Derived Geometric Satake Notes
Course Taught by Victor Ginzburg

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Preface

There will be some gaps in explanation, either due to the lecturer's admission or my own lack of understanding. In particular, many "proofs" are sketches of proofs. Gaps due to my own misunderstanding will be indicated by three red question marks: ???. More generally, my own questions about the material will also be in red. Things like "**Question**" will be questions posed by the lecturer. Feel free to reach out to me with explanations.

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1.1 Geometric Satake

We let $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. Let G be a complex reductive group. We have groups $G(\mathcal{K})$ and $G(\mathcal{O})$, and their quotient is the **affine Grassmannian** $Gr_G = Gr = G(\mathcal{K})/G(\mathcal{O})$. This is an analogue of the partial flag manifold for G. There is a clear action of $G(\mathcal{O})$ on Gr by left translation. There is a $\mathbb{C}^\times$-action called **loop rotation** induced by $(z \cdot f)(t) = f(zt)$ on $\mathcal{K}$. If we think of an element of $G(\mathcal{K})$ as a loop γ in G, i.e. we take $t = \exp(i\theta)$, and we take $z = \exp(i\alpha)$, then $(z \cdot \gamma)(t) = \gamma(\exp(i + \alpha))$, which explains the name loop rotation.

One of the main objects of Geometric Satake is $D_{\mathbb{C}^\times \ltimes G(\mathcal{O})}(Gr)$, the $\mathbb{C}^\times \ltimes G(\mathcal{O})$-equivariant constructible derived category. In terms of stacks, we may write this as $D_{\mathbb{C}^\times}(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O}))$. When written in this form, it is evident that, just as with biinvariant functions, there is a monoidal (convolution) structure $\star$ on this category.

Theorem 1.1 (Derived Satake)**.** *There is a monoidal equivalence of triangulated categories between $(D_{\mathbb{C}^\times}(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})), \star)$ and the derived category of Harish-Chandra $\mathcal{U}_\hbar(\check{\mathfrak{g}})$-bimodules (with the tensor product as the monoidal operation).*

Here, $\mathcal{U}_\hbar(\check{\mathfrak{g}})$ is the Rees algebra of the enveloping algebra of the Langlands dual Lie algebra $\check{\mathfrak{g}}$. $\hbar$ corresponds to loop rotation. If we set $\hbar = 0$, we get the quasi-classical specialization, which is a monoidal equivalence between $(D(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})), \star)$ and the derived category of $\check{G}$-equivariant Sym$\check{\mathfrak{g}}$-modules. More geometrically, this module category is the derived category of $QCoh^{\check{G}}(\check{\mathfrak{g}}^*)$.

Abelian Satake is a restriction of the above quasi-classical equivalence. Namely, it is a monoidal equivalence between the category of $G(\mathcal{O})$-equivariant perverse sheaves on Gr (still with $\star$) and the category of quasi-coherent sheaves of the form $\mathcal{O}_{\check{\mathfrak{g}}^*} \otimes_{\mathbb{C}} V$, where V is a finite dimensional representation of $\check{G}$. This can be further simplified to the category of finite dimensional representations of $\check{G}$, but on the derived level one needs this description of the quasi-coherent sheaves.

Remark.

1. $G(\mathcal{O})$-equivariance of a perverse sheaf is a property, not additional data.
2. Abelian Satake is used in the construction of the derived equivalences, but the statements of the derived equivalences make no mention of perverse sheaves.
3. Derived Satake is related to the representation theory of $\check{\mathfrak{g}}$. In particular, there are connections to Verma modules, the Grothendieck-Springer resolution, and the universal centralizer for $\check{G}$.

4. There is no ∞-categorical analogue of the derived Satake equivalence with loop rotation (at least, one involving quantization); the monoidal structure does not lift.

The **classical Satake** theorem is "secretly" taking complexification of the Grothendieck groups on both sides of the quasi-classical derived equivalence. However, one deals with $\mathcal{K}$ being a local field. The equivalence looks like $C_c(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})) \cong \mathbb{C}[\check{G}]^{\mathrm{Ad}\check{G}}$.

1.2 Equivariant Stuff

Let G be either a compact Lie group over $\mathbb{R}$, or a linear algebraic group over $\mathbb{C}$. According to these cases, let X be a "reasonable" topological space, namely a smooth manifold or an algebraic variety over $\mathbb{C}$. For us, nonreduced structure plays no role. We want G to act on X in either a smooth or algebraic manner, again according to our two cases. We want to define $D_G(X)$ so that we can associate to the objects $\mathcal{F}$ some "equivariant cohomology" $H^\bullet_G(\mathcal{F})$.

Theorem 1.2. *Let G be connected complex reductive. Let T be a maximal torus of G with Weyl group W. Then for $mcF \in D_G(X)$, we have $H^\bullet_G(\mathcal{F}) = H_T(\mathcal{F})^W$.*

Theorem 1.3 (Localization)**.** *There is a correspondence(???)between $H^\bullet_T(\mathcal{F})$ and $H^\bullet(\mathcal{F}|_{X^T})$.*

There is a "refined" version of this theorem called GKM (Goresky-Kottwitz-MacPherson). Another important result is (Braden's) hyperbolic localization. There is also a result that gives equivariant Ext groups between IC sheaves.

Now, we want to discuss how $D_G(X)$ is defined. If we use stacks, then it is "simple": $D_G(X) = D(X/G)$. However, this is useless for more refined questions, e.g. formality (degeneration of spectral sequences) and IC sheaves. We instead look at a classical approach based on classifying spaces.

Definition 1.1. A **universal G-bundle** is a principal G-bundle $EG \to BG$ such that EG is weakly contractible, i.e. (for us) $\mathbb{Q} \otimes \pi_i(EG) = 0$ for all i.

