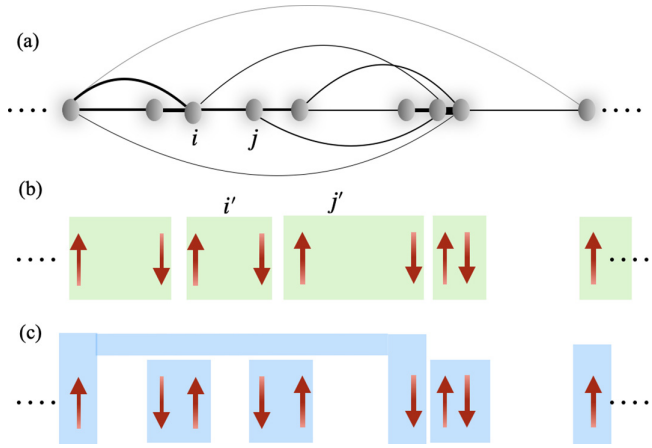

FIG. 1. (a) This diagram shows a long-range bond disorder spin chain where spins are randomly positioned along a lattice. The couplings are represented by solid lines J_{ij}, while stronger bonds are indicated by thicker lines. (b) The initial state is the Néel state, represented as $|\Psi_0\rangle = |\uparrow\downarrow \ldots \uparrow\downarrow\rangle$, where each arrow represents the spin direction. The green shaded box illustrates a naive clustering of spins into clusters of size 2. (c) Clustering inspired by the real-space renormalization group, as explained in the text.

exactly and only the interaction between clusters is approximated on a mean-field level. In the limit of a single cluster encompassing the whole system, cTWA is equivalent to an exact solution. Conversely, in the limit of clusters of single spins, cTWA reduces to regular TWA. Thus, cTWA offers a tuning parameter to steadily tune between TWA and an exact solution by means of increasing the cluster size. In order to be self-contained, we provide an overview of this method. For a more detailed introduction, we refer the reader to the paper by Wurtz *et al.* [47].

To illustrate the cTWA method, consider a system of interacting spins-$\frac{1}{2}$ described by the Hamiltonian

$$H = \sum_{ij} J^{ij}_{ab} \hat{\sigma}^i_a \hat{\sigma}^j_b + \sum_j B^j_a \hat{\sigma}^j_a. \tag{4}$$

Here, $a, b \in x, y, z$ are the indices of Pauli matrices, and i, j denote distinct spins on the lattice. The couplings J^{ij}_{ab} and fields B^j_a can be either short or long range.

The following steps outline the implementation of the operator cTWA:

(i) Divide the system into clusters indexed by $[i']$, as shown in Fig. 1(b). Define a complete operator basis $\{\hat{X}^{[i']}_p\}$, $p = 0, \ldots, D^2 - 1$ for the Hilbert space of each cluster, where $D = 2^n$ is the dimension of the Hilbert space and n the number of spins making up the cluster. Ensure that the basis operators are trace orthogonal and satisfy $\mathrm{Tr}[\hat{X}^{[i']}_p \hat{X}^{[j']}_q] = D\delta_{pq}\delta^{[i'][j']}$. Then any operator $O^{[i']}$ inside a cluster $[i']$ can be written as linear combination of the basis operators $\hat{O}^{[i']} = \sum_p o_p \hat{X}^{[i']}_p$.

(ii) Define structure constants f_{pqr} as

$$\left[\hat{X}^{[i']}_p, \hat{X}^{[j']}_q\right] = i f_{pqr} \delta_{[i'][j']} \hat{X}^{[i']}_r, \tag{5}$$

which project commutators onto the basis spanned by $\{\hat{X}^{[i']}_p\}$.

(iii) Express the Hamiltonian in terms of cluster operators $\hat{X}^{i'}_\alpha$.

$$\hat{H} = \sum_{[i'][j']} \mathbf{J}^{[i'][j']}_{pq} \hat{X}^{[i']}_p \hat{X}^{[j']}_q + \sum_{[j']} \mathbf{B}^{[j']}_p \hat{X}^{[j']}_p. \tag{6}$$

The interactions $\mathbf{J}$ and fields $\mathbf{B}$ generally differ from the original parameters J^{ij}_{ab}, B^j_a. For instance, local fields now encompass connections among spins residing within a particular cluster, given that an operator $\sigma^i_a \sigma^j_p$ becomes linear in $\hat{X}^{[j']}_p$ when both spins i and j are part of the same cluster $[j']$.

(iv) Associate basis operators $\hat{X}^{[i']}_p$ with classical phase-space variables $x^{[i']}_\alpha$ satisfying canonical Poisson bracket relation $\{x^{[i']}_p, x^{[j']}_q\} = i f_{pqr} \delta_{[i'][j']} x^{[i']}_r$ defined by the structure constants

$$\hat{X}^{[i']}_p \to x^{[i']}_p - \frac{i}{2} x^{[i']}_q f_{pqr} \frac{\partial}{\partial x^{[i']}_r}. \tag{7}$$

(v) Represent the Hamiltonian and observables as functions of classical phase-space variables.

$$\hat{O}^{[i']} = \sum_p o_p \hat{X}^{[i']}_p \to O_{\mathbf{W}}(\{x\}) = \sum_p o_p x^{[i']}_p \tag{8}$$

with $o_p = \frac{1}{D}\mathrm{Tr}[\hat{O}\hat{X}^{[i']}_p]$, and

$$\hat{H} \to H_{\mathbf{W}} = \sum_{[i'][j']} \mathbf{J}^{[i'][j']}_{pq} x^{[i']}_p x^{[j']}_q + \sum_{[j']} \mathbf{B}^{[j']}_p x^{[j']}_p, \tag{9}$$

where index $\mathbf{W}$ indicates that this is the Weyl symbol corresponding to symmetric operator ordering.

(vi) Find or approximate the Weyl symbol of the initial state, i.e., its Wigner function. While it can assume negative values, we require that it is completely positive and thus can be thought of as a probability distribution. We present two possible definitions for the Wigner function below this implementation guide.

(vii) Solve the classical equations of motion for the phase-space variables:

$$\frac{dx^{[i']}_p(t)}{dt} = -\left\{x^{[i']}_p, H_{\mathbf{W}}\right\} = f_{pqr} \frac{\partial H_{\mathbf{W}}}{\partial x^{[i']}_q} x^{[i']}_r. \tag{10}$$

(viii) Find expectation values of observables by averaging the corresponding classical functions over phase-space points sampled from the Wigner function $\langle \hat{O}(t) \rangle = \lim_{M\to\infty} \frac{1}{M} \sum_m^M O_{\mathbf{W}}(\{x(t)\}_m)$, where M denotes the number of samples.

1. Gaussian Wigner function

Wurtz *et al.* [47] defined an approximate Gaussian Wigner function $\mathcal{W}(\{x\})$ describing the initial conditions for the system with the only requirement that the initial state factorizes between clusters such that $\mathcal{W}(\{x\}) = \prod_{[i']} \mathcal{W}^{[i']}(\{x^{[i']}_\alpha\})$, where

$$\mathcal{W}^{[i']}(x^{[i']}) = \frac{1}{Z} \exp\left[\left(x_p - \rho^{[i']}_p\right) \Sigma^{[i']}_{pq} \left(x_p - \rho^{[i']}_q\right)\right] \tag{11}$$