Week 5

An Introduction to Logic, Propositions & Logical Operators

I repeat here several of the tables you studied in week 1. In week 1 we reminded ourselves of a few basic properties/laws in algebra. Our study of sets and basic set laws convinced us that many properties of sets and basic algebra were similar. That is their Mathematical structures were similar. Take a few minutes to glance at tables 1 and 2 to remind you or the similarities.

Table 1: Some Basic Algebra Laws

$a + b = b + a$	Commutative Laws $a \cdot b = b \cdot a$
$(a + b) + c = a + (b + c)$	Associative Laws $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
	Distributive Law $a \cdot (b + c) = a \cdot b + a \cdot c$
$a + 0 = a$	Identity Laws $a \cdot 1 = a$
$a + (-a) = 0$	Inverse Laws $a \cdot (a^{-1}) = 1$
$-(-a) = a$	Involution Law $(a^{-1})^{-1} = a$

Table 1

Table 2: Basic Set Laws

Commutative Laws	
(1) $A \cup B = B \cup A$	(1)' $A \cap B = B \cap A$
Associative Laws	
(2) $A \cup (B \cup C) = (A \cup B) \cup C$	(2)' $A \cap (B \cap C) = (A \cap B) \cap C$
Distributive Laws	
(3) $A \cap (B \cup C) =$ $(A \cap B) \cup (A \cap C)$	(3)' $A \cup (B \cap C) =$ $(A \cup B) \cap (A \cup C)$
Identity Laws	
(4) $A \cup \varnothing = \varnothing \cup A = A$	(4)' $A \cap U = U \cap A = A$
Complement Laws	
(5) $A \cup A^c = U$	(5)' $A \cap A^c = \phi$
Idempotent Laws	
(6) $A \cup A = A$	(6)' $A \cap A = A$
Null Laws	
(7) $A \cup U = U$	(7)' $A \cap \varnothing = \varnothing$
Absorption Laws	
(8) $A \cup (A \cap B) = A$	(8)' $A \cap (A \cup B) = A$
DeMorgan's Laws	
(9) $(A \cup B)^c = A^c \cap B^c$	(9)' $(A \cap B)^c = A^c \cup B^c$
Involution Law	
(10) $(A^c)^c = A$	

Table 2

This week we will begin our study of logic. Part of our study will be to determine the properties of the structure/algebra of logic. Although we do not know the meaning of the symbols in the table below a quick glance shows similarities.

In regular algebra we have $a + b = b + a$.

In the algebra of sets we have $A \cup B = B \cup A$.

In the algebra of logic we have $p \vee q \Leftrightarrow q \vee p$

They are the same “except for language”. Compare the laws given in the tables 1, 2 and 3.

Table 3: Basic Logical Laws

	Commutative Laws
$p \vee q \Leftrightarrow q \vee p$	$p \wedge q \Leftrightarrow q \wedge p$
	Associative Laws
$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
	Distributive Laws
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
	Identity Laws
$p \vee 0 \Leftrightarrow p$	$p \wedge 1 \Leftrightarrow p$
	Negation Laws
$p \wedge \sim p \Leftrightarrow 0$	$p \vee \sim p \Leftrightarrow 1$
	Idempotent Laws
$p \vee p \Leftrightarrow p$	$p \wedge p \Leftrightarrow p$
	Null Laws
$p \wedge 0 \Leftrightarrow 0$	$p \vee 1 \Leftrightarrow 1$
	Absorption Laws
$p \wedge (p \vee q) \Leftrightarrow p$	$p \vee (p \wedge q) \Leftrightarrow p$
	DeMorgan’s Laws
$\sim (p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$	$\sim (p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$
	Involution Law
	$\sim (\sim p) \Leftrightarrow p$

Table 3

In our study of arithmetic and algebra the very first thing that we learned is that the **objects** that we were working with were called numbers or more specifically real numbers. Then we learned how to add and multiply two real numbers. The operations of addition and multiplication deal with adding and multiplying **two** numbers and therefore are referred to **binary operations.** Next we learned what $-a$ and a^{-1} meant. [**Note**: here we are operating on just one number, namely a, hence these operations are called **unary operations**]. So, we suspect that in our study of logic the **very first definition** we will see is the definition of the **objects** in the Algebra of Logic. The objects in logic are called **propositions**. See the text for a definition. Then we will study the binary operation of logic, namely, AND and OR. More on this later.

