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Technical Note

A Note on the Nomenclature in Neutron Multiplicity Mathematics

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Abstract — *The purpose of this technical note is to consolidate the notations used for describing parameters that pertain to neutron multiplicity mathematics relevant to various applications including nonproliferation, international safeguards, and criticality safety among others. The nomenclatures used in these techniques vary widely depending on the origin of the work and their applications. We aim to consolidate many of the previously used notations in a single document to enhance past, present, and future technical exchanges pertaining to neutron multiplicity. This will help avoid confusion in future publications and will facilitate wider application-independent advancements and utility of peer-reviewed findings. A brief introduction and history of neutron multiplicity counting is presented, followed by a summary of commonly used techniques in a variety of different applications. In each section, we present the notations used in previous publications for the reader's reference.*

Keywords — *Neutron multiplicity counting, reactor noise analysis, criticality safety, nuclear nondestructive assay.*

I. INTRODUCTION

Zero-power neutron noise analysis techniques are well-established methods that observe the fluctuations in the neutron population to infer information that pertains to the nuclear assembly of interest. The fluctuations are measured by detecting the multiplicities of neutrons that are emitted from fission chains and are studied by analyzing the statistics of the observed time/count distributions; this is commonly referred to as neutron multiplicity counting (NMC). The applications for NMC can range from reactivity measurements for criticality safety or benchmarking purposes[1,2] to nuclear nondestructive assay (NDA) measurements for nuclear safeguards.[3] Many NMC techniques have been derived from common seminal works; however, depending on the origin of the work and its relevant field of applications, the notations used to describe similar, if not the same, parameters vary widely. Furthermore, the advances and developments made in one field (e.g., dead-time corrections, new detector materials, data and uncertainty analyses, etc.) may be of benefit to others.[4] We are motivated to provide a document that consolidates many of the previously used notations to prevent potential confusion in future publications in hopes to facilitate wider application-independent developments and utility of peer-reviewed findings.

The structure of this note is as follows. In Sec. II we discuss the nuclear fission process and establish the timescales of different courses of de-excitation. In Sec. III we describe neutron multiplication and its relationship to the neutron multiplication factor. In Sec. IV we describe the underlying principles of the point kinetics model and how different aspects of the model are implemented based on the application. In Sec. V we discuss NMC for reactivity measurements and briefly describe two commonly used methods. In Sec. VI we describe NMC for applications in NDA and also describe different gate generation techniques. In Sec. VII we discuss the implications and potential benefits of new detector technology. Last, we draw our conclusions on the significance of this document in Sec. VIII. Sections III through VI include a table that compares the nomenclature used for all relevant parameters.

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II. NUCLEAR FISSION PROCESS

When certain heavy nuclides undergo fission, each fission event produces several neutron-rich unstable fission fragments. These fission fragments will emit multiplicities of neutrons (order of 10^{-18} to 10^{-15} s) and gamma rays (order of 10^{-14} to 10^{-10} s) following the initial split to disperse the energy created in the process to reach a stable state; these emitted particles are known as prompt neutrons and prompt gamma rays.[5] However, even after the emission of these prompt particles, the fission fragments may still be unstable and will undergo radioactive decay (order of $>10^{-3}$ s), which also emits neutrons and gamma rays. These emitted particles are known as delayed neutrons and gamma rays as they are emitted much later in time.[5] Both prompt and delayed neutrons can initiate subsequent fission events. These self-propagating fissions are known as fission chains and describe the multiplicative properties of a nuclear system. The multiplication of an assembly (described in Sec. III) is an important characteristic quantity for both criticality safety and safeguards applications.

III. NEUTRON MULTIPLICATION

The definition of neutron multiplication involves two different descriptions, which are the leakage multiplication (also known as the escape or net multiplication) and the total multiplication. The total multiplication refers to the cumulative number of neutrons produced within the system from the induced fissions due to a single initiating neutron whereas the leakage multiplication considers only these "multiplied" neutrons that leak out of the assembly. The definition of total and leakage multiplication was first described by Serber.[6] The leakage multiplication of a multiplying assembly, which considers the loss mechanisms of neutron (i.e., parasitic and fission captures), is defined as

$$M_L = 1 + Q_{F,n}\left(\overline{v} - 1 - \frac{\Sigma_c}{\Sigma_f}\right) , \tag{1}$$

where

$Q_{F,n}$ = number of induced fissions caused by a single initiating neutron (i.e., induced fission rate per neutron)

$\overline{v}$ = mean number of neutrons emitted from an induced fission event

Σ_c/Σ_f = ratio of capture cross section to fission cross section.

The total multiplication, which describes the cumulative neutron production, takes the form

$$M_T = 1 + Q_{F,n}\overline{v} . \tag{2}$$

As one would expect, the net multiplication will always be less than or equal to the total multiplication depending on the geometry of the active material. The total multiplication can be related to the effective neutron multiplication factor k_{eff}, which takes the form

$$M_T = \frac{1}{1 - k_{eff}} = \frac{1}{1 - p_{if}\overline{v}} ,\ k_{eff} < 1 , \tag{3}$$

with

$$k_{eff} = p_{if}\overline{v} , \tag{4}$$

where p_{if} is the probability of a neutron causing an induced fission. The prompt neutron multiplication factor is related to the effective neutron multiplication factor through

$$k_p \approx k_{eff}\left(1 - \beta_{eff}\right) , \tag{5}$$

where $ß_{eff}$ is the effective delayed neutron fraction. It has been shown that if the timescale over which neutrons are observed is much shorter than the average delayed neutron period then we can assume that the observed fluctuations are dominated by the behavior of only prompt neutrons and the approximation of

$$\overline{v} = \overline{v_p} = v_{if,1} \tag{6}$$

and

$$k_p \approx k_{eff} \tag{7}$$

can be made, where v_p is the number of prompt neutrons emitted per fission, and $v_{if,1}$ is the mean number of prompt neutrons emitted per fission (i.e., first-order factorial moment of induced fission prompt neutron multiplicity distribution).[6–9] Table I summarizes the nomenclature used in this work and previous works describing parameters related to neutron multiplication.

