MATH 7211 Homework 10

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1 Problem 1

Let N_1, N_2 be left R-modules. Prove that an R-module N_1, N_2 is isomorphic to the direct sum $N_1 \oplus N_2$ iff there exist R-module maps $\iota_j : N_j \to N$ and $\pi_j : N \to N_j$ for $j = 1, 2$ such that $\pi_i \circ \iota_j = \delta_{ij}\mathrm{id}_{N_j}$ for $i, j = 1, 2$ and $\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = \mathrm{id}_N$.

Proof. ($\to$) Let $\varphi : N \to N_1 \oplus N_2$ be an isomorphism. Let $\iota'_j : N_j \to N_1 \oplus N_2$ and $\pi'_j : N_1 \oplus N_2 \to N_j$ be the maps $\iota'_1 : n_1 \mapsto (n_1, 0)$; $\iota'_2 : n_2 \mapsto (0, n_2)$; $\pi'_1 : (n_1, n_2) \mapsto n_1$; $\pi'_2 : (n_1, n_2) \mapsto n_2$. These are all clearly R-module maps by definition of $N_1 \oplus N_2$. Then let $\iota_j = \varphi^{-1} \circ \iota'_j$ and $\pi_j = \pi'_j \circ \varphi$. Then $\pi_i \circ \iota_j = \pi'_i \circ \varphi \circ \varphi^{-1} \circ \iota'_j = \pi'_i \circ \iota'_j$, so it suffices to show $\pi'_i \circ \iota'_j = \delta_{ij}\mathrm{id}_{N_j}$. We have

$$\begin{aligned}
(\pi'_1 \circ \iota'_1)(n_1) &= \pi'_1(n_1, 0) = n_1, \\
(\pi'_1 \circ \iota'_2)(n_2) &= \pi'_1(0, n_2) = 0, \\
(\pi'_2 \circ \iota'_1)(n_1) &= \pi'_2(n_1, 0) = 0, \\
(\pi'_2 \circ \iota'_2)(n_2) &= \pi'_2(0, n_2) = n_2.
\end{aligned}$$

This computation proves $\pi'_i \circ \iota'_j = \delta_{ij}\mathrm{id}_{N_j}$. Next, we have $\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = \varphi^{-1} \circ \iota'_1 \circ \pi'_1 \circ \varphi + \varphi^{-1} \circ \iota'_2 \circ \pi'_2 \circ \varphi = \varphi^{-1} \circ (\iota'_1 \circ \pi'_1 + \iota'_2 \circ \pi'_2) \circ \varphi$. Then it suffices to prove $\iota'_1 \circ \pi'_1 + \iota'_2 \circ \pi'_2 = \mathrm{id}_{N_1 \oplus N_2}$, since $\varphi^{-1} \circ \mathrm{id}_{N_1 \oplus N_2} \circ \varphi = \varphi^{-1} \circ \varphi = \mathrm{id}_N$. We have

$$\begin{aligned}
(\iota'_1 \circ \pi'_1 + \iota'_2 \circ \pi'_2)(n_1, n_2) &= (\iota'_1 \circ \pi'_1)(n_1, n_2) + (\iota'_2 \circ \pi'_2)(n_1, n_2) \\
&= \iota'_1(n_1) + \iota'_2(n_2) = (n_1, 0) + (0, n_2) = (n_1, n_2),
\end{aligned}$$

which is exactly what we want. This completes the proof of this direction.

($\leftarrow$) Let $\varphi : N \to N_1 \oplus N_2$ be the map $\varphi(n) = (\pi_1(n), \pi_2(n))$. We aim to show that φ is an isomorphism. Since π_1, π_2 are R-module maps, φ is also an R-module map. Suppose $n \in \ker \varphi$, so that $\pi_1(n) = \pi_2(n) = 0$. Then $n = \mathrm{id}_N(n) = (\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2)(n) = \iota_1(0) + \iota_2(0) = 0 + 0 = 0$. Thus φ is injective. Next, given any $(n_1, n_2) \in N_1 \oplus N_2$, let $n = \iota_1(n_1) + \iota_2(n_2)$. We claim $\varphi(n) = (n_1, n_2)$, so that φ is surjective. Indeed, $\varphi(n) = (\pi_1(\iota_1(n_1) +$

$\iota_2(n_2)), \pi_2(\iota_1(n_1) + \iota_2(n_2))) = (\mathrm{id}_{N_1}(n_1) + 0(n_1), 0(n_2) + \mathrm{id}_{N_2}(n_2)) = (n_1, n_2)$ as desired. Thus φ is an isomorphism. $\square$

2 Problem 2

Let N_1, N_2 be left R-modules. Prove that an R-module N is isomorphic to the direct sum $N_1 \oplus N_2$ iff there exist R-module maps $\iota_j : N_j \to N$ for $j = 1, 2$ which satisfy the following universal property: for any pair of R-module maps $i_j : N_j \to M$ for $j = 1, 2$, there exists a unique R-module map $\phi : N \to M$ such that $\phi \circ \iota_j = i_j$.

