Geometric Representation Theory Notes
Course taught by Victor Ginzburg

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Preface

There will be some gaps in explanation, either due to the lecturer's admission or my own lack of understanding. In particular, many "proofs" are sketches of proofs. Gaps due to my own misunderstanding will be indicated by three red question marks: ???. More generally, my own questions about the material will also be in red. Things like "**Question**" will be questions posed by the lecturer. Feel free to reach out to me with explanations.

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1.1 Quantization

Consider a family $A(c)$ of $\mathbb{C}$-algebras depending on $c \in \mathbb{C}$, such that

- $A(0)$ is commutative.
- $A(c) \cong A(1)$ for all $c \neq 0$.

Main Question. Can we get info on representations of $A(1)$ from the geometry of $\operatorname{Spec}(A(0))$?

Example 1.1. Let $\mathfrak{g}$ be a Lie algebra. Consider the family $\mathbb{C}$-indexed family of Lie algebras $\mathfrak{g}_c$, where $\mathfrak{g}_c$ has the same underlying vector space as $\mathfrak{g}$, and the bracket is given by $[x, y]_c = c[x, y]$. Let $A(c) = \mathcal{U}(\mathfrak{g}_c)$. In particular, $A(0) \cong \operatorname{Sym}\mathfrak{g} \cong \mathbb{C}[\mathfrak{g}^*]$ and $A(c) \cong \mathcal{U}\mathfrak{g}$ for $c \neq 0$.

Now let $R = \mathbb{C}[\hbar]$. Let $\mathcal{U}_\hbar\mathfrak{g} = T(\mathfrak{g} \oplus \mathbb{C} \cdot \hbar)/(\hbar \text{ is central}, x \otimes y - y \otimes x = \hbar[x, y]$. Then $\mathcal{U}_\hbar\mathfrak{g}$ is an R-algebra, and $\mathcal{U}_\hbar\mathfrak{g}/(\hbar - c) \cong \mathcal{U}\mathfrak{g}_c$ for all $c \in \mathbb{C}$.

Claim 1.1. *(1) $\mathcal{U}_\hbar\mathfrak{g}$ is flat over R, i.e. $\hbar - c$ is not a zero divisor.*

(2) $\mathbb{C}[\hbar^{\pm 1}] \otimes_R \mathcal{U}_\hbar\mathfrak{g} \cong \mathbb{C}[\hbar^{\pm 1}] \otimes_{\mathbb{C}} \mathcal{U}\mathfrak{g}$.

Proof. Give $\mathcal{U}\mathfrak{g}$ the PBW filtration and consider the Rees algebra $\operatorname{Rees}(\mathcal{U}\mathfrak{g}) = \sum_{i \geq 0} \hbar^i \cdot \mathcal{U}_{\leq i}\mathfrak{g} \subset \mathbb{C}[\hbar] \otimes_{\mathbb{C}} \mathcal{U}\mathfrak{g}$. The map $\mathfrak{g} \to \operatorname{Rees}(\mathcal{U}\mathfrak{g})$ given by $x \mapsto \hbar x$ extends to an isomorphism $\mathcal{U}_\hbar\mathfrak{g} \cong \operatorname{Rees}(\mathcal{U}\mathfrak{g})$, from which the claim follows.

Let $Z\mathfrak{g} = Z(\mathcal{U}\mathfrak{g})$, and fix a central character $\chi : Z\mathfrak{g} \to \mathbb{C}$. Let $A = \mathcal{U}\mathfrak{g}/\ker\chi$. The PBW filtration induces a filtration on A; let $A_\hbar$ be the associated Rees algebra. It is flat over R. Similar to above, we have $\mathbb{C}[\hbar^{\pm 1}] \otimes_R A_\hbar \cong \mathbb{C}[\hbar^{\pm 1}] \otimes_{\mathbb{C}} A$. Let $\overline{A} = A_\hbar/\hbar$.

Proposition 1.1. *$\overline{A} \cong \operatorname{Sym}\mathfrak{g}/I$, where $I = (\operatorname{Sym}\mathfrak{g})^G_+$ is the augmentation/irrelevant ideal of the invariants $(\operatorname{Sym}\mathfrak{g})^G$, and G is any connected Lie/algebraic group with Lie algebra $\mathfrak{g}$.*

Note. Since $\operatorname{Sym}\mathfrak{g} \cong \mathbb{C}[\mathfrak{g}^*]$, the ideal I corresponds to an ideal $J \subset \mathbb{C}[\mathfrak{g}^*]$. Thus, the proposition tells us that $\overline{A} = \mathbb{C}[\mathcal{N}]$ for some closed subscheme $\mathcal{N} \subset \mathfrak{g}^*$.

Example 1.2. Let $\mathfrak{g} = \mathfrak{sl}_2\mathbb{C}$. Identify it with $\mathbb{C}^3$, and identify $\mathfrak{g}^* \cong \mathfrak{g}$ via an invariant form. Then $\mathcal{N}$ is given by $xy = z^2$; it is the nilpotent cone. It actually looks like two (infinite) cones put together. TODO: include an image. (Is the N in the note above always the nilpotent cone?)

Now, fix $p \in \mathbb{C}[t]$. Let $A_{\hbar,p}$ be a $\mathbb{C}[\hbar]$-algebra with generators x, y, z and relations $zx - xz = 2\hbar x, zy - yz = -2\hbar y, xy = p(z + \frac{\hbar}{2}), yx = p(z - \frac{\hbar}{2})$.

Claim 1.2. *$\mathcal{U}_\hbar(\mathfrak{sl}_2)/\ker\chi \cong A_{\hbar,p}$ for some quadratic p.*

Example 1.3. Let $p = t^{n+1}$, and specialize to $\hbar = 0$. Then $A_{\hbar,p} = \mathbb{C}[x,y,z]/(xy - z^{n+1}) = \mathbb{C}[X_n]$, where X_n is the type A_n-Kleinian singularity. As you vary p across monic $(n+1)$-degree polynomials, you get universal deformation of Kleinian singularities.

Let $\mu : \mathbb{C}^2 \to \mathfrak{sl}_2\mathbb{C}$ be $(u,v) \mapsto \begin{pmatrix} uv & u^2 \\ -v^2 & -uv \end{pmatrix}$; this is the moment map for the standard action of SL_2 on $\mathbb{C}^2$. The image of μ is $\mathcal{N}$. The fiber over 0 is $(0,0)$, and the fibers over non-zero elements consist of two points each. Now let $\mathbb{Z}/2$ act on $\mathbb{C}^2$ by negation. Then there is an isomorphism $\mathbb{C}^2/(\mathbb{Z}/2) \xrightarrow{\sim} \mathcal{N}$ of varieties, corresponding to $\mu^* : \mathbb{C}[\mathcal{N}] \xrightarrow{\sim} \mathbb{C}[u,v]^{\mathbb{Z}/2}$.

We can generalize the above idea by replacing $\mathbb{Z}/2$ with any finite subgroup $\Gamma \subset SL_2\mathbb{C}$. For instance, the "type A_n subgroup" would be $\Gamma = \mathbb{Z}/(n+1)$, explicitly given by the diagonal matrices $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ for $(n+1)$th roots of unity ζ. But, consider any finite subgroup Γ for now. Let $\mathbb{C}\langle x,y\rangle \rtimes \Gamma$ be the quotient of $\mathbb{C}\langle x,y\rangle \otimes \mathbb{C}[\Gamma]$ by the relations given by the Γ action. Let $c : \Gamma \to \mathbb{C}$ be a class function. Let $H_{\hbar,c}$ be the quotient of $(\mathbb{C}\langle x,y\rangle \rtimes \Gamma)[\hbar]$ by the relation $yx - xy = \hbar + \sum_{1 \neq \gamma \in \Gamma} c(\gamma)\gamma$. Let $e = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma$. Finally, let $A_{\hbar,c} = eH_{\hbar,c}e$.

Claim 1.3. *For Γ the subgroup of type A_n, there is a bijection between the class functions c and degree $(n+1)$ polynomials p for which $A_{\hbar,c} \cong A_{\hbar,p}$.*

1.2 Singular Support

Let A be "some algebra from our previous constructions". Recall that $R = \mathbb{C}[\hbar]$, and $A_\hbar$ is the Rees algebra. Let M be a finitely generated A-module. Then $\mathbb{C}[\hbar^{\pm 1}] \otimes_{\mathbb{C}} M$ is a module over $\mathbb{C}[\hbar^{\pm 1}] \otimes_{\mathbb{C}[\hbar]} A_\hbar$. Pick a finitely generated $A_\hbar$-submodule $M_\hbar \subset \mathbb{C}[\hbar^{\pm 1}] \otimes M$ so that (I think?) $\mathbb{C}[\hbar^{\pm 1}] \otimes_R M_\hbar \cong \mathbb{C}[\hbar^{\pm 1}] \otimes_{\mathbb{C}} M$. Then $\overline{M} = M_\hbar/\hbar$ is a finitely generated module over $\overline{A} = A_\hbar/\hbar$.

Definition 1.1 (Singular support)**.** The **singular support** of M is $SS(M) = \mathrm{Supp}(\overline{M}) \subset \mathrm{Spec}(\overline{A})$.

Question. Can we get info on category $\mathcal{O}$ from this perspective?

Let $\mathfrak{g}$ be a semisimple Lie algebra. Let $\mathfrak{b}$ be a Borel subalgebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{b}$, and let $\mathfrak{n} = [\mathfrak{b},\mathfrak{b}]$ so that $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$. Fix a central character $\chi : Z\mathfrak{g} \to \mathbb{C}$, and let $\mathcal{U}_\chi = \mathcal{U}\mathfrak{g}/\ker\chi$.

Definition 1.2 (Category $\mathcal{O}_\chi$)**.** The category $\mathcal{O}_\chi$ has objects M such that:

1) M is a finitely generated $\mathcal{U}_\chi$-module.

2) $\dim(\mathcal{U}\mathfrak{b} \cdot m) < \infty$ for all $m \in M$.

3) For all $m \in M$ and $n \in \mathfrak{n}$, there is $i \gg 0$ such that $n^i(m) = 0$.

Definition 1.3 (Verma module)**.** For all $\lambda \in \mathfrak{h}^*$, let M_λ be the quotient of $\mathcal{U}_\chi$ by the left ideal generated by $\mathfrak{n}$ and the elements $h - \lambda(h)$ for $h \in \mathfrak{h}$. This is the **Verma module** associated to λ.

Claim 1.4. *(1) If $M_\lambda \neq 0$, then $SS(M_\lambda) = \mathfrak{b}^\perp \subset \mathcal{N} = \mathrm{Spec}(\overline{\mathcal{U}_\chi}) \subset \mathfrak{g}^*$.*

(2) For all $M \in \mathcal{O}_\chi$, $SS(M) \subset \mathfrak{b}^\perp$.

For $\mathfrak{g} = \mathfrak{sl}_2\mathbb{C}$, the choice of $\mathfrak{b}$ is equivalent to the choice of $\mathfrak{b}^\perp$, which is a line (through the origin) on $\mathcal{N}$ (cf. Example 1.2).

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3.1 From Last Time

Setup:

- A is a non-negatively filtered k-algebra with filtration $A_{\leq i}$.
- $\overline{A} = \mathrm{gr}A$ is non-negatively graded, with $\overline{A}_0 = k$.
- $\overline{A} = k[X]$ for an affine variety X.
- X is reduced, irreducible, and Cohen-Macaulay (e.g. smooth).
- $\mathfrak{h}$ is a Lie algebra.
- $\mu : \mathcal{U}\mathfrak{h} \to A$ is an algebra map such that $\mu(\mathfrak{h}) \subset A_{\leq 1}$.

μ takes the PBW filtration on $\mathcal{U}\mathfrak{h}$ to filtration on A, so we may take the associated graded map $\overline{\mu} : \mathrm{gr}(\mathcal{U}\mathfrak{h}) = k[\mathfrak{h}^*] \to \overline{A} = k[X]$. This is the same data as a map $X \to \mathfrak{h}^*$, which we also call $\overline{\mu}$. We have the map

$$\overline{A}/\overline{A} \cdot \overline{\mu}(\mathfrak{h}) \twoheadrightarrow \mathrm{gr}(A/A \cdot \mu(\mathfrak{h})) \tag{$\star$}$$

We want to know when this is an isomorphism.

Let $\xi_1, \dots, \xi_d$ be a basis of $\mathfrak{h}$. $\mathrm{Var}(\overline{A} \cdot \overline{\mu}(\mathfrak{h})) = \{\overline{\mu}(\xi_i) = 0\} = \overline{\mu}^{-1}(0) =$ scheme theoretic zero fiber. Let $\overline{\xi}_i = \overline{\mu}(\xi_i)$.

Proposition 3.1. *If $\overline{\xi}_1, \dots, \overline{\xi}_d$ form a regular sequence in $\overline{A}$, then $(\star)$ is an isomorphism.*

Corollary 3.1. *If $\dim \overline{\mu}^{-1}(0) = \dim X - \dim \mathfrak{h}$, then:*

1) $\overline{\mu}$ is flat with all fibers of dimension equal to $\dim X - \dim \mathfrak{h}$.

2) $(\star)$ is an isomorphism.

