Money and Competing Means of Payment

Geromichalos, Athanasios and Wang, Yijing

University of California, Davis, Massey University

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Athanasios Geromichalos Yijing Wang

Abstract

In monetary theory, money is typically introduced as an object that can help agents bypass frictions, such as anonymity and limited commitment. Consequently, common wisdom suggests that if agents had access to more unsecured credit these frictions would become less severe and welfare would improve. In similar spirit, common wisdom suggests that as societies get access to more alternative (to money) payment instruments, i.e., more ways to *bypass* the aforementioned frictions, welfare would also increase. We show that for a large variety of settings and market structures this common wisdom is not accurate. If the alternative means of payment is sufficient to cover all the liquidity needs of the economy, then indeed the economy will reach maximum welfare. However, if access to this alternative payment system is relatively low to begin with, increasing it can hurt the economy's welfare, and we characterize in detail the set of parameters for which this result can arise. Our model offers a simple explanation to a recent empirical literature suggesting that increased access to credit is often followed by declined economic activity.

1 Introduction

In monetary theory, money is typically introduced as an object that can help agents carry out transactions in markets characterized by frictions, such as anonymity and lack of commitment, which preclude unsecured credit (see Kiyotaki and Wright (1989) and Kocherlakota (1998)). For example, if trade is bilateral and a consumer cannot commit to repaying her/his debt, then the transaction has to be set in a *quid pro quo* manner, and money is usually the means that allows such a transaction to take place, by helping bypass the friction. Consequently, common wisdom suggests that if agents had access to more unsecured credit (i.e., to a commitment device that allows them to credibly promise repayment of a debt), the frictions in the economy would become less severe and welfare would improve. Similarly, common wisdom suggests that as societies get access to more payment instruments/systems, i.e., more ways to *bypass* the aforementioned frictions, welfare would also increase.

The goal of this paper is to examine whether the introduction of alternative means of payment, like credit, financial assets, or secondary markets where agents can boost their liquidity, is (always) welfare improving. We show that for a large variety of settings and market structures, the common wisdom described in the previous paragraph does not turn out to be accurate. While our paper establishes this surprising result for four different settings (or alternative payment systems), the intuition can perhaps be best described in a simple environment with money and (unsecured) credit. If every agent in the economy has access to perfect credit, indeed the economy will reach maximum welfare, since this would be world without any frictions. However, if access to credit is low to begin with, increasing it can actually hurt the economy's welfare, i.e., increasing the friction in the economy makes people better off.

What gives rise to this counter-intuitive result? Our model exhibits the following interesting feature: agents need to pay a cost to carry money/liquidity (that cost is no other than inflation), and they decide how much money to carry before they know whether they will actually need it for transactions or whether they can use credit. *Ex post*, more credit is certainly good for welfare because it means that transactions will not be hindered by the lack of liquidity. But *ex ante*, easier access to credit diminishes the demand for money and hinders trade in bilateral meetings where credit is *not available*. Obviously, this describes a situation with two opposing forces fighting each other. In the paper, we analyze the dynamic general equilibrium model and describe precisely the set of parameter values for which the second, negative force can dominate, so that ultimately an increase in credit availability (a decrease in frictions) can be welfare improving.

We then generalize the result by considering a specification of the model, where the alternative (to money) system/means of payment can be a financial asset, as opposed to credit. We also consider the case where money is the only direct medium of exchange, but agents have access to a secondary market where they can boost their liquidity either by obtaining an unsecured loan or by selling assets for cash. In each case, we are able to show that there exists a set of parameters for which increased access to the respective alternative payment method can be welfare decreasing. Therefore, we conclude that access to more (and more advanced) payment systems alternative to money is not always welfare improving. Our model offers a simple and intuitive explanation to the recent empirical literature suggesting that increased access to credit is often followed by recessions and decline in economic activity; see for example Schularick and Taylor (2012) and Jordà et al. (2013).

Our paper is related to a large literature that studies the coexistence of money and alternative means of payment. Papers such as Telyukova and Wright (2008), Gu et al. (2013), and Gu et al. (2016) study the coexistence of money and various types of credit (secured or unsecured). However, none of these papers examines whether higher availability of credit can hurt welfare. Our paper is also related to the growing literature that studies the coexistence of money and other financial assets as means of payment. Examples include Geromichalos et al. (2007), Lagos (2011), Lester et al. (2012), Nosal and Rocheteau (2012), Andolfatto et al. (2013), and Hu and Rocheteau (2015). In these papers the liquidity properties of assets are 'direct', in the sense that assets serve as a media of exchange or collateral, helping agents (directly) facilitate trade in decentralized goods markets.

Our paper is also related to the strand of literature that studies the welfare effect of using alternative means of payment (to money). For example, several papers, including Berentsen et al. (2014), Geromichalos and Herrenbrueck (2016), Huber and Kim (2019), and Geromichalos et al. (2021), have studied the welfare effect of accessing a secondary financial market. These papers show that improving financial market accessibility/matching probability is not always welfare-improving.

Other papers have studied the welfare effect of credit acceptability. Rojas Breu (2013) shows that an increase in credit can be welfare-worsening due to inefficient consumption-risk sharing. Lotz and Zhang (2016) study a similar welfare effect of using credit under costly record-keeping technology and limited commitment. In Chiu et al. (2018), increasing credit usage can hurt welfare through "price effect", i.e., higher consumption from credit users drive up the price, which in turn hurts the consumption from non-credit users. Dong and Huangfu (2021) consider that when credit settlement is delayed, credit usage is also subject to inflation distortion, which can further hurt welfare. Our model predicts the non-monotonic impact of credit availability on welfare through a different channel: access to credit discourages *ex-ante* money holding.

Our paper contributes to this strand of literature by providing a generalized framework that studies welfare effect of different types of alternative means of payment to money, including credit, asset, and access to financial markets. In addition, by using a quadratic utility function, we are able to deliver closed-form solutions, which are not discussed in the other papers.

In Section 4 we consider the case where agents, who receive an idiosyncratic liquidity shock, can boost their liquidity in a secondary market. This idea builds on the work of Berentsen et al. (2007), where agents with different liquidity needs visit a competitive banking sector to rebalance their positions. In our model agents can visit a secondary market and boost their liquidity holdings either by obtaining unsecured loans (Section 4.1) or by selling assets (Section 4.2). Thus, money does not have a 'direct' competitor as a medium of exchange (all transactions in the goods market must be settled with money), but it has an 'indirect' competitor, in the sense that assets (in Section 4.2) can be sold for money in the secondary market, and so they are indirectly liquid. This empirically relevant approach to asset liquidity has also been explored in Berentsen et al. (2014, 2016), Mattesini and Nosal (2016), Geromichalos and Herrenbrueck (2016), Herrenbrueck and Geromichalos (2017), Herrenbrueck (2019), Madison (2019), Wang (2022), and Geromichalos et al. (2023).

2 The Model

2.1 Environment

The economy has infinite horizon and time is discrete. In each period, there are two sub-markets that open for different economics activities: a decentralized market (henceforth DM), and a centralized market (henceforth CM). The CM is the settlement market of Lagos and Wright (2005). Access to this market together with quasi-linear preferences helps keep the model tractable. In the DM, agents meet and trade a special good in anonymous bilateral meetings where perfect commitment might not be available. This gives rise to a need for a medium of exchange, and we will discuss various cases in which one or more payment methods are recognizable/acceptable in DM trades. In later sections, we will extend the model to include an additional sub-market, a secondary over-the-counter (henceforth OTC) asset market, where agents with different liquidity needs in the upcoming DM can trade with each other to rebalance their portfolio. More precisely, agents who have an urgent need for cash in the DM can boost their money holdings either by obtaining a loan or by selling assets.

Agents discount future at a rate of $\beta \in (0, 1)$, but there is no discounting between sub-markets. There are two types of agents, consumers and producers, characterized by their roles in the DM which remain permanent. The measure of each type of agents is normalized to 1.

In the DM, consumers will consume a special goods q, and producers will produce it. The quantity of goods that the producer produces, and payment that consumer pays in exchange, will be determined through bargaining. More precisely, the terms of trade are determined by proportional bargaining, following Kalai (1977). We will let θ denote the bargaining power of consumers and $1-\theta$ be the bargaining power of producers. To consume the special goods, there are three objects that could potentially serve as a proper means of payment: money, a real asset, and credit. Money is fiat, storable, and perfectly divisible, with supply of $M_{t+1} = (1+\mu)M_t$ controlled by the monetary authority through lump-sum transfer in the CM. When $\mu > 0$, new money is introduced into the economy; when $\mu < 0$, money is withdrew. Consumers can obtain money in the CM at the ongoing price of φ. (The CM is a Walrasian market, hence, all market participants take its price as given.) The second payment method is a one-period physical asset with fixed supply A. Each share of the assets can be purchased at a price of ψ in CM_t. It will pay 1 unit of numeraire good as dividend in DM_{t+1}, and then the asset dies and gets replaced by an identical set of assets. The other payment method is credit, with which consumers in the DM can purchase special goods from the producer by promising to pay back in the following CM. Thus, credit here is unsecured.

In the CM, both consumers and producers supply labor and consume a general good. The technology transforms 1 unit of labor input into 1 unit of the general good. Consumers' and producers' utility in a given period are given by $\mathcal{U}(X, H, x) = U(X) - H + u(q)$ and $\mathcal{V}(X, H, q) = U(X) - H - q$ respectively, where X is the consumption of numeraire good in the CM, H is the labor supply in the CM, and q is the special good consumed/produced in the DM. We assume that u and U are twice continuously differentiable with $u(0) = 0$, $u' > 0$, $u'(0) = \infty$, and $u'' < 0$. In later sections, we will also consider the special case of a quadratic utility function, which allows us to sharply characterize some of our results by deriving closed form solutions for the key equilibrium variables. In this case the Inada condition will be relaxed. Let q^* be the optimal level of output in the DM, i.e. $u'(q^*) = 1$. Also assume that there exists $X^* \in (0, \infty)$ such that $U'(X^*) = 1$.

In the DM, there are three objects that could potentially serve the role of a medium of exchange: money, a real asset, and credit. With this setting, we will discuss four different environments. In the first two, credit (Section 3.1) and then assets (Section 3.2) will serve as direct means of payment, in the sense that a subset of producers will be able to accept these alternative payment methods. Then, we extend the model (in Section 4) to incorporate an OTC secondary market in which we allow consumers with different liquidity needs to trade with each other in order to rebalance their liquidity holdings. In this section, money is the ultimate medium of exchange in the DM, but it has 'indirect' competition from loans (Section 4.1) or assets (Section 4.2). Considering all these cases allows us to conclude that our finding, namely, the idea that increased access to new alternative payment methods can sometimes be welfare decreasing, is not a coincidence, but a robust result in this class of models.

3 Money and competing media of exchange

We start the analysis with two versions of the model in which money has a direct competitor as a medium of exchange. That competitor is first unsecured credit and then a real asset.

3.1 Money and Credit

In this section, the two forms of payment that could potentially be used in the DM bilateral trade are money and credit. We will let σ be the probability that a producer recognize credit and has the ability to enforce a payment from consumers in the CM. We will refer to these types of producers as type-0 producers. Then, $1 - \sigma$ is the probability that a producer does not have the ability of identifying or accepting unsecured credit; these will be the type-1 producers.[1]

3.1.1 Value Functions and Bargaining Solutions

We start by describing the value functions in the CM. For a typical consumer entering the CM, the state variables are d and m. d is the amount of debt that agents took in the preceding DM for special good consumption, and m is the amount of money agents brought into the CM. The value function is given by

$$W(m,d) = \max_{X,H,\hat{m}} \Big\{ U(X) - H + \beta V(\hat{m}) \Big\}$$
$$\text{s.t. } X + \varphi \hat{m} = \varphi m + H - d + T$$

where $V(\hat{m})$ is the value function of next DM, and $\hat{m}$ is the amount of money that the consumer chooses to bring into next period. Substituting H from the budget constraint allows us to rewrite the value function as

$$W(m,d) = \varphi m - d + \Lambda \tag{1}$$

where $\Lambda = U(X^*) - X^* + T + \max_{\hat{m}}\{-\varphi\hat{m} + \beta V(\hat{m})\}$.

Next, consider a producer's CM value function. Notice that producers will never leave the CM with a positive amount of money because money is costly to hold, in equilibrium, and producers will never have the need to use it (precisely due to their permanent identity as producers in the DM). Of course, producers may enter the CM with some money that they received as means of payment in the preceding DM. Hence for a type-0 producer who

[1] As a mnemonic rule, a type-i, $i = 0,1$, producer is a producer who requires i "assets" or "objects" as media of exchange in order to trade in the DM; of course, in the case of type-0 producers no medium of exchange is required, since that producer accepts unsecured credit.

accepts credit, the value function is

$$W^{S0}(d) = \max_{X,H}\{U(X) - H\}$$
$$\text{s.t. } X = H + d$$

By substituting H from the budget constraint into the value function, it can be rewritten as

$$W^{S0}(d) = \Lambda^S + d \tag{2}$$

where $\Lambda^S = U(X^*) - X^*$.

For a type-1 producer, the value function is

$$W^{S1}(m) = \max_{X,H}\{U(X) - H\}$$
$$\text{s.t. } X = H + \varphi m$$

which can again be rewritten as

$$W^{S1}(m) = \Lambda^S + \varphi m \tag{3}$$

Having established the CM value functions, we can now discuss the bargaining between a consumer and a producer in the DM. Consumers and producers negotiate over the quantity q to be produced by the producer and the payment to be made to the producer, conditional on which payment(s) are being accepted in this particular meeting. In a type-0 meeting (where producers accept credit), the bargaining problem is given by

$$\max_{q,d}\; u(q) + W(m,d) - W(m,0) \tag{4}$$
$$\text{s.t. } u(q) + W(m,d) - W(m,0) = \frac{\theta}{1-\theta}\{-q + W^{S0}(d) - W^{S0}(0)\}$$

Substituting equations (1) and (2) into equation (4), the bargaining problem can be simplified as

$$\max_{q,d} \; u(q) - d$$
$$\text{s.t. } u(q) - d = \frac{\theta}{1-\theta}(d-q)$$

Lemma 1. *Define $z(q) = (1-\theta)u(q) + \theta q$. The solution to the bargaining problem is:*

$$q = q^* \tag{5}$$

$$d = z(q^*) \tag{6}$$

Proof. The proof is obvious, and hence is omitted. □

The bargaining solution is straightforward, which states that in meetings where producers accept unsecured credit, the first-best quantity q^* should be exchanged, and the producer should be promised a payment d (to take place in the CM) that satisfies the Kalai surplus-splitting constraint.

Now, consider a type-1 meeting. Without credit, consumers will face an additional budget constraint. Whether they will be able to achieve the first-best level of DM consumption now depends on the money they have carried into the DM. Hence in a type-1 meeting, let x be the amount of money that changes hands, the bargaining problem is given by:

$$\max_{q,x} \; u(q) + W(m-x,0) - W(m,0) \tag{7}$$
$$\text{s.t. } \; -q + W^{S1}(x) - W^{S1}(0) = \frac{\theta}{1-\theta}[u(q) + W(m-x,0) - W(m,0)]$$
$$x \leq m$$

From the Kalai constraint, we have $\varphi x = (1-\theta)u(q) + \theta q \equiv z(q)$. By substituting the simplified bargaining constraint, as well as equation (3) into equation (7), we can rewrite the bargaining problem as follows:

$$\max_{q,x} \ \theta[u(q)-q]$$
$$\text{s.t.} \ \varphi x = (1-\theta)u(q) + \theta q$$
$$x \leq m$$

And the bargaining solution can be summarized by Lemma 2.

Lemma 2. *Define $m^* \equiv \{m : \varphi m = z(q^*)\}$, which is the amount of money that allows the consumer to afford the optimal consumption q^*. Then the bargaining solution is given by:*

$$q = \begin{cases} q^*, \text{ if } m \geq m^* \\ \tilde{q}(m) \equiv \{q : \varphi m = z(q)\}, \text{ if } m < m^* \end{cases} \tag{8}$$

$$x = \begin{cases} m^*, & \text{if } m \geq m^* \\ m, & \text{if } m < m^* \end{cases} \tag{9}$$

Proof. The proof is obvious, and hence is omitted. □

The bargaining solution states that in meeting where producers accept money only, if the consumer carries enough money m^*, optimal consumption q^* can be achieved. However, if consumer does not bring enough money, i.e. $m < m^*$, she will give up all her money balance, in exchange for the amount of goods $q(m)$ that satisfies the Kalai surplus-splitting constraint.

Next, we describe the value functions of the DM. Upon entering the DM, the expected value function of a consumer with money holdings m at the beginning of the DM is given by

$$\begin{aligned} V(m) &= (1-\sigma)\Big[u\Big(\tilde{q}(m)\Big) + W(m - x(m), 0)\Big] + \sigma\Big[u(q^*) + W(m, z(q^*))\Big] \\ &= (1-\sigma)\Big[u\Big(\tilde{q}(m)\Big) + \varphi x(m)\Big] + \sigma\Big[u(q^*) - z(q^*)\Big] + W(m, 0) \end{aligned} \tag{10}$$

where the second equality is obtained by substituting the bargaining solutions (5), (6), (8), (9), and CM value functions (2), and (3) into the first equality.

The first part of the value function represents that consumer trade with a type-1 producer, which happens with probability $1-\sigma$, and the second part of the value function captures the trade with a type-0 producer, with probability σ.

