Derived Geometric Satake Notes
Course Taught by Victor Ginzburg

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Preface

There will be some gaps in explanation, either due to the lecturer's admission or my own lack of understanding. In particular, many "proofs" are sketches of proofs. Gaps due to my own misunderstanding will be indicated by three red question marks: ???. More generally, my own questions about the material will also be in red. Things like "**Question**" will be questions posed by the lecturer. Feel free to reach out to me with explanations.

Contents

1 Mar 25

1.1 Geometric Satake

We let $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. Let G be a complex reductive group. We have groups $G(\mathcal{K})$ and $G(\mathcal{O})$, and their quotient is the **affine Grassmannian** $Gr_G = Gr = G(\mathcal{K})/G(\mathcal{O})$. This is an analogue of the partial flag manifold for G. There is a clear action of $G(\mathcal{O})$ on Gr by left translation. There is a $\mathbb{C}^\times$-action called **loop rotation** induced by $(z \cdot f)(t) = f(zt)$ on $\mathcal{K}$. If we think of an element of $G(\mathcal{K})$ as a loop γ in G, i.e. we take $t = \exp(i\theta)$, and we take $z = \exp(i\alpha)$, then $(z \cdot \gamma)(t) = \gamma(\exp(i + \alpha))$, which explains the name loop rotation.

One of the main objects of Geometric Satake is $D_{\mathbb{C}^\times \ltimes G(\mathcal{O})}(Gr)$, the $\mathbb{C}^\times \ltimes G(\mathcal{O})$-equivariant constructible derived category. In terms of stacks, we may write this as $D_{\mathbb{C}^\times}(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O}))$. When written in this form, it is evident that, just as with biinvariant functions, there is a monoidal (convolution) structure $\star$ on this category.

Theorem 1.1 (Derived Satake)**.** *There is a monoidal equivalence of triangulated categories between $(D_{\mathbb{C}^\times}(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})), \star)$ and the derived category of Harish-Chandra $\mathcal{U}_\hbar(\check{\mathfrak{g}})$-bimodules (with the tensor product as the monoidal operation).*

Here, $\mathcal{U}_\hbar(\check{\mathfrak{g}})$ is the Rees algebra of the enveloping algebra of the Langlands dual Lie algebra $\check{\mathfrak{g}}$. $\hbar$ corresponds to loop rotation. If we set $\hbar = 0$, we get the quasi-classical specialization, which is a monoidal equivalence between $(D(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})), \star)$ and the derived category of $\check{G}$-equivariant $\mathrm{Sym}\check{\mathfrak{g}}$-modules. More geometrically, this module category is the derived category of $QCoh^{\check{G}}(\check{\mathfrak{g}}^*)$.

Abelian Satake is a restriction of the above quasi-classical equivalence. Namely, it is a monoidal equivalence between the category of $G(\mathcal{O})$-equivariant perverse sheaves on Gr (still with $\star$) and the category of quasi-coherent sheaves of the form $\mathcal{O}_{\check{\mathfrak{g}}^*} \otimes_{\mathbb{C}} V$, where V is a finite dimensional representation of $\check{G}$. This can be further simplified to the category of finite dimensional representations of $\check{G}$, but on the derived level one needs this description of the quasi-coherent sheaves.

Remark.

1. $G(\mathcal{O})$-equivariance of a perverse sheaf is a property, not additional data.
2. Abelian Satake is used in the construction of the derived equivalences, but the statements of the derived equivalences make no mention of perverse sheaves.
3. Derived Satake is related to the representation theory of $\check{\mathfrak{g}}$. In particular, there are connections to Verma modules, the Grothendieck-Springer resolution, and the universal centralizer for $\check{G}$.

4. There is no ∞-categorical analogue of the derived Satake equivalence with loop rotation (at least, one involving quantization); the monoidal structure does not lift.

The **classical Satake** theorem is "secretly" taking complexification of the Grothendieck groups on both sides of the quasi-classical derived equivalence. However, one deals with $\mathcal{K}$ being a local field. The equivalence looks like $C_c(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})) \cong \mathbb{C}[\check{G}]^{\mathrm{Ad}\check{G}}$.

1.2 Equivariant Stuff

Let G be either a compact Lie group over $\mathbb{R}$, or a linear algebraic group over $\mathbb{C}$. According to these cases, let X be a "reasonable" topological space, namely a smooth manifold or an algebraic variety over $\mathbb{C}$. For us, nonreduced structure plays no role. We want G to act on X in either a smooth or algebraic manner, again according to our two cases. We want to define $D_G(X)$ so that we can associate to the objects $\mathcal{F}$ some "equivariant cohomology" $H_G^\bullet(\mathcal{F})$.

Theorem 1.2. *Let G be connected complex reductive. Let T be a maximal torus of G with Weyl group W. Then for $mcF \in D_G(X)$, we have $H_G^\bullet(\mathcal{F}) = H_T(\mathcal{F})^W$.*

Theorem 1.3 (Localization)**.** *There is a correspondence(???)between $H_T^\bullet(\mathcal{F})$ and $H^\bullet(\mathcal{F}|_{X^T})$.*

There is a "refined" version of this theorem called GKM (Goresky-Kottwitz-MacPherson). Another important result is (Braden's) hyperbolic localization. There is also a result that gives equivariant Ext groups between IC sheaves.

Now, we want to discuss how $D_G(X)$ is defined. If we use stacks, then it is "simple": $D_G(X) = D(X/G)$. However, this is useless for more refined questions, e.g. formality (degeneration of spectral sequences) and IC sheaves. We instead look at a classical approach based on classifying spaces.

Definition 1.1. A **universal G-bundle** is a principal G-bundle $EG \to BG$ such that EG is weakly contractible, i.e. (for us) $\mathbb{Q} \otimes \pi_i(EG) = 0$ for all i.

The spaces EG, BG are infinite dimensional, but we can write them as direct limits of spaces E_nG, B_nG, where we have a principal G-bundle $E_nG \to B_nG$ for each n, subject to the following conditions:

1. $\mathbb{Q} \otimes \pi_i(E_nG) = 0$ for $i \leq n$
2. For fixed $k \geq 0$, the maps $\cdots \to H_k(B_nG) \to H_k(B_{n+1}G) \to \cdots$ stabilize.
3. If G is a compact Lie group, then the E_nG, B_nG are smooth manifolds and B_nG are compact. If G is linear algebraic over $\mathbb{C}$, then E_nG, B_nG are smooth algebraic varieties, and B_nG are projective.

There are many constructions that satisfy these criteria. We will follow one particular construction.

Example 1.1. Let $G = \mathbb{C}^\times$. Then $E_nG = \mathbb{C}^n - \{0\}$ with obvious dilation G-action. Thus $B_nG = \mathbb{P}^{n-1}$. We call $BG = \varinjlim_n \mathbb{P}^{n-1}$ by $\mathbb{P}^\infty$. We have $H^*(\mathbb{P}^{n-1}) = \mathbb{C}[u]/(u^n)$, so $H^*(\mathbb{P}^\infty) = \varprojlim_n \mathbb{C}[u]/(u^n) = \mathbb{C}[u]$. Here u is a degree 2 element.

Example 1.2. $G = T$ is a torus, which is an r-fold product of $\mathbb{C}^\times$. Then BT is the r-fold product of $\mathbb{P}^\infty$, and the cohomology ring $H^*(BT)$ is $\mathbb{C}[u_1, \ldots, u_r] = \mathbb{C}[\mathfrak{t}]$ for $\mathfrak{t} = \mathrm{Lie}T$.

Example 1.3. Let $G = GL_N$ over $\mathbb{C}$. Let $E_nG = M^{reg}_{N\times n}$ be the space of full rank $N \times n$ matrices. For n large, $B_nG = Gr_N(\mathbb{C}^n)$. In particular, $BG = Gr_N(C^\infty)$.

If H is a closed subgroup of G, then $(EG)/H$ is a model for BH. The map $BH = (EG)/H \to (EG)/G = BG$ is a fibration with fiber G/H. We get maps $H^*(BG) \to H^*(BH)$.

Example 1.4. Let G be a linear algebraic group over $\mathbb{C}$. Pick a closed embedding $G \hookrightarrow GL_N$. Then we take $BG = \varinjlim_n (M^{reg}_{N\times n}/G)$.

Remark.

1. If G is unipotent, then it is isomorphic to an affine space, i.e. it is contractible. Then BG and EG are also contractible, so the equivariant theory for unipotent groups is trivial.

2. If G/H is weakly contractible, then BG is (weakly?) homotopic to BH, so the equivariant theories are the same. For instance, the equivariant theory for a reductive group G is the same as that of its maximal compact subgroup G_c. As a concrete example, if $G = \mathbb{C}^\times$ and $G_c = S^1$, we get $S^\infty/S^1 = \mathbb{P}^\infty$.

2 Mar 27

Let X be a G-space. We have two maps $G \times X \to X$, namely the action a and the projection pr. An **equivariant structure** on a sheaf $\mathcal{F}$ of vector spaces on X is an isomorphism $\alpha : a^*\mathcal{F} \xrightarrow{\sim} \mathrm{pr}^*\mathcal{F}$ such that $\alpha|_{1_G \times X} = \mathrm{id}$ and a certain cocycle condition is satisfied (condition omitted by lecturer).

Example 2.1. A constant sheaf C_X has a natural equivariant structure via the isomorphisms $C_{G\times X} \to \mathrm{pr}^* C_X$ and $C_{G\times X} \to a^* C_X$.

We denote by $Sh(Y)$ the abelian category of sheaves on any space Y. We denote by $Sh_G(X)$ the abelian category of G-equivariant sheaves on a G-space X.

Lemma 2.1. *Let $q : \widetilde{Y} \to Y$ be a G-bundle. Then $q^* : Sh(Y) \to Sh_G(\widetilde{Y})$ is an equivalence with inverse given by $\widetilde{\mathcal{F}} \mapsto (q_*\mathcal{F})^G$ (taking G-invariants stalk-wise).*

Now let X be a G-space and $E \to B$ a G-bundle. Let $E \times^G X = (E \times X)/G$. We have a diagram

$$\begin{tikzcd} X & E \times X \arrow[l, "p"'] \arrow[r, "q"'] \arrow[d] & E \times^G X \arrow[d, "\pi"'] \\ & B \times X \arrow[r] & B \end{tikzcd}$$

where q is a G-bundle and π is a locally trivial fibration on B with fiber X. Let $\mathcal{F}_X \in Sh_G(X)$. Then $p^*\mathcal{F}_X$ is G-equivariant. By Lemma 2.1, we can uniquely (up to iso) write $p^*\mathcal{F}_X \cong q^*\widetilde{\mathcal{F}}$ for $\tilde{\mathcal{F}} \in Sh(E \times^G X)$. Then for any fiber $\pi^{-1}(b)$, which is non-canonically isomorphic to X, we have $\widetilde{\mathcal{F}}|_{\pi^{-1}(b)}$ is non-canonically isomorphic to $\mathcal{F}_X$.

We apply the above to the bundle being $EG \to BG$. Let $X_G = EG \times^G X$. Let $H^\bullet_G(X) = H^\bullet(X_G)$. Here is a picture relating our setup, on the left, to stacks, on the right, where $\sim$ denotes weak homotopy equivalence:

$$\begin{tikzcd} EG \times X \arrow[d, "q"] & \sim & X \arrow[d] \\ X_G \arrow[d] & \sim & X/G \arrow[d] \\ BG & \sim & pt/G \end{tikzcd}$$

We have $H^\bullet_G(pt) = H^\bullet(BG)$, which acts as our "base ring", and this comes with a map to $H^\bullet_G(X) = H^\bullet(X_G)$.

Now consider the diagram above in our universal bundle setting:

$$\begin{tikzcd} X & EG \times X \arrow[l, "p"'] \arrow[r, "q"'] \arrow[d] & X \arrow[d, "\pi"'] \\ & BG \times X \arrow[r] & BG \end{tikzcd}$$

For any space Y, we write $D(Y)$ for the ordinary derived category of $Sh(Y)$. We "define" (this definition is not technically correct, and will be remedied later) the equivariant derived category $D_G(X)$ to have objects given by a triple $\mathcal{F} = (\mathcal{F}_X, \mathcal{F}_{X_G}, \alpha)$, where $\mathcal{F}_X \in D(X)$, $\mathcal{F}_{X_G} \in D(X_G)$, and $\alpha : p^*\mathcal{F}_X \xrightarrow{\sim} q^*\mathcal{F}_{X_G}$ is an isomorphism in $D(EG \times X)$. We define $H^\bullet_G(\mathcal{F}) = H^\bullet(\mathcal{F}_{X_G})$. Morphisms in $D_G(X)$ are given by a pair of morphisms between the relevant objects of the non-equivariant derived categories (we will often call these sheaves) that satisfy a natural compatibility diagram. This is a triangulated category, where the distinguished triangles are pairs of distinguished triangles with compatibility condition. We have a forgetful functor $D_G(X) \to D(X)$ which sends a triple to its first element. We have a functor $Sh_G(X) \to D_G(X)$ defined by $\mathcal{F}_X \mapsto (\mathcal{F}_X, \widetilde{\mathcal{F}}, \alpha)$ for some naturally defined α. This extends to a functor $D(Sh_G(X)) \to D_G(X)$, which is almost never an equivalence.

