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Week 11

Solving Systems of Equations Using Matrices and Network Analysis

This topic is quite lengthy and has several parts. I consider this extra material which we will cover if we have time. The examples in Part 3 are quite interesting and the material ideas presented have applications in virtually every field.

Part 1 Solving systems of linear equations using matrices (pages 1 through 10). This is the key of all the material in the following pages.

Part 2. Solving systems of linear equations which do not have a unique solution. (pages 11 through 16) This is used in Part 3.

Part 3. Network Analysis. (pages 16 through 51) Fourteen examples, some from students, which depict some of the variety of applications of systems of equations.

Part 4 , The Row Reduction Method for Determining the Inverse of a Matrix (pages 52 through 55)

Part 1. Solving Systems of Linear Equations Using Matrices

1) Consider the following system of equations:

$x_1 + x_3 = 1$
$x_1 – x_2 = 0$
$2x_2 + x_3 = 2$

or

$x_1 + 0\,x_2 + x_3 = 1$
$x_1 - x_2 + 0x_3 = 0$ Why?
$0x_1 + 2x_2 + x_3 = 2$

We will solve this system using a procedure, which will lend itself to a solution using matrices, which is called the Gauss-Jordan elimination method. But first, two systems of equations are called equivalent if they have the same (set of) solutions. We will see that the above system of equations is equivalent to, as the same solutions, as the system

$1x_1 + 0\,x_2 + 0x_3 = 1$
$0x_1 +1\,x_2 + 0x_3 = 1$
$0x_1 + 0\,x_2 + 1x_3 = 0$

Therefore we can "read off" the solutions directly from the above system, namely

$x_1 = 1$
$x_2 = 1$
$x_3 = 0$

The reader should check by substitution into the original system that these are indeed the solutions.

The method of reducing any system of equations to a simpler system where we can more easily "read off" the solutions is based on three simple rules which apply to any system of equations. These rules we incorporate in to the following Theorem.

Theorem 1. (Elementary Operations on Equations) If any sequence of the following operations is performed on a system of equations, the resulting system is equivalent to (has the same solutions as) the original system:

a) Interchange any two equations in the system.
b) Multiply both sides of any equation by a nonzero constant.
c) Multiply both sides of any equation by a nonzero constant and add the result to a second equation in the system, with the sum replacing the latter equation.

We will now apply the above Theorem to the original system given above. The original system is:

Example 1.

$x_1 + x_3 = 1$
$x_1 - x_2 = 0$
$2x_2 + x_3 = 2$

In order to get a clearer idea how the procedure works we will insert the "missing terms" and number the equations to obtain:

(1) $x_1 + 0\,x_2 + x_3 = 1$
(2) $x_1 - x_2 + 0x_3 = 0$
(3) $0x_1 + 2x_2 + x_3 = 2$

Multiply both sides of equation (1) by –1 and add the result to equation (2) to obtain:

(1) $x_1 + 0\,x_2 + x_3 = 1$
(2) $0x_1 - x_2 - x_3 = -1$ **<u>Note</u>**: Equation (1) did not change.
(3) $0x_1 + 2x_2 + x_3 = 2$

Multiply both sides of equation (2) by –1 to obtain:

(1) $x_1 + 0\,x_2 + x_3 = 1$
(2) $0x_1 + x_2 + x_3 = 1$
(3) $0x_1 + 2x_2 + x_3 = 2$

Multiply both sides of equation (2) by –2 and add the result to equation (3) to obtain:

(1) $x_1 + 0\,x_2 + x_3 = 1$
(2) $0x_1 + x_2 + x_3 = 1$ **Note**: Equation (2) did not change.
(3) $0x_1 + 0x_2 - x_3 = 0$

Multiply both sides of equation (3) by –1 to obtain:

(1) $x_1 + 0\,x_2 + x_3 = 1$
(2) $0x_1 + x_2 + x_3 = 1$
(3) $0x_1 + 0x_2 + x_3 = 0$

Multiply both sides of equation (3) by –1 and add the result to equation (1) to obtain:

(1) $x_1 + 0\,x_2 + 0x_3 = 1$
(2) $0x_1 + x_2 + x_3 = 1$ **Note**: Equation (3) did not change.
(3) $0x_1 + 0x_2 + x_3 = 0$

Multiply both sides of equation (3) by –1 and add the result to equation (2) to obtain:

(1) $x_1 + 0\,x_2 + 0x_3 = 1$
(2) $0x_1 + x_2 + 0x_3 = 1$ **Note**: Equation (3) did not change.
(3) $0x_1 + 0x_2 + x_3 = 0$

Therefore the solution to the system is: $x_1 = 1$, $x_2 = 1$ and $x_3 = 0$.

If you think about the step-by-step changes in the above equivalent systems the changes from system to system is in the numbers involved, that is, in the coefficients of the x's and the constants. The only purpose that the variables serve is to ensure that is that of keeping the coefficients (and the constants) in the appropriate location. We can effect this using matrices. We will write the original system in matrix notation as:

$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 2 \end{bmatrix}$ or $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 2 \end{array}\right]$. The only purpose of the vertical line in the latter version of this matrix is to separate the coefficients of the system from the constants for easier readability. Both ways of writing the matrix of the system are used. Since the first **three** columns of this matrix are the coefficients of the given system of equations the matrix consisting of the first three columns is called the **coefficient matrix**. That is, the

coefficient matrix is $\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.The "complete matrix" above, $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 2 \end{array}\right]$, is referred to as the **augmented matrix**. So one way of using the tool of matrices to solve systems of equations is to take Theorem 1 above and to replace the word equation by row and the word system by matrix, that is, another version of Theorem 1 is:

Theorem 1. (Elementary Row Operations) If any sequence of the following operations is performed on a matrix, the resulting matrix is equivalent to the original.

a) Interchange any two rows in the matrix.
b) Multiply any row of the matrix by a nonzero constant.
c) Multiply both sides of any row by a nonzero constant and add the result to a second row, with the sum replacing the latter row.

If we use the convention R_i to stand for row i of a matrix an the symbol $\longrightarrow$ to stand for row equivalent then A $\xrightarrow{cR_i+R_j}$ B means that the matrix B is obtained from the matrix A by multiplying the ith row of A by c and adding it to the jth row of A. Remember for our purposes here if two matrices are row equivalent then they represent equivalent systems of equations.

We now redo example 1 using matrices.

Example 1 revisited.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 2 \end{array}\right] \xrightarrow{(-1)R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 2 & 1 & 2 \end{array}\right]$$

__Note__: Row 1 (R_1) did not change. Step 1

$$\xrightarrow{(-1)R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 \end{array}\right]$$

Step 2

$$\xrightarrow{(-2)R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

__Note__: Row 2 (R_2) did not change Step 3

$$\xrightarrow{(-1)R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Step 4

$$\xrightarrow{(-1)R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$ **Note**: Row 3 (R_3) does not change. Step 5

$$\xrightarrow{(-1)R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$ Step 6

Note, one may prefer to use a different sequence of steps in solving the above.
Study the above sequence of steps in example 1 revisited. The **strategy** in solving a system of equations using matrices should become clear after studying this and the other examples below.

Strategy

1. Get a 1 in the first row first column by interchanging rows or dividing all of row 1 by some nonzero number. In example 1 we already have a 1.
2. Get 0's for the other entries in column 1. See step 1 above.
3. Get a 1 in row 2 column 2. See step 2 above.
4. Use the 1 in row 2 column 2 as a pivot to get 0's for the other entries in column 2. See step 3 above.
5. Get a 1 in row 3 column 3. See step 4 above
6. Use the 1 in row 3 column 3 as a pivot to get 0's for the other entries in column 3. See steps 5 and 6 above.

Example 2. Solve the system. Recall that this means that we want to find all real numbers x_1, x_2, and x_3 which will satisfy each equation in the system.

$$4x_1 + 2x_2 + x_3 = 1$$
$$2x_1 + x_2 + x_3 = 4$$
$$2x_1 + 2x_2 + x_3 = 3$$

The (augmented) matrix of the system is:

$\begin{bmatrix} 4 & 2 & 1 & 1 \\ 2 & 1 & 1 & 4 \\ 2 & 2 & 1 & 3 \end{bmatrix}$ or if you prefer inserting the vertical line $\left[\begin{array}{ccc|c} 4 & 2 & 1 & 1 \\ 2 & 1 & 1 & 4 \\ 2 & 2 & 1 & 3 \end{array}\right]$.

$$\begin{bmatrix} 4 & 2 & 1 & 1 \\ 2 & 1 & 1 & 4 \\ 2 & 2 & 1 & 3 \end{bmatrix} \xrightarrow{\frac{1}{4} R_1} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 2 & 1 & 1 & 4 \\ 2 & 2 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{-2 R_1 + R_2} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & 7/2 \\ 2 & 2 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{-2 R_1 + R_3} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & 7/2 \\ 0 & 1 & \frac{1}{2} & 5/2 \end{bmatrix}$$

$$\xrightarrow[R_2 \text{ and } R_3]{\text{Interchange}} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & \frac{1}{2} & 5/2 \\ 0 & 0 & \frac{1}{2} & 7/2 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2} R_2 + R_1} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & \frac{1}{2} & 5/2 \\ 0 & 0 & \frac{1}{2} & 7/2 \end{bmatrix}$$

$$\xrightarrow{2 R_3} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & \frac{1}{2} & 5/2 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2} R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

We now write in the variables and the equality symbols to obtain the system:

$$x_1 + 0x_2 + 0x_3 = -1$$
$$0x_1 + x_2 + 0x_3 = -1$$
$$0x_1 + 0x_2 + x_3 = 7$$

and read off the solution to the original system as $x_1 = -1$, $x_2 = -1$ and $x_3 = 7$.

I encourage the reader to substitute these values in the system to verify that they are indeed
the solutions to the given system of equations.

Example 3. Solve the system $1x_1 + 2x_2 = 1$
$2x_1 + x_2 = 4$

Recall that this means that we want to find all real numbers $x_1$and x_2 which will satisfy each equation in the system.
Using matrices we have

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & 4 \end{array}\right] \xrightarrow{(-2)R_1 + R_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & 2 \end{array}\right]$$
Obtain a 0 in the 2nd row 1st column

$$\xrightarrow{(-\frac{1}{3})R_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -\frac{2}{3} \end{array}\right]$$
Obtain a 1 in the 2nd row 2nd column

$$\xrightarrow{(-2)R_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & \frac{7}{3} \\ 0 & 1 & -\frac{2}{3} \end{array}\right]$$
Obtain a 0 in the 1st row 2nd column

So the solution is $x_1 = \frac{7}{3}$ and $x_2 = -\frac{2}{3}$

Now solve the system,
$1x_1 + 2x_2 = 2$
$2x_1 + x_2 = 3$
Note we can use the same steps that we used in solving the above system because the coefficient matrix is the same

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & 1 & 3 \end{array}\right] \xrightarrow{(-2)R_1 + R_2} \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & -3 & -1 \end{array}\right]$$

$$\xrightarrow{(-\frac{1}{3})R_2} \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 1 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{(-2)R_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & \frac{4}{3} \\ 0 & 1 & \frac{1}{3} \end{array}\right]$$

So the solution is $x_1 = \frac{4}{3}$ and $x_2 = \frac{1}{3}$

For additional examples google “Using matrices to Solve Systems of Equations”.

Another much faster way of solving the two systems in example 3

If we knew ahead of time that we wanted to solve the above two systems of equations and we noticed that they had the same coefficient matrix we could save considerable time by augmenting the coefficient matrix by, not one column of constants as above but by both columns of constants. That is, we could solve both systems of equations simultaneously as below.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{array}\right] \xrightarrow{(-2)R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 0 & -3 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{(-\frac{1}{3})\,R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 0 & 1 & \frac{-2}{3} & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{(-2)R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{7}{3} & \frac{4}{3} \\ 0 & 1 & \frac{-2}{3} & \frac{1}{3} \end{array}\right]$$

Note, this gives us the solutions of both systems of equations as described earlier.

Example 4. Solve the two systems of equations "simultaneously" as in the second half of example 3. Note, for simplicity I kept the same coefficient matrix as in example 3.

$$\begin{aligned} 1x_1 + 2x_2 &= 1 \\ 2x_1 + \ \ x_2 &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 1x_1 + 2x_2 &= 0 \\ 2x_1 + \ \ x_2 &= 1 \end{aligned}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{(-2)R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{(-\frac{1}{3})\,R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{-1}{3} \end{array}\right]$$

$$\xrightarrow{(-2)R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{-1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & \frac{-1}{3} \end{array}\right]$$

So the solution of the first system are the values in the 3rd column, namely,
$x_1 = \frac{-1}{3}$ and $x_2 = \frac{2}{3}$

The solution to the second systems are the numbers in the 4th column namely,
$x_1 = \frac{2}{3}$ and $x_2 = \frac{-1}{3}$

Example 5. Consider the coefficient matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ above. If we used the formula for finding the inverse of A we would obtain $A^{-1} = \begin{bmatrix} \frac{-1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{bmatrix}$. Please verify this.

The **definition** of the inverse of any n x n matrix is: Given an n x n A if there exists an n x n matrix B such that AB = BA = I then B is the inverse of the matrix A and it is denoted by the symbol A^{-1} which is read A inverse. If we let A = $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and take B or A^{-1} as $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$. The equation AB = I becomes $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} .= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Multiply the left side of this and equate both sides of the result and you will obtain 2 systems of two equations 2 unknowns. In fact, you will obtain the 2 systems in example 4 (different variable names). These systems we solved by using the matrix $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$ and row reducing it to $\left[\begin{array}{cc|cc} 1 & 0 & \frac{-1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & \frac{-1}{3} \end{array}\right]$. This gives us $A^{-1} = \begin{bmatrix} \frac{-1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{bmatrix}$.

This works for any n x n matrix whose inverse exists. WOW!

That is, if we want to find the inverse (if it exists) of any n x n matrix A. Take A, augment it by the matrix I then row reduce that to "I" augmented by "what it becomes" and "what it becomes" will be A^{-1}. In symbols

$$[A \mid I] \xrightarrow{\text{row reduce}} . \ . \ . \xrightarrow{\text{row reduce}} [I \mid A^{-1}]$$

More example of find the inverse of a matrix this way are at the end of week 7, or just "google it".