3.1.2 Objective Function and Optimal Money Choice

After describing the value functions, we now describe the consumer's choice of $\hat{m}$ in the CM. To derive the consumer's objective function in the CM, we first lead the DM value function (10) by one period, and then substitute it into the CM value function (1). The maximization problem over money choice $\hat{m}$ becomes:

$$\max_{\hat{m}} \Big\{ -\varphi \hat{m} + \beta V(\hat{m}) \Big\} \tag{11}$$

$$= \max_{\hat{m}} \Big\{ -\varphi \hat{m} + (1-\sigma)\beta\Big[u\Big(q(\hat{m})\Big) - \hat{\varphi} x(\hat{m})\Big] + \sigma\beta[u(q^*) - z(q^*)] + \beta W(\hat{m}, 0)\Big\}$$

We collect all items that contain $\hat{m}$, and call the resulting expression $J(\hat{m})$, or the agent's 'objective function'. After simplifying the expression, one can verify that $J(\hat{m})$ adopts the following form:

$$J(\hat{m}) = (-\varphi + \beta\hat{\varphi})\hat{m} + (1-\sigma)\beta\Big[u\Big(\tilde{q}(\hat{m})\Big) - \hat{\varphi} x(\hat{m})\Big] \tag{12}$$

Notice that since it is costly to carry money when the economy is away from the Friedman rule, the consumer will never carry $\hat{m} \geq m^*$. So we can substitue that $q(\hat{m}) = \tilde{q}(\hat{m})$ and the objective function can be rewritten as:

$$J(\hat{m}) = (-\varphi + \beta\hat{\varphi})\hat{m} + (1-\sigma)\beta[u\Big(\tilde{q}(\hat{m})\Big) - \hat{\varphi}x(\hat{m})] \tag{13}$$

In order to simplify things, we focus on the special case where $\theta = 1$, or equivalently, consumer makes take-it-or-leave-it (TIOLI) offer in DM bargaining. The bargaining solution (8) and (9) thus can be simplified as $z(q) = q$ and $\tilde{q}(m) \equiv \{q : \varphi m = q\}$.

Obtaining the first-order condition from the objective function $J(\hat{m})$ yields:

$$\varphi = \beta\hat{\varphi} + \beta(1-\sigma)\Big[\hat{\varphi}u'(\hat{\varphi}\hat{m}) - \hat{\varphi}\Big] = \beta\hat{\varphi}\Big\{1 + (1-\sigma)\Big[u'(\hat{\varphi}\hat{m}) - 1\Big]\Big\}$$

The discussion of the model will focus on steady-state equilibrium, hence the equilibrium condition is given by:

$$\frac{1+\mu-\beta}{\beta} = i = (1-\sigma)[u'(q_1) - 1] \tag{14}$$

where q_1 is the real money balance hence consumption in a type-1 meeting, and i is the interest rate on a perfectly safe, yet illiquid asset.[2]

Definition 1. *Let q_i stands for the quantity of special good traded in a type-i meeting, with $i = \{0, 1\}$. A steady state equilibrium can be summarized by a pair (q_0, q_1), where in any equilibrium $q_0 = q^*$, and q_1 is given by the solution to equation (14).*

We have $q_0 = q^*$, since in every type-0 meeting that accepts credit, consumers can always consume first-best quantity q^*. In turn, q_1 is determined by equation (14), so it depends on parameters of the model, including the policy parameter i.

[2] See Geromichalos and Herrenbrueck (2022) for more details.

3.1.3 Welfare Analysis

In this section, we study the the effect of policy parameter, i, and credit acceptability parameter, σ, on social welfare. We define the utilitarian welfare as:

$$\mathcal{W} = \sigma\Big[u(q_0) - q_0\Big] + (1-\sigma)\Big[u(q_1) - q_1\Big] = \sigma\Big[u(q^*) - q^*\Big] + (1-\sigma)\Big[u(q_1) - q_1\Big] \tag{15}$$

One observation from Equation (14) is that, q_1 decreases in i. This is intuitive. A higher interest rate depresses real money balance, which further constrains DM transaction in meetings where credit is not available. Thus for the welfare equation, this implies that that $\frac{\partial \mathcal{W}}{\partial i} \leq 0$. This is the traditional channel that rising interest rate hurts welfare by depressing real money holdings, and hence consumption.

But what about $\frac{\partial \mathcal{W}}{\partial \sigma}$? How would a change in σ, i.e. the probability that a producer accepts credit, affect the welfare? One may expect that, higher acceptability of credit should better facilitate DM trade, hence raise welfare. But in what follows, we will show that this is not always true.

First notice that

$$\frac{\partial \mathcal{W}}{\partial \sigma} = \Big[u(q^*) - q^*\Big] - \Big[u(q_1) - q_1\Big] + (1-\sigma)\Big[u'(q_1) - 1\Big]\frac{dq_1}{d\sigma}$$

and with $q_1 \equiv \{q : i = (1-\sigma)[u'(q) - 1]\}$, we can easily verify that $dq_1/d\sigma = i/(1-\sigma)^2 u''(q_1)$.

Hence, the derivative of welfare with respect to σ becomes:

$$\frac{\partial \mathcal{W}}{\partial \sigma} = \Big[u(q^*) - q^*\Big] - \Big[u(q_1) - q_1\Big] + \left(\frac{i}{1-\sigma}\right)^2 \frac{1}{u''(q_1)} \tag{16}$$

The first terms of equation (16), $[u(q^*) - q^*] - [u(q_1) - q_1]$, is clearly positive, since q^* is the unique maximizer of the surplus $u(q) - q$. And with the economy away from the Friedman rule (i.e., for any $i > 0$), $q_1 < q^*$. However, the last term in equation (16) will be negative since the agent's utility is strictly concave, i.e., $u'' < 0$. This is already providing some intuition about the channels at work. A higher σ benefits welfare as it increases the fraction of meetings in which unsecured credit is accepted and, therefore, the first-best quantity is traded. However, a higher σ also gives agents the (accurate) impression that they will not be needing money as often, which reduces their equilibrium real balances and hurts welfare in meetings where credit is not available. Which of these two forces prevails depends on parameters, but we can provide a sharper characterization of equilibrium if we focus on a specific functional form.

Thus, in what follows, we focus on the quadratic utility function, and assume that $u(q) = -q^2/2 + (1+\gamma)q$. With this quadratic utility function, we can easily verify that $q^* = \gamma$, and $u(q^*) - q^* = \gamma^2/2$. Also, $q_1 = \gamma - i/(1-\sigma)$ and $dq_1/d\sigma = i/(1-\sigma)^2$.

With quadratic utility function, first notice that $u'(q_1) - 1 = \gamma - q_1$, and equation (14) becomes $i = (1-\sigma)(\gamma - q_1)$. Hence for any given level of i, there exists a σ such that $q_1 = 0$ and the equilibrium becomes non-monetary. We define $\bar{\sigma} \equiv 1 - i/\gamma$ as the cutoff level of probability that the producer accepts credit. Then for all $\sigma \in [0, \bar{\sigma})$, equilibrium is monetary, such that consumers will still carry money to consume in the type-1 meeting. When $\sigma \geq \bar{\sigma}$, the cost of carrying money is too high. DM consumption $q_1 \leq 0$, and consumers will not consume in meetings where only money is accepted. We carry out the welfare analysis in both monetary and non-monetary equilibrium.

- If $\sigma \in [0, \bar{\sigma})$, the equilibrium is monetary. In this parameter range, $\partial \mathcal{W}/\partial \sigma < 0$ and $\partial^2 \mathcal{W}/\partial \sigma^2 < 0$, hence welfare is decreasing and concave in σ. We thus verify that in the monetary equilibrium, increasing in the acceptability of credit can actually hurt the welfare.

- If $\sigma \in [\bar{\sigma}, 1]$, the equilibrium is non-monetary. In this parameter range, there will be no trade in type-1 meeting, and the welfare function can be reduced to

$$\mathcal{W}(\sigma) = \sigma[u(\gamma) - \gamma] = \sigma \frac{\gamma^2}{2}$$

In such non-monetary equiloibrium, welfare is linear and increasing in σ. And when all trade accepts credit, i.e., $\sigma = 1$, we have $\mathcal{W}(1) = \gamma^2/2 \geq \mathcal{W}(0)$, with equality only at the Friedman Rule.

We summarize the findings in Proposition 1 and Figure 1.

Proposition 1. *Define $\bar{\sigma} \equiv 1 - \frac{i}{\gamma}$. For $\sigma \in [0, \bar{\sigma})$, the equilibrium is monetary, and welfare is decreasing and concave in σ (the acceptability of credit in the economy); for $\sigma \in [\bar{\sigma}, 1]$, the equilibrium is non-monetary, and welfare is increasing and linear in σ.*

Inspection of the figure makes it obvious that an increase in σ need not be welfare increasing, as common wisdom may suggest. Indeed, if this economy could achieve unsecured credit in every DM meeting (i.e., $\sigma = 1$), welfare would be maximized. However, if the economy starts with a small measure of producers who accept credit (any $\sigma < \bar{\sigma}$), an increase in credit availability (i.e., a small increase in σ) would certainly hurt welfare.

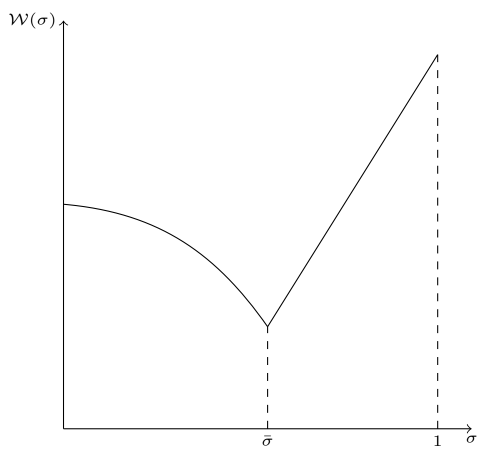


Figure 1: Welfare as a function of σ

3.2 Money and Assets

3.2.1 Value Functions and Bargaining Solution

We now move to the second type of competing medium of exchange: assets. We assume that there exists an asset, with fixed supply A, that pays 1 unit of numeraire good in period

$t+1$. Consumers can purchase such asset in the CM of period t at the given price ψ. We will maintain the σ notation, but this time it will stand for the fraction of producers who accept both money and assets as media of exchange. Then, $1-\sigma$ will be the number of producers that accept only money.[3]

The CM value functions for both consumers and producers are similar to section (3.1), in the sense that they are linear in all arguments. In this version of the model, there are two types of DM meetings: in type-1 meetings, producers accept only money; while in type-2 meetings, producers accept both money and assets.[4] With similar set up as in section (3.1), consumer's CM value function is

$$W(m,a) = \varphi m + a + \Lambda \tag{17}$$

where $\Lambda = U(X^*) - X^* + T + \max_{\hat{m},\hat{a}}\{-\varphi\hat{m} + \beta V(\hat{m},\hat{a})\}$.

Type-1 meeting producer's CM value function is

$$W^{S1}(m) = \Lambda^S + \varphi m \tag{18}$$

And type-2 producer's CM value function is

$$W^{S2}(m,a) = \Lambda^S + \varphi m + a \tag{19}$$

With consumers making TIOLI offer in the DM, the bargaining solution is summarized in Lemma (3).

Lemma 3. *In type-1 meeting, where only money is accepted as the proper medium of*

[3] This version of the model coincides with the model of Lester et al. (2012). However the authors in that paper do not examine how welfare is affected by the fraction of producers who accept assets, which the central question for us.

[4] Thus, the mnemonic rule remains the same as the one described in footnote 1: the index 1 or 2 stands for the number of assets traded in this type of meeting.

exchange, the bargaining solution are as follows

$$q = \begin{cases} q^*, & \text{if } m \geq \frac{q^*}{\varphi} \\ \varphi m, & \text{if } m < \frac{q^*}{\varphi} \end{cases} \tag{20}$$

$$d = \begin{cases} m^*, & \text{if } m \geq \frac{q^*}{\varphi} \\ m, & \text{if } m < \frac{q^*}{\varphi} \end{cases} \tag{21}$$

In type-2 meetings, both money and assets can be used as medium of exchange. Let d_a and d_m be the amount of assets and money changed hands in the DM meeting, respectively. The bargaining solution is given by

$$\text{If } a + \varphi m \geq q^*, \begin{cases} q = q^* \\ d_a + \varphi d_m = q^* \end{cases} \tag{22}$$

$$\text{If } a + \varphi m < q^*, \begin{cases} q = a + \varphi m \\ d_m = m \\ d_a = a \end{cases} \tag{23}$$

Proof. The proof is straightforward, hence omitted. □

Notice that in a type-2 meeting, since both money and assets can be used to purchase DM goods, what matters is whether the total liquidity (money and asset together) is enough to allow for the first-best level of consumption, q^*. When the total liquidity is abundant, consumers are willing to give any combination of money and asset to exchange for q^*. If total liquidity is scarce, consumers will give up all the money and asset in exchange for the equal amount of DM consumption.

3.2.2 Objective Function and Optimal Behavior

The objective function of the typical consumer is

$$\begin{aligned} J(\hat{m}, \hat{a}) =& (-\varphi + \beta\hat{\varphi})\hat{m} + (-\psi + \beta)\hat{a} + \beta(1-\sigma)\Big[u\Big(q_1(\hat{m})\Big) - \hat{\varphi}d(\hat{m})\Big] \\ &+ \beta\sigma\Big[u\Big(q_2(\hat{m}, \hat{a})\Big) - d_a(\hat{m}, \hat{a}) - \hat{\varphi}d_m(\hat{m}, \hat{a})\Big] \end{aligned} \tag{24}$$

where $q_1(\hat{m})$ is the amount of DM consumption when meeting a type-1 producer, and $q_2(\hat{m}, \hat{a})$ is the DM consumption when meeting a type-2 producer.

As discussed in section (3.1.2), it is sub-optimal to bring $\hat{m} > q^*/\hat{\varphi}$, i.e., more money than what is needed to achieve first-best consumption q^*, so we always have $\hat{\varphi}\hat{m} \leq q^*$. But it is possible that total liquidity from both money and asset together is greater than q^*. Hence we discuss consumers' optimal behavior under two cases: (1) total liquidity is scarce, i.e., $\hat{a} + \hat{\varphi}\hat{m} < q^*$; (2) total liquidity is plentiful, i.e., $\hat{a} + \hat{\varphi}\hat{m} \geq q^*$. Eventually, which of the two cases is relevant will depend on parameters of the model, and we provide more details below.

Lemma 4. *Taking prices $(\hat{\varphi}, \hat{\psi})$ as given. The optimal choice of a representative consumer in case 1 satisfies the following conditions:*

$$\varphi = \beta\hat{\varphi}\Big\{1 + (1-\sigma)\Big[u'(\hat{\varphi}\hat{m}) - 1\Big] + \sigma\Big[u'(\hat{a} + \hat{\varphi}\hat{m}) - 1\Big]\Big\} \tag{25}$$

$$\psi = \beta\Big\{1 + \sigma\Big[u'(\hat{a} + \hat{\varphi}\hat{m}) - 1\Big]\Big\} \tag{26}$$

The optimal choice of a representative consumer in case 2 satisfies the following conditions:

$$\varphi = \beta\hat{\varphi}\Big\{1 + (1-\sigma)\Big[u'(\hat{\varphi}\hat{m}) - 1\Big]\Big\} \tag{27}$$

$$\psi = \beta \tag{28}$$

Proof. See Appendix A.1. □

Same as in the previous sections, we define $z \equiv \varphi m$ as the real money balances, which is determined by the policy rate i. In a type-1 meeting which only accepts money as the medium of exchange, total DM consumption is determined as $q_1 = z$. In a type-2 meeting which accepts both money and assets, after imposing the market clearing condition of $\hat{a} = A$, DM consumption is determined as $q_2 = z + a$ if total liquidity is scarce, or $q_2 = q^*$ if total liquidity is plentiful. Our discussion again focuses on the steady state equilibrium.

Definition 2. *A steady state equilibrium is a pair* (z,ψ)*. If* $z + A \geq q^*$*,* (z,ψ) *solves:*

$$i = (1-\sigma)\Big[u'(z) - 1\Big] \tag{29}$$

$$\psi = \beta \tag{30}$$

If $z + A < q^*$*,* (z,ψ) *solves:*

$$i = (1-\sigma)\Big[u'(z) - 1\Big] + \sigma\Big[u'(A+z) - 1\Big] \tag{31}$$

$$\psi = \beta\Big\{1 + \sigma\Big[u'(A+z) - 1\Big]\Big\} \tag{32}$$

Here we focus on the interesting case where $A < q^*$, so that asset alone is not enough to allow consumers to consume q^*. Since real money balance z (hence the total liquidity) depends on the policy parameter i, now the question is, for what parameter values are we in each of the equilibrium cases described in Definition (2)? For any given level of A, the critical level of z is which the total liquidity is just enough to purchase optimal q^*, i.e., $q_2 = A + z \to q^*$ or $u'(A+z) \to 1$. To find such z, we look at equation (29) and define

$$\tilde{i} \equiv (1-\sigma)[u'(q^* - A) - 1] \tag{33}$$

which is the cutoff value of interest rate i such that the real money balance z, together with assets, is just enough to allow for optimal consumption q^*.

For any given $A < q^*$, if $i \leq \tilde{i}$, then cost of holding money is relatively low, and consumers' total liquidity (real money balance and assets) is enough for consuming q^*. In this case, the marginal benefit of carrying additional unit of asset is 1, hence asset is always priced at the fundamental value. Hence if $i \leq \tilde{i}$, (z, ψ) are determined by equations (29) and (30). Oppositely, if $i > \tilde{i}$, the real money balance is too low and (z, ψ) are determined by equations (31) and (32).

To study welfare with a closed-form solution, we again focus on the quadratic utility function $u(q) = -q^2/2 + (1+\gamma)q$. Under the stated utility function, the cutoff level of interest rate is now $\tilde{i} = A(1-\sigma)$.

As discussed before, we analyze welfare implication in both monetary and non-monetary equilibrium. To do so, we start by finding $\bar{i}$, the upper bound of i for which monetary equilibrium exists. In other word, we want to find $\bar{i}$, such that if $i \geq \bar{i}$, $z = 0$. By analyzing equation (31), as this is the relevant condition for when z is small, the corresponding $\bar{i}$ under quadratic utility function is $\bar{i} = \gamma - \sigma A$. Since $\tilde{i} < \bar{i}$ by definition, we must have that $A < \gamma$. Under such parameter restriction, the total liquidity in type-2 meeting can be either plentiful or scarce in monetary equilibrium:

Plentiful: If $i \in (0, \tilde{i}]$, the equilibrium is monetary, and total liquidity allows consumers to consume first-best q^* in type-2 DM meeting. Hence

$$\psi = \beta \tag{34}$$

$$z = \gamma - \frac{i}{1-\sigma} \tag{35}$$

Scarce: If $i \in (\tilde{i}, \bar{i})$, the equilibrium is monetary, but total liquidity is scarce so that consumers consume less than q^* in type-2 DM meeting. Hence

$$\psi = \beta\Big\{1 + \sigma\Big[i - A(1-\sigma)\Big]\Big\} \tag{36}$$

$$z = \gamma - i - \sigma A \tag{37}$$

In what follows, we assume the case that $i < \gamma$, or otherwise there is no monetary equilibrium even when $A = 0$, which would not be interesting for analysis.

