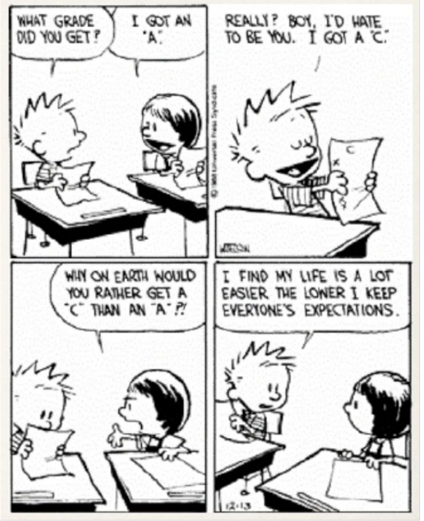
LECTURE 11

Expected Value



Computing and simulating the expected value of a discrete random variable

CSCI 3022 @ CU Boulder

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Content Credit: Lisa Yan, Chris Piech, Mehran Sahami, and Jerry Cain

Announcements

- HW 5 Has Been Published on Canvas - Due next Thursday
  - https://canvas.colorado.edu/courses/101142/assignments/1893692

Road Map

Expected Value

- Simulation
- Definition
- Expected Value of Common Distributions
  - Bernoulli
  - Binomial
  - Poisson

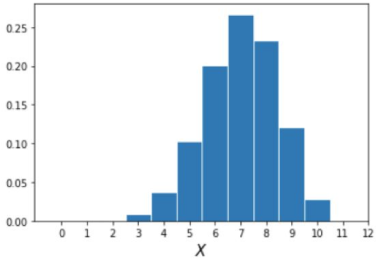
Descriptive Properties of Random Variables

There are several ways to describe a random variable:

Distribution Table
$P(X = x)$

Distribution function

Histogram



Random Variable X

Definition (fully describes)

Properties (summarizes)

Expectation
the "average value" of X
$\mathbb{E}[X]$

Variance
the "spread" of X
$\mathrm{Var}(X)$

The expectation and variance of a random variable are **numerical summaries** of X.

They are numbers and are not random!

Should you gamble?

A Las Vegas roulette board contains 38 numbers (00, 0, 1, 2, ….36}.

Players bet on which red or black numbered compartment of a revolving wheel a small ball (spun in the opposite direction) will come to rest within.

You can place bets on various number/color combinations and each type of bet pays out a different rate. For example:

- If you bet $1 on any particular number and win, then your net winnings is $35 (i.e the casino gives you your original dollar back plus $35)

Definition: Net Winnings:

Amount you received after game) minus (amount you paid to play)

You decide to bet $1 on the number 7.

You repeat this 100 times and write down your net winnings for each trial, and then take the **average of your net winnings.**

Poll: What number below is the closest to the number you'd expect to get as the **average net winnings** out of 100 trials?

A) $38 B). $ 10 C). $ -1

D). -$0.05 E). none of these

Simulation!

A Las Vegas roulette board contains 38 numbers $\{0, 00, 1, 2, \ldots, 36\}$. Of the non-zero numbers, 18 are red and 18 are black. You can place bets on various number/color combinations and each type of bet pays-out at a different rate. For example:

- If you bet $\$1$ on red (or black) and win, then you win $\$1$ (i.e. you get your original dollar back, plus another dollar).
- If you bet $\$1$ any particular number and win, then you win $\$35$ (i.e you get your original dollar back, plus $\$35$).
- If you bet $\$1$ on the first dozen (or second dozen, or third dozen) nonzero numbers and win, then you win $\$2$ (i.e. you get your original dollar back, plus another $\$2$).

You decide to bet \$1 on the number 7.

You repeat this 100 times and write down your net winnings for each trial, and then take the **average of your net winnings.**

Let's simulate this process!

Expected Value: Definition

Expected Value

- Simulation
- **Definition**
- Expected Value of Common Distributions
  - Bernoulli
  - Binomial
  - Poisson

Definition of Expectation: Discrete Random Variables

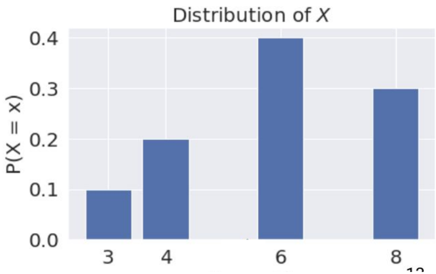
The **expectation (or expected value)** of a random variable X, denoted $\mathbb{E}[X]$ or $\boldsymbol{\mu}$ is the **weighted average** of the values of X, where the weights are the probabilities of the values:

$$\boxed{\mu = \mathbb{E}[X] = \sum_k k\, P(X = k)}$$

- Other names: mean, expected value, weighted average, center of mass, first moment

Ex). Consider the random variable X with distribution:

k	P(X = k)
3	0.1
4	0.2
6	0.4
8	0.3



What is $\mathbb{E}[X]$?

