Symplectic Duality and Coulomb Branches
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Preface

There will be some gaps in explanation, either due to the lecturer's admission or my own lack of understanding. In particular, many "proofs" are sketches of proofs. Gaps due to my own misunderstanding will be indicated by three red question marks: ???. More generally, my own questions about the material will also be in red. Things like "**Question**" will be questions posed by the lecturer. Feel free to reach out to me with explanations.

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1 Jan 6

We fix G reductive connected over $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra, M an affine normal Poisson variety, generically symplectic. Let G have Hamiltonian action on M with moment map $\mu : M \to \mathfrak{g}^*$. There is a scaling action $\mathbb{C}^\times$ on M which commutes with the G-action and with μ.

When we introduce the Langlands dual group $G^\vee$, we want some $M^\vee$ which plays the role of M. This is known basically only in the case where $M = T^*X$ for X a smooth affine G-variety. The main problem is specifically to find a class of "good" M such that M is good implies $M^\vee$ is good, $(M^\vee)^\vee = M$, and all T^*X are good.

Now fix a Borel B. Let X be a smooth affine G-variety, and let M be as in the first paragraph.

Definition 1.1.

1. X is **spherical** if X contains an open dense B-orbit.
2. M is **hyperspherical** if for all $f_1, f_2 \in \mathbb{C}[M]^G$, we have $\{f_1, f_2\} = 0$.

Theorem 1.1. *X is spherical iff T^*X is hyperspherical.*

We will prove this later on.

Theorem 1.2. *Let M be a hyperspherical variety. Then:*

1. *The map $\overline{\mu} : M//G \to \mathfrak{g}^*//G$ on categorical quotients is finite, i.e. $\mathbb{C}[M]^G$ is a finitely generated module over $\mathbb{C}[\mathfrak{g}^*]^G = (\mathrm{Sym}\mathfrak{g})^G$.*
2. *The image $\mathrm{im}(\overline{\mu})$ of $\overline{\mu}$ is closed in $\mathfrak{g}^*//G$.*
3. *The composite $\nu : M \to M//G \xrightarrow{\overline{\mu}} \mathfrak{g}^*//G$ has the property that all irreducible components of all of its non-empty fibers have the same dimension.*
4. *Each irreducible component of the generic fibers of ν is the closure of a G-orbit.*

Note. "Generic" here means it is true in a Zariski open subset.

Corollary 1.1. *If M is hyperspherical, then* $\dim M \leq \dim G + \dim(\mathfrak{g}^*//G) = \dim G + \mathrm{rk} G$.

From now on, we consider M to be smooth and symplectic.

Let $\mathfrak{b} = \mathrm{Lie} B$. The composite $\mu_B : M \xrightarrow{\mu} \mathfrak{g}^* \to \mathfrak{b}^*$ is the moment map for the B-action. Let $\Lambda_M = \mu_B^{-1}(0) = \mu^{-1}(\mathfrak{b}^\perp)$.

Example 1.1. If $M = T^*X$, then Λ_M is the union of the conormal bundles $T^*_O X$ to B-orbits $O \subset X$.

Theorem 1.3. *If X is spherical, then X is a finite union of B-orbits. (???)*

Corollary 1.2. *If X is spherical and $M = T^*X$, then Λ_M is Lagrangian in M.*

Proof. Each conormal bundle is Lagrangian.

Theorem 1.4. *Let M be smooth and symplectic. If Λ_M is Lagrangian, then M is hyperspherical.*

Conjecture: if M is good symplectic hyperspherical, then Λ_M is Lagrangian, and there is a bijection between the irreducible components of Λ_M and the irreducible components of $\Lambda_{M^\vee}$.

Let $\mathcal{B} = G/B$ be the flag variety, let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{g}^*$, let $\widetilde{\mathcal{N}} = T^*\mathcal{B} \xrightarrow{\pi} \mathcal{N}$ be the Springer resolution, and let $\mathrm{St}_G = \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$ be the Steinberg variety. It is known that St_G is Lagrangian in $T^*(\mathcal{B} \times \mathcal{B})$, and $H^{BM}_{top}(\mathrm{St}_G)$ has a natural algebra structure, isomorphic to the group algebra of the Weyl group W.

Now let M be hyperspherical, and assume that Λ_M is Lagrangian. Let $\mathrm{St}_M = M \times_{\mathfrak{g}^*} \widetilde{\mathcal{N}}$. As a subvariety of $M \times \widetilde{\mathcal{N}}$, it is stable under the diagonal G-action. We have $\mathrm{St}_M \cong G \times^B \Lambda_M$. If $M = T^*X$, then $M \times \widetilde{\mathcal{N}} = T^*(X \times \mathcal{B})$, and St_M is the union of conormal bundles to G-orbits.

By analyzing the fiber product conditions, we see that there is a convolution $\mathrm{St}_M \circ \mathrm{St}_G = \mathrm{St}_M$. In particular, two pairs $(\eta, \xi) \in \mathrm{St}_M$ and $(\xi, \xi') \in \mathrm{St}_G$ give a new pair (η, ξ') in St_M. This gives $H^{BM}_{top}(\mathrm{St}_M)$ the structure of a $H^{BM}_{top}(\mathrm{St}_G)$-module, i.e. it is a representation of W.

Conjecture: There is an isomorphism of W-reps $H^{BM}_{top}(\mathrm{St}_M) \cong H^{BM}_{top}(\mathrm{St}_{M^\vee})$.

Example 1.2. Now we tabulate results when I'm not lazy I will make this look nice.. Row 1: $G = T$ is a torus, $M = T^*(T/T_1)$ for a subtorus T_1. Then $M^\vee = T^*(T_1^\vee)$.
Spherical T-variety is a toric variety; for it to be smooth, it would be affine. So in particular (row 2), if $G = (\mathbb{C}^\times)^n$ and $M = T^*(\mathbb{C}^n)$, then $G^\vee = G$ and $M^\vee = M$.
Next (row 3) consider the group $G \times G$ and $M = T^*G$, where $G \times G$ acts by left and right translations. Then the dual group is $G^\vee \times G^\vee$ and $M^\vee = T^*(G^\vee)$. Note that G is spherical in this case, since it has the open $B \times B$ orbit given by Bw_0B, where $w_0 \in W$ is the longest element.
Row 4: if the group is just G and $M = T^*G$, then $M^\vee = \mathcal{N}_{G^\vee}$.
Row 5: Let $U = [B, B]$ be max unipotent. Consider the group $G \times T$, where T is a maximal torus in G. Let M be the affine closure of $T^*(G/U)$. Then $M^\vee$ is the affine closure of $T^*(G^\vee/U^\vee)$. This is related to Eisenstein series. (note: possibly incorrect)

Row 6: consider the same M but for the group G. Then $M^\vee = \overline{T^*(G^\vee/U^\vee)}/W$, where the W-action is by Gelfand-Graev (it is not an obvious action).
Row 7: Let the group be G, and let M be a point. Then $M^\vee = T^*_\psi(G^\vee/U^\vee) = (T^*G^\vee)//_\psi U^\vee$ (Hamiltonian reduction), the Whittaker potential bundle for a nondegenerate character $\psi : U^\vee \to \mathbb{C}^\times$.
Row 8: $G = GL_n \times GL_n$, $M = T^*(\mathbb{C}^n \otimes \mathbb{C}^n) = T^*M_n$, where GL_n acts by left and right translations. This group is self dual, and $M^\vee = T^*(GL_n \times \mathbb{C}^n) = T^*(G \times^{GL_n} \mathbb{C}^n)$. This duality is classical and known in automorphic forms; in one direction it is Rankin-Selberg, and in the other it is Godement-Jacquet.
Row 9: $G = GL_n$, $M = T^*(\mathbb{C}^n)$, $M^\vee = T^*M_n//_\psi U$.
Row 10: $G = GL_{2n}$, $M = T^*(G/(GL_n \times GL_n))$ (block diagonal embedding), $M^\vee = T^*(G \times^{Sp_{2n}} \mathbb{C}^{2n})$.
Row 11: $G = GL_{2n}$, $M = T^*(G/Sp_{2n})$, $M^\vee = T^*_\phi(G/Q)$, where Q is the subgroup of block $(n+n) \times (n+n)$ upper triangular matrices, where the two diagonal blocks are equal, and ϕ takes such a matrix to $e^{tr(a)}$, where a is the upper right block.

2 Jan 8 - Geometry of spherical and hyperspherical varieties

Theorem 2.1 (Rosenlicht)**.** *Let H be a connected algebraic group acting on an irreducible variety X. Then there is an H-stable Zariski open $X^\circ \subset X$ such that:*

1. *H-orbits in X° has maximal dimension.*
2. *There is a smooth surjective morphism $X^\circ \to Y$ such that each fiber is a single orbit. Y is called the **geometric quotient** X°/H.*

Corollary 2.1. *Let $\mathbb{C}(X)$ be the field of rational functions on X. Then $\mathbb{C}(X)^H = \mathbb{C}(X^\circ)^H = \mathbb{C}(X^\circ/H)$, and X has an open H-orbit iff $\mathbb{C}(X) = \mathbb{C}$.*

Resume the usual setup (G, B, T, U). If $\lambda \in X^*(T)$ is a character of T, we may lift it to a character of B by letting λ act trivially on U.

Now let G act on an affine variety X. We can decompose $\mathbb{C}[X]$ into isotypic components corresponding to highest weight irreducible representations:

$$\mathbb{C}[X] = \bigoplus_{V_\lambda \in \mathrm{Irr}(G)} \mathbb{C}[X]_\lambda.$$

We may do this because G is reductive and the action of G on $\mathbb{C}[X]$ is locally finite. Let

$$\mathbb{C}[X]^{U,\lambda} = \{f \in \mathbb{C}[X] \mid b(f) = \lambda(b)f, \forall b \in B\}.$$

These are the B-semiinvariants of weight λ. It follows that the multiplicity $m(\mathbb{C}[X] : V_\lambda)$ of V_λ in $\mathbb{C}[X]$ is $\dim \mathbb{C}[X]^{U,\lambda}$.

Theorem 2.2. *X has an open B-orbit iff $m(\mathbb{C}[X] : V_\lambda) \leq 1$ for all λ.*

Proof. To prove this, we need a lemma.

Lemma 2.1. *Let $f \in \mathbb{C}(X)$. Then $f \in \mathbb{C}(X)^B$ iff there exist λ and and $\varphi, \psi \in \mathbb{C}[X]^{U,\lambda}$ such that $f = \varphi/\psi$.*

Proof of lemma. If $f = \varphi/\psi$, then $b(f) = b(\varphi)/b(\psi) = (\lambda(b)\varphi)/(\lambda(b)\psi) = \varphi/\psi = f$, so f is invariant. Conversely, let $f \in \mathbb{C}(X)^B$. Write $f = \varphi'/\psi'$ for arbitrary $\varphi', \psi' \in \mathbb{C}[X]$. The span $\langle B\psi' \rangle$ of $B\psi'$ is finite dimensional. By Lie's theorem, there is a λ and nonzero B-semiinvariant $\psi \in \langle B\psi' \rangle$ of weight λ. Consequently, write $\psi = \sum_i c_i b_i(\psi')$. Define $\varphi = \sum_i c_i b_i(\varphi')$. For all $b \in B$, we have $b(\varphi') = b(f)b(\psi') = fb(\psi')$. Thus $\varphi = \sum c_i b_i(\varphi') = \sum c_i f b_i(\psi') = f\psi$. Thus $f = \varphi/\psi$, and since f is invariant and ψ is semiinvariant of weight λ, φ must also be semiinvariant of weight λ.