The spaces EG, BG are infinite dimensional, but we can write them as direct limits of spaces E_nG, B_nG, where we have a principal G-bundle $E_nG \to B_nG$ for each n, subject to the following conditions:

1. $\mathbb{Q} \otimes \pi_i(E_nG) = 0$ for $i \leq n$
2. For fixed $k \geq 0$, the maps $\cdots \to H_k(B_nG) \to H_k(B_{n+1}G) \to \cdots$ stabilize.
3. If G is a compact Lie group, then the E_nG, B_nG are smooth manifolds and B_nG are compact. If G is linear algebraic over $\mathbb{C}$, then E_nG, B_nG are smooth algebraic varieties, and B_nG are projective.

There are many constructions that satisfy these criteria. We will follow one particular construction.

Example 1.1. Let $G = \mathbb{C}^\times$. Then $E_nG = \mathbb{C}^n - \{0\}$ with obvious dilation G-action. Thus $B_nG = \mathbb{P}^{n-1}$. We call $BG = \varinjlim_n \mathbb{P}^{n-1}$ by $\mathbb{P}^\infty$. We have $H^*(\mathbb{P}^{n-1}) = \mathbb{C}[u]/(u^n)$, so $H^*(\mathbb{P}^\infty) = \varprojlim_n \mathbb{C}[u]/(u^n) = \mathbb{C}[u]$. Here u is a degree 2 element.

Example 1.2. $G = T$ is a torus, which is an r-fold product of $\mathbb{C}^\times$. Then BT is the r-fold product of $\mathbb{P}^\infty$, and the cohomology ring $H^*(BT)$ is $\mathbb{C}[u_1, \ldots, u_r] = \mathbb{C}[\mathfrak{t}]$ for $\mathfrak{t} = \mathrm{Lie}T$.

Example 1.3. Let $G = GL_N$ over $\mathbb{C}$. Let $E_nG = M^{reg}_{N\times n}$ be the space of full rank $N \times n$ matrices. For n large, $B_nG = Gr_N(\mathbb{C}^n)$. In particular, $BG = Gr_N(C^\infty)$.

If H is a closed subgroup of G, then $(EG)/H$ is a model for BH. The map $BH = (EG)/H \to (EG)/G = BG$ is a fibration with fiber G/H. We get maps $H^*(BG) \to H^*(BH)$.

Example 1.4. Let G be a linear algebraic group over $\mathbb{C}$. Pick a closed embedding $G \hookrightarrow GL_N$. Then we take $BG = \varinjlim_n (M^{reg}_{N\times n}/G)$.

Remark.

1. If G is unipotent, then it is isomorphic to an affine space, i.e. it is contractible. Then BG and EG are also contractible, so the equivariant theory for unipotent groups is trivial.

2. If G/H is weakly contractible, then BG is (weakly?) homotopic to BH, so the equivariant theories are the same. For instance, the equivariant theory for a reductive group G is the same as that of its maximal compact subgroup G_c. As a concrete example, if $G = \mathbb{C}^\times$ and $G_c = S^1$, we get $S^\infty/S^1 = \mathbb{P}^\infty$.

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Let X be a G-space. We have two maps $G \times X \to X$, namely the action a and the projection pr. An **equivariant structure** on a sheaf $\mathcal{F}$ of vector spaces on X is an isomorphism $\alpha : a^*\mathcal{F} \xrightarrow{\sim} \mathrm{pr}^*\mathcal{F}$ such that $\alpha|_{1_G \times X} = \mathrm{id}$ and a certain cocycle condition is satisfied (condition omitted by lecturer).

Example 2.1. A constant sheaf C_X has a natural equivariant structure via the isomorphisms $C_{G\times X} \to \mathrm{pr}^* C_X$ and $C_{G\times X} \to a^* C_X$.

We denote by $Sh(Y)$ the abelian category of sheaves on any space Y. We denote by $Sh_G(X)$ the abelian category of G-equivariant sheaves on a G-space X.

Lemma 2.1. *Let $q : \widetilde{Y} \to Y$ be a G-bundle. Then $q^* : Sh(Y) \to Sh_G(\widetilde{Y})$ is an equivalence with inverse given by $\widetilde{\mathcal{F}} \mapsto (q_*\mathcal{F})^G$ (taking G-invariants stalk-wise).*

Now let X be a G-space and $E \to B$ a G-bundle. Let $E \times^G X = (E \times X)/G$. We have a diagram

$$\begin{tikzcd} X & E \times X \arrow[l, "p"'] \arrow[r, "q"'] \arrow[d] & E \times^G X \arrow[d, "\pi"'] \\ & B \times X \arrow[r] & B \end{tikzcd}$$

where q is a G-bundle and π is a locally trivial fibration on B with fiber X. Let $\mathcal{F}_X \in Sh_G(X)$. Then $p^*\mathcal{F}_X$ is G-equivariant. By Lemma 2.1, we can uniquely (up to iso) write $p^*\mathcal{F}_X \cong q^*\widetilde{\mathcal{F}}$ for $\tilde{\mathcal{F}} \in Sh(E \times^G X)$. Then for any fiber $\pi^{-1}(b)$, which is non-canonically isomorphic to X, we have $\widetilde{\mathcal{F}}|_{\pi^{-1}(b)}$ is non-canonically isomorphic to $\mathcal{F}_X$.

