

# Fargues Fontaine Curve

## Course by Akhil Mathew

Typed by Andrea Bourque

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### Preface

There will be some gaps in explanation, either due to the lecturer's admission or my own lack of understanding. In particular, many "proofs" are sketches of proofs. Gaps due to my own misunderstanding will be indicated by three red question marks: ???. More generally, my own questions about the material will also be in red. Things like "**Question**" will be questions posed by the lecturer. Feel free to reach out to me with explanations.

As a course specific note, at some point I start using the abbreviation CACNA for a complete algebraically closed non-archimedean field.

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# 1 Jan 5 - Perfect Rings and Tilting

## 1.1 Perfect Rings

**Definition 1.1.** Let  $R$  be an  $\mathbb{F}_p$  algebra. We say  $R$  is **perfect** if the Frobenius map  $\varphi : R \rightarrow R, x \mapsto x^p$  is an isomorphism.

**Example 1.1.** 1. Perfect (characteristic  $p$ ) field.

2.  $\mathbb{F}_p[x^{1/p^\infty}] = \bigcup_{n \geq 0} \mathbb{F}_p[x^{1/p^n}]$ ; the free perfect ring on 1 generator. (All monomials have  $p$ th roots.)
3. Any limit or colimit of perfect  $\mathbb{F}_p$  algebras (It's enough to check for pull-backs, arbitrary products, pushouts, and arbitrary coproducts (tensor products). In all of these cases, it's enough to check on generators, in which case the  $p$ th roots are obvious.)
4. Any integrally closed domain whose fraction field is perfect (e.g. algebraically closed) ( $x^p - a$  is monic and has a solution in the fraction field).
5. If  $R$  is perfect,  $I$  is a finitely generated ideal, then  $R_I^\wedge$  (the  $I$ -completion) is perfect. (Can try to take  $p$ th roots of a compatible system of elements in  $R/I^n \dots$  or topologically, show  $\varphi : R \rightarrow R$  is  $I$ -adically continuous.)

The inclusion of the category of perfect  $\mathbb{F}_p$ -algebras into the category of all  $\mathbb{F}_p$ -algebras has both adjoints. The left adjoint  $R \mapsto R_{perf}$  is the colimit of  $R \xrightarrow{\varphi} R \xrightarrow{\varphi} \dots$ . The right adjoint  $R \mapsto R^{perf}$  is the limit of  $\dots \xrightarrow{\varphi} R \xrightarrow{\varphi} R$ .

The idea of the colimit perfection is to add all  $p$ -power roots, and the idea of the limit perfection is to pick things that have all  $p$ -power roots, although these ideas can sometimes not match reality

The adjunction means that if  $R$  is an  $\mathbb{F}_p$ -algebra and  $P$  is a perfect  $\mathbb{F}_p$ -algebra, we have  $\text{Hom}(R_{perf}, P) = \text{Hom}(R, P)$  and  $\text{Hom}(P, R) = \text{Hom}(P, R^{perf})$ .

To see the first adjunction, note that by the definition of a colimit,  $f \in \text{Hom}(R_{perf}, P)$  corresponds to the data of maps  $f_n : R \rightarrow P$  where  $f_n = f_{n+1} \circ \varphi_R$ . But maps of  $\mathbb{F}_p$ -algebras are “Frobenius equivariant”, so  $f_{n+1} \circ \varphi_R = \varphi_P \circ f_{n+1}$ . Since  $P$  is perfect,  $\varphi_P$  is an isomorphism, so  $f_{n+1} = \varphi_P^{-1} \circ f_n$ . Thus the sequence is determined by  $f_1$  via  $f_n = \varphi_P^{-n+1} \circ f_1$ .

To see the second adjunction, note that an element  $x \in P$  must have all  $p$ -power roots, so it must be sent to an element  $y \in R$  that also has all  $p$ -power roots, since if  $f(x) = y$ , then  $f(x^{1/p})^p = f(x) = y$ . Alternatively, since  $R^{perf}$  is a limit, we know  $f \in \text{Hom}(P, R^{perf})$  corresponds to the data of maps  $f_n : P \rightarrow R$  where  $f_n = \varphi_R \circ f_{n+1}$ . As before, we can use  $\varphi_R \circ f_{n+1} = f_{n+1} \circ \varphi_P$ , and then since  $\varphi_P$  is an isomorphism, we get  $f_{n+1} = f_n \circ \varphi_P^{-1}$ . Thus  $f_1$  determines the sequence  $f_n$  via  $f_n = f_1 \circ \varphi_P^{-n+1}$ .

- Example 1.2.** 1.  $\mathbb{F}_p[x^{1/p^\infty}] = \mathbb{F}_p[x]_{perf}$ . As said above, when you take the colimit, the data involved is a ring with an element with a  $p$ th root, a  $p^2$ th root, and so on.
2.  $\mathbb{F}_p[x]^{perf} = \mathbb{F}_p$ . As said above, the data of the limit is the choice of an element in  $\mathbb{F}_p[x]$  that has all  $p$ -power roots; only the constants satisfy this. In general, if  $R$  is Noetherian, then  $R^{perf}$  is a finite product of fields.
3. If  $\varphi$  is surjective ( $R$  is called semiperfect) then we get a surjection  $R^{perf} \twoheadrightarrow R$ , so  $R^{perf}$  is “larger” than  $R$ , which is somewhat contrary to the above intuition of “taking the things which have all  $p$ -power roots”.
4. If  $P$  is perfect and  $I$  is a finitely generated ideal, then  $R = P/I$  is semiperfect, and  $R^{perf} = P_I^\wedge$ . (Exercise: prove this, and also find  $R_{perf}$ .) Solution:  $R$  is semiperfect since the composition  $P \xrightarrow{\varphi} P \rightarrow P/I$  is surjective with kernel  $\varphi^{-1}(I) \supset I$ , so it descends to a surjective map  $R \xrightarrow{\varphi} R$ . To prove  $R^{perf} = P_I^\wedge$ , we first note that we can compute  $P_I^\wedge$  as  $\varprojlim_n P/I^{p^n}$ . Then, because  $I$  is finitely generated,  $\phi : P/I \rightarrow P/I^p$  is an isomorphism. Then we can identify the corresponding limit diagrams using iterations of Frobenius:

$$\begin{array}{ccccc} \cdots & \xrightarrow{\varphi} & P/I & \xrightarrow{\varphi} & P/I \\ \varphi^n \downarrow & & \downarrow \varphi & & \downarrow = \\ \cdots & \longrightarrow & P/I^p & \longrightarrow & P/I \end{array}$$

Here the arrows with double line marks are meant to be isomorphisms. The desired isomorphism  $R^{perf} = P_I^\wedge$  follows. To compute  $R_{perf}$ , the idea is that the colimit is modding out by elements  $x \in P$  such that  $\phi^n(x) \in I$  eventually. But the sequence  $\phi^n(x) = x^{p^n}$  is cofinal with the sequence  $x^n$ , so we are modding out by the radical  $\text{rad}(I)$ . Thus  $R_{perf} = P/\text{rad}(I)$ .  
(this may be wrong)

**Example 1.3.** As a concrete example of the previous case, let  $P = \mathbb{F}_p[x^{1/p^\infty}]$ ,  $I = (x)$ ,  $R = \mathbb{F}_p[x^{1/p^\infty}]/(x)$ . Then  $R^{perf} = \mathbb{F}_p[x^{1/p^\infty}]_x^\wedge$ , which consists of sums

$$\sum_{i \in \mathbb{Z}[1/p]_{\geq 0}} a_i x^i,$$

with  $a_i \in \mathbb{F}_p$ , and for all  $N \geq 0$ , there are only finitely many  $i \leq N$  with  $a_i \neq 0$ . So  $x + x^{p+1/p} + x^{p^2+1/p^2} + \cdots$  is allowed, while  $x + x^{1/p} + x^{1/p^2} + \cdots$  isn't.

## 1.2 Witt Vectors

Perfect  $\mathbb{F}_p$ -algebras have a unique lift to characteristic 0.

**Theorem 1.1.** *Given any perfect  $\mathbb{F}_p$ -algebra  $R$ , there is a unique (up to unique isomorphism)  $p$ -adically complete and  $p$ -torsion free ring  $\tilde{R}$  equipped with an isomorphism  $\tilde{R}/p \xrightarrow{\sim} R$ . This construction is functorial in  $R$ , and  $\tilde{R}$  is the ring  $W(R)$  of Witt vectors of  $R$ .*

*Proof.* See chapter 2 of Serre's "Local Fields".



**Example 1.4.** 1.  $R = \mathbb{F}_p$  gives  $W(R) = \mathbb{Z}_p$ .

2.  $R = \mathbb{F}_p[x^{1/p^\infty}]$  gives  $W(R) = \mathbb{Z}_p[x^{1/p^\infty}]_p^\wedge$ .

3.  $R = \mathbb{F}_p[x^{1/p^\infty}]_x^\wedge$  gives  $W(R) = \mathbb{Z}_p[x^{1/p^\infty}]_{(p,x)}^\wedge$ . The elements of this ring are series  $\sum_{i \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}} a_i x^i$ , where  $a_i \in \mathbb{Z}_p$  and for all  $N \geq 0$  and for all  $\varepsilon > 0$ , there are finitely many  $i \leq N$  such that  $|a_i| > \varepsilon$ .

4. For  $R_1, R_2$  perfect,  $W(R_1 \otimes R_2) = (W(R_1) \otimes W(R_2))_p^\wedge$ . **why?**

Structure of  $W(R)$ : There is a unique multiplicative (but not additive) map  $[-] : R \rightarrow W(R)$  that is a section of the projection  $W(R) \rightarrow R$ .

