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Week 4

Laws of Addition, Combinations, the Binomial Coefficient

Laws of Addition

Example 1. A student must select a project from one of three lists provided by the professor There are 5 projects on the first list, 10 on the second and 12 on the third. How many possible projects are there to choose from?

Solution: From the first list the student can choose as a project any one of 5 projects, from the second any one of 10 and from the third list any one of 12.

So, by the sum rule we have $5 + 10 + 12 = 27$ projects to choose from.

This example illustrates the **Sum Rule**, which is one of the basic laws of addition. In this example notice the student had to choose a project from one of three separate or disjoint lists. The student must choose a project from list I **OR** from list 2 **OR** from list 3. A use of the word or directly or implicitly usually "flags" that the sum rule applies. When a situation can be partitioned into disjoint parts the basic **Sum Rule** applies.

Sum Rule: If two operations must be performed, and if the first operation can always be performed p_1 different ways and the second operation can always be performed p_2 different ways, **and if these operations cannot be done at the same time** then there are $p_1 + p_2$ different ways that the two operations can be performed. This can be extended to any number of operations/tasks. This basic law of addition is stated in more general form in section 2.3 after example 2.3.2.

Example 2.3.5 illustrates **a second law of addition**, called the Inclusion-Exclusion Laws. Read the notes for a statement of these laws. I will illustrate one of them in example 2.

Example 2. I will state the Inclusion-Exclusion Law and then will give the example.
Let A_1 and A_2 be finite sets then $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

How many bit strings of length 8 begin with a 0 or end with a 1?

First, a couple of comments. This means how many bit strings of length 8 begin with a 0 (we know the solution here is 2^7) or end with a 1 (we know the solution here is 2^7) or both begin with a 0 **and** end with a 1 (solution here is 2^6).
If we let A_1 stand for the bit strings of length 8 which begin with a 0 and
let A_2 stand for the bit strings of length 8 which begin with a 0

The first part of section 6.3 in the text is on "permutations". You may recall that back in week 2 I asked you to do exercise 25 of this section using the **Rule of Products**. I wanted to show you that **every single "permutation problem" can done, in fact, more easily, using the more natural counting technique, the Rule of Products.** Permutations problems are just a subset of

"Rule of Products" problems. Every problem that the text does using "permutations" can just as easily (in fact easier) using the rule of products. Try it!

Recall from week 3 that a Permutation is an arrangement of elements, an **ordered** listing of elements. There are "counting problems" where the ordering of the elements does not matter.

Example 3. Suppose that someone has 3 different coins in his pocket; a nickel, a dime and a quarter. He withdraws two coins, looks and them and returns them to his pocket. He continues this process. How many different sums of money can he have.
Answer 3 different sums
.Certainly if he withdraws a nickel and a dime and the later a dime and a nickel he has the same sum in each case, 15 cents. So the order of choosing the coins is immaterial.

Combinations

A combination is an **unordered** listing. I explain the difference between an ordered listing and an **un**ordered list in the following two examples:

Example 3. Suppose that three **different** prizes (1^{st} prize \$1,000 or 2^{nd} prize \$500 or 3^{rd} prize \$100) are going to be awarded to three of the 6 finalists in a drawing. Assume that a person can only win one prize.
(Note, I am assuming that it makes a difference which prize you are awarded.) How many ways are there to award these prizes?

The 1^{st} prize can be awarded to any one of 6 people so there are 6 choices for the 1^{st} prize. There are 5 people left so the 2^{nd} prize can be awarded to any one of 5 people. Next there are only 4 people left so there are 4 people to choose from to award prize 3. Therefore, there are, by the rule of products $6 \cdot 5 \cdot 4 = 120$ ways of awarding the three prizes. Or, if you prefer to use the Permutation Rule the solution is P(6,3) which of course equals $6 \cdot 5 \cdot 4 = 120$.

Example 4. Now suppose that three prizes are all the same prize. That is three \$200 prizes will be awarded. Assume that a person can only win one prize. Here it does not make any difference whether your name is the first name drawn for a prize or the second or the third. All three receive the same prize. The ordering of the names drawn does not make a difference.

The solution is: $\binom{6}{3} = 20.$

The **r-Combinations** Rule is given by Theorem 2. This rule is similar to the Permutation rule and makes use of the formula $C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

The **Binomial Theorem** is defined by Theorem 1 in section 2.4. Several properties of this theorem can be found as corollaries in the text.

