AWS 2025 Lecture Notes

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Preface

There will be some gaps in explanation, either due to the lecturer's admission or my own lack of understanding. In particular, many "proofs" are sketches of proofs. Gaps due to my own misunderstanding will be indicated by three red question marks: ???. More generally, my own questions about the material will also be in red. Things like "**Question**" will be questions posed by the lecturer. Almost always, "Claim" will mean conjecture. Feel free to reach out to me with explanations.

Also, note that the lectures appear here in an order that does not match the order in which they were given. Unfortunately, some lectures refer to lectures (by another lecturer) that has already been given. See the schedule page on the AWS website for the order of lectures.

Contents

1 Jessica Fintzen: Representations of p-adic Groups

1.1 Lecture 1: Basic Background

We fix prime p, non-archimedean local field F, valuation ν_F, integer ring $\mathcal{O}$, uniformizer ϖ, and $q = |\mathcal{O}/\varpi\mathcal{O}| = p^r$.

Definition 1.1. A ***p*-adic group** is the F-points $G = \underline{G}(F)$ of a connected reductive group $\underline{G}$ defined over F. Common examples are $GL_n(F)$, $SL_n(F)$, $SO_n(F)$, $Sp_{2n}(F) = \{A \in SL_{2n}(F) \mid A^t JA = J\}$ where J is the $2n$ by $2n$ symplectic matrix, and products of these.

From now on, G will denote a p-adic group. The topology on F gives a topology on G. Here are some properties of this topology:

1) G has a basis of open neighborhoods of 1 consisting of compact open subgroups. For instance, if $G = GL_n(\mathbb{Q}_p)$, we have $GL_n(\mathbb{Z}_p) \supset 1 + pM_n(\mathbb{Z}_p) \supset 1 + p^2 M_n(\mathbb{Z}_p) \supset \cdots$. This is related to the Moy-Prasad filtration. Each of these groups is normal in the next.

2) G is totally disconnected; the only connected subspaces are singletons. One can think of this as being fractal-like.

Let C denote an algebraically closed field, e.g. $\mathbb{C}$, $\overline{\mathbb{F}_\ell}$, $\overline{\mathbb{F}_p}$, where ℓ is a prime different from p. In these lectures, C will not be characteristic p.

Definition 1.2. A **smooth representation** of G is a representation $(\pi, V) = V$ (I will usually be lazy and write V even though the lecturer writes (π, V)) such that for all $v \in V$, there is a compact open subgroup $K \subset G$ with $Kv = v$.

Example 1.1. (1) $V = C$, $\pi : G \to 1 \in C^\times$. This is the **trivial representation**, also denoted triv.

(2) $G = SL_2(\mathbb{Q}_p)$, B the *(Borel)* subgroup consisting of upper triangular matrices, V the space of locally constant functions $f : B\backslash G \to C$, i.e. locally constant functions $\mathbb{P}^1(\mathbb{Q}_p) \to C$, and $\pi(g)(f)(x) = f(xg)$. Here $B\backslash G$ means the quotient space; we can also take locally constant functions $f : G \to C$ that are left B-invariant, i.e. $f(bx) = f(x)$ for all $b \in B$.

Definition 1.3. A **parabolic subgroup** P of $GL_n(F)$ (or $SL_n(F), \dots$) is a subgroup conjugate to a subgroup consisting of matrices in block upper triangular form.

Remark. There is a more general definition of parabolic subgroup, but it is not so helpful to think about.

These subgroups P have a *Levi decomposition* into a semidirect product of subgroups that are (conjugates of) block diagonals and block upper unitriangular matrices. The first subgroup is the *Levi subgroup* of P, often denoted M; it is a p-adic group. The second subgroup is called the *unipotent radical* of P, often denoted N. The semidirect product is $P = M \ltimes N$.

Definition 1.4. Let $P = M \ltimes N$ be a parabolic subgroup of G, and let (σ, V_σ) be a smooth representation of M. The **parabolic induction** is the representation $(\mathrm{Ind}_P^G \sigma, \mathrm{Ind}_P^G V_\sigma)$ defined as follows:

- $\mathrm{Ind}_P^G V_\sigma$ consists of functions $f : G \to V_\sigma$ such that $f(mng) = \sigma(m)(f(g))$ for any $m \in M, n \in N, g \in G$, and such that there exists an open compact subgroup $K_f \subset G$ with $f(gk) = f(g)$ for all $k \in K_f$.
- $(\mathrm{Ind}_P^G \sigma)(g)(f)(x) = f(xg)$.

It is a smooth representation of G.

Remark. The representation (2) in Example 1.1 is $\mathrm{Ind}_B^G \mathrm{triv}$.

Remark. If $C = \mathbb{C}$, there is also a *normalized* parabolic induction, denoted with n-Ind_P^G, where we have the condition $f(mng) = \delta_P^{1/2}(m)\sigma(m)(f(g))$ with δ_P the *modulus character* $\delta_P(m) = |\det(\mathrm{Ad}_{\mathrm{Lie}(N)}(m))|_p$. This normalized induction preserves unitarity. The other lecturers may use this implicitly (i.e. without distinguishing it from the induction we use here).

From now on, assume $C = \mathbb{C}$.

Definition 1.5. A **supercuspidal representation** V of G is a (smooth) irreducible representation of G such that there is no injection $V \hookrightarrow \mathrm{Ind}_P^G V_\sigma$ for $P = M \ltimes N$ a proper parabolic subgroup and V_σ a (smooth) irreducible representation of M.

Fact: If V is an irreducible representation, then there is a parabolic $P = M \ltimes N$ and a supercuspidal representation V_σ of M such that $V \hookrightarrow \mathrm{Ind}_P^G V_\sigma$.

The category $\mathrm{Rep}(G)$ of smooth representations of G decomposes into a product of **Bernstein blocks** $\mathrm{Rep}(G)_{[M,\sigma]}$ over *intertial* equivalence classes of pairs (M, σ). Here M is a Levi subgroup of some parabolic subgroup, σ is a supercuspidal representation of M, and two pairs (M, σ), (M', σ') are equivalent if $M' = gMg^{-1}$ and $\sigma' = \sigma(g^{-1} - g) \otimes \chi$ for some $g \in G$ and some *unramified* character $\chi : M' \to \mathbb{C}^\times$, which is a character that is trivial on all compact subgroups of G. The block $\mathrm{Rep}(G)_{[M,\sigma]}$ consists of the smooth representations whose irreducible subquotients embed into $\mathrm{Ind}_{P'}^G \sigma'$ for P' a parabolic with Levi M', and (M', σ') is equivalent to (M, σ).

Example 1.2. Let $G = SL_2(F)$.

(a) If $M = G$, then $\mathrm{Rep}(G)_{[M,\sigma]}$ consists of direct sums of copies of σ.

(b) If $P = B$ is the upper triangular matrices, then $M = T$ is the diagonal matrices, and $\mathrm{Rep}(G)_{[T,\mathrm{triv}]}$ is called the **principal block**. The representation $\mathrm{Ind}_B^G \mathrm{triv}$ contains triv, and the quotient St is the **Steinberg representation**.

Remark. Ind_P^G is right adjoint to the *Jacquet functor* $J_M : V \mapsto V_N = V/(v - nv)$.

1.2 Lecture 2: Moy-Prasad Filtration and Bruhat-Tits Theory

We will suppose G is *split* (we won't define it, but GL_n, SL_n, Sp_{2n} are split).

Definition 1.6. A **maximal split torus** of $GL_n(F)$ (or $SL_n(F), \dots$) is a subgroup conjugate to the diagonal matrices.

Definition 1.7. A **BT triple** is a triple $(T, \{x_\alpha\}_{\alpha \in \Phi(G,T)}, x_{BT})$ where T is a maximal split torus in G, $\Phi(G,T)$ is the set of roots for T, x_α is a non-zero element in the root space $\mathrm{Lie}(G)_\alpha$, the $\{x_\alpha\}$ is a Chevalley system (roughly meaning it plays nicely with Weyl group action), and x_{BT} is any element of $X_*(T) \otimes \mathbb{R}$.

Example 1.3. For $G = GL_n(F)$, a Chevalley system is given by the Kronecker delta matrices δ_{ij} as i, j range over distinct numbers in $\{1, \dots, n\}$.

We now fix a BT triple $x = (T, \{x_\alpha\}, x_{BT})$.

Let $T_0 = \{t \in T \mid \nu_F(\chi(t)) = 0, \forall \chi \in X^*(T)\}$. This is a maximal compact subgroup of T. Then for any $r \in \mathbb{R}_{>0}$, we let $T_r = \{t \in T_0 \mid \nu_F(\chi(t) - 1) \geq r, \forall \chi \in X^*(T)\}$.

Example 1.4. If $G = SL_2(\mathbb{Q}_p)$, then T_r consists of diagonal matrices $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ such that $t \in 1 + p^{\lceil r \rceil}\mathbb{Z}_p$.

For a root α, we have a root group U_α given by exponentiating $\mathrm{Lie}(G)_\alpha$.

Example 1.5. For $G = GL_n(F)$ and α the root given by $i \neq j$, then U_α consists of matrices of the form $1 + x\delta_{ij}$ for $x \in F$.

There is an isomorphism $F \to U_\alpha$ whose derivative sends 1 to x_α. We abuse notation by calling this isomorphism x_α. For $r \geq 0$, we let $U_{\alpha,x,r} = x_\alpha(\varpi^{\lceil r - \alpha(x_{BT}) \rceil}\mathcal{O}_F)$.

Example 1.6. Let $G = SL_2(\mathbb{Q}_p)$.

(a) Let $x_1 = (T = \left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}, 0)$. There are two roots $\pm\alpha \in \Phi(G,T)$, where $\alpha : \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^2$, and $-\alpha$ sends that to t^{-2}. The root group U_α is the upper unitriangular matrices, while $U_{-\alpha}$ is the lower unitriangular matrices. The filtration pieces are $U_{\alpha,x_1,r} = \left\{\begin{pmatrix} 1 & p^{\lceil r \rceil}\mathbb{Z}_p \\ 0 & 1 \end{pmatrix}\right\}$ and similarly for $-\alpha$.

(b) Let x_2 be the BT triple with the same T and Chevalley system as above, but with $x_{BT} = \frac{1}{4}\check{\alpha}$, where $\check{\alpha}$ maps t to $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. We have $\alpha(x_{BT}) = \frac{1}{2}$. Then $U_{\alpha,x_2,r} = \left\{\begin{pmatrix} 1 & p^{\lceil r - 1/2 \rceil}\mathbb{Z}_p \\ 0 & 1 \end{pmatrix}\right\}$, and $U_{-\alpha,x_2,r}$ is similar but with $r + \frac{1}{2}$. Thus, already for $r = 0$, there is an antisymmetry appearing.

Now for $r \geq 0$, we let $G_{x,r}$ be the subgroup generated by T_r and $U_{\alpha,x,r}$ for all roots α. This is the **Moy-Prasad filtration** for G.

Example 1.7. Let $G = SL_2(\mathbb{Q}_p)$. We match the BT triples from the previous example.

(a) $G_{x_1,0} = SL_2(\mathbb{Z}_p)$, and for $r > 0$, $G_{x_1,r}$ consists of the determinant one matrices of the form $\begin{pmatrix} 1+p^{\lceil r \rceil}\mathbb{Z}_p & p^{\lceil r \rceil}\mathbb{Z}_p \\ p^{\lceil r \rceil}\mathbb{Z}_p & 1+p^{\lceil r \rceil}\mathbb{Z}_p \end{pmatrix}$.

(b) $G_{x_2,0}$ consists of determinant one matrices on the form $\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$. For $r > 0$, $G_{x_2,r}$ consists of determinant one matrices of the form $\begin{pmatrix} 1+p^{\lceil r \rceil}\mathbb{Z}_p & p^{\lceil r-1/2 \rceil}\mathbb{Z}_p \\ p^{\lceil r+1/2 \rceil}\mathbb{Z}_p & 1+p^{\lceil r \rceil}\mathbb{Z}_p \end{pmatrix}$.

Remark. $G_{x,0}$ is an example of a *parahoric* subgroup.

Let $G_{x,r+} = \bigcup_{s>r} G_{x,s}$.

Some properties of the Moy-Prasad filtration:

(i) $G_{x,r}$ is normal in $G_{x,0}$.

(ii) $G_{x,0}/G_{x,0+}$ is isomorphic to the $\mathbb{F}_q$ points of a reductive group. For instance, if $G = SL_2(\mathbb{Q}_p)$, then $G_{x_1,0}/G_{x_1,0+} \cong SL_2(\mathbb{F}_p)$ and $G_{x_2,0}/G_{x_2,0+} = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbb{F}_p^\times \right\}$

(iii) $[G_{x,r}, G_{x,s}] \subset G_{x,r+s}$. Thus $G_{x,r}/G_{x,r+}$ is abelian for $r > 0$.

Definition 1.8 (Non-traditional Definition)**.** The **(reduced) Bruhat-Tits building** $\mathcal{B}(\underline{G}, F)$ is, as a set, equivalence classes of BT triples, where x_1 is equivalent to x_2 if $G_{x_1,r} = G_{x_2,r}$ for all $r \geq 0$. Thus we can safely write $G_{x,r}$ for $x \in \mathcal{B}(\underline{G}, F)$.

Properties of $\mathcal{B}(\underline{G}, F)$:

(i) G acts on $\mathcal{B}(\underline{G}, F)$ so that $G_{gx,r} = gG_{x,r}g^{-1}$.

(ii) $\mathcal{B}(\underline{G}, F)$ can be equipped with a polysimplicial structure such that x, y are in the interior of the same complex iff $G_{x,0} = G_{y,0}$.

(iii) There are *apartments* $\mathcal{A}(T, F)$ that are in bijection with T; BT triples in the same apartment have the same T. In fact, you can fix the Chevalley system as well. $\mathcal{A}(T, F)$ is an affine space over $X_*(T) \otimes \mathbb{R}/(X_*(Z(G)) \otimes \mathbb{R})$.

Example 1.8. $\mathcal{B}(SL_2, \mathbb{Q}_3)$

Definition 1.9. Let (π, V) be a smooth irreducible representation of G. The **depth** of V is the smallest $r \geq 0$ such that $V^{G_{x,r+}} \neq 0$ for some $x \in \mathcal{B}(\underline{G}, T)$.

We now briefly discuss the case where $\underline{G}$ is not split. Suppose $\underline{G} \times_F E$ is split for E/F tamely ramified. Then we define $G_{x,r} = \underline{G}(E)_{x,r}^{\mathrm{Gal}(E/F)}$ and $\mathcal{B}(\underline{G}, F) = \mathcal{B}(\underline{G}, E)^{\mathrm{Gal}(E/F)}$. (The lecturer wrote $r > 0$ here; is that important or just a typo?)

1.3 Lecture 3: Construction of Supercuspidal Representations

We begin by recalling Theorem 3.3 from Kaletha's Lecture 2. We note that compact induction was not defined:

Definition 1.10. (Compact Induction) If K is a compact subgroup of G with representation (ρ, V_ρ), then $\text{c-Ind}_K^G(V_\rho)$ is the G-representation consisting of functions $f : G \to V_\rho$ with $f(kg) = \rho(k)f(g)$ for all $k \in K$ and whose support is compactly supported mod K. G acts by right translations.

Assume from now on that G splits over a tamely ramified extension.

Definition 1.11. We call $\underline{G}' \subset \underline{G}$ a **(tame) twisted Levi subgroup** if there is a (tamely ramified) field extension E/F such that $\underline{G}' \times_F E$ is a Levi subgroup of $\underline{G}$.

Example 1.9. Let $G = SL_2(\mathbb{Q}_p)$, with $p \neq 2$. Let $T_{an} = G'$ be the subgroup consisting of matrices of the form $\begin{pmatrix} a & b \\ pb & a \end{pmatrix}$. Let $E = \mathbb{Q}_p(\sqrt{p})$. Then $\underline{G}'(E)$ consists of matrices of the same form (but now in $SL_2(E)$), and is conjugate to the subgroup of diagonal matrices.