$D_G(X)$ has a notion of a constant sheaf, namely $(C_X, C_{X_G}, \mathrm{id}_{C_{EG\times X}})$. Its sheaf cohomology recovers $H^\bullet_G(X)$.

We now retroactively impose the condition "constructible" everywhere. Because of this, we have to assume that G has finitely many connected components. This assumption implies $\pi_1(BG)$ is finite, and if G is connected, then BG is simply connected. This follows by a homotopy sequence $\pi_1(EG) \to \pi_1(BG) \to \pi_0(G)$ and $\pi_1(EG) = 1$.

We note that anytime we write sheaf functors, particularly pushforward, we mean their derived versions, unless otherwise specified. As in the abelian case, $\mathcal{F}_{X_G}$ restricted to a fiber of π is isomorphic to $\mathcal{F}_X$. If we take the stalk at $b \in BG$ of $\mathcal{H}^k(\pi_*\mathcal{F}_{X_G})$, we get $H^k(\mathcal{F}_X)$. Thus $\mathcal{H}^k(\pi_*\mathcal{F}_{X_G})$ is a locally constant sheaf on BG. In particular, if G is connected, then this cohomology sheaf is constant.

We have a Leray spectral sequence $E_2^{p,q} = H^p(BG, \mathcal{H}^q(\pi_*\mathcal{F}_X))$, which converges to $H^\bullet_G(\mathcal{F}_X)$. If G is connected, then $E_2^{p,q} = H^p(BG) \otimes H^q(\mathcal{F}_X)$. In particular, if $\mathcal{F}_X$ is constant, $E_2^{p,q} = H^p(BG) \otimes H^q(X)$, and this converges to $H^\bullet_G(X)$.

Let $C \in D_G(X)$ be constant. Then $\mathrm{Ext}^k_{D_G(X)}(C, \mathcal{F}) = \mathrm{Hom}_{D_G(X)}(C, \mathcal{F}[k]) = H^k_G(\mathcal{F})$.

Example 2.2. Let X be a point, and let G be connected. Then $Sh_G(pt)$ is Vect, the category of vector spaces. Thus $D(Sh_G(pt)) = D(\mathrm{Vect})$. We have $\mathrm{Ext}^k_D(C, C) = \mathbb{C}$ if $k = 0$, and 0 otherwise. On the other hand, $\mathrm{Ext}^k_{D_G(pt)}(C, C) = H^k_G(C) = H^k_G(pt) = H^k(BG)$. If $G = \mathbb{C}^\times$, then $H^k(BG) \neq 0$ for all positive even integers k. If follows that $D(Sh_G(pt))$ is very different from $D_G(pt)$.

We now take care of the issue of EG, BG being infinite dimensional and correct the definition of $D_G(X)$. Recall $EG = \varinjlim_n E_nG$, $BG = \varinjlim_n B_nG$, with G-bundles $E_nG \to B_nG$. Let $i_n : E_nG \to E_{n+1}G$. We have a commutative diagram

$$\begin{tikzcd}
& E_nG \times X \arrow[r, "q_n"'] \arrow[dl, "p_n"'] \arrow[dd, "i_n \times \mathrm{id}"] & X_{n,G} \arrow[dd, "\overline{i_n}"] \\
X & & \\
& E_{n+1}G \times X \arrow[r, "q_{n+1}"'] \arrow[ul, "p_{n+1}"] & X_{n+1,G}
\end{tikzcd}$$

An object $\mathcal{F}$ of $D^b_G(X)$ is a collection consisting of $\mathcal{F}_X \in D^b(X)$, $\mathcal{F}_n \in D^b(X_{n,G})$ for each n, $\alpha_n : p_n^*\mathcal{F}_X \xrightarrow{\sim} q_n^*\mathcal{F}_n$ for each n, and $\nu_n : \mathcal{F}_n \xrightarrow{\sim} \overline{i_n}^*\mathcal{F}_{n+1}$ for all n, satisfying the following compatibility diagrams for all n:

$$\begin{tikzcd}
p_n^*\mathcal{F}_X \arrow[rr, "\mathrm{id}"] \arrow[d, "\alpha_n"'] & & (i_n \times \mathrm{id})^* p_{n+1}^*\mathcal{F}_X \arrow[d, "(i_n \times \mathrm{id})^*(\alpha_{n+1})"] \\
q_n^*\mathcal{F}_n \arrow[r, "q_n^*(\nu_n)"] & q_n^*\overline{i_n}^*\mathcal{F}_{n+1} \arrow[r, "\mathrm{id}"] & (i_n \times \mathrm{id})^* q_{n+1}^*\mathcal{F}_{n+1}
\end{tikzcd}$$

Example 2.3. Let X be an algebraic variety. We define $IC_G(X) \in D^b_G(X)$ by $(\mathcal{F}_X = IC(X), \mathcal{F}_n = IC(X_{n,G})[-\dim B_nG])$ with natural compatibility maps. For any Zariski open $U \subset X$, we have $IC_G(X)|_U = C_U[\dim X]$.

We now need to rigorously define $H^\bullet_G(\mathcal{F})$. We have maps $\pi_n : X_{n,G} \to B_nG$ which are locally trivial fibrations with fiber X. We have as before $H^\bullet(\mathcal{F}_n) = H^\bullet((\pi_n)_*\mathcal{F}_n)$ and a spectral sequence $E_2^{p,q} = H^p(\mathcal{H}^q((\pi_n)_*\mathcal{F}_n))$ converging to it. Recall that the maps in cohomology between the B_nG stabilize. Then we get maps between spectral sequences that also stabilize, whence the sheaf cohomologies stabilize.

3 Apr 1

Let $K \leq G$ be a subgroup. Let X be a K-space. Then we can form the *induced* space $G \times^K X = (G \times X)/K$. We have an *induction functor* $\mathrm{Ind}_K^G : D_K(X) \to D_G(G \times^K X)$ defined via the isomorphism $X_K = EG \times^K X \xrightarrow{\sim} EG \times^G (G \times^K X)$. This functor is an equivalence. Note that we have chosen EG as a model for EK. We have $H_G(\mathrm{Ind}_K^G \mathcal{F}) \cong H_K(\mathcal{F})$.

If X is a G-space, then we have a *restriction functor* $\mathrm{Res}_K^G : D_G(X) \to D_K(X)$ given by pullback along the map $p : X_K \to X_G$. Equivalently, you can pullback along the action map $G \times^K X \to X$ and then use the inverse to induction: $D_G(X) \to D_G(G \times^K X) \to D_K(X)$.

Let G be connected, and either a compact Lie group or a complex reductive algebraic group. Let $V \in \mathrm{Rep}(G)$. Consider the vector bundle V_{BG} with fiber V given by $EG \times^G V \to BG$. Its Chern classes are certain classes $c_i(V_{BG}) \in H^{2i}(BG)$. The Chern character is $ch(V_{BG}) = \sum_{i \geq 0} \frac{1}{i!} c_i(V_{BG})$. We will use this only when $G = T$ is a torus. Let $\mathfrak{t} = \mathrm{Lie} T$. Let $X^*(T) = \mathrm{Hom}(T, \mathbb{C}^\times)$ be the character lattice. We have (after complexifying) $\mathfrak{t}^* = \mathbb{C} \otimes_{\mathbb{Z}} X^*(T)$. Recall $BT \simeq (\mathbb{P}^\infty)^r$ for $r = \dim T$. For $\lambda \in X^*(T)$ we have a line bundle $ET \times^T \mathbb{C}_\lambda = \mathbb{C}_{ET,\lambda} \to BT$. Since it is a line bundle, it only has a c_1. The map $\lambda \mapsto c_1(\mathbb{C}_{ET,\lambda})$ gives a group homomorphism $X^*(T) \to H^2(BT)$, which extends to a $\mathbb{C}$-linear $\mathfrak{t}^* \to H^2(BT, \mathbb{C})$. This further extends to an algebra homomorphism $\mathbb{C}[\mathfrak{t}] = \mathrm{Sym}\,\mathfrak{t}^* \to H^\bullet(BT, \mathbb{C})$ (which doubles the degrees).

Return to the case of G as in the previous paragraph, and let T be a maximal torus with Weyl group W. Via the construction above, we have the following commutative diagram with $p : BT \to BG$:

$$\begin{tikzcd}
& \mathrm{Rep}T \arrow[r, "ch"] & \mathbb{C}[[\mathfrak{t}]] = \prod_{i \geq 0} H^{2i}(BT) & \\
\mathrm{Rep}G \arrow[ur, "\mathrm{Res}_T^G"] \arrow[dr, "char"'] & & & \\
& \mathbb{C}[G]^G \arrow[r, "\exp^*"'] & \mathbb{C}[[\mathfrak{g}]]^G \arrow[r, "ch"'] & \prod_{i \geq 0} H^{2i}(BG) \arrow[uul, "p^*"']
\end{tikzcd}$$

Now we want to generalize this to sheaves. Let X be a G-space. We have maps $X_T \xrightarrow{p} X_{N(T)} \xrightarrow{q} X_G$, where p is a W-bundle and q is a $G/N(T)$ bundle. Let $\mathcal{F} = \mathcal{F}_{X_G}$ be the $D(X_G)$-component of an element in $D_G(X)$ (we will often treat elements of the derived category in this way). Then $p^* q^* \mathcal{F} \in D(X_T)$, so $p^* q^* : D_G(X) \to D_T(X)$.

Theorem 3.1. *$H_G(\mathcal{F}) = H(X_G, \mathcal{F}_{X_G}) \to H_T(\mathcal{F}) = H(X_T, p^* q^* \mathcal{F}_{X_G})$ is an isomorphism onto $H(X_T, p^* q^* \mathcal{F}_{X_G})^W$.*

Lemma 3.1. *$H^i(G/N(T), \mathbb{C}) = \mathbb{C}$ if $i = 0$ and 0 otherwise.*

Proof. Suppose $f : \widetilde{Y} \to Y$ is a Galois covering with finite Galois group W. For $\mathcal{E} \in D(Y)$, we have $H(\widetilde{Y}, f^*\mathcal{E}) = H(Y, f_* f^*\mathcal{E})$. W acts on the RHS, and the projection formula gives $H(Y, \mathcal{E}) = H(Y, f_* f^*\mathcal{E})^W$. We apply this to the map $G/T \to G/N(T)$. Then $H(G/N(T), \mathbb{C}) = H(G/T, \mathbb{C})^W$. We conclude via the following Theorem of Borel.

Theorem 3.2 (Borel)**.** *$W \curvearrowright H^\bullet(G/T)$ is the regular representation.*

Proof of Theorem 3.1. $H(X_{N(T)}, q^*\mathcal{F}) = H(X_G, q_* q^*\mathcal{F})$. The Leray spectral sequence gives $E_2^{i,j} = H^i(X_G, \mathcal{H}^j q_* q^*\mathcal{F})$ converging to $H(X_G, q_* q^*\mathcal{F})$. By equivariance, $\mathcal{H}^j q_* q^*\mathcal{F}$ restricted to a fiber of q, which is $G/N(T)$, is constant. Then, by Lemma 3.1, the stalks of $\mathcal{H}^j q_* q^*\mathcal{F}$ are $\mathbb{C}$ for $j = 0$ and 0 otherwise. Then the spectral sequence gives us $H(X_G, \mathcal{F}) \cong H(X_{N(T)}, q^*\mathcal{F})$, and then this is $H(X_T, p^* q^*\mathcal{F})^W$ using the fact that p is a Galois covering.

Corollary 3.1. *$H(BG) \cong H(BT)^W \cong \mathbb{C}[\mathfrak{t}]^W \cong \mathbb{C}[\mathfrak{g}]^G$, where the last isomorphism is due to Chevalley. In particular, $H^{2i+1}(BG) = 0$.*

Proof. Apply the Theorem in the case of a constant sheaf.

Definition 3.1. $\mathcal{F} \in D_G(X)$ is called **equivariantly formal** if $\pi_*\mathcal{F} \cong \bigoplus_i \mathcal{H}^i \pi_*\mathcal{F}$ in $D(BG)$, where $\pi : X_G \to BG$.

Corollary 3.2. *If $\mathcal{F}$ is equivariantly formal, then $H_G(\mathcal{F}) \cong H(BG) \otimes H(X, \mathcal{F}_X)$.*

Proof. Since we are still assuming G is connected, we have BG is simply connected. This implies that $\mathcal{H}^i(\pi_*\mathcal{F})$ is a constant sheaf with stalk $H^i(X, \mathcal{F}_X)$. Formality gives $H(\pi_*\mathcal{F}) \cong \bigoplus_{i,j} H^i(BG, \mathcal{H}^j \pi_*\mathcal{F}) = \bigoplus_{i,j} H^i(BG) \otimes H^j(X, \mathcal{F}_X)$.

