

EMBEDDINGS OF REAL PROJECTIVE SPACES

DONALD M. DAVIS

1. STATEMENT OF RESULTS

The question of finding the smallest Euclidean space in which real projective space P^n can be (differentiably) embedded was the subject of intense investigation during the 1960s and 1970s. The purpose of this paper is to survey the status of the question, and add a little bit to our knowledge by proving one new family of embeddings, using old methods of obstruction theory. Our new result is given in the following theorem.

Theorem 1.1. *If $n = 2^i + 3 \geq 11$, then P^n can be embedded in \mathbf{R}^{2n-4} .*

As far as I can tell, this improves on previous embeddings by 1 dimension. Indeed, Berrick's 1979 table ([4]) lists the best embedding for P^n to be in \mathbf{R}^{2n-3} when $n = 2^i + 3$, from [19], and I know of no embedding results for P^n proved subsequent to Berrick's table. (There are, however, subsequent nonembedding results, notably those of [3].)

The following table lists the best nonembedding and embedding results for P^n of which I am aware, for $n \leq 63$. Of course, most of these results fit into infinite families. Here we use the symbol \subset to refer to differentiable embeddings.

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n	$\not\subset$ ref	\subset ref	n	$\not\subset$ ref	\subset ref
2	3 [20]	4 [11]	32	63 [20]	64 [11]
3	4 [14]	5 [11]	33	64 [14]	65 [11]
4	7 [20]	8 [11]	34	64 [14]	65 [18]
5	8 [14]	9 [11]	35	64 [14]	66 new
6	8 [14]	11 [8]	36	65 [3]	69 [15]
7	8 [14]	12 [17]	37	70 [2]	71 [15]
8	15 [20]	16 [11]	38	70 [2]	71 [19]
9	16 [14]	17 [11]	39	70 [2]	71 [19]
10	16 [14]	17 [18]	40	70 [2]	74 [21]
11	16 [14]	18 new	41	77 [1]	79 [15]
12	17 [3]	21 [15]	42	77 [1]	80 [21]
13	22 [2]	23 [15]	43	78 [2]	80 [21]
14	22 [2]	23 [19]	44	78 [2]	82 [21]
15	22 [2]	24 [19]	45	78 [2]	84 [21]
16	31 [20]	32 [11]	46	79 [3]	85 [6]
17	32 [14]	33 [11]	47	79 [3]	86 [5]
18	32 [14]	33 [18]	48	83 [3]	90 [21]
19	32 [14]	34 new	49	93 [1]	95 [15]
20	33 [3]	37 [15]	50	93 [1]	96 [21]
21	38 [2]	39 [15]	51	94 [2]	96 [21]
22	38 [2]	39 [19]	52	94 [2]	98 [21]
23	38 [2]	39 [19]	53	94 [2]	100 [21]
24	38 [2]	42 [21]	54	95 [3]	101 [6]
25	45 [1]	47 [15]	55	95 [3]	102 [5]
26	45 [1]	48 [21]	56	99 [3]	106 [21]
27	46 [2]	48 [21]	57	99 [3]	108 [21]
28	46 [2]	50 [21]	58	103 [7]	110 [21]
29	46 [2]	52 [21]	59	103 [7]	111 [6]
30	47 [3]	53 [6]	60	107 [3]	114 [21]
31	52 [12]	54 [5]	61	107 [3]	116 [21]
			62	107 [3]	117 [5]
			63	114 [12]	117 [5]

Note that some of the nonembedding results (those of [1], [12], and [7]) are actually obtained from nonimmersion results.

2. PROOF OF THEOREM

In this section, we prove Theorem 1.1. Our method is that used by Mahowald in [15]. A main tool is the following result, which was proved in [15], following [8]. In

Section 3, we will add a few details to the argument given in [15]. Let ξ_q denote the Hopf bundle over P^q , and let ε denote the trivial bundle.

