

A Brief Survey of the Curve Complexes of Non-orientable Surfaces of Low Complexity

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1 Introduction

1.1 Purpose

The contemporary study of surfaces commonly focuses on their mapping class groups and the simple closed curves on the surface considered up to isotopy (for the definition of the mapping class group, see section 1.2). In this paper, a simple closed curve on a surface S is an embedding $S^1 \hookrightarrow S$. We often learn more about the structure and properties of a surface by examining what happens when we cut along such curves, the intersection patterns of these curves, and when we act on the curves by mapping class group elements. One tool that has developed is the *curve complex*, a construction that endows the set of simple closed curves of a surface with the structure of a simplicial complex. A vast collection of literature has amassed regarding the curve complex and its properties. Of note about the curve complex is that the mapping class group acts naturally on it and in many cases is isomorphic to its automorphism group [Iva97].

Since the theory of non-orientable surfaces is still relatively undeveloped, we decided to focus on these surfaces and their curve complexes to understand them better. Using the work of Scharlemann [Sch82], which provided a foundation for the curve complex's properties in the non-orientable case, and of Korkmaz [Kor02], which gives a generating set for the mapping class group of a non-orientable surface, we attempted to explicitly construct the curve complex of the twice punctured Klein bottle. Along the way we computed the curve complexes of the projective plane with up to two punctures,

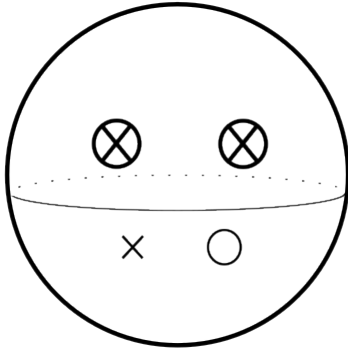


Figure 1: A sphere with two cross caps, one puncture, and one boundary component. We denote this by $N_{2,1,1}$.

as well as that of the Klein bottle with up to one puncture.

We noticed that the curve complexes of the projective plane with zero to two punctures were relatively simple to compute, and with some more effort the curve complex of the punctured Klein bottle was as well, but once we get to the twice punctured Klein bottle or higher genus non-orientable surfaces, the complexity grows immensely. However, we were still able to make some progress towards understanding it locally and having a rough understanding of its simple closed curves. This write-up is a compilation of the various results we came across while investigating this subject.

1.2 Preliminaries

Let S denote a general surface. We denote by $N_{g,b,p}$ the non-orientable surface of genus g , b boundary components, and p punctures. Since this write-up will deal mainly with surfaces with only punctures and no boundaries, we will often suppress b and instead write $N_{g,p}$, or simply N , when the context is clear.

A nonorientable surface is a surface with an embedded Möbius strip. A **crosscap** is a graphical representation of this (see Fig. 1). To form a crosscap in a surface N , remove an open disc, thus leaving a boundary behind, and then identify (glue) the antipodal points of the boundary.

Definition 1 ([Kor02]). Let N be a closed connected surface, possibly with boundaries and punctures. We define the **mapping class group** $\text{Mod}(N)$

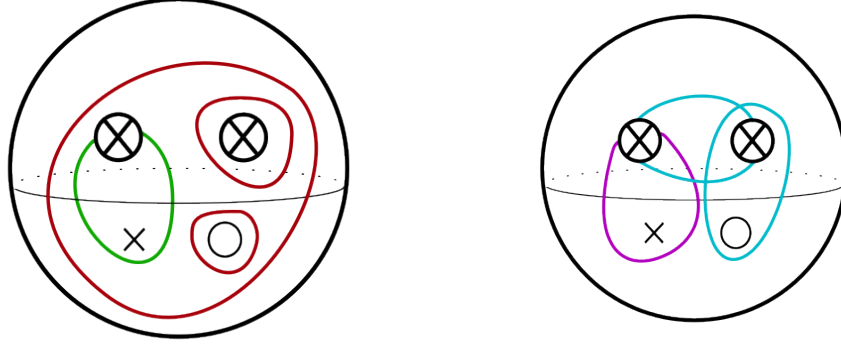


Figure 2: (a): The red curves are two-sided and not essential; the green curve is one-sided and essential. (b): The blue curves are two-sided. The purple curve is one-sided. They are all essential.

as the group of isotopy classes of diffeomorphisms of N that fix the boundary components pointwise. In other words,

$$\text{Mod}(N) = \text{Diff}(N, \partial N) / \text{Diff}_0(N, \partial N)$$

where $\text{Diff}_0(N, \partial N)$ is the subgroup of diffeomorphisms isotopic to the identity.

For a more comprehensive overview on the basics of mapping class groups of non-orientable surfaces, and especially for a more in-depth definition of the mapping class group, we direct the reader to [Par14].

