Całki

Podwójne

4.8 $\displaystyle\int_0^4 dx \int_4^{12} xy\,dy = \int_0^4 x\,dx \cdot \int_4^{12} y\,dy =$

$$= \frac{1}{2}x^2\Big|_0^4 \cdot \frac{1}{2}y^2\Big|_4^{12} = 512$$

4.9 $\displaystyle\int_0^a \int_0^b xy(x-y)\,dx\,dy$

$$\int_0^b xy(x-y)\,dx = \int_0^b x^2y - y^2x\,dx = \int_0^b x^2 y - \int_0^b y^2 x =$$

$$= \frac{x^3}{3}y\Big|_0^b - \frac{x^2y^2}{2}\Big|_0^b = \frac{1}{3}b^3y - \frac{1}{2}b^2y^2$$

① $\displaystyle\frac{1}{3}\int_0^a b^3 y\,dy = \frac{1}{6}b^3a^2$ ② $\displaystyle -\frac{1}{2}\int b^2y^2 = -\frac{1}{6}b^2a^3$

$$① + ② = \frac{1}{6}b^3a^2 - \frac{1}{6}b^2a^3 = \frac{1}{6}(b^3a^2 - b^2a^3) =$$

$$= \frac{1}{6}a^2b^2(b-a)$$

4.10 $\int_0^6 \int_t^{10t} \sqrt{st - t^2}\, ds\, dt$

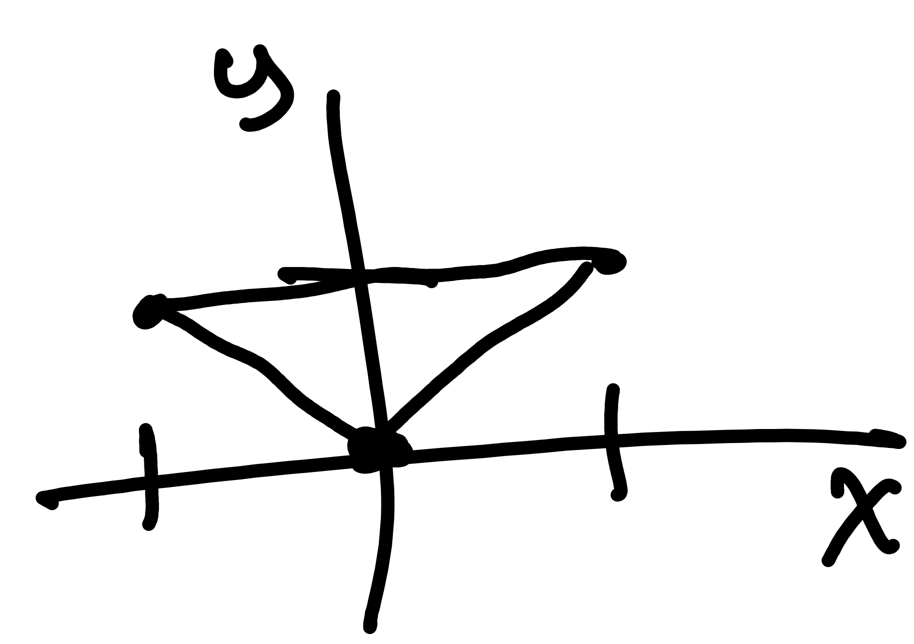
$$\int_t^{10t} (st - t^2)^{\frac{1}{2}}\, ds = \begin{Bmatrix} u = ts - t^2 \\ du = t\, ds \end{Bmatrix} = \frac{1}{t}\int_0^{9t^2} \sqrt{u}\, du =$$

$$= \frac{2u^{\frac{3}{2}}}{3t}\Bigg|_0^{9t^2} = \frac{2(9t^2)^{\frac{3}{2}}}{3t} = 18t^2$$