Theorem 2.1. *Assume that P^q embeds in \mathbf{R}^p with normal bundle ν .*

- *If $\nu \otimes \xi_q$ has n linearly independent sections and P^{n-1} embeds in S^{m-1} , then P^{q+n} embeds in \mathbf{R}^{p+m} .*
- $(\nu \otimes \xi_q) \oplus (q+1)\varepsilon \approx (p+1)\xi_q$.

Here the hypothesized embeddings need only be topological, and the embedding in the conclusion is only topological. We then use the result of Haefliger ([10]) that a topological embedding of an n -manifold in \mathbf{R}^d can be approximated by a differentiable embedding provided $2d \geq 3(n+1)$, in our application of this theorem.

To prove Theorem 1.1, we apply Theorem 2.1 with $q = 2^i + 2$, $p = 2^{i+1} + 1$, $n = 1$, and $m = 1$. Throughout this section, we assume $i \geq 3$. The embedding of P^{2^i+2} in $\mathbf{R}^{2^{i+1}+1}$ was proved by Nussbaum in [18]. Theorem 1.1 will then follow from the following result (with $\theta = \nu \otimes \xi$) together with the embedding of P^0 in S^0 .

Proposition 2.2. *If θ is an orientable $(2^i - 1)$ -plane bundle over P^{2^i+2} such that*

$$\theta \oplus (2^i + 3)\varepsilon \approx (2^{i+1} + 2)\xi,$$

then θ has a nonzero section.

Proof. We apply obstruction theory to the following diagram.

$$\begin{array}{ccc} & S^{2^i-2} & \\ & \searrow & \\ & BSO(2^i - 2) & \\ & \downarrow & \\ P^{2^i+2} & \xrightarrow{\theta} & BSO(2^i - 1) \end{array}$$

The desired lifting of θ to $BSO(2^i - 2)$ is proved using modified Postnikov towers (MPTs), as introduced in [15] and refined in [9]. In the diagram below, $H_n = K(\mathbf{Z}, n)$

and $K_n = K(\mathbf{Z}_2, n)$. We write $\mathbf{Z}/2$ and \mathbf{Z}_2 interchangeably. Each vertical map is part of a fiber sequence with the diagonal maps on either side of it in the diagram; e.g.,

$$K_{2^i} \times K_{2^{i+1}} \rightarrow E_3 \rightarrow E_2 \rightarrow K_{2^{i+1}} \times K_{2^{i+2}}$$

is a fiber sequence.

$$\begin{array}{ccccc}
 & & BSO(2^i - 2) & & \\
 & & \downarrow & & \\
 K_{2^i} \times K_{2^{i+1}} & \searrow & E_3 & \searrow & K_{2^{i+2}} \\
 & & \downarrow & & \\
 K_{2^{i-1}} \times K_{2^{i+1}} & \searrow & E_2 & \searrow & K_{2^{i+1}} \times K_{2^{i+2}} \\
 & & \downarrow & & \\
 H_{2^i-2} & \searrow & E_1 & \searrow k_1 & K_{2^i} \times K_{2^{i+2}} \\
 & & \downarrow & & \\
 P^{2^i+2} & \xrightarrow{\theta} & BSO(2^i - 1) & & \\
 & & & \searrow & H_{2^i-1}
 \end{array}$$

The k -invariants correspond to elements in an Adams resolution of the stable sphere, where we kill the initial \mathbf{Z} all at once. We need the relations which give rise to the k -invariants in the MPT. These are computed by the method initiated in [9] and used in many subsequent papers such as [7] and [13]. It is a matter of building a minimal resolution using Massey-Peterson algebras. The relations are as in the table below, with β denoting the Bockstein.

