Spatial Point Processes

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What is a spatial point pattern?

- A spatial point pattern is a dataset giving the observed spatial locations of things or events.
- One important task is to identify spatial trends in the density of points.

Types of points

- The points in a point pattern may carry all kind of attributes.
- These attributes are called **marks**

Example

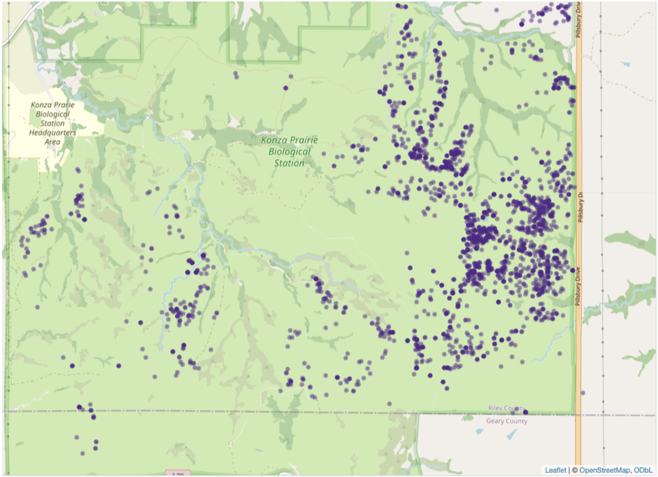


Fig 1. Location points for Grasshopper Sparrows in the Konza Prairie Biological Station located in Kansas from 2013 to 2021.

Definition

Point Process definition

A point process is a random mechanism whose outcome is a point pattern.

- We want to know the mechanism behind the **generation of the points**, not the points themselves.

Basics

- Suppose Area A is a subset of a Euclidean space; Then we can define the event ϵ of a point X_i for $i = 1, 2, ..., n$ as:

$$\epsilon_{X_n}(A) = \begin{cases} 1 & \text{if } X_n \in A \\ 0 & \text{if } X_n \notin A \end{cases}$$

- If we define the counting measure N, we can get the random number of points which fall in the Area A:

$$N(A) = \sum_n \epsilon_{X_n(A)}$$

Intensity

- We define intensity λ as the expected number of points falling in the area A.

$$\mu(A) = EN(A) = \lambda$$

Homogeneous Poisson Point Process (HPPP)

- Also called *Complete Spatial Randomness (CSR)*
- Has two key properties:
    - **Homogeneity:** Points have no preference for any spatial location
    - **Independence:** Points have no influence on the location of other points

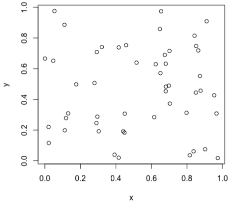
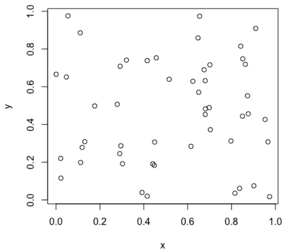


Fig 2. Simulation of *CSR* with 50 points

Building a Homogeneous Point Process

```
set.seed(2024)
x <- runif(50, 0, 1)
y <- runif(50, 0, 1)
plot(x,y)
```



Inhomogeneous Poisson Point Process (IPPP)

- We define a inhomogeneous Poisson point process (IPPP) when intensity λ is **spatially varying**.
- Key properties:
  - **Intensity function:** The expected number of points falling in a region A is the integral $\mu = \int_A \lambda(u)du$
  - **Independence:** Random patterns are independent of each other

