{3}{0}}_{Green} \underbrace{\binom{2}{2}}_{Blue} = 4 \cdot 1 \cdot 1 = 4 \Rightarrow P[(1,0,2)] = \frac{4}{84}$$

$$(R, G) = (0,1): \underbrace{\binom{4}{0}}_{Red} \underbrace{\binom{3}{1}}_{Green} \underbrace{\binom{2}{2}}_{Blue} = 1 \cdot 3 \cdot 1 = 3 \Rightarrow P[(0,1,2)] = \frac{3}{84}$$

The product is:

$$\frac{4}{84} \cdot \frac{3}{84} = \frac{12}{84^2}$$

Case II: $B = 1$

$$B = 1 \Rightarrow R + G = 2$$

$$(R, G) = (2,0): \underbrace{\binom{4}{2}}_{Red} \underbrace{\binom{3}{0}}_{Green} \underbrace{\binom{2}{1}}_{Blue} = 6 \cdot 1 \cdot 2 = 12 \Rightarrow P[(2,0,1)] = \frac{12}{84}$$

$$(R, G) = (1,1): \underbrace{\binom{4}{1}}_{Red} \underbrace{\binom{3}{1}}_{Green} \underbrace{\binom{2}{1}}_{Blue} = 4 \cdot 3 \cdot 2 = 24 \Rightarrow P[(1,1,1)] = \frac{24}{84}$$

$$(R, G) = (0,2): \underbrace{\binom{4}{0}}_{Red} \underbrace{\binom{3}{2}}_{Green} \underbrace{\binom{2}{1}}_{Blue} = 1 \cdot 3 \cdot 2 = 6 \Rightarrow P[(0,2,1)] = \frac{6}{84}$$

The product is:

$$\frac{12}{84}\cdot\frac{24}{84}\cdot\frac{6}{84}=\frac{12^3}{84^3}$$

Case III: $B = 0$

$$B = 0 \Rightarrow R + G = 3$$

$$(R,G) = (3,0): \underbrace{\binom{4}{3}}_{Red}\underbrace{\binom{3}{0}}_{Green}\underbrace{\binom{2}{0}}_{Blue} = 4\cdot 1\cdot 1 = 4 \Rightarrow P[(3,0,0)] = \frac{4}{84}$$

$$(R,G) = (2,1): \underbrace{\binom{4}{2}}_{Red}\underbrace{\binom{3}{1}}_{Green}\underbrace{\binom{2}{0}}_{Blue} = 6\cdot 3\cdot 1 = 18 \Rightarrow P[(2,1,0)] = \frac{18}{84}$$

$$(R,G) = (1,2): \underbrace{\binom{4}{1}}_{Red}\underbrace{\binom{3}{2}}_{Green}\underbrace{\binom{2}{0}}_{Blue} = 4\cdot 3\cdot 1 = 12 \Rightarrow P[(1,2,0)] = \frac{12}{84}$$

$$(R,G) = (0,3): \underbrace{\binom{4}{0}}_{Red}\underbrace{\binom{3}{3}}_{Green}\underbrace{\binom{2}{2}}_{Blue} = 1\cdot 1\cdot 1 = 1 \Rightarrow P[(0,3,0)] = \frac{1}{84}$$

The product is:

$$\frac{4}{84}\cdot\frac{18}{84}\cdot\frac{12}{84}\cdot\frac{1}{84}=\frac{12^2\cdot 6}{84^4}$$

Combining the product from all the cases:

$$\frac{12}{84^2}\cdot\frac{12^3}{84^3}\cdot\frac{12^2\cdot 6}{84^4}=\frac{12^6\cdot 6}{84^9}=\frac{(2^2\cdot 3)^6\cdot 6}{84^9}=\frac{2^{12}\cdot 3^6\cdot 2\cdot 3}{84^9}=\frac{2^{13}\cdot 3^7}{84^9}$$

Substitute $a = 2, b = 13, c = 3, d = 7, k = 9$

$$a + b + c + d + k = 2 + 13 + 3 + 7 + 9 = 34$$

1.2 Equations: Stars and Bars

A. Distinguishability

1.41: Distinguishability

Two objects are distinguishable if you can differentiate between the two objects. This can happen because:

- Objects are numbered.
- Objects are lettered
- Objects have a distinguishing mark such as color, etc.
- Objects have some way to differentiate between them.

1.42: People are distinguishable (Default)

The default assumption is that you can differentiable people from each other:

$$People\ are\ not\ identical = distinguishable$$

Hence, in the absence of information

$$People\ are\ not\ identical = distinguishable$$

Example 1.43

What is the number of ways in which

A. Three people can stand in a row?
B. Five people can stand in a row?

Part A

Three people can stand in a row

$$= 3! = 6$$

If the people are $\{A, B, C\}$, then the ways are:

$$ABC, ACB, BAC, BCA, CAB, CBA$$

Part B

Five people can stand in a row

$$= 5! = 120$$

1.44: Objects in a Row

The number of ways in which you can arrange n distinguishable objects in a row is

$$n!$$

Example 1.45

I have four identical balls. In how many ways can I arrange these balls in a row?

$$\{Ball, Ball, Ball, Ball\} \Rightarrow 1\ way$$

Example 1.46

I need to assign five interns to three different cities: New York, London, and Barcelona. How many ways can I do this?

$$3^5 = 243$$

1.47: Assigning distinguishable balls to distinguishable boxes

The number of ways to assign n distinguishable balls to r distinguishable boxes is:

$$r^n$$

For the first ball, we have:

$$r\ boxes \Rightarrow r\ options$$

For the second ball, we have:

$$r\ boxes \Rightarrow r\ options$$

In general, by the multiplication principle, the total number of choices is:

$$\underbrace{r \cdot r \cdot \ldots \cdot r}_{n\ times} = r^n$$

1.48: Identical Objects

If you do not have a way to distinguish between the objects then they are identical for you.

If you can distinguish between the objects, but their representation is the same, then also they are identical for you.

For instance, when considering ordered pairs

$$(x, y) \neq (y, x) \Rightarrow The\ two\ pairs\ are\ different$$

When considering unordered pairs:

$$(x, y) = (y, x) \Rightarrow Pairs\ are\ identical\ (for\ the\ purpose\ of\ counting\ unordered\ pairs)$$

1.49: Default Identical Objects

Objects which are considered identical unless specified:

- Chocolates
- Balls
- Pencils
- Apples

B. Basics

We begin with a classic kind of question, which is representative of many questions of this type that we will solve in the section. This terminology was introduced by William Feller in his very important book on probability.

Example 1.50

Find the number of ways that 9 *identical* chocolates can be distributed among two children, such that each child gets at least one chocolate.

Since the chocolates are identical, which chocolate is given to which child does not matter.
The only decision point is the number of chocolates that each child gets.

Hence, the above question is equivalent to solving for positive integers a and b:

$$\underbrace{a}_{\substack{Chocolates\ for\\ first\ child}} + \underbrace{b}_{\substack{Chocolates\ for\\ second\ child}} = 9$$

Where

$$a = No.of\ chocolates\ given\ to\ first\ child$$
$$b = No.of\ chocolates\ given\ to\ second\ child$$

The minimum number of chocolates that each child gets must be 1. So, give them those chocolates.
Mathematically, this is a change of variables with A and B being nonnegative integers:

$$A = a - 1, \qquad B = b - 1 \Rightarrow A + B = 7$$

We have seven chocolates to be distributed among two children with no restrictions:

$$\boldsymbol{C\ C\ C\ C\ C\ C\ C}$$

Let us introduce a divider to help us decide how to distribute the chocolates.

$$\underbrace{|}_{\boldsymbol{Divider}}$$

So, in all, we have seven chocolates and one divider:

$$\boldsymbol{C\ C\ C\ C\ C\ C\ C\ |}$$

The position of the divider tells us which child gets how many chocolates. Some sample positions of the divider are given below:

$$\underbrace{\underbrace{\boldsymbol{C\ C}}_{\substack{\boldsymbol{Child\ 1}\\ \boldsymbol{Two\ Chocolates}}} \ |\ \underbrace{\boldsymbol{C\ C\ C\ C\ C}}_{\substack{\boldsymbol{Child\ 2}\\ \boldsymbol{Five\ Chocolates}}}}_{\boldsymbol{Position\ 1}}, \qquad \underbrace{\underbrace{\boldsymbol{C\ C\ C\ C}}_{\substack{\boldsymbol{Child\ 1}\\ \boldsymbol{Four\ Chocolates}}} \ |\ \underbrace{\boldsymbol{C\ C\ C}}_{\substack{\boldsymbol{Child\ 2}\\ \boldsymbol{Three\ Chocolates}}}}_{\boldsymbol{Position\ 2}}$$

Note that once we place the divider, we have only one way to divide the chocolates.

Hence, the number of ways of distributing the chocolates is the same as the number of ways of placing the divider.

$$\text{Number of Chocolates} = 7$$
$$\text{Number of Dividers} = 1$$
$$\text{Total Objects} = 8$$

Number of ways of placing one divider in eight places:

$$= 8 \text{ Choose } 1 = \binom{8}{1} = \frac{8!}{1!\,7!} = 8$$

1.51: Number of Dividers

The number of dividers is one less than the number of variables.
A quick way to remember this is:

$$No.\,of\ Dividers = No.\,of\ Plus\ Signs$$

Example 1.52

Find the number of ways that 9 *identical* chocolates can be distributed among two children, such that each child gets at least one chocolate.

We need to solve for positive integers a and b:

$$a + b = 9 \Rightarrow A + B = 7$$

We have 7 chocolates and 1 divider, giving 8 objects. The bar can be placed among the 8 objects in:

$$= \binom{8}{1} = 8$$

1.53: Distinguishable versus Identical

If the objects are distinguishable (as compared to identical), the logic of counting them also changes.

Example 1.54

Find the numbers of ways that 9 distinguishable chocolates can be distributed among two children, such that each child gets at least one chocolate.

We use complementary counting. With no restrictions, for each chocolate, we have two choices: give to the first child, or give to the second child.
The total number of choices, by the multiplication principle is:

$$\underbrace{2}_{\substack{First\\Chocolate:\\Two\ Choices}} \times \underbrace{2}_{\substack{Second\\Chocolate:\\Two\ Choices}} \times \ldots \times \underbrace{2}_{\substack{Ninth\\Chocolate:\\Two\ Choices}} = 2^9 = 512$$

The choices that we do not want are the ones where a child does not get a single chocolate. This happen if:

- All chocolates are given to the first child: 1 way
- All chocolates are given to the second child: 1 way

Hence, the final answer:

$$= 512 - 2 = 510$$

Example 1.55

Explain what is wrong with the "solution" below:

$$a+b=9, \qquad a \text{ and } b \text{ are positive integers}$$

$$\binom{No.of\ Stars + No.of\ Dividers}{No.of\ Dividers} = \binom{9+1}{1} = \binom{10}{1} = 10$$

Consider the case, where the divider is rightmost:

$$CCCCCCCCC|$$

This corresponds to:

$$\underbrace{CCCCCCCCC}_{9\ Chocolates}\ \underbrace{|}_{\substack{Zero\\Chocolates}}$$

Which is not valid because the minimum value of a and b is 1.

Hence, we need to subtract 1 before we proceed with the stars and bars solution.

1.56: Warning

- Stars and bars will work when solving for nonnegative integers, but not when solving for positive integers.
- If you have an equation in positive integers, convert it to nonnegative integers first.

Example 1.57

How many ways are there to distribute 7 indistinguishable balls to two distinguishable boxes?

$$a+b=7, \qquad a,b \in \mathbb{W}$$

We have seven balls:

$$BBBBBBB$$

Let us introduce a divider to help us decide how to distribute the chocolates.

$$\underbrace{|}_{Divider}$$

This gives us eight objects in all. By placing the divider, we can decide which box gets how many balls.