IV. POINT KINETICS MODEL

Neutron multiplicity counting techniques for both reactivity and NDA measurements are fundamentally derived from the diffusion theory balance equation, which

TABLE I

Summary of Nomenclature Used in this Note to Describe Parameters Related to Neutron Multiplication, Along with Previously Used Notations/Definitions

Definition	Previously Used Nomenclature	Reference
Effective neutron multiplication factor (prompt + delayed)	k_{eff} K' K and k	This work; Refs. 10, 31, and 33 Refs. 3, 6, 8, and 9 Refs. 8, 10, 12, and 34
Prompt neutron multiplication factor	k_p k	This work; Refs. 7 through 12 and 17 Ref. 4
Total multiplication	M_T T M	This work; Refs. 4, 27, and 28 Ref. 6 Refs. 31 and 33 through 37
Leakage (net) multiplication	M_L N M	This work; Refs. 4, 26, 27, 28, and 37 Ref. 6 Refs. 3, 14, 15, 16, and 20
Mean number of neutrons per induced fission (prompt + delayed)	$\bar{\nu} = \overline{\nu_p} + \overline{\nu_d}$ ν_i	This work; Refs. 7, 8, and 10 Ref. 3
Mean number of prompt neutrons per induced fission	$\overline{\nu_p} = \nu_{if,1}$ $\bar{\nu}$ ν_{i1}, $\nu_{i(1)}$, $\overline{\nu_{i(1)}}$	This work; 7, 8, 10, and 12 Refs. 11, 31, 35, and 36 Ref. 4
Mean number of delayed neutrons per induced fission	$\overline{\nu_d}$ ν_{dn}	This work; Ref. 8 Ref. 26
Ratio of capture cross section to fission cross section	$\frac{\Sigma_c}{\Sigma_f}$ α	This work Refs. 6 and 10
Induced fission probability	p_{if} p p_f P_P (prompt neutron-induced probability) P_d (delayed neutron-induced probability)	This work Refs. 3, 14, 15, 21, 23, 24, 31, 35, and 36 Ref. 4 Ref. 8 Ref. 8
Average system fission rate	Q_F (per second) $Q_{F,n}$ (per neutron) S (per second) E_0 (per second) F (per second) E (per second)	This work This work Refs. 7 and 8 Ref. 11 Ref. 12 Ref. 10
Delayed neutron fraction	β f γ	This work; Refs. 10 and 12 Refs. 7 and 8 Ref. 25
Effective delayed neutron factor	β_{eff} γf $\bar{\gamma}\beta$	This work; Refs. 13 and 17 Refs. 7, 8, and 9 Ref. 10

describes the energy-, space-, and time-dependent neutron flux behavior. For practical implementation, it is often useful to condense these equations into a purely time-dependent form by integrating over all space and energy; the resulting formulation is known as the time-dependent point kinetics equation. Let us consider a pointlike multiplying assembly with one neutron-energy group. The point kinetics equation for the change in the neutron population n

(t) can be described by the balance of neutron production and loss terms, and is given by

$$\frac{dn(t)}{dt} = V \cdot n(t) \cdot \left(\nu_{if,1}\Sigma_f - \Sigma_c - DB^2\right) , \tag{8}$$

where

V = average neutron velocity

Σ_f, Σ_c = macroscopic fission and capture cross sections, respectively

DB^2 = product of the diffusion coefficient and the geometric buckling term and is related to the leakage of neutrons.

In Eq. (8), the production term is described by the neutrons produced from fission $Vn(t)\nu_{if,1}\Sigma_f$, while the loss term is described by parasitic captures $Vn(t)\Sigma_c$ of neutrons within the system and leakage $Vn(t)DB^2$ of neutrons out of the system. Equation (8) can be rewritten with the average time behavior of the production and loss terms by defining the mean time to leakage and capture (nonfission) as

$$\tau_L = \frac{1}{VDB^2}$$

and

$$\tau_C = \frac{1}{V\Sigma_C} , \tag{9}$$

which describes the average time for a neutron to "leave," and the mean time to fission (i.e., mean generation time) as

$$\tau_f = \frac{1}{V\Sigma_f} , \tag{10}$$

which describes the average time for a neutron to induce fission. Using Eqs. (9) and (10), the probability per unit time for a neutron to be removed from the system can be described by the mean neutron lifetime τ_0 as the sum of all competing events, and is given by

$$1 = \frac{\tau_0}{\tau_f} + \frac{\tau_0}{\tau_c} + \frac{\tau_0}{\tau_L} = p_{if} + p_c + p_L , \tag{11}$$

where p_{if}, p_c, and p_L are the induced fission, capture, and leakage probability of a neutron, respectively.[4] The prompt neutron multiplication factor is given by the ratio of the production [Eq. (10)] and loss [Eq. (9)] terms, and has the form

$$k_p = \frac{\nu_{if,1}\Sigma_f}{\Sigma_f + \Sigma_c + DB^2} = \frac{\nu_{if,1}\tau_o}{\tau_f} = \nu_{if,1}p_{if} . \tag{12}$$

Using Eqs. (8) through (12), the time behavior of prompt neutrons following an instantaneous injection of initiating neutrons is given by[4,11,12]

$$\begin{aligned}\frac{dn(t)}{dt} &= -\frac{n(t)}{\tau_0} + \frac{n(t)}{\tau_f}\nu_{if,1} \\ &= -\frac{n(t)}{\tau_0} + \frac{n(t)p_{if}\nu_{if,1}}{\tau_0} = n(t)\frac{\left(k_p - 1\right)}{\tau_0} .\end{aligned} \tag{13}$$

Equation (13) is typically rewritten as

$$\frac{dn(t)}{dt} = -\alpha \cdot n(t) \tag{14}$$

with

$$\alpha = \frac{1 - k_p}{\tau_0} , \tag{15}$$

where α is defined as the prompt neutron decay constant.[a,4,11]

Solving Eq. (14) with n_0 initial neutrons at $t = 0$ describes the probable number of neutrons at some later time t and has the form

$$n(t) = n_0 e^{-\alpha t} . \tag{16}$$

It is useful to relate Eq. (16) to the definition of total multiplication described in Eq. (3), and can be derived by first noting that the probable number of fissions dQ_F produced in dt at time t is given by

$$dQ_F = n_0 e^{-\alpha t}\frac{dt}{\tau_f} , \tag{17}$$

which yields the resulting number of neutrons from the fissions produced about dt as $dn = \nu_{if,1} \cdot dQ_F$, or similarly

[a] Note that Orndoff et al. originally define α as the negative of Eq. (15) with a sign change in Eq. (14). However, the sign change is purely conventional and the same relationships can be derived regardless; this is demonstrated with the derivation of the familiar expression for total multiplication M_T using both sign conventions and is shown in the Appendix.

$$dn = \nu_{if,1} n_0 \mathrm{e}^{-\alpha t} \frac{dt}{\tau_f} \ . \tag{18}$$

The total number of subsequent prompt neutrons produced from n_0 neutrons (assuming $n_0 = 1$) emitted at time $t = 0$ is given by

$$\begin{aligned} n_0 + \int_0^{\infty} \nu_{if,1} n_o e^{-\alpha t} \frac{dt}{\tau_f} &= 1 + \frac{\nu_{if,1}}{\tau_f}\left(\frac{1}{\alpha}\right) \\ &= 1 + \frac{\nu_{if,1}}{\tau_f}\left(\frac{\tau_0}{1-k_p}\right) \\ &= 1 + \frac{k_p}{1-k_p} \\ &= \frac{1}{1-k_p} \\ &= M_T \ . \end{aligned} \tag{19}$$

