Week 7

Before we discuss math induction a brief comment. Example 3.7.5 define **prime number**.

A famous conjecture (what does this term mean) is **Goldbach's conjecturewhich** is one of the oldest and best-known unsolved problems in number theory and in all of mathematics. It states: Every even integer greater than 2 can be expressed as the sum of two primes. Can you give several examples?

Mathematical Induction and Predicates, Quantifiers

Part I Mathematical Induction

Study the introductory material in section 3.7 of the text and the procedures outlined in the examples 3.7.1 and 3.7.3. What follows is more explanation and several detailed examples.

Consider the following sentences:

1. S/he is over 6 feet tall. (Assume we are talking about people in this course and you know everyone. This is called the universe of discourse/universal set.)
2. $x + 5 = 6$ Assume the universe of discourse is the set of real numbers, **R** .
3. $1^3 + 2^3 + 3^3 + . . . + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ Assume the universe of discourse is the set of positive integers $n \geq 1$.

Each of these statements contains a variable, hence is not a proposition. But each statement becomes a proposition (true or false) if we replace the variable by an element from its universe of discourse. Sentences of this type are sometimes called open sentences and they are given "propositional names" but with a variable, for example, p(n), q(x), P(n), , Q(x) , ... So I'll call the sentence in number 3 above P(n), that in 2 Q(x) and that in 1 S(m).

Since I'm only 5'9" S(m) of number 1 becomes a false proposition by the substitution of the variable by my name.

If we replace x by 1 in 2 then Q(2) becomes $1 + 5 = 6$ is a true proposition.

If n = 2 in number 3 then P(2) becomes $1^3 + 2^3 = \left(\frac{2(2+1)}{2}\right)^2$ or $9 = 9$, which is a true proposition.

In math induction we frequently work with expression like that in number 3, namely,

$1^3 + 2^3 + 3^3 + . . . + n^3 = \left(\frac{n(n+1)}{2}\right)^2$. Since expressions like this are common in math induction let's make sure we understand what it says. The next 3 examples are intended to explain this and examples like them.

Example 1. Let P(n) stand for the expression $1^3 + 2^3 + 3^3 + . . . + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

(a) What is P(2)?

This we did above and P(2) is $1^3 + 2^3 = \left(\frac{2(2+1)}{2}\right)^2$ which is true.

(b) What is P(3)?

P(3) is $1^3 + 2^3 + 3^3 = \left(\frac{3(3+1)}{2}\right)^2$

The left side gives 1 + 8 + 27 = 36

And the right side is $\left(\frac{3(3+1)}{2}\right)^2 = 6^2 = 36$. So both sides give the same result.

Think of $1^3 + 2^3 + 3^3 + . . . + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ as a formula. If you need to add the first 5 terms on the left to find the sum just replace the n in $\left(\frac{n(n+1)}{2}\right)^2$ by 5 to get 15^2 or 225.

Example 2. Again let P(n) stand for the expression

$1^3 + 2^3 + 3^3 + . . . + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

(a) What is P(k)? This is simple just replace n by k to obtain

$1^3 + 2^3 + 3^3 + . . . + k^3 = \left(\frac{k(k+1)}{2}\right)^2$

(b) What is P(k+1)? This is simple just replace n by k+1 to obtain

$1^3 + 2^3 + 3^3 + . . . + (k+1)^3 = \left(\frac{(k+1)((k+1)+1)}{2}\right)^2$

Example 3. More on the left side of the expression

$1^3 + 2^3 + 3^3 + . . . + (k+1)^3 = \left(\frac{(k+1)((k+1)+1)}{2}\right)^2$. The left side

(i) $1^3 + 2^3 + 3^3 + . . . + (k+1)^3$ can be rewritten as
(ii) $1^3 + 2^3 + 3^3 + . . . + k^3 + (k+1)^3$ or as (Here all I did was to insert the kth term in the series. (i))
(iii) $1^3 + 2^3 + 3^3 + . . . + (k-1)^3 + k^3 + (k+1)^3$ etc. (Here all I did was to insert the (k-1)th term in the series (ii).)

