Homework 6 Solutions **92.386 203 Prob & Stat I** **Spring 2015**

1. Let X be the outcome of throwing a loaded die and suppose $f_X(x) = kx$ where k is a constant.

   (a) Find k.

   $\displaystyle\sum_{k=1}^{6} f_X(x) = \sum_{k=1}^{6} kx = 21k = 1 \text{ so } k = \frac{1}{21}$

   (b) Find $P(X \text{ is even})$.

   $P(X \text{ is even}) = P(X=2) + P(X=4) + P(X=6) = f_X(2) + f_X(4) + f_X(6) = \frac{2}{21} + \frac{4}{21} + \frac{6}{21} = \frac{12}{21} = \frac{4}{7}$

   (c) Find $f_Y(y)$ where $Y = (X-3)^2$.

   Y takes the values $0, 1, 4, 9$ and

   $f_Y(0) = P(Y=0) = P(X=3) = f_X(3) = \frac{3}{21} = \frac{1}{7}$

   $f_Y(1) = P(Y=1) = P(X=2) + P(X=4) = f_X(2) + f_X(4) = \frac{2}{21} + \frac{4}{21} = \frac{6}{21} = \frac{2}{7}$

   $f_Y(4) = P(Y=4) = P(X=1) + P(X=5) = f_X(1) + f_X(5) = \frac{1}{21} + \frac{5}{21} = \frac{6}{21} = \frac{2}{7}$

   $f_Y(9) = P(Y=9) = P(X=6) = f_X(6) = \frac{6}{21} = \frac{2}{7}$

2. Let $f_X(x) = \frac{1}{2}(x+1)$, $|x| < 1$.

   (a) Verify that f_X is a probability density function.

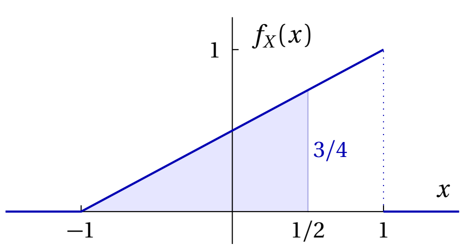
   Clearly $f_X(x) \geq 0$ for all x and $\displaystyle\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-1}^{1} \tfrac{1}{2}(x+1)\,dx = \left[\tfrac{1}{4}(x+1)^2\right]_{-1}^{1} = 1$

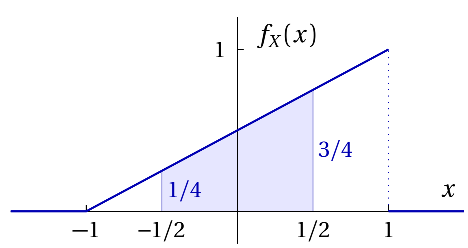
   (b) Draw the graph of f_X and shade the area representing $P(X < 1/2)$. Find this probability.

   $P(X < 1/2) = \frac{1}{2}\frac{3}{2}\frac{3}{4} = \frac{9}{16}$ — first graph below

   (c) Redo (b) for $P(|X| < 1/2)$.

   $P(|X| < 1/2) = \frac{1}{2}\frac{3}{2}\frac{3}{4} - \frac{1}{2}\frac{1}{2}\frac{1}{4} = \frac{1}{2}$ — second graph below

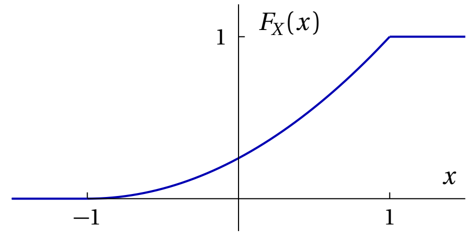




(d) Find F_X and sketch its graph.

For $|x|<1$, $F_X(x)=\int_{-1}^{x}\frac{1}{2}(x+1)\,dx=\left[\frac{1}{4}(x+1)^2\right]_{-1}^{x}=\frac{1}{4}(x+1)^2$ so

$$F_X(x)=\begin{cases}0, & x\le -1\\ (x+1)^2/4, & |x|<1\\ 1, & x\ge 1\end{cases}$$



(e) Find the density and distribution functions for $Y=X^2$.

Clearly $0<Y<1$ and in this interval $f_Y(y)=\sum_{y=x^2}\frac{f_X(x)}{|dy/dx|}=\sum_{x=\pm\sqrt{y}}\frac{\frac{1}{2}(x+1)}{|2x|}=\frac{\sqrt{y}+1}{4\sqrt{y}}+\frac{-\sqrt{y}+1}{4\sqrt{y}}=\frac{1}{2\sqrt{y}}$.

