I rzędu równania różniczkowe $a \log_b c = \log_b c^a$	$5xy' = y$ $5x\frac{dy}{dx} = y$ $5\frac{1}{y}dy = \frac{1}{x}dx \quad \Big| \int$ $\ln\|y\| = \frac{1}{5}\ln\|x\| + c \quad \Big| e^{\wedge}$ $y = e^{\frac{\ln\|x\|}{5} + c}$ $y = e^{\frac{\ln\|x\|}{5}} \cdot C$ $y = Ce^{\ln \sqrt[5]{\|x\|}}$ $y = C\sqrt[5]{\|x\|}$

Przykładowe zadanie

$$5xy' = y$$

$$5x\frac{dy}{dx} = y$$

$$\frac{1}{5x}dx = \frac{1}{y}dy \quad \Big| \int$$

$$\frac{1}{5}\ln|x| + C = \ln|y| \quad \Big| e^{a}$$

$$C\sqrt[5]{x} = y$$

$$2x^2y' + y - 4 = 0$$

$$2x^2 \frac{dy}{dx} + y - 4 = 0$$

$$2x^2 \frac{dy}{dx} = -y + 4$$

$$2x^2 \frac{1}{dx} = (-y+4)\frac{1}{dy}$$

$$\frac{1}{x^2}\,dx = 2\frac{1}{-y+4}\,dy \quad \Big| \int$$

$$\int \frac{1}{-y+4}\,dy = \left\{\begin{matrix} v = -y+4 \\ dv = -dy \\ -dt = dv \end{matrix}\right\} = -\int \frac{1}{v}\,dv = -\ln|v|$$

$$\int \frac{1}{x^2} = -\frac{1}{x} + c$$

$$-2\ln(-y+4) = -\frac{1}{x} + c$$

$$\ln(-y+4) = \frac{1}{2x} + c \quad \Big| e^{\wedge}$$

$$-y + 4 = e^{\frac{1}{2x}}$$

$$-y = ce^{\frac{1}{2x}} - 4$$

$$y = -ce^{\frac{1}{2x}} + 4$$

Przykład 2.

$$2x^2y'+y-4=0$$

$$2x^2\frac{dy}{dx}=-y+4$$

$$\frac{1}{2x^2}dx=\frac{1}{-y+4}\quad dy \mid \int$$

$$-\frac{1}{2x}+C=-\ln|-y+4| \quad e^{\wedge}$$

$$Ce^{\frac{1}{2x}}=\frac{1}{-y+4}$$

$$-Ce^{-\frac{1}{2x}}+4=y$$

$\int u\,dv = uv - \int v\,du$

$$x\sqrt{2+y^2} + y\sqrt{2+x^2}\,\frac{dy}{dx} = 0$$

$$y\sqrt{2+x^2} = -x\sqrt{2+y^2}$$

$$y\frac{1}{\sqrt{2+y^2}}dy = -x\frac{1}{\sqrt{2+x^2}}dx$$

$$\int y\frac{1}{\sqrt{2+y^2}}dy = \left\{\begin{array}{l} u = 2+y^2 \\ du = 2y\,dy \\ \frac{du}{2} = 2y\,dy \end{array}\right\} = \frac{1}{2}\int\frac{1}{\sqrt{u}}du =$$

$$= \sqrt{u} + C = \sqrt{2+2y^2} + C$$

$$\sqrt{2+y^2} = -\sqrt{2+x^2} + C \quad |\wedge 2$$

$$2+y^2 = (-\sqrt{2+x^2} + C)^2$$

$$y^2 = (\sqrt{-2+x^2} + C) - 2$$

$$y = \sqrt{\ldots}$$

$$x\sqrt{2+y^2} + y\sqrt{2+x^2}\,\frac{dy}{dx} = 0$$

$$y\sqrt{2+x^2}\,dy = -x\sqrt{2+y^2}\,dx$$

$$y\frac{1}{\sqrt{2+y^2}}\,dy = x\frac{1}{\sqrt{2+x^2}}\,dx \quad \Big|\int$$

$$\int y\frac{1}{\sqrt{2+y^2}}\,dy = \left|\begin{matrix} u = 2+y^2 \\ du = 2y \end{matrix}\right| = \frac{1}{2}\int\frac{1}{\sqrt{u}} =$$

$$= \frac{1}{2}\int u^{-\frac{1}{2}} = \quad u^{\frac{1}{2}} + C = \sqrt{2+y^2} + C$$

$$\sqrt{2+y^2} \quad = -\sqrt{2+x^2} + C \quad \Big|\ {}^{\wedge}2$$

$$2+y^2 = \left(-\sqrt{2+x^2}+C\right)^2$$

$$y = \sqrt{\left(-\sqrt{2+x^2}+C\right)^2 - 2}$$

$$
\begin{array}{r|l}
 & x+1 \\
\hline
-x+1 & -x^2+4 \\
(-) & -x^2+x \\
\hline
 & -x+4 \\
(-) & -x+1 \\
\hline
 & 3
\end{array}
$$

$y' + x^2 = 4 - y'x \quad , \quad y(0) = 1$

$\frac{dy}{dx} + x^2 = 4 - \frac{dy}{dx}x$

$\frac{dy}{dx}(1+x^2) = 4 - x^2 \;|:(1+x)$

$\frac{dy}{dx} = \frac{4-x^2}{1-x} \;|\cdot dx$

$dy = \frac{4-x^2}{1-x}dx \;|\int$

$\int\frac{4-x^2}{1-x} = \int x + \int 1 + \int\frac{3}{1+x} =$

$= \frac{x^2}{2} + x + 3\ln|x+1| + C$

$y = -\frac{x^2}{2} + x + 3\ln|x+1| + c$

dla $y=0$, $c=1$

$$\begin{array}{r|l} & x-1 \\ \hline 1-x & 4-x^2 \\ (-) & x-x^2 \\ \hline & -x+4 \\ (-) & -1+x \\ \hline & 3 \end{array}$$