To prove the theorem, there exists an open B-orbit in X iff $\mathbb{C}(X)^B = \mathbb{C}$, which by the lemma is true iff $\dim \mathbb{C}[X]^{U,\lambda} \leq 1$ for all λ, which gives the claim (since this dimension is the required multiplicity).

Theorem 2.3. *If X has an open B-orbit, then $\mathbb{C}[T^*X]^G$ is a commutative Poisson algebra.*

Proof. Let $\mathcal{D}(X)$ be the algebra of differential operators on X. Standard facts from X being affine:

1. $\mathbb{C}[X]$ is faithful as a $\mathcal{D}(X)$-module.

2. $gr\mathcal{D}(X) \cong \mathbb{C}[T^*X]$ (where the filtration on $\mathcal{D}(X)$ is by order of differential operators).

3. Since G is reductive, $gr(\mathcal{D}(X)^G) = \mathbb{C}[T^*X]^G$.

Let $a \in \mathcal{D}(X)^G$. Then the action of a on $\mathbb{C}[X]$ commutes with the G-action. Thus a restricts to maps between isotypic components for all weight. By the previous theorem and our hypothesis that X has an open B-orbit, we know that each isotypic component is either V_λ or 0. Thus, by Schur's lemma, a acts by scalars a_λ on all isotypic components. We obtain an algebra map $i : \mathcal{D}(X)^G \to Maps(X^*(T), \mathbb{C})$, where the right hand side consists of arbitrary functions of sets and is equipped with the pointwise algebra structure. Since $\mathcal{D}(X)$ acts faithfully on $\mathbb{C}[X]$, this map i is injective. Since the pointwise algebra structure is commutative, we get that $\mathcal{D}(X)^G$ is commutative. Finally, the associated graded of a commutative algebra is Poisson commutative, so we are done.

Theorem 2.4. *If X has an open B-orbit, then X is a finite union of B-orbits.*

Proof. We do not prove the whole claim, only the following weaker statements:

1. X is a finite union of G-orbits.

2. Each G-orbit contains an open B-orbit.

We need the following lemma:

Lemma 2.2. *If X has an open B-orbit, then for all G-stable closed subvarieties Y, we have $\mathbb{C}(Y)^B = \mathbb{C}$.*

Proof of lemma. Let $f \in \mathbb{C}(Y)^B$. Then there is some λ and some $\varphi, \psi \in \mathbb{C}[Y]^{U,\lambda}$ such that $f = \varphi/\psi$. (I should probably start hyperlinking references to past results). Since Y is a closed subvariety, the restriction map $\mathbb{C}[X] \to \mathbb{C}[Y]$ is surjective. Then each map on isotypic components is surjective. By complete reducibility, this means $\mathbb{C}[Y]_\lambda$ is a direct summand of $\mathbb{C}[X]_\lambda$. The is true of the spaces of semiinvariants, meaning we can lift φ, ψ to semiinvariant functions φ', ψ' of weight λ on all of X. Then $\varphi'/\psi' \in \mathbb{C}(X)^B = \mathbb{C}$, meaning $\varphi/\psi = f$ is also constant.

Now, since X has an open B-orbit, it must also have an open G-orbit O by saturating the open B-orbit. Let X_1 be an irreducible component of $X \setminus O$. Then X_1 is a closed G-stable subvariety of X with strictly smaller dimension. By the lemma, X_1 also has an open B-orbit, so by the same argument, X_1 has an open G-orbit. We may continue in this way, and eventually we the process will end because the dimension is strictly shrinking.

Definition 2.1. Let (E, ω) be a symplectic vector space. Then a subspace $F \subset E$ is **isotropic** if $\omega|_F = 0$. F is **coisotropic** if $F^{\perp_\omega}$ is isotropic. F is **Lagrangian** if it is isotropic and coisotropic.

Definition 2.2. Let (X, ω) be a smooth affine symplectic variety. Then a subvariety $Y \subset X$ is **isotropic/coisotropic** if there is an open smooth $Y^\circ \subset Y$ such that for all $y \in Y^\circ$, $T_y Y$ is isotropic/coisotropic.

Now, let (M, ω) be a smooth affine symplectic variety. Let G have Hamiltonian action on M with moment map $\mu : M \to \mathfrak{g}^*$.

Theorem 2.5. *The following are equivalent:*

1. *The Poisson algebra $\mathbb{C}(M)^G$ is commutative (meaning the bracket vanishes).*
2. *Generic G-orbits in M are coisotropic subvarieties.*
3. *Irreducible components of generic fibers of μ are isotropic.*

Proof. All statements are "generic", so we may assume $M = M^\circ$ in the sense of Rosenlicht's theorem. Any $m \in M$ gives an action map $\mathrm{act}_m : G \to M$, $g \mapsto gm$, and we can differentiate it to get $d_m \mathrm{act}_m : \mathfrak{g} \to \mathfrak{g}m = T_m(Gm) \subset T_m M$. Any $f \in \mathbb{C}(M)$ gives a Hamiltonian vector field ξ_f determined by $df = \omega(\xi_f, -)$. Then $f \in \mathbb{C}(M)^G$ iff f is constant on G-orbits, which is true iff $d_m f|_{\mathfrak{g}m} = 0$ for almost all $m \in M$ (namely, wherever f is defined). But this is true iff $\omega(\xi_f, \mathfrak{g}m) = 0$, i.e. $\xi_f \in (\mathfrak{g}m)^{\perp_\omega}$. For $f_1, f_2 \in \mathbb{C}(M)$, we have $\{f_1, f_2\} = \omega(\xi_{f_1}, \xi_{f_2})$. Thus $\mathbb{C}(M)^G$ is Poisson commutative iff for almost all $m \in M$, the space of $\xi_f(m)$ for $f \in \mathbb{C}(M)^G$ is an isotropic subspace of $T_m M$. But we have computed the space of $\xi_f(m)$; it is $(\mathfrak{g}m)^{\perp_\omega}$. So $\mathbb{C}(M)^G$ is Poisson commutative iff for almost all m, $\mathfrak{g}m = T_m(Gm)$ is coisotropic, which exactly means Gm is coisotropic.

Now observe that the transpose of $d_m\mathrm{act}_m$ is the composite $T_m^*M \xrightarrow{\sim} T_mM \xrightarrow{d_m\mu} \mathfrak{g}^*$, where the first map is the isomorphism given by ω being nondegenerate. (I'm not sure I see why this is true)

Proof to be continued next lecture

3 Jan 13

3.1 Geometry of moment maps

Let G be an algebraic or Lie group (not necessarily reductive), with Hamiltonian action on a symplectic (M, ω). Let $\phi \in \mathfrak{g}^*$ where $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be $\kappa(a, b) = \phi([a, b])$. The radical (those a for which $\kappa(a, b) = 0$ for all b) of κ is $\mathfrak{g}^\phi = \mathrm{Lie}(G^\phi)$, where $G^\phi = \mathrm{Stab}_G(\phi)$, where the stabilizer is taken with respect to the coadjoint action of G on $\mathfrak{g}^*$. THen κ descends to a nondegenerate $\kappa : \mathfrak{g}/\mathfrak{g}^\phi \times \mathfrak{g}/\mathfrak{g}^\phi \to \mathbb{C}$. Note that if $O = \mathrm{Ad}_G \phi$, then $T_\phi O = \mathfrak{g}/\mathfrak{g}^\phi$, so κ is giving rise to something called the Kirillov-Kostant symplectic form on O.

Now let $\alpha \in \mathfrak{g} = (\mathfrak{g}^*)^*$. Let $\mu : M \to \mathfrak{g}^*$ be the moment map, and fix $m \in M$. In the context of the above, we let $\phi = \mu(m)$. Then we have several formulas ("all of this is completely straightforward"):

- $\xi_{\mu^*(a)}(m) = am$, for $a \in \mathfrak{g}$.
- $\omega(am, bm) = \phi([a, b])$, for $a, b \in \mathfrak{g}$.
- Consider the action map $\mathrm{act} : \mathfrak{g} \to T_m M$, $a \mapsto am$. Then the composite $T_m M \xrightarrow{\omega} T_m^* M \xrightarrow{\mathrm{act}^T} \mathfrak{g}^*$ is the differential of μ at m. As a consequence, $\ker(d_m\mu) = (\mathfrak{g}m)^\perp$.
- The radical of $\omega|_{\mathfrak{g}m}$ is $\mathfrak{g}m \cap (\mathfrak{g}m)^\perp = \mathfrak{g}^\phi m$.

In the above (and in the future, unless otherwise specified), any instance of $\perp$ is taken with respect to ω.

Now assume $m \in M$ is sufficiently general, so that in particular m is a smooth point of μ. Take $\phi = \mu(m)$ again. Let $F = \mu^{-1}(\phi)$. By smoothness, $T_m F = \ker(d_m\mu) = (\mathfrak{g}m)^\perp$. So, we have a generic formula for $T_m F$.

Now assume Gm is coisotropic in M (hyperspherical case). Equivalently, $(\mathfrak{g}m)^\perp \subset \mathfrak{g}m$. Recall $\mathrm{rad}(\omega|_{\mathfrak{g}m}) = \mathfrak{g}m \cap (\mathfrak{g}m)^\perp = \mathfrak{g}^\phi m$. Since $\mathfrak{g}m = T_m(Gm)$ and $(\mathfrak{g}m)^\perp \subset \mathfrak{g}m$, we have

$$\mathrm{rad}(\omega_{T_m(Gm)}) = (\mathfrak{g}m)^\perp = \mathfrak{g}^\phi m = T_m F.$$

The last equality has the geometric interpretation that F intersected with some open neighborhood of m is $G^\phi m$. Since m maps to ϕ under a G-equivariant map, we have $T_m F \subset \mathfrak{g}m = T_m(Gm) \xrightarrow{d_m\mu} T_\phi(G\phi) = \mathfrak{g}\phi$. This implies (???)that Gm is an open dense subset of $\mu^{-1}(G\phi)$. This gives another perspective on what hyperspherical means: generic fibers of orbits are generically orbits themselves.