Remark. T_{an} means anisotropic torus; it is compact and becomes a torus after base change.

Definition 1.12. Assume p does not divide the order of the Weyl group. Let G' be a twisted Levi subgroup. Let $T' \subset G'$ be a tame maximal torus, i.e. $\underline{T}' \times_F E$ is a split maximal torus for some tamely ramified extension E/F. A character ϕ of G' is called **(G, G')-generic of depth r** if it is of depth r and $\phi(\mathrm{Nm}_{E/F}(\check{\alpha}((E^\times)_r))) \neq 1$ for all roots $\alpha \in \Phi(\underline{G}_E, \underline{T}'_E) - \Phi(\underline{G}'_E, \underline{T}'_E)$.

Idea: this is the dual notion to regular semisimple.

Example 1.10. Let $G = GL_2(\mathbb{Q}_p)$. Let $G' = T'$ be the usual diagonal matrices. We have $(F^\times)_r = 1 + \varpi^r \mathcal{O}_F$ for $r > 0$. The allowed roots are $\pm\alpha$ where $\alpha : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto ab^{-1}$, so $\check{\alpha}(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.

Example 1.11. Let $G = GL_2(\mathbb{Q}_p)$ for $p > 2$. Let $G' = T'$ be the torus of diagonal matrices. Let $\psi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be non-trivial on $1 + p\mathbb{Z}_p$, but trivial on $1+p^2\mathbb{Z}_p$. Thus, its depth is 1. The following three characters are (G, G')-generic of depth 1:

- $\phi_1 : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \psi(a)$.
- $\phi_2 : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \psi(b)$.

- $\phi_3 : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \psi(ab^{-1})$.

On the other hand, the following two characters are not (G, G')-generic:

- $\phi_4 : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \psi(ab)$.
- $\phi_5 : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \psi(ab^{1-p})$.

We now discuss (a variant of) "Yu's construction" of supercuspidal representations. This will appear in several lectures. The input is often referred to as "Yu data":

(i) $G^0 \subsetneq G^1 \subsetneq \cdots \subseteq G^n = G$ tame twisted Levi subgroups such that $Z(G^0)/Z(G)$ is compact. (e.g. $G^0 = T_{an} \subset SL_2(\mathbb{Q}_p) = G^1 = G$)

(ii) A vertex $x \in \mathcal{B}(\underline{G}^0, F)$.

(iii) $0 < r_0 < \cdots < r_{n-1}$.

(iv) ϕ_i, for $0 \leq i \leq n_1$, a (G^{i+1}, G^i)-generic depth r_i character of G^i.

(v) ρ^0 an irreducible representation of G^0_x such that $\rho^0|_{G^0_{x,0}}$ is trivial and $\rho^0|_{G^0_{x,0}}$ is a cuspidal representation of $G^0_{x,0}/G^0_{x,0+}$.

Construction: $\tilde{K} = G^0_x G^1_{x,r_0/2} \cdots G^n_{x,r_{n-1}/2}$ and a representation $\tilde{\rho} = \rho^0 \otimes \kappa$ of $\tilde{K}$, where $\rho^0 : \tilde{K} \to \tilde{K}/G^0_{x,0+}G^1_{x,r_0/2} \cdots G^n_{x,r_{n-1}/2} \cong G^0_x/G^0_{x,0+} \to \mathrm{End}(V_{\rho^0})$ and $\kappa = \kappa^{nt} \otimes \varepsilon_{FKS}$ where κ^{nt} is built from the ϕ_i via the theory of Heisenberg-Weil representations and $\varepsilon_{FKS} : \tilde{K} \to G^0_x/G^0_{x,0+} \to \{\pm 1\}$ is constructed by Fintzen-Kaletha-Spice. The depth of this representation is r_{n-1}.

Theorem 1.1 (Yu 2001, Fintzen 2021, Fintzen-Schwein 2025)**.** *If $q \neq 2$, then* c-$\mathrm{Ind}_{\tilde{K}}^G \tilde{\rho}$ *is irreducible supercuspidal.*

Theorem 1.2 (Kim 2007, Fintzen 2021)**.** *If p does not divide the order of the Weyl group, then all supercuspidal representations arise from this construction.*

We now sketch the construction for $n = 1, p \neq 2$. Our input consists of G^0, x, $r_0 = r$, $\phi_0 = \phi$, and ρ^0. The group $\tilde{K}$ is $G^0_x G_{x,r/2}$.

1. Extend $\phi|_{G^0_x}$ to a character $\hat{\phi}$ of $G^0_x G_{x,r/2+}$ by "sending root groups of G outside G^0 to 1".
2. We "extend" $\hat{\phi}$ to $G_{x,r/2}$ as follows. $V_{r/2} = G_{x,r/2}/G^0_{x,r/2}G_{x,r/2+}$ is a $\mathbb{F}_p$-vector space, and the pairing $\langle g, h \rangle = \hat{\phi}(ghg^{-1}h^{-1})$ for $g, h \in G_{x,r/2}$ gives a non-degenerate symplectic form on $V_{r/2}$. The theory of Heisenberg representations says there is a unique irreducible representation (ω, V_ω) of $G_{x,r/2}$ such that $\omega|_{G_{x,r/2+}} = \hat{\phi} \cdot 1$. The dimension of V_ω is $\sqrt{|V_{r/2}|}$.

3. We define a compatible action of G_x^0 on V_ω as follows. First note that G_x^0 acts on $V_{r/2}$ by conjugation, and this preserves the symplectic pairing. Thus we get a map $G_x^0 \to Sp(V_{r/2})$. The theory of Weil representations gives an action of $Sp(V_{r/2})$ on V_ω, so we get an action of G_x^0 on V_ω. We tensor with ϕ to get the representation κ^{nt}.

4. Construct the twist ε_{FKS}.

For general inputs, i.e. $n > 1$, one constructs κ_i for each $i = 1, \ldots, n$, and then takes $\kappa = \bigotimes_i \kappa_i$.

1.4 Lecture 4: Bernstein Blocks, Types, and Hecke Algebras

Any time that we use Yu's construction in this lecture, we will assume as is necessary that $\underline{G}$ splits over a tame extension. Apart from this use-case, we will not need that assumption.

Let $M \subset G$ be a Levi subgroup with supercuspidal representation (σ, V_σ). Recall from the first lecture that associated to such data we have an equivalence class $[M, \sigma]$ and a corresponding Bernstein block in $\mathrm{Rep}(G)$.

Definition 1.13. A pair (K, ρ) consisting of a compact open subgroup $K \subset G$ and a smooth irreducible representation (ρ, V_ρ) of K is an $[M, \sigma]$**-type** if for all $(\pi, V) \in \mathrm{Irr}(G)$, the following are equivalent:

(i) $\pi \in \mathrm{Rep}(G)_{[M,\sigma]}$.

(ii) $\rho \hookrightarrow \pi|_K$, i.e. $\mathrm{Hom}_K(\rho, \pi) \neq 0$.

Example 1.12. Let $G = SL_2(F)$, $M = T$ the diagonal matrices. Then (I, triv) is a $[T, \mathrm{triv}]$-type, where I is the Iwahori subgroup consisting of matrices in $SL_2(\mathcal{O}_F)$ which are upper triangular mod ϖ.

Fact (Bushnell-Kutzko 1998): Let $\tilde{K}$ be a compact-mod-center open subgroup with representation $(\tilde{\rho}, V_{\tilde{\rho}})$ such that $\pi = \text{c-Ind}_{\tilde{K}}^{G} \tilde{\rho}$ is irreducible. Let $\tilde{K}_{cpt}$ be the maximal compact subgroup of $\tilde{K}$, and let ρ be an irreducible representation appearing in $\tilde{\rho}|_{\tilde{K}_{cpt}}$. Then $(\tilde{K}_{cpt}, \rho)$ is a $[G, \pi]$-type.

Theorem 1.3 (Bushnell-Kutzko 1998)**.** *If (K, ρ) is an $[M, \sigma]$-type, then $\mathrm{Rep}(G)_{[M,\sigma]}$ is equivalent to the category of modules for the Hecke algebra $\mathcal{H}(G, K, \rho)$ (or $\mathcal{H}_G(V_\rho)$ in the notation of Herzig's lectures). The functor sends V to the (K, ρ)-isotypic component of V.*

As a reminder, $\mathcal{H}_G(V_\rho)$ consists of compactly supported K-biequivariant functions $G \to \mathrm{End}(V_\rho)$.

Example 1.13. Let $G = SL_2(F)$. In this case, compact-mod-center means compact, so we will ignore the tildes despite using the above fact.

a) Let $M = G, \sigma = \text{c-Ind}_K^G \rho$. Then (K, ρ) is an $[M, \sigma]$-type, so the relevant Hecke algebra is $\mathcal{H}(G, K, \rho) \cong \mathrm{End}_G(\text{c-Ind}_K^G \rho) = \mathrm{End}_G(\sigma) \cong \mathbb{C}$. Thus the theorem tells us that $\mathrm{Rep}(G)_{[G,\sigma]}$ is equivalent to the category of vector spaces, which matches the description given in the first lecture (Example 1.2).

b) Let $M = T, \sigma = \mathrm{triv}$. As in the earlier example, we know that (I, triv) is an $[M, \sigma]$-type, so the relevant Hecke algebra is $\mathcal{H}(G, I, \mathrm{triv})$, the space of compactly supported functions $I\backslash G/I \to \mathbb{C}$. The double coset space $I\backslash G/I \to \mathbb{C} \cong N(T)/T_0 = W_{\mathrm{aff}}$, the affine Weyl group, where T_0 is a maximal compact subgroup in T defined in the second lecture. The affine Weyl group admits the presentation $\langle s_0, s_1 \mid s_i^2 = 1\rangle$. We can write $\mathcal{H}(G, I, \mathrm{triv}) = \bigoplus_{w \in W_{\mathrm{aff}}} \mathbb{C} \cdot T_w$, where the T_w have the following relations. If $w = s_{i_1} \cdots s_{i_n}$ is a minimal length expression for w in terms of the generating reflections, then $T_w = \prod_j T_{s_{i_j}}$. Furthermore, if s is a generating reflection, then $T_s^2 = qT_1 + (q-1)T_s$. We also denote this algebra by $\mathcal{H}_{\mathrm{aff}}(W_{\mathrm{aff}}, q)$.

In order to move past SL_2, we need some method of constructing types.

Definition 1.14. Let K be an open compact subgroup of G, and let K_M be a compact open subgroup of M. Let $(\rho, V_\rho) \in \mathrm{Irr}(K)$ and $(\rho_M, V_{\rho_M}) \in \mathrm{Irr}(K_M)$. We say (K, ρ) is a G-**cover** of (K_M, ρ_M) if for every parabolic $P = MN$ with Levi M and opposite $\overline{P} = M\overline{N}$ (meaning $P \cap \overline{P} = M$) we have:

(i) $K = (K \cap N)(K \cap M)(K \cap \overline{N})$ and $K \cap M = K_M$.

(ii) $\rho|_{K_M} = \rho_M$, $\rho|_{K \cap N} = \mathrm{triv} = \rho|_{K \cap \overline{N}}$.

(iii) For any $(\pi, V) \in \mathrm{Irr}(G)$, the restriction of the map $V \twoheadrightarrow V_N = V/(v - nv)$ to the (K, ρ)-isotypic component is injective.

Example 1.14. Let $G = SL_2(F)$ and $M = T$. Then (I, triv) is a G-cover of (T_0, triv).

Theorem 1.4 (Bushnell-Kutzko 1998)**.** *Let (K_M, ρ_M) be an $[M, \sigma]$-type for M. Let (K, ρ) be a G-cover of (K_M, ρ_M). Then (K, ρ) is an $[M, \sigma]$-type for G.*

Thus, to construct types, we need to construct covers. We demonstrate an analogue of Yu's construction in this direction, due to Kim-Yu. The input data is as follows:

(i) A sequence $G^0 \subsetneq G^1 \subsetneq \cdots \subseteq G^n = G$ of tame twisted Levi subgroups, with a Levi $M^0 \subset G^0$.

(ii) A vertex $x \in \mathcal{B}(\underline{M}^0, F)$.

(iii) $0 < r_0 < \cdots < r_{n-1}$.

(iv) ϕ_i a (G^{i+1}, G^i)-generic depth r_i character of G^i, for $i < n$.

(v) An irreducible representation ρ^0 of $K^0 = G^0_{x,0}(M^0_x)_{cpt}$ such that ρ^0 is trivial on $G^0_{x,0+}$, and such that $\rho^0|_{G^0_{x,0}}$ is a cuspidal representation of $G^0_{x,0}/G^0_{x,0+} \cong M^0_{x,0}/M^0_{x,0+}$.

The output is a compact group $K = K^0 G^1_{x,r_0/2} \cdots G^n_{x,r_{n-1}/2}$ with a representation $\rho = \rho^0 \otimes \kappa$, where κ is constructed in a manner analogous to the κ in Yu's construction.

Theorem 1.5 (Kim-Yu 2017, Finzten 2021)**.** *An output (K, ρ) of the above construction is an $[M, \sigma]$-type. Furthermore, if p does not divide the order of the Weyl group, then for all $[M, \sigma]$ there is a output (K, ρ) of the above construction which is an $[M, \sigma]$-type.*

The next goal is to understand the Hecke algebra $\mathcal{H}(G, K, \rho)$ attached to some "Kim-Yu input data" as above.

Definition 1.15. The **support** $\mathrm{Supp}\mathcal{H}(G, K, \rho)$ of a Hecke algebra $\mathcal{H}(G, K, \rho)$ is the collection of K-double cosets for which there is an $f \in \mathcal{H}(G, K, \rho)$ that is non-trivial on that K-double coset.

Fact: Given Kim-Yu data, we have $\mathrm{Supp}\mathcal{H}(G, K, \rho) = K(\mathrm{Supp}\mathcal{H}(G^0, K^0, \rho^0))K$.

Proposition 1.1 (Adler-Fintzen-Mishra-Ohara (AFMO) Aug 2024)**.** *There is a subgroup $N^\heartsuit \subset N_{G^0}(M^0, (M_x^0)_{cpt})$ such that $K^0\backslash\mathrm{Supp}\mathcal{H}(G^0, K^0, \rho^0)/K^0 \cong N^\heartsuit/(N^\heartsuit \cap (M_x^0)_{cpt}) = W^\heartsuit$.*

In other words, we get a group structure on the support, perhaps reminiscent of the W_{aff} story.

Theorem 1.6 (AFMO Aug 2024)**.** *Let (K, ρ) be obtained via the Kim-Yu construction. Then there is a lift $\tilde{\kappa}$ of $\kappa|_{K\cap M}$ to $N^\heartsuit(K \cap M)$ and an isomorphism $J : \mathcal{H}(G^0, K^0, \rho^0) \xrightarrow{\sim} \mathcal{H}(G, K, \rho)$ given as follows. If $\varphi \in \mathcal{H}(G^0, K^0, \rho^0)$ is supported on K^0nK^0 with $n \in N^\heartsuit$, then $J(\varphi)$ is supported on KnK and $J(\varphi)(n) = d_n\varphi(n) \otimes \tilde{\kappa}(n)$, where d_n is a scalar given by*

$$d_n = \sqrt{\frac{|K^0/(nK^0n^{-1} \cap K^0)|}{|K/(nKn^{-1} \cap K)|}}.$$

Corollary 1.1. *If (K, ρ) is an $[M, \sigma]$-type for G, and we suppose that everything comes from depth 0 via the Kim-Yu construction, then* $\mathrm{Rep}(G)_{[M,\sigma]} \cong \mathrm{Rep}(G^0)_{[M_0,\sigma_0]}$.

