"Discrete Structures"

* Lecture 11 *

* Chapter 4: Relations And Digraphs:-

→ Cartesian Product And Partitions:

Let $A=\{a,b,c\}$, $B=\{1,2\}$

$\therefore$ Cartesian product: $A\times B=\{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\}$

* $|A\times B|=|A|\times|B|$ * If $A=B=R \Rightarrow A\times B=R^2$

* The partition of the set A is a collection of subsets of A, where:
1. $\bigcup_{i=1} A_j = A$, 2. $A_i \cap A_j = \emptyset$ for $i \neq j$

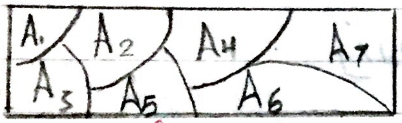


figure 1

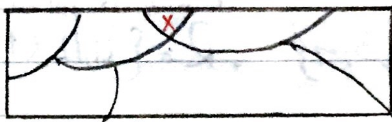


figure 2

⇒ figure 1 shows partition $P=\{A_1,A_2,A_3,A_4,A_5,A_6,A_7\}$ into seven blocks, but figure 2 doesn't show a partition as there is intersection between 2 blocks.

- Example: $A=\{a,b,c,d,e,f,g,h\}$

$A_1=\{a,b,c,d\}$, $A_2=\{a,c,e,f,g,h\}$, $A_3=\{a,c,e,g\}$, $A_4=\{b,d\}$, $A_5=\{f,h\}$

→ $\{A_1,A_2\}$ is not a partition since $A_1\cap A_2 \neq \emptyset$

, $\{A_1,A_5\}$ is not a partition since $e \notin A_1$ and $e \notin A_5$

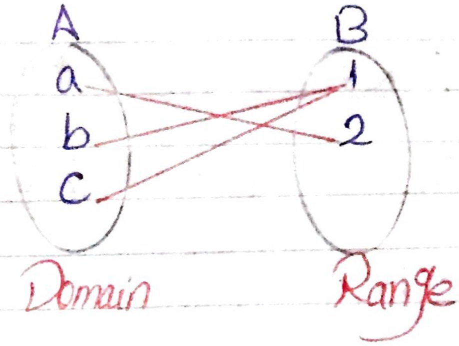
$\therefore$ The collection $P=\{A_3,A_4,A_5\}$ is a partition of A.

- Example 2: $A=\{1,2,3,4\}$, $B=\{1,3,4,5\}$, $R=\{(a,b): a<b\}$

→ $R=\{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5)\}$

, $|A\times B| = 4\times 4 = 16$

* Domain VS Range:



Example: $A=\{1,2,3,4\}$, $B=A$

$R=\{(x,y): x>y\}$

→ $R=\{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$

, $Dom(R)=\{2,3,4\}$

, $Range(R)=\{1,2,3\}$

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* If $A=\{a,b,c,d\}$, $R=\{(a,a),(a,b),(b,c),(c,a),(d,c),(c,d)\}$

→ R relative set of a : $R(a)=\{a,b\}$

, " " " c : $R(c)=\{d\}$

, If $B=\{b,d\}$, $R(B)=\{c\}$

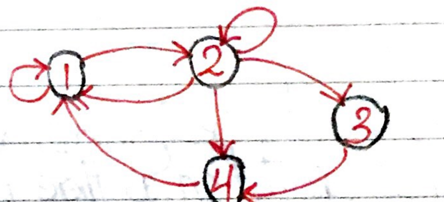
, If $B=\{a,c\}$ ⇒ Restriction of R to B $= R\cap(B\times B) = \{(a,a),(c,a)\}$

Matrix and Digraph of Relations.

→ $A=\{1,2,3,4\}$, $R=\{(1,1),(1,2),(2,1),(2,2),(2,3),(2,4),(3,4),(4,1)\}$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 1 & 0 & 0 & 0 \end{array}$$

matrix of relation.

