MATH 7311 Homework 7

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1 Problem 1

Let g be continuous on $[-a, a]$, $a > 0$. For n such that $\frac{1}{n} < a$, let $I_n = [-1/n, 1/n]$ and let $\varphi_n = \frac{n}{2}\chi_{I_n}$. Show that $\lim_n \int g\varphi_n d\lambda = \lim_n \frac{n}{2}\int_{I_n} g d\lambda = g(0)$.

Proof. Note that for all n, $\int \varphi_n d\lambda = \frac{n}{2}\int_{I_n} d\lambda = 1$. Write $f(x) = g(x) - g(0)$. Then $\int g\varphi_n d\lambda = \int f\varphi_n d\lambda + g(0)\int \varphi_n d\lambda = g(0) + \int f\varphi_n d\lambda$. Thus it suffices to show that $\lim_n \int f\varphi_n d\lambda = 0$. Let $\varepsilon > 0$. By continuity, there is $\delta > 0$ such that $|f(x)| < \varepsilon$ for $x \in [-\delta, \delta]$. Then for $n > 1/\delta$, $|f(x)| < \varepsilon$ for $x \in I_n$. Then $|\int f\varphi_n d\lambda| \leq \int |f|\varphi_n d\lambda \leq \int \varepsilon\varphi_n d\lambda = \varepsilon$ for $n > 1/\delta$, implying that $\lim_n \int f\varphi_n d\lambda = 0$. $\square$

2 Problem 2

Let $f_n(x) = \chi_{[-n,n]}(x)\sin(\pi x/n)$.

a) Find $f(x) = \lim_n f_n(x)$ and show that f_n converges uniformly on compact subsets of $\mathbb{R}$.

Proof. Let $\varepsilon > 0$. By continuity of sin, there is $\delta > 0$ such that $|\sin(x)| < \varepsilon$ for $|x| < \delta$. For any $x \in \mathbb{R}$, there is $n \in \mathbb{N}$ such that $n > x$ and $|\pi x/n| < \delta$. Thus $|f_n(x)| < \varepsilon$, so $f_n(x) \to 0$ for all $x \in \mathbb{R}$.

Let A be a compact subset of $\mathbb{R}$. Then there is some $M \in \mathbb{R}$ such that $A \subset [-M, M]$. Then for all $n > M$ and $x \in A$, $f_n(x) = \sin(\pi x/n)$. Let $\varepsilon > 0$. By continuity of sin, there is $\delta > 0$ such that $|\sin(x)| < \varepsilon$ for $|x| < \delta$. Thus let $n > M\pi/\delta$ and $n > M$. For any $x \in A$, $|x| \leq M$, so $|\pi x/n| \leq \pi M/n < \delta$, so $|\sin(\pi x/n)| < \varepsilon$. Thus f_n converges uniformly on A. $\square$

b) Does f_n converge uniformly to f on all of $\mathbb{R}$?

Proof. No. For any $n \in \mathbb{N}$, we have $f_n(n/2) = 1$, so there is a point which is not arbitrarily close to 0 for all $n \in \mathbb{N}$. $\square$

c) Show that $\int_{\mathbb{R}} f d\lambda = \lim_n \int_{\mathbb{R}} f_n d\lambda$, but there is no integrable function g such that $|f_n(x)| \leq g(x)$ for almost all $x \in \mathbb{R}$.

Proof. Since $f = 0$, $\int_{\mathbb{R}} f d\lambda = 0$. For each n, $\int_{\mathbb{R}} f_n d\lambda = \int_{-n}^{n} \sin(\pi x/n)dx = \frac{n}{\pi}\int_{-\pi}^{\pi} \sin(u)du = 0$. Thus $\int_{\mathbb{R}} f d\lambda = \lim_n \int_{\mathbb{R}} f_n d\lambda$.