Theorem 1.7 (Morris 1993, AFMO Aug 2024)**.** *$W^\heartsuit \cong W(\rho)_{\mathrm{aff}} \rtimes \Omega(\rho)$ and $\mathcal{H}(G, K, \rho) \cong \mathcal{H}_{\mathrm{aff}}(W(\rho)_{\mathrm{aff}}, \{q_s\}) \rtimes \mathbb{C}[\Omega(\rho), \mu]$ for a 2-cocycle $\mu : \Omega(\rho) \times \Omega(\rho) \to \mathbb{C}^\times$.*

Remark. The decomposition here is analogous (and in fact a special case of) the decomposition of an extended affine Weyl group into an affine Weyl group and the length 0 elements.

2 Charlotte Chan: Geometrizations of Representations of p-adic Groups

2.1 Lecture 1: Deligne-Lusztig Theory

We will almost always deal with representations over $\mathbb{C}$.

Cartan and Weyl, in the early 20th century, gave us highest weight theory, which gives a correspondence between irreducible representations of $\underline{G}(\mathbb{C})$ and W-orbits of regular characters of $\underline{T}(\mathbb{C})$, where G is a connected reductive group. Over $\mathbb{F}_q$ or (non-archimedean) local field F, the same kind of thing is *very very roughly* true. Over $\mathbb{F}_q$, the relevant theory is due to Deligne and Lusztig in 1976. Over a local field F, the theory is due to Kaletha in 2019.

Note. A group like $\underline{G}(\mathbb{F}_q)$ is an example of a *finite group of Lie type*.

Our setup is the following. $\mathbb{G}$ will be a connected reductive group over $\overline{\mathbb{F}_q}$. We fix a Frobenius (root) $\sigma : \mathbb{G} \to \mathbb{G}$; this means some power of σ is a Frobenius endomoprhism. We denote $\overline{G} = \mathbb{G}(\overline{\mathbb{F}_q})^\sigma$. We fix a σ-stable maximal torus $\mathbb{T} \hookrightarrow \mathbb{G}$. We also denote $\overline{T} = \mathbb{T}(\overline{\mathbb{F}_q})^\sigma$.

Example 2.1. Let $\mathbb{G} = GL_2$. Let $\mathbb{T}$ be the usual diagonal matrices. Let σ raise all entries of a matrix to the qth power. Then $\overline{G}$ and $\overline{T}$ will be the usual corresponding groups over $\mathbb{F}_q$. However, we could take σ' to be the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d^q & c^q \\ b^q & a^q \end{pmatrix}$. Then $\overline{G}'$ will consist of invertible $\begin{pmatrix} a & b \\ b^q & a^q \end{pmatrix}$ with $a, b \in \mathbb{F}_{q^2}$, and $\overline{T}'$ will consist of $\begin{pmatrix} a & 0 \\ 0 & a^q \end{pmatrix}$ for $a \in \mathbb{F}_{q^2}^\times$. We identify $\overline{T}' \cong \mathbb{F}_{q^2}^\times$.

Consider $\overline{B}$ the upper triangular matrices and $\overline{U}$ the upper unitriangular matrices. We have a map $\mathrm{pr} : \overline{B} \to \overline{T}$. For any character $\theta : \overline{T} \to \mathbb{C}^\times$ we have $\mathrm{Ind}_{\overline{B}}^{\overline{G}}(\theta)$ which consists of $f : \overline{G} \to \mathbb{C}$ such that $f(bg) = \theta(\mathrm{pr}(b))f(g)$, or $f : \overline{U} \backslash \overline{G} \to \mathbb{C}$ such that $f(tg) = \theta(t)f(g)$. We also have the Weyl group $W_{\overline{G}}(\mathbb{T}) = \{g \in \overline{G} \mid g\mathbb{T}g^{-1} = \mathbb{T}\}/\overline{T}$. This consists of two elements, 1 and w. If $\theta \neq \theta^w$, then $\mathrm{Ind}_{\overline{B}}^{\overline{G}}(\theta)$ is irreducible of dimension $q+1$. If $\theta = \theta^w$, then $\theta = \theta_0 \circ \det$ and $\mathrm{Ind}_{\overline{B}}^{\overline{G}}(\theta) = \theta \otimes \mathrm{Ind}_{\overline{B}}^{\overline{G}}(\mathrm{triv})$.

So what about $\mathbb{T}'$? Take X to be the variety defined by $(x^{q+1} - y^{q+1})^{q-1} = 1$. $\overline{G}'$ acts on this in the obvious way, and $\overline{T}'$ acts by the scaling action after identifying $\overline{T}' \cong \mathbb{F}_{q^2}$. We have a virtual $\overline{G}'$ representation given by $H_c^*(X)_\theta = \sum_{i \geq 0} (-1)^i H_c^i(X; \overline{\mathbb{Q}_\ell})_\theta$. Here, the subscript refers to taking the isotypic component. If $\theta \neq \theta^w$ for w the non-trivial element in $W_{\overline{G}'}(\mathbb{T}')$, then this representation is irreducible. Otherwise, it is $\theta \otimes [1 - \mathrm{St}]$ for St the Steinberg representation.

Definition 2.1 (For GL_2)**.** The **DL functors** are $R_{\mathbb{T}}^{\mathbb{G}}(\theta) = \mathrm{Ind}_{\overline{B}}^{\overline{G}}(\theta)$ and $R_{\mathbb{T}'}^{\mathbb{G}}(\theta) = H_c^*(X)_\theta$.

Definition 2.2. Let $\mathbb{B}$ be a Borel containing $\mathbb{T}$, and let $\mathbb{U}$ be its unipotent radical. The **DL Variety** is $X_{\mathbb{T}} = \{g \in \mathbb{G} \mid g^{-1}\sigma(g) \in \mathbb{U}\}$. Again this has $\overline{G}$ and $\overline{T}$ actions. The **DL induction** is $R_{\mathbb{T}}^{\mathbb{G}} : \mathbb{Z}[\mathrm{Irr}(\overline{T})] \to \mathbb{Z}[\mathrm{Irr}(\overline{G})]$, $\theta \mapsto H_c^*(X_{\mathbb{T}})_\theta$.

Theorem 2.1 (Scalar Product Formula)**.** *Let $\mathbb{T}_1, \mathbb{T}_2$ be two maximal tori for the same σ, and let θ_1, θ_2 be characters of $\overline{T}_1, \overline{T}_2$. Then*

$$\langle R_{\mathbb{T}_1}^{\mathbb{G}}(\theta_1), R_{\mathbb{T}_2}^{\mathbb{G}}(\theta_2)\rangle_{\overline{G}} = \sum_{w \in W_{\overline{G}}(\mathbb{T}_1, \mathbb{T}_2)} \langle \theta_1, \theta_2^w \rangle_{\overline{T}_1}.$$

Definition 2.3. We say a character θ is **regular** (or *in general position*) if $\mathrm{Stab}_W(\theta)$ is trivial. We say θ is **nonsingular** if $\theta \circ \mathrm{Nm}|_{\alpha(\mathbb{G}_m)}$ is not trivial for all roots α. Here Nm is the norm map $\mathbb{T}(\overline{\mathbb{F}_q})^{\sigma^n} \to \overline{T}$, where n is the degree of the *splitting field* for $\mathbb{T}$. Regular implies nonsingular, and nonsingular implies regular if the center of $\mathbb{G}$ is connected.

Corollary 2.1. *If θ is regular, then $R_{\overline{T}}^{\mathbb{G}}(\theta)$ is irreducible (or the negative of an irreducible).*

Now we get to main results.

Theorem 2.2 (DL Fixed Point Formula)**.** *Suppose X is separated and finite type. Let g be a finite order automorphism of X, and decompose $g = su = us$ where s has prime-to-p order and u has p-power order. Then*

$$\mathrm{Tr}(g; H_c^*(X)) = \mathrm{Tr}(u; H_c^*(X^s)).$$

Theorem 2.3 (DL Character Formula)**.** *For s semisimple, u unipotent, we have*

$$\Theta_{R_{\mathbb{T}}^{\mathbb{G}}(\theta)}(su) = \frac{1}{|\overline{Z}_{\mathbb{G}}^0(s)|} \sum_{g \in \overline{G}} \theta^g(s) \cdot \Theta_{R_{\mathbb{T}^g}^{\mathbb{Z}_g^0(s)}(1)}(u).$$

The second Θ is a Green function.

Definition 2.4. A representation π of $\overline{G}$ is **cuspidal** (= supercuspidal in this setting) if $\langle \pi, \mathrm{Ind}_{\overline{P}}^{\overline{G}}(\rho)\rangle = 0$ for all σ-stable proper parabolics $\mathbb{P}$ and $\rho \in \mathrm{Irr}(\overline{M})$, with $\mathbb{M}$ the Levi of $\mathbb{P}$.

Theorem 2.4. *If $\mathbb{T}$ is not contained in any σ-stable proper parabolic, then for $\theta : \overline{T} \to \mathbb{C}^\times$ nonsingular, we have $R_{\mathbb{T}}^{\mathbb{G}}(\sigma)$ is cuspidal.*

Example 2.2. $\text{c-Ind}_{SL_2(\mathcal{O}_F)}^{SL_2(F)}(\mathrm{Inf}_{SL_2(\mathbb{F}_q)}^{SL_2(\mathcal{O}_F)}(\pi))$ is irreducible supercuspidal if:

- $\pi = R_{\mathbb{T}'}^{\mathbb{G}}(\theta)$ for θ regular, or
- $\pi \subset R_{\mathbb{T}'}^{\mathbb{G}}(\theta)$ for θ nonsingular but not regular.

2.2 Lecture 2: Lusztig's Conjecture and Positive Depth DL Varieties

$\underline{G}$ connected reductive over F, $\underline{T}$ unramified elliptic maximal torus over F, and $\underline{B}$ Borel containing $\overline{T}$ over F^{ur} (maximal unramified extension), and $\underline{U}$ the unipotent radical of $\underline{B}$.

Claim 2.1 (Conjecture of Lusztig, 1979)**.** *Let* $X_\infty = \{g \in \underline{G}(F^{ur}) \mid g^{-1}\sigma(g) \in \underline{U}(F^{ur})\}/(\underline{U}(F^{ur}) \cap \sigma^{-1}(\underline{U}(F^{ur})))$*. Then:*

- X_∞ *should be an ind-scheme over* $\overline{\mathbb{F}_q}$*.*
- X_∞ *should have homology groups that carry representation theoretic information.*

History:

- 1979: Lusztig wrote down a special case $D^1_{1/n}$.
- 2012: Boyarchenko looked at $D^\times_{1/n}$ and computed $H_i(X_\infty)_\theta$ for θ smallest positive depth.
- 2016: Boyarchenko-Weinstein showed that a piece of X_∞ corresponds to a special affinoid in the Lubin-Tate tower.
- Many other contributions from Weinstein, Imai-Tsushima, Mieda, Tokimoto, ...
- 2016-2020: Chan completed the computation of $H_i(X_\infty)_\theta$ for arbitrary θ.
- 2021-2023: Chan-Ivanov worked with any inner form of GL_n, and found an affine DL variety at infinite level that is isomorphic to X_∞.
- 2016: Fargues conjectured DL stacks.
- 2023: Takamatsu studied X_∞ and affine DL variety at infinite level for GSp_{2n}.
- 2022-2023: Ivanov showed X_∞ is an ind-scheme. Ivanov-Nie showed that if $\underline{T}$ is Coxeter, then there is a decomposition of X_∞ into an infinite dimensional scheme that is a "bounded" or "parahoric" part.
- Future: Ivanov studying homology.

Example 2.3. Let $\underline{G} = GL_2$. Let σ be the twisted Frobenius from last lecture. $\underline{T}, \underline{B}, \underline{U}$ will be the usual matrix groups (upper). Note that $T = L^\times$ for L the degree 2 unramified extension of F. Then $X_\infty = \{g \in GL_2(F^{ur}) : g^{-1}\sigma(g) \in \underline{U}\} = \left\{\begin{pmatrix} \sigma(d) & b \\ \sigma(b) & d \end{pmatrix} \in GL_2(F^{ur}) \mid \det \in F^\times\right\}$. If you mimicked this for $\overline{\mathbb{F}_q}$, you would get the DL variety for $\mathbb{T}'$ from last lecture.

Theorem 2.5. *$X_\infty = \bigsqcup_{\gamma \in GL_2(F)/GL_2(\mathcal{O}_F)} \gamma \cdot X_\infty^\circ$ where $X_\infty^\circ = \left\{ \begin{pmatrix} \sigma(d) & b \\ \sigma(b) & d \end{pmatrix} \in GL_2(\mathcal{O}_{F^{ur}}) \mid \det \in \mathcal{O}_F^\times \right\}$ and $X_\infty^\circ = \varprojlim_r X_r^\circ$ where X_r° has a similar description, but the matrices are over $\mathcal{O}_{F^{ur}}/\varpi^{r+1}$ and the determinant is a unit in $\mathcal{O}_F/\varpi^{r+1}$. There are also group actions similar to ordinary DL varieties.*

It turns out that we have a map $X_r^\circ/((1+\varpi^r)/(1+\varpi^{r+1})) \to X_{r-1}^\circ$ where the fibers are $\mathbb{A}^1$. Furthermore, $\mathbb{F}_{q^2} \cong (1+\varpi^r)/(1+\varpi^{r+1})$. Then $H_i(X_r^\circ)^{\mathbb{F}_{q^2}} = H_i(X_{r-1}^\circ)$. Then $H_i(X_{r-1}^\circ) \hookrightarrow H_i(X_r^\circ)$. Then $H_i(X_\infty^\circ)$ is defined to be $\varinjlim_r H_i(X_r^\circ)$.

Theorem 2.6 (Chan-Ivanov)**.** *Let $\theta : T \to \mathbb{C}^\times$ have depth r. Then $H_*(X_\infty)_\theta =$ c-Ind$_{Z(F)GL_2(\mathcal{O}_F)}^{GL_2(F)}(H_*(X_r)_\theta)$. In the special case $r = 0$, we have $H_*(X_\infty)_\theta =$ c-Ind$_{Z(F)GL_2(\mathcal{O}_F)}^{GL_2(F)}(R_{\mathbb{T}}^{\mathbb{G}}(\theta))$.*

Now return to the setting of algebraic groups from last lecture. Let $\mathbb{G}_r$ be the rth jet scheme for $\mathbb{G}$; it sends A to $\mathbb{G}(A[t]/t^{r+1})$. Let $\overline{G}_r = \mathbb{G}_r(\overline{\mathbb{F}_q})^\sigma$.

Definition 2.5 (Lusztig, 2004-2006)**.** Let $X_{\mathbb{T}_r \subset \mathbb{G}_r} = \{g \in \mathbb{G}_r \mid g^{-1}\sigma(g) \in \mathbb{U}_r\}$. This has actions by $\overline{G}_r$ and $\overline{T}_r$ as usual. Then we can define **positive depth DL induction** as $R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta) = H_c^*(X_{\mathbb{T}_r \subset \mathbb{G}_r})_\theta$, for any $\theta : \overline{T}_r \to \mathbb{C}^\times$.

When $r = 0$, this is exactly ordinary DL theory. What DL theorems hold for $r > 0$?

- $\overline{G}_r$ is a quotient of $G_{x,0}$ only if F has characteristic p. (In this case,?) Stasinski defined a mixed characteristic jet scheme.
- Only some quotients of $G_{x,0}$ arise as $\overline{G}_r$. Chan-Ivanov developed a framework that starts with a BT building.

Claim 2.2 (Conjecture: Positive Depth Scalar Product Formula)**.** *The scalar product formula in Theorem 2.1 generalizes to positive depth.*

This is known to be not true in general! Essentially, after making good choices, the parabolic induction side will grow with depth, but the sum side will not grow. But there are some cases where it is true:

- (θ is?) 0-toral (Lusztig, Stasinski, Chan-Ivanov).
- For general θ and GL_n (Chan-Ivanov).
- Torus is Coxeter with respect to $\mathbb{B}$ (Dudas-Ivanov, Ivanov-Tan-Nie).
- General elliptic $\mathbb{T}^1$, θ^1 Howe factorizable (which holds if p is large) (Chan).