Here's a problem that you did in week 2 using the Rule of Products. It is worth studying again.

Example 5. An ice cream parlor advertises that you may have your choice of five different toppings, and you may choose none, one, two, three, four, or all five toppings. How many choices are there in all?

Solution.

Method 1. Using the Rule of Products that we first studied in week 3. For **each** topping you have 2 choices. Either accept the topping or not. Therefore we have 2^5 choices.

Method 2. Using combinations. We want the number of different ways to select either 0, 1, 2, 3, 4, or 5 toppings from a total of 5 possibilities. We assume that it does not make any difference which topping goes on first or second etc. Also, when we say "we want the number of different ways to select 0, 1, 2, 3, 4, or 5 elements from a total of 5" we mean we must select either 0 toppings OR 1 topping OR 3 toppings and so on we cannot select 2 toppings and also 3 toppings at the same time. We must do one or the other. So the SUM rule applies. So this problem involves both the sum rule and the combination rule. You should take five minutes to compute each of the summands

$\binom{5}{0}, \binom{5}{1}, \ldots, \binom{5}{5}$ and note that they form a palindrome (read the same forwards as backwards). Why? See section 2.4, the Binomial Theorem and Binomial Coefficients and Pascal's triangle.

$$\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 2^5$$

Example 6. A woman wants to invite 5 of her 9 close friends for dinner.

(a) How many ways can she do this with no restrictions? Since we are selecting 5 people out of 9 without ordering the solution is:

$$\binom{9}{5} = \frac{9!}{5!(9-5)!} = \frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{(5\cdot4\cdot3\cdot2\cdot1)(4\cdot3\cdot2\cdot1)} = \frac{9\cdot8\cdot7\cdot6}{(4\cdot3\cdot2\cdot1)} = 126$$

(b) How many ways can she do this if two of her friends do not like each other? If she invites one of them she cannot invite the other.
Let's call the two people who do not like each other person a and person b.

Situation 1. If she selects person a for dinner she cannot select person b.
The selection of person a leaves only 4 people to select out of 7 (a was selected and b can't be) to make up the 5 for dinner. This can be done $\binom{7}{4}$ ways.

Situation 2. This is simply the "flip-flop" of the above. If she selects person b for dinner she cannot select person a. This can be done $\binom{7}{4}$ ways.

Situation 3. She invites neither person a nor person b. Here she must invite 5 out of the remaining 7 friends. This can be done $\binom{7}{5}$ ways.

So the solution for part (b) is: $\binom{7}{4} + \binom{7}{4} + \binom{7}{5}$ or $2\binom{7}{4} + \binom{7}{5}$. Can you explain why this is the solution? You should work out the details

(c) How many ways can she do this if two of her friends are a couple? If she invites one of them she must invite the other. There are two situations here:

Situation 1. She invites the couple. This can be done $\binom{7}{3}$ ways, Why?

Situation 2. She doesn't invite the couple. This can be done $\binom{7}{5}$ ways, Why?

Example 7. Suppose each single character stored in a computer uses eight bits. Then each character is represented by a different sequence of eight 0s and is called a *bit pattern*.

(a) How many different bit patterns contain exactly 3 ones? Before you solve the problem you may want to write several bit patterns that contain 3 ones.
Since if we decide to put ones in the first , third and seventh places it does not make any difference if we put a 1 in the 7^{th} place first and then in the 1^{st} place and finally in the 3^{rd} place. The ordering of placing the 1's is immaterial so the Combination Rule applies so the number bit patterns that contain exactly 3 ones is $\binom{8}{3}$.

(b) How many different bit patterns contain at least 3 ones? To contain at least 3 ones means to contain 3 ones or 4 ones or . . . or 8 ones. This can be done
$\binom{8}{3} + \binom{8}{4} + \binom{8}{5} + \cdots + \binom{8}{8}$ ways. You should be able to explain this. Can this problem be done another way?

