Week 2
More on Sets and Summation

I will comment on the topics of week 1 given in the text. Read the text and then read the following.

Sets:

You can think of week 1 as a combination of learning notation and the structure of the mathematical system, sets. Examples of notation are such items as the notation for the set of integers, **Z**, the notation, for "is an element of", $\in$, the null or empty set, $\varnothing$, etc. We have also studied the definitions of the intersection, $\cap$,the union, $\cup$ and equality of sets, =, subset, $\subseteq$ and the complement of a set. Note, there are at least two different notations for the complement of a set, namely, A^c and $\overline{A}$.

The structure of the mathematical system sets is "how it behaves". What properties or laws does it follow. In week 1 we compared the structure of high school algebra with that of set algebra by comparing some basic properties of high school algebra with those of sets. I list the basic set laws again as a reminder of the key properties of sets.

Basic Set Laws

Commutative Laws	
(1) $A \cup B = B \cup A$	(1)' $A \cap B = B \cap A$
Associative Laws	
(2) $A \cup (B \cup C) = (A \cup B) \cup C$	(2)' $A \cap (B \cap C) = (A \cap B) \cap C$
Distributive Laws	
(3) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	(3)' $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Identity Laws	
(4) $A \cup \varnothing = \varnothing \cup A = A$	(4)' $A \cap U = U \cap A = A$
Complement Laws	
(5) $A \cup A^c = U$	(5)' $A \cap A^c = \phi$
Idempotent Laws	
(6) $A \cup A = A$	(6)' $A \cap A = A$
Null Laws	
(7) $A \cup U = U$	(7)' $A \cap \varnothing = \varnothing$
Absorption Laws	
(8) $A \cup (A \cap B) = A$	(8)' $A \cap (A \cup B) = A$
DeMorgan's Laws	
(9) $(A \cup B)^c = A^c \cap B^c$	(9)' $(A \cap B)^c = A^c \cup B^c$
Involution Law	
(10) $(A^c)^c = A$	

Table 3

However, the algebra of sets is not **exactly** the same as that of regular algebra as example 1 will illustrate a difference between sets and regular algebra in example 1.

In high school algebra two commonly used and very useful properties in solving equations are the so called cancellation laws.

a) If x + y = x + z then y = z for all real numbers x, y and z.
b) If xy = yz then x = y for all real numbers x, y and z where z $\neq$ 0

Example 1.
A property like that in item (a) for sets is:
Let A, B, C be any three sets. Is the following true? If A $\cup$ C = B $\cup$C then B = C .
I claim that this statement is **false.** How do we prove or show that a statement is false? Is the statement “Everyone in this class is over 6 feet tall true?” Is it false? How do we prove it is false? This is easy, all we have to do is give one example where a person in the class is **not** over 6 feet tall. For example me. The example showing a statement is false is called a **counterexample.** Can you give a counterexample to the statement “If A $\cup$ C = B $\cup$C then B = C”? To do this you need examples of three sets A, B, and C where A $\cup$ C = B $\cup$C is true but B = C is false. Be concrete, in your examples of sets a, B, and C. for example let A = {1, 3, 5} and B = ? and C = ? Now show that A $\cup$ C = B $\cup$C is true for your example but B = C is false.

Cartesian Product

Many of the discrete subjects we will review later in the course are based on the concept of the *Cartesian product of sets*. The definition is given in the text in section 1.3. Here are a few more examples.

Example 2.
What is the Cartesian product of A={x , y} and B={2, 4}?
Solution:
A X B = {(x ,2),(x ,4),(y ,2),(y ,4)}. It is important to note that the pairs in A X B are **ordered** pairs. That is, the pair (x, 2) is different than the pair (2, x). This is exactly the same reason that the pair, or point (2, 3) in the xy-plane is not equal to the point, or pair (3, 2).
Is this set the same as the set B X A? NO. Write out the set B X A and notice (x, 2) $\in$ A X B but (x, 2) $\notin$ B X A.