Proof. Since $X = \mathrm{Spec}\ \mathrm{gr}(A)$, it has a $\mathbb{G}_m$-action. Let $o \in X$ be the closed point corresponding to the max ideal $\overline{A}_{>0} = \bigoplus_{i>0} \overline{A}_i$. Then (informally) $\lim_{z \to 0} z \cdot x = o$ for all $x \in X$, where $z \in \mathbb{G}_m$. (The action is contracting to o). The map $\overline{\mu} : X \to \mathfrak{h}^*$ is $\mathbb{G}_m$-equivariant. By semicontinuity, $\dim \overline{\mu}^{-1}(h) \leq \dim \overline{\mu}^{-1}(0)$. But $\dim X - \dim \mathfrak{h}$ is ??? so the given assumption on the zero fiber gives the condition on all fibers.

This plus assumptions on X give $\overline{\mu}$ is flat. It follows that the $\overline{\xi}_i$ form a regular sequence.

Example 3.1. Let $\mathfrak{g}$ be a semisimple Lie algebra, say the Lie algebra of connected G. Let $A = \mathcal{U}\mathfrak{g}$ be with PBW filtration, so $\overline{A} = k[\mathfrak{g}^*]$ (i.e. our X is $\mathfrak{g}^*$). We have (Chevalley) $k[\mathfrak{g}^*]^G = k[\overline{\xi}_1 \dots \overline{\xi}_r]$, with r the rank of $\mathfrak{g}$. We also have (Harish-Chandra) $Z(\mathcal{U}\mathfrak{g}) = (\mathcal{U}\mathfrak{g})^G = k[\xi_1, \dots, \xi_r]$. The notation here is meant to indicate that in fact there is a filtration on $Z(\mathcal{U}\mathfrak{g})$ and the above process takes the ξ_i to the $\overline{\xi}_i$. In particular, let $\mathfrak{h} = k\xi_i \oplus \cdots \oplus k\xi_r$ as an abelian Lie algebra. $U\mathfrak{h} = k[\xi_1, \dots, \xi_r] = Z(\mathcal{U}\mathfrak{g}) \to \mathcal{U}\mathfrak{g} = A$. This doesn't quite fit the above setting, in particular $\mathfrak{h}$ does not map to $A_{\leq 1}$, but the proof idea still works. $\overline{\mu} : k[\mathfrak{g}^*]^G \hookrightarrow k[\mathfrak{g}^*]$, corresponding to $\overline{\mu} : \mathfrak{g}^* \to$??? (probably $\mathfrak{g}^*//G$). $\overline{\mu}^{-1}(0) = \mathcal{N} \subset \mathfrak{g}^*$ is the nilpotent cone. Kostant: $\overline{\mu}$ is flat with fibers of dimension $\dim \mathfrak{g} - r$, so the Corollary above applies. For any max ideal Z_χ of $Z(\mathcal{U}\mathfrak{g})$, we can choose the ξ_i as above so that Z_χ is generated by the ξ_i. Recall the RHS of map ($\star$). We have $\mathcal{U}\mathfrak{g}/\mathcal{U}\mathfrak{g} \cdot Z_\chi = \mathcal{U}_\chi$, the "central reduction". The LHS is $k[\mathcal{N}]$. So we have $k[\mathcal{N}] \xrightarrow{\sim} \mathrm{gr}(\mathcal{U}_\chi)$. (Note: He wrote $\mathbb{C}[\mathcal{N}]$).

Example 3.2. Let H be an algebraic group with Lie algebra $\mathfrak{h}$. Let Y be a smooth affine variety with H-action. We get a map $\mathfrak{h} \to T_Y$, extending to $\mu : \mathcal{U}\mathfrak{h} \to \mathcal{D}(Y) = A$, where $\mathcal{D}(Y)$ is the algebra of differential operators on Y. This has a filtration given by the order of the operators. We get map $\overline{\mu} : k[\mathfrak{h}^*] \to \mathrm{gr}(\mathcal{D}(Y)) = k[T^*Y]$ corresponding to $\overline{\mu} : T^*Y \to \mathfrak{h}^*$ (moment map).

Proof of Proposition. More generally, suppose we have an A-module M. The map $\mu : \mathcal{U}\mathfrak{h} \to A$ makes M a $\mathcal{U}\mathfrak{h}$-module. Consider the Chevalley-Eilenberg complex: $C_\bullet(\mathfrak{h}, M) = M \otimes_k \bigwedge^\bullet \mathfrak{h}$, denoted $M \otimes \bigwedge^\bullet$ for convenience. The differential ∂ does two things to an element $m \otimes \eta_1 \wedge \cdots \wedge \eta_j$.

1. $\mapsto \sum_i \pm(\eta_i \cdot \mu) \otimes \eta_1 \wedge \cdots \widehat{\eta_i} \wedge \cdots \wedge \eta_j$.
2. $\mapsto \sum_{i,\ell} \pm m \otimes [\eta_i, \eta_\ell] \wedge \cdots$.

Then add these up (with an appropriate sign on the second thing). This decreases degree. We define $H_\bullet(\mathfrak{h}, M)$ to be $H(C_\bullet(\mathfrak{h}, M))$. Now suppose M has a "good" filtration as an A-module. The differential ∂ takes $M_{\leq i} \otimes \bigwedge^j \to M_{\leq i+1} \otimes \bigwedge^{j-1} + M_{\leq i} \otimes \bigwedge^{j-1}$. Filter $C_\bullet$ by $(M \otimes \bigwedge^\bullet)_{\leq h} = \sum_{i+j=h} M_{\leq i} \otimes \bigwedge^j$. Then $\mathrm{gr} C_\bullet = \mathrm{gr} M \otimes \bigwedge^\bullet$. The differential ∂ induces $\overline{\partial} = \mathrm{gr}\partial : \mathrm{gr}_i M \otimes \bigwedge^j \to \mathrm{gr}_{i+1} M \otimes \bigwedge^{j-1}$ (the second contribution of ∂ dies). $\mathrm{gr} M$ is a module over $\overline{A} = k[X]$. The complex $(\mathrm{gr} C_\bullet, \overline{\partial})$ is actually the Koszul complex for the $\overline{\xi}_i$.

We have a spectral sequence for the filtered complex $C_\bullet$, with $E_2 = H^\bullet(\mathrm{gr} C_\bullet, \overline{\partial})$ and $E_\infty = \mathrm{gr} H_\bullet(\mathfrak{h}, M)$. If $M_{\leq i} = 0$ for $i \ll 0$, then the spectral sequence converges. If M is complete, the spectral sequence will converge "topologically".

Since the $\overline{\xi}_i$ are a regular sequence, their Koszul complex being acyclic in positive degree, and the H^0 is $\mathrm{gr} M / \overline{\mu}(\mathfrak{h}) \cdot \mathrm{gr} M$. Thus the spectral sequence above degenerates at E_2 and we're done.

3.2 Kirillov Model

Let $\mu : \mathfrak{n} \to \mathbb{C}$ be a Lie algebra character. Suppose as before we have $\mu : \mathcal{U}\mathfrak{n} \to A$. Let $\mathfrak{n}_\psi$ be the Lie subalgebra generated by elements of the form $\xi - \psi(\xi)$. If we want to analyze $A/\mu(\mathfrak{n}_\psi)A$, then we can't use the PBW filtration on $\mathcal{U}\mathfrak{n}$, since it will kill the $\psi(\xi)$ terms and just give $\mathfrak{n}$. Note that $\psi \in \mathfrak{n}^*$. There is a so-called Kazhdan filtration on $\mathcal{U}\mathfrak{n}$ for which ψ is treated as the origin. However, it is not a non-negative filtration and issues arise. The so-called Kirillov model fixes (some?) things. We assume we have a $\mathbb{G}_m$ action on N (where $\mathfrak{n}$ is the Lie algebra of N) and that ψ has positive weight m with respect to this action. Instead of taking fibers over non-zero multiples of ψ, one removes 0 (localizes) and takes $\mathbb{G}_m$ quotient. This is where non-commutative localization shows up. If $\mathfrak{n}$ is a nilpotent Lie algebra, the Ore condition is satisfied.

4 Feb 4

4.1 Poisson Geometry

Let V be a finite dimensional vector space over k. Let $\pi \in \bigwedge^2 V$, thought also as a skew symmetric bilinear form on V^*. We have a map $V^* \xrightarrow{i_\pi} V$, whose image is isomorphic to $V^*/\mathrm{Rad}\pi$. π induces a symplectic form ω on the image by $\omega(i_\pi\alpha, i_\pi\beta) = \langle \pi, \alpha \wedge \beta \rangle$.

Let $\mathcal{A} = k[X]$ be a Poisson algebra. Each element a of $\mathcal{A}$ gives a derivation ξ_a, thought of as a vector field on X. There is π such that $\{a, b\} = \langle \pi, da \wedge db \rangle$. So we apply the above reasoning to the tangent spaces of X (at smooth points).

From now until stated otherwise, restrict to $k = \mathbb{R}$ or $\mathbb{C}$.

Theorem 4.1 (Frobenius)**.** *Let X be smooth (manifold). The collection of spaces* $\mathrm{im}(i_{\pi_x})$ *is integrable.*

This means X is partitioned into symplectic submanifolds Σ called **symplectic leaves**. They are locally nice but not necessarily globally nice. $\{a, b\}_\Sigma$ only depends on $a|_\Sigma, b|_\Sigma$; in particular, it is given by $\omega_\Sigma(\xi_a, \xi_b)$. Thus an element $a \in \mathcal{A}$ is constant on the leaves iff it is in the Poisson center, i.e. $\{a, b\} = 0$ for all $b \in \mathcal{A}$.

A **Poisson ideal** I of $\mathcal{A}$ is an ideal I (using just the commutative ring structure of $\mathcal{A}$) that satisfies

$$\{I, \mathcal{A}\} \subseteq I. \qquad (\star)$$

This holds iff $\xi_a(I) \subseteq I$ for all $a \in \mathcal{A}$. In this case, $\mathcal{A}/I$ inherits a Poisson algebra structure. $\mathrm{Spec}(\mathcal{A}/I) = \mathrm{Var}(I)$ will be a union of leaves. (Which leaves ??? And is this still smooth???)

Example 4.1. Let H be a connected Lie group with Lie algebra $\mathfrak{h}$. Then $\mathrm{gr}(\mathcal{U}\mathfrak{h}) = \mathrm{Sym}\mathfrak{h} = k[\mathfrak{h}^*]$ has a natural Poisson algebra structure. The leaves are exactly the coadjoint H-orbits. Note that if H is not algebraic, the leaves can be "bad", e.g. not locally closed. An ideal $I \subset \mathrm{Sym}\mathfrak{h}$ is a Poisson ideal iff $\mathrm{ad}_h(I) \subseteq I$ for all $h \in \mathfrak{h}$ iff I is H-stable.

Now allow k to be arbitrary (but characteristic 0) again.

Definition 4.1. For any ideal $\mathfrak{a} \subseteq \mathcal{A}$, let $\mathcal{P}(\mathfrak{a})$ be the maximal Poisson ideal contained in $\mathfrak{a}$. Geometrically, one should expect $\mathcal{P}(\mathfrak{a})$ to correspond to the union of leaves which contain the vanishing locus of $\mathfrak{a}$.

Proposition 4.1. *(1) If $I \subset \mathcal{A}$ is a Poisson ideal, then $\sqrt{I}$ and all associated primes are also Poisson ideals.*

(2) If $\mathfrak{a} \subset \mathcal{A}$ is prime, then $\mathcal{P}(\mathfrak{a})$ is also prime.

Lemma 4.1. *Let A be a commutative algebra with derivation ξ.*

(1) *If $I \subset A$ is an ideal with $\xi(I) \subseteq I$, then $\xi(\sqrt{I}) \subseteq \sqrt{I}$.*

(2) *If $\mathfrak{a} \subseteq A$ is prime, then the maximum ξ-stable ideal contained in $\mathfrak{a}$ is also prime.*

Idea of Proof. Consider the derivation of $A[[t]]$ given by $a \mapsto t\xi(a)$. Get an automorphism (???) $a \mapsto a + t\xi(a) + \frac{t^2}{2}\xi^2(a) + \cdots$.

Remark. The integral closure of a Poisson algebra is Poisson. The singular locus of the vanishing locus of an ideal is closed(?) under the various constructions in Proposition 4.1.

Theorem 4.2. *Let $k = \mathbb{C}$, X reduced and irreducible. Assume X has finitely many leaves. Then for all closed points $x \in X$ with corresponding maximal ideal $\mathfrak{m}_x$, the symplectic leaf Σ_x is the regular locus of $\mathrm{Var}(\mathcal{P}(\mathfrak{m}_x))$. In particular, all of the leaves are smooth, locally closed, and connected subvarieties, and their symplectic structure is algebraic.*

Corollary 4.1. *Any irreducible component of $\mathrm{Var}(I)$ for Poisson ideal I is the closure of a leaf.*

Corollary 4.2. *There is a unique Zariski open and dense leaf; hence the Poisson center consists of scalars.*

4.2 Central Reduction

Let $\mathcal{A} = \mathbb{C}[X]$ be a Poisson algebra with Poisson center $\mathcal{Z} = \mathcal{Z}(\mathcal{A})$. Assume $\mathcal{Z}$ is finitely generated as an algebra. Pick a central character $\chi : \mathcal{Z} \to \mathbb{C}$. Then $\mathcal{A}\cdot\ker\chi$ is a Poisson ideal. Let $\mathcal{A}_\chi = \mathcal{A}/\mathcal{A}\cdot\ker\chi$. Then $\mathrm{Spec}(\mathcal{A}_\chi) = \{z = \chi(z)\}$. Ideally $\mathcal{Z}(\mathcal{A}_\chi) = \mathbb{C}$ and $\mathrm{Spec}(\mathcal{A}_\chi)$ has finitely many leaves.

