Math 2331 – Linear Algebra
3.2 Properties of Determinants

Jiwen He

Department of Mathematics, University of Houston

jiwenhe@math.uh.edu
math.uh.edu/∼jiwenhe/math2331

3.2 Properties of Determinants

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Properties of Determinants

Theorem (Elementary Row Operations)

Let A be a square matrix.

a. *If a multiple of one row of A is added to another row of A to produce a matrix B, then* det $A =$ det B.

b. *If two rows of A are interchanged to produce B, then* det $B = -$ det A.

c. *If one row of A is multiplied by k to produce B, then* det $B = k \cdot$ det A.

Theorem still holds if the word *row* is replaced

with _____.

Properties of Determinants: Example

Example

Compute
$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix} = 5 \begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 10 \\ 2 & 7 & 11 \end{vmatrix} = 5 \begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 2 & 7 & 11 \end{vmatrix}$$

$$= 5 \begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -5 \begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{vmatrix} = \underline{\hspace{2cm}} = \underline{\hspace{1cm}}.$$

Properties of Determinants: Example

Theorem (c) indicates that $\begin{vmatrix} * & * & * \\ -2k & 5k & 4k \\ * & * & * \end{vmatrix} = k \begin{vmatrix} * & * & * \\ -2 & 5 & 4 \\ * & * & * \end{vmatrix}$.

Example

Compute $\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -8 & -11 \end{vmatrix}$$

$$= 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -8 & -11 \end{vmatrix} = 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix} = -40$$

Properties of Determinants: Example

Example

Compute $\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix}$ by row reduction and cofac. expansion.

Solution $\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix} = -2 \begin{vmatrix} 2 & 3 & 1 \\ 4 & 7 & 3 \\ 1 & 2 & 4 \end{vmatrix} = -2 \begin{vmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}$

$$= 2 \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 0 & -6 \end{vmatrix} = -2\,(1)\,(-1)\,(-6) = -12.$$

Properties of Determinants: Triangulation

Suppose A has been reduced to

$$U = \begin{bmatrix} \blacksquare & * & * & \cdots & * \\ 0 & \blacksquare & * & \cdots & * \\ 0 & 0 & \blacksquare & \cdots & * \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

by row replacements and row interchanges, then

$$\det A = \begin{cases} (-1)^r \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

Determinant of the Transpose

> **Theorem**
>
> *A square matrix is invertible if and only if $\det A \neq 0$.*

> **Theorem**
>
> *If A is an $n \times n$ matrix, then $\det A^T = \det A$.*

Partial proof (2×2 case)

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \qquad \text{and}$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$$

$$\Rightarrow \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Determinant of the Transpose (cont.)

$(3 \times 3$ case$)$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\det \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = a \begin{vmatrix} e & h \\ f & i \end{vmatrix} - b \begin{vmatrix} d & g \\ f & i \end{vmatrix} + c \begin{vmatrix} d & g \\ e & h \end{vmatrix}$$

$$\Rightarrow \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \det \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}.$$

Multiplicative Property

Theorem (Multiplicative Property)

For $n \times n$ matrices A and B, $\det(AB) = (\det A)(\det B)$.

Example

Compute $\det A^3$ if $\det A = 5$.

Solution: $\det A^3 = \det(AAA) = (\det A)(\det A)(\det A)$

$$= \underline{\hspace{2cm}} = \underline{\hspace{1.5cm}}.$$

Example

For $n \times n$ matrices A and B, show that A is singular if $\det B \neq 0$ and $\det AB = 0$.

Solution: Since $(\det A)(\det B) = \det AB = 0$ and $\det B \neq 0$, then $\det A = 0$.   Therefore A is singular.