A MAP Framework for Support Recovery of Sparse Signals Using Orthogonal Least Squares

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BACKGROUND

- Sampling and Reconstruction
  - *Small number of measurements of* high dimensional sparse signal
  - Applications: Compressive sensing, sparse channel estimation, DNA microarrays
- Notation and model

$$\mathbf{y} = A\mathbf{x} + \mathbf{v},\ A_{ij} \sim \mathcal{N}\left(0, \frac{1}{n}\right)$$

$A \in \mathbb{R}^{n \times m}$: Sensing matrix $(n < m)$

$\mathbf{x} \in \mathbb{R}^m$: High-dimensional sparse signal being measured

$\mathbf{y} \in \mathbb{R}^n$: Compressive measurements

$\mathbf{v} \in \mathbb{R}^m$: Independent Gaussian noise, $\mathcal{N}(0, \sigma^2 I)$

I : Support set of $\mathbf{x}$

S_k : Support set determined in kth iteration

- Goal: Identifying *correct support* of signal $\mathbf{x}$, thus enabling reconstruction from $\mathbf{y}$
- Prior work [1] done on greedy support recovery in a maximum a posteriori (MAP) framework
- Our approach: A MAP framework for *orthogonal least squares (OLS)*
- Leveraging the distributions of the support and sensing matrix to recover support more accurately than conventional greedy algorithms

PROBLEM FORMULATION

- For binary signals, the OLS selection criterion is

$$j_s = \text{argmax}_{j \in \mathcal{I} \setminus S_{k-1}} \Big(\sum_{i \notin S_{k-1} \cup j} x_i \mathbf{a}_i^\top + \mathbf{v}^\top \Big) \frac{P_{k-1}^\perp \mathbf{a}_j}{\|P_{k-1}^\perp \mathbf{a}_j\|_2} + x_j \|P_{k-1}^\perp \mathbf{a}_j\|_2,$$

- Using radial symmetry of $\mathbf{a}_j$ and defining hypotheses $\mathcal{H}_0 : x_j = 0,\ \mathcal{H}_1 : x_j = 1,$ the above argument is equivalent to:

$$\mathcal{H}_0 : z_j^k = \sum_{i \in \mathcal{I} \setminus S_{k-1}} a_i(1) + v(1)$$

$$\mathcal{H}_1 : z_j^k = \sum_{i \in \mathcal{I} \setminus S_{k-1}} a_i(1) + v(1) + \|P_{k-1}^\perp \mathbf{a}_j\|_2$$

- Lemma 1 gives *log-MAP ratio* as:

$$\Lambda_j^k = \log\left(\frac{P(z_j^k | x_j = 1)}{P(z_j^k | x_j = 1)} \frac{P(x_j = 1)}{P(x_j = 0)}\right)$$

$$= \frac{(z_j^k)^2}{2(\frac{K-k+1}{n} + \sigma_n^2)} - \frac{(z_j^k - \mu_k)^2}{2(\frac{K-k}{n} + \sigma_n^2)} + \log\left(\frac{P(x_j = 1)}{P(x_j = 0)}\right).$$

APPROXIMATION OF DISTRIBUTIONS

- Evaluation of log-MAP ratio requires an approximation of the distribution of z_j^k under $\mathcal{H}_1$

Lemma 1:

The cumulative distribution function of $\|P_{k-1}^\perp \mathbf{a}_j\|_2$ is given by

$$F_{\|P_{k-1}^\perp \mathbf{a}_j\|_2}(x) = \frac{\gamma(\frac{n-k+1}{2}, \frac{nx^2}{2})}{\Gamma(\frac{n-k+1}{2})}$$

- Proof's remarks:
  - Radial symmetry of Gaussian distribution
  - $\|P_{k-1}\mathbf{a}_j\|_2$ and $\|\mathbf{a}_j\|_2$ scaled chi distributions
  - $\|P_{k-1}^\perp \mathbf{a}_j\|_2$ difference of above distributions

Approximation:

- To make distribution of z_j^k under $\mathcal{H}_1$ analytically tractable, approximate $\|P_{k-1}^\perp \mathbf{a}_j\|_2$ by the expectation as n grows large, i.e., $\mu_k = \sqrt{\frac{n-k+1}{n}}$
- Assumption: Sparsity level, $K << n$

ALGORITHM

- Using accelerated OLS from [2] for proposed framework, MAP-AOLS

Input: $\mathbf{y}$, A, K, threshold (ϵ)

Output: Support $\mathcal{S}_K$, signal estimate $\hat{\mathbf{x}}_K$

Initialize: $i = 0$, $\mathbf{r}_0 = \mathbf{y}$, $\mathcal{S}_0 = \phi$, $\mathbf{t}_j^{(i)} = \mathbf{a}_j$, $\mathbf{q}_j = \frac{\mathbf{a}_j^\top \mathbf{y}}{\mathbf{a}_j^\top \mathbf{a}_j} \mathbf{a}_j, \forall j$

while $\|\mathbf{r}_i\|_2 > \epsilon$ and $i \leq K$ **do**

Select j_s corresponding to the largest Λ_j^k

$i \leftarrow i + 1$

$\mathcal{S}_i = \mathcal{S}_{i-1} \cup \{j_s\}$

$\mathbf{u}_i = \mathbf{q}_{j_s}$, $\mathbf{r}_i = \mathbf{r}_{i-1} - \mathbf{u}_i$