In order to study the mapping class group, it can be useful to study how it acts on simple closed curves on the surface S considered up to isotopy. First, we distinguish curves in the following ways.

Definition 2. We call a curve c **one-sided** if a regular neighborhood of c is topologically a Möbius strip, and **two-sided** if a regular neighborhood is topologically an annulus (see Figure 2). We say that a curve c is **essential** if c is non-trivial, not homotopic to a boundary component, and does not bound a Möbius band nor a puncture.

It is also useful to examine how the curves on S intersect to understand the surface better. Formally, we define the *geometric intersection number*.

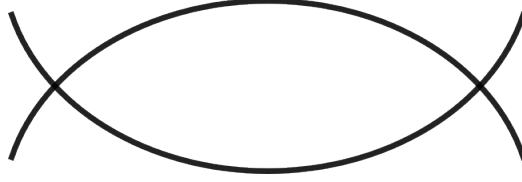


Figure 3: A bigon. Note how when two curves form a bigon, they can be isotoped off each other.

Definition 3. Let α and β represent two isotopy classes of simple closed curves in S . The **geometric intersection number** between α and β is defined as the minimal number of intersection points between a representative for α and β . Explicitly:

$$i(\alpha, \beta) = \min\{|a \cap b| : a \in \alpha, b \in \beta\}.$$

When α and β are given by a, b such that $|a \cap b| = i(\alpha, \beta)$, we say α and β are in **minimal position**.

We will describe a way to compute the geometric intersection number, which we will use in Section 3.

Definition 4. Two transverse simple closed curves α and β form a **bigon** if there is an embedded disk S whose boundary is the union of an arc of α and an arc of β intersecting in exactly two points [FM12, p. 30] (see Fig. 3).

Proposition 1. *Two transverse simple closed curves in a surface S are in minimal position if and only if they do not form a bigon.*

Proof. [FM12, p. 30] ■

We can form a simplicial complex using these simple closed curves called the *curve complex*.

Definition 5. The **curve complex** associated to a surface S , denoted $\mathcal{C}(S)$, is the simplicial complex whose vertices are isotopy classes of simple essential curves in S . A set of $k + 1$ vertices $\{v_0, \dots, v_k\}$ forms a k -simplex if the geometric intersection number $i(v_i, v_j) = 0$ for all $i \neq j$.

Remark 1. It should be noted that the curve complex is unable to distinguish between punctures and boundary components [FM12, p. 96]. That is, the curve complexes of the surfaces N_{g,b_1,p_1} and N_{g,b_2,p_2} are isomorphic as long as $b_1 + p_1 = b_2 + p_2$. However, because elements of the mapping class group are allowed to permute the marked points (that is, send one marked point to another), but must fix the boundary components, the mapping class groups of N_{g,b_1,p_1} and N_{g,b_2,p_2} are not isomorphic.

As stated earlier, there is a natural action of the mapping class group $\text{Mod}(N)$ on the curve complex $\mathcal{C}(N)$, and since the mapping class group preserves intersection numbers between curves, this induces an automorphism of $\mathcal{C}(N)$. Here, by an automorphism $\varphi : \mathcal{C}(N) \rightarrow \mathcal{C}(N)$ we mean that if there exists an edge between vertices $x, y \in \mathcal{C}(N)$, then there also exists an edge between vertices $\varphi(x), \varphi(y)$.

Remark 2. It is important to point out that $\mathcal{C}(N)$ is a **flag** complex, meaning that for any subset of vertices X in the curve complex and any pair of vertices $\{x, y\} \subset X$ is in the complex as well, then X is in the complex as well. It follows that the simplicial structure of $\mathcal{C}(N)$ is contained entirely in its 1-skeleton $\mathcal{C}_1(N)$, and so we only need to define the automorphism on the set of vertices.

The natural action of the mapping class group on the curve complex also allows us to define the natural homomorphism.

Definition 6. Let S be some surface with mapping class group $\text{Mod}(S)$. Denote by $\text{Aut } \mathcal{C}(S)$ the automorphism group of the curve complex of S . Each element f of $\text{Mod}(S)$ induces an automorphism g of $\mathcal{C}(S)$. The natural homomorphism is the map $\phi : \text{Mod}(S) \rightarrow \text{Aut } \mathcal{C}(S)$ such that $\phi(f) = g$.

2 The Real Projective Plane \mathbb{RP}^2

We will now discuss the simple cases of the curve complex of the real projective plane with up to two punctures. Throughout this section, we will refer to real projective plane \mathbb{RP}^2 with p punctures as \mathbb{RP}_p^2 .

We computed these results independently. They can also be found in Section 2.4 of [AK14] and Section 2 of [Irm19].