$$y' + x^2 = 4 - y'x$$

$$\frac{dy}{dx} - x^2 = 4 - \frac{dy}{dx}x$$

$$\frac{dy}{dx} - \frac{dy}{dx}x + x^2 = 4$$

$$\frac{dy}{dx}(1-x) = 4 - x^2$$

$$dy = \frac{4-x^2}{1-x}dx \;\Big|\; \int$$

$$\int \frac{4-x^2}{1-x}dx = \int x - \int dx + \int \frac{3}{1-x}dx$$

$$= \frac{x^2}{2} - x +$$

$$(1+x^2)\frac{dy}{dx} - \sqrt{1-y^2} = 0$$

$$(1+x^2)\frac{dy}{dx} = \sqrt{1-y^2}$$

$$(1+x^2)\frac{1}{dx} = \sqrt{1-y^2}\frac{1}{dy}$$

$$\frac{1}{1+x^2}dx = \frac{1}{\sqrt{1-y^2}}dy \,\Big|\int$$

$$\arcsin(y) = \operatorname{arctgx}(x) + c \,\Big|^{\sin}$$

$$y = \sin(\operatorname{arctgx}(x) + C)$$

$$x(2+e^y)-e^y\frac{dy}{dx}$$

$$-e^y\frac{dy}{dx}=-x(2+e^y)$$

$$e^y\frac{dy}{dx}=x(2+e^y)\Big|\frac{1}{(2+e^y)}$$

$$\frac{e^y}{2+e^y}dy=x\,dx$$

$$\int\frac{e^y}{2+e^y}dy=\left\{\begin{matrix}u=2+e^y\\du=e^y\end{matrix}\right\}=\int\frac{1}{u}du=$$

$$=\ln|u|=\ln|2+e^y|+c$$

$$\frac{1}{2}x^2+C=\ln|2+e^y|$$

$$Ce^{\frac{1}{2}x^2}=2+e^y$$

$$Ce^{\frac{1}{2}x^2}-2=e^y$$

$$y=\ln\left(Ce^{\frac{1}{2}x^2}-2\right)$$

$$y^2 = x\frac{dy}{dx} + y$$

$$y^2 - y = x\frac{dy}{dx}$$

$$\frac{1}{y^2-y}\,dy = \frac{1}{x}\,dx \,/\int$$

$$\int\frac{1}{y^2-y}\,dy = \int\frac{1}{y(y-1)}\,dy = \int\frac{A}{y} + \frac{B}{y-1} =$$

$$A(y-1) + By = 1$$

$$\begin{cases} 0 = A + B \\ 1 = -A \end{cases} \Rightarrow \begin{matrix} A = -1 \\ B = 1 \end{matrix}$$

$$= -\int\frac{1}{y} + \int\frac{1}{y-1} =$$

$$-\ln|y| + \ln|y-1| + C = \ln\left|\frac{y-1}{y}\right| + C$$

$$\ln\left|\frac{y-1}{y}\right| + C = \ln|x| + C$$

$$\frac{y-1}{y} = cx$$

$$y - 1 = Cxy$$

$$y - Cxy = 1$$

$$y(1 - cx) = 1$$

$$y = \frac{1}{1-cx}$$

$$y^2+xy^2+(x^2-yx^2)y'=0$$

$$y^2+xy^2+(x^2-yx^2)\frac{dx}{dy}=0$$

$$(x^2-yx^2)\frac{dy}{dx}=-(y^2+xy^2)$$

$$x^2(1-y)dy=-y^2(1+x)dx$$

$$\frac{(1-y)dy}{y^2}=\frac{-(1+x)dx}{x^2}$$

$$\int\frac{1-y}{y^2}dy=\int\frac{1}{y}-\int\frac{y}{y^2}=-\frac{1}{y}-\ln|y|+c$$

$$-\frac{1}{y}-\ln|y|=\frac{1}{x}-\ln|x|+c$$

postać uwikłana

FILM 2

2) $y' = f(ax + by + c)$

$u = ax + by + c$

3) $y' = f\left(\frac{y}{x}\right)$

$t = \frac{y}{x}$

4) $y' = f\left(\frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}\right)$

TYP 2

$y' = f(ax + by + c)$

$u = ax + by + c$

$y' = 4x - 2y + 5$

$u = 4x - 2y + 5$

$2y = 4x - u + 5$

$y = 2x - \frac{1}{2}u + \frac{5}{2}$

$y' = 2 - \frac{1}{2}u'$

$2 - \frac{1}{2}u' = u \;|\cdot 2$

$4 - u' = 2u$

$4 - \frac{du}{dx} = 2u$ (błąd z −)

$\frac{4}{du} - \frac{1}{dx} = \frac{2u}{du}$

$(4 - 2u)\frac{1}{du} = \frac{1}{dx}$

$\frac{1}{4 - 2u}\,du = dx$

$-\frac{1}{2}\ln|4 - 2u| = x + c \;|\, e^{1}$

$4 - 2u = ce^{-2x}$

? $-2u = ce^{-2x} + 4$

...

$$y' = 4x - 2y + 5$$

$$v = 4x - 2y + 5$$

$$\begin{cases} 2y = 4x - v + 5 \mid :2 \\ y = 2x - \frac{v}{2} + \frac{5}{2} \\ y' = 2 - \frac{1}{2}v' \end{cases}$$

$$2 - \frac{1}{2}v' = v$$

$$\frac{1}{2}v' = 2 - v$$

$$v' = 4 - 2v$$

$$dx = \frac{1}{4-2v}dv \mid \int$$

$$\int \frac{1}{4-2v}dv = \begin{cases} v = 4 - 2v \\ dv = -2 \end{cases} = -\frac{1}{2}\int\frac{1}{v} =$$