For multiplying assemblies that are near critical, it is possible to directly measure α. That is, although τ_0 is quite short for compact fast metal assemblies [order of 20 ns (4)], M_T is very high, thus the observed time behavior is primarily due to the internal kinematics of the system.[4,11,12] In other words, the observed time behavior governed by α is dominated by the rate of decay of the neutrons inside the system. An important aspect for reactivity measurements is that when a near-critical assembly is stimulated, the system can be viewed as having a continual production of neutrons from induced fissions until the population dies away through the loss mechanisms. That is, the system is self-modulated and the dominating source of production in the balance equation [Eq. (13)] is the neutrons born from induced fissions.[4,11,12]

For highly subcritical assemblies where M_T is typically much closer to 1, additional neutron production terms must be considered, mainly neutrons from spontaneous fissions and (α, n) reactions. Considering the additional neutron production terms, it has been shown that a single neutron is multiplied by a factor of M_T (4). Noting that the multiplied neutrons come from fissions induced by neutrons born from spontaneous fission or (α,n) reactions, the induced fission rate (per neutron) is given by[1,4,14–16]

$$\frac{M_T - 1}{\nu_{if,1}} = p_{if} M_T = Q_{F,n} \ . \tag{20}$$

It is often useful to express the total multiplication in relation to the leakage multiplication (i.e., neutrons that leak out of the assembly that are then available for detection). Using the probability of leakage p_L and parasitic capture p_c, the total multiplication can be related to the leakage multiplication by

$$\begin{aligned} M_L &= M_T p_L = M_T\left(1 - p_{if} - p_c\right) \\ &= \frac{1 - p_{if} - p_c}{1 - p_{if}\nu_{if,1}} \ . \end{aligned} \tag{21}$$

If it is assumed that parasitic capture is negligible (i.e., $p_c/p_{if} << 1$), then an approximated expression for $p_{if}M_T$ can be expressed as[1,4,13,15]

$$\frac{M_L - 1}{\nu_{if,1} - 1} = \left[p_{if} - \frac{\frac{p_c}{p_{if}}}{\nu_{if,1} - 1} \right] M_T \approx p_{if} M_T \ . \tag{22}$$

There exist key differences in how the aspects of the point model are applied depending on the state of criticality of the multiplying assembly. The main differences arise from the fact that reactivity measurements are conducted on assemblies that are near critical, while safeguards measurements typically assay highly subcritical assemblies. Croft et al. has summarized how the point model is applied differently for the two types of measurements.[4] The first relates to the dynamics of the multiplying assembly; reactivity measurements treat the assembly as a self-sustaining critical assembly (controlled by the lethargic influence of delayed neutrons[10]), while safeguards measurements treat the assembly as a source-driven highly subcritical system.[4] The second difference is the distinction between an internal detector (i.e., efficiency is defined as detection counts per fission) for near-critical assemblies and a decoupled external detector (i.e., efficiency is defined as detection counts per leaked neutron) for subcritical assemblies. This leads directly into the last aspect, which relates to the distinction between the determination of total multiplication and leakage multiplication.[4] Table II summarizes the nomenclature used in this work and previous works describing parameters related to the point kinetics model.

V. NMC FOR REACTIVITY MEASUREMENTS

Neutron multiplicity counting is used to survey the state of criticality of near-critical assemblies. The criticality of such an assembly is typically expressed in terms of its reactivity, and can be described as

TABLE II

Summary of Nomenclature Used in this Note to Describe Parameters Related to the Point Kinetics Model, Along with Previously Used Notations/Definitions

Definition	Previously Used Nomenclature	Reference
Neutron population	$n(t)$ N	This work; Ref. 10 Refs. 7, 8, and 11
Initial neutron population at $t = 0$	n_0 N_0	This work; Ref. 10 Refs. 7, 8, and 11
Mean prompt neutron lifetime	τ_0 l τ	This work; Refs. 4, 11, and 17 Refs. 10 and 12 Refs. 7 and 8
Mean neutron lifetime for fission (i.e., mean generation time)	τ_f τ_p Λ	This work; Refs. 4 and 11 Ref. 9 Refs. 10 and 12
Mean neutron lifetime for nonfission captures	τ_c	This work; Ref. 4
Mean neutron lifetime for leakage	τ_L	This work; Ref. 4
Nonfission capture probability	p_c	This work; Ref. 4
Leakage probability	p_L l	This work; Ref. 4 Ref. 14
Prompt neutron decay constant	α α_R	This work; Refs. 7–13, 17, 25, 31, 35, and 36 Ref. 4

$$\rho = \frac{k_{eff} - 1}{k_{eff}} \ . \tag{23}$$

Similarly, the prompt reactivity can be described as[10,12]

$$\rho_p = \frac{k_p - 1}{k_p} \ . \tag{24}$$

For reactivity measurements, the parameter of most interest is the prompt neutron decay constant shown in Eq. (15). From Eq. (12), the prompt neutron mean time to fission τ_f is given by[11,12,17]

$$\tau_f = \frac{\nu_{if,1}\tau_0}{k_p} \ . \tag{25}$$

When the multiplying system is at a delayed critical state ($k_{eff} = 1$ and τ_0 at delayed critical $= \tau_f$), the prompt neutron decay constant can be described as[12]

$$\alpha_{dc} = \frac{\beta_{eff}}{\tau_0} = \frac{\beta_{eff}}{\tau_f} \ . \tag{26}$$

Hence, a relationship between α and α_{dc} can be expressed as[12,16]

$$\alpha = \frac{\beta_{eff}}{\tau_f}\left(1 - \frac{\rho}{\beta_{eff}}\right) = \alpha_{dc}(1 - \$) \ , \tag{27}$$

where

$$\$ = \frac{\rho}{\beta_{eff}} \tag{28}$$

is the reactivity expressed in dollars.

As the statistical fluctuations of the neutron population are primarily caused by the variation in the number of neutrons produced per fission, a useful parameter that can describe the prompt neutron emission is the Diven's parameter. The Diven's parameter provides a normalized average for the number of prompt neutrons from induced fissions, and is defined as[12,18]

$$D_\nu = \frac{\overline{\nu_p^2} - \overline{\nu_p}}{\overline{\nu_p}^2} \ . \tag{29}$$

NMC for near-critical assemblies typically involves looking at the time distribution of detected neutrons from correlated events (common fission chain) to infer the characteristic physical properties. The timely detection of the neutrons from a multiplying assembly gives rise to a series of pulses (i.e., pulse trains) that contain useful information pertinent to the length of the fission chains as the prompt burst of neutrons following a fission reaction are expected to be correlated in time (on average). Count-interval distributions can be used to describe the temporal behavior of the neutron population to characterize properties related to the fission chain dynamics of the multiplying assembly. Typical properties such as the mean neutron lifetime and reactivity can be inferred given a well-known detection system; the two most commonly used methods are described in Secs. V.A and V.B.