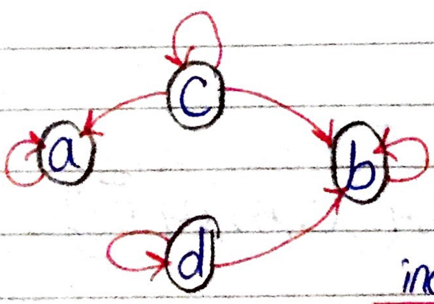


digraph of relation.

Example: $A=\{a,b,c,d\}$, $M_R=$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 0 & 0 \\ b & 0 & 1 & 0 & 0 \\ c & 1 & 1 & 1 & 0 \\ d & 0 & 1 & 0 & 1 \end{array}$$

, find indegree, outdegree.

→ 

	a	b	c	d
indegree	2	3	1	1
outdegree	1	1	3	2

* Relations *

1) Reflexive: Every element is related to itself. $\forall a \in A, (a,a) \in R$
, main diagonal = ones.

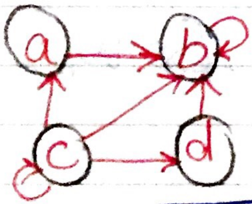
2) IRReflexive: No element is related to itself., main diagonal = zeros.

3) Symmetric: $\forall (a,b) \in R \rightarrow (b,a) \in R$ «at aRb, then bRa»
In digraph: Every arrow goes out must return back.
In matrix : $a_{ij} = a_{ji} = 1$

4) Asymmetric: $\forall (a,b) \in R \rightarrow (b,a) \notin R$ «at aRb, then $b \not R a$»
→ No returning arrow to itself and The arrow that goes out doesn't return back.
→ main diagonal = 0 , if $a_{ij} = 1 \Rightarrow a_{ji} = 0$

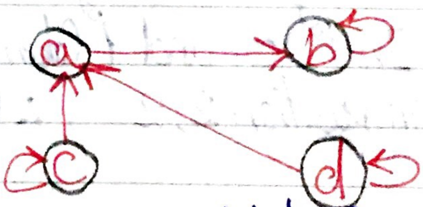
5) AntiSymmetric: $(a,b) \in R$ and $(b,a) \in R \Rightarrow a = b$
→ as Asymmetric but the element could return back to itself.

6) Transitive: $\forall (a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$
→ A Relation R is transitive if and only if its matrix $M_R = [m_{ij}]$:
if $m_{ij} = 1$ and $m_{jk} = 1$, then $m_{ik} = 1$



$\because a \rightarrow b$
$, b \rightarrow b$
✓ $\therefore a \rightarrow b$

Transitive



$\because c \rightarrow a$
$, a \rightarrow b$
x $\therefore c \rightarrow b$

Not Transitive.

* Reflexive Classure: $R \vee I$

→ example: If $R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. find its reflexive classure

→ reflexive classure $= R \vee I$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* Symmetric Classure: $R \vee R'$

→ example: If $R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, find its symmetric classure

→ symmetric classure $= R \vee R^T$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

* Transitive classure: find transitive classure of $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow W_0$

→ step1: In 1^{st} row and 1^{st} Column, W_0 has ones in location 2, So W_1 must have 1 in position 2,2 , $\therefore W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Step 2: In 2^{nd} row and 2^{nd} Column, W_1 has ones in locations 1,2 of Column 2 and locations 1,2,3 of row 2, So W_2 must have 1 in positions: 1,1 ; 1,2 ; 1,3 ; 2,1 ; 2,2 ; 2,3

$\therefore W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Step 3: In 3^{rd} row and 3^{rd} Column, W_2 has ones in locations 1,2 of Column 3 and 4 of row 3, So W_3 must have 1 in positions 1,4 and 2,4

$$\therefore W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

, Finally, In 4^{th} row and 4^{th} Column, W_3 has ones in locations 1,2,3 of Column 4 and no ones in row 4, So no new ones are added and $M_R^{\infty} = W_4 = W_3$