Study the *Principle of Mathematical Induction* and the *Generalized Principle of Mathematical Induction* in section 3.7.

<u>Mathematical Induction</u> :
To prove the statement for all integers $n \geq a$, P(n) is true is a two step process.
Step 1 We first must show that the statement P(n) is true for the first integer mentioned, namely a so we have to show P(a) holds true. This is known as the *basis step*.

Step 2. The induction step. In this step we suppose P(k) is true and show this holds true for P(k +1).

In step 2 all we are doing is using the direct method of proof to prove:
If P(k) is true then P(k +1) is true. To do this we follow the usual procedure:

Assume: P(k) is true and
Prove: P(k +1) is true.

OK, now we are ready for a mathematical induction proof.

Example 4. Prove $1^3 + 2^3 + 3^3 + . . . + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ is true for all positive integers $n \geq 1$

($n \geq 1$ tells you where to start namely at n = 1 or at P(1). This step is called the basis step) .

Proof.(by math induction) Let P(n) stand for $1^3 + 2^3 + 3^3 + . . . + n^3 = \left(\frac{n(n+1)}{2}\right)^2$. Comment: At this point if it is not clear what this equation says replace n by a few values as we did above.. Do not proceed if you do not understand the statement.

Step 1 (Basis Step) Show P(1) is true.
P(1) says $1^3 = \left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1$ which is true

Step 2 (Induction Step)

Assume P(k) is true. This means assume $1^3 + 2^3 + 3^3 + . . . + k^3 = \left(\frac{k(k+1)}{2}\right)^2$ is true. This is called the inductive hypothesis.

To prove: P(k + 1) is true. This means show

$1^3 + 2^3 + 3^3 + \ldots + (k+1)^3 = \left(\frac{(k+1)((k+1)+1)}{2}\right)^2$ is true. To do this I will take the left side of this statement, P(k + 1), and show it is equal to the right side. Note the induction step says use the fact that P(k) is true to prove that P(k+1) is true. So I know I have to use the hypothesis P(k) is true.

OUTLINE

An outline of most induction proofs of this type follows with this example:

Take the left side of what you want to prove

Here $1^3 + 2^3 + 3^3 + \ldots + (k+1)^3$ = rewrite it to use the hypothesis P(k) (1)

.

= use the fact that P(k) is true (2)

.

.

use basic algebra to obtain the right side

Here $\left(\frac{(k+1)((k+1)+1)}{2}\right)^2$ or $\left(\frac{(k+1)(k+2)}{2}\right)^2$ (3)

Now back to step 2

$1^3 + 2^3 + 3^3 + \ldots + (k+1)^3 = [1^3 + 2^3 + 3^3 + \ldots + k^3] + (k+1)^3$ (1)

see example 3. All we did is to rewrite the left side inserting the term k^3

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \qquad (2)$$

The series is brackets is the left side of P(k) and P(k) is assumed to be true so we can replace it by the right side of P(k)

We now want to show this is reducible to right-hand side of the P(k + 1) statement, namely, (3) above. Can you follow these algebra steps

$$= \left(\frac{k(k+1)}{2}\right)^2 + \frac{4(k+1)^3}{4}$$

$$= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}$$

$$= \frac{(k+1)^2\left(k^2 + 4(k+1)\right)}{4}$$

$$= \frac{(k+1)^2\left(k^2 + 4k + 4\right)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)^2$$ This is the same as (3) above.

Here are some additional problems using mathematical induction that you should try. The algebra in example 5 is easier so if you had a tough time with example 4 then you will find example 5 easier to understand.

Example 5. This is easier than example 4. Note, the steps are exactly the same as those in example 4.

Prove $1 + 2 + 3 + . . . + n = \frac{n(n+1)}{2}$ is true for all positive integers $n \geq 1$ (n ≥ 1 tells you where to start. It is the basis step) .