βw_{2^i-2}
$k_{2^i}^1 : (\text{Sq}^2 + w_2)(\beta w_{2^i-2})$
$k_{2^{i+2}}^1 : (\text{Sq}^4 + w_4)(\beta w_{2^i-2})$
$k_{2^{i+1}}^2 : (\text{Sq}^2 + w_2)k_{2^i}^1$
$k_{2^{i+2}}^2 : \text{Sq}^1 k_{2^{i+2}}^1 + (\text{Sq}^2 \text{Sq}^1 + w_3)k_{2^i}^1$
$k_{2^{i+2}}^3 : \text{Sq}^1 k_{2^{i+2}}^2 + (\text{Sq}^2 + w_2)k_{2^{i+1}}^2$

We illustrate how these relations are used, using $k_{2^i+1}^2$. Its relation means that, with μ the action map,

$$(K_{2^i-1} \times K_{2^i+1}) \times E_2 \xrightarrow{\mu} E_2 \xrightarrow{k_{2^i+1}^2} K_{2^i+1}$$

corresponds to the class $1 \otimes k_{2^i+1}^2 + \text{Sq}^2 \iota_{2^i-1} \otimes 1 + \iota_{2^i-1} \otimes w_2$. It also means that in $H^*(E_1)$,

$$(2.3) \quad \text{Sq}^2 k_{2^i}^1 + w_2 k_{2^i}^1 = 0.$$

Since $H^{2^i-1}(P^{2^i+2}; \mathbf{Z}) = 0$, θ lifts to a map $\ell_1 : P^{2^i+2} \rightarrow E_1$. Suppose $\ell_1^*(k_{2^i}^1) = \epsilon_0 x^{2^i}$ and $\ell_1^*(K_{2^i+2}^1) = \epsilon_2 x^{2^i+2}$. By (2.3),

$$0 = \epsilon_0 \text{Sq}^2 x^{2^i} + w_2(\theta) \epsilon_0 x^{2^i} = 0 + \epsilon_0 x^{2^i+2},$$

and so $\epsilon_0 = 0$. If $\epsilon_2 = 1$, then let ℓ'_1 denote the composite

$$P^{2^i+2} \xrightarrow{x^{2^i-2} \times \ell_1} H_{2^i-2} \times E_1 \xrightarrow{\mu} E_1.$$

This will satisfy

$$\begin{aligned} \ell'_1{}^*(k_{2^i+2}^1) &= (x^{2^i-2} \times \ell_1)^*(1 \otimes k_{2^i+2}^1 + \text{Sq}^4 \iota_{2^i-2} \otimes 1 + \iota_{2^i-2} \otimes w_4) \\ &= x^{2^i+2} + \text{Sq}^4 x^{2^i-2} + x^{2^i-2} \cdot w_4(\theta) \\ &= x^{2^i+2} + x^{2^i+2} + 0 = 0. \end{aligned}$$

We don't have to worry about whether varying through H_{2^i-2} changes $\ell^*(k_{2^i}^1)$ because we have already shown that this is 0 for any lifting. Thus a lifting to E_1 can be chosen which sends both level-1 k -invariants to 0, and hence lifts to a map $\ell_2 : P^{2^i+2} \rightarrow E_2$.

To show that $\ell_2^*(k_{2^i+1}^2) = 0$, we need the following result, whose proof will appear at the end of this section.

Lemma 2.4. *The map θ extends to a map $P^{2^i+3} \xrightarrow{\theta'} BSO(2^i - 1)$.*

Since our tower has no k -invariants in dimension 2^i+3 , θ' lifts to a map $\ell'_2 : P^{2^i+3} \rightarrow E_2$ by the same analysis used to lift θ . Note that the MPT was constructed through dimension 2^i+2 , i.e. the space at the top agrees with $BSO(2^i-2)$ through dimension 2^i+2 . Usually we would say that even though the space at the top may not agree with $BSO(2^i-2)$ through dimension 2^i+3 , that is not of concern because in the end we are just using the lifting of P^{2^i+2} ; the (2^i+3) -cell is just used to detect relations

at early stages of the lifting. However, in the case at hand, since $\pi_{2^i+2}(S^{2^i-2}) = 0$, the space at the top of the tower agrees with $BSO(2^i - 2)$ through dimension $2^i + 3$.

