MATH 7520 Homework 4

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Problem 18

Show than an n-connected, n-dimensional CW complex X is contractible.

Proof. If $n = 0$, then X is a point, which is contractible by definition. Thus let $n > 0$. Let $f : X \to \bullet$ be the map to a point. Since X is n-dimensional, $H_k(X) = 0$ for $k > n$. By the Hurewicz theorem (Theorem 4.32 in Hatcher), we have $\tilde{H}_i(X) = 0$ for $i < n + 1$. Thus, f induces isomorphisms $H_i(X) \to H_i(\bullet)$ for all $i > 0$, since the only map $0 \to 0$ is an isomorphism. For $i = 0$, we know that $H_0(X) \cong \mathbb{Z}$ with generator $\bullet \hookrightarrow X$. The image under f_* is $\bullet \hookrightarrow \bullet$, which generates $H_0(\bullet)$. Thus, by Corollary 4.33 in Hatcher, f is a homotopy equivalence, so X is contractible. $\square$

Problem 19

(31) For a fiber bundle $F \to E \to B$ such that the inclusion $F \hookrightarrow E$ is homotopic to a constant map, show that the long exact sequence of homotopy groups breaks up into split short exact sequences giving isomorphisms $\pi_n(B) \cong \pi_n(E) \oplus \pi_{n-1}(F)$.

(32) Show that if $S^k \to S^m \to S^n$ is a fiber bundle, then $k = n-1$ and $m = 2n-1$.

(33) Show that if there were fiber bundles $S^{n-1} \to S^{2n-1} \to S^n$ for all n, then the groups $\pi_i(S^n)$ would be finitely generated free abelian groups computable by induction, and nonzero for $i \geq n \geq 2$.

Proof. (31) Since homotopic maps induce the same maps in homotopy groups, the inclusion $F \hookrightarrow E$ induces the 0 map $\pi_n(F) \to \pi_n(E)$, since a constant map induces the 0 map. Using the definition of exactness, the map $\pi_n(E) \to \pi_n(B)$ has kernel equal to the image of $\pi_n(F) \to \pi_n(E)$, which is 0, so $\pi_n(E) \to \pi_n(B)$ is injective. Similarly, the image of $\pi_n(B) \to \pi_{n-1}(F)$ is the kernel of the 0 map $\pi_{n-1}(F) \to \pi_{n-1}(E)$, which is $\pi_{n-1}(F)$, so $\pi_n(B) \to \pi_{n-1}(F)$ is surjective. Thus the sequence $0 \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to 0$ is exact.

To show that these sequences are split, we construct a left inverse to $\partial : \pi_n(B) \to \pi_{n-1}(F)$ and apply the splitting lemma. All maps will be pointed, but I will not bother writing out basepoints.

Let $f : S^{n-1} \to F$ be a map. Since $F \hookrightarrow E$ is nullhomotopic, f extends to a map $\tilde{f} : (D^n, S^{n-1}) \to (E, F)$ (I developed this construction in the last homework sheet). This extension is well-defined up to homotopy. Using $p : (E, F) \to B$, we get $p\tilde{f} : (D^n, S^{n-1}) \to B$. Since post-composition takes homotopic maps to homotopic maps, the assignment $f \mapsto p\tilde{f}$ is well-defined up to homotopy. We can identify $p\tilde{f}$ with a map $\hat{f} : S^n \to B$ by precomposing with an isomorphism $S^n \cong D^n/S^{n-1}$. Composition with isomorphisms preserves homotopy, so $f \mapsto \hat{f}$ is well-defined up to homotopy. In other words, we have a function $\kappa : \pi_{n-1}(F) \to \pi_n(B)$.

We must check two things. First, that κ is a group homomorphism for $n \geq 2$. Second, that κ is a left inverse to ∂.

I think we also need to show that κ is a homomorphism, but I'm not sure how to do that.

