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Numerical Solution of Richards Equation by Using of Finite Volume Method

Mohamad Azizi Pour, Mohammad Mahmoodian Shoshtari and Arash Adib

Department of Civil Engineering, Faculty of Engineering, Shahid Chamran University, Ahvaz, Iran

Abstract: In this research, the one dimensional mixed-form of Richards Equation (RE) was solved by simple control volume method. The principle of this method is the integrating RE in a domain. This method avoids mass-balance errors and is numerically stable. The resulting tri diagonal matrix was solved by using of Thomas algorithm in each iteration level. The performance of method is assessed on test cases and show capability of mass conservative finite volume.

Key words: Richards equation . finite volume . Thomas algorithm

INTRODUCTION

Attention to the unsaturated zone has increased in recent years because of growing industrial, municipal and agricultural activities that the quality of subsurface environment is being adversely affected by them. In all studies of unsaturated zone, it is assumed that Richards Equation (RE) is governing on subsurface flow. This equation is a mathematical model that made used to describe variably saturated flow and is derived by combining the Darcy's law with the mass conservation equation in porous media [1]. RE is written as one of the following forms:

$$C(\psi)\frac{\partial\psi}{\partial t} - \nabla.K(\psi)\nabla\psi + \frac{\partial K(\psi)}{\partial z} = 0 \quad \text{Pressure-based} \qquad (1)$$

$$\frac{\partial\theta}{\partial t} - \nabla.D(\theta)\nabla\psi + \frac{\partial K(\theta)}{\partial z} = 0 \quad \text{Moisture-based} \qquad (2)$$

$$\frac{\partial\theta}{\partial t} - \nabla.K(\theta)\nabla\psi + \frac{\partial K(\psi)}{\partial z} = 0 \quad \text{Mixed-form} \qquad (3)$$

where Y is the pressure head [L], t is time [T], z is vertical coordinate [L] (positive downward), θ is volumetric water content, C(Y)=(d θ/d Y) is the specific moisture capacity [1/L], K(θ) or K(Y) is the hydraulic conductivity function [L/T] and D(θ)= K(θ)/C(θ) is the diffusivity [L^2/T].

Researchers solved RE analytically [2-4]. These solutions are valid for simple conditions. Because unsaturated flow equation (RE) is high nonlinear, an analytical solution is not possible except for special cases. Therefore numerical methods are typically used to solve the unsaturated flow equation in the last thirty years. Some Finite difference and Finite Element solutions were developed by scientists [5-8]. These solutions often suffer to some degree from mass balance errors as well as from numerical oscillations and dispersion and additional numerical problems may appear when the gravitational term becomes important [9]. Finite elements are advantageous at an irregular geometry in 2 and 3-dimensional flow domains. In one dimension finite difference is advantageous because it needs no mass lumping to prevent oscillations [10, 11]. Researchers presented a mass conservative model for solving mixed form of RE using finite difference method [12]. Other techniques have also been implemented such as mixed finite element [13, 14]. In this paper, we present an accurate numerical solution of RE using finite volume method and then compare results with obtaining results from finite difference and finite element solution presented by other researchers [8, 12].

FINITE VOLUME MODEL

Generation of grid: Dividing the domain to discrete control volumes is the first step in the finite volume method. Consider a number of nodal points in the distance between ground surface and end of soil column. The boundaries of control volumes are positioned mid-way between adjacent nodes. Thus each node is surrounded by a control volume or cell. A general nodal point is identified by P and its neighbors in a one-dimensional geometry, the nodes to the north and south, are identified by N and S respectively. The north side boundary of the control volume is referred to by 'n' and the south side control volume boundary by 's'. A portion of the one-dimensional grid used for the discretization is shown in Fig. 1.

