A. Doerr

Week 3 Basic Counting Techniques The Rule of Products

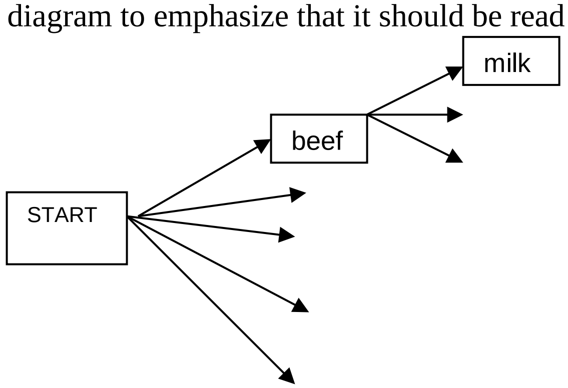
WHAT IS COMBINATORICS?

One of the first concepts our parents taught us was the "art of counting." We were taught to raise three fingers to indicate that we were three years old. The question of "how many" is a natural and frequently asked question. Combinatorics is the "art of counting." It is the study of techniques that will help us to count the number of objects in a set quickly. Counting occurs not only in highly sophisticated applications of mathematics to engineering and computer science but also in many basic applications. Like many other powerful and useful tools in mathematics, the concepts are simple; we only have to recognize when and how they can be applied. The following examples will illustrate that many questions concerned with counting involve the same process.

The Rule of Products

The Product Rule is probably one of the most versatile counting techniques. The first two examples come directly from the text.

Example 1. A snack bar serves five different sandwiches and three different beverages. How many different lunches can a person order? One way of determining the number of possible lunches is by listing or enumerating all the possibilities. One systematic way of doing this is by means of a tree. See trees in the section on *the Basics of counting.* The mathematical concept of trees just follows the ideas of "family trees". Some trees are read from top to bottom , others are written so they are read from left to right as the one below. For readability label the vertex on the extreme left START. I have used arrows in the following diagram to emphasize that it should be read from left to right.



Assume the five sandwich choices are: beef, cheese, chicken ham and bologna. Label the tips of each of the above arrows by these choices. For consistency label them in the order given from the top-most "arrow" to the one at the bottom. Next we have three choices of drink. Let's assume they are: milk, juice and coffee in that order. Draw three arrows from each of the sandwich choices and label them in order milk, juice and coffee. When you fill out the above diagram completely you should have a total of 15 possible choices of lunch. Every path that begins at the position labeled START and goes to the

right can be interpreted as a choice of one of the five sandwiches followed by a choice of one of the three beverages. Note that considerable work is required to arrive at the number fifteen this way, but we also get: more than just a number. The result is a complete list of all possible lunches. For example the top most choice is beef sandwich and milk. A listing of possible lunches a person could have is:
{(BEEF, milk),(BEEF, juice), (BEEF, coffee), ..., (BOLOGNA, coffee)}.

An alternative method of solution for this example is to make the simple observation that there are five different choices for sandwiches and three different choices for beverages, so there are $5 \cdot 3 = 15$ different lunches that can be ordered. This is the **Rule of Products**. NEVER underestimate the power of simple observations.

Example 2. (example 2 of the text) Let A = {a, b, c, d, e} and B = {1, 2, 3}. From Section 1.4 we know how to list the elements in A X B = {(a, 1), (a. 2), (a, 3), .. .,(c, 3)}. The reader is encouraged to imitate the above figure for this example. Since the first entry of each pair can be any one of the five elements a, b, c, d, and e, and since the second can be any one of the three numbers 1, 2, and 3, it is quite clear there are 5 • 3 = 15 different elements in A X B. This we already knew from our discussions on Cartesian product.

Rule Of Products: If two operations must be performed, and if the first operation can always be performed p_1 different ways and the second operation can always be performed p_2 different ways, then there are $p_1 p_2$ different ways that the two operations can be performed. This can be extended to any number of operations/tasks.

Note: it is important that p_2 does not depend on the option that is chosen in the first operation. Another way of saying this is that p_2 is independent of the first operation. If p_2 is dependent on the first operation, then the rule of products does not apply.

Example 3. Assume in Example 1 that coffee is not served with a beef or chicken sandwich, then by inspection of Figure 2.1.1 we see that there are only thirteen different choices for lunch. The rule of products does not apply, since the choice of beverage (p_2) depends on one's choice of
sandwich (p_1).

The **extended rule of products** expands the product rule of two tasks to any number of tasks. It is stated in the text.

More on Bit Strings.

In section 1.4 we discussed the binary representation of positive integers. In that section we learned that a positive integer is represented by a sequence of binary digits (0's or 1's) called **bits**. Such a sequence is usually referred to as a bit string. For example, 01101 is a bit string. The **length** of a bit string is the number of bits in the string. 01101 is a bit string of length 5.

Example 4. Use the **extended rule of products** to answer the following. I'll do a couple of them.

(a) How many bit strings are there of length eight? **Explain.**
Since the bit string is of length it must be of the form
$a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8$ where each a_i is a bit which means it must be a 0 or a 1, two choices. Since there are 2 choices for each of the 8 a_i we have 2x2x2x2x2x2x2x2 (8 (times) = 2^8 = 256 bit strings of length 8.

(b) How many bit strings are there of length eight which begin with a 1? **Explain**
This almost the same as part a except there is only 1 choice for the first bit but 2 choices for the remain 7 bits so the solution is 2^7.

