Symplectic Duality and Coulomb Branches
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Preface

There will be some gaps in explanation, either due to the lecturer's admission or my own lack of understanding. In particular, many "proofs" are sketches of proofs. Gaps due to my own misunderstanding will be indicated by three red question marks: ???. More generally, my own questions about the material will also be in red. Things like "**Question**" will be questions posed by the lecturer. Feel free to reach out to me with explanations.

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We fix G reductive connected over $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra, M an affine normal Poisson variety, generically symplectic. Let G have Hamiltonian action on M with moment map $\mu : M \to \mathfrak{g}^*$. There is a scaling action $\mathbb{C}^\times$ on M which commutes with the G-action and with μ.

When we introduce the Langlands dual group $G^\vee$, we want some $M^\vee$ which plays the role of M. This is known basically only in the case where $M = T^*X$ for X a smooth affine G-variety. The main problem is specifically to find a class of "good" M such that M is good implies $M^\vee$ is good, $(M^\vee)^\vee = M$, and all T^*X are good.

Now fix a Borel B. Let X be a smooth affine G-variety, and let M be as in the first paragraph.

Definition 1.1.

1. X is **spherical** if X contains an open dense B-orbit.
2. M is **hyperspherical** if for all $f_1, f_2 \in \mathbb{C}[M]^G$, we have $\{f_1, f_2\} = 0$.

Theorem 1.1. *X is spherical iff T^*X is hyperspherical.*

We will prove this later on.

Theorem 1.2. *Let M be a hyperspherical variety. Then:*

1. *The map $\overline{\mu} : M//G \to \mathfrak{g}^*//G$ on categorical quotients is finite, i.e. $\mathbb{C}[M]^G$ is a finitely generated module over $\mathbb{C}[\mathfrak{g}^*]^G = (\mathrm{Sym}\mathfrak{g})^G$.*
2. *The image $\mathrm{im}(\overline{\mu})$ of $\overline{\mu}$ is closed in $\mathfrak{g}^*//G$.*
3. *The composite $\nu : M \to M//G \xrightarrow{\overline{\mu}} \mathfrak{g}^*//G$ has the property that all irreducible components of all of its non-empty fibers have the same dimension.*
4. *Each irreducible component of the generic fibers of ν is the closure of a G-orbit.*

Note. "Generic" here means it is true in a Zariski open subset.

Corollary 1.1. *If M is hyperspherical, then* $\dim M \leq \dim G + \dim(\mathfrak{g}^*//G) = \dim G + \mathrm{rk} G$.

From now on, we consider M to be smooth and symplectic.

Let $\mathfrak{b} = \mathrm{Lie} B$. The composite $\mu_B : M \xrightarrow{\mu} \mathfrak{g}^* \to \mathfrak{b}^*$ is the moment map for the B-action. Let $\Lambda_M = \mu_B^{-1}(0) = \mu^{-1}(\mathfrak{b}^\perp)$.

Example 1.1. If $M = T^*X$, then Λ_M is the union of the conormal bundles $T^*_O X$ to B-orbits $O \subset X$.

Theorem 1.3. *If X is spherical, then X is a finite union of B-orbits. (???)*

Corollary 1.2. *If X is spherical and $M = T^*X$, then Λ_M is Lagrangian in M.*

Proof. Each conormal bundle is Lagrangian.

Theorem 1.4. *Let M be smooth and symplectic. If Λ_M is Lagrangian, then M is hyperspherical.*

Conjecture: if M is good symplectic hyperspherical, then Λ_M is Lagrangian, and there is a bijection between the irreducible components of Λ_M and the irreducible components of $\Lambda_{M^\vee}$.

Let $\mathcal{B} = G/B$ be the flag variety, let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{g}^*$, let $\widetilde{\mathcal{N}} = T^*\mathcal{B} \xrightarrow{\pi} \mathcal{N}$ be the Springer resolution, and let $\mathrm{St}_G = \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$ be the Steinberg variety. It is known that St_G is Lagrangian in $T^*(\mathcal{B} \times \mathcal{B})$, and $H^{BM}_{top}(\mathrm{St}_G)$ has a natural algebra structure, isomorphic to the group algebra of the Weyl group W.

Now let M be hyperspherical, and assume that Λ_M is Lagrangian. Let $\mathrm{St}_M = M \times_{\mathfrak{g}^*} \widetilde{\mathcal{N}}$. As a subvariety of $M \times \widetilde{\mathcal{N}}$, it is stable under the diagonal G-action. We have $\mathrm{St}_M \cong G \times^B \Lambda_M$. If $M = T^*X$, then $M \times \widetilde{\mathcal{N}} = T^*(X \times \mathcal{B})$, and St_M is the union of conormal bundles to G-orbits.