Corollary 3.3. *If $\mathcal{F}$ is equivariantly formal, then $H_T(\mathcal{F}) = H(BT) \otimes_{H(BG)} H_G(\mathcal{F})$.*

Proof. Take W invariants on both sides of $H_G(\mathcal{F}) \cong H(BG) \otimes H(X, \mathcal{F}_X)$.

In general, proving that something is equivariantly formal is difficult. However, there are some situations where we are lucky.

One such case is for a projective algebraic G-variety X; then $IC(X)$ is equivariantly formal. Indeed, we recall that $IC(X)$ is defined to be the "limit" of $IC(X_{G,n})$ for varying n. We have maps $\pi_n : X_{G,n} \to BG_n$ whose fibers are X. Since X is projective and BG_n is projective, the map π_n is projective. Thus equivariant formality holds by the *BBD decomposition theorem.*

Another case is if $H^{2i+1}(X, \mathcal{F}_X) = 0$. This follows by an argument with the Leray spectral sequence. Namely, by Corollary 3.1, the spectral sequence collapses since there are no odd degree terms. Namely, $H_G(\mathcal{F}) = H(BG) \otimes H(\mathcal{F}_X)$. Lecturer is actually a little unsure on how to get the full statement of formality from here.

4 Apr 3

4.1 Strategy for Derived Geometric Satake Proof (Without Loop Rotation)

Let G be connected complex reductive with maximal torus T. Let $\mathfrak{g} = \mathrm{Lie}G$ and $\mathfrak{t} = \mathrm{Lie}T$. We have the Langlands dual group and algebras $\check{G}, \check{\mathfrak{g}}, \check{\mathfrak{t}}$. There is a canonical identification $(\check{\mathfrak{t}})^* \cong \mathfrak{t}$. The Weyl groups of both groups are also the same, W. The important computation is $\mathbb{C}[\mathfrak{t}]^W \cong (\mathrm{Sym}\mathfrak{t}^*)^W \cong (\mathrm{Sym}\check{\mathfrak{t}})^W \cong (\mathrm{Sym}\check{\mathfrak{g}})^{\check{G}}$, where the last isomorphism is due to Chevalley.

Let A be a graded (or dg) algebra. Let a reductive group K act on A by graded algebra automorphisms. In the lecturer's course last quarter, a category $HC_{A,K}$ of Harish-Chandra modules was defined. Namely, each object $M \in HC_{A,K}$ is simultaneously a graded A-module and a K-representation, with the compatibility $k(am) = k(a)k(m)$. We assume from here on out that A is commutative (this is where the strategy breaks for loop rotation). Then $HC_{A,K}$ becomes a monoidal category with $\otimes_A$.

We will apply the above formalism to $A = \mathrm{Sym}\check{\mathfrak{g}}$ and $K = \check{G}$ acting on A via adjoint action. Derived Satake is a monoidal equivalence $\Phi : D_{G(\mathcal{O})}(Gr) \xrightarrow{\sim} D(HC_{\mathrm{Sym}\check{\mathfrak{g}},\check{G}})$.

Consider $A = \mathbb{C}$ with trivial K-action. Then $HC_{A,K} = \mathrm{Rep}K$. We have two functors $\mathrm{Rep}K \to \mathrm{Vect}$, namely invariants $M \mapsto M^K$ and forgetful $M \mapsto \underline{M}$. The regular representation $R = \mathbb{C}[K]$ decomposes as $\bigoplus_{V \in \mathrm{Irr}(K)} V \otimes V^* \in \mathrm{Ind}(\mathrm{Rep}K)$. For any $M \in \mathrm{Rep}K$, we have $R \otimes M \cong R \boxtimes M$ as $K \times K$-modules. On $R \otimes M$, the first copy of K acts diagonally, while the second copy of K acts only on R, and on the right. On $R \boxtimes M$, the first copy of K acts only on R and the second copy acts only on M. Thus, in $\mathrm{Rep}K$, where we only take the first copy of K, there is no K-action on M in $R \boxtimes M$. We have $(R \otimes M)^K = R^K \boxtimes M = M$ and $R \otimes R \cong R \boxtimes R$.

G is a G-torsor under right action. If X is an affine G-torsor then we have a coaction map $\mathbb{C}[X] \to \mathbb{C}[X \times G]$. Note $\mathbb{C}[X \times G] \cong \mathbb{C}[X] \otimes \mathbb{C}[G]$. The isomorphism $R \otimes R \cong R \boxtimes R$ corresponds to $\mathbb{C}[X] \otimes \mathbb{C}[X] \cong \mathbb{C}[X] \otimes \mathbb{C}[G]$. (???)(???)

We use triv to denote the trivial 1-dimensional representation in $\mathrm{Rep}K$. It is the monoidal unit of $\mathrm{Rep}K$. We have $M^K = \mathrm{Hom}_K(\mathrm{triv}, M)$.

We can also take K-invariants on $HC_{A,K}$ to get an element of A^K-mod. We have that A is the monoidal unit of $HC_{A,K}$, and $M^K = \mathrm{Hom}_{HC}(A, M)$.

Let $R_A = R \otimes A$. We have $\mathrm{Hom}_{HC}(A, R_A \otimes_A M) \cong (R_A \otimes_A M)^K \cong ((R \otimes A) \otimes_A M)^K \cong (R \otimes M)^K \cong \underline{M}$, giving an internal way to take the forgetful functor.

Example 4.1. Let $A = \mathrm{Sym}\check{\mathfrak{g}}$, $K = \check{G}$. Then $R_A = R \otimes A = \mathbb{C}[\check{G}] \otimes \mathrm{Sym}\check{\mathfrak{g}} = \mathbb{C}[\check{G}] \otimes \mathbb{C}[(\check{\mathfrak{g}})^*] = \mathbb{C}[\check{G} \times (\check{\mathfrak{g}})^*] = \mathbb{C}[T^*\check{G}]$.

Let $\Lambda \subset X^*(\check{T}) \subset \check{\mathfrak{t}}^*$ be the dominant Weyl chamber. For $\lambda \in \Lambda$ we have the finite dimensional $V_\lambda \in \mathrm{Irr}(\check{G})$ with highest weight λ. Then $R = \mathbb{C}[\check{G}] = \bigoplus_{\lambda\in\Lambda} V_\lambda \otimes V_\lambda^*$. Abelian Satake is a monoidal functor $\Psi : (\mathrm{Rep}(\check{G}), \otimes) \to (D_{G(\mathcal{O})}(Gr), \star)$ sending V_λ to IC_λ. Let $\mathcal{R} = \psi(R)$; it decomposes as $\bigoplus_{\lambda\in\Lambda} IC_\lambda \otimes V_\lambda^*$. We get a map $G \to \mathrm{Aut}(\mathcal{R})$ via the action on V_λ^*. The monoidal unit of $\star$ is IC_0.

We will treat $\mathcal{R}$ as R_A, thus giving us a candidate for Φ: $R\mathrm{Hom}_{D_{G(\mathcal{O})}(Gr)}(IC_0, \mathcal{R}\star(-)) = \bigoplus_{\lambda\in\Lambda} V_\lambda^* \otimes R\mathrm{Hom}(IC_0, IC_\lambda \star (-))$. This would mean that A is $R\mathrm{Hom}(IC_0, \mathcal{R})$. Let us see how to give it an algebra structure. $R = \mathbb{C}[\check{G}]$ has a multiplication m, which gives a map $\mu : \mathcal{R} \star \mathcal{R} \to \mathcal{R}$. For $\alpha, \beta \in \mathrm{Hom}(IC_0, \mathcal{R})$, we can form $\alpha \star \beta : IC_0 \star IC_0 \to \mathcal{R} \star \mathcal{R}$. Then precomposing with $IC_0 \cong IC_0 \star IC_0$ and postcomposing with μ gives us a map $IC_0 \to \mathcal{R}$, which we call $\alpha \circ \beta$. Adding in appropriate shifts shows that this works on $R\mathrm{Hom}$ as well. We can also modify this to work for $\beta \in R\mathrm{Hom}(IC_0, \Psi(M))$, giving A-module structures to the image of Ψ.

For any space X and $\mathcal{F} \in D(X)$, we have $C_X \otimes \mathcal{F} \cong \mathcal{F}$ where $C_X \otimes \mathcal{F} = \Delta^*(C_X \boxtimes \mathcal{F})$ for $\Delta : X \hookrightarrow X \times X$. This induces a map $H(C_X) \otimes H(\mathcal{F}) \to H(\mathcal{F})$, i.e. $H(\mathcal{F})$ becomes a module over $H(C_X) = H(X)$. This works equally well in the equivariant setting. In particular, $H_{G(\mathcal{O})}(IC_\lambda)$ is a graded $H_{G(\mathcal{O})}(Gr)$-module.

In order to compute $R\mathrm{Hom}(IC_0, \mathcal{R})$, we need to compute $\mathrm{Ext}(IC_0, IC_\lambda)$. Taking equivariant cohomology gives us a map
$\mathrm{Ext}^i(IC_0, IC_\lambda) \to \mathrm{Hom}_{H_{G(\mathcal{O})}(Gr)}(H^\bullet_{G(\mathcal{O})}(IC_0), H^{i+\bullet}_{G(\mathcal{O})}(IC_\lambda))$.

Theorem 4.1. $\mathrm{Ext}(IC_\mu, IC_\lambda) \xrightarrow{\sim} \mathrm{Hom}_{H_{G(\mathcal{O})}(Gr)}(H_{G(\mathcal{O})}(IC_\mu), H_{G(\mathcal{O})}(IC_\lambda))$.

We have $H_{G(\mathcal{O})}(Gr) = \mathbb{C}[T^*(\mathfrak{t}^*/W)] = \mathbb{C}[T^*\mathfrak{c}] = \mathrm{SymLie}\check{\mathcal{J}}$, universal centralizer (explained last quarter). We also have $\mathrm{Hom}(H(IC_0), H(\mathcal{R})) \cong \mathbb{C}[\check{\mathfrak{c}} \times \check{G}] \cong \mathbb{C}[T^*_\psi(\check{G}/\check{U})]$ (notation also explained last quarter).

5 Apr 8

Cohomology of affine Grassmannian.

6 Apr 10

G connected reductive, $L = G(\mathcal{O})$, $Gr = G(\mathcal{K})/L$. $\pi_0(Gr) \simeq \pi_1(G)$. For any two connected components X, X' of Gr, there is some $g(t) \in G(\mathcal{K})$ such that the map $x \mapsto g(t)x$ is an isomorphism $X \xrightarrow{\sim} X'$. The components are $\mathbb{G}_m \times L$-stable. We fix a component X. We will prove that X is a disjoint union of spaces $X_0, X_1, \ldots$ such that

- Each X_d is $\mathbb{G}_m \times L$-stable.
- Each $X_{\leq d} = X_0 \cup \cdots \cup X_d$ is a projective variety.
- Each X_d is a disjoint union of affine spaces (cells).
- For all $i \geq 0$, there is $d(i) \gg 0$ such that X_d has no cells of dimension $\leq i$ for $d > d(i)$.

As a consequence, the embeddings $X_{\leq d} \hookrightarrow X_{\leq d+1}$ don't add cells of "low" dimension. In particular, for $d > d(i)$, we have isomorphisms $H_i(X_{\leq d}) \xrightarrow{\sim} H_i(X_{\leq d+1})$ and $H^i(X_{\leq d+1}) \xrightarrow{\sim} H^i(X_{\leq d})$. (It follows that?) X_d is equivariantly formal and $H^i_{\mathbb{G}_m \times L}(X_{\leq d+1}) \xrightarrow{\sim} H^i_{\mathbb{G}_m \times L}(X_{\leq d})$. We then define $H^i(Gr)$ to be the stable limit of $H^i(X_{\leq d})$, and similarly for the equivariant cohomology. The odd degree cohomologies vanish and Gr is equivariantly formal.

Let $M = \{g(t) \in L \mid g(0) = 1\}$. There is a SES $1 \to M \to L \to G \to 1$, where the map $L \to G$ sends $g(t)$ to $g(0)$. M is pro-unipotent and there is a group contraction $M \times [0,1] \to M$ defined by $(g(t), c) \mapsto g(ct)$. It follows that in all of our equivariant cohomologies, we can replace L by $L/M = G$.

We now address a point from last time and avoid the use of stacks. Let $\widetilde{Gr} = G(\mathcal{K})/M$. This has an evident map to Gr which has the structure of a principal G-bundle (since $L/M = G$). In particular, $\widetilde{Gr}$ has a left and right action by G. We have $H_{\mathbb{G}_m \times G \times G}(\widetilde{Gr}) \cong H_{\mathbb{G}_m \times G}(Gr)$.

Let $\mathfrak{g} = \mathrm{Lie} G$ have Cartan $\mathfrak{t}$ and Weyl group W. Let $Z = \mathbb{C}[\mathfrak{t}]^W = \mathbb{C}[\mathfrak{t}/W] = H_G(pt)$. We have $(Z \otimes Z)[\hbar] = H_{\mathbb{G}_m \times G \times G}(pt)$. Let $\zeta : (Z \otimes Z)[\hbar] \to H_{\mathbb{G}_m \times G}(Gr)$ be the map induced by $\widetilde{Gr} \to pt$, rewritten with the preceding identifications. Let $\mathcal{A}$ be the subalgebra of $(Z \otimes Z)[\hbar, \hbar^{-1}]$ generated by $(Z \otimes Z)[\hbar]$ and $(z \otimes 1 - 1 \otimes z)\hbar^{-1}$ for $z \in Z$.

