# Hartshorne Section I.1 Exercises

Andrea Bourque

May 16, 2023

We don't do the starred problems. We assume the field $k$ is algebraically closed and characteristic 0.

# 1 Problem 1

(a) Let $Y$ be the plane curve $y = x^2$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over $k$.

*Proof.* Let $\phi : k[x, y] \to k[t]$ be the homomorphism uniquely defined by fixing elements of $k$, sending $x$ to $t$, and sending $y$ to $t^2$. Clearly this is a surjective map, since for any $f(t) \in k[t]$ we have $f(x) \in k[x, y]$ and $\phi(f(x)) = f(t)$. Let us show that $\ker \phi = (y-x^2)$. Clearly $(y-x^2) \subseteq \ker \phi$, since $\phi(y-x^2) = t^2-t^2 = 0$. Suppose $f(x, y) \in \ker \phi$. Up to an element in $(y - x^2)$, we can assume $f(x, y)$ is only a polynomial in $x$, say $f(x, y) = g(x) + (y - x^2)h(x, y)$. Then $\phi(g(x)) = 0$, meaning $g(t) = 0$. Then the coefficients of $g$ are all 0, i.e. $g(x) = 0$. Thus $f(x, y) = (y - x^2)h(x, y)$, so $f(x, y) \in (y - x^2)$. Since $\ker \phi = (y - x^2)$ and $\phi$ is surjective, the first isomorphism theorem says that $\phi$ induces and isomorphism $A(Y) = k[x, y]/(y - x^2) \to k[t]$, as desired. $\square$

(b) Let $Z$ be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over $k$.

*Proof.* Suppose we had an isomorphism $\phi : k[x, y]/(xy - 1) \to k[t]$. Then $\phi(x)\phi(y) = \phi(xy) = \phi(1) = 1$ implies $\phi(x)$ and $\phi(y)$ are units in $k[t]$. The only units in $k[t]$ are the non-zero constant polynomials. Similarly, for any non-zero $a \in k$, we have $\phi(a)\phi(a^{-1}) = \phi(1) = 1$, so $\phi(a)$ is a unit. Thus $\phi(a) \in k$. Then for any $f(x, y) \in k[x, y]/(xy-1)$, we have $\phi(f(x, y)) \in k$, since the images of all the terms of $f(x, y)$ are in $k$. Then $\phi$ can't be surjective, a contradiction. $\square$

## 2 Problem 2

Let $Y \subseteq \mathbb{A}^3$ be the set $Y = \{(t, t^2, t^3) | t \in k\}$. Show that $Y$ is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over $k$.

*Proof.* It is clear to see that $Y = Z(y - x^2, z - x^3)$, so $Y$ is closed. By Corollary 1.4 in Hartshorne chapter I, it suffices to show that $I(Y)$ is prime. This is equivalent to $A(Y) = k[x, y, z]/I(Y)$ being an integral domain, which is implied by $A(Y) = k[x, y, z]/I(Y)$ being isomorphic to a polynomial ring over $k$, which we show later.

By Proposition 1.7 in Hartshorne chapter I, the dimension of $Y$ equals the dimension of $A(Y)$. Thus, once we show that $A(Y)$ is isomorphic to a polynomial ring in one variable over $k$, then we will know the dimension of $Y$ is indeed 1.

By the correspondence between algebraic sets and radical ideals, we see that $I(Y)$ is the radical of $(y - x^2, z - x^3)$. Let us show that $(y - x^2, z - x^3)$ is prime, so that it equals its own radical, and thus equals $I(Y)$. In fact, we will show $k[x, y, z]/(y - x^2, z - x^3)$ is isomorphic to a polynomial ring in one variable over $k$, which then implies that $(y - x^2, z - x^3)$ is prime *and* that $A(Y)$ is isomorphic to a polynomial ring in one variable over $k$.

To that end, let $\phi : k[x, y, z] \to k[w]$ be the unique homomorphism fixing $k$, sending $x$ to $w$, sending $y$ to $w^2$, and sending $z$ to $w^3$. Clearly $\phi$ is surjective, since for $f(w) \in k[w]$, we have $\phi(f(x)) = f(w)$. If we show that $\ker \phi = (x^2 - y, x^3 - z)$, we will be done. Clearly $(x^2 - y, x^3 - z) \subseteq \ker \phi$. For any $f(x, y, z) \in k[x, y, z]$, there is a $g(x) \in k[x, y, z]$ such that $f(x, y, z) - g(x)$ is in $(x^2 - y, x^3 - z)$; namely, $g(x) = f(x, x^2, x^3)$. Thus, if $\phi(f(x, y, z)) = 0$, then $g(x) = 0$, so $f(x, y, z)$ is in $(x^2 - y, x^3 - z)$. $\square$

## 3 Problem 3

Let $Y$ be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that $Y$ is a union of three irreducible components. Describe them and find their prime ideals.