Total Number of Objects

$$= 7\ Balls + 1\ Divider = 8\ Objects$$

And the number of ways of placing the divider uniquely determines the number of ways to distribute the balls:

$$\binom{No.of\ Balls + No.of\ Dividers}{No.of\ Dividers} = \binom{7+1}{1} = \binom{8}{1} = 8$$

Example 1.58

How many ways are there to distribute 7 distinguishable balls to two distinguishable boxes?

$$\underbrace{2}_{\substack{First\\Ball:\\Two\ Choices}} \times \underbrace{2}_{\substack{Second\\Ball:\\Two\ Choices}} \times \dots \times \underbrace{2}_{\substack{Seventh\\Ball:\\Two\ Choices}} = 2^7 = 128$$

Example 1.59

Find the number of ordered triplets (a,b,c), where a, b, and c are positive integers that are solutions to:

$$a+b+c=10$$

Since each of the above variables must be at least one, define new variables such that

$$A = a - 1, \qquad B = b - 1, \qquad C = c - 1$$

Which gives us:

$$A + B + C = 7$$

Objects and Dividers

We have seven stars

$$\boldsymbol{C\,C\,C\,C\,C\,C\,C}$$

We will need two dividers:

$$||$$

This gives nine objects in all:

$$\boldsymbol{C\,C\,C\,C\,C\,C\,C}\,||$$

A sample position of the dividers is given below:

$$\underbrace{\underbrace{\boldsymbol{C}}_{\substack{\text{Child 1}\\ \text{One Chocolates}}} \quad | \quad \underbrace{\boldsymbol{C\,C\,C\,C}}_{\substack{\text{Child 2}\\ \text{Four Chocolates}}} \quad | \quad \underbrace{\boldsymbol{C\,C}}_{\substack{\text{Child 3}\\ \text{Two Chocolates}}}}_{\text{Position 1}}$$

Placing the Dividers

Out of these objects, we need to choose the position of the two dividers, giving us:

$$\binom{No.of\ Chocolates + No.of\ Dividers}{No.of\ Dividers} = \binom{7+2}{2} = \binom{9}{2} = \frac{9!}{7!\,2!} = \frac{9 \times 8}{2} = 36\ Solutions$$

C. Applications

Example 1.60

Three friends have a total of 6 identical pencils, and each one has at least one pencil. In how many ways can this happen? **(AMC 8 2004/17)**

$$a + b + c = 6$$
$$A + B + C = 3$$

Use Stars and Bars with three pencils and two dividers to get a total of five objects:

$$\binom{No.of\ Pencils + No.of\ Dividers}{No.of\ Dividers} = \binom{3+2}{2} = \binom{5}{2} = 10\ Ways$$

Example 1.61 (Re-visited)

Alice has 24 apples. In how many ways can she share them with Becky and Chris so that all of them have at least 2 apples? **(AMC 8 2019/25)**

The questions is equivalent to finding the number of solutions in positive integers to:

$$a + b + c = 24$$

Since we need to give minimum apples to each person, do that, and get a new equation in nonnegative integers:

$$A + B + C = 18$$

$$\underbrace{Apples}_{Alice}, \quad \underbrace{Divider}, \quad \underbrace{Apples}_{Becky}, \quad \underbrace{Divider}, \quad \underbrace{Apples}_{Chris}$$

We have $\underbrace{18\ apples}_{Stars}$ and $\underbrace{two\ dividers}_{Bars}$. We use the 2 bars as dividers to decide which variable the 18 stars are allocated to.

So, the final number of possible arrangements is:

$$\binom{No.of\ Apples\ + No.of\ Dividers}{No.of\ Dividers} = \binom{18+2}{2} = \binom{20}{2} = \frac{20!}{2!\,18!} = \frac{20 \times 19}{2!} = 190$$

Example 1.62

Seven people are to be divided into three teams called X, Y and Z. (Teams can have different number of people, and the number of people in a team can be zero). Find the number of ways in which this can be done.

Since the question is talking about people, we assume (by default), that they are distinguishable.
Also, the teams are named A, B and C. Hence, the teams are distinguishable.

For each person, we have three choices:

$$\underbrace{3}_{\substack{First\\Person:\\Three\ Choices}} \times \underbrace{3}_{\substack{Second\\Person:\\Three\ Choices}} \times \ldots \times \underbrace{3}_{\substack{Seventh\\Person:\\Three\ Choices}} = 3^7 = 2187$$

Example 1.63

You roll three, standard, six-sided dice. How many outcomes are possible, if order is important?

$$6 \times 6 \times 6 = 6^3 = 216$$

Example 1.64

What is the number of ways in which you can get a total of 9 when rolling six, standard, six-sided distinguishable dice.

$$a = Total\ on\ the\ first\ die$$
$$b = Total\ on\ the\ second\ die$$
$$And\ so\ on$$

Each die must have a positive roll:

$$a + b + c + d + e + f = 9$$

Converting to nonnegative variables:

$$A + B + C + D + E + F = 3$$

$$\binom{No.of\ Dice\ Outcomes\ + No.of\ Dividers}{No.of\ Dividers} = \binom{3+5}{5} = \binom{8}{5} = \frac{8!}{3!\,5!} = \frac{8 \times 7 \times 6}{6} = 56$$

Example 1.65

I pick a digit from the decimal system at random. I repeat this process thrice. Digits can be repeated. Determine the number of possible outcomes assuming:

A. Order is important.
B. Order is not important.

Part A

$$\underbrace{10}_{\substack{First\\Digit}} \cdot \underbrace{10}_{\substack{Second\\Digit}} \cdot \underbrace{10}_{\substack{Third\\Digit}} = 10^3 = 1000$$

Part B

Let:

$$x_0 = No.of\ times\ 0\ is\ chosen$$
$$x_1 = No.of\ times\ 1\ is\ chosen$$
$$\vdots$$
$$x_9 = No.of\ times\ 9\ is\ chosen$$

$$x_0 + x_1 + \cdots + x_9 = 3$$

Find the number of solutions of the above equations using Stars and Bars:

$$\binom{No.of\ Digits\ Chosen + No.of\ Dividers}{No.of\ Dividers} = \binom{3+9}{9} = \binom{12}{9} = \binom{12}{3} = 220$$

Example 1.66

I have wooden digits of the decimal system in a bowl. Each digit from 0 to 9 occurs thrice, for a total of 30 digits in the bowl. I pick a digit at random, and place it back. I repeat this process thrice. Determine the number of possible outcomes assuming:

A. Digits of a particular number are identical. So, all digits numbered 1 are identical.
   a. Order is important
   b. Order is not important
B. Digits are distinguishable. For example, two different digits numbered 1 are distinguishable from each other:
   a. Order is important
   b. Order is not important

Part A

Type I: Order is important:

The order of the digits will matter. Hence:

$$10^3 = 1000$$

Type II: Order is not important:

$$x_0 + x_1 + \cdots + x_9 = 3$$

$$\binom{No.of\ Digits\ Chosen + No.of\ Dividers}{No.of\ Dividers} = \binom{3+9}{9} = \binom{12}{9} = \binom{12}{3}$$

Part B

Type I: Order is important:

$$30^3 = 27{,}000$$

Type II: Order is not important:

$$x_1 + x_1 + \cdots + x_{30} = 3$$

$$\binom{No.of\ Digits\ Chosen + No.of\ Dividers}{No.of\ Dividers} = \binom{3+29}{29} = \binom{32}{3}$$

Example 1.67

I pick a capital letter from the English alphabet at random. I repeat this process four times. Letters can be repeated. Determine the number of possible outcomes assuming (write your answer as an expression, the final number is not required):

A. Order is important.
B. Order is not important.

Part A

$$26^4$$

Part B

Let:

$$x_1 = No.\,of\ times\ A\ is\ chosen$$
$$x_2 = No.\,of\ times\ B\ is\ chosen$$
$$\vdots$$
$$x_{26} = No.\,of\ times\ Z\ is\ chosen$$

$$x_1 + x_2 + \cdots + x_{26} = 4$$

$$\binom{No.\,of\ Letters\ Chosen + No.\,of\ Dividers}{No.\,of\ Dividers} = \binom{4+25}{25} = \binom{29}{4}$$

Example 1.68

You draw a card from a standard pack of playing cards, and then put it back. You repeat this three times, to draw three cards. Determine an expression (final number not required) for the number of possible:

A. ordered outcomes
B. unordered outcomes?

Part A

$$52 \times 52 \times 52 = 52^3$$

Part B

Number the 52 cards in the standard deck from 1 to 52 (using any numbering scheme of your choice).

Let

$$c_1 = No.\,of\ times\ card\ 1\ is\ drawn$$
$$c_2 = No.\,of\ times\ card\ 2\ is\ drawn$$
$$\vdots$$
$$c_{52} = No.\,of\ times\ card\ 52\ is\ drawn$$

Since the total number of cards drawn is 3, the total of the above must also be 3:

$$c_1 + c_2 + \cdots + c_{52} = 3$$

To get unordered outcomes we need the number of solutions to the above equation, which using Stars and Bars is:

$$\binom{No.of\ Cards\ Chosen\ + No.of\ Dividers}{No.of\ Dividers} = \binom{3+51}{51} = \binom{54}{51} = \binom{54}{3}$$

Example 1.69

Warning: Compare with previous example carefully

You draw a card from a standard pack of playing cards. You repeat this three times, to draw three cards. Determine an expression (final number not required) for the number of possible:

A. ordered outcomes
B. unordered outcomes?

Part A

$$52 \times 51 \times 50 = 52P3$$

Part B

$$\binom{52}{3}$$

$$3\ toppings: tomatoes + vegetables + tomatoes$$
$$2\ toppings: tomatoes + vegetables$$

Example 1.70

Answer each part independently

PizzaExpress, your favorite pizza chain has a choice of six toppings mentioned in the menu in the following order: cherry tomatoes, chargrilled vegetables, mushrooms, mixed jalapenos, cheese, finely chopped green chilies. In how many ways can you order a pizza if:

A. You have the choice of picking one or more toppings. However, toppings cannot be repeated. The order of picking the toppings does not matter.
B. You ask for six toppings(*one of each type*), and the order of placing the toppings matters. Toppings should not be in the exact same order as in the menu, or reverse of the exact same order as in the menu.
C. Your friend orders six toppings. Double and multiple toppings are allowed (double cherry tomatoes has double the quantity of cherry tomatoes). All six toppings should not all be the same. The order of toppings does *not* matter.

Part A

Toppings (*distinguishable objects*) are to be distributed to two distinguishable boxes (*pick or don't pick*),

$$Cherry\ Tomato: Pick\ or\ Don't\ Pick: 2\ Choices$$
$$Vegetables: Pick\ or\ Don't\ Pick: 2\ Choices$$
$$And\ So\ On$$

$$2^6 = 64$$

$$64 - 1 = 63$$

Part B

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$$
$$720 - 2 = 718$$

Part C

$$a + b + c + d + e + f = 6$$

$$\binom{No. of\ Toppings\ Chosen\ + No. of\ Dividers}{No. of\ Dividers} = \binom{6+5}{5} = \binom{11}{5}$$

From which we need to subtract six solutions for:

$$a = 6, b = 6, \dots, f = 6 \Rightarrow 6\ Solutions$$

Final answer:

$$\binom{11}{5} - 6$$

D. More Variables

Example 1.71

Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected? **(AMC 10A 2003/21)**

$$a + b + c = 6$$

Apply stars and bars with $6\ objects, 2\ dividers$

$$\binom{6+2}{2} = \binom{8}{2} = 28$$

Example 1.72

Bill is sent to a donut shop to purchase exactly six donuts. If the shop has four kinds of donuts and Bill is to get at least one of each kind, how many combinations will satisfy Bill's order requirements? **(MathCounts 2008 State Target)**

The donuts of each type can be assumed identical. Hence, only the number of donuts of each type matters.
Let

$$a = No. of\ donuts\ of\ 1st\ type$$
$$b = No. of\ donuts\ of\ 2nd\ type$$
$$c = No. of\ donuts\ of\ 3rd\ type$$
$$d = No. of\ donuts\ of\ 4th\ type$$

Since the total number of donuts is 6, we must have:

$$a + b + c + d = \underbrace{6}_{No. of\ Donuts}$$

We have to get one of each kind of donut.

$$A + B + C + D = 2$$

Use stars and bars with 2 objects and 3 dividers giving:

$$\binom{No. of\ Donuts\ + No. of\ Dividers}{No. of\ Dividers} = \binom{2+3}{3} = \binom{5}{3} = \frac{5!}{2!\,3!} = 10$$

Example 1.73

How many seven letter words can you make using the vowels from the English language, if the order of the letters is not important.

