TABLE I. Eigensystem of H_{pair}.

State k	Energy E_k	Vector $\|k\rangle$
1	$2-\Delta$	$\sqrt{2}^{-1}(\|\uparrow\downarrow\rangle + \|\downarrow\uparrow\rangle)$
2	Δ	$\|\uparrow\uparrow\rangle$
3	Δ	$\|\downarrow\downarrow\rangle$
4	$-2-\Delta$	$\sqrt{2}^{-1}(\|\uparrow\downarrow\rangle - \|\downarrow\uparrow\rangle)$

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APPENDIX A: DERIVATION OF PAIR PICTURE

Here we derive the pair model of the main text by means of Schrieffer-Wolff transformations [63]. Starting with the full Hamiltonian of the system,

$$\hat{H} = \frac{1}{2}\sum_{i\neq j} J_{ij}\underbrace{\left(\hat{S}_x^{(i)}\hat{S}_x^{(j)} + \hat{S}_y^{(i)}\hat{S}_y^{(j)} + \Delta\hat{S}_z^{(i)}\hat{S}_z^{(j)}\right)}_{\equiv H_{\text{pair}}^{(i)(j)}}. \tag{A1}$$

Suppose without loss of generality that $J_{12} \gg J_{1j}, J_{2j}$ and set $H_0 = J_{12}H_{\text{pair}}^{(1)(2)}$ and $V = H_{XXZ} - H_0$. We label the eigenvectors and eigenenergies of H_{pair} as shown in Table I.

The projectors on these states are consequently named $P_k = |k\rangle\langle k| \otimes \mathbb{1}$, but since the middle two states are degenerate, we need to use the projector on the full eigenspace and call it $P_{23} = P_2 + P_3$.

To first order, only diagonal terms $P_k V P_k$ contribute, which in this case means the pair decouples and only an effective Ising term remains:

$$\hat{H} = \sum_{i,j} J_{ij}\hat{H}_{\text{pair}}^{(i)(j)} \tag{A2}$$

$$\approx J_{12}\hat{H}_{\text{pair}}^{(1)(2)} + \sum_{i,j>2} J_{ij}\hat{H}_{\text{pair}}^{(i)(j)} + \hat{S}_z^{(1)(2)}\sum_{i>2}\tilde{\Delta}_i\hat{S}_z^{(i)} + O(\hat{V}^2), \tag{A3}$$

where $2\hat{S}_z^{(1)(2)} = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| - |\downarrow\downarrow\rangle\langle\downarrow\downarrow|$ is akin to a spin-1 magnetization operator and $\tilde{\Delta}_i = \Delta(J_{1i} + J_{2i})$ is the renormalized Ising coupling. Note that this first order term lifts the apparent degeneracy of the $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ states. This elimination is a good approximation if the interaction within the pair is much stronger than any other interaction between a spin of the pair and some other spin.

We can now repeat this elimination step with remaining spins by incorporating the effective Ising terms into V. This is justified because its coupling is small and is already first-order perturbation theory, and thus including it into the zeroth order of the next pair would mix expansion orders inconsistently.

Further eliminations now generate effective Ising terms between the states $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ of the eliminated pairs. After pairing up all spins, we find

$$\hat{H} = \sum_{i,j} J_{ij}\hat{H}_{\text{pair}}^{(i)(j)} \tag{A4}$$

$$\approx \sum_{\langle i,j\rangle} J_{ij}\hat{H}_{\text{pair}}^{(i)(j)} + \sum_{\langle i,j\rangle,\langle i',j'\rangle}\tilde{\Delta}_{(i,j),(i',j')}\hat{S}_z^{(i)(j)}\hat{S}_z^{(i')(j')} \tag{A5}$$

where the sum over $\langle i,j\rangle$ denotes pairs of spins and $\tilde{\Delta}_{(i,j),(i',j')} = \Delta(J_{i,i'} + J_{j,i'} + J_{i,j'} + J_{j,j'})$.

Also note that with each elimination step, the mean interparticle distance grows and thus the disorder in the system increases [64,65] making it more likely for later elimination steps to be good approximations.

APPENDIX B: PAIR ENTROPY IN A SPECIFIC MAGNETIZATION SECTOR

Averaged over all states, each cut separating a pair gives an average entropy of $\frac{1}{2}$, since two of the pair's eigenstates are fully entangled and the other two possess no entanglement. However, when we consider a sector of fixed magnetization, this simple argument no longer holds as there are now dependencies among the eigenstates given by the external constraint. Sectors around zero magnetization will have more entropy on average and strongly magnetized sectors less, simply because the strongest magnetized eigenstates possess no entropy.

Given N the number pairs of spins where N_+, N_-, and N_0 pairs occupy the states $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, and $|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle$, we find the number of possible configuration with these amounts to be

$$C(N_+, N_-, N_0) = \binom{N}{N_0}\binom{N-N_0}{N_+}2^{N_0}. \tag{B1}$$

In the end, we need the number of configurations $\mathcal{C}(N, r) = \sum_{N_0}\mathcal{C}(N, r, N_0)$ given a total amount of pairs N and a magnetization imbalance $r = N_+ - N_-$, where

$$\mathcal{C}(N, r, N_0) = \sum_{0\leqslant N_+, N_-} C(N_+, N_-, N_0)\delta_{N, N_+ + N_- + N_0}\delta_{r, N_+ - N_-}. \tag{B2}$$

To evaluate this expression, we compute the generating function

$$\mathcal{Z}(x,y,z) = \sum_{N>0} x^N \sum_{-N\leqslant r\leqslant N} y^r \sum_{N_0>0} z^{N_0}\mathcal{C}(N, r, N_0) \tag{B3}$$

$$= \sum_{0\leqslant N_+, N_0, N_-} x^{N_+ + N_0 + N_-} y^{N_+ - N_-} z^{N_0} C(N_+, N_-, N_0) \tag{B4}$$

$$= \sum_{0\leqslant N_-}\left(\frac{x}{y}\right)^{N_-}\sum_{0\leqslant N_+}(xy)^{N_+}\binom{N_+ + N_-}{N_+} \times \sum_{N_0}\binom{N}{N_0}(2z)^{N_0} \tag{B5}$$

$$= \frac{y}{y - 2xyz - xy^2 - x}, \tag{B6}$$

where we used the fact that $(1-x)^{-k-1} = \sum_n \binom{n+k}{k}x^n$ twice and then a geometric series.

From that, it follows directly that

$$\mathcal{Z}(x,y,1) = \sum_{N>0} x^N \sum_{-N\leqslant r\leqslant N} y^r \mathcal{C}(N, r) \tag{B7}$$

$$= \frac{y}{y - 2xy - xy^2 - x} \tag{B8}$$