

Fargues Fontaine Curve

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Preface

There will be some gaps in explanation, either due to the lecturer’s admission or my own lack of understanding. In particular, many “proofs” are sketches of proofs. Gaps due to my own misunderstanding will be indicated by three red question marks: **???**. More generally, my own questions about the material will also be in red. Things like “**Question**” will be questions posed by the lecturer. Feel free to reach out to me with explanations.

As a course specific note, at some point I start using the abbreviation CACNA for a complete algebraically closed non-archimedean field.

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1 Jan 5 - Perfect Rings and Tilting

1.1 Perfect Rings

Definition 1.1. Let R be an \mathbb{F}_p algebra. We say R is **perfect** if the Frobenius map $\varphi : R \rightarrow R, x \mapsto x^p$ is an isomorphism.

Example 1.1. 1. Perfect (characteristic p) field.

2. $\mathbb{F}_p[x^{1/p^\infty}] = \bigcup_{n \geq 0} \mathbb{F}_p[x^{1/p^n}]$; the free perfect ring on 1 generator. (All monomials have p th roots.)
3. Any limit or colimit of perfect \mathbb{F}_p algebras (It's enough to check for pullbacks, arbitrary products, pushouts, and arbitrary coproducts (tensor products). In all of these cases, it's enough to check on generators, in which case the p th roots are obvious.)
4. Any integrally closed domain whose fraction field is perfect (e.g. algebraically closed) ($x^p - a$ is monic and has a solution in the fraction field).
5. If R is perfect, I is a finitely generated ideal, then R_I^\wedge (the I -completion) is perfect. (Can try to take p th roots of a compatible system of elements in $R/I^n \dots$ or topologically, show $\varphi : R \rightarrow R$ is I -adically continuous.)

The inclusion of the category of perfect \mathbb{F}_p -algebras into the category of all \mathbb{F}_p -algebras has both adjoints. The left adjoint $R \mapsto R_{perf}$ is the colimit of $R \xrightarrow{\varphi} R \xrightarrow{\varphi} \dots$. The right adjoint $R \mapsto R^{perf}$ is the limit of $\dots \xrightarrow{\varphi} R \xrightarrow{\varphi} R$.

The idea of the colimit perfection is to add all p -power roots, and the idea of the limit perfection is to pick things that have all p -power roots, although these ideas can sometimes not match reality.

The adjunction means that if R is an \mathbb{F}_p -algebra and P is a perfect \mathbb{F}_p -algebra, we have $\text{Hom}(R_{perf}, P) = \text{Hom}(R, P)$ and $\text{Hom}(P, R) = \text{Hom}(P, R^{perf})$.

To see the first adjunction, note that by the definition of a colimit, $f \in \text{Hom}(R_{perf}, P)$ corresponds to the data of maps $f_n : R \rightarrow P$ where $f_n = f_{n+1} \circ \varphi_R$. But maps of \mathbb{F}_p -algebras are “Frobenius equivariant”, so $f_{n+1} \circ \varphi_R = \varphi_P \circ f_n$. Since P is perfect, φ_P is an isomorphism, so $f_{n+1} = \varphi_P^{-1} \circ f_n$. Thus the sequence is determined by f_1 via $f_n = \varphi_P^{-n+1} \circ f_1$.

To see the second adjunction, note that an element $x \in P$ must have all p -power roots, so it must be sent to an element $y \in R$ that also has all p -power roots, since if $f(x) = y$, then $f(x^{1/p})^p = f(x) = y$. Alternatively, since R^{perf} is a limit, we know $f \in \text{Hom}(P, R^{perf})$ corresponds to the data of maps $f_n : P \rightarrow R$ where $f_n = \varphi_R \circ f_{n+1}$. As before, we can use $\varphi_R \circ f_{n+1} = f_{n+1} \circ \varphi_P$, and then since φ_P is an isomorphism, we get $f_{n+1} = f_n \circ \varphi_P^{-1}$. Thus f_1 determines the sequence f_n via $f_n = f_1 \circ \varphi_P^{-n+1}$.

Example 1.2. 1. $\mathbb{F}_p[x^{1/p^\infty}] = \mathbb{F}_p[x]_{perf}$. As said above, when you take the colimit, the data involved is a ring with an element with a p th root, a p^2 th root, and so on.

2. $\mathbb{F}_p[x]^{perf} = \mathbb{F}_p$. As said above, the data of the limit is the choice of an element in $\mathbb{F}_p[x]$ that has all p -power roots; only the constants satisfy this. In general, if R is Noetherian, then R^{perf} is a finite product of fields.
3. If φ is surjective (R is called semiperfect) then we get a surjection $R^{perf} \twoheadrightarrow R$, so R^{perf} is “larger” than R , which is somewhat contrary to the above intuition of “taking the things which have all p -power roots”.
4. If P is perfect and I is a finitely generated ideal, then $R = P/I$ is semiperfect, and $R^{perf} = P_I^\wedge$. (Exercise: prove this, and also find R_{perf} .) Solution: R is semiperfect since the composition $P \xrightarrow{\varphi} P \rightarrow P/I$ is surjective with kernel $\varphi^{-1}(I) \supset I$, so it descends to a surjective map $R \xrightarrow{\varphi} R$. To prove $R^{perf} = P_I^\wedge$, we first note that we can compute P_I^\wedge as $\varprojlim_n P/I^{p^n}$. Then, because I is finitely generated, $\phi : P/I \rightarrow P/I^p$ is an isomorphism. Then we can identify the corresponding limit diagrams using iterations of Frobenius:

$$\begin{array}{ccccc} \dots & \xrightarrow{\varphi} & P/I & \xrightarrow{\varphi} & P/I \\ \varphi^n \downarrow & & \downarrow \varphi & & \downarrow = \\ \dots & \longrightarrow & P/I^p & \longrightarrow & P/I \end{array}$$

Here the arrows with double line marks are meant to be isomorphisms. The desired isomorphism $R^{perf} = P_I^\wedge$ follows. To compute R_{perf} , the idea is that the colimit is modding out by elements $x \in P$ such that $\phi^n(x) \in I$ eventually. But the sequence $\phi^n(x) = x^{p^n}$ is cofinal with the sequence x^n , so we are modding out by the radical $\text{rad}(I)$. Thus $R_{perf} = P/\text{rad}(I)$. (this may be wrong)

Example 1.3. As a concrete example of the previous case, let $P = \mathbb{F}_p[x^{1/p^\infty}]$, $I = (x)$, $R = \mathbb{F}_p[x^{1/p^\infty}]/(x)$. Then $R^{perf} = \mathbb{F}_p[x^{1/p^\infty}]_x^\wedge$, which consists of sums

$$\sum_{i \in \mathbb{Z}[1/p]_{\geq 0}} a_i x^i,$$

with $a_i \in \mathbb{F}_p$, and for all $N \geq 0$, there are only finitely many $i \leq N$ with $a_i \neq 0$. So $x + x^{p+1/p} + x^{p^2+1/p^2} + \dots$ is allowed, while $x + x^{1/p} + x^{1/p^2} + \dots$ isn't.

1.2 Witt Vectors

Perfect \mathbb{F}_p -algebras have a unique lift to characteristic 0.

Theorem 1.1. Given any perfect \mathbb{F}_p -algebra R , there is a unique (up to unique isomorphism) p -adically complete and p -torsion free ring \tilde{R} equipped with an isomorphism $\tilde{R}/p \xrightarrow{\sim} R$. This construction is functorial in R , and \tilde{R} is the ring $W(R)$ of Witt vectors of R .

Proof. See chapter 2 of Serre's "Local Fields".



Example 1.4. 1. $R = \mathbb{F}_p$ gives $W(R) = \mathbb{Z}_p$.

2. $R = \mathbb{F}_p[x^{1/p^\infty}]$ gives $W(R) = \mathbb{Z}_p[x^{1/p^\infty}]_p^\wedge$.

3. $R = \mathbb{F}_p[x^{1/p^\infty}]_x^\wedge$ gives $W(R) = \mathbb{Z}_p[x^{1/p^\infty}]_{(p,x)}^\wedge$. The elements of this ring are series $\sum_{i \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}} a_i x^i$, where $a_i \in \mathbb{Z}_p$ and for all $N \geq 0$ and for all $\varepsilon > 0$, there are finitely many $i \leq N$ such that $|a_i| > \varepsilon$.

4. For R_1, R_2 perfect, $W(R_1 \otimes R_2) = (W(R_1) \otimes W(R_2))_p^\wedge$. **why?**

Structure of $W(R)$: There is a unique multiplicative (but not additive) map $[-] : R \rightarrow W(R)$ that is a section of the projection $W(R) \rightarrow R$.

Corollary 1.1. For any $x \in W(R)$, there is a unique sequence $x_0, x_1, x_2, \dots, \in R$ such that $x = \sum_{i \geq 0} [x_i] p^i$. So, as a set, $W(R) \cong \prod_{i \geq 0} R$.

Proof. Let π be the isomorphism $W(R)/p \rightarrow R$. Then $[-]$ being a section of $W(R) \rightarrow W(R)/p \xrightarrow{\pi} R$ means $\pi([a] + (p)) = a$ for all $a \in R$. Now, given $x \in W(R)$, suppose $a \in R$ satisfies $x - [a] \in (p)$. Then $\pi(x + (p)) = \pi([a] + (p)) = a$, so such an a is determined by x . Conversely, we have $x - [\pi(x + (p))] \in (p)$ since $\pi([\pi(x + (p))] + (p)) = \pi(x + (p))$, so that $x - [\pi(x + (p))]$ is in the kernel of $W(R) \rightarrow W(R)/p \xrightarrow{\pi} R$, which is (p) . Thus, for any $x \in W(R)$, there is a unique $x_0 \in R$ such that $x - [x_0] \in (p)$. Write $x - [x_0] = px'$. Such an x' is unique since $W(R)$ is p -torsion free. Then we may define $x_1 = x'_0$ and repeat the process forever.



Note that in the course of this proof we showed that $x \mapsto x_0$ is a ring homomorphism, in particular it is the projection $W(R) \rightarrow W(R)/p \xrightarrow{\sim} R$. The kernel of this map is (p) , and in particular we have $[a] + [b] - [a+b] \in (p)$ for all $a, b \in R$.

Exercise: the image of $[-]$ is the set of elements of $W(R)$ that have all p -power roots. Solution: Since R is perfect, it has all p -power roots, and $[-]$ is multiplicative, so anything in $[R]$ has all p -power roots; explicitly, if $x = [a]$, then $x = [a^{1/p^n}]^{p^n}$. Conversely, we claim that if $x \in W(R)$ has a p^n th root for $n \geq 1$, then $[x_i] = 0$ for $1 \leq i \leq n$. This implies that if x has all p -power roots, it must equal $[x_0]$. To prove the claim, we induct on n . In the base case, let $y = x^{1/p}$ and write $y = [y_0] + py'$. Then

$$x = y^p = [y_0^p] + p^2 \left(p^{p-2}(y')^p + \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} [y_0^i] p^{p-i-1} (y')^{p-i} \right),$$

where we use $p \geq 2$ and $p \mid \binom{p}{i}$ for $0 < i < p$. This computation shows that if x has a p th root, then we can write $x = [x_0] + p^2 x'$. Inductively, if x has a p^n th root, we can write its p th root y in the form $y = [y_0] + p^n y'$ and then expand y^p to show that $[x_n] = 0$.

Analogy: $W(R)$ is like a ring of power series over R with variable p .

How to add Witt vectors: Suppose $x = \sum [x_i]p^i$ and $y = \sum [y_i]p^i$. We know $x + y = \sum [z_i]p^i$ for some $z_i \in R$. In fact, for each n , z_n can be expressed as an element of

$$\mathbb{F}_p[x_0^{1/p^\infty}, \dots, x_n^{1/p^\infty}, y_0^{1/p^\infty}, \dots, y_n^{1/p^\infty}].$$

Key fact: the polynomial expressing z_n in this way is homogeneous of degree 1, where each x_i, y_i has degree 1.

Example 1.5. $z_0 = x_0 + y_0$. This follows from the fact that $x \mapsto x_0$ is a homomorphism.