V.A. Feynman-Y (Variance-to-Mean) Method

The basic principle of the Feynman-Y method is to use the statistics of detected pairs of neutrons to characterize the time behavior of the neutron population and, in turn, infer the multiplicative properties.[7–9,19] The idea is to obtain a time-dependent behavior of the relative variance (ratio of variance to mean) by calculating the variance and mean of the detected neutron number distribution within increasing values of T_G. If c represents the number of counts in the interval T_G, then the mean number of pairs of counts in the interval T_G is given by

$$\begin{aligned}\frac{\overline{c^2}-\overline{c}}{2} &= \int_{t_2=0}^{T_G}\int_{t_1=0}^{t_2} p(t_1,t_2)dt_1dt_2\\ &= \frac{Q_F^2\varepsilon_F^2T_G^2}{2}+\frac{Q_F\varepsilon_F^2D_vk_p^2T_G}{2\left(1-k_p\right)^2}\\ &\times\left(1-\frac{1-e^{-\alpha T_G}}{\alpha T_G}\right) ,\end{aligned} \tag{30}$$

where

ε_F = detection efficiency (per fission)

Q_F = system fission rate

$p(t_1,t_2)$ = total probability of detecting all pairs of (uncorrelated + correlated) neutrons.[6,7,12]

Noting that the mean count rate is given by

$$\overline{c} = Q_F\varepsilon_F T_G , \tag{31}$$

then Eq. (31) can be used to rearrange Eq. (30) into the familiar form

$$\begin{aligned}Y(T_G) &= \frac{\overline{c^2}-\overline{c}^2}{\overline{c}}-1\\ &= \frac{\varepsilon_F k_p^2 D_v}{\left(1-k_p\right)^2}\left[1-\frac{1-e^{-\alpha T_G}}{\alpha T_G}\right] ,\end{aligned} \tag{32}$$

where $Y(T_G)$ is the Feynman-Y parameter that quantifies the excess from unity to quantify the deviation from a purely uncorrelated Poisson source.[6,7,12] When T_G is large, the ratio becomes time independent indicating that the observed fluctuations are constant. However, for smaller values of T_G, the ratio will exhibit a time-dependent trend that is directly related to the prompt neutron decay constant. If the prompt reactivity is defined as

$$\rho_p = \frac{\left(k_p-1\right)}{\left(k_p\right)} \approx \rho-\beta_{eff} , \tag{33}$$

then Eq. (32) can also be written as

$$Y(T_G) = \frac{\varepsilon_F D_v}{\rho_p^2}\left[1-\frac{1-e^{-\alpha T_G}}{\alpha T_G}\right] . \tag{34}$$

To extract the neutron time behavior, the Feynman-Y is calculated for various lengths of T_G up to some large value of T_G such that the term in the bracket approaches unity. Fitting the resulting time-dependent Feynman-Y curve allows for an estimate of α.

V.B. Rossi-Alpha (Covariance-to-Mean) Method

The Rossi-Alpha method is quite similar to the Feynman-Y method. The main difference is that it utilizes the covariance-to-mean ratio as the metric to estimate the mean neutron lifetime. That is, the measurement involves investigating the probability of detecting a neutron at time t in a time gate interval dt, given that a neutron was detected at time $t = 0$. Consider when an initial neutron count is detected at time $t = 0$ there exists a probability of detecting another neutron at some time t later; this neutron will either be random (i.e., born from a different fission chain) or correlated (i.e., born from the same fission chain that produced the initial neutron count at $t = 0$). Given a neutron count at time $t = 0$, the probability of detecting a random neutron within dt is

$$p_r(t)dt = Q_F\varepsilon_F dt , \tag{35}$$

while the probability of detecting a correlated neutron within dt is[11,12]

$$p_c(t)dt = \frac{\varepsilon_F k_p}{\tau_0}\left[\frac{\left(\overline{\nu_p^2} - \overline{\nu_p}^2 k_p\right)}{2\overline{\nu_p}^2\left(1-k_p\right)} + \frac{\delta}{\overline{\nu_p}}\right]e^{-\alpha t}dt$$
$$= \frac{\varepsilon_F D_\nu k_p^2}{2\left(1-k_p\right)\tau_0}e^{-\alpha t}dt \;, \tag{36}$$

where δ is a correction term to account for the possibility of a fission occurring and producing the count at exactly $t = 0$; however, Orndoff states that this correction in practice is at most a few percent and does not need to be evaluated precisely.[11,12] The total probability of detecting a neutron at time t within interval dt given an initial count at $t = 0$ is the sum of p_r and p_c and has the form

$$p_{r+c}(t)dt = Q_F\varepsilon_F dt + \frac{\varepsilon_F D_\nu k_p^2}{2\left(1-k_p\right)\tau_0}e^{-\alpha t}dt \;. \tag{37}$$

Using Uhrig's assumption to neglect the δ correction term,[12] Eq. (37) is often represented as

$$p(t)dt = Adt + Be^{-\alpha t}dt \;, \tag{38}$$

with

$$A = Q_F\varepsilon_F \tag{39}$$

and

$$B = \frac{\varepsilon_F D_\nu k_p^2}{2\left(1-k_p\right)\tau_0} = \frac{\varepsilon_F D_\nu k_p^2}{2\alpha\tau_0^2} \;, \tag{40}$$

where Q_F is the total fission rate and τ_0 is the mean neutron lifetime. Equation (38) is the functional form that is used to fit the measured Rossi-Alpha distribution, which can then be used to obtain, in principle, any two of the five quantities (Q_F, ε_F, D_ν, k_p, or τ_0) if the other three are known a priori.[11,12] In practice, Rossi-Alpha measurements are typically used to measure neutron lifetimes of systems. These types of measurements can be used to measure the reactivity of critical and subcritical systems or, additionally, to compare the hardness of neutron spectrum between critical systems.[13] An example of a practical reactivity measurement involves comparing the inverse count rate $1/\overline{c}$ and α for various well-known subcritical assembly configurations. This relationship can provide a fitted trend that can then be extrapolated to assess the expected α when the assembly is delayed critical and prompt critical.[13] Table III summarizes the nomenclature used in this work and previous works describing parameters related to NMC for reactivity measurements.

VI. NMC FOR NDA MEASUREMENTS

The purpose for NMC in NDA measurements is to quantify the amount of fissile material in a sample that is typically far from critical.[3] Within a set of assumptions, physical parameters of the sample can be analytically unfolded by using the statistics of the detected neutron count distribution. The first- (mean), second- (variance), and third-order (skew) reduced moments of the detected neutron count distribution are calculated to extract the singles, doubles, and triples count rates, respectively. The physical parameters of interest include the spontaneous fission *sf* rate, the leakage (and in turn the total) multiplication, and the contribution of neutrons from (α,n) reactions. Although equations for active NDA analysis have previously been derived,[20] we will focus on the formulism for passive NDA. The assumptions governing the reliability of passive NDA analysis have been previously outlined by Bohnel and Cifarelli and Hage[3,14,16,21] and consist of the following:

1. All neutrons from induced fissions *if* are emitted simultaneously with the original initiating *sf* or (α,n) reaction. This is commonly referred to as the superfission concept introduced by Bohnel and is quite valid for highly subcritical assemblies where the fission chains are very short.