Now, suppose there is an integrable function g such that $|f_n(x)| \leq g(x)$ for almost all $x \in \mathbb{R}$. Fix some $n \in \mathbb{N}$. We know that $f_n(n/2) = 1$. For $1 > \varepsilon > 0$, there is $\frac{\pi}{2} > \delta > 0$ such that $\cos(\delta) > 1 - \varepsilon$. Then $f_n(n/2 \pm n\delta/\pi) = \sin(\pi/2 \pm \delta) = \cos(\delta) > 1-\varepsilon$. Furthermore, for $|x| \leq n\delta/\pi$, $f_n(n/2+x) \geq f_n(n/2+n\delta/\pi)$. Thus $f_n(x) > 1 - \varepsilon$ for $x \in [n/2 - n\delta/\pi, n/2 + n\delta/\pi] = I_n$. Since $g(x) \geq |f_n(x)|$ for almost all x, this implies that $\int_{I_n} g d\lambda \geq (1 - \varepsilon)2n\delta/\pi > 0$. Since this holds for all $n \in \mathbb{N}$, there are infinitely many non-zero contributions of size at least $(1 - \varepsilon)2\delta/\pi$, which is a nonzero constant, to $\int |g|$, contradicting the fact that g is integrable. $\square$

3 Problem 3

Let f be real valued and measurable on a finite measure space X. Show that $\lim_n \int_X \cos^{2n}(\pi f(x)) d\mu = \mu(f^{-1}(\mathbb{Z}))$.

Proof. Let $f_n(x) = \cos^{2n}(\pi f(x))$ and let $S = f^{-1}(\mathbb{Z})$. If $x \in S$, then $f_n(x) = 1$, since $\cos^2(\pi n) = 1$ for $n \in \mathbb{Z}$. Otherwise, $\cos^2(\pi f(x)) < 1$, so $f_n(x) \to 0$. Thus $f_n \to \chi_S$. The function $g(x) = 1$ is integrable on X, since $\int |g| = \mu(X) < \infty$. We have $f_n(x) \leq 1$ for all n and x, so $|f_n(x) \leq g(x)$ for all n and x. Thus by LDCT, $\lim_n \int_X f_n d\mu = \int_X \chi_S d\mu = \mu(S)$ as desired. $\square$

4 Problem 4

For each of the following, check if the limit exists. If so, find its value.

a) $\lim_n \int_{[0,\infty)} (1+\frac{x}{n})^{-n} \sin(x/n) d\lambda$, $n \geq 2$.

Proof. The function is continuous, so we can evaluate the integral by standard calculus methods. Using $u = x/n$ gives $n\int_0^\infty (1+x)^{-n} \sin(x) dx$. Using $|\int f| \leq \int |f|$, we have $|\int_0^\infty (1+x)^{-n} \sin(x)| \leq \int_0^\infty (1+x)^{-n} dx$, since $(1+x)^{-n} \geq 0$ for $x \in [0,\infty)$ and $|\sin(x)| \leq 1$. $\int_0^\infty (1+x)^{-n} dx = \frac{1}{n} 2^{-n+1}$ for $n \geq 2$. Thus $|\int_{[0,\infty)} (1+\frac{x}{n})^{-n} \sin(x/n) d\lambda| \leq 2^{-n+1}$, implying that $\lim_n \int_{[0,\infty)} (1+\frac{x}{n})^{-n} \sin(x/n) d\lambda = 0$. $\square$

b) $\lim_n \int_{[0,n]} \frac{\sin(x)}{1+nx^2} d\lambda$.

Proof. We have $|\int_{[0,n]} \frac{\sin(x)}{1+nx^2} d\lambda| \leq \int_{[0,n]} \frac{1}{1+nx^2} d\lambda = \frac{1}{\sqrt{n}} \tan^{-1}(n\sqrt{n}) \leq \frac{\pi}{2\sqrt{n}}$. The first inequality follows $|\int f| \leq \int |f|$ and the fact that $|\sin(x)| \leq 1$ and $|1+nx^2| = 1+nx^2$. Since $1/(1+nx^2)$ is a continuous function, we can evaluate the Lebesgue integral using standard calculus. The last inequality follows from $\tan^{-1}(x) \leq \pi/2$. Since $\lim_n \frac{\pi}{2\sqrt{n}} = 0$, we have $\lim_n \int_{[0,n]} \frac{\sin(x)}{1+nx^2} d\lambda = 0$. $\square$