$$= 18\int_0^6 t^2\, dt = 6\,6^3$$

4.13 $D = \{(x,y) : y \in \langle 0, 1 \rangle ; x \in \langle y, -y \rangle\}$

$$\int_0^1 \int_{-y}^{y} 2x + 1\, dx\, dy.$$

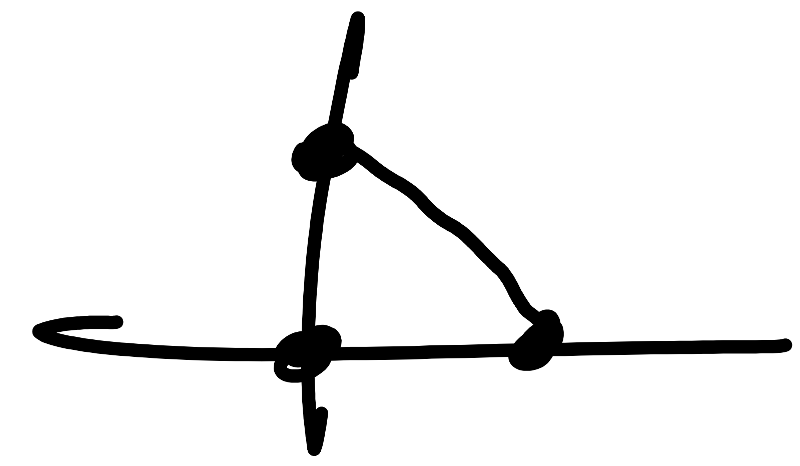


$$x^2 + x\Big|_{-y}^{y} = y^2 + y - y^2 + y = 2y$$

$$\int_0^1 2y\, dy = y^2\Big|_0^1 = 1$$

4.12.

na odwrót!



$$\int_a^{a-x}\int_0^a \sin(x)\cos(y)\,dx\,dy =$$

$$\int -\cos(x)\cos(y)\Big|_0^a = \cos(y) - \cos(y)\cos(a) \int$$

$$\int_a^{a-x} \cos(y) - \cos(y)\cos(a)\,dy =$$

$$= \int_a^{a-x} \cos(y)\,dy - \cos(a)\int_a^{a-x} \cos(y)\,dy =$$

$$\sin(x)\Big|_a^{a-x} + \cos(a)\sin(y)\Big|_a^{a-x} =$$

$$\sin(a) - \sin(a-x) + \cos(a)\sin(a) - \cos(a)\sin(a-x)$$

4.11 $\displaystyle\int_0^{\frac{1}{2}\pi} d\theta \int_{a\cos\theta}^{a} r^4\,dr \quad , a > 0$

$$\int_0^{\frac{\pi}{2}} d\theta \left(\frac{1}{5} r^5 \Big|_{a\cos\theta}^{a} \right) = \int_0^{\frac{\pi}{2}} \frac{1}{5}a^5 - \frac{1}{5}a^5\cos^5\theta \, d\theta =$$

$$= \frac{1}{5}a^5 \left(\int_0^{\frac{\pi}{2}} d\theta - \int_0^{\frac{\pi}{2}} \cos^5\theta \, d\theta \right)$$

$$\int_0^{\frac{\pi}{2}} \cos^5\theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos^4\theta \cos\theta \, d\theta = \left\{ \begin{array}{l} t = \sin\theta \\ dt = \cos\theta \, d\theta \end{array} \right\} =$$

$$= \left\{ \int \left(1 - 2t^2 + 4t^4\right) dt \right\} = \left(\sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta \right) \Big|_0^{\frac{\pi}{2}} =$$