*Proof.* Note that $xz - x = 0$ implies $x = 0$ or $z = 1$. In the first case, $x^2 - yz = 0$ implies $yz = 0$, which implies $y = 0$ or $z = 0$. On the other hand, if $z = 1$, then $x^2 - yz = 0$ implies $x^2 - y = 0$. Thus $Y = Z(x, z) \cup Z(x, y) \cup Z(z - 1, y - x^2)$. Geometrically, we can describe $Y$ as the union of the $y$-axis, the $z$-axis, and the parabola $y = x^2$ sitting in the $z = 1$ plane. The corresponding prime ideals are $(x, z)$, $(x, y)$, and $(y - x^2, z - 1)$. $\square$

## 4 Problem 4

Show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

*Proof.* The diagonal $\{(x,x)|x \in k\}$ is closed in $\mathbb{A}^2$, since it equals $Z(x-y)$. A basic fact from topology is that a space $X$ is Hausdorff iff the diagonal is closed in $X \times X$. Since $\mathbb{A}^1$ is not Hausdorff, the diagonal is not closed in $\mathbb{A}^1 \times \mathbb{A}^1$. Thus, the topologies on $\mathbb{A}^2$ and $\mathbb{A}^1 \times \mathbb{A}^1$ are different. $\square$

## 5 Problem 5

Show that a $k$-algebra $B$ is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some $n$, iff $B$ is a finitely generated $k$-algebra with no nilpotent elements.

*Proof.* Suppose $B \cong A(Y)$ for an algebraic set $Y$ in $\mathbb{A}^n$. Then $I(Y)$ is a radical ideal, so by definition, $k[x_1, ..., x_n]/I(Y)$ has no nilpotent elements. $B$ is generated by the elements $x_i + I(Y)$, so it is finitely generated.

In the other direction, it is a fact of commutative algebra that a finitely generated $k$-algebras are isomorphic to a quotient of $k[x_1, ..., x_n]$. It is also a fact of commutative algebra that a quotient ring $R/I$ has no nilpotent elements if and only if $I$ is radical. Thus, $B$ is isomorphic to a quotient of $k[x_1, ..., x_n]$ by a radical ideal $I$. Then the algebraic set $Z(I)$ has ideal $I$, and thus its coordinate ring $k[x_1, ..., x_n]/I$ is isomorphic to $B$. $\square$

# 6 Problem 6

Any nonempty open subset of an irreducible topological space is dense and irreducible. If $Y$ is a subset of a topological space $X$, which is irreducible in the subspace topology, then the closure $\overline{Y}$ is also irreducible.

*Proof.* Let $U$ be a nonempty open subset of an irreducible topological space $X$. Then $\overline{U}$ and $X - U$ are closed and union to $X$. It follows that one of these sets equals $X$. Since $U$ is nonempty, $X - U \neq X$. Thus $\overline{U} = X$, i.e. $U$ is dense.

Now suppose $U$ is reducible, i.e. $U = Z_1 \cup Z_2$ for two proper subsets of $U$, which are closed in $U$. Then $Z_1 = U \cap Z_1'$, $Z_2 = U \cap Z_2'$ for closed subsets $Z_1', Z_2'$ of $X$. The sets $Z_1', Z_2'$ must also be proper, since if we had $Z_i' = X$, then $Z_i = U$, which goes against the assumption that $Z_1, Z_2$ are proper in $U$. Since $X$ is irreducible, $Z_1' \cup Z_2'$ cannot equal $X$, but it is a closed set containing $U$. Thus $X = (X - U) \cup (Z_1' \cup Z_2')$, a contradiction. Thus, $U$ is irreducible.