Recall that in general $(\mathfrak{g}m)^\perp = \{\xi_f \mid f \in \mathbb{C}(M)^G\}$. If M is hyperspherical, then all such ξ_f commute, and the above space is $\mathfrak{g}^\phi m$. Then $\mathfrak{g}^\phi/\mathfrak{g}^m$ is abelian. Thus the corresponding connected group $(G^\phi)^\circ/G^m$ is abelian ($\circ$ denotes connected component of identity). Furthermore, this group is independent of the choice of element in (an irreducible component of) F, i.e. for any $g \in G^\phi$ we have a canonical isomorphism $(G^\phi)^\circ/G^m \cong (G^\phi)^\circ/G^{gm}$. This is called the universal stabilizer of an irreducible component of F.

Now let G be reductive. Fix Borels B and $\mathfrak{b}$, and fix $\phi \in \mathfrak{g}^*$. Let $\phi_\mathfrak{b} = \phi|_\mathfrak{b}$, and we assume it is a character, i.e. it vanishes on $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. Thus we get a functional on the Cartan $\mathfrak{b}/\mathfrak{n} \cong \mathfrak{t}$. Recall that the composite of $\mu : M \to \mathfrak{g}^*$ with restriction to $\mathfrak{b}$ is exactly the moment map μ_B with respect to the B-action. $\phi_\mathfrak{b}$ being a character means exactly that it is B-stable, so its fiber $\mu_\mathfrak{b}^{-1}(\phi_\mathfrak{b}) = \mu^{-1}(\phi + \mathfrak{b}^\perp)$ is also B-stable. If we identify $\mathfrak{g} \cong \mathfrak{g}^*$ with some invariant nondegenerate form, so $\phi \in \mathfrak{g}$, then the fiber may be expressed as $\mu^{-1}(\phi + \mathfrak{n})$.

Consider the map $\mathfrak{g}^* \to \mathfrak{g}^*//G$. By Chevalley, $\mathfrak{g}^*//G = \mathfrak{t}^*/W$, which we identify with $\mathfrak{t}/W$. Inside of $\mathfrak{g}^*$, we have $\mathfrak{n}^\perp$, which is identified with $\mathfrak{b}$, which maps to $\mathfrak{b}/\mathfrak{n} = \mathfrak{t}$. These identifications are compatible with the quotient maps, i.e. the map $\mathfrak{b} \to \mathfrak{t} \to \mathfrak{t}/W$ is the same as $\mathfrak{b} \cong \mathfrak{n}^\perp \hookrightarrow \mathfrak{g}^* \to \mathfrak{g}^*//G$. What this means is that for a given $c \in \mathfrak{g}^*//G$, there are only finitely many $\phi \in \mathfrak{t}$ which map to c. Hence (???)there are only finitely many spaces $\phi + \mathfrak{b}^\perp = \phi + \mathfrak{n}$. Now, $\mu(M)$ is a G-stable subset of $\mathfrak{g}^*$, and we can take the composite $\mu(M) \hookrightarrow \mathfrak{g}^* \to \mathfrak{g}^*//G$. For $m \in M$ general, let c be the image of m under this composite. Then there are finitely many $\phi \in \mathfrak{t}^*$ such that $\phi + \mathfrak{b}^\perp$ meets $G\phi$ (where $\phi = \mu(m)$ (???)(lecturer refused to understand my confusion)). The following diagram illustrates some of the relevant geometry: (although I missed the lecturer's comments on its significance)

$$
\begin{tikzcd}
M & Gm \arrow[l, hook'] \arrow[d] & Gm \cap \mu^{-1}(\phi + \mathfrak{b}^\perp) \arrow[l, hook'] \arrow[d] \\
\mathfrak{g}^* \arrow[d] & G\phi \arrow[l, hook'] \arrow[d] & G\phi \cap \mu^{-1}(\phi + \mathfrak{b}^\perp) \arrow[l, hook'] \\
\mathfrak{g}^*//G & \{c\} \arrow[l, hook'] &
\arrow[from=1-1, to=2-1, "\mu"']
\end{tikzcd}
$$

Theorem 3.1. *M is hyperspherical iff for general $m \in M$ and $\phi = \mu(m)$, then each irreducible component $\mu^{-1}(\phi + \mathfrak{b}^\perp)$ is Lagrangian in M.*

Conjecture: (I should make a conjecture environment) $\mu^{-1}(\mathfrak{b}^\perp)$ is Lagrangian in M.

Assume there is a $\mathbb{C}^\times$-action on M such that:

- The $\mathbb{C}^\times$-action commutes with the G-action.
- μ is $\mathbb{C}^\times$-equivariant.
- The image of $\mu(M)$ in $\mathfrak{g}^*//G$ contains 0, i.e. $\mu(M)$ intersects the nilpotent cone $\mathcal{N}$ in $\mathfrak{g}^*$.

If $m \in M$ is such that $\mu(m)$ maps to 0 in $\mathfrak{g}^*//G$, then $\mu^{-1}(\phi + \mathfrak{b}^\perp) = \mu^{-1}(\mathfrak{b}^\perp)$. By semicontinuity, the conjecture implies $\dim \mu^{-1}(\phi + \mathfrak{b}^\perp) \leq \frac{1}{2} \dim M$ for all ϕ. A theorem from Chriss-Ginzburg says that $(\mu_{\mathfrak{b}})^{-1}(\phi_{\mathfrak{b}})$ is coisotropic in M. So, together with the conjecture, we get $\mu^{-1}(\phi + \mathfrak{b}^\perp)$ is Langrangian for all $\phi \in \mu(M)$.

Sketch of proof of Theorem 3.1. Fix general $m \in M$, let $\phi = \mu(m)$, and let $F = \mu^{-1}(\phi)$. Let Λ be the irreducible component of $\mu^{-1}(\phi + \mathfrak{b}^\perp)$ containing m. We want to show $T_m\Lambda$ is Lagrangian in T_mM. Let $O = G\phi = \mu(Gm)$. Let $\Sigma = O \cap (\phi + \mathfrak{b}^\perp)$. Then $T_\phi\Sigma = \{a\phi \mid a \in \mathfrak{g}, a\phi \in \mathfrak{b}^\perp\}$ (how is $\mathfrak{g}$ acting on $\mathfrak{g}^*$?) The end result is that $T_m\Lambda = T_mF + \{am \mid a \in \mathfrak{g}, a\phi \in \mathfrak{b}^\perp\}$. By the previously mentioned theorem in Chriss and Ginzburg, we know $T_m\Lambda$ is coisotropic in T_mM, so we need to show it is isotropic. Let $v_1, v_2 \in T_mF$ and $a, b \in \mathfrak{g}$ satisfying $a\phi, b\phi \in \mathfrak{b}^\perp$. Then we need to compute $\omega(v_1 + am, v_2 + bm)$. The cross terms vanish by facts stated earlier in the lecture (related to Kirillov-Kostant). M being hyperspherical implies F is isotropic, which implies $\omega(v_1, v_2) = 0$. The last term is $\phi([a, b]) = \kappa(a, b)$. From Chriss and Ginzburg, Σ is Lagrangian in $G\phi$, where $G\phi$ has the symplectic form κ. Thus $\kappa(a, b) = 0$.

4 Jan 15

4.1 Global symplectic duality

Global symplectic duality is the same as what is known as relative (geometric) Langlands duality. We recall what non-relative Langlands duality is.

Let G be reductive with dual group $G^\vee$. Let Σ be a fixed smooth projective curve. There are two sides, automorphic and spectral, related by duality. TODO: table would be nice here. On the automorphic side, the key object is $\mathrm{Bun} = \mathrm{Bun}_G(\Sigma)$, the G-bundles on Σ. The corresponding object on the spectral side is $\mathrm{Loc} = \mathrm{LocSys}_{G^\vee}(\Sigma)$, the $G^\vee$-local systems on Σ. Technically, both sides should also be restricted to objects with "nilpotent singular support", but we will ignore this. On each side we consider some family of objects related to Bun or Loc. There are also various topological (in particular, cohomological) settings through which we can view things: Betti, de Rham, and étale. In the (Spectral,Betti) is $\mathrm{Hom}(\pi_1(\Sigma), G^\vee)/G^\vee$. (Spectral,de Rham) is $G^\vee$-bundles with a flat connection. (Spectral,étale) is not well-understood (if I understood the lecturer correctly.) (Automorphic, Betti) is constructible sheaves in the analytic topology. (Automorphic, de Rham) is D-modules. (Automorphic, étale) is constructible sheaves in the étale topology. The Langlands duality asserts the existence of a duality map $\mathbb{L}$ from the spectral side to the automorphic side.

Now consider X smooth affine spherical G-variety with a root system "not of type N". Let $M = T^*X$ and assume $M^\vee = T^*X^\vee$. Let P be a G-bundle on Σ. Then we can form $X_P = P \times^G X \to \Sigma$. Via this construction, we consider a space $\mathrm{Bun}^X = \mathrm{Bun}_G^X(\Sigma)$ of pairs (P, s), where P is a G-bundle and s is a section of X_P. Technically there should be some twist somewhere but we are ignoring it. For $X = \mathfrak{g}^*$, Bun^X is the Higgs bundles (up to twist issues). There is an obvious projection $p : \mathrm{Bun}^X \to \mathrm{Bun}$.

Similarly, we may consider $\mathrm{Loc}^{X^\vee} = \mathrm{Loc}_{G^\vee}^{X^\vee}(\Sigma)$, which in the de Rham setting consists of pairs $(P^\vee, s)$ where $P^\vee$ is a $G^\vee$-bundle on Σ with a flat connection, and s is a section of $X^\vee_{P^\vee}$ which is horizontal with respect to the induced connection on $X^\vee_{P^\vee}$. In the Betti setting, $\mathrm{Loc}^{X^\vee}$ consists of $G^\vee$-equivalence classes of pairs (ρ, x), where $\rho : \pi_1(\Sigma) \to G^\vee$ and $x \in (X^\vee)^{\rho(\pi_1(\Sigma))}$. One can see where this comes from/how it relates to the de Rham setting by instead looking at local systems as glued trivial bundles. In particular, the data of a local system is just a cover $\{U_i\}$ of Σ together with some elements $g_{ij} \in G^\vee$ satisfying a cocycle condition. The trivial bundles on U_i are glued using the g_{ij}. Then a horizontal section of $X^\vee_{P^\vee}$ is a set of elements $x_i \in X^\vee$ with $x_i = g_{ij}x_j$, so everything is determined by one point. We again have a projection $p^\vee : \mathrm{Loc}^{X^\vee} \to \mathrm{Loc}$.

Relative Langlands asserts $p_* \underline{\mathbb{C}}_{\mathrm{Bun}^X} = \mathbb{L}(p^\vee_! \omega_{\mathrm{Loc}^{X^\vee}})$, where ω denotes the dualizing complex in $D^b\mathrm{Coh}(\mathrm{Loc}^{X^\vee})$ and $\underline{\mathbb{C}}$ is the constant sheaf in the constructible derived category. (Note: by lecturer's admission, the constant and dualizing sheaves might need to be swapped.)

