Symplectic Duality and Coulomb Branches
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Preface

There will be some gaps in explanation, either due to the lecturer's admission or my own lack of understanding. In particular, many "proofs" are sketches of proofs. Gaps due to my own misunderstanding will be indicated by three red question marks: ???. More generally, my own questions about the material will also be in red. Things like "**Question**" will be questions posed by the lecturer. Feel free to reach out to me with explanations.

Contents

1 Jan 6

We fix G reductive connected over $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra, M an affine normal Poisson variety, generically symplectic. Let G have Hamiltonian action on M with moment map $\mu : M \to \mathfrak{g}^*$. There is a scaling action $\mathbb{C}^\times$ on M which commutes with the G-action and with μ.

When we introduce the Langlands dual group $G^\vee$, we want some $M^\vee$ which plays the role of M. This is known basically only in the case where $M = T^*X$ for X a smooth affine G-variety. The main problem is specifically to find a class of "good" M such that M is good implies $M^\vee$ is good, $(M^\vee)^\vee = M$, and all T^*X are good.

Now fix a Borel B. Let X be a smooth affine G-variety, and let M be as in the first paragraph.

Definition 1.1.

1. X is **spherical** if X contains an open dense B-orbit.
2. M is **hyperspherical** if for all $f_1, f_2 \in \mathbb{C}[M]^G$, we have $\{f_1, f_2\} = 0$.

Theorem 1.1. *X is spherical iff T^*X is hyperspherical.*

We will prove this later on.

Theorem 1.2. *Let M be a hyperspherical variety. Then:*

1. *The map $\overline{\mu} : M//G \to \mathfrak{g}^*//G$ on categorical quotients is finite, i.e. $\mathbb{C}[M]^G$ is a finitely generated module over $\mathbb{C}[\mathfrak{g}^*]^G = (\mathrm{Sym}\mathfrak{g})^G$.*
2. *The image $\mathrm{im}(\overline{\mu})$ of $\overline{\mu}$ is closed in $\mathfrak{g}^*//G$.*
3. *The composite $\nu : M \to M//G \xrightarrow{\overline{\mu}} \mathfrak{g}^*//G$ has the property that all irreducible components of all of its non-empty fibers have the same dimension.*
4. *Each irreducible component of the generic fibers of ν is the closure of a G-orbit.*

Corollary 1.1. *If M is hyperspherical, then* $\dim M \leq \dim G + \dim(\mathfrak{g}^*//G) = \dim G + \mathrm{rk} G$.

From now on, we consider M to be smooth and symplectic.

Let $\mathfrak{b} = \mathrm{Lie}B$. The composite $\mu_B : M \xrightarrow{\mu} \mathfrak{g}^* \to \mathfrak{b}^*$ is the moment map for the B-action. Let $\Lambda_M = \mu_B^{-1}(0) = \mu^{-1}(\mathfrak{b}^\perp)$.

Example 1.1. If $M = T^*X$, then Λ_M is the union of the conormal bundles $T^*_O X$ to B-orbits $O \subset X$.

Theorem 1.3. *If X is spherical, then X is a finite union of B-orbits. (???)*

Corollary 1.2. *If X is spherical and $M = T^*X$, then Λ_M is Lagrangian in M.*

Proof. Each conormal bundle is Lagrangian.

Theorem 1.4. *Let M be smooth and symplectic. If Λ_M is Lagrangian, then M is hyperspherical.*

Conjecture: if M is good symplectic hyperspherical, then Λ_M is Lagrangian, and there is a bijection between the irreducible components of Λ_M and the irreducible components of $\Lambda_{M^\vee}$.

Let $\mathcal{B} = G/B$ be the flag variety, let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{g}^*$, let $\widetilde{\mathcal{N}} = T^*\mathcal{B} \xrightarrow{\pi} \mathcal{N}$ be the Springer resolution, and let $\mathrm{St}_G = \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$ be the Steinberg variety. It is known that St_G is Lagrangian in $T^*(\mathcal{B} \times \mathcal{B})$, and $H^{BM}_{top}(\mathrm{St}_G)$ has a natural algebra structure, isomorphic to the group algebra of the Weyl group W.