Now suppose $Y$ is an irreducible subset of an arbitrary topological space $X$. Suppose $\overline{Y}$ is reducible, so that there exist proper closed subsets $Z_1, Z_2$ of $\overline{Y}$ which satisfy $\overline{Y} = Z_1 \cup Z_2$. By definition, there are closed subsets $Z_1', Z_2'$ of $X$ such that $Z_i = \overline{Y} \cap Z_i'$ for $i \in \{1, 2\}$. Furthermore, since $Z_1$ and $Z_2$ are proper in $\overline{Y}$, we must have $Z_1'$ and $Z_2'$ are proper in $X$. Now,

$$Y \subseteq \overline{Y} \subseteq Z_1' \cup Z_2',$$

so

$$Y = Y \cap (Z_1' \cup Z_2') = (Y \cap Z_1') \cup (Y \cap Z_2').$$

Since $Y$ is irreducible, we must have $Y \cap Z_i' = Y$ for some $i \in \{1, 2\}$. Thus $Y \subseteq Z_i'$. Since $Z_i'$ is closed, we then have $\overline{Y} \subseteq Z_i'$, so that $\overline{Y} = Z_i$, a contradiction. Thus, $\overline{Y}$ must be irreducible. $\square$

## 7 Problem 7

(a) Show that the following conditions are equivalent for a topological space $X$:

(i) $X$ is noetherian ($X$ satisfies the descending chain condition for closed subsets);

(ii) every nonempty family of closed subsets has a minimal element;

(iii) $X$ satisfies the ascending chain condition for open subsets;

(iv) every nonempty family of open subsets has a maximal element.

*Proof.* By taking complements, it is clear that (i) and (iii) are equivalent, and it is also clear that (ii) and (iv) are equivalent. Thus, it suffices to show that (i) and (ii) are equivalent.

Assume (i). This is a standard scenario where one applies Zorn's lemma. Let $\mathcal{F}$ be a family of closed subsets of $X$. We must show that any descending chain of elements of $\mathcal{F}$ has a lower bound in $\mathcal{F}$. This is immediate from our hypothesis: since a descending chain of closed subsets must stabilize, the lower bound is taken to be the element which all elements of the bottom chain are equal to. By Zorn's lemma, $\mathcal{F}$ has a minimal element. Thus, (ii) holds.

Assume (ii). Given a descending chain of closed subsets of $X$, say $Z_1 \supseteq Z_2 \supseteq ...$, let $\mathcal{F}$ be the family of the $Z_i$. Then $\mathcal{F}$ has a minimal element, say $Z_j$ for some $j$. Since $Z_j$ is minimal in $\mathcal{F}$ and $Z_j \supseteq Z_k$ for $k \geq j$, we must have that $Z_j = Z_k$ for $k \geq j$. Thus, every descending chain of closed subsets of $X$ stabilizes, so (i) holds. □

(b) A noetherian topological space is quasi-compact[1] (every open cover has a finite subcover)

*Proof.* Let $X$ be a noetherian topological space. Let $\{U_i\}$ be an open cover. Construct a chain $U_{i_1} \subset U_{i_1} \cup U_{i_2} \subset ...$ where at each step we choose an element of the cover which makes the next element in the chain strictly bigger. This procedure cannot continue indefinitely, by part (a). Thus, at some point, we will have a set $U_{i_1} \cup ... \cup U_{i_m}$ that contains every $U_i$ in the cover. Therefore, it contains the union of all the $U_i$, which is $X$. Thus, there is a finite subset of $\{U_i\}$ which covers $X$, as desired. □

(c) Any subset of a noetherian topological space is noetherian in its subspace topology.

*Proof.* We begin with a reminder of a topological fact on closure in the subspace topology. To avoid confusion, let $\mathrm{cl}_X(A)$ denote the closure of a subset $A$ of the space $X$. If $A \subseteq Y \subseteq X$, we let $\mathrm{cl}_Y(A)$ denote the closure of $A$ in $Y$, i.e. the

[1] You may be familiar with this term as just "compact". However, some people reserve the word "compact" for only Hausdorff spaces.

smallest subset of $Y$ which is closed in the subspace topology and contains $A$. It is a basic fact from topology that

$$\mathrm{cl}_Y(A) = \mathrm{cl}_X(A) \cap Y.$$

In particular, if $A$ is a closed subset of $Y$, then $A = \mathrm{cl}_X(A) \cap Y$. Thus if $A, B$ are two closed subsets of $Y$, and $\mathrm{cl}_X(A) = \mathrm{cl}_X(B)$, then

$$A = Y \cap \mathrm{cl}_X(A) = Y \cap \mathrm{cl}_X(B) = B.$$

This is the fact we need.