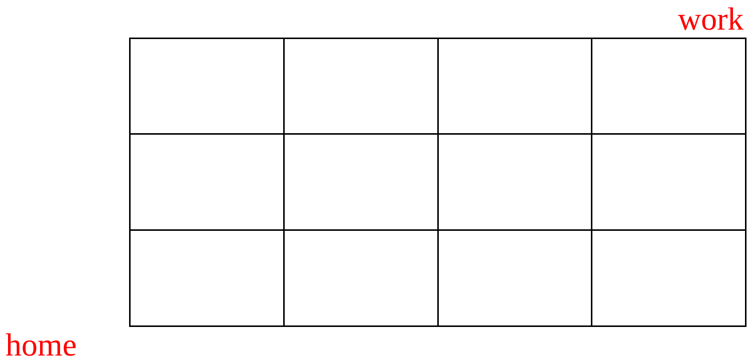
More Counting problems (Some from *The Magic of Numbers* by Gross)

Example 8. Suppose that you are playing scrabble and you have in your rack the letters

"E, E, E, E, N, N, N". How many ways are there of arranging these tiles in your rack? That is, how many seven-letter words (strings of length 7 of these letters) can be formed that contain exactly four Es and three Ns?

This problem is simple. To determine or write any seven-letter word with these letters all we have to specify which of the 4 places the Es are to fill (or equivalently which of the 3 places the Ns are to fill). $\binom{7}{4} = \binom{7}{3} = 35$

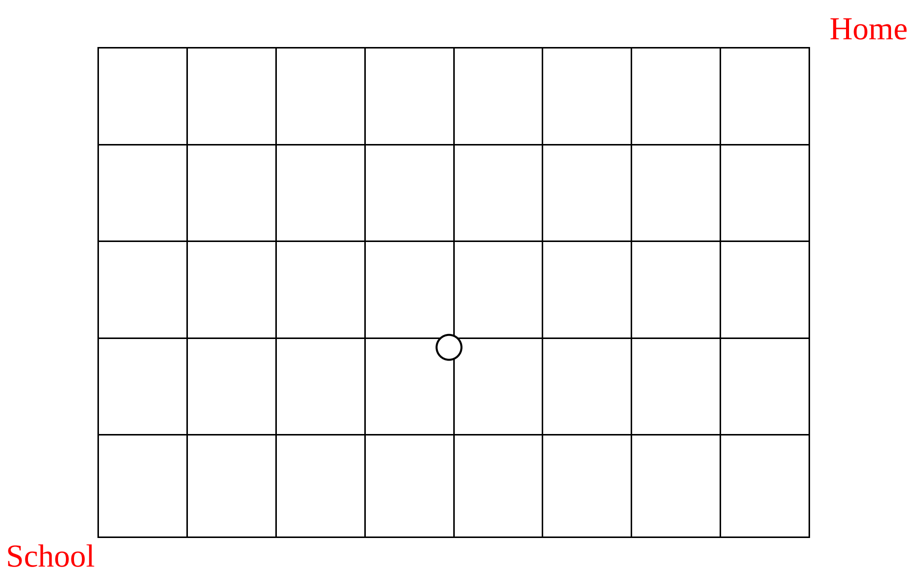
Example 9.



Suppose we live in a city where the streets are arrayed in a rectangular grid as above. How many different paths from home to work can we take? For example, using N for north and E for east one path is E, E, N, E, N, N, E. That is, we need seven moves that contain exactly four Es and three Ns. This is the above problem so the answer is 35 as in example 8.

In general if we have a k by l rectangular grid the number of different paths from the lowest left corner to the top right corner is $\binom{k+l}{k}$.

Example 10.



(a) Consider the grid above. How many paths of shortest possible length are there from the point labeled “school” to the point labeled “home”?

Solution: As above, any path from the point “school” to the point “home” will contain a total of 13 N’s (north) and E’s (east). We have a 5 blocks north (N’s) and 8 blocks east (E’s). So we only have to specify which of the 5 places (of the 13) we wish to fill with N’s (the directions north), $\binom{13}{5}$. Or we could specify which of the 13 places we could specify the easterly direction, $\binom{13}{8}$. In either case the answer is 1,287.

(b) Now suppose, a student going from school to home must first go to work, which is at the point indicated by the circle. How many paths are there of minimum length going from school to home stopping at work? Is the following correct?
$\binom{6}{4}+\binom{7}{4}$ or $\binom{6}{2}+\binom{7}{3}$ or ?

(c) Now suppose that you have called in sick at work and you do not want to drive through that intersection. How many paths are there from school to home omitting that intersection?

Example 11. Assume that your job as a senior on campus is to assign students to dorm rooms. Assume you have nine students to assign to rooms and the rooms are a quad, a triple and a double. How many ways can you make the assignments?

