Homework 8 Solutions **92.386 203 Prob & Stat I** **Spring 2015**

1. Let X_1, X_2 and X_3 each be the result of randomly choosing a 1 or a 2. Find the density functions for

(a) $Y_1 = 3X_1$;

y	3	6
$f_{Y_1}(y)$	$\frac{1}{2}$	$\frac{1}{2}$

(b) $Y_2 = 2X_1 + X_2$;

y	3	4	5	6
$f_{Y_2}(y)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

(c) $Y_3 = X_1 + X_2 + X_3$.

y	3	4	5	6
$f_{Y_3}(y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

2. Let X be a random integer from 1 to 4. Let Y be a random integer from 1 to X.

(a) Find the density function for $W = X + Y$.

W has range $2, 3 \ldots 8$ and

$$f_W(2) = P(W=2) = P(X=1, Y=1) = P(X=1)P(Y=1) = \frac{1}{4}1 = \frac{1}{4}$$

$$f_W(3) = P(W=3) = P(X=2, Y=1) = \frac{1}{4}\frac{1}{2} = \frac{1}{8}$$

$$f_W(4) = P(W=4) = P(X=3, Y=1) + P(X=2, Y=2) = \frac{1}{4}\frac{1}{3} + \frac{1}{4}\frac{1}{2} = \frac{5}{24}$$

$$W(5) = P(W=5) = P(X=4, Y=1) + P(X=3, Y=2) = \frac{1}{4}\frac{1}{4} + \frac{1}{4}\frac{1}{3} = \frac{7}{48}$$

$$f_W(6) = P(W=6) = P(X=4, Y=2) + P(X=3, Y=3) = \frac{1}{4}\frac{1}{4} + \frac{1}{4}\frac{1}{3} = \frac{7}{48}$$

$$f_W(7) = P(W=7) = P(X=4, Y=3) = \frac{1}{4} = \frac{1}{16}$$

$$f_W(8) = P(W=8) = P(X=4, Y=4) = \frac{1}{4} = \frac{1}{16}$$

(b) Find $f_{X|Y=3}(x)$.

$$f_Y(3) = P(X=3, Y=3) + P(X=4, Y=3) = \frac{1}{4}\frac{1}{3} + \frac{1}{4}\frac{1}{4} = \frac{7}{48}$$

If $Y = 3$ then $X = 3$ or $X = 4$ and

$$f_{X|Y=3}(3) = \frac{f_{X,Y}(3,3)}{f_Y(3)} = \frac{(1/4)(1/3)}{7/48} = \frac{4}{7}$$

$$f_{X|Y=3}(4) = \frac{f_{X,Y}(4,3)}{f_Y(3)} = \frac{(1/4)(1/4)}{7/48} = \frac{3}{7}$$

3. Let X, Y be independent random variables with $f_X(x) = 1$, $0 < x < 1$ and $f_Y(y) = \frac{c}{1+y^2}$, $y \in \mathbb{R}$.

(a) Find c so that f_Y is a probability density function.

$$\int_{-\infty}^{\infty} f_Y(y)\,dy = \int_{-\infty}^{\infty} \frac{c}{1+y^2}\,dy = \Big[c\tan^{-1} y\Big]_{-\infty}^{\infty} = \pi c = 1 \text{ so } c = \frac{1}{\pi}$$

(b) Find $f_W(w)$ where $W = X + Y$.

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\,dx = \int_0^1 \frac{1/\pi}{1+(w-x)^2}\,dx$$

$$= \Big[-\frac{1}{\pi}\tan^{-1}(w-x)\Big]_0^1 = \frac{1}{\pi}\big[\tan^{-1} w - \tan^{-1}(w-1)\big],\ w \in \mathbb{R}$$

(c) Find $f_{XY}(w)$.

$$f_{XY}(w) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_X(x) f_Y(w/x)\,dx = \int_0^1 \frac{1}{x}\frac{1/\pi}{1+(w/x)^2}\,dx = \frac{1}{\pi}\int_0^1 \frac{x}{x^2+w^2}\,dx = \frac{1}{2\pi}\ln\frac{1+w^2}{w^2},\ w \in \mathbb{R}$$

(d) Find $f_{X|Y=3}(x)$.

Since X and Y are independent, $f_{X|Y=3}(x) = f_X(x)$.

Bonus.

4. (a) If 3 real numbers are chosen at random in the interval $[-1, 1]$, what is the probability that the largest one is (i) positive? (ii) greater than $1/2$? (iii) greater than 0.8?

(i) $P(\text{largest one} > 0) = 1 - P(\text{all } 3 < 0) = 1 - (1/2)^3 = 7/8$

(ii) $P(\text{largest one} > 1/2) = 1 - P(\text{all } 3 < 1/2) = 1 - (3/4)^3 = 37/64$

(iii) $P(\text{largest one} > 0.8) = 1 - P(\text{all } 3 < 0.8) = 1 - (0.9)^3 = 0.271$

(b) Let $X_1, X_2 \ldots X_n$ be a random sample of continuous random variables with pdf f_X and cdf F_X. Let $X_{\max}$ be the largest outcome in the sample. Show $f_{X_{\max}}(x) = n\big[F_X(x)\big]^{n-1} f_X(x)$.

Hint. Start by finding $F_{X_{\max}}(x)$.

$f_{X_{\max}}(x) = P(X_{\max} \le x) = P(\text{all } X_i \le x) = P(X_i \le x)^n = F_X(x)^n$

so $\quad f_{X_{\max}}(x) = \frac{d}{dx}\big[F_X(x)^n\big] = n\big[F_X(x)\big]^{n-1} f_X(x)$. $\quad X_{\max}$ is called an *order statistic.*