We now illustrate a difficulty in the cotangent situation. First, suppose $X = V$ is just a representation of G. Then $M = T^*V = V \oplus V^* = T^*(V^*)$. But $p_*\underline{\mathbb{C}}_{\mathrm{Bun}^V}$ and $p_*\underline{\mathbb{C}}_{\mathrm{Bun}^{V^*}}$ (or possibly you do something on the spectral side when working with V^*, it wasn't clear) may disagree by some twist or sign or some other obstruction. To me this seems completely reasonable and not like a difficulty, so I'm not sure what the issue is; yes the M is the same but everything above was defined in terms of X and its G-action.

Now fix a maximal torus T and Borel B with $T \subset B \subset G$. Let W be the Weyl group relative to T. Then W permutes the set of Borels containing T. There is a B-stable Lagrangian subspace $L \subset M$, and for $w \in W$, we have that $w(L)$ is a $w(B)$-stable Lagrangian subspace of M. To these Lagrangian subspaces we may associate so-called Coulomb branches $\mathcal{C}_L$ (to be defined later in the course). The W-action gives maps $\mathcal{C}_L \to \mathcal{C}_{w(L)}$.

Let $G = SL_2, W = \{1, s\}$. Then $\mathcal{C}_L \xrightarrow{s} \mathcal{C}_{s(L)} \xrightarrow{s} \mathcal{C}_L$ is not the identity, but a sign. But if you replace W by $\mathbb{Z}/4$, it works. In fact, for general G, there is a Tits group $\widetilde{W} \subset N(T)$ such that the induced map $\widetilde{W} \to W$ is surjective with kernel some finite product of copies of $\mathbb{Z}/2$, and the lecturer's belief is that working with $\widetilde{W}$ will make things work.

Now take $\Sigma = D = \mathrm{Spec}\mathbb{C}[[z]]$ the formal disk and $G = GL_n$, so $G^\vee = G$. Then Loc in the de Rham setting consists of rank n vector bundles $\mathcal{V} \to D$ with flat connection ∇, up to conjugation. On D, all vector bundles are trivial, so we write $\mathcal{V} = V \times D$. The connection can be written $\nabla = d + A$, where $A \in \mathfrak{gl}_n(D)dz$. We take this data up to $G(D)$. Then what you get is just $\mathfrak{g}/G$. If we were careful and did this for general G, we would get $(\mathfrak{g}^\vee)^*/G^\vee$, which appears on one side of derived geometric Satake.

Now we discuss a strategy to define $(T^*X)^\vee$ for X an affine smooth G-variety. It is related to geometric Satake, so we briefly discuss the setup. Let $\mathcal{K} = \mathbb{C}((z)), \mathcal{O} = \mathbb{C}[[z]]$. The affine Grassmannian is $Gr_G = G(\mathcal{K})/G(\mathcal{O})$. This has a left action by $G(\mathcal{O})$, so we consider the equivariant derived category $D^b_{G(\mathcal{O}),c}(Gr)$, defined to be the Satake category Sat. It is monoidal; let $\star$ be the monoidal product. On the other hand, we consider $D((\mathrm{Sym}\mathfrak{g}^\vee, G^\vee)-\mathrm{mod})$, which is also monoidal with respect to the tensor product over $\mathrm{Sym}\mathfrak{g}^\vee$. Geometric Satake asserts that there is a monoidal equivalence Φ between these categories.

Now, given our space X, we claim there is some corresponding commutative ring object $\mathcal{A}_X \in \mathrm{Sat}$. Then $\Phi(\mathcal{A}_X)$ is a commutative algebra, and as it is still a ring object, it admits a map from $\mathrm{Sym}\mathfrak{g}^\vee$. We obtain $M^\vee := \mathrm{Spec}(\Phi(\mathcal{A}_X))$, which one can show is a symplectic $G^\vee$-variety, with a map to $(\mathfrak{g}^\vee)^*$, which is the moment map.

5 Jan 20

As we saw at the end of the last lecture, we want to find a ring object associated to a G-space X. We begin with some preliminary setup.

Let X be a (complex) variety. Let $d = \dim_{\mathbb{R}} X = 2\dim_{\mathbb{C}} X$. Let $D(X)$ be the constructible (probably bounded) derived category. Let ω_X be the dualizing complex. If X is smooth, then $\omega_X = \underline{\mathbb{C}}_X[d]$. Given $f : X \to Y$, we get a "shriek pullback" $f^! : D(Y) \to D(X)$ satisfying $f^!\omega_Y = \omega_X$. In the special case where $f : X \to Y$ is a closed embedding of smooth varieties, we have $\omega_X = f^!\underline{C}_Y[d_Y]$.

We define the Borel-Moore homology to be $H_i^{BM}(X) = H^{-i}(\omega_X)$. We have $H_i^{BM}(\mathbb{R}^d) = 0$ unless $i = d$, in which case it is isomorphic to the coefficient field. For any X, we have a fundamental class $[X] \in H_d^{BM}(X)$.

5.1 Restriction with support

Consider the following pullback square of spaces:

$$\begin{tikzcd} W = M \times_N V \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{g}"'] & V \arrow[d, "g"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Suppose we have $A \in D(M), B \in D(N)$. Suppose we have a map $\phi : f^*B \to A$. Then we can define a morphism

$$g^!B \xrightarrow{adj.} g^!f_*f^*B = \widetilde{f}_*\widetilde{g}^!f^*B \xrightarrow{\phi} \widetilde{f}_*\widetilde{g}^!A,$$

caled the restriction of ϕ with support. The first map above is given by the (f_*, f^*) adjunction, and the "equality" is a base change isomorphism. We apply this in the case where $A = \underline{\mathbb{C}}_M$, $B = \underline{\mathbb{C}}_N$. In this case, $f^*\underline{\mathbb{C}}_N = \underline{\mathbb{C}}_M$, and we let φ be the identity on $\underline{\mathbb{C}}_M$. Then we get a map $g^!\underline{\mathbb{C}}_N \to \widetilde{f}_*\widetilde{g}^!\underline{\mathbb{C}}_M$. In the case where M, N are smooth, we have $g^!\underline{\mathbb{C}}_N = g^!\omega_N[-d_N] = \omega_V[-d_N]$ and $\widetilde{f}_*\widetilde{g}^!\underline{\mathbb{C}}_M = \widetilde{f}_*\omega_W[-d_M]$. Using these identifications in the restriction with support map, and moving shifts to one side, we get a map $\omega_V \to \widetilde{f}_*\omega_W[d_N - d_M]$. The induced maps on cohomology of this map are then induced maps on Borel-Moore homology: $H_i^{BM}(V) \to H_{i+d_M-d_N}^{BM}(W)$.

5.2 Convolution in BM-homology

Suppose we have a proper map of varieties $\widetilde{Y} \to Y$, where $\widetilde{Y}$ is smooth. Let $Z = \widetilde{Y} \times_Y \widetilde{Y}$ be the Steinberg variety. We can form the fiber product $Z \times_Y Z$, where the two maps $Z \to Y$ are the different compositions $Z \to \widetilde{Y} \to Y$; it consists of 4-tuples of elements in $\widetilde{Y}$ that all map to the same thing in Y. Then there is a map $Z \times_Y Z \to Z$ which simply forgets the middle two elements. This gives a convolution algebra structure on BM homology of Z, namely maps $H_i^{BM}(Z) \times H_j^{BM}(Z) \to H_{i+j-d}^{BM}(Z)$, where $d = \dim_{\mathbb{R}} \widetilde{Y}$. We want a sheafy version of this.

Fix a variety $\mathcal{B}$, and consider the three maps $p_{ij} : \mathcal{B}^3 \to \mathcal{B}^2$, where $1 \le i < j \le 3$ and p_{ij} forgets the kth coordinate, where k is the unique element of $\{1, 2, 3\} \setminus \{i, j\}$. Then, given $A_1, A_2 \in D(\mathcal{B}^2)$, we define $A_1 * A_2 = (p_{13})_*(p_{12}^* A_1 \otimes p_{23}^* A_2)$. Here the tensor product is the derived tensor product. Going forward, we will assume $\mathcal{B}$ is smooth and proper.

Now, as before, let Y be arbitrary, but let $\widetilde{Y}$ be a smooth closed subvariety of $\mathcal{B} \times Y$. Then not only do we have a map $\widetilde{Y} \to Y$ as before, but we also have a map $p : \widetilde{Y} \to \mathcal{B}$. The Steinberg $Z = \widetilde{Y} \times_Y \widetilde{Y}$ can be considered as a subset of $\mathcal{B}^2 \times Y$; namely it consists of $(b_1, b_2, y) \in \mathcal{B}^2 \times Y$ such that $(b_i, y) \in \widetilde{Y}$ for each i. Thus we have a map $p^2 : Z \to \mathcal{B}^2$ which forgets the Y coordinate. Then $H_i^{BM}(Z) = H^{-i}(\omega_Z) = H^{-i}(p_*^2 \omega_Z)$.

Claim 5.1. *$\mathcal{A} := p_*^2 \omega_Z$ is a ring object in $D(\mathcal{B}^2)$ with respect to convolution, i.e. there is a morphism $\mathcal{A} * \mathcal{A} \to \mathcal{A}$ subject to various conditions.*

(Note from the future: this claim is missing a shift) We will prove this claim next time.

So far we have not involved any groups. Let G be algebraic with subgroup H, and let $\mathcal{B} = G/H$. We have the following diagram, called a convolution diagram:

$$\begin{tikzcd}
G \times \mathcal{B} \arrow[r, "pr_2"] \arrow[d, "H"] \arrow[dd, bend right=60, "f"'] & \mathcal{B} \\
G \times^H \mathcal{B} \arrow[d, "m"] & \\
\mathcal{B} &
\end{tikzcd}$$

We let G act on $\mathcal{B}^2$ diagonally and consider $D_G(\mathcal{B}^2)$, the equivariant constructible derived category. We can embed $\mathcal{B}$ into $\mathcal{B}^2$ by $gH \mapsto (gH, H)$. This induces an equivalence $i^* : D_G(\mathcal{B}^2) \to D_H(\mathcal{B})$. We define a convolution product on $D_H(\mathcal{B})$ as follows. Let $\mathcal{F}_1, \mathcal{F}_2 \in D_H(\mathcal{B})$. We can first pullback in two ways and tensor to get $f^*\mathcal{F}_1 \otimes pr_2^*\mathcal{F}_2 \in D_H(G \times \mathcal{B})$. Since this is H-equivariant on an H-torsor, it descends to an object $f^*\mathcal{F}_1 \widetilde{\otimes} pr_2^*\mathcal{F}_2 \in D(G \times^H \mathcal{B})$. We finally define $\mathcal{F}_1 * \mathcal{F}_2$ to be $q_*(f^*\mathcal{F}_1 \widetilde{\otimes} pr_2^*\mathcal{F}_2)$. I'm not sure I see why this is equivariant

Claim 5.2. *The equivalence $i^* : D_G(\mathcal{B}^2) \to D_H(\mathcal{B})$ respects the two convolutions, i.e. for $A_1, A_2 \in D_G(\mathcal{B}^2)$, we have $i^*(A_1 * A_2) = (i^*A_1) * (i^*A_2)$.*

Now we let G act on a variety V, let Y be an H-stable subvariety of Y, and let $\widetilde{Y} = G \times^H Y$. This has a map $\widetilde{Y} \to V \times \mathcal{B}$ given by $(g, y) \mapsto (gy, gH)$.