$$\frac{1}{2}\ln|v| + c = \frac{1}{2}\ln|4-2v| + c$$

$$x = \frac{1}{2}\ln|4-2v| + c \mid \cdot -2$$

$$-2(x+c) = \ln|4-2v| \quad e^{\wedge}$$

$$e^{-2(x+c)} = 4 - 2v$$

$$\frac{e^{-4(x+c)} - 4}{2} = v \quad \ldots$$

TYP 2

Nie trzeba 3 składników do podstawienia

$$y' = \frac{1}{2x-y} + 2x - y + 2$$

$$u = 2x - y$$

$$y = 2x - u$$

$$y' = 2 - u'$$

$$2 - u' = \frac{1}{u} + u + 2$$

$$-u' = \frac{1}{u} + u$$

$$-u' = \frac{u^2+1}{u}$$

$$-\frac{du}{dx} = \frac{u^2+1}{u}$$

$$dx = \frac{u}{u^2+1} du$$

$$-x = \frac{1}{2}\ln|u^2+1| + c \quad / e^{\wedge}$$

$$ce^{-2x} = u^2 + 1$$

$$ce^{-2x} - 1 = u^2$$

$$\pm\sqrt{ce^{-2x}-1} = u$$

$$\pm\sqrt{ce^{-2x}-1} = 2x - y$$

$$2x \pm \sqrt{ce^{-2x}-1} = y$$

TYP 2	$y' = \sin(x-y)$ $u = x-y$ $y = x-u$ $y' = 1-u'$ $1-u' = \sin(u)$ $u' = 1-\sin(u)$ $\frac{du}{dx} = 1-\sin(u) \quad \vert \cdot dx \quad \vert : \sin(u)$ $\frac{1}{1-\sin(u)}\, du = dx$ $\int \frac{1}{1-\sin(u)}\, du = -\frac{2}{tg(\frac{u}{2})-1} + C$ $-\frac{2}{tg(\frac{u}{2})-1} = x + C$ . . .

TYP 2	$2x-y+(6x-3y+1)y'=0$ $2x-y+(3(2x-y)+1)\cdot y'=0$ $u=2x-y$ $y=2x-u$ $y'=2-u'$ $u+(3u+1)(2-u')=0$ $u+6u-3u-3u\frac{du}{dx}+2-\frac{du}{dx}=0$ $-\frac{du}{dx}(3u+1)=7u+2$ $-\frac{du}{dx}=\frac{7u+2}{3u+1}$ $dx=\frac{3u+1}{7u+2}du \mid \int$ $\int\frac{3u+1}{7u+2}du=\int\frac{3}{7}du+\frac{1}{7}\int\frac{1}{7u+2}du=$ $\frac{3}{7}u+\frac{1}{49}\ln\|7u+2\|+c=x+C$

podstawienie, uwikłane.

TYP 3 $y = f\left(\frac{y}{x}\right)$ $y' = f\left(\frac{y}{x}\right)$	$y' = \frac{y}{x} + tg\,\frac{y}{x}$ $u = \frac{y}{x} \quad / \cdot x$ $y = ux$ $y' = (ux)' = u' \cdot x + u$ $u'x + u = u + tg(u)$ $\frac{du}{dx}x + u = u + tg(u)$ $\frac{du}{dx}x = tg(u)$ $\frac{1}{x}dx = \frac{1}{tg(u)}du \quad / \int$ $\ln\lvert x\rvert = \ln\lvert\sin(u)\rvert + c$ $cx = \sin(u) \quad / \cdot \arcsin(u)$ $u = \arcsin(cx)$ $\frac{y}{x} = \arcsin(cx)$

$$y = x\arcsin(cx)$$

$$x+2y+x\frac{dy}{dx}=0 \quad \Big| : x$$

$$1+2\frac{y}{x}+\frac{dy}{dx}=0$$

$$u=\frac{y}{x}$$

$$y=ux$$

$$y'=u'x+u$$

$$1+2u+u'\cdot x+u=0$$

$$\frac{du}{dx}x=-3u-1$$

$$-\frac{1}{3}\ln|-3u-1|=\ln|x|+C \quad \Big| \cdot(-3)$$

$$-3u-1=ce^{\ln|x|^{-3}}$$

$$-3u=c\cdot\frac{1}{x^3}+1 \quad \Big| : (-3)$$

$$u=\frac{c}{x^3}-\frac{1}{3}$$

$u = \frac{y}{x}$

$y = ux$

$y' = u'x + u$

$$y' = \frac{y^2 - xy}{x^2 - xy + y^2} \quad | : x^2$$

$$y' = \frac{\left(\frac{y}{x}\right)^2 - \frac{y}{x}}{1 - \frac{y}{x} + \left(\frac{y}{x}\right)^2}$$

$$u'x + u = \frac{-u + u^2}{1 - u + u^2}$$

$$\frac{du}{dx} x = \frac{-u + u^2}{1 - u + u^2} - u$$

$$\frac{du}{dx} x = \frac{-u^3 + 2u^2 - 2u}{1 - u + u^2}$$

$$\ln|x| = \int \frac{1 - u + u^2}{-u^3 + 2u^2 - 2u} du$$

...

$u=\frac{y}{x}$ $y=ux$ $y'=u'x+u$	$(x+y)\frac{dy}{dx}+3y=0 \mid :x$ $(1+\frac{y}{x})\frac{dy}{dx}+3\frac{y}{x}=0$ $(1+u)(u'x+u)+3u=0$
	$\frac{dy}{dx}=\frac{y}{x}+\frac{x}{y}=\frac{y}{x}+\frac{1}{\frac{y}{x}}$
	$x\frac{dy}{dx}=y+\sqrt{x^2-y^2} \mid :x$ $\frac{dy}{dx}=\frac{y}{x}+\frac{\sqrt{x^2-y^2}}{x}$ $\frac{dy}{dx}=\frac{y}{x}+\frac{\sqrt{x^2-y^2}}{\sqrt{x^2}}$
	$y(1+\ln y+\ln x)+x\frac{dy}{dx}=x$ $\frac{y}{x}(1+\ln\frac{y}{x})+\frac{dy}{dx}=1$

Równania liniowe pierwszego rzędu.

$$(*)\ p(x)\cdot y' + q(x)\cdot y = r(x)$$

schemat:

1° Rozwiązujemy równanie $p(x)+y'+q(x)\cdot y=0$
Jest to równanie zmiennych rozdzielnych.
Mamy rozwiązanie w postaci $y=C\cdot C\ldots)$

2° W równaniu $\overset{(**)}{y}= C\cdot(\ldots)$ "uzupełniamy stałą" i mamy $y=C(x)\cdot(\ldots)$.