We apply the above to the bundle being $EG \to BG$. Let $X_G = EG \times^G X$. Let $H^\bullet_G(X) = H^\bullet(X_G)$. Here is a picture relating our setup, on the left, to stacks, on the right, where $\sim$ denotes weak homotopy equivalence:

$$\begin{tikzcd} EG \times X \arrow[d, "q"] & \sim & X \arrow[d] \\ X_G \arrow[d] & \sim & X/G \arrow[d] \\ BG & \sim & pt/G \end{tikzcd}$$

We have $H^\bullet_G(pt) = H^\bullet(BG)$, which acts as our "base ring", and this comes with a map to $H^\bullet_G(X) = H^\bullet(X_G)$.

Now consider the diagram above in our universal bundle setting:

$$\begin{tikzcd} X & EG \times X \arrow[l, "p"'] \arrow[r, "q"'] \arrow[d] & X \arrow[d, "\pi"'] \\ & BG \times X \arrow[r] & BG \end{tikzcd}$$

For any space Y, we write $D(Y)$ for the ordinary derived category of $Sh(Y)$. We "define" (this definition is not technically correct, and will be remedied later) the equivariant derived category $D_G(X)$ to have objects given by a triple $\mathcal{F} = (\mathcal{F}_X, \mathcal{F}_{X_G}, \alpha)$, where $\mathcal{F}_X \in D(X)$, $\mathcal{F}_{X_G} \in D(X_G)$, and $\alpha : p^*\mathcal{F}_X \xrightarrow{\sim} q^*\mathcal{F}_{X_G}$ is an isomorphism in $D(EG \times X)$. We define $H^\bullet_G(\mathcal{F}) = H^\bullet(\mathcal{F}_{X_G})$. Morphisms in $D_G(X)$ are given by a pair of morphisms between the relevant objects of the non-equivariant derived categories (we will often call these sheaves) that satisfy a natural compatibility diagram. This is a triangulated category, where the distinguished triangles are pairs of distinguished triangles with compatibility condition. We have a forgetful functor $D_G(X) \to D(X)$ which sends a triple to its first element. We have a functor $Sh_G(X) \to D_G(X)$ defined by $\mathcal{F}_X \mapsto (\mathcal{F}_X, \widetilde{\mathcal{F}}, \alpha)$ for some naturally defined α. This extends to a functor $D(Sh_G(X)) \to D_G(X)$, which is almost never an equivalence.

$D_G(X)$ has a notion of a constant sheaf, namely $(C_X, C_{X_G}, \mathrm{id}_{C_{EG\times X}})$. Its sheaf cohomology recovers $H^\bullet_G(X)$.

We now retroactively impose the condition "constructible" everywhere. Because of this, we have to assume that G has finitely many connected components. This assumption implies $\pi_1(BG)$ is finite, and if G is connected, then BG is simply connected. This follows by a homotopy sequence $\pi_1(EG) \to \pi_1(BG) \to \pi_0(G)$ and $\pi_1(EG) = 1$.

We note that anytime we write sheaf functors, particularly pushforward, we mean their derived versions, unless otherwise specified. As in the abelian case, $\mathcal{F}_{X_G}$ restricted to a fiber of π is isomorphic to $\mathcal{F}_X$. If we take the stalk at $b \in BG$ of $\mathcal{H}^k(\pi_*\mathcal{F}_{X_G})$, we get $H^k(\mathcal{F}_X)$. Thus $\mathcal{H}^k(\pi_*\mathcal{F}_{X_G})$ is a locally constant sheaf on BG. In particular, if G is connected, then this cohomology sheaf is constant.

We have a Leray spectral sequence $E_2^{p,q} = H^p(BG, \mathcal{H}^q(\pi_*\mathcal{F}_X))$, which converges to $H^\bullet_G(\mathcal{F}_X)$. If G is connected, then $E_2^{p,q} = H^p(BG) \otimes H^q(\mathcal{F}_X)$. In particular, if $\mathcal{F}_X$ is constant, $E_2^{p,q} = H^p(BG) \otimes H^q(X)$, and this converges to $H^\bullet_G(X)$.

Let $C \in D_G(X)$ be constant. Then $\mathrm{Ext}^k_{D_G(X)}(C, \mathcal{F}) = \mathrm{Hom}_{D_G(X)}(C, \mathcal{F}[k]) = H^k_G(\mathcal{F})$.

Example 2.2. Let X be a point, and let G be connected. Then $Sh_G(pt)$ is Vect, the category of vector spaces. Thus $D(Sh_G(pt)) = D(\mathrm{Vect})$. We have $\mathrm{Ext}^k_D(C, C) = \mathbb{C}$ if $k = 0$, and 0 otherwise. On the other hand, $\mathrm{Ext}^k_{D_G(pt)}(C, C) = H^k_G(C) = H^k_G(pt) = H^k(BG)$. If $G = \mathbb{C}^\times$, then $H^k(BG) \neq 0$ for all positive even integers k. If follows that $D(Sh_G(pt))$ is very different from $D_G(pt)$.

We now take care of the issue of EG, BG being infinite dimensional and correct the definition of $D_G(X)$. Recall $EG = \varinjlim_n E_nG$, $BG = \varinjlim_n B_nG$, with G-bundles $E_nG \to B_nG$. Let $i_n : E_nG \to E_{n+1}G$. We have a commutative diagram

$$\begin{tikzcd}
 & E_nG \times X \arrow[r, "q_n"'] \arrow[dl, "p_n"'] \arrow[dd, "i_n \times \mathrm{id}"] & X_{n,G} \arrow[dd, "\overline{i_n}"] \\
X & & \\
 & E_{n+1}G \times X \arrow[ul, "p_{n+1}"] \arrow[r, "q_{n+1}"'] & X_{n+1,G}
\end{tikzcd}$$