Remark

Each system of equations will (usually) have its own set of solutions. The purpose of example 4 and exercises 3 & 4 of the notes is to show that since the coefficient matrix of exercise 3 and that of 4 are the same the same elementary row operations could be used to solve each system. So if we were given 2 systems with the same coefficient matrix instead of solving them separately we could save time by solving them together by augmenting the coefficient by not one but 2 columns. Then use the usual process to row reduce the matrix. If all goes well the numbers in the first (added) column become the solution of the first system and those in the second (added) column become the solutions of the second system. Exercise 5 is an example where you can do this. One intent of the discussion is to lead people to thinking about what exercise 6 means and eventually to why the method of finding the inverse of a matrix (see example 5) in the next set of notes works.

So the matrix of problem six is really the matrix for solving 3 systems of 3 equations and 3 unknowns.

A key part of the definition of the inverse of a matrix A is to find a matrix B such that AB = I. If A is the 3x3 coefficient matrix given in problem 6, and if B (of the definition of inverse) is a 3x3 matrix of variables (since we are looking for B). Then AB = I becomes 3 systems of 3 equations and 3 unknowns, all with the same coefficient matrix. So we can solve all 3 systems simultaneously. The matrix of the 3 systems is that of problem 6. So if you solve problem 6 you are really finding the inverse (of the coefficient matrix).

Exercises (It is important that you do each of the following exercises.)

1. Write the systems of equations that the following matrices represent.

a) $\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 2 \end{array}\right]$ b) $\left[\begin{array}{ccc|c} 4 & 2 & 1 & 1 \\ 2 & 1 & 1 & 4 \\ 2 & 2 & 1 & 3 \end{array}\right]$

c) $\left[\begin{array}{cc|c} 2 & 1 & 4 \\ 1 & -1 & 1 \end{array}\right]$ d) $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 1 & 2 & 1 & -1 \end{array}\right]$ e) $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 1 & 2 & 1 & 0 \end{array}\right]$

2. Solve each of the systems of equations parts a through c in exercise 1 using the Gauss-Jordan technique. **Verify** your solutions by substituting your solutions in the original equations. Note, you will be able to solve 1(d) and 1(e) after you study the next section, “Systems of Equations Which Do Not Have A Unique Solution” .

3. Solve the following system using the Gauss- Jordan technique. **Verify** your solution by substitution in the original system.

$$x_1 + x_3 = 1$$
$$x_1 - x_2 = -1$$
$$2x_2 + x_3 = 3$$

4. Could you have used the same steps in doing exercise 3 that we used in example 1? Why?

5. Can you solve the following two systems of equations simultaneously using the **same** matrix? (Hint: Augment the coefficient matrix by **2** columns.)

$x_1 + x_2 = 0$
$-x_1 + x_2 + x_3 = 1$
$-1x_2 + x_3 = 2$

and

$x_1 + x_2 = 1$
$-x_1 + x_2 + x_3 = -1$
$-1x_2 + x_3 = 3$

6. The matrix $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$ can be viewed as a matrix which allows one to solve three systems of three equations and three unknowns. Write out the three systems of equations and use the given matrix to solve the three systems simultaneously.

7. Look up the **definition** (not the formula) for the inverse of a matrix. What does exercise 6 give us?

Part 2: Systems of Equations Which Do Not Have A Unique Solution

On the previous pages we learned how to solve systems of equations using Gaussian elimination. In each of the examples and exercises (except for execise 1 parts d and e) the systems of equations had a **unique** solution. That is, a single value for each of the variables. In 2-space, the xy-plane, we have the geometric bonus of being able to draw a picture of the solutions to a system of two equations two unknowns. Clearly, if we were asked to draw the graphs of two lines in the xy-plane we have 3 basic choices:

1. Draw the two lines so they intersect. This point of intersection can only happen once for a given pair of lines. That is, the two lines intersect in a **unique** point. There is a unique common solution to the system of equations.

2. Draw the two lines so that one is on "top of" the other. In this case there are an infinite number of common points, an infinite number of solutions to the given system.

3. Draw two parallel lines. In this case there are no points common to both lines. There is no solution to the system of equations that describe the lines.

The 3 cases above apply to any system of equations.

Theorem 1. For any system of m equations with n unknowns ($m \leq n$) one of the following cases applies:

1. There is a unique solution to the system.
2. There is an infinite number of solutions to the system.
3. There are no solutions to the system.

Again, in this section of the notes we will illustrate cases 2 and 3. To solve systems of equations where these cases apply we use the matrix procedure developed previously.

Example 6. Solve the system

$$\begin{aligned} x + 2y &= 1 \\ 2x + 4y &= 2 \end{aligned}$$

It is probably already clear to the reader that the second equation is really the first in disguise. (Simply divide both sides of the second equation by 2 to obtain the first). So if we were to draw the graph of both we would obtain the same line, hence have an infinite number of points common to both lines, an infinite number of solutions. However it would be helpful in solving other systems where the solutions may not be so apparent to do the problem algebraically, using matrices. The matrix of the system with its simplification follows. Recall, we try to express the matrix $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4 & 2 \end{array}\right]$ in the form $\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 \end{array}\right]$ from which we can read off the solution. However after one step we note that $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4 & 2 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$. It should be clear to the reader that no matter what further elementary row operations we perform on the matrix $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$ we cannot change it to the form we hoped for, namely, $\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 \end{array}\right]$. To understand what our result means simply write the system of equations that the matrix $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$ represents, that is,

$$\begin{aligned} x + 2y &= 1 \\ 0x + 0y &= 0. \end{aligned}$$

The first equation tells us that x = 1 - 2y (or equivalently y = 1/2(1 - x)), so that (1,0), (-1,1), (-5,3) etc. are three of the infinite number of possible solutions of the first equation. The second equation places no restrictions on what values x and y can assume, hence there are an infinite number of solutions to **both** equations, to the system. Any pair of real numbers of the form (1 - 2y,y) where y can be any real number is a solution to the given system of equations. The solutions of the system can also be expressed in the form (x,1/2(1 - x) where x can be any real number.

Warning. When there are an infinite number of solutions to a system there are frequently several "different-looking" ways to describe the solutions, as in the above example.

Example 7. Determine the solutions of the system of equations whose matrix is row equivalent to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Give three examples of the solutions.

If we use the variables x_1, x_2, and x_3 the system of equations which is represented by this matrix is $x_1 = 1$, and $x_2 - x_3 = 0$ (or $x_3 = x_2$)

There are an infinite number of solutions and any triple of the form $(1, x_2, x_2)$ where x_2 can be any real number is a solution. (1,1,1), (1,3,3) and (1,-1,-1) are three examples of solutions.

Example 8. Solve the system of equations whose matrix is $\left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -3 & 2 & -1 \end{array}\right]$. Give three examples of the solutions.

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -3 & 2 & -1 \end{array}\right] \xrightarrow[\text{and then } -1R_1]{\text{interchange rows 1 and 2}} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -1 & 0 & 1 \\ 0 & -3 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 3 & -2 & 1 \\ 0 & -3 & 2 & -1 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

At this point we could row-reduce the last matrix further by but this is really not necessary. If we call the variables x_1, x_2 and x_3 the system of equations that this last matrix represents is:

$$x_1 - 2x_2 + x_3 = 0$$
$$3x_2 - 2x_3 = 1.$$

From the latter equation we can say $x_3 = ½ (3x_2 - 1)$. If we substitute this expression for x_3 in the first equation we obtain $x_1 - 2x_2 + ½ (3x_2 - 1) = 0$ or $x_1 = 2x_2 - ½ (3x_2 - 1)$ which can be simplified to $x_1 = ½(x_2 + 1)$. If we replace x_2 by 0 then one solution is $x_1 = 1/2$, $x_2 = 0$ and $x_3 = -1/2$. Another solution is (3/2, 2, 5/2). Why? We ask the reader to substitute these solutions into the original system to verify that they are solutions, and to find two more solutions.

Another situation that one encounters is that when the number of unknowns is greater than the number of equations. For example: $x_1 + x_2 - 3x_3 = -1$
$x_2 - x_3 = 0.$

But upon closer inspection this is simply another form of the above examples . We illustrate as follows

Example 9. Solve the system of equations

$$x_1 + x_2 - 3x_3 = -1$$
$$x_2 - x_3 = 0.$$

This system is the same as the system

$x_1 + x_2 - 3x_3 = -1$
$0x_1 + x_2 - x_3 = 0$
$0x_1 + 0x_2 + 0x_3 = 0,$

So we can represent the above system by the matrix $\left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 0 \end{array}\right]$ or the matrix $\left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Clearly no matter what elementary row operations we perform on this matrix we cannot change it to the form $\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$. For this reason it is common practice to rewrite the matrix without the rows which contain all zeros, so that the matrix of the given system is $\left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 0 \end{array}\right]$ and the reader can show that this matrix can be reduced to $\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & -1 & 0 \end{array}\right]$ which gives us the system

$x_1 - 2x_3 = -1$
$x_2 - x_3 = 0$

From the second equation, $x_3 = x_2$

Substitute this value of x_2 in first equation to obtain

$$x_1 = 2x_2 - 1$$

So all solutions of the given system are ordered triples of the form $(2x_2 – 1, x_2, x_2)$ where x_2 takes on all real numbers. Examples of solutions are:
$x_2 = 0$ gives (-1, 0, 0) as an example of one solution and $x_3 = 2$ gives (3, 2, 2) as another solution.

Keep in mind that there are often many different ways to describe the solutions of the same system. For example, I claim that the solutions of the system given in this example 9 can also be described as any ordered triple of the form $(x_1, ½(x_1+ 1), ½(x_1+ 1))$ where x_1 is any real number. So if $x_1= 1$ we obtain the triple (1, 1, 1) as a solution, which certainly satisfies the original system of equations. To obtain my form of the solution set just take the equation $x_1 = 2x_2 – 1$ and solve for x_2 and then for x_3.

Exercises

1. Determine the solutions of the system of equations whose matrix is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Give three examples of the solutions. Verify that your solutions satisfy the original system of equations.

2. Determine the solutions of the system of equations whose matrix is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Give three examples of the solutions. Verify that your solutions satisfy the original system of equations.

3. Determine the solutions of the system of equations whose matrix is row equivalent to

$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Give three examples of the solutions. Verify that your solutions satisfy the original system of equations.

4. Determine the solutions of the system of equations whose matrix is row equivalent to $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Give three examples of the solutions. Verify that your solutions satisfy the original system of equations.

5. Determine the solutions of the system of equations whose matrix is row equivalent to $\begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Give three examples of the solutions. Verify that your solutions satisfy the original system of equations.

Part 3: Network Analysis

One application of systems of linear equations is an area sometimes referred to as network analysis. I will give you a number of different applications, many from students, to give you an idea of the variety of these types of problems.
You should be prepared to make up a meaningful example like the following.

Directed graphs can be used as models for a variety of situations in disciplines such as: computer science, traffic analysis, electrical engineering and economics. A directed graph is also referred to as a **network** and the analysis of the given problem as **network analysis**. Directed graphs are composed of **nodes** (also called vertices or junctions), and directed **edges**. In the graph below, figure 1, the nodes are the "circles" labeled 1,2,3 and 4. "Dots", with labels, are also used in place of circles. The edges are the "arrows" in the graph.

We assume in the following flow examples that the sum of the flow into any (intermediate) vertex is equal to the sum of the flow out of that vertex. Below are several examples.

Example 1.

The flow of traffic (in the number of vehicles per hour) through a network of streets is shown in the figure 1:
Solve this system for x_i, i = 1, 2, 3, 4.
Find the traffic flow when $x_3 = 0$
Find the traffic flow when $x_3 = 100$

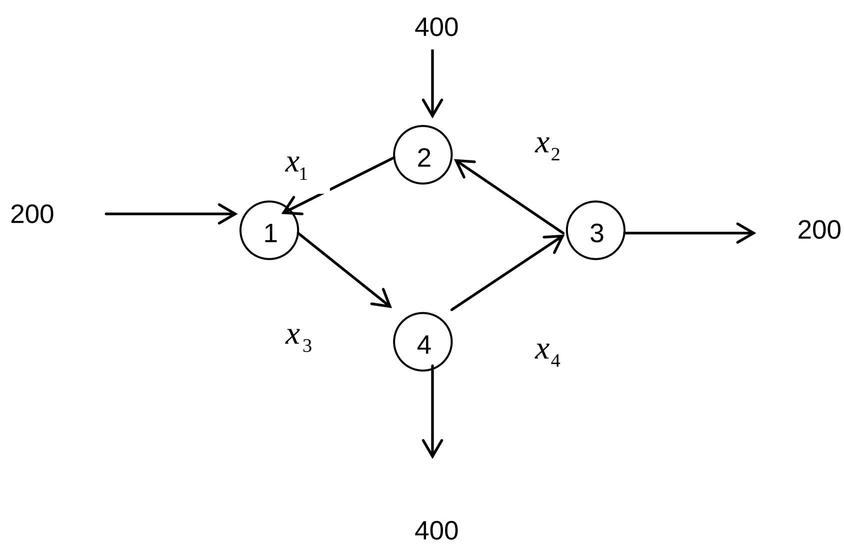


FIGURE 1

Solution
a) From figure 1, we have the following linear equations:

Junction 1: $x_1 + 200 = x_3$
Junction 2: $x_2 + 400 = x_1$
Junction 3: $x_2 + 200 = x_4$
Junction 4: $x_4 + 400 = x_3$

The augmented matrix for this system is:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -200 \\ -1 & 1 & 0 & 0 & -400 \\ 0 & 1 & 0 & -1 & -200 \\ 0 & 0 & -1 & 1 & -400 \end{array}\right]$$

Gauss-Jordan elimination produces the matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 200 \\ 0 & 1 & -1 & 0 & -600 \\ 0 & 0 & 1 & -1 & 400 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix gives us the following equations:
$x_1 - x_4 = 200$
$x_2 - x_3 = -600$
$x_3 - x_4 = 400$

Note you could have obtained these equations directly from the given system of equations.

So
$x_1 = 200 - x_4$
$x_2 = -200 + x_4$ Why?
$x_3 = 400 + x_4$. Since x_4 (and therefore the other variables) stands for a number of vehicles it must be an integer. Can it be negative? Zero?

