ELSEVIER

Discrete Mathematics 193 (1998) 201–224

DISCRETE MATHEMATICS

t, q-Catalan numbers and the Hilbert scheme

Mark Haiman [*,1]

Department of Mathematics, University of California, 9500 Gilman Drive, La Jolla, CA 92093-0112, USA

Received 3 April 1995; received in revised form 14 August 1995; accepted 5 March 1998

In honor of Adriano Garsia

Mathematics subject classification 1991: Primary 14M15, secondary 05E15.

Keywords: Catalan numbers; Hilbert scheme; Sheaf cohomology; Lefschetz formula

1. Introduction

In [9], A. Garsia and the present author introduced a new bivariate 'q-analog' of the familiar Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{1.1}$$

Our (t,q)-*Catalan numbers*

$$C_n(t,q) \tag{1.2}$$

are defined through a peculiar formula, (1.10) below, which expresses them as rational functions of t and q. In [9], making heavy use of the theory of Macdonald polynomials, we were able to establish that the specializations [2]

$$C_n(1,q) \quad \text{and} \quad q^{\binom{n}{2}}C_n(q^{-1},q) \tag{1.3}$$

* E-mail: mhaiman@macaulay.ucsd.edu.

[1] Supported in part by N.S.F. Mathematical Sciences grants DMS-9119355 and DMS-9400934.

[2] As written, the defining formula would have vanishing denominators for these specializations, which should be understood as limits — but the point is moot since are going to prove that $C_n(t,q)$ is actually a polynomial.

PII S0012-365X(98)00141-1

reduce to well-known q-analogues of the Catalan numbers: the first to the Carlitz–Riordan [5] q-Catalan numbers defined by

$$C_n(q) = \sum_{k=0}^{n-1} q^k C_k(q) C_{n-1-k}(q), \qquad C_0(q) = 1, \tag{1.4}$$

and the second to

$$\frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q, \tag{1.5}$$

where as is customary we set $[k]_q = (1-q^k)/(1-q)$, $[k]_q! = [1]_q[2]_q \cdots [k]_q$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q = [n]_q!/[k]_q![n-k]_q!$.

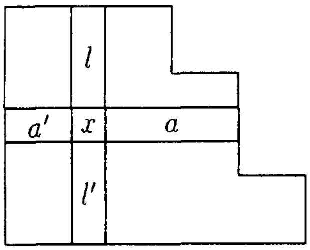
More generally, we define for each $m \geqslant 0$ a (t,q)-Catalan-like sequence

$$C_n^{(m)}(t,q), \tag{1.6}$$

in which our original (t,q)-Catalan numbers are the case $m = 1$. Then we show in [9] that $C_n^{(m)}(1,q)$ counts lattice paths that stay above the main diagonal in an $n \times mn$ rectangle, according to the area below the path (this is what the Carlitz–Riordan numbers $C_n(q)$ count in the case of an $n \times n$ square), and that

$$q^{m\binom{n}{2}} C_n^{(m)}(q^{-1}, q) = \frac{1}{[mn+1]_q} \begin{bmatrix} (m+1)n \\ n \end{bmatrix}_q. \tag{1.7}$$

We now recall the definition of $C_n^{(m)}(t,q)$ from [9]. For each partition μ of n, and each square x in the Ferrers diagram of μ, we define the leg $l(x)$, the arm $a(x)$, the co-leg $l'(x)$, and the co-arm $a'(x)$ of x to be respectively the numbers of squares above, to the right of, below, and to the left of x, with the diagram oriented in the French manner as shown here.



(1.8)

Note that $l'(x)$ and $a'(x)$ are simply the row and column coordinates of x, indexed from $(0,0)$, while $l(x)+a(x)+1$ is the hook length of x. Associated with the partition μ is a statistic $n(\mu)$ which may be variously defined as

$$n(\mu) = \sum_{x \in \mu} l(x) = \sum_{x \in \mu} l'(x) = \sum_i (i-1)\mu_i = \sum_i \binom{\mu_i'}{2}. \tag{1.9}$$

Here and below, μ' denotes the partition conjugate to μ.

This given, we set

$$C_n^{(m)}(t,q) = \sum_{|\mu|=n} \frac{t^{mn(\mu)}q^{mn(\mu')}(1-t)(1-q)\left(\prod_{x\in\mu\setminus(0,0)}(1-t^{l'(x)}q^{a'(x)})\right)\left(\sum_{x\in\mu}t^{l'(x)}q^{a'(x)}\right)}{\prod_{x\in\mu}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})}, \tag{1.10}$$

where μ ranges over all partitions of n.

Our formula for $C_n^{(m)}(t,q)$ originated in connection with a series of conjectures on *diagonal harmonics* [13]. Beginning with the polynomial ring

$$\mathbf{Q}[X,Y] = \mathbf{Q}[x_1,y_1,x_2,y_2,\ldots,x_n,y_n], \tag{1.11}$$

let the symmetric group S_n act diagonally, that is, by

$$\sigma p(x_1,y_1,\ldots,x_n,y_n) = p(x_{\sigma(1)},y_{\sigma(1)},\ldots,x_{\sigma(n)},y_{\sigma(n)}). \tag{1.12}$$

Next letting I be the ideal generated by all S_n-invariant polynomials without constant term, consider the quotient ring

$$R_n = \mathbf{Q}[X,Y]/I. \tag{1.13}$$

The space of *diagonal harmonics* is the *Macaulay inverse system* to R_n:

$$H_n = \{f(X,Y)\colon p(\partial X,\partial Y)f(X,Y) = 0 \text{ for all } p(X,Y)\in I\}, \tag{1.14}$$

where $p(\partial X,\partial Y)$ denotes the differential operator obtained by substituting for the variables $x_1,y_1,\ldots,x_n,y_n$ the corresponding partial derivative operators $\partial x_1,\partial y_1,\ldots,\partial x_n,\partial y_n$.

R_n and H_n are finite-dimensional S_n modules, doubly graded by x-degree and y-degree. One easily shows [13] that the two are isomorphic as doubly graded S_n modules. Among several conjectures of a combinatorial nature concerning H_n is that its subspace H_n^ε of *S_n-alternating* elements — that is, its isotypic component corresponding to the sign character ε of S_n — has dimension equal to the Catalan number C_n.[3] Taking into account the grading, we may define a Hilbert polynomial

$$D_n(t,q) = \sum_{h,k\geqslant 0} t^h q^k \dim(H_n^\varepsilon)_{h,k}, \tag{1.15}$$

where $(H_n^\varepsilon)_{h,k}$ denotes the doubly homogeneous component of bi-degree (h,k) in H_n^ε. This polynomial $D_n(t,q)$ should then be a (t,q)-analog of C_n.

Now it develops that there is a connection, pointed out originally by C. Procesi, to whose insight this work owes a great deal, between the diagonal harmonics and

[3] Another conjecture bearing on the special significance of the the alternating elements is that H_n^ε minimally generates H_n as an inverse system.

the *Hilbert scheme* $\mathrm{Hilb}^n(A^2)$. Pursued to its end — modulo some as yet unproven geometric hypotheses — this connection suggests a formula analogous to (1.10) for the *entire* doubly graded character of R_n and H_n, and not only its alternating component. In [9] we have shown that all the earlier combinatorial conjectures follow from this one master formula, a fact which, although proving none of the conjectures, tends strongly to confirm their validity.

Specializing to the sign character yields (1.10), and with it the conjecture that

$$C_n(t,q) = D_n(t,q), \tag{1.16}$$

and in particular that $C_n(t,q)$ is a polynomial with non-negative integer coefficients.

Our purpose here is to work out the necessary geometry to explain how formula (1.10) comes about. In the process, we shall prove that

for all integers $m \geqslant 0$ and $n \geqslant 1$, $C_n^{(m)}(t,q)$ is a polynomial in t and q,

and

for m sufficiently large (n fixed), this polynomial has non-negative coefficients.

We shall also state, but unfortunately we cannot yet prove, the precise cohomology vanishing theorem needed to extend the second statement to all $m \geqslant 0$, and in particular to the original (t,q)-Catalan case $m = 1$.

As we shall see, (1.10) is one side of an identity known as the Atiyah–Bott Lefschetz formula [1], which equates it with an Euler characteristic of traces for a torus group action on certain sheaf cohomology modules. In this case they are the cohomology modules of ample line bundles on the *zero fiber* $\boldsymbol{H}_0^n$ of the *punctual Hilbert scheme* $\boldsymbol{H}^n = \mathrm{Hilb}^n(A^2)$. We confine attention to (1.10) rather than the more general formula associated with the whole space of diagonal harmonics, because the latter would involve the introduction of an additional scheme whose required properties are as yet merely conjectural.

To complete this introduction, we give a brief outline of the development to follow, hoping to ease the reader's task of retaining perspective in the midst of the details.

Let $\mathfrak{A}$ denote the ideal in $\boldsymbol{Q}[X,Y]$ generated by all S_n-alternating polynomials, and let $\mathfrak{m}$ be the homogeneous maximal ideal $(x_1, y_1, \ldots, x_n, y_n)$. One easily proves that the space $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ — the minimal generating space for the ideal $\mathfrak{A}$ — is isomorphic to the space of S_n-alternating diagonal harmonics H_n^{ε}. More generally, we may consider the space

$$\mathfrak{A}^m/\mathfrak{m}\mathfrak{A}^m, \tag{1.17}$$

and it is the Hilbert polynomial of this space, for large m, that we shall identify as $C_n^{(m)}(t,q)$.

The first step is to review the structure of the Hilbert scheme, its natural torus action, and its local description near the torus fixed points, all in very explicit local coordinates, as these will be essential ingredients in the final formula as well as in the subsequent steps of the argument.

The second step is to observe that the Hilbert scheme is a blow-up of the symmetric power $\mathrm{Sym}^n(A^2)$, and as such is equipped with an ample line bundle $\mathcal{O}(1)$, corresponding to a square root of the exceptional divisor for the blow-up. This ultimately provides the link to the spaces $\mathfrak{A}^m/\mathfrak{m}\mathfrak{A}^m$.