Example 3.
Recall from the text that **R** stands for the set of all real numbers, that is, all numbers on the number line. So we can visualize the x-axis as **R**, and also the y-axis as **R**.
So **R** X **R** = $\{(x, y) \mid x \in R,\ y \in R\}$. That is, **R** X **R** “is the algebraic” description of the xy-plane. **R** X **R** is often written as $\mathbf{R}^2$ which is read as “**R** two”, or two space, not **R** squared.
What is $\mathbf{R}^3$?

Example 4.
Let A = $\{x \in R \mid 0 \le x \le 1\}$

(a) Draw the set A on the x-axis as part of the xy-plane.

(b) Draw the same set on the y-axis. x is a “dummy variable” so now call it y if you wish.

(c) Describe A^2 geometrically on the xy-plane Keep in mind that $A^2 = A \times A$. Answer; A^2 is the interior and the border of the unit square, that is, the square with vertices at (0,0),(1,0),(0,1) and (1,1).

(d) What is A^3? Answer: The interior and the border of the unit cube. Can you visualize this?

Hopefully the above examples and the text examples illustrate the power and simplicity of set notation.

Note those who have taken Calculus recognize the set A in example 4 as usually written in the notation [0, 1]. So I could have said find $[0, 1]^2$ and $[0, 1]^3$.

Power Set

Example 5. I will give procedure to find the power set of any set through an example.
Let A = {a, b, c}.
By definition the power set of A = P(A) = { of all subsets of A}
= { of all subsets which contain o elements, all subsets which contain 1 elements, all subsets which contain 2 elements, all subsets which contain 3 elements, etc }
For the above set A we have
{ ∅, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c} }

Also note, for this example $|P(A)| = 2^3 = 8$

Binary Representation. See the text

Sequences

Definition: A sequence is a function from the set of positive integers {1, 2, 3, . . .} or from the set of nonnegative integers {0,1,2,3, . . .} to the set of real numbers.

The definition of sequence indicates that there are two common notations for terms of a sequence, one using the **positive** integers, for example,
$x_1, x_2, x_3, \ldots$ and the other using the **nonnegative** integers, for example,
$x_0, x_1, x_2, \ldots$

Both notations have their own advantages. Using the notation $x_1, x_2, x_3, \ldots$ the 15th term is clearly x_{15}, simple. Where in the notation $x_0, x_1, x_2, \ldots$ the 15th term is x_{14}, a little awkward. On the other hand, certain applications lend themselves to the latter notation. Examples, your initial deposit in a bank of $1,000 at time t = 0 so x_0 is 1000. Applications in biology when the initial population of bacteria is x_0, in physics where the initial velocity is v_0.

Example 6. The sequence 1, 3, 5, 7, . . . is quite simple and if we were asked to write out several terms we could easily do so and get 1, 3, 5, 7, 9, 11, 13, ... it is clear that to find

any term in the sequence just add 2 to the previous term. So the nth term is 2 + (n – 1)th term. We can say this in notation as follows $x_n = 2 + x_{n-1}$. This formula describes the given sequence **once we know where to start.** So to describe the above sequence we need to know two things: the starting “point” and the formula connecting the terms of the sequence. The sequence of this example can be described as:
$x_n = 2 + x_{n-1}$ and $x_1 = 1$.

Example 7. Write the first three terms of the sequence described by $x_n = 5 + x_{n-1}$ and $x_1 = 2$. In this notation the first three terms are: $x_1, x_2, x_3, \ldots$ where $x_1 = 2$, $x_2 = 5 + x_1 = 5 + 2 = 7$ and $x_3 = 5 + x_2 = 5 + 7 = 12$. So the sequence is
2 , 7, 12, . . .

The sequences described in examples 1 and 2 are called recurrence relations. A recursive description is one where one or more terms are defined in terms of previous terms. The term difference equation is also used for recurrence relation.