Proof.(by math induction) Let P(n) stand for $1 + 2 + 3 + . . . + n = \frac{n(n+1)}{2}$. Comment: At this point if it is not clear what this equation says replace n by a few values as we did above.. Do not proceed if you do not understand the statement.

Step 1 (Basis Step) Show P(1) is true.

P(1) says $1 = \frac{1(1+1)}{2}$ which is true

Step 2 (Induction Step)

Assume P(k) is true. This means assume $1 + 2 + 3 + . . . + k = \frac{k(k+1)}{2}$ is true. This is called the inductive hypothesis.

To prove: P(k + 1) is true. This means show

$1 + 2 + 3 + . . . + (k + 1) = \frac{(k+1)((k+1)+1)}{2}$ is true. To do this I will take the left side of this statement and show it is equal to the right side.

OUTLINE

An outline of this induction proof of this type follows. You should try to write the specifics.

Take the left side of what you want to prove

Here $1 + 2 + 3 + . . . + (k + 1) =$ rewrite it to use the hypothesis P(k) (1)

.

= use the fact that P(k) is true (2)

.

.

use basic algebra to obtain the right side

$$\text{Here } \frac{(k+1)((k+1)+1)}{2} \text{ or } \frac{(k+1)(k+2)}{2} \qquad (3)$$

Math Induction problems involving matrices and recurrence relations.

Skip 1 through 5 until we cover matrices.

1. If $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then what is I^n for all integers $n \geq 1$? Prove your answer using mathematical induction.

2. What is D^n if $D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$. Proof?

3. Let A, B and D be n x n matrices whose entries are real numbers. Assume that B is invertible. If $A = BDB^{-1}$, prove by induction that $A^m = BD^mB^{-1}$ for all integers $m \geq 1$.

4. Let A and B be two k x k matrices whose entries are real numbers. If AB = BA prove, by induction on n, that $AB^n = B^nA$ for all integers $n \geq 1$.

5. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ prove by induction that for all integers $n \geq 1$

$A^n = \begin{bmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$. Try this first then look at the solution below if you get stuck.

Solution for number 5.

You are given the following: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$.

As usual we begin by letting P(n) stand for what we are trying to prove, namely,

$A^n = \begin{bmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$. Note this is “both sides” of the statement.

1. Basis Step Prove P(1) is true.

P(1) says $A^1 = \begin{bmatrix} 1 & 1 & \frac{1(1-1)}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ which is given.

2. Induction Step:

Assume P(n) is true, that is assume $A^n = \begin{bmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$

To prove: $A^{n+1} = \begin{bmatrix} 1 & (n+1) & \frac{(n+1)((n+1)-1)}{2} \\ 0 & 1 & (n+1) \\ 0 & 0 & 1 \end{bmatrix}$.

The procedure I will use is to start with the left side of what I wish to prove and the eventually show it is equal to the right side. NB the induction step essentially says **“use the fact that P(n) is true to show that P(n+1) is true”.** So I know I want to get A^n so I can “use it”.

The format of the prove is:

$A^{n+1} =$

= (use the fact that P(n) is true)

=

.

.

.

$$= \begin{bmatrix} 1 & (n+1) & \frac{(n+1)((n+1)-1)}{2} \\ 0 & 1 & (n+1) \\ 0 & 0 & 1 \end{bmatrix}$$

Here is the induction step:

$A^{n+1} = A^n A^1$ The law of exponents for matrices

$$= \begin{bmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Since P(n) is true we can replace A^n by what it is equal to. Also replace A by what is given above.