(c) How many bit strings are there of length eight which begin with a 0 and end with a 1? **Explain** solution 2^6

(d) How many bit strings are there of length eight which begin with three 1's? **Explain** solution 2^5

(e) How many bit strings are there of length eight which contain an even number of 1's? **Explain** solution 2^7

Example 4 part a could have been stated the following way. Computers use bit strings of length 8, called bytes, to represent the characters (letters both upper case and lower case, punctuation symbols, [, {, the integers 0 through 9 etc) on a key board. The Extended ASCII code is one such coding system. Some examples of this code are: "a" is represented by 01100001, "A" is represented by 01000001 and "{" is represented by 01111011 and the number 1 is represented by 00000001. How may such symbols can be described using a byte?

Go to the exercises in section 2.2 of the and do problems 1, 2, 3, and 5 using the product rule only. Do not read the section just do the exercises. You will find out that every single "permutation problem" given in section 2.2 can be done (more easily) using the Rule of Products. The permutation formula is not necessary to do the exercises in section 2.2. We mention the formula so we can use it to derive the **Combination formula** given in section 2.4. So every single "permutation problem" can be done using the Rule of Products but the converse is not true. It is not true that every permutation problem is a Rule of Products problem. The following simple problem illustrates the difference.

Example 5. How many 3-digit numbers can be formed using the numbers 1, 2, 3, 4, 5.

a) If repetition of digits is allowed.
Here since repetition of digits is allowed the first digit can be any one of the 5 numbers, also the second digit can be any one of the 5 numbers and the same with the third digit. So by the product rule we can form 5 x 5 x 5 = 125 three digit numbers if repetition of numbers is allowed. Note, we used the Rule of

Products to solve this problem but because repetition is allowed we will see the "Permutation formula" cannot be used.

b) If no repetition of digits is allowed.
Here since repetition of digits is not allowed the first digit can be any one of the 5 numbers. This uses one of the 5 numbers so the second digit can be any one of the remaining 4 numbers and the third digit can be any one of the remaining 3 numbers. So by the product rule we can form 5 x 4 x 3 = 60 three digit numbers if repetition of numbers is not allowed. Again, we used the Rule of Products. Since this is an ordered arrangement of 3 numbers the Permutation formula can be used.

Permutations

The definition of permutation is given on the first page of section 2.2. I repeat the definition here investigate what it says.

Definition: Permutation. An ordered arrangement of k elements selected from a set of n elements $0 < k \leq n$, where no two elements of the arrangement are the same, is call a permutation of n objects taken k at a time. The total number of such permutations is $P(n; k)$.

Here are two more examples just like example 4.

Example 6. How many three digit numbers can be formed using the 9 digits 1 through 9, where repetition of digits is **not** allowed.

Method 1. **Using the Rule of Products.** The solution is $9 \cdot 8 \cdot 7$ since there are 9 choices for the first digit, 8 choices for the second digit and 7 for the third .

Method 2. The **Permutation formula** can also be used and the solution can be obtained by a Corollary 1 of Theorem 1, section 6.3:

In general, where $0 \leq r \leq n$ we have $P(n\,,\,r) = \dfrac{n!}{(n - r)!}$

So for this example

$$P(9,3) = \frac{9!}{(9-3)!} = \frac{9!}{6!} = 9 \cdot 8 \cdot 7.$$

Example 7. How many three digit numbers can be formed using the 9 digits 1 through 9, where repetition of digits **is** allowed.

Method 1. **Using the Rule of Products** the solution is $9 \cdot 9 \cdot 9$. Why?

Method 2. The **Permutation formula** can**not** be used because repetition of digits is **not** allowed using the Permutation formula. Notice in Theorem 1 the Permutation formula applies only if we have "n **distinct** elements". This says that repetition of digits is **not** allowed

The second key counting technique of section 6.3 is the **Combination Rule** (Theorem 2). Read this material carefully. For rule of product and permutation problems **order** or **arrangement** is important. In fact the word to permute means to arrange, to order. A

Here are a variety of problems in no particular order. How many of them can you do just using the rule of products?

1. A family of 5 consisting of the parents and 3 children are going to be arranged in a row by a photographer. How many ways are there to arrange the 5 members of the family (no restrictions)? If the parents are to be next to each other, how many arrangements are possible?
   (parents = 2 . 1 ways)(children = 3 . 2 . 1 ways)(4)

   The 4 comes from the parents can stand in any one of 4 places first, between each of the children, and last, that is, P- C_1-P-C_2-P-C_3-P

2. How many ways can the manager of a baseball team select a pitcher and a catcher for a game if there are 5 pitchers and 3 catchers on the team?
   ans = 15

Note as in exercise 3 section 2.2 by the term "words" we mean a sequence or string of letters. The "word" does not have to make sense in any language. For example abdec is a "word" since it is a sequence of 5 letters.

3. How many different three-letter words can be formed using the five letters a, b, c, d and e if repetitions are not allowed?
   ans = 60

4. How many different three-letter words can be formed using the five letters a, b, c, d and e if repetitions are allowed?
   ans = 125

5. How many different three-letter words (using the five letters a, b, c, d and e) without repetitions can be formed whose middle letter is *a*?
   ans = 12

6. An ice cream parlor advertises that you may have your choice of five different toppings, and you may choose none, one, two, three, four, or all five toppings. How many choices are there in all?

   Solution.

   Method 1. Use the Rule of Products For each topping you have 2 choices accept the topping or not. Therefore you have 2^5 choices.

   Method 2. We will do this problem another way in section 2.4.