$\mathbf{t}_j^i = \mathbf{t}_j^{i-1} - \frac{\mathbf{t}_j^{i-1\top} \mathbf{u}_i}{\|\mathbf{u}_i\|_2^2} \mathbf{u}_i$

end while

$\hat{\mathbf{x}}_K = A_{\mathcal{S}_K}^\dagger \mathbf{y}$

- For computing the log-MAP ratio, note that $z_j^k = \|\mathbf{q}_j^{(k)}\|_2$

Connection to OLS:

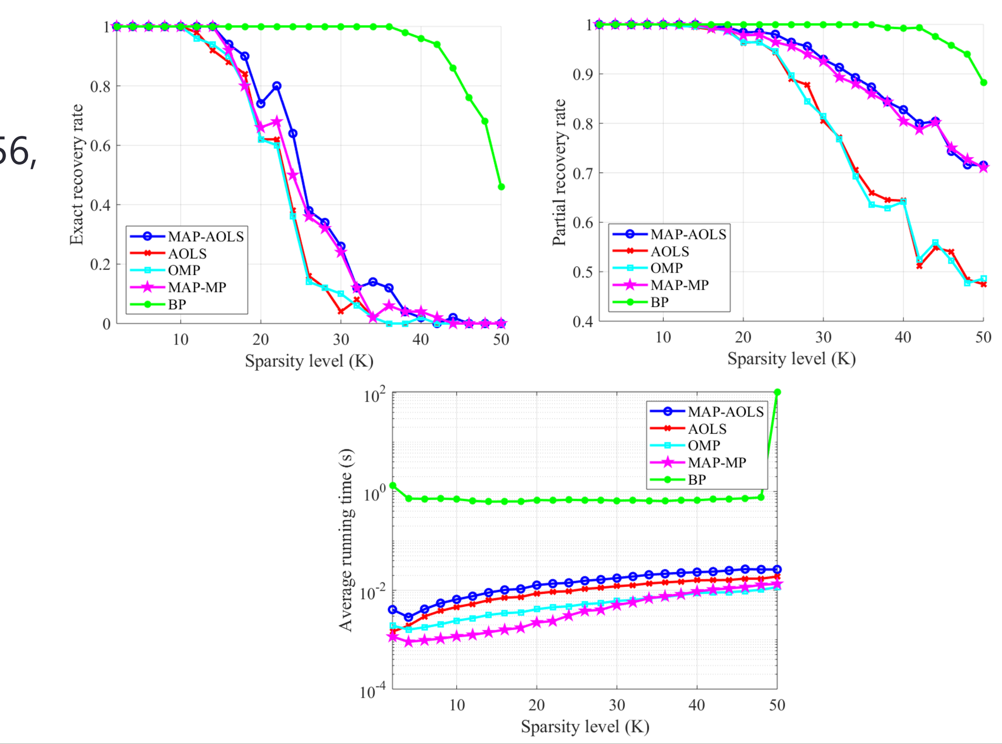
In the low-SNR regime, the variance under $\mathcal{H}_0$ and $\mathcal{H}_1$ will be approximately equal. Then, the log-MAP ratio will be

$$\Lambda_j^k = \frac{(z_j^k)^2}{2\sigma^2} - \frac{(z_j^k - \mu_k)^2}{2\sigma^2} = \frac{2z_j^k \mu_k - \mu_k^2}{2\sigma^2}.$$