Now, let $X$ be a noetherian space, and let $Y$ be a subspace. Let $Z_1 \supseteq Z_2 \supseteq ...$ be a chain of closed subsets of $Y$. This induces a chain $\mathrm{cl}_X(Z_1) \supseteq \mathrm{cl}_X(Z_2) \supseteq ...$ of closed subsets of $X$. By assumption, this chain stabilizes, say $\mathrm{cl}_X(Z_m) = \mathrm{cl}_X(Z_n)$ for some $m$ and $n \geq m$. By the above result, we then know $Z_m = Z_n$ for some $m$ and $n \geq m$, i.e. the chain stabilizes. Thus, $Y$ is noetherian, as desired. $\square$

(d) A noetherian space which is also Hausdorff must be a finite set with the discrete topology.

*Proof.* We first show that a noetherian and Hausdorff space must have the discrete topology. By (c) and (b), every subset is quasi-compact. In a Hausdorff space, quasi-compact subsets are closed. Thus, every subset is closed, so the topology is discrete.

Now, we show that an infinite set with the discrete topology is not noetherian. Indeed, one can construct an ascending chain of open sets by including one point at a time, and this process does not stabilize.

Thus, a noetherian and Hausdorff space must be a finite set with the discrete topology. $\square$

## 8 Problem 8

Let $Y$ be an affine variety of dimension $r$ in $\mathbb{A}^n$. Let $H$ be a hypersurface in $\mathbb{A}^n$, and assume that $Y \not\subseteq H$. Show that every irreducible component of $Y \cap H$ has dimension $r-1$.

*Proof.* Let $H = Z(f)$ for an irreducible polynomial $f \in k[x_1, ..., x_n]$. Let $Y = Z(\mathfrak{a})$ for a prime ideal $\mathfrak{a} \subseteq k[x_1, ..., x_n]$. By Theorem 1.8A(b) of Hartshorne chapter I, the height of $\mathfrak{a}$ is $n-r$. The condition $Y \not\subseteq H$ is equivalent to $f \notin \mathfrak{a}$. We have $Y \cap H = Z((f) \cup \mathfrak{a})$. We can assume that we are not in the vacuous case $Y \cap H = \varnothing$. Thus, the ideal generated by $f$ and $\mathfrak{a}$ is not the unit ideal.

Since quotients of noetherian rings are noetherian, we have $k[x_1, ..., x_n]/\mathfrak{a}$ is noetherian. Since $\mathfrak{a}$ is prime, this ring is an integral domain. Since $f \notin \mathfrak{a}$, we have that the element $\overline{f} = f + \mathfrak{a}$ is not zero, and therefore not a zero-divisor. It is also not a unit; otherwise, $f$ and $\mathfrak{a}$ would generate the unit ideal (so $Y \cap H = \varnothing$). By Theorem 1.11A of Hartshorne chapter I,[2] the minimal primes of $k[x_1, ..., x_n]/\mathfrak{a}$ which contain $\overline{f}$ have height 1. Passing to $k[x_1, ..., x_n]$, we have that the minimal primes containing $f$ and $\mathfrak{a}$ have height $n-r+1$. These prime ideals correspond to the irreducible components of $Y \cap H = Z((f) \cup \mathfrak{a})$. The dimension of these components is then $n-(n-r+1) = r-1$, as desired. $\square$

[2]Also known as Krull's *Hauptidealsatz* or Krull's princicipal ideal theorem.

## 9 Problem 9

Let $\mathfrak{a} \subseteq A = k[x_1, ..., x_n]$ be an ideal which can be generated by $r$ elements. Then every irreducible component of $Z(\mathfrak{a})$ has dimension $\geq n - r$.

*Proof.* By applying Theorem 1.8A(b) of Hartshorne chapter I, the equivalent algebraic statement is: the minimal elements among the collection of prime ideals containing $\mathfrak{a}$ have height $\leq r$. This is stated as Corollary 11.16 in Atiyah-MacDonald. $\square$

## 10 Problem 10

(a) If $Y$ is any subset of a topological space $X$, then $\dim Y \leq \dim X$.

