product state, discrete TWA (dTWA) exploits this to dramatically improve accuracy [35] and to capture quantum revivals [16,36–45].

Another extension aims at incorporating more quantum interactions into the equations of motion, which in traditional TWA are essentially mean-field equations for single particles [46]. This so-called cluster TWA (cTWA) does so by grouping spins together into clusters and then assigning classical variables to all degrees of freedom within these clusters [47]. Thus, all quantum interaction within a cluster a treated exactly and only interactions between clusters are approximated semiclassically. In the limit of clusters consisting of single spins, cTWA is identical to (d)TWA while in the opposing limit, where the whole system constitutes a single cluster the exact quantum evolution is recovered. Thus, one has a tuning parameter to assess the convergence of the simulation which the usual (d)TWA lacks. While in principle cTWA is compatible with the discrete phase-space formulation, literature on their combination is quite sparse. A conceptual precursor, dubbed GDTWA, exists in [48] where the discrete sampling was extended to larger SU(N) spins. In a recent paper a variant of discrete sampling is applied to a Bose-Hubbard model [49].

In this paper, we present a generalization of both cTWA and dTWA combining the discrete sampling scheme of the latter with the capability of treating clusters of spins of the former, which we term dcTWA. We then systematically evaluate the performance of these methods in the context of quench dynamics for bond disordered XX and XXZ long-range interacting spin-$\frac{1}{2}$ models. More precisely, we study the dynamics of an initial Néel state by means of the decay of the staggered magnetization and the buildup of Rényi entropy in a two-spin subsystem for different interaction ranges and disorder strengths and compare the results from the semi-classical methods to exact diagonalization. While in the weakly disordered regime a bigger cluster size is beneficial generally, we find that in the strongly disordered regime the physics is well captured by clusters of size 2 if they are chosen following a pairing rule known from the real-space renormalization group (RSRG) approach to bond-disordered systems. Our analysis of the statistical uncertainties reveals that although the averaged results from cTWA and dcTWA are similar, dcTWA shows less sampling noise and thus converges faster.

II. MODEL AND METHODS

A. Model

We study the behavior of a disordered spin chain with long-range interactions, described by the Hamiltonian

$$H = \sum_{i<j} J_{ij}\left(\hat{s}_i^x\hat{s}_j^x + \hat{s}_i^y\hat{s}_j^y + \Delta\hat{s}_i^z\hat{s}_j^z\right), \tag{1}$$

where N spins ($\hat{s}_i = \frac{1}{2}\hat{\sigma}_i$) are randomly positioned at locations r_i along a lattice of length L with lattice spacing a, resulting in a density $f = N/L$. The interactions J_{ij} between pairs of sites i and j are long range, characterized by a power-law decay with parameter α: $J_{ij} = J_0|(r_i - r_j)/a|^{-\alpha}$. Throughout our study we set $J_0 = 1$ and $a = 1$, and employ open boundary conditions.

The disorder in this model arises from the random arrangement of spins along the chain, leading to different spin-spin couplings. Previous studies of this model with $\Delta = 0$ focused on the entanglement entropy in the ground state [50] and excited states [51] as well as the dynamical growth of the entanglement entropy after a quench [52]. These studies found good agreement between numerically exact results and analytical calculations based on the real-space renormalization group for low density ($f = 10\%$) and interaction exponents $\alpha \gtrsim 1.8$.

We explore the system dynamics by initializing it in a Néel state and subsequently computing dynamic observables. These observables encompass the staggered magnetization $M^{\text{st}}(t) = \sum_i (-1)^i \langle\hat{\sigma}_i^z(t)\rangle/N$ and the Rényi-2 entropy $S_2(t)$ evaluated over a two-spin subsystem. The Rényi-2 entropy belongs to a continuum of entropy measures defined as $S_\gamma[\hat{\rho}_A(t)] = \frac{1}{1-\gamma}\log_2(\text{tr}[\hat{\rho}_A(t)^\gamma])$, where $\gamma \geqslant 1$. In this context, $\hat{\rho}_A(t) \equiv \text{Tr}_B\hat{\rho}(t)$ signifies the reduced density matrix associated with a subsystem A, and $\hat{\rho}(t)$ represents the density matrix of the entire system. Expanding the two-site reduced density matrix $\hat{\rho}_{ij} = \frac{1}{4}\sum_{\alpha\beta}\langle\hat{\sigma}_i^\alpha\hat{\sigma}_j^\beta\rangle\hat{\sigma}_i^\alpha\hat{\sigma}_j^\beta$ in a basis of Pauli strings gives a clear recipe for extracting the Rényi-2 entropy from the expectation values of observables:

$$S_2[\hat{\rho}_{ij}(t)] = -\log_2(\text{Tr}[\hat{\rho}_{ij}(t)]^2) \tag{2}$$

$$= 2 - \log_2\left(\sum_{\alpha\beta}\left\langle\hat{\sigma}_i^\alpha\hat{\sigma}_j^\beta\right\rangle^2\right), \tag{3}$$

where we used the trace orthogonality of the Pauli strings. This expression has a clear physical meaning: The more correlations the subsystem retains after tracing out the environment, the weaker the entanglement is.

B. Cluster truncated Wigner approximation (cTWA)

Phase-space methods are powerful tools for simulating quantum system dynamics close to the classical limit. These methods have applications across various scientific domains, including quantum chemistry, optics, and condensed matter physics [28,34]. Among them, the TWA maps quantum degrees of freedom onto classical phase-space variables following classical equations of motion as in a mean-field treatment. Quantum fluctuations are taken into account by Monte Carlo sampling of initial conditions from the Wigner function, which guarantees accuracy on short timescales. However, for quantum systems close to the classical limit, e.g., highly occupied bosonic modes or collective spin models, TWA has been found to yield accurate results even at late times [28].

When applying TWA to spin systems, usually one considers 3 degrees of freedom per spin: its X, Y, and Z magnetization [35]. Mapping these to classical variables treats all quantum interactions between spins on a mean-field level, which is justified if the interactions are either weak or very long range and thus average out [33]. One avenue of incorporating more quantum effects into the dynamics, known as cluster TWA (cTWA), uses the degrees of freedom of clusters of spins instead of just the single-spin ones [47]. In effect, this means all quantum dynamics within a cluster is computed