Proof. ($\rightarrow$) Let $\varphi : N_1 \oplus N_2 \to N$ be an isomorphism. Let $\iota_1' : N_1 \to N_1 \oplus N_2$ be the map $n_1 \mapsto (n_1, 0)$, and let $\iota_2' : N_2 \to N_1 \oplus N_2$ be the map $n_2 \mapsto (0, n_2)$. Then let $\iota_j = \varphi \circ \iota_j'$. Let $i_j : N_j \to M$ be two R-module maps. By linearity, an R-module map $\phi' : N_1 \oplus N_2 \to M$ is uniquely determined by where it maps elements of the form $(n_1, 0)$ and $(0, n_2)$, i.e., by the pair of maps $\phi' \circ \iota_j'$. Using the isomorphism φ, a map $\phi : N \to M$ is uniquely determined by the corresponding map $\phi' = \phi \circ \varphi^{-1} : N_1 \oplus N_2 \to M$. Furthermore, $\phi \circ \iota_j = \phi' \circ \varphi^{-1} \circ \varphi \circ \iota_j' = \phi' \circ \iota_j'$. By our previous observation, ϕ' is uniquely determined by defining the maps $\phi' \circ \iota_j'$. Thus ϕ is also uniquely defined by defining the maps $\phi \circ \iota_j$, concluding the proof of this direction.

($\leftarrow$) Suppose we are given maps $\iota_j : N_j \to N$ satisfying the universal property. Consider the pair of maps id_{N_1} and $0 : N_1 \to N_2$. By the universal property, there is a unique R-module map $\pi_1 : N \to N_1$ such that $\pi_1 \circ \iota_1 = \mathrm{id}_{N_1}$ and $\pi_1 \circ \iota_2 = 0$. Similarly, there is a unique R-module map $\pi_2 : N \to N_2$ such that $\pi_2 \circ \iota_1 = 0$ and $\pi_2 \circ \iota_2 = \mathrm{id}_{N_2}$. Applying the universal property to the maps ι_1, ι_2 themselves, there is a unique map $\phi : N \to N$ for which $\phi \circ \iota_j = \iota_j$. Clearly id_N has this property. However, $(\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2) \circ \iota_1 = \iota_1 \circ \pi_1 \circ \iota_1 + \iota_2 \circ \pi_2 \circ \iota_1 = \iota_1 \circ \mathrm{id}_{N_1} + \iota_2 \circ 0 = \iota_1$, and similarly, $(\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2) \circ \iota_2 = \iota_2$. Since the universal property gives uniqueness, we must have $\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = \mathrm{id}_N$. Thus, by exercise 1, we are done. $\square$

3 Problem 3

Let M be a left R-module, and $f : M_1 \to M_2$ a left R-module map. We define $\mathrm{Hom}(f, M) : \mathrm{Hom}_R(M_2, M) \to \mathrm{Hom}_R(M_1, M)$ by $\mathrm{Hom}(f, M)(g) = g \circ f$.

(a) Show that $\mathrm{Hom}(f' \circ f, M) = \mathrm{Hom}(f, M) \circ \mathrm{Hom}(f', M)$ for any R-module maps $f : M_1 \to M_2$, $f' : M_2 \to M_3$.

Proof. $\mathrm{Hom}(f' \circ f, M)(g) = g \circ (f' \circ f) = (g \circ f') \circ f = \mathrm{Hom}(f, M)(g \circ f') = (\mathrm{Hom}(f, M) \circ \mathrm{Hom}(f', M))(g)$. $\square$

(b) Show that $\mathrm{Hom}(\mathrm{id}_N, M) = \mathrm{id}_{\mathrm{Hom}_R(N,M)}$ for any left R-module N.

Proof. $\mathrm{Hom}(\mathrm{id}_N, M)(g) = g \circ \mathrm{id}_N$ is the composition $n \mapsto n \mapsto g(n)$, which is $n \mapsto g(n)$, so $\mathrm{Hom}(\mathrm{id}_N, M)(g) = g$. $\square$

(c) Show that if $f : M_1 \to M_2$ is surjective, then $\mathrm{Hom}(f, M)$ is injective.

