Exam 1 Solutions **92.386 201 Prob & Stat I** **Fall 2015**

1. Elements are chosen at random from $S = \{1, 2, 3 \dots 20\}$. Let $A = \{s \in S \mid s \le 8\}$ and $B = \{s \in S \mid s \text{ is even}\}$.

   (a) $P(A) = \frac{8}{20} = \frac{2}{5}$

   (b) $P(B) = \frac{10}{20} = \frac{1}{2}$

   (c) $P(A') = 1 - P(A) = \frac{3}{5}$

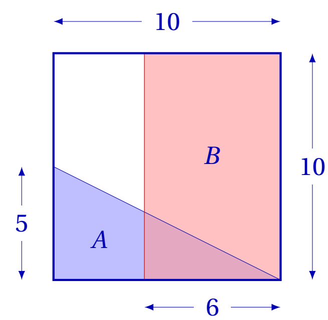
   (d) $P(A \cap B) = \frac{4}{20} = \frac{1}{5}$

   (e) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{2}{5} + \frac{1}{2} - \frac{1}{5} = \frac{7}{10}$

   (f) $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/5}{1/2} = \frac{2}{5}$

   (g) Are A and B independent? Yes … $P(A \mid B) = P(A)$

2. A dart is thrown at the 10×10 board shown. Let E_A be the event that the dart lands in the triangular region A and E_B the event that it lands in the rectangle B.



   (a) Find $P(E_A)$.

   $$P(E_A) = \frac{\text{Area of } A}{\text{Area of board}} = \frac{\frac{1}{2}(10)(5)}{100} = \frac{1}{4}$$

   (b) Find $P(E_B)$.

   $$P(E_B) = \frac{\text{Area of } B}{\text{Area of board}} = \frac{(6)(10)}{100} = \frac{3}{5}$$

   (c) Find $P(E_A \cap E_B)$.

   $$P(E_A \cap E_B) = \frac{\text{Area of } A \cap B}{\text{Area of board}} = \frac{\frac{1}{2}(6)(3)}{100} = \frac{9}{100}$$

   (d) Find $P(E_A \cup E_B)$.

   $$P(E_A \cup E_B) = P(E_A) + P(E_B) - P(E_A \cap E_B) = \frac{1}{4} + \frac{3}{5} - \frac{9}{100} = \frac{76}{100} = \frac{19}{25}$$

   (e) Find $P(E_A \mid E_B)$.

   $$P(E_A \mid E_B) = \frac{P(E_A \cap E_B)}{P(E_B)} = \frac{9/100}{3/5} = \frac{3}{20}$$

   (f) Are the events E_A and E_B independent?

   No … $P(E_A \mid E_B) = \frac{3}{20} \ne P(E_A) = \frac{1}{4}$

3. Two four-sided dice, one green and one red, are rolled. If the number on the green die is less than or equal to the number on the red die, the green die is rolled one more time. Let G and R be the final numbers showing on the green and red die respectively.

   (a) Find $P(R=3)$.

   $P(R=3)=\frac{1}{4}$

   (b) Find $P(G \le R \mid R=3)$.

   $P(G \le R \mid R=3)=P(\text{1st green die }=1, 2\text{ or }3)P(\text{2nd green die }=1, 2\text{ or }3)=\left(\frac{3}{4}\right)^2=\frac{9}{16}$

   (c) Find $P(G \le R)$. *Hint.* Law of Alternatives.

   $$P(G \le R)=\sum_{i=1}^{4} P(G \le R \mid R=i)P(R=i)=\sum_{i=1}^{4} P(G \le i)P(R=i)$$
   $$=\sum_{i=1}^{4}\left(\frac{i}{4}\right)^2\frac{1}{4}=\left(\frac{1}{4}\right)^2\frac{1}{4}+\left(\frac{2}{4}\right)^2\frac{1}{4}+\left(\frac{3}{4}\right)^2\frac{1}{4}+\left(\frac{4}{4}\right)^2\frac{1}{4}=\frac{30}{64}=\frac{15}{32}$$

   (d) Find $P(R=3 \mid G \le R)$. *Hint.* Bayes Theorem.

   $P(R=3 \mid G \le R)=\frac{P(G \le R \mid R=3)P(R=3)}{P(G \le R)}=\frac{(9/16)(1/4)}{15/32}=\frac{3}{10}$ using (a), (b) and (c)

4. A fair coin is tossed 8 times. Find the probability that

   (a) heads comes up exactly twice.

   There are 2^8 possible sequences and $\binom{8}{2}$ ways to choose the heads so $\binom{8}{2}/2^8=7/64$

   (b) heads comes up at least twice.

   Use complements ... $1-\binom{8}{0}/2^8-\binom{8}{1}/2^8=247/256$

   (c) the number of heads and tails is the same.

   This means four heads which has probability $\binom{8}{4}/2^8=35/128$

   (d) there are more heads than tails.

   This means five or more heads: $\binom{8}{5}/2^8+\binom{8}{6}/2^8+\binom{8}{7}/2^8+\binom{8}{8}/2^8=93/256$

   Here's a shorter way: call the answer ... then $2P+35/128=1$ (why?) so $P=93/256$

   (e) the third head appears on 8th toss.

   The first seven tosses have exactly two heads and there are $\binom{7}{2}$ ways this can happen so $\binom{7}{2}/2^8=21/256$