Now, simply multiply these two matrices to obtain:

$$= \begin{bmatrix} 1 & (n+1) & \frac{(n+1)((n+1)-1)}{2} \\ 0 & 1 & (n+1) \\ 0 & 0 & 1 \end{bmatrix}$$

Recurrence Relations and Math Induction

Example 6. The sequence 1, 3, 5, 7, . . . is quite simple and if we were asked to write out several terms we could easily do so and get 1, 3, 5, 7, 9, 11, 13, ... it is clear that to find any term in the sequence just add 2 to the previous term. So the nth term of the sequence, x_n is $2 + x_{n-1}$, the (n – 1)th term. We can say this in notation as follows $x_n = 2 + x_{n-1}$. This formula describes the given sequence **once we know where to start.** So to describe the above sequence we need to know two things: the starting "point" and the formula connecting the terms of the sequence. The sequence of this example can be described as:
$x_n = 2 + x_{n-1}$ and $x_1 = 1$.

There are two common notations for the terms of a sequence, one using the **positive** integers, for example,
$x_1, x_2, x_3, \ldots, x_n$ and the other using the **nonnegative** integers, for example,
$x_0, x_1, x_2, \ldots, x_n$

Both notations have their own advantages. Using the notation $x_1, x_2, x_3, \ldots$ the 15^{th} term is clearly x_{15}, simple. Where in the notation $x_0, x_1, x_2, \ldots$ the 15^{th} term is x_{14}, a little awkward. On the other hand, certain applications lend themselves to the latter notation. Examples, your initial deposit in a bank of $1,000 at time $t = 0$ so x_0 is 1000. Applications in biology when the initial population of bacteria is x_0,
in physics where the initial velocity is v_0.

It is common in the study of recurrence relations to use the later notation above recurrence relation as: $x_0, x_1, x_2, \ldots$

Example 7. Write the first three terms of the sequence described by $x_n = 5 + x_{n-1}$ and $x_1 = 2$. In this notation the first three terms are: $x_1, x_2, x_3, \ldots$ where $x_1 = 2$, $x_2 = 5 + x_1 = 5 + 2 = 7$ and $x_3 = 5 + x_2 = 5 + 7 = 12$. So the sequence is
2, 7, 12, . . .

The sequences described in examples 1 and 2 are called recurrence relations. A recursive description is one where one or more terms are defined in terms of previous terms. The term difference equation is also used for recurrence relation.

Example 8. Write out the first five terms of the sequence described by $x_n = x_{n-1} + x_{n-2}$. Where $x_1 = 1$ and $x_2 = 1$. The first three terms are: $x_1, x_2, x_3, \ldots$ where $x_1 = 1$, $x_2 = 1$ and $x_3 = x_2 + x_1 = 1 + 1 = 2$, $x_4 = x_3 + x_2 = 2 + 1 = 3$,
$x_5 = x_4 + x_3 = 3 + 2 = 5$. So the sequence is: 1, 1, 2, 3, 5, 8, ... This sequence is called the Fibonacci sequence. Google Fibonacci sequence to find its many interesting applications. Another interesting sequence is that referred to as “the Tower of Hanoi”. Google “Tower of Hanoi puzzle”.

I include several examples of recurrence relations our section for math induction because it helps us to understand recurrence a little better and secondly because there is a natural connection between recurrence relation and math induction.

In the notes of week 6 on sequences I gave you three examples (examples 1,2,and 3) of sequences which were a special type of sequence call a **recurrence relation**.

One example was the sequence (recurrence relation) $x_n = x_{n-1} + 2$ where the initial term is $x_1 = 1$ so that the sequence is: 1, 3, 5, 7, . . .

Also recall there are two common notations for the terms of a sequence, one using the **positive** integers, for example,
$x_1, x_2, x_3, \ldots$ and the other using the **nonnegative** integers, for example,
$x_0, x_1, x_2, \ldots$

It is common in the study of recurrence relations to use the later notation above recurrence relation as: $x_0, x_1, x_2, \ldots$
In this notation we would write the above recurrence relation as: $x_n = x_{n-1} + 2$ where the initial term is $x_0 = 1$. Note the only thing that changes is the notation for the initial term.

Definition - Recurrence Relation. A **recurrence relation** for a sequence $x_0, x_1, x_2, \ldots$ is a formula which relates each term of the sequence to one or more of its predecessor terms. The **initial conditions** are the one or more terms x_0, x_1 etc. necessary to write out the sequence.