$z_1 = x_1 + y_1 - \sum_{0 < i < p} \frac{1}{p} \binom{p}{i} x_0^{i/p} y_0^{1-i/p}$. Exercise: prove this. Solution: Writing $x = [x_0] + px'$ and $y = [y_0] + py'$, we have that z_1 is the 0 component of $\frac{1}{p}([x_0] + [y_0] - [x_0 + y_0]) + x' + y'$. Since taking the 0 component is a homomorphism, and since by definition $x_1 = x'_0$, we are reduced to computing the 0 component of $\frac{1}{p}([x_0] + [y_0] - [x_0 + y_0])$, or equivalently, the 1 component of $[x_0] + [y_0] - [x_0 + y_0]$. To simplify notation, we use a, b instead of x_0, y_0 . Write $[a] + [b] - [a + b] = p[c(a, b)] + p^2c'$. So

$$\begin{aligned} [a] + [b] &\equiv [a + b] + p[c(a, b)] \pmod{p^2} \\ &\text{raise both sides to } p \text{ power} \\ ([a] + [b])^p &\equiv ([a + b] + p[c(a, b)])^p \pmod{p^2} \\ [a^p] + [b^p] + p \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} [a^i b^{p-i}] &\equiv [a + b]^p \pmod{p^2} \\ [a^p] + [b^p] + p \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} [a^i b^{p-i}] &\equiv [a^p + b^p] \pmod{p^2} \\ [a^p] + [b^p] - [a^p + b^p] &\equiv -p \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} [a^i b^{p-i}] \pmod{p^2} \\ p[c(a^p, b^p)] &\equiv -p \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} [a^i b^{p-i}] \pmod{p^2} \\ [c(a^p, b^p)] &\equiv - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} [a^i b^{p-i}] \pmod{p}. \end{aligned}$$

Now, we can apply the isomorphism $W(R)/p \rightarrow R$, which we know sends $[a] + (p)$ to a , to get

$$\begin{aligned} c(a^p, b^p) &= - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} a^i b^{p-i}; \\ c(a, b) &= - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} a^{i/p} b^{1-i/p}. \end{aligned}$$

1.3 Tilting

Definition 1.2. For any ring R , its **tilt** R^\flat is $(R/p)^{perf}$.

Proposition 1.1. If R is p -complete, then $\varprojlim(\cdots \xrightarrow{\varphi} R \xrightarrow{\varphi} R) \xrightarrow{\sim} R^\flat$.

Proof. The inverse limit is being taken in multiplicative monoids, since φ is not necessarily additive on R (In fact, the lecturer does not use φ , and I only found out the reason for this after the fact). The isomorphism is in one direction easy to define: take a compatible sequence of elements in R and reduce mod p . In the other direction, if we start with some sequence (a_i) of mod p elements, we send it to $b_i = \lim_n \tilde{a}_{i+n}^{p^n}$, where \tilde{a}_i are arbitrary lifts of a_i . todo: show this is the inverse 

Example 1.6. 1. $R = \mathbb{Z}_p$ has $R^\flat = \mathbb{F}_p$, since $R/p = \mathbb{F}_p$ is already perfect.

2. $R = \mathbb{Z}_p[p^{1/p^\infty}]_p^\wedge = (\mathbb{Z}_p[x^{1/p^\infty}]/(x-p))_p^\wedge$. Then $R/p = \mathbb{F}_p[x^{1/p^\infty}]/(x)$ and $R^\flat = \mathbb{F}_p[x^{1/p^\infty}]_x^\wedge$.

Remark. If R is p -complete, there is a natural multiplicative map $(-)^{\sharp} : R^\flat \rightarrow R$, coming from the isomorphism in the proposition. For example, in the second example above, $x^{\sharp} = p$.

Theorem 1.2. The functor W from perfect \mathbb{F}_p -algebras to p -complete rings is left adjoint to the tilting functor from p -complete rings to perfect \mathbb{F}_p -algebras; if A is perfect/ \mathbb{F}_p and R is p -complete, then $\text{Hom}(W(A), R) = \text{Hom}_{\mathbb{F}_p}(A, R^\flat)$.

Remark. If A is perfect/ \mathbb{F}_p and R is p -complete, then there is a natural map $\text{Hom}(W(A), R) \rightarrow \text{Hom}_{\mathbb{F}_p}(A, R/p)$ given by reducing mod p . Since A is perfect, the second hom set can be identified with $\text{Hom}_{\mathbb{F}_p}(A, R^\flat)$. So, the claim is that this composite map $\text{Hom}(W(A), R) \rightarrow \text{Hom}_{\mathbb{F}_p}(A, R^\flat)$ is an isomorphism. We will prove this next time.

2 Jan 7

2.1 Witt vectors and tilting

Theorem 2.1. *The functor W from perfect \mathbb{F}_p -algebras to p -complete rings is left adjoint to the tilting functor from p -complete rings to perfect \mathbb{F}_p -algebras; if A is perfect/ \mathbb{F}_p and R is p -complete, then $\text{Hom}(W(A), R) = \text{Hom}_{\mathbb{F}_p}(A, R^\flat)$.*

Proof. Suppose $A = \mathbb{F}_p[x^{1/p^\infty}]$. Then $W(A) = \mathbb{Z}_p[x^{1/p^\infty}]_p^\wedge$. Then $\text{Hom}(W(A), B) = \varprojlim_{\phi} B$ and $\text{Hom}(A, B^\flat) = B^\flat$. The natural map $\varprojlim_{\phi} B \rightarrow B^\flat$ is an isomorphism, so we have proved the claim for this specific choice of A . We want to reduce to this case. It will be enough to show that W preserves colimits. Recall that for perfect \mathbb{F}_p -algebras A_1, A_2 , we have $W(A_1 \otimes A_2) = (W(A_1) \otimes W(A_2))_p^\wedge$. More generally, we need to show that if we have a diagram $A_1 \leftarrow A \rightarrow A_2$ of perfect \mathbb{F}_p -algebras, where A is arbitrary, then the natural map $(W(A_1) \otimes_{W(A)} W(A_2))_p^\wedge \rightarrow W(A_1 \otimes_A A_2)$ is an isomorphism. It is an isomorphism mod p , so the difficulty is in showing that $W(A_1) \otimes_{W(A)} W(A_2)$ is p -torsion free. There are two ways to justify this. One is using the following theorem:

Theorem 2.2 (Bhatt-Scholze). *Given a diagram as above, $\text{Tor}_i^A(A_1, A_2) = 0$ for $i > 0$.*

Another justification is as follows. $W(A)$ is a perfect δ -ring, so that $W(A_1) \otimes_{W(A)} W(A_2)$ is also a perfect δ -ring. It is then a general fact that perfect δ -rings are  p -torsion free.

2.2 The map θ

The counit of the adjunction is a map $\theta : W(B^\flat) \rightarrow B$. Explicitly, for $b \in B^\flat$, we have $\theta([b]) = b^\sharp$.

Remark. If B/p is semiperfect, then $B^\flat \rightarrow B/p$ is surjective, so by Nakayama, θ is surjective.

Example 2.1. If $B = \mathbb{Z}_p[p^{1/p^\infty}]_p^\wedge$, then $B^\flat = \mathbb{F}_p[x^{1/p^\infty}]_x^\wedge$ and $W(B^\flat) = \mathbb{Z}_p[x^{1/p^\infty}]_{(p,x)}^\wedge$. Then $\theta(x^{1/p^n}) = p^{1/p^n}$.

2.3 Perfectoid rings

Definition 2.1. Let A be a p -complete ring. A is **perfectoid** if $A \cong W(P)/\xi$, where P is a perfect \mathbb{F}_p -algebra and $\xi = \sum [\xi_i] p^i$ with $\xi_1 \in P^\times$.

Remark. 1. If $A = W(P)/\xi$, then $A/p = P/\xi_0$, so WLOG we may assume P is ξ_0 -adically complete. In this case, $P = A^\flat$. Thus we may equivalently define perfectoid rings as p -complete rings such that $\theta : W(A^\flat) \rightarrow A$ is surjective and $\ker(\theta)$ is generated by ξ with $\xi_1 \in (A^\flat)^\times$.

2. Any perfect \mathbb{F}_p -algebra is perfectoid by taking $\xi = p$.

3. ξ is not a zero-divisor. Proof: Suppose $\xi x = 0$. Note that ξ maps to a unit in $W(P[1/\xi_0])$, so x maps to 0 in $W(P[1/\xi_0])$. In other words, if we write $x = \sum [x_i]p^i$, then all x_i are ξ_0 -power torsion. But in P , ξ_0 -power torsion is the same as ξ_0 -torsion: if $\xi_0^N y = 0$, then multiply by some power of ξ_0 on both sides to get $\xi_0^{p^n} y = 0$, then since Frobenius is an isomorphism, we can take p^n th roots to get $\xi_0 y^{1/p^n} = 0$, and then we may multiply by an appropriate power of y to get $\xi_0 y = 0$. So $\xi_0 x_i = 0$ for all i , or $[\xi_0]x = 0$. Hence $\xi x = (\sum_{i \geq 1} [\xi_i]p^i)x$, which is px times a unit since ξ_1 is a unit, so $px = 0$, so $x = 0$.
4. $A = W(P)/\xi$ is p -torsion free iff P is ξ_0 -torsion free. To show this we use the torsion exchange lemma: if B is a ring with nonzerodivisors x, y , then $(B/x)[y] \cong (B/y)[x]$ (where brackets denote torsion). This is true because both are H_1 of a Koszul complex on x, y . Applying the torsion exchange lemma to $B = W(P)$, $x = p$, and $y = \xi$, we get $A[p] = (W(P)/\xi)[p] = P[\xi_0]$.

Note. The fact that ξ is not a zero-divisor implies that derived completions are well-behaved.

Example 2.2. Let $P = \mathbb{F}_p[x^{1/p^\infty}]$ and $\xi = p - [x]$. Then $W(P)/\xi = \mathbb{Z}_p[p^{1/p^\infty}]_p^\wedge$.

Example 2.3. $P = \mathbb{F}_p[q^{1/p^\infty}]_{(q-1)}^\wedge$, $\xi = 1 + [q^{1/p}] + \cdots + [q^{(p-1)/p}]$. We claim P is ξ_0 -adically complete and that $\xi_1 \in P^\times$. Indeed, $\xi_0 = 1 + q^{1/p} + \cdots + q^{(p-1)/p} = (q^{1/p} - 1)^{p-1}$, so by definition P is ξ_0 -complete. To see $\xi_1 \in P^\times$, we may quotient by the Jacobson radical, hence set all powers of q to be 1, so $\xi = p$ so $\xi_1 = 1$. Thus we can form the perfectoid ring $W(P)/\xi = \mathbb{Z}_p[\zeta_{p^\infty}]_p^\wedge$. Note that $(q^{1/p^n})^\sharp = \zeta_{p^n}$.

Example 2.4. Non-examples:

1. \mathbb{Z}_p is not perfectoid, because θ is an isomorphism.
2. \mathbb{Z}/p^n is not perfectoid for $n > 1$, since $\ker(\theta) = (p^n)$.

Claim 2.1. Let R be perfectoid. Then there exists $u \in R^\times$ and $\alpha \in R^\flat$ such that $pu = \alpha^\sharp$; in other words, pu has a compatible system of p -power roots.

Proof. Write $R = W(R^\flat)/\xi$. We have $\xi = [\xi_0] + p \cdot (\text{unit})$. Thus $\theta([\xi_0]) = \xi_0^\sharp = p \cdot (\text{unit})$.

Example 2.5. If R is perfectoid, then $R\langle x^{1/p^\infty} \rangle = (R \otimes_{\mathbb{Z}} \mathbb{Z}_p[x^{1/p^\infty}])_p^\wedge$ is perfectoid.

3 Jan 9

Proposition 3.1. *Let R be p -torsion free containing ω such that*

1. $\omega^p \in pR^\times$.
2. $R/\omega \xrightarrow{\varphi} R/\omega^p = R/p$ is an isomorphism.

Then R_p^\wedge is perfectoid.