Let us recall the definition of ∂. ∂ takes (the homotopy class of) a map $f : S^n \to B$, precomposes by the quotient isomorphism $S^n \cong D^n/S^{n-1}$ to get $f : (D^n, S^{n-1}) \to B$, lifts (which exists and is unique up to homotopy by Hatcher -2-27-78-637 + 73–) to $\overline{f} : (D^n, S^{n-1}) \to (E, F)$, and then restricts to

$\partial\overline{f} : S^{n-1} \to F$.

Now we show that $\partial\kappa = \mathrm{id}$. Since ∂ is well-defined up to homotopy, it suffices to show that $[f] = \partial[\hat{f}]$. When ∂ calls for us to lift $\hat{f} : (D^n, S^{n-1}) \to B$, we note that $\hat{f}$ is the projection of $\tilde{f}$, so we can take the lift to be $\tilde{f}$. Then we restrict $\tilde{f}$ to S^{n-1}, and by definition this is f. Thus, the short exact sequences split.

(32) We first discard trivial cases.

Suppose $n = 0$. If $m > 0$, then the map $S^m \to S^0$ must be constant, which it can't be ($E \to B$ must be surjective). Thus $m = 0$, and the map $S^0 \to S^0$ is a homeomorphism. It follows that the fiber would have to be a singleton, which no sphere is. Thus $n > 0$.

If $m = 0$, then $S^0 \to S^n$ can't be surjective, since S^n has infinitely many points. Thus $m > 0$, and S^n and S^m are path-connected.

If $k = 0$, then by Example 4.42 in Hatcher, $S^m \to S^n$ is a covering space, with $|S^0| = 2$ sheets. By Proposition 1.32 of Hatcher, $p_*(\pi_1(S^m))$ has index 2 in $\pi_1(S^n)$. But $\pi_1(S^m)$ (resp. $\pi_1(S^n)$) is either $\mathbb{Z}$ or 0, according to whether $m = 1$ (resp. $n = 1$) or $m > 1$ (resp. $n > 1$). If either of these groups are 0, then it is impossible to have the index 2 condition. The only possibility is $m = n = 1$. In this case, we have $k = n - 1$ and $m = 2n - 1$ as desired. We henceforth assume $k > 0$.

Pick an open subset U in S^n so that $p^{-1}(U) \cong U \times S^k$. S^n is locally homeomorphic to $\mathbb{R}^n$, so that U is as well. Then $p^{-1}(U)$ is locally homeomorphic to $\mathbb{R}^{n+k}$. But as an open subset of S^m, it is also locally homeomorphic to $\mathbb{R}^m$. By comparing compactly supported cohomology, we then have $m = n + k$. Since $n, k > 0$, we get $n, k < m$.

Since $k < m$, the map $S^k \hookrightarrow S^m$ is homotopic to a constant map by cellular approximation. Thus, by the previous problem, we have $\pi_n(S^n) \cong \pi_n(S^m) \oplus \pi_{n-1}(S^k)$. The left hand side is $\mathbb{Z}$. Since $n < m$, $\pi_n(S^m) = 0$, so $\pi_{n-1}(S^k) = \mathbb{Z}$. Thus $n - 1 \geq k$.

Assume $n - 1 > k$. Then from the long exact sequence of homotopy groups, we have $\pi_{k+1}(S^n) \to \pi_k(S^k) \to \pi_k(S^m)$ is exact. But $k + 1 < n$ and $k < m$, so this sequence reads $0 \to \mathbb{Z} \to 0$, which is not exact. Thus $n - 1 = k$. Combining with $m = n + k$ gives $m = 2n - 1$.

(33) Let us ignore the case $S^0 \to S^1 \to S^1$, so that $n > 1$, $2n - 1 > n$, and all the spheres are path-connected. We know that $\pi_i(S^n) = 0$ for $i < n$ by cellular approximation. For $i \geq 2$, we have by problem (31) group isomorphisms $\pi_i(S^n) \cong \pi_i(S^{2n-1}) \oplus \pi_{i-1}(S^{n-1})$. Putting $i = n$, we get $\pi_n(S^n) \cong \pi_{n-1}(S^{n-1})$.

