SZEREGI
POTĘGOWE

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \dots$$

$$a_n = \frac{1}{n^2}, \quad a_{n+1} = \frac{1}{(n+1)^2}$$

$$R = \frac{1}{g}$$

$$g = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{n^2}{n^2+2n+1} = \lim_{n \to \infty} \frac{1}{1+\underbrace{\tfrac{2}{n}}_{\to 0}+\underbrace{\tfrac{1}{n^2}}_{\to 0}} = 1$$

Przedział zbieżności

$$-R < x < R \qquad -1 \leq x < 1$$

$x = -1$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ – bezwzględnie zbieżny

$x = 1$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^2}$ – szereg harmoniczny zbieżny

$$\sum_{n=1}^{\infty} n x^n = x + 2x^2 + 3x^3$$

$$R = \frac{1}{g} \quad g = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}, \quad a_n = n \quad a_{n+1} = n+1$$

$$g = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$$

$$-R < x < R \qquad -1 < x < 1$$

$x = 1$ $\sum_{n=1}^{\infty} n$ – rozbieżny

$X = -1$ $\sum_{n=1}^{\infty} (-1)^n \cdot n$ – rozbieżny

33
28 29 32 33 34 35
38
39
40

11.28 $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

$R = \frac{1}{q}$ $q = \lim \frac{a_{n+1}}{a_n} = \lim \frac{(n+1)^{\frac{1}{2}}}{n^{\frac{1}{2}}} = 1$

$-1 < x < 1$ $\frac{\cancel{\sqrt{n}}\sqrt{1+\frac{1}{n}}}{\cancel{\sqrt{n}}}$ $\frac{1}{n} \to 0$

$x = -1$ $\sum \frac{-1}{\sqrt{n}}$ – zbieżny

$x = 1$ $\sum \frac{1}{\sqrt{n}}$ – zbieżny zd. 29 $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 5^n}$

$R = \frac{1}{g}$ $g = \frac{a_{n+1}}{a_n} = \frac{(n+1)\,5^{(n+1)}}{n \cdot 5^n} =$

$= \frac{\cancel{n}(1+\underbrace{\frac{1}{n}}_{\to 0}) \cdot 5 \cdot \cancel{5^n}}{\cancel{n} \cdot \cancel{5^n}} = \frac{1}{5}$ $R = 5$

$-5 < x < 5$

$x = 5$ $\sum_{n=1}^{\infty} \frac{\cancel{5^n}}{n \cdot \cancel{5^n}}$ – rozbieżny

$x = -5$ $\sum_{n=1}^{\infty} \frac{(-5)^n}{n \cdot 5^n}$ – zbieżny

11.32 $\sum_{n=1}^{\infty} \frac{5^n}{n^n} x^n$

$R = \frac{1}{q}$ $q = \lim \sqrt[n]{\left(\frac{5}{n}\right)^n} = \lim \frac{5}{n} = 0$

$R = \infty$

11.33 $\sum_{n=0}^{\infty} \frac{\sqrt{n!}}{10^n} x^n$

$\frac{\sqrt{n!}}{10^n}$ $\frac{\cancel{10^n}\ \sqrt{(n+1)!}}{\cancel{\sqrt{n!}}\ 10^{n+1}} = \frac{\sqrt{n+1}}{10} = \infty$

0

11.34 $\sum_{n=0}^{\infty} (\cos n)\, x^n$

$\sum_{n=0}^{\infty} |\cos n| \cdot |x^n| \quad |\cos n| \cdot |x^n| \leqslant |x^n|$

$|x| < 1$ - bezwzględnie zbieżny

$x = 1 \quad \sum_{n=0}^{\infty} \cos n$ - rozbieżny

11.35 $\sum_{n=1}^{\infty} \left(\sin^2 \frac{1}{n}\right) x^n$

11.36 $\sum_{n=2}^{\infty} \frac{2^{3n-1} \cdot n^3}{(n^2-1)5^n} x^n$

$R = \frac{1}{q}$ $q = \lim \frac{2^{3n-1} \cdot n^3}{(n^2-1)5^n}$

$\frac{(n^2-1) \cdot 5^n}{}$

11.38 $\sum_{n=1}^{\infty} \frac{(n^2+n)\,2^{2n}}{3^{3n}} x^n$

$$\frac{\cancel{3^{3n}} \cdot \left((n+1)^2+(n+1)\right) \cdot \cancel{2^{2n}}^{+2}}{(n^2+n)\,\cancel{2^{2n}} \cdot \cancel{3^{3n}}^{+3}} =$$

$$= \frac{(n^2+3n+2)\,2^2}{(n^2+n)\cdot 3^3} = \frac{\cancel{(n+1)}(n+2)\cdot 2^2}{n\cancel{(n+1)}\cdot 3^3} =$$

$$= \frac{4n+8}{27n} = \frac{4+\frac{8}{n}}{27} = \frac{4}{27}$$

$$-\frac{27}{4} < x < \frac{27}{4}$$

$x = \frac{27}{4}$ $\sum \frac{(n^2+n)\cancel{2^{2n}}}{\cancel{3^{3n}}} \cancel{\left(\frac{27}{4}\right)^n}$ roz

$x = -\frac{27}{4}$ $\sum \frac{(n^2+n)\cdot\cancel{2^{2n}}}{\cancel{3^{3n}}} \cancel{\left(-\frac{27}{4}\right)^n} \cdot -1$ roz

$27^n = 3^{3n}$

11.39 $\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$

$R = \frac{1}{e}$ $\frac{\cancel{n!}\,(n+1)^{(n+1)}}{n^n\,\cancel{(n+1)!}} = \frac{(n+1)^n}{n^n} =$