**Corollary 1.1.** For any  $x \in W(R)$ , there is a unique sequence  $x_0, x_1, x_2, \dots \in R$  such that  $x = \sum_{i \geq 0} [x_i] p^i$ . So, as a set,  $W(R) \cong \prod_{i \geq 0} R$ .

*Proof.* Let  $\pi$  be the isomorphism  $W(R)/p \rightarrow R$ . Then  $[-]$  being a section of  $W(R) \rightarrow W(R)/p \xrightarrow{\pi} R$  means  $\pi([a] + (p)) = a$  for all  $a \in R$ . Now, given  $x \in W(R)$ , suppose  $a \in R$  satisfies  $x - [a] \in (p)$ . Then  $\pi(x + (p)) = \pi([a] + (p)) = a$ , so such an  $a$  is determined by  $x$ . Conversely, we have  $x - [\pi(x + (p))] \in (p)$  since  $\pi([\pi(x + (p))] + (p)) = \pi(x + (p))$ , so that  $x - [\pi(x + (p))]$  is in the kernel of  $W(R) \rightarrow W(R)/p \xrightarrow{\pi} R$ , which is  $(p)$ . Thus, for any  $x \in W(R)$ , there is a unique  $x_0 \in R$  such that  $x - [x_0] \in (p)$ . Write  $x - [x_0] = px'$ . Such an  $x'$  is unique since  $W(R)$  is  $p$ -torsion free. Then we may define  $x_1 = x'_0$  and repeat the process forever.



Note that in the course of this proof we showed that  $x \mapsto x_0$  is a ring homomorphism, in particular it is the projection  $W(R) \rightarrow W(R)/p \xrightarrow{\sim} R$ . The kernel of this map is  $(p)$ , and in particular we have  $[a] + [b] - [a + b] \in (p)$  for all  $a, b \in R$ .

Exercise: the image of  $[-]$  is the set of elements of  $W(R)$  that have all  $p$ -power roots. Solution: Since  $R$  is perfect, it has all  $p$ -power roots, and  $[-]$  is multiplicative, so anything in  $[R]$  has all  $p$ -power roots; explicitly, if  $x = [a]$ , then  $x = [a^{1/p^n}]^{p^n}$ . Conversely, we claim that if  $x \in W(R)$  has a  $p^n$ th root for  $n \geq 1$ , then  $[x_i] = 0$  for  $1 \leq i \leq n$ . This implies that if  $x$  has all  $p$ -power roots, it must equal  $[x_0]$ . To prove the claim, we induct on  $n$ . In the base case, let  $y = x^{1/p}$  and write  $y = [y_0] + py'$ . Then

$$x = y^p = [y_0^p] + p^2 \left( p^{p-2} (y')^p + \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} [y_0^i] p^{p-i-1} (y')^{p-i} \right),$$

where we use  $p \geq 2$  and  $p \mid \binom{p}{i}$  for  $0 < i < p$ . This computation shows that if  $x$  has a  $p$ th root, then we can write  $x = [x_0] + p^2 x'$ . Inductively, if  $x$  has a  $p^n$ th root, we can write its  $p^n$ th root  $y$  in the form  $y = [y_0] + p^n y'$  and then expand  $y^p$  to show that  $[x_n] = 0$ .

Analogy:  $W(R)$  is like a ring of power series over  $R$  with variable  $p$ .

How to add Witt vectors: Suppose  $x = \sum [x_i]p^i$  and  $y = \sum [y_i]p^i$ . We know  $x + y = \sum [z_i]p^i$  for some  $z_i \in R$ . In fact, for each  $n$ ,  $z_n$  can be expressed as an element of

$$\mathbb{F}_p[x_0^{1/p^\infty}, \dots, x_n^{1/p^\infty}, y_0^{1/p^\infty}, \dots, y_n^{1/p^\infty}].$$

Key fact: the polynomial expressing  $z_n$  in this way is homogeneous of degree 1, where each  $x_i, y_i$  has degree 1.

**Example 1.5.**  $z_0 = x_0 + y_0$ . This follows from the fact that  $x \mapsto x_0$  is a homomorphism.

$z_1 = x_1 + y_1 - \sum_{0 < i < p} \frac{1}{p} \binom{p}{i} x_0^{i/p} y_0^{1-i/p}$ . Exercise: prove this. Solution: Writing  $x = [x_0] + px'$  and  $y = [y_0] + py'$ , we have that  $z_1$  is the 0 component of  $\frac{1}{p}([x_0] + [y_0] - [x_0 + y_0]) + x' + y'$ . Since taking the 0 component is a homomorphism, and since by definition  $x_1 = x'_0$ , we are reduced to computing the 0 component of  $\frac{1}{p}([x_0] + [y_0] - [x_0 + y_0])$ , or equivalently, the 1 component of  $[x_0] + [y_0] - [x_0 + y_0]$ . To simplify notation, we use  $a, b$  instead of  $x_0, y_0$ . Write  $[a] + [b] - [a + b] = p[c(a, b)] + p^2 c'$ . So

$$\begin{aligned} [a] + [b] &\equiv [a + b] + p[c(a, b)] \pmod{p^2} \\ &\text{raise both sides to } p \text{ power} \\ ([a] + [b])^p &\equiv ([a + b] + p[c(a, b)])^p \pmod{p^2} \\ [a^p] + [b^p] + p \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} [a^i b^{p-i}] &\equiv [a + b]^p \pmod{p^2} \\ [a^p] + [b^p] + p \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} [a^i b^{p-i}] &\equiv [a^p + b^p] \pmod{p^2} \\ [a^p] + [b^p] - [a^p + b^p] &\equiv -p \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} [a^i b^{p-i}] \pmod{p^2} \\ p[c(a^p, b^p)] &\equiv -p \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} [a^i b^{p-i}] \pmod{p^2} \\ [c(a^p, b^p)] &\equiv - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} [a^i b^{p-i}] \pmod{p}. \end{aligned}$$


Now, we can apply the isomorphism  $W(R)/p \rightarrow R$ , which we know sends  $[a] + (p)$  to  $a$ , to get

$$\begin{aligned} c(a^p, b^p) &= - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} a^i b^{p-i}; \\ c(a, b) &= - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} a^{i/p} b^{1-i/p}. \end{aligned}$$

### 1.3 Tilting

**Definition 1.2.** For any ring  $R$ , its **tilt**  $R^b$  is  $(R/p)^{perf}$ .

**Proposition 1.1.** If  $R$  is  $p$ -complete, then  $\varprojlim (\cdots \xrightarrow{\varphi} R \xrightarrow{\varphi} R) \xrightarrow{\sim} R^b$ .

*Proof.* The inverse limit is being taken in multiplicative monoids, since  $\varphi$  is not necessarily additive on  $R$  (In fact, the lecturer does not use  $\varphi$ , and I only found out the reason for this after the fact). The isomorphism is in one direction easy to define: take a compatible sequence of elements in  $R$  and reduce mod  $p$ . In the other direction, if we start with some sequence  $(a_i)$  of mod  $p$  elements, we send it to  $b_i = \lim_n \tilde{a}_{i+n}^{p^n}$ , where  $\tilde{a}_i$  are arbitrary lifts of  $a_i$ . **todo: show this is the inverse** 

**Example 1.6.** 1.  $R = \mathbb{Z}_p$  has  $R^b = \mathbb{F}_p$ , since  $R/p = \mathbb{F}_p$  is already perfect.

2.  $R = \mathbb{Z}_p[p^{1/p^\infty}]_p^\wedge = (\mathbb{Z}_p[x^{1/p^\infty}]/(x-p))_p^\wedge$ . Then  $R/p = \mathbb{F}_p[x^{1/p^\infty}]/(x)$  and  $R^b = \mathbb{F}_p[x^{1/p^\infty}]_x^\wedge$ .

*Remark.* If  $R$  is  $p$ -complete, there is a natural multiplicative map  $(-)^{\sharp} : R^b \rightarrow R$ , coming from the isomorphism in the proposition. For example, in the second example above,  $x^{\sharp} = p$ .

**Theorem 1.2.** The functor  $W$  from perfect  $\mathbb{F}_p$ -algebras to  $p$ -complete rings is left adjoint to the tilting functor from  $p$ -complete rings to perfect  $\mathbb{F}_p$ -algebras; if  $A$  is perfect/ $\mathbb{F}_p$  and  $R$  is  $p$ -complete, then  $\text{Hom}(W(A), R) = \text{Hom}_{\mathbb{F}_p}(A, R^b)$ .

*Remark.* If  $A$  is perfect/ $\mathbb{F}_p$  and  $R$  is  $p$ -complete, then there is a natural map  $\text{Hom}(W(A), R) \rightarrow \text{Hom}_{\mathbb{F}_p}(A, R/p)$  given by reducing mod  $p$ . Since  $A$  is perfect, the second hom set can be identified with  $\text{Hom}_{\mathbb{F}_p}(A, R^b)$ . So, the claim is that this composite map  $\text{Hom}(W(A), R) \rightarrow \text{Hom}_{\mathbb{F}_p}(A, R^b)$  is an isomorphism. We will prove this next time.


