Inverse Z transform

$Definition$: $F(z)$ **be a function of complex variable** z **such that**

$$F(z) = Z\{f(k)\}$$

then the sequence $f(k)$ **is called inverse Z transform of** $F(z)$**.**

Written as,

$$\mathrm{Z}^{-1}\{\mathrm{F(z)}\} = f(k)$$

Methods to find Inverse Z transform:

1. **Power series Method**
2. **Partial Fractions**
3. **Inversion Integral Method**

Method 1: Power series Method

This Method also known as Long division Method.

$F(z)$ is expressed as a power series in z

$$F(z) = a_0 z^{-0} + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}$$

$$= \sum_{n=0}^{\infty} a_n z^{-n}$$

Working Rule:

step1: observe the expression and try to obtain standard form

e.g. $\frac{1}{1-x}$ or $\frac{1}{1+x}$

step2: Consider ROC and expand in power series.

Note the coefficients

Solved Examples:

e.g. 1) Obtain $Z^{-1}\left\{\frac{1}{z-a}\right\}$ where $|z| < |a|$

Soln: $\frac{1}{z-a} = -\frac{1}{a}\frac{1}{1-\frac{z}{a}}$ given $|z| < |a| \rightarrow \left|\frac{z}{a}\right| < 1$

$$= -\frac{1}{a}\left(1 + \frac{z}{a} + \left(\frac{z}{a}\right)^2 + \cdots\right)$$

$$= -\frac{1}{a} - \frac{z}{a^2} - \frac{z^2}{a^3} - \cdots$$

$$= -\sum_{r=0}^{\infty}\frac{z^r}{a^{r+1}} \qquad (1)$$

By definition of Z transform: $F(z) = \sum_{k=0}^{\infty} f(k)z^{-k}$ $\qquad (2)$

Comparing (1) and (2) we get, $r = -k \; and \; r: 0 \; to \; \infty \rightarrow k: -\infty \; to \; 0$ i.e. $k \leq 0$

Coefficient of z^{-k} is $-a^{-r-1} = -a^{k-1} = f(k)$

Thus, $f(k) = -a^{k-1}, k \leq 0$

e.g. 2) Obtain $Z^{-1}\left\{\frac{1}{z-a}\right\}$ where $|z| > |a|$

Soln: $\frac{1}{z-a} = \frac{1}{z\left(1-\frac{a}{z}\right)}$ given $|z| > |a| \rightarrow \left|\frac{a}{z}\right| < 1$

$$= \frac{1}{z}\left(1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \cdots\right)$$

$$= \frac{1}{z} + \frac{a}{z^2} + \frac{a^2}{z^3} + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{a^r}{z^{r+1}} \qquad (1)$$

By definition of Z transform: $F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$ (2)

Comparing (1) and (2)we get, $r + 1 = k$ and $r: 0$ to $\infty \rightarrow k: 1$ to ∞ i.e. $k \geq 1$

Coefficient of z^{-k} is $a^r = a^{k-1} = f(k)$. Thus, $f(k) = a^{k-1}, k \geq 1$

Exercise: 1) Obtain $Z^{-1}\left\{\frac{z}{z-a}\right\}$ where $|z| > |a|$ Ans: $f(k) = a^k, k \geq 0$

2) Obtain $Z^{-1}\left\{\frac{z}{z-a}\right\}$ where $|z| < |a|$ Ans: $f(k) = -a^k, k < 0$

Method 2: Partial fraction Method

Recall from Partial fractions:

Form of the Rational Function	Form of the Partial Fraction
$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$
Where x^2+bx+c cannot be factorised further	

Working Rule:

step1: observe the expression and try to obtain standard form

e.g. $\frac{1}{1-x}$ or $\frac{1}{1+x}$

step2: Consider ROC and expand in power series.