The third step is to study the zero fiber $\boldsymbol{H}_0^n$ in the Hilbert scheme. Our main new geometric result, Proposition 2.10, is that $\boldsymbol{H}_0^n$ is isomorphic to a complete intersection subscheme of the universal scheme $\boldsymbol{H}_+^n$. As such, $\boldsymbol{H}_0^n$ is Cohen–Macaulay, and of crucial importance for us, its structure sheaf has an explicit locally $\mathcal{O}_{H^n}$-free resolution, providing another key ingredient for the final formula.

The fourth step is to prove that $\mathcal{O}(1)$ is the highest exterior power of the tautological bundle, contributing the last ingredient of the formula.

With these steps completed, we observe (Proposition 2.13) that there is a natural map

$$\mathfrak{A}^m/\mathfrak{m}\mathfrak{A}^m \to H^0(\boldsymbol{H}_0^n, \mathcal{O}(m)), \tag{1.18}$$

which is an isomorphism for m large. For m large we also have, by Serre's theorem,

$$H^i(\boldsymbol{H}_0^n, \mathcal{O}(m)) = 0 \quad \text{for all } i > 0, \tag{1.19}$$

since $\mathcal{O}(1)$ is ample for the projective variety $\boldsymbol{H}_0^n$.

This given, the Atiyah–Bott theorem yields us a formula for the Hilbert polynomial $D_n^{(m)}(t,q)$ of the doubly graded space $\mathfrak{A}^m/\mathfrak{m}\mathfrak{A}^m$. The formula can be fully evaluated with the aid of the preceding steps, and reduces to none other than (1.10). This proves

$$C_n^{(m)}(t,q) = D_n^{(m)}(t,q) \tag{1.20}$$

for all sufficiently large m (n held fixed). If we had isomorphism in (1.18) and the higher cohomology vanishing (1.19) for all m, we could further conclude that $C_n^{(m)}(t,q) = D_n^{(m)}(t,q)$ for all m. With (1.19) alone, we could still conclude that $C_n^{(m)}(t,q)$ is a polynomial with positive integer coefficients, the Hilbert series of $H^0(\boldsymbol{H}_0^n, \mathcal{O}(m))$, whether or not the latter space is isomorphic to $\mathfrak{A}^m/\mathfrak{m}\mathfrak{A}^m$.

2. Description of the Hilbert scheme

In order to explicitly evaluate the Atiyah–Bott formula in Section 3, we need considerable preliminary groundwork on the Hilbert scheme and its zero fiber, which we carry out in this section. We start by reviewing the definition and some well-known facts about the Hilbert scheme, making the details a bit more explicit than is customary, as we shall need quite a precise picture later on. As we turn to the steps of the argument outlined at the end of Section 1, we come to results which we believe to be new — notably, the Cohen–Macaulayness of the zero fiber (which is also the Hilbert scheme of the local ring at a point) and the realization of the Hilbert scheme as a particular blow-up of the scheme $\mathrm{Sym}^n(A^2)$.

Let A^2 = Spec $k[x,y]$ be the affine plane over an algebraically closed field. All schemes considered will be quasi-projective over k, and we take the 'classical' view that the underlying set of a scheme is its set of closed points, for instance, the underlying set of A^2 is $k \times k$. We shall assume whenever dealing with the Hilbert scheme of order n that the characteristic of the ground field is either zero or is greater than n. This is necessary in order for Weyl's theorem on the ring of invariants $k[X,Y]^{S_n}$ to hold — see the proofs of Propositions 2.2 and 2.10.

The *punctual Hilbert scheme of the plane*, $\boldsymbol{H}^n = \mathrm{Hilb}^n(A^2)$, is the set of all ideals $I \subseteq k[x,y]$ such that $\dim_k(k[x,y]/I) = n$. It is a scheme, in a manner that we shall clarify in a moment. Viewed another way, $\boldsymbol{H}^n$ parametrizes subschemes $S \subseteq A^2$ for which $S = \mathrm{Spec}(k[x,y]/I)$ is zero-dimensional, of length n. Generically, such a subscheme S is just a set of n points in A^2, regarded as a reduced subscheme. Subschemes S of this form describe a dense open subset of $\boldsymbol{H}^n$, and one may think of the Hilbert scheme as a kind of compactification of the space of n-point subsets of A^2, which retains extra information in the limit when some or all of the points coincide.

Hilbert schemes were defined — in much greater generality than here — by Grothendieck [12]. The punctual Hilbert scheme of the plane has received particular attention [4,6–8] because of its special properties. Namely, it is smooth and irreducible, neither of which is true of $\mathrm{Hilb}^n(A^m)$ for general m [7,15]. For an excellent survey of the subject to 1985, see [16].

For our purposes we need to describe the scheme structure of $\boldsymbol{H}^n$ via explicit coordinates on open affine subsets, indexed by partitions μ of n. Given such a partition μ, let

$$\mathscr{B}_\mu = \{x^h y^k : (h,\, k) \in \mu\}. \tag{2.1}$$

Recall that our indexing convention for a square $(h,\, k)$ in the diagram of μ is that h and k are its co-leg and co-arm, that is, its row and column indices, numbered starting with zero.[4] Thus $\mathscr{B}_{(4,4,2,2)}$, for example, contains the monomials

$$\begin{matrix} x^3 & x^3y & & \\ x^2 & x^2y & & \\ x & xy & xy^2 & xy^3 \\ 1 & y & y^2 & y^3. \end{matrix} \tag{2.2}$$

We now define

$$U_\mu = \{I \in \boldsymbol{H}^n : \mathscr{B}_\mu \text{ spans } k[x,y]/I\}. \tag{2.3}$$

Here we really mean that image of $\mathscr{B}_\mu$ modulo I spans $k[x,y]/I$. Of course this makes $\mathscr{B}_\mu$ a basis modulo I, since $\dim_k(k[x,y]/I) = n$. Since $\mathscr{B}_\mu$ is a basis, for each monomial

[4] For readers accustomed to English diagrams, note that in French and English alike the rows represent the parts of the partition, so the abscissa coordinate is the row index while the ordinate is the column index. A truly Cartesian diagram convention would represent the parts by the columns.

$x^r y^s$ and ideal $I \in U_\mu$ there is a unique expansion

$$x^r y^s \equiv \sum_{(h,k)\in\mu} c^{rs}_{hk}(I) x^h y^k \pmod{I}, \tag{2.4}$$

whose coefficients depend on I and thus define a collection of functions c^{rs}_{hk} on U_μ.

Proposition 2.1. *The sets U_μ are open affine subvarieties*[5] *which cover $\boldsymbol{H}^n$. The affine coordinate ring $\mathcal{O}_{U_\mu}$ is generated by the functions c^{rs}_{hk}, for $(h, k) \in \mu$ and all (r,s).*

Proof. The sets U_μ cover $\boldsymbol{H}^n$ because of the following fact, often regarded as a part of Gröbner basis theory but actually going back to Gordan [10]: for every ideal I in a polynomial ring there is a basis $\mathscr{B}$ modulo I, consisting of monomials, such that every divisor of a monomial in $\mathscr{B}$ is also in $\mathscr{B}$. For $I \in \boldsymbol{H}^n$ it is clear that such a basis must be $\mathscr{B}_\mu$ for some partition μ of n.

Since every $k[x,y]/I$ has one of the sets $\mathscr{B}_\mu$ as a basis, we see that for $N \geqslant n-1$, the set of M_N of all monomials of degree at most N spans $k[x,y]/I$, and thus I determines an element of the Grassmann variety $\boldsymbol{G}^n(kM_N)$ of n-dimensional quotients of the linear span of M_N. For N sufficiently large (in fact, for $N \geqslant n$ [11]) Grothendieck's construction shows that the resulting map

$$\boldsymbol{H}^n \to \boldsymbol{G}^n(kM_N) \tag{2.5}$$

is injective, its image is locally closed, and the induced reduced subscheme structure is independent of N. This defines the structure of $\boldsymbol{H}^n$ as a scheme.

Now we see immediately that the sets U_μ are the preimages under the embedding (2.5) of standard affines on $\boldsymbol{G}^n(kM_N)$ and that the standard coordinates on these affines reduce to the functions c^{rs}_{hk}. The image of U_μ is closed in the corresponding standard affine on $G^n(kM_N)$, so the functions c^{rs}_{hk} generate $\mathcal{O}_{U_\mu}$. $\square$

For each $I \in \boldsymbol{H}^n$, the scheme $S = \operatorname{Spec}(k[x,y]/I)$ has a finite number of points. If we assign each point $p \in S$ a multiplicity m_p equal to the length of the local ring $\mathcal{O}_{p,S} = (k[x,y]/I)_p$ then these multiplicities sum to n. In this way we associate with I an n-element *multiset* $\pi(I) \subseteq \boldsymbol{A}^2$.

The n-element multisets contained in $\boldsymbol{A}^2$ form an affine variety $\operatorname{Sym}^n(\boldsymbol{A}^2)$. To make this precise, let

$$(\boldsymbol{A}^2)^n = \boldsymbol{A}^2 \times \boldsymbol{A}^2 \times \cdots \times \boldsymbol{A}^2 = \operatorname{Spec} k[x_1, y_1, \ldots, x_n, y_n] \tag{2.6}$$

be the variety of ordered n-tuples of points in $\boldsymbol{A}^2$. The symmetric group S_n acts on $(\boldsymbol{A}^2)^n$ by permuting the factors. Note that the corresponding action on $k[x_1, y_1, \ldots, x_n, y_n]$ is the diagonal one given by (1.12). Identifying each multiset with an *unordered* n-tuple

[5] In general U_μ is not an affine *cell* — but compare Corollary 2.8.

of points in A^2, we have

$$\mathrm{Sym}^n(A^2) = (A^2)^n/S_n = \mathrm{Spec}\, k[X,Y]^{S_n}, \tag{2.7}$$

where $k[X,Y]^{S_n}$ denotes the ring of invariants for the diagonal action.