3° liczymy y'

4° y' i y wstawiamy do równania $p(x)$. Składniki z $C(x)$ powinny się skrócić. Wyznaczamy $C'(x)$.

5° Związek $C(x)$ obustronnie całkujemy. Mamy $C(x)=\ldots$

6° $C(x)$ w 5° wstawiamy do $(**)$ i mamy rozwiązanie.

Przykład: $y' - y = 4e^x$

1° $y' - y = 0$

$$\frac{dy}{dx} - y = 0$$

$$\frac{dy}{dx} = y \quad | \cdot dx \quad | : y$$

$$\frac{dy}{y} = dx \quad | \int$$

$$\ln|y| = x + C \quad | e^{\wedge}$$

$$y = Ce^x$$

2° $y = C(x) \cdot e^x$

3° $y' = C'(x) \cdot e^x + C(x)e^x$

4°
$$\cancel{C'(x)e^x} + C(x)e^x - \cancel{C(x)e^x} = 4e^x$$

5° $C(x)e^x = 4e^x$

$$C'(x) = 4 \quad | \int$$

$$C(x) = 4x + C$$

6° $y = (4x + C) \cdot e^x$

$y' - 4y = -1$

1° $y' - 4y = 0$

$\frac{dy}{dx} = 4y$

$\frac{dy}{4y} = dx \quad | \int$

$\ln|y| = 4x \quad | e^{\wedge}$

$y = ce^{4x}$

2° $y = C(x)\, e^{4x}$

3° $y' = c'(x)\, e^{4x} + 4C(x)\, e^{4x}$

4° $c'(x) e^{4x} + \cancel{4C(x) e^{4x}} - \cancel{4C(x) e^{4x}} = -1$

$c'(x) e^{4x} = -1 \quad | : e^{4x}$

5° $c'(x) = -\frac{1}{e^{4x}} \quad | \int \qquad -\int \frac{1}{e^{4x}} = -\int e^{-4x} = \frac{1}{4} e^{-4x} + c$

$C(x) = \frac{1}{4} e^{-4x} + c$

6° $y = \left(\frac{1}{4} e^{-4x} + c\right) e^{4x} = \frac{1}{4} + ce^{4x}$

$$y' - 4xy = x^3 - x$$

1° $y' - 4xy = 0$

$$\frac{dy}{dx} - 4xy = 0$$

$$\frac{1}{y}\,dy = 4x\,dx \quad \Big| \int$$

$$\ln|y| = 2x^2 + c \quad | e^\wedge$$

$$y = ce^{2x^2}$$

2° $y = c(x)\,e^{2x^2}$

3° $y' = c'(x)\,e^{2x^2} + 4c(x)e^{2x^2}$

4° $c'(x)e^{2x^2} + \cancel{4xc(x)e^{2x^2}} - \cancel{4(c(x)e^{2x^2})x} = x^3 - x$

$$c'(x)\,e^{2x^2} = x^3 - x \quad \Big| : e^{2x^2}$$

$$c'(x) = \frac{x^3 - x}{e^{2x^2}} \quad \Big| \int$$

5° $\displaystyle\int \frac{x^3 - x}{e^{2x^2}}\,dx = \int (x^3 - x)\,e^{-2x^2}dx = \int x(x^2 - 1)\,e^{-2x^2} \left\{ \begin{array}{l} u = x^2 \\ du = 2x\,dx \\ \frac{du}{2x} = dx \end{array} \right|$

$= \frac{1}{2}\displaystyle\int (u - 1)e^{-2u} = \left\{ \begin{array}{ll} \int u\,dv = uv - \int v\,du & \\ U = u - 1 & V = -\frac{1}{2}e^{-2u} \\ dU = 1 & dV = e^{-2u} \end{array} \right\} = \frac{(u-1)e^{-2u}}{2} + \frac{1}{2}\int e^{-2u}du =$

$= \dfrac{(u-1)e^{-2u}}{2} - \dfrac{e^{-2u}}{4} = \dfrac{(2x^2 - 1)e^{-2x^2}}{8} + c = C(x)$

6. $y = \left(\dfrac{(2x^2 - 1)e^{-2x^2}}{8} + c \right) e^{2x^2}$

$$\frac{dy}{dx} + \frac{xy}{1+x^2} = \frac{1}{x(1+x^2)}$$

$$\left(U(x)^n\right)' = n \cdot U(x)^{n-1} \cdot U'(x)$$

1° $\frac{dy}{dx} + \frac{xy}{1+x^2} = 0$

$$\frac{dy}{dx} = -\frac{xy}{1+x^2} \quad | \cdot (1+x^2)$$

$$1+x^2 \ \frac{dy}{dx} = -xy \quad | : x$$

$$\frac{1+x^2}{x} \ \frac{dy}{dx} = -y$$

$$\frac{x}{1+x^2} dx = -\frac{1}{y} dy \quad | \int$$

$$\int \frac{x}{1+x^2} = \left\{ \begin{matrix} u = x^2 \\ du = 2x \end{matrix} \right\} = \frac{1}{2} \int \frac{1}{1+u} du = \frac{1}{2} \ln|u+1| = \frac{1}{2} \ln|x^2+1|$$

$$\frac{1}{2} \ln|x^2+1| + C = \ln|y| \quad | e^{\wedge}$$

$$y = c e^{-\frac{1}{2} \ln|x^2+1|} = C \cdot \frac{1}{\sqrt{x^2+1}}$$

2° $y = C(x) \cdot \frac{1}{\sqrt{x^2+1}}$

3° $y' = C'(x) \frac{1}{\sqrt{x^2+1}} + C(x) \frac{C(x)\, x}{\sqrt{1+x^2}} - \frac{C(x)}{(1+x^2)\sqrt{1+x^2}}$

4° -''-

5° $C'(x) = \frac{\sqrt{1+x^2}}{x(1+x^2)}$

. . .