If $(\ell'_2)^*(k_{2^i+1}^2) = \epsilon_1 x^{2^i+1}$ and $(\ell'_2)^*(k_{2^i+2}^2) = \epsilon_2 x^{2^i+2}$, then by the relation for $k_{2^i+2}^3$, we have

$$0 = \epsilon_2 \text{Sq}^1 x^{2^i+2} + \epsilon_1 \text{Sq}^2 x^{2^i+1} + x^2 \cdot \epsilon_1 x^{2^i+1} = 0 + 0 + \epsilon_1 x^{2^i+3},$$

and so $\epsilon_1 = 0$. This is where we need Lemma 2.4. If $\epsilon_2 \neq 0$, then varying ℓ_2 through K_{2^i+1} will yield a new lifting which sends $k_{2^i+2}^2$ to 0, thanks to the term $\text{Sq}^1 k_{2^i+2}^1$ in the relation $k_{2^i+2}^2$.

Thus there is a lifting $\ell''_2 : P^{2^i+2} \rightarrow E_2$ sending both k -invariants trivially, and hence a lifting $\ell_3 : P^{2^i+2} \rightarrow E_3$. If ℓ_3 sends the lone k -invariant at height 3 nontrivially, then varying ℓ_3 through K_{2^i+1} gives a new lifting which sends the k -invariant to 0, and hence lifts to $BSO(2^i - 2)$, as desired. ■

Proof of Lemma 2.4. By [13, 3.1], $(2^{i+1} + 2)\xi_{2^i+3}$ has $2^i + 4$ linearly independent sections, and hence there is a map $f' : P^{2^i+3} \rightarrow BSO(2^i - 2)$ which classifies $(2^{i+1} + 2)\xi$. Let $f = i \circ f'$, where $i : BSO(2^i - 2) \rightarrow BSO(2^i - 1)$ is the inclusion. By the hypothesis of Proposition 2.1, our map θ and $f|_{P^{2^i+2}}$ differ by a map δ which can be factored as

$$P^{2^i+2} \rightarrow V_{2^i-1} \rightarrow BSO(2^i - 1),$$

since V_{2^i-1} is the fiber of $BSO(2^i - 1) \rightarrow BSO$. By Lemma 2.5, any such δ extends over P^{2^i+3} , and hence $\theta = f + \delta$ is the sum of two maps which extend over P^{2^i+3} . ■

Lemma 2.5. *The restriction*

$$[P^{2^i+3}, V_{2^i-1}] \rightarrow [P^{2^i+2}, V_{2^i-1}]$$

of sets of homotopy classes of maps is surjective.

Proof. We will show that the target group is $\mathbf{Z}/8$, generated by the class of the composite

$$P^{2^i+2} \xrightarrow{c} P_{2^i-1}^{2^i+2} \hookrightarrow V_{2^i-1},$$

with c the collapse map. This composite clearly extends to

$$P^{2^i+3} \rightarrow P_{2^i-1}^{2^i+3} \hookrightarrow V_{2^i-1},$$

which will establish the lemma.

We begin by observing that $\pi_*(V_{2^i-1}) \approx \pi_*^s(P_{2^i-1})$, the stable homotopy groups of a stunted projective space, through dimension at least $2^i + 3$. These groups can be computed by the Adams spectral sequence, as done for example in [16, Table 8.16], to begin $\pi_j(V_{2^i-1}) \approx \mathbf{Z}_2$ if $j = 2^i - 1$, $2^i + 1$, or $2^i + 2$, with Adams filtrations 0, 1, and 2, respectively, and $\pi_{2^i}(V_{2^i-1}) = 0$. By obstruction theory, this implies that the order of $[P^{2^i+2}, V_{2^i-1}]$ is no greater than 8. The claim is that these three classes extend cyclically.