Proof. WLOG, suppose R is p -complete. Condition two implies that R is semiperfect, so $R^\flat \rightarrow R/p$ is surjective. Thus, lift $\omega \bmod p$ to some α , so that $\alpha^\sharp = \omega + py \in \omega R^\times$. Then, WLOG, we have $\omega = (\omega^\flat)^\sharp$. Now, letting \bar{u} be the unit such that $\omega^p = p\bar{u}$, we have $\theta([\omega^\flat]^p) = \omega^p = p\bar{u}$. Lift \bar{u} to $u \in W(R^\flat)^\times$. Then $[\omega^\flat]^p - pu \in \ker \theta$, and this is a valid choice of ξ to show R is perfectoid. So we want to show $W(R^\flat)/([\omega^\flat]^p - pu) \xrightarrow{\theta} R$ is an isomorphism. Both sides are p -complete and p -torsion free; the LHS is p -torsion free because $\omega^{\flat p}$ is a nonzerodivisor. Thus, to show the map is an isomorphism, we may work mod p and show $R^\flat/\omega^{\flat p} \xrightarrow{\sim} R/p$. Explicitly, this means that we want to show that, given a sequence $(x_0, x_1, \dots) \in R$ with $x_{i+1}^p = x_i$ for all i , then the following are equivalent:

1. $p \mid x_0$.
2. $\omega^p \mid x_0$.
3. $\omega^{p^{1-i}} \mid x_i$ for all i .

Clearly conditions 1 and 2 are equivalent since $\omega^p \in pR^\times$. Since Frobenius is an isomorphism $R/\omega \rightarrow R/\omega^p$, we get $\omega^p \mid x_0$ iff $\omega \mid x_1$. In fact, we can quotient the isomorphism by $\omega^{1/p^{n+1}}$ to obtain $\omega^{1/p^{n+1}} \mid y$ iff $\omega^{1/p^n} \mid y^p$. This gives the equivalence between 2 and 3, so we are done. 

I'm confused by a lot of the steps in this proof, it's so over

3.1 p -root closure

Definition 3.1 (P. Roberts). Let R be p -torsion free. R is **p -root closed** if for all $x \in R[\frac{1}{p}]$, we have $x^p \in R$ implies $x \in R$.

Example 3.1. 1. Any integrally closed domain is p -root closed.

2. $\mathbb{Z} \oplus (px) \subset \mathbb{Z}[x]$ is p -root closed.
3. $\mathbb{Z}[px, x^p]$ is not p -root closed.

Definition 3.2. Let A be a p -torsion free ring. Let $A^{+,p}$ (notation possibly not standard) be the set of $x \in A[\frac{1}{p}]$ such that $x^{p^n} \in A$ for large enough n . Then $A^{+,p}$ is the smallest p -root closed subring of $A[\frac{1}{p}]$ containing A , and it is called the **p -root closure** of A .

Exercise: prove $A^{+,p}$ is a ring. (Hint: use divisibility properties of binomial coefficients.)

Claim 3.1. Let R be a p -torsion free ring with $\omega \in R$ such that $\omega^p \in pR^\times$. Then the following are equivalent:

1. R is p -root closed.
2. $R/\omega \xrightarrow{\phi} R/\omega^p$ is injective.

Proof. (1 \Rightarrow 2) Fix $x \in R$ such that $\omega^p \mid x^p$. Then $x/\omega \in R[\frac{1}{p}]$, so by assumption, $x/\omega \in R$, i.e. $\omega \mid x$.

(2 \Rightarrow 1) Fix $y \in R[\frac{1}{p}]$ such that $y^p \in R$. Write $y = z/\omega^i$ for i minimal and $z \in R$. If $i > 0$, then $\omega^{ip} \mid z^p$, so $\omega^p \mid z^p$, so $\omega \mid z$, meaning i is not minimal. Hence $i = 0$ so $y \in R$. 

Definition 3.3. Let P be a perfect \mathbb{F}_p -algebra. Let $\xi = \sum [\xi_i]p^i \in W(P)$. Say ξ is **distinguished** if P is ξ_0 -complete and $\xi_1 \in P^\times$.

Claim 3.2. 1. If ξ is distinguished and $u \in W(P)^\times$, then ξu is also distinguished.

2. If ξ, ξ' are distinguished and $\xi \mid \xi'$, then $\xi'/\xi \in W(P)^\times$.

Theorem 3.1. Let R be p -torsion free and p -complete. Then R is perfectoid iff the following are all true:

1. There is $\omega \in R$ such that $\omega^p \in pR^\times$.
2. R is p -root closed.
3. R/p is semiperfect.

Proof. Assume 1-3 hold. Conditions 1 and 2 imply $R/\omega \rightarrow R/\omega^p$ is injective, and condition 3 implies it is surjective. Then, by Proposition 3.1, R is perfectoid.

Thus assume R is perfectoid. It is enough to show condition 2. Choose $\omega^\flat \in R^\flat$ such that $((\omega^\flat)^\sharp)^p = p\bar{u}$ for $\bar{u} \in R^\times$, and let $\omega = (\omega^\flat)^\sharp$. It is enough to show $R/\omega \rightarrow R/\omega^p$ is an isomorphism. Lift \bar{u} to $u \in W(R^\flat)^\times$, so that $\xi = [\omega^\flat]^p - pu$ is distinguished and in $\ker \theta$. Then $R = W(R^\flat)/\xi$, so $R/\omega \rightarrow R/\omega^p$ is $R^\flat/\omega^\flat \rightarrow R^\flat/\omega^{\flat p}$, and this is an isomorphism since R^\flat is perfect. 

Example 3.2. Let R be a domain. Let R^+ be the absolute integral closure, i.e. the integral closure in the algebraic closure of the fraction field of R . Then $(R^+)_p^\wedge$ is perfectoid. Applying this construction to \mathbb{Z}_p , we obtain that $\mathcal{O}_{\mathbb{C}_p}$ is perfectoid.

3.2 Category of perfectoid rings

There is no initial object in the category of perfectoid rings.

The category of perfectoid rings is equivalent to the category of “perfect prisms”, i.e. pairs (P, I) , where P is a perfect ring and I is an ideal of $W(P)$ generated by a distinguished element. Given a perfectoid R , we send it to $(R^\flat, \ker(\theta))$. Given a perfect prism (P, I) , we send it to $W(P)/I$.

Note. There is no category of rings equivalent to the category of all prisms.

Corollary 3.1. *The category of perfectoid rings has pushouts; in particular, given $R_1 \leftarrow R \rightarrow R_2$ maps of perfectoids, we have $(R_1 \otimes_R R_2)^\wedge_p$ is perfectoid.*

Proof. By the “rigidity of prisms”, we may write $R = W(R^\flat)/I$, $R_1 = W(R_1^\flat)/I$, $R_2 = W(R_2^\flat)/I$ for the same ideal I . Then $(R_1 \otimes_R R_2)^\wedge_p = (W(R_1^\flat) \otimes_{W(R^\flat)} W(R_2^\flat))/I = W(R_1^\flat \otimes_{R^\flat} R_2^\flat)/I$. 

4 Jan 12

4.1 Tilting equivalence

Proposition 4.1. *Let R be perfectoid, $\xi \in W(R^\flat)$ generates $\ker \theta$. Then there is an equivalence between the category of perfectoid R -algebras and the category of ξ_0 -adically complete perfect R^\flat -algebras, given by sending a perfectoid R -algebra R' to its tilt R'^\flat .*

Proof. Under the equivalence of the category of perfectoid rings with perfect prisms, the perfectoid R -algebras are sent to perfect prisms (P, I) with a map from $(R^\flat, \ker \theta)$. Since I is required to be generated by a distinguished element, and since $\xi \in \ker(\theta)$ is mapped into I , we automatically get that such P are ξ_0 -adically complete. 

Example 4.1. The category of perfectoid $\mathbb{Z}_p[p^{1/p^\infty}]_p^\wedge$ -algebras is equivalent to the category of t -complete $\mathbb{F}_p[t^{1/p^\infty}]_t^\wedge$ -algebras.

Theorem 4.1. *Let R be perfectoid, ξ generates $\ker \theta$. There is an equivalence between the category of finite étale $R[\frac{1}{p}]$ -algebras and the category of finite étale $R^\flat[\frac{1}{\xi_0}]$ -algebras, given by taking an algebra T , taking the integral closure \bar{R}_T of R in T , and then forming $\bar{R}_T^\flat[\frac{1}{\xi_0}]$. Thus, the étale fundamental groups of $R[\frac{1}{p}]$ and $R^\flat[\frac{1}{\xi_0}]$ are identified.*

The proof is outside of the scope of the course.

Example 4.2. As a special case, $\text{Gal}(\mathbb{Q}_p(p^{1/p^\infty})) \cong \text{Gal}(\mathbb{F}_p((t)))$.

Example 4.3. We have the following special classes of perfectoid rings:

1. Perfect \mathbb{F}_p -algebras.
2. p -torsion free perfectoid rings.

The following proposition says that any perfectoid ring can be glued together from these.

Proposition 4.2. *If R is perfectoid, then $R/R[p^\infty]$ is p -torsion free and perfectoid, and there is a pullback diagram*

$$\begin{array}{ccc} R & \twoheadrightarrow & R/R[p^\infty] \\ \downarrow & & \downarrow \\ (R/p)_{red} & \twoheadrightarrow & (R/(R[p^\infty], p))_{red} \end{array}$$

where all maps are surjective and the bottom two rings are perfect \mathbb{F}_p -algebras.

Proof. For any ring A and ideals I, J such that $I \cap J = 0$, there is a pullback

$$\begin{array}{ccc} A & \longrightarrow & A/I \\ \downarrow & & \downarrow \\ A/J & \longrightarrow & A/(I+J) \end{array}$$

We apply this to the ring R^\flat , $I = \xi_0$ -power torsion (equivalently, the $x \in R^\flat$ with $\xi_0^{1/p^n} x = 0$ for all n), $J = \bigcup_{n \geq 0} (\xi_0^{1/p^n})$. Then $I \cap J = 0$ and we can apply the remark. Then “untilting” (apply W and mod out by the same ξ everywhere) gives the desired pullback diagram (“it’s a bit of work to do so”). (an R^\flat -algebra T untilts to a p -torsion free ring iff ξ_0 is not a zero divisor, and T untilts to a (perfect?) \mathbb{F}_p -algebra iff $\xi_0 = 0$) 

Example 4.4. A perfectoid ring that is neither a perfect \mathbb{F}_p -algebra nor p -torsion free: $\mathbb{Z}_p[p^{1/p^\infty}, t^{1/p^\infty}]_p^\wedge / (p^a t^b, a > 0, b > 0)$.

4.2 Valuation rings

Definition 4.1. A **valuation ring** is a domain $V \neq 0$ such that for all nonzero $x, y \in V$, we have $x | y$ or $y | x$.

As a consequence, all finitely generated ideals of a valuation ring are principal.

Example 4.5. 1. Any discrete valuation ring.

2. $\mathcal{O}_{\mathbb{C}_p}$, or more generally, if K is a field with a non-archimedean absolute value, then the set of elements \mathcal{O}_K with absolute value ≤ 1 .
3. $\mathbb{Z}_p \oplus t\mathbb{Q}_p[[t]]$ as a subring of $\mathbb{Q}_p[[t]]$.

Remark. 1. V is local, $\text{Spec}(V)$ is totally ordered, and all radical ideals are prime.

2. In any scheme, specialization can be tested via valuation rings. Algebraically, if A is a local domain, then there is an inclusion $A \hookrightarrow A' \hookrightarrow \text{Frac}(A)$ such that A' is a valuation ring and $\mathfrak{m}_A \hookrightarrow \mathfrak{m}_{A'}$.

Definition 4.2. Let K be a field. A **valuation** on K consists of a totally ordered group Γ and a map $\nu : K \rightarrow \Gamma \cup \{\infty\}$ such that

1. $\nu(x) = \infty$ iff $x = 0$.
2. $\nu(xy) = \nu(x) + \nu(y)$.
3. $\nu(x+y) \geq \min(\nu(x), \nu(y))$.

The **value group** is $\nu(K^\times)$.

Example 4.6. If we have a field K with valuation ν , then we can form a valuation ring by taking $V = \{x \in K \mid \nu(x) \geq 0\}$. Conversely, if we have a valuation ring V , we have the field $K = \text{Frac}(V)$ and a valuation $K \rightarrow \Gamma = K^\times/V^\times \cup \{\infty\}$, where Γ is given an ordering by the divisibility relation. Γ is exactly the value group of this valuation.