Note the coefficients

From the formula sheet for inverse Z transform in last slide:

$$Z^{-1}\left\{\frac{z}{z-a}\right\} = a^k, k \geq 0 \quad \text{if ROC is } |z| > |a|$$

$$Z^{-1}\left\{\frac{z}{z-a}\right\} = -a^k, k < 0 \quad \text{if ROC is } |z| < |a|$$

In many cases the Z Transform can be written as a ratio of polynomials as shown below.

$$F(z) = \frac{A(z)}{B(z)}$$

If the denominator $B(z)$ can be factorized then $\frac{F(z)}{z}$ can be expanded into partial fractions as shown below.

$$\frac{F(z)}{z} = \frac{A}{z-a} + \frac{B}{z-b} + \cdots$$

Once Coefficients A, B, … are identified we can find inverse Z transform of $F(z) = A\frac{z}{z-a} + B\frac{z}{z-b} + \cdots$ using formula mentioned above.

Refer examples solved in next slide

Solved examples:

1) Obtain Inverse Z transform for $\frac{z}{(z-1)(z-2)}$, $|z| \geq 2$

Soln: Consider, $\frac{F(z)}{z} = \frac{1}{(z-1)(z-2)}$

Using Partial fractions $\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$

$$\Rightarrow \frac{1}{(z-1)(z-2)} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)}$$

$\Rightarrow$ $A(z-2)+B(z-1)=1$ which gives $A=-1 \; and \; B=1$

$Thus,$ $\frac{F(z)}{z} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$ $\Rightarrow$ $F(z) = \frac{-z}{(z-1)} + \frac{z}{(z-2)}$

Applying inverse Z transform we get,

$$\Rightarrow f(k) = Z^{-1}\left\{\frac{-z}{(z-1)} + \frac{z}{(z-2)}\right\} = -Z^{-1}\left\{\frac{z}{(z-1)}\right\} + Z^{-1}\left\{\frac{z}{(z-2)}\right\}$$

$$= -(1)^k + (2)^k, \qquad k \geq 0$$

Solved examples:

2) Obtain Inverse Z transform for $\frac{3z^2+2z}{z^2-3z+2}$, $1 < |z| < 2$

Soln: Consider, $\frac{F(z)}{z} = \frac{3z+2}{(z-1)(z-2)}$

Using Partial fractions $\frac{3z+2}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$

$$\Rightarrow \frac{3z+2}{(z-1)(z-2)} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)}$$

$$\Rightarrow A(z-2)+B(z-1) = 3z+2$$

Put $z = 1$ $\Rightarrow$ $A(1-2)+B(1-1) = 3(1)+2$ $\Rightarrow$ $-A = 5$

Put $z = 2$ $\Rightarrow$ $A(2-2)+B(2-1) = 3(2)+2$ $\Rightarrow$ $B = 8$

which gives $A = -5 \; and \; B = 8$

……continued in next slide

$$Thus,\quad \frac{F(z)}{z} = \frac{-5}{(z-1)} + \frac{8}{(z-2)}$$

$$F(z) = -5\frac{z}{(z-1)} + 8\frac{z}{(z-2)}$$

Applying inverse Z transform we get,

$$f(k) = Z^{-1}\left\{-5\frac{z}{(z-1)} + 8\frac{z}{(z-2)}\right\}$$

$$= -5Z^{-1}\left\{\frac{z}{(z-1)}\right\} + 8Z^{-1}\left\{\frac{z}{(z-2)}\right\}$$

$Now,\quad |z| > 1 \Rightarrow Z^{-1}\left\{\frac{z}{(z-1)}\right\} = (1)^k, k \geq 0$

and $|z| < 2 \Rightarrow Z^{-1}\left\{\frac{z}{(z-2)}\right\} = -(2)^k, k < 0$

Thus, $f(k) = \begin{matrix} -5(1)^k, k \geq 0 \\ -8(2)^k, k < 0 \end{matrix}$

3) Find the inverse Z – transform of $\dfrac{8z^2}{(2z-1)\ (4z-1)}$

$$\text{Let } F(z) = \frac{8z^2}{(2z-1)\ (4z-1)} = \frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}$$

$$\text{Then } \frac{F(z)}{z} = \frac{z}{(z-\frac{1}{2})(z-\frac{1}{4})}$$

Now,

$$\frac{z}{(z-\frac{1}{2})(z-\frac{1}{4})} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-\frac{1}{4}}$$

