Binary Relations

Any set of ordered pairs defines a *binary relation.*

We express a particular ordered pair, $(x, y) \in R$, where R is a binary relation, as *xRy*.

Properties

Properties of a binary relation R on a set X:

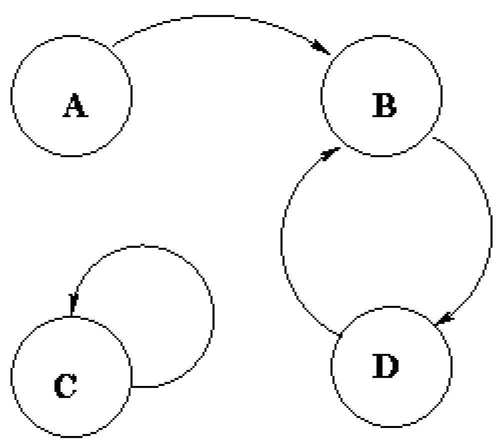
a. *reflexive*: if for every $x \in X$, xRx holds, i.e. $(x, x) \in R$.

b. *symmetric*: if for every x and y in X, whenever xRy holds, then yRx holds.

c. *transitive*: if for every x, y, and z in X, whenever xRy holds and yRz holds, then xRz holds.

Graphical Representation

Suppose that X = {A, B, C, D} and that R = {(A, B), (B, D), (C, C), (D, B)}. We can represent the binary relation R as a set, as a graph or as a Boolean matrix. Graphically, we have



Boolean Matrix Representation

As a Boolean matrix we have

	A	B	C	D
A	0	1	0	0
B	0	0	0	1
C	0	0	1	0
D	0	1	0	0

What would a Boolean matrix look like if it represented a reflexive binary relation, a symmetric binary relation, a transitive binary relation, any combination of the above?

Reflexive Relation

	A	B	C	D
A	-	1	-	-
B	-	-	-	1
C	-	-	1	-
D	-	1	-	-

Symmetric Relation

	A	B	C	D
A	-	1	-	-
B	-	-	-	1
C	-	-	1	-
D	-	1	-	-

Transitive Relation

	A	B	C	D
A	-	1	-	-
B	-	-	-	1
C	-	-	1	-
D	-	1	-	-

Click here for Answers

Composition of Relations:

Let R be a relation from X to Y, and S be a relation from Y to Z. Then a relation written as **R o S** is called a *composite relation* of R and S where

$$R \circ S = \{ (x, z) \mid x \in X \wedge z \in Z \wedge (\exists y) (y \in Y \wedge (x, y) \in R \wedge (y, z) \in S) \}$$

We can also write the composition as

$$R \circ S = \{ (x, z) \mid x \in X \wedge z \in Z \wedge (\exists y) (y \in Y \wedge xRy \wedge ySz) \}$$

Note: Relational composition can be realized as matrix multiplication. For example, let M_R and M_S represent the binary relations R and S, respectively. Then R o S can be computed via $M_R\, M_S$.

e.g.

The Parent Relation

$$x\ P\ y$$

means that x is the parent of y. From this binary relation we can compute: child, grandparent, sibling

child: $x\ C\ y = y\ P\ x$

We say that x is the child of y if y is the parent of x.

grandparent: $x\ GP\ y = x\ P\ z \wedge z\ P\ y$ or $GP = P \circ P$

We say that x is the grandparent of y if x is the parent of z and z is the parent of y.

sibling: $x\ S\ y = z\ P\ x \wedge z\ P\ y$

$x\ S\ y = x\ P^{-1}\ z \wedge z\ P\ y$ or $S = P^{-1} \circ P$

We say that x is the sibling of y if z is the parent of x and z is the parent of y.

Define the following relations: aunt/uncle, cousin

Suppose that

P = {(a, b), (a, c), (b, d), (b, e), (c, f), (f, g)}

Compute the children, grandparents, siblings, aunt/uncle, and cousins for this parent relation.

Closure

Reflexive Closure:

for every $x \in X$, then $(x, x) \in R$

or

for every $x \in X$, then xRx holds

Symmetric Closure:

for every $(x, y) \in R$, then $(y, x) \in R$

or

for every xRy , then yRx

Transitive Closure:

for every (x, y) and $(y, z) \in R$, then $(x, z) \in R$

or

for every xRy and yRz, then xRz

$$R^{\wedge} = R \cup R^2 \cup R^3 \cup \ldots\ldots$$

Let M represent the binary relation R, $R^{\wedge}$ represents the transitive closure of R, and $M^{\wedge}$ represent the transitive closure. Then representing the transitive closure via Boolean matrices, we have

$$M^{\wedge} = M + M^2 + M^3 + \ldots$$

e.g.

Suppose R = {(1,2), (1,5), (2,3), (3,5), (4,3), (5,4)}. Compute the reflexive closure, symmetric closure, and transitive closure.

Reflexive Closure

	1	2	3	4	5
1	-	1	-	-	1
2	-	-	1	-	-
3	-	-	-	-	1
4	-	-	1	-	-
5	-	-	-	1	-

Symmetric Closure

	1	2	3	4	5
1	-	1	-	-	1
2	-	-	1	-	-
3	-	-	-	-	1
4	-	-	1	-	-
5	-	-	-	1	-

Transitive Closure

	1	2	3	4	5
1	-	1	-	-	1
2	-	-	1	-	-
3	-	-	-	-	1
4	-	-	1	-	-
5	-	-	-	1	-

1	2	3	4	5
-	-	-	-	-
-	-	-	-	-
-	-	-	-	-
-	-	-	-	-
-	-	-	-	-

Click here for answers

Warshall's Algorithm:

	1	2	3	4	5
1	-	1	-	-	1
2	-	-	1	-	-
3	-	-	-	-	1
4	-	-	1	-	-
5	-	-	-	1	-

Click here for the answer

ANSWERS:

Reflexive Binary Relation

	A	B	C	D
A	1	1	-	-
B	-	1	-	1
C	-	-	1	-
D	-	1	-	1

Symmetric Binary Relation

	A	B	C	D
A	-	1	-	-
B	1	-	-	1
C	-	-	1	-
D	-	1	-	-

Reflexive, Symmetric Binary Relation

	A	B	C	D
A	1	1	-	-
B	1	1	-	1
C	-	-	1	-
D	-	1	-	1

Return

Reflexive Closure

	1	2	3	4	5
1	1	1	-	-	1
2	-	1	1	-	-
3	-	-	1	-	1
4	-	-	1	1	-
5	-	-	-	1	1

Symmetric Closure

	1	2	3	4	5
1	-	1	-	-	1
2	1	-	1	-	-
3	-	1	-	1	1
4	-	-	1	-	1
5	1	-	1	1	-

Transitive Closure

	1	2	3	4	5	1	2	3	4	5
1	-	1	-	-	1	-	-	1	1	-
2	-	-	1	-	-	-	-	-	-	1
3	-	-	-	-	1	-	-	-	1	-
4	-	-	1	-	-	-	-	-	-	1
5	-	-	-	1	-	-	-	1	-	-

Warshall's Algorithm

	1	2	3	4	5
1	-	1	1	1	1
2	-	-	1	1	1

3	-	-	1	1	1
4	-	-	1	1	1
5	-	-	1	1	1