In the examples below we need only one initial condition to write the sequence.
Let me state exercise 6 and then explain the details

6. Define the sequence L by $L_0 = 5$ and for $k \geq 1$, $L_k = 2L_{k-1} - 7$.
Write out the first four terms of this sequence.
Prove by induction that $L_k = 7 - 2^{k+1}$ is a closed formed expression for the above recurrence relation.

Example 9. We understand the sequence $L_0 = 5$ and for $k \geq 1$, $L_k = 2L_{k-1} - 7$.
And you should be able to write out the first 4 terms: $L_0, L_1, L_2, L_3, \ldots$
$L_0 = 5$ this is given
$L_1 = 2L_{1-1} - 7 = 2L_0 - 7 = 2(5) - 7 = 3$ this is from the given recurrence relation $L_k = 2L_{k-1} - 7$
$L_2 = 2L_{2-1} - 7 = 2L_1 - 7 = 2(3) - 7 = -1$ again this is from the given recurrence relation $L_k = 2L_{k-1} - 7$
$L_3 = 2L_{3-1} - 7 = 2L_2 - 7 = 2(-1) - 7 = -9$ again this is from the given recurrence relation $L_k = 2L_{k-1} - 7$
So the first four terms of the sequence are: 5, 3, -1, -9,
Can you find L_4? L_5?

Example 10. To find the term L_4 in the above sequence you need to first find all preceding terms L_0, L_1, L_2, L_3, . This is time consuming.

$L_k = 7 - 2^{k+1}$ is a closed formed expression (or the solution) for the recurrence relation $L_k = 2L_{k-1} - 7$. One benefit of the solution/equation $L_k = 7 - 2^{k+1}$ is that you can find any term of the sequence directly. For example $L_4 = 7 - 2^{4+1} = 7 - 2^5 = 7 - 32 = -25$.

For problem 6 (stated right above example 6) we want to prove that (b) $L_k = 7 - 2^{k+1}$ is a closed formed expression (or the solution) for the recurrence relation (a) $L_k = 2L_{k-1} - 7$.
A partial proof follows

Partial solution

First Let P(k) stand for what you are **trying to prove,** namely, (b) $L_k = 7 - 2^{k+1}$ is a solution for (a).

Step 1. Prove the basis step, that is, prove P(1) is true
In the induction part of the proof we want to prove that
If P(k) is true that is if $L_k = 7 – 2^{k+1}$ is true
then P(k + 1) is true, that is, $L_{k+1} = 7 – 2^{k+1+1}$ is true. $L_{k+1} = 7 – 2^{k+1+1}$

L_{k+1}	$= 2L_{((k+1)-1)} – 7$	This is from the given recurrence relation
	$= 2L_k – 7$	
	$= 2(7 – 2^{k+1}) – 7$	This is the induction hypothesis
	$= 14 - 2^{k+2} – 7$	Basic algebra
	$= 7 – 2^{k+1+1}$	

So we are done with the induction part. Now, write up the complete solution, the basis step etc

Exercise 7. Define the sequence B recursively by $B_0 = 2$ and for $k \geq 1$, $B_k = B_{k-1} + 3$.
(a) Write out the first four terms of this sequence.
(b) Prove by induction that $B_k = 3k + 2$ for $k \geq 0 \geq 1$, is a closed formed expression for this recurrence relation.

Exercise 8. Consider the following sequence: $a_0 = 0$ and for $n \geq 0$, $a_n = 2a_{n-1} + 1$. (a) Write out the first four terms of this sequence.
(b) Verify **by example** that $a_n = 2^n - 1$ is a closed formed expression for the above recurrence relation.
(c) Prove by induction that $a_n = 2^n - 1$ is a closed formed expression for the recurrence relation given in 8. (Note some of you may recognize this as the classic Towers of Hanoi recurrence relation).

