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Preface

There will be some gaps in explanation, either due to the lecturer's admission or my own lack of understanding. In particular, many "proofs" are sketches of proofs. Gaps due to my own misunderstanding will be indicated by three red question marks: ???. More generally, my own questions about the material will also be in red. Things like "**Question**" will be questions posed by the lecturer. Feel free to reach out to me with explanations.

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1 Jan 5 - Perfect Rings and Tilting

1.1 Perfect Rings

Definition 1.1. Let R be an $\mathbb{F}_p$ algebra. We say R is **perfect** if the Frobenius map $\varphi : R \to R, x \mapsto x^p$ is an isomorphism.

Example 1.1.
1. Perfect (characteristic p) field.
2. $\mathbb{F}_p[x^{1/p^\infty}] = \bigcup_{n \geq 0} \mathbb{F}_p[x^{1/p^n}]$; the free perfect ring on 1 generator. (All monomials have pth roots.)
3. Any limit or colimit of perfect $\mathbb{F}_p$ algebras (It's enough to check for pullbacks, arbitrary products, pushouts, and arbitrary coproducts (tensor products). In all of these cases, it's enough to check on generators, in which case the pth roots are obvious.)
4. Any integrally closed domain whose fraction field is perfect (e.g. algebraically closed) ($x^p - a$ is monic and has a solution in the fraction field).
5. If R is perfect, I is a finitely generated ideal, then $R_I^\wedge$ (the I-completion) is perfect. (Can try to take pth roots of a compatible system of elements in R/I^n ... or topologically, show $\varphi : R \to R$ is I-adically continuous.)

The inclusion of the category of perfect $\mathbb{F}_p$-algebras into the category of all $\mathbb{F}_p$-algebras has both adjoints. The left adjoint $R \mapsto R_{perf}$ is the colimit of $R \xrightarrow{\varphi} R \xrightarrow{\varphi} \cdots$. The right adjoint $R \mapsto R^{perf}$ is the limit of $\cdots \xrightarrow{\varphi} R \xrightarrow{\varphi} R$.

The idea of the colimit perfection is to add all p-power roots, and the idea of the limit perfection is to pick things that have all p-power roots, although these ideas can sometimes not match reality

The adjunction means that if R is an $\mathbb{F}_p$-algebra and P is a perfect $\mathbb{F}_p$-algebra, we have $\mathrm{Hom}(R_{perf}, P) = \mathrm{Hom}(R, P)$ and $\mathrm{Hom}(P, R) = \mathrm{Hom}(P, R^{perf})$.

To see the first adjunction, note that by the definition of a colimit, $f \in \mathrm{Hom}(R_{perf}, P)$ corresponds to the data of maps $f_n : R \to P$ where $f_n = f_{n+1} \circ \varphi_R$. But maps of $\mathbb{F}_p$-algebras are "Frobenius equivariant", so $f_{n+1} \circ \varphi_R = \varphi_P \circ f_{n+1}$. Since P is perfect, φ_P is an isomorphism, so $f_{n+1} = \varphi_P^{-1} \circ f_n$. Thus the sequence is determined by f_1 via $f_n = \varphi_P^{-n+1} \circ f_1$.

To see the second adjunction, note that an element $x \in P$ must have all p-power roots, so it must be sent to an element $y \in R$ that also has all p-power roots, since if $f(x) = y$, then $f(x^{1/p})^p = f(x) = y$. Alternatively, since R^{perf} is a limit, we know $f \in \mathrm{Hom}(P, R^{perf})$ corresponds to the data of maps $f_n : P \to R$ where $f_n = \varphi_R \circ f_{n+1}$. As before, we can use $\varphi_R \circ f_{n+1} = f_{n+1} \circ \varphi_P$, and then since φ_P is an isomorphism, we get $f_{n+1} = f_n \circ \varphi_P^{-1}$. Thus f_1 determines the sequence f_n via $f_n = f_1 \circ \varphi_P^{-n+1}$.

Example 1.2. 1. $\mathbb{F}_p[x^{1/p^\infty}] = \mathbb{F}_p[x]_{perf}$. As said above, when you take the colimit, the data involved is a ring with an element with a pth root, a p^2th root, and so on.

2. $\mathbb{F}_p[x]^{perf} = \mathbb{F}_p$. As said above, the data of the limit is the choice of an element in $\mathbb{F}_p[x]$ that has all p-power roots; only the constants satisfy this. In general, if R is Noetherian, then R^{perf} is a finite product of fields.

3. If φ is surjective (R is called semiperfect) then we get a surjection $R^{perf} \twoheadrightarrow R$, so R^{perf} is "larger" than R, which is somewhat contrary to the above intuition of "taking the things which have all p-power roots".

