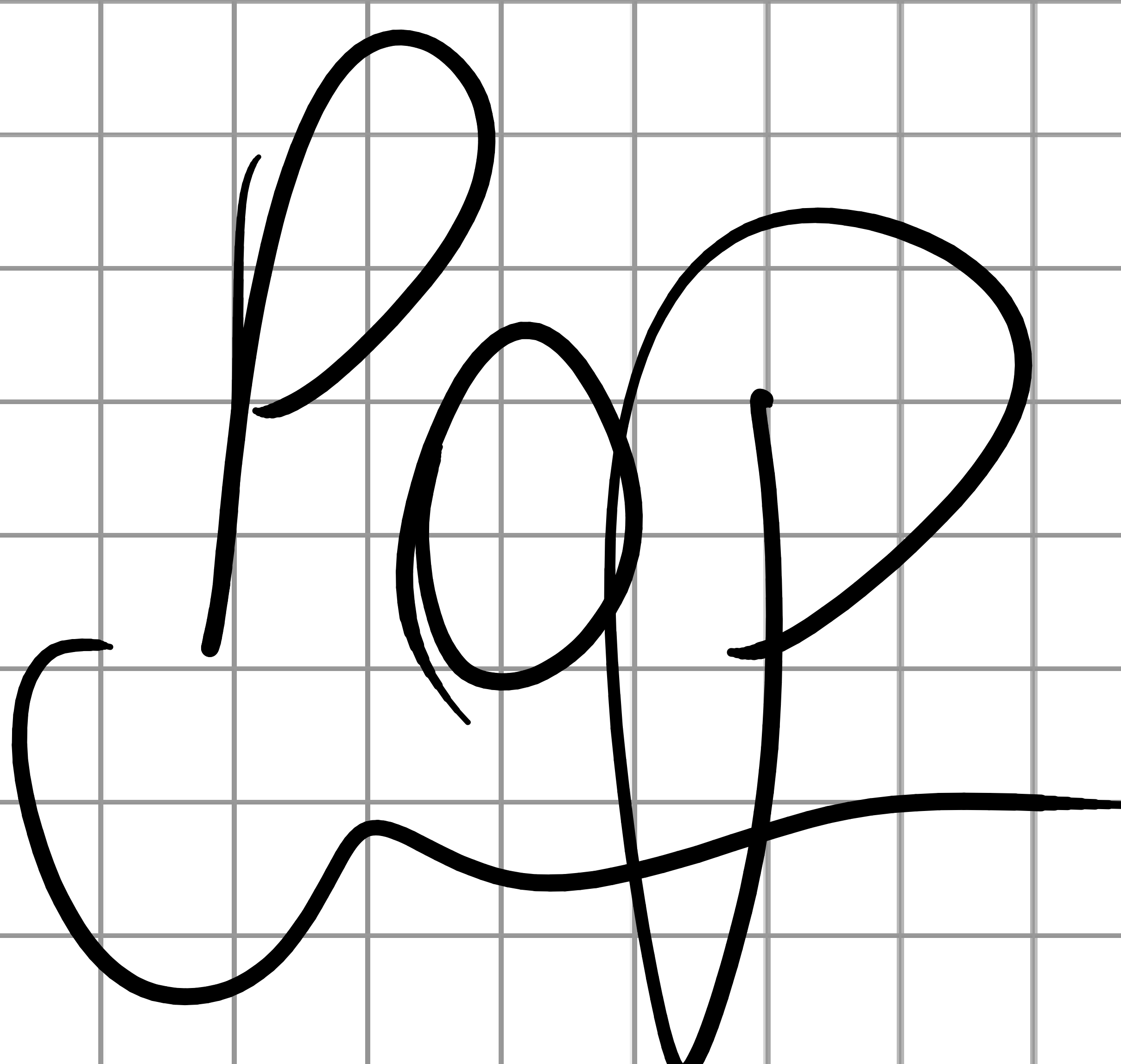
M 2



Definicja granicy ciągu

$g = \lim_{n \to \infty} a_n$ iff $\forall_{\varepsilon > 0} \exists_{n_\varepsilon} \forall_{n > n_\varepsilon} |g - a_n| < \varepsilon$

$a_n = \frac{1}{n}$

$\lim a_n = 0$, 60

$\left|\frac{1}{n} - 0\right| < \varepsilon$

$\frac{1}{n} < \varepsilon \; / \cdot n \;\; : \varepsilon$

$n_\varepsilon = \frac{1}{\varepsilon} < n$

$$b_n = \frac{1}{n2^n}$$

$$\lim_{n\to\infty} b_n = 0 \text{ , bo:}$$

$$\left|\frac{1}{n2^n} - 0\right| < \varepsilon$$

$$\frac{1}{n2^n} < \varepsilon$$

$$\frac{1}{\varepsilon} < n2^n$$

jeżeli $\underbrace{\frac{1}{\varepsilon}}_{= M_\varepsilon} < n < n2^n$

$$M_\varepsilon = \frac{1}{\varepsilon}$$

Twierdzenie o działaniach na granicach

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

jeżeli 2 z tych granic istnieją, to 3 też

$$\lim (a_n \cdot b_n) = \lim a \cdot \lim b$$

$$\lim \left(\frac{a_n}{b_n}\right) = \frac{\lim a_n}{\lim b_n}, \text{ gdy } b_n \neq 0$$