Example 5.1. We compare to the setting of the Springer resolution. Let G be reductive, fix a Borel B, and let $N = [B, B]$. These have algebra counterparts; we let $V = \mathfrak{g}$ and $Y = \mathfrak{n} = \mathrm{Lie} N$. We let $\mathcal{B} = G/B$ be the usual flag variety. G has its adjoint action on V, and Y is B-stable. Then $\widetilde{Y} = G \times^B \mathfrak{n} = \widetilde{\mathcal{N}}$ is the usual Springer resolution.

6 Jan 22

6.1 History

In physics, symplectic duality is known as 3D mirror symmetry, which was introduced by Seiberg-Witten around 2005. They were motivated by the Seiberg-Witten prepotential, which is a smooth function related to Toda integrable system. In mathematics, Kostant showed that the Toda system is the universal centralizer. For Hitchin systems, SW prepotentials are functions on the Hitchin base.

$N = 2$ SUSY (supersymmetry) corresponds to Kähler manifolds with $\partial, \overline{\partial}$. Lefschetz did something for $\mathfrak{sl}_2$. $N = 4$ SUSY corresponds to hyperKähler manifolds (so quaternions, not just complex numbers, act on tangent spaces), which also correspond to holomorphic symplectic manifolds. The cohomology admits an action by $\mathfrak{sp}(4, \mathbb{R})$, which is the analogue to $\mathfrak{sl}_2$ in the $N = 2$ case. There is a conjectured relation between these hyperKähler manifolds and hyperspherical varieties. Donaldson constructed a hyperKähler metric on T^*G for G a complex reductive group.

We discuss some mathematical advances in the literature:

- In 2015, Nakajima wrote "towards a mathematical definition of Coulomb branches in 3-dim $N = 4$ gauge theory", in which he found a variety whose Betti numbers matched the physics story. However, the algebra structures did not match.
- Then Braverman-Finkelberg-Nakajima (BFN) wrote a sequel paper in which they replaced H^* by H_*^{BM} on a slightly different variety, and the convolution structure matched the physics.
- BFN also wrote a paper (2017, but has been updated as recently as 2024) titled "Ring objects in the equivariant derived Satake category arising from Coulomb branches", which will be relevant to us.
- Ben-Zvi, Sakellaridis, Venkatesh (BZSV) wrote in 2024 a paper titled "Relative Langlands duality", which first proposed the titled duality.
- Teleman wrote "the role of Coulomb branches in 2D gauge theory".
- Braverman, Dhillon, Finkelberg, Raskin, and Travkin wrote "Coulomb branches of noncotangent type".
- Teleman also wrote "Coulomb branches for quaternionic representations".

We revisit the setup from last lecture, but with slightly different notation. Let L be an algebraic group with subgroup H. Let $\mathcal{B} = L/H$, which we assume to be proper. Let L act on a variety V, and let Y be an H-stable subvariety. Let $\mathcal{Y} = L \times^H Y = \{(\overline{\ell}, v) \in \mathcal{B} \times V \mid v \in \overline{\ell}(Y)\}$. Let $p : \mathcal{Y} \to \mathcal{B}$ be the first projection. $\mathcal{Y}$ is a subbundle of the trivial bundle $\mathcal{B} \times V$. Let Z be the Steinberg variety $\mathcal{Y} \times_V \mathcal{Y} = \{(\bar{\ell}_1, \bar{\ell}_2, v) \in \mathcal{B}^2 \times V \mid v \in \bar{\ell}_1(Y) \cap \ell_2(Y)\}$. Let $p^2 : Z \to \mathcal{B}^2$ be the induced projection. L acts diagonally on $\mathcal{B}^2$ on various varieties, like $\mathcal{B}^2, \mathcal{Y}^2$, and Z.

Example 6.1 (Springer resolution)**.** Let $L = G$ be reductive, $H = B$ a Borel, so that $\mathcal{B}$ is the flag variety. Take $V = \mathcal{N}$, and take $Y = \mathfrak{n} = \mathfrak{b}^\perp$. Then $\mathcal{Y} = \widetilde{\mathcal{N}} = T^*\mathcal{B} \subset \mathcal{B} \times \mathcal{N}$. Z is the usual Steinberg $\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}} \subset \widetilde{\mathcal{N}}^2 = T^*(\mathcal{B}^2)$. Z decomposes as a disjoint union of $T^*_O(\mathcal{B}^2)$ indexed over diagonal G-orbits $O \subset \mathcal{B}^2$.

Example 6.2 (Affine Grassmannian)**.** Let $\mathcal{K} = \mathbb{C}((z))$, $\mathcal{O} = \mathbb{C}[[z]]$. Let $L = G(\mathcal{K})$ for a reductive G, and let $H = G(\mathcal{O})$. Then $\mathcal{B} = Gr_G$ is the affine Grassmannian. This does not literally match our previous setup, since $\mathcal{B}$ is not proper; however, it is ind-proper. For a (smooth) affine G-variety X, let $V = X(\mathcal{K})$ and $Y = X(\mathcal{O})$. Then $\mathcal{Y} = G(\mathcal{K}) \times^{G(\mathcal{O})} X(\mathcal{O})$ sits inside $Gr \times X(\mathcal{K})$ and $Z = \mathcal{Y} \times_{X(\mathcal{K})} \mathcal{Y}$ sits inside $Gr \times Gr \times X(\mathcal{K})$. The group L is in some way "too big".

Recall that the embedding of the singleton H/H into $\mathcal{B}$ induces a monoidal equivalence $D_L(\mathcal{B}^2) \cong D_H(\mathcal{B})$. Recall we have $p^2 : Z \to \mathcal{B}^2$. It is L-equivariant, as is $pr_2 : \mathcal{B}^2 \to \mathcal{B}$. Let $Z' = (pr_2 \circ p^2)^{-1}(H/H)$. Then $Z \cong L \times^H Z'$. Explicitly, $Z' = \{(\bar{\ell}, v) \in \mathcal{B} \times V \mid \bar{\ell}^{-1} v \in Y\}$. We can realize Z' as the intersection of the two subbundles $\mathcal{Y}$ and $\mathcal{B} \times Y$ inside of $\mathcal{B} \times V$.

Example 6.3 (Springer)**.** Recall Z decomposes over G-orbits in $\mathcal{B}^2$. There is a corresponding decomposition for Z' as a disjoint union of $T^*_{O'}(\mathcal{B})$ indexed over B-orbits $O' \subset \mathcal{B}$.

Example 6.4 (Affine Grassmannian)**.** In this setting, $Z' = \{(\dot{g}, \dot{x}) \in \mathcal{Y} \mid \dot{g}^{-1}\dot{x} \in X(\mathcal{O})\}$. BFN call Z' the "variety of triples". BZSV call Z' the "relative Grassmannian" and denote it Gr_G^X.

In the general setting for $p^2 : Z \to \mathcal{B}^2$, we define $A = p^2_* \omega_Z$. In BZSV, the ring structure comes from interpreting A as $\mathrm{Hom}(\mathcal{P}, \mathcal{P}) = \mathcal{P}^* \otimes \mathcal{P}$ for some object $\mathcal{P}$ in some large (infinity-) category.

Claim 6.1. *$A \in D_L(\mathcal{B}^2)$ is a ring object; in particular there is an "associative" map $A * A \to A[d_Y]$.*

Now let $p' : Z' \hookrightarrow Y \to \mathcal{B}$. Let $\mathcal{A} \in D_H(\mathcal{B})$ be $p'_* \omega_{Z'}$. Then the claim above implies $\mathcal{A}$ is also a ring object.

Example 6.5 (Affine Grassmannian)**.** We argue that $L = G(\mathcal{K})$ is too large; in particular, the category $D_L(Z)$ cannot exist, since the L-equivariant cohomology of its objects would be modules over $H^*_L(pt) = H^*(BL)$, but there is no meaningful classifying stack BL for L. However, by restricting to Z', there is a way to define an $\mathcal{A}$ which will be $G(\mathcal{O})$-equivariant; things will need to be renormalized and ind-completed.

7 Jan 27

We recall the affine Grassmannian setting from last time:

Let $\mathcal{K} = \mathbb{C}((z))$, $\mathcal{O} = \mathbb{C}[[z]]$. Let $L = G(\mathcal{K})$ for a reductive G, and let $H = G(\mathcal{O})$. For a (smooth) affine G-variety X, let $V = X(\mathcal{K})$ and $Y = X(\mathcal{O})$. Then $\mathcal{Y} = G(\mathcal{K}) \times^{G(\mathcal{O})} X(\mathcal{O})$ and $Z' = \{(\dot{g}, \dot{x}) \in \mathcal{Y} \mid \dot{g}\dot{x} \in X(\mathcal{O})\}$. Recall we have a map $p : \mathcal{Y} \to X(\mathcal{K})$ which is just the map induced by the action map.

Going forward, we will refer to Z' as just Z, and we no longer use the original Z.
We have a map $\pi : Z \to Gr$ given by sending $(\dot{g}, \dot{x})$ to the $G(\mathcal{O})$ coset of $\dot{g}$. Let $\mathcal{A} = \pi_* \omega_Z[-\dim X(\mathcal{O})]$. This is an element of $\mathrm{Ind}D_{G(\mathcal{O})}(Gr)$, where Ind denotes ind-completion. (I assume we will discuss what that means later)

Claim 7.1. *There is a "ring structure" $\mathcal{A} * \mathcal{A} \to \mathcal{A}$.*

Under some assumptions on X we will also show that the ring structure is commutative, and in what follows, we will assume that we are in this case.

We have $H^\bullet_{G(\mathcal{O})}(\mathcal{A}) = H^{BM}_{G(\mathcal{O})}(Z)[-\dim X(\mathcal{O})]$ acquires a commutative algebra structure over $H^\bullet_{G(\mathcal{O})}(pt) \cong H^\bullet_G(pt) \cong \mathbb{C}[\mathfrak{g}]^G$. We define the Coulomb branch $\mathcal{C}_{X,G}$ to be $\mathrm{Spec}(H^\bullet_{G(\mathcal{O})}(\mathcal{A}))$. The map $\mathbb{C}[\mathfrak{g}]^G \to H^\bullet_{G(\mathcal{O})}(\mathcal{A})$ induces a map $\mathcal{C}_{X,G} \to \mathfrak{g}//G = \mathfrak{c}$.