Solution for 8c
8 c. Prove that $a_n = 2^n – 1$ for all $n \geq 0$.
Let P(n) stand for $a_n = 2^n – 1$, since this is what we what to prove by induction.

1. **The Basis Step** (prove P(n) is true for the first integer requested, namely n = 0, that is, prove P(0) is true).

P(0) is: $a_0 = 2^0 – 1 = 1$. This is true because $a_0 = 0$ is given in the first line of 8.

2. **The Induction Step**

Prove: If P(n) is true then P(n + 1) is true. To prove any If … then … assume the premise true and prove the conclusion true as in the following:

Assume P(n) is true, that is, assume $a_n = 2^n – 1$ is a true statement.
Prove P(n + 1) is true, that is, prove $a_{n+1} = 2^{n+1} - 1$. In order to prove this we will start with the left side of this expression (namely a_{n+1}) and eventually show it is equal to the right side.

First note, $a_{n+1} = 2a_{(n+1)-1} + 1$ This is from the recurrence relation given in part a which is true for any integer so it is true for the integer n + 1. That is, substitute n + 1 for n in the recurrence relation given in 8a.

$a_{n+1} = 2a_n + 1$	because $(n+1) - 1 = n$
$= 2(2^n - 1) + 1$	Since P(n) is assumed to be true we can replace a_n by $2^n - 1$
$= 2^{n+1} - 2 + 1$	Rewrite the above so we can add.
$= 2^{n+1} - 1$	Which is what we wanted to prove.

(c) Prove by induction that $a_n = 2^n - 1$ is a closed formed expression for the recurrence relation given in (a). (Note some of you may recognize this as the classic Towers of Hanoi recurrence relation. Google Towers of Hanoi to see what you get.).

9. Assume that p_i is a proposition for all i. If $\neg(p_1 \wedge p_2) \Leftrightarrow (\neg p_1 \vee \neg p_2)$ then prove for all integers $n \geq 2$ that $\neg(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \Leftrightarrow (\neg p_1 \vee \neg p_2 \vee \ldots \vee \neg p_n)$

More induction problems for the experts.

10. Use Mathematical induction to prove that if a set A has n elements then P(A), the power set of A, has 2^n elements.

11. Prove that any restaurant bill of \$n, n greater than or equal 5, can be paid exactly using only \$2 and \$5 bills.

12. Use induction to prove: If p is a prime and $p \mid a^n$ for some integer n greater than 1, then $p \mid a$.

13. Use induction and #12 to prove that if gcd (a, b) =1 then gcd (a^n, b^n) =1 for all n greater than or equal to 1.

Part II Predicates and Quantifiers

Assume that the **universe of discourse** is all the people who are participating in this course. Also, let us assume that we know each person in the course. Consider the following statement: "She/he is over 6 feet tall". This statement is not a proposition since we cannot say that it either true or false until we replace the variable (she/he) by a person's name. The statement "She/he is over 6 feet tall" may be denoted by the symbol P(n) where n stands for the variable and P, **the predicate**, "is over six feet tall". The symbol P (or lower case p) is used because once the variable is replaced (by a person's name in this case) the above statement becomes a proposition.

For example, if we know that Jim is over 6 feet tall, the statement "Jim is over six feet tall" is a (true) proposition. The truth set of a predicate is all values in the domain that make it a true statement. Another example, consider the statement, "for all real numbers x, $x^2 - 5x + 6 = (x - 2)(x - 3)$". We could let Q(x) stand for $x^2 - 5x + 6 = (x - 2)(x - 3)$. Also, we note that the truth values of Q(x) are indeed all real numbers.

Quantifiers:

There are two quantifiers used in mathematics: "for all" and "there exists". The symbol used "for all" is an upside down A, namely, $\forall$. The symbol used for "there exists" is a backwards E, namely, $\exists$. We realize that the standard, every day usage of the English language does not necessarily coincide with the Mathematical usage of English, so we have to clarify what we mean by the two quantifiers.