2.1 No punctures: \mathbb{RP}^2

Up to isotopy, there is only a single essential curve on \mathbb{RP}^2 , specifically, the core of the embedded Mobius strip. Hence the curve complex, denoted $\mathcal{C}(\mathbb{RP}^2)$ consists of a single point coinciding with the single isotopy class of the one-sided curve a in the diagram below.

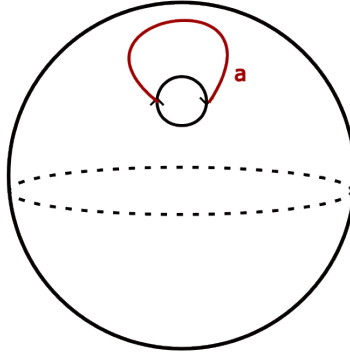


Figure 4: Here, \mathbb{RP}^2 is represented by a cross cap taken out of a sphere.

2.2 One Puncture: \mathbb{RP}_1^2

The curve complex $\mathcal{C}(\mathbb{RP}_1^2)$, is still just a single point, corresponding to the isotopy class of the one-sided curve a in the diagram below.

2.3 Two Punctures: \mathbb{RP}_2^2

Up to isotopy, there are two essential curves on \mathbb{RP}_2^2 , both one-sided. Thus, the curve complex $\mathcal{C}(\mathbb{RP}_2^2)$, consists of two disconnected points, corresponding to the two one-sided curves a and b in the diagram below.

3 Once-Punctured Klein Bottle

The Klein bottle is the next level up in surface complexity with genus $g = 2$. In this section we will mainly discuss the once-punctured Klein bottle and its

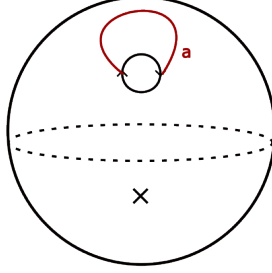


Figure 5: Note this is the same curve as in Fig. 4. If we had instead wrapped a around the puncture, that would be the same as sliding a around the sphere.

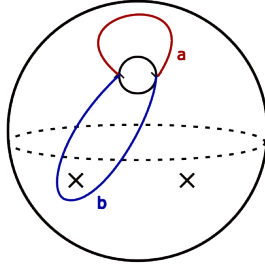


Figure 6: Note that it does not matter whether we wrap b around the left or right puncture.

curve complex, and we will provide a proof of the structure of the complex. However, before getting to the case of 1 puncture, we will first quickly note the case of the closed Klein bottle $K = N_{2,0}$. Both of these results are also in Section 2.4 of [AK14] and Section 2 of [Irm19].

To discuss these two curve complexes, we will introduce one piece of machinery:

Definition 7. Let $a : S^1 \rightarrow N$ be a two-sided circle in N , let $C = S^1 \times [0, 1]$, and let $T : C \rightarrow C$ be the diffeomorphism $T(z, t) = (e^{2i\pi t} z, t)$. Moreover, let $\phi : S^1 \times [0, 1] \rightarrow N$ be an embedding such that $\phi(C) \cap \partial N = \emptyset$ and $\phi(z, 1/2) = a(z)$ for all $z \in S^1$. Then we define the **Dehn twist along a**

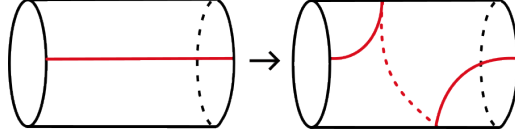


Figure 7: Dehn twist along the center circle of a cylinder.

the isomotopy class in $\text{Mod}(N)$ represented by the map

$$T_a(x) = \begin{cases} \phi \circ T \circ \phi^{-1} & \text{if } x \in \text{Im}(C) \\ x & \text{if } x \notin \text{Im}(C) \end{cases}.$$

Observe an illustration of a Dehn twist in Fig. 7. The curve complex $\mathcal{C}(K)$ consists of three vertices, two of which are connected by an edge. The two connected vertices correspond to the isotopy classes of one-sided simple closed curves a and b , while the one isolated vertex corresponds to two-sided simple closed curve c , all pictured below. Note that $t_c(a) = b$ and $t_c(b) = a$.

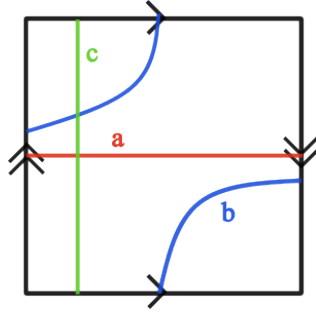


Figure 8: Curves a and b are both one-sided with $i(a, b) = 0$. Curve c is two-sided and intersects both a and b nontrivially.