Corresponding Author: Dr. Arash Adib, Department of Civil Engineering, Faculty of Engineering, Shahid Chamran University, Ahvaz, Iran

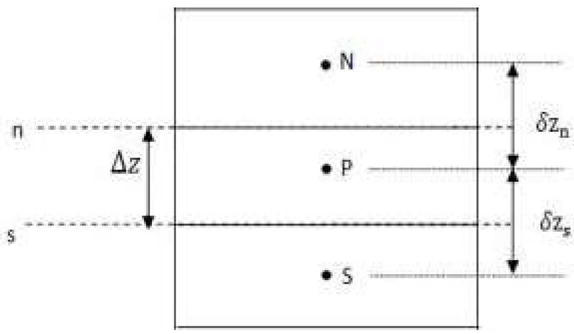

Fig. 1: A portion of 1-D grid

Discretization of equation: Predominant direction of water flow in the vadose zone is vertical and can be simulated as 1-D flow in many applications (Romano *et al.*, 1998). The equation of one-dimensional fluid flow in porous media is:

$$\frac{\partial\theta}{\partial t}-\frac{\partial}{\partial z}\left(K\frac{\partial\psi}{\partial z}+k\right)=0 \tag{4}$$

The control volume integration, which forms the key step of the finite volume method that distinguishes it from finite difference technique, for the control volume defined above, yields the following form:

$$\int_n^s\int_t^{t+\Delta t}\frac{\partial\theta}{\partial t}dtdz-\int_t^{t+\Delta t}\int_n^s\frac{\partial}{\partial z}\left(K\frac{\partial\psi}{\partial z}+K\right)dzdt=0 \tag{5}$$

Integrating from first term of this equation respect to time

$$\int_n^s\int_t^{t+\Delta t}\frac{\partial\theta}{\partial t}dtdz=\int_n^s(\theta^{t+\Delta t}-\theta^t)dz \tag{6}$$

In finite volume method, the whole specification of fluid and media is constant in a control volume. Therefore, equation 6 converts to follow equation:

$$\int_n^s\int_t^{t+\Delta t}\frac{\partial\theta}{\partial t}dtdz=(\theta_p^{t+\Delta t}-\theta_p^t)\Delta z \tag{7}$$

After integrating from second term of equation 5, the result would be:

$$\int_t^{t+\Delta t}\int_n^s\frac{\partial}{\partial z}\left(K\frac{\partial\psi}{\partial z}+K\right)dzdt=\int_t^{t+\Delta t}\begin{bmatrix}\left(K\frac{\partial\psi}{\partial z}+K\right)_s\\-\left(K\frac{\partial\psi}{\partial z}+K\right)_n\end{bmatrix}dt \tag{8}$$

For implicit solution, the discretized form of (4) is:

$$(\theta_p^{t+\Delta t}-\theta_p^t)\Delta z=\begin{bmatrix}\left(K^t\frac{\partial\psi^{t+\Delta t}}{\partial z}+K^t\right)_s\\-\left(K^t\frac{\partial\psi^{t+\Delta t}}{\partial z}+K^t\right)_n\end{bmatrix}\Delta t \tag{9}$$

where θ^t and K^t denotes the value of θ and K at time t, respectively, Dt is the time step, Y^{t+Dt} denotes the Y at time t+Dt and the solution is assumed to be known at time level t and unknown at time level t+Dt. Because the velocity of water flow in soil is very small, it is logical to consider hydraulic conductivity is constant during a time step.

It is a very attractive feature of the finite volume method that the discretized equation has a clear physical interpretation. Equation (9) states that the flux of water leaving the south face minus the water flux entering the north face is equal to the reserved water in a control volume, i.e. it constitutes a balance equation for water over the control volume.

In order to derive useful forms of the discretized equations, the interface hydraulic conductivity and gradient jY/jz at north 'n' and south 's' are required To calculate gradients at the control volume faces an approximate distribution of properties between nodal points is used. Linear approximations seem to be the obvious and the simplest way of calculating interface values and the gradients. For evaluation K_n and K_s, harmonic mean, arithmetic mean and geometric mean may be used that K_n is average between hydraulic conductivity at N,P and K_s is average between hydraulic conductivity at N,S. Applying central difference, right terms of (9) are evaluated as:

$$\left(K^t\frac{\partial\psi^{t+\Delta t}}{\partial z}+K^t\right)_s=K_s^t\frac{\psi_s^{t+\Delta t}-\psi_p^{t+\Delta t}}{\delta z_s}+K_s^t \tag{10}$$

$$\left(K^t\frac{\partial\psi^{t+\Delta t}}{\partial z}+K^t\right)_n=K_n^t\frac{\psi_p^{t+\Delta t}-\psi_n^{t+\Delta t}}{\delta z_n}+K_n^t \tag{11}$$