The map $\pi : \boldsymbol{H}^n \to \mathrm{Sym}^n(A^2)$ defined above is called the *Chow morphism.*

Proposition 2.2. *The Chow morphism* $\pi : \boldsymbol{H}^n \to \mathrm{Sym}^n(A^2)$ *is a projective morphism.*

Proof. This is well-known but it may be instructive to review the proof.

For $I \in U_\mu$, let M_x and M_y be the matrices of multiplication by x and y in $k[x,y]/I$, taken with respect to the basis $\mathscr{B}_\mu$. The entries of M_x and M_y are regular functions of I, in fact they are instances of the functions c_{hk}^{rs}.

The ring $k[x,y]/I$ is the direct product of its local rings $(k[x,y]/I)_p$, and the only eigenvalues of M_x and M_y on the local ring at $p = (\xi,\zeta)$ are ξ and ζ, respectively. Since M_x and M_y commute it follows that $\mathrm{tr}(M_x^r M_y^s) = \sum_p m_p \xi^r \zeta^s$. By the definition of π, this is equal to $p_{r,s}(\pi(I))$, where

$$p_{r,s} = \sum_i x_i^r y_i^s \in k[X,Y]^{S_n} \tag{2.8}$$

is a *polarized power sum.* By a theorem of Weyl [20], the polarized power sums generate $k[X,Y]^{S_n}$. Since we have shown that $\pi^* p_{r,s}$ is regular on $\boldsymbol{H}^n$, π is a morphism.

The morphism π is projective because it extends to a morphism $\tilde{\pi}: \mathrm{Hilb}^n(\boldsymbol{P}^2) \to \mathrm{Sym}^n(\boldsymbol{P}^2)$ of projective varieties, under which the preimage of $\mathrm{Sym}^n(A^2)$ is $\boldsymbol{H}^n$. $\square$

The two-dimensional torus group

$$\boldsymbol{T}^2 = \{(t,q) : t,q \in k^*\} \tag{2.9}$$

acts algebraically on A^2 by $(t,q)\cdot(\xi,\zeta) = (t\xi, q\zeta)$, or equivalently on $k[x,y]$ by $(t,q)\cdot x = tx$, $(t,q)\cdot y = qy$. There is an induced action on $\boldsymbol{H}^n$ which, since (2.4) must remain invariant, is given by $(t,q)\cdot c_{hk}^{rs} = t^{r-h}q^{s-k}c_{hk}^{rs}$. One must take care in computing $(t,q)\cdot I$ for $I \in \boldsymbol{H}^n$ to remember that this means the pullback of I via the homomorphism $(t,q): k[x,y] \to k[x,y]$, given by $(t,q)\cdot I = \{p(t^{-1}x, q^{-1}y) : p(x,y) \in I\}$.

More generally, as long as we keep our constructions homogeneous, the torus action is implicit in the x- and y-degrees: f is doubly homogeneous of bi-degree (h,k) if and only if f is a torus eigenfunction with $(t,q)\cdot f = t^h q^k f$. With this understanding, we need never write down the torus action in equations.

An ideal $I \in \boldsymbol{H}^n$ is a $\boldsymbol{T}^2$ fixed point if and only if I is doubly homogeneous, that is to say, if and only if I is spanned by monomials. Such an ideal I must clearly be of the form

$$I_\mu = (x^h y^k : (h,k) \notin \mu) \tag{2.10}$$

for some partition μ of n. Note that the subscheme of $\boldsymbol{A}^2$ defined by such an ideal I_μ is concentrated at the origin, the sole $\boldsymbol{T}^2$ fixed point of $\boldsymbol{A}^2$.

Lemma 2.3. *Every ideal $I \in \boldsymbol{H}^n$ has a torus fixed point in the closure of its orbit.*

More precisely, let $<$ denote the lexicographic ordering of monomials $1 < x < x^2 < \cdots < y < xy < x^2y < \cdots$, and for $p \in k[x,y]$ let $\lambda(p)$ denote the greatest monomial appearing with non-zero coefficient in p. Let $\mathrm{in}(I) = k \cdot \{\lambda(p) : p \in I\}$ be the initial ideal of I. Then

$$\lim_{a\to 0}\lim_{b\to 0}(a,b)\cdot I = \mathrm{in}(I) = I_\mu \tag{2.11}$$

for some partition μ of n.

Proof. Note that $x\lambda(p) = \lambda(xp)$ and $y\lambda(p) = \lambda(yp)$, which shows that $\mathrm{in}(I)$ is an ideal. Let $\mathscr{B}$ be the set of monomials not belonging to $\mathrm{in}(I)$. These monomials are linearly independent modulo I, else there would be a non-zero polynomial $p \in k\mathscr{B} \cap I$, which would force $\lambda(p) \in \mathscr{B} \cap \mathrm{in}(I)$. The monomials in $\mathscr{B}$ also span modulo I. Indeed, every monomial either belongs to $\mathscr{B}$ or is $\lambda(p)$ for some $p \in I$, so in either case is $\lambda(p)$ for some $p \in k\mathscr{B}+I$. This readily implies by induction on $<$ that every monomial belongs to $k\mathscr{B} + I$.

Having observed that $\mathscr{B}$ is a basis modulo I we see that in particular $\mathscr{B}$ has n elements, so the monomial ideal $\mathrm{in}(I)$ is I_μ for some partition μ of n, and $\mathscr{B} = \mathscr{B}_\mu$ for the same μ. As a consequence we can also conclude that $I \in U_\mu$.

Since $\mathrm{in}(I) = I_\mu$, the leading term of

$$x^r y^s - \sum_{(h,k)\in\mu} c^{rs}_{hk}(I)x^h y^k \in I \tag{2.12}$$

for $(r,s) \notin \mu$ must be $x^r y^s$, which implies that $c^{rs}_{hk} = 0$ unless $x^r y^s > x^h y^k$. In other words, $c^{rs}_{hk} = 0$ unless either (i) $k < s$ or (ii) $k = s$, $h < r$. For h, k, r, s satisfying these conditions we have

$$\lim_{a\to 0}\lim_{b\to 0}(a,b)c^{rs}_{hk} = \lim_{a\to 0}\lim_{b\to 0} a^{r-h}b^{s-k}c^{rs}_{hk} = 0, \tag{2.13}$$

so $\lim_{a\to 0}\lim_{b\to 0}(a,b)\cdot I = I_\mu$ as claimed. □

Finally, we have the following remarkable facts, peculiar to the two-dimensional setting.

Proposition 2.4. *The punctual Hilbert scheme $\boldsymbol{H}^n$ of $\boldsymbol{A}^2$ is smooth and irreducible, of dimension $2n$.*

Proof. Again, this is well-known but we give the proof since we want to have a completely explicit local system of regular parameters at I_μ.

It suffices to verify smoothness locally near each T^2 fixed point I_μ. The reason for this is that the singular locus is closed and T^2 stable, hence by Lemma 2.3 it must either be empty or contain some I_μ.

Under the Chow morphism π, it is easy to see that the image $\pi(U_\mu)$ is dense in $\mathrm{Sym}^n(A^2)$. Therefore U_μ has dimension at least $2n$. The maximal ideal $\mathfrak{m}$ of I_μ in $\mathcal{O}_{U_\mu}$ is given by

$$\mathfrak{m} = (c_{hk}^{rs} : (h, k) \in \mu,\ (r,s) \notin \mu). \tag{2.14}$$

(For $(r,s) \in \mu$, we have $c_{hk}^{rs} = 0$ identically for $(h, k) \neq (r,s)$, and $c_{rs}^{rs} = 1$, so we omit these c_{hk}^{rs} from the ideal.) We shall now find $2n$ of the coordinate functions c_{hk}^{rs} which span the cotangent space $\mathfrak{m}/\mathfrak{m}^2$. This will show that $\dim \mathfrak{m}/\mathfrak{m}^2 = 2n$, so H^n is smooth at I_μ.

We single out two special coordinate functions c_{hk}^{rs} for each square $(h, k) \in \mu$. Let (f,k) be the top square in column k and let (h,g) be the last square in row h. This given, let

$$u_{hk} = c_{f,k}^{h,g+1}, \qquad d_{hk} = c_{h,g}^{f+1,k}. \tag{2.15}$$

These will be our spanning parameters for $\mathfrak{m}/\mathfrak{m}^2$.

Multiplying (2.4) through by x, then expanding each term on the right by (2.4) again and comparing coefficients yields the identity

$$c_{hk}^{r+1,s} = \sum_{(h',k')\in\mu} c_{h'k'}^{rs} c_{hk}^{h'+1,k'} \tag{2.16}$$

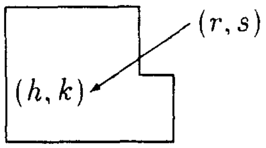
for all $(h, k) \in \mu$ and all (r,s). Proceeding similarly with y in place of x yields

$$c_{hk}^{r,s+1} = \sum_{(h',k')\in\mu} c_{h'k'}^{rs} c_{hk}^{h',k'+1}. \tag{2.17}$$

Modulo $\mathfrak{m}^2$, the terms $c_{h'k'}^{rs} c_{hk}^{h'+1,k'}$ on the right-hand side of (2.16) reduce to zero for $(h'+1,k') \notin \mu$ and for $(h'+1,k') \in \mu$, $(h'+1,k') \neq (h, k)$. The remaining term is $c_{h-1,k}^{rs}$, or zero if $h = 0$. Corresponding reductions apply to the right-hand side of (2.17). Thus in $\mathfrak{m}/\mathfrak{m}^2$ we have

$$\begin{aligned} c_{hk}^{r+1,s} &= c_{h-1,k}^{rs}, \quad \text{or } 0 \text{ if } h = 0; \\ c_{hk}^{r,s+1} &= c_{h,k-1}^{rs}, \quad \text{or } 0 \text{ if } k = 0. \end{aligned} \tag{2.18}$$

It is convenient to depict each c_{hk}^{rs} by an arrow from (r,s) to (h, k), as shown below:



(2.19)

Eqs. (2.18) say that we may move these arrows horizontally or vertically without changing their values modulo $\mathfrak{m}^2$, provided we keep the head inside μ and the tail outside. More generally, as long as we keep the tail in the first quadrant and outside μ, we may even move the head across the x-or y-axis. When this is possible, the value of the arrow is zero.