Now let A be a $\mathbb{Z}$-filtered algebra. Suppose $\overline{A} = \mathrm{gr}A$ is a commutative Poisson algebra. If $J \subset A$ is a left ideal, then $\mathrm{gr}J$ is stable under the Poisson bracket, as is (by Gabber's theorem) $\sqrt{\mathrm{gr}J}$. If $I \subset A$ is a two-sided ideal, then $\mathrm{gr}I$ is a Poisson ideal, as is (by the above Proposition 4.1) $\sqrt{\mathrm{gr}I}$. Furthermore, $\mathrm{gr}(A/I) = \overline{A}/\mathrm{gr}I$.

Example 4.2. Let H be a connected algebraic group with Lie algebra $\mathfrak{h}$. Let $A = \mathcal{U}\mathfrak{h}$. A left ideal J is two-sided iff $\mathrm{ad}_x(J) \subseteq J$ for all $x \in \mathfrak{h}$ iff J is H-stable. Now let M be a finitely generated $\mathcal{U}\mathfrak{h}$-module. Then $SS(M) \subseteq \mathfrak{h}^*$. $\mathrm{Ann}M$ is two-sided an $SS(A/\mathrm{Ann}M)$ is H-stable. If J is a left ideal and $M = \mathcal{U}\mathfrak{h}/J$, then $SS(M) = \mathrm{Var}(\mathrm{gr}(J))$. $\mathrm{Ann}M$ is the maximal two-sided ideal contained in J, i.e. the maximal H-stable ideal contained in J. This is given by $\bigcap_{h\in H}\mathrm{Ad}_h(J)$.

Example 4.3. Let $\mathfrak{g}$ be a semisimple Lie algebra. Let M be a simple $\mathcal{U}\mathfrak{g}$-module. By Schur's lemma, $Z = Z(\mathcal{U}\mathfrak{g})$ acts on M by scalars, i.e. there is a character $\xi : Z \to \mathbb{C}$ such that $\ker\chi \subseteq \mathrm{Ann}M$. Thus the action of $\mathcal{U}\mathfrak{g}$ on M factors through an action of $\mathcal{U}_\chi = \mathcal{U}\mathfrak{g}/\ker\chi$. We have $\mathrm{gr}(\mathcal{U}_\chi) = \mathcal{N} \subset \mathfrak{g}^*$ is the nilpotent cone, which has finitely many nilpotent orbits. We have $SS(M) \subseteq$

$SS(\mathcal{U}_\chi/\mathrm{Ann}M) \subset \mathcal{N}$. The leftmost thing is coisotropic, and the middle thing is a union of orbits. In fact:

Theorem 4.3. *1) $SS(\mathcal{U}_\chi/\mathrm{Ann}M)$ is the closure of a single orbit.*

2) $\dim SS(M) \geq \frac{1}{2} \dim SS(\mathcal{U}_\chi/\mathrm{Ann}M)$.

Remark. Part 2) of this theorem was used in geometric Langlands.

5 Feb 6

5.1 Hamiltionian Reduction

Let (X,ω) be smooth symplectic manifold (variety (???)). Let Σ be smooth connected closed subvariety (submanifold (???)). For $x \in X$ and $V \subseteq T_xX$, write $V^\perp = \{\xi \in T_xX : \omega(\xi,\sigma) = 0 \forall \sigma \in V\}$.

Claim 5.1. *Σ is coisotropic iff $(T_x\Sigma)^\perp \subseteq T_x\Sigma$ for all $x \in \Sigma$. Equality iff Lagrangian.*

From now on (???)we assume Σ is coisotropic.

Claim 5.2. *Let I_Σ be the ideal of Σ (e.g. assuming X affine). $f \in \mathcal{O}_X$ constant on Σ iff $df|_{T\Sigma} = 0$. In case Σ is coisotropic, those two conditions hold iff $\xi_f \in (T_\Sigma)^\perp$. Then $(T_x\Sigma)^\perp$ consists of the $(\xi_f)_x$ for $f \in I_\Sigma$. By coisotropic, I_Σ is a Lie algebra, and by Frobenius theorem (4.1), $(T_x\Sigma)^\perp$ forms an integrable distribution (take Hamiltonian flows of the ξ_f for $f \in I_\Sigma$). The corresponding foliation is called the **nil-foliation**.*

ω induces a nondegenerate bilinear form on $T_x\Sigma/(T_x\Sigma)^\perp$, which are the tangent spaces of... some quotient ... which may or may not exist. If it does exist, this implies that it is symplectic.

$N(I_\Sigma) = \{a \in k[X] \mid \{a, I_\Sigma\} \subseteq I_\Sigma\}$ is the **Poisson normalizer**. It is a Poisson algebra which contains I_Σ as a Lie ideal, so the quotient $N(I_\Sigma)/I_\Sigma$ is naturally a Poisson algebra as well.

For $f \in \mathcal{O}_X$, f is constant along the leaves of the nil-foliation iff $f \in N(I_\Sigma)$.

Let A be an associative algebra with subalgebra B and embedding $\mu : B \hookrightarrow A$ (later on we will not need that it is an embedding). Let J be a two-sided ideal of B. Let $N(J) = \{a \in A \mid aJ \subseteq JA\} = \{a \in A \mid aj - ja \in JA \forall j \in J\}$. It is a subalgebra of A which contains JA as a two-sided ideal, so $N(J)/JA$ is an associative algebra; call it $A//\mu J$ (quantum Hamiltonian reduction). Let $E = A/JA$. It is naturally a right A-module.

Claim 5.3. *The map $N(J)/JA \to \mathrm{End}_A E$ sending a to the mapping $x + JA \mapsto ax + JA$ is an isomoprhism.*

We see then that E is a left $A//\mu J$-module. We have functors $A{-}\mathrm{mod} \to A//\mu J{-}\mathrm{mod}$ and (similarly for right modules) given by $M \mapsto E \otimes_A M = M/JM$ and $M \mapsto \mathrm{Hom}_A(E, M) = M^J = \{m \mid jm = 0 \forall j\}$ respectively.

Want to match this with Poisson story.

Let $\mathcal{A} = k[X]$ be a Poisson algebra. Let $\mathcal{B} = k[Y]$ be a Poisson algebra with Poisson ideal J. Let $Z = \mathrm{Var}(J)$; it is a union of leaves. Let $\mu : \mathcal{B} \to \mathcal{A}$ be a

Poisson morphism. The ideal of $\mu^{-1}(Z)$ is $\mathcal{A}\mu(J)$. $\mu^{-1}(A)$ is coisotropic but not necessarily union of leaves; i.e. $\mathcal{A}\mu(J)$ is a Lie subalgebra bot not necessarily a Lie ideal.

Let H be a connected algebraic group with Lie algebra $\mathfrak{h}$. Let A be an associative algebra. Let $\mu : \mathcal{U}\mathfrak{h} \to A$ be a morphism of algebras. Let J be a two-sided ideal in $\mathcal{U}\mathfrak{h}$. (I don't know where this was going (???))

Let $\mathcal{B} = \mathrm{Sym}\mathfrak{h} = k[\mathfrak{h}^*]$, which maps to some Poisson algebra $\mathcal{A} = k[X]$. Have $\mu : X \to \mathfrak{h}^*$. Analog of J will be a closed coadjoint orbit O (recall these are the leaves). Let $I = I_O$ be its ideal. Assume $d_x\mu$ is surjective at every $x \in \mu^{-1}(O)$. This makes $\mu^{-1}(O)$ smooth submanifold.

Assume in particular that H acts on X such that for any $h \in \mathfrak{h}$, the vector field $\alpha(h)$ induced by h is the same as $\xi_{\mu(h)}$.

Lemma 5.1. *Let H_λ be the stabilizer of $\lambda \in \mathfrak{h}^*$.*

1) $\mu : X \to \mathfrak{h}^*$ is equivariant.

2) For any $\lambda \in O$, the action of H_λ on $\mu^{-1}(\lambda)$ is free.

3) The nil-foliation on $\mu^{-1}(O)$ exists, and in particular, the leaves are the H_λ orbits in the fibers.

Note. Ginzburg: "I got lost, I might/will come back to this"

Remark. The condition that $d\mu$ is surjective along $\mu^{-1}(O)$ is equivalent to the condition that H_λ acts freely on each fiber.

$\mu^{-1}(O)/H = \mu^{-1}(\lambda)/H_\lambda$ allows you to restrict to when orbits are points.

5.2 Slodowy Slices

Let G be a semisimple group with Lie algebra $\mathfrak{g}$. Let $\mu : \mathcal{U}\mathfrak{g} \to A$ be an algebra morphism. Ideologically, one wants to relate simple representations appearing in A to the fibers of $\mathrm{Spec}(\mathrm{gr}A) \to \mathfrak{g}^*$ (where A is filtered and μ respects filtration), but this isn't really possible. Instead think about representations supported on complement of regular elements. Well this might be too coarse – Slodowy slices allow you to control what to remove.

Historically, wanted to understand representations of e.g. $G(\mathbb{F}_q)$ or $G(\mathbb{Q}_p)$. In both cases there is a suitable meaning of "generic". Let U be the maximal unipotent subgroup, ψ additive character. Then $\int_U f(gu)\psi(u)du$ allows you to look at what to throw out.

Remark. Reason geometric Langlands is easier than normal Langlands: "cotangent directions" allow for more refined invariants.

For graded: $SS(M)$ is a point iff M is finite dimensional. In particular, $\dim SS(M)$ measures “how infinite dimensional” M is; related to Hilbert polynomial (how exactly (???)he said look at Atiyah-Macdonald).

Harish-Chandra: Character of infinite dimensional representation of real group exists as a distribution, well-defined function on regular semisimple.

6 Feb 11

6.1 Slodowy Slices

Let $\mathfrak{g}$ be a semisimple Lie algebra, with G a connected complex group whose Lie algebra is $\mathfrak{g}$. By Jacobson-Morosov, any nilpotent element $e \in \mathfrak{g}$ can be completed (non-uniquely) to an $\mathfrak{sl}_2$-triple (e, h, f). (These are elements in $\mathfrak{g}$ that span a Lie subalgebra isomorphic to $\mathfrak{sl}_2$.) h is a semisimple element. We get a decomposition $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ where $\mathfrak{g}_i$ is the subspace where ad_h acts by i. In particular, $e \in \mathfrak{g}_2, f \in \mathfrak{g}_{-2}, h \in \mathfrak{g}_0$. (Insert weight picture here) We get a direct sum decomposition $\mathfrak{g} = \mathfrak{g}_f \oplus [e, \mathfrak{g}] = \ker(\mathrm{ad}_f) \oplus \mathrm{im}(\mathrm{ad}_e)$. Now, recall from last time that we want some way to get rid of bad parts of orbits. We have $T_e(\mathrm{Ad}_G e) = [e, \mathfrak{g}]$, so the affine space $e + \mathfrak{g}_f$ is a "slice" through Ge at e. (insert picture)

$\mathfrak{g}_0 = \mathfrak{g}_h$ is a Levi subalgebra. $G_0 = G_h =$ stabilizer of h has Lie algebra $\mathfrak{g}_0$. $\mathfrak{g}_{\leq 0} = \bigoplus_{i \leq 0} \mathfrak{g}_i$ is parabolic. It contains $\mathfrak{g}_f$. $\mathfrak{u} = \mathfrak{g}_f \cap \mathfrak{g}_{<0}$ is a nilpotent ideal in $\mathfrak{g}_f$. There is a unipotent $U \subseteq G$ with Lie algebra $\mathfrak{u}$. U is normal in G_f. $\mathfrak{g}_f \cap \mathfrak{g}_0 =$ centralizer of h and $f =$ centralizer of the copy of $\mathfrak{gl}_2$, denoted $Z_{\mathfrak{g}}(\mathfrak{sl}_2)$. It is also the direct sum of the trivial $\mathfrak{sl}_2$ representations (insert picture).

Proposition 6.1. *1) $Z_G(\mathfrak{sl}_2)$ is a maximal reductive subgroup of $Z_G(e)$, and U is the unipotent radical of $Z_G(e)$. Thus $Z_G(e) = Z_G(\mathfrak{sl}_2) \ltimes U$.*

2) The G_0-orbit of e (resp. f) is Zariski open and dense subset of $\mathfrak{g}_2$ (resp. $\mathfrak{g}_{-2}$).

Sketch of Proof of Proposition Part 1). $Z_{\mathfrak{g}}(\mathfrak{sl}_2)$ has a nondegenerate invariant form by restriction, and therefore is reductive. Thus $Z_G(\mathfrak{sl}_2)$ is reductive.