We will now move on to the case of the once-punctured Klein bottle $N_{2,1}$. Denote by a and b the following curves on $N_{2,1}$: We will show that any simple closed curve on $N_{2,1}$ is either b or a Dehn twist of a . To do so, we introduce another technique: the fundamental group. Since the conjugacy classes of the fundamental group correspond to free homotopy classes of curves, we can consider oriented curves on $N_{2,1}$ as conjugacy classes of $\pi_1(N_{2,1})$. Therefore, since we have not given a an orientation, choose a to be oriented rightward on Fig. 9. Then we can consider a and b as conjugacy classes in $\pi_1(N_{2,1})$. We will see why this is useful in the following definitions and statements.

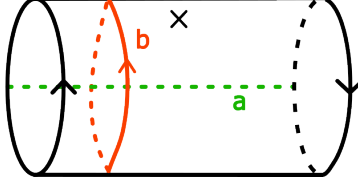


Figure 9: We depict $N_{2,1}$ by a cylinder whose ends are identified oppositely. The curve b is embedded in the orientation depicted.

Definition 8. Let N be a surface. We give an action of $\text{Mod}(N)$ on the set of curves in N via $\varphi \cdot \alpha = \varphi(\alpha)$. For any curves α, β on N , if there exists $\varphi \in \text{Mod}(N)$ such that $\varphi \cdot \alpha = \beta$, we say α and β are **mapping class equivalent**.

Example 1. The curves a and b in Fig. 8 are mapping class equivalent, as $T_c(a) = b$.

Note for later that since $\text{Mod}(N)$ acts on the set of curves in N , each element in $\text{Mod}(N)$ gives an outer automorphism of $\pi_1(N)$.

Now we return to $N_{2,1}$.

Theorem 1. *The simple closed essential curves on $N_{2,1}$ consist of ab^n and b .*

Proof. From [Gom18], all simple closed curves on $N_{2,1}$ are mapping class equivalent to an element of $\{a, b, a^2, ab^{-1}a^{-1}b^{-1}\}$. Since a^2 bounds a Möbius band and $ab^{-1}a^{-1}b^{-1}$ is isotopic to a point, only a and b are essential. Thus, every simple closed essential curve on $N_{2,1}$ is mapping class equivalent to a or b . These curves are ab^n and b . ■

We have the following result.

Theorem 2. $\mathcal{C}(N_{2,1})$ consists of two components. One component is an infinite line bijective with \mathbb{Z} as in Fig. 10. This component represents all the one-sided curves on the surface. The other component is a single point corresponding to the two-sided nonseparating curve b .

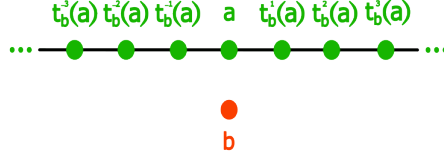


Figure 10: The curve complex of $N_{2,1}$. Here, t_b denotes the Dehn twist along b .

Proof. Consider how b intersects with a in Fig. 9. Since a and b intersect only once, there cannot be any bigons. Thus, they are in minimal position by Proposition 1 and we have $i(a, b) = 1$. Then $i(t^n(a), t^n(b)) = i(ab^n, b) = 1$ for all n , so there is no edge connecting b with any other curve.

Next, a does not intersect ab , as depicted in Fig. 11 below, so $i(ab^n, ab^{n+1}) = 0$. Therefore, there is an edge connecting every ab^n and ab^{n+1} . Additionally, if $n \geq 2$, $i(a, ab^n) \neq 0$ because given any two consecutive intersection points between them, the arcs form the same curve as b , and thus does not bound a disk.

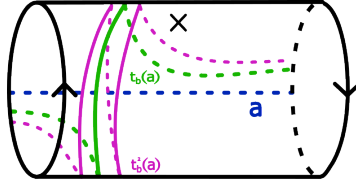


Figure 11: Dehn twists of a along b . Note $t_b(a) = ab$, $t_b^2(a) = ab^2$.

Therefore, these are the only arcs of a and ab^n that intersect in exactly two points, and they do not form bigons. We conclude $i(a, ab^n) \neq 0$. Applying Dehn twists, $i(ab^j, ab^k) \neq 0$ for all $j - k \geq 2$. Hence, there is exactly one edge between each ab^n and ab^{n+1} , and no others. ■

Remark 3. The natural homomorphism $\phi : \text{Mod}(N_{2,1}) \rightarrow \text{Aut } \mathcal{C}(N_{2,1})$ is surjective by an argument practically identical to that in [Kor02]. Since the disconnected components of $\mathcal{C}(N_{2,1})$ are a line and a single point, $\text{Aut } \mathcal{C}(N_{2,1})$ is isomorphic to the infinite dihedral group. The infinite dihedral group is generated by translations, which are induced by Dehn twists, and reflections, which are induced by point slides (defined in Section 4.3).