If order were important:

$$AAAAEEE \neq EEEAAAA$$

However, order is not important, and hence:

$$AAAAEEE = EEEAAAA$$

So, we are only concerned with how many times we are taking each vowel.

$$a+b+c+d+e=\underbrace{7}_{No.of\ Letters}$$

$$\binom{No.of\ Letters\ +No.of\ Dividers}{No.of\ Dividers}=\binom{7+4}{4}=\binom{11}{4}=\frac{11\times 10\times 9\times 8}{2\times 3\times 4}=330$$

E. Non-Increasing Numbers

Example 1.74

Consider numbers made from the digits {2,4,5,8} where every digit occurs maximum once. How many numbers can be made if the digits are arranged in ascending(increasing) order

Case I: Four Digit Number

$$2458 \Rightarrow 1\ Number$$

Case II: Three Digit Numbers

We can list out the numbers which are possible:

$$\{245,458,248,258\} \Rightarrow 4\ Numbers$$

Alternatively, we can arrive at the answer of 4 by:

$$\underbrace{\binom{4}{3}}_{\substack{Choosing\\3\ Digits}}\times\underbrace{1}_{\substack{Arranging\\3\ Digits}}=4$$

Case III: Two Digit Numbers

$$\underbrace{\binom{4}{2}}_{\substack{Choosing\\2\ Digits}}\times\underbrace{1}_{\substack{Arranging\\2\ Digits}}=6$$

Case IV: Two Digit Numbers

$$\underbrace{\binom{4}{1}}_{\substack{Choosing\\1\ Digits}}\times\underbrace{1}_{\substack{Arranging\\1\ Digits}}=4$$

Total

$$=1+4+6+4=15$$

Example 1.75

What is the number of five-digit numbers where the digits (from left to right) are non-increasing?

Once we have chosen the digits, they can be arranged in non-increasing order in exactly one way. This is

because the digit to the right of a digit must either be the same value or less.
Consider some examples below:

$$9,2,8,7,3 \Rightarrow 98732$$
$$9,2,2,7,3 \Rightarrow 97322$$
$$9,2,0,7,3 \Rightarrow 97320$$
$$2,0,0,0,0 \Rightarrow 20,000$$
$$0,0,0,0,0 \Rightarrow 00,000 = 0 \Rightarrow \mathit{Not\ a\ five\ digit\ number}$$

Hence, we only need to determine the number of ways that we can select the digits. Arranging the digits can be done in exactly one way:

$$\underbrace{a_0}_{\substack{No.of\\Zeroes}} + \underbrace{a_1}_{\substack{No.\ of\\Ones}} + \cdots + \underbrace{a_9}_{\substack{No.of\\Nines}} = \underbrace{5}_{\substack{No.of\ Digits\\in\ the\ Number}}$$

Apply Stars and Bars with 5 Digits and 9 Dividers

$$\binom{No.of\ Digits\ in\ the\ Number\ + No\ of\ Dividers}{No.of\ Dividers} = \binom{5+9}{2} = \binom{14}{9} = \frac{14 \times 13 \times 12 \times 11 \times 10}{2 \times 3 \times 4 \times 5} = 2002$$

From the above, we need to subtract the case where the digits are all zeroes:

$$2002 - 1 = 2001$$

Example 1.76
AIME Reference.
More Questions

F. Probability

1.77: Probability

$$Probability = \frac{Successful\ Outcomes}{Total\ Outcomes}$$

We will use the formula above, and count both types of outcomes using counting rules.
If you need to understand probability, you can look at the Notes on Probability.

G. Restriction: Sum of Numbers on Dice

Example 1.78
Rolling Two Dice to get a sum of 5

Example 1.79
What is the number of ways to distribute nine chocolates to six children such that every child gets a minimum of one chocolate and a maximum of six chocolates?

For positive integers a, b, c (which are the outcomes of the dice), we must have:

$$a + b + c = 9, \qquad 1 \le a, b, c \le 6$$

Use a change of variable. Subtract 1 from each roll. For nonnegative integers A, B, C, we must have:

$$A + B + C = 6, \qquad 0 \le A, B, C \le 5$$

Use Stars and Bars with 6 objects and 2 dividers:

$$\binom{No. of\ Objects\ + No. of\ Dividers}{No. of\ Dividers} = \binom{6+2}{2} = \binom{8}{2} = \frac{8 \times 7}{2} = 28$$

But the above also counts all the solutions where the values of the variables are 6. Hence, we have to count and remove these:

$$(A, B, C) = (6,0,0), (0,6,0), (0,0,6) \Rightarrow 3\ Cases$$

Hence, subtract these to get:

$$28 - 3 = 25$$

Example 1.80

If three standard, six-faced dice are rolled, what is the probability that the sum of the three numbers rolled is 9? Express your answer as a common fraction. **(MathCounts 2009 State Sprint)**

Sucessful Outcomes

$$a + b + c = 9, \qquad 1 \le a, b, c \le 6$$

As above, the number of solutions is:

$$28 - 3 = 25$$

Total Outcomes

By the multiplication Rule, the number of outcomes when you roll three dice is:

$$6 \times 6 \times 6 = 216$$

Probability

$$P = \frac{Successful\ Outcomes}{Total\ Outcomes} = \frac{25}{216}$$

Example 1.81

Rolling Four Dice

H. Missing Numbers

Example 1.82

A fair, twenty-faced die has 19 of its faces numbered from 1 through 19 and has one blank face. Another fair, twenty-faced die has 19 of its faces numbered from 1 through 8 and 10 through 20 and has one blank face. When the two dice are rolled, what is the probability that the sum of the two numbers facing up will be 24? Express your answer as a common fraction. **(MathCounts 2008 Chapter Sprint)**

Successful Outcomes

$$a + b = 24, \qquad a \le 19, b \le 20, b \ne 9$$

$$(a, b) = (4,20), (6,18), \dots, (19,5) \Rightarrow 19 - 4 + 1 = 16$$

Also, we cannot have

$$(15,9)$$

Hence, final number of succesful outcomes is:

$$16 - 1 = 15$$

Total Outcomes

$$20 \times 20 = 400$$

Probability

$$P = \frac{Successful\ Outcomes}{Total\ Outcomes} = \frac{15}{400} = \frac{3}{80}$$

I. Working with Remainders

Example 1.83

Suzanne went to the bank and withdrew \$800. The teller gave her this amount using \$20 bills, \$50 bills, and \$100 bills, with at least one of each denomination. How many different collections of bills could Suzanne have received? **(AMC 2023 10B/11)**

For positive integers a, b, and c:

$$20a + 50b + 100c = 800$$

Divide by 10 both sides:

$$\underbrace{2a + 5b + 10c = 80}_{Equation\ I}$$
$$2a + 5b$$
$$2a + 5b = \underbrace{80 - 10c}_{\substack{Multiple \\ of\ 10}}$$

Hence:

$$2a + 5b\ is\ a\ multiple\ of\ 10$$

Since 10 is even, $b\ must\ be\ even$, and a must be a multiple of 5.
Hence, substitute $a = 5A, b = 2B, c = C$ for positive integers A, B, C in Equation I

$$2(5A) + 5(2B) + 10C = 80$$

Divide by 10 both sides:

$$A + B + C = 8$$

Substitute $A = A' + 1, B = B' + 1 C = C' + 1$

$$(A' + 1) + (B' + 1) + (C' + 1) = 8$$
$$A' + B' + C' = 5$$

Apply stars and bars with 5 objects and 2 dividers:

$$\binom{5\ objects + 2\ \ dividers}{2\ dividers} = \binom{7}{2} = 21\ ways$$

J. Multiples

Example 1.84

Brady is stacking 600 plates in a single stack. Each plate is colored black, gold, or red. Any black plates are always stacked below any gold plates, which are always stacked below any red plates. The total number of black plates is always a multiple of two, the total number of gold plates is always a multiple of three, and the total number of red plates is always a multiple of six. For example, the plates could be stacked with:

- 180 black plates below 300 gold plates below 120 red plates, or
- 450 black plates below 150 red plates, or
- 600 gold plates

In how many different ways could Brady stack the plates **(Gauss 8/2017/25)**

Let the number of:

$$black\ plates = 2b, \qquad gold\ plates = 3g, \qquad red\ plates = 6r$$

We want to determine the number of nonnegative integer solutions to

$$2b + 3g + 6r = 600$$

Rearrange and note that since the RHS is even, the LHS must also be even:

$$\underbrace{2b}_{Even} + 3g = \underbrace{600 - 6r}_{Even}$$

Since $2b$ is even $3g$ must also be even:

$$3g\ is\ even \Rightarrow g\ is\ a\ multiple\ of\ 2 \Rightarrow g = 2G \Rightarrow 3g = 6G$$

Again, look at the equation:

$$2b + 6G = 600 - 6r$$
$$b = 3(100 - r - G)$$

Since the RHS is a multiple of 3, the LHS must also be a multiple of 3.

$$b\ is\ a\ multiple\ of\ 3 \Rightarrow b = 3B \Rightarrow 2b = 6B$$

Hence, with a further substitution $r = R$ our equation becomes:

$$6B + 6G + 6R = 600$$
$$B + G + R = 100$$

We need non-negative integer solutions for the above. Use stars and bars with:

$$100\ Stars, 2\ Bars$$

We have to choose the position of the 2 bars among the 100 stars, which is:

$$\binom{102}{2} = \frac{102!}{2!\,100!} = \frac{101(102)}{2} = 5151$$

K. Gap Method Questions

Example 1.85: Warm-Up

What is the number of ways to pick two distinct integers from 1 to 15 (inclusive) such that no two are consecutive.

The total number of ways to pick two numbers from 15 is:

$$\binom{15}{2} = \frac{15(14)}{2} = 15(7) = 105$$

Of the above, we do not want the two numbers to be consecutive, which can be done in:

$$(1,2), (2,3), \dots, (14,15) \Rightarrow 14\ Ways$$

The final answer is:

$$105 - 14 = 91\ Ways$$

1.86: Gap Method

> ➢ The gap method requires placing objects into gaps, and counting the number of ways to do so.
> ➢ It can be applied to a range of scenarios, ranging from direct to difficult.

Example 1.87

What is the number of ways to pick two distinct integers from 1 to 15 (inclusive) such that no two are consecutive.