2. In point kinetics model approximation, the sample behaves as if it had point-geometry as outlined in Sec. IV. Consequently, the neutron detection efficiency and the probability of fission are uniform throughout the sample volume.

3. The detection efficiency, probability for inducing fission, and the induced fission multiplicity are independent of the initiating neutron [*sf* or (α,n)].

4. Parasitic neutron capture without causing fission is considered a negligible loss term. This allows the total multiplication to be inferred from the estimated leakage multiplication.

5. There is no correlation in the emitted neutron energy and multiplicity.

6. The slowing-down time behavior of neutrons in the moderator-detector system can be described as an exponential function with a decay constant λ.

TABLE III

Summary of Nomenclature Used in this Note to Describe Parameters Related to NMC for Reactivity Measurements, Along with Previously Used Notations/Definitions

Definition	Previously Used Nomenclature	Reference
Reactivity	ρ	This work; Refs. 10, 12, and 31
Prompt reactivity	ρ_p	This work; Refs. 10 and 12
Prompt neutron decay constant at delayed critical	α_{dc} (α_{DC}) α_c	This work; Refs. 10, 13, and 17 Ref. 12
Diven's parameter	D_v D_S D	This work; Ref. 12 Ref. 25 Ref. 10
Time gate interval	T_G τ t Δ T G	This work; Refs. 4, 26, 27, and 28 Refs. 13, 24, and 25 Refs. 7, 8, and 11 Ref. 12 Refs. 10, 12, 25, 31, 35, and 36 Refs. 3 and 20
Average number of counts in time gate interval	$\bar{c}$ (per fission) C $\overline{N}$	This work; Refs. 7, 8, 9, 12, 31, 35, and 36 Ref. 10 Ref. 25
Probability of detecting random pairs of neutrons	$p_r(t)$	This work; Ref. 12
Probability of detecting correlated pairs of neutrons	$p_c(t)$ Λ_n	This work; Ref. 12 Ref. 35
Total probability of detecting all pairs of neutrons	$p_{r+c}(t)$ $p(t)$	This work Ref. 12
Neutron detection efficiency per fission	ε_F ε E	This work Refs. 7–10, and 12 Refs. 11 and 17
Neutron detection efficiency per neutron	ε_n ε ε_f ε_{tot}	This work Refs. 3,14, 15, 16, 18, 21, 22, 24, 26, 27, 28, 31, 35, and 36 Ref. 20 Ref. 30

The equations in Secs. VI.A and VI.B relate the detected multiplicity rates to the fission rate, the (α,n) neutron contribution, and the leakage multiplication.

VI.A. Formalism for Factorial Moments and Neutron Count Rates

The effective number of neutron combinations that can be grouped together from a single fission event is an important quantity for passive NDA analysis. These are known as the reduced moments of the emitted neutron multiplicity distributions and have been previously derived analytically and measured experimentally.[15,18] For a given neutron multiplicity distribution, one can calculate the first-, second-, and third-order factorial moments of the *sf* and *if* distributions using

$$\nu_{sf,m} = \sum_{\nu=m}^{\infty} \binom{\nu}{m} P_{sf}(\nu) \tag{41}$$

and

$$\nu_{if,m} = \sum_{\nu=m}^{\infty} \binom{\nu}{m} P_{if}(\nu) \ , \tag{42}$$

respectively, where m denotes the order of the moments and ν denotes the number of emitted neutrons. These factorial moments are considered to be fundamental nuclear data and are generally determined by independent measurements.

It is useful to establish the factorial moments of all the initiating events, which include the neutrons emitted from (α,n) reactions. These moments are then related to the moments of the induced events by consideration of the internal multiplication within the sample. The primary difference between the two initiating events [sf and (α,n) reactions] is the number of neutrons emitted from each type of reaction. That is, (α,n) reactions only emit a single neutron per reaction whereas sf reactions will emit multiplicities of neutrons per reaction. Therefore, the neutrons from (α,n) reactions are random in time and are uncorrelated to one another while the neutrons from sf reactions are correlated to the initial fission event. This allows the detecting system to distinguish between the two types of neutrons through time correlation analysis of the neutron pulses. To derive expressions for the factorial moment of the initiating events, we first define the total reaction rate of the two initiating events Q_S as

$$Q_S = Q_\alpha + Q_{sf} \ , \tag{43}$$

where Q_α and Q_{sf} denote the reaction rate of (α,n) and sf reactions, respectively. Equation (43) can be expressed as a joint distribution in terms of the two neutron emission rates and has the form

$$P_{Q_S}(\nu) = \frac{Q_{sf}}{Q_S} P_{sf}(\nu) + \frac{Q_\alpha}{Q_S} \delta_{1,\nu} \ , \tag{44}$$

where $\delta_{1,\nu}$ is the kronecker delta symbol representing the single neutron emission for (α,n) reaction.[21] To relate the neutrons emitted from the (α,n) and sf reactions, a useful parameter α [not to be confused with α in Eq. (13)] is introduced and is defined as the ratio of neutrons from (α,n) reactions to those from sf reactions, which is expressed mathematically as

$$\alpha = \frac{Q_\alpha}{\nu_{sf,1} Q_{sf}} \ . \tag{45}$$

Equation (45) is one of the sample parameters that are analytically estimated. Using Eq. (45), Eq. (43) can be rewritten in terms of α, Q_{sf}, and fundamental nuclear data $\nu_{sf,1}$, and has the form

$$Q_S = Q_{sf}\left(1 + \alpha \nu_{sf,1}\right) \ . \tag{46}$$

And by substituting Eq. (46) into Eq. (44), the probability distribution for the neutron emission from all initiating events can be written as

$$P_{Q_S}(\nu) = \frac{\alpha \nu_{sf(1)} \delta_{1,\nu} + P_{sf}(\nu)}{1 + \alpha \nu_{sf(1)}} \ . \tag{47}$$

With the probability distribution in Eq. (47) and using the properties of probability generating functions,[21,22] it has been shown that the reduced factorial moments for the neutrons emitted from the initiating events have the form

$$\nu_{Q_S,1} = \frac{\nu_{sf,1}(1+\alpha)}{\left(1 + \alpha \nu_{sf,1}\right)} \tag{48}$$

for the first-order factorial moment and

$$\nu_{Q_S,n} = \frac{\nu_{sf,n}}{\left(1 + \alpha \nu_{sf,n}\right)} , \ n \neq 1 \tag{49}$$

for the higher-order moments. Last, to derive expressions for the factorial moments of the total emitted neutron distribution Q_T (including the additional neutrons from if reactions), Eqs. (48) and (49) are related to the leakage multiplication given by Eq. (21), and thus, the first-, second-, and third-order factorial moments of the total emitted neutron distribution have the form

$$\nu_{Q_T,1} = M_L \nu_{Q_T,1} = \frac{M_L}{\left(1 + \alpha \nu_{sf,1}\right)} \nu_{sf,1}(1+\alpha) \ , \tag{50}$$