An object $\mathcal{F}$ of $D^b_G(X)$ is a collection consisting of $\mathcal{F}_X \in D^b(X)$, $\mathcal{F}_n \in D^b(X_{n,G})$ for each n, $\alpha_n : p_n^*\mathcal{F}_X \xrightarrow{\sim} q_n^*\mathcal{F}_n$ for each n, and $\nu_n : \mathcal{F}_n \xrightarrow{\sim} \overline{i_n}^*\mathcal{F}_{n+1}$ for all n, satisfying the following compatibility diagrams for all n:

$$\begin{tikzcd}
p_n^*\mathcal{F}_X \arrow[rr, "\mathrm{id}"] \arrow[d, "\alpha_n"'] & & (i_n \times \mathrm{id})^* p_{n+1}^*\mathcal{F}_X \arrow[d, "(i_n \times \mathrm{id})^*(\alpha_{n+1})"] \\
q_n^*\mathcal{F}_n \arrow[r, "q_n^*(\nu_n)"] & q_n^*\overline{i_n}^*\mathcal{F}_{n+1} \arrow[r, "\mathrm{id}"] & (i_n \times \mathrm{id})^* q_{n+1}^*\mathcal{F}_{n+1}
\end{tikzcd}$$

Example 2.3. Let X be an algebraic variety. We define $IC_G(X) \in D^b_G(X)$ by $(\mathcal{F}_X = IC(X), \mathcal{F}_n = IC(X_{n,G})[-\dim B_nG])$ with natural compatibility maps. For any Zariski open $U \subset X$, we have $IC_G(X)|_U = C_U[\dim X]$.

We now need to rigorously define $H^\bullet_G(\mathcal{F})$. We have maps $\pi_n : X_{n,G} \to B_nG$ which are locally trivial fibrations with fiber X. We have as before $H^\bullet(\mathcal{F}_n) = H^\bullet((\pi_n)_*\mathcal{F}_n)$ and a spectral sequence $E_2^{p,q} = H^p(\mathcal{H}^q((\pi_n)_*\mathcal{F}_n))$ converging to it. Recall that the maps in cohomology between the B_nG stabilize. Then we get maps between spectral sequences that also stabilize, whence the sheaf cohomologies stabilize.

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Let $K \leq G$ be a subgroup. Let X be a K-space. Then we can form the *induced* space $G \times^K X = (G \times X)/K$. We have an *induction functor* $\mathrm{Ind}_K^G : D_K(X) \to D_G(G \times^K X)$ defined via the isomorphism $X_K = EG \times^K X \xrightarrow{\sim} EG \times^G (G \times^K X)$. This functor is an equivalence. Note that we have chosen EG as a model for EK. We have $H_G(\mathrm{Ind}_K^G \mathcal{F}) \cong H_K(\mathcal{F})$.

If X is a G-space, then we have a *restriction functor* $\mathrm{Res}_K^G : D_G(X) \to D_K(X)$ given by pullback along the map $p : X_K \to X_G$. Equivalently, you can pullback along the action map $G \times^K X \to X$ and then use the inverse to induction: $D_G(X) \to D_G(G \times^K X) \to D_K(X)$.

Let G be connected, and either a compact Lie group or a complex reductive algebraic group. Let $V \in \mathrm{Rep}(G)$. Consider the vector bundle V_{BG} with fiber V given by $EG \times^G V \to BG$. Its Chern classes are certain classes $c_i(V_{BG}) \in H^{2i}(BG)$. The Chern character is $ch(V_{BG}) = \sum_{i \geq 0} \frac{1}{i!} c_i(V_{BG})$. We will use this only when $G = T$ is a torus. Let $\mathfrak{t} = \mathrm{Lie}T$. Let $X^*(T) = \mathrm{Hom}(T, \mathbb{C}^\times)$ be the character lattice. We have (after complexifying) $\mathfrak{t}^* = \mathbb{C} \otimes_{\mathbb{Z}} X^*(T)$. Recall $BT \simeq (\mathbb{P}^\infty)^r$ for $r = \dim T$. For $\lambda \in X^*(T)$ we have a line bundle $ET \times^T \mathbb{C}_\lambda = \mathbb{C}_{ET,\lambda} \to BT$. Since it is a line bundle, it only has a c_1. The map $\lambda \mapsto c_1(\mathbb{C}_{ET,\lambda})$ gives a group homomorphism $X^*(T) \to H^2(BT)$, which extends to a $\mathbb{C}$-linear $\mathfrak{t}^* \to H^2(BT, \mathbb{C})$. This further extends to an algebra homomorphism $\mathbb{C}[\mathfrak{t}] = \mathrm{Sym}\mathfrak{t}^* \to H^\bullet(BT, \mathbb{C})$ (which doubles the degrees).

Return to the case of G as in the previous paragraph, and let T be a maximal torus with Weyl group W. Via the construction above, we have the following commutative diagram with $p : BT \to BG$:

$$\begin{tikzcd}
& \mathrm{Rep}T \arrow[r, "ch"] & \mathbb{C}[[\mathfrak{t}]] = \prod_{i\geq 0} H^{2i}(BT) & \\
\mathrm{Rep}G \arrow[ur, "\mathrm{Res}^G_T"] \arrow[dr, "char"'] & & & \\
& \mathbb{C}[G]^G \arrow[r, "\exp^*"'] & \mathbb{C}[[\mathfrak{g}]]^G \arrow[r, "ch"'] & \prod_{i\geq 0} H^{2i}(BG) \arrow[uul, "p^*"']
\end{tikzcd}$$

Now we want to generalize this to sheaves. Let X be a G-space. We have maps $X_T \xrightarrow{p} X_{N(T)} \xrightarrow{q} X_G$, where p is a W-bundle and q is a $G/N(T)$ bundle. Let $\mathcal{F} = \mathcal{F}_{X_G}$ be the $D(X_G)$-component of an element in $D_G(X)$ (we will often treat elements of the derived category in this way). Then $p^*q^*\mathcal{F} \in D(X_T)$, so $p^*q^* : D_G(X) \to D_T(X)$.