We need to pick two distinct numbers from 1 to 15. Let these numbers be

$$N_1, N_2$$

Since the number that we pick are not consecutive, there must be numbers between N_1 and N_2, and between N_2 and N_3:

$$N_1\ g_1\ N_2\ , \qquad g_1 > 0$$

Where

$$g_1 = No.of\ numbers\ between\ N_1\ and\ N_2$$

And we can (optionally) have numbers before N_1 and after N_2:

$$G_2\ N_1\ g_1\ N_2\ G_3, \qquad g_1 > 0, \qquad G_3 \geq 0$$

There are 15 numbers, out of which we are picking two numbers. Hence, the remaining number of numbers must be:

$$15 - 2 = 13$$

These thirteen numbers must be placed into the gaps in such a way that:

$$G_2 + g_1 + G_3 = 13$$

Since $g_1 > 0$, substitute $G_1 = g_1 + 1$

$$G_2 + (G_1 + 1) + G_3 = 13$$
$$G_2 + G_1 + G_3 = 12$$

Now, apply stars and bars with

$$\binom{12\ objects + 2\ Dividers}{2\ Dividers} = \binom{14}{2} = \frac{14(13)}{2} = 7(13) = 91$$

Example 1.88

What is the number of ways to pick three distinct integers from 1 to 10 (inclusive) such that no two are consecutive.

We need to pick three distinct numbers from 1 to 10. Let these numbers be

$$N_1, N_2, N_3$$

Since the number that we pick are not consecutive, there must be numbers between N_1 and N_2, and between N_2 and N_3:

$$N_1\ g_1\ N_2\ g_2 N_3, \qquad g_1, g_2 > 0$$

Where

$$g_1 = No.of\ numbers\ between\ N_1\ and\ N_2$$
$$g_2 = No.of\ numbers\ between\ N_2\ and\ N_3$$

And we can (optionally) have numbers before N_1 and after N_3:

$$G_3\ N_1\ g_1\ N_2\ g_2\ N_3\ G_4, \qquad g_1, g_2 > 0, \qquad G_3, G_4 \geq 0$$

There are 10 numbers, out of which we are picking three numbers. Hence, the remaining number of numbers must be:

$$10 - 3 = 7$$

These 7 numbers must be placed into the gaps in such a way that:

$$G_3 + g_1 + g_2 + G_4 = 7$$

Since $g_1, g_2 > 0$, substitute $G_1 = g_1 + 1, G_2 = g_2 + 1$

$$G_3 + (G_1 + 1) + (G_2 + 1) + G_4 = 7$$
$$G_3 + G_1 + G_2 + G_4 = 5$$

Now, apply stars and bars with

$$\binom{5\ objects + 3\ Dividers}{3\ Dividers} = \binom{8}{3} = 56$$

1.3 Inequalities: Stars and Bars

A. Slack Variable

1.89: Slack Variable

A slack variable is used to convert an inequality into an equality.

Consider the inequality:

$$a < 7, \qquad a \text{ is a nonnegative integer}$$

Convert into

$$a \le 6$$

Which can be converted into an equality by introducing a slack variable which takes nonnegative integer values:

$$a + Slack = 6$$

Where

$$Slack = 6 - a$$

Example 1.90

Find the number of positive integer solutions for:

$$a + b < 7$$

Use a change of variable to convert from positive integers to non-negative integers. Let $a = A + 1, b = B + 1$:

$$A + B < 5$$

Convert the inequality to also include an equality condition:

$$A + B \le 4$$

If $A + B$ is less than 4, then we can asssign the leftover
Introduce a new variable $Slack = 4 - A - B$, and convert the inequality into an equation.

$$A + B + Slack = 4$$

The equation that we get has the same number of solutions as the inequality that we started with:

Consider cases for Slack:

C	4	3	2	1	0	Total
$A + B$	0	1	2	3	4	
Solutions	1	2	3	4	5	15

(Important Method) Example 1.91 (Stars and Bars)

Find the number of positive integer solutions for:

$$a + b < 7$$

As before

$$A + B + Slack = 4$$

$$CCCC||$$

Now, we have 4 objects and two dividers, giving us:

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{4+2}{2} = \binom{6}{2} = \frac{6 \times 5}{2} = 15$$

Example 1.92

Find the number of positive integer solutions for:

$$a + b < 9$$

Convert to nonnegative integer solutions:

$$A + B < 7$$

Convert to a $\le$ inequality:

$$A + B \le 6$$

Introduce the slack variable C:

$$A + B + C = 6$$

Apply stars and bars with 6 Objects and 2 Dividers:

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{6+2}{2} = \binom{8}{2} = \frac{8 \times 7}{2} = 28$$

B. Questions with $\le$

Example 1.93

Find the number of positive integer solutions:

$$a + b \le 9$$

$$A + B \le 7$$
$$A + B + C = 7$$

Use Stars and Bars with 7 Objects and 2 Dividers:

$$\binom{7+2}{2} = \binom{9}{2} = \frac{9 \times 8}{2} = 36$$

Example 1.94

Find the number of ways in which I can distribute up to ten pencils among my two friends, such that each friend gets at least one pencil.

$$a + b \le 10$$
$$A + B \le 8$$
$$A + B + C = 8$$

We now have 8 Objects, and 2 Dividers, giving us:

$$\binom{No.of\ Pencils\ + No.of\ Dividers}{No.of\ Dividers} = \binom{8+2}{2} = \binom{10}{2} = \frac{10 \times 9}{2} = 45$$

Example 1.95

Find the number of ways in which I can share ten pencils with two of my friends, such that each pencil is allocated to a maximum of one person, and each person gets at least one pencil.

Each pencil is given to a maximum of one person, which means a pencil can be given to zero people also. Which means we have:

$$\leq 10\ Pencils\ to\ be\ shared$$

The pencils are shared (and not distributed), and hence I am also eligible to get pencils. In all, there are three people.

$$a + b + c \leq 10$$

Convert to nonnegative integers:

$$A + B + C \leq 7$$

Introduce a slack variable:

$$A + B + C + D = 7$$

We now have 7 Objects, and 3 Dividers, giving us:

$$\binom{No.of\ Pencils\ + No.of\ Dividers}{No.of\ Dividers} = \binom{7+3}{3} = \binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{2 \cdot 3} = 120$$

1.96: Positive Integer Solutions with $\leq$

$$a + b \leq n \text{ has } \frac{(n)(n-1)}{2}\ positive\ integer\ solutions$$

Note: Don't memorize this result. Instead, follow the process every time.

$$a + b \leq n, \qquad n \in \mathbb{N}, n \geq 2$$

$$A + B \leq n - 2$$
$$A + B + C = n - 2$$

Now, we have $n - 2$ objects and 2 dividers, giving us:

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{(n-2)+2}{2} = \binom{n}{2} = \frac{(n)(n-1)}{2}$$

Example 1.97

A charity worker has twenty food packets to be distributed in two flood hit areas such that each area receives at least two packets, and at least ten packets are distributed overall. Find the number of ways in which the distribution can take place.

Let the number of packets

$$Distributed\ in\ the\ first\ area = a, \qquad a \in \mathbb{W}$$
$$Distributed\ in\ the\ second\ area = b, \qquad b \in \mathbb{W}$$

We need to meet

$$10 \leq a + b \leq 20, \qquad a \geq 2, \qquad b \geq 2$$

Substitute $a = A + 2, b = B + 2$L:

$$10 \leq A + 2 + B + 2 \leq 20$$
$$6 \leq A + B \leq 16$$

First find the solution to:

$$A + B \leq 16$$
$$A + B + C = 16$$

Use Stars and Bars with 16 Objects and 2 Dividers:

$$\binom{16+2}{2} = \binom{18}{2} = \frac{18 \times 17}{2} = 153$$

Out of the above, we subtract the solutions to:

$$A + B \leq 5$$
$$A + B + C = 5$$

Use Stars and Bars with 5 Objects and 2 Dividers:

$$\binom{5+2}{2} = \binom{7}{2} = \frac{7 \times 6}{2} = 21$$

$$153 - 21 = 132$$

C. Back Calculations

Example 1.98

The number of solutions to the inequality $a + b < n$ for positive integer values of a and b is 6. Find the value of n.

$$a + b < n, \qquad n\ is\ a\ positive\ integer$$

$$A + B < n - 2$$
$$A + B \leq n - 3$$
$$A + B + C = n - 3$$

Now, we have $n - 3$ objects and 2 dividers, giving us:

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{(n-3)+2}{2} = \binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$$

From the question, we know that the number of solutions is 6. Hence:

$$\frac{(n-1)(n-2)}{2} = 6$$
$$(n-1)(n-2) = 12 = 3 \times 4$$
$$n = 5$$

D. Non-Negative Integer Solutions

1.99: Non-Negative Integers

Non-negative integers are integers which are not negative, that is

$$\mathbb{W} = \{0,1,2,3, \ldots\}$$

They are also called whole numbers.

Example 1.100

Solve for non-negative integer values:

$$a + b < 6$$

$$a + b \le 5$$
$$a + b + c = 5$$

Use Stars and Bars with 5 Objects and 2 Dividers:

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{5+2}{2} = \binom{7}{2} = \frac{7 \times 6}{2} = 21$$

Example 1.101

What is the number of ways to distribute a maximum of eleven chocolates to two children?

$$a + b \le 11$$

Introduce a slack variable:

$$a + b + c = 11$$

Use Stars and Bars with 11 Objects and 2 Dividers:

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{11+2}{2} = \binom{13}{2} = \frac{13 \times 12}{2} = 78$$

1.102: Non-Negative Integer Solutions

$$a + b < n \text{ has } \frac{(n+1)(n)}{2}\ non-negative\ integer\ solutions$$

Note: Don't memorize this result. Instead, follow the process every time.

$$a + b < n, \qquad n \in \mathbb{N}, n \ge 2$$
$$a + b \le n - 1$$
$$a + b + c = n - 1$$

Now, we have n stars and 2 dividers, giving us:

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{n-1+2}{2} = \frac{(n+1)(n)}{2}$$

Example 1.103

Solve for non-negative integer values:

$$a + b \le 9$$

$$a + b + c = 9$$

Use Stars and Bars with 9 Stars and 2 Dividers:

$$\binom{No.of\ Objects\ + No.of\ Dividers}{No.of\ Dividers} = \binom{9+2}{2} = \binom{11}{2} = \frac{11 \times 10}{2} = 55$$

1.104: Non-Negative Integer Solutions

$$a + b \le n \text{ has } \frac{(n+2)(n+1)}{2}\ non-negative\ integer\ solutions$$

Note: Don't memorize this result. Instead, follow the process every time.

$$a+b\le n,\qquad n\in\mathbb{N}, n\ge 2$$

$$a+b+c=n$$

Now, we have n stars and 2 dividers, giving us:

$$\binom{No.of\ Objects\ +No.of\ Dividers}{No.of\ Dividers}=\binom{n+2}{2}=\frac{(n+2)(n+1)}{2}$$

E. "More than One" Minimum

Example 1.105

In how many ways can Alice give upto 24 apples to Becky and Chris so that each has at least 2 apples?

$$b+c\le 24$$

Now, each one of them needs at least two apples. So, give them the two apples:

$$B+C'\le 20$$
$$A+B+C=20$$

Use Stars and Bars with 20 Objects and 2 Dividers

$$\binom{No.of\ Objects\ +No.of\ Dividers}{No.of\ Dividers}=\binom{20+2}{2}=\binom{22}{2}=\frac{22\times 21}{2}=231$$

F. Negative Integer Ranges

Example 1.106

What is the number of ways to distribute a maximum of ten dollars to three children if it is also possible to distribute "negative dollars" upto the maximum amount (which is one dollar) that the children have in their pockets. Dollars gained from distribution of "negative dollars" are valid to be distributed to other children.