$$x\frac{dy}{dx}+y=x\sin(x)$$

$$\int x\sin(x)=\left\{\begin{array}{ll}\int u\,dv=uv-\int v\,du & \\ u=x & v=-\cos(x)\\ du=1 & dv=\sin(x)\end{array}\right\}=$$

$$=-x\cos(x)+\int\cos(x)\,dx=$$

$$=x\sin(x)+\sin(x)$$

1° $x\frac{dy}{dx}+y=0$

$$y=-x\frac{dy}{dx}$$

$$\frac{1}{y}dy=-\frac{1}{x}dx \,\Big|\int$$

$$\ln|y|=-\ln|x|+c \,\Big|e^{\wedge}$$

$$y=ce^{\ln|x|^{-1}}=\frac{c}{x}$$

2° $y=\frac{c(x)}{x}$

3° $y'=\frac{c'(x)x-c(x)}{x^2}$

4°

$$\frac{c'(x)x-c(x)}{x}+\frac{c(x)}{x}=x\sin(x)$$

$$c'(x)x\cancel{-c(x)}\cancel{+c(x)}=x^2\sin(x)$$

$$c'(x)=x\sin(x)\,\Big|\int$$

5° $c(x)=-x\cos(x)+\sin(x)$

6° $y=\frac{-x\cos(x)+\sin(x)+c}{x}$

Równania Bernoliego

(TYP 4)

$$y^{1-n} = z$$

$$\frac{dy}{dx} + p(x)y + q(x)y^n = 0$$

$y' + y + y^2 \sin(x) = 0 \quad | : y^2$

Bernouli dla $n = 2$ ↑

$$\frac{y'}{y^2} + \frac{1}{y} + \sin(x) = 0$$

$$y^{-1} \stackrel{\text{:}}{=} z$$

$$-\frac{1}{y^2} \cdot \frac{dy}{dx} = \frac{dz}{dx}$$

$$xy' - 4y = x^2\sqrt{y}$$

$$xy' - 4y = x^2 y^{1/2}$$

$$z = y^{1-\frac{1}{2}} = y^{1/2} \,|\, {}^\wedge 2$$

$$z^2 = y$$

$$y' = 2z \cdot z' \; ???$$

$$y^{\frac{1}{2}} = (z^2)^{\frac{1}{2}} = z$$

$$2xzz' - 4z^2 = x^2 z \,|\, :z$$

$$2xz' - 4z = x^2$$

1° $2x\frac{dz}{dx} - 4z = 0$

$$2\frac{1}{x}dx = \frac{1}{z}dz \,|\, \int$$

$$2\ln|x| + c = \ln|z| \,|\, e^\wedge$$

$$cx^2 = z$$

2° $c(x)x^2 = z$

3° $z' = c'(x)x^2 + 2c(x)x$

4° $2x^3c'(x) = x^2 \,|\, 2x^3$

$$c'(x) = \frac{1}{2x} \,|\, \int$$

$$c(x) = \frac{1}{2}\ln|x| + c$$

5° $z = \left(\frac{1}{2}\ln|x| + c\right)x^2$

$$y^{\frac{1}{2}} = \left(\frac{1}{2}\ln|x| + c\right)x^2 \,|\, {}^\wedge 2$$

$$y = \left(\left(\frac{1}{2}\ln x + c\right)x^2\right)^2$$

$$xy' + y - xy^2 \ln(x) = 0$$

$$\left\{\begin{array}{l} y^{-1} = z \quad z^{-1} = y \\ y^2 = z^{-2} \\ y' = -z^{-2} \cdot z' \end{array}\right\}$$

$$x(-z^{-2} \cdot z') + z^{-1} - xz^{-2} = x\ln^2(x) z^{-2} \;\Big|\; : z^{-2}$$

$0°$ $xz' + z = x\ln(x)$

$1°$ $-xz' + z = 0$

$$z = x\frac{dz}{dx}$$

$$\frac{1}{z}dz = \frac{1}{x}\,\cancel{dx} \;\Big|\; \int$$

$$\ln|z| = \ln|x| + c \;\Big|\; e^{\wedge}$$

$$z = cx$$

$2°$ $z = c(x)x$

$3°$ $z' = c'(x)x + c(x)$

$4°$ $-x(c'(x)x + c(x)) + c(x)x = x\ln(x)$

$$-x^2c'(x) \cancel{- xc(x)} \cancel{= c(x)x} = x\ln(x)$$

$$-x^2c'(x) = x\ln(x) \;\Big|\; : x^2$$

$$c'(x) = -\frac{\ln(x)}{x} \;\Big|\; \int$$

$$c(x) = -\frac{1}{2}\ln^2(x) + C$$

$5°$ $z = \left(-\frac{1}{2}\ln^2(x) + C\right)x \;\Big|\; \wedge^{-1}$

$6°$ $y = \dfrac{1}{\left(-\frac{1}{2}\ln^2(x) + c\right)x}$

$$xy' + y - xy^2\ln(x) = 0$$

$$xy' + y = xy^2\ln(x)$$

$$\left\{\begin{aligned} z &= y^{1-2} = y^{-1} \\ y &= z^{-1} \\ y' &= -z^{-2}\cdot z' \\ y^2 &= z^{-2} \end{aligned}\right\}$$

$$x(-z^{-2}\cdot z') + z^{-1} = xz^{-2}\ln(x) \,\Big|\, : z^{-2}$$

0° $-z'x + z = x\ln(x)$

1° $-z'x + z = 0$

$$z = z'x$$

$$1 = z'\frac{x}{z}$$

$$\frac{1}{x}dx = \frac{1}{z}dz \,\Big|\, \int$$

$$\ln|x| + c = \ln|z| \quad |e^{\wedge}$$

$$cx = z$$

2° $c(x)x = z$

3° $z' = c'(x)x + c(x)$

4°

$$-x(c'(x)x + c(x)) + c(x)x = x\ln(x)$$

$$-x^2c'(x) \cancel{- xc(x)} + \cancel{xc(x)} = x\ln(x)$$

$$-xc'(x) = \ln(x)$$

$$c'(x) = -\frac{\ln(x)}{x} \,\Big|\, \int$$

$$-\int\frac{\ln(x)}{x} = \left\{\begin{aligned} u &= \ln(x) \\ du &= \frac{1}{x} \end{aligned}\right\} = -\int u = -\frac{\ln^2(x)}{2}$$