Substitution of equations (10) and (11) into (9) gives:

$$(\theta_p^{t+\Delta t}-\theta_p^t)\Delta z=\begin{bmatrix}\left(K_s^t\frac{\psi_s^{t+\Delta t}-\psi_p^{t+\Delta t}}{\delta z_s}+K_s^t\right)\\-\left(K_n^t\frac{\psi_p^{t+\Delta t}-\psi_n^{t+\Delta t}}{\delta z_n}+K_n^t\right)\end{bmatrix}\Delta t \tag{12}$$

For solution (10) an iterative method must be used. Considering 'm' as iteration level and using Picard iterative method, gives:

$$(\theta_p^{t+\Delta t,m+1} - \theta_p^t)\Delta z = \left[\begin{array}{l} \left(K_s^t \dfrac{\psi_s^{t+\Delta t,m+1} - \psi_p^{t+\Delta t,m+1}}{\delta z_s} + K_s^t\right) \\ -\left(K_n^t \dfrac{\psi_p^{t+\Delta t,m+1} - \psi_n^{t+\Delta t,m+1}}{\delta z_n} + K_n^t\right) \end{array}\right]\Delta t \quad (11)$$

There are two unknowns vector θ and Y in 'm+1'th iteration level. As proposed by researchers, for predicting $\theta_p^{t+Dt,m+1}$, a Taylor series is expanded as follow [12]:

$$\theta_p^{t+\Delta t,m+1} = \theta_p^{t+\Delta t,m} + \left(\frac{d\theta}{d\psi}\right)_p^t (\psi_p^{t+\Delta t,m+1} - \psi_p^{t+\Delta t,m}) \quad (12)$$

In a uniform grid $Dz = dz_s = dz_n$ and (10) can be re-arranged as:

$$\alpha\psi_n^{t+\Delta t} + \beta\psi_p^{t+\Delta t} + \gamma\psi_s^{t+\Delta t} = \lambda \quad (13)$$

where

$$\alpha = K_n^t\left(\frac{\Delta t}{\Delta z^2}\right)$$

$$\beta = -\left[(K_n^t + K_s^t)\left(\frac{\Delta t}{\Delta z^2}\right) + C_p^t\right]$$

$$\gamma = K_s^t\left(\frac{\Delta t}{\Delta z^2}\right)$$

and

$$\lambda = \left[(K_n^t - K_s^t)\left(\frac{\Delta t}{\Delta z}\right)\right] - C_p^t\psi_p^{t+\Delta t,m} + (\theta_p^{t+\Delta t,m} - \theta_p^t)$$

Boundary conditions: For control volumes that are adjacent to the domain boundaries the general discretized equation is modified to incorporate boundary conditions. In this section, Dirichlet and Neumann boundaries are considered and equation (13) is modified.

Dirichlet boundary condition: Dirichlet nodes that pressure head are known. In upper boundary control volume,a=0and:

$$\lambda = \left[(K_n^t - K_s^t)\left(\frac{\Delta t}{\Delta z}\right)\right] - C_p^t\psi_p^{t+\Delta t,m} + (\theta_p^{t+\Delta t,m} - \theta_p^t) - 2K_n^t\left(\frac{\Delta t}{\Delta z^2}\right)\psi_A$$

Also in lower boundary control volume $\gamma = 0$ and

$$\lambda = \left[(K_n^t - K_s^t)\left(\frac{\Delta t}{\Delta z}\right)\right] - C_p^t\psi_p^{t+\Delta t,m} + (\theta_p^{t+\Delta t,m} - \theta_p^t) - 2K_s^t\left(\frac{\Delta t}{\Delta z^2}\right)\psi_B$$

Neumann boundary condition

At Neumann boundary nodes, value of normal flux is specified. In upper boundary control volume, equation (11) modified as:

$$(\theta_p^{t+\Delta t,m+1} - \theta_p^t)\Delta z = \left[\begin{array}{l} \left(K_s^t \dfrac{\psi_s^{t+\Delta t,m+1} - \psi_p^{t+\Delta t,m+1}}{\delta z_s} + K_s^t\right) \\ -(q_n) \end{array}\right]\Delta t \quad (14)$$

where q_n is normal and constant flux at upper boundary of domain. Also in lower boundary control volume, equation (11) can be written as:

$$(\theta_p^{t+\Delta t,m+1} - \theta_p^t)\Delta z = \left[\begin{array}{l} (q_s) - \\ \left(K_n^t \dfrac{\psi_p^{t+\Delta t,m+1} - \psi_n^{t+\Delta t,m+1}}{\delta z_n} + K_n^t\right) \end{array}\right]\Delta t \quad (15)$$

That q_s is normal and constant flux at lower boundary of domain.