For integers $a,b,c\ge -1$:

$$a+b+c\le 10$$

Substitute $a=A-1, b=B-1, c=C-1$

$$A-1+B-1+C-1\le 10$$
$$A+B+C\le 13$$
$$A+B+C+D=13$$

Now, we have 13 objects and 3 dividers, giving us:

$$\binom{No.of\ Objects\ +No.of\ Dividers}{No.of\ Dividers}=\binom{13+3}{3}=\frac{13\cdot 12\cdot 11}{6}=286$$

Example 1.107

Solve for integer values:

$$a+b<5,\qquad a,b>-4$$

Substitute $a = A - 3, b = B - 3$:

$$A - 3 + B - 3 < 5$$
$$A + B < 11$$
$$A + B \le 10$$
$$A + B + C = 10$$

Use Stars and Bars with 12 Objects and 2 Dividers

$$\binom{No. of\ Objects\ + No. of\ Dividers}{No. of\ Dividers} = \binom{10 + 2}{2} = \binom{12}{2} = \frac{12 \times 11}{2} = 66$$

G. Back Calculations

Example 1.108

Find the natural number A such that there are A integer solutions to $x + y \ge A$ where:

$$0 \le x \le 6\ and\ 0 \le y \le 7$$

(CMIMC Combinatorics 2023/2)

Method I: Table

$$x + y \ge 13 \Rightarrow 1\ solution$$
$$x + y \ge 12 \Rightarrow 1 + 2 = 3\ solutions$$
$$x + y \ge 11 \Rightarrow 1 + 2 + 3 = 6\ solutions$$
$$x + y \ge 10 \Rightarrow 1 + 2 + 3 + 4 = 10\ solutions$$

$$A = 10$$

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	8
2	2	3	4	5	6	7	8	9
3	3	4	5	6	7	8	9	10
4	4	5	6	7	8	9	10	11
5	5	6	7	8	9	10	11	12
6	6	7	8	9	10	11	12	13

Method II: Diophantine[2]

Because of the *symmetry* that we observe in Method I, we can rewrite the above inequality as:

$$x + y \le 13 - A\ has\ A\ solutions$$

$$x + y + z = 13 - A$$

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	8
2	2	3	4	5	6	7	8	9
3	3	4	5	6	7	8	9	10
4	4	5	6	7	8	9	10	11
5	5	6	7	8	9	10	11	12
6	6	7	8	9	10	11	12	13

Use stars and bars

$$\binom{13 - A\ Objects + 2\ Dividers}{2\ Dividers} = A$$

$$\binom{15 - A}{2} = A$$
$$\frac{(15 - A)(14 - A)}{2} = A$$
$$A \in \{10,21\}$$

Since $x + y < 13$, reject the solution outside the range:

$$A = 10$$

H. Three Variables

Example 1.109

A total of n ordered quadruples (a, b, c, d) satisfy $a + b + c + d < 24$, where a, b, c, and d are positive

[2] This is a difficult solution...

integers. What are the last two digits of n? **(Stormersyle Mock AMC 8, 2018)**

$$a+b+c+d<24$$

Convert the constraint that each variable must be positive by subtracting one from each variable (and hence 4 from the RHS):

$$A+B+C+D<20$$
$$A+B+C+D\le 19$$

Convert the inequality into an equation by introducing a new variable:

$$A+B+C+D+E=19$$

Use stars and bars with 19 objects and we need 4 dividers.

$$\binom{19+4}{4}=\binom{23}{4}=\frac{23\times 22\times 21\times 20}{2\times 3\times 4}=23\times 11\times 7\times 5=8{,}855$$
$$Last\ two\ digits=55$$

I. Probability

Example 1.110: Using Tables/Enumeration

Oh no! While playing Mario Party, Theo has landed inside the Bowser Zone. If his next roll is between 1 and 5 inclusive, Bowser will shoot his "Zero Flame" that sets a player's coin and star counts to zero. Fortunately, Bowser has a double dice block, which lets him roll two fair 10-sided dice labeled 1-10 and take the sum of the rolls as his roll. If he uses his double dice block, what is the probability that he escapes the Bowser zone without losing his coins and stars? **(CMIMC Combinatorics 2023/1)**

Method I: Conceptual

Since there are two ten sided dice, we can list out the outcomes for the sums, and highlight the successful outcomes.

	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	
2	3	4	5	6	7	8	9	10	11	
3	4	5	6	7	8	9	10	11	12	
4	5	6	7	8	9	10	11	12	13	
5	6	7	8	9	10	11	12	13	14	
6	7	8	9	10	11	12	13	14	15	
7	8	9	10	11	12	13	14	15	16	
8	9	10	11	12	13	14	15	16	17	
9	10	11	12	13	14	15	16	17	18	
10										

The complementary probability (which is the probability of not escaping the Bowser zone) is:

$$\frac{10}{100}=\frac{1}{10}$$

Hence, the probability that we want is:

$$P(Escape)=1-P(Not\ Escaping)=1-\frac{1}{10}=\frac{9}{10}$$

Method II: Exam Oriented

In the exam, you do not need to put the values inside the table. And you do not even need to write all of the outcomes on the dice.

	1	2	3	4
1				
2				
3				
4				

Example 1.111: Using Stars and Bars

Oh no! While playing Mario Party, Theo has landed inside the Bowser Zone. If his next roll is between 1 and 5 inclusive, Bowser will shoot his "Zero Flame" that sets a player's coin and star counts to zero. Fortunately, Bowser has a double dice block, which lets him roll two fair 10-sided dice labeled 1-10 and take the sum of the rolls as his roll. If he uses his double dice block, what is the probability that he escapes the Bowser zone without losing his coins and stars? **(CMIMC Combinatorics 2023/1)**

Total Outcomes

$$10^2 = 100$$

Successful Outcomes

Count unsuccessful outcomes and subtract.
For positive integers a and b:

$$a + b \leq 5$$

Let $A = a - 1 \Rightarrow a = A + 1, B = b - 1 \Rightarrow b = B + 1$:

$$(A + 1) + (B + 1) \leq 5$$
$$A + B \leq 3$$

Introduce a slack variable to convert the inequality into an equality:

$$A + B + C = 3$$

Use Stars and Bars with 3 objects and 2 dividers:

$$\binom{3\ objects + 2\ Dividers}{2\ dividers} = \binom{5}{2} = 10$$

The number of sucessful outcomes

$$= 100 - 10 = 90$$

The probability is

$$\frac{90}{100} = \frac{9}{10}$$

Example 1.112: Using Stars and Bars

Oh no! While playing Mario Party, Theo has landed inside the Bowser Zone. If his next roll is between 1 and 6 inclusive, Bowser will shoot his "Zero Flame" that sets a player's coin and star counts to zero. Fortunately, Bowser has a triple dice block, which lets him roll three fair 8-sided dice labeled 1-8 and take the sum of the rolls as his roll. If he uses his triple dice block, what is the probability that he escapes the Bowser zone without losing his coins and stars? **(CMIMC Combinatorics 2023/1, Adapted)**

Total Outcomes

$$8^3 = 512$$

Unsuccessful Outcomes

Count unsuccessful outcomes and subtract.
For positive integers a, b, and c:

$$a + b + c \leq 6$$

Let $a = A + 1, b = B + 1, c = C + 1$:

$$(A + 1) + (B + 1) + (C + 1) \le 6$$
$$A + B + C \le 3$$

Introduce a slack variable to convert the inequality into an equality:

$$A + B + C + D = 3$$

Use Stars and Bars with 3 objects and 2 dividers:

$$\binom{3\ objects + 3\ Dividers}{3\ dividers} = \binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{6} = 20\ ways$$

The number of sucessful outcomes

$$= 512 - 20 = 492$$

The probability is

$$\frac{492}{512} = \frac{123}{128}$$

1.4 Inequalities: Enumerating Solutions

A. Pattern Method

1.113: Positive Integers

Positive integers are integers which are greater than zero, that is

$$\mathbb{N} = \{1,2,3, \dots\}$$

They are also called natural numbers.

(Important Method) Example 1.114

$$a + b < 7$$

Find the number of

A. ordered pairs $(a, b), a, b \in \mathbb{N}$ which are solutions for the above inequality.
B. unordered pairs $(a, b), a, b \in \mathbb{N}$ which are solutions for the above inequality.

Part A

Minimum value of

$$a = 1, b = 1 \Rightarrow a + b = 2$$

So, we need to find solutions for $2 \le a + b < 7$:

						Total
$a + b$	2	3	4	5	6	
Solutions	1	2	3	4	5	$= \frac{5 \times 6}{2} = 15$
	(1,1)	(1,2) (2,1)	(3,1) (2,2) (1,3)	(4,1) (2,3) (3,2) (1,4)	(5,1) (4,2) (3,3) (2,4) (1,5)	

Part B

						Total
$a+b$	2	3	4	5	6	
Solutions	1	1	2	2	3	$=9$
	$(\mathbf{1},\mathbf{1})$	(1,2) (2,1)	(3,1) $(\mathbf{2},\mathbf{2})$ (1,3)	(4,1) (2,3) (3,2) (1,4)	(5,1) (4,2) $(\mathbf{3},\mathbf{3})$ (2,4) (1,5)	

B. Multiples

Example 1.115

Find the number of positive integer solutions for a and b such that:

$$2a+3b<30$$

Method I

Consider the values that b can take:

$$b=\{1,2,3,4,5,6,7,8,9\}\Rightarrow 3b=\{3,6,9,12,15,18,21,24,27\}$$

We tabulate:

b	$3b$	$30-3b$	$2A$	A
1	3	27	$\{2,4,\dots,26\}$	13
2	6	24	$\{2,4,\dots,24\}$	12
3	9	21	$\{2,4,\dots,20\}$	10
4	12	18	$\{2,4,\dots,18\}$	9
5	15	15	$\{2,4,\dots,14\}$	7
6	18	12	$\{2,4,\dots,12\}$	6
7	21	9	$\{2,4,\dots,8\}$	4
8	24	6	$\{2,4,6\}$	3
9	27	3	$\{2\}$	1
			Total	65

Method II

Solve for a:

$$2a<30-3b$$
$$a<15-\frac{3b}{2}$$

b	$15-\frac{3b}{2}$	a	a
1	13.5	$\{1,2,\dots,13\}$	13
2	12	$\{1,2,\dots,12\}$	12
3	10.5	$\{1,2,\dots,10\}$	10
4	9	$\{1,2,\dots,9\}$	9
5	7.5	$\{1,2,\dots,7\}$	7
6	6	$\{1,2,\dots,6\}$	6
7	4.5	$\{1,2,\dots,4\}$	4
8	3	$\{1,2,\dots,3\}$	3

9	1.5	$\{1\}$	1
		Total	65

Example 1.116

Find the number of positive integer solutions for a and b:

$$5a + 7b < 100$$

Example 1.117

Find the number of ordered triples (a, b, c) of positive integers such that: $30a + 50b + 70c \leq 343$ **(IOQM 2019/17)**

Divide both sides by 10:

$$3a + 5b + 7c \leq 34.3$$

Since a, b, c are integers, we must have:

$$3a + 5b + 7c \leq 34$$

Use a change of variable to convert from positive integers to non-negative integers.
Let $a = x + 1, b = y + 1, c = z + 1$:

$$3(x + 1) + 5(y + 1) + 7(z + 1) \leq 34$$

Expand and rearrange:

$$3x + 5y + 7z + 3 + 5 + 7 \leq 34$$

Move the numbers to the RHS:

$$3x + 5y + 7z \leq 19$$

Split into cases based on the values of z, starting with the largest possible value of z is $z = 2$.