$\mathbb{E}[X] =$

Expectation

- The expectation E(X) is the **long-run average** value of X.
- Why can't we just average together the values of X?
  - Some values are more likely to occur than others
  - So, we have to take a weighted average:

$$E(X) = \sum_{x \in \mathbb{X}} x \cdot P(X = x)$$

- Read this as: average together the values that X can take on, giving more weight to values that appear more often.

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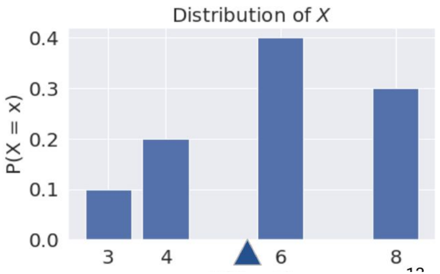
$$\begin{aligned}\mathbb{E}[X] &= 3 \cdot 0.1 + 4 \cdot 0.2 + 6 \cdot 0.4 + 8 \cdot 0.3 \\ &= 0.3 + 0.8 + 2.4 + 2.4 \\ &= 5.9\end{aligned}$$



Definition of Expectation: Discrete Random Variables

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Expectation is a number, not a random variable!

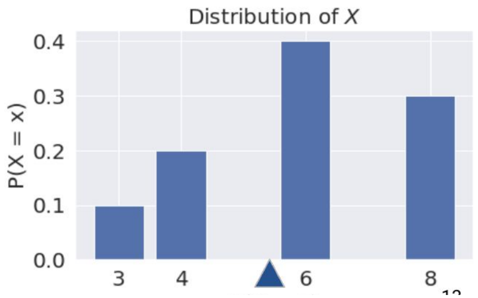
- **Analogous to the sample mean $\bar{x}$**
  - Suppose we have 10 students and their proportions observed exactly mirrored the probabilities given by the probability distribution of X:
    i.e. suppose our data is 3, 4, 4, 6, 6, 6, 6, 8, 8, 8

$$\bar{x} = \frac{3+4+4+6+6+6+6+8+8+8}{10}$$

$$= \frac{3(1)+4(2)+6(4)+8(3)}{10}$$

$$= 3\frac{1}{10} + 4\frac{2}{10} + 6\frac{4}{10} + 8\frac{3}{10} = 3(0.1)+4(0.2)+6(0.4)+8(0.3) = 5.9$$

- **Is the center of gravity or balance point of the probability histogram.**



The expectation of X does not need to be in the support of X

Should you gamble?

A Las Vegas roulette board contains 38 numbers $\{0, 00, 1, 2, \ldots, 36\}$. Of the non-zero numbers, 18 are red and 18 are black. You can place bets on various number/color combinations and each type of bet pays-out at a different rate. For example:

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Definition: Net Winnings:

Amount you received after game) minus (amount you paid to play)

Let X be your **net winnings** if you bet \$1 on any particular number and play once.

a). What is the probability distribution of X?

b). What is the expectation, $E[X]$?

$$\mu = \mathbb{E}[X] = \sum_k k\, P(X = k)$$

Dice Is the Plural; Die Is the Singular

Let X be the outcome of a single die roll.

X is a random variable.

$$P(X = x) = \begin{cases} 1/6 & \text{if } x \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{otherwise} \end{cases}$$

What is the expectation, $E[X]$?

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What is the expectation, $E[X]$?

$$\begin{aligned} \mathbb{E}[X] &= 1(1/6) + 2(1/6) + 3(1/6) + 4(1/6) + 5(1/6) + 6(1/6) \\ &= (1/6)(1+2+3+4+5+6) = \frac{7}{2} \end{aligned}$$

Expected Value of Common Distributions

Expected Value

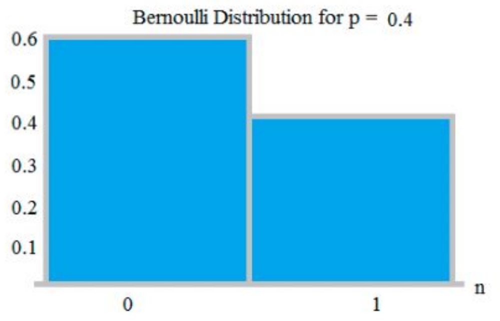
- Simulation
- Definition
- **Expected Value of Common Distributions**
  - **Bernoulli**
  - **Binomial**
  - **Poisson**