It is easy to see that a strictly southwest-pointing arrow can always be moved southeast until its head crosses the x-axis, so its value is zero. Moving a weakly northwest-pointing arrow as far northwest as it will go either carries its head across the y-axis or leaves it with its head at the top of a column in μ and its tail just outside the end of a row. At this point the arrow can neither move north nor west, and it represents one of the parameters u_{hk}. In similar fashion every weakly southeast-pointing arrow either can have its head moved across the x-axis, or else it becomes one of the parameters d_{hk}. Since there are no northeast-pointing arrows to begin with, this completes the demonstration that the u_{hk} and d_{hk} span $\mathfrak{m}/\mathfrak{m}^2$.

Finally, Lemma 2.3 shows that $\boldsymbol{H}^n$ is connected and since it is smooth it is therefore irreducible. □

From the proof we also have

Corollary 2.5. *The parameters u_{hk} and d_{hk} defined by* (2.15) *form a local system of regular parameters at I_μ.*

Remark. The set

$$C_\mu = \{I \in \boldsymbol{H}^n : \mathrm{in}(I) = I_\mu\} \tag{2.20}$$

is closed in U_μ and defined by the vanishing of all c_{hk}^{rs} for $x^r y^s < x^h y^k$, as is clear from the proof of Lemma 2.3. Locally at I_μ this reduces to the vanishing of the parameters $d_{h,k}$ for (h, k) not the last square in its row. This leaves $n + l(\mu)$ parameters free, where $l(\mu)$ is the number of rows, so $\dim C_\mu = n + l(\mu)$. In fact C_μ is an affine cell $C_\mu \cong \boldsymbol{A}^{n+l(\mu)}$. Ellingsrud and Strömme [6] used this cell decomposition to determine the cohomology of the Hilbert scheme over $\boldsymbol{C}$ as a complex manifold.

Up to here we have mainly reviewed known properties of the Hilbert scheme. At this point we sally into new territory.

First, we construct $\boldsymbol{H}^n$ as a blow-up of $\mathrm{Sym}^n(\boldsymbol{A}^2)$, permitting us to express the coordinate functions c_{hk}^{rs} in a useful way as ratios of S_n-alternating polynomials, analogous to Schur functions. This blow-up also provides a natural line bundle $\mathcal{O}(1)$, very ample for $\boldsymbol{H}^n$ as a scheme projective over $\mathrm{Sym}^n(\boldsymbol{A}^2)$.

Next, we investigate the *universal scheme* $\boldsymbol{H}_+^n \subseteq \boldsymbol{H}^n \times \boldsymbol{A}^2$, and the *zero fiber* $\boldsymbol{H}_0^n = \pi^{-1}(\underline{0})$. Both of these schemes are Cohen–Macaulay, and $\boldsymbol{H}_0^n$ is the isomorphic image of a complete intersection in $\boldsymbol{H}_+^n$. Using this we give an explicit resolution of the structure sheaf of $\boldsymbol{H}_0^n$.

Let $A = \{f \in k[X,Y] : \sigma f = \varepsilon(\sigma) \forall \sigma \in S_n\}$ be the space of S_n-alternating polynomials in $k[X,Y] = k[x_1,y_1,\dots,x_n,y_n]$. For each n-element subset $D = \{(p_1,q_1),\dots,(p_n,q_n)\}$ of $\mathbf{N} \times \mathbf{N}$, the determinant

$$\Delta_D(X,Y) = \det \left[x_i^{p_j} y_i^{q_j}\right]_{i,j=1}^n \tag{2.21}$$

is well-defined up to a change of sign and belongs to A. Moreover, if $\mathscr{D}$ denotes the collection of all n-element subsets $D \subseteq \mathbf{N} \times \mathbf{N}$, then

$$\{\Delta_D : D \in \mathscr{D}\} \tag{2.22}$$

is a basis of A. When D is the diagram of a partition μ we set

$$\Delta_\mu = \Delta_D. \tag{2.23}$$

Now let A^d be the space spanned by all products $f_1 f_2 \cdots f_d$, with the f_i in A, and for $d = 0$ take $A^0 = k[X,Y]^{S_n}$. Then for all d and e, $A^d A^e \subseteq A^{d+e}$, so there is a graded $k[X,Y]^{S_n}$-algebra

$$R = A^0 \oplus A^1 \oplus A^2 \oplus \cdots. \tag{2.24}$$

Proposition 2.6. *The scheme* $\operatorname{Proj} R$ *is isomorphic to* $\mathbf{H}^n$, *in such a way that the natural morphism* $\theta : \operatorname{Proj} R \to \operatorname{Spec} A^0 = \operatorname{Sym}^n(\mathbf{A}^2)$ *coincides with the Chow morphism* π.

Proof. Let Y be the open subset of $\operatorname{Sym}^n(\mathbf{A}^2)$ consisting of multisets $S \subseteq \mathbf{A}^2$ with n distinct elements. For each such S, there is a unique ideal $I \in \mathbf{H}^n$ with $\pi(I) = S$, namely, the defining ideal of S as a reduced subscheme of $\mathbf{A}^2$. Thus π maps $Y_1 = \pi^{-1}(Y) \subseteq \mathbf{H}^n$ bijectively onto Y.

Suppose $I \in U_\mu \cap Y_1$, with $S = \pi(I) = \{(x_1,y_1),\dots,(x_n,y_n)\}$. The monomials $x^h y^k \in \mathscr{B}_\mu$ must describe linearly independent functions on S, so $\Delta_\mu(x_1,y_1,\dots,x_n,y_n) \neq 0$. Although $\Delta_\mu(x_1,y_1,\dots,x_n,y_n)$ depends on the ordering chosen for the elements of S, $(\Delta_D/\Delta_\mu)(x_1,y_1,\dots,x_n,y_n)$ does not, for any $D \in \mathscr{D}$, and $\pi^*(\Delta_D/\Delta_\mu) = \pi^*(\Delta_D \Delta_\mu/\Delta_\mu^2)$ is a regular function on $U_\mu \cap Y_1$.

Setting $\mu = \{(h_1,k_1),\dots,(h_n,k_n)\}$, the coefficients $c_{hk}^{rs}(I)$ satisfy

$$\left[x_i^{h_j} y_i^{k_j}\right]_{i,j=1}^n \cdot \begin{pmatrix} c_{h_1k_1}^{rs} \\ c_{h_2k_2}^{rs} \\ \vdots \\ c_{h_nk_n}^{rs} \end{pmatrix} = \begin{pmatrix} x_1^r y_1^s \\ x_2^r y_2^s \\ \vdots \\ x_n^r y_n^s \end{pmatrix}. \tag{2.25}$$

Given a diagram $D = \{(p_i,q_i)\} \in \mathscr{D}$, combining Eqs. (2.25) for $(r,s) = (p_i,q_i)$ into a single matrix equation yields

$$\left[x_i^{h_j} y_i^{k_j}\right]_{i,j=1}^n \cdot \left[c_{h_jk_j}^{p_kq_k}\right]_{j,k=1}^n = \left[x_i^{p_k} y_i^{q_k}\right]_{i,k=1}^n. \tag{2.26}$$

Taking determinants of this gives the identity

$$\pi^* \frac{\Delta_D}{\Delta_\mu} = \det \left[c_{h_j k_j}^{p_k q_k} \right]_{j,k=1}^{n}, \tag{2.27}$$

on $U_\mu \cap Y_1$, showing that $\pi^*(\Delta_D/\Delta_\mu)$ extends to a regular function f_D on all of U_μ.

For every two diagrams $D_1, D_2 \in \mathscr{D}$, the identity

$$\pi^* \Delta_{D_1} \Delta_{D_2} = f_{D_1} f_{D_2} \pi^* \Delta_\mu^2 \tag{2.28}$$

holds on $U_\mu \cap Y_1$ and therefore, since Y_1 is dense, on all of U_μ. This shows that $\pi^* A^2$ is locally the principal ideal (Δ_μ^2) in $\mathcal{O}_{U_\mu}$.

Now $\operatorname{Proj} R \cong \operatorname{Proj} R^{(2)} = \operatorname{Proj}(A^0 \oplus A^2 \oplus A^4 \oplus \cdots)$ is the same as the blow-up of $\operatorname{Sym}^n(\boldsymbol{A}^2)$ along the subscheme defined by the ideal $A^2 \subseteq k[X,Y]^{S_n}$. From the universal property of blowing up it follows that there is a unique morphism $\alpha : \boldsymbol{H}^n \to \operatorname{Proj} R$ such that $\theta \circ \alpha = \pi$. This morphism is surjective, since $\operatorname{Proj} R$ is irreducible (R is an integral domain) and birational to $\operatorname{Sym}^n(\boldsymbol{A}^2)$, while $\boldsymbol{H}^n$ is projective over $\operatorname{Sym}^n(\boldsymbol{A}^2)$. To show it is an isomorphism, it remains only to prove that α is an embedding, i.e., that the natural sheaf homomorphism $\alpha^* \mathcal{O}_{\operatorname{Proj} R} \to \mathcal{O}_{\boldsymbol{H}^n}$ is surjective.

Solving Eqs. (2.25) by Cramer's rule gives

$$c_{hk}^{rs}(I) = \frac{\Delta_D}{\Delta_\mu}(S) \tag{2.29}$$

with $D = \mu \setminus (h,k) \cup (r,s)$. Thus on $U_\mu \cap Y_1$, $c_{hk}^{rs} = \pi^*(\Delta_D/\Delta_\mu) = \alpha^* \theta^* (\Delta_D/\Delta_\mu)$. Since the c_{hk}^{rs} generate $\mathcal{O}_{U_\mu}$, it follows that α restricted to the closure of Y_1 is an embedding. But $\boldsymbol{H}^n$ is irreducible, so Y_1 is dense. □

Remark. (1) It seems quite likely that A^2 is the ideal of the complement Y^c of Y in $\operatorname{Sym}^n(\boldsymbol{A}^2)$, that is, of the locus where two or more points of the multiset coincide. It is easy to see that the radical $\sqrt{A^2}$ is the ideal of Y^c, but we have not managed to prove that A^2 is a radical ideal.