## 2 Jan 7

### 2.1 Witt vectors and tilting

**Theorem 2.1.** *The functor  $W$  from perfect  $\mathbb{F}_p$ -algebras to  $p$ -complete rings is left adjoint to the tilting functor from  $p$ -complete rings to perfect  $\mathbb{F}_p$ -algebras; if  $A$  is perfect/ $\mathbb{F}_p$  and  $R$  is  $p$ -complete, then  $\mathrm{Hom}(W(A), R) = \mathrm{Hom}_{\mathbb{F}_p}(A, R^\flat)$ .*

*Proof.* Suppose  $A = \mathbb{F}_p[x^{1/p^\infty}]$ . Then  $W(A) = \mathbb{Z}_p[x^{1/p^\infty}]_p^\wedge$ . Then  $\mathrm{Hom}(W(A), B) = \varprojlim_\phi B$  and  $\mathrm{Hom}(A, B^\flat) = B^\flat$ . The natural map  $\varprojlim_\phi B \rightarrow B^\flat$  is an isomorphism, so we have proved the claim for this specific choice of  $A$ . We want to reduce to this case. It will be enough to show that  $W$  preserves colimits. Recall that for perfect  $\mathbb{F}_p$ -algebras  $A_1, A_2$ , we have  $W(A_1 \otimes A_2) = (W(A_1) \otimes W(A_2))_p^\wedge$ . More generally, we need to show that if we have a diagram  $A_1 \leftarrow A \rightarrow A_2$  of perfect  $\mathbb{F}_p$ -algebras, where  $A$  is arbitrary, then the natural map  $(W(A_1) \otimes_{W(A)} W(A_2))_p^\wedge \rightarrow W(A_1 \otimes_A A_2)$  is an isomorphism. It is an isomorphism mod  $p$ , so the difficulty is in showing that  $W(A_1) \otimes_{W(A)} W(A_2)$  is  $p$ -torsion free. There are two ways to justify this. One is using the following theorem:

**Theorem 2.2** (Bhatt-Scholze). *Given a diagram as above,  $\mathrm{Tor}_i^A(A_1, A_2) = 0$  for  $i > 0$ .*

Another justification is as follows.  $W(A)$  is a perfect  $\delta$ -ring, so that  $W(A_1) \otimes_{W(A)} W(A_2)$  is also a perfect  $\delta$ -ring. It is then a general fact that perfect  $\delta$ -rings are  $p$ -torsion free. 

### 2.2 The map $\theta$

The counit of the adjunction is a map  $\theta : W(B^\flat) \rightarrow B$ . Explicitly, for  $b \in B^\flat$ , we have  $\theta([b]) = b^\sharp$ .

*Remark.* If  $B/p$  is semiperfect, then  $B^\flat \rightarrow B/p$  is surjective, so by Nakayama,  $\theta$  is surjective.

**Example 2.1.** If  $B = \mathbb{Z}_p[p^{1/p^\infty}]_p^\wedge$ , then  $B^\flat = \mathbb{F}_p[x^{1/p^\infty}]_x^\wedge$  and  $W(B^\flat) = \mathbb{Z}_p[x^{1/p^\infty}]_{(p,x)}^\wedge$ . Then  $\theta(x^{1/p^n}) = p^{1/p^n}$ .

### 2.3 Perfectoid rings

**Definition 2.1.** Let  $A$  be a  $p$ -complete ring.  $A$  is **perfectoid** if  $A \cong W(P)/\xi$ , where  $P$  is a perfect  $\mathbb{F}_p$ -algebra and  $\xi = \sum [\xi_i]p^i$  with  $\xi_1 \in P^\times$ .

*Remark.* 1. If  $A = W(P)/\xi$ , then  $A/p = P/\xi_0$ , so WLOG we may assume  $P$  is  $\xi_0$ -adically complete. In this case,  $P = A^\flat$ . Thus we may equivalently define perfectoid rings as  $p$ -complete rings such that  $\theta : W(A^\flat) \rightarrow A$  is surjective and  $\ker(\theta)$  is generated by  $\xi$  with  $\xi_1 \in (A^\flat)^\times$ .

2. Any perfect  $\mathbb{F}_p$ -algebra is perfectoid by taking  $\xi = p$ .



3.  $\xi$  is not a zero-divisor. Proof: Suppose  $\xi x = 0$ . Note that  $\xi$  maps to a unit in  $W(P[1/\xi_0])$ , so  $x$  maps to 0 in  $W(P[1/\xi_0])$ . In other words, if we write  $x = \sum [x_i]p^i$ , then all  $x_i$  are  $\xi_0$ -power torsion. But in  $P$ ,  $\xi_0$ -power torsion is the same as  $\xi_0$ -torsion: if  $\xi_0^N y = 0$ , then multiply by some power of  $\xi_0$  on both sides to get  $\xi_0^{p^n} y = 0$ , then since Frobenius is an isomorphism, we get can take  $p^n$ th roots to get  $\xi_0 y^{1/p^n} = 0$ , and then we may multiply by an appropriate power of  $y$  to get  $\xi_0 y = 0$ . So  $\xi_0 x_i = 0$  for all  $i$ , or  $[\xi_0]x = 0$ . Hence  $\xi x = (\sum_{i \geq 1} [\xi_i]p^i)x$ , which is  $px$  times a unit since  $\xi_1$  is a unit, so  $px = 0$ , so  $x = 0$ .
4.  $A = W(P)/\xi$  is  $p$ -torsion free iff  $P$  is  $\xi_0$ -torsion free. To show this we use the torsion exchange lemma: if  $B$  is a ring with nonzerodivisors  $x, y$ , then  $(B/x)[y] \cong (B/y)[x]$  (where brackets denote torsion). This is true because both are  $H_1$  of a Koszul complex on  $x, y$ . Applying the torsion exchange lemma to  $B = W(P)$ ,  $x = p$ , and  $y = \xi$ , we get  $A[p] = (W(P)/\xi)[p] = P[\xi_0]$ .

*Note.* The fact that  $\xi$  is not a zero-divisor implies that derived completions are well-behaved.


**Example 2.2.** Let  $P = \mathbb{F}_p[x^{1/p^\infty}]$  and  $\xi = p - [x]$ . Then  $W(P)/\xi = \mathbb{Z}_p[p^{1/p^\infty}]_p^\wedge$ .

**Example 2.3.**  $P = \mathbb{F}_p[q^{1/p^\infty}]_{(q-1)}^\wedge$ ,  $\xi = 1 + [q^{1/p}] + \dots + [q^{(p-1)/p}]$ . We claim  $P$  is  $\xi_0$ -adically complete and that  $\xi_1 \in P^\times$ . Indeed,  $\xi_0 = 1 + q^{1/p} + \dots + q^{(p-1)/p} = (q^{1/p} - 1)^{p-1}$ , so by definition  $P$  is  $\xi_0$ -complete. To see  $\xi_1 \in P^\times$ , we may quotient by the Jacobson radical, hence set all powers of  $q$  to be 1, so  $\xi = p$  so  $\xi_1 = 1$ . Thus we can form the perfectoid ring  $W(P)/\xi = \mathbb{Z}_p[\zeta_{p^\infty}]_p^\wedge$ . Note that  $(q^{1/p^n})^\# = \zeta_{p^n}$ .

**Example 2.4.** Non-examples:

1.  $\mathbb{Z}_p$  is not perfectoid, because  $\theta$  is an isomorphism.
2.  $\mathbb{Z}/p^n$  is not perfectoid for  $n > 1$ , since  $\ker(\theta) = (p^n)$ .

**Claim 2.1.** Let  $R$  be perfectoid. Then there exists  $u \in R^\times$  and  $\alpha \in R^\flat$  such that  $pu = \alpha^\#$ ; in other words,  $pu$  has a compatible system of  $p$ -power roots.

*Proof.* Write  $R = W(R^\flat)/\xi$ . We have  $\xi = [\xi_0] + p \cdot (\text{unit})$ . Thus  $\theta([\xi_0]) = \xi_0^\# = p \cdot (\text{unit})$ . 

**Example 2.5.** If  $R$  is perfectoid, then  $R\langle x^{1/p^\infty} \rangle = (R \otimes_{\mathbb{Z}} \mathbb{Z}_p[x^{1/p^\infty}])_p^\wedge$  is perfectoid.

### 3 Jan 9

**Proposition 3.1.** *Let  $R$  be  $p$ -torsion free containing  $\omega$  such that*

1.  $\omega^p \in pR^\times$ .
2.  $R/\omega \xrightarrow{\varphi} R/\omega^p = R/p$  is an isomorphism.

*Then  $R_p^\wedge$  is perfectoid.*

*Proof.* WLOG, suppose  $R$  is  $p$ -complete. Condition two implies that  $R$  is semiperfect, so  $R^b \rightarrow R/p$  is surjective. Thus, lift  $\omega \bmod p$  to some  $\alpha$ , so that  $\alpha^\sharp = \omega + py \in \omega R^\times$ . Then, WLOG, we have  $\omega = (\omega^b)^\sharp$ . Now, letting  $\bar{u}$  be the unit such that  $\omega^p = p\bar{u}$ , we have  $\theta([\omega^b]^p) = \omega^p = p\bar{u}$ . Lift  $\bar{u}$  to  $u \in W(R^b)^\times$ . Then  $[\omega^b]^p - pu \in \ker \theta$ , and this is a valid choice of  $\xi$  to show  $R$  is perfectoid.

So we want to show  $W(R^b)/([\omega^b]^p - pu) \xrightarrow{\theta} R$  is an isomorphism. Both sides are  $p$ -complete and  $p$ -torsion free; the LHS is  $p$ -torsion free because  $\omega^{bp}$  is a nonzerodivisor. Thus, to show the map is an isomorphism, we may work mod  $p$  and show  $R^b/\omega^{bp} \xrightarrow{\sim} R/p$ . Explicitly, this means that we want to show that, given a sequence  $(x_0, x_1, \dots) \in R$  with  $x_{i+1}^p = x_i$  for all  $i$ , then the following are equivalent:

1.  $p \mid x_0$ .
2.  $\omega^p \mid x_0$ .
3.  $\omega^{p^{1-i}} \mid x_i$  for all  $i$ .