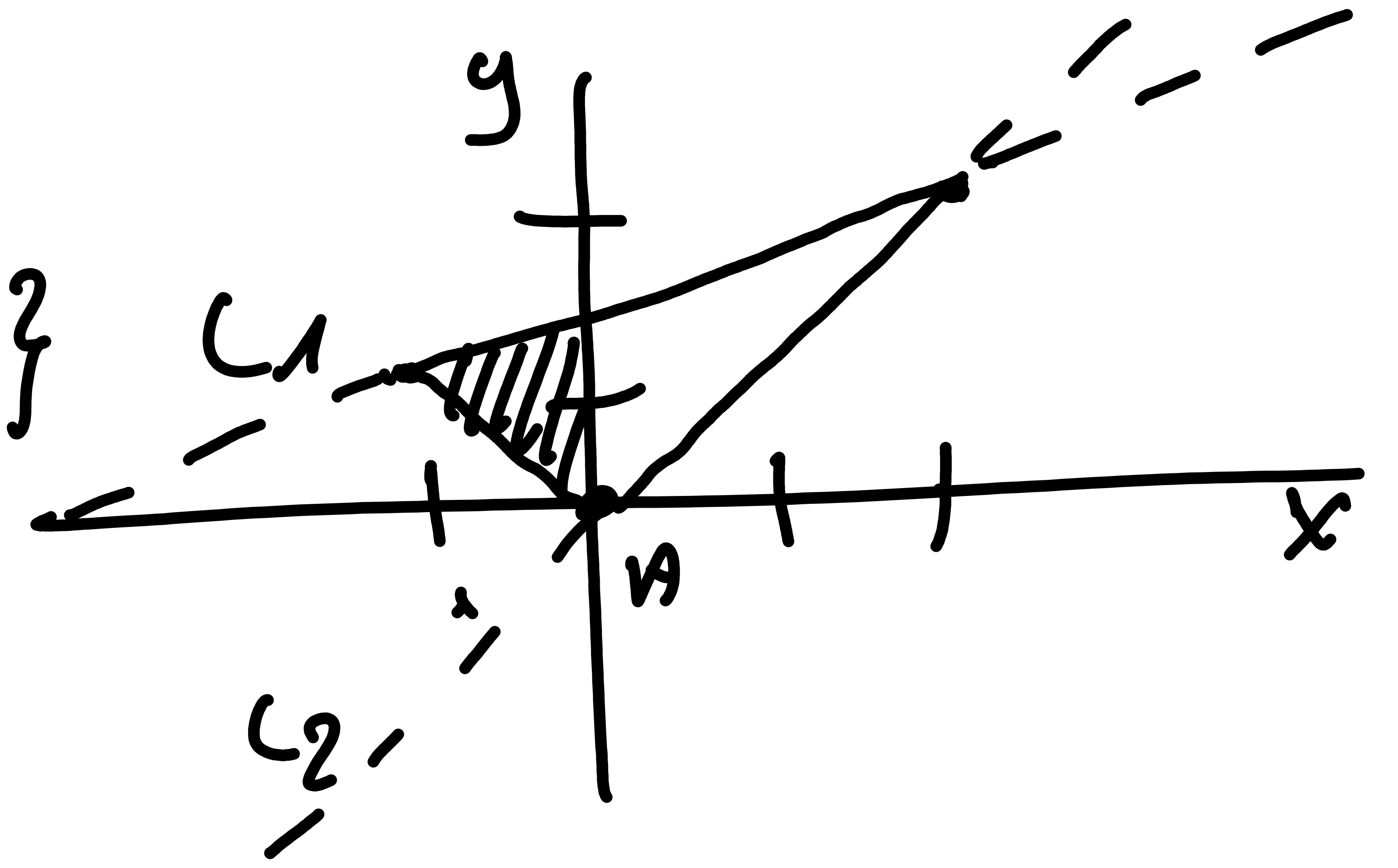
$$= 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}$$

$$\cos^4\theta = \cos^2\theta \cos^2\theta = \left(1 - \sin^2\theta\right)^2$$

$$\frac{1}{5}a^5 \left(\frac{\pi}{2} - \frac{8}{15} \right)$$

4.14 $\iint_D x+2y \, dx\, dy$, $A(0,0)$, $B(2,2)$, $C(-1,1)$

$D=\{(x,y): y \in \langle \quad \rangle; x \in \langle \quad \rangle\}$



$\begin{cases} 1=-a+b \implies b=1+a \\ 2=2a+b \end{cases}$ $\quad a=\frac{1}{3}, b=\frac{1}{3}$

$c_1=\frac{1}{3}x+\frac{4}{3}$

$c_2 = y=x \qquad c_3= y=-x$

$D_1: \begin{cases} 0\leqslant x\leqslant 2 \\ x\leqslant y \leqslant \frac{1}{3}x+\frac{4}{3} \end{cases}$

$D_2: \begin{cases} -1\leqslant x \leqslant 0 \\ -x\leqslant y \leqslant \frac{1}{3}x+\frac{4}{3} \end{cases}$

$\int_0^2\int_x^{\frac{1}{3}x+\frac{1}{3}}(x+2y)\,dy\,dx+\int_{-1}^{0}\int_{-x}^{\frac{1}{3}x+\frac{4}{3}}(x+2y)\,dy\,dx$