Twierdzenie o zupełności liczb rzeczywistych

Ciąg monotoniczny i ograniczony z granicą.

$$e_n = (1 + 1/n)^n$$

1) $e_n \geq 2$ $\quad e_1 = 2,\ e_n \geq 2$

2) $e_n < 3$

$$\left(1 + 1/n\right)^n = \binom{n}{0} + \binom{n}{1}\frac{1}{n} \dots + \binom{n}{n}\frac{1}{n^n}$$

$$= 1 + 1 + \frac{n(n-1)}{2n^2} + \frac{n(n-1)(n-2)}{2 \cdot 3 \cdot n^3} \dots$$

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$$

Lemat Bernoulliego

jeśli $a > -1$, to $(1+a)^n \geq 1+na$

$n=1$ $L \geq P$

załóżmy, że $(1+a)^n \geq 1+na$

$(1+a)^n \geq 1+na \mid \cdot (1+a)$

$(1+a)^{n+1} = 1+(n+1)a + na^2 \geq 1+(n+1)a$

$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$

$\left(\frac{n+1}{n}\right)^n \overset{?}{<} \left(\frac{n+2}{n+1}\right)^n \cdot \left(\frac{n+2}{n+1}\right) \Big| \frac{n+1}{n+2} \cdot \left(\frac{n}{n+1}\right)^n$

$\frac{n+1}{n+2} \overset{?}{<} \left(\frac{(n+2)n}{(n+1)^2}\right)^n$

$\left(\frac{n^2+2n}{n^2+2n+1}\right)^n > \frac{n+1}{n+2}$

verte →

$$\left(1+\frac{-1}{n^2+2n+1}\right)^n > 1-\frac{1}{n+2}$$

$$\left(1+\frac{-1}{n^2+2n+1}\right)^n \geq \frac{-n}{n^2+2n+1} > \frac{-n}{n^2+2n} = 1-\frac{1}{n+2}$$

Granice niewłaściwe

1) $a_n \to +\infty$, gdy $\forall_M \exists_{M_\varepsilon} \forall_{n>M_\varepsilon} a_n > M$

2) $a_n \to -\infty$, gdy $\forall_M \exists_{M_\varepsilon} \forall_{n>M_\varepsilon} a_n < M$

Symbole niewłaściwe:

$\infty-(+\infty)$, $\frac{\pm\infty}{\pm\infty}$, $(+\infty)^0$, $1^{+\infty}$, $0\cdot\infty$

$\frac{0}{0}$, 0^0

$$\lim_{x\to 0} \frac{\arcsin(x)}{x} = \left[\frac{0}{0}\right] \overset{H}{=} \lim_{x\to 0} \frac{1}{\sqrt{1-x^2}} = 1$$

$$\lim_{x\to 0} \frac{\arcsin(x)-x}{x^3} \left[\frac{0}{0}\right] = \lim \frac{\frac{1}{\sqrt{1-x^2}}-1}{3x^2} =$$

$$= \lim_{x=0} \frac{1-\sqrt{1-x^2}}{3x^2\sqrt{1-x^2}} = \lim_{x\to 0} \left(\frac{1}{\sqrt{1-x^2}}\right)^{-1} \cdot \frac{1-\sqrt{1-x^2}}{3x^2}$$

$$\left[\frac{0}{0}\right] = \lim \frac{\_\ \_\ \_}{\_\ \_\ \_}$$

Twierdzenie o reszcie Peano ???

$$f(x) = f(0) + f'(0) + \frac{x^2}{2} f''(0) + \ldots =$$

$$= \lim_{x \to 0} \frac{o(x^4)}{x^4} = 0 \quad ??$$

$$\lim \frac{\operatorname{arctg}(x)-x}{x^3} = \lim \frac{\frac{1}{x^2+1}-1}{3x^2} =$$

$$= \lim \frac{\frac{1-x^2-1}{x^2+1}}{3x^2} = \lim \frac{-x^2}{(x^2+1)3x^2} =$$

$$= \frac{-1\cancel{x^2}}{(x^2+1)3\cancel{x^2}} = -\frac{1}{3}$$

Peano ??? ? ? ? ? ?

$$\lim \frac{\operatorname{arctg}(x) - x}{x^3}$$

$$(\operatorname{arctg}(x))' = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

$$(\operatorname{arctg}(x))' = \sum_{n=0}^{\infty} \binom{-1}{n} \cdot (x^2) = \sum_{n=0}^{\infty}$$

$$\binom{-1}{n} = \frac{-1 \cdot -2 \cdot \ldots \cdot -n}{n!}$$