3.2.3 Welfare Analysis

Now we are ready to discuss the impact of asset acceptability, σ, on welfare. With the definition of q_1 and q_2 the same as in section (3.2.2), the welfare function is

$$\mathcal{W} = (1-\sigma)\Big[u(q_1) - q_1\Big] + \sigma\Big[u(q_2) - q_2\Big] \Rightarrow \mathcal{W}(\sigma) = (1-\sigma)\Big[u(z) - z\Big] + \sigma\Big[u(z+A) - (z+A)\Big]$$

and we are interested in $\partial\mathcal{W}/\partial\sigma$. First of all, when σ changes, there is a direct effect on welfare through the numbers of various meetings. But there is also an indirect effect, through money demand. Formally,

$$\begin{aligned}\frac{\partial\mathcal{W}}{\partial\sigma} &= -[u(z) - z] + (1-\sigma)[u'(z) - 1]\frac{dz}{d\sigma} + [u(z+A) - (z+A)] + \sigma[u'(z+A) - 1]\frac{dz}{d\sigma} \\ &= \underbrace{\Big[u(z+A) - (z+A)\Big] - \Big[u(z) - z\Big]}_{+} + \underbrace{\Big\{(1-\sigma)\Big[u'(z) - 1\Big] + \sigma\Big[u'(z+A) - 1\Big]\Big\}}_{+}\underbrace{\frac{dz}{d\sigma}}_{-}\end{aligned}$$

Similar to the discussion in Section (3.1) with money and credit, the impact on welfare from the increasing in σ might not be positive as many would predict. To see the behavior of $\partial\mathcal{W}/\partial\sigma$, we again focus on quadratic utility form, and discuss it for different range of σ value.

We start with the value of $\sigma = 0$, in which case asset cannot serve the liquidity role in DM at all. In this case, $\bar{i} = \gamma$, and equilibrium will be monetary for all value of i, i.e., $i \in [0, \gamma)$.

$\tilde{i}$ now is irrelevant because the amount of liquidity is always scarce when money is the only acceptable medium of exchange. Here z is determined by equation (37), or $z = \gamma - i$. Hence the welfare when $\sigma = 0$ is:

$$\mathcal{W}(0) = u(\gamma - i) - (\gamma - i) = \gamma(\gamma - i) - \frac{(\gamma - 1)^2}{2} = \frac{\gamma^2 - i^2}{2}$$

As σ increases, asset starts to play a liquidity role. In the cases where σ takes positive values, we first examine the behavior at the other extreme, i.e. $\sigma = 1$, and then come back to the interior value of $\sigma \in (0,1)$. With $\sigma = 1$, $\mathcal{W}(1) = u(z + A) - (z + A)$, $\tilde{i} = 0$, and $\bar{i} = \gamma - A$. In order to calculate $\mathcal{W}(1)$, we need to know the value of real money balance, z, and hence discuss both cases when the equilibrium is monetary and when the equilibrium is non-monetary.

With $\sigma = 1$, when $i \leq \bar{i}$, the equilibrium is monetary. But since $\tilde{i} = 0$, for all $i \in [0, \bar{i})$, total liquidity is scarce. Hence real money balance, z, is determined by equation (37), or $z = \gamma - i - A$. In this case, the welfare is given as

$$\mathcal{W}(1) = u(\gamma - i) - (\gamma - i) = \frac{\gamma^2 - i^2}{2} = \mathcal{W}(0)$$

When $i > \bar{i}$, the equilibrium is non-monetary. And the welfare is given as

$$\mathcal{W}(1) = u(A) - A = A(\gamma - \frac{A}{2})$$

and we claim that, in this case, $\mathcal{W}(1) > \mathcal{W}(0)$.

Proof. Define $G(i) \equiv \mathcal{W}(1) - \mathcal{W}(0) = A(\gamma - \frac{A}{2}) - \frac{\gamma^2 - i^2}{2}$. $G(i)$ is continuous and increasing in i, and $G(\gamma - A) = A\gamma - \frac{A^2}{2} - \frac{\gamma^2 - (\gamma - A)^2}{2} = 0$. Hence $G(i) > 0$ for all $i > \gamma - A = \bar{i}$, and $\mathcal{W}(1)|_{NME} > \mathcal{W}(0)$. □

The lesson is that, if i is small enough, the cost of carrying money is relatively small. Thus even when all producers accept assets, i.e. $\sigma = 1$, consumers could still want to hold some money, and $\mathcal{W}(0) = \mathcal{W}(1)$. Then as i increases, both $\mathcal{W}(1)$ and $\mathcal{W}(0)$ decreases. However, for $\mathcal{W}(1)$ there is a lower bound on how much it can decrease as i increases: if i becomes so high that a monetary equilibrium ceases to exist (for $\sigma = 1$), consumers can still use assets for consumption, hence $\mathcal{W}(1)$ cannot go below $u(A) - A$. But when assets are not accepted at all, i.e. $\sigma = 0$, consumers rely solely on money for DM consumption. Hence as i increases, the value of z, as well as $\mathcal{W}(0)$, keep going down. And this is precisely why $\mathcal{W}(1) > \mathcal{W}(0)$ when $i > \bar{i}$.

Now that the extremes are discussed, we study the interior value of σ. Of course, what happens in the middle depends on whether the equilibrium is plentiful/scarce/non-monetary, which further depends on the value of parameters. Hence to see the behavior of welfare on σ, we discuss that, for any given level of i, what are the cut-off values of σ that divide the economy into different types of equilibria.

Using the similar notation, we use $\tilde{\sigma}$ to denote the threshold value of σ that separates monetary equilibrium into plentiful case and scarce case. By investigating equations (35) and (37), we can verify that such $\tilde{\sigma} = 1 - i/A$. Since σ is the fraction of producers who accept both asset and money, σ must be greater than 0. Hence we define $\tilde{\sigma} \equiv \max\{0, (A - i)/A\}$. Similarly, we use $\bar{\sigma}$ to denote the cut-off value of σ that separates the economy into monetary equilibrium and non-monetary equilibrium. By investigating equation (37), we can verify that such $\bar{\sigma} = (\gamma - i)/A$. Hence, given the definition of σ, we define $\bar{\sigma} \equiv \min\{1, (\gamma - i)/A\}$.

Given the definition of $\tilde{\sigma}$ and $\bar{\sigma}$, first notice that $\bar{\sigma} > \tilde{\sigma}$. In addition, if $\sigma \geq \tilde{\sigma}$, the equilibrium is monetary and plentiful; if $\sigma \in (\tilde{\sigma}, \bar{\sigma})$, the equilibrium is monetary and scarce; if $\sigma \geq \bar{\sigma}$, equilibrium is non-monetary. And there exist four possible cases that satisfy $\bar{\sigma} > \tilde{\sigma}$ with $\bar{\sigma} \in [0, 1]$ and $\tilde{\sigma} \in [0, 1]$.

Case 1. $0 < \tilde{\sigma} < \bar{\sigma} < 1$. This is true if $A > \gamma/2$ and $\gamma - A < i < A$.

Case 2. $0 < \tilde{\sigma} < \bar{\sigma} = 1$. This is true if (1) $A > \gamma/2$ and $i \leq \gamma - A$, or (2) $A < \gamma/2$ and $i < A$.

Case 3. $0 = \tilde{\sigma} < \bar{\sigma} < 1$. This is true if (1) $A > \gamma/2$ and $i \geq A$, or (2) $A < \gamma/2$ and

$i > \gamma - A$.

Case 4. $0 = \tilde{\sigma} < \bar{\sigma} = 1$. This is true if $A \leq \gamma/2$ and $i \in [A, \gamma - A]$ and $\sigma \in [0, 1]$.

Proof. See Appendix A.2. □

After specifying the four cases, we study the behavior of $\mathcal{W}(\sigma)$ for all $\sigma \in (0, 1)$. The impact of improved asset acceptability on welfare is summarized in Proposition 2, which again shows that a better asset acceptability does not necessarily improve welfare.

Proposition 2. *Define* $\bar{\sigma} \equiv \min\{1, (\gamma - i)/A\}$, $\tilde{\sigma} \equiv \max\{0, (A - i)/A\}$. *The equilibrium can be summarized as the following four cases*

- ***Case 1:*** *For all* $A > \gamma/2$ *and* $i \in (\gamma - A, A)$, *if*
  - $\sigma \in [0, \tilde{\sigma}]$, *equilibrium is plentiful;* $\mathcal{W}(\sigma)$ *is decreasing and concave in* σ*;*
  - $\sigma \in (\tilde{\sigma}, \bar{\sigma})$, *equilibrium is scarce; furthermore, if*
    - *(i)* $\tilde{\sigma} < 1/2 < \bar{\sigma}$, $\mathcal{W}$ *has a unique minimizer at* $\sigma = 1/2$*;*
    - *(ii)* $1/2 < \tilde{\sigma} < \bar{\sigma}$, $\mathcal{W}$ *has a unique minimizer at* $\sigma = \tilde{\sigma}$*;*
    - *(iii)* $\tilde{\sigma} < \bar{\sigma} < 1/2$, $\mathcal{W}$ *has a unique minimizer at* $\sigma = \bar{\sigma}$.
  - $\sigma \in [\bar{\sigma}, 1]$, $\mathcal{W}$ *is linear and increasing in* σ.
- ***Case 2:*** *For all* $A > \gamma/2$ *&* $i \leq \gamma - A$, *or* $A < \gamma/2$ *&* $i < A$, *if*
  - $\sigma \in [0, \tilde{\sigma}]$, *equilibrium is plentiful;* $\mathcal{W}$ *is decreasing and concave in* σ*;*
  - $\sigma \in (\tilde{\sigma}, 0]$, *equilibrium is scarce;* $\mathcal{W}$ *is convex in* σ, *and if*
    - *(i)* $\tilde{\sigma} < 1/2$, $\mathcal{W}$ *has a unique minimizer at* $\sigma = 1/2$*;*
    - *(ii)* $\tilde{\sigma} > 1/2$, $\mathcal{W}$ *has a unique minimizer at* $\sigma = \tilde{\sigma}$
- ***Case 3:*** *For all* $A > \gamma/2$ *&* $i \geq A$, *or* $A < \gamma/2$ *&* $i > \gamma - A$, *and if*
  - $\sigma \in [0, \bar{\sigma})$, *equilibrium is scarce;* $\mathcal{W}$ *is convex in* σ, *and if*

(i) $\bar{\sigma} > 1/2$*,* $\mathcal{W}$ *has a unique minimizer at* $\sigma = 1/2$*;*

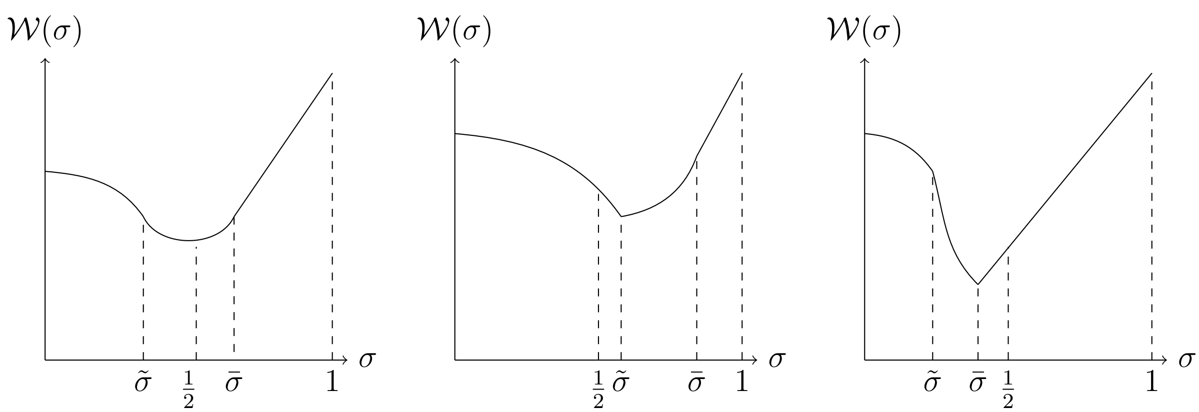
(ii) $\bar{\sigma} < 1/2$*,* $\mathcal{W}$ *has a unique minimizer at* $\sigma = \bar{\sigma}$*;*

- $\sigma \in [\bar{\sigma}, 1]$*, equilibrium is non-monetary;* $\mathcal{W}$ *is increasing and linear in* σ*.*

- ***Case 4:*** *For all* $A \leq \gamma/2$ *and* $i \in [A, \gamma - A]$*, and for all* $\sigma \in [0, 1]$*, equilibrium is scarce;* $\mathcal{W}$ *is convex in* σ*, with a unique minimizer at* $\sigma = 1/2$*.*

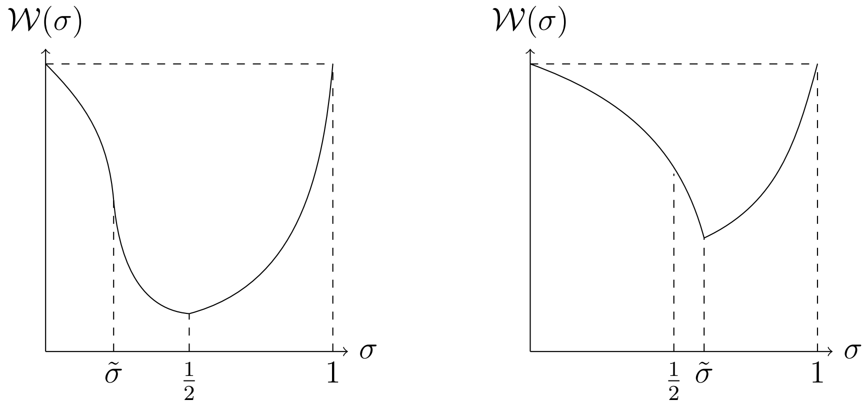
Proof. See Appendix A.3. □

Proposition 2 shows that asset acceptability is not always welfare improving. Welfare as a function of asset acceptability in the four different cases are illustrated by Figure 2, 3, 4, and 5 respectively.



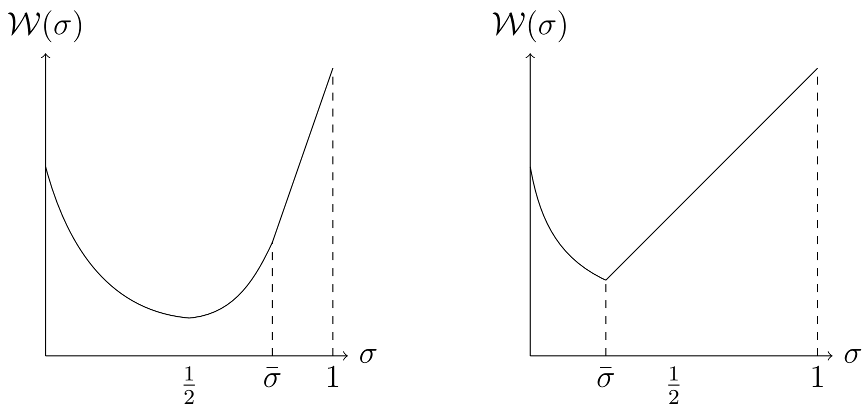
(a) Case 1.1 (b) Case 1.2 (c) Case 1.3

Figure 2: Case 1 Welfare



(a) Case 2.1 (b) Case 2.2

Figure 3: Case 2 Welfare



(a) Case 3.1 (b) Case 3.2

Figure 4: Case 3 Welfare

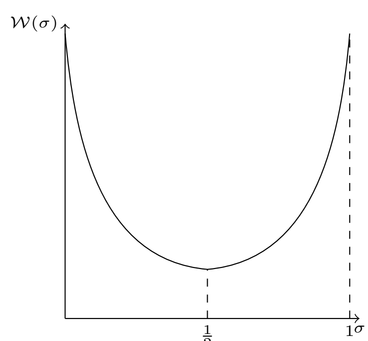


Figure 5: Case 4 Welfare

4 Model with a Secondary Market

In this section, we extend the model to include a secondary asset market. We consider the scenario where a consumption shock is realized at the beginning of each period. With the realization of this shock, ℓ fraction of consumers will have consumption opportunity (henceforth called C-type) in the DM, while the remaining $1 - \ell$ fraction do not have such opportunity (henceforth called the N-type). The consumers find out about the consumption opportunity after they choose their portfolio. Since C-type consumers have additional consumption opportunity in the DM, they need extra medium of exchange. Hence, we add a secondary asset market, which opens before DM, to allow consumers to rebalance their portfolio.

In this setting, we consider two types of secondary markets: secondary market for assets, and secondary market for loans. With the asset secondary market, C-type sells bonds for cash; with the loan secondary market, C-type would receive an unsecured loan. Both types of secondary markets will be over-the-counter market (henceforth OTC market) with bilateral meetings. We conduct welfare analysis with respect to the probability of matching in the secondary market.

4.1 Uncollateralized Loans

4.1.1 Value Functions and Bargaining Solutions

We first consider the scenario where C-type can obtain uncollateralized loans in the OTC market. Let d be the amount of debt that C-type takes in the OTC, which need to be paid back in the upcoming CM. Let $\mathbb{E}\Omega(\hat{m})$ be the expected OTC value function next period when the consumer decides to carry $\hat{m}$ amount of money. Then the Bellman equation of a typical consumer in the CM is given by:

$$W(m,d) = \max_{\hat{m},X,H}\{X - H + \beta\mathbb{E}\Omega(\hat{m})\}$$
$$\text{s.t. } X + \varphi\hat{m} + d = \varphi m + T + H$$

As is standard in the literature, the choice variables will be independent of the current state variables, and the CM value function adopts the form:

$$W(m,d) = \varphi m - d + \Lambda \tag{38}$$

where $\Lambda = U(X^*) - X^* + T + \max_{\hat{m}}\{-\varphi\hat{m} + \beta\mathbb{E}\Omega(\hat{m})\}$.

Now considering a producer entering the CM with m amount of money balance that she obtained from the DM production. The producer's CM value function adopts the form:

$$W^S(m) = \varphi m + \Lambda^S \tag{39}$$

where $\Lambda^S = U(X^*) - X^*$.