5° $z = -\frac{\ln^2(x)}{2}x$

6° $y = \frac{2}{\ln^2(x)}x$

$$y' = \frac{y^2 - x}{2y(x+1)}$$

$$-\int \frac{x}{(x-1)^2} = \left| \begin{array}{l} u = x-1 \\ x = u+1 \\ du = 1 \end{array} \right| = -\int \frac{u+1}{u^2} du =$$

$$= -\int \frac{1}{u} - \int \frac{1}{u^2} = -\ln|u| + \frac{1}{u} + D$$

$$y' \cdot 2y \cdot (x+1) = y^2 - x \mid : y$$

$$y' \cdot 2(x+1) - y = -xy^{-1}$$

$$\left\{ \begin{array}{ll} z = y^2 & y' = \frac{1}{2} z^{-\frac{1}{2}} \cdot z' \\ y = z^{\frac{1}{2}} & y^{-1} = z^{-\frac{1}{2}} \end{array} \right\}$$

$$\cancel{z^{-\frac{1}{2}}} \cdot z'(x-1) - \cancel{z^{\frac{1}{2}}} \cdot z = -x \cancel{z^{-\frac{1}{2}}}$$

$0°$ $z'(x-1) - z = -x$

$1°$ $z'(x-1) - z = 0$

$$\frac{dz}{dx}(x-1) = z$$

$$\frac{1}{(x-1)} = \frac{1}{z} dz \mid \int$$

$$\ln|x-1| + c = \ln|z| \mid e^{\wedge}$$

$$C(x-1) = z$$

$2°$ $z = C(x) \cdot (x-1)$

$3°$ $z' = C'(x)(x-1) + C(x)$

$4°$ $\left(C'(x)(x-1) + \cancel{C(x)}\right)(x-1) - \cancel{C(x) \cdot (x-1)} = -x$

$$C'(x)(x-1)^2 = -x$$

$$C'(x) = -\frac{x}{(x-1)^2} \mid \int$$

$$C(x) = \ln|x-1| + \frac{1}{x-1} + D$$

$5°$ $z = \left(\ln|x-1| + \frac{1}{x-1} + D\right)(x-1)$

$6°$ $y = \sqrt{\ldots\ldots\ldots}$

$$xy' - 4y = x^2\sqrt{y}$$

$$xy' - 4y = x^2 y^{\frac{1}{2}}$$

$$\left\{\begin{aligned} &z = y^{1-\frac{1}{2}} = y^{\frac{1}{2}} \\ &y = z^2 \\ &y' = 2z \cdot z' \end{aligned}\right|$$

$$x(2z \cdot z') - 4z^2 = x^2 z$$

$$2xz \cdot z' - 4z^2 = x^2 z \,|:z$$

0° $2xz' - 4z = x^2$

1° $2x\,z' - 4z = 0$

$$2x\,z' = 4z$$

$$\frac{1}{2}\,x z' = z$$

$$\frac{1}{z}\frac{dz}{dx} = 2\,\frac{1}{x}$$

$$\frac{1}{z}dz = 2\frac{1}{x}dx \,\Big|\int$$

$$\ln|z| = 2\ln|x| + c \,|\, e^{\wedge}$$

$$z = cx^2$$

2° $z = c(x)x^2$

3° $z' = c'(x)x^2 + 2xc(x)$

4° $2x\left(c'(x)x^2 + 2xc(x)\right) - 4c(x)x^2 = x^2$

$$2x^3c'(x) + \cancel{4x^2c(x)} - \cancel{4c(x)x^2} = x^2 \,|:x^2$$

$$2xc'(x) = 1$$

$$c'(x) = \frac{1}{2x} \,\Big|\int$$

5° $c(x) = \frac{1}{2}\ln|x|$

6° $z = \frac{1}{2}\ln|x|\,x^2$

$$y^{\frac{1}{2}} = \frac{1}{2}\ln|x|x^2$$

$$y = \left(\frac{1}{2}\ln|x|x^2\right)^2$$

$$xy' - 2y = x\sqrt{y}$$

$$xy' - 2y = xy^{\frac{1}{2}}$$

$$\left\{ \begin{array}{l} z = y^{1-\frac{1}{2}} = y^{\frac{1}{2}} \\ y = z^2 \\ y' = 2z \cdot z' \\ y^{\frac{1}{2}} = z \end{array} \right|$$

$$x(2z \cdot z') - 2z^2 = xz \quad |:z$$

0° $2xz' - 2z = x$

1° $2xz' - 2z = 0$

$$xz' = z$$

$$\frac{1}{x}dx = \frac{1}{z}dz \quad |\int$$

$$c + \ln|x| = \ln|z| \quad |e^{\wedge}$$

$$cx = z$$

2° $c(x)x = z$

3° $z' = c'(x)x + c(x)$

4° $2x(c'(x)x + c(x)) - 2c(x)x = x \quad |:x$

$$2c'(x) + \cancel{2c(x)} - \cancel{2c(x)} = 0$$

$$2c'(x)x = 0$$

$$c'(x) = \frac{1}{2x} \quad |\int$$

$$c(x) = \frac{1}{2}\ln|x| + D$$

5° $\frac{1}{2}x\ln|x| + D = z$

6° $\frac{1}{2}x\ln|x| + Dx = y^{\frac{1}{2}} \quad |\wedge 2$

$$\left(\frac{1}{2}x\ln|x| + Dx\right)^2 = y$$

$$y' + 2y\,tg(x) = 4y^2\,tg(x)$$

$$y' = \frac{y^2 - x}{2y(x+1)} \Big| \cdot 2y(x+1)$$

$$\int \frac{x}{(x+1)^2} = \begin{Bmatrix} u = x+1 \\ x = u-1 \end{Bmatrix} = \int \frac{u-1}{u^2} =$$

$$\int \frac{1}{u^2} - \int \frac{1}{u} = \ln|u| + \frac{1}{u} + c =$$

$$y' 2y(x+1) = y^2 - x \,|:y$$

$$y' 2(x+1) = y - \frac{x}{y}$$

$$y' 2(x+1) - y = -xy^{-1}$$

Bernoulli

$$\left| z = y^2,\ z^{\frac{1}{2}} = y,\ y' = \frac{1}{2} z^{-\frac{1}{2}} \cdot z',\ y^{-1} = z^{-\frac{1}{2}} \right|$$

$$\frac{1}{2} z^{-\frac{1}{2}} \cdot 2(x+1) \cdot z' - z^{\frac{1}{2}} = -x z^{-\frac{1}{2}} \Big| : z^{-\frac{1}{2}}$$