The inclusion $\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ can be exponentiated to $SL_2 \to G$. Restrict it to the diagonals (identified with $\mathbb{C}^*$) to get a map $\gamma : \mathbb{C}^* \to G$. The differential of this map sends 1 to h. Since $\mathrm{ad}_h e = 2e$, we get $\mathrm{Ad}_{\gamma(z)} e = z^2 e$. Let $\widetilde{G}_e = \{(g, z) \in G \times \mathbb{C}^* \mid \mathrm{Ad}_g e = ze\}$. The natural projection $\widetilde{G}_e \to \mathbb{C}^*$ fits into a short exact sequence $1 \to G_e \to \widetilde{G}_e \to \mathbb{C}^* \to 1$. In particular, $\widetilde{G}_e$ is generated by $\gamma(\mathbb{C}^*)$ and G_e. ("Almost a product, but there is some finite intersection")

$h + \mathfrak{u}$ is a G_e-stable affine subspace of $\mathfrak{g}$. We claim:

Claim 6.1. *$h + \mathfrak{u} = Uh$.*

Proof of Claim. $T_h(Uh) = \mathrm{ad}_{\mathfrak{u}} h = \mathfrak{u}$, since by definition $\mathfrak{u}$ contains only negative weights for h. By translation, $\mathfrak{u} = T_h(h + \mathfrak{u})$. Thus Uh is open in $h + \mathfrak{u}$ (???). But, a general fact is that unipotent orbits of affine varieties are closed. Thus $Uh = h + \mathfrak{u}$.

Continuation of Sketch of Proof of Proposition Part 1). Let R be a maximal reductive subgroup of G_e. Then R acts on $h + \mathfrak{u}$ with a fixed point (pass to maximal compact subgroup to get a fixed point via averaging (this is how one proves cohomology vanishing in... some setting(???))). By the above Claim 6.1, we can conjugate R by an element of U to obtain a maximal reductive R' whose fixed point is h (why does this keep us inside of G_e (???)). But, by definition,

Sketch of Proof of Proposition Part 2). Check $G_0 e$ is open in $\mathfrak{g}_2$. We have $\mathrm{ad}_{\mathfrak{g}_0} e = \mathrm{ad}_e \mathfrak{g}_0 = \mathfrak{g}_2$.

Recall the map $\gamma : \mathbb{C}^* \to G$. Define a action of $\mathbb{C}^*$ on $\mathfrak{g}$ by $z \circ x = z^2 \mathrm{Ad}_{\gamma(z^{-1})} x$. Then this action fixes e. $\mathfrak{g}_i$ has weight $2 - i$ with respect to this action. Thus $z\mathfrak{g}_{\leq 1} \to 0$ as $z \to 0$. The slice $e + \mathfrak{g}_f$ is stable under this action and $z(e + \mathfrak{g}_f) \to e$ as $z \to 0$.

Lemma 6.1. *$e + \mathfrak{g}_f$ meets a G-orbit O iff $Ge \subseteq \overline{O}$. In this case, $e + \mathfrak{g}_f$ meets O transversely.*

Idea. Near e the first part is clear (no not really (???)). Then use the contracting action to pull things towards e. Now suppose $x \in (e + \mathfrak{g}_f) \cap O$. Consider $a_y : \mathfrak{g} \oplus (e + \mathfrak{g}_f) \to \mathfrak{g}$ by $(\xi, \eta) \mapsto [y, \xi] + \eta$. We have a_e is surjective, so a_y is surjective for all y close to e.

Identify $\mathfrak{g} \cong \mathfrak{g}^*$ using Killing form $\langle -, - \rangle$. Let ψ denote the functional corresponding to e.

Lemma 6.2. *The bilinear form $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathbb{C}$ given by $(x, y) \mapsto \psi([x, y])$ is nondegenerate.*

Proof. $\mathrm{ad}_e : \mathfrak{g}_{-1} \xrightarrow{\sim} \mathfrak{g}_1$ by symmetry. Suppose $x \in \mathfrak{g}_{-1}$ is such that $\psi([x, y]) = 0$ for all $y \in \mathfrak{g}_{-1}$. In particular, $0 = \langle e, [x, y] \rangle = \pm \langle [e, x], y \rangle$. Furthermore, the Killing form only pairs $\mathfrak{g}_i$ with $\mathfrak{g}_{-i}$. Thus, since the Killing form is nondegenerate, we get $[e, x] = 0$. By the above isomorphism, we get $x = 0$.

Pick a Lagrangian subspace $\ell \subset \mathfrak{g}_{-1}$. Then $\mathfrak{m} = \ell \oplus \mathfrak{g}_{\leq -2}$ is a Lie subalgebra of $\mathfrak{g}_{\leq -1}$.

Remark. Sometimes $\mathfrak{m}$ is denoted $\mathfrak{g}_{\leq -3/2}$.

The algebra $\mathfrak{m}$ is nilpotent; let M be the corresponding unipotent group.

Lemma 6.3. *1) $\psi_{[\mathfrak{m}, \mathfrak{m}]} = 0$ i.e. $\psi|_{\mathfrak{m}}$ is a character.*

2) $\dim M = \frac{1}{2} \dim G\psi$. (In fact $M\psi$ is a Lagrangian submanifold of $G\psi$, but we will not use this.)

Proof. 1) Since $e \in \mathfrak{g}_2$, $\psi|_{\mathfrak{g}_i} = 0$ unless $i = -2$. By construction, $[\mathfrak{m}, \mathfrak{m}] \cap \mathfrak{g}_{-2} = [\ell, \ell]$. The only way ℓ can be Lagrangian is if $\psi|_{[\ell, \ell]} = 0$ (???).

2) By construction, $\dim \mathfrak{m} = \dim(\mathfrak{g}_{<0}) - \frac{1}{2}\mathfrak{g}_{-1}$. We have $\dim(\mathfrak{g}_{<0}) = \frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{g}_0)$ by symmetry. By looking at the weight space picture, $\dim \mathfrak{g}_0 + \mathfrak{g}_{-1} = \mathfrak{g}_e$. Thus $\dim \mathfrak{m} = \frac{1}{2}(\dim G - \dim G_e) = \frac{1}{2} \dim G\psi$.

Let s be the subset of $\mathfrak{g}^*$ corresponding to $e + \mathfrak{g}_f$. We have $pr : \mathfrak{g}^* \to \mathfrak{m}^*$. ψ is an element of $\mathfrak{m}^*$ that is fixed by M. We have $pr^{-1}(\psi) = \psi + \mathfrak{m}^\perp$, which is M-stable and contains s. Next time, we will look at:

Proposition 6.2. *The action of M gives an isomorphism $M \times s \xrightarrow{\sim} \psi + \mathfrak{m}^\perp$. Thus s is the quotient we want in the context of Hamiltonian reduction.*

7 Feb 13

Lemma 7.1. *Let X, X' be smooth affine varieties with contracting $\mathbb{G}_m$ actions; let x, x' be the points "of contraction". Let $f : X \to X'$ be a $\mathbb{G}_m$-equivariant morphism, so that $f(x) = x'$. Then f is an isomorphism iff $d_x f$ is an isomorphism.*

Proof. Let $A = k[X]$, and let I be the maximal ideal of A corresponding to x. Then $T_x^* X = I/I^2$ and $k[T_x X] = Sym(I/I^2)$. Under the smoothness assumption, $Sym(I/I^2) \cong \bigoplus_{k \geq 0} I^k/I^{k+1}$. The $\mathbb{G}_m$ action on X induces a grading $A = \bigoplus_{m \geq 0} A_m$, where $A_0 = k$ and $A_{>0} = I$. Since $\mathbb{G}_m$ contracts X to x, all weights $\lambda_1, \ldots, \lambda_p$ of the $\mathbb{G}_m$-action on $T_x^* X$ are positive. Then, since A is finitely generated, the weight spaces in A are finite dimensional.

Now, for any $\mathbb{G}_m$-representation $V = \bigoplus_m V_m$, let $\chi(V) = \chi(V)(t) = \sum_m t^m \dim V_m$. Then $\chi(I^i) = \chi(I^{i+1}) + \chi(I^i/I^{i+1})$ and $\chi(A) = \chi(\mathrm{gr} A) = \chi(k[T_x X])$. (???)

Assume $d_x f$ is an isomorphism. The image of f contains some Zariski open neighborhood of x'. Then $f^* : k[X'] \to k[X]$ is an injective map of $\mathbb{G}_m$-representations. The equivariant isomorphism $T_x X \to T_{x'} X'$ gives $\chi(k[T_x X]) = \chi(k[T_{x'} X'])$. Then $\chi(k[X]) = \chi(k[X'])$, so combining with injectivity of f^* gives that f is an isomorphism.

Recall from last time that we had a semisimple Lie algebra $\mathfrak{g}$ with an $\mathfrak{sl}_2$-triple e, h, f. This induces a grading $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ where $e \in \mathfrak{g}_2$, $h \in \mathfrak{g}_0$, $f \in \mathfrak{g}_{-2}$. We used the Killing form $\langle -, - \rangle$ to identify $\mathfrak{g}$ with $\mathfrak{g}^*$. In particular, $\psi = \langle e, - \rangle$ and $s \subset \mathfrak{g}^*$ corresponds to the Slodowy slice $e + \mathfrak{g}_f$. We also saw that $\mathfrak{g}_{-1}$ is symplectic via the form $(x, y) \mapsto \psi([x, y])$. We chose a Lagrangian subspace ℓ and formed $\mathfrak{m} = \ell \oplus \mathfrak{g}_{\leq -2}$, which is a nilpotent Lie subalgebra of $\mathfrak{g}$. We let M be the corresponding unipotent subgroup of G.

Proposition 7.1. *The action of M gives an isomorphism $M \times s \xrightarrow{\sim} \psi + \mathfrak{m}^\perp$. Thus s is the quotient we want in the context of Hamiltonian reduction.*

Proof. Recall that $\mathfrak{g}_i \perp \mathfrak{g}_j$ with respect to the Killing form unless $i + j = 0$. For $k \geq 0$, we get $(\mathfrak{g}_{\leq -k})^\perp = \mathfrak{g}_{\leq k-1}$. (Then there is a nice picture explaining the situation in this proposition that I should add.)

Let us do some dimension computation. $\dim M = \dim \mathfrak{m} = \frac{1}{2} \dim Ge$ (???), $\dim s = \dim \mathfrak{g} - \dim Ge$, $\dim(\psi + \mathfrak{m}^\perp) = \dim \mathfrak{m}^\perp = \dim \mathfrak{g} - \dim m = \dim \mathfrak{g} - \frac{1}{2} \dim Ge$. Thus $\dim(M \times s) = \dim M + \dim s = \dim(\psi + \mathfrak{m}^\perp)$.

Under the Killing form identification, $\psi + \mathfrak{m}^\perp$ corresponds to $e + [e, \ell] + \mathfrak{g}_{\leq 0}$ (some of this is explained in the aforementioned picture). Let $f : M \times (e + \mathfrak{g}_f) \to e + [e, \ell] + \mathfrak{g}_{\leq 0}$ be the action map. Then $d_{(1_M, e)} f : \mathfrak{m} \oplus \mathfrak{g}_f \to [e, \ell] + \mathfrak{g}_{\leq 0}$ sends (y, x) to $[y, e] + x$. Note that $[e, \ell] \subset \mathfrak{g}_1$ so we actually have a direct sum on the right hand side. Thus if (y, x) is in the kernel, we have $[y, e] = 0$ and $x = 0$.

But $[y,e]=0$ implies $y \in \mathfrak{g}_{\geq 0}$, so $y=0$. By our dimension computation, we find that $d_{(1_M,e)}f$ is an isomorphism.

Last time we defined a $\mathbb{G}_m$-action on $\mathfrak{g}$ by $z \circ x = z^2 \mathrm{Ad}_{\gamma(z^{-1})} x$, and this contracts to e. We define a $\mathbb{G}_m$-action on M by $z \cdot m = \gamma(z^{-1}) m \gamma(z)$. Then f defined above is $\mathbb{G}_m$-equivariant, and Lemma 7.1 completes the proof.

We have $\mathfrak{g}_{\leq -2} \subset \mathfrak{m} = \mathfrak{g}_{\leq -2} \oplus \ell \subset \mathfrak{g}_{\leq -1} = \mathfrak{n} \subset \mathfrak{g}_{\leq 0} = \mathfrak{p}$, with the latter being parabolic and the others being nilpotent. We have the corresponding (unipotent) groups $G_{-2} \lhd M \lhd N$ (and $G_{-2} \lhd N$). We have $\mathfrak{g}_{\leq -1}/\mathfrak{g}_{\leq -2} = \mathfrak{g}_{-1}$ as Lie algebras, where the latter is given the trivial bracket. On groups, we have $N/G_{-2} \cong \mathfrak{g}_{-1}$, where the latter is taken as an additive group.