(2) In general, $\operatorname{Hilb}^n(\boldsymbol{A}^m)$ is not irreducible. However, if Y_1 denotes the set of ideals $I \in \operatorname{Hilb}^n(\boldsymbol{A}^m)$ corresponding to reduced subschemes of $\boldsymbol{A}^m$ with n points, then the closure of Y_1 is an irreducible component of $\operatorname{Hilb}^n(\boldsymbol{A}^m)$. Proposition 2.6 applies to this *principal component* of the Hilbert scheme, with R constructed from $k[X^{(1)}, X^{(2)}, \ldots, X^{(m)}]$ in place of $k[X,Y]$. We suspect that the principal component of the Hilbert scheme should enjoy better geometric properties than the Hilbert scheme itself. For example, we have verified using Macaulay [3] that $\operatorname{Hilb}^4(\boldsymbol{A}^3)$, which is equal to its principal component, has Gorenstein singularities. By contrast, $\operatorname{Hilb}^n(\boldsymbol{A}^m)$ in general is not even equidimensional.

Proposition 2.6 provides us with the following useful representation of the coordinate ring of U_μ.

Corollary 2.7. *We have*

$$\mathcal{O}_{U_\mu} \cong k\left[\frac{\Delta_D}{\Delta_\mu} : D \in \mathscr{D}\right]. \tag{2.30}$$

It may be instructive to work everything out in the case of $U_{(1^n)}$, the set of ideals modulo which $\{1,x,\dots,x^{n-1}\}$ is a basis. Such an ideal I is generated by two polynomials

$$\begin{aligned} &x^n - e_1x^{n-1} + e_2x^{n-2} - \cdots \pm e_{n-1}x \mp e_n,\\ &y - (a_0 + a_1x + \cdots + a_{n-1}x^{n-1}). \end{aligned} \tag{2.31}$$

When $I \in U_{(1^n)}$ is the ideal of a reduced subscheme $S = \{(x_1,y_1),\dots,(x_n,y_n)\}$, the x-coordinates x_i are all distinct, and the first polynomial above, which must vanish when $x = x_i$, becomes

$$\prod_i (x - x_i), \tag{2.32}$$

so the coefficients e_r are the elementary symmetric functions $e_r(X)$. The second polynomial in (2.31) is $y - \phi_a(x)$, where $\phi_a(x)$ is a *Langrange interpolation polynomial* — the unique polynomial of degree $n-1$ satisfying $\phi_a(x_i) = y_i$ for $i = 1,\dots,n$.

For $\mu = (1^n)$, Δ_μ is the Vandermonde determinant $\Delta(X) = \prod_{i<j}(x_i - x_j)$. Given any $\Delta_D \in A$, we may substitute into it $y_i = \phi_a(x_i)$ for each i to obtain a polynomial $g(x_1,\dots,x_n,a_0,\dots,a_{n-1})$ which is alternating in the variables x_i and is therefore of the form $\Delta(X)f(e_1,\dots,e_n,a_0,\dots,a_{n-1})$. Thus the isomorphism (2.30) is given explicitly by

$$\frac{\Delta_D}{\Delta_{(1^n)}}(x_1,y_1,\dots,x_n,y_n) = f(e_1,\dots,e_n,a_0,\dots,a_{n-1}). \tag{2.33}$$

In particular, this formula relates the local parameters u_{hk} and d_{hk} given by Corollary 2.5 to the parameters a_r and e_r. We may summarize as follows.

Corollary 2.8. *The open affine set $U_{(1^n)}$ is an affine cell* Spec $k[a_0,\dots,a_{n-1},e_1,\dots,e_n]$. *The isomorphism of Corollary 2.7 carries e_r to the rth elementary symmetric function $e_r(x_1,\dots,x_n)$ and a_0 through a_{n-1} to the coefficients of the Lagrange interpolation polynomial defined by $y_i = a_0 + a_1x_i + \cdots + a_{n-1}x_i^{n-1}$ for all i. The local parameters of Corollary 2.5 become $u_{r0} = \sum_{j=0}^{r} a_{n-1-j}h_{r-j}$ and $d_{r0} = (-1)^{n-r+1}e_{n-r}$, where h_{r-j} is the complete homogeneous symmetric function of degree $r-j$, regarded as a polynomial in the elementary symmetric functions e_i.*

Definition. The *universal scheme* $\boldsymbol{H}^n_+$ is the unique subscheme of $\boldsymbol{H}^n \times \boldsymbol{A}^2$ such that the projection

$$\eta : \boldsymbol{H}^n_+ \to \boldsymbol{H}^n \tag{2.34}$$

is flat, and for each $I \in \boldsymbol{H}^n$ the scheme-theoretic fiber $\eta^{-1}(I)$ is the subscheme $\operatorname{Spec}(k[x,y]/I)$ of $\boldsymbol{A}^2$.

The universal scheme exists, and derives its name from the universal property of $\boldsymbol{H}^n$, namely, given any flat family F of length n subschemes of $\boldsymbol{A}^2$ parametrized by a scheme T, there is a unique morphism $T \to \boldsymbol{H}^n$ such that $F = T \times_{\boldsymbol{H}^n} \boldsymbol{H}^n_+$. We summarize some elementary properties of $\boldsymbol{H}^n_+$ as follows.

Proposition 2.9. *The scheme $\boldsymbol{H}^n_+$ is the reduced subscheme*

$$\boldsymbol{H}^n_+ = \{(I, p) \in \boldsymbol{H}^n \times \boldsymbol{A}^2 : p \in \pi(I)\}. \tag{2.35}$$

It is Cohen–Macaulay, flat and finite over $\boldsymbol{H}^n$. Its ideal sheaf as a subscheme of $\boldsymbol{H}^n \times \boldsymbol{A}^2$ is generated locally on $U_\mu \times \boldsymbol{A}^2$ by the equations

$$x^r y^s - \sum_{(h,k)\in\mu} c^{rs}_{hk} x^h y^k, \tag{2.36}$$

where x, y are the coordinate functions on $\boldsymbol{A}^2$ and c^{rs}_{hk} are those on U_μ.

Proof. The projection $\eta: \boldsymbol{H}^n_+ \to \boldsymbol{H}^n$ is flat and finite by definition, and since $\boldsymbol{H}^n$ is smooth, this implies that $\boldsymbol{H}^n_+$ is Cohen–Macaulay.

The identity (2.35) clearly holds set-theoretically. As before, let $Y_1 \subseteq \boldsymbol{H}^n$ be the set of ideals corresponding to reduced subschemes. Then $\boldsymbol{H}^n_+$ is reduced on the open set $\eta^{-1}(Y_1)$. Since η is flat, Y_1 is dense, and $\boldsymbol{H}^n$ is reduced, it follows that $\boldsymbol{H}^n_+$ is reduced.

Let for the moment X denote the subscheme defined locally on $U_\mu \times \boldsymbol{A}^2$ by Eqs. (2.36). These equations when specialized to $c^{rs}_{hk} = c^{rs}_{hk}(I)$ generate I, so the scheme-theoretic fiber of X over I is $\mathrm{Spec}(k[x, y]/I)$. In particular, X is well-defined where different sets U_μ overlap and flat over $\boldsymbol{H}^n$. Hence X is equal to $\boldsymbol{H}^n_+$. □

Remark. In general, the universal scheme $\mathrm{Hilb}^n_+(\boldsymbol{A}^m)$ need not be reduced, a phenomenon related to the reducibility of $\mathrm{Hilb}^n(\boldsymbol{A}^m)$.

Definition. The *zero fiber* $\boldsymbol{H}^n_0$ is the reduced subscheme $\pi^{-1}(\underline{0}) \subseteq \boldsymbol{H}^n$, where $\underline{0}$ denotes the multiset $\{n \cdot (0, 0)\}$.

The projection $\eta : \boldsymbol{H}^n_+ \to \boldsymbol{H}^n$ maps the fiber scheme $\eta^{-1}(\boldsymbol{H}^n_0)$ bijectively onto $\boldsymbol{H}^n_0$, but not isomorphically, since $\eta^{-1}(\boldsymbol{H}^n_0)$ is not reduced (its fiber over I is $\mathrm{Spec}\, k[x, y]/I$). The corresponding *reduced* subscheme $\eta^{-1}(\boldsymbol{H}^n_0)^{\mathrm{red}}$, however, *does* map isomorphically onto $\boldsymbol{H}^n_0$. Much more is true, as the following proposition shows.

Proposition 2.10. *The projection η maps the reduced fiber scheme $\eta^{-1}(\boldsymbol{H}^n_0)^{\mathrm{red}}$ isomorphically onto $\boldsymbol{H}^n_0$. Moreover, $\eta^{-1}(\boldsymbol{H}^n_0)^{\mathrm{red}}$ is a complete intersection in $\boldsymbol{H}^n_+$, defined locally on $\eta^{-1}(U_\mu)$ by the ideal*

$$\mathscr{J} = (x, y, p_{r,s} : (r, s) \in \mu \setminus (0, 0)), \tag{2.37}$$

where $p_{r,s} \in k[X,Y]^{S_n}$ is the polarized power sum defined in (2.8). *In particular $\boldsymbol{H}_0^n$ is Cohen–Macaulay. Viewed as the fiber scheme $\pi^{-1}(\underline{0})$, $\boldsymbol{H}_0^n$ is scheme-theoretically reduced and irreducible.*

Proof. By definition, for every $I \in \boldsymbol{H}_0^n$, the subscheme $\operatorname{Spec} k[x,y]/I \subseteq \boldsymbol{A}^2$ is concentrated at the origin, so $\eta^{-1}(\boldsymbol{H}_0^n)^{\mathrm{red}}$ is contained in the reduced subscheme $\boldsymbol{H}^n \times \{0\}$ of $\boldsymbol{H}^n \times \boldsymbol{A}^2$. As a reduced subscheme of $\boldsymbol{H}^n \times \{0\}$, it is obviously equal to $\boldsymbol{H}_0^n \times \{0\}$ and thus projects isomorphically onto $\boldsymbol{H}_0^n$.