Praca domowa: 4.15 – 4.17

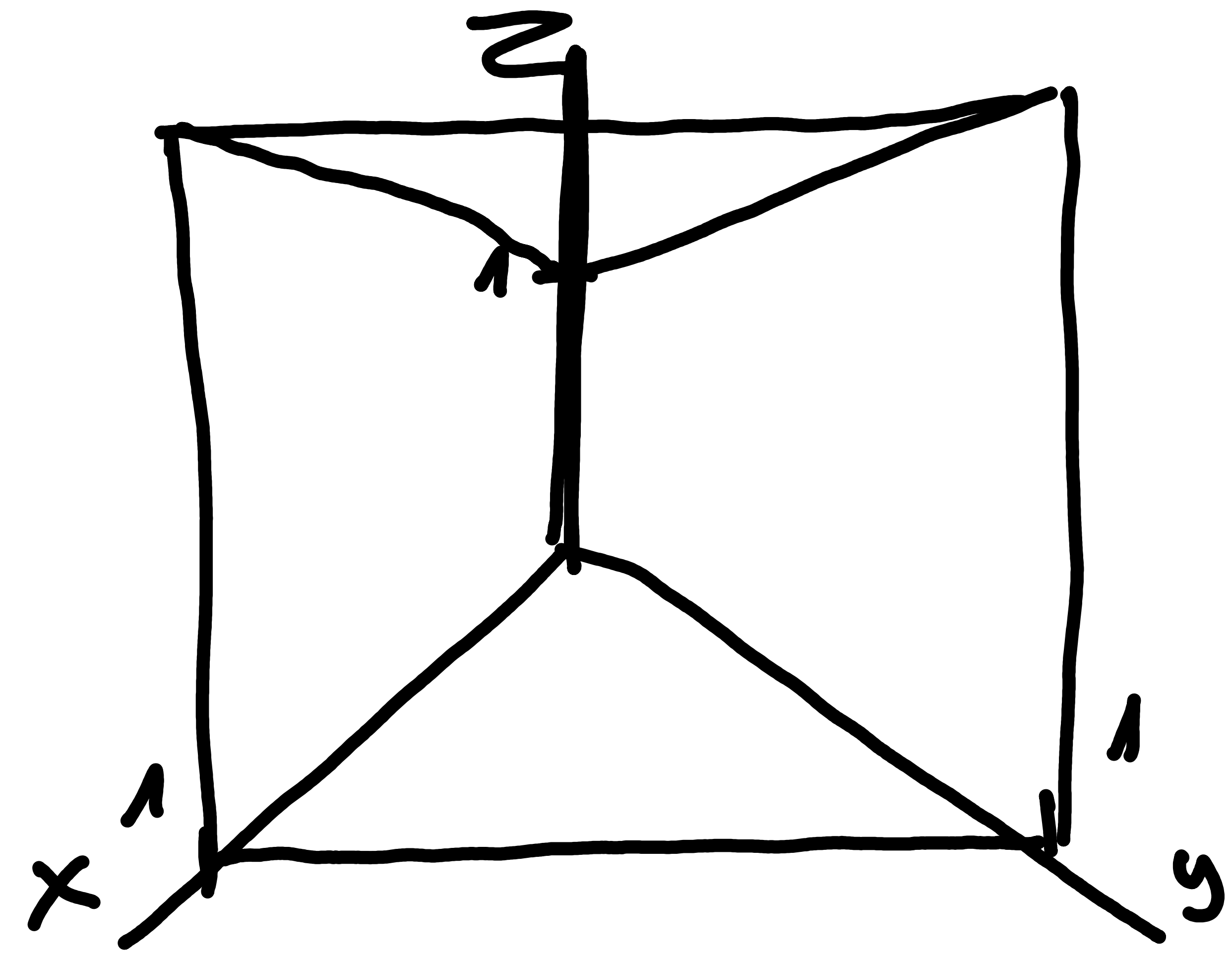
4.17. $z = 1 + x + y$

$x = 0$

$y = 0$

$z = 0$

$x + y = 1$



$D = \{ x \in \langle 0,1 \rangle, y \in \langle 0, 1-x \rangle, z \in \langle 0, 1+x+y \rangle \}$