4. If P is perfect and I is a finitely generated ideal, then $R = P/I$ is semiperfect, and $R^{perf} = P_I^\wedge$. (Exercise: prove this, and also find R_{perf}.) Solution: R is semiperfect since the composition $P \xrightarrow{\varphi} P \to P/I$ is surjective with kernel $\varphi^{-1}(I) \supset I$, so it descends to a surjective map $R \xrightarrow{\varphi} R$. To prove $R^{perf} = P_I^\wedge$, we first note that we can compute $P_I^\wedge$ as $\varprojlim_n P/I^{p^n}$. Then, because I is finitely generated, $\phi : P/I \to P/I^p$ is an isomorphism. Then we can identify the corresponding limit diagrams using iterations of Frobenius:

$$\begin{tikzcd} \cdots \arrow[r, "\varphi"] \arrow[d, "\varphi^n"'] & P/I \arrow[r, "\varphi"] \arrow[d, "\varphi"] & P/I \arrow[d, "="] \\ \cdots \arrow[r] & P/I^p \arrow[r] & P/I \end{tikzcd}$$

Here the arrows with double line marks are meant to be isomorphisms. The desired isomorphism $R^{perf} = P_I^\wedge$ follows. To compute R_{perf}, the idea is that the colimit is modding out by elements $x \in P$ such that $\phi^n(x) \in I$ eventually. But the sequence $\phi^n(x) = x^{p^n}$ is cofinal with the sequence x^n, so we are modding out by the radical $\mathrm{rad}(I)$. Thus $R_{perf} = P/\mathrm{rad}(I)$. (this may be wrong)

Example 1.3. As a concrete example of the previous case, let $P = \mathbb{F}_p[x^{1/p^\infty}]$, $I = (x)$, $R = \mathbb{F}_p[x^{1/p^\infty}]/(x)$. Then $R^{perf} = \mathbb{F}_p[x^{1/p^\infty}]_x^\wedge$, which consists of sums

$$\sum_{i \in \mathbb{Z}[1/p]_{\geq 0}} a_i x^i,$$

with $a_i \in \mathbb{F}_p$, and for all $N \geq 0$, there are only finitely many $i \leq N$ with $a_i \neq 0$. So $x + x^{p+1/p} + x^{p^2+1/p^2} + \cdots$ is allowed, while $x + x^{1/p} + x^{1/p^2} + \cdots$ isn't.

1.2 Witt Vectors

Perfect $\mathbb{F}_p$-algebras have a unique lift to characteristic 0.

Theorem 1.1. *Given any perfect $\mathbb{F}_p$-algebra R, there is a unique (up to unique isomorphism) p-adically complete and p-torsion free ring $\widetilde{R}$ equipped with an isomorphism $\widetilde{R}/p \xrightarrow{\sim} R$. This construction is functorial in R, and $\widetilde{R}$ is the ring $W(R)$ of Witt vectors of R.*

Proof. See chapter 2 of Serre's "Local Fields".

Example 1.4.

1. $R = \mathbb{F}_p$ gives $W(R) = \mathbb{Z}_p$.
2. $R = \mathbb{F}_p[x^{1/p^\infty}]$ gives $W(R) = \mathbb{Z}_p[x^{1/p^\infty}]^\wedge_p$.
3. $R = \mathbb{F}_p[x^{1/p^\infty}]^\wedge_x$ gives $W(R) = \mathbb{Z}_p[x^{1/p^\infty}]^\wedge_{(p,x)}$. The elements of this ring are series $\sum_{i \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}} a_i x^i$, where $a_i \in \mathbb{Z}_p$ and for all $N \geq 0$ and for all $\varepsilon > 0$, there are finitely many $i \leq N$ such that $|a_i| > \varepsilon$.
4. For R_1, R_2 perfect, $W(R_1 \otimes R_2) = (W(R_1) \otimes W(R_2))^\wedge_p$. why?

Structure of $W(R)$: There is a unique multiplicative (but not additive) map $[-] : R \to W(R)$ that is a section of the projection $W(R) \to R$.

Corollary 1.1. *For any $x \in W(R)$, there is a unique sequence $x_0, x_1, x_2, \ldots, \in R$ such that $x = \sum_{i \geq 0} [x_i] p^i$. So, as a set, $W(R) \cong \prod_{i \geq 0} R$.*

Proof. Let π be the isomorphism $W(R)/p \to R$. Then $[-]$ being a section of $W(R) \to W(R)/p \xrightarrow{\pi} R$ means $\pi([a] + (p)) = a$ for all $a \in R$. Now, given $x \in W(R)$, suppose $a \in R$ satisfies $x - [a] \in (p)$. Then $\pi(x + (p)) = \pi([a] + (p)) = a$, so such an a is determined by x. Conversely, we have $x - [\pi(x + (p))] \in (p)$ since $\pi([\pi(x + (p))] + (p)) = \pi(x + (p))$, so that $x - [\pi(x + (p))]$ is in the kernel of $W(R) \to W(R)/p \xrightarrow{\pi} R$, which is (p). Thus, for any $x \in W(R)$, there is a unique $x_0 \in R$ such that $x - [x_0] \in (p)$. Write $x - [x_0] = px'$. Such an x' is unique since $W(R)$ is p-torsion free. Then we may define $x_1 = x'_0$ and repeat the process forever.

Note that in the course of this proof we showed that $x \mapsto x_0$ is a ring homomorphism, in particular it is the projection $W(R) \to W(R)/p \xrightarrow{\sim} R$. The kernel of this map is (p), and in particular we have $[a] + [b] - [a+b] \in (p)$ for all $a, b \in R$.

Exercise: the image of $[-]$ is the set of elements of $W(R)$ that have all p-power roots. Solution: Since R is perfect, it has all p-power roots, and $[-]$ is multiplicative, so anything in $[R]$ has all p-power roots; explicitly, if $x = [a]$, then $x = [a^{1/p^n}]^{p^n}$. Conversely, we claim that if $x \in W(R)$ has a p^nth root for $n \geq 1$, then $[x_i] = 0$ for $1 \leq i \leq n$. This implies that if x has all p-power roots, it must equal $[x_0]$. To prove the claim, we induct on n. In the base case, let $y = x^{1/p}$ and write $y = [y_0] + py'$. Then

$$x = y^p = [y_0^p] + p^2 \left(p^{p-2}(y')^p + \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} [y_0^i] p^{p-i-1} (y')^{p-i} \right),$$

where we use $p \geq 2$ and $p \mid \binom{p}{i}$ for $0 < i < p$. This computation shows that if x has a pth root, then we can write $x = [x_0] + p^2 x'$. Inductively, if x has a p^nth root, we can write its pth root y in the form $y = [y_0] + p^n y'$ and then expand y^p to show that $[x_n] = 0$.