N acts on $e + \mathfrak{g}_1$ by $\exp(x) : e + y \mapsto [x,e] + y$ for $x \in \mathfrak{n}$. This factors to a $\mathfrak{g}_{-1}$ action by the same formula. This is no coincidence: since $\mathfrak{g}_{-1}$ is symplectic with form $\omega(x,y) = \langle e, [x,y] \rangle$, we get a Heisenberg Lie algebra $\mathcal{H} = kz + \mathfrak{g}_{-1}$ where z is central and $[x,y]_{\mathcal{H}} = \omega(x,y)z$ for $x, y \in \mathfrak{g}_{-1}$. The coadjoint action of the Heisenberg group is really just an action of $\mathfrak{g}_{-1}$, because z is central. The coadjoint orbits are planes $z^* \neq 0$ and points with $z^* = 0$. Note that $\mathfrak{g}_{-1} \cong \mathfrak{g}_{-1}^*$ by symplectic form and $\mathfrak{g}_{-1}^* \cong \mathfrak{g}_1$ by Killing form. The inclusions $\mathfrak{g}_{\leq -2} \subset \mathfrak{m} \subset \mathfrak{n}$ give, by taking orthogonal complements, inclusions $\mathfrak{g}_{\leq 1} \supset \mathfrak{m}^\perp \supset \mathfrak{g}_{\leq 0}$. By taking quotient by $\mathfrak{g}_{\leq 0}$ we get $\mathfrak{g}_1 \supset [e, \ell]$. Something something action and translating along ℓ (???).

Let G_0 be the Levi subgroup corresponding to $\mathfrak{g}_0$. We have $G_0 \curvearrowright \mathfrak{g}_2$ and $G_0 e$ is open dense in $\mathfrak{g}_2$. The stabilizer is $Z_G(\mathfrak{sl}_2)$, denote it by R, which is reductive. $\Omega = G_0 e \cong G_0/R$ is an affine variety. Then $\mathfrak{g}_2 - \Omega$ is a G_0-stable and $\mathbb{G}_m$-stable divisor. Then there is a unique (up to constant factor) homogeneous G_0-semiinvariant polynomial $p \in k[\mathfrak{g}_2]$ such that $\mathfrak{g}_2 - \Omega = p^{-1}(0)$.

Now enlarge the isomorphism in Proposition 7.1 to $N \times s \xrightarrow{\sim} (\mathfrak{g}_{\leq -2})^\perp = \mathfrak{g}_{\leq 1}$. (uhhhh there's some diagram that I need to add and don't understand) (???)

8 Feb 18

8.1 Principal Nilpotents

Let G be a connected semisimple complex algebraic group. Fix a triangular decomposition of its Lie algebra, $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}_-$. We have opposite Borel algebras $\mathfrak{b} = \mathfrak{n} + \mathfrak{t}$ and $\mathfrak{b}_- = \mathfrak{n}_- \oplus \mathfrak{t}$, as well as the corresponding Borel groups B, B_-. Their intersection is a maximal torus T with Lie algebra $\mathfrak{t}$. Let R_+ be the set of positive roots, i.e. the roots in $\mathfrak{n}$. Fix simple roots $\alpha_i \in R_+$ for $i = 1, \ldots, r = \dim T = \text{rank of } \mathfrak{g}$. Let $\mathfrak{g}_\alpha$ denote the root space for a root α; it is one-dimensional. Let $h \in \mathfrak{t}$ satisfy $\alpha_i(h) = 2$ for all simple roots α_i. Then h is regular and $Z_{\mathfrak{g}}(h) = \mathfrak{t}$. We get a grading $\mathfrak{g} = \bigoplus_{i \in 2\mathbb{Z}} \mathfrak{g}_i$, where $\mathfrak{g}_i$ is the subspace where ad_h acts by i. Pick non-zero elements $e_i \in \mathfrak{g}_{\alpha_i}$ for all simple roots α_i, and let $e = \sum_i e_i$. It is an element of $\mathfrak{n}$.

Definition 8.1. Let $\mathfrak{g}_r$ be the subspace consisting of x with $\dim Z_{\mathfrak{g}}(x) = r$. It is open in $\mathfrak{g}$ and contains the regular semisimple elements $\mathfrak{g}_{rs}$. It is also G-stable.

Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone.

Proposition 8.1. *$\mathcal{N} \cap \mathfrak{g}_r$ is the unique open dense G-orbit in $\mathcal{N}$.*

Proof. Since $\mathfrak{g}_r$ is G-stable, we have that $\mathcal{N} \cap \mathfrak{g}_r$ is a union of orbits. Recall the Springer resolution $T^*(G/B) = \widetilde{\mathcal{N}} \twoheadrightarrow \mathcal{N}$. We have $\dim \widetilde{\mathcal{N}} = 2\dim(G/B) = 2\dim \mathfrak{n} = \dim \mathfrak{g} - r$, so $\dim \mathcal{N} \le \dim \mathfrak{g} - r$. On the other hand, we can write $k[\mathfrak{g}]^G = k[p_1, \ldots, p_r]$ and $\mathcal{N} = \{p_i = 0\}$. This gives $\dim \mathcal{N} \ge \dim \mathfrak{g} - r$. Thus $\dim \mathcal{N} = \dim \mathfrak{g} - r$. Since $\mathcal{N}$ is the image of the irreducible variety $\widetilde{\mathcal{N}}$, we get that it is also irreducible. We use the fact that the number of G-orbits in $\mathcal{N}$ is finite. Thus there is a unique open dense G-orbit, say Gx. We have $\dim G - r = \dim \mathfrak{g} - r = \dim \mathcal{N} = \dim Gx = \dim G - \dim G_x$, so $r = \dim G_x = \dim Z_{\mathfrak{g}}(x)$, so x is regular. Conversely, any orbit of a regular nilpotent element must have dimension equal to $\dim \mathcal{N}$. Since $\mathcal{N}$ is irreducible, we are done.

Let $\alpha \in R_+$. Then we can write $\alpha = \sum_i c_i \alpha_i$ for $c_i \in \mathbb{Z}_{\ge 0}$. The height of α is $\mathrm{ht}(\alpha) = \sum_i c_i$. We have $\mathfrak{n} = \bigoplus_{i>0} \mathfrak{g}_{2i}$ and $\mathfrak{g}_{2i} = \bigoplus_{\mathrm{ht}(\alpha)=i} \mathfrak{g}_\alpha$. We have $[\mathfrak{n}, \mathfrak{n}] = \bigoplus_{i>1} \mathfrak{g}_{2i}$, so $\mathfrak{n} = \mathfrak{g}_2 \oplus [\mathfrak{n}, \mathfrak{n}]$. Each $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{2i}$ is T-stable. In particular, $\mathfrak{g}_2 = \bigoplus_{i=1}^r \mathfrak{g}_{\alpha_i}$. Let x_i be a coordinate on $\mathfrak{g}_{\alpha_i}$, and let $\varphi = \prod_i x_i$ (a polynomial on $\mathfrak{g}_2$). Let $\Omega = \mathfrak{g}_2 - \varphi^{-1}(0)$; it is a T-orbit. Furthermore, the element e defined earlier is in Ω. Thus $\Omega = Te$.

Lemma 8.1. $\mathfrak{n} \cap \mathfrak{g}_r = \Omega + [\mathfrak{n}, \mathfrak{n}]$.

Proof. By Proposition 8.1, $\mathcal{N} \cap \mathfrak{g}_r$ is an open dense G-orbit in $\mathcal{N}$. Thus $\mathfrak{n} \cap \mathfrak{g}_r$ is open dense in $\mathfrak{n}$ and B-stable. Since $\Omega + [\mathfrak{n}, \mathfrak{n}]$ is also open dense in $\mathfrak{n}$, there is some regular element $e' \in \Omega + [\mathfrak{n}, \mathfrak{n}]$. By acting by T, we can assume $e' = e + x$ for $x \in [\mathfrak{n}, \mathfrak{n}]$. We claim:

Claim 8.1. *$e + [\mathfrak{n}, \mathfrak{n}] = Ne$, where N is the group with Lie algebra $\mathfrak{n}$ and $B = TN$.*

Proof of Claim. We first show that $e+[\mathfrak{n},\mathfrak{n}]$ is N-stable. Since N is unipotent, we have $\exp : \mathfrak{n} \xrightarrow{\sim} N$. For $n \in \mathfrak{n}$ and $x \in [\mathfrak{n},\mathfrak{n}]$, we have $\exp(n)(e+x) = e + x + \mathrm{ad}_n(e+x) + \cdots \in e + [\mathfrak{n},\mathfrak{n}]$ as desired. Now, we use the general fact that unipotent orbits of affine varieties are closed; in particular, the N-orbits in $e+[\mathfrak{n},\mathfrak{n}]$ are closed. I got behind, have sketched the argument in comments Let $e+x \in e+[\mathfrak{n},\mathfrak{n}]$ be regular. Then $\dim Z_N(e+x) \leq \dim Z_G(e+x) = r$, so that $\dim N(e+x) = \dim N - \dim Z_N(e+x) \geq \dim \mathfrak{n} - r = \dim(e+[\mathfrak{n},\mathfrak{n}])$. Since $N(e+x)$ is closed in $e+[\mathfrak{n},\mathfrak{n}]$, we must have $N(e+x) = e+[\mathfrak{n},\mathfrak{n}]$. In particular, since this orbit contains e, it is also Ne.

Continued Proof of Lemma. not really sure how we conclude from here.

Thus, e is regular.

We can find $f_i \in \mathfrak{g}_{-\alpha_i}$ so that if $f = \sum_i f_i$, then e, h, f is a $\mathfrak{sl}_2$-triple (???). This is called the "principal" $\mathfrak{sl}_2$. The G-centralizer of it is the center of G. We can then continue the Slodowy slice discussion from last time. In particular, because e is regular, we call $e+\mathfrak{g}_f$ a Kostant slice. Since $\mathfrak{g}_{\leq -1} = \mathfrak{g}_{\leq -2}$ in this case, we have no need for the ℓ and $\mathfrak{m}$ from last time. In particular, $\ell = 0$ and $\mathfrak{m} = \mathfrak{n}_-$. We then have $\psi + \mathfrak{m}^\perp$ corresponds to $e + \mathfrak{b}_-$. Then from last time we have

- $N_- \times (e+\mathfrak{g}_f) \xrightarrow{\sim} e + \mathfrak{b}_-$.
- If the center of G is trivial, then $B_- \times (e+\mathfrak{g}_f) \xrightarrow{\sim} \Omega + \mathfrak{b}_-$.

Theorem 8.1. *The composite $e+\mathfrak{g}_f \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\!/G = \mathrm{Spec}(k[\mathfrak{g}]^G)$ is an isomorphism.*

Note that the statement of the theorem gives the freedom for us to assume that G has no center (the center acts trivially, so it doesn't change invariants. but why does G and $G/Z(G)$ have same Lie algebra?)

Claim 8.2. *The map $\zeta : \mathfrak{t} \xrightarrow{e+(-)} e+\mathfrak{t} \hookrightarrow \Omega + \mathfrak{b}_- \twoheadrightarrow (\Omega+\mathfrak{b}_-)/B_-$ factors through the Weyl group action on $\mathfrak{t}$.*

Proof of Claim. We must show that for all $a \in \mathfrak{t}$ and $w \in W$, we have $e + w(a) \in B_-(e+a)$. It suffices to show this when $w = s_\alpha$ is a reflection corresponding to a simple root α. There is a unique $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[e, f_\alpha] = \check{\alpha}$. By definition, $s_\alpha(a) = a - \alpha(a)\check{\alpha}$. Then the claim is that $e + s_\alpha(a) = \exp(-\alpha(a)\mathrm{ad}_{f_\alpha})(e+a) = \mathrm{Ad}_{\exp(-\alpha(a)f_\alpha)}(e+a)$, and that $\exp(-\alpha(a)f_\alpha) \in B_-$. To show this, one can realize the action of s_α as $\exp(\mathrm{ad}_{f_\alpha})\exp(-\mathrm{ad}_{e_\alpha})\exp(\mathrm{ad}_{f_\alpha})$, where e_α is the $\mathfrak{g}_\alpha$ component of e that was chosen earlier. One then works out the actions. I sketched some of his sketch in comments, but I don't understand enough of it to confidently put it into writing.

Proof of Theorem. Let $\overline{\zeta} : \mathfrak{t}/W \to (\Omega + \mathfrak{b}_-)/B_-$. Chevalley gives an isomorphism $\mathfrak{g}/\!/G \to \mathfrak{t}/W$. Take the composite $\mathfrak{g}/\!/G \to \mathfrak{t}/W \xrightarrow{\overline{\zeta}} (\Omega+\mathfrak{b}_-)/B_- \cong$

$e + \mathfrak{g}_f \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}//G$. We claim it is identity. It suffices to show that $p(a) = p(e+a)$ for $p \in k[\mathfrak{g}]^G$, "which is true" (???). Now, on coordinate rings, we have $k[\mathfrak{g}]^G \xrightarrow{\mathrm{res}^*} k[e + \mathfrak{g}_f] \xrightarrow{\varphi} k[\mathfrak{g}]^G$, where φ comes from Chevalley and $\overline{\zeta}$, and this composite is identity. Thus φ is surjective. Let Y be the closed subscheme of $k[e + \mathfrak{g}_f]$ given by $\ker \varphi$. We have $k[Y] \xrightarrow{\sim} k[\mathfrak{g}]^G$ so $\dim Y = r = \dim(e + \mathfrak{g}_f)$. Thus $Y = e + \mathfrak{g}_f$, so $\ker \varphi = 0$, so φ is an isomorphism. It follows that res^* is an isomorphism as well.

9 Feb 20

9.1 Applications of the Kostant Slice

Let $p \in k[\mathfrak{g}^*]^G$ for G adjoint and semisimple. Then $dp : x \mapsto d_x p$ is a G-equivariant map. Then G_x fixes $d_x p$.