Theorem 3.1. *$H_G(\mathcal{F}) = H(X_G, \mathcal{F}_{X_G}) \to H_T(\mathcal{F}) = H(X_T, p^*q^*\mathcal{F}_{X_G})$ is an isomorphism onto $H(X_T, p^*q^*\mathcal{F}_{X_G})^W$.*

Lemma 3.1. *$H^i(G/N(T), \mathbb{C}) = \mathbb{C}$ if $i = 0$ and 0 otherwise.*

Proof. Suppose $f : \widetilde{Y} \to Y$ is a Galois covering with finite Galois group W. For $\mathcal{E} \in D(Y)$, we have $H(\widetilde{Y}, f^*\mathcal{E}) = H(Y, f_*f^*\mathcal{E})$. W acts on the RHS, and the projection formula gives $H(Y, \mathcal{E}) = H(Y, f_*f^*\mathcal{E})^W$. We apply this to the map $G/T \to G/N(T)$. Then $H(G/N(T), \mathbb{C}) = H(G/T, \mathbb{C})^W$. We conclude via the following Theorem of Borel.

Theorem 3.2 (Borel)**.** *$W \curvearrowright H^\bullet(G/T)$ is the regular representation.*

Proof of Theorem 3.1. $H(X_{N(T)}, q^*\mathcal{F}) = H(X_G, q_*q^*\mathcal{F})$. The Leray spectral sequence gives $E_2^{i,j} = H^i(X_G, \mathcal{H}^j q_*q^*\mathcal{F})$ converging to $H(X_G, q_*q^*\mathcal{F})$. By equivariance, $\mathcal{H}^j q_*q^*\mathcal{F}$ restricted to a fiber of q, which is $G/N(T)$, is constant. Then, by Lemma 3.1, the stalks of $\mathcal{H}^j q_*q^*\mathcal{F}$ are $\mathbb{C}$ for $j = 0$ and 0 otherwise. Then the spectral sequence gives us $H(X_G, \mathcal{F}) \cong H(X_{N(T)}, q^*\mathcal{F})$, and then this is $H(X_T, p^*q^*\mathcal{F})^W$ using the fact that p is a Galois covering.

Corollary 3.1. *$H(BG) \cong H(BT)^W \cong \mathbb{C}[\mathfrak{t}]^W \cong \mathbb{C}[\mathfrak{g}]^G$, where the last isomorphism is due to Chevalley. In particular, $H^{2i+1}(BG) = 0$.*

Proof. Apply the Theorem in the case of a constant sheaf.

Definition 3.1. $\mathcal{F} \in D_G(X)$ is called **equivariantly formal** if $\pi_*\mathcal{F} \cong \bigoplus_i \mathcal{H}^i\pi_*\mathcal{F}$ in $D(BG)$, where $\pi : X_G \to BG$.

Corollary 3.2. *If $\mathcal{F}$ is equivariantly formal, then $H_G(\mathcal{F}) \cong H(BG) \otimes H(X, \mathcal{F}_X)$.*

Proof. Since we are still assuming G is connected, we have BG is simply connected. This implies that $\mathcal{H}^i(\pi_*\mathcal{F})$ is a constant sheaf with stalk $H^i(X, \mathcal{F}_X)$. Formality gives $H(\pi_*\mathcal{F}) \cong \bigoplus_{i,j} H^i(BG, \mathcal{H}^j\pi_*\mathcal{F}) = \bigoplus_{i,j} H^i(BG) \otimes H^j(X, \mathcal{F}_X)$.

Corollary 3.3. *If $\mathcal{F}$ is equivariantly formal, then $H_T(\mathcal{F}) = H(BT) \otimes_{H(BG)} H_G(\mathcal{F})$.*

Proof. Take W invariants on both sides of $H_G(\mathcal{F}) \cong H(BG) \otimes H(X, \mathcal{F}_X)$.

In general, proving that something is equivariantly formal is difficult. However, there are some situations where we are lucky.

One such case is for a projective algebraic G-variety X; then $IC(X)$ is equivariantly formal. Indeed, we recall that $IC(X)$ is defined to be the "limit" of $IC(X_{G,n})$ for varying n. We have maps $\pi_n : X_{G,n} \to BG_n$ whose fibers are X. Since X is projective and BG_n is projective, the map π_n is projective. Thus equivariant formality holds by the *BBD decomposition theorem.*

Another case is if $H^{2i+1}(X, \mathcal{F}_X) = 0$. This follows by an argument with the Leray spectral sequence. Namely, by Corollary 3.1, the spectral sequence collapses since there are no odd degree terms. Namely, $H_G(\mathcal{F}) = H(BG) \otimes H(\mathcal{F}_X)$. Lecturer is actually a little unsure on how to get the full statement of formality from here.

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4.1 Strategy for Derived Geometric Satake Proof (Without Loop Rotation)

Let G be connected complex reductive with maximal torus T. Let $\mathfrak{g} = \mathrm{Lie}G$ and $\mathfrak{t} = \mathrm{Lie}T$. We have the Langlands dual group and algebras $\check{G}, \check{\mathfrak{g}}, \check{\mathfrak{t}}$. There is a canonical identification $(\check{\mathfrak{t}})^* \cong \mathfrak{t}$. The Weyl groups of both groups are also the same, W. The important computation is $\mathbb{C}[\mathfrak{t}]^W \cong (\mathrm{Sym}\mathfrak{t}^*)^W \cong (\mathrm{Sym}\check{\mathfrak{t}})^W \cong (\mathrm{Sym}\check{\mathfrak{g}})^{\check{G}}$, where the last isomorphism is due to Chevalley.