The irreducibility of $\boldsymbol{H}_0^n$ is proved in [4], together with the fact that $\dim \boldsymbol{H}_0^n = n-1$. Since η is finite, the dimension of $\boldsymbol{H}_+^n$ is that of $\boldsymbol{H}^n$, namely $2n$. The ideal $\mathscr{I}$ is given by $n+1$ generators, so if we show it locally defines $\eta^{-1}(\boldsymbol{H}_0^n)^{\mathrm{red}}$, even if only set-theoretically, then it must be a complete intersection ideal.

Since $\mathscr{I}$ contains x and y, the subscheme $V(\mathscr{I})$ that it defines is contained in $\boldsymbol{H}^n \times \{0\}$, and under the trivial isomorphism $\boldsymbol{H}^n \times \{0\} \cong \boldsymbol{H}^n$, $V(\mathscr{I})$ corresponds to the subscheme of $\boldsymbol{H}^n$ defined locally on U_μ by $(p_{r,s} : (r,s) \in \mu \setminus (0,0))$. Likewise, letting $\mathscr{I}'$ denote the ideal of the same form (2.37) as $\mathscr{I}$ but without the restriction $(r,s) \in \mu$, $V(\mathscr{I}')$ projects isomorphically onto the subscheme of $\boldsymbol{H}^n$ defined by the ideal $(p_{r,s} : (r,s) \neq (0,0))$. Weyl's theorem, mentioned earlier in the proof of Proposition 2.2, implies that this latter is the ideal of $\underline{0}$ in $k[X,Y]^{S_n}$ and thus also of the scheme-theoretic fiber $\pi^{-1}(\underline{0})$ in $\boldsymbol{H}^n$.

Suppose we can show that $\mathscr{I} = \mathscr{I}'$ on $\eta^{-1}(U_\mu)$. Then $V(\mathscr{I}) = V(\mathscr{I}')$ projects isomorphically onto $\pi^{-1}(\underline{0})$, which as pointed out above makes $V(\mathscr{I})$ a complete intersection. Being isomorphic to a local complete intersection in $\boldsymbol{H}_+^n$, $\pi^{-1}(\underline{0})$ must then be Cohen–Macaulay. Now in the explicit coordinates on $U_{(1^n)}$ given by Corollary 2.8, $\pi^{-1}(\underline{0})$ is easily seen to be defined by the ideal $(a_0, e_1, \ldots, e_n)$, so it is generically reduced, hence reduced, hence equal to $\boldsymbol{H}_0^n$, establishing everything asserted.

What remains to prove is that the additional generators $p_{r,s}$, $(r,s) \notin \mu$ of $\mathscr{I}'$ are redundant. For this, note that when $(x_i, y_i) \in \pi(I)$ we must have $x_i^r y_i^s = \sum_{(h,k)\in\mu} c_{hk}^{rs}(I)\, x_i^h y_i^k$. Summing on i from 1 to n we find that

$$p_{r,s} = \sum_{(h,k)\in\mu} c_{hk}^{rs} p_{h,k}, \tag{2.38}$$

identically on $\boldsymbol{H}^n$ and therefore also on $\boldsymbol{H}^n \times \boldsymbol{A}^2$. This equation is not quite enough to eliminate $p_{r,s}$, since it contains a term $c_{00}^{rs} p_{0,0}$ which is not obviously in the given ideal. But we also have on $\boldsymbol{H}_+^n$ the identity

$$x^r y^s = \sum_{(h,k)\in\mu} c_{hk}^{rs} x^h y^k. \tag{2.39}$$

The left-hand side and all terms except $(h,k) = (0,0)$ on the right belong to the ideal (x,y), so the remaining term c_{00}^{rs} belongs to (x,y) as well, completing the proof. □

Corollary 2.11. *The structure sheaf $\mathcal{O}_{H_0^n}$, regarded as an $\mathcal{O}_{H^n}$ module, has a local T^2 equivariant minimal free resolution at I_μ*

$$0 \to F_{n+1} \to \cdots \to F_1 \to F_0 \to \mathcal{O}_{H_0^n} \to 0 \tag{2.40}$$

with $F_i = B \otimes \bigwedge^i(B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q)$, where B is a free module with basis $\mathscr{B}_\mu$, B' is the same with the basis element $x^0 y^0$ omitted, and $\mathcal{O}_t$, $\mathcal{O}_q$ denote $\mathcal{O}_{H^n}$ with trivial torus action multiplied by the 1-dimensional characters t and q, respectively.

Proof. By Proposition 2.9, $\eta_* \mathcal{O}_{H_+^n} \cong B$ locally. The free module on the local generators of the ideal in (2.37) is clearly isomorphic to $B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q$. Since the ideal is a complete intersection, the Koszul resolution is a minimal free resolution, whose terms are the given F_i. □

Remark. The free resolution in (2.40) can easily be made global. We already have the 'tautological' bundle $B = \eta_* \mathcal{O}_{H_+^n}$ defined globally, and the finite morphism η gives us a trace map $B \to \mathcal{O}$ that splits the natural map $\mathcal{O} \to B$ (recall we are assuming the characteristic of k does not divide n). The complementary summand $B/\mathcal{O}$ is the required B', as the trace map sends the section of B represented by $x^h y^k$ to $p_{h,k}$.

To conclude, we study the line bundle $\mathcal{O}(1)$ associated to our explicit representation of H^n as a blow-up.

Proposition 2.12. *Let $\mathcal{O}(1)$ be the ample line bundle arising from the projective embedding $H^n = \operatorname{Proj} R$ of H^n as a variety projective over $\operatorname{Sym}^n(A^2)$ given by Proposition 2.6. Let $B = \eta_*(\mathcal{O}_{H_+^n})$ be the tautological bundle. Then $\mathcal{O}(1)$ is isomorphic to the highest exterior power $\bigwedge^n B$.*

Proof. For each μ, the element Δ_μ of A^1 defines a global section s_μ of $\mathcal{O}(1)$, which generates $\mathcal{O}(1)$ locally on $U_\mu = \operatorname{Spec} k[\Delta_D/\Delta_\mu : D \in \mathscr{D}]$. Where different open affines U_μ, U_ν overlap, these sections are of course related by

$$s_\nu = (\Delta_\nu/\Delta_\mu) s_\mu; \tag{2.41}$$

note that both (Δ_ν/Δ_μ) and its inverse are regular on $U_\mu \cap U_\nu$.

We need to define corresponding sections t_μ of $\bigwedge^n B$. To this end, let us fix the squares in the diagram of μ to be $(h_1, k_1), \ldots, (h_n, k_n)$ in some definite order, and agree to fix the sign of Δ_μ by taking the columns of the determinant in this order. On U_μ, the monomials $x^{h_i} y^{k_i}$ represent a basis of sections of B, and so

$$t_\mu = x^{h_1} y^{k_1} \wedge x^{h_2} y^{k_2} \wedge \cdots \wedge x^{h_n} y^{k_n} \tag{2.42}$$

represents a generating section of $\bigwedge^n B$. Note that an alternative choice of ordering would produce identical sign changes in both t_μ and s_μ; thus there is a well-defined local isomorphism $\beta_\mu : \mathcal{O}(1) \to \bigwedge^n B$ sending s_μ to t_μ on U_μ.

We must now prove that the local isomorphisms β_μ are compatible, which is to say, that

$$t_\nu = (\Delta_\nu/\Delta_\mu)t_\mu \tag{2.43}$$

on $U_\mu \cap U_\nu$. Now t_ν/t_μ is computed on U_μ as follows. For each $(r_i, s_i) \in \nu$, apply the identity

$$x^{r_i}y^{s_i} = \sum_{(h_j,k_j)\in\mu} c^{r_i s_i}_{h_j k_j} x^{h_j} y^{k_j}, \tag{2.44}$$

valid on B, to each factor of the exterior product

$$t_\nu = x^{r_1}y^{s_1} \wedge x^{r_2}y^{s_2} \wedge \cdots \wedge x^{r_n}y^{s_n} \tag{2.45}$$

and expand. This gives

$$t_\nu/t_\mu = \det\left[c^{r_i s_i}_{h_j k_j}\right]^n_{i,j=1}, \tag{2.46}$$

which we have seen in (2.27) is equal to Δ_ν/Δ_μ. □

Every S_n-alternating polynomial Δ defines a global section of $\mathcal{O}(1)$, and this correspondence is actually an isomorphism of A with $H^0(\boldsymbol{H}^n, \mathcal{O}(1))$. The reason for this is that $H^0(\boldsymbol{H}^n, \mathcal{O}(1))$ is the degree 1 part of the integral closure of the ring $R = A^0 \oplus A^1 \oplus \cdots$. It seems very probable that R is already integrally closed, but at any rate the ring $R' = A^0 \oplus A^1 \oplus A^0 \oplus A^1 \oplus \cdots$ is isomorphic to the ring of invariants $k[X, Y, s]^{S_n}$, where the S_n action is extended from $k[X, Y]$ by letting $\sigma(s) = \varepsilon(\sigma)s$ for $\sigma \in S_n$. Thus R' is integrally closed, and the two rings R and R' have the same degree 1 part.

If the polynomial Δ belongs to JA, where

$$J = (p_{h,k} : (h, k) \neq (0, 0)) \tag{2.47}$$

is the ideal of $\underline{0}$ in $k[X, Y]^{S_n}$, then the corresponding global section of $\mathcal{O}(1)$ must vanish upon restriction to the zero fiber $\boldsymbol{H}^n_0$. Thus there is a well-defined map $A/JA \to H^0(\boldsymbol{H}^n_0, \mathcal{O}(1))$, which we expect, but do not prove, to be an isomorphism. Similarly there are maps $A^m/JA^m \to H^0(\boldsymbol{H}^n_0, \mathcal{O}(m))$ which definitely are isomorphisms for large m. We summarize this as follows.

