Homework 5 Solutions **92.386 203 Prob & Stat I** **Spring 2015**

1. Using the digits $1-9$, how many five digit numbers are there

   (a) in total?

   $9^5 = 59{,}049$

   (b) which begin and end with an odd digit?

   $5 \times 9^3 \times 5 = 18{,}225$

   (c) which use only odd digits?

   $5^5 = 3{,}125$

   (d) which use three odd and two even digits in the order odd-even-odd-even-odd?

   $5 \times 4 \times 5 \times 4 \times 5 = 2{,}000$

   (e) which use three odd and two even digits in any order?

   $\binom{5}{3} 5^3 4^2 = 20{,}000$ — the $\binom{5}{3}$ chooses where to put the three odd numbers

   (f) which use three odd and two even digits in any order with no repetitions allowed?

   $\binom{5}{3}\binom{4}{2} 5! = 7{,}200$ — the $\binom{5}{3}$ chooses where to put the three odd numbers

   (g) if the digits must all be different?

   $9 \times 8 \times 7 \times 6 \times 5 = 15{,}120$

   (h) if the digits must be different and in increasing order?

   $\binom{9}{5} = 126$

   Bonus. What if the digits must be in increasing order but repetitions are allowed?

   A brute force way is to count the different patterns of repetitions separately —

5 different	2 of a kind	3 of a kind	4 of a kind	5 of a kind	2 pair	full house
$\binom{9}{5}$	$\binom{9}{1}\binom{8}{3}$	$\binom{9}{1}\binom{8}{2}$	$\binom{9}{1}\binom{8}{1}$	$\binom{9}{1}$	$\binom{9}{2}\binom{7}{1}$	$\binom{9}{2}\binom{2}{1}$

   Adding these 7 terms gives 1,287.

   There's a *stars and bars* (google it) formula for this: $\binom{9+5-1}{5} = \binom{13}{5} = 1{,}287$ where the 9 is how many different digits there are and the 5 is how many we need to choose.

2. Throw two dice and let X be the absolute difference in the numbers showing.

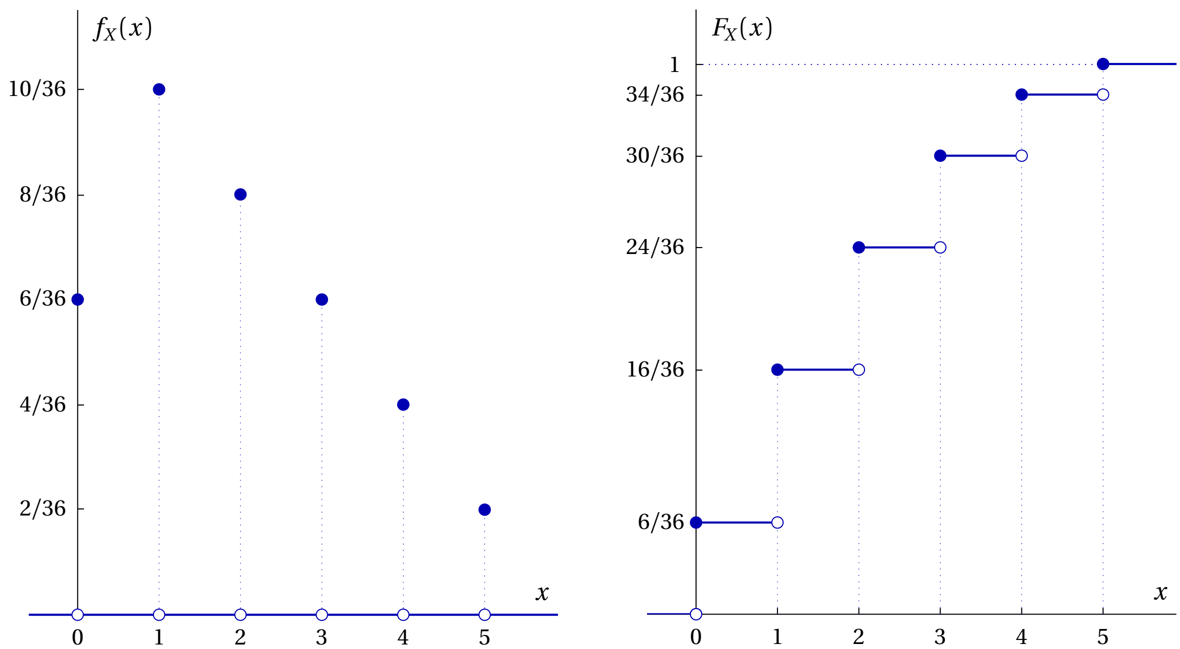
(a) What are the possible values of X?

$0, 1, 2, 3, 4, 5$

(b) Find and sketch the graph of the probability density function f_X.

X	0	1	2	3	4	5
f_X	6/36	10/36	8/36	6/36	4/36	2/36

(c) Carefully sketch the graph of the cumulative distribution function F_X.



3. Let X be the number showing when a die is thrown. Find the density function for $Y = (X-3)^2$.

x	1	2	3	4	5	6
$f_X(x)$	1/6	1/6	1/6	1/6	1/6	1/6

X	1	2	3	4	5	6
Y	4	1	0	1	4	9

y	0	1	4	9
$f_Y(y)$	1/6	2/6	2/6	1/6

For example, $f_Y(4) = P(Y=4) = P(X=1) + P(X=5) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$

Bonus. Assume everyone in a population of size n buys one lottery ticket each week. Use these hints to find the probability someone will win twice before you win once.

(a) Let X be the event that you win in week X and nobody has won twice. Find $P(X=k)$, $k=1,2,3...n$.

$X=k$ means $k-1$ of $n-1$ people won in weeks $1,2,3...k{-}1$ and then you won — this can happen in $\binom{n-1}{k-1}$ ways. The total number of different drawings is n^k so

$$P(X=k)=\binom{n-1}{k-1}\frac{1}{n^k}, \quad k=1,2,3...n$$

(b) Show the probability you win once before someone else wins twice is $\frac{1}{n+1}\left(1+\frac{1}{n}\right)^n$.

Since the events $\{X=k\}$ are exclusive

$$P(\text{you win})=\sum_{k=1}^{n} P(X=k)=\sum_{k=1}^{n}\binom{n-1}{k-1}\frac{1}{n^k}=\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{1}{n^{k+1}}=\frac{1}{n}\left(1+\frac{1}{n}\right)^{n-1}=\frac{1}{n+1}\left(1+\frac{1}{n}\right)^{n}$$

We have re-indexed the sum and used the Binomial Theorem

(c) What happens to $\left(1+\frac{1}{n}\right)^n$ as $n\to\infty$? What can you conclude about the lottery?

From Calc I $\left(1+\frac{1}{n}\right)^n \to e$ as $n\to\infty$ so for large n, $P(\text{you win})\sim\frac{e}{n+1}$ — for even modest values of n it's almost guaranteed that someone will win the lottery twice before you win it once