Analogy: $W(R)$ is like a ring of power series over R with variable p.

How to add Witt vectors: Suppose $x = \sum [x_i]p^i$ and $y = \sum [y_i]p^i$. We know $x + y = \sum [z_i]p^i$ for some $z_i \in R$. In fact, for each n, z_n can be expressed as an element of

$$\mathbb{F}_p[x_0^{1/p^\infty}, \ldots, x_n^{1/p^\infty}, y_0^{1/p^\infty}, \ldots, y_n^{1/p^\infty}].$$

Key fact: the polynomial expressing z_n in this way is homogeneous of degree 1, where each x_i, y_i has degree 1.

Example 1.5. $z_0 = x_0 + y_0$. This follows from the fact that $x \mapsto x_0$ is a homomorphism.
$z_1 = x_1 + y_1 - \sum_{0<i<p} \frac{1}{p}\binom{p}{i} x_0^{i/p} y_0^{1-i/p}$. (Exercise: prove this; use $F = W(\varphi)$)

Note. Multiplying Witt vectors, on the other hand, is simple: it's just the usual multiplication of power series, since $[-]$ is multiplicative.

1.3 Tilting

Definition 1.2. For any ring R, its **tilt** $R^\flat$ is $(R/p)^{perf}$.

Proposition 1.1. *If R is p-complete, then* $\varprojlim(\cdots \xrightarrow{\varphi} R \xrightarrow{\varphi} R) \xrightarrow{\sim} R^\flat$.

Proof. The inverse limit is being taken in multiplicative monoids, since φ is not necessarily additive on R (In fact, the lecturer does not use φ, and I only found out the reason for this after the fact). The isomorphism is in one direction easy to define: take a compatible sequence of elements in R and reduce mod p. In the other direction, if we start with some sequence (a_i) of mod p elements, we send it to $b_i = \lim_n \widetilde{a}_{i+n}^{p^n}$, where $\widetilde{a}_i$ are arbitrary lifts of a_i. todo: show this is the inverse

Example 1.6. 1. $R = \mathbb{Z}_p$ has $R^\flat = \mathbb{F}_p$, since $R/p = \mathbb{F}_p$ is already perfect.

2. $R = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge_p = (\mathbb{Z}_p[x^{1/p^\infty}]/(x-p))^\wedge_p$. Then $R/p = \mathbb{F}_p[x^{1/p^\infty}]/(x)$ and $R^\flat = \mathbb{F}_p[x^{1/p^\infty}]^\wedge_x$.

Remark. If R is p-complete, there is a natural multiplicative map $(-)^\sharp : R^\flat \to R$, coming from the isomorphism in the proposition. For example, in the second example above, $x^\sharp = p$.

Theorem 1.2. *The functor W from perfect $\mathbb{F}_p$-algebras to p-complete rings is left adjoint to the tilting functor from p-complete rings to perfect $\mathbb{F}_p$-algebras; if A is perfect/$\mathbb{F}_p$ and R is p-complete, then* $\mathrm{Hom}(W(A), R) = \mathrm{Hom}_{\mathbb{F}_p}(A, R^\flat)$.

Remark. If A is perfect/$\mathbb{F}_p$ and R is p-complete, then there is a natural map $\mathrm{Hom}(W(A), R) \to \mathrm{Hom}_{\mathbb{F}_p}(A, R/p)$ given by reducing mod p. Since A is perfect, the second hom set can be identified with $\mathrm{Hom}_{\mathbb{F}_p}(A, R^\flat)$. So, the claim is that this composite map $\mathrm{Hom}(W(A), R) \to \mathrm{Hom}_{\mathbb{F}_p}(A, R^\flat)$ is an isomorphism. We will prove this next time.

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2.1 Witt vectors and tilting

Theorem 2.1. *The functor W from perfect $\mathbb{F}_p$-algebras to p-complete rings is left adjoint to the tilting functor from p-complete rings to perfect $\mathbb{F}_p$-algebras; if A is perfect/$\mathbb{F}_p$ and R is p-complete, then* $\mathrm{Hom}(W(A), R) = \mathrm{Hom}_{\mathbb{F}_p}(A, R^\flat)$.

Proof. Suppose $A = \mathbb{F}_p[x^{1/p^\infty}]$. Then $W(A) = \mathbb{Z}_p[x^{1/p^\infty}]^\wedge_p$. Then $\mathrm{Hom}(W(A), B) = \varprojlim_\phi B$ and $\mathrm{Hom}(A, B^\flat) = B^\flat$. The natural map $\varprojlim_\phi B \to B^\flat$ is an isomorphism, so we have proved the claim for this specific choice of A. We want to reduce to this case. It will be enough to show that W preserves colimits. Recall that for perfect $\mathbb{F}_p$-algebras A_1, A_2, we have $W(A_1 \otimes A_2) = (W(A_1) \otimes W(A_2))^\wedge_p$. More generally, we need to show that if we have a diagram $A_1 \leftarrow A \to A_2$ of perfect $\mathbb{F}_p$-algebras, where A is arbitrary, then the natural map $(W(A_1) \otimes_{W(A)} W(A_2))^\wedge_p \to W(A_1 \otimes_A A_2)$ is an isomorphism. It is an isomorphism mod p, so the difficulty is in showing that $W(A_1) \otimes_{W(A)} W(A_2)$ is p-torsion free. There are two ways to justify this. One is using the following theorem:

Theorem 2.2 (Bhatt-Scholze)**.** *Given a diagram as above,* $\mathrm{Tor}_i^A(A_1, A_2) = 0$ *for $i > 0$.*

Another justification is as follows. $W(A)$ is a perfect δ-ring, so that $W(A_1) \otimes_{W(A)} W(A_2)$ is also a perfect δ-ring. It is then a general fact that perfect δ-rings are p-torsion free.