We get,

$$\frac{F(z)}{z} = \frac{2}{z-\frac{1}{2}} - \frac{1}{z-\frac{1}{4}}$$

Therefore,

$$F(z) = 2\frac{z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{4}}$$

Inverting, we get

$$f_n = Z^{-1}\{F(z)\} = 2\,Z^{-1}\left\{\frac{z}{z-\frac{1}{2}}\right\} - Z^{-1}\left\{\frac{z}{z-\frac{1}{4}}\right\}$$

i.e, $f_n = 2\,(1/2)^n - (1/4)^n,\quad n = 0, 1, 2, \ldots\ldots$

4) Obtain Inverse Z transform for $\frac{z}{z^2+7z+10}$

$$\frac{z}{z^2 + 7z + 10}$$

$$\text{Let } F(z) = \frac{z}{z^2 + 7z + 10}$$

$$\text{Then } \frac{F(z)}{z} = \frac{1}{z^2 + 7z + 10} = \frac{1}{(z+2)(z+5)}$$

$$\text{Now, consider } \frac{1}{(z+2)(z+5)} = \frac{A}{z+2} + \frac{B}{z+5}$$

$$= \frac{1}{3}\,\frac{1}{z+2} - \frac{1}{3}\,\frac{1}{z+5}$$

$$\text{Therefore, } F(z) = \frac{1}{3}\,\frac{z}{z+2} - \frac{1}{3}\,\frac{z}{z+5}$$

Inverting, we get

$$= \frac{1}{3}(-2)^n - \frac{1}{3}(-5)^n$$

Method3: Inversion Integral Method

Suppose $F(z) = \frac{A(z)}{B(z)}$ such that, $B(z) = (z-a)(z-b)^{m+1}$

i.e. $z = a$ is simple pole (linear factor) and $z = b$ pole of order m+1

Then, inverse Z transform $f(k) = \frac{1}{2\pi i}\int_C F(z)Z^{k-1}dz$ *where C: Region of Convergence*

$$= sum\ of\ the\ residues\ at\ each\ pole$$

$$= Res_a\{F(z)z^{k-1}\} + Res_b\{F(z)z^{k-1}\}$$

Working Rule:

Step1: equate denominator to zero and identify poles and their order.

Step2:calculate residues of $F(z)z^{k-1}$at each pole using formula given below:

(same as residue done in complex analysis Unit-III)

Residue at simple pole:

$$Res_a\{F(z)z^{k-1}\}=(z-a)F(z)z^{k-1} \quad at\ z = a$$

Residue at pole of order m+1:

$$Res_b\{F(z)z^{k-1}\} = \frac{1}{m!}\frac{d^m}{dz^m}[(z-b)^{m+1}F(z)z^{k-1}] \quad at\ z = b$$

Step3: write sum of all residues as final answer.

Solved Examples:

1) Obtain $Z^{-1}\left\{\frac{3z^2+2z}{z^2-3z+2}\right\}, |z| \leq 2$

Soln: $F(z) = \frac{3z^2+2z}{z^2-3z+2} \Longrightarrow F(z)z^{k-1} = \frac{3z^{k+1}+2z^k}{(z-1)(z-2)}$

$z = 1 \; and \; 2 \; both \; are \; simple \; poles$

$$Res_1\{F(z)z^{k-1}\} = \frac{3z^{k+1}+2z^k}{(z-2)} \quad at \; z = 1$$

$$= \frac{3(1)^{k+1}+2(1)^k}{(1-2)} = -5(1)^k$$

$$Res_2\{F(z)z^{k-1}\} = \frac{3z^{k+1}+2z^k}{(z-1)} \quad at \; z = 2$$

$$= \frac{3(2)^{k+1}+2(2)^k}{(2-1)} = \frac{6(2)^k+2(2)^k}{(2-1)} = 8(2)^k$$

$$Z^{-1}\left\{\frac{3z^2+2z}{z^2-3z+2}\right\} = -5(1)^k + 8(2)^k, k < 0 \; as \; ROC \; is \; |z| \leq 2$$