Corollary 2.2. *If $\mathbb{T}^1$ is elliptic and θ^1 is regular, then $R_{\mathbb{T}_r^1}^{\mathbb{G}_r}(\theta^1)$ is irreducible.*

Theorem 2.7 (Chan-Oi)**.** *Let $s, u \in \overline{G}_r$ commute, such that s has prime to p order and u has p-power order. Then there is a positive-depth version of the DL character formula in Theorem 2.3.*

From the end of last lecture, we know (I'm fairly certain the lecturer did not mention this) $\text{c-Ind}_{Z(F)G_{x,0}}^{G(F)}(R_{\mathbb{T}}^{\mathbb{G}}(\theta))$ is depth 0 supercuspidal. One may ask about the positive depth generalization of this. Work in this direction:

- Chan-Ivanov: $G = GL_n$.
- Chen-Stasinski: 0-toral.
- Nie: General θ.
- Ivanov-Nie-Tan: T Coxeter.
- *Chan-Oi: 0-toral, regular supercuspidal, $q \gg 0$.

*: The first four works are of a more geometric nature, while the work of Chan-Oi is more analytic.

2.3 Lecture 3: Very Regular Elements

Definition 2.6. A regular semisimple element $\gamma \in G$ is called **tame very regular** if

- The identity component of the centralizer of γ in $\underline{G}$, denoted $\underline{T}_\gamma$, is a tamely ramified maximal torus.
- $\alpha(\gamma) \neq 1 \bmod \mathfrak{p}_{\overline{F}}$ for any root of $\underline{T}_\gamma$.

Example 2.4. Let $\underline{G} = GL_2$.

- Let $\underline{T}$ be unramified elliptic. Then $T \cong L^\times$ for L/F a degree 2 unramified extension. Then $\gamma \in L^\times$ is regular semisimple if $\gamma \in L^\times - F^\times$. $\gamma \in \mathcal{O}_L^\times$ is very regular $\overline{\gamma} \in \mathbb{F}_{q^2}^\times - \mathbb{F}_q^\times$.
- Let $\underline{T}$ be ramified elliptic. Then $T \cong E^\times$ for E/F degree 2 ramified extension. The condition for regular semisimple is the same, but $\gamma \in E^\times$ is very regular if $\nu_F(\gamma)$ is odd.

Theorem 2.8 (Chan-Oi 2025)**.** *Assume $q \gg 0$. Let $\underline{T}$ be unramified elliptic and θ a regular character. Then there is at most one irreducible representation π of $G_{x,0}$ such that*

$$\Theta_\pi(\gamma) = \pm \sum_{w \in W_{G_{x,0}}(\underline{T}_\gamma, \underline{T})} \theta^w(\gamma)$$

for all tame very regular γ.

Theorem 2.9 (Chan-Oi 2025)**.** *Assume $q \gg 0$. Let $\mathbb{T}$ be any maximal torus with regular character θ. Then there exists at most one irreducible representation of $\overline{G}$ such that*

$$\Theta_\pi(\gamma) = \pm \sum_{w \in W_{\overline{G}}(\mathbb{T})} \theta^w(\gamma)$$

for regular semisimple γ.

Note that this formula matches the regular semisimple case of the DL character formula (Theorem 2.3) for $R_{\mathbb{T}}^{\mathbb{G}}(\theta)$. So the statement above actually says there is *exactly* one representation, namely $\pm R_{\mathbb{T}}^{\mathbb{G}}(\theta)$.

Example 2.5. Consider $\overline{T}, \overline{T}'$ in $GL_2(\mathbb{F}_q)$, with regular characters θ, θ' (The T and T' are as in the first lecture). Then $\mathrm{Ind}_{\overline{B}}^{\overline{G}}(\theta)$ is the unique irreducible representation of $\overline{G}$ such that

$$\Theta(\gamma) = \begin{cases} \theta\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \theta\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} & \gamma \sim \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a \neq b \\ 0 & \gamma \text{ has distinct eigenvalues in } \mathbb{F}_{q^2}^\times - \mathbb{F}_q^\times \end{cases}.$$

This holds for $q^2 - 3q + 2 > 4$. Similarly, $R_{\mathbb{T}'}^{\mathbb{G}}(\theta')$ is the unique irreducible representation of $\overline{G}$ such that

$$\Theta(\gamma) = \begin{cases} 0 & \gamma \sim \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a \neq b \\ \theta(\gamma) + \theta(\gamma^2) & \gamma \text{ has distinct eigenvalues in } \mathbb{F}_{q^2}^\times - \mathbb{F}_q^\times \end{cases}.$$

This holds for $q^2 - q > 4$.

Theorem 2.10 (Chan-Oi 2023)**.** *Let $(\underline{T}, \theta)$ be a tame elliptic regular pair. If T has enough very regular elements, then the associated (FKS-twisted) regular supercuspidal representation is the unique supercuspidal representation with character*

$$\Theta(\gamma) = \pm \sum_w (\cdots)\theta^w(\gamma)$$

for γ very regular. The $(\cdots)$ factor appears only if $\underline{T}$ is ramified.

Theorem 2.11. *Let $q \gg 0$. Then a supercuspidal π is unipotent (i.e. depth zero, and the depth zero piece appears in an $R_{\mathbb{T}}^{\mathbb{G}}(1)$ for some $\mathbb{T}$) iff*

(i) $\Theta_\pi|_{T_{vreg}}$ *is constant for any T.*

(ii) $\Theta_\pi|_{T_{vreg}} \neq 0$ *for a maximally unramified elliptic maximal torus T.*

Proof of Theorem 2.8. Assume π, π' are smooth irreducible representations of $G_{x,0}$ that have the same character formula as in the statement of the theorem. Our goal is to show $\langle \pi, \pi' \rangle \neq 0$. We have $\langle \pi, \pi' \rangle = \langle \pi, \pi' \rangle_{vreg} + \langle \pi, \pi' \rangle_{nvreg}$. By Cauchy-Schwarz,

$$|\langle \pi, \pi' \rangle_{nvreg}| \leq \langle \pi, \pi \rangle_{nvreg}^{1/2} \langle \pi', \pi' \rangle_{nvreg}^{1/2}.$$

By assumption, $\langle \pi, \pi \rangle = 1 = \langle \pi', \pi' \rangle$. Splitting these into vreg and nvreg loci, and using the assumption that we have the same character formula on vreg, we find $\langle \pi, \pi \rangle_{(n)vreg} = \langle \pi', \pi' \rangle_{(n)vreg}$. It now suffices to show $\langle \pi, \pi \rangle_{vreg} > \frac{1}{2}$. Now we do some computation:

$$\begin{aligned} \langle \pi, \pi \rangle_{vreg} &= \frac{1}{|G_{x,0}|} \sum_{\gamma \in (G_{x,0})_{vreg}} \sum_{w, w' \in W(T_\gamma, T)} \theta^w(\gamma) \overline{\theta^{w'}(\gamma)} \\ &= \frac{1}{|G_{x,0}|} \cdot \frac{|G_{x,0}|}{|N(T)|} \sum_{t \in T_{vreg}} \sum_{w, w' \in W = W(T)} \theta^w(t) \overline{\theta^{w'}(t)} \\ &= \frac{1}{|N(T)|} \sum_{w, w' \in W} \left(|T| \langle \theta^w, \theta^{w'} \rangle_T - \sum_{t \in T_{nvreg}} \theta^w(t) \overline{\theta^{w'}(t)} \right) \\ &\geq \frac{1}{|N(T)|} \sum_{w, w' \in W} (|T| \langle \theta^w, \theta^{w'} \rangle_T - |T_{nvreg}|) = 1 - \frac{T_{nvreg}}{T_0} |W|. \end{aligned}$$

The condition $\langle \pi, \pi \rangle_{vreg} > \frac{1}{2}$ then becomes $|\overline{T}_0| / |\overline{T}_{nvreg}| > 2|W|$. This is an inequality on q.

Remark. A similar proof works for positive depth.

By applying the positive depth DL character formula (Theorem 2.7), we find that the representation in Theorem 2.8 that is unique if it exists, actually exists.

Now we discuss an implication of this work to the representation theory of p-adic groups.

From Kaletha's Lecture 3, we have a procedure to go from a pair $(\underline{T} \subset \underline{G}, \theta)$ consisting of an elliptic maximal torus and a regular character on that torus to a supercuspidal representation $\pi^{alg}_{(\underline{T}\subset\underline{G},\theta)}$. This procedure involves both Howe factorization and Yu's construction. There is also the FKS twisted version of this construction, producing from the pair a representation $\pi^{alg}_{(T,\theta\cdot\xi)} \cong \pi^{alg\times FKS}_{(T,\theta)}$, where ξ is an appropriate twist.

We want to know how this algebraic construction fits into our more geometric framework of positive depth DL theory. An unramified elliptic maximal torus $\underline{T} \subset \underline{G}$ corresponds to a point $x \in \mathcal{B}(G)$, hence a parahoric $G_{x,0}$. Similarly, the character theta gives us a depth r, hence a Moy-Prasad filtration subgroup $G_{x,r+}$. One can then construct an algebraic group $\mathbb{G}_r$ over $\overline{\mathbb{F}_q}$, equipped with an automorphism σ, a σ-stable $\mathbb{T}_r$, and $\mathbb{B}_r$, $\mathbb{U}_r$, all with the condition that $\overline{G}_r = G_{x,0}/G_{x,r+}$ and $\overline{T}_r$ is a subquotient of T. This all means that we can run the positive depth DL machinery to get a representation $\pi^{geo}_{(T,\theta)} = \text{c-Ind}^G_{ZG_{x,0}}(R^{\mathbb{G}_r}_{\mathbb{T}_r}(\theta))$.

Theorem 2.12 (Chan-Oi)**.** *For θ regular, $p \gg 0$, $q \gg 0$, we have $\pi^{geo}_{(T,\theta)} \cong \pi^{alg}_{(T,\theta\cdot\xi)} \cong \pi^{alg\times FKS}_{(T,\theta)}$.*

The benefit of this theorem is that it tells us that $\pi^{geo}_{(T,\theta)}$ "respects Langlands phenomena", which is also the reason why the FKS twist is so important. Somehow, without explicitly incorporating a twist, the geometric procedure obtained the "correct" representation.

Proof. $\pi^{alg\times FKS}_{(T,\theta)} = \text{c-Ind}^G_{\tilde{K}}(\tau^{FKS}_{(T,\theta)}) = \text{c-Ind}^G_{G_{x,0}}(\text{Ind}^{G_{x,0}}_{\tilde{K}}(\tau^{FKS}_{(T,\theta)}))$. Now compute character on very regular elements and apply the appropriate Chan-Oi theorem.

2.4 Lecture 4: Character Sheaves

So far in these lectures, the way geometry has interfaced with representation theory has been through the action of groups on spaces, and hence on their cohomology. Another important way that geometry can inform representation theory is through sheaves.

Suppose X is a scheme over $\mathbb{F}_q$ equipped with an automorphism σ, which is something like a Frobenius. Suppose $\mathcal{F}$ is a complex of constructible $\overline{\mathbb{Q}_\ell}$-sheaves on X with a σ-action, i.e. an isomorphism $\sigma^*\mathcal{F} \to \mathcal{F}$. Then for each closed point $x \in \overline{X} = X^\sigma$, the geometric Frobenius σ_x acts on the stalk $\mathcal{F}_x$. This produces something like a character, namely a map $\Theta_{\mathcal{F},\sigma} : \overline{X} \to \overline{\mathbb{Q}_\ell}$ which sends x to $\mathrm{Tr}(\sigma_x; H_c^*(\mathcal{F}_x))$.

The philosophy is Grothendieck's sheaf-function correspondence. Here is a table that gives a sort of dictionary between things in the world of sheaves and things in the world of functions:

Sheaves	Functions
Skyscraper	Indicator
Multiplicative local system	Multiplicative character
Pullback	Pullback
Pushforward	Average
Base change	Change of variables
Projection formula	Factoring out
Fourier transform	Fourier transform
Convolution	Convolution

Let $\mathbb{G}$ be a connected reductive group. The Grothendieck-Springer resolution $\tilde{\mathbb{G}} = \{(g, h\mathbb{B}) \in \mathbb{G} \times \mathbb{G}/\mathbb{B} \mid h^{-1}gh \in \mathbb{B}\}$ comes with maps $f : \tilde{\mathbb{G}} \to \mathbb{T}$ and $\pi : \tilde{\mathbb{G}} \to \mathbb{G}$ defined as follows. π is just the first projection. The map f sends $(g, h\mathbb{B})$ to $\mathrm{pr}(h^{-1}gh)$, where $\mathrm{pr} : \mathbb{B} \to \mathbb{T}$.

Definition 2.7 (Geometric Parabolic Induction)**.** $\text{p-Ind}_{\mathbb{T}}^{\mathbb{G}}(\mathcal{L}) = \pi_! f^*\mathcal{L}$.

Example 2.6. Let σ act on $\mathbb{G}$, and suppose both $\mathbb{B}$ and $\mathbb{T}$ are σ-stable. Let $\theta : \overline{T} \to \mathbb{C}^\times$ and let $\mathcal{L}_\theta$ be the corresponding local system on $\mathbb{T}$. Then

$$\Theta_{\text{p-Ind}_{\mathbb{T}}^{\mathbb{G}}(\mathcal{L}_\theta)}(g) = \sum_{h\overline{B} \in \overline{G}/\overline{B}, h^{-1}gh \in \overline{B}} \theta(\mathrm{pr}(h^{-1}gh)) = \Theta_{\mathrm{Ind}_{\overline{B}}^{\overline{G}}(\theta)}(g).$$

Theorem 2.13 (Lusztig)**.** *Suppose $\theta : \overline{T} \to \mathbb{C}^\times$ where $\mathbb{T}$ is σ-stable, but $\mathbb{B}$ need not be σ-stable.*

a) (1985): $\text{p-Ind}_{\mathbb{T}}^{\mathbb{G}}(\mathcal{L}_\theta)$ is a perverse sheaf, and it is simple if θ is regular (analogous to $R_{\mathbb{T}}^{\mathbb{G}}(\theta)$ being irreducible if θ is regular).

b) (1990): $\Theta_{\text{p-Ind}_{\mathbb{T}}^{\mathbb{G}}(\mathcal{L}_\theta)} = \pm\Theta_{R_{\mathbb{T}}^{\mathbb{G}}(\theta)}$.

Part a) gives first examples of a theory of character sheaves, while part b) gives compatibility of that theory with other constructions.

In 2004, Lusztig conjectured that there should be a theory of character sheaves for unipotent groups, as well as the jet schemes $\mathbb{G}_r$, at least "generically". The unipotent case was worked on by Boyarchenko and Boyarchenko-Drinfeld, while Lusztig himself worked on the jet scheme case for $r = 1$ and $r = 3$.

We can run the Grothendieck-Springer resolution and geometric parabolic induction by putting an r everywhere.

Claim 2.3 (Lusztig). *Let $\theta : \overline{T}_r \to \mathbb{C}^\times$ be $(\underline{T}, \underline{G})$-generic (cf. Fintzen's Lecture 3, Definition 1.12, although I think the order of things in the pair is swapped?) (this property is also known as 0-toral). Then $\text{p-Ind}_{\mathbb{T}_r}^{\mathbb{G}_r}(\mathcal{L}_\theta)$ is a simple perverse sheaf.*

One of the difficulties is that, while the resolution map $\pi : \tilde{\mathbb{G}_r} \to \mathbb{G}_r$ is proper and *small* when $r = 0$, neither of these properties hold for $r > 0$. It even fails to be semi-small.

Theorem 2.14 (Bezrukavnikov-Chan 2024). *a) Claim 2.3 is true.*

b) $\Theta_{\text{p-Ind}_{\mathbb{T}_r}^{\mathbb{G}_r}(\mathcal{L}_\theta)} = \pm \Theta_{R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta)}$.