2.2 The map θ

The counit of the adjunction is a map $\theta : W(B^\flat) \to B$. Explicitly, for $b \in B^\flat$, we have $\theta([b]) = b^\sharp$.

Remark. If B/p is semiperfect, then $B^\flat \to B/p$ is surjective, so by Nakayama, θ is surjective.

Example 2.1. If $B = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge_p$, then $B^\flat = \mathbb{F}_p[x^{1/p^\infty}]^\wedge_x$ and $W(B^\flat) = \mathbb{Z}_p[x^{1/p^\infty}]^\wedge_{(p,x)}$. Then $\theta(x^{1/p^n}) = p^{1/p^n}$.

2.3 Perfectoid rings

Definition 2.1. Let A be a p-complete ring. A is **perfectoid** if $A \cong W(P)/\xi$, where P is a perfect $\mathbb{F}_p$-algebra and $\xi = \sum [\xi_i] p^i$ with $\xi_1 \in P^\times$.

Remark.
1. If $A = W(P)/\xi$, then $A/p = P/\xi_0$, so WLOG we may assume P is ξ_0-adically complete. In this case, $P = A^\flat$. Thus we may equivalently define perfectoid rings as p-complete rings such that $\theta : W(A^\flat) \to A$ is surjective and $\ker(\theta)$ is generated by ξ with $\xi_1 \in (A^\flat)^\times$.
2. Any perfect $\mathbb{F}_p$-algebra is perfectoid by taking $\xi = p$.

3. ξ is not a zero-divisor. Proof: Suppose $\xi x = 0$. Note that ξ maps to a unit in $W(P[1/\xi_0])$, so x maps to 0 in $W(P[1/\xi_0])$. In other words, if we write $x = \sum [x_i]p^i$, then all x_i are ξ_0-power torsion. But in P, ξ_0-power torsion is the same as ξ_0-torsion: if $\xi_0^N y = 0$, then multiply by some power of ξ_0 on both sides to get $\xi_0^{p^n} y = 0$, then since Frobenius is an isomorphism, we get can take p^nth roots to get $\xi_0 y^{1/p^n} = 0$, and then we may multiply by an appropriate power of y to get $\xi_0 y = 0$. So $\xi_0 x_i = 0$ for all i, or $[\xi_0]x = 0$. Hence $\xi x = (\sum_{i \geq 1} [\xi_i]p^i)x$, which is px times a unit since ξ_1 is a unit, so $px = 0$, so $x = 0$.

4. $A = W(P)/\xi$ is p-torsion free iff P is ξ_0-torsion free. To show this we use the torsion exchange lemma: if B is a ring with nonzerodivisors x, y, then $(B/x)][y] \cong (B/y)[x]$ (where brackets denote torsion). This is true because both are H_1 of a Koszul complex on x, y. Applying the torsion exchange lemma to $B = W(P)$, $x = p$, and $y = \xi$, we get $A[p] = (W(P)/\xi)[p] = P[\xi_0]$.

Note. The fact that ξ is not a zero-divisor implies that derived completions are well-behaved.

Example 2.2. Let $P = \mathbb{F}_p[x^{1/p^\infty}]$ and $\xi = p - [x]$. Then $W(P)/\xi = \mathbb{Z}_p[p^{1/p^\infty}]_p^\wedge$.

Example 2.3. $P = \mathbb{F}_p[q^{1/p^\infty}]_{(q-1)}^\wedge$, $\xi = 1 + [q^{1/p}] + \cdots + [q^{(p-1)/p}]$. We claim P is ξ_0-adically complete and that $\xi_1 \in P^\times$. Indeed, $\xi_0 = 1 + q^{1/p} + \cdots + q^{(p-1)/p} = (q^{1/p} - 1)^{p-1}$, so by definition P is ξ_0-complete. To see $\xi_1 \in P^\times$, we may quotient by the Jacobson radical, hence set all powers of q to be 1, so $\xi = p$ so $\xi_1 = 1$. Thus we can form the perfectoid ring $W(P)/\xi = \mathbb{Z}_p[\zeta_{p^\infty}]_p^\wedge$. Note that $(q^{1/p^n})^\sharp = \zeta_{p^n}$.

Example 2.4. Non-examples:

1. $\mathbb{Z}_p$ is not perfectoid, because θ is an isomorphism.

2. $\mathbb{Z}/p^n$ is not perfectoid for $n > 1$, since $\ker(\theta) = (p^n)$.

Claim 2.1. *Let R be perfectoid. Then there exists $u \in R^\times$ and $\alpha \in R^\flat$ such that $pu = \alpha^\sharp$; in other words, pu has a compatible system of p-power roots.*

Proof. Write $R = W(R^\flat)/\xi$. We have $\xi = [\xi_0] + p \cdot (unit)$. Thus $\theta([\xi_0]) = \xi_0^\sharp = p \cdot (unit)$.

