Homework 4 Solutions **92.386 203 Prob & Stat I** **Spring 2015**

1. Continuing Hw3 Q2 in which balls are selected one at a time with replacement from a box containing 4 red and 6 white balls.

   Let R_i be the event that the ith ball is red

   (a) If the third ball is red, what is the probability the first ball was red?

   $P(R_1 \mid R_3) = P(R_1) = 4/10 = 2/5$

   (b) If the third ball is red, what is the probability the first two balls were red?

   $P(R_1 \cap R_2 \mid R_3) = P(R_1 \cap R_2) = 4/25$

2. Redo Q1 if the balls are selected without replacement.

   (a) $P(R_1 \mid R_3) = \dfrac{P(R_3 \mid R_1)P(R_1)}{P(R_3)} = \dfrac{1/3 \times 2/5}{2/5} = \dfrac{1}{3}$

   (b) Bayes: $P(R_1 \cap R_2 \mid R_3) = \dfrac{P(R_3 \mid R_1 \cap R_2)P(R_1 \cap R_2)}{P(R_3)} = \dfrac{1/4 \times 2/15}{2/5} = \dfrac{1}{12}$

3. One of a die and a coin (sides 0 and 1) is selected at random and tossed.

   (a) If the outcome is 0, what is the probability the coin was chosen?

   Since the die has no 0 the probability is 1

   (b) If the outcome is 1, what is the probability the coin was chosen?

   Bayes: $P(\text{coin} \mid 1) = \dfrac{P(1 \mid \text{coin})P(\text{coin})}{P(1 \mid \text{coin})P(\text{coin}) + P(1 \mid \text{die})P(\text{die})} = \dfrac{(1/2)(1/2)}{(1/2)(1/2) + (1/6)(1/2)} = \dfrac{3}{4}$

4. An HIV test is known to be 98% accurate (i.e. the probability of a correct diagnosis is 0.98) and 0.5% of the population is known to be HIV positive.

   (a) What is the probability that a random person will test positive?

   Let A and B be the events that the person is HIV positive and HIV negative respectively, and let C be the event that the person tested positive. By the Law of Alternatives Theorem 1-10

   $$P(C) = P(C \mid A)P(A) + P(C \mid B)P(B) = 0.98 \times 0.005 + 0.02 \times 0.995 = 0.0248$$

   (b) If a person selected at random does test positive, what is the probability that he is infected?

   Bayes

   $$P(A \mid C) = \frac{P(C \mid A)P(A)}{P(C)} = \frac{0.98 \times 0.005}{0.0248} \simeq 0.198$$

   Why so low? Because even though only a small *proportion* of healthy people test positive, the much larger number of people who are healthy overwhelms the number of HIV positive people who test positive.

Bonus. A gambler has three coins in his pocket, one is normal, one is head-head and the third is tail-tail. He picks one at random and tosses it and it shows heads. The gambler says that since you now know the coin is either head-head or head-tail the odds that the other side is tails is 1/2. Is this true?

Let A_1 be the event that the coin is normal, A_2 be the event that the coin is head-head and A_3 be the event that it is tail-tail and let A the event that you see a head. Then $P(A_i) = 1/3$, $P(A) = 1/2$ (if this is not obvious use Theorem 1-10) and by Bayes Theorem 1-11

$$P(A_1 \mid A) = \frac{P(A \mid A_1)P(A_1)}{P(A)} = \frac{1/2 \times 1/3}{1/2} = \frac{1}{3}$$

This question is actually ambiguous — what would the gambler have done if the coin had shown tails? If he would have tossed again until he got a head, then $P(A_1) = P(A_2) = 1/2$, $P(A_3) = 0$ and instead

$$P(A_1 \mid A) = \frac{P(A \mid A_1)P(A_1)}{P(A)} = \frac{1/2 \times 1/2}{1/2} = \frac{1}{2}$$

Ambiguities like this contribute (justifiably) to the reputation probability has for being difficult.