Example 8. Write out the first five terms of the sequence described by $x_n = x_{n-1} + x_{n-2}$. Where $x_1 = 1$ and $x_2 = 1$. The first three terms are: $x_1, x_2, x_3, \ldots$ where $x_1 = 1$, $x_2 = 1$ and $x_3 = x_2 + x_1 = 1 + 1 = 2$, $x_4 = x_3 + x_2 = 2 + 1 = 3$,
$x_5 = x_4 + x_3 = 3 + 2 = 5$. So the sequence is: 1, 1, 2, 3, 5, 8, ... This sequence is called the Fibonacci sequence. Google Fibonacci sequence to find its many interesting applications. Another interesting sequence is that referred to as “the Tower of Hanoi”. Google “Tower of Hanoi puzzle”.

Series/Summation

A sum of numbers $x_1 + x_2 + x_3 + \ldots + x_n$ is called a series, in fact, a finite series since the sum terminate at a definite number, namely x_n. The three dots . . . in the “middle” of the above terms just indicates continue with the format given and stop at x_n. Series appear frequently in a variety of applications so a notation has been developed to write such sums compactly. The Greek letter “S” which is $\sum$ for sum is used. $\sum$ is read as sigma.

So the series $x_1 + x_2 + x_3 + \ldots + x_n$ be replaced by $\sum_{i=1}^{i=n} x_i$. The number i = 1 is called the lower limit and i = n is called upper limit and i itself is referred to as the index of summation. So $\sum_{i=1}^{i=n}$(some expression involving i) means replace I in the expression by all **integers** starting with up to i = n. Note, the upper limit is frequently just replace by n.

Example 9. Write out the given series in summation notation.
$\sum_{i=1}^{i=n} i^3$ so all we have to do is replace i by 1, then a + sign, then replace i by 2 etc. giving

$\sum_{i=1}^{i=n} i^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3$. The top index is commonly written as simply n but it means i = n

Example 10. Write out the given series in summation notation and simplify.

$\sum_{i=1}^{3}(2i+3)$. First note that the top index is simply written as 3 not i = 3. This is common, however the bottom limit always has i or some other variable mentioned. We know we must replace i by 1 then by 2 and then by 3.

$\sum_{i=1}^{3}(2i+3) = (2(1) + 3) + (2(2) + 3) + (2(3) + 3) = 21.$

Example 11. Write out the series $\sum_{i=1}^{4} 3$ in summation notation and simplify. This is tricky but example 5 gives us a hint on how to do this example. In example 5 we notice that 3 is added 3 times. Why?

Again, this is like example 5 except we have to add 4 terms, not 3, and there is no 2i to contend with. So we just add 3 four times. That is, $\sum_{i=1}^{4} 3 = 3 + 3 + 3 + 3 = 12$. I claim $\sum_{i=0}^{4} 3 = 15$. Why?

Example 12. Write out the given series in summation notation and simplify.

$\sum_{i=0}^{2}\sum_{j=1}^{3}(2i+j)$. When you have two or more summations to evaluate work from the "inside-out". Do the inside summation first. That is, $\sum_{i=0}^{2}\sum_{j=1}^{3}(2i+j)$ really means $\sum_{i=0}^{2}\left(\sum_{j=1}^{3}(2i+j)\right)$.

$\sum_{i=0}^{2}\sum_{j=1}^{3}(2i+j) = \sum_{i=0}^{2}\big((2i+1) + (2i+2) + (2i+3)\big)$ Replace j by 1 then 2 and then 3,

Next, replace I by 0 then 1 then 2 to get

$= \sum_{i=0}^{2}(6i+6)$ Combine like terms (1)

$= 6(0) + 6 + 6(1) + 6 + 6(2) + 6$

$= 36$

Note, we could have simplified equation (1) as follows:

$$\sum_{i=0}^{2}(6i+6) = \sum_{i=0}^{2}6(i+1) = 6\sum_{i=0}^{2}(i+1) = 6\big((0+1)+(1+1)+(2+1)\big) = 6(6) = 36$$

Example 13. I suggest that you try the following. Expand and simplify $\sum_{i=1}^{n}(a_{i+1} - a_i)$