Week 1: Introduction and Sets

The student should read the *Preface* and the note *To the Student*. These pages, with a review of the table of contents will give an idea of what Discrete Mathematics covers. It is a variety of topics, which should assist you in new problem solving techniques. I follow the ACM/IEEE recommendations for the topics for the course Discrete Structures. I recommend that each person google these recommendations and spend a few minutes reading them. Appendix 2A of the recommendations is a good summary.

Our course is called *Discrete Structures* for a reason. Not only do we want to learn the topics in Discrete Mathematics but also we want to determine the relationships between these topics. This is the **glue,** which binds these topics together. I should **forewarn** you that even though you may have seen some of these topics before read the textbook and your notes carefully.

Three of the topics that are covered in the first five chapters are: Sets, Logic, and Matrices. We could refer to these three topics as: the Algebra of Sets, the Algebra of Logic and the Algebra of Matrices. These three algebras have a mathematical structure similar to that of regular arithmetic and regular algebra that we studied in elementary and high school. To determine how easy will it be to work with these new "algebras" we will first refresh our mind on the structure of arithmetic and that of regular algebra. We learn new structures by thinking in terms of a mathematical system about which we already know a significant amount, namely high school algebra. So let's think back to **elementary school arithmetic and high school algebra, and use our knowledge of these concepts to help us learn about** the structures of sets, logic, and matrices. That is, how they behave or "misbehave".

Consider the following problem:

Example 1. Add: 23 + 19 + 7
Certainly we could by add these three numbers by **brute force** proceeding from the left to the right, namely 23 + 19 + 7 = 42 + 7 = 49. But since 23 and 7 are easier to add we may decide to add these two numbers first (and obtain 30) and add 19 to the result to get the solution of 49. What makes this example, and arithmetic and algebra in general easy to work with? What are some of the rules (or properties or basic laws) that make high school algebra so easy to work with? Can we formalize some of these rules?

Let us reconsider example 1 and write in the formal names for the rules we used.

23 + (19 + 7) = 23 + (7 + 19) by the **commutative law** (which says; Let a and b be any two real numbers then a + b = b + a.)
= (23 + 7) + 19 by the **associative law** (which says; Let a, b, and c be any three real numbers then (a + b) + c = a + (b + c).)

The main idea is that high school algebra is easy to work with because it satisfies certain basic laws or properties. It "behaves". If another mathematical system (algebra) satisfies many of the same laws we can predict that it also will be easy to work with, moreover we can use our knowledge of high school algebra to be able to make an intelligent guess on what a law in a second algebra may be.

Example 2. In the above example I mentioned the **commutative law** for addition, which says; Let a and b be any two real numbers then a + b = b + a. We may have forgotten or never learned the name of this property but we use this it all the time without thinking much about it. Certainly 2 + 3 us the same result as 3 + 2. If I told you that in sets we have a similar property, namely that $A \cup B = B \cup A$ (table 3) we immediately see the resemblance and we are not surprised that the name of this law is the commutative law. In table 1 list several commonly used laws in regular algebra/arithmetic can you give an example of each law?

Example 3. Another basic law in regular algebra is:

Let a, b and c be any three real numbers then $a \cdot (b + c) = a \cdot b + a \cdot c$. This is called the **distributive law.** Note, this is nothing more than "factoring out" the a.

In what follows don't worry about what the symbols $\cup, \cap, \vee$ and $\wedge$ **mean just look at the similarity between the different statements**

In the distributive law $a \cdot (b + c) = a \cdot b + a \cdot c$ make the following substitutions. Replace a, b and c by p, q and r respectively. Next, replace $\cdot$ by $\wedge$ and + by $\vee$, and = by $\Leftrightarrow$. You will obtain: $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$. This is the **distributive law** for Logic (see table 2).

Now take the distributive law for high school algebra and make the following substitutions. Replace a, b and c by A, B and C respectively. Next, replace $\cdot$ by $\cap$ and + by $\cup$. You will obtain $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. This is the **distributive law** for Sets (table 3).

Table 1, below lists a few basic laws from high school algebra. The object of the two exercises below is to get you to compare the basic high school laws in algebra to those of both logic and sets. In many cases you note that the laws are the same except for notation. So in your reading of the text and the notes keep this in mind. That is, that the **structure** of these new algebras is **similar** to that of high school.

Exercise 1.

List the commutative laws (note there are two of them.), the associative laws (again two of them) and the distributive laws for Logic and also for Sets. At this point it is not necessary that you understand what the laws say or why they are true but, note how these laws resemble those of high school algebra. Check your solutions by referring to tables 2 and 3 below.

Exercise 2.

Read the identity laws for basic algebra in table 1. Note, there are two of them, one for addition and one for multiplication. Note also that 0 is the additive identity in algebra and 1 is the multiplicative identity. Now compare the identity laws for the algebras of logic and sets below to that of regular algebra. Note that $\vee$ in logic behaves like + in algebra. What does the number “0” in algebra translate to in the algebra of logic? What operation in regular algebra does $\wedge$ behave like? Compare the identity laws in the algebra of sets to that of regular algebra and that of the algebra of logic.