Now let M be hyperspherical, and assume that Λ_M is Lagrangian. Let $\mathrm{St}_M = M \times_{\mathfrak{g}^*} \widetilde{\mathcal{N}}$. As a subvariety of $M \times \widetilde{\mathcal{N}}$, it is stable under the diagonal G-action. We have $\mathrm{St}_M \cong G \times^B \Lambda_M$. If $M = T^*X$, then $M \times \widetilde{\mathcal{N}} = T^*(X \times \mathcal{B})$, and St_M is the union of conormal bundles to G-orbits.

By analyzing the fiber product conditions, we see that there is a convolution $\mathrm{St}_M \circ \mathrm{St}_G = \mathrm{St}_M$. In particular, two pairs $(\eta, \xi) \in \mathrm{St}_M$ and $(\xi, \xi') \in \mathrm{St}_G$ give a new pair (η, ξ') in St_M. This gives $H^{BM}_{top}(\mathrm{St}_M)$ the structure of a $H^{BM}_{top}(\mathrm{St}_G)$-module, i.e. it is a representation of W.

Conjecture: There is an isomorphism of W-reps $H^{BM}_{top}(\mathrm{St}_M) \cong H^{BM}_{top}(\mathrm{St}_{M^\vee})$.

Example 1.2. Now we tabulate results when I'm not lazy I will make this look nice.. Row 1: $G = T$ is a torus, $M = T^*(T/T_1)$ for a subtorus T_1. Then $M^\vee = T^*(T_1^\vee)$.
Spherical T-variety is a toric variety; for it to be smooth, it would be affine. So in particular (row 2), if $G = (\mathbb{C}^\times)^n$ and $M = T^*(\mathbb{C}^n)$, then $G^\vee = G$ and $M^\vee = M$.
Next (row 3) consider the group $G \times G$ and $M = T^*G$, where $G \times G$ acts by left and right translations. Then the dual group is $G^\vee \times G^\vee$ and $M^\vee = T^*(G^\vee)$. Note that G is spherical in this case, since it has the open $B \times B$ orbit given by Bw_0B, where $w_0 \in W$ is the longest element.
Row 4: if the group is just G and $M = T^*G$, then $M^\vee = \mathcal{N}_{G^\vee}$.
Row 5: Let $U = [B, B]$ be max unipotent. Consider the group $G \times T$, where T is a maximal torus in G. Let M be the affine closure of $T^*(G/U)$. Then $M^\vee$ is the affine closure of $T^*(G^\vee/U^\vee)$. This is related to Eisenstein series. (note: possibly incorrect)
Row 6: consider the same M but for the group G. Then $M^\vee = \overline{T^*(G^\vee/U^\vee)}/W$, where the W-action is by Gelfand-Graev (it is not an obvious action).

Row 7: Let the group be G, and let M be a point. Then $M^\vee = T^*_\psi(G^\vee/U^\vee) = (T^*G^\vee)//_\psi U^\vee$ (Hamiltonian reduction), the Whittaker potential bundle for a nondegenerate character $\psi : U^\vee \to \mathbb{C}^\times$.
Row 8: $G = GL_n \times GL_n$, $M = T^*(\mathbb{C}^n \otimes \mathbb{C}^n) = T^*M_n$, where GL_n acts by left and right translations. This group is self dual, and $M^\vee = T^*(GL_n \times \mathbb{C}^n) = T^*(G \times^{GL_n} \mathbb{C}^n)$. This duality is classical and known in automorphic forms; in one direction it is Rankin-Selberg, and in the other it is Godement-Jacquet.
Row 9: $G = GL_n$, $M = T^*(\mathbb{C}^n)$, $M^\vee = T^*M_n//_\psi U$.
Row 10: $G = GL_{2n}$, $M = T^*(G/(GL_n \times GL_n))$ (block diagonal embedding), $M^\vee = T^*(G \times^{Sp_{2n}} \mathbb{C}^{2n})$.
Row 11: $G = GL_{2n}$, $M = T^*(G/Sp_{2n})$, $M^\vee = T^*_\phi(G/Q)$, where Q is the subgroup of block $(n+n) \times (n+n)$ upper triangular matrices, where the two diagonal blocks are equal, and ϕ takes such a matrix to $e^{tr(a)}$, where a is the upper right block.