Theorem 6.1. *Let $X \subset Gr$ be the connected component of $1 \cdot L/L$. Then the map ζ is injective and extends uniquely to a degree doubling algebra isomorphism $\mathcal{A} \to H_{\mathbb{G}_m \times G}(Gr)$.*

Proof Strategy. By Chevalley, $Z = \mathbb{C}[p_1, \ldots, p_r]$ for homogeneous polynomials p_i of degree d_i. Then $Z \otimes Z = \mathbb{C}[p_i \otimes 1, 1 \otimes p_i]$. Let $\xi_i = (p_i \otimes 1 - 1 \otimes p_i)\hbar^{-1}$. Then $\mathcal{A} = \mathbb{C}[\hbar, p_i \otimes 1, \xi_i]$. Note that $\deg(\xi_i) = d_i - 1$. If $\pi_1(G) = 1$, then $d_i \geq 2$ (for reasons related to tori), so $\mathcal{A}$ is non-negatively graded and $\mathcal{A}_0 = \mathbb{C}$. It is free over $(Z \otimes 1)[\hbar]$; the basis is given by the ξ_i. By formality, $H_{\mathbb{G}_m \times G}(Gr)$ is also free over $(Z \otimes 1)[\hbar]$; it is generated by the non-equivariant cohomology of Gr.

Now we "forget about loop rotation": $\zeta|_{\hbar=0} : Z \otimes Z \to H_{\mathbb{G}_m \times G}(Gr)/(\hbar)$. By formality, the codomain is $H_G(Gr)$. We will show that this map sends $z \otimes 1 - 1 \otimes z$ to 0. This will imply, by flatness, that ζ extends to $\widetilde{\zeta}\mathcal{A} \to H_{\mathbb{G}_m \times G}(Gr)$. We can again restrict this map to $\hbar = 0$. We recall $\mathcal{A}/\hbar\mathcal{A} = \mathbb{C}[T(\mathfrak{t}/W)]$. So $\widetilde{\zeta}$ gives a map $\mathbb{C}[T(\mathfrak{t}/W)] \to H_G(Gr)$. As a side note, the generators ξ in $\mathcal{A}$ give rise to partial derivatives in $\mathcal{A}/\hbar\mathcal{A}$.

Let $\mathcal{J} \subset Z$ be the augmentation ideal. If we mod out by $\mathcal{J}$ on both sides, we get a map $\mathbb{C}[\xi_i] \to H(Gr)$. We will prove that this map is an isomorphism. To do so, we will show that it is injective and that Poincare polynomials agree. Once we show this, we will have ζ is an isomorphism by flatness and Nakayama.

We will now change notation; denote $G_{\mathbb{C}}$ by the complex reductive group and G by a maximal compact subgroup. $G_{\mathbb{C}}$ contracts to G so there is no topological information lost. We can choose a Borel B and maximal complex torus $T_{\mathbb{C}}$ such that $T = G \cap B$ is a maximal torus in G. There is an anti-holomorphic anti-involution $G_{\mathbb{C}} \to G_{\mathbb{C}}$, $g \mapsto g^*$, such that $G = \{g \in G_{\mathbb{C}} \mid g^* = g^{-1}\}$. The Iwasawa decomposition states $G_{\mathbb{C}} = GB$. The embedding $G \hookrightarrow G_{\mathbb{C}}$ induces an isomorphism $G/T \xrightarrow{\sim} G_{\mathbb{C}}/B$, i.e. the flag manifold and flag variety can be identified.

We want an Iwasawa decomposition analogue for loop groups. Fix an embedding $G_{\mathbb{C}} \hookrightarrow GL_n(\mathbb{C})$ such that $G = G_{\mathbb{C}} \cap U_n$. Let $LG = G[z^{\pm 1}]$, which we can think of as algebraic maps $S^1 \to G$.

Lemma 6.1. *$LG = \{g(z) \in G_{\mathbb{C}}[z^{\pm 1}] \mid g(z)^* = g(z^{-1})\}$.*

Example 6.1. $G_{\mathbb{C}} = C^\times$, $G = S^1$. Note that $G_{\mathbb{C}}[z^{\pm}] \cong \mathbb{C}^\times \times \mathbb{Z}$, and $\mathbb{Z} \cong \pi_1(G_{\mathbb{C}})$. The $g(z) = cz^n$ that are in LG must satisfy $|c| = 1$. More generally, if $G_{\mathbb{C}} = T_{\mathbb{C}}$ is a torus, then $G = T$ is a torus, $T_{\mathbb{C}}[z^{\pm 1}] \cong T_{\mathbb{C}} \times X_*(T_{\mathbb{C}})$, $T[z^{\pm 1}] \cong T \times X_*(T)$, where X_* denotes the cocharacter lattice, and $X_*(T) = \pi_1(T)$.

We can now state Iwasawa decomposition for loop groups.

- $G_{\mathbb{C}}(\mathcal{K}) = LG \cdot G_{\mathbb{C}}(\mathcal{O})$.
- $LG \cap G_{\mathbb{C}}(\mathcal{O}) = G$, which we interpret as constant maps $S^1 \to G$.

This is equivalent to $LG/G \xrightarrow{\sim} G_{\mathbb{C}}(\mathcal{K})/G_{\mathbb{C}}(\mathcal{O}) = Gr$. The form LG/G is how topologists think of the affine Grassmannian.

7 Apr 15

7.1 Cohomology of Loop Groups

7.1.1 Based Loop Groups

Let G be a compact connected Lie group. Recall LG consists of the polynomial maps $S^1 \to G$. Inside this we have ΩG, the loops based at the identity (i.e. identity is sent to identity). There is a clear isomorphism $\Omega G \times G \xrightarrow{\sim} LG$ given by translation. We can also think of this as embedding $G \hookrightarrow LG$ via constant loops, and then $LG/G \cong \Omega G$. G still has an action on ΩG, namely by conjugation. This can be seen as coming from the left G-action on LG after the right-action has been killed.

Let T be a maximal torus (say rank r) in G. From last time we have $LT = X_*(T) \times T$, so we see $\Omega T = X_*(T) = \mathrm{Hom}(S^1, T) \cong \mathbb{Z}^r$. We have embeddings $LT \hookrightarrow LG$, $\Omega T \hookrightarrow \Omega G$, and in fact $(LG)^T = LT$ and $(\Omega G)^T = \Omega T$. For a cocharacter λ, we write z^λ for the corresponding loop (or its image of a point $z \in S^1$).

S^1 acts on LG by loop rotation, and we ask what happens on ΩG. Note that $(LG)^{S^1} = G$, the constant loops. If $\gamma \in \Omega G$ and $z_0 \in S^1$, then as free loops, $(z_0\gamma)(z) = \gamma(zz_0)$. This evidently sends $z = 1$ to $\gamma(z_0)$. So loop rotation on ΩG sends γ to the map $z \mapsto \gamma(zz_0)\gamma(z_0)^{-1}$. Then we see $(\Omega G)^{S^1} = \mathrm{Hom}(S^1, G)$. Note that for all $\gamma \in \mathrm{Hom}(S^1, G)$, there is $g \in G$ and $\lambda \in X_*(T)$ such that $\gamma(z) = \mathrm{Ad}_g(z^\lambda)$. Now let W be the Weyl group and let X_+ be the dominant Weyl chamber in $X_*(T)$. Then

$$\mathrm{Hom}(S^1, G) = \bigsqcup_{\lambda \in X_+} G/\mathrm{Stab}(\lambda).$$

Furthermore, for each $\lambda \in X_+$, there is a parabolic subgroup P_λ of the corresponding complex group $G_{\mathbb{C}}$ such that $G/\mathrm{Stab}(\lambda) \cong G_{\mathbb{C}}/P_\lambda$.

7.1.2 Algebraic Topology Review

Let X be either a finite connected CW complex or a smooth manifold. Fix a basepoint $x_0 \in X$. In the first case of X, let $\Omega^c X$ be the space of continuous loops in X based at x_0. In the second case, let $\Omega^\infty X$ be the space of smooth loops in X based at x_0. If X is simply connected, then in either case, $\Omega^c X$ or $\Omega^\infty X$ is homotopic to a CW-complex with finitely many cells of each dimension. We note that ΩG is homotopic to such a space, as well as being homotopic to $\Omega^c G$ and $\Omega^\infty G$ (assuming G is simply connected, which the lecturer says is the reasonable thing to do). In the future we will just write ΩX instead of $\Omega^c X$ or $\Omega^\infty X$.

The based path space PX consists of continuous maps $[0,1] \to X$, where 0 is sent to the basepoint x_0. It is contractible and there is a fibration $PX \to X$

with fiber ΩX given by sending a map to its image at 1.

Concatenation of loops makes ΩX into an H-space, which is kind of like a group. The above fibration $PX \to X$ can be thought of as a "principal ΩX-bundle". Since PX is contractible, we see that X is behaving like the classifying space, i.e. $B\Omega X = X$. We also have $\pi_i(\Omega X) \cong \pi_{i+1}(X)$. We have that $\mathbb{C} \otimes \pi_\bullet(\Omega X) = \bigoplus_i \mathbb{C} \otimes \pi_i(\Omega X)$ has the structure of a graded Lie superalgebra (with the Whitehead/Samelson bracket). On the other hand, concatenation makes $H_\bullet(\Omega X)$ into a graded associative algebra with the Pontryagin product. In fact, the diagonal map gives a coproduct Δ that upgrades $H_\bullet(\Omega X)$ to the structure of a graded Hopf algebra. The Hurewicz homomorphism $\mathbb{C} \otimes \pi_i(\Omega X) \to H_i(\Omega X)$ sends the Whitehead/Samelson bracket to the commutator bracket.

Note. The lecturer will generally complexify the homotopy groups as we did above, but it may be omitted from the notation for convenience or on accident.

An element $a \in H_\bullet(\Omega X)$ is called primitive if $\Delta a = a \otimes 1 + 1 \otimes a$. Let $Prim$ be the set of primitive elements. They form a Lie subalgebra. They are the images of the Hurewicz maps. A result of Milnor-Moore says that $H_\bullet(\Omega X) \cong \mathcal{U}(Prim)$.

7.1.3 Return to Groups

We want to apply the above ideas to $X = G$, still a compact connected Lie group. We have that ΩG has a group structure given by pointwise multiplication. By comparing this to the H-space structure given by concatenation, and in particular the induced structures on homology, the Eckmann-Hilton argument gives us that $H_\bullet(\Omega G)$ is a commutative algebra. By again using the diagonal map, we get that $H_\bullet(\Omega G)$ is a commutative and cocommutative Hopf algebra. Then the Milnor-Moore result can be refined to $H_\bullet(\Omega G) \cong \mathrm{Sym}(Prim_0) \otimes \bigwedge(Prim_1)$, where $Prim_0$ and $Prim_1$ are the even and odd parts of $Prim$. A similar decomposition holds for cohomology

Note that the bundles $PG \xrightarrow{\Omega G} G$ and $EG \xrightarrow{G} BG$ give $\pi_i(\Omega G) \cong \pi_{i+1}(G) \cong \pi_{i+2}(BG)$. Using the fact that cohomology of BG only exists in even degrees, we find the same is true of ΩG. In particular, there is no odd part coming from the Milnor-Moore result; it is the symmetric algebra of something. But we know $H^{2\bullet}(BG) = \mathbb{C}^\bullet[\mathfrak{t}]^W = \mathbb{C}[p_1, \dots, p_r]$ with $\deg p_i = d_i$. The shift in degrees tells us $H^{2\bullet}(\Omega G) = \mathbb{C}^\bullet[\xi_1, \dots, \xi_r]$ with $\deg \xi_i = d_i - 1$ (multiply this by 2 to get the shift by 2 in degree). This is a complete description of the cohomology of ΩG. We now look into how the ξ_i are related to the p_i.

There is a "transgression" map which sends p_i as above to some $\eta_i \in H_{2d_i-1}(G)$ (The lecturer wrote this as homology, but I suspect it should be cohomology). This map comes from the Leray spectral sequence for the map $EG \xrightarrow{G} BG$. There is also an evaluation map $\mathrm{ev} : S^1 \times \Omega G \to G$. Pullback gives a map $H^\bullet(G) \to H^\bullet(S^1) \otimes H^\bullet(\Omega G)$. By integrating the image over S^1, we get a map $H^\bullet(G) \to H^{\bullet-1}(\Omega G)$. This map sends η_j as above to ξ_j.

There is a more direct way to obtain the mapping $p_i \mapsto \xi_i$. There is a so-called clutching construction which produces a G-bundle $E_\gamma \to S^2$ from a (based?) loop $\gamma : S^1 \to G$. In particular, one glues trivial bundles on two hemispheres via γ on the equator. By collecting all of these E_γ we get a G-bundle $E \to S^2 \times \Omega G$. Using the universality of $EG \to BG$, there is a map $f : S^2 \times \Omega G \to BG$ such that $E \to S^2 \times \Omega G$ is the pullback of $EG \to BG$. Composing the pullback along f and integration along S^2 gives a map $H^\bullet(BG) \to H^{\bullet-2}(\Omega G)$, and this map sends p_i to ξ_i.