Clearly conditions 1 and 2 are equivalent since  $\omega^p \in pR^\times$ . Since Frobenius is an isomorphism  $R/\omega \rightarrow R/\omega^p$ , we get  $\omega^p \mid x_0$  iff  $\omega \mid x_1$ . In fact, we can quotient the isomorphism by  $\omega^{1/p^{n+1}}$  to obtain  $\omega^{1/p^{n+1}} \mid y$  iff  $\omega^{1/p^n} \mid y^p$ . This gives the equivalence between 2 and 3, so we are done. 🇵🇷

I'm confused by a lot of the steps in this proof, it's so over

#### 3.1 $p$ -root closure

**Definition 3.1** (P. Roberts). Let  $R$  be  $p$ -torsion free.  $R$  is  **$p$ -root closed** if for all  $x \in R[\frac{1}{p}]$ , we have  $x^p \in R$  implies  $x \in R$ .

**Example 3.1.** 1. Any integrally closed domain is  $p$ -root closed.

2.  $\mathbb{Z} \oplus (px) \subset \mathbb{Z}[x]$  is  $p$ -root closed.
3.  $\mathbb{Z}[px, x^p]$  is not  $p$ -root closed.


**Definition 3.2.** Let  $A$  be a  $p$ -torsion free ring. Let  $A^{+,p}$  (notation possibly not standard) be the set of  $x \in A[\frac{1}{p}]$  such that  $x^{p^n} \in A$  for large enough  $n$ . Then  $A^{+,p}$  is the smallest  $p$ -root closed subring of  $A[\frac{1}{p}]$  containing  $A$ , and it is called the  **$p$ -root closure** of  $A$ .

Exercise: prove  $A^{+,p}$  is a ring. (Hint: use divisibility properties of binomial coefficients.)

**Claim 3.1.** *Let  $R$  be a  $p$ -torsion free ring with  $\omega \in R$  such that  $\omega^p \in pR^\times$ . Then the following are equivalent:*

1.  $R$  is  $p$ -root closed.
2.  $R/\omega \xrightarrow{\phi} R/\omega^p$  is injective.

*Proof.* (1  $\Rightarrow$  2) Fix  $x \in R$  such that  $\omega^p \mid x^p$ . Then  $x/\omega \in R[\frac{1}{p}]$ , so by assumption,  $x/\omega \in R$ , i.e.  $\omega \mid x$ .

(2  $\Rightarrow$  1) Fix  $y \in R[\frac{1}{p}]$  such that  $y^p \in R$ . Write  $y = z/\omega^i$  for  $i$  minimal and  $z \in R$ . If  $i > 0$ , then  $\omega^{ip} \mid z^p$ , so  $\omega^p \mid z^p$ , so  $\omega \mid z$ , meaning  $i$  is not minimal. Hence  $i = 0$  so  $y \in R$ . 

**Definition 3.3.** Let  $P$  be a perfect  $\mathbb{F}_p$ -algebra. Let  $\xi = \sum [\xi_i]p^i \in W(P)$ . Say  $\xi$  is **distinguished** if  $P$  is  $\xi_0$ -complete and  $\xi_1 \in P^\times$ .


**Claim 3.2.** 1. *If  $\xi$  is distinguished and  $u \in W(P)^\times$ , then  $\xi u$  is also distinguished.*

2. *If  $\xi, \xi'$  are distinguished and  $\xi \mid \xi'$ , then  $\xi'/\xi \in W(P)^\times$ .*

**Theorem 3.1.** *Let  $R$  be  $p$ -torsion free and  $p$ -complete. Then  $R$  is perfectoid iff the following are all true:*

1. *There is  $\omega \in R$  such that  $\omega^p \in pR^\times$ .*
2.  *$R$  is  $p$ -root closed.*
3.  *$R/p$  is semiperfect.*

*Proof.* Assume 1-3 hold. Conditions 1 and 2 imply  $R/\omega \rightarrow R/\omega^p$  is injective, and condition 3 implies it is surjective. Then, by Proposition 3.1,  $R$  is perfectoid.

Thus assume  $R$  is perfectoid. It is enough to show condition 2. Choose  $\omega^b \in R^b$  such that  $((\omega^b)^\sharp)^p = p\bar{u}$  for  $\bar{u} \in R^\times$ , and let  $\omega = (\omega^b)^\sharp$ . It is enough to show  $R/\omega \rightarrow R/\omega^p$  is an isomorphism. Lift  $\bar{u}$  to  $u \in W(R^b)^\times$ , so that  $\xi = [\omega^b]^p - pu$  is distinguished and in  $\ker \theta$ . Then  $R = W(R^b)/\xi$ , so  $R/\omega \rightarrow R/\omega^p$  is  $R^b/\omega^b \rightarrow R^b/\omega^{bp}$ , and this is an isomorphism since  $R^b$  is perfect. 

**Example 3.2.** Let  $R$  be a domain. Let  $R^+$  be the absolute integral closure, i.e. the integral closure in the algebraic closure of the fraction field of  $R$ . Then  $(R^+)_p^\wedge$  is perfectoid. Applying this construction to  $\mathbb{Z}_p$ , we obtain that  $\mathcal{O}_{\mathbb{C}_p}$  is perfectoid.


### 3.2 Category of perfectoid rings

There is no initial object in the category of perfectoid rings.

The category of perfectoid rings is equivalent to the category of “perfect prisms”, i.e. pairs  $(P, I)$ , where  $P$  is a perfect ring and  $I$  is an ideal of  $W(P)$  generated by a distinguished element. Given a perfectoid  $R$ , we send it to  $(R^\flat, \ker(\theta))$ . Given a perfect prism  $(P, I)$ , we send it to  $W(P)/I$ .

*Note.* There is no category of rings equivalent to the category of all prisms.


**Corollary 3.1.** *The category of perfectoid rings has pushouts; in particular, given  $R_1 \leftarrow R \rightarrow R_2$  maps of perfectoids, we have  $(R_1 \otimes_R R_2)_p^\wedge$  is perfectoid.*

*Proof.* By the “rigidity of prisms”, we may write  $R = W(R^\flat)/I$ ,  $R_1 = W(R_1^\flat)/I$ ,  $R_2 = W(R_2^\flat)/I$  for the same ideal  $I$ . Then  $(R_1 \otimes_R R_2)_p^\wedge = (W(R_1^\flat) \otimes_{W(R^\flat)} W(R_2^\flat))/I = W(R_1^\flat \otimes_{R^\flat} R_2^\flat)/I$ . 

## 4 Jan 12

### 4.1 Tilting equivalence

**Proposition 4.1.** *Let  $R$  be perfectoid,  $\xi \in W(R^\flat)$  generates  $\ker \theta$ . Then there is an equivalence between the category of perfectoid  $R$ -algebras and the category of  $\xi_0$ -adically complete perfect  $R^\flat$ -algebras, given by sending a perfectoid  $R$ -algebra  $R'$  to its tilt  $R'^\flat$ .*

*Proof.* Under the equivalence of the category of perfectoid rings with perfect prisms, the perfectoid  $R$ -algebras are sent to perfect prisms  $(P, I)$  with a map from  $(R^\flat, \ker \theta)$ . Since  $I$  is required to be generated by a distinguished element, and since  $\xi \in \ker(\theta)$  is mapped into  $I$ , we automatically get that such  $P$  are  $\xi_0$ -adically complete. 

**Example 4.1.** The category of perfectoid  $\mathbb{Z}_p[p^{1/p^\infty}]_p^\wedge$ -algebras is equivalent to the category of  $t$ -complete  $\mathbb{F}_p[t^{1/p^\infty}]_t^\wedge$ -algebras.

**Theorem 4.1.** *Let  $R$  be perfectoid,  $\xi$  generates  $\ker \theta$ . There is an equivalence between the category of finite étale  $R[\frac{1}{p}]$ -algebras and the category of finite étale  $R^\flat[\frac{1}{\xi_0}]$ -algebras, given by taking an algebra  $T$ , taking the integral closure  $\overline{R}_T$  of  $R$  in  $T$ , and then forming  $\overline{R}_T^\flat[\frac{1}{\xi_0}]$ . Thus, the étale fundamental groups of  $R[\frac{1}{p}]$  and  $R^\flat[\frac{1}{\xi_0}]$  are identified.*

The proof is outside of the scope of the course.

**Example 4.2.** As a special case,  $\text{Gal}(\mathbb{Q}_p(p^{1/p^\infty})) \cong \text{Gal}(\mathbb{F}_p((t)))$ .

**Example 4.3.** We have the following special classes of perfectoid rings:

1. Perfect  $\mathbb{F}_p$ -algebras.
2.  $p$ -torsion free perfectoid rings.

The following proposition says that any perfectoid ring can be glued together from these.


**Proposition 4.2.** *If  $R$  is perfectoid, then  $R/R[p^\infty]$  is  $p$ -torsion free and perfectoid, and there is a pullback diagram*

$$\begin{array}{ccc} R & \longrightarrow & R/R[p^\infty] \\ \downarrow & & \downarrow \\ (R/p)_{\text{red}} & \longrightarrow & (R/(R[p^\infty], p))_{\text{red}} \end{array}$$

where all maps are surjective and the bottom two rings are perfect  $\mathbb{F}_p$ -algebras.

*Proof.* For any ring  $A$  and ideals  $I, J$  such that  $I \cap J = 0$ , there is a pullback

$$\begin{array}{ccc} A & \longrightarrow & A/I \\ \downarrow & & \downarrow \\ A/J & \longrightarrow & A/(I+J) \end{array}$$

We apply this to the ring  $R^b$ ,  $I = \xi_0$ -power torsion (equivalently, the  $x \in R^b$  with  $\xi_0^{1/p^n} x = 0$  for all  $n$ ),  $J = \bigcup_{n \geq 0} (\xi_0^{1/p^n})$ . Then  $I \cap J = 0$  and we can apply the remark. Then “untilting” (apply  $W$  and mod out by the same  $\xi$  everywhere) gives the desired pullback diagram (“it’s a bit of work to do so”). (an  $R^b$ -algebra  $T$  untilts to a  $p$ -torsion free ring iff  $\xi_0$  is not a zero divisor, and  $T$  untilts to a (perfect?)  $\mathbb{F}_p$ -algebra iff  $\xi_0 = 0$ ) 

**Example 4.4.** A perfectoid ring that is neither a perfect  $\mathbb{F}_p$ -algebra nor  $p$ -torsion free:  $\mathbb{Z}_p[p^{1/p^\infty}, t^{1/p^\infty}]_p^\wedge / (p^a t^b, a > 0, b > 0)$ .

