Week 12

An Introduction to the Concept of Relations

For weeks 12 and 13 we will be following the text closely. Here is some additional material.

Think of your family tree. It consists of your brothers, sisters, cousins, second cousins, aunts, uncles, parents, grandparents, great grandparents etc. On this set of relatives there is defined a relation (ship), that is, person a appears above person b in your family tree if and only if person a is an ancestor of person b. Think of an ancestor, say your great grandmother, and picture your family tree diagram with her listed at the top of the page and all her descendants listed below. This is an example of a “mathematical” diagram or graph called a tree or a *partial ordering* diagram. The ordering is called partial because, for example, you and your siblings and your cousins are all on the same level. Note there is a direction in this graph, namely from the top down so this is a *directed* graph.

Another mental image. Think of the administrative structure of a business, from the CEO at the top to the next layer of VP’s to the Managers etc. Again, this is a *partial ordering* under the relation; person appears above person b if and only if person a is the boss of person b. Note there is a direction in this graph, namely from the top down so this is a *directed* graph.

A third image. Think of all the major field courses you need to take for your degree. Certainly some of them are prerequisites of others. Draw a”tree”; here it might be more readable to draw the tree from the bottom to the top. That is, course x appears below course y in this list of courses tree if and only if course x is a prerequisite of course y.

Each of the above are graphs or models of different situations. But, on reflection we can visualize similarities in the graphs of these diverse situations. In the next several sections we would like to learn how to describe the above and other situations mathematically.

One of the key notations we will use in the next several weeks is that of the **Cartesian product** of two sets. Let’s recall the definition of this concept before we proceed.

Definition: Let A and B be any two sets the **Cartesian product** of A and B, denoted A X B (read as A cross B), is the set of all ordered pairs (a, b) where a takes on all elements from the set a and b takes on all elements from the set B.

In notation: $A \text{ X } B = \left\{ (a,b) \middle| a \in A \text{ , } b \in B \right\}$

Some reminders/comments:

1. If A = {1, 2} and B = {a, b, c} then
   A X B = {(1, a), (2,a), (1, b), 2, b), (1, c), 2, c)} and
   |A X B | = |A| x |B| = 2x3 = 6
2. In general, if |A| = n and |B| = m then |A X B| = nm
3. **R** X **R** is the “algebraic” description of the xy-plane, and **R** X **R** is also written as $\mathbf{R}^2$ which is read as R two.

Before you begin to read the following, think again about the concepts of functions and that of graphs of functions. Let's recall some basic facts about functions.

1. In high school we first learned how to manipulate linear equations algebraically. Next, we learned through an example, like y = 2x + 3, what this meant and then how to graph this function.
2. Next, you probably learned the "functional notation", for example f(x) = 2x + 3.
3. A third notation for the same function you learned in an earlier part of this course, namely, define f: $\mathbf{R} \longrightarrow \mathbf{R}$ by the formula f(x) = 2x + 3. In this notation concepts like, **domain**, **codomain**, **composition** and the precise definition of a function were discussed.
4. We could talk about this very same function the following way. A function, f from **R** to **R** is the subset of the xy- plane, that is of **R** x **R** where $f = \{(x,y) | x \in \mathbf{R} \text{ and } y = 2x + 3\}$. Here the focus is that f is a set of ordered pairs of real numbers (x, y) with certain restrictions. What are they?

So discussions about the function f(x) = 2x + 3 centered about **how to describe it algebraically** and then **how to graph it**.

In **discrete** graph theory our discussions parallel that of the above (let's call the above continuous graph theory). We will define the analogue of the definition of a function namely that of a **relation,** then graph it. The resulting graph is called a **directed graph** of the relation. Precise definitions, examples and illustrations follow in the notes below and in the text. Compare the definition of relation given below to item 4 above. Is every function (from A to B) a relation (from A to B)? YES. Is every relation (from A to B) a function (from A to B)? NO

A Procedure for studying the sections on relations from the text

1. In section 9.1 of the text, study examples 1 through 6. Temporarily skip *properties of relations* in section 1. Study this when you cover partial ordering relations in section 6.

2. **Skip section 9.2 of the text**. Study section 9.3 of the text, **representing relations using digraphs** through example 9. At this point you should be able to write the relation as a set of ordered pairs of each of the graphs in exercises 23 through 28 of section 9.3. This is also covered in section **6.2 of the notes** "Graphs of relation" which begins on page 8 below.

   Here are a couple of examples which are in the text:

Example 1. I will describe the graph given in figure 6.3.6, part iii, as a set of ordered pairs. Keep in mind I must give you 2 things a set A, and the relation, r, which acts on A. Looking at the graph we see that the set A is the set {1, 2, 3}. Next we have to define relation r on A (Note, that r is a subset of A x A, that is, r is a relation from A to A) and there are 3 directed edges ("arrows") in the directed graph so there must be 3 ordered pairs in r. So here is the description. Define r on A by r = {(1, 2), (1, 3), (3, 1)}. Can you represent this relation by a 3 by 3 yes/no matrix? study

Example 2. Describe the graph given in figure 6.3.6, part vi of the text as a set of ordered pairs. I first note there are 10 directed edges so I should list 10 ordered pairs. I also not there are four nodes, namely, 1, 2, 3, and 4 so A = {1, 2, 3, 4}. Can you list the ordered pairs in A? Can you write the yes/no matrix of R?

