MATH 7520 Homework 2

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Problem 6

Distinguish the spaces $S^2 \times S^4$ and $\mathbb{C}P^3$ using cohomology rings.

Proof. Recall that as graded rings, $H^*(\mathbb{C}P^3) \cong \mathbb{Z}[\alpha]/(\alpha^4)$, with $|\alpha| = 2$. In particular, $\alpha \smile \alpha \neq 0$. On the other hand, let β generate $H^2(S^2)$, and let 1 generate $H^0(S^4)$. Since the homology and cohomology groups of spheres are free abelian and finitely generated, we have as graded rings $H^*(S^2 \times S^4) \cong H^*(S^2) \otimes H^*(S^4)$. In particular, $H^2(S^2 \times S^4)$ is generated by $\beta \times 1$. Since $H^i(S^2) = 0$ for $i > 2$, we have $\beta \smile \beta = 0$. Thus, $(\beta \times 1) \smile (\beta \times 1) = (\beta \smile \beta) \times (1 \smile 1) = 0 \times 1 = 0$. Since any element in $H^2(S^2 \times S^4)$ is a multiple of $\beta \times 1$, it must square to 0. Thus, the cohomology rings $H^*(\mathbb{C}P^3)$ and $H^*(S^2 \times S^4)$ cannot be isomorphic, since one has a degree 2 element with non-zero square, while the other has no such element. It follows that $S^2 \times S^4$ and $\mathbb{C}P^3$ are not homotopy equivalent. $\square$

Problem 7

Using cup products, show that every map $S^{k+l} \to S^k \times S^l$ induces the trivial homomorphism in top-dimensional homology, assuming $k > 0$ and $l > 0$.

Proof. Since the homology and cohomology groups of spheres are free abelian and finitely generated, the homology and cohomology groups of products of spheres are also free abelian and finitely generated. It follows that maps in homology are exactly dual to maps in cohomology. In particular, it suffices to show that a map $f : S^{k+l} \to S^k \times S^l$ induces the trivial homomorphism in top-dimensional cohomology.

Let α generate $H^k(S^k)$, and let β generate $H^l(S^l)$. Let p_1 be the projection $S^k \times S^l \to S^k$, and let p_2 be the projection $S^k \times S^l \to S^l$. Then by Künneth's theorem, $H^{k+l}(S^k \times S^l)$ is generated by $\alpha \times \beta = p_1^*\alpha \smile p_2^*\beta$. Since S^{k+l} has 0 cohomology in dimensions k and l, we have $f^*p_1^*\alpha = 0$ and $f^*p_2^*\beta = 0$. Thus, f^* sends $\alpha \times \beta$ to $0 \smile 0 = 0$. Then f^* is the 0 map on $k+l$ dimensional cohomology. The dual of 0 is 0, so f_* is the 0 map of $k + l$ dimensional homology. $\square$

Problem 8

What can you say about the cohomology ring $H^*((S^1)^{\times n};\mathbb{Z})$ of the n-dimensional torus?

Proof. By induction and Künneth's theorem, it is the n-fold tensor product of an exterior algebra (over $\mathbb{Z}$) with one generator, denoted by $\mathbb{Z}\langle a\rangle$. We claim that this ring is isomorphic to an exterior algebra with n generators, denoted by $\mathbb{Z}\langle a_1,\ldots,a_n\rangle$. By induction, it suffices to prove

$$\mathbb{Z}\langle a_1,\ldots,a_n\rangle \cong \mathbb{Z}\langle b_1,\ldots,b_{n-1}\rangle \otimes \mathbb{Z}\langle b_n\rangle.$$

We want to define a map $\phi : \mathbb{Z}\langle a_1,\ldots,a_n\rangle \to \mathbb{Z}\langle b_1,\ldots,b_{n-1}\rangle \otimes \mathbb{Z}\langle b_n\rangle$ by $\phi(a_j) = b_j \otimes 1$ for $j < n$ and $\phi(a_n) = 1 \otimes b_n$. To make sure this gives a well-defined map, we need to check that $\phi(a_i)\phi(a_j) + \phi(a_j)\phi(a_i) = 0$ and $\phi(a_i)^2 = 0$ for all i, j. In the following, assume $i, j < n$. Then

$$\begin{aligned}\phi(a_i)\phi(a_j) + \phi(a_j)\phi(a_i) &= (b_i \otimes 1)(b_j \otimes 1) + (b_j \otimes 1)(b_i \otimes 1)\\ &= b_ib_j \otimes 1 + b_jb_i \otimes 1 = (b_ib_j + b_jb_i) \otimes 1 = 0 \otimes 1 = 0,\end{aligned}$$