$$\begin{aligned} \nu_{Q_T,2} &= M_L^2 \left\{ \nu_{Q_S,2} + \left(\frac{M_L - 1}{\nu_{i(1)} - 1} \right) \nu_{Q_S,1} \nu_{if,1} \right\} \\ &= \frac{M_L^2}{\left(1 + \alpha \nu_{sf,1}\right)} \\ &\times \left\{ \nu_{sf,2} + \left(\frac{M_L - 1}{\nu_{if,1} - 1} \right) \nu_{sf,1}(1+\alpha) \nu_{if,2} \right\} \ , \end{aligned} \tag{51}$$

and

$$\nu_{Q_T,3} = M_L^3\left\{\nu_{Q_S,3} + \left(\frac{M_L-1}{\nu_{if,1}-1}\right)\left(3\nu_{Q_S,2}\nu_{if,2} + \nu_{Q_S,1}\nu_{if,3}\right) + 3\left(\frac{M_L-1}{\nu_{if,1}-1}\right)^2 \nu_{Q_S,1}\nu_{if,2}^2\right\}$$
$$= \frac{M_L^3}{\left(1+\alpha\nu_{sf,1}\right)}\left[\nu_{sf,3} + \left(\frac{M_L-1}{\nu_{if,1}-1}\right)\left[3\nu_{sf,2}\nu_{if,2} + \nu_{sf,1}(1+\alpha)\nu_{if,3}\right] + 3\left(\frac{M_L-1}{\nu_{if,1}-1}\right)^2 \nu_{sf,1}(1+\alpha)\nu_{if,2}^2\right] . \quad (52)$$

Equations (50), (51), and (52) describe the factorial moments of total emitted neutron distribution per initiating event. To relate Eqs. (50), (51), and (52) to the source intensity of the active sample, Q_S is multiplied to each factorial moment and can be written in terms of Q_{sf} by substituting Eq. (46) for Q_S. Last, the detection intensity of single, double, and triple (k = 1, 2, and 3, respectively) neutrons is given by

$$C_k = Q_s\frac{\varepsilon_n^k \nu_{Q_T,k}}{k!} = Q_{sf}\left(1+\alpha\nu_{sf,1}\right)\frac{\varepsilon_n^k \nu_{Q_T,k}}{k!} , \quad (53)$$

which considers all possible combinations of coupling k-tuplet detections from any higher-order multiplet.[16,22] In Eq. (53), ε_n is the neutron detection efficiency (per emitted neutron). Substituting Eqs. (50), (51), and (52) into Eq. (53) gives the detection rates for singles (S), doubles (D), and triples (T) neutron counts, and has the form

$$S \equiv C_1 = Q_{sf}\varepsilon_n M_L \nu_{sf,1}(1+\alpha) , \quad (54)$$

$$D \equiv C_2 = \frac{Q_{sf}\varepsilon_n^2 M_L^2}{2}\left[\nu_{sf,2} + \left(\frac{M_L-1}{\nu_{if,1}-1}\right)\nu_{sf,1}(1+\alpha)\nu_{if,2}\right] , \quad (55)$$

and

$$T \equiv C_3 = \frac{Q_{sf}\varepsilon_n^3 M_L^3}{6}\left\{\nu_{sf,3} + \left(\frac{M_L-1}{\nu_{if,1}-1}\right)\left[3\nu_{sf,2}\nu_{if,2} + \nu_{sf,1}(1+\alpha)\nu_{if,3}\right] + 3\left(\frac{M_L-1}{\nu_{if,1}-1}\right)^2 \nu_{sf,1}(1+\alpha)\nu_{if,2}^2\right\} , \quad (56)$$

and is given in terms of known or measurable quantities and three unknown quantities (Q_{sf}, α, and M_L) that pertain to the fissile sample. It has been shown that M_L can be analytically solved from Eqs. (54), (55), and (56) (3, 14, 16, and 22), after which Q_{sf} and α can also be obtained analytically through algebraic substitution. Table IV summarizes the nomenclature used in this work and previous works describing parameters related to NMC for NDA measurements.

VI.B. Gate Generation Techniques for Extracting Neutron Count Rates

There are various gate generation techniques that are used to extract the observed neutron number distribution throughout the measured neutron pulse train. The first gate generation technique uses a series of randomly triggered time gates to extract the number distribution and will be referred to as random trigger inspection[14,23,24,26–28] (RTI). The second technique uses signal-triggered time gates to extract the number distribution; this mode will be referred to as signal trigger inspection (STI). It has been shown in previous works that one can extract the S, D, and T using RTI, STI, or a combination of both (i.e., multiplicity shift register).[23,26–28] Suppose that we have a pulse train that is to be analyzed through either gate generation technique with an inspection time gate T_G. The number distribution from RTI and STI analysis is given by $B_x(T_G)$ and $N_x(T_G)$, where $B_x(T_G)$ is the number of events with x signals inside K RTI intervals and $N_x(T_G)$ is the number of events with x signals inside N_T STI intervals during a total observation period. Both number distributions can be expressed as frequencies by normalizing by the number of inspection intervals, which is given by

$$b_x(T_G) = \frac{B_x(T_G)}{K} \quad (57)$$

TABLE IV

Summary of Nomenclature Used in this Note to Describe Parameters Related to NMC for NDA Measurements, Along with Previously Used Notations/Definitions

Definition	Previously Used Nomenclature	Reference
Probability of emitting ν neutrons for spontaneous fission reactions	$P_{sf}(\nu)$ $P_{s\nu}$ $P_{s\nu}$ $p_{sf}(\nu)$ P_ν $q_{sp}(\nu)$ $P(\nu)$ $P_S(\nu)$ C_ν^{sp}	This work Ref. 14 Refs. 15, 23, and 24 Ref. 16 Ref. 18 Ref. 21 Ref. 22 Refs. 27 and 28 Refs. 31, 35, and 36
Probability of emitting ν neutrons for induced fission reactions	$P_{if}(\nu)$ $P_{I\nu}(p)$ P_ν $q(\nu)$ C_ν	This work Ref. 14 Refs. 10 and 15 Ref. 21 Refs. 31, 35, and 36
Joint probability of emitting ν neutrons from initiating events [spontaneous fission + (α,n) reactions]	$P_{Q_S}(\nu)$ $p_s(n)$ $q_s(\nu)$	This work Refs. 16 and 22 Ref. 21
Total reaction rate from initiating events [spontaneous fission + (α,n) reactions]	Q_S s	This work; Refs. 16 and 37 Ref. 21
(α,n) reaction rate	Q_α S_α	This work Refs. 14, 15, 21, 23, 24, and 31
Spontaneous fission rate	Q_{sf} F_s Q_f F S_{sp}	This work Refs. 14, 15, 23, 24, 31, 35, and 36 Ref. 16 Refs. 16, 20, 26, and 37 Ref. 21
Ratio of neutrons from (α,n) reactions to those from spontaneous fission reactions	α A	This work; Refs. 3, 14, 15, 22, 26, 27, 28, and 37 Refs. 31, 35, and 36
m'th-order reduced moment of the spontaneous fission neutron multiplicity distribution	$\nu_{sf,m}$ $\nu_{sf(m)}$, $\overline{\nu_{s(m)}}$, ν_{sm} ν_m D_{ms}	This work Refs. 3, 4, 14, 15, 16, 24, 31, and 37 Ref. 26 Refs. 31, 35, and 36
m'th-order reduced moment of the induced fission neutron multiplicity distribution	$\nu_{if,m}$ $\nu_{if(m)}$, $\overline{\nu_{i(m)}}$, ν_{im}, $\overline{\nu}_m$, $\nu_{(m)}$ D_m	This work Refs. 3, 4, 14, 15, 16, 20, 23, 24, and 37 Refs. 31, 35, and 36
m'th-order reduced moments of the emitted neutron multiplicity distribution from initiating events [spontaneous fission + (α,n) neutron-initiated]	$\nu_{Q_S(m)}$ $M_{(q)m}$ ν_{sm}, $\nu_{s,m}$	This work Ref. 21 Refs. 16 and 22
m'th-order reduced moments of the total emitted neutron multiplicity distribution	$\nu_{Q_T(m)}$ $M_{(R)m}$ ν_m	This work Ref. 21 Refs. 16, 22, and 37