## 4.2 Valuation rings

**Definition 4.1.** A **valuation ring** is a domain  $V \neq 0$  such that for all nonzero  $x, y \in V$ , we have  $x \mid y$  or  $y \mid x$ .

As a consequence, all finitely generated ideals of a valuation ring are principal.

**Example 4.5.** 1. Any discrete valuation ring.

2.  $\mathcal{O}_{\mathbb{C}_p}$ , or more generally, if  $K$  is a field with a non-archimedean absolute value, then the set of elements  $\mathcal{O}_K$  with absolute value  $\leq 1$ .
3.  $\mathbb{Z}_p \oplus t\mathbb{Q}_p[[t]]$  as a subring of  $\mathbb{Q}_p[[t]]$ .

*Remark.* 1.  $V$  is local,  $\text{Spec}(V)$  is totally ordered, and all radical ideals are prime.

2. In any scheme, specialization can be tested via valuation rings. Algebraically, if  $A$  is a local domain, then there is an inclusion  $A \hookrightarrow A' \hookrightarrow \text{Frac}(A)$  such that  $A'$  is a valuation ring and  $\mathfrak{m}_A \hookrightarrow \mathfrak{m}_{A'}$ .

**Definition 4.2.** Let  $K$  be a field. A **valuation** on  $K$  consists of a totally ordered group  $\Gamma$  and a map  $\nu : K \rightarrow \Gamma \cup \{\infty\}$  such that

1.  $\nu(x) = \infty$  iff  $x = 0$ .
2.  $\nu(xy) = \nu(x) + \nu(y)$ .
3.  $\nu(x + y) \geq \min(\nu(x), \nu(y))$ .

The **value group** is  $\nu(K^\times)$ .

**Example 4.6.** If we have a field  $K$  with valuation  $\nu$ , then we can form a valuation ring by taking  $V = \{x \in K \mid \nu(x) \geq 0\}$ . Conversely, if we have a valuation ring  $V$ , we have the field  $K = \text{Frac}(V)$  and a valuation  $K \rightarrow \Gamma = K^\times/V^\times \cup \{\infty\}$ , where  $\Gamma$  is given an ordering by the divisibility relation.  $\Gamma$  is exactly the value group of this valuation.


**Definition 4.3.** A valuation ring  $V$  has **rank one** if any of the following (equivalent) conditions hold:

1.  $\mathfrak{m}_V = \text{rad}(f)$  for  $f$  a nonzero nonunit (a “pseudouniformizer”).
2.  $V$  has exactly 2 prime ideals, namely 0 and  $\mathfrak{m}_V$ .
3. There is a non-trivial non-archimedean absolute value on  $K = \text{Frac}(V)$  such that  $V = \{x \in K \mid |x| \leq 1\}$ .

**Example 4.7.** Any DVR has rank one, and  $\mathbb{F}_p[[t]]_{\text{perf}}$  is a non-DVR with rank one.

*Note.* In general, rank  $n$  valuation rings are those with exactly  $n + 1$  prime ideals.

**Proposition 4.3.** *Let  $A$  be perfectoid. Then  $A$  is a valuation ring iff  $A^\flat$  is a valuation ring. In this case  $\text{Frac}(A)^\times/A^\times \cong \text{Frac}(A^\flat)^\times/A^\flat$ , where the map from right to left sends  $\alpha$  to  $\alpha^\sharp$ .*

*Proof.* The forward direction follows from the fact that  $A^\flat = \varprojlim_{\varphi} A$  is an isomorphism of multiplicative monoids, so the divisibility condition can be checked directly. For the backwards direction, show that any  $x \in A$  can be written as  $\alpha^\sharp u$  for some  $\alpha \in A^\flat$  and some  $u \in A^\times$ . 

**Definition 4.4** (Scholze’s original definition). A **perfectoid field** is a complete non-archimedean field  $K$  such that

1.  $|p| < 1$ .
2.  $\mathcal{O}_K/p$  is semiperfect.
3. The value group is not discrete.

Exercise: check that there is a correspondence between perfectoid fields and perfectoid valuation rings of rank one that are complete for a pseudouniformizer.

**Example 4.8.** Let  $C$  be complete, algebraically closed, non-archimedean field with  $|p| < 1$ . Then  $\mathcal{O}_C$  is perfectoid.

Exercise: Find the rank (lecturer’s claim: 2) and value group of  $\mathbb{Z}_p \oplus t\mathbb{Q}_p[[t]]$  as a subring of  $\mathbb{Q}_p[[t]]$ .

## 5 Jan 14

### 5.1 Perfectoidization

Let  $R$  be an  $\mathbb{F}_p$ -algebra. Then

1. There is a universal perfect ring with a map from  $R$ , namely  $R_{perf}$ .
2. If  $R$  is semiperfect ( $\varphi$  surjective) then  $R_{perf} = R_{red} = R/\text{rad}(0)$ , so  $R \twoheadrightarrow R_{perf}$ .

If there is an initial perfectoid ring with a map from a ring  $R$ , then we call it the **perfectoidization**  $R_{perfd}$ . This need not exist, e.g.  $(\mathbb{Z}_p)_{perfd}$  does not exist.

**Theorem 5.1.** 1. If  $R$  is integral over a perfectoid ring  $R_0$ , then  $R_{perfd}$  exists.

2. If  $R$  is **semiperfectoid** ( $R$  is a quotient of a perfectoid ring), then  $R_{perfd}$  exists and the map  $R \rightarrow R_{perfd}$  is surjective.

*Remark.* Part 2 of the theorem was originally believed to be false.

*Remark.*  $R$  being semiperfect is equivalent to the conditions that  $R$  is an algebra over a perfectoid ring  $R_0$  and  $R/p$  is semiperfect. For instance, if the two conditions hold, then  $\theta : W(R^b) \rightarrow R$  is surjective, and then  $R_0 \widehat{\otimes} W(R^b) \rightarrow R$  is also surjective.

**Example 5.1.**  $\mathbb{Z}/p^2$  is not semiperfectoid, but its mod  $p$  ring is semiperfect.

### 5.2 $p$ -complete arc topology

**Definition 5.1.** An **extension** of rank 1 valuation rings is an inclusion  $V \hookrightarrow V'$  such that a pseudouniformizer for  $V$  is mapped to one for  $V'$ . Equivalently, it is an isometric inclusion of non-archimedean fields, where the valuation rings are the unit balls.

**Definition 5.2.** A map of rings  $A \rightarrow B$  is a  **$p$ -complete arc cover** if, given any  $A \rightarrow V$  where  $V$  is a  $p$ -complete rank 1 valuation ring, there is an extension  $V \hookrightarrow V'$  and a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ V & \hookrightarrow & V' \end{array}$$

Equivalently, any multiplicative seminorm  $|\cdot|$  on  $A$  such that  $|A| \leq 1$  and  $|p| < 1$  extends to one on  $B$  with  $|B| \leq 1$ .

*Remark.* 1. This notion of covering gives a Grothendieck topology on  $(\text{Rings})^{op}$ .




2. The topology has a basis given by products of rings of the form  $\mathcal{O}_C$ , where  $C$  is a complete, algebraically closed, non-archimedean field with  $|p| < 1$ . In other words, for any ring  $R$ , there is a  $p$ -complete arc cover  $R \rightarrow \prod_{i \in I} \mathcal{O}_{C_i}$ .
3. Any faithfully flat map is a  $p$ -complete arc cover. For instance, given  $f, g \in R$ , the map  $R \rightarrow R[x]/(fx - g) \times R[y]/(gy - f)$  is a  $p$ -complete arc cover.

**Theorem 5.2.** *On perfectoid rings, the identity is a sheaf with no higher cohomology for the  $p$ -complete arc-topology. Explicitly, if  $A \rightarrow B$  is a  $p$ -complete arc cover, then  $A \xrightarrow{\sim} \varprojlim (B \rightrightarrows B \hat{\otimes}_A B \cdots)$ .*

**Corollary 5.1.** *Let  $R \rightarrow R'$  be a map of rings. Then  $R'$  is a perfectoidization of  $R$  iff the following are true:*

1.  $R'$  is perfectoid.
2. For any complete, algebraically closed non-archimedean field  $C$  with  $0 < |p| < 1$  and any map  $R \rightarrow \mathcal{O}_C$ , there is a unique extension to a map  $R' \rightarrow \mathcal{O}_C$ .
3.  $(R/p)_{\text{perf}} \xrightarrow{\sim} (R'/p)_{\text{perf}}$ ; equivalently, allow  $|p| = 0$  in the above condition.

*Proof.* We want  $\text{Hom}(R', T) \xrightarrow{\sim} \text{Hom}(R, T)$  for  $T$  perfectoid. By descent, we reduce to  $T = \prod \mathcal{O}_C$ , and since products commute with  $\text{Hom}$ , we reduce to  $T = \mathcal{O}_C$ . 


**Example 5.2.** Let  $C = \mathbb{C}_p$  and  $R = \mathcal{O}_C[x]/(x^p - 1)$ . Then there is a map of rings  $R \rightarrow \text{Fun}(\mu_p(\mathcal{O}_C), \mathcal{O}_C)$  given by sending  $x$  to the identity function. This is an isomorphism, in particular the Fourier transform, after inverting  $p$ .