Recall the derived Satake equivalence $\Phi : D_{G(\mathcal{O})}(Gr) \to \text{dg-mod}(\mathrm{Sym}\mathfrak{g}^\vee, G^\vee)$. Note that $\mathrm{Sym}\mathfrak{g}^\vee = \mathbb{C}[\mathfrak{g}^{\vee *}]$. There is an equivalence $\mathrm{QCoh}^{G^\vee}(\mathfrak{g}^{\vee *}) \to (\mathrm{Sym}\mathfrak{g}^\vee, G^\vee)$-mod given by taking global sections. We define $(T^*X)^\vee$ to be $\mathrm{Spec}(\Phi(\mathcal{A}))$. Note that $\Phi(\mathcal{A})$ is a commutative $\mathbb{C}[\mathfrak{g}^{\vee *}]$-algebra, which induces a map $(T^*X)^\vee \xrightarrow{p} \mathfrak{g}^{\vee *}$. This map is $G^\vee$-equivariant.

Let $y = \dot{g}G(\mathcal{O})$ be a point in Gr. The fiber $\pi^{-1}(y)$ is $\dot{g}X(\mathcal{O}) \cap X(\mathcal{O})$. Let ε_y be the embedding of the point y into Gr. Then, by base change, $H^\bullet(\varepsilon_y^! \mathcal{A}) = H^{BM}_\bullet(\dot{g}X(\mathcal{O}) \cap X(\mathcal{O}))[-\dim X(\mathcal{O})]$. On the other hand, $\varepsilon_y^* \mathcal{A}$ doesn't make sense because of infinite dimensionality issues.

Example 7.1. We work out everything in a very simple case.

Fix an integer k. Let $G = \mathbb{C}^*$ act on $X = \mathbb{C}$ with weight k, i.e. $z \cdot t = z^k t$. Then $Gr = \mathcal{K}^\times / \mathcal{O}^\times \cong \mathbb{Z}$. Note that $G(\mathcal{O}) = \mathcal{O}^\times$ has trivial action on Gr. We have $H^\bullet_{G(\mathcal{O})}(pt) = \mathbb{C}[\xi]$ with ξ in degree 2. We will denote this algebra by S

An object $\mathcal{F} \in Sh_{G(\mathcal{O})}(Gr)$ may be represented as a set of finite dimensional $\mathbb{C}$-vector spaces $\mathcal{F}_\lambda$ for $\lambda \in \mathbb{Z}$, such that $F_\lambda \neq 0$ for only finitely many λ. We have $H^\bullet_{G(\mathcal{O})}(Gr, \mathcal{F}) = \bigoplus_\lambda S \otimes F_\lambda$, which is a free finite rank $\mathbb{Z}$-graded S-module.

Now, $G^\vee = (\mathbb{C}^\times)^\vee = \mathbb{C}^\times = G$, so we can ignore all Langlands duals. The derived Satake equivalence is between $D_{G(\mathcal{O})}(Gr)$ and dg-modules over (S, G), and it is just given by taking $G(\mathcal{O})$-equivariant cohomology. The output is G-equivariant since a G-action is the same as a $\mathbb{Z}$-grading. Namely, for sheaves $\mathcal{F}$, we have $\Phi(\mathcal{F}) = \bigoplus_\lambda S \otimes \mathcal{F}_\lambda = H^\bullet(Gr, \mathcal{F})$, and $z \in \mathbb{C}^\times$ acts on $S \otimes \mathcal{F}_\lambda$ by multiplication by z^λ.

The (modified) Steinberg Z is $\{(t^\lambda, x(t) \in \mathcal{O}) \mid t^{k\lambda}x(t) \in \mathcal{O}\}$. It splits as a disjoint union of spaces $Z_\lambda = \mathcal{O} \cap t^{-k\lambda}\mathcal{O}$, with the identification via the second projection. These are infinite dimensional $\mathbb{C}$-vector spaces. Let d_λ be $-k\lambda$ if $-k\lambda \geq 0$ and 0 otherwise; then $Z_\lambda = t^{d_\lambda}\mathcal{O}$. We have $H^{BM}_{G(\mathcal{O})}(Z_\lambda) = S \cdot [Z_\lambda]$, where $[Z_\lambda]$ is the fundamental class, which sits in degree $\dim_{\mathbb{R}} Z_\lambda$, which is infinite... So we need to do some renormalization.

We have that $t^{d_\lambda}\mathcal{O} \subset \mathcal{O}$, and in fact the quotient vector space $\mathcal{O}/t^{d_\lambda}\mathcal{O}$ is finite dimensional over $\mathbb{C}$, with dimension d_λ. So we reexpress our degrees relative to $\dim \mathcal{O}$. Namely, our renormalized $H^{BM}_{G(\mathcal{O})}(Z_\lambda)$ is $S \cdot r_\lambda$, where $\deg r_\lambda = -d_\lambda$. We have $r_0 = 1 \in S$, which is the unit of $R = H^\bullet_{\mathcal{G}(O)}(\mathcal{A})$, where $\mathcal{A}$ is the object defined earlier. In fact, our renormalized BM homology is exactly this R.

We want to see exactly what $r_\lambda r_\mu$ is. From the geometry, we know that $r_\lambda r_\mu = c_{\lambda,\mu} r_{\lambda+\mu}$ for some $c_{\lambda,\mu} \in S$. To compute $c_{\lambda,\mu}$ we need a general digression.

Let V be a finite dimensional $G = \mathbb{C}^\times$ representation. Let Z, Z' be G-stable subspaces. Then $[Z] \cap [Z'] = e(V/(Z+Z'))[Z \cap Z']$, where $e \in H^\bullet_G(pt) = S$ is the equivariant Euler class. Now, if $z \in G$ acts on $V/(Z+Z')$ by $\operatorname{diag}(z^{a_1}, \dots, z^{a_n})$, then $e = \xi^{a_1+\cdots+a_n}$.

Now, for $p, q \in \mathbb{Z}$, define $d(p,q)$ to be 0 if p, q have the same sign, and $\min(|p|, |q|)$ otherwise. Then $c_{\lambda,\mu}$ from above is $\xi^{d(k\lambda, k\mu)}$.

R is generated by three elements: $x = r_1$, $y = r_{-1}$, and ξ. They satisfy the relation $xy = \xi^k$ and nothing else. Thus $R \cong \mathbb{C}[x, y, \xi]/(xy - \xi^k)$. Then $\operatorname{Spec}(R)$ is (at the level of closed points) $\{(x, y, \xi) \in \mathbb{C}^3 \mid xy - \xi^3 = 0\}$, and this also describes the Coulomb branch $\mathcal{C}_{\mathbb{C},\mathbb{C}^\times} = (T^*\mathbb{C})^\vee$.

This result has the following interpretation. Let $\Gamma \cong \mathbb{Z}/k$ be the group of kth roots of unity in $\mathbb{C}$. Then Γ acts on $\mathbb{C}^2$ via $(u, v) \mapsto (\zeta u, \zeta^{-1} v)$. The algebra $\mathbb{C}[u, v]^\Gamma$ has generators $x = u^k$, $y = v^k$, and $\xi = uv$. Thus the Coulomb branch is $\mathbb{C}^2/\Gamma$. This space is known as a Kleinian singularity of type A_k; it is smooth away from the origin.

8 Jan 29

Today we consider the case where our group G is a torus T over $\mathbb{C}$. We have a cocharacter lattice $X_* = X_*(T) = \mathrm{Hom}(\mathbb{C}^\times, T)$ and character lattice $X^* = X^*(T) = \mathrm{Hom}(T, \mathbb{C}^\times)$. They have a perfect pairing $\langle -, - \rangle : X_* \times X^* \to \mathbb{Z}$ given by $\langle \lambda, \alpha \rangle = \deg(\alpha \circ \lambda)$.

Example 8.1. If $T = (\mathbb{C}^\times)^n$, we may identify both lattices with $\mathbb{Z}^n$, and the pairing between $\lambda = (\lambda_i)$ and $\alpha = (\alpha_i)$ is just the dot product $\sum \lambda_i \alpha_i$.

For a commutative ring probably should be a $\mathbb{C}$-algebra R, the R-points of T may be identified with $R^\times \otimes_{\mathbb{Z}} X_*$ via the isomorphism which sends $r \otimes \lambda$ to $\lambda(r)$. I'm not sure how to make sense of $\lambda(r)$.

We have Lie$T = \mathfrak{t} = \mathbb{C} \otimes X_*$.

Now consider the affine Grassmannian setup, so we have $\mathcal{K}$ and $\mathcal{O}$ as usual. Then $T(\mathcal{K}) \cong \mathcal{K}^\times \otimes X_*$ and $T(\mathcal{O}) \cong \mathcal{O}^\times \otimes X_*$. We have an embedding $X_* \hookrightarrow T(\mathcal{K})$ given by $\lambda \mapsto t^\lambda$.

Example 8.2. If $T = (\mathbb{C}^\times)^n$ and $\lambda = (\lambda_i)$, then t^λ sends z to (z^{λ_i}).

Now, $Gr = T(\mathcal{K})/T(\mathcal{O}) \cong (\mathcal{K}^\times \otimes X_*)/(\mathcal{O}^\times \otimes X_*) \cong (\mathcal{K}^\times/\mathcal{O}^\times) \otimes X_* \cong \mathbb{Z} \otimes X_* \cong X_*$. So it is a discrete abelian group with points t^λ, where $t^\lambda t^\mu = t^{\lambda+\mu}$.

$T(\mathcal{O})$ has trivial action on Gr. Thus, as in last lecture, $\mathcal{F} \in Sh_{T(\mathcal{O})}(Gr)$ is simply a collection of finite dimensional $\mathbb{C}$-vector spaces $\mathcal{F}_\lambda$ indexed by $\lambda \in X_*$, with only finitely many nonzero $\mathcal{F}_\lambda$. We have a convolution $(\mathcal{F}' * \mathcal{F}'')_\lambda = \bigoplus_{\lambda' + \lambda'' = \lambda} \mathcal{F}'_{\lambda'} \otimes_{\mathbb{C}} \mathcal{F}''_{\lambda''}$.

We have $H^\bullet_{T(\mathcal{O})}(pt) = H^\bullet_T(pt) = \mathbb{C}[\mathfrak{t}]$, which again we call S. We have $H^\bullet_{T(\mathcal{O})}(\mathcal{F}) = \bigoplus_\lambda S \otimes \mathcal{F}_\lambda$, a X_*-graded free S-module of finite rank.

Now we need the dual torus. Note that $T \cong \mathbb{C}^\times \otimes_{\mathbb{Z}} X_*(T)$. We define $T^\vee = \mathbb{C}^\times \otimes_{\mathbb{Z}} X^*(T)$, so that $X_*(T^\vee) \cong X^*(T)$ and $X^*(T^\vee) \cong X_*(T)$. It has a Lie algebra $\mathfrak{t}^\vee$ that is canonically isomorphism to $\mathfrak{t}^*$. Thus $(\mathfrak{t}^\vee)^*$ is canonically isomorphic to $\mathfrak{t}$.