$= \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n = e$

$-\frac{1}{e} < x < \frac{1}{e}$

$x = -\frac{1}{e}$ $\sum_{n=1}^{\infty} \frac{n^n}{n!}\left(-\frac{1}{e}\right)^n = \sum (-1)^n \frac{n^n}{e^n n!} = \frac{(n+1)^n}{e} < 1$

$x = \frac{1}{e}$ $\sum_{n=1}^{\infty} \frac{n^n}{n!\,e^n} =$ $\left|\frac{\sqrt{2\pi n}\cdot n^n}{e^n \cdot n!} - 1\right| < \varepsilon$

$\frac{1-\varepsilon}{2\pi n} < \frac{n^n}{e^n \cdot n!} < \frac{1+\varepsilon}{\sqrt{2\pi n}}$ ← twierdzenie o 3 ciągach, granica = 0

???

$\lim \frac{\sqrt{2\pi n}\, n^n}{e^n \cdot n!} = 1$

minoranta rozbieżna

szereg rozbieżny

11.40 $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n)!} x^n$

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38
43

$$R = \frac{1}{9} \quad \frac{\cancel{n!}\,(2n+2)!}{(2n)!\;\cancel{(n+1)!}} =$$

$$\frac{(2n)!\,(n+1)!}{(2n+2)!\cdot n!} = \frac{n+1}{(2n+1)(2n+2)} = \frac{1}{(2n+2)} = 0$$

$$11.43 \quad \sum_{n=1}^{\infty} \frac{(3n)!}{n^n (2n)!} x^n \Rightarrow a_n = \frac{(3n)!}{n^n (2n)!}$$

$$D = \lim_{n\to\infty} D_n \Rightarrow D_n = \frac{(3n+3)!}{(n+1)^n (n+1)(2n+2)!} \cdot \frac{n^n \cdot (2n)!}{(3n)!} =$$

$$= \frac{\cancel{(3n)!} \cdot (3n+1)(3n+2)(3n+3)}{(n+1)^n (n+1)\cancel{(2n)!} \cdot (2n+1)(2n+2)} \cdot \frac{n^n \cdot \cancel{(2n)!}}{\cancel{(3n!)}} =$$

$$= \frac{\cancel{n^3} \ldots}{\cancel{n^3} \ldots} = \frac{\left(3+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{3}{n}\right)}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)\left(1+\frac{1}{n}\right)^n}$$

$$\xrightarrow[n\to\infty]{} \frac{27}{4e}$$

$$e^n = \frac{e^{3n}}{e^{2n}}$$

dla $x = \frac{4e}{27}$

$$\sum_{n=1}^{\infty} \frac{(3n)!}{n^n (2n)!} \cdot \left(\frac{4e}{27}\right)^n = \sum_{n=1}^{\infty} \frac{(3n)!}{n^n (2n)!} \cdot \frac{2^{2n} \cdot e^n}{3^{3n}} =$$

$$\sum_{n=1}^{\infty} \frac{(3n)!}{n^n (2n)!} \cdot \frac{2^{2n}}{3^{3n}} \cdot \frac{e^{3n}}{e^{2n}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(3n)!}{n^n \cdot (2n)!} \cdot \frac{2^{2n}}{3^{3n}} \cdot \frac{e^{3n}}{e^{2n}} =$$

$$= \lim_{n\to\infty} \frac{e^{3n}(3n)!}{3^{3n}} \cdot \lim_{n\to\infty} \frac{2^{2n}}{e^{2n} \cdot (2n)!} \cdot \lim \frac{1}{n^n} = 0$$

$$\frac{(3n)! \cdot 2^{2n} \cdot e^{3n}}{(2n)! \cdot 3^{3n} \cdot e^{2n}} \longrightarrow 1$$

$$G_n =$$

$$1 - \varepsilon < G_n < 1 + \varepsilon < 2$$

$$a_n < 2 \cdot \frac{1}{n^n}$$

$$\left[-\frac{4e}{27}, \frac{4e}{27}\right.$$

Jeżeli moduł szeregów jest

$$\sum a_n (x-x_0)^n \quad \lim \frac{a_{n+1}}{a_n} = \frac{1}{R}$$

$$[x-R \,,\, x+R]$$

$$\sum a_n (\cancel{x_0}+R-x_0)^n$$

11.54 $\sum_{n=1}^{\infty} \frac{n^5}{(n+1)!}(x+5)^{2n+1}$

$$R = \frac{1}{q} \quad q = \frac{a_{n+1}}{a_n}$$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{n^5}{(n+1)!} \cdot \frac{(n+2)!}{(n+1)^5} = \lim \frac{n^5 \cdot (n+2)}{(n+1)^5}$$

$$= \infty \qquad x \in (-\infty;\infty) \vee x \in R$$

$$\sum a_n (2x+1)^n = \sum a_n 2^n \left(x+\frac{1}{2^n}\right)^n$$

$$\sum \left| a_n (ax - x_0)^{x_n} \right|$$

$$\lim_{n\to\infty} \frac{|a_{n+1}||ax-x_0|^{n+1}}{|a_n||ax-x_0|^n} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} |ax - x_0| =$$

$$= |ax - x_0| \cdot \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$$

$$\lambda |ax - x_0| < 1$$

$$-R = -\frac{1}{\lambda} < ax - x_0 < \frac{1}{\lambda} = R$$

$$x_0 - R < ax < R + x_0$$

$$\frac{x_0 - R}{a} < x < \frac{x_0 + R}{a}$$