**Proposition 5.1.** *The perfectoidization of the ring  $R$  described above is the ring of functions  $\mu_p(\mathcal{O}_C) \rightarrow \mathcal{O}_C$  that are constant mod the maximal ideal of  $\mathcal{O}_C$ . In other words, it is*

$$\text{Fun}(\mu_p(\mathcal{O}_C), \mathcal{O}_C) \times_{\text{Fun}(\mu_p(\mathcal{O}_C), \overline{\mathbb{F}_p})} \overline{\mathbb{F}_p}.$$

*Proof.* **lecturer is going way too fast** conditions 1 and 3 of Corollary 5.1 can be checked ... call the claimed perfectoidization  $R'$ . Suppose we have a map  $R \rightarrow \mathcal{O}_{C'}$ , where  $C'$  is a field such as in condition 2 of Corollary 5.1. We can extend the map  $R \rightarrow C'$  to  $R'$ , since  $C' = \mathcal{O}_{C'}[\frac{1}{p}]$  and  $R[\frac{1}{p}] = R'[\frac{1}{p}]$ . Now note that  $R' \subset \frac{1}{p}R$ . We are done by the following lemma:

**Lemma 5.1.** *Let  $A \hookrightarrow B$  be a map of  $p$ -torsion free rings. Suppose  $A \subset B \subset \frac{1}{p^n}A$ . Then any map  $A \rightarrow \mathcal{O}_C$  extends uniquely to  $B$ .*

*Proof of lemma.* As above, we can extend  $A \rightarrow C$  to  $B \rightarrow C$ . But  $B \subset \frac{1}{p^n}A$  means that  $B \rightarrow C$  has bounded image. The only bounded subrings of  $C$  are contained in the unit ball, so we indeed get a map  $B \rightarrow \mathcal{O}_C$ . 



**Example 5.3.** Let  $R = \mathcal{O}_C \langle x^{1/p^\infty} \rangle / (x - 1)$ . Then

$$R_{perf} = \mathrm{Fun}(\mathbb{Z}_p(1)(\mathcal{O}_C), \mathcal{O}_C) \times_{\mathrm{Fun}(\mathbb{Z}_p(1)(\mathcal{O}_C), \overline{\mathbb{F}_p})} \overline{\mathbb{F}_p}.$$

However, this is very nontrivial. Furthermore, the map  $R \rightarrow R_{perf}$  is surjective. Next time we will show how to construct a nonzero element of the kernel.

## 6 Jan 15

Today we will discuss the following result.

**Theorem 6.1.** *Let  $R$  be a semiperfectoid ring (recall that this means a quotient of a perfectoid ring). Then it has a perfectoidization  $R_{\text{perf}}$ , and the map  $R \rightarrow R_{\text{perf}}$  is surjective.*

*Remark.* It is enough to consider the case where  $R = R_0/f$  is the quotient of a perfectoid by a principal ideal, since perfectoids are closed under arbitrary relative tensor products.

**Example 6.1.** Suppose furthermore that  $f \in R_0$  had a compatible system of  $p$ -power roots. Then we claim  $R_{\text{perf}} = (R_0/\bigcup_{n \geq 0} f^{1/p^n})_p^\wedge$ . First of all, the right hand side is perfectoid since it may be expressed as  $R_0 \otimes_{R[x^{1/p^\infty}]}^\wedge R_0$ , where  $R[x^{1/p^\infty}]$  acts on the first copy of  $R_0$  by  $x^{1/p^n} \mapsto f^{1/p^n}$ , and it acts on the second copy by  $x^{1/p^n} \mapsto 0$ . check this, I guess. Second, since perfectoids are reduced\*, any map  $R_0/f \rightarrow S$  with  $S$  perfectoid must vanish on all  $f^{1/p^n}$ , hence factoring through a map from the desired ring.

*Remark.* \*To justify that perfectoids are reduced, we give two arguments:

1. By arc descent, any perfectoid ring embeds in a product of  $p$ -complete valuation rings, which are reduced.
2. By Proposition 4.2, any perfectoid embeds in  $A \times B$  where  $A$  is  $p$ -torsion free perfectoid and  $B$  is a perfect  $\mathbb{F}_p$ -algebra. Certainly  $B$  is reduced, so we need to show  $A$  is reduced. Suppose  $x$  is nilpotent, so that  $x^{p^n} = 0$  for some  $n$ . Then, for any  $i$ , the element  $x/p^i \in A[\frac{1}{p}]$  satisfies  $(x/p^i)^{p^n} = 0 \in A$ . Thus, since  $p$ -torsion free perfectoids are  $p$ -root closed by Theorem 3.1, we have  $x/p^i \in A$  for all  $i$ . By  $p$ -completeness, this means  $x = 0$ .

To recap, we reduced Theorem 6.1 to the case where  $R = R_0/f$ , and we handled this case under the further assumption that  $f$  has  $p$ -power roots. On the other hand, if  $f$  doesn't have  $p$ -power roots, things are much less clear.

**Example 6.2.** For example, suppose our ring was  $R = \mathcal{O}_C \langle x^{1/p^\infty} \rangle / (x - 1)$ , where  $C = \mathbb{C}_p$ . What is  $R_{\text{perf}}$ ? By choosing a compatible system of  $p$ -power roots of unity  $\zeta_{p^n} \in \mathcal{O}_C$ , we get a map

$$\begin{aligned} \mathcal{O}_C \langle x^{1/p^\infty} \rangle / (x - 1) &\rightarrow \text{Fun}_{\text{cts}}(\mathbb{Z}_p, \mathcal{O}_C), \\ x^{1/p^n} &\mapsto (i \mapsto \zeta_{p^n}^i). \end{aligned}$$

This map lands inside the fiber product

$$\text{Fun}_{\text{cts}}(\mathbb{Z}_p, \mathcal{O}_C) \times_{\text{Fun}_{\text{cts}}(\mathbb{Z}_p, \overline{\mathbb{F}_p})} \overline{\mathbb{F}_p},$$

which is perfectoid. We claim that it is the perfectoidization.

For this, it suffices to check at each “ $p$ -power depth”, i.e. for each  $n$  we have a perfectoidization

$$\mathcal{O}_C\langle x^{1/p^n} \rangle / (x - 1) \rightarrow \mathrm{Fun}(\mathbb{Z}/p^n, \mathcal{O}_C) \times_{\mathrm{Fun}(\mathbb{Z}/p^n, \overline{\mathbb{F}_p})} \overline{\mathbb{F}_p}, \quad (1)$$

and we did this last time by comparing mod  $p$  perfection and maps into other  $\mathcal{O}_{C'}$ . What is not clear is the following:

1. The map 1 is surjective. We will discuss this.
2. What are nontrivial elements of the kernel of 1? We know that there must be something, since the domain is not perfectoid.

Both of these facts are addressed in Fresnel and de Mathan, “Sur la transformation de Fourier  $p$ -adique”.

The surjectivity of  $R \rightarrow R_{\mathrm{perfd}}$  will be a corollary of André’s lemma.

## 6.1 André’s flatness lemma

**Theorem 6.2.** *Let  $R_0$  be a perfectoid ring and let  $f \in R_0$ . Then there is an  $R_0$ -algebra  $T_0$  such that*

1.  $T_0$  is perfectoid.
2.  $R_0/p^n \rightarrow T_0/p^n$  is faithfully flat for each  $n$ .
3.  $f$  has a compatible system of  $p$ -power roots in  $T_0$ .

It is also possible to choose  $T_0$  absolutely integrally closed.

*Remark.* The surjectivity of  $R_0/f \rightarrow (R_0/f)_{\mathrm{perfd}}$  now follows from flat descent along  $R_0 \rightarrow T_0$ . check

In the rest of the lecture, we will explain in concrete terms the kernel of  $R \rightarrow R_{\mathrm{perfd}}$  when  $R$  is  $p$ -torsion free and semiperfectoid.

**Definition 6.1.** Let  $R$  be  $p$ -torsion free and  $p$ -complete. We say that  $x \in R$  has **spectral radius zero** if for all  $N$ , we have  $x^i \in p^{iN}R$  for large enough  $i$ .

*Remark.* The collection of spectral radius zero elements in  $R$  is  $p$ -complete. It contains all nilpotent elements.

**Example 6.3.**  $\mathcal{O}_C$  has no elements of spectral radius zero why?

**Theorem 6.3.** *Let  $R$  be a  $p$ -torsion free semiperfectoid ring. Then the kernel of  $R \rightarrow R_{\mathrm{perfd}}$  is exactly the set of elements with spectral radius zero.*

*Remark.* Let  $S$  be a  $p$ -torsion free perfectoid ring. Then  $S$  has no nonzero elements of spectral radius zero. To see this, choose  $\omega \in S$  with  $\omega^p \in pS$ . Then one can prove inductively that  $p^i \mid x^p$  iff  $\omega^i \mid x$  (do this), and this implies the claim (why?).

*Note.* My friend's notes claim that any perfectoid should have no spectral radius zero elements, but there is no proof given. My feeling is that you “decompose” a general perfectoid into a  $p$ -torsion free one and a perfect  $\mathbb{F}_p$ -algebra. In characteristic  $p$ , the notion of spectral radius zero becomes nilpotency, and perfect algebras are reduced.

The remark implies that the spectral radius zero elements are contained in the kernel of perfectoidization. To prove the other direction of the theorem, we need a “spectral radius formula”.