2) Find Inverse Z transform for $\frac{z(z+1)}{(z-2)^3}$

Soln: $F(z) = \frac{z^2+z}{(z-2)^3}$ $\quad F(z)z^{k-1} = \frac{z^{k+1}+z^k}{(z-2)^3}$

$z = 2$ is pole of order 3

$$Res_2\{F(z)z^{k-1}\} = \frac{1}{2!}\frac{d^2}{dz^2}(z^{k+1}+z^k) \quad at\ z = 2$$

$$= \frac{1}{2}\frac{d}{dz}\left((k+1)z^k + kz^{k-1}\right) \quad at\ z = 2$$

$$= \frac{1}{2}\left(k(k+1)z^{k-1} + k(k-1)z^{k-2}\right) \quad at\ z = 2$$

$$Z^{-1}\left\{\frac{z^2+z}{(z-2)^3}\right\} = \frac{1}{2}\left(k(k+1)(2)^{k-1} + k(k-1)(2)^{k-2}\right)$$

$$= \frac{1}{2}(2)^{k-2}(2k(k+1) + k(k-1))$$

$$= \frac{1}{8}(2)^k(2k^2 + 2k + k^2 - k) = \frac{1}{8}(3k^2 + k)(2)^k$$

Practice questions

1. Find $Z^{-1}\{\frac{z}{z-5}\}$ if $|Z|<5$

2. Find $Z^{-1}\{\frac{z}{(z-2)(z-3)}\}$ if $|Z|\geqslant 2$

3. Find $Z^{-1}\{\frac{z(z+1)}{z^2-2z+1}\}$ if $|Z|>1$

4. Using inversion integral method obtain $Z^{-1}\{\frac{z^3}{(z-\frac{1}{2})^2(z^2-4)}\}$

5. Using inversion integral method obtain $Z^{-1}\{\frac{z}{(4z-1)(3z-1)}\}$

Answers:

1. $-5^k, k<0$
2. $3^k-2^k, k>0$
3. $1+2k, k\geqslant 0$
4. $\frac{2}{3}2^k+\frac{2}{5}(-2)^k-\frac{1}{15}(\frac{1}{2})^k, k\geqslant 0$
5. $(\frac{1}{3})^k-(\frac{1}{4})^k, k\geqslant 0$

Formula sheet Inverse Z Transform

Note: $U(k)$ denotes the unit step function defined as $U(k) = 1, k \geq 0$ and 0 $Otherwise$

S. No.	F(z)	$\|z\|>\|a\| k\geq 0$	$\|z\|<\|a\| k<0$
1	$\frac{z}{z-a}$	$a^k U(k)$	$-a^k$
2	$\frac{z^2}{(z-a)^2}$	$(k+1)a^k U(k)$	$-(k+1)a^k$
3	$\frac{z^n}{(z-a)^n}$	$\frac{1}{(n-1)!}(k+1)(k+2)......(k+n-1)a^k U(k)$	$\frac{-1}{(n-1)!}(k+1)(k+2)......(k+n-1)a^k$
4	$\frac{1}{z-a}$	$a^{k-1}U(k-1)$	$-a^{k-1}U(-k)$
5	$\frac{1}{(z-a)^2}$	$(k-1)a^{k-2}U(k-2)$	$-(k-1)a^{k-2}U(-k+1)$
6	$\frac{1}{(z-a)^3}$	$\frac{1}{2}(k-2)(k-1)a^{k-3}U(k-3)$	$\frac{-1}{2}(k-2)(k-1)a^{k-3}U(-k+2)$
7	$\frac{z}{z-1}$	U(k)	-----
8	$\frac{[z(z-\cos\alpha)]}{(z^2-2z\cos\alpha+1)};\|z\|>1$	$\cos\alpha k$	----
9	$\frac{[z\sin\alpha]}{(z^2-2z\cos\alpha+1)};\|z\|>1$	$\sin\alpha k$	----