1° $(x+1)z' - z = -x$

2° $(x+1)z' - z = 0$

$$(x+1)\frac{dz}{dx} = z$$

$$\frac{1}{x+1}\,dx = \frac{1}{z}\,dz$$

$$\ln|x+1| + c = \ln|z| \,|\, e^{\wedge}$$

3° $c(x+1) = c(x) \cdot (x+1) = z$

4° $z' = c'(x) \cdot (x+1) + c(x)$

5° $(x+1) \cdot (c'(x)(x+1) + \cancel{c(x)}) - \cancel{c(x)(x+1)} = -x$

$$c'(x)(x+1)^2 = -x$$

$$c'(x) = -\frac{x}{(x+1)^2} \Big| \int$$

6° $z = \left(-\ln(x+1) - \frac{1}{x+1} + c\right)(x+1)$

$$z = -(x+1)\ln|x+1| - 1 + c(x+1)$$

$$y^2 = \ldots$$

$$y = \sqrt{\ldots}$$

Równania różniczkowe

2 rzędu

$F(x, y, y', y'') = 0 \leftarrow$ jednorodne

$ay'' + by' + cy = r(x) \leftarrow$ równanie II rzędu liniowe, o stałych współczynnikach.

I Metoda przewidywań

$$\underline{y = y_j + y_p}$$

1° $ay'' + by' + cy = 0$ – r-nie "jednorodne"

$ar^2 + br + c = 0$

$\Delta = \ldots$

$y_j = C_1 e^{r_1 x} + C_2 e^{r_2 x}$ $\quad \Delta > 0$

$y_j = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$ $\quad \Delta < 0$

12.45

$$y'' = -y$$

$$\left\{ y' = v(y) \right\}$$

$$\frac{dv}{dy} v(y) = -y$$

$$dv \; v(y) = -y \, dy \,\Big|\int$$

$$\frac{v^2}{2} = -\frac{y^2}{2} + C$$

$$v^2 = -y^2$$

$$\frac{dy}{dx} = \sqrt{-y^2 + c}$$

$$dx = \frac{1}{\sqrt{-y^2 + c}} dy \,\Big|\int$$

$$\int \frac{1}{\sqrt{-y^2+c}} dy = \arcsin \frac{y}{\sqrt{c}} + D$$

$$x = \arcsin \frac{y}{\sqrt{c}} + D$$

$$x - D = \arcsin \frac{y}{\sqrt{c}} \,\Big|\cdot \sin$$

$$\sin(x - D) = \frac{y}{\sqrt{c}} \,\Big|\cdot \sqrt{c}$$

$$\sqrt{c} \sin(x - D) = y$$

12.55 $y'' = y'^3 \ln(y)$

$\{ y' = v(y) \}$

$$\frac{dv}{dy} v(y) = v(y)^3 \ln(y)$$

$$dv \; v(y)^{-2} = (\ln(y))\,dy \;\Big|\int$$

$$v(y) = -\frac{1}{y} + c$$

$$\frac{dy}{dx} = -\frac{1}{y} + c$$

$$\frac{1}{-\frac{1}{y}+c}\,dy = dx \;\Big|\int$$

$$\int \frac{1}{-\frac{1}{y}+c}\,dy = \left\{ \begin{array}{l} v = cx - 1 \\ dv = c\,dy \\ y = \frac{v+1}{c} \end{array} \right\} = \frac{1}{c^2}\int \frac{v+1}{v}\,dv =$$

$$= \frac{1}{c^2}\left(\ln(v) + v + D\right) \triangleq \frac{1}{c^2}\left(\ln(cx-1 \; \ldots\right.$$

12.69

$$2y''(4-y)=1+y'^2$$

$$\{y'=u(y)\}$$

$$2\frac{du}{dy}u(y)(4-y)=1+u(y)^2$$

$$2\frac{u(y)}{1+u(y)^2}\,du=\frac{1}{(4-y)}dy \quad \int$$

$$2\int\frac{u(y)}{1+u(y)^2}\,du=\left\{\begin{array}{l}v=1+u(y)^2\\ dv=2u(y)\end{array}\right\}=\int\frac{1}{v}=$$

$$\ln|v|+c=\ln|1+u(y)^2|+c$$

$$\ln|4-y|=\ln|1+u(y)^2|+c \quad e^{a}$$

$$4-y=(1+u(y)^2)c$$

$$y=4-(1+u(y)^2)c$$

$$x=\int\frac{1}{\sqrt{\frac{4-y}{c}-1}}\,dy=$$

12.94

13.86

7.62

12. 82

$$y y'' - \underline{y}'^2 = 0$$

$$\begin{cases} y = e^u \\ y' = \underline{e^u u'} \\ y'' = e^u(u'^2 + u'') \end{cases}$$

$$\cancel{e^{2u}}(u'^2 + u'') - (\cancel{e^u} u')^{\cancel{2}} = 0$$

$$\cancel{u'^2} + u'' - \cancel{u'^2} = 0$$

$$u'' = 0$$

$$u = ax + \text{б}$$

Z rzędu prewidywań

Etap I

$y'' + y' - 2y = 4x$

$r^2 + r - 2 = 0$

$\Delta = 9$

$r_1 = -2, \ r_2 = 1$

$y_j = c_1 e^{-2x} + c_2 e^{x}$

Etap II

$r(x) = 4x$

$y_p = Ax + B$

4.8 $\int_0^4 dx \int_4^{12} xy\,dy = \int_0^4 x\,dx \cdot \int_4^{12} y\,dy =$

$= \frac{1}{2} x^2 \Big|_0^4 \cdot \frac{1}{2} y^2 \Big|_4^{12} = 512$

4.9 $\int_0^a \int_0^b xy(x-y)\,dx\,dy$

$\int_0^b xy(x-y)\,dx = \int_0^b x^2y - y^2x\,dx = \int_0^b x^2 y\,\cancel{dx} - \int_0^b y^2\,\cancel{x\,dx} =$