For a C-type consumer entering the DM with portfolio (m,d), let $q(m)$ be the amount of special goods that producer produces, and $p(m)$ be the amount of money that C-type pays in return. We will see later that both q and p are functions of C-type consumer's money holding, m. C-type consumer's DM value function is:

$$V(m,d) = u\Big(q(m)\Big) + W\Big(m - p(m), d\Big) \tag{40}$$

Since the producer never carries any money, producer's DM value function is:

$$V^S = -q(m) + W^S\Big(p(m)\Big)$$

One thing to notice before further discussion is that, since it is costly to carry money, consumers will never carry more money than needed, i.e., the remaining real money balance after DM consumption is always 0.

The DM bargaining solution is determined by C-type consumer making a TIOLI offer. The bargaining problem is:

$$\begin{aligned}
&\max_{q,p} \; u(q) + W(m-p,d) - W(m,d) \\
&\text{s.t.} \; -q + W^S(p) - W^S(0) = 0 \\
&\qquad p \leq m
\end{aligned}$$

The bargaining solution is summarized in Lemma 5.

Lemma 5. *Define $\varphi m^* = q^*$ as the amount of real money balance that is required for optimal DM consumption q^*. The DM bargaining solution is as follows*

$$q = \begin{cases} q^*, & \text{if } m \geq m^* = \frac{q*}{\varphi} \\ \varphi m, & \text{if } m < m^* \end{cases}$$

$$p = \begin{cases} m^*, & \text{if } m \geq m^* \\ m, & \text{if } m < m^* \end{cases}$$

Proof. The proof is obvious, hence, omitted. □

After discussing the DM bargaining solution, we proceed to the bilateral meetings in the OTC market. Since C-type and N-type consumers have different liquidity needs, they would like to rebalance their portfolio in the OTC market. More specifically, C-type consumer is going to take loans to finance the upcoming DM consumption, while N-type consumer who does not have consumption need in the upcoming DM will provide liquidity. Hence the bilateral meetings take place between a C-type buyer and an N-type buyer.

Denote C-type's money holding as m, and N-type's money holding as $\tilde{m}$. In the OTC market, C-type and N-type bargain over the amount of loan (money), x, that C-type takes from N-type, and the quantity, d, to be paid back to N-type in the CM. We assume that the bargaining solution in OTC market is determined by C-type making a TIOLI offer. Hence the OTC bargaining problem is to maximize C-type's bargaining surplus, subject to N-type's participation constraint, and N-type's money constraint:

$$\begin{aligned} &\max_{x,d} V(m+x,d) - V(m,0) \\ \text{s.t. } &W(\tilde{m}-x,-d) - W(\tilde{m},0) = 0 \\ &x \leq \tilde{m} \end{aligned}$$

By plugging equations (38) and (40) into the bargaining problem, it can be re-written as:

$$\begin{aligned} &\max_{x,d} u(\varphi(m+x)) - u(\varphi m) - d \\ \text{s.t. } &d = \varphi x \\ &x \leq \tilde{m} \end{aligned}$$

The bargaining solution depends on whether N-type's money constraint binds or not, hence can be discussed in two cases, which are summarized in Lemma 6.

Lemma 6. *Given the same definition of m^* and q^*, the OTC bargaining solution is as follows*

$$x = \begin{cases} m^* - m, \ \ \textit{if } m + \tilde{m} \geq m^* \\ \tilde{m}, \ \ \textit{if } m + \tilde{m} < m^* \end{cases}$$

$$d = \begin{cases} q^* - \varphi m, & \textit{if } m + \tilde{m} \geq m^* \\ \varphi \tilde{m}, & \textit{if } m + \tilde{m} < m^* \end{cases}$$

Proof. Proof is obvious, and hence omitted. $\square$

The bargaining solution is intuitive. If money is abundant, i.e., $\tilde{m} + m \geq m^*$, money constraint is not binding. In this case, C-type would borrow just enough money from N-type to consume the first-best in DM, and the solution is $x = m^* - m$ and $d = q^* - \varphi m$. However, if money is scarce, i.e., $\tilde{m} \leq m^* - m$, N-type does not have enough money to lend to C-type for first-best consumption. Then C-type would want to acquire all money that N-type carries, so $x = \tilde{m}$, and in exchange, C-type will pay back the same amount of real balance $d = \varphi\tilde{m}$ in the upcoming CM.

Given the bargaining solutions, these two cases are not hard to analyze. However, later when we study the model with secondary *asset* market, the analysis would become more complicated. Thus to make further analysis simpler, and to keep symmetry between secondary credit market and secondary asset market, we assume that money constraint never binds, i.e. $m + \tilde{m} \geq m^*$. Given this assumption, the bargaining solution would be restrained to the abundant case only. Hence in all secondary credit market meeting, first-best amount of money will change hands, and C-type will be able to consume first-best, q^*, in all DM meeting. But we will show in later section that this is not always true if the only available secondary market is for assets trading instead of uncollateralized loan. More specifically, C-type might not be able to consume first-best quantity q^* even with plentiful money in the economy. The quantity of special goods consumption would depend also on the asset supply.

With the bargaining solutions in the OTC and DM, as well as the value functions discussed, we now derive the OTC value function of a typical consumer entering the market with money holding m. Let $f(\ell, 1-\ell)$ be the matching function between C-types and N-types, hence also the number of matches in the OTC market. Thus the probability of a typical C-type getting a match in the OTC is $f(\ell, 1-\ell)/\ell$. Thus the value function of a typical consumer entering OTC with money holding m is:

$$\mathbb{E}\Omega(m) = \ell\Big[\frac{f}{\ell}V\Big(m+x, d\Big)+\Big(1-\frac{f}{\ell}\Big)V\Big(m, 0\Big)\Big]+(1-\ell)\Big[\frac{f}{1-\ell}W\Big(m-\tilde{x}, \tilde{d}\Big)+\Big(1-\frac{f}{1-\ell}\Big)W\Big(m, 0\Big)\Big] \tag{41}$$

where variables with tilde denote the quantities being exchanged when the consumer turns out to be an N-type, supplying liquidity in the OTC market. Since it is never optimal for consumers to bring more than enough money into next period, i.e. $m \leq m^*$, DM value function $V(m,d) = u(\varphi m) + W(0,d)$. By substituting equations (38) and (40) into (41), the expected OTC value function becomes:

$$\begin{aligned}\mathbb{E}\Omega(m) =& f\Big[u(q^*) - q^* + \varphi m\Big] + (\ell - f)u(\varphi m) + \Lambda \\ &+ f\Big[\varphi(m - m^* + \tilde{m}) + q^* - \varphi\tilde{m}\Big] + (1 - \ell - f)\varphi m\end{aligned}$$

where the four terms in order represent the benefit of holding money m if the consumer turns out to be a matched C-type, an unmatched C-type, a matched N-type, and an unmatched N-type. By substituting this expression into equation (38), and collecting all terms that contain choice variable $\hat{m}$, the objective function of the typical consumer, $J(\hat{m})$, adopts the following form:

$$J(\hat{m}) = -\varphi\hat{m} + \beta f\Big[u(q^*) - q^* + \hat{\varphi}\hat{m}\Big] + \beta(\ell - f)u(\hat{\varphi}\hat{m}) + \beta(1 - \ell)\hat{\varphi}\hat{m} \tag{42}$$

Lemma 7. *Taking price (φ,$\hat{\varphi}$) as given, the optimal choice of money should satisfy the following condition*

$$\varphi = \beta\hat{\varphi}\Big[f + 1 - \ell\Big] + \beta(\ell - f)u'\Big(\hat{\varphi}\hat{m}\Big)\hat{\varphi} = \beta\hat{\varphi}\Big\{1 + (\ell - f)\Big[u'(\hat{\varphi}\hat{m} - 1)\Big]\Big\}$$

And focusing on steady state equilibrium, the real money balance, z, is determined by

$$i = (\ell - f)[u'(z) - 1]$$

Proof. The proof is straightforward, and hence omitted. □

Before we discuss the impact of matching probability on welfare, there are some key issues we need to address. Recall that we made the assumption that money constraint is not a concern for consumption, i.e. $\hat{m} + \tilde{m} \geq m^*$. In the symmetric equilibrium, we then have $z \geq q^*/2$, or $i \leq \gamma(\ell - f)/2$ with the quadratic utility function. But if this is true, then not only will we never reach a non-monetary equilibrium, but we will never even get close to it. And this might be a problem given how crucial the non-monetary equilibrium was in the previous section. So to make sure that we do not overlook the welfare implication in the non-monetary equilibrium, we first analyze the model without imposing the money abundance assumption.

Let m be C-type's money position, and $\tilde{m}$ be C-type's belief of N-type trading partner's money position. Then from Lemma 6, we know that $x = \min\{m^* - m, \tilde{m}\}$, and $d = \min\{\varphi\tilde{m}, q^* - \varphi m\}$, and the expected OTC value function is

$$\begin{aligned}\mathbb{E}\Omega(m) =& f\Big[u\Big(\varphi(m + x(m,\tilde{m}))\Big) - d(m,\tilde{m})\Big] + (\ell - f)u\Big(\varphi m\Big) \\ &+ f\Big[\varphi\Big(m - x(\tilde{m}, m)\Big) + d(\tilde{m}, m)\Big] + (1 - \ell - f)\varphi m + \Lambda\end{aligned}$$

And the objective function is now

$$J(\hat{m}) = -\varphi\hat{m} + \beta f\Big[u\Big(\hat{\varphi}(\hat{m} + x(\hat{m},\tilde{m}))\Big) - d(\hat{m},\tilde{m})\Big] + \beta(\ell - f)u(\hat{\varphi}\hat{m}) + \beta(1 - \ell)\hat{\varphi}\hat{m}$$

Given the belief $\tilde{m}$ and price $\hat{\varphi}$, we discuss the optimal money choice in two cases, i.e. money constraint binds, and money constraint does not bind. Let q_1 be the DM consumption quantity when C-type is not matched with an N-type in the OTC, and q_2 be the DM consumption quantity when C-type is matched with an N-type.

Definition 3. *In the symmetric steady state equilibrium, we have $\hat{\varphi}\hat{m} = \hat{\varphi}\tilde{m} = z$. An equilibrium is a list $\{z, q_1, q_2\}$ that satisfies the following conditions:*

1. $q_1 = z$, $q_2 = \min\{2z, q^*\}$.

2. When money constraint does not bind (case 1), i.e., $\hat{m} + \tilde{m} \geq q^*$, z solves

$$i = (\ell - f)\Big[u'(z) - 1\Big]. \tag{43}$$

3. When money constraint binds (case 2), i.e., $\hat{m} + \tilde{m} < q^*$, z solves

$$i = f\Big[u'(2z) - 1\Big] + (\ell - f)\Big[u'(z) - 1\Big]. \tag{44}$$

Proof. See Appendix A.4. □

As the real money balance z is determined by the policy rate i, we define $\bar{i}$ as the cutoff interest rate. At $i = \bar{i}$, real money balance is just enough to allow for optimal consumption q^*, hence $q_2 = 2z = q^*$. By observing equation (43), we can verify that such cutoff interest rate satisfy $\bar{i} = (\ell - f)[u'(q^*/2) - 1]$.

If the interest rate is below $\bar{i}$, then it is not very costly to carry money, hence agents would bring enough money to consume first-best q^* in the DM. In this (plentiful) case, we have $q_2 = q^*$, and z solves equation (43). If interest rate goes beyond $\bar{i}$, then agents would not bring enough money to consume q^*. In this (scarce) case, we have $q_2 = 2z$, and z solves equation (44).

With quadratic utility function, we have $\bar{i} = \gamma(\ell - f)/2$. If $i \leq \bar{i}$, the equilibrium is plentiful, we have $q_2 = \gamma$ and $z = \gamma - \frac{i}{\ell - f}$. If $i > \bar{i}$, we have $q_2 = 2z$ where $z = \frac{\gamma\ell - i}{\ell + f}$. Also, we observe that as $i \to \bar{i}$, $z \to \gamma/2$ as it should.

Notice that with Inada condition, we will always have $z > 0$ even when i is huge. However, this is not true under quadratic utility function. Hence there exists an upper bound for i such that if i exceeds such upper bound, the equilibrium becomes non-monetary. If can be easily verify that such upper bound is $i = \gamma\ell$. As the assumption for our model in this section is that the only acceptable medium of exchange in DM is money, non-monetary equilibrium is not very meaningful for analysis, and hence we are going to focus on the monetary equilibrium only.

4.1.2 Welfare Analysis

For OTC market for uncollateralized loan, we are interested in finding out how the matching probability in OTC affects welfare, i.e., $\partial \mathcal{W}(f)/\partial f$. To do this, we discuss that for any given level of interest rate i, how the value of f might get the equilibrium into plentiful (scarce) case. The idea is that, z decreases in f. If f becomes too large (above a certain threshold $\bar{f}$), equilibrium will switch from the 'plentiful' to 'scarce'.

By investigating equation (43), we can verify that $\bar{f} = \ell - 2i/\gamma$, which decreases in i. This result is intuitive. When $i = 0$, $\bar{f} = \ell$. Then for all admissible f, we have $f \leq \bar{f} = \ell$, and equilibrium is always the plentiful case since the cost of carrying money is low. As i increases, $\bar{f}$ decreases, and the equilibrium starts to shift to the 'scarce' case as the real money balance becomes smaller. Also notice that, if $i \geq \ell\gamma/2$, $\bar{f} \leq 0$ and all admissible values of f satisfy $f \geq \bar{f}$, and equilibrium is always in the the scarce case, independent of the value of f.

With the discussion, we define the welfare function as

$$\mathcal{W}(f) = (\ell - f)\Big[u(z) - z\Big] + f\Big[u(q_2) - q_2\Big]$$

hence we have

$$\frac{\partial \mathcal{W}}{\partial f} = u(q_2) - q_2 - \Big[u(z) - z\Big] + \Big\{f\Big[u'(q_2) - 1\Big]\frac{dq_2}{dz} + (\ell - f)\Big[u'(z) - 1\Big]\Big\}\frac{dz}{df}$$

We analyze the welfare behavior in 'plentiful' and 'scarce' equilibrium. The results are summarized in Proposition 3.

Proposition 3. *Define $\bar{f} = \frac{l-2i}{\gamma}$ as the cut-off level of matching probability that divides the economy into 'scarce' and 'plentiful' equilibrium.*

- *If $i < \frac{\gamma\ell}{2}$, then $\bar{f} \in (0, \ell)$,*

  (a) For the value of $f < \bar{f}$, the economy is in 'plentiful' equilibrium, and $\mathcal{W}$ is decreasing and concave in f

  (b) For $\bar{f} \leq f \leq l$, the economy is in 'scarce' equilibrium; moreover, if

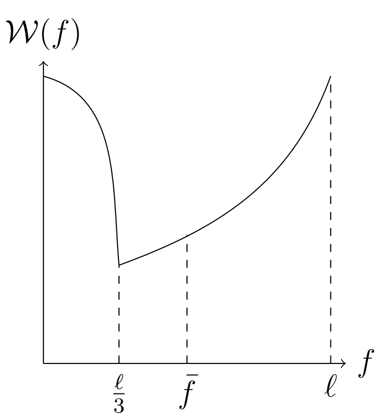
    - *$i \leq \frac{\gamma\ell}{3}$, then $\mathcal{W}$ is increasing in f*
    - *$i \in (\frac{\gamma\ell}{2}, \frac{\gamma\ell}{2})$, then $\mathcal{W}$ is decreasing in f for all $f \in [\bar{f}, \frac{\ell}{2})$; and $\mathcal{W}$ is increasing in f for all $f \in [\frac{\ell}{3}, \ell]$*

- *If $\frac{\gamma l}{3} < i < \frac{\gamma l}{2}$, then $\bar{f} \leq 0$. The economy is in 'scarce' equilibrium, and $\mathcal{W}$ is convex in f, with a unique minimizer at $f = \frac{l}{3}$.*

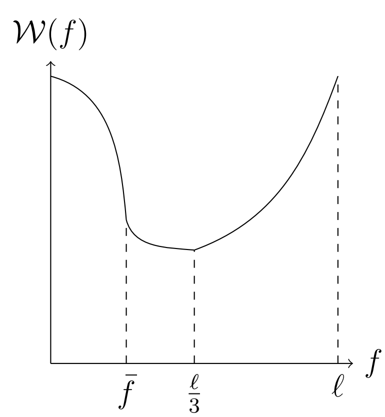
- *$\mathcal{W}(0) = \mathcal{W}(l)$ in both 'plentiful' and 'scarce' equilibrium.*

Proof. See the appendix A.5. □

Figures 6 and 7 give a visual representation of the welfare behavior as a function of matching probability f. Proposition 3 shows that, contrary to the common wisdom, reducing frictions in the OTC market (represented by a higher matching probability) is not always welfare improving.



(a) Case 1.1: $\frac{\ell}{3} < \bar{f}$



(b) Case 1.2: $\frac{\ell}{3} > \bar{f}$

Figure 6: Case 1 Welfare

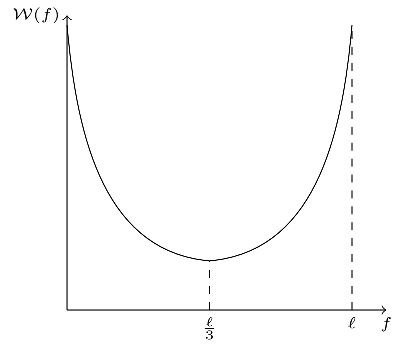


Figure 7: Case 2 Welfare

4.2 Secondary asset market

4.2.1 Value Functions and Bargaining Solution

In this section, we consider the case where the secondary market opens for assets trading, instead of secondary market for uncollateralized loan. C-type consumers can sell assets in the OTC market in exchange for more money, and N-type consumers purchase the asset and provide liquidity in the OTC. In this case, the state variables for a typical consumer entering the CM would be: m, which is the amount of money leftover from previous DM, and a, the amount of asset they consumer carries from last period. And consumers choose the quantity of money and asset, $\hat{m}$ and $\hat{a}$ to bring into next period. So the Bellman equation in the CM is given by

$$W(m,a) = \max_{\hat{m},\hat{a},X,H} \{X - H + \beta \mathbb{E}\Omega(\hat{m})\}$$
$$\text{s.t. } X + \varphi\hat{m} + \varphi\hat{a} = \varphi m + a + T + H$$

and similar to the previous sections, the CM value function of a typical consumer adopts the form

$$W(m,a) = \varphi m + a + T + \max_{\hat{a},\hat{m}} \{-\varphi\hat{m} - \psi\hat{a} + \beta\mathbb{E}\Omega(\hat{m},\hat{a})\} \tag{45}$$

Producer's CM value function is the same as equation (39).