$$\begin{aligned}\phi(a_i)\phi(a_n) + \phi(a_n)\phi(a_i) &= (b_i \otimes 1)(1 \otimes b_n) + (1 \otimes b_n)(b_i \otimes 1)\\ &= b_i \otimes b_n + (-1)^{1^2} b_i \otimes b_n = 0,\end{aligned}$$

$$\phi(a_i)^2 = (b_i \otimes 1)(b_i \otimes 1) = b_i^2 \otimes 1 = 0 \otimes 1 = 0,$$

$$\phi(a_n)^2 = (1 \otimes b_n)(1 \otimes b_n) = 1 \otimes b_n^2 = 1 \otimes 0 = 0.$$

Therefore, ϕ is well-defined. The $\phi(a_i)$ generate $\mathbb{Z}\langle b_1,\ldots,b_{n-1}\rangle \otimes \mathbb{Z}\langle b_n\rangle$, since the b_i for $i < n$ generate $\mathbb{Z}\langle b_1,\ldots,b_{n-1}\rangle$ and b_n generates $\mathbb{Z}\langle b_n\rangle$. Thus, ϕ is surjective. The $\phi(a_i)$ also do not satisfy any relations other than those imposed by the a_i, since the only condition imposed on the b_i for $i < n$ and b_n by the tensor product is the graded commutativity, which makes the exterior commutativity of the a_i, as we showed above. Thus, ϕ is also injective, meaning ϕ is an isomorphism. $\square$

Problem 11

(6) (a) Show that if M_1 and M_2 are closed then there are isomorphisms $H_i(M_1 \# M_2) \cong H_i(M_i) \oplus H_i(M_2)$ for $0 < i < n$, with one exception: If both M_1 and M_2 are nonorientable, then $H_{n-1}(M_1 \# M_2)$ is obtained by replacing a $\mathbb{Z}/2\mathbb{Z}$ summands by a $\mathbb{Z}$ summand.

(6) (b) Show that $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \chi(S^n)$ if M_1 and M_2 are closed.

Proof. (6)(b) We first compare Betti numbers at each dimension. For dimension 0, since all the manifolds are connected, we get $1 = 1 + 1 - 1$. For $0 < i < n - 1$, the ith Betti number of S^n is 0, and from part (a) we get $b_i(M_1 \# M_2) = b_i(M_1) + b_i(M_2)$. The complication now comes at $i = n - 1$ and $i = n$. Recall that $b_{n-1}(S^n) = 0$ and $b_n(S^n) = 1$.

Suppose both M_1, M_2 are orientable. Then $M_1 \# M_2$ is orientable, so $b_n(M_1) = b_n(M_2) = b_n(M_1 \# M_2) = 1$, so that we get $1 = 1 + 1 - 1$ at the top level of comparing the Euler characteristics. Furthermore, we also have from part (b) that $b_{n-1}(M_1) + b_{n-1}(M_2) = b_{n-1}(M_1 \# M_2)$, so we are done in this case.

Now suppose M_1 is non-orientable and M_2 is orientable. Then $M_1 \# M_2$ is non-orientable, so we have $b_n(M_1) = b_n(M_1 \# M_2) = 0$ and $b_n(M_2) = 1$. We get $0 = 0 + 1 - 1$ at the top level of comparing the Euler characteristics. From part (a), we know $b_{n-1}(M_1 \# M_2) = b_{n-1}(M_1) + b_{n-1}$, which gives us the comparison in the $n - 1$ level. We are done in this case.

Now suppose M_1 and M_2 are non-orientable. Then $M_1 \# M_2$ is non-orientable, and the nth Betti numbers of these manifolds are all 0. Comparing nth Betti numbers, we have 0 and $0 + 0 - 1$. On the other hand, from part (a) we know $b_{n-1}(M_1 \# M_2) = 1 + b_{n-1}(M_1) + b_{n-1}(M_2)$. The extra 1 on the $n - 1$ level of the left hand side cancels the -1 on the n level of the right hand side, so the formula is true in this case. $\square$

Problem 12

(16) Show that $(\alpha \frown \varphi) \frown \psi = \alpha \frown (\varphi \smile \psi)$ for all k-chains α and ℓ, m-cochains φ, ψ. Deduce that $\frown$ makes H_* into a right H^* module.

(17) Show that a direct limit of exact sequences is exact. More generally, show that homology commutes with direct limits.