Solution, $\binom{9}{4}\cdot\binom{5}{3}\cdot\binom{2}{2}=\binom{9}{4}\cdot\binom{5}{3}=1,260$. Why? Note, $\binom{2}{2}=\frac{2!}{0!2!}=1$, and in general $\binom{n}{n}=\frac{n!}{0!n!}=1$.

But wait, what if you assigned the rooms in a different order? That is, assigned 2 students to the double first and the 4 to the quad and the remaining 3 to the triple? $\binom{9}{2}\cdot\binom{7}{4}$. Is the answer the same? Let’s look at both results again.

$$\binom{9}{4}\cdot\binom{5}{3}\cdot\binom{2}{2}=\frac{9!}{4!5!}\cdot\frac{5!}{3!2!}\cdot 1=\frac{9!}{4!3!2!}$$

$$\binom{9}{2}\cdot\binom{7}{4}\cdot\binom{3}{3}=\frac{9!}{2!7!}\cdot\frac{7!}{4!3!}\cdot 1=\frac{9!}{4!3!2!}\ \binom{9}{2}\cdot\binom{7}{4}\cdot\binom{3}{3}=\frac{9!}{2!7!}\cdot\frac{7!}{4!3!}\cdot 1=\frac{9!}{4!3!2!}$$
$$\binom{9}{2}\cdot\binom{7}{4}\cdot\binom{3}{3}=\frac{9!}{2!7!}\cdot\frac{7!}{4!3!}\cdot 1=\frac{9!}{4!3!2!}.$$

This type of problem appears frequently and deserves more thought. In each of the above, we had 9 students which we wanted to assign to three rooms so we assign the first group of students to one room (the quad) the second group to a second room (the triple) and the remaining students to the only room left (a double).

In general if we have a number n and three numbers a, b, and c **that add up to** n, the number of ways of distributing n objects into 3 collections of sizes a, b and c is $\frac{n!}{a!b!c!}$.

This number is called a *multinomial coefficient* and it is denoted by the symbol $\binom{n}{a,\ b,\ c}$.

That is, $\binom{n}{a,\ b,\ c}=\frac{n!}{a!b!c!}$. This is Theorem 4 in section 5.5.

The derivation of the *multinomial coefficient* formula is as follows:

We first choose which "a" of our "n" objects go into the first group. This can be done $\binom{n}{a}=\frac{n!}{a!(n-a)!}$ ways. Next, we choose which "b" of the remaining n – a objects go in the second group. This can be done $\binom{n-a}{b}=\frac{(n-a)!}{b!(n-a-b)!}$ ways. The remaining c objects go into the third group. This can be done in only one way. So by the product rule we have:

$$\binom{n}{a}\cdot\binom{n-a}{b}\cdot 1=\frac{n!}{a!(n-a)!}\cdot\frac{(n-a)!}{b!(n-a-b)!}$$

Note "the remaining c objects" is n - a – b = c. So the above becomes.

$$=\frac{n!}{a!(n-a)!}\cdot\frac{(n-a)!}{b!c!}$$
$$=\frac{n!}{a!b!c!}.$$

Fill in the following blank: The number of ways of distributing n objects into groups of sizes a_1, a_2, a_3, a_4, …, a_k is ________________

The number is called a *multinomial coefficient* and it is denoted by the symbol $\binom{n}{a,\ b,\ c}$.

The number $\binom{n}{a,\ b,\ c} = \dfrac{n!}{a!b!c!}$ is called a **multinomial coefficient** since it a coefficient of a multinomial.

Example 12. For example we know that if to expand the trinomial $(x + y + z)^6$ to multiply $(x + y + z)\ (x + y + z)\ (x + y + z)\ (x + y + z)\ (x + y + z)\ (x + y + z)$. One of the terms of this expansion is $x^2\ y\ z^3$. Note the powers of the x, y and z add up to 6. Will this always happen for this expansion? The coefficient of the term $x^2\ y\ z^3$ is $\dfrac{6!}{1!2!3!}$. This is so since we are doing exactly what we did above, namely, choose x from 2 factors, y from a different factor and z from the remaining 3 factors or $\dfrac{6!}{2!1!3!} = \dfrac{6!}{1!2!3!}$.

What is the coefficient of $x^1\ y\ z^4$ in the above expansion?