Let A be a graded (or dg) algebra. Let a reductive group K act on A by graded algebra automorphisms. In the lecturer's course last quarter, a category $HC_{A,K}$ of Harish-Chandra modules was defined. Namely, each object $M \in HC_{A,K}$ is simultaneously a graded A-module and a K-representation, with the compatibility $k(am) = k(a)k(m)$. We assume from here on out that A is commutative (this is where the strategy breaks for loop rotation). Then $HC_{A,K}$ becomes a monoidal category with $\otimes_A$.

We will apply the above formalism to $A = \mathrm{Sym}\check{\mathfrak{g}}$ and $K = \check{G}$ acting on A via adjoint action. Derived Satake is a monoidal equivalence $\Phi : D_{G(\mathcal{O})}(Gr) \xrightarrow{\sim} D(HC_{\mathrm{Sym}\check{\mathfrak{g}},\check{G}})$.

Consider $A = \mathbb{C}$ with trivial K-action. Then $HC_{A,K} = \mathrm{Rep}K$. We have two functors $\mathrm{Rep}K \to \mathrm{Vect}$, namely invariants $M \mapsto M^K$ and forgetful $M \mapsto \underline{M}$. The regular representation $R = \mathbb{C}[K]$ decomposes as $\bigoplus_{V\in\mathrm{Irr}(K)} V \otimes V^* \in \mathrm{Ind}(\mathrm{Rep}K)$. For any $M \in \mathrm{Rep}K$, we have $R \otimes M \cong R \boxtimes M$ as $K \times K$-modules. On $R\otimes M$, the first copy of K acts diagonally, while the second copy of K acts only on R, and on the right. On $R \boxtimes M$, the first copy of K acts only on R and the second copy acts only on M. Thus, in $\mathrm{Rep}K$, where we only take the first copy of K, there is no K-action on M in $R \boxtimes M$. We have $(R \otimes M)^K = R^K \boxtimes M = M$ and $R \otimes R \cong R \boxtimes R$.

G is a G-torsor under right action. If X is an affine G-torsor then we have a coaction map $\mathbb{C}[X] \to \mathbb{C}[X \times G]$. Note $\mathbb{C}[X \times G] \cong \mathbb{C}[X] \otimes \mathbb{C}[G]$. The isomorphism $R\otimes R \cong R\boxtimes \underline{R}$ corresponds to $\mathbb{C}[X]\otimes\mathbb{C}[X] \cong \mathbb{C}[X]\otimes\mathbb{C}[G]$. (???)(???)

We use triv to denote the trivial 1-dimensional representation in $\mathrm{Rep}K$. It is the monoidal unit of $\mathrm{Rep}K$. We have $M^K = \mathrm{Hom}_K(\mathrm{triv}, M)$.

We can also take K-invariants on $HC_{A,K}$ to get an element of A^K-mod. We have that A is the monoidal unit of $HC_{A,K}$, and $M^K = \mathrm{Hom}_{HC}(A, M)$.

Let $R_A = R \otimes A$. We have $\mathrm{Hom}_{HC}(A, R_A \otimes_A M) \cong (R_A \otimes_A M)^K \cong ((R \otimes A) \otimes_A M)^K \cong (R \otimes M)^K \cong \underline{M}$, giving an internal way to take the forgetful functor.

Example 4.1. Let $A = \mathrm{Sym}\check{\mathfrak{g}}$, $K = \check{G}$. Then $R_A = R \otimes A = \mathbb{C}[\check{G}] \otimes \mathrm{Sym}\check{\mathfrak{g}} = \mathbb{C}[\check{G}] \otimes \mathbb{C}[(\check{\mathfrak{g}})^*] = \mathbb{C}[\check{G} \times (\check{\mathfrak{g}})^*] = \mathbb{C}[T^*\check{G}]$.

Let $\Lambda \subset X^*(\check{T}) \subset \check{\mathfrak{t}}^*$ be the dominant Weyl chamber. For $\lambda \in \Lambda$ we have the finite dimensional $V_\lambda \in \mathrm{Irr}(\check{G})$ with highest weight λ. Then $R = \mathbb{C}[\check{G}] = \bigoplus_{\lambda\in\Lambda} V_\lambda \otimes V_\lambda^*$. Abelian Satake is a monoidal functor $\Psi : (\mathrm{Rep}(\check{G}), \otimes) \to (D_{G(\mathcal{O})}(Gr), \star)$ sending V_λ to IC_λ. Let $\mathcal{R} = \psi(R)$; it decomposes as $\bigoplus_{\lambda\in\Lambda} IC_\lambda \otimes V_\lambda^*$. We get a map $G \to \mathrm{Aut}(\mathcal{R})$ via the action on V_λ^*. The monoidal unit of $\star$ is IC_0.

We will treat $\mathcal{R}$ as R_A, thus giving us a candidate for Φ: $R\mathrm{Hom}_{D_{G(\mathcal{O})}(Gr)}(IC_0, \mathcal{R}\star(-)) = \bigoplus_{\lambda\in\Lambda} V_\lambda^* \otimes R\mathrm{Hom}(IC_0, IC_\lambda \star (-))$. This would mean that A is $R\mathrm{Hom}(IC_0, \mathcal{R})$. Let us see how to give it an algebra structure. $R = \mathbb{C}[\check{G}]$ has a multiplication m, which gives a map $\mu : \mathcal{R} \star \mathcal{R} \to \mathcal{R}$. For $\alpha, \beta \in \mathrm{Hom}(IC_0, \mathcal{R})$, we can form $\alpha \star \beta : IC_0 \star IC_0 \to \mathcal{R} \star \mathcal{R}$. Then precomposing with $IC_0 \cong IC_0 \star IC_0$ and postcomposing with μ gives us a map $IC_0 \to \mathcal{R}$, which we call $\alpha \circ \beta$. Adding in appropriate shifts shows that this works on $R\mathrm{Hom}$ as well. We can also modify this to work for $\beta \in R\mathrm{Hom}(IC_0, \Psi(M))$, giving A-module structures to the image of Ψ.