Case I: $z = 2$

$$3x + 5y + 14 \leq 19$$
$$3x + 5y \leq 5$$

$$(3x, 5y) = (0,0), (0,5), (3,0) \Rightarrow 3\ values$$

Case II: $z = 1$

$$3x + 5y + 7 \leq 19$$
$$3x + 5y \leq 12$$

$$5y = 0 \Rightarrow 3x \in \{0,3,6,9,12\} \Rightarrow 5\ values$$
$$5y = 5 \Rightarrow 3x \in \{0,3,6\} \Rightarrow 3\ values$$
$$5y = 10 \Rightarrow 3x \in \{0\} \Rightarrow 1\ value$$
$$Total = 5 + 3 + 1 = 9$$

Case III: $z = 0$

$$3x + 5y \leq 19$$

$$5y = 0 \Rightarrow 3x \in \{0,3, \dots ,18\} \Rightarrow 7\ values$$

$$5y = 5 \Rightarrow 3x \in \{0,3,6,9,12\} \Rightarrow 5 \ values$$
$$5y = 10 \Rightarrow 3x \in \{0,3,6,9\} \Rightarrow 4 \ values$$
$$5y = 15 \Rightarrow 3x \in \{0,3\} \Rightarrow 2 \ values$$
$$Total = 7 + 5 + 4 + 2 = 18$$

$$3 + 9 + 18 = 30$$

(Calculator) Example 1.118

Consider positive integers $a \le b \le c \le d \le e$. There are N lists a, b, c, d, e with a mean of 2023, and a median of 2023, in which the integer 2023 appears more than once, and in which no other integer appears more than once. What is the sum of the digits of N? **(CEMC Pascal 2023/25)**

The mean of the numbers is 2023.

$$\frac{a + b + c + d + e}{5} = 2023$$
$$a + b + c + d + e = 5(2023)$$

The median for the numbers is 2023. Since there are five numbers, the middle value is the median, and that must be $c = 2023$:

$$a \le b \le 2023 \le d \le e$$

The integer 2023 appears more than once. We consider cases based on the number of times that it appears.

Case I: All five numbers are 2023

This is a trivial case.

$$a = b = c = d = e = 2023 \Rightarrow 1 \ List$$

Case II: Four numbers are 2023

Let the numbers be:

$$2023,2023,2023,2023, x$$

The sum will be:

$$2023(4) + x = 5(2023) \Rightarrow x = 2023 \Rightarrow Contradiction \Rightarrow No \ Solution$$

Case III: Three numbers are 2023

We need to consider sub-cases.

Case IIIA: $\boldsymbol{a, b, 2023, 2023, 2023}$

Since $a \le b < 2003$

$$a < 2023, b < 2023 \Rightarrow a + b < 2023(2)$$

The sum of the numbers will be:

$$a + b + (2023)(3) = 5(2023) \Rightarrow a + b = 2(2023) \Rightarrow Contradiction \Rightarrow No \ Solution$$

Case IIIB: $\boldsymbol{2023, 2023, 2023, d, e}$

Since $2003 < d \le e$

$$2023 < d, 2023 < e \Rightarrow 2023(2) < d + e$$

The sum of the numbers will be:

$$d + e + (2023)(3) = 5(2023) \Rightarrow d + e = 2(2023) \Rightarrow Contradiction \Rightarrow No\ Solution$$

Case IIIC: $\boldsymbol{a, 2023, 2023, 2023, e}$

$$a + (2023)(3) + e = 5(2023)$$
$$a + e = 2(2023)$$
$$a + e = 4046$$

$$(a, e) = (1,4045), (2,4044), \dots, (2022,2024) \Rightarrow 2022\ Solutions$$

Case IV: Two numbers are 2023

We need to consider sub-cases.

Case IVA: $\boldsymbol{a, b, 2023, 2023, e}$

The sum of the numbers will be:

$$a + b + (2023)(2) + e = 5(2023)$$
$$a + b + e = 6069$$

Consider cases for a. The smallest value that a can take is 1. Since no integer other than 2023 appears more than once, $a < b$:

$$a = 1: b + e = 6068 \Rightarrow (b, e) = (2,6067), (3,6066), \dots, (2022,4046) \Rightarrow 2021\ solutions$$
$$a = 2: b + e = 6067 \Rightarrow (b, e) = (3,6064), (4,6063), \dots, (2022,4045) \Rightarrow 2020\ solutions$$
$$a = 3: b + e = 6066 \Rightarrow (b, e) = (4,6062), (4,6061), \dots, (2022,4044) \Rightarrow 2019\ solutions$$

.
.
.

$$a = 2021: b + e = 4048 \Rightarrow (b, e) = (2022,2025) \Rightarrow 1\ solutions$$

The total number of solutions from Case IIIA is:

$$1 + 2 + \dots + 2021 = \frac{2021(2022)}{2} = 2021(1011) = 2{,}043{,}231$$

Case IVB: $\boldsymbol{a, 2023, 2023, d, e}$

The sum of the numbers will be:

$$a + (2023)(2) + d + e = 5(2023)$$
$$a + d + e = 6069$$

Consider cases for a:

$$a = 1: d + e = 6068 \Rightarrow (d, e) = (2024,4044), (2025,4043), \dots, (3033,3035), \underbrace{(3034,3034)}_{Not\ Valid}$$
$$3033 - 2024 + 1 = 1010\ solutions$$

$$a = 2: d + e = 6067 \Rightarrow (d, e) = (2024,4043), (2025,4042), \dots, (3033,3034)$$
$$\Rightarrow 3033 - 2024 + 1 = 1010\ solutions$$

Note that $a = 1$ and $a = 2$ both result in the same number of solutions. Similarly, $a = 3$ and $a = 4$ will both result in

$$1009\ solutions$$

$$2(1 + 2 + \dots + 1010) = 2 \cdot \frac{1010(1011)}{2} = 1010(1011) = 1{,}021{,}110$$

The final answer from all the cases is:

$$= \underbrace{1}_{Case\ I} + \underbrace{0}_{Case\ II} + \underbrace{2022}_{Case\ III} + \underbrace{2{,}043{,}231}_{Case\ IV A} + \underbrace{1{,}021{,}110}_{Case\ IV B} = 3{,}066{,}364$$

The sum of the digits of the above is:

$$3 + 0 + 6 + 6 + 3 + 6 + 4 = 28$$

2. Distinguishability

2.1 Review

A. Subsets of Indistinguishable Objects

2.1: Distinguishability

- If I can distinguish between two objects, they are distinguishable.
- If two objects are identical, I cannot distinguish between them.

Example 2.2

In how many ways can I pick two balls (order of picking the balls is not important), if I have:

A. Five balls colored red, blue, green, yellow, and black.
B. Five balls colored red, and numbered from 1 to 5.
C. Five identical balls
D. Three balls colored green, yellow, and black, respectively, and two red balls

Part A

$$R, B, G, Y, B$$

The color scheme lets me distinguish between the balls.

If order is not important, I can choose the balls in:

$$\binom{5}{2} = \frac{5 \times 4}{2} = 10 \; ways$$

If order is important, I can pick the balls in:

$$5 \times 4 = 20 \; Ways$$

Part B

$$R_1, R_2, R_3, R_4, R_5$$

The numbering scheme lets me distinguish between the balls.

The answers are the same as Part A

$$Order\ is\ not\ important: 10\ Ways$$
$$Order\ is\ important: 20\ Ways$$

Part C

$$R, R, R, R, R$$

Since there is no way for me to distinguish between the balls I pick, whether order is important or not, I can do it in only

$$One\ way$$

Example 2.3

In how many ways can I pick two balls, if the order of picking the balls is not important, from three balls colored green, yellow, and black, respectively, and two red balls.

$$G, Y, B, R, R$$

Order is not important:

Two Red Balls

$$1\ Way$$

Less than two red balls:

$$\binom{4}{2} = 6$$

Total

$$= 1 + 6 = 7$$

Example 2.4
In how many ways can I pick two balls, if the order of picking the balls is important, from three balls colored green, yellow, and black, respectively, and two red balls.

Two Red Balls

$$1\ Way$$

Less Than Two Red Balls

$$4 \times 3 = 12$$

Total

$$= 1 + 12 = 13$$

B. Fruit Baskets

Example 2.5
In how many ways can I make a non-empty fruit basket with five identical apples.

$$Pick\ one\ apple\ out\ of\ 5{:}\ 1\ way$$
$$Pick\ two\ apples\ out\ of\ 5{:}\ 1\ way$$
$$Pick\ three\ apples\ out\ of\ 5{:}\ 1\ way$$
$$Pick\ four\ apples\ out\ of\ 5{:}\ 1\ way$$
$$Pick\ five\ apples\ out\ of\ 5{:}\ 1\ way$$

Total:

$$1 + 1 + 1 + 1 + 1 = 5\ ways$$

Example 2.6
In how many ways can I make a non-empty fruit basket with four identical oranges.

The number of oranges can be

$$\{1,2,3,4\} \Rightarrow 4\ Ways$$

Example 2.7
In how many ways can I make a non-empty fruit basket with six identical oranges.

$$Pears\ can\ range\ from\ 1\ to\ 6 \Rightarrow \{1,2,3,4,5,6\} \Rightarrow\ 6\ Choices$$

Example 2.8
I have five identical apples and four identical oranges. In how many ways can I make a nonempty fruit basket.

In this case, count the number of choices if we do not take any apples also:

$$Apples\ can\ range\ from\ 0 - 5{:}\ 6\ Choices$$

$$Oranges\ can\ range\ from\ 0-4{:}\ 5\ Choices$$

$$\underbrace{6}_{\substack{Choices\ for\\ Apples}} \times \underbrace{5}_{\substack{Choices\ for\\ Oranges}} = 30$$

But this also includes the case where there are zero apples and zero oranges.
Hence, the final answer is

$$30-1=29$$

Example 2.9
I have five identical apples and six identical pears. In how many ways can I make a nonempty fruit basket.

$$6\times 7-1=42-1=41$$

Example 2.10
I have four identical oranges and six identical pears. In how many ways can I make a nonempty fruit basket.

$$5\times 7-1=35-1=34$$

Example 2.11
I have five identical apples, four identical oranges and six identical pears. In how many ways can I make a non-empty fruit basket.

$$6\times 5\times 7-1=210-1=209$$

Example 2.12
In the above examples, a hotel manager comes along and numbers the apples, oranges, and pears. Now, answer all parts of the above question again.
First answer with the condition that the fruit basket must be non-empty

Apples
There are five apples. For the first apple, you have a choice, place it in the fruit basket, or do not place it. This gives you

$$2\ Choices$$

For the second apple also, you have a choice, place it in the fruit basket, or do not place it. This gives you

$$2\ Choices$$

In general, you have:

$$2\times 2\times 2\times 2\times 2=2^5=32$$

But, you cannot the choice where none of the apples have been chosen, so the final answer is:

$$32-1=31$$

Oranges

$$2^4-1=16-1=15$$

Pears

$$2^6-1=64-1=63$$

Apples and Oranges

$$2^9 - 1 = 64 - 1 = 63$$

Apples and Pears

$$2^{11} - 1 = 2048 - 1 = 2047$$

Oranges and Pears

$$2^{10} - 1 = 1024 - 1 = 1023$$

Apples, Oranges and Pears

$$2^{15} - 1$$

Example 2.13

Now answer with the condition that there must be one fruit of every eligible type.

Parts A, B, C

Same as above: $\{31, 15, 63\}$

Part D

We can do this using complementary counting.