Since this is valid for all $n \geq 2$, we get $\pi_n(S^n) \cong \pi_1(S^1) \cong \mathbb{Z}$ by induction.

Since $\pi_i(S^1) = 0$ for $i > 1$, putting $n = 2$ gives $\pi_i(S^2) \cong \pi_i(S^3)$ for $i \geq 3$.

Putting $n = 3$ gives $\pi_i(S^3) \cong \pi_i(S^5) \oplus \pi_{i-1}(S^2)$. Assuming $i \geq 3$ and using the previous isomorphism, we get $\pi_i(S^2) \cong \pi_i(S^5) \oplus \pi_{i-1}(S^2)$. Since we know $\pi_i(S^5)$ for $i \leq 5$, we get $\pi_i(S^2)$ (and $\pi_i(S^3)$) for $i \leq 5$. By induction on the isomorphisms $\pi_i(S^n) \cong \pi_i(S^{2n-1}) \oplus \pi_{i-1}(S^{n-1})$ for $n > 3$, and using $n+2 < 2n-1$ for $n > 3$, we acquire $\pi_i(S^n)$ for all (i,n) such that $i \leq n+2$.

Suppose we know $\pi_i(S^n)$ for all (i,n) such that $i \leq n+k$, and $k \geq 2$. Then in order to get to know $\pi_i(S^n)$ for all (i,n) such that $i \leq n+k+1$, we need to know the groups $\pi_{k+3+j}(S^{2j+3})$ for $j \in \{1, \ldots, k-1\}$. These groups are not trivially 0 or $\mathbb{Z}$, since $k+3+j > 2j+3$. But each of these groups have already been computed, since $k+3+j \leq 2j+3+k$.

Thus, we can inductively compute any $\pi_i(S^n)$. Since they are all obtained by direct summation, and our base cases were 0 and $\mathbb{Z}$, they are all finitely generated and free abelian.

To establish the last claim, we run through our procedure again, this time using the that $\pi_i(S^n)$ is finitely generated and free abelian; we denote its rank by $r(i,n)$. We only need to show non-vanishing, so we will not bother with the exact values (if non-zero). By our preliminary work, we have $r(i,n) \geq 0$ for all (i,n), $r(n,n) > 0$ for all n, and $r(i,2) = r(i,3)$ for $i \geq 3$. The splitting gives the recurrence relation $r(i,n) = r(i,2n-1) + r(i-1,n-1)$. Consider the claim: $r(i,n) > 0$ for $i \geq n \geq 2$ such that $i - n \leq k$, with $k \geq 0$. The case $k = 0$ has already been proven. Assume it is true for some fixed k. We want to show that $r(n+k+1,n) > 0$ for $n \geq 2$. For the base case, we have $r(3+k,2) = r(3+k,3) > 0$. Then the inductive step follows from the recurrence relation: $r(n+k+2,n+1) = r(n+k+2,2n+1) + r(n+k+1,n) \geq r(n+k+1,n) > 0$. Thus $\pi_i(S^n)$ is non-zero for $i \geq n \geq 2$. $\square$

Problem 20

If $F \to E \to B$ is a fiber bundle with fiber F, and $F \hookrightarrow E$ is a retract, how are the homotopy groups $\pi_n(E), \pi_n(F), \pi_n(B)$ related?

Proof. Let $r : E \to F$ be such that the composition $F \hookrightarrow E \xrightarrow{r} F$ is the identity on F. Then the map $\pi_n(F) \to \pi_n(E)$ has left inverse r_*, so it must be injective. This means that the maps $\pi_{n+1}(B) \to \pi_n(F)$ are 0, so we get short exact sequences $0 \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to 0$. By the splitting lemma, the existence of the left inverse r_* implies that these short exact sequences split, so $\pi_n(E) \cong \pi_n(B) \oplus \pi_n(F)$. $\square$