$= \frac{x^3}{3} y \Big|_0^b - \frac{x^2y^2}{2} \Big|_0^b = \frac{1}{3} b^3 y - \frac{1}{2} b^2 y^2$

① $\frac{1}{3}\int_0^a b^3 y\,dy = \frac{1}{6} b^3 a^2$ ② $-\frac{1}{2}\int b^2 y^2 = -\frac{1}{6} b^2 a^3$

① + ② $= \frac{1}{6} b^3 a^2 - \frac{1}{6} b^2 a^3 = \frac{1}{6}(b^3a^2 - b^2a^3) =$

$= \frac{1}{6} a^2 b^2 (b-a)$

4.10 $\int_0^6 \int_t^{10t} \sqrt{st - t^2}\, ds\, dt$

$\int_t^{10t} (st - t^2)^{\frac{1}{2}} ds = \begin{cases} u = ts - t^2 \\ du = t\,ds \end{cases} = \frac{1}{t} \int_0^{9t^2} \sqrt{u}\, du =$

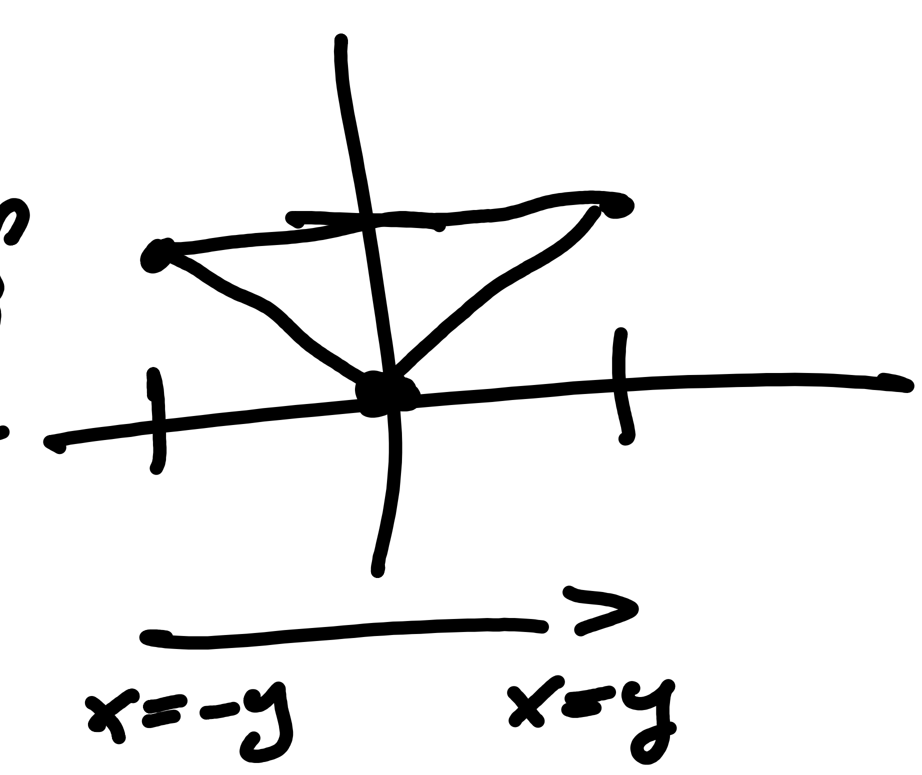
$= \left. \frac{2u^{\frac{3}{2}}}{3t} \right|_0^{9t^2} = \frac{2(9t^2)^{\frac{3}{2}}}{3t} = 18t^2$

$= 18 \int_0^6 t^2 dt = 6 \cdot 6^3$

4.13 $A(-1,1)$ $B(1,1)$ $C(0,0)$

$\int_0^1 \int_{-y}^{y} 2x + 1\, dx\, dy.$

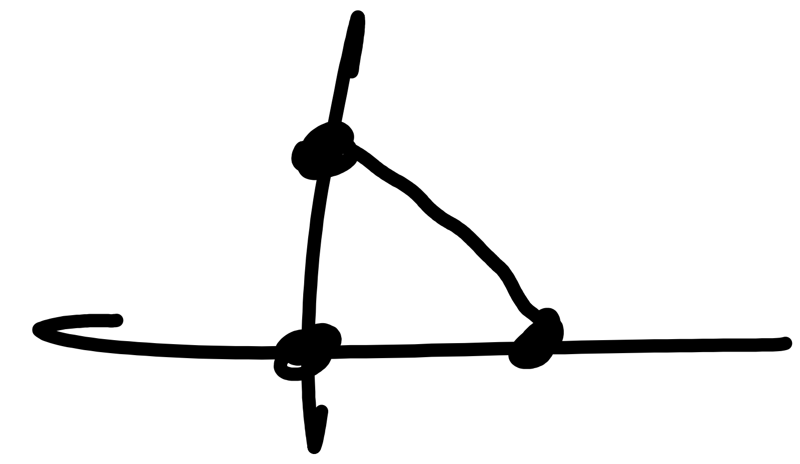
0,1 {



$x^2 + x \Big|_{-y}^{y} = y^2 + y - y^2 + y = 2y$

$\int_0^1 2y\, dy = y^2 \Big|_0^1 = 1$

4.12.



$$\int_a^{a-x}\int_0^a \sin(x)\cos(y)\,dx\,dy =$$

$$\left\{ -\cos(x)\cos(y)\Big|_0^a = \cos(y) - \cos(y)\cos(a) \right\}$$

$$\int_a^{a-x} \cos(y) - \cos(y)\cos(a)\,dy =$$

$$= \int_a^{a-x} \cos(y)\,dy - \cos(a)\int_a^{a-x}\cos(y)\,dy =$$

$$\sin(x)\Big|_a^{a-x} + \cos(a)\sin(y)\Big|_a^{a-x} =$$

$$\sin(a) - \sin(a-x) + \cos(a)\sin(a) - \cos(a)\sin(a-x)$$