RESULTS

In this section, the performance of model is assessed by solving vertical infiltration problems.

In the first problem, the soil is composed of sand. This example modeling data from [5] and verify θ and Y predictions of this model during unsteady flow with a flux boundary condition. The initial pressure head profile is specified as $\psi(z,0) = 61.5$cm. A constant flux is imposed at the soil surface ($q(0,t) = 0.003808$cm/s). Also pressure head is maintained constant at the lower end of the column ($\psi(70,t) = 61.5$cm). The soil is parameterized by the constitutive relationships:

$$\theta = \theta_r + \frac{A(\theta_s - \theta_r)}{A + \psi^B} \quad (17)$$

$$K(\psi) = K_s \frac{D}{D + \psi^C} \quad (18)$$

where θ_s=0.287, θ_r=0.075, A=1.175*10^6, B=3.96, K_s=0.00944 cm/s, C=4.94, D=1.611*10^6.

In this problem Δt=1s (period of time) and Δz=1cm (distance between two nodes). Figure 2 illustrates distribution of water content, at 0.1h time interval. As shown in Fig. 2, experimentally observed θ values and calculated θ profile are very close to one another.

Considering Y(0,t)=20.7cm, Y(70,t)=61.5cm as boundary conditions and same initial condition, water content and pressure head profiles are shown in Fig. 3. In Fig. 3 the continues line illustrates the finite volume