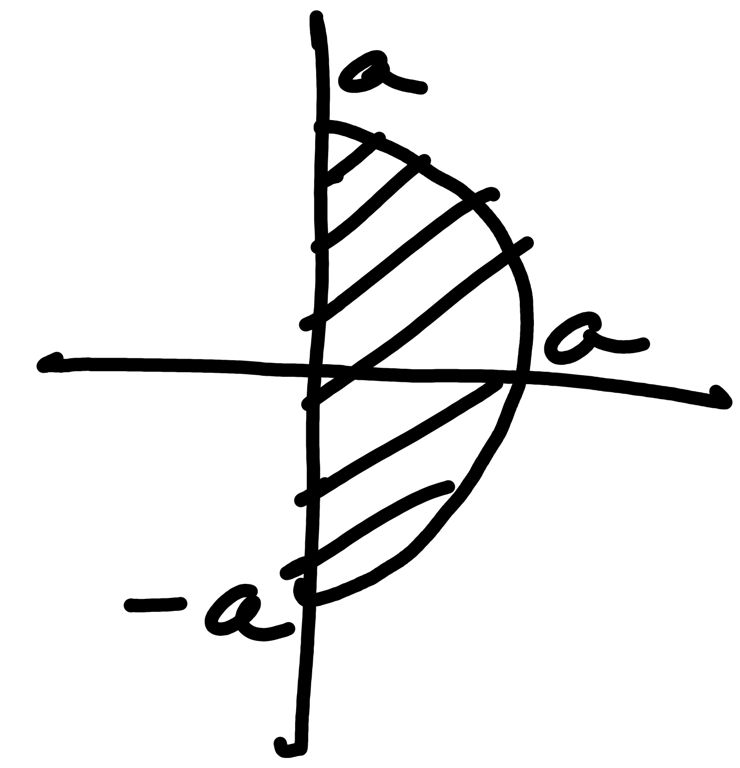
$$\int_0^1 dx \int_0^{1-x} dy \int_0^{1+x+y} dz = \int_0^1 dx \int_0^{1-x} (1+x+y)\, dy =$$

$$= \int_0^1 \left((1-x) + x(1-x) + \frac{1}{2}(1-x)^2 \right) dx =$$

$$= \int_0^1 (1-x)\left(1 + x + \frac{1}{2}(1-x)\right) dx = \int_0^1 \left(-\frac{x^2}{2} - x + \frac{3}{2} \right) dx =$$

$$= \frac{1}{2}\left(-\frac{x^3}{3} - x^2 + 3x \right)\Big|_0^1 = -\frac{1}{6} - \frac{1}{2} + \frac{3}{2} = \frac{5}{6}$$

11.15 $\iint \frac{dx\,dy}{\sqrt{ax-x^2}}$



$y^2 = -ax + a^2$

$D = \{(x,y) : 0 < x < a, -\sqrt{a^2-xy} < y < \sqrt{a^2-xy}$

$$\int_0^a dx \int_{\sqrt{\ }}^{\sqrt{\ }} \frac{1}{\sqrt{ax-x^2}}\, dy = \int_0^a \frac{2\sqrt{x\cancel{(a-x)}}}{\sqrt{x\cancel{(a-x)}}}\, dx =$$

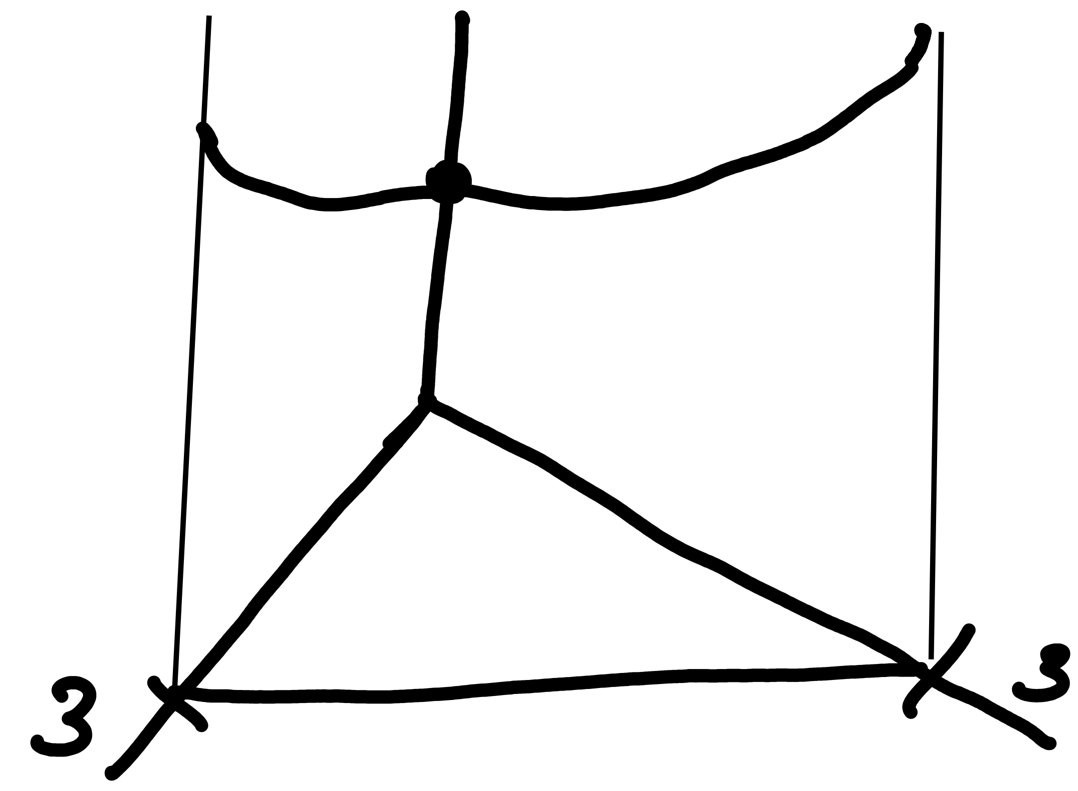
$$= 2\sqrt{x} \int_0^a \frac{dx}{\sqrt{x}} = 4a$$