If there are no restrictions, the number of ways to do this is:

$$2^5 \times 2^4 = 2^9 = 512$$

We cannot have the case where no apples are selected, which can be done in:

$$\underbrace{1}_{\substack{Select\ Zero \\ Apples}} \times \underbrace{2^4}_{\substack{Select\ 0-4\ of \\ the\ Oranges}} = 16\ Ways$$

We cannot have the case where no oranges are selected, which can be done in :

$$\underbrace{2^5}_{\substack{Select\ 0-5\ of \\ the\ Apples}} \times \underbrace{1}_{\substack{Select\ Zero \\ Oranges}} = 32\ Ways$$

However, both the cases above count the situation where no apple and no orange is selected, which can be done in exactly

$$1\ Way$$

Hence, the final answer is

$$512 - 16 - 32 + 1 = 465$$

Part E

C. Multiplication Principle with Repetition

2.14: Distinguishable Balls in Distinguishable Boxes

The number of ways that r distinguishable balls can be distributed among n distinguishable boxes, such that each box has a nonnegative number of objects is:

$$n^r$$

The number of choices for the first object

$$= n$$

The number of choices for the second object

$$= n$$

In general, the number of choices is:

$$\underbrace{n \cdot n \cdot \ldots \cdot n}_{r\ times} = n^r$$

Example 2.15

A. A professor needs to assign grades to 12 of his students. Grades are a number from 0 to 10 (inclusive). Determine an expression for the number of possible grades sheets that the professor could have?
B. Connect this with balls and boxes

$$n^r = 11^{12}$$

You are assigning each student to belong to a particular capability, or grade.

$$Grades: Boxes, \qquad Students: Balls$$

Example 2.16

My class of 25 students needs to decide between going to Starbucks, and Café Coffee Day for their afternoon coffee. Each student will visit exactly one coffee shop and make the decision independently. Connect this to balls and boxes, and then find the answer to how many ways the students can visit the coffee shops.

$$2^{25}$$
$$Students: Balls: 25$$
$$Boxes: cafe: 2$$

Example 2.17

I am going to categorize my seven favorite restaurants into three categories so that I have them handy when my seven friends visit. The three categories are: must-visit, might-visit, not-important-to-visit. Each restaurant will be classified into exactly one category. Connect this to balls and boxes, and then find the answer to how many ways the categorization can be done.

$$3^7$$
$$Restaurants: Balls: 7$$
$$Categories: Boxes: 3$$

Example 2.18: Balls-Distinguishable, Boxes-Distinguishable

Find the number of ways that 5 distinguishable balls can be distributed among 3 distinguishable boxes.

This is a direct application of the multiplication principle:
For the first ball, we have five choices.
For the second ball, we have five choices.
In fact, for each ball, we have five choices, and these choices are independent.

Hence, the total number of choices is:

$$\underbrace{3}_{\substack{First\\Ball}} \times \underbrace{3}_{\substack{Second\\Ball}} \times \underbrace{3}_{\substack{Third\\Ball}} \times \underbrace{3}_{\substack{Fourth\\Ball}} \times \underbrace{3}_{\substack{Fifth\\Ball}} = 3^5 = 243$$

D. Permutations

2.19: Distinguishable Balls assigned to Distinguishable Boxes

For $n \geq r$, the number of ways to assign r distinguishable balls to n distinguishable boxes such that each box has at most one ball assigned to it:

$$nPr = \frac{n!}{r!}$$

The first ball has

$$n\ boxes = n\ choices$$

The second ball

$$(n-1)\ boxes = (n-1)\ choices$$

.
.
.

The r^{th} ball will have

$$n - r + 1\ choices$$

The final answer, by the multiplication principle is:

$$n(n-1)\ldots(n-r+1) = \frac{n!}{r!} = nPr$$

Example 2.20

The incoming class in Euler High has 23 freshman, each of whom must be assigned to one of five classrooms. They must also be assigned a roll number out of the numbers 1 to 23. How many ways can you assign the:

A. classrooms
B. roll numbers

$$5^{23}$$
$$23!$$

E. Stars and Bars/Diophantine Equations

2.21: Identical Balls assigned to Distinguishable Boxes

The number of ways to assign r identical balls to n distinguishable boxes is:

$$\binom{r+n-1}{n-1} = \binom{r+n-1}{r}$$

Let $x_i = No.of\ balls\ in\ box\ i$. Then:

$$x_1 + x_2 + \cdots + x_n = r$$

Apply stars and bars with r *balls*, $(n-1)$ *dividers* resulting in:

$$\binom{r+n-1}{n-1}$$

And using $\binom{a}{b} = \binom{a}{a-b}$:

$$= \binom{r+n-1}{r}$$

Example 2.22: Balls-Identical, Boxes-Distinguishable

Find the number of ways that 5 identical balls can be distributed among 3 distinguishable boxes.

We have $\underbrace{\textit{three distinguishable boxes}}_{\underline{Box\ 1}\ \underline{Box\ 2}\ \underline{Box\ 3}}$, among which we put two dividers:

$$\underline{\mathbf{Box\ 1}}\ \ \underline{\mathbf{Divider\ 1}}\ \ \underline{\mathbf{Box\ 2}}\ \ \underline{\mathbf{Divider\ 2}}\ \ \underline{\mathbf{Box\ 3}}$$

If we add $\underbrace{\textit{two dividers}}_{\underline{||}}$ among $\underbrace{\textit{five identical balls}}_{\underline{0}\,\underline{0}\,\underline{0}\,\underline{0}\,\underline{0}}$, then the dividers divide the balls into three parts. For example

$$\underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{||}$$

corresponds to putting all five balls in the first box:

$$\underbrace{\underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}}_{First\ Box}\ \underline{|}\ \underbrace{\underline{Empty}}_{Second\ Box}\ \underline{|}\ \underbrace{\underline{Empty}}_{Third\ Box}$$

Similarly

$$\underline{0}\ \underline{0}\underline{|}\ \underline{0}\underline{|}\ \underline{0}\ \underline{0}$$

corresponds to two balls in the first box, one ball in the second box, and two balls in the third box:

$$\underbrace{00}_{First\ Box}\ \ \underline{|}\ \ \underbrace{0}_{Second\ Box}\ \ \underline{|}\ \ \underbrace{00}_{Third\ Box}$$

In general, the number of ways we can distribute the balls among the boxes is the way we can select the positions of the two dividers out of the seven objects that we have:

$$^{5\ Balls\ +2\ Dividers}C_{2\ Positions} = {}^{5+2}C_2 = {}^{7}C_2 = \frac{7 \times 6}{2} = 21$$

Example 2.23

Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected? **(AMC 10A 2003/21)**

Method I: Formula

The number of cookies is

$$r = 6$$

Imagine each cookie of a particular type is to be placed in a box of its specific type. Then:

$$n = 3$$

Substitute $r = 6, n = 3$ in

$$\binom{r+n-1}{r} = \binom{6+3-1}{6} = \binom{8}{6} = \binom{8}{2} = 28$$

Method II: Stars and Bars

$$a + b + c = 6$$

Apply stars and bars with $6\ objects, 2\ dividers$

$$\binom{6+2}{2} = \binom{8}{2} = 28$$

2.2 Distinguishability

A. Partitions of a Set

A set is a collection of elements.

2.24: Partition of a Set

A partition of a set divides a set into one or more non-empty subsets sets such that each element belongs to exactly one subset.

Example 2.25

List the partitions of the set $A = \{1,2\}$.

$$\{1,2\} \Rightarrow 1\ way$$
$$\{1\}\{2\} \Rightarrow 1\ way$$

Total

$$1 + 1 = 2$$

Example 2.26

List the partitions of the set $A = \{1,2,3\}$.

$$\{1,2,3\} \Rightarrow 1\ way$$
$$\{1,2\}\{3\}, \quad \{1,3\}\{2\}, \quad \{2,3\}\{1\} \Rightarrow 3\ way$$
$$\{1\}\{2\}\{3\} \Rightarrow 1\ way$$

Total

$$1 + 3 + 1 = 5$$

Example 2.27

List the partitions of the set $A = \{1,2,3,4\}$.

All four elements in one set (4):

$$\{1,2,3,4\} \Rightarrow 1\ way$$

Three elements in one set, and the fourth element separate $(3 + 1)$:

$$\{2,3,4\}\{1\}, \quad \{1,3,4\}\{2\}, \quad \{1,2,4\}\{3\}, \quad \{1,2,3\}\{4\} \Rightarrow 4\ ways$$

Two elements in one set, and two elements in a second set $(2 + 2)$:

$$\{1,2\}\{3,4\}, \quad \{1,3\}\{2,4\}, \quad \{1,4\}\{2,3\} \Rightarrow 3\ ways$$

Two elements in one set, and two sets of a single element each$(2 + 1 + 1)$:

$$\{1,2\}\{3\}\{4\}, \quad \{1,3\}\{2\}\{4\}, \quad \{1,4\}\{2\}\{3\}$$
$$\{2,3\}\{1\}\{4\}, \quad \{2,4\}\{1\}\{3\}$$

$$\{3,4\}\{1\}\{2\}$$
$$3+2+1=6\ ways$$

Four singleton sets of a single element each $(1+1+1+1)$:

$$\{1\}\{2\}\{3\}\{4\} \Rightarrow 1\ way$$

Total

$$= 1+4+3+6+1 = 15$$

Example 2.28

A. I have four balls colored red, green, yellow, and blue. List the ways in which I can put them into identical boxes.

B. List the partitions of the set $A = \{R, G, Y, B\}$.

All four elements in one set (4):

$$\{R, G, Y, B\} \Rightarrow 1\ way$$

Three elements in one set, and the fourth element separate $(3+1)$:

$$\{G, Y, B\}\{R\}, \quad \{R, Y, B\}\{G\}, \quad \{R, G, B\}\{Y\}, \quad \{R, G, Y\}\{B\} \Rightarrow 4\ ways$$

Two elements in one set, and two elements in a second set $(2+2)$:

$$\{R, G\}\{Y, B\}, \ \{R, Y\}\{G, B\}, \quad \{R, B\}\{G, Y\} \Rightarrow 3\ ways$$

Two elements in one set, and two sets of a single element each$(2+1+1)$:

$$\{R, G\}\{Y\}\{B\}, \ \{R, Y\}\{G\}\{B\}, \quad \{R, B\}\{G\}\{Y\}$$
$$\{G, Y\}\{R\}\{B\}, \quad \{G, B\}\{R\}\{Y\}$$
$$\{Y, B\}\{R\}\{G\}$$
$$3+2+1=6\ ways$$

Four singleton sets of a single element each $(1+1+1+1)$:

$$\{R\}G\{Y\}\{B\} \Rightarrow 1\ way$$

Total

$$= 1+4+3+6+1 = 15$$

Example 2.29

List the partitions of the set $A = \{1,2,3,4,5\}$.

All five elements in one set:

$$\{1,2,3,4,5\} \Rightarrow 1\ way$$

Four elements in one set, and the fifth element separate:

$$\{2,3,4,5\}\{1\}, \quad \{1,3,4,5\}\{2\}, \quad \{1,2,4,5\}\{3\}, \quad \{1,2,3,5\}\{4\}, \quad \{1,2,3,4\}\{5\} \Rightarrow 5\ ways$$