Definition 4.3. A valuation ring V has **rank one** if any of the following (equivalent) conditions hold:

1. $\mathfrak{m}_V = \text{rad}(f)$ for f a nonzero nonunit (a “pseudouniformizer”).
2. V has exactly 2 prime ideals, namely 0 and \mathfrak{m}_V .
3. There is a non-trivial non-archimedean absolute value on $K = \text{Frac}(V)$ such that $V = \{x \in K \mid |x| \leq 1\}$.

Example 4.7. Any DVR has rank one, and $\mathbb{F}_p[[t]]_{perf}$ is a non-DVR with rank one.

Note. In general, rank n valuation rings are those with exactly $n + 1$ prime ideals.

Proposition 4.3. Let A be perfectoid. Then A is a valuation ring iff A^\flat is a valuation ring. In this case $\text{Frac}(A)^\times/A^\times \cong \text{Frac}(A^\flat)^\times/A^\flat$, where the map from right to left sends α to α^\sharp .

Proof. The forward direction follows from the fact that $A^\flat = \varprojlim_\varphi A$ is an isomorphism of multiplicative monoids, so the divisibility condition can be checked directly. For the backwards direction, show that any $x \in A$ can be written as $\alpha^\sharp u$ for some $\alpha \in A^\flat$ and some $u \in A^\times$. 

Definition 4.4 (Scholze’s original definition). A **perfectoid field** is a complete non-archimedean field K such that

1. $|p| < 1$.
2. \mathcal{O}_K/p is semiperfect.
3. The value group is not discrete.

Exercise: check that there is a correspondence between perfectoid fields and perfectoid valuation rings of rank one that are complete for a pseudouniformizer.

Example 4.8. Let C be complete, algebraically closed, non-archimedean field with $|p| < 1$. Then \mathcal{O}_C is perfectoid.

Exercise: Find the rank (lecturer’s claim: 2) and value group of $\mathbb{Z}_p \oplus t\mathbb{Q}_p[[t]]$ as a subring of $\mathbb{Q}_p[[t]]$.

5 Jan 14

5.1 Perfectoidization

Let R be an \mathbb{F}_p -algebra. Then

1. There is a universal perfect ring with a map from R , namely R_{perf} .
2. If R is semiperfect (φ surjective) then $R_{perf} = R_{red} = R/\text{rad}(0)$, so $R \twoheadrightarrow R_{perf}$.

If there is an initial perfectoid ring with a map from a ring R , then we call it the **perfectoidization** $R_{perf\,d}$. This need not exist, e.g. $(\mathbb{Z}_p)_{perf\,d}$ does not exist.

Theorem 5.1. 1. If R is integral over a perfectoid ring R_0 , then $R_{perf\,d}$ exists.

2. If R is **semiperfectoid** (R is a quotient of a perfectoid ring), then $R_{perf\,d}$ exists and the map $R \rightarrow R_{perf\,d}$ is surjective.

Remark. Part 2 of the theorem was originally believed to be false.

Remark. R being semiperfect is equivalent to the conditions that R is an algebra over a perfectoid ring R_0 and R/p is semiperfect. For instance, if the two conditions hold, then $\theta : W(R^\flat) \rightarrow R$ is surjective, and then $R_0 \widehat{\otimes} W(R^\flat) \rightarrow R$ is also surjective.

Example 5.1. \mathbb{Z}/p^2 is not semiperfectoid, but its mod p ring is semiperfect.

5.2 p -complete arc topology

Definition 5.1. An **extension** of rank 1 valuation rings is an inclusion $V \hookrightarrow V'$ such that a pseudouniformizer for V is mapped to one for V' . Equivalently, it is an isometric inclusion of non-archimedean fields, where the valuation rings are the unit balls.

Definition 5.2. A map of rings $A \rightarrow B$ is a **p -complete arc cover** if, given any $A \rightarrow V$ where V is a p -complete rank 1 valuation ring, there is an extension $V \hookrightarrow V'$ and a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ V & \hookrightarrow & V' \end{array}$$

Equivalently, any multiplicative seminorm $|\cdot|$ on A such that $|A| \leq 1$ and $|p| < 1$ extends to one on B with $|B| \leq 1$.

Remark. 1. This notion of covering gives a Grothendieck topology on $(\text{Rings})^{op}$.

2. The topology has a basis given by products of rings of the form \mathcal{O}_C , where C is a complete, algebraically closed, non-archimedean field with $|p| < 1$. In other words, for any ring R , there is a p -complete arc cover $R \rightarrow \prod_{i \in I} \mathcal{O}_{C_i}$.
3. Any faithfully flat map is a p -complete arc cover. For instance, given $f, g \in R$, the map $R \rightarrow R[x]/(fx - g) \times R[y]/(gy - f)$ is a p -complete arc cover.

Theorem 5.2. *On perfectoid rings, the identity is a sheaf with no higher cohomology for the p -complete arc-topology. Explicitly, if $A \rightarrow B$ is a p -complete arc cover, then $A \xrightarrow{\sim} \varprojlim(B \rightrightarrows B \widehat{\otimes}_A B \cdots)$.*

Corollary 5.1. *Let $R \rightarrow R'$ be a map of rings. Then R' is a perfectoidization of R iff the following are true:*

1. R' is perfectoid.
2. For any complete, algebraically closed non-archimedean field C with $0 < |p| < 1$ and any map $R \rightarrow \mathcal{O}_C$, there is a unique extension to a map $R' \rightarrow \mathcal{O}_C$.
3. $(R/p)_{\text{perf}} \xrightarrow{\sim} (R'/p)_{\text{perf}}$; equivalently, allow $|p| = 0$ in the above condition.

Proof. We want $\text{Hom}(R', T) \xrightarrow{\sim} \text{Hom}(R, T)$ for T perfectoid. By descent, we reduce to $T = \prod \mathcal{O}_C$, and since products commute with Hom, we reduce to $T = \mathcal{O}_C$. 

Example 5.2. Let $C = \mathbb{C}_p$ and $R = \mathcal{O}_C[x]/(x^p - 1)$. Then there is a map of rings $R \rightarrow \text{Fun}(\mu_p(\mathcal{O}_C), \mathcal{O}_C)$ given by sending x to the identity function. This is an isomorphism, in particular the Fourier transform, after inverting p .

Proposition 5.1. *The perfectoidization of the ring R described above is the ring of functions $\mu_p(\mathcal{O}_C) \rightarrow \mathcal{O}_C$ that are constant mod the maximal ideal of \mathcal{O}_C . In other words, it is*

$$\text{Fun}(\mu_p(\mathcal{O}_C), \mathcal{O}_C) \times_{\text{Fun}(\mu_p(\mathcal{O}_C), \overline{\mathbb{F}_p})} \overline{\mathbb{F}_p}.$$

Proof. lecturer is going way too fast conditions 1 and 3 of Corollary 5.1 can be checked ... call the claimed perfectoidization R' . Suppose we have a map $R \rightarrow \mathcal{O}_{C'}$, where C' is a field such as in condition 2 of Corollary 5.1. We can extend the map $R \rightarrow C'$ to R' , since $C' = \mathcal{O}_{C'}[\frac{1}{p}]$ and $R[\frac{1}{p}] = R'[\frac{1}{p}]$. Now note that $R' \subset \frac{1}{p}R$. We are done by the following lemma:

Lemma 5.1. *Let $A \hookrightarrow B$ be a map of p -torsion free rings. Suppose $A \subset B \subset \frac{1}{p^n}A$. Then any map $A \rightarrow \mathcal{O}_C$ extends uniquely to B .*

Proof of lemma. As above, we can extend $A \rightarrow C$ to $B \rightarrow C$. But $B \subset \frac{1}{p^n}A$ means that $B \rightarrow C$ has bounded image. The only bounded subrings of C are contained in the unit ball, so we indeed get a map $B \rightarrow \mathcal{O}_C$. 



Example 5.3. Let $R = \mathcal{O}_C\langle x^{1/p^\infty} \rangle / (x - 1)$. Then

$$R_{perf} = \text{Fun}(\mathbb{Z}_p(1)(\mathcal{O}_C), \mathcal{O}_C) \times_{\text{Fun}(\mathbb{Z}_p(1)(\mathcal{O}_C), \overline{\mathbb{F}_p})} \overline{\mathbb{F}_p}.$$

However, this is very nontrivial. Furthermore, the map $R \rightarrow R_{perf}$ is surjective. Next time we will show how to construct a nonzero element of the kernel.

6 Jan 15

Today we will discuss the following result.

Theorem 6.1. *Let R be a semiperfectoid ring (recall that this means a quotient of a perfectoid ring). Then it has a perfectoidization R_{perf} , and the map $R \rightarrow R_{perf}$ is surjective.*

Remark. It is enough to consider the case where $R = R_0/f$ is the quotient of a perfectoid by a principal ideal, since perfectoids are closed under arbitrary relative tensor products.

Example 6.1. Suppose furthermore that $f \in R_0$ had a compatible system of p -power roots. Then we claim $R_{perf} = (R_0 / \bigcup_{n \geq 0} f^{1/p^n})^\wedge_p$. First of all, the right hand side is perfectoid since it may be expressed as $R_0 \otimes_{R[x^{1/p^\infty}]}^{\wedge} R_0$, where $R[x^{1/p^\infty}]$ acts on the first copy of R_0 by $x^{1/p^n} \mapsto f^{1/p^n}$, and it acts on the second copy by $x^{1/p^n} \mapsto 0$. check this, I guess. Second, since perfectoids are reduced*, any map $R_0/f \rightarrow S$ with S perfectoid must vanish on all f^{1/p^n} , hence factoring through a map from the desired ring.

Remark. *To justify that perfectoids are reduced, we give two arguments:

1. By arc descent, any perfectoid ring embeds in a product of p -complete valuation rings, which are reduced.
2. By Proposition 4.2, any perfectoid embeds in $A \times B$ where A is p -torsion free perfectoid and B is a perfect \mathbb{F}_p -algebra. Certainly B is reduced, so we need to show A is reduced. Suppose x is nilpotent, so that $x^{p^n} = 0$ for some n . Then, for any i , the element $x/p^i \in A[\frac{1}{p}]$ satisfies $(x/p^i)^{p^n} = 0 \in A$. Thus, since p -torsion free perfectoids are p -root closed by Theorem 3.1, we have $x/p^i \in A$ for all i . By p -completeness, this means $x = 0$.

To recap, we reduced Theorem 6.1 to the case where $R = R_0/f$, and we handled this case under the further assumption that f has p -power roots. On the other hand, if f doesn't have p -power roots, things are much less clear.

Example 6.2. For example, suppose our ring was $R = \mathcal{O}_C\langle x^{1/p^\infty} \rangle/(x - 1)$, where $C = \mathbb{C}_p$. What is R_{perf} ? By choosing a compatible system of p -power roots of unity $\zeta_{p^n} \in \mathcal{O}_C$, we get a map

$$\begin{aligned} \mathcal{O}_C\langle x^{1/p^\infty} \rangle/(x - 1) &\rightarrow \text{Fun}_{cts}(\mathbb{Z}_p, \mathcal{O}_C), \\ x^{1/p^n} &\mapsto (i \mapsto \zeta_{p^n}^i). \end{aligned}$$

This map lands inside the fiber product

$$\text{Fun}_{cts}(\mathbb{Z}_p, \mathcal{O}_C) \times_{\text{Fun}_{cts}(\mathbb{Z}_p, \overline{\mathbb{F}_p})} \overline{\mathbb{F}_p},$$

which is perfectoid. We claim that it is the perfectoidization.

For this, it suffices to check at each “ p -power depth”, i.e. for each n we have a perfectoidization

$$\mathcal{O}_C \langle x^{1/p^n} \rangle / (x - 1) \rightarrow \text{Fun}(\mathbb{Z}/p^n, \mathcal{O}_C) \times_{\text{Fun}(\mathbb{Z}/p^n, \overline{\mathbb{F}_p})} \overline{\mathbb{F}_p}, \quad (1)$$

and we did this last time by comparing mod p perfections and maps into other $\mathcal{O}_{C'}$. What is not clear is the following:

1. The map 1 is surjective. We will discuss this.
2. What are nontrivial elements of the kernel of 1? We know that there must be something, since the domain is not perfectoid.

Both of these facts are addressed in Fresnel and de Mathan, “Sur la transformation de Fourier p -adique”.