(24) Let M be a closed connected 3-manifold, and write $H_1(M;\mathbb{Z})$ as $\mathbb{Z}^r \oplus F$, with F finite. Show that $H_2(M;\mathbb{Z})$ is $\mathbb{Z}^r$ is M is orientable and $\mathbb{Z}^{r-1} \oplus (\mathbb{Z}/2\mathbb{Z})$ otherwise. In particular, $r \geq 1$ when M is non-orientable. Using Exercise 6, construct examples showing there are no other restrictions on the homology groups of closed 3-manifolds.

(25) Show that if a closed orientable manifold M of dimension $2k$ has $H_{k-1}(M;\mathbb{Z})$ torsion-free, then $H_k(M;\mathbb{Z})$ is also torsion-free.

Proof. (16) To show two chains are equal, it suffices to show that they have the same value under all cochains (for instance, one can check on duals to the simplices). Thus, let η be a $k-\ell-m$ cochain. In Hatcher page 249, we are given a formula for evaluating a cochain on a cap product. Using this formula, we have

$$\eta((\alpha \frown \varphi) \frown \psi) = (\psi \smile \eta)(\alpha \frown \varphi) = (\varphi \smile \psi \smile \eta)(\alpha);$$
$$\eta(\alpha \frown (\varphi \smile \psi)) = (\varphi \smile \psi \smile \eta)(\alpha).$$

We have implicitly used associativity of cup product in both lines. Since the two chains agree under all cochains, they are equal. Since this equation is true for all (co)chains, it is true when passing to (co)homology. This equation is precisely the condition which makes H_* into a right H^* module.

(17) Exactness of a chain complex means that all homology is 0. Since $0 \cong \varinjlim 0$, the first part of the exercise is truly a special case of the latter part. Thus, we do not assume that the our complexes are exact, and show that homology commutes with direct limits.

Let $(A^i_\bullet)_i$ be a directed system of chain complexes with transfer maps $f^{i,i'}_j : A^i_j \to A^{i'}_j$ if $i \leq i'$. The chain maps $A^i_j \to A^i_{j-1}$ are denoted by d^i_j. For each j, let A_j be the direct limit of A^i_j. Let f^i_j be the induced map $A^i_j \to A_j$. We first construct maps $d_j : A_j \to A_{j-1}$ which make $A_\bullet$ into a chain complex. To do so, it suffices to give a compatible (with respect to the transfer maps $f^{i,i'}_j$) family of maps $\hat{d}^i_j : A^i_j \to A_{j-1}$. We define $\hat{d}^i_j = f^i_{j-1} d^i_j$ for all i, j. For $i \leq i'$, we have

$$\hat{d}^{i'}_j f^{i,i'}_j = f^{i'}_{j-1} d^{i'}_j f^{i,i'}_j \overset{(1)}{=} f^{i'}_{j-1} f^{i,i'}_{j-1} d^i_j \overset{(2)}{=} f^i_{j-1} d^i_j = \hat{d}^i_j.$$

Equality (1) follows from the fact (by definition) that our transfer maps are chain maps, and equality (2) follows from compatibility (by definition) of the limit maps with the transfer maps. By the universal property of direct limit, we get a unique map $d_j : A_j \to A_{j-1}$ such that $\hat{d}^i_j = d_j f^i_j$. Explicitly, if $x \in A_j$ is represented by some $x^i \in A^i_j$, then $d_j x = \hat{d}^i_j x^i = f^i_{j-1} d^i_j x^i$. Compatibility of the maps ensures that this is independent of the choice of representative.

Let us check that $(A_\bullet, d_\bullet)$ is a chain complex, i.e. $d_j d_{j+1} = 0$. Let $x \in A_{j+1}$ be represented by $x^i \in A^i_{j+1}$. Then

$$d_j d_{j+1} x = d_j f^i_j d^i_{j+1} x^i \overset{(1)}{=} f^i_{j-1} d^i_j d^i_{j+1} x^i = 0,$$

where equality (1) follows from $d_j f^i_j = \hat{d}^i_j = f^i_{j-1} d^i_j$, and the last equality follows from $A^i_\bullet$ being a chain complex.

We now know that $H_j(A_\bullet)$ is defined. Each transfer map $f^{i,i'}_j$ induces a map in homology, say $g^{i,i'}_j : H_j(A^i_\bullet) \to H_j(A^{i'}_\bullet)$. Taking induced maps in homology commutes with composition, so the maps $g^{i,i'}_j$ make $(H_j(A^i_\bullet))_i$ a directed system. In particular, we have the direct limit $\varinjlim H_j(A^i_\bullet)$. We want to show $\varinjlim H_j(A^i_\bullet) \cong H_j(A_\bullet)$. To do so, we show that $H_j(A_\bullet)$ satisfies the universal property of direct limits.