In the DM, C-type consumer makes a TIOLI offer to the producer. They bargain over quantity q to be produced, and d amount of money to be paid to the producer. The value function of a C-type consumer entering the DM with portfolio (m,a) is hence

$$V(m,a) = u(q) + W(m-d,a) \tag{46}$$

And producer's DM value ufnction is

$$V^S = -q + W^S(d)$$

The bargaining problem is given as

$$\begin{aligned} &\max_{q,d}\ u(q) + W(m-d,a) - W(m,a) \\ &\text{s.t.}\ -q + W^S(d) - W^S(0) = 0 \\ &\quad d \leq m \end{aligned}$$

The DM bargaining solution is summarized in Lemma 8.

Lemma 8. *The DM bargaining solution is as follows*

$$q = \begin{cases} q^*, & \textit{if } m \geq m^* = \frac{q*}{\varphi} \\ \varphi m, & \textit{if } m < m^* \end{cases}$$

$$d = \begin{cases} m^*, & \textit{if } m \geq m^* \\ m, & \textit{if } m < m^* \end{cases}$$

Proof. The proof is obvious, hence, omitted. □

In the OTC market, C-type consumer makes a TIOLI offer to an N-type consumer trading counterpart. They bargain over x and χ, which are the quantity of money and asset to be exchanged respectively. Using again $f(l, 1-l)$ as the matching function in the OTC, the value function of a typical consumer entering the OTC with portfolio (m,a) is

$$\begin{aligned}
\mathbb{E}\Omega(m,a) =& \ell\Big[\frac{f}{\ell}V(m+x,a-\chi)+(1-\frac{f}{\ell})V(m,a)\Big] \\
&+(1-\ell)\Big[\frac{f}{1-\ell}W(m-\tilde{x},a+\tilde{\chi})+(1-\frac{f}{1-\ell})W(m,a)\Big] \\
=& f\Big[u\Big(\varphi(m+x)\Big)-\chi\Big]+(\ell-f)u\Big(\varphi m\Big)+f\Big(\tilde{\chi}-\varphi\tilde{x}\Big)+(1-\ell)\varphi m+a+\Lambda
\end{aligned}$$

The OTC bargaining take places between a C-type with portfolio (m, a) and an N-type with portfolio $(\tilde{m}, \tilde{a})$. The bargaining problem is to maximize C-type consumer's bargaining surplus, subject to N-type consumer's participation constraint, and portfolio constraint, i.e., the amount of money exchanged cannot exceed N-type consumer's money position, and the amount of asset exchanged cannot exceed the amount of asset that C-type consumer has.

$$\begin{aligned}
&\max_{x,\chi} V(m+x,a-\chi)-V(m,a) \\
\text{s.t. } & W(\tilde{m}-x,\tilde{a}+\chi)-W(\tilde{m},\tilde{a})=0 \\
& x\leq\tilde{m},\ \chi\leq a
\end{aligned}$$

By substituting equations (45) and (46) into the bargaining problem, it can be verified that the bargaining solution is summarized in Lemma 9. And we can see that, depending on whether the portfolio constraints bind or not, we can end up with different sets of bargaining solutions.

Lemma 9. *Consider a meeting in the OTC market between a C-type and an N-type with portfolios (m,a) and ($\tilde{m}$,$\tilde{a}$), respectively, and define the cutoff level of asset holdings as* $\bar{a} \equiv \min\{\varphi\tilde{m}, q^* - \varphi m\}$. *Then the solution to the bargaining problem in the OTC market is given by*

$$x = \begin{cases} \min\{\tilde{m}, m^* - m\}, & \textit{if } a \geq \bar{a} \\ \frac{a}{\varphi}, & \textit{if } a < \bar{a} \end{cases}$$

$$\chi = \begin{cases} \bar{a}, & \textit{if } a \geq \bar{a} \\ a, & \textit{if } a < \bar{a} \end{cases}$$

Proof. See the appendix A.6. $\square$

We can observe from the bargaining solution that, what matters for the determination of x and χ are: $m, \tilde{m}$, and a. More specifically, N-type's asset position does not matter. Hence both x and χ are potentially functions of $(m, \tilde{m}, a)$, i.e. $x = x(m, \tilde{m}, a)$, and $\chi = \chi(m, \tilde{m}, a)$. Using these notations, the expected OTC value function is given by:

$$\begin{aligned} \mathbb{E}\Omega(m,a) =& f\Big[u(\varphi(m + x(m,\tilde{m},a))) - \chi(m,\tilde{m},a)\Big] + (\ell - f)u\Big(\varphi m\Big) \\ &+ f\Big[\chi(m,\tilde{m},a) - \varphi x(m,\tilde{m},a)\Big] + (1-\ell)\varphi m + a + \Lambda \end{aligned} \tag{47}$$

and the first term in line 2 equals 0 regardless of which region the equilibrium is in (as N-type's bargaining surplus is always 0 under TIOLI bargaining protocol). By substituting equation (47) into equation (45), we get the objective function of a typical consumer, $J(\hat{m}, \hat{a})$, is given by

$$\begin{aligned} J(\hat{m},\hat{a}) =& -\varphi\hat{m} - \psi\hat{a} + \beta\hat{a} + \beta f\Big[u\Big(\hat{\varphi}(\hat{m} + x(\hat{m},\tilde{m},\hat{a}))\Big) - \chi(\hat{m},\tilde{m},\hat{a})\Big] \\ &+\beta(\ell - f)u\Big(\hat{\varphi}\hat{m}\Big) + \beta(1-\ell)\hat{\varphi}\hat{m} \end{aligned}$$

As the bargaining solution depends on whether the money and asset constraints bind or not, we have three potential sets of OTC bargaining solutions. Hence we study the objective function and the pricing functions for each set of bargaining solution. The optimal porfolio choice of a representative consumer for each of the three regions, are summarized in Lemma

10.

Lemma 10. *Using the superscript $i = 1, 2, 3$ to denote the three potential regions, determined by the set of bargaining solution. Taking prices $(\varphi,\hat{\varphi},\psi,\hat{\psi})$ and belief $\tilde{m}$ as given, the optimal choice of a representative agent with portfolio (m,a) satisfies:*

$$\beta^{-1}J_m^1(\hat{m},\hat{a}) = -\frac{\varphi}{\beta} + \hat{\varphi}\Big\{1 + (\ell - f)\Big[u'(\hat{\varphi}\hat{m}) - 1\Big]\Big\}$$
$$\beta^{-1}J_a^1(\hat{m},\hat{a}) = -\frac{\psi}{\beta} + 1$$
$$\beta^{-1}J_m^2(\hat{m},\hat{a}) = -\frac{\varphi}{\beta} + \hat{\varphi}\Big\{1 + f\Big[u'(\hat{\varphi}\hat{m} + \hat{a}) - 1\Big] + (\ell - f)\Big[u'(\hat{\varphi}\hat{m}) - 1\Big]\Big\}$$
$$\beta^{-1}J_a^2(\hat{m},\hat{a}) = -\frac{\psi}{\beta} + \Big\{1 + f\Big[u'(\hat{\varphi}\hat{m} + \hat{a}) - 1\Big]\Big\}$$
$$\beta^{-1}J_m^3(\hat{m},\hat{a}) = -\frac{\varphi}{\beta} + \hat{\varphi}\Big\{1 + f\Big[u'(\hat{\varphi}(\hat{m} + \tilde{m})) - 1\Big] + (l - f)\Big[u'(\hat{\varphi}\hat{m}) - 1\Big]\Big\}$$
$$\beta^{-1}J_a^3(\hat{m},\hat{a}) = -\frac{\psi}{\beta} + 1$$

Proof. See the appendix A.7. □

Again, we focus on symmetric steady state equilibrium, i.e., C-type and N-type carry same amount of assets since they are ex-ante identical. Hence in equilibrium, $\hat{\varphi}\hat{m} = \hat{\varphi}\tilde{m} = z$, $\hat{a} = \tilde{a} = A$. Before we analyze the welfare, we give a characterization of how the regions are divided given the aggregate asset supply A and policy rate i.

Figure 8 shows the aggregate region division on a $(z,\ A)$ plane. If $z + A \geq q^*$, then the asset constraint never binds and C-type gets optimal consumption in the DM, hence the equilibrium is in region 1. If $z + A < q^*$ and $z > q^*/2$, then total money balance is enough to afford q^*, but C-type's assets are not enough to purchase the desired amount of money from N-type, hence equilibrium is in region 2. If $z + A < q^*$ and $z < q^*/2$, assets constraint binds and money is scarce, so $q_2 < q^*$, hence equilibrium is in region 3.

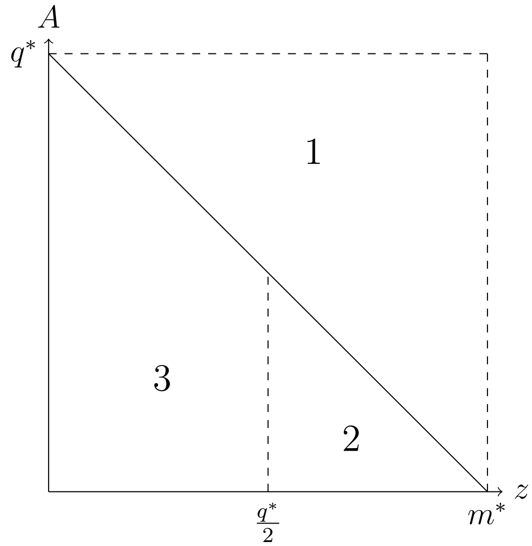


Figure 8: Aggregate Regions on (z,A) plane

As z is further determined by policy rate i, we further analyze the region division on a (i,A) plane. The border that divides region 1 and region 2 is characterized by $z + A \to q^*$. In addition, regions 1 and 2 are cases when the total money balance is abundant, hence the relevant equation that determines z is $i = (\ell - f)[u'(z) - 1]$. Given this, the boundary can be re-written as $(u')^{-1}\Big[1 + i/(\ell - f)\Big] + A = q^*$, which has a positive slope. The intuition is straightforward. Increasing in interest rate i depresses real money balance z, hence it requires a higher A to consume optimal q^*. When $i = 0$, the required amount of asset to consume q^* is $A = 0$. Because at Friedman Rule, it is costless to carry money, agents always hold enough money to consume q^* and does not rely on asset for DM consumption. When $A = q^*$, regardless of the value of i, total balance is always enough to consume q^*, i.e. $z + A = q^*$.

For the boundary between region 2 and region 3, it is defined by the critical point where $z = (u')^{-1}\Big[1 + i/(\ell - f)\Big] \to q^*/2$. This boundary is a perpendicular line since the boundary is independent of A, and the corresponding i that defines such boundary $\tilde{i} \equiv \Big\{i : (u')^{-1}\Big[1 + i/(\ell - f)\Big] = q^*/2\Big\}$.

Figure 9 gives a visual representation of the aggregate regions on a (i, A) plane.

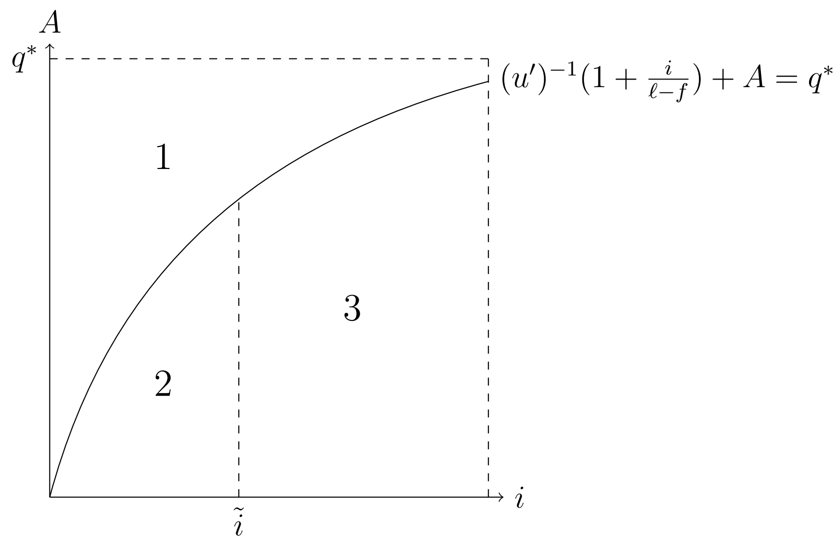


Figure 9: Aggregate Regions on (i,A) plane

Definition 4. *A symmetric steady state equilibrium is a list of* $\{z, q_1, q_2, \psi\}$ *with* $q_1 = z$. *And*

- *If* $A + (u')^{-1}(1 + \frac{i}{\ell-f}) \geq q^*$, *then* $q_2 = q^*$, $\psi = \beta$, *and* z *solves* $i = (\ell - f)[u'(z) - 1]$.
- *If* $A + (u')^{-1}(1 + \frac{i}{\ell-f}) < q^*$ *and* $i \leq \tilde{i}$, *then* $q_2 = z + A$, $\psi = \beta\{1 + f[u'(z + A) - 1]\}$, *and* z *solves* $i = f[u'(z + A) - 1] + (\ell - f)[u'(z) - 1]$.
- *If* $A + (u')^{-1}(1 + \frac{i}{\ell-f}) < q^*$ *and* $i > \tilde{i}$, *then* $q_2 = 2z$, $\psi = \beta$, *and* z *solves* $i = f[u'(2z) - 1] + (\ell - f)[u'(z) - 1]$.

Focusing on the quadratic utility function, the relevant z that separate region 1 and 2 is $z = \gamma - i/(\ell - f)$, and the condition $A + z \geq q^* = \gamma$ is satisfied if $A \geq i/(\ell - f)$. The equilibrium under quadratic utility is defined as follows.

Definition 5. *A symmetric steady state equilibrium is a list of* $\{z, q_1, q_2, \psi\}$ *with* $q_1 = z$. *And*

- *If* $A \geq \frac{i}{\ell-f}$, *then* $q_2 = \gamma$, $\psi = \beta$, *and* $z = \gamma - \frac{i}{\ell-f}$.
- *If* $A < \frac{i}{\ell-f}$ *and* $i \leq \tilde{i} \equiv \frac{\gamma(\ell-f)}{2}$, *then* $q_2 = \gamma - \frac{i}{\ell} + A(\frac{\ell-f}{\ell})$, $z = i - \frac{i+fA}{\ell}$, *and* $\psi = \beta\Big[1 + \frac{f}{\ell}\Big(i - (\ell - f)A\Big)\Big]$.

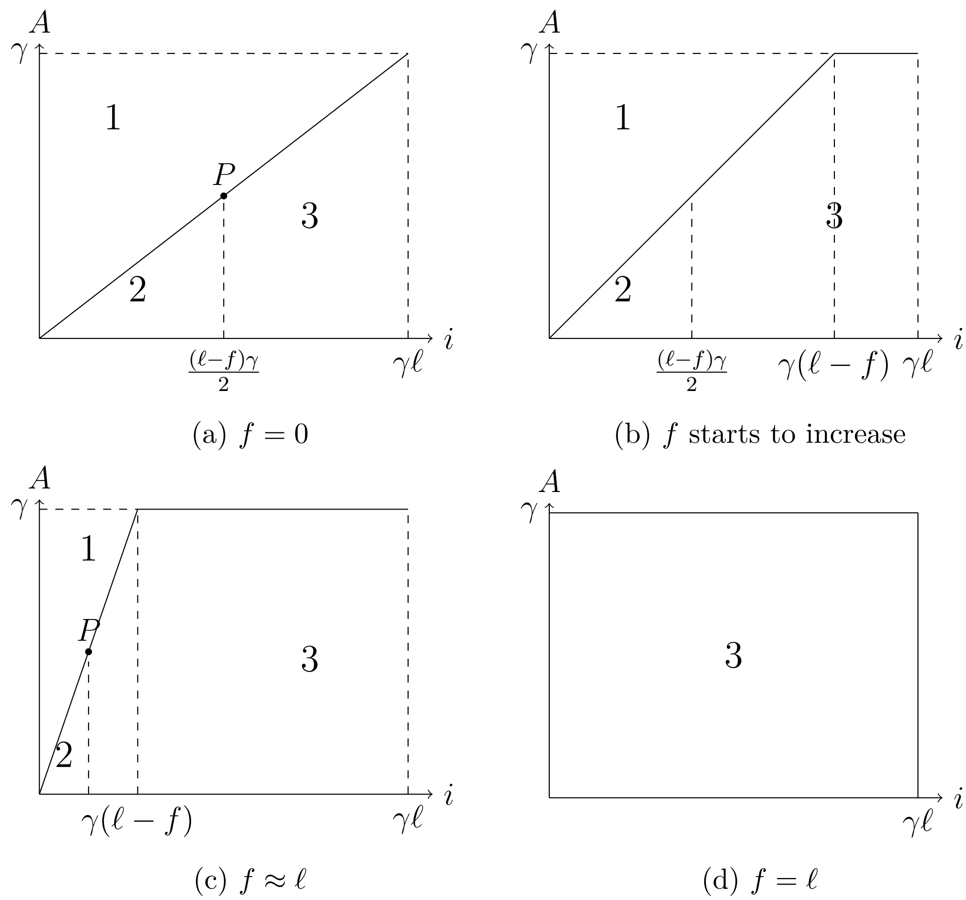
- *If* $A < \frac{i}{\ell - f}$ *and* $i > \frac{\gamma(\ell - f)}{2}$*, then* $q_2 = 2z$*,* $z = \frac{\ell\gamma - i}{f + \ell}$*, and* $\psi = \beta$*.*

4.2.2 Welfare Analysis

After specifying the aggregate regions, now we proceed to analyze welfare. Same as in section (4.1), the impact of f on welfare is given by

$$\frac{\partial \mathcal{W}}{\partial f} = u(q_2) - q_2 - \Big[u(z) - z\Big] + \Big\{f\Big[u'(q_2) - 1\Big]\frac{dq_2}{dz} + (\ell - f)\Big[u'(z) - 1\Big]\Big\}\frac{dz}{df}$$

To see how change in f affects welfare, we need to figure out how the different regions are affected by the change in f. Before detailed discussion, we make a couple of clarifying notes. z decreases in f for all regions, hence it is more likely that the economy is in 'scarce' equilibrium as f increases. This is illustrated by Figure 10. For any given level of A, as f increases, the equilibrium is more likely to be 'scarce'. For example, point P in panel (a) corresponds to a 'plentiful' equilibrium, while the same (i, A) value would corresponds to a 'scarce' equilibrium in panel (c) due to the increase in f.