The surjectivity of $R \rightarrow R_{perf d}$ will be a corollary of André’s lemma.

6.1 André’s flatness lemma

Theorem 6.2. *Let R_0 be a perfectoid ring and let $f \in R_0$. Then there is an R_0 -algebra T_0 such that*

1. T_0 is perfectoid.
2. $R_0/p^n \rightarrow T_0/p^n$ is faithfully flat for each n .
3. f has a compatible system of p -power roots in T_0 .

It is also possible to choose T_0 absolutely integrally closed.

Remark. The surjectivity of $R_0/f \rightarrow (R_0/f)_{perf d}$ now follows from flat descent along $R_0 \rightarrow T_0$. **check**

In the rest of the lecture, we will explain in concrete terms the kernel of $R \rightarrow R_{perf d}$ when R is p -torsion free and semiperfectoid.

Definition 6.1. Let R be p -torsion free and p -complete. We say that $x \in R$ has **spectral radius zero** if for all N , we have $x^i \in p^{iN}R$ for large enough i .

Remark. The collection of spectral radius zero elements in R is p -complete. It contains all nilpotent elements.

Example 6.3. \mathcal{O}_C has no elements of spectral radius zero **why?**

Theorem 6.3. *Let R be a p -torsion free semiperfectoid ring. Then the kernel of $R \rightarrow R_{perf d}$ is exactly the set of elements with spectral radius zero.*

Remark. Let S be a p -torsion free perfectoid ring. Then S has no nonzero elements of spectral radius zero. To see this, choose $\omega \in S$ with $\omega^p \in pS$. Then one can prove inductively that $p^i \mid x^p$ iff $\omega^i \mid x$ (**do this**), and this implies the claim (**why?**).

Note. My friend's notes claim that any perfectoid should have no spectral radius zero elements, but there is no proof given. My feeling is that you “decompose” a general perfectoid into a p -torsion free one and a perfect \mathbb{F}_p -algebra. In characteristic p , the notion of spectral radius zero becomes nilpotency, and perfect algebras are reduced.

The remark implies that the spectral radius zero elements are contained in the kernel of perfectoidization. To prove the other direction of the theorem, we need a “spectral radius formula”.

Let A be a \mathbb{Q}_p -Banach algebra; it is a \mathbb{Q}_p -algebra with a subadditive submultiplicative norm $\|\cdot\| : A \rightarrow \mathbb{R}_{\geq 0}$ such that $\|\lambda a\| = |\lambda|_p \|a\|$ and A is complete with respect to its norm. Let A_0 be its closed unit ball. This is a p -complete, p -torsion free ring such that $A = A_0[\frac{1}{p}]$. Conversely, if B_0 is a p -complete p -torsion free ring, then $B = B_0[\frac{1}{p}]$ can be given a norm in terms of p -divisibility, which makes it a \mathbb{Q}_p -Banach algebra with closed unit ball B_0 . There is an equivalence between the category of \mathbb{Q}_p -Banach algebras with continuous maps and the category of p -complete p -torsion free rings up to isogeny ([what does isogeny mean in this context?](#)).

Definition 6.2. Let A be a commutative \mathbb{Q}_p -Banach algebra. Let $\mathcal{M}(A)$ be the set of bounded (by the Banach norm) multiplicative seminorms on A , i.e. functions $|\cdot| : A \rightarrow \mathbb{R}_{\geq 0}$ with $|0| = 0$, $|1| = 1$, $|a+b| \leq |a| + |b|$, $|ab| = |a||b|$, and $|a| \leq \|a\|$. This is called the **Berkovich spectrum**. It can be given the topology of a compact Hausdorff space.

Note. $\mathcal{M}(A)$ can be identified with the set of maps from A_0 to rank 1 valuation rings where p is a pseudouniformizer, up to isomorphism on the right hand side.

We use the following facts:

1. If $A \neq 0$, then $\mathcal{M}(A) \neq \emptyset$.
2. For all $a \in A$, we have

$$\sup_{|\cdot| \in \mathcal{M}(A)} |a| = \lim_{n \rightarrow \infty} \|a^n\|^{1/n},$$

which is the “spectral radius formula”. The equal quantity is called the spectral radius of a . An element $a \in A_0$ has spectral radius zero (in the sense discussed earlier) iff its spectral radius (in the current sense) is zero.

An equivalent formulation of property 1 is that if B_0 is a p -complete p -torsion free ring, then there is a map $B_0 \rightarrow \mathcal{O}_C$, where C is a complete algebraically closed non-archimedean field with $0 < |p| < 1$.

It then follows from this reformulation and from the spectral radius formula that $a \in A_0$ has spectral radius zero iff a is in the kernel of any map from A_0 to a rank 1 valuation ring with p a pseudouniformizer.

We can now finish proving Theorem 6.3. Recall that we need to show that, for R p -torsion free and semiperfectoid, the kernel of $R \rightarrow R_{perf}$ is the set of spectral radius zero elements. By the universal property of R_{perf} , this kernel is the joint kernel of all maps from R to a perfectoid. By arc descent, this is the joint kernel of all maps from R to rank 1 valuation rings where $|p| < 1$. The case where $0 < |p| < 1$ is then covered by our previous work. The characteristic p case is left to be checked.

Today's material raises two questions:

1. Is there a direct proof that $R \rightarrow R_{perf}$ is surjective for R p -torsion free and semiperfectoid? Namely, a proof that does not use base change.
2. As a corollary of our argument, we also see that if R is p -torsion free semiperfectoid with no elements of spectral radius zero, then R is perfectoid. Is there a direct proof of this?

7 Jan 21

Definition 7.1. Let C be a CACNA of characteristic p . (For example, take C to be the completion of $\overline{\mathbb{F}_p((t))}$ with respect to the induced power series valuation.) An **untilt** of C consists of a perfectoid field (Definition 4.4) F and an isomorphism $\iota : F^\flat \rightarrow C$, or equivalently an isomorphism $\mathcal{O}_F^\flat \rightarrow \mathcal{O}_C$.

Note. An untilt $(F, \iota : F \rightarrow C)$ gives a multiplicative map $\sharp : C \rightarrow F$ how?.

Example 7.1. If C is characteristic 0, then C is an untilt of itself.

Remark. 1. The notion of untilt makes sense for any perfectoid field. For instance, $\mathbb{Q}_p(p^{1/p^\infty})_p^\wedge$ and \mathbb{Q}_p^{cyc} (the cyclotomic extension of \mathbb{Q}_p adjoin all p -power roots of unity) are untilts of $\mathbb{F}_p((t^{1/p^\infty}))$.

2. Any untilt of an algebraically closed field is algebraically closed.

There is a natural notion of isomorphism of untilts, so we get a set $\text{Untilt}(C)$. The key idea is that $\text{Untilt}(C)$ is analogous to the rigid open unit disk, which we now try to introduce.

We now let C be any CACNA.

Definition 7.2. The **Tate algebra** $C\langle T \rangle$ is the algebra of power series $\sum a_i T^i$ in $C[[T]]$ such that $a_i \rightarrow 0$ as $i \rightarrow \infty$.

The idea is that $C\langle T \rangle$ is the algebra of analytic functions on the closed unit disk $D_{\leq 1} = \{\alpha \in C \mid |\alpha| \leq 1\}$. For any $\alpha \in D_{\leq 1}$, we have an evaluation map $\text{ev}_\alpha : C\langle T \rangle \rightarrow C$ given by $T \mapsto \alpha$.

Definition 7.3. The **Gauss norm** on $C\langle T \rangle$ is the map $\|\cdot\| : C\langle T \rangle \rightarrow \mathbb{R}_{\geq 0}$ given by $\|\sum a_i T^i\| = \max |a_i|$.

Remark. 1. $\|f + g\| \leq \|f\| + \|g\|$.

2. $\|fg\| = \|f\| \cdot \|g\|$; this is Gauss' lemma!

3. $C\langle T \rangle$ with the Gauss norm is a Banach algebra. It is in fact the completion of $C[T]$ with respect to the Gauss norm.

The Tate algebra also admits a purely algebraic definition; we have $C\langle T \rangle = (\mathcal{O}_C[T])_\pi^\wedge[\frac{1}{\pi}]$, where $\pi \in C$ has $|\pi| = 1$.

There is also a variant of the Tate algebra as follows. Let $r \in |C^\times| \subset \mathbb{R}_{>0}$.

Definition 7.4. Let $C\langle r^{-1}T \rangle$ consist of power series in T whose coefficients satisfy $|a_i|r^i \rightarrow 0$ as $i \rightarrow \infty$. It is called the **Tate algebra of radius r** .

We have $C\langle T \rangle \cong C\langle r^{-1}T \rangle$ via $T \mapsto \pi T$, where $|\pi| = r$.

Theorem 7.1. $C\langle T \rangle$ is a PID. Its maximal ideals are exactly $(T - \alpha)$ for $\alpha \in D_{\leq 1}$. The residue fields of the maximal ideals are all isomorphic to C via the evaluation maps.

We will prove this later.

Remark. For any n there is a ring $C\langle T_1, \dots, T_n \rangle$, and its maximal ideals are exactly the ideals $(T_1 - \alpha_1, \dots, T_n - \alpha_n)$ for $\alpha_i \in D_{\leq 1}$.

The theorem implies that a nonzero $f \in C\langle T \rangle$ has only finitely zeros on $D_{\leq 1}$. We can count the roots using a valuation.

Let ν be the valuation on C associated to its absolute value. Equip $\mathbb{R} \times \mathbb{Z}$ with lexicographic order. We define a valuation ν_+ on $C\langle T \rangle$ by $\nu_+(\sum a_i T^i) = \min_i(\nu(a_i), -i)$ for nonzero elements. The first coordinate of $\nu_+(f)$ is $\min_i \nu(a_i)$, which is the Gauss valuation $-\log ||f||$. Saying that $\nu_+(f) = (r, -n)$ means the following three things:

1. $\nu(a_i) \geq r$ for all i .
2. $\nu(a_n) = r$.
3. $\nu(a_m) > r$ for $m > n$.

As an example, $\nu_+(T) = (0, -1)$, so “ T is just outside the valuation ring”.

Remark. This is a rank 2 valuation on $C\langle T \rangle$. The elements of valuation ≥ 0 are $\mathcal{O}_C \oplus \mathfrak{m}_C\langle T \rangle$.

Proposition 7.1. *If $\nu_+(f) = (r, -n)$, then f has exactly n zeros (with multiplicity) in $D_{\leq 1}$.*

We will prove Theorem 7.1 and Proposition 7.1 using “Weierstrass preparation”:

Theorem 7.2 (Weierstrass preparation). *Given nonzero $f \in C\langle T \rangle$, there is a monic polynomial $g \in \mathcal{O}_C[T]$ such that $f \in gC\langle T \rangle^\times$. In particular, if the second coordinate of $\nu_+(f)$ is $-n$, then we can choose g to be degree n .*

Proof. Since the statement is clearly unaffected by scaling f by constants (as they are units and do not affect the second coordinate of $\nu_+(f)$), we may assume $||f|| = 1$, i.e. $f \in \mathcal{O}_C\langle T \rangle$. By the assumption on $\nu_+(f)$, we know that $|a_n| = 1$ and $|a_i| < 1$ for $i > n$, where $f = \sum a_i T^i$. Thus, we can write $f = f_1 + \pi f_2$ where $f_1 \in \mathcal{O}_C[T]$ has degree n and leading coefficient in \mathcal{O}_C^\times , $f_2 \in \mathcal{O}_C\langle T \rangle$, and π is a generator of \mathfrak{m}_C . It follows that $(\mathcal{O}_C\langle T \rangle/f)/\pi \cong (\mathcal{O}_C/\pi)[T]/\bar{f}_1$. Since \bar{f}_1 is degree n and monic up to a unit, we have that $(\mathcal{O}_C/\pi)[T]/\bar{f}_1$ is free on $\{1, T, \dots, T^{n-1}\}$. By Nakayama and π -completeness, we can lift this and say that $\mathcal{O}_C\langle T \rangle/f$ is also free on $\{1, T, \dots, T^{n-1}\}$. As a result, there exist $q \in \mathcal{O}_C\langle T \rangle$ and $r \in \mathcal{O}_C[T]$ of degree at most $n-1$ such that $T^n = qf + r$. If we reduce mod \mathfrak{m}_C , so that we are in a polynomial ring over a field and f becomes a degree n polynomial, then we see that q must reduce to a constant. Thus $q \in \mathcal{O}_C^\times \oplus T\mathfrak{m}_C < T >$, so q is a unit. By setting $g = T^n - r$, we are done. 