In **table 4** below we outline the similarities between logic, sets and that of "regular" algebra.

Next, we learn that = is used to "connect" two statements, for example, in the following sentence $a + b = b + a$ the "=" symbol connects $a + b$ and $b + a$. For this reason the = symbol is referred to as a **connective**. See table 4 for a comparison of the connectives in the different algebras. Finally algebra is easy to work with because it satisfies a number of basic laws. Table 1 above mentions a few of them. We suspect that any other structure is or is not easy to work with depending on how closely it "measures up" to basic algebra laws. In the above tables compare the commutative laws for the binary operations $+$, $\cup$ and $\vee$.

Example 1. Similarly**,** $\leq$**,** $\subseteq$ and $\Rightarrow$ behave the same. As you can see from the following three true statements:

a) Let x, y, and z be any three real numbers then If $x \leq y$ and $y \leq z$ then $x \leq z$.
b) Let A, B, and C be any three sets then If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.
c) Let p, q, and r be any three propositions then If $p \to q$ and $q \to r$ then $p \to r$.

Read over the "similarities" table below.

Table 4: Similarities Between "Algebras"

	High School Algebra	**(Algebra of) Logic**	**(Algebra of) Sets**	**Matrix Algebra**
Objects	Real numbers	Propositions	Sets	Matrices
Binary Operations	Addition, + Mutiplication, •	OR, $\vee$, + AND, $\wedge$, •	Union, $\cup$ Intersection, $\cap$	Addition, + Multiplication, •
Unary Operations	Additive inverse - Multiplicative Inverse -1	Negation, $\neg$ or $\approx$	Complementation $'$ or c or -	Additive Inverse – Multiplicative Inverse -1
Other Connectives	$\leq$ =	$\Rightarrow$ $\Leftrightarrow$	$\subseteq$ =	=
Usual Constant/ Variable Names	a, b, c,…. x, y, z,….	0, 1 p, q, r,…	A, B, C,… X, Y, Z,…	A, B, C,… X, Y, Z,…
Some Basic Laws	See table	See table	See table	See table

Table 4

In this table and in the text note the following:

In logic we will translate the English words and phrases into a precise (mathematical) meaning. Since all (natural) languages are at best imprecise we must define our words carefully. Hence the definitions of AND, OR, NOT, IF ... THEN... etc in the text.

There are several notations for many of the above operations. For example, in Boolean Algebra OR is written as + but it is still read as "or". In our text the symbol, $\vee$, is used for the operation "or". Also, the multiplication symbol $\cdot$ is used in Boolean Algebra and it is read as "and". We will use the symbol $\wedge$ for the operation AND. Part of our study of discrete structures is to be aware of the different notations used by different disciplines.

More on AND and OR:

I repeat the definitions of AND and OR given in the text.

Definition of AND, $\wedge$

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

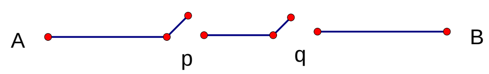
Definition of OR, $\vee$

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

The above definitions are motivate by the following examples

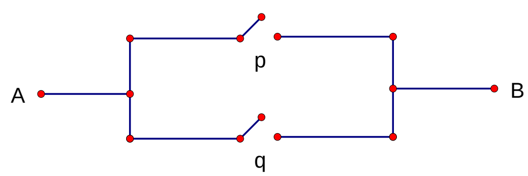
Example 2.

Assume the following is an electrical network containing two on/off switches, p and q. When will current go from point A to point B? Surely, this happens only when both switches are on, that is, when $p \wedge q$ is on or true. This motivates our definition of $p \wedge q$, the AND statement.



Example 3.

Now, consider the switching circuit:

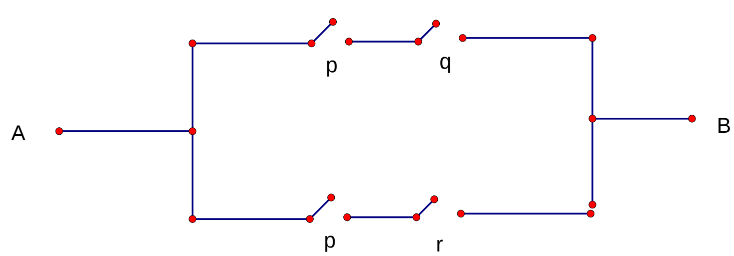


When will current go from point A to point B? Here, current will flow when switch p is on or switch q is on or both switches are on. That is, when $p \vee q$ is true. This motivates our mathematical definition of the word OR, which is of course different than the regular English usage of "or". Here "or" is the inclusive or not the standard English exclusive or. Indeed the "or" is exactly the same as that used in the definition of set union, $\cup$.