Let e, h, f be a principal $\mathfrak{sl}_2$-triple. Recall the $\mathbb{G}_m$-action $\circ$ on $\mathfrak{g}$ given by $z \circ x = z^2 \gamma(z^{-1})x$ where $\gamma : \mathbb{G}_m \hookrightarrow SL_2 \to G$, with the embedding by $z \mapsto \operatorname{diag}(z, z^{-1})$, and the map $SL_2 \to G$ induced by the triple. Let h decompose $\mathfrak{g}$ into $\mathfrak{g}_i$ as before. Then $\mathfrak{g}_i$ has weight $2 - i$ with respect to $\circ$. We get an induced $\circ$ action on $k[\mathfrak{g}^*]$, and the homogeneous polynomials of degree i in $k[\mathfrak{g}^*]^G$ have weight $2i$ with respect to this action.

Now let $k[\mathfrak{g}^*]^G = k[p_1, \dots, p_r]$ with p_i homogeneous of degree d_i. The map dp_i has $\circ$-weight $2(d_i - 1)$.

Proposition 9.1. *The differentials $d_x p_1, \dots d_x p_r$ form a k-basis of $\mathfrak{g}_x$ iff x is regular.*

Proof. By last time, we proved $\varpi : s \hookrightarrow \mathfrak{g}^* \twoheadrightarrow \mathfrak{g}^*//G$, where s is the Killing-identification of $e + \mathfrak{g}_f$. The p_i form a coordinate system on $\mathfrak{g}^*//G$, so their pullbacks $\varpi^* p_i$ form a coordinate system on s. Similarly, the differentials of the p_i form a basis for the cotangent space of $\mathfrak{g}^*//G$, so the $(\varpi^* dp_i)_x$ form a basis of $T_x^* s = \mathfrak{g}_f$. Since $d_x p_i \in \mathfrak{g}_x$ and $\dim \mathfrak{g}_x = r$ iff x is regular, the result follows from the fact that the slice consists of regular elements.

Remark. Let $\mathfrak{c} = \mathfrak{g}^*//G$ and let $c \in \mathfrak{c}$. The above argument shows that, canonically, $T_c^* \mathfrak{c} \cong \mathfrak{g}_x$ for any $x \in \mathfrak{g}^*_{reg}$ such that $\pi(x) = c$ for $\pi : \mathfrak{g}^* \twoheadrightarrow \mathfrak{g}^*//G$.

Proposition 9.2. *Let $\mathfrak{g} = \bigoplus \rho_i$ be the decomposition into irreps for the adjoint representation of the principal $\mathfrak{sl}_2$ triple. Then*

(1) There are r irreps appearing.

(2) $\dim \rho_i = 2d_i - 1 = 2\deg(p_i) - 1$.

Example 9.1. If $\mathfrak{g} = \mathfrak{sl}_2$, then $k[\mathfrak{sl}_2^*]^{SL_2} = k[\Delta]$ for Δ a quadratic. There is one irrep, namely the whole thing, which has dimension $3 = 2 * 2 - 1$.

Proof. (1) ad_h has even eigenvalues in $\mathfrak{g}$, so for each ρ_i, the zero weight space wrt ad_h has dimension 1 (???). Thus (???)the number of ρ_i is $\dim \mathfrak{g}_h = r$.

(2) $d_e p_i$ is a highest weight vector (???)of some irrep ρ_i. It has weight $2(d_i - 1)$, so the corresponding dimension is $2d_i - 1$ by the classification of $\mathfrak{sl}_2$ irreps.

Let $\mathcal{N}$ be the zero scheme of the invariant polynomials with no constant term.

Theorem 9.1. *$\mathcal{N}$ is a normal reduced complete intersection.*

Proof. Last time we showed $\dim \mathcal{N} = \dim \mathfrak{g} - r$, and this shows that $\mathcal{N}$ is a complete intersection. The $d_x p_i$ are linearly independent on $\mathcal{N} \cap \mathfrak{g}_{reg} = Ge$, which is open and dense in $\mathcal{N}$. This implies Ge is in the smooth locus of $\mathcal{N}$. Thus $\mathcal{N}$ is generically reduced. Combining with complete intersection gives reduced. Finally, $\mathcal{N}$ is a finite union of G-orbits, each of which is even-dimensional. Since Ge is in the smooth locus, we see that $\mathcal{N}$ is regular in codimension 2, so $\mathcal{N}$ is normal.

Let $\pi : T^*(G/B) : \widetilde{\mathcal{N}} \to \mathcal{N}$ be the Springer resolution.

Theorem 9.2. *$\pi^* : k[\mathcal{N}] \to \Gamma(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}})$ is an isomorphism.*

Proof. We have $\Gamma(\widetilde{\mathcal{N}}, \mathcal{O}_{\widetilde{\mathcal{N}}}) = \Gamma(\mathcal{N}, \pi_*\mathcal{O}_{\widetilde{\mathcal{N}}})$, so it suffices to show $k[\mathcal{N}] \to \Gamma(\mathcal{N}, \pi_*\mathcal{O}_{\widetilde{\mathcal{N}}})$ is an isomorphism. Since π is proper, $\pi_*\mathcal{O}_{\widetilde{\mathcal{N}}}$ is a coherent $\mathcal{O}_{\mathcal{N}}$-module. Thus $\Gamma(\mathcal{N}, \pi_*\mathcal{O}_{\widetilde{\mathcal{N}}})$ is a finitely generated $k[\mathcal{N}]$-module. We have an embedding $k[\mathcal{N}] \hookrightarrow \Gamma(\mathcal{N}, \pi_*\mathcal{O}_{\widetilde{\mathcal{N}}})$. Since π is birational, the fraction fields of the two sides are equal. Since $\mathcal{N}$ is normal, $k[\mathcal{N}]$ is integrally closed, so it is all of $\Gamma(\mathcal{N}, \pi_*\mathcal{O}_{\widetilde{\mathcal{N}}})$.

Let $e \in \mathcal{N}$ be an arbitrary nilpotent and complete it to an $\mathfrak{sl}_2$ triple. In general, $\overline{Ge}$ is not necessarily normal. Let X be its normalization.

Theorem 9.3. *X has symplectic singularities, i.e. there is a resolution of singularities $\pi : \widetilde{X} \to X$ such that $\pi^*\omega$ extends to a regular 2-form on $\widetilde{X}$, where ω is the symplectic form on Ge. It follows that X has rational singularities, i.e. $R^{>0}\pi_*\mathcal{O}_{\widetilde{X}} = 0$. These results also hold for any resolution of singularities of X.*

Proof. Ge is stable under scaling action. This implies the following: let ξ be the Euler vector field $\sum_i x_i \partial_i$. Let $\lambda = \iota_\xi \omega$. Then $\omega = d\lambda$ (at least up to a constant). The proof is by Cartan's "magic" formula. In fact, $\lambda_x = \langle x, - \rangle$ (Killing form).

Now, decompose $\mathfrak{g}$ into weight spaces for h. $\mathfrak{g}_{\geq 0}$ is the Lie algebra of a parabolic subgroup P. The Lie algebra of a Levi of P is $\mathfrak{g}_0$. $G_0 e$ is open dense in $\mathfrak{g}_2$. Similarly Pe is open dense in $\mathfrak{g}_{\geq 2}$.

Define $\widetilde{X} = G \times_P \mathfrak{g}_{\geq 2}$ it has an obvious map π' to $\mathfrak{g}$. Then π' is proper with image $G(\mathfrak{g}_{\geq 2}) = G(\overline{Pe}) = \overline{Ge}$. Since $\widetilde{X}$ is smooth, π' lifts to a map $\pi : \widetilde{X} \to X$. One can check explicitly that the pullback of ω extends.

9.2 Resolution of Slodowy Slices

Let $\mathcal{S} = s \cap \mathcal{N}$. Let $\widetilde{\mathcal{S}} = \pi^{-1}(\mathcal{S})$ for π the Springer resolution. Recall that $\psi \in \mathcal{S}$ is the element corresponding to e.

Proposition 9.3. *1) $\widetilde{\mathcal{S}}$ is a smooth symplectic subvariety of $\widetilde{\mathcal{N}}$.*

2) $\pi : \widetilde{\mathcal{S}} \to \mathcal{S}$ is a symplectic resolution.

3) $\pi^{-1}(\psi)$ is a Lagrangian subvariety.

10 Feb 25

Setting: $\mathfrak{sl}_2$-triple $e, h, f \in \mathfrak{g} = \bigoplus_i \mathfrak{g}_i$, $\mathfrak{m} = \ell + \mathfrak{g}_{\leq -2}$, $\ell \subseteq \mathfrak{g}_{-1}$ Lagrangian, $\psi = \langle e, - \rangle$. M is the unipotent subgroup corresponding to $\mathfrak{m}$. Identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ identifies $e + \mathfrak{g}_f$ with s.

Lemma 10.1. *Let Y be a smooth G-variety, where G is adjoint with Lie algebra $\mathfrak{g}$, and $\pi : Y \to \mathfrak{g}^*$ a G-equivariant map. Then any point of $\pi(Y) \cap (\psi + \mathfrak{m}^\perp) \subseteq \mathfrak{g}^*$ is a regular value of π, i.e. for all $y \in \pi^{-1}(\psi + \mathfrak{m}^\perp)$, the map $d_y\pi : T_yY \to \mathfrak{g}^*$ is surjective.*

Proof. We know that s is a slice to $G\psi$ and in fact transversal to all orbits it hits, i.e. for all $\xi \in s$, we have $T_\xi s + T_\xi(G\xi) = \mathfrak{g}^*$. We have (cite) $M \times s \xrightarrow{\sim} \psi + \mathfrak{m}^\perp$. Thus for all $\xi \in Y + \mathfrak{m}^\perp$, we have $T_\xi(\psi + \mathfrak{m}^\perp) + T_\xi(G\xi) = \mathfrak{m}^\perp + T_\xi(G\xi) = \mathfrak{g}^*$. A bit lost here, sketch in comments

Corollary 10.1. *Let Y be a symplectic manifold with a Hamiltonian G-action with moment map $\pi : Y \to \mathfrak{g}^*$. Then $\pi^{-1}(\psi + \mathfrak{m}^\perp)$ is a smooth coisotropic submanifold of Y with nil-foliation given by M-orbits. Moreover, $M \times \pi^{-1}(s) \to \pi^{-1}(\psi + \mathfrak{m}^\perp)$ is an isomorphism of algebraic varieties, so $\pi^{-1}(\psi + \mathfrak{m}^\perp)/M \cong \pi^{-1}(s)$ is a smooth symplectic manifold.*

Proof. $Y \xrightarrow{\pi} \mathfrak{g}^* \xrightarrow{\mathrm{pr}} \mathfrak{m}^*$ is the moment map for the M-action on Y. $\psi + \mathfrak{m}^\perp = \mathrm{pr}^{-1}(\psi)$. By Lemma 10.1, ψ is a regular value of $\mathrm{pr} \circ \pi$, so $(\mathrm{pr} \circ \pi)^{-1}(\psi) = \pi^{-1}(\psi + \mathfrak{m}^\perp)$ is a smooth coisotropic submanifold of Y with nil-foliation given by M-orbits. Then the following commutative diagram (add it!) shows the last part (???). Note that the action of M on the top right component is free, as a result of regular value condition (and M is unipotent).

Example 10.1. Let $Y = \widetilde{\mathcal{N}} = T^*(G/B) \xrightarrow{\pi} \mathcal{N} \subseteq \mathfrak{g}^* \supseteq s$. We have $\pi(Y) \cap s = \mathcal{S} = \mathcal{N} \cap s$. By Corollary 10.1, $\widetilde{\mathcal{S}} = \pi^{-1}(\mathcal{S})$ is a smooth symplectic submanfiold of $\widetilde{\mathcal{N}}$. The projection $\pi_{\mathcal{S}} : \widetilde{\mathcal{S}} \to \mathcal{S}$ is a symplectic resolution of singularities. $\pi^{-1}(\psi)$ is a Springer fiber. Now recall the $\circ$-action of $\mathbb{G}_m$ (maybe I should add a label to this since we reference it so much). The same formula defines an action on $\widetilde{\mathcal{N}}$. Then π respects this action, so since the action contracts $\mathcal{S}$ to ψ, it "contracts" $\widetilde{\mathcal{S}}$ to $\pi^{-1}(\psi)$ (We call this the central fiber).

Claim 10.1. *(1) $\pi^{-1}(\psi)$ is a typically singular reducible Lagrangian subvariety of $\widetilde{\mathcal{S}}$.*

(2) $H^*(\widetilde{\mathcal{S}}) \cong H^*(\pi^{-1}(\psi))$.