Proposition 2.13. *Let $\mathfrak{A}$, $\mathfrak{m}$ be respectively the ideal generated by all S_n-alternating polynomials, and the maximal ideal $(x_1, y_1, \ldots, x_n, y_n)$ in $k[X, Y]$. Then there are natural maps*

$$\mathfrak{A}^m/\mathfrak{m}\mathfrak{A}^m \to H^0(\boldsymbol{H}^n_0, \mathcal{O}(m)), \tag{2.48}$$

and they are isomorphisms for all sufficiently large m. Of course we also have

$$H^i(\boldsymbol{H}^n_0, \mathcal{O}(m)) = 0 \quad \textit{for all } i > 0, \tag{2.49}$$

for all sufficiently large m.

Proof. The last part is Serre's vanishing theorem, because $\mathcal{O}(1)$ is ample for the projective variety $\boldsymbol{H}_0^n$.

Since we have shown that $\boldsymbol{H}_0^n$, viewed as $\pi^{-1}(\underline{0})$, is scheme-theoretically reduced, we have $\boldsymbol{H}_0^n = \mathrm{Proj}(R/JR)$, where J is the ideal in (2.47). This implies that for all sufficiently large m we have $H^0(\boldsymbol{H}_0^n, \mathcal{O}(m)) \cong (R/JR)_m = A^m/JA^m$.

Finally, note that $\mathfrak{A}^m$ is obviously generated by its S_n-alternating or S_n-invariant elements, respectively, depending whether m is odd or even. For simplicity let us refer to these elements as having *correct parity*. One easily proves by induction on m that A^m is the set of elements with correct parity in $\mathfrak{A}^m$. It follows that the natural map $A^m/JA^m \to \mathfrak{A}^m/\mathfrak{m}\mathfrak{A}^m$ is surjective. One also proves easily that JA^m is the set of elements with correct parity in $\mathfrak{m}\mathfrak{A}^m$, and therefore the map $A^m/JA^m \to \mathfrak{A}^m/\mathfrak{m}\mathfrak{A}^m$ is also injective. □

Remark. Conditions (2.48) and (2.49) must hold for *all* $m \geqslant 0$ if the ring R/JR or, what is the same, the Rees algebra fiber

$$S = k[X, Y, s\mathfrak{A}]/\mathfrak{m}k[X, Y, s\mathfrak{A}], \tag{2.50}$$

is a Cohen–Macaulay ring. To see this, note that $\boldsymbol{H}_0^n = \mathrm{Proj}\ S$ implies that, apart from a possible embedded component at the homogeneous maximal ideal $\mathfrak{n}$, Spec S is the affine cone over $\boldsymbol{H}_0^n$ in its projective embedding induced by $\mathcal{O}(1)$. Therefore by the long exact sequence for local cohomology [14], the local cohomology modules $H_\mathfrak{n}^i(S)$ are given by

$$0 \to H_\mathfrak{n}^0(S) \to S \to \bigoplus_{m \in \boldsymbol{Z}} H^0(\boldsymbol{H}_0^n, \mathcal{O}(m)) \to H_\mathfrak{n}^1(S) \to 0; \tag{2.51}$$

$$H_\mathfrak{n}^{i+1}(S) \cong \bigoplus_{m \in \boldsymbol{Z}} H^i(\boldsymbol{H}_0^n, \mathcal{O}(m)) \quad \text{for } i > 0. \tag{2.52}$$

The vanishing of $H_\mathfrak{n}^0(S)$ and $H_\mathfrak{n}^1(S)$ then gives (2.48), while the vanishing of $H_\mathfrak{n}^i(S)$ for $i < n$ gives (2.49) for $i < n - 1$.

To obtain (2.49) for $i = n - 1$, we must show that the local cohomology module $H_\mathfrak{n}^n(S)$ vanishes in degrees zero and above. To this end let

$$h(m) = \dim(\mathfrak{A}^m/\mathfrak{m}\mathfrak{A}^m) = \dim\,(S)_m\,; \qquad g(m) = \dim(H_\mathfrak{n}^n(S))_m \tag{2.53}$$

be the Hilbert functions of S and $H_\mathfrak{n}^n(S)$ respectively. Then, still assuming S is Cohen–Macaulay, $h(m) + (-1)^{n-1}g(m)$ is given by a polynomial in m for all $m \in \boldsymbol{Z}$ [19]. Our main theorem, Theorem 2 below, holds for all large m and so shows that this polynomial must be

$$h(m) + (-1)^{n-1}g(m) = C_n^{(m)}(1,1) = \frac{1}{mn+1}\binom{(m+1)n}{n}. \tag{2.54}$$

Now this vanishes for $m = -1$, and of course $h(m)$ vanishes for $m < 0$, so we have $g(-1) = 0$, which implies that $H_\mathfrak{n}^m(S)$ must vanish in degrees -1 and above.

If S is Cohen–Macaulay, then $\boldsymbol{H}_0^n$ must also obey the *Kodaira vanishing theorem*

$$H^i(\boldsymbol{H}_0^n, \mathcal{O}(-m)) = 0 \quad \text{for all } i < n-1,\ m > 0. \tag{2.55}$$

Conversely, conditions (2.48) and (2.49) for all $m \geqslant 0$, together with (2.55), imply that S is Cohen–Macaulay.

3. The Atiyah–Bott Lefschetz formula

We are now ready to use the results of Section 2 to write down explicitly the Atiyah–Bott formula for the Euler characteristic of the line bundle $\mathcal{O}(m)$ on the zero fiber of the Hilbert scheme.

We begin with a statement of the theorem in the form we need. Here we shall assume that the ground field k is $\boldsymbol{C}$, which is sufficient for our application.

Theorem 1. *Let X be a smooth d-dimensional complex projective variety on which the torus group $T = \boldsymbol{T}^l$ acts algebraically. Assume that X possesses a T-equivariant ample sheaf $\mathcal{O}(1)$. Assume also that the fixed point set X^T is finite. For each point $x \in X^T$ let $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ denote the structure sheaf of the point x, regarded as an $\mathcal{O}_X$ module, and let $C(x) = \mathfrak{m}_x/\mathfrak{m}_x^2$ denote the cotangent space to X at x.*

For every T-equivariant coherent sheaf of $\mathcal{O}_X$ modules E we have, as an identity of rational functions of $\tau \in T$,

$$\sum_{i=0}^{d} (-1)^i \operatorname{tr}_{H^i(X,E)}(\tau) = \sum_{x \in X^T} \frac{\sum_{i=0}^{d} (-1)^i \operatorname{tr}_{\operatorname{Tor}_i(k(x),E)}(\tau)}{\det_{C(x)}(1-\tau)}. \tag{3.1}$$

Proof. If the sheaf E is locally free, this is the original theorem stated in [1]. By the long exact sequences for cohomology and Tor, each side of (3.1) is additive, in the sense that for an exact sequence

$$0 \to E' \to E \to E'' \to 0, \tag{3.2}$$

the expression for E is the sum of the corresponding expressions for E' and E''. This additivity reduces the theorem to the locally free case, provided that E has a finite T-equivariant locally free resolution.

We can construct the desired resolution along the lines of the usual proof of the syzygy theorem for a smooth projective variety. For some large enough m, $E \otimes \mathcal{O}(m)$ is generated by its global sections. Its space of global sections is a finite-dimensional representation of T (as we are assuming $\mathcal{O}(1)$ T-equivariant), so it has a basis of T-eigensections. Each eigensection gives a T-equivariant homomorphism $\mathcal{O}(-m) \to E$, and combining them we have a T-equivariant epimorphism

$$\mathcal{O}(-m)^r \to E \tag{3.3}$$

for some r.

Now replace E by the kernel of (3.3), which is again T-equivariant, and iterate the construction. By the syzygy theorem for a regular local ring, the d-th kernel is itself locally free, and we are finished. □

Remark. By reduction to a finite cyclic group action, one can eliminate the need to assume that $\mathcal{O}(1)$ is T-equivariant. The same reduction and the results of Baum et al. [3] allow one to generalize the theorem to arbitrary characteristic, replacing the k-valued trace with a formal character. For locally free sheaves one can also generalize to singular varities, although the terms on the right-hand side then become less simple and explicit.

We now come to our main theorem.

Theorem 2. *The (t,q)-Catalan numbers $C_n^{(m)}(t,q)$ defined by (1.10) are equal to*

$$C_n^{(m)}(t,q) = \sum_{i=0}^{n-1}(-1)^i \operatorname{tr}_{H^i(\boldsymbol{H}_0^n,\mathcal{O}(m))}(t,q). \tag{3.4}$$

In particular, for $m \geqslant 0$ they are polynomials in t and q with integer coefficients, and when (2.49) holds, $C_n^{(m)}(t,q)$ is the Hilbert polynomial of $H^0(\boldsymbol{H}_0^n,\mathcal{O}(m))$.

Proof. We apply the Atiyah–Bott formula to the projective variety $X = \operatorname{Hilb}^n(\boldsymbol{P}^2)$, $T = \boldsymbol{T}^2$ and the coherent sheaf $E = \mathcal{O}_{\boldsymbol{H}_0^n}(m) = \mathcal{O}_{\boldsymbol{H}_0^n} \otimes \mathcal{O}(m)$, whose cohomology groups are $H^i(\boldsymbol{H}_0^n,\mathcal{O}(m))$. We must show that the right-hand side of (3.1) amounts to our formula (1.10). Although X has some T fixed points outside $\operatorname{Hilb}^n(\boldsymbol{A}^2)$, their contribution is zero, since the sheaf $\mathcal{O}_{\boldsymbol{H}_0^n}$ is zero there. So we only have to consider terms corresponding to the fixed points I_μ of $\boldsymbol{H}^n$.