8 Apr 17

We begin with a discussion of equivariant cohomology of S^2. Consider the standard embedding of $S^2 \hookrightarrow \mathbb{R}^3$, and let S^1 act on S^2 by rotation around the z-axis. If we think of S^2 as $\mathbb{CP}^1$, then the two fixed points are 0 and ∞. Restriction to the fixed points gives a map $\iota^* : H_{S^1}(S^2) \to H_{S^1}(0) \oplus H_{S^1}(\infty) = \mathbb{C}[\hbar] \oplus \mathbb{C}[\hbar]$. Here $\hbar$ has degree 2. Integration over S^2 gives a map $H^\bullet_{S^1}(S^2) \to H^{\bullet-2}_{S^1}(pt) = \mathbb{C}^{\bullet-2}[\hbar]$.

Proposition 8.1. *(1) ι^* is injective.*

(2) $\mathrm{im}(\iota^*) = \{(f_1, f_2) \mid f_1(0) = f_2(0)\}$.

(3) $\int_{S^2}(f_1, f_2) = (f_1 - f_2)/\hbar$.

Let T be a maximal torus in G, and let λ be a cocharacter of T, which we think of as a based loop in T. Using the clutching construction and this loop, we get a T-bundle E_λ on S^2. Since T is abelian, the fiber product $E_\lambda \times_T ET$ has a well-defined free T-action by acting on the first factor. We have the following diagram:

$$\begin{tikzcd} & E_\lambda \times_T ET \arrow[dl] \arrow[dr] & \\ S^2 = E_\lambda/T & & BT \end{tikzcd}$$

We have $H_T(E_\lambda \times_T ET) = H_{T\times T}(E_\lambda) = H(S^2) \otimes H_T(pt)$. This admits a map from $H_T(BT) = H_T(pt) \otimes H_T(pt)$, via pullback along the right arrow in the above diagram. We can include the S^1 action on S^2 everywhere to get $\eta : H_T(pt) \otimes H_T(pt) \otimes H_{S^1}(pt) \to H_{S^1}(S^2) \otimes H_T(pt)$. The domain can be rewritten as $\mathbb{C}[\mathfrak{t}] \otimes \mathbb{C}[\mathfrak{t}] \otimes \mathbb{C}[\hbar]$, and the codomain can be rewritten as $\{(f_1, f_2) \in \mathbb{C}[\hbar] \oplus \mathbb{C}[\hbar] \mid f_1(0) = f_2(0)\} \otimes \mathbb{C}[\mathfrak{t}]$.

The $S^1 \times T \times T$ action on E_λ is given by $(z_0, t_1, t_2) \cdot x = \lambda(z_0) t_1 t_2^{-1} x$ (here x is thought of as an element of T, i.e. the action is fiberwise). In particular, the subgroup $\{t_2 = \lambda(z_0)t_1\}$ is the stabilizer. By looking at the Lie algebra of the stabilizer, we deduce that $H_{S^1}(S^2) \otimes \mathbb{C}[\mathfrak{t}] \cong (\mathbb{C}[\mathfrak{t}] \otimes \mathbb{C}[\mathfrak{t}])[\hbar]/(f(x+\hbar\lambda) \otimes 1 - 1 \otimes f(x) \mid f \in \mathbb{C}[\mathfrak{t}])$. Note that we are treating λ as an element of $\mathfrak{t}$ by embedding $X_*(T)$ as a lattice in $\mathfrak{t}$.

Let W be the Weyl group. We always have $H_G(-) = H_T(-)^W$. In particular, $H_G(pt) = \mathbb{C}[\mathfrak{t}]^W$, which we denote by Z. Recall we have $\zeta : Z \otimes Z \otimes \mathbb{C}[\hbar] \to H_{S^1 \times G}(Gr) = H_{S^1 \times G}(\Omega G)$. We also (will?) construct $\zeta_T : \mathbb{C}[\mathfrak{t}] \otimes Z \otimes \mathbb{C}[\hbar] \to H_{S^1 \times T}(Gr) = H_{S^1 \times T}(\Omega G)$.

Recall the G-bundle $E \to S^2 \times \Omega G$ constructed last time. By repeating what we have done today, we have $Z \otimes Z \otimes \mathbb{C}[\hbar] = H_{G\times G\times S^1}(E) \cong H_{G\times S^1}(E/G) = H_{G\times S^1}(S^2 \times \Omega G) \xrightarrow{\int_{S^2}} H_{G\times S^1}(\Omega G)$. This is the map ζ above. We can similarly obtain ζ_T, and then ζ is obtained by taking W-invariants. The cohomology $H_{G\times S^1}(S^2\times\Omega G)$ is similar to what we had before; it is $\{(f_1,f_2)\in\mathbb{C}[\mathfrak{t}\times\mathbb{A}^1]^2 \mid f_1(x,0)=f_2(x,0)\}$. Integration sends (f_1,f_2) to $(f_1-f_2)/\hbar$ (here $\hbar$ is the coordinate on $\mathbb{A}^1$). Recall $\mathcal{A}$ is the subalgebra of $(Z\otimes Z)[\hbar,\hbar^{-1}]$ generated by $(Z\otimes Z)[\hbar]$ and elements $(f\otimes 1-1\otimes f)\hbar^{-1}$. The map ζ extends to $\mathcal{A}\to H_{S^1\times G}(\Omega G)$. We know $\mathcal{A}^{2\bullet}/\hbar\mathcal{A}\cong\mathbb{C}^\bullet[p_1,\dots,p_r,\xi_1,\dots,\xi_r]$ with $\deg p_i=d_i=1+\deg\xi_i$, and $H_G(\Omega G)$ has the exact same description.

Recall $\mathcal{A}/\hbar\mathcal{A}=\mathbb{C}[T(\mathfrak{t}/W)]$. Recall $\check{\mathfrak{t}}\cong\mathfrak{t}^*$ as W-representations, and that $\mathfrak{t}/W\cong\check{\mathfrak{t}}^*/W\cong\check{\mathfrak{g}}^*//\check{G}=\mathfrak{c}$. Let $\check{\mathfrak{g}}^*_{reg}$ be the regular locus, i.e. consisting of those ϕ with $\dim\mathrm{Stab}_{\check{G}}(\phi)=r$, the rank of $\check{\mathfrak{g}}$. The adjoint quotient map restricted to the regular locus, $\pi:\check{\mathfrak{g}}^*_{reg}\to\mathfrak{c}$, is a smooth morphism, and the fibers are $\check{G}$-orbits. Then $\mathbb{C}[\mathfrak{c}]=\mathbb{C}[\check{\mathfrak{g}}]^{\check{G}}$. For $p\in\mathbb{C}[\mathfrak{c}]$ we can take its differential at a point $c\in\mathfrak{c}$ to get $d_cp\in T^*_c\mathfrak{c}$. If $x\in\check{\mathfrak{g}}^*_{reg}$ and $\pi(x)=c$, then $d\pi^*$ sends d_cp to $d_x(\pi^*p)\in T^*_x(\check{\mathfrak{g}}^*)=\check{\mathfrak{g}}$.

Let $Z_{reg}=\{(x,g)\mid x\in\check{\mathfrak{g}}^*,\mathrm{Ad}_g(x)=x\}$. Let $\mathcal{J}=Z_{reg}/\check{G}$. It is a smooth group scheme over $\mathfrak{c}$. $\mathrm{Lie}\mathfrak{J}\to\mathfrak{c}$ is a vector bundle of rank r. The fiber of this bundle over c is the Lie algebra of the stabilizer of some $x\in\pi^{-1}(c)$. We can make identifications of this with $\check{\mathfrak{g}}_x=\{d_x\pi^*p\mid p\in\mathbb{C}[\mathfrak{c}]\}=T^*_c\mathfrak{c}$. The first equality here is due to Kostant (or using the Kostant slice?).

9 Apr 22

Consider the usual Langlands dual setup. For $x \in \check{\mathfrak{g}}^*$ we let $Z_x = \mathrm{Stab}_{\check{G}}(x)$ and $\mathfrak{z}_x = \mathrm{Lie} Z_x$. We say x is regular if $\dim Z_x = \dim \check{\mathfrak{t}}$. If this holds, then Z_x is abelian. We have that $Z = \{(x, g) \mid x \in \check{\mathfrak{g}}^*_{reg}, g \in Z_x\}$ is a smooth group scheme over $\check{\mathfrak{g}}^*_{reg}$, and the fiber over any x is Z_x. The Lie algebra $\mathfrak{z} = \mathrm{Lie} Z$ is a vector subbundle of $T^*(\check{\mathfrak{g}}^*_{reg})$.

Recall $\mathfrak{c} = \check{\mathfrak{t}}^*/W = \mathfrak{t}/W \cong \check{\mathfrak{g}}^*//\check{G}$. Let χ be the restricted quotient map $\check{\mathfrak{g}}^*_{reg} \to \mathfrak{c}$. Then there is a smooth group scheme called the universal centralizer $\mathcal{J} \to \mathfrak{c}$ such that $\chi^* \mathcal{J}$ is isomorphic to Z, and this isomorphism is compatible with maps to $\check{\mathfrak{g}}^*_{reg}$. It follows that $\mathfrak{z} \cong \chi^*(\mathrm{Lie} \mathcal{J})$.

Lemma 9.1. *Let $x \in \check{\mathfrak{g}}^*_{reg}$ and let $c = \chi(x)$. Then $T^*_c \mathfrak{c} \cong \mathfrak{z}_x$, and in particular, $\mathrm{Lie} \mathcal{J}$ is isomorphic to $T^* \mathfrak{c}$ as vector bundles.*

Proof. We have $d_x \chi : \check{\mathfrak{g}}^* \to T_c \mathfrak{c}$ and $d_x \chi^* : T^*_c \mathfrak{c} \to \check{\mathfrak{g}}$. The image of $d_x \chi^*$ consists of those $\varphi \in \check{\mathfrak{g}}$ such that $\langle \varphi, T_x(\check{G}x) \rangle = 0$. We have $T_x(\check{G}x) = \mathrm{ad}_{\check{\mathfrak{g}}} x$. Now, use the Killing form $(-,-)$ to identify $\check{\mathfrak{g}}$ and its linear dual. Then $x \in \check{\mathfrak{g}}$ and $T_x(\check{G}x) = [\check{\mathfrak{g}}, x]$. Then the image of $d_x \chi^*$ is those $\varphi \in \check{\mathfrak{g}}$ with $(\varphi, [y, x]) = 0$ for all $y \in \check{\mathfrak{g}}$. But $(\varphi, [y, x]) = \pm(y, [\varphi, x])$, so this condition is equivalent to $\varphi \in \mathfrak{z}_x$. need some geometry to show that $d_x \chi^*$ is injective

Here is another approach. Let $p \in \mathbb{C}[\mathfrak{c}]$. Then $\chi^* p \in \mathbb{C}[\check{\mathfrak{g}}^*]^{\check{G}}$. The collection of $d_c p$ over all p is $T^*_c \mathfrak{c}$. Now, $d_x(\chi^* p) \in (\check{\mathfrak{g}}^*)^* = \check{\mathfrak{g}}$. The whole differential $d(\chi^* p) \in (\mathbb{C}[\check{\mathfrak{g}}^*_{reg}] \otimes \check{\mathfrak{g}})^{\check{G}}$. Since we have the $\check{G}$ invariance, we actually have $d_x(\chi^* p) \in \mathfrak{z}_x$.

We have $\mathbb{C}[\mathfrak{c}] = \mathbb{C}[p_1, \ldots, p_r]$. Then $T^* \mathfrak{c}$ has a basis of sections given by $\xi_i = dp_i$. Then $\mathbb{C}[T^* \mathfrak{c}] = \mathbb{C}[p_i, \xi_i]$. Now $H = \mathbb{C}[T \mathfrak{c}] \cong \mathrm{Sym}(T^* \mathfrak{c}) \cong \mathrm{Sym}(\mathrm{Lie} \mathcal{J}) \cong \mathcal{U}(\mathrm{Lie} \mathcal{J})$. H has a coproduct $\Delta : H \to H \otimes_{\mathbb{C}[\mathfrak{c}]} H$ such that, for $\xi \in \mathrm{Lie} \mathcal{J}$, we have $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi$. Note that $\mathrm{Lie} \mathcal{J}$ consists of the primitive elements. The ξ_i are primitive and a basis.