Let  $A$  be a  $\mathbb{Q}_p$ -Banach algebra; it is a  $\mathbb{Q}_p$ -algebra with a subadditive submultiplicative norm  $|| \cdot || : A \rightarrow \mathbb{R}_{\geq 0}$  such that  $||\lambda a|| = |\lambda|_p ||a||$  and  $A$  is complete with respect to its norm. Let  $A_0$  be its closed unit ball. This is a  $p$ -complete,  $p$ -torsion free ring such that  $A = A_0[\frac{1}{p}]$ . Conversely, if  $B_0$  is a  $p$ -complete  $p$ -torsion free ring, then  $B = B_0[\frac{1}{p}]$  can be given a norm in terms of  $p$ -divisibility, which makes it a  $\mathbb{Q}_p$ -Banach algebra with closed unit ball  $B_0$ . There is an equivalence between the category of  $\mathbb{Q}_p$ -Banach algebras with continuous maps and the category of  $p$ -complete  $p$ -torsion free rings up to isogeny (what does isogeny mean in this context?).

**Definition 6.2.** Let  $A$  be a  $\mathbb{Q}_p$ -Banach algebra. Let  $\mathcal{M}(A)$  be the set of bounded (by the Banach norm) multiplicative seminorms on  $A$ . This is called the **Berkovich spectrum**. It can be given the topology of a compact Hausdorff space.

*Note.*  $\mathcal{M}(A)$  can be identified with the set of maps from  $A_0$  to rank 1 valuation rings where  $p$  is a pseudouniformizer, up to isomorphism on the right hand side.

We use the following facts:

1. If  $A \neq 0$ , then  $\mathcal{M}(A) \neq \emptyset$ .
2. For all  $a \in A$ , we have

$$\sup_{|\cdot| \in \mathcal{M}(A)} |a| = \lim_{n \rightarrow \infty} ||a^n||^{1/n},$$

which is the “spectral radius formula”. The equal quantity is called the spectral radius of  $a$ . An element  $a \in A_0$  has spectral radius zero (in the sense discussed earlier) iff its spectral radius (in the current sense) is zero.

An equivalent formulation of property 1 is that if  $B_0$  is a  $p$ -complete  $p$ -torsion free ring, then there is a map  $B_0 \rightarrow \mathcal{O}_C$ , where  $C$  is a complete algebraically closed non-archimedean field with  $0 < |p| < 1$ .

It then follows from this reformulation and from the spectral radius formula that  $a \in A_0$  has spectral radius zero iff  $a$  is in the kernel of any map from  $A_0$  to a rank 1 valuation ring with  $p$  a pseudouniformizer.

We can now finish proving Theorem 6.3. Recall that we need to show that, for  $R$   $p$ -torsion free and semiperfectoid, the kernel of  $R \rightarrow R_{\text{perfd}}$  is the set of spectral radius zero elements. By the universal property of  $R_{\text{perfd}}$ , this kernel is the joint kernel of all maps from  $R$  to a perfectoid. By arc descent, this is the joint kernel of all maps from  $R$  to rank 1 valuation rings where  $|p| < 1$ . The case where  $0 < |p| < 1$  is then covered by our previous work. The characteristic  $p$  case is left to be checked.

Today's material raises two questions:

1. Is there a direct proof that  $R \rightarrow R_{\text{perfd}}$  is surjective for  $R$   $p$ -torsion free and semiperfectoid? Namely, a proof that does not use base change.
2. As a corollary of our argument, we also see that if  $R$  is  $p$ -torsion free semiperfectoid with no elements of spectral radius zero, then  $R$  is perfectoid. Is there a direct proof of this?

## 7 Jan 21

**Definition 7.1.** Let  $C$  be a CACNA of characteristic  $p$ . (For example, take  $C$  to be the completion of  $\overline{\mathbb{F}_p((t))}$  with respect to the induced power series valuation.) An **untilt** of  $C$  consists of a perfectoid field (Definition 4.4)  $F$  and an isomorphism  $\iota : F^\flat \rightarrow C$ , or equivalently an isomorphism  $\mathcal{O}_F^\flat \rightarrow \mathcal{O}_C$ .

*Note.* An untilt  $(F, \iota : F \rightarrow C)$  gives a multiplicative map  $\sharp : C \rightarrow F$  **how?**.

**Example 7.1.** If  $C$  is characteristic 0, then  $C$  is an untilt of itself.

*Remark.* 1. The notion of untilt makes sense for any perfectoid field. For instance,  $\mathbb{Q}_p(p^{1/p^\infty})_p^\wedge$  and  $\mathbb{Q}_p^{cyc}$  (the cyclotomic extension of  $\mathbb{Q}_p$  adjoin all  $p$ -power roots of unity) are untilts of  $\mathbb{F}_p((t^{1/p^\infty}))$ .

2. Any untilt of an algebraically closed field is algebraically closed.

There is a natural notion of isomorphism of untilts, so we get a set  $\text{Untilt}(C)$ . The key idea is that  $\text{Untilt}(C)$  is analogous to the rigid open unit disk, which we now try to introduce.

We now let  $C$  be any CACNA.

**Definition 7.2.** The **Tate algebra**  $C\langle T \rangle$  is the algebra of power series  $\sum a_i T^i$  in  $C[[T]]$  such that  $a_i \rightarrow 0$  as  $i \rightarrow \infty$ .

The idea is that  $C\langle T \rangle$  is the algebra of analytic functions on the closed unit disk  $D_{\leq 1} = \{\alpha \in C \mid |\alpha| \leq 1\}$ . For any  $\alpha \in D_{\leq 1}$ , we have an evaluation map  $\text{ev}_\alpha : C\langle T \rangle \rightarrow C$  given by  $T \mapsto \alpha$ .

**Definition 7.3.** The **Gauss norm** on  $C\langle T \rangle$  is the map  $\|\cdot\| : C\langle T \rangle \rightarrow \mathbb{R}_{\geq 0}$  given by  $\|\sum a_i T^i\| = \max |a_i|$ .

*Remark.* 1.  $\|f + g\| \leq \|f\| + \|g\|$ .

2.  $\|fg\| = \|f\| \cdot \|g\|$ ; this is Gauss' lemma!

3.  $C\langle T \rangle$  with the Gauss norm is a Banach algebra. It is in fact the completion of  $C[T]$  with respect to the Gauss norm.

The Tate algebra also admits a purely algebraic definition; we have  $C\langle T \rangle = (\mathcal{O}_C[T])_\pi^\wedge[\frac{1}{\pi}]$ , where  $\pi \in C$  has  $|\pi| = 1$ .

There is also a variant of the Tate algebra as follows. Let  $r \in |C^\times| \subset \mathbb{R}_{>0}$ .

**Definition 7.4.** Let  $C\langle r^{-1}T \rangle$  consist of power series in  $T$  whose coefficients satisfy  $|a_i| r^i \rightarrow 0$  as  $i \rightarrow \infty$ . It is called the **Tate algebra of radius  $r$** .

We have  $C\langle T \rangle \cong C\langle r^{-1}T \rangle$  via  $T \mapsto \pi T$ , where  $|\pi| = r$ .

**Theorem 7.1.**  $C\langle T \rangle$  is a PID. Its maximal ideals are exactly  $(T - \alpha)$  for  $\alpha \in D_{\leq 1}$ . The residue fields of the maximal are all isomorphic to  $C$  via the evaluation maps.

*Remark.* For any  $n$  there is a ring  $C\langle T_1, \dots, T_n \rangle$ , and its maximal ideals are exactly the ideals  $(T_1 - \alpha_1, \dots, T_n - \alpha_n)$  for  $\alpha_i \in D_{\leq 1}$ .

need to finish this



## 8 Jan 23

Let  $(C, |\cdot|)$  be a complete algebraically closed non-archimedean (CACNA) field. Let  $r \in |C^\times|$ .

**Definition 8.1.** Let  $C\langle r^{-1}T \rangle$  consist of power series in  $T$  whose coefficients satisfy  $|a_i|r^i \rightarrow 0$  as  $i \rightarrow \infty$ . It is called the **Tate algebra of radius  $r$** .

**Theorem 8.1.**  $C\langle r^{-1}T \rangle$  is a PID whose maximal ideals are  $(T - \alpha)$ , where  $\alpha \in C$  has  $|\alpha| \leq r$ .


**Definition 8.2.** We have an  $r$ -Gauss norm  $\|\cdot\|_r$  given by taking the maximum of  $|a_i|r^i$ . It is a Banach norm, strictly multiplicative.

For any  $r$ , the Tate algebra is the completion of  $C[T]$  with respect to the  $r$ -Gauss norm. If  $r < 1$ , then it is also the completion of  $\mathcal{O}_C[[T]]\left[\frac{1}{\pi}\right]$ , where  $\pi$  is a pseudouniformizer, with respect to the  $r$ -Gauss norm. If  $r = 1$ , we have  $C\langle T \rangle = (\mathcal{O}_C[[T]])_\pi^\wedge\left[\frac{1}{\pi}\right]$ . Similarly, if  $r < 1$ , then  $C\langle r^{-1}T \rangle = (\mathcal{O}_C[[T]]\left[\frac{T}{\pi}\right])_\pi^\wedge\left[\frac{1}{\pi}\right]$ , where we pick  $\pi$  with  $|\pi| = r$ .

**Definition 8.3.** Let  $f \in C\langle r^{-1}T \rangle$ . Say  $\|f\|_r = \lambda$ . Say  $f = \sum a_i T^i$ . Define  $\mu_r(f)$  to be the maximum index  $i$  that satisfies  $\|a_i T^i\|_r = \lambda$ .

**Theorem 8.2.** 1.  $\mu_r(f)$  is the number of roots of  $f$  in  $D_{\leq r}$ .

2. The map  $f \mapsto (-\log \|f\|_r, -\mu_r(f))$  defines a valuation on  $C\langle r^{-1}T \rangle$  valued in  $\mathbb{R} \times \mathbb{Z}$  equipped with lexicographic order.