This duality tells us that a $X_*(T)$-graded $\mathbb{C}[\mathfrak{t}]$-module is the same thing as a $X^*(T^\vee)$-graded $\mathbb{C}[(\mathfrak{t}^\vee)^*]$-module. Furthermore, grading by the character lattice is the same as giving an action, so a $X^*(T^\vee)$-graded $\mathbb{C}[(\mathfrak{t}^\vee)^*]$-module is the same thing as a $(\mathbb{C}[(\mathfrak{t}^\vee)^*], T^\vee)$-module. In particular, this is true of $H^\bullet_{T(\mathcal{O})}(\mathcal{F})$. It happens that in this case, the derived Satake equivalence sends $\mathcal{F}$ to $H^\bullet_{T(\mathcal{O})}(\mathcal{F}) = \bigoplus_\lambda S \otimes \mathcal{F}_\lambda$.

Now let X be a vector space of dimension d. Let $T \to GL(X)$ be a representation. As before, we get a variety of triples $Z = Z_X$ with a map p to

$Gr = X_*(T)$. Then Z is a disjoint union of fibers $Z_\lambda = p^{-1}(t^\lambda)$, which are all infinite dimensional vector spaces. As before, we renormalize Borel-Moore homology so that each fundamental class $[Z_\lambda]$ becomes a class r_λ in a finite degree, and $R = H^{BM,rn}_{T(\mathcal{O})}(Z) \bigoplus_\lambda Sr_\lambda$. As before, $r_\lambda r_\mu = c(\lambda,\mu)r_{\lambda+\mu}$ for some $c(\lambda,\mu) \in S$. We want to find c in terms of the combinatorics of the representation X of T.

We know $X = \bigoplus_{i=1}^d X_{\alpha_i}$, where each $\alpha_i \in X^*(T)$ and X_{α_i} is a one-dimensional vector space where T acts with weight α_i. Given integer $j \in \mathbb{Z}$, let j_+ be j if $j \geq 0$ and 0 otherwise. Let $j_- = j - j_+$. For $\lambda, \mu \in X_*(T)$ and each α_i, let $m_i(\lambda,\mu) = \langle \alpha_i, \lambda\rangle_+ + \langle \alpha_i, \mu\rangle_+ - \langle \alpha_i, \lambda+\mu\rangle_+$. Then $c(\lambda,\mu) = \prod_{i=1}^d (d\alpha_i)^{m_i(\lambda,\mu)}$, where $d\alpha_i$ is the differential of α_i, so it is a map $\mathfrak{t} \to \mathbb{C}$, i.e. an element of $S = \mathbb{C}[\mathfrak{t}]$.

From the computation we see that R is commutative. We define the Coulomb branch $(T^*X)^\vee = \mathcal{C}_{T,X}$ to be Spec$R$. We have $R_0 = \mathbb{C}[\mathfrak{t}]r_0 = \mathbb{C}[(\mathfrak{t}^\vee)^*]r_0$ a subalgebra of R, so we get a map $\pi : \mathrm{Spec}R \to (\mathfrak{t}^\vee)^*$. The $X^*(T^\vee)$-grading on R gives a $T^\vee$-action on Spec$R$, and the map π is equivariant with respect to this action.

We will now show that $\mathcal{C}_{T,X}$ is a $T^\vee$-hypertoric variety.

8.1 Hypertoric varieties

Going forward, all mentions of "variety" is meant to be "affine variety".

Recall that a spherical T-variety is an T-variety with an open (dense) T-orbit. Let X be such a variety. For $x \in X$, let $T_x = \mathrm{Stab}_T(x)$. Since T is abelian, we have $T_{tx} = T_x$ for all $t \in T$. If x is in the open orbit O, then T_x acts trivially on O, and by continuity, T_x acts trivially on X. Thus we may replace T by $T' = T/T_x$, so that X is a T'-variety with an open dense T'-orbit, on which T' acts freely.

Definition 8.1. A **toric** T-variety is a T-variety with an open dense T-orbit on which T acts freely.

A general fact (which we do not prove) is:

Theorem 8.1. *In a toric T-variety, the number of T-orbits is finite.*

Example 8.3. For $T = \mathbb{C}^\times$, the smooth (affine) toric T-varieties are just $\mathbb{C}^\times$ and $\mathbb{C}$. More generally for $T = (\mathbb{C}^\times)^n$, the only smooth (affine) toric T-varieties are $(\mathbb{C}^\times)^{n_1} \times \mathbb{C}^{n_2}$ where $n_1 + n_2 = n$.

Now fix torus T, integer $d \geq 1$, and cocharacters $\alpha_1, \ldots, \alpha_d \in X_*(T)$. Let $H = (\mathbb{C}^\times)^d$. Then the α_i give a map $\alpha : H \to T$ where $h = (h_i)$ is mapped to $\prod \alpha_i(h_i)$. We can differentiate each α_i, giving maps $d\alpha_i : \mathbb{C} \to \mathfrak{t}$, which we identify with elements a_i in $(\mathfrak{t}^*)^*$. We let $H \times T$ act on $\mathbb{C}^d \times T$ by $(h,t)\cdot(u,t') = (h^{-1}u, \alpha(h)tt')$. This induces a Hamiltonian action of $H \times T$ on $T^*(\mathbb{C}^d \times T) = \mathbb{C}^d \times \mathbb{C}^d \times T \times \mathfrak{t}^*$ which we identify as follows. We denote an element in this cotangent bundle by (u, v, t', ξ). Then $(h,t)\cdot(u,v,t',\xi) = (h^{-1}u, hv, \alpha(h)tt', \xi)$. The moment map for the H action is $\mu(u,v,t,\xi) = (-u_i v_i + a_i(\xi))$, where we identify $(\mathrm{Lie} H)^* = \mathfrak{h}^*$ with $\mathbb{C}^d$. Given a fixed $c \in \mathfrak{h}^*$, we let $X_{\alpha,c} = \mu^{-1}(c)//H$. A hypertoric variety is a variety obtained in this way (you can also allow $//$ to be GIT quotient).

Note. It is an open question whether or not one can intrinsically characterize hypertoric varieties.

9 Feb 3

9.1 Hypertoric Geometry

Fix $d \geq 0$, $H = (\mathbb{C}^\times)^d$, $\mathfrak{h} = \mathrm{Lie} H = \mathbb{C}^d$. Let H act on $\mathbb{C}^d$ with weight -1 in all coordinates; we write $h \cdot u = h^{-1}u$.

Let T be a torus with Lie algebra $\mathfrak{t}$. Fix $\alpha : H \to T$ and let H act on $\mathbb{C}^d \times T$ by $h \cdot (u, t) = (h^{-1}u, \alpha(h)t)$. This gives a Hamiltonian action on $T^*(\mathbb{C}^d \times T)$ with moment map μ_H. Let $X = \mu^{-1}(0)//H$. Then X is an affine Poisson variety.

Let T act on $\mathbb{C}^d \times T$ with trivial action on the first factor and by translation on the second factor. Then the T and H actions commute, giving X a Hamiltonian T-action with moment map $\mu = \mu_T$.

Theorem 9.1.

1. *X is a reduced irreducible and Cohen-Macaulay with* $\dim X = 2 \dim T$. *In fact, X is birational to $T^*(T)$.*
2. *μ is flat and every fiber of μ is a finite union of T-orbits. In particular, every irreducible component of every fiber of μ is a toric T-variety.*

Proof Sketch. One checks that the fibers of μ are finite unions of T-orbits. This implies that $\dim X \leq \dim \mu^{-1}(x) + \dim \mathfrak{t}^* \leq 2 \dim T$.

Later, we will construct a map $\phi : T^*(T) \to X$ such that $\phi^* : \mathbb{C}[X] \to \mathbb{C}[T \times \mathfrak{t}^*]$ is injective. This implies $\mathbb{C}[X]$ is a domain, which implies X is reduced and irreducible.

Now, $\alpha : H \to T$ must be of the form $\alpha((h_i)) = \prod \alpha_i(h_i)$ for $\alpha_i : \mathbb{C}^\times \to T$. These give $d\alpha_i : \mathbb{C} \to \mathfrak{t}$, which give $a_i = d\alpha_i(1) \in \mathfrak{t} = (\mathfrak{t}^*)^*$. Recall the H action on $T^*(\mathbb{C}^d \times T) = \mathbb{C}^d \times (\mathbb{C}^d)^* \times T \times \mathfrak{t}^*$ has the explicit form $h \cdot (u, v, t, \xi) = (h^{-1}u, hv, \alpha(h)t, \xi)$. Similarly, the moment map μ_H sends (u, v, t, ξ) to $(-u_i v_i + a_i(\xi))$. Thus $X = \{(u, v, t, \xi) \mid u_i v_i = a_i(\xi)\}//H$. The moment map μ_T sends (the equivalence class of) (u, v, t, ξ) to ξ.

Now consider $a_i^\perp = \{\xi \mid a_i(\xi) = 0\}$, a hyperplane in $\mathfrak{t}^*$. Let $\mathfrak{t}^*_{reg} = \mathfrak{t}^* \setminus (\bigcup_i a_i^\perp)$. Thus if $(u, v, t, \xi) \in \mu_H^{-1}(0)$, then $u_i v_i = a_i(\xi) \neq 0$ for all i, so $v_i = a_i(\xi)/u_i$ for all i. Then the H-orbit of (u, v, t, ξ) is closed and isomorphic to H. Thus, generically, $\dim X = 2 \dim T$, meaning that $\dim X \geq 2 \dim T$. But we proved that $\dim X \leq 2 \dim T$, so we have $\dim X = 2 \dim T$.

Let $B = \mathbb{C}[u, v, t, \xi]$ be the coordinate ring of $T^*(\mathbb{C}^d \times T)$. Inside this we have $A = B^H$, which contains $\mathbb{C}[\xi]$ as well as each $u_i v_i$. Let $f_i = u_i v_i - a_i(\xi) \in A$. Then $\mathbb{C}[X] = (B/B\{f_i\})^H$, which by Hilbert is $A/A\{f_i\}$. A is a domain so $Y = \mathrm{Spec} A$ is reduced and irreducible. The moment map μ_H descends to a map $\overline{\mu}_H : Y \to \mathfrak{h}^*$, and $X = \overline{\mu}_H^{-1}(0)$.

Hochster-Roberts theorem says (according to wikipedia the original statement is slightly different) that if $\widetilde{Y} \to Y$ is surjective and $\widetilde{Y}$ has rational singularities, then Y has rational singularities. As a consequence, the categorical quotient of a Cohen-Macaulay variety is Cohen-Macaulay. (This requires characteristic 0.) Since $T^*(\mathbb{C}^d \times T)$ is smooth, this implies Y is Cohen-Macaulay. Considerations with $\mathfrak{t}^*_{reg}$ shows $\dim Y = d + 2\dim T$. Then we obtain $\dim X = \dim \overline{\mu}_H^{-1}(0) = \dim Y - \dim \mathfrak{h}^*$, so X is a complete intersection in Y. A complete intersection in a Cohen-Macaulay variety is again Cohen-Macaulay, so X is Cohen-Macaulay.