Table 1: Some Basic Algebra Laws

(here a, b and c can be any real numbers except in the 2nd involution law $a \neq 0$)

$a + b = b + a$	Commutative Laws $a \cdot b = b \cdot a$
$(a + b) + c = a + (b + c)$	Associative Laws $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
	Distributive Law $a \cdot (b + c) = a \cdot b + a \cdot c$
$a + 0 = a$	Identity Laws $a \cdot 1 = a$
$a + (-a) = 0$	Inverse Laws $a \cdot (a^{-1}) = 1$
$-(-a) = a$	Involution Law $(a^{-1})^{-1} = a$

Table 1

Table 2: Basic Logical Laws

$p \vee q \Leftrightarrow q \vee p$	**Commutative Laws** $p \wedge q \Leftrightarrow q \wedge p$
$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	**Associative Laws** $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	**Distributive Laws** $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
$p \vee 0 \Leftrightarrow p$	**Identity Laws** $p \wedge 1 \Leftrightarrow p$
$p \wedge \sim p \Leftrightarrow 0$	**Negation Laws** $p \vee \sim p \Leftrightarrow 1$
$p \vee p \Leftrightarrow p$	**Idempotent Laws** $p \wedge p \Leftrightarrow p$
$p \wedge 0 \Leftrightarrow 0$	**Null Laws** $p \vee 1 \Leftrightarrow 1$
$p \wedge (p \vee q) \Leftrightarrow p$	**Absorption Laws** $p \vee (p \wedge q) \Leftrightarrow p$
$\sim (p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$	**DeMorgan’s Laws** $\sim (p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$
	Involution Law $\sim (\sim p) \Leftrightarrow p$

Table 2

Table 3: Basic Set Laws

Commutative Laws	
(1) $A \cup B = B \cup A$	(1)' $A \cap B = B \cap A$
Associative Laws	
(2) $A \cup (B \cup C) = (A \cup B) \cup C$	(2)' $A \cap (B \cap C) = (A \cap B) \cap C$
Distributive Laws	
(3) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	(3)' $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Identity Laws	
(4) $A \cup \varnothing = \varnothing \cup A = A$	(4)' $A \cap U = U \cap A = A$
Complement Laws	
(5) $A \cup A^c = U$	(5)' $A \cap A^c = \phi$
Idempotent Laws	
(6) $A \cup A = A$	(6)' $A \cap A = A$
Null Laws	
(7) $A \cup U = U$	(7)' $A \cap \varnothing = \varnothing$
Absorption Laws	
(8) $A \cup (A \cap B) = A$	(8)' $A \cap (A \cup B) = A$
DeMorgan's Laws	
(9) $(A \cup B)^c = A^c \cap B^c$	(9)' $(A \cap B)^c = A^c \cup B^c$
Involution Law	
(10) $(A^c)^c = A$	

Table 3

In our study of arithmetic and algebra the very first thing that we learned is that the **objects** that we were working with were called numbers or more specifically real numbers. Then we learned how to add and multiply two real numbers. The operations of addition and multiplication deal with adding and multiplying **two** numbers and therefore are referred to **binary operations.** Next we learned what $-a$ and a^{-1} meant. [**Note**: here we are operating on just one number, namely a, hence these operations are called **unary operations**].

Next, we learn that = is used to "connect" two statements, for example, in the sentence a + b = b + a. For this reason the = symbol could be referred to as a **connective**. Finally algebra is easy to work with because it satisfies a number of basic laws. Table 1 above mentions a few of them.

So, we suspect that in our study of sets and logic the **very first definition** we will see is the definition of the **objects** in the algebra. The object in the algebra of sets are called **sets** (text section 1.1) and the objects in logic are called **propositions** section 3.1).

In table 4 below we outline the similarities between logic and that of “regular” algebra.

Table 4: Similarities Between “Algebras”

	High School Algebra	**(Algebra of) Logic**	**(Algebra of) Sets**	**Matrix Algebra**
Objects	Real numbers	Propositions	Sets	Matrices
Binary Operations	Addition, + Mutiplication, •	OR, $\vee$, + AND, $\wedge$, •	Union, $\cup$ Intersection, $\cap$	Addition, + Multiplication, •
Unary Operations	Additive Inverse - Multiplicative Inverse $^{-1}$	Negation, $\neg$ or $\approx$	Complementation ′ or c or -	Additive Inverse – Multiplicative Inverse $^{-1}$
Other Connectives	$\leq$ =	$\Rightarrow$ $\Leftrightarrow$	$\subseteq$ =	=
Usual Constant/ Variable Names	a, b, c,…. x, y, z,….	0, 1 p, q, r,…	A, B, C,… X, Y, Z,…	A, B, C,… X, Y, Z,…
Some Basic Laws	See table 1	See table 2	See table 3	See the text section 5.5

Table 4

As you study chapter 1 and later chapters 3 and 5 revisit the comparison chart in table to see the similarities with high school algebra.

Let’s see how good you are. Again, don’t worry about the meaning of the symbols just think in terms of the above “language”.

In algebra.
Let x, y and z be any three real numbers, then in algebra we have:
If $x \leq y$ and $y \leq z$ then $x \leq z$.
In sets.
From the above chart since $\subseteq$ behaves like $\leq$ we suspect the following is true:
If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$ for any three sets A, B and C.

So in logic. (Use the comparable symbol given in table 4 to fill in the blanks of the following.)
If p__q and q__r then p__r.