Recall Ω consists of the based-at-identity polynomial loops in compact Lie group G. We have $H_{S^1 \times G}(\Omega) \cong \mathcal{A} \subset \mathbb{C}[\mathfrak{c} \times \mathfrak{c}][\hbar^{\pm 1}]$ and $H_G(\Omega) \cong \mathcal{A}/\hbar \mathcal{A} \cong \mathbb{C}[T \mathfrak{c}] \cong \mathrm{Sym}(\mathrm{Lie} \mathcal{J})$, or what we just called H above. Let $m : \Omega \times \Omega \to \Omega$ be pointwise multiplication. Then $\Delta = m^*$ sends $H_G(\Omega)$ to $H_G(\Omega \times \Omega) \cong H_G(\Omega) \otimes_{H_G(pt)} H_G(\Omega) \cong \mathrm{Sym} \mathcal{J} \otimes_{\mathbb{C}[\mathfrak{c}]} \mathrm{Sym} \mathcal{J}$. After identifying $H_G(\Omega) \cong \mathrm{Sym} \mathcal{J}$, this is the same as the coproduct of H mentioned above.

Fix T and consider $X_*(T) = \Omega^T$. For $\lambda \in X_*(T)$, let $i_\lambda : \{\lambda\} \hookrightarrow \Omega$. Then $i^*_\lambda : H_T(\Omega) \to H_T(pt) = \mathbb{C}[\mathfrak{t}]$. Recall that $H_G(\Omega) = H_T(\Omega)^W$, so we can look at i^*_λ on $H_G(\Omega) \cong \mathbb{C}[T \mathfrak{c}] = \mathbb{C}[p_i, \xi_i]$. We have that $p_i \mapsto p_i$ and $\xi_i \mapsto \langle \xi_i, \lambda \rangle$. The pairing is due to the fact that $\xi_i \in (\mathbb{C}[\mathfrak{t}] \otimes \mathfrak{t}^*)^W$ and λ can be identified with a point in $\mathfrak{t}$, so we can pair them to get an element of $\mathbb{C}[\mathfrak{t}]$. The fact that $i^*_\lambda(\xi_i)$ is linear in λ is very important, and we now discuss its importance.

If $\lambda, \mu \in X_*(T) = \Omega^T$, then the multiplication map $m : \Omega \times \Omega \to \Omega$ sends (λ, μ) to $\lambda + \mu$. We have the following commutative diagram:

$$\begin{tikzcd} H_G(\Omega) \arrow[r, "i^*_{\lambda+\mu}"] \arrow[d, "m^*"'] & H_G(\lambda+\mu) \arrow[d] \\ H_G(\Omega) \otimes_{H_G(pt)} H_G(\Omega) \arrow[r, "i^*_\lambda \otimes i^*_\mu"'] & H_G(\lambda) \otimes_{H_G(pt)} H_G(\mu) \end{tikzcd}$$

The second vertical map is also a "stupid" pullback. If we start with ξ_i in the top-left corner, then we get

$$\begin{tikzcd} \xi_i \arrow[r] \arrow[d] & i^*_{\lambda+\mu}\xi_i \\ \xi_i \otimes 1 + 1 \otimes \xi_i \arrow[r] & i^*_\lambda \xi_i \otimes 1 + 1 \otimes i^*_\mu \xi_i \end{tikzcd}$$

(Yes, the second vertical arrow is missing. I'm not sure what the natural description of its image would be.) The commutativity of the diagram is equivalent to $\lambda \mapsto i^*_\lambda \xi_i$ being linear, and without this, ξ_i would not be primitive. (???)

Now we look at what happens with loop rotation. There are two embeddings $\mathbb{C}[\mathfrak{c}] \hookrightarrow \mathcal{A}$, namely $p \mapsto p \otimes 1$ and $1 \otimes p$. The difference between these maps is related to the differential. We have that $\mathfrak{c} \leftarrow \mathrm{Spec}\mathcal{A} \to \mathfrak{c}$ is a commutative Hopf algebroid, namely tensor products need to be taken over $\mathbb{C}[\mathfrak{c}][\hbar]$. We have $\mathcal{A} = \mathbb{C}[p_i, \xi_i, \hbar]$ where $\xi_i = (p_i \otimes 1 - 1 \otimes p_i)\hbar^{-1}$ in $(\mathbb{C}[\mathfrak{c}] \otimes \mathbb{C}[\mathfrak{c}])[\hbar^{\pm 1}]$. Note however that $p_i \otimes 1 = 1 \otimes p_i$ in $\mathcal{A} \otimes_{\mathbb{C}[\mathfrak{c}][\hbar]} \mathcal{A}$, or when $\hbar = 0$. The coproduct sends $p_i \otimes 1$ to $p_i \otimes 1 \otimes 1$, sends $1 \otimes p_i$ to $1 \otimes 1 \otimes p_i$, sends ξ_i to $\xi_i \otimes 1 + 1 \otimes \xi_i$, and sends $\hbar$ to $\hbar$.

10 May 1

Let G be a linear algebraic group. For commutative $\mathbb{C}$-algebras R, the functor $R \mapsto G(R[[t]])$ is representable by an affine group scheme $G(\mathcal{O})$. We also have ind-affine group schemes $G(\mathcal{O}_-)$ and $G(\mathcal{K})$ representing the functors $R \mapsto G(R[t^{-1}])$ and $R \mapsto G(R((t)))$, respectively.

Example 10.1. Let $G = GL_n$. For $d \geq 1$ we define $(L_d GL_n)(R)$ to be the set of $\sum_{i \geq -d} a_i t^i \in M_n(R)((t))$ that are invertible. This is defines an open affine subscheme in $\mathbb{A}^\infty$.

Theorem 10.1 (Chevalley)**.** *If G is linear algebraic, there is an embedding $G \hookrightarrow GL(V)$ and $v \in V$ such that $G = \{g \in GL(V) \mid g(\mathbb{C}v) \subset \mathbb{C}v\}$. If G has no characters, then $G = \{g \in GL(V) \mid gv = v\}$.*

We can use this theorem to prove that the functors mentioned at the start of this lecture are representable, assuming that G has no characters. Take an embedding as in the theorem. Then $G(\mathcal{O}) = \{g \in GL(V[[t]]) \mid gv = v\}$. We define $GL_n(R((t)))$ to be $\coprod_{d \geq 1} L_d GL_n(R)$. Define Gr_d for GL_n as $R \mapsto$ the set of projective $R[[t]]$-submodules $L \subset R((t))^n$ such that $t^d R[[t]]^n \subset L \subset t^{-d} R[[t]]^n$. Each $Gr_d(R)$ is projective, so the limit $Gr(R)$ is ind-projetive. $GL_n(R((t)))$ acts on $Gr(R)$. We define $GL_n(R[[t]])$ to be the $GL_n(R((t)))$-stabilizer of the standard lattice $L_0 = R[[t]]^n$.

We now discuss the determinant bundle on Gr. Any time G is used in this paragraph, it is probably meant to be GL_n. For any two lattices L_1, L_2 and any $L \subset L_1 \cap L_2$, we define the determinant $\det(L_1, L_2)$ to be $\bigwedge^{top}(L_1/L) \otimes (\bigwedge^{top}(L_2/L))^*$. This is independent of L (and it will be important that we can choose different L). If we fix one the two lattices, say $L_2 = L_0$, then we can define the determinant of any lattice L to be $\det(L) = \det(L, L_0)$. The determinant bundle is a line bundle with fiber $\det(L)$ over L. It is $G(\mathcal{O})$-equivariant, and also equivariant with respect to loop rotation. There is no action of $G(\mathcal{K})$ on it, but there is an action of $\widehat{G(\mathcal{K})}$, a Kac-Moody group. The first Chern class is (up to scalar) the differential of a quadratic Casimir element.

I kinda stopped paying attention here, he discussed $L^{--}G$, which is the kernel of evaluation $G(\mathcal{O}) \to G$ where one sets $t^{-1} = 0$ (???).

11 May 6

11.1 Torus Actions

Let X be a projective variety with a cell decomposition, i.e. $X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$ with each $X_\lambda \cong \mathbb{A}^{N(\lambda)}$ for some non-negative integers $N(\lambda)$. We are interested in cases where X is acted on by a torus T, and the T-action on each cell is linear (in particular, each cell must have a fixed point). In fact, given a T-action on any projective variety X with finitely many fixed points, there is such a decomposition of X into cells. Indeed, first choose a one-parameter subgroup $\mathbb{C}^\times \hookrightarrow T$ such that $X^T = X^{\mathbb{C}^\times} = \{x_\lambda\}_{\lambda \in \Lambda}$. Then, for all x, there is some λ such that $\lim_{z \to 0} zx = x_\lambda$. We define $X_\lambda = \{x \in X \mid \lim_{z \to 0} zx = x_\lambda\}$. This is called a Bialynicki-Birula (BB) decomposition. BB proved that there is a T-equivariant isomorphism $X_\lambda \to T_{x_\lambda} X_\lambda$.

Example 11.1. The affine Grassmannian Gr is the limit of $G(\mathcal{O})$-stable projective varieties g_d. The torus T of G acts via the inverse image of evaluation at 0, and we know that the T-fixed points of Gr is $X_*(T)$. In particular, the g_d have a BB decomposition indexed by $X_*(T)$. We can also include loop rotation everywhere. If we do, then the one-parameter subgroup we choose inside $\mathbb{C}^\times \times T$ should be one that contracts Lie$I$ to 0, where I is the Iwahori, the preimage of Borel B under evaluation at 0. If we take this one-parameter subgroup, then the pieces $g_{d,\lambda}$ are exactly the Iwahori orbits Ix_λ. One can prove this claim by showing that there is an equality of tangent spaces. As a corollary, we obtain the Iwahori decomposition of Gr, $Gr = \bigsqcup_{\lambda \in X_*(T)} It^\lambda$.

Lemma 11.1. *If X has a decomposition as described in the first paragraph, then its odd cohomology groups vanish, and $\dim H^{2i}(X)$ is the number of cells of dimension i.*

Corollary 11.1. *If X is furthermore connected, then there is a unique 0-dimensional cell.*

A related result is that there is a unique open dense cell.

Now, suppose we have a connected projective T-variety X with a BB decomposition. Let $i_\lambda : X_\lambda \hookrightarrow X$. We let $D_T(X, \Lambda)$ be the full subcategory of $D_T(X)$ whose objects $\mathcal{F}$ satisfy the property $\mathcal{H}^m(i_\lambda^* \mathcal{F})$ is a constant sheaf for all $m \in \mathbb{Z}$ and all $\lambda \in \Lambda$. If this holds, then the similar statement holds with shriek pullback.

Definition 11.1. We say $\mathcal{F} \in D_T(X, \Lambda)$ is ***-even**, resp. ***-odd**, if $\mathcal{H}^{odd}(i_\lambda^* \mathcal{F}) = 0$, resp. $\mathcal{H}^{even}(i_\lambda^* \mathcal{F}) = 0$, for all λ. One can similarly define !-even and !-odd. We say something is even if it is both *-even and !-even, and we similarly define odd objects. We say $\mathcal{F}$ is **parity** if it is even or odd.

We will only consider sheaves of the same parity.

For fixed λ, let $j : X_\lambda \hookrightarrow \overline{X}_\lambda$, and let i be the complementary embedding. Let $i_{\leq\lambda} : \overline{X}_\lambda \hookrightarrow X$. We use the partial order $\leq$ given by which strata are contained in the closures of other strata, together with induction, to prove almost everything. If $\mathcal{F}$ is parity on X, then $i_\lambda^{*,!}\mathcal{F}$ is parity. For $\overline{\mathcal{F}} \in D_T(\overline{X}_\lambda, \Lambda_{\leq\lambda})$, we have two ("open-closed") distinguished triangles:

$$i_!i^!\overline{\mathcal{F}} \longrightarrow \overline{\mathcal{F}} \longrightarrow j_*j^*\overline{\mathcal{F}}$$

$$j_!j^!\overline{\mathcal{F}} \longrightarrow \overline{\mathcal{F}} \longrightarrow i_*i^*\overline{\mathcal{F}}$$

Note that the terminology in these is redundant, as $j^! = j^*$ and $i_! = i_*$. If $\overline{\mathcal{F}}$ is parity, then the induced LES in cohomology become short exact sequences:

$$0 \longrightarrow H^\bullet(i^!\overline{\mathcal{F}}) \longrightarrow H^\bullet(\overline{\mathcal{F}}) \longrightarrow H^\bullet(j^*\overline{\mathcal{F}}) \longrightarrow 0$$

$$0 \longrightarrow H_c^\bullet(j^*\overline{\mathcal{F}}) \longrightarrow H^\bullet(\overline{\mathcal{F}}) \longrightarrow H^\bullet(i^*\overline{\mathcal{F}}) \longrightarrow 0$$

If $\overline{\mathcal{E}}$ and $\overline{\mathcal{F}}$ are parity (of the same parity, as usual), then there is a filtration on $\mathrm{Ext}(\overline{\mathcal{E}}, \overline{\mathcal{F}})$ such that the associated graded is $\bigoplus_{\mu\leq\lambda} \mathrm{Ext}(i_\mu^*\overline{\mathcal{E}}, i_\mu^!\overline{\mathcal{F}})$. As a corollary, $\mathrm{Ext}^{odd}(\overline{\mathcal{E}}, \overline{\mathcal{F}}) = 0$. A corollary of this is that the cohomology, not just the cohomology sheaves, of a parity sheaf vanish in the opposite parity. I.e., if $\overline{\mathcal{F}}$ is even, then $H^{odd}(\overline{\mathcal{F}}) = 0$, and vice versa.