By analyzing the fiber product conditions, we see that there is a convolution $\mathrm{St}_M \circ \mathrm{St}_G = \mathrm{St}_M$. In particular, two pairs $(\eta, \xi) \in \mathrm{St}_M$ and $(\xi, \xi') \in \mathrm{St}_G$ give a new pair (η, ξ') in St_M. This gives $H^{BM}_{top}(\mathrm{St}_M)$ the structure of a $H^{BM}_{top}(\mathrm{St}_G)$-module, i.e. it is a representation of W.

Conjecture: There is an isomorphism of W-reps $H^{BM}_{top}(\mathrm{St}_M) \cong H^{BM}_{top}(\mathrm{St}_{M^\vee})$.

Example 1.2. Now we tabulate results when I'm not lazy I will make this look nice.. Row 1: $G = T$ is a torus, $M = T^*(T/T_1)$ for a subtorus T_1. Then $M^\vee = T^*(T_1^\vee)$.
Spherical T-variety is a toric variety; for it to be smooth, it would be affine. So in particular (row 2), if $G = (\mathbb{C}^\times)^n$ and $M = T^*(\mathbb{C}^n)$, then $G^\vee = G$ and $M^\vee = M$.
Next (row 3) consider the group $G \times G$ and $M = T^*G$, where $G \times G$ acts by left and right translations. Then the dual group is $G^\vee \times G^\vee$ and $M^\vee = T^*(G^\vee)$. Note that G is spherical in this case, since it has the open $B \times B$ orbit given by Bw_0B, where $w_0 \in W$ is the longest element.
Row 4: if the group is just G and $M = T^*G$, then $M^\vee = \mathcal{N}_{G^\vee}$.
Row 5: Let $U = [B, B]$ be max unipotent. Consider the group $G \times T$, where T is a maximal torus in G. Let M be the affine closure of $T^*(G/U)$. Then $M^\vee$ is the affine closure of $T^*(G^\vee/U^\vee)$. This is related to Eisenstein series. (note: possibly incorrect)

Row 6: consider the same M but for the group G. Then $M^\vee = \overline{T^*(G^\vee/U^\vee)}/W$, where the W-action is by Gelfand-Graev (it is not an obvious action).
Row 7: Let the group be G, and let M be a point. Then $M^\vee = T^*_\psi(G^\vee/U^\vee) = (T^*G^\vee)//_\psi U^\vee$ (Hamiltonian reduction), the Whittaker potential bundle for a nondegenerate character $\psi : U^\vee \to \mathbb{C}^\times$.
Row 8: $G = GL_n \times GL_n$, $M = T^*(\mathbb{C}^n \otimes \mathbb{C}^n) = T^*M_n$, where GL_n acts by left and right translations. This group is self dual, and $M^\vee = T^*(GL_n \times \mathbb{C}^n) = T^*(G \times^{GL_n} \mathbb{C}^n)$. This duality is classical and known in automorphic forms; in one direction it is Rankin-Selberg, and in the other it is Godement-Jacquet.
Row 9: $G = GL_n$, $M = T^*(\mathbb{C}^n)$, $M^\vee = T^*M_n//_\psi U$.
Row 10: $G = GL_{2n}$, $M = T^*(G/(GL_n \times GL_n))$ (block diagonal embedding), $M^\vee = T^*(G \times^{Sp_{2n}} \mathbb{C}^{2n})$.
Row 11: $G = GL_{2n}$, $M = T^*(G/Sp_{2n})$, $M^\vee = T^*_\phi(G/Q)$, where Q is the subgroup of block $(n+n)\times(n+n)$ upper triangular matrices, where the two diagonal blocks are equal, and ϕ takes such a matrix to $e^{tr(a)}$, where a is the upper right block.

2 Jan 8 - Geometry of spherical and hyperspherical varieties

Theorem 2.1 (Rosenlicht)**.** *Let H be a connected algebraic group acting on an irreducible variety X. Then there is an H-stable Zariski open $X^\circ \subset X$ such that:*

1. *H-orbits in X° has maximal dimension.*
2. *There is a smooth surjective morphism $X^\circ \to Y$ such that each fiber is a single orbit. Y is called the **geometric quotient** X°/H.*

Corollary 2.1. *Let $\mathbb{C}(X)$ be the field of rational functions on X. Then $\mathbb{C}(X)^H = \mathbb{C}(X^\circ)^H = \mathbb{C}(X^\circ/H)$, and X has an open H-orbit iff $\mathbb{C}(X) = \mathbb{C}$.*

Resume the usual setup (G, B, T, U). If $\lambda \in X^*(T)$ is a character of T, we may lift it to a character of B by letting λ act trivially on U.