(a) $f = 0$ (b) f starts to increase

(c) $f \approx \ell$ (d) $f = \ell$

Figure 10: Regions division as f increases

Now, we describe each region with parameters A and i, and analyze the welfare.

Proposition 4. *The equilibrium welfare depends on the value of A and i, which are divided into four cases:*

- *Case 1: If $A > \frac{\gamma}{2}$ and $A > \frac{i}{l}$, then there exists a unique cutoff level of matching probability $f_{13} = l - \frac{2i}{\gamma}$ such that if $f \in [0, f_{13})$, the equilibrium is in Region 1; if $f \in [f_{13}, l]$, equilibrium is in Region 3. Furthermore,*
  - *If $i > \frac{\gamma l}{3}$, then $\frac{\partial \mathcal{W}}{\partial f} < 0$ for all $f \in [0, f_{13})$; $\frac{\partial \mathcal{W}}{\partial f} > 0$ for all $f \in [f_{13}, l]$.*
  - *If $i < \frac{\gamma l}{3}$, then $\frac{\partial \mathcal{W}}{\partial f} < 0$ for all $f \in [0, \frac{l}{3})$; $\frac{\partial \mathcal{W}}{\partial f} > 0$ for all $f \in [\frac{l}{3}, l]$.*
- *Case 2: If $A < \frac{\gamma}{2}$, $A > \frac{i}{\gamma}$, and $i < \frac{\gamma l}{2}$, then there exist cutoff levels of matching probability $f_{12} = l - \frac{i}{A}$ and $f_{23} = \frac{\gamma l - i}{A} - l$ such that if $f \in [0, f_{12})$, equilibrium is in*

Region 1; if $f \in [f_{12}, f_{23})$*, equilibrium is in Region 2; if* $f \in [f_{23}.l]$*, equilibrium is in Region 3. Furthermore,*

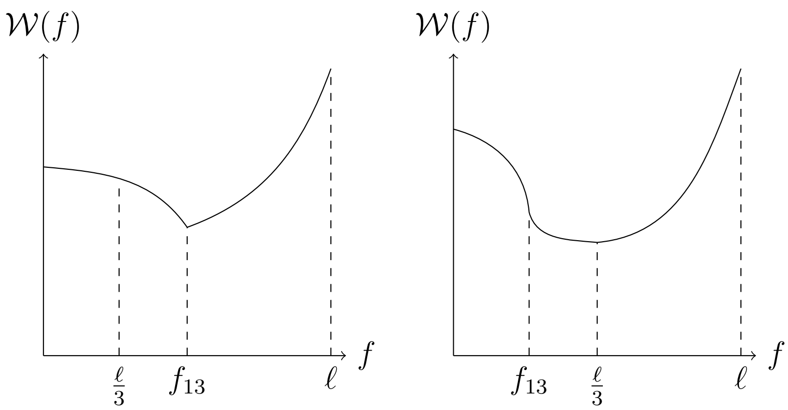
- *If* $\frac{l}{2} < f_{12}$*, then* $\frac{\partial \mathcal{W}}{\partial f} < 0$ *for all* $f \in [0, f_{12}]$*; and* $\frac{\partial \mathcal{W}}{\partial f} > 0$ *for all* $f \in (f_{12}, l]$*.*
- *If* $\frac{l}{2} \in (f_{12}, f_{23})$*, then* $\frac{\partial \mathcal{W}}{\partial f} < 0$ *for all* $f \in [0, \frac{l}{2}]$*;* $\frac{\partial \mathcal{W}}{\partial f} > 0$ *for all* $f \in (\frac{l}{2}, l]$*.*
- *If* $f_{12} < \frac{l}{3} < f_{23} < \frac{l}{2}$*, then* $\frac{\partial \mathcal{W}}{\partial f} < 0$ *for all* $f \in [0, f23]$*;* $\frac{\partial \mathcal{W}}{\partial f} > 0$ *for all* $f \in (f_{23}, l]$*.*
- *If* $f_{12} < f_{23} < \frac{l}{3} < \frac{l}{2}$*, then* $\frac{\partial \mathcal{W}}{\partial f} < 0$ *for all* $f \in [0, \frac{l}{3}]$*;* $\frac{\partial \mathcal{W}}{\partial f} > 0$ *for all* $f \in (\frac{l}{3}, l]$*.*

- *Case 3: If* $A < \frac{i}{l}$ *and* $i < \frac{\gamma l}{2}$*, there exists a cutoff level of matching probability* $f_{23} = \frac{\gamma l - i}{A} - l$ *such that if* $f \in [0, f_{23})$*, equilibrium is in Region 2; if* $f \in [f_{23}, l]$*, equilibrium is in Region 3. Furthermore,*
  - *If* $\frac{l}{2} < f_{23}$*, then* $\frac{\partial \mathcal{W}}{\partial f} < 0$ *for all* $f \in [0, \frac{l}{2}]$*; and* $\frac{\partial \mathcal{W}}{\partial f} > 0$ *for all* $f \in (\frac{l}{2}, l]$*.*
  - *If* $\frac{l}{3} < f_{23} < \frac{l}{2}$*, then* $\frac{\partial \mathcal{W}}{\partial f} < 0$ *for all* $f \in [0, f_{23})$*; and* $\frac{\partial \mathcal{W}}{\partial f} > 0$ *for all* $f \in [f_{23}, l]$*.*
  - *If* $f_{23} < \frac{l}{3}$*, then* $\frac{\partial \mathcal{W}}{\partial f} < 0$ *for all* $f \in [0, \frac{l}{3})$*; and* $\frac{\partial \mathcal{W}}{\partial f} > 0$ *for all* $f \in [\frac{l}{3}, l]$*.*

- *Case 4: If* $A \leq \frac{i}{l}$ *and* $i \geq \frac{\gamma l}{2}$*, then equilibrium is always in Region 3. Furthermore,* $\frac{\partial \mathcal{W}}{\partial f} < 0$ *for all* $f \in [0, \frac{l}{3})$*, and* $\frac{\partial \mathcal{W}}{\partial f} > 0$ *for all* $f \in [\frac{l}{3}, l]$*.*

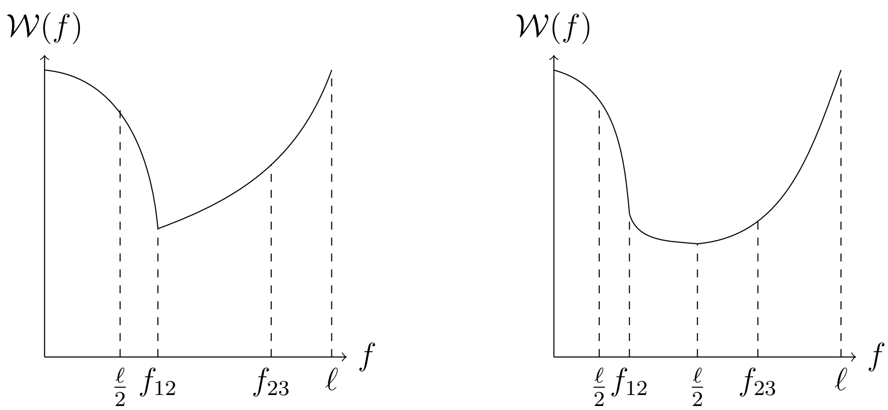
Proof. See the appendix A.8. □

Figures 11, 12, 13, and 14 give a visual representation of the welfare behavior as a function of matching probability f for all four cases. Proposition 4 again shows that, a better matching technique in the secondary asset market is not always welfare improving.

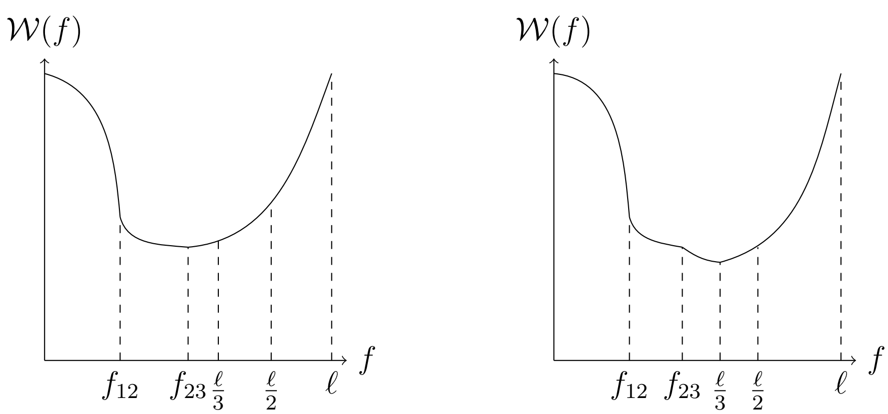


(a) Case 1.1: $\frac{\ell}{3} \leq f_{13}$ (b) Case 1.2: $\frac{\ell}{3} > f_{13}$

Figure 11: Case 1 Welfare

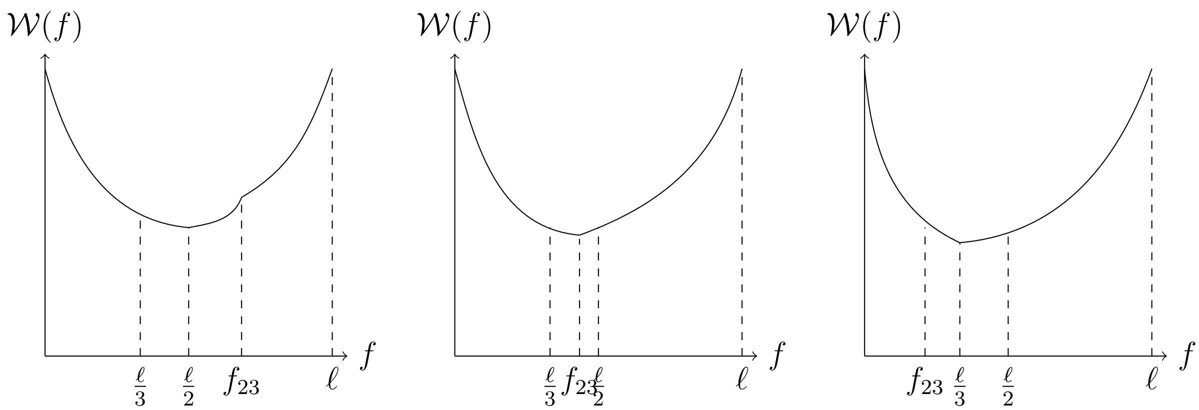


(a) Case 2.1 (b) Case 2.2



(c) Case 2.3 (d) Case 2.4

Figure 12: Case 2 Welfare



(a) Case 3.1 (b) Case 3.2 (c) Case 3.3

Figure 13: Case 3 Welfare

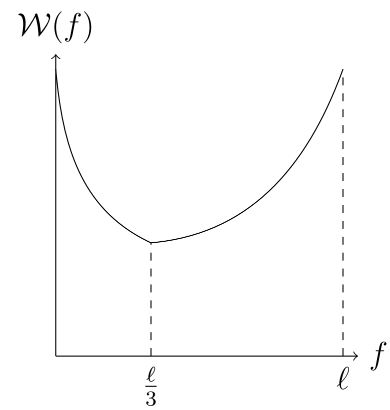


Figure 14: Case 4 Welfare

5 Conclusion

In this paper, we study whether more unsecured credit and other alternative means of payment to money can improve the welfare. To do this, we provide a general framework which agents can use credit and secondary asset (loan) markets to rebalance their liquidity position. The model delivers a result that, contrary to the common wisdom, having access to such credit and market opportunity is not always welfare improving. More specifically, if access to credit or alternative means of payments is low to begin with, increasing the access can hurt the welfare. Our model offers an explanation that, more credit/market access *ex* *post* means that transactions will not be hindered by lack of liquidity, however *ex ante*, easier access to credit/secondary market means agents have less incentive to hold money, which will hurt transactions in bilateral meetings where credit is not accepted.

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A Appendix

A.1 Proof of Lemma 4.

Case 1: $\hat{\varphi}\hat{m} + \hat{a} < q^*$. Let subscript 1 denote the objective function under case 1. Since total liquidity is scarce, according to equation (23), $q_2 = \hat{a} + \hat{\varphi}\hat{m}$. Hence equation (24) becomes

$$J_1(\hat{m}, \hat{a}) = (-\varphi + \beta\hat{\varphi})\hat{m} + (-\psi + \beta)\hat{a} + \beta(1-\sigma)[u(\hat{\varphi}\hat{m}) - \hat{\varphi}\hat{m}] + \beta\sigma[u(\hat{a} + \hat{\varphi}\hat{m}) - \hat{a} - \hat{\varphi}\hat{m}]$$

Let subscript m (and a) denote the derivative of the objective function with respect to money (and asset). The FOCs are as follows

$$J_{1m} = 0 \Rightarrow \varphi = \beta\hat{\varphi}\Big\{1 + (1-\sigma)[u'(\hat{\varphi}\hat{m}) - 1] + \sigma[u'(\hat{a} + \hat{\varphi}\hat{m}) - 1]\Big\}$$
$$J_{1a} = 0 \Rightarrow \psi = \beta\Big\{1 + \sigma[u'(\hat{a} + \hat{\varphi}\hat{m}) - 1]\Big\}$$

Case 2: $\hat{\varphi}\hat{m} + \hat{a} \geq q^*$. Under case 2, the total liquidity is plentiful in a type-2 meeting for consumers to consume the first-best quantity, i.e., $q_2 = q^*$. Hence equation (24) becomes

$$J_2(\hat{m}, \hat{a}) = (-\varphi + \beta\hat{\varphi})\hat{m} + (-\psi + \beta)\hat{a} + \beta(1-\sigma)[u(\hat{\varphi}\hat{m}) - \hat{\varphi}\hat{m}] + \beta\sigma[u(q^*) - q^*]$$

The FOCs with respect to $\hat{m}$ and $\hat{a}$ are as follows:

$$J_{2m} = 0 \Rightarrow \varphi = \beta\hat{\varphi}\Big\{1 + (1-\sigma)[u'(\hat{\varphi}\hat{m}) - 1]\Big\}$$
$$J_{2a} = 0 \Rightarrow \psi = \beta$$

A.2 Proof of Case Division.

Case 1: $0 < \tilde{\sigma} < \bar{\sigma} < 1$. This is true when: (1) $\tilde{\sigma} > 0$ or $i < A$, and (2) $\bar{\sigma} < 1$ or $i > \gamma - A$. For this to be possible, we need $A > \gamma - A$ or equivalently $A > \gamma/2$. When these conditions are satisfied, we have $0 < \tilde{\sigma} < \bar{\sigma} < 1$, and the equilibrium is

$$\begin{cases} \text{Plentiful } \forall \sigma \in [0, \tilde{\sigma}] \\ \text{Scarce } \forall \sigma \in (\tilde{\sigma}, \bar{\sigma}) \\ \text{Non-monetary } \forall \sigma \in [\bar{\sigma}, 1] \end{cases}$$

Case 2: $0 < \tilde{\sigma} < \bar{\sigma} = 1$. This is true when: (1) $\tilde{\sigma} > 0$ or $i < A$, and (2) $\bar{\sigma} = 1$ or $i \leq \gamma - A$. This can happen under two circumstances:

a) If $A > \gamma/2$ (or just $A > \gamma - A$), then we need $i \leq \gamma - A$

b) If $A < \gamma/2$ (or $A < \gamma - A$), we need $i < A$

If either of these two conditions happens, then $0 < \tilde{\sigma} < \bar{\sigma} = 1$, and the equilibrium is

$$\begin{cases} \text{Plentiful } \forall \sigma \in [0, \tilde{\sigma}] \\ \text{Scarce } \forall \sigma \in (\tilde{\sigma}, 1] \end{cases}$$

Case 3: $0 = \tilde{\sigma} < \bar{\sigma} < 1$. This is true when: (1) $\tilde{\sigma} = 0$ or $i \geq A$, and (2) $\bar{\sigma} < 1$ or $i > \gamma - A$. This can happen under 2 circumstances:

a) If $A > \gamma/2$ (or $A > \gamma - A$), we need $i \geq A$

b) If $A < \gamma/2$ (or $A < \gamma - A$), we need $i > \gamma - A$

When one of these circumstances happen, we have $0 = \tilde{\sigma} < \bar{\sigma} < 1$ and the equilibrium is

$$\begin{cases} \text{Scarce } \forall \sigma \in [0, \bar{\sigma}) \\ \text{Non-monetary } \forall \sigma \in [\bar{\sigma}, 1] \end{cases}$$

Case 4: $0 = \tilde{\sigma} < \bar{\sigma} = 1$. This is true when: (1) $i \geq A$, and (2) $i \leq \gamma - A$, which requires $A \leq \gamma/2$. So the economy is in case 4 when $A \leq \gamma/2$ and $i \in [A, \gamma - A]$. The equilibrium is scarce (but still monetary) for all $\sigma \in [0, 1]$.

Figure (15) and Figure (16) give a visual illustration of the parameter range of the four cases. And from the figures, we verify that the discussion covers all possible parameter values.

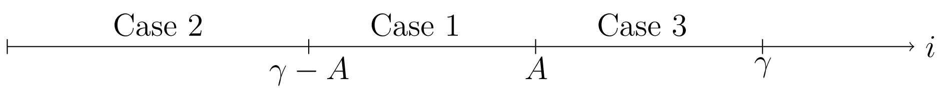

Figure 15: $A > \frac{\gamma}{2}$

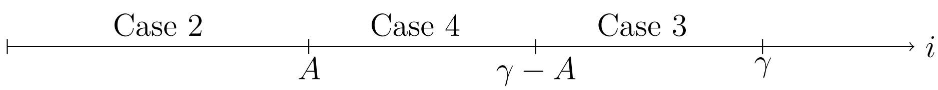

Figure 16: $A \leq \frac{\gamma}{2}$

A.3 Proof of Proposition 2.