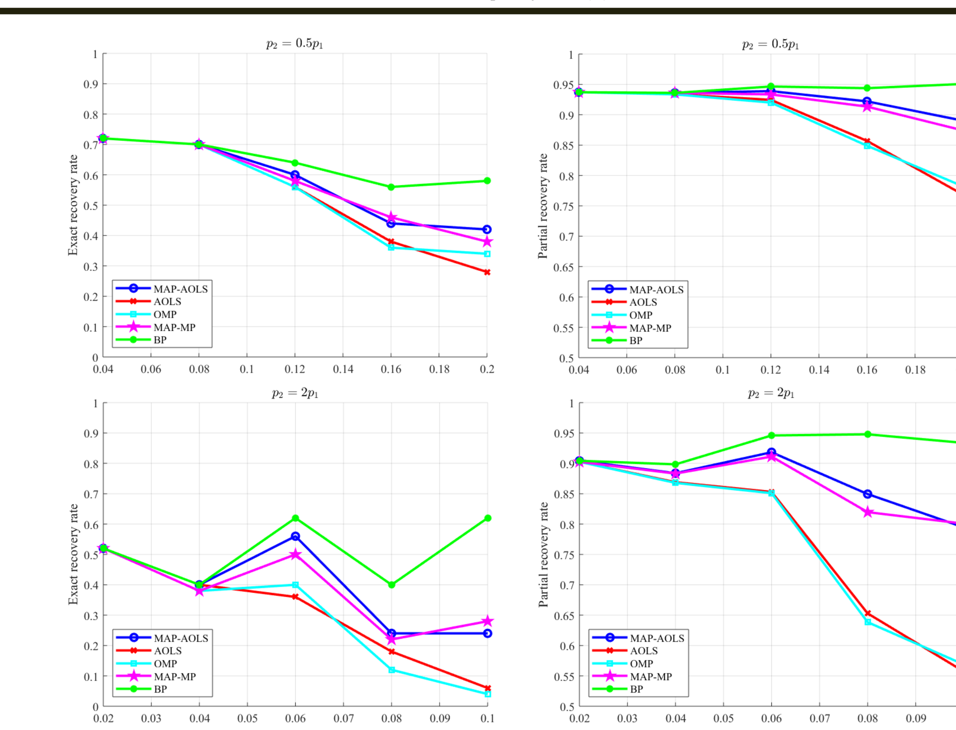
This is equivalent to the selection criterion of OLS, so OLS is optimal in the MAP sense in the low-SNR scenario.

RESULTS

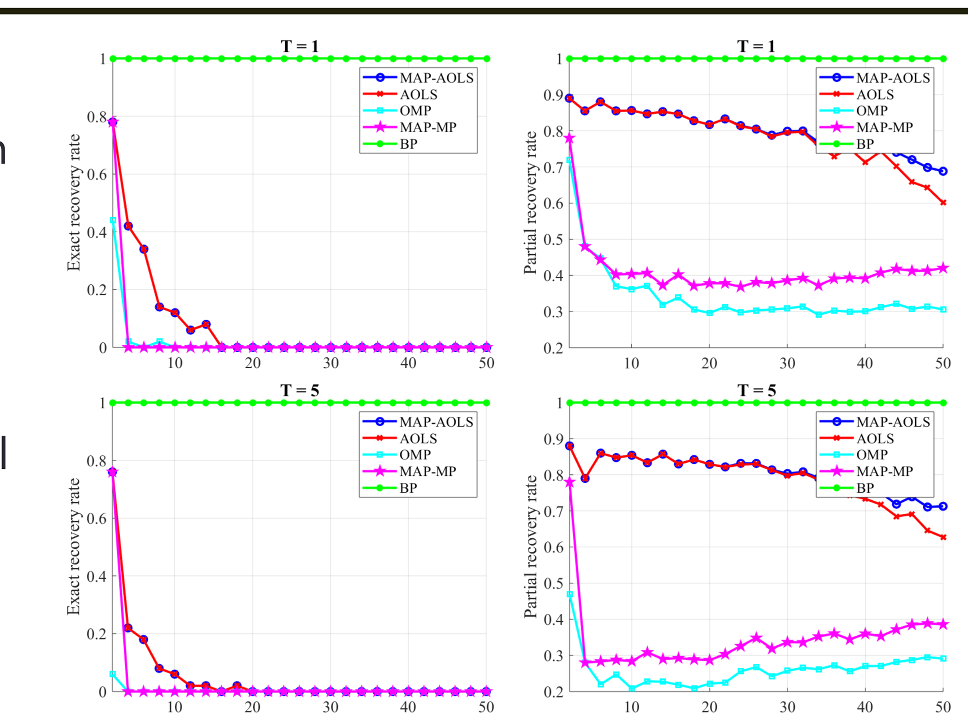
- Binary signals with m = 256, n = 128
- MAP-MP and MAP-AOLS provide *best PRR and ERR*
- MAP-AOLS only slightly slower than other greedy algorithms



- Support sets with p_1 from $|K_1|$ = m/4 and p_2 from $|K_2|$
- n = 256, m = 128
- K estimated as $p_1|K_1|+p_2|K_2|$
- MAP methods *exhibit best PRR and ERR*
- Non-monotonicity due to each missed index greater effect of missed index for small p_1



- Sensing matrix, A, drawn from hybrid dictionary
- OLS and MAP-AOLS *robust to correlation* in columns of A
- OMP-based schemes fail with hybrid dictionaries



CONCLUSION

- Our contributions:
  - Framework for support recovery that is optimal in MAP sense
  - Proposed framework empirically shown to show improved ERR and PRR
  - Potentially superior recovery with hybrid dictionaries
- Future work: Proof of optimality of proposed algorithm and speedup by selecting multiple indices per iteration

- [1] N. Lee, ""Map support detection for greedy sparse signal recovery algorithms in compressive sensing," , June 2016.
- [2] A. Hashemi et al., "Accelerated orthogonal least-squares for large-scale sparse reconstruction," Nov 2018.