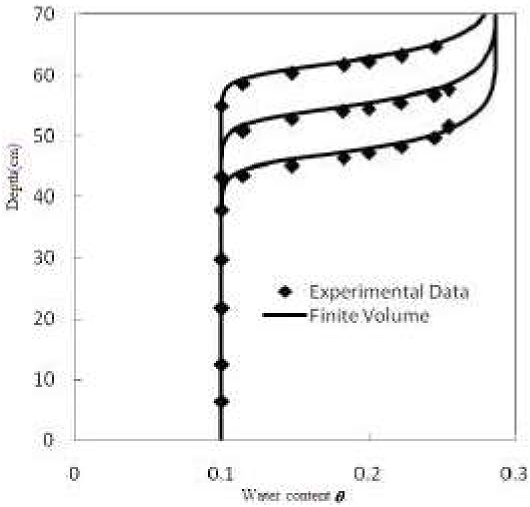

Fig. 2: Calculated and experimental water content profile

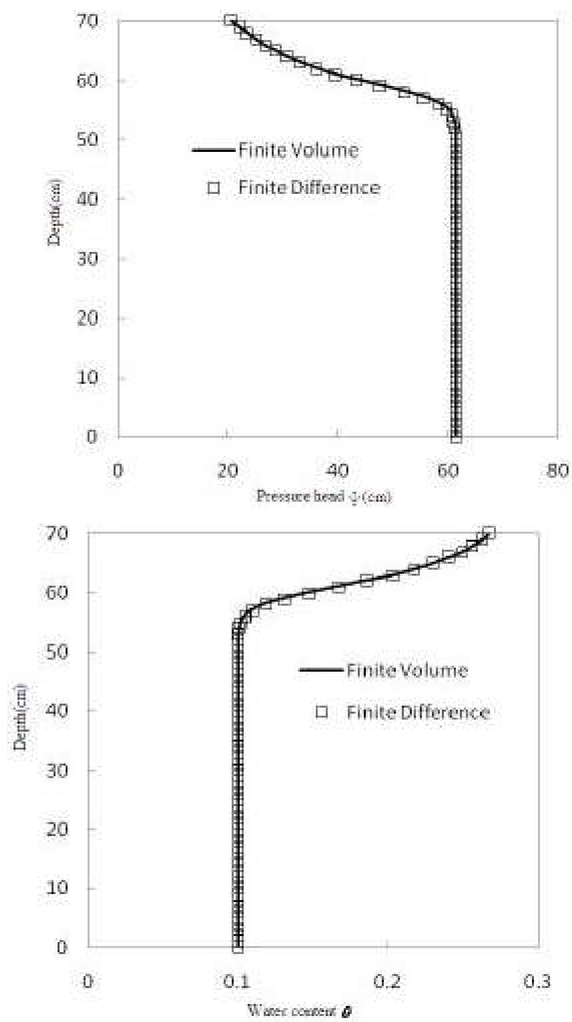

Fig. 3: Calculated pressure head and water content profile using FD and FV

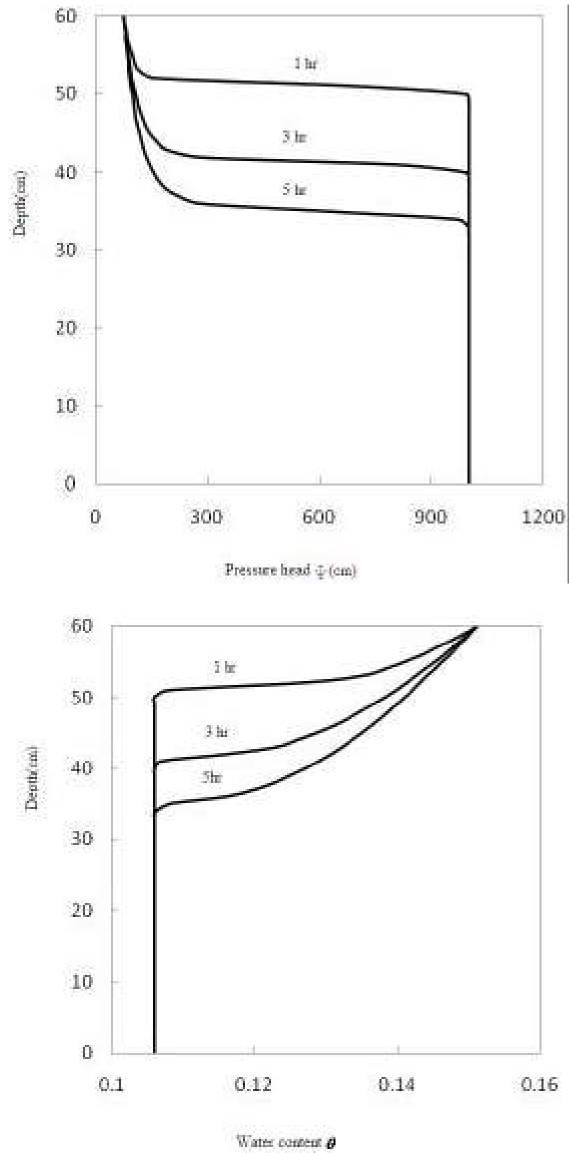

Fig. 4: Calculated pressure head and water content profiles

solution and squares denote the finite difference solution, present in [12]. The agreement between two solutions is clear.

In the second problem, New Mexico soil is used for simulation the pressure head and water content profile. The Van Genuchten functions are considered for hydraulic conductivity and water characteristic curve:

$$\theta = \theta_r + \frac{\theta_s - \theta_r}{(1 + (a\psi)^n)^m} \tag{19}$$

$$K(\psi) = K_s \frac{\left\{1 - (\alpha\psi)^{n-1}[1 + (a\psi)^n]^{-m}\right\}^2}{[1 + (a\psi)^n]^{m/2}} \tag{20}$$

where θ_s = 0.368, θ_r = 0.102, are respectively saturated and residual water contents. A = 0.0335cm^{-1}, K_s = 0.00922cm/s and n = 2, m = 0.5. The initial pressure profile is specified as Y(z,0) = 1000cm. Also boundary conditions are Y(0,t) = 75cm and Y(70,t) = 1000cm. Figure 4 illustrates the water content and pressure head profiles using Δt = 10s and Δz = 1cm at time 1, 3 and 5 hours.

CONCLUSION

A finite volume method has been developed and assessed for modeling unsaturated flow thorough porous media. This method is based on integral formulation of equation. Finite volume is inherently mass conservative and performs well over three problems.

In first problem, the infiltration in sand column considered and capability of model demonstrated by performance against published experimental data. In second test case, the boundary conditions changed and results are in good agreement with the finite difference proposed by Celia *et al*. In last problem, the infiltration column taken from [12].

The results show that the finite volume is a mass conservative and is a good method for solving RE.

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