Integrating gives $F_Y(y)=\begin{cases}0, & y\le 0\\ \sqrt{y}, & 0<y<1\\ 1, & x\ge 1.\end{cases}$

Alternatively we could start with the distribution function … for $0<y<1$

$$\begin{aligned}F_Y(y)&=P(Y\le y)=P(X^2\le y)=P(-\sqrt{y}\le X\le\sqrt{y})\\ &=F_X(\sqrt{y})-F_X(-\sqrt{y})=\tfrac{1}{4}(\sqrt{y}+1)^2-\tfrac{1}{4}(-\sqrt{y}+1)^2=\sqrt{y}\end{aligned}$$

and differentiate to find the density function.

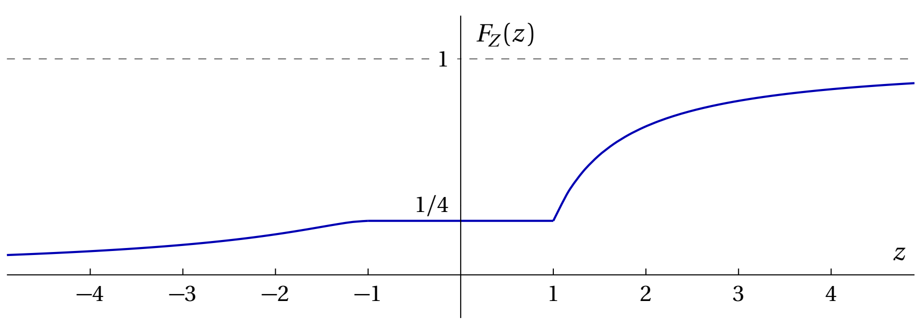
(f) Find the density function for $Z=\dfrac{1}{X}$.

$|Z|>1$ and $f_Z(z)=\dfrac{f_X(x)}{|dz/dx|}=\dfrac{\frac{1}{2}(x+1)}{|-1/x^2|}=\dfrac{\frac{1}{2}(1/z+1)}{z^2}=\dfrac{z+1}{2z^3}$.

Bonus. Find and sketch F_Z.

$\int\frac{z+1}{2z^3}\,dz=-\frac{2z+1}{4z^2}+C$ for $|z|>1$ but we have to be careful with the C — it's different in the two intervals $z<1$ and $z>1$. Since $F_Z(z)\to 0$ as $z\to-\infty$ we get $C=0$ for $z<-1$. On the other hand, $F_Z(z)\to 1$ as $z\to\infty$ so $C=1$ when $z>1$. Finally, $F_Z(z)$ is constant for $|z|\le 1$ and its value is $F_Z(-1)=F_Z(1)=1/4$. Putting it all together …

$$F_Z(z)=\begin{cases}-\dfrac{2z+1}{4z^2}, & z<0\\ \dfrac{1}{4}, & -1\le z\le 1\\ 1-\dfrac{2z+1}{4z^2}, & z>1\end{cases}$$



Bonus. Let X have density function $f_X(x) = 3e^{-3x}$, $x > 0$. Find $P(\sin X \le 1/2)$.

$\left\{\sin X \le \frac{1}{2}\right\} = \bigcup_{k=-\infty}^{\infty}\left[\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi\right]$ and the union is disjoint so

$$\begin{aligned}
P\left(\sin X \le \frac{1}{2}\right) &= \sum_{k=-\infty}^{\infty} P\left(\frac{5\pi}{6} + 2k\pi \le X \le \frac{13\pi}{6} + 2k\pi\right) \\
&= \sum_{k=-\infty}^{\infty} \int_{5\pi/6+2k\pi}^{13\pi/6+2k\pi} f_X(x)\,dx \\
&= \int_0^{\pi/6} 3e^{-3x}\,dx + \sum_{k=0}^{\infty} \int_{5\pi/6+2k\pi}^{13\pi/6+2k\pi} 3e^{-3x}\,dx \\
&= \left[-e^{-3x}\right]_0^{\pi/6} + \sum_{k=0}^{\infty}\left[-e^{-3x}\right]_{5\pi/6+2k\pi}^{13\pi/6+2k\pi} \\
&= 1 - e^{-\pi/2} + \sum_{k=0}^{\infty}\left[e^{-5\pi/2-6k\pi} - e^{-13\pi/2-6k\pi}\right] \\
&= 1 - e^{-\pi/2} + \left(e^{-5\pi/2} - e^{-13\pi/2}\right)\sum_{k=0}^{\infty} e^{-6k\pi} \\
&= 1 - e^{-\pi/2} + \frac{e^{-5\pi/2} - e^{-13\pi/2}}{1 - e^{-6\pi}} \quad \text{by the geometric formula} \\
&= 1 - 0.20788 + 0.00038802 = 0.7925
\end{aligned}$$

The third term here took a lot of effort for very little gain — we could have anticipated this since it is obviously less than

$$\int_{5\pi/6}^{\infty} 3e^{-3x}\,dx = \left[-e^{-3x}\right]_{5\pi/6}^{\infty} = e^{-5\pi/2} = 0.00038803$$

This is because the density function is decaying so quickly. The probability being calculated is the area under the blue graph where the red graph is under the green line. Only the first portion contributes significantly.