$$= \frac{1}{1 - x\frac{(y+1)^2}{y}} \tag{B9}$$

$$= \sum_{0\leqslant k} x^k \left(\frac{(y+1)^2}{y}\right)^k \tag{B10}$$

$$= \sum_{0\leqslant k} x^k \sum_{0\leqslant l\leqslant 2k} y^{l-k}\binom{2k}{l}, \tag{B11}$$

and thus by identification of terms

$$\mathcal{C}(N, r) = \binom{2N}{r+N} \quad . \tag{B12}$$

Singling out a specific pair and asking how often it is in one of the entangled states given a set of configurations described by values for (N_+, N_0, N_-), we find that its the case in

$$S(N_+, N_-, N_0) = 2C(N_+, N_-, N_0 - 1) = \frac{N_0}{N} C(N_+, N_-, N_0) \tag{B13}$$

configurations. Again we want to find this number for a total amount of pairs N and an magnetization imbalance $r = N_+ - N_-$. Fortunately, we can find the generating function $\mathcal{Z}_\mathcal{S}(x, y, z)$ of $\mathcal{S}(N, r, N_0) = \frac{N_0}{N} C(N, r, N_0)$ by means of $\mathcal{Z}$:

$$\mathcal{Z}_\mathcal{S}(x, y, z) = \int \frac{\mathrm{d}x}{x} z \frac{\partial}{\partial z} \mathcal{Z}(x, y, z). \tag{B14}$$

So, we compute

$$\mathcal{Z}_\mathcal{S}(x, y, z = 1) = \sum_N x^N \sum_r y^r \mathcal{S}(N, r) \tag{B15}$$

$$= \int \frac{\mathrm{d}x}{x} \frac{2xy^2}{(y - x(y+1))^2} \tag{B16}$$

$$= \frac{2y^2}{(y+1)^2} \frac{1}{y - x(y+1)^2} \tag{B17}$$

$$= 2\sum_k x^k \sum_l y^{l-k+1} \binom{2k-2}{l} \tag{B18}$$

$$\Rightarrow \mathcal{S}(N, r) = 2\binom{2N-2}{r+N-1}. \tag{B19}$$

Thus, cutting a single pair contributes

$$\bar{S}(N, r) = \frac{\mathcal{S}(N, r)}{\mathcal{C}(N, r)} \tag{B20}$$

$$= 2\frac{N^2 - r^2}{4N^2 - 2N} \tag{B21}$$

bits of entropy, on average, over all states in a given magnetization sector (cf. Fig. 7).

For the prediction of the average entropy in Fig. 4, we extracted the size of the pairs from the position data, which directly determines how many times a pair is cut, when moving along the chain. The number of cut pairs is then divided by the number of cuts made—which equal the number of spins—and multiplied by the average entropy contributed by cutting a pair, computed here.

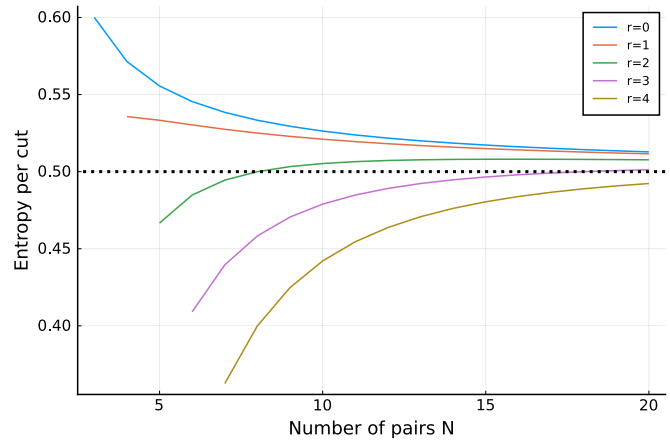

FIG. 7. Entropy value of a single cut for different magnetization sectors.

APPENDIX C: DRAWING BLOCKADED POSITIONS

In the following, we restrict ourselves to N spins in $d = 1$ dimension and measure every distance in units of the blockade radius r_b. We define the density of spins $0 \leqslant \rho = \frac{1}{2W} \leqslant 1$, the corresponding volume of the space $L = \frac{N}{\rho}$, and set out to construct a scheme to efficiently generate a set of independently drawn positions $\{x_i\}$, that respect the blockade condition

$$|x_i - x_j| \geqslant r_b \quad \forall i \neq j. \tag{C1}$$

A priori, all positions are drawn i.i.d. from a uniform distribution over the full space $\mathcal{U}[0, L]$ and the naive way would be to just draw N positions and reject the sample if the blockade condition [Eq. (C1)] is violated. This is essentially equivalent to a random sequential adsorption process where the expected density in $d = 1$ is given by Renyi's parking constant $m \approx 0.748$ [57]. It directly follows that the rejection rate will become essentially 1 for any $\rho > m$ and we certainly will not get close to the fully ordered regime.

To circumvent this problem, we parameterize the positions like

$$x_i = is + \sigma_i, \tag{C2}$$

where $s = \frac{1}{\rho} = 2W$ is the mean interspin distance and $\sigma_i \sim \mathcal{U}[-\sigma, \sigma]$ are i.i.d. random variables. For $\sigma = \frac{L}{2}$, this ansatz is certainly equivalent to the naive scheme.

Note that, in the highly ordered case $\rho = 1 - \epsilon$, where ϵ is small, each realization of the experiment looks essentially like a regularly spaced chain with $s = \frac{1}{1-\epsilon} \approx r_b(1 + \epsilon)$ where each site has small fluctuations around the mean. This means, in this limit, we get away with choosing $\sigma \approx \epsilon$.