Here is a takeaway. We have a relationship between $\text{p-Ind}_{\mathbb{T}_r}^{\mathbb{G}_r}(\mathcal{L}_\theta)$ and $R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta)$ given by taking trace. If $\underline{T}$ is an unramified elliptic maximal torus, then we have a relationship between $R_{\mathbb{T}_r}^{\mathbb{G}_r}(\theta)$ and the supercuspidal $\pi_{(T,\theta)}^{alg \times FKS}$ via c-Ind. Recall that the idea of supercuspidal is "does not come from parabolic induction"; however, we see here that a supercuspidal representation corresponds geometrically to a parabolic induction! This is "something magical about the perspective of sheaves".

We now discuss the strategy for Theorem 2.14. One general phenomenon that happens here and elsewhere is that it is easier to prove properties of the functor $\text{p-Ind}_{\mathbb{T}_r}^{\mathbb{G}_r}$ than it is to prove properties of its outputs. In particular, to show $\text{p-Ind}_{\mathbb{T}_r}^{\mathbb{G}_r}(\mathcal{L}_\theta)$ is perverse, we show $\text{p-Ind}_{\mathbb{T}_r}^{\mathbb{G}_r}$ is t-exact on some "generic subcategory".

Definition 2.8. Suppose $\psi : \mathfrak{t} \to \mathbb{C}^\times$ is $(\underline{T}, \underline{G})$-generic, where $\mathfrak{t} = \ker(\mathbb{T}_r \to \mathbb{T}_{r-1})$. We define $D_{\mathbb{T}_r}^{\psi}(\mathbb{T}_r)$ to be the subcategory of $D_{\mathbb{T}_r}(\mathbb{T}_r)$ (the bounded derived category of equivariant constructible sheaves) consisting of objects that are $(\mathfrak{t}, \psi)$-equivariant with respect to the multiplication action of $\mathfrak{t}$ on $\mathbb{T}_r$. You can also define $D_{\mathbb{G}_r}^{\psi}(\mathbb{G}_r)$, which is an "averaged" variant of the above.

Theorem 2.15 (Bezrukavnikov-Chan). $\text{p-Ind}_{\mathbb{T}_r}^{\mathbb{G}_r} : D_{\mathbb{T}_r}(\mathbb{T}_r) \to D_{\mathbb{G}_r}^{\psi}(\mathbb{G}_r)$ *is a t-exact equivalence of categories.*

Furthermore, for ψ a $(\underline{G}', \underline{G})$-generic character, one can obtain a t-exact equivalence $\text{p-Ind}_{\mathbb{G}'_r}^{\mathbb{G}_r} : D_{\mathbb{G}'_r}^{\psi}(\mathbb{G}'_r) \to D_{\mathbb{G}_r}^{\psi}(\mathbb{G}_r)$, where $\text{p-Ind}_{\mathbb{G}'_r}^{\mathbb{G}_r}$ is an appropriate Levi version of the geometric parabolic induction. We now take inspiration from Yu's construction of supercuspidal representations. We define geometric Yu data to consist of the following:

- A chain of twisted Levis $\underline{G}^0 \subsetneq \underline{G}^1 \subsetneq \cdots \subset G^n = G$ that are genuine Levis over F^{ur}.
- Vertex $x \in \mathcal{B}(\underline{G}^0, F)$.
- A character sheaf $\mathcal{F}$ on $\mathbb{G}_0^0$.
- Positive integers $r_0 < r_1 < \cdots \leq r_n = r$.
- Multiplicative local systems $\mathcal{L}_i$ on $\mathbb{G}_{r_i}^i$, with a genericity condition controlled by the next one up.

We repeatedly apply "generic" geometric parabolic induction to obtain a character sheaf on $\mathbb{G}_r^n$. Namely, start with $\mathcal{F}$ on $\mathbb{G}_0^0$, tensor with $\mathcal{L}_0$ to get something on $\mathbb{G}_{r_0}^0$, use the genericity hypothesis to induce to something on $\mathbb{G}_{r_0}^1$, and repeat. Here is a wonderful illustration of this process, where blue is tensoring and red is inducing:

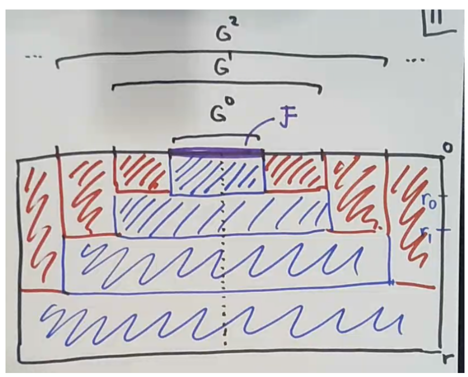

Figure 1: Make this look nicer

A further remark about this process is that if we start with a character $\theta : \overline{T}_r \to \mathbb{C}^\times$ with a Howe factorization (cf. Kaletha's Lecture 3, Theorem 3.7), then we can get associated geometric Yu data (with $\mathcal{F} = \text{p-Ind}_{\mathbb{T}}^{\mathbb{G}^0}(\mathcal{L}_{-1})$), and thus a character sheaf on $\mathbb{G}_r$.

3 Tasho Kaletha: Characters of Representations of Reductive p-adic Groups

3.1 Lecture 1: Characters of Admissible Representations

If G is any group and $V = (\pi, V)$ is any finite dimensional representation (say over $\mathbb{C}$), then its character is just given by taking the trace; $\theta_\pi : G \to \mathbb{C}$, $g \mapsto \mathrm{Tr}\pi(g)$.

We will consider G to a p-adic group over non-archimedean local field F, and throughout (I think) we will use Γ to denote the absolute Galois group of F. For such a G, the interesting (e.g. smooth irreducible) representations are often not finite dimensional, so we cannot take trace. Instead, Harish-Chandra defined a character distribution, $\theta_\pi : \mathcal{C}_c^\infty(G) \to \mathbb{C}$, $f \mapsto \mathrm{Tr}\pi(f)$, where $\pi(f) : V \to V$, $v \mapsto \int_G f(g)\pi(g)v dg$. Here $\mathcal{C}_c^\infty(G)$ consists of locally constant functions $f : G \to \mathbb{C}$ with compact support and such that there is a compact open subgroup K such that f is bi-K-invariant. For such f and K, the image of $\pi(f)$ is inside of the K-invariants V^K. Importantly, V^K is finite dimensional, so taking the trace is well-defined. We also had to choose a Haar measure dg on G; one always exists and is unique up to scalar. For a p-adic group G, left and right Haar measures agree (i.e. G is unimodular).

Definition 3.1. A **distribution** on G is a linear functional $\mathcal{C}_c^\infty(G) \to \mathbb{C}$. We let $D(G)$ be the space of all distributions.

Any locally integrable function ϕ on G gives rise to a distribution $f \mapsto \int_G \phi(g)f(g)dg$, and this is an injective correspondence (i.e. different functions give rise to different distributions). Such a distribution is called **representable** (by ϕ).

Theorem 3.1 (Harish-Chandra's Representability Theorem)**.** *The character distribution θ_π is representable by a function, which we also call θ_π, such that*

- *θ_π is locally constant on the regular semisimple locus G_{rs}.*
- *$\theta_\pi \cdot |D_G|_F^{1/2}$ is bounded.*

Here, $g \in G_{rs}$ iff $Z_G(g)^0$ is a torus iff g lies in a unique maximal torus, and $D_G(g) = \prod_{\alpha \in R(T)}(1 - \alpha(g))$, where $R(T)$ is the set of roots with respect to a maximal torus T.

Facts:

1) If $\pi_1, \dots, \pi_n$ are distinct irreducible representations, then their characters (as functions or as distributions) are linearly independent.
2) Two irreducible representations are isomorphic iff they have the same character.
3) The character is additive; if you have an exact sequence $0 \to \pi_1 \to \cdots \to \pi_n \to 0$, then $\sum(-1)^i\theta_{\pi_i} = 0$.

Proposition 3.1. *Suppose $\pi = \text{n-Ind}_P^G(\sigma)$, where $P = M \ltimes N$ is parabolic. Fix a compact open subgroup K of G that satisfies $G = PK$ and choose measures on G, M, N, K such that $\int_G f(g)dg = \int_M \int_N \int_K f(mnk)dmdndk$. Then $\theta_\pi(f) = \theta_\sigma(f^{(P)})$, where $f^{(P)}(m) = \delta_P(m)^{1/2} \int_N \int_K f(k^{-1}mnk)dkdn$. Furthermore,*

$$\theta_\pi(x) = \sum_{\substack{g \in M\backslash G \\ gxg^{-1} \in M}} \frac{\theta_\sigma(gxg^{-1})}{|D_{G/M}(gxg^{-1})|^{1/2}}.$$

Let $P_0 = M_0 N_0$ be a minimal parabolic. For any *standard* parabolic P, i.e. one containing P_0, we have $\mathcal{C}^\infty(P\backslash G) \subset \mathcal{C}^\infty(P_0\backslash G)$. Let Σ_0 be the sum of the spaces $\mathcal{C}^\infty(P\backslash G)$ as P ranges over all standard parabolics that are not P_0. Then the **Steinberg representation** is $\mathrm{St} = \mathcal{C}^\infty(P_0\backslash G)/\Sigma_0$.

Remark. St is square-integrable.

Let I_t be the direct sum of $\mathcal{C}^\infty(P\backslash G)$ over standard parabolics of rank t. For instance, $I_0 = \mathcal{C}^\infty(P_0\backslash G)$ and $I_r = \text{triv}$ for r the rank of G. Then there is a **Borel-Serre resolution**

$$0 \to I_r \to \cdots \to I_0 \to \mathrm{St} \to 0.$$

This leads to a character formula for St:

$$\Theta_{\mathrm{St}}(g) = (-1)^{\dim(A_0)} \sum_{(M,P)} (-1)^{\dim(A_M)} \delta_P(g)^{-1/2} |D_{G/M}(g)|^{-1/2}.$$

Here the sum is over standard parabolics P with Levi M, while A_0 is the split center of P_0 and A_M is the split center of M (the maximal split torus which is central in M).

The character formulas can get horrendous, even for SL_2. But the lecturer says, don't panic! We look to the "guiding light of real groups".

Theorem 3.2 (Harish-Chandra)**.** *Let G be a real reductive group.*

1) *G has discrete series representations iff G has an elliptic (compact modulo center of G) maximal torus (e.g. SL_2 but not SL_3).*

2) *There is a bijection between discrete series representations and G-conjugacy classes of triples (S, B, θ), where S is an elliptic maximal torus, $B \subset G_{\mathbb{C}}$ is a Borel containing S, and $\theta : S \to \mathbb{C}^\times$ is a regular character such that $d\theta$ is "B-dominant".*

3) *The representation π corresponding to (S, B, θ) is uniquely characterized by*

$$\Theta_\pi(s) = (-1)^{q(G)} \sum_{w \in W_S} \frac{\theta(s^w)}{\prod_{\alpha >_B 0}(1 - \alpha(s^w)^{-1})},$$

where $q(G) = \frac{1}{2}\dim(G/K)$ for a maximal compact K, W_S is the Weyl group for S, and $\alpha >_B 0$ means α is a positive root with respect to B.

3.2 Lecture 2: Regular Depth Zero Supercuspidal Representations

We start by recalling the character distribution attached to an admissible representation. We note that the distribution depends on the Haar measure, but the associated function does not. We also will write θ_π for $\theta_\pi \cdot |D_G|^{1/2}$.

The main ideas of this lecture are in DeBacker-Reeder (2009).

Theorem 3.3 (Moy-Prasad 1996)**.** *π is an irreducible depth zero supercuspidal representation iff $\pi = \text{c-Ind}_{G_x}^{G}\sigma$, where* c-Ind *is compact induction, x is a vertex of the Bruhat-Tits building $\mathcal{B}(G)$, G_x is the stabilizer of x, $\sigma \in \mathrm{Irr}(G_x/G_{x,0+})$, and σ restricted to $G_{x,0}/G_{x,0+}$ is cuspidal.*

There is a so-called Kottwitz homomorphism $G \to \pi_1(G)_I^{\mathrm{Fr}}$, where π_1 is the "algebraic" fundamental group, I is the inertia group (we take coinvariants with respect to it), and Fr is Frobenius. We call the kernel of this map G^0. There is a norm map $\pi_1(G)_I^{\mathrm{Fr}} \to \pi_1(G)^\Gamma$, and we define G^1 to be the kernel of the composite map $G \to \pi_1(G)^\Gamma$. It contains G^0.

Example 3.1. If $G = GL_n(F)$, then $G^0 = G^1$ consists of the matrices whose determinant is in $\mathcal{O}_F^\times$. For $G = PGL_2(F)$, then $G^1 = G$, while G^0 is an index 2 subgroup consisting of elements whose determinant can be represented by an element of $\mathcal{O}_F^\times$.

Now, associated to a vertex $x \in \mathcal{B}(G)$, we have the stabilizers $G_x^0 \subset G_x^1 \subset G_x$. There are group schemes $\mathcal{G}_x^0 \hookrightarrow \mathcal{G}_x^1 \hookrightarrow \mathcal{G}_x$ defined over $\mathcal{O}_F$ such that $\mathcal{G}_x^*(\mathcal{O}_F) = G_x^*$. Their special fibers $\overline{\mathcal{G}}_x^*$ are group schemes over the residue field k_F. They are affine algebraic groups, but not necessarily reductive, so we take their reductive quotients, denoted by $\overline{G}_x^*$. $\overline{G}_x^0$ is the identity component of the usually disconnected $\overline{G}_x^1$; these are both affine algebraic groups. On the other hand, $\overline{G}_x$ is a smooth group scheme, but generally not an affine algebraic group. The center Z of G gives a closed subgroup $\overline{Z} \subset \overline{G}_x$. We have $[\overline{Z} \cdot \overline{G}_x^0 : \overline{G}_x] < \infty$. This is saying that $\overline{G}_x$ is not too big.

Fact: Let S be a maximal torus of G, and let S' be a maximal unramified subtorus of S. Then the following are equivalent:

1. S' is a maximal unramified (sub)torus of G.
2. $S = Z_G(S')$.
3. $S \times_F F^u$ is a minimal Levi in $G \times_F F^u$.

Definition 3.2. A torus S satisfying the conditions above is called **maximally unramified**.

Let S be such a maximally unramified elliptic (compact modulo center of G) maximal torus. Then $\mathcal{A}(S' \times_F F^u) \subset \mathcal{B}(G \times_F F^u)$ and $\mathcal{A}(S' \times_F F^u)^{\mathrm{Fr}} = \{x\}$. This point x is called **the point of** S.

Proposition 3.2. *x is a vertex of $\mathcal{B}$.*

Remark. Any vertex of the building for SL_2 is possible as a point for some S.

The embedding $S \hookrightarrow G$ extends to embeddings $\mathcal{S} \hookrightarrow \mathcal{G}_x$ of integral models and $\overline{S} \hookrightarrow \overline{G}_x$ of reductive quotients. $\overline{S'}$ is a maximal torus in $\overline{G}_x^0$. The identity component of $\overline{S}$ is $\overline{S'}$. After passing to the algebraic closure $\overline{k}_F$ of the residue field k_F, we have $\overline{S} = \overline{Z} \cdot \overline{S'}$.

Let $\theta : S \to \mathbb{C}^\times$ be depth zero; this happens iff θ is trivial on S_{0+}. Then we get a character $\overline{\theta}$ of $\overline{S}$.

Definition 3.3. $\overline{\theta}$ is regular if $\theta \neq \theta^w$ for any non-trivial $w \in N_{\overline{G}_x}(\overline{S})/\overline{S}$.

Choose a $\overline{k}_F$-Borel satisfying $\overline{S}^0 \subset \overline{B} \subset \overline{G}_x^0$. Let $Y = \{g \in \overline{G}_x \mid g^{-1}\mathrm{Fr}(g) \in U \cdot \mathrm{Fr}(U)\}$. Then $\overline{G}_x$ and $\overline{S}$ act on the ℓ-adic compactly supported cohomology of Y.

