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Winter 2026

Preface

There will be some gaps in explanation, either due to the lecturer's admission or my own lack of understanding. In particular, many "proofs" are sketches of proofs. Gaps due to my own misunderstanding will be indicated by three red question marks: ???. More generally, my own questions about the material will also be in red. Things like "**Question**" will be questions posed by the lecturer. Feel free to reach out to me with explanations.

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1 Jan 5 - Perfect Rings and Tilting

1.1 Perfect Rings

Definition 1.1. Let R be an $\mathbb{F}_p$ algebra. We say R is **perfect** if the Frobenius map $\varphi : R \to R, x \mapsto x^p$ is an isomorphism.

Example 1.1.
1. Perfect field.
2. $\mathbb{F}_p[x^{1/p^\infty}] = \bigcup_{n\geq 0} \mathbb{F}_p[x^{1/p^n}]$; the free perfect ring on 1 generator.
3. Any limit or colimit of perfect $\mathbb{F}_p$ algebras.
4. Any integrally closed domain whose fraction field is perfect (e.g. algebraically closed).
5. If R is perfect, I is fg ideal, then $R_I^\wedge$ (the I-completion) is perfect.

Example 1.2. The inclusion of the category of perfect $\mathbb{F}_p$-algebras into the category of all $\mathbb{F}_p$-algebras has both adjoints. The left adjoint $R \mapsto R_{perf}$ is the colimit of $R \xrightarrow{\varphi} R \xrightarrow{\varphi} \cdots$. The right adjoint $R \mapsto R^{perf}$ is the limit of $\cdots \xrightarrow{\varphi} R \xrightarrow{\varphi} R$.

Example 1.3.
1. $\mathbb{F}_p[x^{1/p^\infty}] = \mathbb{F}_p[x]_{perf}$.
2. $\mathbb{F}_p[x]^{perf} = \mathbb{F}_p$. In general, if R is Noetherian, then R^{perf} is a finite product of fields.
3. If φ is surjective (R is called semiperfect) then we get a surjection $R^{perf} \twoheadrightarrow R$, so R^{perf} is "larger" than R.
4. If P is perfect and I is a fg ideal, then $R = P/I$ is semiperfect, and $R^{perf} = P_I^\wedge$. (idea: frobenius iterates of I are cofinal with powers of I) (Exercise: prove this, and also find R_{perf}.)

Example 1.4. As a concrete example, let $R = \mathbb{F}_p[x^{1/p^\infty}]/(x)$. Then $R^{perf} = \mathbb{F}_p[x^{1/p^\infty}]_x^\wedge$, which consists of sums

$$\sum_{i\in\mathbb{Z}[1/p]_{\geq 0}} a_i x^i,$$

with $a_i \in \mathbb{F}_p$, and for all $N \geq 0$, there are only finitely many $i \leq N$ with $a_i \neq 0$. So $x + x^{p+1/p} + x^{p^2+1/p^2} + \cdots$ is allowed, while $x + x^{1/p} + x^{1/p^2} + \cdots$ isn't.

1.2 Witt Vectors

Perfect $\mathbb{F}_p$-algebras have a unique lift to characteristic 0.

Theorem 1.1. *Given any perfect $\mathbb{F}_p$-algebra R, there is a unique (up to unique isomorphism) p-adically complete and p-torsion free ring $\widetilde{R}$ equipped with an isomorphism $\widetilde{R}/p \xrightarrow{\sim} R$. This construction is functorial in R, and $\widetilde{R}$ is the ring $W(R)$ of Witt vectors of R.*

Proof. See chapter 2 of Serre's "Local Fields".

Example 1.5.
1. $R = \mathbb{F}_p[x^{1/p^\infty}]$ gives $\widetilde{R} = \mathbb{Z}_p[x^{1/p^\infty}]^\wedge_p$.

2. $R = \mathbb{F}_p[x^{1/p^\infty}]^\wedge_x$ gives $\widetilde{R} = \mathbb{Z}_p[x^{1/p^\infty}]^\wedge_{(p,x)}$. add description of elements in this ring

3. For R_1, R_2 perfect, $W(R_1 \otimes R_2) = (W(R_1) \otimes W(R_2))^\wedge_p$.

Structure of $W(R)$: There is a unique multiplicative but not additive map $[-] : R \to W(R)$ that is a section of the projection $W(R) \to R$.

Corollary 1.1. *For any $x \in W(R)$, there is a unique sequence $x_0, x_1, x_2, \ldots, \in R$ such that $x = \sum_{i \geq 0} [x_i] p^i$. So, as a set, $W(R) \cong \prod_{i \geq 0} R$.*

Exercise: the image of $[-]$ is the elements of $W(R)$ that are p^n powers for all n.

Analogy: $W(R)$ is like a ring of power series over R with variable p.

How to add Witt vectors: Suppose $x = \sum [x_i] p^i$ and $y = \sum [y_i] p^i$. We know $x + y = \sum [z_i] p^i$ for some $z_i \in R$. In fact, for each n, $z_n \in \mathbb{F}_p[x_0^{1/p^\infty}, \ldots, x_n^{1/p^\infty}, y_0^{1/p^\infty}, \ldots, y_n^{1/p^\infty}]$. Key fact: the polynomial expressing z_n in this way is homogeneous of degree 1, where each x_i, y_i has degree 1.

Example 1.6. $z_0 = x_0 + y_0$.
$z_1 = x_1 + y_1 - \sum_{0<i<p} \frac{1}{p} \binom{p}{i} x_0^{i/p} y_0^{1-i/p}$. (Exercise: prove this; use $F = W(\varphi)$)

1.3 Tilting

Definition 1.2. For any ring R, its **tilt** $R^\flat$ is $(R/p)^{perf}$.

Proposition 1.1. *If R is p-complete, then $\varprojlim(\cdots \xrightarrow{\varphi} R \xrightarrow{\varphi} R) \xrightarrow{\sim} R^\flat$.*

Example 1.7.
1. $R = \mathbb{Z}_p$ has $R^\flat = \mathbb{F}_p$.

2. $R = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge_p = (\mathbb{Z}_p[x^{1/p^\infty}]/(x-p))^\wedge_p$. Then $R/p = \mathbb{F}_p[x^{1/p^\infty}]/(x)$ and $R^\flat = \mathbb{F}_p[x^{1/p^\infty}]^\wedge_x$.

Remark. If R is p-complete, there is a natural multiplicative map $(-)^\sharp : R^\flat \to R$. For example, in the second example above, $x^\sharp = p$.

Theorem 1.2. *The functor W from perfect $\mathbb{F}_p$-algebras to p-complete rings is left adjoint to the tilting functor from p-complete rings to perfect $\mathbb{F}_p$-algebras; if A is perfect/$\mathbb{F}_p$ and R is p-complete, then $\mathrm{Hom}(W(A), R) = \mathrm{Hom}_{\mathbb{F}_p}(A, R^\flat)$.*

Remark. If A is perfect/$\mathbb{F}_p$ and R is p-complete, then there is a natural map $\mathrm{Hom}(W(A), R) \to \mathrm{Hom}_{\mathbb{F}_p}(A, R/p)$. Since A is perfect, the second hom set can be identified with $\mathrm{Hom}_{\mathbb{F}_p}(A, R^\flat)$.