Example 2.5. If R is perfectoid, then $R\langle x^{1/p^\infty}\rangle = (R \otimes_{\mathbb{Z}} \mathbb{Z}_p[x^{1/p^\infty}])_p^\wedge$ is perfectoid.

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Proposition 3.1. *Let R be p-torsion free containing ω such that*

1. $\omega^p \in pR^\times$.
2. $R/\omega \xrightarrow{\varphi} R/\omega^p = R/p$ *is an isomorphism.*

Then $R_p^\wedge$ is perfectoid.

Proof. WLOG, suppose R is p-complete. Condition two implies that R is semiperfect, so $R^\flat \to R/p$ is surjective. Thus, lift $\omega \mod p$ to some α, so that $\alpha^\sharp = \omega + py \in \omega R^\times$. Then, WLOG, we have $\omega = (\omega^\flat)^\sharp$. Now, letting $\overline{u}$ be the unit such that $\omega^p = p\overline{u}$, we have $\theta([\omega^\flat]^p) = \omega^p = p\overline{u}$. Lift $\overline{u}$ to $u \in W(R^\flat)^\times$. Then $[\omega^\flat]^p - pu \in \ker\theta$, and this is a valid choice of ξ to show R is perfectoid. So we want to show $W(R^\flat)/([\omega^\flat]^p - pu) \xrightarrow{\theta} R$ is an isomorphism. Both sides are p-complete and p-torsion free; the LHS is p-torsion free because $\omega^{\flat p}$ is a nonzerodivisor. Thus, to show the map is an isomorphism, we may work mod p and show $R^\flat/\omega^{\flat p} \xrightarrow{\sim} R/p$. Explicitly, this means that we want to show that, given a sequence $(x_0, x_1, \dots) \in R$ with $x_{i+1}^p = x_i$ for all i, then the following are equivalent:

1. $p \mid x_0$.
2. $\omega^p \mid x_0$.
3. $\omega^{p^{1-i}} \mid x_i$ for all i.

Clearly conditions 1 and 2 are equivalent since $\omega^p \in pR^\times$. Since Frobenius is an isomorphism $R/\omega \to R/\omega^p$, we get $\omega^p \mid x_0$ iff $\omega \mid x_1$. In fact, we can quotient the isomorphism by $\omega^{1/p^{n+1}}$ to obtain $\omega^{1/p^{n+1}} \mid y$ iff $\omega^{1/p^n} \mid y^p$. This gives the equivalence between 2 and 3, so we are done.

I'm confused by a lot of the steps in this proof, it's so over

3.1 p-root closure

Definition 3.1 (P. Roberts)**.** Let R be p-torsion free. R is **p-root closed** if for all $x \in R[\frac{1}{p}]$, we have $x^p \in R$ implies $x \in R$.

Example 3.1.

1. Any integrally closed domain is p-root closed.
2. $\mathbb{Z} \oplus (px) \subset \mathbb{Z}[x]$ is p-root closed.
3. $\mathbb{Z}[px, x^p]$ is not p-root closed.

Definition 3.2. Let A be a p-torsion free ring. Let $A^{+,p}$ (notation possibly not standard) be the set of $x \in A[\frac{1}{p}]$ such that $x^{p^n} \in A$ for large enough n. Then $A^{+,p}$ is the smallest p-root closed subring of $A[\frac{1}{p}]$ containing A, and it is called the **p-root closure** of A.

Exercise: prove $A^{+,p}$ is a ring. (Hint: use divisibility properties of binomial coefficients.)

Claim 3.1. *Let R be a p-torsion free ring with $\omega \in R$ such that $\omega^p \in pR^\times$. Then the following are equivalent:*

1. *R is p-root closed.*
2. *$R/\omega \xrightarrow{\phi} R/\omega^p$ is injective.*

Proof. ($1 \Rightarrow 2$) Fix $x \in R$ such that $\omega^p \mid x^p$. Then $x/\omega \in R[\frac{1}{p}]$, so by assumption, $x/\omega \in R$, i.e. $\omega \mid x$.

($2 \Rightarrow 1$) Fix $y \in R[\frac{1}{p}]$ such that $y^p \in R$. Write $y = z/\omega^i$ for i minimal and $z \in R$. If $i > 0$, then $\omega^{ip} \mid z^p$, so $\omega^p \mid z^p$, so $\omega \mid z$, meaning i is not minimal. Hence $i = 0$ so $y \in R$.

Definition 3.3. Let P be a perfect $\mathbb{F}_p$-algebra. Let $\xi = \sum [\xi_i] p^i \in W(P)$. Say ξ is **distinguished** if P is ξ_0-complete and $\xi_1 \in P^\times$.

Claim 3.2.
1. *If ξ is distinguished and $u \in W(P)^\times$, then ξu is also distinguished.*
2. *If ξ, ξ' are distinguished and $\xi \mid \xi'$, then $\xi'/\xi \in W(P)^\times$.*

Theorem 3.1. *Let R be p-torsion free and p-complete. Then R is p-complete iff the following are all true:*

1. *There is $\omega \in R$ such that $\omega^p \in pR^\times$.*
2. *R is p-root closed.*
3. *R/p is semiperfect.*

Proof. Assume 1-3 hold. Conditions 1 and 2 imply $R/\omega \to R/\omega^p$ is injective, and condition 3 implies it is surjective. Then, by the proposition earlier (cite), R is perfectoid.

