TP 5 : Projection sur un polyèdre et application en débruitage

Q1) Le lagrangien s'écrit : $L(x,\lambda) = \overbrace{\frac{1}{2}\|x-p\|^2}^{f(x)} + \sum_{i=1}^{k} \overbrace{\lambda_i\left(a_i^T x - b\right)}^{\lambda_i c_i}$

$$= \frac{1}{2}\|x-p\|^2 + \langle \lambda, Ax-b \rangle$$

$$\min_{x\in\mathbb{R}^d} L(x,\lambda) = \min_x f_\lambda(x)$$

f_λ est convexe ($\nabla^2 f_\lambda = 2I_k$), donc son minimiseur est le point x_λ t.q $\nabla f_\lambda(x_\lambda) = 0$

$$\nabla f_\lambda = (x-p) + A^T\lambda \quad \Rightarrow \quad \boxed{x_\lambda = p - A^T\lambda}$$

Ainsi $(D) = \max_{\lambda\in M} \min_{x\in\mathbb{R}^d} L(x,\lambda)$

$$= -\min_{\lambda\in M} -f_\lambda(x_\lambda)$$

$$= -\min_{\lambda\in M} -\frac{1}{2}\|p - A^T\lambda - p\|^2 - \lambda^T(Ap - AA^T\lambda) + \lambda^T b$$

$$h(\lambda) = -\frac{1}{2}\|A^T\lambda\|^2 - \lambda^T Ap + \|A^T\lambda\|^2 + \lambda^T b$$

$$= \frac{1}{2}\|A^T\lambda\|^2 - \lambda^T Ap + \lambda^T b$$

$$\boxed{h(\lambda) = \frac{1}{2}\|A^T\lambda - p\|^2 - \frac{1}{2}\|p\|^2 + \lambda^T b}$$

$$\nabla h(\lambda) = A(\overbrace{A^T\lambda - p}^{-x_\lambda}) + b$$

$$= b - Ax_\lambda.$$

Q2)

Algorithme d'Uzawa .

$\nabla^2 h(\lambda) = AA^T \succcurlyeq 0$ donc <u>h est convexe</u>

Si λ^* t.q $S_\tau(\lambda^*) = \lambda^*$

$\iff P_{\mathbb{R}_+^k}(\lambda^* - \tau \nabla h(\lambda^*)) = \lambda^*$

$$\iff \begin{cases} \lambda^* \in \mathbb{R}_+^k \\ \lambda^* - \tau \nabla h(\lambda^*) = \lambda^* . \end{cases}$$

<u>ie</u> <u>$\nabla h(\lambda^*) = 0$.</u>