For any space X and $\mathcal{F} \in D(X)$, we have $C_X \otimes \mathcal{F} \cong \mathcal{F}$ where $C_X \otimes \mathcal{F} = \Delta^*(C_X \boxtimes \mathcal{F})$ for $\Delta : X \hookrightarrow X \times X$. This induces a map $H(C_X) \otimes H(\mathcal{F}) \to H(\mathcal{F})$, i.e. $H(\mathcal{F})$ becomes a module over $H(C_X) = H(X)$. This works equally well in the equivariant setting. In particular, $H_{G(\mathcal{O})}(IC_\lambda)$ is a graded $H_{G(\mathcal{O})}(Gr)$-module.

In order to compute $R\mathrm{Hom}(IC_0, \mathcal{R})$, we need to compute $\mathrm{Ext}(IC_0, IC_\lambda)$. Taking equivariant cohomology gives us a map
$\mathrm{Ext}^i(IC_0, IC_\lambda) \to \mathrm{Hom}_{H_{G(\mathcal{O})}(Gr)}(H^\bullet_{G(\mathcal{O})}(IC_0), H^{i+\bullet}_{G(\mathcal{O})}(IC_\lambda))$.

Theorem 4.1. $\mathrm{Ext}(IC_\mu, IC_\lambda) \xrightarrow{\sim} \mathrm{Hom}_{H_{G(\mathcal{O})}(Gr)}(H_{G(\mathcal{O})}(IC_\mu), H_{G(\mathcal{O})}(IC_\lambda))$.

We have $H_{G(\mathcal{O})}(Gr) = \mathbb{C}[T^*(\mathfrak{t}^*/W)] = \mathbb{C}[T^*\mathfrak{c}] = \mathrm{SymLie}\check{\mathcal{J}}$, universal centralizer (explained last quarter). We also have $\mathrm{Hom}(H(IC_0), H(\mathcal{R})) \cong \mathbb{C}[\check{\mathfrak{c}} \times \check{G}] \cong \mathbb{C}[T^*_\psi(\check{G}/\check{U})]$ (notation also explained last quarter).

5 Apr 8

Cohomology of affine Grassmannian.

6 Apr 10

G connected reductive, $L = G(\mathcal{O})$, $Gr = G(\mathcal{K})/L$. $\pi_0(Gr) \simeq \pi_1(G)$. For any two connected components X, X' of Gr, there is some $g(t) \in G(\mathcal{K})$ such that the map $x \mapsto g(t)x$ is an isomorphism $X \xrightarrow{\sim} X'$. The components are $\mathbb{G}_m \times L$-stable. We fix a component X. We will prove that X is a disjoint union of spaces $X_0, X_1, \ldots$ such that

- Each X_d is $\mathbb{G}_m \times L$-stable.
- Each $X_{\leq d} = X_0 \cup \cdots \cup X_d$ is a projective variety.
- Each X_d is a disjoint union of affine spaces (cells).
- For all $i \geq 0$, there is $d(i) \gg 0$ such that X_d has no cells of dimension $\leq i$ for $d > d(i)$.

As a consequence, the embeddings $X_{\leq d} \hookrightarrow X_{\leq d+1}$ don't add cells of "low" dimension. In particular, for $d > d(i)$, we have isomorphisms $H_i(X_{\leq d}) \xrightarrow{\sim} H_i(X_{\leq d+1})$ and $H^i(X_{\leq d+1}) \xrightarrow{\sim} H^i(X_{\leq d})$. (It follows that?) X_d is equivariantly formal and $H^i_{\mathbb{G}_m \times L}(X_{\leq d+1}) \xrightarrow{\sim} H^i_{\mathbb{G}_m \times L}(X_{\leq d})$. We then define $H^i(Gr)$ to be the stable limit of $H^i(X_{\leq d})$, and similarly for the equivariant cohomology. The odd degree cohomologies vanish and Gr is equivariantly formal.

Let $M = \{g(t) \in L \mid g(0) = 1\}$. There is a SES $1 \to M \to L \to G \to 1$, where the map $L \to G$ sends $g(t)$ to $g(0)$. M is pro-unipotent and there is a group contraction $M \times [0,1] \to M$ defined by $(g(t), c) \mapsto g(ct)$. It follows that in all of our equivariant cohomologies, we can replace L by $L/M = G$.

We now address a point from last time and avoid the use of stacks. Let $\widetilde{Gr} = G(\mathcal{K})/M$. This has an evident map to Gr which has the structure of a principal G-bundle (since $L/M = G$). In particular, $\widetilde{Gr}$ has a left and right action by G. We have $H_{\mathbb{G}_m \times G \times G}(\widetilde{Gr}) \cong H_{\mathbb{G}_m \times G}(Gr)$.

Let $\mathfrak{g} = \mathrm{Lie} G$ have Cartan $\mathfrak{t}$ and Weyl group W. Let $Z = \mathbb{C}[\mathfrak{t}]^W = \mathbb{C}[\mathfrak{t}/W] = H_G(pt)$. We have $(Z \otimes Z)[\hbar] = H_{\mathbb{G}_m \times G \times G}(pt)$. Let $\zeta : (Z \otimes Z)[\hbar] \to H_{\mathbb{G}_m \times G}(Gr)$ be the map induced by $\widetilde{Gr} \to pt$, rewritten with the preceding identifications. Let $\mathcal{A}$ be the subalgebra of $(Z \otimes Z)[\hbar, \hbar^{-1}]$ generated by $(Z \otimes Z)[\hbar]$ and $(z \otimes 1 - 1 \otimes z)\hbar^{-1}$ for $z \in Z$.