<u>Case 1:</u> $A > \gamma/2$ and $i \in (\gamma - A, A)$. We already know that, $\mathcal{W}(0) = (\gamma^2 - i^2)/2 < A(\gamma - A/2) = \mathcal{W}(1)$. Now, for all $\sigma \in [0, \tilde{\sigma}]$ (plentiful equilibrium), we have

$$\begin{aligned}\frac{\partial \mathcal{W}}{\partial \sigma} &= u(\gamma) - \gamma - \Big[u\Big(\gamma - \frac{i}{1-\sigma}\Big) - (\gamma - \frac{i}{1-\sigma})\Big] + \Big\{(1-\sigma)[u'(\gamma - \frac{i}{1-\sigma}) - 1] + \sigma\Big\}[-\frac{i}{(1-\sigma)^2}] \\ &= -\frac{1}{2}(\frac{i}{1-\sigma})^2\end{aligned}$$

Notice that for all $\sigma \in [0, \tilde{\sigma}]$, $\partial \mathcal{W}/\partial \sigma < 0$, and $\partial^2 \mathcal{W}/\partial \sigma^2 = -i^2/(1-\sigma)^3 < 0$, so the welfare is decreasing and concave in σ.

Next, for all $\sigma \in (\tilde{\sigma}, \bar{\sigma})$ (scarce equilibrium), we have:

$$\begin{aligned}\frac{\partial \mathcal{W}}{\partial \sigma} &= u(\gamma - i + A(1-\sigma)) - (\gamma - i + A(1-\sigma)) - \Big[u(\gamma - i - \sigma A) - (\gamma - i - \sigma A)\Big](-A) \\ &= [\gamma - i + A(1-\sigma)]\Big[\gamma - \frac{\gamma - i + A(1-\sigma)}{2}\Big] - [\gamma - i - A\sigma]\Big[\gamma - \frac{\gamma - i - A\sigma}{2}\Big] \\ &\quad + \Big\{(1-\sigma)(i + \sigma A) + \sigma(i + A(1-\sigma))\Big\}(-A) \\ &= (\sigma - \frac{1}{2})A^2\end{aligned}$$

This result could be positive or negative, depending on the value of σ relative to $\frac{1}{2}$. And $\partial^2\mathcal{W}/\partial\sigma^2 = A^2 > 0$ implies that welfare is convex in this region.

Finally, for all $\sigma \in [\bar{\sigma}, 1]$ (non-monetary equilibrium), we have $\mathcal{W}(\sigma) = \sigma[u(A) - A]$, so that $\mathcal{W}$ is increasing and linear in σ, with $\mathcal{W}(1) > \mathcal{W}(0)$.

To sum up case 1, we have that $\mathcal{W}$ is:

- decreasing and concave in σ for $\sigma \in [0, \tilde{\sigma}]$
- convex in σ for $\sigma \in (\tilde{\sigma}, \bar{\sigma})$, and
- increasing and linear in σ for $\sigma \in [\bar{\sigma}, 1]$

In order to plot it, we still need to figure out the sign of $\partial\mathcal{W}/\partial\sigma$ for $\sigma \in (\tilde{\sigma}, \bar{\sigma})$, which depends on the value of $\bar{\sigma}$ and $\tilde{\sigma}$ relative to the stationary point $\sigma = \frac{1}{2}$, and there are three possibilities:

Case 1.1: $\tilde{\sigma} < 1/2 < \bar{\sigma}$. The corresponding i is in the range $A/2 < i < \gamma - A/2$. Hence if $i \in \Big(\max\{\gamma - A, A/2\}, \min\{\gamma - A/2, A\}\Big)$, and the minimizer of $\mathcal{W}$ is $\sigma = 1/2$.

Case 1.2: $\tilde{\sigma} > \frac{1}{2}$ (and necessarily $\bar{\sigma} > \frac{1}{2}$) The corresponding i must be that $i < A/2$, which can only happen if $A/2 > \gamma - A$ or $A \in (\frac{2}{3}\gamma, \gamma)$ and $i \in (\gamma - A, \frac{A}{2})$. The minimizer of $\mathcal{W}$ when $\frac{1}{2} < \tilde{\sigma} < \bar{\sigma}$ is $\sigma = \tilde{\sigma}$.

Case 1.3: $\bar{\sigma} < \frac{1}{2}$ (and necessarily $\tilde{\sigma} < 1/2$). Hence $i > \gamma - A/2$. This can only happen if $A \in (2\gamma/3, \gamma)$ and $i \in (\gamma - A/2, A)$. The minimizer in this case would be $\sigma = \bar{\sigma}$.

Case 2: $A > \gamma/2$ & $i \leq \gamma - A$, or $A < \gamma/2$ & $i < A$.

- For all $\sigma \in [0, \tilde{\sigma}]$ (plentiful equilibrium), $\partial\mathcal{W}/\partial\sigma = -i^2/2(1-\sigma)^2 < 0$.
- For all $\sigma \in (\tilde{\sigma}, 1]$ (scarce equilibrium), $\partial\mathcal{W}/\partial\sigma = (\sigma - 1/2)A^2$, which is convex but the sign can be positive or negative, depending on the value of σ relative to $1/2$. So we discuss this in two sub-cases.

  Case 2.1: $\tilde{\sigma} < 1/2$. The corresponding i must be that $i > A/2$. With the parameter range for case 2, this means that we can potentially be in the following two cases.

  - $A > \gamma/2$ and $i < \gamma - A$. Hence $A \in (\gamma/2, 2\gamma/3)$ and $i \in (A/2, \gamma - A)$
  - $A < \gamma/2$ and $i < A$. Hence $A < \gamma/2$ and $i \in (A/2, A)$

  Case 2.2: $\tilde{\sigma} > 1/2$. The corresponding i must be that $i < A/2$. Again, for parameter range of case 2, we have two cases.

  - $A > \gamma/2$ and $i < \gamma - A$. If $A/2 > \gamma - A$ or $A > 2\gamma/3$, then $i < A/2$ is automatically satisfied. If $A \in (\gamma/2, 2\gamma/3)$, then $A/2 < \gamma - A$, and $i < A/2$ becomes meaningful.
  - $A < \gamma/2$ and $i < A$. We need $A < \gamma/2$ and $i < A/2$

Case 3: $A > \gamma/2$ & $i \geq A$, or $A < \gamma/2$ & $i > \gamma - A$.

- For all $\sigma \in [0, \bar{\sigma})$ (scarce equilibrium), $\partial \mathcal{W}/\partial\sigma = (\sigma - 1/2)A^2$, which is convex but can be positive or negative, depending on the value of σ relative to $1/2$. This will be discussed in the following two sub-cases.

  - Case 3.1: $\bar{\sigma} > 1/2$. Hence $i < \gamma - A/2$. Given the parameter values in case 3, we potentially have the following two cases:
    - $A > \gamma/2$ and $i \geq A$, which means that $A \in (\gamma/2, 2\gamma/3)$ and $i \in [A, \gamma - A/2)$.
    - $A < \gamma/2$ and $i > \gamma - A$. which means that $A < \gamma/2$ and $i \in (\gamma - A, \gamma - A/2)$
  - Case 3.2: when $i \geq \gamma - A/2$. And we potentially have the following two cases:
    - $A > \gamma/2$ and $i \geq A$. If $A > \gamma - A/2$, then $i \geq \gamma - A/2$ is automatically satisfied, which implies that all we need is $A > 2\gamma/3$ and $i \geq A$. If $A \in (\gamma/2, 2\gamma/3)$, then $\gamma - A/2 > A$, and all we need is $A \in (\gamma/2, 2\gamma/3)$ and $i \geq \gamma - A/2$
    - $A < \gamma/2$ and $i > \gamma - A$. Of course, here it is guaranteed that $\gamma - A/2 > \gamma - A$, which means that all we need is $A < \gamma/2$ and $i \geq \gamma - A/2$.

- For all $\sigma \in [\bar{\sigma}, 1]$ (non-monetary equilibrium), $\mathcal{W}(\sigma) = \sigma[u(A)]$, which is linear and increasing in σ.

Case 4: $A \leq \gamma/2$ and $i \in [A, \gamma - A]$. For all $\sigma \in [0, 1)$ (scarce equilibrium), $\partial \mathcal{W}/\partial \sigma = (\sigma - 1/2)A^2$, and $\mathcal{W}$ has a unique minimizer at $\sigma = 1/2$.

A.4 Proof of Definition 3.

Region 1. If $\hat{m} + \tilde{m} \geq m^*$, and call this region 1, then the objective function $J_1(\hat{m})$ and pricing function take the form of

$$J_1(\hat{m}) = -\varphi\hat{m} + \beta f\Big[u(q^*) - q^* + \hat{\varphi}\hat{m}\Big] + \beta(\ell - f)u\Big(\hat{\varphi}\hat{m}\Big) + \beta(1-\ell)\hat{\varphi}\hat{m}$$

$$\varphi = \beta\hat{\varphi}\Big\{1 + (\ell - f)\Big[u'(\hat{\varphi}\hat{m}) - 1\Big]\Big\}$$

In steady state, real money balance, z, is determined by

$$\frac{\varphi M(1+\mu)}{\beta\hat{\varphi}\hat{M}} - 1 = \frac{1-\mu}{\beta} - 1 = i = (\ell - f)\Big[u'(z) - 1\Big]$$

Region 2. If $\hat{m} + \tilde{m} < m^*$, and call this region 2, then $J_2(\hat{m})$ term and pricing functions are

$$J_2(\hat{m} =) - \varphi\hat{m} + \beta f\Big[u\Big(\hat{\varphi}(\hat{m} + \tilde{m})\Big) - \hat{\varphi}\tilde{m}\Big] + \beta(\ell - f)u\Big(\hat{\varphi}\hat{m}\Big) + \beta(1-\ell)\hat{\varphi}\hat{m}$$

$$\varphi = \beta\hat{\varphi}\Big\{1 + f\Big[u'(\hat{\varphi}(\hat{m} + \tilde{m})) - 1\Big] + (\ell - f)\Big[u'(\hat{\varphi}\hat{m}) - 1\Big]\Big\}$$

In steady state, the real money balance is determined by

$$\frac{\varphi M(1+\mu)}{\beta\hat{\varphi}\hat{M}} - 1 = \frac{1-\mu}{\beta} - 1 = i = f\Big[u'(2z - 1\Big] + (\ell - f)\Big[u'(z) - 1\Big]$$

A.5 Proof of Proposition 3.

Case 1: $i \in (0, \frac{\ell\gamma}{2})$, then $\bar{f} \in (0, \ell)$.

- For all $f \in [0, \bar{f})$, which is the plentiful equilibrium,

$$\begin{aligned}\frac{\partial \mathcal{W}}{\partial f} =& u(\gamma) - \gamma - [u(\gamma - \frac{i}{\ell - f}) - (\gamma - \frac{i}{\ell - f})] + (\ell - f)[u'(\gamma - \frac{i}{\ell - f}) - 1]\frac{d[\gamma - \frac{i}{\ell - f}]}{dz} \\ =& -\frac{1}{2}(\frac{i}{\ell - f})^2 < 0\end{aligned}$$

  So in this range of matching probability, increasing in f reduces welfare. Also, the function is concave in such parameter range.

- For all $f \in [\bar{f}, \ell]$, which is the scarce equilibrium, we have $z = (\gamma\ell - i)/(\ell + f)$ and $q_2 = 2z$,

$$\frac{\partial \mathcal{W}}{\partial f} = u(2z) - 2z - [u(z) - z] - \{2f(\gamma - 2z) + (\ell - f)(\gamma - z)\}\frac{\gamma\ell - i}{(\ell + f)^2} = \frac{3f - \ell}{\ell + f}\frac{z^2}{2}$$

Welfare is convex in f. But the sign of $\partial \mathcal{W}/\partial f$ depends on $3f - \ell$. If $f > \ell/3$, then $\partial \mathcal{W}/\partial f > 0$. Also recall that, here we are discussing the case where $f \geq \bar{f} \equiv \ell - 2i/\gamma$, and $i < \ell\gamma/2$. Depending on how the value of $\ell/3$ compare with $\bar{f}$, we can have two cases, which are summarized by Figure 6. If $\ell/3 < \bar{f}$, then $f > \bar{f} > \ell/3$, and welfare increases in f. If $\ell/3 > \bar{f}$, then for all $f \in (\ell/3, \bar{f})$, welfare decreases in f; and for all $f \in [\ell/3, \ell]$, welfare increases with matching probability f.

The figure on the left is the scenario when $\bar{f} > \frac{\ell}{3}$ or $i < \ell\gamma/3$. We already assumed that $i < \ell\gamma/2$, but this does not necessarily mean that we automatically have $i < \frac{\ell\gamma}{3}$. So we need to discuss the sign in terms of i in two cases:

<u>Case 1.1:</u> $i < \ell\gamma/3$, hence $\bar{f} > \frac{\ell}{3}$. Then $\partial \mathcal{W}/\partial f > 0 \; \forall f \in [\bar{f}, \ell]$, which is illustrated by the figure on the left panel of Figure 6.

<u>Case 1.2:</u> $i \in (\ell\gamma/3, \ell\gamma/2)$, hence $\bar{f} < \ell/3$. Then $\partial \mathcal{W}/\partial f < 0 \; \forall f \in [\bar{f}, \ell/3)$ and $\partial \mathcal{W}/\partial f > 0 \; \forall f \in (\ell/3, \ell]$, which corresponds to the figure on the right panel. In this case, $\mathcal{W}$ has a unique minimum at $f = \ell/3$.

<u>Case 2:</u> $i \in (\ell\gamma/2, \ell\gamma)$, which means $\bar{f} \le 0$, or in other words, all possible fs will satisfy $f > \bar{f}$, and so all equilibirum will be in the scarce case. In "scarce" equilibrium, we have $q_2 = 2z$, and $z = (\ell\gamma - i)/(\ell + f)$. and as already shown, $\partial\mathcal{W}/\partial f = z^2(3f - \ell)/2(\ell + f)$ or $\partial\mathcal{W}/\partial f = (3f - \ell)(\ell\gamma - i)^2/2(\ell + f)^3$. Case 2 is summarized by Figure 7.

Notice that we know about the convexity, but we never really compare $\mathcal{W}(0)$ and $\mathcal{W}(\ell)$, so let's do a quick comparison. First, at $f = \ell$, we are always at the scarce case, and hence $\mathcal{W}(\ell) = \left[(\ell\gamma)^2 - i^2\right]/2\ell$.

What about $\mathcal{W}(0)$? If we are in the scarce equilibrium (case 2), then $z = (\gamma\ell - i)/\ell$, and $\mathcal{W}(0) = \left[(\ell\gamma)^2 - i^2\right]/2\ell = \mathcal{W}(\ell)$. So in the scarce case, we have $\mathcal{W}(0) = \mathcal{W}(\ell)$. If the equilibrium is plentiful as in case 1, we have $z = \gamma - \frac{i}{\ell - f} = \gamma - \frac{i}{\ell}$, and hence $\mathcal{W}(0) = \ell[u(z) - z] = \ell z(\gamma - z/2) = \left[(\ell\gamma)^2 - i^2\right]/2\ell = \mathcal{W}(\ell)$. So we always have $\mathcal{W}(0) = \mathcal{W}(\ell)$, regardless of whether the economy is in plentiful equilibrium or scarce equilibrium.

A.6 Proof of Lemma 9.

<u>Case 1:</u> a and $\tilde{m}$ are huge, such that the asset constraints do not bind, i.e. $m + \tilde{m} \ge m^*$ and $a \ge q^* - \varphi m$. Then if we plug in $\chi = \varphi x$, the bargaining problem becomes

$$\max_x u(\varphi(m + x)) - u(\varphi m) - \varphi x$$

and the bargaining solution is $x = m^* - m$ and $\chi = q^* - \varphi m$.

<u>Case 2:</u> $m + \tilde{m} \ge m^*$ (so a C-type would like to have $x = m^* - m$), but $a \le q^* - \varphi m$. In this case, a C-type woud be willing to give all her assets to N-type, in exchange for the right amount of money. So the bargaining solution is given by $\chi = a$ and $x = \frac{a}{\varphi}$.

<u>Case 3:</u> $a \ge q^* - \varphi\tilde{m}$, but money is limited, i.e. $m + \tilde{m} < m^*$. In this case, C-type wants to obtain all of N-type's money, so $x = \tilde{m}$, and pay just enough asset, $\chi = \varphi\tilde{m}$, in exchange.

<u>Case 4:</u> $m + \tilde{m} < m^*$ (so C-type wants to have all of N-type's money $\tilde{m}$), but C-type does not have enough assets to buy this amount, i.e. $a \le \varphi\tilde{m}$. In this case, the bargaining

solution is given by $\chi = a$ and $x = \frac{a}{\varphi}$, which are the same as in case 2.

Figure 17 summarizes these four cases.

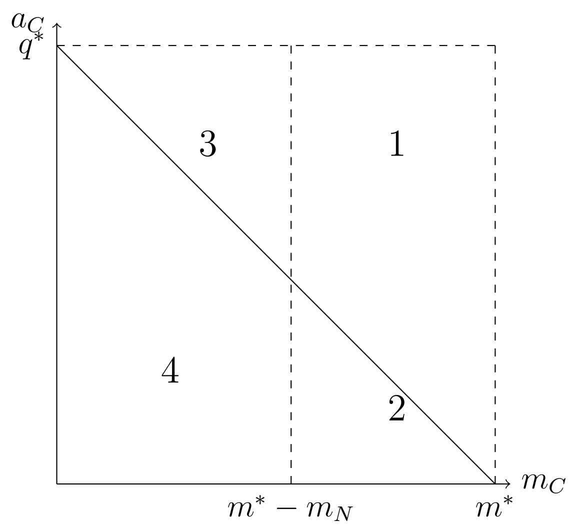

Figure 17: Bargaining Solution

A.7 Proof of Lemma 10.