and

$$n_x(T_G) = \frac{N_x(T_G)}{N_T} \ , \tag{58}$$

where $b_x(T_G)$ and $n_x(T_G)$ are the normalized (i.e., the probability) of the RTI and STI number distributions, respectively. The reduced moments from the resulting RTI and STI distributions can then be extracted using the following relationship:

$$m_{b(\mu)} = \sum_{x=\mu}^{\infty} \binom{x}{\mu} b_x(T_G) \tag{59}$$

and

$$m_{n(\mu)} = \sum_{x=\mu}^{\infty} \binom{x}{\mu} n_x(T_G) \ , \tag{60}$$

where $m_{b(\mu)}$ and $m_{n(\mu)}$ represent the μ'th factorial moments for the normalized RTI and STI number distributions.

The first-, second-, and third-order factorial moments from Eqs. (59) or (60) are then used to quantify the *S, D*, and *T* count rates. Considering the point-kinetics model detailed in Sec. IV, these neutron count rates become self-contained expressions that can be inverted to estimate physical properties of the highly subcritical assembly. Depending on the type of gate generation technique implemented, the formalism for the *S, D*, and *T* count rates differ; however, it has been shown that they indeed extract the same count rates.[27,28] If only the RTI gates are used, the *S, D*, and *T* count rates can be calculated using

$$S = \frac{m_{b(1)}}{T_G w_1} \ , \tag{61}$$

$$D = \frac{1}{T_G w_2}\left[m_{b(2)} - \frac{1}{2} m_{b(1)}^2\right] \ , \tag{62}$$

and

$$T = \frac{1}{T_G w_3}\left[m_{b(3)} - m_{b(2)} m_{b(1)} + \frac{1}{3} m_{b(1)}^3\right] \ . \tag{63}$$

w_μ is the gate utilization factor for randomly triggered gates and has the form

$$w_\mu = \sum_{j=0}^{\mu-1} \binom{\mu-1}{j} (-1)^j \left(\frac{1 - e^{-jT_G\lambda}}{jT_G\lambda}\right),$$
$$w_1 = 1 \ , \tag{64}$$

where λ is the inverse system die-away time, which describes the neutron slowing-down time in the detector-moderator system. When using STI gates, information from the RTI gates is still required to estimate the rate of uncorrelated counts. If STI analysis is used, then the *S, D*, and *T* count rates are calculated by

$$S = \frac{N_T}{T_M} m_{n(0)} = \frac{N_T}{T_M} \ , \tag{65}$$

$$D = \frac{N_T}{T_M f}\left[m_{n(1)} - m_{b(1)}\right] \ , \tag{66}$$

and

$$\begin{aligned} T &= \frac{N_T}{T_M f^2}\left[m_{n(2)} - m_{b(2)} - m_{b(1)}\left(m_{n(1)} - m_{b(1)}\right)\right] \\ &= \frac{N_T}{T_M f^2}\Bigg[m_{n(2)} - m_{b(1)}\left(m_{n(1)} - m_{b(1)}\right) \\ &\quad \times \left(1 + \frac{w_2}{f}\right) - \frac{1}{2} m_{b(1)}^2\Bigg] \ , \end{aligned} \tag{67}$$

where T_M is the total observation period (total measurement time), and *f* is the gate utilization factor for signal-triggered gates and has the form

$$f = \left(e^{-T_{PD}\lambda}\left(1 - e^{-T_G\lambda}\right)\right) \ , \tag{68}$$

where T_{PD} is the predelay between the trigger and the start of observation.[3,14]

There is also a method that involves the use of both RTI and STI simultaneously and is a type of signal autocorrelation analysis that aims to quantify the number distribution in two inspection intervals of equal length separated by a long delay. The first inspection interval is commonly referred to as the "reals plus accidental" gate (R + A) and is opened following a signal and a short predelay. The inspection length T_G is chosen to ensure that the observed counts during this interval are correlated in time with the initial signal trigger. A second inspection interval, known as the "accidental" gate (A), is opened after a long delay (long enough to ensure that no correlation exists to the initial signal trigger) to estimate the number distribution of uncorrelated signals. The S, D, and T count rates are given by

$$S = S \cdot m_{n(0)} = \frac{m_{b(1)}}{T_G} = \frac{N_T}{T_M} \ , \tag{69}$$

$$D = \frac{S}{f} \cdot \left[m_{n(1)} - m_{b(1)}\right] = \frac{S}{f}\left[m_{n(1)} - S \cdot T_G\right] \ , \tag{70}$$

and

$$\begin{aligned} T &= \frac{S}{f^2}\left[m_{n(2)} - m_{b(2)} - m_{b(1)}\left(m_{n(1)} - m_{b(1)}\right)\right] \\ &= \frac{S}{f^2}\left[\left(m_{n(2)} - m_{b(2)}\right) - \left(D \cdot T_G\right)\right] \ , \end{aligned} \tag{71}$$

noting that the number of randomly triggered gates K is equal to and governed by the signal triggers N_T. In other words, the number distribution given by the (A) gates can be thought of as an oversampling of the number distribution given by the independent RTI distribution as N_T is typically much greater than K (4). The S, D, and T count rates in Eqs. (61), (62), (63), (65), (66), (67), (69), (70), and (71) can be related to Eqs. (54), (55), and (56) to unfold physical properties of the multiplying assembly. Table V summarizes the nomenclature used in this work and previous works describing parameters related to gate generation techniques for extracting neutron count rates.

