Homework 11 Solutions **92.386 203 Prob & Stat I** **Spring 2015**

1. Find $E(X)$ and $\mathrm{Var}(X)$ if

   (a) $f_X(x) = \frac{1}{24}x^2$, $x \in \{-2, 2, 4\}$;

   $E(X) = \sum_x x\, f_X(x) = (-2) f_X(-2) + (2) f_X(2) + (4) f_X(4) = (-2)\frac{4}{24} + (2)\frac{4}{24} + (4)\frac{16}{24} = \frac{8}{3}$

   $E(X^2) = \sum_x x^2 f_X(x) = (-2)^2 f_X(-2) + (2)^2 f_X(2) + (4)^2 f_X(4) = (4)\frac{4}{24} + (4)\frac{4}{24} + (16)\frac{16}{24} = 12$

   $\mathrm{Var}(X) = E(X^2) - E(X)^2 = 12 - \left(\frac{8}{3}\right)^2 = \frac{44}{9}$

   (b) $f_X(x) = \frac{1}{24}x^2$, $x \in (-2, 4)$.

   $E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_{-2}^{4} \frac{x^3}{24}\, dx = \left[\frac{1}{96}x^4\right]_{-2}^{4} = \frac{5}{2}$

   $E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_{-2}^{4} \frac{x^4}{24}\, dx = \left[\frac{1}{120}x^5\right]_{-2}^{4} = \frac{44}{5}$ so $\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{44}{5} - \left(\frac{5}{2}\right)^2 = \frac{51}{20}$

2. The life expectancy of a chip is 4 years.

   (a) What are the expected value and standard deviation of the life of a unit which comes with one chip installed and one replacement chip?
   Hint. See Homework 10 Q4 (b).

   By Homework 10 Q4 (b) the life expectancy X of the unit follows a gamma distribution with $\lambda = 1/4$ and $n = 2$ so $E(X) = n/\lambda = 8$ years and $\sigma_X = \sqrt{n}/\lambda = 4\sqrt{2}$ years

   (b) Redo (a) if a unit comes with two chips installed and which operates as long as one chip is functional.
   Hint. This is the maximum of two exponential variables … see Homework 8 Q4 (b).

   Let Y be the life expectancy of the unit. Then Y is the maximum of two independent exponential random variables with parameter $\lambda = 1/4$ so, from the Bonus in Homework 8 Q4 (b) Y has density

   $$f_Y(y) = 2\left(1 - e^{-\lambda y}\right)\lambda e^{-\lambda y} = \frac{1}{2}\left(e^{-y/4} - e^{-y/2}\right).$$

   So the expected life of the unit is

   $$E(Y) = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy = \int_0^{\infty} y \frac{1}{2}\left(e^{-y/4} - e^{-y/2}\right) dy = 6 \text{ years}$$

   and, since

   $$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_0^{\infty} y^2 \frac{1}{2}\left(e^{-y/4} - e^{-y/2}\right) dy = 56$$

   $\mathrm{Var}(Y) = E(Y^2) - E(Y)^2 = 20$. The standard deviation $\sigma_X = 2\sqrt{5}$ years.

   (c) In a shipment of 200 chips, how many are expected to fail within 5 years?

   By Homework 10 Q4 (a) the probability that a chip fails in the first 5 years is $1 - 0.287 = 0.713$ so $200 \times 0.713 = 143$ of the 200 are expected to fail within 5 years

3. Find the moment generating functions for the random variables of Homework 6.

(a) $f_X(x) = \frac{1}{21}x$, $x = 1,2,3,4,5,6$.

$$M_X(t) = \sum_{x=1}^{6} f_X(x)e^{tx} = \sum_{x=1}^{6} \frac{x}{21}e^{tx} = \frac{1}{21}\left(e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}\right)$$

(b) $f_X(x) = \frac{1}{2}(x+1)$, $|x| < 1$.

$$M_X(t) = \int_{-\infty}^{\infty} f_X(x)e^{tx}\,dx = \int_{-1}^{1} \tfrac{1}{2}(x+1)e^{tx}\,dx = \left[\frac{xt+t-1}{2t^2}e^{tx}\right]_{-1}^{1} = \frac{2t-1}{2t^2}e^t + \frac{1}{2t^2}e^{-t}$$

Bonus. Find the expected value and variance of the random variable X in Homework 9 Q1.

The discrete and continuous parts must be treated separately ...

$$E(X) = 6P(X=6) + \int_1^4 x f_X(x)\,dx = 6\times\frac{1}{16} + \int_1^4 \frac{2}{x^2}\,dx = \frac{3}{8} + \left[-\frac{2}{x}\right]_1^4 = \frac{3}{8} + \frac{3}{2} = \frac{15}{8}$$

and

$$E(X^2) = 6^2P(X=6) + \int_1^4 x^2 f_X(x)\,dx = 36\times\frac{1}{16} + \int_1^4 \frac{2}{x}\,dx = \frac{9}{4} + \left[2\ln x\right]_1^4 = \frac{9}{4} + 4\ln 2$$

so

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{9}{4} + 4\ln 2 - \left(\frac{15}{8}\right)^2 = 4\ln 2 - \frac{81}{64} \simeq 1.5$$