By Corollary 2.5 the local parameters u_{hk} and d_{hk} are a basis of $\boldsymbol{T}^2$ eigenfunctions for the cotangent space of $\boldsymbol{H}^n$ at I_μ. For a square $x = (h, k)$ in μ, we see from (2.15) that d_{hk} has bi-degree $(1 + l(x), -a(x))$, or torus character $t^{1+l(x)}q^{-a(x)}$, while u_{hk} has character $t^{-l(x)}q^{1+a(x)}$. These eigenvalues account for the product in the denominator in (1.10).

For the numerator we use the resolution of $\mathcal{O}_{\boldsymbol{H}_0^n}$ in Corollary 2.11, noting that

$$\sum_i(-1)^i \operatorname{tr}_{Tor_i(k(I_\mu),\mathcal{O}_{\boldsymbol{H}_0^n})}(\tau) = \sum_i(-1)^k \operatorname{tr}_{F_i(I_\mu)}(\tau), \tag{3.5}$$

where $F_i(I_\mu) = F_i \otimes k(I_\mu)$ is the fiber of F_i at I_μ.

Each F_i involves the tautological bundle B as a tensor factor. The trace of (t,q) on $B(I_\mu)$ is

$$\sum_{(h,k)\in\mu} t^h q^k = \sum_{x\in\mu} t^{l'(x)}q^{a'(x)}, \tag{3.6}$$

since $\{x^h y^k : (h,k) \in \mu\}$ is a basis of $B(I_\mu) = k[x,y]/I_\mu$. The remaining factor in F_i is the ith exterior power of $B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q$. Now it is easy to see that for any space V and linear endomorphism τ we have

$$\sum_i (-1)^i \operatorname{tr}_{\wedge^i V}(\tau) = \det{}_V(1-\tau). \tag{3.7}$$

In our case V is the fiber of $B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q$ at I_μ, which has the $\boldsymbol{T}^2$ eigenvalues t, q, and $\{t^h q^k : (h,k) \in \mu, (h,k) \neq (0,0)\}$. Thus the above determinant becomes

$$(1-t)(1-q) \prod_{x \in \mu \setminus (0,0)} (1 - t^{l'(x)} q^{a'(x)}). \tag{3.8}$$

Finally, we are tensoring $\mathcal{O}_{\boldsymbol{H}_0^n}$ by $\mathcal{O}(m)$, which multiplies the term for the fixed point I_μ by the torus character of $\mathcal{O}(m) \otimes k(I_\mu)$. Since $\mathcal{O}(1)$ is the nth exterior power of B, the character of its fiber at I_μ is

$$\prod_{x \in \mu} t^{l'(x)} q^{a'(x)} = t^{n(\mu)} q^{n(\mu')}, \tag{3.9}$$

and that of its mth tensor power $\mathcal{O}(m)$ is

$$t^{mn(\mu)} q^{mn(\mu')}. \tag{3.10}$$

Multiplying together (3.6), (3.8) and (3.10) we obtain the numerator in (1.10), and we are finished.

Or almost finished, since what this really shows is only that $C_n^{(m)}(t,q)$ is a *Laurent polynomial* in t and q. However, by expanding (1.10) as a series in t with coefficients functions of q, we can easily see that this Laurent polynomial in fact contains no negative powers of t, and by symmetry, neither does it contain negative powers of q. □

We close with a few comments concerning the cohomology vanishing hypothesis (2.49). We can prove, by explicit computation on $U_{(1^n)} \cup U_{(n)}$, using Corollary 2.8, that the canonical sheaf of $\boldsymbol{H}^n$ is isomorphic to its structure sheaf $\mathcal{O}$. Knowing this, we can apply the criterion of Mehta–Ramanathan [18] to prove that $\boldsymbol{H}^n$ has a *Frobenius splitting* in prime characteristic. This implies the vanishing of higher cohomology for all the sheaves $\mathcal{O}(m)$, $m > 0$, on $\boldsymbol{H}^n$, both in positive characteristics and characteristic zero. It is also known that the higher cohomology of $\mathcal{O}_{\boldsymbol{H}^n}$ vanishes.

If the zero fiber $\boldsymbol{H}_0^n$ were also Frobenius split, we would immediately have the result we desire. Unfortunately, the Mehta–Ramanathan criterion (adapted to singular Cohen–Macaulay varieties using duality theory) can be used in reverse to prove that for all but a few small values of n, $\boldsymbol{H}_0^n$ *does not have* a Frobenius splitting in any characteristic.

An alternative approach would be to prove higher cohomology vanishing on $\boldsymbol{H}^n$ of the vector bundles $\mathcal{O}(m) \otimes B \otimes \bigwedge^k(B)$ for all m and k, and so deduce the vanishing for $\mathcal{O}_{\boldsymbol{H}_0^n}(m)$ from the global form of the resolution in Corollary 2.11. We have suceeded in this for $m = 0$, showing that the higher cohomology of $\mathcal{O}_{H_0^n}$ vanishes. This yields the identity $C_n^{(0)}(t,q) = 1$, which was also proved in [9] by elementary means.

Recalling that $\mathcal{O}(1)$ is an exterior power of B, and that exterior powers are summands of tensor powers, all the needed cohomology vanishing would follow from

Conjecture 3.1. $H^i(H^n, B^{\otimes k}) = 0$ for all $i > 0$ and all k, where $B^{\otimes k}$ denotes the kth tensor power $B \otimes B \otimes \cdots \otimes B$ of the tautological bundle.

This conjecture would imply higher cohomology vanishing for the tensor powers of B on H_0^n, and hence for any Schur functor [17] applied to B or its summand $B/\mathcal{O}$, since the Schur functors are summands of the tensor powers. This leads to the following vast generalization of our (t,q)-Catalan formula. For any symmetric function $f(z_1,\ldots,z_{n-1})$, let $f[B_\mu^0]$ denote f evaluated with $z_i = t^{h_i}q^{k_i}$, where $\{(h_i, k_i) : i = 1,\ldots,n-1\}$ is the set of squares in the diagram of μ, excepting $(0,0)$. Then on the validity of Conjecture 3.1 the expression

$$C_n^{\langle f\rangle}(t,q) = \sum_{|\mu|=n} \frac{f[B_\mu^0](1-t)(1-q)\left(\prod_{x\in\mu\setminus(0,0)}(1-t^{l'(x)}q^{a'(x)})\right)\left(\sum_{x\in\mu} t^{l'(x)}q^{a'(x)}\right)}{\prod_{x\in\mu}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})}, \tag{3.11}$$

must be a polynomial with positive integer coefficients whenever f is a Schur function or a non-negative linear combination of Schur functions. Our $C_n^{(m)}(t,q)$ is the special case $f = e_{n-1}^m$, where e_{n-1} is the elementary symmetric function of degree $n-1$. The expression in (3.11) can of course be readily evaluated for reasonable values of n with the aid of a computer, and such experiments invariably confirm its positivity. For this reason it seems very likely that Conjecture 3.1 holds true.

Acknowledgements

Many thanks to A. Iarrobino and the two anonymous referees for suggestions leading to numerous improvements in the final draft.

References

[1] M.F. Atiyah, R. Bott, A Lefschetz fixed point formula for elliptic differential operators, Bull. A.M.S. 72 (1966) 245–250.
[2] D. Bayer, M. Stillman, MACAULAY: A computer algebra system for algebraic geometry, version 3.0, Public domain computer program distributed by Harvard University, 1989.
[3] P. Baum, W. Fulton, G. Quart, Lefschetz–Riemann–Roch for singular varieties, Acta Math. 143 (1979) 193–211.
[4] J. Briançon, Description de Hilb$^n C\{x, y\}$, Invent. Math. 41 (1977) 45–89.

[5] L. Carlitz, J. Riordan, Two element lattice permutation numbers and their q-generalization, Duke J. Math. 31 (1964) 371–388.
[6] G. Ellingsrud, S.A. Strömme, On the homology of the Hilbert scheme of points in the plane, Invent. Math. 87 (1987) 343–352.
[7] J. Fogarty, Algebraic families on an algebraic surface, Amer. J. Math. 90 (1968) 511–521.
[8] J. Fogarty, Algebraic families on an algebraic surface, II. The Picard scheme of the punctual Hilbert scheme, Amer. J. Math. 95 (1973) 660–687.
[9] A.M. Garsia, M. Haiman, A remarkable q,t-Catalan sequence and q-Lagrange inversion, J. Algebra Combin. (to appear).
[10] M. Gordan, Les invariants des formes binaires, J. Math. Pures et Appl. (Liouville's J.) 6 (1900) 141–156.
[11] G. Gotzmann, Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes, Math. Z. 158 (1978) 61–70.
[12] A. Grothendieck, Techniques de construction et théorèmes d'existence en géométrie algébrique, IV: Les schemas de Hilbert, Séminaire Bourbaki 221, IHP, Paris, 1961.
[13] M. Haiman, Conjectures on the quotient ring by diagonal invariants, J. Algebra Combin. 3 (1994) 17–76.
[14] R. Hartshorne, Local cohomology, Lecture Notes in Mathematics 41, Springer, Berlin, 1967.
[15] A. Iarrobino, Reducibility of the family of 0-dimensional subschemes on a variety, Invent. Math. 15 (1972) 72–77.
[16] A. Iarrobino, Hilbert scheme of points: overview of last ten years, Algebraic Geometry, Bowdoin 1985, in: S.J. Bloch, (Ed.), Proceedings of Symposia in Pure Mathematics 46 part 2, A.M.S., Providence RI, 1987, 297–320.
[17] I.G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., Clarendon Press, Oxford, England, 1995.
[18] V.B. Mehta, A. Ramanathan, Frobenius splitting and cohomology vanishing for Schubert varieties, Ann. Math. 122 (1985) 27–40.
[19] R.P. Stanley, Combinatorics and Commutative Algebra, Progr. Math. 41, Birkhäuser, Boston, Mass, 1983.
[20] H. Weyl, The Classical Groups, Their Invariants and Representations, 2nd Ed., Princeton Univ. Press, Princeton, N.J, 1946.