Homework 10 Solutions **92.386 203 Prob & Stat I** **Spring 2015**

1. Let X be the number of rolls of a die until you get a 6.

   (a) Find f_X.

   $f_X(x) = \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{x-1}$, $x = 1, 2, 3\ldots$

   (b) Find $P(X > 10)$.

   Using the geometric formula

   $$\begin{aligned} P(X > 10) &= P(X = 11) + P(X = 12) + P(X = 13) + \ldots \\ &= \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{10} + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{11} + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{12} + \ldots = \left(\frac{5}{6}\right)^{10} = 0.162 \end{aligned}$$

   (c) Find the smallest n so that $P(X > n) < 0.01$.

   Replacing 10 with n in (b) shows $\left(\frac{5}{6}\right)^{n} < 0.01$ or $n > \frac{\ln 0.01}{\ln 5/6} = 25.26$ so $n = 26$

2. Let X be the result of a die toss and Y the number of heads when a coin is tossed X times. Find a formula for the joint density function $f_{X,Y}$.

   $P(Y = y \mid X = x) = \binom{x}{y}\left(\frac{1}{2}\right)^{x}$ is the probability of getting y heads from x tosses of the coin — this is just $B(x, 1/2)$. Using the conditional probability formula

   $$f_{X,Y}(x, y) = P(X = x, Y = y) = P(Y = y \mid X = x)P(X = x) = \frac{1}{6}\binom{x}{y}\left(\frac{1}{2}\right)^{x}, \; x = 1, 2\ldots 6, \; y = 0, 1\ldots x$$

3. A typist has probability 0.01 of typing a word incorrectly.

   (a) Use the Binomial distribution to find the probability of typing 2 or fewer words incorrectly on a page of 250 words.

   Let X be the number of errors on the page. Then $X \sim B(250, 0.01)$ and

   $$P(X \le 2) = \binom{250}{0}(0.01)^0(0.99)^{250} + \binom{250}{1}(0.01)^1(0.99)^{249} + \binom{250}{2}(0.01)^2(0.99)^{248} = 0.543$$

   (b) Redo (a) using the Poisson distribution.

   The expected number of errors per page is $\lambda = 250 \times 0.01 = 2.5$ so $X \sim P(\lambda)$ gives

   $$P(X \le 2) = \frac{2.5^0}{0!}e^{-2.5} + \frac{2.5^1}{1!}e^{-2.5} + \frac{2.5^2}{2!}e^{-2.5} = 0.544$$

   It is no coincidence that these answers are close.

4. The life expectancy of a memory chip is 4 years.

   (a) Find the probability that a chip will last more than 5 years. *Hint.* This is exponential with $\lambda = 1/4$.

   If X is the time for the chip to fail then

   $$P(X > 5) = \int_5^\infty \lambda e^{-\lambda x}\, dx = \left[-e^{-\lambda x}\right]_5^\infty = e^{-5\lambda} = e^{-1.25} = 0.287$$

(b) If a second chip is used to replace the first when it fails, what is the probability you will still have a good chip after 5 years? *Hint.* This is a gamma distribution.

If Y be the time to fail of the second chip then Y follows the gamma distribution with $n=2$ and

$$P(Y>5)=\int_5^\infty \lambda^2 y e^{-\lambda y}\, dy = \left[-\lambda y e^{-\lambda y} - e^{-\lambda y}\right]_5^\infty = (1+5\lambda)e^{-5\lambda} = 2.25e^{-1.25} = 0.645$$

(c) If two chips start being used at the same time, what is the probability at least one of them is still good after 5 years?

This is the complement of the probability that they both fail within 5 years which, assuming independence and using (a) is $P = 1-(1-0.287)^2 = 0.491$