3. There are 3 key properties listed in section 6.3 of the text. The properties are: **reflexive, antisymmetric and transitive**. Study these definitions. The reason for theses definitions is to introduce a special type of relation called a **partial ordering relation.** When a relation is a partial ordering relation we can draw a much cleaner version of its graph called a **partially ordered** graph or a **Hasse graph.** This topic is covered in the text in section 6.3.

4. **Relations Using Matrices.** This is covered in section 6.4.. The remark given in the notes, Section 6.4, after Theorem 6.4.1 gives a convenient, and much faster way of determining the yes/no matrix of a relation. I will write the (yes/no) matrix of example 1 above using this method.

Example 3. Let A = {a, b, c} and let r be the relation defined on A as r = {(a, a), (b, b), (c, c), (a, c), (b, a), (b, c)}. Write the yes/no or Boolean matrix of the relation r. First from the notes we realize that there is a 1 (a true) for each ordered pair (or directed edge) in the relation R. Since R contains 6 ordered pairs there are 6 1's in the matrix. I will use the notation for the matrix of r as M_R. Since the domain and the codomain are the same set A = {a, b, c} we will prefix both the rows and the columns of M_R by the elements in A in the order listed, namely a, b, c. For ease in typing I will first write the matrix in table form as

	a	b	c
a	1	0	1
b	1	1	1
c	0	0	1

$$\text{so} \quad M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

In section 6.4 we discuss Boolean products of matrices. Here is more information on this operation if you need it.

The Boolean product of matrices

Introduction

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. If we compute the product AB using regular arithmetic we obtain the matrix $AB = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix}$. The entry in the 1st row 2nd column is computed as $1 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 2$,

Boolean Matrices. The above matrices can be considered yes/no matrices. Yes/no matrices contain only 0's and 1's and are called by a variety of names depending on the context: 0/1 matrices, adjacency matrices and Boolean matrices. The term Boolean matrix is used if the operation we want to perform on them is Boolean arithmetic.

Boolean Arithmetic. Most of what follows was discussed in our notes on logic. When we studied logic we defined the following binary operations:

Definition of AND, $\wedge$

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Definition of OR, $\vee$

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

I also mentioned at that time that the notation for OR ($\vee$) in Boolean Algebra is + but it is still read as "or". Also, the multiplication symbol, $\cdot$, is used in Boolean Algebra for the logical symbol for AND, $\wedge$. Note in Boolean Algebra the symbol $\cdot$ is read as "and". Replace the $\wedge$ and $\vee$ symbols in the above tables by $\cdot$ and $+$ respectively to obtain

Definition of AND, $\wedge$, $\cdot$

p	q	$p \cdot q$
0	0	0
0	1	0
1	0	0
1	1	1

Definition of OR, $\vee$, +

p	q	$p + q$
0	0	0
0	1	1
1	0	1
1	1	1

Look at the AND table in the context of using the symbol $\cdot$ for AND. We note that this table is exactly the same as that of regular multiplication, for example, $0 \cdot 1 = 0$, $0 \cdot 0 = 0$, etc. **Boolean Arithmetic under • is exactly the same as regular arithmetic under multication.**

Now look at the OR table in the context of using the symbol + for OR. We note that this table is almost the same as that of regular addition, **except** 1 + 1 = 1, So in this notation 0 + 1 = 1 0 + 0 = 0 and 1 + 0 = 1. **Boolean Arithmetic under + is the same as regular arithmetic under addition except 1 + 1 = 1.**

Let's take the two matrices A and B above and find their Boolean product. That is, find their product but using Boolean arithmetic.

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Using Boolean arithmetic the entry in the 1st row 2nd column in AB is computed as $1 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 1$. The entry in the 1st row 1st column of the matrix AB is $1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 1$. The reader should verify that the Boolean product of A and

B is AB = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. To distinguish the Boolean product of matrices from the usual product the symbol $\odot$ is sometimes used. So AB = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ becomes $A \odot B$ = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Exercise.

Let A = $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and let B = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Show that the Boolean product

AB = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

One of the first tables I gave you in the notes which compares the different algebras I repeat below. Now we could add the column Boolean Algebra. It would be the same as that of logic except we would use the symbols $\cdot$ and $+$.

Table : Similarities Between "Algebras"

	High School Algebra	**(Algebra of) Logic**	**(Algebra of) Sets**	**Matrix Algebra**
Objects	Real numbers	Propositions	Sets	Matrices
Binary Operations	Addition, + Mutiplication, •	OR, $\vee$, + AND, $\wedge$, •	Union, $\cup$ Intersection, $\cap$	Addition, + Multiplication, •
Unary Operations	Additive inverse - Multiplicative Inverse -1	Negation, $\neg$ or $\approx$	Complementation ′ or c or -	Additive Inverse – Multiplicative Inverse -1
Other Connectives	$\leq$ =	$\Rightarrow$ $\Leftrightarrow$	$\subseteq$ =	=
Usual Constant/ Variable Names	a, b, c,…. x, y, z,….	0, 1 p, q, r,…	A, B, C,… X, Y, Z,…	A, B, C,… X, Y, Z,…
Some Basic Laws	See table	See table	See table	See table