Expectation of Bernoulli Random Variable

Consider an experiment with two outcomes: "success" and "failure".

def A Bernoulli random variable X maps "success" to 1 and "failure" to 0.
Other names: indicator random variable, Boolean random variable

$X \sim \text{Ber}(p)$	PMF	$P(X=1) = p(1) = p$ $P(X=0) = p(0) = 1-p$
	Expectation	$E[X] = $
Support: $\{0,1\}$		



Examples:

- Coin flip
- Random binary digit
- Whether Doris barks

$$\textbf{Expected Value} = \sum_k k\, P(X=k)$$

$$=$$

Properties of Expectation

Recall definition of expectation: $$\mu = \mathbb{E}[X] = \sum_k k\, P(X = k)$$

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Properties:

1. Linearity:

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

- Let X = 6-sided dice roll, $Y = 2X - 1$.
- $E[X] = 3.5$
- $E[Y] = 6$

Properties of Expectation

Recall definition of expectation: $$\mu = \mathbb{E}[X] = \sum_k k\, P(X = k)$$

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2. Expectation of a sum = sum of expectation:

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

- Let X = 6-sided dice roll,
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Sum of two dice rolls:

- Let X = roll of die 1
  Y = roll of die 2
- $E[X + Y] = 3.5 + 3.5 = 7$

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2. Expectation of a sum = sum of expectation:

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

3. Expected value of a function of X:

$$\mathbb{E}[g(X)] = \sum_{all\ k} g(k) P(X = k)$$

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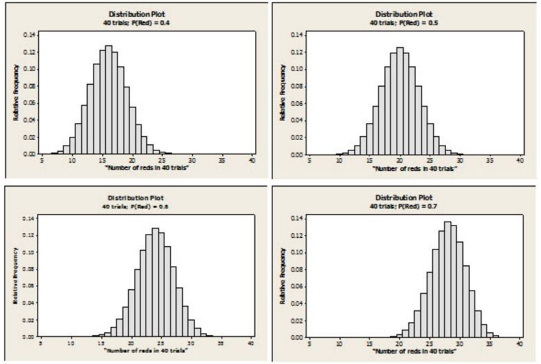
Sum of two dice rolls:

- Let X = roll of die 1, Y = roll of die 2
- $E[X + Y] = 3.5 + 3.5 = 7$

See proofs in Appendix

Expectation of Binomial Random Variable



Consider an experiment: n independent trials of Ber(p) random variables.

def A Binomial random variable ***Y*** is the number of successes in n trials.

$$Y \sim \text{Bin}(n, p)$$

Support: $\{0, 1, \dots, n\}$

PMF $k = 0, 1, \dots, n$:

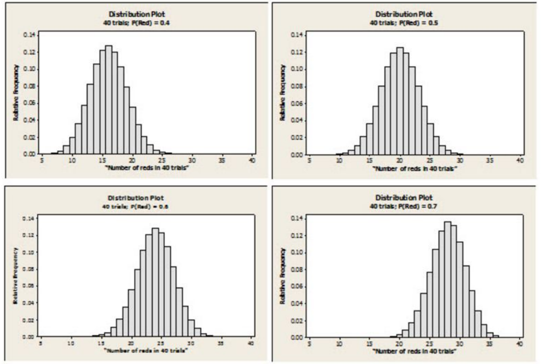
$$P(Y=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Expectation $E[Y] = \boxed{}$

Examples:

- # heads in n coin flips
- # of 1's in randomly generated length n bit string
- # of disk drives crashed in 1000 computer cluster (assuming disks crash independently)

Expectation of Binomial Random Variable



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Examples:
- # heads in n coin flips
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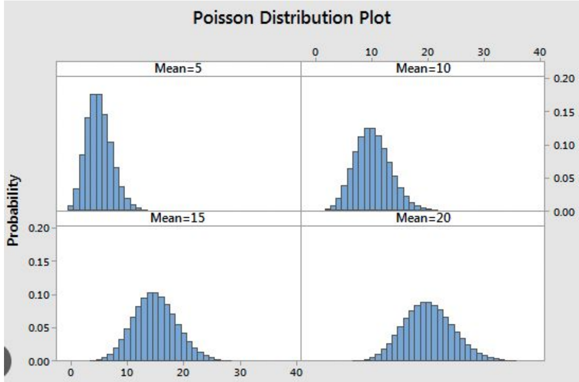
We can write: $Y = \sum_{i=1}^{n} X_i$

A count is a **sum** of 0's and 1's.

- X_i is the indicator of success on trial i. X_i = 1 if trial i is a success, else 0.
- All X_i s are **IID** (independent and identically distributed) and **Bernoulli**(p).