4.16. $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$D = \{$

4.18 $z = 4x^2 + 2y^2 + 1$

$x + y = 3$



$$D = \{ x \in \langle 0,3 \rangle, y \in \langle 0, 3-x \rangle, z \in \langle 0, 4x^2+2y^2+1 \rangle \}$$

$$\int_0^3 dx \int_0^{3-x} dy \int_0^{4x^2+2y^2+1} dz = \int_0^3 dx \int_0^{3-x} (4x^2+2y^2+1)\, dy = \int_0^3 \left(4x^2(3-x) + \frac{2}{3}(3-x)^3 + (3-x)\right) dx$$

$$= \int_0^3 12x^2 - \int_0^3 4x^3 + \frac{2}{3}\int_0^3 (3-x)^3 + \int_0^3 3 - \int_0^3 x =$$

$$\left\{ \int_0^3 (3-x)^3 dx = \begin{Bmatrix} u = 3-x \\ du = -1 \end{Bmatrix} = \int_3^0 u^3 du = \frac{u^4}{4}\Big|_3^0 = \frac{81}{4} \right\}$$

$$4x^3\Big|_0^3 - x^4\Big|_0^3 + \frac{2}{3}\cdot\frac{81}{4} + 9 - \frac{9}{2} = 45$$

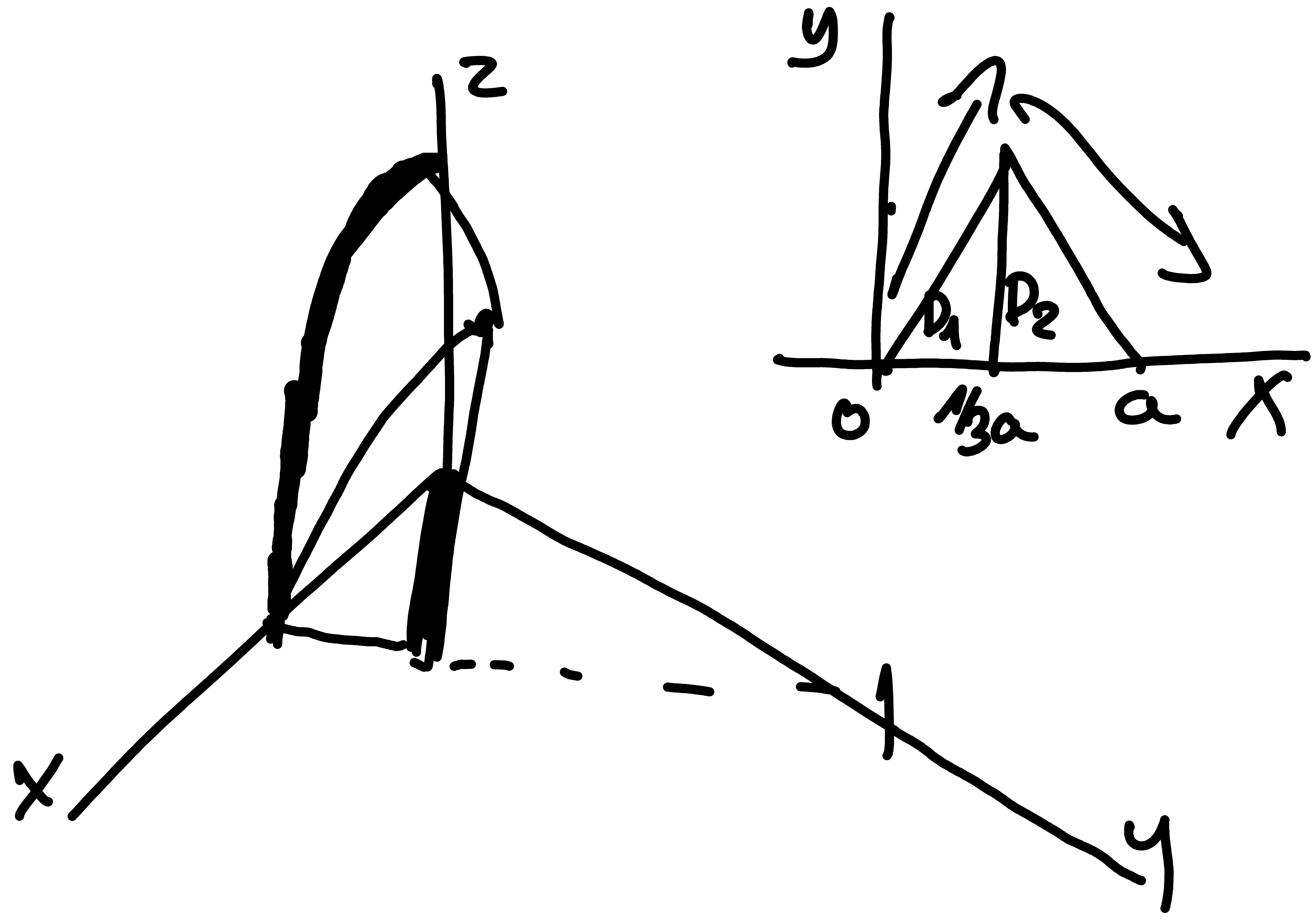