Three elements in one set, and two elements separate

$$\{1,2\}\{3,4,5\}, \quad \{1,3\}\{2,4,5\}, \quad \{1,4\}\{2,3,5\}, \quad \{1,5\}\{2,3,4\} \Rightarrow 4\ ways$$
$$\{2,3\}, \{1,4,5\}, \quad \{2,4\}\{1,3,5\}, \quad \{2,5\}\{1,3,4\} \Rightarrow 3\ ways$$
$$\{3,4\}, \{1,2,5\}, \quad \{3,5\}\{1,2,4\} \Rightarrow 2\ ways$$
$$\{4,5\}, \{1,2,3\} \Rightarrow 1\ ways$$

$$4+3+2+1=10\ ways$$

Three elements in set one, and two sets with a single element each:

$$\{1\}\{2\}\{3,4,5\},\quad \{1\}\{3\}\{2,4,5\},\quad \{1\}\{4\}\{2,3,5\},\quad \{1\}\{5\}\{2,3,4\} \Rightarrow 4\ ways$$
$$\{2\}\{3\},\{1,4,5\},\quad \{2\}\{4\}\{1,3,5\},\quad \{2\}\{5\}\{1,3,4\} \Rightarrow 3\ ways$$
$$\{3\}\{4\},\{1,2,5\},\quad \{3\}\{5\}\{1,2,4\} \Rightarrow 2\ ways$$
$$\{4\}\{5\},\{1,2,3\} \Rightarrow 1\ ways$$

$$4+3+2+1=10\ ways$$

Two elements in one set, and three separately:

$$\{1,2\}\{3\}\{4\}\{5\},\quad \{1,3\}\{2\}\{4\}\{5\},\quad \{1,4\}\{2\}\{3\}\{5\},\quad \{1,5\}\{2\}\{3\}\{4\} \Rightarrow 4\ ways$$
$$\{2,3\},\{1\}\{4\}\{5\},\quad \{2,4\}\{1\}\{3\}\{5\},\quad \{2,5\}\{1\}\{3\}\{4\} \Rightarrow 3\ ways$$
$$\{3,4\},\{1\}\{2\}\{5\},\quad \{3,5\}\{1\}\{2\}\{4\} \Rightarrow 2\ ways$$
$$\{4,5\},\{1\}\{2\}\{3\} \Rightarrow 1\ ways$$

$$4+3+2+1=10\ ways$$

Two elements in one set, two elements in a second set, and one element in a set by itself:

$$\{1,2\}\{3,4\}\{5\},\quad \{1,2\}\{3,5\}\{4\},\quad \{1,2\}\{4,5\}\{3\} \Rightarrow 3\ ways$$
$$\{1,3\}\{2,4\}\{5\},\quad \{1,3\}\{2,5\}\{4\},\quad \{1,3\}\{4,5\}\{2\} \Rightarrow 3\ ways$$
$$\{1,4\}\{2,3\}\{5\},\quad \{1,4\}\{2,5\}\{3\},\quad \{1,4\}\{3,5\}\{2\} \Rightarrow 3\ ways$$
$$\{1,5\}\{2,3\}\{4\},\quad \{1,5\}\{2,4\}\{3\},\quad \{1,5\}\{3,4\}\{2\} \Rightarrow 3\ ways$$
$$\{2,3\}\{4,5\}\{1\},\quad \{2,4\}\{3,5\}\{1\},\quad \{2,5\}\{3,4\}\{1\} \Rightarrow 3\ ways$$

$$3+3+3+3+3=15$$

Five singleton sets of a single element each:

$$\{1\}\{2\}\{3\}\{4\}\{5\} \Rightarrow 1\ way$$

Final Answer

$$=1+5+10+10+10+15+1=52$$

B. Casework

2.30: Identical Balls assigned to Identical Boxes

When identical balls are assigned to identical boxes, there is nothing to distinguish either the balls or the boxes, and

$$there\ is\ no\ direct\ formula$$

This has to be done using casework.

Example 2.31

5 identical parcels are to be transported from New York to London via plane. The parcels are important, and hence for their security, then can be transported in upto three planes. The timing of the plane, or the type does not matter. Only the distribution of the parcels across different planes is important. In other words, the planes are identical. In how many ways can the parcels be transported.

	Distribution	Meaning

All in one plane	$5-0-0$	Five balls in one box and zero in the other two
Two planes getting used	$4-1-0$	Four balls in one box, and one ball in one ball
	$3-2-0$	Three Balls in one box, and two balls in one box
Three planes used	$3-1-1$	Three balls in one box, and one ball each in the remaining boxes
	$2-2-1$	Two balls each in two boxes, and the remaining ball in the remaining box
	$5\ Ways$	

Example 2.32

Find the number of ways that 5 identical balls can be distributed among 3 identical boxes.

Distribution	Meaning
$5-0-0$	Five balls in one box and zero in the other two
$4-1-0$	Four balls in one box, and one ball in one ball
$3-2-0$	Three Balls in one box, and two balls in one box
$3-1-1$	Three balls in one box, and one ball each in the remaining boxes
$2-2-1$	Two balls each in two boxes, and the remaining ball in the remaining box
$5\ Ways$	

C. Casework with Selections

Example 2.33

5 parcels colored red, green, yellow, white, and black are to be transported from New York to London via plane. The parcels are important, and hence for their security, then can be transported in upto three planes. The timing of the plane, or the type does not matter. Only the distribution of the parcels across different planes is important. In other words, the planes are identical. In how many ways can the parcels be transported.

Example 2.34

Find the number of ways that 5 distinguishable balls can be distributed among 3 identical boxes.

$5-0-0$	${}^5C_5 = 1$	Choice of box does not matter here. All five balls go in one box.
$4-1-0$	$\binom{5}{4} = \binom{5}{1} = 5$	Four balls in one box, and one ball in another box. The choice of ball is going to matter, since the single ball that you pick is placed differently from the rest.
$3-2-0$	$\binom{5}{3} = \binom{5}{2} = 10$	Three Balls in one box, and two balls in one box
$3-1-1$	$\binom{5}{3} = \binom{5}{2} = 10$	Three balls in one box, and one ball each in the remaining boxes

$2-2-1$	$\frac{{}^5C_2 \times {}^3C_2}{2} = \frac{10 \times 3}{2} = 15$	Two balls each in two boxes, and the remaining ball in the remaining box
$5\ Ways$	$1+5+10+10+15=41$	

Suppose that the balls are distinguishable on the basis on color: Red(R), Blue(B), Green(G), Yellow(Y), Magenta(M)

We need to first select two balls from the five available, giving us:

$$ {}^5C_2 = \frac{5 \times 4}{2} = 10 $$

Let us say we get Red and Blue as the balls that go into the first box. We are now left with three balls, from which we must choose two:

$$ {}^3C_2 = \frac{3 \times 2}{2} = 3 $$

Let's say we Green and Yellow. Now, we don't have any choice with respect to the last ball, because it is already selected for us.

Hence, the total number of choices is

$$ {}^5C_2 \times {}^3C_2 = 10 \times 3 = 30 $$

But, this counts as two different arrangements the same situation:

$$ \{R,B\}+\{G,Y\}, \qquad \{G,Y\}+\{R,B\} $$

Hence, the number that we found above must be divided by 2, giving us:

$$ \frac{30}{2} = 15 $$

Example 2.35

Find the number of ways in which the O'Hara quadruplets can go for a wedding in four limousines if

A. Mother can distinguish both quadruplets and limousines
B. Butler can distinguish quadruplets, but not between cars
C. Chauffer can distinguish cars, but not between quadruplets
D. Usher can't distinguish between either

Part A: Multiplication Principle

The first quadruplet can sit in any of the four limousines:

$$ 4\ Choices $$

The second quadruplet can sit in any of the four limousines:

$$ 4\ Choices $$

The third quadruplet can sit in any of the four limousines:

$$ 4\ Choices $$

The fourth quadruplet can sit in any of the four limousines:

$$ 4\ Choices $$

Each of the choices is independent. Hence, the total number of choices is

$$ 4 \times 4 \times 4 \times 4 = 256 $$

Part B

$4-0-0$	$1\ Choice$
$3-1-0$	$4\ Choices$
$2-2-0$	$\binom{4}{2}$

$2-1-1$	$\binom{4}{2}$

Part C

$$\underbrace{a+b+c+d=4}_{\substack{4\ Stars \\ 3\ Bars}} \Rightarrow \binom{4+3}{3} = \binom{7}{3} = 35$$

Part D

$4-0-0$	
$3-1-0$	
$2-2-0$	
$2-1-1$	

D. Climbing Stairs

Example 2.36

I want to climb a nine-step staircase. At one time, I can only take a number of steps which is a nonnegative integer power of 2. The order of the steps does not matter. That is:

$$(4,4,1) = (4,1,4) = (1,4,4)$$

Find the number of ways in which I can reach the top of the stairs (9^{th} step) from the ground.

Preparatory Work

The powers of 2 are:

$$2^0 = 1, \quad 2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8 \Rightarrow \{1,2,4,8\}$$

Negative numbers cannot be achieved as nonnegative integer powers of 2, and hence you can only go up. Not go up and down.

Listing the Cases

Largest Number is 8:

$$8+1$$

Largest Number is 4:

$$4+4+1$$
$$4+2+2+1$$
$$4+1\times 5$$
$$4+2+1\times 3$$

Largest Number is 2:

$$2\times 4+1$$
$$2\times 3+1\times 3$$
$$2\times 2+1\times 5$$
$$2+1\times 7$$

Largest Number is 1:

$$1\times 9$$

(Continuation) Example 2.37

In the above example, find the number of ways if the order does matter. That is {(4,4,1), (4,1,4), (1,4,4)} are three different ways to climb the stairs.

Note that the (1,4,4) has two 4's which are identical.

Largest Number is 8:

$$8+1 \Rightarrow (8,1),(1,8) \Rightarrow 2\ ways$$

Largest Number is 4:

$$4+4+1 \Rightarrow Choose\ position\ for\ the\ 1: \binom{3}{1} = 3\ ways$$
$$4+2+2+1 \Rightarrow Choose\ the\ position\ of\ the\ 4\ and\ 1 : 4 \times 3 = 12\ ways$$
$$4+2+1 \times 3 \Rightarrow Choose\ the\ position\ of\ the\ 4\ and\ the\ 2: 5 \times 4 = 20\ ways$$
$$4+1 \times 5 \Rightarrow Choose\ the\ position\ of\ the\ 4: \binom{6}{1} = 6\ ways$$

Largest Number is 2:

$$2 \times 4 + 1: Choose\ the\ position\ of\ the\ 1: \binom{5}{1} = 5\ ways$$
$$2 \times 3 + 1 \times 3: Choose\ the\ position\ of\ the\ three\ 2's: \binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{6} = 20$$
$$2 \times 2 + 1 \times 5: Choose\ the\ position\ of\ the\ two\ 2's: \binom{7}{2} = 21$$
$$2 + 1 \times 7: Choose\ the\ position\ of the\ 2: \binom{8}{1} = 8$$

Largest Number is 1:

$$1 \times 9 \Rightarrow 1\ way$$

$$2+3+12+20+6+5+20+21+8+1 = 98$$

E. Different kinds of Identical Objects

Example 2.38

The identical Patil twins, Padma and Parvati, have two identical *Cleansweep* brooms and two identical *Comet* brooms. In how many ways can they ride home on the brooms?

Note: There are no restrictions on how people/objects a broom can accommodate.

List out the ways in which they can ride home. Let P represent a Patil twin.

	$Cleansweep$	$Cleansweep$	$Comet$	$Comet$
Both on a single broom	PP			
			PP	
On two different brooms	P	P		
			P	P
		P	P	

From the above, we can see that there are 5 ways in which they can ride home.

Example 2.39

Does it matter if Padma sits in front on the broom, and Parvati sits behind, or Parvati sits in front, and Padma sits behind.

The twins are identical, and hence the two arrangements are indistinguishable.
So, they are the same arrangement for us.

(Continuation) Example 2.40

When riding back home, their father presents the Parvati twins with an owl. In how many ways can they ride back to Hogwarts on the brooms with their owl?

Notes:

1. There are no restrictions on how people/objects a broom can accommodate.
2. An owl is perfectly capable of flying a broom by itself.
3. The position of the owl on the broom does not matter. What matters is how many people the brooms is transporting.

From the previous example, we know that the number of ways in which they can go home is:

	$Cleansweep$	$Cleansweep$	$Comet$	$Comet$
Both on a single broom	PP			
			PP	
On two different brooms	P	P		
			P	P
		P	P	

In the above table, we can add the owl in different positions and see how many ways we can get:

	$Cleansweep$	$Cleansweep$	$Comet$	$Comet$	
Both on a single broom	$PP, \boldsymbol{O_1}$	$\boldsymbol{O_2}$	$\boldsymbol{O_3}$		3
		$\boldsymbol{O_1}$	$PP, \boldsymbol{O_2}$	$\boldsymbol{O_3}$	3
On two different brooms	$P, \boldsymbol{O_1}$	P	$\boldsymbol{O_2}$		2
	$\boldsymbol{O_1}$		$P, \boldsymbol{O_2}$	P	2
	$\boldsymbol{O_1}$	$P, \boldsymbol{O_2}$	$P, \boldsymbol{O_3}$	$\boldsymbol{O_4}$	4
					14

F. Applications

Example 2.41

2.3 Further Topics

42 Examples