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Winter 2026

Preface

There will be some gaps in explanation, either due to the lecturer's admission or my own lack of understanding. In particular, many "proofs" are sketches of proofs. Gaps due to my own misunderstanding will be indicated by three red question marks: ???. More generally, my own questions about the material will also be in red. Things like "**Question**" will be questions posed by the lecturer. Feel free to reach out to me with explanations.

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1 Jan 5 - Perfect Rings and Tilting

1.1 Perfect Rings

Definition 1.1. Let R be an $\mathbb{F}_p$ algebra. We say R is **perfect** if the Frobenius map $\varphi : R \to R, x \mapsto x^p$ is an isomorphism.

Example 1.1.

1. Perfect field.
2. $\mathbb{F}_p[x^{1/p^\infty}] = \bigcup_{n \geq 0} \mathbb{F}_p[x^{1/p^n}]$; the free perfect ring on 1 generator.
3. Any limit or colimit of perfect $\mathbb{F}_p$ algebras.
4. Any integrally closed domain whose fraction field is perfect (e.g. algebraically closed).
5. If R is perfect, I is fg ideal, then $R_I^\wedge$ (the I-completion) is perfect.

Example 1.2. The inclusion of the category of perfect $\mathbb{F}_p$-algebras into the category of all $\mathbb{F}_p$-algebras has both adjoints. The left adjoint $R \mapsto R_{perf}$ is the colimit of $R \xrightarrow{\varphi} R \xrightarrow{\varphi} \cdots$. The right adjoint $R \mapsto R^{perf}$ is the limit of $\cdots \xrightarrow{\varphi} R \xrightarrow{\varphi} R$.

Example 1.3.

1. $\mathbb{F}_p[x^{1/p^\infty}] = \mathbb{F}_p[x]_{perf}$.
2. $\mathbb{F}_p[x]^{perf} = \mathbb{F}_p$. In general, if R is Noetherian, then R^{perf} is a finite product of fields.
3. If φ is surjective (R is called semiperfect) then we get a surjection $R^{perf} \twoheadrightarrow R$, so R^{perf} is "larger" than R.
4. If P is perfect and I is a fg ideal, then $R = P/I$ is semiperfect, and $R^{perf} = P_I^\wedge$. (idea: frobenius iterates of I are cofinal with powers of I) (Exercise: prove this, and also find R_{perf}.)

Example 1.4. As a concrete example, let $R = \mathbb{F}_p[x^{1/p^\infty}]/(x)$. Then $R^{perf} = \mathbb{F}_p[x^{1/p^\infty}]_x^\wedge$, which consists of sums

$$\sum_{i \in \mathbb{Z}[1/p]_{\geq 0}} a_i x^i,$$

with $a_i \in \mathbb{F}_p$, and for all $N \geq 0$, there are only finitely many $i \leq N$ with $a_i \neq 0$. So $x + x^{p+1/p} + x^{p^2+1/p^2} + \cdots$ is allowed, while $x + x^{1/p} + x^{1/p^2} + \cdots$ isn't.

1.2 Witt Vectors

Perfect $\mathbb{F}_p$-algebras have a unique lift to characteristic 0.

Theorem 1.1. *Given any perfect $\mathbb{F}_p$-algebra R, there is a unique (up to unique isomorphism) p-adically complete and p-torsion free ring $\widetilde{R}$ equipped with an isomorphism $\widetilde{R}/p \xrightarrow{\sim} R$. This construction is functorial in R, and $\widetilde{R}$ is the ring $W(R)$ of Witt vectors of R.*

Proof. See chapter 2 of Serre's "Local Fields".

Example 1.5.
1. $R = \mathbb{F}_p[x^{1/p^\infty}]$ gives $\widetilde{R} = \mathbb{Z}_p[x^{1/p^\infty}]^\wedge_p$.
2. $R = \mathbb{F}_p[x^{1/p^\infty}]^\wedge_x$ gives $\widetilde{R} = \mathbb{Z}_p[x^{1/p^\infty}]^\wedge_{(p,x)}$. add description of elements in this ring
3. For R_1, R_2 perfect, $W(R_1 \otimes R_2) = (W(R_1) \otimes W(R_2))^\wedge_p$.

Structure of $W(R)$: There is a unique multiplicative but not additive map $[-] : R \to W(R)$ that is a section of the projection $W(R) \to R$.