Now let G act on an affine variety X. We can decompose $\mathbb{C}[X]$ into isotypic components corresponding to highest weight irreducible representations:

$$\mathbb{C}[X] = \bigoplus_{V_\lambda \in \mathrm{Irr}(G)} \mathbb{C}[X]_\lambda.$$

We may do this because G is reductive and the action of G on $\mathbb{C}[X]$ is locally finite. Let

$$\mathbb{C}[X]^{U,\lambda} = \{f \in \mathbb{C}[X] \mid b(f) = \lambda(b)f, \forall b \in B\}.$$

These are the B-semiinvariants of weight λ. It follows that the multiplicity $m(\mathbb{C}[X] : V_\lambda)$ of V_λ in $\mathbb{C}[X]$ is $\dim \mathbb{C}[X]^{U,\lambda}$.

Theorem 2.2. *X has an open B-orbit iff $m(\mathbb{C}[X] : V_\lambda) \leq 1$ for all λ.*

Proof. To prove this, we need a lemma.

Lemma 2.1. *Let $f \in \mathbb{C}(X)$. Then $f \in \mathbb{C}(X)^B$ iff there exist λ and and $\varphi, \psi \in \mathbb{C}[X]^{U,\lambda}$ such that $f = \varphi/\psi$.*

Proof of lemma. If $f = \varphi/\psi$, then $b(f) = b(\varphi)/b(\psi) = (\lambda(b)\varphi)/(\lambda(b)\psi) = \varphi/\psi = f$, so f is invariant. Conversely, let $f \in \mathbb{C}(X)^B$. Write $f = \varphi'/\psi'$ for arbitrary $\varphi', \psi' \in \mathbb{C}[X]$. The span $\langle B\psi' \rangle$ of $B\psi'$ is finite dimensional. By Lie's theorem, there is a λ and nonzero B-semiinvariant $\psi \in \langle B\psi' \rangle$ of weight λ. Consequently, write $\psi = \sum_i c_i b_i(\psi')$. Define $\varphi = \sum_i c_i b_i(\varphi')$. For all $b \in B$, we have $b(\varphi') = b(f)b(\psi') = fb(\psi')$. Thus $\varphi = \sum c_i b_i(\varphi') = \sum c_i f b_i(\psi') = f\psi$. Thus $f = \varphi/\psi$, and since f is invariant and ψ is semiinvariant of weight λ, φ must also be semiinvariant of weight λ.

To prove the theorem, there exists an open B-orbit in X iff $\mathbb{C}(X)^B = \mathbb{C}$, which by the lemma is true iff $\dim \mathbb{C}[X]^{U,\lambda} \leq 1$ for all λ, which gives the claim (since this dimension is the required multiplicity).

Theorem 2.3. *If X has an open B-orbit, then $\mathbb{C}[T^*X]^G$ is a commutative Poisson algebra.*

Proof. Let $\mathcal{D}(X)$ be the algebra of differential operators on X. Standard facts from X being affine:

1. $\mathbb{C}[X]$ is faithful as a $\mathcal{D}(X)$-module.
2. $gr\mathcal{D}(X) \cong \mathbb{C}[T^*X]$ (where the filtration on $\mathcal{D}(X)$ is by order of differential operators).
3. Since G is reductive, $gr(\mathcal{D}(X)^G) = \mathbb{C}[T^*X]^G$.

Let $a \in \mathcal{D}(X)^G$. Then the action of a on $\mathbb{C}[X]$ commutes with the G-action. Thus a restricts to maps between isotypic components for all weight. By the previous theorem and our hypothesis that X has an open B-orbit, we know that each isotypic component is either V_λ or 0. Thus, by Schur's lemma, a acts by scalars a_λ on all isotypic components. We obtain an algebra map $i : \mathcal{D}(X)^G \to Maps(X^*(T), \mathbb{C})$, where the right hand side consists of arbitrary functions of sets and is equipped with the pointwise algebra structure. Since $\mathcal{D}(X)$ acts faithfully on $\mathbb{C}[X]$, this map i is injective. Since the pointwise algebra structure is commutative, we get that $\mathcal{D}(X)^G$ is commutative. Finally, the associated graded of a commutative algebra is Poisson commutative, so we are done.