4.19. $z = a^2 - x^2$

$y = 2x$

$x + y = a$

$z = 0$

$y = 0$



$$D_1 = \{ x \in \langle 0, \tfrac{1}{3}a \rangle, y \in \langle 0, 2x \rangle, z \in \langle 0, a^2 - x^2 \rangle \}$$

$$D_2 = \{ x \in \langle \tfrac{1}{3}a, a \rangle, y \in \langle 0, 2-x \rangle, z \in \langle 0, a^2 - x^2 \rangle \}$$

D_1 $$\int_0^{\frac{a}{3}} dx \int_0^{2x} (a^2 - x^2)\, dy = \int_0^{\frac{a}{3}} 2xa - 2x^3\, dx = \frac{a^4}{9} - \frac{a^4}{168}$$

D_2 $$\int_{\frac{a}{3}}^{a} dx \int_0^{2-x} (a^2 - x^2)\, dy = \int_{\frac{a}{3}}^{a} \left(a^2(2-x) - x^2(2-x) \right) dx =$$

$$= \int x^3\, dx - a \int x^2\, dx + a^2 \int (a - x)\, dx =$$

$$= a^3 x - \frac{1}{2} a^2 x^2 - \frac{1}{3} a x^3 + \frac{x^4}{4} \Big|_{\frac{a}{3}}^{a} = \frac{4a^4}{27}$$