Half-proof for (1). Let $\mathcal{B} = G/B$. Let G act on $\mathcal{B} \times \mathcal{B}$ diagonally. We get $\pi_\Delta : T^*(\mathcal{B} \times \mathcal{B}) \to \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$, where the first map is the product of maps $\pi : T^*\mathcal{B} \to \mathfrak{g}^*$, and the second map takes the difference. Note then that $\pi_\Delta^{-1}(0)$ is a diagonal. We have $\pi^{-1}(0)$ is the union of $T^*_{\mathcal{O}}(\mathcal{B} \times \mathcal{B})$ as $\mathcal{O}$ ranges over the G-orbits of $\mathcal{B} \times \mathcal{B}$. The dimension of every irreducible component of $\pi_\Delta^{-1}(0)$ is $\frac{1}{2} \dim T^*(\mathcal{B} \times \mathcal{B}) = \dim T^*\mathcal{B}$. Now, thinking of G/B as the space of Borel subalgebras, we have explicitly $\widetilde{\mathcal{N}} = \{(\xi, \mathfrak{b}) \mid \xi \in \mathfrak{b}\}$ so $\pi_\Delta^{-1}(0) = \{(\xi, \mathfrak{b}_1, \mathfrak{b}_2) \mid \xi \in \mathfrak{b}_1 \cap \mathfrak{b}_2\}$ (In each of these, $\xi \in \mathcal{N}$). This is called the **Steinberg variety**. Now decompose $\pi_\Delta^{-1}(0)$ into $Z_{\mathcal{O}} = \{(\xi, \mathfrak{b}_1, \mathfrak{b}_2) \mid \xi \in \mathfrak{b}_1 \cap \mathfrak{b}_2 \cap \mathcal{O}\}$ according to the G-orbits of $\mathcal{N}$. This is a G-stable decomposition. We have obvious projection maps $Z_{\mathcal{O}} \to \mathcal{O}$. The fiber of this map over some ξ is clearly $\pi^{-1}(\xi) \times \pi^{-1}(\xi)$. Pairs of irreducible components of $\pi^{-1}(\xi)$ lead to at most an irreducible component of $Z_{\mathcal{O}}$. By taking the same component twice, we find that the dimension of each irreducible component of $\pi^{-1}(\xi)$ is $\leq \frac{1}{2}(\dim \mathcal{N} - \dim \mathcal{O})$. This shows isotropic, but coisotropic is a fairly different argument that we do not do.

Example 10.2. Now by (2) of the above claim, we have a map $H_\bullet(\pi^{-1}(\psi)) \to H_\bullet^{BM}(\widetilde{\mathcal{S}})$, where the right hand side is Borel-Moore homology. Let e be a subregular nilpotent, meaning Ge is the unique open orbit in $\mathcal{N} - \mathcal{N}_{reg}$. Then $\dim \mathcal{N} - \dim Ge = 2$. $\mathcal{S}$ is a surface with an isolated singularity (at ψ). In fact it is a Kleinian singularity. The fiber $\pi^{-1}(\psi)$ is a chain of $\mathbb{P}^1$s, and the graph of how they meet is the Dynkin graph associated to $\mathfrak{g}$. Furthermore, if we enter topology land and move these $\mathbb{P}^1$s around as cycles so that they are transverse, the intersection matrix is the Cartan matrix. There is some intersection form coming from $H_\bullet(\widetilde{\mathcal{S}}) \to H_\bullet^{BM}(\widetilde{\mathcal{S}})$.

Example 10.3. Stuff I had to draw, add later

11 Feb 27

Setting (same as last time): $\mathfrak{sl}_2$-triple $e, h, f \in \mathfrak{g} = \bigoplus_i \mathfrak{g}_i$, $\mathfrak{m} = \ell + \mathfrak{g}_{\leq -2}$, $\ell \subseteq \mathfrak{g}_{-1}$ Lagrangian, $\psi = \langle e, - \rangle$. M is the unipotent subgroup corresponding to $\mathfrak{m}$. Identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ identifies $e + \mathfrak{g}_f$ with s.

Let $G \curvearrowright G$ on the *right*, i.e. $g : g' \mapsto g'g^{-1}$. We use *left* translations to trivialize T^*G by left-invariant sections, giving $T^*G \cong G \times \mathfrak{g}^*$. The right-action on T^*G corresponds to the diagonal G-action on $G \times \mathfrak{g}^*$. The moment map $\mu_r : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ is just second projection, and this is equivariant.

Now we just use the right M-action. We have $T^*G \xrightarrow{\mu_r} \mathfrak{g}^* \xrightarrow{\mathrm{pr}} \mathfrak{m}^*$. The **Hamiltonian reduction** of T^*G with respect to (M, ψ) is $(\mathrm{pr} \circ \mu_r)^{-1}(\psi)/M = \mu_r^{-1}(\psi + \mathfrak{m}^\perp)/M = (G \times (\psi + \mathfrak{m}^\perp))/M = G \times^M (\psi + \mathfrak{m}^\perp)$. This space comes with an obvious map p to G/M by taking the M-coset of the first component. It also has a left G-action. This is called the **twisted cotangent bundle** $T^*_\psi(G/M)$. If we ran the similar procedure with 0 instead of ψ, we would get $G \times^M \mathfrak{m}^\perp = T^*(G/M)$. The twisted cotangent bundle is a smooth symplectic algebraic variety, and the left G-action is Hamiltonian. We denote the moment map by π.

Now, identify $\psi + \mathfrak{m}^\perp$ with the fiber of p over the trivial coset.

Claim 11.1.

- *Under this identification, the slice s is a symplectic submanifold of $T^*_\psi(G/M)$.*
- *Using Proposition 7.1, we have $T^*_\psi(G/M) \cong G \times^M (M \times s) = G \times s$. Thus it is an affine variety with free G-action.*
- *Similarly, $k[T^*_\psi(G/M)]^G = k[G \times s]^G = k[s] = k[\psi + \mathfrak{m}^\perp]^M$.*

According to our decomposition of $\mathfrak{g}$, we have $\mathfrak{g}_0 = \mathfrak{g}_h$ is a Levi subalgebra. Let $G_0 = Z_G(h)$ be the corresponding Levi subgroup. It acts on $\mathfrak{g}_2$ and $\Omega = G_0 e$ is open dense in $\mathfrak{g}_2$. Recall that $\psi + \mathfrak{m}^\perp$ corresponds to $e + [e, \ell] + \mathfrak{g}_{\leq 0}$. Embed that into $\Omega + \mathfrak{g}_{\leq 1}$, which corresponds to $G_0 \psi + \mathfrak{g}^\perp_{\leq -2}$. Let $P = G_{\leq 0}$ be parabolic, and let $R = Z_G(\mathfrak{sl}_2)$; this is reductive and sits inside G_0. We have $\Omega \cong G_0/R$. Let $\mathcal{X} = G \times^P (G_0 \psi + \mathfrak{g}^\perp_{\leq -2})$. This has left G-action, maps to G/P and $\mathfrak{g}^*$, and it also has an R-action on the right that commutes with the left G-action (Lecturer: "I won't explain the R-action"). We have an analogue of Proposition 7.1, namely $P \times^R s \xrightarrow{\sim} G_0 \psi + \mathfrak{g}^\perp_{\leq -2}$ (perhaps this is where the R-action comes from). Using this, we have $\mathcal{X} = G \times^R s$. Note that in the case of a principal nilpotent (and G adjoint), R is trivial and this is the same as $T^*_\psi(G/M)$. All of this is related to "hyperspherical varieties" and "S duality", but we will not use it.

Now suppose e is a principal nilpotent. Recall that in this case we have a triangular decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}_-$ and no odd degree weight spaces.

We have $\mathfrak{m} = \mathfrak{n}_-$ and $M = N_-$. $\psi + \mathfrak{m}^\perp$ is identified with $e + \mathfrak{b}_-$. Now $T^*_\psi(G/N_-) \cong G \times (e + \mathfrak{g}_f) \cong G \times \mathfrak{c}$, where $\mathfrak{c} = \mathfrak{g}^*//G \cong \mathfrak{g}//G$. The moment map π is identified with the action map $G \times (e + \mathfrak{g}_f) \to \mathfrak{g}$; its image is the regular locus.

We have $G_0 = T$ is a torus. Assume G is adjoint so $R = 1$. Recall e is a sum of root vectors $e_1, \ldots, e_r$. We have $\Omega = Te = \{\sum c_i e_i \mid \prod c_i \neq 0\}$. Recall that $\mathfrak{g}_2$ is identified with $(\mathfrak{n}_-/[\mathfrak{n}_-, \mathfrak{n}_-])^*$.

Definition 11.1. A character of $\mathfrak{n}_-$ is called **nondegenerate** if it corresponds to an element of Ω.

insert diagram In this diagram we write $\varpi : \mathfrak{g}^*_{reg} \to \mathfrak{c}$ and claim $\varpi \circ \pi$ is a G-torsor.

$Z_{reg} = \{(\xi, g) \in \mathfrak{g}^*_{reg} \times G \mid g\xi = g\}$ has first projection to $\mathfrak{g}^*_{reg}$. The fiber over any ξ is the centralizer G_ξ. Z_{reg} is a smooth group scheme over $\mathfrak{g}^*_{reg}$.

Claim 11.2. *The moment map $\pi : T^*_\psi(G/N_-) \to \mathfrak{g}^*_{reg}$ is a Z_{reg}-torsor.*

Proof. Fix a G-orbit $O = G\xi \subset \mathfrak{g}^*_{reg}$. Let $c = \varpi(\xi)$. Then $\pi^{-1}(O) = (\varpi \circ \pi)^{-1}(c) \cong G \xrightarrow{\pi} O = G/G_\xi$. Thus $\pi^{-1}(\xi) = G_\xi$ so G_ξ acts freely transitively on the fiber $\pi^{-1}(\xi)$. (???)(???)(???)

Remark. Let $T^*Bun_G \to \mathbb{A}$ be the Hitchin fibration, general fibers are abelian varieties. It has a Hitchin section $\mathbb{A} \to T^*Bun_G$, and composing with the projection $T^*Bun_G \to Bun_G$ gives a constant map. This somehow relates to freeness of action on $T^*_\psi(G/N_-)$. It also corresponds to the fact that the S-dual of a point is $T^*_\psi(G/N_-)$.

Claim 11.3. *There is a smooth group scheme $\mathcal{J} \to \mathfrak{c}$ called the **universal centralizer** such that $\varpi^*\mathcal{J} \cong Z_{reg}$.*

For $\xi \in \mathfrak{t}_{reg}$, we have $G_\xi = T$, and $\pi^{-1}(\xi) \hookrightarrow \mathcal{X} \twoheadrightarrow G/B_-$ is injective. The first embedding is via the isomorphism $T^*_\psi(G/N_-) \cong \mathcal{X}$. The closure of the image of the fiber is a smooth toric variety whose fan is given by decomposition of $\mathfrak{t}^*$ into Weyl chambers. In particular, the toric variety is W-stable. If we run the same game for any $\xi \in \mathfrak{g}_{reg}$, we get fibers of the compactification of Z_{reg} (or something like that (???)).

12 Mar 4

12.1 Quantization of Slodowy Slices

Same setup as before. With $\mathrm{pr} : \mathfrak{g}^* \to \mathfrak{m}^*$, we have the algebraic isomorphism $k[s] \cong k[\psi + \mathfrak{m}^\perp]^M$ is equivalent to the geometric isomorphism $s \cong \mathrm{pr}^{-1}(\psi)/M$. We want to quantize $k[s]$ using a similar geometric isomorphism.

We have $k[\mathfrak{g}^*] = \mathrm{Sym}\mathfrak{g}$ and $k[\psi + \mathfrak{m}^\perp] = \mathrm{Sym}\mathfrak{g}/(m - \psi(m))_{m \in \mathfrak{m}}$. We want to replace $\mathrm{Sym}\mathfrak{g}$ with $\mathcal{U}\mathfrak{g}$ and get something similar for $\mathfrak{m}$. Let $\varepsilon : \mathfrak{m} \to \mathcal{U}\mathfrak{g}$ be $m \mapsto m - \psi(m)$, and denote the image by $\mathfrak{m}_\psi$. It is a Lie subalgebra of $\mathcal{U}\mathfrak{g}$, isomorphic to $\mathfrak{m}$, but it sits in $\mathcal{U}\mathfrak{g}$ differently. Let $E_\psi = \mathcal{U}\mathfrak{g}/(\mathcal{U}\mathfrak{g} \cdot \mathfrak{m}_\psi)$. Let $N_\psi = N(\mathcal{U}\mathfrak{g} \cdot \mathfrak{m}_\psi) = \{u \in \mathcal{U}\mathfrak{g} \mid \mathfrak{m}_\psi \cdot u \subseteq \mathcal{U}\mathfrak{g} \cdot \mathfrak{m}_\psi\} = \{u \in \mathcal{U}\mathfrak{g} \mid [u, \mathfrak{m}_\psi] \subseteq \mathcal{U}\mathfrak{g} \cdot \mathfrak{m}_\psi\}$. Clearly N_ψ contains $\mathcal{U}\mathfrak{g} \cdot \mathfrak{m}_\psi$; let A_ψ be the quotient. It is isomorphic to $(\mathrm{End}_{\mathcal{U}\mathfrak{g}} E_\psi)^{op}$. Also, using the last expression for N_ψ, it is isomorphic to $E_\psi^{\mathfrak{m}_\psi}$, where the superscript refers to the left action. It is also isomorphic to $E_\psi^{\mathrm{ad}\mathfrak{m}}$, with no ψ in the superscript. It is also isomorphic E_ψ^M. This all shows that A_ψ quantizes $k[\psi + \mathfrak{m}^\perp]^M$.

Remark. A_ψ is an example of a finite $\mathcal{W}$-algebra.