Corollary 1.1. *For any $x \in W(R)$, there is a unique sequence $x_0, x_1, x_2, \ldots, \in R$ such that $x = \sum_{i \geq 0} [x_i] p^i$. So, as a set, $W(R) \cong \prod_{i \geq 0} R$.*

Exercise: the image of $[-]$ is the elements of $W(R)$ that are p^n powers for all n.

Analogy: $W(R)$ is like a ring of power series over R with variable p.

How to add Witt vectors: Suppose $x = \sum [x_i] p^i$ and $y = \sum [y_i] p^i$. We know $x + y = \sum [z_i] p^i$ for some $z_i \in R$. In fact, for each n, $z_n \in \mathbb{F}_p[x_0^{1/p^\infty}, \ldots, x_n^{1/p^\infty}, y_0^{1/p^\infty}, \ldots, y_n^{1/p^\infty}]$. Key fact: the polynomial expressing z_n in this way is homogeneous of degree 1, where each x_i, y_i has degree 1.

Example 1.6. $z_0 = x_0 + y_0$.
$z_1 = x_1 + y_1 - \sum_{0 < i < p} \frac{1}{p} \binom{p}{i} x_0^{i/p} y_0^{1-i/p}$. (Exercise: prove this; use $F = W(\varphi)$)

1.3 Tilting

Definition 1.2. For any ring R, its **tilt** $R^\flat$ is $(R/p)^{perf}$.

Proposition 1.1. *If R is p-complete, then $\varprojlim(\cdots \xrightarrow{\varphi} R \xrightarrow{\varphi} R) \xrightarrow{\sim} R^\flat$.*

Example 1.7.
1. $R = \mathbb{Z}_p$ has $R^\flat = \mathbb{F}_p$.
2. $R = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge_p = (\mathbb{Z}_p[x^{1/p^\infty}]/(x-p))^\wedge_p$. Then $R/p = \mathbb{F}_p[x^{1/p^\infty}]/(x)$ and $R^\flat = \mathbb{F}_p[x^{1/p^\infty}]^\wedge_x$.

Remark. If R is p-complete, there is a natural multiplicative map $(-)^\sharp : R^\flat \to R$. For example, in the second example above, $x^\sharp = p$.

Theorem 1.2. *The functor W from perfect $\mathbb{F}_p$-algebras to p-complete rings is left adjoint to the tilting functor from p-complete rings to perfect $\mathbb{F}_p$-algebras; if A is perfect/$\mathbb{F}_p$ and R is p-complete, then* $\mathrm{Hom}(W(A), R) = \mathrm{Hom}_{\mathbb{F}_p}(A, R^\flat)$.

Remark. If A is perfect/$\mathbb{F}_p$ and R is p-complete, then there is a natural map $\mathrm{Hom}(W(A), R) \to \mathrm{Hom}_{\mathbb{F}_p}(A, R/p)$. Since A is perfect, the second hom set can be identified with $\mathrm{Hom}_{\mathbb{F}_p}(A, R^\flat)$.

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2.1 Witt vectors and tilting

Theorem 2.1. *The functor W from perfect $\mathbb{F}_p$-algebras to p-complete rings is left adjoint to the tilting functor from p-complete rings to perfect $\mathbb{F}_p$-algebras; if A is perfect/$\mathbb{F}_p$ and R is p-complete, then* $\mathrm{Hom}(W(A), R) = \mathrm{Hom}_{\mathbb{F}_p}(A, R^\flat)$.

Proof. Suppose $A = \mathbb{F}_p[x^{1/p^\infty}]$. Then $W(A) = \mathbb{Z}_p[x^{1/p^\infty}]^\wedge_p$. Then $\mathrm{Hom}(W(A), B) = \varprojlim_\phi B$ and $\mathrm{Hom}(A, B^\flat) = B^\flat$. The natural map $\varprojlim_\phi B \to B^\flat$ is an isomorphism, so we have proved the claim for this specific choice of A. We want to reduce to this case. It will be enough to show that W preserves colimits. Recall that for perfect $\mathbb{F}_p$-algebras A_1, A_2, we have $W(A_1 \otimes A_2) = (W(A_1) \otimes W(A_2))^\wedge_p$. More generally, we need to show that if we have a diagram $A_1 \leftarrow A \rightarrow A_2$ of perfect $\mathbb{F}_p$-algebras, where A is arbitrary, then the natural map $(W(A_1) \otimes_{W(A)} W(A_2))^\wedge_p \to W(A_1 \otimes_A A_2)$ is an isomorphism. It is an isomorphism mod p, so the difficulty is in showing that $W(A_1) \otimes_{W(A)} W(A_2)$ is p-torsion free. There are two ways to justify this. One is using the following theorem:

Theorem 2.2 (Bhatt-Scholze)**.** *Given a diagram as above,* $\mathrm{Tor}_i^A(A_1, A_2) = 0$ *for $i > 0$.*

Another justification is as follows. $W(A)$ is a perfect δ-ring, so that $W(A_1) \otimes_{W(A)} W(A_2)$ is also a perfect δ-ring. It is then a general fact that perfect δ-rings are p-torsion free.

2.2 The map θ

The counit of the adjunction is a map $\theta : W(B^\flat) \to B$. Explicitly, for $b \in B^\flat$, we have $\theta([b]) = b^\sharp$.

Remark. If B/p is semiperfect, then $B^\flat \to B/p$ is surjective, so by Nakayama, θ is surjective.

Example 2.1. If $B = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge_p$, then $B^\flat = \mathbb{F}_p[x^{1/p^\infty}]^\wedge_x$ and $W(B^\flat) = \mathbb{Z}_p[x^{1/p^\infty}]^\wedge_{(p,x)}$. Then $\theta(x^{1/p^n}) = p^{1/p^n}$.

2.3 Perfectoid rings

Definition 2.1. Let A be a p-complete ring. A is **perfectoid** if $A \cong W(P)/\xi$, where P is a perfect $\mathbb{F}_p$-algebra and $\xi = \sum [\xi_i] p^i$ with $\xi_1 \in P^\times$.

Remark.

1. If $A = W(P)/\xi$, then $A/p = P/\xi_0$, so WLOG we may assume P is ξ_0-adically complete. In this case, $P = A^\flat$. Thus we may equivalently define perfectoid rings as p-complete rings such that $\theta : W(A^\flat) \to A$ is surjective and $\ker(\theta)$ is generated by ξ with $\xi_1 \in (A^\flat)^\times$.
2. Any perfect $\mathbb{F}_p$-algebra is perfectoid by taking $\xi = p$.

3. ξ is not a zero-divisor. Proof: Suppose $\xi x = 0$. Note that ξ maps to a unit in $W(P[1/\xi_0])$, so x maps to 0 in $W(P[1/\xi_0])$. In other words, if we write $x = \sum [x_i] p^i$, then all x_i are ξ_0-power torsion. But in P, ξ_0-power torsion is the same as ξ_0-torsion: if $\xi_0^N y = 0$, then multiply by some power of ξ_0 on both sides to get $\xi_0^{p^n} y = 0$, then since Frobenius is an isomorphism, we get can take p^nth roots to get $\xi_0 y^{1/p^n} = 0$, and then we may multiply by an appropriate power of y to get $\xi_0 y = 0$. So $\xi_0 x_i = 0$ for all i, or $[\xi_0] x = 0$. Hence $\xi x = (\sum_{i \geq 1} [\xi_i] p^i) x$, which is px times a unit since ξ_1 is a unit, so $px = 0$, so $x = 0$.

4. $A = W(P)/\xi$ is p-torsion free iff P is ξ_0-torsion free. To show this we use the torsion exchange lemma: if B is a ring with nonzerodivisors x, y, then $(B/x)][y] \cong (B/y)[x]$ (where brackets denote torsion). This is true because both are H_1 of a Koszul complex on x, y. Applying the torsion exchange lemma to $B = W(P)$, $x = p$, and $y = \xi$, we get $A[p] = (W(P)/\xi)[p] = P[\xi_0]$.

Note. The fact that ξ is not a zero-divisor implies that derived completions are well-behaved.

Example 2.2. Let $P = \mathbb{F}_p[x^{1/p^\infty}]$ and $\xi = p - [x]$. Then $W(P)/\xi = \mathbb{Z}_p[p^{1/p^\infty}]_p^\wedge$.