*Proof.* Apparently this was discussed last time. Prove 2 directly, and then it implies 1 by “Weierstrass preparation”, which means that anything in the Tate algebra is the product of a unit and a polynomial. 

Our goal is to describe a mixed characteristic analog of  $C\langle r^{-1}T \rangle$  for  $r < 1$  and  $C$  characteristic  $p$ . In everything that follows, assume  $r < 1$


Last lecture we discussed the notion of untilting  $C$ , i.e. a pair  $(F, i : F^\flat \cong C)$ .  $|p|_F \in [0, 1)$ , and if  $x \in C$ ,  $|x^\sharp|_F = |x|_C$ . We give some analogies between the two sides. (make table.)  $D_{\leq r}$  corresponds to untilts where  $|p|_F \leq r$ .  $\mathcal{O}_C[[T]]$  corresponds to  $W(\mathcal{O}_C)$ .  $C[[T]]$  corresponds to  $W(C)$ .  $C\langle r^{-1}T \rangle$  will correspond to something called  $B_{[0, r]}$ .

**Definition 8.4.** We define  $B_{[0, r]}$  to be the subset of  $W(C)$  consisting of  $\sum [a_i]p^i$  where  $|a_i|r^i \rightarrow 0$  as  $i \rightarrow \infty$ .

**Lemma 8.1.** Given  $\lambda > 0$  and  $N \in \mathbb{N}$ , define  $A_{\lambda, N}$  to be the subset of  $W(C)$  consisting of those  $\sum [a_i]p^i$  such that  $|a_i| \leq \lambda$  for  $i \leq N$ . Then  $A_{\lambda, N}$  is a subgroup.

*Proof.* This is proved by direct computation, using the discussion of how addition of Witt vectors works from earlier lectures. (add citation, maybe more detail) 


**Corollary 8.1** (Corollary of the Lemma). *Let  $\lambda : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  be a nondecreasing function. Then the set  $A_\lambda$  of  $\sum [a_i]p^i$  such that  $|a_i| \leq \lambda(i)$  for all  $i$  is a subgroup.*

*Proof.* It is the intersection of  $A_{\lambda(N),N}$  for all  $N$ . 

We can now show that  $B_{[0,r]}$  is a subgroup. Let  $\lambda_{\varepsilon,M}(i) = \max(M, \varepsilon r^{-i})$ . Then  $B_{[0,r]} = \bigcap_{\varepsilon > 0} \bigcup_{M \in \mathbb{N}} A_{\lambda_{M,\varepsilon}}$ .

**Definition 8.5.** Given  $f = \sum [a_i]p^i \in W(C)$ , define its  **$r$ -Gauss norm** to be  $\|f\|_r = \sup |a_i| r^i$  (this can be  $\infty$ , but is finite if  $f \in B_{[0,r]}$ ).

**Proposition 8.1.** *If  $f, g \in W(C)$ , then  $\|f + g\|_r \leq \max(\|f\|_r, \|g\|_r)$ .*

*Proof.* **maybe?** write  $\|f\|_r, \|g\|_r \leq \lambda$ , write this in terms of coefficients, then do stuff with coefficients of the sum. 

## 9 Jan 26

Let  $C$  be a CACNA of characteristic  $p$ . Recall that for  $r \in (0, 1) \cap |C^\times|$ , we defined  $B_{[0,r]} \subset W(C)$  to be those  $x = \sum [x_i]p^i$  with  $|x_i|p^i \rightarrow 0$  as  $i \rightarrow \infty$ .

**Proposition 9.1.**  $B_{[0,r]}$  is a subring of  $W(C)$ .

*Proof.* To prove this we start with a lemma.

**Lemma 9.1.** Let  $\lambda : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  be a nondecreasing function. Let  $A_\lambda$  be the set of  $\sum [a_i]p^i$  such that  $|a_i| \leq \lambda(i)$  for all  $i$ . Suppose  $\lambda, \lambda'$  are two such functions, and that  $x \in A_\lambda, y \in A_{\lambda'}$ . Then

1.  $x + y \in A_{\max(\lambda, \lambda')}$ .
2.  $xy \in A_{\lambda * \lambda'}$ , where  $(\lambda * \lambda')(i) = \max_{a+b=i} \lambda(a)\lambda'(b)$ .

*Proof of Lemma.* For part 1, we use the rule for addition in  $W(C)$ . Namely if  $x = \sum [a_i]p^i$  and  $y = \sum [b_i]p^i$ , then  $x + y = \sum [c_i]p^i$ , where  $c_i$  is a homogeneous degree 1 polynomial in the  $p$ -power roots of  $a_0, \dots, a_i, b_0, \dots, b_i$ . By assumption, all of these are bounded by  $\max(\lambda(i), \lambda'(i))$ , and then so are all of the possible degree 1 monomials. Since the absolute value is non-archimedean, we are done.

For part 2, we have  $xy = \sum_{i,j} [a_i b_j]p^{i+j}$ . One can show that each summand is in  $A_{\lambda * \lambda'}$ , so the entire sum is in  $A_{\lambda * \lambda'}$  (by part 1, finite sums are ok; infinite sums are ok because the condition to be checked only involves finitely many terms at a time.)

fill in proof that the lemma implies the proposition.

**Definition 9.1.** On  $B_{[0,r]}$ , define the  $r$ -Gauss norm to be  $\|\sum [a_i]p^i\|_r = \sup_i |a_i|r^i$ .

**Proposition 9.2.** For any  $f, g \in B_{[0,r]}$ , we have  $\|f + g\|_r \leq \max(\|f\|_r, \|g\|_r)$  and  $\|fg\|_r \leq \|f\|_r \|g\|_r$ .

*Proof.* Use the fact that  $\|[a_i]p^i\| \leq \delta$  iff  $|a_i| \leq \delta r^{-i}$  in conjunction with the Lemma above.

**Proposition 9.3.**  $B_{[0,r]}$  is complete for the  $r$ -Gauss norm, so it is a Banach algebra.

*Proof.* Check that if a sequence in  $B_{[0,r]}$  is Cauchy, then each sequence of  $p^i$  components is Cauchy, so you can take the limit pointwise. Then do some bookkeeping to ensure that the limit lands back in  $B_{[0,r]}$ .

**Proposition 9.4.** If  $f, g \in B_{[0,r]}$ , then  $\|fg\|_r = \|f\|_r \|g\|_r$ .

Recall that in  $C\langle r^{-1}T \rangle$  we had a similar result, and it let us define a Gauss valuation  $\nu_r(\sum a_i T^i) = \inf_i (-\log(|a_i|r^i))$ . This refines to  $\nu_{r,\pm}(\sum a_i T^i) = \inf_i (-\log(|a_i|r^i), \mp i) \in \mathbb{R} \times \mathbb{Z}$ , and it let us count roots.


**Theorem 9.1.** We can define  $\nu_{r,\pm}$  on  $B_{[0,r]}$  in the same way, and they also define valuations.


*Proof.* The problem reduces to looking at monomials. In particular, suppose one has a finite collection  $S$  of monomials  $[a]p^n$  in  $W(C)$  that have the same  $r$ -Gauss norm  $\delta$ . Suppose there is a unique monomial in  $S$  that maximizes (resp. minimizes) the degree  $n$ . Then the sum of the monomials has  $r$ -Gauss norm  $\delta$ .

**Lemma 9.2.** Let  $x = \sum [a_i]p^i$ ,  $y = \sum [b_i]p^i$  in  $W(C)$ . Let  $z = x + y = \sum [c_i]p^i$ . Fix  $n \geq 0$  and  $M \in \mathbb{R}_{>0}$  such that

1.  $|a_i| < M$  for  $i \leq n$ .
2.  $|b_i| \leq M$  for  $i \leq n$ .
3.  $|b_n| = M$ .

Then  $|c_n| = M$ .

*Proof of Lemma.*  $c_n = a_n + b_n$  plus cross terms involving  $a_i, b_j$  for  $i, j < n$ . Everything is homogeneous of degree 1, so everything works out by non-archimedean property. 

One can use this Lemma to complete the proof of the Theorem. 

**Example 9.1.** Consider  $[a]p^i$ ,  $[b]p^j$ , both have Gauss norm  $\delta$ . Show their sum has Gauss norm  $\delta$ . todo

Recall the following results for  $C\langle r^{-1}T \rangle$ . Fix  $\pi \in C$  with  $|\pi| = r$ .

1.  $C\langle r^{-1}T \rangle$  is the completion of  $C[T]$  with respect to the  $r$ -Gauss norm. It is also the completion of  $\mathcal{O}_C[[T]] \otimes_{\mathcal{O}_C} C = \mathcal{O}_C[[T]][\frac{1}{\pi}]$  (the power series with bounded coefficients) with respect to the  $r$ -Gauss norm.
2.  $C\langle r^{-1}T \rangle = \mathcal{O}_C[\frac{T}{\pi}]^\wedge[\frac{1}{\pi}]$ .

We have the following similar results for  $B_{[0,r]}$ :

1.  $B_{[0,r]}$  is the completion of  $W(\mathcal{O}_C)[\frac{1}{[\pi]}]$  with respect to the  $r$ -Gauss norm.
2.  $B_{[0,r]} = W(\mathcal{O}_C)[\frac{p}{[\pi]}]^\wedge[\frac{1}{[\pi]}]$ .

*Remark.* The ring  $W(\mathcal{O}_C)[\frac{p}{[\pi]}]$  is contained in the unit ball of  $B_{[0,r]}$ . It can also be expressed as  $W(\mathcal{O}_C)[x]/([\pi]x - p)$ , because  $[\pi], p$  form a regular sequence in  $W(\mathcal{O}_C)$ .