Another way of describing the solutions is to use a dummy variable, say s as used below.

Let $x_4 = s$ we have:

$$x_1 = 200 + s \quad x_2 = -200 + s \quad x_3 = 400 + s$$

where s is any real number. Thus this system has an infinite number of solutions. (More about this later.) **Note** the solutions could have been described without using s (see the above).

b) If $x_3 = 0$ then $0 = 400 + s$ so that $x_4 = s = -400$. This tells us that $x_1 = -200$, $x_2 = -600$. *What this means is when the traffic flow along street x_3 is restricted to 0, that is, when this street is closed the flow along the other streets is: $x_1 = -200$, $x_2 = -600$ and $x_4 = -400$. Which of course does not make any sense. Why?*

If: $x_3 = 100$ then $100 = 400 + s$ so that Systems of Equations Which Do Not Have A Unique Solution $= s = -300$. This tells us that $x_1 = -100$, $x_2 = -500$ and $x_4 = -300$, which still does not make any sense.

Can you determine the minimum value of s (or x_4) which will make sense in this problem? That is, what is the smallest value of x_4 which will make the other variables nonnegative?

In addition, for this problem to be realistic we should have a maximum capacity for each edge. That is, each road or edge cannot handle an infinite number of cars per hour. The number of vehicles that can travel along a road is determined by several factors, for example: the width of the road, its surface (how smooth etc.), and how straight it is, speed limit etc. So there are a maximum number of cars per hour that can travel along each road. You should make up a realistic maximum capacity for each edge.

Example 2

The flow of traffic (in vehicles per hour) through a network of streets is shown in the figure 3:
Solve this system for x_i, i = 1, 2,..,5.
Find the traffic flow when $x_3 = 0$ and $x_5 = 50$
Find the traffic flow when $x_3 = x_5 = 50$

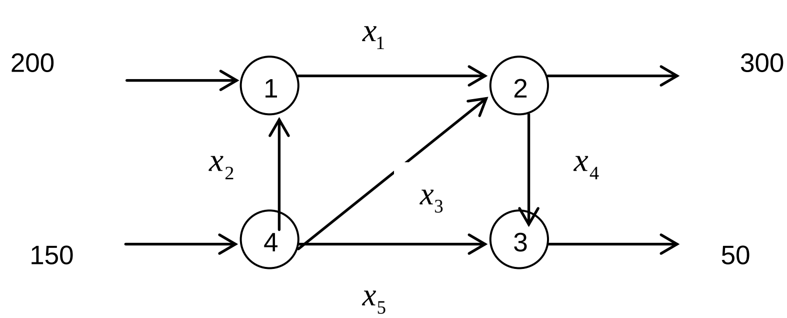


FIGURE 3

Solution
a) From figure 3, we have the following linear equations:

Junction 1: $x_2 + 200 = x_1$

Junction 2: $x_1 + x_3 = 300 + x_4$

Junction 3: $x_4 + x_5 = 50$

Junction 4: $x_2 + x_3 + x_5 = 150$

The augmented matrix for this system is:

$$\left[\begin{array}{ccccc|c} -1 & 1 & 0 & 0 & 0 & -200 \\ 1 & 0 & 1 & -1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 50 \\ 0 & 1 & 1 & 0 & 1 & 150 \end{array}\right]$$

Gauss-Jordan elimination produces the matrix

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & -200 \\ 0 & 1 & 1 & -1 & 0 & 100 \\ 0 & 0 & 0 & 1 & 1 & 50 \\ 0 & 0 & 0 & 0 & 0 & 00 \end{array}\right]$$

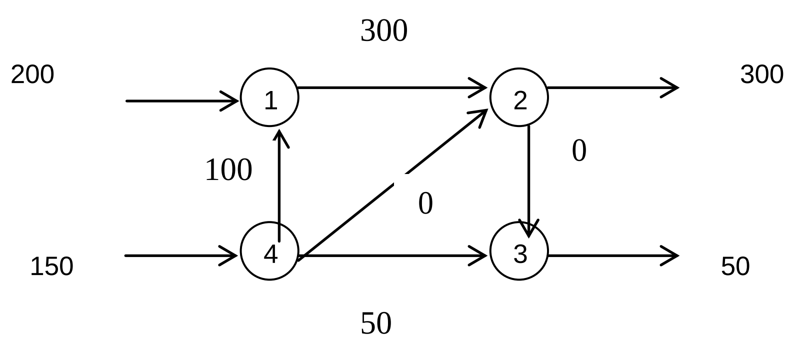
If we let $x_5 = s$ and $x_3 = t$ we have:

$$x_1 = 350 - s - t \quad x_2 = 150 - s - t \quad x_4 = 50 - s$$

where s and t are integers.

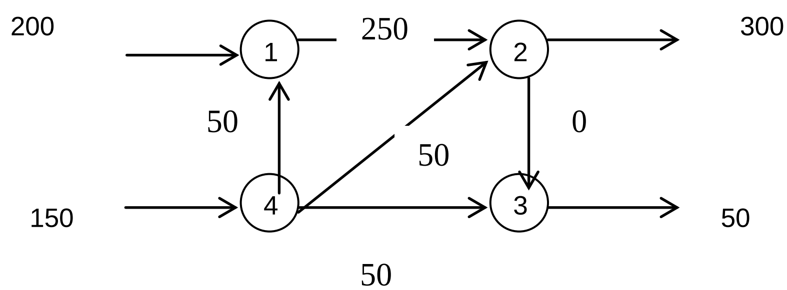
b) For an example if we let: t or $x_3 = 0$ and let s or $x_5 = 50$

we obtain for the other variables: $x_1 = 300$, $x_2 = 100$ and $x_4 = 0$



c) If we let: $x_3 = x_5 = 50$

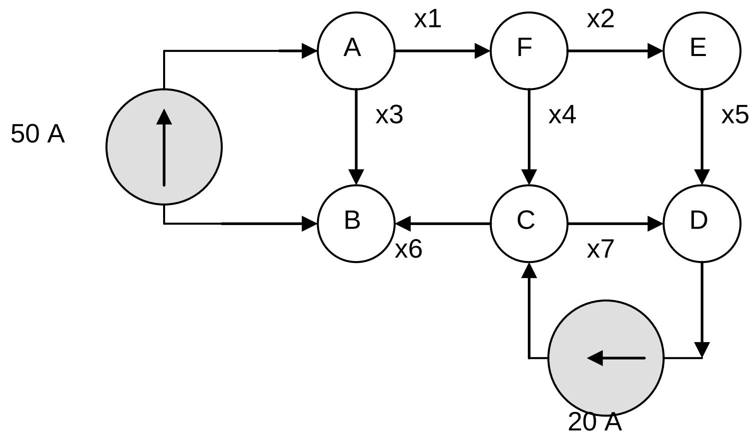
$\Rightarrow t = s = 50$



Follow the same rationale we did on example 2 so that the results of this problem make sense in the "real world".

Example 4. A student example (good if you know electric circuits).

A particular circuit has current sources connected across ports ab and cd. If Iac=50 Amps and Icd=20 amps the flow of current in the circuit is demonstrated in the figure.



With the sources fixed at their particular values. Find the flow between each node when the path between F & E is an open circuit ($x_2 = 0$) and when the path is a short circuit at 10 amps.

By Kirchoff's current law, the sum of the voltages entering a node is equal to that leaving the node, so: (NOTE: Two of the following equations are wrong. Which two? Check the flow equations for nodes C and D.)

$50 = x_1 + x_3$
$50 = x_3 + x_6$
$x_1 - x_4 = x_2$
$x_2 = x_5$
$x_5 = x_7 + 20$
$20 = x_4 + x_4$

The augmented matrix for this system is:

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 50 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 50 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 20 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 20 \end{array}\right]$$

Gauss-Jordan elimination produces:

$$\left[\begin{array}{ccccccc|c} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 50 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -20 \end{array}\right]$$

In other words

$x_3 = 50$
$x_6 = 0$
$x_1 = 0$
$x_2 + x_4 = 0$
$x_4 + x_5 = 0$
$x_3 + x_3 = -20$

This implies that the current from the 50A source follows the shortest path possible and does not go over to the other side of circuit. For the open circuit case $x_2 = 0$ This results in:

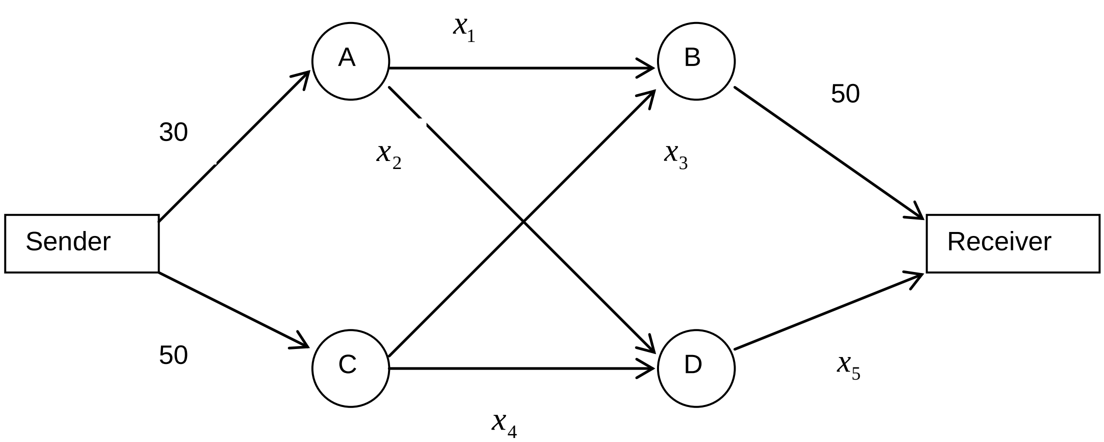
$x_4 = 0$, $x_5 = 0$, and $x_7 = -20$ as all the current flows between C and D.

When the path x_2 permits 10 Amps the results become:

$x_4 = -10$, $x_5 = 10$, and $x_7 = -10$ this is an example of a current division as half flows through one path and the remaining current flows through another.

Example 5.

The internet is a network of servers; servers receive requests of information from client PCs (i.e. individual's PCs), or send the requested information to the client PCs. A person (the sender) sends a large file over the internet to second person (the receiver). Before the file is sent, it is split into small packets. Then the packets are routed through the network using routers to avoid congestion; routers are set up to regulate the flow of data. In this problem, the network contains four routers named A, B, C, and D. The bandwidth of the network is measured in megabytes per second (Data is represented in binary inside a computer, 0 or 1, which is called a bit. A unit of byte contains of 8 bits. A unit of kilobyte contains 1024 bytes. A unit of megabytes contains 1000 kilobytes). Suppose the file contains 80 megabytes of data, which are sent to the routers A and C as illustrated below. The complete network is illustrated below:



According to the above figure, the 80-megabytes file is split into 2 packets, 30-megabytes and 50-megabytes. The two packets are then routed through the routers, and split into more packets. At the routers B and D, these small packets are combined and sent to the receiver. Note that the received file must contain exactly 80 megabytes of data such as when it is sent by the sender.

Solve this system the data flow represented by x_i, i = 1, 2, ..5.

Find the network flow pattern when $x_2 = 0$

Find the network flow pattern when $x_2 = 30$

Solution

a) From the figure , we have the following linear equations:

Junction A: $x_1 + x_2 = 30$
Junction B: $x_1 + x_3 = 50$
Junction C: $x_3 + x_4 = 50$
Junction D: $x_2 + x_4 = x_5$
Junction RECEIVER: $x_5 + 50 = 80$

The augmented matrix for this system is:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 30 \\ 1 & 0 & 1 & 0 & 0 & 50 \\ 0 & 0 & 1 & 1 & 0 & 50 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 30 \end{array}\right]$$

Gauss-Jordan elimination produces the matrix:

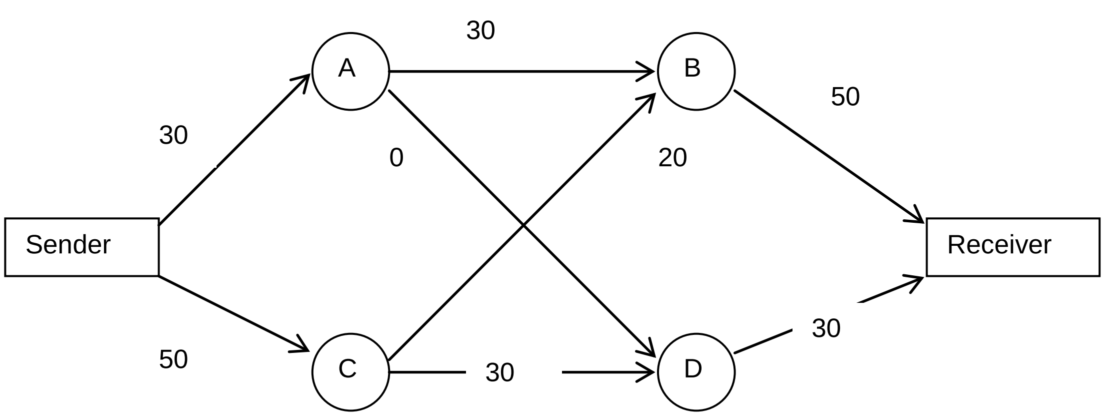
$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 30 \\ 0 & 1 & 0 & 1 & 0 & 30 \\ 0 & 0 & 1 & 1 & 0 & 50 \\ 0 & 0 & 0 & 0 & 1 & 30 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_4 = s$ we have:

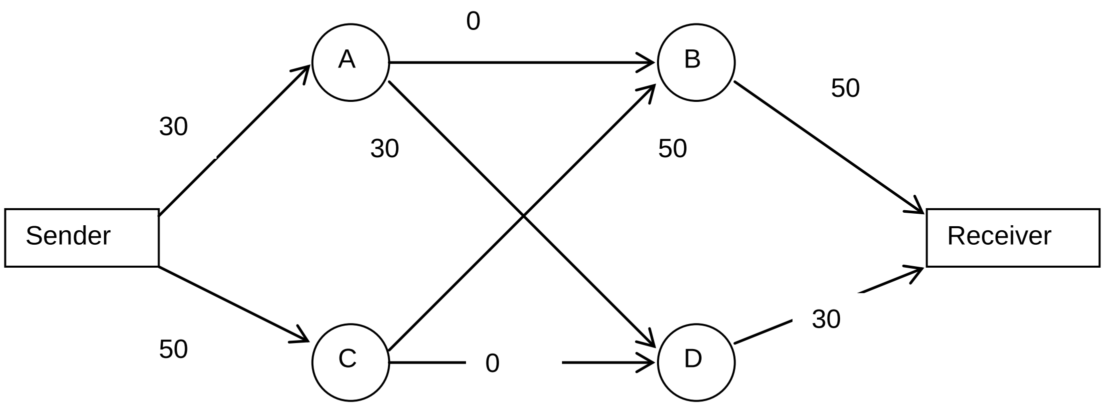
$$x_1 = s \quad x_2 = 30 - s \quad x_3 = 50 - s \quad x_5 = 30$$

where s is real numbers. Thus this system has an infinite number of solutions.

b) If $x_2 = 0$, then we have the following graph:

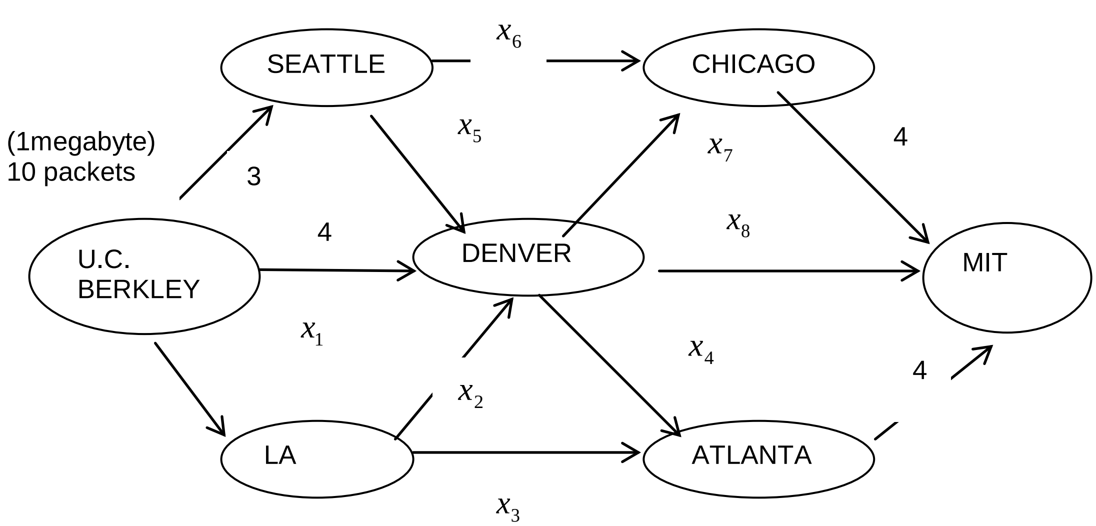


d) If $x_2 = 30$, then we have the following graph:



Example 6.

Now, we understand the concept of how a file is sent through the network of internet. We can consider the next problem. A student at MIT wants to download a 1-megabyte graphic computer file located on a server at U. C. Berkley in Oakland, CA. If the file is sent out as a sequential data stream, it will take too long. So the file is broken up into ten 100-kilobytes "packets" and routed via multiple servers located in different cities around the country, and reassembled at MIT into one file.



Solution

a) From the figure, we have the following linear equations:

Junction BERKLEY: $3+4+x_1 = 10$
Junction SEATTLE: $x_5 + x_6 = 3$
Junction CHICAGO: $x_6 + x_7 = 4$
Junction LA: $x_2 + x_3 = x_1$
Junction ATLANTA: $x_3 + x_4 = 4$
Junction DENVER: $x_2 + x_5 + 4 = x_4 + x_7 + 2$
Junction MIT: $4 + x_8 + 4 = 10$

The augmented matrix for this system is:

$$\left[\begin{array}{cccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 4 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Gauss-Jordan elimination produces the matrix:

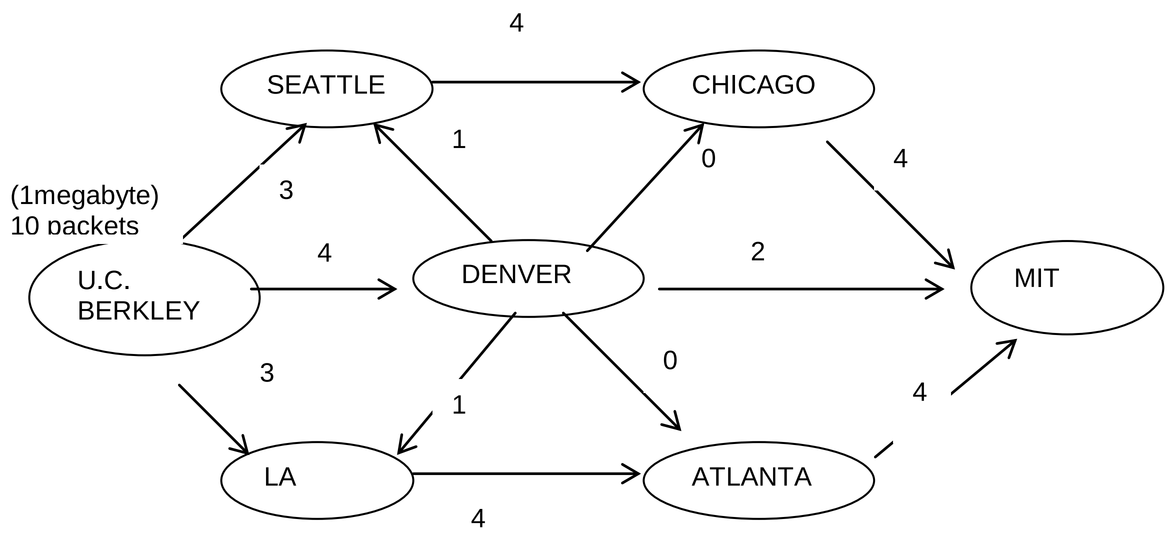
$$\left[\begin{array}{cccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_4 = t$ and $x_7 = s$, we have:

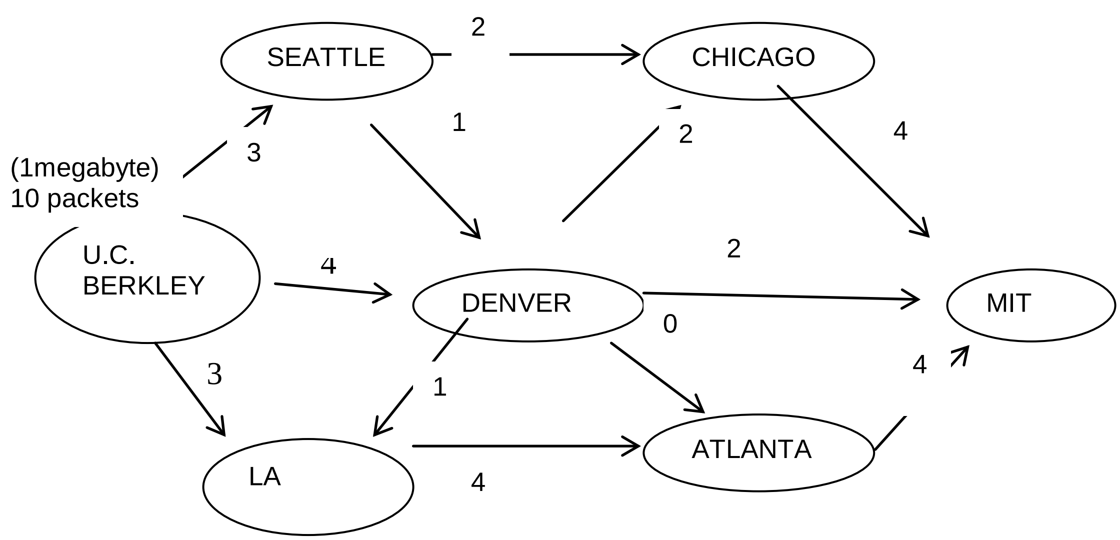
$$x_1 = 3 \quad x_2 = -1+t \quad x_3 = 4-t \quad x_5 = -1+s \quad x_6 = 4-s \quad x_8 = 2$$

where s and t are real numbers. Thus this system has an infinite number of solutions.

b) If $x_4 = x_7 = 0$, then we have the following graph:

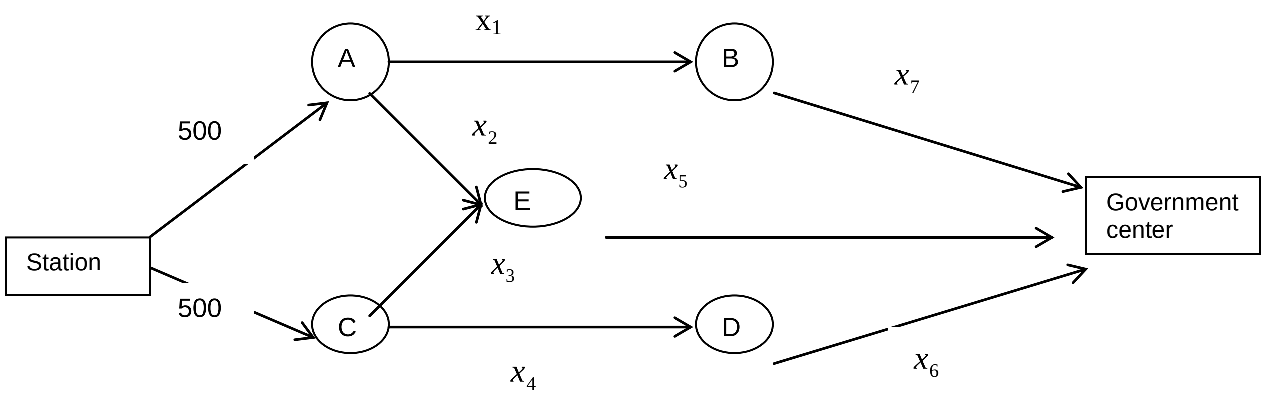


c) If $x_3 = 4$ and $x_6 = 2$, then we have the following graph:



Example 7.

A train carries 1000 people into Boston. Once there everybody must immediately get to the Government Center to go to work. There are several different subways, which go to the Government Center. The network of the subways in this problem is illustrated below:



Let the nodes A, B, C, D, and E represent the different subways, A, B, C, D, and E.

Solution

a) From the figure, we have the following linear equations:

Junction A:	$x_1 + x_2 = 500$
Junction B:	$x_1 = x_7$
Junction C:	$x_3 + x_4 = 500$
Junction D:	$x_4 = x_6$
Junction E:	$x_2 + x_3 = x_5$

The augmented matrix for this system is:

$$\left[\begin{array}{ccccccc|c} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 500 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 500 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \end{array}\right]$$

Gauss-Jordan elimination produces the matrix:

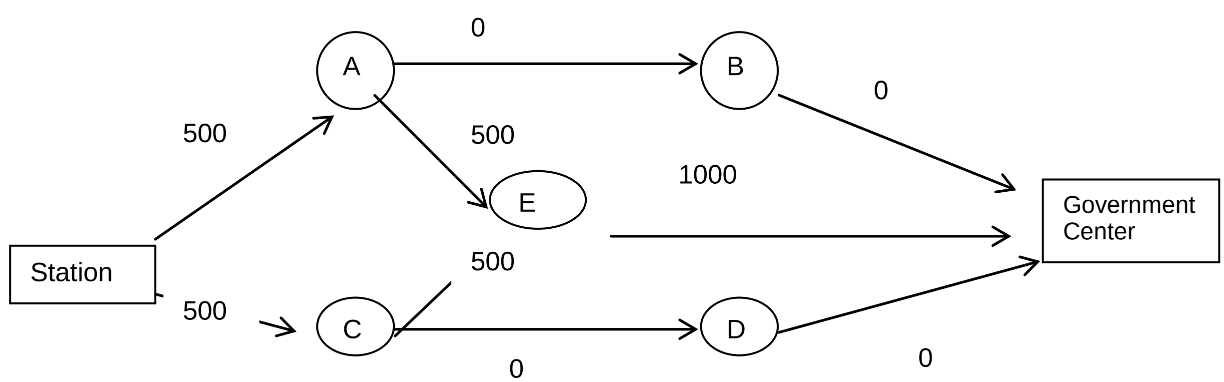
$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 500 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 500 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1000 \end{array}\right]$$

Let $x_6 = s$ and $x_7 = t$, we have:

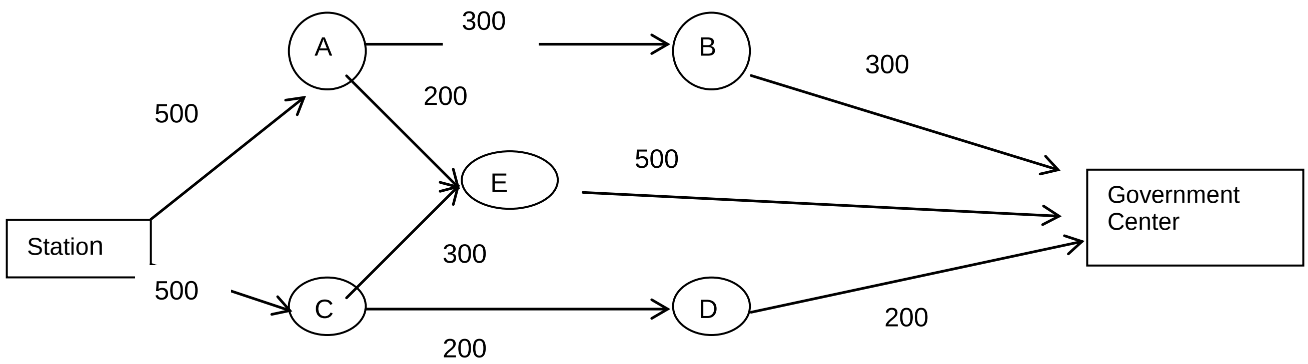
$$x_1 = t \quad x_2 = 500 - t \quad x_3 = 500 - s \quad x_4 = s \quad x_5 = 1000 - s - t$$

where s and t are real numbers. Thus this system has an infinite number of solutions.

b) If $x_6 = 0$ and $x_7 = 0$, then we have the following graph:

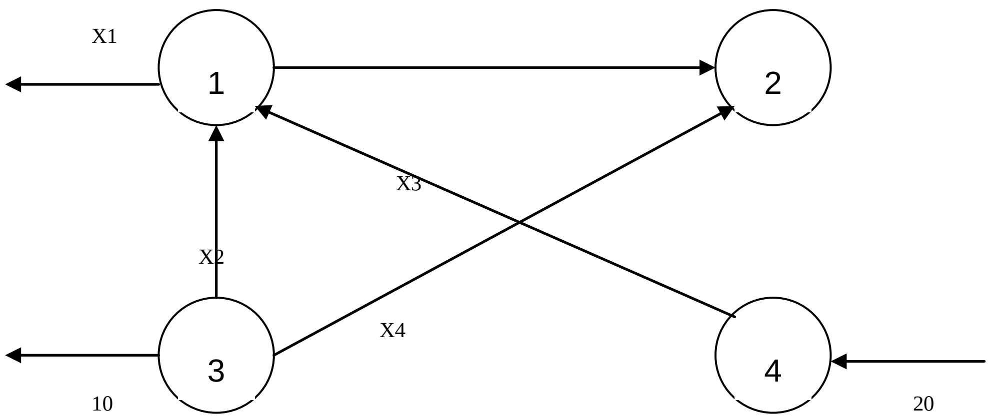


If $x_6 = 200$ and $x_7 = 300$ then we have the following graph:



Example 8.