$\forall$	For all	For every	For each	For any
$\exists$	There exists at least one	There exists	There is	Some

The table indicates that the mathematical meaning of the universal quantifier, for all, coincides with our everyday usage of this term. However, the mathematical meaning of the existential quantifier does not. When we use the word "some" in everyday language we ordinarily mean two or more; yet, in mathematics the word "some" means at least one, which is true when there is exactly one.

The Negation of the "For all "Quantifier:

Consider the statement "All people in this course are over 6 feet tall." Assume it is false (I am not over six feet tall). How do we prove it is false? All we have to do is to point to one person to prove the statement is false. That is, all we need to do is give one **counterexample**.

We need only show that **there exists at least one** person in this class who is **not** over 6 feet tall. Here is a more formal procedure.

Example 1:

Let P(n)stand for "people in this course are over 6 feet tall", then the sentence "All people in this course are over 6 feet tall" can be written as: "$\forall$ n P(n)". The negative, "$\neg$ ($\forall$ n P(n))", is equivalent to: "$\exists$ n($\neg$ P(n))". So, in English the negative is, "There is (there is at least one/ there exists/ some) a person in this room who is not over 6 feet tall."

Example 2:

How would one negate the sentence: "Every person in this course is over six feet tall **and** is taking the course C Programming".

Answer:

Let P(n) stand for "people in this course are over 6 feet tall" and let Q(n) stand for "person n is taking the course C Programming". Then, the given sentence in symbolic form is: $\forall$ n(P(n) $\wedge$ Q(n)).

Next, $\neg (\forall n(P(n) \wedge Q(n))) \Leftrightarrow \exists n(\neg (P(n) \wedge Q(n)))$ Why?

$\Leftrightarrow \exists n(\neg P(n) \vee \neg Q(n))$ [**Note**: the use of DeMorgan's law]

In English we have:

There is a person in this course who is not over six feet tall **or** who is not taking the course C Programming **OR** Some people in this course are not over six feet tall **or** are not taking the course C Programming.

The Negation of the "There exists" Quantifier:

Example 3:

How would you prove that the statement " There is a person in this course over 7 feet tall." is **false**?

Answer:

Surely, one would have to demonstrate that no one in the room is over seven feet tall. In other words in logical symbols we have:

$\neg (\exists$ (nP(n))) is equivalent to $\forall n(\neg P(n))$.

Can you write the negatives of the following expressions using logic?

1. All people in this course are interesting and informative.
2. Some people in this course are having fun.
3. All people in this course are over 6 feet tall or wear glasses.
4. For all real numbers x, the equation $x^2 + 6x + 5 = 0$ is true.

Some thoughts on using multiple quantifiers

In week 1 we reviewed some of the basic laws we saw in high school algebra. For example, we saw that for all real numbers a and b, a + b = b + a (the **commutative law** for addition). Of course, what this means is that for every real number a and for every real number b, we have a + b = b + a. If it is understood that a and b are real numbers, we could use quantifier notation and say $\forall$ a and $\forall$ b, a + b = b + a. So, using the "for all" quantifier multiple times is a easy process. The following two examples illustrate that "mixing" quantifiers makes life more difficult.

Example 4.

In the following assume a and b are real numbers:

"$\forall$ a, $\exists$ b where a + b = 1".

In English this says: "For all real numbers a there exists a real number b such that a + b = 1." This statement is **true** since, for example, if a = 5 there does exist a real number b, which makes a + b = 1 true. Just take b = -4 the 5 + (-4) = 1. Of course this can be done for any real number a, just take b = -(a -1). and a + b = 1 becomes a + (-(a – 1)) = 1, which is of course true.

Example 5.

In the following assume a and b real numbers.

"$\exists$ b, $\forall$ a where a + b = 1".

In English this says "There exists a real number b, such that, for **all** real numbers a we have a + b = 1". This statement is **false** since, for example, if b = 4 then a + b = 1 becomes a + 4 = 1. Is this true for every single real number a? The equation a + 4 = 1 is **not** true for all real numbers a it is true only for the real number a = -3