*Proof.* For any chain $Z_0 \subset ... \subset Z_m$ of closed irreducible subsets of $Y$, which are also irreducible subsets of $X$, we have the chain $\overline{Z}_0 \subset ... \subset \overline{Z}_m$ of closed subsets of $X$. Furthermore, by Problem 6, the subsets $\overline{Z}_i$ are irreducible. Thus, for every chain of closed irreducible sets in $Y$, we get a chain of closed irreducible sets of the same length in $X$. It follows that $\dim Y \leq \dim X$. □

(b) If $X$ is a topological space with an open cover $\{U_i\}$, then $\dim X = \sup \dim U_i$.

*Proof.* By part (a), we know $\dim U_i \leq \dim X$ for all $U_i$. Thus $\sup \dim U_i \leq \dim X$. It remains to show $\dim X \leq \sup \dim U_i$.

Given a chain of closed irreducible subsets $Z_0 \subset ... \subset Z_m$ of $X$, pick an $x \in Z_0$ and then a $U_i$ which contains $x$. Then $U_i$ intersects each $Z_j$, so that we have a chain $U_i \cap Z_0 \subset ... \subset U_i \cap Z_m$ of closed subsets of $U_i$. Furthermore, $U_i \cap Z_j$ is a nonempty open subset of $Z_j$, so by Problem 6 it is irreducible. Thus, for any chain of closed irreducible subsets of $X$, there is a chain of closed irreducible subsets of some $U_i$ of the same length. This proves $\dim X \leq \sup \dim U_i$, as desired. □

(c) Give an example of a topological space $X$ with a dense open subset $U$ with $\dim U < \dim X$.

*Proof.* Let $n \geq 1$. Let $X = \{0, 1, ..., n\}$ with topology given by closed sets $\varnothing$ and $\{0, ..., m\}$ for $m \in \{0, 1, ..., n\}$. Then the set $U = \{n\}$ is open since it is the complement of the closed set $\{0, ..., n-1\}$, and the smallest closed set containing $U$ is $X$. Thus $U$ is a dense and open subset of $X$. The dimension of $U$ is clearly 0, since there it has only one closed and irreducible subset: itself. The dimension of $X$ is $n$, since there is the chain $\{0\} \subset \{0, 1\} \subset ... \subset X$ of $n+1$ closed irreducible subsets, and these are all of the closed irreducible subsets. Thus $\dim U = 0 < n = \dim X$. □

(d) If $Y$ is a closed subset of an irreducible finite-dimensional topological space $X$, and if $\dim Y = \dim X$, then $Y = X$.

*Proof.* Since $Y$ is closed, any subset of $Y$ that is closed in the subspace topology is closed in $X$. If $Y \neq X$, then for any chain of closed irreducible subsets of $Y$, say $Y_0 \subset Y_1 \subset ... \subset Y_m$, we have the strictly bigger chain $Y_0 \subset Y_1 \subset ... \subset Y_m \subset X$. This contradicts the assumption $\dim Y = \dim X$, so we must have $Y = X$. □

(e) Give an example of a noetherian topological space of infinite dimension.

*Proof.* Let $X = \{0, 1, ...\}$ with closed sets given by $\varnothing$, $\{0, ..., m\}$ for all $m \in X$, and $X$. Then $\dim X$ is infinite, since $\{0\} \subset \{0, 1\} \subset ...$ forms an infinite sequence of distinct irreducible closed subsets of $X$.

To show that $X$ is noetherian, note that any descending chain of closed subsets is in one of two cases. The first case is that the chain consists only of $X \supset X \supset X \supset ...$, in which case it of course is stable. The second case is that somewhere in the chain, there is a proper closed subset, say $\{0, ..., m\}$. Then there are finitely closed subsets of $X$ which are contained in $\{0, ..., m\}$, so the sequence cannot strictly decrease indefinitely. Thus, $X$ is noetherian. $\square$

## 11 Problem 12

Give an example of an irreducible polynomial $f \in \mathbb{R}[x, y]$, whose zero set $Z(f)$ in $\mathbb{A}^2_{\mathbb{R}}$ is not irreducible.

*Proof.* The polynomial $x^2 + y^2 + 1$ is a fine example, since the empty set is by definition not irreducible. To justify that $x^2 + y^2 + 1$ is irreducible in $\mathbb{R}[x, y] = (\mathbb{R}[x])[y]$, we apply Eisenstein's criterion with the prime $x^2 + 1$, since it should be familiar that $x^2 + 1$ is irreducible (hence prime) in $\mathbb{R}[x]$. $\square$