One way to see this begins by noting that the desired group equals $[P^{2^i+2}, P_{2^i-1}^{2^i+3}]$ and is in the stable range. There is a degree-1 map $P_{2^i-1}^{2^i+3} \rightarrow S^{2^i}$. Let F be its fiber. We will show that in the exact sequence

$$(2.6) \quad [P^{2^i+2}, F] \rightarrow [P^{2^i+2}, P_{2^i-1}^{2^i+3}] \rightarrow [P^{2^i+2}, S^{2^i}]$$

the first group is 0, and the third group is $\mathbf{Z}/8$. Since we have already observed that the middle group has order no greater than 8, and has an element of Adams filtration 0, the claim made in the first line of this proof will be established.

The third group in (2.6) is isomorphic to $\pi_{-1}(\Sigma^{2^i} P_{-2^i-3})$, using S -duality. This is $\mathbf{Z}/8$ by an elementary Adams spectral sequence calculation; it is the group in the column labeled 2 in [16, Table 8.14]. It is not difficult to check that $H^*(P_{-2^i-3} \wedge F)$ is, through dimension 0, a free module over the Steenrod algebra on classes of dimension -4 , -2 , and 0. Thus the Adams spectral sequence or elementary obstruction theory implies that $\pi_{-1}(P_{-2^i-3} \wedge F)$, which equals the first group in (2.6), is 0. ■

3. REVIEW OF MAHOWALD PROOF

In this section, we prove Theorem 2.1, merely fleshing out a few details in the proof given in [15]. The second part of 2.1 is elementary:

$$\begin{aligned} (\nu \otimes \xi) \oplus (q+1)\varepsilon &\approx (\nu \oplus (q+1)\xi) \otimes \xi \\ &\approx (\nu \oplus \tau \oplus \varepsilon) \otimes \xi \\ &\approx (p+1)\varepsilon \otimes \xi \\ &\approx (p+1)\xi \end{aligned}$$

The first part utilizes the following result. Throughout this section, embeddings (\subset) are not necessarily differentiable.

Proposition 3.1. *If $n\xi_q \subset \mathbf{R}^p$ and $P^{n-1} \subset S^{m-1}$, then $P^{q+n} \subset \mathbf{R}^{p+m}$.*

Proof of first part of Theorem 2.1. By hypothesis, $n\varepsilon \subset \nu \otimes \xi$. Tensoring with ξ yields the first part of

$$n\xi \subset \nu \subset \mathbf{R}^p,$$

with the second part clear since ν can be taken to be a tubular neighborhood of P^q in \mathbf{R}^p . The theorem now follows immediately from Proposition 3.1. ■

To prove Proposition 3.1, we will need the description of P^{n+q} given in the following lemma.

Lemma 3.2. *There is a homeomorphism*

$$(3.3) \quad P^{n-1} \cup_g D(n\xi_q) \approx P^{n+q},$$

where $g : S(n\xi_q) \rightarrow P^{n-1}$ is defined by $g([x, y]) = [x]$. Here we use the model $S(n\xi_q) = (S^{n-1} \times S^q)/(x, y) \sim (-x, -y)$.

Proof. The space on the LHS of (3.3) is

$$(3.4) \quad D^n \times S^q / ((x, y) \sim (-x, -y), (x, y) \sim (x, y') \text{ if } |x| = 1).$$

The desired homeomorphism sends $[(x, y)]$ to $[(x, \sqrt{1 - |x|^2}y)]$ in $S^{n+q}/z \sim -z$. ■

Proof of Proposition 3.1. Let f denote the embedding of $D(n\xi_q)$ in \mathbf{R}^p , and h the embedding of P^{n-1} in S^{m-1} . Using the descriptions of P^{n+q} given by the LHS of (3.3) and (3.4), we define our embedding by

$$\begin{aligned} \phi([(x, y)]) &= (\phi_1(x, y), \phi_2(x, y)) \\ \phi_1(x, y) &= (1 - |x|)f(x, y) \\ \phi_2(x, y) &= |x|h([x/|x|]). \end{aligned}$$