Now we prove Theorem 7.1:

Proof Theorem 7.1. One can check (I guess I should do this) that the map $C\langle T \rangle / (T - \alpha) \rightarrow C$ induced by ev_α , for $\alpha \in D_{\leq 1}$, is an isomorphism. This implies that each $(T - \alpha)$ is maximal. By Weierstrass preparation, we see that all non-zero principal ideals will factor into these maximals. Finally, (this is my guess; nothing is said about it in the lecture notes) one can show that all ideals are principal by considering the minimal degree of the polynomials in the ideal. 

And we prove Proposition 7.1:

Proof Proposition 7.1. By Weierstrass preparation, we reduce to the case where $f \in \mathcal{O}_C[T]$ is monic and degree n , since units in $C\langle T \rangle$ cannot have roots on $D_{\leq 1}$. Clearly f has n roots in total, but we need to check that they are in $D_{\leq 1}$. Indeed, if $\alpha \in C$ has $|\alpha| > 1$, then $|f(\alpha)| = \alpha^n$ by the non-archimedean property. Thus, we are done. 

8 Jan 23

Let $(C, |\cdot|)$ be a complete algebraically closed non-archimedean (CACNA) field. Let $r \in |C^\times|$.

Definition 8.1. Let $C\langle r^{-1}T \rangle$ consist of power series in T whose coefficients satisfy $|a_i|r^i \rightarrow 0$ as $i \rightarrow \infty$. It is called the **Tate algebra of radius r** .

Theorem 8.1. $C\langle r^{-1}T \rangle$ is a PID whose maximal ideals are $(T - \alpha)$, where $\alpha \in C$ has $|\alpha| \leq r$.

Definition 8.2. We have an r -**Gauss norm** $\|\cdot\|_r$ given by taking the maximum of $|a_i|r^i$. It is a Banach norm, strictly multiplicative.

For any r , the Tate algebra is the completion of $C[T]$ with respect to the r -Gauss norm. If $r < 1$, then it is also the completion of $\mathcal{O}_C[[T]][\frac{1}{\pi}]$, where π is a pseudouniformizer, with respect to the r -Gauss norm. If $r = 1$, we have $C\langle T \rangle = (\mathcal{O}_C[T])_\pi^\wedge[\frac{1}{\pi}]$. Similarly, if $r < 1$, then $C\langle r^{-1}T \rangle = (\mathcal{O}_C[[T]][\frac{T}{\pi}])_\pi^\wedge[\frac{1}{\pi}]$, where we pick π with $|\pi| = r$.

Definition 8.3. Let $f \in C\langle r^{-1}T \rangle$. Say $\|f\|_r = \lambda$. Say $f = \sum a_i T^i$. Define $\mu_r(f)$ to be the maximum index i that satisfies $\|a_i T^i\|_r = \lambda$.

Theorem 8.2. 1. $\mu_r(f)$ is the number of roots of f in $D_{\leq r}$.

2. The map $f \mapsto (-\log \|f\|_r, -\mu_r(f))$ defines a valuation on $C\langle r^{-1}T \rangle$ valued in $\mathbb{R} \times \mathbb{Z}$ equipped with lexicographic order.

Proof. Apparently this was discussed last time. Prove 2 directly, and then it implies 1 by “Weierstrass preparation”, which means that anything in the Tate algebra is the product of a unit and a polynomial. 

Our goal is to describe a mixed characteristic analog of $C\langle r^{-1}T \rangle$ for $r < 1$ and C characteristic p . In everything that follows, assume $r < 1$

Last lecture we discussed the notion of untilting C , i.e. a pair $(F, i : F^\flat \cong C)$. $|p|_F \in [0, 1)$, and if $x \in C$, $|x^\sharp|_F = |x|_C$. We give some analogies between the two sides. (make table.) $D_{\leq r}$ corresponds to untilts where $|p|_F \leq r$. $\mathcal{O}_C[[T]]$ corresponds to $W(\mathcal{O}_C)$. $C[[T]]$ corresponds to $W(C)$. $C\langle r^{-1}T \rangle$ will correspond to something called $B_{[0,r]}$.

Definition 8.4. We define $B_{[0,r]}$ to be the subset of $W(C)$ consisting of $\sum [a_i]p^i$ where $|a_i|r^i \rightarrow 0$ as $i \rightarrow \infty$.

Lemma 8.1. Given $\lambda > 0$ and $N \in \mathbb{N}$, define $A_{\lambda,N}$ to be the subset of $W(C)$ consisting of those $\sum [a_i]p^i$ such that $|a_i| \leq \lambda$ for $i \leq N$. Then $A_{\lambda,N}$ is a subgroup.

Proof. This is proved by direct computation, using the discussion of how addition of Witt vectors works from earlier lectures. (add citation, maybe more detail) 

Corollary 8.1 (Corollary of the Lemma). *Let $\lambda : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be a nondecreasing function. Then the set A_λ of $\sum [a_i]p^i$ such that $|a_i| \leq \lambda(i)$ for all i is a subgroup.*

Proof. It is the intersection of $A_{\lambda(N), N}$ for all N . 

We can now show that $B_{[0, r]}$ is a subgroup. Let $\lambda_{\varepsilon, M}(i) = \max(M, \varepsilon r^{-i})$. Then $B_{[0, r]} = \bigcap_{\varepsilon > 0} \bigcup_{M \in \mathbb{N}} A_{\lambda_{M, \varepsilon}}$.

Definition 8.5. Given $f = \sum [a_i]p^i \in W(C)$, define its **r -Gauss norm** to be $\|f\|_r = \sup |a_i|r^i$ (this can be ∞ , but is finite if $f \in B_{[0, r]}$).

Proposition 8.1. *If $f, g \in W(C)$, then $\|f + g\|_r \leq \max(\|f\|_r, \|g\|_r)$.*

Proof. **maybe?** write $\|f\|_r, \|g\|_r \leq \lambda$, write this in terms of coefficients, then do stuff with coefficients of the sum. 

9 Jan 26

Let C be a CACNA of characteristic p . Recall that for $r \in (0, 1) \cap |C^\times|$, we defined $B_{[0,r]} \subset W(C)$ to be those $x = \sum [x_i]p^i$ with $|x_i|p^i \rightarrow 0$ as $i \rightarrow \infty$.

Proposition 9.1. $B_{[0,r]}$ is a subring of $W(C)$.

Proof. To prove this we start with a lemma.

Lemma 9.1. Let $\lambda : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be a nondecreasing function. Let A_λ be the set of $\sum [a_i]p^i$ such that $|a_i| \leq \lambda(i)$ for all i . Suppose λ, λ' are two such functions, and that $x \in A_\lambda, y \in A_{\lambda'}$. Then

1. $x + y \in A_{\max(\lambda, \lambda')}$.
2. $xy \in A_{\lambda * \lambda'}$, where $(\lambda * \lambda')(i) = \max_{a+b=i} \lambda(a)\lambda'(b)$.

Proof of Lemma. For part 1, we use the rule for addition in $W(C)$. Namely if $x = \sum [a_i]p^i$ and $y = \sum [b_i]p^i$, then $x + y = \sum [c_i]p^i$, where c_i is a homogeneous degree 1 polynomial in the p -power roots of $a_0, \dots, a_i, b_0, \dots, b_i$. By assumption, all of these are bounded by $\max(\lambda(i), \lambda'(i))$, and then so are all of the possible degree 1 monomials. Since the absolute value is non-archimedean, we are done. 

For part 2, we have $xy = \sum_{i,j} [a_i b_j] p^{i+j}$. One can show that each summand is in $A_{\lambda * \lambda'}$, so the entire sum is in $A_{\lambda * \lambda'}$ (by part 1, finite sums are ok; infinite sums are ok because the condition to be checked only involves finitely many terms at a time.) 

To finish the proposition, note that $x = \sum [a_i]p^i \in W(C)$ is in $B_{[0,r]}$ iff there is a nondecreasing λ such that $x \in A_\lambda$ and $\lambda(i)r^i \rightarrow 0$ as $i \rightarrow \infty$. It is clear that given such λ , x is in $B_{[0,r]}$. Conversely, if $x \in B_{[0,r]}$, then we define $\lambda(i) = \sup_{j \leq i} |a_j|$. Finally, the set of λ that satisfy $\lambda(i)r^i \rightarrow 0$ is closed under max and $*$. 

Definition 9.1. On $B_{[0,r]}$, define the **r -Gauss norm** to be $\|\sum [a_i]p^i\|_r = \sup_i |a_i|r^i$.

Proposition 9.2. For any $f, g \in B_{[0,r]}$, we have $\|f + g\|_r \leq \max(\|f\|_r, \|g\|_r)$ and $\|fg\|_r \leq \|f\|_r \|g\|_r$. 

Proof. Use the fact that $\|[a_i]p^i\| \leq \delta$ iff $|a_i| \leq \delta r^{-i}$ in conjunction with the Lemma above. 

Proposition 9.3. $B_{[0,r]}$ is complete for the r -Gauss norm, so it is a Banach algebra.

Proof. Check that if a sequence in $B_{[0,r]}$ is Cauchy, then each sequence of p^i components is Cauchy, so you can take the limit pointwise. Then do some bookkeeping to ensure that the limit lands back in $B_{[0,r]}$. 

Proposition 9.4. If $f, g \in B_{[0,r]}$, then $\|fg\|_r = \|f\|_r \|g\|_r$.

Recall that in $C\langle r^{-1}T \rangle$ we had a similar result, and it let us define a Gauss valuation $\nu_r(\sum a_i T^i) = \inf_i (-\log(|a_i|r^i))$. This refines to $\nu_{r,\pm}(\sum a_i T^i) = \inf_i (-\log(|a_i|r^i), \mp i) \in \mathbb{R} \times \mathbb{Z}$, and it let us count roots.

Theorem 9.1. *We can define $\nu_{r,\pm}$ on $B_{[0,r]}$ in the same way, and they also define valuations.*

Proof. The problem reduces to looking at monomials. In particular, suppose one has a finite collection S of monomials $[a]p^n$ in $W(C)$ that have the same r -Gauss norm δ . Suppose there is a unique monomial in S that maximizes (resp. minimizes) the degree n . Then the sum of the monomials has r -Gauss norm δ .

Lemma 9.2. *Let $x = \sum [a_i]p^i$, $y = \sum [b_i]p^i$ in $W(C)$. Let $z = x + y = \sum [c_i]p^i$. Fix $n \geq 0$ and $M \in \mathbb{R}_{>0}$ such that*

1. $|a_i| < M$ for $i \leq n$.
2. $|b_i| \leq M$ for $i \leq n$.
3. $|b_n| = M$.

Then $|c_n| = M$.

Proof of Lemma. $c_n = a_n + b_n$ plus cross terms involving a_i, b_j for $i, j < n$. Everything is homogeneous of degree 1, so everything works out by non-archimedean property. 

One can use this Lemma to complete the proof of the Theorem. 

Example 9.1. Consider $x = [a]p^i$, $y = [b]p^j$, with $i < j$, and both have r -Gauss norm δ . Then $|a| = \delta r^{-i}$ and $|b| = \delta r^{-j}$. The sum $x + y = \sum [c_k]p^k$ has expansion $[a]p^i + [b]p^j$, from which it is clear that its r -Gauss norm is also δ , since $|c_k|r^k$ is either δ when $k \in \{i, j\}$ or 0 otherwise.

Recall the following results for $C\langle r^{-1}T \rangle$. Fix $\pi \in C$ with $|\pi| = r$.

1. $C\langle r^{-1}T \rangle$ is the completion of $C[T]$ with respect to the r -Gauss norm. It is also the completion of $\mathcal{O}_C[[T]] \otimes_{\mathcal{O}_C} C = \mathcal{O}_C[[T]][\frac{1}{\pi}]$ (the power series with bounded coefficients) with respect to the r -Gauss norm.
2. $C\langle r^{-1}T \rangle = \mathcal{O}_C[\frac{T}{\pi}]^\wedge[\frac{1}{\pi}]$.