RAVMAK Corporation developed a DSP (digital signal processor) that can translate voice over IP (Internet protocol) signals. The DSP processor can take a data signal, a fax in, and a voice input from one port and can send it out over IP. The way the processor works is it has internal currents that translate analog to digital signals, and digital to analog signals. The processor takes these currents at different frequencies and combines them and translates them out back to the port, and then sends it out over IP. Within the processor there are 4 digital to analog converters. These converters handle the entire analog to digital conversions. These converters are labeled 1, 2, 3, and 4 on the flow chart. The current coming into the processor is 20 mA at converter 4, and the current that leaves converters 1 and 3 is 10 mA. The currents combine at converters 1 and 3 to the total current that originally entered the circuit that was 20 mA. There is one limitation to the circuit design, it has a limiting factor of circuit, and the circuit can only handle a maximum of 25 mA per node, so the input and combination of currents at any node can never exceed 25 mA. The complete network is illustrated below:



Just as a second reminder the DSP circuit cannot exceed 25mA per a node and the total input at converter 4 must equal to sum of the outputs of converters 1 and 3.

A.) Solve this system for current flow for Xi, i = 1, 2, 3, 4.

<u>Solution:</u>

From the figure, we have the following linear equations:

Converter 1: X2 + X3 – X1 = 10

Converter 2: X1 + X4 = 0

Converter 3: X2 + X4 = -10

Converter 4: X3 = 20

The augmented matrix for this system is:

$$\left(\begin{array}{cccc|c} -1 & 1 & 1 & 0 & 10 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 10 \\ 0 & 0 & 1 & 0 & 20 \end{array}\right)$$

Gauss-Jordan elimination produces the matrix:

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 10 \\ 0 & 0 & 1 & 0 & 20 \end{array}\right)$$

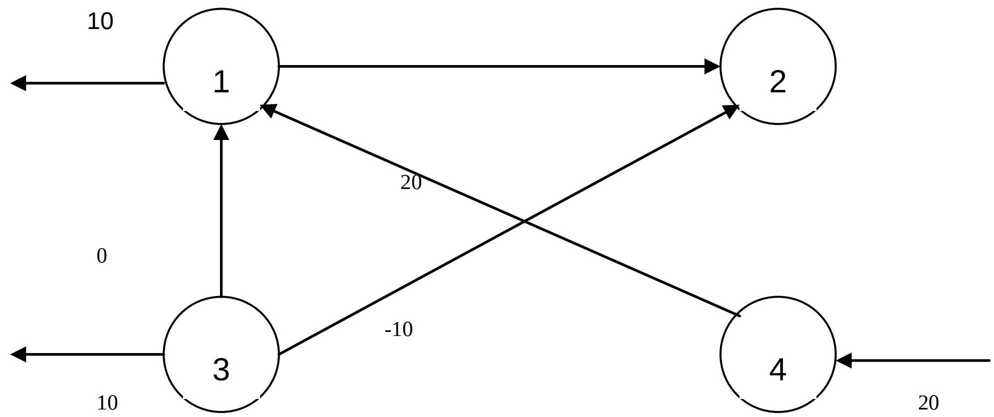
Let X4 = s we have:

X1 = -s   -X2 – 10 = s   X3 = 20

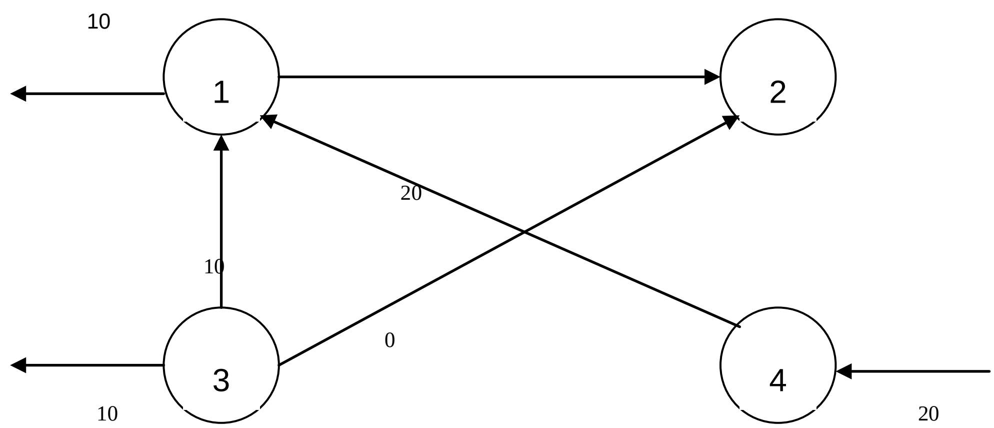
where s is a real number. Thus this system has an infinite number of solutions.

Find the current flow pattern when X2 = 0

If X2 = 0, then we have the following graph:

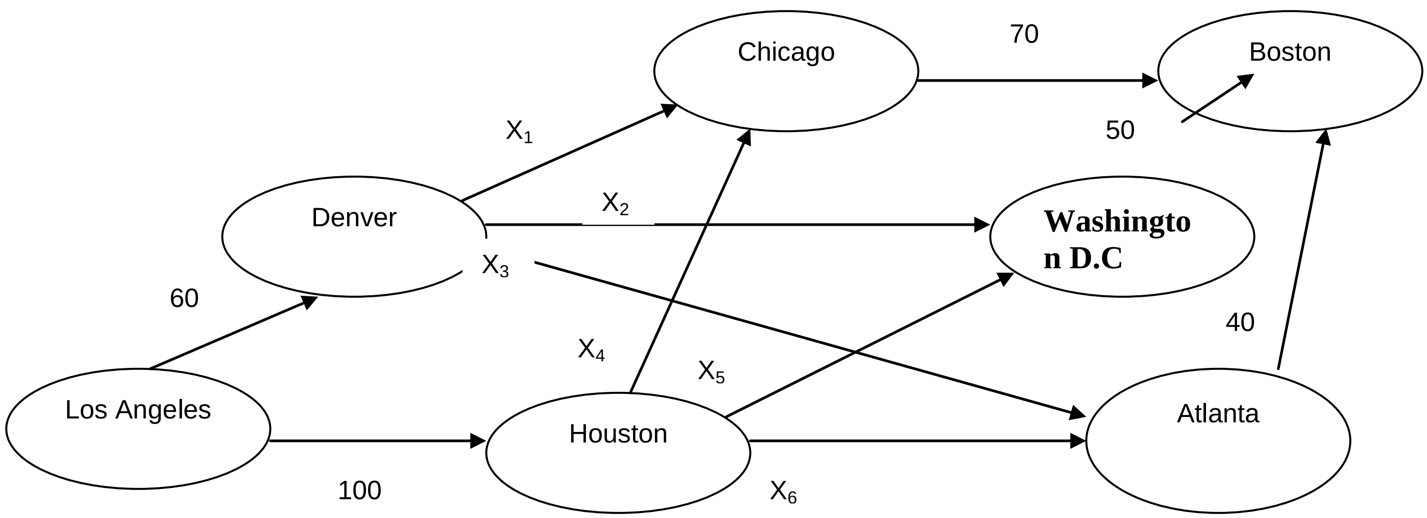


C.) Find the current flow pattern when X2 = 10
If X2 = 10, then we have the following graph:



Example 9

The following figure shows flights per day from selected airports in various cities to other cities' airports. The routes shown can be interpreted as alternative ways to travel from Los Angeles airport to Boston airport. In this problem, we will analyze the effects of changing the number of flights to and from intermediate cities' airports. The results can be used as a means to decide which path(s) to take when flying from Los Angeles to Boston. The more number of flights might lead to increased delay times which might prevent a traveler from choosing that route (edge). On the other hand, the more number of flights means more frequent flights per day, which might make impatient travelers prefer this route.



<u>Solving the system</u>

From the figure, we have the following linear equations:

Junction Denver: $60 = x_1 + x_2 + x_3$

Junction Houston: $100 = x_4 + x_5 + x_6$

Junction Chicago: $x_1 + x_4 = 70$

Junction Atlanta: $x_3 + x_6 = 40$

Junction DC: $x_2 + x_5 = 50$

Re-organizing the equations, we have:

$x_1 + x_2 + x_3 = 60$

$x_4 + x_5 + x_6 = 100$

$x_1 + x_4 = 70$

$x_3 + x_6 = 40$

$x_2 + x_5 = 50$

The augmented matrix for this system is:

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 60 \\ 0 & 0 & 0 & 1 & 1 & 1 & 100 \\ 1 & 0 & 0 & 1 & 0 & 0 & 70 \\ 0 & 0 & 1 & 0 & 0 & 1 & 40 \\ 0 & 1 & 0 & 0 & 1 & 0 & 50 \end{array}\right]$$

Gauss-Jordan elimination produces the matrix:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 70 \\ 1 & 1 & 0 & 0 & 0 & -1 & 20 \\ 0 & 0 & 1 & 0 & 0 & 1 & 40 \\ 0 & 1 & 0 & 0 & 1 & 0 & 50 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix gives us the following equations:

$x_1 + x_4 = 70$

$x_1 + x_2 – x_6 = 20$

$x_3 + x_6 = 40$

$x_2 + x_5 = 50$

Further simplification of the equations leads to the following:

$x_6 = x_1 + x_2 – 20 = (70 – x_4) + (50 – x_5) – 20 = 100 – x_4 – x_5 \Rightarrow x_4 = 100 – x_5 – x_6$

$x_1 = 70 – x_4 = 70 – 100 + x_5 + x_6 = x_5 + x_6 – 30$

Putting the variables in order, we have:

$x_1 = x_5 + x_6 - 30$

$x_2 = 50 – x_5$

$x_3 = 40 – x_6$

$x_4 = 100 – x_5 – x_6$

Letting x_5 = s and x_6 = t, we get:

$x_1 = s + t –30$, $x_2 = 50 – s$, $x_3 = 40 – t$, $x_4 = 100 – s – t$

where s and t are real numbers. Thus this system has an infinite number of solutions.

Using the equations that are just found, we are now ready to analyze the traffic under different circumstances.

a) What is the traffic load when there is no flights from Houston to Atlanta and only 30 flights per day from Houston to Washington DC?

This problem states that s = 30, t = 0

$x_1 = 30 + 0 – 30 = 0$

$x_2 = 50 – 30 = 20$

$x_3 = 40 – 0 = 40$

$x_4 = 100 – 30 – 0 = 70$

b) What is the traffic load when there is no flights from Houston to Washington DC and only 35 flights from Houston to Atlanta?

According to the problem, we have s = 0, t = 35

$x_1 = 0 + 35 – 30 = 5$

$x_2 = 50 – 0 = 50$

$x_3 = 40 – 35 = 5$

$x_4 = 100 – 0 – 35 = 65$

c) How is the traffic when there is no flights from Denver to Chicago, but no other restrictions flights to other cities?

From the problem statement, we have $x_1 = 0$

$0 = x_5 + x_6 – 30 \Rightarrow x_5 + x_6 = 30$

$x_2 = 50 – x_5$

$x_3 = 40 – x_6$

$x_4 = 100 – x_5 – x_6 = 100 – (x_5 + x_6) = 100 – (30) = 70$

as a result of no flights from Denver to Chicago, we see that the number of flights from Houston to Chicago must be 70 per day and the total number of flights from Houston to Washington DC and Atlanta must be 30 per day.

d) What if there were no flights to Washington DC?

This states that $x_2 = x_5 = 0$,

$x_1 = 0 + x_6 – 30$

$0 = 50 – 0 \Rightarrow 0 = 50$

here, we see that there is no solution to this problem. This is because there are 50 flights per day going from Washington DC to Boston but if the incoming flights are restricted, there is no way 50 flights can be going out.

e) How about the case when there is no flights from Houston to Atlanta and the number of flights from Houston to Washington DC is restricted to 10?

From this statement, we have s = 10, t = 0
$x_1 = 10 – 30 = -20$

$x_2 = 50 – 10 = 40$

$x_3 = 40 – 0 = 40$

$x_4 = 100 – 10 = 90$

In this case we see that x_1 is a negative number. This doesn't make sense because it is impossible to have a negative number of flights from a city to another one. It might be possible to have a number of flights in the reverse direction however, that is not what is being analyzed here, that is, we are analyzing flights going in one direction.

As for the capacity of the routes (edges), there is theoretically no limit since infinite number of planes can be in the air simultaneously (assuming there is no limit to the sky) however, this does not mean that the airports can handle all the traffic. So the capacity limits are the result of airport limitations and not the routes themselves.

f) What are the ranges of s and t, assuming that negative numbers don't make sense?