First, we must show that there is a compatible (with respect to the transfer maps $g^{i,i'}_j$) family of maps $g^i_j : H_j(A^i_\bullet) \to H_j(A_\bullet)$. Given $[x] \in H_j(A^i_\bullet)$, represented by some $x \in \ker d^i_j \subset A^i_j$, we define $g^i_j([x]) = [f^i_j(x)]$. Let us check that this is well-defined. First, we need $f^i_j(x) \in \ker d_j$. Indeed, $d_j f^i_j x = f^i_{j-1} d^i_j x = 0$. Next, we need that f^i_j sends boundaries to boundaries. Indeed, $f^i_j d^i_{j+1} x = d_{j+1} f^i_{j+1} x$. Thus g^i_j is well-defined.

Next, we must show that the maps g^i_j are compatible. Indeed,

$$g^{i'}_j g^{i,i'}_j [x] = g^{i'}_j [f^{i,i'}_j x] = [f^{i'}_j f^{i,i'}_j x] = [f^i_j x] = g^i_j [x].$$

Now we need to show that $H_j(A_\bullet)$, together with the maps g^i_j, satisfies the universal property of direct limit. Unwrapping the definition, let $h^i_j : H_j(A^i_\bullet) \to X$ be a compatible family of maps. We want to show that there is a unique map $h_j : H_j(A_\bullet) \to X$ such that $h_j g^i_j = h^i_j$.

Let $[x] \in H_j(A_\bullet)$ be represented by $x \in \ker d_j \subset A_j$. In turn, x is represented by some $x^i \in A^i_j$. However, x^i may not be closed. We have

$$0 = d_j x = d_j f^i_j x^i = f^i_{j-1} d^i_j x_i,$$

showing only that $d^i_j x_i$ represents $0 \in A_{j-1}$. However, this does mean that for some $i' \geq i$, we have $f^{i,i'}_{j-1} d^i_j x^i = 0$. Using that the transfer maps are chain maps, we have $d_j^{i'} f_j^{i,i'} x^i = 0$. Let $x^{i'} = f_j^{i,i'} x^i$. Then $x^{i'}$ is closed, and it represents x:

$$f_j^{i'} x^{i'} = f_j^{i'} f_j^{i,i'} x^i = f_j^i x^i = x.$$

For simplicity, we rename i' by i. The point is that x can be represented by a closed $x^i \in A_j^i$. Fixing such i, we now define $h_j[x] = h_j^i[x^i]$. We must show that this is well-defined in terms of the choice of representative for $[x]$ and in terms of the representative x^i for x. To show that the definition is independent of the choice of representative for $[x]$, it suffices (assuming the second well-definedness condition) to show that boundaries in A_j have a representative which is a boundary. To that end, let $x = d_{j+1} y$, and let $y = f_{j+1}^i y^i$. Then

$$x = d_{j+1} f_{j+1}^i y^i = f_j^i d_{j+1}^i y^i,$$

so x is represented by the boundary $d_{j+1}^i y^i$. Now suppose $x^i, x^{i'}$ are two closed representatives for x. By definition, this means there is $k \geq i, i'$ such that $f_j^{i,k}(x^i) = f_j^{i',k}(x^{i'})$ in A_j^k. Call this common value by x^k. By compatibility, x^k represents x. Since the transfer maps are chain maps, x^k is closed. Then we claim that

$$h_j^i[x^i] = h_j^k[x^k] = h_j^{i'}[x^k].$$

By symmetry, it suffices to show the first equality. We have

$$h_j^k[x^k] = h_j^k[f_j^{i,k} x^i] = h_j^k g_j^{i,k}[x^i] = h_j^i[x^i],$$

as desired. Therefore, the map h_j is well-defined.

Now we must show that h_j satisfies $h_j g_j^i = h_j^i$, and that h_j is the unique map with this property. Given $[x^i] \in H_j(A_\bullet^i)$ with $x^i \in \ker d_j^i$, we have

$$h_j g_j^i[x^i] = h_j[f_j^i x^i] = h_j^i[x^i],$$

since x^i is a closed representative for $f_j^i x^i$. Now, suppose some other map h_j' satisfies $h_j' g_j^i = h_j^i$. Given $[x] \in H_j(A_\bullet)$, let x^i be a closed representative for x. Then

$$h_j'[x] = h_j'[f_j^i x^i] = h_j' g_j^i[x^i] = h_j^i[x^i] = h_j[x],$$

as desired. This completes the proof that $H_j(A_\bullet) \cong \varinjlim H_j(A_\bullet^i)$.