However, ϕ is not injective, since the cross cap slide (defined in Section 4.3) is nontrivial in $\text{Mod}(N_{2,1})$, but acts trivially on $\mathcal{C}(N_{2,1})$.

4 Twice-Punctured Klein Bottle

Next, we treat the case of the twice-punctured Klein Bottle. This case is particularly interesting since in [Kor02], where it is proven that for connected nonorientable surfaces $N_{g,n}$ with $g+n \geq 5$, $\Phi : \text{Mod}(N) \rightarrow \text{Aut}(\mathcal{C}(N))$ is an isomorphism. However, we do not know if this is true for $N_{2,2}$. We mimic the method in [Gom18]. First, we need the following definition.

Definition 9. Let N be a surface. For any curves α, β in N , define $\alpha \sim \beta$ if $N \setminus \alpha \cong N \setminus \beta$. The equivalence classes divide the curves on N into **topological types**.

Note that for all curves α, β with the same topological type, there exists $\varphi \in \text{Mod}(N)$ such that $\varphi \cdot \alpha = \beta$.

4.1 Notation

We will set up notation for the curves on $N_{2,2}$. To do so, consider the fundamental group $\pi_1(N_{2,2}) \cong F_3$ generated by the following curves a, b, c in the below diagram (Fig. 12). Note that in the diagram, $d = c^{-1}baba^{-1}$.

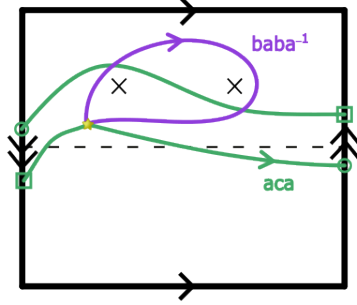


Figure 13: The curves $baba^{-1}$ and aca mentioned in Theorem 3.

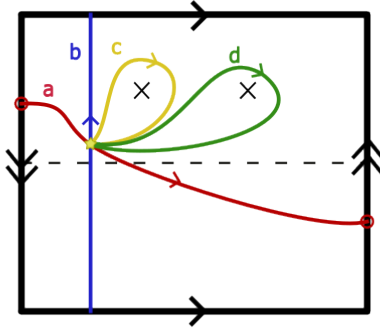


Figure 12: $\pi_1(N_{2,2}) = \langle a, b, c \rangle$.

4.2 Mapping Class Equivalents

Theorem 3. *All simple closed essential curves on $N_{2,2}$ are mapping class equivalent to one of the curves $\{a, b, baba^{-1}, aca\}$.*

Proof. We can identify the topological type of a curve by the resulting surface from cutting out that curve. Thus, the proof is by Euler Characteristics, which is unaffected by removing a simple closed curve. Recall $\chi(N_{2,2}) = 2 - 2 - 2 = -2$.

Given a curve γ on $N_{2,2}$, we identify the possibilities for what $N_{2,2}$ can be.

1. If γ is separating, we have the cases:

- (i) γ separates $N_{2,2}$ into two orientable surfaces. This is not possible, as the connected sum of orientable surfaces is orientable.
- (ii) γ separates $N_{2,2}$ into one orientable surface S of genus g_S and one non-orientable surface N of genus g_N . Both S and N have a boundary component given by γ , with two punctures shared between them. Thus, $-2 = \chi(N) + \chi(S) = -g_N - 2g_S$, so $g_N + 2g_S = 2$, where $g_N > 0$ because N is not orientable. Then we must have $g_N = 2, g_S = 0$.

If N has both punctures, then S is a disk, which implies γ is not essential.

If S and N contain one puncture each, then S is a punctured disk. Thus, γ bounds a puncture, so is not essential.

If S contains both punctures, then the curve $baba^{-1}$ accomplishes this.

- (iii) γ separates $N_{2,2}$ into two non-orientable surfaces N_1 and N_2 with genus g_1 and g_2 respectively. Then $g_1 + g_2 = 2, g_1 > 0$ and $g_2 > 0$, so as that would contradict the genus of $N_{2,2}$. Thus, we must have $g_1 = g_2 = 2$. Hence, γ divides $N_{2,2}$ into two Möbius bands. If either N_1 or N_2 contains no punctures, then γ bounds a Möbius band. Hence, either N_1 and N_2 contain one puncture each. The curve aca accomplishes this.

2. Non-separating:

- (i) if the resulting surface is orientable of genus g , then $2g + i = 2$, where $i > 0$ is the number of boundary components introduced from removing γ . Then $g = 0$ and $i = 2$. This is given by the curve b .
- (ii) If the resulting surface is non-orientable of genus g , then $g + i = 2, g > 0, i > 0$. The only option is $g = i = 1$. That is, the resulting surface is a twice-punctured Möbius band. This is given by the curve a .