Since negative numbers don't make sense, the minimum either variable can be is 0 (zero), that is, $s_{min} = t_{min} = 0$.

Letting $s = s_{min} = 0$, we get:

$x_1 = x_6 – 30$ $\quad$ $x_2 = 50 – 0 = 50$

$x_3 = 40 – x_6$ $\quad$ $x_4 = 100 – x_6$

From the equations, we see that in order for x_1 to make sense (not be negative), we need x_6 to be greater than or equal to 30. for x_3 to make sense, we need x_6 to be less than or equal to 40. this gives us our range for x_6

$40 \geq x_6 \geq 30$

Letting $t = t_{min} = 0$, we get:

$x_1 = x_5 - 30$ $\quad$ $x_2 = 50 – x_5$

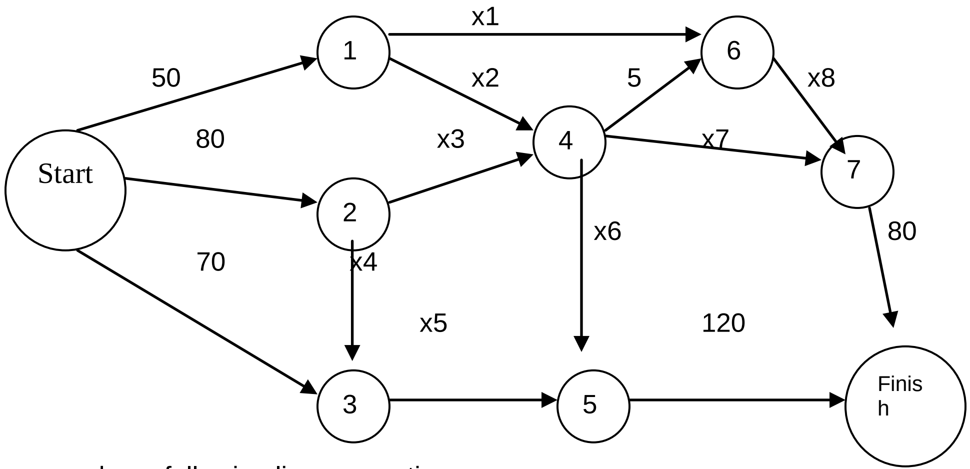
$x_3 = 40 – 0 = 0$ $x_4 = 100 – x_5$

From the equations, we see that in order for x_1 to make sense, we need x_5 to be greater than or equal to 30. in order for x_2 to make sense, we need x_5 to be less than or equal to 50, giving us the range for x_5

$50 \geq x_6 \geq 30$

Example 10.

The following figure shows hiking trails from a parking lot to a summit of a mountain. The starting point is indicated as Start and the summit as Finish. The number of people hiking to the summit is 200. We assume that all hikers who go out reach the summit.



In the figure we have following linear equations:

Junction 1:	50 = x1 + x2
Junction 2:	80 = x4 + x3
Junction 3:	70 + x4 = x5
Junction 4:	x2 + x3 = x6 + x7 + 5
Junction 5:	x5 + x6 = 120
Junction 6:	x1 + 5 = x8
Junction 7:	x8 + x7 = 80

After reorganizing the equations:

x1 + x2 = 50
x3 + x4 = 80
x4 - x5 = -70
x2 + x3 -x6 - x7 = 5
x5 + x6 = 120
x1 - x8 = -5
x7 + x8 = 80

The augmented matrix for that system is:

$$\left(\begin{array}{ccccccccc|c}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 50 & \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 80 & \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -70 & \\
0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 5 & \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & & 120 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -5 & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 80 &
\end{array}\right)$$

R4 = R4 + R5

$$\left(\begin{array}{ccccccccc|c}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 50 & \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 80 & \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -70 & \\
\mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{125} & \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & & 120 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -5 & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 80 &
\end{array}\right)$$

R4 = R4 + R3

$$\left(\begin{array}{ccccccccc|c}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 50 & \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 80 & \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -70 & \\
\mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{55} & \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & & 120 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -5 & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 80 &
\end{array}\right)$$

R4 = R4 – R2

$$\left(\begin{array}{ccccccccc|c}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 50 & \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 80 & \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -70 & \\
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{-25} & \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & & 120 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -5 & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 80 &
\end{array}\right)$$

R4 = R4 – R1

$$\left(\begin{array}{ccccccccc|c}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 50 & \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 80 & \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -70 & \\
\mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{-75} & \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & & 120 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -5 & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 80 &
\end{array}\right)$$

R4 = R4 + R6

$$\left(\begin{array}{ccccccccc|c}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 50 & \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 80 & \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -70 & \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{-1} & \mathbf{-80} & \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & & 120 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -5 &
\end{array}\right)$$

0 0 0 0 0 0 1 1 80

R4 = R4 + R7

$$\left(\begin{array}{ccccccccc|c}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 50 & \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 80 & \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -70 & \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & & 120 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -5 & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 80 &
\end{array}\right)$$

Exchanging row positions:

$$\left(\begin{array}{ccccccccc|c}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 50 & \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -5 & \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 80 & \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -70 & \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & & 120 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 80 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right)$$

The matrix gives following equations:

x1 + x2 = 50
x1 – x8 = -5
x3 + x4 = 80
x4 – x5 = -70
x5 + x6 = 120
x7 + x8 = 80

Further simplification of the equations leads to the following:

x1 = x8 – 5
x2 = 50 – x1
x3 = 80 – x4
x4 = x5 – 70
x5 = 120 – x6
x7 = 80 – x8

Letting x8 = s and x6 = t (s and t are non-negative integers), we get:

x1 = s – 5
x2 = 50 – s + 5 = **55 – s**
x5 = 120 – t
x4 = 120 – t – 70 = **50 - t**
x3 = 80 – 50 + t = **30 + t**
x7 = 80 – s

Thus this system has an “infinite number of solutions” with the restrictions mentioned below.

Several examples for the above system are:

a) Suppose that in last hurricane severally damaged paths x6 and x8. After an inspection, path x6 was closed for public and path x8 was allowed for use only by 5 people per day. That leaves us with t = 0 and s = 5.

Therefore,

$x1 = 5 – 5 = 0$ (the path was closed as well, even though it wasn't damaged)
$x2 = 55 – 5 = 50$
$x3 = 30 – 0 = 30$
$x4 = 50 – 0 = 50$
$x5 = 120 – 0 = 120$
$x6 = 0$
$x7 = 80 – 5 = 75$
$x8 = 5$

b) What happens when a large hiking group decides to go for the summit and the fact becomes widely known? Smaller groups and individual hikers take other routs to avoid the crowded paths, which are x3 and x7. Let's say that it is a group of 75 hikers ($x3 = x7 = 75$), and the total number people going for the summit is still 200.

Therefore,

$x3 = 30 + t = 75$, $t = 75 – 30 = 45$

$x7 = 80 – s = 75$, $s = 80 – 75 = 5$

$x1 = s – 5 = 5 – 5 = 0$
$x2 = 55 – s = 55 – 5 = 50$
$x5 = 120 – t = 120 – 45 = 75$
$x4 = 50 – t = 50 – 45 = 5$
$x3 = 30 + t = 30 + 45 = 75$
$x7 = 80 – s = 80 – 5 = 75$

The ranges of s and t are as follows:

- Since those numbers represent number of people taking routs x6 and x8 to the summit, those numbers are integers and cannot be negative.
  Meaning, $s \geq 0$ and $t \geq 0$.
- Since there is a path to junction 6 that 5 people hike everyday, there should be a way out for at least 5 people. Therefore, $s \geq 5$.
- The maximum number of people who can pass thru junction 6 is 55.
  Therefore, s cannot be bigger than that number ($s \leq 55$).
- On the same token, he maximum number of people who can pass thru junction 4 is 130. 5 out of those 130 must go thru junction 6. Therefore, $t \leq (130 – 5) =125$.

Applying that to other equations, we get range values for other routes:

$0 \leq x1 \leq 50$
$0 \leq x2 \leq 50$
$0 \leq x3 \leq 80$
$0 \leq x4 \leq 80$
$70 \leq x5 \leq 150$
$0 \leq x6 \leq 125$
$0 \leq x7 \leq 125$
$5 \leq x8 \leq 125$

Example 11.

People get sick. Sometimes in order to receive the proper treatment and medication they must go through certain channels. The following example will outline a scenario in which 150 sick people go to their regular doctor's office while another 200 sick people go to a specialist's office with a specific problem over the course of a year. These two groups of people represent the input into the system. A certain number of people who go to the doctor are then referred to the specialist's office; this is the variable x_1. Another group of people going to the doctor are given Drug A; this is the variable x_3. The people seeing the specialist are then tested to determine which drug is appropriate for them; this is x_2. After this test, 200 people are given Drug B, while the rest are prescribed Drug A; this is x_4. There are then 150 people with Drug A. This network is illustrated below:

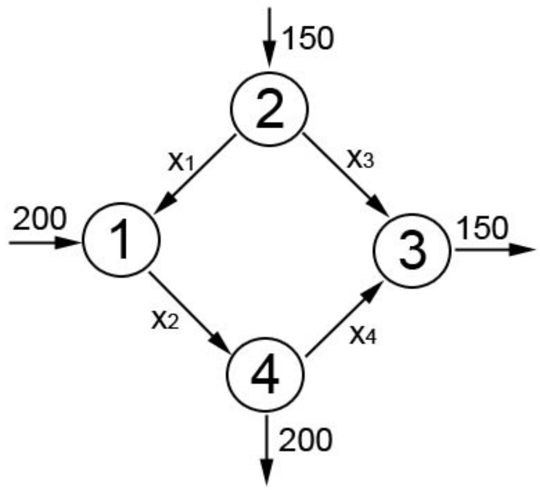


Figure 1

Solve this system for x_i, i = 1, 2, 3, 4
Find the flow of people when $x_4 = 50$
Find the flow of people when $x_4 = 10$

Solution

a) From Figure 1, we have the following linear equations:
Junction 1: $200 + x_1 = x_2$
Junction 2: $x_1 + x_3 = 150$
Junction 3: $x_3 + x_4 = 150$
Junction 4: $200 + x_4 = x_2$

The augmented matrix for this system is:

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 150 \\ -1 & 1 & 0 & 0 & 200 \\ 0 & 1 & 0 & -1 & 200 \\ 0 & 0 & 1 & 1 & 150 \end{array}\right)$$

Gauss-Jordan elimination produces the matrix:

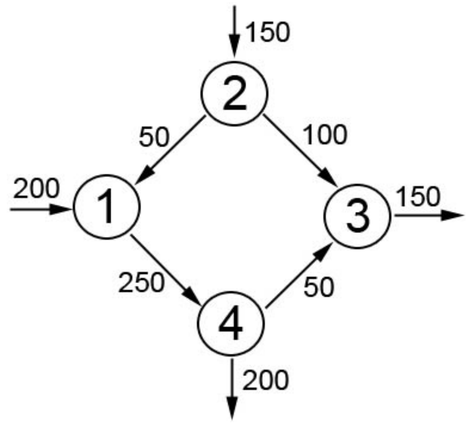
$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 150 \\ 0 & 1 & 0 & -1 & 200 \\ 0 & 0 & 1 & 1 & 150 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This matrix gives the following equations:

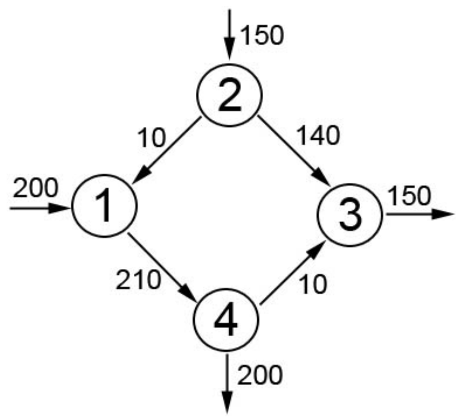
$x_1 + x_3 = 150$
$x_2 – x_4 = 200$
$x_3 + x_4 = 150$

By substitution we can find that $x_1 = x_4$.
We can also describe this with a dummy variable, s, where $x_4 = s$. This gives us:
$x_1 = s$ $x_2 = 200 + s$ $x_3 = 150 - s$
where s is any real number. This shows that the system has an infinite number of solutions.

b) Let $s = x_4 = x_1 = 50$
Then $x_2 = 250$ and $x_3 = 100$



c) Let $s = x_4 = x_1 = 10$
Then $x_2 = 210$ and $x_3 = 140$



d) These results make sense for the real world. When $x_2 = 0$, s has a negative value. This zero flow would occur when the specialist's office is closed and not conducting the tests. The only limit on the maximum capacity of each edge would be how many patients the doctor and specialist are able to see over a period of time. The number of 350 people being seen in a year is a reasonable maximum capacity.

Example 12.

Product Flow Network Analysis

The following illustration describes the flow of product through a particular piece of equipment used in the fabrication of semiconductor chips. The automated wet clean station uses robotic manipulation to move semiconductor wafers from a load area (A) to a selection of chemical baths(B,C or D) to a drying chamber (E) and finally to an unload area (F). See figure 1.