VII. IMPLICATION OF NEW DETECTOR TECHNOLOGY

Recently, there have been many works that investigate the feasibility of using scatter-based organic scintillators for fast-NMC and have described the potential improvements that can be made from the widely used capture-based thermal NMC systems (i.e., ^{3}He systems).[29–32] Scatter-based organic scintillators for fast-NMC do not require moderating material between the detector and multiplying assembly, which in turn reduces the system die-away time drastically. Furthermore, these scintillators are also sensitive to gamma rays and allow for additional capabilities unavailable to thermal NMC systems. There are many works that investigate the potential benefits of using organic scintillators for characterizing multiplying assemblies.[33,34] The use of organic scintillators has prompted a reevaluation of the well-established NMC mathematics summarized in Secs. III through VI due to the fundamentally different detection mechanism compared to capture-based thermal NMC systems.

An example of why NMC mathematics has been recently revisited specifically for organic scintillators includes the necessity of accounting for neutron cross-talk effects. Neutron cross talk occurs when a single neutron scatters and deposits enough energy in multiple detectors, which in turn increases the observed count rates. Several works have extended NMC mathematics to include neutron cross-talk effects in the context of Feynman-Y analysis[34,35] and passive NDA analysis.[36] Although the nomenclature used in these works differ, it is shown in 35 that with a simple variable transformation the same conclusions can be made. These works show that as detector technology advances, the fundamental mathematics must also be addressed.

VIII. CONCLUSIONS

This note consolidates the previously used notations for NMC in the context of both reactivity and passive nondestructive assay measurements. Specifically, this note explores the concepts and areas of neutron multiplication, point kinetics model, reactivity measurements, NDA, gate generation techniques, and implications of new detector technology. We hope that the readers will utilize this document to compare and understand neutron multiplicity work, which has been performed in a variety of application areas, to facilitate wider application-independent advancements to address the challenges pertinent to both reactivity and passive NDA techniques.

APPENDIX

In this appendix, we demonstrate that regardless of the sign convention used to describe the prompt neutron decay constant α given in Eq. (15), the derivation of the familiar expression for the total neutron multiplication (which is related to the multiplication factor k_p) are the same. We begin the derivation with the negative of Eq. (15) given by

$$\alpha = \frac{k_p - 1}{\tau_0} \ , \tag{A.1}$$

and thus the time-dependent behavior of the prompt neutron population is described by

$$\frac{dn(t)}{dt} = \alpha \cdot n(t) \ . \tag{A.2}$$

TABLE V

Summary of Nomenclature Used in this Note to Describe Parameters Related to Gate Generation Techniques for Extracting Neutron Counts, Along with Previously Used Notations/Definitions

Definition	Previously Used Nomenclature	Reference
Number of detected events with x signals inside inspection interval T_G using RTI analysis	$B_x(T_G)$ B_x $B(x)$	This work; Refs. 24, 31, 35, and 36 Ref. 23 Refs. 27 and 28
Number of detected events with x signals inside inspection interval T_G using STI analysis	$N_x(T_G)$ N_x $N(x)$	This work; Ref. 24 Ref. 23 Refs. 27 and 28
Probability distribution for x number of detected events inside inspection interval T_G using RTI analysis	$b_x(T_G)$ b_x	This work; Refs. 14, 31, 35, and 36 Refs. 23 and 24
Probability distribution for x number of detected events inside inspection interval T_G using STI analysis	$n_x(T_G)$ n_x	This work; Ref. 14 Refs. 23 and 24
μ'th order–reduced moments of the detected neutron distribution using RTI analysis	$m_{b(\mu)}$	This work; Refs. 4, 14, 24, 27, and 28
μ'th order–reduced moments of the detected neutron distribution using STI analysis	$m_{n(\mu)}$	This work; Refs. 4, 14, 27, and 28
Number of RTI intervals for RTI analysis	K N_b	This work; Refs. 4, 14, 27, and 28 Ref. 23
Number of STI intervals for STI analysis	N_T	This work; Refs. 4, 14, 23, 27, and 28
Total measurement time	T_M	This work; Refs. 4, 14, 23, 24, 27, and 28
Delay between trigger and acquisition (predelay)	T_{PD} T T_P PD P	This work Ref. 14 Refs. 26, 27, and 28 Ref. 3 Ref. 20
Neutron singles count rate corrected for finite coincidence gating structure	S R_1	This work; Refs. 3, 16, 20, 26, 27, 28, and 37 Refs. 14, 23, 31, 35, and 36
Neutron doubles count rate corrected for finite coincidence gating structure	D R_2	This work; Refs. 3, 16, 20, 26, 27, 28, and 37 Refs. 14 and 23
Neutron triples count rate corrected for finite coincidence gating structure	T R_3	This work; Refs. 3, 16, 20, 26, 27, 28, and 37 Refs. 14 and 23
Fundamental mode decay constant of detection system (system die-away decay constant)	λ τ T^{-1} λ^{-1}	This work; Refs. 4, 14, 23, and 24 Refs. 20, 26, 27, and 28 Refs. 3 Refs. 31, 35, and 36
Gate utilization factor for RTI analysis	w_μ	This work; Refs. 4, 14, 26, 27, and 28
Gate utilization factor for STI analysis	f	This work; Refs. 3, 16, 22, 26, 27, 28, and 37

Following the derivation presented in this note, we have the neutron population at some time t given by

$$n(t) = n_0 e^{\alpha t} \quad , \tag{A.3}$$

and the probable number of fissions about dt is given by

$$dQ_F = n_0 e^{\alpha t} \frac{dt}{\tau_f} \quad . \tag{A.4}$$

The number of subsequent neutrons resulting from the fissions produced about dt is given by multiplying $\nu_{if,1}$ to Eq. (A.4) and yields

$$dn = \ \nu_{if,1} n_0 e^{\alpha t} \frac{dt}{\tau_f} \quad . \tag{A.5}$$

The total number of subsequent neutrons produced from $n_0 = 1$ initial neutron emitted at time $t = 0$ is given by integrating Eq. (A.5) through time:

$$\begin{aligned} n_0 + \int_0^\infty \nu_{if,1} n_o e^{\alpha t} \frac{dt}{\tau_f} &= 1 + \frac{\nu_{if,1}}{\tau_f}\left(-\frac{1}{\alpha}\right) \\ &= 1 + \frac{\nu_{if,1}}{\tau_f}\left(\frac{\tau_0}{1-k_p}\right) \\ &= 1 + \frac{k_p}{1-k_p} \\ &= \frac{1}{1-k_p} = M_T \ , \end{aligned} \tag{A.6}$$

which is the same result given by Eq. (19). Note that defining α using Eq. (15), negative values of α describe the region of supercriticality and positive values describe the region of subcriticality, whereas defining α using Eq. (A.1) results in the opposite description. That is,

$$\alpha = \frac{1-k_p}{\tau_0} \ is \begin{cases} >0, \ sub-prompt\ critical \\ =0, \ prompt\ critical \\ <0, \ super-prompt\ critical \end{cases} \tag{A.7}$$

and

$$\alpha = \frac{k_p - 1}{\tau_0} \ is \begin{cases} >0, \ super-prompt\ critical \\ =0, \ prompt\ critical \\ <0, \ sub-prompt\ critical \end{cases} \quad . \tag{A.8}$$

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