Theorem 2.4. *If X has an open B-orbit, then X is a finite union of B-orbits.*

Proof. We do not prove the whole claim, only the following weaker statements:

1. X is a finite union of G-orbits.
2. Each G-orbit contains an open B-orbit.

We need the following lemma:

Lemma 2.2. *If X has an open B-orbit, then for all G-stable closed subvarieties Y, we have $\mathbb{C}(Y)^B = \mathbb{C}$.*

Proof of lemma. Let $f \in \mathbb{C}(Y)^B$. Then there is some λ and some $\varphi, \psi \in \mathbb{C}[Y]^{U,\lambda}$ such that $f = \varphi/\psi$. (I should probably start hyperlinking references to past results). Since Y is a closed subvariety, the restriction map $\mathbb{C}[X] \to \mathbb{C}[Y]$ is surjective. Then each map on isotypic components is surjective. By complete reducibility, this means $\mathbb{C}[Y]_\lambda$ is a direct summand of $\mathbb{C}[X]_\lambda$. The is true of the spaces of semiinvariants, meaning we can lift φ, ψ to semiinvariant functions φ', ψ' of weight λ on all of X. Then $\varphi'/\psi' \in \mathbb{C}(X)^B = \mathbb{C}$, meaning $\varphi/\psi = f$ is also constant.

Now, since X has an open B-orbit, it must also have an open G-orbit O by saturating the open B-orbit. Let X_1 be an irreducible component of $X \setminus O$. Then X_1 is a closed G-stable subvariety of X with strictly smaller dimension. By the lemma, X_1 also has an open B-orbit, so by the same argument, X_1 has an open G-orbit. We may continue in this way, and eventually we the process will end because the dimension is strictly shrinking.

Definition 2.1. Let (E, ω) be a symplectic vector space. Then a subspace $F \subset E$ is **isotropic** if $\omega|_F = 0$. F is **coisotropic** if $F^{\perp_\omega}$ is isotropic. F is **Lagrangian** if it is isotropic and coisotropic.

Definition 2.2. Let (X, ω) be a smooth affine symplectic variety. Then a subvariety $Y \subset X$ is **isotropic/coisotropic** if there is an open smooth $Y^\circ \subset Y$ such that for all $y \in Y^\circ$, $T_y Y$ is isotropic/coisotropic.

Now, let (M, ω) be a smooth affine symplectic variety. Let G have Hamiltonian action on M with moment map $\mu : M \to \mathfrak{g}^*$.

Theorem 2.5. *The following are equivalent:*

1. *The Poisson algebra $\mathbb{C}(M)^G$ is commutative (meaning the bracket vanishes).*

2. *Generic G-orbits in M are coisotropic subvarieties.*

3. *Irreducible components of generic fibers of μ are isotropic.*

Proof. All statements are "generic", so we may assume $M = M^\circ$ in the sense of Rosenlicht's theorem. Any $m \in M$ gives an action map $\mathrm{act}_m : G \to M$, $g \mapsto gm$, and we can differentiate it to get $d_m \mathrm{act}_m : \mathfrak{g} \to \mathfrak{g}m = T_m(Gm) \subset T_m M$. Any $f \in \mathbb{C}(M)$ gives a Hamiltonian vector field ξ_f determined by $df = \omega(\xi_f, -)$. Then $f \in \mathbb{C}(M)^G$ iff f is constant on G-orbits, which is true iff $d_m f|_{\mathfrak{g}m} = 0$ for almost all $m \in M$ (namely, wherever f is defined). But this is true iff $\omega(\xi_f, \mathfrak{g}m) = 0$, i.e. $\xi_f \in (\mathfrak{g}m)^{\perp_\omega}$. For $f_1, f_2 \in \mathbb{C}(M)$, we have $\{f_1, f_2\} = \omega(\xi_{f_1}, \xi_{f_2})$. Thus $\mathbb{C}(M)^G$ is Poisson commutative iff for almost all $m \in M$, the space of $\xi_f(m)$ for $f \in \mathbb{C}(M)^G$ is an isotropic subspace of $T_m M$. But we have computed the space of $\xi_f(m)$; it is $(\mathfrak{g}m)^{\perp_\omega}$. So $\mathbb{C}(M)^G$ is Poisson commutative iff for almost all m, $\mathfrak{g}m = T_m(Gm)$ is coisotropic, which exactly means Gm is coisotropic.

Now observe that the transpose of $d_m \mathrm{act}_m$ is the composite $T_m^* M \xrightarrow{\sim} T_m M \xrightarrow{d_m \mu} \mathfrak{g}^*$, where the first map is the isomorphism given by ω being nondegenerate. (I'm not sure I see why this is true)

Proof to be continued next lecture