Proof. Suppose $\mathrm{Hom}(f, M)(g) = g \circ f = 0$. For any $y \in M_2$, there is an $x \in M_1$ such that $y = f(x)$. Then $g(y) = g(f(x)) = (g \circ f)(x) = 0(x) = 0$. Since y is arbitrary, this means $g = 0$, so $\mathrm{Hom}(f, M)$ is indeed injective. $\square$

4 Problem 4

Let M be a left R-module, and $f : M_1 \to M_2$ a left R-module map. We define $\mathrm{Hom}(M, f) : \mathrm{Hom}_R(M, M_1) \to \mathrm{Hom}_R(M, M_2)$ by $\mathrm{Hom}(M, f)(g) = f \circ g$.

(a) Show that $\mathrm{Hom}(f' \circ f, M) = \mathrm{Hom}(M, f') \circ \mathrm{Hom}(M, f)$ for any R-module maps $f : M_1 \to M_2, f' : M_2 \to M_3$.

Proof. $\mathrm{Hom}(f' \circ f, M)(g) = (f' \circ f) \circ g = f' \circ (f \circ g) = f' \circ (\mathrm{Hom}(f, M)(g)) = \mathrm{Hom}(f', M)(\mathrm{Hom}(f, M)(g)) = (\mathrm{Hom}(f', M) \circ \mathrm{Hom}(f, M))(g)$. $\square$

(b) Show that $\mathrm{Hom}(M, \mathrm{id}_N) = \mathrm{id}_{\mathrm{Hom}_R(M,N)}$ for any left R-module N.

Proof. We have $\mathrm{Hom}(M, \mathrm{id}_N)(g) = \mathrm{id}_N \circ g$, and $(\mathrm{id}_N \circ g)(m) = \mathrm{id}_N(g(m)) = g(m)$, so $\mathrm{Hom}(M, \mathrm{id}_N)(g) = g$ for any $g \in \mathrm{Hom}_R(M, N)$, proving $\mathrm{Hom}(M, \mathrm{id}_N) = \mathrm{id}_{\mathrm{Hom}_R(M,N)}$. $\square$

(c) Show that if f is injective, then $\mathrm{Hom}(M, f)$ is injective.

Proof. Suppose $\mathrm{Hom}(M, f)(g) = f \circ g$ is 0. Then the image of g is contained in the kernel of f, which is 0 since f is injective. Thus the image of g is 0, meaning g is 0. Thus the kernel of $\mathrm{Hom}(M, f)$ is 0, which means $\mathrm{Hom}(M, f)$ is injective. $\square$

5 Problem 5

Let V be a finite dimensional vector space over a field F. Then for any vector space W over F, prove that $\mathrm{Hom}_F(V, W) \cong V^* \otimes W$ as vector spaces over F.

Proof. Choose a basis $\{v_1, ..., v_n\}$ for V and a basis $\{w_1, ..., w_m\}$ for W. Let $\{v^1, ..., v^n\}$ be the dual basis for V^* and let $\{w^1, ..., w^m\}$ be the dual basis for W^*. Define a function $\Psi : \mathrm{Hom}_F(V, W) \to V^* \otimes W$ by $\Psi(\varphi) = \sum_{i,j} w^j(\varphi(v_i))(v^i \otimes w_j)$. We show that Ψ is a linear isomorphism. To show that it is linear, it suffices to show that the assignment $\varphi \mapsto w^j(\varphi(v_i))$ is linear for each i, j; we have $w^j((a\varphi + \phi)(v_i)) = w^j(a\varphi(v_i) + \phi(v_i)) = w^j(a\varphi(v_i)) + w^j(\phi(v_i)) = aw^j(\varphi(v_i)) + w^j(\phi(v_i))$. By a previous homework, we know that the set of $v^i \otimes w_j$ over each i, j is a basis for $V^* \otimes W$. Thus if $\Psi(\varphi) = 0$, we must have $w^j(\varphi(v_i)) = 0$ for each i, j. Then $\varphi(v_i) = 0$ for each i, so $\varphi = 0$. Thus Ψ is injective. Again using the fact that $v^i \otimes w_j$ is a basis, to show Ψ is surjective, it suffices to show that given mn constants $a_{ij} \in F$, we can find a linear map φ for which $w^j(\varphi(v_i)) = a_{ij}$. Of course, to determine a linear map out of V, it suffices to specify the map on basis elements. Thus we define $\varphi(v_i) = \sum_k a_{ik} w_k$, so that $w^j(\varphi(v_i)) = a_{ij}$ as desired. Thus Ψ is an isomorphism. $\square$