Thus assume R is perfectoid. It is enough to show condition 2. Choose $\omega^\flat \in R^\flat$ such that $((\omega^\flat)^\sharp)^p = p\overline{u}$ for $\overline{u} \in R^\times$, and let $\omega = (\omega^\flat)^\sharp$. It is enough to show $R/\omega \to R/\omega^p$ is an isomorphism. Lift $\overline{u}$ to $u \in W(R^\flat)^\times$, so that $\xi = [\omega^\flat]^p - pu$ is distinguished and in $\ker\theta$. Then $R = W(R^\flat)/\xi$, so $R/\omega \to R/\omega^p$ is $R^\flat/\omega^\flat \to R^\flat/\omega^{\flat p}$, and this is an isomorphism since $R^\flat$ is perfect.

Example 3.2. Let R be a domain. Let R^+ be the absolute integral closure, i.e. the integral closure in the algebraic closure of the fraction field of R. Then $(R^+)^\wedge_p$ is perfectoid. Applying this construction to $\mathbb{Z}_p$, we obtain that $\mathcal{O}_{\mathbb{C}_p}$ is perfectoid.

3.2 Category of perfectoid rings

There is no initial object in the category of perfectoid rings.

The category of perfectoid rings is equivalent to the category of "perfect prisms", i.e. pairs (P, I), where P is a perfect ring and I is an ideal of $W(P)$ generated by a distinguished element. Given a perfectoid R, we send it to $(R^\flat, \ker(\theta))$. Given a perfect prism (P, I), we send it to $W(P)/I$.

Note. There is no category of rings equivalent to the category of all prisms.

Corollary 3.1. *The category of perfectoid rings has pushouts; in particular, given $R_1 \leftarrow R \rightarrow R_2$ maps of perfectoids, we have $(R_1 \otimes_R R_2)_p^\wedge$ is perfectoid.*

Proof. By the "rigidity of prisms", we may write $R = W(R^\flat)/I, R_1 = W(R_1^\flat)/I, R_2 = W(R_2^\flat)/I$ for the same ideal I. Then $(R_1 \otimes_R R_2)_p^\wedge = (W(R_1^\flat) \otimes_{W(R^\flat)} W(R_2^\flat))/I = W(R_1^\flat \otimes_{R^\flat} R_2^\flat)/I$.

4 Jan 12

4.1 Tilting equivalence

Proposition 4.1. *Let R be perfectoid, $\xi \in W(R^\flat)$ generates $\ker\theta$. Then there is an equivalence between the category of perfectoid R-algebras and the category of ξ_0-adically complete perfect $R^\flat$-algebras, given by sending a perfectoid R-algebra R' to its tilt $R'^\flat$.*

Proof. Under the equivalence of the category of perfectoid rings with perfect prisms, the perfectoid R-algebras are sent to perfect prisms (P, I) with a map from $(R^\flat, \ker\theta)$. Since I is required to be generated by a distinguished element, and since $\xi \in \ker(\theta)$ is mapped into I, we automatically get that such P are ξ_0-adically complete.

Example 4.1. The category of perfectoid $\mathbb{Z}_p[p^{1/p^\infty}]_p^\wedge$-algebras is equivalent to the category of t-complete $\mathbb{F}_p[t^{1/p^\infty}]_t^\wedge$-algebras.

Theorem 4.1. *Let R be perfectoid, ξ generates $\ker\theta$. There is an equivalence between the category of finite étale $R[\frac{1}{p}]$-algebras and the category of finite étale $R^\flat[\frac{1}{\xi_0}]$-algebras, given by taking an algebra T, taking the integral closure $\overline{R}_T$ of R in T, and then forming $\overline{R}_T^\flat[\frac{1}{\xi_0}]$. Thus, the étale fundamental groups of $R[\frac{1}{p}]$ and $R^\flat[\frac{1}{\xi_0}]$ are identified.*

The proof is outside of the scope of the course.

Example 4.2. As a special case, $\mathrm{Gal}(\mathbb{Q}_p(p^{1/p^\infty})) \cong \mathrm{Gal}(\mathbb{F}_p((t)))$.

Example 4.3. We have the following special classes of perfectoid rings:

1. Perfect $\mathbb{F}_p$-algebras.

2. p-torsion free perfectoid rings.

The following proposition says that any perfectoid ring can be glued together from these.

Proposition 4.2. *If R is perfectoid, then $R/R[p^\infty]$ is p-torsion free and perfectoid, and there is a pullback diagram*

$$\begin{tikzcd}
R \arrow[r, two heads] \arrow[d, two heads] & R/R[p^\infty] \arrow[d, two heads] \\
(R/p)_{red} \arrow[r, two heads] & (R/(R[p^\infty], p))_{red}
\end{tikzcd}$$

where all maps are surjective and the bottom two rings are perfect $\mathbb{F}_p$-algebras.

Proof. For any ring A and ideals I, J such that $I \cap J = 0$, there is a pullback

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A/I \arrow[d] \\ A/J \arrow[r] & A/(I+J) \end{tikzcd}$$

We apply this to the ring $R^\flat$, $I = \xi_0$-power torsion (equivalently, the $x \in R^\flat$ with $\xi_0^{1/p^n} x = 0$ for all n), $J = \bigcup_{n \geq 0} (\xi_0^{1/p^n})$. Then $I \cap J = 0$ and we can apply the remark. Then "untilting" (apply W and mod out by the same ξ everywhere) gives the desired pullback diagram ("it's a bit of work to do so"). (an $R^\flat$-algebra T untilts to a p-torsion free ring iff ξ_0 is not a zero divisor, and T untilts to a (perfect?) $\mathbb{F}_p$-algebra iff $\xi_0 = 0$)

Example 4.4. A perfectoid ring that is neither a perfect $\mathbb{F}_p$-algebra nor p-torsion free: $\mathbb{Z}_p[p^{1/p^\infty}, t^{1/p^\infty}]_p^\wedge/(p^a t^b, a > 0, b > 0)$.