Lemma 3.1. *If $\overline{\theta}$ is regular, then $H_c^*(Y)_{\overline{\theta}}$ (subscript meaning isotypic component) vanishes away from middle degree, where it is an irreducible cuspidal representation of $\overline{G}_x$.*

Definition 3.4. $\text{c-Ind}_{G_x}^G \sigma$, where σ is the inflation to G_x of the nonzero $\kappa = H_c^i(Y)_{\overline{\theta}}$, is a regular depth zero supercuspidal representation.

Theorem 3.4. *There is a bijection between depth zero supercuspidal representations and G-conjugacy classes of pairs (S, θ) of S elliptic maximally unramified maximal torus and θ regular depth zero character of S.*

Now we assume G splits over a tame extension, F is characteristic 0, and the prime p is sufficiently large.

Proposition 3.3. *The character of σ at a regular semisimple $\gamma \in G_x$ is*

$$(-1)^{r_G - r_S} |Z^0_{\overline{G}_x(\gamma_s)}| \sum_{h \in \overline{G}_x^0, h^{-1}\gamma_s h \in \overline{S}} \overline{\theta}(h^{-1}\gamma_s h) Q^c_{h\overline{S}^0 h^{-1}}(\gamma_u),$$

where $\gamma = \gamma_s \gamma_u$ is the topological Jordan decomposition; γ_s has finite prime-to-p order, and γ_u has pro-p order. Q is a Green function.

A conjecture of Springer, proved by Kazhdan, states that Q is the Fourier transform of an explicit orbital integral, up to a sign. Using this, the above character formula leads to:

Theorem 3.5 (DeBacker-Reeder Character Formula)**.**

$$\Theta_{\pi(S,\theta)}(\gamma) = (-1)^{r_G - r_J} \sum_{g \in S \backslash G/J, g\gamma_s \in S} \theta(g\gamma_s) \hat{\imath}^J_{X^g}(\log \gamma_u)$$

where J is the identity component of the centralizer of γ_s in G, $X \in \mathrm{Lie}^*(S)_0$ is regular mod p, and $\hat{\imath}$ is defined later.

Corollary 3.1. *If $\gamma \in G$ is topologically semisimple (and regular semisimple), then $\theta_{\pi(S,\theta)}(\gamma) = 0$ if γ is not conjugate to an element in S, and otherwise*

$$\Theta_{\pi(S,\theta)}(\gamma) = (-1)^{r_G - r_S} \sum_{w \in W(G,S)} \theta(\gamma^w).$$

Given a finite dimensional F-vector space V and a non-trivial character $\Lambda : F \to \mathbb{C}^\times$, then there is a Fourier transform $\mathcal{C}_c^\infty(V) \to \mathcal{C}_c^\infty(V^*)$ given by $\hat{f}(\xi) = \int_V f(x)\Lambda(\langle x, \xi \rangle)dx$. Slogan: the bigger the support of f, the more constant $\hat{f}$ is. If d is a distribution, we define its Fourier transform by $\hat{d}(f) = d(\hat{f})$.

For $\mathfrak{g} = \mathrm{Lie}(G)$ and $x \in \mathfrak{g}^*$, the orbital integral O_x is a distribution on $\mathcal{C}_c^\infty(\mathfrak{g}^*)$, sending f to $\int_{Gx} f(y)dy$.

Theorem 3.6 (Harish-Chandra)**.** *Let $x \in \mathfrak{g}^*$ be regular semisimple. Then $\hat{O}_x$ is represented by a function $\hat{\mu}_x : \mathfrak{g} \to \mathbb{C}$ which is locally constant on $\mathfrak{g}_{rs}$ and bounded after multiplying by $|D_{\mathfrak{g}}|^{1/2}$.*

We then define $\imath_x(y) = |D_{\mathfrak{g}}(x)|^{1/2} |D_{\mathfrak{g}}(y)|^{1/2} \hat{\mu}_x(y)$; this appeared in the DBR character formula.

3.3 Lecture 3: General Depth Regular Supercuspidal Representations

We begin by emphasizing that, even if you are ultimately interested in connected groups, there will be disconnectedness showing up. For instance, if $G = \underline{G}(F)$ for $\underline{G}$ a connected reductive group over a p-adic field F, and if $x \in \mathcal{B}(G)$, then the group $G_x/G_{x,0+}$ is isomorphic to the k_F-points of an algebraic k_F-group, which may very well be disconnected. You don't need to look far for such an example; if you take $G = PGL_2$ and x to be the midpoint of an edge in $\mathcal{B}$, then you end up with a finite group of Lie type with two connected components. Thus, you really need machinery that can deal with disconnectedness.

Recall that over $F = \mathbb{R}$, there is a bijection between regular discrete series representations and G-conjugacy classes of pairs (S, θ), where S is an elliptic maximal torus and θ is a regular character of S. There is an explicit character formula, cf. part 3) of Theorem 3.2. Furthermore, S may not exist, but if it does, it is unique up to conjugation.

On the other hand, over $F/\mathbb{Q}_p$, there is a bijection between regular supercuspidal depth zero representations and G-conjugacy classes of pairs (S, θ), where S is a maximally unramified elliptic maximal torus and θ is a regular depth zero character of S. There is an explicit character formula, cf. Corollary 3.1. S always exists, but it is rarely unique up to conjugation.

We note the similarities in the two character formulas mentioned above. Today we want to bridge the gap between these two stories.

We recall Yu's construction of supercuspidal representations, described in Finzten's lecture 3 (1.3). The input data (or "Yu data") is $G^0 \subset \cdots G^d = G$, a depth zero representation π_{-1}, and characters ϕ_i of G_i that satisfy a genericity condition. The output is a supercuspidal representation π of G. Work of Kim in 2007 and Finzten in 2021 showed that this construction is surjective when p does not divide the order of the Weyl group. Hakim-Murnaghan 2008 described exactly the fibers of this construction, i.e. the failure of injectivity. In particular, they introduced an explicit equivalence relation on the set of Yu data such that the equivalence classes are the fibers of Yu's construction. Part of this equivalence relation is the notion of refactorization.

Definition 3.5. The representation π is **regular** iff its depth zero part π_{-1} is regular.

Theorem 3.7. *Let S be a tame maximal torus with character θ, and suppose p does not divide the order of the Weyl group. Then there exists $S = G^{-1} \subset G^0 \subset \cdots \subset G^d = G$ and characters ϕ_i of G^i for $i = -1, \ldots, d$, where ϕ_i is G^{i+1}-generic for $i = 0, \ldots, d-1$, such that $\theta = \prod_{i=-1}^{d} \phi_i|_S$. This is called **Howe factorization**. The data is unique up to "refactorization".*

Theorem 3.8. *There is a bijection between regular supercuspidal representations and G-conjugacy classes of pairs (S, θ), where S is a elliptic maximal torus, θ is a regular character of S, and S is maximally unramified in G^0. There is also a root datum interpretation of this.*

Consider the one-step Yu data $(S \subset G, \phi_0 = \theta)$, where θ is a generic character of positive depth. There is a theorem of Adler-Spice, which I cannot see well, and the lecturer advised that we do not worry about copying it down. It is in the lecturer's notes, for the curious. Now, recall our main tool:

Main tool: "Don't panic!"

We can try to induct on the formula of Adler-Spice... PANIC! There was an error in Yu's construction, due to Yu copying an error from another paper. This is why the FKS twist was needed.

Theorem 3.9 (Spice)**.** *Vague: One can unwind the induction and collect all orbital integrals.*

We can reinterpret the roots of unity appearing in the formula of Adler-Spice. We say a root α is **symmetric** if $-\alpha$ is a Galois conjugate of α. In this case, there is a quadratic extension $F_\alpha/F_{\pm\alpha}$, where F_α is the field of definition of F. We say a symmetric root is **ramified** or **unramified** according to whether $F_\alpha/F_{\pm\alpha}$ is ramified or unramified. There is also a cohomological invariant $\mathfrak{f}$ which takes a symmetric root to either 1 or -1. Then for symmetric α, the set $\mathrm{ord}_x(\alpha)$ appearing in the formula of Adler-Spice is $e_\alpha^{-1}\mathbb{Z}$ if α is either ramified, or unramified with $\mathfrak{f}(\alpha) = 1$, and $e_\alpha^{-1}(\mathbb{Z} + \frac{1}{2})$ otherwise.

Theorem 3.10. *Character formula.*

Corollary 3.2. *For $\gamma \in S$ topologically semisimple, there is a special case of the above.*

More character formulas.

Let $F = \mathbb{R}$. Let ρ be half the sum of positive roots. It is not a function on S. We take a double cover $S_\pm$ where it is a function, but it is not a double cover as an algebraic group, only as a topological group. It comes with a function $a_S : S_\pm \to \{\pm 1\}$ that appears in the character formulas. There is a way to generalize this to work over any characteristic 0 local field. This leads to a character formula that agrees for all characteristic 0 local fields.

3.4 Lecture 4: Local Langlands Correspondence

Recall that from last time, if F is *any* local field of char 0, i.e. allowing $\mathbb{R}$, then we have a bijection between irreducible regular discrete series representations of G (although, the meaning of discrete series is slightly different depending on $F = \mathbb{R}$ or $F/\mathbb{Q}_p$) and G-conjugacy classes of pairs $(S, \theta_\pm)$, where S is elliptic maximal torus and $\theta_\pm$ is a regular character of the double cover $S_\pm$. Moreover, we have a character formula

$$\Theta_{\pi(S,\theta_\pm)}(\gamma) = e(G)\varepsilon(T_G - S)\sum_w [a_S \cdot \theta_\pm](\gamma^w)$$

for $\gamma \in S$ with prime-to-p order. This formula uniquely determines the representation in the following two cases:

- $F = \mathbb{R}$; this is due to Harish-Chandra.
- $F/\mathbb{Q}_p$, S unramified, $q \gg 0$; this is due to Chan-Oi.

We now discuss the "basic" local Langlands correspondence (LLC). We continue assuming F is a characteristic 0 local field. Let ω_F be the Weil group. The (local) Langlands group $\mathcal{L}_F$ is ω_F if $F = \mathbb{R}$ and $\omega_F \times SL_2(\mathbb{C})$ otherwise.

Claim 3.1 (Basic LLC)**.** *There is a finite-to-one map* $\mathrm{Irr}(G) \to \Phi(G)$*, where* $\Phi(G)$ *is the set of* $\hat{G}$*-conjugacy classes of Langlands parameters, which are homomorphisms* $\varphi : \mathcal{L}_F \to {}^L G$*. The fibers* Π_φ *are called L-packets. Furthermore, there are expectations about this map:*

- *Elements of an L-packet have the same local L-factor.*
- *If one element of an L-packet* Π_φ *is tempered, then all elements of* Π_φ *are tempered, and this happens iff* φ *is tempered, meaning it has bounded image.*
- *If one element of an L-packet* Π_φ *is an essential discrete series, then all elements of* Π_φ *are essential discrete series, and this happens iff* φ *is discrete, meaning it doesn't factor through any proper Levi.*

Definition 3.6. Let G_{sr} be the locus of *strongly*(???)regular semisimple elements. A function $f : G_{sr} \to \mathbb{C}$ is **stable** if $f(g) = f(g')$ whenever g, g' are conjugate in $\mathbb{G}(\overline{F})$ (this is stronger than being conjugate in G).

Claim 3.2 (Conjecture of Atomic Stability)**.** *Assume* φ *is tempered. Then there should exist a* $\mathbb{C}$*-linear combination* $S\Theta_\varphi = \sum_{\pi \in \Pi_\varphi} z_\pi \Theta_\pi$ *which is non-zero and stable,* ***and*** *(this is where "atomic" comes in) no proper subset of* Π_φ *has this property.* $S\Theta_\varphi$ *is called the* ***stable character*** *of* φ*.*

Fact: $S\Theta_\varphi$ is unique up to rescaling (if it exists), and the z_π are all non-zero. Also, observe that Π_φ and $S\Theta_\varphi$ (up to rescaling) determine each other.

This fact leads to a characterization of the basic LLC: it is enough to specify $S\Theta_\varphi$ in terms of φ. Arthur did this for classical groups via twisted endoscopy. Today we will exhibit an explicit formula (for arbitrary groups) when φ is supercuspidal, p does not divide the order of the Weyl group, and G is tame (it splits over a tame extension of F). From now on we assume these conditions.

Definition 3.7. A Langlands parameter φ is **supercuspidal** if it is discrete and trivial on SL_2.

Lemma 3.2. *$\hat{S} = Z_{\hat{G}}(Z_{\hat{G}}(\varphi(I_F)^0))$ is a maximal torus of $\hat{G}$ normalized by φ.*

Let $\mathcal{S} = \hat{S} \cdot \varphi(\omega_F) \subseteq {}^L G$. Then

- $\omega_F \xrightarrow{\varphi} \mathcal{S} \subseteq {}^L G$ (this is a tautology).
- We have an exact sequence $1 \to \hat{S} \to \mathcal{S} \to \Gamma \to 1$ where Γ is the Galois group of F.

So it "looks like $\mathcal{S}$ wants to be the L group of the maximal torus".

Magic: There is a canonical isomorphism $\mathcal{S} \xrightarrow{\sim} {}^L S_\pm$. By LLC, we get a genuine character $\theta_\pm : S_\pm \to \mathbb{C}^\times$.

You may be asking what S is; we defined $\hat{S}$ and $\mathcal{S}$, but not S. The extension $1 \to \hat{S} \to \mathcal{S} \to \Gamma \to 1$ gives an action of Γ on $\hat{S}$. A complex torus with an action of the Galois group is the dual of an F-torus, which is what we call S. Furthermore, S is just "floating in space". It comes with a set of *admissible embeddings* $S \hookrightarrow G$ that forms a *stable conjugacy class*, i.e. they are all conjugate over $\overline{F}$. The upshot is that we don't just get one representation arising from φ in this procedure; we get one representation *per admissible embedding*, and this is where the L-packet comes from.

Claim 3.3.

$$S\Theta_\varphi(\gamma) = e(G)\varepsilon(T - S) \sum_{w \in (N_{\mathbb{G}}(\mathbb{S})/\mathbb{S})(F)} [a_S \cdot \theta_\pm](\gamma^w)$$

for $\gamma \in S$ with prime-to-p order.

There is a more general version of this formula where we don't impose a condition on γ. We also emphasize that $(N_{\mathbb{G}}(\mathbb{S})/\mathbb{S})(F)$ is larger than $N_G(S)/S$.

Remark. Very recently, Chi-Heng Lo and Cheng-Chiang Tsai have been working to use this to give a similar description of LLC for non-supercuspidal parameters φ.

We now assume G is quasi-split, meaning it has a Borel subgroup B defined over F.

Claim 3.4. *Let $S_\varphi = Z_{\hat{G}}(\varphi)$. There is a bijection $i : \Pi_\varphi \leftrightarrow \mathrm{Irr}(\pi_0(S_\varphi / Z_{\hat{G}}^\Gamma))$.*

But notice that there is a distinguished element on the right hand side, namely the trivial representation. What should correspond to it on the left hand side?

Claim 3.5 (Shahidi's Generic/Temepred Packet Conjecture)**.** *Fix a generic character $\psi : U \to \mathbb{C}^\times$, where U is the unipotent radical of B. Then there is a unique $\pi \in \Pi_\varphi$ which is ψ-generic, meaning* $\mathrm{Hom}_U(\pi, \psi) \neq 0$ *("think of this as non-zero Fourier coefficients").*

We now enhance the earlier claim:

Claim 3.6. *Let $S_\varphi = Z_{\hat{G}}(\varphi)$, and let ψ be a generic character on U. There is a bijection $i_\psi : \Pi_\varphi \leftrightarrow \mathrm{Irr}(\pi_0(S_\varphi/Z_{\hat{G}}^\Gamma))$ that sends the unique ψ-generic π to the trivial representation.*

Let $s \in S_\varphi$ be semisimple, let $\hat{H} = Z_{\hat{G}}(s)^0$, and let $\mathcal{H} = \hat{H} \cdot \varphi(\omega_F)$. Then we have the exact same phenomena as with $\mathcal{S}$; the parameter factors through $\mathcal{H}$ (obviously) and $1 \to \hat{H} \to \mathcal{H} \to \Gamma \to 1$ is exact. Furthermore, there is a quasi-split H/F dual to $\hat{H}$ (although more work is needed to define it), a (topological) double cover $H_\pm \to H$ (again harder to do), and a canonical isomorphism $\mathcal{H} \to {}^L H_\pm$. We then get a parameter φ' for $H_\pm$.