Recall the example

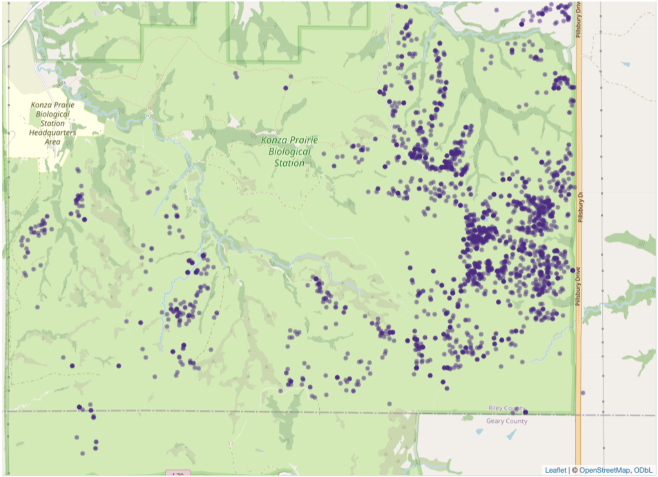
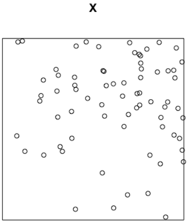


Fig 1. Location points for Grasshopper Sparrows in the Konza Prairie Biological Station located in Kansas from 2013 to 2021.

Building a Inhomogeneous Point Process

- Lewis-Shedler rejection method (1979):
  - We start by simulating a homogeneous Poisson process with intensity λ_{max}.
  - We retain each event ϵ of the homogeneous process with probability $\frac{\lambda(\epsilon)}{\lambda_{\text{max}}}$.



Building a Inhomogeneous Point Process

- First, we will create our intensity function `lambda`. In this case, $(xy)^2$:

```
set.seed(2024)
lambda <- function(x,y){
(x*y)*(x*y)
}
```

Building a Inhomogeneous Point Process

- We create a matrix `points` for the points that we will generate:

```
points <- matrix(NA, ncol = 2, n = 50)
```

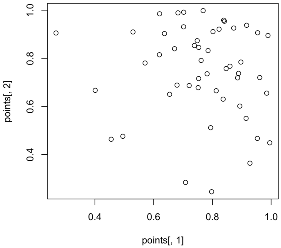
- We generate the for loop to create our points:

```
for(i in 1:50){
repeat{
x <- runif(1, 0, 1)
y <- runif(1, 0, 1)
if(runif(1)<lambda(x,y)){
points[i,] <- c(x,y)
break}
} }
```

Building a Inhomogeneous Point Process

- We plot matrix `points` for the points that we generate:

```
plot(points[,1],points[,2])
```



Getting into the Point Process itself

- We know that our points are being generated from an intensity λ. Can we visualize our intensity function?
- We will start with a 10 x 10 grid:

```
# creating grid
x <- seq(from = 0.1, to = 1, by = 0.1)
y <- seq(from = 0.1, to = 1, by = 0.1)

# giving values to all grids
newp <- outer(lambda(x,1), lambda(1,y))
newp <- apply(newp, 2, rev)
newp
```

Getting into the Point Process itself

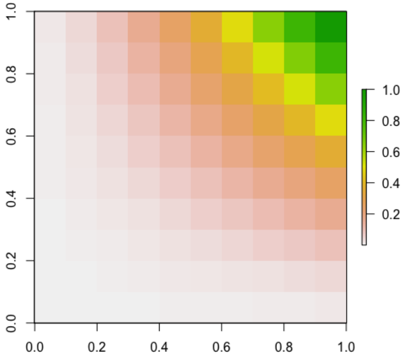
- Now we use the `raster` package:

```
# loading library
library(raster)
r <- raster(nrows=10, ncols=10, xmn=0,
xmx=1, ymn=0, ymx=1)
vals <- newp
r <- setValues(r, vals)
```