Theorem 6.1. *Let $X \subset Gr$ be the connected component of $1 \cdot L/L$. Then the map ζ is injective and extends uniquely to a degree doubling algebra isomorphism $\mathcal{A} \to H_{\mathbb{G}_m \times G}(Gr)$.*

Proof Strategy. By Chevalley, $Z = \mathbb{C}[p_1, \ldots, p_r]$ for homogeneous polynomials p_i of degree d_i. Then $Z \otimes Z = \mathbb{C}[p_i \otimes 1, 1 \otimes p_i]$. Let $\xi_i = (p_i \otimes 1 - 1 \otimes p_i)\hbar^{-1}$. Then $\mathcal{A} = \mathbb{C}[\hbar, p_i \otimes 1, \xi_i]$. Note that $\deg(\xi_i) = d_i - 1$. If $\pi_1(G) = 1$, then $d_i \geq 2$ (for reasons related to tori), so $\mathcal{A}$ is non-negatively graded and $\mathcal{A}_0 = \mathbb{C}$. It is free over $(Z \otimes 1)[\hbar]$; the basis is given by the ξ_i. By formality, $H_{\mathbb{G}_m \times G}(Gr)$ is also free over $(Z\otimes 1)[\hbar]$; it is generated by the non-equivariant cohomology of Gr.

Now we "forget about loop rotation": $\zeta|_{\hbar=0} : Z \otimes Z \to H_{\mathbb{G}_m \times G}(Gr)/(\hbar)$. By formality, the codomain is $H_G(Gr)$. We will show that this map sends $z \otimes 1 - 1 \otimes z$ to 0. This will imply, by flatness, that ζ extends to $\widetilde{\zeta}\mathcal{A} \to H_{\mathbb{G}_m \times G}(Gr)$. We can again restrict this map to $\hbar = 0$. We recall $\mathcal{A}/\hbar\mathcal{A} = \mathbb{C}[T(\mathfrak{t}/W)]$. So $\widetilde{\zeta}$ gives a map $\mathbb{C}[T(\mathfrak{t}/W)] \to H_G(Gr)$. As a side note, the generators ξ in $\mathcal{A}$ give rise to partial derivatives in $\mathcal{A}/\hbar\mathcal{A}$.

Let $\mathcal{J} \subset Z$ be the augmentation ideal. If we mod out by $\mathcal{J}$ on both sides, we get a map $\mathbb{C}[\xi_i] \to H(Gr)$. We will prove that this map is an isomorphism. To do so, we will show that it is injective and that Poincare polynomials agree. Once we show this, we will have ζ is an isomorphism by flatness and Nakayama.

We will now change notation; denote $G_{\mathbb{C}}$ by the complex reductive group and G by a maximal compact subgroup. $G_{\mathbb{C}}$ contracts to G so there is no topological information lost. We can choose a Borel B and maximal complex torus $T_{\mathbb{C}}$ such that $T = G \cap B$ is a maximal torus in G. There is an anti-holomorphic anti-involution $G_{\mathbb{C}} \to G_{\mathbb{C}}$, $g \mapsto g^*$, such that $G = \{g \in G_{\mathbb{C}} \mid g^* = g^{-1}\}$. The Iwasawa decomposition states $G_{\mathbb{C}} = GB$. The embedding $G \hookrightarrow G_{\mathbb{C}}$ induces an isomorphism $G/T \xrightarrow{\sim} G_{\mathbb{C}}/B$, i.e. the flag manifold and flag variety can be identified.

We want an Iwasawa decomposition analogue for loop groups. Fix an embedding $G_{\mathbb{C}} \hookrightarrow GL_n(\mathbb{C})$ such that $G = G_{\mathbb{C}} \cap U_n$. Let $LG = G[z^{\pm 1}]$, which we can think of as algebraic maps $S^1 \to G$.

Lemma 6.1. *$LG = \{g(z) \in G_{\mathbb{C}}[z^{\pm 1}] \mid g(z)^* = g(z^{-1})\}$.*

Example 6.1. $G_{\mathbb{C}} = C^{\times}$, $G = S^1$. Note that $G_{\mathbb{C}}[z^{\pm}] \cong \mathbb{C}^{\times} \times \mathbb{Z}$, and $\mathbb{Z} \cong \pi_1(G_{\mathbb{C}})$. The $g(z) = cz^n$ that are in LG must satisfy $|c| = 1$. More generally, if $G_{\mathbb{C}} = T_{\mathbb{C}}$ is a torus, then $G = T$ is a torus, $T_{\mathbb{C}}[z^{\pm 1}] \cong T_{\mathbb{C}} \times X_*(T_{\mathbb{C}})$, $T[z^{\pm 1}] \cong T \times X_*(T)$, where X_* denotes the cocharacter lattice, and $X_*(T) = \pi_1(T)$.

We can now state Iwasawa decomposition for loop groups.

- $G_{\mathbb{C}}(\mathcal{K}) = LG \cdot G_{\mathbb{C}}(\mathcal{O})$.
- $LG \cap G_{\mathbb{C}}(\mathcal{O}) = G$, which we interpret as constant maps $S^1 \to G$.

This is equivalent to $LG/G \xrightarrow{\sim} G_{\mathbb{C}}(\mathcal{K})/G_{\mathbb{C}}(\mathcal{O}) = Gr$. The form LG/G is how topologists think of the affine Grassmannian.