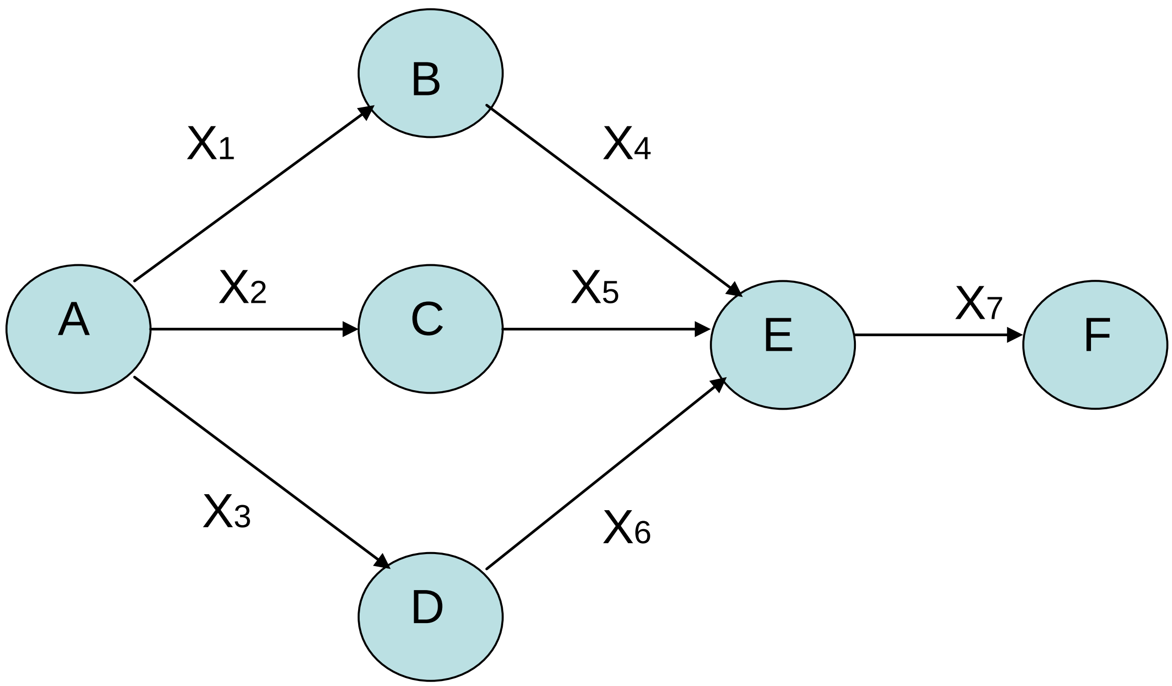


Figure 1

On a typical day 3000 wafers are processed by this machine.
All 3000 wafers move through nodes A, E and F.
Node B processes 1200.
Node C processes 800.
Node D processes 1000.

A: $x_1 + x_2 + x_3 = 3000$
B: $x_4 = x_1 = 1200$
C: $x_5 = x_2 = 800$
D: $x_6 = x_3 = 1000$
E: $x_7 = x_4 + x_5 + x_6 = 3000$

x1	x2	x3	x 4	x5	x6	x7	
1	1	1	0	0	0	0	3000
0	0	0	1	0	0	0	1200
0	0	0	0	1	0	0	800
0	0	0	0	0	1	0	1000
0	0	0	0	0	0	1	3000

Recent testing has shown that increased cleaning effectiveness can be achieved on some process steps by adding a step to the process. We want to move some product from node B and run through node C prior to the drying chamber (E). This will change our process flow represented by Figure 2. The added steps to the process are represented by the addition of flow line $x_{8.}$

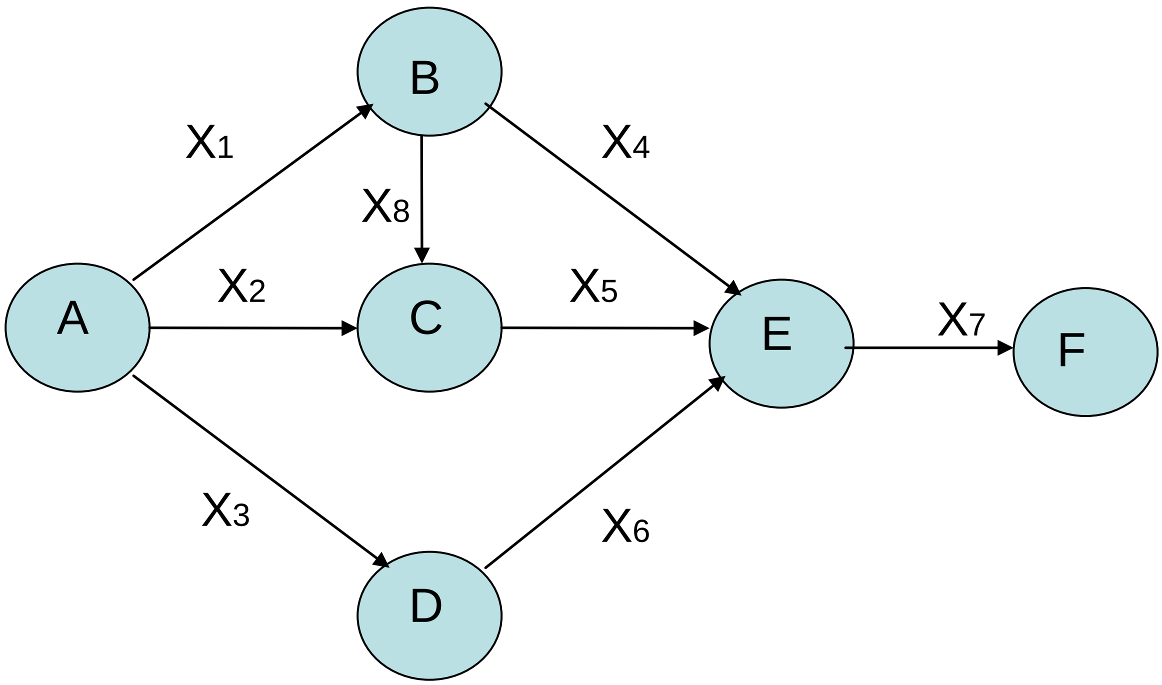


Figure 2

A: $x_1 + x_2 + x_3 = 3000$
B: $x_4 = x_1 - x_8$
C: $x_5 = x_2 + x_8$
D: $x_6 = x_{3\,=}\ 1000$
E: $x_7 = x_4 + x_5 + x_6 = 3000$

x1	x2	x3	x4	x5	x6	x7	x8	
1	1	1	0	0	0	0	0	A = 3000
0	0	0	1	0	0	0	-1	B =?
0	0	0	0	1	0	0	1	C =?
0	0	0	0	0	1	0	0	D = 1000
0	0	0	0	0	0	1	0	E = 3000

A pilot experiment will be implemented which sets the number of wafers processed through the new path to be 50. This makes $x_8 = 50$. The impact will be felt by node B and node C. This is represented by the following:

As previously mentioned:

On a typical day 3000 wafers are processed by this machine.
All 3000 wafers move through nodes A, E and F.
Node B processes 1200.
Node C processes 800.
Node D processes 1000.

A: $x_1 + x_2 + x_3 = 3000$
B: $x_4 = x_1 – 50 = 1200 – 50 = 1150$
C: $x_5 = x_2 + 50 = 800 + 50 = 850$
D: $x_6 = x_3 = 1000$
E: $x_7 = x_4 + x_5 + x_6 = 3000$

x1	x2	x3	x4	x5	x6	x7	x8	
1	1	1	0	0	0	0	0	A = 3000
0	0	0	1	0	0	0	-1	B =1150
0	0	0	0	1	0	0	1	C = 850
0	0	0	0	0	1	0	0	D = 1000
0	0	0	0	0	0	1	0	E = 3000

In conclusion, we did not use Gauss-Jordan elimination techniques as it would not be of benefit to understand the situation. The elements of the network analysis, such as network models, system of linear equations and matrices help us see the effect of the change in process flow.

Example 13.

Introduction:
This past Saturday, I worked at the "Accepted Engineering Students Reception" here at UMass Lowell. The basic idea of the day was to provide tours and presentations to explain to prospect students what the engineering programs here at UMass Lowell have to offer. This reception is the basis of my network analysis project, and a directed graph was made to represent the activities of the day. The reception tours consisted of three major parts:

- The introduction speeches
- A choice of two tours of the engineering programs
- The conclusion speeches

Theory/Calculations:
Approximately 3200 prospect students and parents attended the reception, all of which attended both the introduction and the conclusion speeches. The attendees were also given two choices for their first tour (nodes 1 & 2), and three choices for their second tour (nodes 3→4), before returning for the conclusion speeches. Figure 1 shows the directed graph of the flow of people between the tours and speeches throughout the day.

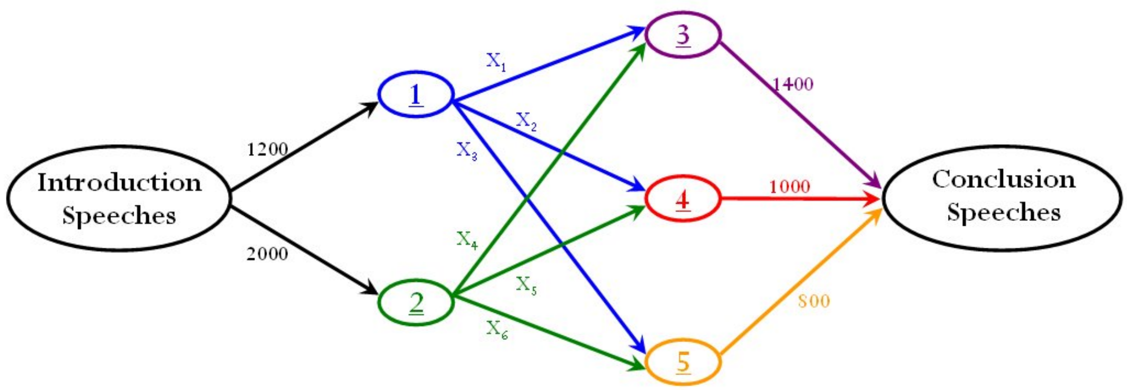


Fig. 1: Network Representation of the Accepted Students Reception Tours

From Figure 1, a system of linear algebraic equations was formed, one for each junction:

Junction 1: $1200 = x_1 + x_2 + x_3$

Junction 2: $2000 = x_4 + x_5 + x_6$

Junction 3: $1400 = x_1 + x_4$

Junction 4: $1000 = x_2 + x_5$

Junction 5: $800 = x_3 + x_6$

These five equations were then put into an augmented matrix, and solved using Gauss-Jordan Elimination.

Initial Augmented Matrix: Gauss-Jordan Solution Matrix:

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 1200 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2000 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1400 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 0 & 0 & 1 & 800 \end{array}\right] \qquad \left[\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 0 & -1 & 400 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1400 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 0 & 0 & 1 & 800 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

From the Gauss-Jordan Solution Matrix, the following system of equations can be written

$$x_1 + x_2 - x_6 = 400 \qquad x_1 + x_4 = 1400 \qquad x_2 + x_5 = 1000 \qquad x_3 + x_6 = 800$$

By rearranging and combining these equations, all of the variables can be expressed in terms of x_5 and x_6.

$$x_6 = x_1 + x_2 - 400 = (1400 - x_4) + (1000 - x_5) - 400 = 2000 - x_4 - x_5$$
$$x_4 = 2000 - x_5 - x_6$$

$$x_1 = 1400 - x_4 = 1400 - (2000 - x_5 - x_6)$$
$$x_1 = x_5 + x_6 - 600$$

Placing the variables in order yields

$$x_1 = x_5 + x_6 - 600 \qquad x_2 = 1000 - x_5 \qquad x_3 = 800 - x_6 \qquad x_4 = 2000 - x_5 - x_6$$

If the variables 5 and 6 are then represented as positive integers (you can’t have half of a person OR a negative person) such that

$$\begin{matrix} x_5 = n \\ x_6 = k \end{matrix} \quad \text{where} \quad \begin{Bmatrix} 0 \le n \le 1000 \\ 0 \le k \le 800 \end{Bmatrix}$$

These ranges for the integers are a result of the maximum number of people that can attend tours #4 and #5. The four variable equations can then be rewritten in the forms of

$$\boxed{\begin{aligned} x_1 &= n + k - 600 \\ x_2 &= 1000 - n \\ x_3 &= 800 - k \\ x_4 &= 2000 - n - k \\ x_5 &= n \\ x_6 &= k \end{aligned}}$$

Discussion of Results:

EXAMPLE 1:

First, assume that a lot of people from tour #2 were interested in tour #4, but no one was interested in tour #5. To satisfy these conditions $n = 800$ and $k = 0$. Using the equations above, values for Figure 2 were calculated. It is seen that because no one from tour 2 went to tour 5, tours 3 and 4 were mostly filled up with the people from tour 2.

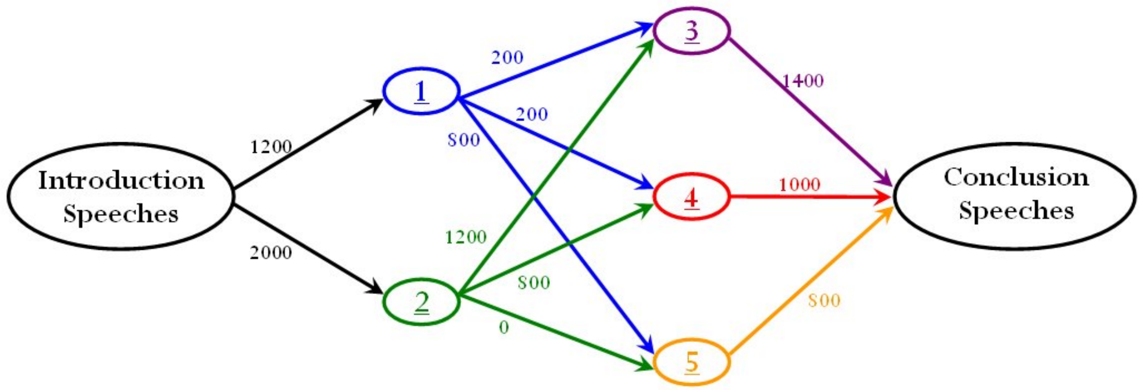


Fig. 2: Directed Graph Behavior for $n = x_5 = 800$ and $k = x_6 = 0$

EXAMPLE 2:

Now, assume the opposite as before, that no one from tour #2 wants to see tour #4 and a lot of people want to see tour #5. This is represented by $n = 0$ and $k = 800$. Figure 3 shows the flow behavior for this case. In this extreme case, it was shown that no one from tour #1 was able to see tour #5, because it was full with people from tour #2.