We have the following similar results for $B_{[0,r]}$:

1. $B_{[0,r]}$ is the completion of $W(\mathcal{O}_C)[\frac{1}{\pi}]$ with respect to the r -Gauss norm.
2. $B_{[0,r]} = W(\mathcal{O}_C)[\frac{p}{\pi}]^\wedge[\frac{1}{\pi}]$.

Remark. The ring $W(\mathcal{O}_C)[\frac{p}{\pi}]$ is contained in the unit ball of $B_{[0,r]}$. It can also be expressed as $W(\mathcal{O}_C)[x]/([\pi]x - p)$, because $[\pi], p$ form a regular sequence in $W(\mathcal{O}_C)$.

10 Jan 28

As before, we fix a characteristic p CACNA C , some $r \in |C^\times|$ with $r < 1$, and consider the ring $B_{[0,r]}$. Today we want to show that $B_{[0,r]}$ is “functions on untilts where $|p| \leq r$ ”, which is meant to be analogous to $C\langle r^{-1}T \rangle$ consisting of analytic functions on $D_{\leq r}$.

Fix an untilt $(F, \iota : C \cong F^\flat)$. We have a map $\theta : W(\mathcal{O}_C) \cong W(\mathcal{O}_F^\flat) \rightarrow \mathcal{O}_F$ which is surjective, and $\ker(\theta)$ is generated by a distinguished $\xi = [\xi_0] + pu$, where $\xi_0 \in \mathfrak{m}_C$ and u is a unit.

Let $W_{prim}(\mathcal{O}_C)$ be the set of distinguished elements in $W(\mathcal{O}_C)$. Then $\text{Untilt}(C) \cong W_{prim}(\mathcal{O}_C)/W(\mathcal{O}_C)^\times$, where given a distinguished ξ , we can form the untilt $C_\xi = \text{Frac}(W(\mathcal{O}_C)/\xi)$.

Given the untilt C_ξ , we can compute $|p|_{C_\xi}$. Namely, if we write $\xi = [\xi_0] + pu$, then in C_ξ , we have $\xi_0^\sharp = pu'$, where u' is a unit. It follows that $|p|_{C_\xi} = |\xi_0^\sharp|_{C_\xi}$. The isomorphism ι gives a preferred absolute value on an untilt such that the sharp map preserves absolute value; thus $|p|_{C_\xi} = |\xi_0|_C$. We obtain a map $|p| : \text{Untilt}(C) \rightarrow [0, 1]$ which sends the class of C_ξ to $|\xi_0|_C$.

Now, consider a fixed untilt (F, ι) of C with $|p|_F \leq r$. There is a map $\text{ev}_F : B_{[0,r]} \rightarrow F$ which sends $\sum [a_i]p^i$ to $\sum a_i^\sharp p^i$. We note that this is the exact same formula which defines the map $\theta : W(\mathcal{O}_C) \rightarrow \mathcal{O}_F$. The series $\sum a_i^\sharp p^i$ converges exactly because $|p|_F \leq r$ and $|a_i|r^i \rightarrow 0$. To show that this is a ring map, we take a different approach. Recall that if $\pi \in \mathcal{O}_C$ has $|\pi| = r$, then $B_{[0,r]} = W(\mathcal{O}_C)[\frac{p}{[\pi]}]_{[\pi]}^\wedge[\frac{1}{[\pi]}]$. The assumption that $|p|_F \leq r$ implies that π^\sharp divides p in \mathcal{O}_F . This allows one to extend θ . This approach agrees with the previous approach because they are both continuous maps extending θ .

Proposition 10.1. *Let $\xi = [\xi_0] + pu \in W_{prim}(\mathcal{O}_C)$ and assume $|\xi_0|_C \leq r$, so $|p|_{C_\xi} \leq r$. Thus we have a map $\text{ev}_{C_\xi} : B_{[0,r]} \rightarrow C_\xi$ constructed above. Then it induces an isomorphism $B_{[0,r]}/\xi \xrightarrow{\sim} C_\xi$.*

Note. As an aside, note that distinguished ξ can be expressed in the form $p - [\xi_0]$, which makes this proposition look analogous to the similar result for the Tate algebra.

Proof. We know $W(\mathcal{O}_C)/\xi \xrightarrow{\sim} \mathcal{O}_{C_\xi}$ by definition. The key point is that $B_{[0,r]}\xi$ is a closed subspace of $B_{[0,r]}$, because the r -Gauss norm is multiplicative and $||\xi||_r \neq 0$. As a result, the induced map $B_{[0,r]}/\xi \rightarrow C_\xi$ is a map of Banach spaces, and it has norm at most 1, which can be seen by analyzing the r -Gauss norm and using $|p|_{C_\xi} \leq r$. Construct a map in the other direction by using Banach space trickery. fill in diagram 

Theorem 10.1. $B_{[0,r]}$ is a PID, and its maximal ideals are in bijection with the units of C where $|p| \leq r$, namely the ideals of the form (ξ) where $\xi \in W_{\text{prim}}(\mathcal{O}_C)$ and $|\xi_0| \leq r$.

The statement is analogous to Theorem 7.1, although it is more difficult, and the proof will possibly span multiple lectures.

Recall from last time that we have a valuation $\nu_{r,+} : B_{[0,r]} \rightarrow \mathbb{R} \times \mathbb{Z}$.

Claim 10.1. Let $\xi \in W_{\text{prim}}(\mathcal{O}_C)$ have $|\xi_0| \leq r$. Then $\nu_{r,+}(\xi) = (-\log r, -1)$.

Proof. todo 

As a result, if $x \in B_{[0,r]}$ has $\nu_{r,+}(x) = (y, -n)$, then x is divisible by at most n primitive elements ξ with $|\xi_0| \leq r$. Then it remains to show that any nonunit in $B_{[0,r]}$ is divisible by at least one of these ξ .

11 Jan 30

Recall that our goal is Theorem 10.1, and that we reduced it to the following proposition:

Proposition 11.1. *Let $x \in B_{[0,r]}$. Suppose that for all $F \in \text{Untilt}(C)$ where $|p|_F \leq r$, we have $\text{ev}_F(x) \neq 0$. Then $x \in B_{[0,r]}^\times$.*

In order to prove this, we will use the Berkovich spectrum machinery.

Recall for a commutative Banach algebra A , we have the Berkovich spectrum $\mathcal{M}(A)$, see Definition 6.2. We have $\mathcal{M}(A) \subset \prod_{a \in A} [0, ||a||]$. We equip $\mathcal{M}(A)$ with the weakest topology such that the functions $\text{ev}_a : |\cdot| \mapsto |a|$ from $\mathcal{M}(A)$ to $\mathbb{R}_{\geq 0}$ are continuous. This topology is compact. We use the following facts; cf. Berkovich's book:

1. If $A \neq 0$, then $(A) \neq \emptyset$.
2. $x \in A$ is a unit iff ev_x does not vanish on $\mathcal{M}(A)$; in this case, ev_x takes a minimal positive value. In one direction, one proves that if x is not a unit, then $A/\overline{Ax} \neq 0$, so $\mathcal{M}(A/\overline{Ax}) \neq \emptyset$. The map $A \rightarrow A/\overline{Ax}$ induces a map $\mathcal{M}(A/\overline{Ax}) \rightarrow \mathcal{M}(A)$, and ev_x will vanish on the image.
3. If $A \rightarrow B$ is a bounded map of Banach algebras, then there is an induced map $\mathcal{M}(B) \rightarrow \mathcal{M}(A)$.
4. For all $a \in A$, we have

$$\sup_{|\cdot| \in \mathcal{M}(A)} |a| = \lim_{n \rightarrow \infty} ||a^n||^{1/n} = \inf ||a^n||^{1/n},$$

which is the spectral radius formula. The equal quantity is the spectral radius $\rho(a)$, which is a seminorm, but not necessarily multiplicative. One does get $\rho(a^n) = \rho(a)^n$ though.

Definition 11.1. A subset of $\mathcal{M}(A)$ is **Zariski closed** if it is the vanishing locus of some subset $S \subset A$, i.e. it consists of those $|\cdot| \in \mathcal{M}(A)$ with $|s| = 0$ for all $s \in S$.

Remark. Consider non-archimedean Banach algebras A which are **Tate**, meaning there is $\pi \in A^\times$ with $||\pi|| < 1$, i.e. there is a topologically nilpotent unit. Also assume $||\pi^{-1}|| = ||\pi||^{-1}$. Sometimes these are called analytic units. Let $A^{\leq 1}$ be the unit ball in A . Then it is a π -complete and π -torsion free ring, and $A = A^{\leq 1}[\frac{1}{\pi}]$. Conversely, if B° is a π -complete and π -torsion free ring, one can make $B = B^\circ[\frac{1}{\pi}]$ into a Tate Banach algebra with $B^{\leq 1} = B^\circ$.

In this case, we can think of elements of $\mathcal{M}(A)$ as maps $A^{\leq 1} \rightarrow V$, where V is a rank 1 valuation ring, and such that π is mapped to a pseudouniformizer. Furthermore, $|\pi| = ||\pi||$ for all $|\cdot| \in \mathcal{M}(A)$.

Theorem 11.1. Let C be a CACNA of any characteristic. Consider $C\langle T \rangle$ with its Gauss norm. Then $\mathcal{M}(C\langle T \rangle)$ has 4 types of points:

1. For $\alpha \in D_{\leq 1}$, the map sending $f \in C\langle T \rangle$ to $|f(\alpha)|_C$ is a point in $\mathcal{M}(C\langle T \rangle)$. These are called **classical points**.
2. For any disk $D(a_0, r) \subset D_{\leq 1}$ with $r \in |C^\times|$, then there is an r -Gauss norm centered as a_0 , namely $f \mapsto \sup_{\alpha \in D(a_0, r)} |f(\alpha)|_C$.
3. Similar to above, but with $r \in (0, 1) \setminus |C^\times|$. Can think of these as limits of points of type 2.
4. Suppose we have a sequence $D_{\leq 1} \supseteq D_0 \supseteq D_1 \supseteq \dots$ with $\bigcap D_i = \emptyset$. Then consider $f \mapsto \lim_{n \rightarrow \infty} \sup_{\alpha \in D_n} |f(\alpha)|_C$. Two sequences give the same point if they are cofinal with each other.

Proof. A point $|\cdot| \in \mathcal{M}(C\langle T \rangle)$ is determined by its values on $C[T]$ by density, hence on polynomials of the form $T - \alpha$ for $\alpha \in D_{\leq 1}$ since we can factor, and α with $|\alpha| > 1$ doesn't contribute by the triangle inequality.

Fix a point in $\mathcal{M}(C\langle T \rangle)$. Given a disk $D(a_0, r_0) \subset D_{\leq 1}$ with $r_0 \in |C|$ (namely $r_0 = 0$ allowed), ask "does $T \in D(a_0, r_0)$ ", i.e. is $|T - a_0| \leq r_0$? The collection of disks that "contain T " is a totally ordered (under inclusion) collection of sub-disks of $D_{\leq 1}$, which is also stable under infinite intersections. If this collection has a minimal element, then we are in type 1 or 2 depending on the radius of the minimal element (radius 0 for type 1, positive radius for type 2). Otherwise, consider the intersection of this family. If it is non-empty we are in type 3, and if it is empty we are in type 4.

Conversely, any nested collection \mathcal{C} of disks gives a point in $\mathcal{M}(C\langle T \rangle)$ given by $f \mapsto \inf_{D \in \mathcal{C}} \sup_{\alpha \in D} |f(\alpha)|_C$.

Remark. The Zariski closed subsets of $\mathcal{M}(C\langle T \rangle)$ are exactly finite sets of classical points.

12 Feb 2

Recall that our current goal is Theorem 10.1, which we reduced to Proposition 11.1, which we aim to tackle with the Berkovich spectrum. Last time we classified points in the Berkovich spectrum of a Tate algebra.

Let C be a CACNA of characteristic p and let $r \in (0, 1) \cap |C^\times|$.

Definition 12.1. Given an untilt F of C with $|p|_F \leq r$, there is a corresponding point in $\mathcal{M}(B_{[0,r]})$ given by $y \mapsto |\text{ev}_F(y)|_F$. Points of this form are the **classical points**.