Example 2.3. $P = \mathbb{F}_p[q^{1/p^\infty}]_{(q-1)}^\wedge$, $\xi = 1 + [q^{1/p}] + \cdots + [q^{(p-1)/p}]$. We claim P is ξ_0-adically complete and that $\xi_1 \in P^\times$. Indeed, $\xi_0 = 1 + q^{1/p} + \cdots + q^{(p-1)/p} = (q^{1/p} - 1)^{p-1}$, so by definition P is ξ_0-complete. To see $\xi_1 \in P^\times$, we may quotient by the Jacobson radical, hence set all powers of q to be 1, so $\xi = p$ so $\xi_1 = 1$. Thus we can form the perfectoid ring $W(P)/\xi = \mathbb{Z}_p[\zeta_{p^\infty}]_p^\wedge$. Note that $(q^{1/p^n})^\sharp = \zeta_{p^n}$.

Example 2.4. Non-examples:

1. $\mathbb{Z}_p$ is not perfectoid, because θ is an isomorphism.

2. $\mathbb{Z}/p^n$ is not perfectoid for $n > 1$, since $\ker(\theta) = (p^n)$.

Claim 2.1. *Let R be perfectoid. Then there exists $u \in R^\times$ and $\alpha \in R^\flat$ such that $pu = \alpha^\sharp$; in other words, pu has a compatible system of p-power roots.*

Proof. Write $R = W(R^\flat)/\xi$. We have $\xi = [\xi_0] + p \cdot (unit)$. Thus $\theta([\xi_0]) = \xi_0^\sharp = p \cdot (unit)$.

Example 2.5. If R is perfectoid, then $R\langle x^{1/p^\infty} \rangle = (R \otimes_{\mathbb{Z}} \mathbb{Z}_p[x^{1/p^\infty}])_p^\wedge$ is perfectoid.

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Proposition 3.1. *Let R be p-torsion free containing ω such that*

1. $\omega^p \in pR^\times$.
2. $R/\omega \xrightarrow{\varphi} R/\omega^p = R/p$ *is an isomorphism.*

Then $R_p^\wedge$ is perfectoid.

Proof. WLOG, suppose R is p-complete. Condition two implies that R is semiperfect, so $R^\flat \to R/p$ is surjective. Thus, lift $\omega \mod p$ to some α, so that $\alpha^\sharp = \omega + py \in \omega R^\times$. Then, WLOG, we have $\omega = (\omega^\flat)^\sharp$. Now, letting $\overline{u}$ be the unit such that $\omega^p = p\overline{u}$, we have $\theta([\omega^\flat]^p) = \omega^p = p\overline{u}$. Lift $\overline{u}$ to $u \in W(R^\flat)^\times$. Then $[\omega^\flat]^p - pu \in \ker\theta$, and this is a valid choice of ξ to show R is perfectoid. So we want to show $W(R^\flat)/([\omega^\flat]^p - pu) \xrightarrow{\theta} R$ is an isomorphism. Both sides are p-complete and p-torsion free; the LHS is p-torsion free because $\omega^{\flat p}$ is a nonzerodivisor. Thus, to show the map is an isomorphism, we may work mod p and show $R^\flat/\omega^{\flat p} \xrightarrow{\sim} R/p$. Explicitly, this means that we want to show that, given a sequence $(x_0, x_1, \dots) \in R$ with $x_{i+1}^p = x_i$ for all i, then the following are equivalent:

1. $p \mid x_0$.
2. $\omega^p \mid x_0$.
3. $\omega^{p^{1-i}} \mid x_i$ for all i.

Clearly conditions 1 and 2 are equivalent since $\omega^p \in pR^\times$. Since Frobenius is an isomorphism $R/\omega \to R/\omega^p$, we get $\omega^p \mid x_0$ iff $\omega \mid x_1$. In fact, we can quotient the isomorphism by $\omega^{1/p^{n+1}}$ to obtain $\omega^{1/p^{n+1}} \mid y$ iff $\omega^{1/p^n} \mid y^p$. This gives the equivalence between 2 and 3, so we are done.

I'm confused by a lot of the steps in this proof, it's so over

3.1 p-root closure

Definition 3.1 (P. Roberts)**.** Let R be p-torsion free. R is **p-root closed** if for all $x \in R[\frac{1}{p}]$, we have $x^p \in R$ implies $x \in R$.

Example 3.1.

1. Any integrally closed domain is p-root closed.
2. $\mathbb{Z} \oplus (px) \subset \mathbb{Z}[x]$ is p-root closed.
3. $\mathbb{Z}[px, x^p]$ is not p-root closed.

Definition 3.2. Let A be a p-torsion free ring. Let $A^{+,p}$ (notation possibly not standard) be the set of $x \in A[\frac{1}{p}]$ such that $x^{p^n} \in A$ for large enough n. Then $A^{+,p}$ is the smallest p-root closed subring of $A[\frac{1}{p}]$ containing A, and it is called the **p-root closure** of A.