4.2 Valuation rings

Definition 4.1. A **valuation ring** is a domain $V \neq 0$ such that for all nonzero $x, y \in V$, we have $x \mid y$ or $y \mid x$.

As a consequence, all finitely generated ideals of a valuation ring are principal.

Example 4.5.
1. Any discrete valuation ring.
2. $\mathcal{O}_{\mathbb{C}_p}$, or more generally, if K is a field with a non-archimedean absolute value, then the set of elements $\mathcal{O}_K$ with absolute value ≤ 1.
3. $\mathbb{Z}_p \oplus t\mathbb{Q}_p[[t]]$ as a subring of $\mathbb{Q}_p[[t]]$.

Remark.
1. V is local, $\operatorname{Spec}(V)$ is totally ordered, and all radical ideals are prime.
2. In any scheme, specialization can be tested via valuation rings. Algebraically, if A is a local domain, then there is an inclusion $A \hookrightarrow A' \hookrightarrow \operatorname{Frac}(A)$ such that A' is a valuation ring and $\mathfrak{m}_A \hookrightarrow \mathfrak{m}_{A'}$.

Definition 4.2. Let K be a field. A **valuation** on K consists of a totally ordered group Γ and a map $\nu : K \to \Gamma \cup \{\infty\}$ such that

1. $\nu(x) = \infty$ iff $x = 0$.
2. $\nu(xy) = \nu(x) + \nu(y)$.
3. $\nu(x + y) \geq \min(\nu(x), \nu(y))$.

The **value group** is $\nu(K^\times)$.

Example 4.6. If we have a field K with valuation ν, then we can form a valuation ring by taking $V = \{x \in K \mid \nu(x) \geq 0\}$. Conversely, if we have a valuation ring V, we have the field $K = \operatorname{Frac}(V)$ and a valuation $K \to \Gamma = K^\times/V^\times \cup \{\infty\}$, where Γ is given an ordering by the divisibility relation. Γ is exactly the value group of this valuation.

Definition 4.3. A valuation ring V has **rank one** if any of the following (equivalent) conditions hold:

1. $\mathfrak{m}_V = rad(f)$ for f a nonzero nonunit (a "pseudouniformizer").
2. V has exactly 2 prime ideals, namely 0 and $\mathfrak{m}_V$.
3. There is a non-trivial non-archimedean absolute value on $K = \operatorname{Frac}(V)$ such that $V = \{x \in K \mid |x| \leq 1\}$.

Example 4.7. Any DVR has rank one, and $\mathbb{F}_p[[t]]_{perf}$ is a non-DVR with rank one.

Note. In general, rank n valuation rings are those with exactly $n+1$ prime ideals.

Proposition 4.3. *Let A be perfectoid. Then A is a valuation ring iff $A^\flat$ is a valuation ring. In this case* $\operatorname{Frac}(A)^\times/A^\times \cong \operatorname{Frac}(A^\flat)^\times/A^\flat$*, where the map from right to left sends α to $\alpha^\sharp$.*

Proof. The forward direction follows from the fact that $A^\flat = \varprojlim_\varphi A$ is an isomorphism of multiplicative monoids, so the divisibility condition can be checked directly. For the backwards direction, show that any $x \in A$ can be written as $\alpha^\sharp u$ for some $\alpha \in A^\flat$ and some $u \in A^\times$.

Definition 4.4 (Scholze's original definition)**.** A **perfectoid field** is a complete non-archimedean field K such that

1. $|p| < 1$.
2. $\mathcal{O}_K/p$ is semiperfect.
3. The value group is not discrete.

Exercise: check that there is a correspondence between perfectoid fields and perfectoid valuation rings of rank one that are complete for a pseudouniformizer.

Example 4.8. Let C be complete, algebraically closed, non-archimedean field with $|p| < 1$. Then $\mathcal{O}_C$ is perfectoid.

Exercise: Find the rank (lecturer's claim: 2) and value group of $\mathbb{Z}_p \oplus t\mathbb{Q}_p[[t]]$ as a subring of $\mathbb{Q}_p[[t]]$.

5 Jan 14

5.1 Perfectoidization

Let R be an $\mathbb{F}_p$-algebra. Then

1. There is a universal perfect ring with a map from R, namely R_{perf}.
2. If R is semiperfect (φ surjective) then $R_{perf} = R_{red} = R/\mathrm{rad}(0)$, so $R \twoheadrightarrow R_{perf}$.

If there is an initial perfectoid ring with a map from a ring R, then we call it the **perfectoidization** R_{perfd}. This need not exist, e.g. $(\mathbb{Z}_p)_{perfd}$ does not exist.

Theorem 5.1. *1. If R is integral over a perfectoid ring R_0, then R_{perfd} exists.*

*2. If R is **semiperfectoid** (R is a quotient of a perfectoid ring), then R_{perfd} exists and the map $R \to R_{perfd}$ is surjective.*

Remark. Part 2 of the theorem was originally believed to be false.