Claim 3.7 (Endoscopic Character Identities)**.**

$$S\Theta_{\varphi'}(f') = \sum_{\pi \in \Pi_\varphi} \mathrm{Tr}(i_\psi(\pi)(s))\Theta_\pi(f),$$

stated at the level of distributions; here f' is some "transfer" of f. Denote the right hand distribution by $\Theta^\psi_{\varphi,s}$. This identity is equivalent to

$$\sum_{\gamma \in H/\sim_{\mathrm{St}}} \Delta(\dot{\gamma}, \delta) S\Theta_{\varphi'}(\dot{\gamma}) = \Theta^\psi_{\varphi,s}(\delta),$$

where $\delta \in G$ is regular semisimple, $\sim_{\mathrm{St}}$ means stable conjugacy, and Δ is a transfer factor/kernel function.

Fact: The above conjecture uniquely determines i_ψ.

We now discuss the global setting. Let $G/\mathbb{Q}$ be connected reductive. Consider $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$, where $\mathbb{A}$ is the ring of adeles for $\mathbb{Q}$. This L^2 space is a module for $G(\mathbb{A})$. Any (irreducible?) π in it decomposes as a (restricted) tensor product of local factors $\pi_p \in \mathrm{Irr}(G(\mathbb{Q}_p))$. We further restrict ourselves to $L^2_{dt} = L^2_{disc,temp}$, where temp stands for tempered and means that each π_p is tempered, and disc stands for discrete and means the locus where things decompose into direct sums. We have $L^2_{dt} = \widehat{\bigoplus}_{\pi \in \mathrm{Irr}_{adm}(G(\mathbb{A}))} \pi^{m(\pi)}$, where $m(\pi)$ is the automorphic multiplicity; if $m(\pi) > 0$, we call π automorphic. A major question is if we can compute $m(\pi)$.

Claim 3.8 (Kottwitz)**.** *For any $\pi \in \mathrm{Irr}_{adm}(G(\mathbb{A}))$, we have*

$$m(\pi) = \sum_{\varphi : \mathcal{L}_\mathbb{Q} \to {}^L G, \pi_p \in \Pi_{\varphi_p}} m_{S_\varphi}(\mathrm{triv}, \bigotimes_p i_\psi(\pi_p)).$$

Here $\mathcal{L}_{\mathbb{Q}}$ is the Langlands group of $\mathbb{Q}$; its existence is in itself conjectural. It admits maps $\mathcal{L}_{\mathbb{Q}_p} \hookrightarrow \mathcal{L}_{\mathbb{Q}}$ which "mirror" the inclusion of the Galois group for a local field into the Galois group for the global field. The φ_p is obtained by precomposing with these maps. Each S_{φ_p} contains S_φ, which allows you to restrict the factors $i_\psi(\pi_p)$ to representations of S_φ and take the tensor product.

For $f \in \mathcal{C}_c^\infty(G(\mathbb{A}))$, we have

$$\operatorname{Tr}(f|L^2_{dt}) = \sum_\pi m(\pi)\Theta_\pi(f).$$

We may try choosing f wisely, or varying f, in order to isolate $m(\pi)$. Let us first continue manipulating this expression using the conjecture of Kottwitz:

$$\begin{aligned}\sum_\pi m(\pi)\Theta_\pi(f) &= \sum_\pi \sum_\varphi |S_\varphi|^{-1} \sum_{s\in S_\varphi} \prod_p \operatorname{Tr}(i_\psi(\pi)(s))\Theta_{\pi_p}(f) \\ &= \sum_\varphi \prod_p \sum_{\pi_p} \operatorname{Tr}(i_\psi(\pi_p)(s))\Theta_{\pi_p}(f).\end{aligned}$$

Now we reindex via endoscopy; pairs (φ, s) are replaced with triples $(H_\pm, s, \varphi')$. The pairs $(H_\pm, s)$ are called elliptic endoscopic. The sum becomes

$$\sum_{(H_\pm,s)} i(G,H) \sum_{\varphi'} |S_{\varphi'}|^{-1} S\Theta_{\varphi'}(f').$$

The first factor is a constant, while the inner sum is the stable trace of f' on $L^2([H])$.

4 Florian Herzig: Mod-p Representations of p-adic Groups

4.1 Lecture 1: Serre Weights and Induced Characters

F will be a p-adic local field (finite extension of $\mathbb{Q}_p$) with integers $\mathcal{O}_F$ and uniformizer ϖ; let $q = |\mathcal{O}_F/\varpi| = p^f$. Let $G = GL_n(F)$; often we will take $n = 2$. Let $K = GL_n(\mathcal{O}_F)$; it is a maximal compact subgroup. We have the nested compact open subgroups $K_i = 1 + \varpi^i M_n(\mathcal{O}_F)$. They are all pro-$p$ groups, i.e. limits of finite p-groups. We have $K_r/K_{r+1} \cong M_n(\mathbb{F}_q)$, where $1 + \varpi^r A$ is mapped to the reduction $\overline{A}$.

Fix algebraically closed field C of characteristic p. We have the notions of smooth and admissible representation V; smooth means $V = \bigcup_{K \leq G} V^K$ for K compact open, and admissible means each V^K has finite dimension over C.

Classical local Langlands conjectures give a correspondence between irreducible smooth representations of G over $\mathbb{C}$ and Galois representations $\mathrm{Gal}(\overline{F}/F) \to GL_n(\mathbb{C})$ or $GL_n(\overline{\mathbb{Q}_\ell})$. Similarly, the mod p local Langlands conjectures are about a correspondence between irreducible smooth representations of G over $\overline{\mathbb{F}_p}$ and Galois representations valued in $GL_n(\overline{\mathbb{F}_p})$.

Some challenges of mod p representations:

- There is no C-valued Haar measure, hence no analytic tools.
- Open compact subgroups do not act semisimply.
- Taking invariants with respect to an open compact subgroup is not an exact functor.

Example 4.1 (Warmup: $n = 1$)**.** $G = GL_1(F) = F^\times = \varpi^{\mathbb{Z}} \times \mathbb{F}_q^\times \times (1 + \varpi\mathcal{O}_F)$. Since G is abelian, any irreducible smooth representation is one-dimensional, $\chi : G \to C^\times$. Smoothness is equivalent to $\ker\chi$ being open. (*) Note that, since K_1 is pro-p, we have $\chi(K_1)$ is a finite p-group. But inside $C^\times$, such a thing must be trivial.

We generalize the observation (*):

Lemma 4.1 (p-group Lemma)**.** *If V is a smooth representation of a pro-p group H, then $V^H \neq 0$.*

Idea. One reduces to the case where H is a finite p-group, $C = \mathbb{F}_p$, and V is finite dimensional. The map $H \to GL(V)$ has image inside of a p-Sylow subgroup, which is the upper unitriangular matrices. These fix the first basis element.

Corollary 4.1. *If π is a smooth representation of G, and H is an open pro-p subgroup, then π is admissible iff π^H is finite dimensional.*

We illustrate the usefulness of the p-group lemma in proving irreducibility:

Example 4.2. Consider $\Gamma = GL_2(\mathbb{F}_q)$ acting on $V = \mathrm{Sym}^r(C^2)$, where C^2 refers to the standard representation of 2 by 2 matrices on 2D column vectors. We can realize V as the space of degree r homogeneous polynomials in $C[X, Y]$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on polynomial f by $(gf)(X, Y) = f(aX + cY, bX + dY)$.

Claim 4.1. *If $0 \leq r \leq q - 1$, then V is irreducible.*

Note. This is actually incorrect if $q > p$, and a correction was given at the start of the next lecture.

Sketch. Let Δ be the lower unitriangular matrices in Γ. It is a Sylow subgroup. Then one can calculate that V^Δ is one-dimensional, in fact given by $C \cdot Y^r$, and that V is generated by $\Gamma \cdot V^\Delta$. If W is a non-trivial subrepresentation, then the p-group lemma (4.1) gives $W^\Delta \neq 0$. But $W^\Delta \subset V^\Delta$, and V^Δ is one-dimensional, so $W^\Delta = V^\Delta$. Thus V is generated by $\Gamma \cdot W^\Delta$, which is contained in W, so $V = W$.

Corollary 4.2. *The irreducible representations of Γ are $\mathrm{Sym}^r(C^2) \otimes \det^s$ for $0 \leq r \leq q - 1$ and $0 \leq s \leq q - 2$. The* Sym *factors are called **Serre weights**.*

Example 4.3. Consider $G = GL_2(F)$. Since K_1 is a normal subgroup of K, we have that $V^{K_1} = V$ for any irreducible smooth representation of K. Thus V is an irreducible representation of $K/K_1 \cong \Gamma$, and it is a Serre weight.

Example 4.4. Again consider $G = GL_2(F)$. Let B be the upper triangular matrices. We have $B = T \ltimes U$ for T the diagonal matrices and U the upper unitriangular matrices. Let $\theta = \theta_1 \otimes \theta_2$ be a smooth character of T, where θ_i is the component of θ on the (i, i) entry of elements of T. Then $\mathrm{Ind}_B^G(\theta)$ consists of locally constant functions $f : G \to C$ such that $f(\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} g) = \theta_1(\alpha)\theta_2(\delta)f(g)$, and it is an admissible smooth representation of G.

Theorem 4.1 (Barthel-Livné)**.** *If $\theta_1 \neq \theta_2$, then $\pi = \mathrm{Ind}_B^G(\theta)$ is irreducible.*

Idea. Assume the restrictions of the θ_i to $\mathcal{O}_F^\times$ are different. Then $\pi|_K$ contains a unique irreducible K-subrepresentation V. To show this, one uses $\pi|_K = \mathrm{Ind}_{B \cap K}^K(\theta)$ and Frobenius reciprocity. Also, π is generated by $G \cdot V$. Now suppose we have a non-trivial subrepresentation π'. By the uniqueness of V, we have $\pi'|_K = V$, and then π is generated by something contained in π', hence $\pi = \pi'$.

Add last page

4.2 Lecture 2: Mod p Satake Isomorphism and Applications

We begin by talking about Hecke algebras. Let C be any field, and let H be compact open in G. Let V be a finite dimensional representation of H over C, and let π be a smooth representation of G. Then $\mathrm{Hom}_H(V, \pi|_H) = \mathrm{Hom}_G(\text{c-Ind}_H^G V, \pi)$ where $\text{c-Ind}_H^G V \cong C[G] \otimes_{C[H]} V$. The Hecke algebra $\mathcal{H}_G(V)$ is $\mathrm{End}_G(\text{c-Ind}_H^G V)$; it acts on the prior Hom space on the right.

Lemma 4.2. *There is an algebra isomorphism between $\mathcal{H}_G(V)$ and the space of compactly supported functions $\varphi : G \to \mathrm{End}_C V$ that satisfy $\varphi(hgh') = h\varphi(g)h'$ for $h, h' \in H$ and $g \in G$. This space has a convolution action that makes it into a ring, but this will not be written here.*

Idea. By Frobenius reciprocity, $\mathrm{End}_G(\text{c-Ind}_H^G V) = \mathrm{Hom}_H(V, \text{c-Ind}_H^G V)$. Elements of the space on the right are functions $V \to (G \to V)$, which can be identified with functions $G \to (V \to V)$. One keeps track of the relevant conditions to show that the lemma holds.

Remark. If V is the trivial representation, then the Hom space above is π^H, and the Hecke algebra can be realized as finitely supported functions $H\backslash G/H \to C$. This relates to Hecke operators in modular forms.

If $C = \mathbb{C}$, $H = K$ is maximal compact, and V is the trivial representation, then there is the *Satake isomorphism* $\mathcal{H}_G^C(V) \xrightarrow{\sim} \mathcal{H}_T^C(V)^W$. The map sends φ to the map $t \mapsto \delta(t)^{-1/2} \sum_{u \in (U \cap K)\backslash U} \varphi(ut)$. Here δ is the modulus character, and it is needed to make sure the image is W-invariant. Note that δ takes values in powers of p, which is a no-go if we want to study Hecke algebras in characteristic p. The map does not depend on choice of Borel.

Now we let C be characteristic p, and let $H = K = GL_n(\mathcal{O}_F)$. We want to develop a mod p Satake isomorphism. Let V be a Serre weight.

Fact: The coinvariants $V_{U \cap K}$ are one-dimensional. This is a $T \cap K$ representation. Also, it is the same as $V^{\overline{U} \cap K}$, where $\overline{U}$ is the opposite unipotent radical (lower triangular).

Let $p_U : V \twoheadrightarrow V_{U \cap K}$. There is a Satake map $S^G : \mathcal{H}_G(V) \to \mathcal{H}_T(V_{U \cap K})$ which sends φ to the map which sends t to $\sum_{u \in (U \cap K)\backslash U} p_U \circ \varphi(ut)$. This map depends on choice of Borel. Let T^+ consist of $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ where $\nu_F(x) \geq \nu_F(y)$ (this definition is only stated here for GL_2, but there is a T^+ for any GL_n, and the following theorem holds for any GL_n).

Theorem 4.2 (Herzig)**.** *S^G is an injective algebra homomorphism with image $\mathcal{H}_T^+$, which consists of $\psi \in \mathcal{H}_T$ supported on T^+.*

Idea for GL_2. (1) One checks that S^G is an algebra homomorphism.

(2) Use the Cartan decomposition $G = \bigsqcup_{a \geq b} K \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} K$ and show that for $a \geq b$, there is a unique $\varphi_{a,b} \in \mathcal{H}_G$ that is supported on the corresponding piece of the decomposition and satisfies $\varphi_{a,b} \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix}$ is idempotent. Thus $\mathcal{H}_G = \bigoplus_{a \geq b} C \cdot \varphi_{a,b}$. Similarly one shows $\mathcal{H}_T = \bigoplus_{a,b} C \cdot \psi_{a,b}$ and $\mathcal{H}_T^+ = \bigoplus_{a \geq b} C \cdot \psi_{a,b}$.

(3) Show $S^G(\mathcal{H}_G)$ is contained in $\mathcal{H}_T^+$. This uses the fact that we are working in characteristic p.

(4) Use a triangular basis argument: $S^G(\varphi_{a,b}) = \psi_{a,b} + \sum C_{a',b'} \psi_{a',b'}$, where $a > a' \geq b'$ and $a + b = a' + b'$. Since this is a finite sum, we are done by linear algebra.

Corollary 4.3. *$\mathcal{H}_G(V)$ is commutative.*

If π is an admissible representation of G and V is a Serre weight, then $\mathrm{Hom}_K(V, \pi) = \mathrm{Hom}_K(V, \pi^{K_1})$, which is finite dimensional. Since $\mathcal{H}_G(V)$ is commutative, we can then decompose this Hom space into generalized eigenspaces.

Lemma 4.3. *If $\theta : T \to C^\times$ is a smooth character, then the isomorphism $\mathrm{Hom}_K(V, \mathrm{Ind}_B^G \theta) \cong \mathrm{Hom}_{T \cap K}(V_{U \cap K}, \theta)$ is compatible with the Satake map, in the sense that $\mathcal{H}_G(V)$ acting on the left Hom space agrees with $\mathcal{H}_T(V_{U \cap K})$ acting on the right Hom space via the Satake map.*

Corollary 4.4. *If $\mathrm{Ind}_B^G \theta \cong \mathrm{Ind}_B^G \theta'$, then $\theta = \theta'$.*

Corollary 4.5. *$\varphi_{1,0} \in \mathcal{H}_G(V)$ acts invertibly on $\mathrm{Hom}_K(V, \mathrm{Ind}_B^G \theta)$.*

Idea. $S^G(\varphi_{1,0}) = \psi_{1,0}$, which is invertible in $\mathcal{H}_T$.