Everything we've been saying for $\overline{X}_\lambda$ can be applied to X by taking X_λ to be the unique open dense stratum.

Let $\Delta : X \hookrightarrow X \times X$ be the diagonal map. Let C_X be a constant sheaf on X (with stalks $\mathbb{C}$, probably). The map $C_X \boxtimes \mathcal{F} \to \Delta_*\Delta^*(C_X \boxtimes \mathcal{F}) = \Delta_*(C_X \otimes \mathcal{F}) = \Delta_*\mathcal{F}$ gives an $H(X)$-module structure on $H(\mathcal{F})$.

Theorem 11.1. *If $\mathcal{E}, \mathcal{F}$ are parity, then the map $\mathrm{Ext}^k(\mathcal{E}, \mathcal{F}) = H^k(R\mathrm{Hom}(\mathcal{E}, \mathcal{F})) \to \mathrm{Hom}^k_{H(X)}(H(\mathcal{E}), H(\mathcal{F})) = \mathrm{Hom}_{H(X)}(H^\bullet(\mathcal{E}), H^{\bullet+k}(\mathcal{F}))$ is an isomorphism.*

Proof. We treat the special case where $\mathcal{E} = \mathbf{1}$ is a skyscraper at the unique 0-dimensional stratum x. Let i be the embedding of x into X. Let K be the kernel of $i^* : H(X) \to H(x) = H(pt)$, so that the $H(X)$-module structure on $H(\mathbf{1})$ is just $H(X)/K$. On the one hand, we have $\mathrm{Hom}^k_{H(X)}(H(\mathbf{1}), H(\mathcal{F})) = \mathrm{Hom}^k_{H(X)}(H(X)/K, H(\mathcal{F})) = H^k(\mathcal{F})^K$. On the other hand, we have $\mathrm{Ext}^k(\mathbf{1}, \mathcal{F}) = \mathrm{Hom}^k_{H(x)}(C_x, i^!\mathcal{F})$ by adjunction, which is then just $H^k(i^!\mathcal{F})$. There is an obvious map $H^k(i^!\mathcal{F}) \to H^k(\mathcal{F})^K$. This map is injective under the parity assumption by the short exact sequences coming from open-closed triangle. The main hurdle is to prove that the map is surjective. We do this by induction on the closure order (although skipping the base case). Fix λ. We assume X_λ is not the 0-dimensional stratum. Let $i_{\leq\lambda}$ be as before. Let j_λ be j from before, and let $i_{<\lambda}$ be the i from before. Let $\overline{\mathcal{F}} = i^!_{\leq\lambda}\mathcal{F}$. The closed point stratum $\{x\}$ sits inside the complement of X_λ in $\overline{X}_\lambda$. We have an exact sequence $0 \to H(i^!_{<\lambda}\overline{\mathcal{F}})^K \xrightarrow{\varphi} H(\overline{\mathcal{F}})^K \to H(j^*_\lambda\overline{\mathcal{F}})^K$. We want to show that φ is an isomorphism. We have a fundamental class $[X_\lambda] \in H_c(X_\lambda)$, which we can consider as an element of $H(X)$. Then we have the following commutative diagram, where the top row is exact:

$$\begin{tikzcd}
0 \arrow[r] & H(i_!\overline{\mathcal{F}}) \arrow[r] & H(\overline{\mathcal{F}}) \arrow[r, "j^*_\lambda"] \arrow[d, "{[X_\lambda]\smile -}"'] & H(j^*_\lambda\overline{\mathcal{F}}) \arrow[r] \arrow[d, "{[X_\lambda]\smile -}"] & 0 \\
 & & H(\overline{\mathcal{F}}) & H_c(j^*_\lambda\overline{\mathcal{F}}) \arrow[l, "(j_\lambda)_!"] &
\end{tikzcd}$$

The verical map on the right is an isomorphism essentially by classical Poincaré duality, since we have a constant sheaf on an affine space. The horizontal map on the bottom is injective. The claim follows from some diagram chasing that I'm too tired to think through carefully.

12 May 8

12.1 Convolution

Let G be a finite group with unit e and multiplication m. Then there is a convolution operation on the space $\mathbb{C}(G)$ of functions on G, namely

$$(f_1 * f_2)(g) = \sum_{g_1 \in G} f_1(g_1) f_2(g_1^{-1} g).$$

The unit of this convolution operation is the indicator function of the unit e. Similarly, if we define a sheaf on G to be a collection of finite dimensional vector spaces indexed by elements of G, then there is a convolution operation on the space $Sh(G)$ of sheaves on G:

$$(\mathcal{F}_1 * \mathcal{F}_2)_g = \bigoplus_{g_1 \in G} \mathcal{F}_{g_1} \otimes \mathcal{F}_{g_1^{-1} g}.$$

Once again, this convolution operation has a unit given by an "indicator" sheaf consisting of the 0 vector space on non-identity elements, and a one-dimensional vector space on the identity. There is a functor $\Gamma : Sh(G) \to \mathbb{C}(G)$-mod sending $\mathcal{F}$ to $\bigoplus_{g \in G} \mathcal{F}_g$. Here, $\mathbb{C}(G)$ is considered as an algebra with pointwise multiplication, and this gives an obvious way that $f \in \mathbb{C}(G)$ acts on $\Gamma(\mathcal{F})$, namely $fx = (f_g x_g)_g$ for $x = (x_g)_g \in \Gamma(\mathcal{F})$. It turns out that $\Gamma(\mathcal{F}_1 * \mathcal{F}_2) \cong \Gamma(\mathcal{F}_1) \otimes_{\mathbb{C}} \Gamma(\mathcal{F}_2)$. Under this identification, a function $f \in \mathbb{C}(G)$ acts on the tensor product by $m^*(f)$, where $m^* : \mathbb{C}(G) \to \mathbb{C}(G) \otimes_{\mathbb{C}} \mathbb{C}(G)$ is induced by multiplication on G.

Now, let G be a topological group. We replace $\mathbb{C}(G)$ with $H(G)$ and $Sh(G)$ by $D(G)$. The sheaf convolution becomes $\mathcal{F}_1 * \mathcal{F}_2 = m_*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$. The functor Γ is sheaf cohomology $H(G, -)$, which does return an $H(G)$-module. We have a Künneth isomorphism $H(\mathcal{F}_1 * \mathcal{F}_2) \cong H(\mathcal{F}_1) \otimes H(\mathcal{F}_2)$, and again some $f \in H(G)$ acts on the right-hand side by $m^*(f)$ under this identification.

Now we return to the case of G being a finite group. Let K be a subgroup. Consider the space of functions on $K \backslash G / K$, i.e. those $f \in \mathbb{C}(G)$ whose coefficients (or outputs) are constant on double cosets for K. This space is stable under convolution, but it does not in general contain the indicator 1_e. Instead, it has an indicator of K, which satisfies $1_K * 1_K = |K| 1_K$. Thus, $\frac{1}{|K|} 1_K$ is a monoidal unit for $C(K \backslash G / K)$. We may also consider sheaves on $K \backslash G / K$ and the corresponding "indicator" sheaf C_K. However, $C_K * C_K \cong \mathbb{C}(K) \otimes C_K$, and there is no way to divide by $\mathbb{C}(K)$ to get a unit.

In the case of a topological group, we *can* get a monoidal unit. We consider $D(K \backslash G / K)$ to be $D_{K \times K}(G)$. We need to slightly modify our definition of convolution so that it respects equivariance. K^4 acts on $G \times G$ in the obvious way. We embed $K^3 \hookrightarrow K^4$ by letting the middle copy of K in K^3 sit diagonally in the middle two copies of K in K^4. This gives an action of K^3 on $G \times G$.

We also let K^3 act on G by ignoring the middle copy of K, i.e. $(k_1, k_2, k_3)g = k_1 g k_3^{-1}$. We do this so that $m : G \times G \to G$ is K^3-equivariant. We can then define convolution on $D_{K \times K}(G)$ by the following diagram:

$$\begin{tikzcd}
D_{K\times K}(G) \times D_{K\times K}(G) \arrow[r, "\boxtimes"] \arrow[d, "*"'] & D_{K^4}(G \times G) \arrow[r] & D_{K^3}(G \times G) \arrow[d, "m^*"] \\
D_{K\times K}(G) & & D_{K^3}(G \times G) \arrow[ll, "\text{forget middle } K"]
\end{tikzcd}$$

Now, let C_K be the constant sheaf on K, and let $i : K \hookrightarrow G$. Then $i_* C_K \in D_{K \times K}(G)$. The lecturer explained why $i_* C_K$ is a monoidal unit, although I didn't fully understand the reasoning. K acts simply transitively on the fibers of $m : K \times K \to K$, so $H_K(\text{fiber of } m) = H(pt) = \mathbb{C}$. We have a K-bundle $EK \to BG$, and $H(BG)$ "is kind of like" $1/H(G)$.

12.2 Convolution on $D_{G(\mathcal{O})}(Gr)$

We have *congruence subgroups* $L^i = \ker(G(\mathcal{O}) \to G(\mathcal{O}/t^i))$ with $G(\mathcal{O}) = L^0 \supset L^1 \supset \cdots$. For each $i \geq 1$, we have L^i / L^{i+1} is isomorphic to the underlying additive group of $\mathfrak{g} = \mathrm{Lie}(G)$. Let $L_i = G(\mathcal{O})/L^i$.

We know that Gr is the direct limit of finite dimensional projective $G(\mathcal{O})$-varieties Gr_i. Let g_i be the preimage of Gr_i under the quotient $G(\mathcal{K}) \to G(\mathcal{K})/G(\mathcal{O}) = Gr$. We have $G(\mathcal{O}) \subset g_i$ and $G(\mathcal{O}) g_i \subseteq g_i$ for all i.

For all $i, j \geq 0$, there are m, k large enough (depending on i, j) such that $g_i Gr_j \subseteq Gr_m$ and L^k acts trivially on Gr_j and Gr_m. The $G(\mathcal{K})$-action gives a well-defined map $\mu : (g_i / L^k) \times Gr_j \to Gr_m$. L_k acts on g_i / L^k on the right and on Gr_j on the left. Now, g_i / L^k is a finite dimensional variety, and the $G(\mathcal{O})$-action on it factors through some L_ℓ-action. We have $D_{G(\mathcal{O})}(Gr) = \varinjlim_{i,j} D_{L_i}(Gr_j)$. We have the following diagram (???):

$$\begin{array}{c} D_{L_\ell \times L_k}(g_i/L^k) \times D_{L_k}(Gr_j) \\ \big\downarrow \\ D_{L_\ell \times L_k \times L_k}((g_i/L^k) \times Gr_j) \\ \big\downarrow \\ D_{L_\ell \times L_k}((g_i/L^k) \times Gr_j) \\ \big\downarrow{\scriptstyle \mu_*} \\ D_{L_\ell \times L_k}(Gr_m) \\ \big\downarrow \\ D_{L_\ell}(Gr_m) \end{array}$$

The map $g_i/L^k \to g_i/G(\mathcal{O}) = Gr_i$ is an L_k-torsor, so we have $D_{G(\mathcal{O})}(Gr_i) \cong D_{G(\mathcal{O}) \times L_k}(g_i/L^k)$.

Now, recall from last that the T-fixed points of Gr are points t^λ labeled by $\lambda \in X_*(T)$. For each λ we let $Gr_\lambda = G(\mathcal{O})t^\lambda$. For the Iwahori subgroup $I \subset G(\mathcal{O})$ we have $Gr_\lambda = \bigsqcup_{w \in W} It^{w(\lambda)}$. Thus, if *dom* denotes the dominant Weyl chamber in $X_*(T)$, we have $Gr = \bigsqcup_{\lambda \in X_*(T)} It^\lambda = \bigsqcup_{\lambda \in dom} Gr_\lambda$.

Let $\rho \in \mathfrak{t}^*$ be the half-sum of positive roots. Then, for $\lambda \in dom$, we have $\dim Gr_\lambda = 2\langle \rho, \lambda \rangle$. The parity of $2\langle \rho, \lambda \rangle$ depends only on λ mod the coroot lattice. It follows that the parity of $\dim Gr_\lambda$ depends only on the connected component of Gr in which it lives.

We define the Satake category $Sat = Sat(Gr)$ to be the category of $G(\mathcal{O})$-equivariant perverse sheaves on Gr. Its simple objects are $IC_\lambda = IC(\overline{Gr_\lambda})$. We state two facts:

1. $IC_\lambda * IC_\mu \in Sat$.
2. IC_λ is parity.

The lecturer claimed (and painfully struggled to give a proof) that Sat is semisimple. This amounts to an Ext computation between IC sheaves.

13 May 13

13.1 Monoidal Categories

Let $(\mathcal{C}, \otimes)$ be a (we will always assume symmetric) monoidal category with unit $\mathbf{1}$.

Definition 13.1. $\mathcal{C}$ is **rigid** if for any $M \in \mathcal{C}$, there is a dual object $\check{M}$ so that $\mathrm{Hom}(M, -) \cong \mathrm{Hom}(\mathbf{1}, \check{M} \otimes -)$ and the canonical morphism $M \to \check{\check{M}}$ is an isomorphism. We also have/assume $\check{\mathbf{1}} \cong \mathbf{1}$.