Now, the fibers of $\mu^{-1}(x)$ are at most $\dim T$. This implies (???)all fibers have exactly dimension $\dim T$. Since X is CM, $\mathfrak{t}^*$ is smooth, and the fibers of μ all have dimension $\dim X - \dim \mathfrak{t}^*$, "miracle flatness" (Stacks Project Tag 00R4) implies that μ is flat.

From this proof we extract the algebra $A = \mathbb{C}[u, v, t, \xi]^H$. We know it contains ξ as well as the $u_i v_i$. For $\lambda \in X^*(T)$, we write t^λ instead of $\lambda(t)$. Let $f(u, v, t) = t^\lambda \prod u_i^{m_i} v_i^{n_i}$. An element $h \in H$ sends f to $f \prod h_i^{\langle \alpha_i, \lambda\rangle + n_i - m_i}$, so $f \in A$ iff $m_i - n_i = \langle \alpha_i, \lambda \rangle$ for all i. Define w_i to be $u_i^{-\langle \alpha_i, \lambda\rangle}$ if $\langle \alpha_i, \lambda\rangle < 0$, and $v_i^{\langle \alpha_i, \lambda\rangle}$ if $\langle \alpha_i, \lambda \rangle \geq 0$. Then define $r_\lambda = t^\lambda \prod w_i$. We may also write this as $t^\lambda \prod u_i^{\langle \alpha_i, \lambda\rangle_-} v_i^{\langle \alpha_i, \lambda\rangle_+}$, where the plus and minus notation was defined in the last lecture add hyperref. Then A is spanned by (possibly has basis given by) the elements $f(\xi) \prod (u_i v_i)^{\ell_i} r_\lambda$ for $\ell_i \geq 0$ and $\lambda \in X^*(T)$. Then $r_\lambda r_\mu = r_{\lambda+\mu} \prod (u_i v_i)^{\langle \alpha_i, \lambda\rangle_+ + \langle \alpha_i, \mu\rangle_+ - \langle \alpha_i, \lambda+\mu\rangle_+}$, which matches the Coulomb branch calculation. The similarity between these computations is currently a mystery.

We note that $\mathbb{C}[X] = A/(A(u_i v_i - a_i(\xi)))$ is generated by $\mathbb{C}[\xi]$ and r_λ for $\lambda \in X^*(T)$, again matching the labeling for the Coulomb branch.

Let $\tau : T^*(\mathbb{C}^d \times T) \to T^*(\mathbb{C}^d \times T)$ be $\tau(u, v, t, \xi) = (v, u, t^{-1}, \xi)$. Since the moment map μ_H does not care about t, and it is symmetric in u, v, we see that τ restricts to an involution on the zero fiber $\mu_H^{-1}(0)$. We will show that in fact it descends to an involution on X which sends r_λ to $r_{-\lambda}$. This is supposed to be a kind of Fourier transform, in a sense which we will explain later.

Define a map $T^*(T) \to T^*(\mathbb{C}^d \times T)$ by $(t, \xi) \mapsto (a_i(\xi), 1, t, \xi)$. The image is contained in $\mu_H^{-1}(0)$, so it induces a map $\varphi : T^*(T) \to X$, and it is a birational isomorphism. In particular, $\phi^* : \mathbb{C}[X] \to \mathbb{C}[T^*(T)]$ is injective and it is an isomorphism on function fields. Let $\widetilde{\tau} = \phi^* \tau (\phi^*)^{-1} : \mathbb{C}(T^*(T)) \to \mathbb{C}(T^*(T))$. Next time we will discuss the following theorem:

Theorem 9.2. $\varphi^*(\mathbb{C}[X]) = \mathbb{C}[T^*(T)] \cap \widetilde{\tau}(\mathbb{C}[T^*(T)])$.

10 Feb 5

Recall our setup; we have $\alpha = (\alpha_i) : H = (\mathbb{C}^\times)^d \to T$, $a_i = d\alpha_i(1) \in \mathfrak{t}$, which we instead think of as elements of $(\mathfrak{t}^*)^*$. We have a Hamiltonian H action on $T^*(\mathbb{C}^d \times T)$ with moment map $\mu_H(u, v, t, \xi) = (-u_i v_i + a_i(\xi))$, and $X = \mu_H^{-1}(0)//H$. X is a Hamiltonian T-variety with moment map $\mu = \mu_T : (u, v, t, \xi) \mapsto \xi$. We have $\mathbb{C}[X] = \sum_{\lambda \in X^*(T)} \mathbb{C}[\xi] r_\lambda$, where $r_\lambda = t^\lambda \prod u_i^{\langle \alpha_i, \lambda \rangle_-} v_i^{\langle \alpha_i, \lambda \rangle_+}$.

At the end of last lecture, we introduced an involution τ on $T^*(\mathbb{C}^d \times T)$ which sends (u, v, t, ξ) to (v, u, t^{-1}, ξ). We note some properties of this map:

1. For $h \in H$, $\tau(h\eta) = h^{-1}\tau(\eta)$.
2. For $t \in T$, $\tau(t\eta) = t^{-1}\tau(\eta)$.
3. τ restricts to an involution on $\mu_H^{-1}(0)$.
4. $\tau(r_\lambda) = r_{-\lambda}$.
5. τ descends to an involution on X.

The first four properties follow from direct computation. The last property can be seen in two ways. From property 1, we see that τ respects H-orbits, and in particular closed H-orbits. Alternatively, since τ does nothing to ξ, we see that it fixes $\mathbb{C}[\xi]$, and thus by property 4 and the description of $\mathbb{C}[X]$, we see τ acts on $\mathbb{C}[X]$.

Recall we have hyperplanes $a_i^\perp$ in $\mathfrak{t}^*$ and the regular locus $\mathfrak{t}^*_{reg} = \mathfrak{t}^* \setminus \bigcup a_i^\perp$. We let $\mu_{H,reg}^{-1}(0)$ be the subset of $\mu_H^{-1}(0)$ where $\xi \in \mathfrak{t}^*_{reg}$. We let $X_{reg} = \mu^{-1}(\mathfrak{t}^*_{reg})$.

Define $\widetilde{\varphi} : T \times \mathfrak{t}^* \to \mu_H^{-1}(0)$ by $(t, \xi) \mapsto (1, a_i(\xi), t, \xi)$. The map $H \times T \times \mathfrak{t}^*_{reg} \to \mu_{H,reg}^{-1}(0)$ sending (h, t, ξ) to $h\widetilde{\varphi}(t, \xi) = (h^{-1}, ha(\xi), \alpha(h)t, \xi)$ is an isomorphism, since the first coordinate determines h, the fourth coordinate determines ξ, and the third coordinate together with h determines t. Since the u_i, v_i are nonzero when dealing with the regular locus, there is no problem about lack of injectivity.

ADD DIAGRAMS

We let φ be the composite $T \times \mathfrak{t}^* \xrightarrow{\widetilde{\varphi}} \mu_H^{-1}(0) \to X$. Then $\varphi^* r_\lambda = t^\lambda \prod_{\langle \alpha_i, \lambda \rangle > 0} a_i(\xi)^{\langle \alpha_i, \lambda \rangle}$.

Example 10.1. Let $H = T = \mathbb{C}^\times$, and let $\alpha(h) = h^k$ for $k \in \mathbb{Z}$. Then $\mu_H^{-1}(0) = \{(u, v, t, \xi) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^\times \times \mathbb{C} \mid uv = k\xi\}$. The H-invariant functions are generated by $\xi, x = r_1, y = r_{-1}$. write explicit form of x, y. Then $xy = (uv)^{|k|} = (k\xi)^{|k|}$. Thus we can write $X = \{(x, y, \xi) \in \mathbb{C}^3 \mid xy = (k\xi)^{|k|}\}$. The map φ sends (t, ξ) to $(1, k\xi, t, \xi)$. The corresponding (x, y) depends on $k \geq 0$ or $k < 0$; for $k \geq 0$ we have $(x, y) = (t, (k\xi)^k t^{-1})$ and for $k < 0$ we have

$(x, y) = ((k\xi)^{-k}t, t^{-1})$. Going forward we just assume $k \geq 0$.

For $\xi \neq 0$, we have φ restricts to an isomorphism $\mathbb{C}^\times \times \{\xi\} \to \mu^{-1}(\xi)$, where the fiber is a single T-orbit. For $\xi = 0$, we get $\mu^{-1}(0) = \{(x, y) \mid xy = 0\}$, which is three T-orbits (two punctured axes and the origin). φ restricts to an isomorphism $\mathbb{C}^\times \times \{0\} \to \{(x, 0) \mid x \neq 0\}$.

Since $x = u^k t$ and $y = v^k t^{-1}$ (again we are assuming $k \geq 0$), we see that $\tau(x) = v^k t^{-1} = y$ and thus $\tau(y) = x$. In fact this is true for $k < 0$ as well.

Now specialize to $k = 1$. Then $X = \{(x, y, \xi) \mid xy = \xi\}$ is isomorphic to $\mathbb{C}^2$ via the first two coordinates. The map φ sends (t, ξ) to $(t, \xi t^{-1})$, which is not the standard embedding.

Return to $k > 0$ arbitrary. We know $\varphi : T \times \mathfrak{t}^*_{reg} = \mathbb{C}^\times \times \mathbb{C}^\times \xrightarrow{\sim} X_{reg}$. For $\xi \neq 0$ we have $\tau(\varphi(t, \xi)) = \tau(1, k\xi, t, \xi) = (k\xi, 1, t^{-1}, \xi)$. But we are in X, so we are free to act by H. Take $h = k\xi$ to get $(k\xi, 1, t^{-1}, \xi) = (1, k\xi, (k\xi)^k t^{-1}, \xi)$. Since $\xi \neq 0$ we can invert φ to get $(\gamma(\xi)t^{-1}, \xi)$, where $\gamma(\xi) = (k\xi)^k$. Thus let $\tau' = \varphi^{-1}\tau\varphi$, which only makes sense when $\xi \neq 0$, and it sends (t, ξ) to $(\gamma(\xi)t^{-1}, \xi)$.

We return to the setting where $H = (\mathbb{C}^\times)^d$ and T is arbitrary. We have used the notation implicitly before, but let $a : \mathfrak{t}^* \to \mathfrak{h}^*$ be the map $\xi \mapsto (a_i(\xi))$. We see that $a(\mathfrak{t}^*_{reg}) \subset H$, so we can form the composite $\gamma = \alpha \circ a$. This gives a rational map, also called γ, from $\mathfrak{t}^*$ to T. We can then define τ' by the exact same formula as above.

Example 10.2. Let $H = T = (\mathbb{C}^\times)^d$ and $\alpha = \mathrm{id}$. Then $X = \mathbb{C}^{2d}$. The map γ (on $\mathfrak{t}^*_{reg}$) is also the identity. The graph of γ, $\Gamma(\gamma)$, sits inside $\mathfrak{t}^* \times T = T^*(T)$.

very confused at the end, need to add some stuff