Expectation of Poisson RV



Consider an experiment that lasts a fixed interval of time.

def A **Poisson** random variable X is the number of successes over the experiment duration, assuming **the time each success occurs is independent** and the average number of successes per interval is a constant, λ

$X \sim \text{Poi}(\lambda)$

Support: $\{0, 1, 2, \dots\}$

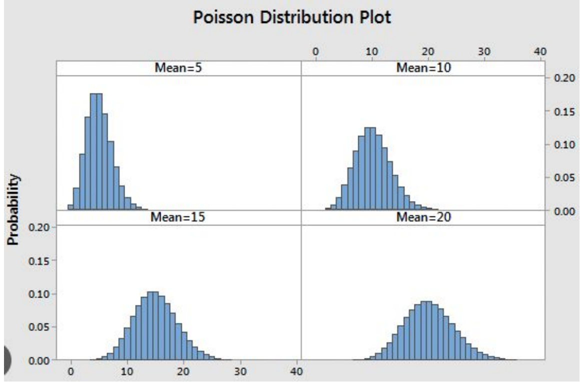
PMF $$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

Expectation $E[X] = \boxed{}$

Examples:

- # earthquakes per year
- # server hits per second
- # of emails per day

Expectation of Poisson RV



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Examples:
- # earthquakes per year
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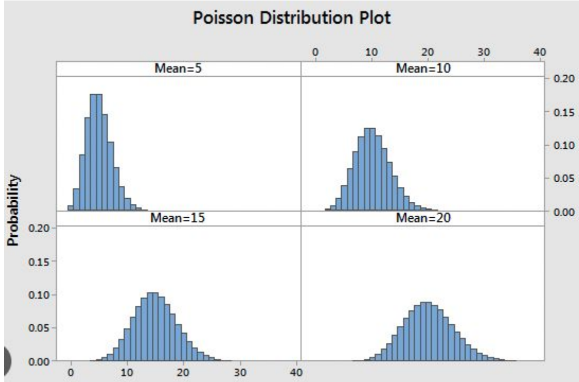
$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k\, P(X = k) = \sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!} = \frac{1\lambda^1 e^{-\lambda}}{1!} + \frac{2\lambda^2 e^{-\lambda}}{2!} + \frac{3\lambda^3 e^{-\lambda}}{3!} + \cdots \frac{n\lambda^n e^{-\lambda}}{n!} + \cdots$$

Expectation of Poisson RV



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Expectation $E[X] =$ $\boxed{}$

Examples:

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$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k\, P(X = k) = \sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!} = \frac{1\lambda^1 e^{-\lambda}}{1!} + \frac{2\lambda^2 e^{-\lambda}}{2!} + \frac{3\lambda^3 e^{-\lambda}}{3!} + \cdots \frac{n\lambda^n e^{-\lambda}}{n!} + \cdots$$

$$= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$=$$

[Extra Slides] Derivations

Properties of Expectation #1

Recall definition of expectation:

$$\mathbb{E}[X] = \sum_x xP(X = x)$$

2. **Expectation is linear**:

(intuition: summations are linear)

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

Proof:

$$\begin{aligned}\mathbb{E}[aX + b] &= \sum_x (ax + b)P(X = x) = \sum_x \big(axP(X = x) + bP(X = x)\big) \\ &= a\sum_x xP(X = x) + b\sum_x P(X = x) \\ &= a\mathbb{E}[X] + b \cdot 1\end{aligned}$$

Expectation of Sum intuition

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$

$$E[X+Y] = E[X] + E[Y]$$

we'll prove this in a few lectures

Intuition for now:

X	Y	$X+Y$
3	6	9
2	4	6
6	12	18
10	20	30
-1	-2	-3
0	0	0
8	16	24

Average:

$$\frac{1}{n}\sum_{i=1}^{n} x_i + \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}\sum_{i=1}^{n}(x_i + y_i)$$

$$\frac{1}{7}(28) + \frac{1}{7}(56) = \frac{1}{7}(84)$$

Properties of Expectation #3

$$E[g(X)] = \sum_x g(x)p(x)$$ Expectation of $g(X)$

Let $Y = g(X)$, where g is a real-valued function.

$$E[g(X)] = E[Y] = \sum_j y_j p(y_j)$$

$$= \sum_j y_j \sum_{i:g(x_i)=y_j} p(x_i)$$

$$= \sum_j \sum_{i:g(x_i)=y_j} y_j\, p(x_i)$$

$$= \sum_j \sum_{i:g(x_i)=y_j} g(x_i)\, p(x_i)$$

$$= \sum_i g(x_i)\, p(x_i)$$

For you to review so that you can sleep tonight!