Region 1: Both money and asset are plentiful, thus the objective functions adopts the following form:

$$J^1(\hat{m}, \hat{a}) = -\varphi\hat{m} - \psi\hat{a} + \beta\hat{a} + \beta f[u(q^*) - q^* + \hat{\varphi}\hat{m}] + \beta(\ell - f)u(\hat{\varphi}\hat{m}) + \beta(1-\ell)\hat{\varphi}\hat{m}$$

And the asset pricing functions are:

$$\begin{aligned} J^1_m =& 0 \Rightarrow \varphi = \beta\hat{\varphi}(f + 1 - \ell) + \beta(\ell - f)\hat{\varphi}u'(\hat{\varphi}\hat{m}) = \beta\hat{\varphi}\Big\{1 + (\ell - f)\Big[u'(\hat{\varphi}\hat{m} - 1)\Big]\Big\} \\ J^1_a =& 0 \Rightarrow \psi = \beta \end{aligned}$$

Region 2: Total money allows for first-best consumption q^*, but total assets are not enough for C-type to exchange for optimal amount of money.

$$J^2(\hat{m},\hat{a}) = -\varphi\hat{m} - \psi\hat{a} + \beta\hat{a} + \beta f\Big[u(\hat{\varphi}\hat{m}+\hat{a}) - \hat{a}\Big] + \beta(\ell - f)u(\hat{\varphi}\hat{m}) + \beta(1-\ell)\hat{\varphi}\hat{m}$$

Hence asset pricing functions are:

$$\begin{aligned} J^2_m = 0 \Rightarrow \varphi =& \beta\hat{\varphi}\Big\{1 + f\Big[u'(\hat{\varphi}\hat{m}+\hat{a}) - 1\Big] + (\ell - f)\Big[u'(\hat{\varphi}\hat{m}) - 1\Big]\Big\} \\ J^2_a = 0 \Rightarrow \psi =& \beta\Big\{1 + f\Big[u'(\hat{\varphi}\hat{m}+\hat{a}) - 1\Big]\Big\} \end{aligned}$$

Region 3: C-type's asset is enough to buy all of N-type's money, but total money is not enough to allow for optimal consumption q^*.

$$J^3(\hat{m},\hat{a}) = -\varphi\hat{m} - \psi\hat{a} + \beta\hat{a} + \beta f[u(\hat{\varphi}(\hat{m}+\tilde{m})) - \hat{\varphi}\tilde{m}] + \beta(\ell - f)u(\hat{\varphi}\hat{m}) + \beta(1-\ell)\hat{\varphi}\hat{m}$$

the pricing functions in this regions is given by

$$\begin{aligned} J^3_m = 0 \Rightarrow \varphi =& \beta\hat{\varphi}\Big\{1 + f\Big[u'(\hat{\varphi}(\hat{m}+\tilde{m})) - 1\Big] + (\ell - f)\Big[u'(\hat{\varphi}\hat{m}) - 1\Big]\Big\} \\ J^3_a = 0 \Rightarrow \psi =& \beta a \end{aligned}$$

A.8 Proof of Proposition 4.

Case 1: $A > \gamma/2$ and $A > i/\ell$. There exists a $f_{13} \in (0,\ell)$ such that given (A,i), equilibrium is in region 1 if $f \in [0, f_{13})$, and in region 3 if $f \in [f_{13}, \ell]$, where f_{13} solves $\gamma - i/(\ell - f_{13}) = \gamma/2$. Hence $f_{13} = \ell - 2i/\gamma$.

- If equilibrium is in region 1, $q_2 = \gamma, z = \gamma - i/(\ell - f)$, and

$$\frac{\partial \mathcal{W}}{\partial f} = u(\gamma) - \gamma - [u(z) - z] + (\ell - f)(\gamma - z)\frac{d\Big[\gamma - i(\ell - f)^{-1}\Big]}{df} = -\frac{1}{2}\Big(\frac{i}{\ell - f}\Big)^2 < 0$$

- If equilibrium is in region 3, $z = (\ell\gamma - i)/(\ell + f), q_2 = 2z$, and

$$\begin{aligned}\frac{\partial \mathcal{W}}{\partial f} =& [u(2z) - 2z] - [u(z) - z] + [2f(\gamma - 2z) + (\ell - f)(\gamma - z)]\frac{d\big[(\ell\gamma - i)(f + \ell)^{-1}\big]}{df} \\ =& \frac{3f - \ell}{f + \ell}\frac{1}{2}(\frac{\gamma f - i}{\ell + f})^2\end{aligned}$$

Whether this expression is positive or negative depends on the value of $3f - \ell$. Since in region 3 we have $f \in [\ell - 2i/\gamma, \ell]$, if the lower bound of f is greater than $\ell/3$, then all f in this case would yield a positive result. Hence the sign depends on how $\ell - 2i/\gamma$ compares with $\ell/3$, or if $i > \ell\gamma/3$.

Case 1.1: If $i \geq \gamma\ell/3$, then $f_{13} \geq \ell/3$. Hence $\partial\mathcal{W}/\partial f > 0 \; \forall f \in [f_{13}, \ell]$.

Case 1.2: If $i < \gamma\ell/3$, then $f_{13} < \ell/3$. Hence $\partial\mathcal{W}/\partial f < 0 \; \forall \; f \in [0, f_{13})$; and $\partial\mathcal{W}/\partial f > 0 \; \forall \; f \in [\frac{\ell}{3}, \ell]$. $\mathcal{W}$ has a unique minimum at $f = \ell/3$.

Case 2: $A < \gamma/2$, $A > i/\ell$, and $i < \ell\gamma/2$. This corresponds to the lower half of the aggregate regions, in which the interest rate is not too large, and the equilibrium could be in any of the 3 regions. Hence there exist $f_{12} = \ell - i/A$ and $f_{23} = (\gamma\ell - i)/A - \ell$ with $0 < f_{12} < f_{23} < \ell$, such that equilibrium is in region 1, if $f \in [0, f_{12})$; in region 2, if $f \in [f_{12}, f_{23})$; and in region 3, if $f \in [f_{23}, \ell]$.

- If equilibrium is in region 1, $z = \gamma - i/(\ell - f), q_2 = z$, and

$$\frac{\partial \mathcal{W}}{\partial f} = u(\gamma) - \gamma - [u(z) - z] + (\ell - f)(\gamma - z)\frac{d(i - \frac{i}{\ell - f})}{df} = -\frac{1}{2}(\frac{i}{\ell - f})^2 < 0$$

- If equilibrium is in region 2, $z = \gamma - (i + \gamma A)/\ell, q_2 = \gamma - i/\ell + A(\ell - f)/\ell$.

$$\frac{\partial \mathcal{W}}{\partial f} = u(q_2) - q_2 - \Big[u(z) - z\Big] + \Big[f(\gamma - q_2)\frac{dq_2}{dz} + (\ell - f)(\gamma - z)\Big]\frac{d\big[\gamma - \frac{i + fA}{\ell}\big]}{df} = A^2(\frac{f}{\ell} - \frac{1}{2})$$

and the sign depends on how f/ℓ compares to 1/2. Recall that in region 2, $f \in [f_{12}, f_{23})$, so there are 3 possibilities:

1. $f_{12} > \ell/2$. In this case, $i/A < \ell/2$, hence $A > 2i/\ell$, and any admissible f will be greater than $\ell/2$. $\partial\mathcal{W}/\partial f > 0$ for all fs in this range.

2. $\ell/2 > f_{23}$. In this case, $A > 2(\gamma\ell - i)/3\ell$. Then any admissible fs will be smaller than $\ell/2$, and $\partial\mathcal{W}/\partial f < 0$ for all fs.

3. $f_{12} < \ell/2 < f_{23}$. Given the parameter range specified for case 2, $A < 2i/\ell$ and $A < 2(\gamma\ell - i)/3\ell$, hence $\partial\mathcal{W}/\partial f < 0$ for all $f \in [f_{12}, \frac{\ell}{2})$; $\partial\mathcal{W}/\partial f > 0$ for all $f \in [\frac{\ell}{2}, f_{23})$.

- If equilibrium is in region 3, and we have $z = (\ell\gamma - i)/(\ell + f), q_2 = 2z$, and

$$\frac{\partial\mathcal{W}}{\partial f} = \frac{3f - \ell}{f + \ell}\frac{1}{2}(\frac{\gamma\ell - i}{\ell + f})^2$$

The sign depends on the value of $3f - \ell$, or how the value of f_{23} compares to $\ell/3$.

1. $f_{23} > \ell/3$, then $A < 3(\gamma\ell - i)/4\ell$, and all admissible fs are greater than $\ell/3$, so $\partial\mathcal{W}/\partial f > 0$ for all fs.

2. $f_{23} < \ell/3$, then $A > 3(\gamma\ell - i)/4\ell$, and $\partial\mathcal{W}/\partial f < 0$ for all $f \in [f_{23}, \ell/3)$; $\partial\mathcal{W}/\partial f > 0$ for all $f \in [\ell/3, \ell]$.

Thus taking parameters f and ℓ, and asset supplies (A, i) as given, the equilibrium could potentially have 6 cases as region 2 and region 3 each has 3 and 2 subcases respectively. But 2 cases will be ruled out: (1) $\ell/2 < f_{12}$ and $\ell/3 > f_{23}$; (2) $\ell/2 \in (f_{12}, f_{23})$ and $\ell/3 > f_{23}$. Next we put together everything we learnt about case 2:

<u>Case 2.1:</u> $\ell/2 < f_{12}(< f_{23})$ hence $A > \frac{2i}{\ell}$. Then for all $f \in [0, f_{12}]$ $\partial\mathcal{W}/\partial f < 0$; for all $f \in (f_{12}, f_{23})$, $\partial\mathcal{W}/\partial f > 0$; for all $f \in [f_{23}, \ell]$, $\partial\mathcal{W}/\partial f > 0$

<u>Case 2.2:</u> $\ell/3 < \ell/2 \in (f_{12}, f_{23})$. Then for all $f \in [f_{12}, \ell/2]$, $\partial\mathcal{W}/\partial f < 0$; for all $f \in [\ell/2, f_{23})$, $\partial\mathcal{W}/\partial f > 0$; for all $f \in [f_{23}, \ell]$, $\partial\mathcal{W}/\partial f > 0$. Besides the general parameter specification of case 2, this subcase also requires $A < 2i/\ell$, $A < 2(\gamma\ell - i)/3\ell$.

Case 2.3: $f_{12} < \ell/3 < f_{23} < \ell/2$. Then for all $f \in [f_{12}, f_{23}]$, $\partial\mathcal{W}/\partial f < 0$; for all $f \in (f_{23}, \ell]$, $\partial\mathcal{W}/\partial f > 0$. This requires $A > 2(\gamma\ell - i)/3\ell$ and $A < 3(\gamma\ell - i)/4\ell$, which can be satisfied with $3(\gamma\ell - i)/4\ell > 2(\gamma\ell - i)/3\ell$.

Case 2.4: $f_{12} < f_{23} < \ell/3 < \ell/2$. Then for all $f \in [f_{12}, f_{23}]$, $\partial\mathcal{W}/\partial f < 0$; for all $f \in (f_{23}, \ell/3)$, $\partial\mathcal{W}/\partial f < 0$; for all $f \in (\ell/3, \ell]$, $\partial\mathcal{W}/\partial f > 0$, with a (smooth) minimum at $\ell/3$. This region requires $A > 3(\gamma\ell - i)/4\ell$, which also guarantees that $\ell/2 > f_{23}$.

Figure 18 shows how the aggregate regions are dividend given the values of (A,i).

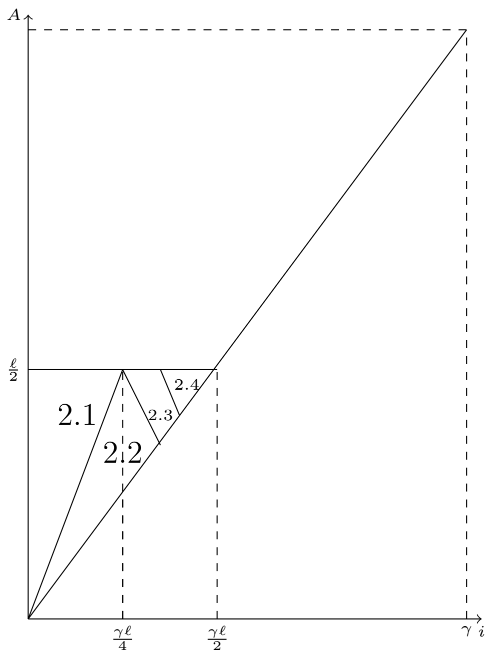

Figure 18: Aggregate Regions of Case 2

Figure 12 summarizes the welfare in case 2.

Case 3: $A \leq i/\ell$, $i < (\gamma\ell)/2$. This is the case where the asset supply A is scarce, hence equilibrium could be in either region 2 or region 3. Given the definition of f_{23}, if $f \in [0, f_{23})$, equilibrium is in region 2; if $f \in [f_{23}, \ell]$, equilibrium is in region 3.

- If the equilibrium is in region 2, $z = \gamma - (i + \gamma A)/\ell$, $q_2 = z + A$. Then

$$\frac{\partial\mathcal{W}}{\partial f} = A^2(\frac{f}{\ell} - \frac{1}{2})$$

The sign depends on how the value of f compares with $\ell/2$, hence we discuss this with the following two cases.

1. $\ell/2 > f_{23}$. Then $A > 2(\gamma\ell - i)/3\ell$, and all admissible fs are smaller than $\ell/2$, hence $\partial\mathcal{W}/\partial f < 0$ for all fs in region 2.

2. $\ell/2 < f_{23}$. Then $A < 2(\gamma\ell - i)/3\ell$. $\partial\mathcal{W}/\partial f < 0$ for all $f \in [0, \frac{\ell}{2})$; and $\partial\mathcal{W}/\partial f > 0$ for all $f \in [\ell/2, f_{23})$.

- For all $f \in [f_{23}, \ell]$, the equilibrium is in region 3 and

$$\frac{\partial\mathcal{W}}{\partial f} = \frac{3f - \ell}{f + \ell}\frac{1}{2}(\frac{\gamma\ell - i}{f + \ell})^2$$

  and the sign depends on whether how the value of $\ell/3$ compare with f_{23}, hence we discuss this with the following two cases.

1. $\ell/3 < f_{23}$. Then $A < 3(\gamma\ell - i)/4\ell$, and all admissible fs are greater than $\ell/3$, hence $\partial\mathcal{W}/\partial f > 0$ for all fs in region 3.

2. $\ell/3 > f_{23}$. Then $A > 3(\gamma\ell - i)/4\ell$. $\partial\mathcal{W}/\partial f < 0$ for all $f \in [f_{23}, \ell/3)$; $\partial\mathcal{W}/\partial f > 0$ for all $f \in [\ell/3, \ell]$.

Hence given the parameter range, there could potentially be four cases in total. But one case can be ruled out since it is impossible to have $\ell/2 < f_{23}$ and $\ell/3 > f_{23}$ at the same time. We summarize the remaining three cases as follows.

Case 3.1: $\ell/3 < \ell/2 < f_{23}$. Then for all $f \in [0, \ell/2)$, $\partial\mathcal{W}/\partial f < 0$; for all $f \in [\ell/2, f_{23})$, $\partial\mathcal{W}/\partial f > 0$; for all $f \in [f_{23}, \ell]$, $\partial\mathcal{W}/\partial f > 0$.

Case 3.2: $\ell/3 < f_{23} < \ell/2$. Then for all $f \in [0, f_{23})$, $\partial\mathcal{W}/\partial f < 0$; for all $f \in [f_{23}, \ell]$, $\partial\mathcal{W}/\partial f > 0$. Also, besides the general parameter specification of case 3, this subcase also requires $\ell/2 > f_{23}$ or $A < 3(\gamma\ell - i)/4\ell$.

Case 3.3: $f_{23} < \ell/3 < \ell/2$. Then for all $f \in [0, f_{23})$, $\partial\mathcal{W}/\partial f < 0$; for all $f \in [f_{23}, \ell/3)$, $\partial\mathcal{W}/\partial f < 0$; for all $f \in [\ell/3, \ell]$, $\partial\mathcal{W}/\partial f > 0$. This subcase again requires additional parameter restriction that $\ell/3 > f_{23}$ or $A > 3(\gamma\ell - i)/4\ell$.

Figure 19 shows how the aggregate regions are dividend given the values of (A,i) for case 3.

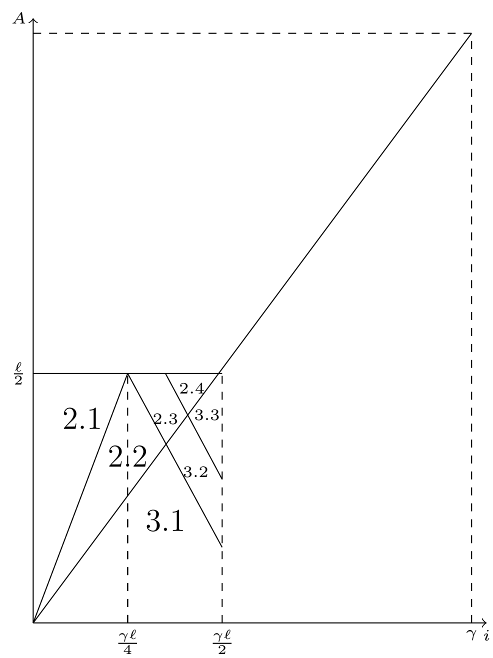

Figure 19: Regions Specification of Case 3

Figure 13 shows a visual representation of case 3.

Case 4: $A \leq i/\ell, i \geq \gamma\ell/2$. Given this parameter values, the equilibrium is always in region 3. Thus for all $f \in [0, \ell]$,

$$\frac{\partial\mathcal{W}}{\partial f} = \frac{1}{2}\frac{3f - \ell}{f + \ell}(\frac{\gamma\ell - i}{f + \ell})^2$$

which is positive if $f > \ell/3$, and is negative if $f < \ell/3$.

We use the Figure 20 to summarize the parameter values of all the regions:

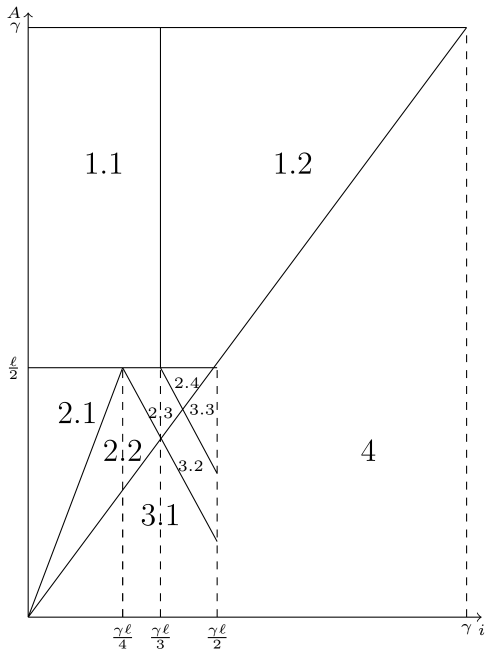

Figure 20: Summary of all regions

Figure 14 shows a visual representation of welfare behavior in case 4.