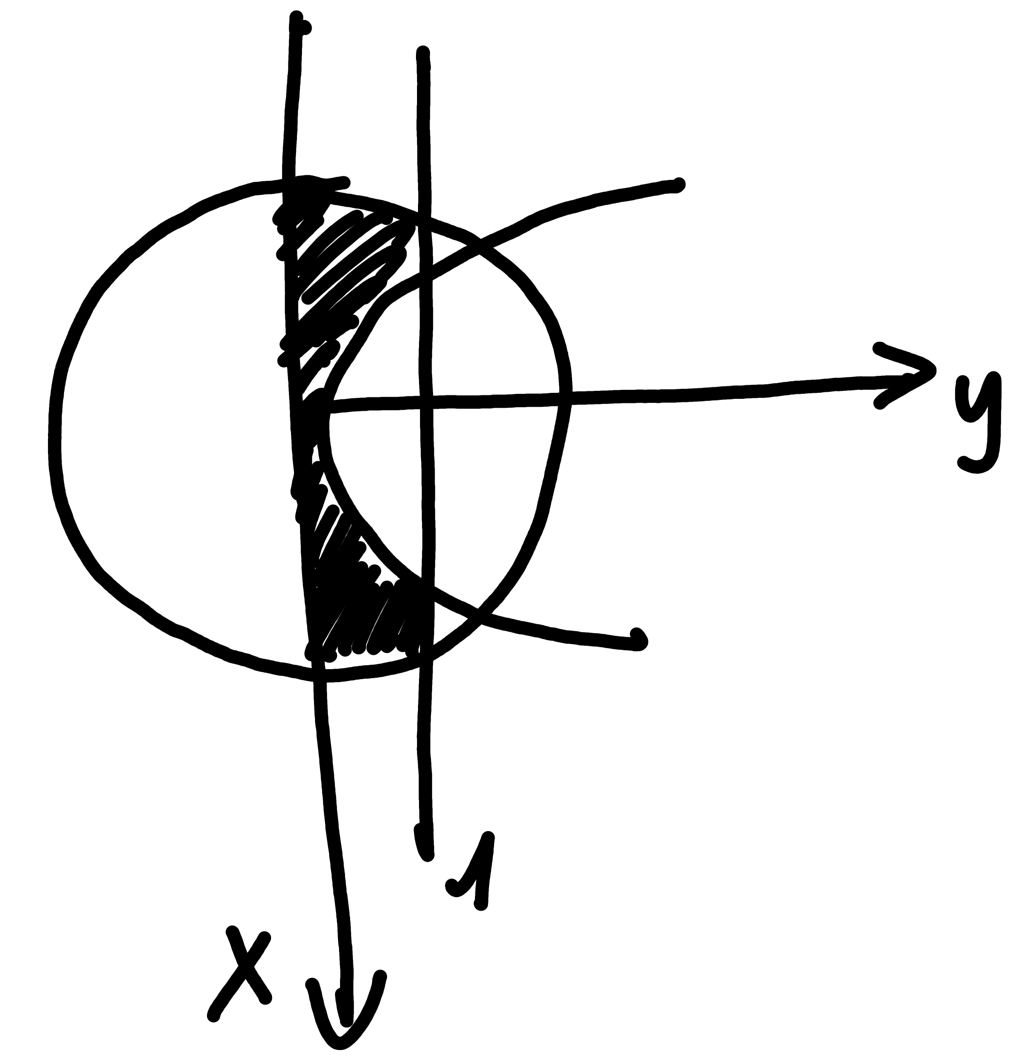
$$D_1 + D_2 = \frac{a^4}{9} - \frac{a^4}{168} + \frac{4a^4}{27}$$

4.20 $y = x^2$

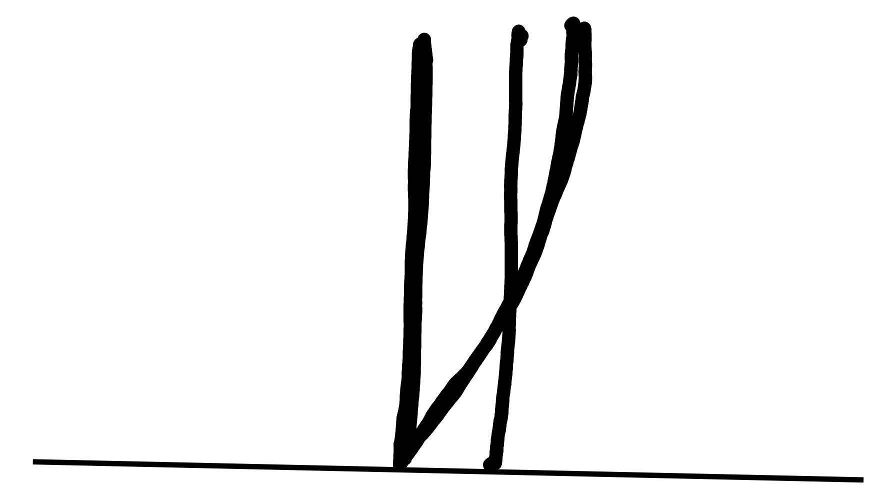
$z = x^2 + y^2$

$y = 1$

$z = 0$



$1 = x^2 + y^2$



$D \in \{ <$

$x \quad 0, 1$