Therefore, there are four distinct topological types of curves on $N_{2,2}$, and we can take representatives $a, b, baba^{-1}$, and aca . The curves a

and b are pictured in Fig. 12, while the curves $baba^{-1}$ and aca are pictured in 13.

■

In other words, given any simple closed essential curve α on $N_{2,2}$, there exists some $\varphi \in \text{Mod}(N_{2,2})$ such that $\varphi(\alpha)$ is a mapping class element. Therefore, we can use the action of $\text{Mod}(N_{2,2})$ on the curves a , b , $baba^{-1}$, and aca to generate all the simple essential curves on $N_{2,2}$.

4.3 Mapping Class Actions

The generators of $\text{Mod}(N_{2,2})$ are t, v, σ, y , which we will define below [Kor02]. Here, t denotes a Dehn twist as in Definition 7.

The homeomorphism v is a *point slide* (also referred to as a boundary slide), and it is obtained by sliding the marked point along the green one-sided curve in the diagram (Fig. 14) below.

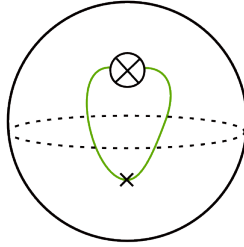


Figure 14: Point slide

The homeomorphism σ is an *elementary braid*, and it is achieved by permuting the two marked points as in the diagram (Fig. 15) below.

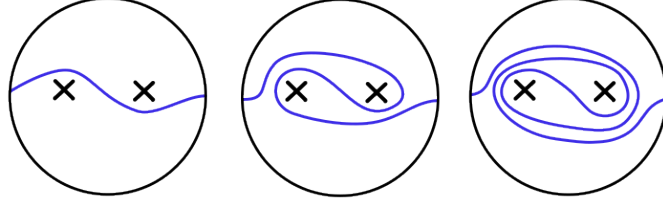


Figure 15: Elementary braid

The cross cap slide y is only supported on a surface of genus at least two. It is obtained by sliding a cross cap along the blue curve below (Fig. 16) (which is actually given by b).

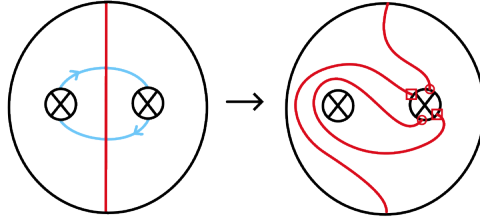


Figure 16: Cross cap slide

As before, the elements of $\text{Mod}(N_{2,2})$ act on the conjugacy classes of $\pi_1(N_{2,2})$ by composition, yielding an action $\text{Mod}(N_{2,2}) \rightarrow \text{Out}(\pi_1(N_{2,2}))$. To compute the action of a mapping class on a curve, we need only compute how each generator of $\text{Mod}(N_{2,2})$ acts on each generator of $\pi_1(N_{2,2}) = \langle a, b, c \rangle$. To do so, we fix representatives t, b, σ, y for the generators of $\text{Mod}(N_{2,2})$ and compute their actions on a, b, c .

Here are the actions we know:

- $t(a) = ab, t(b) = b, t(c) = c$
- $t^{-1}(a) = ab^{-1}, t^{-1}(b) = b, t^{-1}(c) = c$
- $v(a) = ca, v(b) = a^{-1}c^{-1}ba, v(c) = c^{-1}$
- $v^{-1}(a) = ca, v^{-1}(b) = a^{-1}bac^{-1}, v^{-1}(c) = c^{-1}$

- $\sigma(a) = a, \sigma(b) = b, \sigma(c) = baba^{-1}c^{-1}$
- $\sigma^{-1}(a) = a, \sigma^{-1}(b) = b, \sigma^{-1}(c) = c^{-1}baba^{-1}$
- $y(a) = a^{-1}, y(b) = b, y(c) = a^{-1}b^{-1}cba$
- $y^{-1}(a) = a^{-1}, y^{-1}(b) = b, y^{-1}(c) = ba^{-1}cab^{-1}$

Then given any word $w \in \pi_1(N_{2,2})$ and any generator $[\varphi] \in \text{Mod}(N_{2,2})$, the action of $[\varphi]$ on a conjugacy class $[w]$ can be computed by first computing the action of φ on a . That is, we apply the action of φ to each generator in w separately. The result yields a representative for the conjugacy class $[\varphi] \cdot [w]$.

