Week 6
Equivalence, Implication and the Laws of Logic

Equivalences and Implications

In section 3.3 there are several key terms; **tautology, contradiction**, **equivalence and implication.** Study the definitions of these terms before reading further.

Examples of tautologies in mathematics, statements in mathematics which are always true are everywhere. In the text we state there are two types of tautologies **equivalences** and **implications.**

Examples of equivalences

i. $x + (y + z) = (x + y) + z$ for all real numbers x, y, and z.
ii. $x + 0 = x$ for any real number x.
iii. $x^2 - 4 = (x - 2)(x + 2)$ for any real number x.
iv. $\sin^2 x + \cos^2 x = 1$ for any real number x.

In each of the above equivalences the left side of each equation can be replaced by the right side and the right side by the left. Each of them is an "identity". They go "both ways".

Examples of implications

In example 1 in the notes of week 5 I gave you three examples of implications, namely,

i. Let x, y, and z be any three real numbers then If $x \leq y$ and $y \leq z$ then $x \leq z$.
ii. Let A, B, and C be any three sets then If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.
iii. Let p, q, and r be any three propositions then If $p \to q$ and $q \to r$ then $p \to r$.

So i. states if $x \leq y$ and $y \leq z$ is true then you can say $x \leq z$. Each of these is an implication, the left implies the right. Each goes "one way".

If a biconditional statement is true we use $\Leftrightarrow$ in place of $\leftrightarrow$. Notice the use of $\Leftrightarrow$ in all the basic logic laws. Frankly if the distinction between the two symbols $\Leftrightarrow$ and $\leftrightarrow$ is not clear to you just use $\Leftrightarrow$.

Similarly, if a conditional statement is true, is an implication we use $\Rightarrow$ in place of $\to$. In example 3.3.7 since $p \to (p \vee q)$ is a tautology we should write it as $p \Rightarrow (p \vee q)$. Again,let's not be too concerned about the difference.

Some laws of Logic and More Equivalences and Implications

Example 1. (a) Use truth tables to prove **[**(p ∨ q) ∧ ~p] ⇒ q is a tautology. (b) Is the converse a tautology?

(a) There are only two variables so there are only four rows in the truth table for this statement. The truth table is:

		1	2	3
p	q	$p \vee q$	$(p \vee q) \wedge \neg p$	$[(p \vee q) \wedge \neg p] \Rightarrow q$
0	0	0	0	1
0	1	1	1	1
1	0	1	0	1
1	1	1	0	1

The truth values in the AND statement in column 2 are true only when both parts of the statement $(p \vee q) \wedge \neg p$ are true. The statement in column 3 is a conditional. It is false only when the premise, $(p \vee q) \wedge \neg p$ is true and the conclusion, q, is false. So the implication in column 3 is a tautology.

(b) Is the converse a tautology? The converse is: q ⇒ **[**(p ∨ q) ∧ ~p]. I claim no. Can you prove it?

Before we look at example 2, note, there is a different notation for negation, namely ~ in table 2, below, compared to that given in the text. (in the text ¬ is used). Both notations are used.

Example 2. Each of the logic laws in tables 2 and 3 below are tautologies. You should be able to prove each as I did in example 1. I'll prove the first of the Negation laws, namely, $(p \wedge \neg p) \Leftrightarrow 0$ .Two remarks are in order here. First, the "0" on the right side stands for a proposition which is always false. Secondly, on the left side we have only one proposition, p. By the product rule since we have two choices for the proposition p, 0 or 1, and only one choice for the proposition 0, namely 0 we have 2 times 1 = 2 rows in the truth table for $(p \wedge \neg p) \Leftrightarrow 0$.

1		2	3	4
p	$\neg p$	0	$(p \wedge \neg p)$	$(p \wedge \neg p) \Leftrightarrow 0$
0	1	0	0	1
1	0	0	0	1

. Since columns 2 and 3 have the same truth values in each row the definition of the biconditional, ⇔ is true in each row in column 4. This is a tautology. Note, the second of the Negation laws mentions the proposition 1. "1" stand for the proposition which is always true, contains all 1's.

Over the past weeks we think of logic, or more precisely the "algebra of logic" as behaving **similar** to (but not necessarily **exactly the same** as) that of high school algebra. We have the logic laws in section 3.4 which I also list below which exhibits this similarity.

But there are also some “dissimilarities”, differences. For example, in logic there are two distributive laws the first distributes AND over OR and the second OR over AND. In regular algebra we have only one distributive law. Multiplication over addition, that is,
$a \cdot (b + c) = a \cdot b + a \cdot c$.
Also, in high school algebra we use the “cancellation law” frequently. If a + c = b + c then a = b, we can cancel the c’s to make the equation easier to work with. Is a similar statement true in logic? In the following can we cancel the r’s in $(p \vee r) \Leftrightarrow (q \vee r)$ to obtain $p \Leftrightarrow q$? That is, is the following true, if $(p \vee r) \leftrightarrow (q \vee r)$ then $p \leftrightarrow q$. Check to see if
$[(p \vee r) \leftrightarrow (q \vee r)] \Rightarrow (p \leftrightarrow q)$ is a tautology. There are 8 rows and you can show that this is not a tautology but the converse is.