Getting into the Point Process itself

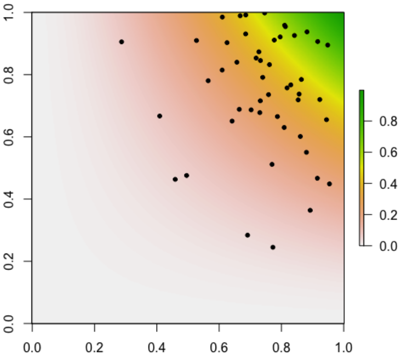
- And we plot!

```
plot(r)
```

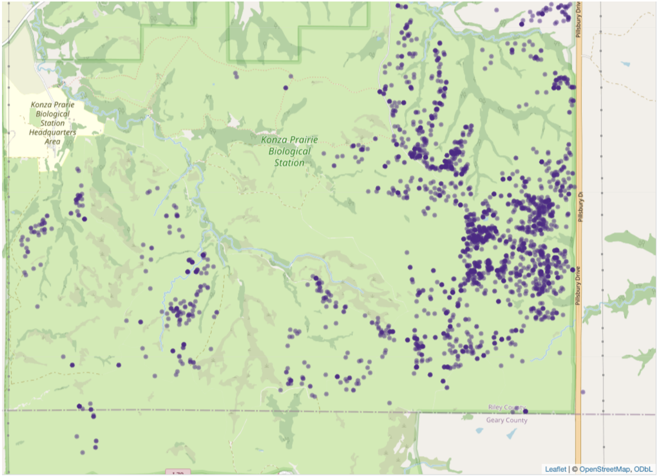


Getting into the Point Process itself

- The more grids, the smoother it gets!



Recall the example...once again



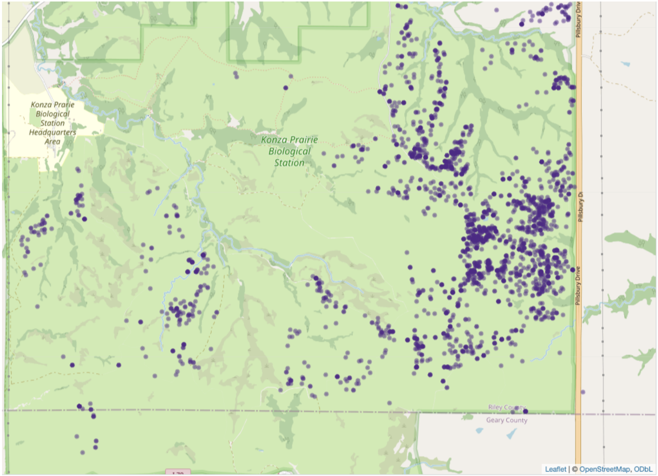
Recall the example...once again

$$\theta_i \sim \text{Bern}(p_i)$$

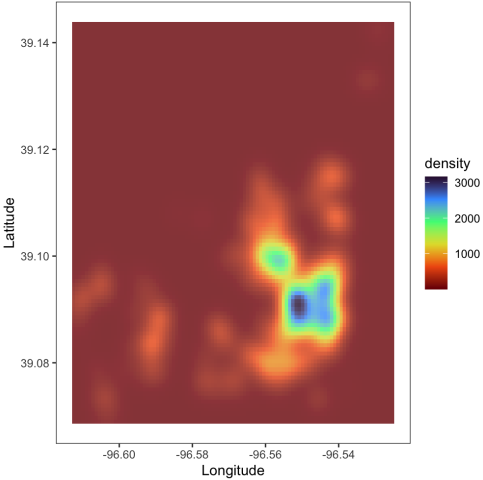
$$p_i \sim \text{IPPP}(\lambda_i)$$

$$\ln \lambda_i = \beta_0 + \beta_1 X_i + \beta_2 Y_i$$

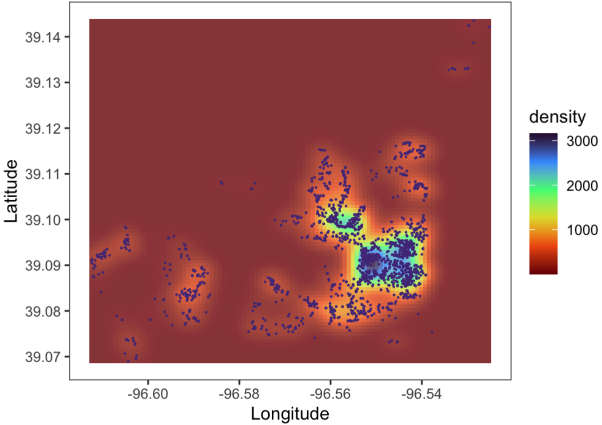
Recall the example...once again



Intensity function with a kernel density function



Our full model...for now



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