For example, consider the action of $[vy\sigma^{-1}t]$ on $[abc]$. First, $t(abc) = t(a)t(b)t(c) = ab^2c$. Then $\sigma^{-1}(ab^2c) = ab^2c^{-1}baba^{-1}$. We are free to cyclically reduce this result as $b^3c^{-1}ba$, then continue $y(b^3c^{-1}ba) = b^3a^{-1}b^{-1}c^{-1}baba^{-1}$. We can finally compute that $v(b^3a^{-1}b^{-1}c^{-1}baba^{-1})$ is the word

$$a^{-1}c^{-1}(bc^{-1})^3a^{-1}b^{-1}caca^{-1}c^{-1}babcb^{-1}.$$

Note that since $[tv] \cdot [a] = [abc]$, this long word represents a curve that is the same topological type as a .

This suggests a method of classifying the simple essential curves on $N_{2,2}$. Since every such curve on $N_{2,2}$ is given by some action of $\text{Mod}(N_{2,2})$ on the curves $\mathcal{A} := \{a, b, baba^{-1}, aca\}$, it remains to determine which actions are superfluous. That is, fixing $x \in \mathcal{A}$, for which elements $[\varphi], [\psi] \in \text{Mod}(N_{2,2})$ do we have that $[\varphi] \cdot [x]$ and $[\psi] \cdot [x]$ yield the same conjugacy class up to inverses?

In other words, we can let $\text{Mod}(N_{2,2})$ act on the equivalence classes of $\pi_1(N_{2,2})$ where conjugates and inverses are identified. Then for any $x \in \mathcal{A}$, the distinct curves of the same topological type as x are given by the cosets of the stabilizer of the action on x . In particular, since a is the only one-sided curve in \mathcal{A} , the cosets of the stabilizer of a give all the one-sided curves on $N_{2,2}$. However, we have not been able to determine these stabilizers. It is simple to write a program to quickly calculate actions, cyclically reduce, and compare words, so large amounts of computational data may provide insights.

5 Local Structure of $\mathcal{C}(N_{2,2})$

In this section, we will discuss the local structure of $\mathcal{C}(N_{2,2})$. First, recall from the previous section that all simple closed essential curves on $N_{2,2}$ are mapping class equivalent to one of $\mathcal{A} = \{a, b, baba^{-1}, aca\}$. Since every other curve in $N_{2,2}$ will be topologically equivalent to a curve in \mathcal{A} , it is possible to extend the local structure at each of these vertices to sketch and understanding of the entire curve complex $\mathcal{C}(N_{2,2})$.

The idea here is to cut along these four curves to obtain different surfaces whose curve complexes are already known. The curve complexes of these new surfaces will be subcomplexes of $N_{2,2}$, and be disjoint from the curve we cut along to obtain the new surface. That is, if we cut along the curve α , and denote the resulting surface as N_α , then we get an edge connecting the vertex α in $\mathcal{C}(N_{2,2})$ to each of the vertices in $\mathcal{C}(N_\alpha)$, each of which injects to a vertex in $\mathcal{C}(N_{2,2})$. This produces a local picture of the vertex α . We now demonstrate this process by cutting along each of the curves in \mathcal{A} .

Case 1. Consider the curve a . Cutting along a gives us a surface of genus $g - 1$ with $b + 1$ boundary components. So, we have $N_{1,1,2}$. The curve complex of this surface is isomorphic to that of $N_{1,3}$. An important result is that Szepietowski proved in [Sze19] that $\mathcal{C}(N_{1,3})$ is quasi-isomorphic to a simplicial tree.

Case 2. Now consider b . Cutting along b gives us a sphere with two marked points and two punctures, and its curve complex is isomorphic to that of $S_{0,4}$, the sphere with four boundaries. It is a well-known result that the curve complex of this surface is isomorphic to the Farey complex if we allow two vertices to have an edge if their intersection number is minimal rather than 0.

If we enforce the rule that the intersection number must be 0, $\mathcal{C}(S_{0,4})$ consists of just the vertices of the Farey complex, which is bijective with $\mathbb{Q} \cup \{\infty\}$.

Case 3. If we cut along $baba^{-1}$, we get two surfaces since this curve separates. One is a disk with two punctures, and the other is $N_{2,1}$, a Klein bottle with one boundary component. The curve complex of $N_{2,1}$ has already been discussed. Recall that it consists of two disconnected components: an infinite line bijective with \mathbb{Z} and an isolated point.

Case 4. If we cut along the other separating curve aca , we obtain two Möbius strips, each with a single marked point. The curve complex of each of these surfaces is isomorphic to that of $N_{1,2} = \mathbb{RP}_2^2$, which also has already been found to consist of two disconnected vertices.