Remark. R being semiperfect is equivalent to the conditions that R is an algebra over a perfectoid ring R_0 and R/p is semiperfect. For instance, if the two conditions hold, then $\theta : W(R^\flat) \to R$ is surjective, and then $R_0 \widehat{\otimes} W(R^\flat) \to R$ is also surjective.

Example 5.1. $\mathbb{Z}/p^2$ is not semiperfectoid, but its mod p ring is semiperfect.

5.2 p-complete arc topology

Definition 5.1. An **extension** of rank 1 valuation rings is an inclusion $V \hookrightarrow V'$ such that a pseudouniformizer for V is mapped to one for V'. Equivalently, it is an isometric inclusion of non-archimedean fields, where the valuation rings are the unit balls.

Definition 5.2. A map of rings $A \to B$ is a **p-complete arc cover** if, given any $A \to V$ where V is a p-complete rank 1 valuation ring, there is an extension $V \hookrightarrow V'$ and a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ V & \hookrightarrow & V' \end{array}$$

Equivalently, any multiplicative seminorm $|\cdot|$ on A such that $|A| \leq 1$ and $|p| < 1$ extends to one on B with $|B| \leq 1$.

Remark. 1. This notion of covering gives a Grothendieck topology on $(\mathrm{Rings})^{op}$.

2. The topology has a basis given by products of rings of the form $\mathcal{O}_C$, where C is a complete, algebraically closed, non-archimedean field with $|p| < 1$. In other words, for any ring R, there is a p-complete arc cover $R \to \prod_{i \in I} \mathcal{O}_{C_i}$.

3. Any faithfully flat map is a p-complete arc cover. For instance, given $f, g \in R$, the map $R \to R[x]/(fx - g) \times R[y]/(gy - f)$ is a p-complete arc cover.

Theorem 5.2. *On perfectoid rings, the identity is a sheaf with no higher cohomology for the p-complete arc-topology. Explicitly, if $A \to B$ is a p-complete arc cover, then $A \xrightarrow{\sim} \varprojlim(B \rightrightarrows B\widehat{\otimes}_A B \cdots)$.*

Corollary 5.1. *Let $R \to R'$ be a map of rings. Then R' is a perfectoidization of R iff the following are true:*

1. *R' is perfectoid.*

2. *For any complete, algebraically closed non-archimedean field C with $0 < |p| < 1$ and any map $R \to \mathcal{O}_C$, there is a unique extension to a map $R' \to \mathcal{O}_C$.*

3. *$(R/p)_{perf} \xrightarrow{\sim} (R'/p)_{perf}$; equivalently, allow $|p| = 0$ in the above condition.*

Proof. We want $\mathrm{Hom}(R', T) \xrightarrow{\sim} \mathrm{Hom}(R, T)$ for T perfectoid. By descent, we reduce to $T = \prod \mathcal{O}_C$, and since products commute with Hom, we reduce to $T = \mathcal{O}_C$.

Example 5.2. Let $C = \mathbb{C}_p$ and $R = \mathcal{O}_C[x]/(x^p - 1)$. Then there is a map of rings $R \to \mathrm{Fun}(\mu_p(\mathcal{O}_C), \mathcal{O}_C)$ given by sending x to the identity function. This is an isomorphism, in particular the Fourier transform, after inverting p.

Proposition 5.1. *The perfectoidization of the ring R described above is the ring of functions $\mu_p(\mathcal{O}_C) \to \mathcal{O}_C$ that are constant mod the maximal ideal of $\mathcal{O}_C$. In other words, it is*

$$\mathrm{Fun}(\mu_p(\mathcal{O}_C), \mathcal{O}_C) \times_{\mathrm{Fun}(\mu_p(\mathcal{O}_C), \overline{\mathbb{F}_p})} \overline{\mathbb{F}_p}.$$

Proof. lecturer is going way too fast conditions 1 and 3 of Corollary 5.1 can be checked ... call the claimed perfectoidization R'. Suppose we have a map $R \to \mathcal{O}_{C'}$, where C' is a field such as in condition 2 of Corollary 5.1. We can extend the map $R \to C'$ to R', since $C' = \mathcal{O}_{C'}[\frac{1}{p}]$ and $R[\frac{1}{p}] = R'[\frac{1}{p}]$. Now note that $R' \subset \frac{1}{p}R$. We are done by the following lemma:

Lemma 5.1. *Let $A \hookrightarrow B$ be a map of p-torsion free rings. Suppose $A \subset B \subset \frac{1}{p^n}A$. Then any map $A \to \mathcal{O}_C$ extends uniquely to B.*

Proof of lemma. As above, we can extend $A \to C$ to $B \to C$. But $B \subset \frac{1}{p^n}A$ means that $B \to C$ has bounded image. The only bounded subrings of C are contained in the unit ball, so we indeed get a map $B \to \mathcal{O}_C$.

Example 5.3. Let $R = \mathcal{O}_C\langle x^{1/p^\infty}\rangle/(x-1)$. Then

$$R_{perfd} = \mathrm{Fun}(\mathbb{Z}_p(1)(\mathcal{O}_C), \mathcal{O}_C) \times_{\mathrm{Fun}(\mathbb{Z}_p(1)(\mathcal{O}_C), \overline{\mathbb{F}_p})} \overline{\mathbb{F}_p}.$$

However, this is very nontrivial. Furthermore, the map $R \to R_{perfd}$ is surjective. Next time we will show how to construct a nonzero element of the kernel.

6 Jan 15

I will be absent, but I will at least transcribe the lecturer's handwritten notes.