Table 2: Basic Logical Laws

$p \vee q \Leftrightarrow q \vee p$	**Commutative Laws** $p \wedge q \Leftrightarrow q \wedge p$
$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	**Associative Laws** $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	**Distributive Laws** $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
$p \vee 0 \Leftrightarrow p$	**Identity Laws** $p \wedge 1 \Leftrightarrow p$
$p \wedge \sim p \Leftrightarrow 0$	**Negation Laws** $p \vee \sim p \Leftrightarrow 1$
$p \vee p \Leftrightarrow p$	**Idempotent Laws** $p \wedge p \Leftrightarrow p$
$p \wedge 0 \Leftrightarrow 0$	**Null Laws** $p \vee 1 \Leftrightarrow 1$
$p \wedge (p \vee q) \Leftrightarrow p$	**Absorption Laws** $p \vee (p \wedge q) \Leftrightarrow p$
$\sim (p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$	**DeMorgan’s Laws** $\sim (p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$
	Involution Law $\sim (\sim p) \Leftrightarrow p$

Table 2

Table 3: Some Common Implications and Equivalences

Detachment $(p \rightarrow q) \wedge p \Rightarrow q$
Indirect Reasoning $(p \rightarrow q) \wedge \sim q \Rightarrow \sim p$
Disjunctive Addition $p \Rightarrow (p \vee q)$
Conjunctive Simplification $(p \wedge q) \Rightarrow p$ and $(p \wedge q) \Rightarrow q$
Disjunctive Simplification $(p \vee q) \wedge \sim p \Rightarrow q$ and $(p \vee q) \wedge \sim q \Rightarrow p$
Chain Rule $(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow (p \rightarrow r)$
Conditional Equivalence $(p \rightarrow q) \Leftrightarrow (\sim p \vee q)$
Biconditional Equivalences $(p \leftrightarrow q) \Leftrightarrow ((p \rightarrow q) \wedge (q \rightarrow p)) \Leftrightarrow ((p \wedge q) \vee (\sim p \wedge \sim q))$
Contrapositive $(p \rightarrow q) \Leftrightarrow (\sim q \rightarrow \sim p)$

Table 3

Table 3 lists a few more frequently used tautologies. Two of the most important are the **biconditional equivalence**, namely $(p \leftrightarrow q) \Leftrightarrow ((p \rightarrow q) \wedge (q \rightarrow p))$ and the **contrapositive law** which states $(p \rightarrow q) \Leftrightarrow (\sim q \rightarrow \sim p)$. Consider the following sentence/Theorem from high school algebra:

"Let a and b be any two real numbers, ab = 0 iff a = 0 or b = 0".

First the symbol $\Leftrightarrow$ is frequently translated as "if and only if" as explained in the text. The phrase, "if and only if", is commonly abbreviated as iff. So the biconditional equivalence tells us that we can write the above theorem as two if … then… statements, namely: "If ab = 0 then a = 0 or b = 0 and If a = 0 or b = 0 then ab = 0". (**or** of course as If ab = 0 then a = 0 or b = 0 and conversely).

Example 3:

Try to use your knowledge of truth tables to prove the following "if P then C" statement: $(p \vee \sim q) \rightarrow (\sim p)$ is **not a tautology.**

Solution:

p	q	~q	p∨~q	~p	(p∨~q) → (~p)
0	0	1	1	1	1
0	1	0	0	1	1
1	0	1	1	0	0
1	1	0	1	0	0

Since the column $(p \vee \sim q) \rightarrow (\sim p)$ does not contain all 1's this is not a tautology.

The reader should by now be thoroughly familiar with the terms: converse, contrapositive and iff. You should be able to use a truth table to prove that an If … then …. statement may not be equivalent to its converse. Can you show that the **law of detachment** or the **law of indirect reasonin**g or any of the other laws given in table 4 are tautologies?

More on Converse and Contrapositive

Can you answer the following?

1. Assume the following statement is true: If today is September 29th then it is my birthday. What is the converse of this statement? What is the contrapositive? Are they both true?
2. Assume the following statement is true: If she is on the cross country track team then she likes to run long distances.
   What is the converse of this statement? What is the contrapositive? Are they both true?
3. Let a and b be any two real numbers. Write the converse of the statement
   If ab = 0 then a = 0 or b = 0. Is it true? [**Note**: the converse is true].
4. Write the converse of the statement If a < ½ and b < ½ then a + b < 1. Is it true?
   Note the converse is false. Why?
5. Write the contrapositive of the statement If ab = 0 then a = 0 or b = 0. Is it true? (yes).
   Be careful, what does De Morgan's law say about the negation of an **OR** statement?
   Solution, the converse is: If $a \neq 0$ **and** $b \neq 0$ then $ab \neq 0$
6. Write the contrapositive of the statement If a < ½ and b < ½ then a + b < 1. Is it true?
   (yes) this is similar to #5 above.

Solution to exercise 6. I will do this in several steps:

(i) Let P stand for "a < ½ and b < ½" and let C stand for "a + b < 1". So the given statement is of the form If P then C.

(ii) The contrapositive is If **not** C then **not** P.

(iii) So far we have If **not** a + b < 1 then **not** (a < ½ and b < ½)

(iv) Next, we have to apply De Morgans law, namely, $(\sim(p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q))$

to **not** (a < ½ and b < ½).

So **not** (a < ½ and b < ½) becomes "**not** a < ½ **or not** b < ½.

(v) If we put (iii) and (iv) together we have

If **not** a + b < 1 then **not** a < ½ **or not** b < ½ or in more graceful English

If $a + b \geq 1$ then $a \geq ½$ **or** $b \geq ½$.

7. Express the statement: “n is an odd integer iff n^2 is an odd integer” as two if .. then … statements.
8. Rewrite the above statement using the term conversely.
9. Let a and b be any two real numbers. Express the statement

”ab = 0 iff a = 0 or b = 0”. As two if. . . then. . . statements. Convince yourelf with several examples that this iff statement is true by showing that each part is true.