Ideally, we would like to understand the global structure of the complex $\mathcal{C}(N_{2,0,2})$, but as we have already seen this is quite the challenge. However, since we have an understanding locally, we may try to extend this to get a view of the bigger picture. By referring to Figures 12 and 13, we can argue using bigons that the curves in \mathcal{A} form a line that from left to right is $b, baba^{-1}, a, aca$.

6 Injectivity of Natural Homomorphism

We now turn to a discussion of the natural homomorphism between the mapping class group of a surface and the automorphism group of its curve complex. Recall the definition of the natural homomorphism:

Definition 10. Let S be some surface with mapping class group $\text{Mod}(S)$. Denote by $\text{Aut } \mathcal{C}(S)$ the automorphism group of the curve complex of S . Each element f of $\text{Mod}(S)$ induces an automorphism g of $\mathcal{C}(S)$. The natural homomorphism is the map $\phi : \text{Mod}(S) \rightarrow \text{Aut } \mathcal{C}(S)$ such that $\phi(f) = g$.

Further, recall the definition of a mapping class group:

Definition 11 ([Kor02]). Let N be a closed connected surface, possibly with boundaries and punctures. We define the **mapping class group** $\text{Mod}(N)$ as the group of isotopy classes of diffeomorphisms of N that fix the boundary components pointwise. In other words,

$$\text{Mod}(N) = \text{Diff}(N, \partial N) / \text{Diff}_0(N, \partial N)$$

where $\text{Diff}_0(N, \partial N)$ is the subgroup of diffeomorphisms isotopic to the identity.

In [AK14], written by Atalan and Korkmaz, it seemed like they used a slightly different definition for the mapping class group, defining $\text{Mod}(S)$ without the restriction of fixing the boundary components pointwise. The case where the boundary components are fixed pointwise is defined to be

the pure mapping class group, $PMod(N)$. From their definition of the mapping class group, they then proved that $\phi : Mod(N_{g,b}) \rightarrow Aut \mathcal{C}(N_{g,b})$ is an isomorphism for $g + b \geq 5$. In the final section of [AK14], they discussed some sporadic cases for when $g + b < 5$, but do not discuss the cases where $g + b = 4$.

In [Sze19], Szepietowski proved that the natural homomorphism ϕ is an isomorphism for $N_{1,3}$ (the projective plane with three holes). Using this result, we were able to produce the following theorem by following the definition of $Mod(S)$ given in [AK14].

Theorem 4. *Let $N = N_{g,b}$ be a nonorientable surface of genus g and b boundary components such that $g+b = 4$. Then the natural map $\phi : Mod(N) \rightarrow Aut \mathcal{C}(N)$ is injective.*

Proof. The proof of this fact is almost identical to the injectivity proof in [AK14]. It uses the same trick of cutting along a one sided curve to obtain a surface of one more boundary component and one lower genus.

First we will prove the case of $N = N_{2,2}$. Let $f \in Mod(N)$ such that f acts trivially on $\mathcal{C}(N)$. Since f acts trivially on $\mathcal{C}(N)$, it fixes all vertices of $\mathcal{C}(N)$. Let F be a diffeomorphism representing f (so $F \in f$).

We will show that F is isotopic to the identity on N , which will imply that ϕ is an injective homomorphism.

Let a be a one-sided simple closed curve on N . Then N_a , the surface obtained from N by cutting along a , is a surface of genus $g - 1$ with $b + 1$ boundary components.

Let α be the isotopy class of a ; that is, $a \in \alpha$. We then have that $f(\alpha) = \alpha$. Therefore, $F(a)$ is isotopic to a .

Choose a diffeomorphism G , isotopic to the identity, such that $G(F(a)) = a$. Let $H = G \circ F$. Let $H_a : N_a \rightarrow N_a$ be the diffeomorphism on N_a induced by H . We can see that the isotopy class of H_a acts trivially on $\mathcal{C}(N_a)$.

Notice that $N_a = N_{1,3}$. By Szepietowski's result in [Sze19], the natural homomorphism between $Mod(N_a)$ and $Aut \mathcal{C}(N_a)$ is an isomorphism, and is therefore injective. Therefore, the diffeomorphism H_a is isotopic to the identity on $N_a = N_{1,3}$.

We can then choose the isotopy so that it induces an isotopy $N \rightarrow N$ between H and the identity. We have that F is isotopic to H .

Since H is isotopic to the identity, and F is isotopic to H , we have that F is isotopic to the identity. Thus, the theorem is proved for $N_{2,2}$.

Repeating the above argument, except using $N_{2,2}$ in place of $N_{1,3}$, we arrive at the same result for $N_{3,1}$.

Thus, the theorem is proved. ■

We do not know if the natural homomorphism is surjective. Therefore, we do not know if the isomorphism result in [AK14] can be extended.

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