Let $\Phi : \mathcal{C} \to \mathcal{C}'$ be a functor between monoidal categories.

Definition 13.2. Φ is called **lax monoidal** if there are functorial morphisms $\Phi_{M,N} : \Phi(M) \otimes \Phi(N) \to \Phi(M \otimes N)$ for all $M, N \in \mathcal{C}$. Φ is called **monoidal** if each $\Phi_{M.N}$ is an isomorphism. We also assume Φ sends unit to unit.

Definition 13.3. Suppose $\mathcal{C}, \mathcal{C}'$ are abelian. Φ is a **fiber functor** if it is monoidal, exact, and conservative (meaning $M \neq 0$ implies $\Phi(M) \neq 0$).

Let $\mathrm{End}\Phi$ be the collection of natural transformations from Φ to itself. Inside here we have $\mathrm{End}^{\otimes}\Phi$, those natural transformations which respect the tensor structure in some sense (lecturer omitted the condition), and inside those we have the natural isomorphisms, $\mathrm{Aut}^{\otimes}\Phi$. Under "reasonable" conditions, $\mathrm{Aut}^{\otimes}\Phi$ is an algebraic group.

Let $\Phi : \mathcal{C} \to \mathrm{Vect}$ be a fiber functor. Then the group $G = \mathrm{Aut}^{\otimes}\Phi$ acts on $\Phi(M)$ for all $M \in \mathcal{C}$, so Φ upgrades to a fiber functor $\Phi : \mathcal{C} \to \mathrm{Rep}(G)$.

Theorem 13.1. *If $\mathcal{C}$ is rigid symmetric monoidal abelian category such that $\mathrm{End}\mathbf{1} \cong k$, then the above functor $\Phi : \mathcal{C} \to \mathrm{Rep}(G)$ is a monoidal equivalence.*

Properties of $\mathcal{C}$ transfer to properties of G. For instance, if $\mathcal{C}$ is semisimple, then G is reductive.

Let $X \to B$ be given. Then X is a G-**torsor** iff $G \times X \xrightarrow{\alpha} X \times_B X$ given by $(g, x) \mapsto (gx, x)$ is an isomorphism. If $X = \mathrm{Spec}(A)$ and $B = \mathrm{Spec}(S)$, then the condition becomes that $A \otimes_S A \xrightarrow{\alpha^*} \mathbb{C}[G] \otimes A$ is an S-algebra isomorphism. A G-torsor is flat. Now let $proj(S)$ denote the category of finite rank projective S-module. If $X \to B$ is a G-torsor, then $A \in \mathrm{Ind}(proj(S))$. We have a monoidal functor $\mathrm{Rep}(G) \to proj(S)$ given by $V \mapsto \Gamma(X \times^G V) = (A \otimes_k V)^G$, where invariants are taken with respect to diagonal action.

Proposition 13.1. *Any fiber functor $\Phi : \mathrm{Rep}(G) \to proj(S)$ is isomorphic to the functor $V \mapsto \Gamma(X \times^G V)$ for some G-torsor $X \to B$.*

Proof. Let $R = \mathbb{C}[G] \in \mathrm{Ind}(\mathrm{Rep}(G))$, considered with G-action on the right. We have a multiplication $m : R \otimes R \to R$ that makes R a commutative ring object. Then $\Phi(R)$ is a commutative ring object. Thus it is a commutative flat

S-algebra. Let $X = \mathrm{Spec}\Phi(R)$. X has a G-action coming from the left G-action on R, which furthermore acts along fibers of the projection $X \to B$ (since the G-action comes from an action on $\Phi(R)$ as an S-module). For any $V \in \mathrm{Rep}(G)$, we have $R \otimes_G V \cong R \otimes_{\mathbb{C}} \underline{V}$. The left action on R "becomes" (???)the G-action on V. Apply this to $V = R$ to get $\Phi(R) \otimes_S \Phi(R) \cong \Phi(R) \otimes_{\mathbb{C}} \underline{R}$. This isomorphism shows $X \to B$ is a G-torsor.

For any G-torsor $X \to B$, we have $\mathrm{Aut}(X/B) = g$, a group scheme and locally $G \times B$ over B. For all $V \in \mathrm{Rep}(G)$, g acts on $X \times^G V = V_X$, so it acts on its sections $\Gamma(V_X)$. Also, $X \to B$ is a g-torsor. The group of automorphisms of X/B as a g-torsor is G. If $\Phi : \mathrm{Rep}(G) \to proj(S)$ sends V to $\Gamma(V_X)$, then the action of g on each $\Gamma(V_X)$ gives an isomorphism $g \cong \mathrm{Aut}^{\otimes}(\Phi)$ of group schemes over B.

The above story has a Lie algebra version. Let $Der^{\otimes}\Phi$ be the collection of functor endomorphisms such that $\psi_{M \otimes N}$ is identified with $\psi_M \otimes \mathrm{id} + \mathrm{id} \otimes \psi_N$ under the identification of $\Phi(M \otimes N)$ with $\Phi(M) \otimes \Phi(N)$. The corresponding statement is that $\mathrm{Lie}(g) \cong Der^{\otimes}\Phi$.

13.2 Satake Category

Now we return to studying the Satake category.

Theorem 13.2. *The convolution on Sat "is" symmetric.*

Proof. The precise statement is that there are functorial isomorphisms $\sigma_{M,N} : M * N \to N * M$. We recall from last time that Sat is semisimple. As a consequence, any object in Sat is $G(\mathcal{O})$-equivariantly formal, so there is an equivariant Künneth formula. Whenever we write $H(-)$, we will mean $H_{G(\mathcal{O})}$, unless stated otherwise. Let $C = H(pt) = \mathbb{C}[\mathfrak{t}]^W$. The equivariant Künneth says that the natural morphism $H(M) \otimes_C H(N) \to H(m_*(M \boxtimes N)) = H(M * N)$ is an isomorphism. We have $\mathrm{Hom}_{Sat}(M * N, N * M) = \mathrm{Ext}^0_{D_{G()}(Gr)}(M * N, N * M) \cong \mathrm{Hom}_{H(Gr)}(H(M * N), H(N * M)) = \mathrm{Hom}_{H(Gr)}(H(M) \otimes_C H(N), H(N) \otimes_C H(M))$. $H(Gr)$ is a cocommutative Hopf algebra, so there is a natural isomorphism in the last Hom. We take $\sigma_{M,N}$ to be the corresponding element of the first Hom. The isomorphism out of Ext^0 is a theorem proved previously, and relies on our sheaves being parity. Technically we must look component-by-component for parity to work.

14 May 15

Let $S = H_{G(\mathcal{O})}(pt) = H_G(pt) = \mathbb{C}[\mathfrak{t}]^W = \mathbb{C}[\check{\mathfrak{t}}^*]^W = \mathbb{C}[\check{\mathfrak{g}}^*/G]$. Let $\mathfrak{c} = \mathrm{Spec}(S)$. From now on let $H(-)$ be $H_{G(\mathcal{O})}(-)$. If $\mathcal{F}$ is an equivariantly formal sheaf, then $H(\mathcal{F})$ is a free S-module, so it is $\Gamma(V)$ for some vector bundle V on $\mathfrak{c}$. The ordinary (non-equivariant) cohomology of $\mathcal{F}$ is the fiber over 0 of this vector bundle, which we write $H_{c=0}(\mathcal{F})$, which is also $H(\mathcal{F})/S_+H(\mathcal{F})$.

Cohomology gives a fiber functor $Sat \to proj(S)$. Furthermore, $H_{c=0}$ gives a conservative fiber functor $Sat \to \mathrm{Vect}$. By Tanakian reconstruction, there is an equivalence $\Psi : \mathrm{Rep}(\check{G}) \to Sat$ for *some* group $\check{G}$ (we do not yet claim it is the Langlands dual). The composition $H \circ \Psi$ is then of the form $V \mapsto \Gamma(X \times^{\check{G}} V)$ for some $\check{G}$-torsor $X \to \mathfrak{c}$. Recall that we can construct the torsor X is as follows. Let $R = \mathbb{C}[\check{G}]$ be the regular representation in $\mathrm{Ind}(\mathrm{Rep}(\check{G}))$. Let $\mathcal{R} = \Psi(R)$ and $A = H(\mathcal{R})$. A is a flat S-algebra. Then $X = \mathrm{Spec}(A)$ is our desired torsor.

Let $g = \mathrm{Aut}^{\otimes}(H \circ \Psi)$. Then $g \cong \mathrm{Aut}_{\check{G}}(X/\mathfrak{c})$. It is a flat and smooth group scheme over $\mathfrak{c}$. Whenever $\check{G}$ acts on some Y, we get a torsor $Y_X = X \times^{\check{G}} Y$ over $\mathfrak{c}$. If we apply this to the adjoint action of $\check{G}$ on itself, we get $\check{G}_X \cong g$ canonically. In particular, $\check{G}$ is the fiber over 0 of g. More generally, for $c \in \mathfrak{c}$, we have g_c is the collection of maps $f : X_c \to \check{G}$ such that $f(ax) = af(x)a^{-1}$ for all $a \in \check{G}$ and all $x \in X_c$ (and this further generalizes to the case of the fiber of a torsor Y_X).

Let $H^* = H(Gr)$. It is a commutative and cocommutative Hopf algebra over S. Let Δ be the comultiplication. We have $H^* \cong \mathrm{Sym}(\Gamma(T^*_{\mathfrak{c}})) \cong \mathrm{Sym}(\Gamma(\mathrm{Lie}(\mathcal{J})))$. We recall that $\mathrm{Lie}(\mathcal{J}) \to \mathfrak{c}$ is a locally free $\mathcal{O}_{\mathfrak{c}}$-module. Let $prim = prim(H^*) = \Gamma(\mathrm{Lie}(\mathcal{J}))$. These are the elements satisfying $\Delta(h) = h \otimes 1 + 1 \otimes h$. If we write $S = \mathbb{C}[p_1, \ldots, p_r]$, then $\Gamma(T^*_{\mathfrak{c}})$ has a basis $dp_1, \ldots, dp_r$.

Let $\check{\mathfrak{g}} = \mathrm{Lie}(\check{G})$. Then $\mathrm{Lie}(g) = \check{\mathfrak{g}}_X$ as a $\check{G}$ torsor over $\mathfrak{c}$. The dual bundle is $(m\check{f}g^*)_X$. We note that $Z^g_{reg}/(\mathrm{Ad}g) \cong \mathcal{J}$, where Z^g_{reg} is a twisted version of Z_{reg}.

$H : Sat \to proj(S)$ extends to a functor $S \to H^*$-mod, but the outputs generally lack any good properties (e.g. flat), and the functor itself is not monoidal. However, we do have that $h \in H^*$ acts on $H(\mathcal{F} * \mathcal{F}') \cong H(\mathcal{F}) \otimes_S H(\mathcal{F}')$ by $\Delta(h)$. In particular, a primitive element h lives in $Der^{\otimes}(H \otimes \Psi) = \Gamma(\mathrm{Lie}(g))$. This gives a morphism $\mathrm{Lie}(\mathcal{J}) \to \mathrm{Lie}(g)$ of Lie algebras over $\mathfrak{c}$. Our goal is to construct a morphism $\mathcal{J} \to g$ of group schemes over $\mathfrak{c}$.

Note. I am very confused about the use and non-use of Γ.

14.1 Equivariant Homology

For any G-variety X, let $\omega_X = \mathbb{D}(C_X)$. Then $H_\bullet(X) = H^\bullet(\omega_X)$. This works for non-equivariant and equivariant settings. In order to apply this to Gr, we write Gr as the increasing union of projective $G(\mathcal{O})$-varieties Gr_d, and we define $H_* = H(\omega_{Gr}) = \varinjlim_d H(\omega_{Gr_d})$. This is $\mathbb{Z}$-graded, and can have stuff in *all* degrees. It is a module over H^*. We have $H_* \cong \mathrm{Hom}_S(H^*, S)$. We have a pairing $H^* \otimes H_* \to S$. Dualizing the coproduct of H^*, we get a Pontryagin product $\circ$ on H_*. It also has a coproduct via the diagonal embedding. Thus H_* is a Hopf algebra over S, in some sense dual to H^*.

Let $\pi : \mathcal{J} \to \mathfrak{c}$. Let $F : \mathrm{Rep}(\check{G}) \to proj(S)$ be the map $V \mapsto V_X$ (technically we need Γ, but it is "better to think of this more geometrically). Then we form $\pi^* F : \mathrm{Rep}(\check{G}) \to proj(\mathbb{C}[\mathcal{J}])$ by $V \mapsto \mathcal{J} \times_{\mathfrak{c}} V_X$. The monoidal automorphisms of this functor are $\Gamma(\pi^* g)$. Such a section is equivalent to the data of a morphism $\mathcal{J} \to g$ over $\mathfrak{c}$. We will construct a special automorphism of $\pi^* F$, and hence get a special map $\mathcal{J} \to g$.