Then our goal, which will Proposition 11.1, is the following:

Proposition 12.1. *Zariski closed sets in $\mathcal{M}(B_{[0,r]})$ are finite sets of classical points.*

To prove this, we need more general machinery, together with the observation that if $\pi \in C$ has $|\pi| = r$, then $B_{[0,r]} = (W(\mathcal{O}_C)[p/\pi])_{[\pi]}^\wedge[1/\pi]$.

Let A_0 be an f -torsion free, f -complete ring, where $f \in A_0$. Then $A = A_0[1/f]$ can be made into a Banach algebra such that its unit ball is A_0 , $\|f\| = r$, and $\|f^{-1}\| = r^{-1}$, for any $r \in (0, 1)$. This is well-defined up to equivalence/isogeny (powers of f). Then $\mathcal{M}(A)$ is the set of equivalence classes of maps $A_0 \rightarrow V$, where V is a rank 1 valuation ring, and where f is mapped to a pseudouniformizer. We can even take V to be perfectoid. Call this set of equivalence classes $\mathcal{M}_f(A_0)$.

Note. The Berkovich spectrum doesn't actually need the norm, i.e. it can be defined not just for Banach algebras, but more generally for things called Tate rings. These are topological rings with an open subring A_0 that has the f -adic topology.

Now suppose A_0 is a perfectoid ring which is π -complete and π -torsion free, where $\pi \in A_0$ has $\pi = (\pi')^\sharp$ for some π' (hence we can take $\pi' = \pi^\flat$), and $\pi^p | p$. We can form $A = A_0[\frac{1}{\pi}]$ and $A^\flat = A_0^\flat[\frac{1}{\pi^\flat}]$.

Theorem 12.1. *There is a homeomorphism $\flat : \mathcal{M}_\pi(A_0) = \mathcal{M}(A) \xrightarrow{\sim} \mathcal{M}(A^\flat) = \mathcal{M}_{\pi^\flat}(A_0^\flat)$ defined as follows. Given $|\cdot| : A \rightarrow \mathbb{R}_{\geq 0}$, we define $\flat(|\cdot|) = |\cdot|^\flat : A^\flat \rightarrow \mathbb{R}_{\geq 0}$ by $|a'|^\flat = |a'^\sharp|$.*

Proof. The fact that the map is a bijection follows from tilting equivalence (???). The fact that it is a homeomorphism is more difficult, since not everything in A is in the image of \sharp . We will not need that it is a homeomorphism, so we will not prove it. 

Proposition 12.2. *Under $\mathcal{M}(A) \cong \mathcal{M}(A^\flat)$, Zariski closed subsets correspond.*

Proof. Zariski closed subsets of $\mathcal{M}(A)$ are things of the form $\mathcal{M}_\pi(\overline{A}_0)$, where \overline{A}_0 is a quotient of A_0 . But $\mathcal{M}_\pi(\overline{A}_0) = \mathcal{M}((\overline{A}_0)_{perf})$ (because they're controlled

by maps into perfectoid things), so in fact the Zariski closed subsets correspond to perfectoid quotients. Now the perfectoid quotients of A_0 , A_0^\flat correspond and the spectra match (why?). 

The unit ball in $B_{[0,r]}$ is $(W(\mathcal{O}_C)[x]/([\pi]x-p))_{[\pi]}^\wedge$. We can map this to $W(\mathcal{O}_C)[x^{1/p^\infty}]_{[\pi]}^\wedge/([\pi]x-p)$, which is the same as $W(\mathcal{O}_C\langle x^{1/p^\infty} \rangle)/([\pi][x]-p)$, which is perfectoid. We call it $B_{[0,r]}^{\leq 1,\infty}$. The induced map in Berkovich spectra is surjective. Thus $\mathcal{M}(B_{[0,r]})$ is a quotient of $\mathcal{M}_{[\pi]}(B_{[0,r]}^{\leq 1,\infty})$, which, by tilting, is $\mathcal{M}_\pi(\mathcal{O}_C\langle x^{1/p^\infty} \rangle)$, or $\mathcal{M}(C\langle x^{1/p^\infty} \rangle)$. But $\mathcal{M}(C\langle x^{1/p^\infty} \rangle) = \mathcal{M}(C\langle T \rangle)$ as topological spaces (though they have different Zariski closed sets), since knowing what an absolute value does on T determines what it does on p power roots. This gives $\mathcal{M}(C\langle x^{1/p^\infty} \rangle)$ the notion of classical points, as well as points of types 2-4.

To finish our argument, we need to show that this map $\mathcal{M}(C\langle x^{1/p^\infty} \rangle) \rightarrow \mathcal{M}(B_{[0,r]})$ sends classical points to classical points, and that the Zariski closed sets in $\mathcal{M}(C\langle x^{1/p^\infty} \rangle)$ consist only of classical points. Towards the first point, we need to untilt everything explicitly and unwind definitions. One ends up finding that classical points on both sides correspond to things that are quotients of $W(\mathcal{O}_C)$. For the second point, one notes that nonzero elements of $C\langle x^{1/p^\infty} \rangle$ cannot vanish at points of type 2 or 3, because the vanishing condition forces the series to be 0. Then one can enlarge the field C to get rid of type 4 points. The preimage of a type 4 point will consist of type 2 points in the larger field.

Note. Lecturer is curious in an example of an element in $C\langle x^{1/p^\infty} \rangle$ such that one can explicitly compute its vanishing locus and see that it is infinite.

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Let C be a characteristic p CACNA. We introduce the notation \mathcal{Y}^{cl} for the set of untilts of C up to isomorphism. Our guiding heuristic is that \mathcal{Y}^{cl} is like $D_{<1}$ in C . In particular, we defined an algebra $B_{[0,r]}$ of “functions on $\mathcal{Y}_{|p|\leq r}^{cl}$ ” and showed that $B_{[0,r]}$ is a PID whose maximal spectrum can be identified with $\mathcal{Y}_{|p|\leq r}^{cl}$.

There is an induced Frobenius map on \mathcal{Y}^{cl} , as follows. The Frobenius maps on $W(\mathcal{O}_C)$ and $W(C)$ give rise to an isomorphism $\varphi : B_{[0,r]} \rightarrow B_{[0,r^p]}$. Since we are working with PIDs, this gives a map on max spectra, $\mathcal{Y}_{|p|\leq r^p}^{cl} \rightarrow \mathcal{Y}_{|p|\leq r}^{cl}$. If we take unions as $r \rightarrow 1$, we get an automorphism φ on \mathcal{Y}^{cl} .

Recall that \mathcal{Y}^{cl} can be identified with $W_{prim}(\mathcal{O}_C)/W(\mathcal{O}_C)^\times$. If a point $* \in \mathcal{Y}^{cl}$ is represented by $\xi \in W_{prim}(\mathcal{O}_C)$, then $\varphi(*)$ is represented by $\varphi^{-1}(\xi)$.

If we consider the map $|p| : \mathcal{Y}^{cl} \rightarrow [0, 1]$, then $|p|(\varphi(*)) = (|p|(*))^{1/p}$. In terms of our heuristic that \mathcal{Y}^{cl} is like $D_{<1}$, we see that the Frobenius is like $\alpha \mapsto \alpha^{1/p}$.

Definition 13.1. Let $Y^{cl} = \mathcal{Y}^{cl} \setminus \{C\}$.

This is analogous to a punctured disk, and it is preserved by φ . In fact, the group $\varphi^{\mathbb{Z}}$ acts freely on it, since we have removed the only untilt where $|p| = 0$, and when $|p| \neq 0$, a nontrivial power of φ strictly changes the radius.

Fargues-Fontaine construct:

1. “Analytic spaces” \mathcal{Y} and $Y \subset \mathcal{Y}$ containing \mathcal{Y}^{cl} and Y^{cl} as “classical points”, together with automorphisms φ on Y and \mathcal{Y} . The notion of analytic space here is Huber adic space. Y and \mathcal{Y} are supposed to be thought of as p -adic Riemann surfaces.
2. $\varphi^{\mathbb{Z}}$ acts freely on Y , and there is an adic space $X = Y/\varphi^{\mathbb{Z}}$.
3. There is a scheme X^{alg} with a proper map of locally ringed spaces $X \rightarrow X^{alg}$. X is the “analytification” of X^{alg} . There is a GAGA-like result relating the theory of vector bundles on X and X^{alg} . X^{alg} is the (algebraic) Fargues-Fontaine curve.
4. X^{alg} is a regular 1-dimensional scheme, and its closed points are $Y^{cl}/\varphi^{\mathbb{Z}}$.
5. There is a line bundle $\mathcal{O}(1)$ on X^{alg} (or X , by GAGA) which is trivial on Y , with $\varphi^{\mathbb{Z}}$ -equivariance given by multiplication by p . Heuristically, $\mathcal{O}(1)$ is “ample”.
6. Consider the ring B of functions on Y . B has a Frobenius automorphism, and $\Gamma(X, \mathcal{O}(1)) = B^{\varphi=p}$. This leads to a definition of X^{alg} , namely $X^{alg} = \text{Proj}(\bigoplus \Gamma(X, \mathcal{O}(n))) = \text{Proj}(\bigoplus B^{\varphi=p^n})$.

13.1 Functions on punctured open disk

The ring B discussed above is analogous to the ring of functions on the punctured open unit disk, $\mathcal{O}(D_{<1}^\times)$. Namely this is the ring of Laurent series with coefficients in C , $\sum_{i=-\infty}^{\infty} a_i z^i$, such that for all $r \in (0, 1)$, we have $|a_i|r^i \rightarrow 0$ as $i \rightarrow \pm\infty$.

Definition 13.2. The ring of “analytic functions” on $\{r \leq |z| \leq s\} \subset C$ is the set of Laurent series $\sum a_i z^i$ such that $|a_i|r^i \rightarrow 0$ and $|a_i|s^i \rightarrow 0$ as $i \rightarrow \pm\infty$. We denote this by $C\langle \frac{r}{z}, \frac{z}{s} \rangle$.

It is not clear a priori that this is in fact a ring.

Assume $r = |\pi_r|$ and $s = |\pi_s|$. We give alternative constructions of $C\langle \frac{r}{z}, \frac{z}{s} \rangle$:

1. Complete $C[z^{\pm 1}]$ with respect to the maximum of the r -Gauss norm and the s -Gauss norm.
2. Take any pseudouniformizer π and π -complete $\mathcal{O}_C[\frac{z}{\pi_s}, \frac{\pi_r}{z}]$ and then invert π .

Theorem 13.1. $C\langle \frac{r}{z}, \frac{z}{s} \rangle$ is a PID, and its maximal spectrum consists of $(z - \alpha)$ for $\alpha \in C$ with $r \leq |\alpha| \leq s$.

Remark. Let $s \leq 1$. Then $\mathcal{M}(C\langle \frac{r}{z}, \frac{z}{s} \rangle) \subset \mathcal{M}(C(z))$, and it is exactly the subset of absolute values where $r \leq |z| \leq s$. This can be used to prove the theorem.

Remark. Given $f \in C\langle \frac{r}{z}, \frac{z}{s} \rangle$ and $t \in [r, s]$, let $N_{t,\pm}(f)$ be the largest/smallest degree of a monomial in f among those maximizing the t -Gauss norm. Then:

Proposition 13.1. The number of zeroes of f in $\{r \leq |z| \leq s\}$ is $N_{s,+}(f) - N_{r,-}(f)$.

Observe that $\mathcal{O}(D_{<1}^\times) = \varprojlim_{[r,s] \subset (0,1)} C\langle \frac{r}{z}, \frac{s}{z} \rangle$, which is a Frechet algebra. We can also characterize this as the completion of $C[z^{\pm 1}]$ with respect to all r -Gauss norms for $r \in (0, 1)$.

Given $f \in \mathcal{O}(D_{<1}^\times)$, let $\text{div}(f) = \sum_{q \in D_{<1}^\times} (\text{ord}_q f)[q]$. This is an infinite sum, but finite in any annulus.

Proposition 13.2. $f \mid g$ iff $\text{div}(f) \leq \text{div}(g)$.

Question: For any locally finite divisor D in $D_{<1}^\times$, is there $f \in \mathcal{O}(D_{<1}^\times)$ such that $\text{div}(f) = D$?

Answer (Lazard, '62): Yes iff C is spherically complete.