MATH 7311 Homework 1

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September 2021

1 Problem 1

Show directly that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{B}_4) = \sigma(\mathcal{B}_8)$.

Proof. Recall that $\mathcal{B}_4 = \{[a,b] | a < b\}$ and $\mathcal{B}_8 = \{[a,\infty) | a \in \mathbb{R}\}$. Note that $[a,\infty) = \bigcup_{n=0}^{\infty}[a, b+n] \in \sigma(\mathcal{B}_4)$, implying that $\sigma(\mathcal{B}_8) \subset \sigma(\mathcal{B}_4)$. On the other hand, $(b,\infty) = \bigcup_{n=1}^{\infty}[b+\frac{1}{n},\infty) \in \sigma(\mathcal{B}_8)$, so $[a,b] = [a,\infty)\backslash(b,\infty) \in \sigma(\mathcal{B}_8)$. Therefore, $\sigma(\mathcal{B}_4) \subset \sigma(\mathcal{B}_8)$, and so $\sigma(\mathcal{B}_4) = \sigma(\mathcal{B}_8)$.

We now show $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{B}_4)$. We have $[a,b] = \bigcap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n}) \in \mathcal{B}(\mathbb{R})$, so $\sigma(\mathcal{B}_4) \subset \mathcal{B}(\mathbb{R})$. On the other hand, any open set is a countable union of open intervals, and open intervals are countable unions of closed intervals: $(a,b) = \bigcup_{n \geq m}[a+\frac{1}{n}, b-\frac{1}{n}]$, where m is chosen such that $m > \frac{2}{b-a}$, to ensure that $a + \frac{1}{n} < b - \frac{1}{n}$ for $n \geq m$. By definition, countable unions of closed sets are in $\sigma(\mathcal{B}_4)$, so open intervals are in $\sigma(\mathcal{B}_4)$. Then open sets, being countable unions of open intervals, are also in $\sigma(\mathcal{B}_4)$. Thus $\mathcal{B}(\mathbb{R}) \subset \sigma(\mathcal{B}_4)$. $\square$

2 Problem 2

Let $\{E_n\}$ be a sequence of subsets of Y.

a) Show that $\underline{\lim} E_n \subset \overline{\lim} E_n$.

Proof. Let $x \in \underline{\lim} E_n = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} E_k$, so there exists n such that $x \in \bigcap_{k \geq n} E_k$. That is, $x \in E_n, x \in E_{n+1}, ...$ For $j = 1, ..., n-1, n$, $E_n \subset \bigcup_{k \geq j} E_k$, so $x \in \bigcup_{k \geq 1} E_k, ..., x \in \bigcup_{k \geq n} E_k$. For $j \geq n+1$, $E_j \subset \bigcup_{k \geq j} E_k$, so $x \in \bigcup_{k \geq n+1} E_k, \bigcup_{k \geq n+2} E_k, ...$ Thus $x \in \bigcup_{k \geq n} E_k$ for all $n = 1, 2, ...$ Thus $x \in \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k = \overline{\lim} E_n$. □

b) Give an example where $\underline{\lim} E_n = \emptyset, \overline{\lim} E_n = X$.

Proof. Consider the alternating sequence $E_{2n} = X, E_{2n+1} = \emptyset$. Then $\bigcap_{k \geq n} E_k = \emptyset$ for all n, since there will always be a $\emptyset$ term. Similarly, $\bigcup_{k \geq n} E_k = X$ for all n, since there will always be a X term. Thus $\underline{\lim} E_n = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} E_k = \bigcup_{n=1}^{\infty} \emptyset = \emptyset$ and $\overline{\lim} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k = \bigcap_{n=1}^{\infty} X = X$. □

3 Problem 3

Let $f, g : X \to \overline{\mathbb{R}}$ be measurable. Show that $\{x \in X | f(x) = g(x)\}$ is measurable.

Proof. Since measurable functions form a vector space, $h = f - g$ is measurable. Then $\{x \in X | f(x) = g(x)\} = \{x \in X | h(x) = 0\} = h^{-1}(\{0\})$. $\{0\}$ is measurable in $\overline{\mathbb{R}}$, since $\{0\} = \bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$. Thus $h^{-1}(\{0\})$ is measurable since h is measurable. $\square$

4 Problem 4

Let $f : X \to \mathbb{R}$ be measurable. For $M > 0$ define $f_M(x)$ to be $f(x)$ when $|f(x)| \leq M$, M if $f(x) > M$, and $-M$ if $f(x) < -M$. Show f_M is measurable.

Proof. Since $[-M, M]$ is measurable in $\mathbb{R}$, $E = f^{-1}([-M, M])$ is measurable in X. Similarly, $F = f^{-1}((M, \infty))$ and $G = f^{-1}((-\infty, -M))$ are measurable. Then $f_M = f\chi_E + M\chi_F - M\chi_G$, where χ_Y denotes the indicator function on $Y \subset X$. Since indicator functions are measurable, and the product and linear combinations of measurable functions are measurable, it follows that f_M is measurable. $\square$