Our goal now is to construct a filtration on A_ψ such that $\mathrm{gr} A_\psi \cong k[s]$.

Decompose $\mathrm{Sym}\mathfrak{g}$ into weight spaces $(\mathrm{Sym}\mathfrak{g})_{(i)}$ with respect to the ad_h action. The adjoint action respects symmetric powers, so $\mathrm{Sym}^j\mathfrak{g} \cap (\mathrm{Sym}\mathfrak{g})_{(i)} = (\mathrm{Sym}^j\mathfrak{g})_{(i)}$. Define the **Kazhdan grading** on $\mathrm{Sym}\mathfrak{g}$ by saying that the "K-degree" n piece is $\bigoplus_{i+2j=n} (\mathrm{Sym}^j\mathfrak{g})_{(i)}$. The K-degree of $\mathfrak{g}_i$ is then $2 + i$. Then $\mathfrak{g}/\mathfrak{m}$ has no K-graded components of K-degree ≤ 0.

Remark. The scaling by 2 here and in many places (i.e. derived geometric Satake) is because the nilpotent element is in $\mathfrak{g}_2$.

Definition 12.1 (Kazhdan Filtration)**.** Let $M = \bigoplus_{i \in \mathbb{Z}} M_{(i)}$ be a $\mathbb{Z}$-graded vector space with non-negative increasing filtration $F_j M$. Assume that each $F_j M$ is a graded subspace. Then the **Kazhdan filtration** on M is $K_n M = \bigoplus_{i+j \leq n} (F_j M)_{(i)}$. Note that the associated gradeds $\mathrm{gr}^F M$ and $\mathrm{gr}^K M$ are isomorphic as *non-graded* objects.

We apply this in the case of $\mathcal{U}\mathfrak{g}$. We want the filtration F_j to only exist for even j, in which case F_{2j} is the jth part of the PBW filtration. Then $\mathfrak{g}_i \subseteq K_{i+2}(\mathcal{U}\mathfrak{g})$. Note that if $x \in \mathfrak{g}_m$ and $y \in \mathfrak{g}_n$, then $[x, y] \in K_{m+n+2} = K_{(m+2)+(n+2)-2}$. Extending this, we see that if $u \in K_m(\mathcal{U}\mathfrak{g})$ and $v \in K_n(\mathcal{U}\mathfrak{g})$, then $uv - vu \in K_{m+n-2}(\mathcal{U}\mathfrak{g})$. This shows the associated graded is commutative. Also, the map $\mathfrak{g} \to \mathrm{gr}^K(\mathcal{U}\mathfrak{g})$ extends to a graded algebra map $\mathrm{Sym}\mathfrak{g} \to \mathrm{gr}^K(\mathcal{U}\mathfrak{g})$, where $\mathrm{Sym}\mathfrak{g}$ is given the Kazhdan grading.

The key thing that makes everything work is that $\mathfrak{m}_\psi$ is homogeneous with respect to the Kazhdan grading: since ψ is 0 outside of $\mathfrak{g}_{-2}$, the elements

$m - \psi(m)$ for m homogeneous are either m, if m is not in $\mathfrak{g}_{-2}$, or $m - \psi(m)$, if m is in $\mathfrak{g}_{-2}$. In the latter case, both m and $\psi(m)$ have K-degree 0. We get $\mathrm{gr}^K E_\psi = \mathrm{Sym}\mathfrak{g}/(\mathrm{Sym}\mathfrak{g} \cdot \mathfrak{m}_\psi)$, where again $\mathrm{Sym}\mathfrak{g}$ has the Kazhdan grading, and this is the same as $k[\psi + \mathfrak{m}^\perp]$. Note $K_{\leq 0} E_\psi = 0$.

Before stating the following theorem, note that the image of the obvious map $Z(\mathcal{U}\mathfrak{g}) \to E_\psi$ is contained in the center of A_ψ.

Theorem 12.1. *1) There is a graded algebra isomorphism $\mathrm{gr}^K A_\psi \cong k[s]$.*

2) The map $Z(\mathcal{U}\mathfrak{g}) \to Z(A_\psi)$ is an isomorphism.

Lemma 12.1. *Let M be any unipotent group. Let M act on $k[M]$ by left translation, so that it becomes a $\mathfrak{m}$-module by differentiation. Then $H^n(\mathfrak{m}, k[M])$ is zero in positive degrees, and $k[M]^{\mathfrak{m}} = k[M]^M = k$ in degree zero.*

Proof. Recall the Chevalley-Eilenberg complex $C^\bullet(\mathfrak{m}, k[M]) = k[M] \otimes \bigwedge^\bullet \mathfrak{m}^*$. For any group, this can actually be identified with $\Omega^\bullet(M)$, where the differential of the CE complex is identified with the de Rham differential. Crucially, since M is unipotent, it is an affine space (via the exponential), hence the de Rham cohomology is trivial as desired.

Proof of Part 1) of Theorem 12.1. The isomorphism $M \times s \xrightarrow{\sim} \psi + \mathfrak{m}^\perp$ gives an isomorphism of M-modules, $k[\psi + \mathfrak{m}^\perp] \cong k[M] \otimes k[s]$. On the right, M acts only on $k[M]$. Then, by Lemma 12.1, $H^n(\mathfrak{m}, k[\psi + \mathfrak{m}^\perp])$ is nonzero in positive degrees, and $k[s]$ in degree 0.

Now define the Kazhdan filtration on $C^\bullet(\mathfrak{m}, E_\psi)$ by $K_p C^n = \bigoplus_{q+i_1+\cdots+i_n \leq p} K_q E_\psi \otimes \mathfrak{m}^*_{i_1} \wedge \cdots \wedge \mathfrak{m}^*_{i_n}$. Then $\mathrm{gr}^K C^\bullet(\mathfrak{m}, E_\psi) = C^\bullet(\mathfrak{m}, k[\psi + \mathfrak{m}^\perp])$. There is a spectral sequence giving $H^\bullet(\mathfrak{m}, k[\psi + \mathfrak{m}^\perp])$ converging to $\mathrm{gr} H^\bullet(\mathfrak{m}, E_\psi)$. We have seen that the first term exists only in degree zero, so we actually get an isomorphism, and then the degree 0 filtrations agree by definition. (???)

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We continue from last time.

Remark. The Kazhdan filtration $K_\bullet$ and PBW filtration $F_\bullet$ are the same on $Z(\mathcal{U}\mathfrak{g})$.

Corollary 13.1. *If e is a principal nilpotent, then $Z(\mathcal{U}\mathfrak{g}) \to A_\psi$ is an isomorphism.*

Proof. The map is compatible with K-filtrations, so it suffices to show we have an isomorphism on gr. By the above remark, the map becomes $\mathrm{gr}^F Z(\mathcal{U}\mathfrak{g}) \to \mathrm{gr}^K A_\psi$, i.e. $(\mathrm{Sym}\mathfrak{g})^G = k[\mathfrak{g}^*//G] \to k[s]$ by part 1) of Theorem 12.1. But we have seen that $s \hookrightarrow \mathfrak{g}^* \twoheadrightarrow \mathfrak{g}^*//G$ is an isomorphism, so we are done.

Theorem 13.1. *$k[\mathfrak{g}^*]$ is free as a $k[\mathfrak{g}^*]^G$ module. Furthermore, $\varpi : \mathfrak{g}^* \to \mathfrak{g}^*//G$ is flat with equidimensional fibers, and ϖ restricted to regular elements is smooth.*

Corollary 13.2. *All fibers of $\varpi_s : s \hookrightarrow \mathfrak{g}^* \to \mathfrak{g}^*//G$ have the same dimension, hence ϖ_s is flat.*

Corollary 13.3. *A_ψ is a flat $Z(\mathcal{U}\mathfrak{g})$-module.*

Proof. Reduce to gr^K.

Recall the slice s has a Poisson structure with symplectic leaves given by the intersection with G-orbits.

Let $\chi : Z(\mathcal{U}\mathfrak{g}) \to \mathbb{C}$ be a character. Let $\mathcal{U}_\chi = \mathcal{U}\mathfrak{g}/(\ker \chi)$. We can run the same constructions as before to get $E_{\psi,\chi}$, $A_{\psi,\chi}$. We have seen that $\mathrm{gr}^F \mathcal{U}_\chi = k[\mathcal{N}]$, where F is the PBW filtration and $\mathcal{N}$ is the nilpotent cone. As a nongraded algebra, this is the same as $\mathrm{gr}^K \mathcal{U}_\chi$.

Theorem 13.2. $\mathrm{gr}^K(A_{\psi,\chi}) \cong k[\mathcal{N} \cap s] = k[\mathcal{S}_\psi]$.

This is nicer because $\mathcal{N} \cap s$ has finitely many symplectic leaves.

13.1 something different

Let H be an algebraic group with Lie algebra $\mathfrak{h}$. Let $\psi \in \mathfrak{h}^*$ be fixed under coadjoint action. Let A be an associative algebra.

Definition 13.1 (Hamiltonian action)**.** A **Hamiltonian action** of H on A is the following data/conditions:

- An algebraic action $\mathrm{act}_A : H \to \mathrm{Aut}A$.
- An H-equivariant algebra homomorphism $\mu : \mathcal{U}\mathfrak{h} \to A$, called the comoment map.

- $d\mathrm{act}_A(h) = \mathrm{ad}_{\mu(h)}$ for all $h \in \mathfrak{h}$.

Given such an action, let $\mathfrak{h}_\psi$ be the image of the map $\mathfrak{h} \to A$, $h \mapsto \mu(h) - \psi(h)$. We can then define $E_\psi = A/A\mathfrak{h}_\psi$, $A_\psi = E_\psi^H$ as before.

Definition 13.2. An (A, H, ψ)-module is the following data/conditions:

- A left A-module V.
- An algebraic action $\mathrm{act}_V : H \to GL(V)$.
- $\mathrm{act}_V(h)(av) = \mathrm{act}_A(h)(a)\mathrm{act}_V(h)(v)$.
- $\mu(\xi)v = (d\mathrm{act}_V)(\xi)(v) + \psi(\xi)v$.

Example 13.1. Let X be a smooth affine H-variety. Let $A = D(X)$ be the algebra of differential operators. Then $D(X)$-modules are equivalent to D-modules on X; $(D(X), H, \psi = 0)$-modules are equivalent to strongly equivariant D-modules on X; $(D(X), H, \psi)$-modules are equivalent to ψ-monodromic D-modules on X.

Example 13.2. E_ψ is an $(A, H, \psi) - module$.

Define a Hamiltonian reduction functor $\mathbb{H} : (A, H, \psi)-\mathrm{mod} \to A_\psi-\mathrm{mod}$ by $V \mapsto V^H$. Note that $\mathfrak{h}_\psi$ acts on V^H by zero, so $A \times V \to V$ factors through $(A/A\mathfrak{h}_\psi) \times V \to V$. This gives an A_ψ-action on V^H. Furthermore, $V^H = \mathrm{Hom}_{(A,H,\psi)-\mathrm{mod}}(E_\psi, V)$. Then the functor $\mathbb{H}$ has a left adjoint given by $F \mapsto E_\psi \otimes_{A_\psi} F$.

We now specialize to the case $A = \mathcal{U}\mathfrak{g}$ and $H = M$ is unipotent. A $\mathcal{U}\mathfrak{g}$-module V has the structure of a $(\mathcal{U}\mathfrak{g}, M, \psi)$-module iff the action of $\mathfrak{m}_\psi$ on V is locally nilpotent, and in this case, the structure is unique.

Assume $V \in (\mathcal{U}\mathfrak{g}, M, \psi)$-mod is finitely generated as a $\mathcal{U}\mathfrak{g}$-module. Then there is a finite dimensional M-stable subspace V_0 such that $\mathcal{U}\mathfrak{g} \cdot V_0 = V$. Define $K_\bullet V = K_\bullet(\mathcal{U}\mathfrak{g}) \cdot V_0$. This filtration is M and $\mathfrak{m}$ stable, so $\mathrm{gr}^K V$ is a $\mathrm{Sym}\mathfrak{g}$-module with an M-action. By checking actions, we see that in fact $\mathrm{gr}^K V$ is a $k[\psi + \mathfrak{m}^\perp]$-module. In geometric terms, $\mathrm{gr}^K V$ is an M-equivariant coherent sheaf on $\psi + \mathfrak{m}^\perp \cong M \times s$.

Lemma 13.1. $H^{>0}(\mathfrak{m}, V) = 0$.

Proof. Similar to cohomology computations last time.

Theorem 13.3 (Skryabin Theorem)**.** *The functor* $\mathbb{H} : (\mathcal{U}\mathfrak{g}, M, \psi)-\mathrm{mod} \to A_\psi-\mathrm{mod}$ *is an equivalence.*

Proof. The previous Lemma shows that $\mathbb{H}$ is exact. We claim $V = \mathcal{U}\mathfrak{g} \cdot (V^M)$. Let $C = V/(\mathcal{U}\mathfrak{g} \cdot (V^M))$. Applying M-invariants to the quotient SES gives $C^M = 0$. But by unipotence, this implies $C = 0$. A similar argument shows the map $E_\psi \otimes_{A_\psi} V^M \to V$ is an isomorphism, and thus that $\mathbb{H}$ is an equivalence.