(24) By Corollary 3.37 of Hatcher, the Euler characteristic of M is 0. By Theorem 3.26 of Hatcher, we know that $H_3(M;\mathbb{Z})$ is $\mathbb{Z}$ if M is orientable and 0 otherwise. Since M is connected, $H_0(M;\mathbb{Z}) = \mathbb{Z}$. Letting r' denote the rank of $H_2(M;\mathbb{Z})$ and i denote the rank of $H_3(M;\mathbb{Z})$, we get

$$0 = 1 - r + r' - i.$$

Thus, $r' = r$ if M is orientable and $r' = r - 1$ otherwise. By Corollary 3.28 of Hatcher, the torsion part of $H_2(M; \mathbb{Z})$ is trivial if M is orientable and $\mathbb{Z}/2\mathbb{Z}$ otherwise. Thus, we get the desired claim that $H_2(M; \mathbb{Z})$ is $\mathbb{Z}^r$ if M is orientable and $\mathbb{Z}^{r-1} \oplus (\mathbb{Z}/2\mathbb{Z})$ otherwise.

Now let us show that for any $r \geq 0$ and any finite abelian group F, there is a connected closed orientable 3-manifold with $H_1(M; \mathbb{Z}) = \mathbb{Z}^r \oplus F$. By Exercise 6 of the same section, and by the classification of finite abelian groups, it suffices to find connected closed orientable 3-manifolds M and M_n for $n > 1$ such that $H_1(M; \mathbb{Z}) = \mathbb{Z}$ and $H_1(M_n; \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$. We can take $M_n = L_n(n-1, n+1)$ and $M = S^2 \times S^1$.

Now let us show that for any $r \geq 1$ and any finite abelian group F, there is a connected closed non-orientable 3-manifold with $H_1(M; \mathbb{Z}) = \mathbb{Z}^r \oplus F$. Since the connected sum of an orientable and non-orientable manifold is non-orientable, we need only find a connected closed non-orientable 3-manifold with $H_1(M; \mathbb{Z}) = \mathbb{Z}$; the torsion part can come from the orientable lens spaces from before.

We claim that a suitable choice of M is the quotient space of $S^2 \times I$ where $S^2 \times \{0\}$ and $S^2 \times \{1\}$ are identified via a reflection. This space is certainly a closed connected 3-manifold. To show it is non-orientable, it suffices to show that $H_3(M; \mathbb{Z}) = 0$. Let U, V be two open sets in M which are both homeomorphic to $S^2 \times \mathbb{R}$, and whose intersection is the disjoint union of two open sets, each homeomorphic to $S^2 \times \mathbb{R}$. We then have

$$H_i(U) = H_i(V) = \begin{cases} \mathbb{Z} & i = 0, 2 \\ 0 & \text{else,} \end{cases}$$

$$H_i(U \cap V) = \begin{cases} \mathbb{Z}^2 & i = 0, 2 \\ 0 & \text{else.} \end{cases}$$

To compute $H_i(M)$, we use the reduced Mayer-Vietoris sequence:

$$0 \to H_3(M) \to \mathbb{Z}^2 \to \mathbb{Z}^2 \to H_2(M) \to 0,$$
$$0 \to H_1(M) \to \mathbb{Z} \to 0.$$

The map $\mathbb{Z}^2 \to \mathbb{Z}^2$ can be interpreted as follows. Each copy of $\mathbb{Z}$ in the domain corresponds to a connected component of $U \cap V$, and each copy of $\mathbb{Z}$ in the codomain corresponds to either U or V. Since M is not $S^2 \times S^1$, we must think of V as having a "twist" in it, relative to U. In particular, the orientation of one component of $U \cap V$ will give the same orientation in both U and V, but the orientation of the other component will give opposite orientations in U and V. Thus, the map $\mathbb{Z}^2 \to \mathbb{Z}^2$ can be described by $(1, 0) \mapsto (1, 1)$, $(0, 1) \mapsto (1, -1)$. This map is injective, meaning $H_3(M) = 0$. Thus M is non-orientable. The latter part of the sequence gives $H_1(M) = \mathbb{Z}$ as desired.

(25) By UCT and Poincaré duality, we have

$$H_k(M) \cong \mathrm{Ext}(H_{k-1}(M), \mathbb{Z}) \oplus \mathrm{Hom}(H_k(M), \mathbb{Z}).$$

Since $H_{k-1}(M)$ is torsion free, the Ext term vanishes, and we are left with $H_k(M) \cong \mathrm{Hom}(H_k(M), \mathbb{Z})$. Since $\mathrm{Hom}(H_k(M), \mathbb{Z})$ is the free part of $H_k(M)$, we must have $H_k(M)$ torsion free.

$\square$