Exam 2 Solutions 92.386 201 Prob & Stat I **Fall 2015**

1. Let X be a random integer from 1 to 3 and Y a random integer from 1 to X (inclusive).

   (a) Find $P(X=2)$.

   $P(X=2)=1/3$

   (b) Find $P(Y=2)$.

   Use the Law of Alternatives

   $$P(Y=2)=P(Y=2\,|\,X=1)P(X=1)+P(Y=2\,|\,X=2)P(X=2)+P(Y=2\,|\,X=3)P(X=3)$$
   $$=(0)(1/3)+(1/2)(1/3)+(1/3)(1/3)=5/18$$

   (c) Find $P(Y=2\,|\,X=2)$.

   $P(Y=2\,|\,X=2)=1/2$

   (d) Find $P(X=2\,|\,Y=2)$.

   Use Bayes Theorem

   $$P(X=2\,|\,Y=2)=\frac{P(Y=2\,|\,X=2)P(X=2)}{P(Y=2)}=\frac{(1/2)(1/3)}{5/18}=\frac{3}{5}$$

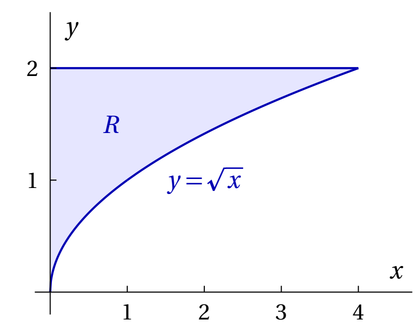
   (e) Find $P(X=2\cap Y=2)$.

   Use the Successive Conditioning $P(X=2\cap Y=2)=P(X=2)P(Y=2\,|\,X=2)=(1/3)(1/2)=1/6$

   (f) Are $P(X=2)$ and $P(Y=2)$ independent?

   No ... $P(X=2\cap Y=2)\neq P(X=2)P(Y=2)$ or $P(X=2\,|\,Y=2)\neq P(X=2)$ or $P(Y=2\,|\,X=2)\neq P(Y=2)$

2. A random point is chosen in the region R. Let Y be the y coordinate of the chosen point.



   (a) Find the area of the region R.

   $$\text{Area}(R)=\int_0^4\left(2-\sqrt{x}\right)\,dx=\left[2x-\tfrac{2}{3}x^{3/2}\right]_0^4=8-\frac{16}{3}=\frac{8}{3}\quad\text{or}$$

   $$\text{Area}(R)=\int_0^2 y^2\,dy=\left[\tfrac{1}{3}y^3\right]_0^2=\frac{8}{3}$$

   (b) Find $P(Y\le 1)$.

   $$P(Y\le 1)=\frac{1}{\text{Area}(R)}\int_0^1\left(1-\sqrt{x}\right)\,dx=\frac{1}{8/3}\left[x-\tfrac{2}{3}x^{3/2}\right]_0^1=\frac{3}{8}\left(1-\frac{2}{3}\right)=\frac{1}{8}$$

   (c) Find the distribution function F_Y.

   Clearly $F_Y(y)=0$ for $y\le 0$ and $F_Y(y)=1$ for $y\ge 2$; for $0<y<2$

   $$F_Y(y)=P(Y\le y)=\frac{1}{\text{Area}(R)}\int_0^{y^2}\left(y-\sqrt{x}\right)\,dx=\frac{1}{8/3}\left[yx-\tfrac{2}{3}x^{3/2}\right]_0^{y^2}=\frac{3}{8}\left(y^3-\frac{2}{3}y^3\right)=\frac{1}{8}y^3$$

   Note that the answer in (b) is just $F_Y(1)$

   (d) Find the density function f_Y.

   $f_Y(y)=F_Y'(y)=\frac{3}{8}y^2,\ 0<y<2$

3. The probability density function f_X of the discrete random variable X is given in the table. Find

x	1	2	3	4	5
$f_X(x)$	0.2	0.3	0.2	0.1	0.2

(a) $P(X=0)$

$P(X=0)=0$

(b) $P(X \leq 4)$

$P(X \leq 4)=1-P(X=5)=1-0.2=0.8$

(c) $P(X^2 \leq 4)$

$P(X^2 \leq 4)=P(X=1)+P(X=2)=0.2+0.3=0.5$

(d) $E(X)$

$E(X)=1\times 0.2+2\times 0.3+3\times 0.2+4\times 0.1+5\times 0.2=2.8$

(e) $E(X^2)$

$E(X^2)=1^2\times 0.2+2^2\times 0.3+3^2\times 0.2+4^2\times 0.1+5^2\times 0.2=9.8$

(f) the density function for $Y=|X-3|$.

Y takes the values $0,1,2$ and $f_Y(0)=P(Y=0)=P(X=3)=0.2$

$f_Y(1)=P(Y=1)=P(X=2)+P(X=4)=0.4$

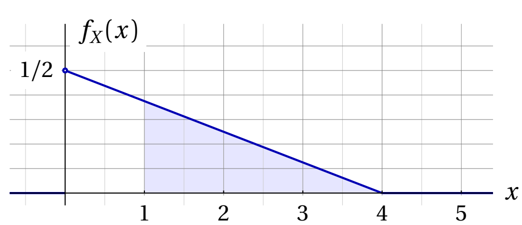
$f_Y(2)=P(Y=2)=P(X=1)+P(X=5)=0.4$

4. The density function of the continuous random variable X is $f_X(x)=\frac{1}{2}-\frac{1}{8}x$, $0<x<4$.

(a) Find $P(X>1)$.

From the graph below $P(X>1)=$ shaded area $=\frac{1}{2}(3)(3/8)=9/16$

(b) Sketch the graph of f_X and shade the area indicated by the probability in (a).



(c) Find $E(X)$.

$$E(X)=\int_{-\infty}^{\infty} x f_X(x)\,dx=\int_0^4 x\left(\tfrac{1}{2}-\tfrac{1}{8}x\right)dx=\left[\tfrac{1}{4}x^2-\tfrac{1}{24}x^3\right]_0^4=\frac{16}{4}-\frac{64}{24}=\frac{4}{3}$$

(d) Find $E(X^2)$.

$$E(X^2)=\int_{-\infty}^{\infty} x^2 f_X(x)\,dx=\int_0^4 x^2\left(\tfrac{1}{2}-\tfrac{1}{8}x\right)dx=\left[\tfrac{1}{6}x^3-\tfrac{1}{32}x^4\right]_0^4=\frac{64}{6}-\frac{256}{32}=\frac{8}{3}$$

(e) Find Var(X).

$$\mathrm{Var}(X)=E(X^2)-E(X)^2=\tfrac{8}{3}-\left(\tfrac{4}{3}\right)^2=\frac{8}{3}-\frac{16}{9}=\frac{8}{9}$$