Exercise: prove $A^{+,p}$ is a ring. (Hint: use divisibility properties of binomial coefficients.)

Claim 3.1. *Let R be a p-torsion free ring with $\omega \in R$ such that $\omega^p \in pR^\times$. Then the following are equivalent:*

1. *R is p-root closed.*
2. *$R/\omega \xrightarrow{\phi} R/\omega^p$ is injective.*

Proof. ($1 \Rightarrow 2$) Fix $x \in R$ such that $\omega^p \mid x^p$. Then $x/\omega \in R[\frac{1}{p}]$, so by assumption, $x/\omega \in R$, i.e. $\omega \mid x$.

($2 \Rightarrow 1$) Fix $y \in R[\frac{1}{p}]$ such that $y^p \in R$. Write $y = z/\omega^i$ for i minimal and $z \in R$. If $i > 0$, then $\omega^{ip} \mid z^p$, so $\omega^p \mid z^p$, so $\omega \mid z$, meaning i is not minimal. Hence $i = 0$ so $y \in R$.

Definition 3.3. Let P be a perfect $\mathbb{F}_p$-algebra. Let $\xi = \sum [\xi_i] p^i \in W(P)$. Say ξ is **distinguished** if P is ξ_0-complete and $\xi_1 \in P^\times$.

Claim 3.2.

1. *If ξ is distinguished and $u \in W(P)^\times$, then ξu is also distinguished.*
2. *If ξ, ξ' are distinguished and $\xi \mid \xi'$, then $\xi'/\xi \in W(P)^\times$.*

Theorem 3.1. *Let R be p-torsion free and p-complete. Then R is p-complete iff the following are all true:*

1. *There is $\omega \in R$ such that $\omega^p \in pR^\times$.*
2. *R is p-root closed.*
3. *R/p is semiperfect.*

Proof. Assume 1-3 hold. Conditions 1 and 2 imply $R/\omega \to R/\omega^p$ is injective, and condition 3 implies it is surjective. Then, by the proposition earlier (cite), R is perfectoid.

Thus assume R is perfectoid. It is enough to show condition 2. Choose $\omega^\flat \in R^\flat$ such that $((\omega^\flat)^\sharp)^p = p\overline{u}$ for $\overline{u} \in R^\times$, and let $\omega = (\omega^\flat)^\sharp$. It is enough to show $R/\omega \to R/\omega^p$ is an isomorphism. Lift $\overline{u}$ to $u \in W(R^\flat)^\times$, so that $\xi = [\omega^\flat]^p - pu$ is distinguished and in $\ker\theta$. Then $R = W(R^\flat)/\xi$, so $R/\omega \to R/\omega^p$ is $R^\flat/\omega^\flat \to R^\flat/\omega^{\flat p}$, and this is an isomorphism since $R^\flat$ is perfect.

Example 3.2. Let R be a domain. Let R^+ be the absolute integral closure, i.e. the integral closure in the algebraic closure of the fraction field of R. Then $(R^+)^\wedge_p$ is perfectoid. Applying this construction to $\mathbb{Z}_p$, we obtain that $\mathcal{O}_{C_p}$ is perfectoid.

3.2 Category of perfectoid rings

There is no initial object in the category of perfectoid rings.

The category of perfectoid rings is equivalent to the category of "perfect prisms", i.e. pairs (P, I), where P is a perfect ring and I is an ideal of $W(P)$ generated by a distinguished element. Given a perfectoid R, we send it to $(R^\flat, \ker(\theta))$. Given a perfect prism (P, I), we send it to $W(P)/I$.

Corollary 3.1. *The category of perfectoid rings has pushouts; in particular, given $R_1 \leftarrow R \rightarrow R_2$ maps of perfectoids, we have $(R_1 \otimes_R R_2)_p^\wedge$ is perfectoid.*

Proof. By the "rigidity of prisms", we may write $R = W(R^\flat)/I, R_1 = W(R_1^\flat)/I, R_2 = W(R_2^\flat)/I$ for the same ideal I. Then $(R_1 \otimes_R R_2)_p^\wedge = (W(R_1^\flat) \otimes_{W(R^\flat)} W(R_2^\flat))/I = W(R_1^\flat \otimes_{R^\flat} R_2^\flat)/I$.