Here $x \in D^n$ and $y \in S^q$. We first show ϕ is well-defined. Because of the $(1 - |x|)$ -factor, the second relation in (3.4) is not a problem, and because of the $|x|$ -factor, we have continuity at $x = 0$, even though $x/|x|$ is not defined. Injectivity is proved as

follows. If $\phi_2(x, y) = \phi_2(x', y')$, then $x' = \pm x$. If also $\phi_1(x, y) = \phi_1(x', y')$, then either $|x| = 1$ and hence $[(x, y)] = [(\pm x, y')]$, or $(x', y') = (\pm x, \pm y)$ so that $[(x, y)] = [(x', y')]$.

■

REFERENCES

1. J. Adem and S. Gitler, *Nonimmersion theorems for real projective spaces*, Bol. Soc. Mat. Mex. **9** (1964) 37-50.
2. J. Adem, S. Gitler, and M. Mahowald, *Embedding and immersion of projective spaces*, Bol. Soc. Mat. Mex. **10** (1965) 84-88.
3. L. Astey, *A cobordism obstruction to embedding manifolds*, Ill. Jour. Math. **31** (1987) 344-350.
4. A. J. Berrick, *Projective space immersions, bilinear maps, and stable homotopy groups of spheres*, SV Lecture Notes in Mathematics **788** (1979) 1-22.
5. ———, Oxford Univ. D. Phil. thesis (1973).
6. ———, *The Smale invariants of an immersed projective space*, Math. Proc. Camb. Phil. Soc. **86** (1979) 401-412.
7. D. M. Davis, *Some new immersions and nonimmersions of real projective spaces*, AMS Contemporary Math. **19** (1983) 51-64.
8. D. B. A. Epstein and R. L. E. Schwarzenberger, *Imbeddings of real projective spaces*, Annals of Math. **76** (1962) 180-184.
9. S. Gitler and M. E. Mahowald, *The geometric dimension of real stable vector bundles*, Bol. Soc. Mat. Mex. **11** (1966) 85-107.
10. A. Haefliger, *Plongements différentiable dans le domaine stable*, Comm. Math. Helv. **37** (1962) 155-176.
11. H. Hopf, *Systeme symmetrischer Bilinearformen und euklidische Modelle der projektiven Räume*, Vierteljschr. Naturforsch. Gesellschaft Zurich **85** (1940) 165-177.
12. I. M. James, *On the immersion problem for real projective spaces*, Bull. Amer. Math. Soc. **69** (1963) 231-238.
13. K. Y. Lam, *Sectioning vector bundles over real projective spaces*, Quar. Jour. Math. Oxford **23** (1972) 97-106.
14. M. E. Mahowald, *On the embeddability of the real projective spaces*, Proc. Amer. Math. Soc. **13** (1962) 763-764.
15. ———, *On obstruction theory in orientable fibre bundles*, Trans. Amer. Math. Soc. **110** (1964) 315-349.
16. ———, *The metastable homotopy of S^n* , Mem. Amer. Math. Soc. **72** (1967).
17. M. E. Mahowald and R. J. Milgram, *Embedding real projective spaces*, Annals of Math. **87** (1968) 411-422.
18. F. Nussbaum, *Obstruction theory of possibly nonorientable fibrations*, Northwestern Univ. PhD thesis (1970).
19. E. Rees, *Embeddings of real projective spaces*, Topology **10** (1971) 309-312.
20. R. Thom, *Espaces fibrés en spheres et carrés de Steenrod*, Ann. Sci. Ecole Norm. Sup. **69** (1952) 109-182.
21. E. Thomas, *Embedding manifolds in Euclidean space*, Osaka Jour. Math. **13** (1976) 163-186.

LEHIGH UNIVERSITY, BETHLEHEM, PA 18015
E-mail address: dmd1@lehigh.edu