Definition 4.1. An irreducible admissible representation π of G is **supersingular** if (for $n = 2$) $\varphi_{1,0}$ acts nilpotently on $\mathrm{Hom}_K(V, \pi)$ for one (and then all) V.

"What can we say about supersingular representations? For $GL_2(\mathbb{Q}_p)$ it is easy, and for every other group it is extremely complicated."

Suppose π is an irreducible smooth representation of G. Pick $V \subset \pi|_K$. Denote eigenvalues with $\chi : \mathcal{H}_G(V) \to C$. $\mathrm{Hom}_K(V, \pi)$ contains an $\mathcal{H}_G(V)$-eigenvector f. We have c-$\mathrm{Ind}_K^G V \twoheadrightarrow \pi$. On the left, φ acts by $\chi(\varphi)$. Then we get a G-linear map c-$\mathrm{Ind}_K^G V \otimes_{\mathcal{H}_G(V), \chi} C \twoheadrightarrow \pi$. If π is supersingular, then $\chi(\varphi_{1,0}) = 0$.

Theorem 4.3 (Breuil, 2003)**.** *If $G = GL_2(\mathbb{Q}_p)$, then for all V and χ such that $\chi(\varphi_{1,0}) = 0$, we have c-$\mathrm{Ind}_K^G V \otimes_{\mathcal{H}_G V, \chi} C$ is irreducible and admissible.*

Corollary 4.6. *For $G = GL_2(\mathbb{Q}_p)$, we obtain a classification of irreducible semisimple representations.*

Theorem 4.4 (Herzig)**.** *Let P be the parabolic corresponding to a partition $n_1, \ldots, n_r$. Suppose σ_i are irreducible admissible representations of $GL_{n_i}(F)$ such that, for each i, one (depending on i) of the following two conditions holds:*

(a) σ_i *is supersingular and* $n_i > 1$.

(b) $\sigma_i = \mathrm{Sp}_{Q_i} \otimes (\eta_i \otimes \det)$, *where* Sp *is a generalized Steinberg representation,* Q_i *is a subgroup of* $GL_{n_i}(F)$ *satisfying what?, and* η_i *is a smooth character of* F.

Further suppose that $\eta_i \neq \eta_{i+1}$ if σ_i, σ_{i+1} are both of type (b). Then $\mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ is irreducible and admissible.

4.3 Lecture 3: Too Many Supersingular Representations

We fix $G = GL_2(F)$ for $F/\mathbb{Q}_p$ a nontrivial unramified extension of degree f (so $q = p^f$), and $p > 2$. We let $K = GL_2(\mathcal{O}_F)$. Inside K is the Iwahori subgroup I consisting of those elements congruent to an upper triangular matrix mod p. Inside I is a normal subgroup I_1, consisting of those elements congruent to an upper unitriangular matrix mod p. It is pro-p. We have $I/I_1 \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$. We also let $Z = Z(G) = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\}$.

Our main idea is to use the (transitive) action of G on the Bruhat-Tits tree $\mathcal{T}$. Its vertices are homothety (scaling) classes of $\mathcal{O}_F$-lattices Λ in F^2. There is an edge between $[\Lambda]$ and $[\Lambda']$ iff $p\Lambda \subsetneq \Lambda' \subsetneq \Lambda$. We fix x_0 to be the class of $\mathcal{O}_F^2$ and x_1 the class of $\mathcal{O}_F \oplus p\mathcal{O}_F$, which is $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} x_0$. We let m be the midpoint of the edge between x_0 and x_1, and e the half-edge between x_0 and m. We have $\mathrm{Stab}_G(x_0) = KZ$, $\mathrm{Stab}_G(e) = IZ$, and $\mathcal{N} = \mathrm{Stab}_G(m) = IZ \sqcup IZ \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$. The matrix $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ exchanges x_0 and x_1.

add picture of the tree for $q = 2$

Theorem 4.5 (Serre)**.** *$KZ *_{IZ} \mathcal{N} \xrightarrow{\sim} G$ ($*$ = pushout in the category of groups).*

Definition 4.2. A **diagram** is (D_0, D_1, r) where D_0 is a smooth KZ-representation, D_1 is a smooth $\mathcal{N}$-representation, and $r : D_1 \hookrightarrow D_0$ is IZ-linear. For convenience, we also impose the conditions that D_0 is admissible and $p \in Z$ acts trivially on it. There are unstated notions of morphisms of diagrams and irreducible diagrams.

Remark. Very often, $D_1 = D_0^{I_1}$. (after restricting to I?)

Example 4.5. If π is an admissible G-representation and $p \in Z$ acts trivially, then we have two diagrams $(\pi|_{KZ}, \pi|_{\mathcal{N}}, \mathrm{id})$ and $(\pi^{K_1}, \pi^{I_1}, \mathrm{incl.})$.

Definition 4.3. The K-**socle** $\mathrm{soc}_K \pi$ is the largest semisimple K-subrepresentation of π. This is contained in π^{K_1}, so it is finite dimensional.

Theorem 4.6 (Pa?)**.** *Given a diagram (D_0, D_1, r), there is an admissible G-representation π such that*

(i) (D_0, D_1, r) is a subdiagram of $(\pi|_{KZ}, \pi|_{\mathcal{N}}, \mathrm{id})$.

(ii) $\mathrm{soc}_K D_0 = \mathrm{soc}_K \pi$.

(iii) π is generated by G acting on D_0.

Moreover, if the diagram is irreducible, then π is irreducible.

The proof is highly non-canonical. The idea is as follows:

Idea. Let $\pi' = \mathrm{inj}_K D_0$ be the K-injective envelope of D_0. It is the largest smooth K-representation whose K-socle is $\mathrm{soc}_K D_0$, and the smallest injective smooth K-representation containing D_0. It is unique up to isomorphism (but not unique isomorphism – this is the non-canonical part of the proof). The action of I on π' extends to an action of $\mathcal{N}$ in such a way that when you restrict to D_0 and then to D_1, you get the given action of $\mathcal{N}$ (The lecturer "cannot explain at the moment" why this is true). By taking amalgams, we get π' is a G-representation satisfying parts (i) and (ii). Then we let π be the G-subrepresentation of π' generated by D_0.

Before going to applications of this theorem, we make an aside on multiplicity free representations.

Definition 4.4. Let H be any group, and let W be a finite dimensional H-representation. W is **mutiplicity free** if all Jordan-Hölder factors occur with multiplicity one.

Fact: Given (H, W) as above, there is a partial order $\leq$ on the set $JH(W)$ of Jordan-Hölder factors such that there is a bijection between H-subrepresentations of W and "downward closed" subsets of $JH(W)$, i.e. $X \subset JH(W)$ such that if $b \in X$ and $a \leq b$, then $a \in X$. The bijection sends W' to $JH(W')$.

insert picture. Arrows in the Hasse diagram indicate non-split extensions. For instance, if we took b and a, the corresponding subrepresentation contains a as a subrepresentation, but not b. We see $\mathrm{soc}_H W \cong a$ and $\mathrm{cosoc}_H W \cong b \oplus d$.

Definition 4.5. The **extension graph**, due to LLLM, has as vertices the Serre weights of $\Gamma = GL_2(\mathbb{F}_q)$, and an edge between V and V' iff $\mathrm{Ext}^1_\Gamma(V, V') \neq 0$. This is a subgraph of $\mathbb{Z}^f$.

insert drawings of these

Example 4.6. We look at supersingular representations for $f = 2$. We use extension graphs and the partial order described above. For $\lambda \in C^\times$ we take the diagram $D(\lambda) = (D_0, D_0^{I_1}, \mathrm{id})$ where D_0 is constructed by choosing representations as follows: insert picture. More over, the actions of I/I_1 on the one-dimensional spaces $\sigma_i^{I_1}$ and $(\sigma_i')^{I_1}$ are interchanged via $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$. D_1 is 8-dimensional and splits into the invariant spaces. Pick basis v_i, v_i'. We define the $\mathcal{N}$-action on D_1 by saying $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} : v_i \mapsto v_i'$ if $i \neq 4$ and $\lambda v_i'$ if $i = 4$. This diagram $D(\lambda)$ is irreducible, so by Theorem 4.6, we get an irreducible admissible G-representation π_λ. We have $\mathrm{soc}_K \pi_\lambda = \sigma_1 \oplus \cdots \oplus \sigma_4$. By comparing to the Barthel-Livné classification, we find that π_λ is supersingular. Furthermore, if $\pi_\lambda \cong \pi_{\lambda'}$, then $\lambda = \lambda'$.

4.4 Lecture 4: Global Picture

The basic idea of how the global picture can inform the local picture is as follows. Let E be a number field and let G/E be reductive. Then we can consider the space of automorphic forms on G in the modern sense, which is roughly $\mathcal{C}(G(E)\backslash G(\mathbb{A}_E), \mathbb{C})$. This space has an action on the right by the full adelic group $G(\mathbb{A}_E)$, but since we want local information, we can also just look at the action by some $G(E_p)$. Later on, we will also quotient $G(E)\backslash G(\mathbb{A}_E)$ on the right by some level subgroup U^p away from p. Since we are also interested in mod p representations, we should replace $\mathbb{C}$ by $\overline{\mathbb{F}_p}$.

We fix E a totally real number field in which the prime p is inert. We also let D be a quaternion algebra over E such that $D \otimes_E E_\nu$ is nonsplit (hence isomorphic to Hamilton's quaternions) at infinite places ν.

Inside $(D \otimes_E \mathbb{A}_E^\infty)^\times = \prod'_{\nu \nmid \infty}(D \otimes_E E_\nu)^\times$ we fix a level $U = \prod_{\nu \nmid \infty} U_\nu = U^p \times U_p$, namely every U with or without decoration is a compact open subgroup of the relevant space.

For k any field, we let $S(U, k) = \{f : D^\times \backslash (D \otimes_E \mathbb{A}_E^\infty)^\times / U \to k\}$. The double coset space appearing here is actually finite. We can also take $S(U^p, k) = \varinjlim_{U_p} S(U^p U_p, k)$, which consists of locally constant functions $D^\times \backslash (D \otimes_E \mathbb{A}_E^\infty)^\times / U^p \to k$. This has a smooth admissible action on the right by $(D \otimes_E E_p)^\times$, because the U_p invariants are the finite dimensional space $S(U^p U_p, k)$.

Now, assume D splits at p. Then $(D \otimes_E E_p)^\times \cong GL_2(E_p) = G$. We introduce Hecke actions by first excluding a finite set Σ of places. Σ consists of p, infinite places, places where $(D \otimes_E E_\nu)^\times \cong GL_2(E_\nu)$, and places where $U_\nu \neq GL_2(\mathcal{O}_\nu)$. For $\nu \notin \Sigma$, we have a Hecke algebra $\mathcal{H}_\nu = k[GL_2(\mathcal{O}_\nu)\backslash GL_2(E_\nu)/GL_2(\mathcal{O}_\nu)]$, and like in the classical modular forms setting, this is $k[T_\nu, S_\nu^{\pm 1}]$. We then form $\mathbb{T} = \bigotimes'_{\nu \notin \Sigma} \mathcal{H}_\nu$. This acts on $S(U^p, k)$, and the action commutes with the action of G. We will use this action, along with "Galois information", to cut out a piece of the G-representation given by $S(U^p, k)$.

Suppose $f \in S(U^p, k)$ is a $\mathbb{T}$-eigenvector; it has local Hecke eigenvalues $T_\nu f = \lambda_\nu f$ and $S_\nu f = \mu_\nu f$.

Fact: If $k = \overline{\mathbb{Q}_p}$ or $\overline{\mathbb{F}_p}$, we get a unique semisimple continuous Galois representation $\rho_f : \mathrm{Gal}(\overline{E}/E) \to GL_2(k)$ satisfying:

(i) ρ_f is unramified at $\nu \notin \Sigma$ (if you restrict to the corresponding inertia group, it is trivial).

(ii) For $\nu \notin \Sigma$, $\rho_f(\mathrm{Fr})$ is well-defined and has characteristic polynomial $X^2 - \lambda_\nu X + q_\nu \mu_\nu$, where q_ν is the cardinality of the residue field at ν.

(iii) $\rho_f(c) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, where c is complex conjugation.

If $k = \overline{\mathbb{F}_p}$ and $\overline{r}$ is a Galois representation satisfying the above conditions for some λ_ν, μ_ν, then we have the maximal ideal $\mathfrak{m}_{\overline{r}} = (T_\nu - \lambda_\nu, S_\nu - \mu_\nu)$ in $\mathbb{T}$. We cut down the space $S(U^p, k)$ by passing to its $\mathfrak{m}_{\overline{r}}$-torsion points; we call the resulting space $\pi(\overline{r})$.

Hope: $\pi(\overline{r})$ corresponds to $\overline{r}|_{D_p}$ under the mod p local Langlands correspondence. Here D_p is the decomposition group.

When $E = \mathbb{Q}$ (and in a slightly different context) this was proved by Emerton in 2011.

We now look at evidence towards this hope. We assume $\overline{r}$ is modular, i.e. $\pi(\overline{r}) \neq 0$.

The weight part of Serre's conjecture can be interpreted as saying $\mathrm{soc}_K \pi(\overline{r}) = \bigoplus_{\sigma \in W(\overline{r})} \sigma$, where $W(\overline{r})$ is a finite set depending only on $\overline{r}|_{D_p}$.

If we look at K_1-invariants, we have $\pi(\overline{r})^{K_1} \cong D_0(\overline{r})$ as representations of $K/K_1 = \Gamma = GL_2(\mathbb{F}_q)$, where $D_0(\overline{r})$ is the largest Γ-representation such that $\mathrm{soc}_K D_0(\overline{r}) = \bigoplus_{\sigma \in W(\overline{r})} \sigma$ *and* the multiplicity of each of these σ in $D_0(\overline{r})$ is just 1.

We note that $D_0(\overline{r})$ was defined along with a family of diagrams $(D_0(\overline{r}), D_0(\overline{r})^{I_1}, \mathrm{id})$. Among these, there is one of the form $(\pi(\overline{r})^{K_1}, \pi(\overline{r})^{I_1}, \mathrm{id})$, and it depends only on $\overline{r}|_{D_p}$.

Remark. In the above results, the Taylor-Wiles method was crucial.

We now discuss work towards showing the existence of admissible supersingular representations. The idea is to show that there is a global $\overline{r}$ such that $\pi(\overline{r})$ contains a supersingular. In other words, for Serre weight V, we want $\varphi_{1,0} \in \mathcal{H}_G(V)$ to act nilpotently on $\mathrm{Hom}_K(V, \pi(\overline{r}))$. We have $\mathrm{Hom}_K(V, \pi(\overline{r})) = \mathrm{Hom}_K(V, S(U^p, k)[\mathfrak{m}_{\overline{r}}]) = S(U, V^*)[\mathfrak{m}_{\overline{r}}]$, where $U = U^p GL_2(\mathcal{O}_p)$. Next, we lift the Serre weight V to a characteristic 0 representation $\tilde{V}$ by replacing the instances of $\mathbb{F}_p$ by $\mathbb{Q}_p$. Similarly, we lift the local Hecke eigenvalues. We have $S(U, \tilde{V}^*)$, which is more or less like a space of classical modular/automorphic forms. The system of eigenvalues then produces a Galois representation $r : \mathrm{Gal}(\overline{E}/E) \to GL_2(\overline{\mathbb{Q}_p})$ that lifts $\overline{r}$. Furthermore, we know that $r|_{D_p}$ is crystalline, and its Hodge-Tate weights correspond to $\tilde{V}$. We now carry out the following steps:

(1) Construct a modular $\overline{r}$ such that $\overline{r}|_{D_p}$ is irreducible.

(2) By way of contradiction, suppose $\varphi_{1,0}$ had a non-zero eigenvalue. Then one can lift it to characteristic 0 and use p-adic Hodge theory to deduce that $r|_{D_p}$ is reducible, contradicting the choice made in the first step.