The following example connects the algebra of logic with that of the practical, switching circuits. Currently this all described in terms of gates but I think switching circuit diagrams are easier to follow. If you are interested to learn more go to chapter 13.5 of our text.

Exercise 4.

(a) Write out the expression in logic which is illustrated by the following switching circuit diagram.



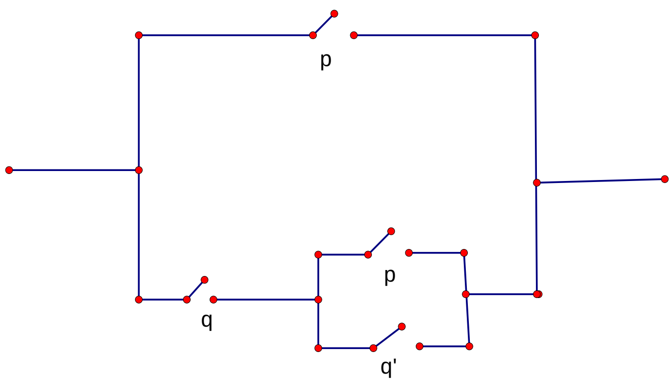
Solution: $(p \wedge q) \vee (p \wedge r) \Leftrightarrow p \wedge (p \vee r)$ by the distributive law for logic, table 3.

(b) Try to write a simpler switching circuit of the above, one using one less switch.

[**Note**: you have just reduced the hardware costs of the network.]

Exercise 5. This might be a little tough at this time but several of you can do it.

(a) Can you write the logical expression of the following switching circuit diagram?



[**Note**, q' means ¬q, that is, not q. The software I'm using does not allow the ¬ symbol in the diagram.]

(b) Simplify the logical expression you found in part a using basic logic laws.

Solution: The logical expression simplifies to p. Try it.

(c) Write the switching circuit diagram of the simplified expression you found in part b.

Example 3 above gives us a clearer idea why in mathematical English the word OR frequently means the "inclusive" or. That is, current will flow from A to B when switch p is on or if switch q is on or if both switches are on, as is stated in the definition of $\vee$ (OR) above . Keep this in mind when you study the mathematical definition of the conditional in table 3.1.4 in the text. That is definition 3.1.4 is a mathematical definition, not a Standard English definition.

The reader should note that the arrangement of the true and false symbols for p and q in the above table, namely 00, 01, 10 and 11. This arrangement of true and false is different than that of the text. There is a reason for this arrangement, which gives a "bonus" for this 0 and 1 notation. The "strings" 00, 01, 10, and 11 are the 2-digit binary integers of 0, 1, 2 and 3 respectively. To form a truth table with 3 variables, say p, q, and r just repeat (in a column 00,01,10 and11 twice and simply prefix the first four of them by 0's and the second four of them by 1's. This will give you 000, 001, 010, 011, 100, 101, 110 and 111. See the list below. These are the eight 3-digit binary integers for 0,1,2,3,4,5,6 and 7. What are the four digit binary integers for the integers 0,1,2,…,15?

0 0 0

0 0 1

0 1 0

0 1 1

1 0 0

1 0 1

1 1 0

1 1 1

Section 3.2 of the text is on truth tables. Read this section and make sure you can construct truth tables as required in the assignment.

You should be aware that if there are 2 logical variables, say p and q, in a expression then in a truth table of that expression there are 4 rows. This is so because there are 2 choices for the variable p, namely 0 and 1 and 2 choices for the variable q so by the product rule there are

2 x 2 = 4 choices or **rows**. For an expression of three logical variables there are 8 rows. Why? For an expression of n logical variables there are 2^n rows.

Example. A truth table for the expression $p \wedge \neg q$ is

p	q	$\neg q$	$p \wedge \neg q$
0	0	1	0
0	1	0	0
1	0	1	1
1	1	0	0

Study and do the assignments for sections 3.1 and 3.2.