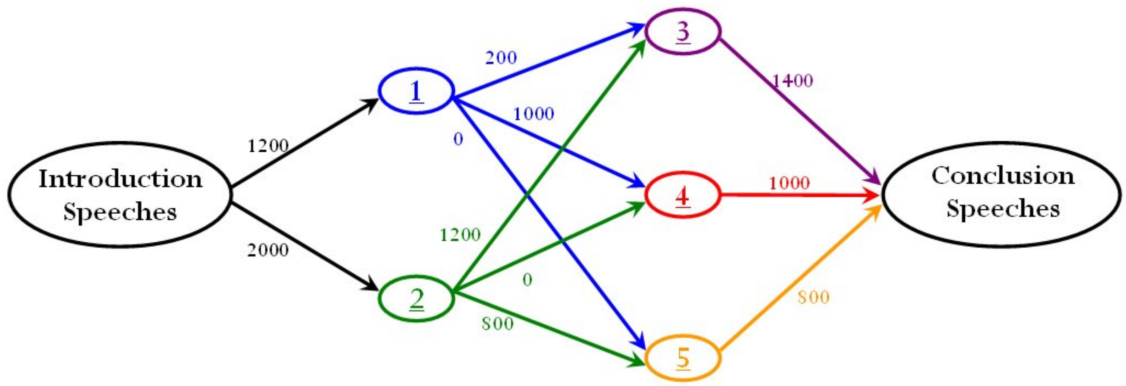


Fig. 3 Directed Graph Behavior for $n = x_5 = 0$ and $k = x_6 = 800$

EXAMPLE 3:

Finally, let's assume that about equal amounts of people from tour #2 want to take tours #4 and 5. This can be accomplished by setting $n = 500$ and $k = 500$. Figure 4 shows the visual representation of this case. It is seen that when people from tour #2 start to spread themselves out between the second three tours, the people from tour #1 also start to spread out between the tours.

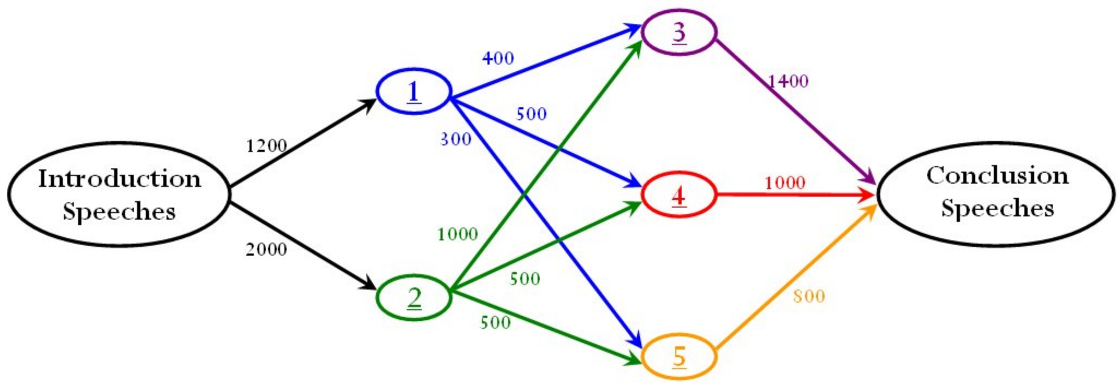


Fig. 4: Directed Graph Behavior for $n = x_5 = k = x_6 = 500$

Conclusion:

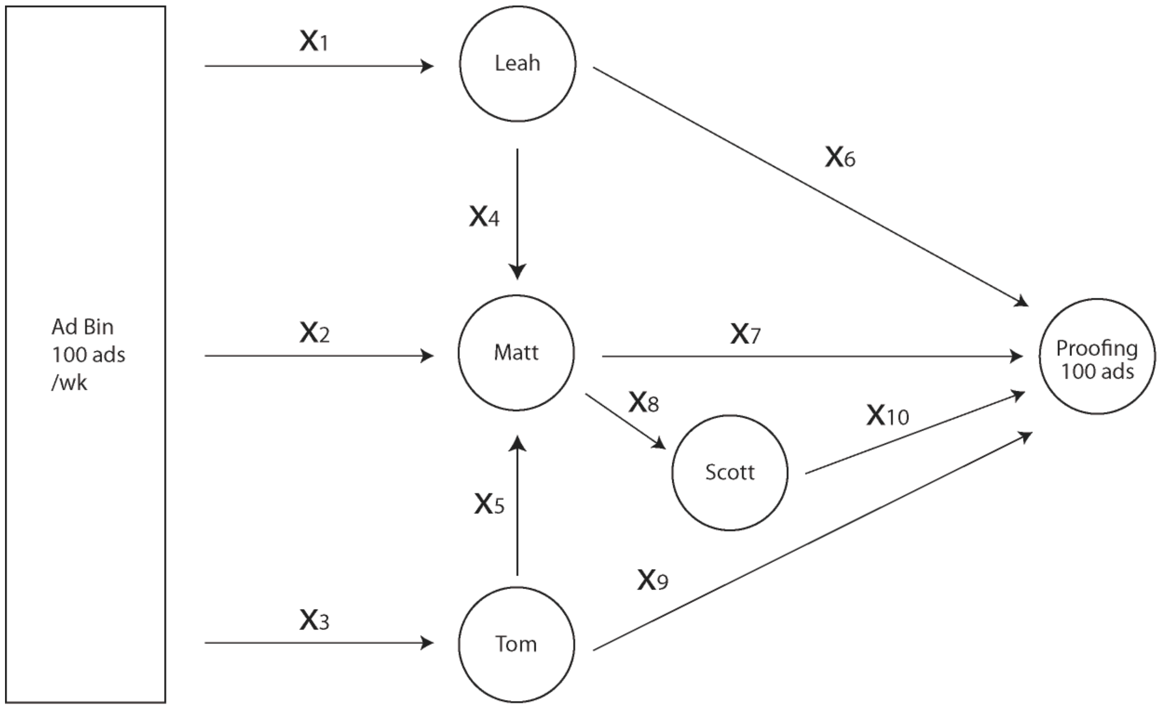
It was shown how network analysis, directed graphs, and Gauss-Jordan Elimination techniques could all be used in conjunction to analyze fairly complex network systems. This procedure can be used over a wide range of disciplines, and the movement of crowds (as discussed above) is only a small part of the possibilities with this method.

Example 14. (student example)

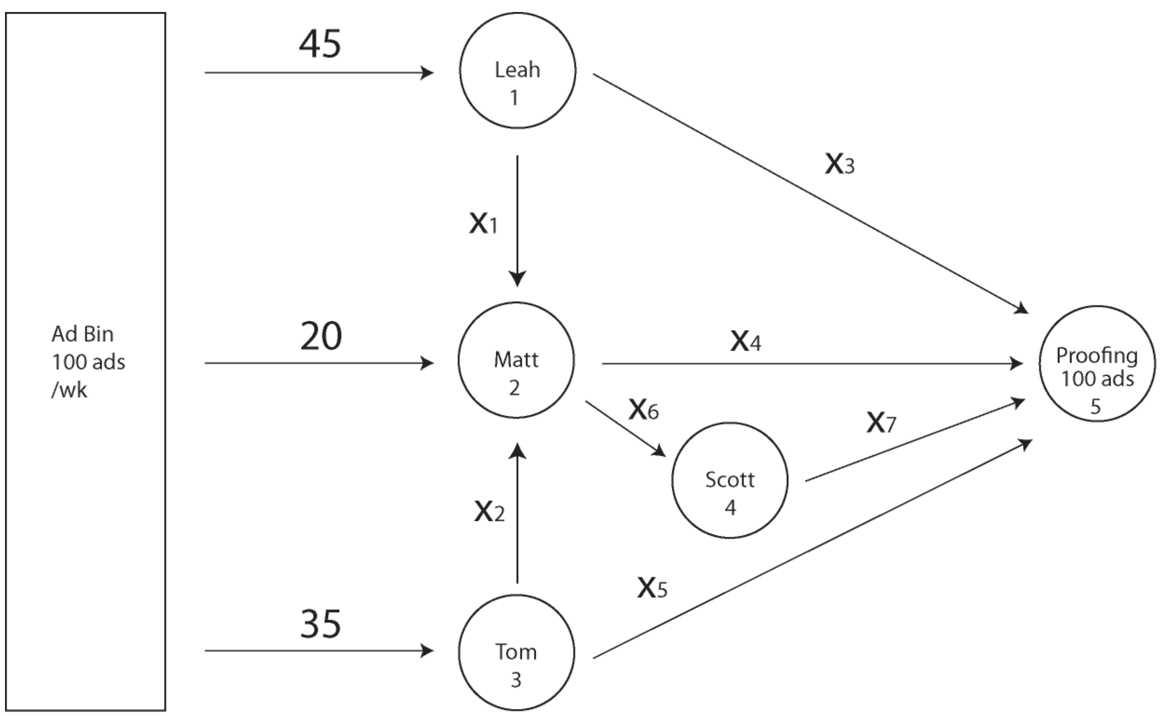
I work as a production manager for a publication. Part of my departments' job entails creating display ads that go into our magazines. In a typical peak season week, we work on 100 display ads. 2 production staff members (Leah, Tom) pick ads up out of a bin. I handle any overflow (which means I take from the bin when it's busy). Production staff can hand "problem" ads off to me (Matt). I hand some ads that require additional design work off to our designer, Scott. These ads may be some of the "problem" ads I received from others. I can hand off anywhere from 0-10 ads to him in a normal week. All ads leave the production workflow by being submitted to proofreading by the deadline.

I'm curious to see what the workflow is like when I'm handling a large or small number of ads myself. This value will wind up being x_7 (or x_4 on the cleaned up version on the next page), so I'd likely plug in multiple values for this variable once everything is defined.

Here's a directed graph of the network without any values plugged in:



I’ve plugged in some values based on a typical week.



From this graph, we get the following equations:
Junction 1 (Leah) $x_1 + x_3 = 45$
Junction 2 (Matt) $x_4 + x_6 - x_1 - x_2 = 20$
Junction 3 (Tom) $x_2 + x_5 = 35$

Junction 4 (Scott) $x_6 = x_7$
Junction 5 (Proofing) $x_3 + x_4 + x_5 + x_7 = 100$

The augmented matrix is:

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 45 \\ -1 & -1 & 0 & 1 & 0 & 1 & 0 & 20 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 35 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 100 \end{array}\right]$$

We could do a Gauss Jordan elimination, but it seems like a waste of time given the circumstances.

Simplifying the original equations we can arrive at:
$X_1 = 45 - x_3$
$x_2 = 35 - x_5$
$x_6 = x_7$

$100 - x_3 - x_5 = x_4 + x_6$
$20 + x_1 + x_2 = x_4 + x_6$
$120 + x1 + x2 - x3 - x5 / 2 = x_4 + x_6$

There are an infinite number of solutions. At this point, I realize that I have left too many variables open, and that I cannot solve for the two I have chosen without defining at least one more variable. I'll define x_1 so as to get a solution.

Let $x_1 = 5$
Let $x_4 = 40$
Let $x_6 = 0$

Plugging these into the above equations, this would mean:
$x_1 = 5$
$x_2 = 15$
$x_3 = 40$
$x_4 = 40$
$x_5 = 20$
$x_6 = 0$
$x_7 = 0$

Other numbers to try:

Let $x_1 = 5$
Let $x_4 = 30$
Let $x_6 = 5$

Plugging these into the above equations, this would mean:
$x_1 = 5$
$x_2 = 10$
$x_3 = 40$
$x_4 = 30$
$x_5 = 25$
$x_6 = 5$
$x_7 = 5$

Other numbers to try:

Let $x_1 = 15$
Let $x_4 = 35$
Let $x_6 = 5$

Plugging these into the above equations, this would mean:
$x_1 = 15$
$x_2 = 0$
$x_3 = 30$
$x_4 = 30$
$x_5 = 35$
$x_6 = 5$
$x_7 = 5$

f. As all ads are done by deadline, even if that means overtime, there is no official maximum capacity, though there must be some human limit. There are also other duties for the staff, which limits the amount of time dedicated towards ad production. My guess is that someone dedicated to working primarily on ads in a given week could get 65 done.

Part 4: The Row Reduction Method for Determining the Inverse of a Matrix

(This is extra material we will cover it only if time permits)

In week 5 in the notes we defined the inverse of an *n x n* matrix. We noted that not all matrices have inverses, but when the inverse of a matrix exists, it is unique. This enables us to define the inverse of an *n x n* matrix *A* as the unique matrix *B* such that
AB = *BA* = I, where I is the *n x n* identity matrix. In order to obtain some practical experience, we developed a formula that allowed us to determine the inverse of invertible 2 x 2 matrices. We will now use the Gauss-Jordan procedure for solving systems of linear equations to compute the inverses, when they exist, of *n x n* matrices, $n \geq 2$. The following procedure for a 3 x 3 matrix can be generalized for *n x n* matrices, $n \geq 2$.

From exercise 6, week 6 we know that the matrix $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$ can be interpreted as solving 3 systems of three equations three unknowns. By now you should have written out the systems.

If A = $\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ and if we use the above definition to find the inverse of A. Then we want a 3 X 3 matrix B such that AB = I. Assume B = $\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ then the equation AB =I becomes

$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Write out the three systems of 3 equations 3 unknowns that this equation produces. These should be the same systems that you wrote previously. You should know understand why the procedure given in the following examples works.

1) Find the inverse of $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-1)R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

Note: Row 1 (R_1) does not change

$$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

Note: Row 3 (R_3) does not change.

$$\xrightarrow{(-2)R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right]$$

Note: Row 2 (R_2) does not change.

$$\xrightarrow{(-1)R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right]$$

Note: Row 3 (R_3) does not change.

So $A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 1 & 1 \\ 2 & -2 & -1 \end{bmatrix}$

Check that $A A^{-1} = I$.

2) Find the inverse of $B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-1)R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{interchange } R_2 \text{ and } R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$. So B^{-1} = ? Check?

3) Find the inverse of the matrix $C = \begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 1 & 0 \\ 0 & -1 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 0 & 0 \\ 0 & 1 & 4 & 2 & 1 & 0 \\ 0 & 0 & 7 & 3 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{2}{7}R_3+R_1}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-1}{7} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 4 & 2 & 1 & 0 \\ 0 & 0 & 7 & 3 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{4}{7}R_3+R_2}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-1}{7} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \frac{2}{7} & \frac{3}{7} & \frac{-1}{7} \\ 0 & 0 & 1 & \frac{3}{7} & \frac{1}{7} & \frac{1}{7} \end{array}\right]. \quad C^{-1} = ? \text{ Check?}$$

Exercises:

Use the above procedure to determine the inverses of the following matrices if they exist. Check your solutions.

1. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ -1 & 1 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 1 & 0 \\ -4 & -1 & -3 \\ 3 & 1 & 3/2 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$